Two-Qubit Pulse Gate for the Three-Electron Double Quantum Dot Qubit

Sebastian Mehl\textsuperscript{1,2} \thanks{E-mail: sebastian.mehl@rwth-aachen.de}

\textsuperscript{1}Peter Gr"{u}nberg Institute (PGI-2), Forschungszentrum J"{u}lich, D-52425 J"{u}lich, Germany
\textsuperscript{2}JARA-Institute for Quantum Information, RWTH Aachen University, D-52056 Aachen, Germany

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The three-electron configuration of gate-defined double quantum dots encodes a promising qubit for quantum information processing. We propose a two-qubit entangling gate using a pulse-gated manipulation procedure. The requirements for high-fidelity entangling operations are equivalent to the requirements for the pulse-gated single-qubit manipulations that have been successfully realized for Si QDs. Our two-qubit gate completes the universal set of all-pulse-gated operations for the three-electron double-dot qubit and paves the way for a scalable setup to achieve quantum computation.

I. INTRODUCTION

The name hybrid qubit (HQ) was coined for the encoded qubit in a three-electron configuration on a gate-defined double quantum dot (DQD)\textsuperscript{1,2} The HQ is a spin qubit in its idle configuration, but it is a charge qubit during the manipulation procedure. Recently, impressive progress was made for the single-qubit control of a HQ in Si.\textsuperscript{3,4} It was argued that single-qubit gates were implemented, whose fidelities exceed 85 \% for X-rotations and 94 \% for Z-rotations.\textsuperscript{3} These manipulations rely on the transfer of one electron between quantum dots (QDs)\textsuperscript{2,3} Subnanosecond gate pulses were successfully applied to transfer the third electron between singly occupied QDs. A two-qubit entangling gate for HQs was suggested theoretically that uses electrostatic couplings.\textsuperscript{2} If the charge configuration of one HQ is changed, then Coulomb interactions modify the electric field at the position of the other HQ. Note the equivalent construction for a controlled phase gate (CPHASE) for singlet-triplet qubits in two-electron DQDs.\textsuperscript{5}

Using Coulomb interactions for entangling operations can be critical. Even though electrostatic couplings are long-ranged, they are generally weak and they are strongly disturbed by charge noise.\textsuperscript{6} We propose an alternative two-qubit gate. Two HQs brought into close proximity enable the transfer of electrons. We construct a two-qubit gate that works similar to the pulse-gated single-qubit manipulations. Our approach requires fast control of the charge configurations on the four QDs through subnanosecond pulse times at gates close to the QDs. A two-qubit manipulation scheme of the same principle as for the single-qubit gates is highly promising because single-qubit pulse gates have been implemented with great success.\textsuperscript{3,4}

The central requirement of the entangling operation is the tuning of one two-qubit state (here $|0_L0_R\rangle$) to a degeneracy point with one leakage state $|E\rangle$. $|0_L0_R\rangle$ picks up a nontrivial phase, while all other two-qubit states evolve trivially. Note that a similar construction for an entangling operation has been implemented with impressive fidelities for superconducting qubits. The couplings to other leakage states must be avoided during the operation. We propose a two-step procedure. First, we tune $|1_L1_R\rangle$ and $|0_L1_R\rangle$ away from the initial charge configuration to protect these states from leakage. $|1_L0_R\rangle$ and $|0_L0_R\rangle$ remain unchanged at the same time. We have then reached the readout regime of the second HQ. The second part of the tuning procedure corrects the passage of $|1_L0_R\rangle$ through the anticrossing with $|E\rangle$, at a point where $|1_L0_R\rangle$ is degenerate with the leakage state $|L\rangle$. We call this anticrossing degenerate Landau-Zener crossing (DLZC) because it is described by a generalization of the Landau-Zener model.\textsuperscript{1,11,12}

We focus on pulse-gated entangling operations for HQs in gate-defined Si QDs. Even though our entangling operation is not specifically related to the material and the qubit design, gate-defined Si QDs are the first candidate where our two-qubit pulse gate might be implemented because Si QDs were used for single-qubit pulse gates.\textsuperscript{3,4} We discuss therefore specifically the noise sources that are dominant for experiments involving gate-defined Si QDs. The proposed two-qubit pulse gates can be directly implemented with the existing methods of the single-qubit pulse gates. It will turn out that high-fidelity two-qubit entangling operations require low charge noise.

The organization of this paper is as follows. Sec. \textbf{II} introduces the model to describe a pair of three-electron DQDs. Sec. \textbf{III} constructs the two-qubit gate. Sec. \textbf{IV} discusses the noise properties of the entangling operation, and Sec. \textbf{V} summarizes all the results.

II. SETUP

We consider an array of four QDs, which are labeled by QD\textsubscript{1}−QD\textsubscript{4} [cf. Fig. \textbf{1}]. One qubit is encoded using a three-electron configuration on two QDs. QD\textsubscript{1} and QD\textsubscript{2} encode HQ\textsubscript{L}, and QD\textsubscript{3} and QD\textsubscript{4} encode HQ\textsubscript{R}. We describe the system by a Hubbard model, which includes two orbital states at each QD. The transfer of electrons between neighboring QDs is possible but weak, unless the system is biased using electric gates. A large global magnetic field is applied, which separates states of different $s_z$ energetically. The $S = \frac{1}{2}$, $s_z = \frac{1}{2}$ spin subspace of three electrons encodes a qubit.\textsuperscript{13}

The single-qubit states for HQ\textsubscript{L} are $|1_L\rangle = \sqrt{\frac{2}{3}} |\downarrow T_+\rangle - \sqrt{\frac{1}{3}} |\uparrow T_0\rangle$ and $|0_L\rangle = |\uparrow S\rangle$. The first
entry in the state notation labels electrons at QD$_1$, and the second entry labels electrons at QD$_2$. QD$_1$ is singly occupied, but two electrons are paired at QD$_2$. $|S⟩ = c_{i1}^{\dagger} c_{i1}^{\dagger} |0⟩$ is the two-electron singlet state at QD$_1$, $|T_+⟩ = c_{i1}^{\dagger} c_{i2}^{\dagger} |0⟩$, $|T_0⟩ = \frac{1}{\sqrt{2}} (c_{i1}^{\dagger} c_{i2}^{\dagger} + c_{i2}^{\dagger} c_{i1}^{\dagger}) |0⟩$, and $|T_-⟩ = c_{i1}^{\dagger} c_{i2}^{\dagger} |0⟩$ are triplet states at QD$_1$. $c_{iσ}^{\dagger}$ is the creation operator of one electron in state $|i⟩$ of QD$_1$, with spin $σ$, $|i⟩$ and $|T⟩$ are the ground state and the first excited state at QD$_1$, respectively, and $|0⟩$ is the vacuum state. Similar considerations hold for HQ$_L$ and HQ$_R$. The four red QDs encode two HQs; we call them HQ$_L$ and HQ$_R$. Black dots represent electrons. We apply voltages to gates close to the QDs providing universal single-qubit control and realizes a CPHASE gate by the transfer of single electrons between the QDs. We describe the manipulation protocols in the text. The encoding scheme can be scaled up trivially, as shown by the blue QDs.

### III. TWO-QUBIT PULSE GATE

Two-qubit operations are constructed using the transfer of electrons between neighboring QDs. We describe the charge transfer between (1, 2, 1, 2) and (1, 2, 2, 1) by $\mathcal{H}_{34} = \sum_{σ \in \{↑, ↓\}} (c^{\dagger}_1 c_4 σ + H.c.)$, where $\mathcal{T}_1$, $\mathcal{T}_2$ are tunnel couplings between states from neighboring QDs, and H.c. labels the Hermitian conjugate of the preceding term. $\epsilon_{34} = eV_2 - eV_3$ describes the transfer of electrons through applied voltages at gates close to QD$_2$ and QD$_4$. Lowering the potential at QD$_2$ compared to QD$_4$ favors (1, 2, 2, 1) ($\epsilon_{34} > 0$), but (1, 2, 1, 2) is favored for the opposite case ($\epsilon_{34} < 0$). (1, 2, 1, 2) and (1, 2, 2, 1) have identical energies at $\epsilon_{34} = Δ_{34} > Ω_{L,R}$. Similar considerations hold for the manipulation between (1, 2, 1, 2) and (1, 1, 2, 2), which is described by $\epsilon_{23} = eV_2 - eV_3$ and $\mathcal{H}_{23} = \sum_{σ \in \{↑, ↓\}} (c^{\dagger}_1 c_3 σ + H.c.) + \sum_{σ \in \{↑, ↓\}} (c^{\dagger}_2 c_3 σ + H.c.).$ (1, 2, 1, 2) and (1, 1, 2, 2) have identical energies at $\epsilon_{23} = 0 = Δ_{23} > Ω_{L,R}.$

We construct an entangling operation in a two-step manipulation procedure, which is shown in Fig. 2. In the first step, we tune $\epsilon_{43}$ and pulse from (1, 2, 1, 2) towards (1, 2, 2, 1). Only $|1_R⟩$ is transferred to $|B⟩ = |S⟩ |↑⟩_R$ because $|1_R⟩$ is unfavored energetically compared to $|0_R⟩$, which remains in (1, 2). The tuning uses a rapid pulse to $\epsilon_{43} = Δ_{43} > Ω_{L,R}$. $\mathcal{H}_{34}$ couples $|1_R⟩$ and $|B⟩$ by $\sqrt{\frac{3}{2}} \mathcal{T}_2$. The occupations of $|1_R⟩$ and $|B⟩$ swap after the waiting time $t_1 = \frac{h}{2 \sqrt{6} \epsilon_{34}}$. Afterwards, we pulse to $\epsilon_{43} = \epsilon_{13}^*$, which is far away from all anticrossings. $|B⟩$ and $|0_R⟩$ have the energy difference $Ω_{R}$ at $\epsilon_{34} = \epsilon_{13}^*$. Note that $\epsilon_{43} = \epsilon_{13}^*$ is in the readout regime of HQ$_R$: $|1_R⟩$ is in (2, 1), but $|0_R⟩$ is in (1, 2).

In the second step, gate pulses modify $\epsilon_{23}$ at fixed $\epsilon_{43} = \epsilon_{13}^*$. The charge configuration is pulsed towards (1, 1, 2, 2). States in (1, 2, 2, 1) remain unchanged because they need the transfer of two electrons to reach (1, 1, 2, 2). We introduce the states:

$$|L⟩ = \left[\sqrt{\frac{1}{6}} |↑⟩_T |0⟩_T + \sqrt{\frac{3}{2}} |↑⟩_T |↑⟩_T + \frac{1}{2\sqrt{3}} |↓⟩_T |T_+⟩\right] \otimes |S⟩,$$

$$|β⟩ = \frac{1}{2} \left[\sqrt{2} |↑⟩_T |0⟩_T + |↑⟩_T |↓⟩ + |↓⟩_T |↑⟩\right] \otimes |S⟩.$$  

(1)

(2)

$$|E⟩ = |↑↑SS⟩$$

is the ground state in (1, 1, 2, 2) with $s_z = 1$. $\mathcal{H}_{23}$ couples $|0_L0_R⟩$, $|1_L0_R⟩$, $|L⟩$, and $|E⟩$, while $|β⟩$ is decoupled. When approaching (1, 1, 2, 2), first the anticrossing of $|1_L0_R⟩$, $|L⟩$, and $|E⟩$ is reached at $\epsilon_{23} = Δ_{23} - Ω_{L}:$

$$\mathcal{H}_{23} (\epsilon_{23}) \approx \begin{pmatrix}
Ω_L & 0 & \frac{1}{\sqrt{6}} \\
0 & Ω_L & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} & Δ_{23} - \epsilon_{23}
\end{pmatrix}.$$  

(3)
The passage through the anticrossing at $\epsilon_{23} = \Delta_{23} - \Omega_L$ is critical for the construction of the entangling operation. $\mathcal{H}_{23}$ describes within the subspace $\{ |1_L0_R, |L1 L R \rangle, \} \rangle$ a DLZC [cf. Eq. (3)]. A basis transformation partially diagonalizes $\mathcal{H}_{23}$: $|T_1\rangle = \frac{1}{\sqrt{3}} |1_L0_R\rangle - \frac{2\sqrt{2}}{3} |L\rangle$ and $|E\rangle$ have the overlap $\sqrt{2/3}t_4$, but $|T_2\rangle = \frac{2\sqrt{2}}{3} |1_L0_R\rangle + \frac{1}{\sqrt{3}} |L\rangle$ is decoupled. $|T_1\rangle$ and $|E\rangle$ swap at $\epsilon_{23} = \Delta_{23} - \Omega_L$ after $t_2 = \frac{h}{2\sqrt{6}\tau_3}$. We introduce the waiting time $t_w$ at $\epsilon_{23} = \epsilon_{23}$, where $|E\rangle$ has the energy $\Omega_L/2$. $t_w$ must compensate after the full cycle the relative phase evolution between $|T_1\rangle$ and $|T_2\rangle$, as a consequence, $|1_L0_R\rangle$ does not leak to $|L\rangle$. Simple mathematics shows that this is the case for $t_w = h \left( \frac{\pi}{\sqrt{2}t_3} - \frac{\pi}{t_3} \right) > 0$ with $n \in \mathbb{N}$.

The time evolution at $\epsilon_{23} = \Delta_{23}$ constructs the central part of the entangling gate. $\mathcal{H}_{23}$ couples $|0_L0_R\rangle$ and $|E\rangle$ by $\tau_3$. The states on the subspace $\{ |0_L0_R\rangle, |E\rangle \}$ pick up a $\pi$-phase factor after the waiting time $t_\pi = \frac{h}{2\sqrt{6}t_3}$: $e^{-i\tau_3} = -1$. All other states of the computational basis evolve trivially with the energies $\Omega_L$, $\Omega_R$, and $\Omega_L + \Omega_R$. Finally the setup is tuned back to the initial configuration with swaps at $\epsilon_{23} = \Delta_{23} - \Omega_L$ and $\epsilon_{43} = \Delta_{43} - \Omega_R$ generated after the waiting time $t_2 = \frac{h}{2\sqrt{6}t_3}$ and $t_1 = \frac{h}{2\sqrt{6}t_3}$.

In total, the described pulse cycle realizes a CPHASE gate in the basis $\{ |L11 R\rangle, |L10 R\rangle, |0_L1 R\rangle, |0_L0 R\rangle \}$ when permitting additional single-qubit phase gates:

$$U_{43} \Delta_{43} - \Omega_R(t_1) U_{23} \Delta_{23} - \Omega_L(t_2) U_{23} \Delta_{23} - \Omega_L(t_\pi) \frac{e^{i\pi\phi}}{2} Z_L^p \frac{e^{i\phi}}{2} Z_R^p \text{ CPHASE},$$

with $Z_i^\phi = e^{-i2\pi\sigma_i^y}$, $p_1 = \Omega_R\left( \frac{1}{t_3} - \frac{2\sqrt{2/3}}{t_4} - \frac{4n}{W_L} \right)$, and $p_2 = \Omega_L\left( \frac{1}{t_3} - \frac{2\sqrt{2/3}}{t_4} \right)$. $U_\epsilon(t)$ describes the time evolution at $\epsilon$ for the waiting time $t$. We have constructed a phase shift on HQ$_R$ conditioned on the state of HQ$_L$.

IV. GATE PERFORMANCE AND NOISE PROPERTIES

In general, two-qubit pulse gates are fast. The only time consuming parts of the entangling gate are the waiting times at $\epsilon_{43} = \Delta_{43} - \Omega_R$, $\epsilon_{23} = \Delta_{23} - \Omega_L$, $\epsilon_{23} = \epsilon_{23}$, and $\epsilon_{23} = \Delta_{23}$. The overall gate time is on the order of $O\left( \frac{h}{\tau_3}, \frac{h}{\tau_3}, \frac{h}{\tau_3} \right)$. It was shown that tunnel couplings between QDs of a DQD in Si reach $3 \mu$eV. Two DQDs might be some distance apart from each other; nevertheless, $\mu$eV tunnel couplings seem possible. An entangling gate will take only few nanoseconds but requires subnanosecond pulses.

The setup provides a rich variety of leakage states. Appx. B introduces an extended state basis in $s_z = 1$. We consider the charge configurations $(1, 2, 1, 2)$, $(2, 2, 2, 1)$,
and \((1, 1, 2, 2)\), but we neglect doubly occupied triplets at QD\(_1\) and QD\(_3\). We assumed in the gate construction that the tunnel couplings are only relevant around state degeneracies, which is justified for vanishing \(\tau_i, i = 1, \ldots, 4\), compared to \(\Omega_L\) and \(\Omega_R\). In reality, \(\tau_i\) are small compared to \(\Omega_L\) and \(\Omega_R\), but they are not negligible. As a consequence, modifications from the anticrossings partially lift the neighboring state crossings (cf. the blue and purple circles in Fig. 2) and modify the energy levels and anticrossings. Fig. 3 shows that high-fidelity gates having only small leakage are possible, when the waiting times and the waiting positions introduced earlier are adjusted numerically. Small leakage errors and minor deviations from a CPHASE gate are reached for \(\tau_i/\Omega_{L,R} < 5\%\), \(i = 1, \ldots, 4\). We use \(\Omega/h = \Omega_L/h = \Omega_R/h = 15\) GHz and \(\tau/h = \delta\) GHz, \(i = 1, \ldots, 4\) in the following noise analysis (cf. Ref. [17] for a similar noise discussion).

\[
\begin{align*}
|G_1(G_2)| \approx 0.1 & \quad \text{for } \Omega = \Omega_L = \Omega_R \\
|P_{\text{Leak}}| & \approx \frac{|U_{PQ}|^2}{2} \\
\tau/\Omega & \approx 0.02 \quad \text{to } 0.1 \\
\end{align*}
\]

Figure 3. Numerically optimized gate sequences according to Eq. (4) for \(\Omega = \Omega_L = \Omega_R\) and \(\tau = \tau_i, i = 1, \ldots, 4\). We minimize numerically the deviations of the Makhlin invariants from \(G_1 = 0\) and \(G_2 = 1\) and the leakage errors \(P_{\text{Leak}}\) by adjusting the waiting times and waiting positions. \(P_{\text{Leak}} = |U_{PQ}|^2\) is the transition probability from the computational subspace \(P\) to the leakage subspace \(Q\). The points describe single numerical results; the solid lines are a polynomial fit. Note that small \(\tau/\Omega\) permit better gates.

A. Charge Noise

Charge traps of the heterostructure introduce low-frequency electric field fluctuations\(^{14,20}\). Their influence is weak for spin qubits, but it increases for charge qubits.\(^{21,22}\) Consequently, HQs are protected from charge noise only in the idle configuration. We model charge noise by an energy fluctuation between different charge configurations. We introduce no fluctuations during one gate simulation, but use modifications between successive runs. The fluctuations follow a Gaussian probability distribution of rms \(\delta\epsilon\). Note that we simulate the numerically optimized gate sequence of Eq. (4).

Fig. 4 shows the gate fidelity \(F\), which is defined in Appx. A, while \(\delta\epsilon\) is varied. \(F\) decreases rapidly with \(\delta\epsilon\). A Gaussian decay is seen for small \(\delta\epsilon\). The decay constant shows that \(\tau\) is the relevant energy scale of the entangling gate. The coherence is lost if \(\delta\epsilon\) increases beyond \(\tau\) because a typical gate misses the anticrossings of Fig. 2. Noisy gate sequences keep only the diagonal entries of the density matrix, but they remove all off-diagonal entries leading to \(F = 0.25\).

Charge noise was measured in GaAs QDs to cause energy fluctuations of the magnitude \(1\) \(\mu\text{eV}/h \approx 0.2\) GHz. For high-fidelity pulse-gated entangling operations, \(\delta\epsilon\) must be smaller than \(\tau\) that reaches typically a few \(\mu\text{eV}\) in Si HQs.

B. Hyperfine Interactions

Nuclear spins couple to HQs, and they cause low-frequency magnetic field fluctuations.\(^{5,23}\) The applied magnetic fields \(|E_z/h > 3\) GHz \((>100\) mT)\] are larger than the uncertainty in the magnetic field from the nuclear spins in typical QD experiments in Si \(|\delta E_z/h < 3\) MHz \((<100\) \(\mu\text{T})\]. We restrict the error analysis to the total \(s_z = 1\) subspace. We simulate the numerically optimized pulse sequence of Eq. (4) under magnetic field fluctuations. The variations of the magnetic fields at every QD are determined by a Gaussian probability distribution with the rms \(\delta E_z\) (in energy units).

Fig. 3 shows that \(F\) decreases rapidly with \(\delta E_z\). Again, a Gaussian decay is observed with a decay constant determined by \(\tau\) for small \(\delta E_z\). The influence of hyperfine interactions differs from charge noise. Local magnetic fields lift the state crossings that are protected by the spin-selection rules (cf. blue markings in Fig. 2). Not only is the coherence lost for large \(\delta E_z\), but leakage further suppresses \(F\). We can approximate the limit of large \(\delta E_z\) with \(F = 9/64\). All off-diagonal entries of the density matrix are removed. Additionally, some states are mixed with leakage states. \(|\uparrow_L\downarrow_R\rangle\) goes to a mixed state with three other states; \(|\uparrow_L\uparrow_R\rangle\) and \(|\downarrow_L\uparrow_R\rangle\) mix with one other state each.

Si is a popular QD material because the number of finite-spin nuclei is small.\(^{24}\) Nevertheless, noise from nuclear spins was identified to be dominant in the first spin qubit manipulations of gate-defined Si QDs.\(^{16}\) \(\delta E_z/h = 7.5 \cdot 10^{-4}\) GHz in natural Si (cf. Ref. [25]) is sufficient for nearly perfect two-qubit pulse gates. The fluctuations of the nuclear spins decrease further for isotopically purified Si instead of natural Si.

V. CONCLUSION

We have constructed a two-qubit pulse gate for the HQ - an encoded qubit in a three-electron configuration on a gate-defined DQD. Applying fast voltage pulses at gates close to the QDs enables the transfer of single electrons between QDs. We tune the setup to the anticrossing of \(|\uparrow_L\uparrow_R\rangle\) with the leakage state \(|\uparrow\rangle\). \(|\downarrow_L\uparrow_R\rangle\) picks up a nontrivial phase without leaking to \(|\uparrow\rangle\), while all other
two-qubit states accumulate trivial phases. The main challenge of the entangling gate is to avoid leakage to other states. We use a two-step procedure. (1) we pulse the right HQ to the readout configuration. Here, $|1_R\rangle$ goes to (2,1), but $|0_R\rangle$ stays in (1,2). (2) $|0_L1_R\rangle$ passes through a DLZC during the pulse cycle. The pulse profile is adjusted to avoid leakage after the full pulse cycle. Note that an adiabatic manipulation protocol can substitute the pulse-gated manipulation.

Cross-couplings between anticrossings, charge noise, and nuclear spin noise introduce errors for our pulse-gated two-qubit operation. Cross-couplings of anticrossings are problematic as they open state crossings. Also these mechanism slightly influence the energy levels and the sizes of the anticrossings. Reasonably small values of $\tau/\Omega \lesssim 5\%$ still permit excellent gates through pulse shaping. Charge noise is problematic because the gate tunes the HQs between different charge configurations. Current QD experiments suggest that charge noise is critical for our pulse-gated entangling operation. Nuclear spins are unimportant for the pulse-gated entangling operation of HQs in natural Si and, even more, for isotopically purified Si. We are hopeful that material improvements and advances in fabrication techniques for Si QDs still allow an experimental realization of this gate in the near future.

Pulse gates provide universal control of HQs through single-qubit operations, which have been implemented experimentally and our two-qubit entangling gate. Because this setup can be scaled up trivially (cf. Fig. 1), further experimental progress should be stimulated to realize all-pulse-gated manipulations of HQs.

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Appendix A: Fidelity Description of Noisy Gates

We describe a noisy operation $U^\xi_n$ with a parameter $\xi$ that modifies the gate between different runs of the experiment and obeys a classical probability distribution $f(\xi)$. The entanglement fidelity is a measure for the gate performance:

$$
F(\xi) = \text{tr} \left\{ \rho_R^{RS} 1_R \otimes [U_{-1}^{-1}U_n^\xi]_S \rho^{RS} 1_R \otimes [(U_n^\xi)^{-1}U_{-1}]_S \right\}.
$$

$U_i$ describes the ideal time evolution. The state space is doubled to two identical Hilbert spaces $R$ and $S$. $\rho^{RS} = |\psi\rangle\langle\psi|$ is a maximally entangled state on the larger Hilbert space, e.g. $|\psi\rangle = (|0000\rangle + |0110\rangle + |1001\rangle + |1111\rangle)/\sqrt{2}$. The gate fidelity $F$ is calculated by averaging Eq. (A1) over many instances of $U^\xi_n$, giving $F = \int d\xi f(\xi) F(\xi)$. $F = 1$ for perfect gates. This definition captures also leakage errors.

Appendix B: Extended Basis

Tab. [I] provides an extended state basis in $s_z = 1$ for the description of two HQs in (1,2,1,2), (1,2,2,1), and (1,1,2,2). We neglect states with a doubly occupied configuration, giving in total $|L\rangle$, $|1_LB\rangle$, and $|0_LB\rangle$ are partially filled during the manipulation procedure. All other states are leakage states that are ideally unfiltered during the manipulation. The states describe the spin configurations at QD$i$, $i = 1,\ldots,4$ of the array of four QDs, and they are grouped into subspaces of equal energy.

It is straightforward to prove that the 23 states in Tab. I are a complete set to describe the six-electron spin problem of two HQs. Note that the discussion is restricted to total $s_z = 1$. One needs two additional spin-$1/2$ electrons compared to the spin-$1$ electrons in the (1,2,1,2) configuration, giving in total $\binom{6}{3} = 15$ choices. In the (1,2,2,1) and (1,1,2,2) configurations, the electrons at QD$_2$ and at QD$_4$ are always paired to a singlet (because it is strongly unfavored to reach a triplet at these QDs), giving $\binom{4}{3} = 4$ choices to reach $s_z = 1$. 

Figure 4. Fidelity analysis for numerically optimized CPHASE gates under charge noise (black) and nuclear spin noise (red) at $\Omega_L/h = \Omega_R/h = 15$ GHz and $\tau/h = \tau_i/h = 0.5$ GHz, $i = 1,\ldots,4$. Energy fluctuations $\delta\epsilon$ between different charge configurations model charge noise. Nuclear spins cause local, low-frequency magnetic field fluctuations of the energy $\delta E_z$. We describe both noise sources by a classical probability distribution with the rms $\delta\epsilon$ (for charge noise) and $\delta E_z$ (for nuclear spin noise). The fidelity $F$ is extracted from 1000 gate simulations according to Eq. (I). Increasing uncertainties suppress $F$ strongly till it saturates at 0.25 (for charge noise) and 9/64 (for nuclear spin noise) (cf. the vertical lines). The initial decay of $F$ is described by a Gaussian decay law (cf. the dotted lines).
Further details are given in the text. We include all relevant states for the electron configurations Table I. Extended state basis with the total spin quantum number

\[ |1_{L1_R}⟩ = \left( \sqrt{\frac{1}{2}} |↓ T_+⟩ - \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) |↑ S⟩ \]
\[ |α_1⟩ = \left( \sqrt{\frac{1}{2}} |↑ T_+⟩ + \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) \left( \sqrt{\frac{1}{2}} |↓ T_+⟩ - \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) \]
\[ |α₂⟩ = \left( \sqrt{\frac{1}{2}} |↑ T_+⟩ - \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) \left( \sqrt{\frac{1}{2}} |↓ T_+⟩ + \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) \]
\[ |α₃⟩ = \left( \sqrt{\frac{1}{2}} |↑ T_+⟩ + \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) \left( \sqrt{\frac{1}{2}} |↓ T_+⟩ + \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) \]
\[ |α₄⟩ = |↑ T_− T_+⟩ \]
\[ |α₅⟩ = |↑ T_+ T_−⟩ \]
\[ |α₆⟩ = |↑ T_+ T̅₁ T₀⟩ \]
\[ |α₇⟩ = |↓ T₀ T₃ T₀⟩ \]

\[ (|1_{2,1}_R⟩ = \left[ \sqrt{\frac{1}{2}} |↓ T_+⟩ - \sqrt{\frac{1}{2}} |↑ T_0⟩ \right] |↑ S⟩ \]
\[ |L⟩ = \left[ \frac{1}{2} |↑ T_+ t⟩ - \sqrt{\frac{1}{2}} |↑ T_+ d⟩ + \frac{1}{2} |↓ T_+ t⟩ \right] |S⟩ \]
\[ |β⟩ = \left( \frac{1}{2} |↑ t_{T_+} d⟩ + \frac{1}{2} |↓ T_+ t⟩ \right) \left( \sqrt{\frac{1}{2}} |↑ T_0 t⟩ \right) |S⟩ \]
\[ |0_{L1_R}⟩ = |↑ S⟩ \left( \sqrt{\frac{1}{2}} |↓ T_+⟩ - \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) \]
\[ |γ₁⟩ = |↑ S⟩ \left( \sqrt{\frac{1}{2}} |↓ T_+⟩ + \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) \]
\[ |γ₂⟩ = |↓ S T₄ T₃⟩ \]
\[ |0_{L0_R}⟩ = |↑ S T₄⟩ \]

\[ |1_{L1_B}⟩ = \left( \sqrt{\frac{1}{2}} |↓ T_+⟩ - \sqrt{\frac{1}{2}} |↑ T_0⟩ \right) |S t⟩ \]
\[ |δ₁⟩ = \left( \sqrt{\frac{1}{2}} |↓ T_+ t⟩ + \sqrt{\frac{1}{2}} |↑ T_0 t⟩ \right) |S t⟩ \]
\[ |δ₂⟩ = |↑ T_+ S d⟩ \]
\[ |0_{L1_B}⟩ = |↑ S d t⟩ \]
\[ |1_{2,2}⟩ \]
\[ |μ₁⟩ = |↑ t S T₀⟩ \]
\[ |μ₂⟩ = |↑ d S T₀⟩ \]
\[ |1_{1,1}⟩ \]
\[ |E⟩ = |↑ t S t⟩ \]

Table I. Extended state basis with the total spin quantum number \( s_z = 1 \) for the setup of six electrons distributed over four QDs. Each entry of the states describes a spin configuration at one of the QDs with the notation \( |QD_1, QD_2, QD_3, QD_4⟩ \). We include all relevant states for the electron configurations \( (n_{QD_1}, n_{QD_2}, n_{QD_3}, n_{QD_4}) = (1, 2, 1, 2), (1, 2, 2, 1), \) and \( (1, 1, 2, 2) \). Further details are given in the text.
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