Abstract: We compute the superconformal index of 3d $\mathcal{N} = 2$ superconformal theories obtained from $N$ M5-branes wrapped on a hyperbolic 3-manifold. Exploiting the 3d-3d correspondence we use perturbative invariants of $SL(N, \mathbb{C})$ Chern-Simons theory to determine the superconformal index in the large $N$ limit, including logarithmic in $N$ corrections. The leading order partition function provides a microscopic foundation for the entropy function of the dual rotating asymptotically AdS$^4$ black holes. We also verify that the supergravity one-loop contribution to the log $N$ term coincides with the field theoretic result. We also propose a 3d-3d formulation for the refined topologically twisted index and provide strong evidence in support of its vanishing which agrees with the fact that the expected dual rotating magnetically charged black hole does not exist, thus providing an interesting link between gravity and a tantalizing mathematical result.
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1 Introduction

One of the most basic quantum properties of black holes is that they have an entropy [1–5]. Bekenstein and Hawking understood that at leading order, the entropy is fixed in terms of the horizon area by the simple formula $S_{BH} = A/4G_N$ where $G_N$ is the Newton constant. This formula is universal: it applies in all dimensions, and for all types of smooth black holes. One of the great successes of string theory was to reproduce the Bekenstein-Hawking entropy of certain asymptotically-flat BPS black holes through a microscopic computation [6].

In the context of asymptotically $\text{AdS}_4$ black holes, string theory in its AdS/CFT guise has again provided a microscopic explanation for the Bekenstein-Hawking entropy. Namely, it has been shown that the topologically twisted index of certain field theory precisely matches the entropy function of the dual magnetically charged asymptotically $\text{AdS}_4$ black holes [7]. Similar microscopic explanations have now been extended to a wide ranges of contexts [8–12] (see the review [13] for a complete list of references).

For rotating electrically charged asymptotically $\text{AdS}_4$ black holes an impressive result was reported based on field-theoretic computations of the superconformal index [14]; more recently an explanation based on localization was provided in [15]. In this manuscript our goal is to approach a class of rotating, electrically charged asymptotically $\text{AdS}_4$ black holes via their dual field theory. We concentrate on configurations that are related via dual descriptions of a stack of $N$ $\text{M5}$-branes. An advantage of this approach is that it allows us to make use of the 3d-3d correspondence. Recall that the 3d-3d correspondence states an equivalence between certain 3d $\mathcal{N} = 2$ superconformal field theory $\mathcal{T}_N[M]$ and Chern-Simons theory with gauge group $\text{SL}(N, \mathbb{C})$ on a hyperbolic three-manifold $M$. This correspondence has the potential to provide an exact in $N$ expression for the partition function of the field theory through known Chern-Simons results. Most of the field theoretic approaches to partition functions or indices ultimately reduce a problem to a Matrix model whose exact solutions has not yet been established, the exactness of the 3d-3d correspondence becomes a crucial advantage particularly in studies of subleading orders.

More precisely, in this manuscript we present a computation of the superconformal index of the field theory and show that it produces the entropy function that generates the entropy of the dual rotating asymptotically $\text{AdS}_4$ black holes through a Legendre transformation. We also compute the one-loop effective action in 11-d supergravity for M5-branes wrapped on a hyperbolic 3-space. We match the gravity result with the field theory answer: $- \log N$. Moreover, we demonstrate that the agreement persists when we consider hyperbolic 3-manifolds that are more general than those previously considered in the literature, including nontrivial first Betti number. Our subleading computation has the potential to distinguish between the various supersymmetric observables that have been shown to yield the same entropy function at leading order.

We also use the 3d-3d correspondence to evaluate the refined topologically twisted index. We find that it vanishes in the large $N$ limit, in fact, we provide strong evidence that it vanishes at finite $N$. This result is a mathematical curiosity that has been entertained previously and it agrees nicely with the fact that there do not exist rotating magnetically
charged dual black holes in the universal supergravity sector that we discussed.

The rest of the paper is organized as follows. In section 2 we provide a 3d-3d perspective for various indices of the corresponding 3d $\mathcal{N} = 2$ superconformal theories. In section 3 we compute the indices in the large $N$ limit. Section 4 discusses properties of the supergravity description of wrapped M5 branes on hyperbolic 3-manifolds, including some of the relevant solutions. Some details regarding the one-loop correction to logarithmic in $N$ terms from the supergravity point of view are presented in section 5. We conclude in section 6 where we also highlight a number of interesting problems. In appendix A we provide explicit examples of state-integrals for complex Chern-Simons theory from which perturbative Chern-Simons invariants can be computed.

**Note added:** While we were preparing this manuscript for submission we received [16] which has some overlap regarding the leading behavior of the superconformal index.

## 2 3d-3d relations for 3d indices

In this section, we study the 3d-3d relations for three types of 3d supersymmetric indices: the superconformal index, the refined topologically twisted index on $S^2$, and the topologically twisted index on a general Riemann surface $\Sigma_g$ of genus $g$. Schematically, the 3d-3d relations take the following form:

$$A \text{ supersymmetric index of the 3d } \mathcal{N} = 2 \ T_N[M_3] \text{ theory } = \text{ An invariant of } SL(N,\mathbb{C}) \text{ Chern-Simons theory on } M_3 \quad (2.1)$$

In subsections 2.1 and 2.2 we will explain the basic definitions and properties of the two quantum field theories — a supersymmetric theory and a bosonic topological theory, respectively — appearing in the relations. Then, in subsection 2.3, we will give concrete statements for the 3d-3d relations for the three types of indices. The part on the superconformal index will basically be a review of previous work [17–19], with some additional clarifications. On the other hand, we will propose the 3d-3d relation for the refined version of the twisted index, with non-trivial supporting evidence. Motivated by that, we will also propose a 3d-3d relation for the twisted index on general Riemann surfaces, which generalizes previous work [20] to cover more general classes of 3-manifolds $M_3$. Finally, in subsection 2.4 we will provide non-trivial consistency checks for the proposed 3d-3d relations, confirming the integral properties of the various indices.

### 2.1 3d $\mathcal{T}_N[M_3]$ theory

We start with the definition and basic properties of the 3d $\mathcal{T}_N[M_3]$ theory appearing on the LHS of the 3d-3d relation (2.1). We define

$$\mathcal{T}_N[M_3] := \left( \text{ the 3d } \mathcal{N} = 2 \text{ superconformal field theory obtained from } \right. \\
\text{ a twisted compactification of the 6d } A_{N-1} (2,0) \text{ theory on } M_3 \left. \right), \quad (2.2)$$

$$M_3 = \text{ (a closed hyperbolic 3-manifold) } = \mathbb{H}^3/\Gamma.$$
We use an $SO(3)$ subgroup of the $SO(5)$ R-symmetry of the 6d theory in the partial topological twist along $M_3$. The resulting 3d theory $\mathcal{T}_N [M_3]$ has 3d $\mathcal{N} = 2$ supersymmetry, with no (non-R) flavor symmetries at sufficiently large $N$. The $U(1) = SO(2)$ R-symmetry of the 3d theory comes from the $SO(2) \subset SO(2) \times SO(3) \subset SO(5)$ subgroup of the 6d R-symmetry and thus its charge $R$ is quantized:

$$R \in \mathbb{Z} \quad \text{in } \mathcal{T}_N [M_3]\text{ theory}.$$  \hspace{1cm} (2.3)

At small $N$, there could be an accidental flavor symmetry in $\mathcal{T}_N [M_3]$ as studied in [21, 22]. In that case, the infra-red superconformal R-symmetry could be different from the R-symmetry originated from the $SO(2) \subset SO(5)$ R-symmetry. The field theoretic construction of the $\mathcal{T}_N [M_3]$ was proposed in [22–24] based on a Dehn-surgery representation of $M_3$ using a link $L \subset S^3$ and an ideal triangulation of the link complement $S^3 \setminus L$. The $\mathcal{T}_N [M_3]$ is expected to be invariant under the choices of Dehn-surgery representation and ideal triangulation. Such an invariance gives a geometrical understanding of certain dualities among 3d $\mathcal{N}=2$ gauge theories.

### 2.2 $SL(N, \mathbb{C})$ Chern-Simons theory on $M_3$

The most general action for $SL(N, \mathbb{C})$ Chern-Simons theory on $M_3$ is given by

$$S_{k,\sigma} [A, \tilde{A}; M_3] = \frac{(k + i\sigma)}{8\pi} \int_{M_3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A^3 \right) + \frac{(k - i\sigma)}{8\pi} \int_{M_3} \text{Tr} \left( \tilde{A} \wedge d\tilde{A} + \frac{2}{3} \tilde{A}^3 \right)$$

$$= \frac{2\pi i}{\hbar} \text{CS}[A; M_3] + \frac{2\pi i}{\tilde{\hbar}} \text{CS}[\tilde{A}; M_3].$$  \hspace{1cm} (2.4)

We have introduced (anti-)holomorphic couplings and the Chern-Simons functional:

$$\hbar := \frac{4\pi i}{k + i\sigma}, \quad \tilde{\hbar} := \frac{4\pi i}{k - i\sigma},$$

$$\text{CS}[A; M_3] := \frac{1}{4\pi} \int_{M_3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A^3 \right).$$  \hspace{1cm} (2.5)

For gauge invariance and unitarity of the theory, it is required that

$$k \in \mathbb{Z}, \quad \sigma \in \mathbb{R} \text{ or } i\mathbb{R}.$$  \hspace{1cm} (2.6)

The $SL(N, \mathbb{C})$ Chern-Simons partition function is defined by the following path-integral:

$$Z^{SL(N,\mathbb{C})}_{k,\sigma}[M_3; \{n_{\alpha\beta}\}] = \int_{\Gamma(\{n_{\alpha\beta}\})} [DA][D\tilde{A}] \ e^{iS_{k,\sigma}[A, \tilde{A}; M_3]},$$

$$\Gamma(\{n_{\alpha\beta}\}) := \sum_{A^\alpha, \tilde{A}^\beta \in \chi(M_3;\mathbb{C})} n_{\alpha\beta} C^\alpha(A) \times C^\beta(\tilde{A}) ,$$  \hspace{1cm} (2.7)

where $\chi(M_3;\mathbb{C})$ represents

$$\chi(M_3;\mathbb{C}) := \text{ (set of gauge-inequivalent } SL(N,\mathbb{C}) \text{ flat connections on } M_3).$$  \hspace{1cm} (2.8)
Moreover, $C^\alpha(\mathcal{A})$ denotes the absolutely converging integration cycle (Lefschetz thimble) in the configuration space of gauge fields $\mathcal{A}$, associated to a flat connection $\mathcal{A}^\alpha$. (See, for example, [25] for details on the integration cycle $C^\alpha(\mathcal{A})$.) Thus, to specify a quantum $SL(N, \mathbb{C})$ Chern-Simons theory, we need to specify the choice of a consistent integral cycle $\{n_{\alpha\beta}\}$ as well as the CS levels $(k, \sigma)$. The partition function can be factorized as

$$Z^{SL(N,\mathbb{C})}_{k,\sigma}[M_3; \{n_{\alpha\beta}\}] = \sum_{\mathcal{A}^\alpha, \tilde{\mathcal{A}}^\beta \in \chi(M_3; N)} \frac{1}{|\text{Stab}(\alpha, \beta)|} n_{\alpha\beta} B^\alpha_M(q; N) B^\beta_M(q; N) . \quad (2.9)$$

Here the holomorphic block $B^\alpha_M(q)$ is defined as

$$B^\alpha_M(q := e^{ih}; N) = \int_{C^\alpha(\mathcal{A})} \frac{[DA]}{(\text{gauge})} e^{-\frac{2\pi}{\hbar} \text{CS}[\mathcal{A}; M_3]} . \quad (2.10)$$

We defined the exponentiated holomorphic and anti-holomorphic couplings as

$$q := e^{ih} = e^{\frac{4\pi i}{\hbar} k + \sigma}, \quad \tilde{q} := e^{ih} = e^{\frac{4\pi i}{\hbar} k - \sigma}. \quad (2.11)$$

The prefactor $|\text{Stab}(\alpha, \beta)|$ is the volume of the stabilizer of $(\mathcal{A}^\alpha, \tilde{\mathcal{A}}^\beta)$, i.e., the volume of the subgroup of the gauge group that preserves the flat connections. In an asymptotic $\hbar \to 0$ limit, the holomorphic block $B^\alpha_M(q)$ can be perturbatively expanded:

$$B^\alpha_M(q) \xrightarrow{\hbar \to 0} \exp \left( \frac{1}{\hbar} S_0^\alpha(M_3; N) + S_1^\alpha(M_3; N) + \ldots + \hbar^{n-1} S_n^\alpha(M_3; N) + \ldots \right) . \quad (2.12)$$

For example

$$S_0^\alpha = -2\pi \text{CS}[\mathcal{A}^\alpha; M_3] , \quad S_1^\alpha = -\frac{1}{2} \text{Tor}[\mathcal{A}^\alpha; M_3] , \quad (2.13)$$

where $\text{Tor}[\mathcal{A}^\alpha; M_3]$ is the analytic Ray-Singer torsion twisted by the flat connection $\mathcal{A}^\alpha$ defined as follows [26, 27]:

$$\text{Tor}[\mathcal{A}^\alpha; M_3] := \frac{[\det' \Delta_1(\mathcal{A}^\alpha)]^{1/4}}{[\det' \Delta_0(\mathcal{A}^\alpha)]^{3/4}} . \quad (2.14)$$

Above, $\Delta_n(\mathcal{A}^\alpha)$ is the Laplacian acting on $\mathfrak{sl}(N, \mathbb{C})$-valued $n$-forms twisted by a flat connection $\mathcal{A}^\alpha$:

$$\Delta_n(\mathcal{A}) = d_A \ast d_A \ast + \ast d_A \ast d_A , \quad d_A = d + \mathcal{A} \wedge . \quad (2.15)$$

The, $\det'(\Delta)$ denotes the zeta-function regularized product of non-zero eigenvalues of $\Delta$. To define the Laplacian, one needs to introduce a smooth metric on $M_3$. The torsion is however independent of the metric choice and is a topological invariant.

## 2.3 3d-3d relations

We consider three types of supersymmetric indices of 3d $\mathcal{N} = 2$ gauge theories: the superconformal index, the refined topologically-twisted index on $S^2$, and twisted index on $M_3$. The notation $\mathcal{A}^\alpha$ explicitly denotes the flat connection $\mathcal{A}^\alpha$. For example

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The, $\det'(\Delta)$ denotes the zeta-function regularized product of non-zero eigenvalues of $\Delta$. To define the Laplacian, one needs to introduce a smooth metric on $M_3$. The torsion is however independent of the metric choice and is a topological invariant.
general closed Riemann surfaces $\Sigma_g$ of genus $g$. Let us define
\[
\mathcal{I}_{\text{sci}}(q; T_N[M]) = (\text{Superconformal index of } T_N[M] \text{ theory on } S^2) \\
= \text{Tr}_{H_{\text{sci}}(S^2)} (-1)^R q^{\frac{R}{2}} j_3, \\
\mathcal{I}_{\text{top}}(q; T_N[M]) = (\text{Refined topologically twisted index of } T_N[M] \text{ theory on } S^2) \\
= \text{Tr}_{H_{\text{top}}(S^2)} (-1)^R q^{j_3},
\]
(2.16)
\[
\mathcal{I}_{\Sigma_g}(T_N[M]) = (\text{Topologically twisted index of } T_N[M] \text{ theory on } \Sigma_g) \\
= \text{Tr}_{H_{\text{top}}(\Sigma_g)} (-1)^R.
\]
Here $R$ is the R-charge corresponding the compact $SO(2)$ R-symmetry subgroup of $SO(5)$, while $j_3$ is the Cartan generator of the $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ isometry of $S^2$. Instead of $(-1)^R$ we use $(-1)^{2j_3}$ which typically appears in the context of 3d-3d correspondences. Then $H_{\text{sci}}(S^2)$ is the radially-quantized Hilbert space of $T_N[M]$, whose elements are linear combinations of local operators. Only 1/4 BPS operators satisfying $\Delta = R + j_3$, where $\Delta$ is the conformal dimension, contribute to the index. On the other hand, $H_{\text{top}}(\Sigma_g)$ is the Hilbert space of the topologically twisted $T_N[M]$ theory on a closed Riemann surface $\Sigma_g$. In the topological twisting, we turn on a background gauge field $A_R^{\text{background}}$ coupled to the $SO(2) \cong U(1)$ R-symmetry along $\Sigma_g$ in such a way that
\[
A_R^{\text{background}} = \omega(\Sigma_g) \Rightarrow \frac{1}{4\pi} \int_{\Sigma_g} dA_R^{\text{background}} = (1 - g) .
\]
(2.17)
Here $\omega(\Sigma_g)$ is the spin-connection on $\Sigma_g$. Due to the background magnetic flux, the following Dirac quantization conditions are required:
\[
(g - 1) R(\mathcal{O}) \in 2\mathbb{Z}
\]
(2.18)
for all gauge-invariant operators $\mathcal{O}$ in the theory. Thanks to (2.3), the quantization condition is automatically satisfied for $T_N[M]$ theories and we can consider the twisted index for arbitrary $g$. When $g = 0$, i.e., for $\Sigma_g = S^2$, there is an additional $\mathfrak{su}(2)$ rotational symmetry in the topologically twisted background, and we can refine the twisted index by turning on a fugacity $q$ associated to the Cartan generator $j_3$ of $\mathfrak{su}(2)$. As typical for a Witten index, the twisted index only gets contributions from ground states in $H_{\text{top}}(\Sigma_g)$, whose number is finite.

### 2.3.1 Superconformal index

The 3d-3d relation for the superconformal index was studied in [17–19, 28, 29], and the following proposal was made:
\[
\mathcal{I}_{\text{sci}}(q; T_N[M]) = Z^{SL(N,\mathbb{C})}_{k=0, \sigma = \frac{2\pi}{B_{M3}}} [M_3, \{n_{\alpha\beta} = \delta_{\alpha\beta}\}] \\
= \sum_{A^\alpha \in \chi_{\text{irred}}^+} \frac{1}{|\text{Stab}(\alpha)|} B^{\alpha}_{M_3}(q; N) B^{\alpha}_{M_3}(q^{-1}; N) \\
= \sum_{A^\alpha \in \chi_{\text{irred}}(M_3; N)} \frac{1}{|\text{Stab}(\alpha)|} B^{\alpha}_{M_3}(q; N) B^{\alpha}_{M_3}(q^{-1}; N).
\]
(2.19)
For an reducible flat connection $A^\alpha$, its stablizer subgroup is non-compact, $|\text{Stab}(A^\alpha)| = \infty$ and thus it does not contribute to the path-integral — see [30, 31] for more detailed discussions on this issue. This is one crucial difference between Chern-Simons theory with compact and non-compact gauge group. Therefore, we only need to sum over a subset $\chi_{\text{irred}}(M_3; N)$ of flat connections defined as

$$\chi_{\text{irred}}(M_3; N) := \text{set of irreducible gauge-inequivalent } SL(N, \mathbb{C}) \text{ flat connections on } M_3.$$  \hspace{1cm} (2.20)

In (2.19), $A^\overline{\alpha}$ denotes the complex conjugation of $A^\alpha$:

$$A^\alpha \rightarrow (A^\alpha)^*.$$  \hspace{1cm} (2.21)

The 3d-3d relation (2.19) can be further simplified using the following fact:

$$B_{M_3}^\alpha(q; N) = B_{M_3}^{\alpha}\bigg|_{\text{if } A^{\alpha_1} \text{ and } A^{\alpha_2} \text{ are related to each other by tensoring with a } \mathbb{Z}_N \text{ flat-connection.}}$$  \hspace{1cm} (2.22)

Here $\mathbb{Z}_N$ is the center of $SL(N, \mathbb{C})$. Let us explain this tensoring more explicitly. Each $SL(N, \mathbb{C})$ flat-connection $A^\alpha$ determines a group homomorphism $\rho^\alpha$

$$\rho^\alpha \in \text{Hom}(\pi_1(M_3) \rightarrow SL(N, \mathbb{C}))/SL(N, \mathbb{C})$$  \hspace{1cm} (2.23)

up to conjugation. Similarly, a $\mathbb{Z}_N$ flat-connection determines a homomorphism $\eta$

$$\eta \in \text{Hom}(\pi_1(M_3) \rightarrow \mathbb{Z}_N) = \text{Hom}(H_1(M_3, \mathbb{Z}) \rightarrow \mathbb{Z}_N).$$  \hspace{1cm} (2.24)

By tensoring with $\eta$, we can obtain from a flat connection $A^\alpha$ another flat connection $A^{\eta \otimes \alpha}$ whose $SL(N, \mathbb{C})$ holonomies are given by

$$\rho^{\eta \otimes \alpha}(c) = \eta(c) \cdot \rho^\alpha(c), \quad \forall c \in \pi_1(M_3).$$  \hspace{1cm} (2.25)

Here $\cdot$ is the group multiplication in $SL(N, \mathbb{C})$. Since there are only adjoint fields in $SL(N, \mathbb{C})$ Chern-Simons theory, we expect the equality in (2.22) to hold. Using (2.22) and the fact that

$$|\text{Stab}(A^\alpha \in \chi_{\text{irred}}(M_3; N))| = |\text{Center of } SL(N, \mathbb{C})| = N,$$  \hspace{1cm} (2.26)

we obtain the following final version of the 3d-3d relation for the superconformal index:

$$I_{\text{sci}}(q; T_N[M_3]) = Z^\text{SL(N,C)}_{k=0, \sigma = \frac{2\pi}{\log q}} \left[ M_3, \{ n_{\alpha \beta} = \delta_{\alpha \beta} \} \right]$$

$$= \frac{|\text{Hom}[\pi_1(M_3) \rightarrow \mathbb{Z}_N]|}{N} \sum\limits_{[A^\alpha] \in \chi_{\text{irred}}(M_3; N)} \sum\limits_{[A^\alpha] \in \text{Hom}[\pi_1(M_3) \rightarrow \mathbb{Z}_N]} B_{M_3}^{\alpha}(q; N) B_{M_3}^{\overline{\alpha}}(q^{-1}; N).$$  \hspace{1cm} (2.27)

Here $[A^\alpha]$ is an equivalence class in $\chi_{\text{irred}}(M_3; N)$ under the equivalence relation defined by tensoring with $\mathbb{Z}_N$ flat connections:

$$[A^\alpha] = [A^{\eta \otimes \alpha}], \quad \forall \eta \in \text{Hom}[\pi_1(M_3) \rightarrow \mathbb{Z}_N].$$  \hspace{1cm} (2.28)
2.3.2 Refined twisted index

From the factorization property of the refined index \cite{32,33},\footnote{The refined index for a tetrahedron (\(\Delta\)) theory \(T_{\Delta}\) \cite{22} can be written as \(B_\Delta(q,z)B_\Delta(\hat{q},\hat{z})\) with \(\hat{q} = q^{-1}\) and \(\hat{z} = z\). In the 3d-3d relation, \(q = e^{\frac{2\pi i}{N}}\) and \(\hat{q} = e^{\frac{2\pi i}{N}}\). So the relation \(\hat{q} = q^{-1}\) implies \(k = 0\). Then \(z\) and \(\hat{z}\) parameterize boundary \(SL(2,\mathbb{C})\) holonomies of the gauge fields \(A\) and \(\bar{A}\), respectively. Thus the relation \(\hat{z} = z\) implies \(n_{\alpha\beta} = \delta_{\alpha\beta}\).} we naturally propose the following 3d-3d relation:

\[
\mathcal{I}_{\text{top}}(q; T_N[M_3]) = Z_{k=0,\sigma}^\text{SL(N,C)}[M_3, \{n_{\alpha\beta} = \delta_{\alpha\beta}\}] \bigg|_{n=0,\sigma=0} = \left[\frac{\text{Hom}[\pi_1(M_3) \to Z_N]}{N}\right] \sum_{[A^\alpha] \in \chi_{\text{irred}}(M_3;N)} B^\alpha_M(q; N) B^\alpha_{M_3}(q^{-1}; N) .
\]

This looks similar to \(2.27\) but, as opposed to the former, this formula has no complex conjugation of the holomorphic blocks. A study of the refined twisted index in the context of the 3d-3d correspondence already appeared in \cite{34}. However, the relation we propose is different and complementary to theirs. In \cite{34}, the twisted index was related to \textit{homological} blocks labelled by Abelian flat connections. On the other hand, we propose a relation with \textit{holomorphic} blocks labelled by irreducible flat connections.

As an initial consistency check of our proposal, we can obtain the 3d-3d relation for the (unrefined) twisted index at \(g = 0\) put forward in \cite{20}, by taking the \(q \to 1\) limit of the refined index:

\[
\mathcal{I}_{\text{top}}(q; T_N[M_3]) \bigg|_{q \to 1} = \left[\frac{\text{Hom}[\pi_1(M_3) \to Z_N]}{N}\right] \sum_{[A^\alpha] \in \chi_{\text{irred}}(M_3;N)} B^\alpha_M(q; N) B^\alpha_{M_3}(q^{-1}; N) \bigg|_{q \to 1} = \left[\frac{\text{Hom}[\pi_1(M_3) \to Z_N]}{N}\right] \sum_{[A^\alpha]} \exp \left( \sum_{n=0}^\infty h^{n-1} S^\alpha_n \right) \exp \left( \sum_{n=0}^\infty (-h)^{n-1} S^\alpha_n \right) \bigg|_{h \to 0} \sum_{[A^\alpha]} \exp (2S^\alpha_1(M_3; N)) .
\]

Notice that only the 1-loop part contributes in the unrefined limit. The 3d-3d relation for twisted indices proposed in \cite{20} is

\[
\mathcal{I}_{\text{top}}(\mathcal{T}_N[M_3]) = \sum_{\alpha \in \chi_{\text{irred}}(M_3; N)} (N \times \exp(-2S^\alpha_1))^{g-1} ,
\]

for \(M_3\) with trivial \(H_1(M_3, \mathbb{Z}_N)\).

Such a relation was indirectly derived from combining several recent technical developments: i) field theoretic constructions of \(T_N[M_3] \cite{22–24}\); ii) field theory localization formulas of twisted indices \cite{32,35,36}; iii) mathematical tools \cite{37–39} for computing the Ray-Singer
torsion $e^{-2S^a}$ from state-integral models for $SL(N, \mathbb{C})$ Chern-Simons theory. The 3d-3d relation (2.31) with $g = 0$ precisely matches the $q \to 1$ limit (2.30) of the proposed 3d-3d relation (2.29).

2.3.3 Twisted indices

Let us compare the 3d-3d relation (2.29) with the following universal expression for the twisted index $I_{\Sigma^g=0}$ of general 3d $\mathcal{N} = 2$ theories $\mathcal{T}$:

$$I_{\Sigma^g=0}(\mathcal{T}) = \sum_{\alpha \in \mathcal{S}_{BE}(\mathcal{T})} (\mathcal{H}_\alpha)^{-1}. \quad (2.32)$$

Here $\mathcal{S}_{BE}(\mathcal{T})$ denotes the set of vacua\(^2\) of the 3d $\mathcal{N} = 2$ theory obtained by extremizing the effective twisted superpotential on $\mathbb{R}^2 \times S^1$ after summing all 1-loop contributions from infinitely many massive Kaluza-Klein modes along $S^1$. The order of the set is equal to the Witten index $[40, 41]$ of the 3d theory $\mathcal{T}$.

$$|\mathcal{S}_{BE}(\mathcal{T})| = (\text{Witten index of } \mathcal{T}). \quad (2.33)$$

On the other hand, $\mathcal{H}_\alpha$ is the so-called handle-gluing operator.

We naturally conjecture that

$$\mathcal{S}_{BE}(\mathcal{T}) = \frac{\chi_{\text{irred}}(M_3; N)}{\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]}, \quad \mathcal{H}_\alpha = \left| \text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N] \right| \exp(-2S^a_1(M_3; N)) . \quad (2.34)$$

This identification leads to following generalized 3d-3d relation for twisted indices:

$$I_{\Sigma^g}(\mathcal{T}[M_3]) = \sum_{\{\mathcal{A}^r\} \in \chi_{\text{irred}}(M_3; N)} \left( \frac{N}{\left| \text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N] \right|} \exp(-2S^a_1) \right)^{g-1} \quad (2.35)$$

for arbitrary closed hyperbolic 3-manifolds $M_3$.

While in [20] the 3d-3d relation was proposed for 3-manifolds with vanishing $H_1(M_3, \mathbb{Z}_N)$, here we generalize it to arbitrary closed hyperbolic 3-manifolds.

2.4 Integrality of indices at finite $N$

To test our proposals, in this subsection we verify a few integrality properties that the indices must satisfy.

\(^2\)The notation BE stands for “Bethe Equations”, because the 2d F-term equations from the effective twisted superpotential look similar to Bethe ansatz equations.
2.4.1 Integer Laurent series expansion of the refined twisted index

Since the $R$-charge is quantized as in (2.3) and $j_3 \in \frac{1}{2}\mathbb{Z}$, we expect that

$$I_{\text{top}}(q; T_N[M_3]) \in \mathbb{Z}[q^{1/2}, q^{-1/2}] ,$$

namely that the refined twisted index is a finite Laurent series in $q^{1/2}$ with integer coefficients. Indeed, the (refined) twisted index is a Witten index, and it only gets contributions from ground states in the twisted Hilbert space $\mathcal{H}_{\text{top}}(S^2)$ which is finite. Since the index is a finite Laurent polynomial in $q^{1/2}$, it converges in the limit $q = e^h \to 1$ and we can systematically study it using perturbative methods in complex Chern-Simons theory. The 3d-3d relation (2.29) then translates this rather obvious statement into the following highly non-trivial prediction on the perturbative invariants $\{S_n(M_3; N)\}$:

$$\left| \frac{[\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]]}{N} \sum_{[A^0] \in X_{\text{irred}}(M_3; N)} \exp \left( 2 \sum_{n=0}^{\infty} h^{2n} S_{2n+1}^0(M_3; N) \right) \right| \in \mathbb{Z}[e^{h/2}, e^{-h/2}] \quad \text{as formal expansion in } h .$$

In the following we will check that, in fact, an even stronger statement holds true: the LHS of (2.37) actually vanishes for many closed hyperbolic 3-manifolds $M_3$ (at least up to some high power of $h$).

**Example: $M_3 = (S^3 \setminus 4_1)_{P/Q=5}$ and $N = 2$.** See (A.1) for our notation of Dehn surgery representation of closed 3-manifolds. Here $4_1$ is the figure-eight knot. The hyperbolic volume of this 3-manifold is approximately

$$\text{vol}(M_3) \simeq 0.981369 .$$

The corresponding 3d gauge theory for $N = 2$ is [21]

$$T_{N=2}[M_3] = (U(1)_{k= -7/2} \text{ coupled to a single chiral } \Phi \text{ of charge } +1) .$$
The perturbative invariants are:

\[
S_0 \left( (S^3 \setminus \mathbb{A}_1)_5; N = 2 \right) = \text{Li}_2(e^{-z}) + i\pi(1 + 2\ell_z)Z - \frac{3}{2}Z^2 ,
\]

\[
S_1 \left( (S^3 \setminus \mathbb{A}_1)_5; N = 2 \right) = \log z - \frac{1}{2}\log(3z - 4) + \frac{1}{2}\log 2 ,
\]

\[
S_2 \left( (S^3 \setminus \mathbb{A}_1)_5; N = 2 \right) = \frac{27z^3 - 54z^2 + 92z - 64}{24(3z - 4)^3} ,
\]

\[
S_3 \left( (S^3 \setminus \mathbb{A}_1)_5; N = 2 \right) = \frac{5z(21z^3 - 49z^2 + 36z - 8)}{2(4 - 3z)^6} ,
\]

\[
S_4 \left( (S^3 \setminus \mathbb{A}_1)_5; N = 2 \right) = \frac{z\left(-19683 z^7 + 318087 z^6 + 75762 z^5 - 2103318 z^4 + 2881056 z^3 \right)}{720(3z - 4)^9}
\quad + \frac{z\left(-105608 z^2 - 265152z + 169856 \right)}{720(3z - 4)^9} ,
\]

\[
S_5 \left( (S^3 \setminus \mathbb{A}_1)_5; N = 2 \right) = \frac{z\left(-11907z^9 + 671895z^8 - 1057914z^7 - 2768940z^6 + 8509962z^5 \right)}{24(4 - 3z)^{12}}
\quad + \frac{z\left(-7816248 z^4 + 2119680z^3 + 809536z^2 - 511872z + 55808 \right)}{24(4 - 3z)^{12}} .
\]

(2.40)

Here \( z := e^Z \) and \( \ell_z \in \mathbb{Z} \) take different values on each irreducible flat connection. There are 4 irreducible \( SL(2, \mathbb{C}) \) flat connections on \( M_3 \), which correspond to the following 4 Bethe vacua:

\[
\exp \left( \partial_Z S_0 \right) \bigg|_{z = \log z} = 1 \quad \Rightarrow \quad S_{\text{BE}} \left( T_{N=2} [M_3 = (S^3 \setminus \mathbb{A}_1)_5] \right) = \left\{ z \left| \frac{1 - z}{z^4} = 1 \right. \right\} .
\]

(2.41)

The manifold \( M_3 \) has vanishing \( H_1(M_3, \mathbb{Z}_2) \) and there are no \( \mathbb{Z}_2 \) flat connections. This is compatible with the Witten index computation [41]:

\[
\text{Witten index of } (U(1)_k \text{ + fundamental } \Phi) = |k| + 1/2
\quad \Rightarrow \quad \text{Witten index of } T_{N=2} [M_3 = (S^3 \setminus \mathbb{A}_1)_5] = 4.
\]

For each solution to the Bethe vacua equation, there exists a solution \( Z \) satisfying \( \partial_Z S_0 = 0 \) for a proper choice of \( \ell_z \in \mathbb{Z} \): this fixed \( \ell_z \). Except for the classical part \( S_0 \), the perturbative invariants are independent of the choice of \( \ell_z \). The numerical values of \( Z, \ell_z \) on the Bethe vacua and their classical actions are

\[
\begin{align*}
\mathcal{A}_{\text{geom}} : Z &= 0.061412 + 1.33528i \quad (\ell_z = 0), \quad S_0 = -1.52067 - 0.981369i, \\
\mathcal{A}_{\text{geom}} : Z &= 0.061412 - 1.33528i \quad (\ell_z = -1), \quad S_0 = -1.52067 + 0.981369i, \\
\mathcal{A}_{3\text{rd}} : Z &= 0.199461 + 3.14159i \quad (\ell_z = 1), \quad S_0 = -15.5579, \\
\mathcal{A}_{4\text{th}} : Z &= -0.322285 \quad (\ell_z = -1), \quad S_0 = 2.14992.
\end{align*}
\]

(2.43)

Note that \( \text{Im} [S_0^{\text{geom}}] = -\text{Im} [S_0^{\text{geom}}] = \text{vol}(M_3) \). The \( \mathcal{A}_{3\text{rd}} \) and \( \mathcal{A}_{4\text{th}} \) are irreducible flat connections in \( SU(2) \subset SL(2, \mathbb{C}) \) and have vanishing \( \text{Im}[S_0] \). Using the perturbative invariants,
one can check that
\[
\mathcal{I}_{\text{top}} \left( q := e^{\theta}; T_{N=2}[M_3 = (S^3 \setminus \{4\})_{P/Q=5}] \right) \\
= \frac{1}{2} \sum_{1/z^i=1} \exp \left( 2S_1(M_3; N) + 2S_3(M_3; N)h^2 + 2S_5(M_3; N)h^4 + \mathcal{O}(h^6) \right) \bigg|_{M_3=(S^3 \setminus \{4\})_{P/Q=5}}^{N=2} = 0 + \mathcal{O}(h^6). \tag{2.44}
\]

Surprisingly, the refined twisted index vanishes (at least up to a certain power of \(h\)) in a highly non-trivial way. We have checked that the vanishing persists up to \(\mathcal{O}(h^{10})\).

**Example:** \(M_3 = (\text{Weeks manifold}) := (S^3 \setminus \{5\})_{P_1/Q_1=-5, P_2/Q_2=-5/2}\). This manifold is the smallest orientable closed hyperbolic 3-manifold [42], and its volume is
\[
\text{vol}(M_3) \simeq 0.94271. \tag{2.45}
\]

The corresponding 3d gauge theory for \(N = 2\) is [21]
\[
T_{N=2}[M_3] = (U(1)_{k=-5/2} \text{ coupled to a single chiral } \Phi \text{ of charge } +1). \tag{2.46}
\]

The Witten index (2.42) for the theory is 3 and there are 3 irreducible flat connections given by following Bethe equation:
\[
\exp \left( \partial_Z S_0 \right) \bigg|_{Z \to \log z} = 1 \Rightarrow \frac{z-1}{z^3} = 1. \tag{2.47}
\]

The manifold has vanishing \(H_1(M_3, \mathbb{Z}_2)\) and there are no \(\mathbb{Z}_2\) flat connections. The perturbative invariants \(\{S_n\}\) up to \(n = 5\) for the irreducible flat connections are
\[
S_0(\text{Weeks}; N = 2) = \text{Li}_2(e^{-Z}) - Z^2 + 2\pi i \ell, \\
S_1(\text{Weeks}; N = 2) = -\frac{1}{2} \log \left( 1 - \frac{3}{2z} \right), \\
S_2(\text{Weeks}; N = 2) = \frac{46z^2 - 72z + 27}{24(2z - 3)^3}, \\
S_3(\text{Weeks}; N = 2) = \frac{z(5z^4 + 9z^3 - 41z^2 + 36z - 9)}{2(3 - 2z)^6}, \\
S_4(\text{Weeks}; N = 2) = \frac{(z-1)z(988z^6 + 45700z^5 - 9642z^4 - 149328z^3 + 140103z^2)}{720(2z - 3)^9}, \\
\quad + \frac{(z-1)z(-18225z - 9477)}{720(2z - 3)^9}, \\
S_5(\text{Weeks}; N = 2) = \frac{(1-z)z(4z^9 - 4022z^8 - 34158z^7 + 64404z^6 + 74400z^5 - 209358z^4)}{24(3 - 2z)^{12}}, \\
\quad + \frac{(1-z)z(121041z^3 + 1296z^2 - 15795z + 2187)}{24(3 - 2z)^{12}}. \tag{2.48}
\]
Using the perturbative invariants, one can check that

\[
\mathcal{I}_{\text{top}} \left( q := e^h; T_{N=2}[M_3 = \text{Weeks}] \right) \\
= \frac{1}{2} \sum_{z \in \mathbb{Z} \cap 1} \exp \left( 2S_1(M_3; N) + 2S_3(M_3; N)h^2 + 2S_5(M_3; N)h^4 + O(h^6) \right) \bigg|_{M_3 = \text{Weeks}}^{N=2} \\
= 0 + O(h^6). \tag{2.49}
\]

Again, the refined index vanishes in a highly non-trivial way — we have checked the vanishing up to \( O(h^{10}) \).

**Example:** \( M_3 = (S^3 \setminus 5T_1)_{P_1/Q_1 = -6, P_2/Q_2 = -3/2} \). This manifold is the 3rd smallest orientable closed hyperbolic 3-manifold [42], and its volume is

\[
\text{vol}(M_3) \simeq 1.01494. \tag{2.50}
\]

The corresponding 3d gauge theory for \( N = 2 \) is [21]

\[
T_{N=2}[M_3] = (U(1)_{k=-3/2} \text{ coupled to a single chiral } \Phi \text{ of charge } +1). \tag{2.51}
\]

Using the state-integral model for the 3-manifold, we have checked that the refined index computed using the 3d-3d relation (2.29) vanishes up to \( O(h^{10}) \).

### 2.4.2 Integer expansion of the twisted index

On a generic Riemann surface \( \Sigma_g \), thus with no \( q \)-refinement, we expect

\[
\mathcal{I}_{\Sigma_g}(T_{N=2}[M_3]) \in \mathbb{Z}. \tag{2.52}
\]

According to the 3d-3d relation (2.35), the (unrefined) twisted index is determined by the 1-loop invariants \( S_1^n[M_3; N] \). The 1-loop part is simply related to the mathematical quantity called analytic Ray-Singer torsion, see (2.13). Computing such topological quantity using its definition (2.14) is a quite challenging task, since we need to know the full spectrum of the Laplacians on the 3-manifold. There are two simpler alternative ways to compute the Ray-Singer torsion: i) using a state-integral model for complex Chern-Simons theory, and ii) using Cheeger-Muller theorem. See Appendix A for a brief introduction to the state-integral model. Cheeger-Muller theorem claims equivalence between the analytic Ray-Singer torsion and the combinatorial Reidermeister torsion. The combinatorial torsion can be computed from so-called Fox calculus on the \( SL(N, \mathbb{C}) \) representations of \( \pi_1(M_3) \). In [38], it is explicitly checked that the two simpler approaches give the same answer in a large number of examples. In this subsection, we give the explicit expression for the combinatorial Reidermeister torsion \( \text{Tor}[A^\alpha; M_3] \) (equal to the analytic Ray-Singer torsion) twisted by \( SL(2, \mathbb{C}) \) irreducible flat connections \( A^\alpha \) on \( M_3 = (S^3 \setminus 4T_1)_{P/Q} \) and check the integrality of the \( \mathcal{I}_{\Sigma_g}(T_{N=2}[M_3]) \) in (2.35).
Example: \((S^3\setminus 4)_{P/Q}\). These manifolds are hyperbolic for all \(P/Q \in \mathbb{Q} \cup \{\infty\}\) but the following 10 exceptional slopes:

Exceptional slopes of \(S^3\setminus 4\) : \(P/Q = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \infty\}\). \hspace{1cm} (2.53)

The fundamental groups of these 3-manifolds are 

\[
\pi_1(S^3\setminus 4) = \langle a, b : ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b \rangle, \\
\pi_1(M_3) = \langle a, b : ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b, mP^Q = 1 \rangle, \text{ where } m := a, \quad 1 := ab^{-1}aba^{-2}bab^{-1}a^{-1}. \hspace{1cm} (2.54)
\]

The set \(\chi_{\text{irred}}(M_3 = (S^3\setminus 4)_{P/Q}, N = 2)\) can be obtained by solving the following matrix equations:

\[
\chi_{\text{irred}}(M_3, N = 2)\big|_{M_3 = (S^3\setminus 4)_{P/Q}} = \left\{ A = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, B = \begin{pmatrix} m & 0 \\ y & m^{-1} \end{pmatrix} : \text{matrix equations} \right\}/\mathbb{Z}_2, \hspace{1cm} (2.55)
\]

where the matrix equations are

\[
AB^{-1}A^{-1}BA = BAB^{-1}A^{-1}B \text{ and } M^P L^Q = I \text{ with } M := A, \quad L = AB^{-1}ABA^{-2}BAB^{-1}A^{-1}.
\]

Indeed \(a\) and \(b\) are related to each other by a conjugation in \(\pi_1(M)\) and their corresponding holonomy matrices \(A\) and \(B\) are related by a conjugation of \(SL(2, \mathbb{C})\). Using the \(SL(2, \mathbb{C})\) conjugation, we fix the matrices \(A\) and \(B\) as above. There is a residual \(\mathbb{Z}_2\) gauge symmetry acting as

\[
\mathbb{Z}_2 : (m, y) \leftrightarrow (m^{-1}, y). \hspace{1cm} (2.56)
\]

The matrix equations give algebraic equations for \(m\) and \(y\), which can be solved numerically. When \(P/Q = 5\), we have the following 4 solutions which can be identified with 4 Bethe vacua in (2.43):

\[
(m, y) = (1.20331 + 0.780836i, 0.992448 + 0.513116i) \leftrightarrow \mathcal{A}^{\text{geom}} \\
(m, y) = (1.20331 - 0.780836i, 0.992448 - 0.513116i) \leftrightarrow \mathcal{A}^{\text{geom}} \\
(m, y) = (0.164478 + 0.986381i, 3.49022) \leftrightarrow \mathcal{A}^{3\text{rd}} \\
(m, y) = (-0.452583 + 0.891722i, 2.52489) \leftrightarrow \mathcal{A}^{4\text{th}} \hspace{1cm} (2.57)
\]

The torsion can be computed from the following formula (see the appendix of [20] for the derivation):

\[
\exp(-2S^3_M(M_3, N = 2))\big|_{M_3 = (S^3\setminus 4)_{P/Q}} = P \left( \ell - \frac{1}{\ell} \right) \frac{m^4}{(m^4 - 1)(4 - 2m^2 + 4m^4) + Q} \left( \frac{2m^2 + 2}{m^2} - 2 \right) \frac{1}{(1 - m^2 R^{2S})(1 - m^2 R^{-2S})}. \hspace{1cm} (2.58)
\]
Here \( \ell \) is defined from the following relation:

\[
L = \begin{pmatrix}
\ell & 0 \\
0 & \ell^{-1}
\end{pmatrix}.
\] (2.59)

After solving the matrix equations, the two \( SL(2, \mathbb{C}) \) matrices \( M \) and \( L \) commute since \( ml = lm \) in \( \pi_1(S^3 \backslash 4_1) \). So \( L \) takes the above upper triangular form after solving the matrix equations. Co-prime integers \( (R, S) \) are chosen such that \( PS - QR = 1 \). For given \( (P, Q) \), the choice is not unique but the above torsion is independent of the choice since the matrix equation \( M^P L^Q = I \) implies \( m^P \ell^Q = 1 \). By applying the formula to \( P/Q = 5 \), we find that

\[
\exp \left( -2S_1^\alpha(M_3, N = 2) \right) \bigg|_{\alpha = \text{geom}}^{M_3 = (S^3 \backslash 4_1)_{P/Q = 5}} = 1.90538 - 0.568995i,
\]

\[
\exp \left( -2S_1^\alpha(M_3, N = 2) \right) \bigg|_{\alpha = 3\text{rd}}^{M_3 = (S^3 \backslash 4_1)_{P/Q = 5}} = -2.57085,
\]

\[
\exp \left( -2S_1^\alpha(M_3, N = 2) \right) \bigg|_{\alpha = 4\text{th}}^{M_3 = (S^3 \backslash 4_1)_{P/Q = 5}} = -1.73992.
\] (2.60)

One can check that the numerical values are equal to \( e^{-2S_1^\alpha} \) in (2.40) for the Bethe vacua (2.43) computed from the state-integral model.

Combining the torsion computations in (2.55) and (2.58) with the 3d-3d relation in (2.35), we obtain the following concrete expression for the twisted index:

\[
\mathcal{I}_{\Sigma_g}(T_{N=2}[M_3]) = \frac{1}{|\text{Hom}(\pi_1(M_3) \to \mathbb{Z}_2)|}
\times \sum_{(m, y) \in \chi_{\text{irred}} \text{ in (2.55)}} \left( \frac{2}{|\text{Hom}(\pi_1(M_3) \to \mathbb{Z}_2)|} (\exp(-2S_1) \text{ in (2.58)}) \right)^{g-1}
\]

for \( M_3 = (S^3 \backslash 4_1)_{P/Q} \).

\[
\mathcal{I}_{\Sigma_g}(T_{N=2}[M_3])
\]

(2.61)

Note that

\[
|\text{Hom}(\pi_1(M_3) \to \mathbb{Z}_2)| = \begin{cases} 1 & \text{for odd } P, \\
2 & \text{for even } P. 
\end{cases}
\] (2.62)

Using the expression above, one can check that \( \mathcal{I}_{\Sigma_g}(T_{N=2}[M_3]) \in \mathbb{Z} \), which is quite non-trivial. For example, using the torsion in (2.60), one obtains

\[
\mathcal{I}_{\Sigma_g}(T_{N=2}[M_3]) \bigg|_{M_3 = (S^3 \backslash 4_1)_{P/Q = 5}} = \sum_{\alpha} 2^{g-1} \exp \left( -2(g - 1)S_1^\alpha(M_3, N = 2) \right) \bigg|_{M_3 = (S^3 \backslash 4_1)_{P/Q = 5}}
\]

\[
= \{ 0_{g=0}, 4_{g=1}, -1_{g=2}, 65_{g=3}, -97_{g=4}, \ldots \}
\] (2.63)
For $P/Q = 6$, there are 6 irreducible flat connections in $\chi_{\text{irred}}(M_3; N = 2)$ but there are only 3 equivalence classes $[A^\alpha] \in \frac{\chi_{\text{irred}}(M_3; N = 2)}{\text{Hom}(\pi_1(M_3) \to \mathbb{Z}_N)}$. The numerical values of the torsion for the 3 classes are

$$\{\exp\left(-2S_1^\alpha(M_3; N = 2)\right)\}_{[A^\alpha] \in \frac{\chi_{\text{irred}}(M_3; N = 2)}{\text{Hom}(\pi_1(M_3) \to \mathbb{Z}_N)}}$$

for $M_3 = (S^3 \setminus \mathcal{A}_1)_{P/Q=6}$

$$= \{4.30835 - 1.99637i, 4.30835 + 1.99637i, -2.61671i\}.$$ (2.64)

Then, we see that

$$\mathcal{I}_{\Sigma_g}(T_{N=2}[M_3]) \bigg|_{M_3=(S^3 \setminus \mathcal{A}_1)_{P/Q=6}} \sum_{[A^\alpha] \in \frac{\chi_{\text{irred}}(M_3; N = 2)}{\text{Hom}(\pi_1(M_3) \to \mathbb{Z}_N)}} \exp\left(-2(g - 1)S_1^\alpha(M_3; N = 2)\right) \bigg|_{M_3=(S^3 \setminus \mathcal{A}_1)_{P/Q=6}}$$

$$= \{0_g=0, 3_g=1, 6_g=2, 36_g=3, 39_g=4, \ldots\}.$$ (2.65)

This integrality gives a non-trivial consistency check for all the subtle factors in the 3d-3d relation (2.35). These factors will play an important role in the computation of the log $N$ subleading corrections to the twisted indices at large $N$.

### 3 Large $N$ limit of the indices

We consider now the large $N$ limit of three types of indices of $T_N[M_3]$ theory using the 3d-3d relations in (2.27),(2.29) and (2.35). Through the 3d-3d relations, they are all related to the holomorphic blocks (2.10) of $SL(N, \mathbb{C})$ Chern-Simons theory on $M_3$. The blocks can be perturbatively expanded in the holomorphic coupling. We have two expansion parameters, $1/N$ and $\hbar$. We first consider the perturbative expansion in $\hbar$, and then take the large $N$ limit of the perturbative expansion coefficients $\{S_n(M_3; N)\}$.

#### 3.1 Perturbative Chern-Simons invariants

There are two canonical $SL(N, \mathbb{C})$ irreducible flat connections on an hyperbolic 3-manifold $M_3$, denoted $A_N^{\text{geom}}$ and $A_N^{\text{geom}}$:

$$A_N^{\text{geom}} = \rho_N(\omega + ie), \quad A_N^{\text{geom}} = \rho_N(\omega - ie).$$ (3.1)

According to Mostow’s rigidity theorem [43], there is a unique hyperbolic metric on $M_3$ satisfying $R_{\mu\nu} = -2g_{\mu\nu}$. Here $e$ and $\omega$ are dreibein and spin connection of the hyperbolic structure, respectively. Both of them can be considered as $\mathfrak{so}(3)$-valued 1-forms and they form two $SL(2, \mathbb{C})$ flat connections, $\omega \pm ie$. On the other hand, $\rho_N : SL(2) \to SL(N)$ is the principal embedding. A characteristic property of the two irreducible flat connections is

$$\text{Im}[S_0^{\text{geom}}] \leq \text{Im}[S_0^\alpha] \leq \text{Im}[S_0^{\text{geom}}] \quad \forall \ [A^\alpha] \in \frac{\chi_{\text{irred}}(M_3; N)}{\text{Hom}(\pi_1(M_3) \to \mathbb{Z}_N)}.$$ (3.2)

The equality holds for only for $\alpha = \text{geom}$ or $\alpha = \overline{\text{geom}}$. The classical and 1-loop parts around the flat connections can be expressed in terms of the hyperbolic volume of the
3-manifold, vol($M_3$), and the complex length spectrum $\{\ell_C(\gamma)\}$:

$$\text{Im}[S_{0}^{\text{geom}}(M_3; N)] = \frac{(N^3 - N)}{6} \text{vol}(M_3),$$

$$\text{Re}[S_{1}^{\text{geom}}(M_3; N)] = -\frac{\text{vol}(M_3)}{12\pi} (2N^3 - N - 1) - \frac{1}{2} \sum_{\gamma} \sum_{m=1}^{N-1} \sum_{k=m+1}^{\infty} \log |1 - e^{-k\ell_C(\gamma)}|. \quad (3.3)$$

Here we defined

$$\text{vol}(M_3) := \text{(volume of } M_3 \text{ measured in the unique hyperbolic metric)}. \quad (3.4)$$

Moreover $\sum_{\gamma}$ is the sum over the non-trivial primitive geodesics $\gamma \in M_3$, and $\ell_C(\gamma)$ is the complexified geodesic length of $\gamma$, which is defined by the following relation:

$$\text{Tr Pexp} \left( -\oint_{\gamma} A_3^{\text{geom}} N = 2 \right) = e^{\frac{1}{2} \ell_C(\gamma)} + e^{-\frac{1}{2} \ell_C(\gamma)}, \quad \text{Re}[\ell_C] > 0. \quad (3.5)$$

The real part of $\ell_C$ measures the geodesic length. The classical part of the Chern-Simons action can be obtained using the following computation ($h_1$ and $h_2$ are non-zero elements in $\mathfrak{sl}(2, \mathbb{C})$):

$$\text{Im}[S_{0}^{\text{geom}}(M_3; N)] = \text{Im}(-2\pi \text{CS}[A_N^{\text{geom}}, M_3])$$

$$= \frac{\text{Tr}(\rho_N(h_1) \rho_N(h_2))}{\text{Tr}(\rho_{N=2}(h_1) \rho_{N=2}(h_2))} - \text{Im}(-2\pi \text{CS}[A_{N=2}^{\text{geom}}, M_3])$$

$$= \frac{(N^3 - N)}{6} \text{Im}(-2\pi \text{CS}[A = \omega + i\epsilon, M_3]) = -\frac{(N^3 - N)}{24} \int_{M_3} \sqrt{g}(R + 2) R_{\mu\nu} = -2 g_{\mu\nu}$$

$$= \frac{(N^3 - N)}{6} \int_{M_3} \sqrt{g} = \frac{(N^3 - N)}{6} \text{vol}(M_3). \quad (3.6)$$

The expression for the 1-loop part $S_{1}^{\text{geom}}$ is derived in [20] from results [44, 45] in mathematics using Selberg’s trace formula. To see the $1/N$ expansion more explicitly, we can write

$$\sum_{\gamma} \sum_{m=1}^{N-1} \sum_{k=m+1}^{\infty} \log |1 - e^{-k\ell_C(\gamma)}|$$

$$= -\text{Re} \sum_{\gamma} \sum_{s=1}^{\infty} \frac{1}{s} \left( \frac{e^{-s\ell_C}}{1 - e^{-s\ell_C}} \right)^2 + \text{Re} \sum_{\gamma} \sum_{s=1}^{\infty} \frac{1}{s} \left( \frac{e^{-\frac{s(N+1)}{2} \ell_C}}{1 - e^{-\frac{s(N+1)}{2} \ell_C}} \right)^2. \quad (3.7)$$

The first term is $N$-independent while the second one is exponentially suppressed at large $N$. 

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The higher perturbative invariants \( S_n^{\text{geom}} \) (\( n \geq 2 \)) are conjectured to behave at large \( N \) as \([46]^{\text{3}}\)
\[
\text{Im}[S_2^{\text{geom}}(M_3; N)] = -\frac{N^3}{24\pi^2} \text{vol}(M_3) + \text{subleading} ,
\]
\[
\lim_{N \to \infty} \frac{1}{N^3} \text{Re}[S_{2n+1}^{\text{geom}}(M_3; N)] = 0 \quad (n \geq 1) ,
\]
\[
\lim_{N \to \infty} \frac{1}{N^3} \text{Im}[S_{2n}^{\text{geom}}(M_3; N)] = 0 \quad (n \geq 2) .
\]
(3.8)

Unlike \( S_0^{\text{geom}} \) and \( S_1^{\text{geom}} \), we only know the leading \( O(N^3) \) terms at large \( N \). The large \( N \) behavior of \( S_n^{\text{geom}} \) can also be obtained using the following general fact:
\[
S_n^\pi = (S_n^\pi)^* .
\]
(3.9)

The conjecture (3.8) has been numerically tested in many examples up to \( n = 3 \). One way to understand the conjecture is the S-duality of complex Chern-Simons theory \([47]\) relating \( h \leftrightarrow -\frac{4\pi^2}{N^2} \). At finite \( N \), the S-duality is only seen after Borel-Padé resummation \([48]\). At large \( N \), on the other hand, the perturbative expansion becomes a convergent series as far as the leading \( O(N^3) \) behavior is concerned, and one can see the S-duality directly in \( O(N^3) \) coefficients of the perturbative invariants.

3.2 Superconformal index

3.2.1 Leading \( N^3 \) behavior

Combining the the 3d-3d relation (2.27) with the large \( N \) behavior (3.3) and (3.8), and setting \( q := e^{-\omega} \) with \( \text{Im} \omega > 0 \) and \( |\omega| \ll 2\pi \), we find
\[
\mathcal{I}_{\text{sci}}(q; \mathcal{T}_N[M_3]) = \frac{\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]}{N} \sum_{[A^n] \in ^{\text{geom}}\text{Hom}^n(M_3; \mathbb{Z}_N)} B^n(q) B^n(q^{-1})
\]
\[
\lim_{N \to \infty} \frac{|\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]|}{N} B_{\text{geom}}(q) B_{\text{geom}}(q^{-1}) + \ldots
\]
\[
= \frac{\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]}{N} \exp \left( \sum_{n=0}^{\infty} (-\omega)^n S_n^{\text{geom}} \right) \exp \left( \sum_{n=0}^{\infty} \omega^{n-1} (S_n^{\text{geom}})^* \right) + \ldots
\]
\[
= \frac{|\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]|}{N} \exp \left( -i \frac{N^3 \text{vol}(M_3)}{3\omega} - i \frac{N^3 \text{vol}(M_3)}{3\pi} + i \frac{N^3 \text{vol}(M_3)}{12\pi^2} + \omega + \text{subleading} \right) + \ldots
\]
(3.10)

where dots stand for exponentially smaller terms. Thus, we have (for \( \text{Im} \omega > 0 \) and \( |\omega| \ll 2\pi \))
\[
\log \mathcal{I}_{\text{sci}}(q = e^{-\omega}; \mathcal{T}_N[M_3]) = -i \frac{N^3 \text{vol}(M_3)}{3} \left( \frac{1}{\omega} - i \frac{\omega}{4\pi^2} \right) + \text{(subleading in } 1/N) \]
\[
= i \frac{L^2}{8G_4\omega} (\omega + 2\pi i)^2 + \text{(subleading in } G_4 \text{ and } 1/L) .
\]
(3.11)
Here $L$ is the radius of AdS$_4$ and $G_4$ is the 4d Newton constant. We used the relation [20]

$$L^2 G_4 = \frac{2N^3}{3\pi^2} \text{vol}(M_3).$$

(3.12)

The large $N$ free energy nicely matches with the entropy function in (4.18).

### 3.2.2 Logarithmic subleading corrections

The large $N$ logarithmic subleading correction, proportional to $\log N$, is independent of the continuous deformation parameter $\omega$. So we can compute it in the $\omega \to 0$ limit, which is determined by the overall factor $|\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]|/N$ and the perturbative expansion of $S_{0,1}^{\text{geom}}$.

$$\log |\mathcal{I}_{\text{sc}}(q = e^{-\omega}; T_N[M_3])|_{\omega \to 0, N \gg 1} = -\frac{2i}{\omega} \text{Im}[S_{0}^{\text{geom}}(M_3; N)] + 2 \text{Re}[S_{1}^{\text{geom}}(M_3; N)] + \log \frac{|\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]|}{N} + O(\omega).$$

(3.13)

From (3.3) we see that there is no $\log N$ corrections from the first two terms, $\text{Im}[S_{0}^{\text{geom}}]$ and $\text{Re}[S_{1}^{\text{geom}}]$. The $\log N$ term only comes from the last term which is

$$\log \frac{|\text{Hom}[\pi_1(M_3) \to \mathbb{Z}_N]|}{N} \sim \log N \frac{b_1(M_3)}{N} \sim (b_1(M_3) - 1) \log N.$$

(3.14)

This matches the supergravity analysis in (5.3) with $g = 0$. The comparison between the computations of logarithmic corrections in $SL(N, \mathbb{C})$ Chern-Simons theory and in supergravity is summarized in Table 1.

| $- \log N$ | Supergravity on AdS$_4 \times M_3 \times S^4$ |
|-----------|----------------------------------|
| $\frac{1}{N} \log \frac{b_1(M_3)}{N}$ | Two-form zero modes on AdS$_4$ from 3-form $C(3)$ in the 11d supergravity |

Table 1. Comparison of logarithmic correction computations for superconformal index of $T_N[M_3]$ from 3d-3d dual $SL(N, \mathbb{C})$ Chern-Simons theory and from holographic dual supergravity.

### 3.3 Twisted indices for $g > 1$

A remarkable aspect of the 3d-3d relation for twisted indices (2.35) is that only the 1-loop invariant $S^0_\eta$ appears in the relation. Since we know the full perturbative $1/N$ expansion of the 1-loop part, we can compute the full perturbative $1/N$ corrections to the twisted
the contributions from the two dominant Bethe vacua, where dots stand for exponentially smaller terms. The last term comes from summing over 3-manifolds with vanishing $H_1(M_3,Z_N)$. With the more general 3d-3d relation for twisted indeces (2.35), we can repeat the large $N$ analysis for general closed hyperbolic 3-manifold $M_3$. Interestingly, the logarithmic subleading correction $\log N$ can detect the first Betti number $b_1(M_3)$ of the 3-manifold. See [49] for previous studies on full perturbative $1/N$ corrections to the twisted index for different classes of 3d theories.

3.3.1 All orders in $1/N$

Combining the 3d-3d relation (2.35) with the large $N$ behavior (3.3) and (3.8), we obtain for $g > 1$:

$$I_{\Sigma_3}(T_N[M_3]) \xrightarrow{gN \to \infty} \frac{N^{g-1} \exp\left( -2(g-1)S^\text{geom}(M_3;N) \right)}{|\Hom(\pi_1(M_3) \to Z_N)|^{g-1}} + \text{c.c.} + \ldots$$

$$= \exp\left( (g-1)\frac{\text{vol}(M_3)}{6\pi}(2N^3 - N - 1) - (g-1)\text{Re} \sum_{\tau} \sum_{s=1}^{\infty} \frac{1}{s} \left( \frac{e^{-st\tau}}{1 - e^{-st\tau}} \right)^2 \right)$$

$$- (g-1) \log \frac{|\Hom(\pi_1(M_3) \to Z_N)|}{N} \xrightarrow{N \to \infty} (g-1)(1 - b_1(M_3)) \log N. \quad (3.16)$$

where dots stand for exponentially smaller terms. The last term comes from summing over the contributions from the two dominant Bethe vacua, $\alpha = \text{geom}$ and $\alpha = \text{geom}$, which differ only by a phase factor $e^{i(g-1)\theta_{N,M_3}}$. For $M_3$ with vanishing $H_1(M_3,Z_N)$, the large $N$ analysis was already done in [20] and it was checked that the leading $O(N^3)$ term nicely matches the entropy of magnetically-charged universal black holes in the holographic dual AdS$_4$.

The logarithmic correction is

$$-(g-1) \log \frac{|\Hom(\pi_1(M_3) \to Z_N)|}{N} \xrightarrow{N \to \infty} (g-1)(1 - b_1(M_3)) \log N. \quad (3.17)$$

This also matches the supergravity analysis in (5.3).

3.4 Refined index: no $N^3$ behavior

From (2.29), (3.3) and (3.8) we obtain

$$I_{\text{top}}(q = e^{-\omega}; T_N[M_3]) = \frac{|\Hom(\pi_1(M_3) \to Z_N)|}{N} \sum_{[A^\alpha] \in X_{n\alpha}(M_3;N)} B^\alpha(q) B^\alpha(q^{-1})$$

$$= \frac{|\Hom(\pi_1(M_3) \to Z_N)|}{N} B^\text{geom}(q) B^\text{geom}(q^{-1}) + \text{c.c.} + \text{(contributions from other } \alpha)$$

$$= \frac{|\Hom(\pi_1(M_3) \to Z_N)|}{N} \exp \left( 2 \sum_{n=0}^{\infty} (-\omega)^{2n} \frac{S^\text{geom}_{2n+1}}{S_{2n+1}} \right) + \ldots$$

$$= \frac{|\Hom(\pi_1(M_3) \to Z_N)|}{N} \exp \left( -\frac{2N^3 \text{vol}(M_3)}{3\pi} + \text{subleading} \right) + \ldots. \quad (3.17)$$
This time dots stand for contributions from other $\alpha$. We see that for the refined index, unlike for the superconformal index, the contributions from $A_{\text{geom}}$ and $\hat{A}_{\text{geom}}$ are exponentially small in the large $N$ limit. This is compatible with the curious observation in section 2.4.1 that the refined twisted index seems to actually vanish at finite $N$. This is also compatible with the non-existence of magnetically charged black hole with $\text{AdS}_2 \times S^2$ horizon in the universal sector as will be discussed in subsection 4.2.

4 M5 Branes wrapped on hyperbolic $M_3$ and $\mathcal{N} = 2$ Gauged Supergravity

To motivate the holographic description of the field theories discussed in the previous sections, we start in eleven dimensions where the M5-brane naturally resides. Since the AdS/CFT dictionary identifies the conformal algebra in the field theory with the isometries of the spacetime on the gravity side, we expect a gravitational solution given by a back-reacted AdS solution where M5-branes are partly wrapped on $M_3$.

The study of various solutions representing the supergravity description of M5 branes wrapping hyperbolic 3-manifolds has a distinguished history starting with [50]. Although much of the emphasis has been on the construction of explicit solutions, starting with AdS$_4$ vacuum solutions and expanding into more general black hole solutions, there are fruitful efforts in the direction of consistent truncations [46, 51]. Our strategy for presentation of the consistent truncation is to present the AdS$_4$ vacuum solution in the 11d context and then to indicate how the setup generalizes to yield a consistent truncation of 11d supergravity into 4d $\mathcal{N} = 2$ gauged supergravity with just the gravi-photon field.

The central intuition for understanding the 11d origin of M5 branes wrapping hyperbolic 3-manifolds is provided by the Pernici-Sezgin solution to 7d gauged supergravity originally constructed in [52]; roughly it discusses solutions of the type AdS$_4 \times M_3$. The original 11d point of view was discussed in [51]. In this section, however, we will follow closely the notation and presentation of [53] which is a rewriting of the more geometric $\text{SU}(2)$-structure based approach [54]. In 11d the solutions has the form of $\text{AdS}_4 \times Y_7$ where $Y_7$ is an $S^4$ fibration over the hyperbolic 3-manifold $M_3$ [53]:

$$ds_{11}^2 = \lambda^{-1} ds^2(\text{AdS}_4) + ds^2_4(\mathcal{M}_{\text{SU}(2)}) + \hat{\omega} \otimes \hat{\omega} + \lambda^2 \left( \frac{d\rho^2}{1 - \lambda^3 \rho^2} + \rho^2 d\psi^2 \right), \quad (4.1)$$

where $\lambda$ is the warp factor, $\mathcal{M}_{\text{SU}(2)}$ is a 4d space with $\text{SU}(2)$-structure, $\hat{\omega}$ is a one-form, $\rho$ is the interval coordinate and $\psi \in [0, 2\pi]$ is a coordinate on the circle $S^1_R$. The associated Killing vector $\partial/\partial \psi$ is dual to the field-theoretic $U(1)$ R-symmetry.

The particular form of the Pernici-Sezgin solution [52] is (see also [55]):

$$ds^2_4(\mathcal{M}_{\text{SU}(2)}) + \hat{\omega} \otimes \hat{\omega} = f^2(\rho)DY^aDY_a + g^2(\rho)e^ae^a, \quad (4.2)$$

where $Y^a, a = 1, 2, 3$ are coordinates on $S^2$, $Y^aY_a = 1$, and $e^a$ are vielbeins for the 3-manifold $M_3$. The form of the covariant derivative is

$$DY^a = dY^a + \omega^a_bY^b, \quad (4.3)$$
where $\omega^{ab}$ is the spin connection of $M_3$. The supersymmetry conditions dictate:

$$
\lambda^3 = \frac{2}{8 + \rho^2}, \quad f = \frac{\sqrt{1 - \lambda^3\rho^2}}{2\sqrt{\lambda}}, \quad g = \frac{1}{2^{3/2}\sqrt{\lambda}}.
$$

(4.4)

The coordinates $Y^a, \rho, \psi$ would have formed a round $S^4$ in the un-deformed case but here they form a space we denote by $X_4$. A concrete way to realize the hyperbolic manifold $M_3$ is by taking a quotient of $H^3$ by a discrete subgroup $\Gamma \in PSL(2, \mathbb{C})$: $\mathbb{H}^3/\Gamma$.

The background is:

$$
\begin{align*}
\sqrt{s_{11}}^2 &= \frac{1}{4}\lambda^{-1}\left[4ds^2(AdS_4) + \frac{dx^2 + dy^2 + dz^2}{s^2} + \frac{8 - \rho^2}{8 + \rho^2}DY^aDY^a + \frac{1}{2}\left(\frac{d\rho^2}{8 - \rho^2} + \frac{\rho^2}{8 + \rho^2}d\psi^2\right)\right],
G_4 &= \frac{1}{4}d\psi \wedge d\left(\lambda^{-1/2}\sqrt{1 - \lambda^3\rho^2}J_3\right).
\end{align*}
$$

(4.5)

The slightly more general class of solutions involves the standard two-forms $(J_1, J_2, J_3)$ defining the $SU(2)$ structure [54]. The Ansatz for the structure is

$$
\begin{align*}
\hat{w} &= gY^a e^a, \\
J_1 &= fgDY^a \wedge e^a, \\
J_2 &= fg\epsilon^{abc}Y^aDY^b \wedge e^c, \\
J_3 &= \frac{1}{2}\epsilon^{abc}Y^a\left(f^2DY^b \wedge Dy^c - g^2e^b \wedge e^c\right)
\end{align*}
$$

(4.6)

To understand the quantization condition of the flux and the identification with the number of M5 branes we need to consider the 4-form field $G_4$ of 11d supergravity when restricted to the internal space $X_4$,

$$
G_4|_{X_4} = \frac{1}{4}d\psi \wedge d\left(\lambda^{-1/2}\sqrt{1 - \lambda^3\rho^2}J_3\right).
$$

(4.7)

The free energy can be easily computed [54] to yield

$$
\mathcal{F} = \frac{\pi^5\text{Vol}(M_3)}{2(2\pi l_p)^4}.
$$

(4.8)

The quantization of the four-form flux leads to a relation between Planck’s constant and $N$:

$$
N = \frac{1}{(2\pi l_p)^3} \int_{X_4} G_4 = \frac{1}{8\pi l_p^3}.
$$

(4.9)

Substituting, we write the free energy in a more field theoretic way:

$$
\mathcal{F} = \frac{N^3}{3\pi}\text{Vol}(M_3).
$$

(4.10)

This result was generalized to account for the free energy corresponding to a squashed 3-sphere $S^3_b$ is [56, 57]

$$
\mathcal{F}_b = \frac{1}{4}\left(b + \frac{1}{b}\right)^2 \mathcal{F}.
$$

(4.11)
The steps to obtain the 11d embedding of $\mathcal{N} = 2$ gauged supergravity are intuitively clear. Having identified the Killing vector $\partial/\partial \psi$ as the dual to the $U(1)$ R-symmetry we simply shift in the metric given in equation (4.5) $d\psi \rightarrow d\psi - A$, where $A$ is the 4d one-form gauge field. The general Ansatz for $G_4$ is somehow more involved containing the field strength $F = dA$ as well as its Hodge dual; it can be found from, for example, [46, 51]. For our purpose the most important property is that this generalization does not affect the quantization condition above.

The Einstein-graviphoton part of the action is simply
\begin{equation}
I = \frac{1}{16\pi G_{(4)}} \int d^4x \sqrt{-g} \left( R + \frac{6}{L^2} - \frac{L^2}{4} F^2 \right).
\end{equation}

Note that in the above expression we have explicitly restored the radius of the vacuum AdS solution which we have set to one in the previous expressions. The theory given by equation (4.12) is precisely the universal sector discussed recently in [10] in the context of microscopic counting of AdS$_4$ black hole entropy. Here the crucial property is the embedding into M-theory and, more precisely, the scaling of Newton’s constant with the number of branes $N$. In the action above $F$ is the field strength for $U(1)$ gauge field in AdS$_4$ which couples to the $U(1)$ R-symmetry in the boundary CFT. The 4d Newton constant $G_{(4)}$ after the consistent truncation is related to $N$ in the following way [46]:
\begin{equation}
G_{(4)}/L^2 = \frac{3\pi^2}{2N^2 \text{vol}(M_3)}.
\end{equation}

4.1 Rotating electrically charged AdS$_4$ black hole

The original placement of the rotating solutions in the context of gauged $\mathcal{N} = 2$ supergravity was presented in [58, 59]. In this manuscript we follow the notation of [60] in which the background takes the following form:
\begin{align}
&ds^2 = -\frac{\Delta_r}{W} \left( dt - \frac{a}{\Xi} \sin^2 \theta d\phi \right)^2 + W \left( \frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta_\theta \sin^2 \theta}{W} \left( a dt - \frac{\rho^2 + a^2}{\Xi} \right)^2, \\
&\rho = r + 2m \sinh \delta, \\
&\Delta_r = r^2 + a^2 - 2mr + g^2 (\rho^2 + a^2), \\
&W = \rho^2 + a^2 \cos^2 \theta, \\
&\Xi = 1 - a^2 g^2, \\
&A = \frac{4m \sinh \delta \cosh \delta}{W} \rho \left( dt - \frac{a}{\Xi} \sin^2 \theta d\phi \right).
\end{align}

The parameter $g$ simply characterizes the strength of the coupling in gauged supergravity and is inversely proportional to the radius of AdS, $L$. The physics of this solution is parametrized by $(m, a, \delta)$ which roughly characterize the energy, the angular momentum and the non-extremality or charged of the solution. More precisely:
\begin{align}
&E = \frac{m}{\Xi^2} (1 + 2 \sinh^2 \delta), \\
&J = \frac{ma}{\Xi^2} (1 + 2 \sinh^2 \delta), \\
&Q = \frac{m \sinh \delta \cosh \delta}{2\Xi}.
\end{align}
This solution has, of course, been generalized to include its embedding in gauged $\mathcal{N} = 8$ gauged supergravity with $U(1)^4$. This fact makes it a prime object in the context of AdS/CFT correspondence due to its relation with ABJM theory. If the four charges in $U(1)^4$ are set equal pairwise, one obtains a solution which is naturally embedded into 4d $\mathcal{N} = 4$ gauged supergravity with $U(1) \times U(1)$ – this is the solution that has been widely discussed in [60] focusing in the supersymmetric limit of the non-extremal solution originally presented in [61]. Technically, in this paper we are concerned with the limit of the solution discussed in [60] when $\delta_1 = \delta_2$ in their notation which we simply denote by $\delta$. Versions of these solutions have recently been reviewed in [62, 63] with emphasis on the entropy function.

Having a potential microscopic description of these solutions, we are naturally interested in entropy in the BPS limit where one verifies that [62, 63]:

$$S = \frac{\pi J_{BPS}}{4G(4)Q_{BPS}}.$$  \hspace{1cm} (4.16)

The entropy can more explicitly be written as:

$$S_{BH} = \frac{\pi}{2G(4)} \left( \sqrt{1 + 4G_4^2Q^2} - 1 \right).$$  \hspace{1cm} (4.17)

The entropy function can be computed from the on-shell action of the solution and takes the following form [62, 63]

$$\frac{N^3}{12\pi^2} \frac{(\omega + 2\pi i)^2 \text{vol}(M_3)}{2\omega},$$  \hspace{1cm} (4.18)

where we used the explicit form of the four-dimensional Newton’s constant $G(4)$. This expression perfectly matches the field theory computation and justifies the use of the field theory description as a microscopic explanation for the entropy of the rotating electrically charged asymptotically AdS$_4$ black holes described above.

### 4.2 Rotating magnetically charged AdS$_4$ black holes

There have been many studies regarding rotating magnetically charged black holes that are the natural candidates for the refined topologically twisted index. In particular, there is one recent publication that discusses these issues in detail and contains a comprehensive list of references [64]. A common theme, since the very beginning of the explorations of rotating magnetically charged black holes, is that within the confines of the universal sector described by the action in equation (4.12) spherically symmetric black holes do not exist [59]. Indeed, the need to couple to matter was recognized very early on [65].

The absence of such gravity solutions is compatible with the result on the field theory side that the refined topologically twisted index vanishes. It would be interesting to explore the refined topologically twisted index for the hyperbolic two-plane in which case there are supergravity solutions that one might hope to match.
5 Logarithmic corrections from one-loop gravity

Logarithmic corrections to the extremal black hole entropy can be computed purely in terms of the low energy data, that is, from the spectrum of massless fields in a given gravity background. These IR corrections are expected to be reproduced by any candidate to a UV complete description of the gravity theory. This IR window into the UV physics has been exploited by Ashoke Sen and collaborators in the context of asymptotically flat black holes whose microscopic descriptions lie in string theory [66, 67]. For asymptotically AdS black holes the microscopic UV complete theory is provided by the dual field theory. It is, therefore, natural to study if the logarithmic in $N$ terms coming from the field theory partition functions are reproduced by one-loop supergravity around the black hole backgrounds. There have been some developments in matching the gravity computation to the coefficient of $\log(N)$ term on the field theory side [20, 68–74]. In this section we compute the one-loop logarithmic corrections from the gravity side and confront them with the field-theoretic (UV) results. One of the advantages of working in the context of the 3d-3d correspondence is that the logarithmic term is derived analytically without the standard recourse to numerical methods such as [68, 72, 73].

Let us start by recalling a number of important facts regarding the one-loop effective actions of supergravity backgrounds. Our setup is 11d supergravity where we have an explicit embedding of the solutions describing spinning black holes. We highlight that in odd dimensional spaces only zero modes and boundary terms can contribute to the logarithmic expression. We make the assumption that the whole contribution to the action comes from the asymptotic $AdS_4$ region as was the case in [75] for the $AdS_4$ solution and in [70] for the magnetically charged asymptotically $AdS_4$ black hole case; more directly related to our description of M5 branes wrapping hyperbolic 3-manifolds is the recent discussion in [20].

On very general grounds of diffeomorphic invariance, it can be argued that in odd-dimensional spacetimes, the top Seeley-De Witt coefficient $a_{d/2}$ vanishes [76]. Therefore, the only contribution to the heat kernel comes from the zero modes. Applied to our case, the one-loop contribution due to 11d supergravity comes from the analysis of zero modes. Similar properties have, in fact, been already exploited in the context of the logarithmic corrections to BMPV black holes in [67] whose logarithmic contributions come from an effectively 5d theory. Similarly, the authors of [75] successfully matched the logarithmic term in the large $N$ expansion of the ABJM free energy on $S^3$ with a gravity computation performed in 11d sugra which essentially reduced to the contribution of a two-form zero mode. Along these lines other matches for magnetically charged asymptotically $AdS_4$ black holes dual to the topologically twisted index was performed in [20, 70].

We will not reproduce all the details of the computation here, the interested reader is referred to [20, 70] for details. We briefly sketch the derivation of the one-loop effective action. Given that the only zero mode in $AdS_4$ is a 2-form and assuming that the solution is roughly of the form of warped products of $AdS_4 \times M_3 \times \tilde{S}^4$ we need to decompose the kinetic operator along these three subspaces. For the 2-form zero mode of $AdS_4$ to survive we need to have the corresponding part of the kinetic Laplace-like operator also vanishing.
The number of zero modes depends on the topology of the full space. When integrating over zero modes there is a factor of $L^{\pm \beta_A}$ for each zero mode. The total contribution to the partition function from the zero modes is

$$L^{\pm \beta_A n^0_A},$$

(5.1)

where $n^0_A$ is the number of zero modes of the kinetic operator $A$ and the sign depends on whether the operator is fermionic or bosonic. Typically, zero modes are associated with certain asymptotic symmetries. For example, with gauge transformations that do not vanish at infinity. The key idea in determining $\beta_A$ is to find the right variables of integrations and to count the powers of $L$ that such integration measure contributes when one starts from fields that would naturally be present in the action.

Therefore, in our case, the computation of the one-loop effective action reduces to the computation of scaling of the zero mode $\beta_A$ and counting the number of zero modes, $n^0_A$. The most important ingredient in formulating the answer is the number of two-form zero modes. A simple way to determine the number of 2-form zero modes is by computing the Euler characteristic of the black hole. In [20, 70] it was established that $n^0_2 = 2(1 - g)$ for a black hole of horizon given by a genus $g$ Riemann surface. It is worth pausing over this result. Note that this number is computed using the non-extremal branch of the solution. Moreover, it is independent of the charges of the black holes. Therefore, be it for the magnetically charged or the electrically charged black holes we obtain the same result.

The full contribution to the logarithmic terms of the one-loop effective action is thus given only by the 2-form zero modes and we have:

$$\log Z_{1\text{-loop}} = (2 - \beta_2) n^0_2 \log L = (2 - 7/2)2(1 - g) \log L = -(1 - g) \log N,$$

(5.2)

where according to the structure of the M5 brane solution we have $L^3 \sim N$. When restricting to spherically symmetric horizons ($g = 0$) we find perfect agreement with the field theory result.

In [20] we furthered discussed the case when $M_3$ admits one-form zero modes. It this case there are contributions coming from the supergravity 3-form potential as one can construct a zero mode by combining the 2-form zero mode in AdS$_4$ and a 1-form zero mode in $M_3$. The total contribution to the one-loop partition function takes the form$^4$:

$$\log Z_{1\text{-loop}} = (g - 1)(1 - b_1) \log N.$$

(5.3)

This results perfectly agrees with the field theory computation of the coefficient of the logarithmic in $N$ term. It thus provides strong evidence that the superconformal index captures the black hole entropy beyond the leading order.

### 6 Conclusions

In this manuscript we have presented an explicit recipe for and computation of the superconformal index of $\mathcal{N} = 2$ supersymmetric field theories denoted by $\mathcal{T}_N[M_3]$. Our recipe

$^4$This expression corrects a sign error in [20].
identifies the superconformal index with certain perturbative invariants of $SL(N, \mathbb{C})$ Chern-Simons theory via the 3d-3d correspondence. One advantage of this setup is that it provides analytical results for the leading and even for the logarithmic in $N$ term. Recent approaches to the microscopic counting in rotating electrically charged asymptotically AdS$_4$ black holes have reduced the field-theoretic object to a matrix model where the leading order can be easily found but the sub-leading in $N$ corrections seem like a daunting task [14, 15]. We show that the leading order in the superconformal index perfectly matches the entropy of rotating electrically charged black holes in the universal sector of $\mathcal{N} = 2$ gauge supergravity. More importantly, we demonstrated that the logarithmic in $N$ terms also match on both sides of the AdS/CFT correspondence.

We have also studied the refined topologically twisted index and found agreement between its vanishing in the field theory side and the absence of the dual rotating magnetically charged black holes in the universal sector of $\mathcal{N} = 2$ gauged supergravity. Our approach provides strong evidence in favor of the refined index vanishing also for finite $N$. This observation has powerful mathematical implications and it would be quite interesting to pursue this tantalizing relation between number-theoretical objects and black hole physics.

Our sub-leading result is a powerful tool that could be applied in other situations where various observables are vying for the genuine description of black hole microstates. The situation with various, a priori, different observables yielding the same leading order expression for the black hole entropy in AdS$_4$ is quite similar to the situation in AdS$_5$. In the context of rotating electrically charged asymptotically AdS$_5$ black holes there are various observables that, at leading order, yield the same entropy function and, therefore, compete for the microscopic description of the entropy. These observables include: a localization computation on complex backgrounds [77], the superconformal index computation via Bethe Ansatz [78] and a free-field theory approach [79]. It is quite possible that these observables are equivalent; it is also possible that a sub-leading in $N$ study has the chance to break this degeneracy. It would be quite appropriate to study this problem in more detail.

6.1 Future directions

Curious observation $I_{\text{top}}(q; T_N[M_3]) = 0$ We checked that $I_{\text{top}}(q; T_N[M_3]) = 0$ up to some power in $\hbar$ for some examples of closed hyperbolic 3-manifold $M_3$. Is it just a coincidence? It would be interesting to check if the zeroness holds for arbitrary closed hyperbolic 3-manifold $M_3$. Rigorous mathematical proof of the zeroness for infinitely many closed hyperbolic 3-manifolds at $q = 1$ will be reported in [80]. If the answer is positive, we need to understand the non-trivial property either from $SL(N, \mathbb{C})$ Chern-Simons theory or from the physics of wrapped M5-branes. In the viewpoint of wrapped M5-branes, the zeroness implies that

$$\mathcal{Z}[6d \ A_{N-1} (2,0) \text{ theory on } S^1 \times M_3 \times S^2] = 0.$$  \hspace{1cm} (6.1)$$

Along the $M_3 \times S^2$, we perform a partial topological twisting using $SO(3) \times SO(2)$ subgroup of $SO(5)$ R-symmetry. We also turn on the omega deformation determined by $\omega = - \log q$ using the rotational isometry on the $S^2$. 
Going beyond universal sector in 3d-3d correspondence. We only consider the 3d-3d correspondence for closed 3-manifold $M_3$ without any codimension-two defect. The resulting 3d $\mathcal{N} = 2$ theory at sufficiently large $N$ only has $SO(2)$ R-symmetry without any flavor symmetry. Holographic duals for these theories are well-established \cite{50, 52} and we could perform 1-loop computations in the gravity side. The system can be generalized by introducing codimension-two defects in the 3d-3d set-up \cite{2.2}. The 6d $A_{N-1}$ $(2,0)$ theory has regular supersymmetric codimension-two defects. Putting the defect along $(3d$ space-time$) \times (a$ knot $K$ inside $M_3)$, we can engineer more general classes of 3d theories labelled by $M_3$, $K$ and the types of the regular defect. We have concrete field theoretic construction of the 3d theory only when the codimension-two defect is maximal type \cite{23, 24}. To extend our work to understand more general classes of supersymmetric black holes in $AdS_4$, we need a better understanding on the codimension-two defects in the context of 3d-3d correspondence from both of gauge theory side and supergravity side. It would be quite interested to pursue this direction elaborating on the progress reported in \cite{53, 81}.

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A Perturbative invariants $\{S^\alpha_n[M_3, N = 2]\}$ for some $M_3$

**Dehn surgery representation** Every orientable closed 3-manifold can be obtained by Dehn surgery along a link $L$ inside 3-sphere $S^3$. The 3-manifold $M_3$ obtained by a Dehn surgery along a link $L$ with slopes $\{P_i/Q_i\}_{i=1}^{|L|}$ is defined as

$$M_3 = (S^3 \setminus L)_{P_i/Q_i, \ldots, P_i/L_i/Q_i(L_i)} \mathrel{:=} \left( (S^3 \setminus L) \bigcup (|L| \text{ solid-tori}) \right) / \sim \quad \text{where the gluing determined by the slopes}
$$

$$P_i \mu_i + Q_i \lambda_i \in H_1 \left( \partial(S^3 \setminus L), \mathbb{Z} \right) \sim (\text{shrinkable boundary cycle of } i\text{-th solid torus}) .$$

(A.1)

Here $|L|$ is the number of components of the link $L$ and $S^3 \setminus L$ denotes the link complement obtained from removing tubular neighborhood of the link $L$ from 3-sphere $S^3$. The boundary of the link complement consists of $|L|$ tori. For each boundary torus, there is a canonical choice of basis 1-cycle called meridian $\mu$ and longitude $\lambda$. $\mu_i$ and $\lambda_i$ are the meridian and the longitude respectively in the $i$-th boundary torus. $P_i$ and $Q_i$ are co-prime integers and the $P_i/Q_i \in \mathbb{Q} \cup \{\infty\}$ is called the $i$-th slope of Dehn surgery. The procedure of gluing solid
tori back to the link complement is called Dehn filling. After the Dehn filling, the resulting 3-manifold $M_3$ is a closed 3-manifold without any boundary.

**Perturbative invariants from State-integral model** Using the topological nature of Chern-Simons theory, its path-integral can be reduced to a finite dimensional integral which are called state-integral model. See [37–39, 82–85] for state-integral models of $SL(N, \mathbb{C})$ Chern-Simons theory from which the perturbative invariants $\{S^n_{\alpha}[M_3; N]\}$ can be efficiently computed. The state-integral model is turned out to be equal to the localization integral for $T_N[M_3]$ on squashed 3-sphere $S^3_b$ with the identification $\hbar = 2\pi ib^2$. Here we give concrete expression for the state integral model for $N=2$ with 3 smallest closed hyperbolic 3-manifolds, Weeks manifold:= $(S^3 \setminus 5_2^1)_{P_1/Q_1=-5, P_2/Q_2=-5/2}$, $(S^3 \setminus 5_1^2)_{P_1/Q_1=-6, P_2/Q_2=-3/2}$ and $(S^3 \setminus 4_1^1)_{P_1/Q_1=-5, P_2/Q_2=-5/2}$. Here $4_1$ represents the figure-eight knot and $5_2^1$ represents the Whitehead link.

The state-integral is schematically written as

$$Z^S_{\hbar}(M_3; N) = \int \frac{d^s\vec{Z}}{(2\pi\hbar)^{s/2}} I_{\hbar}(\vec{Z}; M_3, N),$$

and the integrand can be asymptotic expanded as

$$\log I_{\hbar}(\vec{Z}; M_3, N) \xrightarrow{\hbar \to 0} \sum_{n=0}^{\infty} W_n(\vec{Z}, \vec{\ell}_z; M_3, N).$$

(A.2)

Here $\vec{\ell}_z \in \mathbb{Z}^s$ comes from the brach-cut structure of the log $I_{\hbar}$ in the asymptotic limit. $W_0$ and $W_1$ depends on the choice in the following way

$$W_0(\vec{\ell}_z) = W_0(\vec{\ell}_z = \vec{0}) + 2\pi i \vec{\ell}_z \cdot \vec{Z},$$

$$W_1(\vec{\ell}_z) = W_1(\vec{\ell}_z = \vec{0}) - 2\pi i \sum_{i=1}^{2} (\vec{\ell}_z)_i,$$

(A.3)

and $W_{n>2}$ are independent of the choice. Integration of the state-integral model along a proper cycle will give the squashed 3-sphere partition function of $T_N[M_3]$ theory. The classical part $W_0$ is equal to the twisted superpotential and the Bethe-vacua of the $T_N[M_3]$ is obtained by extremizing it

$$S_{BE}(T_N[M_3]) = \left\{ \vec{z} : \exp \left( \partial_{\vec{z}_i} W_0(\vec{Z}, \vec{\ell}_z; M_3, N) \right) \right|_{\vec{Z}_i = \log z_i} = 1, \text{ for } i = 1, \ldots, s \right\}. \quad (A.4)$$

For each $\vec{z}^\alpha \in S_{BE}(T_N[M_3])$, there is a corresponding saddle point $\vec{Z}^\alpha$

$$\partial_{\vec{z}_i} W_0(\vec{Z}, \vec{\ell}_z; M_3, N) = 0, \quad i = 1, \ldots, s.$$

(A.5)

with a proper choice of $\vec{\ell}_z \in \mathbb{Z}^s$. Using the 3d-3d relation (2.34), each saddle point can be identified with a $SL(N, \mathbb{C})$ flat connection (modulo tensoring with $\mathbb{Z}_N$ flat connections) on $M_3$:

$$\vec{Z}^\alpha \leftrightarrow [\mathcal{A}^\alpha] = \frac{\chi_{\text{irred}}[N; M_3]}{\text{Hom}(\pi_1(M_3) \to SL(N, \mathbb{C}))}.$$

(A.6)
Then, the perturbative invariants \( \{ S^n(M_3; N) \} \) for a flat connection \( \mathcal{A} \) can be obtained from formal perturbative expansion of the state integral around the corresponding saddle point \( \bar{Z} \). The precise relation is
\[
Z^\text{SI}_h(M_3; N) \overset{\text{expansion around saddle point } \bar{Z}}{\longrightarrow} \sqrt{\frac{\text{Hom}(\pi_1(M_3) \to \mathbb{Z}_N)}{N}} \exp \left( \sum_n h^{n-1} S^n(M_3; N) \right) .
\]  

(A.7)

**Example :** \( M_3 = (S^3 \setminus 4_1)_{P/Q=5} \). \( 4_1 \) denotes the figure-eight knot. The corresponding 3d gauge theory is given in (2.39) and the corresponding state-integral model is given as
\[
Z^\text{SI}_h(M_3 = (S^3 \setminus 4_1)_5; N = 2) = \int \frac{dZ}{\sqrt{2\pi h}} \psi_h(Z) \exp \left( -\frac{3}{2\hbar} Z^2 + \frac{1}{\hbar} (i\pi h/2) Z \right) .
\]

(A.8)

Here \( \psi_h(Z) \) is a special functional called quantum dilogarithm (QDL).
\[
\psi_h(Z) := \begin{cases} \prod_{r=1}^\infty \frac{1 - q^r e^{-Z}}{1 - q^{-r+1} e^{-Z}} & \text{if } |q| < 1 , \\ \prod_{r=1}^\infty \frac{1 - q^r e^{-\bar{Z}}}{1 - q^{-r+1} e^{-\bar{Z}}} & \text{if } |q| > 1 , \end{cases}
\]

(A.9)

with
\[
q := e^{2\pi ib^2} , \quad \bar{q} := e^{2\pi ib^{-2}} , \quad \bar{Z} := \frac{1}{b^2} Z .
\]

(A.10)

The asymptotic expansion when \( \hbar = 2\pi ib^2 \to 0 \) is given by
\[
\log \psi_h(Z) \overset{\hbar \to 0}{\longrightarrow} \sum_{n=0}^\infty \frac{B_n h^{n-1}}{n!} \tilde{\text{Li}}_{2-n}(e^{-Z}, \ell_z) .
\]  

(A.11)

Here \( B_n \) is the \( n \)-th Bernoulli number with \( B_1 = 1/2 \).
\[
\tilde{\text{Li}}_2(e^{-Z}, \ell_z) = \text{Li}_2(e^{-Z}) + 2\pi i\ell_z Z \\
\tilde{\text{Li}}_1(e^{-Z}, \ell_z) = \text{Li}_1(e^{-Z}) - 2\pi i\ell_z \\
\tilde{\text{Li}}_{2-n}(e^{-Z}, \ell_z) = \text{Li}_{2-n}(e^{-Z}) \quad (n \geq 2)
\]

(A.12)

\( \ell_z \in \mathbb{Z} \) are locally constant function on \( Z \) which comes from branch-cut structure of \( \text{Li}_{1,2} \).

Using the asymptotic expansion of the QDL, we can asymptotically expand the integrand as follows
\[
I_h(Z) \overset{\hbar \to 0}{\longrightarrow} \exp \left( \sum_{n=0}^\infty h^{n-1} \mathcal{W}_n(Z, \ell_z) \right) , \quad \text{where}
\]
\[
\mathcal{W}_0(Z; \ell_z) = \text{Li}_2(e^{-Z}) + i\pi(1 + 2\ell_z)Z - \frac{3}{2} Z^2 , \\
\mathcal{W}_1(Z; \ell_z) = \text{Li}_1(e^{-Z}) - 2\pi i\ell_z + \frac{1}{2} Z , \\
\ldots
\]

(A.13)
The $W_0$ is the twisted superpotential and the Bethe-vacua is given by following algebraic equation in $z := e^z$

$$\exp (\partial_z W_0) = 1 \quad \Rightarrow \quad \frac{1-z}{z^4} = 1.$$  \hspace{1cm} (A.14)

For each Bethe-vacua $z^\alpha$, there is a corresponding $Z^\alpha$ with a proper choice of $\ell_z^\alpha$ such that

$$\partial_z W_0|_{Z=Z^\alpha, \ell_z=\ell_z^\alpha} = 0.$$  \hspace{1cm} (A.15)

The Bethe-vacua are in one-to-one correspondence with irreducible $SL(2, \mathbb{C})$ flat connections on $M_3$. We can perturbatively expand the state-integral around each saddle point $Z^\alpha$ as follows:

$$\frac{1}{\sqrt{2}}\exp \left( \sum_{n=0}^{\infty} \hbar^{n-1} S_n^\alpha (M_3; N = 2) \right)$$

$$= \int \frac{d(\delta Z)}{\sqrt{2\pi}} \mathcal{I}_h \left( Z^\alpha + \hbar^{1/2} \delta Z; M_3 \right)$$

$$= \exp \left( \sum_{n=0}^{\infty} W_n(Z^\alpha) \right) \int \frac{d(\delta Z)}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \Pi^\alpha (\delta Z)^2 + \sum_{n=1}^{\infty} \sum_{1 \leq m \leq n+2; m-n \in 2\mathbb{Z}} h^n/2 C_{n,m}^\alpha (\delta Z)^m \right)$$

$$= \exp \left( \frac{1}{2} \log \Pi^\alpha + \sum_{n=0}^{\infty} W_n(Z^\alpha) \right) \left( 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{3n} h^n D_{n,m}^\alpha G_m^\alpha \right).$$  \hspace{1cm} (A.16)

Here we define

$$\Pi^\alpha := -\partial^2_z W_0 \bigg|_{Z=Z^\alpha} \quad \text{(propagator)},$$

$$C_{n,m}^\alpha := \frac{1}{m!} \partial^m_z W_{n-m+1} \bigg|_{Z=Z^\alpha} \quad \text{(vertices)},$$

$$D_{n,m}^\alpha := \text{coefficient of } \hbar^n(\delta Z)^{2m} \text{ in } \exp \left( \sum_{n=1}^{\infty} \sum_{1 \leq m \leq n+2; m-n \in 2\mathbb{Z}} h^n/2 C_{n,m}^\alpha (\delta Z)^m \right),$$

$$G_m^\alpha := \sqrt{\Pi^\alpha} \int \frac{d(\delta Z)}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \Pi^\alpha (\delta Z)^2 \right) (\delta Z)^m = \partial^2_{\mu} \exp \left( \frac{1}{2} (\Pi^\alpha)^{-1} \mu^2 \right) \bigg|_{\mu=0}.$$  \hspace{1cm} (A.17)

The perturbative coefficients $\{S_n^\alpha(M_3 = (S^3\setminus S^1_4)_P/Q=5, N = 2)\}$ computed as above is given in (2.40) up to $n = 5$.

Example: $M_3 = \text{Weeks manifold} = (S^3\setminus S^1_4)_{P_1/Q_1=-5, P_2/Q_2=-5/2}$ Here $S^1_4$ denote the Whitehead link. The corresponding 3d gauge theory is given in (2.46). Corresponding state integral is given as

$$Z_{h}^{SI} (\text{Weeks}) = \int \frac{dZ}{\sqrt{2\pi h}} \psi_h(Z) \exp \left( \frac{Z^2}{h} \right).$$  \hspace{1cm} (A.18)
Example: \( M_3 = (S^3 \setminus 5_2^1)_{p_1/Q_1=-6, p_2/Q_2=-3/2} \). Here \( 5_2^1 \) denote the Whitehead link. The corresponding 3d gauge theory is given in (2.51). Corresponding state integral is given as

\[
Z_{h}^{SI} (M_3 = (S^3 \setminus 5_2^1)_{p_1/Q_1=-6, p_2/Q_2=-3/2}) = \int \frac{dZ}{\sqrt{2\pi\hbar}} \psi_h(Z) \exp \left( -\frac{Z^2}{2\hbar} \right) . \tag{A.19}
\]

References

[1] J. D. Bekenstein, “Black holes and the second law,” *Lett. Nuovo Cim.* 4 (1972) 737–740.

[2] J. D. Bekenstein, “Black holes and entropy,” *Phys. Rev.* D7 (1973) 2333–2346.

[3] J. D. Bekenstein, “Generalized second law of thermodynamics in black hole physics,” *Phys. Rev.* D9 (1974) 3292–3300.

[4] S. W. Hawking, “Black hole explosions?,” *Nature* 248 (1974) 30–31.

[5] S. W. Hawking, “Particle Creation by Black Holes,” *Commun. Math. Phys.* 43 (1975) 199–220.

[6] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” *Phys. Lett.* B379 (1996) 99–104, arXiv:hep-th/9601029.

[7] F. Benini, K. Hristov, and A. Zaffaroni, “Black hole microstates in AdS4 from supersymmetric localization,” *JHEP* 05 (2016) 054, arXiv:1511.04085 [hep-th].

[8] F. Benini, K. Hristov, and A. Zaffaroni, “Exact microstate counting for dyonic black holes in AdS4,” *Phys. Lett.* B771 (2017) 462–466, arXiv:1608.07294 [hep-th].

[9] A. Cabo-Bizet, V. I. Giraldos-Rivera, and L. A. Pando Zayas, “Microstate counting of AdS4 hyperbolic black hole entropy via the topologically twisted index,” *JHEP* 08 (2017) 023, arXiv:1701.07893 [hep-th].

[10] F. Azzurli, N. Bobev, P. M. Crichigno, V. S. Min, and A. Zaffaroni, “A universal counting of black hole microstates in AdS4,” *JHEP* 02 (2018) 054, arXiv:1707.04257 [hep-th].

[11] F. Benini, H. Khachatryan, and P. Milan, “Black hole entropy in massive Type IIA,” *Class. Quant. Grav.* 35 no. 3, (2018) 035004, arXiv:1707.06886 [hep-th].

[12] S. M. Hosseini, K. Hristov, and A. Passias, “Holographic microstate counting for AdS4 black holes in massive IIA supergravity,” *JHEP* 10 (2017) 190, arXiv:1707.06884 [hep-th].

[13] A. Zaffaroni, “Lectures on AdS Black Holes, Holography and Localization,” 2019. arXiv:1902.07176 [hep-th].

[14] S. Choi, C. Hwang, and S. Kim, “Quantum vortices, M2-branes and black holes,” arXiv:1908.02470 [hep-th].

[15] J. Nian and L. A. Pando Zayas, “Microscopic Entropy of Rotating Electrically Charged AdS4 Black Holes from Field Theory Localization,” arXiv:1909.07943 [hep-th].

[16] N. Bobev and P. M. Crichigno, “Universal Spinning Black Holes and Theories of Class \( \mathcal{R} \),” arXiv:1909.05873 [hep-th].

[17] T. Dimofte, D. Gaiotto, and S. Gukov, “3-Manifolds and 3d Indices,” *Adv. Theor. Math. Phys.* 17 no. 5, (2013) 975–1076, arXiv:1112.5179 [hep-th].

[18] S. Lee and M. Yamazaki, “3d Chern-Simons Theory from M5-branes,” *JHEP* 12 (2013) 035, arXiv:1305.2429 [hep-th].
[19] J. Yagi, “3d TQFT from 6d SCFT,” *JHEP* **08** (2013) 017, arXiv:1305.0291 [hep-th].

[20] D. Gang, N. Kim, and L. A. Pando Zayas, “Precision Microstate Counting for the Entropy of Wrapped M5-branes,” *arXiv:1905.01559 [hep-th]*.

[21] D. Gang, Y. Tachikawa, and K. Yonekura, “Smallest 3d hyperbolic manifolds via simple 3d theories,” *Phys. Rev. D* **96** no. 6, (2017) 061701, arXiv:1706.06292 [hep-th].

[22] D. Gang and K. Yonekura, “Symmetry enhancement and closing of knots in 3d/3d correspondence,” *arXiv:1803.04009 [hep-th]*.

[23] T. Dimofte, D. Gaiotto, and S. Gukov, “Gauge Theories Labelled by Three-Manifolds,” *Commun. Math. Phys.* **325** (2014) 367–419, arXiv:1108.4389 [hep-th].

[24] T. Dimofte, M. Gabella, and A. B. Goncharov, “K-Decompositions and 3d Gauge Theories,” *JHEP* **11** (2016) 151, arXiv:1301.0192 [hep-th].

[25] E. Witten, “Analytic Continuation Of Chern-Simons Theory,” *AMS/IP Stud. Adv. Math.* **50** (2011) 347–446, arXiv:1001.2933 [hep-th].

[26] D. B. Ray and I. M. Singer, “R-torsion and the laplacian on riemannian manifolds,” *Advances in Mathematics* **7** no. 2, (1971) 145–210.

[27] S. Gukov and H. Murakami, “sl(2,c) chern–simons theory and the asymptotic behavior of the colored Jones polynomial,” *Letters in Mathematical Physics* **86** no. 2-3, (2008) 79–98.

[28] C. Beem, T. Dimofte, and S. Pasquetti, “Holomorphic Blocks in Three Dimensions,” *JHEP* **12** (2014) 177, arXiv:1211.1986 [hep-th].

[29] E. Witten, “Analytic Continuation Of Chern-Simons Theory,” *AMS/IP Stud. Adv. Math.* **50** (2011) 347–446, arXiv:1001.2933 [hep-th].

[30] H.-J. Chung, T. Dimofte, S. Gukov, and P. SuÅĆkowski, “3d-3d Correspondence Revisited,” *JHEP* **04** (2016) 140, arXiv:1405.3663 [hep-th].

[31] T. Dimofte, “Perturbative and nonperturbative aspects of complex ChernâĂŞSimons theory,” *J. Phys.* **A50** no. 44, (2017) 443009, arXiv:1608.02961 [hep-th].

[32] F. Benini and A. Zaffaroni, “A topologically twisted index for three-dimensional supersymmetric theories,” *JHEP* **07** (2015) 127, arXiv:1504.03698 [hep-th].

[33] F. Nieri and S. Pasquetti, “Factorisation and holomorphic blocks in 4d,” *JHEP* **11** (2015) 155, arXiv:1507.00261 [hep-th].

[34] S. Gukov, D. Pei, P. Putrov, and C. Vafa, “BPS spectra and 3-manifold invariants,” arXiv:1701.06567 [hep-th].

[35] F. Benini and A. Zaffaroni, “Supersymmetric partition functions on Riemann surfaces,” *Proc. Symp. Pure Math.* **96** (2017) 13–46, arXiv:1605.06120 [hep-th].

[36] C. Closet and H. Kim, “Comments on twisted indices in 3d supersymmetric gauge theories,” *JHEP* **08** (2016) 059, arXiv:1605.06531 [hep-th].

[37] T. Dimofte, S. Gukov, J. Lenells, and D. Zagier, “Exact Results for Perturbative Chern-Simons Theory with Complex Gauge Group,” *Commun. Num. Theor. Phys.* **3** (2009) 363–443, arXiv:0903.2472 [hep-th].

[38] T. D. Dimofte and S. Garoufalidis, “The Quantum content of the gluing equations,” *Geom. Topol.* **17** (2013) 1253–1316, arXiv:1202.6268 [math.GT].
[39] D. Gang, M. Romo, and M. Yamazaki, “All-Order Volume Conjecture for Closed 3-Manifolds from Complex Chern-Simons Theory,” arXiv:1704.00918 [hep-th].

[40] H.-C. Kim and S. Kim, “Supersymmetric vacua of mass-deformed M2-brane theory,” Nucl. Phys. B839 (2010) 96–111, arXiv:1001.3153 [hep-th].

[41] K. Intriligator and N. Seiberg, “Aspects of 3d N=2 Chern-Simons-Matter Theories,” JHEP 07 (2013) 079, arXiv:1305.1633 [hep-th].

[42] D. Gabai, R. Meyerhoff, and P. Milley, “Minimum volume cusped hyperbolic three-manifolds,” Journal of the American Mathematical Society 22 no. 4, (2009) 1157–1215.

[43] G. D. Mostow, “Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms,” Publications Mathématiques de l’IHÉS 34 (1968) 53–104.

[44] W. Müller, “The asymptotics of the ray-singer analytic torsion of hyperbolic 3-manifolds,” in Metric and differential geometry, pp. 317–352. Springer, 2012.

[45] J. PARK, “Reidemeister torsion, complex volume, and zograf infinite product for hyperbolic 3-manifolds,”.

[46] D. Gang, N. Kim, and S. Lee, “Holography of 3d-3d correspondence at Large N,” JHEP 04 (2015) 091, arXiv:1409.6206 [hep-th].

[47] T. Dimofte and S. Gukov, “Chern-Simons Theory and S-duality,” JHEP 05 (2013) 109, arXiv:1106.4550 [hep-th].

[48] D. Gang and Y. Hatsuda, “S-duality resurgence in SL(2) Chern-Simons theory,” arXiv:1710.09994 [hep-th].

[49] S. M. Hosseini, Black hole microstates and supersymmetric localization. PhD thesis, Milan Bicocca U., 2018-02. arXiv:1803.01863 [hep-th].

[50] J. P. Gauntlett, N. Kim, and D. Waldram, “M Five-branes wrapped on supersymmetric cycles,” Phys. Rev. D63 (2001) 126001, arXiv:hep-th/0012195.

[51] A. Donos, J. P. Gauntlett, N. Kim, and O. Varela, “Wrapped M5-branes, consistent truncations and AdS/CMT,” JHEP 12 (2010) 003, arXiv:1009.3805 [hep-th].

[52] M. Pernici and E. Sezgin, “Spontaneous Compactification of Seven-dimensional Supergravity Theories,” Class. Quant. Grav. 2 (1985) 673.

[53] I. Bah, M. Gabella, and N. Halmagyi, “BPS M5-branes as Defects for the 3d-3d Correspondence,” JHEP 11 (2014) 112, arXiv:1407.0403 [hep-th].

[54] M. Gabella, D. Martelli, A. Passias, and J. Sparks, “N = 2 supersymmetric AdS4 solutions of M-theory,” Commun. Math. Phys. 325 (2014) 487–525, arXiv:1207.3082 [hep-th].

[55] J. P. Gauntlett, O. A. P. Mac Conamhna, T. Mateos, and D. Waldram, “AdS spacetimes from wrapped M5 branes,” JHEP 11 (2006) 053, arXiv:hep-th/0605146.

[56] D. Martelli, A. Passias, and J. Sparks, “The gravity dual of supersymmetric gauge theories on a squashed three-sphere,” Nucl. Phys. B864 (2012) 840–868, arXiv:1110.6400 [hep-th].

[57] D. Gang, N. Kim, and S. Lee, “Holography of wrapped M5-branes and Chern–Simons theory,” Phys. Lett. B733 (2014) 316–319, arXiv:1401.3595 [hep-th].

[58] V. A. Kostelecky and M. J. Perry, “Solitonic black holes in gauged N=2 supergravity,” Phys. Lett. B371 (1996) 191–198, arXiv:hep-th/9512222.
[59] M. M. Caldarelli and D. Klemm, “Supersymmetry of Anti-de Sitter black holes,” *Nucl. Phys. B545* (1999) 434–460, arXiv:hep-th/9808097.

[60] M. Cvetic, G. W. Gibbons, H. Lu, and C. N. Pope, “Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines,” arXiv:hep-th/0504080.

[61] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, “Charged rotating black holes in four-dimensional gauged and ungauged supergravities,” *Nucl. Phys. B717* (2005) 246–271, arXiv:hep-th/0411045.

[62] S. Choi, C. Hwang, S. Kim, and J. Nahmgoong, “Entropy functions of BPS black holes in AdS$_4$ and AdS$_6$,” arXiv:1811.02158 [hep-th].

[63] D. Cassani and L. Papini, “The BPS limit of rotating AdS black hole thermodynamics,” *JHEP 09* (2019) 079, arXiv:1906.10148 [hep-th].

[64] K. Hristov, S. Katmadas, and C. Toldo, “Rotating attractors and BPS black holes in AdS$_4$,” *JHEP 01* (2019) 199, arXiv:1811.00292 [hep-th].

[65] D. Klemm, “Rotating BPS black holes in matter-coupled AdS$_4$ supergravity,” *JHEP 07* (2011) 019, arXiv:1103.4699 [hep-th].

[66] A. Sen, “Logarithmic Corrections to N=2 Black Hole Entropy: An Infrared Window into the Microstates,” *Gen. Rel. Grav. 44* no. 5, (2012) 1207–1266, arXiv:1108.3842 [hep-th].

[67] A. Sen, “Logarithmic Corrections to Rotating Extremal Black Hole Entropy in Four and Five Dimensions,” *Gen. Rel. Grav. 44* (2012) 1947–1991, arXiv:1109.3706 [hep-th].

[68] J. T. Liu, L. A. Pando Zayas, V. Rathee, and W. Zhao, “Toward Microstate Counting Beyond Large N in Localization and the Dual One-loop Quantum Supergravity,” *JHEP 01* (2018) 026, arXiv:1707.04197 [hep-th].

[69] I. Jeon and S. Lal, “Logarithmic Corrections to Entropy of Magnetically Charged AdS$_4$ Black Holes,” *Phys. Lett. B774* (2017) 41–45, arXiv:1707.04208 [hep-th].

[70] J. T. Liu, L. A. Pando Zayas, V. Rathee, and W. Zhao, “One-Loop Test of Quantum Black Holes in anti-de Sitter Space,” *Phys. Rev. Lett. 120* no. 22, (2018) 221602, arXiv:1711.01076 [hep-th].

[71] K. Hristov, I. Lodato, and V. Reys, “On the quantum entropy function in 4d gauged supergravity,” *JHEP 07* (2018) 072, arXiv:1803.05920 [hep-th].

[72] J. T. Liu, L. A. Pando Zayas, and S. Zhou, “Subleading Microstate Counting in the Dual to Massive Type IIA,” arXiv:1808.10445 [hep-th].

[73] L. A. Pando Zayas and Y. Xin, “The Topologically Twisted Index in the ’t Hooft Limit and the Dual AdS$_4$ Black Hole Entropy,” arXiv:1908.01194 [hep-th].

[74] K. Hristov, I. Lodato, and V. Reys, “One-loop determinants for black holes in 4d gauged supergravity,” arXiv:1908.05696 [hep-th].

[75] S. Bhattacharyya, A. Grassi, M. Marino, and A. Sen, “A One-Loop Test of Quantum Supergravity,” *Class. Quant. Grav. 31* (2014) 015012, arXiv:1210.6057 [hep-th].

[76] D. V. Vassilevich, “Heat kernel expansion: User’s manual,” *Phys. Rept. 388* (2003) 279–360, arXiv:hep-th/0306138.
[77] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, “Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric $\text{AdS}_5$ black holes,” \texttt{arXiv:1810.11442 [hep-th]}.

[78] F. Benini and P. Milan, “Black holes in 4d $\mathcal{N} = 4$ Super-Yang-Mills,” \texttt{arXiv:1812.09613 [hep-th]}.

[79] S. Choi, J. Kim, S. Kim, and J. Nahmgoong, “Large AdS black holes from QFT,” \texttt{arXiv:1810.12067 [hep-th]}.

[80] D. Gang, S. Kim, and S. Yoon, “To appear.”

[81] D. Gang, N. Kim, M. Romo, and M. Yamazaki, “Aspects of Defects in 3d-3d Correspondence,” \textit{JHEP} 10 (2016) 062, \texttt{arXiv:1510.05011 [hep-th]}.

[82] K. HIKAMI, “Hyperbolic structure arising from a knot invariant,” \textit{International Journal of Modern Physics A} \textbf{16} no. 19, (Jul, 2001) 3309–3333.

[83] K. Hikami, “Generalized volume conjecture and the a-polynomials: The neumannâ€”zagier potential function as a classical limit of the partition function,” \textit{Journal of Geometry and Physics} \textbf{57} no. 9, (Aug, 2007) 1895–1940.

[84] T. Dimofte, “Quantum Riemann Surfaces in Chern-Simons Theory,” \textit{Adv. Theor. Math. Phys.} \textbf{17} no. 3, (2013) 479–599, \texttt{arXiv:1102.4847 [hep-th]}.

[85] J.-B. Bae, D. Gang, and J. Lee, “3d $\mathcal{N} = 2$ minimal SCFTs from Wrapped M5-branes,” \textit{JHEP} \textbf{08} (2017) 118, \texttt{arXiv:1610.09259 [hep-th]}.

[86] N. Hama, K. Hosomichi, and S. Lee, “SUSY Gauge Theories on Squashed Three-Spheres,” \textit{JHEP} \textbf{05} (2011) 014, \texttt{arXiv:1102.4716 [hep-th]}.