Beta-Function Identities via Probabilistic Approach

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Abstract

Using a probabilistic approach, we derive several interesting identities involving beta functions. Our results generalize certain well-known combinatorial identities involving binomial coefficients and gamma functions.

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1 Introduction

There are several interesting combinatorial identities involving binomial coefficients, gamma functions, hypergeometric functions (see, for example, Riordan (1968), Petkovsek et al (1996), Bagdasaryan (2015), Srinivasan (2007) and Vellaisamy and Zeleke (2017), etc. One of these is the famous identity that involves the convolution of the central binomial coefficients:

\[
\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n. \quad (1.1)
\]

In recent years, researchers have provided several proofs of (1.1). A proof that uses generating functions can be found in Stanley (1997). The combinatorial proofs can also be found, for example, in Sved (1984), De Angelis (2006) and Mikić (2016). A computer generated proof
using the WZ method is given by Petkovsek, Wilf and Zeilberger (1996). Chang and Xu (2011) extended the identity in (1.1) and presented a probabilistic proof of the identity

\[ \sum_{k_1 \geq 0, 1 \leq j \leq m; \sum_{j=1}^{m} k_j = n} \binom{2k_1}{k_1} \binom{2k_2}{k_2} \cdots \binom{2k_m}{k_m} = \frac{4^n \Gamma(n + \frac{m}{2})}{n! \Gamma(\frac{m}{2})}, \]  

for positive integers \( m \) and \( n \), and Mikić (2016) presented a combinatorial proof of this identity based on the method of recurrence relations and telescoping.

A related identity for the alternating convolution of central binomial coefficients is

\[ \sum_{k=0}^{n} (-1)^k \binom{2k}{k} \binom{2n - 2k}{n - k} = \begin{cases} 2^n \left( \frac{n}{2} \right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd}. \end{cases} \]  

The combinatorial proofs of the above identity can be found in, for example, Nagy (2012), Spivey (2014) and Mikić (2016). Recently, Pathak (2017) has given a probabilistic of the above identity.

Unfortunately, in the literature, there are only a few identities that involve beta functions are available. Our goal in this paper is to establish several interesting beta-function identities, similar to the ones given in (1.1) and (1.3). Interestingly, our results generalize all the above-mentioned identities, including the main result (1.2) of Chang and Xu (2011). Our approach is based on probabilistic arguments, using the moments of the sum or the difference of two gamma random variables.

## 2 Identities Involving the Beta Functions

The beta function, also known as the Euler first integral, is defined by

\[ B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0. \]  

It was studied by Euler and Legendre and is related to the gamma functions by

\[ B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \]  

where \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, x > 0 \). The beta function is symmetric, i.e. \( B(x, y) = B(y, x) \) and satisfies the basic identity

\[ B(x, y) = B(x, y+1) + B(x+1, y), \quad \text{for } x, y > 0. \]
Using a probabilistic approach, we generalize, in some sense, the above basic identity in (2.3) in two different directions.

Let $X$ be a gamma random variable with parameter $p > 0$, denoted by $X \sim G(p)$, and density

$$f(x|p) = \frac{1}{\Gamma(p)} e^{-x^p x^{-1}}, \quad x > 0, \quad p > 0.$$ 

Then, it follows (see Rohatgi and Saleh (2002), p. 212) that

$$E(X^n) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-x^p x^{-1}} dx = \frac{\Gamma(p+n)}{\Gamma(p)}. \quad (2.4)$$

Let $X_1 \sim G(p_1)$ and $X_2 \sim G(p_2)$ be two independent gamma distributed random variables, with parameters $p_1$ and $p_2$, respectively. Then it is known that $Y = X_1 + X_2$ follows a gamma distribution with parameters $(p_1 + p_2)$, i.e, $Y \sim G(p_1 + p_2)$. We compute the $n$-th moment $E(Y^n)$ in two different ways, and equating them gives us an identity involving beta functions.

**Theorem 2.1** Let $p_1, p_2 > 0$. Then for any integer $n \geq 0$,

$$\sum_{k=0}^n \binom{n}{k} B(p_1 + k, p_2 + n - k) = B(p_1, p_2). \quad (2.5)$$

**Proof.** Since $X_1 + X_2 = Y \sim G(p_1 + p_2)$, we get, from (2.4),

$$E(Y^n) = \frac{\Gamma(p_1 + p_2 + n)}{\Gamma(p_1 + p_2)}. \quad (2.6)$$

Alternatively, since $X_1 \sim G(p_1), \ X_2 \sim G(p_2)$ and are independent, we have by using the binomial theorem

$$E(Y^n) = E(X_1 + X_2)^n = E\left(\sum_{k=0}^n \binom{n}{k} X_1^k X_2^{n-k}\right)$$

$$= \sum_{k=0}^n \binom{n}{k} E(X_1^k) E(X_2^{n-k})$$

$$= \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(p_1 + k)\Gamma(p_2 + n - k)}{\Gamma(p_1)\Gamma(p_2)}, \quad (2.7)$$

using (2.4).

Equating (2.6) and (2.7), we obtain

$$\frac{\Gamma(p_1 + p_2 + n)}{\Gamma(p_1 + p_2)} = \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(p_1 + k)\Gamma(p_2 + n - k)}{\Gamma(p_1)\Gamma(p_2)}$$
which leads to
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(p_1 + k)\Gamma(p_2 + n - k)}{\Gamma(p_1 + p_2 + n)} = \frac{\Gamma(p_1)\Gamma(p_2)}{\Gamma(p_1 + p_2)}.
\]
This proves the result.

**Remarks 2.1**

(i) Indeed, one could also consider \(X_1 \sim G(\alpha, p_1)\) and \(X_2 \sim G(\alpha, p_2), \alpha > 0\), the two-parameter gamma random variables. But, the result does not change as as the powers of \(\alpha\) cancel out.

(ii) When \(p_1 = p_2 = \frac{1}{2}\), we get,
\[
\sum_{k=0}^{n} \binom{n}{k} B\left(\frac{1}{2} + k, \frac{1}{2} + n - k\right) = \pi, \text{ for all } n \geq 0,
\]
which is an interesting representation for \(\pi\). Also, for this case, it is shown later (see Remark 3.2) that (2.5) reduces to (1.1).

(iii) When \(p_1 = x, p_2 = y\) and \(n = 1\), we get,
\[
B(x, y) = B(x, y + 1) + B(x + 1, y),
\]
the basic identity in (2.3). Thus, Theorem 2.1 extends both the identities in (1.1) and in (2.3).

Our next result is an interesting identity that relates binomial coefficients and beta functions on one side and to a simple expression on the other side. The proof relies on the binomial inversion formula (see, for example, Aigner (2007)) which we include here for ease of reference.

**The Binomial Inversion Formula.** For a positive sequence of \(\{a_n\}_{n \geq 0}\) of real numbers, define the real sequence \(\{b_n\}_{n \geq 0}\) by
\[
b_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_j.
\]
Then, for all \(n \geq 0\), the binomial inversion of \(b_n\) is given by
\[
a_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} b_j.
\]
**Theorem 2.2** Let $s > 0$ and $n \geq 0$ be an integer. Then,

$$
\sum_{j=0}^{n} (-1)^j \binom{n}{j} B(j+1, s) = \frac{1}{s+n}. \tag{2.8}
$$

**Proof.** The following binomial identity is known:

$$
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( \frac{s}{s+j} \right) = \prod_{j=1}^{n} \left( \frac{j}{s+j} \right). \tag{2.9}
$$

Recently, Peterson (2013) and Vellaisamy (2015) gave a probabilistic proof of the above identity. Note that the right hand side of (2.9) can also be written as

$$
\prod_{j=1}^{n} \left( \frac{j}{s+j} \right) = \frac{\Gamma(n+1)\Gamma(s+1)}{\Gamma(s+n+1)} = \frac{\Gamma(n+1)\, s\Gamma(s)}{\Gamma(s+n+1)} = sB(n+1, s).
$$

Then, the identity in (2.8) becomes

$$
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( \frac{1}{s+j} \right) = B(n+1, s). \tag{2.10}
$$

Applying the binomial inversion formula to (2.10) with $a_j = \frac{1}{s+j}$ and $b_n = B(n+1, s)$, we get

$$
\sum_{j=0}^{n} (-1)^j \binom{n}{j} B(j+1, s) = \frac{1}{s+n},
$$

which proves the result. \(\blacksquare\)

**Remark 2.1** When $n = 1$, equation (2.8) becomes

$$
B(1, s) - B(2, s) = \frac{1}{s+1} = B(1, s+1),
$$

which coincides with the basic beta-function identity in (2.3), when $x = 1$. Thus, (2.8) can be viewed as another generalization of (2.3) in the case when $x$ is a positive integer. Also, when $n = 2$,

$$
B(1, s) - 2B(2, s) + B(3, s) = \frac{1}{s+2},
$$

which can be verified using the basic identity in (2.3). It may be of interest to provide a different proof of Theorem 2.2 based on induction or combinatorial arguments.
It is easy to see that the derivative of the beta function is
\[
\frac{\partial}{\partial y} B(x, y) = B(x, y) \left( \psi(y) - \psi(x + y) \right),
\]
where \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \) is the digamma function. The following result is a consequence of Theorem 2.2.

**Theorem 2.3** For \( s > 0 \) and an integer \( n \geq 0 \),
\[
\sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^j \binom{n}{j} \frac{B(j + 1, s)}{s + i} = \frac{1}{(s + n)^2}.
\]

**Proof.** The proof proceeds by taking the derivative of both sides of (2.7) with respect to \( s \).

From the right-hand side, we get,
\[
\frac{\partial}{\partial s} \left( \frac{1}{s + n} \right) = \frac{-1}{(s + n)^2}.
\]

Also, form the left-hand side,
\[
\frac{\partial}{\partial s} \sum_{j=0}^{n} (-1)^j \binom{n}{j} B(j + 1, s) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\partial}{\partial s} B(j + 1, s)
\]
\[
= \sum_{j=0}^{n} (-1)^j \binom{n}{j} B(j + 1, s) \left( \psi(s) - \psi(j + 1 + s) \right),
\]
using (2.11). Further, it is known that the digamma function \( \psi(x) \) satisfies
\[
\psi(x + 1) - \psi(x) = \frac{1}{x}.
\]

Using (2.15) iteratively leads to
\[
\psi(x + j + 1) - \psi(x) = \sum_{i=0}^{j} \frac{1}{x + i},
\]
for a nonnegative integer \( j \).

Using (2.13) (2.14) and (2.16), we get
\[
\frac{-1}{(s + n)^2} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} B(j + 1, s) \left( \psi(s) - \psi(j + 1 + s) \right)
\]
\[
= \sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^j \binom{n}{j} B(j + 1, s)(-1) \frac{1}{s + i},
\]
which proves the result.

Our aim next is to extend the identity in (1.3).
Theorem 2.4 Let \( p > 0 \) and \( n \) be a positive integer. Then
\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} B(p + k, p + n - k) = \begin{cases} 
n! \Gamma(p) \Gamma(p + \frac{n}{2}) & \text{if } n \text{ is even} \\
\Gamma(\frac{n}{2} + 1) \Gamma(2p + n) & \text{if } n \text{ is odd.}
\end{cases}
\] (2.17)

Proof. To prove the result, we consider the rv \( X = X_1 - X_2 \), where \( X_1 \) and \( X_2 \) are as before independent gamma \( G(p) \) variables. Note first that since \( X_1 \) and \( X_2 \) are independent and identically distributed, we have \( X_1 - X_2 \overset{d}{=} X_2 - X_1 \). Here \( \overset{d}{=} \) denotes the equality in distributions. That is, \( X \) and \( -X \) have the same distributions on \( \mathbb{R} \), which implies the density of \( X \) is symmetric about zero. Hence, \( E(X^n) = 0 \) if \( n \) is an odd integer.

Next, we compute the even moments of \( X \). The earlier approach of finding the moments of \( X \) using the probability density function is tedious. This is because the density of \( X \) is very complicated and it involves Whittaker’s W-function (see Mathai (1993)). Therefore, we use the moment generating function (MGF) approach to find the moments of \( X \).

It is known (see Rohatgi and Saleh (2002, p. 212) that the MGF of \( X_1 \) is \( M_{X_1}(t) = E(e^{tX_1}) = (1 - t)^{-p} \). Hence, the MGF of \( X \) is
\[
M_X(t) = M_{X_1 - X_2}(t) = M_{X_1}(t)M_{X_2}(-t) = (1 - t)^{-p}(1 + t)^{-p} = (1 - t^2)^{-p}
\] (2.18)

which exits for \(|t| < 1\).

Using the result, for \( p > 0 \) and \(|q|<1\), that
\[
(1 - q)^{-p} = \sum_{n=0}^{\infty} \frac{\Gamma(n + p)q^n}{\Gamma(n + 1)\Gamma(p)},
\]
we have
\[
M_X(t) = (1 - t^2)^{-p} = \sum_{n=0}^{\infty} \frac{\Gamma(n + p)t^{2n}}{\Gamma(n + 1)\Gamma(p)}.
\] (2.19)

Hence, for \( n \geq 1 \), we have from (2.19)
\[
E(X^{2n}) = M_X^{(2n)}(t)|_{t=0} = \frac{\Gamma(n + p)(2n)!}{\Gamma(n + 1)\Gamma(p)},
\] (2.20)

where \( f^{(k)} \) denotes the \( k \)-th derivative of \( f \). Thus, we have shown that
\[
E(X^n) = \begin{cases} 
n! \Gamma(\frac{n}{2} + p) & \text{if } n \text{ is even} \\
\Gamma(\frac{n}{2} + 1)\Gamma(p) & \text{if } n \text{ is odd.}
\end{cases}
\] (2.21)
Next we compute the moments of $X$, using the binomial theorem. Note that

$$E(X^n) = E((X_1 - X_2)^n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E(X_1^k) E(X_2^{n-k})$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(p+k) \Gamma(p+n-k)}{\Gamma(p)} \frac{\Gamma(p)}{\Gamma(p)}.$$  \hspace{1cm} (2.22)

Equating (2.21) and (2.22), we get

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \Gamma(p+k) \Gamma(p+n-k) = \begin{cases} 
\frac{n! \Gamma\left(\frac{n}{2} + p\right) \Gamma\left(p\right)}{\Gamma\left(\frac{n}{2} + 1\right)} , & \text{if } n \text{ is even} \\
0 , & \text{if } n \text{ is odd.} 
\end{cases}$$  \hspace{1cm} (2.23)

which is an interesting identity involving gamma functions and binomial coefficients. Dividing both sides of (2.23) by $\Gamma(2p + n)$, the result follows.

We next show that the identity in (1.2) follows as a special case.

**Corollary 2.1** Let $n$ be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-2k) \binom{2n}{n-k} = \begin{cases} 
2^n \binom{n}{2} , & \text{if } n \text{ is even} \\
0 , & \text{if } n \text{ is odd.} 
\end{cases}$$  \hspace{1cm} (2.24)

**Proof.** Let $p = \frac{1}{2}$ in (2.23) and it suffices to consider the case $n$ is even. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(n-k + \frac{1}{2}\right) = \frac{n! \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}$$

That is,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma\left(k + \frac{1}{2}\right) \Gamma\left(n-k + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n}{2} + 1\right)} = \frac{n! \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n}{2} + 1\right)}. \hspace{1cm} (2.25)$$

Note that,

$$\frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{\left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \cdots \left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{(2n-1) \cdot (2n-3) \cdots 3 \cdot 1}{2^n} = \frac{(2n)!}{n! 4^n}. \hspace{1cm} (2.26)$$
Using (2.26) in (2.25), we get
\[
\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2k)!}{4^k k!} \frac{(2n-2k)!}{4^{n-k}(n-k)!} = \frac{n!n!}{4^n (n/2)! (n/2)!}
\]
That is,
\[
\sum_{k=0}^{n} (-1)^k \binom{2k}{k} \binom{2n-2k}{n-k} = 2^n \binom{n}{n/2},
\]
which proves the result.

3 An Extension

In this section, we discuss an extension of the beta-function identity given in Theorem 2.1. This result in particular extends the main result of Chang and Xu (2011). Let \( p_i > 0, 1 \leq i \leq m \), and consider the beta function of \( m \)-variables defined by
\[
B(p_1, \ldots, p_m) = \int_{\mathbb{T}} x_1^{p_1-1} x_2^{p_2-1} \cdots x_m^{p_m-1} (1 - p_1 - p_2 - \cdots - p_m)^{p_m-1} dx_1 \cdots dx_m,
\] (3.1)
where \( \mathbb{T} = (0, 1) \times \cdots \times (0, 1) \). It is well known that \( B(p_1, \ldots, p_m) \) can be expressed as a ratio of gamma functions as
\[
B(p_1, \ldots, p_m) = \frac{\Gamma(p_1) \cdots \Gamma(p_m)}{\Gamma(p_1 + \cdots + p_m)}.
\] (3.2)

**Theorem 3.1** Let \( p_1, \ldots, p_m \geq 0 \). Then for any non-negative integer \( n \),
\[
\sum_{k_j \geq 0; \ 1 \leq j \leq n; \ \sum_{j=1}^{m} k_j = n} \binom{n}{k_1, \ldots, k_m} B(p_1 + k_1, \ldots, p_m + k_m) = B(p_1, \ldots, p_m) \] (3.3)

where \( \binom{n}{k_1, \ldots, k_m} = \frac{n!}{k_1! \cdots k_m!} \) denotes the multinomial coefficient.

**Proof.** Let \( X_1, \ldots, X_m \) be \( m \) independent gamma random variables, where \( X_i \sim G(p_i), 1 \leq i \leq m \). Then it is known that
\[
Y = \sum_{i=1}^{m} X_i \sim G(p_1 + \cdots + p_m).
\]
Also, from (2.4),
\[
E(Y^n) = \frac{\Gamma(p_1 + p_2 + \cdots + p_m + n)}{\Gamma(p_1 + \cdots + p_m)}. \quad (3.4)
\]
Since \(X_i\)'s are independent, we have by multinomial theorem,

\[
E(X_1 + \cdots + X_m)^n = E \left[ \sum_{k_j \geq 0, 1 \leq j \leq m} \left( \sum_{k_j = 0}^{n} \binom{n}{k_1, \ldots, k_m} X_1^{k_1} \cdots X_m^{k_m} \right) \right]
\]

\[
= \sum_{k_j \geq 0, 1 \leq j \leq m} \binom{n}{k_1, \ldots, k_m} E(X_1^{k_1}) \cdots E(X_m^{k_m})
\]

\[
= \sum_{x_j \geq 0, 1 \leq j \leq m} \binom{n}{k_1, \ldots, k_m} \frac{\Gamma(p_1 + k_1) \cdots \Gamma(p_m + k_m)}{\Gamma(p_1) \cdots \Gamma(p_m)}. \quad (3.5)
\]

Equating (3.4) and (3.5), we obtain

\[
\sum_{k_j \geq 0, 1 \leq j \leq m} \binom{n}{k_1, \ldots, k_m} \frac{\Gamma(p_1 + k_1) \cdots \Gamma(p_m + k_m)}{\Gamma(p_1) \cdots \Gamma(p_m)} = \frac{\Gamma(p_1 + p_2 + \cdots p_m + n)}{\Gamma(p_1 + \cdots + p_m)}.
\]

That is,

\[
\sum_{k_j \geq 0, 1 \leq j \leq m} \binom{n}{k_1, \ldots, k_m} \frac{\Gamma(p_1 + k_1) \cdots \Gamma(p_m + k_m)}{\Gamma(p_1 + p_2 + \cdots p_m + n)} = \frac{\Gamma(p_1) \cdots \Gamma(p_m)}{\Gamma(p_1 + \cdots + p_m)},
\]

from which the result follows.

**Remark 3.1** Obviously, when \(m = 2\), the identity in (3.3) reduces to

\[
\sum_{k_1 \geq 0, 1 \leq j \leq 2; k_1 + k_2 = n} \binom{n}{k_1, k_2} B(p_1 + k_1, p_2 + k_2) = \sum_{k=0}^{n} \binom{n}{k} B(p_1 + k, p_2 + n - k) = B(p_1, p_2),
\]

which is (2.5).

Our next result shows that the identity in (1.2) follows as a special case.

**Corollary 3.1** When \(p_1 = p_2 = \cdots = p_m = \frac{1}{2}\), the identity in (3.3) reduces to

\[
\sum_{k_j \geq 0, 1 \leq j \leq m} \binom{2k_1}{k_1} \cdots \binom{2k_m}{k_m} = \frac{4^n \Gamma(n + m)}{n! \Gamma\left(\frac{n + m}{2}\right)}, \quad (3.6)
\]

for all integers \(m, n \geq 1\).
Proof. Putting \( p_1 = p_2 = \cdots p_m = \frac{1}{2} \) in (3.3), we obtain,

\[
\sum_{k_j \geq 0, \ 1 \leq j \leq m} \binom{n}{k_1, \cdots, k_m} B\left(\frac{1}{2} + k_1, \cdots, \frac{1}{2} + k_m\right) = B\left(\frac{1}{2}, \cdots, \frac{1}{2}\right)
\]

This implies,

\[
\sum_{k_j \geq 0, \ 1 \leq j \leq m} \binom{n}{k_1, \cdots, k_m} \frac{\Gamma\left(\frac{1}{2} + k_1\right) \cdots \Gamma\left(\frac{1}{2} + k_m\right)}{\Gamma\left(n + \frac{m}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right) \cdots \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}.
\]

or, equivalently,

\[
\sum_{k_j \geq 0, \ 1 \leq j \leq m} \binom{n}{k_1, \cdots, k_m} \frac{\Gamma\left(\frac{1}{2} + k_1\right) \cdots \Gamma\left(\frac{1}{2} + k_m\right)}{\Gamma\left(\frac{n + m}{2}\right)} = \frac{\Gamma\left(n + \frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}.
\]

Using (2.26), we get

\[
\sum_{k_j \geq 0, \ 1 \leq j \leq m} \binom{n}{k_1, \cdots, k_m} \frac{(2k_1)! \cdots (2k_m)!}{4^{k_1}(k_1)! \cdots 4^{k_m}(k_m)!} = \frac{\Gamma\left(n + \frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}. \tag{3.7}
\]

We can write (3.7) as

\[
\sum_{k_j \geq 0, \ 1 \leq j \leq m} \binom{2k_1}{k_1} \frac{\binom{2k_2}{k_2} \cdots \binom{2k_m}{k_m}}{n!} = \frac{4^n}{n!} \left(\frac{\Gamma\left(n + \frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right), \tag{3.8}
\]

which proves the corollary.

Remark 3.2 Note that when \( m = 2 \), (3.6) reduces to (1.1). This implies also that when \( p = \frac{1}{2} \), (2.5) reduces to (1.1).

Remark 3.3 (i) Let \( m \) be even so that \( m = 2l \) for some positive integer \( l \). Then the right hand side of (3.6) is

\[
\frac{4^n \Gamma(n + l)}{n! \Gamma(l)} = 4^n \binom{n + l - 1}{n} = 4^n \binom{n + \frac{m}{2} - 1}{n}.
\]
Similarly, when \( m \) is odd, say \( m = 2l + 1 \),
\[
\frac{4^n \Gamma(n + \frac{m}{2})}{n! \Gamma(\frac{m}{2})} = \frac{4^n \Gamma(n + l + \frac{1}{2})}{n! \Gamma(l + \frac{1}{2})} = \frac{4^n \left( \Gamma(n + l + \frac{1}{2}) \right)}{n! \left( \Gamma(l + \frac{1}{2}) \right)} = \left( \frac{2n+2l}{2n} \right) \left( \frac{(2n)!}{n! n!} \right) \left( \frac{l! n!}{(n+l)!} \right) \quad (\text{using (2.6)})
\]
\[
= \left( \frac{2n+2l}{2n} \right) \left( \frac{2n}{n} \right) \left( \frac{n+l}{n} \right)
\]
\[
= \left( \frac{2n+m-1}{2n} \right) \left( \frac{2n}{n} \right) \left( \frac{n+m-1}{n} \right)
\]

since \( 2l = m - 1 \). Thus, we have from (3.6),
\[
\sum_{k_j \geq 0, \sum k_j = n} \binom{2k_1}{k_1} \binom{2k_2}{k_2} \cdots \binom{2k_m}{k_m} = \begin{cases} 
4^n \binom{n + \frac{m}{2} - 1}{n}, & \text{if } m \text{ is even} \\
\frac{(2n+m-1)(2n)}{(n+m-1)^2}, & \text{if } m \text{ is odd}
\end{cases}
\]

which is equation (3) of Mikić (2016). Indeed, Mikić (2016) provided a combinatorial proof of the above result based on recurrence relations.

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