Solutions to Knizhnik-Zamolodchikov equations with coefficients in non-bounded modules

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Abstract

We explicitly write down integral formulas for solutions to Knizhnik-Zamolodchikov equations with coefficients in non-bounded – neither highest nor lowest weight – $sl_{n+1}$-modules. The formulas are closely related to WZNW model at a rational level.

1 Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\hat{\mathfrak{g}}$ the corresponding non-twisted affine Lie algebra. Let $\lambda$ be a weight of $\mathfrak{g}$, $M(\lambda, k)$ ($M(\lambda, k)^c$) be the Verma (contragredient Verma) module over $\hat{\mathfrak{g}}$ with the central charge $k$; for a $\mathfrak{g}$-module $V$ be denote by $V((z))$ the module of formal Laurent series in $z$ with coefficients in $V$, regarded as a $\hat{\mathfrak{g}}$-module with the central charge equal to 0.

Vertex operator is a $\hat{\mathfrak{g}}$-linear map

$$\Phi(z) : M(\lambda_1, k) \rightarrow M(\lambda_2, k)^c \otimes V((z)).$$ (1)

If highest weights $(\lambda_1, k), \cdots (\lambda_{N+1}, k)$ are generic then $M(\lambda_i, k) \approx M(\lambda_i, k)^c$, $1 \leq i \leq N+1$ and one may consider a product of vertex operators $\Phi_N(z_N) \cdots \Phi_1(z_1)$. Matrix element $\langle v_{\lambda_{N+1}}^*, \Phi_N(z_N) \cdots \Phi_1(z_1)v_{\lambda_1} \rangle$
related to vacuum vectors is called a correlation function. One of the central
results of conformal field theory (see [1]) is that a correlation function satisfies
a remarkable system of Knizhnik-Zamolodchikov equations. We prepare no-
tations in order to write down the trigonometric form of Knizhnik-Zamolodchikov
equations.

Let

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

be a root space decomposition. Fix an invariant inner product on $\mathfrak{g}$ and a basis
$\{h_i \in \mathfrak{h}, g_\alpha \in \mathfrak{g}_\alpha : 1 \leq i \leq n, \alpha \in \Delta\}$ of $\mathfrak{g}$ so that $(h_i, h_j) = \delta_{i,j}$, $(g_\alpha, g_\beta) = \delta_{\alpha, -\beta}$. For each $\mu \in \mathfrak{h}^*$ denote by $h_\mu$ an element of $\mathfrak{h}$ satisfying (and uniquely
determined) by the condition $(h_\mu, h) = \mu(h)$.

Set

$$r = \frac{1}{2} \sum_{i=1}^{n} h_i \otimes h_i + \sum_{\alpha \in \Delta} g_\alpha \otimes g_{-\alpha}.$$

Being an element of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ $r$ naturally acts on a tensor product of 2
$\mathfrak{g}$–modules. There are $N^2$ different ways to make it act on a tensor product of
$N$ $\mathfrak{g}$–modules via the following $N^2$ embeddings of $U(\mathfrak{g})^{\otimes 2}$ in $U(\mathfrak{g})^{\otimes N}$: each of
them is associated to a pair of numbers $1 \leq i, j \leq N$ and sends

$$U(\mathfrak{g})^{\otimes 2} \ni \omega \mapsto \omega_{ij} \in U(\mathfrak{g})^{\otimes N},$$

so that

$$r(z_i, z_j) = \frac{n_{ij}z_i + n_{ji}z_j}{z_i - z_j}. $$

For a pair $1 \leq i, j \leq N$ introduce the following function in 2 complex
variables with values in $U(\mathfrak{g})^{\otimes N}$:

$$r(z_i, z_j) = \frac{n_{ij}z_i + n_{ji}z_j}{z_i - z_j}.$$

**Theorem 1.1 (Knizhnik, Zamolodchikov)** The correlation function

$$\Psi(z) = \langle v_{\lambda_{N+1}}^* \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1)v_{\lambda_1} \rangle$$

satisfies the following system of differential equations

$$\begin{align*}
(k + h^\vee)z_i \frac{\partial \Psi}{\partial z_i} &= \left\{ \sum_{j \neq i} r_{ij}(z_i, z_j) - \frac{1}{2}(\lambda_1 + \lambda_{N+1} + 2\rho)^{(i)} \right\} \Psi, \\
1 \leq i \leq N,
\end{align*}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and for each $\mu \in \mathfrak{h}^*$ $\mu^{(i)}$ stands for the
operator acting on $V_1 \otimes \cdots \otimes V_N$ as $h_\mu$ applied to the $i$–th factor of $V_1 \otimes \cdots \otimes V_N$. 
Solutions to KZ equations

To keep track of the parameters we will be referring to (2) as $KZ(\lambda_{N+1}, \lambda_1)$.

The deep theory of KZ equations has been developed by several authors (see e.g. [2, 3, 4, 5]) in the case when $V_i$ are highest weight modules. It has also been realized that this theory is relevant to physics applications in the case when $(\lambda_i, k)$ is either integral or generic. Indeed, if conflicting with the above assumptions, some of $M(\lambda_i, k)$ are reducible then the product $\Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1)$ does not exist unless each of the operators $\Phi_i(z_i)$ can be pushed down to a map

$$\Phi_i(z_i) : L(\lambda_i, k) \to L(\lambda_{i+1}, k) \otimes V_i((z_i)),$$

where $L(\lambda, k)$ stands for an irreducible highest weight module with the highest weight $(\lambda, k)$. In the case when $(\lambda, k)$ is an admissible weight (for example, dominant integral weight) the last condition reduces to the singular vector decoupling condition: matrix elements of $\Phi_i(z_i)$ related to singular vectors of $M(\lambda_i)$ vanish. It is known that if each $(\lambda_i, k)$ is dominant integral then everything goes through nicely, in particular, the Schechtman-Varchenko integral solutions to (2) come from products of vertex operators (3). However if the central charge is not integral it has been realized (see also [8, 9]) that the singular vector decoupling condition implies that $V_i$ is neither highest nor lowest weight module. Though some results for such models were obtained in [8, 9], where in particular the connection to quantum hamiltonian reduction was revealed, not much is known about KZ equations in this case.

In [11] a new method of constructing solutions to (2) was proposed which seems to be relevant to the problem. Let $G$ be a complex Lie group related to $g$, $F = G/B$ be a flag manifold and $F^0 \subset F$ be the big cell. There is a family of embeddings of $g$ into the algebra of order 1 differential operators on $F^0$

$$\pi_\mu : g \to Diff^1(F^0), \ \mu \in \mathfrak{h}^*.$$ 

This makes the space of analytic functions on $F^0$ into a huge $g$–module. Different $g$–closed subspaces give realization of different $g$–modules. For example, contragredient Verma modules are realized in the space of polynomials on $F^0$, $\mu$ being the highest weight and a constant function being a highest weight vector; this observation has been extensively used recently with regards to Wakimoto modules [12, 13, 14]. The spaces of multi-valued functions give modules with quite different properties, the simplest example being that of $\mathfrak{sl}_2$: in this case the big cell is $\mathbb{C}$, contragredient Verma modules are realized in $\mathbb{C}[x]$; the space $x^\nu \mathbb{C}[x, x^{-1}], \ \nu \in \mathbb{C}$ is also closed under the action of $\mathfrak{sl}_2$ and the embedding $\pi_\mu, \ \mu \in \mathbb{C}$ makes it into generically irreducible $\mathfrak{sl}_2$–module. This module is transparently neither highest nor lowest weight one.

Regarding $V$ in (2) as a $g$–module realized in functions on the big cell one identifies elements of $V((z))$ with functions of 2 groups of variables: $x$ and $z$, where $x$ stands for a (vector) coordinate on the big cell and $z$ is a coordinate on $\mathbb{C}$. Likewise, the correlation function

$$\Psi(z) = \langle v_\lambda^{\ast_{N+1}}, \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) v_{\lambda_1} \rangle$$
is identified with a function of \(x^{(1)}, \ldots, x^{(N)}; z_1, \ldots, z_N\) where \(x^{(i)}\) is a coordinate on the \(i\)-th copy of the big cell, \(z_i \in \mathbb{C}\), \(1 \leq i \leq N\). One of the advantages of this functional realization is that the embedding \(\pi_\lambda : \mathfrak{g} \to Diff_1(F^0)\) lifts to the mapping of the group \(G\): for \(g \in \mathfrak{g}\) the exponent \(\exp(-tg)\) is a well-defined operator.

Let \(W\) be the Weyl group of \(\hat{\mathfrak{g}}\), \(w = r_{m_1}r_{m_2} \cdots r_{m_l} \in W\) be a decomposition (not necessarily reduced), where \(r_m\) denotes the reflection at the corresponding simple root. Set,

\[
\beta_j = \frac{2(r_{m_{i+2}} \cdots r_{m_{i+1}}, \lambda_1, \alpha_{m_{i+1}})}{(\alpha_{m_{i+1}}, \alpha_{m_{i+1}})} + 1, \quad 1 \leq j \leq l.
\]

Given

\[
\Psi_{\text{old}}(z) = \langle v_{\lambda_{N+1}}, \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1)v_{\lambda_1} \rangle,
\]

set

\[
\Psi_{\text{new}} = \prod_{j=1}^l \Gamma(-\beta_j)^{-1} \int \{ \exp(-t_1 F_{m_1}) \cdots \exp(-t_l F_{m_l}) \Psi_{\text{old}} \} \prod_{j=1}^l t_j^{-\beta_j-1} dt_1 \cdots dt_l,
\]

where the integration is carried out over an arbitrary cycle of the highest homology group related to the multi-valued integrand. In (4) it is set that \(E_i, F_i, H_i, 0 \leq i \leq rk \mathfrak{g}\) are canonical Cartan generators of \(\hat{\mathfrak{g}}\) and \(E_i, F_i, H_i, 1 \leq i \leq rk \mathfrak{g}\) are the ones coming from the inclusion \(\mathfrak{g} \subset \hat{\mathfrak{g}}\).

**Theorem 1.2** \([11]\) \(\Psi_{\text{new}}\) is a solution to \(KZ(\lambda_{N+1}, w \cdot \lambda_1)\).

Theorem 1.2 works as follows: given a solution to \(KZ\) it generates new ones labelled by elements of the affine Weyl group. In our notations the simplest solution to \(KZ(\lambda_{N+1}, \lambda_1)\) is given by

\[
^\circ \Psi_{\text{old}} = \prod_{i,j} (z_i - z_j)^{2(\mu_i, \mu_j)/(k+h^\vee)}(z_i z_j)^{-(\mu_i, \mu_j)/(k+h^\vee)} \times \prod_{i} z_i^{(\lambda_i + \lambda_{N+1} + 2p, \mu_i)/2(k+h^\vee)},
\]

where \(\mu_i\) is a highest weight of \(V_i, 1 \leq i \leq N\). In particular, \(^\circ \Psi_{\text{old}}\) is independent of \(x\)'s. The purpose of this paper is to explicitly write down the integral

\[
\Psi_{\text{new}} = ^\circ \Psi_{\text{old}} \times \prod_{j=1}^l \Gamma(-\beta_j)^{-1} \int \{ \exp(-t_1 F_{m_1}) \cdots \exp(-t_l F_{m_l}) \} t_j^{-\beta_j-1} dt_1 \cdots dt_l,
\]

for \(\mathfrak{g} = sl_{n+1}\), generalizing the calculation carried out in \([11]\) for \(sl_2\).
Remark. The integral representation of $\Psi_{new}$ in Theorem 1.2 is nothing but the conventional definition of $F_{m_1}^{\beta_1} \cdots F_{m_l}^{\beta_l} \cdot \Psi_{old}$. The latter comes from looking at the “matrix element”

$$\langle \phi_{N+1}^* \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) F_{m_1}^{\beta_1} \cdots F_{m_l}^{\beta_l}, \Psi_{old} \rangle.$$

Though the expression $F_{m_1}^{\beta_1} \cdots F_{m_l}^{\beta_l} \cdot \Psi_{old}$ is not understood as an element of $M(\lambda_1, k)$, the powers are chosen in such a way that it formally satisfies the singular vector conditions [11, 15], which makes the statement of Theorem 1.2 almost obvious.

One can similarly consider an expression

$$\langle E_{m_1}^{\beta_1'} \cdots E_{m_l}^{\beta_l'} v_{\lambda_1}^* \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) v_{\lambda_1}, \Psi_{old} \rangle,$$

for appropriate $\beta_1', \cdots, \beta_l'$ and write down another solution in the form close to (4) but with $F$’s replaced with $E$’s or combine both methods or, finally, apply them to other solutions obtained in [4, 8, 9].

As to relation of our solution (4) to correlation functions, we have been able to verify in simplest cases that (4) indeed gives a matrix element of a product of vertex operators and hope that (4) will prove useful for investigation of other rational level models.

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2 Integral formulas for solutions of $KZ(\lambda_{N+1}, \lambda_1)$

2.1 Main result

Here we are going to write down the integral (3) in the case of $g = sl_{n+1}$, $\hat{g} = sl_{\infty+1}$. In this case there are $3(n+1)$ Cartan generators $E_i, F_i, H_i, 0 \leq i \leq n$, where $E_i, F_i, H_i, 1 \leq i \leq n$ are the ones coming from the inclusion $g \subset \hat{g}$. Explicitly the generators are described as follows. If $e_{ij} = (a_{ij})$ is an $(n+1)$ matrix then $E_i = e_{ii+1}, F_i = e_{i+1i}, H_i = e_{ii} - e_{i+1i+1}, 1 \leq i \leq n$ and $E_0 = e_{n+11} \otimes z, F_0 = e_{1n+1} \otimes z^{-1}$ (see [16] for details). The $g$–weight $\mu$ is considered as a vector $(\mu_1, \ldots, \mu_n)$, $\mu_i = \mu(H_i)$. The embedding

$$\pi_\mu : sl_{n+1} \rightarrow Diff^1(F^0), \mu = (\mu_1, \ldots, \mu_n)$$

is calculated in [12] (see also [13]). To recall this result we choose coordinates of the big cell $F^0$ to be $\{x_{ij} : 1 \leq i < j \leq n\}$ identifying as usual the big cell
Solutions to KZ equations

with the subgroup of matrices

\[
\begin{pmatrix}
1 & x_{11} & \cdots & \cdots & x_{1n} \\
1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 1 & 1
\end{pmatrix}
\]

For \(1 \leq i \leq n\) set

\[\partial x_{ij} := \frac{\partial}{\partial x_{ij}}.\]

Then \(\pi_\mu\) acts on Cartan generators by

\[E_i \mapsto -\partial x_{ii} - \sum_{j=i+1}^n x_{i+1,j} \partial x_{ij},\]

\[F_i \mapsto x_{ii} \left( \sum_{j=1}^i x_{ji} \partial x_{ji} - \sum_{j=1}^{i-1} x_{ji-1} \partial x_{ji-1} \right) - \sum_{j=i+1}^n x_{ij} \partial x_{i+1,j} + \sum_{j=1}^{i-1} x_{ji} \partial x_{ji-1} + \mu_i x_{ii}.\]

Here \(x_{ij} = 0\) unless \(1 \leq i \leq j \leq n\).

The matrix \(e_{1n+1}\) may be written as

\[e_{1n+1} := [\cdots [E_1, E_2], \cdots], E_n].\]

Using the above formulas one proves the following

**Lemma 2.1**

\[\pi_\mu(e_{1n+1}) = -\partial x_{1n}.\]

From now on till the end of this section we omit writing \(\pi_\mu\) identifying Lie algebra elements with their images under \(\pi_\mu\).

The action of the Lie algebra \(\hat{g}\) on a function on \(F^0 \times \mathbb{C}^*\) is determined by the evaluation map \(g \otimes z^k \mapsto z^k g\). In particular

\[F_0 = e_{1n+1} \otimes z^{-1} \mapsto -z^{-1} \partial x_{1n}.\]

The result of exponentiation of these formulas is given by

**Lemma 2.2**
(i) If $\mu = 0$ then

1) $\exp(-tF_0) : x_{kl} \mapsto \begin{cases} z^{-1}t + x_{1n} & \text{for } l = i \\ x_{kl} & \text{otherwise} \end{cases}$

2) $\exp(-tF_i) : x_{kl} \mapsto \begin{cases} -(x_{ki} - x_{k-1}x_{ii})t + x_{ki-1} & \text{for } l = i - 1 \\ x_{il} + x_{i+1l} & \text{for } k = i + 1 \\ x_{kl} & \text{otherwise} \end{cases}$

(ii) Generically

$$\exp(-tF_i)\cdot\psi(x) = (1 + x_{ii}t)^{\mu_i}\psi(x'), \ 1 \leq i \leq n$$

where $x'$ is given by the substitution of the item (i) while action of $F_0$ is independent of $\mu$.

**Proof**

If $\mu = 0$ then all $F$’s are vector fields. The problem of evaluating an exponent of a vector field is, actually, a problem of the theory of ordinary differential equations: the exponent of a vector field is an element of a 1-parametric family of diffeomorphisms generated by the vector field and, therefore, is given by a general solution to the corresponding system of o.d.e.’s. In our case the system and the solution are (resp.)

1):

$$\dot{x}_{kl} = \delta_{l1}\delta_{kn}z^{-1} \implies x_{1n} = z^{-1}t + x_{1n};$$

2):

$$\begin{align*}
\dot{x}_{ii} &= x_{ii}^2 \\
\dot{x}_{ji} &= x_{ii}x_{ji} \\
\dot{x}_{ji-1} &= x_{ji} - x_{ji-1}x_{ii} \\
\dot{x}_{i+1j} &= -x_{ij} x_{ij} = 0 & \text{for } j \geq i + 1
\end{align*}$$

which completes proof of the item (i). As to the item (ii), one shows that any order 1 differential operator is conjugated to a vector field by the multiplication by a function it annihilates. This implies (ii) since $F_i \cdot (x_{ii}^{-\mu_i}) = 0$. Q.E.D.

Now by using all this one can calculate the integrand of (5). But to formulate the result it is convenient to give some more notations.

Set

$$T = (t_{ij}) := \begin{pmatrix} x_{11} & \cdots & \cdots & x_{1n} \\
1 & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \\
0 & \cdots & 1 & x_{nn} \end{pmatrix}.$$
for $1 \leq i_k \leq i_{k-1} \leq \cdots \leq i_1 \leq j$, $j + k \leq n + 1$,

$$I^j_{i_1, i_2, \ldots, i_k} := \{ j + 1 - i_1, j + 2 - i_2, \ldots, j + k - i_k \}$$

$$J^j_k := \{ j, j + 1, \ldots, j + k - 1 \}$$

$$T^j_{i_1, i_2, \ldots, i_k} := (i_{j})_{i \in I^j_{i_1, i_2, \ldots, i_k}, j \in J^j_k}$$

$$Q^j_{i_1, i_2, \ldots, i_k} := \left\{ \begin{array}{ll}
\det(T^j_{i_1, i_2, \ldots, i_k}) & \text{for } k > 0 \\
1 & \text{for } k = 0
\end{array} \right.$$ 

Introduce a collection of functions on the big cell along with an ordering on it.

**Definition** We write $Q^j Q^i \xrightarrow{F} Q'$

$$\Leftrightarrow \exp(-tF_l)Q^j_{i_1, i_2, \ldots, i_k} = \left\{ \begin{array}{ll}
\frac{1}{1 + x_{i_l} t} \left\{ Q' t + Q^j_{i_1, i_2, \ldots, i_k} \right\} & \text{for } l = j \\
Q' t + Q^j_{i_1, i_2, \ldots, i_k} & \text{for } l \neq j
\end{array} \right.$$ 

In the definition it is assumed that $\mu = 0$.

**Lemma 2.3**

1) $Q^j_{i_1, i_2, \ldots, i_k} \xrightarrow{F_{j+r-1-i_r}} Q^j_{i_1, i_2, \ldots, i_r+1, \ldots, i_k}$ for $1 \leq r \leq k$

2) $Q^j_{i_1, i_2, \ldots, i_k} \xrightarrow{F_{j+k}} Q^j_{i_1, i_2, \ldots, i_k, 1}$ for $k \geq 0$

3) $Q^j_{i_1, i_2, \ldots, i_{n+1-j}} \xrightarrow{F_{n-j}} (-1)^{n-j} Q^j_{i_2-1, \ldots, i_{k'}, -1} z^{-1}$ where $k' - 1 := \sharp \{ r : r > 2, i_r > 1 \}$

Otherwise, $Q \xrightarrow{F_i} 0$

The proof of this lemma is a standard calculation of linear algebra using Lemma 2.2, in particular we use Laplace expansion of a certain determinant to prove 2).

The above definition suggests to introduce the following $n + 1$-colored graph $\Gamma$. The set of vertices of $\Gamma$ is the set of all $Q \neq 0$ such that

$$1 \xrightarrow{F_{j_1}} Q_1 \xrightarrow{F_{j_2}} Q_2 \rightarrow \cdots \rightarrow Q_{r-1} \xrightarrow{F_{j_r}} Q$$

for some $j_1, \ldots, j_r$. It follows from Lemma 2.3 that each vertex is of the form $(-1)^{(n-j)r} Q^j_{i_1, i_2, \ldots, i_k} z^{-r}$. Define a function on the set of vertices by

$$l((-1)^{(n-j)r} Q^j_{i_1, i_2, \ldots, i_k} z^{-r}) = (n + 1)r + \sum_{p=1}^{k} i_p.$$
Solutions to KZ equations

Two vertices $P, Q$ are connected by an edge of the color $i$ if and only if

$$P \xrightarrow{E_i} Q.$$  

With any vertex $Q \in \Gamma$ associate a set $\mathcal{P}(Q)$ of all oriented paths connecting 1 and $Q$.

**Lemma 2.4** All $\gamma \in \mathcal{P}((-1)^{(n-j)r}Q^j_{i_1, i_2, \ldots i_k} z^{-r})$ are of the same length $l((-1)^{(n-j)r}Q^j_{i_1, i_2, \ldots i_k} z^{-r})$.

**Proof**

Lemma 2.3 shows that if there is an edge going from $P$ to $Q$ then $l(Q) = l(P) + 1$. The lemma now follows from the obvious remark that $l(1) = 0$.

Q.E.D

We are in a position to write down the integral (5). Recall that $W$ is the Weyl group of $\hat{g}$ and $w = r_{m_1} r_{m_2} \cdots r_{m_l} \in W$ is a decomposition (not necessarily reduced), where $r_m$ denotes the reflection at the corresponding simple root $(\alpha_m)$. $m$ can be viewed as a map from $I_1 (= \{1, 2, \ldots, l\})$ to $I_2 (= \{0, 1, \ldots, n\})$. Therefore, $m^{-1}(j), j \in I_2$, is a subset of $I_1$. Set,

$$\beta_j = \frac{2(r_{m_{i+2-j}} r_{m_i} \cdot \lambda_1, \alpha_{m_{i+1-j}})}{(\alpha_{m_{i+1-j}}, \alpha_{m_{i+1-j}})} + 1, 1 \leq j \leq l$$

and

$$K_w(t_1, t_2, \cdots, t_l) = \prod_{j=1}^{l} \Gamma(-\beta_j)^{-1} \times \{ \exp(-t_l F_{m_1}) \cdots \exp(-t_1 F_{m_1}) \prod_{j=1}^{l} t_j^{-\beta_j-1} \},$$

where 1 is viewed as an element of $V_1 \otimes \cdots \otimes V_N$ equal to the unit function on the product of $N$ copies of the flag manifold. With any path

$$\gamma : 1 \xrightarrow{F_{i_1}} Q_1 \xrightarrow{F_{i_2}} Q_2 \longrightarrow \cdots \longrightarrow Q_{r-1} \xrightarrow{F_{i_{r-1}}} Q, r = l(Q)$$

associate a polynomial in $t$'s:

$$f_{\gamma}(t) = \sum_{p_1 < \cdots < p_{r}, p_i \in m^{-1}(j_i)} t_{p_1} t_{p_2} \cdots t_{p_r}.$$  

( This is the only point where the decomposition $w = r_{m_1} r_{m_2} \cdots r_{m_l}$ enters the calculation.) Denote by $\Gamma^j$ the subgraph of $\Gamma$ consisting of all vertices connected with $Q^j_1$ by an oriented path. It is equivalently defined as a subgraph generated by all vertices $(-1)^{(n-j)r}Q^j_{i_1, i_2, \ldots i_k} z^{-r}$ with the fixed superscript $j$. Set

$$P^j_w(x, z; t_1, t_2, \cdots, t_l) = \sum_{\gamma \in \mathcal{P}(Q)} Q \sum_{\gamma \in \mathcal{P}(Q)} f_{\gamma}(t).$$  

(6)
Theorem 2.5

\[ K_w(t_1, t_2, \cdots, t_l) = \prod_{j=1}^{l} \Gamma(-\beta_j)^{-1} \prod_{p=1}^{N} \prod_{j=1}^{n} \{ P_{wp}(x(p), z_p, t_1, t_2, \cdots, t_l) \}^{\mu(p)} \prod_{j=1}^{l} t_j^{-\beta_j - 1}, \]

(7)

where \( \mu^{(p)} = (\mu_1^{(p)}, \ldots, \mu_n^{(p)}) \), \( 1 \leq p \leq N \), is a highest weight of \( V_p \) and \( x^{(p)}, 1 \leq p \leq N \), is a coordinate in the \( p \)-th copy of the flag manifold.

This theorem can be proved by induction on \( l \) using Lemma 2.3 and Lemma 2.4.

Let \( M \) be the local system of continuous branches of \( K_w(t_1, \cdots, t_l) \) over the Domain of \( K_w(t_1, \cdots, t_l) \) (say \( D \)). Then finally we obtain

Theorem 2.6

For any \( \sigma \in H_l(M, D) \), the integral

\[ \Psi_\sigma \int_{(t_1, t_2, \cdots, t_l)} K_w(t_1, t_2, \cdots, t_l) dt_1 dt_2 \cdots dt_l \]

satisfies the system \( KZ(\lambda_{N+1}, w, \lambda_1) \)

Remark. Theorem 2.6 gives solutions as an integral over a certain cycle depending on parameters \((x, z)\). These cycles belong to a homology group of a complement to a collection of hypersurfaces \( K_w(t_1, t_2, \cdots, t_l) = 0 \) with coefficients in a local system defined over this complement. Note that generically \( (l > 2) \), and much unlike the case of Schechtman-Varchenko integral formulas, \( K_w(t_1, t_2, \cdots, t_l) = 0 \) is a union of hypersurfaces not isomorphic to hyperplanes and, therefore, investigation of the integral cannot be carried out by usual methods. We have already encountered with the same phenomenon in a different but related framework. As we argued in the Introduction, our integral formulas are intimately related to \( \hat{\mathfrak{g}} \) or \( \mathfrak{g} \)-modules extended by complex powers of a Lie algebra generators. Rigorous treatment of such modules requires consideration of a Lie algebra action on sections of a local system defined over a complement to – highly non-linear – set of “shifted” Schubert cells on a flag manifold; for details see [11].

Note also that if \( l = 1, 2 \) then \( K_w(t_1, t_2, \cdots, t_l) = 0 \) is isomorphic to a union of affine hyperplanes and the number of cycles can be calculated using results of [3].

2.2 Some examples

Theorem 2.5 produces rather an algorithm to write down the kernel of the integral \((7)\) than a completely explicit formula for it: \((7)\) relies on \((6)\), while the latter is a linear combination of explicitly given polynomials \( Q^j_{i_1, i_2, \ldots, i_k} z^{-r} \) with coefficients in the form \( \sum_{p, p_1} t_{p_1} \cdots t_{p_r} \) determined by the combinatorial data. We have been able to “resolve” the combinatorial part of the formula in the cases
Solutions to KZ equations

$g = \mathfrak{sl}_2, \mathfrak{sl}_3$. Although the $\mathfrak{sl}_2$--case was treated in [15], we discuss here both in a unified way for completeness.

**The $\mathfrak{sl}_2$--case.** In this case the flag manifold is $\mathbb{CP}^1$, the big cell is $\mathcal{C} \subset \mathbb{CP}^1$. Fix a coordinate $x$ on $\mathcal{C}$. Then the matrix $T$ (via which the polynomials $Q_{i_1,\ldots,i_k}$ are defined) is given by $T = (x)$. The set of all $Q_{i_1,\ldots,i_k}$ consists of 2 elements: $Q^1 = 1, Q^1_1 = x$. The vertices of the graph $\Gamma$ are all of the form: $A^\alpha_{ij}(x,z) = z^{-i}x^\epsilon, \epsilon = 0, 1, i = 0, 1, 2, 3, \ldots$ Further, $\Gamma$ coincides with $\Gamma^1$ and is given by

$$
1 \xrightarrow{F_1} x \xrightarrow{F_0} z^{-1} \xrightarrow{F_1} z^{-1}x \xrightarrow{F_0} z^{-2} \xrightarrow{F_1} z^{-2}x, \ldots
$$

Observe that the Weyl group of $\hat{\mathfrak{sl}}_2$ is a free group generated by 2 reflections $r_0, r_1$ and, therefore, each element is uniquely expanded as either

$$r_0r_1 \cdots$$

or

$$r_1r_0 \cdots$$

the second one being relevant to our calculation. Setting

$$w = \underbrace{r_1r_0 \cdots r_{\epsilon}}_{l},$$

one obtains

$$P_w(x, z; t_1, \ldots, t_l) = P^1_w(x, z; t_1, \ldots, t_l) = \sum_{\epsilon=0}^{l-\epsilon} \sum_{i=0}^{z^{-i}x^\epsilon} \sigma_{i+\epsilon}(t_1, \ldots, t_l),$$

$$\sigma_j(t_1, \ldots, t_l) = \sum_{0 \leq i_1 < i_2 < \cdots < i_j < l/2} t_{2i_1+1}t_{2i_2+1} \cdots t_{2i_j+1},$$

completing the $\mathfrak{sl}_2$--case.

**The $\mathfrak{sl}_3$--case.** The big cell is $\mathbb{C}^3$ with coordinates $x_{11}, x_{12}, x_{22}$. The matrix $\tilde{\Gamma}$ is given by $\tilde{T} = \begin{pmatrix} x_{11} & x_{12} \\ 1 & x_{22} \end{pmatrix}$. The set of all $Q_{i_1,\ldots,i_k}$ consists of 5 elements:

$$Q^1 = 1, Q^j = x_{jj} (j = 1, 2) Q^{11} = x_{11}x_{22} - x_{12}, Q^2 = x_{12}$$

The graph $\tilde{\Gamma}$ and its subgraphs $\tilde{\Gamma}^1, \tilde{\Gamma}^2$ are given by

$$\tilde{\Gamma}^1: \quad Q^1_1 \xrightarrow{F_1} Q^1_1 \xrightarrow{F_0} -z^{-1} \xrightarrow{F_1} -z^{-1}Q^1_1 \xrightarrow{F_2} -z^{-1}Q^1_1, \ldots$$

$$\tilde{\Gamma}^2: \quad Q^2_1 \xrightarrow{F_1} Q^2_2 \xrightarrow{F_0} z^{-1} \xrightarrow{F_2} z^{-1}Q^2_1 \xrightarrow{F_1} z^{-1}Q^2_1, \ldots$$
The Weyl group \( W \) of \( \hat{\mathfrak{sl}}_3 \) is realized as a group generated by reflections at a certain collection of affine lines on the plane (see [16]). These lines produce a covering of the plane by triangles, called alcoves, which \( W \) acts on effectively. Looking at this action one obtains a collection of elements of \( W \) so that a reduced decomposition of any element of \( W \) is contained in it:

Put \( c := r_0 r_1 r_2 (c \) is called a Coxeter element), then any \( w \in W \) can be written as \( w = s e r_2 r_1 r_2 \), \( u = e r_2 r_1 t = r_0 r_1 r_0 r_1 r_0 r_1 r_0 \) if \( kl \neq 0 \) and if \( kl = 0 \), then \( t \) can also be equal to \( e \).

Further one obtains

\[
P_w^j(x,z; t_1, t_2, \cdots, t_l) = \sum_{l' = 0}^{l} A^j_{l'} f^j_{l'}(t), \ j = 1, 2,
\]

where

\[
f^j_{l'}(t) = \sum_{p_1 < p_2 < \cdots < p_{l'}, p_i \in m^{-1}(ji)_3} t_{p_1} t_{p_2} \cdots t_{p_{l'}},
\]

\((k)_3 \in \{0, 1, 2\}\) signifies the residue of \( k \) modulo 3 and \( m : \{1, \ldots, l\} \to \{0, 1, 2\}\) is a function determining a reduced decomposition of \( w \),

\[
A^1_{l'} = \left\{ \begin{array}{ll}
(-z)^{-q} & \text{if } l' = 3q \\
(-z)^{-q} x_{11} & \text{if } l' = 3q + 1 \\
(-z)^{-q} (x_{11} x_{22} - x_{12}) & \text{if } l' = 3q + 2,
\end{array} \right.
\]

\[
A^2_{l'} = \left\{ \begin{array}{ll}
z^{-q} & \text{if } l' = 3q \\
z^{-q} x_{22} & \text{if } l' = 3q + 1 \\
z^{-q} x_{12} & \text{if } l' = 3q + 2.
\end{array} \right.
\]

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Solutions to KZ equations

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