Large N QCD – Continuum Reduction and Chiral Condensate

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Continuum reduction in large N QCD enables one to extract physical quantities in the $N \to \infty$ limit of QCD by working in small physics volumes. The computation of chiral condensate is an example of such a calculation.

Keywords: Large N QCD; Continuum Reduction; Chiral Condensate.

1. Introduction

QCD with massless quarks does not have any free parameters. ’t Hooft\cite{1} pointed out that one might consider the size of the gauge group, N, as a parameter. The initial hope was to exactly solve QCD in the $N \to \infty$ limit and compute physical quantities as a power series in $1/N$. Although the exact solution of large N QCD is not yet available, significant progress has been in the area of large N phenomenology by a careful study of large N counting\cite{2}. This analysis has extensively shown that $1/N$ is a good expansion parameter and that many experimental results can be reproduced by just studying one or two leading terms in $1/N$. In addition lattice studies of pure gauge theories\cite{3} have shown that large N limits of quantities like string tension and deconfinement temperature are reached as early as $N = 5$ and $N = 6$. If one also solves the theory in the $N \to \infty$ limit, then one can compute physical observables as a power series in $1/N$ with no other free parameter.

It seems quite likely that one can solve for the meson spectra in the large N limit of QCD using existing numerical techniques and moderate computing power\cite{4,5,6,8}. The basic idea behind the numerical solution is the concept of continuum reduction. Continuum reduction enables one to work in a finite physical volume and compute observables in the infinite volume theory without any finite volume effects. One first takes the $N \to \infty$ limit at a fixed physical volume and the resulting theory does not depend on the physical volume. The numerical study of large N QCD is further simplified by the fact that fermions in the fundamental representation are naturally quenched. This is the case as long as the theory only has a finite number of fermion flavors in the fundamental representation.
1.1. Continuum reduction

Eguchi and Kawai\textsuperscript{[3]} made the observation that the space-time lattice in the limit of $N \to \infty$ can be reduced to a single point provided the global $U^d(1)$ symmetries that multiplies Polyakov loops by $U(1)$ phases are not broken. This symmetry is broken for $d > 2$ and the equivalence of the loop equations used by Eguchi and Kawai is ruined\textsuperscript{[10]}. But, the arguments of Eguchi and Kawai also hold for an $L^d$ lattice where $L > 1$. That is to say, the infinite space-time lattice can be reduced to a finite $L^d$ lattice in the limit of $N \to \infty$ provided the global $U^d(1)$ symmetries that multiplies Polyakov loops by $U(1)$ phases are not broken on the $l^d$ lattice.

The only parameter in the lattice theory is the 't Hooft gauge coupling $b = \frac{1}{96N}$. The location of the transition point $b_c(L)$ where one of the $U^d(1)$ symmetries is broken scales properly with $L$\textsuperscript{[4]} One can therefore define a critical size $l_c$ in the continuum such that the $U^d(1)$ symmetries remain unbroken for $l > l_c$. The argument of Eguchi and Kawai will hold for all $L$ as long as we keep $b < b_c(L)$ and there will be no dependence on $L$. There will be no dependence on the box size $l$ as long as $l > l_c$ and this is referred to as continuum reduction. Physical results on the lattice can be extracted by working on an $L^d$ lattice and keeping $b$ just below $b_c(L)$. Computations should be done on two or three different values of $N$ to study the infinite $N$ limit. Once this is done, it is sufficient to work with two or three different $L$ values to study the effect of finite lattice spacing and one will be able to extract the results for infinite volume QCD in the limit of large $N$.

There is one lattice artifact to be taken into account when setting the parameters $\{L, b < b_c(L)\}$. There is a bulk transition on the lattice in the large $N$ limit at $b_B = 0.36$ that is associated with the spectrum of the single plaquette $(1 \times 1$ Wilson loop).\textsuperscript{[5]} One has to set $b > b_B$ to obtain the proper continuum limit. The scaling of the critical coupling $b_c(L)$ in perturbation theory up to two loops is given by

$$\Lambda L = \left[\frac{11}{48\pi^2 b_c(L)}\right]^{\frac{7\pi}{24}} e^{\frac{2\pi^2 b_c(L)}{11}}$$

Lattice study of the phase transition where one goes from $0c$ (phase where $b > b_B$ and all $U^d(1)$ are unbroken) to $1c$ (phase where $b > b_B$ and one of the $U^d(1)$ is broken) shows that a value of $\Lambda = 3.85 \pm 0.2$ in Eqn.\textsuperscript{[11]} fits the data well with $b_c(L)$ replaced by its tadpole improved value.

2. Chiral condensate

QCD with massless quarks has a chiral symmetry that implies $\langle \bar{\psi} \psi \rangle = 0$ as long as one is in a finite physical volume. But the massless limit and the infinite volume do

\textsuperscript{[a]}There is no phase transition for finite $N$ but there is a cross-over that gets stronger with increasing $N$.\textsuperscript{[6]}
not commute. In particular,
\[
\lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \langle \bar{\psi} \psi(m) \rangle = \Sigma \neq 0 \tag{2}
\]
in QCD at zero temperature indicating that chiral symmetry is spontaneously broken. The theory does not depend on the physical volume in the large N limit as long as \( l > l_c \). Therefore, the massless limit has to commute with the infinite volume limit. How does chiral symmetry break in large N QCD? The answer lies in the \( N \to \infty \) limit. This limit does not commute with the massless limit. In particular,
\[
\lim_{m \to 0} \lim_{N \to \infty} \frac{1}{N l^4} \langle \bar{\psi} \psi(m, N, l) \rangle = \Sigma \neq 0 \tag{3}
\]
and this \( \Sigma \) does not depend on \( l \) as long as \( l > l_c \).

Overlap fermions enable one to study chiral symmetry breaking away from the continuum limit. The computation of the chiral condensate was performed as follows. Gauge fields were generated at a fixed \( L, N \) and coupling \( b = \frac{1}{\sigma N} \). Gauge fields were generated in the \( Q = 0 \) and \( Q = 1 \) topological sectors. Two lowest non-zero eigenvalues of the massless hermitian overlap Dirac operator, \( \lambda_1 \) and \( \lambda_2 \), were computed on each configuration.\(^b\) If chiral symmetry is spontaneously broken as \( N \to \infty \) at a fixed \( L \) and \( b \), then the distributions of \( z_i = \lambda_i \Sigma N L^4 \) should be given by universal functions described by chiral random matrix theory.\(^{11} \) It was shown that the distribution \( p(z_1/z_2) \) was universal as predicted by chiral random matrix theory as \( N \) gets large at a fixed \( L \) and \( b \). Furthermore, we determined \( \Sigma \) such that both \( p_1(z_1) \) and \( p_2(z_2) \) followed universal distribution functions. We showed that \( \Sigma \) did not depend on \( L \) after one has taken the large N limit at a fixed \( b \). This confirmed that continuum reduction hold for fermionic operators. Furthermore, \( \Sigma(b) \) obeyed the usual scaling law of QCD and \( \Sigma(b)L_c^3(b) \) was a constant. The detailed analysis resulted in the following estimate for the chiral condensate.
\[
\frac{N_c^3}{\langle \bar{\psi} \psi \rangle_{\overline{\text{MS}}} (2 \text{ GeV})} \approx (0.65)^3 \tag{4}
\]

3. 't Hooft model

It is interesting to consider two dimensional QCD where one has a chiral condensate. Continuum reduction is trivial in \( d = 0 \) since \( l_c = 0 \). Therefore, one should be able to extract the chiral condensate by working on a \( 1^2 \) lattice. The only degrees of freedom are two SU(N) matrices we will call \( U \) and \( V \). The \( 2N \times 2N \) hermitian overlap Dirac matrix, \( H_o \), is a dense matrix arising out of the denseness of \( U \) and \( V \). This model looks very close to a chiral random matrix model with the complication arising out of the fact that one has two \( N \times N \) matrices. We are not aware of an exact solution of this model that results in universal distributions of the eigenvalues of \( H_o \).

\(^b\)More eigenvalues can be computed but this increases the computational cost and two are sufficient to obtain an estimate of the chiral condensate.
Let $\pm \lambda_i$, $i = 1, \ldots, N$ be the $2N$ eigenvalues of $H_0$. Then the joint distribution of $z_i = \frac{\lambda_i N}{\sqrt{6\pi b}}$ should obey the universal functions given by chiral RMT since $\Sigma = \frac{1}{\sqrt{6\pi b}}$.

We performed a numerical simulation of this model for $b = 1$ and several different $N$ values. We computed all eigenvalues of $H_0$ and obtained an estimate of $\Sigma_i$ from the two lowest eigenvalues by forcing the average of $z_i$ to be the same as the one dictated by chiral RMT. Fig. 1 shows $\Sigma_1$ and $\Sigma_2$ as a function of $N$ and one can clearly see the approach to the correct value as $N \to \infty$. Furthermore, the average of the ratio, $r = \frac{\lambda_1}{\lambda_2}$, also approaches the correct value as shown in Fig. 1. Fig. 2 shows the distribution of $p_1(z_1)$, $p_2(z_2)$ and $p(r)$ obtained from the numerical simulation and the comparison with the universal distributions given by chiral RMT. The distributions approach the expected results as $N \to \infty$.

4. Conclusions

Continuum reduction in the large $N$ limit of QCD enables one to compute physical observables at infinite volume by performing numerical simulations on finite physical volumes. Computation of chiral condensate was used as an example of such a calculation. It would be nice to obtain an analytic derivation of the connection between chiral RMT and 't Hooft model shown by numerical simulations in section 3.

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Fig. 1. Extraction of $\Sigma_1$ and $\Sigma_2$ from numerical simulations along with the average value of $r = \frac{\lambda_1}{\lambda_2}$ as a function of $N$. 
Fig. 2. Comparison of the distributions of the scaled eigenvalues $z_1$ and $z_2$ with the chiral RMT distributions for the same. Also shown is the comparison of $p(r)$ with that of chiral RMT.