Effect of the intrinsic Josephson coupling on the pancake lattice in layered
superconductors.

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We study the pancake lattice system induced by application of transverse magnetic field to the
layered superconductor. A simple statistical field theory for the pancake lattice is derived. It
incorporates effects of the magnetic interaction between pancakes as well as those of the interlayer
Josephson coupling. The proposed description enables us to estimate the pancake crystal melting
temperature from above. Also, it allows us to compare directly the effects of the magnetic and
Josephson energies on the statistical properties of the pancake lattice. We demonstrate that even
for such an anisotropic material as BSCCO the pancake interaction induced by the Josephson
coupling is of the same order as the magnetic interaction.

I. INTRODUCTION

Magnetic field penetrates a sample of layered superconductor forming pancake vortices in every layer of the sample. Calculation of statistical properties of the pancake matter is a very complicated question for several reasons of which the following is the most important. The pancakes interact through two mechanisms, magnetic and Josephson. While magnetic mechanism is reduced to pairwise interaction of the pancakes, potential energy due to the interlayer Josephson coupling cannot be brought down to such simple form. Available methods lead to a non-local description [1] or rely on the duality transformation [2, 3] which maps the pancake system onto a system of quasi-two-dimensional Coulomb-like plasma. Both approaches are technically complicated and difficult to generalize.

Alternatively, many authors neglect Josephson mechanism assuming that for extremely anisotropic superconductors its influence is small compared to the effects of the magnetic interaction [4, 5, 6, 7, 8].

In this paper we show that under certain circumstances it is possible to include Josephson coupling into consideration with the help of general physical devices. At sufficiently high field the Josephson and magnetic interlayer coupling get smaller than the intralayer interaction. In such a situation the system is viewed as a collection of weakly coupled 2D layers. If pancakes in every layer form a structure with short range crystalline order both magnetic and Josephson mechanisms could be accounted for. The resultant model is a version of the sine-Gordon field theory. It could be analyzed with the help of usual field theoretical tools.

The proposed approach allows one to find an upper bound for the pancake lattice melting temperature. Another interesting application is that it becomes possible to compare contributions of the magnetic and Josephson interactions to the statistical properties of the pancake matter. Thus, we could determine under what circum-
stances it is possible to drop the Josephson coupling from the description and when it cannot be omitted.

The paper is organized as follows. In Sect. II starting from the vortex Hamiltonian for the anisotropic superconductor we reproduce the well-known result for the magnetic potential energy of the pancake lattice. In Sect. III the energy due to the Josephson coupling is derived. In Sect. IV we compare Josephson and magnetic interaction energies. The purpose of the latter comparison is to determine when it is permissible to ignore the Josephson mechanism and when it is not. In Sect. V we apply the derived Hamiltonian to the problem of pancake lattice melting. Sect. VI contains conclusions.

II. MAGNETIC COUPLING

We start with the vortex Hamiltonian $H$ [9, 10]:

$$H = \frac{\Phi_0^2}{8\pi} \int_q \left\{ \mathcal{K}_z(q) |v_z(q)|^2 + \mathcal{K}_\parallel(q) |v_\parallel(q)|^2 \right\},$$

(1)

$$\mathcal{K}_z(q) = \frac{1 + \lambda_z^2 q^2}{(1 + \lambda_{ab}^2 q^2) \left( 1 + \lambda_z^2 q^2 + \lambda_{ab}^2 q^2 \right)} ,$$

(2)

$$\mathcal{K}_\parallel(q) = \frac{1}{1 + \lambda_z^2 q^2 + \lambda_{ab}^2 q^2} .$$

(3)

The symbol $\int_q \ldots$ stands for $\int (2\pi)^{-3} d^3 q \ldots$; vector field $\mathbf{v}$ denotes vorticity. The component $v_z$ in the $n$th superconducting layer coincides with the pancake density in that layer:

$$v_z(r, n) = \rho(r, n) = \sum_\mu \delta^{(2)}(r - r_\mu(n)) ,$$

(4)

where $r_\mu(n)$ is the position of the $\mu$th vortex in the $n$th layer.

The parallel components of the vorticity $v_\parallel$ correspond to the Josephson vortices between superconducting layers. The Josephson vorticity between $n$th and $(n-1)$th layer is equal to:

$$v_\parallel(r, n) = \frac{1}{\delta_c} \sum_\mu \{ r_\mu(n) - r_\mu(n-1) \} \delta(\mathbf{r}) ,$$

(5)
\[ \delta_{\mu}(r) = \int_{0}^{1} d\sigma \delta^{(2)}(r - r_{\mu}(n - 1) - [r_{\mu}(n) - r_{\mu}(n - 1)]) \sigma. \]

Here \( d_c \) is the spacing between neighboring layers, \( r \) is 2D coordinate vector. Function \( \delta_{\mu} \) is defined in such a way that it is localized on a straight line connecting \( n \) and another pancake in \( n \). Physically, it is non-zero on the Josephson vortex which connects a pancake in \( n \)th layer and another pancake in \( (n-1) \)th layer.

Since a vortex cannot terminate inside the superconductor we have

\[ \text{div } \mathbf{v} = 0, \quad (7) \]

or, equivalently, in Fourier space

\[ q_{z}v_{z} = -q_{\parallel}v_{\parallel}. \quad (8) \]

Let us rewrite the kernel \( \mathcal{K} \) in the following manner:

\[ \mathcal{K}(q) = \mathcal{K}_{\text{mag}} + \delta \mathcal{K}, \quad (9) \]

\[ \mathcal{K}_{\text{mag}} = \frac{1}{\lambda_{ab} q_{\parallel}^{2}} \left( 1 - \frac{1}{1 + \lambda_{ab} q_{\parallel}^{2}} \right), \quad (10) \]

\[ \delta \mathcal{K} = -\frac{q_{z}^{2}}{q_{\parallel}^{2}} \left( 1 + \lambda_{ab} q_{\parallel}^{2} \right). \quad (11) \]

Note, that \( \mathcal{K}_{\text{mag}} \) contains no \( \lambda_{c} \). In other words, it does not depend on the Josephson coupling between layers. It describes magnetic interaction between pancakes. On the contrary, kernel \( \delta \mathcal{K} \) vanishes when the Josephson energy vanishes.

Now, using 3D-transversality of the vorticity (eq.(8)), we establish the following:

\[ \delta \mathcal{K} |v_{z}|^{2} = -\frac{1}{1 + \lambda_{c}^{2} q_{\parallel}^{2} + \lambda_{ab} q_{\parallel}^{2}} \frac{|(q_{\parallel}v_{\parallel})|^{2}}{q_{\parallel}^{2}}. \quad (12) \]

This allows us to rewrite the Hamiltonian (1):

\[ H = \frac{\Phi_{0}^{2}}{8\pi} \int_{q} \mathcal{K}_{\text{mag}} |v_{z}|^{2} + \mathcal{K} |v_{\parallel}|^{2}, \quad (13) \]

\[ \mathcal{K} = \frac{1}{1 + \lambda_{c}^{2} q_{\parallel}^{2} + \lambda_{ab} q_{\parallel}^{2}}, \quad (14) \]

\[ v_{\parallel} = v_{\parallel} - q_{\parallel} \frac{(q_{\parallel}v_{\parallel})}{q_{\parallel}^{2}}. \quad (15) \]

In other words, only the 2D-transverse part \( v_{\parallel} \) of the Josephson vorticity \( v_{\parallel} \) contributes to the interaction energy.

Magnetic part of the energy is the easiest to handle. Imagine that in every layer the pancakes form 2D crystal with short range hexagonal lattice characterized by the lattice constant \( a_{0} \). We assume that it is possible to choose the length scale \( \xi_{0} \gg a_{0} \) in such a way that within \( \xi_{0} \) the lattice translational invariance is undisturbed. Although, we do not need the exact value of \( \xi_{0} \) for our calculations, yet, our ability to set such a scale is imperative.

If the above prerequisite is met a pancake lattice in \( n \)th layer could be described locally by a 2D displacement vector field \( u_{n} \). This field does not change much on the scale \( \xi_{0} \). The pancake density is expressed as:

\[ v_{z}(r,n) = \rho_{0} \left( 1 + \text{div}_{2D} u_{n} + \sum_{q_{1} \neq 0} e^{iq_{1}\cdot r + iq_{1}\cdot u_{n}} \right). \quad (16) \]

The sum is upon vectors of the reciprocal lattice. The 2D divergence is defined as:

\[ \text{div}_{2D} u = \frac{\partial u_{x}}{\partial x} + \frac{\partial u_{y}}{\partial y}, \quad (17) \]

The quantity \( \rho_{0} \) denotes the average density of the pancakes:

\[ \rho_{0} = \frac{B}{\Phi_{0}}, \quad (18) \]

Observe, that two first terms of (16) vary slowly on the scale of the lattice constant \( a_{0} \) while the third term quickly oscillates as function of \( r \):

\[ v_{z} = \tilde{\rho} + \delta \rho, \quad (19) \]

\[ \tilde{\rho} = \rho_{0} \left( 1 + \text{div}_{2D} u_{n} \right), \quad (20) \]

\[ \delta \rho = \rho_{0} \sum_{q_{1} \neq 0} e^{iq_{1}\cdot r + iq_{1}\cdot u_{n}}. \quad (21) \]

We have for the magnetic interaction energy:

\[ H_{\text{mag}} = \frac{\Phi_{0}^{2} d_{c}^{2}}{8\pi} \sum_{n,m} \int_{r,r'} \mathcal{K}_{\text{mag}}(r - r', n - m) v_{z} v_{z}' = \quad (22) \]

\[ \frac{\Phi_{0}^{2} d_{c}^{2}}{8\pi} \sum_{n,m} \int_{r,r'} \mathcal{K}_{\text{mag}}(r - r', n - m) (\tilde{\rho} \tilde{\rho}' + \delta \rho \delta \rho' ), \]

where the symbol \( \int \ldots \) stands for \( \int d^{2}r \ldots \). The term proportional to \( \tilde{\rho} \tilde{\rho}' \) describes the statics of the longitudinal displacements. We will not be interested in them for they are suppressed [11]. The remaining parts are:

\[ H_{\text{mag}} = \frac{\Phi_{0}^{2}}{8\pi} \rho_{0}^{2} d_{c}^{2} \times \quad (23) \]

\[ \int_{rr'} \left\{ \sum_{n} \mathcal{K}_{\text{mag}}(r - r', 0) \sum_{q_{1} \neq 0} e^{iq_{1}(r - r') + iq_{1}(u_{n} - u_{n}')} + \sum_{n \neq m} \mathcal{K}_{\text{mag}}(r - r', n - m) \sum_{q_{1} \neq 0} e^{iq_{1}(r - r') + iq_{1}(u_{n} - u_{m}')} \right\}. \]

The notation

\[ u_{n}' = u_{n}(r') \quad (24) \]

was adopted in the formula above. To avoid clutter we omit the subscript ‘T’ which denoted the transverse displacement mode: we write here \( u_{n} \) rather than \( u_{n,T} \). We
will continue to do so, yet, one has to remember that the displacement vector field we consider is pure shear.

The first term in the above expression gives usual $C_{66}$:

$$C_{66} = \frac{\Phi^2 d_c}{(8\pi \lambda_{ab})^2} \rho_0. \quad (25)$$

To proceed further with the second term we need to calculate $K_{\text{mag}}$:

$$K_{\text{mag}}(q, n) = \frac{1}{\lambda_{ab}^2 q^2} \int_{q} e^{iq \cdot d_c \cdot n} \left(1 - \frac{1}{1 + \lambda_{ab}^2 q^2}\right) \cdot \frac{1}{\lambda_{ab}^2 q^2} \left(d_c^{-1} \delta_{n,0} - \frac{e^{-\gamma n d_c \cdot |n|}}{2 \lambda_{ab}^2 q^2}\right). \quad (26)$$

If $\lambda_{ab}q \gg 1$ which is true for $B > H_{c1}$ the above expression for $K_{\text{mag}}$ could be simplified:

$$K_{\text{mag}} \approx \frac{1}{\lambda_{ab}^2 q^2} \left(d_c^{-1} \delta_{n,0} - \frac{e^{-\gamma n d_c \cdot |n|}}{2 \lambda_{ab}^2 q^2}\right). \quad (27)$$

Thus, we get for the second term in (23):

$$\sum_{q \neq 0} \int_{\mathbb{R}^2} K_{\text{mag}}(r - r', n - m) e^{iq \cdot (r - r') + i \rho_{n - m}} = \sum_{q \neq 0} \int_{\mathbb{R}^2} K_{\text{mag}}(\Delta r, n - m) e^{iq \cdot \Delta r} = \sum_{q \neq 0} e^{-\gamma n d_c \cdot |n - m|} \int_{\mathbb{R}^2} e^{iq \cdot (u_n - u_m)}. \quad (28)$$

Here we neglected the weak dependence of $u(r + \Delta r)$ on $\Delta r$ since $|\Delta r| \sim a_0 \ll \xi_0$. Finally:

$$H_{\text{mag}} = \sum_{n} \int \frac{C_{66}}{2} |\nabla u_n|^2 - \sum_{n \neq m} \frac{\Phi^2 d_c^2}{16 \pi \lambda_{ab}^4 q^2} \sum_{q \neq 0 \overline{q}^2} e^{-\gamma n d_c \cdot |n - m|} e^{iq \cdot (u_n - u_m)}. \quad (29)$$

Here $q_0$ is the magnitude of the smallest reciprocal lattice vectors:

$$q_0 = \frac{4\pi}{\sqrt{3} a_0}, \quad (30)$$

$$q_0^2 = \frac{8\pi^2}{\sqrt{3}} q_0. \quad (31)$$

Properties of the Hamiltonian similar to (29) was investigated in [4, 5, 6, 7, 8].

At finite temperature the above expression could be simplified. It is enough to notice that at $T > 0$ contribution of a term $\exp(iq \cdot (u_n - u_m))$ is proportional to $\exp(-q_0^2/2)$. Thus, when fluctuations of the lattice are substantial it is permissible to retain in (29) lowest $|q_0^2|$ terms only. Consequently, it is not necessary to sum over all possible $q_0^2$ in the above formula. It is sufficient to keep only six terms corresponding to six elementary reciprocal lattice vectors whose absolute values are equal to $q_0$. Namely, the last term of (29) might be written as:

$$\frac{\Phi^2 d_c^2}{16 \pi \lambda_{ab}^4 q_0^2} \times \sum_{m \neq n} \int_{\mathbb{R}^2} \sum_{|q_0^2| = q_0} e^{-\gamma n d_c \cdot |n - m|} \cos(q_0^2 (u_n - u_m)). \quad (32)$$

Our model becomes a vector version of a sine-Gordon field theory.

### III. JOSEPHSON COUPLING

Now we add the Josephson coupling to our Hamiltonian. This means that we have to include the Josephson vorticity as well. At first it seems like an impossible task since we have to account for the Josephson vorticity fluctuations. We will demonstrate that under rather broad conditions these fluctuations could be included by simple renormalization of the Josephson coupling parameter.

![FIG. 1: A fluctuating Josephson vortex (solid line) connects two pancakes, one in the $n$th layer (closed circle), another in the $(n-1)$th layer (open circle). The deformed vortex is represented by a solid line. Fluctuations create an outgrowth on the vortex line. The size of the outgrowth is $\xi_{\text{loop}}$. The characteristic size of the loop is $\xi_{\text{loop}}$. A straight line and a loop (both broken lines).](image)

On fig.1 an example of Josephson vorticity fluctuation is shown. In the absence of fluctuations two pancake vortices in neighboring layers would be connected by a straight Josephson vortex. Fluctuations create an outgrowth on the vortex line. The size of the outgrowth is $\xi_{\text{loop}}$. The deformed vortex is represented by a solid line. The deformed vortex could be “decomposed” into a vortex loop and a straight vortex connecting two pancakes. Thus, the problem of the Josephson vorticity fluctuations could be reduced to the problem of the thermally induced vortex loop gas.
Thermal fluctuations infuse superconductor with loops of Josephson vortices of typical size $\xi_{\text{loop}}$. This size $\xi_{\text{loop}}$ diverges at the superconducting transition temperature $T_c$ and vanishes at $T = 0$. It is finite at $T_c > T > 0$. Loosely speaking, $\xi_{\text{loop}}$ characterizes a spatial scale above which “mean-field” Josephson coupling could be defined and fluctuations are unimportant. Therefore, if we are in the regime

$$\xi_{\text{loop}} < a_0$$

(33)

the fluctuations might be omitted. In such a case fluctuations result in small vibrations of the Josephson vortex. The only consequence of the fluctuations would be renormalization of effective value of the Josephson coupling, which is equivalent to renormalization of the penetration depth from its bare value $\lambda_c^0$ to experimentally measurable value of $\lambda_c$.

Of course, it is desirable to generalize the argumentation beyond (33). How this could be done is discussed in Appendix A.

Once we settle the issue of the fluctuations we could proceed with the derivation of the Josephson contribution to the energy of the pancake system.

As in the previous Section we assume that in every layer the pancakes form a 2D structure with (at least) short range crystalline order. Remember also that the longitudinal displacements of the pancakes are suppressed. In absence of the longitudinal displacements the Josephson vorticity is a periodic function of vector $s_n$ defined as (fig.2):

$$s_n(r) = u_n(r) - u_{n-1}(r).$$

(34)

Two-dimensional vector field $v_\parallel(r,n)$ could be expressed as a sum of smooth and oscillating parts:

$$v_\parallel = \bar{v}_\parallel + \delta v_\parallel,$$

(35)

where the smooth part is equal to:

$$\bar{v}_\parallel = \frac{\rho_0}{d_c} g(s_n).$$

(36)

Vector function $g$ is defined like so:

$$g(s) = s - l(s).$$

(37)

The lattice vector $l(s)$ is chosen to deliver minimum to the expression $|s - l|$ (see fig.(3)). Function $g(s)$ is invariant under lattice translations. For any $s$ vector $g$ always lies within the primitive lattice cell. Physically, eq.(36) means that when two lattices are shifted only slightly with respect to each other $\bar{v}_\parallel = \rho_0 s/d_c$. However, when $s$ does not fit into the primitive cell the Josephson vortices rearrange themselves to minimize their own length. In this case $|g| < |s|$. For example, if $s$ is the lattice vector then $g = 0$.

Next, we calculate the oscillating part. Full Josephson vorticity field between $n$th and $(n-1)$th layers is given by eq.(5) with $r_{\mu}(n-1) = l_{\mu} + u_{n-1}$ ($l_{\mu}$ is the lattice vector which shows the undisturbed position of the $\mu$th pancake) and $r_{\mu}(n) - r_{\mu}(n-1) = g$. The Fourier transform of the vorticity vector field due to a single vortex $\mu$ is:

$$v_{\parallel\mu}(q|n) = \frac{g(s_n)}{d_c} \int_0^1 d\sigma e^{i q |u_{n-1} + l_{\mu} + g(s_n)\sigma} =$$

$$\frac{g(s_n)}{i|q|d_c} (e^{i q |u_{n-1} + g} - e^{i q |u_{n-1}}) e^{i |q| l_{\mu}}.$$  

(38)

The total vorticity is:

$$v_{\parallel}(q|n) = \sum_{\mu} v_{\parallel\mu}(q|n).$$

(39)

FIG. 2: The Josephson vorticity generated by the relative displacement of pancake lattices in the neighboring layers. A hexagonal pancake lattice in the $n$th layer (closed circles) is shifted with respect to the lattice in the $(n-1)$th layer (open circles). Vector $s$, eq.(34), is shown in a box.

FIG. 3: To the definition of vector $g$. When two pancake lattices in two neighboring layers are shifted with respect to each other pancakes got connected by Josephson vortices in such a way as to minimize the length of the latter. The optimal choice is shown by solid lines. Two other possible connections (broken line and dotted line) require longer vortices which, obviously, costs more energy.
It is non-zero only when $q_{||}$ is the reciprocal lattice vector. If $q_{||}$ belong to the reciprocal lattice, one prove

$$e^{i q_{||} \cdot \mathbf{s}(n)} = e^{i q_{||} \cdot s_n}.$$

(40)

Thus, we have for the oscillating part:

$$\delta v_{||}(r, n) = \frac{\rho_0}{d_e} \sum_{q_{||} \neq 0} \frac{q_{||}}{q_{||}^2} \mathbf{g}(s_n) \left( e^{i q_{||} u_n(r)} - e^{i q_{||} u_{n-1}(r)} \right) e^{-i q_{||} r},$$

where the sum runs over the reciprocal lattice vectors. The above expression is explicitly periodic with respect to both $u_n$ and $u_{n-1}$. This means that if one of the layers is shifted by a lattice vector the Josephson vorticity remains unaffected by such a shift.

To calculate the energy we need the transverse component of the Josephson vorticity:

$$\delta v_{||T}(r, n) = \frac{\rho_0}{d_e} \sum_{q_{||} \neq 0} \frac{q_{||}}{q_{||}^2} \mathbf{g}(s_n) \times$$

$$\left( e^{i q_{||} u_n(r)} - e^{i q_{||} u_{n-1}(r)} \right) e^{-i q_{||} r}.\tag{42}$$

(Note: in 2D vector product is a scalar, not a vector.) As for $v_{||}$, it is transverse as long as $u_n$’s are transverse.

We are in position now to write down the Josephson energy:

$$H_J = H_{J1} + H_{J2} + H_{J3}.\tag{43}$$

$$H_{J1} = \frac{\Phi_0^2 \gamma^2}{8 \pi} \sum_{n,m} \int_{r,r'} \mathcal{K}_J(r-r',n-m) \mathbf{g}(s_n) \cdot \mathbf{g}(s_m) \mathbf{(44)}.$$

$$H_{J2} = \frac{\Phi_0^2 \gamma^2}{8 \pi} \sum_n \int_{r,r'} \mathcal{K}_J(q_{||},0) \left| \frac{q_{||}}{q_{||}^2} \mathbf{g}(s_n) \times \left( 2 - 2 \cos q_{||} s_n \right) \right|^2 \times \tag{45}$$

$$H_{J3} = \frac{\Phi_0^2 \gamma^2}{8 \pi} \sum_{n \neq m} \int_{r,r'} \mathcal{K}_J(q_{||},n-m) \times \left| \frac{q_{||}}{q_{||}^2} \mathbf{g}(s_n) \times \mathbf{g}(s_m) \right|^2 \times \tag{46}$$

$$\left( e^{i q_{||} u_n} - e^{i q_{||} u_{n-1}} \right) \left( e^{-i q_{||} u_n} - e^{-i q_{||} u_{n-1}} \right).$$

where we used the notation $s_n' = u_n(r') - u_{n-1}(r')$. The term $H_{J1}$ describes the kinetic energy of supercurrents induced by the Josephson vorticity. The kernel $\mathcal{K}_J(\Delta r, n)$ is non-zero for $|\Delta r| \approx \lambda_c$ (see Appendix C) since such currents spread over large regions. Therefore, two Josephson vortices apart from each other interact via these currents. The second term is purely local. It describes the increase of the free energy of an interlayer Josephson junction due to insertion of vortices. The third term describes similar effect: the modification of the free energy of an interlayer Josephson junction by a vorticity in another layer. As we will show below, this effect is small.

At low temperature, where it is possible to write $\mathbf{g}(s) \approx s$, Hamiltonian $H_J$ is responsible for the Josephson contribution to the tilt modulus $C_{44}$ of the vortex lattice [11]. The first term corresponds to the $B^2$ piece of $C_{44}$ while the second and the third terms contribute to the so-called single-vortex part ($\propto B$) of the tilt modulus.

At sufficiently high temperature we cannot approximate $\mathbf{g}$ by $s$. Instead we must follow different type of analysis. The term

$$\int_{r,r'} \mathcal{K}_J(r-r',n-m) \mathbf{g}(s_n) \cdot \mathbf{g}(s_m) \mathbf{(48)}$$

may be disregarded right away if $|n-m| \geq 2$ since at low Josephson coupling we could neglect correlations of the displacement fields in different layers: $\langle u_n u_m \rangle = 0$. Thus, for $|n-m| \geq 2$

$$\mathbf{g}(s_n) \cdot \mathbf{g}(s_m) \approx \mathbf{g}(s_n) \mathbf{(49)}$$

since $\langle \mathbf{g}(s) \rangle = 0$. The latter statement is obviously true for $\mathbf{g}$ is an odd function.

We have:

$$H_{J1} \propto \sum_{m_n, p_n} \int_{r,r'} \mathcal{K}_J \mathbf{g}(s_n) \cdot \mathbf{g}(s_m) \mathbf{(50)}$$

$$- \sum_n \sum_{q_{||} \neq 0} \int_{r,r'} \left\{ \mathcal{K}_J \hat{g}_{q_{||}} \hat{g}_{p_{||}} e^{i q_{||} (u_n-u_{n-1})} \cdot \mathbf{i} p_{||} (u_n-u_{n-1}) \right\},$$

where notation (24) is used and $\hat{g}_{q_{||}} = -\mathbf{g}_{-q_{||}}$ are Fourier transform coefficients of the odd periodic function $\mathbf{g}(s)$. As one notices from this formula, $H_{J1}$ is a product of four exponent of the form $\exp(i q_{||} u)$. It will be shown that $H_{J2}$ depends on two exponents only. At high enough temperatures $H_{J1}$ may be neglected in comparison with $H_{J2}$. This is because $H_{J2}$ is proportional to $\exp(i q_{||} u)^2$ while $H_{J1}$ is proportional to $\exp(i q_{||} u)^4$ and the expectation value of the exponent vanishes when $T$ grows.

One could adopt a more formal approach. As shown in Appendix B at sufficiently large $T$ operator $H_{J1}$ becomes irrelevant in the renormalization group sense while $H_{J2}$ remains relevant up until higher temperature. In this temperature interval it is possible to neglect $H_{J1}$: its contribution to the thermodynamics is purely perturbative while the contribution of the relevant second term is singular.

The term $H_{J3}$ could be also disregarded for the following reason. The kernel

$$\mathcal{K}_J(q_{||}) = \frac{1}{1 + \lambda_c^2 q_{||}^2 + \lambda_{ab}^2 q_{||}^2} \approx \frac{1}{\lambda_c^2 (q_{||}^2 + \gamma^2 q_{||}^2)}, \quad \gamma = \frac{\lambda_c}{\lambda_{ab}} \tag{51}$$

$$\tag{52}$$
may be simplified provided that the Josephson length is bigger than the lattice constant:
\[
\lambda_J = \gamma d_c \gg a_0.
\]  
(53)

Namely, we can write:
\[
K_j(q) \approx \frac{1}{\lambda_J^2 q_{||}} \left( 1 - \frac{q_{||}^2}{\gamma^2 q_{||}^4} \right).
\]  
(54)

The term \(q_{||}^2/\gamma^2 q_{||}^4\) is smaller than unity as long as (53) is valid. Indeed, \(q_{||} < \pi/d_c\) and \(q_{||} > a_0^{-1}\). This means that \(K_j(q_{||}, n)\) is smaller than \(K_j(q_{||}, 0)\). Therefore, \(H_{J3}\) (corresponds to \(K_j(q_{||}, n)\)) is much less than \(H_{J2}\) (corresponds to \(K_j(q_{||}, 0)\)).

Thus, we retain \(H_{J2}\) only:
\[
H_j \approx \frac{\Phi_0^2}{8\pi} \sum_n \sum_{q_{||} \neq 0} K_j(q_{||}, 0) \left| \frac{\hat{q_{||}} \times \hat{g}(s_n)}{|\hat{q_{||}}||\hat{g}(s_n)|} \right|^2 \times (2 - 2 \cos q_{||} s_n),
\]  
(55)
\[
K_j = \frac{1}{d_c \lambda_J^2 q_{||}^2}.
\]  
(56)

We rewrite this expression in a more compact way:
\[
H_j = \frac{3\Phi_0^2}{256\pi^3 \lambda_c^2 d_c} \sum_n \int_\mathbf{r} h(\mathbf{u}_n(\mathbf{r}) - \mathbf{u}_{n-1}(\mathbf{r})),
\]  
(57)
\[
h(s) = \sum_{q_{||} \neq 0} \frac{g_0^4}{q_{||}^4} \left| \frac{\hat{q_{||}} \times \hat{g}(s)}{|\hat{q_{||}}||\hat{g}(s)|} \right|^2 (1 - \cos q_{||} s).
\]  
(58)

Function \(h\) is dimensionless and periodic with the period of the hexagonal lattice. Its contour graph is shown on fig.4. Due to its periodicity \(h\) can be expanded in a Fourier series:
\[
h(s) = h_0 + \sum_{q_{||} \neq 0} \hat{h}_{q_{||}} \cos q_{||} s.
\]  
(59)

Using this expansion it is possible to simplify the expression for \(H_j\) at finite temperatures with the help of the trick already discussed at the end of Sec.II:
\[
H_j = \frac{3|\hat{h}_{q_0}|\Phi_0^2}{256\pi^3 \lambda_c^2 d_c} \sum_n \int_\mathbf{r} \left| \frac{1 - \cos(q_{||} [\mathbf{u}_n - \mathbf{u}_{n-1}])}{|\hat{q_{||}}||\hat{g}(s)|} \right|,
\]  
(60)
\[
\hat{h}_{q_0} \approx -3.7,
\]  
(61)
\[
\frac{3|\hat{h}_{q_0}|}{256\pi^3} \approx 1.4 \times 10^{-4},
\]  
(62)

Here the sum over vectors \(q_{||}\) runs only over the shortest of them. The number \(\hat{h}_{q_0}\) is the Fourier coefficient of \(h(s)\) corresponding to such \(q_{||}\) vector(s) as \(q_{||}\) is maximum at the corners of the primitive cell.

### IV. COUPLING CONSTANTS

Let us estimate coupling constants of our Hamiltonian. That way we can understand what are the largest interaction in our system and when the perturbative treatment of interactions is permissible.

First, we calculate the elastic constant \(\varepsilon_{el}\). To find it we measure area in units of \(\rho_0^{-1}\). Thus:
\[
\varepsilon_{el} = \frac{C_{66}}{\rho_0} = \frac{\Phi_0^2 d_c}{64\pi^2 \lambda_{ab}^2}.
\]  
(63)

For BSCCO \(\varepsilon_{el} = 182\) K assuming:
\[
d_c = 1.5\, \text{nm},
\]  
(64)
\[
\lambda_{ab} = 200\, \text{nm}.
\]  
(65)

The Josephson constant corresponding to \(H_{J2}\):
\[
\varepsilon_j = \frac{3\Phi_0^2}{256\pi^3 \lambda_c^2 d_c \rho_0} |\hat{h}_{q_0}|.
\]  
(66)

The coupling constant corresponding to \(H_{J1}\):
\[
\varepsilon_{j} = \frac{\Phi_0^2}{8\pi} \frac{1}{4\pi \lambda_{ab} \lambda_c} g_0^2 \sqrt{\rho_0}.
\]  
(67)

Using the estimate for the Fourier coefficients \(g^2 \approx 1/2q_0^2\) we find:
\[
\varepsilon_{j}' \approx \frac{3\Phi_0^2 a_0}{1024\pi^3 \lambda_{ab} \lambda_c}.
\]  
(68)

Ratios of these two coupling constants to the elastic energy scale:
\[
\frac{\varepsilon_j}{\varepsilon_{el}} = \frac{3|\hat{h}_{q_0}|}{4\pi^3 \gamma \lambda_c d_c \rho_0} = \frac{12|\hat{h}_{q_0}| a_0^2}{\sqrt{3\pi^2} \lambda_j^2} \approx 2.6 \frac{a_0^2}{\lambda_j^2}.
\]  
(69)

**FIG. 4:** Contour plot of the function \(h(s) = h(x, y)\), defined by eq. (58). Function \(h\) has a minimum at the origin and six maximums at the corners of the primitive cell.
and
\[
\frac{\varepsilon'_1}{\varepsilon_{el}} = \frac{\alpha_0}{16\pi^2 \lambda_j} \approx 0.02 \frac{\alpha_0}{\lambda_j}
\]  (70)
are both smaller than unity when (53) holds. This explains the physical significance of (53). The latter is essentially a criterion for our vortex system to be viewed as a quasi-2D pancake gas. If (53) is violated the system is better described in terms of the Abrikosov vortices rather than pancakes.

The interlayer magnetic interaction constant could be estimated directly from (29):
\[
\varepsilon_{mag} = \frac{3\Phi_0^2 d_c}{8\pi \lambda_{ab}^3 q_0^3} = \frac{3\Phi_0^2 d_c}{512\pi^3 \lambda_{ab}^3 \rho_0}.
\]  (71)
The quantity $1/q_0d_c$ is an estimate for number of layers coupled by the magnetic interaction with a given layer. The ratio of the magnetic and the Josephson constants is:
\[
\frac{\varepsilon_1}{\varepsilon_{mag}} = \frac{2\hat{h}_{q_0}^2 |q_0| \lambda_{ab}^2}{\gamma^2 d_c^2} = 2\hat{h}_{q_0}^2 |q_0| \lambda_{ab}^2 \gamma^2 d_c^2 \approx 7.4 \frac{\lambda_{ab}^2}{\lambda_j^2}. 
\]  (72)
The last result is important for it allows one to judge when the Josephson coupling could be neglected and when it must be retained. As one can see from the above formula, the ratio is independent of the magnetic field and, in that sense, it is a “material constant”. If we take for BSCCO:
\[
\gamma = 300.
\]  (73)
then the ratio equals to 1.5. That is, even for such an extremely anisotropic material the Josephson coupling is of the same order as magnetic.

V. UPPER BOUND FOR THE MELTING TEMPERATURE

In this section we find the upper bound $T_u$ on a melting temperature $T_m$. The gist of the following calculations is that at sufficiently high temperature the interlayer coupling becomes irrelevant and the system could be thought of as a collection of 2D layers decoupled from each other. In these layers no long-range crystalline order is possible. The temperature found in this fashion is not necessary a true melting point for melting could occur at even lower temperatures through a first order phase transition.

In certain situation our upper bound could serve as a reasonable estimate for the melting temperature. This happens if the Josephson coupling is sufficiently strong and the difference between the solid and the liquid phases of the pancake matter is small. When this difference is exceeded by the bare Josephson energy scale the phase transition is controlled by the Josephson energy. Since the latter vanishes at $T_u$ the melting line lies close to $T_u$.

We assume that the temperature is large so that Josephson term $H_{J1}$ is irrelevant (see Appendix B). Only $H_{J2}$ part of the Josephson coupling and the magnetic coupling, eq.(32), have to be accounted for. In such a regime it is possible to get $T_u$ using simple perturbative argument which we are ready to present.

The crystalline phase is characterized by a non-zero order parameter:
\[
\Delta = \langle e^{i u_n b_n} \rangle. 
\]  (74)
With the help of $\Delta$ the interlayer interactions (32) and (45) could be written in a mean-field manner as
\[
\cos q_0 (u_n - u_m) \approx \Delta \cos q_0 u_n. 
\]  (75)
The above transformations reduces our multilayer problem to a problem for a single layer with the Hamiltonian:
\[
H_{mf} = \sum_n \int_r \left[ \frac{C_{66}}{2} |\nabla u_n|^2 - \alpha \Delta \sum |q_0| \cos q_0 u_n \right],
\]  (76)
\[
\alpha = \rho_0 (\varepsilon_1 + \varepsilon_{mag}). 
\]  (77)
It is presumed that $\alpha$ is small:
\[
\alpha \ll C_{66}. 
\]  (78)
This guarantee the applicability of approximation (75). It is equivalent to (53).

Let us transform the interlayer interaction term as follows:
\[
- \alpha \Delta \cos q_0 u_n \approx - \alpha \Delta^2 \left( 1 - \frac{1}{2} \langle q_0 u_n \rangle^2 \right) = \frac{\alpha \Delta^2}{2} \langle q_0 u_n \rangle^2 = \frac{\tilde{\alpha}}{2} \langle q_0 u_n \rangle^2,
\]  (79)
\[
\tilde{\alpha} = \alpha \rho_0^2 \langle \cos q_0 u_n \rangle^2 = \alpha \rho_0^2 \exp \left( - \langle q_0 u_n \rangle^2 \right). 
\]  (80)
After such substitution the effective Hamiltonian becomes quadratic. Thus, the problem could be solved exactly. However, when solving it we must keep in mind the self-consistency condition:
\[
\langle (q_0 u_n)^2 \rangle = \int_k \frac{T(q_0 k)^2}{k^2 (C_{66} k^2 + \tilde{\alpha})} = \int_k \frac{T_k^2}{k^2} \ln \left[ \frac{C_{66} k^2 + \tilde{\alpha}}{\tilde{\alpha}} + \text{const.} \right] = \frac{T_0^2}{8\pi C_{66}} \left[ \ln \frac{C_{66} k^2 + \tilde{\alpha}}{\tilde{\alpha}} + \langle (q_0 u_n)^2 \rangle + \text{const.} \right].
\]  (81)
The last equation has positive solution for $\langle (q_0 u_n)^2 \rangle$ only when
\[
T < T_u = \frac{8\pi C_{66}}{\tilde{\alpha}} = \sqrt{3} \frac{\Phi_0 d_c}{64\pi^3 \lambda_{ab}^2}. 
\]  (82)
Above $T_u$ the order parameter $\Delta$ is zero.
Since $\lambda_{ab}$ is a function of temperature itself, to find $T_u$ it is in fact necessary to solve the equation:

$$T_u = \frac{\sqrt{3} \Phi_0^2 d_c}{64\pi^3 \lambda_{ab}^2(T_u)}.$$  

(83)

Assume that the temperature dependence of $\lambda_{ab}$ is given by a phenomenological formula:

$$\lambda_{ab}^{-2}(T) = \lambda_{ab}^{-2}(0) (1 - T^2/T_c^2),$$  

(84)

where the superconducting transition temperature $T_c$ equals to 100 K for BSCCO. Eq.(83), thus, gives:

$$T_u \approx 62 \text{ K.}$$  

(85)

Let us reiterate, that this result is “high-field”. Namely, it is derived under the assumption that (53) is satisfied. The latter inequality is equivalent to:

$$B \gg \frac{\Phi_0}{\lambda_3^2}. $$  

(86)

Thus, there is no contradiction if at low fields ($B < \Phi_0/\lambda_3^2$) the melting temperature is higher than $T_u$ found above.

We can also find the contribution of the interlayer (Josephson and magnetic) coupling to the total free energy density of the pancake lattice:

$$F_{\text{inter}}/S = -\alpha \langle \cos(q_u \mathbf{u}) \rangle^2$$  

(87)

$$= -\alpha \exp \left( -\langle \langle q_u \mathbf{u} \rangle^2 \rangle \right) = -C_{66} \left( \frac{\alpha}{C_{66}} \right)^{\frac{q_0^2 T}{8\pi C_{66} - q_0^2 T}} ,$$

where the following expression for $\langle \langle q_u \mathbf{u} \rangle^2 \rangle$ was used:

$$\langle \langle q_u \mathbf{u} \rangle^2 \rangle = \frac{\alpha}{C_{66}} \ln \left( \frac{C_{66}}{\alpha} \right). $$  

(88)

The latter expression is a trivial consequence of (81).

Note that $\alpha/C_{66} \sim (\varepsilon_1 + \varepsilon_{\text{max}})/\varepsilon_{\text{el}}$ and $q_0^2 T/8\pi C_{66} = T/T_u$. Thus, the formula for $F_{\text{inter}}$ could be rewritten:

$$F_{\text{inter}}/S \sim -C_{66} \left( \frac{3|\hat{q}_{ab}|}{4\pi^3 \gamma^2 d_c^2 \rho_0} + \frac{3}{8\pi^3 \lambda_{ab}^2 \rho_0} \right)^{1/(1-T/T_u)}.$$  

(89)

Interlayer contribution to the free energy, as expected, vanishes at $T = T_u$. This result is useful only if the lattice remains stable close to $T_u$, that is, if $T_m$ is close to $T_u$.

VI. DISCUSSION

In this paper we proposed a description of the quasi-2D pancake lattice. The description is a version of an elasticity theory: the Hamiltonian is a functional of the displacement field only. Such a transparent structure allows easy generalization. The most obvious extension is for the case of non-zero pinning. This is an important direction for future studies.

Another advantage of the method is that at high magnetic field (eq.(86)) all non-harmonic terms are small. Therefore, consistent perturbative analytical tools are applicable.

At present we do not include dislocations and vacancies into our model. Yet, to certain extent the effect of both dislocations and vacancies may be accounted for even at this stage. Unlike the interlayer coupling, these two disorder the lattice. The interlayer coupling tries to eliminate them by binding into a topologically neutral pairs. In turn, dislocations and vacancies attempt to destroy the interlayer coupling by breaking translational order. Which side wins this competition could be determined through comparison of corresponding singular corrections to the free energy. At least conceptually, this is a straightforward task.

The main goal of this paper is the derivation of the model. However, we’d like to discuss the most elementary of its consequences here.

First of all, our model allows to compare quantitatively the magnitude of the magnetic and Josephson pancake interactions. We have seen above that even for BSCCO ($\gamma \sim 300$) both of these are of the same order. Therefore, models which neglect the Josephson interaction are only qualitatively accurate, at best.

Next, let us briefly discuss the application of formula (83) for pancake lattice melting in real layered superconductors. Probably, it has no physical significance for BSCCO due to its extremely small Josephson coupling. The total interlayer coupling is also small since the magnetic interaction is of the same order as the Josephson. The transition into the liquid phase occurs because the free energy of the liquid becomes less than the free energy of the solid. Small contribution of the interlayer interaction could change the melting line slightly but otherwise is insignificant.

We would like to conjecture cautiously that (83) may be of relevance for Pb-doped BSCCO. This superconductor has much smaller anisotropy $\gamma \sim 70$ [12]. Presumably, this indicates that its Josephson coupling parameter is higher than that of pure BSCCO.

In [13] Pb-doped BSCCO was studied by means of magneto-optic technique. A depinning transition was observed at $T^* \approx 54$ K. The authors of [13] interpreted the transition as a pancake lattice melting. Remarkably, they see that this transition is field independent. This is a strong argument in favor of (83) being applicable since $T_u$ does not depend on the magnetic field. Another field-independent melting transition, 2D dislocation unbinding, is an unlikely candidate for it occurs at much smaller temperatures.

Pb-doped BSCCO has $T_c \approx 91$ K and $d_c \approx 1.9$ nm. In [14] $\lambda_{ab}$ was reported to be of the order of 180 nm. Thus:

$$T_u \approx 68 K. $$  

(90)

This is close to the transition at $T^* \approx 54$ K reported in
The fact, that \( T_u \) and \( T^* \) differs by about 25% is not by itself discouraging since penetration depth \( \lambda_{ab} \) shows strong sample-to-sample variations [15]. It is very well possible that they are responsible for the deviation of the experimental transition temperature from its theoretical estimate. Yet, the final judgment about the applicability of the presented theory is to be postponed until further investigation.

In conclusion, we proposed a simple model for the pancake lattice. It could be used to study pinning and thermodynamics of the pancake matter at sufficiently high fields and temperatures.

### VII. ACKNOWLEDGMENTS

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### APPENDIX A: THERMALLY INDUCED JOSEPHSON VORTICITY

In this Appendix we will try to estimate the characteristic size \( \xi_{\text{loop}}(T) \) of the thermally induced Josephson vortex loops at given temperature \( T \). We will also discuss what happens when \( \xi_{\text{loop}} \) grows bigger than \( a_0 \).

To achieve the first goal we calculate the energy of a single Josephson vortex loop \( E_{\text{loop}}(R) \) as a function of its radius \( R \) and compare this energy to the temperature. We will assume that \( \xi_{\text{loop}} \) is determined by the condition

\[
E_{\text{loop}}(\xi_{\text{loop}}) = T. \tag{A1}
\]

To proceed with the calculation we first find the Fourier transformation of the vorticity field of a single loop whose radius is \( R \):

\[
v_{\|}(q) = \int_{r=R} e^{i\mathbf{q} \cdot \mathbf{r}} \mathbf{\hat{z}} \cdot d\mathbf{r} = 2\pi R J_1(q_{\parallel} R) \frac{[\hat{z} \times q_{\|}]}{q_{\parallel}} \tag{A2}
\]

The energy is given by:

\[
E_{\text{loop}}(R) = \frac{\Phi_0^2}{8\pi} R^2 \int_{q_{\|} = 1} \frac{4\pi^2 J_1^2(q_{\|} R)}{1 + \lambda_{ab}^2 q_{\|}^2 + \lambda_{ab}^2 q_{\|}^2} \tag{A3}
\]

The integral in this formula could be transformed as follows:

\[
I = \int_{q_{\|} = 1} \frac{4\pi^2 J_1^2(q_{\|} R)}{1 + \lambda_{ab}^2 q_{\|}^2 + \lambda_{ab}^2 q_{\|}^2} \sim (A4)
\]

\[
2 \int_0^{1/\xi} \frac{J_1^2(q_{\|} R)}{\lambda_{ab} \sqrt{1 + \lambda_{ab}^2 q_{\|}^2}} \arctg \left( \frac{\pi \lambda_{ab}}{d c \sqrt{1 + \lambda_{ab}^2 q_{\|}^2}} \right) q_{\|} dq_{\|}.
\]

where \( \xi \) is the superconducting coherence length. After a substitution \( x = q_{\|} R \) the integral becomes:

\[
I = \frac{2}{\lambda_{ab} \lambda_c R} \int_0^{R/\xi} J_1^2(x) \arctg \left( \frac{\pi R}{\lambda_c x} \right) dx. \tag{A5}
\]

In deriving the above equation we used the approximation \( \sqrt{1 + \lambda_c q_{\|}^2} \approx \lambda_c q_{\|} \). This is accurate for \( q_{\|} \gg \lambda_c^{-1} \) or, equivalently, \( R \ll \lambda_c \).

Finally, confining ourselves to the range \( \xi \ll R \ll \lambda_c \) we write:

\[
I \approx \frac{2}{\lambda_{ab} \lambda_c R} \left( \frac{\pi R}{\lambda c} \right) \int_0^\infty J_1^2(x) \frac{dx}{x} = \frac{\pi}{\lambda_c^2 d_e} \tag{A6}
\]

Therefore:

\[
E_{\text{loop}} \approx \frac{\Phi_0^2}{8\lambda_{ab}^2 \lambda_c^2} \left( \frac{R}{\lambda c} \right)^2 = \frac{8\pi^3}{\sqrt{3} T_u} \left( \frac{R}{\lambda c} \right)^2, \tag{A7}
\]

where \( T_u \) is given by (83). Using (A1) one determines:

\[
\xi_{\text{loop}} \sim 0.08 \lambda c \sqrt{\frac{T}{T_u}}. \tag{A8}
\]

At \( T = T_u \) which is the highest temperature at which the model proposed in this paper is still applicable we have

\[
\xi_{\text{loop}} \lesssim 0.08 \lambda c. \tag{A9}
\]

This give us the upper bound for the characteristic size of the thermally induced Josephson vorticity loops.

As it was discussed in the body of the paper we would like to have \( \xi_{\text{loop}} \ll a_0 \) (eq.(33)). Consequently,

\[
0.08 \lambda_c \ll a_0. \tag{A10}
\]

It is not difficult to notice that the latter condition is not always compatible with the requirement \( a_0 \ll \lambda_c \) which has to be imposed to observe the quasi-2D effects of the pancake physics (eq.(53)). What does this mean for the analysis undertaken in this paper?

The answer to this question is that it is possible to recover virtually all of the physics described in the paper even if (33) is not satisfied as long as \( \xi_{\text{loop}} \ll \xi_0 \). (The quantity \( \xi_0 \) was defined in the paper as a size within which the 2D pancake lattice could be viewed as almost ideal.) Let us examine how this is done.

Presumably, the violation of (33) might lead to problems with (57). It was calculated under assumption that it is possible to define a single-vortex line tension on a scale \( \sim a_0 \). When vorticity fluctuation scale \( \xi_{\text{loop}} \) exceeds \( a_0 \) it becomes difficult to assign a well-defined line tension to a Josephson vortex. Thus, the analysis at the scale of \( a_0 \) fails.

It is, however, possible to study the system at the scale of \( \xi_0 \). At this scale we need not to be concerned with the properties of a single Josephson vortex. Instead, we will think in terms of average local vorticity which pierces interlayer Josephson junction.
Imagine first the situation when pancake lattices in the
nth and (n − 1)th layers coincide at r = r₀. This
implies that for any r, |r − r₀| < ξ₀, the average Joseph-
sen vorticity between these two layers is absent. Thus,
the local density of the Josephson energy is zero. Now
we shift the lattices with respect to each other: sₙ =
uₙ(r) − uₙ₋₁(r) ≠ 0. The average vorticity ⟨v₀(s)⟩ is
no longer zero as well. We have to disentangle its con-
tribution from contribution of the fluctuation-induced vort-
icity δv║ in the situation where fluctuations locally are
strong: ⟨|δv║|⟩ > ⟨|v║⟩. To do so we coarse-grain the sys-
tem description up to the scale ξloop. After this coarse-
graining δv║ ≈ 0 and the bare value of the penetration
depth measured experimentally.) Thus, at the scale
ξ0 ≫ ξloop the vorticity fluctuations could be
neglected and it is possible to work with the effective
Josephson coupling instead of including fluctuations exp-
plicitly.

The “mean-field” vorticity ⟨v₀(r)⟩ is a periodic func-
tion of sₙ. This implies that the pancake lattice energy
density due to Josephson coupling is a periodic function
of sₙ as well:

\[ H₁ = \sumₙ \int_{\mathbb{R}} \varepsilon₁ ρ₀ \sum_{|qₙ| = q₀} [1 - \cos(qₙsₙ)]. \]  
(A11)

This expression is completely general: it is the lowest q||
terms of periodic function Fourier series.

The final question now is the estimation of ε₁. The
latter, however, is fairly trivial. As it was discussed in
the body of the paper H₁ comes from the increase of the
interlayer Josephson junction energy due to non-zero
⟨v₀(r)⟩. Thus, we have match (A11) and the interlayer
Josephson junction free energy:

\[ F₁(r₀) = \frac{Φ₀²}{16π³λₖ²d_c} \left[1 - \cos(φₙ(r₀) - φ_{n-1}(r₀))\right]. \]  
(A12)

When sₙ = 0 both are zero. When |sₙ| ∼ a₀/2 the
Josephson junction energy F₁ is no longer zero but rather
equals to Φ₀²/16π³λₖ²d_c. It is because the cos term in
the above expression effectively vanishes due to fast oscil-
lation of the phase difference. This implies that

\[ \varepsilon₁ρ₀ \max_{s} \left\{ \sum_{|qₙ| = q₀} [1 - \cos(qₙsₙ)] \right\} \sim \frac{Φ₀²}{16π³λₖ²d_c}. \]  
(A13)

Since maximum of the sum in this formula is equal to 9
the coupling constant ε₁ is:

\[ \varepsilon₁ρ₀ \sim \frac{Φ₀²}{144π³λₖ²d_c} = 2.2 \times 10^{-4} \frac{Φ₀²}{λₖ²d_c}. \]  
(A14)

Equations (A11) and (A14) reproduce (45) up to a nu-
merical constant of order unity.

The derivation of (45) developed in this Appendix is
more general than that found in the body of the paper for
it does not rely on ξloop ≪ a₀ condition. The advantage
of the latter, however, it its ability to give the numerical
constant accurately. The fact that numerical coefficients
of (45) and (A14) are very close is pure luck. Obviously,
the estimates like (A13) have “order of magnitude” pre-
cision only. Numerical constant in (A14) should not be
taken too seriously.

We re-derived H₁₂ above. Operator H₁₁ may be re-
derived in the similar fashion.

APPENDIX B: SCALING DIMENSIONS

In this Appendix we determine the scaling dimension
(SD) of the operators (50) and (60). We start by solving
an auxiliary problem: we calculate SD of an exponent
exp(iq∥u). The easiest way to find SD is to evaluate the
correlation function:

\[ \langle e^{iq∥u(r)}e^{-iq∥u(r')}⟩ = e^{-\frac{1}{4}(|q∥u-u')²}. \]  
(B1)

The correlation function of the displacement field is
found:

\[ \langle (q∥[u - u']²) \rangle = \int_k \frac{T(q∥k)²}{C₆₆k^4} \left(1 - e^{i k(r - r')}\right). \]  
(B2)

The integral over k was evaluated in [16] (eq. 9.3.33)
and problem (9.9) of the latter reference):

\[ \int_k \frac{kjk∥}{k^4} (1 - e^{i k r}) = \frac{1}{4π} \delta_{ij} \ln(|r]/a₀) + \ldots, \]  
(B3)

where ellipsis stand for terms which remain bound at
|r| → ∞. Consequently:

\[ \langle (q∥[u - u']²) \rangle = \frac{Tq∥²}{4πC₆₆} \ln |r - r'|/a₀ + \ldots. \]  
(B4)

Therefore:

\[ \langle e^{iq∥u(r)}e^{-iq∥u(r')}⟩ \propto \left(\frac{a₀}{|r - r'|}\right)^{2T/T_u} . \]  
(B5)

From this equation we can read off SD of the exponent:

\[ e^{iq∥u} = T/T_u, \]  
(B6)

where Tₜ is given by (82). Thus, SD of (60) is equal to
2T/T_u since it is a product of two exponents. This SD
becomes equal to 2 when T = T_u. At T > T_u op-
terator (60) is irrelevant, otherwise it is relevant. As usual,
this mark the transition point from ordered to disordered
phase driven by (60). Eq. (82) establishes the same fact
derived by other means.

As for (50), its operator density is proportional to:

\[ \int \frac{d²R∥}{R∥} g(s(r))g(s(r + R∥)). \]  
(B7)
Here the factor $1/R_{\|}$ comes from the kernel $K_J$. The kernel $K_J$ is calculated in Appendix C.

SD of $g$ is $2T/T_u$ as it is obvious from (50), SD of the integration measure is -2, SD of $1/R_{\|}$ is 1. Totally, we have

$$\left[ \int \frac{d^2 R_{\|} g g}{R_{\|}} \right] = 4T/T_u - 1. \quad (B8)$$

Equating this SD with 2 we discover that for $T > 3T_u/4$ operator (50) is irrelevant. Therefore, is is permissible to neglect it for these temperatures. At the same time, operator (60) remains relevant up until the temperature has grown to $T_u$. Thus, the discussion of the Sect.V is valid for

$$\frac{3T_u}{4} < T < T_u. \quad (B9)$$

**APPENDIX C: KERNEL $K_J$**

In this Appendix we calculate the kernel $K_J(r, 0)$ required for eq. (50).

We must evaluate the integral:

$$K_J(r, 0) = \int_{k_1}^{\pi/dc} \int_{-\pi/dc}^{\pi/dc} dk_z \frac{e^{i k_z r}}{2\pi + \lambda^2 k_z^2 + \lambda_{ab}^2 k_z^2} = \frac{1}{\pi \lambda_{ab}} \int_{k_1}^{\pi/dc} e^{i k_z r} \frac{\pi \lambda_{ab}}{\sqrt{1 + \lambda^2 k_z^2}} \arctg \left( \frac{\pi \lambda_{ab}}{\lambda_{ab} k_z} \right). \quad (C1)$$

In most layered superconductors the screening length $\lambda_c$ is very big. Therefore, we are interested in distances $r \ll \lambda_c$. For such $r$ it is possible to neglect unity under the square root and write:

$$K_J \approx \frac{1}{\pi \lambda_{ab} \lambda_c} \int_{k_1}^{\pi/dc} e^{i k_z r} \arctg \left( \frac{\pi}{\lambda_{ab} k_z} \right). \quad (C2)$$

Next we perform the angular integration:

$$K_J \approx \frac{1}{2\pi^2 \lambda_{ab} \lambda_c} \int_0^{1/\alpha_0} J_0(k_\| r) \arctg \left( \frac{\pi}{\lambda_{ab} k_\|} \right) \, dk_\| = (C3)$$

$$\frac{1}{2\pi^2 \lambda_{ab} \lambda_c r} \int_0^{r/\alpha_0} J_0(x) \arctg \left( \frac{\pi r}{\lambda_{bc} x} \right) \, dx.$$ 

The upper limit in this integral is assumed to be large: $r \gg \alpha_0$.

If $r \ll \lambda_J$ then $\arctg(\pi r/\lambda_J x)$ could be approximated by $\pi r/\lambda_J x$ for almost whole integration interval. The contribution of the interval where this approximation is invalid could be evaluated separately:

$$K_J \approx \frac{1}{2\pi^2 \lambda_{ab} \lambda_c r} \left\{ \int_0^{r/\alpha_0} J_0(x) \arctg \left( \frac{\pi r}{\lambda_{bc} x} \right) \, dx \right\}. \quad (C4)$$

The first integral is $O(\pi r/\lambda_J)$. We neglect it. The second integral must be integrated by parts:

$$\int_0^{r/\alpha_0} J_0(x) \frac{\pi r}{\lambda_J x} \, dx \approx - \ln \frac{\pi r}{\lambda_J} + \int_0^{\infty} J_1(x) \ln x \, dx. \quad (C5)$$

Consequently:

$$K_J \approx \frac{1}{2\pi^2 \lambda_{ab} \lambda_c r} \left[ \ln \frac{\pi r}{\lambda_J} + \text{const.} \right] \quad (C6)$$

In the opposite limit $r \gg \lambda_J$ we define a function:

$$f(x) = \int_0^x dy J_0(y). \quad (C7)$$

and write:

$$K_J \approx \frac{1}{2\pi^2 \lambda_{ab} \lambda_c r} \int_0^{r/\alpha_0} \arctg \left( \frac{\pi r}{\lambda_J x} \right) \, df = \frac{1}{2\pi^2 \lambda_{ab} \lambda_c r} \left( \frac{\pi r}{\lambda_J} \right) \int_0^{r/\alpha_0} f(x) \, dx \quad (C8)$$

The integral may be transformed in the following manner:

$$\int_0^{r/\alpha_0} f(x) \, dx = \int_0^1 f(x) \, dx + \int_1^{r/\alpha_0} f(x) \, dx \approx \int_1^{r/\alpha_0} dx \quad (C9)$$

$$= \int_0^1 f(x) \, dx + \int_1^{r/\alpha_0} \frac{dx}{x^2 + (\pi r/\lambda_J)^2}.$$ 

In the last line the first integral is smaller than the second one. Indeed, the first one is $O((\lambda_J/r)^2)$ while the second is $O(\lambda_J/r)$. Therefore:

$$\int_0^{r/\alpha_0} \frac{f(x) \, dx}{x^2 + (\pi r/\lambda_J)^2} \approx \int_1^{r/\alpha_0} \frac{dx}{x^2 + (\pi r/\lambda_J)^2} \approx f(\infty) \frac{\lambda_J}{2r}. \quad (C11)$$

Note that:

$$f(\infty) = \int_0^{\infty} J_0(x) \, dx = 1. \quad (C12)$$

Thus, for $r \ll \lambda_J$ we have:

$$K_J \approx \frac{1}{4\pi \lambda_{ab} \lambda_{bc} r}. \quad (C13)$$
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