On the Stability of Compactified D=11 Supermembranes

I. Martín

Universidad Simón Bolívar, Departamento de Física
Caracas 89000, Venezuela.
e-mail: isbeliam@usb.ve

A. Restuccia

Universidad Simón Bolívar, Departamento de Física
Caracas 89000, Venezuela.
e-mail: arestu@usb.ve

R. Torrealba

Depto. de Matemáticas, Decanato de Ciencias,
Universidad Centro Occidental "Lisandro Alvarado", Apartado 400,
Barquisimeto, Venezuela.
e-mail: rtorre@compaq.ucla.edu.ve

Abstract

We prove $D = 11$ supermembrane theories wrapping around in an irreducible way over $S^1 \times S^1 \times M^9$ on the target manifold, have a hamiltonian with strict minima and without infinite dimensional valleys at the minima for the bosonic sector. The minima occur at monopole connections of an associated $U(1)$ bundle over topologically non trivial Riemann surfaces of arbitrary genus. Explicit expressions for the minimal connections in terms of membrane maps are presented. The minimal maps and corresponding connections satisfy the BPS condition with half SUSY.

June 1997

1
1 Introduction

Minkowski $D = 11$ supermembrane theories are thought to be unstable at the quantum level because of the presence of string like configurations that, with supersymmetry, renders the spectrum continuous. There the problem of the stability of quantum supermembranes was addressed by approximating the supermembrane action by a $SU(N)$ super Yang Mills theory as $N \to \infty$. Even when their result looks plausible, the whole proof depends on whether the $N \to \infty$ is applicable. From the point of view of D-brane theories, however, what it should be relevant is not the supermembrane with 11-dimensional Minkowskian space as target space but a compactified version, where at least one dimension is compactified to $S^1$. In recent works by [4], the $SU(N)$ super Yang Mills theory approach to the compactified supermembrane is taken but without a clear conclusion on the problem of stability. In [2] it is argued that winding over more than one compact dimension may remove the continuity of the spectrum. In [1] however the opposite argument is given based on the qualitative feature that classical membranes with winding can still have string like configurations of arbitrary length without increasing its area. But the problem remains with no solution since a generalization of the $SU(N)$ supersymmetric matrix model regularization to the compactified version was not found.

In this paper, we analyze the bosonic part of the Hamiltonian of the compactified supermembrane from a different point of view. Tackling the problem geometrically in terms of $U(1)$ connections over non trivial bundles defined on Riemann surfaces of arbitrary genus $g$.

We make use of an irreducibility condition on the winding of the supermembrane on the compactified directions in the target space. When this condition is not satisfied it is straightforward to show the existence of infinite dimensional valleys in the bosonic sector of the supermembrane Hamiltonian leading to a continuous spectrum as argued in [1].

We prove that area preserving diffeomorphisms on the supermembrane may be lifted to gauge transformations of particular $U(1)$ bundles. So that the problem of minima in the mass operator of the supermembrane may be addressed from the point of view of an equivalent Maxwell action. We conclude that the minima space is finite dimensional, with no minimal valleys that go to infinity, at least for the bosonic contribution of the Hamiltonian. Also, we construct the explicit solutions of these minima corresponding to
filaments of $U(1)$ monopoles.

In section 2, we briefly present the problem of the existence of infinite dimensional valleys at the minimum of the Hamiltonian. In section 3, we formulate the problem in term of connections constructed from membrane maps and introduce the irreducibility condition. Next, we explicitly construct the minimal connections in terms of monopole potentials. It is shown that they provide a minima of the Hamiltonian. In section 5, we show that the space of minimal maps reconstructed from minimal connections is finite dimensional.

2 The unstable Supermembrane

The hamiltonian of the $D = 11$ Supermembrane, in the light cone gauge, takes the form

$$H = \frac{P_{0a}^2}{2P_0^+} + \frac{M^2}{2P_0^+}$$

where $P_{0a}$ are the transverse momenta of the center of mass and $M$ is the Supermembrane mass operator.

The bosonic contribution to the mass operator is

$$M^2 = \int_\Sigma d^2\sigma \sqrt{g} [P_a^2 + V(x)]$$

where $P_a$ are the transverse momenta of the center of mass and $M$ is the Supermembrane mass operator.

The bosonic contribution to the mass operator is

$$M^2 = \int_\Sigma d^2\sigma \sqrt{g} [P_a^2 + V(x)]$$

where $\sigma^r, r=1,2$, denote local coordinates on the 2-dimensional world volume $\Sigma, X^a, P_a a = 1, \ldots, 9$ are the transverse coordinates and conjugate momenta of the supermembrane in the light cone coordinates.

The world volume $\Sigma$ is usually taken to be a Riemann surface of genus $g$. The analysis in $\Sigma$ is based on an expansion of coordinates and momenta in terms of a complete set of real orthonormal functions $Y^A(\sigma), A = 1, \ldots, \infty$. This set represent the generators of a $SU(N)$ gauge theory in the limit when $N$ tends to infinity. In particular, the coordinates are expressed as

$$X^a(\sigma) = \sum_A X_A^a Y^A(\sigma)$$
The mass operator then takes the form

\[ M^2 = (P^A_a) + (F_{ABC}X^B_a X^B_b)^2 - if_{ABC}^{\theta^A_a X^B_a \theta^C} (2.4) \]

where \( \theta^A \) are the fermionic coordinates, \( f_{ABC} \) are the structure constants of the \( SU(N) \) gauge theory that approximates, in the limit \( N \to \infty \), to the invariant subgroup of the area preserving transformations of the supermembrane. In fact, the group of area preserving diffeomorphisms for spherical membranes and for toroidal membranes can be regarded as the limit of \( SU(N), N \to \infty \). The approach of [1] is to regularize the theory by replacing this infinite dimensional group by \( SU(N) \).

An important property of (2.2) is that it vanishes if the coordinates \( X^a_A \) take values in some abelian subalgebra. The classical theory is then unstable but quantum mechanical effects turn it stable. This is in fact the case for the bosonic membrane. It is classically unstable because the potential has zero valleys along \( X_a = 0 \) for all but one coordinate that may have an arbitrary value. However, the quantum mechanical oscillators perpendicular to the valley directions give rise to a zero point energy, inducing an effective potential barrier which confines the wave function. When supersymmetry is introduced, the situation changes because SUSY harmonic oscillators have no zero point energy so that the confining potential vanishes. Explicit calculations [1] show indeed that the wave function can go to infinity along the valleys of zero classical energy, so that the spectrum of the supermembrane is continuous.

The main ingredients leading to an unstable problem are then the existence of valleys with zero energy extending to infinity for any arbitrary map \( X(\sigma^1, \sigma^2) \) with all other coordinates \( X^a \) zero in (2.2), and SUSY which eliminates the zero point energy of the bosonic oscillators perpendicular to the valley directions. In the next section we will consider the problem of stability for compactified supermembranes.

### 3 Minima of the Potential

On what follows we will address the problem of instability from another point of view, taking advantage of the physics of \( U(1) \) bundles. We first consider the bosonic contribution to the supermembrane potential. To analyze the minima of the potential (2.2), we rewrite it putting the momenta equal to
zero since it will be satisfied by all minimal configurations, in the following way

\[ < V(x) >_\sigma \equiv \int_\Sigma d^2\sigma \sqrt{g} V(x) = \int_\sigma \ast F_{ab} F_{ab} \]  

(3.1)

where

\[ F_{ab} \equiv d(X_a dX_b), \quad a, b = 1, \ldots, 9. \]  

(3.2)

d being the exterior derivative on the world volume \( \sigma \), which is taken to be a Riemann surface of arbitrary genus \( g \). \( \ast F_{ab} \) denotes the Hodge dual of the 2-form \( F_{ab} \). The latter may be regarded as different ‘Maxwell fields’ for every pair of numbers \( ab \).

We assume the supermembrane wraps up in an irreducible way over the torus \( S^1 \times S^1 \) in the target space. Let \( X(\sigma) \) and \( Y(\sigma) \) denote the compactified coordinates on the target manifold. \( X(\sigma) \) defines a map from the world volume \( \Sigma \) to \( S^1 \). \( dX \) may then always be expressed as

\[ dX = -ig^{-1}dg \]  

(3.3)

where

\[ g = \exp(i\phi) \]  

(3.4)
\( \phi \) being an angular coordinate over \( S^1 \). \( dX \) satisfies the following properties for a 1-form \( L \),

\[ dL = 0 \]  

(3.5)

\[ \oint_{C_i} L = 2\pi n^i \]  

(3.6)

where \( C_i \) denotes a basis of the integer homology of dimension one over the worldvolume. The converse is also valid [4], given a globally defined 1-form \( L \) over \( \Sigma \) satisfying (3.5) and (3.6) there exists a map \( X : \Sigma \rightarrow S^1 \) for which \( L = dX \).

We will say that the supermembrane wraps up in a nontrivial way when at least one of the \( n^i \) is different from zero.

We also say that the supermembrane wraps up in an irreducible way over \( S^1 \times S^1 \) when

\[ \frac{1}{2\pi} \int_\Sigma dX \wedge dY \neq 0 \]  

(3.7)
If the supermembrane wraps up in an irreducible way it does it in a nontrivial way over each of the $S^1$ in $S^1 \times S^1$. Moreover,

$$\frac{1}{2\pi} \int_{\Sigma} dX \wedge dY = 2\pi N$$

(3.8)

where $N$ is also an integer obtained as the product of the $n^i$, associated to conjugate pairs on a canonical basis over $\Sigma$.

We look now for the stationary points of (3.1) over the space of maps defining supermembranes with irreducible wrapping over $S^1 \times S^1$. It is straightforward to see in this case that the minimal configurations occur when all but $X, Y$ maps are zero. Associated to this space we may introduce an $U(1)$ principle bundle. We proceed by noting that

$$F = \frac{1}{2\pi} dX \wedge dY$$

(3.9)

are closed 2-forms globally defined over $\Sigma$ satisfying (3.8). By Weil’s Theorem [6], [5], there exists a $U(1)$ principle bundle and a connection over it such that its pull back by sections over $\Sigma$ are 1-form connections with curvatures given by (3.9). The stationary points of (3.1) with respect to the space of 1-form connections satisfy

$$d^* F = 0.$$  

(3.10)

now, since $*F$ is a 0-form,

$$*F = constant.$$  

(3.11)

using (3.8) for the compactified directions, we get

$$*F = \frac{2\pi N}{Vol \Sigma} = N.$$  

(3.12)

Although there may be additionally stationary points when taking variations with respect to the maps, the only relevant ones are determined by (3.12). In fact, we will now show that these are the only strict minima of the potential (3.1).

Let $A_0$ be a 1-form connection satisfying (3.12) and $A_1$ another 1-form connection on the same $U_1$ principle bundle characterized by $N$, then the
curvature of any connection $A$ on it must satisfy

\begin{align}
    dF &= 0 \quad (3.13) \\
    \int_{\Sigma} F &= 2\pi N \quad (3.14)
\end{align}

Consider now that the following connection is also a 1-form connection on the same principle bundle

\[(1 - \lambda)A_0 + \lambda A_1, \quad (3.15)\]

with $\lambda$ an arbitrary real parameter.

We then have

\[
    \int_{\Sigma} \ast F_{\lambda} \wedge \ast F_{\lambda} = (1 - \lambda^2)2\pi N^2 + \lambda^2 \int_{\Sigma} \ast F(A_1) \wedge F(A_1), \quad (3.16)
\]

where $F_{\lambda}$ is the curvature of (3.15).

From (3.16) we obtain

\[
    \int_{\Sigma} \ast F_{\lambda} \wedge \ast F_{\lambda} = 2\pi N^2 + \lambda^2 \int_{\Sigma} [\ast F(A_1) \wedge F(A_1) - \ast F(A_0) \wedge F(A_0)], \quad (3.17)
\]

since the left hand member is positive and (3.17) is valid for all $\lambda$ we get

\[
    \int_{\Sigma} \ast F(A_1) \wedge F(A_1) \geq \int_{\Sigma} \ast F(A_0) \wedge F(A_0) \quad (3.18)
\]

Moreover, the equality is obtained if and only if

\[A_1 = A_0 + d\Lambda \quad (3.19)\]

where $\Lambda$ is a closed 1-form.

The stationary points satisfying (3.12) are then the only minima of the potential, modulo gauge transformations in the $U(1)$ bundle. The question of whether there exist maps $X$ and $Y$ for which a minimal connection (3.12) can be constructed as

\[A_0 = XdY. \quad (3.20)\]

still remains to be answered.
Furthermore, we have to determine how many maps yield a minimal connection. It may happen that there are infinite independent maps, all of them giving rise to the same minimal connection, that is the old problem of valleys extending to infinite. Before discussing these points we will completely characterize the minimal connections.

4 Minimal Connections: Monopoles over Riemann Surfaces

We will show in this section that the minimal connections are the Dirac-Hopf monopole connections on $S^2$ and its generalization to topologically non-trivial Riemann surfaces found in [7][8]. We briefly discuss here their construction, for a more extensive analysis see [7][8].

The explicit expression of the monopole connections is obtained in terms of the abelian differential $d\Phi$ of the third kind over the compact Riemann surface $\Sigma$ of genus $g$. $d\Phi$ is a meromorphic 1-form with poles of residue +1 and -1 at points $a$ and $b$, with real normalization. $\Phi$ is the abelian integral, its real part $G(z, \bar{z}, a, b, t)$ is a harmonic univalent function over $\Sigma$ with logarithmic behavior around $a$ and $b$

$$
\ln\left(\frac{1}{|z + a|}\right) + \text{ regular terms }, \\
\ln |z - b| + \text{ regular terms },
$$

(4.1)

It is a conformally invariant geometrical object. $z$ denotes the local coordinate over $\Sigma$ and $t$ the set of $3g - 3$ parameters describing the moduli space of Riemann surfaces. We are considering maps from $\Sigma \rightarrow S^1 \times S^1 \times M^7$ for a given $\Sigma$, so the parameters $t$ are kept fixed. They show however that the construction of minimal connections is a conformally invariant one.

Let $a_i, i = 1, ..., m$ be $m$ points over the compact Riemann surface. We associate to them integer weights $\alpha_i, i = 1, ..., m$, satisfying

$$
\sum_{i=1}^{m} \alpha_i = 0
$$

(4.2)
We define
\[ \phi = \sum_{i=1}^{m} \alpha_i G(z, \bar{z}, a_i, b, t). \] (4.3)

and have
\[ \phi \to -\infty \text{ at } a_i \text{ with negative weights} \]
\[ \phi \to +\infty \text{ at } a_i \text{ with positive weights}. \]

\( \alpha_i \) are integers in order to have univalent transition functions over the non-trivial fiber bundle that we consider.

We denote \( \tilde{\Phi} \) the abelian integral with real part \( \phi \). Its imaginary part \( \varphi \) is also harmonic but multivalued over \( \Sigma \),
\[ \tilde{\Phi} = \phi + i\varphi. \] (4.4)

Let us consider the curve \( C \) over \( \Sigma \) defined by
\[ \phi = \text{constant}. \] (4.5)

It is a closed curve homologous to zero. It divides the Riemann surface into two regions \( U_+ \) and \( U_- \), where \( U_+ \) contains all the points \( a_i \) with negative weights and \( U_- \) the ones with positive weights.

We define over \( U_+ \) and \( U_- \) the connection 1-forms
\[ A_+ = \frac{1}{2} (1 + \tanh(\phi)) d\varphi \]
\[ A_- = \frac{1}{2} (-1 + \tanh(\phi)) d\varphi \] (4.6)

respectively. \( A_+ \) is regular in \( U_+ \) and \( A_- \) in \( U_- \). In the overlapping \( U_+ \cap U_- \) we have
\[ A_+ = A_- + d\varphi \] (4.7)

\( g = \exp(i\varphi) \) defines the transition function on the overlapping \( U_+ \cap U_- \), and because of the integer weights it is univalued over \( U_+ \cap U_- \).

The base manifold \( \Sigma \), the transition function \( g \) and the structure group \( U(1) \) have a unique class of equivalent \( U(1) \) principle bundles over \( \Sigma \) associated to them. (4.6) defines a 1-form connection over \( \Sigma \) with curvature
The $U(1)$ principle bundles are classified by the sum of the positive integer weights $\alpha_i$

$$N = \sum \alpha_i^+, \alpha_i^+ > 0.$$  \hspace{1cm} (4.9)

which is the only integer determining the number of times $\varphi$ wraps around $C$. All the bundles with the same $N$ are equivalent. (4.8) satisfies (3.14). Moreover they also satisfy (3.12). In fact, since $\varphi$ and $\phi$ are harmonic over $\Sigma$, the metric is

$$d^2 s = \frac{1}{\cosh^2 \phi} (d^2 \varphi + d^2 \phi),$$ \hspace{1cm} (4.10)

and then (3.12) follows directly.

We have explicitly constructed all the $U(1)$ principle bundles over compact Riemann surfaces $\Sigma$, and 1-form connections (4.6) which describe strings of monopoles, generalizing the Dirac monopole construction. The different connections obtained by choosing different sets of $a_i$, are all regular over $\Sigma$ and are different descriptions (using different metrics over $\Sigma$) of the same physical string of monopoles. All metrics are on the same conformal class, but with different number of points $a_i$, at which they become zero. This picture is a generalization of the construction of the Dirac-Hopf connection over $S^2$ in terms of the metric $d^2 s = d^2 \theta + \text{sen}^2 \theta d^2 \varphi$ whose determinant becomes zero at the north and south pole. In the particular case of $\Sigma$ being $S^2$ with only one $a$ with weight $N$, (4.10) reduces to the standard $F = N \text{sen} \theta d\theta \wedge d\varphi$.

## 5 Reconstruction of Membrane maps

Given two minimal connections on the same principle bundle they differ by a closed 1-form, locally we have

$$A_1 = A_0 + d\Lambda.$$ \hspace{1cm} (5.1)

We will consider now the residual area preserving diffeomorphisms left after the light cone gauge fixing, and show that we can locally generate
any exact \( d\Lambda \) by such a transformation, i.e the residual symmetry on the membrane induces, at least locally, an U(1) gauge transformation on the space of connections.

The area preserving diffeomorphisms are generated by the first class constraints.

\[
d(P_a dX_a) = 0 \tag{5.2}
\]
\[
\oint_{C_i} P_a dX_a = 0 \tag{5.3}
\]

where \( P_a \) and \( X_a \) are the transverse components of the membrane maps in the light cone coordinates. \( C_i \) is a basis of homology over the Riemann surface \( \Sigma \). In the right hand member of (5.3) we may add \( 2\pi n \) if we allow \( X^- \) to take values on \( S(1) \), in this case the homology must be integral. The infinitesimal gauge transformations generated by (5.2) and (5.3) may be expressed in terms of the Poisson bracket

\[
\delta F = \{ F, \langle d\xi \wedge P dX \rangle_\Sigma \} . \tag{5.4}
\]

where \( \xi \) is the infinitesimal parameter of the transformation. \( d\xi \) is a closed 1-form, so \( \xi \) may be multivalued over \( \Sigma \). The point is that we may reexpress

\[
\langle d\xi \wedge P dX \rangle_\Sigma = \langle -\xi dP \wedge dX \rangle_\Sigma + \langle d(\xi P dX) \rangle_\Sigma , \tag{5.5}
\]

the first term in the right hand side is the gauge transformation generated by (5.2), while the second term may be rewritten in terms of the transition of \( \xi \) only, and represents then the gauge transformation generated by the global constraint (5.3). We will restrict \( d\xi \) to have integer periods in order to preserve the bundle structure we have introduced in previous sections. We will only use the gauge transformations generated by the local constraint.

We define over \( U^+ \) and \( U^- \) on the membrane, \( X_0^+ \) and \( Y_0^- \), a pair of maps into the compactified directions of the target manifold, with

\[
X_0^+ = \frac{1}{2}(\tanh(\phi) + 1)
\]
\[
X_0^- = \frac{1}{2}(\tanh(\phi) - 1) \tag{5.6}
\]

respectively, and
\[
dY_0 = d\varphi, \quad (5.7)
\]
where \( \phi \) and \( \varphi \) are the two harmonic functions introduced in section 4. \( X_0 \) may then be expressed as a sum of a harmonic function
\[
\frac{1}{2}(\phi \pm 1) \quad (5.8)
\]
with a jump at \( U^+ \cap U^- \), giving rise to a harmonic differential, plus functions yielding an exact 1-form. We will call the pair \((X_0, Y_0)\) a minimal map. The minimal connection \( \hat{A} \) may then be expressed in terms of minimal maps as
\[
\hat{A} = X_0 dY_0. \quad (5.9)
\]

Let us now consider the change of \( \hat{A} \) under area preserving diffeomorphisms. We obtain
\[
\delta \hat{A}_r = \partial_r (-\epsilon^{st} \partial_t \xi \hat{A}_s - \frac{1}{2} \xi^* \hat{F}), \quad (5.10)
\]
where \( \xi(\sigma^1, \sigma^2) \) is the infinitesimal parameter of the transformation. (5.8) is only valid for a minimal connection \( \hat{A} \), since \( \hat{F} \) is then constant and may be included in the total derivative. Given \( \xi \), a gauge transformation, induced on the connection space, is defined by
\[
\Lambda^+ = -\epsilon^{st} \partial_t \xi \hat{A}_s^+ - \frac{1}{2} \xi^* \hat{F}, \quad (5.11)
\]
\[
\Lambda^- = -\epsilon^{st} \partial_t \xi \hat{A}_s^- - \frac{1}{2} \xi^* \hat{F} \quad (5.12)
\]
at \( U^+ \) and \( U^- \) respectively.

Conversely, given \( \Lambda^+ \) and \( \Lambda^- \) defined on \( U^+ \) and \( U^- \) respectively, there exists a differentiable function \( \xi \) satisfying (5.11) and (5.12). In fact, let us denote \( C \) a closed curve over \( U^+ \cap U^- \) dividing \( \Sigma \). This curve may be taken to be \( \phi = 0 \) without a lost of generality. On \( C \), we take the boundary condition
\[
\xi \mid_C = -\frac{(\Lambda^+ + \Lambda^-)}{\hat{F}}, \quad (5.13)
\]
is different from zero because of the irreducibility condition. We then solve (5.11) with boundary condition (5.13) on $U^+$ and (5.12) with boundary condition (5.13) on $U^-$. The solution $\xi(\sigma^1, \sigma^2)$ is then differentiable with continuous derivatives on $C$.

We thus conclude that the space of all minimal connections may be generated by considering the minimal connection (5.9) in terms of the minimal maps plus a representative of each real cohomology class of 1-forms over $\Sigma$.

We will now show that the space of maps which give rise to minimal connections is finite dimensional.

The general map $(X, Y)$ leading to a minimal connection $A$ may be written in terms of a minimal map $(\hat{X}, \hat{Y})$ as

$$X = \hat{X} + \tilde{x},$$
$$Y = \hat{Y} + \tilde{y},$$

and the corresponding connection as

$$A = \hat{A} + (\hat{X}d\tilde{y} + \tilde{x}d\hat{Y} + \tilde{x}d\tilde{y})$$

For $A$ to be minimal, the parenthesis in (5.15) must be a closed 1-form. Hence the pair $(\tilde{x}, \tilde{y})$ is restricted by that condition. Let us assume that the term in the parenthesis in (5.15) is a closed 1-form. $\tilde{x}$ must then be an univalued map. If we keep $\tilde{y}$ fixed and consider $\tilde{x}_1$ a map giving rise to closed 1-form $(\hat{X}d\tilde{y} + \tilde{x}_1d\hat{Y} + \tilde{x}_1d\tilde{y})$ in the same cohomology class then

$$(\tilde{x} - \tilde{x}_1)(d\hat{Y} + d\tilde{y}),$$

must be exact, and hence it may be annihilated by an area preserving diffeomorphism as shown previously. We may then ask how many $\tilde{x}_1$ are solutions of

$$(\tilde{x} - \tilde{x}_1)(d\hat{Y} + d\tilde{y}) = 0.$$  

The only solution not violating the irreducibility condition is $\tilde{x} = \tilde{x}_1$. The same argument may be performed leaving $\tilde{x}$ fixed and varying $\tilde{y}$.

We then come to the conclusion that the space of maps giving rise to a minimal connection is a finite dimensional space related to the space of cohomology classes of 1-forms. This means that there are no valleys at the minima since the latter correspond to an infinite dimensional space. We
have not exhausted the residual gauge freedom related to the global constraint (5.3), we have only used the gauge transformation generated by the local constraint. The global constraint with the restriction we have assumed generates large gauge transformations, it transforms connections by adding elements of the integer cohomology class. Hence the space of minima may be further reduced to $H^1(\Sigma, R)/H^1(\Sigma, Z)$.

6 Conclusions

We showed that the Hamiltonian of the $D = 11$ membrane satisfying the irreducibility condition, introduced in section 3, has isolated minima. The space of these minima are classified by integer numbers defined by the irreducibility condition. For each $N$ there is an associated $U(1)$ bundle over the 2-dimensional worldvolume $\Sigma$ of arbitrary genus, and the minima are the monopole connections over Riemann surfaces $\mathbb{R}$ . We explicitly constructed the minimal connections in terms of the membrane maps from $\Sigma \mapsto S^1 \times S^1 \times M^7$.

The minimal maps are strictly defined over a punctured compact Riemann surface, however the minimal connections are regular over the surface. The Hamiltonian density is consequently regular on it as well. We also showed, using area preserving diffeomorphisms, that there are no infinite dimensional valleys at the minima. In distinction to the Hamiltonian of the $D = 11$ supermembrane with target space $M^{11}$ or $S^1 \times M^{10}$ where there are infinite dimensional valleys at the minima giving rise to a continuous spectrum.

Outside the minima of the potential, it is possible to construct explicit configurations with $C^\infty$ maps satisfying the global irreducibility condition, and with a non uniform distribution of the winding density. Allowing for open neighborhoods on the membrane where there is no winding and where supersymmetric valley configurations could arise, i.e. solutions with thick hairs that go to infinity. The existence of these maps would then give a continuous spectrum as first argued in $[4]$.

An interesting point to analyze now would be to completely characterize the potential of the irreducible membranes and supermembranes in a general way, since our study has been concerned only with the behavior of the potential near the minima. It will then be possible to answer the question of the continuity or discretness of the spectrum for the case of perhaps ‘non degen-
erate irreducible supermembranes since our analysis shows that the global irreducibility condition is not enough to forbid the existence of valleys outside the minima, but it shows anyhow that it is a good condition to prohibit the valleys at the minima. Also, it points to the requirement of a non degeneracy condition on the 2-form integrated in the irreducibility condition. This 2-form being closed and non degenerate would define a symplectic geometry on the space of maps with winding.

We would like to relate in general the structure of minimal maps and connections to the central charge analysis of the SUSY algebra recently discussed in [4]. In fact, it is easy to see that our global irreducibility condition is directly related to the central charges of the SUSY algebra. Also, the minimal maps and corresponding minimal connections satisfy the BPS condition with $\frac{1}{2}$ SUSY as in [4] proving that they are supersymmetric BPS states.

Acknowledgements

I. Martin likes to thank B. de Wit and J. Plefka for helpful conversations.

References

[1] B. de Wit, M. Lüscher and H. Nicolai, Nucl. Phys. B320 (1989) 135

[2] J.G. Russo, Nucl. Phys. B492 (1997) 205, R. Kallosh, hep-th/9612004

[3] J. Polchinski, S. Chaudhuri and C.V. Johnson, hep-th/9602052, J. Polchinski, hep-th/9611050.

[4] B. de Wit, K. Peeters and J. Plefka, hep-th/9705225.

[5] M. Caicedo, I. Martin and A. Restuccia, hep-th/9701010.

[6] A. Weil, Variétés Kaehleriennes, Hermann (1957).

[7] I. Martin and A. Restuccia, Lett. Math. Phys. 39 (4) (1997).

[8] F. Ferrari, hep-th/9310024.

[9] K. Ezawa, Y. Matsuo and K. Murakami, hep-th/9706002.