ON THE RAYLEIGH-BÉNARD-MARANGONI PROBLEM: THEORETICAL AND NUMERICAL ANALYSIS

JHEAN E. PÉREZ-LÓPEZ
DIEGO A. RUEDA-GÓMEZ AND ÉLDER J. VILLAMIZAR-ROA*

Universidad Industrial de Santander
Escuela de Matemáticas
Bucaramanga, A.A. 678, Colombia

(Communicated by Yajuan Sun)

Abstract. This paper is devoted to the theoretical and numerical analysis of the non-stationary Rayleigh-Bénard-Marangoni (RBM) system. We analyze the existence of global weak solutions for the non-stationary RBM system in polyhedral domains of $\mathbb{R}^3$ and the convergence, in the norm of $L^2(\Omega)$, to the corresponding stationary solution. Additionally, we develop a numerical scheme for approximating the weak solutions of the non-stationary RBM system, based on a mixed approximation: finite element approximation in space and finite differences in time. After proving the unconditional well-posedness of the numerical scheme, we analyze some error estimates and establish a convergence analysis. Finally, we present some numerical simulations to validate the behavior of our scheme.

1. Introduction. The problem of instabilities in the fluid movements by temperature gradients is a classical subject in fluid mechanics [11, 24]. It is well known that two different effects are responsible for the onset of motion when the temperature difference becomes larger than a certain threshold: gravity and capillary forces. When the effects of buoyancy and surface tension are taken into account, the problem is called Rayleigh-Bénard-Marangoni (RBM) problem [22]. In order to describe the RBM problem, we consider a container with a horizontal fluid layer driven by a temperature gradient from below, with the bottom surface and the lateral walls rigid and the upper surface open to the atmosphere. Due to heating, the fluid in the bottom surface expands and it becomes lighter than the fluid in the upper surface; thus, by effect of the buoyancy, the liquid is potentially unstable and it tends to redistribute. However, this natural tendency will be controlled by its own viscosity. On the other hand, the temperature gradients in the upper surface, which is free to the atmosphere, produces changes in its surface tension. Therefore, it is expected

2020 Mathematics Subject Classification. Primary: 35Q35, 76D03, 76D05, 35D30; Secondary: 65L60, 65L70.

Key words and phrases. Rayleigh-Bénard-Marangoni system, global weak solutions, finite elements, convergence rates, error estimates.

The first and third authors were supported by Vicerrectoría de Investigación y Extensión of the Universidad Industrial de Santander (UIS), proyecto Capital semilla-2412. The second author was supported by Vicerrectoría de Investigación y Extensión-UIS.

* Corresponding author: Élder J. Villamizar-Roa.
that the temperature gradient exceeds a critical value, above which the instability
can be manifested (see for example \cite{14}, and references therein).

The first experiments showing the pattern formation in fluids were developed
by Henri Bénard in 1900 (cf. \cite{1}). In his experiments, Bénard considered a thin
layer of a variety of fluids in a metal plate maintained at a constant temperature.
The upper surface was open to the air with a lower temperature. Bénard found
that with the temperature growth, at some time, a state of instability was gener-
ated forming patterns of hexagonal cells, all alike and correctly aligned. From a
theoretical point of view, the first interpretation of thermal convection was given
by Lord Rayleigh in 1916 (see \cite{26}). Motivated by Bénard’s experiment, Rayleigh
considered a physical set-up in which a fluid is filling a container confined between
two horizontal thermally conductive plates and the fluid was being heated from
below. Rayleigh considered that the origin of the instability was due uniquely to
the buoyancy effect, finding a theoretical match with the experiments reported by
Bénard; nevertheless, it is known now that the Rayleigh theory is not adequate for
explaining the convective mechanism observed by Bénard. Indeed, Pearson \cite{24}
in 1958 discovered a second destabilizing mechanism: the changes in surface tension,
that were actually causing the Bénard convective cells. Rayleigh, using a plate in
the upper surface, was eliminating this second mechanism. The changes in the sur-
fase tension may depend on the temperature or external elements in the surface,
and this dependence is called capillarity or Marangoni effect (see \cite{15, 16, 19}
and references therein). The importance of the Marangoni effect in Bénard’s experi-
ments was established by Block in \cite{2} from an experimental point of view, and by
Pearson \cite{24} theoretically. Now is recognized that the Marangoni effect is the main
cause of instability and convection in the Bénard original experiments \cite{23}. Previous
considerations originated the so called Rayleigh-Bénard-Marangoni problem.

In order to establish the mathematical model describing the RBM problem, we
consider a horizontal layer of fluid of height \(d\) (\(x_3\)-coordinate), in a container of
length \(L\) (\(x_2\)-coordinate) and width \(l\) (\(x_1\)-coordinate) (see \cite{23, 28}). Thus, the
domain where the fluid is confined is given by \(\Omega = [0, l] \times [0, L] \times [0, d]\), with
boundary \(\partial \Omega = \Gamma_0 \cup \Gamma_1\), where \(\Gamma_1 = \partial \Omega \cap \{x_3 = d\}\), and \(\Gamma_0 = \partial \Omega \setminus \Gamma_1\). The
bottom of the container and the side faces are rigid and the the upper surface
is free to the atmosphere. The equations that describe the RBM problem are
given by a coupling between the Navier-Stokes system and the energy equation.
According to the Oberbeck-Boussinesq approximations \cite{4}, which establish that all
thermodynamic coefficients (viscosity, heat capacity, conductivity) are considered
constants except in the case of density in the flotation therm (\(\rho \mathbf{g}\)), it holds that the
only forcing term in the fluid equation is given by \(\rho_0 [1 - \alpha (T - T_a)]\), where \(\rho_0\) is
the density measured at room temperature \(T_a\), \(\alpha > 0\) is the volumetric expansion
coefficient, and \(T\) is the temperature. It is assumed that surface tension is a function
of temperature, and is approximated by \(\sigma(T) = \sigma_0 - \gamma (T - T_a)\), where \(\sigma_0\) is the
surface tension measured at \(T_a\), and \(\gamma > 0\) is the ratio of change of surface tension
with temperature. Finally, we consider that the free surface is not distorted, that
is, the vertical component of the velocity on the free surface will always be zero.
Thus, the equations that govern the phenomenon are given by

\[
\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho_0 (\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \pi - \mu \Delta \mathbf{u} = \rho_0 [1 - \alpha (T - T_a)] \mathbf{g} \quad \text{in } \Omega \times (0, T),
\]

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T),
\]
ON THE RAYLEIGH-BÉNARD-MARANGONI PROBLEM

\[
\rho_0 \hat{C}_p \frac{\partial T}{\partial t} + \rho_0 \hat{C}_p (\mathbf{u} \cdot \nabla T) = K \Delta T \quad \text{in } \Omega \times (0, T).
\]

(3)

Here, the unknowns are \( \mathbf{u}(t, \vec{x}) = (u_1(t, \vec{x}), u_2(t, \vec{x}), u_3(t, \vec{x})) \), \( T(t, \vec{x}) \) and \( \pi(t, \vec{x}) \), denoting the velocity, the temperature, and the pressure of the fluid, respectively, at \( \vec{x} \in \Omega \) and \( t \in (0, T) \). \( \hat{C}_p \) represents the heat capacity per unit of mass of the fluid, \( K \) the thermal conductivity, and \( \mu \) the viscosity coefficient. In order to express the variables in a dimensionless form, the following change of variables is considered (cf. [23]):

\[
x'_i = \frac{x_i}{d}, \quad u'_i = \frac{du_i}{\kappa} \quad \text{for} \quad i = 1, 2, 3, \quad T' = \frac{T - T_a}{T_a}, \quad \pi' = \frac{d^2 \pi}{\rho_0 \mu \kappa}, \quad t' = \frac{kt}{d^2},
\]

where \( T_c \) is the bottom temperature of the container, \( T_a = T_c - T_a, \kappa = \frac{K}{\rho_0 \hat{C}_p} \) and \( \nu = \frac{\mu}{\rho_0} \). Taking into account this change of variables in the equations (1)-(3) and omitting the primes, one gets the system

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = Pr \left[ (b + RT) \mathbf{e}_3 - \nabla \pi + \Delta \mathbf{u} \right] \quad \text{in } \Omega \times (0, T),
\]

(4)

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T),
\]

(5)

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T \quad \text{in } \Omega \times (0, T),
\]

(6)

where \( \Omega = [0, l/d] \times [0, L/d] \times [0, 1] \), \( Pr = \frac{\nu}{\kappa} \), \( R = \frac{g \alpha T_a d^3}{\kappa \nu} \), \( b = -\frac{gd^3}{\kappa \nu} \) and \( \mathbf{e}_3 = (0, 0, 1) \) (the unit vector in the third direction). The number \( Pr \) is known as the Prandtl number, and represents the relationship between the diffusion rate of amount of movement and the diffusion rate of heat in the fluid; \( R \) is known as the Rayleigh number and measures the effect of flotation, and \( g \) is the acceleration due to gravity. The following boundary conditions are now imposed on the system

\[
u_i \big|_{\Gamma_0} = 0, \quad i = 1, 2 \quad u_3 \big|_{\partial \Omega} = 0 \quad \text{in } (0, T),
\]

(7)

\[
\left( \frac{\partial u_i}{\partial \bar{n}} + M \frac{\partial T}{\partial x_i} \right) \bigg|_{\Gamma_1} = 0, \quad i = 1, 2 \quad \text{in } (0, T),
\]

(8)

\[
\left. \frac{\partial T}{\partial \bar{n}} \right|_{\Gamma_0 \setminus \{x_3 = 0\}} = 0, \quad \left. \left( \frac{\partial T}{\partial \bar{n}} + B T \right) \right|_{\Gamma_3} = 0, \quad T \big|_{\{x_3 = 0\}} = 1 \quad \text{in } (0, T),
\]

(9)

where \( \bar{n} = (n_1, n_2, n_3) \) is the exterior normal vector, \( M = \frac{\gamma T_a d}{\rho_0 \mu k} \), \( B = \frac{hd}{K} \) and \( h \) is the heat exchange coefficient of the surface with the atmosphere. We also consider initial conditions

\[
\mathbf{u}(\vec{x}, 0) = \mathbf{u}_0(\vec{x}) \quad \text{and} \quad T(\vec{x}, 0) = T_0(\vec{x}) \quad \text{in } \Omega,
\]

(10)

where \( \mathbf{u}_0(\vec{x}) \) and \( T_0(\vec{x}) \) are given functions, with \( \nabla \cdot \mathbf{u}_0(\vec{x}) = 0 \). The boundary conditions for velocity in (7) indicate that there is no liquid slip on the rigid surfaces and there is no deformation on the free surface. The condition (8) takes into account the Marangoni effect, as explained in [15]. The conditions for temperature in (9) say that there is no heat flow (adiabatic) on the side surfaces, heat flow is allowed on the free surface, and the lower surface is maintained at isothermal temperature equal to 1 \((T_c \text{ in the original form})\). When only is considered the buoyancy as the destabilizing mechanism (neglecting the surface tension) the model is commonly
called Rayleigh-Bénard convection (RB) problem. For details about the current understanding of this problem see [5, 13, 20].

The main mathematical difficulty of the RBM in comparison with the RB model is due to the presence of tangential derivatives of the temperature and normal derivatives of the velocities field (condition (8)). There are results for the existence of solutions, including the existence of bifurcations, stability of steady solutions, related to the RB problem (see [7, 17, 18, 25, 27] and references therein). In relation to the RBM problem, the literature is scarce. Some numerical results on the stationary solutions and the existence of bifurcations in different domains have been obtained in [6, 11, 12]; however not much is known from the theoretical point of view [23, 28]. In [23] the authors prove the existence of strong solutions for the stationary RBM problem in a bounded domain flat on the top, bifurcating from the basic heat conductive state. To deal with the mathematical difficulty brought on by the surface tension in the boundary conditions, the authors introduce a weak formulation where the boundary integral is transformed into an equivalent integral defined on the whole domain. In [28] the authors study a boundary control problem associated to the stationary RBM problem with controls for the velocity and the temperature on parts of the boundary. They analyze the existence, uniqueness and regularity of weak solutions for the stationary RBM system in polyhedral domains, and then, they prove the existence of an optimal solution. In this paper, we want to complete the existing gap about theoretical and numerical analysis of the non-stationary RBM problem. We first develop a detailed analysis on the existence of global weak solutions for the non-stationary RBM model and the convergence, at infinity, to the basic solution of the stationary RBM system. After that, we show a numerical scheme for approximating the weak solutions of the non-stationary RBM system, based on a mixed approximation by using finite element approximations in space and finite differences in time. We analyze the unconditional well-posedness of the numerical scheme, prove some error estimates and establish a convergence analysis. Finally, we present some numerical simulations to validate the behavior of our scheme.

This work is divided into five sections. In Section 2, we give some definitions and preliminary results. In Section 3, we consider the stationary RBM model, recalling some previous results related to the existence and uniqueness of stationary weak solutions. In Section 4, we analyze the existence of weak solutions of the non-stationary RBM model and establish a stability result, which shows that, as $t$ goes to $\infty$, the weak solution of the evolution problem converges in the $L^2(\Omega)$-norm to the only solution of the associated stationary problem, provided that the parameters $M$, $Pr$, $B$ and $R$ satisfy some smallness conditions. In Section 5, we develop a numerical scheme for approximating the weak solutions of the non-stationary RBM system and, finally, in Section 6, we provide some numerical simulations in agreement with the theoretical results.

2. Preliminaries. In this section we recall some definitions and establish some preliminary results which will support our analysis. As usual, the Sobolev space $W^{m,p}(\Omega)$, $(1 \leq p \leq \infty$ and $m$ integer) is the Banach space of the classes of measurable functions $f$, such that $f$ and all its distributional derivatives of order $\leq m$, belong to $L^p(\Omega)$. In particular, for $p = 2$, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$, which is a Hilbert space with inner product given by
Thus, we conclude that by integrating the second equation in (12) twice with respect to \( x \) we obtain the system (4)-(6) is reduced to a basic solution which will be important for the development of the work. We are tempted to find a solution where \( u \in H^1(\Omega) \) such that \( u|_{\Gamma} = 0 \) where \( \Gamma \subset \partial \Omega \) is such that \( |\Gamma| \neq 0 \).

Let \( 1 \leq p \leq \infty \), \( T > 0 \) and \( X \) a Banach space. We denote by \( L^p(0,T;X) \) the Banach space of all classes of measurable functions \( f : [0,T] \rightarrow X \) such that \( \|f(\cdot)\|_X \in L^p(0,T) \).

We recall the following generalized Poincaré inequality which we will use to control the \( H^1(\Omega) \)-norm in terms of the \( L^2(\Omega) \)-norm of the gradient of functions \( u \in H^1(\Omega) \) such that \( u|_{\Gamma} = 0 \) where \( \Gamma \subset \partial \Omega \) is such that \( |\Gamma| \neq 0 \).

**Theorem 2.1.** (Friedrichs Inequality). Let \( \Omega \subset \mathbb{R}^n \) a bounded domain with Lipschitz boundary \( \partial \Omega \), and let \( \Gamma \subset \partial \Omega \) be such that \( |\Gamma| \neq 0 \). Then, for \( u \in W^{1,2}(\Omega) \) it follows that

\[
\|u\|_{W^{1,2}(\Omega)} \leq C \left( \int_{\Gamma} |u|^2 \, ds + \|\nabla u\|_{L^2(\Omega)}^2 \right)^{1/2},
\]

where \( C \) is independent of \( u \).

Finally, the letter \( C \) denotes different positive constants (independent of discrete parameters) which may change from line to line (or even within the same line).

When we consider a \( L^p \) or \( W^{k,p} \)-norm on the whole domain \( \Omega \), we simply denote \( \| \cdot \|_{L^p(\Omega)} \) instead of \( \| \cdot \|_{L^p}, \| \cdot \|_{W^{k,p}(\Omega)} \).

3. Preliminary analysis of the steady problem.

3.1. Basic solution of the steady problem. This subsection deals with the stationary problem associated to the system (4)-(6). We first recall the existence of a basic solution which will be important for the development of the work.

Due to physical and experimental considerations (at least at low temperatures), we are tempted to find a solution where \( u = 0 \), and where the temperature depends only on the \( x_3 \)-variable. Under these considerations, the stationary version of the system (4)-(6) is reduced to

\[
([b + RT] \vec{c}_3 - \nabla \pi) = 0 \text{ and } \frac{\partial^2 T}{\partial x_3^2} = 0 \text{ in } \Omega. \quad (12)
\]

By integrating the second equation in (12) (twice) with respect to \( x_3 \), we conclude that \( T \) has the form

\[
T(x_1, x_2, x_3) = c_2 x_3 + c_1.
\]

From the boundary conditions (7)-(9) it is easy to see that \( c_1 = 1 \) and \( c_2 = -\frac{B}{1 + B} \). Thus, \( T(x_1, x_2, x_3) = -\frac{B}{1 + B} x_3 + 1 \), and replacing this value of \( T \) in the first equation of (12) we obtain the system

\[
\frac{\partial \pi}{\partial x_1} = 0, \quad \frac{\partial \pi}{\partial x_2} = 0 \text{ and } \frac{\partial \pi}{\partial x_3} = b + R - \frac{RB}{1 + B} x_3. \quad (13)
\]

Integrating (13) with respect to \( x_3 \), we get

\[
\pi = (b + R) x_3 - \frac{RB}{1 + B} \frac{x_3^2}{2} + F(x_1, x_2),
\]

but using (13) and (13) we conclude that \( F(x_1, x_2) = C_\pi \), for some constant \( C_\pi \). Since at \( x_3 = 1 \) the pressure in the fluid is due only to the atmospheric pressure, the constant \( C_\pi \) must verify

\[
C_\pi = \pi_a - \left( b + R - \frac{RB}{2(1 + B)} \right),
\]

where \( (\cdot, \cdot) \) stand for the inner product in \( L^2(\Omega) \).
where \( \pi_a \) is the atmospheric pressure. The above considerations give the following solution of the stationary version of the system (4)-(6):

\[
(u_0, T_b, \pi_b) = \left(0, 1 - \frac{B}{1 + B} x_3, (b + R) x_3 - \frac{RB}{1 + B} \frac{x_3^2}{2} + C \pi\right),
\]

which is called the basic solution of the stationary system.

### 3.2. Perturbed stationary problem

In order to calculate the stability changes of the basic state, we introduce the perturbation fields \( u = u_0 + \bar{u}, T = T_b + \theta \) and \( \pi = \pi_b + \bar{\pi} \) in the stationary version of the system (4)-(6) and boundary conditions (7)-(9). For convenience in the notation we will omit the “bars”. Thus, the equations for the perturbations are

\[
\begin{align*}
\mathbf{u} \cdot \nabla \mathbf{u} &= Pr \left[ R \theta \bar{c}_3 - \nabla \pi + \Delta \mathbf{u} \right] \quad \text{in} \ \Omega, \\
\mathbf{u} \cdot \nabla \theta &= \Delta \theta + \frac{B}{1 + B} u_3 \quad \text{in} \ \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in} \ \Omega,
\end{align*}
\]

with homogeneous boundary conditions

\[
\begin{align*}
\mathbf{u} i |_{\Gamma_0} &= 0, & i = 1, 2 & u_3 |_{\partial \Omega} = 0, \\
\left( \frac{\partial u_i}{\partial \mathbf{n}} + M \frac{\partial \theta}{\partial x_i} \right) |_{\Gamma_1} &= 0, & i = 1, 2, \\
\frac{\partial \theta}{\partial \mathbf{n}} |_{\Gamma_0 \setminus \{x_3 = 0\}} &= 0, & \left( \frac{\partial \theta}{\partial \mathbf{n}} + B \theta \right) |_{\Gamma_1} &= 0, & \theta |_{\{x_3 = 0\}} = 0.
\end{align*}
\]

We begin by describing the function spaces necessary to define a weak solution to the perturbed stationary problem. Let \( H_{0, \Gamma}^1(\Omega) = \{ u \in H^1(\Omega) ; u |_{\Gamma \cap \partial \Omega} = 0 \} \); for convenience we denote \( H_{0, \Gamma}^1(\Omega) \) as \( H_0^1(\Omega) \). Let

\[
X = [H_{0, \Gamma}^1(\Omega)]^2 \times H_0^1(\Omega),
\]

with inner product and associated norm given respectively by

\[
\langle \mathbf{u}, \mathbf{v} \rangle_X = \sum_{i=1}^{3} \int_{\Omega} \nabla u_i \cdot \nabla v_i, \quad \| \mathbf{u} \|_X = \langle \mathbf{u}, \mathbf{u} \rangle_X^{1/2}.
\]

Thanks to the inequality (11) with \( \Gamma = \Gamma_0 \), and the definition of the norm in \( X \), is direct to prove that the norm induced in \( X \) by \( [H^1(\Omega)]^3 \) and the norm (19) are equivalent; so, \( X \) is a Hilbert space. Let now \( X_0 = \{ \mathbf{u} \in X : \nabla \cdot \mathbf{u} = 0 \} \). It follows that \( X_0 \) is a closed subspace of \( X \); therefore, \( X_0 \) is also a Hilbert space. Now, we denote \( Y = H_{0, \{x_3 = 0\}}^1(\Omega) \) with inner product and norm

\[
\langle \theta, \nu \rangle_Y = \int_{\Omega} \nabla \theta \cdot \nabla \nu + B \int_{\Gamma_1} \theta \nu, \quad \| \theta \|_Y = \langle \theta, \theta \rangle_Y^{1/2}.
\]

The space \( Y \) is a closed subspace of \( H^1(\Omega) \), and therefore, it is a Hilbert space; in addition, the inherited norm and the norm defined in (20) are equivalent. Finally, let \( X_0 = X_0 \times Y \), with inner product and norm

\[
\langle \mathbf{U}, \mathbf{V} \rangle_X = Pr \langle \mathbf{u}, \mathbf{V} \rangle_X + \langle \theta, \nu \rangle_Y, \quad \| \mathbf{U} \|_X = \langle \mathbf{U}, \mathbf{U} \rangle_X^{1/2},
\]

where \( X = X \times Y \). In order to establish the variational formulation of the problem, the following lemma will be necessary.
Lemma 3.1. [23]. Let \( \Omega \subset \mathbb{R}^3 \) a domain with Lipschitz boundary \( \partial \Omega \), such that \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), with \( \Gamma_1 \subset \{ x_3 = \text{constant} \} \). If \( \theta \in H^2(\Omega) \), then
\[
\int_{\Gamma_1} \frac{\partial \theta}{\partial x_1} v_1 + \frac{\partial \theta}{\partial x_2} v_2 = \int_\Omega \nabla \theta \cdot \frac{\partial \nu}{\partial x_3}, \quad \forall \nu \in X_0.
\]

Now we will establish the variational formulation of our problem; for that we proceed as usual, that is, we multiply the equations (15) and (16) by the components of a test function \( \nu = (v, \sigma) \in X_0 \). This puts us in position to establish the variational formulation of the problem (15)-(18).

Definition 3.2. (Weak Formulation). A vector \( \mathbf{U} = (u, \theta) \in X_0 \) is said a weak solution of (15)-(16) if
\[
\sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial u_j}{\partial x_i} v_j + Pr \int_{\Omega} \nabla \theta \cdot \frac{\partial \nu}{\partial x_3} + \sum_{i=1}^3 \int_{\Omega} \nabla u_i \cdot \nabla v_i = PrR \int_\Omega \theta_0 \cdot v,
\]
\[
\sum_{i=1}^3 \int_{\Omega} u_i \frac{\partial \theta}{\partial x_i} + \int_{\Gamma_1} B \theta \sigma + \int_{\Omega} \nabla \theta \cdot \nabla \sigma = \frac{B}{1+B} \int_\Omega u_3 \sigma,
\]
for all \( \mathbf{V} = (v, \sigma) \in X_0 \).

From Definition 3.2, we want to formulate problem (21) in operators form. To do this, we define the bilinear forms \( a, \Pi : X_0 \times X_0 \to \mathbb{R} \), and the trilinear form \( b : X_0 \times X_0 \times X_0 \to \mathbb{R} \) by
\[
a(\mathbf{U}, \mathbf{V}) = Pr \sum_{i=1}^3 \int_{\Omega} \nabla u_i \cdot \nabla v_i + \int_{\Omega} \nabla \theta \cdot \nabla \sigma + B \int_{\Gamma_1} \theta \sigma + MP \int_{\Omega} \nabla \theta \cdot \frac{\partial \nu}{\partial x_3},
\]
\[
\Pi(\mathbf{U}, \mathbf{V}) = PrR \int_\Omega \theta_0 + \frac{B}{1+B} \int_\Omega u_3 \sigma,
\]
\[
b(\mathbf{U}, \mathbf{V}, \mathbf{W}) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial w_j}{\partial x_i}, + \sum_{i=1}^3 \int_{\Omega} w_i \frac{\partial \sigma}{\partial x_i},
\]
where \( \mathbf{U} = (u, \theta), \mathbf{V} = (v, \sigma) \) and \( \mathbf{W} = (w, \Theta) \) are elements of \( X_0 \). With the previous notation, we establish the following definition of a weak solution for the perturbed stationary problem (15)-(17).

Definition 3.3. We say that \( \mathbf{U} \in X_0 \) is a weak solution of (15)-(16), with boundary conditions (18), if
\[
a(\mathbf{U}, \mathbf{V}) + b(\mathbf{U}, \mathbf{U}, \mathbf{V}) - \Pi(\mathbf{U}, \mathbf{V}) = 0, \quad \forall \mathbf{V} \in X_0.
\]

3.3. Existence and uniqueness of the perturbed stationary problem. The existence of nontrivial weak solutions of the perturbed stationary problem was obtained in [23] under a set of hypotheses. It is an open problem to show if such hypotheses hold true (see also [28]). As noted above, for the variational equation (21) we have that \( \mathbf{U} = (0, 0) \in X_0 \) is a solution. We are now interested in studying the conditions under which one has that this solution is unique. In order to demonstrate uniqueness, we need the following lemmas.

Lemma 3.4. ([23] Lemma 4.4) The bilinear form \( a : X_0 \times X_0 \to \mathbb{R} \) is continuous. Moreover, if \( M \sqrt{Pr} < 2 \), then for any \( B > 0 \), the bilinear form \( a \) is coercive.
Thus, \(b \text{ continuous. Moreover, } b(U, V, V) = 0 \) for all \(U, V \in X_0\). As a consequence, we obtain that \(b(U, V, U) = -b(U, U, V)\).

**Lemma 3.5.** ([23] Lemma 4.5) The trilinear form \(b : X_0 \times X_0 \times X_0 \rightarrow \mathbb{R}\) is continuous. Moreover, \(b(U, V, V) = 0\) for all \(U, V \in X_0\). As a consequence, we obtain that \(b(U, V, U) = -b(U, U, V)\).

**Lemma 3.6.** The bilinear form \(\Pi : X_0 \times X_0 \rightarrow \mathbb{R}\) is continuous.

**Proof.** For \(U = (u, \theta)\) and \(V = (v, \sigma)\) in \(X_0\), by using the Hölder inequality, Sobolev embeddings and (11), we have

\[
\|\Pi(U, V)\| \leq \left| PrR \int_\Omega \theta v_2 \right| + \left| \frac{B}{1 + B} \int_\Omega u_3 \sigma \right|
\leq PrR \|\theta\|_{L^2} \|v_3\|_{L^2} + \frac{B}{1 + B} \|u_3\|_{L^2} \|\sigma\|_{L^2}
\leq PrR \|\theta\|_{W^{1,2}} \|v_3\|_{W^{1,2}} + \frac{B}{1 + B} \|u_3\|_{W^{1,2}} \|\sigma\|_{W^{1,2}}
\leq C_1 PrR \|\nabla \theta\|_{L^2} \|\nabla v_3\|_{L^2} + C_2 \frac{B}{1 + B} \|\nabla u_3\|_{L^2} \|\nabla \sigma\|_{L^2}
\leq C_1 PrR \|\theta\|_Y \|v\|_X + C_2 \frac{B}{1 + B} \|u\|_X \|\sigma\|_Y
\leq C \left( R + \frac{B}{Pr(1 + B)} \right) \|U\|_X \|V\|_X.
\]

\[\square\]

We are now in a position to prove the uniqueness of solution for the perturbed stationary problem.

**Proposition 1.** Let \(M \sqrt{Pr} < 2\) and \(R, B\) small enough or \(M \sqrt{Pr} < 2, \ R \text{ small and } Pr \text{ big enough}\). Then, the unique solution to the problem (21) is the trivial solution.

**Proof.** Suppose that \(U^* \in X_0\) is a solution of (21). This implies that

\[a(U^*, V) + b(U^*, U^*, V) - \Pi(U^*, V) = 0, \quad \forall V \in X_0.\]

In particular, taking \(V = U^*\) we get

\[a(U^*, U^*) + b(U^*, U^*, U^*) - \Pi(U^*, U^*) = 0. \quad (22)\]

Using Lemma 3.5, equation (22) takes the form

\[a(U^*, U^*) - \Pi(U^*, U^*) = 0.\]

Now, from Lemmas 3.4 and 3.6 we have

\[C_1 \|U^*\|_X^2 \leq a(U^*, U^*) \quad \text{and} \quad \Pi(U^*, U^*) \leq C_2 \left( R + \frac{B}{Pr(1 + B)} \right) \|U^*\|_X^2.\]

Thus,

\[\left( C_1 - C_2 \left( R + \frac{B}{Pr(1 + B)} \right) \right) \|U^*\|_X^2 \leq 0. \quad (23)\]

Since \(C_1 - C_2 \left( R + \frac{B}{Pr(1 + B)} \right) > 0\) owing to the conditions imposed on the dimensionless numbers and from (23), we conclude that \(\|U^*\|_X^2 = 0\), this is, \(U^* = (0, 0)\). \(\square\)

**Remark 1.** Indirectly, Proposition 1 shows that the basic solution is the only solution to the stationary system related to (4)-(6) with boundary conditions (7)-(9).
4. **Theoretical analysis of the evolution problem.** In this section we consider the RBM problem in the evolution case (4)-(10). First we consider the perturbed evolution problem by the basic solution, and then we analyze the behaviour at infinity.

4.1. **Perturbed evolution problem.** If \((u, T, p)\) is a solution of (4)-(10), denoting by \((\bar{u}, \bar{\theta}, \bar{p})\) the difference \((u - u_0, T - T_b, p - p_b)\), formally we have that \((\bar{u}, \bar{\theta}, \bar{p})\) solves the following system:

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} &= Pr \left[ R \bar{\theta} \bar{e}_3 - \nabla \bar{\theta} + \Delta \bar{u} \right] \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \bar{\theta}}{\partial t} + \bar{u} \cdot \nabla \bar{\theta} &= \Delta \bar{\theta} + \frac{B}{1 + B} \bar{u}_3 \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot \bar{u} &= 0 \quad \text{in } \Omega \times (0, T), \\
\end{align*}
\]

with homogeneous boundary conditions

\[
\begin{align*}
\bar{u}_i \mid_{\Gamma_0} &= 0 \quad i = 1, 2, \quad \bar{u}_3 \mid_{\partial \Omega} = 0 \quad \text{in } (0, T), \\
\left. \frac{\partial \bar{u}}{\partial n} \right|_{\Gamma_1 \setminus \{x_3 = 0\}} &= 0, \quad \left. \frac{\partial \bar{\theta}}{\partial n} \right|_{\Gamma_1} = 0, \quad \theta \mid_{\{x_3 = 0\}} = 0 \quad \text{in } (0, T),
\end{align*}
\]

and initial conditions

\[
\begin{align*}
\bar{u}(\bar{x}, 0) &= u_0(\bar{x}) - u_b(\bar{x}) \quad \text{in } \Omega, \\
\bar{\theta}(\bar{x}, 0) &= T_0(\bar{x}) - T_b(\bar{x}) \quad \text{in } \Omega.
\end{align*}
\]

For convenience, in that follows we omit the bars. Proceeding in a similar way to the stationary case, we obtain that if \(U(t) \in X_0\) is a solution for (24)-(28), then we must have that

\[
\frac{d}{dt} \langle U(t), V \rangle + a(U(t), V) + b(U(t), U(t), V) = \Pi(U(t), V) \quad \forall V \in X_0,
\]

with initial condition

\[
U(\bar{x}, 0) = U_0(\bar{x}) = (u(\bar{x}, 0), \bar{\theta}(\bar{x}, 0)).
\]

**Definition 4.1. (Weak solution)** Let \(u_0 \in L^2(\Omega)\) with \(\nabla \cdot u_0 = 0\) and \(\theta_0 \in L^2(\Omega)\). A vector function \(U(x, t) = (u(x, t), \theta(x, t))\) is called a weak solution to the perturbed evolution problem (24)-(28) if

\[
U \in L^2(0, T; X_0) \cap L^\infty(0, T; L^2), \quad \text{for all } T > 0,
\]

and (29) and (30) are verified.

Following [30], Ch. 3, if \(U \in L^2(0, T; X_0)\) and satisfies (29), then \(U\) is almost everywhere equal to some continuous function, so that (30) is meaningful. Related to the existence of a weak solution, we have the following theorem.

**Theorem 4.2.** Let \(u_0 \in L^2(\Omega)\) with \(\nabla \cdot u_0 = 0\), \(\theta_0 \in L^2(\Omega)\), \(M \sqrt{Pr} < 2\) and \(R, B\) small enough (or \(R\) small enough and \(Pr\) big enough). Then, the perturbed evolution problem (24)-(28) has a solution \(U(x, t)\) in the sense of Definition 4.1.

**Proof.** The proof is based on the Faedo-Galerkin method. For this, let \(\{W^n\}_{n=1}^\infty\) a Hilbert basis for \(X_0\), and let \(X^n_0\) the space generated by the first \(m\) elements of
this basis. For \( m \in \mathbb{N} \) fixed, we define the following approximated problem: find \( U^m(t) \in L^2(0,T;X^m) \) such that

\[
\frac{d}{dt}(U^m(t),W) + a(U^m(t),W) + b(U^m(t),\bar{U}^m(t),W) = \Pi(U^m(t),W) \tag{31}
\]

for all \( j = 1,2,...,m \), with initial condition \( U^m(\bar{x},0) = U^m_0(\bar{x}) \), where \( U^m_0 \) is the projection of \( U_0 \) on \( X^m_0 \). Thus, we look for \( U^m(t) \) in \( X^m_0 \) in the form

\[
U^m(t) = \sum_{i=1}^{m} g_{im}(t)W^i,
\]

where the \( g_{im} \) are functions defined on \([0,T]\). Then, \( (31) \) is reduced to

\[
\sum_{i=1}^{m} \frac{dg_{im}(t)}{dt}(W^i,W^j) + \sum_{i=1}^{m} g_{im}(t)a(W^i,W^j) + \sum_{i,k=1}^{m} g_{im}(t)g_{km}(t)b(W^k,W^i,W^j) = \sum_{i=1}^{m} g_{im}(t)\Pi(W^i,W^j),\tag{32}
\]

which is a nonlinear system of ordinary differential equations with initial conditions

\[
g_{im}(0) = r_j, \tag{33}
\]

where the coefficients \( r_j \) are such that \( U_0^m = \sum_{i=1}^{m} r_i W^i \).

By the Peano Theorem, it follows that \( (32) \), with initial conditions \( (33) \), has a maximal solution defined in some interval \([0,t_m]\). If \( t_m < T \) we must have that \( |g_{jm}(t)| \rightarrow \infty \) as \( t \rightarrow t_m \), which implies that \( \|U^m(t)\|_{L^2(\Omega)} \rightarrow \infty \) as \( t \rightarrow t_m \); however, the \textit{apriori} estimates shown below prove that actually \( t_m = T \). Now we prove the \textit{apriori} estimates. Multiplying equations \( (31) \) by \( g_{jm}(t) \), adding these equations for each \( j = 1,2,...,m \), and using Lemma 3.5 we obtain

\[
\frac{d}{dt}\|U^m(t)\|_{L^2(\Omega)}^2 + 2a(U^m(t),\bar{U}^m(t)) = 2\Pi(U^m(t),U^m(t)).
\]

Now, using Lemmas 3.4 and 3.6, we get

\[
\frac{d}{dt}\left(\|U^m(t)\|_{L^2(\Omega)}^2 + 2\left(C_1 - C_2\left(R + \frac{B}{Pr(1+B)}\right)\right)\|U^m(t)\|_X^2\right) \leq 0. \tag{34}
\]

From the previous inequality and the conditions in the dimensionless numbers, we conclude that

\[
\frac{d}{dt}\left(\|U^m(t)\|_{L^2(\Omega)}^2\right) \leq 0.
\]

Thus, we can conclude that

\[
\|U^m\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{35}
\]

where \( C \) does not depend on \( m \). On the other hand, integrating \( (34) \) and taking into account that \( \|U^m(0)\|_{L^2}^2 \leq \|U(0)\|_{L^2}^2 \), we obtain

\[
\|U^m(T)\|_{L^2}^2 + 2\left(C_1 - C_2\left(R + \frac{B}{Pr(1+B)}\right)\right)\int_0^T \|U^m(t)\|_X^2 dt \leq \|U(0)\|_{L^2}^2.
\]

Thus,

\[
\int_0^T \|U^m(t)\|_X^2 dt \leq \frac{\|U(0)\|_{L^2(\Omega)}^2}{2\left(C_1 - C_2\left(R + \frac{B}{Pr(1+B)}\right)\right)} = C, \tag{36}
\]
this is,
\[ \|U^m\|_{L^2(0,T;X_0)} \leq C, \]  
where \( C \) does not depend on \( m \). On the other hand, defining the operators \( \Pi U, A U \) and \( B U \) as
\[ \langle \Pi U, V \rangle \equiv \Pi(U, V), \quad \langle A U, V \rangle \equiv a(U, V) \quad \text{and} \quad \langle B U, V \rangle \equiv b(U, U, V), \]  
the problem (31) can be rewritten as
\[ \langle U^m_t(t), W^j \rangle + \langle A U^m(t), W^j \rangle + \langle B U^m(t), W^j \rangle = \langle \Pi U^m(t), W^j \rangle \quad \forall j = 1, 2, \ldots, m. \]
From Lemmas 3.4, 3.5 and 3.6 we know that these operators are linear and continuous on \( X_0 \). Moreover,
\[ \langle U^m_t(t), V \rangle = \langle \Pi U^m(t) - A U^m(t) - B U^m(t), V \rangle \quad \forall V \in X_0, \]  
which implies that
\[ U^m_t(t) = \Pi U^m(t) - A U^m(t) - B U^m(t). \]
In particular \( U^m_t(t) \in X_0 \) and
\[
\begin{align*}
\|U^m_t(t)\|_{X_0} & \leq \|\Pi U^m(t)\|_{X_0} + \|A U^m(t)\|_{X_0} + \|B U^m(t)\|_{X_0} \\
& \leq C_1 \|U^m(t)\|_{X} + C_2 \|U^m(t)\|_{X} + C_3 \|U^m(t)\|_{X}^2 \\
& \leq \frac{(C_1 + C_2)^2}{2} + \frac{\|U^m(t)\|_{X}^2}{2} + C_3 \|U^m(t)\|_{X}^2 \\
& \leq \frac{(C_1 + C_2)^2}{2} + \left(\frac{1}{2} + C_3\right) \|U^m(t)\|_{X}^2.
\end{align*}
\]
Integrating the previous inequality between 0 and \( T \) and using the estimate (36) we conclude that
\[ \|U^m_t\|_{L^1(0,T;X_0)} \leq C. \]  
The bounds (35) and (37) allow us to ensure the existence of a vector \( U \in L^2(0,T;X_0) \cap L^\infty(0,T;L^2(\Omega)) \) and a not relabeled subsequence \( \{U^m\} \) such that
\[ U^m \rightharpoonup U \quad \text{weakly in} \quad L^2(0,T;X_0), \]  
and
\[ U^m \rightharpoonup U \quad \text{weakly * in} \quad L^\infty(0,T;L^2(\Omega)). \]  
Moreover, from (37), (39) and the well known Aubin–Lions lemma of compactness (see [29]) we conclude that
\[ U^m \rightarrow U \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \]
Now we want to take the limit in \( m \). To do this, multiply the equation (31) by \( \psi(t) \in D(0,T) \), and integrate by parts in \( (0,T) \) to obtain
\[
- \int_0^T \langle U^m(t), \psi'(t)W^j \rangle + \int_0^T a(U^m(t), \psi(t)W^j) \\
+ \int_0^T b(U^m(t), U^m(t), \psi(t)W^j) = \int_0^T \Pi(U^m(t), \psi(t)W^j).
\]
Now we are going to analyze the term-to-term convergence in the previous equation. Since \( \psi'(t) \mathbf{W}^j \) belongs to \( L^1(0,T;L^2(\Omega)) \), we have that
\[
\int_0^T (\mathbf{U}(t), \psi'(t) \mathbf{W}^j) = \int_0^T (\mathbf{U}(t), \psi'(t) \mathbf{W}^j)_{L^2(\Omega)}
\]
and the convergence in \( (41) \) implies that
\[
\langle \mathbf{U}(t), \psi'(t) \mathbf{W}^j \rangle_{L^\infty(0,T;L^2(\Omega))} \rightarrow \langle \mathbf{U}(t), \psi'(t) \mathbf{W}^j \rangle_{L^\infty(0,T;L^2(\Omega))}.
\]
Now, for \( \mathbf{W} \in \mathbf{X}_0 \) fixed, we define \( A' \mathbf{W} : \mathbf{X}_0 \rightarrow \mathbb{R} \) by \( \langle A' \mathbf{W}, \mathbf{U} \rangle = a(\mathbf{U}, \mathbf{W}) \); then \( A' \mathbf{W} \in \mathbf{X}_0 \), and thus,
\[
\int_0^T a (\mathbf{U}(t), \psi(t) \mathbf{W}^j) = \int_0^T \langle A' \psi(t) \mathbf{W}^j, \mathbf{U}(t) \rangle_{\mathbf{X}_0', \mathbf{X}_0} = \langle A' \psi(t) \mathbf{W}^j, \mathbf{U}(0) \rangle_{L^2(\Omega)}.
\]
Since \( A' \psi(t) \mathbf{W}^j \in L^2(0,T;\mathbf{X}_0') \), by the convergence \( (40) \) it follows that
\[
\langle A' \psi(t) \mathbf{W}^j, \mathbf{U}(0) \rangle_{L^2(\Omega)} \rightarrow \langle A' \psi(t) \mathbf{W}^j, \mathbf{U}(0) \rangle_{L^2(\Omega)}.
\]
A similar analysis shows that
\[
\int_0^T \Pi (\mathbf{U}(t), \psi(t) \mathbf{W}^j) \rightarrow \int_0^T \Pi (\mathbf{U}(t), \psi(t) \mathbf{W}^j).
\]
To establish the convergence in the trilinear form, we assume that the elements \( \mathbf{W}^j \) are at least in \( C^1(\bar{\Omega}) \) (the general case follows by density) and use the Lemma 3.5. We have
\[
b (\mathbf{U}^m, \mathbf{U}^m, \psi(t) \mathbf{W}^j) = -b (\mathbf{U}^m, \psi(t) \mathbf{W}^j, \mathbf{U}^m)
\]
\[
= -\sum_{i,k=1}^3 \int_{\Omega} u_i^m \frac{\partial \psi(t) w_k^j}{\partial x_i} u_k^m - \sum_{i=1}^3 \int_{\Omega} u_i^m \frac{\partial \psi(t) \Theta^j}{\partial x_i} \theta^m.
\]
Each term \( \int_{\Omega} u_i^m \frac{\partial w_k^j}{\partial x_i} u_k^m \) verifies
\[
\left| \int_{\Omega} u_i^m u_k^m \frac{\partial \psi(t) w_k^j}{\partial x_i} - \int_{\Omega} u_i u_k \frac{\partial \psi(t) w_k^j}{\partial x_i} \right|
\]
\[
\leq \int_{\Omega} u_i^m (u_k^m - u_k) \frac{\partial \psi(t) w_k^j}{\partial x_i} + \int_{\Omega} u_k (u_i^m - u_i) \frac{\partial \psi(t) w_k^j}{\partial x_i}.
\]
Since \( \frac{\partial \psi(t) w_k^j}{\partial x_i} \in L^\infty(\Omega) \), using the Hölder inequality we obtain
\[
\int_{\Omega} u_i^m (u_k^m - u_k) \frac{\partial \psi(t) w_k^j}{\partial x_i} \leq \|u_i^m\|_{L^2} \|u_k^m - u_k\|_{L^2} \left\| \frac{\partial \psi(t) w_k^j}{\partial x_i} \right\|_{L^\infty}.
\]
Therefore, noting that \( \frac{\partial \psi(t) u^m}{\partial x_i} \in L^\infty(0, T; L^\infty(\Omega)) \) and using the Hölder inequality (in time) it follows that
\[
\int_0^T \int_\Omega \left| u^m_k (u^m_k - u_k) \right|^{1/2} \partial \psi(t) \frac{\partial \psi(t) u^m}{\partial x_i} \leq \int_0^T \| u^m_k \|_{L^2} \| u^m_k - u_k \|_{L^\infty} \| \frac{\partial \psi(t) u^m}{\partial x_i} \|_{L^\infty}
\]
\[
\leq \| u^m_k \|_{L^2(0, T; L^2(\Omega))} \| u^m_k - u_k \|_{L^2(0, T; L^2(\Omega))} \| \frac{\partial \psi(t) u^m}{\partial x_i} \|_{L^\infty(0, T; L^\infty(\Omega))}.
\]
We know that \( \{u^m_k\} \) is bounded in \( L^2(0, T; L^2(\Omega)) \) and \( u^m_k \to u_k \) in \( L^2(0, T; L^2(\Omega)) \); then we have that the whole term on the right side goes to zero as \( m \) tends to infinity. A similar analysis shows that
\[
\int_0^T b \left( U^m, U^m, \psi(t) W^j \right) \to \int_0^T b \left( U, U, \psi(t) W^j \right).
\]
Therefore,
\[
\int_0^T \left( U_t(t), W^j \right) \psi(t) + \int_0^T a \left( U(t), W^j \right) \psi(t)
\]
\[
+ \int_0^T b \left( U(t), U(t), W^j \right) \psi(t) = \int_0^T \Pi \left( U(t), W^j \right) \psi(t).
\]
By linearity we also have that
\[
\int_0^T \left( U_t(t), V \right) \psi(t) + \int_0^T a \left( U(t), V \right) \psi(t)
\]
\[
+ \int_0^T b \left( U(t), U(t), V \right) \psi(t) = \int_0^T \Pi \left( U(t), V \right) \psi(t), \tag{43}
\]
for any \( V \) that is a finite combination of the \( W^j \). In addition to the above, each of the terms in the equation depends linearly and continuously on the norm of \( X_0 \); therefore equality is still valid for each \( V \) in \( X_0 \) by density. Finally, equality (43) is valid for any \( \psi \) in \( D(0, T) \); thus, the following equality
\[
\frac{d}{dt} \left( U(t), V \right) + a \left( U(t), V \right) + b \left( U(t), U(t), V \right) = \Pi \left( U(t), V \right),
\]
is true in the sense of the distributions \( D(0, T) \), for each \( V \in X_0 \).

4.2. **Convergence to the steady solution.** In this subsection we will show that when \( t \to \infty \), the solution of the evolution problem converges in the norm of \( L^2(\Omega) \) to the only stationary solution studied in the Subsection 3.2, or equivalently, that when \( t \to \infty \) the solution of the perturbed evolution problem converges to zero in the norm of \( L^2(\Omega) \). In fact, in the notation of Subsection 3.2 we have that
\[
\left( U_t(t), V \right) + a \left( U(t), V \right) + b \left( U(t), U(t), V \right) = \Pi \left( U(t), V \right) \quad \forall V \in X_0.
\]
In particular, taking \( V = U \) in the last equality and applying Lemma 3.5 we can obtain
\[
\frac{d}{dt} \| U \|_{L^2}^2 + 2a \left( U, U \right) = 2\Pi \left( U, U \right).
\]
Thus, from Lemmas 3.4 and 3.6 we can conclude that
\[
\frac{d}{dt} \| U \|_{L^2}^2 \leq \left( 2C_2 \left( R + \frac{B}{Pr(1 + B)} \right) - 2C_1 \right) \| U \|_{X}^2.
\]
For $R$, $B$ small enough (or $R$ small enough and $Pr$ big enough) and using the Sobolev embeddings, it follows that
\[
\frac{d}{dt} \|U\|_{L^2}^2 \leq -C_3 \|U\|_{X}^2 \leq -C_4 \|U\|_{L^2}^2.
\] (44)
So, from (44) we conclude that $\|U(t)\|_{L^2}^2$ verifies the inequality
\[
\|U(t)\|_{L^2}^2 \leq \|U(0)\|_{L^2}^2 \exp(-C_4 t).
\]
Therefore, when $t \to \infty$ we have $U(t) \to (0, 0)$ in the norm of $L^2(\Omega)$.

5. Numerical analysis. In this section, we construct a numerical approximation for the weak solutions of the perturbed evolution problem (24)-(28). Notice that the weak formulation (29) eliminates the pressure variable due to the nature of the space $X_0$. However, it is well-known that the divergence-free condition can not be imposed on the finite element space for $u$ (cf. [8]); therefore, in order to establish a fully discrete numerical scheme, it will be necessary to consider the pressure variable.

In this scheme we consider finite element approximations in space and finite differences in time (considered for simplicity on a uniform partition of $[0, T]$ with time step $\Delta t = T/N$ : $(t_n = n\Delta t)_{n=0}^N$). In order to deal with the velocity trilinear form and the nonlinear convective term for the temperature equation, we will use the skew-symmetric forms $A$ and $\bar{A}$ given in (53)-(54) below. Concerning the space discretization, we consider finite element spaces: $X_u \times X_\pi \times X_0 \subset X \times L^2(\Omega) \times Y$ corresponding to a family of shape-regular and quasi-uniform triangulations of $\Omega$, $\{T_h\}_{h>0}$, made up of tetrahedra $K$, so that $\Omega = \bigcup_{K \in T_h} K$, where $h = \max_{K \in T_h} h_K$, with $h_K$ being the diameter of $K$. Along this section, the letter $C$ will denote different positive constants independent of the discrete parameters $\Delta t$ and $h$.

We assume that $X_u$ and $X_\pi$ satisfy the following discrete inf-sup condition: There exists a constant $\beta > 0$, independent of $h$, such that
\[
\sup_{v \in X_u \setminus \{0\}} \frac{-(w, \nabla \cdot v)}{\|v\|_{X_u}} \geq \beta \|w\|_{X_\pi}, \quad \forall w \in X_\pi.
\] (45)
We consider the hydrostatic Stokes projector $(P_u, P_\pi) : X_0 \times L^2(\Omega) \to X_u \times X_\pi$ such that $[P_u u, P_\pi \pi] \in X_u \times X_\pi$ satisfies
\[
\begin{cases}
Pr(\nabla(P_u u - u), \nabla v) - (P_\pi \pi - \pi, \nabla \cdot v) = 0, & \forall v \in X_u, \\
(\nabla \cdot (P_u u - u), \bar{\pi}) = 0, & \forall \bar{\pi} \in X_\pi,
\end{cases}
\] (46)
and the following stability and approximation properties (of order $O(h^r)$) hold [9] (for $r = 1$ or 2):
\[
\|u - P_u u\|_{H^1} + \frac{1}{h} \|u - P_u u\|_{L^2} \leq Ch^r \|u\|_{H^{r+1}}, \quad \forall u \in H^{r+1}(\Omega) \cap X_0,
\] (47)
\[
\|\pi - P_\pi \pi\|_{L^2} \leq Ch^r \|\pi\|_{H^r}, \quad \forall \pi \in H^r(\Omega) \cap L^2(\Omega),
\] (48)
\[
\|P_u u\|_{W^{1,6}} \leq C \|u\|_{H^2} \quad \text{and} \quad \|P_\pi \pi\|_{L^6} \leq C \|\pi\|_{H^1}.
\] (49)
Remark 2. (Choice of the discrete spaces) The finite element space $X_\theta$ is approximated by $P_l$-continuous, with $l \geq 1$. For the spaces $[X_u, X_\pi]$, we can consider the Taylor-Hood approximation ($r = 2$) $P_2 \times P_1$, or the approximation $P_1 - bubble \times P_1$ ($r = 1$) (cf. [8]; see also [3, 9] for these approximations in models with homogeneous Dirichlet boundary conditions only on a part of the boundary).
Moreover, we consider the following projection operator: \( P_\theta : Y \to \mathcal{X}_\theta \), such that for all \( \theta \in Y \), \( P_\theta \theta \in \mathcal{X}_\theta \) satisfies

\[
\nabla (P_\theta \theta - \theta), \nabla \sigma \rangle = 0, \quad \forall \sigma \in \mathcal{X}_\theta.
\]

Observe that from Lax-Milgram Theorem, we have that the projection operator \( P_\theta \) is well defined. Moreover, it is well known that the following stability and approximation properties hold:

\[
\| \theta - P_\theta \theta \|_{H^1} + \frac{1}{h} \| \theta - P_\theta \theta \|_{L^2} \leq Ch^l \| \theta \|_{H^{l+1}}, \quad \forall \theta \in H^{l+1}(\Omega) \cap Y,
\]

\[
\| P_\theta \theta \|_{H^1} \leq \| \theta \|_{H^1} \quad \text{and} \quad \| P_\theta \theta \|_{W^{1,6}} \leq \| \theta \|_{H^2}.
\]

Finally, we consider the following skew-symmetric trilinear forms which will be used in the formulation of the numerical scheme:

\[
\tilde{A}(v_1, v_2, v_3) = \frac{1}{2} \left[ ((v_1 \cdot \nabla)v_2, v_3) - (v_1 \cdot \nabla)v_3, v_2 \right]
\]

for all \( v_1, v_2, v_3 \in H^1(\Omega) \), and

\[
A(v, w_1, w_2) = \frac{1}{2} \left[ ((v \cdot \nabla)w_1, w_2) - (v \cdot \nabla)w_2, w_1 \right]
\]

for all \( v \in H^1(\Omega) \), \( w_1, w_2 \in H^1(\Omega) \). It is easy to verify that

\[
\tilde{A}(v_1, v_2, v_3) = \left( (v_1 \cdot \nabla)v_2, v_3 \right), \quad \forall v_1 \in X_0, \quad v_2, v_3 \in H^1(\Omega),
\]

\[
A(v_1, w_1, w_2) = \left( (v_1 \cdot \nabla)w_1, w_2 \right), \quad \forall v_1 \in X_0, \quad w_1, w_2 \in H^1(\Omega),
\]

\[
\tilde{A}(v_1, v_2, v_2) = 0, \quad \forall v_1, v_2 \in H^1(\Omega),
\]

\[
A(v, w, w) = 0, \quad \forall w \in H^1(\Omega), \quad v \in H^1(\Omega).
\]

The skew-symmetric forms \( A \) and \( \tilde{A} \) verifying the properties (55)-(58) will be important in order to obtain good estimations in the convergence analysis, since they conserve the alternance property of the trilinear forms of the continuous problem.

### 5.1. Definition and well-posedness of the scheme

We consider the following first order in time, linear and semi-coupled scheme:

**Initialization:** Let \( [u_h^0, \theta_h^0] \in X_u \times X_\theta \) be a suitable approximation of \( (u_0, \theta_0) \) as \( h \to 0 \).

**Time step \( m \):** Given the vector \( [u_h^{m-1}, \theta_h^{m-1}] \in X_u \times X_\theta \), compute \( [u_h^m, \pi_h^m, \theta_h^m] \in X_u \times X_\pi \times X_\theta \) such that for each \( [v, \bar{\pi}, \sigma] \in X_u \times X_\pi \times X_\theta \) it holds:

\begin{enumerate}
  \item \( (\delta_t u_h^m, v) + \tilde{A}(u_h^{m-1}, u_h^m, v) + Pr(\nabla u_h^m, \nabla v) - (\pi_h^m, \nabla \cdot v) \)
  \begin{align*}
    = -PrM(\nabla \theta_h^{m-1}, \frac{\partial \bar{\pi}}{\partial x_3}) + PrR(\theta_h^{m-1}, v_3),
  \end{align*}
  \item \( (\nabla \cdot u_h^m, \bar{\pi}) = 0 \),
  \item \( (\delta_t \theta_h^m, \sigma) + (\nabla \theta_h^m, \nabla \sigma) + B \int_{\Gamma_1} \theta_h^m \sigma + A(u_h^{m-1}, \theta_h^m, \sigma) = \frac{B}{1+B}(u_{\theta_h}^{m-1}, \sigma),
\end{enumerate}

where, in general, we denote \( \delta_t z_h^m = \frac{z_h^m - z_h^{m-1}}{\Delta t} \). Now, we prove the well-posedness of the scheme (59).
Theorem 5.1. (Unconditional well-posedness) The numerical scheme (59) is well-posed, that is, there exists a unique \([u^m_h, \pi^m_h, \theta^m_h] \in X_u \times X_\pi \times X_\theta\) solution of the scheme (59).

Proof. Taking into account that the scheme (59) is an algebraic linear system, it suffices to prove the uniqueness. For that, suppose that there exist \([u^m_{h,1}, \pi^m_{h,1}, \theta^m_{h,1}]\), \([u^m_{h,2}, \pi^m_{h,2}, \theta^m_{h,2}]\) \(\in X_u \times X_\pi \times X_\theta\) two solutions of the scheme (59). Then defining \(u^m_h = u^m_{h,1} - u^m_{h,2}\), \(\pi^m_h = \pi^m_{h,1} - \pi^m_{h,2}\), and \(\theta^m_h = \theta^m_{h,1} - \theta^m_{h,2}\), we have that \([u^m_h, \pi^m_h, \theta^m_h] \in X_u \times X_\pi \times X_\theta\) satisfies

\[
\begin{align*}
\int_{\Omega} ((\nabla \cdot u^m_h, v) + \Delta t \hat{A}(u^m_{h-1}, u^m_h, v) + \Delta t P_R(\nabla u^m_h, \nabla v) - \Delta t (\pi^m_h, \nabla \cdot v) &= 0, \\
(\nabla \cdot u^m_h, \pi) &= 0, \\
\int_{\Gamma_1} (\theta^m_h, \sigma) + \Delta t (\nabla \theta^m_h, \nabla \sigma) + \Delta t A(u^m_{h-1}, \theta^m_h, \sigma) + \Delta t B \int_{\Gamma_1} \sigma &= 0,
\end{align*}
\]

for all \([v, \pi, \sigma] \in X_u \times X_\pi \times X_\theta\). Thus, taking \([v, \pi, \sigma] = [u^m_h, \Delta t \pi^m_h, \theta^m_h]\) in (60), using the properties (57)-(58) and adding, we get

\[
\begin{align*}
||u^m_h||^2 + \Delta t P_R ||\nabla u^m_h||^2 = 0, \\
||\theta^m_h||^2 + \Delta t ||\nabla \theta^m_h||^2 + \Delta t B ||\theta^m_h||_{L^2(\Gamma_1)} = 0,
\end{align*}
\]

which implies that \([u^m_h, \theta^m_h] = [0, 0]\). Finally, using that \(u^m_h = 0\) in (60), and using the discrete inf-sup condition (45) we deduce that \(\pi^m_h = 0\). \(\square\)

5.2. Convergence analysis. The aim of this subsection is to obtain some \textit{a priori} error estimates for any solution \([u^m_h, \pi^m_h, \theta^m_h]\) of the scheme (59), with respect to the sufficiently regular solution \([u, \pi, \theta]\) of (24)-(28). We start by recalling the following discrete Gronwall lemma:

Lemma 5.2. ([10, p. 369]) Assume that \(\Delta t > 0\) and \(B, b^k, d^k, g^k, h^k \geq 0\) satisfy:

\[
d^{m+1} + \Delta t \sum_{k=0}^{m} b^{k+1} \leq \Delta t \sum_{k=0}^{m} g^k d^k + \Delta t \sum_{k=0}^{m} h^k + B, \quad \forall m \geq 0.
\]

Then, it holds

\[
d^{m+1} + \Delta t \sum_{k=0}^{m} b^{k+1} \leq \exp \left( \Delta t \sum_{k=0}^{m} g^k \right) \left( \Delta t \sum_{k=0}^{m} h^k + B \right), \quad \forall m \geq 0.
\]

Now, we introduce the following notations for the errors at \(t = t_m\): \(e^m_u = u^m_h - u^m_h\), \(e^m_\pi = \pi^m_h - \pi^m_h\) and \(e^m_\theta = \theta^m_h - \theta^m_h\) where \(z^m\) denote, in general, the value of \(z\) at time \(t_m\). Then, subtracting the scheme (59) to (29) (including the pressure terms and decoupling the weak formulation for the unknowns) at \(t = t_m\), we obtain that \([e^m_u, e^m_\pi, e^m_\theta]\) satisfies

\[
\begin{align*}
(\delta_t e^m_u, v) + P_R(\nabla e^m_u, \nabla v) &= (\omega^m_u, v) - A(u^m_h - u^m_{h-1} + e^m_u, u^m_h, v) \\
&+ (e^m_\pi, \nabla \cdot v) - A(u^m_{h-1}, e^m_u, v) - P_R M(\nabla \theta^m - \theta^{m-1}) \frac{\partial v}{\partial x_3} \\
&- P_R M(\nabla e^m_{\theta^m-1} \cdot \nabla v) + P_R R(\theta^m - \theta^{m-1}, v_3) + P_R R(e^m_{\theta^m-1}, v_3), \\
(\nabla \cdot e^m_u, \pi) &= 0,
\end{align*}
\]
(δtε_θ^m, σ) + (∇ε_θ^m, ∇σ) + B \int_{Γ_t} e_θ^m σ = (ω_θ^m, σ) - A(u_h^{m-1}, ε_θ^m, σ) - A(u^{m-1} - u_\theta^m, σ) + \frac{B}{1 + B}((u^{m-1} - u_\theta^m)_{3} + (e_\theta^{m-1})_{3}, σ), \quad (63)
for all [v, \pi, σ] ∈ X_\pi × X_\theta × X_σ, where ω_\theta^m and ω_θ^m are the consistency errors associated to the scheme (59), that is, ω_\theta^m = δ_θ u^{m} - (u_\theta)^m and ω_\theta^m = δ_\theta^m - (\theta_\theta)^m.

1. Error estimate for the temperature θ

Considering the projection operators \mathbb{P}_u and \mathbb{P}_\theta defined in (46) and (50) respectively, we decompose the total errors ε_\theta^m and e_\theta^m as follows:

\begin{align*}
e_\theta^m &= (\theta^m - \mathbb{P}_\theta \theta^m) + (\mathbb{P}_\theta \theta^m - \theta_h^m) := \zeta_\theta^m + \zeta_\theta^m, \quad (64) \\
e_\theta^m &= (u^m - \mathbb{P}_u u^m) + (\mathbb{P}_u u^m - u_h^m) := \zeta_u^m + \zeta_u^m, \quad (65)
\end{align*}
where e_\theta^m is the interpolation error and \xi_\theta^m is the discrete error of θ (ident for u). Then, taking into account (50), (64) and (65), from (63) we have

\begin{align*}
(\delta_t \xi_\theta^m, σ) + (∇\xi_\theta^m, ∇σ) + A(u_h^{m-1}, \xi_\theta^m, σ) + B \int_{Γ_t} (\zeta_\theta^m + \zeta_\theta^m) σ = (ω_\theta^m, σ) \\
- (δ_t \xi_\theta^m, σ) - A(u_h^{m-1}, \xi_\theta^m, σ) - A(u^m - u_\theta^{m-1} + \xi_u^{m-1} + \zeta_u^{m-1}, \theta^m, σ) + \frac{B}{1 + B}((u^m - u_\theta^m)_{3}, σ) + \frac{B}{1 + B}((\zeta_u^{m-1} + \zeta_u^{m-1})_{3}, σ).
\end{align*}
Taking σ = ξ_\theta^m in (66) and using (58) as well as the Friedrichs inequality (11) (since ξ_\theta^m ∈ X_\theta ⊆ Y), we get

\begin{align*}
\frac{1}{2} \delta_t \|\xi_\theta^m\|_{L^2}^2 + \frac{Δt}{2}||δ_t \xi_\theta^m\|_{L^2}^2 + \|\xi_\theta^m\|_{H^1}^2 + B \|\xi_\theta^m\|_{L^2(Γ_t)}^2 = (ω_\theta^m, ξ_\theta^m) - (δ_t \xi_\theta^m, ξ_\theta^m) - A(u^m - u_\theta^{m-1}, \theta^m, ξ_\theta^m) - A(u_h^{m-1}, \xi_\theta^m, \zeta_\theta^m) - A((\xi_u^{m-1} + \zeta_u^{m-1}), \theta^m, \zeta_\theta^m) + \frac{B}{1 + B}((u^m - u_\theta^m)_{3} + (\zeta_u^{m-1} + \zeta_u^{m-1})_{3}, ζ_\theta^m) - B \int_{Γ_t} \zeta_u^{m-1} ζ_\theta^m = \sum_{k=1}^{7} I_k.
\end{align*}
Then, using the Hölder, Young and Friedrichs inequalities, the Sobolev embedding H^1(Ω) → L^p(Ω), as well as (47), (49), (51) and (52), we control the terms on the right hand side of (67) as follows

\begin{align*}
I_1 &\leq \frac{1}{10} \|\xi_\theta^m\|_{H^1}^2 + C \|\omega_\theta^m\|_{H^{1,1}}^2 \leq \frac{1}{10} \|\xi_\theta^m\|_{H^1}^2 + C \Delta t \int_{t_{m-1}}^{t_m} \|\theta_\theta(t)\|_{H^{1,1}}^2 dt, \quad (68) \\
I_2 &\leq \|\xi_\theta^m\|_{L^2} \|\mathcal{I} - \mathbb{P}_\theta\| \|δ_t \theta^m\|_{L^2} \leq \frac{1}{10} \|\xi_\theta^m\|_{H^1}^2 + C h^{2(l+1)} \|δ_t \theta^m\|_{H^{l+1}}^2 \leq \frac{1}{10} \|\xi_\theta^m\|_{H^1}^2 + \frac{C h^{2(l+1)}}{Δt} \int_{t_{m-1}}^{t_m} \|\theta_t\|_{H^{l+1}}^2 dt, \quad (69) \\
I_4 &= A((\xi_u^{m-1}, \xi_\theta^m, \zeta_\theta^m) - A(\mathbb{P}_u u^{m-1}, \xi_\theta^m, \zeta_\theta^m) \\
&\leq \|\xi_\theta^m\|_{L^2} \|\mathbb{P}_u u^{m-1}\|_{L^\infty \rightarrow W^{1,3}} \|\xi_\theta^m\|_{H^1} + \|\mathbb{P}_u u^{m-1}\|_{L^\infty} \|\nabla \xi_\theta^m\|_{L^2} + \|\xi_\theta^m\|_{L^2} \|\mathbb{P}_u u^{m-1}\|_{L^3} \|\zeta_\theta^m\|_{L^2} + \|\xi_\theta^m\|_{L^2} \|\mathbb{P}_u u^{m-1}\|_{L^2} \|\zeta_\theta^m\|_{L^2} \leq \frac{1}{10} \|\xi_\theta^m\|_{H^1}^2 + C \|θ^m\|_{H^2}^2 \|\zeta_\theta^m\|_{H^2}^2 + C h^{2(l+1)} \|\xi_\theta^m\|_{H^{l+1}}^2, \quad (70)
\end{align*}
\[ I_3 + I_5 \leq (\|u^m - u^{m-1}\|_{L^2} + \|\xi_u^{m-1}\|_{L^2} + \|\xi_u^{m-1}\|_{L^2}) \|\theta^m\|_{L^\infty W^{1,3}} \|\xi_\theta^m\|_{H^1} \]

\[ \leq \frac{1}{10} \|\xi_u^m\|_{H^1}^2 + C \|u^m - u^{m-1}\|_{H^1}^2 \|\theta^m\|_{H^2}^2 \]

\[ + C(h^{2(r+1)}\|u^m - u^{m-1}\|_{H^{r+1}}^2 + \|\xi_u^{m-1}\|_{L^2}^2) \|\theta^m\|_{H^2}^2, \]  

(71)

\[ I_6 \leq \frac{B}{1 + B} (\|u^m - u^{m-1}\|_{L^2} + \|\xi_u^{m-1}\|_{L^2} + \|\xi_u^{m-1}\|_{L^2} \|\xi_\theta^m\|_{L^2} \]

\[ \leq \frac{1}{10} \|\xi_\theta^m\|_{H^1}^2 + C(\|u^m - u^{m-1}\|_{L^2}^2 + h^{2(r+1)}\|u^m - u^{m-1}\|_{H^{r+1}}^2 + \|\xi_u^{m-1}\|_{L^2}^2), \]  

(72)

\[ I_7 \leq \frac{B}{2} \|\xi_\theta^m\|_{L^2(\Gamma_1)}^2 + C B \|\xi_\theta^m\|_{L^2(\Gamma_1)}^2 \]

\[ \leq \frac{B}{2} \|\xi_\theta^m\|_{L^2(\Gamma_1)}^2 + C B \|\theta^m\|_{H^{r+1}}^2 (h^{2(l+1)} + h^{2l}), \]  

(73)

where the trace inequality \( \|\xi_\theta^m\|_{L^2(\Gamma_1)}^2 \leq C \|\xi_\theta^m\|_{L^2} \|\xi_\theta^m\|_{H^1} \) was used in the last inequality. Therefore, from (67)-(73) we get

\[ \left. \frac{1}{2} \delta_t \|\xi_\theta^m\|_{H^1}^2 + \frac{1}{2} \|\xi_\theta^m\|_{H^1}^2 + \frac{B}{2} \|\xi_\theta^m\|_{L^2(\Gamma_1)}^2 \leq C \Delta t \int_{t_{m-1}}^{t_m} \|\theta_u(t)\|_{H^1}^2 dt \right. 

\[ + C \frac{h^{2(l+1)}}{\Delta t} \int_{t_{m-1}}^{t_m} \|\theta_u\|_{H^{r+1}}^2 dt + C h^{2(l+1)} \|\theta^m\|_{H^{r+1}}^2 \|u^m - u^{m-1}\|_{H^2}^2 \]

\[ + C(\|\xi_u^{m-1}\|_{L^2}^2 + \|u^m - u^{m-1}\|_{L^2}^2 + h^{2(r+1)}\|u^m - u^{m-1}\|_{H^{r+1}}^2) \|\theta^m\|_{H^2}^2 \]

\[ + C(\|u^m - u^{m-1}\|_{L^2}^2 + h^{2(r+1)}\|u^m - u^{m-1}\|_{H^{r+1}}^2 + \|\xi_u^{m-1}\|_{L^2}^2 + B h^{2l}\|\theta^m\|_{H^{r+1}}^2). \]  

(74)

2. Error estimate for the velocity \( u \)

Now, taking into account (46), (64) and (65), from (61)-(62), we have

\[ \left. (\delta_t \xi_u^m, v) + Pr(\nabla \xi_u^m, \nabla v) = (\omega_u^m, v) - (\delta_t \xi_u^m, v) - \bar{A}(u^m - u^{m-1}, u^m, v) \right. 

\[ - \bar{A}(\xi_u^{m-1} + \bar{c}^m, u^m, v) - \bar{A}(u_h^{m-1}, \xi_u^m + \bar{c}^m, v) + (\bar{c}^m, \nabla \cdot v) \]

\[ - Pr M(\nabla(\theta^m - \theta^{m-1}), \frac{\partial v}{\partial x_3}) - Pr M(\nabla \xi_u^{m-1}, \frac{\partial v}{\partial x_3}) - Pr M(\nabla \xi_u^m, \frac{\partial \bar{c}^m}{\partial x_3}) \]

\[ + Pr R(\theta^m - \theta^{m-1}, v_3) + Pr R(\xi_u^{m-1}, v_3) + Pr R(\bar{c}^m, v_3), \]  

(75)

\[ \left(\bar{u}, \nabla \cdot \xi_u^m\right) = 0. \]  

(76)

Taking \( v = \xi_u^m \) in (75), \( \bar{u} = \xi_u^m \) in (76), using (57) as well as the Friedrichs inequality (11) (since \( \xi_u^m \in X_u \subseteq X \)), and adding the resulting expressions, we obtain

\[ \left. \frac{1}{2} \delta_t \|\xi_u^m\|_{L^2}^2 + \frac{\Delta t}{2} \|\xi_u^m\|_{L^2}^2 + Pr \|\xi_u^m\|_{H^1}^2 = (\omega_u^m, \xi_u^m) - (\delta_t \xi_u^m, \xi_u^m) \right. 

\[ - \bar{A}(u^m - u^{m-1}, u^m, \xi_u^m) - \bar{A}(\xi_u^{m-1} + \bar{c}^m, u^m, \xi_u^m) - \bar{A}(u_h^{m-1}, \xi_u^m, \xi_u^m) \]

\[ - Pr M(\nabla(\theta^m - \theta^{m-1}) + \nabla \xi_u^{m-1}, \frac{\partial \bar{c}^m}{\partial x_3}) - Pr M(\nabla \xi_u^m, \frac{\partial \bar{c}^m}{\partial x_3}) \]

\[ + Pr R(\theta^m - \theta^{m-1} + \xi_u^{m-1} + \bar{c}^{m-1}, v_3) + Pr R(\xi_u^{m-1}, v_3) = \sum_{k=1}^{8} L_k. \]  

(77)
Using the Hölder, Young and Friedrichs inequalities, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, as well as \((47), (49)\) and \((51)\), we control the terms on the right hand side of \((77)\) as follows

\[
L_1 \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + \frac{C}{Pr} \|\omega_m\|^2_{(H^1)^\gamma} \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + \frac{C}{Pr} \Delta t \int_{t_{m-1}}^{t_m} \|u_{tt}(t)\|^2_{(H^1)^\gamma} dt, \tag{78}
\]

\[
L_2 \leq \|\xi u\|_{L^2} \|I - P_u\| \|u\|_{L^2} \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + \frac{C}{Pr} \xi \|u\|^2_{H^{r+1}}, \tag{79}
\]

\[
L_3 + L_4 \leq (\|u^m - u^{m-1}\|_{L^2} + \|\xi u^{m-1}\|_{L^2} + \|\xi u\|_{L^2}) \|u^m\|_{L^\infty \cap W^{1,3}} \|\xi u\|_{H^1} \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + \frac{C}{Pr} \|u^m - u^{m-1}\|_{L^2}^2 + \xi \|u^m - u^{m-1}\|_{H^{r+1}}^2 \|u^m\|_{L^2}^2 + \frac{C}{Pr} \|\xi u\|_{H^1}^2 \|u^m\|_{L^2}^2, \tag{80}
\]

\[
L_5 = A(\xi u, \xi u, \xi u) - \tilde{A}(P_u u^{m-1}, \xi u, \xi u) \leq C(\|\xi u\|_{L^2} \|\xi u\|_{L^\infty \cap W^{1,3}} \|\xi u\|_{H^1}) \|P_u u^{m-1}\|_{L^\infty \cap W^{1,3}} \|\xi u\|_{L^2} \|\xi u\|_{H^1} \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + \frac{C}{Pr} \|u^m\|_{H^2} + \frac{C}{Pr} \|u^m - u^{m-1}\|_{L^2}^2 + \frac{C}{Pr} \|u^m - u^{m-1}\|_{H^{r+1}}^2 \|u^m\|_{L^2}^2, \tag{81}
\]

\[
L_6 \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + Pr M^2 C(\|\nabla (\theta^m - \theta^{m-1})\|_{L^2}^2 + \xi \|\theta^m - \theta^{m-1}\|_{H^{r+1}}^2), \tag{82}
\]

\[
L_7 \leq Pr M \|\xi u\|_{H^1} \|\nabla \theta^{m-1}\|_{L^2} \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + \frac{Pr M^2}{2} \|\xi u\|_{H^1}^2 \tag{83}
\]

\[
L_8 \leq Pr R (\|\theta^m - \theta^{m-1}\|_{L^2} + \|\xi u^m\|_{L^2} + \|\xi u\|_{L^2}) \|\xi u\|_{L^2} \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + Pr R^2 C(\|\theta^m - \theta^{m-1}\|_{L^2}^2 + \xi \|\theta^m - \theta^{m-1}\|_{H^{r+1}}^2 + \|\xi u^m\|_{L^2}^2). \tag{84}
\]

Therefore, from \((77)-(84)\) we get

\[
\frac{1}{2} \|\xi u\|_{L^2}^2 + \frac{Pr}{4} \|\xi u\|_{H^1}^2 \leq \frac{Pr}{24} \|\xi u\|_{H^1}^2 + \frac{Pr M^2}{2} \|\xi u\|_{H^1}^2 + \frac{1}{\Delta t} \|\theta^m - \theta^{m-1}\|_{H^{r+1}}^2 dt + C(\|u^m - u^{m-1}\|_{L^2}^2 + \|\xi u\|_{L^2}^2 + \|\xi u\|_{L^2}^2) \|u^m\|_{L^2}^2 + C(\|u^m - u^{m-1}\|_{H^{r+1}}^2 + \|\theta^m - \theta^{m-1}\|_{H^{r+1}}^2 + \|\xi u^m\|_{L^2}^2) \|u^m\|_{H^{r+1}}^2 + C(\|\xi u^m\|_{L^2}^2 + \|\xi u\|_{L^2}^2). \tag{85}
\]

3. Estimate for the terms $\|u^m - u^{m-1}\|_{L^2}$ and $\|\theta^m - \theta^{m-1}\|_{H^1}$.

Notice that

\[
\|\omega^m\|_{H^1} \leq \|\delta \theta^m - (\theta_t)^m\|_{H^1},
\]

\[
= \frac{1}{\Delta t} (\theta^m - \theta^{m-1} - (\theta_t)^m, \tag{86}
\]

\[
\leq C(\Delta t)^{1/2} \left( \int_{t_{m-1}}^{t_m} \|\theta^m(t)\|^2_{H^1} dt \right)^{1/2},
\]

\[
\leq C(\Delta t)^{1/2} \left( \int_{t_{m-1}}^{t_m} \|\theta^m(t)\|^2_{H^1} dt \right)^{1/2},
\]

\[
\leq C(\Delta t)^{1/2} \left( \int_{t_{m-1}}^{t_m} \|\theta^m(t)\|^2_{H^1} dt \right)^{1/2},
\]

\[
\leq C(\Delta t)^{1/2} \left( \int_{t_{m-1}}^{t_m} \|\theta^m(t)\|^2_{H^1} dt \right)^{1/2},
\]
where the last inequality was obtained as in (68) with the norm $H^1$ instead of $(H^1)'$.

Therefore, we deduce

$$\Delta t \sum_{m=1}^{n} \| \theta^m - \theta^{m-1} \|_{H^1}^2 \leq C(\Delta t)^4 \| \theta_t \|_{L^2(H^1)}^2 + C(\Delta t)^2 \| \theta_t \|_{L^2(H^1)}^2. \quad (86)$$

Analogously, we get

$$\Delta t \sum_{m=1}^{n} \| u^m - u^{m-1} \|_{L^2}^2 \leq C(\Delta t)^4 \| u_t \|_{L^2(L^2)}^2 + C(\Delta t)^2 \| u_t \|_{L^2(L^2)}^2. \quad (87)$$

Collecting the previous estimates, we prove the following theorem:

**Theorem 5.3.** Assume that $[u_0, \theta_0] \in H^1(\Omega) \times H^1(\Omega)$ and consider $[u^0, \theta^0] = [\mathbb{P} u_0, \mathbb{P} \theta_0] \in X_u \times X_\theta$. Moreover, assume that there exists a sufficiently regular solution of (24)-(28). If $M \sqrt{Pr} < 1$, the following error estimate for the discrete errors holds

$$\| [e_u^m, e_\theta^m] \|_{\mathcal{L}^{\infty}(L^2) \cap \mathcal{L}^2(H^1)} \leq C(T) \left( \Delta t + \max\{ h^l, h^{r+1} \} \right),$$

where the constant $C(T) > 0$ is independent of $m$, $\Delta t$ and $h$.

**Proof.** The proof follows adding the estimates (74) and (85), multiplying the resulting expression by $\Delta t$, adding from $m = 1$ to $m = n$, using (86)-(87), applying the discrete Gronwall Lemma 5.2 and recalling that $[\xi^0_u, \xi^0_\theta] = [0,0]$. \qed

From Theorem 5.3 and taking into account the approximation properties (47) and (51), we can easily deduce the following result:

**Corollary 1.** Under the hypotheses of Theorem 5.3, the following error estimate for the total errors holds

$$\| [e_u^m, e_\theta^m] \|_{\mathcal{L}^{\infty}(L^2) \cap \mathcal{L}^2(H^1)} \leq C(T) \left( \Delta t + \max\{ h^l, h^{r} \} \right),$$

where the constant $C(T) > 0$ is independent of $m$, $\Delta t$ and $h$.

6. **Numerical simulations.** In this section, we present two numerical experiments: the first one is used to verify that under the smallness hypotheses in the dimensionless numbers, the solution of the evolution problem converges to the trivial solution, or equivalently, the original problem converges to the basic state; and the second one has been considered in order to show that if we consider no small dimensionless numbers, then we obtain that the solution of the evolution problem converges to a non-trivial solution forming patterns. These numerical experiments have been computed by using the software Freefem++. We have considered finite element spaces for $u$, $\theta$ and $\pi$ generated by $\mathcal{P}_2, \mathcal{P}_2, \mathcal{P}_1$-continuous, respectively. In both experiments, we have considered the following initial conditions:

$$u_0(x_1, x_2, x_3) = 0, \quad \pi_0(x_1, x_2, x_3) = 0, \quad \theta_0(x_1, x_2, x_3) = x_3 - x_3^2.$$

**Test 1.** The numerical solution is computed with time step $\Delta t = 1.10 e^{-2}$ and a mesh with 21870 tetrahedra. We consider the parameters values $Pr = 1$, $R = 1$, $M = 2$ and $B = 0.05$. We show simulation results for the times $t = 0$, $t = 5.5 e^{-2}$, $t = 2.2 e^{-1}$, $t = 3.3 e^{-1}$, $t = 5.5 e^{-1}$ and $t = 2.2$. In Figure 1, we show the behaviour of the discrete variables; each image contains information of both, temperature and velocity field. Initially, the fluid is at rest and have a parabolic temperature profile; then, the fluid shows a slight movement and the temperature profile change. We see that all the variables (perturbed) go quickly to zero (the trivial solution).
Test 2: The numerical solution in this case is computed with time step $\Delta t = 5.5e^{-1}$ and a mesh with 21870 tetrahedra. Here, we consider the parameters values $Pr = 890$, $R = 2228$, $M = 105$ and $B = 1.25$. In Figure 2, we show the behaviour of the discrete variables in the times $t = 0$, $t = 1.1$, $t = 2.26$, $t = 5.51$, $t = 9.92$ and $t = 16.53$. As before, the fluid is in rest and have a parabolic temperature profile; then, the fluid evolves showing instability and when the time grows the fluid forms well defined patterns.
Acknowledgments. The authors would like to thank the anonymous referees for useful remarks and suggestions. The authors thank the Vicerrectoría de Investigación y Extensión-UIS by the support.

REFERENCES

[1] H. Bénard, Les tourbillons cellulaires dans une nappe liquide, *Revue Gén. Sci. Pures Appl.*, 11 (1900), 1261–1271, 1309–1328.

[2] M. J. Block, Surface tension as the cause of Bénard cells and surface deformation of a liquid film, *Nature*, 178 (1965), 650–651.

[3] T. Chacón-Rebollo and F. Guillén-González, An intrinsic analysis of the hydrostatic approximation of Navier-Stokes equations, *C. R. Acad. Sci. Paris Sér. I Math.*, 330 (2000), 841–846.

[4] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, The International Series of Monographs on Physics, Clarendon Press, Oxford, 1961.

[5] P. Colinet, J. C. Legros and M. G. Velarde, *Nonlinear Dynamics of Surface-Tension-Driven Instabilities*, Wiley-VCH Verlag Berlin GmbH, Berlin, 2001.

[6] P. C. Dauby and G. Lebon, Bénard-Marangoni instability in rigid rectangular containers, *J. Fluid Mech.*, 329 (1996), 25–64.

[7] L. C. F. Ferreira and E. J. Villamizar-Roa, On the stability problem for the Boussinesq equations in weak-$L^p$ spaces, *Commun. Pure Appl. Anal.*, 9 (2010), 667–684.

[8] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.

[9] F. Guillén-González and M. V. Redondo-Neble, Convergence and error estimates of viscosity-splitting finite-element schemes for the primitive equations, *Appl. Numer. Math.*, 111 (2017), 219–245.

[10] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second order time discretization, *SIAM J. Numer. Anal.*, 27 (1990), 353–384.

[11] S. Hoyas, H. Herrero and A. M. Mancho, Thermal convection in a cylindrical annulus heated laterally, *J. Phys. A*, 35 (2002), 4067–4083.

[12] S. Hoyas, *Estudio Teórico y Numérico de un Problema de Convección de Bénard-Marangoni en un Anillo*, Ph.D thesis, U. Complutense de Madrid, 2003.

[13] E. L. Koschmieder, *Bénard Cells and Taylor Vortices*, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, New York, 1993.

[14] A. M. Kvarving, T. Bjontegaard and E. M. Ronquist, On pattern selection in three-dimensional Bénard-Marangoni flows, *Commun. Comput. Phys.*, 11 (2012), 893–924.

[15] M. Lappa, *Thermal Convection: Patterns, Evolution and Stability*, John Wiley & Sons, Ltd., Chichester, 2010.

[16] G. Lebon, D. Jou and J. Casas-Vázquez, *Understanding Non-equilibrium Thermodynamics. Foundations, Applications, Frontiers*, Springer-Verlag, Berlin, 2008.

[17] S. A. Lorca and J. L. Boldrini, Stationary solutions for generalized Boussinesq models, *J. Differential Equations*, 124 (1996), 389–406.

[18] S. A. Lorca and J. L. Boldrini, The initial value problem for the generalized Boussinesq model, *Nonlinear Anal.*, 36 (1999), 457–480.

[19] C. Marangoni, Sull’espansione delle gocce di un liquido gallegianti sulla superficie di altro liquido. Pavia: Tipografia dei fratelli Fusi, *Ann. Phys. Chem.*, 143 (1871), 337–354.

[20] I. Mutabazi, J. E. Wesfreid and E. Guyon, *Dynamics of Spatio-Temporal Cellular Structures. Henri Bénard Centenary Review*, Springer, New York, 2006.

[21] J. Necas, *Direct Methods in the Theory of Elliptic Equations*, Springer Monographs in Mathematics, Springer, Heidelberg, 2012.

[22] D. A. Nield, Surface tension and buoyancy effects in cellular convection, *J. Fluid Mech.*, 19 (1964), 341–352.

[23] R. Pardo, H. Herrero and S. Hoyas, Theoretical study of a Bénard-Marangoni problem, *Journal of Mathematical Analysis and Applications*, 376 (2011), 231–246.

[24] J. R. A. Pearson, On convection cells induced by surface tension, *J. Fluid Mech.*, 4 (1958), 489–500.

[25] P. H. Rabinowitz, Existence and nonuniqueness of rectangular solutions of the Bénard problem, *Arch. Ration. Mech. Anal.*, 29 (1968), 32–57.
[26] L. Rayleigh, On convection currents in horizontal layer of fluid when the higher temperature is on the under side, *Philos. Mag. Ser. (6)*, 32 (1916), 529–546.

[27] M. A. Rodríguez-Bellido, M. A. Rojas-Medar and E. J. Villamizar-Roa, The Boussinesq system with mixed nonsmooth boundary data, *C. R. Math. Acad. Sci. Paris*, 343 (2006), 191–196.

[28] D. A. Rueda-Gómez and E. J. Villamizar-Roa, On the Rayleigh-Bénard-Marangoni system and a related optimal control problem, *Computers and Mathematics with Applications*, 74 (2017), 2969–2991.

[29] J. Simon, Compact sets in the space $L^p(0; T; B)$, *Ann. Mat. Pura Appl. (4)*, 146 (1987), 65–96.

[30] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, RI, 2001.

Received November 2019; January 2020.

*E-mail address:* jhean.perez@uis.edu.co

*E-mail address:* diaruego@uis.edu.co

*E-mail address:* jvillami@uis.edu.co