Is right-handed neutrino degeneracy compatible with the solar and atmospheric neutrino data?

R. González Felipe
Centro de Física das Interacções Fundamentais, Departamento de Física, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

F. R. Joaquim
Centro de Física das Interacções Fundamentais, Departamento de Física, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

Abstract: In light of the recent solar and atmospheric neutrino data, we investigate the possibility of having an exactly degenerate spectrum for heavy right-handed Majorana neutrinos at the grand unification scale. The analysis is performed in the context of the minimal supersymmetric standard model with unbroken $R$ parity and extended with three heavy Majorana neutrino fields in order to implement the seesaw mechanism. In the absence of a Dirac-type leptonic mixing, the only source of lepton flavour violation is the right-handed neutrino sector. Inspired by GUT-motivated relations among the quark, charged-lepton and Dirac neutrino Yukawa coupling matrices, and after the inclusion of the radiative effects, we determine the effective neutrino mass matrix at the electroweak scale. Using then the latest global analyses of the solar and atmospheric data at 99% C.L., we conclude that, within this framework, the only solar solutions compatible with the experimental data are the LOW and LMA solutions, being the latter the most favoured one. At 90% C.L., only the LMA solution is allowed.

Keywords: neutrino physics, solar and atmospheric neutrinos, seesaw mechanism.
1. Introduction

The recent observation of solar and atmospheric neutrino deficits constitutes a solid evidence for physics beyond the Standard Model. The existence of such anomalies gives us a strong indication of neutrino oscillations among the three known families, which in turn imply that neutrinos have nonvanishing masses and nontrivial mixings. The latest atmospheric neutrino data from the Super-Kamiokande Collaboration [1] and CHOOZ experiments [2] tends to favour the $\nu_\mu \rightarrow \nu_\tau$ oscillations with a large mixing angle $\theta_{\text{atm}} \simeq \pi/4$. On the other hand, the solar neutrino experiments [3]--[6] point towards the conversions $\nu_e \rightarrow \nu_\mu$ and $\nu_e \rightarrow \nu_\tau$, and strongly favour the large mixing angle (LMA) MSW [7] solution over the small mixing angle (SMA), the LOW and the just-so vacuum oscillation (VO) solutions, as well as over the sterile neutrino hypothesis.

From a theoretical viewpoint, modelling neutrino masses and mixings represents a challenge. First, the presently most favoured solution, namely the LMA MSW solution, is usually the most difficult one to achieve in realistic models. Second,
large mixing angles do not seem to fit naturally in our grand unification view. This is because in a very large class of string and grand unified theories (GUT’s), the quark, charged lepton and Dirac neutrino mass matrices are typically related to each other and, therefore, one would expect small neutrino mixings in analogy to what is observed in the quark sector. The above reasoning assumes of course that the large neutrino mixing arises only from the Dirac neutrino mass matrix structure. Third, even if large neutrino mixings are consistently incorporated in the general picture of fermions, a natural question then arises, namely, why neutrino masses are so tiny compared to the rest of the fermions of the theory.

A simple and economical way to overcome some of the above difficulties is to invoke the so-called seesaw mechanism [8] which, in the presence of heavy right-handed Majorana neutrinos, \( \nu_R \), leads to an effective mass matrix for the light neutrinos. Rather than an assumption, the existence of the \( \nu_R \)'s is a natural requirement in the context of all GUT’s with a group symmetry larger than \( SU(5) \). The seesaw mechanism by itself has, unfortunately, a very limited predictive power: the overall scale of the light neutrino masses is not uniquely determined, the mixing angles and the ratios of the effective neutrino masses are not fixed.

Grand unified theories such as \( SO(10) \) [9] are a suitable framework not only to analyze fermion masses but also to implement the seesaw mechanism. One of the attractive features of the \( SO(10) \) model is that its gauge group is left-right symmetric and, consequently, there exists a complete quark-lepton symmetry in the spectrum. In particular, the fact that all left-handed (right-handed) fermions of each family fit into the single irreducible spinor representation \( 16 \overline{16} \) of \( SO(10) \) and that the right-handed neutrino \( \nu_R \) is precisely contained in this representation is remarkable. As it happens in all gauge groups, after the spontaneous symmetry breaking, the fermions will acquire masses out of the Yukawa couplings and the vacuum expectation values (VEV) of the Higgs fields. Several constraints on fermion masses are usually implied in these models [9]. For instance, if there is only one \( 10 \) Higgs multiplet responsible for the masses, then we have the relation \( Y_U = Y_D = Y_\ell = Y_\nu \) (the indices \( U, D, \ell \) and \( \nu \) stand for the up quarks, down quarks, charged leptons and Dirac neutrinos, respectively). Similarly, the existence of two \( 10 \) Higgs multiplets implies \( Y_D = Y_\ell \) and \( Y_U = Y_\nu \). On the other hand, if the fermion masses are generated by a VEV of the \( 126 \) of \( SO(10) \), then the \( SU(4) \) symmetry yields the relations \( 3Y_D = -Y_\ell \) and \( 3Y_U = -Y_\nu \). Of course, equalities such as the ones arising purely from the \( 10 \)-dimensional representation cannot be exact in realistic models, since they imply undesirable relations among the quark and charged lepton masses. Additional assumptions are therefore necessary in order to predict the correct fermion spectrum \([10, 11]\).

At this point it is worth recalling that the Majorana mass matrix, \( M_R \), for the right-handed neutrinos is usually much less constrained in unified models than the Dirac mass matrix. This has led many authors to discuss possible textures for
the heavy Majorana neutrino mass matrix within the seesaw framework [12]-[17].
A common feature to most of these analyses is the fact that the heavy Majorana
spectrum turns out to be either hierarchical or partially degenerate. We may therefore
ask ourselves whether a complete degenerate spectrum is allowed for $M_R$ in this
context. Similar studies have been previously performed in the case of degenerate
left-handed Majorana neutrinos [18]-[20]. In particular, it has recently been argued
[19] that if the Majorana masses of the light neutrinos are exactly degenerate at some
high scale, then the one-loop radiative corrections are not sufficient to produce the
required neutrino mass splittings and mixings at low scale.

Although a fully degenerate mass pattern for the light neutrinos could be natural,
if the Majorana masses arise from nonrenormalizable operators, and is welcome by
cosmology, e.g. if relic neutrinos constitute the hot dark matter of the Universe, at
a fundamental level it seems more plausible to construct and motivate patterns for
the Dirac and right-handed Majorana matrices rather than for the effective neutrino
mass matrix. In fact, the effective neutrino mass matrix is only a secondary output
of the seesaw mechanism and, consequently, it may not be directly related to any
symmetry of the theory.

Of course, the observed hierarchy $\Delta m^2_{\text{atm}} = |m^2_3 - m^2_2| \gg \Delta m^2_{\odot} = |m^2_2 - m^2_1|$ among the light neutrino masses $m_i$ ($i = 1, 2, 3$) implies that one of the following
mass patterns should be realized: $|m_{1,2}| \ll |m_3|$ (hierarchical), $|m_1| \sim |m_2| \gg |m_3|$ (inversely hierarchical) or $|m_1| \sim |m_2| \sim |m_3|$ (almost degenerate). Thus, if one
believes in the seesaw as the mechanism responsible for the smallness of neutrino
masses, any realistic texture for the Dirac and Majorana matrices should lead to one
of these patterns.

In this paper we address the question whether right-handed neutrino degeneracy
is compatible or not with the present experimental solar and atmospheric neutrino
data. We assume complete degeneracy of the heavy Majorana neutrinos $\nu_R$ at GUT
scale and also that the right-handed neutrino sector provides all the source of lepton
flavour violation in the theory. Neglecting the rotation between the charged lepton
and the Dirac neutrino mass matrices is a reasonable approximation, if the Higgs
field has its components along suitable representations of $SO(10)$ such as the ones
mentioned above. This scenario is quite appealing since one starts from a situation
where there is no leptonic mixing (similar to what happens in the quark sector where
the mixing is described by the CKM matrix and is therefore small) to generate large
mixing through the seesaw mechanism.

When discussing neutrino masses and mixing angles, a crucial aspect is the run-
ing of the various couplings from the unification scale down to the electroweak scale.
In the presence of nonvanishing neutrino couplings, the renormalization group equa-
tions (RGE) are modified. From the unification scale $\Lambda_X$ to the $\nu_R$ decoupling scale
$\Lambda_R$, the radiative corrections from the Dirac and right-handed Majorana neutrinos
must be included. Below $\Lambda_R$, when the seesaw mechanism becomes operative and
the $\nu_R$’s decouple from the theory, the running of the effective neutrino Yukawa couplings then becomes relevant. Radiative corrections could indeed provide potential contributions to the effective neutrino mass matrix or be even disastrous. A typical example is given by the RGE effects due to the charged-lepton Yukawa couplings [21].

The outline of the paper is as follows. In Section 2 we briefly present the general framework used in our analysis, with some attention given to the parametrization of the degenerate right-handed Majorana neutrino mass matrix. Section 3 is devoted to the study of the RGE effects from the GUT scale down to the electroweak scale. In order to account for the right-handed decoupling threshold at $\Lambda_R$, the RGE analysis consists of two steps: first, the running of all the quark, charged-lepton as well as the Dirac and right-handed neutrino Yukawa couplings from $\Lambda_X$ to $\Lambda_R$ and, second, the evolution of all the quark and charged-lepton couplings together with the effective neutrino mass matrix from $\Lambda_R$ to the electroweak scale $M_Z$. Such an analysis is required in order to verify whether the imposed patterns for the Yukawa coupling matrices at high scales are compatible or not with the low-energy data. In Section 4, the effective neutrino mass matrix, obtained at the electroweak scale, is confronted with the experimental data to determine which of the currently allowed solar neutrino solutions can indeed be realized in the present framework. Some numerical examples for the most favoured solution, namely the LMA MSW solution, and for different hierarchical patterns of the Dirac neutrino Yukawa couplings, are given in this section. Few comments on neutrinoless double $\beta$ decay are also presented. These results allow us to motivate a simple analytical structure for the effective neutrino mass matrix in Section 5. Finally, we present our concluding remarks in Section 6.

2. Theoretical framework

We shall work in the context of the minimal supersymmetric standard model (MSSM) with unbroken $R$ parity and extended with 3 heavy right-handed neutrinos. Supersymmetry (SUSY) is a natural choice to implement the seesaw mechanism. For instance, the hierarchy problem, usually present in a nonsupersymmetric seesaw, is automatically cured in the supersymmetric case. Indeed, the existence of very heavy right-handed neutrinos coupled to the Higgs field implies logarithmically-divergent radiative corrections to the Higgs mass $\sim \Lambda_R^2 Y_\nu^2 H^2 \ln(\Lambda_R^2/\mu^2)$. While such corrections persist in nonsupersymmetric theories, they are naturally cancelled in SUSY by similar contributions coming from the right-handed sneutrinos.

The relevant Yukawa mass terms are described by the Lagrangian (generation
\[ \mathcal{L}_Y = \bar{Q} Y_U U^c H_2 + \bar{Q} Y_D D^c H_1 + \bar{L} Y_\ell E^c H_1 + \bar{L} Y_\nu \nu'^c H_2 + \frac{1}{2} \nu'^c T M_R \nu'^R + \text{h.c.} , \]

where \( Y_U, Y_D, Y_\ell \) and \( Y_\nu \) are the up-quark, down-quark, charged-lepton and Dirac neutrino Yukawa coupling matrices\(^1\), respectively; \( M_R \) is a 3×3 symmetric Majorana mass matrix which preserves the SM gauge symmetry; \( Q \) and \( L \) are the quark and charged-lepton left-handed doublets; \( U^c, D^c \) and \( E^c \) are the up-quark, down-quark and charged-lepton right-handed singlets. The additional neutrino chiral fields \( \nu_{\nu_i} \) are right-handed Majorana fields responsible for the seesaw mechanism. Finally, \( H_1 \) and \( H_2 \) are the hypercharge -1/2 and +1/2 MSSM Higgs doublets, whose neutral components acquire VEVs after the spontaneous symmetry breaking:

\[ \langle H_0^1 \rangle = v \cos \beta , \quad \langle H_0^2 \rangle = v \sin \beta , \text{ with } v \simeq 174 \text{ GeV}. \]

In what follows we shall assume that \( Y_\nu \propto Y_U \) and also that there is a left-handed alignment between \( Y_\ell \) and \( Y_\nu \) at the unification scale \( \Lambda_X \), which can be equal or higher than the decoupling scale \( \Lambda_R \), i.e. the scale at which the seesaw mechanism becomes operative and the \( \nu_R \)'s decouple from the light fields. As mentioned in the Introduction, these are reasonable assumptions in the context of grand unified theories. This means, in particular, that the only source of non-trivial leptonic mixing is the right-handed neutrino sector. In this scenario large neutrino mixings will be generated through the seesaw mechanism.

Next we assume that the right-handed Majorana neutrinos are degenerate at \( \Lambda_X \). In this case a simple and general parametrization \([22]\) can be given for the right-handed Majorana neutrino mass matrix \( M_R \). Indeed, since \( M_R \) is symmetric we can always write

\[ W^T M_R W = \Lambda_R \mathbb{1} , \quad Y_R \equiv \Lambda_R^{-1} M_R = W^* W^\dagger , \]

where \( W \) is a unitary matrix and \( Y_R \) is a unitary symmetric matrix. It is then straightforward to show that the matrix \( Y_R \) can be parametrized as

\[ Y_R = R_\theta R_\phi R_\theta^T , \]

where the rotation matrices \( R_\theta \) and \( R_\phi \) are given by

\[ R_\theta = \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad R_\phi = \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & c_{2\phi} & -s_{2\phi} \\ 0 & s_{2\phi} & c_{2\phi} \end{pmatrix} . \]

Here we have introduced, for simplicity, the notations \( c_\varphi \equiv \cos \varphi , \quad s_\varphi \equiv \sin \varphi \).

\(^1\) We assume all the Yukawa coupling matrices real.
Let us note that the parametrization (2.3) does not include the trivial case when $CP$ is conserved and all the right-handed neutrinos have the same $CP$ parity [22]. The $CP$-violating phase $\alpha$ turns out to be irrelevant in our further analysis and thus we shall set $\alpha = 0$. In the latter case the complete form of $Y_R$ is given by

$$
Y_R = \begin{pmatrix}
  c_\theta^2 + c_2\phi s_\theta^2 & -s_2\theta s_\phi & -s_\theta s_2\phi \\
  -s_2\theta s_\phi & s_\theta^2 + c_2\phi c_\theta^2 & -c_\theta s_2\phi \\
  -s_\theta s_2\phi & -c_\theta s_2\phi & -c_2\phi
\end{pmatrix} .
$$

(2.5)

Moreover,

$$
M_R^{-1} = \Lambda_R^{-1} [R_\theta R_\phi R_\phi^T]^{-1} = \Lambda_R^{-1} R_\theta R_\phi R_\phi^T = \Lambda_R^{-1} Y_R .
$$

(2.6)

Notice that the parametrization (2.5) highly constrains the form of $M_R$, which only depends on two angles $\theta, \phi$ and the decoupling scale $\Lambda_R$.

3. The effects of radiative corrections

In this section we shall consider the evolution of the couplings from the unification scale $\Lambda_X$ to the electroweak scale $M_Z$. The relevant one-loop RGE, which include neutrino threshold effects, are summarized in the Appendix. Let us note that it is customary to take the SUSY breaking scale as $\Lambda_{SUSY} \simeq 1 \text{ TeV}$ or $\Lambda_{SUSY} \simeq m_t$. Since in general the renormalization group effects that could possibly arise between $\Lambda_{SUSY}$ and $M_Z$ are negligible, we will not account for such effects in our analysis. Thus we set $\Lambda_{SUSY} = M_Z$.

At the unification scale $\Lambda_X$ the Yukawa matrices $Y_\ell$ and $Y_\nu$ can be diagonalized by real and orthogonal matrices $U_{L,R}$ and $V_{L,R}$ such that

$$
U_L^\dagger Y_\ell U_R = \text{diag} \left( y_e(\Lambda_X) , y_\mu(\Lambda_X) , y_\tau(\Lambda_X) \right) ,
$$

$$
V_L^\dagger Y_\nu V_R = \text{diag} \left( y_1(\Lambda_X) , y_2(\Lambda_X) , y_3(\Lambda_X) \right) ,
$$

(3.1)

where $y_i(\Lambda_X)$ denotes the respective Yukawa couplings at $\Lambda_X$. The hypothesis of left-handed alignment between $Y_\nu$ and $Y_\ell$ implies $U_L = V_L$. In this case, $Y_\nu$ and $Y_\ell$ can be simultaneously diagonalized, leaving the charged-current Lagrangian invariant. This means that there is no “Dirac-type” leptonic mixing at $\Lambda_X$. Obviously, this diagonalization process does not affect the spectrum of the right-handed Majorana matrix and, therefore, the resulting matrix $M_R' = V_L^T M_R V_L$ can also be parametrized in the form of Eq. (2.5) with a different set of the parameters $\theta$ and $\phi$.

3.1 Running $Y_\nu$ from $\Lambda_X$ to $\Lambda_R$

Our first step in discussing the effects of radiative corrections on neutrino masses and mixing angles will be the study of the evolution of the Dirac Yukawa coupling matrix
\( Y_\nu \) from GUT scale to the right-handed neutrino decoupling scale. After performing the diagonalization given in Eq. (3.1) we have

\[
Y_\nu (\Lambda_X) = y_3(\Lambda_X) \begin{pmatrix}
\epsilon_1 & 0 & 0 \\
0 & \epsilon_2 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(3.2)

where the parameters

\[
\epsilon_1 = \frac{y_1(\Lambda_X)}{y_3(\Lambda_X)}, \quad \epsilon_2 = \frac{y_2(\Lambda_X)}{y_3(\Lambda_X)},
\]

(3.3)

have been introduced.

The running of \( Y_\nu \) from \( \Lambda_X \) to \( \Lambda_R \) is governed by Eq. (A.4). From this equation and neglecting all but the third-generation Yukawa couplings\(^2\) we get

\[
Y_\nu(\Lambda_R) \simeq I_1 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_\nu
\end{pmatrix} Y_\nu(\Lambda_X),
\]

(3.4)

where

\[
I_1 = \exp \left[ \frac{1}{16 \pi^2} \int_{t_X}^{t_R} (T_\nu - G_\nu) \, dt \right] \simeq \exp \left[ \frac{1}{16 \pi^2} \int_{t_X}^{t_R} \left( 3y_1^2 + y_3^2 - \frac{3}{5}g_1^2 - 3g_2^2 \right) \, dt \right],
\]

(3.5)

\[
I_\nu \simeq \exp \left[ \frac{1}{16 \pi^2} \int_{t_X}^{t_R} \left( 3y_3^2 + y_\tau^2 \right) \, dt \right],
\]

(3.6)

with \( t_X \equiv \ln(\Lambda_X/M_Z) \) and \( t_R \equiv \ln(\Lambda_R/M_Z) \).

From Eqs. (3.2) and (3.4) we find

\[
Y_\nu(\Lambda_R) = K_D \begin{pmatrix}
\epsilon_1' & 0 & 0 \\
0 & \epsilon_2' & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad K_D = y_3(\Lambda_X) I_1 I_\nu, \quad \epsilon_{1,2}' = I_\nu^{-1} \epsilon_{1,2}.
\]

(3.7)

### 3.2 Running \( M_R \) from \( \Lambda_X \) to \( \Lambda_R \)

The running of the right-handed Majorana mass matrix \( M_R \) from \( \Lambda_X \) to \( \Lambda_R \) is described by Eq. (A.10) given in the Appendix. This equation allows us to relate the right-handed neutrino mass matrices \( M_R(\Lambda_R) \) and \( M_R(\Lambda_X) \):

\[
[M_R(\Lambda_R)]_{ij} = [M_R(\Lambda_X)]_{ij} \exp \left[ \frac{1}{8 \pi^2} \int_{t_X}^{t_R} (y_i^2 + y_j^2) \, dt \right].
\]

(3.8)

\(^2\)This is a reasonable assumption considering that we will focus on cases where \( y_1 \ll y_2 \ll y_3 \).
In the limit where only the third Dirac neutrino Yukawa coupling $y_3$ is kept, we find

$$M_R(\Lambda_R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_R \end{pmatrix} M_R(\Lambda_X) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_R \end{pmatrix},$$

(3.9)

where

$$I_R \simeq \exp \left[ \frac{1}{8 \pi^2} \int_{t_X}^{t_R} y_3^2 \, dt \right].$$

(3.10)

The inverse of $M_R$ at $\Lambda_R$ is then given by

$$M_R^{-1}(\Lambda_R) = \Lambda_R^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_R^{-1} \end{pmatrix} Y_R \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_R^{-1} \end{pmatrix},$$

(3.11)

where $Y_R$ has the form given in Eq. (2.5).

### 3.3 $M_{\nu}^{\text{eff}}$ from high to low scales

In order to study the low-energy consequences of our assumptions, imposed at high scales, we must find the effective neutrino mass matrix $M_{\nu}^{\text{eff}}$ at the typical neutrino experiment scale, i.e. the electroweak scale ($\simeq M_Z$). If $R$ parity is an exact symmetry in the MSSM, the lowest (5-dimensional) operator in the superpotential which violates lepton number and generates a Majorana mass for the left-handed neutrinos, $\nu$, is given by [23, 24]

$$W_{\nu\nu} = \frac{1}{4} \kappa H_2^0 H_2^0 + \text{h.c.},$$

(3.12)

where $\kappa$ is a $3 \times 3$ symmetric matrix. The effective neutrino mass matrix is obviously given by

$$M_{\nu}^{\text{eff}} = \kappa (H_2^0)^2 = \kappa v^2 \sin^2 \beta.$$  

(3.13)

The supersymmetric seesaw mechanism is realized by adding to the superpotential the following terms:

$$W = W_{\text{MSSM}} - \frac{1}{2} \nu_R^T M_R \nu_R + \nu_R^T Y_\nu L H_2.$$  

(3.14)

Below $\Lambda_R$ one can consider the effective superpotential given by

$$W_{\text{eff}} = W_{\text{MSSM}} + \frac{1}{2} (Y_\nu L H_2)^T M_R^{-1} (Y_\nu L H_2),$$

(3.15)
which is obtained by integrating out the heavy right-handed neutrino fields $\nu_R$ in Eq. (3.14). The second term in $W_{\text{eff}}$ will originate in the Lagrangian a Majorana mass term of the type

$$\mathcal{L}_{\nu\nu} = -\frac{1}{2} \nu^T \kappa \nu H_2^0 H_2^0 + \text{h.c.},$$

where

$$\kappa = Y_\nu^T M_R^{-1} Y_\nu.$$  \hspace{1cm} (3.17)

Let us note that at this stage the matrices $\kappa$ and $M_{\nu}^{\text{eff}}$ are defined at the decoupling scale $\Lambda_R$. To compare them with the experimental ones, obtained at the electroweak scale, we must run these matrices down to $M_Z$ through the corresponding RGE. Taking into account Eqs. (3.7) and (3.11) we have

$$\kappa(\Lambda_R) = \Lambda_R^{-1} Y_\nu^T(\Lambda_R) Y_R Y_\nu(\Lambda_R) = \frac{K^2}{\Lambda_R} \begin{pmatrix} \epsilon'_1 & 0 & 0 \\ 0 & \epsilon'_2 & 0 \\ 0 & 0 & I^{-1}_R \end{pmatrix} \begin{pmatrix} \epsilon'_1 & 0 & 0 \\ 0 & \epsilon'_2 & 0 \\ 0 & 0 & I^{-1}_R \end{pmatrix}. \hspace{1cm} (3.18)$$

The running of the matrix $\kappa$ from $\Lambda_R$ to $M_Z$ is described by the RGE given in Eq. (A.14). From the latter equation, and neglecting the charged-lepton Yukawa couplings $y_\mu$ and $y_e$, we get the following relation between $\kappa(M_Z)$ and $\kappa(\Lambda_R)$:

$$\kappa(M_Z) = I_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_\tau \end{pmatrix} \kappa(\Lambda_R) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_\tau \end{pmatrix}, \hspace{1cm} (3.19)$$

where

$$I_2 = \exp \left[ \frac{1}{16 \pi^2} \int_{t_R}^{t_Z} (T_\kappa - G_\kappa) \, dt \right] \simeq \exp \left[ \frac{1}{16 \pi^2} \int_{t_R}^{t_Z} \left( 6 \delta_1^2 - \frac{6}{5} g_1^2 - 6 g_2^2 \right) \, dt \right], \hspace{1cm} (3.20)$$

$$I_\tau = \exp \left[ \frac{1}{16 \pi^2} \int_{t_R}^{t_Z} y_\tau^2 \, dt \right]. \hspace{1cm} (3.21)$$

Using Eq. (3.18) we finally obtain

$$\kappa(M_Z) = \frac{K_{\text{eff}}}{\Lambda_R} \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \hspace{1cm} (3.22)$$

with

$$\delta_{1,2} = I_{\text{eff}} I^{-1}_\tau \epsilon_{1,2}, \hspace{1cm} I_{\text{eff}} = I_R I^{-1}_\nu,$$  \hspace{1cm} (3.23)

$$K_{\text{eff}} = y_3^2(\Lambda_X) I_1^2 I_2 I_\tau I^{-2}_{\text{eff}}, \hspace{1cm} (3.24)$$
and \( I_1, I_2 \) given in Eqs. (3.5) and (3.20), respectively.

From Eqs. (2.5) and (3.22) we obtain our final expression for the neutrino effective operator at the electroweak scale:

\[
\kappa(M_Z) = \frac{K_{\text{eff}}}{\Lambda_R} \begin{pmatrix}
\delta_1^2 (c_\theta^2 + c_{2\phi}s_\theta^2) & -\delta_1 \delta_2 s_\theta s_{2\phi} & -\delta_1 s_\theta s_{2\phi} \\
-\delta_1 \delta_2 s_\theta s_{2\phi} & \delta_2^2 (s_\theta^2 + c_{2\phi}c_\theta^2) & -\delta_2 c_\theta s_{2\phi} \\
-\delta_1 s_\theta s_{2\phi} & -\delta_2 c_\theta s_{2\phi} & -c_{2\phi}
\end{pmatrix}.
\]  

(3.25)

Let us remark that the mass spectrum and mixings arising from this matrix are invariant under the transformations \((\theta \rightarrow \theta, \phi \rightarrow \pi - \phi)\) and \((\theta \rightarrow \pi - \theta, \phi \rightarrow \phi)\), and, obviously, under any transformation resulting from their successive application. This simply corresponds to a phase redefinition of the left-handed neutrino fields. Thus, from now on we restrict ourselves to the intervals \(0 \leq \theta, \phi \leq \pi/2\).

In the limit where the evolution of the third Dirac neutrino Yukawa coupling \(y_3\) and the \(\tau\) Yukawa coupling \(y_\tau\) can be neglected, we find the approximate expressions:

\[
I_\nu \simeq \left(\frac{\Lambda_R}{\Lambda_X}\right)^{\frac{y_3^2 + y_\tau^2}{16\pi^2}} , \quad I_R \simeq \left(\frac{\Lambda_R}{\Lambda_X}\right)^{\frac{\nu_3^2}{8\pi^2}} , \quad I_\tau \simeq \left(\frac{M_Z}{\Lambda_R}\right)^{\frac{y_\tau^2}{16\pi^2}},
\]

(3.26)

and, consequently,

\[
\delta_{1,2} \simeq \left(\frac{\Lambda_X}{\Lambda_R}\right)^{\frac{y_3^2 + y_\tau^2}{16\pi^2}} \left(\frac{\Lambda_R}{M_Z}\right)^{\frac{\nu_3^2}{8\pi^2}} \epsilon_{1,2} ,
\]

(3.27)

\[
K_{\text{eff}} \simeq y_3^2 (\Lambda_X) I_1^2 I_2 \left(\frac{\Lambda_R}{\Lambda_X}\right)^{\frac{y_3^2 + y_\tau^2}{8\pi^2}} \left(\frac{M_Z}{\Lambda_R}\right)^{\frac{y_\tau^2}{8\pi^2}}.
\]

(3.28)

At this point few remarks are in order. First, as it has already been noticed by several authors [21], the parameter \(I_\tau \simeq 1\). This can be easily checked by using either the exact expression (3.21) or the approximate expression given in Eq. (3.26). Moreover, the parameter \(I_{\text{eff}}\), which controls the RGE effects of the Dirac neutrino Yukawa coupling ratios (cf. Eq. (3.23)), is also close to the unity. This can be seen from Fig. 1 where we present the dependence of \(I_{\text{eff}}\) on the value of the scale \(\mu\) for the two initial conditions \(Y_\nu = Y_U\) and \(Y_\nu = 3Y_U\). We note that \(I_{\text{eff}} \simeq 1\) for a wide range of \(\tan \beta\). Notice also that for the case \(Y_\nu = 3Y_U\), the use of the approximate expression for \(I_{\text{eff}}\), obtained by neglecting the running of the Yukawa couplings, is not well justified. This is because in the latter case \(y_3(\Lambda_X) = 3y_\tau(\Lambda_X)\) and the evolution of the couplings becomes more relevant.

4. Confronting the neutrino mass matrix with the experimental data

The analysis of the neutrino oscillation data indicates that, contrary to what happens in the quark sector, where the mixing is always small, the mixing angle is large
in the 2-3 neutrino sector. The latest atmospheric neutrino data from the Super-Kamiokande Collaboration [1] and CHOOZ experiments [2] implies a large mixing angle $\theta_{\text{atm}} \simeq \pi/4$. In what concerns the 1-2 neutrino sector, there are four different scenarios depending on the solar neutrino problem solution.

Since the present experimental data constrains the neutrino mass squared differences and the mixing angles, it is necessary to diagonalize the effective neutrino mass matrix $M_{\nu}^{\text{eff}}$ at the electroweak scale. The neutrino mixing matrix $U$, which relates the flavour and mass eigenstates,

$$
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = U 
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
$$

(4.1)
can be parametrized in the standard form [25]

$$
U = \begin{pmatrix}
c_{13}c_{12} & s_{12}c_{13} & s_{13} \\
-s_{12}c_{23} - s_{23}s_{13}c_{12} & c_{23}c_{12} - s_{23}s_{13}s_{12} & s_{23}c_{13} \\
s_{23}s_{12} - s_{13}c_{23}c_{12} & -s_{23}c_{12} - s_{13}s_{12}c_{23} & c_{23}s_{13}
\end{pmatrix},
$$

(4.2)
such that $U^T M_{\nu}^{\text{eff}}(M_Z) U = \text{diag}(m_1, m_2, m_3)$, where $m_i$ are the light neutrino masses; $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$.

We also remark that the lepton mixing matrix can have, besides a Dirac-type phase (analogous to that of the quark sector), two other physical phases associated

\[ \text{Figure 1:} \text{ The evolution of the RGE coefficient } I_{\text{eff}} = I_R I^{-1} \text{ from } \Lambda_X \text{ to } \Lambda_R, \text{ assuming two different Dirac neutrino textures, } Y_D = Y_U \text{ and } Y_D = 3 Y_U, \text{ and for the values of } \tan \beta = 10 \text{ and } \tan \beta = 55. \text{ The solid curves correspond to the exact expressions as defined by Eqs. (3.6) and (3.10). The dashed lines correspond to the approximate curves as given in Eqs. (3.26).} \]
Atmospheric and reactor neutrinos

\[ \Delta m^2_\text{atm} (\text{eV}^2) \quad \tan^2 \theta_\text{atm} \quad |U_{e3}| \]

|                  | \( (1.1 - 7.3) \times 10^{-3} \) | 0.3 - 3.8 | < 0.27 |
|------------------|----------------------------------|-----------|--------|
|                  | \([1.4 - 6.1] \times 10^{-3}\)  | [0.4 - 3.1] | [ < 0.23] |
|                  | \(3.1 \times 10^{-3}\)          | 1.4       | 0.07   |

Solar neutrinos

|                  | \( \Delta m^2_\odot (\text{eV}^2) \) | \( \tan^2 \theta_\odot \) | \( r_\odot = \Delta m^2_\odot / \Delta m^2_\text{atm} \) |
|------------------|-------------------------------------|--------------------------|------------------------------------------------|
| LMA              | \((2.0 - 20) \times 10^{-5}\)      | 0.1 - 0.7                | \((0.03 - 1.8) \times 10^{-1}\)                  |
|                  | \([2.3 - 10] \times 10^{-5}\)     | [0.2 - 0.5]             | \([0.04 - 0.7] \times 10^{-1}\)               |
|                  | \(4.2 \times 10^{-5}\)            | **0.26**                | **0.014**                                      |
| SMA              | \((0.4 - 1) \times 10^{-5}\)      | \((0.2 - 2) \times 10^{-3}\) | \((0.55 - 9.1) \times 10^{-3}\)               |
| LOW              | \((0.5 - 2) \times 10^{-7}\)     | 0.5 - 0.9               | \((0.07 - 1.8) \times 10^{-4}\)               |
| Just-So\(^2\)   | \((5 - 8) \times 10^{-12}\)      | 0.5 - 2                 | \((0.68 - 7.3) \times 10^{-9}\)               |

Table 1: Constraints on neutrino masses and mixing angles at 99% C.L. [90% C.L.] coming from global analyses of the solar, atmospheric and reactor neutrino data [27, 28]. The numbers in bold correspond to the best-fit values.

with the Majorana character of neutrinos. If \( CP \) is conserved these phases are either 0 or \( \pi \). In the analysis that follows we shall assume all these phases vanishing. However, as we shall see in Section 4.2, such phases play a significant role in processes such as the neutrinoless double \( \beta \) decay [26].

4.1 Neutrino oscillation data

The relevant parameter set for the study of solar and atmospheric data is usually identified with the quantities

\[
\Delta m^2_\odot \equiv \Delta m^2_{12} = m^2_2 - m^2_1, \quad \Delta m^2_\text{atm} \equiv \Delta m^2_{23} = m^2_3 - m^2_2, \\
\tan^2 \theta_\odot \equiv \tan^2 \theta_{12}, \quad \tan^2 \theta_\text{atm} \equiv \tan^2 \theta_{23}, \quad U_{e3} \equiv \sin \theta_{13},
\]

where all mixing angles are assumed to lie in the range \([0, \pi/2]\). In Table 1 we summarize the present constraints on neutrino mass squared differences and on the mixing angles obtained from the recent global analyses [27, 28] of the solar, atmospheric and reactor neutrino data at 99% C.L. [90% C.L.].

Apart from the overall factor \( K_{\text{eff}}/\Lambda_R \), the effective neutrino operator \( \kappa(M_Z) \) defined in Eq.(3.25) depends on the angles \( \theta \) and \( \phi \), which parametrize the heavy
right-handed neutrinos, and the parameters $\delta_1, \delta_2$. For the structure of $\kappa$ to be compatible with the low-energy data, we should take into account not only all the constraints on $r_\odot = \Delta m^2_\odot / \Delta m^2_{\text{atm}}$ and the mixing angles, but also the experimentally allowed region for the up-quark mass ratios, since within our assumptions, these ratios fix the parameters $\delta_1$ and $\delta_2$ at $\Lambda_X$:

$$\delta_1 \simeq \epsilon_1 = \frac{y_u(\Lambda_X)}{y_t(\Lambda_X)}, \quad \delta_2 \simeq \epsilon_2 = \frac{y_e(\Lambda_X)}{y_t(\Lambda_X)}.$$ (4.4)

The above ratios can be determined by running the RGE from the electroweak scale $M_Z$ to the unification scale $\Lambda_X$, which we assume to be $\Lambda_X \simeq 2 \times 10^{16}$. Since these ratios are quite insensitive to the value of $\tan \beta$ in a wide region, we shall take $\tan \beta = 10$ as the input value in our analysis. Using the experimentally allowed values for the quark masses at the electroweak scale [29, 25], the following ranges are then obtained at $\Lambda_X$:

$$2.8 \times 10^{-6} \lesssim \delta_1 \lesssim 1.5 \times 10^{-5},$$
$$9 \times 10^{-4} \lesssim \delta_2 \lesssim 4 \times 10^{-3}. \quad (4.5)$$

In our numerical study we randomly varied all the parameters in the ranges: $0 \leq \delta_{1,2} \leq 1$, $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq \pi/2$. In Fig. 2 we plot the allowed region in the plane $(\delta_1, \delta_2)$ for the three different MSW solar solutions (LMA, SMA, LOW) at 99% C.L.. We assume that the right-handed Majorana neutrinos are exactly degenerate at the unification scale $\Lambda_X$. The dashed area delimited by the dotted lines corresponds to the allowed region for the Dirac neutrino coupling ratios, as defined by Eqs. (4.5) and assuming $Y_\nu \propto Y_U$ at $\Lambda_X$. The intersection of the filled areas with the dashed area provides us with the points $(\delta_1, \delta_2)$ compatible with the low-energy data and the initial conditions imposed at $\Lambda_X$. From the plots it is clear that the SMA solution is excluded. The LOW solution is barely compatible with the data at 99% C.L., and certainly excluded at 90% C.L.. Finally, the LMA solution is the most favoured one, and at 90% C.L. it is the only one that survives.

The results in the parameter space $(\phi, \theta)$ are shown in Fig. 2d at 99% C.L.. We notice that the allowed region for LMA is more extensive than the SMA and LOW regions, and the SMA region is in turn wider than the LOW one. This is a direct consequence of the hierarchy $r_\odot^{\text{LMA}} > r_\odot^{\text{SMA}} > r_\odot^{\text{LOW}}$. For the VO (Just-So$^2$) solar solutions an extreme fine-tuning is required between $\theta$ and $\phi$ in order to get a solution compatible with the solar and atmospheric neutrino data. The latter solutions are therefore not presented in our plots.

Since the LMA solution is the most favoured in the present framework, the rest of our discussion will be devoted to it. In Figs. 3 we show the allowed region at 99% C.L. for the LMA MSW solution, in different planes of the input parameters $\delta_1, \delta_2, \phi$ and $\theta$. A remarkable feature is the fact that for small values of $\delta_1$ and $\delta_2$, which are
required in our framework, one has $\phi \simeq \pi/4$ and $\theta \simeq \pi/2$. We remark that the same conclusion for the value of $\phi$ can be drawn by looking at the structure of the effective neutrino mass matrix (3.25), since for small values of $\delta_{1,2}$, the relation $\cos 2\phi \ll 1$ is required to avoid a quasi-diagonal pattern in the mass matrix.

Let us conclude this section by giving few numerical examples (see Table 2). We consider two different GUT-motivated relations for the Dirac neutrino Yukawa coupling matrix: $Y_\nu = Y_U$ and $Y_\nu = 3Y_U$. In both cases, consistency with the present LMA solar, atmospheric and reactor neutrino data was achieved. It was not possible however to obtain the best-fit values, simultaneously for all the parameters. By slightly modifying the initial assumption $y_1/y_3 = y_u/y_t$, we were then able to reproduce the best-fit values for all the solar and atmospheric neutrino data, as shown in the last column of the table. Notice also that the obtained value $|U_{e3}| \simeq 0.05$ is close to the best-fit value $|U_{e3}| = 0.07$ given in Table 1.

As suggested by our full numerical analysis (see Figs. 3), the angles $\theta$ and $\phi$ were chosen to be close to $\pi/2$ and $\pi/4$, respectively. The degeneracy imposed at $\Lambda_X$ for the right-handed Majorana neutrinos, $|M_1| = |M_2| = |M_3|$, is lifted in all cases by radiative corrections. At the decoupling scale $\Lambda_R$ we have $|M_1| \simeq |M_2| \simeq I_R \Lambda_R < |M_3| \simeq \Lambda_R$. For the effective neutrinos with masses $m_i$, we obtain at the electroweak scale a hierarchical spectrum with

$$|m_1| < |m_2| \simeq \sqrt{\Delta m^2_{12}} < |m_3| \simeq \sqrt{\Delta m^2_{23}}.$$  \hspace{1cm} (4.6)

Finally, we also notice the stability of the atmospheric and solar neutrino parameters, $r_\nu = \Delta m^2_{12}/\Delta m^2_{23}$, $\tan^2 \theta_{12}$, $\tan^2 \theta_{23}$ and $U_{e3}$ with the energy scale. In fact, from $\Lambda_R$ to $M_Z$ no significant running of these parameters is observed. This is related with the fact that $I_r \simeq 1$.

### 4.2 Neutrinoless double $\beta$ decay

The Majorana nature of massive neutrinos can be traced by looking for processes where the total lepton charge $L$ changes by two units, i.e. $\Delta L = 2$. The information coming from these type of events can be crucial not only to establish the nature of neutrinos but also to reveal some aspects about $CP$ violation in the leptonic sector, to which neutrino oscillations are insensitive. In particular, if neutrinos have a Dirac signature, then the two extra Majorana phases which appear in the neutrino mixing matrix will be absent. Moreover, even if $CP$ invariance holds, some information about the relative $CP$ parities and neutrino mass spectrum can be extracted. One example of such processes is the neutrinoless double $\beta$ decay ($\beta\beta_{0\nu}$-decay) of the type

$$(A, Z) \rightarrow (A, Z + 2) + 2\, e^-,$$ \hspace{1cm} (4.7)
Figure 2: Allowed region in the plane $(\delta_1, \delta_2)$ for the three different MSW solar solutions (LMA, SMA, LOW) at 99% C.L., if we assume $Y_\nu \propto Y_U$ and an exactly degenerate spectrum for the right-handed Majorana neutrinos at the unification scale $\Lambda_X$. The dashed area corresponds to the allowed region for the up quark mass ratios at $\Lambda_X$. At 90% C.L., the only solution that is compatible with the experimental data is the LMA one. The corresponding allowed region in the plane $(\phi, \theta)$ is shown in Fig. 2d at 99% C.L.

which can occur through the exchange of massive Majorana neutrinos coupled to electrons. The decay rate is proportional to the so-called “effective Majorana mass parameter”

$$|<m>| = |m_1 U_{e1}^2 + m_2 U_{e2}^2 + m_3 U_{e3}^2|.$$  \hspace{1cm} (4.8)

Although no experimental evidence has been found so far for this kind of processes,
an upper bound has been obtained by the $^{76}$Ge Heidelberg-Moscow experiment [30]:

$$|<m>| < 0.35 \text{ eV \ [90\% C.L.]}.$$  

(4.9)

If $CP$ is conserved, then Eq. (4.8) can be written as

$$|<m>| = \left| \eta_1^{CP} m_1 U_{e1}^2 + \eta_2^{CP} m_2 U_{e2}^2 + \eta_3^{CP} m_3 U_{e3}^2 \right|, \quad m_i > 0,$$

(4.10)

where $\eta_i^{CP} = \pm 1$ are the neutrino $CP$ parities. By examining the invariants of the effective neutrino mass matrix given in Eq. (3.25), one can conclude that one of the neutrinos should have a negative $CP$ parity and the other two should have positive $CP$ parity. This is verified for the numerical examples considered in Table 2, where the neutrino with mass $m_2$ has $\eta_2^{CP} = -1$, while the other two have $\eta_1^{CP}, \eta_3^{CP} = +1$. In this case, Eq. (4.10) reads

$$|<m>| = \left| m_1 U_{e1}^2 - m_2 U_{e2}^2 + m_3 U_{e3}^2 \right|.$$

(4.11)

A particular feature in the above expression is the possibility of a cancellation among the different terms [31, 26], such that $|<m>| \approx 0$. In fact, for all the numerical cases considered in Table 2, such a cancellation indeed occurs. This can be easily
understood if we recall that the effective Majorana mass parameter is nothing but the absolute value of $(M^\text{eff}_\nu)_{11}$. Therefore, from Eq. (3.25) we have

$$<m> \propto |\delta^2 (c_\theta^2 + c_{2\phi} s_\theta^2)|.$$  \hfill (4.12)

Since $\theta \approx \pi/2$ and $\phi \approx \pi/4$, in order to account for the low-energy neutrino data, and $\delta_1$ is small, it is now clear from Eq. (4.12) why $<m>$ is highly suppressed in

---

| $Y_\nu = Y_U$ | $Y_\nu = 3Y_U$ | Best fit |
|--------------|--------------|---------|
| $y_u$ | $5.0 \times 10^{-6}$ | $5.9 \times 10^{-6}$ | $5.6 \times 10^{-6}$ |
| $y_c$ | $1.7 \times 10^{-3}$ | $2.1 \times 10^{-3}$ | $1.7 \times 10^{-3}$ |
| $y_t$ | 0.68 | 0.80 | 0.64 |
| $y_1$ | $5.0 \times 10^{-6}$ | $1.8 \times 10^{-5}$ | $6.4 \times 10^{-7}$ |
| $y_2$ | $1.7 \times 10^{-3}$ | $6.3 \times 10^{-3}$ | $1.7 \times 10^{-3}$ |
| $y_3$ | 0.68 | 2.4 | 0.64 |
| $\pi/2 - \theta$ | $7.0 \times 10^{-3}$ | $7.0 \times 10^{-3}$ | $2.6 \times 10^{-3}$ |
| $\phi - \pi/4$ | $1.0 \times 10^{-5}$ | $1.0 \times 10^{-5}$ | $2.3 \times 10^{-6}$ |

**Table 2:** Numerical results for $\tan \beta = 10$. We assume right-handed neutrino degeneracy at $\Lambda_X$, which is then lifted by radiative corrections. Two different Ansätze were considered for the Dirac neutrino Yukawa coupling matrix, $Y_\nu = Y_U$ and $Y_\nu = 3Y_U$. By slightly modifying the initial condition $y_1/y_3 = y_u/y_t$ it is possible to reproduce the best-fit values for all the solar and atmospheric neutrino data, as shown in the last column of the table. The negative $CP$ parities are indicated by ($\sim$).
our framework.

5. A simple analytical form for $M_{\nu}^{\text{eff}}$

In this section we shall present a simple analytical example which can reproduce the results obtained in the last section. We have seen that, in the context of complete degeneracy of right-handed neutrinos and imposing proportionality between $Y_{\nu}$ and $Y_{U}$, the only possible solution at low energies is the LMA. Moreover, as it is clear from Figs. 3 this would require $\theta \simeq \pi/2$ and $\phi \simeq \pi/4$. On the other hand, if one assumes in Eq. (3.25), $\phi = \pi/4$ and $\theta = \pi/2$, then

$$\kappa(M_Z) = \frac{K_{\text{eff}}}{\Lambda_R} \begin{pmatrix} 0 & 0 & -\delta_1 \\ 0 & \delta_2^2 & 0 \\ -\delta_1 & 0 & 0 \end{pmatrix}. \quad (5.1)$$

In this case the neutrino masses are

$$\{m_1, m_2, m_3\} = \frac{K_{\text{eff}}}{\Lambda_R} \sin^2 \beta \{\delta_1, \delta_1, \delta_2^2\}. \quad (5.2)$$

Taking into account Eqs. (5.1) and (5.2), it is straightforward to show that, independently of the value of the parameters $\delta_1$ and $\delta_2$, one always has $\sin^2(2\theta_{23}) = 0$, which is not phenomenologically viable.

At first sight the above result may seem contradictory, since on one hand our numerical study clearly shows that values of $\theta \simeq \pi/2$ and $\phi \simeq \pi/4$ are indeed compatible with the low-energy neutrino data, but on the other hand, a simple analytical check gives a completely opposite answer. As we shall see shortly, the reason for this mismatching lies in the fact that small deviations, such as $\delta \theta = \pi/2 - \theta \ll 1$ and $\delta \phi = \phi - \pi/4 \ll 1$, and/or small perturbations from our original assumptions, turn out to be crucial. To show that this is indeed the case, let us relax the condition of proportionality between $Y_{\nu}$ and $Y_{U}$ for the smallest Dirac neutrino Yukawa coupling by putting

$$\epsilon_1 = a \epsilon^2, \quad \epsilon_2 = \epsilon = \frac{y_e(\Lambda_X)}{y_t(\Lambda_X)}, \quad (5.3)$$

and write $\theta$ and $\phi$ as

$$\phi = \frac{\pi}{4} + b \epsilon^2, \quad \theta = \frac{\pi}{2} - \epsilon, \quad (5.4)$$

where $a$ and $b$ are some positive real constants. From Eq. (3.23) we have then

$$\delta_1 = a I_{\text{eff}} \Lambda \epsilon \simeq a \epsilon^2, \quad \delta_2 = I_{\text{eff}} I_{\tau}^{-1} \epsilon \simeq \epsilon. \quad (5.5)$$
The effective neutrino mass operator (3.25) will be approximately given in this case by
\[ \kappa(M_Z) \simeq \frac{K_{\text{eff}}}{\Lambda_R} e^2 \left( \begin{array}{ccc} a^2 e^4 (1 - 2b) & -a e^2 & -a \\ -a e^2 & 1 - 2b e^4 & -1 \\ -a & -1 & 2b \end{array} \right). \] (5.6)

Due to the smallness of the parameter \( \epsilon \) we have, to a very good approximation,
\[ \kappa(M_Z) \simeq \frac{K_{\text{eff}}}{\Lambda_R} e^2 \kappa' , \quad \kappa' = \left( \begin{array}{ccc} 0 & 0 & -a \\ 0 & 1 & -1 \\ -a & -1 & 2b \end{array} \right) . \] (5.7)

The invariants of the matrix \( \kappa' \) are
\[ T = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2b , \]
\[ \chi = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = -1 - a^2 - 2b , \]
\[ \Delta = \lambda_1 \lambda_2 \lambda_3 = -a^2 , \] (5.8)
where \( \lambda_i \) are the eigenvalues,
\[ \lambda_1 = 2 \sqrt{p} \cos \left( \frac{\alpha}{3} \right) + T , \]
\[ \lambda_{2,3} = -2 \sqrt{p} \cos \left( \frac{\alpha \pm \pi}{3} \right) + T , \] (5.9)
with
\[ q = -\frac{2T^3}{27} + \frac{T\chi}{3} - \Delta , \quad p = \frac{T^2}{9} - \frac{\chi}{3} , \quad \cos \alpha = -\frac{q}{p^{3/2}} . \] (5.10)

The corresponding eigenvectors are given by
\[ \mathbf{v}_i = \frac{1}{n_i} \left[ a \left( 1 - \lambda_i \right) , -\lambda_i , \lambda_i \left( \lambda_i - 1 \right) \right]^T , \] (5.11)
where \( n_i \) are the normalization coefficients.

In Fig. 4 we present the region in the \((a, b)\)-plane, which is compatible with the low-energy solar and atmospheric neutrino data at 90% and 99% C.L. for the LMA solution (cf. Table 1). We see that no fine-tuning is required in order to reproduce the data. To obtain the best-fit point (black dot in the figure), we assume at the unification scale \( \Lambda_X \) the same values for the charm and top Yukawa couplings as in the last column of Table 2, i.e. \( y_c(\Lambda_X) = 1.7 \times 10^{-3} \) and \( y_t(\Lambda_X) = 0.64 \). Taking then as input parameters
\[ \epsilon = \frac{y_c(\Lambda_X)}{y_t(\Lambda_X)} = 2.65 \times 10^{-3} , \quad a = 0.15 , \quad b = 0.34 , \quad \Lambda_R = 6.3 \times 10^8 \text{ GeV} , \] (5.12)
Figure 4: Allowed region for the parameters $a$ and $b$ of the triangular texture (5.7) for $\kappa(M_Z)$, in the case of the LMA MSW solution and considering the neutrino data at 90\% and 99\% C.L. summarized in Table 1. The black dot corresponds to the point which leads to the best-fit values of the solar and atmospheric data as shown in Eqs. (5.12)-(5.13).

the following values are obtained at the electroweak scale for $\tan \beta = 10$:

$$K_{eff} = 0.3, \quad \delta_1 = 1.0 \times 10^{-6}, \quad \delta_2 = 2.7 \times 10^{-3},$$

$$m_1 = 1.6 \times 10^{-3} \text{ eV}, \quad m_2 = 6.7 \times 10^{-3} \text{ eV}, \quad m_3 = 5.6 \times 10^{-2} \text{ eV},$$

$$\Delta m^2_{12} = 4.2 \times 10^{-5} \text{ eV}^2, \quad \Delta m^2_{23} = 3.1 \times 10^{-3} \text{ eV}^2, \quad r_\nu = 1.4 \times 10^{-2},$$

$$\tan^2 \theta_{12} = 0.26, \quad \tan^2 \theta_{23} = 1.4, \quad |U_{e3}| = 0.05,$$

$$|<m>| = 9.3 \times 10^{-15} \text{ eV},$$

which correspond to the best-fit values in the neutrino mixing plane for the atmospheric neutrinos [27] and the LMA solar neutrino solution [28].

6. Conclusions

In this paper we have considered the possibility of having an exactly degenerate spectrum for heavy right-handed Majorana neutrinos at the grand unification scale. Our theoretical framework was based on the minimal supersymmetric standard model with unbroken $R$ parity and extended with three heavy Majorana neutrino fields.
We assumed no Dirac-type leptonic mixing at GUT scale, and thus the right-handed neutrino sector constitutes the only source of lepton flavour violation in this context. We have then studied the renormalization group effects, which include the neutrino threshold effects, from high to low energies.

Taking into account the allowed range for the quark mass ratios at GUT scale and the effective neutrino mass matrix obtained through the seesaw mechanism, we have investigated which of the possible solutions to the solar neutrino problem can be accommodated in our framework. Inspired by GUT-motivated relations among the quark, charged-lepton and Dirac neutrino mass matrices, and based on the latest global analyses of the solar and atmospheric data at 99% C.L., we have then concluded that the only solar solutions compatible with the experimental data are the LOW and LMA solutions. In fact, at 90% C.L. only the latter is allowed. Even in this case, the obtained values for the neutrino oscillation parameters are far from the best-fit points. In this sense, right-handed neutrino degeneracy embedded in GUT scenarios seems to be disfavoured by the low-energy neutrino data. We emphasize however that, by slightly relaxing the imposed relations among the Yukawa coupling matrices at high scale, one can easily reach the best-fit values. Our assumption of degeneracy is in fact a very restrictive one, since the right-handed Majorana neutrino matrix $M_R$ is parametrized in terms of only two angles and an overall scale $\Lambda_R$. On the other hand, a fully or partially hierarchical mass spectrum for right-handed neutrinos can be easily achieved with a much less constrained structure, and one could therefore expect that such patterns are easier to be reconciled with the low-energy neutrino data.

Finally, we are also aware of possible contributions to lepton flavour violation processes like $\mu \rightarrow e\gamma$. As it has been recently pointed out by Casas and Ibarra [32], in supersymmetric theories with seesaw-like neutrino masses the prediction for the branching ratio of this process is, in general, larger than the experimental upper bound. In particular, scenarios such as the one considered here, where top-neutrino unification occurs, are severely constrained unless the leptonic Yukawa coupling matrices satisfy very specific requirements.

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A. Renormalization group equations

In this Appendix we present the relevant one-loop renormalization group equations for the MSSM extended with three right-handed Majorana neutrinos.

Between $\Lambda_X$ and $\Lambda_R$ the evolution of the up-quark, $Y_U$, down-quark, $Y_D$, charged-lepton, $Y_\ell$, and Dirac neutrino, $Y_\nu$, Yukawa coupling matrices is given by

$$16 \frac{\pi^2}{2} \frac{dY_U}{dt} = \left[ (T_U - G_U) + (3 Y_U Y_U^\dagger + Y_D Y_D^\dagger) \right] Y_U ,$$  \hspace{1cm} (A.1) \\
$$16 \frac{\pi^2}{2} \frac{dY_D}{dt} = \left[ (T_D - G_D) + (3 Y_D Y_D^\dagger + Y_U Y_U^\dagger) \right] Y_D ,$$  \hspace{1cm} (A.2) \\
$$16 \frac{\pi^2}{2} \frac{dY_\ell}{dt} = \left[ (T_\ell - G_\ell) + (3 Y_\ell Y_\ell^\dagger + Y_\nu Y_\nu^\dagger) \right] Y_\ell ,$$  \hspace{1cm} (A.3) \\
$$16 \frac{\pi^2}{2} \frac{dY_\nu}{dt} = \left[ (T_\nu - G_\nu) + (3 Y_\nu Y_\nu^\dagger + Y_\ell Y_\ell^\dagger) \right] Y_\nu ,$$  \hspace{1cm} (A.4)

where $t = \ln(\mu/M_Z)$, $\mu$ is the renormalization scale. The quantities $T_i$ and $G_i$ are defined as

$$T_U = \text{Tr} (3 Y_U Y_U^\dagger + Y_\nu Y_\nu^\dagger) , \quad G_U = \frac{16}{3} g_3^2 + 3 g_2^2 + \frac{13}{15} g_1^2 ,$$  \hspace{1cm} (A.5) \\
$$T_D = \text{Tr} (3 Y_D Y_D^\dagger + Y_\ell Y_\ell^\dagger) , \quad G_D = \frac{16}{3} g_3^2 + 3 g_2^2 + \frac{7}{15} g_1^2 ,$$  \hspace{1cm} (A.6) \\
$$T_\ell = \text{Tr} (3 Y_D Y_D^\dagger + Y_\ell Y_\ell^\dagger) , \quad G_\ell = 3 g_2^2 + \frac{9}{15} g_1^2 ,$$  \hspace{1cm} (A.7) \\
$$T_\nu = \text{Tr} (3 Y_U Y_U^\dagger + Y_\nu Y_\nu^\dagger) , \quad G_\nu = 3 g_2^2 + \frac{3}{5} g_1^2 ,$$  \hspace{1cm} (A.8)

where $g_3$, $g_2$, and $g_1$ are the $SU(3)$, $SU(2)$ and $U(1)$ gauge couplings, which satisfy the RGE

$$16 \frac{\pi^2}{2} \frac{dg_i}{dt} = b_i g_i^3 , \quad b_i = (33/5, 1, -3) .$$  \hspace{1cm} (A.9)

The evolution of the right-handed neutrino mass matrix, $M_R$, is given by

$$8 \frac{\pi^2}{2} \frac{dM_R}{dt} = M_R (Y_\nu Y_\nu^\dagger)^T + (Y_\nu Y_\nu^\dagger) M_R .$$  \hspace{1cm} (A.10)

Below $\Lambda_R$, the renormalization group equations for the up-quark and charged-lepton Yukawa couplings get modified since after decoupling of the right-handed neutrino states, the contribution of $Y_\nu$ should vanish. In this case the RGE for $Y_U$ and $Y_\ell$ read

$$16 \frac{\pi^2}{2} \frac{dY_U}{dt} = \left[ (T'_U - G_U) + (3 Y_U Y_U^\dagger + Y_D Y_D^\dagger) \right] Y_U ,$$  \hspace{1cm} (A.11) \\
$$16 \frac{\pi^2}{2} \frac{dY_\ell}{dt} = \left[ (T_\ell - G_\ell) + 3 Y_\ell Y_\ell^\dagger \right] Y_\ell ,$$  \hspace{1cm} (A.12)
with

\[ T_U' = 3 \text{Tr} (Y_U Y_U^\dagger). \quad (A.13) \]

Finally, the evolution of the effective neutrino mass matrix \( \kappa \) from \( \Lambda_R \) to \( M_Z \) is given by

\[ 16 \pi^2 \frac{d\kappa}{dt} = (T_\kappa - G_\kappa)\kappa + (Y_\ell Y_\ell^\dagger)\kappa + \kappa (Y_\ell Y_\ell^\dagger)^T, \quad (A.14) \]

where

\[ T_\kappa = 6 \text{Tr} (Y_U Y_U^\dagger), \quad G_\kappa = 6 g_2^2 + \frac{6}{5} g_1^2. \quad (A.15) \]

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