The Schwarzschild black hole in f(R) can exist superradiation phenomenon

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Abstract  
This paper explores the mechanism by which Schwarzschild black holes may exhibit superradiance when Bell’s theorem is considered a fundamental principle. By integrating quantum field theory with general relativity under the constraints imposed by Bell’s inequalities, we develop a comprehensive model that extends the classical understanding of black hole dynamics. The study further draws parallels between the superradiant phenomena in Schwarzschild black holes and those observed in f(R) gravity theories. Our findings suggest a profound mathematical and conceptual linkage between Bell’s theorem and f(R) gravity, particularly in the manifestation of irreducible singularities and their associated residues. This work provides a novel framework bridging quantum non-locality with modified theories of gravity, offering potential insights into the quantum nature of spacetime and black hole physics.

Keywords: Bell’s Theorem; Schwarzschild Black Hole; Superradiance; f(R) Gravity; Singularity; Quantum Residues

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1 Introduction

In 1972, Press and Teukolsky proposed that by adding a mirror outside a black hole, it is possible to create a “black hole bomb,” a concept involving a scattering process rooted in both classical and quantum mechanics [25]. When a boson wave impacts a rotating black hole, if the wave frequency falls within the superradiance region, the wave reflected from the event horizon will be amplified:

\[ 0 < \omega < m \Omega_H, \quad \Omega_H = \frac{a}{r_+^2 + a^2}, \]

where \( m \) is the azimuthal number of the bosonic wave mode, and \( \Omega_H \) is the angular velocity of the black hole’s horizon. This amplification is known as superradiant scattering. Consequently, the rotational energy of the black hole can be extracted through the superradiance process. If a mirror is placed between the black hole’s horizon and infinity, the amplified wave will scatter back and forth, growing exponentially and leading to an unstable superradiance of the black hole.

The paper discusses a standard no-hair theorem and reveals that, within the framework of the Einstein-scalar theory, an asymptotically flat black hole with a regular horizon cannot support a static scalar field configuration. This intriguing physical property also characterizes black hole spacetime, particularly when the scalar field is non-minimally coupled with the Ricci curvature scalar.

Interestingly, recent works have confirmed that in spherically symmetric, asymptotically flat rotating black hole spacetimes, a massless scalar field configuration cannot be supported when non-minimally coupled with the spacetime curvature after preset boundary conditions are applied [6]. This phenomenon, termed the spontaneous scaling of black holes, has been explored in the literature. For a large mass scalar field configuration, it is not minimally coupled with the spacetime of rotating black holes.

If the coupling function of the nontrivial scalar field is characterized by the weak-field behavior \( C(\phi) = 1 + \alpha \phi^2 + o(\phi^4) \), then the Kerr black hole spacetime is effectively a solution to the field equations under the ordinary \( \phi \equiv 0 \) limit. This is an ideal property of the non-minimally coupled Einstein-scalar theory. Here, \( \alpha > 0 \) is a dimensionless physical parameter that determines the coupling strength between the scalar field and the preset boundary conditions in the spacetime of a rotating black hole.

Numerical results indicate that for a given value of the dimensionless rotation-to-mass ratio \( a/m \), the nontrivial coupled Einstein-scalar system has a critical threshold \( \alpha = \alpha(\mu) \). This threshold marks the boundary between a multi-hairy Kerr black hole with a large mass scalar field configuration and a bald (scalar-free) Kerr black hole, where \( \mu \) is the appropriate mass of the non-minimally coupled scalar field. The critical threshold corresponds to a regularized, linearized field configuration supported by the central Kerr black hole. The term “scalar cloud” is commonly used to describe these linearized scalar field configurations on the system’s critical existence line [6].

In a recent study, it was proposed to use information flow from simulation models to gravitational scenarios to understand superradiance from the perspective of Bose-Einstein condensation. The reverse analogy is also compelling. Here, we reconsider the stability of quantized vortices in Bose-Einstein condensates (BECs) and examine the specific instabilities that arise in non-uniform BEC flows analogous to hydrodynamic parallel shear flows.

Bekenstein discovered that the area of a black hole can be interpreted as its entropy, leading to the formulation of four laws of black hole dynamics that parallel the laws of
thermodynamics. The entropy of a black hole is seen as a Noether charge related to the diffeomorphism invariance of gravity. Einstein’s equations have been derived from the first law of thermodynamics and extended to nonequilibrium thermodynamics of spacetime \[17\]. This connection has also been explored in modified gravity theories, such as \( f(R) \) theory \[7\], Lanczos-Lovelock gravity, and scalar-Gauss-Bonnet gravity. It has been shown that the equations of motion for these generalized gravity theories are equivalent to the thermodynamic relation \( \delta Q = T \delta S \). In spherically symmetric spacetimes, Padmanabhan proposed a generalized understanding of horizon thermodynamics, where the near-horizon structure of Einstein’s equations can be expressed as a thermodynamic identity under a virtual displacement of the horizon.

The existence of nonlinear dispersion relations complicates the definition of energy regions and event horizons. This issue is related to what is known as the trans-Planckian problem in Hawking radiation. In Hawking’s original derivation, radiation emitted to future infinity experiences an infinite blue shift, suggesting a dependence on the validity of quantum field theory at arbitrarily high energies in curved spacetime—an assumption without experimental support \[23\]. Beyond the Planck scale, the microscopic physics of spacetime might significantly differ, potentially diverging from general relativity. Gravitational analogs provide a framework to study possible deviations from Lorentz-invariant curved spacetime physics. For instance, dispersion relations of fluctuations might be modified at high frequencies, as observed in Bogoliubov dispersion, with the healing length playing a role similar to the Planck length. Hawking radiation, modified under these conditions, has been extensively studied, showing that dispersion effects strongly influence its thermal properties while retaining its fundamental characteristics \[9\].

In BECs, the scattering of Hawking radiation by acoustic waves differs markedly from gravitational scenarios. It has been demonstrated that in Bose-Einstein condensates, even far from the hydrodynamic limit, the microscopic mechanisms underlying pair formation are conceptually similar \[10\]. This gravitational analogy facilitates the study of field theory in curved spacetime, leveraging the quantum characteristics of BECs’ low temperatures. We explain in detail how the Bogoliubov excitation field relates to the Klein-Gordon field and discuss how the differences may highlight features of the gravitational analogy rather than errors.

For the relationship between the limit \( y \) of the incident particle under the superradiance of the preset boundary \( (\mu = y \omega) \), we find a connection between black hole thermodynamics and superradiance, using boundary conditions to establish a relationship between \( y \) and \( R \). It is found that the black hole’s energy-momentum tensor can be transformed into an effective potential. When this effective potential has a barrier, the Schwarzschild black hole in \( f(R) \) gravity exhibits the phenomenon of superradiance.

Bell’s theorem serves as a cornerstone in quantum mechanics, elucidating the inherent non-locality of quantum entanglement and challenging the classical notions of locality and realism \[4\]. On the other hand, Schwarzschild black holes, as solutions to Einstein’s field equations in general relativity, represent some of the simplest black hole models, characterized by their lack of charge and angular momentum \[5\]. Superradiance, a phenomenon where waves are amplified upon scattering off a rotating black hole, has been extensively studied in the context of Kerr black holes \[6\]. However, the possibility of superradiance in non-rotating Schwarzschild black holes remains an intriguing subject, especially when quantum mechanical principles such as Bell’s theorem are incorporated into the gravitational framework.

Furthermore, \( f(R) \) gravity, a generalization of Einstein’s general relativity, introduces
modifications to the Einstein-Hilbert action by replacing the Ricci scalar \( R \) with a general function \( f(R) \) \[7\]. This theory has garnered significant attention for its potential to explain cosmic acceleration without invoking dark energy and for its rich phenomenology in black hole physics \[8\].

This paper aims to investigate the conditions under which Schwarzschild black holes can exhibit superradiance when Bell’s theorem is considered a fundamental principle. Additionally, we explore the mathematical and physical connections between Bell’s theorem and \( f(R) \) gravity, particularly focusing on the emergence of irreducible singularities and their residues. By developing a more mathematically rigorous framework, we aim to provide deeper insights into the interplay between quantum mechanics and modified theories of gravity.

2 Theoretical Framework

2.1 Bell’s Theorem

Bell’s theorem demonstrates that no local hidden variable theories can reproduce all the predictions of quantum mechanics \[4\]. The theorem is often encapsulated in the form of Bell inequalities, which are violated by quantum mechanical systems exhibiting entanglement. The most commonly discussed Bell inequality is the CHSH (Clauser-Horne-Shimony-Holt) inequality, given by:

\[
|\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle| \leq 2, \tag{2}
\]

where \( A, A' \) and \( B, B' \) are observables measured by two separate parties, typically denoted as Alice and Bob.

In quantum mechanics, the expectation values \( \langle AB \rangle \) can exceed the classical bound of 2, reaching up to \( 2\sqrt{2} \), as dictated by the Tsirelson bound \[24\]:

\[
|\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle| \leq 2\sqrt{2}. \tag{3}
\]

This violation signifies the presence of quantum entanglement and the failure of local realism.

2.2 Schwarzschild Black Hole and Superradiance

The Schwarzschild metric describes the spacetime geometry surrounding a non-rotating, uncharged black hole \[5\]:

\[
ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\Omega^2, \tag{4}
\]

where \( G \) is the gravitational constant, \( M \) is the mass of the black hole, \( c \) is the speed of light, and \( d\Omega^2 \) represents the metric on the unit 2-sphere.

Superradiance typically occurs in rotating black holes (Kerr black holes) where the presence of an ergosphere allows for the extraction of rotational energy \[6\]. The condition for superradiance in Kerr black holes is given by:

\[
\omega < m\Omega_H, \tag{5}
\]
where $\omega$ is the frequency of the incident wave, $m$ is the azimuthal quantum number, and $\Omega_H$ is the angular velocity of the black hole horizon.

For Schwarzschild black holes, the absence of rotation implies that the traditional mechanism for superradiance is absent [25]. However, under certain quantum mechanical constraints, such as those imposed by Bell’s theorem, it is hypothesized that a similar amplification of waves might occur.

### 2.3 $f(R)$ Gravity Theory

$f(R)$ gravity generalizes Einstein’s general relativity by modifying the Einstein-Hilbert action [7]:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_m, \quad (6)$$

where $\kappa = 8\pi G/c^4$, $R$ is the Ricci scalar, $f(R)$ is a function of the Ricci scalar, and $S_m$ represents the matter action.

The field equations derived from this action are:

$$f_R(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f_R(R) = \kappa T_{\mu\nu}, \quad (7)$$

where $f_R(R) = \frac{df(R)}{dR}$, $\Box = \nabla^\mu \nabla_\mu$, and $T_{\mu\nu}$ is the energy-momentum tensor.

$f(R)$ gravity introduces additional degrees of freedom compared to general relativity, allowing for richer phenomenology, including the potential for black hole superradiance even in non-rotating spacetimes.

### 2.4 Quantum Field Theory in Curved Spacetime

To investigate superradiance under the constraints of Bell’s theorem, we employ quantum field theory (QFT) in curved spacetime [9]. The dynamics of a scalar field $\phi$ in the Schwarzschild background are governed by the Klein-Gordon equation:

$$\Box \phi - m^2 \phi = 0, \quad (8)$$

where $m$ is the mass of the scalar field, and $\Box$ is the d’Alembert operator in the Schwarzschild metric.

Expanding the scalar field in terms of spherical harmonics:

$$\phi(t, r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\psi_l(r)}{r} Y_{lm}(\theta, \phi)e^{-i\omega t}, \quad (9)$$

leads to a radial equation for each mode $(l, m)$:

$$\frac{d^2\psi_l}{dr_*^2} + \left[\omega^2 - V_l(r)\right] \psi_l = 0, \quad (10)$$

where $r_*$ is the tortoise coordinate defined by:

$$dr_* = \frac{dr}{1 - \frac{2GM}{c^2r}}, \quad (11)$$

and the effective potential $V_l(r)$ is given by:
\[ V_l(r) = \left( 1 - \frac{2GM}{c^2 r} \right) \left( \frac{l(l+1)}{r^2} + \frac{2GM}{c^2 r^3} + m^2 \right). \]  

(12)

Effective Potential \( V_l(r) \) for \( l = 1 \) and \( m = 0 \)

![Graph of effective potential](image)

Figure 1: Effective potential \( V_l(r) \) for angular momentum quantum number \( l = 1 \) and scalar field mass \( m = 0 \).

3 Description of the System

We utilize the method of horizon thermodynamics in 4-dimensional Einstein gravity to explain its workings. Consider the spacetime of a static spherically symmetric black hole, the geometry of which is given by \([2]\):

\[
 ds^2 = -g(r) dt^2 + \frac{1}{g(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]  

(13)

where the event horizon is located at \( r = r_+ \), which is the largest positive root of \( g(r_+) = 0 \) satisfying \( g'(r_+) \neq 0 \). Assuming minimal coupling with matter, the stress-energy tensor \( T_{\mu\nu} \), the radial Einstein equation yields

\[
 8\pi T_r^r|_{r_+} = G_r^r|_{r_+} = \frac{g'(r_+)}{r_+} - \frac{1 - g(r_+)}{r_+^2},
\]  

(14)

We take units in which \( G = c = \hbar = 1 \) throughout this paper. We have \( P = T_r^r \) and identify

\[
  T = \frac{g'(r_+)}{4\pi}
\]  

(15)

as the temperature \( T \).

Here \( M \) and \( a \) are respectively the black hole’s mass and its electric charge. The black hole horizon radii \( \{r_+, r_-\} \) are determined by the polynomial equation \( h(r = r_{\pm}) = 0 \).
The system composed of a Schwarzschild black hole under $f(R)$ gravity-coupled-massive-scalar-field is characterized by the action

$$S = \int d^4x \sqrt{-g} \left[ R - 2\nabla_{\alpha}\phi\nabla^{\alpha}\phi - 2\mu^2\phi^2 - C(\phi)\mathcal{T} \right],$$

(16)

where $\mathcal{T}$ represents the trace of the stress-energy tensor.

The coupling function $C(\phi)$ of the supported massive scalar field configurations is characterized by the universal quadratic behavior $[1, 3]$

$$C(\phi) = 1 + \alpha\phi^2,$$

(17)

in the weak-field regime, where the dimensionless expansion constant $\alpha$ is the physical coupling parameter of the composed black-hole-field theory. We shall henceforth assume $\alpha > 0$.

Hawking radiation is described by the equation

$$\frac{d^2\psi}{dr^2} + V\psi = 0,$$

(18)

where it is easy to obtain the asymptotic behavior of the new potential $V$ as

$$\lim_{r\to\infty} V = \omega^2 - \mu^2.$$  

(19)

Then the radial wave equation has the following asymptotic solutions

$$r \to \infty \text{ (or } r_* \to \infty) \Rightarrow R_{lm} \sim \frac{1}{r} e^{-\sqrt{\mu^2 - \omega^2} r_*},$$

(20)

when

$$\omega^2 - \mu^2 > 0,$$

(21)

there is a bound state of the scalar field.

4 New Gravitational Coupling Equation

We can preset the boundary conditions $\mu = y\omega$ $[1,3]$.

Previous literature $[1,3]$ indicates that the superradiance effect is a process of entropy subtraction.

Spherical quantum solution in vacuum state.

In this theory, the general relativity field equation is written completely as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}.$$  

(22)

The Ricci tensor satisfies $T_{\mu\nu} = 0$ in the vacuum state:

$$R_{\mu\nu} = 0.$$ 

(23)

The proper time of spherical coordinates is

$$dr^2 = A(t, r)dt^2 - \frac{1}{c^2} [B(t, r)dr^2 + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2],$$

(24)
where \( A(t, r) \) and \( B(t, r) \) are metric functions.

Using the above equation, we obtain the Ricci-tensor equations:

\[
R_{tt} = -\frac{A''}{2B} + \frac{A'B'}{4B^2} - \frac{A'}{Br} + \frac{A'^2}{4AB} + \frac{B}{2B} - \frac{\dot{B}^2}{4B^2} - \frac{\dot{A}B}{4AB} = 0,
\]

\( (25) \)

\[
R_{rr} = \frac{A''}{2A} - \frac{A'^2}{4A^2} - \frac{A'B'}{4AB} - \frac{B'}{2A} - \frac{\dot{B}}{2A} + \frac{\dot{A}B}{4A^2} + \frac{\dot{B}^2}{4AB} = 0,
\]

\( (26) \)

\[
R_{\theta\theta} = -1 + \frac{1}{B} - \frac{rB'}{2B^2} + \frac{rA'}{2AB} = 0,
\]

\( (27) \)

\[
R_{\phi\phi} = R_{\theta\theta} \sin^2 \theta = 0,
\]

\( (28) \)

\[
R_{tr} = -\frac{\dot{B}}{Br} = 0,
\]

\( (29) \)

\[
R_{t\theta} = R_{t\phi} = R_{r\theta} = R_{r\phi} = 0.
\]

\( (30) \)

From \( R_{tr} = 0 \), we obtain:

\[
\dot{B} = 0.
\]

\( (31) \)

Hence, \( B \) is time-independent.

We see that,

\[
\frac{R_{tt}A + R_{rr}B}{AB} = -\frac{1}{Br} \left( \frac{A'}{A} + \frac{B'}{B} \right) = -\frac{(AB)'}{rAB^2} = 0.
\]

\( (32) \)

Therefore,

\[
A = \frac{1}{B}.
\]

\( (33) \)

Substituting into \( R_{\theta\theta} \):

\[
R_{\theta\theta} = -1 + \frac{1}{B} - \frac{rB'}{2B^2} + \frac{rA'}{2AB} = -1 + \left( \frac{r}{B} \right)' = 0.
\]

\( (34) \)

Solving the equation:

\[
\frac{r}{B} = r + C \quad \Rightarrow \quad \frac{1}{B} = 1 + \frac{C}{r}.
\]

\( (35) \)

When \( r \) tends to infinity, and we set \( C = ye^{-y} \), therefore,

\[
A = \frac{1}{B} = 1 - \frac{y}{r} \Sigma,
\]

\( (36) \)

where \( \Sigma = e^{-y} \).

Thus,

\[
dr^2 = \left( 1 - \frac{y}{r} \Sigma \right) dt^2.
\]

\( (37) \)

In this context, if particles’ masses are \( m_i \), the fusion energy is \( E \),

\[
E = Mc^2 = m_1c^2 + m_2c^2 + \ldots + m_nc^2 + T.
\]

\( (38) \)
5 The Energy of Black Holes in $f(R)$ Theory and Superradiation

As shown in the previous section, the first law of the new horizon works well in Einstein’s theory. A natural question arises: does it still apply to other modified gravitational theories? Here we consider this problem in the $f(R)$ theory. The general function of gravity theory in four dimensions is

$$I = \int d^4x \sqrt{-g} \left[ \frac{f(R)}{2k^2} + L_m \right],$$

(39)

where $k^2 = 8\pi$, $f(R)$ is a general function of the Ricci scalar $R$, and $L_m$ is the matter Lagrangian.

One of the modes under $f(R)$ gravity, where the movement of gravity and the material part of the expression is considered, is

$$f(R) = R - \alpha R^n,$$

(40)

where $\alpha$ and $n$ are constants, $\alpha > 0$ and $0 < n < 1$. Since $f(0) = 0$, the model has a Schwarzschild solution.

$T_{\mu\nu}$ is the stress-energy tensor of the effective curvature fluid, given by

$$T_{\mu\nu} = \frac{1}{F(R)} \left[ \frac{1}{2} g_{\mu\nu} (f - RF) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F \right],$$

(41)

where $F = \frac{df}{dR}$ and $\Box = \nabla^\gamma \nabla_\gamma$.

Using the relations $\Box F = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu F]$ and assuming the geometry of a static spherically symmetric black hole, we take $g(r) = 1 - \frac{2m}{r} + \beta r$, $f(R_0) = \frac{3\mu - y}{r}$, and

$$T^1_{1} = \frac{1}{F} \left[ \frac{1}{2} (f - RF) - \frac{1}{2} g' F' - \frac{2}{r} g F' \right].$$

(42)

We find a special solution such that the special solution has a potential barrier, letting $y$ be the function $y(1/r)$ of $1/r$.

We find the derivative of the energy and momentum with respect to $y$:

$$X = \frac{dT^1_{1}}{dy} = \frac{1}{2} \left( -y - \frac{e^y}{-e^y + e^y} \right).$$

(43)

We find the derivative of $X$ with respect to $y$:

$$\frac{dX}{dy} = \frac{1}{2} \left( -\frac{2e^y(-e + 2e^y)^2}{(-e^y + e^y)^2} + \frac{2e^2y^2}{(-e^y + e^y)^2} + \frac{4e^y(-e + 2e^y)}{(-e^y + e^y)^2} - \frac{2e}{-e^y + e^y} \right).$$

(44)

6 Superradiance Mechanism under Bell’s Theorem Constraints

6.1 Incorporating Bell’s Theorem into Quantum Field Dynamics

Considering Bell’s theorem as a fundamental constraint implies that the quantum states of the field $\phi$ must exhibit entanglement properties that violate Bell inequalities [3].
Specifically, the quantum state $\Psi$ of the field must satisfy:

$$\langle \Psi | B | \Psi \rangle \leq 2\sqrt{2}, \quad (45)$$

where $B$ is the Bell operator corresponding to the CHSH inequality.

To enforce this constraint within the QFT framework, we introduce a Lagrange multiplier $\lambda$ to incorporate the Bell inequality as a constraint in the action:

$$L = L_{KG} - \lambda \left( \langle \Psi | B | \Psi \rangle - 2\sqrt{2} \right), \quad (46)$$

where $L_{KG}$ is the Lagrangian density for the Klein-Gordon field.

Varying the action with respect to $\phi$ yields the modified Klein-Gordon equation:

$$\Box \phi - m^2 \phi + \frac{\delta \langle \Psi | B | \Psi \rangle}{\delta \phi} = 0. \quad (47)$$

### 6.2 Enhanced Superradiance through Quantum Entanglement

The presence of the Lagrange multiplier $\lambda$ introduces additional interactions mediated by the quantum entanglement constraints imposed by Bell’s theorem. This modification alters the effective potential in the radial equation, potentially enabling conditions favorable for superradiance even in non-rotating Schwarzschild black holes.

To explore this possibility, we consider the perturbative expansion of the potential $V_l(r)$ under the influence of the Bell constraint. The modified effective potential can be expressed as:

$$V_l^{\text{eff}}(r) = V_l(r) + \lambda \Delta V_l(r), \quad (48)$$

where $\Delta V_l(r)$ encapsulates the contributions from the Bell constraint.

![Modified Effective Potential](image.png)

Figure 2: Modified effective potential $V_l^{\text{eff}}(r)$ with an example perturbation $\lambda \Delta V_l(r) = 0.1/x^4$. 


The superradiance condition can then be redefined in the context of the modified potential:

\[ \omega < \sqrt{V_{l}^{\text{eff}}(r)}. \tag{49} \]

Superradiance Condition \( \omega < \sqrt{V_{l}^{\text{eff}}(r)} \)

![Graphical representation of the superradiance condition \( \omega < \sqrt{V_{l}^{\text{eff}}(r)} \).](image)

This condition suggests that, under the influence of quantum entanglement, even non-rotating black holes could facilitate energy amplification of incident waves, akin to the superradiance observed in rotating black holes.

### 6.3 Mathematical Derivation of Superradiant Amplification

To quantify the superradiant amplification, we analyze the scattering process of scalar waves off the Schwarzschild black hole under the Bell constraint. The reflection and transmission coefficients, \( R \) and \( T \), respectively, are determined by the asymptotic behavior of the radial solution \( \psi_l(r) \) at spatial infinity \( (r \to \infty) \) and near the event horizon \( (r \to 2GM/c^2) \).

The asymptotic form of the radial equation (10) in the limits \( r \to \infty \) and \( r \to 2GM/c^2 \) can be approximated as:

\[
\begin{align*}
  r \to \infty : & \quad \psi_l(r) \sim Ae^{-i\omega r} + Be^{i\omega r}, \\
  r \to 2GM/c^2 : & \quad \psi_l(r) \sim Ce^{-i\omega r}.
\end{align*}
\tag{50, 51}
\]

The reflection coefficient \( R \) and transmission coefficient \( T \) are defined as:

\[ R = \frac{|B|^2}{|A|^2}, \quad T = \frac{|C|^2}{|A|^2}. \tag{52} \]

The conservation of energy implies that:

\[ |R|^2 + |T|^2 = 1. \tag{53} \]
However, under the Bell constraint, this relationship is modified to allow for energy amplification:

$$|R|^2 - |T|^2 = \Gamma,$$

(54)

where $\Gamma$ represents the amplification factor due to superradiance.

By solving the modified radial equation with the effective potential (48), one can derive expressions for $R$ and $T$ that satisfy the superradiant condition (49).

### 6.4 Numerical Analysis of Superradiant Modes

To substantiate the theoretical predictions, numerical solutions to the modified radial equation (10) with the effective potential (48) are required. We employ the WKB approximation to estimate the amplification factor $\Gamma$ for various modes $(l, m)$ and field masses $m$.

The WKB condition for superradiance can be expressed as:

$$\frac{d^2 \psi_l}{dr^2} + \left[ \omega^2 - V_{\text{eff}}^l(r) \right] \psi_l = 0,$$

(55)

where the effective potential includes contributions from both the Schwarzschild geometry and the Bell constraint.

The results indicate that the amplification factor increases with the strength of the Bell constraint parameter $\lambda$, suggesting that quantum entanglement can enhance superradiant amplification even in non-rotating black holes. Additionally, higher angular momentum modes exhibit greater amplification, aligning with expectations from superradiant behavior.
7 Connection Between Bell’s Theorem and $f(R)$ Gravity

7.1 Singularities and Quantum Residues

In general relativity, singularities are points in spacetime where the curvature becomes infinite, such as the center of a Schwarzschild black hole ($r = 0$) \cite{15,16}. In $f(R)$ gravity, the modification to the action introduces additional curvature terms that can alter the nature of these singularities.

Bell’s theorem, when integrated into the quantum field dynamics near singularities, imposes constraints on the quantum states that can exist in such extreme environments. Specifically, the presence of entangled quantum states near singularities may give rise to irreducible quantum residues that affect the spacetime geometry \cite{9}.

The quantum residue $Z$ near the singularity can be defined as:

$$Z = \lim_{r \to 0} \int_{\mathcal{M}} L_{\text{quantum}} \sqrt{-g} \, d^4x,$$

where $L_{\text{quantum}}$ is the Lagrangian density of the quantum field constrained by Bell’s theorem, and $\mathcal{M}$ represents the spacetime manifold.

The constraint from Bell’s theorem \cite{15} imposes:

$$|Z| \leq 2\sqrt{2}.$$  \hfill (57)

This constraint ensures that the quantum non-locality does not lead to unbounded residues that could destabilize the spacetime geometry.

7.2 Mathematical Correspondence with $f(R)$ Gravity

The modifications introduced by $f(R)$ gravity can be expressed as a series expansion around $R = 0$ \cite{2}:

$$f(R) = R + \alpha R^2 + \beta R^3 + \mathcal{O}(R^4),$$  \hfill (58)

where $\alpha$ and $\beta$ are coefficients characterizing the deviations from general relativity.

Substituting this expansion into the field equations \cite{7} yields:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \alpha \left( 2RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 - \nabla_\mu \nabla_\nu R + g_{\mu\nu} \Box R \right) + \mathcal{O}(R^3) = \kappa T_{\mu\nu}.$$  \hfill (59)

The additional terms proportional to $\alpha$ and $\beta$ introduce higher-order curvature contributions that can modify the behavior of spacetime near singularities.

Comparing this with the quantum residue constraint (57), we observe that both frameworks impose bounds on the curvature contributions near singularities. Specifically, the Bell theorem-induced quantum residues act similarly to the $f(R)$ corrections, suggesting a mathematical correspondence between the two theories.

7.3 Unified Framework for Singularities

By equating the quantum residue constraint with the $f(R)$ curvature corrections, we propose a unified framework where Bell’s theorem and $f(R)$ gravity jointly govern the behavior of spacetime near singularities.
\[ \alpha \left( 2RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 - \nabla_\mu \nabla_\nu R + g_{\mu\nu} \Box R \right) \sim Z, \tag{60} \]

where \( Z \) is bounded by Bell’s theorem constraints.

This correspondence suggests that the quantum non-locality encapsulated by Bell’s theorem may play a role analogous to the curvature modifications in \( f(R) \) gravity, providing a potential pathway to reconcile quantum mechanics with gravitational singularities.

8 Mathematical Models and Analysis

8.1 Quantum Field with Bell Constraint in Schwarzschild Background

Considering the modified Klein-Gordon equation (47), we express the quantum field dynamics in the Schwarzschild background under the influence of the Bell constraint. The equation becomes:

\[
\left( -\frac{1}{1 - \frac{2GM}{c^2 r}} \frac{\partial^2}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \left( 1 - \frac{2GM}{c^2 r} \right) \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \nabla^2 S - m^2 \right) \phi + \lambda \frac{\delta \langle |B| \Psi \rangle}{\delta \phi} = 0. \tag{61} \]

Assuming a separable solution as in (9), and focusing on a single mode \((l,m)\), the radial equation becomes:

\[
\frac{d^2 \psi_l}{dr_*^2} + \left[ \omega^2 - V_{l\text{eff}}(r) \right] \psi_l = \lambda \Delta V_l(r) \psi_l. \tag{62} \]

Here, the effective potential now includes a term proportional to the Lagrange multiplier \( \lambda \) representing the Bell constraint:

\[
\Delta V_l(r) = \frac{\delta \langle |B| \Psi \rangle}{\delta \phi}. \tag{63} \]

8.2 Perturbative Approach to Solving the Modified Radial Equation

To solve the modified radial equation (62), we employ perturbation theory, treating the Bell constraint term as a small perturbation to the classical potential \( V_l(r) \).

Let us define the perturbative parameter \( \epsilon = \lambda \Delta V_l(r) \), assuming \( \epsilon \ll 1 \). The solution can be expanded as:

\[
\psi_l(r_*) = \psi_l^{(0)}(r_*) + \epsilon \psi_l^{(1)}(r_*) + \mathcal{O}(\epsilon^2). \tag{64} \]

Substituting this expansion into (62) and equating terms of equal powers in \( \epsilon \) leads to:

\[
\mathcal{O}(1) : \frac{d^2 \psi_l^{(0)}}{dr_*^2} + \left[ \omega^2 - V_l(r) \right] \psi_l^{(0)} = 0, \tag{65} \]

\[
\mathcal{O}(\epsilon) : \frac{d^2 \psi_l^{(1)}}{dr_*^2} + \left[ \omega^2 - V_l(r) \right] \psi_l^{(1)} = \Delta V_l(r) \psi_l^{(0)}. \tag{66} \]

The zeroth-order equation is the standard radial equation without the Bell constraint, while the first-order equation captures the modifications due to the Bell constraint.
8.3 Green’s Function Approach

To solve the inhomogeneous first-order equation, we utilize the Green’s function method. The Green’s function $G(r_*, r'_*)$ satisfies:

$$\frac{d^2 G}{dr_*^2} + \left[\omega^2 - V_l(r)\right] G = \delta(r_* - r'_*).$$  \hfill (67)

The first-order correction $\psi^{(1)}_l(r_*)$ is then given by:

$$\psi^{(1)}_l(r_*) = \int_{-\infty}^{\infty} G(r_*, r'_*) \Delta V_l(r') \psi^{(0)}_l(r'_*) \, dr'_*. \hfill (68)$$

8.4 Amplification Factor Calculation

The amplification factor $\Gamma$ can be computed by analyzing the behavior of the solution at spatial infinity and near the event horizon. Specifically, the ratio of outgoing to incoming fluxes provides the measure of energy amplification.

The energy flux associated with the scalar field is given by:

$$F = \frac{1}{2i} (\phi^* \partial_r \phi - \phi \partial_r \phi^*).$$  \hfill (69)

At spatial infinity ($r_* \to \infty$), the flux for the outgoing and incoming waves are:

$$F_{\text{out}} = \frac{1}{2i} (B^* i \omega B - B(-i \omega) B^*) = \omega |B|^2,$$  \hfill (70)

$$F_{\text{in}} = \frac{1}{2i} (A^* (-i \omega) A - A i \omega A^*) = -\omega |A|^2.$$  \hfill (71)

The amplification factor is defined as:

$$\Gamma = \frac{F_{\text{out}}}{|F_{\text{in}}|} - 1 = \frac{|B|^2}{|A|^2} - 1. \hfill (72)$$

Under the modified energy conservation condition (54), the amplification factor is related to the transmission coefficient $T$ by:

$$\Gamma = |R|^2 - 1 = -|T|^2.$$  \hfill (73)

Thus, a positive $\Gamma$ indicates superradiant amplification.

8.5 Numerical Simulation Results

To validate the theoretical predictions, we perform numerical simulations of the modified radial equation (62) using the finite difference method. The amplification factor $\Gamma$ is computed for various values of the Bell constraint parameter $\lambda$, angular momentum numbers $(l, m)$, and field mass $m$.

The results indicate that the amplification factor increases with the strength of the Bell constraint parameter $\lambda$, suggesting that quantum entanglement can enhance superradiant amplification even in non-rotating black holes. Additionally, higher angular momentum modes exhibit greater amplification, aligning with expectations from superradiant behavior.
Angular Momentum

Amplification Factor $\Gamma$

Amplification Factor $\Gamma$ vs Angular Momentum $l$

$\bullet m = 0, \lambda = 0.1$

Figure 5: Amplification factor $\Gamma$ as a function of angular momentum quantum number $l$ for $m = 0$ and $\lambda = 0.1$.

9 Mathematical Connection Between Bell’s Theorem and $f(R)$ Gravity

9.1 Equivalence of Quantum Residues and $f(R)$ Curvature Corrections

By equating the quantum residue constraint (57) with the $f(R)$ curvature corrections (59), we establish an equivalence between the two frameworks:

$$\alpha \left( 2 RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 - \nabla_\mu \nabla_\nu R + g_{\mu\nu} \Box R \right) \sim Z. \quad (74)$$

This equivalence implies that the quantum constraints imposed by Bell’s theorem can be interpreted as effective curvature corrections in $f(R)$ gravity. Consequently, the modified gravitational dynamics near singularities in $f(R)$ gravity can be understood as manifestations of quantum non-locality.

9.2 Implications for Singularity Resolution

The unified framework suggests that the incorporation of Bell’s theorem into quantum field dynamics may provide a natural mechanism for resolving singularities in $f(R)$ gravity. The bounded quantum residues prevent the curvature from becoming infinite, effectively smoothing out the singularity.

Mathematically, the condition (74) imposes a constraint on the form of $f(R)$ near singularities. Specifically, the function $f(R)$ must satisfy:

$$|Z| \leq 2\sqrt{2} \quad \Rightarrow \quad |\alpha (2 RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 - \nabla_\mu \nabla_\nu R + g_{\mu\nu} \Box R)| \leq 2\sqrt{2}. \quad (75)$$
This constraint restricts the permissible forms of \( f(R) \), ensuring that the curvature corrections do not lead to unbounded residues that violate Bell’s theorem.

### 9.3 Extended Mathematical Correspondence

Further analysis reveals that the higher-order terms in the \( f(R) \) expansion correspond to higher-order quantum corrections in the presence of the Bell constraint. For instance, the \( R^2 \) and \( R^3 \) terms in (58) can be associated with two-particle and three-particle entangled states, respectively.

The correspondence can be generalized as:

\[
f(R) = R + \sum_{n=2}^{\infty} \alpha_n R^n \leftrightarrow \mathcal{L}_{\text{quantum}} = \mathcal{L}_{\text{KG}} + \sum_{n=2}^{\infty} \lambda_n \mathcal{B}_n,
\]

where \( \alpha_n \) are the coefficients in the \( f(R) \) expansion, and \( \lambda_n \) are the corresponding Lagrange multipliers enforcing Bell-like constraints \( \mathcal{B}_n \) for \( n \)-particle entangled states.

This extended correspondence underscores the deep mathematical and conceptual link between quantum non-locality and modified gravitational dynamics.

### 10 Discussion

#### 10.1 Theoretical Implications

The integration of Bell’s theorem into the study of black hole superradiance represents a novel approach to bridging quantum mechanics with general relativity. The enhanced superradiant amplification in Schwarzschild black holes under Bell constraints suggests that quantum entanglement can influence gravitational phenomena in non-trivial ways \[10\].

Moreover, the mathematical correspondence between Bell’s theorem and \( f(R) \) gravity provides a potential pathway for unifying quantum mechanics with modified theories of gravity. This unified framework offers a new perspective on the nature of spacetime singularities, proposing that quantum non-locality may play a role in their resolution.

#### 10.2 Experimental and Observational Prospects

While the theoretical framework presented is compelling, empirical validation remains a significant challenge. Observing superradiance in Schwarzschild black holes would require precise measurements of wave amplification in the vicinity of non-rotating black holes, a task that is currently beyond our technological capabilities \[11\].

However, advancements in gravitational wave astronomy may offer indirect evidence for such phenomena. The detection of amplified gravitational waves or electromagnetic signals from regions near black holes could provide support for the theoretical predictions \[12\].

Furthermore, the correspondence with \( f(R) \) gravity suggests that cosmological observations, such as those related to dark energy and cosmic acceleration, could offer insights into the interplay between quantum mechanics and modified gravity \[7\].
10.3 Future Research Directions

Future research should focus on several key areas:

1. **Detailed Numerical Simulations**: Advanced numerical methods should be employed to solve the modified radial equations with higher precision, exploring a broader parameter space for the Bell constraint and $f(R)$ corrections.

2. **Higher-Order Corrections**: Investigating higher-order terms in both the $f(R)$ expansion and the Bell constraint framework could reveal additional layers of the correspondence between quantum non-locality and gravitational dynamics.

3. **Extension to Rotating Black Holes**: Extending the analysis to rotating black holes (Kerr black holes) would provide a more comprehensive understanding of superradiance under quantum constraints [13].

4. **Quantum Gravity Integration**: Integrating this framework with approaches to quantum gravity, such as loop quantum gravity or string theory, could further elucidate the quantum nature of spacetime and singularities [14].

5. **Experimental Proposals**: Developing feasible experimental setups or observational strategies to test the predictions of enhanced superradiance in non-rotating black holes.

11 Conclusion

This study has explored the intriguing possibility that Schwarzschild black holes can exhibit superradiance when constrained by Bell’s theorem, a fundamental principle of quantum mechanics. By incorporating quantum entanglement constraints into the dynamics of scalar fields in the Schwarzschild background, we have demonstrated that superradiant amplification may occur even in non-rotating spacetimes.

Furthermore, the mathematical correspondence established between Bell’s theorem and $f(R)$ gravity suggests a deep-seated connection between quantum non-locality and modified gravitational dynamics. This unified framework has significant implications for our understanding of spacetime singularities and the potential resolution mechanisms that bridge quantum mechanics with gravity.

While the theoretical foundations laid in this paper are promising, empirical validation remains a formidable challenge. Future research endeavors should aim to refine the mathematical models, perform comprehensive numerical simulations, and develop observational strategies to test the theoretical predictions presented herein.
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