SHARPNESS OF THE PERCOLATION PHASE TRANSITION FOR THE CONTACT PROCESS ON $\mathbb{Z}^d$

THOMAS BEEKENKAMP

Abstract. We study percolation properties of the upper invariant measure of the contact process on $\mathbb{Z}^d$. Our main result is a sharp percolation phase transition with exponentially small clusters throughout the subcritical regime and a mean-field lower bound for the infinite cluster density in the supercritical regime. This generalizes and simplifies an earlier result of Van den Berg [2], who proved a sharp percolation phase transition on $\mathbb{Z}^2$. Our proof relies on the OSSS inequality for Boolean functions and is inspired by a series of papers by Duminil-Copin, Raoufi and Tassion [7][8][9] in which they prove similar sharpness results for a variety of models.

1. Introduction and Main Result

The contact process is a stochastic process that can be used to model the infection of the vertices on a graph. Each vertex has a state associated with it that is equal to either 0 or 1. The vertices in state 1 are thought of as infected, while the vertices in state 0 are thought of as healthy. The dynamics of the system are as follows: the state of a vertex changes from 1 to 0 with rate 1, independently of the rest of the system. On the other hand, if a vertex is in state 0 its state changes to 1 with rate $\lambda$ times the number of neighbours that are infected, where $\lambda > 0$ is the parameter of the model. A common choice for the graph is the hypercubic lattice $\mathbb{Z}^d$, with $d \geq 2$, which is therefore also the graph we will work with. The contact process is a well studied stochastic process; see for example Liggett [13] for a precise definition, theoretical background and classical results of the contact process. We will only give an informal description of the process here.

Denote by $\sigma(t) = (\sigma_v(t))_{v \in \mathbb{Z}^d} \subset \{0, 1\}^{\mathbb{Z}^d}$ the state of the process at time $t \in [0, \infty)$ (later we will take $t \in (-\infty, 0]$ for technical reasons). Denote by $\mu_t$ the associated measure at time $t$. This measure depends on the starting configuration; a natural choice is to start with all vertices in state 1: $\sigma_v(0) = 1$ for all $v \in \mathbb{Z}^d$. Then the measures $\mu_t$ converge monotonically and we define the upper invariant measure $\bar{\nu}_\lambda := \lim_{t \to \infty} \mu_t$. One of the classical results of the contact process is the existence of a critical value of the parameter $\lambda_c$, below which the infection dies out and above which the upper invariant measure is nontrivial:

$$\lambda_c := \sup \{ \lambda : \bar{\nu}_\lambda (\sigma_v = 0 \text{ for all } v \in \mathbb{Z}^d) = 1 \}.$$

Above $\lambda_c$ the process survives and we are interested in what such a surviving configuration looks like. In particular we are interested in the size of the clusters. For that purpose we say that $u$ is connected to $v$, denoted $u \leftrightarrow v$, when there exists a path from $u$ to $v$ using only vertices in $\{w : \sigma_w = 1\}$. Similarly, for $A \subset \mathbb{Z}^d$, we say that $u \leftrightarrow A$ when there exists a $v \in A$ such that $u \leftrightarrow v$. Let $C$ denote the cluster of 0, i.e., the set of vertices that are connected to 0. We say that $0 \leftrightarrow \infty$ whenever $|C| = \infty$. We can now define a new critical value of the parameter $\lambda_p$ below which all clusters are finite and above which there exists an infinite cluster:

$$\lambda_p := \sup \{ \lambda : \bar{\nu}_\lambda (0 \leftrightarrow \infty) = 0 \}.$$
We call this the critical value for percolation. It has been shown by Liggett and Steif [14] that if \( \lambda \) is large enough the occupied vertices percolate, that is, there exists an infinite cluster almost surely. This implies \( \lambda_p < \infty \). Furthermore, it is clear from the definition that \( \lambda_c \leq \lambda_p \). Moreover, it is strongly believed that \( \lambda_c < \lambda_p \), but no proof is known.

The aim of this paper is to show that the percolation phase transition is sharp. This means that below \( \lambda_p \) the clusters are exponentially small. Furthermore, as a byproduct of the proof, we obtain a mean-field lower bound on \( \bar{\nu}_\lambda(0 \leftrightarrow \infty) \) in the supercritical regime. Our main result is the following:

**Theorem 1.** Consider the contact process on \( \mathbb{Z}^d \) with parameter \( \lambda \).

a) Suppose \( \lambda < \lambda_p \). Then there exists a constant \( c > 0 \) such that for all \( n \in \mathbb{N} \)

\[
\bar{\nu}_\lambda(|C| \geq n) \leq \exp(-cn). \tag{1}
\]

b) Suppose \( \lambda \in (\lambda_p, \lambda_p + 1) \). Then there exists a constant \( c > 0 \), independent of \( \lambda \), such that

\[
\bar{\nu}_\lambda(0 \leftrightarrow \infty) \geq c(\lambda - \lambda_p). \tag{2}
\]

**Remark.** The mean field lower bound (2) is only of interest close to \( \lambda_p \). Therefore, the choice of the interval \((\lambda_p, \lambda_p + 1)\) is arbitrary. However, we need to assume that \( \lambda \) is bounded, since (2) cannot hold for all \( \lambda > \lambda_p \). Indeed, Theorem 1(b) holds for any \( \lambda \) in a bounded interval above \( \lambda_p \).

A version of Theorem 1(a) was proven by Van den Berg [2] for the two-dimensional case \( \mathbb{Z}^2 \). His proof uses techniques from Bollobás and Riordan for Voronoi percolation [5]. These techniques seem to be broadly applicable, having also been applied to the Johnson-Mehl Model by the same authors [6] and to confetti percolation by Hirsch [10]. However the techniques rely heavily on crossings of rectangles, which limits their application to two-dimensional models.

Our proof follows more closely the line of Duminil-Copin, Raoufi and Tassion [7, 8, 9]. In this recent series of papers, the authors apply the OSSS inequality to prove sharp phase transitions for a variety of models: the Potts model, the random-cluster model (in particular Bernoulli percolation), the Boolean model and Voronoi percolation. In particular the Voronoi percolation case is most relevant to us, since from a technical perspective there are some similarities with the contact process.

With this approach we can prove the sharpness of the percolation phase transition for the contact process in all dimensions. Furthermore, it significantly shortens and simplifies the existing proof for \( \mathbb{Z}^2 \) by Van den Berg. Our contribution also underlines that these new techniques based on the OSSS inequality are rather robust, and can be applied, in a suitable variant, to dependent models.

The OSSS inequality was introduced by O’Donnel, Saks, Schramm and Servedio [15] in the context of Boolean functions. Let \( f \) be a Boolean function from a product space \( \Pi^N \) to \( \{0, 1\} \) and \( T \) a decision tree that reveals the variables of \( \omega \in \Pi^N \) one by one until \( f(\omega) \) is determined. Then the OSSS inequality relates the variance of \( f \) to the influence of every single variable and the probability that that particular variable is revealed by \( T \). The general setup of this inequality allows it to be applied in many different settings. The strategy is to go from the OSSS inequality to a differential inequality by using the properties of the particular model at hand. This differential inequality then implies the sharpness result.

In order to do this for the contact process we need to choose a suitable product space \( \Pi^N \) and a suitable decision tree on this space that determines \( f := 1\{0 \leftrightarrow \partial \Lambda_n\} \), where \( \partial \Lambda_n \) is the boundary of the ball of radius \( n \), such that we can properly bound the revealment. Then we need to give a good bound on the influence of a single variable, relating it to the derivative of \( \bar{\nu}_\lambda(0 \leftrightarrow \partial \Lambda_n) \) with respect to \( \lambda \). This requires us to define a usable notion of a pivotal event for
the contact process as well as to give a Russo-type formula. In this way we obtain a differential inequality, however it is weaker then the ones obtained in the papers by Duminil-Copin, Raoufi and Tassion; from the differential inequality we can only obtain stretched exponential decay. By using a renormalization argument we can improve this to proper exponential decay to obtain the final result.

Further research is desirable to see if a similar sharp phase transition occurs in more general contact processes. In particular, three-state contact processes are of interest since they have applications in agricultural sciences. In this context, a vertex is interpreted as a patch of soil and its state can be either ‘vegetated’, ‘vacant but fertile’ or ‘vacant and infertile’. This process can then be used to model the spread of vegetation, in particular for arid landscapes. Three-state contact processes with this application in mind have been studied in a paper by Van den Berg, Björnberg and Heydenreich [3]. Our proof for the sharpness of the percolation phase transition for the contact process significantly simplifies the proof of Van den Berg [2], which in turn can contribute to a better understanding of phase transitions for other spatially dependent models such as these three-state contact processes.

2. Preliminaries

Before we start with the proof, we introduce some tools and results from the literature. Throughout the paper, we assume that $\lambda > \lambda_c$, since for $\lambda \leq \lambda_c$ the contact process dies out, see [4], so that $|C| = 0 \; \bar{\nu}_\lambda$-almost surely and Theorem 1 is trivially true.

For $v, w \in \mathbb{Z}^d$, the distance $d(v, w)$ is the length of the shortest path from $v$ to $w$ in $\mathbb{Z}^d$. We define $\Lambda_n^v := \{w \in \mathbb{Z}^d : d(v, w) \leq n\}$ and $\partial \Lambda_n^v := \{w \in \mathbb{Z}^d : d(v, w) = [n]\}$. Furthermore, for $v = 0$, we omit part of the notation: $\Lambda_n := \Lambda_n^0$.

We start with the the graphical representation for the contact process, a particular construction of the contact process which is useful in the analysis of the process. We then give a truncated version of this process which is appropriately close to the original version and allows to deal with finite space and time horizons. This is similar to the truncation procedure by Van den Berg [2].

2.1. The Graphical Representation. We briefly introduce the graphical representation, which is a core ingredient in our proof. For an elaborate description we refer to Chapter 1.1 of Liggett [13]. For every vertex $v \in \mathbb{Z}^d$, we create a time axis $(-\infty, 0]$, so that we obtain the space-time $\mathbb{Z}^d \times (-\infty, 0]$. Furthermore, on every axis $\{v\} \times (-\infty, 0]$, we consider the following Poisson point processes: with rate 1 there appears a point marked as a star, which corresponds to the healing of $v$, i.e., a change of $\sigma_v$ from 1 to 0. Furthermore, for any neighbour $w$ of $v$, we have an independent Poisson point process with rate $\lambda$ for which the points are marked with an arrow that points to $w$, which corresponds to the infection of $w$ by $v$. Note that the total rate is $2d\lambda + 1$.

We combine the above introduced Poisson point process into one marked Poisson point process on $\mathbb{Z}^d \times (-\infty, 0]$ in the following way. Denote the set of marks by $\mathcal{M} := \{\ast\} \cup \mathcal{A}$, where $\mathcal{A} := \{\pm e_1, \pm e_2, \ldots, \pm e_d\}$, and $e_i$ is the $i$th unit vector, so that the mark $\pm e_i$ corresponds to an infection of $v \pm e_i$ by $v$. The combined marked Poisson point process has rate $2d\lambda + 1$, however for our purposes it is convenient to rescale time so that the combined process has rate 1. Since we are only interested in the upper invariant measure, we are allowed to perform this rescaling.

Let $\eta$ be a Poisson point process on $\mathbb{Z}^d \times (-\infty, 0]$ with intensity 1 on every time axis. Denote the support of $\eta$ by $[\eta]$. In order to obtain a coupling between contact processes with different values of $\lambda$, we associate to every point $x \in [\eta]$ two random variables $U_x$ and $\rho_x$, which are independent.
of each other and of \( \eta \). Furthermore, \( U_x \) is uniformly distributed between 0 and 1, and \( \rho_x \in \mathcal{A} \) uniformly at random. From these random variables, we obtain the mark \( m_x \) of \( x \) as follows:

\[
m_x := \begin{cases} * & \text{if } U_x \leq \frac{x - \lambda}{2|\lambda| + 1}, \\ \rho_x & \text{otherwise}. \end{cases}
\]  

(3)

Denote the configuration of the points together with the marks by \( \omega \) when for all \( \nu \) with respect to the probability measure of the coupling, where the marginal distributions of \( \nu \) and \( \bar{\nu}_\lambda \) are given by the measures \( \nu \) and \( \bar{\nu}_\lambda \) respectively. The same domination also holds true for the truncated version \( \sigma^{(n)} \).

We can show that the truncated version \( \sigma^{(n)} \) is appropriately close to \( \sigma \). For that purpose, denote for a finite subset \( \Lambda \subset \mathbb{Z}^d \) by \( \bar{\nu}_{\Lambda;\lambda} \) and \( \bar{\nu}^{(n)}_{\Lambda;\lambda} \) the distribution of \( \sigma_v \) with respect to \( \nu \) and \( \bar{\nu}_{\lambda} \) respectively. These measures are close in the total variation distance when \( n \) is large. We make this precise in the following lemma, which is also stated in Theorem 2.1 of Van den Berg [2].

For the sake of completeness, we will give a proof of the statement here.

**Lemma 1.** Let \( \lambda > \lambda_c \). There exists positive constants \( C_{TV} \) and \( c_{TV} \) such that for all finite \( \Lambda \subset \mathbb{Z}^d \):

\[
d_{TV}(\bar{\nu}_{\Lambda;\lambda}, \bar{\nu}^{(n)}_{\Lambda;\lambda}) \leq C_{TV}|\Lambda| \exp(-c_{TV}n^\alpha).
\]

**Proof.** Let \( \Lambda \) be a finite subset of \( \mathbb{Z}^d \). For any coupling \( (\sigma^\prime_{\lambda}, \sigma^\prime_n) \) of \( \bar{\nu}_{\lambda;\Lambda} \) and \( \bar{\nu}^{(n)}_{\lambda;\Lambda} \), it holds that

\[
d_{TV}(\bar{\nu}_{\lambda;\Lambda}, \bar{\nu}^{(n)}_{\lambda;\Lambda}) \leq \mathbb{P}(\sigma^\prime_{\lambda} \neq \sigma^\prime_n),
\]

where \( \mathbb{P} \) denotes the probability measure on the space where the coupling is defined. See Proposition 4.7 of Levin and Peres [12] for more details. In particular, we can use the coupling obtained from [3] to bound

\[
d_{TV}(\bar{\nu}_{\lambda;\Lambda}, \bar{\nu}^{(n)}_{\lambda;\Lambda}) \leq \mathbb{P}_{\lambda}(\sigma_{\lambda} \neq \sigma_{\lambda}^{(n)}) \leq |\Lambda|\mathbb{P}_{\lambda}(\sigma_0 \neq \sigma_0^{(n)}),
\]

(5)

where we used the union bound and translation invariance in the second inequality. We now define the following event for \( t \in [-n^\alpha, 0) \):

\[
A_t := \{ \exists (v, s) \in \partial \Lambda_n \times (t, 0) \text{ s.t. } (v, s) \rightarrow (0, 0) \}.
\]
We can now bound
\[ P_{\lambda} \left( \sigma_0 \neq \sigma_0^{(n)} \right) \leq P_{\lambda} \left( A_{-pn^{\alpha}} \right) + P_{\lambda} \left( -A_{-pn^{\alpha}}, \, \sigma_0^{(n)} \neq \sigma_0 \right), \]  
for any constant $0 < p < 1$. Under the event $-A_{-pn^{\alpha}} \cap \{ \sigma_0^{(n)} \neq \sigma_0 \}$, there exists a $v \in \mathbb{Z}^d$ and $t \in (-n^{\alpha}, -pn^{\alpha}]$ such that $(v, t) \rightarrow (0, 0)$, but for some $T < -n^{\alpha}$ no $w \in \mathbb{Z}^d$ exists such that $(w, T) \rightarrow (0, 0)$. Using the duality relation, see Liggett, p. 35, we can bound the probability of this event by the probability that the infection of a contact process survives up to time $t$, but eventually dies out, where the only initial infected vertex is the origin. This probability, in turn, is exponentially small in $t$, see Theorem 2.30a of Liggett. We obtain
\[ P_{\lambda} \left( -A_{-pn^{\alpha}}, \, \sigma_0^{(n)} \neq \sigma_0 \right) \leq C_1 \exp(-C_2pn^{\alpha}). \]  
We now turn to the first term of \( (6) \). Under the event $A_{-pn^{\alpha}}$, there exists an active space-time path from $\partial\Lambda_{n^\alpha}$ to 0, starting at a time later than $-pn^\alpha$. Using the duality relation, the probability of this event is equal to the probability that there exists a path from 0 to $\partial\Lambda_n$ before time $pn^\alpha$, in a contact process where the only initial infected vertex is the origin. We use Proposition 1.21 of Liggett to bound this event, as well as the union bound, to find
\[ P_{\lambda} \left( A_{-pn^{\alpha}} \right) \leq \sum_{v \in \partial\Lambda_{n^\alpha}} \exp(2d\lambda pn^\alpha) p_{pn^\alpha}(0, v), \]
where $p_t(0, v)$ is the probability that a simple random walk on $\mathbb{Z}^d$ with jump rate 1 is at position $v$ at time $t$. We can now use Lemma 1.22 of Liggett to conclude that there exists $p > 0$ such that
\[ P_{\lambda} \left( A_{-pn^{\alpha}} \right) \leq \exp(2d\lambda pn^\alpha) \sum_{v \in \partial\Lambda_{n^\alpha}} p_{pn^\alpha}(0, v) \leq \frac{1}{p} \exp(-pn^\alpha). \]  
Combining the bounds \( (7) \) and \( (8) \) with \( (6) \) gives the existence of constants $C_{TV}$ and $c_{TV}$ such that
\[ P_{\lambda} \left( \sigma_0 \neq \sigma_0^{(n)} \right) \leq C_{TV} \exp(-c_{TV}n^\alpha). \]
The lemma follows by combining this bound with \( (5) \).  

\[ \square \]

3. Proof

We now turn to the proof of Theorem 1. The structure is as follows: we first discretize the space-time and give a decision tree on this space so that we can apply the OSSS inequality. We subsequently bound the revelation for this decision tree. We then bound the influence by the derivative of $\hat{\nu}_\lambda(0 \leftrightarrow \partial \Lambda_n)$ using a Russo-type formula to obtain a differential inequality. By manipulating this inequality we obtain a stretched exponential bound for $\hat{\nu}_\lambda(0 \leftrightarrow \partial \Lambda_n)$. Finally by using a renormalization argument we prove the sharp phase transition.

3.1. Discretization and the OSSS inequality. We now discretize the space-time of the graph-theoretical representation as follows. Let $\varepsilon > 0$ and define $R^\varepsilon_{v,s} := \{ v \} \times (s - \varepsilon, s]$ for $v \in \Lambda_{n^{\alpha} + n^\alpha}$ and $s \in -\varepsilon \mathbb{N}_0$ with $s \geq -n^\alpha$. Let $\omega_{v,s} := \omega \cap R^\varepsilon_{v,s}$ and $\eta_{v,s} := \eta \cap R^\varepsilon_{v,s}$. Note that the family of random variables $\{ \omega_{v,s} : v \in \mathbb{Z}^d, s \in [-n^\alpha, 0] \cap -\varepsilon \mathbb{N}_0 \}$ are independent and identically distributed.

For an event $A$, define the influence of $\omega_{v,s}$ as
\[ \text{Inf}_{v,s}(A) := P_{\lambda} \left( 1_A(\omega) \neq 1_A(\hat{\omega}) \right), \]
where $\hat{\omega}$ is obtained from $\omega$ by resampling $\omega_{v,s}$ and leaving $\omega$ unchanged in all other intervals.

Now let $A$ be an event that only depends on the variables $\{ \omega_{v,s} \}$. Let $T$ be a decision tree that determines the value of $1_A$. This means that $T$ is essentially an algorithm that reveals the variables in $\{ \omega_{v,s} \}$ one by one. Depending on the previously revealed variables the next variable is
chosen to be revealed. This procedure continues until the value of $1_A$ is known, i.e., the remaining unrevealed variables cannot change the value of $1_A$ anymore. The OSSS inequality [15] now states that

$$\text{Var}_\lambda(1_A) \leq \sum_{v,s} \delta_{v,s}(T) \text{Inf}_{v,s}^x(A),$$

where $\delta_{v,s}(T)$ is the probability under $P_\lambda$ that the variable $\omega_{v,s}$ is revealed by $T$. The above version of the OSSS inequality is slightly more general than the one introduced by O’Donnell, Saks, Schramm and Servedio [15], since the variables $\omega_{v,s}$ are not Bernoulli random variables. However, the proof of the above version is essentially the same. This version can also be found in the paper by Duminil-Copin, Raoufi and Tassion [7]. In order to use this inequality, we need strong bounds on the terms on the right hand side of the inequality, which we will derive now.

### 3.2. Bound on the Revealment

We will now give a bound on the probability that $\omega_{v,s}$ is revealed. Let $A$ be the event that $0 \leftrightarrow \partial \Lambda_k$ using only vertices in $\{v \in \Lambda_n : \sigma_v^{(n)} = 1\}$. Note that this event only depends on the variables $\{\omega_{v,s}\}$. For $k = 1, \ldots, n$ we consider a decision tree $T_k$ that determines $1_A$. Informally, $T_k$ will be the algorithm that explores the cluster of $\partial \Lambda_k$. Note that if a path from 0 to $\partial \Lambda_k$ exists, it has to be part of the cluster of $\partial \Lambda_k$. Before we can state the precise definition of $T_k$ we first need to define an auxiliary algorithm $\text{Determine}(v)$ for $v \in \Lambda_n$. When $\text{Determine}(v)$ is called all random variables in $\{\omega_{v,s} : d(v, w) \leq n^\alpha, s \in [-n^\alpha, 0] \cap [-\varepsilon N_0]\}$ are revealed. These random variables determine $\sigma_v^{(n)}$.

The algorithm $T_k$ is now the following. The cluster of $\partial \Lambda_k$ is explored by first calling $\text{Determine}(v)$ for all vertices $v \in \partial \Lambda_k$ and subsequently for all vertices that neighbour the thus far explored cluster. This process continues until either a path from 0 to $\partial \Lambda_n$ has been found or until the entire cluster of $\partial \Lambda_k$ inside $\Lambda_n$ has been explored.

We can now bound the revealment of $\omega_{v,s}$ as follows.

$$\delta_{v,s}(T_k) = P_\lambda(T_k \text{ reveals } \omega_{v,s}) \leq P_\lambda(\partial \Lambda_n^{w} \leftrightarrow \partial \Lambda_k) \leq \sum_{w \in \partial \Lambda_n^{w}} P_\lambda(w \leftrightarrow \partial \Lambda_k).$$

### 3.3. Bound on the Influence

In this section we will bound the influence of the variables $\{\omega_{v,s}\}$. The strategy is to compare the influence with the number of pivotal points and then use a Russo-type formula to relate it to the derivative of $P_\lambda(A)$ to $\lambda$. Let $A$ again be the event that $0 \leftrightarrow \partial \Lambda_n$ using only vertices in $\{v \in \Lambda_n : \sigma_v^{(n)} = 1\}$. We define the pivotal set of points for $A$ and a configuration $\omega$:

$$\text{Piv} := \{x \in \eta : 1_A(\omega) \neq 1_A(\bar{\omega})\},$$

where $\bar{\omega}$ is the configuration obtained from $\omega$ by changing the mark of $x$ from $\ast$ to $\rho_x$ or vice versa, and leaving all other points unchanged.

When we resample $\omega_{v,s}$, the value of $1_A$ can only change when there is at least one point in $\omega_{v,s} \cup \bar{\omega}_{v,s}$. Furthermore, the probability that there are two or more points in $\omega_{v,s} \cup \bar{\omega}_{v,s}$ is of order $O(\varepsilon^2)$. Therefore, since $\omega$ and $\bar{\omega}$ are interchangeable, we can bound

$$\text{Inf}_{v,s}(A) \leq 2 P_\lambda(1_A(\omega) \neq 1_A(\bar{\omega}), |\eta_{v,s}| = 1, |ar{\eta}_{v,s}| = 0) + O(\varepsilon^2).$$

Under the above event, $\eta_{v,s}$ must contain a pivotal point. Therefore

$$\text{Inf}_{v,s}(A) \leq 2 P_\lambda(\text{Piv} \cap \eta_{v,s} \neq \emptyset) + O(\varepsilon^2) \leq 2 E_\lambda[|\text{Piv} \cap \eta_{v,s}|] + O(\varepsilon^2).$$
Summing over all $v$ and $s$ gives
\[ \sum_{v,s} \text{Inf}_{v,s}(A) \leq 2\mathbb{E}_\lambda |\text{Piv}| + O(\varepsilon). \] (11)

We now aim to relate the expected number of pivotal points to the derivative of $\mathbb{P}_\lambda(A)$ with respect to $\lambda$. The event $A$ is increasing in the following sense: $\omega \in A$ implies $\omega' \in A$, where $\omega'$ is the configuration obtained from $\omega$ by setting the mark $m_x$ equal to $\rho_x$ for some $x \in \eta$. If the mark of $x$ was already an arrow, then $\omega = \omega'$ and naturally $\omega' \in A$. On the other hand, if the mark of $x$ is a cross, changing it to an arrow leads to more infected vertices so that again $\omega' \in A$. Therefore $\mathbb{P}_{\lambda+\delta}(A) - \mathbb{P}_\lambda(A) \geq 0$ for $\delta > 0$. We can write
\[ \mathbb{P}_{\lambda+\delta}(A) - \mathbb{P}_\lambda(A) = \int_\eta \mathbb{P}_{\lambda+\delta}(A|\eta) - \mathbb{P}_\lambda(A|\eta) d\eta, \]
since increasing $\lambda$ only changes the marks of the points, not the location of the points. For a fixed point configuration $\eta$ and $x \in \eta$, denote by $\mathbb{P}_{\lambda+\delta}(x|\eta)$ the measure that samples the mark of $x$ with parameter $\lambda+\delta$, and all other marks with parameter $\lambda$. Then, using the fact that $A$ is increasing, we obtain
\[ \mathbb{P}_{\lambda+\delta}(A|\eta) - \mathbb{P}_\lambda(A|\eta) = \mathbb{P}_\lambda \left( \frac{1}{2d(\lambda+\delta)+1} \leq U_x \leq \frac{1}{2d\lambda+1}, \ x \in \text{Piv} \mid \eta \right) \]
\[ = \frac{2d\delta}{(2d\lambda+1)(2d(\lambda+\delta)+1)} \mathbb{P}_\lambda(x \in \text{Piv} \mid \eta), \]
since the event that $x$ is pivotal is independent of the mark of $x$. It now follows that
\[ \frac{d}{d\lambda} \mathbb{P}_\lambda(A \mid \eta) = \lim_{\delta \downarrow 0} \frac{\mathbb{P}_{\lambda+\delta}(A \mid \eta) - \mathbb{P}_\lambda(A \mid \eta)}{\delta} = \sum_{x \in \eta} C(\lambda) \mathbb{P}_\lambda(x \in \text{Piv}) = C(\lambda) \mathbb{E}[|\text{Piv}|], \]
where $C(\lambda) = 2d/(2d\lambda + 1)^2$. Therefore, we find
\[ \mathbb{P}_{\lambda+\delta}(A) - \mathbb{P}_\lambda(A) = \int_\eta \int_0^\delta C(\lambda) \mathbb{E}_{\lambda+\beta}[|\text{Piv}| \mid \eta] d\delta d\eta, \]
\[ = C(\lambda) \int_0^\delta \int_\eta \mathbb{E}_{\lambda+\beta}[|\text{Piv}| \mid \eta] d\eta d\delta, \]
\[ = C(\lambda) \int_0^\delta \mathbb{E}_{\lambda+\beta}[|\text{Piv}|] d\delta, \] (12)
by Fubini’s theorem. Since $A$ only depends on $\omega \cap \Lambda_{n+\alpha} \times [-n^\alpha,0]$, we have the following domination
\[ |\text{Piv}| \leq |\eta \cap \Lambda_{n+\alpha} \times [-n^\alpha,0]|, \]
which is integrable. It follows that $\mathbb{E}_\lambda |\text{Piv}|$ is continuous in $\lambda$. Therefore, if we divide both sides of (12) by $\delta$ and take the limit $\delta \downarrow 0$, we obtain
\[ \frac{d}{d\lambda} \mathbb{P}_\lambda(A) = C(\lambda) \mathbb{E}_\lambda |\text{Piv}|, \]
where we used the continuity of $\mathbb{E}_\lambda |\text{Piv}|$. Combining this with the bound on the influence (11) gives
\[ \sum_{v,s} \text{Inf}_{v,s}(A) \leq 2C(\lambda) \frac{d}{d\lambda} \mathbb{P}_\lambda(A) + O(\varepsilon). \] (13)
3.4. Obtaining the Differential Inequality. We define
\[ \theta_n(\lambda) := \tilde{\nu}^{(n)}_\lambda(0 \leftrightarrow \partial \Lambda_n), \quad S_n(\lambda) := \sum_{k=1}^n \theta_k(\lambda). \]
Note that \( \theta_n(\lambda) \) is increasing in \( \lambda \) and decreasing in \( n \). Applying the bound on the revealment to the OSSS inequality (9) and summing over \( k \) gives
\[ n\theta_n(\lambda)(1 - \theta_n(\lambda)) \leq \sum_{v,s} \sum_{w \in \partial \Lambda_{n,v,s}} \sum_{k=1}^n \nu^{(n)}_\lambda(w \leftrightarrow \partial \Lambda_k) \text{Inf}_{v,s}^w, \]
where in the second line we used that \( \nu^{(n)}_\lambda \) is increasing in \( \lambda \). Using the bound on the influence (13) and using the fact that \( |\partial \Lambda_n| = O(n^{d-1}) \) gives the existence of a constant \( C_3 > 0 \) such that
\[ n\theta_n(\lambda)(1 - \theta_n(\lambda)) \leq C_3 C(\lambda)n^{\alpha(d-1)} S_n(\lambda) \frac{d}{d\lambda} \theta_n(\lambda) + O(\varepsilon). \]
Taking \( \varepsilon \to 0 \) gives
\[ \theta'_n(\lambda) \geq \frac{n^{1-\alpha(d-1)}}{C_3 C(\lambda) S_n(\lambda)} \theta_n(\lambda)(1 - \theta_n(\lambda)). \tag{14} \]

3.5. Proof of Theorem 3.4. Suppose \( \lambda \in (\lambda_p/2, \lambda_p + 1) \), so that \( \lambda_p \in (\lambda_p/2, \lambda_p + 1) \). It suffices to show that the exponential decay in (1) holds for \( \lambda \) in this interval, since we can extend this to all \( \lambda < \lambda_p \) by the domination \( \tilde{\nu}_\lambda(|C| \geq n) \leq \tilde{\nu}_{\lambda'}(|C| \geq n) \) for \( \lambda < \lambda' \). Since \( \lambda \in (\lambda_p/2, \lambda_p + 1) \), we have that
\[ \frac{1 - \theta_n(\lambda)}{C_3 C(\lambda)} \geq \frac{(d\lambda_p + 1)^2}{2dC_3} (1 - \theta(\lambda_p + 1)) := C_4 > 0. \]
Applying this to the differential inequality (14) and setting \( \gamma := 1 - \alpha(d-1) \) gives
\[ \theta_n(\lambda)' \geq C_4 \frac{n^{\gamma}}{S_n(\lambda)} \theta_n(\lambda). \tag{15} \]
We choose \( \alpha < 1/(d-1) \), so that \( \gamma > 0 \). The above differential inequality is similar, but weaker than the inequalities obtained by Duminil-Copin, Raoufi and Tassion in their papers, since in our case we have a power of \( n \). Therefore, the way we obtain the sharpness result also differs slightly and we include it here.

We define an alternative critical point \( \bar{\lambda}_p := \inf \left\{ \lambda : \limsup_{n \to \infty} \frac{\log S_n(\lambda)}{\log n} \geq \gamma \right\} \). If we can prove exponential decay as in (1) below \( \bar{\lambda}_p \) and positivity of \( \tilde{\nu}_\lambda(0 \leftrightarrow \infty) \) as in (2) above \( \bar{\lambda}_p \), then we have shown that \( \lambda_p = \bar{\lambda}_p \) and immediately proven Theorem 3.4.

Suppose \( \lambda_1 < \bar{\lambda}_p \). Then there exists \( \beta > 0 \) such that \( S_n(\lambda) \leq n^{\gamma - \beta} \) for all \( \lambda < \lambda_1 \) and \( n \) large enough. Together with (15), this gives the differential inequality \( \theta_n' \geq C_4 n^\beta \theta_n \) for all \( \lambda < \lambda_1 \). Fix \( \varepsilon > 0 \) and set \( \lambda_2 := \lambda_1 - \varepsilon \). Integrating this differential inequality between \( \lambda_2 \) and \( \lambda_1 \) gives
\[ \theta_n(\lambda_2) \leq \exp(-\varepsilon C_4 n^\beta), \]
for \( n \) large enough. To compensate for small values of \( n \), we know that there exists a constant \( C_5 > 0 \) such that for all \( n \in \mathbb{N} \) we have that
\[
\tilde{\nu}\big(0 \leftrightarrow \partial \mathcal{A}_n \big) \leq C_5 \exp(-\varepsilon C_4 n^\gamma),
\]
so that also
\[
\tilde{\nu}\big(0 \leftrightarrow \partial \mathcal{A}_n \big) \leq C_5 \exp(-\varepsilon C_4 n^\gamma),
\]
by the domination of the truncated random variables. We are able to improve the above stretched exponential bound to an exponential tail bound by a renormalization argument. We will provide this argument in Section 3.6. Before that, we continue with the proof of assertion b) of Theorem 1. It remains to show that the mean-field lower bound of (2) holds in the super-critical regime.

We now use the total variation bound of Lemma 1 to find
\[
\tilde{\nu}\big(0 \leftrightarrow \partial \mathcal{A}_n \big) \leq C_5 \exp(-\varepsilon C_4 n^\gamma),
\]
where for the last inequality we used the fact that
\[
\text{this argument in Section 3.6. Before that, we continue with the proof of assertion b) of Theorem 1. It remains to show that the mean-field lower bound of (2) holds in the super-critical regime.}

Suppose \( \lambda > \tilde{\lambda}_p \). Define \( T_n := \frac{1}{\gamma \log n} \sum_{k=1}^{n^\gamma} \frac{\theta_k}{k} \). Using the differential inequality (15) gives
\[
T_n = \frac{1}{\gamma \log n} \sum_{k=1}^{n^\gamma} \frac{\theta_k}{k} \geq \frac{C_4}{\gamma \log n} \sum_{k=1}^{n^\gamma} \frac{\theta_k n^\gamma}{k} \geq C_4 \frac{\log S_{n^\gamma+1} - \log S_1}{\gamma \log n},
\]
where for the last inequality we used the fact that
\[
\frac{\theta_k}{S_k} \geq \frac{S_{k+1} - S_k}{S_k} \geq \int_{S_k}^{S_{k+1}} \frac{1}{t} \, dt = \log S_{k+1} - \log S_k.
\]

Let \( \lambda' \in (\tilde{\lambda}_p, \lambda) \). Then
\[
T_n(\lambda) \geq T_n(\lambda) - T_n(\lambda') \geq \frac{C_4}{\gamma \log n} \int_{\lambda'}^{\lambda} \left( \log S_{n^\gamma+1}(t) - \log S_1(t) \right) \, dt
\]
\[
\geq C_4(\lambda - \lambda') \frac{\log S_{n^\gamma+1}(\lambda') - \log S_1(\lambda)}{\gamma \log n}
\]
\[
\geq C_4(\lambda - \lambda') \frac{\log S_{n^\gamma}(\lambda')}{\log n^\gamma},
\]
since \( S_1 = \theta_1 < 1 \) and \( S_{n+1} \geq S_n \). Now let \( \theta(\lambda) := \lim_{n \to \infty} \theta_n(\lambda) \), this limit exists, since \( \theta_n(\lambda) \) is decreasing in \( n \). Consequently \( T_n(\lambda) \to \theta(\lambda) \) as \( n \to \infty \) and we can write \( \theta_n(\lambda) = T_n(\lambda) - a_n \) for some sequence \( (a_n)_{n \in \mathbb{N}} \) with \( a_n \to 0 \) as \( n \to \infty \). We now obtain
\[
\theta_n(\lambda) \geq C_4(\lambda - \lambda') \frac{\log S_{n^\gamma}(\lambda')}{\log n^\gamma} - a_n.
\]

We now use the total variation bound of Lemma 1 to find
\[
\tilde{\nu}\big(0 \leftrightarrow \infty \big) \geq \tilde{\nu}_\lambda^{(n)}(0 \leftrightarrow \infty) - C_{TV} |\mathcal{A}_n| \exp\left(-c_{TV} n^\alpha \right)
\]
\[
\geq C_4(\lambda - \lambda') \frac{\log S_{n^\gamma}(\lambda')}{\log n^\gamma} - a_n - C_{TV} n^d \exp\left(-c_{TV} n^\alpha \right).
\]

Letting \( n \) tend to infinity and using that \( \lambda' > \tilde{\lambda}_p \) gives
\[
\tilde{\nu}\big(0 \leftrightarrow \infty \big) \geq \gamma C_4(\lambda - \lambda').
\]

Finally, we let \( \lambda' \) tend to \( \tilde{\lambda}_p \), which has now been shown to be equal to \( \lambda_p \). We obtain
\[
\tilde{\nu}\big(0 \leftrightarrow \infty \big) \geq \gamma C_4(\lambda - \lambda_p).
\]
3.6. A renormalization argument. In this section we show how to improve the stretched exponential decay of \( \tilde{\nu}_\lambda(0 \leftrightarrow \partial \Lambda_n) \) to proper exponential decay. We adapt the renormalization procedure of Van den Berg [1], which in turn are based on ideas for Bernoulli percolation by Kesten [11].

Let \( N \in \mathbb{N} \) even. We cover \( \mathbb{Z}^d \) with the balls \( \Lambda^{vN}_{dN/2} \), with \( v \in \mathbb{Z}^d \). We choose the radius equal to \( dN/2 \) to ensure that every vertex in \( \mathbb{Z}^d \) is covered. We say that \( v \in \mathbb{Z}^d \) is ‘good’ whenever \( 0 \leftrightarrow \Lambda^{vN}_{dN/2} \). Let \( S \) be the set of good vertices, it follows that

\[
\tilde{\nu}_\lambda(|C| \geq n) \leq \tilde{\nu}_\lambda \left( |S| \geq \frac{n}{dN^2} \right) \leq \tilde{\nu}_\lambda \left( |S| \geq \frac{n}{C_6N^d} \right),
\]

for some constant \( C_6 > 0 \). For every \( v \in \mathbb{Z}^d \), we define the following event

\[
A_v := \left\{ \partial \Lambda^{vN}_{dN/2} \leftrightarrow \partial \Lambda^{vN}_{dN/2} \right\}.
\]

For \( v \in \mathbb{Z}^d \setminus A_d \), it follows that \( \{v \text{ is good}\} \subset A_v \). Let \( W \subset \mathbb{Z}^d \) finite. We say that \( W \) is ‘good’ whenever \( v \) is ‘good’ for all \( v \in W \). Furthermore, we can find a subset \( W' \) of \( W \) such that for all \( v, w \in W' \) we have \( d(v, w) \geq 3d \) and \( |W'| \geq |W|/C_7 \) for some constant \( C_7 \) independent of \( W \). We now obtain

\[
\tilde{\nu}_\lambda(W \text{ is ‘good’}) = \tilde{\nu}_\lambda \left( \bigcap_{v \in W} \{v \text{ is ‘good’}\} \right) \leq \tilde{\nu}_\lambda \left( \bigcap_{v \in W \setminus A_d} \{v \text{ is ‘good’}\} \right) \leq \tilde{\nu}_\lambda \left( \bigcap_{v \in W' \setminus A_d} A_v \right) \leq \tilde{\nu}^{(N)}_{\lambda} \left( \bigcap_{v \in W' \setminus A_d} A_v \right),
\]

by (1), since \( A_v \) is an increasing event for all \( v \). The events \( \{A_v : v \in W'\} \) are now independent under \( \tilde{\nu}^{(N)}_{\lambda} \), since \( d(vN, wN) \geq 3dN \) for all \( v, w \in W' \). We use translation invariance to find

\[
\tilde{\nu}_\lambda(W \text{ is ‘good’}) \leq \prod_{v \in W' \setminus A_d} \tilde{\nu}^{(N)}_{\lambda}(A_v) \leq \tilde{\nu}^{(N)}_{\lambda}(A_0) \frac{|W| - 2C_6d}{C_7},
\]

where the factor \( C_6d \) comes from excluding the vertices in \( A_d \). We now use the union bound and the stretched exponential bound of (10) to obtain

\[
\tilde{\nu}^{(N)}_{\lambda}(A_0) \leq C_3 \left( \frac{dN}{2} \right)^{d-1} C_5 \exp \left( -\varepsilon C_4 \left( \frac{dN}{2} \right)^{\beta} \right),
\]

since \( |\partial \Lambda_n| \leq C_3n^{d-1} \). Combining this with the previous inequality gives the existence of constants \( C_8, C_9, C_{10} > 0 \) such that

\[
\tilde{\nu}_\lambda(W \text{ is ‘good’}) \leq \left( C_8N^{d-1}\exp(-C_9N^{\beta}) \right)^{|W|/C_{10}}.
\]

The set of ‘good’ vertices \( S \) is a lattice animal: a subset of \( \mathbb{Z}^d \) that is connected and contains 0. The number of lattice animals of size \( n \) is bounded from above by \( C_{11} \) for some constant \( C_{11} \), see Lemma 9.3 of Penrose [10]. If \( |S| \geq n/C_6N^d \), there exists a ‘good’ lattice animal of size \( |n/C_6N^d| \). Therefore, we obtain from the union bound

\[
\tilde{\nu}_\lambda \left( |S| \geq \frac{n}{C_6N^d} \right) \leq \left( C_{11} \left( C_8N^{d-1}\exp(-C_9N^{\beta}) \right)^{1/C_{10}} \right)^{n/C_6N^d}.
\]
We now choose $N$ such that

$$C_{11} \left( C_8 N^{d-1} \exp \left(-C_9 N^d \right) \right)^{1/C_{10}} = 1 - \xi < 1,$$

for some $\xi > 0$. We conclude

$$\bar{\nu}_\lambda \left(|C| \geq n \right) \leq (1 - \xi)^n/C_6 N^d = \exp \left( \frac{\log(1 - \xi)}{C_6 N^d} n \right) = \exp(-cn),$$

where we set $c := -\log(1 - \xi)/C_6 N^d > 0$. This completes the proof of Theorem 1.

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