Voter models on subcritical inhomogeneous random graphs

JOHN FERNLEY and MARCEL ORTGIÈSE

Abstract

The voter model is a classical interacting particle system modelling how consensus is formed across a network. We analyse the time to consensus for the voter model when the underlying graph is a subcritical scale-free random graph. Moreover, we generalise the model to include a ‘temperature’ parameter. The interplay between the temperature and the structure of the random graph leads to a very rich phase diagram, where in the different phases different parts of the underlying geometry dominate the time to consensus. Finally, we also consider a discursive voter model, where voters discuss their opinions with their neighbours. Our proofs rely on the well-known duality to coalescing random walks and a detailed understanding of the structure of the random graphs.

2010 Mathematics Subject Classification: Primary 60K35, Secondary 05C80, 05C81, 82C22

Keywords: voter model; inhomogeneous random graphs; scale-free networks; interacting particle systems; random walks on random graphs

1 Introduction

Voter models are a classical example of an interacting particle system defined on a graph that models how consensus is formed e.g. across a social network. In the standard model, voters can have one of two opinions and at rate 1 a vertex updates and copies the opinion from one of its neighbours.

Classically, the underlying graph is $\mathbb{Z}^d$, see e.g. [Lig85], and typical questions study the existence of invariant measures. When considered on a (connected) finite graph, the invariant measures become trivial and the main question is how long it takes to reach consensus. One example where this question can be answered is the complete graph on $N$ vertices, where the voter model is a variation of the Moran model from evolutionary biology and it is well known that consensus is reached in time of order $N$.

More recently, the voter model has also been studied on random graphs, see e.g. [Dur07]. Here, however the results are often very much dependent on the underlying model. Moreover, the case when the underlying graph is inhomogeneous in the sense that its empirical degree distribution shows power law behaviour has not been treated systematically. In the nonrigorous literature, this analysis has been carried out by [SAR08] for a mean-field model.

1Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, United Kingdom, j.d.fernley@bath.ac.uk, m.ortgiese@bath.ac.uk.
It is well known that voter models are dual to coalescing random walks (which in some sense trace back where opinions came from). In particular, any paper that bounds the time until all random walks have coalesced into a single walker has also bounded the consensus time in the voter model. Examples include \cite{Oli13, KMTS16}.

In this paper, we consider the voter model on subcritical inhomogeneous random graphs showing power law behaviour. The reason that we focus on subcritical random graphs is that, as we will see below, the behaviour observed here cannot be captured by mean-field methods. Moreover, in our model the random graphs are disconnected, but the components are still of polynomial order in the number of vertices and have the fractal-like structure seen in Figure 1. Therefore, the asymptotics of the consensus time (defined as the first time that all component reach consensus) depends on a subtle interplay of the different structures of the components.

Furthermore, we also introduce a “temperature” parameter $\theta \in \mathbb{R}$ into the model, so that vertices update at rate proportional to $d(v)^\theta$, where $d(v)$ denotes the degree of vertex $v$. This extra parameter leads to interesting phase transitions in $\theta$, where in the different phases different structural elements of the underlying random graphs dominate the consensus time.

Finally, we also consider a variation of the voter model, also considered by \cite{MBS18} and similar to the “oblivious” model of \cite{CDFR18}, which we will refer to as the discursive voter model. In this model, vertices update at rate $d(v)^\theta$, but then they ‘discuss’ with a randomly chosen neighbour and agree on one opinion chosen at random from their respective opinions. Surprisingly, this model gives a very different phase diagram and unlike the classical voter model the polynomial order of the consensus speed does not decrease with $\theta$.

Our proofs rely on duality to coalescing random walkers and by using the right tools to bound the coalescing times. This is combined with a very fine analysis of the structure of subcritical inhomogeneous random graphs models that are not readily available in the literature.

**Notation.** Throughout the paper, we will use the following notation. For sequences of positive random variables $(X_N)_{N \geq 1}$ and $(Y_N)_{N \geq 1}$, we write $X_N = O_{\mathbb{P}}^\log N (Y_N)$ if there exists $K > 0$ such that

$$\Pr (X_N \leq Y_N (\log N)^K) \to 1$$

as $N \to \infty$. Similarly, we write $X_N = \Omega_{\mathbb{P}}^\log N (Y_N)$ if $Y_N = O_{\mathbb{P}}^\log N (X_N)$. If both bounds hold we write $X_N = \Theta_{\mathbb{P}}^\log N (Y_N)$.

Throughout we write $[N] = \{1, \ldots, N\}$. For any graph $G$, we write $V(G)$ for its vertex set and $E(G)$ for its edge set. Moreover, if $v, w \in V(G)$, we write $v \sim w$ if $v$ and $w$ are neighbours, i.e. if $\{v, w\} \in E(G)$. Also, for $v \in V(G)$, we denote by $d(v)$ its degree (i.e. the number of neighbours).
In this paper, we will consider the following two variants of the voter model.

Definition 2.1. Let $G = (V,E)$ be a (simple) finite graph. Given $\eta \in \{0,1\}^V$, define for $i \neq j \in V$,

$$\eta^{i \leftrightarrow j}(k) = \begin{cases} 
\eta(j) & \text{if } k = i \in V, \\
\eta(k) & \text{if } k \in V \setminus \{i\}.
\end{cases}$$

The voter model is a Markov process $(\eta_t)_{t \geq 0}$ with state space $\{0,1\}^V$ and with the following update rules depending on a parameter $\theta \in \mathbb{R}$:
(a) In the classical voter model, for any neighbours $i$ and $j \in V$, the state $\eta \in \{0,1\}^V$ is replaced by $\eta^i \leftarrow j$ at rate $d(i)^{\theta-1}$.

(b) For the discursive voter model, for any neighbours $i$ and $j \in V$, the state $\eta \in \{0,1\}^V$ is replaced by $\eta^i \leftarrow j$ at rate $\frac{1}{2}(d(i)^{\theta-1} + d(j)^{\theta-1})$.

The classical voter model has the interpretation that each vertex $i$ updates its opinion at rate $d(i)^{\theta}$ by copying the opinion of a uniformly chosen neighbour. In the discursive model, each vertex $i$ becomes active at rate $d(i)^{\theta}$, then it chooses a neighbour uniformly at random and then both neighbours agree on a common opinion by picking one of their opinions randomly.

Adding the ‘temperature’ parameter $\theta$ leads to a very interesting phase diagram. Note also that $\theta = 0$ corresponds to the ‘standard’ voter model, where vertex $i$ updates its opinion at rate 1.

We consider a general class of inhomogeneous random graphs, which are close to a special case of the Chung-Lu model. The latter has vertex set $[N]$ and each edge is present independently with probability

$$p_{ij} = \frac{\beta N^{2\gamma-1}}{i^\gamma j^\gamma} \wedge 1. \quad (1)$$

We generalize this model as follows.

**Definition 2.2.** Fix $\beta, \gamma > 0$ such that $\beta + 2\gamma < 1$. We say that a sequence of (simple) random graphs $(G_N)_{N \geq 1}$, where $V(G_N) = [N]$, is in the class $G_{\beta,\gamma}$ of subcritical inhomogeneous random graph models with parameters $\beta$ and $\gamma$, if for any $N$ there exists a symmetric array $(q_{i,j})_{i,j \in [N]}$ of numbers in $(0, \frac{1}{2})$ such that each edge $\{i, j\}$, $i \neq j$, is present in $G_N$ independently of all others with probability $q_{i,j}$. Moreover, for $(p_{i,j})_{i,j \in [N]}$ as in (1), we require that

$$\lim_{N \to \infty} \sum_{i \neq j} \frac{(p_{i,j} - q_{i,j})^2}{p_{i,j}} = 0.$$

**Remark 2.3.** (a) Our definition includes various well-known models of inhomogeneous random graphs additionally to the Chung-Lu (CL) model, where $q_{i,j} = p_{i,j}$. Other models are the simple Norros-Reittu random graph (SNR) with $q_{i,j} = 1 - e^{-p_{i,j}}$ obtained from the Norros-Reittu model by flattening all multiedges, see also Section 4.1 below. It also includes the Generalised Random Graph (GRG) with $q_{i,j} = \frac{p_{i,j}}{1+P_{i,j}}$, which has the distribution of a particular configuration model conditioned to be simple [vdH16, Theorem 7.18].

(b) In the following, we will slightly abuse notation and write $G_N \in G_{\beta,\gamma}$ if we mean that $G_N$ is the random graph with vertex set $[N]$ in a sequence of random graphs in $G_{\beta,\gamma}$.
(c) Any two representatives of the class $G_{\beta,\gamma}$ are asymptotically equivalent, see [vdH16, Theorem 6.18], so that if a statement about one particular model holds for one with high probability it also holds for the other one. In the following we can thus choose any particular model.

(d) By [BJR07] we know that the regime $\beta + 2\gamma < 1$ is the complete subcritical region, so in particular our class of random graphs does not have a giant component. Also, by [CL06] it is known that these models have power-law exponent $\tau = 1 + 1/\gamma$.

Note that our network model is typically disconnected, so then the voter model $(\eta_t)_{t \geq 0}$ can hit an absorbing state without being in global consensus. So, let $C_1, \ldots, C_k$ be the components of a graph $G$ on the vertex set $[N]$. The consensus time is the first time that there is consensus on each component, i.e.

$$\tau_{\text{cons}} = \inf\{t \geq 0 : \eta_t|_{C_i} \text{ is constant for each } i \in [k]\}.$$ 

We can now state our first main theorem on the expected consensus time (conditional on the random graph). Here and in Theorem 2.5 we find regimes where the consensus time is dominated by the slow-meeting effect on the largest components of order $N^{\gamma}$, and regimes where consensus time is driven by the slow-mixing effect on a double star component of order $N^{2-2\gamma}$.

**Theorem 2.4.** Suppose $G_N \in G_{\beta,\gamma}$ for some $\beta + 2\gamma < 1$ and that the initial conditions are chosen according to $\mu_u$ such that each initial opinion is a an independent Bernoulli($u$) random variable, for some $u \in (0,1)$. Then, for the classical voter model with parameter $\theta \in \mathbb{R}$, we have

$$E_{\mu_u}^\theta(\tau_{\text{cons}}|G_N) = \Theta_p^{\log N}(N^c)$$

where the exponent $c = c(\gamma, \theta)$ is given as

$$c = \begin{cases} 
\gamma & \theta \geq 1, \\
\gamma \theta & \frac{1}{2-2\gamma} < \theta < 1, \\
\frac{\gamma}{2-2\gamma} & 0 \leq \theta \leq \frac{1}{2-2\gamma}, \\
\frac{\gamma}{2(1-\theta)} & \theta < 0.
\end{cases}$$

Note that we are only taking the expectation over the voter model dynamics, so the right hand side in (2) is still random (depending only on the realization of the particular random graph).

For $\theta = 0$ then we do find a sublinear consensus time $N^{\frac{\gamma}{2-2\gamma}} = N^{\frac{1}{2-2\gamma}} = o(\sqrt{N})$. This is fundamentally different from any mean-field model as one can see by comparing with the linear consensus times in [SAR08].

Unlike for the classical model, where large positive $\theta$ slowed down consensus, for the discursive model we see that large $\theta$ accelerates consensus by accelerating mixing.
Figure 2: This figure shows the typical shapes by setting $\gamma = 1/3$. Somewhat surprisingly, for any subcritical $(\beta, \gamma)$ parameters the function $c(\gamma, \theta)$ is not monotonic in $\theta$. On the left we see that the most popular model $\theta = 0$ is one of the fastest parameter values for consensus, as part of the optimal interval $\theta \in [0, 1/(2 - 2\gamma)]$. Conversely, the discursive model has monotonicity in the exponent.

**Theorem 2.5.** Suppose $G_N \in G_{\beta, \gamma}$ for some $\beta + 2\gamma < 1$ and that the initial conditions are chosen according to $\mu_u$ such that each initial opinion is an independent Bernoulli($u$) random variable, for some $u \in (0, 1)$. Then, for the discursive voter model with parameter $\theta \in \mathbb{R}$, we have

$$E^\theta_{\mu_u}(\tau_{\text{cons}}|G) = \Theta_p^\log N (N^c)$$

where the exponent $c = c(\gamma, \theta)$ is given as

$$c = \begin{cases} \frac{\gamma}{2 - 2\gamma} & \theta \geq \frac{3 - 4\gamma}{2 - 2\gamma}, \\ \gamma(2 - \theta) & 1 < \theta < \frac{3 - 4\gamma}{2 - 2\gamma}, \\ \frac{\gamma(2 - \theta)}{2 - 2\gamma} & 2\gamma \leq \theta \leq 1, \\ \theta < 2\gamma. & \end{cases}$$

Here, for each $\gamma$, $c(\gamma, \theta)$ is non-increasing in $\theta$ as might be expected.

**Remark 2.6** (Transition in the power law). For illustration, we rephrase the previous claims by fixing $\theta$ and varying the tail exponent $\tau = 1 + 1/\gamma$. For $G_N \in G_{\beta, \gamma}$ and for the classical dynamics we obtain for $\theta \in (\frac{3}{2}, 1)$,

$$E^\theta_{\mu_u}(\tau_{\text{cons}}|G) = \Theta_p^\log N \begin{cases} N^{\frac{1}{2\tau - 1}} & \tau \leq 3 + 2 \left( \frac{1 - \theta}{2\theta - 1} \right), \\ N^{\frac{1}{\tau - 1}} & \text{otherwise}, \end{cases}$$

6
and for the discursive dynamics with \( \theta \in (1, \frac{3}{2}) \), this translates to
\[
E^\theta_{\mu_u}(\tau_{\text{cons}}|G) = \Theta_p^{\log N} \left\{ \begin{array}{ll}
N^{\frac{1}{\tau-1}} & \tau \leq 3 + 2 \left( \frac{\theta-1}{2\theta} \right), \\
N^{\frac{2-\theta}{2-\tau}} & \text{otherwise}.
\end{array} \right.
\]

From the proof one can see that in both these cases the consensus time on the largest component is dominant only for small \( \tau \). If \( \theta \in (0, 1) \) then for the discursive dynamic we have that
\[
E^\theta_{\mu_u}(\tau_{\text{cons}}|G) = \Theta_p^{\log N} \left\{ \begin{array}{ll}
N^{\frac{1}{\tau-1}} & \tau \leq 3 + 2 \left( \frac{1-\theta}{\theta} \right), \\
N^{\frac{2-\theta}{2-\tau}} & \text{otherwise}.
\end{array} \right.
\]

In this case, one can see from the proofs that the asymptotics for the largest component dominate for the large \( \tau \) values.

**Remark 2.7** (The supercritical Regime). On the complete graph the model reduces to the one-dimensional Wright-Fisher model, an example of a well mixed regime. On a supercritical scale free network, too, we expect fast mixing times so that in many cases the consensus time has order
\[
\Theta_p \left( \frac{1}{\sum q \pi} \right)
\]
where \( \pi, q \) are the stationary distribution and vertex jump rate of the random walk obtained by tracing back opinions, see Section 3 for precise definitions.

The order in Equation (3) is that of the mean-field model, see [SAR08] for mean-field approach and [CCC10] for related results that work in great generality (under suitable assumptions on the mixing times).

For the Erdős-Rényi graph we have detailed mixing time and structural results, see [BKW14]. Also on configuration models we have some mixing results but under the assumption of subpolynomial maximum degree in [BLPS18], or a degree lower bound [ACF12]. The conjecture is that these results do extend to general configuration models with power-law degree sequence and so
\[
t_{\text{mix}} = \Theta_p \left( \log^2 N \right)
\]
which would, if known, give the correct order up to polynomial exponents from results in [CCC16] and [KMTS16] in the supercritical case.

In fact, [Dur10] conjectures via Aldous “Poisson Clumping Heuristic” [Ald13] that the order of the mean consensus time is really the exact polynomial without logarithmic corrections as found by the mean-field approximation in [SAR08]. A structural result like that in [DLP14] for the rank one scale-free network would solve the open question of mixing time, but also potentially give a direct handle on meeting time without the logarithmic factors.

The remaining paper is organised as follows: In Section 3 we describe the (classical) duality of the voter model to coalescing random walks and then develop various tools.
for the coalescing random walks. This section works whenever the random walks follow the dynamics of a reversible Markov chain, so apply to both our models. In Section 3, we derive structural results for subcritical inhomogeneous random graphs. In Section 4, we combine the structural results with the bounds in Section 3 to complete the proofs of Theorems 2.4 and 2.5.

3 Duality and bounds on the coalescence time

In this section, we consider the voter model on an arbitrary finite state space. Moreover, we will discuss the main tool to analyse the voter model, which is the duality to a system of coalescing random walks. In the remaining section, we will then show various bounds on the coalescence time of a system of general random walks.

We will describe a general voter model, where the voters are indexed by \( n = \{1, \ldots, n\} \) for some \( n \in \mathbb{N} \) and the dynamics are governed by a matrix \( Q = (Q(i, j))_{i,j \in [n]} \) which is the generator of a continuous-time, reversible Markov chain \((X_t)_{t \geq 0}\) on \([n]\).

Let \( O \) be the set of possible opinions, then the \( Q \)-voter model \((\eta_t)_{t \geq 0}\) with \( \eta_t \in O^{[n]} \) evolves as follows: for all \( i \neq j \in [n] \) at rate \( Q(i, j) \) the current state \( \eta \in O^{[n]} \) is replaced by

\[
\eta^i \leftarrow j(k) = \begin{cases} 
\eta(j) & \text{if } k = i \\
\eta(k) & \text{if } k \in [n] \setminus \{i\}.
\end{cases}
\]

In other words, at rate \( Q(i, j) \) the voter \( i \) copies the opinion from voter \( j \).

It is classical that the voter model is dual to a system of coalescing random walks, see [Lig85]. The duality can be described via a graphical construction. We start with the graph \( \{(j, t) : j \in [n], t \geq 0\} \) and independent Poisson point processes \((N_{i,j}(t))_{t \in \mathbb{R}}, i \neq j\) (with rates \( Q(i, j) \) respectively). If \( t_k \) denotes a jump of \( N_{i,j} \) we draw an arrow from \((t_k, j)\) to \((t_k, i)\). Given any initial condition \( \eta_0 \in O^{[n]} \), we then let the opinions flow upwards starting at time 0 and if they encounter an arrow following the direction of the arrow and replacing the opinion of the voter found at the tip of the arrow. Now, we fix a time horizon \( T > 0 \) and we start with \( n \) random walkers located at each of the points \((j, T), j \in [n]\), then the trajectories follow the graph downwards, following each arrow and if two walkers meet they coalesce. Denote by \( \xi^T_t \subseteq [n] = \{\xi^T_t(j), j \in [n]\} \) the set of positions of these walkers at time \( t \geq 0 \), where thus \( \xi^T_0 = [n] \). From this construction, it follows that each walker follows the dynamics of the Markov chain \( X \), so we obtain a system of coalescing Markov chains/random walks. Moreover, one can immediately see that the voter model at time \( T \) can be obtained, by tracing the random walk paths backwards, i.e. for any \( j \in [n] \),

\[
\eta_T(j) = \eta_0(\xi^T_T(j)).
\]

We are interested in general reversible Markov chains, so we do not necessarily assume that the Markov chain is irreducible. However, since \( X = (X_t)_{t \geq 0} \) is reversible, we
can decompose the state space into its irreducible components, which we will denote by $C_1, \ldots, C_k$, so that $X$ restricted to $C_j$ is irreducible. In this case, For any $j \in [k]$, we denote the consensus time on the $j$th component by

$$\tau_{\text{cons}}(C_j) = \inf\{t \geq 0 : \eta_t|_{C_j} \text{ is constant}\}.$$ 

Then, define the overall consensus time

$$\tau_{\text{cons}} = \max_{j \in [k]} \tau_{\text{cons}}(C_j).$$

Our main interest in this article, is in the case when $O = \{0, 1\}$ and the initial conditions $\eta_0$ are distributed according to $\mu_u$, the product of Bernoulli($u$) measures for some $u \in [0, 1]$. Then, we set

$$t_{\text{cons}}^{(u)} = \mathbb{E}_{\mu_u}(\tau_{\text{cons}}).$$

For the duality, it will be easier to consider the voter model where each voter starts with a different opinion, i.e. $\eta_0 = [n]$, then we define

$$t_{\text{cons}}^* = \mathbb{E}_{[n]}(\tau_{\text{cons}}).$$

For the system of coalescing random walks, we define for each irreducible component $C_j, j \in [k]$,

$$\tau_{\text{coal}}(C_j) = \inf\{t \geq 0 : |\xi^T_t| = 1\},$$

i.e. the first time all walkers in this component have coalesced into a single walker. Moreover, we then define

$$t_{\text{coal}} = \mathbb{E}_{[n]}(\tau_{\text{coal}}), \quad \text{where } \tau_{\text{coal}} = \sup_{j \in [k]} \tau_{\text{coal}}(C_j).$$

By duality, we have that if the voter model starts in $\eta_0 = [n]$, then

$$\mathbb{P}_{[n]}(\tau_{\text{coal}} \leq T) = \mathbb{P}_{[n]}(\tau_{\text{cons}} \leq T),$$

so $\tau_{\text{coal}}$ and $\tau_{\text{cons}}$ agree in distribution and in particular $t_{\text{coal}} = t_{\text{cons}}^*$. As the following lemma shows, we can also get a bound for $t_{\text{cons}}^{(u)}$.

**Lemma 3.1** (Duality). For all $u \in (0, 1)$

$$2u(1-u)t_{\text{coal}} \leq t_{\text{cons}}^{(u)} \leq t_{\text{coal}}$$

**Proof.** By recolouring we can see that we reach consensus from the binomial measure $\mu_u$ before we do from unique colours, and hence

$$t_{\text{cons}}^{(u)} \leq t_{\text{cons}}^* = t_{\text{coal}}.$$
For the other direction, observe from the duality relation (1) that
\[ P_{\mu_u} (\eta_T \text{ constant}) = P (\mu_u \text{ constant on } \xi_T^T) = E \left( u|\xi_T^T| + (1 - u)|\xi_T^T| \right) \]
which we can crudely upper bound by considering the event \( \{|\xi_T^T| = 1\} \)
\[ E \left( u|\xi_T^T| + (1 - u)|\xi_T^T| \right) \leq P (|\xi_T^T| = 1) + (u^2 + (1 - u)^2) P (|\xi_T^T| \geq 2) \]
\[ = 1 + 2u(1 - u)P (|\xi_T^T| \geq 2) \]
and therefore we have
\[ t_{cons}^{(u)} = \int_0^\infty 1 - E \left( u|\xi_T^T| + (1 - u)|\xi_T^T| \right) dT \]
\[ \geq 2u(1 - u) \int_0^\infty P (|\xi_T^T| \geq 2) dT = 2u(1 - u)t_{coal}. \]

We will control the time \( t_{hit} \) until all random walkers have coalesced using the following two bounds in terms of the following two auxiliary quantities. First of all, let \( X = (X_t)_{t \geq 0} \) and \( Y = (Y_t)_{t \geq 0} \) be two independent reversible Markov chains with generator \( Q \). Then, define the (expected) meeting time for \( j \in [k] \) as
\[ t_{meet}(C_j) = \max_{x,y \in C(j)} E_{x,y}(\tau_{meet}), \quad \text{where } \tau_{meet} = \inf\{t \geq 0 : X_t = Y_t\}. \]
Moreover, an important role will be played by the (expected) hitting time defined for \( j \in [k] \) as
\[ t_{hit}(C_j) = \max_{x,y \in C_j} E_x(T_y), \quad \text{where } T_y = \inf\{t \geq 0 : X_t = y\}. \]
Both these quantities give bounds on the coalescence time and thus on the consensus time.

**Proposition 3.2.** With the notation as above, we have that
\[ \sup_{j \in [k]} t_{meet}(C_j) \leq t_{coal} \leq e(\log n + 2) \sup_{j \in [k]} t_{meet}(C_j). \]
Moreover, for any \( j \in [k] \),
\[ t_{meet}(C_j) \leq t_{hit}(C_j). \]

**Remark 3.3.** Recall that \( t_{coal} \) is defined as
\[ t_{coal} = E_{|n|} \left( \sup_{j \in [k]} \tau_{coal}(C_j) \right) \]
so the non-standard part of the statement is that we can take the supremum out of the expectation. For irreducible chains, the statement is Proposition 14.11 in [AF02].
However, their proof does not really need this extra assumption. For the convenience of the reader, we repeat the proof below. Furthermore, we note that for reducible chains the first bound is shown in [Oli12] without the \( \log n \) factor. The stronger bound does not hold without the assumption of irreducibility. Indeed, by looking at a Markov chain with \( n \) components of size 2 each (e.g. with transition rates 1 within these components), it becomes obvious that the factor \( \log n \) in the proposition is sharp.

**Proof.** The reversible Markov chain decomposes into irreducible recurrence classes - write \( \mathcal{C}(i) \) for the class containing the state \( i \). As in the proof of [AF02, Proposition 14.11], consider each walker \( W^{(i)} \) independently, labelled by its starting position. We have \( n^2/2 \) meeting time variables

\[
\tau_{i,j}^{\text{meet}} := \inf \left\{ t \geq 0 : W^{(i)}_t = W^{(j)}_t \right\}
\]

for the walkers \( i \) and \( j \), where \( \inf \emptyset := \infty \). Define a function \( f \) which maps all elements in a recurrence class \( \mathcal{C}(i) \) to a label \( \min \mathcal{C}(i) \) which is of lowest index in that component.

\[
f : i \mapsto \min \mathcal{C}(i)
\]

Then we can say for the random coalescence time,

\[
\tau_{\text{coal}} := \max_{i=1}^n \tau_{\text{coal}}(\mathcal{C}(i)) \leq n \max_{i=1}^n \tau_{i,f}^{\text{meet}}.
\]

We then apply a result for the general exponential tails of hitting times on finite Markov chains [AF02, Equation 2.20]: from arbitrary initial distribution \( \mu \) and for a continuous time reversible chain, for any subset \( A \subset V \)

\[
\mathbb{P}_\mu(T_A > t) \leq \exp \left( - \frac{t}{e \max_v E_v T_A} \right).
\]

For the meeting time variables this leads to

\[
\mathbb{P}(\tau_{\text{meet}} > t) \leq \exp \left( - \frac{t}{et_{\text{meet}}} \right)
\]

\[
\Rightarrow \mathbb{P}(\tau_{\text{coal}} > t) \leq \sum_{i=1}^n \mathbb{P}(\tau_{i,f}^{\text{meet}} > t) \leq n \exp \left( - \frac{t}{et_{\text{meet}}} \right).
\]

And we conclude just as in [AF02, Proposition 14.11] by integrating to get

\[
t_{\text{coal}} \leq \int_0^{\infty} 1 \wedge \left( ne \exp \left( - \frac{t}{et_{\text{meet}}} \right) \right) \, dt = e(2 + \log n) t_{\text{meet}}.
\]

The final claim is [AF02, Proposition 14.5].

In particular, Proposition 3.2 allows us to to bound the consensus time by bounding either hitting times or meeting times for an irreducible chain. We start by collecting the bounds on hitting times in Section 3.1 and continue with the bounds on meeting times in Section 3.2.
3.1 Bounds on hitting times

Throughout the following two sections $X = (X_t)_{t \geq 0}$ will be a reversible, irreducible Markov chain with state space $[n] = \{1, \ldots, n\}$ and transition rates given by a matrix $Q$. Moreover, we denote by $\pi = (\pi(i))_{i \in [n]}$ the invariant measure of $X$.

In this case, as their is only one irreducible component the (expected) hitting time is defined as

$$t_{\text{hit}} = \max_{k,j \in [n]} \mathbb{E}_k(T_j).$$

For our bound on the hitting time, we will make use of the well-known correspondence between Markov chains and electric networks, see e.g. [AF02, WLP09]. In this context, we associate to $Q$ a graph $G_Q$ with vertex set $[n]$ and connect $i$ and $j$ by an edge, written $i \sim j$ if and only if the conductance defined as

$$c(\{i,j\}) := \pi(i)Q(i,j) = \pi(j)Q(j,i) > 0.$$  \hspace{1cm} (5)

This is also known as the ergodic flow of the edge. Moreover, the interpretation as an electric network lets us define the effective resistance between two vertices $i,j \in [n]$, denoted $\mathcal{R}(i \leftrightarrow j)$, as in [WLP09, Chapter 9].

To state the following proposition, we also define the $\text{diam}(Q)$ for the diameter in the graph theoretic sense to the graph obtained from $Q$ as above. The proof uses the representation of the effective resistances in terms of the Markov chains, combined with Thomson’s principle.

**Proposition 3.4** (Conductance bounds). Let $(X_t)_{t \geq 0}$ be a reversible, irreducible Markov chain on $[n]$ with associated conductances $c$. Let $P_{i,j}$ be a path from $i$ to $j$ in $G_Q$ and denote by $E(P_{i,j})$ the set of edges in $P_{i,j}$. Then

$$\mathbb{E}_i T_j + \mathbb{E}_j T_i \leq \sum_{e \in E(P_{i,j})} \frac{1}{c(e)}$$

In particular, we have

$$t_{\text{hit}} \leq \text{diam}(Q) \max_{{i \sim j \in [n]}} \frac{1}{c(\{ij\})},$$

**Proof.** Let $T_{i^+} = \inf\{t > 0 : X_t = i, \lim_{s \uparrow t} X_s \neq i\}$ be the return time to state $i$. From [WLP09, Proposition 9.5],

$$\mathcal{R}(i \leftrightarrow j) = \frac{1}{c(i)\mathbb{P}_i(T_j < T_{i^+})},$$

where $c(i) = \sum_{j \sim i} c(ij)$ is the conductance around a vertex. We further have from [AF02, Corollary 2.8 (continuous time version)]

$$\mathbb{E}_i T_j + \mathbb{E}_j T_i = \frac{1}{\pi(i)q(i)\mathbb{P}_i(T_j < T_{i^+})}.$$
where \( q(i) = -Q(i, i) \) is the walker speed at \( i \), and by the choice of \( c \) these expressions are equal. Finally by Thompson’s theorem (essentially by monotonicity of effective resistances with respect to edge resistances)

\[
\mathbb{E}_i T_j + \mathbb{E}_j T_i = \mathcal{R}(i \leftrightarrow j) \leq \mathcal{R}(i \leftrightarrow j \text{ through } P_{i,j}) \leq \sum_{\{u,v\} \in E(P_{i,j})} \frac{1}{c(uv)}.
\]

### 3.2 Bounds on meeting times

In this setting, we continue to use the notation from the beginning of Section 3.1. In particular, \((X_t)_{t \geq 0}\) is a reversible, irreducible Markov chain on \([n]\) with transition rates given by \(Q\) and invariant measure \(\pi\).

It will often be easier to work with the (expected) meeting time when both chains are started in the invariant measure, i.e. we define

\[
t_\pi^{\text{meet}} := \sum_{i,j \in [n]} \pi(i) \pi(j) \mathbb{E}_{i,j}(\tau_\text{meet}).
\]

In order to make the connection to \(t_\text{meet}\), we will need the time it takes to reach stationarity, i.e. we consider the mixing time \(t_\text{mix}\) defined as

\[
t_\text{mix} = \max_{i \in [n]} t_\text{mix}(i), \quad \text{where } t_\text{mix}(i) := \min \left\{ t \geq 0 : \left\| p_i(t) - \pi \right\|_1 \leq \frac{1}{2} \right\}.
\]

Closely related to the mixing time is the relaxation time

\[
t_\text{rel} := \max \left\{ \frac{1}{\lambda} : \lambda \text{ a positive eigenvalue of } -Q \right\}.
\]

**Proposition 3.5.** (a)

\[
t_\pi^{\text{meet}} \geq \frac{1 - \sum_{i \in [n]} \pi(i)^2)^2}{\sum_{i \in [n]} q(i) \pi(i)^2},
\]

where \( q(i) = -Q(i, i) \).

(b) There exists an absolute constant \( c_{\text{cond}} > 0 \) such that

\[
t_\text{meet} \geq c_{\text{cond}} \left( \max_{A \subset [N]} \frac{\pi(A) \pi(A^c)}{\sum_{x \in A} \sum_{y \in A^c} c(x, y)} - 1 \right).
\]

**Proof.** For Part (a) see Remark 3.5 in [CCC16].
From the coupling bound for mixing times seen in [WLP09, Corollary 5.3] and restated in continuous time by [AF02]

\[ d(t) \leq \max_{i,j} \mathbb{P}\left(t_{\text{meet}}^{i,j} > t\right) \leq \exp\left(-\frac{t}{e t_{\text{meet}}}\right) \leq \exp\left(1 - \frac{t}{e t_{\text{meet}}}\right) \]

and so by taking the integral

\[
\frac{1}{4} t_{\text{mix}} \leq \int_{0}^{\infty} d(t) \, dt \leq e^{2 t_{\text{meet}}}
\]

which, combined with Lemma 3.1 and Proposition 3.2 gives

\[ t_{\text{mix}} \leq \frac{2 e^{2 u(1 - u)}}{u(1 - u)} \text{cons.} \]

Because \( c(xy) = \pi(x)Q(x, y) \), by [AF02, Corollary 4.37],

\[ \max_{A \subseteq \mathbb{N}} \frac{\pi(A)\pi(A^c)}{\sum_{x \in A} \sum_{y \in A^c} c(x, y)} \leq t_{\text{rel}}. \]

and then by [WLP09, Theorem 12.4]

\[ t_{\text{mix}} \geq (t_{\text{rel}} - 1) \log 2. \]

[PPSS17, Corollary 1.2] has the consequence that, for some universal \( C > 0 \), \( t_{\text{mix}} \leq C \min_i t_{\text{hit}}(i) \). We present a simple proof of this consequence.

**Lemma 3.6.** For any \( i \in [n] \),

\[ t_{\text{mix}}(i) \leq 2E_{\pi}(T_i). \]

**Proof.** Let \( i \in [n] \), then by Cauchy-Schwarz we have that

\[
\left\| p_{i}^{(t)} - \pi \right\|_{1}^{2} = \sum_{j \in [n]} \left| \frac{p_{ij}^{(t)}}{\pi(j)} - 1 \right| \pi(j) \leq \left( \sum_{i \in [n]} \left| \frac{p_{ij}^{(t)}}{\pi(j)} - 1 \right|^{2} \pi(j) \right)^{\frac{1}{2}} \leq \left\| \frac{p_{i}^{(t)}}{\pi} - 1 \right\|_{\pi}^{2}.
\]

To simplify the right hand side, we use reversibility to obtain

\[
\left\| \frac{p_{i}^{(t)}}{\pi} - 1 \right\|_{\pi}^{2} = -1 + \sum_{j} \left( \frac{p_{ij}^{(t)}}{\pi(j)} \right)^{2} = -1 + \frac{1}{\pi(i)} \sum_{j} p_{ij}^{(t)} p_{ji}^{(t)} = -1 + p_{ii}^{(2t)}/\pi(i).
\]

Now, by [AF02, Lemma 2.11], we have that for any \( t \geq 0 \),

\[
E_{\pi}(T_i) = \int_{0}^{\infty} \left( -1 + \frac{p_{ii}^{(s)}}{\pi(i)} \right) \, ds \geq 2t \left( -1 + \frac{p_{ii}^{(2t)}}{\pi(i)} \right),
\]
because the integrand is non-increasing \[\text{AF02, Equation 3.40}.\]

Combining these inequalities, we have for \(t > 0\),

\[\left\| p_i^{(t)} - \pi \right\|_1 \leq \left\| \frac{p_i^{(t)}}{\pi} - 1 \right\|_\pi \leq \sqrt{\frac{\mathbb{E}_\pi (T_i)}{2t}}.\]

Hence, if \(t\) is such that

\[\frac{\mathbb{E}_\pi (T_i)}{2t} \leq \frac{1}{4},\]

then we can deduce that \(t_{\text{mix}}(i) \leq t\), which completes the proof.

**Corollary 3.7.** For two independent copies \((X_t)_t\) and \((Y_t)_t\) of a Markov Chain on \([n]\), and any state \(s \in [n]\), we find

\[t_{\text{mix}} \leq 16t_{\text{hit}}(s)\]

and further we can construct a time for the product chain with

\[\mathbb{E}_{x,y}(S) \leq 188t_{\text{hit}}(s)\]

which is a strong stationary time in the sense that for any \(t \geq 0\)

\[\mathcal{L}(X_{t+S}, Y_{t+S}) = \pi \otimes \pi\]

and this is independent of the value of \(S\).

**Proof.** We will first construct a time \(S_X\) for \((X_t)_t\). Recall from Theorem 3.6 that

\[t_{\text{mix}}(s) \leq 2\mathbb{E}_\pi (T_s) \leq 2t_{\text{hit}}(s)\]

but in fact we will halve the permitted \(L^1\) distance to \(\frac{1}{4}\). Thus at the mixing time \(M_2\) with

\[\frac{\mathbb{E}_\pi (T_s)}{2M_2} = \frac{1}{16}\]

we have total variation distance at most \(\frac{1}{8}\). Then, to show that we have hit \(s\) with probability at most \(\frac{1}{8}\) we use Markov’s inequality. We choose a time \(M_1\) such that

\[\max_x \mathbb{P}_x(T_s \geq M_1) \leq \max_x \frac{\mathbb{E}_x(T_s)}{M_1} = \frac{t_{\text{hit}}(s)}{M_1} = \frac{1}{8}\]

and so we can mix via \(s\) over these hitting and mixing periods to bound

\[t_{\text{mix}} \leq M_1 + M_2 = 8t_{\text{hit}}(s) + 8\mathbb{E}_\pi (T_s) \leq 16t_{\text{hit}}(s).\]

By Theorem 1.1 in \[\text{Fil91}\] we can construct a strong stationary time with

\[\mathbb{P}_s (S_X > t) = \text{sep}_s(t) =: 1 - \min_{j \in [n]} \frac{p_{s,j}^{(t)}}{\pi(j)}\]
then we recover several definitions and results from [AF02]

\[ d(t) := \frac{1}{2} \max_{x \in [n]} \| p_x(t) - \pi(\cdot) \|_1 \]

\[ \tilde{d}(t) := \frac{1}{2} \max_{x,y \in [n]} \| p_x(t) - p_y(t) \|_1. \]

These various definitions of distance from stationarity are easily comparable

\[ \text{sep}_s(2t) \leq \max_{v \in [N]} \text{sep}_v(2t) < 2\tilde{d}(t) \leq 4d(t) \]

and so we translate alternative definitions of the mixing time

\[ \tau_1 := \min \left\{ t : \tilde{d}(t) \leq \frac{1}{2} \right\} \leq \min \left\{ t : d(t) \leq \frac{1}{4} \right\} = t_{\text{mix}} \]

then we use that \( \tilde{d} \) is submultiplicative to obtain

\[ \tilde{d}(t) \leq 2^{-\lfloor t/\tau_1 \rfloor} \leq 2^{-\lfloor t/t_{\text{mix}} \rfloor} \]

and therefore we can bound the expectation of the time to stationarity

\[ \mathbb{E}_x(\mathcal{S}_X) = 2 \int_0^\infty \text{sep}_s(2t)dt < 4 \int_0^\infty 2^{1-t/t_{\text{mix}}}dt = \frac{8t_{\text{mix}}}{\log 2} \leq \frac{64t_{\text{hit}}(s)}{\log 2}. \]

This becomes a strong stationary time for \((X_t)\) with \(X_0 = x\) by constructing another time \(\tilde{S}_X\) which simply waits for the event \(\mathcal{T}_s\) when the walker hits \(s\), and then waits for \(\mathcal{S}_X\). Thus

\[ \mathbb{E}_x(\tilde{S}_X) \leq t_{\text{hit}}(s) + \frac{64t_{\text{hit}}(s)}{\log 2} < 94t_{\text{hit}}(s). \]

We construct the symmetric time \(\tilde{S}_Y\) for \((Y_t)\) and then finally our object is the time

\[ \mathcal{S} := \tilde{S}_X \vee \tilde{S}_Y \]

so that

\[ \mathbb{E}_{x,y}(\mathcal{S}) \leq \mathbb{E}_x(\tilde{S}_X) + \mathbb{E}_y(\tilde{S}_Y) \leq 188t_{\text{hit}}(s). \]

\[ \square \]

**Proposition 3.8.** For any state \(s \in [n]\)

\[ t_{\text{meet}} \leq \frac{189t_{\text{hit}}(s)}{\pi(s)}. \]
Proof. From any configuration of two walkers, we can apply Corollary 3.7 to construct a strong stationary time \( S \) with \( \mathbb{E}(S) \leq 188t_{\text{hit}}(s) \). Then, wait for \((X_t)_t\) to hit \( s \), which we expect to require at most an additional time period of length \( t_{\text{hit}}(s) \).

On hitting, \((Y_t)_t\) is still in independent stationarity, and so we have exactly probability \( \pi(s) \) to meet at that instant. Otherwise, we repeat the stationarity and hitting periods to get another chance to meet at \( s \).

\[ \square \]

Remark 3.9. We find the following illustrative bound in [KMTS16]

\[ t_{\text{meet}}^\pi = O\left( \frac{t_{\text{mix}}}{||\pi||_2^2} \right) \]

which, while appearing a better bound, is commonly not so for Markov chains on trees. The mixing time for a Markov chain on a tree (which must always be a reversible chain) is always the hitting time of a central vertex, i.e. one with

\[ \mathbb{E}_\pi T_c = \min_{v \in [n]} \mathbb{E}_\pi T_v \]

\[ t_{\text{mix}} = \Theta(t_{\text{hit}}(c)) \]

and so because \( ||\pi||_2^2 \leq ||\pi||_\infty \). Theorem 3.8 will often give a tighter bound.

The following large deviations result is given in [SC97].

**Theorem 3.10.** For any finite, irreducible continuous-time Markov chain \((X_t)_t\) with initial stationary distribution \( \pi \), and any function on the state space \( f \) with

\[ \langle f, \pi \rangle = 0, \quad ||f||_\infty \leq 1, \]

we have

\[ \mathbb{P}_q \left( \frac{1}{t} \int_0^t f(X_s)ds > \gamma \right) \leq ||q/\pi||_2 \exp \left( -\frac{\gamma^2 t}{10t_{\text{rel}}} \right) \]

where \( q(i) = -Q(i, i) \) is the rate at \( i \) of the Markov chain.

We now use a concept of the chain observed on a subset \( V \subset N \) described in Section 2.7.1 in [AF02]:

- The partially observed chain \((P_t)_t\) on \( V \) “jumps” from \( P_0 = v \in V \) at rate \( q(v) = -Q(v, v) \) so, if \( X_0 = v \), we can couple the first jump for both chains.

- Let \((J_n)_n\) denote the discrete time “jump chain” for \((X_t)_t\). Given first jump time \( j \) for \((X_t)_t\),

\[ k := \min \{ n \geq 1 : J_n \in V \} \]

\[ P_j \overset{(d)}{=} J_k \]

and because \((P_t)_t\) is Markov we can restart this description from its second state.

Note \( P_j = P_{j-} \) is possible so the true jump rate for \((P_t)_t\) may be less than \( q \).
• The partially observed chain has the natural stationary distribution

\[
\frac{\pi(\cdot) \mathbbm{1}_V(\cdot)}{\pi(V)}.
\]

Then we can define the random subset meeting time \(\tau^T_{\text{meet}}(A)\) as \(\tau^T_{\text{meet}}\) except for the partially observed product chain on \(A \times A\), and \(t^T_{\text{meet}}(A) = \mathbb{E}(\tau^T_{\text{meet}}(A))\).

**Theorem 3.11.**

\[
t_{\text{meet}} \leq 188t_{\text{hit}}(s) + \frac{2t_{\text{meet}}(A)}{\pi(A)^2} + \frac{1280t_{\text{hit}}(s)}{\pi(A)^4}.
\]

**Proof.** We first claim

\[
t^T_{\text{meet}} \leq \frac{2t_{\text{meet}}(A)}{\pi(A)^2} + \frac{80t_{\text{mix}}}{\pi(A)^4}.
\]

Consider two independent copies of the stationary chain \((X_t, Y_t)_t\)

\[
\forall t \geq 0, \mathcal{L}(X_t, Y_t) = \pi \otimes \pi
\]

and define

\[
U(t) := \int_0^t 1_{A \times A}(X_s, Y_s) \, ds
\]

\[
U^{-1}(t) := \sup\{x \geq 0 : U(x) = t\}
\]

we find a stochastic domination, because the first meeting may or may not be seen in \(A\),

\[
U^{-1}(\tau^T_{\text{meet}}(A)) \succeq \tau^T_{\text{meet}}.
\]

Then \(U\) and \(\tau^T_{\text{meet}}(A)\) are not independent but, by conditioning,

\[
\mathbb{P}(U^{-1}(\tau^T_{\text{meet}}(A)) > t) \leq \mathbb{P}(\tau^T_{\text{meet}}(A) > \frac{\pi(A)^2}{2}t) + \mathbb{P}(U^{-1}\left(\frac{\pi(A)^2}{2}t\right) > t)
\]

so that

\[
\mathbb{E}(U^{-1}(\tau^T_{\text{meet}}(A))) = \int_0^\infty \mathbb{P}(U^{-1}(\tau^T_{\text{meet}}(A)) > t) \, dt
\]

\[
\leq \frac{2t_{\text{meet}}(A)}{\pi(A)^2} + \int_0^\infty \mathbb{P}(U^{-1}\left(\frac{\pi(A)^2}{2}t\right) > t) \, dt.
\]

It remains to bound the second term. Note

\[
U^{-1}\left(\frac{\pi(A)^2}{2}t\right) > t \Leftrightarrow \exists y > t : U(y) = \frac{\pi(A)^2}{2}t \Rightarrow U(t) \leq \frac{\pi(A)^2}{2}t
\]

then apply a Chernoff bound in Theorem 3.10 to bound this term. Specifically, we apply this theorem to the function

\[
f := 1_{A \times A} - \pi(A)^2
\]
to obtain
\[ P_\pi \left( \frac{1}{t} \int_0^t 1_{A \otimes A} (X_s, Y_s) \, ds - \pi(A)^2 < -\gamma \right) \leq \exp \left( -\frac{\gamma^2 t}{10t_{rel}} \right) \]
and hence
\[ P \left( U^{-1} \left( \frac{\pi(A)^2}{2} t \right) > t \right) \leq P \left( U(t) \leq \frac{\pi(A)^2}{2} t \right) \]
\[ = P \left( \frac{U(t)}{t} - \pi(A)^2 \leq -\frac{\pi(A)^2}{2} \right) \]
\[ \leq \exp \left( -\frac{\pi(A)^4}{40t_{rel}} \right) \]
from which by integration we obtain
\[ t_{meet}^\pi \leq \frac{2t_{meet}(A)}{\pi(A)^2} + \frac{40t_{rel}}{\pi(A)^4}. \]

We now want to put \( t_{rel} \) in terms of the more practical \( t_{mix} \). By our definition
\[ d(t_{mix}) = \frac{1}{4} \Rightarrow \bar{d}(t_{mix}) \leq \frac{1}{2} \]
then by the submultiplicativity of \( \bar{d} \)
\[ \bar{d}(2t_{mix}) \leq \frac{1}{4} < \frac{1}{e} \]
which tells us from [AF02, Chapter 4] that \( 2t_{mix} \geq t_{rel} \) and the first claim follows.
For the statement of the theorem, recall in Corollary 3.7 we found
\[ t_{mix} \leq 16t_{hit}(s) \]
and that the stationary time constructed in this corollary gives the natural bound
\[ t_{meet} \leq \max_{x,y \in [n]} E_{x,y}(S) + t_{meet}^\pi. \]

4 Structural results for subcritical random graphs

In this section, we collect some of the structural results on subcritical inhomogeneous random graphs that we will need later on in Section 5. Some of these results are known, but as the literature on subcritical inhomogeneous random graphs is less developed than for supercritical random graphs, we have to prove the more specialised ones.
Let $G_N \in \mathcal{G}_{\beta,\gamma}$. Recall that $\text{Comp}(G_N)$ denotes the set of (connected) components of $G_N$. For any $\mathcal{C} \in \text{Comp}(G_N)$ we write the graph as $(V(\mathcal{C}), E(\mathcal{C}))$ and denote by $|\mathcal{C}| := |V(\mathcal{C})|$ the number of vertices in $\mathcal{C}$. Moreover, we let $\mathcal{C}(i)$ denote the component containing vertex $i$. Throughout this section, we will use the notation

$$K_\gamma := N^{\frac{1 - \beta}{2 + 2\gamma}} \log N$$

and a call a component $\mathcal{C} \in \text{Comp}(G_N)$ big if $\mathcal{C} = \mathcal{C}(i)$ for some $i \leq K_\gamma$. Otherwise, the component is called small. Moreover, we define the collection of all vertices lying in big components as

$$V_{\text{big}} := \bigcup_{i \leq K_\gamma} V(\mathcal{C}(i)).$$

The first proposition gives standard results on the size of components and the diameter – the diameter is easy to bound in the subcritical case.

**Proposition 4.1.** For $G_N \in \mathcal{G}_{\beta,\gamma}$ with $\beta + 2\gamma < 1$, we have that

$$\sup_{\mathcal{C} \in \text{Comp}(G_N)} \text{diam}(\mathcal{C}) = O_p(\log N).$$

As we will see later on, for the classical voter model, the invariant measure of the associated random walk is normalized by $\sum_{z \in \mathcal{C}(k)} d(z)$, so that in the following we collect various bounds on $\sum_{z \in \mathcal{C}(k)} d(z)$.

**Proposition 4.2.** For $G_N \in \mathcal{G}$ with $\beta + 2\gamma < 1$, with high probability,

(a) $$\max_{k \leq K_\gamma} \frac{\sum_{z \in \mathcal{C}(k)} d(z)}{(N/k)^{\gamma}} \leq \log N.$$

(b) $$\max_{v \notin V_{\text{big}}} \sum_{z \in \mathcal{C}(i)} d(v) = O_p \left( N^{\frac{1}{2 + 2\gamma}} \right).$$

As a next result, we need that the large degrees $d(i)$ are well approximated by their means $w_i \sim \frac{\beta}{1 - \gamma} \left( \frac{N}{i} \right)^{\gamma}$. Also, we need to know that for each of the vertices with large degree, a positive proportion of its neighbours has degree 1. One of the challenges in the proof is that we need these bounds uniformly over all big components.

**Proposition 4.3.** For $G_N \in \mathcal{G}_{\beta,\gamma}$ with $\beta + 2\gamma < 1$ the following statements hold:

(a) $$\min_{k \leq K_\gamma} \frac{d(k)}{(N/k)^{\gamma}} = \Omega_p(1), \quad \max_{k \leq K_\gamma} \frac{d(k)}{(N/k)^{\gamma}} = O_p(1),$$
(b) For any $k \in [N]$, let $L_k$ be the number of neighbours of $k$ of degree 1, then we have

$$\min_{k \leq K, \ d(k)} \frac{|L_k|}{d(k)} = \Omega_P(1)$$

**Definition 4.4.** For $G_N \in \mathcal{G}_{\beta, \gamma}$ and any component $\mathcal{C} \in \text{Comp}(G_N)$, we define the set of branches $B(\mathcal{C})$ of $\mathcal{C}$ as the set of connected components of the subgraph of $\mathcal{C}$ induced by the vertex set $V(\mathcal{C}) \setminus \{i\}$, where $i = \min V(\mathcal{C})$.

The next lemma states that big components have branches that are small (at least when compared to the degree of vertices with small index, which is of order $N^{\gamma}$).

**Lemma 4.5.** For $G_N \in \mathcal{G}_{\beta, \gamma}$ with $\beta + 2\gamma < 1$, with high probability every big component is a tree. On this event we have that

$$\max_{k \leq K, \ d(k)} \max_{B \in B(k)} \sum_{v \in B} d(v) = \Theta_P^{\log N} \left( N^{\frac{\gamma}{2-2\gamma}} \right).$$

The following claim for the empirical moment of the degree distribution of $\mathcal{C}(1)$ will allow us to demonstrate a lower bound on this component for certain parameters of both the classical and discursive models.

**Lemma 4.6.** For $G_N \in \mathcal{G}_{\beta, \gamma}$ whenever $\eta \geq 1$ and $\beta + 2\gamma < 1$

$$\sum_{v \in \mathcal{C}(1)} d(v)^\eta = \Theta_P^{\log N} \left( N^{\gamma \eta} \right).$$

Two of our lower bounds require the existence of a ‘double star’ component together with a suitable bound on the empirical moment.

**Proposition 4.7.** For $G_N \in \mathcal{G}_{\beta, \gamma}$ with $\beta + 2\gamma < 1$ there exists with high probability a tree component containing two adjacent vertices $x, y \in K_\gamma$ such that

$$d(x), d(y) = \Theta_P^{\log N} \left( N^{\frac{\gamma}{2-2\gamma}} \right)$$

and further for any $\eta \geq 1$ it has

$$\sum_{v \in \mathcal{C}(\{x,y\})} d(v)^\eta = \Theta_P \left( N^{\frac{\gamma \eta}{2-2\gamma}} \right).$$

The final proposition of this section states that we can always find a “long double star” in $\mathcal{G}_{\beta, \gamma}$, i.e. two vertices with degree of order at least $N^{\gamma/(2-2\gamma)}$ that are connected via a short path with two intermediate vertices of degree 2 each. The path having length at least 3 is important for the discursive voter model dynamic.

**Proposition 4.8.** With high probability any $G_N \in \mathcal{G}_{\beta, \gamma}$ with $\beta + 2\gamma < 1$ contains a path $\mathcal{P} = (v_1, v_2, v_3, v_4)$ such that:
(a) $d(v_2) = d(v_3) = 2$.

(b) $\{v_1, v_4\} \subset [K_\gamma]$ (and hence the component is a tree whp.)

(c) $d(v_1), d(v_4) = \Theta^\log N \left( N^{\frac{\gamma}{1-\gamma}} \right)$.

In the remaining part of this section, we will prove these results. An essential tool will be a coupling with a branching process that we set up in Section 4.1. Then in Section 4.2 we will prove the structural results stated above.

### 4.1 Coupling with a branching process

By Remark 2.3(c), we have some flexibility for which model in the class $G_{\beta,\gamma}$ to show our results. For most of our proofs, we will prove the statements for the simple Norros-Reittu (SNR) model, i.e. where edges are present independently with probabilities

$$q_{i,j} = 1 - e^{-p_{i,j}}.$$  \hspace{1cm} (6)

The reason for this choice is the close relation with the standard multigraph Norros-Reittu (MNR) model.

In the NR multigraph $G^{NR}_N$ each vertex $i \in [N]$ has weight $w(i) > 0$ and independently for each pair $\{i, j\}$ with $i, j \in [N]$, the number of edges between $i$ and $j$ has the distribution

$$\text{Pois} \left( \frac{w(i)w(j)}{w([N])} \right),$$

where $w([N]) = \sum_{i=1}^{N} w(i)$ is the total weight. Note this graph model not only has multiple edges, but also allows for self-loops.

The SNR model with edge probability as in (6) is then obtained by first choosing

$$w(i) := \sum_{j=1}^{N} \beta N^{2\gamma-1} i^{-\gamma} j^{-\gamma} \sim \frac{\beta N^\gamma}{i^{\gamma}},$$

and then collapsing all multi-edges to simple edges and deleting the loops.

The MNR model is particularly nice, because it allows for an exact coupling with a two-stage Galton-Watson process with thinning and cycle creation. Our construction here extends the coupling introduced in [NR06] (see also [vdH16]) by also keeping track of the number of edges, so that we can also control when we create cycles.

Define mark distribution $M$ to be the random variable on $[N]$ which chooses a vertex biased proportional to its weight

$$\mathbb{P}(M = m) \propto w(m) 1_{m \in [N]} \propto m^{-\gamma} 1_{m \in [N]}$$
so that if $W_N$ is the empirical weight distribution in the network, the weight of a typical neighbour in our local picture will be simply the size-biased version of $W_N$, denoted $W_N^*$

$$w(M) \overset{(d)}{=} W_N^*.$$ 

Fix $k \in [N]$, we now describe the (marked) branching process that describes the cluster exploration when started from vertex $k$. To describe branching process, we label the tree vertices using the standard Ulam-Harris notation, in particular we denote by $\emptyset$ the root of the tree, by 1 the first offspring of the root, by 11 the first offspring of tree vertex 1 etc. We will write $v < w$ if $v$ comes first in the lexicographic ordering induced by this labelling.

For the root of the branching process, we define

$$M_{\emptyset} = k, \ X_{\emptyset} \sim \text{Pois} (w(k))$$

Then, we define independent random variables $(X_v)_{v \neq \emptyset}$ in two stages: we first choose marks $(M_v)_{v \neq \emptyset}$ which are i.i.d. with the same distribution as $M$. Then, conditionally on $M_v$, let $X_v \sim \text{Pois} (w(M_v))$.

Then, if we take $X_v$ to be the children of vertex $v$ (if it exists in the tree), this construction can be used to define a (marked) Galton-Watson tree $T^k$ (where only the root has a different offspring distribution).

To obtain the cluster at $k$ from $T^k$, we introduce a thinning procedure. We set $\emptyset$ to be unthinned and then explore the tree in lexicographic order (according to the Ulam-Harris labelling) and thin a tree vertex $w$ if either of the tree vertices in the unique path between $\emptyset$ and $w$ has been thinned or if there exists an unthinned $v < w$ with $M_v = M_w$.

Now, denote for $i \in [N]$, $X_v(i)$ to be the number of children of $v$ with mark $i$. If $v$ and $w$ are unthinned tree vertices, then we define

$$E(M_v, M_w) = \begin{cases} 
X_v(M_w) & \text{if } v < w, \\
X_w(M_v) & \text{if } w \leq v.
\end{cases} \quad (7)$$

We can the define the multigraph $T_{\text{thin}}^k$ with vertex set $\{M_v : v \text{ unthinned}\}$ and the number of edges given by $\{E(M_v, M_w) : v, w \text{ unthinned}\}$.

Similarly, we can define a forest $(T^1, T^2, \ldots, T^n)$ of independent trees constructed as above, where the root of the $k$th forest has mark $k$. Then, we can define the same thinning operation as above, starting in the tree $T^1$ and moving to the left, where now also the roots of the trees may be thinned if their label has appeared in a previous tree. If we define the edges as in (7), then we obtain a multigraph $(T^1, T^2, \ldots, T^n)_{\text{thin}}$ with vertex set $\{M_v : v \text{ unthinned}\} = [N]$ and the number of edges between $i$ and $j$ given as $E(M_v, M_w)$, where $v$ and $w$ are the unique unthinned vertices $v, w$ with $M_v = i$ and $M_w = j$.

With this construction, we have the following proposition.
Proposition 4.9. Let $G^\text{NR}_N$ be a realization of a Norros-Reittu multigraph. For any fixed vertex $k \in [N]$, we have for the component $C(k)$ in $G^\text{NR}_N$ containing $k$,

$$C(k) \overset{d}{=} T^k_{\text{thin}}.$$ 

Moreover,

$$G^\text{NR}_N \overset{d}{=} (T^1, T^2, \ldots, T^n)_{\text{thin}}.$$ 

This proposition can be proved in the same way as Prop. 3.1 in [NR06]. The only difference is that we explicitly keep track of the number of edges.

Remark 4.10. Note that for the second construction, if the root of the $k$th tree $T^k$ is not unthinned, then any vertex in the subtree that receives mark $j \leq k$ will be thinned. So to get a stochastic upper bound on the number of vertices in the component $C(k)$, we can replace $T^k$ by $T^k_k$, where the marks are chosen independently with distribution

$$M_k^{(d)} = \begin{cases} M & \text{if } M > k \\ \dagger & \text{otherwise} \end{cases}$$

so that the offspring variables are given as $\text{Pois}(W^*_{N,k})$ with $W^*_{N,k} \overset{d}{=} w(M_k)$, defining $w(\dagger) = 0$. The error in this upper bound comes from thinning within $T^k$ (of which we have neglected all except with the root $k$), and also that thinned vertices are included as leaves of zero weight rather than simply being removed.

4.2 Proofs for the simple Norros-Reittu network

When $\beta + 2\gamma < 1$, the network has arbitrarily many components of order $N^\gamma$ in probability. We can ignore the labels and thinning in the initial construction of these components and still get an algorithm which terminates almost surely with $N$ components. Then, retrospectively, we see the effects of the thinning.

Proof of Proposition 4.1. The SNR network is constructed as a thinned and glued version of a Galton-Watson tree with offspring distribution

$$D \sim \text{Pois}(W^*_N)$$

which has offspring mean

$$\mathbb{E}(D) \rightarrow \frac{\beta}{1 - 2\gamma} < 1.$$ 

Therefore when we explore a branch of a component neglecting thinning it is the Galton-Watson tree $(Z_k)_{k \geq 0}$ with offspring distribution $D$ and $Z_0 = 1$. For any $\rho \in \left(\frac{\beta}{1 - 2\gamma}, 1\right)$ and $N$ large enough we have

$$\forall k \geq 0 \quad \mathbb{E}(Z_k) \leq \rho^k$$

24
and hence by Markov’s inequality we have an exponential tail
\[ \forall k \geq 0 \quad P(Z_k \neq 0) \leq \rho^k \]
so a maximum depth over \( O_p(N) \) such explorations required to explore from every vertex in \([N]\) is \( O_p(\log N)\). Note that thinning, whether simply by deletion or by adding cycles within a component, can only reduce the diameter of the resultant graph. \( \square \)

In the following we will write \( X \succ Y \) if the random variable \( X \) stochastically dominates the random variable \( Y \).

**Lemma 4.11.** For any \( \alpha > 1, \gamma < \frac{1}{2} \) and \( N \) sufficiently large, we have
\[ \alpha W^* \succ W^*_N \]
where \( W^* \) is the weak limit of \( (W^*_N)_N \) with density
\[ \frac{P(W^* \in dx)}{dx} = 1_{x > \frac{1}{1-\gamma}} \frac{1-\gamma}{\gamma} \left( \frac{\beta}{1-\gamma} \right)^{\frac{1}{1-\gamma} \gamma} x^{-\frac{1}{\gamma}}. \] (8)

**Proof.** Recall the MNR weights \((w(i))_i\)
\[ w(i) := \beta N^{2\gamma-1} i^{-\gamma} \sum_{j=1}^{N} j^{-\gamma} \leq \frac{\beta}{1-\gamma} \left( \frac{N}{i} \right)^{\gamma} =: \lambda(i) \]
and the mark distribution
\[ P(M = k) = \frac{k^{-\gamma}}{\sum_{i=1}^{N} i^{-\gamma}} \]
so that the exploration algorithm gives weight distribution
\[ W^*_N \overset{(d)}{=} w(M) \prec \lambda(M) \]
Now we can write down the distribution functions for each variable at \( x \in \mathbb{R} \)
\[ P(W_N^* \geq x) = P(M \leq w^{-1}(x)) \leq P(M \leq \lambda^{-1}(x)) < \frac{(\lambda^{-1}(x))^{1-\gamma}}{(1-\gamma) \sum_{i=1}^{N} i^{-\gamma}}. \]
and
\[ P(\alpha W^* \geq x) = \left( \frac{1-\gamma}{\beta} \frac{x}{\alpha} \right)^{1-1/\gamma} = \alpha^{1/\gamma-1} \left( \frac{\lambda^{-1}(x)}{N} \right)^{1-\gamma} \]
Therefore
\[ \inf_{x \geq 0} \frac{P(\alpha W^* \geq x)}{P(W_N^* \geq x)} \geq \alpha^{1/\gamma-1} \frac{(1-\gamma) \sum_{i=1}^{N} i^{-\gamma}}{N^{1-\gamma}} \rightarrow \alpha^{1/\gamma-1} > 1. \] \( \square \)
Lemma 4.12. If we pick $\alpha$ with

$$1 < \alpha < \frac{1 - 2\gamma}{\beta}$$

then $D_\alpha \sim \text{Pois} \left( \alpha W^* \right)$ has

$$p_k := \mathbb{P}(D_\alpha = k) = \Theta(k^{-1/\gamma}).$$

Proof. From (8) we know that $\alpha W^*$ has density $f$ where $f(x) = \frac{C}{(x^{1-1/\gamma})} e^{-x}$ for some $C > 0$ that makes this a probability measure.

Therefore,

$$p_{k-1} = \frac{C}{(k-1)!} \int_{\alpha^{\beta/(1-\gamma)}}^{\infty} x^{k-1-1/\gamma} e^{-x} dx = C \frac{\Gamma(k-1/\gamma) - E_k}{\Gamma(k)}$$

and we can bound the error term for $k \geq 1/\gamma$ using the small $\alpha$ assumption

$$E_k = \int_0^{\alpha^{\beta/(1-\gamma)}} x^{k-1-1/\gamma} e^{-x} dx \leq \left( \frac{\alpha^{\beta}}{1-\gamma} \right)^{k-1/\gamma} \leq 1$$

so $E_k = \Theta(1)$. Now we use $\Gamma(k) = \Theta \left( k^{k-1/2} e^{-k} \right)$ to rearrange

$$p_{k-1} = \Theta \left( \frac{(k-1/\gamma)^{k-1-1/2} e^{1/\gamma-k}}{k^{k-1/2} e^{-k}} \right) = \Theta \left( \left( k - \frac{1}{\gamma} \right)^{-1/\gamma} \right)$$

by using the classical limit $\left( 1 - \frac{k}{k^{1/\gamma}} \right)^{k-1/2} \to e^{-1/\gamma}$. \qed

In, for example, [DSL04], we see that the variance of the total size for a subcritical tree is expressible in terms of the offspring variance. In fact, this connection follows for arbitrary moments.

Theorem 4.13 ([JS11] Theorem 4.1). A subcritical GW tree $T$ with offspring distribution

$$p_k = \Theta \left( k^{1-\tau} \right),$$

for some $\tau > 3$, has

$$\mathbb{P}(|T| = k) = \Theta \left( k^{1-\tau} \right).$$

Lemma 4.14. The statement of Proposition 4.2 (a),

$$\max_{k \leq K^\gamma} \sum_{z \in C(k)} d(z) \leq \log N,$$

remains true if we construct the components as in Section 4.1 but neglect the thinning.
Proof of Proposition 4.2 (a) and Lemma 4.14. We construct the SNR network in $G_{\beta,\gamma}$ via the MNR, as in Section 4.1. Before thinning and before collapsing multi-edges each vertex has a Poisson degree, therefore

$$d(k) \leq \text{Pois}(w(k))$$

and classical large deviations upper bounds on these independent Poisson distributions give $C_1 > 0$ such that with high probability

$$\max_{k \leq K_\gamma} \frac{d(k)}{(N/k)^\gamma} \leq C_1.$$  

Then, grow an iid Galton-Watson tree with offspring distribution $D_\alpha \sim \text{Pois}(\alpha W^*)$ from each neighbour of a seed, where $1 < \alpha < \frac{1-2\gamma}{\beta}$. Observe

$$\mathbb{E}(D_\alpha) = \alpha \mathbb{E}(W^*) = \frac{\alpha \beta}{1-2\gamma} < 1$$

to conclude from Theorem 4.13 and Lemma 4.12 that the total size of the $D_\alpha$-Galton-Watson tree has heavy tails with the power law $-\frac{1}{\gamma}$.

Now such heavy tailed variables are subexponential [EKM13] in the sense that, writing $(T_1, T_2, \ldots)$ for the numbers of vertices in an i.i.d. sequence of $D_\alpha$-Galton-Watson trees

$$\lim_{n \to \infty} \sup_{x \geq \gamma n} \left| 1 - \frac{\mathbb{P}(\sum_{k=1}^n (T_k - \mathbb{E}(T_k)) > x)}{n \mathbb{P}(T_1 > x)} \right| = 0.$$  

Write $\mathcal{C}_d^\alpha$ for the random tree which has one vertex of degree $d$ and attaches an i.i.d. $D_\alpha$-Galton-Watson tree to each of its incident edges. By the thinning construction of Section 4.1 and the domination in Lemma 4.11 we have

$$|\mathcal{C}(k)| \leq |\mathcal{C}_d^\alpha(k)|.$$  

Subexponentiality implies that for any $\epsilon > 0$ there is a $y \geq 1$ so components with seed degree $d \geq y$ have

$$\mathbb{P}(|\mathcal{C}_d^\alpha| > x) < (1 + \epsilon) d \mathbb{P}(T_1 > x - \mathbb{E}(|\mathcal{C}_d^\alpha|))$$

where we identify from the Galton-Watson tree mean

$$\mathbb{E}(|\mathcal{C}_d^\alpha|) = 1 + \frac{d}{1 - \frac{\alpha \beta}{1-2\gamma}}.$$  

Therefore we have some $C_2 > 0$ such that

$$\max_{k \leq K_\gamma} \frac{\mathbb{E}(|\mathcal{C}_d^\alpha(k)|)}{(N/k)^\gamma} \leq C_2.$$
Therefore in the unthinned, unlabelled skeleton of $D_{\alpha}$-Galton-Watson trees generated from the vertex $k$ we have, with high probability for any $k \leq K_\gamma$,

$$P\left(|\mathcal{C}(k)| > \left(\frac{N}{k}\right)^\gamma \log N\right) \leq P\left(|\mathcal{C}^\alpha_{d(k)}| > \left(\frac{N}{k}\right)^\gamma \log N\right)$$

$$\leq P\left(|\mathcal{C}^\alpha_{C_1(\frac{N}{k})}\gamma| > \left(\frac{N}{k}\right)^\gamma \log N\right)$$

$$\leq (1 + \epsilon)C_1 \left(\frac{N}{k}\right)^\gamma P\left(T_1 > \left(\frac{N}{k}\right)^\gamma \log N - C_2 \left(\frac{N}{k}\right)^\gamma\right)$$

$$\leq (1 + 2\epsilon)C_1 \left(\frac{N}{k}\right)^\gamma P\left(T_1 > \left(\frac{N}{k}\right)^\gamma \log N\right)$$

where the last inequality holds for $N$ sufficiently large because we know $T_1$ has a power law tail. Hence by the union bound

$$P\left(\exists k < K_\gamma : |\mathcal{C}(k)| > \left(\frac{N}{k}\right)^\gamma \log N\right)$$

$$\leq (1 + 2\epsilon)C_1 N^\gamma \sum_{k<K_\gamma} k^{-\gamma} P\left(T_1 > \left(\frac{N}{k}\right)^\gamma \log N\right)$$

which for $N$ large enough we can bound via some constant $C_3 > 0$

$$\leq (1 + 2\epsilon)C_1 N^\gamma \sum_{k<K_\gamma} k^{-\gamma} C_3 \left(\left(\frac{N}{k}\right)^\gamma \log N\right)^{\frac{\gamma - 1}{2\gamma - 1}}$$

$$\leq (1 + 2\epsilon)C_1 \left(C_3 N^{\gamma - 1} \log \frac{N}{k}\right) N^\gamma \sum_{k<K_\gamma} k^{-\gamma} k^{1-\gamma}$$

$$= (1 + 2\epsilon)C_1 C_3 N^{2\gamma - 1} \log \frac{N}{k} \sum_{i<K_\gamma} i^{1-2\gamma}$$

$$\leq (1 + \epsilon)C_1 C_3 N^{2\gamma - 1} \log \frac{N}{k} \int_{0}^{K_\gamma} t^{1-2\gamma} dt$$

$$= \frac{(1 + \epsilon)C_1 C_3}{2 - 2\gamma} \log^{3-\frac{1}{\gamma}-2\gamma} N = o(1).$$

So we have the claimed bound on the number of vertices in thinned or unthinned components. In fact, because $\mathcal{C}^\alpha_{d}$ was a tree we have

$$\sum_{v \in \mathcal{C}^\alpha_{d}} d(v) = 2 |\mathcal{C}^\alpha_{d}| - 2$$

and so because thinning can only reduce the total degree, we conclude the bound on the total degree (also for either $\mathcal{C}^\alpha_{d(k)}$ or the real object $\mathcal{C}(k)$) is satisfied with high probability over $\{k < K_\gamma\}$.
Proposition 4.15. On $G_N \in \mathcal{G}_{\beta, \gamma}$ with $\beta + 2\gamma < 1$ we have the following uniform bound on the degrees of vertices with larger index:

$$\max_{k > K_\gamma} d(k) = O_P(N^{\frac{2\gamma}{2-\gamma}}).$$

Proof. To obtain the SNR network in $G_{\beta, \gamma}$ we use the branching construction of Section 4.1 to first build the MNR, and so we have the stochastic domination

$$d(k) \preceq \text{Pois}(w(k))$$

where

$$w(k) = \sum_{j=1}^{N} \beta N^{2\gamma - 1} k^{-\gamma} j^{-\gamma} \leq \beta N^{2\gamma - 1} k^{-\gamma} \int_{j=0}^{N} j^{-\gamma} dj = \frac{\beta}{1 - \gamma} \left( \frac{N}{k} \right)^{\gamma}.$$  

Then we use a Chernoff bound

$$\log \mathbb{P}\left( d(k) \geq N^{\frac{2\gamma}{2-\gamma}} \right) \leq \log \mathbb{P}\left( \text{Pois}(w(k)) \geq N^{\frac{2\gamma}{2-\gamma}} \right) \leq \log \mathbb{E}\left(e^{\text{Pois}(w(k))} \right) - N^{\frac{2\gamma}{2-\gamma}} = w(k)(e - 1) - N^{\frac{2\gamma}{2-\gamma}}$$

so that by the union bound

$$\mathbb{P}\left( \exists k > K_\gamma : d(k) > N^{\frac{2\gamma}{2-\gamma}} \right) \leq e^{-N^{\frac{2\gamma}{2-\gamma}}} \sum_{k > K_\gamma} e^{w(k)(e-1)} \leq (N - K_\gamma + 1) e^{w(K_\gamma)(e-1) - N^{\frac{2\gamma}{2-\gamma}}} \leq N \exp\left( N^{\frac{2\gamma}{2-\gamma}} \left( -1 + \frac{\beta(e - 1) - 1}{1 - \gamma} \right) \log \gamma N \right) = o(1).$$

Proof of Proposition 4.2 (b). We will use Remark 4.10 so define the truncated size-biased weight distribution $W_{N,z}^*$ as that which forbids indices of $z$ or less. That is, if $M$ has the mark distribution

$$\mathbb{P}(M = i) = \frac{j^{-\gamma}}{\sum_{j=1}^{N} j^{-\gamma}}$$

and $w(i) = \sum_{j=1}^{N} \beta N^{2\gamma - 1} i^{-\gamma} j^{-\gamma}$ is the weight function, then

$$W_{N,z}^* \xrightarrow{(d)} w(M) 1_{M > z}.$$  

Write also $(T_{z}^{M}(s))_{s \in \mathbb{N}}$ to denote the simple Galton-Watson process from a single ancestor $T_{k}^{M}(0) = 1$ which has offspring distribution

$$\text{Pois}(W_{N,z}^*)$$

29
and first generation size $T^M_z(1) = M$ by conditioning.

After realising the “big” components, those containing a label in $\lfloor K_{\gamma} \rfloor$, and then further the component of every vertex in some $\lfloor z - 1 \rfloor$, we may find $z$ has not been explored and seek to explore $C(z)$. Edges pointing to the mark set $\lfloor z \rfloor$ will not be pointing downwards in the ordering and hence will not be accepted. So, this component with initial label $z > K_{\gamma}$ has

$$|C_z| \leq \sum_{s=0}^{\infty} T^M_z(s)$$

with high probability, by the high probability degree bound in Proposition 4.15 which uses

$$M := \left\lfloor N^\frac{\theta-z}{\gamma} \right\rfloor.$$ 

We now see exploration of the Galton-Watson tree as a supermartingale walk on $\mathbb{N}$. After “exploring” the root of the tree, we find (at most) $M$ vertices to be explored.

When a vertex is explored, we find an iid $\text{Pois}(W^*_{N,z})$ children beyond it and we do not have to explore it any more. Therefore if $S_i$ denotes the number of vertices active after $i$ have been explored as in [AS04]

$$S_1 = M$$

$$S_i - S_{i-1} \text{iid } \sim \text{Pois} (W^*_{N,z}) - 1$$

and note for $D \sim \text{Pois}(W^*_N)$

$$\mathbb{E}(S_i - S_{i-1}) \leq \mathbb{E}(D - 1) \to \mathbb{E}(W^* - 1) < 0.$$ 

Using the random walk, see see that for $x \in \mathbb{N}$

$$\mathbb{P}\left(\sum T^M_z = x\right) = \mathbb{P}\left(\min\{i : S_i = 0\} = x\right)$$

so that if we define $L := \frac{2M}{1-\mathbb{E}(W^*_N)}$, then for any $z \in \mathbb{N}$

$$L := \frac{2M}{1-\mathbb{E}(W^*_N)} \geq \frac{2M}{1-\mathbb{E}(W^*_{N,z})},$$

and thus

$$\mathbb{P}\left(\sum T^M_z > L\right) \leq \mathbb{P}(S_{L+1} \geq 0) = \mathbb{P}\left(M + \sum_{i=1}^{L} (\text{Pois} (W^*_{N,z}) - 1) \geq 0\right).$$

To clarify notation, define the centred mixed Poisson variables

$$\left(X_i^{(z)}\right)_{i=1}^{\infty} \text{iid } \sim \text{Pois} (W^*_{N,z}) - \mathbb{E}(W^*_{N,z})$$ 

(10)
to rewrite for some $C_1 > 0$

$$\mathbb{P} \left( \sum_{z} T^M_z > L \right) \leq \mathbb{P} \left( \sum_{i=1}^{L} X_i^{(z)} \geq L \left( 1 - \mathbb{E} \left( W_{N,z} \right) \right) - M \right)$$

$$\leq \mathbb{P} \left( \sum_{i=1}^{L} X_i^{(z)} \geq M \right) \leq \frac{\mathbb{E} \left| \sum_{i=1}^{L} X_i^{(z)} \right|^r}{M^r}$$

$$C_1 \left( L \sum_{i=1}^{L} X_i^{(z)} \right)^{r \left(3 - \frac{1}{\gamma} \right)^+} + L \left( z \right)^{r + 1 - \frac{1}{\gamma}} \leq \frac{\mathbb{P} \left( L \sum_{i=1}^{L} X_i^{(z)} \right)^{r \left(3 - \frac{1}{\gamma} \right)^+}}{M^r} \leq \frac{C_1}{M^r} \left( L \sum_{i=1}^{L} X_i^{(z)} \right)^{r \left(3 - \frac{1}{\gamma} \right)^+}$$

(11)

for any $r > 2 \vee \left( \frac{1}{\gamma} - 1 \right)$ and $N$ sufficiently large, by Markov’s inequality and calculations which we defer to Lemma 4.16. We will need to fix a larger $r$ in the course of the proof, in fact we will assume that $r$

$$r > \frac{4 - 4\gamma}{\gamma \wedge (1 - 2\gamma)}.$$

By summing the expression in (11) to use the union bound, we find for $C_2 > 0$

$$\sum_{z > K_{\gamma}} \mathbb{P} \left( \sum_{z} T^M_z > L \right) \leq C_2 \sum_{z > K_{\gamma}} \frac{N \left( z \right)^{r \left(3 - \frac{1}{\gamma} \right)^+}}{z^2 \left(3 - \frac{1}{\gamma} \right)^+} \leq \frac{C_1}{M^r} \left( L \sum_{i=1}^{L} X_i^{(z)} \right)^{r \left(3 - \frac{1}{\gamma} \right)^+}$$

and we require that this sum tends to 0. For the first term, observe

$$\frac{N \left( z \right)^{r \left(3 - \frac{1}{\gamma} \right)^+}}{z^2 \left(3 - \frac{1}{\gamma} \right)^+} \leq \frac{N \left( z \right)^{r \left(3 - \frac{1}{\gamma} \right)^+}}{z^{2-2\gamma} \left(3 - \frac{1}{\gamma} \right)^+} = \left( \frac{N \left( z \right)^{r \left(3 - \frac{1}{\gamma} \right)^+}}{z^{3-2\gamma}} \right)^\frac{r}{3-2\gamma}$$

and because

$$(3\gamma - 1)^+ - \gamma = \begin{cases} 2\gamma - 1 & \gamma > \frac{1}{3} < 0 \\ -\gamma & \gamma \leq \frac{1}{3} \end{cases}$$

the exponent can be made less than $-1$ for $r$ sufficiently large. For the second term, if $r$ is also large enough such that $2 - \gamma r - \gamma < 0$, we have

$$\sum_{z > K_{\gamma}} \frac{1}{z^{\gamma r + \gamma - 1}} = O \left( N^{\frac{1}{2-2\gamma}(2-\gamma r - \gamma)} \right)$$

so that

$$\frac{N^{\gamma r + \gamma - 1 + (1-r)\frac{1}{2-2\gamma}}}{\log^{r-1} N} \sum_{z > K_{\gamma}} \frac{1}{z^{\gamma r + \gamma - 1}} = O \left( \frac{N^{\gamma r + \gamma - 1 + (1-r)\frac{1}{2-2\gamma}}}{\log^{r-1} N} \right) = O \left( \log^{-1} N \right) = O(1)$$

and we conclude that every tree had size at most $L$ by the union bound. These trees will be thinned further and some of that thinning will create cycles but the total degree cannot increase by these alterations, which gives the result. \qed
Lemma 4.16. We have for the variables defined in Equation (10)

$$\mathbb{E}\left[\sum_{i=1}^{L} X_i^{(z)}\right]^r \leq C\left(L^7w(z)^{\frac{3-\frac{1}{r}}{2}} + Lw(z)^{r+\frac{1}{r}}\right)$$

whenever $r > 2 \vee \left(\frac{1}{\gamma} - 1\right)$ and $z > 1$, with $C > 0$ depending only on $r$ and $\gamma$.

Proof. These calculations use a similar strategy to [Jan08]. We write as before

$$X^{(z)} := \text{Pois}\left(\mathbb{E}\left(X^{(z)}|W_{N,z}^*\right)\right)$$

and require two moments of $X^{(z)}$. First

$$\mathbb{E}\left(\left(X^{(z)}\right)^2\right) = \text{Var}\left(\text{Pois}\left(W_{N,z}^*\right)\right) = \text{Var}\left(\mathbb{E}\left(X^{(z)}|W_{N,z}^*\right)\right) + \mathbb{E}\left(\text{Var}\left(X^{(z)}|W_{N,z}^*\right)\right)$$

$$= \text{Var}\left(W_{N,z}^*\right) + \mathbb{E}\left(W_{N,z}^*\right) \leq \frac{\beta}{1-2\gamma} + \mathbb{E}\left(\left(W_{N,z}^*\right)^2\right)$$

then for some constant $C_1 > 0$ independent of $N$ and any $\alpha > 1$

$$\mathbb{E}\left(\left(W_{N,z}^*\right)^2\right) = \int_0^\infty 2x d\mathbb{P}(W_{N,z}^* > x) = \int_0^{w(z)} 2x d\mathbb{P}(W_{N,z}^* > x)$$

$$\leq \int_0^{w(z)} 2x d\mathbb{P}(\alpha W^* > x) \leq C_1 \int_0^{w(z)} x^{2-\frac{1}{r}} dx$$

$$\leq C_1 w(z)\left(3-\frac{1}{r}\right)^{\frac{1}{2}}.$$

We now have to estimate $\mathbb{E}\left(\left(X^{(z)}\right)^r\right)$ for some $r$ large enough. We claim that

$$\sup_{\lambda \geq \beta \frac{1}{1-\gamma}} \frac{\mathbb{E}(\text{Pois}(\lambda)^{\frac{1}{r}})}{\lambda^r} < \infty.$$

Indeed, since the Poisson distribution has an exponential moment, we know

$$\lambda \mapsto \frac{\mathbb{E}(\text{Pois}(\lambda)^{\frac{1}{r}})}{\lambda^r}$$

is finite and continuous on $\left[\beta \frac{1}{1-\gamma}, \infty\right)$. Further

$$\left\|\frac{\text{Pois}(\lambda)}{\lambda}\right\|_r \leq \frac{\lambda}{\lambda} + \left\|\text{Pois}(\lambda) - \lambda\right\|_r = 1 + \left\|\frac{\text{Pois}(\lambda) - \lambda}{\sqrt{\lambda}}\right\|_r \to 1$$

as $\lambda \to \infty$, by identifying the Gaussian central limit. This proves the claim.
Hence we can say because $X^{(z)} \leq \text{Pois}\left(W_{N,z}^{+}\right)$ we have some $C_2 > 0$ such that

$$E \left( \left( X^{(z)} \right)^r \right) \leq C_2 E((W_{N,z}^{+})^r)$$

for $N$ sufficiently large. We have, given $r > \frac{1}{\gamma} - 1$,

$$\frac{E((W_{N,z}^{+})^r)}{z^r E(M = z) w(z)^r} = \sum_{j=z}^{N} \frac{1 - j^{-\gamma} \cdot j^{-\gamma r}}{z^{-\gamma} z^{-\gamma r}} \leq z^{\gamma + \gamma r - 1} \int_{z-1}^{\infty} j^{-\gamma} \cdot j^{-\gamma r} \, dj = \frac{1}{\gamma + \gamma r - 1} \left( 1 - \frac{1}{z} \right)^{1-\gamma-\gamma r} \leq \frac{1}{\gamma + \gamma r - 1} \left( 1 - \frac{1}{N} \right)^{1-\gamma-\gamma r} = O(1)$$

as $N \to \infty$, noting that this bound holds for the supremum over all $z > 1$. Thus there is some constant $C_3$ such that for every $z > 1$

$$E \left( \left( X^{(z)} \right)^r \right) < C_3 z^r E(M = z) w(z)^r.$$

Therefore by Rosenthal’s inequality [Gut13, Chapter 3, Theorem 9.1] we obtain for $r > 2$

$$E \left( \sum_{i=1}^{L} \left( X_i^{(z)} \right)^r \right) \leq C_3 L^{r/2} E \left( \left( X^{(z)} \right)^2 \right)^{r/2} + C_4 L E \left( \left( X^{(z)} \right)^r \right) \leq C_5 L^{r/2} \left( 1 + w(z)^{\left(3-\frac{1}{\gamma}\right)^+} \right)^{r/2} + C_6 L z^r E(M = z) w(z)^r \leq C_7 L^{r/2} w(z)^{\left(3-\frac{1}{\gamma}\right)^+} + C_8 L \frac{z^{1-\gamma} w(z)^r}{N^{1-\gamma}}.$$

Proof of Proposition 4.3. Again we construct the SNR network in $G_{\beta,\gamma}$ via the MNR and so have a Poisson upper bound on degrees. Recall by the large deviations principle there is some universal constant $C > 0$ such that for $\lambda$ large enough

$$P \left( \left| \text{Pois} (\lambda) - \lambda \right| \leq \frac{\lambda}{2} \right) \leq e^{-C\lambda} \tag{12}$$

so that for the unthinned degrees we can immediately compare $d(k)$ to $(N/k)^{\gamma}$. The upper bound $d(k) \leq (N/k)^{\gamma}$ follows because thinning can only decrease the degree.

From Lemma 4.14 we have with high probability (and before thinning these components)

$$\max_{k \leq K} \frac{|\mathcal{E}(k)|}{(N/k)^{\gamma}} \leq \max_{k \leq K} \sum_{z \in \mathcal{E}(k)} \frac{d(z)}{(N/k)^{\gamma}} \leq \log N.$$
and so by summation we have whp
\[ \left| \bigcup_{i \leq K \gamma} V(\mathcal{G}(i)) \right| = |V_{\text{Big}}| \leq \frac{1}{1 - \gamma} \sqrt{N \log N}. \]

We can bound the expected total amount of thinning by the double sum over probabilities of indices coinciding (for independent copies \( M, M' \) of the mark distribution) at
\[
\frac{1}{1 - \gamma} \sqrt{N \log N} \mathbb{P}(M \leq K \gamma) + \frac{1}{(1 - \gamma)^2 N \log N} \mathbb{P}(M = M')
= O \left( \log^{2 - \gamma} N + \log^{2} N \right)
\]

And hence by Markov’s inequality we do not see more than \( \log^{3} N \) thinned vertices in total. This is insignificant compared to the polynomially large degrees and thus we obtain the lower bound \( d(k) \gtrsim (N/k)^{\gamma} \) by using the other side of the large deviations claim \( \textbf{(12)} \).

Now each of these neighbours which was not thinned has offspring \( D \sim \text{Pois}(W^{*}_N) \) and no children with independent probability
\[
p_0 = \mathbb{P}(D = 0) = \mathbb{E}(e^{-W^{*}_N}) \geq e^{-\mathbb{E}(W^{*}_N)} \rightarrow e^{-\frac{\beta}{1-2\gamma}} > 0
\]
by using Jensen’s inequality. This gives the bound \( |L_k| \gtrsim d(k) \) easily by binomial concentration. \( \square \)

**Proof of Lemma 4.5.** Use the tree construction of \( G_{\text{SNR}} \in \mathcal{G}_{\beta, \gamma} \) with edge probabilities \( q_{i,j} = 1 - e^{-p_{i,j}} \). By Lemma 4.14 we know that, jointly with high probability, every vertex \( k \leq K \gamma \) before thinning has
\[
|\mathcal{C}(k)| \leq \left( \frac{N}{k} \right)^{\gamma} \log N. \tag{13}
\]

Thinning between components cannot create cycles, and nor can thinning between the root vertex and one of its children. Observe that two independent labels \( W, W' \) from the mark distribution \( W^{*}_N \) are equal with probability
\[
\mathbb{P}(W = W') = \sum_{i=1}^{N} \left( \frac{i-\gamma}{\sum_{j=1}^{N} j^{1-\gamma}} \right)^{2} = \Theta \left( \frac{N^{1-2\gamma}}{N^{2-2\gamma}} \right) = \Theta \left( \frac{1}{N} \right)
\]
and so the expected surplus (the number of edges more than a tree) of a component is easily bounded on the high probability assumptions in \( \textbf{(13)} \)
\[
\mathbb{E}(|\text{surplus}(\mathcal{G}(k))|) \leq \mathbb{P}(W = W') \left( \frac{N}{k} \right)^{2\gamma} \log^{2} N.
\]

34
which lets us use the first moment method

\[ E(\text{surplus } \mathcal{C}([|K_i|])) \leq \sum_{i=1}^{[K_i]} E(\text{surplus}(\mathcal{C}(i))) \leq P(W = W') N^{2\gamma} \log^2 N \int_0^{K_i} i^{-2\gamma} \, di \]

\[ = O^{\log N} \left( N^{2\gamma-1} N^\frac{(1-2\gamma)^2}{2-2\gamma} \right) \]

\[ = O^{\log N} \left( N^{\frac{2\gamma-1}{2-2\gamma}} \right) = o(1) \]

so by Markov’s inequality the big components form a forest with high probability.

Now that we know each component is a tree, it makes sense to talk about branches of the seed vertices. With high probability, thinning can only make these branches smaller, so they are bounded by their unthinned size. Before thinning though, we are just talking about i.i.d. \( W^*_N \)-GW trees and there at most

\[ O \left( \sum_{i=1}^{K_i} \left( \frac{N}{i} \right)^\gamma \right) = O \left( \sqrt{N} \log^{1-\gamma} N \right) \]

such trees. Finally, take \( \alpha \in \left( 1, \frac{1-2\gamma}{\beta} \right) \). The maximum of \( M = \left[ \sqrt{N} \log N \right] \) independent \( \text{Pois}(\alpha W^*) \)-Galton-Watson trees \( (T_i)_i \) has

\[ P \left( \max_{i=1}^{M} |T_i| > N^{\frac{1-2\gamma}{\beta}} \log N \right) \]

\[ = 1 - P \left( |T_1| \leq N^{\frac{1-2\gamma}{\beta}} \log N \right)^M \]

\[ = 1 - \left( 1 - \Theta \left( \left( N^{\frac{1-2\gamma}{\beta}} \log N \right)^{1-1/\gamma} \right) \right)^M \]

\[ = 1 - \left( 1 - \Theta \left( N^{-1/2} \log N \right) \right)^M = 1 - o(1) \]

Further, because we saw in Lemma 4.11 that \( \alpha W^* > W^*_N \), we can say the same bound holds for a maximum of \( M \) independent \( W^*_N \)-trees.

\[ \square \]

**Proof of Lemma 4.6.** We work with the SNR network. Let \( A \) be the event that \( \mathcal{C}(1) \) in this construction uses unique labels and so is not thinned at all.

Conditionally on \( A \) the lower bound is trivial, because

\[ d(1) \overset{(d)}{=} \text{Pois} \left( \beta N^{2\gamma - 1} \sum_{i=2}^{N} i^{-\gamma} \right) = \Theta_{\mathbb{P}}(N^\gamma) . \]

35
Still conditionally on \( \mathcal{A} \), \( \mathcal{C}(1) \) is generated algorithmically by this root degree and sequence of i.i.d. variables. Generate a finite set of these variables \( (D_i, i = 1, \ldots, N^\gamma \log N) \) with common distribution \( D_i \sim 1 + \text{Pois}(W_N^*) \). Since

\[
\mathbb{E}((W_N^*)^{\eta}) = \Theta_p \begin{cases} 
N^{\eta \gamma - 1} & \eta > \frac{1}{\gamma} \\
1 & \eta \leq \frac{1}{\gamma}
\end{cases}
\]

we have some universal \( C > 0 \) such that

\[
\mathbb{E}((1 + \text{Pois}(W_N^*))^{\eta}) \leq 2^\eta \mathbb{E}(1 \vee \text{Pois}(W_N^*)) \leq C \mathbb{E}((W_N^*)^{\eta})
\]

so we can bound the mean of the sum of the list \( (D_i)_i \),

\[
\mathbb{E} \sum_{i=1}^{N^\gamma \log N} D_i^{\eta} = O^{\log N} \left(N^{(\eta \gamma - 1)^+ \gamma}\right) \leq O^{\log N} (N^{\eta \gamma})
\]

and hence by Markov’s inequality

\[
\sum_{i=1}^{N^\gamma \log N} D_i^{\eta} = O^{\log N} (N^{\eta \gamma}).
\]

Note by Lemma 4.14 that before thinning

\[
|\mathcal{C}(1)| \leq N^\gamma \log N
\]

with high probability, and so conditionally we expect a number repeated labels of order

\[
O^{\log N} (N^\gamma \mathbb{P}(M = 1) + N^{2\gamma} \mathbb{P}(M = M')) = O^{\log N} (N^{2\gamma - 1}) = o(1)
\]

so we conclude by Markov’s inequality that \( \mathcal{A} \) holds with high probability.

Finally, conditionally on both high probability events \( \mathcal{A} \) and (14), we can say

\[
\sum_{v \in \mathcal{C}(1)} d(v)^{\eta} \leq d(1)^{\eta} + \sum_{i=1}^{N^\gamma \log N} D_i^{\eta}.
\]

\( \square \)

**Proof of Proposition 4.7.** Consider the index set

\[
I := \left[N^{\frac{1-2\gamma}{2\gamma}}, N^{\frac{1-2\gamma}{2\gamma}} \log N\right] \cap \mathbb{N} \subset [K_\gamma].
\]

We calculate the Multigraph Norros-Reittu weight

\[
w(I) \sim \sum_{k \in I} \frac{\beta}{(1 - \gamma) \left(\frac{N}{k}\right)^\gamma} = \Theta \left(\sqrt{N} \log^{1-\gamma} N\right)
\]

36
and so the expected number of edges on the subgraph induced on $I$ is

$$\frac{w(I)^2}{w([N])} = \Theta (\log^{2-2\gamma} N) = \omega(1)$$

which, because this is Poisson distributed, must then be nonzero with high probability. After collapsing any multi-edges to arrive at the SNR model it must still be nonzero.

We can take any such edge $(x, y) \in I^2$ to create a double star, which by Lemma 4.5 is a tree and by Proposition 4.3 (a) has

$$d(x), d(y) = \Theta (\log N) \left( N^{\frac{\gamma}{2-2\gamma}} \right).$$

For the final claim of the Proposition we consider the empirical moment. This is done in the same way as the previous proof of Lemma 4.6, by taking an i.i.d. sequence

$$(D(i))_{i=1}^{N^{\frac{\gamma}{2-2\gamma}} \log N}$$

with common distribution

$$D(i) \sim 1 + \text{Pois} (W^*_N).$$

and the result follows just as in that proof, in addition to the fact that with high probability we do not see any thinning at all on this double star.

**Proof of Proposition 4.8.** Define the set $V$ of vertices with weight (defined in Section 4.1) less than 1

$$V := \{v \in [N] : w(v) < 1\}$$

we will also need the same low index set of the previous existence proof

$$I := \left[ N^{\frac{1-2\gamma}{2-2\gamma}}, N^{\frac{1-2\gamma}{2-2\gamma}} \log N \right] \cap N \subset [K_{\gamma}].$$

Define further:

$$V^{\text{even}} := V \cap (2N) ; \quad V^{\text{odd}} := V \cap (2N + 1) ;$$

$$I^{\text{even}} := I \cap (2N) ; \quad I^{\text{odd}} := I \cap (2N + 1).$$

Note

$$[N] \setminus \left[ N \left( \frac{\beta}{1-\gamma} \right)^{1/\gamma} \right] \subseteq V$$

so because

$$\frac{\beta}{1-\gamma} < \frac{1-2\gamma}{1-\gamma} < 1$$

we have $|V| = \Theta(N)$, so there is $\Theta(N)$ weight in both halves of $V$,

$$\frac{w(V^{\text{even}})}{N} \sim \frac{w(V^{\text{odd}})}{N} \rightarrow \rho > 0.$$
We also calculate

\[ w([N]) = \sum_{i,j} \beta N^{2\gamma-1} y^{-\gamma} j^{-\gamma} \sim \frac{\beta N}{(1-\gamma)^2} = \Theta(N) \]

and finally for the large degree sets

\[ w(\mathcal{I}^{\text{even}}) \sim w(\mathcal{I}^{\text{odd}}) \sim \frac{1}{2} \sum_{k \leq I} \frac{\beta}{1-\gamma} \left( \frac{N}{k} \right)^{\gamma} \sim \frac{\beta}{2(1-\gamma)^2} N\gamma K^{1-\gamma}. \]

Now, each vertex \( x \in V^{\text{even}} \) has a number of edges (in the multigraph) to the other half with exact distribution

\[ E(x, V^{\text{odd}}) \overset{(d)}{=} \text{Pois}(\Lambda_1(x)) \]

for the mean \( \Lambda_1(x) \sim \frac{w(x)\rho(1-\gamma)^2}{\beta} \). So, because \( w(V) \subset \left[ \frac{\rho(1-\gamma)^2}{\beta}, 1 \right] \), we have

\[ \lim_{N \to \infty} \min_{x \in V^{\text{even}}} P \left( E(x, V^{\text{odd}}) = 1 \right) \geq \min_{\lambda \in \left[ \rho(1-\gamma), \frac{\rho(1-\gamma)^2}{\beta} \right]} \lambda e^{-\lambda} > 0. \]

By binomial concentration, this means we have some set \( \mathcal{E} \subset V^{\text{even}} \) and some disjoint set \( \mathcal{O} \subset V^{\text{odd}} \) such that

- \( |\mathcal{O} \cup \mathcal{E}| = \Theta_P(N) \),
- The subgraph induced on \( \mathcal{O} \cup \mathcal{E} \) has a perfect matching.

It remains to consider all the other potential edges. First of all, we discard vertices in \( \mathcal{O} \) with edges into \([N] \setminus (I^{\text{even}} \cup V^{\text{even}})\)

which, because vertices in \( V \) have MNR weight at most 1, occurs independently for each vertex with probability at most \( 1 - e \).

We can make a symmetric claim for vertices in \( \mathcal{E} \), so that in the end we keep at least a proportion \( e^{-2} \) of the original pairs in the matching of \( \mathcal{O} \cup \mathcal{E} \).

Finally, a remaining adjacent pair \((x, y) \in \mathcal{O} \times \mathcal{E}\) has an exact Poisson distribution of desired edges

\[ E(x, I^{\text{odd}}) \overset{(d)}{=} \text{Pois}(\Lambda_2(x)), \]

\[ E(y, I^{\text{even}}) \overset{(d)}{=} \text{Pois}(\Lambda_2(y)), \]

where the mean has asymptotic

\[ \Lambda_2(x) \sim \frac{w(x)}{2} \left( \frac{K_1}{N} \right)^{1-\gamma}. \]

38
and each of these variables is equal to 1 with probability that is uniformly $\Theta((K_\gamma/N)^{1-\gamma})$ over $x, y \in V$. Therefore, conditioning on the high probability event that we so far have $\Theta_F(N)$ pairs remaining, the expected number of these pairs which both send a single edge into a disjoint half of $I$ has order

$$N \cdot \left( \frac{K_\gamma}{N} \right)^{1-\gamma} = \log^{2-2\gamma} N \to \infty$$

and therefore with high probability we will find at least one such path $P$. Because this path leads

$$I_{\text{odd}} \leftrightarrow V_{\text{odd}} \leftrightarrow V_{\text{even}} \leftrightarrow I_{\text{even}}$$

and each of these sets is disjoint, we know that after collapsing multi-edges the path will still exist, and will then satisfy the criteria for our “double star”. 

5 Voter models

In this section, we will prove the two main theorems about the asymptotics of the consensus time. In Section 5.1 we will consider the classical voter model and prove Theorem 2.4. Then, in Section 5.2 we will prove Theorem 2.5 for the discursive voter model. Throughout we will use the duality of the voter model to a system of coalescing random walks as described in Section 3. We will also use the notation regarding various random walks statistics from that section.

5.1 Consensus time for the classical voter model

In this section, we will consider the classical voter model as defined in Definition 2.1(a). Throughout, let $G_N \in G_{\beta, \gamma}$ for $\beta + 2\gamma < 1$ be the underlying graph. We note that this version of the voter model fits into the general setting of a $Q$-voter model of Section 3 if for $\theta \in \mathbb{R}$ we consider

$$Q_\theta(i, j) = d(i)^{\theta-1} \text{ if } i \sim j \text{ in } G_N.$$  \hspace{1cm} (15)

As before, we write $\mathbb{P}_\theta$ for the law of (and $\mathbb{E}_\theta$ for the expectation with respect to) the coalescing random walks with generator $Q_\theta$.

If we denote by $C_1, \ldots, C_k$ the connected components of $G_N$, then these also correspond to the irreducible components of the Markov chain with generator $Q_\theta$. So if we let $\pi = (\pi(z), z \in V(G_N))$ be defined via

$$\pi(z) = \frac{d(z)^{1-\theta}}{\sum_{y \in C_j} d(y)^{1-\theta}}, \text{ for } z \in C_j,$$

for $j \in [k]$, then $\pi|_{C_j}$ is the invariant measure of the $Q_\theta$ Markov chain restricted to $C_j$. 

39
Before the main proof, we show an elementary bound on the meeting time of two independent random walks, when the component contains a star, i.e., if there exists a vertex $k$ with a set of neighbours $L_k$, each of degree 1 (compare Proposition 4.3).

**Lemma 5.1.** Let $k$ be as above and denote by $L_k$ the non-empty set of its neighbours of degree 1. Let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be independent Markov chains on $\mathcal{C}(k)$ with generator $Q^\theta$. Then, for the product chain observed on $\{k\} \cup L_k$ (as defined before Theorem 3.11), we have

$$t_{\text{meet}}^\pi(k \cup L_k) \leq \frac{3 + d(k)^\theta}{2}.$$

**Proof.** Let $S_t$ count how many of the two walkers are currently in the leaf set $L_k$. Then $(S_t)_{t \geq 0}$ is Markov chain on $\{0, 1, 2\}$ with transition rates $(s_{ij})$, where in particular

$$s_{21} = 2, \quad s_{10} \geq 1, \quad s_{12} = |L_k|d(k)^{\theta-1} \leq d(k)^\theta,$$

using that $|L_k| \leq d(k)$.

Now, note that $S_t = 0$ implies that $\tau_{\text{meet}} \leq t$. In particular, if $T_i = \inf\{t \geq 0 : S_t = i\}$ for $i \in \{0, 1, 2\}$, then we have that $T_0 \geq \tau_{\text{meet}}$.

From the explicit transition rates, we can see that $E(T_1) = \frac{1}{2}$ and if we write $s(1) = s_{10} + s_{12}$, then

$$E(T_0) = 1 + \frac{s_{12}}{s(1)} \left(1 + E(T_0)\right),$$

so that

$$E(T_0) = 1 + \frac{s_{12}}{s(1)} \leq 1 + \frac{s_{12}}{2}.$$

We conclude that

$$\sup_{v, w \in V(S_k)} E((v, w) \tau_{\text{meet}}) \leq \max\{E(T_0), E(T_0)\} \leq \frac{1}{2} + 1 + \frac{s_{12}}{2} \leq \frac{3 + d(k)^\theta}{2},$$

as claimed. \qed

**Proof of Theorem 2.4.** We will start by showing the upper bounds. For the cases $\theta \geq 1$, we use that by Proposition 3.2 we can bound

$$E_{\mu_x}(\tau_{\text{cons}} | G_{\beta, \gamma}) \leq e(2 + \log N) t_{\text{hit}}(G_N),$$

where $t_{\text{hit}} = \sup_{j \in [k]} t_{\text{hit}}(C_j)$ for $C_1, \ldots, C_k$ the components of $G_N$. Note in particular that the right hand side is still random and the expectation is only over the random walks.

We recall that the random walk associated to the classical voter model has transition rates $Q^\theta(x, y) = d(x)^{\theta-1} \mathbf{1}_{x \sim y}$ and $\pi(x) \propto d(x)^{1-\theta}$. In particular, for any component $\mathcal{C} \in \text{Comp}(G_N)$ the conductances as defined in (3) are

$$c(x, y) = \pi(x)Q^\theta(x, y) = \frac{1}{\sum_{z \in \mathcal{C}} d(z)^{1-\theta}} \mathbf{1}_{x \sim y}, \quad \text{for any } x, y \in \mathcal{C}.$$
Hence, by Proposition 3.4, we have for any component \( C \),

\[
t_{\text{hit}}(C) \leq \text{diam}(C) \sum_{z \in \theta} d(z)^{1-\theta}.
\] (16)

Because \( \theta \geq 1 \), we have that \( \sum_{z \in \theta} d(z)^{1-\theta} \leq |\theta| \). Therefore, by Proposition 4.2 (a) and Proposition 4.1, we get that

\[
\sup_{C \in \text{Comp}(G_N)} t_{\text{hit}}(C) \leq \text{diam}(C) \times \max_{C \in \text{Comp}(G_N)} |C| = O_p^{\log N} (N^\gamma).
\]

For \( \theta \leq 0 \), we first deal with the small components, where we recall that the vertex set of the ‘small’ components is defined as

\[
V_{\text{small}} := [N] \setminus V_{\text{big}}, \quad V_{\text{big}} := \bigcup_{k \leq K_\gamma} V(C(k))
\]

where \( K_\gamma = N^{1-2\gamma} \log N \). By Proposition 4.2 (b) we know that

\[
\max_{k \in V_{\text{small}}} \sum_{x \in \theta(k)} \frac{d(x)}{d(k)} = O_p^{\log N} \left( N^{-2\gamma} \right)
\]

In particular, we get from (16) using \( \sum_{i} x_i^p \leq (\sum_{i} x_i)^p \) for any \( p \geq 1 \) and \( x_i \geq 0 \) that

\[
\max_{k \in V_{\text{small}}} t_{\text{hit}}(C(k)) \leq \text{diam}(G_N) \text{max}_{k \in V_{\text{small}}} \sum_{x \in \theta(k)} d(x)^{1-\theta}
\]

\[
\leq \text{diam}(G_N) \text{max}_{k \in V_{\text{small}}} \left( \sum_{x \in \theta(k)} d(x) \right)^{1-\theta} = O_p^{\log N} \left( N^{\frac{\gamma(1-\theta)}{2-2\gamma}} \right),
\]

where we also used Proposition 4.1 to bound the diameter.

To bound the consensus time on large components, we use that by Proposition 3.8 for any \( k \leq K_\gamma \),

\[
t_{\text{meet}}(C(k)) \leq 95 \frac{t_{\text{hit}}(k)}{\pi(k)^{1-\theta}},
\]

and find a suitable upper bound on the right hand side, which in turn gives us by Proposition 3.2 an upper bound on \( \mathbb{E}_{\mu_\pi}(\tau_{\text{cons}}(C(k)) \mid G_N) \). In order to bound the invariant measure, we note that since \( \theta \leq 0 \), we have from Proposition 4.2 (a)

\[
\pi(k) = \frac{d(k)^{1-\theta}}{\sum_{z \in \theta(k)} d(v)^{1-\theta}} \geq \left( \frac{d(k)}{\sum_{z \in \theta(k)} d(z)} \right)^{1-\theta} = \Omega_p^{\log N} (1).
\]

In order to bound the hitting time \( t_{\text{hit}}(k) \) we apply the same argument as for the small component, but for the random walk restricted to each branch of \( C(k) \) (see also Definition 4.4 above for the formal definition of a branch). The bound on the sum of degrees comes from Lemma 4.3. Together, we obtain that

\[
\sup_{k \leq K_\gamma} t_{\text{meet}}(C(k)) = O_p^{\log N} \left( N^{\frac{\gamma(1-\theta)}{2-2\gamma}} \right).
\]
Combined with the bound on the small components, this completes the upper bound in the case $\theta \leq 0$.

We complete the upper bounds by showing for $\theta \in (0, 1)$ that

$$
\max_{k \in [N]} t_{\text{meet}}(C(k)) = O_p^{\log N} \left( N^{\gamma \theta} + N^{\frac{1}{\gamma - 2}} \right).
$$

The upper bound on the consensus time then follows by Proposition 3.2 and by noting that in each of the two different regimes one of the summands dominates.

For $k \in V_{\text{small}}$, we use similar strategy as above and obtain by (16) that

$$
t_{\text{hit}}(C(k)) \leq \frac{\sum_{z \in C(k)} d(z)^{-1}}{\sum_{z \in C(k)} d(z)} \leq \frac{|L_k|}{\sum_{z \in C(k)} d(z)}.
$$

which if we combine Proposition 4.2 (b) and Proposition 4.1 is seen to be $O_p^{\log N} \left( N^{\frac{1}{\gamma - 2}} \right)$ uniformly in $k \in V_{\text{small}}$.

For the bound on the large components, define for $k \leq K_{\gamma}$ the set $L_k$ as the neighbours of $k$ that have degree 1. By Proposition 4.3 (b) we have that

$$
\min_{k \leq K_{\gamma}} |L_k| \geq \frac{\sum_{x \in L_k} d(x)^{-1}}{\sum_{x \in C(k)} d(x)} \geq \frac{|L_k|}{\sum_{x \in C(k)} d(x)}.
$$

Thus by (18) and Proposition 4.2 (a) we have

$$
\pi(L_k \cup \{k\}) \geq \frac{\sum_{x \in L_k} d(x)^{-1}}{\sum_{x \in C(k)} d(x)} \geq \frac{|L_k|}{\sum_{x \in C(k)} d(x)}.
$$

Because we have a large stationary mass in $L_k \cup \{k\}$, Theorem 3.11 gives us that

$$
T_{\text{meet}}(C(k)) = O_p^{\log N} \left( T_{\text{meet}}(L_k \cup \{k\}) + t_{\text{hit}}(k) \right),
$$

where we recall that $T_{\text{meet}}(L_k \cup \{k\})$ is the meeting time for the Markov chain observed on $\{k\} \cup L_k$ (see also the definition just before Proposition 3.11). We bound this by Lemma 5.1 as

$$
\max_{k \leq K_{\gamma}} T_{\text{meet}}(\{k\} \cup L_k) \leq \max_{k \leq K_{\gamma}} \frac{3 + d(k)^{\theta}}{2} = O_p(N^{\gamma \theta})
$$

and by Lemma 1.5

$$
t_{\text{hit}}(k) \leq \max_{B \in \mathcal{B}(\gamma)} \text{diam}(B) \sum_{v \in B} d(v)^{1-\theta} \leq \text{diam}(G_N) \max_{B \in \mathcal{B}(\gamma)} \sum_{v \in B} d(v) = O_p^{\log N} \left( N^{\frac{1}{\gamma - 2 \gamma}} \right).
$$
So, substituting both bounds into (20), we obtain

\[
\max_{k \leq K} t_{\text{meet}}(\mathcal{C}(k)) = O_p^2 \left( N^{\gamma} + N^{\frac{\gamma^2}{2\gamma}} \right),
\]

By combining this with the bound on the small components, we have completed the proofs for the upper bounds in all cases.

We continue with the **lower bounds**.

For the first part we consider \( \mathcal{C}(1) \) when \( \theta > 0 \). By Lemma 3.1 and Proposition 3.2

\[
\mathbb{E}_{\mu_k} (\tau_{\text{cons}}(\mathcal{C}(1))) \geq 2u(1 - u) t_{\text{meet}}(\mathcal{C}(1)) \geq 2u(1 - u) t^{\pi}_{\text{meet}}(\mathcal{C}(1)),
\]

where the last inequality follows from the definitions. To bound the right hand side, we recall from Lemma 3.5 (a) that

\[
\sum_{x \in \mathcal{C}(1)} \pi(x)^2 \leq \max_{x \in \mathcal{C}(1)} \pi(x) \sum_{v \in \mathcal{C}(1)} \sum_{z \in \mathcal{C}(1)} d(z) \frac{q(z)}{\pi(z)} = O_p(1).
\]

To estimate the denominator in (21), we note that for \( \theta \geq 1 \),

\[
\sum_{v \in \mathcal{C}(1)} q(v)\pi(v)^2 = \frac{\sum_{v \in \mathcal{C}(1)} d(v)^{1-\theta} d(v)^{2-2\theta}}{\left( \sum_{v \in \mathcal{C}(1)} d(v)^{1-\theta} \right)^2} \leq \frac{\sum_{v \in \mathcal{C}(1)} d(v)}{\left( \sum_{v \in \mathcal{C}(1)} d(v)^{1-\theta} \right)^2} \leq O_p \left( \frac{N^{\gamma(1-\theta)}}{N^{\gamma}} \right) = O_p(N^{-\gamma}).
\]

Similarly, if \( \theta \in (0, 1) \),

\[
\sum_{v \in \mathcal{C}(1)} q(v)\pi(v)^2 \leq \frac{\left( \sum_{v \in \mathcal{C}(1)} d(v)^{1-\theta} \right)^2}{\left( \sum_{v \in \mathcal{C}(1)} d(v) \right)^2} \leq O_p \left( \frac{N^{2-\theta}}{N^{2\gamma}} \right) = O_p(N^{-\theta}).
\]
Hence, we obtain from (21) for $\theta > 0$

$$t_{\text{meet}}^\pi(\mathcal{C}(1)) = \Omega \mathbb{P}(N^{\gamma} \wedge N^{\gamma \theta}).$$

(22)

For the second of the part of the lower bound, we use a component that contains a sufficiently large “double star” structure and consider parameters $\theta < 1$. More precisely, by Proposition 4.7, we have a tree component constructed with two adjacent vertices $\{x, y\}$ such that

$$d(x), d(y) \text{ and } \sum_{v \in \mathcal{C}(\{x, y\})} d(v) \text{ are } \Theta_{\mathbb{P}} \log N \left( N^{\frac{1}{1-\theta}} \right)$$

(23)

Now, let $A_x$ be the set of vertices in $\mathcal{C}(\{x, y\})$ that are closer to $x$ than to $y$, and $A_y$ the complement. Then, we will use that by Proposition 3.5 (b)

$$\mathbb{E}_{\mu_u}(\tau_{\text{cons}} | G_N) = \Omega \left( \frac{\pi(A_x) \pi(A_y)}{\sum_{v \in A_x} \sum_{w \in A_y} c(v, w)} - 1 \right).$$

(24)

Because for $\theta \in (0, 1)$

$$\frac{d(x)}{\sum_{v \in \mathcal{C}(\{x, y\})} d(v)} \leq \frac{\sum_{v \in A_x} d(v)^{1-\theta}}{\sum_{v \in A_y} d(v)^{1-\theta}} \leq \frac{\sum_{v \in \mathcal{C}(\{x, y\})} d(v)}{d(y)}$$

we obtain from (23) that

$$\pi(A_x) \pi(A_y) = \left( \sqrt{\frac{\sum_{v \in A_x} d(v)^{1-\theta}}{\sum_{v \in A_y} d(v)^{1-\theta}}} + \sqrt{\frac{\sum_{v \in A_y} d(v)^{1-\theta}}{\sum_{v \in A_x} d(v)^{1-\theta}}} \right)^{-2} = \Omega_{\mathbb{P}} \log N (1).$$

Furthermore, since $\mathcal{C}(\{x, y\})$ is a tree the denominator in (24) reduces to

$$c(x, y) = \frac{1}{\sum_{v \in \mathcal{C}(\{x, y\})} d(v)^{1-\theta}} \leq \frac{1}{|\mathcal{C}(\{x, y\})|} = O_{\mathbb{P}} \log N \left( N^{\frac{1}{1-\theta}} \right).$$

We finally consider the case $\theta \leq 0$ on this double star. Here the stationary mass is slightly harder to control, but from the claim in Proposition 4.7 we have

$$d(x)^{1-\theta} \leq \sum_{v \in A_x} d(v)^{1-\theta} = \Theta_{\mathbb{P}} \left( N^{\frac{1}{1-\theta}} \right)$$

so because further

$$d(x) = \Theta_{\mathbb{P}} \left( N^{\frac{1}{1-\theta}} \right) \implies d(x)^{1-\theta} = \Theta_{\mathbb{P}} \left( N^{\frac{1}{2-2\gamma}} \right)$$

44
we again have $\pi(A_x)\pi(A_y) = \Omega_P(1)$, but now

$$c(x, y) = \frac{1}{\sum_{v \in \mathcal{E}(\{x, y\})} d(v)^{1 - \theta}} = \Theta_p^{\log N} \left( N^{-\frac{1-\theta}{2-2\gamma}} \right)$$

so that by using the conductance bound \((24)\) we conclude

$$\mathbb{E}_{\mu_x} (\tau_{\text{cons}} | G_N) = \begin{cases} 
\Omega_P^{\log N} (N^{\frac{1-\theta}{2-2\gamma}}) & \text{if } \theta \in (0, 1) \\
\Omega_P^{\log N} (N^{\frac{\gamma(1-\theta)}{2-2\gamma}}) & \text{if } \theta \in (-\infty, 1].
\end{cases} \quad (25)$$

Combining Equations \((25)\) and \((22)\) completes the proof of Theorem \(2.4\) by giving all the required lower bounds.

5.2 Consensus time for the discursive voter model

In this section, we will consider the discursive voter model as defined in Definition \(2.1\)(b). This version of the voter model fits into the general setting of a \(Q\)-voter model of Section 3 if for \(\theta \in \mathbb{R}\) we consider \(Q = Q^\theta\) defined as

$$Q^\theta(i, j) = \frac{d(i)^{\theta-1} + d(j)^{\theta-1}}{2} \quad \text{if } i \sim j \text{ in } G_N. \quad (26)$$

As before, we write \(P^\theta\) for the law of \((\text{and } E^\theta\) for the expectation with respect to) the coalescing random walks with generator \(Q^\theta\).

If we denote by \(C_1, \ldots, C_k\) the connected components of \(G_N\), then define \(\pi = (\pi(z), z \in V(G_N))\) via

$$\pi(z) = \frac{1}{|C_j|}, \quad \text{for } z \in C_j,$$

for \(j \in [k]\). Then, \(\pi|_{C_j}\), i.e. the uniform measure on \(C_j\) is the invariant measure of the \(Q^\theta\) Markov chain restricted to \(C_j\).

First we require another application of Corollary \(3.11\) which is simpler than for the classical voter model, but covers a wider range of cases.

**Lemma 5.2.**

$$t_{\text{meet}}(G_N) = \sup_{j \in [k]} t_{\text{meet}}(C_i) = \begin{cases} 
O_p^{\log N} (N^{\frac{1-\theta}{2-2\gamma}}) & \theta > \frac{3-4\gamma}{2-2\gamma} \\
O_p^{\log N} (N^{\gamma(2-\theta)}) & 1 < \theta \leq \frac{3-4\gamma}{2-2\gamma} \\
O_p^{\log N} (N^{\gamma}) & 2\gamma \leq \theta \leq 1 \\
O_p^{\log N} (N^{\frac{\gamma(2-\theta)}{2-2\gamma}}) & \theta < 2\gamma
\end{cases}$$

**Proof.** Work in the high probability set where all big components are unthinned trees. Take any low index seed with

$$k \leq K_\gamma := N^{\frac{1-\gamma}{2-2\gamma}} \log N.$$
and again denote by $L_k$ the set of degree 1 vertices adjacent to $k$. By Proposition 4.3 (b) and because the stationary distribution is always uniform

$$\min_{k \leq K_\gamma} \pi(L_k) = \min_{k \leq K_\gamma} \frac{|L_k|}{|\mathcal{G}(k)|} = \Omega_p \log N(1).$$

Then by exchangeability, coalescence observed in $L_k$ is just complete graph (Wright-Fisher) coalescence except sometimes both walkers move simultaneously, which doesn’t change anything. Thus coalescence occurs for the partially observed process at rate

$$\frac{1 + d(k)^{\theta - 1}}{|L_k|}$$

and we conclude

$$\max_{k \leq K_\gamma} t^\pi_{\text{meet}}(L_k) \leq \max_{k \leq K_\gamma} \frac{|L_k|}{1 + d(k)^{\theta - 1}} = O_p \log N \bigg( \begin{cases} N^{\gamma(2-\theta)} & \theta > 1 \\ N^{\gamma} & \theta \leq 1 \end{cases} \bigg).$$

Now, we let $S$ be the collection of small components and branches in large components. Then, by Proposition 3.3

$$\max_{S \in \mathcal{S}} t_{\text{hit}}(S) \leq \max_{S \in \mathcal{S}} \min_{x,y \in \mathcal{P}_x,y} \sum_{\{u,v\} \in E(P_{x,y})} \frac{2|S|}{d(u)^{\theta - 1} + d(v)^{\theta - 1}}$$

$$\leq \max_{S \in \mathcal{S}} |S| \text{diam}(S) \max_{v \in S} \left( d(v)^{1-\theta} \right)$$

$$\leq \left( \max_{S \in \mathcal{S}} |S| \right) \text{diam}(G_{\beta,\gamma}) \max_{v > K_\gamma} \left( d(v)^{1-\theta} \right)$$

$$= O_p \log N \bigg( \begin{cases} N^{\frac{2-\theta}{2-2\gamma}} & \theta < 1 \\ N^{\gamma} & \theta \geq 1 \end{cases} \bigg).$$

This gives a bound on the consensus time for small components by Proposition 3.2. It also provides the final ingredient of Theorem 3.11, which we apply to obtain

$$\max_{k \leq K_\gamma} t_{\text{meet}}(\mathcal{G}(k)) = \begin{cases} O_p^{\log N} \left( N^{\gamma(2-\theta)} + N^{\frac{2-\theta}{2-2\gamma}} \right) & \theta > 1, \\ O_p^{\log N} \left( N^\gamma + N^{\frac{\gamma(2-\theta)}{2-2\gamma}} \right) & \theta \leq 1. \\ O_p^{\log N} \left( N^{\frac{\gamma}{2-2\gamma}} \right) & \theta > \frac{3-4\gamma}{2-2\gamma}, \\ O_p^{\log N} \left( N^{\gamma(2-\theta)} \right) & 1 < \theta \leq \frac{3-4\gamma}{2-2\gamma}, \\ O_p^{\log N} \left( N^\gamma \right) & 2\gamma \leq \theta \leq 1, \\ O_p^{\log N} \left( N^{\frac{\gamma(2-\theta)}{2-2\gamma}} \right) & \theta < 2\gamma. \end{cases}$$
Proof of Theorem 2.5. The upper bound for all four cases follows immediately from Lemma 5.2 and Proposition 3.2, so it only remains to prove the lower bounds. For these, it will be very useful that the stationary distribution \( \pi \) on each component is always uniform.

The first lower bound is for the case \( \theta \geq \frac{3 - 4\gamma}{2 - 2\gamma} \) for which we must consider the long double star component of Proposition 4.8. Separating the long double star on the edge \((v_2, v_3)\) which we know has low conductance

\[
c(v_2, v_3) = O_{\log N} \left( N^{-\frac{2}{1 - 2\gamma}} \right)
\]

and yet by Propositions 4.2 (a) and 4.3 (a)

\[
d(v_1), d(v_4) \text{ and } |C(\{v_1, v_4\})| = \Theta_{\log N} \left( N^{-\frac{2}{1 - 2\gamma}} \right)
\]

which implies that we have \( \Theta_{\log N} (1) \) stationary mass on each side (by a similar argument as before). Hence by Theorem 3.5 (b) we have consensus time \( \Omega_{\log N} \left( N^{-\frac{2}{1 - 2\gamma}} \right) \).

For the lower bound when \( 2\gamma < \theta < \frac{3 - 4\gamma}{2 - 2\gamma} \), we apply Corollary 3.5 (a) to \( C(1) \)

\[
t_{\text{meet}}^\pi (C(1)) \geq \left( 1 - \sum_{v \in C(1)} \pi(v)^2 \right)^2 \sim \frac{|C(1)|^2}{\sum_{v \in C(1)} q(v) \pi(v)} = \frac{|C(1)|^2}{\sum_{v \in C(1)} d(v)^\theta}.
\]

Recall the moment calculation in Lemma 4.6 to see that when \( \theta \geq 1 \)

\[
\sum_{v \in C(1)} d(v)^\theta = \Theta_{\log N} \left( N^{\gamma \theta} \right)
\]

whereas for \( \theta \in (2\gamma, 1) \) we instead have by Proposition 4.2 (a)

\[
\sum_{v \in C(1)} d(v)^\theta \leq \sum_{v \in C(1)} d(v) = O_{\log N} (N^\gamma).
\]

so by combination

\[
\frac{|C(1)|^2}{\sum_{v \in C(1)} d(v)^\theta} = \Omega_{\log N} \left( N^{(2 - \theta)\gamma} \right)
\]

which is a lower bound for the consensus time by Lemma 3.1 and Proposition 3.2.

For the final case, when \( \theta < 2\gamma \), we require another double star component, but this one must be that without a path, whose existence is stated in Proposition 4.7. This double star is a tree structure with two adjacent “star” vertices \( \{x, y\} \) with

\[
d(x), d(y) \text{ and } |C(\{x, y\})| = \Theta_{\log N} \left( N^{-\frac{2}{1 - 2\gamma}} \right)
\]

so that we have balanced stationary mass of \( \Theta_{\log N} (1) \) in the vertices closest to \( x \) and in those closest to \( y \).
We note
\[ Q^\theta(x, y) = \mathcal{O}_p^{\log N} \left( N^{-\frac{\gamma(2-\theta)}{2-2\gamma}} \right) \]
and because the stationary distribution is uniform
\[ \pi(x) = \mathcal{O}_p^{\log N} \left( N^{-\frac{\gamma}{2-2\gamma}} \right), \]
so that by the definition of the conductance \[ c(x, y) = \pi(x) Q^\theta(x, y) = \mathcal{O}_p^{\log N} \left( N^{-\frac{\gamma(2-\theta)}{2-2\gamma}} \right). \]

Hence, by Theorem 3.5 (b),
\[ t_{\text{meet}}(\mathcal{E}(1)) = \Omega_p^{\log N} \left( N^{-\frac{\gamma(2-\theta)}{2-2\gamma}} \right), \]
which gives the last remaining lower bound.

Acknowledgements. We would like to thank Peter Mörters and Alexandre Stauffer for many useful discussions. JF is supported by a scholarship from the EPSRC Centre for Doctoral Training in Statistical Applied Mathematics at Bath (SAMBa), under the project EP/L015684/1.

References

[ACF12] M. Abdullah, C. Cooper, and A. Frieze. Cover time of a random graph with given degree sequence. *Discrete Mathematics*, 312(21):3146–3163, 2012.

[AF02] D. Aldous and J. A. Fill. Reversible Markov chains and random walks on graphs, 2002. Unfinished monograph, recompiled 2014, available at http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[Ald13] D. Aldous. *Probability approximations via the Poisson clumping heuristic*, volume 77. Springer Science & Business Media, 2013.

[AS04] N. Alon and J. H. Spencer. *The probabilistic method*. John Wiley & Sons, 2004.

[BJR07] B. Bollobás, S. Janson, and O. Riordan. The phase transition in inhomogeneous random graphs. *Random Structures & Algorithms*, 31(1):3–122, 2007.

[BKW14] I. Benjamini, G. Kozma, and N. Wormald. The mixing time of the giant component of a random graph. *Random Structures & Algorithms*, 45(3):383–407, 2014.

[BLPS18] N. Berestycki, E. Lubetzky, Y. Peres, and A. Sly. Random walks on the random graph. *The Annals of Probability*, 46(1):456–490, 2018.
[CCC16] Y.-T. Chen, J. Choi, and J. T. Cox. On the convergence of densities of finite voter models to the wright–fisher diffusion. In Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, volume 52, pages 286–322. Institut Henri Poincaré, 2016.

[CFR18] C. Cooper, M. Dyer, A. Frieze, and N. Rivera. Discordant voting processes on finite graphs. SIAM Journal on Discrete Mathematics, 32(4):2398–2420, 2018.

[CL06] F. Chung and L. Lu. Complex graphs and networks. Number 107. American Mathematical Soc., 2006.

[DLP14] J. Ding, E. Lubetzky, and Y. Peres. Anatomy of the giant component: The strictly supercritical regime. European Journal of Combinatorics, 35:155–168, 2014.

[DSL04] K. S. Dorman, J. S. Sinsheimer, and K. Lange. In the garden of branching processes. SIAM review, 46(2):202–229, 2004.

[Dur07] R. Durrett. Random graph dynamics, volume 200. Cambridge university press Cambridge, 2007.

[Dur10] R. Durrett. Some features of the spread of epidemics and information on a random graph. Proceedings of the National Academy of Sciences, 107(10):4491–4498, 2010.

[EKM13] P. Embrechts, C. Klüppelberg, and T. Mikosch. Modelling extremal events: for insurance and finance, volume 33. Springer Science & Business Media, 2013.

[Fil91] J. A. Fill. Time to stationarity for a continuous-time Markov chain. Probability in the Engineering and Informational Sciences, 5(1):61–76, 1991.

[Gut13] A. Gut. Probability: a graduate course, volume 75. Springer Science & Business Media, 2013.

[vdH16] R. van der Hofstad. Random graphs and complex networks, Cambridge University Press, 2016.

[Jan08] S. Janson. The largest component in a subcritical random graph with a power law degree distribution. The Annals of Applied Probability, 18(4):1651–1668, 2008.

[JS11] T. Jonsson and S. Ö. Stefánsson. Condensation in nongeneric trees. Journal of Statistical Physics, 142(2):277–313, 2011.

[KM16] V. Kanade, F. Mallmann-Trenn, and T. Sauerwald. On coalescence time in graphs—when is coalescing as fast as meeting? arXiv preprint arXiv:1611.02460, 2016.

[Lig85] T. M. Liggett. Interacting particle systems. Springer, 1985.

[MBS18] A. Moinet, A. Barrat, and R. P. Satorras. Generalized voter-like model on activity driven networks with attractiveness. arXiv preprint arXiv:1804.07476, 2018.
[NR06] I. Norros and H. Reittu. On a conditionally Poissonian graph process. Advances in Applied Probability, 38(1):59–75, 2006.

[Oli12] R. Oliveira. On the coalescence time of reversible random walks. Transactions of the American Mathematical Society, 364(4):2109–2128, 2012.

[Oli13] R. I. Oliveira. Mean field conditions for coalescing random walks. The Annals of Probability, 41(5):3420–3461, 2013.

[PSSS17] Y. Peres, T. Sauerwald, P. Sousi, and A. Stauffer. Intersection and mixing times for reversible chains. Electronic Journal of Probability, 22, 2017.

[SAR08] V. Sood, T. Antal, and S. Redner. Voter models on heterogeneous networks. Physical Review E, 77(4):041121, 2008.

[SC97] L. Saloff-Coste. Lectures on finite Markov chains. In Lectures on probability theory and statistics, pages 301–413. Springer, 1997.

[WLP09] E. Wilmer, D. A. Levin, and Y. Peres. Markov chains and mixing times. American Mathematical Soc., Providence, 2009.