Performance and safety of Bayesian model predictive control: Scalable model-based RL with guarantees

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Abstract

Despite the success of reinforcement learning (RL) in various research fields, relatively few algorithms have been applied to industrial control applications. The reason for this unexplored potential is partly related to the significant required tuning effort, large numbers of required learning episodes, i.e. experiments, and the limited availability of RL methods that can address high dimensional and safety-critical dynamical systems with continuous state and action spaces. By building on model predictive control (MPC) concepts, we propose a cautious model-based reinforcement learning algorithm to mitigate these limitations. While the underlying policy of the approach can be efficiently implemented in the form of a standard MPC controller, data-efficient learning is achieved through posterior sampling techniques. We provide a rigorous performance analysis of the resulting ‘Bayesian MPC’ algorithm by establishing Lipschitz continuity of the corresponding future reward function and bound the expected number of unsafe learning episodes using an exact penalty soft-constrained MPC formulation. The efficiency and scalability of the method are illustrated using a 100-dimensional server cooling example and a nonlinear 10-dimensional drone example by comparing the performance against nominal posterior MPC, which is commonly used for data-driven control of constrained dynamical systems.

1 Introduction

Driven by a constantly increasing research and development effort in the field of autonomous systems, including e.g. autonomous driving, service robotics, or various production processes in chemical or biological industry branches, the number of challenging control problems is growing steadily. Together with the ever increasing complexity of such systems, including physical constraints and safety specifications, this motivates research efforts towards automated and efficient synthesis procedures of high-performance control algorithms.

While significant progress in this context has been made with learning-based control for systems with continuous state and action spaces in the areas of machine learning and in particular reinforcement learning (RL), see e.g. [1], only few methods support data-efficient learning of control policies that can satisfy system constraints. In addition, when compared to classical control strategies such as simple PID or state-feedback control [2], the implementation effort and the required expert knowledge for tuning high performance RL algorithms based on, e.g. deep learning techniques, is potentially limiting and can hinder wide-spread adoption in industrial control applications.

Control design using machine learning tools has also been approached from control theoretic perspectives, see for example [3, 4] for an overview. Particularly in the case of general, complex, and safety-critical control problems, model predictive control (MPC) techniques [5, 6, 7] have shown significant impact on both, industrial and research-driven applications, see also Figure [1]
Due to its principled controller synthesis procedures and professional software tools \cite{9,10,11}, MPC offers an important framework for learning-based control, see, e.g., the recent reviews \cite{12,13,4}.

MPC can be seen as an approximate solution to an optimal control problem, which is intractable to solve exactly. The central mechanism is based on solving an open-loop optimal control problem in the form of an optimization problem, the MPC problem, at discrete time instances. More precisely, based on the currently measured system state, the sequence of future control actions is optimized at every time step in real time using a model to predict the evolution of the underlying system. To compensate for uncertainties in the prediction model and external disturbances, only the first element of the computed optimal action sequence is applied to the system and the procedure is repeated at every time step.

By construction, this mechanism heavily relies on a sufficiently accurate prediction and reward model of the system, which typically results in time-consuming system modeling and identification procedures. As a result, research in learning-based MPC mainly focuses on automatically improving the model quality, either by relying on available system data \cite{4} Section 3 or through active data collection using exploration-exploitation mechanisms similar to RL see e.g. \cite{12} for an overview. While passive approaches often allow efficient implementation \cite{14,15}, they rely on sufficiently informative data with respect to the optimal system behavior, which is usually unknown beforehand. Passive methods therefore run into the risk of converging to suboptimal operation regimes.

This limitation is addressed in so-called dual MPC schemes that provide effective exploration-exploitation strategies by approximating the information gain of future data. By relying on approximate stochastic dynamic programming \cite{16}, these MPC-based techniques are closely related to reinforcement learning concepts, see e.g. \cite{17,18}. In case of episodic tasks, an alternative strategy is to optimize model or cost function parameters of an MPC using automatic differentiation \cite{19}, sensitivity analysis \cite{20}, or Bayesian optimization \cite{21,22}. While the underlying concepts are promising, their theoretical properties still need to be investigated and the techniques often have limited scalability.

By relying on a posterior sampling framework for model-based reinforcement learning, a Bayesian MPC scheme was proposed in \cite{23} with the goal of enabling practical and scalable reinforcement learning for industrial applications using concepts from model predictive control. We extend this basic idea to a theoretical framework for a variety of learning-based MPC controllers by analyzing the theoretical properties of Bayesian MPC and propose a modification that introduces cautiousness w.r.t. the constraints leading to the following main contributions.

**Performance:** In \cite{23}, a fundamental regularity assumption of the future reward is adopted \cite{24} Section 6.1, which is a central ingredient to transfer well-known Thompson sampling analysis in a Bayesian optimization setting to RL and characterizes the resulting regret bound. In Section 3.1 we show that the regularity assumption is always satisfied in the important special case of linear mean transitions, concave rewards, and polytopic state and action constraints. For more general nonlinear mean transition functions and rewards, we provide relatively weak sufficient conditions in Section 3.2 that ensure the required regularity and thereby provide an intuition for cases in which Bayesian MPC works and potentially fails in terms of cumulative regret bounds.

**Safety:** The cautious Bayesian MPC formulation uses a simple state constraint tightening that allows to rigorously relate the expected number of unsafe learning episodes to the cumulative performance regret bound in Section 3.3.

1.1 Preliminaries

**Notation:** We denote the $i$-th element of a vector $c \in \mathbb{R}^n$ as $c_i$. A vector with $n$ elements equal to 1 is denoted by $1_n$ and if it is clear from the context we write 1.
with a compact set of admissible states of the form $\mathcal{S} := \{ s \in \mathbb{R}^n | g_{s}(s) \leq 1 \}$ and compact set of admissible actions $\mathcal{A} := \{ a \in \mathbb{R}^m | g_{a}(a) \leq 1 \}$ where $g_{s}: \mathbb{R}^n \rightarrow \mathbb{R}^{n_s}$ and $g_{a}: \mathbb{R}^m \rightarrow \mathbb{R}^{n_a}$. The state transition probability at time $t$ is described by $s(t+1) \sim F(s(t), a(t))$ over the state space $\mathcal{S}$ starting from a random initial condition $s(0) \sim S^0$. We restrict our attention to time-invariant transition models of the form

$$s(t+1) = f(s(t), a(t); \theta_F) + \epsilon_F(t), \quad t \in \mathbb{N}$$

(2)

that are parametric w.r.t. $\theta_F$ and subject to $\sigma_F$-sub-Gaussian zero mean i.i.d. noise $\epsilon_F(t)$. The reward signal at time $t \in \mathbb{N}$ is distributed according to $r(t) \sim R(t, s(t), a(t))$ over $\mathbb{R}$ with $s(t) \in \mathcal{S}$ and $a(t) \in \mathcal{A}$, i.e. its distribution may change over time. Similarly as for the state transitions, we focus on reward models of the form

$$r(t, s(t), a(t); \theta_R) = \ell(t, s(t), a(t); \theta_R) + \epsilon_R(t)$$

(3)

where $\epsilon_R(t)$ is $\sigma_R$-sub-Gaussian zero mean i.i.d. noise and $\theta_R$ parameterizes the mean function.

In case of perfectly known transition and reward parameters $\theta_R$ and $\theta_F$, the goal is to find a control policy $\pi: \mathbb{N} \times \mathcal{S} \rightarrow \mathcal{A}$ such that application of $a(t) = \pi(t, s(t))$ maximizes the time-varying sum of reward signals starting from a given initial condition $s(0) \sim S^0$ over a finite horizon of $T$ time steps:

$$\max_{\pi} \mathbb{E}_E \left[ \sum_{t=0}^{T-1} \ell(t, s(t), a(t); \theta_R) \right] \text{ subject to (2) with } E := [\epsilon_F(0), ..., \epsilon_F(T-2)].$$

(4)

Importantly, maximization of (4) needs to be performed while taking into account state and action constraints, i.e. $s(t) \in \mathcal{S}$ and $a(t) \in \mathcal{A}$ for all $t = 0, ..., T-1$. It should be noted that this yields a challenging control problem, even for small-scale systems with state dimension $n < 5$, perfectly known parameters $\theta_F, \theta_R$, and noise-free rewards and transitions.

### 1.2 Soft-constrained model predictive control as an approximate optimal control policy

In the idealized case of perfect system parameter knowledge, an approximate policy $\pi$ to maximize (4) can be obtained by repeatedly solving a simplified open-loop optimal control problem, a so-called model predictive control (MPC) problem, initialized at the currently measured state $s(t)$. While MPC formulations vary greatly in their complexity, a simple formulation as originally proposed by [25] provides sufficient practical properties in terms of performance and constraint satisfaction for many applications. Thereby, we optimize over an action sequence $\{ \tilde{a}_{kt} \}$ subject to the system constraints while neglecting zero mean additive disturbances. The resulting MPC problem is given by

$$J^0_{\ell}(s) := \max_{\{a_{kt}\}} \sum_{k=t}^{T-1} \ell(k, \tilde{s}_{kt}, \tilde{a}_{kt}; \theta_R) - I(\rho_{kt})$$

(5a)

subject to

\begin{align}
&\tilde{s}_{kt} = s, \quad \rho_{kt} \geq 0, \\
&\tilde{s}_{k+1|t} = f(\tilde{s}_{kt}, \tilde{a}_{kt}; \theta_F), \quad k = t, ..., T-2, \\
&\tilde{s}_{kt} \in \bar{\mathcal{S}}(\rho_{kt}), \quad k = t, ..., T-1, \\
&\tilde{a}_{kt} \in \mathcal{A}, \quad k = t, ..., T-1,
\end{align}

(5b, 5c, 5d)

and can be efficiently solved online based on the current system state $s(t)$ using tailored MPC solvers [9, 10, 11]. Ideally, the prediction horizon $T$ equals the task length, yielding a shrinking horizon MPC. For long task horizons $T$, another common approximation in MPC is to select a smaller prediction horizon $T'$, another common approximation in MPC is to select a smaller prediction horizon $T$.

Different from the original formulation as proposed by [25] and different from [23], we enforce a modified state constraint (5d). First, we use a tightened state constraint common in MPC for uncertain systems to foster closed-loop constraint satisfaction, see e.g. [26] and [4, Section 3] for an overview. By optimizing state trajectories subject to a tightened state constraint set, i.e. $g_{s}(s) \leq (1-\epsilon)^T$ with $0 < \epsilon < 1$, we gain a safety margin to compensate for uncertain model parameters $\theta_F$ and unknown external disturbances $\epsilon_F$ before state constraint violation occurs at some time step $t$ in the future, i.e. $g_{s}(s(t)) \notin 1$. As a second modification, we soften the tightened state constraint in (5d) and include the extra negative reward term $-I(\rho)$ on the constraint relaxation in (5d) to ensure feasibility of problem [5] as similarly done in [27]. The penalty is selected to realize a so-called
exact penalty function as proposed by [28]. The resulting cautious soft-constraint formulation is given as $S_\delta(\rho) = \{ s \in \mathbb{R}^n | \rho(s) \leq (1 - \delta) \mathbb{I} + \rho \}$ with parameter $\delta \in \mathbb{R}$, $\delta > 0$ defining the degree of cautiousness, slack variable $\rho \in \mathbb{R}^n$, $\rho \geq 0$, and the exact penalty $I(\rho) = c_1 \rho + c_2 \rho^\top \rho$ for sufficiently large linear penalty weights $c_1 \in \mathbb{R}^n$, $c_1 > 0$ [28]. Importantly, the tightening of the constraints using $\delta > 0$ will be the main mechanism allowing to upper bound the expected number of unsafe learning episodes in Section 3.3.

Using an imperfect estimate $\hat{\theta} = (\hat{\theta}_R, \hat{\theta}_F)$ of the true system parameters $\theta := (\theta_R, \theta_F)$ in the MPC problem [5], we denote the expected closed-loop future reward, including a weighted constraint violation penalty, at time $t$ and state $s$ as

$$V^\theta_{\hat{\theta},t}(s) := E \left[ \sum_{j=t}^{T-1} r(j, x(j), u(j); \theta_R) - I(\rho(j)) \right]$$

(6)

with $E := [\epsilon_R(t), \epsilon_F(t), \epsilon_F(T - 1), \epsilon_F(T - 2)]$ and $\bar{a}^*(j, s(j); \hat{\theta}) := \min_{\rho \geq 0} \rho \text{ s.t. } s(j) \in \bar{S}_\delta(\rho)$,

1.3 Reinforcement learning problem

For MDPs of the form (1), we consider the case of unknown transition and reward distributions that are parametric according to (2). The corresponding reinforcement learning problem is to improve the MPC policy, which is based on solving (5) at every time step, through learning episodes that lead to reward maximization in a data-efficient manner. During each learning episode $e = 0, 1, \ldots, N - 1$ we therefore need to provide a control policy that trades-off information extraction and knowledge exploitation when applied to the MDP (1) at each time step $t = 0, 1, \ldots, T - 1$ starting from $s(0) \sim S_0$. We assume access to prior information about the MDP parameterization, such as production tolerances, to be given as $(\theta_F, \theta_R) \sim Q_0$. Collected data up to $N$ episodes is denoted by $\mathcal{D}_N := \{(t, s_t, e, a_t, e, s_{t+1}, e, r_t, e)_{t=0}^{T-1} \}_{e=0}^{N-1}$.

Conditioned on collected data (7), the corresponding posterior distribution up to episode $e$ is denoted by $\theta_e \sim Q_{\theta|\mathcal{D}_e}$.

Based on the acquired data over $N$ episodes, the performance of the RL algorithm is measured in terms of the expected Bayesian cumulative regret

$$CR(N) := E_{\theta, \theta_e, \mathcal{D}_e} \left[ \sum_{e=0}^{N-1} \Delta_e \right]$$

with episodic regret $\Delta_e := E_e \left[ V^\theta_{\theta_e,t}(s) - V^\theta_{\theta_e,t}(s) \right]$. (8)

Using the notation of the expected future reward in (6), the cumulative regret (8) quantifies the expected performance deviation between the MPC-based RL algorithm using episodically updated model parameters $\theta_e$ and the optimal MPC-based policy with access to the true parameters $\theta$ of the underlying MDP (1).

2 Scalable model-based RL: The Bayesian MPC algorithm

Following the concept introduced in [23], we propose to combine model-based RL using posterior sampling as introduced in [29] and investigated by [30, 31, 24] with a cautious model predictive control policy parametrization as described in Section 1.2 to obtain a new class of model-based RL policies with safety guarantees, called Bayesian MPC. At the beginning of each learning episode $e$ we sample transition and reward parameters $\theta_e$ according to their posterior distribution that results from the prior distribution $Q_0$ together with observed data $\mathcal{D}_e$.

| Bayesian MPC algorithm |
|------------------------|
| **Data:** Parametric model $f, \ell$; Prior $Q_0$ |
| Initialize $D_0 = \emptyset$ |
| for episodes $e = 0, 1, \ldots, N - 1$ do |
| sample $\theta_e \sim Q_{\theta|\mathcal{D}_e}$ |
| for time steps $t = 0, 1, \ldots, T - 1$ do |
| apply $a(t) = \bar{a}^*(t, x(t); \theta_e)$ |
| measure objective and state |
| end |
| extend data set to obtain $\mathcal{D}_{e+1}$ |
| end |
The sampled parameters yield an MPC problem parametrization [5] that would correspond to an MDP with parameters $\theta_e$. However, since $\theta_e \neq \theta$, such an MPC policy might be inconsistent with the underlying system to be controlled, in particular if the posterior parameter variance is large. In this case, the sampled policy is likely to cause explorative closed-loop behavior, producing information-rich data. Compared to using, e.g., the current maximum a-posteriori estimate of the parameters as done in most learning-based MPC approaches, which we refer to as nominal posterior MPC [4], the algorithm therefore generates explorative behavior in case of large posterior parameter uncertainties. As soon as the task-relevant parameter distributions begin to cumulate around the corresponding true classes. Instead of the instant regret

\[ r\sum_{t=0}^{T-1} \mathbb{E}[r(t, s(t), a(t); \theta_e)] - r(t, s(t), a(t); \theta) \]  

and is therefore scalable to arbitrary dimensions.

Since the Bayesian MPC algorithm can conceptually be used to enhance any existing MPC application for a wide variety of MDPs, we briefly recap general sufficient conditions from [23] in this section to reformulate the regret in terms of the expected learning progress of the algorithm in Section 3.3. These results enable us together with cautious soft constraints from Section 1.2 to derive a bound on the expected number of unsafe learning episodes under application of the Bayesian MPC algorithm in Section 3.3.

We start by reviewing the main steps of model-based RL based on posterior sampling arguments as presented in [23] [24] to reformulate the regret in terms of the expected learning progress of the transition and reward function. By using a regularity assumption on the expected future reward under the sampled MPC controllers this then allows us to bound the cumulative regret using the so-called Eluder dimension, which expresses the learning complexity for different mean and reward function classes. Instead of the instant regret $\Delta_e$ in (9), which includes the unknown optimal future reward $V_{\theta_e,0}(s)$, we formulate the regret in terms of the sampled MPC controller applied to the corresponding sampled system, for which it is optimal, i.e.

\[ \mathbb{E}_{\theta, \theta_e, s, D_e} \left[ \Delta_e \right] = \mathbb{E}_{\theta, s, D_e} \left[ \mathbb{E}_{\theta_e} \left[ V_{\theta_e,0}^\theta(s) - V_{\theta_e,0}(s) \right] \right], \]  

where $V_{\theta_e,0}(s)$ is known based on the sample $\theta_e$ and $V_{\theta_e,0}(s)$ can be observed. Using posterior sampling arguments we can verify that $\mathbb{E}_{\theta, \theta_e, s, D_e} \left[ \Delta_e - \Delta_e \right] = 0 \Rightarrow \mathbb{E}_{\theta, \theta_e, s, D_e} \left[ \Delta_e \right] = \mathbb{E}_{\theta, \theta_e, s, D_e} \left[ \Delta_e \right].$

The reformulated regret allows another reformulation based on the Bellman operator as originally proposed by [31] for discrete states and actions and sketched in [24] [23] for the continuous case to end up with a regret bound of the form

\[ \mathbb{E}[\Delta_e] \leq \mathbb{E} \sum_{t=0}^{T-1} \mathbb{E}_{e(t)} \left[ V_{\theta_e,t+1}(f(s(t), a(t); \theta_e)) + \epsilon_F(t) \right] - V_{\theta_e,t+1}(f(s(t), a(t); \theta) + \epsilon_F(t))] \]  

\[ + \mathbb{E} \sum_{t=0}^{T-1} |r(t, s(t), a(t); \theta_e) - r(t, s(t), a(t); \theta)|, \]  

where $\epsilon_F(t)$ denotes the noise in the estimated future reward. This bound is tight as it matches the regret bound obtained in [31] for discrete states and actions and sketched in [24] for the continuous case. The bound can be interpreted as an upper bound on the expected number of unsafe learning episodes under application of the Bayesian MPC algorithm in Section 3.3.
with expectation over \(\theta, \theta_e, s, D_e\). The second term in \(10\) can be bounded via the conditional posterior through \(\mathbb{E}_{\theta, s, D_e}[\sum_{t=0}^{T-1} r(t, s(t), a(t); \theta_e) - r(t, s(t), a(t); \theta)] \mid \theta, s, D_e\]. The first term, however, requires a regularity assumption that quantifies how errors in the expected one-step-ahead prediction \(f(s(t), a(t); \theta_e) - f(s(t), a(t); \theta)\) cause deviations w.r.t. the one-step-ahead expected future reward \(V_{\theta_e, t+1}(\cdot)\) as follows.

**Assumption 1.** For all \(\theta_e \in \mathbb{R}^{n_e}\) and \(s^+, \hat{s}^+ \in \mathbb{S}\) there exists a constant \(L_V > 0\) such that

\[
E_{\epsilon_F(t)} \left[ |V_{\theta_e, t+1}(s^+ + \epsilon_F(t)) - V_{\theta_e, t+1}(\hat{s}^+ + \epsilon_F(t))| \right] \leq L_V \|s^+ - \hat{s}^+\|_2. \quad (11)
\]

Note that if the expected future reward could vary arbitrarily, even for very similar states \(s^+\) and \(\hat{s}^+\), it might be impossible to provide any kind of regret bound. While previous literature only required this assumption, we will theoretically investigate its justification in the case of MPC-based policies in Section 3.1 and Section 3.2.

The relationship between the regret and the mean deviation between the true and sampled reward and transition function as described previously allows us to derive a Bayesian regret bound using statistical measures. The first bound relates to the complexity of the respective mean function also known as the Kolmogorov dimension \(\dim_K\), see also \([30]\). In an online learning setup it is additionally necessary to quantify the difficulty of extracting information and accurate predictions based on observed data, which is measured in terms of the Eluder dimension \(\dim_E\) \([50]\). These measures further require boundness of the mean reward and transition function as follows.

**Assumption 2.** There exist constants \(c_R\) and \(c_F\) such that for all admissible \(s \in \mathbb{R}^n, a \in \mathbb{R}^m, \theta \in \mathbb{R}^{n_e}\), and \(t = 0, 1, .., T - 1\) it holds \(|f(t, s, a; \theta_R)| \leq c_R\), and \(||f(s, a; \theta_F)|| \leq c_F\).

Following \([24, 23]\), we can combine these measures to obtain the following regret bound for the Bayesian MPC algorithm as an immediate consequence from Theorem 1 in \([24]\) with \(\mathcal{O}\) neglecting terms that are logarithmic in \(N\).

**Theorem 3.1.** If Assumptions 1 and 2 hold then it follows that

\[
\text{CR}(N) \leq \mathcal{O} \left( \sigma_R \sqrt{\dim_K(\ell)} \dim_E(\ell)TN + L_V \sigma_F \sqrt{\dim_K(f)} \dim_E(f)TN \right). \quad (12)
\]

Specific bounds for different parametric function classes can be found, e.g., in \([30, 24]\).

### 3.1 Regularity of the value function for large-scale linear transitions and concave rewards

Regularity of the future reward as required by Assumption 1 is a central ingredient for the performance analysis and essentially determines the shape of the regret bound in Theorem 3.1. While explicit bounds on the Kolmogorov- and Eluder dimensions are available for relevant parametric function classes \([30, 24]\), we provide a bound on \(L_V\) that holds under application of the Bayesian MPC algorithm. In this section we begin by focusing on the control relevant case of linear time-invariant transitions \(2\) of the form

\[
s(t + 1) = A(\theta_F)s(t) + B(\theta_F)a(t) + \epsilon_F(t) \quad (13)
\]

and reward models \(6\) that are either affine or quadratic and concave in the states and actions for each time step \(t = 0, 1, .., T - 1\). Furthermore, we restrict our attention to state and action spaces that are polytopic of the form \(\mathbb{S} := \{s \in \mathbb{R}^n \mid A_s s \leq b_s\}\) and \(\mathbb{A} := \{a \in \mathbb{R}^m \mid A_a a \leq b_a\}\). Based on these assumptions we establish Lipschitz continuity of the optimizer of the MPC Problem \((5)\). Combining Lipschitz continuity of \(\tilde{a}_{\ell}^g\) with Lipschitz continuity of the mean transition and reward model allows us to establish Assumption 1.

**Theorem 3.2.** Consider MPC problem \((5)\). If the mean transition \((5c)\) is linear, the state \((5d)\) and action \((5e)\) constraints are polytopic, and the mean reward function \((5a)\) is linear or quadratic and strictly concave for all time steps, then under application of the Bayesian MPC algorithm it follows that Assumption 1 holds.

The proof together with a detailed construction of \(L_V\) according to Assumption 1 can be found in Appendix B.1. Combining this result with Corollary 3.1 and the specific bounds on the Eluder- and Kolmogorov dimensions from \([30, 24]\) provides the following performance bound without explicit need of Assumption 1.
Corollary 3.3. Under the same assumptions of Theorem 3.2, the cumulative Bayesian regret of the Bayesian MPC algorithm is bounded by

$$CR(N) \leq \tilde{O} \left( \sigma_{F} n_{\ell} \sqrt{2TN} + L_{V} \sigma_{F} n \sqrt{n(n+m)TN} \right)$$

with  \( n_{\ell} \) mean reward parameters and  \( L_{V} \) according to (B.3) in Appendix B.1.

Proof. The statement is an immediate consequence of Theorem 3.1, Theorem 3.2, and [24] Proposition 2.3. □

3.2 Extension of regularity towards nonlinear transitions and rewards

While the linear case as considered in the previous section covers a large portion of control applications, the increasing availability and performance of nonlinear MPC solvers motivates the extension to nonlinear reward and transition models. We therefore extend the analysis from Section 3.1 to the more general case of transition and reward functions  \( f \) and  \( \ell \) that are nonlinear and non-convex as well as more general state and action spaces of the form  \( S := \{ s \in \mathbb{R}^{n} | g_{s}(s) \leq 1 \} \) and  \( A := \{ a \in \mathbb{R}^{m} | g_{a}(a) \leq 1 \} \).

To this end, we use a similar line of reasoning as in the proof of Theorem 3.2 to provide sufficient conditions on the resulting nonlinear MPC problem (5) that ensure Assumption 1. In particular, we utilize results from [35] to analyze local continuity properties of KKT-based solutions of the MPC problem (5) as follows.

Theorem 3.4. Let Assumption 2 hold and consider the MPC problem (5) with  \( f, \ell, I, g_{s}, \) and  \( g_{a} \) continuously differentiable and Lipschitz continuous. If the linear independence (LI) and the strong second-order sufficient condition (SSOSC) according to (33) 3(a) and 3(d) hold for all admissible  \( s \in \mathbb{R}^{n}, \theta \in \mathbb{R}^{n_{\theta}}, \) and  \( t = 0, 1, ..., T - 1 \), then it follows that Assumption 1 holds.

The main steps of the proof can be found in Appendix B.2. The linear independence (LI) condition refers to the linear independence of the gradients of the active constraints at an optimum of (5) with respect to the decision variables and ensure necessity of the corresponding KKT conditions. In addition, the strong second-order sufficient condition (SSOSC) guarantees sufficiency of the KKT conditions and uniqueness of local solutions through a local positive definiteness condition of the Hessian of the Lagrangian w.r.t. the decision variables, also depending on the active constraints at the optimum. Consequently, if these conditions do not hold, situations where small deviations of  \( s \) cause a ‘jumping behavior’ between different local optimal solutions of the MPC problem (5) can occur and potentially lead to a non-Lipschitz continuous future reward function.

While the imposed assumptions are difficult to verify, note that the LI condition is a common requirement for nonlinear solvers and that the SSOSC is the weakest condition to ensure existence and local uniqueness of local solutions of the MPC problem (5) for small perturbations of the initial condition  \( s \), see [36]. Importantly, note that, e.g., a normally distributed  \( \epsilon_{F} \) helps to smooth the future regret through the expectation operator in (11) and Assumption 1 may still be satisfied.

3.3 Bounding the expected number unsafe learning episodes

While a tightened MPC formulation using the true parameters  \( \theta \) typically provides state constraint satisfaction in expectation for many practical applications, the parameter samples  \( \theta_{e} \) during application of the Bayesian MPC algorithm can vary significantly during initial learning episodes. It can therefore happen that the constraints are violated, even in expectation. In such learning episodes, however, the amount of constraint violation can partially be observed through the regret due to the exact penalty in the future reward function (6). As a consequence, if the MPC using the true system parameters  \( \hat{\theta} \) provides satisfaction of the tightened constraints in expectation, i.e.  \( s(t) \in \mathbb{S}_{d}(0) \), we can use regularity of the expected future reward to show that a converging parameter estimate yields a converging future reward and therefore converging constraint satisfaction. In other words, since the stage cost function is bounded and the constraints are tightened, a sufficiently large soft constraint penalty ensures observability and a bound on state constraint violations, which can be formalized as follows.

We first derive an upper bound on the instant regret  \( \Delta_{e} \) as defined in (5) implying constraint satisfaction in Appendix B.5. By combining this intermediate result with the regret bound from
Theorem 3.1 we bound the cumulative expected number of unsafe learning episodes in Theorem 3.5. To streamline notation we denote the state, action, and slack variable sequence in the expected future reward (6) for a given initial state $s \sim S^D$ as $s_D^\delta(j)$, $a_D^\delta(j)$, and $\rho_D^\delta(j)$ for $j = 0, \ldots, T - 1$ in the following.

**Theorem 3.5.** Let the conditions of Theorem 3.1 hold and consider a weighting factor in the exact penalty term $I(\cdot)$ in (5a) that satisfies $\min_i(c_{1,i} \geq \frac{2T c_6 + c_7}{\delta})$ for some $c_\delta > 0$. If $\mathbb{E}_{E,s}[\rho_D^\delta(j)] = 0$, then the total number of $N_{\text{unsafe}}$ episodes, for which there exists a $j \in \mathbb{N}$, $0 \leq j \leq T - 1$ such that $\mathbb{E}_{E,s}[s_D^\delta(j) \notin \mathcal{S}]$ is bounded in terms of the cumulative regret by $N_{\text{unsafe}} \leq \lceil CR(N)c_\delta^{-1} \rceil$.

A sublinear cumulative regret bound therefore ensures a decreasing ratio between the number of episodes with constraint violation and the total number of learning episodes, which vanishes at the rate of $c(1/N)$ for $N \to \infty$ and some positive constant $c$. This can be seen as a first step towards combined finite time safety and performance guarantees in case of model-based RL in continuous state and action spaces. Note that the upper bound $c_\delta$ and consequently also the lower bound on the exact penalty scaling could potentially be improved in the corresponding proof (Appendix B.3), e.g. by exploiting the concrete structure of $I$ including quadratic terms.

### 4 Numerical examples

We first consider the task of efficiently controlling a large-scale network with 100 cooling units, e.g. a server farm or production machines in manufacturing plants, that are arranged in a grid structure and have a strong thermal coupling with respect to locally neighboring units. The system dynamics and reward satisfy the assumptions of Section 3.1, see Appendix A.1 for further details. The goal is to find an energy efficient control policy that obeys maximal allowable temperatures despite parametric uncertainties in the system. In Figure 2 (Left), we compare the proposed Bayesian MPC algorithm against commonly used nominal posterior MPC, i.e. selecting $\theta_c := \mathbb{E}[\theta|D_c]$ [4, Section 3], using 100 different system realizations. While both algorithms show reasonable learning performance and provide constraint satisfaction at all times, Bayesian MPC is able to significantly reduce the cumulative regret by almost 50% compared to nominal posterior MPC.

In addition, we consider a generic drone search task falling into the problem class of Section 3.2. The goal is to collect information about an a-priori unknown position of interest using a quadrotor drone. While the prior of the 10-dimensional drone dynamics are selected according to [37], we additionally simulate strong winds in different altitudes, which adds strong nonlinear effects to the dynamics. Once the target position is reached, the drone collects information before it returns to the base station for analysis and recharge. The overall goal therefore is to learn the drone dynamics, winds in different altitudes and the most informative search position. The safety-critical constraints are a maximum range of the drone together with a minimum altitude that need to be satisfied under physical actuator limitations, see Appendix A.2 for further details. By comparing nominal posterior MPC against Bayesian MPC over 100 different experiments in terms of expected constraint satisfaction, we notice from Figure 2 (Middle) that Bayesian MPC causes explorative behavior during initial episodes, which yields higher constraint violations compared to nominal posterior MPC. However, this behavior enables safety of future episodes and bounded cumulative regret Figure 2 (Right) compared to posterior nominal MPC, which has unbounded cumulative regret.

Figure 2: Simulation results of numerical examples for 100 different experiments. Thin lines depict experiment samples and thick lines show the corresponding mean. **Left:** Cumulative regret of the large-scale thermal application detailed in Section A.1. **Middle:** Maximum value of $\rho(j)$ as defined in (6) over one episode. **Right:** Cumulative regret of exploration task as described in Section A.2.
**Broader Impact**

The proposed RL algorithm is tailored to solve modern control engineering problems, i.e. problems where high-performance control under system constraints is essential. Nowadays, such applications are typically driven by model predictive control (MPC) techniques. Since the proposed RL algorithm provides an automated way of improving MPC controllers, any existing MPC application can potentially be enhanced through the presented method addressing a key challenge and development cost factor in industry. Prominent example systems and can be found in aerospace [38], automotive [39], or process manufacturing [40], where control methods help to reduce energy consumption by optimizing and coordinating processes. As for most control techniques, which act on a low-level planning instance, ethical and societal aspects mainly depend on the specific application in which they are used.

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Figure 3: Cooling network structure: Each unit \( i \) has a measured temperature state \( s_i \) and cooling action \( a_i \) and is affected by the temperature states of the top/down/left/right neighboring units arranged in a grid.

### Appendix A  Additional information: Numerical examples

All examples are implemented using the Casadi framework \[41\] together with the IPOPT solver \[42\].

#### A.1 Large scale thermal application

We consider the task of efficiently controlling a large-scale network of 100 cooling units, e.g. a server farm or production machines in manufacturing plants, that are arranged in a grid structure and have strong thermal couplings with respect to locally neighboring units, see Figure 3.

Actions \( a(t) \in A \subset \mathbb{R}^{100} \) describe the applied cooling power to each cooling, which are subject to physical limitations \( A := \{a \in \mathbb{R}^{100} | a \leq \bar{a} \} \). The system state is defined by the temperatures \( s(t) \in \mathbb{R}^{100} \) of each unit that needs to be below a given threshold \( s(t) \in S := \{s \in \mathbb{R}^{100} | s \leq 100 \} \) for all times. The thermodynamics of the plant are given as an MDP with linear mean dynamics such that each unit \( i \) has unknown dynamics of the form

\[
    s_i(t+1) = A_{ii}s_i(t) + B_ia_i(t) + \sum_{j \in N_i} A_{ij}s_j(t) + C_i + \epsilon_F(t)
\]

with neighboring units indexed by \( j \), known Gaussian parameter prior distribution \( (A,B,C) \sim Q_{\theta_F} \) and Gaussian process noise \( \epsilon_F(t) \). Note that the resulting overall dynamics can be stated in the form \( (13) \) by extending the state space.

The goal is to minimize the overall expected energy consumption \( \sum_i L_ia_i(t) + \epsilon_R \) by considering thermal couplings while keeping the temperature of each cooling unit below a specified maximum temperature, starting form a temperature level below 100 degrees. The energy efficiency of each unit is described through parameters \( L \in \mathbb{R}^{100} \) that are sampled from a known Gaussian prior distribution \( l \sim Q_{\theta_R} \) plus additive Gaussian measurement noise \( \epsilon_R \). The overall plant consists partly of new cooling units with known efficiency and older cooling units with uncertain efficiency factors that are worse in expectation. Due to these different efficiency levels that are provided through the prior distribution, explorative behavior can be beneficial to exploit more efficient units. The exact numerical values and prior parametrisation can be found in the function `server_experiment.m` in the provided source code for the example.

#### A.2 Drone search application

We consider the task depicted in Figure 4, with a quadrotor system as described in \[37\] with a 10 dimensional state space. The dynamics are of the form

\[
    s(t+1) = As(t) + Ba(t) + C\Phi(s(t)) + \epsilon_F,
\]

where \( A \) and \( B \) matrices describe the unknown dynamics around the hovering state and \( C\Phi(s(t)) \) models strong winds in different altitudes as \( \Phi(s(t)) = [k(s_3(t), W_1), k(s_3(t), W_2), k(s_3(t), W_3)]^T \) using radial basis functions \( k(.,.) \) \[43\] with given hyper-parameters \( W_i \) and unknown parameters \( C \). The system has
Figure 4: Illustration of the drone search application (Section A.2). **Left:** Conceptual drawing of the task including wind in different altitudes, the maximum range of the drone, and the unknown optimal position for surveillance. **Right:** Sample drone trajectory during one episode. Starting from the base station in the middle, the drone quickly approaches the position of maximum information gain.

a three dimensional action space that allows to control the desired pitch and roll as well as vertical acceleration of the drone.

The system constraints are given by physical action constraints and a box constraint on the position states, which describes the minimal altitude and maximum range of the drone. Furthermore, the use of a linear model is only valid around the hovering state, yielding additional absolute pitch and roll constraints of $30 \, \text{deg}$ to the system.

The reward signal corresponds to the information gained at the final position at the end of an episode and is modeled as the sum of equally spread radial-basis-functions at positions $p_i \in \mathbb{R}^3, i = 1, \ldots, 9$, i.e. $r(T-1, s(T-1), a(T-1)) = \epsilon_R + \sum_{i=1}^9 \theta_R, k(s_{1-3}(T-1), p_i)$ see Figure 4 (Right) for an example illustration. The unknown system dynamics parameters $\theta_F := C$, dynamics process noise $\epsilon_F$, reward parameters $\theta_R := \{p_i\}$, and reward noise $\epsilon_R$ are normally distributed. The exact numerical values can be found in the function `quadrotor_example.m` in the provided source code for the example.

**Appendix B   Proofs**

**B.1 Proof of Theorem 3.2**

It is sufficient to show global Lipschitz continuity of $V_{\theta_e,t}^{s+}(s)$ since existence of an $L_{V_{t}} > 0$ such that

$$
\mathbb{E}_{\epsilon_F} \left[ V_{\theta_e,t+1}^{s+} + \epsilon_F \right] - \mathbb{E}_{\epsilon_F} \left[ V_{\theta_e,t}^{s+} + \epsilon_F \right] \leq L_{V_{t}} \left| s + \epsilon_F - s + \epsilon_F \right|
$$

implies the desired result. Let $\ell(j, s, a) := \ell(j, s, a; \theta_e) - I(\rho(s))$ with $\rho(s) := \min_{\rho \geq 0 \text{ s.t. } s \in \mathbb{S}_a(\rho)}$ according to (9), which is Lipschitz continuous in $s$ and $a$ since $\mathbb{S}$ and $\mathbb{A}$ are polytopic. To streamline notation we denote the state and action sequence in the expected future reward (6) with $\tilde{\ell} = \theta_e$ as $s(j, s, E)$ and $a(j, s)$ for $j = 0, \ldots, T-1$ with $s(0, s, E) = s$ and $E := [\epsilon_R(t), \ldots, \epsilon_R(T-1), \epsilon_F(t), \ldots, \epsilon_F(T-2)]$ in the following. We have due to linearity of the expectation operator, Jensen’s
We show that
\[ |V_{\theta_e,t+1}^{\theta_e}(s^+) - V_{\theta_e,t}^{\theta_e}(s^+)| \]
\[ = E \left[ \sum_{j=t+1}^{T-1} \tilde{\ell}(j, s(j, s^+, E), a(j, s(j, s^+, E))) - \tilde{\ell}(j, s(j, \tilde{s}^+, E), a(j, s(j, \tilde{s}^+, E))) \right] \]
\[ \leq E \left[ \sum_{j=t+1}^{T-1} \tilde{\ell}(j, s(j, s^+, E), a(j, s(j, s^+, E))) - \tilde{\ell}(j, s(j, \tilde{s}^+, E), a(j, s(j, \tilde{s}^+, E))) \right] \]
\[ \leq E \left[ \sum_{j=t+1}^{T-1} L_{\ell,j} \left[ (s(j, s^+, E)^\top - s(j, \tilde{s}^+, E)^\top, a(j, s(j, s^+, E))^\top - a(j, s(j, \tilde{s}^+, E))^\top) \right]^\top \right] \]

It therefore remains to show that \( s \) and \( a \) are Lipschitz continuous in their second argument. Since \( a(j, s) \) is the first element of the optimal action sequence according to (5) and (5) is guaranteed to be feasible due to the soft-constraint reformulation, it follows from (44, Thm. 1.8) for affine \( \ell \) and from (44, Thm. 1.12) for strictly concave quadratic \( \ell \) for any \( j \) and \( s, \tilde{s} \in \mathbb{R}^n \) that there exists a \( K \in \mathbb{R}^+ \) such that
\[ ||a(j, s) - a(j, \tilde{s})|| \leq K ||s - \tilde{s}||. \] (B.1)

It remains to show that there exists an \( L_s(j) \) such that
\[ ||s(j, s^+, E) - s(j, \tilde{s}^+, E)|| \leq L_s(j)||s^+ - \tilde{s}^+|| \] (B.2)

We show that
\[ ||s(j, s^+, E) - s(j, \tilde{s}^+, E)|| \leq L_s(j)||s^+ - \tilde{s}^+|| \]
\[ \Rightarrow ||s(j + 1, s^+, E) - s(j + 1, \tilde{s}^+, E)|| \leq L_s(j + 1)||s^+ - \tilde{s}^+|| \]
with \( L_s(j) = L_s(j - 1)(||A|| + ||B||K) \) and \( L_s(0) = 1 \) by induction.

Induction start:
\[ j = 0 : \quad ||s(0, s^+, E) - s(0, \tilde{s}^+, E)|| = ||s^+ - \tilde{s}^+|| \leq L(0)||s^+ - \tilde{s}^+|| \quad \text{with} \quad L(0) = 1 \]
implying
\[ ||s(1, s^+, E) - s(1, \tilde{s}^+, E)|| = ||As^+ + Ba(0, s^+) + \epsilon_F(0) - A\tilde{s}^+ + Ba(0, \tilde{s}^+) - \epsilon_F(0)|| \]
\[ \leq ||A(s^+ - \tilde{s}^+) + B(a(0, s^+) - a(0, \tilde{s}^+))|| \]
\[ \leq (||A|| + ||B||K)1||s^+ - \tilde{s}^+||. \]

Induction step:
For any \( j > 0 \)
\[ ||s(j, s^+, E) - s(j, \tilde{s}^+, E)|| \leq L_s(j)||s^+ - \tilde{s}^+|| \]
we have
\[ ||s(j + 1, s^+, E) - s(j + 1, \tilde{s}^+, E)|| \]
\[ = ||As(j, s^+, E) + Ba(j, s(j, s^+, E)) + \epsilon_F(j) - As(j, \tilde{s}^+, E) - Ba(j, s(j, \tilde{s}^+, E)) - \epsilon_F(j)|| \]
\[ = ||A(s(j, s^+, E) - s(j, \tilde{s}^+, E)) + B(a(j, s(j, s^+, E)) - a(j, s(j, \tilde{s}^+, E)))|| \]
\[ \leq (||A|| + ||B||K)||s(j, s^+, E) - s(j, \tilde{s}^+, E)|| \]
\[ \leq (||A|| + ||B||K)L(j)||s^+ - \tilde{s}^+|| \quad \text{(induction step hypothesis)} \]
\[ \leq (||A|| + ||B||K)L(j)(1 + \tilde{K}) \]
Combining these results yields
\[ L_V = \sum_{j=t+1}^{T-1} L_{\ell,j}(L_s(j)(1 + \tilde{K})) < \infty. \] (B.3)
B.2 Proof outline of Theorem 3.4

For any initial state \( s_0 \) there exists a corresponding optimal solution \( \tilde{a}_{t}^* \) to (5) due to the soft-constraint formulation, Lipschitz continuity of the objective, Lipschitz continuity of the constraints in (5), and the compactness of the input constraints. Together with Theorem 3.7 it follows from the given assumptions that there exists a unique function \( y(t, s) \rightarrow \tilde{a}_{t}^* \) that is Lipschitz continuous w.r.t. all initial conditions \( s \in B(s_0, r) \) with \( B(s_0, r) := \{ s \in \mathbb{R}^n \mid ||s - s_0|| \leq r \} \) and \( s_0 \) fulfilling the KKT conditions corresponding to (5). Due to the SSOSC, the KKT conditions imply optimality of \( y(t, s) \) and we conclude existence of a local Lipschitz constant \( L_a(t, s_0) > 0 \) such that for all \( s, \tilde{s} \in B(r, s_0) \) it holds \( ||a(t, s) - a(t, \tilde{s})|| = ||y(\tilde{s}) - y(t)|| \leq L_a(t, s_0)||s - \tilde{s}|| \). Since the action space is compact it also follows boundedness of

\[
L_a = \max_{s, \tilde{s}, ||s - \tilde{s}|| > r} \frac{||a(t, s) - a(t, \tilde{s})||}{||s - \tilde{s}||}, \tag{B.4}
\]

allowing us to select \( \bar{K} := \max\{L_a, L_a(t, s_0)\} \). From here we can proceed analogously to the proof of Lemma 3.2 using \( ||s(j + 1, s^+, E) - s(j + 1, \tilde{s}^+, E)|| \leq (L_{fa} + L_{fs} \bar{K})||s(j, s^+, E) - s(j, \tilde{s}^+, E)|| \) with \( L_{fa} \) and \( L_{fs} \) being the Lipschitz constants of \( f \) w.r.t. the state \( s \) and action \( a \).

B.3 Bounding the expected number of unsafe learning episodes

**Lemma B.1.** Let Assumption 2 hold. Consider the expected future reward in (6) for a constraint tightening \( \delta > 0 \) and \( s \in \mathbb{S}_\delta(0) \). If \( \mathbb{E}_E \left[ \rho_0^0(j) \right] = 0 \) for all \( j = 0, \ldots, T - 1 \) and the weighting factor of the exact penalty term \( I(\cdot) \) in (5a) satisfies \( \min_{i, (c_1, i)} \geq \frac{2T_{c_0} + \bar{c}}{\bar{c}} \) for some \( c_0 > 0 \) then it holds

\[
|V^0_{\theta, 0}(s) - V^0_{\bar{\theta}, 0}(s)| < c_3 \Rightarrow \mathbb{E}_E \left[ s^0_\theta(\tilde{j}) \in \mathbb{S} \right] \text{ for all } j = 0, 1, \ldots, T - 1. \tag{B.5}
\]

**Proof.** For a proof by contradiction, consider the case \( |V^0_{\theta, 0}(s) - V^0_{\bar{\theta}, 0}(s)| < c_3 \) and \( \mathbb{E}_E [s^0_\theta(\tilde{j}) \notin \mathbb{S}] \) for some \( 0 \leq \tilde{j} \leq T - 1 \). It holds \( \max_{i} \mathbb{E}_E [\rho_i^0(\tilde{j})] > \delta \) and \( \mathbb{E}_E [I(\rho_0^0(\tilde{j}))] \geq \min_{i, (c_1, i)} \max_{i} \mathbb{E}_E [\rho_i^0(\tilde{j})] \)

Next, we derive a lower bound on the absolute expected reward difference

\[
|V^0_{\theta, 0}(s) - V^0_{\bar{\theta}, 0}(s)| = \mathbb{E}_E \left[ \sum_{j=0}^{T-1} \ell^0_\theta(j) - \ell^0_{\bar{\theta}}(j) + I(\rho^0_\theta(j)) \right]
\]

with \( \ell^0_\theta(j) := (j, s^0_\theta(j), \rho^0_\theta(j); \theta_R) \) to show the contradiction. We distinguish two cases:

Case \( \mathbb{E}_E [\ell^0_\theta(j) - \ell^0_\theta(\tilde{j})] \geq 0 \) for all \( j \): It follows directly that \( |V^0_{\theta, 0}(s) - V^0_{\bar{\theta}, 0}(s)| \geq \min_{i, (c_1, i)} \max_{i} \mathbb{E}_E [\rho_i^0(\tilde{j})] \geq c_3 \).

Case \( \mathbb{E}_E [\ell^0_\theta(j) - \ell^0_\theta(\tilde{j})] < 0 \) with \( j \in J \) for some index set \( J \subseteq \{0, \ldots, T - 1\} \):

\[
|V^0_{\theta, 0}(s) - V^0_{\bar{\theta}, 0}(s)| \geq \mathbb{E}_E \left[ \ell^0_\theta(\tilde{j}) - \ell^0_\theta(\tilde{j}) + \sum_{i \in \{0, \ldots, T - 1\} \setminus \{\tilde{j}\}} \ell^0_\theta(j) - \ell^0_{\bar{\theta}}(j) + I(\rho^0_\theta(j)) \right] \]

\[
\geq \mathbb{E}_E \left[ (T - 1)2c_R + c_3 + \sum_{j \in \{0, \ldots, T - 1\} \setminus \{\tilde{j}\}} \ell^0_{\theta}(j) - \ell^0_{\bar{\theta}}(j) + I(\rho^0_{\theta}(j)) \right] \]

\[
\geq \mathbb{E}_E \left[ (T - 1)2c_R + c_3 + \sum_{j \in \mathbb{S}(\tilde{j})} \ell^0_{\theta}(j) - \ell^0_{\bar{\theta}}(j) + I(\rho^0_{\theta}(j)) + \sum_{j \notin \mathbb{S}(\tilde{j})} \ell^0_{\theta}(j) - \ell^0_{\bar{\theta}}(j) + I(\rho^0_{\theta}(j)) \right] \]
where we use linearity of the expectation operator, \( \min_i (c_{1,i}) \geq 2Tc_R + c_\delta \), and the fact that \( \ell_\theta^j - \ell_\theta^{\tilde{\theta}}(j) > -2c_R \). Similarly, \( \sum_{j \in S \setminus \{\bar{j}\}} (\ell_\theta^j - \ell_\theta^{\tilde{\theta}}(j)) \geq -|J|c_R \geq -(T - 1)2c_R \) and \( I(\rho_\theta^j(j)) \geq 0 \) yielding

\[
|V_{\theta,0}^{\tilde{\theta}}(s) - V_{\tilde{\theta},0}^{\tilde{\theta}}(s)| \geq c_\delta + \sum_{j \notin S \setminus \{\bar{j}\}} (\ell_\theta^j - \ell_\theta^{\tilde{\theta}}(j) + I(\rho_\theta^j(j))) \geq c_\delta.
\]

The lower bound implies

\[
c_\delta > |V_{\theta,0}^{\tilde{\theta}}(s) - V_{\tilde{\theta},0}^{\tilde{\theta}}(s)| \geq c_\delta
\]

yielding the contradiction.

B.4 Proof of Theorem 3.5

Let \( \mathcal{N}_{\text{unsafe}} \) be the set of episode indices such that \( E_\theta \left[ |V_{\theta,0}^{\tilde{\theta}}(s) - V_{\tilde{\theta},0}(s)| \right] > c_\delta \) for \( e \in \mathcal{N}_{\text{unsafe}} \), i.e., potentially unsafe episodes for which there exists some \( j \) s.t. \( E_{\theta,0} \left[ \rho_\theta^j(j) \notin S \right] \) out of a total of \( N \) episodes. By summing up potentially unsafe episodes \( e \in \mathcal{N}_{\text{unsafe}} \) we get

\[
\sum_{e \in \mathcal{N}_{\text{unsafe}}} E_\theta \left[ |V_{\theta,0}^{\tilde{\theta}}(s) - V_{\tilde{\theta},0}(s)| \right] \geq N_{\text{unsafe}}c_\delta \tag{B.6}
\]

by Lemma B.1 with \( N_{\text{unsafe}} \) such that \( |\mathcal{N}_{\text{unsafe}}| = N_{\text{unsafe}} \). By definition, the cumulative regret provides a bound for the sum in (B.6), and we therefore end up with \( CR(N) \geq N_{\text{unsafe}}c_\delta \), which proves the desired statement. \( \square \)