On “Do the attractive bosons condense?” by

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Abstract

Using Perron-Frobenius theorem, we prove that the results by Wilkin, Gunn
and Smith \[1\] for the ground states of \( N \) Bose atoms rotating at the angular
momentum \( L \) in a harmonic atomic trap with frequency \( \omega \) interacting via
attractive \( \delta^2(r) \) forces, are valid for a broad class of predominantly attractive
interactions \( V(r) \), not necessarily attractive for any \( r \). The sufficient condition
for the interaction is that all the two-body matrix elements
\[ \langle \bar{z}_1^k \bar{z}_2^l | V | \bar{z}_2^m \bar{z}_1^n \rangle \]
allowed by the conservation of angular momentum \( k+l=m+n \), are negative.
This class includes, in particular, the Gaussian attraction of arbitrary ra-
dius, \( \frac{1}{r} \)-Coulomb and \( \log(r) \)-Coulomb forces, as well as all the short-range
\( R \ll \omega^{-1/2} \) interactions satisfying inequality
\( \int d^2r V(r) < 0 \). There is no con-
densation at \( L \gg 1 \), and the angular momentum is concentrated in the col-
lective “center-of-mass” mode.
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Recently, Wilkin, Gunn and Smith have shown in Ref. [1] that the minimum energy
states at given angular momentum \( L \) of a system of \( N \) bosons in a 2d spherically symmetric
harmonic trap with frequency \( \omega = 1 \), interacting via attractive \( \delta^2(r) \)-forces have, in the weak
coupling limit, the form

\[
\Psi_L = G \left( \sum_{k=1}^{k=N} z_k \right)^L, \quad G = e^{-\sum_{i=1}^{i=N} |z_i|^2}, \quad z_k = x_k + iy_k
\]  

(1)

These non-degenerate ground states reveal no condensation at large \( L \) [1]. Here we show that
their result is valid for a broad class of forces, predominantly attractive, but not necessarily
everywhere attractive.

We reformulate the arguments of Ref. [1] in the way that they can be viewed as a particular case of Perron-Frobenius theorem [2]: if a matrix \( M_{\alpha\beta} \) is irreducible (A) and its entries
are non-negative (B), then its positive eigenvector \( \sum c_\alpha |\alpha\rangle \) (all \( c_\alpha > 0 \)) has maximum eigen-
value, which is non-degenerate. Irreducibility means that there is a chain \( |1\rangle \rightarrow |2\rangle \rightarrow \ldots |p\rangle \),
connecting all the basis states, such that \( M_{\alpha,\alpha+1} \neq 0 \) for any \( |\alpha\rangle \rightarrow |\alpha+1\rangle \). In this case, the
matrix can not be expressed in block-diagonal form by means of permutations of rows and
columns.

In the weak coupling limit, the Hilbert space \( \mathcal{H} \) of the problem is spanned by the vectors

\[
|\alpha\rangle \equiv \left[ l_1, l_2, \ldots, l_N \right] \equiv GP_S z_1^{l_1} z_2^{l_2} \ldots z_N^{l_N}, \quad \sum_{i=0}^{N} l_i = L.
\]  

(2)

Each vector corresponds to a given partition of integer \( L \). Here, \( P_S \) denotes symmetrization.
The state \( \Psi_L = \sum c_\alpha |\alpha\rangle \) has all \( c_\alpha > 0 \) in this basis [1], and \( \Psi_L \) is eigenstate of any interaction
\( \sum_{i>j} V(|\vec{r}_i - \vec{r}_j|) \) in \( \mathcal{H} \) [3] with eigenvalue

\[
\frac{1}{2} (N^2 - N) \int_0^\infty rdre^{-r^2/2}V(r).
\]

Therefore, the state (1) must be non-degenerate ground state of any interaction \( V \), whose
matrix \( M_{\alpha,\beta} = -V_{\alpha,\beta} \) obeys the conditions of the Perron-Frobenius theorem, (A) and (B).
Since operation \( M_{\alpha\alpha} \rightarrow M_{\alpha\alpha} + const \) does not affect eigenvectors, (B) reads \( M_{\alpha\neq\beta} \geq 0 \). The
off-diagonal matrix elements are given by
\[ -M_{\alpha\beta} = V_{\alpha\beta} = \sum_{klmn(m > l)} S^{\alpha,\beta}_{kl,mn} V_{kl,mn}, \quad S^{\alpha,\beta}_{kl,mn} \geq 0, \quad (3) \]

with

\[ V_{kl,mn} \equiv \langle \bar{z}_1^k z_2^l V z_2^m z_1^n \rangle = \int d^2z_1 \int d^2z_2 \bar{z}_1^k z_2^l V \left( \sqrt{|z_1 - z_2|^2} \right) z_2^m z_1^n Q = \delta_{k+l,m+n} V_{klm} \]

the two-body matrix element obeying conservation of the angular momentum. Here, \( Q = e^{-|z_1|^2 - |z_2|^2} \), bar denotes complex conjugation and S are some non-negative quantities [Cf.(2)].

Now, we show that the conditions \( V_{klm}[V] < 0 \quad (4) \)

\((l>m)\) are sufficient for both (A) and (B). Indeed, the connecting chain for the basis states (2)

\[ [L, 0, 0, ..] \rightarrow [L - 1, 1, 0, ..] \rightarrow [L - 2, 2, 0, ..] \rightarrow [L - 2, 1, 1, ..] \rightarrow .. \rightarrow [1, 1, ..1] .. \]

is found keeping the only terms \( V_{kl,l-1k+1} \) in (3). Adding other \( V_{kl,mn} \) can produce no cancellations by virtue of (4). For the same reason, all \( M_{\alpha\beta} \geq 0 \).

In particular, the conditions (4) hold for the attractive \( \delta \)-function, as we have

\[ V_{klm}(-\delta) = \frac{-(k + l)!2^{-(k+l+1)}}{\pi \sqrt{k!!l!!m!!}(k + l - m)!}. \]

In general case, \( (4) \) reads

\[ V_{klm} = \sum_{i,j=0}^{m,j=k} a(-2)^{-(i+j)}(\Delta + i + j)!f \frac{1}{(m - i)!(\Delta + i)!(k - j)!(\Delta + j)!i!j!} < 0, \quad (5) \]

where \( a=\frac{\sqrt{k!!l!!m!!(k+\Delta)!}}{2^{m}} \) and \( \Delta = l - m > 0 \), the function \( f \) is defined by

\[ f = \int_0^{\infty} rdrV(r)L_{\Delta+i+j}(\frac{r^2}{2})e^{-\frac{r^2}{2}} \]

with \( L_{\Delta}(x) \) the Laguerre polynomial [3].

For short-range interactions \( V_R(r) \) with effective radius \( R \) much smaller than the oscillator length, \( R \ll 1 \), (3) is reduced to
\[ \mathcal{V}_{klm}(V_{R \ll 1}) = \mathcal{V}_{klm}(\delta) \int d^2 r V_{R \ll 1}(r) \]

and the sufficiency conditions [4,5] are therefore replaced by the single inequality

\[ \int d^2 r V_{R \ll 1}(r) < 0. \]

Thus, the results (1) hold for short-range \( (R \ll 1 \equiv \sqrt{\frac{k}{m\omega}}) \), interactions which are attractive on average.

The conditions (4) can be seen valid for a wide class of long-range interactions. The Gaussian attractive interaction, \( V = -\frac{e^{-r^2/R^2}}{\pi R^2} \), gives

\[ \mathcal{V}_{klm} \left[ -\frac{e^{-r^2/R^2}}{\pi R^2} \right] = -\frac{1}{\pi \Delta!} \frac{k!m!}{k!m!} \frac{1}{(2+R^2)^{k+m+1}} \frac{1}{(1+R^2)^2} < 0, \quad (6) \]

where \( _2F_1[a, b; c; x] \) is the hypergeometric function [4] which is seen to be positive for any \( R \). The results for log-Coulomb forces, \( V = \log(r) \), and Coulomb forces, \( V = -1/r \), can be obtained from (6). We have

\[ \mathcal{V}_{klm}[\log(r)] = \frac{\pi}{2} \int_0^\infty d(R^2) \mathcal{V}_{klm} \left[ -\frac{e^{-r^2/R^2}}{\pi R^2} \right] \]

and

\[ \mathcal{V}_{klm} \left[ -\frac{1}{r} \right] = 2\sqrt{\pi} \int_0^\infty dR \mathcal{V}_{klm} \left[ -\frac{e^{-r^2/R^2}}{\pi R^2} \right], \]

respectively. It is seen that \( \mathcal{V}_{klm} \leq 0 \) in both cases.

We conclude therefore that the results (1) for the “yrast states” of weakly attractive Bose atoms in harmonic trap, derived by Wilkin, Gunn and Smith in Ref. [1,4] for the case of \( \delta \)-forces, are valid for a wide class of interactions which are predominantly attractive (4,5): There is no condensation at \( L \gg 1 \) (4), and the angular momentum is concentrated in the collective “center-of-mass” mode.

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