Compositional Inverses of AGW-PPs *

Pingzhi Yuan†

Abstract

In this paper, we present two methods to obtain the compositional inverses of AGW-PPs. We improve some known results in this topic.

Keywords: Finite fields, permutation polynomials, AGW criterion, compositional inverses, branch functions.

1 Introduction

Let $q$ be a prime power, $\mathbb{F}_q$ be the finite field of order $q$, and $\mathbb{F}_q[x]$ be the ring of polynomials in a single indeterminate $x$ over $\mathbb{F}_q$. A polynomial $f \in \mathbb{F}_q[x]$ is called a permutation polynomial (PP for short) of $\mathbb{F}_q$ if it induces a one-to-one map from $\mathbb{F}_q$ to itself.

Permutation polynomials over finite fields have been an interesting subject of study for many years, and have applications in coding theory [4], cryptography [11, 12], combinatorial design theory [3], and other areas of mathematics and engineering. Information about properties, constructions, and applications of permutation polynomials may be found in Lidl and Niederreiter [6, 7], and Mullen [8].

In 2011, Akbrary, Ghioca and Wang [1] proposed a powerful method called the AGW criterion for constructing PPs. A PP is called AGW-PP when a PP is constructed using the AGW criterion or it can be interpreted by the AGW criterion. AGW-PPs can be divided into three types: multiplicative type, additive type and hybrid type. It is difficult to obtain the explicit compositional inverse of a random PP, except for several well-known classes. Compositional inverses of different classes of PPs of special forms have been obtained in explicit or implicit forms; see [2, 5, 6, 9, 13, 14, 16, 17, 18, 22, 23, 24] for more details. Recently, Niu, Li, Qu and Wang [9] obtained a general method to finding compositional inverses of AGW-PPs. They obtained the compositional inverses of all AGW-PPs of the type $x^r h(x^s)$ over $\mathbb{F}_q$, where $s/q - 1$. However there are many other classes of AGW-PPs whose compositional inverses are unknown, and it is not easy to follow the framework in [9] to obtain the compositional inverses of AGW-PPs for other types. The purpose of the present paper is to find other general methods to obtain the compositional inverses of AGW-PPs.

The rest of this paper is organized as follows. In Section 2, we prove some results related to the AGW criterion. In particular, we obtain a useful commutative diagram, which is essential for the proofs of our main theorems. In Section 3, we use the dual diagram obtained in Section 2 to finding the compositional

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*Supported By NSF of China No. 12171163
†P. Yuan is with School of of Mathematical Science, South China Normal University, Guangzhou 510631, China (email: yuanpz@scnu.edu.cn).
 inverses of AGW-PPs. We improve some results in \cite{9}. In Section 4, we describe another method to compositional inverses of PPs by using the branch functions.

## 2 AGW criterion and the dual diagram

In this section, we present some results related to the AGW criterion, and we will give the dual diagram when the AGW criterion is applied to a bijective function \( f \).

The following lemma is taken from \cite{12} Lemma 1.1, which is called AGW criterion now.

**Lemma 2.1.** (AGW criterion) Let \( A, S \) and \( S \) be finite sets with \( \#S = \#S \), and let \( f : A \rightarrow A, h : S \rightarrow S \), \( \lambda : A \rightarrow S \), and \( \bar{\lambda} : A \rightarrow \bar{S} \) be maps such that \( \lambda \circ f = h \circ \lambda \). If both \( \lambda \) and \( \bar{\lambda} \) are surjective, then the following statements are equivalent:

(i) \( f \) is a bijection (a permutation of \( A \)); and

(ii) \( h \) is a bijection from \( S \) to \( \bar{S} \) and if \( f \) is injective on \( \lambda^{-1}(s) \) for each \( s \in S \).

We also have other results for the bijection of a map. We have

**Lemma 2.2.** Let \( A, S \) be finite sets, \( f : A \rightarrow A \) a map and \( \lambda : A \rightarrow S \) a surjective map. Then \( f \) is a bijection if and only if

(i) \( f(x) \) is injective on each \( \lambda^{-1}(s) \) for all \( s \in S \).

(ii) If \( \lambda(a) \neq \lambda(b) \), then \( f(a) \neq f(b) \).

Moreover (ii) is equivalent to \( f \left( \lambda^{-1}(s_1) \right) \cap f \left( \lambda^{-1}(s_2) \right) = \emptyset \) for any distinct \( s_1, s_2 \in S \).

**Proof.** If \( f \) is a bijection, then (i) holds trivially. For any distinct \( s_1, s_2 \in S \), since \( \lambda^{-1}(s_1) \cap \lambda^{-1}(s_2) = \emptyset \), we have \( f \left( \lambda^{-1}(s_1) \right) \cap f \left( \lambda^{-1}(s_2) \right) = \emptyset \), and (ii) holds.

Conversely, suppose that (i) and (ii) hold, since \( f(x) \) is injective on each \( \lambda^{-1}(s) \) for all \( s \in S \) and \( f \left( \lambda^{-1}(s_1) \right) \cap f \left( \lambda^{-1}(s_2) \right) = \emptyset \) for any distinct \( s_1, s_2 \in S \), we have

\[
\#f(A) = \sum_{s \in S} \#f \left( \lambda^{-1}(s) \right) = \sum_{s \in S} \#\lambda^{-1}(s) = \#A,
\]

which implies that \( f \) is a surjective, and thus \( f \) is a bijection.

**Lemma 2.3.** Let \( A, S, \bar{S} \) be finite sets with \( \#S = \#S \), \( f : A \rightarrow A \) a map and \( \lambda : A \rightarrow S \) a surjective map. Then \( f : A \rightarrow A \) is a bijection if and only if the following two conditions hold:

(i) \( f(x) \) is injective on each \( \lambda^{-1}(s) \) for all \( s \in S \).

(ii) There exists a maps pair \((\bar{\lambda}, h)\) such that \( \bar{\lambda} : A \rightarrow \bar{S} \) is a surjective, \( h : S \rightarrow \bar{S} \) is a bijection and the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{\lambda} & & \downarrow{\bar{\lambda}} \\
S & \xrightarrow{h} & \bar{S}
\end{array}
\]

**Proof.** (i) is obvious. If \( f : A \rightarrow A \) is a bijection, then, by the proof of Lemma 2.2, \( A \) is a disjointed union of \( \lambda^{-1}(s_i), s_i \in S \), and \( A \) is also a disjointed union of \( f(\lambda^{-1}(s_i)), s_i \in S \), i.e.

\[
A = \bigcup_{s \in S} f \left( \lambda^{-1}(s) \right) = \bigcup_{s \in S} \lambda^{-1}(s).
\]
Now for any bijection \( h : S \to \bar{S}, s \mapsto h(s) \), we define a map \( \lambda : A \to \bar{S} \) by \( \lambda(a_s) = h(s) \) for any \( s \in f(\lambda^{-1}(s)) \). It is easy to check that \( \lambda : A \to \bar{S} \) is a surjective and the diagram in the theorem commutes. This proves that (ii) holds.

If (i) and (ii) holds, then by AGW criterion, \( f \) is a bijection.

\[ \square \]

Remark: It is not difficult to see that there are precisely \( \sharp \hat{S}! \) maps pairs \((\bar{\lambda}, h)\) such that (ii) holds.

We also have the following result by Lemma 2.3.

Corollary 2.1. Let \( A, S, \bar{S} \) be finite sets with \( \sharp S = \sharp \bar{S} \), \( f : A \to A \) a map, \( \lambda : A \to S \) a surjective map and \( h : S \to \bar{S} \) is a bijection. Then \( f : A \to A \) is a bijection if and only if the following two conditions hold:

(i) \( f(x) \) is injective on each \( \lambda^{-1}(s) \) for all \( s \in S \).

(ii) There exists a unique determined surjective map \( \bar{\lambda} : A \to \bar{S} \) such that the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{\lambda} & & \downarrow{\bar{\lambda}} \\
S & \xrightarrow{h} & \bar{S}
\end{array}
\]

Lemma 2.4. Let \( A, S \) be finite sets, \( f : A \to A \) a map and \( \lambda : A \to S \) a surjective map. Suppose that there is a set \( \bar{S} \) and a map \( \bar{\lambda} : A \to \bar{S} \) such that \( \bar{\lambda}(f(a)) = \bar{\lambda}(f(b)) \) for any \( a, b \in A \) with \( \lambda(a) = \lambda(b) \). Then there exists a unique map \( h : S \to \bar{S} \) such that the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{\lambda} & & \downarrow{\bar{\lambda}} \\
S & \xrightarrow{h} & \bar{S}
\end{array}
\]

Furthermore, \( f \) is a bijection if the following two conditions hold

(i) \( f(x) \) is injective on each \( \lambda^{-1}(s) \) for all \( s \in S \).

(ii) \( h \) is an injection.

Proof. Since \( \lambda \) is surjective, so it is easy to check that

\[ h : \lambda(a) \to \bar{\lambda}(f(a)) \]

satisfies \( h(\lambda(a)) = \bar{\lambda}(f(a)) \) for all \( a \in A \), so the above diagram commutes.

If (i) and (ii) hold, then for any \( a, b \in A \) with \( \lambda(a) \neq \lambda(b) \), then \( h(\lambda(a)) \neq h(\lambda(b)) \), and so \( \bar{\lambda}(f(a)) \neq \bar{\lambda}(f(b)) \), which implies that \( f(a) \neq f(b) \). By Lemma 2.2 we conclude that \( f \) is a bijection.

\[ \square \]

Lemma 2.5. Let \( A, S \) be finite sets, \( f : A \to A \) a map and \( \lambda : A \to S \) a surjective map. Then \( f \) is a bijection if and only if the following conditions hold

(i) \( f(x) \) is injective on each \( \lambda^{-1}(s) \) for all \( s \in S \).
(ii) there exist a set $\bar{S}$, an injective $h : S \to \bar{S}$ and a map $\lambda : A \to \bar{S}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
A & \xrightarrow{f^{-1}} & A \\
\downarrow{\lambda} & & \downarrow{\lambda} \\
S & \xrightarrow{h^{-1}} & \bar{S}
\end{array}
$$

Proof. Suppose that $f$ is a bijection, then we take

$$\bar{S} = \{[f(\lambda^{-1}(s)), s \in S]\},$$

$\lambda : f(a) \to [f(\lambda^{-1}(s))]$ for any $a \in \lambda^{-1}(s)$ and $h : s \to [f(\lambda^{-1}(s))]$. Since $f$ is a bijection, so $\lambda$ is a well-defined surjective map. Moreover, $\bar{\lambda}(f(a)) = \bar{\lambda}(f(b)) = [f(\lambda^{-1}(s))]$ if $f(a) = f(b) = s$ and $\sharp(S) = \sharp(\bar{S})$.

The other direction follows from Lemma 2.2.

Lemma 2.6. Let $A, B$ be finite sets, $f : A \to A$ a map and $g : B \to A$ a surjective map. Then $f$ is a bijection if and only if $g(f(x))$ is a surjection.

Proof. Obviously.

The following result is essential in this paper, which will be used in the sequel.

Theorem 2.1. Let the notations be defined as in Lemma 2.1. If $f : A \to A$ is a bijection, $f^{-1}$ and $h^{-1}$ are the compositional inverses of $f$ and $h$, respectively, then the have $\lambda \circ f^{-1} = h^{-1} \circ \bar{\lambda}$, i.e. the following diagram commutes

$$
\begin{array}{ccc}
A & \xrightarrow{f^{-1}} & A \\
\downarrow{\bar{\lambda}} & & \downarrow{\lambda} \\
S & \xrightarrow{h^{-1}} & \bar{S}
\end{array}
$$

Proof. By assumption, we have $\bar{\lambda} \circ f = h \circ \lambda$, hence

$$h^{-1} \circ (\bar{\lambda} \circ f) \circ f^{-1} = h^{-1} \circ (h \circ \lambda) \circ f^{-1},$$

which yields $\lambda \circ f^{-1} = h^{-1} \circ \bar{\lambda}$. This completes the proof.

We call the diagram in Theorem 2.1 the dual diagram of the AGW criterion.

3 Compositional inverses of AGW-PPs

In this section, we present one approach to finding the compositional inverses of AGW-PPs by using the dual diagram of the AGW criterion. Our first result is as follows.

Theorem 3.1. Let $q$ be a prime power, and $S, \bar{S}$ subsets of $\mathbb{F}_q^*$ with $\sharp S = \sharp \bar{S}$. Let $f : \mathbb{F}_q^* \to \mathbb{F}_q^*, g : S \to \bar{S}$, $\lambda : \mathbb{F}_q^* \to S$, and $\bar{\lambda} : \mathbb{F}_q^* \to \bar{S}$ be maps such that both $\lambda$ and $\bar{\lambda}$ are surjective maps and $\lambda \circ f = g \circ \lambda$.

Let $f_1(x)$ be a PP and $f(x) = f_1(x)h(\lambda(x))$ a AGW-PP over $\mathbb{F}_q^*$, and let $f_1^{-1}(x), f^{-1}(x)$ and $g^{-1}(x)$ be the compositional inverses of $f_1(x), f(x)$ and $g(x)$, respectively. Then we have

$$f^{-1}(x) = f_1^{-1}\left(\frac{x}{h(g^{-1}(\lambda(x)))}\right).$$
**Proof.** By assumption and Theorem 2.1, we have the following commutative diagram

\[
\begin{array}{ccc}
F_q^* & f & F_q^* \\
\lambda & \downarrow & \lambda \\
S & g & S
\end{array}
\]

Hence \(\lambda(f^{-1}(x)) = g^{-1}(\lambda(x))\). Since \(f^{-1}(x)\) is the compositional inverse of \(f(x)\), we have \(f(f^{-1}(x)) = x\), that is

\[f_1(f^{-1}(x))h(\lambda(f^{-1}(x))) = x.
\]

It follows that

\[f_1(f^{-1}(x)) = f_1^{-1}\left(\frac{x}{h(g^{-1}(\lambda(x)))}\right).
\]

This completes the proof.

**Remark:** In Theorem 3.1, we use \(F_q^*\) to avoid the case of \(x = 0\). If we use \(F_q\), then we must consider the case of \(x = 0\) independently.

For AGW-PPs in the hybrid case, we have

**Lemma 3.1.** ([1] Theorem 6.3) Let \(q\) be any power of the prime number \(p\), let \(n\) be any positive integer, and let \(S\) be any subset of \(F_q^n\) containing 0. Let \(h, k \in F_q^n\) be any polynomials such that \(h(0) \neq 0\) and \(k(0) = 0\), and let \(\lambda(x) \in F_q^n[x]\) be any polynomial satisfying

1. \(h(\lambda(F_q^n)) \subseteq S\); and
2. \(\lambda(\alpha a) = k(\alpha)\lambda(a)\) for all \(a \in S\) and all \(a \in F_q^n\). Then the polynomial \(f(x) = xh(\lambda(x))\) is a PP for \(F_q^n\) if and only if \(g(x) = xk(h(x))\) induces a permutation of \(\lambda(F_q^n)\).

We apply Theorem 3.1 to \(f_1(x) = x\) to obtain the following corollary, which improves Theorem 22 in [9].

**Corollary 3.1.** Let the symbols be defined as in Lemma 3.1. Let \(f(x) = xh(\lambda(x))\) permute \(F_q^n\) and \(g^{-1}(x)\) be the compositional inverse of \(g(x) = xk(h(x))\) over \(\lambda(F_q^n)\). Then the compositional inverse of \(f(x)\) is given by

\[f^{-1}(x) = \frac{x}{h(g^{-1}(\lambda(x)))}.
\]

The following result was discovered independently by several authors, and it can be proved by AGW criterion.

**Lemma 3.2.** ([10] Theorem 2.3) Let \(q\) be a prime power and \(f(x) = x^r h(x^s) F_q[x]\), where \(s = \frac{q-1}{l}\) and \(l\) is an integer. Then \(f(x)\) permutes \(F_q\) if and only if

1. \(\gcd(r, s) = 1\) and
2. \(g(x) = x^r h(x^s)\) permutes \(\mu_l\).

Applying Theorem 3.1 to \(f_1(x) = x^r\), we obtain the following result, which is a same as in [5] Theorem 2.3.
Corollary 3.2. Let \( f(x) = x^r h(x^a) \in \mathbb{F}_q[x] \) defined in Lemma 3.2 be a permutation over \( \mathbb{F}_q \) and \( g^{-1}(x) \) be the compositional inverse of \( g(x) = x^r h(x^a) \) over \( \mu_r \). If \( \gcd(r, q - 1) = 1 \) and \( a \) and \( b \) are two positive integers satisfying \( br = 1 + a(q - 1) \). Then the compositional inverse of \( f(x) \) in \( \mathbb{F}_q[x] \) is given by

\[
\left( f^{-1}(x) = x^b h(g^{-1}(x^a))^{-b} \right.
\]

Proof. Since \( \gcd(r, q - 1) = 1 \) and \( a \) and \( b \) are two positive integers satisfying \( br = 1 + a(q - 1) \), the compositional inverse of \( x^r \) is \( x^b \). Applying Theorem 3.1 to \( f_1(x) = x^r \), we obtain the desired result.

For the compositional inverses of AGW-PPs in the additive case, we have

Theorem 3.2. Let \( q \) be a prime power, and let \( S, \bar{S} \) be subsets \( \mathbb{F}_q \) with \( S \subseteq \bar{S} \). Let \( f : \mathbb{F}_q \to \mathbb{F}_q, g : S \to \bar{S}, \lambda : \mathbb{F}_q \to S, \) and \( \bar{\lambda} : \mathbb{F}_q \to \bar{S} \) be maps such that both \( \lambda \) and \( \bar{\lambda} \) are surjective maps and \( \bar{\lambda} \circ f = g \circ \lambda \).

Let \( f_1(x) \) be a PP and \( f(x) = f_1(x) + h(\lambda(x)) \) a AGW-PP over \( \mathbb{F}_q \), and let \( f_1^{-1}(x), f^{-1}(x) \) and \( g^{-1}(x) \) be the compositional inverses of \( f(x), f_1(x) \) and \( g(x) \), respectively. Then we have

\[
\left( f^{-1}(x) = f_1^{-1}(x - h(g^{-1}(\bar{\lambda}(x)))) \right).
\]

Proof. By assumption and Theorem 2.1 we have the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{F}_q & \xrightarrow{f} & \mathbb{F}_q \\
\lambda \downarrow & & \bar{\lambda} \downarrow \\
S & \xrightarrow{g} & \bar{S}
\end{array}
\]

Hence \( \lambda(f^{-1}(x)) = g^{-1}(\bar{\lambda}(x)) \). Since \( f^{-1}(x) \) is the compositional inverse of \( f(x) \), we have \( f(f^{-1}(x)) = x \), that is

\[
f_1(f^{-1}(x)) + h(\lambda(f^{-1}(x))) = x.
\]

It follows that \( f_1(f^{-1}(x)) = x - h(\lambda(f^{-1}(x))) = x - h(g^{-1}(\bar{\lambda}(x))) \), which implies

\[
f^{-1}(x) = f_1^{-1}(x - h(g^{-1}(\bar{\lambda}(x)))).
\]

This completes the proof.

Lemma 3.3. ([20 Theorem 6.1]): Assume that \( F \) is a finite field and \( S, \bar{S} \) are finite subsets of \( F \) with \( S \subseteq \bar{S} \) such that the maps \( \lambda : F \to S \) and \( \bar{\lambda} : F \to \bar{S} \) are surjective and \( \bar{\lambda} \) is additive, i.e.,

\[
\bar{\lambda}(x + y) = \bar{\lambda}(x) + \bar{\lambda}(y), x, y \in F.
\]

Let \( g_0 : S \to F \), and \( g : F \to F \) be maps such that

\[
\bar{\lambda} \circ (g + g_0 \circ \lambda) = g \circ \lambda,
\]

\( g(S) = \bar{S} \) and \( \bar{\lambda}(g_0(\lambda(x))) = 0 \) for every \( x \in F \). Then the map \( f(x) = g(x) + g_0(\lambda(x)) \) permutes \( F \) if and only if \( g \) permutes \( F \).

We apply Theorem 3.2 to \( f_1(x) = g(x) \), we obtain Theorem 16 in [9].
**Corollary 3.3.** ([9] Theorem 16]): Let the symbols be defined as in Lemma 3.3. Let \( f(x) = g(x) + g_0(\lambda(x)) \) be a permutation over \( F \) and \( g^{-1}(x) \) be the compositional inverse of \( g(x) \) over \( F \). Then the compositional inverse of \( f(x) \) is given by

\[
f^{-1}(x) = g^{-1}(x - g_0(g^{-1}(\lambda(x)))).
\]

For \( S \subseteq \mathbb{F}_q \), \( \gamma, b \in \mathbb{F}_q \) and a map \( \lambda : \mathbb{F}_q \rightarrow \mathbb{F}_q, \gamma \) is called a \( b \)-linear translator \([1]\) of \( \lambda \) with respect to \( S \) if \( \lambda(x + u\gamma) = \lambda(x) + ub \) for all \( x \in \mathbb{F}_q \) and \( u \in S \).

**Lemma 3.4.** ([1] Theorem 6.4)): Let \( S \subseteq \mathbb{F}_q \) and \( \lambda : \mathbb{F}_q \rightarrow S \) be a surjective map. Let \( \gamma \in \mathbb{F}_q^* \) be a \( b \)-linear translator with respect to \( S \) for the map \( \lambda \). Then for any \( G \in \mathbb{F}_q[x] \) which maps \( S \) into \( S \), we have that \( f(x) = x + \gamma G(\lambda(x)) \) is a PP of \( \mathbb{F}_q \) if and only if \( g(x) = x + bG(x) \) permutes \( S \).

Applying Theorem 3.2 to \( f_1(x) = x \), we get the following corollary, which improves Theorem 27 in [9].

**Corollary 3.4.** Let \( f(x) = x + \gamma G(\lambda(x)) \) defined as in Lemma 3.4 be a PP on \( \mathbb{F}_q \) and \( g^{-1}(x) \) be the compositional inverse of \( g(x) = x + bG(x) \). Then the compositional inverse of \( f(x) \) is given by

\[
f^{-1}(x) = x - \gamma G(g^{-1}(\lambda(x))).
\]

We end this section with another AGW-PP over finite fields, which is the corrected version of Theorem 3.13 in [21].

**Lemma 3.5.** Let \( n \) be a positive integer, \( f(x) \in \mathbb{F}_q[x] \) a linearized polynomial such that \( \gcd(l(x), x^n - 1) \neq 1 \), where \( l(x) \) is the associated polynomial of \( L(x) \). Let \( a \in \mathbb{F}_q^* \) be a solution of \( L(x) = 0 \) and \( h(x) \) is a polynomial with \( h(x)^q = h(x) \). Let \( L_1(x) \in \mathbb{F}_q[x] \) be a linearized polynomial. Then for every \( \delta \in \mathbb{F}_q^* \), the polynomial

\[
f(x) = ah(L(x) + \delta) + L_1(x)
\]

permutes \( \mathbb{F}_q^n \) if and only if \( L_1(x) \) permutes \( \mathbb{F}_q^n \). Moreover, if \( L_1(x) \) permutes \( \mathbb{F}_q^n \), then

\[
f^{-1}(x) = L_1^{-1}(x - ah(L_1^{-1}(L(x)) + \delta)).
\]

**Proof.** By assumption, we have \( h(L(x) + \delta) \in \mathbb{F}_q, x \in \mathbb{F}_q^n \) since \( h^q(x) = h(x) \), hence \( L(ah(L(x) + \delta)) = h(L(x) + \delta)L(a) = 0 \). It follows that the following diagram commutes

\[
\begin{array}{ccc}
\mathbb{F}_q^n & \xrightarrow{f} & \mathbb{F}_q^n \\
\downarrow{L(x)+\delta} & & \downarrow{L(x)} \\
S & \xrightarrow{L_1(x-\delta)} & \tilde{S}
\end{array}
\]

where \( S = \{L(x) + \delta, x \in \mathbb{F}_q^n\} \) and \( \tilde{S} = \{L(x), x \in \mathbb{F}_q^n\} \). By Lemma 3.3, \( f(x) \) permutes \( \mathbb{F}_q^n \) if and only if \( L_1(x) \) permutes \( \mathbb{F}_q^n \). Further, if \( L_1(x) \) permutes \( \mathbb{F}_q^n \), by Theorem 3.2 and the following diagram

\[
\begin{array}{ccc}
\mathbb{F}_q^n & \xrightarrow{f} & \mathbb{F}_q^n \\
\downarrow{L(x)+\delta} & & \downarrow{L(x)} \\
S & \xrightarrow{L_1(x-\delta)} & \tilde{S}
\end{array}
\]

we get

\[
f^{-1}(x) = L_1^{-1}(x - ah(L_1^{-1}(L(x)) + \delta)).
\]

\( \square \)
4 Branch PPs and their compositional inverses

For a permutation polynomial \( f \) by AGW criterion, we have

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\lambda & \downarrow & \lambda \\
S & \xrightarrow{k^{-1}} & S
\end{array}
\]

As \( \lambda \) is a surjective map, we have that \( A \) is a disjointed union of \( \lambda^{-1}(s_i), s_i \in S \), i.e.

\[
A = \bigcup_{s_i \in S} \lambda^{-1}(s_i).
\]

Let \( |S| = m, S = \{s_1, \ldots, s_m\} \),

\[
A_i = \lambda^{-1}(s_i), \quad B_i = f(\lambda^{-1}(s_i)), \quad i = 1, \ldots, m,
\]

\[
f|_{A_i} = f_i : A_i \rightarrow f(A_i) = B_i, \quad i = 1, \ldots, m.
\]

Then \( f : A \rightarrow A \) can be viewed as a branch function of \( f_i(x), i = 1, \ldots, m \). Let \( f_i^{-1} : B_i \rightarrow A_i, i = 1, \ldots, m \), be the local inverse of \( f_i \), \( i = 1, \ldots, m \), i.e. \( f_i^{-1}(f_i(x)) = x, x \in A_i, i = 1, \ldots, m \). For any non-empty subset \( T \) of \( S \), let \( \chi_T(x) \) be the characteristic function on \( T \), i.e.

\[
\chi_T(x) = \begin{cases} 
1, & x \in T \\
0, & \text{otherwise}.
\end{cases}
\]

Then we have

**Theorem 4.1.** Let \( A \) be a non-empty subset of \( \mathbb{F}_q \), and \( f \) is a branch bijection defined as above, then we have

\[
f^{-1}(x) = \sum_{i=1}^{m} \chi_{B_i}(x)f_i^{-1}(x).
\]

**Proof.** For \( x \in A_i, i = 1, \ldots, m \), we have \( f(x) = f_i(x) \in B_i \), hence

\[
(\sum_{i=1}^{m} \chi_{B_i}f_i^{-1}) \circ f(x) = \sum_{i=1}^{m} \chi_{B_i}(f(x))f_i^{-1}(f(x)) = f_i^{-1}(f_i(x)) = x,
\]

which implies that

\[
f^{-1}(x) = \sum_{i=1}^{m} \chi_{B_i}(x)f_i^{-1}(x).
\]

This completes the proof. \( \square \)

For any branch bijection, it is not easy to give the characteristic functions \( \chi_{B_i}(x) \). However, it is easy for the branch functions when we use the cyclotomic cosets as branches.

Let \( \gamma \) be a fixed primitive element of \( \mathbb{F}_q \), \( s|q - 1 \). The integer \( \ell = \frac{q-1}{s} \) is called the index of \( f(x) = x^{\gamma \ell}h(x^{\gamma \ell}) \). Let \( C_0 \) be the set of all non-zero \( \ell \)-th powers, i.e. \( C_0 = \langle \gamma^{\ell} \rangle \). \( C_0 \) is a subgroup of \( \mathbb{F}_q^* \) of index \( \ell \). The cosets of \( C_0 \) are the cyclotomic cosets

\[
C_i := \gamma^i C_0, \quad i = 0, 1, \ldots, \ell - 1.
\]
Let $\mu_\ell$ denote the set of $\ell$-th roots of unity in $\mathbb{F}_q^*$, i.e.
\[
\mu_\ell = \{ x \in \mathbb{F}_q^* | x^\ell = 1 \}.
\]

We have

**Lemma 4.1.** Let $\gamma$ be a fixed primitive element of $\mathbb{F}_q$, $s | q - 1$ and $\ell = \frac{q-1}{s}$. Let $C_0 = \langle \gamma^\ell \rangle$, and
\[
C_i := \gamma^i C_0, \quad i = 0, 1, \ldots, \ell - 1.
\]
Let $h_i(x) = 1 + \left( \frac{x}{\gamma^i} \right)^s + \cdots + \left( \frac{x}{\gamma^i} \right)^{(\ell-1)s}$, $i = 0, 1, \ldots, \ell - 1$. Then we have
\[
\chi_{C_i}(x) = \frac{h_i(x)}{\ell} = \begin{cases} 1, & x \in C_i \\ 0, & \text{otherwise} \end{cases}
\]
for $i = 0, 1, \ldots, \ell - 1$.

**Proof.** The result follows from the following well-known result that
\[
1 + g^s + \cdots + g^{(\ell-1)s} = \begin{cases} \ell, & g \in C_0 \\ 0, & \text{otherwise}. \end{cases}
\]

Now we will the branch function to give two proofs of Theorem 11 in [9].

**Proposition 4.1.** ([9] Theorem 11) Let $f(x) = x^r h(x^s) \in \mathbb{F}_q[x]$ defined in Lemma 3.2 be a permutation over $\mathbb{F}_q$ and $g^{-1}(x)$ be the compositional inverse of $g(x) = x^r h(x^s)$ over $\mu_\ell$. Suppose $a$ and $b$ are two integers satisfying $as + br = 1$. Then the compositional inverse of $f(x)$ in $\mathbb{F}_q[x]$ is given by
\[
f^{-1}(x) = g^{-1}(x^s)^a x^b (g^{-1}(x^s))^{-b}.
\]

**Proof. The first proof:** View $f(x)$ as a branch function of $f_i(x) = x^r h(\gamma^i x^s), x \in C_i = \gamma^i \langle \gamma^\ell \rangle$ for $i = 0, 1, \ldots, \ell - 1$. Since $h(x^s) = h(\gamma^i x^s)$ is a constant for any $x \in C_i$, it is easy to check that $f_i^{-1}(x) = \gamma^{ias} h(\gamma^i x^s)^{-b} x^{b}, i = 0, \ldots, \ell - 1$ satisfy $f_i^{-1}(f_i(x)) = x, x \in C_i, i = 0, \ldots, \ell - 1$. Here $f_i : C_i \to C_i, f(C_i) = C_i$. Since $f(\gamma^{i+di}s) = g(\gamma^{(i+di)s}) = g(\gamma^i x^s)$, we have
\[
f(C_i)^s = g(C_i) = C_i^s, \quad g(\gamma^i x^s) = \gamma^{(i+di)s},
\]
and
\[
f_i^{-1} : C_i \to C_i, \quad x \mapsto \gamma^{ias} h(\gamma^i x^s)^{-b} x^{b}, x \in C_i, i = 0, \ldots, \ell - 1.
\]
Hence $f_i^{-1}(x) = x^b h'(x^s)$, where $h'(\gamma^i x^s) = \gamma^{ias} h(\gamma^i x^s)^{-b}$. Since $g$ is a bijection, we have $g^{-1}(\gamma^{(i+di)s}) = \gamma^{i+di}s$ and $\gamma^{ias} h(\gamma^i x^s)^{-b} = (g^{-1}(\gamma^{(i+di)s}))^a (h(g^{-1}(\gamma^i x^s)))^{-b}$, which implies that
\[
h'(x^s) = (g^{-1}(x^s))^a (h(g^{-1}(x^s)))^{-b}.
\]
Hence $f^{-1}(x) = g^{-1}(x^s)^a x^b (h(g^{-1}(x^s)))^{-b}$, and we are done. \qed
Second proof: From the first proof we have \( f^{-1}(x) = x^bh'(x^s) \), so it suffices to prove that \( h'(x^s) = g^{-1}(x^s)^a x^b h(g^{-1}(x^s))^{-b} \). We use the following commutative diagram to prove this.

\[
\begin{array}{c}
F_q^n \\ x^s \downarrow \\
\mu \downarrow g^{-1} \\
F_q^n \\
\end{array}
\]

By the diagram, we have

\[ g^{-1}(x^s) = x^bh'(x^s), \quad sa + rb = 1. \tag{4.1} \]

Since \( f(f^{-1}(x)) = x \), we obtain

\[ (x^bh'(x^s))^r h(x^bh'(x^s)) = x, \]

i.e. \( (x^bh'(x^s))^r h(g^{-1}(x^s)) = x \), hence

\[ x^b = (x^bh'(x^s))^r h(g^{-1}(x^s))^b. \]

By \((4.1)\), we have

\[ (g^{-1}(x^s))^a = x^{sba} h'(x^s)^sa = x^b h'(x^s) (x^b h'(x^s))^{-rb} = h'(x^s) h(g^{-1}(x^s))^b. \]

It follows that \( h'(x^s) = (g^{-1}(x^s))^a (h(g^{-1}(x^s)))^{-b} \), and we are done.

We end the paper with two results on branch functions.

**Lemma 4.2.** Let \( q \) be an odd prime power, \( \gamma \) a fixed primitive element of \( \mathbb{F}_q \), \( a_1, a_2 \in \mathbb{F}_q^* \). Let \( r_1 \) and \( r_2 \) be two positive integers and let

\[ f(x) = \frac{a_1}{2} x^{r_1} \left( 1 - x^{\frac{a_2}{2}} \right) + \frac{a_2}{2} x^{r_2} \left( 1 + x^{\frac{a_2}{2}} \right) \]

be defined as a branch function, i.e.

\[ f(x) = \begin{cases} 
0 & x = 0, \\
\frac{a_1}{2} x^{r_1} & x \in \gamma < \gamma^2 >, \\
\frac{a_2}{2} x^{r_2} & x < \gamma^2 >. 
\end{cases} \]

Then \( f(x) \) is a PP over \( \mathbb{F}_q \) if and only if \( \gcd(r_1 r_2, \frac{4}{a_2}) = 1 \) and \( \{(-1)^{r_1} a_1^{\frac{a_2-1}{2}}, a_2^{\frac{a_2-1}{2}}\} = \{-1, 1\} \).

**Proof.** Since

\[ x^{\frac{a_2-1}{2}} \circ f(x) = \begin{cases} 
\frac{a_1}{2} x^{\frac{r_1(q-1)}{2}} & x \in \gamma < \gamma^2 >, \\
\frac{a_2}{2} x^{\frac{r_2(q-1)}{2}} & x < \gamma^2 >, 
\end{cases} \]

thus we have the following commutative diagram

\[
\begin{array}{c}
\mathbb{F}_q^n \\ x^{\frac{a_2-1}{2}} \downarrow \\
\{ -1, 1 \} \downarrow h \\
\{ -1, 1 \}
\end{array}
\]
where \((h(-1), h(1)) = (\varepsilon^{-1}a_1, a_2x^2)\). Observe that \(a_1x^r_1\) permutes \(\gamma < \gamma^2 >\) if and only if \(\gcd(r_1, \frac{a_1}{\gamma}) = 1\), and \(a_2x^r_2\) permutes \(\gamma^2 >\) if and only if \(\gcd(r_1, \frac{a_2}{\gamma}) = 1\). Therefore by AGW criterion \(f(x)\) is a PP over \(\mathbb{F}_q\) if and only if \(\gcd(r_1, \frac{a_1}{\gamma}) = 1, a_1a_2 = \{1, 1\}\). This completes the proof.

\[\text{Corollary 4.1.} \quad \text{Let } f(x) \text{ be a AGW-PP defined as in Lemma 4.2. We have}

(i) If \(r_1\) is odd, then \(f(x)\) is a PP if and only if \(\gcd(r_1r_2, \frac{a_1}{\gamma}) = 1\) and \(a_1a_2 = \{1, 1\}\). (ii) If \(r_1\) is even, then \(f(x)\) is a PP if and only if \(\gcd(r_1r_2, \frac{a_2}{\gamma}) = 1\) and \(a_1a_2 = \{1, 1\}\). Moreover, the number of such PPs is \(\varphi(q-1)^2\langle \frac{a_1}{\gamma}\rangle /2\), where \(\varphi(x)\) is the Euler function.

\[\text{Proof.} \quad \text{By Lemma 4.2, the proofs of (i) and (ii) are obvious. For the number of such PPs, it is easy to see that} \quad a_1x^{r_1}, a_1^{r_1}x^{r_2}, 1 \leq r_1, r_2 \leq q - 1, a_1, a_2 \in \mathbb{F}_q^* \quad \text{are distinct bijections on} \quad \gamma < \gamma^2 > \quad \text{if and only if} \quad \gcd(r_1r_2, q - 1) = 1 \quad \text{and} \quad r_1 \equiv r_2 \quad \text{mod} \quad (q - 1)/2. \quad \text{Therefore the number of such bijections on} \quad \gamma < \gamma^2 > \quad \text{is} \quad \varphi(q-1)^2\langle \frac{a_1}{\gamma}\rangle. \quad \text{The same result holds for the bijections on} \quad < \gamma^2 >. \quad \text{Hence, by (i) and (ii), the total number of such PPs is} \quad \varphi(q-1)^2\langle \frac{a_1}{\gamma}\rangle /2.\]

\[\text{Remark:} \quad \text{In fact, all such PPs in Corollary 4.1 form a group with the operation of composition of functions. We denote this group as} \quad G_2, \quad \text{then} \quad \sharp(G_2) = \varphi(q-1)^2\langle \frac{a_1}{\gamma}\rangle /2, \quad \text{and all functions of the form} \quad ax^r, a \in \mathbb{F}_q^*, \gcd(r, q - 1) = 1, 1 \leq r \leq q - 1 \quad \text{form a subgroup of} \quad G_2. \quad \text{Usually,} \quad G_2 \quad \text{is not an abelian group, and thus it is not easy to give all elements of order 2 (the involutions). We will discuss this question in another paper.}\]

\[\text{References}\]

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