Mirror Symmetry in Generalized Calabi–Yau Compactifications

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ABSTRACT

We discuss mirror symmetry in generalized Calabi–Yau compactifications of type II string theories with background NS fluxes. Starting from type IIB compactified on Calabi–Yau threefolds with NS three-form flux we show that the mirror type IIA theory arises from a purely geometrical compactification on a different class of six-manifolds. These mirror manifolds have SU(3) structure and are termed half-flat; they are neither complex nor Ricci-flat and their holonomy group is no longer SU(3). We show that type IIA appropriately compactified on such manifolds gives the correct mirror-symmetric low-energy effective action.

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1 Introduction

In ten space-time dimensions ($D = 10$) there exist two inequivalent type II string theories denoted type IIA and type IIB. Both theories have the maximal amount of 32 local supersymmetries but they differ in their field content [1, 2, 3]. From a phenomenological point of view it is of interest to study their compactifications with less supersymmetry and a space-time background of the form $\mathbb{R}^{1,3} \times Y$. Here $\mathbb{R}^{1,3}$ denotes four-dimensional Minkowski space while $Y$ is a compact six-dimensional Euclidean manifold whose holonomy group determines the amount of supersymmetry which is left unbroken by the background. If the holonomy group is trivial all 32 supercharges are preserved while $SO(6)$ (or any subgroup thereof) breaks all (or some) of the supercharges. Calabi–Yau threefolds are a particularly interesting class of compactification manifolds as their holonomy group is $SU(3)$ and as a consequence they only leave eight supercharges intact [1, 2, 3].

In a compactification on a Calabi–Yau threefold, the light modes of the effective theory all appear as form-field zero modes of the Laplace operator on $Y$. Such harmonic forms are in one-to-one correspondence with non-trivial elements of the cohomology groups $H^{(p,q)}(Y)$. The interactions of the light modes are captured by a low-energy effective Lagrangian $\mathcal{L}_{\text{eff}}$ which can be computed via a Kaluza-Klein (KK) reduction of the ten-dimensional Lagrangian. This low-energy theory is found to be a four-dimensional $N = 2$ supergravity coupled to vector-, tensor- and hypermultiplets [1, 2, 3].

The low-energy effective theories of type IIA and type IIB in $D = 4$ are not unrelated. Mirror symmetry assembles topologically distinct Calabi–Yau threefolds into ‘mirror pairs’ $(Y, \tilde{Y})$ such that type IIA compactified on $Y$ is equivalent to type IIB compactified on the mirror manifold $\tilde{Y}$ [8]. This leads to a relation between even (or odd) cohomology groups on $Y$ and odd (or even) cohomology groups on $\tilde{Y}$. Thus, for instance, the Euler numbers $\chi$ of the mirror pair have opposite signs $\chi(Y) = -\chi(\tilde{Y})$.

Among their massless excitations both type II string theories contain $(p - 1)$-form gauge potentials $C_{p-1}$ with a $p$-form field strength $F_p = dC_{p-1}$. Recently generalized Calabi–Yau compactifications of type II string theories have been considered where background fluxes for the field strengths $F_p$ on $Y$ are turned on [9, 10, 11, 12, 13, 14, 15, 16, 17]. More precisely, one allows $F_p$ of the form

$$F_p = e_i \omega_p^i,$$

where $e_i \omega_p^i$ is a general harmonic $p$-form on $Y$, written in terms of a harmonic basis $\omega_p^i$ of the group $H^p(Y, \mathbb{R})$. The harmonic condition is required to ensure that the Bianchi identity and the equation of motion are left intact so

$$dF_p = 0 = d^\dagger F_p.$$

Note that this implies that the gauge potential $C_{p-1}$ is only locally defined on $Y$. Integrating $F_p$ over the $p$-cycle $\gamma_p^i$ in $Y$ which is Poincaré-dual to $\omega_p^i$ gives

$$\int_{\gamma_p} F_p = e_i.$$
Due to a Dirac quantization condition, the flux $e_i \omega^i$ is quantized in string theory meaning it is actually an element of integral cohomology $H^p(Y, \mathbb{Z})$. Choosing a basis $\omega^i_p$ which is also a basis of $H^p(Y, \mathbb{Z})$ (we ignore here any torsion elements in the integral cohomology) this means that the flux parameters $e_i$ are integers. The number of parameters is simply given by the Betti number which is the dimension of the appropriate cohomology group $H^p(Y, \mathbb{R})$. On a Calabi–Yau manifold, the only odd cohomology group is $H^3(Y)$, while all the even cohomology groups $H^0(Y)$, $H^2(Y)$, $H^4(Y)$ and $H^6(Y)$ are present.

The flux parameters contribute to the energy-momentum tensor and as a consequence the geometry backreacts and a non-trivial warp-factor is induced \cite{21,19,20}. However, in the supergravity approximation the cycles $\gamma^i_p$ are chosen to be large and hence the fluxes parameters are effectively continuous and represent small perturbations of the original Calabi–Yau compactification. This in turn implies that the light modes are still determined by the linear fluctuations around the background values (zero modes) of the theory in the absence of fluxes. In this approximation their induced masses are much smaller than the integrated out heavy states with masses of order the string scale or the compactification scale. Thus the interactions of the light modes continue to be captured by an effective Lagrangian $L_{\text{eff}}$ which now depends continuously on the flux parameters $e_i$. The fluxes appear as gauge or mass parameters and deform the original supergravity into a gauged or massive supergravity. The fluxes introduce a non-trivial potential for some of the massless fields and spontaneously break (part of) the supersymmetry.

$L_{\text{eff}}$ has been computed in various situations. In refs. \cite{10,11,16,18} type IIB compactified on Calabi–Yau threefolds $\tilde{Y}$ in the presence of RR-three-form flux $F_3$ and NS-three-form flux $H_3$ was derived. In refs. \cite{4,5,6} type IIA compactified on the mirror manifold $Y$ with RR-fluxes $F_0$, $F_2$, $F_4$ and $F_6$ present was considered. The resulting low-energy effective action was equivalent to the type IIB action on the mirror manifold $\tilde{Y}$ with $F_3$ non-zero, but $H_3 = 0$ \cite{15}. As expected, given the matching of odd and even cohomologies on mirror pairs, the type IIB RR-fluxes $F_3$ in the third cohomology group $H^3(\tilde{Y})$ are mapped to the type IIA RR-fluxes in the even cohomology groups $H^0(Y)$, $H^2(Y)$, $H^4(Y)$ and $H^6(Y)$ \cite{11,32}.

However, for non-vanishing NS-fluxes the situation is less clear as no obvious mirror symmetric compactification is known. In both type IIA and type IIB on $Y$ an NS three-form $H_3$ exists which can give a non-trivial NS-flux in $H^3(Y)$. However, in neither case are there NS form fields which can give fluxes in the mirror symmetric even cohomologies $H^0(Y)$, $H^2(Y)$, $H^4(Y)$ and $H^6(Y)$. Vafa \cite{33} suggested that the mirror symmetric configuration is related to compactifying on a manifold $\hat{Y}$ which is not complex but only admits an almost complex structure whose Nijenhuis tensor is non-vanishing. The purpose of this paper is to make this proposal more precise.

As a first step we demand that the $D = 4$ effective action continues to have $N = 2$ supersymmetry, that is, eight local supersymmetries. This implies that there is a single globally defined spinor $\eta$ on $\hat{Y}$ so that each of the $D = 10$ supersymmetry parameters gives a single local four-dimensional supersymmetry. As result, the structure group of the bundle of orthonormal frames on $\hat{Y}$ has to reduce from $SO(6)$ to $SU(3)$. If we further demand that the two $D = 4$ supersymmetries are unbroken in a Minkowskian ground state $\eta$ has to be covariantly constant with respect to the Levi-Civita connection $\nabla$ or equivalently the holonomy group has to be $SU(3)$. This second requirement uniquely singles out Calabi–Yau threefolds as the correct compactification manifolds. However, in this paper
we relax this second condition and only insist that a globally defined $SU(3)$-invariant spinor exists. Manifolds with this property have been discussed in the mathematics and physics literature and are known as manifolds with $SU(3)$ structure (see, for example, refs. [34,35,36,37,38,39,40,41,42,43]). They admit an almost complex structure $J$, a metric $g$ which is hermitian with respect to $J$ and a unique $(3,0)$-form $\Omega$. Generically, since $\eta$ is no longer covariantly constant, the Levi-Civita connection now fails to have $SU(3)$-holonomy. However one can always write $\nabla \eta$ in terms of a three-index tensor, $T^0$, contracted with gamma matrices, acting on $\eta$. In the same way $\nabla J$ and $\nabla \Omega$ can be also written in terms of contractions of $T^0$ with $J$ and $\Omega$ respectively. This tensor $T^0$, known as the intrinsic torsion, is thus a measure of the obstruction to having $SU(3)$ holonomy.

Different classes of manifolds with $SU(3)$ structure exist and they are classified by the different elements in the decomposition of the intrinsic torsion into irreducible $SU(3)$ representations. We will mostly consider the slightly non-generic situation where only “electric” flux is present. In this case, we find that mirror symmetry restricts us to a particular class of manifolds with $SU(3)$ structure called half-flat manifolds [39]. They are neither complex, nor Kähler, nor Ricci-flat but they are characterized by the conditions

$$d\Omega^- = 0 = d(J \wedge J), \quad (1.4)$$

where $\Omega^-$ is the imaginary part of the $(3,0)$-form. On the other hand the real part of $\Omega$ is not closed and plays precisely the role of an NS four-form $d\Omega^+ \sim F_4^{NS}$ corresponding to fluxes along $H^4(Y)$ [33]. Thus the ‘missing’ NS-fluxes are purely geometrical and arise directly from the change in the compactification geometry.

Half-flat manifolds also appear from a different point of view. When appropriately fibered over an interval the resulting seven-dimensional manifold always admits a metric of $G_2$ holonomy [39,44]. Physically this corresponds to the fact that the effective four-dimensional $N = 2$ theory has $N = 1$ BPS domain-wall (DW) solutions which are mirror symmetric to the type IIB DW solutions studied in [45]. In fact, all these DW solutions are exact solutions of the full ten-dimensional supergravity theory without any need to assume the relevant NS flux is small. Typically one can only expect the KK-reduction to be consistent in the limit where the flux is a small perturbation. This picture is also related to work in [46]. There it was shown that starting from a type IIB theory with both RR- and NS background fluxes the conjectured mirror symmetric type IIA theory is related to a purely geometrical compactification of M-theory on a $G_2$ manifold.

This paper is organized as follows. In section 2.4 we briefly recall mirror symmetry in Calabi–Yau compactifications with RR-flux. In section 2.2 we discuss properties of manifolds with $SU(3)$ structure and the way they realize supersymmetry in the effective action. These manifolds are classified in terms of irreducible representations of the structure group $SU(3)$ and in section 2.3 we argue that the class of half-flat manifolds are likely to be the mirror geometry of Calabi–Yau manifolds with electric NS-fluxes. We confirm this ‘educated guess’ in section 2.4 by considering a complex six-torus (and implicitly orbifolds thereof [47]) where mirror symmetry is directly related to T-duality

[Manifolds with torsion have also been considered in refs. [19,20,30,37,40,41,42]. However, in these papers the torsion is usually chosen to be completely antisymmetric in its indices or in other words it is a three-form. This turns out to be a different condition on the torsion and these manifolds are not half-flat.]
and thus the mirror manifold can be explicitly constructed. This can be slightly gener-
ized by considering Calabi–Yau manifolds in the SYZ picture where $Y$ is a special 
Lagrangian $T^3$ fibration \( \mathbb{T}^3 \). In this case mirror symmetry is also related to T-duality 
and, in the large complex structure limit, can be carried out explicitly. In both cases we 
discover that half-flat manifolds arise as the mirror symmetric geometry. This is further 
confirmed in section 2.5 where we discuss $N = 1$ BPS-domain wall solutions and their 
relation to manifolds with $SU(3)$ structure and manifolds with $G_2$ holonomy. In section 3 
we perform the KK-reduction of type IIA compactified on $\hat{Y}$, derive the low-energy effective 
action and show that it is mirror symmetric to type IIB compactified on threefolds $Y$ with non-trivial electric NS-flux $H_3$. The effect of the altered geometry is as expected. It turns an ordinary supergravity into a gauged supergravity in that scalar fields become 
charged and a potential is induced. This potential receives contributions from different 
terms in ten-dimensional effective action, one of which arises from the non-vanishing 
Ricci-scalar. This contribution is crucial to obtain the exact mirror symmetric form of the potential.

The derivation of the mirror symmetric effective action including magnetic fluxes is technically more involved due to the appearance of a massive RR two-form. This in 
turn requires a KK-reduction on the ‘democratic’ version of the ten-dimensional type IIA effective action \([57,58]\) and we postpone this study to a separate publication \([59]\). Section 4 contains our conclusions. Some of the technical details are relegated to four appendices. In appendix A we summarize our conventions. In order to make the paper self-contained 
we recall in appendix B.1 the Calabi–Yau compactification of type IIA without fluxes while in appendix B.2 we recall the effective action for type IIB compactification on Calabi–Yau manifolds with non-trivial NS-flux. In appendix C we discuss manifolds 
with $G$-structure from a more mathematical point of view and supply some explicit 
expressions omitted in the main text. In appendix D we compute the Ricci-scalar for 
half-flat manifolds, show that it is non-zero and hence contributes to the scalar potential.

## 2 Generalized mirror manifolds

### 2.1 Mirror symmetry in Calabi–Yau compactifications with flux

Let us begin by reviewing mirror symmetry for Calabi–Yau compactifications with non-
trivial fluxes. Recall that the two ten-dimensional type II theories, compactified on 
a Calabi–Yau manifold $Y$, each lead to a four-dimensional low-energy effective action 
which is an $N = 2$ supergravity coupled to vector-, tensor- and hypermultiplets \([4, 5, 6, 7]\). More precisely, for type IIA the massless spectrum contains $h^{(1,1)}$ vector multiplets, 
$h^{(1,2)}$ hypermultiplets and one tensor multiplet while for type IIB one has $h^{(1,2)}$ vector 
multiplets, $h^{(1,1)}$ hypermultiplets and one tensor multiplet. Here, the Hodge numbers 
h^{(1,1)} and $h^{(1,2)}$ are the dimensions of the cohomology groups $H^{1,1}(Y)$ and $H^{1,2}(Y)$. In appendix A we review some of the details of these compactifications and give explicitly 
the effective action.

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\[5\] This has also been considered recently in refs. \([48, 49]\) and our discussion in section 2.4 overlaps 
with these papers. We thank the authors of \([49]\) for communicating their results prior to publication. 
T-duality in massive supergravities has been discussed in refs. \([50, 51, 52, 53, 54, 55]\).
Calabi–Yau manifolds $Y$ and $\tilde{Y}$ form a mirror pair if compactifying type IIA on $Y$ gives the same theory as compactifying type IIB on $\tilde{Y}$. More precisely one requires that the corresponding string superconformal field theories including quantum corrections are equivalent. This implies, among other things, that they have reversed Hodge numbers,

$$h^{(1,1)}(\tilde{Y}) = h^{(1,2)}(Y), \quad h^{(1,2)}(\tilde{Y}) = h^{(1,1)}(Y),$$

and, that the two effective $D=4$ Lagrangians coincide

$$L^{\text{IIA}}(Y) \equiv L^{\text{IIB}}(\tilde{Y}).$$

In the supergravity limit this symmetry continues to hold when background RR-flux is included on the Calabi–Yau manifolds. Consider first type IIB. The only allowed RR-flux on the internal Calabi–Yau manifold $\tilde{Y}$ is the three-form $F_3 = dC_2$. It must be harmonic and so is parameterized by an element of the cohomology group $H^3(\tilde{Y}, \mathbb{R})$. In string theory, the flux is quantized so is more correctly an element of the integer cohomology $H^3(\tilde{Y}, \mathbb{Z})$. This allows the possibility that $F_3$ includes “torsion” elements, that is non-zero elements of $H^3(\tilde{Y}, \mathbb{Z})$ the image of which in $H^3(\tilde{Y}, \mathbb{R})$ vanishes. (This should not be confused with the notion of torsion of a metric-compatible connection which will be central to later discussions in this section.) Here we will ignore such subtleties and assume such elements vanish. In general, one can introduce a symplectic integral basis $\{\alpha_A, \beta^A\}$ with $A = 0, \ldots, h^{(1,2)}$ for $H^3(\tilde{Y}, \mathbb{R})$. The flux $F_3$ then defines $2(h^{(1,2)} + 1)$ flux parameters $(\tilde{e}_A, \tilde{m}^A)$ according to

$$F_3 = dC_2 + \tilde{m}^A \alpha_A + \tilde{e}_A \beta^A.$$  

The effective action of this compactification is worked out via a Kaluza–Klein reduction in refs. $[10, 11, 16, 18]$. It uses the supergravity limit where the flux parameters are small compared to the string scale and the backreaction of the Calabi–Yau geometry to the presence of the fluxes is assumed to excite only the zero modes of the Calabi–Yau manifold. In other words, a KK reduction is performed on the original Calabi–Yau geometry albeit with the non-vanishing fluxes taken into account. This leads to a potential which induces perturbatively small masses for some of the scalar fields and spontaneously breaks supersymmetry.

It was shown in $[18]$ that this IIB effective action is manifestly mirror symmetric to the one arising from the compactification of massive type IIA supergravity $[11]$ on $Y$ with RR-fluxes turned on in the even cohomology of $Y$. More precisely, in IIA compactifications the RR two-form field strength $F_2$ can have non-trivial flux in $H^2(Y, \mathbb{Z})$ while the four-form field strength $F_4$ has fluxes in $H^4(Y, \mathbb{Z})$. (Again we ignore any torsion elements.) Let $\omega_i$ with $i = 1, \ldots, h^{(1,1)}$ be an integral basis of $H^2(Y, \mathbb{Z})$ and $\tilde{\omega}^i$ be an integral basis of $H^4(Y, \mathbb{R})$. Then there are $2h^{(1,1)}$ IIA RR-flux parameters given by

$$F_2 = dA_1 + m^i \omega_i, \quad F_4 = dC_3 - A_1 \wedge H_3 + e_0 \tilde{\omega}^j.$$  

In addition there are the two extra parameters $m^0$ and $e_0$, where $e_0$ is the dual of the space-time part of the four-form $F_4_{\mu
u\rho\sigma}$ and $m^0$ is the mass parameter of the original ten-dimensional massive type IIA theory $[18]$. Altogether there are $2(h^{(1,1)} + 1)$ real RR-flux parameters $(e_I, m^j), I, J = 0, 1, \ldots, h^{(1,1)}$ which precisely map to the $2(h^{(1,2)} + 1)$

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$^6$It has been argued that the flux is really described by an element of K-theory $[60]$. This differs from $H^3(\tilde{Y}, \mathbb{Z})$ precisely in the subgroup of torsion elements, hence again here we will ignore this distinction.
type IIB RR-flux parameters under mirror symmetry. This is confirmed by an explicit KK-reduction of the respective effective actions and one finds \[ \mathcal{L}^{(IIA)}(Y, e_I, m^I) \equiv \mathcal{L}^{(IIB)}(\tilde{Y}, \tilde{e}_A, \tilde{m}^B) . \] (2.5)

We expect that mirror symmetry continues to hold when one considers fluxes in the NS-sector. However, in this case, the situation is more complicated. In both type IIA and type IIB there is a NS two-form \( B_2 \) with a three-form field strength \( H_3 \), so one can consider fluxes in \( H^3(Y, \mathbb{Z}) \) in IIA and \( H^3(\tilde{Y}, \mathbb{Z}) \) in IIB. However, these are clearly not mirror symmetric backgrounds since mirror symmetry exchanges the even and odd cohomologies. One appears to be missing \( 2(h^{1,1} + 1) \) NS-fluxes, lying along the even cohomology of \( Y \) and \( \tilde{Y} \), respectively. Since the NS fields include only the metric, dilaton and two-form \( B_2 \), there is no candidate NS even-degree form-field strength to provide the missing fluxes. Instead, they must be generated by the metric and the dilaton. Thus we are led to consider compactifications on a generalized class of manifolds \( \hat{Y} \) with a metric which is no longer Calabi–Yau, and perhaps a non-trivial dilaton in order to find a mirror-symmetric effective action. This necessity was anticipated by Vafa in ref. [33].

We now turn to what characterizes this generalized class of compactifications on \( \hat{Y} \). For definiteness, we will pose the problem as one of finding the IIA compactifications on \( \hat{Y} \) mirror to IIB compactifications on the Calabi–Yau manifold \( \tilde{Y} \) with NS flux \( H_3 \). Since the NS sectors of IIA and IIB are identical this is, of course, equivalent to the problem with the roles of IIA and IIB reversed. The low-energy effective action of the IIB theory with NS flux

\[ H_3 = dB_2 + \tilde{m}^A \alpha_A + \tilde{e}_A \beta^A . \] (2.6)

can be easily calculated as is done in appendix B.2. Following the usual convention, we refer to \( \tilde{e}_A \) and \( \tilde{m}^A \) as electric and magnetic fluxes respectively. We will generally consider the IIA dual of the pure electric case where only half the fluxes in (2.6) are excited.

### 2.2 Supersymmetry and manifolds with \( SU(3) \)-structure

The low-energy effective action arising from IIB compactifications with non-trivial \( H_3 \)-flux describes a massive deformation of an \( N = 2 \) supergravity \[10, 11, 16, 18\]. Compactification on the conjectured generalized mirror IIA manifold \( \hat{Y} \) should lead to the same effective action. Thus the first constraint on \( \hat{Y} \) is that the resulting low-energy theory preserves \( N = 2 \) supersymmetry.

Let us first briefly review how supersymmetry is realized in the conventional Calabi–Yau compactification. Ten-dimensional type IIA supergravity has two supersymmetry parameters \( \epsilon^\pm \) of opposite chirality each transforming in a real 16-dimensional spinor representation of the Lorentz group \( Spin(1,9) \). In particular, the variation of the two gravitinos in type IIA is schematically given by \[12\]

\[ \delta \psi_\pm_M = [\nabla_M + (\Gamma \cdot H_3)_M] \epsilon^\pm + [(\Gamma \cdot F_2)_M + (\Gamma \cdot F_4)_M] \epsilon^\mp + \ldots , \] (2.7)
where the dots indicate further fermionic terms. Next one dimensionally reduces on a
six-dimensional manifold $Y$ and requires that the theory has a supersymmetric vacuum
of the form $\mathbb{R}^{1,3} \times Y$ with all other fields trivial. This implies that there are particular
spinors $\epsilon^\pm$ for which the gravitino variations (2.7) with $H_3 = F_2 = F_4 = 0$ vanish. On
$\mathbb{R}^{1,3} \times Y$ the Lorentz group $Spin(1,9)$ decomposes into $Spin(1,3) \times Spin(6)$ and we can
correspondingly write $\epsilon^\pm = \theta^\pm \otimes \eta$. In the supersymmetric vacuum, the vanishing of the
gravitino variations imply the $\theta^\pm$ are constant and $\eta$ is a solution of
\[
\nabla_m \eta = 0 , \quad m = 1, \ldots, 6 .
\] (2.8)
If this equation has a single solution, each $\epsilon^\pm$ gives a Killing spinor and we see that the
background preserves $N = 2$ supersymmetry in four dimensions as required. Equivalently,
if we compactify on $Y$, the low-energy effective action will have $N = 2$ supersymmetry
and admits a flat supersymmetric ground state $\mathbb{R}^{1,3}$.

The condition (2.8) really splits into two parts: first the existence of a non-vanishing
globally defined spinor $\eta$ on $Y$ and second that $\eta$ is covariantly constant. The first
condition implies the existence of two four-dimensional supersymmetry parameters and
hence that the effective action has $N = 2$ supersymmetry. The second condition that $\eta$
is covariantly constant implies that the effective action has a flat supersymmetric ground
state.

The existence of $\eta$ is equivalent to the statement that the structure group of the
tangent bundle is reduced. To see what this means, recall that the structure group refers
to the group of transformations required to patch the tangent bundle (or more precisely
the bundle of orthonormal frames) over the manifold. Thus on a spacetime of the form
$\mathbb{R}^{1,3} \times Y$ the structure group reduces from $SO(1,9)$ to $SO(1,3) \times SO(6)$ and the spinor
representation decomposes accordingly as $16 \rightarrow (2, 4) + (\bar{2}, \bar{4})$. Suppose now that the
structure group of $Y$ reduces further to $SU(3) \subset SO(6) \cong SU(4)$. The $4$ then decomposes
as $3 + 1$ under the $SU(3)$ subgroup. An invariant spinor $\eta$ in the singlet representation of
$SU(3)$ thus depends trivially on the tangent bundle of $Y$ and so is globally defined and
non-vanishing. Conversely, the existence of such a globally defined spinor implies that
the structure group of $Y$ is $SU(3)$ (or a subgroup thereof). Mathematically, one says
that the $Y$ has $SU(3)$-structure. In appendix we review some of the properties of such
manifolds from a more mathematical point of view and for a more detailed discussion
we refer the reader to the mathematics literature [34, 35, 36, 37, 38, 39]. Here we will
concentrate on the physical implications.

The second condition that $\eta$ is covariantly constant has well known consequences (as
reviewed for instance in [4]). It is equivalent to the statements that the Levi–Civita
connection has $SU(3)$ holonomy or similarly that $Y$ is Calabi–Yau. It implies that an
integrable complex structure exists and that the corresponding fundamental two-form
$J$ is closed. In addition, there is a unique closed holomorphic three-form $\Omega$. Together these
structure and integrability conditions imply that Calabi–Yau manifolds are complex,
Ricci-flat and Kähler.

Symmetry with the low-energy IIB theory with $H_3$-flux, implies that compactification
on generalized mirror manifold $\hat{Y}$ still leads to an effective action that is $N = 2$
supersymmetric. However, the IIB theory with flux in general no longer has a flat-space
ground state which preserves all supercharges [11, 12, 13]. From the above discussion,
we see that this implies that we still have a globally defined non-vanishing spinor $\eta$, but
we no longer require that $\eta$ is covariantly constant, so $\nabla_m \eta \neq 0$. In other words, $\hat{Y}$ has $SU(3)$-structure but generically the Levi–Civita connection no longer has $SU(3)$-holonomy, so in general, $\hat{Y}$ is not Calabi–Yau. In particular, as discussed in appendix $\text{D}$, generic manifolds with $SU(3)$-structure are not Ricci flat.

In analogy with Calabi–Yau manifolds let us first use the existence of the globally defined spinor $\eta$ to define other invariant tensor fields. Specifically, one has a fundamental two-form

$$J_{mn} = -i\eta^\dagger \Gamma_7 \Gamma_{mn} \eta,$$

and a three-form

$$\Omega = \Omega^+ + i\Omega^-,$$

where

$$\Omega^+_{mnp} = -i\eta^\dagger \Gamma_{mnp} \eta, \quad \Omega^-_{mnp} = -\eta^\dagger \Gamma_7 \Gamma_{mnp} \eta.$$  \hspace{1cm} (2.11)

By applying Fierz identities one shows

$$J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \Omega,$$

$$J \wedge \Omega = 0,$$

exactly as for Calabi–Yau manifolds. Similarly, raising an index on $J_{mn}$ and assuming a normalization $\eta^\dagger \eta = 1$, one finds

$$J^p_m J^n_p = -\delta^n_m, \quad J^p_m J^n_r g_{pr} = g_{mn},$$

by virtue of the $\Gamma$-matrix algebra. This implies that $J^p_m$ defines an almost complex structure such that the metric $g_{mn}$ is Hermitian with respect to $J^p_m$. The existence of an almost complex structure is sufficient to define $(p,q)$-forms as we review in appendix $\text{C}$. In particular, one can see that $\Omega$ is an $SU(3)$-invariant $(3,0)$-form.

Thus far we have used the existence of the $SU(3)$-invariant spinor $\eta$ to construct $J$ and $\Omega$. One can equivalently characterize manifolds with $SU(3)$-structure by the existence of a globally defined, non-degenerate two-form $J$ and a globally defined non-vanishing complex three-form $\Omega$ satisfying the conditions (2.12). Together these then define a metric [36,14].

The key difference from the Calabi–Yau case is that a generic $\hat{Y}$ does not have $SU(3)$ holonomy since $\nabla_m \eta \neq 0$. Using (2.9) and (2.10) this immediately implies that $J$ and $\Omega$ are also generically no longer covariantly constant $\nabla_m J_{np} \neq 0, \nabla_m \Omega_{npq} \neq 0$. The deviation from being covariantly constant is a measure of the deviation from $SU(3)$ holonomy and thus a measure of the deviation from the Calabi–Yau condition. This can be made more explicit by using the fact that on $\hat{Y}$ there always exists another connection $\nabla^{(T)}$, which is metric compatible (implying $\nabla^{(T)}_m g_{np} = 0$), and which does satisfy $\nabla^{(T)}_m \eta = 0$ [33,36]. The difference between any two metric-compatible connections is a tensor, known as the contorsion $\kappa_{mnp}$, and thus we have explicitly

$$\nabla^{(T)}_m \eta = \nabla_m \eta - \frac{1}{4} \kappa_{mnp} \Gamma^{np} \eta = 0,$$  \hspace{1cm} (2.14)

For Calabi–Yau manifolds these constructions are reviewed, for example, in ref. [1]. For compactifications with torsion they are generalized in ref. [21,19,20,43] and here we closely follow these references.
where $\Gamma^{np}$ is the antisymmetrized product of $\Gamma$-matrices defined in appendix A and $\kappa_{mnp}$ takes values in $\Lambda^1 \otimes \Lambda^2$ ($\Lambda^p$ being the space of $p$-forms). We see that $\kappa_{mnp}$ is the obstruction to $\eta$ being covariantly constant with respect to the Levi-Civita connection and thus for non-vanishing $\kappa$ the manifold $\hat{Y}$ can not be Calabi–Yau. Similarly, using (2.9), (2.10) and (2.14) one shows that $J$ and $\Omega$ are also generically no longer covariantly constant but instead obey

\[
\nabla_m^T J_{np} = \nabla_m J_{np} - \kappa_{mn}^\ r J_{rp} - \kappa_{mp}^\ r J_{nr} = 0 ,
\]

\[
\nabla_m^T \Omega_{nmp} = \nabla_m \Omega_{npq} - \kappa_{mn}^\ r \Omega_{rpq} - \kappa_{mp}^\ r \Omega_{nrq} - \kappa_{mq}^\ r \Omega_{npr} = 0 ,
\]

so again $\kappa$ is measuring the obstruction to $J$ and $\Omega$ being covariantly constant with respect to the Levi-Civita connection. We see that the connection $\nabla^{(T)}$ preserves the $SU(3)$ structure in that $\eta$ or equivalently $J$ and $\Omega$ are constant with respect to $\nabla^{(T)}$.

Let us now analyze the contorsion $\kappa \in \Lambda^1 \otimes \Lambda^2$ in a little more detail. Recall that $\Lambda^2$ is isomorphic to the Lie algebra $so(6)$, which in turn decomposes into $su(3)$ and $su(3)^\perp$, with the latter defined by $su(3) \oplus su(3)^\perp \cong so(6)$. Thus the contorsion actually decomposes as $\kappa^{su(3)} + \kappa^0$ where $\kappa^{su(3)} \in \Lambda^1 \otimes su(3)$ and $\kappa^0 \in \Lambda^1 \otimes su(3)^\perp$. Consider now the action of $\kappa$ on the spinor $\eta$. Since $\eta$ is an $SU(3)$ singlet, the action of $su(3)$ on $\eta$ vanishes, and thus, from (2.14), we see that

\[
\nabla_m \eta = \frac{1}{4} \kappa^0_{mnp} \Gamma^{np} \eta .
\]

From (2.13), one finds that analogous expressions hold for $\nabla_m J_{np}$ and $\nabla_m \Omega_{npq}$. We see that the obstruction to having a covariantly constant spinor (or equivalently $J$ and $\Omega$) is actually measured by not the full contorsion $\kappa$ but by the so-called “intrinsic contorsion” part $\kappa^0$. Eq. (2.16) implies that $\kappa^0$ is independent of the choice of $\nabla^{(T)}$ satisfying (2.14), and thus is a property only of the $SU(3)$-structure. This fact is reviewed in more detail in appendix C.

Mathematically, it is sometimes more conventional to use the torsion $T$ instead of the contorsion $\kappa$; the two are related via $T_{mnp} = \frac{1}{2} (\kappa_{mnp} - \kappa_{nmp})$ and $T_{mnp}$ also satisfies (2.15). Similarly, one usually refers to the corresponding “intrinsic torsion” $T^0_{mnp} = \frac{1}{2} (\kappa^0_{mnp} - \kappa^0_{nmp})$ which also is an element of $\Lambda^1 \otimes su(3)^\perp$ and is in one-to-one correspondence with $\kappa^0$. If $\kappa^0$ and hence $T^0$ vanishes, we say that the $SU(3)$ structure is torsion-free. This implies $\nabla_m \eta = 0$ and the manifold is Calabi–Yau.

Both $\kappa^0$ and $T^0$ can be decomposed in terms of irreducible $SU(3)$ representations, and hence different $SU(3)$ structures can be characterized by the non-trivial $SU(3)$ representations $T^0$ carries. Adopting the notation used in [38, 39] we denote this decomposition by

\[
T^0 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 ,
\]

with the corresponding parts of $T^0$ labeled by $T_i$ with $i = 1, \ldots, 5$ and where the representations corresponding to the different $\mathcal{W}_i$ are given in table 2.4.

The second column of table 2.1 gives an interpretation of each component of $T^0$ in terms of exterior derivatives of $J$ and $\Omega$. The superscripts refer to projecting onto a

\footnote{Note that our terminology will not be very precise in that often we will use the term torsion when in fact we mean by this intrinsic torsion.}
particular \((p, q)\)-type, while the 0 subscript refers to the irreducible \(SU(3)\) representation with any trace part proportional to \(J^n\) removed (see appendix C.3). This interpretation arises since, from (2.15), we have

\[
\begin{align*}
  dJ_{mnp} &= 6 T^0_{[mn} r J_{r|p]} , \\
  d\Omega_{mnpq} &= 12 T^0_{[mn} r \Omega_{r|pq]} .
\end{align*}
\] (2.18)

These can then be inverted to give an expression for each component \(T_i\) of \(T^0\) in terms of \(dJ\) and \(d\Omega\). This is discussed in more detail from the point of view of \(SU(3)\) representations in appendix C.3.

Manifolds with \(SU(3)\) structure are in general not complex manifolds. An almost complex structure \(J\) (obeying (2.13)) necessarily exists but the integrability of \(J\) is determined by the vanishing of the Nijenhuis tensor \(N_{mnp}\). From its definition (C.4) we see that a covariantly constant \(J\) has a vanishing \(N_{mnp}\) and in this situation the manifold is complex and Kähler (as is the case for Calabi–Yau manifolds). However, for a generic \(J\) the Nijenhuis tensor does not vanish and is instead determined by the (con-) torsion using (C.4) and (2.13). Thus \(T^0\) also is an obstruction to \(\hat{Y}\) being a complex manifold. However, one can show [38,39] that \(N_{mn}^p\) does not depend on all torsion components but is determined entirely by the component of the torsion \(T_{1\oplus 2} \in \mathcal{W}_1 \oplus \mathcal{W}_2\), through

\[
N_{mn}^p = 8 (T_{1\oplus 2})_{mn}^p .
\] (2.19)

Before we proceed let us summarize the story so far. The requirement of an \(N = 2\) supersymmetric effective action led us to consider manifolds \(\hat{Y}\) with \(SU(3)\)-structure. Such manifolds admit a globally defined \(SU(3)\)-invariant spinor \(\eta\) but the holonomy group of the Levi-Civita connection is no longer \(SU(3)\). The deviation from \(SU(3)\) holonomy is measured by the intrinsic (con-)torsion, and implies that generically the manifold is neither complex nor Kähler. However, the fundamental two-form \(J\) and the \((3,0)\)-form \(\Omega\) can still be defined; in fact their existence is equivalent to the requirement that \(\hat{Y}\) has \(SU(3)\)-structure. Different classes of manifolds with \(SU(3)\) structure are labeled by the \(SU(3)\)-representations in which the intrinsic torsion tensor resides. In terms of \(J\) and \(\Omega\) this is measured by which components of the exterior derivatives \(dJ\) and \(d\Omega\) are non-vanishing.

### 2.3 Half-flat manifolds

In general, we might expect that there are further restrictions on \(\hat{Y}\) beyond the supersymmetry condition that it has \(SU(3)\)-structure. This would correspond to constraining

| Component | Interpretation | \(SU(3)\)-representation |
|-----------|---------------|--------------------------|
| \(\mathcal{W}_1\) | \(J \wedge d\Omega\) or \(\Omega \wedge dJ\) | \(1 \oplus 1\) |
| \(\mathcal{W}_2\) | \((d\Omega)^{2,2}_0\) | \(8 \oplus 8\) |
| \(\mathcal{W}_3\) | \((dJ)^{2,1}_0 + (dJ)^{1,2}_0\) | \(6 \oplus 6\) |
| \(\mathcal{W}_4\) | \(J \wedge dJ\) | \(3 \oplus 3\) |
| \(\mathcal{W}_5\) | \(d\Omega^{3,1}\) | \(3 \oplus 3\) |

Table 2.1: The five classes of the intrinsic torsion of a space with \(SU(3)\) structure.
the intrinsic torsion so that only certain components in table 2.1 are non-vanishing. We provide evidence for a particular set of constraints in the following subsections. Then, in section 3, we verify that these conditions do lead to the required mirror symmetric type IIA effective action.

Before doing so, however, let us consider two arguments suggesting how these constraints might appear. First, recall that the Kähler moduli on the Calabi–Yau manifold are paired with the \( B_2 \) moduli as an element \( B_2 + i J \) of \( H^2(Y,\mathbb{C}) \) where \( J \) is the Kähler form. Under mirror symmetry, these moduli map to the complex structure moduli of \( \tilde{Y} \) which are encoded in the closed holomorphic \((3,0)\)-form \( \Omega \). Turning on \( H_3 \) flux on the original Calabi–Yau manifold means that the real part of the complex Kähler form \( B_2 + i J \) is no longer closed. Under the mirror symmetry, this suggests that we now have a manifold \( \hat{Y} \) where half of \( \Omega = \Omega^+ + i\Omega^- \), in particular \( \Omega^+ \), is no longer closed. From table 2.1, we see that \( d\Omega^{2,2} \) is related to the classes \( W_1 \) and \( W_2 \) which can be further decomposed into \( W_1^+ \oplus W_1^- \) and \( W_2^+ \oplus W_2^- \) giving

\[
T_{1\oplus 2}^+ \text{ corresponding to } (d\Omega^+)^{2,2},
\]

\[
T_{1\oplus 2}^- \text{ corresponding to } (d\Omega^-)^{2,2}.
\]

Thus, the above result that only \( \Omega^- \) remains closed suggests that,

\[
T_{1\oplus 2}^- = 0.
\]

One might expect that it also implies that half of the \( W_5 \) component vanishes. However, as discussed in [39], \( (d\Omega^+)^{3,1} \) and \( (d\Omega^-)^{3,1} \) are related, so, in fact, all of the component in \( W_5 \) vanishes and we have in addition

\[
T_5 = 0.
\]

The second argument comes from the fact that the intrinsic torsion \( T^0 \) should be such that it supplies the missing \( 2(h^{(1,1)} + 1) \) NS-fluxes. In other words we need the new fluxes to be counted by the even cohomology of the original Calabi–Yau manifold \( Y \). This implies that there should be some well-defined relation between \( \hat{Y} \) and the Calabi–Yau manifold \( Y \). We return to this relation in more detail in section 3.1 but here let us simply make the very naive assumption that we try to match the \( SU(3) \) representations of the \( H^{p,q}(Y) \) cohomology group with the \( SU(3) \) representations of \( T^0 \). This suggests setting

\[
T_4 = T_5 = 0.
\]

since the corresponding \( H^{3,2}(Y) \) and \( H^{3,1}(Y) \) groups vanish on \( Y \). On the other hand \( T_{1,2,3} \) can be non-zero as the corresponding cohomologies do exist on \( Y \).

Taken together, these arguments suggest that the appropriate conditions might be

\[
T_{1\oplus 2}^- = T_4 = T_5 = 0.
\]

This is in fact a known class of manifolds, denoted half-flat [39]. From table 2.1 it is easy to see that the necessary and sufficient conditions can be written as

\[
d\Omega^- = 0,
\]

\[
d(J \wedge J) = 0.
\]
It will be useful in the following to have explicit expressions for the components of
the intrinsic torsion $T_1, T_2$ and $T_3$ which are non-vanishing when the manifold is half-flat.
From table 2.1 we recall that $T_{1\oplus 2}$ is in the same $SU(3)$ representation as a complex
four-form $F^{(2,2)}$ of type (2, 2). Explicitly we have
\[
(T_{1\oplus 2})_{mn}^p = F_{mnrs} \Omega^{rsp} + F_{mnrs} \bar{\Omega}^{rsp}.
\]
(2.26)
The half flatness condition $T^{\perp}_{1\oplus 2} = 0$ just imposes that $F$ is real ($F = \bar{F}$) so that
\[
(T_{1\oplus 2})_{mn}^p = (T_{1\oplus 2}^+)_{mn}^p = 2F_{mnrs} \Omega^{+rsp},
\]
(2.27)
where we have used (2.10). Explicitly, from the relations (2.18) one has that $F$ is related
to $d\Omega$ by
\[
F_{mnrs} = \frac{1}{4 ||\Omega||^2} (d\Omega)_{mnrs}^2 = \frac{1}{4 ||\Omega||^2} (d\Omega^+)_{mnrs}^2.
\]
(2.28)
We will see in section 3 that this plays the role of the NS four-form which precisely
complexifies the RR 4-form background flux in the low-energy effective action. This fact
was anticipated in [33]. However, it will only generate the electric fluxes defined in (2.6),
i.e. half of the missing NS-fluxes. As we said in the introduction, the treatment of the
magnetic fluxes, corresponding to the NS two-form flux is more involved and will be
discussed in a separate publication [59].

Similarly, we see from table 2.1 that the component $T_3$ of the torsion is in the same
representation as a real traceless three-form $A_{0}^{(2,1)} + \bar{A}_{0}^{(1,2)}$ of type (2, 1) + (1, 2) (see also appendix C.3). From (2.18) we see that this form is proportional to $(dJ)^{(2,0)}_0$. Explicitly we have
\[
(T_3)_{mnp} = \frac{1}{4} \left( \delta^m_{m'} \delta^n_{n'} - J_m^{m'} J_n^{n'} \right) J_p^{p'} (dJ)_{m'n'p'} - 2F \left( \Omega^+ \right)_{mnp},
\]
(2.29)
where by $F$ we denoted the trace of $F_{mnpq}$ defined in (D.18).

The remainder of the section focuses on providing evidence that equations (2.25) are
indeed the correct conditions. Before doing so, recall that compactifications on manifolds
with torsion have also been discussed in refs. [19, 20, 21, 30, 37, 40, 41, 42]. The philoso-
phy of these papers was slightly different in that they consider backgrounds where in
addition some of the $p$-form field strength were chosen non-zero and one still satisfied
$\delta \psi_m = 0$. Here instead we want the torsion to generate terms which mimic or rather are
mirror symmetric to NS-flux backgrounds. As a consequence, one finds rather different
conditions. Since in both cases one wants $N = 2$ supersymmetry in four dimensions,
the class of manifolds discussed in [19, 20, 21, 30, 37, 40, 41, 42] are also manifolds with $SU(3)$
structure. However, in these cases the torsion is a traceless real three-form. This im-
plies $T \in \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$, but with the $T_4$ and $T_5$ being no longer independent. Thus
$T_1 = T_2 = 0$ and as a consequence the Nijenhuis tensor vanishes (since it depends only
on $T_{1\oplus 2}$) and the manifolds are complex but not Kähler.

---

10Note, that up to this point, the normalization $\eta^\dagger \eta = 1$ fixed the normalization of $J$ and $\Omega$. In
the following it will be useful to allow an arbitrary normalization of $\Omega$, thus we have included in this
expression the general factor $||\Omega||^2 \equiv \frac{1}{2} \Omega^{\alpha\beta\gamma} \Omega_{\alpha\beta\gamma}$.

11After the completion of this paper we received [82] which discusses this issue in more detail. We
thank K. Dasgupta for pointing out an error in the earlier version of the paper.
2.4 The complex three-torus and the SYZ picture

Obviously the most direct approach to finding the structure of $\hat{Y}$ is to consider a Calabi–Yau manifold where we can do the mirror symmetry explicitly. The simplest example is $T^6$ viewed as a complex three-torus where mirror symmetry is realized by T-duality on $T^3 \subset T^6$. This case we study explicitly in this section and our discussion overlaps with refs. [47,48,49] where also orbifolds of $T^6$ are considered. Furthermore, as we note at the end of the section, given the SYZ conjecture [56], which argues that mirror manifolds can be realized as $T^3$ fibrations, one also gets a picture of how the analysis generalizes to arbitrary Calabi–Yau manifolds.

For a square complex three-torus with unit length sides we can write the metric as

$$ds^2 = dz^1 d\bar{z}^1 + dz^2 d\bar{z}^2 + dz^3 d\bar{z}^3,$$

(2.30)

where we have chosen a complex structure $dz^\alpha = dx^\alpha + idy^\alpha$ for $\alpha = 1, 2, 3$. The Kähler form and holomorphic three-form are given by

$$J = -\frac{i}{2} \delta_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta,$$

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3.$$

(2.31)

Mirror symmetry then corresponds to doing three T-dualities in the $x^\alpha$ directions.

We want to start with some $H_3 \in H^3(T^6, \mathbb{R})$ flux on the torus. Because the torus is such a trivial example of a Calabi–Yau manifold, its cohomology does not have the standard form. For example, on a true Calabi–Yau threefold $H^1(Y, \mathbb{R}) = 0$. However this is clearly not the case on $T^6$. More relevant to us is that for a Calabi–Yau manifold any element $H_3 \in H^3(T^6, \mathbb{R})$ is primitive, meaning that $J \wedge \gamma = 0$. Thus to match the generic case we should ensure that the flux $H_3$ is primitive. (An equivalent statement is that the primitive elements are the ones which survive orbifolding the torus to give a true Calabi–Yau manifold.)

The second point to bear in mind is that T-duality is only a symmetry of consistent string backgrounds, or, more simply, solutions of the IIA and IIB supergravity equations. However the space $\mathbb{R}^{1,3} \times T^6$ with non-zero $H_3 = \gamma \in H^3(T^6, \mathbb{R})$ is not such a solution. Nonetheless it is easy to construct suitable solutions by viewing the flux as coming from a wrapped NS five-brane. As we will see, finding the T-dual solution is then simply a smeared version of the general result [63] that the transverse T-dual of $k+1$ NS five-branes is an ALE space with an $A_k$ singularity.

Recall that the five-brane solution in flat space is given by [64]

$$ds^2 = ds^2_{\mathbb{R}^{1,5}} + V ds^2_{\mathbb{R}^4},$$

$$H_3 = *_4 dV,$$

$$e^{2\Phi} = V,$$

(2.32)

where $ds^2_{\mathbb{R}^{1,5}}$ describes the flat worldvolume, $V ds^2_{\mathbb{R}^4}$ the conformally flat space transverse to the brane and $*_4$ is the Hodge star on $ds^2_{\mathbb{R}^4}$. The function $V$ is harmonic in the same transverse four-dimensional flat space. Smearing the five-brane in three of the transverse
directions corresponds to solution where the harmonic function depends on only one coordinate. We write $V = \lambda \xi$ with $\lambda$ constant and $ds_{\mathbb{R}^4}^2 = d\xi^2 + (d\eta^1)^2 + (d\eta^2)^2 + (d\eta^3)^2$. Let us similarly split off three of the worldvolume directions so $ds_{\mathbb{R}^1,5}^2 = ds_{\mathbb{R}^1,2}^2 + (d\eta^4)^2 + (d\eta^5)^2 + (d\eta^6)^2$. The solution can then be written as

$$
\begin{align*}
ds^2 &= ds_{\mathbb{R}^1,2}^2 + V d\xi^2 + (V(d\eta^1)^2 + V(d\eta^2)^2 + V(d\eta^3)^2 + (d\eta^4)^2 + (d\eta^5)^2 + (d\eta^6)^2) , \\
H_3 &= \lambda d\eta^1 \wedge d\eta^2 \wedge d\eta^3 , \\
e^{2\phi} &= V = \lambda \xi .
\end{align*}
$$

Since this is invariant under translations of all the $\eta^i$ coordinates, these directions can be compactified to form a six-torus. We see that the three-form flux is entirely on this internal $T^6$. In the non-compact four-dimensional space we have a (singular) domain wall located at $\xi = 0$ with a linear dependence on $\xi$. (To include the source one takes $V = \lambda|\xi - \xi_0|$, giving the domain wall at $\xi = \xi_0$.) Geometrically the solution is a $T^6$ fibration over the half line $\mathbb{R}^+$ parameterized by $\xi$. Physically, we have a “stack” of five-branes all wrapping the $(\eta^4, \eta^5, \eta^6)$ torus and smeared in the $(\eta^1, \eta^2, \eta^3)$ directions on the $T^6$. Since two spatial directions of the five-branes are unwrapped, in the non-compact four-dimensional space they appear as domain walls.

As we discuss further below, these solutions generalize to the case of $H_3$ flux on an arbitrary Calabi–Yau manifold $Y$, appearing as BPS domain wall solutions of the four-dimensional effective action \[ \mathbb{R}^4 \] \[ \text{[13, 65, 45]} \].

We are now in a position to consider the action of mirror symmetry on a solution of the form (2.33). There are several ways we could identify the complex structure in (2.33). As an example let us take $(x^1, x^2, x^3) = (\eta^4, \eta^5, \eta^6)$ and $(y^1, y^2, y^3) = (\eta^2, \eta^3, \eta^6)$. Thus the flux is given by the primitive form

$$
H_3 = \lambda dy^1 \wedge dy^2 \wedge dx^3 .
$$

Mirror symmetry is the same as T-duality in the $x^a$ directions. We can realize this explicitly by first choosing a gauge where locally $B_2 = \lambda y^1 dy^2 \wedge dx^3$ independent of the $x^a$ coordinates. T-duality in the $x^1$ and $x^2$ directions is then essentially trivial, simply inverting the size of the $x^1$ and $x^2$ circles. From the usual formulae derived in ref. \[ \text{[60]} \], the mirror solution is given by

$$
\begin{align*}
ds^2 &= ds_{\mathbb{R}^1,2}^2 + V dz^2 \\
&\quad+ \left[ (dx^1)^2 + (dx^2)^2 + V^{-1} (dx^3 - \lambda y^1 dy^2)^2 + V(dy^1)^2 + V(dy^2)^2 + (dy^3)^2 \right] ,
\end{align*}
$$

with $e^{2\phi} = 1$ and $H_3 = 0$. (Note, this same calculation of essentially the T-dual of flat space with constant $H_3$-flux has been considered several times before. Recent related examples are found in \[ \text{[13, 49]} \].)

We see that we again have a domain wall solution, but now it is completely geometrical, with no $H$-flux and a constant dilaton. It also has the form of a fibration of a six-dimensional manifold $\hat{Y}$ over $\mathbb{R}^+$. However, $\hat{Y}$ is not a torus. This matches our expectation: the mirror of $T^6$ with $H_3$ flux is no longer a Calabi–Yau. We now turn to investigating what structure $\hat{Y}$ has at any given fixed value of $V$. 
Geometrically, $\hat{Y}$ has the form $T^3 \times Q$ where $Q$ is a $S^1$ fibration over $T^2$, with $x^3$ the coordinate on $S^1$ and $y^1$ and $y^2$ the coordinates on $T^2$. (More generally one can view it as a special case of a $T^3$ fibration over $T^3$.) This immediately implies a quantization condition. For the fibration to be properly defined, it is easy to see that

$$\lambda \in \mathbb{Z}.$$  \hspace{1cm} (2.36)

Viewed as a $U(1)$ bundle over $T^2$, this is simply the statement that the first Chern class must be integral. This is interesting, since, it reproduces the quantization of the original flux $H_3 \in H^3(Y,\mathbb{Z})$ as a string theory background.

Next we note that we can still introduce a candidate complex structure on $\hat{Y}$. We have a basis of globally defined orthonormal complex one-forms given by

$$e^1 = dx^1 + i\sqrt{V}dy^1,$$

$$e^2 = dx^2 + i\sqrt{V}dy^2,$$

$$e^3 = \frac{1}{\sqrt{V}}(dx^3 - \lambda y^1 dy^2) + idy^3,$$  \hspace{1cm} (2.37)

However, clearly, this cannot be integrated to give complex coordinates $z^\alpha$. Thus this in fact only defines an almost complex structure. We can define the associated Kähler form

$$J = -\frac{i}{2} \delta_{\alpha\beta} e^\alpha \wedge \bar{e}^\beta.$$  \hspace{1cm} (2.38)

We immediately see that (taking the exterior derivative on $\hat{Y}$ only)

$$dJ = -\frac{2\lambda}{\sqrt{V}} dy^1 \wedge dy^2 \wedge dy^3 \neq 0,$$  \hspace{1cm} (2.39)

while on the other hand we do find

$$d(J \wedge J) = 0.$$  \hspace{1cm} (2.40)

We can also globally define a holomorphic three-form

$$\Omega = \Omega^+ + i\Omega^- = e^1 \wedge e^2 \wedge e^3,$$  \hspace{1cm} (2.41)

satisfying

$$d\Omega = -\frac{\lambda}{\sqrt{V}} dx^1 \wedge dx^2 \wedge dy^1 \wedge dy^2,$$  \hspace{1cm} (2.42)

so that

$$d\Omega^+ \neq 0, \hspace{0.5cm} d\Omega^- = 0.$$  \hspace{1cm} (2.43)

In summary, we see that, first, one can still define $J$ and $\Omega$ implying that $\hat{Y}$ does indeed have $SU(3)$-structure as we argued above was necessary for a low-energy $N = 2$ effective action. Secondly, this structure is not Calabi–Yau since $J$ and $\Omega$ are not closed. Instead we have exactly the half-flat conditions (2.25) as suggested above. We note that the T-duality analysis given here can be easily generalized to a class of flux configurations on $T^6$ of the form

$$H_3 = \lambda_1 dy^1 \wedge dy^2 \wedge dx^3 + \lambda_2 dy^2 \wedge dy^3 \wedge dx^1 + \lambda_3 dy^3 \wedge dy^1 \wedge dx^2,$$  \hspace{1cm} (2.44)
as well as more general tori, giving the same set of conditions. Thus, we see that at least the corresponding sub-class of generalized mirror manifolds \( \hat{Y} \) are all half-flat.

Finally, let us comment on how this picture might generalize to arbitrary Calabi–Yau manifolds. Recall the SYZ picture of mirror symmetry [56]. This conjectures that any Calabi–Yau manifold which has a mirror is a \( T^3 \) fibration with, in general, singular fibers. Mirror symmetry is realized as T-duality on the toroidal fibers. In particular, if we start in IIB with the manifold \( \hat{Y} \), the moduli space of D3 branes wrapping the \( T^3 \) fiber of \( \hat{Y} \) must be the same as the moduli space of D0 branes on the mirror IIA manifold \( Y \). But this later space is just the manifold \( Y \) itself. Thus we can construct \( Y \) from the moduli space of D3 branes wrapping the fibers. This space arises both from deformations of the D3 in \( \hat{Y} \) and also the flat \( U(1) \) connections on the D3 brane. As such, classically, it is also a \( T^3 \) fibration over the same base. (It also gets instanton corrections.) The complex torus \( T^6 \) discussed here is a very simple example, realizing the SYZ picture as a trivial \( T^3 \) fibration over \( T^3 \). Without flux, the T-dual of \( T^6 \) is simply another six-torus.

Now consider the case with flux. The point is that the NS two-form \( B_2 \) couples to the D3 brane in the Born–Infeld action. As a result the moduli space is changed. Thus the mirror space is no longer \( Y \), but is a new manifold \( \hat{Y} \). We saw this explicitly in the \( T^6 \) example. The generalized mirror manifold \( \hat{Y} \) was again a \( T^3 \) fibration over \( T^3 \) but unlike \( Y = T^6 \) the fibration was no longer trivial and hence the manifold was not Calabi–Yau. This suggests that, in general, in the SYZ picture, the manifold \( \hat{Y} \) corresponds to the original mirror manifold \( Y \) with some “extra twists” in the \( T^3 \) fibration, so that \( \hat{Y} \) is not Calabi–Yau or Ricci flat.

Just as in [57], one can calculate the T-duality explicitly in the large complex structure, or semi-flat, limit. In this limit the \( T^3 \) fibers are very small compared with the size of the base \( B \) of the fibration. Away from singular fibers, the metric can then be written in a form which depends only on the coordinates \( y^i \) on \( B \)

\[
ds^2 = g_{ij}(y)dy^idy^j + h^{\alpha\beta}(y)\left(dx_\alpha + \omega_\alpha(y)\right)\left(dx_\beta + \omega_\beta(y)\right),
\]

(2.45)

where \( x_\alpha \) parameterize the \( T^3 \) fiber and \( \omega_\alpha \) are locally one-forms on \( B \) describing the twisting of the fiber as one moves over the base. Metrics of this type are described in [57, 58]. They must satisfy certain conditions in order to be Calabi–Yau. As in the \( T^6 \) example let us now consider a primitive harmonic \( H_3 \)-flux on the semi-flat metric of the form

\[H_3 = F_\alpha \wedge dx_\alpha,
\]

(2.46)

where \( F_\alpha \) are a triplet of harmonic two-forms on \( B \). Locally, one can write \( B_2 = A_\alpha \wedge dx^\alpha \), where \( A_\alpha \) are the corresponding one-form potentials for \( F_\alpha \). In this gauge, the background is independent of \( x^\alpha \) and one can make an explicit T-duality transformation along the \( T^3 \). This generates a new metric of the same form (it is actually related to it by a Legendre transform [57]) except now with \( \omega_\alpha \) replaced by \( \omega_\alpha + A_\alpha \). These new terms modify the twisting of the \( T^3 \) fiber and mean that the metric is no longer Calabi–Yau. This is precisely the “extra twisting” discussed above.

### 2.5 Domain walls and fibered \( G_2 \) manifolds

The above discussion can be generalized to arbitrary Calabi–Yau compactifications in the following way. The point is that \( N = 1 \) supersymmetric domain wall solutions exist for
any such compactification with $H_3$ flux. This can be seen directly from the low-energy effective action as discussed in [13, 53, 15]. Just as in the torus case, physically, one can view these solutions as NS five-branes wrapped on special Lagrangian three-cycles on the Calabi–Yau manifold. This leaves two unwrapped spatial dimensions and hence corresponds to a BPS domain wall in four dimensions. As ten-dimensional solutions, the five-branes are not localized in the Calabi–Yau, and so the domain walls correspond in this sense to five-branes smeared within the compact Calabi–Yau manifold. By definition, compactifying IIA on $\hat{Y}$ leads to the same effective action as compactifying IIB on $\tilde{Y}$ with flux $H_3$. Thus the effective IIA theory on $\hat{Y}$ necessarily also admits BPS domain wall solutions. As we will see, this requirement can then be used to constrain the possible form of $\hat{Y}$.

From the point of view of the four-dimensional effective action the $H_3$ flux provides a potential for essentially the complex structure moduli describing $\Omega$, though, in fact, also for the dilaton $\Phi$ and the Kähler modulus describing the overall size of the Calabi–Yau manifold. The domain walls then correspond to a solution where the moduli depend non-trivially on the direction perpendicular to the wall. The ten-dimensional solution in the string frame has the form

$$ds^2 = ds^{2}_{R^{1,2}} + dy^2 + ds^2_{\hat{Y}}(y),$$

$$\phi = \phi(y),$$

$$H_3 \in H^3(Y, \mathbb{R}),$$

where $y$ parameterizes the direction perpendicular to the wall, $ds^{2}_{R^{1,2}}$ is the flat metric on the worldvolume of the domain wall, $ds^2_{\hat{Y}}(y)$ is the metric on the Calabi–Yau $\hat{Y}$, which through the complex structure moduli and the overall volume is a function of $y$, and the flux $H_3$ is a harmonic form in $H^3(Y, \mathbb{R})$. Note this is the same form as (2.33) in the $T^6$ case above except we made a change of variables from $z$ to $y$ for the transverse coordinates to remove the factor multiplying $dz^2$. Geometrically, the solution has the form $\mathbb{R}^{1,2} \times \hat{Z}$ where $\hat{Z}$ is a non-compact seven-dimensional manifold which is a fibration $\hat{Z} \to I$ of the Calabi–Yau manifold $\hat{Y}$ over an interval $I \subset \mathbb{R}$ parameterized by $y$. Again this is just as for the $T^6$ case.

Now consider the mirror of these domain wall solutions. The four-dimensional effective actions will be the same, simply with the role of the complex structure moduli and complexified Kähler moduli exchanged. Thus there will still be supersymmetric domain wall solutions breaking half the supersymmetries, but now these arise from a potential for the complexified Kähler moduli. Let us assume that, as above, that the mirror compactification should be pure geometrical with no $H_3$ flux and trivial dilaton. The mirror solution then still has the domain wall form

$$ds^2 = ds^{2}_{R^{1,2}} + dy^2 + ds^2_{\hat{Y}}(y),$$

but now $H_3$ is zero and $\Phi$ is constant. Thus again we have the structure $\mathbb{R}^{1,2} \times \hat{Z}$ where $\hat{Z}$ is a non-compact seven-dimensional manifold which is a fibration $\hat{Z} \to I$ of $\hat{Y}$ over an interval $I \subset \mathbb{R}$ parameterized by $y$.

Now we use the condition that the domain wall should break half the supersymmetries. First recall that for the low-energy effective action to be supersymmetric the manifold
\( \hat{Y} \) has to have \( SU(3) \) structure. This is equivalent to the existence of the forms \( J \) and \( \Omega = \Omega^+ + i \Omega^- \) everywhere on \( \hat{Y} \). Then, for the domain wall solution (2.48) to be BPS it must describe a supersymmetric manifold. In particular, \( \hat{Z} \) must have \( G_2 \)-holonomy. (As is the case, in the \( T^6 \) example, for the metric (2.35).) There is now an obvious question: what are the conditions on the six-dimensional manifolds \( \hat{Y} \) with \( SU(3) \) structure for \( \hat{Z} \) to have \( G_2 \)-holonomy?

This has been answered in a very interesting paper by Hitchin \[44\] (see also \[39\]). The manifold \( \hat{Z} \) is a \( G_2 \)-manifold iff \( \hat{Y} \) is half-flat. Again we get the same conditions we found in the \( T^6 \) example (2.25). (Note, that, \( \Omega \) is only defined up to an overall phase, thus whether the real or imaginary part or some other combination is closed is purely a choice of conventions. In \[39\], the real part is closed, while here we take the imaginary part to match the conventions used in the \( T^6 \) example above.)

This concludes our analysis of manifolds with \( SU(3) \) structure and in particular of half-flat manifolds. We identified them as promising candidates to supply the missing (electric) NS-fluxes which are demanded by mirror symmetry. In the next section we provide further evidence for this proposal by explicitly compactifying type IIA on half-flat manifolds \( \hat{Y} \).

### 3 The dimensional reduction on \( \hat{Y} \)

Before we launch into the details of the dimensional reduction, recall that we are aiming at the derivation of a type IIA effective action which is mirror symmetric to the type IIB effective action obtained from compactifications on Calabi–Yau threefolds with (electric) NS 3-form flux \( H_3 \) turned on. This effective theory is reviewed in appendix B.2 while the Calabi–Yau compactification of type IIA without fluxes is recalled in appendix B.1. As we have stressed throughout, the central problem is that in IIA theory there is no NS form-field which can reproduce the NS-fluxes which are the mirrors of \( H_3 \) in the type IIB theory. Vafa suggested that the type IIA mirror symmetric configuration is a different geometry where the complex structure is no longer integrable \[33\], so that the compactification manifold \( \hat{Y} \) is not Calabi–Yau. In the previous section we have already collected evidence that half-flat manifolds are promising candidates for \( \hat{Y} \). The additional flux was characterized by the four-form \( F^{(2,2)} \sim d\Omega^{2,2} \). The purpose of this section is to calculate the effective action, in an appropriate limit, for type IIA compactified on a half-flat \( \hat{Y} \), and show that it is exactly equivalent to the known effective theory for the mirror type IIB compactification with electric flux.

The basic problem we face in this section is that so far we have no mathematical procedure for constructing a half-flat manifold \( \hat{Y} \) from a given Calabi–Yau manifold \( Y \). Instead we will give a set of rules for the structure of \( \hat{Y} \) and the corresponding light spectrum by using physical considerations and in particular using mirror symmetry as a guiding principle. Specifically, we will write a set of two-, three- and four-forms on \( \hat{Y} \) which are in some sense “almost harmonic”. By expanding the IIA fields in these forms, we can then derive the four-dimensional effective action which is equivalent to the known mirror type IIB action.
3.1 The light spectrum and the moduli space of $\hat{Y}$

To derive the effective four-dimensional theory we first have to identify the light modes in the compactification such as the metric moduli. Unlike the case of a conventional reduction on a Calabi–Yau manifold, from the IIB calculation we know that the low-energy theory has a potential (3.31) and so not all the light fields are massless. In any dimensional reduction there is always an infinite tower of massive Kaluza–Klein states, thus we need some criterion for determining which modes we keep in the effective action.

Recall first how this worked in the type IIB case. One starts with a background Calabi–Yau manifold $\tilde{Y}$ and makes a perturbative expansion in the flux $H_3$. To linear order, $H_3$ only appears in its own equation of motion, while it appears quadratically in the other equations of motion, such as the Einstein and dilaton equations, so, heuristically,

$$\nabla^m H_{mnp} = \ldots,$$

$$R_{mn} = H^2_{mn} + \ldots. \quad (3.1)$$

In the perturbation expansion we first solve the linear equation on $\tilde{Y}$ which implies that $H_3$ is harmonic. We then consider the quadratic backreaction on the geometry of $\tilde{Y}$ and the dilaton. The backreaction will be small provided $H_3$ is small compared to the curvature of the compactification, set by the inverse size of the Calabi–Yau manifold $1/\tilde{L}$. Recall, however, that in string theory the flux $\int_{\gamma_3} H_3$, where $\gamma_3$ is any three-cycle in $\tilde{Y}$ is quantized in units of $\alpha'$. Consequently $H_3 \sim \alpha'/\tilde{L}^3$ and so for a small backreaction we require $H_3/\tilde{L}^{-1} \sim \alpha'/\tilde{L}^2$ to be small. In other words, we must be in the large volume limit where the Calabi–Yau manifold is much larger than the string length, which anyway is the region where supergravity is applicable. The Kaluza–Klein masses will be of order $1/\tilde{L}$. The mass correction due to $H_3$ is proportional to $\alpha'/\tilde{L}^3$ and so is comparatively small in the large volume limit. Thus in the dimensional reduction it is consistent to keep only the zero-modes on $\tilde{Y}$ which get small masses of order $\alpha'/\tilde{L}^3$ and to drop all the higher Kaluza–Klein modes with masses of order $1/\tilde{L}$.

We would like to make the same kind of expansion in IIA and think of the generalized mirror manifold $\hat{Y}$ as some small perturbation of the original Calabi–Yau $Y$ mirror to $\tilde{Y}$ without flux. The problem we will face throughout this section is that we do not have, in general, an explicit construction of $\hat{Y}$ from $Y$. Thus we can only give general arguments about the meaning of such a limit. From the previous discussion we saw that it is the intrinsic torsion $T^0$ which measures the deviation of $\hat{Y}$ from $Y$. Thus we can only give general arguments about the meaning of such a limit. From the previous discussion we saw that it is the intrinsic torsion $T^0$ which measures the deviation of $\hat{Y}$ from a Calabi–Yau manifold. Thus we would like to think that in the limit where $T^0$ is small $\hat{Y}$ approaches $Y$. The problem is, as we saw for the simple complex torus example in section 2.4, in general $Y$ and $\hat{Y}$ have different topology. Thus, at the best, we can only expect that $\hat{Y}$ approaches $Y$ locally in the limit of small intrinsic torsion. Put another way, the torsion, like $H_3$ is really “quantized” in the sense that, again as we saw in the torus example, it is associated with topological twists in the SYZ fibration structure of $\hat{Y}$. Consequently, it cannot really be put to zero, instead we can only try distorting the space to a limit where locally $T^0$ is small and then locally the manifold looks like $Y$.

This can be made slightly more formal in the following way. It is a general result that the Riemann tensor of any manifold with $SU(n)$ structure has a decomposition as

$$R = R_{\text{CY}} + R_{\perp} \, , \quad (3.2)$$
where the tensor $R_{CY}$ has the symmetry properties of the curvature tensor of a true Calabi–Yau manifold, so that, for instance the corresponding Ricci tensor vanishes. The orthogonal component $R_\perp$ is completely determined in terms of $\nabla T^0$ and $(T^0)^2$. (Note that the corresponding decomposition of the Ricci scalar in the half-flat case is calculated explicitly in appendix [3].) From this perspective, we can think of $R_\perp$ as a correction to the Einstein equation on a Calabi–Yau manifold, analogous to the $H_2^3$ correction in the IIB theory. In particular, if $\hat{Y}$ is to be locally like $Y$ in the limit of small torsion, we require

$$R_{CY}(\hat{Y}) = R(Y). \tag{3.3}$$

What, however, characterizes the limit where the intrinsic torsion is small? Unlike the IIB case the string scale does not appear in $T^0$. Typically both curvatures $R_{CY}$ and $R_\perp$ are of order $1/\hat{L}^2$ where $\hat{L}$ is the size of $\hat{Y}$. Thus making $\hat{Y}$ large will not help us. Instead, we must consider some distortion of the manifold so that $R_\perp \ll R_{CY}$. What this distortion might be is suggested by mirror symmetry. We know that, without flux, a large radius $\hat{Y}$ is mapped to $Y$ with large complex structure. Thus we might expect that we are interested in the large complex structure limit of $\hat{Y}$. It is easy to see that this is what happens for the example of the complex torus. In the half-flat metric (2.35) suppose we now take the $x^\alpha$ torus to be of radius $L_x$ and the $y^\alpha$ torus to be of radius $L_y$. The parameter $\lambda$ in (2.36) is then quantized in units of $L_x/L_y^2$. The intrinsic torsion, measured by $dJ$ and $d\Omega$ is proportional to $\lambda$ and so is suppressed by a factor of a power of $L_x/L_y$ compared with the mass scale set by the volume of $\hat{Y}$. In this sense the intrinsic torsion is small when $L_x/L_y$ is small which is precisely the large complex structure limit.

In this limit, the conjecture is that $R_\perp(\hat{Y})$ becomes a small perturbation, with a mass scale much smaller than the Kaluza–Klein scale set by the average size of $\hat{Y}$. Thus, as in the IIB case, at least locally, the original zero modes on $Y$ become approximate massless modes on $\hat{Y}$ gaining a small mass due to the non-trivial torsion. This suggests it is again consistent in this limit to consider a dimensional reduction keeping only the deformations of $\hat{Y}$ which correspond locally to zero modes of $Y$. This holds both for the ten-dimensional gauge potentials given in case without flux in (B.3) and the deformations of the metric as in (B.6) and (B.12).

Having discussed the approximation, let us now turn to trying to identify this light spectrum more precisely and characterizing how the missing NS flux enters the problem. As discussed, it is the intrinsic torsion of $\hat{Y}$ which characterizes the deviation of $\hat{Y}$ from a Calabi–Yau manifold therefore we expect that this encodes the NS-flux parameters we are looking for. Mirror symmetry requires that these new NS-fluxes are counted by the even cohomology of the “limiting” Calabi–Yau manifold $Y$. As we saw above, in the case of half-flat manifolds this suggests that the real $(2,2)$-form $F \sim d\Omega$ on $\hat{Y}$, introduced in (2.27) and discussed by Vafa [33], can be viewed as specifying some “extra data” on $Y$ which is a harmonic form $\zeta \in H^4(Y, \mathbb{R})$ (or equivalently $H^2(Y, \mathbb{R})$) measuring, at least part of, the missing NS flux.

While we have no explicit construction of $\hat{Y}$ in terms of $Y$ and some given flux $\zeta$, nonetheless, we expect, if mirror symmetry is to hold, that for each pair $(Y, \zeta)$ there is a unique half-flat manifold $\hat{Y}_\zeta$, so that there is a map

$$(Y, \zeta) \leftrightarrow \hat{Y}_\zeta. \tag{3.4}$$
where, in the limit of small torsion (large complex structure), $Y$ and $\hat{Y}_\zeta$ with the corresponding metrics are locally the same. In fact, we can argue two more conditions. First, the identification (3.4) can be applied at each point in the moduli space of $Y$ giving us, assuming uniqueness, a corresponding moduli space of $\hat{Y}_\zeta$. Furthermore, from the torus example, we see that the type IIB $H_3$-flux only effected the topology of $\hat{Y}$ in the sense that all points in the moduli space of $\hat{Y}_\zeta$ for given flux had the same topology. Thus we see that, if mirror symmetry is to hold, the moduli space of metrics $M(Y)$ and $M(\hat{Y}_\zeta)$ of $Y$ and $\hat{Y}$ are locally the same

$$M(\hat{Y}_\zeta) = M(Y), \quad \text{for any given } \zeta , \quad (3.5)$$

where $\zeta$ only effects the topology of $\hat{Y}$. This gives the full moduli space of all $\hat{Y}_\zeta$ the structure of an infinite number of copies of $M(Y)$ labeled by $\zeta$.

More explicitly, the matching of moduli spaces means that for each $(\Omega, J)$ on $Y$, since $\hat{Y}_\zeta$ has $SU(3)$ structure, we have a unique corresponding $(\Omega, J)$ on $\hat{Y}$ and we must have a corresponding expansion in terms of a basis of forms on $\hat{Y}$

$$\Omega = z^A \alpha_A - \mathcal{F}_A \beta^A, \quad A = 0, 1, \ldots, h^{(1,2)}(Y),$$

$$J = v^i \omega_i, \quad i = 1, \ldots, h^{(1,1)}(Y), \quad (3.6)$$

where $z^A = (1, z^a)$ with $a = 1, \ldots, h^{(1,2)}(Y)$ and the $z^a$ are the scalar fields corresponding to the deformations of the complex structure ($\mathcal{F}_A$ is defined in appendix [B.1]), while the $v^i$ are the scalar fields corresponding to the Kähler deformations. The key point here is that although $(\alpha_A, \beta^A)$ form a basis for $\Omega$ and the $\omega_i$ form a basis for $J$ they are not, in general, harmonic, and thus are not bases for $H^3(\hat{Y})$ and $H^{(1,1)}(\hat{Y})$. Locally, however, in the limit of small intrinsic torsion, they should coincide with the harmonic basis of $H^3(Y)$ and $H^{(1,1)}(Y)$ on $Y$. For $*J$ one has an analogous expansion in terms of four-forms on $\hat{Y}$ as in (B.10)

$$*J = 4K_{gij} v^i \tilde{\omega}^j, \quad i = 1, \ldots, h^{(1,1)}(Y), \quad (3.7)$$

where, again, there is no condition on $\tilde{\omega}^i$ being harmonic on $\hat{Y}$, but in the small torsion limit they again locally approach harmonic forms on $Y$.

The above expressions (3.6) and (B.7) have been written in terms of a prepotential $\mathcal{F}$ and a metric $g_{ij}$ which defines the metric on the moduli space just as for $Y$. If the low-energy effective action is to be mirror symmetric we necessarily have that the metrics on the moduli spaces $M(\hat{Y}_\zeta)$ and $M(Y)$ agree. This means that the corresponding kinetic terms in the low-energy effective action agree and implies the conditions

$$\int_{\hat{Y}} \omega_i \wedge \tilde{\omega}^j = \delta^j_i, \quad \int_{\hat{Y}} \alpha_A \wedge \beta^B = \delta^B_A, \quad \int_{\hat{Y}} \alpha_A \wedge \alpha_B = \int_{\hat{Y}} \beta^A \wedge \beta^B = 0, \quad (3.8)$$

exactly as on $Y$ in (B.9) and (B.4).

Now let us return to the flux and the restrictions implied by $\hat{Y}_\zeta$ being half-flat. Recall that we have argued that the four-form $F^{(2,2)} \sim (d\Omega)^{2,2}$ corresponds to a harmonic form

\footnote{We thank Ron Donagi for discussions on this point.}
\( \zeta \in H^4(Y, \mathbb{Z}) \) measuring the flux. Given the map between harmonic four-forms on \( Y \) and the basis \( \tilde{\omega}^i \) introduced in (3.7), we are naturally led to rewrite (2.28) as

\[
F^{(2,2)}_{mnpq} \equiv \frac{1}{4||\Omega||^2}(d\Omega)^{2,2}_{mnpq} \equiv \frac{1}{4||\Omega||^2} e_i \tilde{\omega}_{mnpq}^i, \quad i = 1, \ldots, h^{(1,1)}(Y),
\]

where the \( e_i \) are constants parameterizing the flux. Again, in the limit of small torsion, locally \( F \) is equivalent to a harmonic form on \( Y \), namely \( \zeta \).

Inserting (3.6) into (3.9), we have

\[
d\Omega = z^A d\alpha^A - F_A d\beta^A = e_i \tilde{\omega}^i.
\]

However, we argued that the flux only effects the topology of \( \hat{Y} \) and does not depend on the point in moduli space. Thus, we require that this condition is satisfied independent of the choice of moduli \( z^A = (1, z^a) \). This is only possible if we have

\[
\alpha_0 = e_i \tilde{\omega}^i, \quad \alpha_a = d\beta^A = 0,
\]

where \( \alpha_0 \) is singled out since it is the only direction in \( \Omega \) which is independent of \( z^a \).\footnote{Of course this corresponds to a specific choice of the symplectic basis of \( H^3 \). It is the same choice which is conventionally used in establishing the mirror map without fluxes.}

Furthermore, inserting (3.11) into (3.8) gives

\[
e_i = \int \omega_i \wedge d\alpha_0 = -\int d\omega_i \wedge \alpha_0.
\]

Thus consistency requires

\[
d\omega_i = e_i \beta^0, \quad d\tilde{\omega}^i = 0,
\]

where the second equation follows from (3.11).\footnote{Strictly speaking also \( d\omega_i = e_i \beta^0 + a^A \alpha_A + b_a \beta^a \) for some yet undetermined coefficients \( a^A, b_a \) solves (3.12). However by a similar argument as presented for the exterior derivative of \( \omega_i \) one can see that any non-vanishing such coefficient will produce a non-zero derivative of \( \alpha_a \) or/and \( \beta^A \) contradicting (3.11). From this one concludes that the only solution of (3.12) is (3.13).}

Eqs. (3.11) and (3.13) imply, just as we anticipated above, that neither \( \omega_i \) nor \( \tilde{\omega}^i \) are harmonic. In particular, \( \omega_i \) are no longer closed while the dual forms \( \tilde{\omega}^i \) are no longer coclosed, since at least one linear combination \( e_i \tilde{\omega}^i \) is exact. However, assuming for instance that \( e_1 \) is non-zero, the linear combinations

\[
\omega'_i = \omega_i - \frac{e_i}{e_1} \omega_1, \quad i \neq 1,
\]

are harmonic in that they satisfy

\[
d\omega'_i = d^\dagger \omega'_i = 0,
\]

where we used \( d^\dagger \omega'_i = *d^* \omega'_i \sim *d\tilde{\omega}'^i \). Thus there are still at least \( h^{(1,1)}(Y) - 1 \) harmonic forms \( \omega'_i \) on \( \hat{Y} \). The same argument can be repeated for \( H^3 \) where one finds \( 2h^{(1,2)} \).
harmonic forms or in other words the dimension of $H^3$ has changed by two and we have together
\[ h^{(2)}(\hat{Y}) = h^{(1,1)}(Y) - 1, \quad h^{(3)}(\hat{Y}) = h^{(3)}(Y) - 2. \] (3.16)

Physically this can be understood from the fact that some of the scalar fields gain a mass proportional to the flux parameters and no longer appear as zero modes of the compactification. Similarly, from mirror symmetry we do not expect the occurrence of new zero modes on $\hat{Y}$ as these would correspond to additional new massless fields in the effective action. This is also consistent with our expectation that $\hat{Y}$ is topologically different from $Y$ which stresses the point that $Y$ and $\hat{Y}$ can only be locally close to each other in the large complex structure limit.

Simply from the moduli space of $SU(3)$-structure of $\hat{Y}_\zeta$ and the relation (3.9) we have conjectured the existence of a set of forms on $\hat{Y}_\zeta$ satisfying the conditions (3.11) and (3.13) which essentially encode information about the topology of $\hat{Y}_\zeta$. We should now see if this is compatible with a half-flat structure. In particular we find, given (3.6),
\[ \text{d}J = v^i e_i \beta^0, \]
\[ \text{d}\Omega = e_i \tilde{\omega}^i. \] (3.17)

From the standard $SU(3)$ relation $J \wedge \Omega = 0$ we have that $\omega_i \wedge \alpha^A = \omega_i \wedge \beta^A = 0$ for all $A$ and $i$ and hence in particular $J \wedge \text{d}J = 0$. Furthermore, since the $e_i$ are real, $\text{d}\Omega^- = 0$. Thus we see that (3.11) and (3.13) are consistent with half-flat structure. Furthermore, since $\text{d}J$ and $\text{d}\Omega$ completely determine the intrinsic torsion $T^0$, we see that all the components of $T^0$ are given in terms of the constants $e_i$ without the need for any additional information.

Let us summarize. We proposed a set of rules for identifying the light modes for compactification on $\hat{Y}$ compatible with mirror symmetry and half-flatness. We first argued that in the limit of large complex structure the torsion of $\hat{Y}$ is small, and locally $\hat{Y}$ and $Y$ are metrically equivalent, even though globally they have different topology. In this limit, the light spectrum corresponds to modes on $\hat{Y}$ which locally map to the zero modes of $Y$. This was made more precise by first noting that mirror symmetry implies a one-to-one correspondence between each pair of a Calabi–Yau manifold $Y$ and flux $\zeta \in H^4(Y, \mathbb{Z})$ and a unique half-flat manifold $\hat{Y}_\zeta$. As a consequence the moduli space of half-flat metrics on $\hat{Y}_\zeta$ has to be identical with the moduli space of Calabi–Yau metrics on $Y$. In addition, the metrics on these moduli spaces agree and a basis of forms for $J$ and $\Omega$ exist on $\hat{Y}$ which coincides with the corresponding basis of harmonic forms on $Y$ in the small torsion limit. Identifying the missing NS flux $e_i$ as $F \sim \text{d}\Omega^{2,2} \sim e_i \tilde{\omega}^i$ led to a set of differential relations among this basis of forms in terms of the $h^{(1,1)}(Y)$ flux parameters $e_i$. We further showed that these relation are compatible with the conditions of half-flatness. As we will see more explicitly in the next section these forms give the correct basis for expanding the ten-dimensional fields on $\hat{Y}$ and obtaining a mirror symmetric effective action. We will find that the masses of the light modes are proportional to the fluxes and thus to the intrinsic torsion of $\hat{Y}$.

\[ ^{15} \text{It would be interesting to calculate the moduli space of half-flat metrics on } \hat{Y}_\zeta \text{ directly and see that it agreed with, or at least had a subspace, of the form given by (3.6) and (3.7) together with (3.11) and (3.13).} \]
3.2 The effective action

In this section we present the derivation of the low-energy effective action of type IIA supergravity compactified on the manifold $\hat{Y}$ described in sections 2.3 and 3.1. As argued in the previous section we insist on keeping the same light spectrum as for Calabi–Yau compactifications and therefore the KK-reduction is closely related to the reduction on Calabi–Yau manifolds which we recall in appendix B. The difference is that the differential forms we expand in are no longer harmonic but instead obey

$$
d\alpha_0 = e_i \hat{\omega}^i, \quad d\alpha_a = d\beta^A = 0, \quad d\omega_i = e_i \beta^0, \quad d\hat{\omega}^i = 0. \quad (3.18)
$$

However we continue to demand that these forms have identical intersection numbers as on the Calabi–Yau or in other words obey unmodified (3.8). As we are going to see shortly the relations (3.18) are responsible for generating mass terms in the effective action consistent with the discussion in the previous section.\footnote{Note that we are not expanding in the harmonic forms $\omega'_i$ defined in (3.14) but continue to use the non-harmonic $\omega_i$. The reason is that in the $\omega_i$-basis mirror symmetry will be manifest. An expansion in the $\omega'_i$-basis merely corresponds to field redefinition in the effective action as they are just linear combinations of the $\omega_i$.}

Let us start from the type IIA action in $D = 10$\footnote{In the KK-reduction the ten-dimensional (hatted) fields are expanded in terms of the forms $\omega_i, \alpha_A, \beta^A$ introduced in (3.6)

$$
\hat{\phi} = \phi, \quad \hat{A}_1 = A^0, \quad \hat{B}_2 = B_2 + b' \omega_i,
\end{equation}

where $A^0, A^i$ are one-forms in $D = 4$ (they will populate $h^{(1,1)}$ vector multiplets and contribute the graviphoton to the gravitational multiplet) while $\xi^A, \xi_A, b^i$ are scalar fields in $D = 4$. The $b'$ combine with the Kähler deformations $v^i$ of (3.6) to form the complex scalars $t^i = b^i + iv^i$ sitting in the $h^{(1,1)}$ vector multiplets. The $\xi^a, \xi_a$ together with the complex structure deformations $\varepsilon^a$ of (3.6) are members of $h^{(1,2)}$ hypermultiplets while $\xi^0, \xi_0$ together with the dilaton $\phi$ and $B_2$ form the tensor multiplet.

The difference with Calabi–Yau compactifications results from the fact that the derivatives of $\hat{B}_2, \hat{C}_3$ in (3.20) are modified as a consequence of (3.18) and we find

$$
d\hat{C}_3 = dC_3 + (dA^i) \wedge \omega_i + (d\xi^A) \alpha_A + (d\bar{\xi}_a) \beta^a + (d\bar{\xi}_0 - e_i A^i) \beta^0 + \xi^0 e_i \hat{\omega}^i,
\end{equation}

$$
d\hat{B}_2 = dB_2 + (db') \omega_i + e_i b^i \beta^0. \quad (3.21)
$$

We already see that the scalar $\xi_0$ becomes charged precisely due to (3.18) which is exactly what we expect from the type IIB action. However, on the type IIB side we have ($h^{(1,2)}+1$)
electric flux parameters while in (3.21) only $h^{(1,1)}$ fluxes $e_i$ appear. The missing flux arises from the NS 3-form field strength $\hat{H}_3 = d\hat{B}_2$ in the direction of $\beta^0$. Turning on this additional NS flux amounts to a shift

$$\hat{H}_3 \to \hat{H}_3 + e_0 \beta^0,$$

(3.22)

where $e_0$ is the additional mass parameter. Using (3.21), (3.22) and (B.2) we see that the parameter $e_0$ introduced in this way naturally combines with the other fluxes $e_i$ into

$$\hat{H}_3 = dB_2 + db^i \omega_i + (e_i b^i + e_0) \beta^0,$$

(3.23)

$$\hat{F}_4 = (dC_3 - A^0 \wedge dB_2) + (dA^i - A^0 db^i) \wedge \omega_i + D\xi^A \alpha_A + D\tilde{\xi}_A \beta^A + \xi^0 e_i \tilde{\omega}^i,$$

where the covariant derivatives are given by

$$D\xi_0 = d\xi_0 - e_i (A^i + b^i A^0) - e_0 A^0, \quad D\xi^A = d\xi^A, \quad D\tilde{\xi}_a = d\tilde{\xi}_a. \quad (3.24)$$

This formula is one of the major consequences of compactifying on $\hat{Y}$ (in particular of expanding the ten-dimensional fields in forms which are not harmonic) as one of the scalars, $\xi_0$, becomes charged.

From here on the compactification proceeds as in the massless case by inserting (3.23) into the action (3.19). Except for few differences which we point out, the calculation continues as in appendix B.1 and we are not going to repeat this calculation here. Using (3.3), (3.21) and (3.23) one can see that the parameters $e_0$ and $e_i$ give rise to new interactions coming from the topological term in (3.19)

$$\frac{1}{2} \int_Y \hat{H}_3 \wedge \hat{C}_3 \wedge d\hat{C}_3 = \frac{\xi_0}{2} dB_2 \wedge A^i e_i - \frac{1}{2} dB_2 \wedge \left( \xi^0 (d\xi_0 - e_i A^i) + \xi^a d\tilde{\xi}_a - \tilde{\xi}_A d\xi^A \right)$$

$$+ \frac{\xi_0}{2} e_i db^i \wedge C_3 + \frac{1}{2} db^i \wedge A^j \wedge dA^k K_{ijk}$$

$$- \frac{\xi_0}{2} (e_i b^i + e_0) dC_3 - \frac{1}{2} (e_i b^i + e_0) \wedge C_3 \wedge d\xi^0,$$

(3.25)

where $K_{ijk}$ is defined in (3.3).

The 3-form $C_3$ in 4 dimensions carries no physical degrees of freedom. Nevertheless it cannot be neglected as it may introduce a cosmological constant. Moreover when such a form interacts non-trivially with the other fields present in the theory as in (3.25) its dualization to a constant requires more care. Collecting all terms which contain $C_3$ we find

$$S_{C_3} = -\frac{K}{2} (dC_3 - A^0 \wedge dB_2) \wedge *(dC_3 - A^0 \wedge dB_2) - \xi^0 (e_i b^i + e_0) dC_3.$$

(3.26)

As shown in [71] the proper way of performing this dualization is by adding a Lagrange multiplier $\lambda dC_3$. The 3-form $C_3$ is dual to the constant $\lambda$ which was shown to be mirror symmetric to a RR-flux in ref. [18] and consequently plays no role in the analysis here. Solving for $dC_3$, inserting the result back into (3.26) and in the end setting $\lambda = 0$ we obtain the action dual to (3.24)

$$S_{\text{dual}} = -\frac{(\xi_0^0)^2}{2K} (e_i b^i + e_0)^2 - \xi^0 (e_i b^i + e_0) A^0 \wedge dB_2.$$

(3.27)
Finally, in order to obtain the usual $N = 2$ spectrum we dualize $B_2$ to a scalar field denoted by $a$. Due to the Green-Schwarz type interaction of $B_2$ (the first term in (B.25) and the second term in (B.27)) $a$ is charged, but beside that the dualization proceeds as usual. Putting together all the pieces and after going to the Einstein frame one can write the compactified action in the standard $N = 2$ form

\[ S_{IIA} = \int \left[ -\frac{1}{2} R \ast 1 - g_{i j} d t^i \wedge * d \tilde{e}^j - h_{u v} D q^u \wedge * D q^v \right. \\
+ \frac{1}{2} \Im N_{I J} F^I \wedge * F^J + \frac{1}{2} \Re N_{I J} F^I \wedge F^J - \left. V_{IIA} \ast 1 \right] , \quad (3.28) \]

where the gauge coupling matrix $N_{I J}$ and the metrics $g_{i j}, h_{u v}$ are given in (B.21), (B.7) and (B.22) respectively. As explained in appendix B.4 the gauge couplings can be properly identified after redefining the gauge fields $A^i \rightarrow A^i - b^i A^0$. We have also introduced the notation $I = (0, i) = 0, \ldots, h^{(1,1)}$ and so $A^I = (A^0, A^i)$. Among the covariant derivatives of the hypermultiplet scalars $D q^u$ the only non-trivial ones are

\[ D a = d a - \xi^0 e_I A^I , \quad D \tilde{\xi}_0 = d \tilde{\xi}_0 - e_I A^I . \quad (3.29) \]

We see that two scalars are charged under a Peccei-Quinn symmetry as a consequence of the non-zero $e_I$.

Before discussing the potential $V_{IIA}$ let us note that the action (3.28) already has the form expected from the mirror symmetric action given in appendix B.3. In particular the forms $\alpha_0$ and $\beta^0$ in (3.18) single out the two scalars $\xi^0, \tilde{\xi}_0$ from the expansion of $\hat{C}_3$. $\xi^0$ maps under mirror symmetry to the RR scalar $l$ which is already present in the $D = 10$ type IIB theory while $\tilde{\xi}_0$ maps to the charged RR scalar in type IIB. Moreover, using these identifications one observes that the gauging (3.29) is precisely what one obtains in the type IIB case with NS electric fluxes turned on (3.32).

Finally, we need to check that the potential from (3.28) coincides with the one obtained in the type IIB case (B.31). In the case of type IIA compactified on $\hat{Y}$ one can identify four distinct contributions to the potential: from the kinetic terms of $\hat{B}_2$ and $\hat{C}_3$, from the dualization of $C_3$ in 4 dimensions and from the Ricci scalar of $\hat{Y}$. We study these contributions in turn. We go directly to the four-dimensional Einstein frame which amounts to multiplying every term in the potential by a factor $e^{4 \phi}$ coming from the rescaling of $\sqrt{-g}$, $\phi$ being the four-dimensional dilaton which is related to the ten-dimensional dilaton $\hat{\phi}$ by $e^{-2 \phi} = e^{-2 \hat{\phi}} \mathcal{K}$.

Using (B.23) we see that the kinetic term of $\hat{B}_2$ in (B.13) contributes to the potential

\[ V_1 = \frac{e^{2 \phi}}{4 \mathcal{K}} (s_i b^i + e_0)^2 \int_{\bar{Y}} \beta^0 \wedge * \beta^0 = -\frac{e^{-2 \phi}}{4 \mathcal{K}} (s_i b^i + e_0)^2 \left[ (\Im \mathcal{M})^{-1} \right]^{00} , \quad (3.30) \]

where the integral over $\bar{Y}$ was performed using (B.13), (B.14) and (B.10). Similarly, the kinetic term of $\hat{C}_3$ produces the following piece in the potential

\[ V_2 = e^{4 \phi} \frac{(\xi^0)^2}{8 \mathcal{K}} e_i e_j g^{i j} , \quad (3.31) \]

where $g^{i j}$ arises after integrating over $\hat{Y}$ using (B.10). Furthermore, (B.27) contributes

\[ V_3 = e^{4 \phi} \frac{(\xi^0)^2}{2 \mathcal{K}} (s_i b^i + e_0)^2 . \quad (3.32) \]
Combining (3.30), (3.31) and (3.32) we arrive at

\[ V_{IIA} = V_g + V_1 + V_2 + V_3 \]

\[ = V_g - \frac{e^{2\phi}}{4K} (e_j b^i + c_0)^2 \left[ (\text{Im} \mathcal{M})^{-1} \right]^{00} - \frac{e^{4\phi}}{2} \epsilon_{IJ} e_I e_J \left[ (\text{Im} \mathcal{N})^{-1} \right]^{IJ} , \]

where we used the form of the matrix \((\text{Im} \mathcal{N})^{-1}\) given in (3.23). \(V_g\) is a further contribution to the potential which arises from the Ricci scalar. Since \(\hat{Y}\) is no longer Ricci-flat \(R\) contributes to the potential and in this way provides another sensitive test of the half-flat geometry.

In appendix D we show that for half-flat manifolds the Ricci scalar can be written in terms of the contorsion as

\[ R = -\kappa_{mnp} \kappa^{npm} - \frac{1}{2} \epsilon_{mnpqr} (\nabla_m \kappa_{npq} - \kappa_{mp} \kappa_{nlq}) J_{rs} , \]

which, as expected, vanishes for \(\kappa = 0\). In order to evaluate the above expression we first we have to give a prescription about how to compute \(\nabla_m \kappa_{npq}\). Taking into account that at in the end the potential in the four-dimensional theory appears after integrating over the internal manifold \(\hat{Y}\) we can integrate by parts and ‘move’ the covariant derivative to act on \(J\). This in turn can be computed by using the fact that \(J\) is covariantly constant with respect to the connection with torsion (2.15). Replacing the contorsion \(\kappa\) from (C.18), going to complex indices and using the defining relations for the torsion (2.27), (2.29) and (3.9) one can find after some straightforward but tedious algebra the expression for the Ricci scalar. The calculation is presented in appendix D and here we only record the final result

\[ R = -\frac{1}{8} \epsilon_{IJ} e_I e_J \left[ (\text{Im} \mathcal{M})^{-1} \right]^{00} . \]

Taking into account the factor \(e^{-2\phi}\) which multiplies the Ricci scalar in the ten-dimensional action (3.19) and the factor \(e^{4\phi}\) coming from the four-dimensional Weyl rescaling one obtains the contribution to the potential coming from the gravity sector to be

\[ V_g = -\frac{e^{2\phi}}{16K} \epsilon_{IJ} e_I e_J \left[ (\text{Im} \mathcal{M})^{-1} \right]^{00} . \]

Inserted into (3.33) and using again (3.23) we can finally write the entire potential which appears in the compactification of type IIA supergravity on \(\hat{Y}\)

\[ V_{IIA} = -\frac{e^{4\phi}}{2} \left[ (\xi^0)^2 - \frac{e^{-2\phi}}{2} \left[ (\text{Im} \mathcal{M})^{-1} \right]^{00} \right] e_I e_J \left[ (\text{Im} \mathcal{N})^{-1} \right]^{IJ} . \]

In order to compare this potential to the one obtained in type IIB case (3.31) we should first see how the formula (3.37) changes under the mirror map. We know that under mirror symmetry the gauge coupling matrices \(\mathcal{M}\) and \(\mathcal{N}\) are mapped into one another. In particular this means that\(^1\)

\[ \left[ (\text{Im} \mathcal{M}_A)^{-1} \right]^{00} \leftrightarrow \left[ (\text{Im} \mathcal{N}_B)^{-1} \right]^{00} = -\frac{1}{K_B} . \]

\(^{1}\)In order to avoid confusions we have added the label \(A/B\) to specify the fact that the corresponding quantity appears in type IIA/IIB theory.
where we used the expression for $(\text{Im } \mathcal{N})^{-1}$ from (3.23). With this observation it can be easily seen that the type IIA potential (3.37) is precisely mapped into the type IIB one (3.31) provided one identifies the electric flux parameters $e_I \leftrightarrow \tilde{e}_A$ and the four-dimensional dilatons on the two sides.

To summarize the results obtained in this section, we have seen that the low-energy effective action of type IIA theory compactified on $\hat{Y}$ is precisely the mirror of the effective action obtained in appendix B.2 for type IIB theory compactified on $Y$ in the presence of NS electric fluxes. This is our final argument that the half-flat manifold $\hat{Y}$ is the right compactification manifold for obtaining the mirror partners of the NS electric fluxes of type IIB theory. In particular the interplay between the gravity and the matter sector which resulted in the potential (3.37) provided a highly nontrivial check of this assumption.

4 Conclusions

In this paper we propose that type IIB (or alternatively IIA) compactified on a Calabi–Yau threefold $\hat{Y}$ with electric NS three-form flux is mirror symmetric to type IIA (respectively IIB) compactified on a half-flat manifold $\tilde{Y}$ with $SU(3)$ structure. The manifold $\hat{Y}$ is neither complex nor is it Ricci-flat. Nonetheless, though topologically distinct, it is closely related to the ordinary Calabi–Yau mirror partner $\tilde{Y}$ of the original threefold $\hat{Y}$. In particular, we argued that the moduli space of half-flat metrics on $\hat{Y}$ must be the same as the moduli space of Calabi–Yau metrics on $\tilde{Y}$. Furthermore, it is the topology of $\hat{Y}$ that encodes the even-dimensional NS-flux mirror to the original $H_3$-flux on $\tilde{Y}$.

We established this correspondence first by considering toroidal and orbifold compactification of type IIB where the mirror map is realized as a T-duality transformation and therefore can be performed explicitly. Similarly, in the SYZ picture, where mirror Calabi–Yau threefolds are viewed as special Lagrangian $T^3$-fibrations, in the large complex structure limit, the mirror map is again realized as a simple T-duality. In both cases, starting with a Calabi–Yau background with NS three-form flux, the mirror configuration is purely geometrical, with no $H_3$-flux and trivial dilaton, and the resulting geometry has $SU(3)$-structure and satisfies the half-flat conditions (2.23).

We further strengthened this proposal by deriving the low-energy type IIA effective action in the supergravity limit and showing that it is exactly equivalent to the appropriate type IIB effective action. In particular, the resulting potential delicately depends on the non-vanishing Ricci scalar of the half-flat geometry and thus provided a highly non-trivial check on our proposal.

It is interesting to note that one particular NS flux $e_0$ played a special role in that it did not arise from the half-flat geometry but, as in type IIB, appeared as a NS three-form flux $H_3 \in H^{(3,0)}(\hat{Y})$. In this context, it appears that mirror symmetry only acts on the ‘interior’ of the Hodge diamond in that it exchanges $H^{(1,1)} \leftrightarrow H^{(1,2)}$ but leaves $H^{(3,3)} \oplus H^{(0,0)}$ and $H^{(3,0)} \oplus H^{(0,3)}$ untouched. Put another way, it appears that it is the same single NS electric flux which is associated to both $H^{(3,0)} \oplus H^{(0,3)}$ and $H^{(3,3)} \oplus H^{(0,0)}$ on a given Calabi–Yau manifold.

We found that requirements of mirror symmetry provided a number of conjectures
about the geometry of the half-flat manifold $\hat{Y}$. For instance the cohomology groups of $\hat{Y}$ shrink compared to those of $Y$ in that the Hodge numbers $h^{(1,1)}$ and $h^{(1,2)}$ are reduced by one. In addition, a non-standard KK reduction had to be performed in order to obtain masses for some of the scalar fields. This in turn led us to make a number of assumptions which need to be better understood from a mathematical point of view. One particular conjecture is the following. In general, the electric NS $H_3$-flux maps under mirror symmetry to some element $\zeta \in H^4(Y)$. Mirror symmetry would appear to imply that

$$\text{for all integer fluxes } \zeta \in H^4(Y, \mathbb{Z}) \text{ there should be a unique manifold } \hat{Y}_\zeta \text{ admitting a family of half-flat metrics such that the moduli space of such metrics } M(\hat{Y}_\zeta) \text{ is equal to the moduli space } M(Y) \text{ of Calabi–Yau metrics on } Y.$$ 

In the SYZ picture we expect that all these manifolds $\hat{Y}_\zeta$ and $Y$ appear locally as $T^3$-fibrations over the same base. However, the fibration of $\hat{Y}_\zeta$ is generalized so that the total space is no longer Calabi–Yau. We note that it should be possible to determine this moduli space of half-flat geometries directly from its definition and without relying on the physical relation with Calabi–Yau threefold compactification. Moreover, a more precise mathematical statement about the relationship between a given Calabi–Yau threefold $Y$ and its ‘cousin’ half-flat geometry on $\hat{Y}$ should also be possible. Finally, our analysis only treated electric NS fluxes. The discussion of the magnetic ones is technically more involved as on the type IIB side a massive RR two-form appears which has no obvious counterpart on the type IIA side. We hope to report on all of these issues in the near future.

The relevance of half-flat geometries can also be understood from a different point of view. The four-dimensional type IIB effective action with NS background fluxes admits $N = 1$ BPS domain-wall solutions. In the mirror symmetric type IIA action these domain-walls have to be entirely geometrical with no fluxes turned on. Indeed, as shown in refs. [44, 39] half-flat manifolds when appropriately fibered over an interval always admit a metric of $G_2$ holonomy. It is precisely this geometry which governs the three-dimensional $N = 1$ effective action on the domain-wall. This is closely related to the discussion in [46]. There it was shown that starting from a type IIB theory with both RR- and NS background fluxes the conjectured mirror symmetric type IIA theory is related to a similarly purely geometrical compactification of M-theory on a $G_2$ manifold.

The $N = 1$ BPS domain walls can be conveniently characterized by a holomorphic superpotential $W$ which on the type IIB side takes the form

$$W_B = \int_{\hat{Y}} \Omega \wedge (F_3 + \tau H_3),$$

where $\tau = l + i e^{-\phi}$ is the complex type IIB dilaton. For $H_3 = 0$ the type IIA mirror

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18 This fact could be related to a possible generalization of the proposal by M. Reid where similarly the Hodge numbers decrease due to a resolution of small three-cycles as two-cycles such that the manifold after the ‘topology-changing’ transition is still complex but non-Kähler. (In our terminology these manifolds correspond to $W_1 = W_2 = 0$.)

19 In this respect a generalization of ref. [28] might be useful.
superpotential is given by \[ W^{RR} = \int_Y F_6 + J \wedge F_4 + J^2 \wedge F_2 + J^3 F_0 \] (4.2)

For \( H_3 \neq 0 \) the type IIA mirror \( W \) should have the exact same structure but with complexified fluxes [33]. In this paper we only considered electric fluxes and hence we only discovered the real four-form \( F \sim d\Omega \) in (2.28) and (3.9). Hence the corresponding superpotential should have the form,

\[ W^{NS} = \int_{\hat{Y}} (J + iB) \wedge (dC_3 + ie^{-\phi}d\Omega_+) \]  (4.3)

(Recall that the IIA analog of \( l \) comes from part of the \( C_3 \) light modes, hence the \( dC_3 \) term.) Indeed, it is easy to check that, truncating the theory so we consider only the NS zero modes, the relevant four-dimensional \( N = 1 \) action with such a superpotential does admit appropriate BPS domain walls solutions.

From this perspective it is easy to conjecture that the magnetic fluxes introduce a non-trivial two form with \( W^{NS} \sim \int_{\hat{Y}} J^2 \wedge d\Omega^+ \) being a natural candidate. Note that, this is equivalent, given the action of the Hodge star operator, to a term \( \int_{\hat{Y}} J \wedge d\Omega^- \). This would suggest that magnetic fluxes require a further generalization of the half-flat geometry allowing the possibility of \( d\Omega^- \neq 0 \). In general, for an effective action with \( N = 2 \) supersymmetry all that is required is that one compactifies on a manifold with \( SU(3) \) structure.

We can also make some speculations on the relation of the half-flat geometry of \( \hat{Y} \) to the Calabi–Yau manifold \( Y \). Recall that type IIB theory has an S-duality symmetry exchanging RR and NS three-form flux. The IIA theory must have the same symmetry in four dimensions. Consider first the electric RR flux. \( F_4 \) is then a non-zero harmonic form so that we only have \( F_4 = dC_3 \) locally. (Mathematically \( C_3 \) is the connection on a “gerbe”, a sort of generalization of a bundle, as described for instance in [32].) Without flux, the moduli of \( C_3 \) appear in hypermultiplets with the complex structure moduli of \( \Omega \) on \( Y \), and are paired by the S-duality. In other words the S-dual of \( F_4 \) flux would appear to be that statement that \( d\Omega \) is now non-zero. (Note exactly the same kind of argument was made earlier relating \( dB_2 \) and \( d\Omega_+ \) by mirror symmetry.) If one really takes the analogy seriously not only should the flux \( d\Omega \) be non-zero but \( \Omega \) should only be defined locally on \( Y \). In other words, it appears that the mirror configuration is the same three-fold \( Y \) but now, since \( \Omega \) is not globally defined, we now longer have an \( SU(3) \) structure. However, this is, in fact, not the situation. The actual mirror configuration as we have seen, is to take a different manifold \( \hat{Y} \), in some sense a twisted version of \( Y \), on which \( \Omega \) is globally defined. Essentially we exchange \( Y \) with a non-trivial gerbe for a new manifold \( \hat{Y} \) with a trivial gerbe. On \( \hat{Y} \) the four-form \( F_4 \) is actually dual to a two-form, which in turn is just the field strength of a conventional \( U(1) \) bundle. From this point of view the twisting of \( \Omega \) on \( Y \) may simply be encoded by a some \( U(1) \) bundle, perhaps with a gauge action on the phase of \( \Omega \). From this perspective the new manifold \( \hat{Y} \) might be defined by something like the space of covariantly constant sections (in general multiple covers of \( Y \)) in this bundle. Such a construction might then allow us to define the basis of forms on \( \hat{Y} \) introduced in [3,4] in terms of a twisted cohomology on \( Y \).

Let us end by noting that the ideas in this paper apply to a number of other situations. In general, it appears that whenever one considers compactifications on some
supersymmetric manifold $Z$ with non-trivial flux, one should at the same level allow for compactifications on a generalized manifold $\hat{Z}$. To preserve a supersymmetric effective action, $\hat{Z}$ should have at least the same $G$-structure as $Z$, but it need no longer have special holonomy. It would be nice to know in general what conditions one must impose on $\hat{Z}$. However, just from the current work a number of possibilities can be considered. First, since we are considering NS-fluxes, it is natural to take the conjectures of this paper directly over to type I and heterotic theories. Thus, with trivial gauge fields, half-flat compactifications of the heterotic string should be dual to compactifications with electric $H_3$-flux. Similarly type II compactifications on $G_2$-holonomy manifolds with $H_3$-flux should be dual to compactification on a particular seven-manifold $\hat{Z}$ with $G_2$ structure. Since the manifolds with flux admit BPS domain walls, $\hat{Z}$ fibered over an interval should be a manifold with $\text{Spin}(7)$-holonomy. Using the results of [44], this implies that $\hat{Z}$ has “co-calibrated” $G_2$ structure [8].

Appendix

A Conventions and notations

Throughout the paper we use the conventions from [18] (see appendix A of this paper). Beside this we use the following conventions.

- Indices $m, n, p, \ldots = 1, \ldots, 6$ label real internal coordinates. When we use complex coordinates we label them with $\alpha, \beta = 1, 2, 3$, $\bar{\alpha}, \bar{\beta} = 1, 2, 3$.

- The Riemann curvature tensor is defined as
  \[ R_{mnp}^q = \partial_m \phi_{np}^q - \partial_n \phi_{mp}^q - \phi_{mp}^r \phi_{nr}^q + \phi_{np}^r \phi_{mr}^q , \]  
  where $\phi$ denotes a general connection that contains two contributions $\phi_{mn}^p = \Gamma_{mn}^p + \kappa_{mn}^p$ where $\Gamma_{mn}^p = \Gamma_{nm}^p$ denote the Christoffel symbols and $\kappa_{mn}^p$ is the contorsion which we define more precisely in appendix C. For the Ricci tensor we use $R_{np} = R_{nm}^m$. (Note that differs by a minus sign from the one used in [18])

- We define the $\epsilon$-symbol to be $\epsilon^{123456} = +1$. The indices are lowered with the metric. It follows that in terms of ‘complex indices’ one has, as a result of the $SU(3)$ structure,
  \[ \epsilon^{\alpha\gamma\bar{\alpha}\bar{\gamma}} = -i \epsilon^{\alpha\beta\gamma} \epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}} . \]  
  where similarly $\epsilon^{123} = \epsilon^{123} = +1$.

- For the gamma matrices we use the conventions from [74]. In particular the gamma matrices on the internal space are chosen to be hermitian matrices satisfying
  \[ \{ \Gamma_m, \Gamma_n \} = 2g_{mn} . \]  
  The chirality operator $\Gamma_7$ is defined as
  \[ \Gamma_7 = i \Gamma_1 \ldots \Gamma_6 = \frac{i}{6!} \epsilon_{m_1 \ldots m_6} \Gamma^{m_1} \ldots \Gamma^{m_6} . \]
Majorana spinors on the six-dimensional internal space can be defined if we adopt the following conventions for the charge conjugation matrix $C$

$$C^T = C, \quad \Gamma_m^T = -C\Gamma_mC^{-1}, \quad (A.5)$$

while the Majorana condition on a spinor $\eta$ reads

$$\eta^\dagger = \eta^T C. \quad (A.6)$$

Symmetry properties of the gamma matrices and $C$ with the above conventions imply that for a commuting Majorana spinor $\eta$ the following quantities vanish

$$\eta^\dagger \Gamma_{(n)} \eta = 0, \quad (A.7)$$

where by $\Gamma_{(n)}$ we have denoted the antisymmetric product of $n$ gamma matrices

$$\Gamma_{(n)} = \Gamma_{m_1 \ldots m_n} = \Gamma_{[m_1 \ldots \Gamma_{m_n}}. \quad (A.8)$$

### B Type II theories compactified on a Calabi–Yau 3-folds

In this appendix we recall the known results of type II compactifications on Calabi–Yau threefolds $Y$ in order to make the paper more self-contained and to supplement the discussion and conventions used in section 3.2. In B.1 we recall type IIA compactified on $Y$ without background fluxes while in B.2 we summarize the results of type IIB with NS three-form flux turned on.

#### B.1 Type IIA compactification without fluxes

Calabi–Yau compactifications of type IIA theory were first considered in [6]. We start from the following action in 10 dimensions

$$S = \int e^{-2\hat{\phi}} \left(-\frac{1}{2} \hat{R} * 1 + 2d\hat{\phi} \wedge d\hat{\phi} - \frac{1}{4} \hat{H}_3 \wedge \star \hat{H}_3\right)$$

$$- \frac{1}{2} \int \left(\hat{F}_2 \wedge \star \hat{F}_2 + \hat{F}_4 \wedge \star \hat{F}_4\right) + \frac{1}{2} \int \hat{H}_3 \wedge \hat{C}_3 \wedge d\hat{C}_3, \quad (B.1)$$

where

$$\hat{H}_3 = d\hat{B}_2, \quad \hat{F}_2 = d\hat{A}_1, \quad \hat{F}_4 = d\hat{C}_3 - \hat{A}_1 \wedge \hat{H}_3, \quad (B.2)$$

and $\hat{\phi}$ is the dilaton. (The * is used to denote the fields in $D = 10$.)

In the KK-reduction we expand the ten-dimensional fields in terms of harmonic forms on $Y$

$$\hat{A}_1 = A^0, \quad \hat{C}_3 = C_3 + A^i \wedge \omega_i + \xi^A \alpha_A + \xi_A^i \beta^A, \quad (B.3)$$

$$\hat{B}_2 = B_2 + b^i \omega_i, \quad (B.4)$$

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where $C_3$ is a three-form, $B_2$ a two-form, $(A^0, A^i)$ are one-forms and $b^i, \xi^a, \tilde{\xi}_a$ are scalar fields in $D = 4$. $\omega_i, \ i = 1, \ldots, h^{(1,1)}$ are harmonic $(1, 1)$-forms which form a basis of $H^{(1,1)}(Y, Z)$ while $(\alpha_A, \beta^A)$ are harmonic three-forms which form a real basis of $H^3(Y, Z)$. They are normalized as follows

$$
\int_Y \alpha_A \wedge \beta^B = \delta^B_A = -\int_Y \beta^B \wedge \alpha_A , \quad A, B = 0, \ldots, h^{(1,2)} ,
$$

$$
\int_Y \alpha_A \wedge \alpha_B = \int_Y \beta^A \wedge \beta^B = 0 . \quad (B.4)
$$

Furthermore the deformations of the Calabi–Yau metric can be divided into two classes, Kähler class and complex structure deformations, each producing a set of scalar fields (moduli). The Kähler class moduli $v^i$ are real and in one to one correspondence with the elements of $H^{(1,1)}(Y, Z)$ while the complex structure moduli, $z^a, a = 1, \ldots, h^{(1,2)}$ are complex and are counted by the elements of $H^{(2,1)}(Y, Z)$. The Kähler class moduli are combined with the scalars $b^i$ defined in (B.3) into complex scalar fields $t^i = b^i + iv^i$. These fields together with the one-forms $A^i$ defined in (B.3) combine into $h^{(1,1)}$ vector multiplets $(A^i, t^i)$. The $\xi^a, \tilde{\xi}_a$ together with the complex structure deformations $z^a$ are members of $h^{(1,2)}$ hypermultiplets while $\xi^0, \tilde{\xi}_0$ together with the dilaton $\phi$ and $B_2$ form the tensor multiplet. $A^0$ in (B.3) is the graviphoton which together with the four-dimensional metric $g_{\mu\nu}$ describes the bosonic components of the gravitational multiplet.

The matter part of the four-dimensional low-energy effective action can be obtained by replacing the expansion (B.3) in the ten-dimensional action (B.1) and performing the integrals over the Calabi–Yau space. The integrals we abbreviate as

$$
\mathcal{K} = \frac{1}{6} \int_Y J \wedge J \wedge J , \quad \mathcal{K}_i = \int_Y \omega_i \wedge J \wedge J , \quad (B.5)
$$

$$
\mathcal{K}_{ij} = \int_Y \omega^i \wedge \omega^j \wedge J , \quad \mathcal{K}_{ijk} = \int_Y \omega^i \wedge \omega^j \wedge \omega^k ,
$$

where $J$ is the Kähler form which can be expanded in terms of the basis $\omega_i$ as

$$
J = v^i \omega_i . \quad (B.6)
$$

For the gravitational moduli we here only record the results obtained in the literature [6, 75]. The metric on the complexified Kähler cone is

$$
g_{ij} = \frac{1}{4\mathcal{K}} \int_Y \omega_i \wedge * \omega_j = -\frac{1}{4} \left( \frac{\mathcal{K}_{ij}}{\mathcal{K}} - \frac{1}{4} \frac{\mathcal{K}_i \mathcal{K}_j}{\mathcal{K}^2} \right) , \quad (B.7)
$$

which is Kähler i.e. $g_{ij} = \partial_i \bar{\partial}_j K$ with the Kähler potential $K$ given by

$$
e^{-K} = 8\mathcal{K} . \quad (B.8)
$$

On a Calabi–Yau threefold $H^{(2,2)}(Y)$ is dual to $H^{(1,1)}(Y)$ and it is useful to introduce the dual basis $\tilde{\omega}^i$ normalized by

$$
\int_Y \omega_i \wedge \tilde{\omega}^j = \delta^j_i . \quad (B.9)
$$
With this normalization the following relations hold
\begin{align}
g^{ij} &= 4\mathcal{K} \int_Y \omega^i \wedge \ast \omega^j, \quad \ast \omega_i = 4\mathcal{K} g_{ij} \hat{\omega}^j, \quad \ast \hat{\omega}^i = \frac{1}{4\mathcal{K}} g^{ij} \omega_j, \quad \omega_i \wedge \omega_j \sim \mathcal{K}_{ij\ell} \hat{\omega}^\ell, \quad (B.10)
\end{align}
where the symbol \(\sim\) denotes the fact that the quantities are in the same cohomology class.

For the complex structure deformations the metric \(g_{ab}\) is also Kähler with the Kähler potential given by
\begin{align}
e^{-K} &= i \int_Y \Omega \wedge \bar{\Omega} = \mathcal{K} \||\Omega||^2, \quad (B.11)
\end{align}
where \(\Omega\) is the holomorphic \((3,0)\) form on the Calabi–Yau space and \(\||\Omega||^2 \equiv \frac{1}{3!} \Omega_{\alpha\beta\gamma} \bar{\Omega}^{\alpha\beta\gamma}\). \(\Omega\) can be expanded in terms of \((\alpha_A, \beta^A)\) as
\begin{align}
\Omega &= z^A \alpha_A - F_A \beta^A, \quad A = 0, 1, \ldots, h^{(1,2)}, \quad (B.12)
\end{align}
where \(z^A = (1, z^a)\) are the deformations of the complex structure and \(F_A\) is the derivative of the \(N = 2\) prepotential. This geometry is defined more precisely for example in ref. [75] but for our purpose here we only need to record that \(F_A\) is a function of the \(z^a\).

In order to evaluate the integrals in the reduction we need to recall that the Hodge-dual basis \((\ast \alpha_A, \ast \beta^A)\) is related to \((\alpha_A, \beta^A)\) via
\begin{align}
\ast \alpha_A &= A_A^B \alpha_B + B_{AB} \beta^B, \quad \ast \beta^A = C^{AB} \alpha_B + D_A^B \beta^B, \quad (B.13)
\end{align}
where the matrices \(A, B, C\) are determined by a matrix \(\mathcal{M}\) according to [76, 77]
\begin{align}
A &= (\text{Re} \mathcal{M}) (\text{Im} \mathcal{M})^{-1}, \\
B &= - (\text{Im} \mathcal{M}) - (\text{Re} \mathcal{M}) (\text{Im} \mathcal{M})^{-1} (\text{Re} \mathcal{M}), \\
C &= (\text{Im} \mathcal{M})^{-1}. \quad (B.14)
\end{align}
\(\mathcal{M}\) in turn is determined in terms of the \(N = 2\) prepotential \(\mathcal{F}\) but we do not recall this somewhat involved relation here (see, for example, [76]). We should note that \(\mathcal{M}\) depends non-trivially on the complex structure moduli \(z^a\) and plays the role of the gauge couplings for the case of type IIB theory to which we will turn shortly.

With these expressions we can reduce the different terms appearing in the action
\begin{align}
-\frac{1}{4} \int_{Y_3} \hat{H}_3 \wedge \ast \hat{H}_3 &= -\frac{\mathcal{K}}{4} dB_2 \wedge \ast dB_2 - \mathcal{K} g_{ij} db^i \wedge \ast db^j, \\
-\frac{1}{2} \int_{Y_3} \hat{F}_2 \wedge \ast \hat{F}_2 &= -\frac{\mathcal{K}}{2} da^0 \wedge \ast da^0, \quad (B.15) \\
-\frac{1}{2} \int_{Y_3} \hat{F}_4 \wedge \ast \hat{F}_4 &= -\frac{\mathcal{K}}{2} \left( dc_3 - da^0 \wedge B_2 \right) \wedge \ast \left( dc_3 - da^0 \wedge B_2 \right)
-2\mathcal{K} g_{ij} \left( da^i - A^0 db^j \right) \wedge \ast \left( da^i - A^0 db^j \right)
+\frac{1}{2} \left( \text{Im} \mathcal{M}^{-1} \right)^{AB} \left[ d\xi^A + \mathcal{M}_{AC} d\xi^C \right] \wedge \ast \left[ d\xi_B + \mathcal{M}_{BD} d\xi^D \right], \\
\frac{1}{2} \int_Y \hat{H}_3 \wedge \hat{C}_3 \wedge d\hat{C}_3 &= -\frac{1}{2} dB_2 \wedge \left( \xi^A d\bar{\xi}_A - \bar{\xi}_A d\xi^A \right) + \frac{1}{2} db^i \wedge A^i \wedge dA^k \mathcal{K}_{ijk}.
\end{align}
The dualization of a 3-form $C_3$ in 4 dimensions produces a contribution to the cosmological constant. As shown in [18] this constant can be viewed as a specific RR-flux. Since we are not interested in RR-fluxes here we choose it to be zero and hence discard the contribution of $C_3$ in 4 dimensions. Thus the only thing we still need to do in order to recover the standard spectrum of $N = 2$ supergravity in 4 dimensions is to dualize the 2-form $B_2$ to an axion $a$. The string frame action for $B_2$

$$\mathcal{L}_{H_3} = -\frac{1}{4} e^{-2\phi} H_3 \wedge * H_3 + \frac{1}{2} H_3 \wedge \left( \xi_A d\xi^A - \xi^A d\xi_A \right),$$

(B.16)

produces the following equation of motion for $B_2$

$$d \left( e^{-2\phi} * dB_2 - \xi_A d\xi^A + \xi^A d\xi_A \right) = 0,$$

(B.17)

which can be satisfied by setting $da = \left( e^{-2\phi} * dB_2 - \xi_A d\xi^A + \xi^A d\xi_A \right)$. The equation of motion for $a$ (derived from the Bianchi identity for $H_3$)

$$d * \left( e^{2\phi} da + \xi_A d\xi^A - \xi^A d\xi_A \right) = 0,$$

(B.18)

can in turn be obtained from the action

$$\mathcal{L}_a = -\frac{e^{2\phi}}{4} \left[ da + (\bar{\xi}_A d\xi^A - \xi^A d\bar{\xi}_A) \right] \wedge * \left[ da + (\bar{\xi}_A d\xi^A - \xi^A d\bar{\xi}_A) \right],$$

(B.19)

which is the dual action of (B.16). The usual $N = 2$ supergravity couplings can be read off after redefining the gauge fields $A^i \to A^i - b^i A^0$ and introducing the collective notation $A^I = (A^0, A^i)$ where $I = (0, i) = 0, \ldots, h^{(1,1)}$.

Collecting all terms from (B.15), (B.19) and taking into account the scalars coming from the gravity sector [8] and after going to the Einstein frame the four-dimensional action becomes

$$S_{IIA} = \int \left[ -\frac{1}{2} R^* - g_{ij} dt^i \wedge * dt^j - h_{uv} dq^u \wedge * dq^v 
+ \frac{1}{2} \text{Im} \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{2} \text{Re} \mathcal{N}_{IJ} F^I \wedge F^J \right],$$

(B.20)

where the gauge coupling matrix $\mathcal{N}$ has the form

$$\text{Re} \mathcal{N}_{00} = -\frac{1}{3} \mathcal{K}_{ijk} b^i b^j b^k, \quad \text{Im} \mathcal{N}_{00} = -\mathcal{K} + \left( \mathcal{K}_{ij} - \frac{1}{4} \mathcal{K}_{ii} \mathcal{K}_{jj} \right) b^i b^j, \quad \text{Re} \mathcal{N}_{i0} = \frac{1}{2} \mathcal{K}_{ij} b^i b^k, \quad \text{Im} \mathcal{N}_{i0} = -\left( \mathcal{K}_{ij} - \frac{1}{4} \mathcal{K}_{ii} \mathcal{K}_{jj} \right) b^j, \quad \text{Re} \mathcal{N}_{ij} = -\mathcal{K}_{ijk} b^k, \quad \text{Im} \mathcal{N}_{ij} = \left( \mathcal{K}_{ij} - \frac{1}{4} \mathcal{K}_{ii} \mathcal{K}_{jj} \right),$$

(B.21)

and $h_{uv}$ is the $\sigma$-model metric for the scalars in the hypermultiplets [78].

$$h_{uv} dq^u \wedge * dq^v = d\phi \wedge * d\phi + g_{ab} d\bar{z}^a \wedge * d\bar{z}^b
+ \frac{e^{2\phi}}{4} \left[ da + (\bar{\xi}_A d\xi^A - \xi^A d\bar{\xi}_A) \right] \wedge * \left[ da + (\bar{\xi}_A d\xi^A - \xi^A d\bar{\xi}_A) \right]
- \frac{e^{2\phi}}{2} \left( \text{Im} \mathcal{M}^{-1} \right)^{AB} \left[ d\bar{\xi}_A + \mathcal{M}_{AC} d\xi^C \right] \wedge * \left[ d\bar{\xi}_B + \mathcal{M}_{BD} d\xi^D \right].$$

(B.22)

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In the main part of the paper we also need the form of the inverse gauge couplings which is given by
\[
(\text{Im}\mathcal{N})^{-1} = -\frac{1}{\mathcal{K}} \begin{pmatrix}
1 & b^i \\
b^j & \frac{g_{ij}}{4} + b^i b^j
\end{pmatrix}.
\] (B.23)

### B.2 Type IIB theory with NS flux

This appendix is intended to outline the main features of the low-energy effective action of type IIB supergravity compactified on Calabi–Yau 3-folds in the presence of NS 3-form flux \(H_3\). For only electric NS fluxes the effective action was derived in [10,16] while the potential for both electric and magnetic fluxes appeared in [11]. However, the entire bosonic action for electric and magnetic NS fluxes has not been worked out previously and we fill this gap here. Furthermore, we need this action in order to facilitate the comparison with the mirror version derived in section 3.2 for type IIA compactified on the manifolds \(\hat{Y}\). Since the derivation of this action closely follows the existing literature [7,10,11,16,18] we only highlight the aspects which are relevant for our analysis.

The ten-dimensional bosonic spectrum of type IIB supergravity consists of the metric \(\hat{g}\), an antisymmetric tensor field \(\hat{B}_2\) and the dilaton \(\hat{\phi}\) in the NS-NS sector and an axion \(\hat{l}\), a 2-form \(\hat{C}_2\), and a 4-form \(\hat{A}_4\) with self-dual field strength \(\hat{F}_5 = \hat{F}_5\), in the RR sector. No local covariant action can be written for this theory in 10 dimensions due to the self-duality of \(\hat{F}_5\). Instead one uses the action [3]
\[
S_{IIB}^{(10)} = \int e^{-2\hat{\phi}} \left( -\frac{1}{2} R + 2 d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{4} d\hat{B}_2 \wedge *d\hat{B}_2 \right)
\]
\[
-\frac{1}{2} \int \left( d\hat{l} \wedge *d\hat{l} + (d\hat{C}_2 - ld\hat{B}_2) \wedge *(d\hat{C}_2 - ld\hat{B}_2) + \frac{1}{2} \hat{F}_5 \wedge *\hat{F}_5 \right)
\]
\[
-\frac{1}{2} \int \hat{A}_4 \wedge d\hat{B}_2 \wedge d\hat{C}_2,
\] (B.24)
where
\[
\hat{F}_5 = d\hat{A}_4 - d\hat{B}_2 \wedge \hat{C}_2,
\] (B.25)
and imposes the self-duality of \(\hat{F}_5\) separately.

The compactification proceeds as usual by expanding the ten-dimensional quantities in terms of harmonic forms on the Calabi–Yau manifold
\[
\hat{B}_2 = B_2 + b^i \wedge \omega_i, \quad i = 1, \ldots, h^{(1,1)}
\]
\[
\hat{C}_2 = C_2 + c^i \wedge \omega_i,
\]
\[
\hat{A}_4 = D^i_2 \wedge \omega_i + \rho_i \wedge \tilde{\omega}^i + V^A \wedge \alpha_A - U_A \wedge \beta^A, \quad A = 1, \ldots, h^{(1,2)}
\] (B.26)
where \(B_2, C_2, D^i_2\) are two-forms, \(V^A, U_A\) are one-forms and \(b^i, c^i, \rho_i\) are scalar fields in \(D = 4\). The \(\omega_i\) form a basis for the harmonic \((1, 1)\)-forms and \((\alpha_A, \beta^A)\) form a basis for the harmonic 3-forms as introduced in the previous section. The self-duality of \(\hat{F}_5\) implies that
only half of the fields appearing in the expansion of $\hat{A}_4$ in (B.26) are independent. The four-dimensional spectrum consists of a gravitational multiplet $(g_{\mu \nu}, V_\mu^0)$, a double tensor multiplet $(B_2, C_2, \phi, l)$, $h^{(1,1)}$ tensor multiplets $(D_2^a, v^i, b^i, c^i)$ and $h^{(1,2)}$ vector multiplets $(V^a, z^a)$. The $v^i$ represent the Kähler class moduli while the $z^a$ are the complex structure moduli as introduced in $[3,1]$. In Calabi–Yau compactifications without fluxes all these fields are massless and the tensor and double tensor multiplets can be dualized to $h^{(1,1)} + 1$ hypermultiplets.

Turning on NS fluxes amounts to a modification of $H_3$ according to

$$d\tilde{B}_2 = dB_2 + db^i \wedge \omega_i + \tilde{m}^A \alpha_A - \tilde{e}_A \beta^A .$$  \tag{B.27}

After taking into account the self-duality of $F_5$ one arrives at the following action

$$S^{(4)}_{IIB} = \int \left[ -\frac{1}{2} R \ast 1 - g_{ab} d\omega^a \wedge \ast d\omega^b - g_{ij} dt^i \wedge \ast d\tilde{t}^j - d\phi \wedge \ast d\phi 
- \frac{1}{4} e^{-4\phi} dB_2 \wedge \ast dB_2 - \frac{1}{2} e^{-2\phi} K (dC_2 - ldB_2) \wedge \ast (dC_2 - ldB_2) 
- \frac{1}{2} K e^{2\phi} dl \wedge \ast dl - 2K e^{2\phi} g_{ij} (dc^i - ldb^i) \wedge \ast (dc^j - ldb^j) 
- \frac{e^{2\phi}}{2K} g^{-1ij} \left( d\rho_i - \frac{1}{2} K_{ikl} c^k db^l \right) \wedge \ast \left( d\rho_j - \frac{1}{2} K_{jmn} c^m db^n \right) 
+ 2 (db^i \wedge C_2 + c^i dB_2) \wedge \left( d\rho_i - \frac{1}{2} K_{ikl} c^l db^k \right) + \frac{1}{2} K_{ijk} c^i c^j dB_2 \wedge db^k 
+ \frac{1}{2} \text{Re} M_{AB} \tilde{F}^A \wedge \tilde{F}^B + \frac{1}{2} \text{Im} M_{AB} \tilde{F}^A \wedge \ast \tilde{F}^B + \frac{1}{2} \tilde{e}_A \left( F^A + \tilde{F}^A \right) \wedge C_2 
+ \frac{1}{2} e^{4\phi} \left( l^2 + \frac{e^{-2\phi}}{2K} \right) (\tilde{e} - \tilde{\mathcal{M}} \tilde{m})_A \text{Im} M^{-1AB} (\tilde{e} - \tilde{\mathcal{M}} \tilde{m})_B \ast 1 , \tag{B.28}$$

where $\tilde{F}^A = F^A - \tilde{m}^A C_2$ and the metrics $g_{ab}, g_{ij}$ as well as the other scalar dependent couplings have been defined in the previous appendix. Due to the appearance of $\tilde{F}^A$ in (B.28) the RR 2-form $C_2$ is massive. It was shown $[8]$ that in the case of only RR fluxes the NS 2-form $B_2$ acquired a mass. Due to the $SL(2,\mathbb{R})$ symmetry of the ten-dimensional type IIB effective action which rotates the two 2-forms into one another this is in agreement with the result found here that when NS fluxes are present the RR 2-form $C_2$ becomes massive.

In most parts of this paper we choose to consider $\tilde{m}^A = 0$ and in this case all 2-forms are massless and can be dualized to scalars.$^{[20]}$ After redefining these scalars appropriately $[7]$ the sigma model metric for the hypermultiplets can be brought to the standard

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$^{[20]}$In fact the massive RR two-form is one of the technical reasons that the construction of the mirror symmetric type IIA effective action is more involved. We will come back to this issue in a separate publication $[3]$. 

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quaternionic form of ref. \[78\]. In this basis the action reads
\[
S_{IIB}^{(4)} = \int -\frac{1}{2} R - \frac{g_{ab} d z^a \wedge \ast d \bar{z}^b - h_{uv} D q^u \wedge \ast D q^v - V_{IIB} \ast 1}{2} \\
+ \frac{1}{2} \operatorname{Re} \mathcal{M}_{AB} F^A \wedge F^B + \frac{1}{2} \operatorname{Im} \mathcal{M}_{AB} F^A \wedge \ast F^B ,
\] (B.29)
where the quaternionic metric is given by
\[
h_{uv} D q^u \wedge \ast D q^v = g_{ij} d t^i \wedge \ast d \bar{t}^j + d \phi \wedge \ast d \phi
\] (B.30)
while the potential reads
\[
V_{IIB} = -\frac{1}{2} e^{4 \phi} \left( l^2 + \frac{e^{-2 \phi}}{2 K} \right) \tilde{e}_A \left[ (\operatorname{Im} \mathcal{M})^{-1} \right]^{AB} \tilde{e}_B.
\] (B.31)
The presence of the electric fluxes has gauged some of the isometries of the hyperscalars as can be seen from the covariant derivatives
\[
Da = d a \prec \xi^0 \tilde{e}_A V^A , \quad D \tilde{\xi}_0 = d \tilde{\xi}_0 - \tilde{e}_A V^A , \quad D \tilde{\xi}_i = d \tilde{\xi}_i , \quad D \xi^I = d \xi^I .
\] (B.32)

\section{C \ G-structures}

In this section we assemble a few facts about $G$-structures as taken from the mathematical literature where one also finds the proofs omitted here. (See, for example, \[34, 35, 36, 38, 39, 79, 80\].) We concentrate on the example of manifolds with $SU(3)$-structure.

\subsection{C.1 Almost Hermitian manifolds}

Before discussing $G$-structures in general, let us recall the definition of an almost Hermitian manifold. This allows us to introduce some useful concepts, and, as we subsequently will see, provides us with a classic example of a $G$-structure.

A manifold of real dimension $2n$ is called \textit{almost complex} if it admits a globally defined tensor field $J^m_n$ which obeys
\[
J^m_p J^p_n = -\delta^m_n .
\] (C.1)
A metric $g_{mn}$ on such a manifold is called Hermitian if it satisfies
\[
J^m_p J^p_n g_{pr} = g_{mn} .
\] (C.2)
An almost complex manifold endowed with a Hermitian metric is called an \textit{almost Hermitian manifold}. The relation (C.2) implies that $J^m_m = J^m_p g_{pm}$ is a non-degenerate 2-form which is called \textit{the fundamental form}. 

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On any even-dimensional manifold one can locally introduce complex coordinates. However, complex manifolds have to satisfy in addition that, first, the introduction of complex coordinates on different patches is consistent, and second that the transition functions between different patches are holomorphic functions of the complex coordinates. The first condition corresponds to the existence of an almost complex structure. The second condition is an integrability condition, implying that there are coordinations such that the almost complex structure takes the form

\[ J = \begin{pmatrix} i1_{n \times n} & 0 \\ 0 & -i1_{n \times n} \end{pmatrix}. \]

(C.3)

The integrability condition is satisfied if and only if the Nijenhuis tensor \( N_{mn}^p \) vanishes. It is defined as

\[ N_{mn}^p = J_m^q (\partial_q J_n^p - \partial_n J_q^p) - J_n^q (\partial_q J_m^p - \partial_m J_q^p) \]

\[ = J_m^q (\nabla_q J_n^p - \nabla_n J_q^p) - J_n^q (\nabla_q J_m^p - \nabla_m J_q^p), \]

(C.4)

where \( \nabla \) denotes the covariant derivative with respect to the Levi–Civita connection.

One can also consider an even stronger condition where \( \nabla_m J_{np} = 0 \). This implies \( N_{mn}^p = 0 \) but in addition that \( dJ = 0 \) and means we have a Kähler manifold. In particular, it implies that the holonomy of the Levi–Civita connection \( \nabla \) is \( U(n) \).

Even if there is no coordinate system where it can be put in the form (C.3), any almost complex structure obeying (C.1) has eigenvalues \( \pm i \). Thus even for non-integrable almost complex structures one can define the projection operators

\[ (P^\pm)_m^n = \frac{1}{2}(\delta_m^n \mp iJ_m^n), \]

(C.5)

which project onto the two eigenspaces, and satisfy

\[ P^\pm P^\pm = P^\pm, \quad P^+ P^- = 0. \]

(C.6)

On an almost complex manifold one can define \((p,q)\) projected components \( \omega^{p,q} \) of a real \((p+q)\)-form \( \omega^{p+q} \) by using (C.3)

\[ \omega^{p,q}_{m_1 \ldots m_{p+q}} = (P^+)_{m_1}^{n_1} \ldots (P^+)_{m_p}^{n_p} (P^-)_{m_{p+1}}^{n_{p+1}} \ldots (P^-)_{m_{p+q}}^{n_{p+q}} (P^+)_{n_1 \ldots n_p}^{p,q} \omega^{p+q}_{n_1 \ldots n_{p+q}}. \]

(C.7)

Furthermore, a real \((p+q)\)-form is of the type \((p,q)\) if it satisfies

\[ \omega_{m_1 \ldots m_p n_1 \ldots n_q} = (P^+)_{m_1}^{r_1} \ldots (P^+)_{m_p}^{r_p} (P^-)_{s_1}^{n_1} \ldots (P^-)_{n_q}^{s_q} \omega_{r_1 \ldots r_p s_1 \ldots s_q}. \]

(C.8)

In analogy with complex manifolds we denote the projections on the subspace of eigenvalue \( +i \) with an unbarred index \( \alpha \) and the projection on the subspace of eigenvalue \( -i \) with a barred index \( \bar{\alpha} \). For example the hermitian metric of an almost Hermitian manifold is of type \((1,1)\) and has one barred and one unbarred index. Thus, raising and lowering indices using this hermitian metric converts holomorphic indices into anti-holomorphic ones and vice versa. Moreover the contraction of a holomorphic and an anti-holomorphic index vanishes, i.e. given \( V_m \) which is of type \((1,0)\) and \( W^n \) which is of type \((0,1)\), the product \( V_m W^n \) is zero. Similarly, on an almost hermitian manifold of real dimension \( 2n \) forms of type \((p,0)\) vanish for \( p > n \). Finally, derivatives of \((p,q)\)-forms pick up extra pieces compared to complex manifolds precisely because \( J \) is not constant. One finds

\[ d\omega^{(p,q)} = (d\omega)^{(p-1,q+2)} + (d\omega)^{(p,q+1)} + (d\omega)^{(p+1,q)} + (d\omega)^{(p+2,q-1)}. \]

(C.9)
C.2 $G$-structures and $G$-invariant tensors

An orthonormal frame on a $d$-dimensional Riemannian manifold $M$ is given by a basis of vectors $e_i$, with $i = 1, \ldots, d$, satisfying $e_i^m e_j^n g_{mn} = \delta_{ij}$. The set of all orthonormal frames is known as the frame bundle. In general, the structure group of the frame bundle is the group of rotations $O(d)$ (or $SO(d)$ if $M$ is orientable). The manifold has a $G$-structure if the structure group of the frame bundle is not completely general but can be reduced to $G \subset O(d)$. For example, in the case of an almost Hermitian manifold of dimension $d = 2n$, it turns out one can always introduce a complex frame and as a result the structure group reduces to $U(n)$.

An alternative and sometimes more convenient way to define $G$-structures is via $G$-invariant tensors, or, if $M$ is spin, $G$-invariant spinors. A non-vanishing, globally defined tensor or spinor $\xi$ is $G$-invariant if it is invariant under $G \subset O(d)$ rotations of the orthonormal frame. In the case of almost Hermitian structure, the two-form $J$ is an $U(n)$-invariant tensor. Since the invariant tensor $\xi$ is globally defined, by considering the set of frames for which $\xi$ takes the same fixed form, one can see that the structure group of the frame bundle must then reduce to $G$ (or a subgroup of $G$). Thus the existence of $\xi$ implies we have a $G$-structure. Typically, the converse is also true. Recall that, relative to an orthonormal frame, tensors of a given type form the vector space for a given representation of $O(d)$ (or $Spin(d)$ for spinors). If the structure group of the frame bundle is reduced to $G \subset O(d)$, this representation can be decomposed into irreducible representations of $G$. In the case of almost complex manifolds, this corresponds to the decomposition under the $P^\pm$ projections (C.3). Typically there will be some tensor or spinor that will have a component in this decomposition which is invariant under $G$. The corresponding vector bundle of this component must be trivial, and thus will admit a globally defined non-vanishing section $\xi$. In other words, we have a globally defined non-vanishing $G$-invariant tensor or spinor.

To see this in more detail in the almost complex structure example, recall that we had a globally defined fundamental two-form $J$. Let us specialize for definiteness to a six-manifold, though the argument is quite general. Two-forms are in the adjoint representation $15$ of $SO(6)$ which decomposes under $U(3)$ as

$$15 = 1 + 8 + (3 + \bar{3}) \ .$$ \hspace{1cm} (C.10)

There is indeed a singlet in the decomposition and so given a $U(3)$-structure we necessarily have a globally defined invariant two-form, which is precisely the fundamental two-form $J$. Conversely, given a metric and a non-degenerate two-form $J$, we have an almost Hermitian manifold and consequently an $U(3)$-structure.

In this paper we are interested in $SU(3)$-structure. In this case we find two invariant tensors. First we have the fundamental form $J$ as above. In addition, we find an invariant complex three-form $\Omega$. Three-forms are in the $20$ representation of $SO(6)$, giving two singlets in the decomposition under $SU(3)$,

$$15 = 1 + 8 + 3 + \bar{3} \implies J \ ,$$

(C.11)

$$20 = 1 + 1 + 3 + 3 + 6 + 6 \implies \Omega = \Omega^+ + i\Omega^- \ .$$

In addition, since there is no singlet in the decomposition of a five-form, one finds that

$$J \wedge \Omega = 0 \ .$$ \hspace{1cm} (C.12)

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Similarly, a six-form is a singlet of $SU(3)$, so we also must have that $J \wedge J \wedge J$ is proportional to $\Omega \wedge \bar{\Omega}$. The usual convention is to set
\begin{equation}
J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}, \tag{C.13}
\end{equation}

Conversely, a non-degenerate $J$ and $\Omega$ satisfying (C.12) and (C.13) implies that $M$ has $SU(3)$-structure. Note that, unlike the $U(n)$ case, the metric need not be specified in addition; the existence of $J$ and $\Omega$ is sufficient [44]. Essentially this is because, without the presence of a metric, $\Omega$ defines an almost complex structure, and $J$ an almost symplectic structure. Treating $J$ as the fundamental form, it is then a familiar result on almost Hermitian manifolds that the existence of an almost complex structure and a fundamental form allow one to construct a Hermitian metric.

We can similarly ask what happens to spinors for a structure group $SU(3)$. In this case we have the isomorphism $Spin(6) \cong SU(4)$ and the four-dimensional spinor representation decomposes as
\begin{equation}
4 = 1 + 3 \Rightarrow \eta. \tag{C.14}
\end{equation}
We find one singlet in the decomposition, implying the existence of a globally defined invariant spinor $\eta$. Again, the converse is also true. A metric and a globally defined spinor $\eta$ implies that $M$ has $SU(3)$-structure.

### C.3 Intrinsic torsion

One would like to have some classification of $G$-structures. In particular, one would like a generalization of the notion of a Kähler manifold where the holonomy of the Levi–Civita connection reduces to $U(n)$. Such a classification exists in terms of the intrinsic torsion. Let us start by recalling the definition of torsion and contorsion on a Riemannian manifold $(M, g)$.

Given any metric compatible connection $\nabla'$ on $(M, g)$, i.e. one satisfying $\nabla'_{\mu}g_{\nu \rho} = 0$, one can define the Riemann curvature tensor and the torsion tensor as follows
\begin{equation}
[\nabla'_{\mu}, \nabla'_{\nu}]V_{\rho} = -R^{\rho}_{\mu \nu \sigma}V_{\sigma} - 2T^{\rho}_{\mu \nu}V_{\sigma}, \tag{C.15}
\end{equation}
where $V$ is an arbitrary vector field. The Levi-Civita connection is the unique torsionless connection compatible with the metric and is given by the usual expression in terms of Christoffel symbols $\Gamma^{\rho}_{\mu \nu} = \Gamma^{\rho}_{\nu \mu}$. Let us denote by $\nabla$ the covariant derivative with respect to the Levi-Civita connection while a connection with torsion is denoted by $\nabla^{(T)}$.

Any metric compatible connection can be written in terms of the Levi-Civita connection
\begin{equation}
\nabla^{(T)} = \nabla + \kappa, \tag{C.16}
\end{equation}
where $\kappa^{\rho}_{\mu \nu}$ is the contorsion tensor. Metric compatibility implies
\begin{equation}
\kappa^{\rho}_{\mu \nu} = -\kappa^{\rho}_{\nu \mu}, \quad \text{where} \quad \kappa^{\rho}_{\mu \nu} = \kappa^{\rho}_{\mu \nu}g_{\nu \rho}. \tag{C.17}
\end{equation}
Inserting (C.17) into (C.15) one finds a one-to-one correspondence between the torsion and the contorsion
\begin{equation}
T^{\rho}_{\mu \nu} = \frac{1}{2}(\kappa^{\rho}_{\mu \nu} - \kappa^{\rho}_{\nu \mu}) \equiv \kappa^{\rho}_{[\mu \nu]}, \tag{C.18}
\end{equation}
\begin{equation}
\kappa^{\rho}_{\mu \nu} = T^{\rho}_{\mu \nu} + T^{\rho}_{\nu \mu} + T^{\rho}_{\mu \nu}. \end{equation}
These relations tell us that given a torsion tensor $T$ there exist a unique connection $\nabla^{(T)}$ whose torsion is precisely $T$.

Now suppose $M$ has a $G$-structure. In general the Levi-Civita connection does not preserve the $G$-invariant tensors (or spinor) $\xi$. In other words, $\nabla \xi \neq 0$. However, one can show [36], that there always exist some other connection $\nabla^{(T)}$ which is compatible with the $G$-structure so that

$$\nabla^{(T)} \xi = 0.$$  \hspace{1cm} (C.19)

Thus for instance, on an almost Hermitian manifold one can always find $\nabla^{(T)}$ such that $\nabla^{(T)} J = 0$. On a manifold with $SU(3)$-structure, it means we can always find $\nabla^{(T)}$ such that both $\nabla^{(T)} J = 0$ and $\nabla^{(T)} \Omega = 0$. Since the existence of $SU(3)$-structure is also equivalent to the existence of an invariant spinor $\eta$, this is equivalent to the condition $\nabla^{(T)} \eta = 0$.

Let $\kappa$ be the contorsion tensor corresponding to $\nabla^{(T)}$. From the symmetries (C.17), we see that $\kappa$ is an element of $\Lambda^1 \otimes \Lambda^2$ where $\Lambda^n$ is the space of $n$-forms. Alternatively, since $\Lambda^2 \cong so(d)$, it is more natural to think of $\kappa_{mn}{}^p$ as one-form with values in the Lie-algebra $so(d)$ that is $\Lambda^1 \otimes so(d)$. Given the existence of a $G$-structure, we can decompose $so(d)$ into a part in the Lie algebra $g$ of $G \subset SO(d)$ and an orthogonal piece $g^\perp = so(d)/g$. The contorsion $\kappa$ splits according into

$$\kappa = \kappa^0 + \kappa^g,$$ \hspace{1cm} (C.20)

where $\kappa^0$ is the part in $\Lambda^1 \otimes g^\perp$. Since an invariant tensor (or spinor) $\xi$ is fixed under $G$ rotations, the action of $g$ on $\xi$ vanishes and we have, by definition,

$$\nabla^{(T)} \xi = (\nabla + \kappa^0 + \kappa^g) \xi = (\nabla + \kappa^0) \xi = 0 .$$ \hspace{1cm} (C.21)

Thus, any two $G$-compatible connections must differ by a piece proportional to $\kappa^g$ and they have a common term $\kappa^0$ in $\Lambda^1 \otimes g^\perp$ called the “intrinsic contorsion”. Recall that there is an isomorphism (C.18) between $\kappa$ and $T$. It is more conventional in the mathematics literature to define the corresponding torsion

$$T^0_{\mn}{}^p = \kappa^0_{[mn]}{}^p \in \Lambda^1 \otimes g^\perp ,$$ \hspace{1cm} (C.22)

known as the intrinsic torsion.

From the relation (C.21) it is clear that the intrinsic contorsion, or equivalently torsion, is independent of the choice of $G$-compatible connection. Basically it is a measure of the degree to which $\nabla \xi$ fails to vanish and as such is a measure solely of the $G$-structure itself. Furthermore, one can decompose $\kappa^0$ into irreducible $G$ representations. This provides a classification of $G$-structures in terms of which representations appear in the decomposition. In particular, in the special case where $\kappa^0$ vanishes so that $\nabla \xi = 0$, one says that the structure is “torsion-free”. For an almost Hermitian structure this is equivalent to requiring that the manifold is complex and Kähler. In particular, it implies that the holonomy of the Levi–Civita connection is contained in $G$.

Let us consider the decomposition of $T^0$ in the case of $SU(3)$-structure. The relevant representations are

$$\Lambda^1 \sim 3 \oplus \bar{3} , \quad g \sim 8 , \quad g^\perp \sim 1 \oplus 3 \oplus \bar{3} .$$ \hspace{1cm} (C.23)
Thus the intrinsic torsion, which is an element of $\Lambda^1 \otimes su(3)^\perp$, can be decomposed into the following $SU(3)$ representations

$$\Lambda^1 \otimes su(3)^\perp = (3 \oplus \bar{3}) \otimes (1 \oplus 3 \oplus \bar{3}) = (1 \oplus 1) \oplus (8 \oplus 8) \oplus (6 \oplus \bar{6}) \oplus (3 \oplus \bar{3}) \oplus (3 \oplus \bar{3})'. \quad (C.24)$$

The terms in parentheses on the second line correspond precisely to the five classes $\mathcal{W}_1,\ldots,\mathcal{W}_5$ presented in table 2.1. We label the component of $T^0$ in each class by $T_1,\ldots,T_5$.

In the case of $SU(3)$-structure, each component $T_i$ can be related to a particular component in the $SU(3)$ decomposition of $dJ$ and $d\Omega$. From (C.21), we have

$$dJ_{mnp} = 6T^0_{[mn}^r J^p_{r]} ,$$

$$d\Omega_{mpq} = 12T^0_{[mn}^r \Omega_{r|pq]} . \quad (C.25)$$

Since $J$ and $\Omega$ are $SU(3)$ singlets, $dJ$ and $d\Omega$ are both elements of $\Lambda^1 \otimes su(3)^\perp$. Put another way, the contractions with $J$ and $\Omega$ in (C.24) simply project onto different $SU(3)$ representations of $T^0$. We can see which representations appear simply by decomposing the real three-form $dJ$ and complex four-form $d\Omega$ under $SU(3)$. We have,

$$dJ = [(dJ)^{3,0} + (dJ)^{0,3}] + [(dJ)^{2,1} + (dJ)^{1,2}] + [(dJ)^{1,0} + (dJ)^{0,1}] ,$$

$$20 = (1 \oplus 1) \oplus (6 \oplus \bar{6}) \oplus (3 \oplus \bar{3}) , \quad (C.26)$$

and

$$d\Omega = (d\Omega)^{3,1} + (d\Omega)^{2,2} + (d\Omega)^{0,0} ,$$

$$24 = (3 \oplus \bar{3})' \oplus (8 \oplus 8) \oplus (1 \oplus 1) . \quad (C.27)$$

The superscripts in the decomposition of $dJ$ and $d\Omega$ refer to the $(p,q)$-type of the form. The 0 subscript refers to the irreducible $SU(3)$ representation where the trace part, proportional to $J^\nu$ has been removed. Thus in particular, the traceless parts $(dJ)^{2,1}_0$ and $(d\Omega)^{2,2}_0$ satisfy $J \wedge (dJ)^{2,1}_0 = 0$ and $J \wedge (d\Omega)^{2,2}_0 = 0$ respectively. The trace parts on the other hand, have the form $(dJ)^{1,0} = \alpha \wedge J$ and $(d\Omega)^{0,0} = \beta \wedge J$, with $\alpha \sim *(J \wedge dJ)$ and $\beta \sim *(J \wedge d\Omega)$ respectively. Note that a generic complex four-form has 30 components. However, since $\Omega$ is a $(3,0)$-form, from (C.4) we see that $d\Omega$ has no $(1,3)$ part, and so only has 24 components. Comparing (C.26) and (C.27) with (C.24) we see that

$$dJ \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 , \quad d\Omega \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_5 , \quad (C.28)$$

and as advertised, $dJ$ and $d\Omega$ together include all the components $T_i$. Explicit expressions for some of these relations are given above in (2.26) and (2.29). Note that the singlet component $T_1$ can be expressed either in terms of $(dJ)^{0,3}$, corresponding to $\Omega \wedge dJ$ or in terms of $(d\Omega)^{0,0}$ corresponding to $J \wedge d\Omega$. This is simply a result of the relation (C.12) which implies that $\Omega \wedge dJ = J \wedge d\Omega$.

## D The Ricci scalar of half-flat manifolds

The simplest way to derive the Ricci scalar for the manifold considered in section 2.2 is by using the integrability condition one can derive from the Killing spinor equation

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\[ R^{(T)}_{\mu
u\rho\sigma} \Gamma^{\rho\sigma} \eta = 0, \quad (D.1) \]

where the Riemann tensor of the connection with torsion is given by (A.1)
\[ R^{(T)}_{\mu
u\rho\sigma} = R(\Gamma)^{\mu
u\rho\sigma} + \nabla_m \kappa_{\nu\rho\sigma} - \nabla_n \kappa_{\mu\rho\sigma} - \kappa_{\mu\nu} \kappa_{r\rho\sigma} + \kappa_{\mu\rho} \kappa_{n\sigma}. \quad (D.2) \]

Here \( R(\Gamma)^{\mu
u\rho\sigma} \) represents the usual Riemann tensor for the Levi-Civita connection and the covariant derivatives are again with respect to the Levi-Civita connection. For definiteness we choose the solution of the Killing spinor equation (2.14) to be a Majorana spinor. \(^{21}\) Multiplying (D.1) by \( \Gamma_n \) and summing over \( n \) one obtains
\[ R^{(T)}_{\mu
u\rho\sigma} \Gamma^{\mu\rho\sigma} \eta - 2 R^{(T)}_{\mu\nu} \Gamma^n \eta = 0. \quad (D.3) \]

Contracting from the left with \( \eta^\dagger \Gamma^m \) and using the conventions for the Majorana spinors (A.7) one derives
\[ 2 R^{(T)} = R^{(T)}_{\mu\nu\rho\sigma} \eta^\dagger \Gamma^{\mu\nu\rho\sigma} \eta. \quad (D.4) \]

where \( R^{(T)} \) represents the Ricci scalar which can be defined from the Riemann tensor (D.2). Expressing \( R^{(T)}_{\mu\nu\rho\sigma} \) in terms of \( R(\Gamma)^{\mu\nu\rho\sigma} \) from (D.2), using the Bianchi identity \( R(\Gamma)^{\mu\nu\rho\sigma} = 0 \) and the fact that the contorsion is traceless \( \kappa_{\mu\nu} = \kappa_{\nu\mu} = 0 \) which holds for half flat manifolds one can derive the formula for the Ricci scalar of the Levi-Civita connection
\[ R = -\kappa_{\mu\nu\rho} \kappa^{\rho\mu} - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma\tau\xi}(\nabla_m \kappa_{\nu\rho\sigma} - \kappa_{\nu\rho} \kappa_{m\sigma\tau\xi}) J_{\tau\xi}. \quad (D.5) \]

In order to simplify the formulas we evaluate (D.3) term by term. The strategy will be to express first the contorsion \( \kappa \) in terms of the torsion \( T \) (C.18) and then go to complex indices splitting the torsion in its component parts \( T_{1\oplus 2} \) and \( T_3 \) which are of definite type with respect to the almost complex structure \( J \).

The first term can be written as
\[ A \equiv -\kappa_{\mu\nu\rho} \kappa^{\rho\mu} = -(T_{\mu\nu} + T_{\nu\mu} + T_{\mu\nu}) T^{\mu\nu} = T_{\mu\nu} T^{\mu\nu} - 2 T_{\mu\nu} T^{\mu\nu}. \quad (D.6) \]

Using (2.27) and (2.29) one sees that the first two indices of \( T \) are of the same type and thus one has
\[ A = (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} - 2 (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} + (T_3)_{\alpha\beta\gamma}(T_3)^{\alpha\beta\gamma} + c.c., \quad (D.7) \]

where \( c.c. \) denotes complex conjugation.

The second term can be computed if one takes into account that the four-dimensional effective action appears after one integrates the ten-dimensional action over the internal space, in this case \( \hat{Y} \). Thus the second term in (D.3) can be integrated by parts to give\(^{22}\)
\[ B \equiv -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma\tau\xi}(\nabla_m \kappa_{\nu\rho\sigma} J_{\tau\xi}) \sim \frac{1}{2} \epsilon^{\mu\nu\rho\sigma\tau\xi} \kappa_{\nu\rho\sigma} \nabla_m J_{\tau\xi}. \quad (D.8) \]

\(^{21}\)The results are independent of the choice of the spinor, but the derivations may be more involved.\(^{22}\) Strictly speaking in 10 dimensions the Ricci scalar comes multiplied with a dilaton factor (3.19). However in all what we are doing we consider that the dilaton is constant over the internal space so it still make sense to speak about integration by parts without introducing additional factors with derivatives of the dilaton.
Using (2.13) and (C.18) we obtain after going to complex indices

\[ B = -\epsilon^{mnpqr}T_{mnp}T_{qr}J_{st} \]

\[ = -\epsilon^{\alpha\beta\gamma\delta\epsilon\zeta}(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta}J_{\delta\epsilon\zeta} + c.c. \]  \hspace{1cm} \text{(D.9)}

The six-dimensional \( \epsilon \) symbol splits as

\[ \epsilon^{\alpha\beta\gamma\delta\epsilon\zeta} = -i\epsilon^{\alpha\beta\gamma}\epsilon^{\delta\epsilon\zeta}, \]

and after some algebra involving the three-dimensional \( \epsilon \) symbol one finds

\[ B = -2(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} - 4(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} + 2(T_{3})_{\alpha\beta\gamma}(T_{3})_{\delta\epsilon\zeta} + c.c. \] \hspace{1cm} \text{(D.10)}

In the same way one obtains for the last term

\[ C = \frac{1}{2}\epsilon^{mnpqr}\kappa_{mp}^{\ d}\kappa_{ntq}^{\ s}J_{rs} = 2(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} + 2(T_{3})_{\alpha\beta\gamma}(T_{3})_{\delta\epsilon\zeta} + c.c. \] \hspace{1cm} \text{(D.11)}

Collecting the results from (D.7), (D.11) and (D.12) the formula for the Ricci scalar (D.5) becomes

\[ R = (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} - 6(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} + (T_{3})_{\alpha\beta\gamma}(T_{3})_{\delta\epsilon\zeta} + c.c. \] \hspace{1cm} \text{(D.12)}

The first two terms in the above expression can be straightforwardly computed using (2.27), (3.9). After a little algebra we find

\[ (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} = \frac{e_i e_j}{8||\Omega||^2}(\bar{\omega}^i)_{\alpha\beta\gamma}(\bar{\omega}^j)_{\delta\epsilon\zeta} \]

\[ (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} = -\frac{e_i e_j}{8||\Omega||^2}(\bar{\omega}^i)_{\alpha\beta\gamma}(\bar{\omega}^j)_{\delta\epsilon\zeta} + \frac{e_i e_j}{4||\Omega||^2}(\ast\bar{\omega}^i)_{\alpha\beta\gamma}(\ast\bar{\omega}^j)_{\delta\epsilon\zeta} + \frac{(e_i v^i)^2}{4||\Omega||^2}. \] \hspace{1cm} \text{(D.13)}

In order to obtain the above expressions we have used (B.6) and [81]. After integrating (D.14) over \( \hat{Y} \) we obtain

\[ \int_{\hat{Y}}(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} = \frac{e_i e_j g^{ij}}{8||\Omega||^2}, \]

\[ \int_{\hat{Y}}(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\delta\epsilon\zeta} = -\frac{e_i e_j g^{ij}}{16||\Omega||^2} + \frac{(e_i v^i)^2}{4||\Omega||^2}. \]

Finally, we have to compute the third term in (D.13). For this we note that the expression (2.29) for the \( T_3 \) component of the intrinsic torsion can be written as

\[ (dJ)_{mnp} = 4F(\Omega^-)_{mnp} + 6(T_3)_{[mn}rJ_{r|p]} \]

\[ \hspace{1cm} \text{(D.17)} \]
In order to evaluate this formula we need the expressions for \( F \) and \( dJ \) which correspond to the definition (3.9). Using (D.15) one finds for \( F \)

\[
F \equiv F_{\alpha \beta}^{\alpha \beta} = \frac{e_i v^i}{2|\Omega|^2|K|^2}.
\]

Taking the square of (D.17) we note that the terms on the RHS do not mix as they carry indices of different types. Inserting (D.18) and \( dJ \) of (3.17) we obtain

\[
(e_i v^i)^2 \int_Y \beta^0 \wedge * \beta^0 = 2i \left( \frac{e_i v^i}{|\Omega|^2|K|^2} \right)^2 \int_Y \Omega \wedge \bar{\Omega} + 2 \int_Y (T_3)_{mnp} (T_3)^{mnp}.
\]

The integral which appears on the LHS is given by

\[
\int \beta^0 \wedge * \beta^0 = - \left[ (\text{Im} \ M)^{-1} \right]^{00} = \frac{8}{|\Omega|^2|K|^2},
\]

where the first equation follows from (B.13) and (B.14) while the second equation is less obvious. The simplest way to see this is by using a mirror symmetry argument. We know that under mirror symmetry the gauge couplings \( M \) and \( N \) are mapped into one another. This also means that \((\text{Im} \ M)^{-1}\) is mapped into \((\text{Im} \ N)^{-1}\) and this matrix is given in (B.23) for a Calabi–Yau space. From here one sees that the element \([(\text{Im} \ N)^{-1}]^{00}\) is just the inverse volume of the mirror Calabi–Yau space. Using again mirror symmetry and the fact that the Kähler potential of the Kähler moduli (B.8) is mapped into the Kähler potential of the complex structure moduli (B.11) we end up with the RHS of the above equation.

Now we can write (D.19) as

\[
\int_Y (T_3)_{mnp} (T_3)^{mnp} = \frac{3}{2} \left( \frac{e_i v^i}{|\Omega|^2|K|^2} \right)^2,
\]

or in complex indices

\[
\int_Y (T_3)_{\alpha \beta \gamma} (T_3)^{\alpha \beta \gamma} = \frac{3}{2} \left( \frac{e_i v^i}{|\Omega|^2|K|^2} \right)^2.
\]

Inserting (D.16) and (D.22) into (D.13) and taking into account that all the terms in (D.16) and (D.22) are explicitly real such that the term ‘c.c.’ in (D.13) just introduces one more factor of 2 we obtain the final form of the Ricci scalar

\[
R = -\frac{1}{8} e_i e_j g^{ij} \left[ (\text{Im} \ M)^{-1} \right]^{00},
\]

where we have used again (D.20).

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References

[1] For a review see, for example, M. B. Green, J. H. Schwarz and E. Witten, “Superstring Theory”, Cambridge University Press (1987).

[2] D. Lüst and S. Theisen, “Lectures on string theory”, Springer-Verlag, Berlin, (1989).

[3] J. Polchinski, “String theory”, Cambridge University Press (1998).

[4] B. de Wit and A. Van Proeyen, “Potentials and symmetries of general gauged N=2 supergravity - Yang-Mills models,” Nucl. Phys. B245 (1984) 89;
J. Bagger and E. Witten, “Matter couplings in N=2 supergravity,” Nucl. Phys. B222 (1983) 1;
B. de Wit, P. G. Lauwers, and A. Van Proeyen, “Lagrangians of N=2 supergravity - matter systems,” Nucl. Phys. B255 (1985) 569;
R. D’Auria, S. Ferrara, and P. Fre, “Special and quaternionic isometries: General couplings in N=2 supergravity and the scalar potential,” Nucl. Phys. B359 (1991) 705.

[5] For a review and a more complete list of references see, for example, L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fre, and T. Magri, “N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map,” J. Geom. Phys. 23 (1997) 111, hep-th/9605032.

[6] M. Bodner, A. C. Cadavid, and S. Ferrara, “(2,2) vacuum configurations for type IIA superstrings: N=2 supergravity Lagrangians and algebraic geometry,” Class. Quant. Grav. 8 (1991) 789.

[7] M. Bodner and A. C. Cadavid, “Dimensional Reduction Of Type IIB Supergravity And Exceptional Quaternionic Manifolds,” Class. Quant. Grav. 7 (1990) 829;
R. Böhm, H. Günther, C. Herrmann, and J. Louis, “Compactification of type IIB string theory on Calabi–Yau threefolds,” Nucl. Phys. B569 (2000) 229–246, hep-th/9908007.
For a review see, for example, S. Hosono, A. Klemm and S. Theisen, “Lectures on mirror symmetry”, in the Proceedings of Integrable Models and Strings, eds. A. Alekseev, A. Hietamaki, K. Huitu, A. Morozov, and A. Niemi, Springer-Verlag, Berlin, 1994, hep-th/9403096; A. Klemm, “On the geometry behind N = 2 supersymmetric effective actions in four dimensions”, in the Proceedings of the 33rd Karpacz Winter School Of Theoretical Physics: Duality - Strings And Fields, eds. Z. Hasiewicz, Z. Jaskolski, J. Sobczyk, North-Holland, Amsterdam, 1998, hep-th/9705131, and references therein.

J. Polchinski and A. Strominger, “New vacua for type II string theory,” Phys. Lett. B388 (1996) 736, hep-th/9510227.

J. Michelson, “Compactifications of type IIB strings to four dimensions with non-trivial classical potential,” Nucl. Phys. B495 (1997) 127, hep-th/9610151.

T. Taylor and C. Vafa, “RR Flux on Calabi-Yau and Partial Supersymmetry Breaking,” Phys. Lett. B474 (2000) 130, hep-th/9912152.

P. Mayr, “On supersymmetry breaking in string theory and its realization in brane worlds,” Nucl. Phys. B593 (2001) 99, hep-th/0003198.

G. Curio, A. Klemm, D. Lüst, and S. Theisen, “On the vacuum structure of type II string compactifications on Calabi–Yau spaces with H-fluxes,” Nucl. Phys. B609 (2001) 3, hep-th/0012213.

S. B. Giddings, S. Kachru, and J. Polchinski, “Hierarchies from fluxes in string compactifications,” hep-th/0105097; O. DeWolfe and S. B. Giddings, “Scales and hierarchies in warped compactifications and brane worlds,” hep-th/0208123.

G. Curio, A. Klemm, B. Körs, and D. Lüst, “Fluxes in heterotic and type II string compactifications,” Nucl. Phys. B620 (2002) 237, hep-th/0106155.

G. Dall’Agata, “Type IIB supergravity compactified on a Calabi–Yau manifold with H-fluxes,” JHEP 11 (2001) 005, hep-th/0107264.

G. Curio, B. Körs, and D. Lüst, “Fluxes and Branes in Type II Vacua and M-theory Geometry with G(2) and Spin(7) Holonomy,” Nucl. Phys. B636 (2002) 197, hep-th/0111165.

J. Louis and A. Micu, “Type II Theories Compactified on Calabi–Yau Threefolds in the Presence of Background Fluxes,” Nucl. Phys. B635 (2002) 395, hep-th/0202168.

A. Strominger, “Superstrings with torsion,” Nucl. Phys. B274 (1986) 253.

C. M. Hull, “Superstring Compactifications With Torsion And Space-Time Supersymmetry,” in Turin 1985, Proceedings, Superunification and Extra Dimensions, 347-375.

B. de Wit, D. J. Smit, and N. D. Hari Dass, “Residual supersymmetry of compactified d = 10 supergravity,” Nucl. Phys. B283 (1987) 165.
[22] C. Bachas, “A Way to break supersymmetry,” hep-th/9503030.

[23] K. Dasgupta, G. Rajesh and S. Sethi, “M theory, orientifolds and G-flux,” JHEP 9908 (1999) 023, hep-th/9908088.

[24] J. Louis and A. Micu, “Heterotic string theory with background fluxes,” Nucl. Phys. B626 (2002) 26, hep-th/0110187.

[25] S. Kachru, M. B. Schulz and S. Trivedi, “Moduli stabilization from fluxes in a simple IIB orientifold,” hep-th/0201028.

[26] B. Acharya, M. Aganagic, K. Hori and C. Vafa, “Orientifolds, mirror symmetry and superpotentials,” hep-th/0202208.

[27] K. Becker, M. Becker, M. Haack, and J. Louis, “Supersymmetry breaking and alpha’ corrections to flux induced potentials,” JHEP 0206 (2002) 060, hep-th/0204254.

[28] R. Blumenhagen, V. Braun, B. Körs, and D. Lüst, “Orientifolds of K3 and Calabi–Yau manifolds with intersecting D-branes,” JHEP 0207 (2002) 026, hep-th/0206038.

[29] S. Ferrara and M. Porrati, “N = 1 no-scale supergravity from IIB orientifolds,” Phys. Lett. B545 (2002) 411, hep-th/0207135.

[30] K. Becker and K. Dasgupta, “Heterotic strings with torsion,” hep-th/0209077.

[31] S. Gukov, C. Vafa and E. Witten, “CFT’s from Calabi–Yau four-folds,” Nucl. Phys. B584 (2000) 69, Erratum-ibid. B608 (2001) 477, hep-th/9906070.

[32] S. Gukov, “Solitons, superpotentials and calibrations,” Nucl. Phys. B574 (2000) 169, hep-th/9911011.

[33] C. Vafa, “Superstrings and topological strings at large N”, J. Math. Phys. 42 (2001) 2798, hep-th/0008142.

[34] M. Falcitelli, A. Farinola, and S. Salamon, “Almost-Hermitian Geometry”, Diff. Geo. 4 (1994) 259.

[35] S. Salamon, Riemannian Geometry and Holonomy Groups, Vol. 201 of Pitman Research Notes in Mathematics, Longman, Harlow, 1989.

[36] D. Joyce, “Compact Manifolds with Special Holonomy”, Oxford University Press, Oxford, 2000.

[37] T. Friedrich and S. Ivanov, ”Parallel spinors and connections with skew-symmetric torsion in string theory,” math.dg/0102142.

[38] S. Salamon, “Almost Parallel Structures,” in Global Differential Geometry: The Mathematical Legacy of Alfred Gray (Bilbao, 2000), pp. 162, math.DG/0107146.

[39] S. Chiossi and S. Salamon, “The Intrinsic Torsion of SU(3) and G₂ Structures,” in Differential geometry, Valencia, 2001, pp. 115, math.DG/0202282.
[40] M. Rocek, “Modified Calabi–Yau manifolds with torsion,” in Essays on Mirror Manifolds, ed. S.T. Yau, International Press, Hong Kong, 1992; S. J. Gates, C. M. Hull, and M. Rocek, “Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models,” Nucl. Phys. B248 (1984) 157; S. Lyakhovich and M. Zabzine, “Poisson geometry of sigma models with extended supersymmetry,” Phys. Lett. B 548 (2002) 243 hep-th/0210043.

[41] S. Ivanov and G. Papadopoulos, “Vanishing theorems and string backgrounds,” Class. Quant. Grav. 18 (2001) 1089, math.dg/0010038. “A no-go theorem for string warped compactifications,” Phys. Lett. B497 (2001) 309, hep-th/0008232. G. Papadopoulos, “KT and HKT geometries in strings and in black hole moduli spaces,” hep-th/0201111. J. Gutowski, S. Ivanov, and G. Papadopoulos, “Deformations of generalized calibrations and compact non-Kahler manifolds with vanishing first Chern class,” math.dg/0205012.

[42] J. P. Gauntlett, N.W. Kim, D. Martelli, and D. Waldram, “Fivebranes wrapped on SLAG three-cycles and related geometry,” JHEP 0111 (2001) 018, hep-th/0110034. J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, “G-structures and wrapped NS5-branes,” hep-th/0205050.

[43] P. Kaste, R. Minasian, M. Petrini, and A. Tomasiello, “Kaluza-Klein bundles and manifolds of exceptional holonomy,” JHEP 0209 (2002) 033, hep-th/0206213.

[44] N. Hitchin, “Stable forms and special metrics”, in Global Differential Geometry: The Mathematical Legacy of Alfred Gray, ed. M. Fernandez and J.A. Wolf, Contemporary Mathematics 288, American Mathematical Society, Providence (2001), math.DG/0107101.

[45] K. Behrndt, G. Lopes Cardoso, and D. Lüst, “Curved BPS domain wall solutions in four-dimensional N = 2 supergravity,” Nucl. Phys. B607 (2001) 391, hep-th/0102128.

[46] M. Atiyah, J. M. Maldacena, and C. Vafa, “An M-theory flop as a large N duality,” J. Math. Phys. 42 (2001) 3209, hep-th/0011256.

[47] C. Vafa and E. Witten, “On orbifolds with discrete torsion,” J. Geom. Phys. 15 (1995) 189, hep-th/9409188.

[48] A. Dabholkar and C. Hull, “Duality Twists, Orbifolds, and Fluxes,” hep-th/0210209.

[49] S. Kachru, M. B. Schulz, P. K. Tripathy and S. P. Trivedi, “New supersymmetric string compactifications,” hep-th/0211182.

[50] C. M. Hull, “Massive string theories from M-theory and F-theory,” JHEP 9811 (1998) 027, hep-th/9811021.

[51] N. Kaloper and R. C. Myers, “The O(dd) story of massive supergravity,” JHEP 9905, 010 (1999), hep-th/9901048.
[52] M. Haack, J. Louis and H. Singh, “Massive type IIA theory on K3,” JHEP 0104 (2001) 040, hep-th/0102110.

[53] B. Janssen, “Massive T-duality in six dimensions,” Nucl. Phys. B 610 (2001) 280 hep-th/0105016.

[54] K. Behrndt, E. Bergshoeff, D. Roest and P. Sundell, “Massive dualities in six dimensions,” Class. Quant. Grav. 19 (2002) 2171 hep-th/0112071.

[55] H. Singh, “Romans type IIA theory and the heterotic strings,” Phys. Lett. B 545 (2002) 403 hep-th/0201206.

[56] A. Strominger, S. T. Yau, and E. Zaslow, “Mirror symmetry is T-duality,” Nucl. Phys. B 479 (1996) 243 hep-th/9606040.

[57] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, “New formulations of D = 10 suprersymmetry and D8 - O8 domain walls,” Class. Quant. Grav. 18 (2001) 3359, hep-th/0103233.

[58] S. Gukov and M. Haack, “IIA string theory on Calabi–Yau fourf olds with background fluxes,” Nucl. Phys. B 639 (2002) 95, hep-th/0203267.

[59] S. Gurrieri, J. Louis, A. Micu and D. Waldram, in preparation.

[60] G. W. Moore and E. Witten, “Self-duality, Ramond-Ramond fields, and K-theory,” JHEP 0005 (2000) 032, hep-th/9912279.

[61] L. J. Romans, “Massive N=2a supergravity in ten-dimensions,” Phys. Lett. B 169 (1986) 374.

[62] F. Giani and M. Pernici, “N=2 Supergravity In Ten-Dimensions,” Phys. Rev. D 30, (1984) 325.

[63] H. Ooguri and C. Vafa, “Two-Dimensional Black Hole and Singularities of CY Manifolds,” Nucl. Phys. B 463 (1996) 55, hep-th/9511164.

[64] A. Strominger, “Heterotic Solitons,” Nucl. Phys. B 343 (1990) 167 [Erratum-ibid. B 353 (1991) 565];
C. G. Callan, J. A. Harvey and A. Strominger, “World Sheet Approach To Heterotic Instantons And Solitons,” Nucl. Phys. B 359 (1991) 611.

[65] K. Behrndt, S. Gukov, and M. Shmakova, “Domain walls, black holes, and supersymmetric quantum mechanics,” Nucl. Phys. B 601 (2001) 49, hep-th/0101113.

[66] T. H. Buscher, “A Symmetry Of The String Background Field Equations,” Phys. Lett. B 194 (1987) 59;
T. H. Buscher, “Path Integral Derivation Of Quantum Duality In Nonlinear Sigma Models,” Phys. Lett. B 201 (1988) 466.

[67] N. J. Hitchin, “The Moduli Space of Special Lagrangian Submanifolds,” Asian J. Math. 3 (1999) 77. dg-ga/9711002.
[68] M. Gross, “Special Lagrangian Fibrations II: Geometry,” in *Surveys in differential geometry: differential geometry inspired by string theory* 341, Surv. Differ. Geom., 5, Int. Press, Boston, MA, 1999, math.AG/9809072.

[69] N. J. Hitchin, “Lectures on Special Lagrangian Submanifolds,” in *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)* 151, Stud. Adv. Math. 23 Amer. Math. Soc., Providence, RI, 2001, math-DG/9907034.

[70] I. V. Lavrinenko, H. Lu and C. N. Pope, “Fibre bundles and generalised dimensional reductions,” *Class. Quant. Grav.* 15 (1998) 2239 hep-th/9710243.

[71] P. Binetruy, F. Pillon, G. Girardi and R. Grimm, “The 3-Form Multiplet in Supergravity,” *Nucl. Phys.* B477 (1996) 175 hep-th/9603181.

[72] M. Reid, “The moduli space of 3-Folds with $K = 0$ may Nevertheless be Irreducible”, *Math. Ann.* 278 (1987) 329.

[73] N. Hitchin, “Generalized Calabi–Yau manifolds”, math.DG/0209099.

[74] A. Van Proeyen, “Tools for supersymmetry,” hep-th/9910030.

[75] P. Candelas and X. de la Ossa, “Moduli space of Calabi-Yau manifolds,” *Nucl. Phys.* B355 (1991) 455.

[76] A. Ceresole, R. D’Auria, and S. Ferrara, “The symplectic structure of N=2 Supergravity and its central extension,” *Nucl. Phys. Proc. Suppl.* 46 (1996) 67, hep-th/9509160.

[77] H. Suzuki, “Calabi-Yau compactification of type IIB string and a mass formula of the extreme black holes,” *Mod. Phys. Lett.* A11 (1996) 623, hep-th/9508001.

[78] S. Ferrara and S. Sabharwal, “Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces,” *Nucl. Phys.* B332 (1990) 317.

[79] K. Yano, “Differential geometry on complex and almost complex spaces”, Macmillan, New York, 1965.

[80] P. Candelas, “Lectures On Complex Manifolds,” in *Superstrings ’87*, Proceedings of the 1987 Trieste Spring School, eds. L. Alvarez-Gaume, M.B. Green, M. Grisaru, R. Jengo, E. Sezgin, World Scientific, 1987.

[81] A. Strominger, “Yukawa Couplings In Superstring Compactification,” *Phys. Rev. Lett.* 55 (1985) 2547.
[82] G. L. Cardoso, G. Curio, G. Dall’Agata, D. Lust, P. Manousselis and G. Zoupanos, “Non-Kaehler string backgrounds and their five torsion classes,” hep-th/0211118.