The packing chromatic number of the square lattice is at least 12 *

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Abstract

The packing chromatic number $\chi_{\rho}(G)$ of a graph $G$ is the smallest integer $k$ such that the vertex set $V(G)$ can be partitioned into disjoint classes $X_1, \ldots, X_k$, where vertices in $X_i$ have pairwise distance greater than $i$.

For the 2-dimensional square lattice $\mathbb{Z}^2$ it is proved that $\chi_{\rho}(\mathbb{Z}^2) \geq 12$, which improves the previously known lower bound 10.

Keywords: Packing chromatic number; Square lattice;
ACM 1998 classification: G.2.2 Graph theory

1 Introduction

The concept of packing coloring comes from the area of frequency planning in wireless networks. This model emphasizes the fact that some frequencies have higher throughput and hence they are used more sparsely to avoid an interference.

In our model, the first frequency cannot be assigned to neighbouring nodes. The second frequency cannot be assigned to nodes in distance at most two and so on. In graph theory language we ask for a partition of

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the vertex set of a graph $G$ into disjoint color classes $X_1, \ldots, X_k$ according to the following constraints. Each color class $X_i$ should be an \textit{i-packing}, that is, a set of vertices with the property that any distinct pair $u, v \in X_i$ satisfies $\text{dist}(u, v) > i$. Here $\text{dist}(u, v)$ denotes the shortest path distance between $u$ and $v$. Such partition is called a \textit{packing k-coloring}, even though it is allowed that some sets $X_i$ may be empty. The smallest integer $k$ for which there exists a packing $k$-coloring of $G$ is called the \textit{packing chromatic number} of $G$ and it is denoted by $\chi_p(G)$.

This concept, under the notion \textit{broadcast chromatic number}, was introduced by Goddard et al. \cite{3}. The notion packing chromatic was proposed by Brešar et al. \cite{1}.

Topic of this work is the packing chromatic number of the infinite square lattice $\mathbb{Z}^2$. The question of determining $\chi_p(\mathbb{Z}^2)$ was posed in \cite{3}. Also a lower bound 9 and an upper bound 23 were given there. The upper bound was improved by Schwenk \cite{5} to 22 and later by Holub and Soukal \cite{4} to 17. The lower bound was improved to 10 by Fiala et al. \cite{2}.

We further improve the lower bound from 10 to 12.

\textbf{Theorem 1.} \textit{The packing chromatic number for the square lattice is at least 12.}

The proof relies on computer. In the next section we describe the main idea of the algorithm, which proves the theorem. All necessary code for running the computation is available at \url{http://kam.mff.cuni.cz/~bernard/packing}.

\section{The Result}

The algorithm for proving Theorem 1 is a brute force search through all possible configurations on lattice $15 \times 9$. It is too time consuming to simply check everything. Hence we use the following observation to speed up the computation by avoiding several configurations.

\textbf{Observation 2.} \textit{If there exists a coloring of the square lattice with 11 colors then it is possible to color lattice $15 \times 9$ where color 9 is at position $[5, 5]$.}

Any other color at any other position could be fixed instead of 9 at $[5, 5]$. Color 9 at $[5, 5]$ just sufficiently reduces the number of configurations to check. We do not claim that it is the optimal choice.

If there exists a coloring we simply find any vertex of color 9 and take a piece of the lattice in its neighborhood.

So in the search through the configurations we assume that at position $[5, 5]$ is precolored by 9. The coloring procedure gets a matrix and tries to color the vertices row by row. A pseudocode is given here in this note.
function boolean try_color(lattice, [x,y])
begin
    for color := 1 to 11 do
        if can use color on lattice at [x,y] then
            lattice[x,y] := color
            if [x,y] is the last point return true
        else if try_color(lattice, next([x,y])) then return true
        endif
    endfor
    return false
end

We have two implementations of this procedure. One is in the language C++ and the other is in Pascal. The first one is available online at http://kam.mff.cuni.cz/~bernard/packing We include the full source code as well as descriptions of inputs and outputs. We were checking the outputs of both programs during the computation and we verified that they match. The total number of checked configurations was 43112312093324. The computation took about 120 days of computing time on a single core workstation in year 2009.

The procedure fails to color the matrix 15 × 9 with 9 at position [5, 5]. Hence we conclude that the packing chromatic number for the square lattice is at least 12.

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