Note on Matter Collineations in Kantowski-Sachs, 
Bianchi Types I and III Spacetimes

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September 29, 2018

Abstract

We show that the classification of Kantowski-Sachs, Bianchi Types I and III 
spacetimes admitting Matter Collineations (MCs) presented in a recent paper by 
Camci et al. [Camci, U., and Sharif, M. Matter Collineations in Kantowski-Sachs, 
Bianchi Types I and III Spacetimes, (2003) Gen. Relativ. Grav. 35 97-109] is 
incomplete. Furthermore for these spacetimes and when the Einstein tensor is non-
degenerate, we give the complete Lie Algebra of MCs and the algebraic constraints 
on the spatial components of the Einstein tensor.

KEY WORDS: Matter Collineations;Bianchi I;Bianchi III; Kantowski-Sachs spacetimes.

Introduction

In a recent paper Camci et al. [1] studied Matter Collineations (MCs) in Kantowski-Sachs 
(k = +1), Bianchi Type I (k = 0) and Bianchi Type III (k = −1) spacetimes, which are 
described by the following, non static, hypersurface homogeneous Locally Rotationally 
Symmetric (LRS) metrics:

\[ ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t) \left[ dy^2 + \Sigma^2(y, k)dz^2 \right] \] (1)

where \( \Sigma(y, k) = \sin y, \sinh y, y \) and \( k = 1, -1, 0 \) is the curvature of the Euclidean 2-space 
of constant curvature.

The main conclusions of their interesting work are:

1. There are degenerate cases of \( T_{ab} \) which admit a finite number of proper MCs and 
more specifically 9 (k = 0) or 10 MCs (k = ±1).

2. In case where \( T_{ab} \) is non-degenerate there are either 6 proper MCs or no proper 
MCs.
Unfortunately some of the results of the aforementioned paper are incorrect. Indeed:

a. The first conclusion is true for $k = \pm 1$ but not for $k = 0$.

b. The second conclusion is incomplete, in the sense, that there are cases which the authors did not consider. Therefore the classification they give is not complete. Furthermore the results given for the non-degenerate case $k = 0$ are incorrect.

The purpose of this note is twofold:

A. To show that in the degenerate case and for $k = 0$ the dimension of the Lie Algebra of MCs is 5, therefore there exists (possibly) only one proper MC.

B. For the non-degenerate case to give the correct and complete Lie Algebra of MCs and the differential constraints on the components of the Einstein tensor.

For later use we give the Einstein tensor $G_{ab}$ for the metric (1):

$$G_{00} = G_0 = 2\frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{k}{B^2}$$  \hspace{1cm} (2)

$$G_{11} = G_1 = -A^2 \left[\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{k}{B^2}\right]$$  \hspace{1cm} (3)

$$G_{22} = G_2 = -B^2 \left[\frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A}\dot{B}}{AB}\right] = \Sigma^{-2}(y, k)G_{33}$$  \hspace{1cm} (4)

where a dot denotes differentiation w.r.t. $t$ and $k = 0, 1, -1$.

**The degenerate case**

We shall only consider case (c.i) of [1] for which $G_0 = 0$. Equation (2) for $k = 0$ implies:

$$2\frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{B}}{B}\right)^2 = 0.$$  \hspace{1cm} (5)

Demanding $\dot{B} \neq 0$ (rank$G_{ab} = 3$) equation (5) gives:

$$B = \frac{D}{A^2}$$  \hspace{1cm} (6)

where $D$ is a constant of integration.

Using equation (6) in (3) and (4) we obtain ($k = 0$):

$$G_1 = 4\left(\dddot{A} - 4\dddot{A}^2\right)$$  \hspace{1cm} (7)

$$G_2 = \frac{D^2\left(\dddot{A} - 4\dddot{A}^2\right)}{A^6}.$$  \hspace{1cm} (8)

As correctly stated in [1] the vector fields $\xi(5), \xi(6), \xi(7), \xi(8), \xi(9)$ (equation (41) of [1]) are MCs of the metrics (1) provided the components of $G_{ab}$ satisfy the condition $G_1 = \dots
$\epsilon G_2$ where $\epsilon$ is a constant. Replacing equations (7) and (8) in this condition we find $(\ddot{A}A - 4\dot{A}^2 \neq 0 \Leftrightarrow G_1 \neq 0)$:

$$4 \left( \ddot{A}A - 4\dot{A}^2 \right) = \epsilon \frac{D^2 \left( \ddot{A}A - 4\dot{A}^2 \right)}{A^6}$$

which implies that $A =$constant, therefore the spacetime reduces to Minkowksi spacetime i.e. $G_{ab} = 0$.

The error lies in the solution of equations (11) and (12) of [1]. Indeed from these two equations we obtain the general condition:

$$G_1 = A_1 G_2^\alpha;$$

(9)

where $A_1$ and $\alpha_1$ are constants. Therefore for $\alpha_1 \neq 1$ there exists only the vector field:

$$X = -2\frac{G_1}{G_1} \partial_t + x \partial_x + \frac{y}{\alpha_1} \partial_y$$

(10)

which may be proper MC (what it is will follow from the solution of the constraint equation (9)). Therefore in this case the dimension of the Lie Algebra of MCs is five (one proper) and not nine as the authors claim. The vector field (10) is given in [1] with the contradictory restriction $\alpha_1 = 1$.

It is to be noted that the physical interest in the degenerate case is limited, because it is well known that the only interesting case of degenerate stress-energy tensor is when $\text{rank}G_{ab} = 1$ in which case the matter is either dust fluid or radiation and null Einstein-Maxwell fields [2]. In this case the $C^\infty$ MCs for the metrics (1) form an infinite dimensional Lie Algebra [3].

The non degenerate case

In the non-degenerate case, $\text{rank}G_{ab} = 4$ and $G_{ab}$ can be treated as a metric [3]. This means that the Lie algebra $C$ of MCs is finite dimensional with possible dimension $4,5,6,7,10$. Four of these vectors are the KVs of the metric (1), therefore there can be either $0,1,2,3,6$ proper MCs. The authors have obtained the MCs only for the cases where $\dim C = 0, 6$ and have ommited the rest. However even when $\dim C = 10$ there are problems concerning the forms of the MCs given in [1]. In order to justify our claims we present the following counterexamples.

Counterexample 1

Assuming $B(t) \neq 0$ and using the new time variable $d\tau = \frac{dt}{B(t)}$ we rewrite the metric (1) as:

$$ds^2 = B^2(\tau) \left[ -d\tau^2 + \frac{A^2(\tau)}{B^2(\tau)} dx^2 + dy^2 + \Sigma^2(y,k) dz^2 \right].$$

(11)

Consider a spacetime for which the components of $G_{ab}$ satisfy the relations:

$$G_1 = -c_1^2 \tau^2 \quad G_2 = \pm c_2^2$$

(12)
with:

\[ \tilde{\tau}(\tau) = \int |G_0|^{1/2} d\tau \]  

(13)

where \( k = 0 \) and \( \text{sign}(G_0) > 0 \) (this case corresponds to the case \( \alpha_1 \neq 0, \alpha_2 = 0 \) of [1]).

It is easy to check that for this class of spacetimes (11) we have the following six proper MCs (\( \dim \mathcal{C} = 10 \)):

\[ X_1 = \cosh c_1 x \partial_\tau - \frac{1}{c_1 \tilde{\tau}} \sinh c_1 x \partial_x \]  

(14)

\[ X_2 = \sinh c_1 x \partial_\tau - \frac{1}{c_1 \tilde{\tau}} \cosh c_1 x \partial_x \]  

(15)

\[ X_{2\mu+\nu} = -c_2 f_{(\mu)} f'_{(\nu)} \partial_\tau + \frac{c_2 f_{(\mu)}}{c_1 \tilde{\tau}} x \partial_x - \frac{\tilde{\tau} f'_{(\nu)} [f_{(\mu)}]}{c_2} y \partial_y - \frac{\tilde{\tau} f'_{(\nu)} [f_{(\mu)}]}{y^2 c_2} z \partial_z \]  

(16)

where:

\[ f_{(\mu)} = (y \cos z, y \sin z) \]  

(17)

\[ f'_{(\nu)} = -(\cosh c_1 x, \sinh c_1 x) \]  

(18)

and the non tensorial indices \( \mu = 1, 2 \) and \( \nu = 1, 2 \) count vector fields.

These proper MCs are not given in [1]. Furthermore for \( k = 0 \) the MCs they found (e.g. the vector fields \( \xi_9, \xi_{10} \) in equation (53) of [1]) are equal to zero, because when \( k = 0 \Leftrightarrow T_2 = \text{constant} \) therefore \( \dot{T}_2 = 0 \).

**Counterexample 2**

Consider the spherically/hyperbolic symmetric spacetime \((k = \pm 1)\) in which the metric functions are given by:

\[ A(\tau) = \frac{D_1}{D_2}, \quad B(\tau) = B_1 \sinh^2 \frac{c_2 \tau}{2}, \quad B_1 \sin^2 \frac{c_2 \tau}{2}, \quad B_1 \cosh^2 \frac{c_2 \tau}{2} \]  

(19)

where \( D_1, D_2, B_1 \) are constants of integration.

For these spacetimes the components of \( G_{ab} \) satisfy the relations:

\[ G_1 = -c_1^2 c_2^2, \quad G_2 = \pm c_2^2 \]  

(20)

where \( c_1 = \pm \frac{D_1^2 c_2^2 + 1}{c_2} \) and the signs depend on the forms of the metric function \( B(\tau) \) and \( k \).

Spacetimes (19) are special cases of the class of metrics satisfying (20). They correspond to the case \( \alpha_1 = 0, \alpha_2 = 0 \) of [1] for which the authors state that there do not exist proper MCs. However it is easy to check that the following two vectors are proper MCs (hence \( \dim \mathcal{C} = 6 \)):

\[ X_1 = |G_0|^{-1/2} \partial_\tau, \quad X_2 = |G_0|^{-1/2} c_1 c_2 x \partial_\tau + \frac{\tilde{\tau}(\tau)}{c_1 c_2} \partial_x \]  

(21)

where, as previously:

\[ \tilde{\tau}(\tau) = \int |G_0|^{1/2} d\tau. \]  

(22)
We conclude this note by giving in Tables 1, 2, 3, 4 the complete Lie Algebra of proper MCs for the metrics (1). The results are given in terms of the coordinate \( \tilde{\tau} \) (essentially \( G_0 \)) and some integration constants. In the Tables the first column enumerates the various cases, the second column gives the constant curvature of the spatial 2-space, the third and fourth columns give the corresponding forms of \( G_1, G_2 \), the fifth column gives the dimension of the Lie Algebra of MCs (including the Lie algebra of the four Killing Vectors) and finally the sixth column gives the form of the Collineation vectors.

Table 1. Matter Collineations admitted by the metrics (11). The sign of \( G_1 \) is such that \( \text{sign}(G_0 \cdot G_1) < 0 \).

| Class | k | \( G_1 \) | \( G_2 \) | \( \text{dim } C \) | \( X \) |
|-------|---|----------|----------|----------------|--------|
| \( A_1 \) | 0 | \( \pm c^2 e^{-2\tilde{\tau}/c} \) | \( \pm c^2 e^{-2\tilde{\tau}/c} \) | 5 | \( \alpha_1 \partial_{\tilde{\tau}} + x \partial_x + \alpha_1 y \partial_y \) |
| \( A_2 \) | \( \pm 1 \) | \( \pm c_1 c_2^2 \) | \( \pm c_2^2 \) | 6 | \( c_1 c_2 x \partial_x + \frac{\tilde{\tau}}{c_1 c_2} \partial_x \) |
| \( A_3 \) | 0, \( \pm 1 \) | \( \pm c_2 e^{\frac{2\tilde{\tau}}{c_1 c_2}} \) | \( \pm c_2^2 \) | 6 | \( -ac_2 \partial_{\tilde{\tau}} + x \partial_x \) |
| \( A_4 \) | 0, \( \pm 1 \) | \( \pm c^2 \cosh^2 \frac{\tilde{\tau}}{ac} \) | \( \pm c^2 \) | 6 | \( c \sin \frac{\tilde{\tau}}{a} \partial_{\tilde{\tau}} + \tanh \frac{\tilde{\tau}}{ac} \cos \frac{\tilde{\tau}}{a} \partial_x \\
| | | & c \cos \frac{\tilde{\tau}}{ac} \partial_{\tilde{\tau}} - \tanh \frac{\tilde{\tau}}{ac} \sin \frac{\tilde{\tau}}{a} \partial_x \) |
| \( A_5 \) | 0, \( \pm 1 \) | \( \pm c^2 \sinh^2 \frac{\tilde{\tau}}{ac} \) | \( \pm c^2 \) | 6 | \( c \sinh \frac{\tilde{\tau}}{a} \partial_{\tilde{\tau}} - \coth \frac{\tilde{\tau}}{ac} \cosh \frac{\tilde{\tau}}{a} \partial_x \\
| | | & c \cosh \frac{\tilde{\tau}}{ac} \partial_{\tilde{\tau}} - \coth \frac{\tilde{\tau}}{ac} \sinh \frac{\tilde{\tau}}{a} \partial_x \) |
| \( A_6 \) | 0, \( \pm 1 \) | \( \pm c^2 \cos^2 \frac{\tilde{\tau}}{ac} \) | \( \pm c^2 \) | 6 | \( c \sinh \frac{\tilde{\tau}}{a} \partial_{\tilde{\tau}} + \tan \frac{\tilde{\tau}}{ac} \cosh \frac{\tilde{\tau}}{a} \partial_x \\
| | | & c \cosh \frac{\tilde{\tau}}{ac} \partial_{\tilde{\tau}} + \tan \frac{\tilde{\tau}}{ac} \sinh \frac{\tilde{\tau}}{a} \partial_x \) |
| \( A_7 \) | \( \pm 1 \) | \( \pm \tilde{\tau}^2 \) | \( \pm c^2 \) | 6 | \( \cosh \frac{\tilde{\tau}}{a} \partial_{\tilde{\tau}} - \tilde{\tau}^{-1} \sinh \frac{\tilde{\tau}^{-1}}{a} \partial_x \\
| | | & \sinh \frac{\tilde{\tau}}{a} \partial_{\tilde{\tau}} - \tilde{\tau}^{-1} \cosh \frac{\tilde{\tau}^{-1}}{a} \partial_x \) |
Table 2. Matter Collineations admitted by the metrics (11). The sign of \(G_1\) is such that
\[\text{sign}(G_0 \cdot G_1) < 0.\]

| Class | \( k \) | \( G_1 \) | \( G_2 \) | \( \text{dim} \mathcal{C} \) | \( \mathbf{X} \) |
|-------|--------|---------|---------|-----------------|-------------------------------|
| \( B_1 \) | 1      | \( \pm c_1^2 c^2 \) | \( \pm c^2 \cosh^2 \frac{x}{c} \) | 7                             | \( \begin{aligned} 
\mathbf{X}_{\mu+\nu+3} &= -f(\mu) \left[ f(\nu) \right] \tau \left( \cosh \frac{x}{c} \right)^2 \partial_\tau + \\
&+ \frac{f(\nu)}{c_1^2 \tanh^2 \frac{x}{c}} \partial_x - f(\nu) \left[ f(\nu) \right]_y \partial_y - \frac{f(\nu)}{y^2 c_1^2} \partial_z \\
\end{aligned} \) |
| \( B_2 \) | \(-1\) | \( \pm c_1^2 c^2 \) | \( \pm c^2 \sinh^2 \frac{x}{c} \) | 7                             | \( \begin{aligned} 
\mathbf{X}_{\mu+\nu+3} &= -f(\mu) \left[ f(\nu) \right] \tau \left( \sinh \frac{x}{c} \right)^2 \partial_\tau + \\
&+ \frac{f(\nu)}{c_1^2 \coth^2 \frac{x}{c}} \partial_x - f(\nu) \left[ f(\nu) \right]_y \partial_y - \frac{f(\nu)}{\sinh^2 y} \partial_z \\
\end{aligned} \) |
| \( B_3 \) | \(-1\) | \( \pm c_1^2 c^2 \) | \( \pm c^2 \sin^2 \frac{x}{c} \) | 7                             | \( \begin{aligned} 
\mathbf{X}_{\mu+\nu+3} &= -f(\mu) \left[ f(\nu) \right] \tau \left( \sin \frac{x}{c} \right)^2 \partial_\tau + \\
&+ \frac{f(\nu)}{c_1^2 \cot^2 \frac{x}{c}} \partial_x - f(\nu) \left[ f(\nu) \right]_y \partial_y - \frac{f(\nu)}{\sin^2 y} \partial_z \\
\end{aligned} \) |
| \( B_4 \) | 1      | \( \pm c_1^2 c^2 \sinh^2 \frac{x}{c} \) | \( \pm c^2 \cosh^2 \frac{x}{c} \) | 10                           | \( \begin{aligned} 
\mathbf{X}_{2(\mu+1)+\nu} &= -f(\mu) \left[ f(\nu) \right] \tau \left( \cosh \frac{x}{c} \right)^2 \partial_\tau + \\
&+ \frac{f(\nu)}{c_1^2 \tanh \frac{x}{c}} \partial_x - f(\nu) \left[ f(\nu) \right]_y \partial_y - \frac{f(\nu)}{y^2 c_1^2} \partial_z \\
\end{aligned} \) |
| \( B_5 \) | \(-1\) | \( \pm c_1^2 c^2 \cosh^2 \frac{x}{c} \) | \( \pm c^2 \sinh^2 \frac{x}{c} \) | 10                           | \( \begin{aligned} 
\mathbf{X}_{2(\mu+1)+\nu} &= -f(\mu) \left[ f(\nu) \right] \tau \left( \sinh \frac{x}{c} \right)^2 \partial_\tau + \\
&+ \frac{f(\nu)}{c_1^2 \coth \frac{x}{c}} \partial_x - f(\nu) \left[ f(\nu) \right]_y \partial_y - \frac{f(\nu)}{\sinh^2 y} \partial_z \\
\end{aligned} \) |
| \( B_6 \) | \(-1\) | \( \pm c_1^2 c^2 \cos^2 \frac{x}{c} \) | \( \pm c^2 \sin^2 \frac{x}{c} \) | 10                           | \( \begin{aligned} 
\mathbf{X}_{2(\mu+1)+\nu} &= -f(\mu) \left[ f(\nu) \right] \tau \left( \sin \frac{x}{c} \right)^2 \partial_\tau + \\
&+ \frac{f(\nu)}{c_1^2 \cot \frac{x}{c}} \partial_x - f(\nu) \left[ f(\nu) \right]_y \partial_y - \frac{f(\nu)}{\sin^2 y} \partial_z \\
\end{aligned} \) |
| \( B_7 \) | 0      | \( \pm c_1^2 \) | \( \pm c_2^2 \) | 10                           | \( \begin{aligned} 
\mathbf{X}_{2(\mu+1)+\nu} &= -c_2 f(\mu) \left[ f(\nu) \right] \tau \partial_\tau + \frac{c_2 f(\mu) f(\nu)}{c_1^2} \partial_x - \\
&- \frac{f(\nu) f(\nu)}{c_2} \partial_y - \frac{f(\nu)}{y^2 c_2} \partial_z \\
\mathbf{X}_{9} &= \partial_\tau \\
\mathbf{X}_{10} &= c_1 x \partial_\tau + \frac{\tau}{c_1} \partial_x \\
\end{aligned} \) |
| \( B_8 \) | 0      | \( \pm c_1^2 \tau^2 \) | \( \pm c_2^2 \) | 10                           | \( \begin{aligned} 
\mathbf{X}_{2(\mu+1)+\nu} &= -c_2 f(\mu) f(\nu) \tau \partial_\tau + \frac{c_2 f(\mu) f(\nu)}{c_1^2} \partial_x - \\
&- \frac{\tau f(\nu) f(\nu)}{c_2} \partial_y - \frac{\tau f(\nu) f(\nu)}{y^2 c_2} \partial_z \\
\mathbf{X}_{9} &= \cosh c_1 x \partial_\tau - \frac{\cosh c_1 x}{c_1^2} \sinh c_1 x \partial_x \\
\mathbf{X}_{10} &= \sinh c_1 x \partial_\tau - \frac{\sinh c_1 x}{c_1^2} \cosh c_1 x \partial_x \\
\end{aligned} \) |
Table 3. Matter Collineations admitted by the metrics (11). The sign of $G_1$ is such that $\text{sign}(G_0 \cdot G_1) > 0$.

| Class | $k$ | $G_1$ | $G_2$ | $\text{dim } C$ | $X$ |
|-------|-----|-------|-------|-----------------|-----|
| $A_1$ | 0   | $\pm c^2 e^{-2\varphi/\alpha_1 c}$ | $\pm c^2 e^{-2\varphi/c}$ | 5    | $\alpha_1 c \partial_{\tilde{\tau}} + x \partial_x + \alpha_1 y \partial_y$ |
| $A_2$ | $\pm 1$ | $\pm c_1^2 c_2^2$ | $\pm c_2^2$ | 6    | $c_1 c_2 x \partial_{\tilde{\tau}} - \frac{x}{c_1 c_2} \partial_x$ |
| $A_3$ | 0, $\pm 1$ | $\pm c_1^2 e^{2\varphi}$ | $\pm c_2^2$ | 6    | $-ac_2 \partial_{\tilde{\tau}} + x \partial_x$ |
|       |     |       |       |      | $2ac_2 x \partial_{\tilde{\tau}} - \left( x^2 - \frac{a^2 c_2^2 e^{-2\varphi}}{c_1^2 e^{2\varphi}} \right) \partial_x$ |
| $A_4$ | 0, $\pm 1$ | $\pm c^2 \cos^2 \frac{\varphi}{ac}$ | $\pm c^2$ | 6    | $c \sin \frac{x}{a} \partial_{\tilde{\tau}} + \tan \frac{x}{a} \cos \frac{x}{a} \partial_x$ |
|       |     |       |       |      | $c \cos \frac{x}{a} \partial_{\tilde{\tau}} + \tan \frac{x}{a} \sin \frac{x}{a} \partial_x$ |
| $A_5$ | 0, $\pm 1$ | $\pm c^2 \sinh^2 \frac{\varphi}{ac}$ | $\pm c^2$ | 6    | $c \sin \frac{x}{a} \partial_{\tilde{\tau}} + \coth \frac{x}{a} \cos \frac{x}{a} \partial_x$ |
|       |     |       |       |      | $c \cos \frac{x}{a} \partial_{\tilde{\tau}} - \coth \frac{x}{a} \sin \frac{x}{a} \partial_x$ |
| $A_6$ | 0, $\pm 1$ | $\pm c^2 \cosh^2 \frac{\varphi}{ac}$ | $\pm c^2$ | 6    | $c \sinh \frac{x}{a} \partial_{\tilde{\tau}} - \tanh \frac{x}{ac} \cosh \frac{x}{a} \partial_x$ |
|       |     |       |       |      | $c \cosh \frac{x}{a} \partial_{\tilde{\tau}} - \tanh \frac{x}{ac} \sinh \frac{x}{a} \partial_x$ |
| $A_7$ | $\pm 1$ | $\pm \tilde{\tau}^2$ | $\pm c^2$ | 6    | $\cos x \partial_{\tilde{\tau}} - \tilde{\tau}^{-1} \sin x \partial_x$ |
|       |     |       |       |      | $\sin x \partial_{\tilde{\tau}} + \tilde{\tau}^{-1} \cos x \partial_x$ |
Table 4. Matter Collineations admitted by the metrics (11). The sign of $G_1$ is such that $\text{sign}(G_0 \cdot G_1) > 0$.

| Class | $k$ | $G_1$ | $G_2$ | $\text{dim} \mathcal{C}$ | $X$ |
|-------|-----|-------|-------|-----------------|-----|
| $B_1$ | 1   | $\pm c_1^2 c^2$ | $\pm c^2 \cos^2 \frac{\tilde{\tau}}{c}$ | 7   | $X_{\mu+\nu+3} = f(\mu) \left[ f'_{(\nu)} \right]_{\tilde{\tau}} \left( c \cos \frac{\tilde{\tau}}{c} \right) \partial_{\tilde{\tau}} +$ |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
| $B_2$ | $-1$ | $\pm c_1^2 c^2$ | $\pm c^2 \cosh^2 \frac{\tilde{\tau}}{c}$ | 7   | $X_{\mu+\nu+3} = f(\mu) \left[ f'_{(\nu)} \right]_{\tilde{\tau}} \left( c \cosh \frac{\tilde{\tau}}{c} \right) \partial_{\tilde{\tau}} +$ |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
| $B_3$ | 1   | $\pm c_1^2 c^2$ | $\pm c^2 \sinh^2 \frac{\tilde{\tau}}{c}$ | 7   | $X_{\mu+\nu+3} = f(\mu) \left[ f'_{(\nu)} \right]_{\tilde{\tau}} \left( c \sinh \frac{\tilde{\tau}}{c} \right) \partial_{\tilde{\tau}} +$ |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
| $B_4$ | 1   | $\pm c_1^2 c^2 \sin^2 \frac{\tilde{\tau}}{c}$ | $\pm c^2 \cos^2 \frac{\tilde{\tau}}{c}$ | 10  | $X_{2(\mu+1)+\nu} = f(\mu) \left[ f'_{(\nu)} \right]_{\tilde{\tau}} \left( c \cos \frac{\tilde{\tau}}{c} \right) \partial_{\tilde{\tau}} +$ |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
| $B_5$ | $-1$ | $\pm c_1^2 c^2 \sinh^2 \frac{\tilde{\tau}}{c}$ | $\pm c^2 \cosh^2 \frac{\tilde{\tau}}{c}$ | 10  | $X_{2(\mu+1)+\nu} = f(\mu) \left[ f'_{(\nu)} \right]_{\tilde{\tau}} \left( c \cosh \frac{\tilde{\tau}}{c} \right) \partial_{\tilde{\tau}} +$ |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
| $B_6$ | 1   | $\pm c_1^2 c^2 \cosh^2 \frac{\tilde{\tau}}{c}$ | $\pm c^2 \sinh^2 \frac{\tilde{\tau}}{c}$ | 10  | $X_{2(\mu+1)+\nu} = f(\mu) \left[ f'_{(\nu)} \right]_{\tilde{\tau}} \left( c \sinh \frac{\tilde{\tau}}{c} \right) \partial_{\tilde{\tau}} +$ |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
| $B_7$ | 0   | $\pm c_1^2$ | $\pm c_2^2$ | 10  | $X_{2(\mu+1)+\nu} = f(\mu) \left[ f'_{(\nu)} \right]_{\tilde{\tau}} \partial_{\tilde{\tau}} +$ |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
|       |     |       |       |                 |     |
| $B_8$ | 0   | $\pm c_1^2 \tilde{\tau}^2$ | $\pm c_2^2$ | 10  | $X_{2(\mu+1)+\nu} = f(\mu) \left[ f'_{(\nu)} \right]_{\tilde{\tau}} \partial_{\tilde{\tau}} +$ |
Table 5. Explanations for the quantities $f(\mu), f'(\nu)$ appearing in Table 2. $\mu, \nu = 1, 2, 3$.

| Class | $k$ | $f'(\nu)$ | $f(\mu)$ |
|-------|-----|-----------|-----------|
| $B_1$ | 1   | $(-\tanh \frac{\tau}{e}, 0, 0)$ | $(-\cos y, \sin y \cos z, \sin y \sin z)$ |
| $B_2$ | $-1$ | $(\coth \frac{\tau}{e}, 0, 0)$ | $(\cosh y, \sinh y \cos z, \sinh y \sin z)$ |
| $B_3$ | $-1$ | $(-\cot \frac{\tau}{e}, 0, 0)$ | $(\cosh y, \sinh y \cos z, \sinh y \sin z)$ |
| $B_4$ | 1   | $(\tanh \frac{\tau}{e} \cosh c_1 x, \tanh \frac{\tau}{e} \sinh c_1 x, 0)$ | $(-\cos y, \sin y \cos z, \sin y \sin z)$ |
| $B_5$ | $-1$ | $(-\coth \frac{\tau}{e} \cosh c_1 x, -\coth \frac{\tau}{e} \sin c_1 x, 0)$ | $(\cosh y, \sinh y \cos z, \sinh y \sin z)$ |
| $B_6$ | $-1$ | $(-\cot \frac{\tau}{e} \cosh c_1 x, -\cot \frac{\tau}{e} \sinh c_1 x, 0)$ | $(\cosh y, \sinh y \cos z, \sinh y \sin z)$ |
| $B_7$ | 0   | $-(\tilde{\tau}, c_1 x, 0)$ | $(y \cos z, y \sin z, 0)$ |
| $B_8$ | 0   | $-(\cosh c_1 x, \sinh c_1 x, 0)$ | $(y \cos z, y \sin z, 0)$ |

Table 6. Explanations for the quantities $f(\mu), f'(\nu)$ appearing in Table 4. $\mu, \nu = 1, 2, 3$.

| Class | $k$ | $f'(\nu)$ | $f(\mu)$ |
|-------|-----|-----------|-----------|
| $B_1$ | 1   | $(-\tan \frac{\tau}{e}, 0, 0)$ | $(-\cos y, \sin y \cos z, \sin y \sin z)$ |
| $B_2$ | $-1$ | $(\tan \frac{\tau}{e}, 0, 0)$ | $(\cosh y, \sinh y \cos z, \sinh y \sin z)$ |
| $B_3$ | 1   | $(-\coth \frac{\tau}{e}, 0, 0)$ | $(\cosh y, \sinh y \cos z, \sinh y \sin z)$ |
| $B_4$ | 1   | $(\tanh \frac{\tau}{e} \cosh c_1 x, \tanh \frac{\tau}{e} \sin c_1 x, 0)$ | $(-\cos y, \sin y \cos z, \sin y \sin z)$ |
| $B_5$ | $-1$ | $(-\tanh \frac{\tau}{e} \cos c_1 x, -\tanh \frac{\tau}{e} \sin c_1 x, 0)$ | $(\cosh y, \sinh y \cos z, \sinh y \sin z)$ |
| $B_6$ | 1   | $(-\coth \frac{\tau}{e} \cosh c_1 x, -\coth \frac{\tau}{e} \sinh c_1 x, 0)$ | $(\cosh y, \sinh y \cos z, \sinh y \sin z)$ |
| $B_7$ | 0   | $-(\tilde{\tau}, c_1 x, 0)$ | $(y \cos z, y \sin z, 0)$ |
| $B_8$ | 0   | $-(\cos c_1 x, \sin c_1 x, 0)$ | $(y \cos z, y \sin z, 0)$ |

A systematic and complete study of MCs in hypersurface LRS spacetimes (which includes the present case as a special case) will be discussed in a forthcoming work.

Note added:

In order the spacetimes (1) to admit a MC, the spatial components $G_1, G_2$ of the Einstein tensor must satisfy a first order differential equation whose solution gives $G_1, G_2$ and, consequently, the collineation vectors. A detailed presentation of these differential equations for all hypersurface homogeneous LRS will be given in [4].

Each algebraic constraint (third and fourth column of Tables 1, 2, 3, 4) leads to a system of two differential equations among the metric functions $A(t), B(t)$, which, in general, is difficult to solve explicitly. In the counterexamples for simplicity and in order to present spacetimes which are not immediately ruled out as unphysical (in fact it can be shown that the spacetimes (19) satisfy all the energy conditions) we use the most simple case where $G_1, G_2$ are constants. In this case a class of solutions of the system of differential equations are the metric functions (19) with $D_1, D_2, B_1$ being constants of integration.
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