COUNTING GENERIC QUADRISECANTS OF POLYGONAL KNOTS

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Abstract. Let $K$ be a polygonal knot in general position with vertex set $V$. A generic quadrisecant of $K$ is a line that is disjoint from the set $V$ and intersects $K$ in exactly four distinct points. We give an upper bound for the number of generic quadrisecants of a polygonal knot $K$ in general position. This upper bound is in terms of the number of edges of $K$.

1. Introduction

In this article, we study polygonal knots in three dimensional space that are in general position. Given such a knot $K$, we define a quadrisecant of $K$ as an unoriented line that intersects $K$ in exactly four distinct points. We require that these points are not vertices of the knot, in which case we say that the quadrisecant is generic.

Using geometric and combinatorial arguments, we give an upper bound for the number of generic quadrisecants of a polygonal knot $K$ in general position. This bound is in terms of the number $n \geq 3$ of edges of $K$. More precisely, we prove the following.

Main Theorem 1. Let $K$ be a polygonal knot in general position, with exactly $n$ edges. Then $K$ has at most $U_n = \frac{n}{12}(n-3)(n-4)(n-5)$ generic quadrisecants.

Applying Main Theorem 1 to polygonal knots with few edges, we obtain the following.

1. If $n \leq 5$, then $K$ has no generic quadrisecant.
2. If $n = 6$, then $K$ has at most three generic quadrisecants.
3. If $n = 7$, then $K$ has at most 14 generic quadrisecants.

Using a result of G. Jin and S. Park ([3]), we can prove that the above bound is sharp for $n = 6$. In other words, a hexagonal trefoil knot has exactly three quadrisecants, all of which are generic.

Quadrisecants of polygonal knots in $\mathbb{R}^3$ have been studied by many people, such as E. Pannwitz, H. Morton, D. Mond, G. Kuperberg and E. Denne. The study of quadrisecants started in 1933 with E. Pannwitz’s doctoral dissertation ([7]). There, she found a lower bound for the number of quadrisecants of non-trivial generic polygonal knots. This bound is in terms of the minimal number of boundary singularities for a disk bounded by $K$. Later, H. Morton and D. Mond ([5]) proved that every non-trivial generic knot has a quadrisecant, and G. Kuperberg extended...
their result to non-trivial tame knots and links (\cite{4}). More recently, E. Denne (\cite{1}) proved that essential alternating quadrisecants exist for all non-trivial tame knots.

**Notation.** Unless otherwise stated, all polygonal knots studied in this article are embedded in the three-dimensional Euclidean space $\mathbb{R}^3$. Such a knot will be denoted by $K$. The cardinality of a set $A$ is denoted by $|A|$. Given a set $A$, with $|A| = n$, \( \binom{n}{k} \) denotes the number of subsets of $A$ of cardinality $k$. The symbol $\sqcup$ denotes the disjoint union of sets.

## 2. Preliminaries

It is well-known that a triple of pairwise skew lines $E_1, E_2, E_3$ determines a unique quadric. This quadric is a doubly-ruled surface $S$ that is either a hyperbolic paraboloid, if the three lines are parallel to one plane, or a hyperboloid of one sheet, otherwise (see for example \cite{2}). The lines $E_1, E_2, E_3$ belong to one of the rulings of $S$, and every line intersecting all those three lines belongs to the other ruling of $S$. Further, every point in $S$ lies on a unique line from each ruling (see \cite{1}, \cite{6} and \cite{8}).

We now define the type of polygonal knots that we will consider in this article.

**Definition 2.1.** We say that the polygonal knot $K$ in $\mathbb{R}^3$ is in **general position** if the following conditions are satisfied:

- No four vertices of $K$ are coplanar.
- Given three edges $e_1, e_2, e_3$ of $K$ that are pairwise skew, no other edge of $K$ is contained in the quadric generated by $e_1, e_2, e_3$.

The quadrisecants of knots that we will study are defined as follows.

**Definition 2.2.** Let $K$ be a polygonal knot in general position with vertex set $V$. A **generic quadrisecant** of $K$ is an unoriented line that is disjoint from the set $V$ and intersects $K$ in exactly four distinct points.

In this paper we are interested in giving an upper bound for the number of generic quadrisecants of a polygonal knot $K$ in general position. This upper bound is in terms of the number of edges of $K$. We start by estimating the number of generic quadrisecants that intersect a given collection of four edges of $K$ that are pairwise skew.

**Proposition 2.3.** Let $K$ be a knot in general position. Let $\mathcal{E}_4$ be a collection of four distinct edges of $K$ that are pairwise skew. Then there are at most two generic quadrisecants of $K$ that intersect all edges in $\mathcal{E}_4$.

**Proof.** Let $e_1, e_2, e_3, e_4$ be the four edges in the collection $\mathcal{E}_4$. Each edge $e_i$ generates a line $E_i$ ($i = 1, 2, 3, 4$). Let $S$ be the doubly-ruled quadric generated by $E_1, E_2, E_3$. Since $K$ is in general position, the edge $e_4$ is not contained in $S$. Therefore, $e_4$ intersects the quadric $S$ in at most two points.

Let $\mathcal{Q}_{\mathcal{E}_4}$ be the set of all generic quadrisecants of $K$ that intersect all edges in $\mathcal{E}_4$. For $l \in \mathcal{Q}_{\mathcal{E}_4}$, we define the point $p_l$ as the point of intersection between the edge $e_4$ and the line $l$. Since $l$ intersects all lines $E_1, E_2, E_3$, then it belongs to a ruling $\mathcal{R}$ of $S$. Also, $p_l \in e_4 \cap S$, and so the cardinality of the set $\{p_l : l \in \mathcal{Q}_{\mathcal{E}_4}\}$ is at most two. To complete the proof, we show that the function $l \mapsto p_l$ is one-to-one. Suppose that $p_l = p_{l'}$, where $l \in \mathcal{Q}_{\mathcal{E}_4}$ and $l' \in \mathcal{Q}_{\mathcal{E}_4}$. Then the point $p_l = p_{l'}$ lies in two lines, $l$ and $l'$, that belong to the ruling $\mathcal{R}$ of $S$. Since every point in $S$ lies on a unique line from $\mathcal{R}$, then $l = l'$. \( \square \)
Our next result complements Proposition 2.3.

**Proposition 2.4.** Let \( K \) be a knot in general position. Let \( \mathcal{E}_4 \) be a collection of four distinct edges of \( K \), two of which are coplanar. Then there is at most one generic quadrisecant of \( K \) that intersects all edges in \( \mathcal{E}_4 \).

**Proof.** Let \( e_1, e_2, e_3, e_4 \) be the four edges in the collection \( \mathcal{E}_4 \), and suppose that \( e_1 \) and \( e_2 \) lie in a plane \( P \). By general position, \( e_1 \) and \( e_2 \) are adjacent edges. Arguing toward a contradiction, suppose that \( l_1 \) and \( l_2 \) are two distinct generic quadrisecants of \( K \) that intersect all edges in \( \mathcal{E}_4 \).

Since \( e_1 \) and \( e_2 \) lie in \( P \), then the same is true for \( l_1 \) and \( l_2 \). Since both \( l_1 \) and \( l_2 \) intersect the edge \( e_i \), then so does \( P \) (\( i = 3, 4 \)). By general position, the edge \( e_i \) intersects \( P \) in a single point \( p_i \), which is a point of intersection between the lines \( l_1 \) and \( l_2 \) (\( i = 3, 4 \)). Thus, \( p_3 = p_4 \), and so the edge \( e_3 \) intersects the edge \( e_4 \).

This means that the point \( p_3 = p_4 \) is a vertex of both \( e_3 \) and \( e_4 \), and this vertex is different from those of edges \( e_1 \) and \( e_2 \) (because \( K \) is a knot). This contradicts general position. \( \square \)

### 3. Quadrisecants Intersecting Consecutive Edges of the Knot

To prove some of the results in the next section, we will need to analyze collections of edges of a polygonal knot that have the property defined below.

**Definition 3.1.** Let \( \mathcal{E}' \) be a collection of distinct edges of a polygonal knot \( K \). We will say that the edges in \( \mathcal{E}' \) are *consecutive* if their union (with the subspace topology induced from \( K \)) is connected.

Since two consecutive edges of a polygonal knot are always coplanar, then Proposition 2.4 implies the following.

**Proposition 3.2.** Let \( K \) be a knot in general position. Let \( \mathcal{E}_4 \) be a collection of four distinct edges of \( K \) that contains a pair of consecutive edges. Then there is at most one generic quadrisecant of \( K \) that intersects all edges in \( \mathcal{E}_4 \).

We now investigate the existence of generic quadrisecants intersecting two or three consecutive edges of a polygonal knot.

**Proposition 3.3.** There are no generic quadrisecants of \( K \) intersecting three distinct consecutive edges of \( K \).

**Proof.** Let \( n \) be the number of edges of \( K \). If \( n = 3 \), then the result is clear. Suppose that \( n > 3 \) and that \( l \) is a generic quadrisecant that intersects three distinct consecutive edges of \( K \). Then the plane \( P \) that contains \( l \) and one of the three consecutive edges also contains the other two edges. Since \( n > 3 \), then the endpoints of the three consecutive edges are four distinct vertices of \( K \), and these vertices lie in the plane \( P \). This contradicts that \( K \) is in general position. \( \square \)

Proposition 3.3 has the following immediate corollary.

**Corollary 3.4.** There are no generic quadrisecants of \( K \) intersecting four distinct consecutive edges of \( K \).

The following proposition complements Proposition 3.2.
Proposition 3.5. Let $E_4$ be a collection of four distinct edges of $K$ that contains no pair of consecutive edges. Then there are at most two generic quadrisecants of $K$ that intersect all edges in $E_4$.

Proof. If all edges in $E_4$ are pairwise skew, then Proposition 2.3 implies that there are at most two generic quadrisecants of $K$ intersecting all edges in $E_4$. If the collection $E_4$ contains a pair of coplanar edges, then Proposition 2.4 implies that there is at most one generic quadrisecant of $K$ intersecting all edges in $E_4$. □

4. Combinatorial Results

For a collection $E_4$ of four distinct edges of the knot $K$, the following theorem gives an upper bound for the number of generic quadrisecants of $K$ that intersect all edges in $E_4$.

Theorem 4.1. Let $K$ be a polygonal knot in general position. Given a collection $E_4$ of four distinct edges of $K$, consider the union $X_{E_4}$ of the edges in $E_4$ (with the subspace topology induced from $K$). Let $c$ be the number of connected components of the space $X_{E_4}$.

- If $c = 1$, then there are no generic quadrisecants intersecting all edges in $E_4$.
- If $c = 2$, and one of the connected components of $X_{E_4}$ consists of a single edge of $K$, then there are no generic quadrisecants intersecting all edges in $E_4$.
- If $c = 2$, and each of the connected components of $X_{E_4}$ is the union of exactly two consecutive edges of $K$, then there is at most one generic quadrisecant intersecting all edges in $E_4$.
- If $c = 3$, then there is at most one generic quadrisecant intersecting all edges in $E_4$.
- If $c = 4$, then there are at most two generic quadrisecants intersecting all edges in $E_4$.

Proof. We divide the proof into four cases.

Case 1: $c = 1$. In this case Corollary 3.4 implies the result.
Case 2: $c = 2$. If one of the connected components of $X_{E_4}$ consists of a single edge, then the result follows from Proposition 3.3. Otherwise, the result follows from Proposition 3.2.
Case 3: $c = 3$. Since the collection $E_4$ contains a pair of consecutive edges, then Proposition 3.2 implies the result.
Case 4: $c = 4$. Since $E_4$ contains no pair of consecutive edges, then the result follows from Proposition 3.5. □

To obtain an upper bound for the number of generic quadrisecants of a knot, we need to consider the number of collections of four distinct edges of the knot for each of the cases stated in Theorem 4.1 These numbers are defined as follows.

Definition 4.2. Let $K$ be a polygonal knot in general position with exactly $n$ edges. For a collection $E_4$ of four distinct edges of $K$, consider the union $X_{E_4}$ of the edges in $E_4$ (with the subspace topology induced from $K$).
For $c = 1, 2, 3, 4$, let $S_c(n)(K)$ be the number of collections $\mathcal{E}_4$ of four distinct edges of $K$ such that $X_{\mathcal{E}_4}$ has exactly $c$ connected components.

For $c = 2$ we also define the following.

- Let $S_{2,1}(n)(K)$ be the number of collections $\mathcal{E}_4$ of four distinct edges of $K$ such that $X_{\mathcal{E}_4}$ has exactly two connected components, and one of these components consists of a single edge.

- Let $S_{2,2}(n)(K)$ be the number of collections $\mathcal{E}_4$ of four distinct edges of $K$ such that $X_{\mathcal{E}_4}$ has exactly two connected components, and each of these components is the union of exactly two consecutive edges.

By definition,

$$S_2(n)(K) = S_{2,1}(n)(K) + S_{2,2}(n)(K);$$

$$S_3(n)(K) + S_{2,1}(n)(K) + S_{2,2}(n)(K) + S_3(n)(K) + S_4(n)(K) = \binom{n}{4}. \quad (4.1)$$

By definition,

$$S_{2,1}(n)(K) = \begin{cases} 
0 & \text{if } n = 3 \\
1 & \text{if } n = 4 \\
n & \text{if } n \geq 5; 
\end{cases} \quad (4.2)$$

Combining Theorem 4.1 with Definition 4.2 we obtain an upper bound for the number of generic quadrisecants of a polygonal knot in general position.

**Corollary 4.3.** Let $K$ be a polygonal knot in general position with exactly $n$ edges. Then the number $U_n = S_{2,1}(n)(K) + S_3(n)(K) + 2S_4(n)(K)$ is an upper bound for the number of generic quadrisecants of $K$.

In our next result we find explicit formulas for the numbers $S_c(n)(K)$’s.

**Theorem 4.4.** Let $K$ be a polygonal knot in general position with exactly $n$ edges. Then

$$S_1(n)(K) = \begin{cases} 
0 & \text{if } n = 3 \\
1 & \text{if } n = 4 \\
n & \text{if } n \geq 5; 
\end{cases} \quad (4.3)$$

$$S_{2,1}(n)(K) = \begin{cases} 
0 & \text{if } n \leq 5 \\
n(n-5) & \text{if } n \geq 6; 
\end{cases} \quad (4.4)$$

$$S_{2,2}(n)(K) = \begin{cases} 
0 & \text{if } n \leq 5 \\
n(n-5) & \text{if } n \geq 6; 
\end{cases} \quad (4.5)$$

$$S_3(n)(K) = \begin{cases} 
0 & \text{if } n \leq 6 \\
n(n-5)(n-6) & \text{if } n \geq 7; 
\end{cases} \quad (4.6)$$

$$S_4(n)(K) = \begin{cases} 
0 & \text{if } n \leq 7 \\
\binom{n}{4} - \frac{n(n-5)(n-6)}{2} & \text{if } n \geq 8. 
\end{cases} \quad (4.7)$$

**Proof.** Fix an orientation of $K$ and an edge $e_1$ of $K$. Suppose that $e_1, e_2, \ldots, e_n$ (in that order) are all the distinct edges of $K$ that we encounter when we follow the orientation of $K$, starting and ending at the initial point of $e_1$. For the rest of the proof, the subindices of the edges $e_j$’s are understood modulo $n$. 
Corollary 5.1. for the number of generic quadrisecants of a polygonal knot in general position.

Proof of equation (4.3): Clearly, $S_4^{(n)}(K) = 0$ for $n = 3$ and $S_4^{(n)}(K) = 1$ for $n = 4$. Suppose that $n \geq 5$. Let $\mathcal{E}_4$ be a collection of four distinct edges of $K$ such that $X_{\mathcal{E}_4}$ is connected. The collection $\mathcal{E}_4$ is completely determined by the only integer $i \in \{1, 2, \ldots, n\}$ such that $\mathcal{E}_4 = \{e_i, e_{i+1}, e_{i+2}, e_{i+3}\}$. Since this number $i$ can be chosen in $n$ different ways, then $S_4^{(n)}(K) = n$.

Proof of equation (4.4): If $n \leq 5$, then clearly $S_2^{(n)}(K) = 0$ and $S_2^{(n)}(K) = 0$. For the proof of equations (4.4) and (4.5), we will assume that $n \geq 6$.

Let $\mathcal{E}_4$ be a collection of four distinct edges of $K$ such that $X_{\mathcal{E}_4}$ has exactly two connected components, $X_1$ and $X_2$, with $X_1$ consisting of a single edge of $K$. Let $\mathcal{E}_3$ be the collection of the three consecutive edges in $X_2$. There are $n$ different ways to choose the collection $\mathcal{E}_3$. Once we have chosen the three edges $e_i, e_{i+1}, e_{i+2}$ in $X_2$, the edge in $X_1$ has to be different from the edges $e_{i-1}, e_i, e_{i+1}, e_{i+2}, e_{i+3}$. Thus, given the edges in $X_2$, the edge in $X_1$ can be chosen in $n-5$ different ways. Hence, the number $S_2^{(n)}(K)$ is equal to $n(n-5)$.

Proof of equation (4.5): We may assume that $n \geq 6$. Let $\mathcal{E}_4$ be a collection of four distinct edges of $K$ such that $X_{\mathcal{E}_4}$ has exactly three connected components, $X_1$, $X_2$ and $X_3$, with $X_1$ being the union of exactly two edges of $K$. There are $n$ different ways to choose the collection of edges in $X_1$. Once we have chosen the two edges in $X_1$, the edges in $X_2$ can be chosen in $n-5$ different ways. However we are double-counting, as interchanging the collections $X_1$ and $X_2$ produces the same collection $X_{\mathcal{E}_4}$. Therefore, $S_2^{(n)}(K) = \frac{n(n-5)}{2}$.

Proof of equation (4.6): We may assume that $n \geq 7$. Let $\mathcal{E}_4$ be a collection of four distinct edges of $K$ such that $X_{\mathcal{E}_4}$ has exactly three connected components, $X_1$, $X_2$ and $X_3$, with $X_1$ being the union of exactly two edges of $K$. There are $n$ different ways to choose the collection of edges in $X_1$. Once we have chosen the two edges in $X_1$, the two edges in $X_2 \cup X_3$ can be chosen in $\binom{n-4}{2} - k$ different ways, where $k$ is the number of different ways to choose a collection of two consecutive edges out of $n-4$ edges. Since $k = n-5$, then the collection of edges in $X_2 \cup X_3$ can be chosen in $\binom{n-4}{2} - (n-5) = \frac{(n-5)(n-6)}{2}$ different ways. Hence, the number $S_3^{(n)}(K)$ is equal to $\frac{n(n-5)(n-6)}{2}$.

Proof of equation (4.7): We may assume that $n \geq 8$. By equation (4.2),

$$S_4^{(n)}(K) = \binom{n}{4} - S_1^{(n)}(K) - S_2^{(n)}(K) - S_3^{(n)}(K) - S_4^{(n)}(K).$$

Thus, equation (4.7) follows from equations (4.3) to (4.6).

5. The Main Result

Combining Corollary 4.3 with Theorem 4.4, we obtain an explicit upper bound for the number of generic quadrisecants of a polygonal knot in general position.

Corollary 5.1. Let $K$ be a polygonal knot in general position with exactly $n$ edges.

1. If $n \leq 5$, then $K$ has no generic quadrisecant.
2. If $n = 6$, then $K$ has at most three generic quadrisecants.
3. If $n = 7$, then $K$ has at most 14 generic quadrisecants.
(4) If \( n \geq 8 \), then \( K \) has at most \( \frac{n}{12}(n-3)(n-4)(n-5) \) generic quadrisecants.

**Proof.** By Corollary 4.3, the knot \( K \) has at most \( U_n = S_{2,2}^{(n)}(K) + S_3^{(n)}(K) + 2S_4^{(n)}(K) \) generic quadrisecants.

1. Suppose that \( n \leq 5 \). Then \( S_{2,2}^{(n)}(K) = 0 = S_3^{(n)}(K) = S_4^{(n)}(K) \), and so \( U_n = 0 \).
2. Suppose that \( n = 6 \). Then \( S_{2,2}^{(6)}(K) = 3, S_3^{(6)}(K) = 0 \) and \( S_4^{(6)}(K) = 0 \), so \( U_n = 3 \).
3. Suppose that \( n = 7 \). Then \( S_{2,2}^{(7)}(K) = 7, S_3^{(7)}(K) = 7 \) and \( S_4^{(7)}(K) = 0 \), so \( U_n = 14 \).
4. Suppose that \( n \geq 8 \). By equation (4.2),

\[
S_4^{(n)}(K) = \binom{n}{4} - S_1^{(n)}(K) - S_{2,1}^{(n)}(K) - S_{2,2}^{(n)}(K) - S_3^{(n)}(K).
\]

Thus,

\[
U_n = 2\binom{n}{4} - 2S_1^{(n)}(K) - 2S_{2,1}^{(n)}(K) - 2S_{2,2}^{(n)}(K) - S_3^{(n)}(K).
\]

By Theorem 4.4, equation (5.1) becomes:

\[
U_n = \frac{1}{12}n(n-1)(n-2)(n-3) - 2n - 2n(n-5) - \frac{n(n-5)}{2} - \frac{n(n-5)(n-6)}{2}.
\]

Equation (5.2) can be written as \( \frac{n}{12}(n-3)(n-4)(n-5) \).

Notice that the expression \( \frac{n}{12}(n-3)(n-4)(n-5) \) from Corollary 5.1 is equal to zero for \( n = 3, 4, 5 \); it is equal to three for \( n = 6 \), and it is equal to 14 for \( n = 7 \). This means that Corollary 5.1 can be reformulated as follows.

**Theorem 5.2.** Let \( K \) be a polygonal knot in general position with exactly \( n \) edges. Then \( K \) has at most \( U_n = \frac{n}{12}(n-3)(n-4)(n-5) \) generic quadrisecants.

**5.1. Examples.**

- Let \( K \) be a trivial knot with exactly \( n \leq 5 \) edges. Then \( U_3 = 0 \), so \( K \) has no generic quadrisecants.
- Let \( K \) be a hexagonal trefoil knot. Then \( U_6 = 3 \), so \( K \) has at most three generic quadrisecants. In this case, the upper bound given by \( U_6 \) is sharp, as \( K \) has exactly three generic quadrisecants (see [3]). Figure 1 shows a hexagonal trefoil knot with its three generic quadrisecants.
- Let \( K \) be a heptagonal figure eight knot. We have found that \( K \) has at most 6 generic quadrisecants. On the other hand, the upper bound given by \( U_7 \) may not be sharp in this case, as \( U_7 = 14 \).

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Figure 1. The three quadrisecants of a hexagonal trefoil knot.

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