Two Injective Proofs of a Conjecture of Simion

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Abstract

Simion [9] conjectured the unimodality of a sequence counting lattice paths in a grid with a Ferrers diagram removed from the northwest corner. Recently, Hildebrand [5] and then Wang [11] proved the stronger result that this sequence is actually log concave. Both proofs were mainly algebraic in nature. We give two combinatorial proofs of this theorem.

1 Introduction

In this note we present two injective proofs of a strengthening of a conjecture of Simion [9]. To describe the result, let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be the Ferrers diagram of a partition viewed as a set of squares in English notation. (See any of the texts [1, 8, 10] for definitions of terms that we do not define here.) The shape \( \lambda \) will be fixed for the rest of this paper.

Consider a grid with the vertices labeled \((i, j)\) for \(i, j \geq 0\) as in Figure 1. Place \( \lambda \) in the northwest corner of this array so that its squares coincide with those of the grid.

A northeastern lattice path is a lattice path on the grid in which each step goes one unit to the north or one unit to the east. Let \( N(m, n) \) be the number of northeastern lattice paths from \((m, 0)\) to \((0, n)\),
that do not go inside $\lambda$ (although they may touch its boundary), and let $N(m, n)$ be the set of such paths. In particular, $N(m, n) = 0$ if either the starting or ending point is inside $\lambda$.

Simion [9] conjectured that for all $m, n \geq 0$ the sequence

$$N(0, m + n), N(1, m + n - 1), \ldots, N(m + n, 0)$$

is unimodal. Lattice path techniques for proving unimodality were investigated by Sagan [7], but the conjecture remained open at that point. Recently, Hildebrand [5] proved the stronger result that this sequence is actually log concave by mostly algebraic means. Shortly thereafter, Wang [11] simplified Hildebrand’s proof using results about Polya frequency sequences. In the present work, we will give two injective proofs of the strong version of Simion’s conjecture. The one in Sections 3 and 4 employs ideas from Hildebrand’s proof while the one in Section 5 is more direct. Our injections come from a method of Lindström [6], later popularized by Gessel and Viennot [3, 4], that can be used to prove total positivity results for matrices. For an exposition, see Sagan’s book [8, pp. 158–163]. Bóna [2] has used related ideas to prove the log concavity of a sequence counting $t$-stack sortable permutations.

We end this section by reiterating the statement of the main theorem for easy reference. Notice that when $\lambda = \emptyset$ it specializes to the well-known result that the rows of Pascals’s triangle are log concave.

**Theorem 1 (The Strong Simion Conjecture)** Let $\lambda$ be the Ferrers diagram of a partition and let $N(m, n)$ be the number of northeastern lattice paths in the grid from $(m, 0)$ to $(0, n)$ which do not intersect the interior of $\lambda$. Then for all $m, n \geq 0$ the sequence

$$N(0, m + n), N(1, m + n - 1), \ldots, N(m + n, 0)$$

is log concave.
2 A decomposition of the problem

This preliminary part of the first proof is from [5]. We include it so that our exposition will be self contained. We need to prove that for all \(m, n > 0\) we have

\[ N(m - 1, n + 1)N(m + 1, n - 1) \leq N(m, n)^2. \]

To prove this, it suffices to show that

\[ N(m - 1, n + 1)N(m + 1, n) \leq N(m, n)N(m + 1, n + 1), \]

because then, by symmetry, we also have

\[ N(m + 1, n - 1)N(m, n + 1) \leq N(m, n)N(m + 1, n). \]

Now multiplying the last two equations together and simplifying gives the first.

The second inequality can be proved by demonstrating another pair of equations, namely

\[ N(m, n + 1)N(m + 1, n) \leq N(m, n)N(m + 1, n + 1), \tag{1} \]

and

\[ N(m - 1, n + 1)N(m + 1, n + 1) \leq N(m, n + 1)^2. \tag{2} \]

Multiplying these two equations together and cancelling gives the desired result.

3 The proof of (I)

In this section we prove that (I) holds by constructing an injection

\[ \Psi : \mathcal{N}(m, n + 1) \times \mathcal{N}(m + 1, n) \to \mathcal{N}(m, n) \times \mathcal{N}(m + 1, n + 1). \]

Consider a path pair \((p, q) \in \mathcal{N}(m, n + 1) \times \mathcal{N}(m + 1, n)\). Then \(p\) and \(q\) must intersect. Let \(C\) be their first (most southwestern) intersection point. Say that \(C\) splits \(p\) into parts \(p_1\) and \(p_2\), and splits \(q\) into parts \(q_1\) and \(q_2\). Then the concatenation of \(p_1\) and \(q_2\) is a path in \(\mathcal{N}(m, n)\), and the concatenation of \(q_1\) and \(p_2\) is a path in \(\mathcal{N}(m + 1, n + 1)\). So define \(\Psi(p, q) = (p_1q_2, q_1p_2) = (p', q')\). It is easy to see that the image of \(\Psi\) is exactly all \((p', q') \in \mathcal{N}(m, n) \times \mathcal{N}(m + 1, n + 1)\) such that \(p'\) and \(q'\) intersect. It is also simple to verify that if \(\Psi(p, q) = (p', q')\), then applying the same algorithm to \((p', q')\) recovers \((p, q)\). So \(\Psi\) is injective. See Figure 2 for an example.
4 The proof of (2)

In this section we construct an injection

$$\Phi : \mathcal{N}(m-1, n) \times \mathcal{N}(m+1, n) \to \mathcal{N}(m, n)^2,$$

thus proving (2).

Let $\mathbf{p}, \mathbf{q} \in \mathcal{N}(m-1, n) \times \mathcal{N}(m+1, n)$. If $P = (i, j)$ and $Q = (i, k)$ are vertices of $\mathbf{p}$ and $\mathbf{q}$, respectively, with the same first coordinate, define the **vertical distance from $\mathbf{p}$ to $\mathbf{q}$ at $P$ and $Q$** to be $k - j$. The vertical distance from $\mathbf{p}$ to $\mathbf{q}$ starts at 2 for their initial vertices and ends at 0 for their final ones. Since vertical distance can change by at most one with a step of a path, there must be some vertical distance equal to 1. Let $P$ and $Q$ be the first (most southwest) pair of points with vertical distance one.

It follows from our choice of vertices that $\mathbf{p}$ must enter $P$ with an east step and $\mathbf{q}$ must enter $Q$ with a north step. Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the portions of $\mathbf{p}$ before and after $P$, respectively, and similarly for $\mathbf{q}$.

Now let

$$\Phi(\mathbf{p}, \mathbf{q}) = (\mathbf{p}'_1 \mathbf{q}_2, \mathbf{q}'_1 \mathbf{p}_2)$$

where $\mathbf{p}'_1$ is $\mathbf{p}_1$ moved south one unit and $\mathbf{q}'_1$ is $\mathbf{q}_1$ moved north one unit. Since $P$ and $Q$ are the first pair of points at vertical distance one, $\mathbf{q}'_1$ will not intersect $\lambda$ and the concatenations are valid paths in $\mathcal{N}(m, n)$. In fact, the image of $\Phi$ is exactly all path pairs $(\mathbf{p}', \mathbf{q}') \in \mathcal{N}(m, n)^2$ that have a pair of points at vertical distance -1. Applying the same procedure to the first such pair inverts the map and so $\Phi$ is injective. See Figure 3 for an example.

This completes the first proof of Theorem 1. ◊
The reader may wonder if we can do away with splitting our problem into two parts, that is, equations (1) and (2). The answer is yes, and the necessary injection is just a modification $\Phi$ of the map $\Phi$. This will give us a second, completely combinatorial, proof of our main theorem.

Take a path pair $(p, q) \in \mathcal{N}(m-1, n+1) \times \mathcal{N}(m+1, n-1)$. Notice that $p$ and $q$ must intersect. So before the first intersection there must be a first pair of points $P, Q$ (on $p, q$ respectively) at vertical distance 1. Similarly, after the last intersection there must be a last pair of points $\overline{P}, \overline{Q}$ at horizontal distance 1, where horizontal distance is defined analogously. Let $P$ and $\overline{P}$ divide $p$ into subpaths $p_1, p_2, p_3$ and use the same notation for $q$. Then define

$$\overline{\Phi}(p, q) = (p'_1 q_2, p''_3, q'_1 p_2 q''_3)$$

where $p'_1$ is $p_1$ moved south one unit, $p''_3$ is $p_3$ moved west one unit, and $q'_1, q''_3$ are defined in the analogous way but moving in the opposite directions. It is a simple job to verify that $\overline{\Phi}$ is well-defined and injective just as we did with $\Phi$.

This completes the second proof of Theorem 1.

We have two final remarks. First of all, it is clear from the geometry of the situation that if $\lambda$ is self-conjugate then the sequence in Theorem 1 is also symmetric, but this does not hold in general. One might also wonder if this sequence has the stronger property that the associated polynomial generating function has only real zeros. This is not always true as can be seen by taking $\lambda = (1)$ and $m + n = 4$. In this case the associated polynomial is $x(3x^2 + 5x + 3)$ which has two complex roots. It might be interesting to determine for which shapes the real zero property holds.

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