Factorization of Kazhdan–Lusztig elements for Grassmanians

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Abstract. We show that the Kazhdan-Lusztig basis elements $C_w$ of the Hecke algebra of the symmetric group, when $w \in S_n$ corresponds to a Schubert subvariety of a Grassmann variety, can be written as a product of factors of the form $T_i + f_j(v)$, where $f_j$ are rational functions.

1. Notation

In this section, we briefly list the main facts and notations related to Kazhdan–Lusztig polynomials and their parabolic analogues (see [0], [0]). We use the following notations:

$H$—the Hecke algebra of the symmetric group $S_n$; we consider it as an algebra over the field $\mathbb{Q}(v)$ (the variable $v$ is related to the variable $q$ used by Kazhdan and Lusztig via $v = q^{1/2}$), and we write the quadratic relation in the form $(T_i - v)(T_i + v - 1) = 0$.

$C_w$—KL basis in $H$, which we define by the conditions $C_w = C_w$, $C_w - T_w \in \oplus v\mathbb{Z}[v]T_y$.

For any subset $J \subset \{1, \ldots, n - 1\}$, we denote by $W_J \subset S_n$ the corresponding parabolic subgroup, and by $W_J^-$ the set of minimal length representatives of cosets $S_n/W_J$. We also denote by $M_J$ the $H$-module induced from the one-dimensional representation of $H(W_J)$, given by $T_jm_1 = -v^{-1}m_1, j \in J$. We denote $m_y = T_ym_1, y \in W_J^-$ the usual basis in $M_J$.

We define the parabolic KL basis $C_J^y, y \in W_J$ in $M_J$ by $C_J^y = C_y, C_J^y - m_y \in \oplus_{z \in W_J} v\mathbb{Z}[v]m_z$.

Denote for brevity $C_J = C_{w_0^J}$ the element of KL basis in $H$ corresponding to the element of $w_0^J$ of maximal length in $W_J$. The following result is well-known (see, e.g., [0]).

**Lemma 1.** (i) $C_J = \sum_{w \in W_J} (-v)^{l(w_0^J) - l(w)}T_w$.

(ii) Let $w \in W$ be such that it is an element of maximal length in the coset $WW_J$ (which is equivalent to $w = \tau w_0^J$ for some $\tau \in W_J$). Then $C_w = XC_J$ for some $X \in \oplus_{y \in W_J} \mathbb{Z}[v^{\pm 1}]T_y$.

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(iii) Let $X \in \oplus_{y \in W^J} \mathbb{Z}[v^{\pm 1}]T_y$. Then

$$Xm_1 = C^J_\tau \iff XC^J = C_{\tau w^J}.$$ 

Let us now consider the special case of the above situation. From now on, fix $k \leq n - 1$, and let $J = \{1, \ldots, k-1, k+1, \ldots, n-1\}$ so that $W_J = S_k \times S_{n-k}$ is a maximal parabolic subgroup in $S_n$. In this case, the module $M^J$ can be described as follows:

$$M = \bigoplus_{\varepsilon \in E} Q(\varepsilon),$$

where $E$ is the set of all length $n$ sequences of pluses and minuses which contain exactly $k$ pluses. The relation of this with the previous notation is given by $m_y \leftrightarrow y(1) = T_y(1)$, where

$$1 = (\underbrace{+ \cdots +}_{k} \underbrace{- \cdots -}_{n-k}).$$

In particular, $m_1 \leftrightarrow 1$.

The set of minimal length representatives $W^J$ also admits a description in terms of Young diagrams. Namely, let $\lambda$ be a Young diagram which fits inside the $k \times (n-k)$ rectangle. Define $w_\lambda \in S_n$ by

$$w_\lambda = \prod_{(i,j) \in \lambda} s_{k+j-i},$$

where $(i,j)$ stands for the box in the $i$-th row and $j$-th column, and the product is taken in the following order: we start with the lower right corner and continue along the row, until we get to the first column; then we repeat the same with the next row, and so on until we reach the upper left corner.

**Example 1.** Let $\lambda$ be the diagram shown below, and $k = 7$ (to assist the reader, we put the numbers $k+j-i$ in the diagram).

Then $w_\lambda = s_3 \cdot s_4 \cdot s_7s_6s_5 \cdot s_8s_7s_6 \cdot s_{11}s_{10}s_9s_8s_7$ (for easier reading, we separated products corresponding to different rows by $\cdot$).

The proof of the following proposition is straightforward.

**Proposition 2.** The correspondence $\lambda \mapsto w_\lambda$, where $w_\lambda$ is defined by (3), is a bijection between the set of all Young diagrams which fit inside the $k \times (n-k)$ rectangle and $W^J$. 
2. The main theorem

As before, we fix \( k \leq n - 1 \) and let \( J = \{1, \ldots, k - 1, k + 1, \ldots, n - 1\} \). Unless otherwise specified, we only use Young diagrams which fit inside the \( k \times (n - k) \) rectangle.

For a Young diagram \( \lambda \), we define the shifts \( r_{i,j} \in \mathbb{Z}_{>0} \), \((i, j) \in \lambda \) by the following relation

\[
    r_{ij} = \max(r_{i,j+1}, r_{i+1,j}) + 1,
\]

where we let \( r_{ij} = 0 \) if \((i, j) \notin \lambda\).

Example 2. For the diagram \( \lambda \) from Example 1, the shifts \( r_{ij} \) are shown below.

\[
\begin{array}{cccccc}
6 & 5 & 4 & 3 & 2 & 1 \\
4 & 3 & 2 & & & \\
3 & 2 & 1 & & & \\
2 & & & & & \\
1 & & & & & \\
\end{array}
\]

Next, let us define for each diagram \( \lambda \) an element \( X_\lambda \in H \) by

\[
    X_\lambda = \prod_{(i,j) \in \lambda} \left( T_{k+j-i} - \frac{v^{r_{ij}}}{[r_{ij}]} \right)
\]

where, as usual, \([r] = (v^r - v^{-r})/(v - v^{-1})\), and the product is taken in the same order as in (3).

The main result of this paper is the following theorem.

**Theorem 3.** Let \( \lambda \) be a Young diagram. Then

\[
    X_\lambda 1 = C^J_{w_\lambda}.
\]

Note that by Lemma 1, this is equivalent to

\[
    X_\lambda C_J = C_{w_\lambda w_0}.
\]

We remind the reader that the Kazhdan-Lusztig elements \( C_{ww_0} \), where \( w \in W^J \), and \( W_J \) is a maximal parabolic in \( S_n \) (they are also known as KL elements for Grassmanians), have been studied in a number of papers. A combinatorial description was given in [LS1]; it was interpreted geometrically in [Z], and in terms of representations of quantum \( \mathfrak{gl}_n \) in [FKK]. However, it is unclear how these results are related with the factorization given by the theorem above. A similar factorization was given in [L] for those permutations which correspond to non-singular Schubert varieties—i.e., for those \( w \) such that, for any \( v \in S_n \), the Kazhdan-Lusztig polynomial \( P_{v,w} \) is either 1 or 0.

Note that one can easily check that the elements \( X_\lambda \) are invariant under the Kazhdan–Lusztig involution: \( \overline{X_\lambda} = \overline{X_\lambda} \); thus, all the difficulty is in proving that they are integral and have the right specialization at \( v = 0 \).

A crucial step in proving this theorem is the following proposition.

**Proposition 4.** Theorem 3 holds when \( \lambda \) is the \( k \times (n - k) \) rectangle.

**Proof.** For any \( w \in S_n \), choose a reduced expression \( w = s_{i_1} \ldots s_{i_t} \). Define the element \( \nabla_w \in H \) by

\[
    \nabla_w = \left( T_{i_t} - \frac{v^{r_{i_t}}}{[r_{i_t}]} \right) \ldots \left( T_{i_1} - v \right),
\]
where \( r_1, \ldots, r_\ell \in \mathbb{Z}_+ \) are defined as follows: if \( s_{i_{m-1}} \cdots s_{i_1} (1, \ldots, n) = (\ldots, a, b, \ldots) \) (in \( i_m \)-th, \( (i_m + 1) \)-st places), then \( r_m = b - a \). Then \( \{ \nabla_w, w \in S_n \} \) is a Yang-Baxter basis of the Hecke algebra, and we have (see \cite{DKLLST}, §3):

**Lemma 5.** (i) The element \( \nabla_w \) does not depend on the choice of reduced expression.

(ii) If \( w_0^J \) is the longest element in some parabolic subgroup \( W_J \subset S_n \), then \( \nabla_{w_0^J} = C_J \).

Now, let us prove our proposition, i.e. that \( X_\lambda C_J \) is a KL element for rectangular \( \lambda \). In this case, \( w_\lambda \) is the longest element in \( W_J \):

\[
w_\lambda(1) = (- \cdots - + \cdots +)_{n-k}^{k}.
\]

Let us choose the following reduced expression for the longest element \( w_0 \) in \( S_n \): \( w_0 = w_\lambda w_0^J \), where we take for \( w_\lambda \) the reduced expression given by (3). Then one easily sees that definition (7) in this case gives

\[
\nabla_{w_0} = X_\lambda \nabla_{w_0^J}.
\]

By Lemma 3, we get \( C_{w_0} = X_\lambda C_J \), which is exactly the statement of the proposition.

The proof in the general case is based on the following proposition. Denote

\[
O(v^m) = \{ f \in \mathbb{Q}(v) | f \text{ has zero of order } \geq m \text{ at } v = 0 \}.
\]

**Proposition 6.**

\[
X_\lambda 1 = w_\lambda(1) + \sum_{\varepsilon \in E} O(v)\varepsilon.
\]

A proof of this proposition is given in Section 3.

Now we can give a proof of the main theorem. First, one easily checks the invariance under the bar involution, since

\[
T_i - \frac{v^r}{|r|} = T_i - \frac{v^r}{|r|}.
\]

Combining this with Proposition 1, we see that it remains to show that \( X_\lambda C_J \) are integral, i.e. \( X_\lambda C_J \in \oplus \mathbb{Z}[v^{\pm 1}]T_w \) (note that it is not true that \( X_\lambda \) itself is integral.) This will be done by induction.

Let \( \lambda \) be a Young diagram. Then we claim that any such diagram can be presented as a union \( \lambda = \lambda' \cup \mu \), where \( \mu \) is a rectangle, and \( \lambda' \) is again a Young diagram such that for \( (i, j) \in \lambda' \), the shifts \( r_{(i, j)}^\lambda = r_{(i, j)}^{\lambda'} \). It can be formally proved as follows: if one writes the successive widths and heights of the stairs of the diagram

\[
\infty, (a_1, b_1), (a_2, b_2), \ldots (a_k, b_k), \infty
\]

then there is at least one index \( i \) for which \( a_i \leq b_{i+1} \) and \( b_i \leq a_{i+1} \). In that case, the rectangle \( \mu \) has the lower right corner \( i \).

**Example 3.** For the diagram \( \lambda \) from Example 3, the sequence \( (a_k, b_k) \) is given by \( \infty, (1, 2), (2, 2), (3, 1), \infty \), and the subdiagram \( \mu \) is the shaded \( 2 \times 2 \) square, as
shown below. As before, we also included the shifts $r_{ij}$ in this diagram. The subsets $I^\mu, J^\mu$ in this case are given by $I^\mu = \{6, 7, 8\}, J^\mu = \{6, 8\}$.

\[
\begin{array}{cccccc}
6 & 5 & 4 & 3 & 2 & 1 \\
4 & 3 & 2 \\
3 & 2 & 1 \\
2 \\
1 \\
\end{array}
\]

Let us choose for $\lambda$ the presentation $\lambda = \lambda' \cup \mu$, where $\mu$ is a rectangle, as above. Then $X_\lambda = X_\mu X_{\lambda'}$.

Define the subsets $I^\mu, J^\mu \subset \{1, \ldots, n-1\}$ by $I^\mu = \{k' - a + 1, \ldots, k' + b - 1\}, J^\mu = I^\mu \setminus \{k\}$, where $k' = k - i + j$; $(i, j)$—coordinates of the UL corner of $\mu$, $a$ and $b$ are numbers of rows and columns in $\mu$ respectively.

We need to show that $X_\mu X_\lambda C_J \in \sum \mathbb{Z}[v^{\pm 1}] T_w$. By induction assumption, we may assume that $X_\lambda C_J = C_\sigma$, where we denoted for brevity $\sigma = w_{\lambda'} w_0'$. It is easy to show that if $\mu$ is chosen as before, then $\sigma$ is the maximal length element in the coset $W_\mu \sigma$. Thus, by Lemma \ref{lem:id}, we can write $C_\sigma = C_{J_\sigma} Y$ for some integral $Y \in \mathcal{H}$. Therefore, $X_\mu X_\lambda C_J = X_\mu C_{J_\sigma} Y$. Since $W_{J_\sigma}$ is itself a symmetric group, and $W_{J_\mu}$ is a maximal parabolic subgroup in it, we can use Proposition \ref{prop:reg}, which gives $X_\mu C_{J_\mu} = C_{J_\mu}$, and therefore, $X_\mu X_\lambda C_J = C_{J_\mu} Y \in \sum \mathbb{Z}[v^{\pm 1}] T_w$. \hfill $\Box$

3. Proof of regularity at $v = 0$

In this section we give the proof of Proposition \ref{prop:reg}. Before doing so, let us introduce some notation.

As before, assume that we are given $n, k, \lambda$ and a collection of positive integers $r_{ij}, (i, j) \in \lambda$ (not necessarily defined as in \ref{def:ri}). Let $\varepsilon \in E$ be a sequence of pluses and minuses. We define the weight $r_\lambda(\varepsilon)$ as follows.

Define $a(i), i = 1 \ldots k$ by $a(i) = k + 1 + i + 1$. Equivalently, these numbers can be characterized by saying that $w_\lambda(1)$ has pluses exactly at positions $a(k), \ldots, a(1)$.

Define $r_\lambda(\varepsilon) = \sum_{t=1}^{n} r_t(\varepsilon)$, where $r_t(\varepsilon)$ is defined as follows:

(i) if $t = a(i), \varepsilon_t = -$ then $r_t(\varepsilon) = r_{i, \lambda} - 1$

(ii) if $a(i) < t < a(i + 1), \varepsilon_t = +$ then $r_t(\varepsilon) = r_{i, j}, k + j - i = t$

(iii) otherwise, $r_t(\varepsilon) = 0$

In a sense, $r_\lambda(\varepsilon)$ measures the discrepancy between $\varepsilon$ and $w_\lambda(1)$. Indeed, let us denote the numbers of rows and columns in $\lambda$ by $i, j$ respectively, and let $\varepsilon$ be such that

\[
\varepsilon_t = + \text{ for } t \leq k - i,
\]

\[
\varepsilon_t = - \text{ for } t > k + j.
\]

Then one easily sees that

\[
r_\lambda(\varepsilon) \geq 0, \quad r_\lambda(\varepsilon) = 0 \iff \varepsilon = w_\lambda(1)
\]

**Example 4.** Below we illustrate the calculation of $r_\lambda(\varepsilon)$, where $\lambda$ is the diagram used in Example \ref{ex:reg}. The positions $a(i)$ are shaded (thus, the sequence of colors encodes $w_\lambda(1)$, with “shaded” $\leftrightarrow +$, “unshaded” $\leftrightarrow -$), and we connected unshaded pluses with the corresponding box $(i, j)$, defined in (ii) above. For convenience of
the reader, we also put the numbers \( k + j - i \) (not the shifts \( r_{ij}! \)) in the diagram.

![Diagram with numbers and signs]

**Lemma 7.** Let \( \lambda \) be any Young diagram inside the \( k \times (n - k) \) rectangle, and let \( r_{ij}, (i, j) \in \lambda \), be positive integers satisfying \( r_{ij} > r_{i,j+1}, r_{ij} > r_{i+1,j} \). Define \( \mathcal{L}_\lambda \subset \mathcal{M}^J \) by

\[
\mathcal{L}_\lambda = \sum_{\varepsilon \in E} O(v^{r_{ij}}(\varepsilon)) \varepsilon.
\]

Then

\[
X_\lambda 1 \in \mathcal{L}_\lambda.
\]

Before proving this lemma note that due to \([11]\), this lemma immediately implies Proposition \([3]\).

**Proof.** The proof is by induction. Let \((i, j)\) be a corner of \( \lambda \), and \( \lambda' = \lambda - (i, j) \), so that \( X_\lambda = \left( T_{k-i+j} - \frac{v^{r_{ij}}}{r_{ij}} \right) X_{\lambda'} \). Since \( \frac{v^r}{r!} \in O(v^{2r-1}) \), it suffices to prove that

\[
\left( T_{k-i+j} + O(v^{2r_{ij}-1}) \right) \mathcal{L}_{\lambda'} \subset \mathcal{L}_\lambda.
\]

Since this operation only changes \( \varepsilon_a, \varepsilon_{a+1} \) \((a = k - i + j)\), we need to consider 4 cases: \((++, \cdots, (+-)), (-+, \cdots, (-+)), (---), (+++).\) This is done explicitly. For example, for the \((+-)\) case, we have

\[
(T_m + O(v^{2r_{ij}-1}))(\cdots + \ldots) = (\cdots + \ldots) + O(v^{2r_{ij}-1})(\cdots + \ldots)
\]

In this case, the first summand has the same weight and comes with the same power of \( v \) as the original \( \varepsilon \) (note that in the original \( \varepsilon \), this \((+-)\) didn’t contribute to the weight), so it is in \( \mathcal{L}_\lambda \). As for the second summand, its weight is increased by \( 2r_{ij} - 1 \) (the plus contributes \( r \) and the minus, \( r - 1 \)), but it comes with the factor \( O(v^{2r_{ij}-1}) \), so again, it is in \( \mathcal{L}_\lambda \). The other cases are treated similarly. \( \square \)

4. Divided differences and parabolic Kazhdan-Lusztig bases

In this section, we give a factorization for the dual Kazhdan–Lusztig basis for Grassmannians.

To induce a parabolic module, one can start from the 1-dimensional representation \( T_j \mapsto v \) instead of \( T_j \mapsto -1/v \) which was used in \( \S 1 \). We now denote the corresponding module by \( M' \) and its Kazhdan-Lusztig basis by \( C'_y \) and \( C'_y \) to distinguish from previous case. Note that there exists a natural pairing between \( M \) and \( M' \), and \( C'_y \) and \( C'_y \) are dual bases with respect to this pairing (see, e.g., \([8], [FKK]\)). However, we will not use this pairing.
A simple element $T_i - v$ acts now by

$$M' = \bigoplus_{\varepsilon \in E} Q(v)\varepsilon,$$

(11)

$$(T_i - v)\varepsilon = \begin{cases} 
    s_i \varepsilon - v \varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (\pm), \\
    0, & (\varepsilon_i, \varepsilon_{i+1}) = (--) \text{ or } (++) , \\
    s_i \varepsilon - v^{-1} \varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (+-). 
\end{cases}$$

Consider the space $P(k, n)$ of polynomials in $x_1, \ldots, x_n$ of total degree $n - k$, and of degree at most 1 in each $x_i$. For any partition $\lambda$, denote by $x^{[\lambda]}$ the monomial $w_\lambda(x_{k+1} \cdots x_n)$, the symmetric group acting now by permutation of the $x_i$. In other words, if $w_\lambda(1) = (\varepsilon_1, \ldots, \varepsilon_n)$, then $x^{[\lambda]}$ is the product of the $x_i$'s for those $i$ such that $\varepsilon_i = -$.

Consider the isomorphism of vector spaces

$$M' \simeq P(k, n)$$

$$w_\lambda(1) \mapsto v^{-|\lambda|} x^{[\lambda]}.$$  

(12)

Then $T_i - v$ induces the operator $\nabla_i$, acting only on $x_i, x_{i+1}$ as follows:

$$\begin{cases} 
    \nabla_i(x_i) = vx_{i+1} - v^{-1}x_i, \\
    \nabla_i(1) = \nabla_i(x_i x_{i+1}) = 0, \\
    \nabla_i(x_{i+1}) = -vx_{i+1} + v^{-1}x_i, 
\end{cases}$$

(13)

Therefore $\nabla_i$ is the operator $f \mapsto (vx_{i+1} - v^{-1}x_i) \partial_i(f)$

denoting by $\partial_i$ the divided difference $f \mapsto \frac{f - f^{x_i}}{x_i - x_{i+1}}$

(for a more general action of the Hecke algebra on the ring of polynomials, see [LS2, DKLLST]).

We intend to show that divided differences easily furnish the Kazhdan-Lusztig basis of $P(k, n)$ (i.e. the image of the Kazhdan-Lusztig basis $C'_{y, y} \in W^J$ of $M'$).

To any element $\varepsilon := w_\lambda(1)$ of $E$ one associates a polynomial $Q_\varepsilon$ as follows

1) pair recursively $-, +$ (as one pairs opening and closing parentheses)

2) replace each pair $(-, +)$, where $-$ is in position $i$ and $+$ in position $j$, with $x_i - v^{j-i+1} x_j$.

3) replace each single $-$, in position $i$, by $x_i$

The product of all these factors by $v^{-|\lambda|}$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots$, is by definition $Q_\varepsilon$.

**Theorem 8.** Let $E$ be the set of sequences of $(+, -)$ of length $n$ with $k$ pluses. Then the collection of polynomials $Q_\varepsilon, \varepsilon \in E$, is the Kazhdan-Lusztig basis of the space $P(k, n)$.

**Proof.** We shall show that $Q_\varepsilon = \nabla_j \cdots \nabla_h(x_1 \cdots x_k)$.
when \( \varepsilon = w_{\lambda}(1) \), and when \( s_j \cdots s_k \) is a reduced decomposition of \( w_{\lambda} \). Now, it is clear that the inverse image of \( Q_{\varepsilon} \) in \( M' \) is invariant under involution, and it is easy to check the powers of \( v \) to get that for \( v = 0 \), it specializes to \( \varepsilon \).

Assume by induction that we already know \( Q_{\varepsilon} \). Let us add on the right of \( \varepsilon \) sufficiently many pluses, so that all minuses are now paired (the original polynomial is recovered from the new one by specializing \( x_{n+1}, x_{n+2}, \ldots \) to 0). Take now any simple transposition \( s_i \) such that \( \varepsilon_i = +, \varepsilon_{i+1} = - \). The variables \( x_i, x_{i+1} \) involve two or one factor in \( P_{\varepsilon} \), depending whether \( \varepsilon_i \) is paired or not. The only possible cases for those factors and their images under \( \nabla_i \) are

\[
(x_{i-a} - v^{a+1} x_i)(x_{i+1} - v^{b+1} x_{i+b+1}) \mapsto (x_{i-a} - v^{a+b+2} x_{i+b+1})(v^{-1} x_i - vx_{i+1})
\]

\[
(x_{i+1} - v^{b+1} x_{i+b+1}) \mapsto (v^{-1} x_i - vx_{i+1})
\]

but now the new pairing of \(-, +\) differs from the previous one exactly in the places described by the factors on the right.

\[\square\]

**Corollary 9.** Let \( \sigma_j \cdots \sigma_h \) be a reduced decomposition of \( w \in W^J \). Then the corresponding Kazhdan-Lusztig element \( C_w^J \in M' \) is equal to \( (T_j - v) \cdots (T_h - v)(1) \).

This factorization is equivalent to the one given in [FKK, Theorem 3.1]. One can check on examples that this factorization is compatible, via the duality between the two modules \( M \) and \( M' \), with the factorization given by Theorem 8. However, deducing Theorem 3 from Theorem 8 seems more intricate than proving the two factorization properties directly.

**Example 5.** Let \( \lambda = [5, 3, 2] \) and \( \mu = [5, 3, 3] \). Then one has

\[
\begin{array}{cccccccc}
\text{places} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
w_{\lambda}(1) & + & - & - & + & - & + & - & - & + \\
\hline
\text{pairing} & + & - & - & + & - & + & - & - & + \\
\hline
\text{polynomial} & x_2 & (x_3 - v^2 x_4) & (x_5 - v^2 x_6) & (x_8 - v^2 x_9) & x_7 \\
\hline
w_\mu(1) & + & - & - & - & + & + & - & - & + \\
\hline
\text{pairing} & + & - & - & + & + & - & - & + & - \\
\hline
\text{polynomial} & x_2 & x_3 & (x_4 - v^2 x_5) & -v^4 x_6 & -v^2 x_9 & x_7 \\
\end{array}
\]

and thus

\[
Q_{w_{\lambda}(1)} = v^{-10} x_2 x_7 (x_3 - v^2 x_4) (x_5 - v^2 x_6) (x_8 - v^2 x_9)
\]

\[
Q_{w_{\mu}(1)} = v^{-11} x_2 x_7 (x_4 - v^4 x_6) (x_4 - v^2 x_5) (x_8 - v^2 x_9).
\]

Note that the pairing between \(-, +\), which was a key point in the description of Kazhdan-Lusztig polynomials for Grassmannians in [LS], is provided by divided differences, starting from the monomial \( x_{k+1} \cdots x_n \).
References

[D] V. V. Deodhar, On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan–Lusztig polynomials, J. of Algebra 111 (1987), 483–506.

[DKLLST] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, and J.-Y. Thibon, Euler–Poincaré characteristic and polynomial representations of Iwahori–Hecke algebras, Publ. RIMS, 31 (1995), 179–201.

[FKK] I. B. Frenkel, M. G. Khovanov, and A. A. Kirillov, Jr., Kazhdan–Lusztig polynomials and canonical basis, Transf. Groups 3 (1998), 321–336.

[L] A. Lascoux, Ordonner le groupe symétrique: pourquoi utiliser l’algèbre de Iwahori-Hecke?, ICM Berlin 1998, Documenta Mathematica, vol. III (1998), 355–364.

[LS1] A. Lascoux, M.-P. Schützenberger, Polynômes de Kazhdan & Lusztig pour les grassmaniennes, Astérisque 87–88 (1981), 249–266.

[LS2] ——, Symmetrization operators on polynomial rings, Functional Anal. Appl., 21 (1987), 77–78.

[S] W. Soergel, Kazhdan–Lusztig polynomials and combinatorics for tilting modules, Represent. Theory (electronic journal), 1 (1997), 83–114.

[Z] A. V. Zelevinski, Small resolutions of singularities of Schubert varieties, Functional Anal. Appl., 17 (1983), 142–144.

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