Elliptic Algebra $\mathcal{A}_{q,p}(\hat{\mathfrak{sl}}_2)$ in the Scaling Limit

S. Khoroshkin$^1$, D. Lebedev$^2$

Institute of Theoretical & Experimental Physics
117259 Moscow, Russia

S. Pakuliak$^3$

Bogoliubov Laboratory of Theoretical Physics, JINR
141980 Dubna, Moscow region, Russia

Abstract

The scaling limit $\mathcal{A}_\hbar,\eta(\hat{\mathfrak{sl}}_2)$ of the elliptic algebra $\mathcal{A}_{q,p}(\hat{\mathfrak{sl}}_2)$ is investigated. The limiting algebra is defined in terms of a continuous family of generators being Fourier harmonics of Gauss coordinates of the $L$-operator. Ding-Frenkel isomorphism between $L$-operator’s and current descriptions of the algebra $\mathcal{A}_\hbar,\eta(\hat{\mathfrak{sl}}_2)$ is established and is identified with the Riemann problem on a strip. The representations, coalgebraic structure and intertwining operators of the algebra are studied.
0 Introduction

This paper is devoted to the investigation of the infinite-dimensional algebra \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \) which is supposed to be the algebra of symmetries in integrable models of quantum field theories. Our work arose from the attempts to understand the mathematical background of the results by S. Lukyanov and to combine his methods with the group-theoretical approach to quantum integrable models developed in [2,3] and with the Yangian calculations in [4].

The investigation of the symmetries in quantum integrable models of the two-dimensional field theory started in [5,6] and resulted in the form-factor (bootstrap) approach to these models developed in the most completed form in the works by F.A. Smirnov [7]. This approach was not addressed to investigation of dynamical symmetries in the model but to computation of certain final objects of the theory, form-factors of the local operators and correlation functions of the local operators. It was observed in the papers [8,9,10] that the mathematical structures underlying the success of the bootstrap approach in the massive integrable models are related to the representation theory of infinite-dimensional Hopf algebras. The dynamical symmetries in massive two-dimensional field theories was investigated in [1] in the framework of Zamolodchikov-Faddeev operators [11,12].

The elliptic algebra \( \mathcal{A}_{q,p}(\widehat{\mathfrak{su}_2}) \) was proposed in the works [3] as an algebra of symmetries for the eight-vertex lattice integrable model. The algebra \( \mathcal{A}_{q,p}(\widehat{\mathfrak{su}_2}) \) was formulated in the framework of the “RLL” approach [13] in terms of the symbols \( L_{\epsilon,\epsilon'}^{n} \) (\( n \in \mathbb{Z}, \epsilon, \epsilon' = \pm, \epsilon \epsilon' = (-1)^n \)) gathered using the spectral parameter \( \zeta \) into \( 2 \times 2 \) matrices \( L^\pm(\zeta) \) and the central element \( c \). The generating series \( L^\pm(\zeta) \) satisfy the defining relations:

\[
R_{12}^+(\zeta_1/\zeta_2)L_1^\pm(\zeta_1)L_2^\pm(\zeta_2) = L_2^\pm(\zeta_2)L_1^\pm(\zeta_1)R_{12}^+(\zeta_1/\zeta_2),
\]

\[
R_{12}^+(q^{\epsilon/2}\zeta_1/\zeta_2)L_1^\pm(\zeta_1)L_2^\pm(\zeta_2) = L_2^\pm(\zeta_2)L_1^\pm(\zeta_1)R_{12}^+(q^{-\epsilon/2}\zeta_1/\zeta_2),
\]

\[
q^{\epsilon/2} = L_{++}^+(q^{-1}\zeta)L_{--}^+(\zeta) - L_{+-}^+(q^{-1}\zeta)L_{-+}^+(\zeta),
\]

\[
L_{\epsilon \epsilon'}^-(q^{-1}\zeta) = \epsilon \epsilon' L_{-\epsilon,-\epsilon'}^+(p^{1/2}q^{-\epsilon/2}\zeta),
\]

(0.1)

where

\[
R^\pm(\zeta) = q^{\mp 1/2}\zeta \left[ (q^2 \zeta^{-2};q^4)_\infty(q^2 \zeta^2;q^4)_\infty \right]^{\pm 1} R(\zeta),
\]

(0.2)

and \( R(\zeta) = R(\zeta;p^{1/2},q^{1/2}) \) is the Baxter elliptic \( R \) matrix normalized to satisfy the unitarity and crossing symmetry relations [14]. \( R^* = R(\zeta;p^{1/2},q^{1/2}) \) and \( p^* = pq^{-2\epsilon} \). Unfortunately, there is no description of infinite-dimensional representations of \( \mathcal{A}_{q,p}(\widehat{\mathfrak{su}_2}) \) in terms of free fields.

In this paper we are going to investigate the scaling limit of this algebra when \( q, p \to 1 \). We call this algebra \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \). Let us note that although the algebra \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \) is constructed by means of the trigonometric \( R \)-matrix, it is quite different from the quantum affine algebra which is a degeneration of the elliptic algebra \( \mathcal{A}_{q,p}(\widehat{\mathfrak{su}_2}) \) when \( p = 0 \). The algebra \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \), written in integral relations for usual commutators and anticommutators, conserve many principal properties of the elliptic algebra. For example, it possesses an evaluation homomorphism onto a degenerated Sklyanin algebra [15] for zero central charge and has no imbeddings of finitedimensional quantum groups for \( c \neq 0 \). But due to the more simple structure of \( R \)-matrix comparing with the elliptic case a more detailed study of its algebraical structure is possible.

In particular, one of our achievements is the current description of the algebra \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \), which is equivalent to the factorization of the quantum determinant in the \( L \)-operator approach. This allows us to make a more detailed investigation of the representation theory of the algebra \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \). Starting from basic representation of \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \) in a Fock space we reconstruct precisely the Zamolodchikov–Faddeev algebra, described in [11,12,14]. We carry out this reconstruction from the analysys of the Hopf structure of \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \).

The distinguished feature of the algebra \( \mathcal{A}_{h,\eta}(\widehat{\mathfrak{su}_2}) \) is a presence of analysis in its description. The formal generators of the algebra are Fourier harmonics of the currents labeled by real numbers, and the elements of the algebra are integrals over generators with coefficients being functions with certain conditions on their analyticity and on their asymptotical behaviour.
The paper is organized as follows. In the first section we give a description of the algebra $A_{h,\eta}(\hat{sl}_2)$ in terms of formal generators being Fourier harmonics of the currents. The relations for the formal generators are given in a simple integral form. We assign a precise meaning to the elements of the algebra as to certain integrals over generators and show that the relations are correctly defined in corresponding vector spaces. Moreover, we show that the quadratic integral relations could be interpreted as ordering rules with polylogarithmic coefficients for monomials composed from generators of the algebra. In this section we suppose that central charge is not equal to zero. In the next section we develop the formalism of $L$-operators for $A_{h,\eta}(\hat{sl}_2)$. We show that the $L$-operators $L^\pm(u)$, satisfying the standard relations with $R$-matrices being scaling limits of those from [14], admit the Gauss decomposition. We write down relations for the Gauss coordinates and identify them with generating functions for the generators of $A_{h,\eta}(\hat{sl}_2)$ described in the previous section. Looking to the rational limit $\eta \rightarrow 0$ we find a double of the Yangian but in a presentation different from [17, 18]. We describe also the coalgebraic structure of $A_{h,\eta}(\hat{sl}_2)$. The comultiplication rule looks standard in terms of the $L$-operators, it is compatible with the defining relations, but it sends now the initial algebra into a tensor product of two different algebras which differ by the value of the parameter $\eta$. Nevertheless it is sufficient for the definition of the intertwining operators. We call this structure a Hopf family of algebras.

Section 3 is devoted to the description of algebra $A_{h,\eta}(\hat{sl}_2)$ when $c$ tends to zero. The limit is not trivial, one should look carefully to the asymptotics of the currents in the limit in order to define correct generators for $c = 0$. We describe finite-dimensional representations and the evaluation homomorphism onto the degenerate Sklyanin algebra, which is isomorphic in this case to $U_q(sl_2)$ with $|q| = 1$. In the next section we complete Ding-Frenkel isomorphism [19] and present a description of the algebra in terms of total currents. We show that Ding-Frenkel formulas are equivalent in our case to Sokhotsky-Plemely’s formulas for the Riemann problem on a strip. The relation $[11]$ for $L^\pm$ operators is also natural in the framework of the Riemann problem.

The last two sections are devoted to the study of the basic representation of the algebra $A_{h,\eta}(\hat{sl}_2)$ in a Fock space. The representation of the corresponding Zamolodchikov-Faddeev algebra in this space was recently described in [16, 20]. We start from a bosonization of the total currents for $A_{h,\eta}(\hat{sl}_2)$ and then identify Zamolodchikov-Faddeev algebra with the algebra of type I and type II twisted intertwining operators. The twisting means a presence of a certain involution in the definition of the intertwining operators. The twisting comes from the lack of zero mode operator $(-1)^p$ in the continuous models. There is no motivation to introduce this operator in our case since, to the contrary to discrete models, we have the unique level one module. As a consequence, Zamolodchikov-Faddeev operators commute by means of an $R$ matrix [11, 13] which differs from the one used in the description of $A_{h,\eta}(\hat{sl}_2)$ by certain signs. We check also the correspondence of the Miki’s formulas [21] to the $L$-operator description of the basic representation of $A_{h,\eta}(\hat{sl}_2)$. Note also that the notions of a Fock space and of vertex operators for continuous free boson field require special analytical definition which we suggest in the last section.

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1 Algebra $A_{h,\eta}(\hat{sl}_2)$ ($c \neq 0$)

1.1 The definition

For $\lambda \in \mathbb{R}$ we consider the family of symbols $\hat{\epsilon}_\lambda$, $\hat{f}_\lambda$, $\hat{t}_\lambda$ and $c$ of the formal algebra which satisfy the commutation relations:

$$[c, \text{everything}] = 0,$$  \hspace{1cm} (1.1)

$$[\hat{\epsilon}_\lambda, \hat{f}_\mu] = \text{sh} \left( \frac{\lambda}{2\eta} + \frac{\mu}{2\eta} \right) \hat{t}_{\lambda+\mu},$$ \hspace{1cm} (1.2)

$$[\hat{t}_\lambda, \hat{\epsilon}_\mu] = \frac{t\pi\eta\hbar}{2\pi\eta} \int_{-\infty}^{\infty} d\tau \text{sh} \left( \frac{z}{2\eta} \right)^{-1} \{\hat{t}_{\lambda+\tau}, \hat{\epsilon}_{\mu-\tau}\},$$ \hspace{1cm} (1.3)
The correctness of the definition of the algebra \( \mathcal{A} \) follows from the Lemma 1 and the properties of the kernels of the integral transforms which enter in the r.h.s. of the commutation relations (1.2)–(1.7) which can be treated as equalities in the vector space \( \mathcal{A} \). The correctness of the definition of the algebra \( \mathcal{A}_{\hbar, \eta}(\mathfrak{sl}_2) \) is equal to some number such that \( \hbar c > 0 \) and we identify \( c \) with this number. The case \( c = 0 \) requires a special treatment and will be considered in the next section.

Let us consider the vector space \( \mathcal{A} \) formed by the formal integrals of the type

\[
\int_{-\infty}^{\infty} \prod_k d\lambda_k \prod_i d\mu_i \prod_j d\nu_j \phi(\{\lambda_k\};\{\mu_i\};\{\nu_j\})P(\{\hat{\kappa}_k\};\{\hat{\mu}_j\};\{\hat{t}_j\}) ,
\]

where \( \phi(\{\lambda_k\};\{\mu_i\};\{\nu_j\}) \) is the C-number function of real variables \( \lambda_k, \mu_i \) and \( \nu_j \) which satisfy the conditions of analyticity:

\[
\begin{align*}
\phi(\{\lambda_k\};\{\mu_i\};\{\nu_j\}) & \text{ is analytical in the strip } -\pi \eta < \text{Im} \lambda_k < \pi \eta \quad \forall \lambda_k , \\
\phi(\{\lambda_k\};\{\mu_i\};\{\nu_j\}) & \text{ is analytical in the strip } -\pi \eta' < \text{Im} \mu_i < \pi \eta' \quad \forall \mu_i , \\
\phi(\{\lambda_k\};\{\mu_i\};\{\nu_j\}) & \text{ is analytical in the strip } -\pi \eta' < \text{Im} \nu_j < \pi \eta' \quad \forall \nu_j ,
\end{align*}
\]

and conditions on the asymptotics when \( \text{Re} \lambda_k, \text{Re} \mu_i, \text{Re} \nu_j \to \pm \infty \):

\[
\begin{align*}
\phi(\{\lambda_k\};\{\mu_i\};\{\nu_j\}) & < C e^{-|\text{Re} \lambda_k|} , \\
\phi(\{\lambda_k\};\{\mu_i\};\{\nu_j\}) & < C e^{-|\text{Re} \mu_i|} , \\
\phi(\{\lambda_k\};\{\mu_i\};\{\nu_j\}) & < C e^{-|\text{Re} \nu_j|} ,
\end{align*}
\]

for some real positive \( \alpha, \beta, \gamma \). The notation

\[ P(\{\hat{\kappa}_k\};\{\hat{\mu}_j\};\{\hat{t}_j\}) \]

means monomial which is a product of the formal generators \( \hat{\kappa}_k, \hat{\mu}_j, \hat{t}_j \) in some order.

The space \( \mathcal{A} \) has a natural structure of free (topological) algebra.

By definition the algebra \( \mathcal{A}_{\hbar, \eta}(\mathfrak{sl}_2) \) is identified with \( \mathcal{A} \) factorized by the ideal generated by the commutation relations (1.2)–(1.3) which can be treated as equalities in the vector space \( \mathcal{A} \).

The correctness of the definition of the algebra \( \mathcal{A}_{\hbar, \eta}(\mathfrak{sl}_2) \) follows from the Lemma 1 and the properties of the kernels of the integral transforms which enter in the r.h.s. of the commutation relations (1.2)–(1.7). These relations make also possible to write the monomials in (1.8) in the ordered form (see the next subsection).
Lemma 1. For two functions \( a(\lambda) \) and \( b(\lambda) \) which are analytical in the strips \(-\alpha_1 < \text{Im} \lambda < \alpha_2, -\beta_1 < \text{Im} \lambda < \beta_2 \) respectively for \( \alpha_1, \alpha_2, \beta_1, \beta_2 > 0 \) and have exponentially decreasing asymptotics when \( \text{Re} \lambda \to \pm \infty \) the convolution

\[
(a \ast b)(\lambda) = \int_{-\infty}^{\infty} \delta(\tau) a(\lambda - \tau) b(\lambda + \tau)
\]

is analytical function of \( \lambda \) in the strip \(-\alpha_1 - \beta_1 < \text{Im} \lambda < \alpha_2 + \beta_2 \) and also have exponentially decreasing asymptotics at \( \text{Re} \lambda \to \pm \infty \).

In the sequel we will need following involution of the algebra \( A_{\tilde{b}, \tilde{a}}(\eta_2) \):

\[
\iota(\hat{e}_\lambda) = -\hat{e}_\lambda, \quad \iota(\hat{f}_\lambda) = -\hat{f}_\lambda, \quad \iota(\hat{t}_\lambda) = \bar{t}_\lambda.
\]

1.2 Commutation relations as ordering rules

Assign the meaning to the commutation relations (1.2)–(1.7). One should understand them as the rules to express the product of the formal generators in the form

\[
\hat{a}_\lambda \hat{b}_\mu = \hat{b}_\mu \hat{a}_\lambda + 2 \int_{-\infty}^{\infty} d\tau \hat{b}_{\mu - \tau} \hat{a}_{\lambda + \tau} \times \left( \varphi(\tau) + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\tau_1 \cdots d\tau_n \varphi(\tau - \tau_1) \prod_{k=1}^{n-1} \varphi(\tau_k - \tau_{k+1}) \varphi(\tau_n) \right),
\]

where the function \( \varphi(\tau) \) is

\[
\frac{\text{th}(\pi \eta h)}{2\pi \eta \text{sh}(\pi \eta/2)} \quad \text{for (1.3)} \quad \text{and} \quad -\frac{\text{th}(\pi \eta' h)}{2\pi \eta' \text{sh}(\pi \eta'/2)} \quad \text{for (1.4)}.
\]

The operators of the same type can be ordered according to the ordering of the indeces. Fix \( \lambda > \mu \). Then iterating the commutation relations of the type (1.3), (1.6) or (1.7) we obtain:

\[
\hat{a}_\lambda \hat{a}_\mu = \hat{a}_\mu \hat{a}_\lambda + 2 \int_{0}^{\infty} d\tau \hat{a}_{\mu - \tau} \hat{a}_{\lambda + \tau} \times \left( \varphi(\tau) + \sum_{n=1}^{\infty} \int_{0}^{\infty} d\tau_1 \cdots d\tau_n \varphi(\tau - \tau_1) \prod_{k=1}^{n-1} \varphi(\tau_k - \tau_{k+1}) \varphi(\tau_n) \right),
\]

where

\[
\varphi(\tau; \tau') = \varphi(\tau - \tau') + \varphi(\tau + \tau')
\]

and the function \( \varphi(\tau) \) is

\[
-\frac{\text{th}(\pi \eta h)}{2\pi \eta \text{th}(\pi \eta/2)} \quad \text{for (1.3)}, \quad \frac{\text{th}(\pi \eta' h)}{2\pi \eta' \text{th}(\pi \eta'/2)} \quad \text{for (1.4)}
\]

and \( -\kappa(\eta) \) for (1.7).

**Conjecture 2.** The series in (1.10) and (1.11) are convergent.

There are few remarks in favour if this conjecture. First, we see that if we consider the deformation parameter \( h \) small than these series are series with respect to powers of the small parameter. Second, in the Yangian limit when \( \eta \to 0 \) the series in (1.11) can be summed up to obtain the function \( h e^{\kappa \tau} \).
2 \quad L\text{-Operator Realization of the Algebra } \mathcal{A}_{\hat{h},\eta}(\hat{\mathfrak{sl}}_2)\

2.1 \quad \text{Gauss coordinates of the } L\text{-operator}\

Fix the following \( R \)-matrix \([1, 3]\):

\[
R^+(u, \eta) = \tau^+(u)R(u, \eta), \quad R(u, \eta) = r(u, \eta)\mathcal{R}(u, \eta),
\]

\[
\mathcal{R}(u, \eta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(u, \eta) & c(u, \eta) & 0 \\
0 & c(u, \eta) & b(u, \eta) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
r(z, \eta) = \frac{\Gamma(\hbar \eta) \Gamma(1 + i\eta u) \prod_{p=1}^{\infty} R_p(u, \eta) R_p(i\hbar - u, \eta)}{\Gamma(\hbar \eta + i\eta u) \prod_{p=1}^{\infty} R_p(0, \eta) R_p(i\hbar, \eta)},
\]

\[
R_p(u, \eta) = \frac{\Gamma((2p + 1)\hbar \eta + i\eta u) \Gamma(1 + (2p - 1)\hbar \eta + i\eta u)}{\Gamma(2p\hbar \eta + i\eta u) \Gamma(1 + 2p\hbar \eta + i\eta u)},
\]

\[
b(u, \eta) = \frac{\text{sh} \pi \eta u}{\text{sh} \pi \eta(u - i\hbar)}, \quad c(u, \eta) = \frac{-\text{sh} i\pi \eta h}{\text{sh} \pi \eta(u - i\hbar)}, \quad \tau^+(u) = \text{cth} \left(\frac{\pi u}{2\hbar}\right),
\]

where \( u \) is a spectral parameter.

Let

\[
L(u) = \begin{pmatrix}
L_{++}(u) & L_{+-}(u) \\
L_{-+}(u) & L_{--}(u)
\end{pmatrix}
\]

be a quantum \( L \)-operator which matrix elements are treated as generating functions for the elements of the algebra given by the commutation relations:

\[
R^+(u_1 - u_2, \eta') L_1(u_1, \eta) L_2(u_2, \eta) = L_2(u_2, \eta) L_1(u_1, \eta) R^+(u_1 - u_2, \eta),
\]

\[
q\text{-det}L(u) = 1.
\]

The quantum determinant of the \( L \)-operator is given by

\[
q\text{-det}L(z) = L_{++}(z - i\hbar) L_{--}(z) - L_{+-}(z - i\hbar) L_{-+}(z).
\]

Let

\[
L(u) = \begin{pmatrix}
1 & f(u) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
k_1(u) & 0 \\
0 & k_2(u)
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
e(u) & 1
\end{pmatrix},
\]

be the Gauss decomposition of the \( L \)-operator \([2, 2]\). One can deduce from \(2.3, 2.4\) that

\[
k_1(u) = (k_2(u + i\hbar))^{-1}.
\]

Let

\[
\hat{h}(u) = k_2(u)^{-1} k_1(u), \quad \tilde{h}(u) = k_1(u) k_2(u)^{-1}.
\]

Then, due to \(2.3, 2.4\)

\[
\tilde{h}(u) = \frac{\eta \sin \pi \eta h}{\eta' \sin \pi \eta h} h(u).
\]

We have the following

**Proposition 3.** The Gauss coordinates \( e(u), f(u) \) and \( h(u) \) of the \( L \)-operator \([2, 2]\) satisfy the following commutation relations \((u = u_1 - u_2)\)

\[
e(u_1)f(u_2) - f(u_2)e(u_1) = \frac{\text{sh} i\pi \eta h}{\text{sh} \pi \eta u} h(u_1) - \frac{\text{sh} i\pi \eta h}{\text{sh} \pi \eta u} h(u_2),
\]

\[
\text{sh} \pi \eta(u + i\hbar) h(u_1) e(u_2) - \text{sh} \pi \eta(u - i\hbar) e(u_2) h(u_1) = \text{sh} (i\pi \eta h) \{h(u_1), e(u_1)\},
\]

where \( u \) is a spectral parameter.
\[
\begin{align*}
\text{sh } \pi \eta' (u - i \hbar) h(u_1) f(u_2) - \text{sh } \pi \eta' (u + i \hbar) f(u_2) h(u_1) &= - \text{sh} \{i \pi \eta' \hbar \} \{ h(u_1), f(u_1) \}, \\
\text{sh } \pi \eta (u + i \hbar) e(u_1) e(u_2) - \text{sh } \pi \eta (u - i \hbar) e(u_2) e(u_1) &= \text{sh} \{i \pi \eta \} \{ e(u_1)^2 + e(u_2)^2 \}, \\
\text{sh } \pi \eta' (u - i \hbar) f(u_1) f(u_2) - \text{sh } \pi \eta' (u + i \hbar) f(u_2) f(u_1) &= - \text{sh} \{i \pi \eta' \hbar \} \{ f(u_1)^2 + f(u_2)^2 \}, \\
\text{sh } \pi \eta (u + i \hbar) \text{sh } \pi \eta' (u - i \hbar) h(u_1) h(u_2) &= h(u_2) h(u_1).
\end{align*}
\]

The proof is a direct substitution of the Gauss decomposition of \( L \)-operators \((2.6) \) into \((2.3) \).

### 2.2 The generating integrals for \( A_{\hbar, \eta}(sl_2) \)

Let \( e^\pm(u), f^\pm(u) \) and \( h^\pm(u) \) be the following formal integrals of the symbols \( \hat{\epsilon}_\lambda, \hat{f}_\lambda, \hat{t}_\lambda \) (\( u \in \mathbb{C} \)):

\[
\begin{align*}
e^\pm(u) &= \pm i \frac{\sin \pi \eta h}{\pi \eta} \int_{-\infty}^{\infty} d\lambda \, e^{i \lambda u} \frac{\hat{\epsilon}_\lambda e^{\pm \lambda \eta / 4}}{1 + e^{\pm \lambda \eta}}, \\
f^\pm(u) &= \pm i \frac{\sin \pi \eta' h}{\pi \eta'} \int_{-\infty}^{\infty} d\lambda \, e^{i \lambda u} \frac{\hat{f}_\lambda e^{\pm \lambda \eta / 4}}{1 + e^{\pm \lambda \eta}}, \\
h^\pm(u) &= -i \frac{\sin \pi \eta h}{2 \pi \eta} \int_{-\infty}^{\infty} d\lambda \, e^{i \lambda u} \hat{t}_\lambda e^{\pm \lambda \eta / 2}.
\end{align*}
\]

Here

\[
\eta'' = \frac{2 \eta \eta'}{\eta + \eta'}.
\]

By the direct verification we can check that if the complex number \( u \) is inside the strip

\[
\Pi^+ = \left\{ -\frac{1}{\eta} - \frac{\hbar c}{4} < \text{Im} \, u < -\frac{\hbar c}{4} \right\}
\]

then elements \( e^+(u), f^+(u), h^+(u) \) belong to \( A_{\hbar, \eta}(\widehat{sl}_2) \). If the complex number \( u \) is inside the strip

\[
\Pi^- = \left\{ \frac{\hbar c}{4} < \text{Im} \, u < \frac{\hbar c}{4} + \frac{1}{\eta} \right\}
\]

then the elements

\[
e^-(u) = -e^+(u - i/\eta''), \quad f^-(u) = -f^+(u - i/\eta''), \quad h^-(u) = h^+(u - i/\eta'')
\]

also belong to \( A_{\hbar, \eta}(\widehat{sl}_2) \). Thus we can treat the integrals \( e^\pm(u), f^\pm(u), h^\pm(u) \) as generating functions of the elements of the algebra \( A_{\hbar, \eta}(\widehat{sl}_2) \), analytical in the strips \( \Pi^\pm \). We can state the following

**Proposition 4.** The generating functions \( e(u) = e^+(u), f(u) = f^+(u), h(u) = h^+(u) \) satisfy the commutation relations \((2.8) - (2.13) \) if \( \hat{\epsilon}_\lambda, \hat{f}_\lambda, \hat{t}_\lambda \) satisfy the relations \((1.4) - (1.7) \).

In order to prove this proposition we should use the Fourier transform calculations and fix in \((2.8) - (2.13) \) either \( \text{Im} \, u_1 < \text{Im} \, u_2 \) or \( \text{Im} \, u_1 > \text{Im} \, u_2 \).

The relations \((2.17) \) and analyticity of generation functions in the domains \( \Pi^\pm \) allow one to make an analytical continuation of the relations \((2.8) - (2.13) \) including all possible combinations of the generating integrals. For instance, from \((2.9) \) we have also \(( u = u_1 - u_2) \)

\[
\begin{align*}
\text{sh } \pi \eta (u + i \hbar) h^\pm(u_1) e^\pm(u_2) - \text{sh } \pi \eta (u - i \hbar) e^\pm(u_2) h^\pm(u_1) &= \text{sh} \{i \pi \eta h\} \{ h^\pm(u_1), e^\pm(u_1) \}, \\
\text{sh } \pi \eta (u + i \hbar + i \hbar c / 2) h^+(u_1) e^-(u_2) - \text{sh } \pi \eta (u - i \hbar + i \hbar c / 2) e^+(u_2) h^+(u_1) &= \text{sh} \{i \pi \eta h\} \{ h^+(u_1), e^+(u_1) \}, \\
\text{sh } \pi \eta (u + i \hbar - i \hbar c / 2) h^-(u_1) e^+(u_2) - \text{sh } \pi \eta (u - i \hbar - i \hbar c / 2) e^+(u_2) h^-(u_1) &= \text{sh} \{i \pi \eta h\} \{ h^-(u_1), e^-(u_1) \}.
\end{align*}
\]
Let now
\[ R^-(u) = \tau^-(u) R(u), \quad \tau^-(u) = \text{th} \left( \frac{\pi u}{2\hbar} \right) \]
and \( e^\pm(u), f^\pm(u), h^\pm(u) \) be the Gauss coordinates of the \( L \)-operators \( L^\pm(u) \):
\[
L^\pm(u) = \begin{pmatrix} 1 & f^\pm(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (k^\pm(u + i\hbar))^{-1} & 0 \\ 0 & k^\pm(u) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^\pm(u) & 1 \end{pmatrix}.
\]

One can prove in an analogous manner that the described above mixed relations for generating functions \( e^\pm(u), f^\pm(u), h^\pm(u) \) are equivalent to the following system of equations for the Gauss coordinates of the \( L \)-operators \( u = u_1 - u_2 \):
\[
R^+(u - i\hbar/2, \eta') L^+_1 (u_1, \eta) L^+_2 (u_2, \eta) = L^+_2 (u_2, \eta) L^+_1 (u_1, \eta) R^+(u + i\hbar/2, \eta),
\]
\[
R^\pm (u, \eta') L^\pm_1 (u_1, \eta) L^\pm_2 (u_2, \eta) = L^\pm_2 (u_2, \eta) L^\pm_1 (u_1, \eta) R^\pm (u, \eta).
\]

These equations can be also obtained by means of the formal analytical continuation of the relations (2.3).

The relation (2.17) and the involution (1.9) in the algebra \( A_{\hbar, \eta} (\mathfrak{sl}_2) \) can be obtained by means of the formal analytical continuation from the quasi-periodicity property of the \( R \)-matrices \( R^\pm (u, \eta) \).

Let us note that the formal generators \( \hat{e}_\lambda, \hat{f}_\lambda, \hat{h}_\lambda \) of the algebra \( A_{\hbar, \eta} (\mathfrak{sl}_2) \) can be expressed through their generating integrals using the inverse integral transform
\[
\hat{e}_\lambda = \pm \frac{\eta e^{\pm i\lambda \eta/4} (1 + e^{\pm \lambda/\eta})}{2 \sin \pi \eta h} \int_{\Gamma^\pm} du e^{-i\lambda u} e^\pm(u),
\]
\[
\hat{f}_\lambda = \pm \frac{\eta' e^{\mp i\lambda \eta/4} (1 + e^{\pm \lambda/\eta'})}{2 \sin \pi \eta' h} \int_{\Gamma^\pm} du e^{-i\lambda u} f^\pm(u),
\]
\[
\hat{h}_\lambda = -\frac{\eta e^{\pm i\lambda /2 \eta'}}{\sin \pi \eta h} \int_{\Gamma^\pm} du e^{-i\lambda u} h^\pm(u),
\]

where \( \Gamma^\pm \) are contours which go from \(-\infty\) to \(+\infty\) inside the strips \( \Pi^\pm \).

Using the relations (2.20) one can verify that the defining relations (1.1)–(1.7) for the algebra \( A_{\hbar, \eta} (\mathfrak{sl}_2) \) are equivalent to the relations (2.3)–(2.13) on generating functions of the algebra.

**Remarks.**

1. The formal algebra generated by Gauss coordinates \( e(u), f(u) \) and \( h(u) \) is not completely equivalent to the algebra of coefficients of \( L(u) \) with the relations (2.3)–(2.4) since \( h(u) \) is a quadratic expression of \( k_2(u) \). Naturally, one may consider the corresponding extension of \( A_{\hbar, \eta} (\mathfrak{sl}_2) \), which looks a bit more complicated. Nevertheless, the algebra \( A_{\hbar, \eta} (\mathfrak{sl}_2) \) is sufficient for the description of representations in which we are interested in.

2. Let \( \eta = 1/\xi \). The matrix
\[
S(\beta, \xi) = -(\sigma_z \otimes 1) R(\beta, 1/\xi) (1 \otimes \sigma_z)
\]
was obtained in [11] as an exact \( S \)-matrix of soliton-antisoliton scattering in Sine-Gordon model, where \( \xi \) is related to the coupling constant of the model and we should set \( h = \pi \) (we prefer to keep the free parameter \( h \) for the convenience of taking the classical limit [22]). This \( S \)-matrix satisfies the conditions of unitarity and crossing symmetry
\[
S(\beta, \xi) S(-\beta, \xi) = 1,
\]
\[(C \otimes \text{id}) S(\beta, \xi) (C \otimes \text{id}) = (S(i\pi - \beta, \xi))^t_1\]

with the charge conjugation matrix
\[
C = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Because of the relation \((2.21)\) the \(R\)-matrix \((2.1)\) satisfies the same properties of unitarity and crossing symmetry but with a different charge conjugation matrix
\[
\hat{C} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

Since the square of this matrix is equal to \(-1\) we have to use an unusual definition of the intertwining operators \((5.6)\) in order to have a possibility to identify them as Zamolodchikov–Faddeev operators for the Sine-Gordon model. We will discuss this point in more details in the last section.

### 2.3 The Yangian limit

As follows from the definition of the elements of the algebra \(A_{\hbar, \eta}(\widehat{sl}_2)\) \((1.8)\) and the generating functions \((2.14) - (2.15)\) each substrip of the strips \(\Pi^\pm\) defines a subalgebra of the algebra \(A_{\hbar, \eta}(\widehat{sl}_2)\). In terms of the Fourier components these subalgebras defined by different asymptotics of the functions \(\phi(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\})\) at \(\lambda_k, \mu_i, \nu_j \to \pm \infty\) in \((1.8)\).

Let us consider the substrips \(\Pi^\pm \subset \Pi^\pm\)
\[
\Pi^+ = \left\{ \frac{1}{2\eta} < \text{Im} u < -\frac{\hbar c}{4} \right\}, \quad \Pi^- = \left\{ \frac{\hbar c}{4} < \text{Im} u < \frac{1}{2\eta} \right\}
\]

and restrict the generating functions \(e^\pm(u), f^\pm(u), h^\pm(u)\) onto these strips. Then in the limit \(\eta \to 0\) these generating functions will be defined in the lower and the upper half-planes and the relations \((2.17)\) drops out. The expressions via formal generators turn into the Laplace transform (see Fig. 1.). For example,
\[
e^\pm(u)|_{\eta=0} = \pm \hbar \int_0^\infty d\lambda \ e^{\mp i\lambda(u \mp i\chi\hbar/4)} \hat{e}_{\mp, \lambda},
\]

and the defining relations \((2.18)\) turn into the defining relations of the central extended Yangian double \((17)\).

![Fig. 1.](image)

Although the commutation relation in terms of the generating functions for the limiting algebra \(A_{\hbar, 0}(\widehat{sl}_2)\) coincide with the commutation relations of the central extended Yangian double \(\widehat{DY}(sl_2)\) these two algebras should not be treated as isomorphic. For instance, the algebra \(\widehat{DY}(sl_2)\) has a discrete set of generators and \(A_{\hbar, 0}(\widehat{sl}_2)\) has a continuous family of them. As a consequence, they have different representation theories. It was also pointed out in \([20]\). See details in \([23]\).
2.4 Comultiplication structure

As we already mentioned in the introduction, the algebra $A_{\hbar,\eta}(\mathfrak{sl}_2)$ is not a Hopf algebra in the usual sense. Nevertheless we can assign the Hopf algebra structure to the family of the algebras $A_{\hbar,\eta}(\mathfrak{sl}_2)$ parametrized by the parameter $\eta$. Let us describe this Hopf structure. In this subsection it is convenient to use instead of the parameter $\eta$ its inverse

$$\xi = \frac{1}{\eta}.$$ 

Because of the relation (2.19) we can define the coproduct only for one type of the operator, say, $L^+(u)$. Consider the operation

$$\Delta c = c' + c'' = c \otimes 1 + 1 \otimes c,$$

$$\Delta' L^+(u, \xi) = L^+(u - i\hbar \xi''/4, \xi + \hbar \xi''/4) \otimes L^+(u + i\hbar c'/4, \xi)$$

or in components

$$\Delta L^+_{ij}(u, \xi) = \sum_{k=1}^2 L^+_{ik}(u + i\hbar \xi'/4, \xi) \otimes L^+_{jk}(u - i\hbar c'/4, \xi + \hbar \xi')$$

which defines the coassociative map

$$\Delta : A_{\hbar, \xi}(\mathfrak{sl}_2) \to A_{\hbar, \xi}(\mathfrak{sl}_2) \otimes A_{\hbar, \xi + \hbar \xi''}(\mathfrak{sl}_2)$$

on the family of algebras $A_{\hbar, \xi}(\mathfrak{sl}_2)$. The map $\Delta$ is a morphism of algebras, but it sends one algebra to a tensor product of two different algebras, which we do not identify. So we say that $A_{\hbar, \eta}(\mathfrak{sl}_2)$ form (over the parameter $\eta$) a Hopf family of algebras. Let us also note that because of the relation (2.19) the comultiplication of the $L$-operator $L^-(u)$ is given by

$$\Delta L^-_{ij}(u, \xi) = \sum_{k=1}^2 L^-_{ik}(u - i\hbar \xi'/4, \xi) \otimes L^-_{jk}(u + i\hbar c'/4, \xi + \hbar \xi').$$

In order to save notations below in this subsection and in the Appendix A we will understand the operators $k(u, \xi)$, $e(u, \xi)$, $f(u, \xi)$ as operators $k^+(u, \xi)$, $e^+(u, \xi)$, $f^+(u, \xi)$ and will write $u', \xi' = (u'', \xi'')$ in left (right) components of the tensor product and understand them as $u'' = u + i\hbar \xi''/4$, $\xi'' = 1/\eta$ ($u'' = u - i\hbar c'/4$, $\xi'' = \xi + \hbar c'$).

The comultiplications of the operators $e(u, \xi)$, $f(u, \xi)$ and $h(u, \xi)$ are

$$\Delta e(u, \xi) = e(u', \xi) \otimes 1 + \sum_{p=0}^{\infty} (-1)^p (f(u' - i\hbar, \xi'))^p h(u', \xi') \otimes (e(u'', \xi''))^{p+1}$$

$$\Delta f(u, \xi) = 1 \otimes f(u'', \xi'') + \sum_{p=0}^{\infty} (-1)^p (f(u', \xi'))^{p+1} \otimes h(u'', \xi'') (e(u'' - i\hbar, \xi''))^p$$

$$\Delta h(u, \xi) = \sum_{p=0}^{\infty} (-1)^p [p + 1]_\eta (f(u' - i\hbar, \xi'))^p h(u', \xi') \otimes h(u'', \xi'') (e(u'' - i\hbar, \xi''))^p$$

where we define

$$[p]_\eta = \frac{\sin \pi \hbar p}{\sin \pi \eta \hbar}.$$ 

The proof of these formulas is shifted to the Appendix A. Note that the involution $\iota$ is compatible with coalgebraic structure: $\Delta \iota = (\iota \otimes \iota) \Delta$. 

9
3 The Algebra $A_{\hbar, \eta}(\widehat{\mathfrak{sl}}_2)$ ($c = 0$)

Consider the formal algebra of the symbols $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$ and $S_0$ which satisfy the commutation relations

$$[\hat{e}_\lambda, \hat{f}_\mu] = \delta_{\lambda+\mu} ,$$

$$[S_0, \hat{e}_\mu] = \sin(\pi \eta \hbar) \tan(\pi \eta \hbar) \{\hat{h}_0, \hat{e}_\mu\} ,$$

$$[\hat{h}_\lambda, \hat{e}_\mu] = \frac{\{S_0, \hat{e}_{\lambda+\mu}\}}{\cos \pi \eta \hbar} +$$

$$+ \frac{\tan \pi \eta \hbar}{2 \pi \eta} \int_{-\infty}^{\infty} d\tau \left[ \coth \left( \frac{\tau}{2 \eta} \right) - \coth \left( \frac{\lambda + \tau}{2 \eta} \right) \right] \{\hat{h}_{\lambda+\tau}, \hat{e}_{\mu-\tau}\} ,$$

$$[S_0, \hat{f}_\mu] = -\sin(\pi \eta \hbar) \tan(\pi \eta \hbar) \{\hat{h}_0, \hat{f}_\mu\} ,$$

$$[\hat{h}_\lambda, \hat{f}_\mu] = -\frac{\{S_0, \hat{e}_{\lambda+\mu}\}}{\cos \pi \eta \hbar} -$$

$$- \frac{\tan \pi \eta \hbar}{2 \pi \eta} \int_{-\infty}^{\infty} d\tau \coth \left( \frac{\tau}{2 \eta} \right) \{\hat{e}_{\lambda+\tau}, \hat{e}_{\mu-\tau}\} ,$$

$$[\hat{e}_\lambda, \hat{e}_\mu] = \frac{\tan \pi \eta \hbar}{2 \pi \eta} \int_{-\infty}^{\infty} d\tau \coth \left( \frac{\tau}{2 \eta} \right) \{\hat{e}_{\lambda+\tau}, \hat{e}_{\mu-\tau}\} ,$$

$$[\hat{f}_\lambda, \hat{f}_\mu] = -\frac{\tan \pi \eta \hbar}{2 \pi \eta} \int_{-\infty}^{\infty} d\tau \coth \left( \frac{\tau}{2 \eta} \right) \{\hat{f}_{\lambda+\tau}, \hat{f}_{\mu-\tau}\} ,$$

$$[S_0, \hat{h}_\lambda] = 0 \ , \ [\hat{h}_\lambda, \hat{h}_\mu] = 0 \ ,$$

$$1 = S_0^2 + \sin^2(\pi \eta \hbar) \hat{h}_0^2 .$$

We can assign to these commutation relations the analogous ordering sense as we did it in the section 2.

Consider the free algebra $\mathcal{A}_0$ formed by the formal integrals of the type

$$\int_{-\infty}^{\infty} \prod_k d\lambda_k \prod_i d\mu_i \prod_j d\nu_j \phi_0(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\}) P(\{\hat{e}_{\lambda_k}\}; \{\hat{f}_{\mu_i}\}; \{\hat{h}_{\nu_j}\})$$

where $\phi_0(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\})$ is the $C$-number function of real variables $\lambda_k$, $\mu_i$ and $\nu_j$ which satisfy the conditions of analyticity:

$$\phi_0(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\}) \text{ is analytical in the strip } -\pi \eta < \text{Im}(\lambda_k, \mu_i) < \pi \eta \ \forall \ \lambda_k, \mu_i ,$$

$$\phi_0(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\}) \text{ is analytical in the strip } -2\pi \eta < \text{Im}(\nu_j) < 2\pi \eta \ \forall \ \nu_j ,$$

except the points $\nu_j = 0$ where this function has simple pole with respect to all $\nu_j$. The function $\phi_0(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\})$ has the asymptotics when $\text{Re} \lambda_k$, $\text{Re} \mu_i$, $\text{Re} \nu_j \to \pm \infty$:

$$\phi_0(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\}) < C e^{-|\text{Re} \lambda_k|} ,$$

$$\phi_0(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\}) < C e^{-|\text{Re} \mu_i|} ,$$

$$\phi_0(\{\lambda_k\}; \{\mu_i\}; \{\nu_j\}) < C e^{-|\text{Re} \nu_j|} ,$$

for some real positive $\alpha$, $\beta$, $\gamma$.

The algebra $A_{\hbar, \eta}(\widehat{\mathfrak{sl}}_2)$ at $c = 0$ is identified with $\mathcal{A}_0$ factorized by the ideal generated by the commutation relations (3.1)-(3.5).

Consider the formal integrals

$$e^+(u) = \frac{\sin \pi \eta \hbar}{\pi \eta} \int_{-\infty}^{\infty} d\lambda \ e^{\lambda u} \frac{\hat{e}_\lambda}{1 + e^{\lambda/\eta}} ,$$

(3.10)
\[ f^+(u) = \frac{\sin \pi \eta}{\pi \eta} \int_{-\infty}^{\infty} d\lambda \frac{e^{i\lambda u}}{1 + e^{i\lambda/\eta}}, \quad (3.11) \]
\[ h^+(u) = S_0 + \frac{\sin \pi \eta}{\pi \eta} \int_{-\infty}^{\infty} d\lambda \frac{e^{i\lambda u}}{1 - e^{i\lambda/\eta}}, \quad (3.12) \]
as generating integrals of the elements of the algebra \( A_{\hbar, \eta}(\widehat{sl}_2) \) at \( c = 0 \). We can prove that these generating functions are analytical in the strip \( \Pi^+ = \{-1/\eta < \text{Im} u < 0\} \) and satisfy the commutations relations \((3.8)-(3.13)\), where at \( c = 0 \) we should set \( \eta' = \eta \). In particular, in this case the generating functions \( h^+(u) \) commute.

The different presentation of the generating functions \( h^\pm(u) \) for \( c \neq 0 \) \((2.16)\) and for \( c = 0 \) \((3.12)\) follows from the analysis of their asymptotical behaviour. Indeed, the relations \((2.8)-(2.13)\) imply that
\[ e^+(u) \underset{\text{Re } u \to \pm \infty}{\sim} e^{-\pi \eta |u|}, \quad f^+(u) \underset{\text{Re } u \to \pm \infty}{\sim} e^{-\pi \eta |u|}, \quad h^+(u) \underset{\text{Re } u \to \pm \infty}{\sim} e^{-\pi(q-\eta')|u|}, \quad (3.13) \]
and constant, but different for \( +\infty \) and \( -\infty \) asymptotics of \( h^+(u) \) for \( c = 0 \):
\[ h^+(u) \underset{\text{Re } u \to \pm \infty}{\sim} h^+(\pm \infty) \equiv h_\pm = S_0 \mp i \sin(\pi \eta \hbar_0). \]
Such asymptotics can be achieved by the following Cauchy kernel presentations:
\[ e^+(u) = \frac{\text{sh} i \pi \eta}{2\pi} \int_{-\infty}^{\infty} dv \frac{E(\hat{v})}{\text{sh} \pi \eta(\hat{v} - u)}, \quad (3.14) \]
\[ f^+(u) = \frac{\text{sh} i \pi \eta}{2\pi} \int_{-\infty}^{\infty} dv \frac{F(\hat{v})}{\text{sh} \pi \eta(\hat{v} - u)}, \quad (3.15) \]
\[ h^+(u) = S_0 + \frac{\text{sh} i \pi \eta}{2\pi} \int_{-\infty}^{\infty} dv \frac{H(\hat{v}) \text{cth} \pi \eta(\hat{v} - u)}{H(\hat{v}) \text{cth} \pi \eta(\hat{v} - u)}, \quad (3.16) \]
where \( u \in \Pi^+ = \{-1/\eta < \text{Im} u < 0\} \) and \( \hat{v} \in \mathbb{R} \). Analogous formulas take place for the “−”-generating functions but with the spectral parameter \( u \) in the strip \( \Pi^- = \{0 < \text{Im} u < 1/\eta\} \). The presentations \((3.14)-(3.16)\) are equivalent to the deformed Laplace presentations \((3.10)-(3.12)\) if \( E(\hat{u}), F(\hat{u}) \) and \( H(\hat{u}) \) are Fourier transforms of \( \hat{e}_\lambda, \hat{f}_\lambda \) and \( \hat{h}_\lambda \):
\[ E(\hat{v}) = \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda \hat{v}} \hat{e}_\lambda, \quad F(\hat{v}) = \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda \hat{v}} \hat{f}_\lambda, \quad H(\hat{v}) = \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda \hat{v}} \hat{h}_\lambda. \]
For \( c \neq 0 \), due to \((3.13)\), we use instead of \((3.12)\) and \((3.16)\) the usual Fourier transform \((2.16)\). Note that in the limit \( c \to 0 \) we have the relation
\[ \hat{h}_\lambda = \hat{i}_\lambda \text{sh} \left( \frac{\lambda}{2\eta} \right), \quad (3.17) \]
for all \( \lambda \), so \( \hat{h}_0 \) is well defined whereas \( \hat{i}_0 \) tends to infinity when \( c \to 0 \); to the contrary, \( \hat{h}_0 \) from \((3.17)\) is zero for \( c \neq 0 \) while \( i_0 \neq 0 \) in this case.

The asymptotic generators \( h^\pm \) of \( c = 0 \) algebra \( A_{\hbar, \eta}(\widehat{sl}_2) \) have the following commutation relations with the generating functions \( e^+(u) \) and \( f^+(u) \):
\[ h^\pm e^+(u) h^\pm^{-1} = q^{\mp 2} e^+(u), \quad h^\pm f^+(u) h^\pm^{-1} = q^{\pm 2} f^+(u), \quad q = e^{i\pi \eta \hbar}. \]
and are primitive group-like elements: \( \Delta h^\pm = h^\pm \otimes h^\pm \). Thus their product
\[ h_+ h_- = S_0^2 + \sin^2(\pi \eta \hbar) h_0^2 \]
is central and group-like primitive. Due to this we can put it to be equal to 1. This kills unnecessary representations of level 0.
3.1 Evaluation homomorphism

Let \( e, f \) and \( h \) be the generators of the algebra \( U_{i\pi\eta\hbar}(sl_2) \):

\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = [h]_\eta = \frac{\sin \pi\eta \hbar}{\sin \pi \hbar}.
\]

The following proposition presents two descriptions of the evaluation homomorphism of \( c = 0 \) algebra \( A_{h,\eta}(\widehat{sl}_2) \) onto \( U_{i\pi\eta\hbar}(sl_2) \).

Proposition 5. The algebra \( A_{h,\eta}(\widehat{sl}_2) \) at \( c = 0 \) has the following evaluation homomorphism \( \mathcal{E}v_z \) onto \( U_q(sl_2) \), \( z \in \mathbb{C} \):

\[
\mathcal{E}v_z(S_0) = \cos(\pi \eta \hbar), \\
\mathcal{E}v_z(\hat{e}_\lambda) = e^{-i\lambda z} e^{-h\lambda(h-1)/2} e = e^{-i\lambda z} e^{-\hbar\lambda(h+1)/2}, \\
\mathcal{E}v_z(\hat{f}_\lambda) = e^{-i\lambda z} e^{-h\lambda(h+1)/2} f = e^{-i\lambda z} f e^{-\hbar\lambda(h-1)/2}, \\
\mathcal{E}v_z(\hat{h}_\lambda) = e^{-i\lambda z} e^{-h\lambda(h-1)/2} e f = e^{-i\lambda z} e^{-\hbar\lambda(h+1)/2} f e
\]

or, equivalently, \((u \in \Pi^+)\)

\[
\mathcal{E}v_z(e^+(u)) = -\frac{\sin i \pi \eta \hbar}{\sin \pi \eta (u - z + i \hbar(h + 1)/2)} e = -e^{-i \pi \eta \hbar} \frac{\sin i \pi \eta \hbar}{\sin \pi \eta (u - z + i \hbar(h + 1)/2)}, \\
\mathcal{E}v_z(f^+(u)) = -\frac{\sin i \pi \eta \hbar}{\sin \pi \eta (u - z + i \hbar(h + 1)/2)} f = -f \frac{\sin i \pi \eta \hbar}{\sin \pi \eta (u - z + i \hbar(h + 1)/2)}, \\
\mathcal{E}v_z(h^+(u)) = \cos(\pi \eta \hbar) - \sin i \pi \eta \hbar \left[ \cth \eta (u - z + i \hbar(h + 1)/2) e f - \cth \eta (u - z + i \hbar(h + 1)/2) f e \right].
\]

Let \( V_n \) be \((n+1)\)-dimensional \( U_{i\pi\eta\hbar}(sl_2)\)-module with a basis \( v_k, k = 0, 1, \ldots, n \) where the operators \( h, e \) and \( f \) act according to the rules

\[
h v_k = (2k - n) v_k, \quad e v_k = [k]_\eta v_{k-1}, \quad f v_k = [n - k]_\eta v_{k+1}.
\]

Due to the Proposition 5 we have an action \( \pi_{\alpha}(z) \) of the algebra \( A_{h,\eta}(\widehat{sl}_2) \) in the space \( V_{n,z} = V_n \). Note that the action of \( h^+(u) \) can be simplified in this case as

\[
\pi_{\alpha}(z) (h^+(u)) = \frac{\sin \pi \eta (u - z + i \hbar(n + 1)/2) \sin \pi \eta (u - z + i \hbar(n + 1)/2)}{\sin \pi \eta (u - z + i \hbar(h + 1)/2) \sin \pi \eta (u - z + i \hbar(h + 1)/2)}.
\]

(3.18)

The simplest two-dimensional evaluation representation \( \pi_1(z) \) of the algebra \( A_{h,\eta}(\widehat{sl}_2) \) on the space \( V_z = V_{1,z} \) (we have identified \( v_{+,z} = v_0 \) and \( v_{-,z} = v_1 \)) is

\[
e^+(u)v_{+,z} = 0, \quad f^+(u)v_{-,z} = 0, \quad (3.19)
\]

\[
e^+(u)v_{-,z} = -\frac{\sin i \pi \eta \hbar}{\sin \pi \eta (u - z)} v_{+,z}, \quad f^+(u)v_{+,z} = -\frac{\sin i \pi \eta \hbar}{\sin \pi \eta (u - z)} v_{-,z}, \quad (3.20)
\]

\[
h^+(u)v_{\pm,z} = \sin i \pi \eta \hbar \left[ \cth \pi \eta \hbar \mp \cth \eta (u - z) \right] v_{\pm,z} = \frac{\sin \pi \eta (u - z + \mp \hbar)}{\sin \pi \eta (u - z)} v_{\pm,z}.
\]

(3.21)

The action of “\( -\)\"-generating functions on the space \( V_z \) is given by the same formulas \( (3.19)-(3.21) \) but with \( u \in \Pi^- \). In \( L\)-operator’s terms, the representation described by \( (3.13)-(3.21) \) is equivalent to the standard one:

\[
\pi_1(z) L^\pm(u) = R^\pm(u - z, \eta).
\]

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For $c = 0$ in addition to the evaluation homomorphism we have, analogously to the case of $U_q(\widehat{sl}_2)$, an embedding of a subalgebra, isomorphic to $U_q(\widehat{sl}_2)$. Here $q = e^{i\pi\hbar}$, $|q| = 1$. This subalgebra is generated by the elements $S_0$, $e_0$, $f_0$ and $h_0$ and is given in the form of Sklyanin degenerated algebra [13]:

$$[\hat{e}_0, \hat{f}_0] = h_0, \quad [S_0, h_0] = 0, \quad S_0^2 + \sin^2(\pi\eta\hbar)h_0^2 = 1,$$

$$[S_0, \hat{e}_0] = \sin(\pi\eta\hbar)\tan(\pi\eta\hbar)\{h_0, \hat{e}_0\}, \quad [\hat{h}_0, \hat{e}_0] = \frac{\tan(\pi\eta\hbar)}{\sin(\pi\eta\hbar)}\{S_0, \hat{e}_0\},$$

$$[S_0, \hat{f}_0] = -\sin(\pi\eta\hbar)\tan(\pi\eta\hbar)\{h_0, \hat{f}_0\}, \quad [\hat{h}_0, \hat{f}_0] = -\frac{\tan(\pi\eta\hbar)}{\sin(\pi\eta\hbar)}\{S_0, \hat{f}_0\}.$$  

But, unlike to quantum affine case, this subalgebra is not a Hopf subalgebra. Moreover, this embedding is destroyed when $c \neq 0$. For instance, the generators $\hat{e}_0$ and $\hat{f}_0$ commute for $c \neq 0$.

In the rational limit $\eta \to 0$ this finite-dimensional subalgebra becomes $\widehat{sl}_2$ subalgebra of the Yangian double $\mathcal{A}_{h,0}(\widehat{sl}_2)$.

### 4 Current Realization of $\mathcal{A}_{h,\eta}(\widehat{sl}_2)$

In this section we would like to give another realization of the algebra $\mathcal{A}_{h,\eta}(\widehat{sl}_2)$ which is an analog of the current realization of the affine Lie algebras. The necessity of this realization follows from the construction of infinite-dimensional representations of the algebra $\mathcal{A}_{h,\eta}(\widehat{sl}_2)$ at $c \neq 0$ in terms of free fields. The generating functions $e^\pm(u)$ and $f^\pm(u)$ cannot be realized directly in terms of free fields, but only some combinations of them, called total currents, have a free field realization.

Let us define generating functions (total currents) $E(u)$ and $F(u)$ as formal Fourier transforms of the symbols $\hat{e}_\lambda$ and $\hat{f}_\lambda$:

$$E(u) = \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda u} \hat{e}_\lambda, \quad F(u) = \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda u} \hat{f}_\lambda, \quad u \in \mathbb{C}$$

and put

$$H^\pm(u) = -\frac{\hbar}{2} \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda u} \hat{e}_\lambda e^{\pi\lambda/2\eta'}, \quad h^\pm(u) = \frac{\sin(\pi\eta\hbar)}{\pi\eta\hbar} H^\pm(u).$$

We prove in this section that:

(i) The currents $E(u)$, $F(u)$ and $H^\pm(u)$ satisfy the relations (4.1)-(4.7):

$$H^+(u)H^-(v) = \frac{\sin(\pi\eta(u - v - i\hbar(1 - c/2)))}{\sin(\pi\eta(u - v + i\hbar(1 + c/2)))} H^-(v)H^+(u),$$

$$H^\pm(u)H^\pm(v) = \frac{\sin(\pi\eta(u - v - i\hbar(1 + c/4)))}{\sin(\pi\eta(u - v + i\hbar(1 + c/4)))} H^\pm(v)H^\pm(u),$$

$$H^\pm(u)E(v) = \frac{\sin(\pi\eta(u - v + i\hbar(1 + c/4)))}{\sin(\pi\eta(u - v - i\hbar(1 + c/4)))} E(v)H^\pm(u),$$

$$H^\pm(u)F(v) = \frac{\sin(\pi\eta(u - v + i\hbar(1 + c/4)))}{\sin(\pi\eta(u - v - i\hbar(1 + c/4)))} F(v)H^\pm(u),$$

$$E(u)E(v) = \frac{\sin(\pi\eta(u - v - i\hbar))}{\sin(\pi\eta(u - v + i\hbar))} E(v)E(u),$$

$$F(u)F(v) = \frac{\sin(\pi\eta(u - v + i\hbar))}{\sin(\pi\eta(u - v - i\hbar))} F(v)F(u),$$

$$[E(u), F(v)] = \frac{2\pi}{\hbar} \left[ \delta \left( u - v - \frac{i\hbar}{2} \right) H^+(u - \frac{i\hbar}{4}) - \delta \left( u - v + \frac{i\hbar}{4} \right) H^-(v - \frac{i\hbar}{4}) \right].$$

where the $\delta$-function is defined as

$$2\pi\delta(u) = \lim_{\varepsilon \to 0} \frac{1}{i} \left[ \frac{1}{u - i\varepsilon} - \frac{1}{u + i\varepsilon} \right] = \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda u}, \quad u \in \mathbb{R}. $$
The contours of integration could be deformed to a straight line being the boundary of one of the strips Ding-Frenkel relations [19]:

\[ e^\pm(u) = \sin \pi \eta h \int_C \frac{dv}{2\pi i} \frac{E(v)}{\text{sh} \pi \eta(u-v \pm i\eta/4)}, \quad (4.8) \]

\[ f^\pm(u) = \sin \pi \eta' h \int_{C'} \frac{dv}{2\pi i} \frac{F(v)}{\text{sh} \pi \eta'(u-v \mp i\eta/4)}, \quad (4.9) \]

where the contour \( C' \) goes from \(-\infty\) to \(+\infty\), the points \( u + i\eta h/4 + ik/\eta' (k \geq 0) \) are above the contour and the points \( u - i\eta h/4 - ik/\eta' (k \geq 0) \) are below the contour. The contour \( C \) also goes from \(-\infty\) to \(+\infty\) but the points \( u - i\eta h/4 + ik/\eta (k \geq 0) \) are above the contour and the points \( u + i\eta h/4 - ik/\eta (k \geq 0) \) are below the contour. In the considered case when \( \hbar > 0 \) the form of the contours is shown on the Fig. 2.

\[ C \quad \bullet \quad u + i\eta h/4 \quad C' \]
\[ -\infty \quad \bullet \quad u - i\eta h/4 \quad +\infty \]

Fig. 2.

The total currents \( E(u) \) and \( F(u) \) can be expressed through \( e^\pm(u) \) and \( f^\pm(u) \) by means of the Ding-Frenkel relations [19]:

\[ e^+(u - i\eta h/4) - e^-(u + i\eta h/4) = \frac{\sin \pi \hbar \eta}{\pi \eta} E(u), \quad (4.10) \]

\[ f^+(u + i\eta h/4) - f^-(u - i\eta h/4) = \frac{\sin \pi \eta' h}{\pi \eta'} F(u). \quad (4.11) \]

Let us start from (ii). One can see that in every particular case of \( e^+(u), e^-(u), f^+(u) \) and \( f^-(u) \) the contours of integration could be deformed to a straight line being the boundary of one of the strips \( \Pi^\pm \). Then the relations (2.14)–(2.15) are equivalent to (4.8)–(4.9) via Fourier transform. Moreover, the relations (4.8)–(4.9) say that \( E(u) \) and \( F(u) \) are the differences of boundary values of analytical functions \( e^\pm(u) \) and \( f^\pm(u) \) for the Riemann problem on a strip. Let us demonstrate this for \( e^+(u) \).

When the spectral parameter \( u \in \Pi^+ \) tends to the upper and the lower boundaries of the strip \( \Pi^+ \) we can obtain from (2.14):

\[ \lim_{\epsilon \to +0} (e^+(u_i - i\eta h/4 - i\epsilon) + e^+(u_i - i\eta h/4 + i\epsilon)) = \frac{\text{sh} \frac{i\pi \eta h}{\pi}}{i\pi \eta} E(u), \quad u_i \in \mathbb{R}, \]

\[ \lim_{\epsilon \to +0} (e^+(u_i - i\eta h/4 - i\epsilon) - e^+(u_i - i\eta h/4 + i\epsilon)) = \frac{\text{sh} \frac{i\pi \eta h}{\pi}}{\pi} \int_{-\infty}^{\infty} d\hat{v} \frac{E(\hat{v})}{\text{sh} \pi \eta (\hat{v} - u_i)}. \quad (4.12) \]

These relations are Sokhotsky-Plemely’s formulas associated with the Riemann problem on the strips of the width \( 1/\eta \). Summing the formulas (4.12) and using the analytical continuation with respect to the spectral parameter \( \hat{u} \) we obtain (4.8) where the contour of integration goes from \(-\infty\) to \(+\infty\) in such a way that point \( u + i\eta h/4 + ic \) is above the contour and the point \( u - i\eta h/4 \) is below. The same arguments applied to the generating function \( e^-(u) \) lead to (4.8) but with contour going between points \( u - i\eta h/4 \) and \( u - i\eta h/4 + ic \). Repeating this consideration for the generating functions \( f^+(u) \) and \( f^-(u) \) we obtain (4.9).

The relations (4.10)–(4.11) follow from (4.12). The commutation relations (4.1)–(4.7) are direct corollaries of (4.8) and (4.10)–(4.11).

**Remark.** Note that, as usual for affine algebras, the relations (4.10)–(4.11) should be understood in a sense of analytical continuation. For instance, if the argument of \( e^+(u) \) is inside of its domain of analyticity, the argument of \( e^-(u) \) does not. It means, in particular, that the total currents \( E(u) \) and \( F(u) \)
belong not the algebra $\mathcal{A}_{h,\eta}(\hat{sl}_2)$ but rather to some its analytical extension. Nevertheless, they act on highest weight representations, and the precise definition of the category of highest weight representations should be equivalent to the description of the proper analytical extension of the algebra $\mathcal{A}_{h,\eta}(\hat{sl}_2)$.

5 Representations of the Algebra $\mathcal{A}_{h,\eta}(\hat{sl}_2)$ at Level 1

5.1 Representation of the commutation relations by a free field

This section is devoted to the construction of an infinite dimensional representations of the algebra $\mathcal{A}_{h,\eta}(\hat{sl}_2)$. For simplicity we will consider the representation at level 1 ($c = 1$) since in this case only one free field is sufficient instead of three free fields for the general case. The generalization to an arbitrary level could be done using ideas developed in [24, 25]. Here and till the end of the paper we will understand everywhere $\eta'$ equal $\eta/(1 + \eta h)$.

Define bosons $a_{\lambda}, \lambda \in \mathbb{R}$ which satisfy the commutation relations [1, 16]:

$$[a_{\lambda}, a_{\mu}] = \frac{1}{\hbar^2} \frac{\text{sh}(h\lambda) \text{sh}(h\lambda/2)}{\lambda} \frac{\text{sh}(\lambda/2\eta)}{\text{sh}(\lambda/2\eta')} \delta(\lambda + \mu) = \alpha(\lambda) \delta(\lambda + \mu) .$$  (5.1)

Introduce also bosons $a'_{\lambda}$ related to the initial ones as follows:

$$a'_{\lambda} = \frac{\text{sh}(\lambda/2\eta')}{\text{sh}(\lambda/2\eta)} a_{\lambda} .$$

Consider the generating functions

$$E(u) = e^\gamma : \exp \left( \hbar \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda u} \frac{a'_{\lambda}}{\text{sh}(h\lambda/2)} \right) : ,$$  (5.2)

$$F(u) = e^\gamma : \exp \left( -\hbar \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda u} \frac{a_{\lambda}}{\text{sh}(h\lambda/2)} \right) : ,$$  (5.3)

$$H^\pm(u) = e^{-2\gamma} : E \left( u \pm \frac{i\hbar}{4} \right) F \left( u \mp \frac{i\hbar}{4} \right) : =$$

$$= : \exp \left( \mp 2\hbar \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda u} \frac{a_{\lambda} e^{\mp h\lambda/4}}{1 - e^{\pm h/\eta}} \right) : ,$$  (5.4)

where $\gamma$ is the Euler constant and the product of these generating functions is defined according to [16]

$$: \exp \left( \int_{-\infty}^{\infty} d\lambda \ g_1(\lambda) a_{\lambda} \right) : \cdot : \exp \left( \int_{-\infty}^{\infty} d\mu \ g_2(\mu) a_{\mu} \right) : =$$

$$= \exp \left( \int_{C} d\lambda \ \frac{\ln(-\lambda)}{2\pi i} \ \alpha(\lambda) g_1(\lambda) g_2(-\lambda) \right) : \exp \left( \int_{-\infty}^{\infty} d\lambda \ (g_1(\lambda) + g_2(\lambda)) a_{\lambda} \right) : .$$  (5.5)

The contour $\tilde{C}$ is shown on the Fig. 3.

![Fig. 3.](image_url)
5.2 Representation in a Fock space and twisted intertwining operators

The goal of this subsection is to interpret Zamolodchikov-Faddeev operators \([1, 16]\) following the ideology \([26, 2]\) as twisted intertwining operators for an infinite-dimensional representation of \(A_{h, \eta} (\hat{sl}_2)\). For a description of this infinite-dimensional representation we need a definition of a Fock space generated by continuous family of free bosons together with a construction of vertex operators. We do it below in a slightly more general setting.

Let \(a(\lambda)\) be a meromorphic function, regular for \(\lambda \in \mathbb{R}\) and satisfying the following conditions:

\[
a(\lambda) = -a(-\lambda),
\]

\[
a(\lambda) \sim a_0 \lambda, \quad \lambda \to 0, \quad a(\lambda) \sim e^{a' |\lambda|}, \quad \lambda \to \pm\infty.
\]

Let \(a, \lambda \in \mathbb{R}, \lambda \neq 0\) be free bosons which satisfy the commutation relations

\[
[a_\lambda, a_\mu] = a(\lambda) \delta(\lambda + \mu).
\]

We define a (right) Fock space \(H_{a(\lambda)}\) as follows. \(H_{a(\lambda)}\) is generated as a vector space by the expressions

\[
\langle \text{vac} | \int_{-\infty}^{0} f_n(\lambda_n) a_{\lambda_n} d\lambda_n \ldots \int_{0}^{0} f_1(\lambda_1) a_{\lambda_1} d\lambda_1 | \text{vac} \rangle,
\]

where the functions \(f_i(\lambda)\) satisfy the condition

\[
f_i(\lambda) < Ce^{(a'/2+\epsilon)\lambda}, \quad \lambda \to -\infty,
\]

for some \(\epsilon > 0\) and \(f_i(\lambda)\) are analytical functions in a neighbourhood of \(\mathbb{R}_+\) except \(\lambda = 0\), where they have a simple pole.

The left Fock space \(H^*_{a(\lambda)}\) is generated by the expressions

\[
\langle \text{vac} | \int_{0}^{\infty} g_1(\lambda_1) a_{\lambda_1} d\lambda_1 \ldots \int_{0}^{\infty} g_n(\lambda_n) a_{\lambda_n} d\lambda_n | \text{vac} \rangle,
\]

where the functions \(g_i(\lambda)\) satisfy the conditions

\[
g_i(\lambda) < Ce^{-(a'/2+\epsilon)\lambda}, \quad \lambda \to +\infty,
\]

for some \(\epsilon > 0\) and \(g_i(\lambda)\) are analytical functions in a neighbourhood of \(\mathbb{R}_-\) except \(\lambda = 0\), where they also have a simple pole.

The pairing \((,): H^*_{a(\lambda)} \otimes H_{a(\lambda)} \to \mathbb{C}\) is uniquely defined by the following prescriptions:

\[
(i) \quad (\langle \text{vac}, | | \text{vac} \rangle) = 1,
\]

\[
(ii) \quad (\langle \text{vac}, \int_{0}^{\infty} d\lambda g(\lambda) a_\lambda, \int_{0}^{\infty} d\mu f(\mu) a_\mu | \text{vac} \rangle) = \int_C \frac{d\lambda \ln(-\lambda)}{2\pi i} g(\lambda) f(-\lambda) a(\lambda),
\]

\[
(iii) \quad \text{the Wick theorem.}
\]

The pairing \((,): H^*_{a(\lambda)} \otimes H_{a(\lambda)} \to \mathbb{C}\) is uniquely defined by the following prescriptions:

\[
(i) \quad (\langle \text{vac}, | | \text{vac} \rangle) = 1,
\]

\[
(ii) \quad (\langle \text{vac}, \int_{0}^{\infty} d\lambda g(\lambda) a_\lambda, \int_{0}^{\infty} d\mu f(\mu) a_\mu | \text{vac} \rangle) = \int_C \frac{d\lambda \ln(-\lambda)}{2\pi i} g(\lambda) f(-\lambda) a(\lambda),
\]

\[
(iii) \quad \text{the Wick theorem.}
\]

Let the vacuums \(\langle \text{vac} \rangle\) and \(| \text{vac} \rangle\) satisfy the conditions

\[
a_\lambda \langle \text{vac} \rangle = 0, \quad \lambda > 0, \quad \langle \text{vac} | a_\lambda = 0, \quad \lambda < 0,
\]

and \(f(\lambda)\) be a function analytical in some neighbourhood of the real line with possible simple pole at \(\lambda = 0\) and which has the following asymptotical behaviour:

\[
f(\lambda) < e^{-(a'/2+\epsilon) t}, \quad \lambda \to \pm\infty
\]
for some $\epsilon > 0$. Then, by definition, the operator

$$F = : \exp \left( \int_{-\infty}^{+\infty} d\lambda \, f(\lambda) a_\lambda \right):$$

acts on the right Fock space $\mathcal{H}_{a(\lambda)}$ as follows. $F = F_- F_+$, where

$$F_- = \exp \left( \int_{-\infty}^{0} d\lambda \, f(\lambda) a_\lambda \right) \quad \text{and} \quad F_+ = \lim_{\epsilon \to +0} e^{\epsilon \ln \epsilon f(\cdot) a}, \exp \left( \int_{\epsilon}^{+\infty} d\lambda \, f(\lambda) a_\lambda \right).$$

The action of operator $F$ on the left Fock space $\mathcal{H}^*_{a(\lambda)}$ is defined via another decomposition: $F = \tilde{F}_- \tilde{F}_+$, where

$$\tilde{F}_+ = \exp \left( \int_{0}^{+\infty} d\lambda \, f(\lambda) a_\lambda \right) \quad \text{and} \quad \tilde{F}_- = \lim_{\epsilon \to +0} e^{\epsilon \ln \epsilon f(-\cdot) a}, \exp \left( \int_{-\infty}^{-\epsilon} d\lambda \, f(\lambda) a_\lambda \right).$$

These definitions imply the following statement:

**Proposition 7.**

(i) The defined above actions of the operator

$$F = : \exp \left( \int_{-\infty}^{+\infty} d\lambda \, f(\lambda) a_\lambda \right):$$

on the Fock spaces $\mathcal{H}$ and $\mathcal{H}^*$ are adjoint;

(ii) The product of the normally ordered operators satisfy the property $[F, \Phi(a)] = 0$.

Returning to level one representation of $A_{h,\eta}(\widehat{sl}_2)$ we choose $\mathcal{H} = \mathcal{H}_{a(\lambda)}$ for $a(\lambda)$ defined in (5.1):

$$a(\lambda) = \frac{1}{h} \frac{\text{sh}(\lambda) \text{sh}(\lambda/2)}{\lambda} \frac{\text{sh}(\lambda/2\eta)}{\text{sh}(\lambda/2\eta')}.$$

From the definition of the Fock space $\mathcal{H}$ and from the proposition 6 we have immediately the construction of a representation of $A_{h,\eta}(\widehat{sl}_2)$:

**Proposition 8.** The relations (5.2)–(5.4) define a highest weight level 1 representation of the algebra $A_{h,\eta}(\widehat{sl}_2)$ in the Fock space $\mathcal{H}$.

The highest weight property means that

$$\hat{e}_\lambda |\text{vac}\rangle = 0, \quad \hat{f}_\lambda |\text{vac}\rangle = 0, \quad \lambda > 0 \quad \text{and} \quad |\text{vac}\rangle \hat{e}_\lambda = 0, \quad |\text{vac}\rangle \hat{f}_\lambda = 0, \quad \lambda < 0.$$

Let us define the twisted intertwining operators

$$\Phi(z) : \mathcal{H} \rightarrow \mathcal{H} \otimes V_{z+i\hbar/2}, \quad \Phi^*(z) : \mathcal{H} \otimes V_{z+i\hbar/2} \rightarrow \mathcal{H},$$

$$\Psi^*(z) : V_{z+i\hbar/2} \otimes \mathcal{H} \rightarrow \mathcal{H}, \quad \Psi(z) : \mathcal{H} \rightarrow V_{z+i\hbar/2} \otimes \mathcal{H},$$

as those which commute with the action of the algebra $A_{h,\eta}(\widehat{sl}_2)$ up to the involution (1.9)

$$\Phi(z) \Delta(x) = \Delta(x) \Phi(z), \quad \Phi^*(z) \Delta(x) = \Delta(x) \Phi^*(z),$$

$$\Psi^*(z) \Delta(x) = \Delta(x) \Psi^*(z), \quad \Psi(z) \Delta(x) = \Delta(x) \Psi(z),$$

for arbitrary $x \in A_{h,\eta}(\widehat{sl}_2)$. In the definition of the intertwining operators $V_z$ denotes the two-dimensional evaluation module

$$V_z = V \otimes \mathbb{C}[e^{i\lambda z}] \quad \text{and} \quad V = \mathbb{C} v_+ \oplus \mathbb{C} v_-, \quad \lambda \in \mathbb{R}, \quad z \in \mathbb{C}.$$

The components of the intertwining operators are defined as follows:

$$\Phi(z)v = \Phi_+(z)v \otimes v_+ + \Phi_-(z)v \otimes v_-,$$

$$\Phi^*(z)(v \otimes v_+) = \Phi^*_+(z)v, \quad \Phi^*(z)(v \otimes v_-) = \Phi^*_-(z)v,$$

$$\Psi^*(z)(v_+ \otimes v) = \Psi^*_+(z)v, \quad \Psi^*(z)(v_- \otimes v) = \Psi^*_-(z)v.$$
where \( v \in \mathcal{H} \) and one should understand the components \( \Phi_\varepsilon(z), \Psi^*_\varepsilon(z), \varepsilon = \pm \) as generating functions, for example:

\[
\Phi(z)v = \sum_{\varepsilon = \pm} \int_{-\infty}^{\infty} d\lambda \Phi_{\varepsilon,\lambda} v \otimes v_\varepsilon e^{i\lambda(z+i\hbar/2)}.
\]

To find a free field realization of the intertwining operators we introduce the generating functions

\[
Z(z) = : \exp \left( -\hbar \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda z} \frac{a_\lambda}{\text{sh}(\hbar \lambda)} \right) : ,
\]

\[
Z'(z) = : \exp \left( \hbar \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda z} \frac{a_\lambda}{\text{sh}(\hbar \lambda)} \right) :
\]

Now we are ready to prove the following

**Proposition 9.** The components of the twisted intertwining operators have the free field realization

\[
\Psi^*_\varepsilon(z) = Z(z) , \quad (5.7)
\]

\[
\Psi^*_\varepsilon(z) = \int_{C'} \frac{du}{2\pi} e^\pi(\xi-u) \left[ (\xi)^{1/2} E(u)Z(z) + (\xi)^{-1/2} Z(E(u)) \right] , \quad (5.8)
\]

\[
\Psi_\nu(z) = \Psi^*_\nu z, \quad \nu = \pm , \quad (5.9)
\]

\[
\Phi_\varepsilon(z) = Z'(z) , \quad (5.10)
\]

\[
\Phi^*_\varepsilon(z) = \Phi^*_\varepsilon(z+i\hbar) , \quad \varepsilon = \pm , \quad (5.12)
\]

where the contours \( C \) and \( C' \) are the same as in (4.8) and (4.4).

To prove the proposition 6 we should use the first terms in the comultiplication formulas (2.24), (2.25) and (2.26) specified for the operators \( x = e^\pm(u), f^\pm(u) \) and \( h^\pm(u) \) and the action of these generating functions on the elements of the evaluation two-dimensional module (3.19)–(3.21). The result is the commutation relations of the components of the intertwining operators with the generating functions of the algebra \( \mathcal{A}_\hbar,\nu(s^1_2) \). For example, for the operators \( \Psi^*_\varepsilon(z) \) these defining equations are:

\[
h^\pm(u)\Psi^*_\varepsilon(z) = \frac{\text{sh} \pi \eta(u-z+i\hbar/2 \pm i\hbar/4)}{\text{sh} \pi \eta(u-z-i\hbar/2 \pm i\hbar/4)} \Psi^*_\varepsilon(z) h^\pm(u) , \quad (5.13)
\]

\[
0 = f^\pm(u) \Psi^*_\varepsilon(z) + \Psi^*_\varepsilon(z) f^\pm(u) , \quad (5.14)
\]

\[
\text{sh} i\pi \eta \Psi^*_\varepsilon(z) = \text{sh} \pi \eta(u-z-i\hbar/2 \pm i\hbar/4) e^\pm(u) \Psi^*_\varepsilon(z) + \text{sh} \pi \eta(u-z+i\hbar/2 \pm i\hbar/4) \Psi^*_\varepsilon(z) e^\pm(u) . \quad (5.15)
\]

Because of (4.10) and (4.11) from (5.14) and (5.15) follows that the operator \( \Psi^*_\varepsilon(z) \) anticommute with the generating function \( F(u) \) and has the commutation relation with \( E(u) \) as follows:

\[
\text{sh} \pi \eta(u-z-i\hbar/2) E(u) \Psi^*_\varepsilon(z) = -\text{sh} \pi \eta(u-z+i\hbar/2) E(u) \Psi^*_\varepsilon(z) , \quad (5.16)
\]

It is easy now to verify using formulas of the Appendix B that the generating function \( Z(z) \) satisfy these commutation relations with \( E(u) \), \( F(u) \) and also (5.13).

The representation of \( \Psi^*_\varepsilon(z) \) in integral form follows from (5.14) and (5.16). The analysis of the normal ordering relations of the generating functions \( E(u), Z(z), F(u) \) and \( Z'(z) \) shows that the contours \( C \) and \( C' \) in (5.8) and (5.11) coincide with those in (4.8) and (4.4).

Comparing the formulas (5.7)–(5.11) with the free field representation of the Zamolodchikov-Faddeev operators from (16) we conclude that these operators coincide with twisting intertwining operators. Therefore, they satisfy the Zamolodchikov-Faddeev algebra:

\[
\Psi^*_\varepsilon(z_1) \Psi^*_\varepsilon(z_2) = S_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3} (z_1 - z_2, \xi) \Psi^*_\varepsilon(z_2) \Psi^*_\varepsilon(z_1) , \quad (5.16)
\]
Define also the extended Fock space $\mathcal{H}$ where the can come back to the usual definition introducing additional operators equation (n)\(^\text{ents}\) of the intertwining (Zamolodchikov-Faddeev) operators. This relation can be treated as a quantum equation (5.23) being substituted into (5.8) and (5.11) yield the integral relation between compo-
$H$ and the subspaces $\mathcal{H}$.

In this subsection we follow a well known idea presented in [7, 10] in order to discuss unusual (twisted) definition of the intertwining operators (5.6). The reason for this definition lies in the absence of the zero $g$ and the constants $R$ -matrix given by (2.1) as follows:

\[ \Phi_{\varepsilon z}(z_2)\Phi_{\varepsilon z}(z_1) = \tilde{R}^\varepsilon_{z_2}(z_1 - z_2, \xi + \hbar)\Phi_{\varepsilon z}(z_1)\Phi_{\varepsilon z}(z_2), \quad (5.17) \]

\[ \Psi_{\varepsilon z}(z_1)\Phi_{\varepsilon z}(z_2) = \nu \varepsilon \tanh \left( \frac{\pi}{4} + \frac{i\pi(z_1 - z_2)}{2\hbar} \right)\Phi_{\varepsilon z}(z_2)\Psi_{\varepsilon z}(z_1), \quad (5.18) \]

\[ \Phi_{\varepsilon z}(z)\Phi_{\varepsilon z}^*(z) = g\delta_{z_1z_2} \text{id}, \quad (5.19) \]

\[ \Psi_{\varepsilon z}(z_1)\Psi_{\varepsilon z}^*(z_2) = \frac{g\delta_{z_1z_2} \text{id}}{z_1 - z_2} + o(z_1 - z_2), \quad (5.20) \]

where the $S$-matrix in (5.10) is given by (2.21), the $R$-matrix $\tilde{R}(z, \xi + \hbar)$ in (5.17) is related to the $R$-matrix given by (2.1) as follows:

\[ \tilde{R}(z, \xi + \hbar) = (\sigma_z \otimes 1)R(z, 1/(\xi + \hbar))(1 \otimes \sigma_z), \quad (5.21) \]

and the constants $g, g'$ are equal to:

\[ g = \frac{i\varepsilon^{-3\gamma/2\eta/|2\eta/\eta'\rangle \Gamma_2(2\hbar \mid 2\hbar; 1/\eta)\Gamma_2(2\hbar + 1/\eta | 2\hbar; 1/\eta)}{\Gamma(2\hbar | 2\hbar; 1/\eta)\Gamma(2\hbar + 1/\eta | 2\hbar; 1/\eta)}, \quad (5.22) \]

The proof of the commutation relations (5.14)-(5.21) can be found in [10] and is based on the formulas gathered in the Appendix B. In order to prove (5.18) and (5.20) one should use following operator identities \[ E(u) = e^{\gamma \left(Z \left(u + i\frac{\hbar}{2}\right)Z\left(u - i\frac{\hbar}{2}\right)\right)^{-1}}, \quad F(u) = e^{\gamma \left(Z' \left(u + i\frac{\hbar}{2}\right)Z'\left(u - i\frac{\hbar}{2}\right)\right)^{-1}}. \quad (5.23) \]

The identities (5.23) being substituted into (5.8) and (5.11) yield the integral relation between components of the intertwining (Zamolodchikov-Faddeev) operators. This relation can be treated as a quantum version of the relation between linearly independent solutions to the second order ordinary differential equation $(\partial^2 + u(z))\psi(z) = 0$.

### 5.3 Zero mode discussion

In this subsection we follow a well known idea presented in [7, 10] in order to discuss unusual (twisted) definition of the intertwining operators (5.6). The reason for this definition lies in the absence of the zero mode operator $(-1)^p$ in the bosonization of the algebra $A_\hbar(\mathfrak{sl}_2)$ and twisted intertwining operators. We can come back to the usual definition introducing additional operators $P$ and $I$ such that

\[ [P, a_\lambda] = [I, a_\lambda] = 0, \quad PI = iIP \quad \text{and} \quad I^4 = 1. \]

Define also the extended Fock space \[ \mathcal{H} = \mathcal{H} \otimes \mathbb{C}[Z/4Z] = \mathcal{H}_0 \oplus \mathcal{H}_1 \]

and the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ formed by the elements

\[ \mathcal{H}_0 = \mathbb{C}[v \otimes 1] \oplus \mathbb{C}[v \otimes I^2], \quad \mathcal{H}_1 = \mathbb{C}[v \otimes I] \oplus \mathbb{C}[v \otimes I^3], \quad v \in \mathcal{H}. \]

The generating functions of the currents and the intertwining operators are modified as follows:

\[ \tilde{E}(u) = E(u) \cdot I^2, \quad \tilde{F}(u) = F(u) \cdot I^2, \quad \tilde{H}^\pm(u) = H^\pm(u), \]

\[ \tilde{\Psi}_\pm(z) = \Psi_\pm(z) \cdot I^\pm P, \quad \tilde{\Phi}_\pm(z) = \Phi_\pm(z) \cdot I^\pm P. \]

1The double $\Gamma$-function $\Gamma_2(x \mid \omega_1; \omega_2)$ is defined in the Appendix B.
It is clear now that the commutation relations of the modified intertwining operators with elements of the algebra will be usual, for example:

\[ \Phi(z)x = \Delta(x)\Phi(z), \quad x \in A_{h,\eta}(\tilde{sl}_2), \]

since the action of the operator \( P \) on the elements of the algebra coincide with the action of the involution \( \xi \):

\[ PX^{-1} = P^{-1}xP = \iota(x), \quad \forall x \in A_{h,\eta}(\tilde{sl}_2). \]

The subspaces \( H_0 \) and \( H_1 \) become irreducible with respect to the action of the algebra and the operators \( \Phi(z) \), etc. intertwine these subspaces. Since the known physical models for which the algebra \( A_{h,\eta}(\tilde{sl}_2) \) serves as the algebra of dynamical symmetries have single vacuum states this mathematical construction of two irreducible Fock spaces is unnecessary and this unnesseryness explains the absence of the zero mode operators in the bosonization of the massive models of the quantum field theory.

### 5.4 Miki’s Formulas

We would like to demonstrate now that bosonized expressions for the intertwining operators are in accordance with the \( L \)-operator description of the algebra \( A_{h,\eta}(\tilde{sl}_2) \). The way to do it is to use Miki’s formulas [21]. Consider the \( 2 \times 2 \) operator valued matrices acting in the Fock space \( \mathcal{H}(\varepsilon, \nu = \pm) \)

\[ L_{\varepsilon\nu}^+(u) = \nu \varepsilon \sqrt{\frac{2\hbar e^\gamma}{\pi}} \Psi^*_\nu \left( u - \frac{i\hbar}{4} \right) \Phi_\varepsilon \left( u - \frac{3i\hbar}{4} \right), \quad L_{\varepsilon\nu}^-(u) = \sqrt{\frac{2\hbar e^\gamma}{\pi}} \Phi^*_\varepsilon \left( u - \frac{i\hbar}{4} \right) \Psi_\nu \left( u - \frac{3i\hbar}{4} \right). \quad (5.24) \]

Now it is easy to show that so defined \( L \)-operators satisfy the commutation relations (2.18) if the operators \( \Phi(u) \) and \( \Psi^*(u) \) satisfy the commutation relations of the Zamolodchikov-Faddeev algebra (5.16)–(5.18).

Let us note first that there are many possibilities to choose initial parameters \( \eta = 1/\xi \) and \( \hbar \) in the definition of the algebra \( A_{h,\eta}(\tilde{sl}_2) \). For instance, it is clear from above analysis that the properties of the algebra \( A_{h,\eta}(\tilde{sl}_2) \) or its representations change drastically when the parameter \( \xi = 1/\eta = r\hbar \), where \( r \) is some rational number. The commutation relations (2.8)–(2.13) show that in this case a smaller factoralgebra of \( A_{h,\eta}(\tilde{sl}_2) \) could be defined. It is naturally to assume that this factoralgebra serves the symmetries of \( \Phi_{[1,3]} \)-perturbations of the minimal models of conformal field theories.

### 6 Discussion

To conclude this paper we would like to mention some open problems which, to our opinion, deserve further investigation.

Let us note first that there are many possibilities to choose initial parameters \( \eta = 1/\xi \) and \( \hbar \) in the definition of the algebra \( A_{h,\eta}(\tilde{sl}_2) \). For instance, it is clear from above analysis that the properties of the algebra \( A_{h,\eta}(\tilde{sl}_2) \) or its representations change drastically when the parameter \( \xi = 1/\eta = r\hbar \), where \( r \) is some rational number. The commutation relations (2.8)–(2.13) show that in this case a smaller factoralgebra of \( A_{h,\eta}(\tilde{sl}_2) \) could be defined. It is naturally to assume that this factoralgebra serves the symmetries of \( \Phi_{[1,3]} \)-perturbations of the minimal models of conformal field theories.
It is known from the theory of the Sine-Gordon model that if the parameter $\xi < \hbar$ than the spectrum of the model possesses scalar particles, so called breathers. Our considerations were based on the assumption that $\xi > \hbar$ and it is an open question to investigate the algebra $\mathcal{A}_{h,\nu}(\hat{sl}_2)$ in the regime $\xi < h$ and to apply its representation theory to the Sine-Gordon model in the breather’s regime. We think that the algebra $\mathcal{A}_{h,\nu}(\hat{sl}_2)$, $\nu = 1, 2, \ldots$ is related to the restricted Sine-gordon model in the reflectionless points and representation theory of this algebra can be used for the group-theoretical interpretation of the results obtained recently in [24]. In particular, it is interesting to investigate the simplest case $\xi = h$ which should correspond to the free fermion point of the Sine-Gordon model.

It is also natural from algebraic point of view to put the value of deformation parameter $\hbar$ to be pure imaginary instead of positive real. It could correspond to the Sinh-Gordon theory. Surely, one can also try to apply the known technique [24, 25] for studying $c > 1$ integer level integrable representations of the algebra $\mathcal{A}_{h,\nu}(\hat{sl}_2)$ and to find out possible physical applications.

As we have shown in the Section 2, the definition of the algebra $\mathcal{A}_{h,\nu}(\hat{sl}_2)$ cannot be done in purely algebraical terms. So far the corresponding analytical language should be developed for its representations. It was partially done in Section 6 for the level one Fock space. Nevertheless, the rigorous mathematical description of the space of representation is far from completeness. One needs more detailed topological description of the space, the precise definition of the trace, making the calculations in [1, 16, 20] to be rigorous, the investigation of the irreducibility and so on. Moreover, it would be nice to have an axiomatical description of the category of highest weight representations.

The analysis of the defining commutation relations of the algebra $\mathcal{A}_{h,\nu}(\hat{sl}_2)$ demonstrate that the algebra in question possesses the rich structure of automorphisms. For example, if we allow parameters $\lambda, \mu$ in (1.2)–(1.7) to be complex then these commutation relations can be rewritten in the form of difference commutation relations without integrals in the r.h.s.

The algebra $\mathcal{A}_{h,\nu}(\hat{sl}_2)$ is not a Hopf algebra and even for $c = 0$, when it becomes a Hopf algebra, it does not have a structure of the quantum double [28], as well as its classical counterpart [22]. The double structure can be reconstructed in the Yangian limit $\eta \to 0$ when the algebra $\mathcal{A}_{h,0}(\hat{sl}_2)$ becomes the central extended Yangian double. The representation theory of the latter algebra have been investigated in [4, 23, 18, 20] using two alternative possibilities related to the Riemann problems on the circle and on the line. It is interesting to understand what structure replaces the double structure in $\mathcal{A}_{h,\nu}(\hat{sl}_2)$; whether there exists an analog of the universal $R$-matrix.

It is also interesting to formulate the quantum Sugawara construction corresponding to the algebra $\mathcal{A}_{h,\nu}(\hat{sl}_2)$. The scaling quantum Virasoro algebra can be obtained from the papers [29, 30, 31, 22] where two-parameter deformation of the Virasoro algebra corresponding to the algebra $\mathcal{A}_{\eta,p}(\hat{sl}_2)$ has been investigated.

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### Appendix A. Consistency of Comultiplication Formulas

To prove that the comultiplication formulas (2.22) are in accordance with the commutation relations of the algebra $\mathcal{A}_{h,\nu}(\hat{sl}_2)$ we have to check that the relations (here $L(u) = L^+(u)$)

\[ R^+(u_1 - u_2, \xi + \hbar \Delta c) \Delta L_1(u_1) \Delta L_2(u_2) = \Delta L(u_2) \Delta L(u_1) R^+(u_1 - u_2, \xi), \]

\[ \Delta (q\text{-det} L(u)) = \Delta (q\text{-det} L(u)) \otimes \Delta (q\text{-det} L(u)) \]
follow from (2.3) and the definition of the quantum determinant (2.3). First of all we rewrite the equation (2.3) in components and the comultiplication formulas (2.23) using the short notations. We have

\[ R_{im, kp}(u_1 - u_2, \xi + \hbar c)L_{mj}(u_1, \xi)L_{pl}(u_2, \xi) = L_{kj}(u_2, \xi)L_{ir}(u_1, \xi)R_{rj, ql}(u_1 - u_2, \xi) \]

and

\[ \Delta L_{mj}(u, \xi) = L_{fj}(u', \xi')L_{mj}(u'', \xi'') , \]

where, as it was in the subsection 3.3, prime (double prime) denotes that the corresponding L-operator or its component is in the first (second) component of the tensor product and \( u' = u + i\hbar c'/4, \xi' = \xi + i\hbar c' \). The summation over repeating indexes is always supposed.

\[ R_{im, kp}(u_1 - u_2, \xi + \hbar c + \hbar c') = R_{im, kp}(u_1 - u_2, \xi + \hbar c + \hbar c')L_{fj}(u_1', \xi)L_{mj}(u_1'' , \xi + \hbar c')L_{fj}(u_2', \xi)L_{pl}(u_2'', \xi + \hbar c') \]

\[ = R_{rj, ql}(u_1 - u_2, \xi)L_{qj}(u_1', \xi)L_{rj}(u_1'', \xi)L_{kq}(u_2', \xi + \hbar c')L_{ir}(u_2'', \xi + \hbar c') \]

The primitiveness of the coproduct of the quantum determinant can be proved easily using (A.3), (A.4) and the formula equivalent to (2.3)

\[ \text{q-det} L(u) = D(u - i\hbar)A(u) - C(u - i\hbar)B(u) . \]

Now we are in position to prove the formulas (2.24)–(2.26). The simplest comultiplication relation which follows from (2.22) is

\[ \Delta k(u, \xi) = k(u', \xi') \otimes k(u'', \xi'') + f(u', \xi')k(u', \xi') \otimes k(u'', \xi'')e(u'', \xi'') \]

\[ = [1 \otimes 1 + f(u', \xi') \otimes e(u'' - i\hbar, \xi'')] k(u', \xi') \otimes k(u'', \xi'') \]

\[ = k(u', \xi') \otimes k(u'', \xi'') [1 \otimes 1 + f(u' - i\hbar, \xi') \otimes e(u', \xi'')] . \]

The equivalent form of comultiplication of the operators \( k^\pm(u) \) follows from the operator identities

\[ k(u, \xi)e(u, \xi) - e(u - i\hbar, \xi)k(u, \xi) = 0 , \]

\[ k(u, \xi)f(u - i\hbar, \xi) - f(u, \xi)k(u, \xi) = 0 , \]

which are consequences of the commutation relations (2.3) at the critical point \( u_1 - u_2 = i\hbar \).

Formulas (2.24) and (2.25) easily follows from (2.23) and (A.1), (A.2). The proof of (2.26) is more involved. It follows from the chain of identities

\[ \Delta h(u) = \sum_{p, p' = 0}^{\infty} (-1)^{p + p'} f^{p + p'}(u' - i\hbar)h(u') \otimes k^{-1}(u'')e^p(u'' - i\hbar)k^{-1}(u'' + i\hbar)e^{p'}(u'') \]

\[ = \sum_{p, p' = 0}^{\infty} (-1)^{p + p'} f^{p + p'}(u' - i\hbar)h(u') \otimes k^{-1}(u'')\left([2]_q e^{(u'' - i\hbar)} - e(u'')\right)^p e^{p'}(u'') \]

\[ = \sum_{p = 0}^{\infty} (-1)^{p} f^{p}(u' - i\hbar)h(u') \otimes k^{-1}(u'')\sum_{k = 0}^{p} \left([2]_q e^{(u'' - i\hbar)} - e(u'')\right)^k e^{p-k}(u'') \]

\[ = \sum_{p = 0}^{\infty} (-1)^{p[p + 1]} f^{p}(u' - i\hbar)h(u') \otimes k^{-1}(u'')e^{(u'' - i\hbar)} . \]
Using the formulas (A.1) and (A.2) we can obtain

\[ k(u + i\hbar)e(u - i\hbar)k^{-1}(u + i\hbar) = [2]_\eta e(u - i\hbar) - e(u) \]

and the combinatorial identity

\[ \sum_{k=0}^p ([2]_\eta e(u - i\hbar) - e(u))^k e^{p-k}(u) = [p + 1]_\eta e^p(u - i\hbar) \]

which follows by induction from (2.11).

There is another way to verify the concordance of the formulas (2.23). The comultiplication formulas for the currents follow from comultiplication of \( L \)-operator entries \( L_{12}(u, \xi), L_{21}(u, \xi), L_{22}(u, \xi) \) given by (2.23). The essential part of this calculation is the comultiplication of the inverse operator \( (k(u))^{-1} \). Using the formulas (A.1) and (A.3) we can obtain

\[ \Delta (k(u, \xi))^{-1} = \sum_{p=0}^\infty (-1)^p (f(u' - i\hbar, \xi'))^p (k(u', \xi'))^{-1} \otimes (k(u'', \xi''))^{-1} (e(u'' - i\hbar, \xi''))^p . \]  \[ (A.5) \]

The comultiplication of the entry \( L_{11}(u, \xi) \) also defines the comultiplication of the operator \( (k(u))^{-1} \) so we should prove that these two comultiplication formulas lead to the same result. After some simple algebra we have

\[ \Delta k^{-1}(u + i\hbar) = (k^{-1}(u' + i\hbar) + f(u')k(u')e(u')) \otimes (k^{-1}(u'' + i\hbar) + f(u'')k(u'')e(u'')) + k(u')e(u') \otimes f(u'')k(u'') - \Delta f(u)\Delta k(u)\Delta e(u) \]

\[ = \sum_{p=0}^\infty (-1)^p (f(u'))^p k^{-1}(u' + i\hbar) \otimes k^{-1}(u'' + i\hbar) (e(u''))^p . \]

The last line obviously coincide with (A.5) after shifting \( z \to z - i\hbar \).

### Appendix B. Normal Ordering Relations

The relations below are based on the formulas which can be found in [33, 54]

\[ \int_C \frac{d\lambda \ln(-\lambda)}{2\pi i\lambda} \frac{e^{-x\lambda}}{1 - e^{-\lambda/\eta}} = \ln \Gamma(\eta x) + \left( \eta x - \frac{1}{2} \right) (\gamma - \ln \eta) - \frac{1}{2} \ln 2\pi , \]

\[ \int_C \frac{d\lambda \ln(-\lambda)}{2\pi i\lambda} \frac{e^{-x\lambda}}{(1 - e^{-\lambda/\eta})(1 - e^{-\lambda/\omega})} = \ln \Gamma_2(x | \omega_1, \omega_2) - \frac{\gamma}{2}B_{2,2}(x | \omega_1; \omega_2) , \]

where \( B_{2,2}(x | \omega_1; \omega_2) \) is the double Bernulli polynomial of the second order

\[ B_{2,2}(x | \omega_1; \omega_2) = \frac{1}{\omega_1; \omega_2} \left[ x^2 - x(\omega_1 + \omega_2) + \frac{\omega_1^2 + 3\omega_1\omega_2 + \omega_2^2}{6} \right] . \]

Using these integral representations of the ordinary and double \( \Gamma \)-functions and the definition of the product \( [5, 3] \) we can calculate:

\[ Z(z)E(u) = \frac{\Gamma(i\eta(u - z) - \eta\hbar/2)}{\Gamma(1 + i\eta(u - z) + \eta\hbar/2)} :Z(z)E(u): \]

\[ = \frac{\Gamma(\eta(u - z) - \eta\hbar/2)}{(e^{\eta/\hbar})^{u'/\eta}} , \quad \text{Im}(u - z) < \frac{\hbar}{2} . \]
\[ E(u)Z(z) = \frac{\Gamma(-i\eta(u-z)-\eta^2/2)}{\Gamma(1-i\eta(u-z)+\eta^2/2)} \cdot \frac{Z(z)E(u)}{e^{\eta^2/\eta'}} \cdot \text{Im}(u-z) > \frac{\hbar}{2}, \]

\[ Z(z)F(u) = ie^\gamma(u-z):Z(z)F(u):, \quad \text{Im}(u-z) < 0, \]

\[ F(u)Z(z) = -ie^\gamma(u-z):F(u)Z(z):, \quad \text{Im}(u-z) > 0, \]

\[ Z'(z)F(u) = \frac{\Gamma(\eta(u-z)+\eta^2/2)}{\Gamma(1+\eta(u-z)-\eta^2/2)} \cdot \frac{Z'(z)F(u)}{e^{\eta^2/\eta'}} \cdot \text{Im}(u-z) < \frac{\hbar}{2}, \]

\[ F(u)Z'(z) = \frac{\Gamma(-i\eta(u-z)+\eta^2/2)}{\Gamma(1-i\eta(u-z)-\eta^2/2)} \cdot \frac{Z'(z)F(u)}{e^{\eta^2/\eta'}} \cdot \text{Im}(u-z) > -\frac{\hbar}{2}, \]

\[ Z'(z)E(u) = ie^\gamma(u-z):Z'(z)E(u):, \quad \text{Im}(u-z) < 0, \]

\[ E(u)Z'(z) = -ie^\gamma(u-z):E(u)Z'(z):, \quad \text{Im}(u-z) > 0, \]

\[ Z(z_1)Z'(z_2) = \frac{1}{\sqrt{2\hbar e^{\eta^2/\eta}}} \cdot \frac{\Gamma(\frac{3}{2}+\frac{i(z_2-z_1)}{2\hbar})}{\Gamma(\frac{3}{2}+\frac{i(z_2-z_1)}{2\hbar})} \cdot Z(z_1)Z'(z_2):, \quad \text{Im}(z_1-z_2) > -\frac{\hbar}{2}, \]

where the functions \( g(z) \) and \( g'(z) \) are

\[ g(z) = \exp\left(-\int_C \frac{d\lambda}{2\pi i} \ln(-\lambda) \frac{(1-e^{-\lambda(z+h)})}{(1-e^{-\lambda})e^{(-\lambda)}(1-e^{-2\lambda})} e^{i\lambda z}\right), \]

\( = e^{\gamma^2/2\eta} \frac{\Gamma_2(h-i\eta|2\hbar;1/\eta)\Gamma_2(h+1/\eta-i\eta|2\hbar;1/\eta)}{\Gamma_2(h-i\eta|2\hbar;1/\eta)\Gamma_2(h+1/\eta-i\eta|2\hbar;1/\eta)} \),

\[ g'(z) = \int_C \frac{d\lambda}{2\pi i} \ln(-\lambda) \frac{(1-e^{-\lambda(z+h)})}{(1-e^{-\lambda})e^{(-\lambda)}(1-e^{-2\lambda})} e^{i\lambda z}, \]

\( = e^{\gamma^2/2\eta} \frac{\Gamma_2(h-i\eta|2\hbar;1/\eta)\Gamma_2(1/\eta'-i\eta|2\hbar;1/\eta')}{\Gamma_2(h-i\eta|2\hbar;1/\eta)\Gamma_2(1/\eta'-i\eta|2\hbar;1/\eta')} \).

As usually the normal ordering of all operators is calculated in the regions specified above and then analytically continued to all possible values of the spectral parameters.

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