Valley Bifurcation in an $O(3)$ $\sigma$ Model: Implications for High-Energy Baryon Number Violation

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The valley method for computing the total high-energy anomalous cross section $\sigma_{\text{anom}}$ is the extension of the optical theorem to the case of instanton-antiinstanton backgrounds. As a toy model for baryon number violation in Electroweak theory, we consider a version of the $O(3)$ $\sigma$ model in which the conformal invariance is broken perturbatively. We show that at a critical energy the saddle-point values of the instanton size and instanton-antiinstanton separation bifurcate into complex conjugate pairs. This nonanalytic behavior signals the breakdown of the valley method at an energy where $\sigma_{\text{anom}}$ is still exponentially suppressed.

February 1993

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1. Introduction

Despite intense theoretical effort, the riddle of high-energy baryon number violation remains unsolved nearly four years after the original calculations of Ringwald and Espinosa. The phenomenon is more or less the same in 2-dimensional systems such as the abelian Higgs model, or the $O(3)$ $\sigma$ model which we focus on here, as it is in 4 dimensions in Weinberg-Salam. In each case, one calculates for the inclusive anomalous $2 \to \text{many}$ cross section:

$$\sigma_{\text{anom}} \sim \exp \left( 2S_{\text{cl}} \cdot F_{\text{hg}}(E/E_s) \right),$$

(1)

neglecting sub-exponential effects, which vary much more slowly with energy. Here $S_{\text{cl}}$ is the action of a single (small) instanton, and $F_{\text{hg}}$, the so-called "holy grail function," is a rising function of energy measured in units of a characteristic scale $E_s$ of order the sphaleron mass. $F_{\text{hg}}$ has the general form

$$F_{\text{hg}}(E/E_s) = -1 + c_1 (E/E_s)^{k_1} + (c_2 + l_2 \log(E/E_s)) \cdot (E/E_s)^{k_2} + \cdots.$$  

(2)

Only the constants $c_i$, $l_i$ and $k_i$ are model-dependent. In Weinberg-Salam, $k_1 = 4/3$ with subsequent $k_i$ increasing by $2/3$, while in the $O(3)$ $\sigma$ model $k_1 = 1$ with subsequent $k_i$ increasing by unity.

The riddle in all these models is: Does the holy grail function rise close enough to zero that the exponential suppression is lost and $\sigma_{\text{anom}}$ becomes observable? A closely related question is: What is the mechanism that keeps $F_{\text{hg}}$ from becoming positive, yielding an exponentially large $\sigma_{\text{anom}}$ in flagrant violation of the unitarity bounds of quantum field theory?

A useful approximate* tool for examining these issues is the valley method of Balitsky and Yung, adapted to high-energy scattering by Khoze and Ringwald. In the Khoze-Ringwald approach, $\sigma_{\text{anom}}$ is extracted via the optical theorem as the imaginary part of a nonanomalous forward $2 \to 2$ amplitude in which the intermediate state contains a distorted instanton-antiinstanton ($I\bar{I}$) pair. The set of such $I\bar{I}$ configurations satisfying the appropriately constrained Euler-Lagrange equation ("valley equation") is known

* We comment on multi-instanton and initial-state corrections, ignored in our treatment, at the end.
as the valley. In both the $O(3)$ $\sigma$ model and in Weinberg-Salam, $F_{hg}$ from Eq. (1) is then approximated as the sum of three terms:

$$\sigma_{\text{anom}} \sim \text{Im} \int dR d\rho \exp \{ER - S_{\text{conf}}(R/\rho) - S_{\text{cl}} \cdot 2\rho^2\mu^2\}, \quad (3)$$

where the collective coordinate integrations over $\rho$ (the (anti)instanton size) and $R$ (the $I\bar{I}$ separation) are to be evaluated in saddle-point approximation. Taking the imaginary part in Eq. (3) strips off the factor of $i$ that enters by analytic continuation from the “wrong-sign” Gaussian integral $\int dx e^{+ax^2}$ implicit in the small-fluctuations determinant about the saddle point.

The three terms in Eq. (3) have the following interpretation. The first term on the right-hand side is the proper Euclidean continuation of the phase factor due to pumping energy $E$ into the system through the initial-state quanta. The second and third terms represent a splitting-up of the valley action into a classically conformally invariant piece $S_{\text{conf}}$ depending only on the dimensionless ratio $R/\rho$, and an ad-hoc conformal breaking term $2\rho^2\mu^2$, where $\mu$ is a characteristic mass of order $g^2E_s$ (e.g., $\mu = \frac{1}{2}M_W$ in Weinberg-Salam), and the factor of 2 reflects the identical contributions of the $I$ and the $\bar{I}$. For example, in Weinberg-Salam, $S_{\text{conf}}$ comes from the pure Yang-Mills part of the action, whereas the last term (which is only justified when $\rho \ll M_W^{-1}$, see Ref. [14]) crudely models the effect of the Higgs.

The upshot of the Khoze-Ringwald approach, in both the $O(3)$ $\sigma$ model and in Weinberg-Salam, is the following. In the one-instanton sector of the theory, $F_{hg}$ rises monotonically with energy, starting at $-1$ as per Eq. (2), and hitting zero at some critical energy $E_{KR}$ of order $E_s$. For $E \geq E_{KR}$ the Khoze-Ringwald method breaks down, and extra physics is needed, but this is a moot point. For, the Khoze-Ringwald scenario has not only predicted that $\sigma_{\text{anom}}$ loses its exponential suppression at some finite energy potentially accessed by experiment, it has also provided a putative mechanism for keeping $F_{hg}$ from becoming positive, thus ensuring unitarity.

In this Letter, we improve on the Khoze-Ringwald approach in a definite way. Specifically, for the first time in any model, we promote the ad-hoc conformal breaking term $2\rho^2\mu^2$ in Eq. (3) to a bona fide term in a Lagrangian. In the particular version of the $O(3)$ $\sigma$ model that we examine, we then find an altogether different behavior than the Khoze-Ringwald scenario, namely, a bifurcation in the valley at an energy at which $\sigma_{\text{anom}}$ is still exponentially suppressed. By a bifurcation, we mean that the saddle-point values of $\rho$
and $R$ leave the real axis as complex-conjugate pairs, at which point the optical-theorem justification of the valley method is apparently lost.

This bifurcation scenario was first outlined in Sec. 4 of Ref. [18]. The present work fleshes it out in the context of a specific Lagrangian model. Whether a similar bifurcation holds for other versions of the $O(3)$ $\sigma$ model in which conformal breaking is handled differently, or indeed in Weinberg-Salam when the effect of the Higgs sector on the valley is treated correctly, is anyone’s guess. But it is at least plausible that the phenomenon we exhibit here turns out to be more general, and furthermore, that it prevents $\sigma_{\text{anom}}$ from ever becoming observable at sphaleron energies.

2. Motivating the model

The classically conformally invariant $O(3)$ $\sigma$ model is defined by the Euclidean action

\[ S_{\text{conf}} = \frac{1}{2g^2} \int d^2x \partial_\mu \mathbf{n} \cdot \partial_\mu \mathbf{n} \]

\[ = \frac{1}{2g^2} \int dz d\bar{z} \frac{1}{(1 + w\bar{w})^2} \left( \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} + \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right) \]

where $\mathbf{n}$ lives on the 2-sphere. In the second equality we have passed to the complex representation

\[ w = \frac{n_1 + in_2}{1 - n_3}, \quad z = x_1 + ix_0. \]

While the $O(3)$ symmetry is more obscure in this representation, what becomes manifest is conformal invariance. Specifically, $S_{\text{conf}}$ is invariant under the 1-to-1 conformal mappings

\[ z \to f(z) = \frac{az + b}{cz + d}, \quad w(z) \to w(f(z)). \]

The $I$'s ($\bar{I}$'s) in this model have the simple (anti)analytic form

\[ w_I = \frac{\rho_I e^{i\theta_I}}{z - z_I} + c_I, \quad w_I = \frac{\rho_I e^{i\theta_I}}{\bar{z} - \bar{z}_I} + c_I \]

and action $4\pi/g^2$. Here $\rho_I$ ($\rho_I$) and $z_I$ ($z_I$) are the (anti)instanton’s scale size and location, respectively, while the phases $\theta_I$ ($\theta_I$) and asymptotic constants $c_I$ ($c_I$) fill out the $SL(2, C)$ manifold of collective coordinates. For our purposes, we need also the $\bar{I}I$ valley. In the notation of Ref. [12], it is given by the concentric configuration

\[ w_V(z, \bar{z}) = \frac{is}{u - u^{-1}}z^{-1} + \frac{is^{-1}}{u - u^{-1}}\bar{z}, \]
followed by any of the transformations $z \rightarrow f(z)$ given in Eq. (6) (see Refs. [12]-[13] for details). It then appears that $w_V$ depends on an unmanageably large number of complex parameters. Fortunately, most of them are redundant. To be precise, by rotating the phases of $z$ and $\bar{z}$, and translating and factoring a complex phase from $w_V$, we can actually express $w_V$ in terms of just three real collective coordinates ($\rho_I, \rho_{\bar{I}}, R$):

$$w_V(z, \bar{z}) = \frac{\rho_I}{R} \cdot \frac{z - R/2}{z + R/2} + \frac{\rho_{\bar{I}}}{R} \cdot \frac{\bar{z} + R/2}{\bar{z} - R/2},$$

(9)

where the new parameter $R$ measures the $I\bar{I}$ separation. Such phase redefinitions of $w_V$ are permissible provided that all expressions of interest (e.g., Eq. (4)) depend only on real products such as $w\bar{w}$.

So far as the $S_{\text{conf}}$ contribution to $F_{hg}$ is concerned (albeit not the other two terms in Eq. (3)), the set $(\rho_I, \rho_{\bar{I}}, R)$ is redundant still. In fact, the valley action is

$$S_{\text{conf}}(w_V) = \frac{8\pi}{g^2} \left[ \frac{u^4 + 1}{(u^2 + 1)^2} - \frac{2u^4 \log u^4}{(u^2 - 1)(u^2 + 1)^3} \right],$$

(10)

where the single valley parameter $u$ introduced in Eq. (8) is now reexpressed as

$$u = \left( \frac{R^2 + 2\rho_I\rho_{\bar{I}} + R\sqrt{R^2 + 2\rho_I\rho_{\bar{I}}}}{2\rho_I\rho_{\bar{I}}} \right)^{1/2}. \quad (11)$$

$S_{\text{conf}}(w_V)$ interpolates smoothly between the far-separated regime at the $u \rightarrow \infty$ end of the valley,

$$S_{\text{conf}}(w_V) \rightarrow \frac{8\pi}{g^2} \equiv S_{\text{conf}}(w_I) + S_{\text{conf}}(w_{\bar{I}}) \quad \text{when} \quad R \gg \rho_I, \rho_{\bar{I}},$$

(12)

and the perturbative vacuum as $u \rightarrow 1$:

$$S_{\text{conf}}(w_V) \rightarrow 0 \quad \text{when} \quad R \ll \rho_I, \rho_{\bar{I}}. \quad (13)$$

Rather than vanishing as one would expect, $w_V$ as given in Eq. (9) actually blows up in this latter limit. However, for any configuration, vanishing and blowing up are really the same thing in this model, since $w \rightarrow w^{-1}$ (equivalently $\hat{n}_2 \rightarrow -\hat{n}_2$, $\hat{n}_3 \rightarrow -\hat{n}_3$) is a specific instance of the $O(3)$ symmetry of Eq. (4).

In order to serve as a plausible toy model for Electroweak theory, $S_{\text{conf}}$ needs to be supplemented by an explicit conformal symmetry breaking term $S_{\text{csb}}$. The presence of
$S_{\text{csb}}$ will modify the valley equation, and consequently the valley itself (as well as the $I$’s and $\bar{I}$’s). Unfortunately, solving for the conformally broken valley in any field theory is a formidable numerical task. Recall that for conformally invariant field theories, the concentric valleys are obtained by a series of mathematical tricks that map the problem onto a solvable quantum mechanical model. These tricks become invalid when the conformal symmetry is broken.

To simplify our task, we limit ourselves herein to first order perturbation theory. In other words, we will simply plug the known conformally invariant valley (8)-(9) into $S_{\text{csb}}$. In order to trust this approximation, we will verify \textit{a posteriori} that

$$S_{\text{csb}}[w_V] \ll S_{\text{conf}}[w_V]$$

throughout the energy range of interest. A comparable first-order perturbation theory scheme has been tacitly assumed in previous valley-method calculations.

What to use for $S_{\text{csb}}$? Mottola and Wipf have used

$$S_{\text{csb}}^{\text{MW}} = \frac{4\pi\mu^2}{g^2} \int dz d\bar{z} \frac{1}{1 + w\bar{w}}$$

in their study of sphaleron physics in this model. However, $S_{\text{csb}}^{\text{MW}}$ is unsuitable for our purposes, because it diverges on the instanton (7). Alternatively, Khlebnikov, Rubakov and Tinyakov have used

$$S_{\text{csb}}^{\text{KRT}} = \frac{4\pi\mu^2}{g^2} \int dz d\bar{z} \left( \frac{w\bar{w}}{1 + w\bar{w}} \right)^2$$

which is finite on the subset of instantons (7) for which the asymptotic constant $c_I$ is zero. However, $S_{\text{csb}}^{\text{KRT}}$ still diverges on the valley (9). The reason is that

$$w_V(z, \bar{z}) \rightarrow -\frac{\rho_I + \rho_{\bar{I}}}{R} \text{ as } |z| \rightarrow \infty$$

and this unavoidable asymptotic constant gives rise to an infrared divergence.

We are led to concoct a term with gradients that kill this asymptotic constant. A natural set of such terms are powers of the kinetic energy density:

$$S_{\text{csb}} = \frac{4\pi}{g^2} \sum_{n=1}^{\infty} f_n \mu^{2-2n} \int dz d\bar{z} \left[ \frac{1}{(1 + w\bar{w})^2} \left( \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{z}} \frac{\partial w}{\partial z} \right) \right]^n.$$
Whereas the mass scale $\mu$ is inserted for dimensional reasons, the dimensionless constants $f_n$ can be chosen in any convenient manner. We will restrict this choice by demanding that at low energies, $S_{csb}(w_V)$ reduces to the Khoze-Ringwald form of the conformal breaking term shown in Eq. (I8). Equation (I8) is actually more symmetric than either $S_{csb}^{MW}$ or $S_{csb}^{KRT}$ as it preserves the $O(3)$ invariance. The fact that it is nonrenormalizable does not bother us, as we are focusing exclusively on semiclassical physics.

For guidance in selecting the constants $f_n$ intelligently, we insert the instanton $w_I = \rho/z$ and calculate

$$S_{csb}(w_I = \rho/z) = \frac{4\pi}{g^2} \cdot \sum_{n=1}^{\infty} \frac{\pi f_n}{2n - 1} \cdot (\mu \rho)^{2-2n}. \tag{19}$$

The choice

$$f_n = -\frac{2n - 1}{\pi} \cdot (-\lambda/g^2)^{-n} \tag{20}$$

leads to the geometric series

$$S_{csb}(w_I = \rho/z) = \frac{4\pi}{g^2} \cdot \frac{\mu^2 \rho^2}{1 + (\lambda/g^2)^{-2 \mu \rho^2}}. \tag{21}$$

Of course, what we want is $S_{csb}(w_V)$, not $S_{csb}(w_I)$. But Eq. (21) is suggestive. For, in the low-energy limit, the $I$ and $\bar{I}$ are well separated, and we therefore expect that $S_{csb}(w_V) \rightarrow 2S_{csb}(w_I)$, as happens for $S_{conf}$ (cf. Eq. (I2)). Furthermore, in this limit, the saddle-point value of $\mu^2 \rho^2 \rightarrow 0$, so that the denominator in Eq. (21) approaches unity. Therefore, $S_{csb}(w_V)$ specified by Eqs. (I3) and (I21) pleasingly reduces at low energies to the Khoze-Ringwald form of the conformal breaking term shown in Eq. (I8) (as we explicitly verify below). The reader can check that imposing this low-energy limit forces us to take an infinite number of powers of the kinetic energy density, so that the choice (20) is in a sense a minimal construction. At the same time—and this is the new feature of our calculation—$S_{csb}$ provides a well-defined Lagrangian prescription for extrapolating to higher energies.

The remaining free parameter $\lambda/g^2$ in Eqs. (20)-(21) determines the relative strengths of the $\mu^2 \rho^2$ and $\mu^4 \rho^4$ contributions to $S_{csb}$. In this respect (and motivating our notation), $\lambda/g^2$ is roughly analogous to the ratio $\lambda_{higgs}/g_W^2 \sim M_{higgs}^2/M_W^2$ in Electroweak theory,* where $\lambda_{higgs}$ is the Higgs self-coupling and $g_W$ is the $SU(2)_W$ gauge coupling. Our freedom to tune $\lambda/g^2$ will turn out to be important in ensuring that the perturbation theory criterion (I4) is met.

* See Eqs. (3.27) and (3.4) in Ref. 8
3. Results and Discussion

To recapitulate, our model is defined by

\[ 2S_{cl} \cdot F_{hg} = ER - S_{conf}(w_V) - S_{csb}(w_V) \] (22)

where \( w_V \) is given in Eq. (9), \( S_{conf} \) is given in Eqs. (10)-(11), and \( S_{csb} \) is given in Eqs. (18) and (20).

The values of the \( I \bar{I} \) collective coordinates \( \rho_I, \rho_{\bar{I}} \) and \( R \) used in Eqs. (9) and (22) are to be determined self-consistently from Eq. (22) by saddle-point methods. To simplify this task, we introduce the rescaled dimensionless variables

\[ \theta = \rho/R \]
\[ \zeta = (\mu R)^{-2} \]
\[ \epsilon = g^2 E/4\pi \mu \] (23)

where, as in all previous work on the valley method, we have anticipated that by symmetry

\[ \rho_I = \rho_{\bar{I}} \equiv \rho \] (24)

at the saddle point. Equation (22) then becomes

\[ F_{hg} = \frac{\epsilon}{2\sqrt{\zeta}} - \tilde{S}_{conf}(\theta) - \tilde{S}_{csb}(\theta, \zeta) \] (25)

where \( \tilde{S}_{csb} = (g^2/8\pi)S_{csb} \) and

\[ \tilde{S}_{conf}(\theta) = \frac{u^4 + 1}{(u^2 + 1)^2} - \frac{2u^4 \log u^4}{(u^2 - 1)(u^2 + 1)^2} \cdot u^2 = 1 + \frac{1}{2\theta^2} + \sqrt{\frac{1}{4\theta^4} + \frac{1}{2\theta^2}}. \] (26)

The saddle-point equations are then

\[ 0 = \frac{\partial}{\partial \theta} \tilde{S}_{conf} - \frac{\partial}{\partial \theta} \tilde{S}_{csb}, \] (27)
\[ 0 = \frac{\epsilon}{4\zeta^{3/2}} + \frac{\partial}{\partial \zeta} \tilde{S}_{csb}. \] (28)

The numerical evaluation of \( \tilde{S}_{csb} \) and its derivatives is actually somewhat subtle, and we digress for a paragraph to discuss it. One first performs the sum indicated in Eqs. (18) and (20) in closed form. The angular part of the \( dz \, d\bar{z} \) integration is carried out analytically using contour methods. The subtle point is that when \( \theta \) is small, the resulting
radial integral features a sharp peak or boundary layer, whose contribution almost exactly cancels that of the rest of the integration domain. This near cancellation is carried out numerically to great accuracy with the help of an appropriate rescaling of the boundary layer.

This having been done, we proceed first to Eq. (27), as it is independent of energy. Figure 1 shows the numerical result of this equation for two different values of $\lambda/g^2$, namely .2 and 2. The intercept value $\xi = 1/2$ in the far-separation (and low-energy, see Fig. 2) limit $\theta \to 0$ is no surprise: in this limit the Khoze-Ringwald model (3) becomes a good approximant for (22), and also $S_{\text{conf}} \approx 1 - 2\theta^2$, so that the resulting saddle-point algebra is elementary.

Using Fig. 1 to eliminate $\theta$ in favor of $\xi$, we next solve Eq. (28) to obtain the saddle-point value of $\xi$ as a function of energy $\epsilon$. The numerical results are plotted in Fig. 2, again for $\lambda/g^2 = .2$ and 2. In either case a bifurcation is evident: beyond a critical energy $\epsilon_{\text{crit}}(\lambda/g^2)$ there is no solution for $\xi$. More accurately, for $\epsilon > \epsilon_{\text{crit}}$ the saddle-point value of $\xi$ leaves the real axis as a complex conjugate pair.

Finally, Fig. 3 reassembles the complete holy grail function $F_{hg}$, from Eq. (25), as a function of energy, restricted to the range $0 \leq \epsilon \leq \epsilon_{\text{crit}}(\lambda/g^2)$. By design, the two values of $\lambda/g^2$ we have chosen give very different results. In the “light-Higgs” case $\lambda/g^2 = .2$, $F_{hg}$ rises to zero, and so $\sigma_{\text{anom}}$ loses its exponential suppression. This case is in qualitative agreement with the results of Ref. [13] in the single-instanton sector of the $\sigma$ model, as well as with Ref. [11] in Electroweak theory. However, remembering our first order perturbation theory criterion (14), we calculate that where $F_{hg} \approx 0$ the two terms $S_{\text{csb}}[wV]$ and $S_{\text{conf}}[wV]$ give approximately equal contributions to $F_{hg}$, strongly violating the criterion (14). Therefore, beyond low energies, we have no reason to trust the “light Higgs” result shown in Fig. 3, especially not in the interesting regime where $F_{hg}$ nears zero. For still smaller values of $\lambda/g^2$, $F_{hg}$ rises even faster with energy, becoming considerably greater than zero, but the inequality (14) is even more badly violated. Our calculation in the “light Higgs” regime is not self-consistent, and deserves no further discussion.

On the other hand, for the “heavy Higgs” case $\lambda/g^2 = 2$, $F_{hg}$ only rises around 15% prior to $\epsilon_{\text{crit}}$, so that $\sigma_{\text{anom}}$ remains exponentially suppressed. And in contrast to the “light Higgs” case, here $S_{\text{csb}}[wV]$ never exceeds around 15% of $S_{\text{conf}}[wV]$ in this energy range, so that the criterion (14) is reasonably well obeyed. For still larger values of $\lambda/g^2$ the trend continues: $\sigma_{\text{anom}}$ loses progressively less of its exponential suppression before bifurcating, while first order perturbation theory becomes increasingly reliable.
In sum, we have exhibited a self-consistent one-parameter family of models, parametrized by large values of $\lambda/g^2$ (say, greater than 2), which bifurcate at energies where $\sigma_{\text{anom}}$ is still exponentially suppressed, and for which first-order perturbation theory appears to be a reasonable approximation. Needless to say, a parallel calculation in Electroweak theory, although quite intricate, would be of great interest. In such a calculation, $S_{\text{csb}}$ would not involve an arcane construction such as Eq. (18), but would be given by the Standard Model Higgs Lagrangian.

We conclude with three brief comments about the limitations of our model, and of our understanding:

(i) How does one properly extrapolate $\sigma_{\text{anom}}$ beyond the bifurcation energy? We have no idea. The valley method has been shown\cite{16,17} to provide the analytic continuation in the variable $\rho^2/R^2$ of the so-called “$R$-term method” of Khlebnikov, Rubakov and Tinyakov.\cite{5} This equivalence extends the optical theorem to the case of $I\bar{I}$ backgrounds. But at $\epsilon = \epsilon_{\text{crit}}$ the valley method ceases to be analytic, and consequently, we have no good reason to believe that it has anything to do with the total anomalous cross section $\sigma_{\text{anom}}$. A conservative guess would be that $\epsilon = \epsilon_{\text{crit}}$ marks the end of the exponential rise of $\sigma_{\text{anom}}$. In any event, new physics for $\epsilon \geq \epsilon_{\text{crit}}$ is obviously required.

(ii) What about multi-instanton configurations? These, also, can be thought of as a bifurcation in the $I\bar{I}$ valley. While in the present scenario for $\epsilon \geq \epsilon_{\text{crit}}$ the number of collective coordinate degrees of freedom jumps discontinuously ($\rho$ and $R$ become complex), so too in the multi-instanton scenario of Zakharov\cite{22} and Maggiore and Shifman\cite{23} the number of relevant degrees of freedom increases at some critical energy $\epsilon_{\text{ZMS}}$ to encompass the collective coordinates of long chains of alternating $I$’s and $\bar{I}$’s. Presumably, the important bifurcation is the one that happens first. However, a calculation of $\epsilon_{\text{ZMS}}$ in the present version (22) of the $\sigma$ model is beyond the scope of this Letter (cf. Ref. [13]). We also note the possibility that a bifurcation of the type discussed herein occurs separately in each multi-instanton sector, and prior to the point where the $I\bar{I}I\bar{I}$ contribution (for example) catches up to the $I\bar{I}$ result. If that is the case, then multi-instanton contributions can be safely ignored.

(iii) What about initial-state corrections? Recently much attention has focused on the semiclassical description of initial-state corrections in the Ringwald problem.\cite{24} These corrections are absent in the valley method, except to the extent that the division between final-state and initial-state effects is itself somewhat ambiguous.\cite{25,16} However, if the final-state valley corrections by themselves are understood to cut off the rise of $\sigma_{\text{anom}}$ at
an exponentially suppressed value, through a bifurcation or otherwise, then it is difficult for us to imagine that the additional effect of the overlap of the hard initial state with the valley could enhance $\sigma_{\text{anom}}$ and render it observable.

We thank Nick Dorey for valuable input at all stages of this work.
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Figure Captions

1. Energy-independent relation between the saddle-point values of collective coordinates $\theta$ and $\zeta$, from Eq. (27). Here and in the subsequent figures, the solid line denotes the “heavy Higgs” case $\lambda/g^2 = 2$, while the broken line denotes the “light Higgs” case $\lambda/g^2 = .2$.

2. Saddle-point value of $\zeta$ as a function of energy, from Eq. (28). The open circles at the terminus of the curves mark the bifurcation points, $\epsilon = \epsilon_{\text{crit}}$.

3. The holy grail function $F_{hg}$, from Eq. (25), for energies $\epsilon \leq \epsilon_{\text{crit}}$. 
Figure 1

- $\theta$ vs. $\zeta$
Figure 3