Abstract—Systematic polar codes are shown to outperform non-systematic polar codes in terms of the bit-error-rate (BER) performance. However theoretically the mechanism behind the better performance of systematic polar codes is not yet clear. In this paper, we set the theoretical framework to analyze the performance of systematic polar codes. The exact evaluation of the BER of systematic polar codes conditioned on the BER of non-systematic polar codes involves in $2^{N R}$ terms where $N$ is the code block length and $R$ is the code rate, resulting in a prohibitive number of computations for large block lengths. By analyzing the polar code construction and the successive-cancellation (SC) decoding process, we use a statistical model to quantify the advantage of systematic polar codes over non-systematic polar codes, so called the systematic gain in this paper. A composite model is proposed to approximate the dominant error cases in the SC decoding process. This composite model divides the errors into independent regions and coupled regions, controlled by a coupling coefficient. Based on this model, the systematic gain can be conveniently calculated. Numerical simulations are provided in the paper showing very close approximations of the proposed model in quantifying the systematic gain.

Index Terms—Polar Codes, Systematic Polar Codes, Polar Codes Encoding, Successive Cancellation Decoding, Systematic Polar Gain

I. INTRODUCTION

Polar codes are systematically introduced by Arikan in [1]. It’s shown there that polar codes can achieve the capacity for symmetric binary-input discrete memoryless channels (B- DMC) with a low complexity. The encoding and decoding process (with successive cancellation, SC) can be implemented with a complexity of $O(N \log N)$. The polarization of $N$ channels is realized through two stages: channel combining and splitting. Channels are polarized after these two stages in the sense that bits transmitting in these channels either experience almost noiselss channels or almost completely noisy channels for a large $N$. The idea of polar codes is to transmit information bits on those noiseless channels while fix the information bits on those completely noisy channels. The fixed bits are made known to both the transmitter and receiver. The binary input alphabet in Arikan seminal work [1] is later on extended to non-binary input alphabet [2]–[4]. The construction of polar codes have then been investigated and different procedures are proposed [5]–[8] assuming the original $2 \times 2$ kernel matrix. Polar codes based on the kernel matrices of size $l \times l$ are studied in [9]. Polar codes have also been extended to different scenarios since then [10]–[12].

The rate of polarization of polar codes is studied in [1], [13] without including the effect of the code rate. In works [14]– [16], the authors analyzed the polarization rate considering the effect of both the block length and the code rate. The asymptotic behavior of polar codes reported in these works does not guarantee a good performance in practice when a finite block length is applied. In fact, the performance of polar codes with the SC decoding and finite block lengths are not satisfactory [17] [18]. Different decoding techniques are deployed to improve the performance of polar codes [17]–[24]. The authors of [17]–[21] use belief propagation (BP) in the decoding process in place of the SC decoding. The list decoding procedure of [22] and [23] involves multiple paths instead of a single path as in the SC decoding process. The concatenation of polar codes with LDPC codes are proposed in [20] and [24] to further improve the performance of polar codes. These techniques focus on the improvement in the decoding algorithms while keeping the original coding process as in [1]. The price paid in these improvements is the extra decoding complexity.

Another direction to improve the performance of polar codes is also introduced by Arikan in [25] by using systematic polar codes. If we denote $\mathbf{u}$ as a vector containing source bits and $\mathbf{z}$ as the corresponding codeword obtained by using the normal polar codes construction. Note that in this paper we use non-systematic polar codes and normal polar codes interchangeably without further notice. The basic idea of systematic polar codes is to use some part of the codeword $\mathbf{z}$ to transmit information bits instead of directly using $\mathbf{u}$ to transmit them. The advantage of systematic polar codes is the low decoding complexity: Systematic polar codes require only part of the encoding process (involving only 0s and 1s) after the normal SC decoding is done. This low complexity can be seen from the way $\mathbf{z}$ is estimated: $\hat{\mathbf{z}} = \mathbf{u}G$ where $\hat{\mathbf{u}}$ is the estimation of $\mathbf{u}$ from the normal SC decoding, and $G$ is the generator matrix. In the rest of the paper, we call this indirect, two-step (SC decoding then encoding) decoding process of systematic polar codes the SC-EN decoding.

In [25], it’s shown that systematic polar codes achieve better BER performance than normal polar codes. However, Arikan also noted in [25] that it’s not clear why systematic polar codes achieve better BER performance than non-systematic polar codes even with an indirect decoding procedure (the SC-EN decoding): first decoding $\hat{\mathbf{u}}$ then re-encoding $\hat{\mathbf{z}}$ as $\hat{\mathbf{u}}G$. One would expect that any error in $\hat{\mathbf{u}}$
would be amplified from this re-encoding process \( \hat{x} = \hat{u}G \). However simulation results in [25] as well as simulation results in this paper show that with this two-step decoding procedure, systematic polar codes still achieve better BER performance than non-systematic polar codes.

This paper studies the error performance of systematic polar codes with special focus on characterizing the advantage of systematic polar codes over non-systematic polar codes. We start by simplifying the general encoding process of the systematic polar codes. This is done through proving a theorem on the structure of the generator matrix. Then we discuss the theoretical BER performance of systematic polar codes conditioned on the BER performance of non-systematic polar codes. The general form of this error prediction involves in \( 2^{NR} \) terms which is prohibitive to compute for large block lengths \( N \). It's then proven that for two special cases we can theoretically predict the error rate of systematic polar codes. To understand the general better behavior of systematic polar codes, we further study the basic error patterns of non-systematic polar codes with the SC decoding. A systematic gain is defined to describe the advantage of systematic polar codes over non-systematic polar codes. A composite model is proposed to approximate the mean effect (or the dominant effect) of the error events. This composite model uses the fact that the errors in the SC decoding process are coupled. A coupling coefficient is used to control the level of coupling between the errors. This model facilitates the calculation of the systematic gain and can be used to predict the performance of systems utilizing systematic polar codes.

Following the notations in [1], in the paper, we use \( v_i \) to represent a row vector with elements \( (v_1, v_2, \ldots, v_N) \). We also use \( u \) to represent the same vector for notational convenience. Given a vector \( v_i \), the vector \( v'_i \) is a subvector \( (v_i, \ldots, v_j) \) with \( 1 \leq i, j \leq N \). If there is a set \( A \subseteq \{1, 2, \ldots, N\} \), then \( v_A \) denotes a subvector with elements in \( \{v_i, i \in A\} \).

The rest of the paper is organized as follows. In Section II the background of systematic polar codes is introduced and a theorem on the structure of systematic polar codes is proven. The first part of Sec. III provides a general theoretical formulation of the BER performance of systematic polar codes given the BER performance of non-systematic polar codes. Two special cases are analyzed in this part whose BER performance can be characterized. Section IV studies the basic error patterns and the first error distribution of non-systematic polar codes, followed by the introduction of the systematic gain. In Section V we propose a coupling model which is used to predict the BER performance of systematic polar codes. Simulation results are given in Section VI. Concluding remarks are presented in Section VII.

II. SYSTEMATIC POLAR CODES

For completeness, in the first part of this section, we restate the relevant materials on the construction of normal polar codes and systematic polar codes from [1] [25]. In the second part of this section a theorem on the structure of the normal polar codes is provided which is used to simplify the encoding of the systematic polar codes.

A. Preliminaries of Non-Systematic Polar Codes

Let \( \mathbf{W} \) be any binary discrete memoryless channel (B-DMC) with a transition probability \( \mathbf{W}(y|x) \). The input alphabet \( \mathcal{X} \) takes values in \( \{0, 1\} \) and the output alphabet is \( \mathcal{Y} \). Channel polarization is carried in two phases: channel combining and splitting. Eventually, \( N = 2^n(n \geq 1) \) independent copies of \( \mathbf{W} \) are first combined and then split into \( N \) bit channels \( \{W_N^{(i)}\}_{i=1}^{N} \). This polarization process has a recursive tree structure in [11], which we plot here for the ease of reference. The \( 0\)'s and \( 1\)'s in Fig. 1 refer to the bit channels \( \mathbf{W}' \) and \( \mathbf{W}'' \) respectively in the basic one-step channel transformation defined as \( (\mathbf{W}', \mathbf{W}'') \), where

\[
W'(y_1, y_2 | u_1) = \sum_{u_2} \frac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2) 
\]

(1)

\[
W''(y_1, y_2, u_1 | u_2) = \frac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2)
\]

(2)

The Bhattacharyya parameters of channel \( \mathbf{W}' \) and \( \mathbf{W}'' \) satisfy the following conditions:

\[
Z(\mathbf{W}'') = Z(\mathbf{W})^2 
\]

(3)

\[
Z(\mathbf{W}') \leq 2Z(\mathbf{W}) - Z(\mathbf{W})^2 
\]

(4)

\[
Z(\mathbf{W}') \geq Z(\mathbf{W}) \geq Z(\mathbf{W}'') 
\]

(5)

The label \( 0 \) (the upper branch in the transformation) in Fig. 1 means that the output channel takes the branch \( \mathbf{W}' \) in that specific transformation. Correspondingly, a label \( 1 \) (the lower branch in the transformation) means the output channel takes \( \mathbf{W}'' \) in that transformation. Note that for binary erasure channels (BEC), the Bhattacharyya parameter \( Z(\mathbf{W}') \) has an exact expression \( Z(\mathbf{W}') = 2Z(\mathbf{W}) - Z(\mathbf{W})^2 \), resulting in a recursive calculation of the Bhattacharyya parameters of the final bit channels. Finally, after the channel transformations, the transition probability for bit channel \( i \) is defined as

\[
W_N^{(i)}(y_1, u_1 | u_i) = \sum_{u_{i+1} \in \mathcal{X}^{N-n}} \frac{1}{2^{N-1}} W_N(y_1 | u_1 \oplus G) 
\]

(6)

where \( W_N(\cdot) \) is the underlying vector channel (\( N \) copies of the channel \( W \)) and \( G \) is the generator matrix whose form is to be discussed in the next section.

B. Construction of Systematic Polar Codes

Polar codes in the original format [1] are not systematic. The generator matrix for polar codes is \( G = BF^{\otimes n} \) in [1] where \( B \) is a permutation matrix and \( F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \). The operation \( F^{\otimes n} \) is the \( n \)th Kronecker power of \( F \) over the binary field \( \mathbb{F}_2 \). For systematic polar codes, we focus on a generator matrix without the permutation matrix \( B \), namely \( G = F^{\otimes n} \). With such a matrix, the encoding for normal polar codes is done as \( \mathbf{z} = \mathbf{u}G \).

The indices of the source bits \( \mathbf{u} \) corresponding to the information bits can be set by selecting indices of the bit channels with the smallest Bhattacharyya parameters. Denote \( \mathcal{A} \) as the set consisting of indices for the information bits. Correspondingly, \( \mathcal{A} \) consists of indices for the frozen bits. Both
sets $A$ and $\bar{A}$ are in $\{1, 2, \ldots, N\}$. For any element $i \in A$ and $j \in \bar{A}$, we have $Z(W_N^{(i)}) < Z(W_N^{(j)})$. In this paper, the set $A$ is always sorted in ascending order according to the index values, instead of ordered by their values of Bhattacharyya parameters.

The source bits $u$ can be split as $u = (u_A, u_{\bar{A}})$. The codeword can then be expressed as $x = u_A G_A + u_{\bar{A}} G_{\bar{A}}$, where $G_A$ is the submatrix of $G$ with rows specified by the set $A$. The systematic polar code is constructed by specifying a set of indices of the codeword $x$ as the indices to convey the information bits. Denote this set as $B$ and the complementary set as $\bar{B}$. The codeword $x$ is thus split as $(x_B, x_{\bar{B}})$. With some manipulations, we have

$$(x_B, x_{\bar{B}}) = (u_A G_{AB} + u_{\bar{A}} G_{\bar{A}A}, u_A G_{AB} + u_{\bar{A}} G_{\bar{A}B})$$

(7)

The matrix $G_{AB}$ is a submatrix of the generator matrix with elements $(G_{ij})_{i \in A, j \in B}$. Given a non-systematic encoder $(A, u_A)$, there is a systematic encoder $(B, u_B)$ which performs the mapping $x_B \mapsto x = (x_B, x_{\bar{B}})$. To realize this systematic mapping, $x_B$ needs to be computed for any given information bits $x_B$. To this end, we see from (7) that $x_B$ can be computed if $u_A$ is known. The vector $u_A$ can be obtained as the following

$$u_A = (x_B - u_A G_{AB}) (G_{AB})^{-1}$$

(8)

From (8), it’s seen that $x_B \mapsto u_A$ is one-to-one if $x_B$ has the same elements as $u_A$ and if $G_{AB}$ is invertible. In [25], it’s shown that $B = A$ satisfies all these conditions in order to establish the one-to-one mapping $x_B \mapsto u_A$. In the rest of the paper, the systematic encoding of polar codes adopts this selection of $B$ to be $B = A$. Therefore we can rewrite (7) as

$$(x_A, x_{\bar{A}}) = (u_A G_{AA} + u_{\bar{A}} G_{\bar{A}A}, u_A G_{AA} + u_{\bar{A}} G_{\bar{A}A})$$

(9)

C. Theorem on Polar Coding Construction

In this section, we prove a general theorem on polar codes. In the following, we say that row $i$ intersects with column $j$ of the matrix $G$ if $G_{ij} = 1$. Otherwise, we say row $i$ does not intersect with column $j$.

Theorem 1: For any given index $j \in A$, we divide the elements of $A$ into two sets: $A_i = \{i : i \in A, i < j\}$ and $\bar{A}_j = \{i : i \in A, i > j\}$. For $i \in A_i$, it’s obvious that $G_{i,j} = 0$ since the matrix $G$ is lower triangular. So we only need to prove $G_{i,j} = 0$ for $i \in \bar{A}_j$.

Let $(b^{(1)}_n, b^{(2)}_{n-1}, \ldots, b^{(1)}_1)$ be the $n$-bit binary expansion of the integer $i - 1$ with $i \in \bar{A}_j$, and $b^{(2)}_n$ is the MSB. The bit $b^{(2)}_m$ corresponds to the root channel selection in Fig. 1 and $b^{(1)}_i$ corresponds to the last channel selection. Each bit in the binary vector $(b^{(2)}_n, b^{(2)}_{n-1}, \ldots, b^{(1)}_1)$ defines a channel selection of the corresponding level in the tree of Fig. 1. For example, bit $b^{(2)}_m$ $(m \in \{1, 2, \ldots, n\})$ determines bit channel $i$ at level $m$ takes the upper branch or the lower branch.

Suppose row $i$ intersects with column $j \in A$, equivalent to $G_{i,j} = 1$. We know the entry of the generator matrix $G$ can be calculated as

$$G_{i,j} = \prod_{m=1}^{n} (1 \oplus b^{(2)}_m \oplus b^{(1)}_m b^{(2)}_m)$$

(10)

To have $G_{i,j} = 1$, we must have $b^{(2)}_m = 1$ when $b^{(1)}_m = 1$. Suppose $M_i$ is the last non-zero position of $(b^{(1)}_n, b^{(1)}_{n-1}, \ldots, b^{(1)}_1)$ and $M_j$ is the last non-zero position of $(b^{(2)}_n, b^{(2)}_{n-1}, \ldots, b^{(1)}_1)$. With $i \in \bar{A}_j$, $M_i \geq M_j$. We proceed by discussing two cases: $M_i = M_j$ and $M_i > M_j$. 

Fig. 1. The recursive channel transformation of polar codes.
1) Case 1 $M_i = M_j$: For $M_i = M_j$, we have $(b_i^1, b_i^2, ..., b_i^{M+1}) = (b_j^1, b_j^2, ..., b_j^{M+1}) = 0_i-M_j$. Referring to Fig. 1 it’s seen that the recursive channel transformation from level $n$ to level $M_j+1$ (or $M_i+1$) is the same for both bit channel $i$ and bit channel $j$: they all take the upper branch in each transformation. Then at level $M_j$, both channels involve in the same fashion by taking the lower branch (corresponding to $b_j^1 = b_j^M = 1$). Divide the levels \( \{m : m \leq M_j\} \) of bit channel $j$ into two sets:

\[
M_0 = \{m : m \leq M_j \text{ and } b_j^m = 0\} \quad (11)
\]

\[
M_1 = \{m : m \leq M_j \text{ and } b_j^m = 1\} \quad (12)
\]

With $b_j^m = 1$ whenever $b_j^m = 1$, we can equivalently express $M_1$ as

\[
M_1 = \{m : m \leq M_j \text{ and } b_j^m = b_i^m = 1\} \quad (13)
\]

Define a set $M_{01} = \{m : m \in M_0 \text{ and } b_i^m = 1\}$. This set $M_{01}$ is not empty since there must be at least one $m \in M_0$ at which $b_i^m = 1$ since $i > j$, $M_i = M_j$, and $b_i^m = 1$ whenever $b_j^m = 1$ for $m \in M_1$. When $|M_{01}| > 1$, we select $m^*$ to be the largest in $M_{01}$. At level $m^*$, bit channel $i$ takes the lower branch (corresponding to $b_i^{m^*} = 1$) and bit channel $j$ takes the upper branch (corresponding to $b_j^{m^*} = 0$). Therefore starting from level $m^*$, the Bhattacharyya parameter for bit channel $i$ and bit channel $j$ diverge according to (8) and (9):

\[
Z(W_{N_m^{(k_i)}}^{(k_i)}) = (Z(W_{N_m^{(k_j)}}^{(k_j)}))^2 \quad (14)
\]

\[
Z(W_{N_{m^*}^{(k_i)}}^{(k_i)}) \geq Z(W_{N_{m^*}^{(k_j)}}^{(k_j)}) \geq Z(W_{N_{m^*}^{(k_i)}}^{(k_i)}) \quad (15)
\]

where

\[
N_{m^*} = 2^n-m^* \quad (16)
\]

\[
k_{m^*} = (b_n, b_{n-1}, ..., b_{m^*+1}) \quad (17)
\]

\[
k_i = (b_i^n, b_i^{n-1}, ..., b_i^{m^*+1}, 1) \quad (18)
\]

\[
k_j = (b_j^n, b_j^{n-1}, ..., b_j^{m^*+1}, 0) \quad (19)
\]

The number $k_i$ and $k_j$ is the channel index for bit channel $i$ and bit channel $j$ at level $m^*$, respectively. Starting from the same previous channel $W_{N_{m^*}^{(k_j)}}^{(k_j)}$, it’s obvious that bit channel $i$ has a smaller Bhattacharyya parameter than bit channel $j$ at level $m^*$. For levels $m < m^*$, this advantage of bit channel $i$ continues until the last level because of the recursive channel transformation process defined by the set $M_0$ and $M_1$ in (11) and (13). Therefore if $D_{i,j} = 1$, when $M_i = M_j$, we have $Z(W_{N_{m^*}^{(k_i)}}^{(k_i)}) \leq Z(W_{N_{m^*}^{(k_j)}}^{(k_j)})$ for $j \in A$ and $i \in A$.

2) Case 2 $M_i > M_j$: In this case, we define a set $M_{i1} = \{m : m > M_j \text{ and } b_i^m = 1\}$. This set $M_{i1}$ obviously is not empty since $M_i > M_j$. But the set $M_{01}$ could be empty in this case. If $M_{01} = \emptyset$, the recursive channel transformation for bit channel $i$ and $j$ is the same for levels \( \{m \leq M_j\} \): they take the upper branches at levels in $M_0$ and take lower branches for levels in $M_1$. However, their involving processes differ in at least one level $m^* \in M_{i1}$ because of the existence of the non-empty set $M_{i1}$. When $|M_{i1}| > 1$, we select $m^*$ to be the smallest in $M_{i1}$. Bit channel $i$ takes the lower branch at level $m^*$ while bit channel $j$ takes the upper branch at the same level, resulting in a smaller Bhattacharyya parameter for bit channel $i$ at that level. After level $m^*$, as we already point out, the two channels involving in the same fashion defined by $M_0$ and $M_1$. Therefore the final Bhattacharyya parameters for bit channel $i$ is still smaller than bit channel $j$. If $M_{01} \neq \emptyset$, the advantage of the bit channel $i$ is even more pronounced than the case when $M_{01} = \emptyset$ since bit channel $i$ takes additional lower branches besides taking the same lower branches as bit channel $j$, producing a final channel with an even smaller Bhattacharyya parameter. Therefore as in the case when $M_i = M_j$, we also have $Z(W_{N_{m^*}^{(k_i)}}^{(k_i)}) \leq Z(W_{N_{m^*}^{(k_j)}}^{(k_j)})$ for $j \in A$ and $i \in \bar{A}$ when $M_i > M_j$.

Combining Case 1 and Case 2, we see that if $D_{i,j} = 1$, $Z(W_{N_{m^*}^{(k_i)}}^{(k_i)}) \leq Z(W_{N_{m^*}^{(k_j)}}^{(k_j)})$ for $j \in A$ and $i \in \bar{A}$. But this contradicts with the polar encoding principle that $Z(W_{N_{m^*}^{(k_i)}}^{(k_i)}) > Z(W_{N_{m^*}^{(k_j)}}^{(k_j)})$ for $j \in A$ and $i \in \bar{A}$. Therefore $D_{i,j} = 0$ for $j \in A$ and $i \in \bar{A}$.

Corollary 1: The matrix $G_{A\bar{A}} = 0$.

Proof: The statement of $G_{A\bar{A}} = 0$ is equivalent to say that any column $j \in A$ of the generator matrix $G$ does not intersect with row $i \in \bar{A}$ of $G$, which we already prove in Theorem 1.

Using Corollary 1 the systematic encoding of polar codes can be simplified as

\[
(x_A, x_{\bar{A}}) = (u_AG_{A\bar{A}}, u_AG_{A\bar{A}} + u_AG_{\bar{A}\bar{A}}) \quad (20)
\]

The calculation of $u_A$ in (5) can thus be simplified as $u_A = x_AG_{\bar{A}\bar{A}}^{-1}$.

From the proof of Theorem 1 another corollary is readily available.

Corollary 2: For any $i, j \in A$, if row $i$ intersects with column $j$ of the generator matrix $G$ ($G_{i,j} = 1$), then $Z(W_{N_{m^*}^{(k_i)}}^{(k_i)}) \leq Z(W_{N_{m^*}^{(k_j)}}^{(k_j)})$. Or in other words, bit channel $i$ has a better channel quality than bit channel $j$ when $G_{i,j} = 1$.

D. Generator Matrix with Permutation

The original generator matrix in (11) is $G_0 = BF^{\otimes n}$ where $B$ is the bit-reversal permutation matrix. We use the vector $a$ to represent the sorted elements in $A$ and the vector $b$ the corresponding vector consisting of the indices for the systematic encoding. In [25], it’s pointed out that $B$ is the image of $A$ under the matrix $B$, namely $b = aB$. If the encoding of the normal polar codes is based on $G_p$, then the submatrix $G_{AB}$ in (11) is $G_{AB} = (G_p)_{AB}$. With some manipulations, it can be shown that $(G_p)_{AB} = G_{A\bar{A}}$. Thus, for systematic encoding, the generator matrix $G_p = BF^{\otimes n}$ and $b = aB$ is equivalent to $G = F^{\otimes n}$ and $B = A$. In the sequel, when it comes to the SC decoding, we assume the encoding is based on the generator matrix $G_p$ so that the natural order schedule of the decoding can be applied. This is only for the ease of description and doesn’t affect the performance of systematic polar codes.

III. The Theoretical Performance of Systematic Polar Codes

In this section, we provide a general relationship between the error performance of systematic polar codes and the error performance of the non-systematic polar codes.
Denote the BER of non-systematic polar codes as \(P_b\) and the corresponding BER of systematic polar codes as \(P_{sys,b}\). Define a set \(A_s \subseteq A\) to contain the indices of the information bits in error for non-systematic polar codes. Correspondingly, the set \(A_{sys,t} \subseteq A\) is the indices of the information bits in error for systematic polar codes under the SC-EN decoding. The BER for systematic polar codes can be predicted from the BER of the non-systematic polar codes in the following way:

\[
P_{sys,b} = \frac{\sum_{A_{sys,t} \subseteq A} |A_{sys,t}| \Pr\{A_{sys,t}\}}{\sum_{A_t \subseteq A} |A_t| \Pr\{A_t\}} P_b
\]

where \(|A_t|\) is to take the cardinality of the set \(A_t\) and \(\Pr\{\cdot\}\) is the probability of the inside event.

For any given set \(A_t\), the set \(A_{sys,t}\) can be calculated from it. From (20), we already have \(x_A = u_A G_{AA}\). This says that the values or the errors in \(x_A\) only depend on \(u_A\) and \(G_{AA}\). The values of the frozen bits don’t affect the values or the errors of \(x_A\). Therefore, we can convert the cardinality of the set \(A_t\) and \(A_{sys,t}\) into weight of the following vectors. Let \(v\) be a \(N\)-element vector with 1s in the positions specified by \(A_t\) and 0s elsewhere, namely \(v_{A_t} = 1_{|A_t|}\). Then the cardinality of \(A_t\) is the same as the Hamming weight of the vector \(v\) written as \(w_H(v)\). In the same way, we define a vector \(\tilde{v}\) with \(q_{A_{sys,t}} = 1_{|A_{sys,t}|}\) and 0s elsewhere. We can then have

\[
P_{sys,b} = \frac{\sum_{A_{sys,t} \subseteq A} w_H(q) \Pr\{A_{sys,t}\}}{\sum_{A_t \subseteq A} w_H(v) \Pr\{A_t\}} P_b
\]

The equality \(q = v G\) in equation (23) is because of the re-encoding of \(x = \tilde{u} G\) after the decoding of \(\tilde{u}\). Note that the operation \(q = v G\) only represents the error conversion from \(v\) to \(q\), not the real calculation of \(\tilde{u} = v G\).

The cardinality of \(A\) is \(|A| = NR = K\) where \(R\) is the code rate and \(K\) is the number of information bits in each code block. It’s easy to verify that the number of terms in the denominator of (23) is \(N^R = 2^K\). With a large block length \(N\) and a fixed code rate \(R\), it’s practically impossible to evaluate the error performance for systematic polar codes conditioned on the error performance of the non-systematic polar codes.

In this section, without considering the probabilities of the error events \(\{A_t\}\), we evaluate the error performance of systematic polar codes in two special cases to gain some initial insights of the behavior of the systematic polar codes. These two special cases are: 1) \(v_{A_t} = 1_{|A_t|}\); and 2) The \(e\)th element of \(v\) is one: \(v_e = 1\) with \(e \in A\). Case 1) is the situation where all bits are in error and case 2) says only one bit is in error. The rationale for evaluating case 1) is due to the fact that if one bit \(j \in A\) is in error, then theoretically this error bit could affect all bits after it. This can be seen from the transition probability of bit channel \(i > j\) in (9): bit channel \(i\) has its output \(y^N\) (all received channel samples) and \(u_i^{-1}\) (all previously decoded bits). As for case 2), it’s related to the common assumption of coded systems that errors of the code bits in one codeword are independent and that at high SNR, there is only one bit in error in each codeword, resulting in the relationship \(P_b = P_b / N\), where \(P_b\) is the BER and \(P_b\) is the block error rate.

Before we analyze case 1, we need the following proposition.

**Proposition 1:** For a block length \(N = 2^n\), \(n > 0\), any column \(j\) \((1 \leq j \leq N)\) of the generator matrix \(G = F^{2^n}\) has a Hamming weight of \(2^w j(b_1', b_2', ..., b_n')\), where \((b_1', b_2', ..., b_n')\) is the binary expansion of \(j - 1\), and \(b_i' = b_i' + 1\) or \(0\).

**Proof:** For a fixed column \(j\), the weight of this column is to sum over all possible values of \(i - 1 = (b_1', b_2', ..., b_n')\):

\[
\sum_i G_{i,j} = \sum_i F^{2^n}_{i,j} = \sum_i \prod_{m=1}^{i-1} (1 + b_m' + b_m b_m')
\]

The rest of the proof is readily available.

### A. All Bits in Error

From Proposition 1, it can be inferred that except column \(N\), the weight of all other columns of \(G\) is even. From Theorem 1, we know column \(j \in A\) of the generator matrix \(G\) only has 1s at positions specified by \(A\) since column \(j\) doesn’t intersect with rows in \(A\). Therefore, during the re-encoding process \(q = v G\), the vector \(q_{(A,N)} = 0^{N-1}\) and \(q_N = 1\). Here \(A \setminus N\) means the set \(A\) excluding the last element \(N\). The weight of \(q\) is then \(w_H(q) = 1\). We see almost all the errors in the vector \(v\) are cancelled after the re-encoding process (with only one error remaining). If this is the only error case, then \(P_{sys,b} = \frac{1}{N^R} P_b = \frac{1}{2^K}\). We give an example below to explicitly present this error cancelling process.

Suppose we are dealing with a BEC channel with an erasure probability 0.4 and \(N = 16\). Let the code rate \(R = 1/2\). The code index set can be calculated as \(A = \{8, 10, 11, 12, 13, 14, 15, 16\}\). With all bits in error during the SC decoding process, the vector \(v = 1^8\). The elements of \(q_{A}\) can be calculated from \(q_{A} = (v G)_{A}\). For example,

\[
q_8 = v_8 + v_{16} = 0
\]

\[
q_{10} = v_{10} + v_{12} + v_{14} + v_{16} = 0
\]

\[
q_{11} = v_{11} + v_{12} + v_{15} + v_{16} = 0
\]

With the weight of the columns of \(G\) be even (excluding column \(N\)) and the columns with indices in \(A\) only intersect with rows in \(A\), the elements of \(q\) (excluding \(q_N\)) are essentially summing over even numbers of elements of \(v_{A}\), which eventually resulting in 0s when \(v_{A} = 1^8\). The last element is \(q_{16} = v_{16} = 1\), which is the only error remaining after the vector \(v\) going through the matrix \(G\). The error rate is then \(P_b = 1/2\) and \(P_{sys,b} = \frac{1}{2^8}\).

From this example, it’s seen that the re-encoding process of \(\tilde{u} = v G\) after decoding \(\tilde{u}\) does not amplify the number of errors in \(\tilde{u}\) when all bits of \(\tilde{u}_{A}\) are in error. Actually in this case, the number of errors is already at its maximum and can’t be amplified. But the number of errors doesn’t stay the same, as one would expect in this case, after the re-encoding process. Instead, almost all errors are cancelled after the re-encoding process.
B. One Bit in Error

Now we return to the case with only one error \( v_e = 1 \), \( e \in \mathcal{A} \). Denote the \( e \)th row of \( G \) as \( G_{e,:} \). The indices of the corresponding error bits for systematic polar codes are the indices of the non-zero positions of the subvector \((G_{e,:})_A\). Therefore the number of non-zero positions of \( q_A \) is determined by the weight of this subvector \((G_{e,:})_A\): \( w_H(q) = w_H((G_{e,:})_A) \). Due to the fact that \( G \) is a lower triangular matrix, only elements in \( \{ i : i \in \mathcal{A} \text{ and } i \leq e \} \) of \( q_A \) are affected by this error in \( P \). In this one-error case, the number of errors could be amplified after the re-encoding process \( \hat{P} = \hat{G}P \), depending on the location of the error. The error rate is \( P_b = \frac{1}{N^e} \) and \( P_{\text{sys},b} = \frac{w_H((G_{e,:})_A)}{N^e} P_b \). In the preceding example, instead of \( v_A = 1_{\mathcal{A}} \), if we only have \( v_{16} = 1 \), then \( q_A = 1_{\mathcal{A}} \) since \( w_H((G_{16,:})_A) = 8 \), resulting in \( P_b = \frac{1}{8} \) and \( P_{\text{sys},b} = 1 \). But if we only have \( v_k = 1 \) or \( v_{10} = 1 \), then we also only have the corresponding bit in error \( q_k = 1 \) or \( q_{10} = 1 \) with \( P_b = P_{\text{sys},b} = 1 \). The number of errors in the case \( v_{16} = 1 \) is indeed amplified by 8 times after the re-encoding process while the number of errors with \( v_k = 1 \) or \( v_{10} = 1 \) stays the same after the re-encoding process.

IV. SYSTEMATIC POLAR CODES GAIN

In the discussions from Section III-A and III-B we already noted that the number of errors of polar codes with the SC decoding is not necessarily amplified in the SC-EN decoding process of systematic polar codes. It all depends on how the errors are distributed in the SC decoding process. This section is devoted to the analysis of the behavior of the errors in the SC decoding process and to characterize the advantage of systematic polar codes over non-systematic polar codes. The analysis is based on BEC channels. In Section VI it’s seen that the results in this Section can be extended to AWGN channels as well.

A. Basic Error Patterns

In order to understand how the errors are distributed with the SC decoding, we first look at the basic error patterns. The decoding graph of polar codes with a block length \( N = 2^n \) consists of \( n \) columns of Z-shape sections, with each column having \( N/2 \) Z-shape sections. For the connections of the Z-shape sections in each level, please refer to [1] [17]. In this subsection, we use the natural order schedule for the SC decoding as discussed in Section II-D.

The basic error patterns in the decoding graph are illustrated in Fig. 2 where a node without any label has a correct likelihood ratio (LR) value, a node with a label 1 has a LR value of one, a node with a label X has an incorrect LR value, and a node with a label ? can have a correct or incorrect LR value depending on the context. In the SC decoding, before the first error happens, the LR values of the variable nodes in the Z-shape sections are either correct or 1, represented by (a),(b) and (c) of Fig. 2. We provide the proof of the error pattern Fig. 2c in the Appendix and all other patterns in Fig. 2 can be proved in the same fashion.

The LR value of the first error bit must be one. Again, the proof of this fact is omitted as this is relatively a simple practice. In other words, the first error happens because the decoder takes an incorrect guess, corresponding to the upper left node in Fig. 2(a)(b)(c) and the lower left node in Fig. 2(c).

After the first error, as we already point out, all bits after this error bit could potentially be affected by this error. For example, the lower left nodes in Fig. 2(d)(f)(g) are in error because of the previous errors. These errors are surely the errors propagated (or coupled) from previous errors. But not all bits after the first error bit are in error, simply through observing the basic error patterns in Fig. 2. The first example is Fig. 2(a). If the bit (or the combined bit) corresponding to the upper left node is in error, then the LR value corresponding to the lower left bit is still correct. Actually, the LR of the lower left node is not affected by the upper left node since the upper right node in Fig. 2(a) has a LR value of one. In this case, as long as the lower right node has a correct LR value, the lower left node can always make a correct decision. Another example is Fig. 2(e) in which the upper left node has an incorrect LR value thus with an incorrect bit decision. But the incorrect bit decision cancels the effect of the incorrect LR value of the upper right node when it comes to the decision of the lower left node. Therefore the lower left node can make a correct decision in this case even though the upper left node has an incorrect decision. For a rigorous proof of this pattern, please refer to the Appendix. There are other cases, for example Fig. 2(g)(h), where incorrect LRs due to incorrect previously decoded bits don’t necessarily cause all bits in error after those error bits. Because of these effects, it’s extremely unlikely that after the first error bit, all bits after it are in error, especially with large block lengths. For the same reason, it’s also unlikely that all bits after the first error bits are correct. In other words, one bit error, like all bits in error, is also unlikely.

From the basic error patterns in Fig. 2 one proposition can be easily obtained for BEC channels.

Proposition 2: For polar codes with the SC decoding on BEC channels, the number of nodes with \( LR = 1 \) stays the same in each column of the decoding graph.

B. First Error Distribution

As stated in the previous section, the first error happens because the decoder takes an incorrect guess. All calculations before the first error involve patterns in Fig. 2(a)(b)(c). Note that the question marker in the lower left node should be removed before the first error as there are no errors yet. Of course, there is always a pattern involving two correct nodes which is not shown in Fig. 2.

The probability of bit \( i \) being the first error is determined by the quality of bit channel \( i \), which in turn is determined by its Bhattacharyya parameter. For a rigorous proof, please refer to Section V-B of [1]. In this section, we present simulation results on the first error distribution without further theoretical discussions.

For BEC channels, we can precisely calculate the Bhattacharyya parameter for each bit channel using the recursive expressions given in [1]. Fig. 3 shows the histogram of the
indices of the first error bit and the corresponding average
Bhattacharyya parameter for $N = 2^{10}$ and $R = 1/2$ in a BEC
channel with an erasure probability 0.4. Fig. 3 has two y-axes:
the right axis shows the number of occurrences of the first
error and the left axis shows the value of the corresponding
Bhattacharyya parameters. Seen from Fig. 3 the probability
of the first error is indeed determined by the quality of each
bit channel. Also shown in Fig. 3 is the brick-wall nature
of the first error distribution, which is the reflection of the
polarization effect of the $N$ channels.

At this point, we want to point out the effect of the first
error in non-systematic and systematic polar codes scenarios.
The first error in the SC decoding process could potentially
affect all bits after it (or bits with indices larger than it with
the natural order decoding). This effect can be considered as
a forward error effect. But in the re-encoding process of the SC-
EN decoding of systematic polar codes, the errors (including
the first error) in the decoded vector $\hat{u}$ only affect bits in
$\hat{u}$ before them (or bits with indices smaller than the error
bits), due to the lower triangularity of the generator matrix.
Correspondingly, this effect in the re-encoding process is a
backward error effect.

### C. Systematic Polar Codes Gain

In this section, we extract the first part of the right hand side
of (21) and define the reverse of it as the gain of systematic
polar codes over non-systematic polar codes:

$$\gamma = \frac{\sum_{A_{t} \subseteq A} |A_{t}| \Pr \{A_{t}\}}{\sum_{A_{sys,t} \subseteq A} |A_{sys,t}| \Pr \{A_{sys,t}\}}$$

(27)

From the previous discussion in Section III-A and III-B, we
can safely constrain the systematic gain to be strict for large
$N$: $\frac{1}{NR} < \gamma < NR$.

The analysis in Section III does not include the effect of the
coupling between errors as discussed in Section IV. From the
discussions in Section IV-A we know the errors in previous
decoded bits could affect the bits after them, although not all
bits after the error bits are necessarily in error. With a large
$N$, there are $2^N - 1$ error combinations in the received vector $y_1^N$. Therefore, the errors of the decoded bits after the first
decoded error can be considered as independent and identically
distributed (i.i.d) with probability $p$ when $N$ is large. Based
on this assumption, we can convert the calculation of the
systematic gain in (27) into the analysis of a function involving
the first error distribution and the probability $p$.

Denote $p_i$ as the probability of the first error occurring to
the information bit $i$ and denote this error event as $\xi_i$. As in
Section III we use a vector $v_1^N$ to represent the error positions:
$v_i = 1$ if bit $i$ is in error and $v_i = 0$ otherwise. Then the
probability of all information bits in error conditioned on $\xi_i$ is:

$$\Pr \{v_A = 1_i^K | \xi_i\} = (0, 0, ..., 1, p, p, ..., p)$$

(28)

In (28), the first $(i - 1)$ probabilities are zeros because the first
error at bit $i$ doesn’t affect bits before it (the forward error
effect). After bit $i$, the errors are i.i.d with probability $p$ as
discussed previously. Since the events $\{\xi_i\}_{i=1}^K$ are exclusive,
the probability of the information bits in error is simply the following summation

$$\Pr \{v_A = 1_i^K\} = \sum_{i=1}^{K} \Pr \{v_A = 1_i^K | \xi_i\}$$

(29)
with the individual bit error as $\Pr\{v_i = 1\} = p_i + p \sum_{j=1}^{i-1} t_j$.

Utilizing the brick-wall property of the first error distribution as shown in Fig. 3 we can divide the bits in $\mathcal{A}$ into two groups: group one consisting of the error bits due to the bad bit channel conditions and group two consisting of the error bits purely coupled from group one. Denote these two groups as $\mathcal{A}_I$ and $\mathcal{A}_C$, respectively. Referring to Fig. 3 the set $\mathcal{A}_I$ includes the bits within the brick wall and the set $\mathcal{A}_C$ includes the bits outside the brick wall. Denote $K_I = |\mathcal{A}_I|$. The probabilities can now be expressed as:

$$
\Pr\{v_i = 1\} = \begin{cases} 
p_i + p \sum_{j=1}^{i-1} t_j, & 1 \leq i \leq K_I \\
p K_I \sum_{j=1}^{i} p_j, & K_I < i \leq K
\end{cases}
$$

(30)

And the systematic gain can be calculated using (30) as

$$
\gamma = \frac{\mathbb{E}(\omega_H(v_N^N))}{\mathbb{E}(\omega_H(v_N^N G))}
$$

(31)

The evaluation of (31) involves the distribution of the first error probabilities $\{p_i\}_{i=1}^K$ of the information bits and the probability $p$. The distribution of the probabilities $\{p_i\}_{i=1}^K$ can be approximated by the distribution of the corresponding Bhattacharyya parameters $\{Z(W_N^{(i)})\}$. But the combined effect of the probability $p$ and the distribution of $\{Z(W_N^{(i)})\}$ is not intended to be fully discussed in this paper due to the space limit. Instead, in Section V we establish a simplified statistic model to characterize the probabilities in (30) and this model is used to calculate the systematic gain $\gamma$ in (31).

D. A Qualitative View of the Systematic Gain

Using Corollary 2 we can qualify why the systematic gain $\gamma$ should be generally larger than one. Or at least, the systematic polar codes should perform as well as the non-systematic polar codes. In the re-encoding process, the estimation $\hat{\bar{x}} = \hat{u} G$ is performed. The entry of $\hat{\bar{x}}$, say $\hat{x}_j$ ($j \in \mathcal{A}$), is

$$
\hat{x}_j = \hat{u}_j G_{A,j}
$$

(32)

where $G_{A,j}$ is the $j$th column of $G$ with entries specified by $\mathcal{A}$. The error correction capability of the systematic polar codes comes from this re-encoding process in (32). To understand this capability of systematic polar codes, we first note that the weight of all the columns of the matrix $G$ is even except the last column. This property of $G$ is already stated in the beginning of Section III-A. This is where the theoretical maximum $\gamma = NR$ comes from.

From (32), it’s seen that the errors in $\hat{u}_A$ can only affect $\hat{x}_j$ at positions where $G_{A,j}$ have non-zero entries. From Corollary 2 it’s known that a non-zero entry of column $j$, $G_{i,j} = 1$, means a better bit channel $i$ than $j$. Let’s call the set of bits $\{i : i \neq j, i \in \mathcal{A} \text{ and } G_{i,j} = 1\}$ the compatible bits of the information bit $j$. For bit $\hat{x}_j$, only bit $j$ and its compatible bits affect the decision. Since the compatible bits of bit $j$ transmit at better bit channels than $j$, it’s more likely that bit $j$ is in error and the compatible bits are in error due to the error propagation of bit $j$. In other words, the errors of bit $j$ and its compatible bits are coupled. The re-encoding process $\hat{x}_j = \hat{u}_A G_{A,j}$ is equivalent to sum over bit $j$ and its compatible bits, a process to average out the coupled errors. This mechanism of the re-encoding process leads to the fact that systematic polar codes perform at least as well as non-systematic polar codes, or $\gamma \geq 1$.

V. Composite Error Model

So far we are still short of an efficient way to calculate the systematic gain $\gamma$. In this section, we establish a statistic model to simplify the probabilities of the errors in (30). This simplified model is then used to calculate the systematic gain $\gamma$.

We define a new set $\mathcal{S}$ as the ensemble of the error events $\mathcal{A}_i$:

$$
\mathcal{S} = \bigcup_{\mathcal{A}_i \subset \mathcal{A}} \mathcal{A}_i
$$

(33)

Considering the basic error patterns in Fig. 2 the errors could happen to any bit after the first error bit, no matter which bit channel the bit experiences. Therefore, the set $\mathcal{S}$ can almost surely consist of all the information bits after
the first information bit with a non-negligible Bhattacharyya parameter. For this, we set a threshold \( \alpha \), below which the Bhattacharyya parameter is considered as negligible. Otherwise, the Bhattacharyya parameter is considered as large. Define a set consisting of all the indices of the bit channels with non-negligible Bhattacharyya parameters as

\[
\mathcal{I} = \{i, \ i \in \mathcal{A} \text{ and } Z(W_N^{(i)}) > \alpha \} \tag{34}
\]

As stated in Section II-B, the set \( \mathcal{A} \) is sorted in ascending order according to the index values. So is the set \( \mathcal{I} \). This set \( \mathcal{I} \) is used to define the boundaries of the brick wall in Fig. 3. The first element (with the smallest index value) in \( \mathcal{I} \) is denoted as \( I_1 \). Then \( S \) can be written as

\[
S = \{a : \ a \in \mathcal{A} \text{ and } a \geq I_1 \} \tag{35}
\]

When \( I_1 \) happens to be also the first element of \( \mathcal{A} \), then \( S = \mathcal{A} \). The elements of \( S \) are also sorted according to the index values as the elements in the set \( \mathcal{A} \).

The next part to define \( S \) is to assign each element in \( S \) a probability of being in error. Following the discussions in Section IV-C we perform the following steps to \( S \):

- Divide the set \( S \) into two sections: the first section, denoted as \( S_1 \), being the region where the first error could happen, and the second section, \( S_2 \), being the region where errors are coupled or induced from region one.
- The composite effect of \( \sqcup \mathcal{A}_t \) in the first region \( S_1 \) is denoted by the error probability of the first equation of (36).
- The composite effect of \( \sqcup \mathcal{A}_t \) in the second region \( S_2 \) is denoted by the error probability of the second equation of (36).

From the second equation of (36) it’s known that all bits in \( S_2 \) have the same probability of error from a composite point of view and this probability of error should be larger than the probability of error in \( S_1 \) due to the following observation. In region one, any bit with index \( a_1 \) could be in error at one error event \( \mathcal{A}_t \subseteq \mathcal{A} \) with the first error bit \( e_1 < a_1 \), but will be for sure correctly decoded in another error event \( \mathcal{A}_2 \subseteq \mathcal{A} \) when \( a_1 < e_2 \) with \( e_2 \) being the index of the first error bit in event \( \mathcal{A}_2 \). In region two, any bit can be potentially decoded incorrectly in any error event. So statistically, the bits in region \( S_2 \) have a higher probability of being in error when considering the composite effects of \( \sqcup \mathcal{A}_t \). This condition translates to the way we select the probability \( p \) in (36). However, as we point out in Section IV-C, it’s theoretically difficult to precisely calculate the probability of error for each element in \( S \). With the above observation, we propose the following simplified model in place of the precise model:

\[
S_1 = \{a : \ a \in S \text{, } a \leq I_m, \ \Pr[a \text{ is in error}] = p_0\} \tag{36}
\]

\[
S_2 = \{a : \ a \in S \text{, } a > I_m, \ \Pr[a \text{ is in error}] = 1\} \tag{37}
\]

with a probability \( p_0 \). The rest of the information bits are in error with probability one from a composite point of view. Although in this paper a precise probability \( p_0 \) is not pursued, empirically we find that \( p_0 = 1/2 \) is a very good approximation.

An important parameter of the model in (36) is the boundary \( I_m \). It’s clear that this boundary element \( I_m \) is related to the channel \( W \). For example, with a BEC channel, when the block length \( N \) and the code rate \( R \) is fixed, the boundary \( I_m \) is related to the erasure probability. With a large erasure probability, there are more bits which are in error due to the channel itself and less bits in error due to the forward error effect, and vice versa. Without going into the details of calculating the Bhattacharyya parameters of the bit channels (which is only possible for BEC channels), we can use a coupling coefficient to calculate another boundary element \( I_m \in S \). The coupling coefficient here means the fraction of incorrect information bits due to the previously incorrectly decoded information bits. Denote the coupling coefficient as \( \beta \) and the element \( I_m \) is the \( m' \)th element of \( S \) where

\[
m' = |S| \ast (1 - \beta) \tag{38}
\]

Then we can use \( I_m \) to replace the boundary element \( I_m \) in the model (36). This boundary based on the coupling coefficient is especially useful for bit channels whose Bhattacharyya parameters are not readily available.

Note that this simple model in (36) can approximate the composite effect \( \sqcup \mathcal{A}_t \) only in the statistical sense and it only models the dominant effect (or the mean effect) of \( \sqcup \mathcal{A}_t \). It is not, by any means, an exact error event \( \mathcal{A}_t \subseteq \mathcal{A} \).

### A. Calculation of the Systematic Gain

With the composite error model in (36), we can calculate the systematic gain. Use the same \( N \)-element vector \( v \) as an error indicator vector of \( S \): the \( i \)th entry of \( v \) is zero if \( i \notin S \); otherwise \( v_i \) is one if \( i \in S \) and the \( i \)th bit is in error. The subvector corresponding to region two of \( S \) is \( v_{S_2} = 1_{|S_2|} \) seen from (37). Each element of the subvector \( v_{S_2} \) takes value in \( \{0, 1\} \) with probability \( p_0 \) as shown in (36). The systematic gain from the composite model is thus

\[
\gamma = \frac{\mathbb{E}\{w_H(v)\}}{\mathbb{E}\{w_H(vG)\}} \tag{39}
\]

The mean weight \( w \) can be easily calculated as \( \mathbb{E}\{w_H(v)\} = p_0|S_1| + |S_2| \). Now we need to calculate the mean weight of \( x_S = vG_{SS} \), which can be decomposed as \( (x_{S_1}, x_{S_2}) = vG_{S_1S_1} + vG_{S_2S_2} \). Due to the lower triangularity of the matrix \( G_{SS} \), the weight of \( x_{S_2} \) can be directly calculated as

\[
\omega_H\{x_{S_2}\} = \omega_H\{vG_{SS}\} = \omega_H\{v_{S_2}G_{S_2S_2}\} = 1 \tag{40}
\]

which uses the even weight property of the columns of \( G \) except the last column. The first part \( x_{S_1} = vG_{S_1S_1} \) can be further divided into the summation of two parts:

\[
x_{S_1} = v_{S_1}G_{S_1S_1} + v_{S_2}G_{S_2S_1} \tag{41}
\]

The second part in (41) is a deterministic vector since \( v_{S_2} \) is the all-one vector. With \( G_{S_1S_1} \), an invertible lower triangular
matrix, the vector \( x_{S_1} \) belongs to the row space of the matrix \( G_{S_1,S_1} \). Thus it can be formed by another vector \( \tilde{v}_{S_1} \) in the identity basis of the row space of \( G_{S_1,S_1} \) as:
\[
x_{S_1} = \tilde{v}_{S_1} I
\] (42)

with \( \tilde{v}_{S_1} \) defined in the same way as \( v_{S_1} \). Therefore the mean weight of \( x_{S_1} \) is the same as the mean weight of \( \tilde{v}_{S_1} \) which is
\[
E\{\omega_H\{x_{S_1}\}\} = p_0|S_1|
\] (43)
The systematic gain is then
\[
\gamma = \frac{p_0|S_1| + |S_2|}{1 + p_0|S_1|}
\] (44)

When the cardinality of \( S_1 \) is quite large, the systematic gain can be approximated as:
\[
\gamma \approx 1 + \frac{1}{p_0} \frac{|S_2|}{|S_1|}
\] (45)

An immediate conclusion from (45) is that the systematic gain is greater than one, meaning that systematic polar codes should perform better than the corresponding non-systematic polar codes. Another interpretation of (45) is that the systematic gain is only determined by the ratio of cardinalities of the two sets \( S_1 \) and \( S_2 \). It does not increase with the increase of the block length as one would intuitively expect. This property of the systematic polar codes is verified in the simulations in Section VI.

VI. NUMERICAL RESULTS

In this sections, numerical examples for both BEC channels and AWGN channels are provided to validate the results in Sections V and V. The encoding for BEC channels are done through the selection of the bit channels with the smallest Bhattacharyya parameters. For AWGN channels, we still use the same recursive formula in calculating the Bhattacharyya parameters for BEC channels in encoding. We emphasize that this encoding serves our purpose just as well, as long as it’s consistent for both non-systematic polar codes and systematic polar codes.

Fig. 4 is the result in the BEC channel for \( N = 2^{10} \) and \( R = 1/2 \). Several curves are shown in Fig. 4. The curve of the stared dotted line is the BER of the non-systematic polar codes under the SC decoding. The legend for this curve is ‘SC’. The curve of the dash dotted line with triangles is the BER of the systematic polar codes with the SC-EN decoding for which the legend is ‘SYSTEMATIC’. The circled solid line is the theoretical BER for systematic polar codes from the model in (36) (37). Also shown in Fig. 4 is the BER of the non-systematic polar codes with the belief-propagation (BP) decoding (the curve of the dashed line with diamonds).

The theoretical BER for systematic polar codes in Fig. 4 is generated using two different coupling coefficients: \( \beta = 0.3 \) when the erasure probability is larger than 0.45 and \( \beta = 0.5 \) when the erasure probability is smaller than 0.45. This choice of the coupling coefficient corresponds to the bad channel condition and the good channel condition, respectively. The probability of independent error in \( S_1 \) is \( p_0 = 1/2 \) and it’s used in all of the following theoretical calculations. The threshold \( \alpha \) in determining the set \( S \) in (34) is set to be \( \alpha = 10^{-3} \). Under this setting, the first element of \( S \) is \( I_1 = 192 \) when the erasure probability is 0.4, which is also the first element of \( A \). Thus the composite set is \( S = A \) in this case. The systematic gain calculated from the composite set \( S \) is quite stable. A small number can be used in averaging this systematic gain.

In Fig. 4 only ten realizations are used in calculating the theoretical systematic gain \( \gamma \). The simulated BER and the BER from the model in (36) (37) match quite well, showing that the simple model in (36) (37) can approximate the dominant error events of \( \cup_iA_i \) and thus can be used to calculate the systematic gain.

Also showing in Fig. 4 is the BER for non-systematic polar codes with the BP decoding. BP decoding is generally better than the SC decoding as shown in (17). With a bad channel condition, for example, with an erasure probability larger than 0.45, BP decoding performs almost the same as the SC decoding. Systematic polar codes, however, perform two to three times better than both SC and BP decoding under the same channel conditions, at a cost almost negligible compared to the complexity of the BP decoding. At better channel conditions, BP decoding starts to show its advantage.

We observe the same phenomenon in Fig. 5 as in Fig. 4 for \( N = 10^{12} \) and \( R = 1/2 \). The curves in Fig. 5 have the same style and labels as Fig. 4. The coupling coefficient is set the same as the case \( N = 10 \) and \( R = 1/2 \). Again, the simulated systematic gain embedded in the BER of systematic polar codes matches that calculated using the composite set \( S \).

Fig. 6 is the result in the BEC channel for \( N = 10 \) and \( R = 1/4 \) in the AWGN channel. The composite set is \( S = A \). The systematic gain is set the same as the case \( N = 10 \) and \( R = 1/2 \). Again, the simulated systematic gain matches that with the simulations, showing that the composite model in (36) (37) can also be used for AWGN channels.

From Fig. 4 to Fig. 6 we see that systematic polar codes perform consistently better than non-systematic polar codes, echoing the results in (25). The systematic gain for different block lengths is shown in Fig. 7. The underlying channel \( W \) is a BEC channel with an erasure probability 0.4. The gain represented by the circled line (with a legend ‘Sys Gain Sim’) is simulated. The gain shown by the starred line (with a legend ‘Sys Gain Theoretical’) is calculated using (44). The systematic gain calculated using (44) is accurate when \( N \) is large. It’s seen from Fig. 7 that the systematic gain increases with the increase of the block lengths but saturates at around \( \gamma = 3 \) when \( N \geq 2^9 \). This coincides with the simulation results in Fig. 4. The saturating nature of the systematic gain can be seen from the composite set \( S \). With a fixed code rate \( R \), as the block length \( N \) increases, the cardinality of \( S \) also increases. So the increase in the error-correction capability of the systematic polar codes is counteracted by the increase in the number of error bits, rendering the systematic gain to reach a limit.
Fig. 4. BER for $n = 10, R = 1/2$ in BEC channel.

Fig. 5. BER for $n = 12, R = 1/2$ in BEC channel.

Fig. 6. BER for $n = 10, R = 1/4$ in AWGN channel.
VII. CONCLUSION

In this paper, we analyze the error performance of systematic polar codes with the SC-EN decoding. Through the analysis of the generating matrix of polar codes, the encoding process of systematic polar codes is simplified. We use a parameter, the systematic gain, to characterize the performance of systematic polar codes as compared with the non-systematic polar codes. From the study of the basic error patterns and the first error distribution of the SC decoding, the information bits are divided into two regions and the probability of errors in each region is provided. To further use the properties of these two regions, we propose a composite model to approximate the mean effect of the error events in the SC decoding. Using this composite model, the systematic gain can be calculated. Numerical results are provided and our models are verified in the paper. Systematic polar codes are shown to be around 3 times better than non-systematic polar codes in terms of the BER performance with large block lengths.

APPENDIX

PROOF OF THE ERROR PATTERNS

We provide the proof of the error pattern in Fig. 2c. Let’s assume the two bits at the input to the Z-section is $u_1$ and $u_2$. The output is then $x_1 = u_1 \oplus u_2$ and $x_2 = u_2$. In this pattern, the LR value of $x_1$ is incorrect, namely $LR(x_1) = LR(u_1 \oplus u_2 \oplus 1)$. In estimating $u_1$, we have

$$LR(\hat{u}_1) = \frac{1 + LR(u_1 \oplus u_2 \oplus 1) \ast LR(u_2)}{LR(u_1 \oplus u_2 \oplus 1) + LR(u_2)}$$  \hspace{1cm} (46)

Compared with the true estimation

$$LR(u_1) = \frac{1 + LR(u_1 \oplus u_2) \ast LR(u_2)}{LR(u_1 \oplus u_2) + LR(u_2)}$$  \hspace{1cm} (47)

it’s readily seen that $\hat{u}_1 = u_1 \oplus 1$. Therefore the LR value of the variable node $u_1$ is incorrect, as indicated by a $X$ in the upper left node in Fig. 2c.

After obtaining the estimation of $\hat{u}_1$, the LR value of bit $u_2$ is given by

$$LR(\hat{u}_2) = LR(u_2) \ast LR(u_1 \oplus u_2 \oplus 1)^{1-2\hat{u}_1}$$  \hspace{1cm} (48)

Substituting $\hat{u}_1 = u_1 \oplus 1$ into (48) and using the fact that $LR(u_1 \oplus 1) = LR(u_1)^{-1}$, we obtain the following

$$LR(\hat{u}_2) = LR(u_2) \ast LR(u_1 \oplus u_2)^{-1+2(u_1 \oplus 1)}$$  \hspace{1cm} (49)

Again, comparing with the true estimation of $u_2$

$$LR(u_2) = LR(u_2) \ast LR(u_1 \oplus u_2)^{1-2u_1}$$  \hspace{1cm} (50)

we can verify that (49) and (50) are equivalent, meaning the estimation of $\hat{u}_2$ is the true estimation, which is the lower left node in Fig. 2e.

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