1. Introduction

The goal of spectral graph theory is to study structural properties of graphs by means of eigenvalues and eigenvectors of matrices associated with it. Researchers, motivated by the success of this theory, have studied many hypergraph matrices aiming to develop a spectral hypergraph theory. See for example [1, 11, 13, 14, 15, 16, 17, 18, 19]. In 2012 Cooper and Dutle presented a new approach and, in their paper [4], the authors proposed the study of hypergraphs through tensors, causing a revolution in this area and, consequently, the study of hypergraph from its matrices has been put aside. Because determining the spectrum of a tensor has a high computational (as well as theoretical) cost, the application of this theory has its toll. Therefore, we believe that the study of hypergraphs through matrices remains important.

Let $\mathcal{H}$ be a hypergraph whose incidence matrix is $B(\mathcal{H})$. The signless Laplacian matrix of $\mathcal{H}$ is defined as $Q(\mathcal{H}) = BB^T$. The main goal of this paper is the study of this matrix. We say that the eigenvalues of $Q$ are the signless Laplacian eigenvalues of $\mathcal{H}$. The matrix $Q$ has many interesting properties such as being symmetric, non-negative, semi-definite positive and irreducible. Thus, important theorems such as Perron-Frobenius and Rayleigh Principle can be inherited directly from matrix theory. In this paper, we prove generalizations of some results that this matrix has in the context of graphs and, consequently, it is possible to determine structural properties of the hypergraph from $Q$. For example, we show that the number of edges of the hypergraph can be determined from the sum of its signless Laplacian eigenvalues. We also show that the number of distinct eigenvalues of $Q$ is larger than the diameter of the hypergraph. The spectral radius is bounded by the degrees of the hypergraph and the chromatic number is bounded from the spectral radius. We also show how to determine whether a hypergraph is regular by analyzing its spectral radius, or its principal eigenvector.

One of the most important properties of the signless Laplacian matrix in the context of spectral graph theory is the relation between the eigenvalue zero and the existence of bipartite components in the graph. See Proposition 2.1 of [6]. In an attempt to obtain
similar results, we study the signless Laplacian eigenvalue zero of a hypergraph. We
establish the following result.

**Theorem 1.** Let \( H = (V, E) \) be a \( k \)-graph. If \( \lambda = 0 \) is an eigenvalue of \( Q(H) \), then \( H \) is partially bipartite.

For the definition of a partially bipartite hypergraph see Section 5. The converse
of Theorem 1 is not true. For example \( H = (\{1, 2, 3, 4\}, \{123, 124, 134, 234\}) \), has a
partially bipartition \( V_1 = \{1, 2\}, V_2 = \{3, 4\} \) and \( V_0 = \emptyset \), but the eigenvalues of \( Q \)
are \( \rho = 9 \) and \( \lambda = 1 \) with multiplicity 3. In view of this, we leave here the following
question.

**Question 1.1.** How to characterize uniform hypergraphs with signless Laplacian eigen-
value zero?

As an application of our developed theory, we also study the spectrum of the signless
Laplacian matrix of the class of hypergraphs called power hypergraphs (see definition
in Section 7). We show how to construct the whole spectrum of the power hypergraph
from the signless Laplacian eigenvalues of its base hypergraph.

The remaining of the paper is organized as follows. In Section 2 we present some
basic definitions about hypergraphs and matrices. In Section 3 we study the incidence
matrix and exploit some properties of line and clique multigraphs. In Section 4 we
study the signless Laplacian matrix, extending many classical results of this matrix
to the context of hypergraphs. In Section 5 we study structural characteristics of a
hypergraph such as being regular or partially bipartite, analyzing its signless Laplacian
eigenvalues. In Section 6 we correlate classical and spectral parameters of a hypergraph,
such as chromatic number and diameter, with spectral radius and number of distinct
eigenvalues. In Section 7 we study the spectrum of the signless Laplacian matrix of a
power hypergraph.

2. Preliminaries

In this section, we shall present some basic definitions about hypergraphs and ma-
trices, as well as terminology, notation and concepts that will be useful in our proofs.
More details about hypergraphs can be found in [2].

A **hypergraph** \( H = (V, E) \) is a pair composed by a set of vertices \( V(H) \) and a set
of (hyper)edges \( E(H) \subseteq 2^V \), where \( 2^V \) is the power set of \( V \). \( H \) is said to be a \( k \)-
uniform (or a \( k \)-graph) for \( k \geq 2 \), if all edges have cardinality \( k \). Let \( H = (V, E) \) and
\( H' = (V', E') \) be hypergraphs, if \( V' \subseteq V \) and \( E' \subseteq E \), then \( H' \) is a subgraph of \( H \).

The **neighborhood** of a vertex \( v \in V(H) \), denoted by \( N(v) \), is the multi-set formed
by all vertices, distinct from \( v \), that have some edge in common with \( v \), where the
multiplicity of each element in the multi-set is exactly the number of edges in common
with the vertex \( v \). The **edge neighborhood** of a vertex \( v \in V \), denoted by \( E_{[v]} \), is the set
of all edges that contain \( v \). More precisely, \( E_{[v]} = \{ e : v \in e \in E \} \).

The **degree** of a vertex \( v \in V \), denoted by \( d(v) \), is the number of edges that contain \( v \). More precisely, \( d(v) = |E_{[v]}| \). A hypergraph is **\( r \)-regular** if \( d(v) = r \) for all \( v \in V \).

We define the **maximum**, **minimum** and **average** degrees, respectively, as

\[
\Delta(H) = \max_{v \in V} \{d(v)\}, \quad \delta(H) = \min_{v \in V} \{d(v)\}, \quad d(H) = \frac{1}{n} \sum_{v \in V} d(v).
\]

When we are working with more than one hypergraph, we can use the notation \( d_H(v) \),
to avoid ambiguity.
Let $\mathcal{H}$ be a hypergraph. A walk of length $l$ is a sequence of vertices and edges $v_0e_1v_1e_2\ldots e_lv_l$ where $v_{i-1}$ and $v_i$ are distinct vertices contained in $e_i$ for each $i = 1, \ldots, l$. The distance between two vertices is the length of the shortest walk connecting these two vertices. The diameter of the hypergraph is the largest distance between two of its vertices. The hypergraph is connected, if for each pair of vertices $u, w$ there is a walk $v_0e_1v_1e_2\ldots e_lv_l$ where $u = v_0$ and $w = v_l$. Otherwise, the hypergraph is disconnected.

Let $\mathcal{G}$ and $\mathcal{H}$ be $k$-graphs. We define its union $\mathcal{G} \cup \mathcal{H}$ as the $k$-graph, with the sets of vertices $V(\mathcal{G} \cup \mathcal{H}) = V(\mathcal{G}) \cup V(\mathcal{H})$ and edges $E(\mathcal{G} \cup \mathcal{H}) = E(\mathcal{G}) \cup E(\mathcal{H})$. The cartesian product $\mathcal{G} \times \mathcal{H}$ is the $k$-graph, with the sets of vertices $V(\mathcal{G} \times \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$ and edges $E(\mathcal{G} \times \mathcal{H}) = \{(v, e) : v \in V(\mathcal{G}), e \in E(\mathcal{H})\} \cup \{a \times \{u\} : u \in V(\mathcal{H}), a \in E(\mathcal{G})\}$.

A multigraph is an ordered pair $\mathcal{G} = (V, E)$, where $V$ is a set of vertices and $E$ is a multi-set of pairs of distinct, unordered vertices, called edges. Its adjacency matrix $A(\mathcal{G})$, is the square matrix of order $|V|$, where $a_{ii} = 0$ and if $i \neq j$, then $a_{ij}$ is the number of edges connecting the vertices $i$ and $j$.

Let $M$ be a square matrix of order $n$. We denote its characteristic polynomial by $P_M(\lambda) = \det(\lambda I_n - M)$. Its eigenvalues will be denoted by $\lambda_1(M) \geq \cdots \geq \lambda_n(M)$. If $x$ is an eigenvector from eigenvalue $\lambda$, then the pair $(\lambda, x)$ will be called eigenpair of $M$. The spectral radius $\rho(M)$, is the largest modulus of an eigenvalue.

### 3. Incidence Matrix, Clique and Line Multigraphs

In this section, we will study the incidence matrix of a hypergraph. More specifically, we will analyze the relationship of this matrix with two multigraphs associated with it: the line and clique multigraphs. The results of this section are generalizations of well-known properties of the incidence matrix and line graphs [5, 7].

**Definition 3.1.** Let $\mathcal{H} = (V, E)$ be a hypergraph. The incidence matrix $B(\mathcal{H})$ is defined as the matrix of order $|V| \times |E|$, where $b(v, e) = 1$ if $v \in e$ and $b(v, e) = 0$ otherwise. Its matrix of degrees $D(\mathcal{H})$, is a square matrix of order $|V|$, where $d_{ii} = d(i)$ and if $i \neq j$, then $d_{ij} = 0$.

The clique multigraph $\mathcal{C}(\mathcal{H})$, is obtained by transforming the vertices of $\mathcal{H}$ in its vertices. The number of edges between two vertices of this multigraph is equal the number of hyperedges containing them in $\mathcal{H}$. The line multigraph $\mathcal{L}(\mathcal{H})$, is obtained by transforming the hyperedges of $\mathcal{H}$ in its vertices. The number of edges between two vertices of this multigraph is equal the number of vertices in common in the two respective hyperedges.

**Example 3.2.** The clique and line multigraphs from $\mathcal{H} = (\{1, \ldots, 5\}, \{123, 145, 345\})$, are illustrate in Figure 1.

Our first result is the following observation. We believe it is worth mentioning because it opens the possibility of studying hypergraphs from the spectrum of multigraphs.

**Theorem 2.** Let $\mathcal{H}$ be a $k$-graph, $B$ its incidence matrix, $D$ its degree matrix, $A_L$ and $A_C$ the adjacency matrices of its line and clique multigraphs, respectively. Then

$$B^T B = kI + A_L, \quad \text{and} \quad BB^T = D + A_C.$$  

**Proof.** Let $C = B^T B$. Note that $c_{ij}$ is the number of vertices in common between the hyperedges $e_i$ and $e_j$. So, if $i \neq j$, then $c_{ij}$ is the number of edges between the vertices $i$ and $j$ in the line multigraph $\mathcal{L}(\mathcal{H})$, otherwise $c_{ii} = k$. Therefore, we conclude $C = kI + A_L$. 


Now, let $M = BB^T$. Note that $m_{ij}$ is the number of hyperedges that contains at the same time the vertices $i$ and $j$. So we have $m_{ii} = d(i)$ for all $i \in V$, and if $i \neq j$, then $m_{ij}$ is the number of edges between the vertices $i$ and $j$ in the clique multigraph $\mathcal{C}(\mathcal{H})$. Therefore, we conclude $M = D + A_c$. □

**Proposition 3.** If $\mathcal{H}$ is a $k$-graph, $r$-regular, with $n$ vertices and $m$ edges, then

$$P_{A_c}(\lambda) = (\lambda + k)^{m-n}P_{A_c}(\lambda - r + k).$$

**Proof.** Let $B$ be the incidence matrix of $\mathcal{H}$. Consider the following matrices.

$$U = \begin{bmatrix} \lambda I_n & 0 \\ 0 & -B \end{bmatrix}, \quad V = \begin{bmatrix} I_n & B \\ B^T & \lambda I_m \end{bmatrix} \Rightarrow UV = \begin{bmatrix} \lambda I_n - BB^T & 0 \\ B^T & \lambda I_m \end{bmatrix}, \quad VU = \begin{bmatrix} \lambda I_n - BB^T \\ B^T & \lambda I_m - B^T B \end{bmatrix}.$$  

We know that $\det(VU) = \det(UV)$. So,

$$\lambda^n \det(\lambda I_m - BB^T) = \lambda^m \det(\lambda I_n - BB^T).$$  \hspace{1cm} (1)

Thus,

$$P_{A_c}(\lambda) = \det(\lambda I_m - A_e) = \det((\lambda + k)I_m - B^T B)$$

$$= (\lambda + k)^{m-n} \det((\lambda + k)I_n - BB^T)$$

$$= (\lambda + k)^{m-n} \det((\lambda + k - r)I_n - A_c)$$

$$= (\lambda + k)^{m-n} P_{A_c}(\lambda - r + k).$$

Therefore, the result follows. □

**Lemma 4.** Let $\mathcal{H}$ be a $k$-graph and $\mathcal{L}(\mathcal{H})$ its line graph. If $u \in V(\mathcal{L}(\mathcal{H}))$ is a vertex obtained from the edge $e \in E(\mathcal{H})$, then

$$d_L(u) = \left( \sum_{v \in e} d_H(v) \right) - k.$$

**Proof.** Notice that, for each $v \in e$, there exist other $d_H(v) - 1$ hyperedges containing it. That is, this vertex will generate $d_H(v) - 1$ edges containing $u$ in $\mathcal{L}(\mathcal{H})$. Using the same argument for the other vertices of $e$, we conclude that the degree of the vertex $u$ in the line multigraph, must be $d_L(u) = \sum_{v \in e} (d_H(v) - 1)$.

\hspace{1cm} □

## 4. Signless Laplacian matrix

In this section, we study some properties of the signless Laplacian matrix of a hypergraph, generalizing important results of this matrix in the context of spectral graph theory, whose main results may be found in the series of papers by Cvetković, Rowlinson and Simić [6, 8, 9, 10], and references therein.
Definition 4.1. Let $\mathcal{H}$ a hypergraph and $B$ its incidence matrix. The signless Laplacian matrix is defined as $Q(\mathcal{H}) = BB^T$.

An oriented hypergraph $H = (\mathcal{H}, \sigma)$ is a hypergraph where for each vertex-edge incidence $(v, e)$ it is given a label $\sigma(v, e) \in \{+1, -1\}$. In [16], Reef and Rusnak define the incidence matrix of an oriented hypergraph $B(H)$ by $b(v, e) = \sigma(v, e)$ if $v \in e$ and $b(v, e) = 0$ otherwise. The Laplacian matrix for oriented hypergraphs is defined as, $L(H) = BB^T$. We observe that if $\sigma(v, e) = 1$ for all vertex-edge incidence $(v, e)$, then this definition coincides with the our definition of signless Laplacian matrix.

Remark 4.2. Let $\mathcal{H}$ be a $k$-graph, and $Q$ its signless Laplacian matrix. This matrix has some simple but useful properties, such as being symmetric, non-negative and positive semi-definite. Further, if $\mathcal{H}$ is connected, then $Q$ is irreducible. These properties allow us to conclude directly from matrix theory the Rayleigh principle and Perron-Frobenius Theorem, stated below.

Theorem 5 (Rayleigh principle for hypergraphs). Let $\mathcal{H}$ be a $k$-graph. If $\lambda_1$ is the largest eigenvalue of $Q$, then

$$\lambda_1 = \rho(Q) = \max_{||x|| = 1} \{x^TQx\} \geq 0.$$  

Further, the equality is achieved if and only if $x$ is an eigenvector of $\lambda_1$.

Theorem 6 (Perron-Frobenius Theorem for hypergraphs). Let $\mathcal{H}$ be a $k$-graph. If $\mathcal{H}$ is connected, then $\rho(Q)$ is an algebraically simple eigenvalue, with a positive eigenvector.

We finish this section proving some basic properties of the signless Laplacian matrix for uniform hypergraphs.

Lemma 7. Let $\mathcal{H}$ be a $k$-graph and $Q = (q_{ij})$ its signless Laplacian matrix. Then, for each $i \in V$, we have

$$\sum_{j \in V} q_{ij} = kd(i).$$

Proof. By the characterization of signless Laplacian matrix of Theorem 2, we have

$$\sum_{j \in V} q_{ij} = d(i) + \sum_{j \in N(i)} 1 = d(i) + (k - 1)d(i) = kd(i).$$

Proposition 8. If $\mathcal{H}$ is a $k$-graph with $n$ vertices and $m$ edges, then

$$P_{A_L}(\lambda) = (\lambda + k)^{m-n}P_Q(\lambda + k).$$

Proof. Notice that, by equation (1), we have

$$P_{A_L}(\lambda) = \det(\lambda I_m - A_L) = \det((\lambda + k)I_m - BB^T) = (\lambda + k)^{m-n} \det((\lambda + k)I_n - B^T B) = (\lambda + k)^{m-n}P_Q(\lambda + k).$$

Therefore, the result follows.

Remark 4.3. Here we highlight two interesting consequences of the Proposition 8. First, if $\lambda$ is an eigenvalue of $A_L$, then $\lambda \geq -k$. Second, we see that $\rho(A_L) = \rho(Q) - k$. 
**Proposition 9.** Let $G$ and $G'$ be two $k$-graphs. If $H = G \cup G'$, then

$$\text{Spec}(Q(H)) = \text{Spec}(Q(G)) \cup \text{Spec}(Q(G')).$$

**Proof.** If we first enumerate all the vertices in $G$ and then the vertices in $G'$, the signless Laplacian matrix of $H$, will have the following form $Q(H) = \begin{pmatrix} Q(G) & 0 \\ 0 & Q(G') \end{pmatrix}$. Thus, $\det(\lambda I - Q(H)) = \det(\lambda I - Q(G))\det(\lambda I - Q(G'))$. So, the result follows. \hfill \Box

We now introduce the following notation. Let $H = (V, E)$ be a hypergraph, for each non-empty subset of vertices $\alpha = \{v_1, \ldots , v_t\} \subset V$, given a vector $x = (x_i)$ of dimension $n = |V|$, we denote $x(\alpha) = x_{v_1} + \cdots + x_{v_t}$. Under these conditions we can write,

$$(Qx)_u = d(u)x_u + \sum_{w \in N(u)} x_w = \sum_{e \in E[u]} x(e).$$

**Proposition 10.** If $G$ and $H$ are two $k$-graphs, with signless Laplacian eigenvalues $\mu$ of multiplicity $m_1$ and $\lambda$ of multiplicity $m_2$ respectively, then $\mu + \lambda$ is an eigenvalue of $Q(G \times H)$, with multiplicity $m_1 \cdot m_2$.

**Proof.** Suppose $x$ an eigenvector of $\lambda$ in $Q(H)$ and $y$ an eigenvector of $\mu$ in $Q(G)$. Consider $(v, u)$ a vertex of $G \times H$, define a vector $z$ by $z_{(v,u)} = y_v x_u$. Thus,

$$(Qz)_{(v,u)} = \sum_{\alpha \in E[(v,u)]} z(\alpha) = \sum_{e \in E[u]} y_v x(e) + \sum_{a \in E[v]} y(e) x_u = \lambda y_v x_u + \mu y_v x_u = (\mu + \lambda) z_{(v,u)}.$$ 

Therefore, the result is true. \hfill \Box

The result below, may be seen as a corollary of Proposition 4.4 of [14].

**Proposition 11.** Let $H$ be a $k$-graph with $n$ vertices. For each vector $x \in \mathbb{R}^n$, we have

$$x^T Q x = \sum_{e \in E} [x(e)]^2.$$ 

**Proof.** Notice that, for each edge $e \in E$ is true that $(B^T x)_e = x(e)$. Therefore,

$$x^T Q x = x^T (BB^T) x = (B^T x)^T (B^T x) = \sum_{e \in E} [x(e)]^2.$$ 

\hfill \Box

**Proposition 12.** Let $H$ be a connected $k$-graph. If $H'$ is a subgraph of $H$, then

$$\rho(H') \leq \rho(H).$$

**Proof.** Let $x$ be the principal eigenvector of $Q(H')$. Define a new vector $\overline{x}$ of dimension $n = |V(H)|$, by $\overline{x}_i = x_i$ if $i \in V(H')$ and $\overline{x}_i = 0$ otherwise. Thus,

$$\rho(H) \geq \sum_{e \in E(H)} [\overline{x}(e)]^2 = \sum_{e \in E(H')} [x(e)]^2 = \rho(H').$$  

\hfill \Box
5. Structural and spectral properties

In this section, we will determine structural characteristics of a hypergraph from its signless Laplacian spectrum. More precisely, we will study regular and partially bipartite uniform hypergraphs through their signless Laplacian eigenvalues.

**Theorem 13.** Let $\mathcal{H}$ be a connected $k$-graph. The following statements are equivalent

(a) $\mathcal{H}$ is regular.
(b) $\rho(\mathcal{H}) = kd(\mathcal{H})$.
(c) $\rho(\mathcal{H}) = k\Delta(\mathcal{H})$.
(d) The principal eigenvector of $Q(\mathcal{H})$ is $x = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$, where $n = |V(\mathcal{H})|$.

**Proof.** We will prove the result through the following chain of implications,

(a) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (a).

Suppose $\mathcal{H}$ is $r$-regular, then for each vertex $u$ we have $|E[u]| = r$. Thus,

$$(Q1)_u = \sum_{e \in E[u]} x(e) = kr(1).$$

That is, $1$ is an eigenvector associated with the eigenvalue $kr$, and since $\mathcal{H}$ is regular, then $r = d(\mathcal{H})$. By Perron-Frobenius Theorem 6, we conclude that $\rho(\mathcal{H}) = kd(\mathcal{H})$.

Now, suppose $\rho(\mathcal{H}) = kd(\mathcal{H})$. We notice that the vector $x = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$ solves the following optimization problem.

$$\rho(\mathcal{H}) = \max_{|y|=1} \{y^TQy\} \geq x^TQx = \sum_{i \in V} \frac{kd(i)}{n} = kd(\mathcal{H}).$$

Thus, by Rayleigh principle 5, we conclude that $x$ is the principal eigenvector of $Q$.

Let $x = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$ be the principal eigenvector of $Q$. If $u \in V$ is a vertex of maximum degree, then

$$\rho(\mathcal{H}) \left(\frac{1}{\sqrt{n}}\right) = (Qx)_u = \sum_{e \in E[u]} x(e) = \Delta k \left(\frac{1}{\sqrt{n}}\right), \quad \Rightarrow \quad \rho(\mathcal{H}) = k\Delta(\mathcal{H}).$$

If $\rho(\mathcal{H}) = k\Delta(\mathcal{H})$ is the spectral radius of $Q$ and $x$ its principal eigenvector. Let $u \in V$ be a vertex, such that $x_u \geq x_v$ for all $v \in V$. Thus,

$$kd(x)_u = \sum_{e \in E[u]} x(e).$$

We observe that this equality is only possible if, $d(u) = \Delta$ and $x_v = x_u$ for all $v \in N(u)$. Hence, we conclude that every vertex that has maximum value in the eigenvector $x$, must have maximum degree. Moreover, every vertex that is neighbor of another vertex that has maximum value in the eigenvector, must also have maximum value. By the connectivity of the hypergraph, we conclude that all the vertices have maximum value in the principal eigenvector and therefore maximum degree, i.e. $\mathcal{H}$ is regular. \qed

**Lemma 14.** Let $\mathcal{H}$ be a $k$-graph. Thus, $(0, x)$ is an eigenpair of $Q(\mathcal{H})$ if, and only if, for each edge $e \in E$ we have $x(e) = 0$.

**Proof.** If $(0, x)$ is a signless Laplacian eigenpair of $\mathcal{H}$, then $Qx = 0$. So,

$$x^TQx = x^T0 = 0 \Rightarrow \sum_{e \in E} [x(e)]^2 = 0 \Rightarrow x(e) = 0, \quad \forall e \in E.$$
Conversely, let \( x \) be a vector of dimension \( n = |V(\mathcal{H})| \), such that \( x(e) = 0 \), for each edge \( e \in E \). So,
\[
(Qx)_u = \sum_{e \in E[u]} x(e) = 0, \quad \forall u \in V \quad \Rightarrow \quad (0, x) \text{ is an eigenpair of } Q(\mathcal{H}).
\]

We notice that for a graph, the condition \( x_{v_i} + x_{v_j} = 0 \) for all \( e = \{v_i, v_j\} \in E \), implies a bipartition of vertices. Unfortunately for \( k \geq 3 \), we do not have such a trivial characterization.

**Definition 5.1.** A hypergraph \( \mathcal{H} \) is partially bipartite, if we can separate the set of vertices into three disjoint subsets \( V = V_0 \cup V_1 \cup V_2 \), where \( V_1 \) and \( V_2 \) are non empty and each edge is fully contained in \( V_0 \) or has vertices in both \( V_1 \) and \( V_2 \).

Now, we prove Theorem 1. We state here again for easy reference.

**Theorem 1.** Let \( \mathcal{H} = (V, E) \) be a \( k \)-graph. If \( \lambda = 0 \) is an eigenvalue of \( Q \), then \( \mathcal{H} \) is partially bipartite.

**Proof.** Let \( x \) be a eigenvector of \( \lambda = 0 \). Define
\[
V_1 = \{v \in V : x_v > 0\}, \quad V_2 = \{v \in V : x_v < 0\}, \quad V_0 = \{v \in V : x_v = 0\}.
\]
As \( x(e) = 0 \) for each edge \( e \in E \), then the edge is contained in \( V_0 \), or it must to have some vertices in \( V_1 \) and others in \( V_2 \), i.e. \( \mathcal{H} \) is partially bipartite.

**Definition 5.2.** A hypergraph \( \mathcal{H} \) is balanced partially bipartite, if it is partially bipartite and there exists a constant \( c > 0 \), such that for each edge \( e \not\subseteq V_0 \), it happens \( |e \cap V_1| = |e \cap V_2| = c \).

**Theorem 15.** Let \( \mathcal{H} = (V, E) \) be a \( k \)-graph. If it is balanced partially bipartite, then \( \lambda = 0 \) is an eigenvalue of \( Q(\mathcal{H}) \).

**Proof.** Since \( \mathcal{H} \) is balanced partially bipartite, there is a constant \( c = \frac{|e \cap V_1|}{|e \cap V_2|} \), where \( e \in E(\mathcal{H}) \). So we define a vector \( x \) of dimension \( n = |V| \), by
\[
x_v = \begin{cases} 
1 & \text{if } v \in V_1 \\
0 & \text{if } v \in V_0 \\
-c & \text{if } v \in V_2 
\end{cases}
\]
Thus \( x(e) = 0 \), for each edge \( e \in E \). By Lemma 14, we conclude the result.

**Example 5.3.** We illustrate here that balanced partially bipartite graphs are abundant, by presenting two examples that are easy to find. Let \( \mathcal{H} = (V, E) \) be a \( k \)-graph in which

1. \( \mathcal{H} \) has a vertex \( v \) with the property that each edge containing it also contains another vertex of degree 1;
2. \( \mathcal{H} \) has a couple of vertices which are contained in exactly the same edges;
In both cases \( \mathcal{H} \) is balanced partially bipartite.

6. RELATING CLASSICAL AND SPECTRAL PARAMETERS

In this section, we will relate classic and spectral parameters of a hypergraph. More precisely, we will relate the spectral radius to the degrees and the chromatic number, the number of edges is related to the sum of the eigenvalues, and the diameter is related to the number of distinct eigenvalues of the signless Laplacian matrix.
Theorem 16. If $\mathcal{H}$ is a connected $k$-graph and $\rho(\mathcal{Q})$ is its spectral radius, then
\[
\min_{e \in E} \left\{ \sum_{v \in e} d(v) \right\} \leq \rho(\mathcal{Q}) \leq \max_{e \in E} \left\{ \sum_{v \in e} d(v) \right\}.
\]

Proof. For each $u \in V(\mathcal{L}(\mathcal{H}))$, suppose $e_u \in E(\mathcal{H})$ is the edge that gives vertex $u$. By Lemma 4, we have $d_{\mathcal{L}}(u) = \left( \sum_{v \in e_u} d_{\mathcal{H}}(v) \right) - k$. Now, by Theorem 8, we have $\rho(A_{\mathcal{L}}) = \rho(\mathcal{Q}) - k$. For graphs and multigraphs, we know
\[
\min_{u \in V(\mathcal{L}(\mathcal{H}))} d_{\mathcal{L}}(u) \leq \rho(A_{\mathcal{L}}) \leq \max_{u \in V(\mathcal{L}(\mathcal{H}))} d_{\mathcal{L}}(u).
\]
Therefore,
\[
\min_{\{v_1,\ldots,v_k\} \in E} \left\{ \left( \sum_{i=1}^{k} d(v_i) \right) - k \right\} \leq \rho(\mathcal{Q}) - k \leq \max_{\{v_1,\ldots,v_k\} \in E} \left\{ \left( \sum_{i=1}^{k} d(v_i) \right) - k \right\}.
\]
Adding $k$ in each of the three parts of the inequalities, we obtain the desired result. □

The result below may be seen as a corollary of Propositions 4.7 and 4.12 in [14].

Corollary 17. If $\mathcal{H}$ is a connected $k$-graph and $\rho(\mathcal{H})$ is its spectral radius, then
\[
k d(\mathcal{H}) \leq \rho(\mathcal{H}) \leq k \Delta(\mathcal{H}).
\]

Proof. If $n = |V|$, define $x = \left( \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)$. By Theorems 5 and 16, we have
\[
k d(\mathcal{H}) = x^T \mathcal{Q} x \leq \rho(\mathcal{H}) \leq \max_{e \in E} \left\{ \sum_{v \in e} d(v) \right\} \leq k \Delta(\mathcal{H}).
\]
□

Definition 6.1. For a $k$-graph $\mathcal{H}$, a function $f : V \to \{1, \ldots, r\}$ is a (vertex) $r$-coloring of $\mathcal{H}$, if for every edge $e = \{v_1, \ldots, v_k\}$ there exists $i \neq j$ such that $f(v_i) \neq f(v_j)$. The chromatic number $\chi(\mathcal{H})$, is the minimum integer $r$ such that $\mathcal{H}$ has an $r$-coloring.

Theorem 18. Let $\mathcal{H}$ be a connected $k$-graph. If $\chi(\mathcal{H})$ is its chromatic number, then
\[
\chi(\mathcal{H}) \leq \frac{1}{k} \rho(\mathcal{H}) + 1.
\]

Proof. We will define an order for the vertices of $\mathcal{H}$ as follows. Let $\mathcal{H}(n) = \mathcal{H}$ and $v_n$ be a vertex of minimum degree in $\mathcal{H}(n)$. For each $t = 2, \ldots, n$, let $\mathcal{H}(t-1)$ be the subgraph obtained after removing a vertex, $v_t$ with minimum degree from $\mathcal{H}(t)$.

Let us use the ordering $v_1, v_2, \ldots, v_n$, as input of a greedy coloring algorithm, which paints $v_t$ with the smallest color that makes $\mathcal{H}(t)$ properly colored.

Notice that $\chi(\mathcal{H}(1)) = 1 \leq \frac{1}{k} \rho(\mathcal{H}) + 1$. Inductively, suppose $\mathcal{H}(t-1)$ is properly colored with up to $\frac{1}{k} \rho(\mathcal{H}) + 1$ distinct colors. We see that $v_t$ has a minimum degree in $\mathcal{H}(t)$. Thus, in the worst case, each edge containing $v_t$ has all the other vertices painted with the same color, and each of these edges uses one of the colors $1, 2, \ldots, d_{\mathcal{H}(t)}(v_t)$. So we should paint $v_t$ with the color $d_{\mathcal{H}(t)}(v_t) + 1$. Thus,
\[
d_{\mathcal{H}(t)}(v_t) + 1 = \delta(\mathcal{H}(t)) + 1 \leq \frac{1}{k} \rho(\mathcal{H}(t)) + 1 \leq \frac{1}{k} \rho(\mathcal{H}) + 1.
\]
By induction hypothesis, we have $\chi(\mathcal{H}(t)) \leq \frac{1}{k} \rho(\mathcal{H}) + 1$. So, $\chi(\mathcal{H}) \leq \frac{1}{k} \rho(\mathcal{H}) + 1$. □

Proposition 19. Let $\mathcal{H}$ be a $k$-graph with characteristic polynomial $P_{\mathcal{Q}}(\lambda) = \lambda^n + q_1 \lambda^{n-1} + \cdots + q_{n-1} \lambda + q_n$. The number of edges of of $\mathcal{H}$ is given by $m = -\frac{n}{k}$. 
Proof. If \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) are all eigenvalues of the matrix \( Q \), then

\[
q_1 = -(\lambda_1 + \cdots + \lambda_n) = -Tr(Q) = -km.
\]

\(
\square
\)

**Theorem 20.** Let \( H \) be a \( k \)-graph with diameter \( D \). The number of distinct eigenvalues of the matrix \( Q \) is at least \( D + 1 \).

**Proof.** First we will show the following claim.

**Claim 6.2.** If there is a walk with length \( l \) connecting two distinct vertices \( i \) and \( j \), then \( (Q')_{ij} > 0 \), otherwise \( (Q')_{ij} = 0 \).

The proof is by induction on \( l \). We first notice that if \( l = 1 \) then the signless Laplacian matrix has the desired properties. Now suppose the statement is true for \( l \geq 1 \). Note that,

\[
(Q^{l+1})_{ij} = \sum_{t=1}^{n} (Q^t)_{ii}(Q^t)_{jj}.
\]

Thus, if there is no walk with length \( l + 1 \), linking \( i \) and \( j \), then there can be no walk linking \( i \) to a neighbor of \( j \). This implies that, if \( u \) is a neighbor of \( j \) then \((Q')_{iu} = 0\) and otherwise \((Q)_{uj} = 0\). Therefore, \((Q^{l+1})_{ij} = 0\). On the other hand, assuming there is a walk with length \( l + 1 \), linking \( i \) and \( j \), then there must be a walk with length \( l \), linking \( i \) to a neighbor \( u \) of \( j \). So, \((Q]^l_{ii} > 0\) and \((Q)_{uj} > 0\). Therefore, \((Q^{l+1})_{ij} > 0\).

The claim is proven.

Returning to the proof of the theorem, we let \( \lambda_1, \lambda_2, \ldots, \lambda_r \) be all the distinct eigenvalues of \( Q \). So, \((Q - \lambda_1 I)(Q - \lambda_2 I) \cdots (Q - \lambda_r I) = 0\). Thus, \( Q' + a_1 Q'^{-1} + \cdots + a_r I = 0 \). Suppose, by way of contradiction that \( D \geq t \). Hence, there must exist \( i, j \) such that its distance is \( t \). Thus, \((Q')_{ij} = -a_1(Q'^{-1})_{ij} - \cdots - a_r (I)_{ij} = 0\), because there should be no walk shorter than \( t \) linking the vertices \( i \) and \( j \). This contradicts the claim. Therefore \( t \geq D + 1 \).

\(
\square
\)

**Theorem 21.** Let \( H \) be a \( k \)-graph with more than one edge. If the diameter of \( H \) is \( D \), then

\[
D \leq \left[ 1 + \frac{\log((1 - x_{\min}^2)/x_{\min}^2)}{\log(\lambda_1/\lambda_2)} \right],
\]

where \( \lambda_1 > \lambda_2 \) are the greatest eigenvalues of \( Q \) and \( x_{\min} \) is the smallest entry of the principal eigenvector.

**Proof.** As \( Q \) is real symmetric we may consider the orthonormal eigenvectors \( x_1, \ldots, x_n \) from the eigenvalues \( \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \), respectively. In this case, \( x_1 \) is the principal eigenvector. Let \( i \) and \( j \) be vertices such that, its distance is \( D \). Using the spectral decomposition of \( Q \), for each integer \( t \), we have

\[
(Q')_{ij} = \sum_{l=1}^{n} \lambda_l^t (x_l x_l^T)_{ij} \geq \lambda_1^t (x_1)_{ij} - \sum_{l=2}^{n} \lambda_l^t (x_l x_l^T)_{ij}
\]

\[
\geq \lambda_1^t x_{\min}^2 - \lambda_2^t \left( \sum_{l=2}^{n} (x_l)_{l}^2 \right)^{\frac{1}{2}} \left( \sum_{l=2}^{n} (x_l)_{j}^2 \right)^{\frac{1}{2}}
\]

\[
\geq \lambda_1^t x_{\min}^2 - \lambda_2^t \left( 1 - (x_1)_{l}^2 \right)^{\frac{1}{2}} \left( 1 - (x_1)_{j}^2 \right)^{\frac{1}{2}}
\]

\[
\geq \lambda_1^t x_{\min}^2 - \lambda_2^t \left( 1 - x_{\min}^2 \right).
\]

(2)
Notice that, if the expression (2) is positive, then \((Q^t)_{ij}\) is positive and therefore \(t \geq D\).

\[
\lambda_1 t^2 x_{\min}^2 - \lambda_2 t (1 - x_{\min}^2) > 0 \quad \Rightarrow \quad t > \frac{\log((1 - x_{\min}^2)/x_{\min}^2)}{\log(\lambda_1/\lambda_2)}.
\]

Therefore, the result follows. \(\square\)

7. Power hypergraph

In this section, we will study the spectrum of the class of power hypergraphs, relating its signless Laplacian eigenvalues to those of its base hypergraph. The spectrum of this class has already been studied in the context of tensors. See for example [3, 12].

**Definition 7.1.** Let \(H = (V, E)\) be a \(k\)-graph, let \(s \geq 1\) and \(r \geq ks\) be integers. We define the (generalized) power hypergraph \(H_r^s\) as the \(r\)-graph with the following sets of vertices and edges

\[
V(H_r^s) = \left( \bigcup_{v \in V} \varsigma_v \right) \cup \left( \bigcup_{e \in E} \varsigma_e \right)
\]

and

\[
E(H_r^s) = \{ \varsigma_e \cup \varsigma_{v_1} \cup \cdots \cup \varsigma_{v_k} : e = \{v_1, \ldots, v_k\} \in E\},
\]

where \(\varsigma_v = \{v_1, \ldots, v_s\}\) for each vertex \(v \in V(H)\) and \(\varsigma_e = \{v^1_e, \ldots, v^{r-ks}_e\}\) for each edge \(e \in E(H)\).

Informally, we say that \(H_r^s\) is obtained from a base hypergraph \(H = (V, E)\), by replacing each vertex \(v \in V\) by a set \(\varsigma_v\) of cardinality \(s\), and by adding a set \(\varsigma_e\) with \(r - ks\) new vertices, to each edge \(e \in E\).

**Example 7.2.** The power hypergraph \((P_4)_2^5\) of the path \(P_4\) is illustrated in Figure 2.

![Figure 2. The power hypergraph \((P_4)_2^5\).](image)

Let \(H_r^s\) be a power hypergraph. For each edge \(e = \{i_1, \ldots, i_k\} \in E(H)\), we denote by \(e^*_r = \varsigma_{i_1} \cup \cdots \cup \varsigma_{i_k} \cup \varsigma_e \in E(H_r^s)\) the edge obtained from \(e \in E(H)\). For simplicity, we will write \(H^r = H_r^1\) and \(H_s = H_s^{ks}\). We identify a vertex in each of the sets \(\varsigma_v\) with the vertex \(v\), and say that it is a main vertex of \(H_r^s\), while the other vertices in \(\varsigma_v\) are called copies. The vertices in some of the sets \(\varsigma_e\) will be called additional vertices.

We start this section by proving some algebraic properties of this class.

**Lemma 22.** Let \(H\) be a \(k\)-graph having two vertices \(u\) and \(v\) which are contained exactly in the same edges. If \((\lambda, x)\) is an eigenpair of \(Q\) with \(\lambda > 0\), then \(x_u = x_v\).

**Proof.** We just notice that,

\[
\lambda x_u = \sum_{e \in E(u)} x(e) = \sum_{e \in E(v)} x(e) = \lambda x_v.
\]

Since \(\lambda \neq 0\), then the result is true. \(\square\)
Lemma 23. Let $\mathcal{H}$ be a $k$-graph and $r \geq k$ be an integer. Then, for each eigenpair $(\lambda, x)$ of $Q(\mathcal{H}^r)$, with $\lambda > r - k$, we have,

$$x_u = \frac{x(e)}{\lambda - r + k}, \text{ if } u \text{ is an additional vertex of edge } e^r \in E(\mathcal{H}^r).$$

Proof. Suppose $\zeta_e = \{i^1_e, \ldots, i^{r-k}_e\}$ and denote $u = i^1_e$. By Lemma 22 we know that $x_u = x_{i^1_e} \cdots x_{i^{r-k}_e}$. So, $\lambda x_u = x(e^r) = (r - k)x_u + x(e)$. Thus,

$$(\lambda - r + k)x_u = x(e) \quad \Rightarrow \quad x_u = \frac{x(e)}{\lambda - r + k}.$$

$\square$

Proposition 24. Let $\mathcal{H}$ be a $k$-graph. If $\mu \neq 0$ is a signless Laplacian eigenvalue of $\mathcal{H}$, then $\lambda = \mu + r - k$ is an eigenvalue of $Q(\mathcal{H}^r)$.

Proof. Suppose $y$ is an eigenvector of $Q(\mathcal{H})$, associated with $\mu$. Define a vector $x$ of dimension $|V(\mathcal{H}^r)|$, by

$$x_i = \begin{cases} y_i \\ \frac{y(e)}{\mu} \\ \end{cases} \text{ if } i \text{ is a main vertex,}$$

$$\text{if } i \text{ is an additional vertex of the edge } e^r.$$ 

If $u$ is a main vertex, we have

$$(Q(\mathcal{H}^r)x)_u = \sum_{e^r \in E(\mathcal{H}^r)|u|} x(e^r)$$

$$= \sum_{e \in E(\mathcal{H})|u|} \left( y(e) + (r - k) \frac{y(e)}{\mu} \right)$$

$$= \sum_{e \in E(\mathcal{H})|u|} y(e) + \left( \frac{r - k}{\mu} \right) \sum_{e \in E(\mathcal{H})|u|} y(e)$$

$$= (\mu + r - k)y_u = (\mu + r - k)x_u.$$ 

Now, if $u$ is an additional vertex, we have

$$(Q(\mathcal{H}^r)x)_u = x(e^r) = y(e) + (r - k) \frac{y(e)}{\mu} = (\mu + r - k) \frac{y(e)}{\mu} = (\mu + r - k)x_u.$$ 

Therefore, the result follows. $\square$

Lemma 25. Let $\mathcal{H}$ be a $k$-graph and $s \geq 1$ an integer. If $(\lambda, x)$ is a signless Laplacian eigenpair of $\mathcal{H}_s$, with $\lambda > 0$, then for each edge $e_s \in E(\mathcal{H}_s)$, we have $x(e_s) = sx(e)$.

Proof. By Lemma 22, we have $x(\zeta_u) = x_u + x_{u2} + \cdots + x_{us} = sx_u$. Hence,

$$x(e_s) = x(\zeta_{u1}) + \cdots + x(\zeta_{us}) = s(x_{u1} + \cdots + x_{us}) = sx(e).$$ 

$\square$

Proposition 26. Let $\mathcal{H}$ be a $k$-graph and $s \geq 1$ an integer. If $\mu \neq 0$ is a signless Laplacian eigenvalue of $\mathcal{H}$, then $\lambda = s\mu$ is an eigenvalue of $Q(\mathcal{H}_s)$.

Proof. Suppose $y$ is an eigenvector of $Q(\mathcal{H})$ associated with $\mu$. Define a vector $x$ of dimension $|V(\mathcal{H}_s)|$, by $x_u = y_u$, if $u \in \zeta_v$. Thus,

$$(Q(\mathcal{H}_s)x)_u = \sum_{e_s \in E(\mathcal{H}_s)|u|} x(e_s) = \sum_{e \in E(\mathcal{H})|u|} sx(e) = s\mu x_u.$$ 

$\square$
**Theorem 27.** Let $H$ be a $k$-graph, $s \geq 1$ and $r \geq ks$ be two integers. $\lambda > r - ks$ is an eigenvalue of $Q(H_s^r)$ if and only if there is a signless Laplacian eigenvalue $\mu > 0$ of $H$ such that $\lambda = s(\mu - k) + r$.

**Proof.** If $\mu$ is a signless Laplacian eigenvalue of $H$, then $s\mu$ is an eigenvalue of $Q(H_s)$. So, $\lambda = s\mu + r - ks = s(\mu - k) + k$ is a signless Laplacian eigenvalue of $(H_s)^r = H_s^r$.

Now, let $x$ be an eigenvector associated with $\lambda$ in $Q(H_s^r)$. Thus,

\[
\lambda x_u = \sum_{e^s \in E(H_s^r)_u} x(e^s)
\]

\[
= \sum_{e^s \in E(H_s^r)_u} (r - ks)x_{i^s} + x(e^s)
\]

\[
= \sum_{e^s \in E(H_s)_u} (r - ks) \frac{x(e^s)}{\lambda - r + ks} + x(e^s)
\]

\[
= \left( \frac{\lambda s}{\lambda + ks - r} \right) \sum_{e \in E(H)_{[u]}} x(e).
\]

Therefore,

\[
\sum_{e \in E(H)_{[u]}} x(e) = \frac{\lambda + ks - r}{s} x_u.
\]

That is, $H$ has a signless Laplacian eigenvalue $\mu$, such that

\[
\mu = \frac{\lambda + ks - r}{s} \Rightarrow \lambda = s(\mu - k) + r.
\]

\[
\square
\]

We notice that Theorem 27 characterizes all signless Laplacian eigenvalues greater than $r - ks$ of a power hypergraph $H_s^r$. Now we will study the other eigenvalues.

**Proposition 28.** Let $H$ be a $k$-graph. If $s \geq 1$ is an integer, then the multiplicity of $\lambda = 0$ as eigenvalue of $Q(H_s)$ is $s|V| - t$. Where $t$ is the rank of the matrix $Q(H)$.

**Proof.** If $z$ is an eigenvector of $\lambda = 0$ in $Q(H)$, define a new vector $x$ of dimension $|V(H_s)|$, by $x_v = z_u$ if $v \in s_u$. Notice that,

\[
(Q(H_s)x)_v = \sum_{e^s \in E(H_s)_{[v]}} x(e^s) = s \sum_{e \in E(H)_{[v]}} z(e) = 0.
\]

Hence, for each eigenvector of $\lambda = 0$ in $Q(H)$, we build one for $H_s$, i.e., we construct a family of $|V| - t$ linearly independent eigenvectors.

Now, for each $v \in V(H)$, suppose $s_v = \{v, v_2, \ldots, v_s\}$ and $2 \leq j \leq s$. We can construct the following family of $s - 1$ linearly independent vectors.

\[
x^j = \begin{cases} 
(x^j)_v = 1, \\
(x^j)_{v_j} = -1, \\
(x^j)_u = 0, & \text{for } u \in V(H_s) - \{v_1, v_j\}.
\end{cases}
\]

Notice that these vectors are eigenvectors of $\lambda = 0$ in $Q(H_s)$. Repeating this construction for the other main vertices of $H_s$, we obtain $(s - 1)|V|$ linearly independent eigenvectors. Observe that these vectors are linearly independent from those constructed from the zero eigenvectors of the base hypergraph $H$. To see this, we observe that the
former vectors have constant sign in each $\zeta_u$, while these new vectors have more than one sign in these sets. Therefore we have $s|V| - t$ linearly independent eigenvectors of $\lambda = 0$.

**Proposition 29.** Let $\mathcal{H}$ be a $k$-graph. If $s \geq 1$ and $r > ks$ are two integers, then the multiplicity of $\lambda = 0$ as eigenvalue of $Q(\mathcal{H}_s^r)$ is at least $(r - ks - 1)|E| + s|V|$.

**Proof.** Let $e \in E(\mathcal{H})$ be an edge, suppose $\zeta_e = \{u_1, \ldots, u_{r-ks}\}$ and $2 \leq j \leq r - ks$. Similarly to the proof of Proposition 28, we can construct the following family of $r - ks - 1$ linearly independent vectors.

$$y_j = \begin{cases} 
(y_j)_{u_1} = 1, \\
(y_j)_{u_j} = -1, \\
(y_j)_u = 0, & \text{for } u \in V(\mathcal{H}_s^r) - \{u_1, u_j\}. 
\end{cases}$$

Repeating this construction for the other edges of $\mathcal{H}$, we obtain $(r - ks - 1)|E|$ linearly independent eigenvectors, associated with $\lambda = 0$.

Now, let $w \in V(\mathcal{H}_s^r)$, and consider $e_1, \ldots, e_p$, all edges of $\mathcal{H}_s^r$ that contain the vertex $w$. For each of this, take $w_i \in e_i$ an additional vertex. So we can build the vector

$$z = \begin{cases} 
z_w = 1, \\
z_{w_i} = -1, & \text{for } 1 \leq i \leq p, \\
z_u = 0, & \text{for } u \in V(\mathcal{H}_s^r) - \{w, w_1, \ldots, w_p\}. 
\end{cases}$$

Repeating this construct for the other vertices of $\mathcal{H}_s^r$, we obtain $s|V|$ eigenvectors associated with $\lambda = 0$, linearly independent to each other and with the others previously created. Totalizing $(r - ks - 1)|E| + s|V|$ eigenvectors.

**Theorem 30.** Let $\mathcal{H}$ be a $k$-graph. If $s \geq 1$ and $r > ks$ are integers, then the multiplicity of $\lambda = r - ks$ as eigenvalue of $Q(\mathcal{H}_s^r)$ is $|E| - t$. Where $t$ is the rank of the signless Laplacian matrix $Q(\mathcal{H})$.

**Proof.** Firstly, note that $-k$ is an eigenvalue of multiplicity $|E| - t$ from $A_{\mathcal{L}}$. Let $z = (z_{\epsilon_1}, \ldots, z_{\epsilon_m})$ be an eigenvector of $-k$ in $A_{\mathcal{L}}$. Note that

$$\sum_{v \in V} \left( \sum_{e \in E_v} z_e \right)^2 = \mathbf{z}^T \mathbf{B}^T \mathbf{B} \mathbf{z} = 0, \quad \Rightarrow \quad \sum_{e \in E_v} z_e = 0, \quad \forall v \in V.$$

Now, define a vector $\mathbf{x}$ of dimension $|V(\mathcal{H}_s^r)|$, by

$$\mathbf{x} = \begin{cases} 
x_v = z_e, & \text{if } v \text{ is an additional vertice of } e, \\
x_v = 0, & \text{if } v \text{ is not an additional vertice.} 
\end{cases}$$

If $u$ is an additional vertice, then

$$(Q(\mathcal{H}_s^r)\mathbf{x})_u = x(e) = (r - ks)(z_e) = (r - ks)x_u.$$

If $u$ is a main or copy vertice, then

$$(Q(\mathcal{H}_s^r)\mathbf{x})_u = \sum_{e \in E_u} x(e) = \sum_{e \in E_u} z_e = 0 = (r - ks)x_u.$$ 

Therefore, the result follows.
Remark 7.3. If $H$ is a $k$-graph with $n$ vertices, $m$ edges having signless Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > \lambda_{t+1} = \cdots = \lambda_n = 0$, then the eigenvalues of $Q(H'_s)$ are $s(\lambda_1-k)+r, \ldots, s(\lambda_t-k)+r, r-ks$ with multiplicity $m-t$ and 0 with multiplicity $(r-ks-1)m+sn$.

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References

[1] Banerjee, A. On the spectrum of hypergraphs. arXiv:1711.09365v3 (2019).
[2] Bretto, A. Hypergraph Theory: An Introduction. Springer, 2013.
[3] Cardoso, K., Hoppen, C., and Trevisan, V. The spectrum of a class of uniform hypergraphs. arXiv:1909.00234 (2019).
[4] Cooper, J., and Dutle, A. Spectra of uniform hypergraphs. Linear Algebra Appl. 436 (2012), 3268–3292.
[5] Cvetković, D., Rowlinson, P., and Simić, S. Spectral Generalizations of Line Graphs: On graphs with least eigenvalue -2. Cambridge university press, 2004.
[6] Cvetković, D., Rowlinson, P., and Simić, S. Signless laplacians of finite graphs. Linear Algebra Appl. 423 (2007), 155171.
[7] Cvetković, D., Rowlinson, P., and Simić, S. An introduction to the theory of graph spectra. Cambridge university press, 2010.
[8] Cvetković, D., and Simić, S. Towards a spectral theory of graphs based on the signless laplacian, i. Publ. Inst. Math. (Beograd) 85 (2009), 19–33.
[9] Cvetković, D., and Simić, S. Towards a spectral theory of graphs based on the signless laplacian, ii. Linear Algebra Appl. 432 (2010), 2257–2272.
[10] Cvetković, D., and Simić, S. Towards a spectral theory of graphs based on the signless laplacian, iii. Appl. Anal. Discrete Math. 4 (2010), 156–166.
[11] Feng, K., Ching, W., and Li, W. Spectra of hypergraphs and applications. Journal of number theory 60 (1996), 1–22.
[12] Hu, S., Qi, L., and Shao, J. Cored hypergraphs, power hypergraphs and their laplacian h-eigenvalues. Linear Algebra Appl. 439 (2013), 2980–2998.
[13] Kitouni, O., and Reff, N. Lower bounds for the laplacian spectral radius of an oriented hypergraph. Australasian Journal of Combinatorics 74 (2019), 408422.
[14] Reff, N. Spectral properties of oriented hypergraphs. Electronic Journal of Linear Algebra 27 (2014), 373–391.
[15] Reff, N. Intersection graphs of oriented hypergraphs and their matrices. Australasian Journal of Combinatorics 65 (2016), 108123.
[16] Reff, N., and Rusnak, L. An oriented hypergraphic approach to algebraic graph theory. Linear Algebra Appl. 437 (2012), 2262–2270.
[17] Rodríguez, J. On the laplacian eigenvalues and metric parameters of hypergraphs. Linear and Multilinear Algebra 50 (2002), 1–14.
[18] Rodríguez, J. On the laplacian spectrum and walk-regular hypergraphs. Linear and Multilinear Algebra 51 (2003), 285–297.
[19] Rodríguez, J. Laplacian eigenvalues and partition problems in hypergraphs. Applied Mathematics Letters 22 (2009), 916–921.
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