Krein’s strings whose spectral functions are of polynomial growth

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Abstract In the case of Krein’s strings with spectral functions of polynomial growth a necessary and sufficient condition for the Krein’s correspondence to be continuous is given.

1. Introduction

Let \( \mathcal{M} \) be the totality of nondecreasing, right-continuous functions on \([0, \infty)\) satisfying
\[
m(0-) = 0, \quad m(x) \leq \infty,
\]
and set
\[
\begin{align*}
\ell &= \inf\{x \geq 0; m(x) = \infty\}, \\
a &= \inf\{x \geq 0; m(x) > 0\}.
\end{align*}
\]
For \( m \in \mathcal{M} \) denote \( \varphi_\lambda(x), \psi_\lambda(x) \) the solutions to
\[
\begin{align*}
\varphi_\lambda(x) &= 1 - \lambda \int_0^x (x - y) \varphi_\lambda(y) \, dm(y), \\
\psi_\lambda(x) &= x - \lambda \int_0^x (x - y) \psi_\lambda(y) \, dm(y),
\end{align*}
\]
and define
\[
h(\lambda) = \lim_{x \to \ell} \frac{\psi_\lambda(x)}{\varphi_\lambda(x)} = \int_0^\ell \varphi_\lambda(x)^{-2} \, dx.
\]
Then it is known that there exists a unique measure \( \sigma \) on \([0, \infty)\) satisfying
\[
h(\lambda) = a + \int_0^\infty \frac{1}{\xi - \lambda} \, d\sigma(\xi),
\]
and conversely, \( h \) determines \( m \) uniquely. Conventionally it is understood that for \( m \in \mathcal{M} \) taking \( \infty \) identically on \([0, \infty)\) the \( h \) vanishing identically corresponds, and for \( m \in \mathcal{M} \) vanishing identically on \([0, \infty)\) the \( h \) taking identically \( \infty \) corresponds. This is the theorem obtained by Krein [8], and \( m \) is called Krein’s (regular) string. Later Kasahara [1] established the continuity for the correspondence and applied it to show limit theorems for 1D diffusion processes with

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m their speed measures. Recently Kotani [7] extended Kasahara’s result to a certain kind of singular strings m, namely, to m which is a nondecreasing and right-continuous function on \((−\infty, \infty)\) satisfying

\[ m(−\infty) = 0, \quad m(x) \leq \infty, \]

and

\[ \int_{−\infty}^{a} x^2 \, dm(x) < \infty \]  

for some a. When the condition (1.1) is satisfied, the boundary \(-\infty\) is called the limit circle type for the associated generalized second-order differential operator \(d^2/dm \, dx\). In this case he introduced a new \( h \) by

\[
h(\lambda) = \lim_{x \to −\infty} (x + \varphi_\lambda(x) \int_{x}^{l} \frac{dy}{\varphi_\lambda(y)^2}) = a + \int_{0}^{\infty} \left( \frac{1}{\xi - \lambda} - \frac{\xi}{\xi^2 + 1} \right) d\sigma(\xi),
\]

which satisfies

\[
h'(\lambda) = \int_{−\infty}^{l} \frac{\partial}{\partial \lambda} \varphi_\lambda(x)^{-2} \, dx,
\]

and proved the continuity of the correspondence between \( m \) and \( h \). Probabilistic applications of this result were given by Kasahara and Watanabe [2], [3], and it was interpreted from the point of view of the excursion theory by Yano [9]. In this article we consider \( m \) satisfying a milder condition than (1.1), namely,

\[ \int_{−\infty}^{a} |x| \, dm(x) < \infty, \]

and obtain the continuity result under additional conditions on \( m \), which allows any power growth of the spectral measures at \( \infty \).

2. Preliminaries

Let \( m(x) \) be a nondecreasing and right-continuous function on \((−\infty, \infty)\) satisfying

\[ m(−\infty) = 0, \quad m(\infty) \leq \infty. \]

Set

\[ l = \sup\{ x > −\infty, m(x) < +\infty \}, \quad l_+ = \sup \supp dm, \quad l_- = \inf \supp dm. \]

Note that \( m(l) = \infty \) if \( l < \infty \). Assume that

\[ \int_{−\infty}^{a} |x| \, dm(x) < \infty \]  

with some \( a \in (l_-, l_+) \). Let \( \mathcal{E} \) be the totality of nondecreasing functions \( m \) satisfying (2.1). We exclude \( m \) vanishing identically on \((−\infty, \infty)\) from \( \mathcal{E} \). One can regard \( dm \) as a distribution of weight, and in this case \( m \) works as a string. On the other hand, one can associate a generalized diffusion process with generator \( L \):
\[ L = \frac{d}{dm} \frac{d}{dx} \]

if we impose a suitable boundary condition if necessary. The condition (2.1) is called an entrance condition in 1D diffusion theory developed by W. Feller, so we say that \( m \) satisfies (2.1), a string of entrance type. For an entrance type \( m \), it is easy to show that for \( \lambda \in \mathbb{C} \) an integral equation

\[ \varphi(x) = 1 - \lambda \int_{-\infty}^{x} (x - y) \varphi(y) \, dm(y) \]

has a unique solution, which is denoted by \( \varphi_\lambda(x) \). Introduce a subspace

\[ L^2_0(dm) = \{ f \in L^2(dm); \text{supp}\, f \subset (-\infty, l) \}, \]

and for \( f \in L^2_0(dm) \) define a generalized Fourier transform by

\[ \widehat{f}(\lambda) = \int_{-\infty}^{l} f(x) \varphi_\lambda(x) \, dm(x). \]

Krein’s spectral theory implies that there exists a measure \( \sigma \) on \([0, \infty)\) satisfying

\[ \int_{-\infty}^{l} |f(x)|^2 \, dm(x) = \int_{0}^{\infty} |\widehat{f}(\xi)|^2 \, d\sigma(\xi) \quad \text{for any } f \in L^2_0(dm); \]

\( \sigma \) is called a spectral measure for the string \( m \). The nonuniqueness of such \( \sigma \) occurs if and only if

\[ l_+ + m(l_+) < \infty. \quad (2.3) \]

The number \( l(\geq l_+) \) possesses its meaning only when (2.3) is satisfied, and in this case there exists a \( \sigma \) satisfying (2.2) with the boundary condition

\[ f(l_+) + (l - l_+) f^+(l_+) = 0 \]

at \( l_+ \). Here \( f^+ \) is the derivative from the right-hand side. If \( l = \infty \), then this should be interpreted as

\[ f^+(l_+) = 0. \]

At the left boundary \( l_- \) no boundary condition is necessary if \( l_- = -\infty \), and if \( l_- > -\infty \), then we impose the reflective boundary condition, namely,

\[ f^-(l_-) = 0 \quad \text{the derivative from the left.} \]

Generally, for a string \( m \) of entrance type it is known that for \( \lambda < 0 \) there exists uniquely \( f \) such that

\[
\begin{cases}
-Lf = \lambda f, & f > 0, f^+ \leq 0, f(l-) = 0, \\
f(x)\varphi^+(x) - f^+(x)\varphi_\lambda(x) = 1.
\end{cases}
\]

This unique \( f \) is denoted by \( f_\lambda \) and contains information of the boundary condition we are imposing on \(-L\) at the right boundary \( l_+ \), and \( f_\lambda \) can be represented by \( \varphi_\lambda \) as

\[ f_\lambda(x) = \varphi_\lambda(x) \int_{x}^{l} \frac{dy}{\varphi_\lambda(y)^2}. \quad (2.4) \]
The right-hand-side integral is always convergent for $\lambda < 0$, because if supp $dm \neq \phi$, then choosing $a \in$ supp $dm$, we see for $x > a$,

$$\varphi_{\lambda}(x) \geq 1 - \lambda \int_{-\infty}^{x} (x - y) \, dm(y) \geq 1 - \lambda \int_{-\infty}^{a} (x - y) \, dm(y) \geq 1 - \lambda (x - a) m(a);$$

hence

$$\int_{x}^{l} \frac{dy}{\varphi_{\lambda}(y)^2} \leq \int_{x}^{l} \frac{dy}{(1 - \lambda (y - a) m(a))^2} < \infty$$

for $x > a$. If supp $dm = \phi$, then $l < \infty$ and

$$(2.5) \quad m(x) = \begin{cases} 0 & \text{for } x < l, \\ \infty & \text{for } x > l, \end{cases}$$

which implies

$$\varphi_{\lambda}(x) = \begin{cases} 1 & \text{for } x < l, \\ \infty & \text{for } x > l, \end{cases}$$

and

$$\int_{x}^{l} \frac{dy}{\varphi_{\lambda}(y)^2} = l - x < \infty.$$  

Here note that we have excluded $m = 0$ identically on $(-\infty, \infty)$; hence $l < \infty$. If $m$ is a nondecreasing function of (2.5) the spectral measure vanishes identically on $[0, \infty)$. If $m$ is $\infty$ identically on $(-\infty, \infty)$, then the spectral function $\sigma$ is defined to be $0$ identically on $[0, \infty)$. Conversely, if a spectral measure vanishes identically on $[0, \infty)$, then the associated string $m$ should be of (2.5). We note that $\varphi_{\lambda}(x)$ is an entire function of minimal exponential type as a function of $\lambda$ and the zeros of $\varphi_{\lambda}(x)$ coincide with the eigenvalues of $-L$ defined as a self-adjoint operator on $L^2(dm, (-\infty, x])$ with the Dirichlet boundary condition at $x$, which means that $\varphi_{\lambda}(x)$ has simple zeros on $(0, \infty)$. The Green function $g_{\lambda}$ for $-L$ on $L^2(dm)$ is given by

$$g_{\lambda}(x, y) = g_{\lambda}(y, x) = f_{\lambda}(y) \varphi_{\lambda}(x)$$

for $x \leq y$. The relationship between $\sigma$ and $g_{\lambda}$ is described by an identity

$$\int_{-\infty}^{l} \int_{-\infty}^{l} g_{\lambda}(x, y) f(x) \overline{f(y)} \, dm(x) \, dm(y) = \int_{0}^{\infty} \frac{|\hat{f}(\xi)|^2}{\xi - \lambda} \sigma(d\xi)$$

for any $f \in L^2(dm)$, and

$$g_{\lambda}(x, y) = \int_{0}^{\infty} \varphi_{\xi}(x) \varphi_{\xi}(y) \frac{d\sigma(\xi)}{\xi - \lambda},$$

through which $\sigma$ is determined uniquely from the string $m$. Distinct $m$’s may give an equal $\sigma$; namely, for $a \in \mathbb{R}$ a new string

$$m_{a}(x) = m(x + a)$$

defines the same $\sigma$, because

$$\varphi_{\lambda}^{a}(x) = \varphi_{\lambda}(x + a), \quad f_{\lambda}^{a}(x) = f_{\lambda}(x + a),$$
and hence
\[ g^a_\lambda(x, x) = \varphi^a_\lambda(x) f^a_\lambda(x) = g_\lambda(x + a, x + a) = \int_0^\infty \frac{\varphi_\xi(x + a)^2}{\xi - \lambda} \, d\sigma_\xi(\xi). \]

On the other hand,
\[ g^a_\lambda(x, x) = \int_0^\infty \frac{\varphi^a_\xi(x) \varphi^a_\xi(x)}{\xi - \lambda} \, d\sigma_a(\xi) = \int_0^\infty \frac{\varphi_\xi(x + a)^2}{\xi - \lambda} \, d\sigma_a(\xi); \]

hence an identity
\[ \sigma_a(\xi) = \sigma(\xi) \]
should be held. Conversely, we have the following.

**THEOREM 1 (SEE KOTANI [5], [6])**

*If two strings \( m_1 \) and \( m_2 \) of \( \mathcal{E} \) have the same spectral measure \( \sigma \), then \( m_1(x + c) = m_2(x) \) for \( a \in \mathbb{R} \).*

If we hope to obtain the continuity of the correspondence between \( m \) and \( \sigma \), we have to keep the nonuniqueness in mind. Namely, for \( m \) of \( \mathcal{E} \) a sequence \( \{ m_n \}_{n \geq 1} \) of \( \mathcal{E} \) defined by
\[ m_n(x) = m(x - n) \]
converges to the trivial function 0 as \( n \to \infty \). However, the associated \( \sigma \)'s are independent of \( n \). Therefore, we shall give several alternative definitions of convergence by imposing certain extra conditions (related to tightness) in addition to pointwise convergence. Set
\[ M(x) = \int_{-\infty}^x (x - y) \, dm(y) = \int_{-\infty}^x m(y) \, dy. \]

Then, the condition (2.1) is equivalent to
\[ M(x) < \infty \]
for \( x < l \). Using a convention
\[ [-\infty, a) = (-\infty, a), \quad (a, \infty] = (a, \infty) \quad \text{and so on,} \]
we can see that \( M \) is a nondecreasing convex function on \((-\infty, \infty)\) satisfying
\[ \begin{cases} M(x) = 0 \text{ on } (-\infty, l_-], \\ \text{continuous and strictly increasing on } [l_-, l), \\ M(x) = \infty \text{ on } (l, \infty). \end{cases} \]

For a fixed positive number \( c \), we assume that
\[ 0 \in (l_-, l) \quad \text{and} \quad M(l) \geq c \]
and normalize such an \( m \) by
\[ M(0) = c. \]
Denote by $E^{(c)}$ the set of all elements of $E$ satisfying (2.6), (2.7), and set
\[ E_+ = \bigcup_{c>0} E^{(c)}. \]

In this definition of $E_+$ among functions satisfying (2.1) any function $m$ defined by (2.5) for some $l \leq \infty$ is excluded from $E_+$. Therefore, $E \setminus E_+$ consists of $m$ satisfying (2.5) for some $l < \infty$. The uniqueness of the correspondence between $m$ and $\sigma$ holds under this normalization. Set
\[ S = \text{the set of all spectral measures for strings of } E. \]

Any suitable characterization of $S$ is not known yet; however, any measure on $[0, \infty)$ with polynomial growth at $\infty$ belongs to $S$.

We prepare a basic estimate for $\varphi_\lambda$; $\varphi_\lambda$ can be represented as
\[ (2.8) \quad \varphi_\lambda(x) = \sum_{n=0}^{\infty} (-\lambda)^n \phi_n(x), \]
where $\{\phi_n\}_{n \geq 0}$ are
\[ \phi_n(x) = \int_{-\infty}^{x} (x-y) \phi_{n-1}(y) \, dm(y), \quad \phi_0(x) = 1. \]

Then, the convergence of the above series can be shown by the following lemma.

**LEMMA 1**

$\varphi_\lambda$ is given by an absolute convergent series (2.8) and satisfies
\[ |\varphi_\lambda(x)| \leq \exp(|\lambda|M(x)). \]

**Proof**
First we show that for any $k \geq 0$,
\[ (2.9) \quad \phi_k(x) \leq \frac{M(x)^k}{k!} \]
holds. Observe that
\[ \phi_1(x) = \int_{-\infty}^{x} (x-y) \, dm(y) = M(x). \]

Assuming (2.9) for some $k$, we have
\[ \phi_{k+1}(x) \leq \frac{1}{k!} \int_{-\infty}^{x} (x-y) M(y)^k \, dm(y) \]
\[ = \frac{1}{k!} \int_{-\infty}^{x} \left( M(y) - k(x-y)M'(y) \right) M(y)^{k-1} M'(y) \, dy \]
\[ \leq \frac{1}{k!} \int_{-\infty}^{x} M'(y) M(y)^k \, dy = \frac{M(x)^{k+1}}{(k+1)!}, \]
which proves (2.9) for general $k$. Then the estimate of $\varphi_\lambda$ is clear. \qed
Here we clarify the convergence of a sequence of monotone functions taking value $\infty$. For a nonnegative and nondecreasing function $m$ which may take $\infty$, set
\[
\hat{m}(x) = \frac{2}{\pi} \tan^{-1} m(x), \quad x \in \mathbb{R}.
\]
Then we have
\[
\hat{m}(x) \in [0,1]
\]
and a right-continuous nondecreasing function satisfying
\[
0 \leq \hat{m}(-\infty) \leq \hat{m}(x) \leq \hat{m}(l-) \leq \hat{m}(l) = 1,
\]
if $l < \infty$. A sequence of nonnegative and nondecreasing functions $m_n$ is defined to converge to $m$ as $n \to \infty$ if
\[
(2.10) \quad \hat{m}_n(x) \to \hat{m}(x)
\]
holds at any point of continuity of $\hat{m}(x)$.

**Lemma 2**
Suppose that $m_n \in \mathcal{E}$ converges to $m \in \mathcal{E}$ as $n \to \infty$. Then it holds that
\[
\lim_{n \to \infty} l_n \geq l.
\]

**Proof**
Let $x < l$ be a point of continuity for $\hat{m}$. Then
\[
\hat{m}_n(x) \to \hat{m}(x) < 1;
\]
hence
\[
\hat{m}_n(x) < 1
\]
for every sufficiently large $n$, which implies $x < l_n$ and completes the proof. \(\square\)

The continuity of the correspondence from $\mathcal{E}$ to $\mathcal{S}$ is not hard to show. Let $m_n, m$ be strings of $\mathcal{E}$, and define the convergence of $m_n$ to $m$ by

(A) $m_n(x) \to m(x)$ for every point of continuity of $m$,
(B) $\lim_{x \to -\infty} \sup_{n \geq 1} M_n(x) = 0$.

**Theorem 2**
Suppose that $m_n \in \mathcal{E}$ converge to $m \in \mathcal{E}$. Then, for every $\lambda < 0$ the Green functions $g^{(n)}_{\lambda}(x,y)$ of the string $m_n$ converge to the Green function $g_{\lambda}(x,y)$ of $m$ for any $x,y < l$. In particular the spectral functions $\sigma_n(\xi)$ converge to $\sigma(\xi)$ at every point of continuity of $\sigma$.

**Proof**
Under the conditions it is easy to see that the $\varphi$-functions $\varphi^{(n)}_{\lambda}(x)$ of $m_n$ converge to the $\varphi$-function $\varphi_{\lambda}(x)$ of $m$ compact uniformly with respect to $(x,\lambda) \in$
\((−∞, l) \times \mathbb{C}\) from the uniform bound for \(φ^{(n)}_λ\) due to Lemma 1. Moreover, if \(m(a) > 0\) at some \(a\), a point of continuity of \(m\), then there exists a positive constant \(C\) such that

\[ φ^{(n)}_λ(y) ≥ 1 − λM_n(y) ≥ 1 + C(y − a) \]

holds for any \(y > a\); hence

\[ f^{(n)}_λ(x) = φ^{(n)}_λ(x) \int_x^{l_n} \frac{1}{φ^{(n)}_λ(y)^2} \, dy \]

also converge to \(f_λ(x)\). If \(\text{supp } m = φ\), namely, \(m(x) = 0\) identically on \((−∞, l)\), then

\[ f^{(n)}_λ(x) → l − x \]

if \(l < ∞\). The case \(l = ∞\) is excluded. Consequently, we have

\[ g^{(n)}_λ(x, y) = φ^{(n)}_λ(y) f^{(n)}_λ(x) → φ_λ(y) f_λ(x) = g_λ(x, y) \]

for any \(y ≤ x < l\). The identity

\[ g^{(n)}_λ(x, y) = \int_0^∞ \frac{φ^{(n)}_ξ(x) φ^{(n)}_ξ(y)}{ξ − λ} \, dσ_n(ξ) \]

shows the last statement of the theorem. \(\square\)

3. Scales and estimates by trace

The straight converse statement of Theorem 2 is hopeless to be true, because there is no characterization for a measure \(σ\) on \([0, ∞)\) to be a spectral measure of \(a\) for \(m \in \mathcal{E}\). Therefore we prove the converse continuity of the correspondence by imposing a condition on \(\{σ_n\}\). In the process of the proof we have to estimate \(φ_λ(x)^{-2}\) in terms of \(m\). A better way to investigate \(φ_λ(x)^{-2}\) is to use probabilistic methods. Recall that for each fixed \(a\), \(φ_λ(a)\) has simple zeros \(\{μ_n\}_{n \geq 1}\) which are eigenvalues of \(-L\) on \((−∞, a]\) with Dirichlet boundary condition at \(x = a\). Since the Green function for this operator is

\[ a − (x \vee y), \]

we see that

\[ \sum_{n=1}^{∞} μ_n^{-1} = \text{tr}(-L)^{-1} = \int_{−∞}^a (a − x) \, dm(x) = M(a) < ∞. \]

Choosing a \(b < l\), we denote by \(φ^b_λ(ψ^b_λ)\) the solutions of

\[ \frac{− d}{dm} \frac{d}{dx} f = λf, \quad \text{with } f(b) = 1, f'(b) = 0 \left( f(b) = 0, f'(b) = 1 \right), \] respectively.

Then we see that an identity

\[ φ_λ(x) = φ_λ(b) φ^b_λ(x) + φ'_λ(b) ψ^b_λ(x) \]

holds. Lemma 1 implies that \(φ_λ(b)\) and \(φ'_λ(b)\) are entire functions of at most exponential type \(M(b)\) as functions of \(λ\). On the other hand, \(φ^b_λ(x)\) and \(ψ^b_λ(x)\)
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are entire functions of order at most $1/2$ as functions of $\lambda$. Therefore we know that $\varphi_\lambda(x)$ is an entire function of minimal exponential type, which combined with (3.1) shows that

$$\varphi_\lambda(a) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\mu_n}\right).$$

For the detail refer to [5, p. 441]. Now let $\{X_n\}_{n \geq 1}$ be independent random variables, each of which has an exponential distribution of mean 1. Then, an identity

$$E \exp\left(\lambda \sum_{n=1}^{\infty} \mu_n^{-1} X_n \right) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\mu_n}\right)^{-1} = \varphi_\lambda(a)^{-1}$$

holds. Therefore, letting $\{\tilde{X}_n\}_{n \geq 1}$ be independent copies of $\{X_n\}_{n \geq 1}$ and setting

$$Y_n = X_n + \tilde{X}_n, \quad X = \sum_{n=1}^{\infty} \mu_n^{-1} Y_n,$$

we have

$$(3.2) \quad \varphi_\lambda(a)^{-2} = E \exp(\lambda X).$$

We denote $X = X(a)$ if necessary, because the eigenvalue $\{\mu_n\}_{n \geq 1}$ depends on the boundary $a$.

**LEMMA 3**

Suppose that the spectral measure $\sigma$ of an $m \in \mathcal{E}$ satisfies

$$p(t) = \int_{0}^{\infty} e^{-\xi t} d\sigma(\xi) < \infty \quad \text{for any } t > 0.$$

Then, for any nonnegative Borel measurable function $f$ on $[0, \infty)$,

$$\int_{-\infty}^{l} E f(X(x)) \, dx = \int_{0}^{\infty} p(t) f(t) \, dt$$

holds by permitting the integrals to take the value $\infty$ simultaneously.

**Proof**

From (2.4) it follows that for any $x < l$ and $\lambda < 0$,

$$\int_{x}^{l} E e^{\lambda X(y)} \, dy = E e^{\lambda X(x)} \int_{0}^{\infty} \frac{\varphi_\lambda(x)^2}{\xi - \lambda} d\sigma(\xi) = \int_{0}^{\infty} E e^{\lambda(X(x) + t)} p(t, x, x) \, dt$$

holds, where $p(t, x, y)$ is the transition probability density defined by

$$p(t, x, y) = \int_{0}^{\infty} e^{-t\xi} \varphi_\lambda(x) \varphi_\lambda(y) \, d\sigma(\xi).$$

Hence a functional monotone class theorem shows that the identity below holds for any nonnegative bounded continuous function $f$ on $[0, \infty)$:

$$(3.4) \quad \int_{x}^{l} E(f(X(y))e^{\lambda X(y)}) \, dy = \int_{0}^{\infty} E(f(X(x) + t)e^{\lambda(X(x) + t)})p(t, x, x) \, dt.$$
Since, for $t > 2M(x)$,
\[
p(t, x, x) = \int_0^\infty e^{-\xi t} \varphi_\xi(x)^2 d\sigma(\xi) \leq \int_0^\infty e^{-\xi t} e^{2\xi M(x)} d\sigma(\xi) = p(t - 2M(x))
\]
holds, assuming $f(t) = 0$ for $t < \epsilon$, we see that
\[
\int_{-\infty}^l E(f(X(y))e^{\lambda X(y)}) dy = \int_0^\infty f(t)e^{\lambda t}p(t) dt
\]
by letting $x \to -\infty$. Here we have used the fact that
\[
X(x) \to 0 \quad \text{as} \quad x \to -\infty.
\]
The rest of the proof is routine. □

Now we define a scale function $\phi$ on $[0, 1]$ as a function satisfying the following properties:

(S.1) $\phi$ is strictly increasing, convex and $\phi(0) = 0$, $\phi'(1-) < \infty$;

(S.2) for each $x > 0$,
\[
\lim_{y \downarrow 0} \frac{\phi(xy)}{\phi(y)} < \infty;
\]

(S.3) for each $x \in (0, 1]$,
\[
\lim_{y \downarrow 0} \frac{\phi(xy)}{\phi(y)} > 0.
\]

The property (S.1) enables us to extend $\phi$ linearly to $[1, \infty)$, namely,
\[
\phi(x) = \phi(1) + \phi'(1-)(x - 1)
\]
for $x > 1$. Then $\phi$ becomes a nonnegative, convex, and nondecreasing function on $[0, \infty)$. Throughout the paper $\phi$ is always extended to $[1, \infty)$ linearly in this way. A regularly varying function at 0 satisfies the conditions (S.2) and (S.3). Set
\[
C_+(x) = \sup_{y > 0} \frac{\phi(xy)}{\phi(y)} < \infty, \quad C_-(x) = \inf_{y \in (0, 1]} \frac{\phi(xy)}{\phi(y)} > 0.
\]
Then $C_+$ becomes nonnegative, convex, and nondecreasing on $[0, \infty)$. It satisfies the submultiplicative property
\[
C_+(xy) \leq C_+(x)C_+(y)
\]
for any $x, y > 0$; hence, setting
\[
\alpha_+ = \sup_{x > 1} \frac{\log C_+(e^x)}{x} \in [0, \infty)
\]
we see that
\[
C_+(x) \leq x^{\alpha_+}
\]
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holds for any \( x \geq e \). We note that \( \alpha_+ \) should not be less than 1 due to the convexity of \( C_+ \). Since

\[
\phi(xy) \leq C_+(x)\phi(y)
\]

holds for any \( x, y > 0 \), we have

\[
\phi(x) \geq \frac{\phi(1)}{C_+(1/x)} \geq \phi(1)x^{\alpha_+}
\]

for any \( x \in [0, 1/e] \). Therefore the property (S.2) restricts \( \phi \) not to decay faster than with a power order:

(3.5) \quad \phi(xy) \leq C_+(x)\phi(y).

\( C_- \) satisfies

\[
C_-(xy) \geq C_-(x)C_-(y)
\]

for any \( x, y \in [0, 1] \). Typical examples for functions satisfying (S.1)~(S.3) are

\[
\phi(x) = x^\alpha, \quad x^\alpha(c - \log x),
\]

where \( \alpha \geq 1 \) and \( c \) is a sufficiently large positive constant.

**LEMMA 4**

Let \( \{Y_n\}_{n \geq 1} \) be a sequence of identically distributed nonnegative random variables with mean \( \mu \), let \( \{\lambda_n\}_{n \geq 1} \) be a nonnegative sequence satisfying

\[
\sum_{n=1}^\infty \lambda_n < \infty,
\]

and set

\[
X = \sum_{n=1}^\infty \lambda_n Y_n.
\]

Then, we have the following.

(1) If \( \phi \) satisfies (S.1), then

\[
\phi(EX) \leq E\phi(X).
\]

(2) If \( \phi \) satisfies (S.1) and (S.2), then

\[
E\phi(X) \leq \left( EC_+ \left( \frac{Y_1}{\mu} \right) \right) \phi(EX).
\]

**Proof**

Jensen’s inequality implies the inequality in (1). To show the second inequality we set

\[
m_n = m^{-1}\lambda_n, \quad m = \sum_{k=1}^\infty \lambda_k.
\]
Then (3.5) implies
\[ \phi(X) = \phi\left( m\mu \sum_{n=1}^{\infty} \frac{m_n Y_n}{\mu} \right) \leq \phi(m\mu)C_+ \left( \sum_{n=1}^{\infty} \frac{m_n Y_n}{\mu} \right). \]

Since the function $C_+$ is convex, we have
\[ C_+ \left( \sum_{n=1}^{\infty} \frac{m_n Y_n}{\mu} \right) \leq \sum_{n=1}^{\infty} m_n C_+ \left( \frac{Y_n}{\mu} \right) \]
and
\[ E\phi(X) \leq \phi(m\mu) \sum_{n=1}^{\infty} m_n EC_+ \left( \frac{Y_n}{\mu} \right) = \phi(EX)EC_+ \left( \frac{Y_1}{\mu} \right). \]

Let $X$ be defined as in (3.2), and set
\[ C_\phi = EC_+ \left( \frac{Y_1}{\mu} \right) = \int_0^{\infty} t e^{-t} C_+ (t/2) \, dt < \infty. \]

**Lemma 5**

We have the following.

1. If $\phi$ satisfies (S.1), then
   \[ E(\phi(X)e^{\lambda X}) \geq \varphi_\lambda(a)^{-2}\phi \left( \int_{-\infty}^{a} \varphi_\lambda(x)^2 \, dm(x) \int_x^{a} \varphi_\lambda(y)^{-2} \, dy \right). \]

2. If $\phi$ satisfies (S.1) and (S.2), then
   \[ E(\phi(X)e^{\lambda X}) \leq C_\phi \varphi_\lambda(a)^{-2}\phi \left( \int_{-\infty}^{a} \varphi_\lambda(x)^2 \, dm(x) \int_x^{a} \varphi_\lambda(y)^{-2} \, dy \right). \]

**Proof**

For a fixed $\lambda < 0$ let $Z$ be a nonnegative random variable satisfying
\[ Ee^{\mu Z} = \varphi_{\lambda+\mu}(a)^{-2}\varphi_\lambda(a)^2 = \prod_{n=1}^{\infty} \left( 1 - \frac{\mu}{\mu_n - \lambda} \right)^{-2}. \]

Then, note an identity
\[(3.6) \quad E(\phi(X)e^{\lambda X}) = \varphi_\lambda(a)^{-2}E(\phi(Z)), \]
which can be shown from the observation
\[ \frac{\partial^k}{\partial \lambda^k} \varphi_\lambda(a)^2 = \varphi_\lambda(a)^{-2} \frac{\partial^k}{\partial \mu^k} (\varphi_{\lambda+\mu}(a)^{-2} \varphi_\lambda(a)^2) \bigg|_{\mu=0} \]
for any $k \geq 0$, because this implies the identity when $\phi(x) = x^k$. To apply Lemma 4 to $Z$ we need to compute $EZ$. If we denote the Green operator for $L$ on $(-\infty, a]$ with Dirichlet boundary condition at $a$ by $G_\lambda$, then
\[ G_\lambda(x,y) = \varphi_\lambda(y) \varphi_\lambda(x) \int_x^{a} \varphi_\lambda(z)^{-2} \, dz \quad \text{for } x \geq y. \]
Hence
\[ EZ = \sum_{j=1}^{\infty} \frac{1}{\mu_j - \lambda} = \text{tr} G_\lambda = \int_{-\infty}^{a} \varphi_\lambda(x)^2 \, dm(x) \int_{x}^{a} \varphi_\lambda(y)^{-2} \, dy \]
holds, and we have the inequalities in the statement. \( \square \)

The right-hand side of the inequalities in Lemma 5 can be estimated further.

**Lemma 6**
For \( \lambda < 0 \) the following inequalities are valid:

1. \( \int_{-\infty}^{a} \varphi_\lambda(x)^2 \, dm(x) \int_{x}^{a} \varphi_\lambda(y)^{-2} \, dy \geq M(a) \varphi_\lambda(a)^{-2} \),
2. \( \int_{-\infty}^{a} \varphi_\lambda(x)^2 \, dm(x) \int_{x}^{a} \varphi_\lambda(y)^{-2} \, dy \leq M(a) \wedge \left( \log \varphi_\lambda(a)^{-1} \right) \).

**Proof**
Inequality (1) and the first inequality of (2) follow from the monotonicity of \( \varphi_\lambda(z) \); namely, we have
\[ \int_{x}^{a} \varphi_\lambda(y)^{-2} \, dy \geq \varphi_\lambda(a)^{-2}(a-x), \quad \int_{x}^{a} \varphi_\lambda(y)^{-2} \, dy \leq \varphi_\lambda(x)^{-2}(a-x), \]
which implies
\[ \int_{-\infty}^{a} \varphi_\lambda(x)^2 \, dm(x) \int_{x}^{a} \varphi_\lambda(y)^{-2} \, dy \geq \varphi_\lambda(a)^{-2} \int_{-\infty}^{a} (a-x) \, dm(x) = \varphi_\lambda(a)^{-2} M(a) \]
and
\[ \int_{-\infty}^{a} \varphi_\lambda(x)^2 \, dm(x) \int_{x}^{a} \varphi_\lambda(y)^{-2} \, dy \leq \int_{-\infty}^{a} (a-x) \, dm(x) = M(a). \]
The second inequality of (2) follows by using the equation satisfied by \( \varphi_\lambda(x) \),
\[ d\varphi_\lambda(y) = -\lambda \varphi_\lambda(y) \, dm(y), \]
which yields
\[ -\lambda \int_{-\infty}^{a} \varphi_\lambda(y)^2 \, dm(y) \int_{y}^{a} \varphi_\lambda(z)^{-2} \, dz \]
\[ = \int_{-\infty}^{a} \varphi_\lambda(y) \, d\varphi'_\lambda(y) \int_{y}^{a} \varphi_\lambda(z)^{-2} \, dz \]
\[ = \varphi_\lambda(y) \varphi'_\lambda(y) \int_{y}^{a} \varphi_\lambda(z)^{-2} \, dz \bigg|_{-\infty}^{a} \]
\[ - \int_{-\infty}^{a} \varphi'_\lambda(y)^2 \, dy \int_{y}^{a} \varphi_\lambda(z)^{-2} \, dz + \int_{-\infty}^{a} \frac{\varphi'_\lambda(y)}{\varphi_\lambda(y)} \, dy. \]
Noting
\[ \varphi_\lambda(y) \varphi'_\lambda(y) \int_{y}^{a} \varphi_\lambda(z)^{-2} \, dz \sim_{y \to -\infty} -\lambda m(y)(a-y) \sim_{y \to -\infty} 0, \]
we see that
\[-\lambda \int_{-\infty}^{a} \varphi_\lambda(y)^2 \, dm(y) \int_y^{a} \varphi_\lambda(z)^{-2} \, dz = \log \varphi_\lambda(a) - \int_{-\infty}^{a} \varphi'_\lambda(y)^2 \, dy \int_y^{a} \varphi_\lambda(z)^{-2} \, dz \leq \log \varphi_\lambda(a),\]

which completes the proof. \[\Box\]

As the last lemma in this section, we have the following.

**LEMMA 7**

The following two estimates hold.

1. For any function $\phi$ satisfying (S.1) it holds that
   \[\int_0^{\infty} p(t) \phi(t) e^{\lambda t} \, dt \geq \int_{-\infty}^{l} \phi(M(x) \varphi_\lambda(x)^{-2}) \varphi_\lambda(x)^{-2} \, dx.\]

2. For any function $\phi$ satisfying (S.1), (S.2) it holds that
   \[\int_0^{\infty} p(t) \phi(t) e^{\lambda t} \, dt \leq C_\phi \int_{-\infty}^{l} \phi(M(x)) \varphi_\lambda(x)^{-2} \, dx + C_\phi \frac{-\lambda}{\varphi'_\lambda(a)} \int_0^{\infty} \phi(t) e^{\lambda t} \, dt.\]

**Proof**

All we have to show is an estimate of the integral
\[\int_{a}^{l} \phi \left( \frac{\log \varphi_\lambda(x)}{-\lambda} \right) \varphi_\lambda(x)^{-2} \, dx.\]

Noting the monotonicity of $\varphi_\lambda(x)$, $\varphi'_\lambda(x)$, and $\varphi_\lambda(x) \geq 1$, we see that
\[\int_{a}^{l} \phi \left( \frac{\log \varphi_\lambda(x)}{-\lambda} \right) \varphi_\lambda(x)^{-2} \, dx \]
\[= \int_{\varphi_\lambda(a)}^{\varphi_\lambda(l)} \phi \left( \frac{\log z}{-\lambda} \right) \frac{1}{z^2 \varphi'_\lambda(z)} \, dz \]
\[\leq \frac{1}{\varphi'_\lambda(\varphi_\lambda^{-1}(a))} \int_{\varphi_\lambda(a)}^{\varphi_\lambda(l)} \phi \left( \frac{\log z}{-\lambda} \right) \frac{1}{z^2} \, dz \leq \frac{-\lambda}{\varphi'_\lambda(a)} \int_0^{\infty} \phi(t) e^{\lambda t} \, dt. \quad \Box\]

**4. Continuity of the correspondence from $S$ to $E$**

In this section we give a partial converse of Theorem 2. The lemma below will be useful later.

**LEMMA 8**

Let $m_n \in E$, and let $\sigma_n$ be its spectral function. Suppose that
\[
\text{(4.1)} \quad \lim_{n \to \infty} M_n(l_n) = 0
\]
holds. Then $\sigma_n(\xi) \to 0$ for any $\xi > 0$. 
Proof
Since \( M_n(l_n) < \infty \), we have \( l_n < \infty \). Set \( \tilde{m}_n(x) = m_n(x + l_n) \). Then

\[
(4.2) \quad \tilde{M}_n(0) \to 0,
\]

and its spectral measure coincides with \( \sigma_n \). Since the condition (4.2) implies

\[
\tilde{m}_n(x) \to \begin{cases} 
0 & \text{for } x < 0, \\
\infty & \text{for } x > 0
\end{cases}
\]
in \( \mathcal{E} \), Theorem 2 shows \( \sigma_n \to 0 \). \( \square \)

**Theorem 3**

Let \( m_n \in \mathcal{E} \), and let \( \sigma_n \) be its spectral function satisfying

\[
p_n(t) = \int_0^\infty e^{-t\xi} d\sigma_n(\xi) < \infty \quad \text{for any } t > 0
\]

and

\[
(4.3) \quad \sup_{n \geq 1} \int_0^1 p_n(t)\phi(t) dt < \infty
\]

for a function \( \phi \) satisfying (S.1). Assume that there exists a nontrivial measure \( \sigma \) on \([0, \infty)\) satisfying

\[
\sigma_n(\xi) \to \sigma(\xi)
\]
at every point of continuity of \( \sigma \). Then

\[
\lim_{n \to \infty} p_n(t) = p(t), \quad \lim_{n \to \infty} M_n(l_n) > 0
\]

hold. Choose \( c \) such that

\[
0 < c < \lim_{n \to \infty} M_n(l_n),
\]

and define \( a_n \) by the solution \( M_n(a_n) = c \). Then there exists a unique \( m \in \mathcal{E}^{(c)} \) with spectral measure \( \sigma \), and it holds that \( m_n(c + a_n) \to m \) in \( \mathcal{E} \); hence \( \sigma \in \mathcal{S} \).

Proof
For any \( \epsilon > 0 \) and any \( N > 0 \),

\[
\int_0^\epsilon p_n(t)\phi(t) dt = \int_0^\epsilon \phi(t) dt \int_0^\infty e^{-t\xi} d\sigma_n(\xi) \geq e^{\epsilon N} \int_0^\infty e^{-2\epsilon\xi} d\sigma_n(\xi) \int_0^\epsilon \phi(t) dt
\]
holds, and the condition (4.3) implies that there exists a constant \( C_\epsilon \) such that

\[
\int_N^\infty e^{-2\epsilon\xi} d\sigma_n(\xi) \leq e^{\epsilon N} \int_0^\epsilon p_n(t)\phi(t) dt \int_0^\epsilon \phi(t) dt \leq e^{\epsilon N} C_\epsilon
\]
is valid for any \( n, N \), which yields

\[
(4.4) \quad p_n(t) \to p(t) = \int_0^\infty e^{-t\xi} d\sigma(\xi)
\]
as \( n \to \infty \). Applying (3.3) to \( \phi(t)e^{\lambda t} \) for \( \lambda < 0 \) shows that
\[
\int_{-\infty}^{t_n} E(\phi(X_n(x))e^{\lambda X_n(x)}) \, dx = \int_0^{\infty} p_n(t)e^{\lambda t}\phi(t) \, dt
\]
\[
\leq \int_0^{1} p_n(t)e^{\lambda t}\phi(t) \, dt + p_n(1) \int_1^{\infty} e^{\lambda t}\phi(t) \, dt \leq C
\]
with a constant \( C \), where \( X_n(x) \) is defined by the eigenvalues \( \{\mu_j(x)\}_{j \geq 1} \) corresponding to \( m_n \). Since we assume that the limiting spectral measure \( \sigma \) is non-trivial, Lemma 8 shows
\[
\lim_{n \to \infty} M_n(l_n) > 0.
\]
For \( c \) such that
\[
0 < c < \lim_{n \to \infty} M_n(l_n),
\]
define \( a_n \) by the solution \( M_n(a_n) = c \), and set
\[
\tilde{m}_n(x) = m_n(x + a_n).
\]
Then \( \tilde{m}_n \in \mathcal{E}(c) \) and an inequality (1) of Lemma 7 imply
\[
\int_{-\infty}^{t_n} E(\phi(X_n(x))e^{\lambda X_n(x)}) \, dx \geq \int_{-\infty}^{t_n} \varphi_\lambda^{(n)}(x)^{-2}\phi(\tilde{M}_n(x)\varphi_\lambda^{(n)}(x)^{-2}) \, dx,
\]
and from Lemma 1 we have
\[
\int_{-\infty}^{t_n} \varphi_\lambda^{(n)}(x)^{-2}\phi(\tilde{M}_n(x)\varphi_\lambda^{(n)}(x)^{-2}) \, dx \geq \int_{-\infty}^{t_n} e^{2\lambda\tilde{M}_n(x)}\phi(\tilde{M}_n(x)e^{2\lambda}) \, dx
\]
\[
\geq \int_{-\infty}^{0} e^{2\lambda c}\phi(\tilde{M}_n(x)e^{2\lambda c}) \, dx.
\]
Thus
\[
\int_{-\infty}^{0} e^{2\lambda c}\phi(\tilde{M}_n(x)e^{2\lambda c}) \, dx \leq C
\]
is valid for any \( n \geq 1 \). Therefore, for any \( x < 0 \),
\[
e^{2\lambda c}(-x)\phi(\tilde{M}_n(x)e^{2\lambda c}) \leq \int_{-\infty}^{0} e^{2\lambda c}\phi(\tilde{M}_n(y)e^{2\lambda c}) \, dy \leq C,
\]
which implies that \( \{\tilde{m}_n\}_{n \geq 1} \) has a convergent subsequence in the sense of the convergence in \( \mathcal{E} \), namely, the convergence under conditions (A) and (B). Since we have proved (4.4), the uniqueness of the spectral measure in \( \mathcal{E}(c) \) and Theorem 2 complete the proof. \( \square \)

**COROLLARY 1**
*Suppose that a measure \( \sigma \) on \([0, \infty)\) satisfies*
\[
\int_0^{1} p(t)\phi(t) \, dt < \infty
\]
with a function \( \phi \) on \([0, 1]\) satisfying (S.1). Then, there exists an \( m \in E \) with spectral measure \( \sigma \) in \( S \).

**Proof**
Define \( \sigma_n \) by
\[
\sigma_n(\xi) = \begin{cases} 
\sigma(\xi) & \text{for } \xi < n, \\
\sigma(n) & \text{for } \xi \geq n.
\end{cases}
\]
Then this \( \sigma_n \) satisfies all the conditions of Theorem 3. Applying the theorem we easily obtain the corollary. \( \square \)

This corollary provides a plenty of spectral measures in \( S \) growing faster than any power order at \( \infty \).

**5. Continuity of the correspondence between \( E_\phi \) and \( S_\phi \)**

In this section we give a necessary and sufficient condition for the continuity of the correspondence by restricting the order of growth of spectral measures at \( \infty \). We call \( \phi \) a scale function if it satisfies conditions (S.1), (S.2), and (S.3). For a scale function \( \phi \) set
\[
E_\phi = \left\{ m \in E : \int_{-\infty}^{a} \phi(M(x)) \, dx < \infty \text{ for } \exists a \in (l_-, l_+) \right\},
\]
and
\[
S_\phi = \left\{ \sigma : \int_{1}^{\infty} \tilde{\phi}(\xi) \sigma(d\xi) < \infty \right\},
\]
where
\[
\tilde{\phi}(\xi) = \int_{0}^{\infty} e^{-t\xi} \phi(t) \, dt.
\]
It is easy to see that
\[
\int_{1}^{\infty} \tilde{\phi}(\xi) \sigma(d\xi) < \infty \iff \int_{0}^{1} p(t) \phi(t) \, dt < \infty.
\]
Moreover, from the properties of scales, it is always valid that for \( \sigma \in S_\phi \),
\[
\int_{1}^{\infty} \xi^{-\alpha-1} \sigma(d\xi) < \infty
\]
for an \( \alpha \geq 1 \). On the other hand, if \( m \in E_\phi \), Lemma 7(2) yields
\[
\int_{0}^{\infty} p(t) \phi(t) e^{\lambda t} \, dt \leq C_\phi \int_{-\infty}^{a} \phi(M(x)) \varphi_\lambda(x)^{-2} \, dx + C_\phi \frac{-\lambda}{\varphi_\lambda(a)} \int_{0}^{\infty} \phi(t) e^{\lambda t} \, dt
\]
\[
\leq C_\phi \int_{-\infty}^{a} \phi(M(x)) \, dx + C_\phi \frac{-\lambda}{\varphi_\lambda(a)} \int_{0}^{\infty} \phi(t) e^{\lambda t} \, dt.
\]
Therefore, it holds that
\[
\int_{0}^{\infty} p(t) \phi(t) e^{\lambda t} \, dt < \infty,
\]
which shows $\sigma \in S_\phi$. Conversely, assume $\sigma \in S_\phi$. Then
\[
\int_0^\infty p(t)\phi(t)e^{\lambda t} dt = \int_0^1 p(t)\phi(t)e^{\lambda t} dt + \int_1^\infty p(t)\phi(t)e^{\lambda t} dt \\
\leq \int_0^1 p(t)\phi(t) dt + p(1)\int_1^\infty \phi(t)e^{\lambda t} dt < \infty
\]
for any $\lambda < 0$. Therefore,
\[
\int_{-\infty}^{l} \phi(M(x)\varphi_\lambda(x)^{-2})\varphi_\lambda(x)^{-2} dx < \infty
\]
holds. Since $\phi$ satisfies the condition (S.3)
\[
\phi(M(x)\varphi_\lambda(x)^{-2}) \geq C_-(\varphi_\lambda(x)^{-2})\phi(M(x))
\]
holds. Due to
\[
\varphi_\lambda(x)^{-2} \geq e^{2\lambda M(x)}
\]
and $M(x) \to 0$ as $x \to -\infty$, we easily see that
\[
\int_{-\infty}^{a} \phi(M(x)) dx < \infty,
\]
which implies that $m \in E_\phi$. Therefore, a string belongs to $E_\phi$ if and only if its spectral measure is an element of $S_\phi$.

For $\phi$ let $m_n, m$ be strings of $E_\phi$, and define the convergence of $m_n$ to $m$ in $E_\phi$ by
\[
(C) \lim_{x \to -\infty} \sup_{n \geq 1} \int_{-\infty}^{x} \phi(M_n(y)) dy = 0,
\]
in addition to the condition $(A)$. The convergence of spectral measures in $S_\phi$ is defined by
\[
(A') \quad \sigma_n(\xi) \to \sigma(\xi) \text{ at every point of continuity of } \sigma,
\]
\[
(C') \quad \lim_{N \to \infty} \sup_{n \geq 1} \int_{-\infty}^{\infty} \phi(\xi)\sigma_n(d\xi) = 0.
\]
An equivalent statement is possible by $p(t)$:
\[
(A'') \quad p_n(t) \to p(t) \text{ for any } t > 0,
\]
\[
(C'') \quad \lim_{\epsilon \downarrow 0} \sup_{n \geq 1} \int_{0}^{\infty} p_n(t)\phi(t) dt = 0.
\]
Set
\[
E_\phi^{(c)} = E_\phi \cap \mathcal{E}^{(c)}.
\]

**THEOREM 4**

Let $\{\sigma_n\}_{n \geq 1}$, $\sigma$ be elements of $S_\phi$, and let $m_n, m \in E_\phi^{(c)}$ be the strings corresponding to $\sigma_n$, $\sigma$, respectively. Then, $m_n \to m$ in $E_\phi^{(c)}$ if and only if $\sigma_n \to \sigma$ in $S_\phi$.

**Proof**

Suppose that $m_n \to m$ in $E_\phi^{(c)}$. Then, Theorem 2 shows the validity of the condition $(A')$. Therefore we have only to check the condition $(C'')$. From Lemma 7(2),
\[ \int_0^\infty p_n(t)\phi(t)e^{\lambda t} \, dt \leq C_\phi \int_{-\infty}^0 \phi(M_n(x))\varphi^{(n)}_\lambda(x)^{-2} \, dx + C_\phi \frac{-\lambda}{\varphi^{(n)}_\lambda(0)} \int_0^\infty \phi(t)e^{\lambda t} \, dt \]

is valid. Fix \( \epsilon > 0 \), and choose \( a < 0 \) such that

\[ C_\phi \int_a^{-\infty} \phi(M_n(x)) \, dx < \epsilon \]

for any \( n \geq 1 \). Since \( M_n(x), \varphi^{(n)}_\lambda(x) \) converge to \( M(x), \varphi_\lambda(x) \) uniformly on \((-\infty, 0]\), and the estimate

\[ \varphi^{(n)}_\lambda(0) \geq 1 - \lambda M_n(0) = 1 - \lambda c \]

shows that if \( -\lambda \) is sufficiently large, then

\[ C_\phi \int_a^0 \phi(M_n(x))\varphi^{(n)}_\lambda(x)^{-2} \, dx < \epsilon \]

is valid for any \( n \geq 1 \). Moreover, due to \( c > 0 \),

\[ \lim_{n \to \infty} m_n(0) \geq m(0-) > 0 \]

holds; hence

\[ C_\phi \frac{-\lambda}{\varphi^{(n)}_\lambda(0)} \int_0^\infty \phi(t)e^{\lambda t} \, dt \leq C_\phi \frac{1}{m_n(0)} \int_0^\infty \phi(t)e^{\lambda t} \, dt < \epsilon \]

also holds for any \( n \geq 1 \) if we choose sufficiently large \( -\lambda \), which implies that

\[ \sup_{n \geq 1} \int_0^\infty p_n(t)\phi(t)e^{\lambda t} \, dt \leq 3\epsilon. \]

From

\[ \int_{-1/\lambda}^0 p_n(t)\phi(t) \, dt \leq e \int_0^\infty p_n(t)\phi(t)e^{\lambda t} \, dt \leq 3\epsilon e \]

the condition \((C''')\) is confirmed. Conversely, assume that \( \sigma_n \to \sigma \) in \( S_\phi \). Then Theorem 3 shows that the condition \((A)\) holds. Hence we have only to check the condition \((C)\). From Lemma 7(1),

\[ \int_0^\infty p_n(t)\phi(t)e^{\lambda t} \, dt \geq \int_{-\infty}^{t} \phi(M_n(x)\varphi^{(n)}_\lambda(x)^{-2}) \varphi^{(n)}_\lambda(x)^{-2} \, dx \]

follows. The property \((S.3)\) implies

\[ \phi(M_n(x)\varphi^{(n)}_\lambda(x)^{-2}) \geq C_\sigma (\varphi^{(n)}_\lambda(x)^{-2}) \phi(M_n(x)), \]

and, as was pointed out in the proof of Theorem 3, \( \varphi^{(n)}_\lambda(x) \to 1 \) as \( x \to -\infty \) uniformly with respect to \( n \). Therefore, the condition \((C''')\) guarantees the condition \((C)\). \( \square \)

The last theorem can be restated as the convergence in \( E_\phi \).

**Theorem 5**

Let \( \{\sigma_n\}_{n \geq 1} \), \( \sigma \) be elements of \( S_\phi \), and let \( m_n, m \in E_\phi \) be the strings corresponding
to \( \sigma_n, \sigma \) respectively. Assume that \( \sigma_n \to \sigma \) in \( S_\phi \) and \( \sigma \) is nontrivial. Then, there exist a sequence \( \{a_n\}_{n \geq 1} \) in \( \mathbb{R} \) and \( c > 0 \) with \( m_n(\cdot + a_n) \in \mathcal{E}_\phi^{(c)} \), and
\[
m_n(\cdot + a_n) \to m
\]
holds in \( \mathcal{E}_\phi \).

**Proof**
Since \( \sigma \) is nontrivial, we can apply Lemma 8. The rest of the proof is clear from Theorem 4. \( \square \)

For applications it will be helpful to rewrite the condition (C) as Kasahara and Watanabe did in [4].

**Lemma 9**
Assume that \( \phi \) satisfies (S.1). Then a sequence \( \{m_n\}_{n \geq 1} \) converges to \( m \) in \( \mathcal{E}_\phi \) if and only if \( \{m_n\}_{n \geq 1} \) and \( m \) satisfy the condition below.

\[\text{(D)} \quad \text{For any } x \in \mathbb{R}, \quad \int_{-\infty}^{\xi} \phi(M_n(y)) \, dy \to \int_{-\infty}^{\xi} \phi(M(y)) \, dy.\]

Similarly the set of conditions (A') and (C') is equivalent to (D'), and that of (A'') and (C'') is equivalent to (D'').

\[\text{(D')} \quad \text{For any } \lambda < 0, \quad \int_{0}^{\infty} \tilde{\phi}(\xi - \lambda) \sigma_n(d\xi) \to \int_{0}^{\infty} \tilde{\phi}(\xi - \lambda) \sigma(d\xi).\]

\[\text{(D'')} \quad \text{For any } \lambda < 0, \quad \int_{0}^{\infty} p_n(t) \phi(t)e^{t\lambda} \, dt \to \int_{0}^{\infty} p(t) \phi(t)e^{t\lambda} \, dt.\]

**Proof**
Assume that \( m_n \to m \) in \( \mathcal{E}_\phi \). Then, (C) implies that there exists \( c < l \) such that
\[
\int_{-\infty}^{c} \phi(M_n(y)) \, dy \leq 1.
\]
Hence for any \( x < c \),
\[
\phi(M_n(x))(c - x) \leq \int_{x}^{c} \phi(M_n(y)) \, dy \leq \int_{-\infty}^{c} \phi(M_n(y)) \, dy \leq 1,
\]
which shows that
\[
M_n(x) \leq \phi^{-1}\left(\frac{1}{c - x}\right)
\]
for any \( n \geq 1 \) and \( x < c \). Then it is easy to see that \( M_n(x) \to M(x) \) at every point \( x \), and this together with (C) implies (D). Conversely, for any \( \epsilon > 0 \), choose \( c < l \)
such that
\[ \int_{-\infty}^{c} \phi(M(y)) \, dy < \epsilon. \]
Then, clearly (C) follows from (D). The condition (A) can be derived from (D) by the monotonicity of \( \phi \) and \( M_n \). We omit the proof for (D') and (D''). \( \square \)

6. Application

Typical examples of \( m \) belonging to \( \mathcal{E} \) are

\[
   m_{\alpha}(x) = \begin{cases} 
   C_\alpha x^{-\beta} & x > 0 \text{ if } 0 < \alpha < 1, \\
   e^x & x \in \mathbb{R} \text{ if } \alpha = 1 \\
   C_\alpha (-x)^{-\beta} & x < 0 \text{ if } \alpha > 1,
   \end{cases} \quad \text{with } \beta = \frac{\alpha}{\alpha - 1},
\]

and the spectral measures and \( p(t) \) are

\[
   \sigma_{\alpha}(d\xi) = \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2} d\xi^\alpha, \quad p_{\alpha}(t) = \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)} t^{-\alpha},
\]

where

\[
   C_\alpha = \begin{cases} 
   \left( \frac{1-\alpha}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}}, & 0 < \alpha < 1, \\
   \left( \frac{\alpha}{\alpha - 1} \right)^{-\frac{\alpha}{\alpha - 1}}, & 1 < \alpha.
   \end{cases}
\]

In this section we consider the asymptotic behavior of the spectral measures and the transition probability densities when strings are close to the above typical ones. If \( \alpha \in (0, 2) \), the following results are already known. Here we denote

\[
   f(x) \sim g(x) \quad \text{as } x \uparrow 0 \quad (x \to \infty)
\]

if

\[
   \lim_{x \uparrow 0} \frac{f(x)}{g(x)} = 1 \quad \left( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \right)
\]

hold, respectively. Let \( \varphi \) be a function regularly varying at 0 with exponent \( \alpha - 1 \).

**Theorem 6 (See [1], [4])**

The following asymptotic relationship between \( m \) and \( p \) is valid.

1. If \( \alpha \in (0, 1) \), then

   \[
   m(x) \sim \frac{(-\beta)^\beta}{x\varphi^{-1}(x)} \quad \text{as } x \uparrow \infty
   \]

   holds if and only if

   \[
   p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)} \frac{1}{t} \varphi\left( \frac{1}{t} \right) \quad \text{as } t \to \infty.
   \]

2. If \( \alpha \in (1, 2) \), then

   \[
   m(x) \sim \frac{\beta^\beta}{-x\varphi^{-1}(-x)} \quad \text{as } x \uparrow 0
   \]

   holds if and only if
\[
p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1 + \alpha)} \frac{1}{t} \varphi \left( \frac{1}{t} \right) \quad \text{as } t \to \infty.
\]

They showed an analogous result in the case \( \alpha = 1 \) in Kasahara and Watanabe [4]. In this section we extend their results to the case \( \alpha \geq 2 \) by applying Theorem 4. The basic idea, which was first employed by Kasahara [1], is to use the continuity between \( m \) and \( p \) and the scaling relationship

\[
(6.1) \quad abm(ax) \leftrightarrow \frac{1}{ab} p(b^{-1}t)
\]

for any \( a, b > 0 \). The proof proceeds as in [4], especially in the case \( \alpha = 1 \).

Let \( m \in \mathcal{E} \) be a nondecreasing function with \( \ell = 0 \); namely, let

\[
m(x) < \infty \quad \text{on } (-\infty, 0) \quad \text{and} \quad m(x) = \infty \quad \text{on } (0, \infty).
\]

Let \( \varphi \) be a regularly varying function at 0 with exponent \( \alpha - 1 \), and set

\[
m_\nu(x) = \nu \varphi^{-1}(\nu)m(\nu x).
\]

Then from (6.1) we have

\[
\begin{cases}
M_\nu(x) = \varphi^{-1}(\nu)M(\nu x), \\
p_\nu(t) = \nu^{-1} \varphi^{-1}(\nu)^{-1} p(\varphi^{-1}(\nu)^{-1} t), \\
\sigma_\nu(\xi) = \nu^{-1} \varphi^{-1}(\nu)^{-1} \sigma(\varphi^{-1}(\nu) \xi).
\end{cases}
\]

To consider an extension of Theorem 6 we introduce conditions on \( m \) and \( \sigma \):

\[
(6.3) \quad m(x) \sim \frac{\beta^\beta}{-x \varphi^{-1}(-x)} \quad \text{as } x \uparrow 0,
\]

which means that

\[
(6.4) \quad M(x) \sim \frac{\beta^\beta}{(\beta - 1) \varphi^{-1}(-x)} \quad \text{as } x \uparrow 0,
\]

and

\[
(6.5) \quad p(t) = \int_0^\infty e^{-t \xi} d\sigma(\xi) < \infty \quad \text{for any } t > 0.
\]

**Proposition 1**

*If \( m \in \mathcal{E} \) satisfies (6.3), then it holds that*

\[
(6.6) \quad \sigma(\xi) \sim \frac{\alpha^{2\alpha}}{\Gamma(1 + \alpha)^2} \xi^\alpha \quad \text{as } \xi \downarrow 0.
\]

*Moreover, if \( m \) satisfies (6.5) as well, then (6.7) below holds:*

\[
(6.7) \quad p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1 + \alpha)} \frac{1}{t} \varphi \left( \frac{1}{t} \right) \quad \text{as } t \to \infty.
\]

**Proof**

Since

\[
M_\nu(x) = \int_{-\infty}^x m_\nu(y) dy = \varphi^{-1}(\nu)M(\nu x)
\]
holds, from (6.4) we know that

\[ M_\nu(x) \to \frac{\beta^\beta}{\beta - 1}(-x)^{1-\beta} \quad \text{as} \; \nu \to 0 \]

for any \( x < 0 \), which means that \( \{M_\nu\} \) satisfies the condition (B). Applying Theorem 2 yields

\[ \sigma_\nu(\xi) \to \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2} \xi^\alpha \quad \text{for any} \; \xi > 0 \]

as \( \nu \to 0 \), which is equivalent to (6.6) due to (6.2). If we assume the condition (6.5) as well on \( \sigma \), the Abelian theorem for the Laplace transform shows the property (6.7).

To obtain a converse statement to the above proposition we need the following.

**LEMMA 10**

Assume that \( \sigma \in \mathcal{S} \) satisfies condition (6.5) and a condition

\[ (6.8) \quad \int_0^1 p(t)\phi(t) \, dt < \infty \]

for a positive function \( \phi \) on \([0, 1]\) satisfying

\[ (6.9) \quad \phi(st) \leq Ct^k\phi(s) \quad \text{for any} \; s,t \leq 1 \; \text{for some} \; k > \alpha - 1. \]

Then \( \{p_\nu(t)\} \) satisfies condition (4.3), namely,

\[ (6.10) \quad \sup_{\nu > 0} \int_0^1 p_\nu(t)\phi(t) \, dt < \infty. \]

**Proof**

Since \( \varphi \) is a regularly varying function at 0 with exponent \( \alpha - 1 \), \( t^{-1}\varphi(t^{-1}) \) is regularly varying at \( \infty \) with exponent \( -\alpha \), and there exists a slowly varying function \( l(t) \) such that

\[ \frac{1}{t} \varphi \left( \frac{1}{t} \right) = t^{-\alpha}l(t). \]

Generally a slowly varying function \( l(t) \) has an expression

\[ (6.11) \quad l(t) = c(t) \exp \left( \int_a^t \frac{\epsilon(u)}{u} \, du \right) \]

with a positive constant \( a \) and functions \( c(t) \), \( \epsilon(t) \) behaving as

\[ c(t) \to c > 0, \quad \epsilon(t) \to 0 \quad \text{as} \; t \to \infty. \]

Now we decompose the integral in (6.10) into two parts:

\[ \int_0^1 p_\nu(t)\phi(t) \, dt = \nu^{-1}\varphi^{-1}(\nu)^{-1} \int_0^1 p(\varphi^{-1}(\nu)^{-1}t)\phi(t) \, dt = I_1 + I_2 \]
with
\[
\begin{align*}
I_1 &= \nu^{-1}\varphi^{-1}(\nu)^{-1} \int_{N\varphi^{-1}(\nu)}^{1} p(\varphi^{-1}(\nu)^{-1} t) \phi(t) dt, \\
I_2 &= \nu^{-1}\varphi^{-1}(\nu)^{-1} \int_{0}^{N\varphi^{-1}(\nu)} p(\varphi^{-1}(\nu)^{-1} t) \phi(t) dt,
\end{align*}
\]
where \(N\) is chosen so that
\[
|\epsilon(u)| \leq \delta \quad \text{for any } u \geq N
\]
holds with a positive \(\delta\) satisfying \(\delta < k - (\alpha - 1)\). Since the condition (6.7) implies
\[
0 < \frac{p(\varphi^{-1}(\nu)^{-1} t)}{\varphi^{-1}(\nu)^{\alpha - 1} t} \leq C'
\]
for any \(t \geq N\varphi^{-1}(\nu)\) with some constant \(C'\), we have
\[
I_1 \leq C' \nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha} l(\varphi^{-1}(\nu)^{-1}) \int_{N\varphi^{-1}(\nu)}^{1} t^{-\alpha} \frac{l(\varphi^{-1}(\nu)^{-1} t)}{l(\varphi^{-1}(\nu)^{-1})} \phi(t) dt.
\]
First note that
\[
\nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha} l(\varphi^{-1}(\nu)^{-1}) = \nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha}\varphi^{-1}(\nu)^{-1} \varphi(\varphi^{-1}(\nu)) = 1,
\]
and (6.11) shows, for \(t \geq N\varphi^{-1}(\nu)\), that
\[
\frac{l(\varphi^{-1}(\nu)^{-1} t)}{l(\varphi^{-1}(\nu)^{-1})} = \exp\left(\int_{a}^{\varphi^{-1}(\nu)^{-1} t} \frac{\epsilon(u)}{u} du - \int_{a}^{\varphi^{-1}(\nu)^{-1}} \frac{\epsilon(u)}{u} du\right)
\]
\[
= \exp\left(-\int_{\varphi^{-1}(\nu)^{-1} t}^{\varphi^{-1}(\nu)^{-1}} \frac{\epsilon(u)}{u} du\right)
\]
\[
\leq \exp(\delta \log t^{-1}) = t^{-\delta}.
\]
In (6.9) setting \(s = 1\), we have \(\phi(t) \leq C\phi(1) t^k\); hence
\[
I_1 \leq C \int_{N\varphi^{-1}(\nu)}^{1} t^{-\alpha - \delta + k} \phi(t) dt \leq CC' \phi(1) \int_{0}^{1} t^{-\alpha - \delta + k} dt
\]
is valid. Due to (6.9) \(I_2\) can be estimated as
\[
I_2 = \nu^{-1} \int_{0}^{N} p(s) \phi(\varphi^{-1}(\nu) s) ds
\]
\[
\leq C \nu^{-1}(\varphi^{-1}(\nu))^k \int_{0}^{N} p(s) \phi(s) ds \leq C'' \nu^{-1+\frac{k}{\alpha - 1} - \delta'} \int_{0}^{N} p(s) \phi(s) ds,
\]
where \(\delta' > 0\) can be chosen so that
\[
-1 + \frac{k}{\alpha - 1} - \delta' > 0
\]
holds. Consequently, we have
\[
\int_{0}^{1} p_\nu(t) \phi(t) dt \leq CC' \phi(1) \int_{0}^{1} t^{-\alpha - \delta + k} dt + C'' \nu^{-1+\frac{k}{\alpha - 1} - \delta'} \int_{0}^{N} p(s) \phi(s) ds,
\]
and (6.8) implies the second assertion of (6.10). \(\square\)
REMARK 1
The property (6.9) is satisfied not only by $\phi(t) = t^k$ with $k > \alpha - 1$ but also by subexponential functions: for $p > 1, c > 0$,

$$\phi(t) = \exp\left(-c(-\log t)^p\right).$$

Within the knowledge of the previous sections the best converse statement to Proposition 1 is as follows.

PROPOSITION 2
Let $m \in \mathcal{E}$ be a nondecreasing function with $l = 0$ and $M(0) = \infty$. Assume that $\sigma \in \mathcal{S}$ satisfies the conditions (6.5) and (6.8) with a positive function $\phi$ on $[0,1]$ satisfying (S.1) and (6.9). Then the property (6.6) (equivalently, (6.7)) implies (6.3).

Proof
First note that (6.6) is equivalent to

$$\sigma_\nu(\xi) \rightarrow \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2} \xi^\alpha$$

for any $\xi > 0$, as $\nu \rightarrow 0$. Since we are assuming that $M(0) = \infty$,

$$M_\nu(0) = \varphi^{-1}(\nu)M(0) = \infty$$

holds for any $\nu > 0$, and there exists uniquely $a_\nu < 0$ such that

$$M_\nu(a_\nu) = \frac{\beta^\beta}{\beta - 1} \equiv c.$$

Set

$$\tilde{M}_\nu(x) = M_\nu(x + a_\nu + 1).$$

Then, taking $-1$ instead of 0 as a normalization point, Lemma 10 makes it possible to apply Theorem 3, and we have the following:

$$\tilde{M}_\nu(x) \rightarrow \begin{cases} c(-x)^{1-\beta} & \text{for } x < 0, \\ \infty & \text{for } x > 0 \end{cases}$$

holds in $\mathcal{E}$ as $\nu \rightarrow 0$, from which

$$(6.12) \quad \varphi^{-1}(\nu)M(\nu(x + a_\nu + 1)) \rightarrow \begin{cases} c(-x)^{1-\beta} & \text{for } x < 0, \\ \infty & \text{for } x > 0 \end{cases}$$

follows. To simplify the involved formula (6.12) we take their inverse. Set

$$u = M\left(\nu(x + a_\nu + 1)\right), \quad \lambda = c\varphi^{-1}(\nu)^{-1}.$$

Since $\varphi^{-1}(\nu)M(\nu a_\nu) = c$, we easily see that

$$\varphi(c\lambda^{-1})(x + 1) + M^{-1}(\lambda) = M^{-1}(u).$$

Denoting $y = \lambda^{-1}u$, (6.12) is equivalent to

$$y \rightarrow (-x)^{1-\beta},$$
from which
\[
\frac{M^{-1}(\lambda y) - M^{-1}(\lambda)}{\varphi(c\lambda^{-1})} = x + 1 \to 1 - y^{-(\beta-1)}
\]
follows for any \(y > 0\) as \(\lambda \to \infty\). Since \(\varphi\) is regularly varying at 0 with exponent \(\alpha - 1\),
\[
\frac{\varphi(c\lambda^{-1})}{\varphi(\lambda^{-1})} \to c^{\alpha-1} = (\alpha - 1)^{-1}\alpha^\alpha \quad \text{as} \quad \lambda \to \infty,
\]
and
\[
\lim_{\lambda \to \infty} \frac{M^{-1}(\lambda x) - M^{-1}(\lambda)}{\varphi(\lambda^{-1})} = (\alpha - 1)^{-1}\alpha^\alpha(1 - x^{-(\alpha-1)})
\]
follows. Then Lemma 11 below shows (6.3). \(\square\)

**Lemma 11**

(6.13) implies (6.3).

**Proof**

Assume (6.13). Since \(M^{-1}(x)\) has a monotone density
\[
(M^{-1}(x))' = \frac{1}{m(M^{-1}(x))},
\]
the monotone density theorem implies
\[
\lim_{\lambda \to \infty} \lambda \frac{M^{-1}(\lambda x) - M^{-1}(\lambda)}{\varphi(\lambda^{-1})} = ((\alpha - 1)^{-1}\alpha^\alpha(1 - x^{-(\alpha-1)}))',
\]
which is
\[
\lim_{\lambda \to \infty} \frac{\lambda}{\varphi(\lambda^{-1})m(M^{-1}(\lambda x))} = \alpha^\alpha x^{-\alpha}.
\]

Setting \(x = 1\) and \(u = M^{-1}(\lambda)\), we have
\[
\lim_{u \to 0} \frac{M(u)}{\varphi(M(u))^{-1}m(u)} = \alpha^\alpha.
\]
For any \(\epsilon > 0\) there exists \(\delta > 0\) such that for any \(u \in (-\delta, 0)\),
\[
\alpha^{-\alpha} - \epsilon \leq \frac{\varphi(M(u))^{-1}m(u)}{M(u)} \leq \alpha^{-\alpha} + \epsilon
\]
is valid. Noting \(m(u) = M(u)'\), we see that
\[
(\alpha^{-\alpha} - \epsilon)(-x) \leq \int_0^x \frac{\varphi(M(u))^{-1}}{M(u)} dM(u) \leq (\alpha^{-\alpha} + \epsilon)(-x)
\]
for any \(x \in (-\delta, 0)\); hence
\[
(\alpha^{-\alpha} - \epsilon)(-x) \leq \int_{M(x)}^{\infty} \frac{\varphi(y^{-1})}{y} dy = \int_0^{M(x)^{-1}} \frac{\varphi(z)}{z} dz \leq (\alpha^{-\alpha} + \epsilon)(-x).
\]
Since \(\varphi(z)/z\) is a regularly varying function at 0 with exponent \(\alpha - 2\),
\[
\frac{\varphi(z)}{z} 
\]
\[
\int_0^y \frac{\varphi(z)}{z} \, dz \sim \frac{\varphi(y)}{\alpha - 1} \quad \text{as } y \downarrow 0
\]
is valid, which implies
\[
\frac{\varphi(M(x)^{-1})}{\alpha - 1} \sim \alpha^{-\alpha}(-x) \quad \text{as } x \uparrow 0;
\]
hence
\[
M(x) \sim \frac{\beta^\beta}{\beta - 1} \varphi^{-1}(-x)^{-1} \quad \text{as } x \uparrow 0.
\]
This is equivalent to (6.3).

In the above two propositions we stated the conditions which should be satisfied by \( m \in \mathcal{E} \) in terms of its spectral function \( \sigma \). It may be preferable to describe the result by \( m \) itself directly. To do so, unfortunately we have to impose a more restrictive condition on \( m \), and combining Propositions 1 and 2 we have the following.

**THEOREM 7**

Let \( \alpha \geq 2 \), let \( k > \alpha - 1 \), and let \( \varphi \) be a regularly varying function at 0 with exponent \( \alpha - 1 \). Let \( m \in \mathcal{E} \) be a nondecreasing function with \( l = 0 \) and \( M(0) = \infty \). Assume that \( m \) satisfies
\[
\int_{-\infty}^{M^{-1}} M(x)^k \, dx < \infty.
\]
Then, the property
\[
p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1 + \alpha)} \frac{1}{t} \varphi\left(\frac{1}{t}\right) \quad \text{as } t \to \infty
\]
holds if and only if the asymptotics below is valid:
\[
m(x) \sim \frac{\beta^\beta}{x \varphi^{-1}(-x)} \quad \text{as } x \uparrow 0.
\]

**Proof**
The proof is immediate from the above two propositions if we observe \( \phi(t) = t^k \) satisfies all the requirements needed in Proposition 2.

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