On a Class of P-Kenmotsu Manifolds Admitting Weyl-projective Curvature Tensor of Type (1, 3)

K. L. Sai Prasad¹,*, S. Sunitha Devi², G. V. S. R. Deekshitulu³

¹Department of Mathematics, Gayatri Vidya Parishad College of Engineering for Women, Visakhapatnam, India
²Department of Mathematics, Vignan Institute of Information Technology, Visakhapatnam, India
³Department of Mathematics, Jawaharlal Nehru Technological University, Kakinada, India

*Corresponding author: klsprasad@gvpcew.ac.in

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Abstract We study a class of para-Kenmotsu manifolds admitting Weyl-projective curvature tensor of type (1, 3). At the end, it is shown that an n-dimensional (n > 2) P-Kenmotsu manifold is Ricci semisymmetric if and only if it is an Einstein manifold.

Keywords: para kenmotsu manifold, recurrent manifold, W₂ - Curvature tensor, ricci tensor, einstein manifold

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1. Introduction

In [1,2], Sato introduced the notions of an almost para contact Riemannian manifold. In 1977, Adati and Matsumoto defined para-Sasakian and special para-Sasakian manifolds, which are regarded as a special kind of an almost contact Riemannian manifolds [3]. Para-Sasakian manifolds have been studied by Adati and Miyazawa [4], De and Avijit [5], Matsumoto, Ianus and Mihai [6] and many others. Before Sato, Kenmotsu defined a class of almost contact Riemannian manifolds [7]. In 1995, Sinha and Sai Prasad defined a class of almost para contact metric manifolds namely para-Kenmotsu (briefly P-Kenmotsu) and special para-Kenmotsu (briefly SP-Kenmotsu) manifolds [8].

In 1970, Pokhariyal and Mishra introduced new tensor fields, called W₂ and E tensor fields, on a Riemannian manifold [9]. Later, in [10], Pokhariyal studied some of the properties of these tensor fields on a Sasakian manifold. In 1986, Matsumoto, Ianus and Mihai have extended these concepts to almost para-contact structures and studied para-Sasakian manifolds admitting these tensor fields [6]. These results were further generalised by De and Sarkar, in [5]. Motivated by these studies, in 2015, Sai Prasad and Satyanarayana studied W₂-tensor field in an SP-Kenmotsu manifold [11]. In the present work, we investigate a class of para-Kenmotsu manifolds admitting Weyl-projective curvature tensor W₂ of type (1, 3). The present work is organised as follows: Section 2 is equipped with some prerequisites about P-Kenmotsu manifolds. In Section 3, we define W₂-recurrent and semisymmetric para-Kenmotsu manifolds and shown that W₂-recurrent para-Kenmotsu manifold is a semisymmetric manifold. Further, it is shown that the curvature of W₂-semisymmetric para-Kenmotsu manifold is constant and hence we establish that a W₂-recurrent para-Kenmotsu manifold is an SP-Kenmotsu manifold. Section 4 is devoted to study Ricci semisymmetric P-Kenmotsu manifold.

2. Preliminaries

Let 𝑀ₙ be an n-dimensional differentiable manifold equipped with structure tensors (Φ, ξ, η) where Φ is a tensor of type (1, 1), ξ is a vector field, η is a 1-form such that

\[ \Phi;\Phi = 0, \nabla\xi = 0, \nabla\eta = 0, rank\Phi = n - 1 \]

Then the manifold 𝑀ₙ is called an almost para-contact manifold.

Let g be a Riemannian metric such that, for all vector fields X and Y on 𝑀ₙ

\[ g(X, \xi) = \eta(X) \]
\[ g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y). \]

Then the manifold 𝑀ₙ is called an almost para-contact manifold.

In addition, if (Φ, ξ, η, g) satisfies the conditions

\[ (\nabla_X \eta)Y - (\nabla_Y \eta)X = 0, \]
\[ (\nabla_X \nabla_Y \eta)Z = \left[ -g(X, Z) + \eta(X)\eta(Z) \right]\eta(Y) + \left[ -g(Y, Z) + \eta(Y)\eta(Z) \right]\eta(X), \]
\[ (\nabla_X \Phi)Y = -g(X, \Phi Y)\xi - \eta(Y)\Phi X; \]

\[ \nabla_X \xi = X - \eta(X)\xi, \]
[\[ \nabla_X \Phi = -g(X, \Phi Y)\xi - \eta(Y)\Phi X; \]

then it is called a P-Kenmotsu manifold.
then $M_n$ is called para-Kenmotsu manifold or briefly a P-Kenmotsu manifold [8].

A P-Kenmotsu manifold admitting a 1-form $\eta$ satisfying
\[
(V_x \eta) Y = g(X,Y) - \eta(X)\eta(Y) \\
(V_x \eta) Y = \phi(\overline{X},Y);
\]
(2.4)
where $\phi$ is an associate of $\Phi$, is called special para-Kenmotsu manifold or briefly SP-Kenmotsu manifold [8].

Let $(M_n,g)$ be an $n$-dimensional, $n \geq 3$, differentiable manifold of class $C^m$ and let $V$ be its Levi-Civita connection. Then the Riemannian Christoffel curvature tensor $R$ of type $(1,3)$ is given by [9]:
\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]
(2.5)

The Ricci operator $S$ and the $(0,2)$-tensor $S^2$ are defined by
\[
g(SX,Y) = S(X,Y);
\]
(2.6)
and
\[
S^2(X,Y) = S(SX,Y).
\]
(2.7)

It is known [8] that in a P-Kenmotsu manifold the following relations hold:
\[
S(X,\xi) = -(n-1)\eta(X),
\]
\[
g[R(X,Y)Z,\xi] = \eta[R(X,Y,Z)]
\]
(2.8)
\[
= g(X,Z)\eta(Y) - g(Y,Z)\eta(X),
\]
\[
R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi,
\]
\[
R(X,Y,\xi) = \eta(X)Y - \eta(Y)X;
\]

when $X$ is orthogonal to $\xi$.

An $n$-dimensional $(n > 2)$ Riemannian manifold $M_n$ is said to be Einstein manifold if the Ricci curvature tensor $S(X,Y)$ of the Levi-Civita connection satisfies the condition
\[
S(X,Y) = \lambda g(X,Y)
\]
(2.9)
where $\lambda$ is a constant.

3. $W_2$- Recurrent P-Kenmotsu Manifolds

The Weyl-projective curvature tensor $W_2$ of type $(1,3)$ of a Riemannian manifold $M_n$ with respect to Riemannian connection is given by [9]:
\[
W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1} [g(X,Z)S(Y,U) - g(Y,Z)S(X,U)].
\]
(3.1)

Now, we define a $W_2$-semisymmetric para-Kenmotsu manifold as:

**Definition 3.1:** An $n$-dimensional para-Kenmotsu manifold is called $W_2$-semisymmetric if its $W_2$-curvature tensor satisfies the condition
\[
R(X,Y)W_2 = 0,
\]
(3.2)
where $R(X,Y)$ is considered to be a derivation of the tensor algebra at each point of the manifold for tangent vectors $X$ and $Y$.

It can be easily shown that on a P-Kenmotsu manifold the $W_2$-curvature tensor satisfies the condition
\[
W_2(X,Y,Z,\xi) = 0.
\]
(3.3)

Further, we define a $W_2$-recurrent para-Kenmotsu manifold as:

**Definition 3.2:** An $n$-dimensional para-Kenmotsu manifold with respect to the Levi-Civita connection is called $W_2$-recurrent manifold if its $W_2$-curvature tensor satisfies the condition
\[
(V_U W_2)(X,Y)Z = A(U)W_2(X,Y)Z,
\]
(3.4)
where $A$ is some non-zero 1-form.

Now, let us establish a relation between $W_2$-recurrent and $W_2$-semisymmetric para-Kenmotsu manifolds.

For that, let us suppose that $W_2 \neq 0$. Now, we define a function by
\[
f^2 = g(W_2,W_2).
\]
(3.5)
Using the fact that $V_U g = 0$, from (3.5) we get $2f(Uf) = 2f^2(A(U))$.

Since $f \neq 0$, we have
\[
Uf = f(A(U)).
\]
(3.6)
Then, from (3.6), we get
\[
X(Uf) = \frac{1}{f}(XYf) + (X(Af))f,
\]
(3.7)
and hence, we have
\[
X(Uf) - U(Xf) = [X(Af) - UA(X)]f.
\]
(3.8)
Therefore,
\[
(V_X V_U - V_U V_X - V_{[X,U]})f = [X(Af) - UA(X)]f
\]
(3.9)
\[
= 2[da(X,U)f].
\]

Since the left hand side of (3.9) is zero and $f \neq 0$, we deduce that $dA(X,Y) = 0$ and it shows that the 1-form $A$ is closed.

Then from (3.4), we get that
\[
(V_X V_U W_2)(X,Y)Z = [V_A(U) + A(V)A(U)]W_2(X,Y)Z,
\]
(3.10)
and hence, we get that
\[
(V_V V_U W_2)(X,Y)Z - (V_U V_V W_2)(X,Y)Z
\]
(3.11)
\[
- (V_{[U,V]} W_2)(X,Y)Z = 2 dA(V,U)W_2(X,Y)Z = 0;
\]
i.e., \( R(V, U), W_2 = 0 \), where \( R(V, U) \) is considered to be a derivation of tensor algebra at each point of the manifold for the tangent vectors \( V \) and \( U \).

This shows that a \( W_2 \)-recurrent P-Kenmotsu manifold is \( W_2 \)-semisymmetric and hence we state that:

**Theorem 3.1:** A \( W_2 \)-recurrent para-Kenmotsu manifold is \( W_2 \)-semisymmetric.

Further we determine the curvature value of \( W_2 \)-semisymmetric P-Kenmotsu manifold.

From (3.2), we have

\[
R(X, Y)W_2(Z, U)V - W_2(R(X, Y)Z, U)V
- W_2(Z, R(X, Y)U)V - W_2(Z, U)R(X, Y)V = 0,
\]

which implies

\[
g\left( R(X, Y)W_2(Z, U)V, \xi \right)
- g\left( W_2(R(X, Y)Z, U)V, \xi \right)
- g\left( W_2(Z, R(X, Y)U)V, \xi \right)
- g\left( W_2(Z, U)R(X, Y)V, \xi \right) = 0.
\]

By putting \( X = \xi \) in the above equation, we get

\[
R((\xi, Y)W_2(Z, U)V, \xi)
- W_2(R(\xi, Y)Z, U)V, \xi)
- W_2(Z, R(\xi, Y)U)V, \xi)
- W_2(Z, U)R(\xi, Y)V, \xi) = 0.
\]

Now, by using (2.8) and (3.3), the above equation reduces to:

\[
\eta(Y)\eta(W_2(Z, U)V) - g(Y, W_2(Z, U)V) = 0.
\]

Again on using (3.3), we get that \( W_2(Z, U, V, Y) = 0 \). Therefore, from (3.1) we have

\[
R(X, Y, Z, V) = \frac{1}{n-1}\left[ \frac{g(Y, Z)S(X, V) - g(X, Z)S(Y, V)}{n} \right].
\]

On contracting the above equation, we get

\[
S(Y, Z) = \frac{r}{n}g(Y, Z).
\]

Then, from equations (3.16) and (3.17), we have

\[
R(X, Y, Z, V) = \frac{r}{n(n-1)}\left[ \frac{g(Y, Z)g(X, V) - g(X, Z)g(Y, V)}{n} \right].
\]

This shows that the curvature of \( W_2 \)-semisymmetric P-Kenmotsu manifold is constant.

As it is known [8] that a P-Kenmotsu manifold with constant curvature is an SP-Kenmotsu manifold and using the above shown result, we state that:

**Theorem 3.2:** A \( W_2 \)-semisymmetric P-Kenmotsu manifold is an SP-Kenmotsu manifold.

Therefore, form theorems (3.1) and (3.2), we have the following result:

**Theorem 3.3:** A \( W_2 \)-recurrent P-Kenmotsu manifold is an SP-Kenmotsu manifold.

### 4. Ricci Semisymmetric Para-Kenmotsu Manifolds

**Definition 4.1:** An \( n \)-dimensional Riemannian manifold is said to be Ricci semisymmetric if its Ricci tensor \( S(X, Y) \) of the Levi-Civita connection satisfies the condition

\[
R(X, Y)S = 0.
\]

**Theorem 4.1:** An \( n \)-dimensional (\( n > 2 \)) P-Kenmotsu manifold \( M_n \) is Ricci semisymmetric if and only if it is an Einstein Manifold.

**Proof:** Let us suppose that a P-Kenmotsu manifold be Ricci semisymmetric. Then from (4.1), we have

\[
S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0.
\]

By putting \( X = \xi \) in (4.2), we get

\[
S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0.
\]

Now by using the equations (2.8) (a) and (2.8) (c), the above equation reduces to

\[
\eta(U)\eta(R(X, Y)V) + S(U, R(\xi, Y)V) = 0.
\]

Again by putting \( X = \xi \) in (4.4), we get

\[
S(Y, V) = -(n-1)g(Y, V).
\]

This proves that the manifold \( M_n \) is an Einstein manifold.

As an every Einstein manifold is Ricci semisymmetric, the converse of the theorem is trivial.

This completes the proof.

### Statement of Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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