Unsteady coating flow on a rotating cylinder in the presence of an irrotational airflow with circulation

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ABSTRACT

Unsteady two-dimensional coating flow of a thin film of a viscous fluid on the outside of a uniformly rotating horizontal circular cylinder in the presence of a steady two-dimensional irrotational airflow with circulation is considered. The analysis of this problem by Newell and Viljoen [Phys. Fluids 31(3), 034106 (2019)], who sought to generalize the work of Hinch and Kelmanson [Proc. R. Soc. London, Ser. A 459(2033), 1193–1213 (2003)] to include the effect of the airflow, is revisited. In contrast with the claim of Newell and Viljoen that the flow is conditionally unstable (in the sense that the solution for the film thickness grows without bound for certain values of the physical parameters), it is shown that, in fact, the film remains unconditionally stable in the presence of the airflow.

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I. INTRODUCTION

Since the publication of the seminal papers by Moffatt1 and Pukhnachev2 in 1977, coating flow and rimming flow (i.e., flow of a film of fluid on the outside and inside, respectively, of a rotating solid horizontal cylinder) have come to be regarded as paradigm problems in the study of free-surface flows of viscous fluids. A rather large literature has grown up concerning these flows, including the study of discontinuous “shock” solutions in rimming flow by Johnson, the pioneering numerical investigation of coating flow by Hansen and Kelmanson, the study of “curtain” solutions, which are unbounded at the top and the bottom of the cylinder, by Duffy and Wilson, the study of the effect of capillarity in rimming flow by Ashmore et al., studies of the large-time dynamics of unsteady coating flow by Hinch and Kelmanson, Hinch et al., Kelmanson, and Groh and Kelmanson, the numerical investigations of two- and three-dimensional coating flow by Evans et al., the bifurcation analysis of coating flow by Lin et al., and the discovery of new branches of steady solutions in coating and rimming flow by Lopes et al.. In addition, there have been many extensions to the basic problems, including the studies of the effect of a uniform azimuthal shear stress on the free surface of the film by Black and Villegas-Díaz et al., the studies of coating flow on elliptical cylinders by Hunt and Li et al., the study of thermoviscous effects in coating and rimming flow by Leslie et al., the study of coating flow on patterned cylinders by Li et al., the studies of unsteady and steady coating flow in the presence of an irrotational airflow with circulation by Newell and Viljoen and Mitchell et al., respectively, the study of a “thick-film” model for coating flow by Wray and Cimpeanu, and the study of coating flow of a film laden with colloidal particles in the presence of solvent evaporation by Parrish and Kumar.

Studies of the stability of coating and/or rimming flow in various parameter regimes have been undertaken. Hosoi and Mahadevan found numerically that two-dimensional solutions for rimming flow can be stable even to three-dimensional perturbations in the presence of capillarity and weak inertia. Peterson et al. determined regions of parameter space for which unsteady solutions of the Stokes equation for coating flow approach steady states at large times. O’Brien confirmed the conclusion of Benjamin et al. that subcritical solutions for rimming flow (that is, those for which the mass of fluid is less than a critical maximum value) are neutrally stable to small two-dimensional perturbations, and O’Brien showed subsequently that weak capillarity can render these solutions stable. Villegas-Díaz et al. used kinematic-wave theory to show that subcritical solutions are stable and that solutions with a shock on the rising side of the cylinder are...
stable if the shock is in the lower quadrant but unstable if it is in the upper quadrant. Benilov et al.\textsuperscript{30} found that the linearized equations for rimming flow have non-harmonic solutions that develop singularities in finite time, and Benilov et al.\textsuperscript{11} showed subsequently that the inclusion of capillarity precludes these singular solutions and renders most of the eigenmodes stable. Groh and Kelmanson\textsuperscript{27} revealed previously undiscovered contributions to capillary decay and gravitational drift in coating flow. Pougeth and Frigaard\textsuperscript{3} showed that in rimming flow capillarity may stabilize some modes but destabilize others.

In the present work, we are concerned with the flow of a thin film of a viscous fluid on a moving substrate in the presence of an airflow. As Newell and Viljoen\textsuperscript{23} describe, just such a situation arises in the operation of a novel rotary pesticide applicator for crops, comprising a rotating cylinder covered in a film of fluid (pesticide) that brushes the undersides of leaves of plants as it is moved through the foliage. Similar situations include the jet-wiping (or air-knife) coating process in which impinging jets of air are used to control the thickness of a film of fluid (see, for example, Mendez et al.\textsuperscript{14} and Barreiro-Villaverde et al.\textsuperscript{15}) and the interaction between the airflow within and the film of oil on the inside of the outer shaft of the bearing chamber in a rapidly rotating aeroengine, which is a key element of the overall performance of the engine (see, for example, Farrall et al.\textsuperscript{36}, Noakes et al.\textsuperscript{37} and Williams et al.\textsuperscript{38}).

In the present study, we model the air as inviscid, so that it exerts a non-uniform pressure but no shear stress on the film. Specifically, we investigate unsteady two-dimensional coating flow of a thin film of a viscous fluid on the outside of a uniformly rotating solid horizontal circular cylinder in the presence of a steady two-dimensional irrotational airflow with circulation. Three of the above-mentioned papers are particularly relevant to the present study, namely, those by Hinch and Kelmanson.\textsuperscript{7} Newell and Viljoen,\textsuperscript{23} and Mitchell et al.\textsuperscript{22} Very recently, Mitchell et al.\textsuperscript{22} investigated the steady version of the present problem in the absence of capillarity. They classified the possible steady solutions that can occur and proved (by a straightforward generalization of the argument of O’Brien\textsuperscript{37}) that subcritical solutions remain neutrally stable to small two-dimensional perturbations in the presence of the airflow. Earlier, Hinch and Kelmanson\textsuperscript{7} constructed the asymptotic solution for unsteady coating flow in the absence of an airflow in the case in which the effects of gravity and of capillarity are weak compared with those of viscous shear; in particular, they showed that at very large times the solution decays to a steady state in which the thickness of the film exhibits a gravity-induced phase lag relative to the solid cylinder. More recently, Newell and Viljoen\textsuperscript{15} sought to generalize the work of Hinch and Kelmanson\textsuperscript{7} to include the effect of the airflow; specifically, they sought to obtain the corresponding asymptotic solution to the present problem and found that it is conditionally unstable (in the sense that it grows without bound at large times for certain values of the physical parameters). However, the work of Newell and Viljoen\textsuperscript{15} is compromised by a number of unfortunate errors, and so in the present study we revisit their analysis. In particular, we shall show that, in fact, the film remains unconditionally stable in the presence of the airflow.

**II. PROBLEM FORMULATION**

We consider unsteady two-dimensional coating flow of a thin film of incompressible viscous fluid of constant density $\rho$ and viscosity $\mu$ on a solid horizontal circular cylinder of radius $a$ rotating anticlockwise with uniform angular speed $\Omega$ ($>0$) in the presence of a steady two-dimensional airflow, as sketched in Fig. 1. Specifically, we take the air to be undergoing steady two-dimensional irrotational flow with uniform horizontal velocity $U_\infty$ from left to right and pressure $p_\infty$ in the far field and a circulation $\kappa$ (measured anticlockwise) around the cylinder. In particular, we assume that since the film is thin, the airflow is unaffected by the presence of the film.

For the airflow around the cylinder to be even approximately irrotational, it is necessary that the Reynolds number based on the circumferential speed of the cylinder, $\rho_a a^2 \Omega / \mu$, is large, where $\mu$ is the viscosity of the air, is large, and that the boundary layer that forms in the air remains attached to the cylinder. As Mitchell et al.\textsuperscript{22} describe, the conclusions of a body of analytical and numerical studies of high-Reynolds-number flow around a rotating cylinder without a film of viscous fluid (notably the work by Glauert,\textsuperscript{39} Moore,\textsuperscript{40} Kang et al.,\textsuperscript{41} Stojkovic et al.\textsuperscript{41}, Mittal and Kumar,\textsuperscript{42} and Aljure et al.\textsuperscript{43}) are that the rotation of the cylinder tends to suppress the separation of the boundary layer, that when also the circumferential speed of the cylinder is large compared with the speed of the far-field airflow, $a \Omega \gg U_\infty$, the boundary layer does indeed remain attached all the way around the cylinder, and that the circulation then takes the value $\kappa = 2\pi \alpha^2 \Omega$. The question of what effect the presence of the thin film of viscous fluid may have on the suppression of boundary-layer separation remains open.

Referred to polar coordinates $r, \theta$ with origin on the axis of the cylinder and with $\theta$ measured from the horizontal, the pressure in the film, denoted by $p = p(\theta, t)$, where $t$ denotes time, is given by

$$p = p_\infty + \frac{\sigma}{a} \left( \frac{\partial^2 h}{\partial \theta^2} + h \right) + \frac{\rho_s}{2} \left( \frac{U_\infty^2}{h} - \left( 2 U_\infty \sin \theta - \frac{\kappa}{2 \pi a} \right)^2 \right),$$

(1)

where $h = h(\theta, t)$ is the thickness of the film, $\sigma$ is the constant coefficient of surface tension, and $\rho_s$ is the constant density of the air. The
azimuthal volume flux of fluid per unit axial length in the film, denoted by \( Q = Q(\theta, t) \), is given by

\[
Q = a\Omega h - \frac{h^3}{3\mu} (\rho g \cos \theta + \frac{1}{a} \frac{\partial p}{\partial \theta}),
\]

where \( g \) is the magnitude of the acceleration due to gravity, and the statement of conservation of mass in the film gives the evolution equation for \( h \),

\[
\frac{\partial h}{\partial t} + \frac{1}{a} \frac{\partial Q}{\partial \theta} = 0.
\]

We non-dimensionalize \( t \) with \( \Omega^{-1} \), \( h \) with the (constant) average film thickness \( \bar{h} \), and \( p - p_\infty - \sigma/a \) with \( \rho g a \), and with \( a\Omega \bar{h} \). Then the pressure (1) becomes

\[
p - p_\infty - \sigma/a \text{ with } \rho g a \text{, and with } a\Omega \bar{h}.
\]

The flux (2) becomes

\[
Q = h - \gamma h^3 \left( \cos \theta + \frac{\partial p}{\partial \theta} \right),
\]

that is,

\[
Q = h - \gamma h^3 \cos \theta + 2h^3 \frac{\partial^2 h}{\partial \theta^2} + h - 2\gamma F h \left( K \cos \theta - F \sin 2\theta \right),
\]

and, hence, the governing evolution equation (3) takes the form

\[
\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta} \left[ h - \gamma h^3 \cos \theta + 2h^3 \frac{\partial h}{\partial \theta} + h - 2\gamma F h \left( K \cos \theta - F \sin 2\theta \right) \right] = 0,
\]

where \( \alpha, \gamma, F, \text{ and } K \), which are defined by

\[
\alpha = \frac{\sigma}{3\Omega \mu a}, \quad \gamma = \frac{\rho g a}{3\Omega a} \left( \frac{\bar{h}}{a} \right)^2,
\]

\[
F = \left( \frac{\rho \alpha U_\infty}{\rho g a} \right)^{\frac{1}{2}} \quad \text{and} \quad K = \frac{\kappa}{2\sigma a} \left( \frac{\rho \alpha}{\rho g a} \right)^{\frac{1}{2}}.
\]

are nondimensional measures of surface tension, acceleration due to gravity, the speed of the far-field airflow, and the circulation of the airflow, respectively. Note that \( \alpha \geq 0, \gamma \geq 0, \text{ and } F \geq 0 \), but that \( K \) has the same sign as \( \kappa \), which may be positive, negative, or zero (corresponding to anticlockwise, clockwise, and no circulation, respectively).

The evolution equation (6) is to be solved subject to periodicity conditions on \( h \) and its derivatives, and an initial condition specifying \( h(\theta, 0) \).

In the special case of no far-field airflow, \( F = 0 \), the parameter \( K \) appears in (5) only in a constant contribution \(-K^3/2\) to \( p_0 \), and does not appear in (7) or (8), showing that the evolution of the film is unaffected by a purely circularly symmetric airflow.

Equation (8) is invariant under the transformation

\[
\theta \to -\theta + \pi, \quad K \to -\left( K + \frac{1}{F} \right) \quad (F > 0),
\]

showing that if a free-surface profile \( h(\theta, t) \) is a solution of (8) corresponding to a given value of the circulation \( K \), then the phase-shifted profile \( h(\theta + \pi, t) \) is a solution corresponding to the circulation \(-[K + (1/F)]\). Moreover, although Eq. (8) involves all four of the parameters \( \alpha, F, K, \text{ and } \gamma \), for \( F \neq 0 \), it may be reduced to the form

\[
\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta} \left[ h - 2\gamma F h \left( h^3 \cos \theta - 2h^2 \frac{\partial h}{\partial \theta} + h \right) + \frac{1}{2} \gamma h^3 \sin 2\theta \right] = 0,
\]

involving only \( \alpha \) and two new parameters \( \beta \) and \( \gamma \) defined by

\[
\beta = 1 + 2KF, \quad \gamma = 4F^2 \gamma.
\]

which may be regarded as reduced measures of \( K \) and \( \gamma \), respectively [and we note that the transformation of \( K \) in (10) corresponds simply to \( \beta \Rightarrow -\beta \)]. Despite this reduction in the number of parameters, it is usually more convenient to use the evolution equation in its original form (8), because (11) obscures somewhat the dependence of the film flow on the original parameters \( F \) and \( K \) of the airflow; in particular, (8) makes the comparison between our analysis and that of Hinch and Kelmanson in the case \( F = K = 0 \) more transparent. However, the reduced form (11) is useful when the dependence of results on \( \alpha \) is being discussed.

In the absence of the airflow, \( F = K = 0 \), Eq. (8) reduces to Eq. (3.14) of Pukhnachev, whose parameters \( \mu \) and \( \gamma \) are related to \( \alpha \) and \( \gamma \) by \( \mu = 3\gamma \) and \( \gamma = 3\alpha \). Moreover, Eqs. (5)–(8) reduce to the leading-order versions of Eqs. (35), (52), and (54), respectively, of Evans et al., whose parameters \( Bo_3 \) and \( U_\infty \) are related to \( \alpha \) and \( \gamma \) by \( Bo_3 = \gamma h/(2a) \) and \( U_\infty = 1/(3\gamma) \), and to Eq. (2.13) of Lopes et al., whose parameters \( \Omega_0 \) and \( Bo_0 \) are related to \( \alpha \) and \( \gamma \) by \( \Omega_0 = 1/(3\gamma) \) and \( Bo_0 = \gamma/\alpha \). Note that the terms in the pressure \( p \) in Eqs. (2.2) and (2.3) of Hinck and Kelmanson have the opposite signs from those in (2) and (1), but that this is of no consequence because these differences in sign cancel out in their evolution equation (2.4) [which is identical to the present (8)] with \( F = K = 0 \) on which all of their subsequent analysis is based. All of the results in the present work obtained from (8) agree with those of Hinck and Kelmanson in the absence of the airflow.

For flow in the presence of the airflow, \( F \neq 0 \), Eqs. (1) and (2) reduce to Eqs. (2.1) and (3.3), respectively, of Mitchell et al. in the case of steady flow in the absence of capillarity, \( \alpha = 0 \).

Newell and Vliegen investigated the particular case \( \kappa = 2\alpha^2 \Omega \), while allowing \( U_\infty \) to take values in the range \(-\alpha \Omega < U_\infty < \alpha \Omega \). Note, however, that, irrespective of whether or not a particular choice of \( \kappa \) is made, there are two free parameters associated with the airflow, namely, \( F \) and \( K \) in the present notation, or, correspondingly, the parameters \( w \) and \( \varphi \) in the notation of Newell and Vliegen, which are related to \( F \) and \( K \) by \( w = F/K \) and \( \varphi = 2K^2 \), in the case \( \kappa = 2\alpha^2 \Omega \),

\[
w = \frac{F}{K} = \frac{U_\infty}{\alpha \Omega}, \quad \varphi = 2K^2 = \frac{2\rho \alpha \Omega^2}{\rho g} \quad (\geq 0)
\]
in the present notation. (Newell and Viljoen\textsuperscript{21} omitted the factor 2 from the definition of their parameter $\phi$, but this appears to be simply a typographical error.)

As Mitchell\textit{et al.}\textsuperscript{22} describe, Newell and Viljoen\textsuperscript{21} (evidently following Hinch and Kelman son) have the opposite signs on $p$ in their Eqs. (5) and (6) from those in the present (2) and (1), but unfortunately, unlike for Hinch and Kelman son, these differences in sign do not cancel out in their evolution equation (7) [i.e., their version of the present Eq. (8)], leading to their (7) having the incorrect sign on the term due to the airflow, i.e., the term involving their parameter $\phi$. However, as we shall show in what follows, obtaining the correct description of the behavior of the film is not simply a matter of reversing the sign of the term due to the airflow in the analysis of Newell and Viljoen.\textsuperscript{21}

III. EVOLUTION OF THE FILM THICKNESS

In this section, we revisit the asymptotic analysis of Newell and Viljoen,\textsuperscript{21} who, as we have already described, sought to extend the analysis of Hinch and Kelman son to include the effect of the airflow.

Hinch and Kelman son considered the case $\gamma \ll 1$ and $\alpha \ll 1$ (in the absence of the airflow, $F = K = 0$) and showed that the film evolves on four different timescales, and they posited a two-timescale expansion of the film thickness to reveal the structure of this evolution. Specifically, Hinch and Kelman son analyzed the evolution of an initially uniform film [so that $h(\theta, 0) = 1$] in the regime

$$
y^2 \ll x \ll \gamma \ll 1,
$$

(14)

by seeking a solution of (8) with $F = 0$ for $h$ as an expansion in powers of $\gamma$,

$$
h = 1 + \gamma \psi_1 + \gamma^2 \psi_2 + \gamma^3 \psi_3 + O(\gamma^4),
$$

(15)

where the $\psi_i = \psi_i(\theta, t_0, t_1)$ depend on the two timescales $t_0 = t$ and $t_1 = \gamma^2 t$, so that $\partial \psi_i / \partial t = \partial \psi_i / \partial t_0 + \gamma^2 \partial \psi_i / \partial t_1$, and the asymptotic solution (15) is uniformly valid only for $t < O(\gamma^{-2})$. Note that the $\psi_i$ satisfy the initial conditions $\psi_i(\theta, 0, 0) = 0$.

Based on the observation that the film thickness $h$ in the absence of the airflow is only weakly dependent on $\alpha$, Hinch and Kelman son make the point that, despite the formal restrictions on $\alpha$ in (14), in practice their analysis is valid for $\alpha$ large as $O(1)$. For the present purposes, the main result of Hinch and Kelman son is that the growth rate of the film thickness, which we denote by $s_1$, is given by

$$
s_1 = -\frac{81\alpha}{1 + 1444\alpha^2} \leq 0,
$$

(16)

which, for $\alpha > 0$, is clearly negative, showing that the solution obtained is stable in the sense that it decays to a steady state at very large times and, for $\alpha = 0$, is clearly zero, showing that the solution is neutrally stable in the absence of capillarity. We adopt the same approach to the problem in the presence of the airflow by analyzing the evolution equation (8) for the film with $F \neq 0$ and, in particular, we obtain the corresponding expression for the growth rate $s_1$, to understand how the presence of the airflow affects the evolution of the film. On the assumption that in the presence of the airflow the film thickness $h$ is again only weakly dependent on $\alpha$, we follow Hinch and Kelman son and Newell and Viljoen\textsuperscript{21} and allow $\alpha$ to be as large as $O(1)$.

Some of the algebraic manipulations involved in obtaining the analytical results presented in the next three subsections are rather lengthy, and so as a check on the accuracy of our results we implemented the entire calculation in two independent ways using the symbolic computational systems Maple\textsuperscript{25} and Mathematica.\textsuperscript{26} Two simple but important checks on the validity of the present analysis are that in the absence of the airflow, $F = K = 0$, the results obtained reduce to those of Hinch and Kelman son, and that when the transformation (10) the free-surface profile $h(\theta, 0)$ becomes the phase-shifted profile $h(\theta + \pi, t)$.

A. Solution for $\psi_1$

Substituting (15) into (8) yields

$$
\mathcal{L}\psi_1 = -(1 + 2KF)s_1,0 - 4F^2 c_{1,0}
$$

(17)

at $O(\gamma)$, where the linear operator $\mathcal{L}$, introduced by Hinch and Kelman son, is defined by

$$
\mathcal{L} = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_0} + \frac{\partial^2 h}{\partial t_0^2} + \frac{\partial^2 h}{\partial t_1^2} - \frac{\partial^4 h}{\partial t_0^2 \partial t_1^2} + \frac{\partial^4 h}{\partial t_0^4},
$$

(18)

and $c_{ij}$ and $s_{ij}$ are defined by $c_{ij} = \cos (i \theta - j \phi)$ and $s_{ij} = \sin (i \theta - j \phi)$. As Hinch and Kelman son describe, $2\pi$-periodic solutions $\Psi(\theta, t_0, t_1)$ of the equation $\mathcal{L} \Psi = 0$ are of the form

$$
\Psi(\theta, t_0, t_1) = \sum_{n=1}^{\infty} \left[ A_n(t_1)c_{n,0} + B_n(t_1)s_{n,0} \right] \exp \left[ -n^2(n^2 - 1)\alpha t_0 \right],
$$

(19)

where $A_n(t_1)$ and $B_n(t_1)$ are arbitrary functions of $t_1$, showing that the nth harmonics $c_{n,0}$ and $s_{n,0}$ ($n = 2, 3, 4, \ldots$) decay exponentially quickly on the fast timescale $\alpha t_0$, but that the fundamental ($n = 1$) modes $c_{1,1}$ and $s_{1,1}$ can decay only slowly via the functions $A_1(t_1)$ and $B_1(t_1)$ on the timescale $t = \gamma^2 t_0$.

The solution of (17) is of the form

$$
\psi_1 = \sum_{n=1}^{\infty} \left[ A_n(t_1)c_{n,0} + B_n(t_1)s_{n,0} \right] \exp \left[ -n^2(n^2 - 1)\alpha t_0 \right] + (1 + 2KF) c_{1,0} - \frac{2F^2}{1 + 36\alpha^2} (s_{1,0} + 6s_{2,0}),
$$

(20)

where the $A_n(t_1)$ and $B_n(t_1)$ are required by the initial condition $\psi_1(\theta, 0, 0) = 0$ to satisfy

$$
\begin{align*}
A_{11}(0) &= -(1 + 2KF), \quad B_{11}(0) = 0, \\
A_{12}(0) &= \frac{12F^2}{1 + 36\alpha^2}, \quad B_{12}(0) = \frac{2F^2}{1 + 36\alpha^2}, \\
A_{1n}(0) &= 0, \quad B_{1n}(0) = 0 & \text{for } n = 3, 4, 5, \ldots.
\end{align*}
$$

(21)

Since the contributions from the higher modes ($n \geq 2$) are non-negligible only for small times, we may replace $A_{1n}(t_1)$ and $B_{1n}(t_1)$ with $A_{1n}(0)$ and $B_{1n}(0)$ for $n \geq 2$ (but not for $n = 1$). Thus, (20), becomes, with the subscripts omitted from $A_{1n}$ and $B_{1n}$ for clarity,

$$
\psi_1 = A(t_1)c_{1,1} + B(t_1)s_{1,1} + (1 + 2KF) c_{1,0} - \frac{2F^2}{1 + 36\alpha^2} \left[ (s_{2,0} + 6s_{2,0}) - (s_{2,0} + 6s_{2,0}) \exp \left[ -12\alpha t_0 \right] \right],
$$

(22)

where, from (21), $A(t_1)$ and $B(t_1)$ satisfy $A(0) = -(1 + 2KF)$ and $B(0) = 0$. In Subsection III C, we shall determine $A$ and $B$ by considering secular terms at $O(\gamma^3)$.
Note that the solution for $\psi_1$ given by (22) is in agreement with the solution for $\psi_1$ given by Eqs. (9) and (10) of Newell and Viljoen;\textsuperscript{21} also in the absence of the airflow, $F = K = 0$, it reduces to the solution for $\psi_1$ given by Eq. (3.3) of Hinch and Kelmanson.\textsuperscript{7}

Since modes with $n \geq 2$ decay exponentially quickly on the timescale $t_0$, henceforth we follow Hinch and Kelmanson\textsuperscript{7} and Newell and Viljoen\textsuperscript{21} in dropping them when we consider higher-order terms in $\gamma$.

### B. Solution for $\psi_2$

At $O(\gamma^3)$, Eq. (8) gives

\[
\mathcal{L}\psi_2 = -\frac{3F^2}{1 + 36x^2} \left\{ (A(t_1) - 6zB(t_1))c_{1,1} - [6zA(t_1) + B(t_1)]s_{1,1} \right\} - \frac{6F^2(1 + 2KF)c_{1,0} - 6zs_{1,0}}{1 + 36x^2} - \frac{3(1 + 2KF)c_{1,2} - A(t_1)s_{2,1}}{1 + 36x^2} - 2F^2(1 + 2KF)c_{1,0} - 6zs_{1,0} - 3(1 + 2KF)^2s_{2,0} + 24F^4 \left[ 12c_{4,0} + (1 - 36x^2)s_{4,0} \right] \right\}.
\]

The solution of (23) may be written as a sum $\psi_2 = \psi_{2u} + \psi_{2u}$ of a steady part $\psi_{2u}$ and an unsteady part $\psi_{2u}$, where

\[
\psi_{2u} = -\frac{3F^2}{1 + 36x^2} \left\{ (6zA(t_1) + B(t_1))c_{1,1} - [A(t_1) - 6zB(t_1)]s_{1,1} \right\} + \frac{3(1 + 2KF)c_{1,0} - 6zs_{1,0}}{2(1 + 36x^2)} - \frac{6F^2(1 + 2KF)c_{1,2} - 12zA(t_1) - B(t_1)s_{2,1}}{1 + 144x^2} - 2F^2(1 + 2KF)c_{1,0} - 6zs_{1,0} - 3(1 + 2KF)^2s_{2,0} + 24F^4 \left[ 12c_{4,0} + (1 - 36x^2)s_{4,0} \right] \right\}.
\]

Note that the solution for $\psi_2$ given by Eqs. (24) and (25) does not agree with that given by Eqs. (11) and (12) of Newell and Viljoen;\textsuperscript{21} and moreover that, as previously pointed out, obtaining the correct solution for $\psi_2$ is not simply a matter of reversing the sign of the term due to the airflow in their solution. However, in the absence of the airflow, $F = K = 0$, both of these solutions reduce to that given by Eq. (3.5) of Hinch and Kelmanson,\textsuperscript{7} as they should.

### C. Solution for $\psi_1$

At $O(\gamma^3)$, Eq. (8) gives

\[
\mathcal{L}\psi_1 = -3 \frac{\partial}{\partial \theta} \left\{ \left( \psi_1^2 + \psi_2 \right) \left[ \frac{\partial \psi_1}{\partial \theta} + \frac{2}{3} \frac{\partial^3 \psi_1}{\partial \theta^3} \right] \right\} + 2F^2 \sin 2\theta - (1 + 2KF) \cos \theta \right\} \right\} - 3 \frac{\partial}{\partial \theta} \left\{ \psi_1 \left[ \frac{\partial \psi_1}{\partial \theta} + \frac{2}{3} \frac{\partial^3 \psi_1}{\partial \theta^3} \right] \right\} \right\} - (26)
\]

with $\psi_1$ and $\psi_2$ given by (22), (24), and (25). The expanded form of (26), omitted for brevity, involves secular terms $s_{1,1}$ and $c_{1,1}$; setting the coefficients of these terms to zero leads to a pair of ordinary differential equations for $A(t_1)$ and $B(t_1)$, namely,

\[
\frac{dA}{dt_1} = s_1 A + s_2 B, \quad \frac{dB}{dt_1} = -s_2 A + s_1 B.
\]

where the real constants $s_1$ and $s_2$ are given by

\[
s_1 = -81z \left( \frac{10F^4}{(1 + 36x^2)(1 + 1296x^2)} + \frac{(1 + 2KF)^2}{1 + 144x^2} \right) \leq 0
\]

and

\[
s_2 = \frac{30(1 + 324x^2)F^4}{(1 + 36x^2)(1 + 1296x^2)} + \frac{3(5 + 72x^2)(1 + 2KF)^2}{2(1 + 144x^2)} \geq 0.
\]

Note that these expressions for $s_1$ and $s_2$ are invariant under the transformation of $K$ given in (10), which provides a check on their validity. The solution of (27) and (28) subject to the initial conditions $A(0) = -(1 + 2KF)$ and $B(0) = 0$ is simply
Equation (29) shows that for $x > 0$ the growth rate of the solution, $s_1$, is always negative, and so we arrive at our main result, namely, that the solution (15) for $h$ to $O(x^7)$ is unconditionally stable, decaying exponentially quickly like $\exp(-s_1 t)$ to a steady state at very large times. In particular, in the limit of a slow airflow, $F \to 0$,

\[
s_1 = -\frac{81x}{1 + 144x^2} \frac{324KzF}{1 + 144x^2} + O(F^2) \to -\frac{81x}{1 + 144x^2} \leq 0,
\]

in the limit of a fast airflow, $F \to \infty$,

\[
s_1 = -\frac{810x^2F^4}{(1 + 36x^2)(1 + 1296x^2)} + O(F^2) \to -\infty,
\]

in the case of no circulation, $K = 0$,

\[
s_1 = -\frac{81x}{1 + 144x^2} \left(\frac{10F^2}{(1 + 36x^2)(1 + 1296x^2)} + \frac{1}{1 + 144x^2}\right) \leq 0,
\]

and in the limit of strong circulation, $K \to \pm \infty$,

\[
s_1 = -\frac{324x^2F^2}{1 + 144x^2} K^2 + O(K) \to -\infty.
\]

Furthermore, in the limit of weak capillarity, $x \to 0$,

\[
s_1 = -81 \left((1 + 2KF)^2 + 10F^2\right)x + O(x^3) \to 0^-,
\]

which is consistent with the conclusion of the limited stability analysis of Mitchell et al.\cite{Mitchell2012} that in the absence of capillarity, $x = 0$, the solution is neutrally stable (i.e., $s_1 = 0$).

Figure 2 shows plots of $s_1$ as a function of $F$ for several values of $K$ for the values of $x$ considered by Hinch and Kelmanson\cite{Hinch2011} and Newell and Viljoen,\cite{Newell2021} namely, $x = 0.0048$ and $x = 0.058$. As Fig. 2 illustrates, $s_1$ decreases monotonically with $F$ when $K \geq 0$, but first increases to a negative maximum before decreasing monotonically when $K < 0$; in particular, the behavior of $s_1$ is consistent with the asymptotic results (33)–(37). The dashed curves in Fig. 2 show the locus of the maximum of $s_1$ as $K$ varies from $-\infty$ [corresponding to the point $(F, s_1) = (0, 0)$] to 0 [corresponding to the point $(F, s_1) = (0, -81x/(1 + 144x^2))]$, confirming that $s_1 \geq 0$, and hence that the solution is unconditionally stable, for all values of $F (\geq 0)$ and $K$.

Equations (29) and (30) show that $s_1$ and $s_2$ depend on all three of the parameters $x$, $F$, and $K$; however, the behavior of $s_1$ and $s_2$ can be seen more clearly if (29) and (30) are instead written as equations for the scaled quantities $s_1/F^4$ and $s_2/F^4$, which depend on just the two parameters $x$ and $\beta$,

\[
\frac{s_1}{F^4} = -162x \left(\frac{5}{(1 + 36x^2)(1 + 1296x^2)} + \frac{8\beta^2}{1 + 144x^2}\right) \leq 0
\]

and

\[
\frac{s_2}{F^4} = \frac{30(1 + 324\beta^2)}{(1 + 36x^2)(1 + 1296x^2)} + \frac{24(5 + 72x^2)\beta^2}{1 + 144x^2} \geq 0.
\]

D. Evolution of the free surface of the film

To $O(\gamma)$, we have $h = 1 + \gamma\psi_1$, and so to $O(\gamma^2)$, where $\epsilon = h/\alpha \ll 1$ is the small aspect ratio of the film, the free surface has the form $r = 1 + \epsilon(1 + \gamma\psi_1)$, that is,
As Hinch and Kelmanson\textsuperscript{7} pointed out, in the absence of the airflow, \( F = K = 0 \), the free surface (40) is a circular cylinder of radius 1 + \( \epsilon \) whose center is offset from the axis of the solid cylinder by \( c_f (1 + A \cos \theta - B \sin \theta, A \sin \theta + B \cos \theta) \) at any instant, where \( A = A(t_1) \) and \( B = B(t_1) \) are given by (31) and (32) with \( F = 0 \); thus, for \( \alpha > 0 \) the center of the circular free surface spirals around the point \( (c_f, 0) \), approaching it in the limit \( t \to \infty \). Moreover, rewriting the term \( c_{1,1} \cos (s_2 t_1) - s_{1,1} \sin (s_2 t_1) \) in (40) with \( F = 0 \) in the form \( \cos (\theta - (t_0 - s_2 t_1)) \) shows that the free surface lags the solid cylinder by the amount \( s_2 t_1 \).

In the presence of the airflow, the occurrence of the terms involving \( c_{2,0} \) and \( s_{2,0} \) (i.e., \( \cos 20 \) and \( \sin 20 \)) in (40) and (41) means that the free surface, while still, of course, a cylinder, is no longer simply circular, and that it is no longer possible to identify a uniquely defined lag of the free surface relative to the solid cylinder.

Figure 4 shows examples of snapshots of free surfaces given by (40) at various times, comparing a case in the absence of the airflow, \( F = K = 0 \), with cases with far-field airflow \( F = 1 \) and positive, negative, or zero air circulation \( K \), all plotted for \( \alpha = 0 \) and \( \gamma = 1/15 \) (the latter value being chosen for illustrative purposes). Figure 4 illustrates the effect of the airflow in both distorting the film and modifying its offset from the center of the solid cylinder. Figure 5 shows corresponding plots for \( \alpha = 1/15 \) (and again with \( \gamma = 1/15 \)), illustrating the tendency of capillarity to suppress the distortion of the film. The largest time shown in Figs. 4 and 5, namely, \( t_0 = 5 \), was chosen simply because the free surfaces in Fig. 5 (but not those in Fig. 4) are close to their large-time asymptotic state (41) by this time and change little thereafter.

The successive parts of both Figs. 4 and 5 correspond to the values \( \beta = \infty, -3/4, -1/4, 1/4, 3/4, \) and \( 5/4 \), respectively, and therefore, parts (b) and (e) and parts (c) and (d) provide examples of the phase shift of \( \pi \) under the transformation (10), mentioned in Sec. II; for example, the value of \( r \) given by (40) at any position \( \theta \) at time \( t \) in Fig. 4(b) is the same as the value of \( r \) at position \( \theta + \pi \) at time \( t \) in Fig. 4(e).
IV. CONCLUSIONS

We revisited the analysis of Newell and Viljoen \(^2\) of unsteady two-dimensional coating flow of a thin film of a viscous fluid on the outside of a uniformly rotating solid horizontal circular cylinder in the presence of a steady two-dimensional irrotational airflow with circulation, in the case in which the effects of gravity and of capillarity are weak compared with those of viscous shear. In contrast with the claim of Newell and Viljoen \(^2\) that the solution is unstable for certain values of the physical parameters, we found that the growth rate \(\sigma \), given by (29), in terms of the parameters \(F, K\), and \(\alpha\), representing the speed of the far-field airflow, the circulation of the airflow, and surface tension, respectively, is always non-positive, and so the solution for \(h\) to \(O(\gamma^3)\) is unconditionally stable.

From their study, Newell and Viljoen \(^2\) drew five conclusions concerning the operation of the novel rotary pesticide applicator for crops described in Sec. I. Unfortunately, their erroneous prediction of instability renders their first conclusion incorrect and their third, fourth, and fifth conclusions moot; their second conclusion is correct but concerns only the case of no airflow and so provides no new information about the use of the applicator. On a more positive note, however, the successful use of the pesticide applicator depends on the film being stable—and our results show that this is always the case in the regime considered.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts of interest to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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