D-brane Superpotentials and Ooguri-Vafa Invariants of Compact Calabi-Yau Threefolds

Feng-Jun Xu, Fu-Zhong Yang

College of Physical Sciences, Graduate University of Chinese Academy of Sciences
YuQuan Road 19A, Beijing 100049, China

Abstract

We calculate the D-brane superpotentials for two non-Fermat type compact Calabi-Yau manifolds which are the hypersurface of degree 14 in the weighed projective space $P(1,1,2,3,7)$ and the hypersurface of degree 8 in the weighed projective space $P(1,1,1,2,3)$ in type II string theory respectively. By constructing the open-closed mirror maps, we also compute the Ooguri-Vafa invariants, which are related to the open Gromov-Witten invariants.

May 9, 2013

*Corresponding author  E-mail: fzyang@gucas.ac.cn
1 Introduction

When considering certain $N = 1$ supersymmetric string compactifications of type II string theories with space-filling D-branes, superpotential is an important quantity due to its BPS-property, which is exactly solvable. Superpotential is both important in physics and mathematics. The physical interest is served by facts that superpotential is also known the holomorphic F-term in effective lagrangian. The mathematical application is related to the non-perturbative stringy geometry. Similar to the Closed string theory, there is a geometry parameterizing the moduli space, namely $N=1$ special geometry. From the viewpoint of Mathematics, the presence of a superpotential describe an obstruction to continues deformation of moduli space. And even more surprise results is in enumerative geometry. The superpotentials of A-model at large radius region count the disk invariants $\int$ which are related to open Gromov-Witten invariants [? ,? ,? ,? ,? ].

The $N = 1$, $d = 4$ superpotential term can be computed by the open topological string amplitudes $F_{g,h}$ of the A-model as follows

$$h \int d^4 x d^2 \theta F_{g,h}(G^2)^g(F^2)^{h-1}$$

where $G$ is the gravitational chiral superfield and $F$ is the gauge chiral superfield. The formula (1.1) at $g = 0$, $h = 1$ leads to in F-terms of $N = 1$ supersymetric theories:

$$\int d^4 x d^2 \theta W(\Phi)$$

For non-compact Calabi-Yau manifolds, the refs [3–9] studied the open-closed mirror symmetry and its applications. In particular, the work [3] constructed the classical A-brane geometry with special Lagrangian submanifold and the work [5,6] introduced $N = 1$ spacial geometry and variation of mixed Hodge structure to calculate superpotentials. Motivated and guided by these works, a progress on compact manifolds came from [10–13], which studied a class of involution branes independent of open deformation moduli. Furthermore, there appeared some related works on superpotential for compact Calabi-Yau manifolds depending on open-closed deformation moduli [14–32], where the works [28,29] studied it by the conformal field theory and matrix factorization, the works [14,21] by the direct integration and the others considered with Hodge theoretic method.
In this paper, we calculate D-brane superpotentials for compact non-Fermat Calabi-Yau threefolds by open-closed mirror symmetry and generalized GKZ system [34,37,38,40,41]. There exists a duality between the type II compactification with brane on the threefold and the M/F-theory compactify on the Calabi-Yau fourfold without any branes but with fluxes [17–19,24,27,42]. In the weak decoupling limit $g_s \to 0$, the Gukov-Vafa-Witten superpotentials [35] $W_{GVW}$ of F-theory compactify on this fourfold agrees with superpotentials $\mathcal{W}$ of Type II compactify threefold with branes at lowest order in $g_s$ [18–20,24,27,36]

$$W_{GVW} = \mathcal{W} + \mathcal{O}(g_s) + \mathcal{O}(e^{-1/g_s})$$

Hence, in this limit, we can obtain the flux superpotential $W_{GVW}$ from the superpotential $\mathcal{W}$ which will be given in this paper.

In Sect. 2 and Sect. 3 we give a overview of Superpotentials on Calabi-Yau Threefolds and generalized GKZ system. In sect. 4, we concentrates on the applications with two compact non-Fermat Calabi-Yau threefold with two deformation parameters—$X_{14}(1,1,2,3,7)$ and $X_8(1,1,1,2,3)$. Although the compact Calabi-Yau manifolds are available in many works above, the works about non-fermat case are few. In these manifolds, We consider superpotential, mirror symmetry and Ooguri-Vafa invariants for D-brane with a single open deformation moduli. Sect. 5 is for summary.

2 Some Known Results

In this section, we collect some known results on D-brane superpotenatial in type II string compactification.

2.1 Superpotentials on Calabi-Yau Threefolds

Type II compactification theory is described by an effective $N = 1$ supergravity action with non-trivial superpotentials on the deformation space $\mathcal{M}$ when adding D-branes and background fluxes. For D5-brane wrapped the whole Calabi-Yau threefold, the holomorphic Chern-Simons theory [43]

$$\mathcal{W} = \int_X \Omega^{3,0} \wedge \text{Tr}[A \wedge \bar{\partial}A + \frac{2}{3} A \wedge A \wedge A]$$

(2.1)

gives the brane superpotential $\mathcal{W}_{brane}$, where $A$ is the gauge field with gauge group $U(N)$ for $N$ D6-branes. When reduced dimensionally, the low dimensional brane su-
perpotentials can be obtained as \[3, 44\]

\[
W_{\text{brane}} = N_\nu \int_{\Gamma_\nu} \Omega^{3,0}(z, \hat{z}) = \sum_{\nu} N_\nu \Pi_\nu
\] (2.2)

where \(\Gamma_\nu\) is a special Lagrangian 3-chain and \((z, \hat{z})\) are closed-string complex structure moduli and D-brane moduli from open-string sector, respectively.

The background fluxes \(H^{(3)} = H^{(3)}_{RR} + \tau H^{(3)}_{NS}\), which take values in the integer cohomology group \(H^3(X, \mathbb{Z})\), also break the supersymmetry \(N = 2\) to \(N = 1\). The \(\tau = C^{(0)} + ie^{-\varphi}\) is the complexified Type IIB coupling field. Its contribution to superpotentials is \[45, 46\]

\[
W_{\text{flux}}(z) = \int_X H^{(3)}_{RR} \wedge \Omega^{3,0} = \sum_{\alpha} N_\alpha \cdot \Pi_\alpha(z), \quad N_\alpha \in \mathbb{Z}.
\] (2.3)

The contributions of D-brane and background flux (here the NS-flux ignored) give together the general form of superpotential as follow \[5, 6\]

\[
W(z, \hat{z}) = W_{\text{brane}}(z, \hat{z}) + W_{\text{flux}}(z) = \sum_{\Sigma \in H^3(Z^*, \mathcal{H})} N_\Sigma \Pi_\Sigma(z, \hat{z})
\] (2.4)

where \(N_\Sigma = n_\Sigma + \tau m_\sigma\), \(\tau\) is the dilaton of type II string and \(\Pi_\Sigma\) is a relative periods defined in a relative cycle \(\Gamma \in H_3(X, D)\) whose boundary is wrapped by D-branes.

The off-shell tension of D-branes, \(T(z, \hat{z})\), is equal to the relative period \[5, 6, 47\]

\[
\Pi_\Sigma = \int_{\Gamma_\Sigma} \Omega(z, \hat{z})
\] (2.5)

which measures the difference between the value of on-shell superpotentials for the two D-brane configurations

\[
T(z, \hat{z}) = \mathcal{W}(C^+) - \mathcal{W}(C^-)
\] (2.6)

with \(\partial \Gamma_\Sigma = C^+ - C^-\). The on-shell domain wall tension is \[24\]

\[
T(z) = T(z, \hat{z}) \mid_{\hat{z} = \text{critic points}}
\] (2.7)

where the critic points correspond to \(\frac{d\mathcal{W}}{d\hat{z}} = 0\) \[47\] and the \(C^\pm\) is the holomorphic curves at those critical points. At those critical points, the domain wall tensions are also known as normal function giving the Abel-Jacobi invariants \[11, 22, 24, 48, 49\]

4
In A-model interpretation, the superpotential expressed in terms of flat coordinates \((t, \hat{t})\) is the generating function of the Ooguri-Vafa invariants \([3, 6, 17, 50]\)

\[
\mathcal{W}(t, \hat{t}) = \sum_{\vec{k}, \vec{m}} G_{\vec{k}, \vec{m}} q^{d \vec{k}} \hat{q}^{d \vec{m}} = \sum_{\vec{k}, \vec{m}} \sum_{d} n_{\vec{k}, \vec{m}} q^{d \vec{k}} \hat{q}^{d \vec{m}} \quad (2.8)
\]

where \(q = e^{2\pi it}\), \(\hat{q} = e^{2\pi i\hat{t}}\) and \(n_{\vec{k}, \vec{m}}\) are Ooguri-Vafa invariants \([50]\) counting disc instantons in relative homology class \((\vec{m}, \vec{k})\), where \(\vec{m}\) represents the elements of \(H_1(D)\) and \(\vec{k}\) represents an element of \(H_2(X)\). \(G_{\vec{k}, \vec{m}}\) are open Gromov-Witten invariants. From string world-sheet viewpoint, these terms in the superpotential represent the contribution from instantons of sphere and disk.

### 2.2 Relative periods and Generalized GKZ system

The generalized hypergeometric systems originated from \([34]\) and have been applied in mirror symmetry \([37-41]\). The notation is as follows: \((X^*, X)\) is the mirror pair of compact Calabi-Yau threefold defined as hypersurfaces in toric ambient spaces \((W^*, W)\), respectively. The generators \(l^a\) of Mori cone of the toric variety \([51-54]\) give rise to the charge vectors of the gauged linear sigma model (GLSM) \([55]\). \(\Delta\) is a real four-dimensional reflexive polyhedron. \(W = P_{\Sigma(\Delta)}\) is the toric variety with fan \(\Sigma(\Delta)\) being the set of cones over the faces of \(\Delta^*\). \(\Delta^*\) is the dual polyhedron and \(W^*\) is the toric variety obtained from \(\Sigma(\Delta^*)\). The enhanced polyhedron \(\Delta^*\) constructed from polyhedron \(\Delta^*\) is associated to \(X_4^*\) on which the dual F-theory compactify. The threefold \(X\) on B-model side is defined by \(p\) integral points of \(\Delta^*\) as the zero locus of the polynomial \(P\) in the toric ambient space

\[
P = \sum_{i=0}^{p-1} a_i \prod_{k=0}^{4} X_k^{\nu_k_i}
\]

where the \(X_k\) are coordinates on an open torus \((\mathbb{C}^*)^4 \in W\) and \(a_i\) are complex parameters related to the complex structure of \(X\). In terms of homogeneous coordinates \(x_j\) on the toric ambient space, it can be rewritten as

\[
P = \sum_{i=0}^{p-1} a_i \prod_{\nu \in \Delta} x_j^{(\nu, \nu^*_i) + 1} .
\]

The open-string sector from D-branes can be described by the family of hypersurfaces \(\mathcal{D}\), which is defined as intersections \(P = 0 = Q(\mathcal{D})\). In toric variety, the \(Q(\mathcal{D})\)
can be defined as \[ Q(D) = \sum_{i=p}^{p+p'-1} a_i X_k^{\nu_i,k} \] (2.11)

where additional \( p' \) vertices \( \nu_i^* \) correspond with the monomials in \( Q(D) \).

When considering the dual F-theory compactification on Four-fold \( X_4 \), the relevant "Enhanced polyhedron" consists of extended vertices

\[
\mathcal{V}_i = \begin{cases} 
(\nu_i^*, 0) & i = 0, \ldots, p - 1 \\
(\nu_i^*, 1) & i = p, \ldots, p + p' - 1.
\end{cases}
\] (2.12)

The period integrals can be written as

\[
\Pi_i = \int_{\gamma_i} \frac{1}{P(a, X)} \prod_{j=1}^n dX_j X_j.
\] (2.13)

According to the refs. [40, 41], the period integrals can be annihilated by differential operators

\[
\mathcal{L}(l) = \prod_{l_i > 0} (\partial_{a_i})^{l_i} - \prod_{l_i < 0} (\partial_{a_i})^{l_i}
\]

\[
\mathcal{Z}_k = \sum_{i=0}^{p-1} \nu_i^* \phi_i, \quad \mathcal{Z}_0 = \sum_{i=0}^{p-1} \phi_i - 1
\] (2.14)

where \( \phi_i = a_i \partial_{a_i} \). As noted in refs. [37] [18], the equations \( \mathcal{Z}_k \Pi(a_i) = 0 \) reflex the invariance under the torus action, defining torus invariant algebraic coordinates \( z_a \) on the moduli space of complex structure of \( X \) [24]:

\[
z_a = (-1)^l a_i \prod_{l_i}^{l_{a_i}} a_i
\] (2.15)

where \( l_a, \ a = 1, \ldots, h^{2,1}(X) \) is generators of the Mori cone, one can rewrite the differential operators \( \mathcal{L}(l) \) as [24, 37, 41]

\[
\mathcal{L}(l) = \prod_{k=1}^{l_0} (\partial_0 - k) \prod_{l_i > 0}^{l_{a_i}} (\partial_i - k) - (-1)^{l_0} z_a \prod_{k=1}^{-l_0} (\partial_0 - k) \prod_{l_i < 0}^{l_{a_i}} (\partial_i - k).
\] (2.16)

The solution to the GKZ system can be written as [24, 37, 41]

\[
B_{l_a}(z^a; \rho) = \sum_{n_1, \ldots, n_N \in \mathbb{Z}_0^+} \frac{\Gamma(1 - \sum_{a} l_a n_a + \rho_a)}{\prod_{l_i > 0} \Gamma(1 + \sum_{a} l_a n_a + \rho_a)} \prod_{a} z_a^{n_a + \rho_a}.
\] (2.17)
In this paper we consider the family of divisors $D$ with a single open deformation moduli $\hat{z}$ \footnote{In refs [15, 25] they considered another approach which blows up along the curve C and replaces the pair $(X, C)$ with a non-Calabi-Yau manifold $\hat{X}$.}

\begin{equation}
    x_1^{b_1} + \hat{z}x_2^{b_2} = 0 \tag{2.18}
\end{equation}

where $b_1, b_2$ are some appropriate integers. The relative 3-form $\Omega := (\Omega_X^{3,0}, 0)$ and the relative periods satisfy a set of differential equations \cite{5, 6, 14, 18, 24}

\begin{equation}
    \mathcal{L}_a(\theta, \hat{\theta})\Omega = d\omega^{(2,0)} \Rightarrow \mathcal{L}_a(\theta, \hat{\theta})T(z, \hat{z}) = 0. \tag{2.19}
\end{equation}

with some corresponding two-form $\omega^{(2,0)}$. The differential operators $\mathcal{L}_a(\theta, \hat{\theta})$ can be expressed as \cite{24}

\begin{equation}
    \mathcal{L}_a(\theta, \hat{\theta}) := \mathcal{L}_a^b - \mathcal{L}_a^{bd}\hat{\theta} \tag{2.20}
\end{equation}

for $\mathcal{L}_a^b$ acting only on bulk part from closed sector, $\mathcal{L}_a^{bd}$ on boundary part from open-closed sector and $\hat{\theta} = \hat{z}\partial\hat{z}$. The explicit form of these operators will be given in following model. One can obtain

\begin{equation}
    2\pi i\hat{\theta}T(z, \hat{z}) = \pi(z, \hat{z}) \tag{2.21}
\end{equation}

by reduction to Noether-Lefshetz locus for only the family of divisors $D$ depending on the $\hat{z}$. From above one can obtain differential equation with the inhomogeneous term $f_a(z)$ at the critical points

\begin{equation}
    \mathcal{L}_a^bT(z) = f_a(z) \tag{2.22}
\end{equation}

and

\begin{equation}
    2\pi i f_a(z) = \mathcal{L}_a^{bd}\pi(z, \hat{z})|_{\hat{z}=\text{critic points}} \tag{2.23}
\end{equation}

3 Superpotential of Calabi-Yau $X_{14}(1, 1, 2, 3, 7)$

The $X_{14}(1, 1, 2, 3, 7)$ is defined as a degree 14 hypersurface which is the zero locus of polynomial $P$

\begin{equation}
    P = x_1^{14} + x_2^{14} + x_3^7 + x_3x_4^4 + x_5^2 \tag{3.1}
\end{equation}
The GLSM charge vectors for this manifold are [57]

| l_1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|---|
| 0   | 0 | -2| -2| -4| 1 | 0 | 7 |
| l_2 | -2| 1 | 1 | 2 | 0 | 1 | -3 |

(3.2)

The mirror manifold is [57] \( X^* = \hat{X}/H \), where \( \hat{X} = X_{28}(2, 2, 3, 7, 14) \) and \( H = (h_k^i) = \frac{1}{14}(1, 13, 0, 0, 0), \frac{1}{2}(1, 0, 0, 0, 1) \) which act by \( x_i \to \exp(2\pi i h_i^k) x_i \). We consider the following curves

\[
C_{\alpha, \pm} = \{ x_4 = \xi x_5, x_3 = 0, x_1^7 x_4 + x_2^7 x_5 = 0 \}, \quad \xi^3 = -1
\]

(3.3)

which are on the family of divisor

\[
Q(D) = x_4^3 + z_3 x_5^3
\]

(3.4)

at the critical points \( z_3 = 1 \).

By the generalized GKZ system, the period on the surface \( P = 0 = Q(D) \) has the form

\[
\pi = \frac{c}{2} B_{(i, j)}(u_1, u_2; 0, \frac{1}{2}) = \sum_{n_1, n_2} \frac{c z_1^{n_1} z_2^{n_2} \Gamma(3(n_1 + \frac{1}{2}) + (n_2) + 1)}{\Gamma(2 + 2n_1) \Gamma(n_2) \Gamma(n_1 - 2n_2)}
\]

\[
= -\frac{4c}{\pi^2} \sqrt{u_1 u_2} + O((u_1 u_2)^{3/2})
\]

(3.5)

which vanishes at the critical locus \( u_2 = 0 \) in terms of new parameters as

\[
u_1 = \frac{-z_1}{z_3} (1 - z_3)^2 \quad u_2 = z_2
\]

(3.6)

and \( Z_{1,2} \) are relevant coordinates in the large complex structure limit defined as \((2.15)\). Following the [33], the off-shell superpotentials can be obtained by integrating the \( \pi \):

\[
T_{a}^{\pm}(z_1, z_2) = \frac{1}{2\pi i} \int \pi(z_3) \frac{dz_3}{z_3},
\]

(3.7)

with the appropriate integral constants [24], the superpotentials can be chosen as \( W^+ = -W^- \). In this convention, the on-shell superpotentials can be obtained as

\[
2W^+ = \frac{1}{2\pi i} \int_{-z_3}^{z_3} \pi(z_3) \frac{dz_3}{z_3}, \quad W^+(z_1, z_2) = W^+(z_1, z_2, z_3)|_{z_3=1}
\]

(3.8)
Eventually, the superpotential are
\[
W^\pm(z_1, z_2, z_3) = \mp \sum_{n_1, n_2} \frac{c z_1^{n_1} z_2^{1+n_2} z_3^{-2n_1}}{4\pi (-1 + 4n_1^2)} \frac{\Gamma(3n_1 + n_2 + 1)}{\Gamma(2 + 2n_1) \Gamma(1 + n_2) \Gamma(n_1 - 2n_2)} \Gamma(3n_1 + 1 + n_2 + 1) \Gamma(n_1 + 1) \Gamma(n_1 - 2n_2) \Gamma(n_1 + 1 - 2n_2)
\]  
(3.9)

\{(1 - 2n_1)2F_1(-\frac{1}{2} - n_1, -2n_1, \frac{1}{2} - n_1; z_3) + (1 + 2n_1)z_3^2 F_1((\frac{1}{2} - n_1, -2n_1, \frac{3}{2} - n_1; z_3))\}

and those can divide two parts
\[
W^\pm(z_1, z_2, z_3) = W^\pm(z_1, z_2) + f(z_1, z_2, z_3)
\]  
(3.10)

where the \(f(z_1, z_2, z_3)\) are related to the open-string parameter, \(W\) are the on-shell superpotential defined as
\[
W^\pm = \mp \sum_{n_1, n_2} \frac{c z_1^{n_1} z_2^{1+n_2} \Gamma(3n_1 + (n_2 + \frac{1}{2}) + 1)}{\Gamma(n_2 + \frac{1}{2} + 1) \Gamma(n_1 + 1) \Gamma(n_1 - 2n_2)}
\]  
(3.11)

The additional GLSM charge vectors corresponding to the divisor (4.4) are
\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
l_4 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}
\]  
(3.12)

The classic A-brane in the mirror Calabi-Yau manifold \(X^*\) of \(X\) determined by the additional charge vectors \((0, -1, 1, 0, 0, 0, 0)\) is a special Lagrangian submanifold of \(X^*\) defined as \([3, 4, 8, 9, 17]\)
\[
-|x_4| + |x_5| = \eta
\]  
(3.13)

where \(x_i\) are coordinates on \(X^*\), \(\eta\) is a Kähler moduli parameter with \(\hat{z} = e^\epsilon \eta\) for a phase \(\epsilon\).
The flat coordinates in A-model at large radius regime are related to the flat coordinates of B-model at large complex structure regime by mirror map $t_i = \frac{\omega}{\omega_0}$, $\omega := D_i^{(1)} \omega_0(z, \rho)|_{\rho=0}$

$$q_1 = z_1 - 6 z_1^2 + 63 z_3 - 866 z_1^4 + 68 z_1^3 z_2 + O(z^5)$$
$$q_2 = z_2 + 14 z_1 z_2 - 7 z_1^2 z_2 + 294 z_1^3 z_2 - 96 z_1^2 z_2^2 + O(z^5)$$

and we can obtain the inverse mirror map in terms of $q_i = e^{2\pi i t_i}$

$$z_1 = q_1 + 6 q_1^2 + 9 q_1^3 + 56 q_1^4 - 68 q_1^3 q_2 + O(q^5)$$
$$z_2 = q_2 - 14 q_1 q_2 - 119 q_1^2 q_2 - 924 q_1^3 q_2 + 96 q_1^2 q_2^2 O(q^5).$$

Using the modified multi-cover formula $[10,23]$ for this case

$$\frac{W^\pm(z(q))}{w_0(z(q))} = \frac{1}{(2i\pi)^2} \sum_{k \text{ odd}} \sum_{d_2,d_1 \text{ odd} \geq 0} n_{d_1,d_2}^\pm q_1^{kd_1} q_2^{kd_2/2}$$  \hspace{1cm} (3.16)

the superpotentials $W^\pm$, at the critical points $z_3 = 1$, give Ooguri-Vafa invariants $n_{d_1,d_2}$ for the normalization constants $c = 1$, which are listed in Table.1.

Another interesting thing which should be mentioned is the superpotential $W^+$ encode the information of superpotential of the non-compact geometry $O(-3)_{\mathbb{P}^2}$ in the limit of $q_1 \to 0$. This can be shown by $n_{d_1,0} = n_k$, where $n_k$ is the disc invariants of $O(-3)_{\mathbb{P}^2}$ which were studied in work [58]. See more details in appendix of [24].

4 Superpotential of Calabi-Yau $X_8(1,1,1,2,3)$

The $X_8(1,1,1,2,3)$ is defined as a degree 8 hypersurface which is the zero locus of polynomial $P$

$$P = x_1^8 + x_2^8 + x_3^8 + x_4^4 + x_4 x_5^2$$  \hspace{1cm} (4.1)

The GLSM charge vectors $l_a$ are the generators of the Mori cone as follows [57]

|      | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|------|---|---|---|---|---|---|---|
| $l_1$ | -2 | 0 | 0 | 0 | 0 | 1 | 1 |
| $l_2$ | -2 | 1 | 1 | 1 | 2 | 0 | -3 |

The mirror manifolds $X^*$ is $X^* = \hat{X}/H$, where $\hat{X} = X_8(1,1,1,1,4)$ and $H = (h_1^k) = \frac{1}{8}(7,0,0,1,0), \frac{1}{8}(7,0,1,0,0)$ which act by $x_i \to x_i \exp(2\pi i h_i^k)$. 

10
We consider the following curves
\[ C_{\alpha, \pm} = \{ x_2 = \xi x_1, x_3 = \xi x_4, x_5^2 + \psi x_1 x_2 x_3 x_4 x_5 = 0 \}, \quad \xi^8 = -1 \] (4.3)
which are on the family of divisor
\[ Q(D) = x_2^8 + z_3 x_1^8 \] (4.4)
at the critical points \( z_3 = 1 \).

By the GKZ system, the period on the hypersurface has the form
\[ \pi = c_{\frac{1}{2}} B_{i_1, i_2}(u_1, u_2; 0, \frac{1}{2}) = -\frac{4c}{\pi^2} u_1 \sqrt{u_2} + \mathcal{O}((u_1 u_2)^{3/2}) \] (4.5)
which vanishes at the critical locus \( u_2 = 0 \) expressed in terms of new parameters as
\[ u_1 = z_1 \quad u_2 = \frac{-z_2}{z_3}(1 - z_3)^2 \] (4.6)
Similarly, the off-shell superpotentials can be obtained by integrating the \( \pi \):
\[ \mathcal{T}_a^\pm(z_1, z_2, z_3) = \frac{1}{2\pi i} \int \pi(z) \frac{dz}{z}, \] (4.7)
with the appropriate integral constants [24], the superpotentials can be chosen as \( \mathcal{W}^+ = -\mathcal{W}^- \). In this convention, the on-shell superpotentials can be obtained as
\[ 2\mathcal{W}^+ = \frac{1}{2\pi i} \int_{-z_3}^{z_3} \pi(z_3) \frac{dz_3}{z_3}, \quad \mathcal{W}^\pm(z_1, z_2, z_3) = \mathcal{W}^\pm(z_1, z_2) \big|_{z_3=1} \] (4.8)
Eventually, The superpotential are
\[ \mathcal{W}^\pm(z_1, z_2, z_3) = \mp \sum_{n_1, n_2} \frac{c z_1^{n_1} z_2^{n_2} z_3^{-1 + 2n_1}}{4\pi(-1 + 4n_1^2) \Gamma(2 + 2n_1^2) \Gamma(1 + n_2) \Gamma(\frac{3}{2} + n_2)} \Gamma(2n_1 + 2n_2 + 1) \] (4.9)
\[ \{(1 - 2n_1)2F_1(-\frac{1}{2} - n_1, -2n_1, \frac{1}{2} - n_1; z_3) + (1 + 2n_1)z_32F_1((\frac{1}{2} - n_1, -2n_1, \frac{3}{2} - n_1; z_3))\}\]
and those can divide two parts
\[ \mathcal{W}^\pm(z_1, z_2, z_3) = \mathcal{W}^\pm(z_1, z_2) + f(z_1, z_2, z_3) \] (4.10)
where the \( f(z_3) \) are related to the open-string parameter, \( W \) are the on-shell superpotential defined as
\[ W^\pm(z_1, z_2) = \mp \frac{c}{8} B_{i_1, i_2}((z_1, z_2); 0, \frac{1}{2}) \] (4.11)
substituting the vector $l_1, l_2$ in this hypersurface, the superpotentials are

$$W^\pm = \mp \frac{c}{8} \sum_{n_1, n_2} z_1^{n_1} z_2^{n_2} \Gamma \left( 2n_1 + 2(n_2 + \frac{1}{2}) + 1 \right) \Gamma \left( n_2 + \frac{1}{2} + 1 \right) \Gamma \left( n_1 + 1 \right) \Gamma \left( 2(n_1 + \frac{1}{2}) + 1 \right) \Gamma \left( n_1 - 3(n_2 + \frac{1}{2}) + 1 \right)$$ (4.12)

The additional GLSM charge vectors corresponding to the divisor (4.4) are

$$l_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$ (4.13)

The classic A-brane in the mirror Calabi-Yau manifold $X^*$ of $X$ determined by the additional charge vectors $(0, -1, 1, 0, 0, 0, 0)$ is a special Lagrangian submanifold of $X^*$ defined as

$$- |x_1|^2 + |x_2|^2 = \eta$$ (4.14)

where $x_i$ are coordinates on $X^*$, $\eta$ is a Kähler moduli parameter with $\tilde{\eta} = e^{\epsilon \eta}$ for a phase $\epsilon$.

| $d_1 \mid d_2$ | 1 | 3 | 5 | 7 |
|----|----|----|----|----|
| 0  | 2  | -2 | 10 | -84|
| 1  | -40| 40 | -360| 7232|
| 2  | -180| -200| 5500| -215356|
| 3  | -40| -29720| 66963200| 2314120|
| 4  | -628| -625424| 25006400960| -|

The flat coordinates in A-model at large radius regime are related to the flat coordinates of B-model at large complex structure regime by mirror map $t_i = \frac{\omega_i}{\omega_0}$, $\omega_i := D_i^{(1)} \omega_0(z, \rho)|_{\rho=0}$

$$q_1 = z_1 + 2z_1^2 + 5z_1^3 + 2z_1 z_2 - 12z_1^2 z_2 - 13z_1 z_2^2 + O(z^4)$$
$$q_2 = z_2 - 6z_2^2 + 12z_1 z_2 + 100z_1^2 z_2 - 24z_1 z_2^2 + O(z^4)$$ (4.15)

and we can obtain the inverse mirror map in terms of $q_i = e^{2\pi i t_i}$

$$z_1 = q_1 - 2q_1^2 + 3q_1^3 - 2q_1 q_2 + 48q_1^2 q_2 + 5q_1 q_2^2 + O(q^4)$$
$$z_2 = q_2 + 6q_2^2 + 9q_2^3 - 12q_1 q_2 + 68q_1^2 q_2 - 168q_1 q_2^2 + O(q^4).$$ (4.16)
Using the modified multi-cover formula \[10, 23\] for this case

\[
\frac{W^\pm(z(q))}{w_0(z(q))} = \frac{1}{(2i\pi)^2} \sum_{k \text{ odd}} \sum_{d_2,d_1 \text{ odd} \geq 0} n^\pm_{d_1,d_2} \frac{q_1^{kd_1} q_2^{kd_2/2}}{k^2}
\]  

(4.17)

the superpotentials \(W^\pm\), at the critical points \(\hat{z} = 1\), give Ooguri-Vafa invariants \(n_{d_1,d_2}\) for the normalization constants \(c = 1\), which are listed in Table.1.

In the limit \(Z_1 = 0\) we also give the superpotential of non-compact manifold \(\mathcal{O}(-3)_{\mathbb{P}^2}\) which can be proved by \(n_{0,d_2} = n_k\).

5 Summary

For two compact Calabi-Yau threefolds of non-fermat type constructed by using the toric geometry \[57\], we study superpotentials of the D-branes on the these manifolds by extending the GKZ method to the Calabi-Yau manifolds of the non-Fermat type on the B-model side. Furthermore, with the mirror symmetry, we calculate the non-perturbative superpotential on the A-model side and extract the Ooguri-Vafa invariants for D-brane with a single open deformation moduli.

On the other hand, we will study these problems in this paper from an alternative approach \[15, 16, 19, 25, 27\] that treats the complex structure deformations of the Calabi-Yau threefolds and the open string deformations of the D-branes on an equal footing by studying the complex structure deformations of a non-Calabi-Yau manifold obtained by blowing up the original Calabi-Yau threefold along curve wrapped by the D-brane. It is interesting to compare the results obtained from the two approaches for the same system of the Calabi-Yau threefold and D-branes.

Furthermore, we are going to treat the obstruction problem to the deformations with the quiver gauge theories \[59\] and the \(A_\infty\) -algebraic structure \[60–65\] in the derived category of coherent sheaves.

Acknowledgments

The work is supported by the NSFC (11075204) and President Fund of GUCAS (Y05101CY00).
References

[1] P. Candelas, X. C. DeLaOssa, P. S. Green, and L. Parkes, A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory, Nucl. Phys. B 359 (1991) 21–74.

[2] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Kodaira-Spenc er theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. 165, 311 (1994) [hep-th/9309140].

[3] M. Aganagic and C. Vafa, Mirror symmetry, D-branes and counting holomorphic discs, hep-th/0012041.

[4] M. Aganagic, A. Klemm and C. Vafa, Disk instantons, mirror symmetry and the duality web, Z. Naturforsch. A 57, 1 (2002) [hep-th/0105045].

[5] W. Lerche, P. Mayr and N. Warner, N=1 special geometry, mixed Hodge variations and toric geometry, hep-th/0208039.

[6] W. Lerche, P. Mayr and N. Warner, Holomorphic N=1 special geometry of open - closed type II strings, hep-th/0207259.

[7] M. Aganagic, A. Klemm, M. Marino and C. Vafa, The Topological vertex, Commun. Math. Phys. 254, 425 (2005) [hep-th/0305132].

[8] S. Kachru, S. H. Katz, A. E. Lawrence and J. McGreevy, Open string instantons and superpotentials, Phys. Rev. D 62, 026001 (2000) [hep-th/9912151].

[9] S. Kachru, S. H. Katz, A. E. Lawrence and J. McGreevy, Mirror symmetry for open strings, Phys. Rev. D 62, 126005 (2000) [hep-th/0006047].

[10] J. Walcher, Opening mirror symmetry on the quintic, Commun. Math. Phys. 276, 671 (2007) [hep-th/0605162].

[11] D. R. Morrison and J. Walcher, D-branes and Normal Functions, arXiv:0709.4028 [hep-th].

[12] J. Knapp and E. Scheidegger, Towards Open String Mirror Symmetry for One-Parameter Calabi-Yau Hypersurfaces, arXiv:0805.1013 [hep-th].
[13] D. Krefl and J. Walcher, Real Mirror Symmetry for One-parameter Hypersurfaces, JHEP 0809, 031 (2008) [arXiv:0805.0792 [hep-th]].

[14] H. Jockers and M. Soroush, Effective superpotentials for compact D5-brane Calabi-Yau geometries, Commun. Math. Phys. 290, 249 (2009) [arXiv:0808.0761 [hep-th]].

[15] T. W. Grimm, T. -W. Ha, A. Klemm and D. Klevers, The D5-brane effective action and superpotential in N=1 compactifications, Nucl. Phys. B 816, 139 (2009) [arXiv:0811.2996 [hep-th]].

[16] T. W. Grimm, T. -W. Ha, A. Klemm and D. Klevers, Five-Brane Superpotentials and Heterotic / F-theory Duality, Nucl. Phys. B 838, 458 (2010) [arXiv:0912.3250 [hep-th]].

[17] M. Alim, M. Hecht, P. Mayr and A. Mertens, Mirror Symmetry for Toric Branes on Compact Hypersurfaces, JHEP 0909, 126 (2009) [arXiv:0901.2937 [hep-th]].

[18] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens and M. Soroush, Hints for Off-Shell Mirror Symmetry in type II/F-theory Compactifications, Nucl. Phys. B 841, 303 (2010) [arXiv:0909.1842 [hep-th]].

[19] T. W. Grimm, T. -W. Ha, A. Klemm and D. Klevers, Computing Brane and Flux Superpotentials in F-theory Compactifications, JHEP 1004, 015 (2010) [arXiv:0909.2025 [hep-th]].

[20] H. Jockers, P. Mayr and J. Walcher, On N=1 4d Effective Couplings for F-theory and Heterotic Vacua, Adv. Theor. Math. Phys. 14, 1433 (2010) [arXiv:0912.3265 [hep-th]].

[21] H. Jockers and M. Soroush, Relative periods and open-string integer invariants for a compact Calabi-Yau hypersurface, Nucl. Phys. B 821, 535 (2009) [arXiv:0904.4674 [hep-th]].

[22] S. Li, B. H. Lian and S. -T. Yau, Picard-Fuchs Equations for Relative Periods and Abel-Jacobi Map for Calabi-Yau Hypersurfaces, arXiv:0910.4215 [math.AG].

[23] J. Walcher, Calculations for Mirror Symmetry with D-branes, JHEP 0909, 129 (2009) [arXiv:0904.4905 [hep-th]].
[24] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens and M. Soroush, Type II/F-theory Superpotentials with Several Deformations and N=1 Mirror Symmetry, JHEP 1106, 103 (2011) [arXiv:1010.0977 [hep-th]].

[25] T. W. Grimm, A. Klemm and D. Klevers, “Five-Brane Superpotentials, Blow-Up Geometries and SU(3) Structure Manifolds,” JHEP 1105, 113 (2011) [arXiv:1011.6375 [hep-th]].

[26] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens and M. Soroush, Flat Connections in Open String Mirror Symmetry, arXiv:1110.6522 [hep-th].

[27] D. Klevers, Holomorphic Couplings In Non-Perturbative String Compactifications, Fortsch. Phys. 60, 3 (2012) [arXiv:1106.6259 [hep-th]].

[28] M. Baumgartl, I. Brunner and M. R. Gaberdiel, D-brane superpotentials and RG flows on the quintic, JHEP 0707, 061 (2007) [arXiv:0704.2666 [hep-th]].

[29] M. Baumgartl and S. Wood, Moduli Webs and Superpotentials for Five-Branes, JHEP 0906, 052 (2009) [arXiv:0812.3397 [hep-th]].

[30] M. Baumgartl, I. Brunner and M. Soroush, D-brane Superpotentials: Geometric and Worldsheet Approaches, Nucl. Phys. B 843, 602 (2011) [arXiv:1007.2447 [hep-th]].

[31] H. Fuji, S. Nakayama, M. Shimizu and H. Suzuki, A Note on Computations of D-brane Superpotential, J. Phys. A A 44, 465401 (2011) [arXiv:1011.2347 [hep-th]].

[32] M. Shimizu and H. Suzuki, Open mirror symmetry for Pfaffian Calabi-Yau 3-folds, JHEP 1103, 083 (2011) [arXiv:1011.2350 [hep-th]].

[33] Feng-Jun Xu, Fu-Zhong Yang Type II/F-theory Superpotentials and Ooguri-Vafa Invariants of Compact Calabi-Yau Threefolds with Three Deformations, [hep-th/1206.0445].

[34] I.M.Gel’fand, A.Zelevinski and M.Kapranov, Funct.Anal.Appl. 28, 94 (1989) [arXiv:9308122 [hep-th]].

[35] S. Gukov, C. Vafa and E. Witten, CFT’s from Calabi-Yau four folds, Nucl. Phys. B 584, 69 (2000) [Erratum-ibid. B 608, 477 (2001)] [hep-th/9906070].
P. Berglund and P. Mayr, Non-perturbative superpotentials in F-theory and string duality, hep-th/0504058.

S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces, Commun. Math. Phys. 167, 301 (1995) [hep-th/9308122].

S. Hosono, B. H. Lian and S. T. Yau, GKZ generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces, Commun. Math. Phys. 182, 535 (1996) [alg-geom/9511001].

S. Hosono, B. H. Lian and S. T. Yau, GKZ Systems, Grobner Fans and Moduli Spaces of Calabi-Yau Hypersurfaces, [arXiv:alg-geom/9707003v2].

V. V. Batyrev and D. van Straten, Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties, Commun. Math. Phys. 168, 493 (1995) [alg-geom/9307010].

V. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Alg. Geom. 3, 493 (1994) [alg-geom/9310003].

P. Mayr, N=1 mirror symmetry and open / closed string duality, Adv. Theor. Math. Phys. 5, 213 (2002) [hep-th/0108229].

E. Witten, Chern-Simons gauge theory as a string theory, Prog. Math. 133, 637 (1995) [hep-th/9207094].

W. Lerche, Special geometry and mirror symmetry for open string backgrounds with N = 1 supersymmetry, hep-th/0312326.

P. Mayr, On supersymmetry breaking in string theory and its realization in brane worlds, Nucl. Phys. B 593, 99 (2001) [hep-th/0003198].

T. R. Taylor and C. Vafa, RR flux on Calabi-Yau and partial supersymmetry breaking, Phys. Lett. B 474, 130 (2000) [hep-th/9912152].

E. Witten, Branes and the dynamics of QCD, Nucl. Phys. B 507, 658 (1997) [hep-th/9706109].
[48] H. Clemens, Cohomology and Obstructions II: Curves on K-trivial threefolds, arXiv:math/0206219.

[49] P. Griffiths, A theorem concerning the differential equations satisfied by normal functions associated to algebraic cycles, Am. J. Math. 101, 96 (1979)

[50] H. Ooguri and C. Vafa, Knot invariants and topological strings, Nucl. Phys. B 577, 419 (2000) [hep-th/9912123].

[51] O. Fujino and H. Sato, Introduction to the toric Mori theory, arXiv:math/0307180v2 [math.AG]

[52] O. Fujino, Notes on toric varieties from Mori theoretic viewpoint, arXiv:math/0112090v1 [math.AG]

[53] A. Scaramuzza, Smooth complete toric varieties: an algorithmic approach. Ph.D. dissertation, University of Roma Tre, 2007.

[54] C. V. Renesse, Combinatorial aspects of toric varieties. Ph.D. dissertation, University of Massachusetts Amherst, 2007.

[55] E. Witten, Phases of N=2 theories in two-dimensions, Nucl. Phys. B 403, 159 (1993) [hep-th/9301042].

[56] P. Berglund and P. Mayr, Heterotic string/F theory duality from mirror symmetry, Adv. Theor. Math. Phys. 2, 1307 (1999) [hep-th/9811217].

[57] P. Berglund, S. H. Katz and A. Klemm, Mirror symmetry and the moduli space for generic hypersurfaces in toric varieties, Nucl. Phys. B 456, 153 (1995) [hep-th/9506091].

[58] W. Lerche, P. Mayr and N. P. Warner, Noncritical strings, Del Pezzo singularities and Seiberg-Witten curves, Nucl. Phys. B 499, 125 (1997) [hep-th/9612085].

[59] Paul S. Aspinwall, Lukasz M. Fidkowski, Superpotentials for Quiver Gauge Theories, JHEP 0610:047,(2006) [hep-th/0506041].

[60] Sujay K. Ashok, Eleonora Dell’Aquila, Duiliu-Emanuel Diaconescu, Bogdan Florea, Obstructed D-Branes in Landau-Ginzburg Orbifolds, Adv. Theor. Math. Phys. 8 427-472 (2004) [hep-th/0404167].
[61] Sujay K. Ashok, Eleonora Dell’Aquila, Duiliu-Emanuel Diaconescu, Fractional Branes in Landau-Ginzburg Orbifolds, Adv. Theor. Math. Phys. 8 461-513 (2004) [hep-th/0401135].

[62] Manfred Herbst, Quantum A-infinity Structures for Open-Closed Topological Strings, [hep-th/0602018].

[63] Nils Carqueville, Laura Dowdy, Andreas Recknagel, Algorithmic deformation of matrix factorisations, JHEP 04 (2012) 014 [1112.3352].

[64] Paul S. Aspinwall, Sheldon Katz, Computation of Superpotentials for D-Branes Commun.Math.Phys. 264, 227-253 (2006) [hep-th/0412209].

[65] Gueorgui Todorov, D-branes, obstructed curves, and minimal model superpotentials, [0709.4673].