Abstract

We consider the discrete-time GI/Geo/1 queue with multiple geometric vacations. We derive the joint probability generating function of the length of a busy period, the number of customers served during the busy period, and the residual interarrival time at the instant the busy period ends.

Keywords: Busy Period Analysis, Component, Discrete-time Queue, General Input Queue, Multiple Vacations

1. Introduction

We consider the discrete-time GI/Geo/1 queue with multiple geometric vacations. In recent years, there has been a growing interest in the analyses of discrete-time queueing systems due to their applications to a variety of slotted digital communication systems and other related areas. Takagi presented extensive studies on the discrete-time vacation queueing systems. However, most studies have dealt with the discrete-time Geo/G/1 queues with a variety of vacation policies. Tian and Zhang obtained the stationary distributions of the number of customers and the waiting time of customers in the discrete-time GI/Geo/1 queue with multiple vacations recently. Also, Hunter briefly presented the busy period of the discrete-time GI/Geo/1 queue without vacation. However, to the best of the authors' knowledge, the busy period of the discrete-time GI/Geo/1 queue with vacations has not been studied while many researches for the continuous-time queueing system can be found in. Goswami studied the number of customers served during the busy period in the Geo/Geo/1 queue recently.

In this paper, we obtain the joint Probability Generating Function (PGF) of the length of a busy period, the number of customers served during the busy period and the residual interarrival time at the instant the busy period ends for the GI/Geo/1 queue with multiple geometric vacations. Chae and Kim studied the busy period analysis for GI/M/1 queue with exponential vacations based on Takacs' approach recently. Our research is a new result which covers unique properties of the discrete-time queueing systems. In section 2, we give the model assumptions. In section 3, we derive the main results. And we conclude with a brief discussion in the last section.

2. Model Description

This paper assumes that all events can only occur at exact slot boundaries. Also it assumes that the busy period is not over if new customer comes at the slot when the last customer in the system departs. It is very important for that the idle period can't be zero in that assumption. The interarrival time is Independent and Identically Distributed (I.I.D.) and has a general discrete distribution:

\[ \Pr[I = j] = i, \quad j \geq 1; \quad I(z) = \sum_{j=1}^{\infty} i_j z^j. \]

The mean of the interarrival time is \( \lambda^{-1} \). The service time \( S \) is i.i.d. and has a geometric distribution:

\[ \Pr[S = i] = \mu(1-\mu)^{i-1}, \quad i \geq 1; \quad S(z) = \sum_{i=1}^{\infty} \mu(1-\mu)^{i-1} z^i. \]
And the mean of the service time is $\mu^{-1}$. $\lambda < \mu$ is assumed for the stability condition. The i.i.d random variable vacation time also follows a geometric distribution:

$$\Pr[V = i] = \nu(1 - \nu)^{i-1}, \quad i \geq 1; \quad V(z) = \sum_{i=1}^{\infty} \nu(1 - \nu)^{i-1} z^i; \quad E(V) = \nu^{-1}.$$  

In this system whenever the system empties, the server takes a vacation. When the server returns from the vacation, he starts serving the customers if he finds at least one customer. Otherwise he takes another vacation repeatedly until there are one or more customers. We call this system GI/Geo/1/MGV queue, where MGV means multiple geometric vacations.

### 3. Busy Period Analysis

Let $B_i$, $\Gamma_i$, and $R_i$ denote the length of a busy period, the number of customers served during the busy period, and the residual interarrival time at the instant the busy period ends for GI/Geo/1/MGV. We derive the joint transform of $B_i$, $\Gamma_i$, and $R_i$:

$$Q^\text{MGV}_1 = Q^\text{MGV}_1(z_1, z_2, z_3) = E[z_1^B z_2^\Gamma z_3^R].$$

We add the notation $Q^\text{MGV}_n$, for all $n \geq 1$, to find $Q^\text{MGV}_n$:

$$Q^\text{MGV}_n = Q^\text{MGV}_n(z_1, z_2, z_3) = E[z_1^B z_2^\Gamma z_3^R], \quad (1)$$

where $B_n$, $\Gamma_n$, and $R_n$ denote, respectively, the length of a busy period, the number of customers served during the busy period, and the residual interarrival time at the instant the busy period ends for GI/Geo/1/MGV/n-policy. In the system, the server takes a vacation as soon as the system becomes empty. If the number of waiting customers is less than $N$ when the server returns from the vacation, he takes another vacation. He takes vacations repeatedly until he finds at least $N$ customers on return from a vacation. Finally, the server begins to serve customers if there are $N$ or more customers in the queue.

In the GI/Geo/1/MGV/n-policy system, suppose that a customer arrives when there are $n - 1$ customers in the system and the server is on vacation, where $n \geq 1$. Then busy period starts as soon as the remaining vacation ends. The remaining vacation time, $V^1$, may be zero or not. We can condition $V^1$ into three sections comparing with $A$, where let $A$ denote the interarrival time between the next arrival and the current arrival which sees $n - 1$ customers and the vacationing server in the GI/Geo/1/MGV/n-policy. Suppose that $V^1 = 0$ which happens with probability $\nu$. That is, the current arrival customer brings a new busy period. That is stochastically equivalent to the busy period in the GI/Geo/1/n-policy. We add $\Omega_n$, $n \geq 1$:

$$\Omega_n = \Omega_n(z_1, z_2, z_3) = E[z_1^B z_2^\Gamma z_3^R], \quad (2)$$

where $b_n$, $\tau_n$, and $r_n$ denote, respectively, the length of a busy period, the number of customers served during the busy period and the residual interarrival time at the instant the busy period ends for GI/Geo/1/n-policy. Thus, we have

$$E[z_1^B z_2^\Gamma z_3^R | V^1 = 0] = \Omega_n.$$  

Suppose that $V^1 \geq A$. It means that the next customer sees $n$ customers and the vacationing server. The new busy period starts after the server vacation ends. That is stochastically equivalent to the busy period in the GI/Geo/1/MGV/(n+1)-policy because of the memoryless property of the vacation time that has the geometric distribution. The probability that $V^1 \geq 0$ can be expressed as follows:

$$\Pr(V^1 \geq A) = \sum_{i=1}^{\infty} \Pr(A = i) \Pr(V^1 \geq A | A = i)$$

$$= \sum_{i=1}^{\infty} \Pr(A = i)(1 - \nu)^i = A(1 - \nu).$$

Also, we have

$$E[z_1^B z_2^\Gamma z_3^R | V^1 \geq A] = E[z_1^B z_2^\Gamma z_3^R] = Q^\text{MGV}_{n+1} \quad (3)$$

Finally, suppose that $A - V^1 = i > 0$. In this condition, the server returns from vacation before the next customer arrives and a new busy period starts. Now we focus when the service of the customer that sees $n - 1$ customers ends from the beginning of the busy period. Let $S_n$ denote the service ending time of the customer that sees $n - 1$ customers and from the beginning of the busy period. So, $S_n$ may be less or more than $i$ or equal to $i$. At first, if $S_n$ is less than $i$, the busy period ends before the next customer arrives and the residual interarrival time at the instant the busy period ends is $i - S_n$. It is clear that the number of customers served during the busy period is $n$. We then have the following conditional joint transform:

$$E[z_1^B z_2^\Gamma z_3^R | A - V^1 = i, S_n = k] = z_1^k z_2^{i-k}.$$  

$$n + 1 \leq i \leq A - 1, \quad n \leq k \leq i - 1. \quad (4a)$$
Second, suppose that $S_1$ is equal to $i$. That is, the next customer arrives at the same time the last customer departs the system. In this case, the current busy period goes on because we follow the assumption in section 2. The busy period analysis is divided into two parts. One is the period for $n$ customers service times, $S_n$. The other is the period from the arrival of the next customer. The latter is stochastically equivalent to $\Omega$. So we have the following conditional joint transform:

$$E[z_1^R z_2^R z_3^R | A - V_R = i, S_n = j] = z_1^i z_2^i \Omega_{m-j+1}, \quad 1 \leq i \leq A - 1.$$  

By conditioning (1) on $A$, $V_R$, $S_n$, $N'$, and by using (2)-(4), we have

$$\xi_{MGV}^G = v \Omega + A(1-v) \Omega_{n+1}^{MGV} + \sum_{i=1}^{p} \Pr(A = i) \sum_{j=1}^{q} \sum_{k=1}^{n} (1-v)^{i-j} \mu^k (1-\mu)^{j-n} \mu z_1^k z_2^j z_3^k$$

$$+ \sum_{i=1}^{p} \Pr(A = i) \sum_{j=1}^{q} \sum_{k=1}^{n} (1-v)^{i-j} \mu^k (1-\mu)^{j-n} \mu \Omega$$

$$+ \sum_{i=1}^{p} \Pr(A = i) \sum_{j=1}^{q} \sum_{k=1}^{n} (1-v)^{i-j} \mu^k (1-\mu)^{j-n} \mu z_1^k z_2^j \Omega_{m-j+1}.$$  

We define $\Omega^{MGV}(w)$ and $\Omega(w)$ as:

$$\Omega^{MGV}(w) = \sum_{n=1}^{\infty} w^n \xi_{MGV}^{MGV}, \quad \Omega(w) = \sum_{n=1}^{\infty} w^n \Omega_n,$$

respectively. We multiply both sides of (5) with $w^n$, where $|w| < 1$, and then sum over $n = 1, 2, 3, \cdots$. Then we have

$$\Omega^{MGV}(w) = v \xi(w) + A(1-v) \left( w \Omega^{MGV}(w) - \Omega^{MGV} \right)$$

$$+ \frac{v \omega x w z_2}{(z_3 - r)(1-v)} + \frac{v \omega x w z_3}{(z_3 - r)(1-v)} A(z_3)$$

$$+ \frac{v \omega x w z_3}{(z_3 - r)(1-v)} A(r)$$

$$+ \frac{v w^2 \Omega(w) - \Omega}{1-v-r}.$$  

where $r = z_1 (1-\mu + \mu v z_2)$.

Also, we have

$$\Omega^{MGV}(w) \left\{ 1 - w^{-1} A(1-v) \right\} = -A(1-v) \Omega^{MGV}_{n+1}$$

$$+ \frac{v \omega x w z_2}{(z_3 - r)(1-v)}$$

$$+ \frac{v \omega x w z_3}{(z_3 - r)(1-v)} A(z_3)$$

$$+ \frac{v \omega x w z_3}{(z_3 - r)(1-v)} A(r)$$

$$+ \frac{v w^2 \Omega(w) - \Omega}{1-v-r}.$$  

We have the following equation by substituting $\Omega(w)$ and $\Omega_{n+1}$ in (7) (see Appendix about $r$ and $w$).

$$\Omega^{MGV}(w) \left\{ 1 - w^{-1} A(1-v) \right\} = -A(1-v) \Omega^{MGV}_{n+1}$$

$$+ \frac{v \omega x w z_2}{(z_3 - r)(1-v)}$$

$$+ \frac{v \omega x w z_3}{(z_3 - r)(1-v)} A(z_3)$$

$$+ \frac{v \omega x w z_3}{(z_3 - r)(1-v)} A(r)$$

$$+ \frac{v w^2 \Omega(w) - \Omega}{1-v-r}.$$  

By Rouché’s theorem, it follows that the equation

$$1 - w^{-1} A(1-v) = 0$$

has a unique root in the unit circle $|w| < 1$. So substituting $A(1-v)$ for $w$ in the left hand side in (8), it becomes zero. Let $r$ denote $z_1 (1-\mu + \mu A(1-v) z_2)$. By substituting $w$ with $A(1-v)$ in (8), we have

$$0 = -A(1-v) \Omega_{n+1}^{MGV}$$

$$+ \frac{v \omega x w z_2}{(z_3 - r)(1-v)}$$

$$+ \frac{A(1-v) \left\{ r \omega x w z_2 \right\} A(z_3)}{z_3 - r}$$

$$+ \frac{A(1-v) \left\{ r \omega x w z_3 \right\} A(r)}{z_3 - r}$$

$$+ \frac{A(1-v) \left\{ r \omega x w z_3 \right\} A(z_3)}{z_3 - r}$$

$$+ \frac{A(1-v) \left\{ r \omega x w z_3 \right\} A(r)}{z_3 - r}$$

$$+ \frac{v w^2 \Omega(w) - \Omega}{1-v-r}.$$  

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We have

\[
\Omega_{1}^{MGV} = \frac{v \mu z_2}{w} \left[ \frac{1 - v}{A(1 - v) - A(r_i)} \right] \left\{ \frac{(1 - v)\{A(1-v) - A(r_i)\}}{A(1-v)(1 - v - r_i)} \right\} \times \left[ \frac{A(1-v)\{r_i A(z_3) - z_3 A(r_i)\}}{z_3 - r_i} - \frac{A(1-v)\{r_i A(z_3) - z_3 w_0\}}{z_3 - r_i} \right] - \frac{v \mu z_2}{w} \frac{r_i A(z_3) - z_3 w_0}{z_3 - r_i} \frac{A(1-v) - (1-v)A(r_i)}{z_3 - r_i} + \frac{v \mu z_2}{w} \frac{z_3^2 A(1-v) - (1-v)^2 A(z_3)}{(z_3 - r_i)(1-v)}.
\]

Finally, we have the following result from the equation (9).

\[
\Omega_{1}^{MGV} = \frac{v \mu z_2}{w} \left[ \frac{z_3 w_0 - r_i A(z_3)}{z_3 - r_i} \frac{A(1-v) - (1-v)A(z_3)}{(1-v-r_i)(1-v-z_3)} \right] + \frac{z_3 A(1-v) - (1-v)A(z_3)}{(1-v-r_i)(1-v-z_3)}.
\]

4. Conclusion

In this paper we have developed the analysis not only about the busy period but the number of customers served during the busy period and the residual interarrival time at the instant the busy period ends in the discrete-time GI/Geo/1 queue with vacations. Actually, busy period analysis is seldom appeared. However, researchers can use the results about busy period to study a cycle time and an optimal threshold.

5. References

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Appendix

We consider the discrete-time GI/Geo/1/n-policy queue. And notations about \(b_o, r_o, r_a, \) and \(\Omega_e\) follow the section 3. To find \(\Omega_n\), we have the following conditional joint transform equation considering the time when the \(n\)th customer in the system departs the system:

\[
\Omega_n = \sum_{i=1}^w \sum_{j=0}^{\min(n-1,i)} \mu^i (1-\mu)^{-j} z_1^j z_2^{i+j} \Omega_{i+j}^{Q}
\]

\[
+ \sum_{i=n}^w \sum_{j=0}^{i-1} \mu^{i-1} (1-\mu)^{-j} z_1^{i-j} z_2^{i} \Omega
\]

\[
+ \sum_{i=n+1}^w \sum_{j=n}^{i-1} \mu^{i-1} (1-\mu)^{-j} z_1^{i-j} z_2^{n-j} \Omega
\]

The second term at the right-hand side in (10) means that the busy period doesn't end but continues with one customer when a new customer's arrival and the departure of the last customer occur at the same time. We multiple both sides of (10) with \(w^n\), where \(|w| \leq 1\), and then sum over \(n = 1, 2, 3, \cdots\). Then we have

\[
\Omega(w) = \sum_{n=1}^w \sum_{i=1}^w \sum_{j=0}^{\min(n-1,i)} \mu^i (1-\mu)^{-j} z_1^j z_2^{i+j} \Omega_{i+j}^{Q}
\]

\[
+ \sum_{n=1}^w \sum_{i=n}^w \sum_{j=0}^{i-1} \mu^{i-1} (1-\mu)^{-j} z_1^{i-j} z_2^{i} \Omega
\]

\[
+ \sum_{n=1}^w \sum_{i=n+1}^w \sum_{j=n}^{i-1} \mu^{i-1} (1-\mu)^{-j} z_1^{i-j} z_2^{n-j} \Omega
\]

Finally, we have the following equation with the equation (11).

\[
\Omega(w) = \{w^{-1} \Omega(w) - \Omega)A(r) + \frac{\mu w z_2 z_3 \Omega}{z_3 - r} A(r)
\]

\[
+ \frac{\mu w z_2 z_3}{z_3 - r} \left[ A(z_3) - \frac{z_3}{r} A(r) \right]
\]

\[
w^{-1} \Omega(w)\{w - A(r)\} = A(r) \left[ -\Omega + \frac{\mu w z_2 z_3 \Omega}{z_3 - r} A(r) \right]
\]

\[
+ \frac{\mu w z_2 z_3}{z_3 - r} \left[ A(z_3) - \frac{z_3}{r} A(r) \right]
\]
\[ w^{-1} \Omega(w) \{ w - A(r) \} = w \left\{ \frac{\mu z_3}{z_3 - r} A(z_3) \right\} \]
\[- A(r) \left\{ \frac{z_3}{r} \frac{\mu wz_2}{z_3 - r} + Q \left( 1 - \frac{\mu wz_2}{r} \right) \right\} \]  
(12)

where \( r = z_3 \left( 1 - \mu + \mu wz_2 \right) \).

Let \( w_0 \) denote \( A \left( z_1 \left( 1 - \mu + \mu w_0 z_2 \right) \right) \). To divide the both sides of (12) by \( \{ w_0 - A(r_0) \} \), the following equation is necessary:

\[ \frac{\mu z_3}{z_3 - r_0} A(z_3) = \frac{z_3}{r_0} \frac{\mu w_0 z_2}{z_3 - r_0} + Q_i \left( 1 - \frac{\mu w_0 z_2}{r_0} \right) \]  
(13)

where \( r_0 = z_1 \left( 1 - \mu + \mu w_0 z_2 \right) \). Finally, we have from (13)

\[ Q = \frac{\mu z_3}{(1 - \mu)(z_3 - r_0)} \left\{ r_0 A(z_3) - w_0 z_2 \right\}. \]