Toward a systematic analysis of the fourth-root trick

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Abstract

In this note I briefly discuss ideas related to the so-called fourth-root trick. A decomposition of the “rooted” fermion effective action into Wilson fermions and a nonlocal, lattice spacing suppressed functional is presented, complete with link interactions. Some proposals are given for analytical, nonperturbative studies of the fourth-root trick.

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1 Introduction

Lattice QCD with improved staggered fermions (SF) has recently enjoyed publicity for its ability to correctly reproduce many aspects of hadron physics with reasonable accuracy [1, 2]. However, criticism has been leveled at the approach [3–5] due to, among other things, the use of the fourth-root trick (FRT). In this note, I briefly review aspects of this issue, and mention some ideas in relation to it.

The FRT is used because staggered fermions, or, Kogut-Susskind fermions [6–8], do not entirely overcome the fermion doubling problem. Rather, they reduce the number of continuum modes from 16 to 4. These 4 modes are referred to as tastes, so as to distinguish them from the \( N_f \) flavors in the continuum (target) theory. To estimate the fermion measure of \( N_f \) continuum flavors, one takes the power \( N_f/4 \) of the fermion determinant in the definition of the functional integral. This is what I will refer to as the fourth-root trick, although it is often called the square-root trick since in QCD two light flavors are used.

In the case of free staggered fermions, the eigenvalues \( \omega_k \) of the fermion matrix \( M_{SF} \) are 4-fold degenerate, corresponding to symmetries that relate the 4 tastes. Thus there appears to be nothing wrong with the FRT:

\[
\det M_{SF}^{1/4} = \left( \prod_k \omega_k^4 \right)^{1/4} = \prod_k \omega_k \equiv \det M_{\text{eff}}. \tag{1.1}
\]

That is, one effectively weights by the eigenvalue spectrum of some other “fermion matrix” \( M_{\text{eff}} \). It has been shown by Shamir [9] that RG blocking allows one to write (as suggested in [10])

\[
\det M_{\text{eff}} = \det D_{RG} \det T^{1/4}. \tag{1.2}
\]

Here, \( D_{RG} \) is a local lattice Dirac operator with a single flavor. \( T \) is a local operator that only contains UV modes. Thus nonlocality contained in \( \det T^{1/4} \) factorizes into a harmless overall factor once the UV modes are integrated out. The long-distance effective action is local.

The problem is that one is not interested in free staggered fermions. Rather, in QCD the gauge-covariant staggered fermion operator contains interactions through the link fields \( U_\mu(r) \). For a generic configuration of the link fields, the 4-fold degeneracy of the eigenvalues of the fermion matrix is destroyed:

\[
\prod_k \omega_k^4 \to \prod_k \prod_{\alpha=1}^4 \omega_{k,\alpha}(U). \tag{1.3}
\]
In this circumstance, one has a right to worry about the FRT.

The splitting of eigenvalues occurs due to *taste violating* (TV) interactions. In an expansion of link bosons about the identity, these arise from higher lattice-derivative, irrelevant operators. That is, the effect is suppressed by a power of the lattice spacing $a$ that depends on the level of improvement of the lattice action, and increases with the amplitude of UV components of the link configuration. The asymmetric mixing of tastes that occurs in the presence of a nontrivial link configuration lifts the 4-fold degeneracy. Numerical studies of these splittings have been performed for dynamically generated link configurations. For example, in [11] it was seen that smearing of links—which reduces the TV effects by smoothing out UV link fluctuations—can render the IR eigenvalues “near degenerate (sic).” TV effects are regarded as a source of systematic error that can be reduced with enough effort. Since it has been observed that many physical observables are insensitive to a truncation of UV eigenvalues of the lattice Dirac operator (e.g. [12]), the nonlocalities associated with “rooting” a spectrum with nearly degenerate IR eigenvalues and nondegenerate UV eigenvalues are perhaps harmless. While this sort of argument is somewhat reassuring, clearly it would be preferable to rigorously address the consistency of the “rooted” theory.

Improved staggered fermions are used because they are efficient, cost effective, and seem to give good results. This raises simple questions that we would like to answer: Why does the FRT work in practice? At what point might it fail? For instance, is there a limit on its applicability due to a breakdown of unitarity at some short distance scale?

I now summarize the remainder of this note:

- In §2 I outline various perspectives on the FRT. Firstly, in §2.1 I remark on the effects of the FRT in perturbation theory. I reach the conclusion which has been stated many times: the FRT poses no problem for weak coupling perturbation theory about the $A_\mu = 0$ vacuum. I present details as to why this is a reasonable conclusion. Secondly, in §2.2 I speculate as to analytical, nonperturbative studies that could be done to study the FRT. Thirdly, in §2.3 I present an unusual decomposition of the “rooted” fermion effective action, in terms of Wilson fermions, and a nonlocal TV operator that has explicit suppression by the lattice spacing. Applications of this decomposition, which isolates the taste-violating part of the theory, are left to future work.

- In §3 I discuss some details of the staggered fermion action. A translation into
the taste basis, without recourse to expansion about \( U_\mu = 1 \), is described. The explicit form of the decomposition into Wilson fermions and a TV correction is given in the taste basis.

- In \( \S 4 \) I conclude with a summary. Directions for future research are recapitulated.

## 2 Perspectives

### 2.1 The perturbative view

Let me begin with conventional perturbation theory. One expands about the \( U_\mu(r) = \exp(iagA_\mu(r)) \to 1 \) limit. First I write

\[
M_{SF}(U) = M_{SF}(1) + [M_{SF}(U) - M_{SF}(1)] \equiv M_{SF}(1) + \Delta M_{SF}(U). \tag{2.1}
\]

Note that \( \Delta M_{SF}(U) = O(igA_\mu) \) and contains, among other things, the minimal fermion-boson vertex. With the FRT applied, the fermion measure is represented by the following effective contribution to the action:

\[
S_{eff} = -\frac{N_f}{4} \Tr \ln [M_{SF}(1) + \Delta M_{SF}(U)]
\]

\[
= -\frac{N_f}{4} \Tr \ln \left[ 1 + M_{SF}^{-1}(1)\Delta M_{SF}(U) \right] + \text{const.} \tag{2.2}
\]

One expands \( \exp(-S_{eff}) \) in \( \Delta M_{SF}(U) \) to obtain a series of terms, each \( \Tr \) multiplied by \( N_f/4 \). One further expands \( \Delta M_{SF}(U) \) in terms of \( U_\mu = \exp iagA_\mu \) to obtain weak coupling perturbation theory and the continuum limit. Since each \( \Tr \) corresponds to a fermion loop, the effect of the FRT is seen to be as follows: each diagram of ordinary SF perturbation theory is multiplied by \( (N_f/4)^n \), with \( n \) the number of fermion loops.

If perturbation theory is consistent at \( N_f = 0 \text{ mod } 4 \) (i.e., for ordinary SF perturbation theory), then it is difficult to see how any problems would arise for values of \( N_f \) in-between. E.g., let \( Z_i, m, g, \ldots \) be chosen to renormalize the theory for \( N_f = 0 \text{ mod } 4 \). These then become functions of \( N_f/4 \), with a natural extension to arbitrary \( N_f \). It is conceivable that cancellations between divergent diagrams happen at \( N_f = 0 \text{ mod } 4 \) but not at other values. That would indicate the need for additional counterterms; one might worry whether or not they are always local, given the apparent nonlocal nature of the rooted SF matrix. However, the symmetries of the SF action are still operative with the FRT applied, so the form of counterterms is similarly restricted.
Also, it can be seen that the perturbation series is constructed entirely from local vertex operators, since the only change is a factor of $N_f/4$ for each fermion loop. This is enough to exclude the possibility of nonlocal counterterms.

Finally, renormalization of composite operators could be different when $N_f \neq 0 \mod 4$, due to a lack of certain cancellations. But, I have no example to cite.

The conclusion I draw from these considerations is in agreement with “standard lore”: the FRT poses no problem in perturbation theory.

### 2.2 Nonperturbative ideas

Perhaps nonperturbative effects of the FRT might be accessible through an instanton calculus, or some other semiclassical expansion. Whereas it has been argued above that the FRT poses no problems for perturbation theory about the $A_\mu = 0$ vacuum, perturbation theory about some other background could conceivably yield different conclusions. The point is to compare to results obtained from other approaches in a similar regime: from the continuum, Ginsparg-Wilson fermions, etc. Finding any discrepancy and understanding its origin would teach us valuable things. Finite system volume, perhaps in conjunction with twisted boundary conditions, could help to exert greater theoretical control over such calculations.

It might also be of interest to look at strong coupling expansions with the FRT applied. Some nonperturbative features, such as chiral symmetry breaking, can be studied by this approach; e.g. [13]. One could compare expectations for the spectrum of mesons and baryons for a theory with $N_f$ flavors to what occurs in the strong coupling limit with fermion measure (2.2).

### 2.3 The Wilson fermion decomposition

In this decomposition, I organize the effective fermion measure in such a way that TV effects are isolated for systematic study, and suppression by the lattice spacing is explicit.

Let $M_{WF}$ be a single flavor Wilson fermion (WF) matrix. Then I decompose the staggered matrix as follows:

$$M_{SF} = M_{WF} \otimes 1_4 + aM_{TV},$$

where $M_{TV}$ contains all the TV effects. This decomposition is possible because in the taste (flavor) basis for staggered fermions [14,15], the terms that are not suppressed
by a are just four identical flavors of the naive lattice Dirac matrix. (With gauge 
interactions included, some reinterpretation of link variables is required; c.f. (3.7). 
This will be explained in detail below.) Once this decomposition has been effected, I 
note that
\[ \det M_{SF}^{N_f/4} = \det M_{WF}^{N_f} \exp{\frac{N_f}{4} \text{Tr} \ln \left[ 1 + a(M_{WF}^{-1} \otimes 1_4)M_{TV} \right]} . \] 
(2.4)

It can be seen that this arrangement sequesters all the TV effects and nonlocality into 
\( \mathcal{O}(a) \) correction terms.\(^1\) The fermions of the theory now consist of \( N_f \) degenerate 
flavors, each with a conventional matrix \( M_{WF} \). The correction term is a nonlocal 
functional of the the link fields.

One disadvantage of the Wilson fermion decomposition is that it will typically 
obscure the staggered fermion symmetries, such as the one that prevents additive mass 
renormalization. These are hidden in the induced tranformations of the correction 
term.

Finally, I remark that the Wilson fermion is not the unique choice for the type of 
decomposition described here. Any other fermion that consists of the naive fermion 
plus \( \mathcal{O}(a) \) terms would also work. It remains to be shown that this decomposition 
has a useful application. I will not do that here, but hope to find one in the future.

### 3 Taste basis details

The fermion action I start with is the single-flavor staggered fermion, written in 
the conventions of Kluberg-Stern et al. [15]. It is just (with obvious notations and 
summation conventions)
\[ S_{SF} = -\frac{1}{2} a^3 \alpha_\mu(r) \left[ \bar{\chi}(r)U_\mu(r)\chi(r + \hat{\mu}) + \bar{\chi}(r + \hat{\mu})U_\mu^\dagger(r)\chi(r) \right] \]
+ ima^4 \bar{\chi}(r)\chi(r). 
(3.1)

The phases are, as usual, \( \alpha_\mu(r) = (-1)^{\sum_{\nu<\mu} r_\nu} \). One associates a \( (2a)^4 \) hypercube 
with each site \( r = 2y \), where \( y \) has integral entries. Sites contained in the hypercube 
are labeled on the original lattice by
\[ r = 2y + \eta, \quad \eta \in K \equiv \{(0^4), (1,0^3), (1^2,0^2), (1^3,0), (1^4)\}. \]
\(^1\)I note that the operator \( [(M_{WF}^{-1} \otimes 1_4)M_{TV}]^{1/4} \) is the same one that Adams has studied in [10]. 
In this respect, his proposal is related to the one presented here.
Powers indicate how many times a 0 or 1 appears and underlining indicates that all permutations of entries are to be included.

To proceed further, one defines fermion fields for each point in the hypercube and transforms to the (covariant, position space) taste basis: \( \chi(r), \bar{\chi}(r) \rightarrow q^{\alpha}(y), \bar{q}^{\alpha}(y) \). Since this is all very well-known, I just summarize the ingredients:

\[
\begin{align*}
\chi(2y + \eta) &= (-1)^{\sum_{\mu} y_{\mu}} \chi_{\eta}(y), \quad \bar{\chi}(2y + \eta) = (-1)^{\sum_{\mu} y_{\mu}} \bar{\chi}_{\eta}(y), \\
\Gamma_{\eta} &= \gamma_{1}^{\eta} \gamma_{2}^{\eta} \gamma_{3}^{\eta} \gamma_{4}^{\eta}, \quad \{\gamma_{\mu}, \gamma_{\nu}\} = -2\delta_{\mu\nu}, \\
U_{\mu, \eta}(y) &= U_{\mu}(2y + \eta), \quad \Gamma_{\eta, \eta'}^{\mu} = \frac{1}{4} \text{Tr} (\Gamma_{\eta}^{\dagger} \gamma_{\mu} \Gamma_{\eta'}), \\
U_{\eta}(y) &\equiv U_{1}^{\eta}(2y) U_{2}^{\eta}(2y + \eta_{1}) U_{3}^{\eta}(2y + \eta_{1} + \eta_{2}) U_{4}^{\eta}(2y + \eta_{1} + \eta_{2} + \eta_{3}), \\
\chi_{\eta}(y) &= 2U_{\eta}^{\dagger}(y) \text{Tr} (\Gamma_{\eta} \bar{q}(y)), \quad \bar{\chi}_{\eta}(y) = 2 \text{Tr} (\Gamma_{\eta} \bar{q}(y)) U_{\eta}(y). 
\end{align*}
\]

Using various well-known identities, I have found that (3.1) becomes:

\[
S_{SF} = 2a^{3} \left\{ \bar{q}(y) C_{\mu}(y) q(y + \hat{\mu}) + \bar{q}(y + \hat{\mu}) C_{\mu}^{\dagger}(y) q(y) \\
+ \bar{q}(y) \left( A(y) + A^{\dagger}(y) \right) q(y) + 8i amq(y)(1 \otimes 1)q(y) \right\}. 
\]

I have defined the following link-dependent structures (in these two equations all sums are explicit):

\[
\begin{align*}
A_{\alpha\beta;ab}(y) &= \sum_{\mu} \sum_{\eta, \eta' \in K} \delta_{\eta + \mu, \eta'} \Gamma_{\eta}^{\alpha} \Gamma_{\eta'}^{\dagger \beta} \Gamma_{\eta, \eta'}^{\mu} U_{\eta}(y) U_{\mu, \eta}(y) U_{\eta + \mu}^{\dagger}(y), \\
C_{\mu;ab}(y) &= \sum_{\eta, \eta' \in K} \delta_{\eta - \mu, \eta'} \Gamma_{\eta}^{\alpha} \Gamma_{\eta'}^{\dagger \beta} \Gamma_{\eta, \eta'}^{\mu} U_{\eta}(y) U_{\mu, \eta}(y) U_{\eta - \mu}^{\dagger}(y + \hat{\mu}).
\end{align*}
\]

Greek and Latin indices correspond to spinor and taste labels respectively. These have been suppressed in (3.4), but are easy to put back in. For instance, \( \bar{q}(y) A(y) q(y) = \bar{q}^{\alpha}(y) A_{\alpha\beta;ab}(y) q^{\beta}(y) \).

It is not difficult to show that (on the original lattice) \( A(y) \) is a parallel transporter from \( 2y \) back to \( 2y \) along a sum of paths. These paths traverse the \((2a)^{4}\) hypercube that is associated with the site \( y \) of the doubled lattice. \( A(y) \) thus transforms as a site variable at \( y \) on the doubled lattice. \( C_{\mu}(y) \) transports from \( 2y \) to \( 2(y + \hat{\mu}) \) along a sum of paths in the hypercube. It is thus a link variable from \( y \) to \( y + \hat{\mu} \) on the doubled lattice. This is consistent with the transformation properties of the quark fields appearing in (3.4).

Consider the action obtained by expanding in \( ag \) the links \( U_{\mu}(r) = \exp[iagA_{\mu}(r)] \) that appear implicitly through \( A(y) \) and \( C_{\mu}(y) \) in (3.4). Kluberg-Stern et al. give
this continuum approximation to the interacting theory up to $O(a^2)$ corrections [15].

This exposes the leading irrelevant operators, which are suppressed by a single power of $a$. It is a simple matter to decompose their expression into 4 degenerate tastes of WFs (lattice spacing $2a$) and TV terms:

\[
S_{SF} = S_{WF(4)} + aS_{TV}
\]

\[
eq (2a)^4 \bar{q}(y) \left[ M_{WF(4)} + aM_{TV} \right] q(y),
\]

\[
M_{WF} = \gamma_\mu D_\mu - i a D_\mu D_\mu + im, \quad M_{WF(4)} = M_{WF} \otimes 1_4,
\]

\[
M_{TV} = \sum_\mu \left[ \gamma_5^\dagger \otimes (t^\dagger \mu t^\dagger \mu) + i 1_4 \otimes 1_4 \right] D^2_\mu - \sum_\mu \sum_\nu \frac{i g}{4} T_{\mu \nu} F_{\mu \nu} + O(a),
\]

\[
T_{\mu \nu} = (\gamma_\mu - \gamma_\nu) \otimes 1_4 + \frac{1}{2} \gamma_5^{\dagger} [\gamma_\mu, \gamma_\nu] \otimes ((t_\mu + t_\nu)^{\dagger} t^\dagger_5).
\] (3.6)

Here, $D_\mu$ is the gauge-covariant derivative and $F_{\mu \nu}$ is the field-strength.

In the same spirit, I can make Wilson fermions manifest in (3.4), working entirely in terms of link variables. I define links that connect sites of the doubled lattice:

\[
U_\mu(y) \equiv U_\mu(2y)U_\mu(2y + \hat{\mu}).
\] (3.7)

This permits one to define the action of 4 flavors of WF on the doubled lattice:

\[
S_{WF(4)} = 4a^3 \left\{ \bar{q}(y) \left( (i + \gamma_\mu) \otimes 1 \right) U_\mu(y)q(y + \hat{\mu}) + \bar{q}(y + \hat{\mu}) \left( (i - \gamma_\mu) \otimes 1 \right) U^\dagger_\mu(y)q(y) + i(4ma - 8)\bar{q}(y)(1 \otimes 1)q(y) \right\}. \] (3.8)

If I add and subtract this from (3.4), I then have $S_{SF} = S_{WF(4)} + aS_{TV}$, where

\[
S_{TV} = (2a)^4 \left\{ \bar{q}(y)C_\mu(y)q(y + \hat{\mu}) + \bar{q}(y + \hat{\mu})C^\dagger_\mu(y)q(y) + \bar{q}(y)A(y)q(y) \right\},
\]

\[
A(y) \equiv \frac{1}{8a^2} \left[ A(y) + A^\dagger(y) \right] + \frac{2i}{a^2} (1 \otimes 1),
\]

\[
C_\mu(y) \equiv \frac{1}{8a^2} \left[ C_\mu(y) - \frac{1}{4a^2} \left( (i + \gamma_\mu) \otimes 1 \right) U_\mu(y) \right].
\] (3.9)

4 Discussion

I have given an argument in support of the conclusion that the FRT poses no problem in perturbation theory about the $A_\mu = 0$ vacuum. The consistency of the FRT needs to be studied further at a nonperturbative level.

I have suggested a semiclassical analysis about nontrivial link configurations; for example, ones that have nonzero topological charge. Perhaps that might allow for an
examination of nonperturbative TV vis-à-vis the fourth-root trick. The point is to
compare to results obtained from other approaches in a similar regime. It would also
be interesting to examine the strong coupling limit. Studies in these directions are
currently in progress.

It is not clear to me whether or not the questions of global topology suggested
in [5] can be addressed by semiclassical or strong coupling methods. However, many
other important questions of consistency may be accessible through the techniques
envisaged here.

Finally, I hope to present some useful applications of the Wilson fermion decom-
position at a later date.

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