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The existence
of a semialgebraic continuous
factorization map
for some compact linear groups

It is proved that each of compact linear groups of one special type admits a semialgebraic continuous factorization map onto a real vector space.

Key words: Lie group, factorization map of an action, semialgebraic map.

§ 1. Introduction

In this paper, we prove that each of compact linear groups of one certain type admits a semialgebraic continuous factorization map onto a real vector space. This problem arose from the question when the topological quotient of a compact linear group is homeomorphic to a vector space that was researched in [1, 2, 3, 4, 5].

For convenience, denote by $\mathcal{P}_\mathbb{R}$ (resp. by $\mathcal{P}_\mathbb{C}$) the class of all finite-dimensional Euclidean (resp. Hermitian) spaces. Let $\mathbb{F}$ be one of the fields $\mathbb{R}$ and $\mathbb{C}$. We will write $\mathbb{F}^k$ ($k \geq 0$) for the space $\mathbb{F}^k \in \mathcal{P}_\mathbb{F}$ whose standard basis is orthonormal. For any spaces $W_1, W_2 \in \mathcal{P}_\mathbb{F}$, we will write $W_1 \oplus W_2$ for their orthogonal direct sum $W_1 \oplus W_2 \in \mathcal{P}_\mathbb{F}$ (unless they are linearly independent subspaces of the same space). For any space $W \in \mathcal{P}_\mathbb{F}$, denote by $S(W)$ the real subspace $\{A \in \text{End}_\mathbb{F}(W) : A = A^*\} \subset \text{End}_\mathbb{F}(W)$, by $O(W)$ the compact Lie group $\{C \in \text{GL}_\mathbb{F}(W) : CC^* = E\} \subset \text{GL}_\mathbb{F}(W)$, and by $SO(W)$ its compact subgroup $O(W) \cap \text{SL}_\mathbb{F}(W)$.

The following theorem is the main result of the paper and will be proved in §3.

**Theorem 1.1.** Consider a number $k \in \{1, 2\}$, a space $W \in \mathcal{P}_\mathbb{F}$ such that $\dim_\mathbb{F} W \geq 2-k$, and the representation

$$O(W) : (S(W)/(\mathbb{R}E)) \oplus \text{Hom}_\mathbb{F}(\mathbb{F}^k, W), \quad C : (A + \mathbb{R}E, B) \rightarrow (CAC^{-1} + \mathbb{R}E, CB). \quad (1.1)$$
1) If $k = 1$, then there exist a real vector space $V$ and, in terms of the representation $O(W) : V \oplus \mathbb{F}, C : (v, \lambda) \to (v, \det_{\mathbb{F}} C \cdot \lambda)$, a semialgebraic continuous $(O(W))$-equivariant surjective map $(S(W)/(\mathbb{RE})) \oplus \text{Hom}_{\mathbb{F}}(\mathbb{F}^k, W) \to V \oplus \mathbb{F}$ whose fibres coincide with the $(SO(W))$-orbits.

2) If $k = 2$, then there exist a real vector space $V$ and a semialgebraic continuous surjective map $(S(W)/(\mathbb{RE})) \oplus \text{Hom}_{\mathbb{F}}(\mathbb{F}^k, W) \to V$ whose fibres coincide with the $(O(W))$-orbits.

§ 2. Auxiliary facts

Consider a space $W \in \mathcal{P}_\mathbb{F}$ and the subsets $S_+(W) := \{A \in S(W) : A \geq 0\} \subset S(W)$ and $S_0(W) := (S_+(W)) \setminus (\text{GL}_\mathbb{F}(W)) \subset S_+(W) \subset S(W)$.

Set $M(W) := \{(A, \lambda) \in S_+(W) \times \mathbb{F} : \det_{\mathbb{F}} A = |\lambda|^2\} \subset S(W) \oplus \mathbb{F}$.

The following statement is well-known.

**Statement 2.1.** The fibres of the map $\pi : \text{End}_{\mathbb{F}}(W) \to S(W) \oplus \mathbb{F}, X \to (XX^*, \det_{\mathbb{F}} X)$ coincide with the orbits of the action

$$SO(W) : \text{End}_{\mathbb{F}}(W), C : X \to XC^{-1}, \quad (2.1)$$

and $\pi(\text{End}_{\mathbb{F}}(W)) = M(W) \subset S(W) \oplus \mathbb{F}$.

**Lemma 2.1.** The map $M(W) \to (S(W)/(\mathbb{RE})) \oplus \mathbb{F}, (A, \lambda) \to (A + \mathbb{RE}, \lambda)$ is a bijection.

☐ See Lemma 5.1 in [3].

**Corollary 2.1.** The map $S_0(W) \to S(W)/(\mathbb{RE}), A \to A + \mathbb{RE}$ is a bijection.

**Corollary 2.2.** The map $\text{End}_{\mathbb{F}}(W) \to (S(W)/(\mathbb{RE})) \oplus \mathbb{F}, X \to (XX^* + \mathbb{RE}, \det_{\mathbb{F}} X)$ is surjective, and its fibres coincide with the orbits of the action $\pi(XX^*)$.

☐ Follows from Statement 2.1 and Lemma 2.1

Now consider arbitrary spaces $W, W_1 \in \mathcal{P}_\mathbb{F}$ over the filed $\mathbb{F}$.

**Statement 2.2.** The fibres of the map $\pi : \text{Hom}_{\mathbb{F}}(W_1, W) \to S(W), X \to XX^*$ coincide with the orbits of the action

$$O(W_1) : \text{Hom}_{\mathbb{F}}(W_1, W), C_1 : X \to XC_1^{-1}, \quad (2.2)$$

and $\pi(\text{Hom}_{\mathbb{F}}(W_1, W)) = \{A \in S_+(W) \text{ : rk}_{\mathbb{F}} A \leq \dim_{\mathbb{F}} W_1\} \subset S(W)$. In particular, in the case $\dim_{\mathbb{F}} W_1 = \dim_{\mathbb{F}} W - 1$, we have $\pi(\text{Hom}_{\mathbb{F}}(W_1, W)) = S_0(W) \subset S(W)$.

We omit the proof since it is clear.

**Corollary 2.3.** Suppose that $\dim_{\mathbb{F}} W_1 = \dim_{\mathbb{F}} W - 1$. Then the map $\text{Hom}_{\mathbb{F}}(W_1, W) \to S(W)/(\mathbb{RE}), X \to XX^* + \mathbb{RE}$ is surjective, and its fibres coincide with the orbits of the action $\pi(XX^*)$.

☐ Follows from Statement 2.2 and Corollary 2.1
§ 3. Proof of the main result

In this section, we will prove Theorem 1.1. 

First of all, let us prove the claim for a pair $(k, W)$ $(k \in \{1, 2\}, W \in \mathcal{P}_W, \dim W = 2 - k)$. 

If $k = 2$ and $W = 0$, then the space of the representation (1.1) is trivial. If $k = 1$ and $\dim W = 1$, then the representation (1.1) is isomorphic to the tautological representation of the group $O(W)$ and, consequently, to the representation $O(W): F, C: \lambda \rightarrow \det_\mathbb{F} C \cdot \lambda$, and the subgroup $SO(W) \subset O(W)$ is trivial.

This completely proves the claim for a pair $(k, W)$ $(k \in \{1, 2\}, W \in \mathcal{P}_W, \dim W = 2 - k)$.

Take an arbitrary pair $(k, W)$ $(k \in \{1, 2\}, W \in \mathcal{P}_W, n := \dim W > 2 - k)$ and assume that the claim of Theorem 1.1 holds for any pair $(k, W_1)$ $(W_1 \in \mathcal{P}_W, \dim W_1 = n - 1)$. We will now prove it for the pair $(k, W)$.

Since $n > 2 - k \geq 0$, there exists a space $W_1 \in \mathcal{P}_W$ such that $\dim W_1 = n - 1$. Consider the embedding $R: O(W_1) \hookrightarrow O(W_1 \oplus \mathbb{F}^k), R(C_1): (x_1, x_2) \rightarrow (C_1 x_1, x_2)$. The representations

$$O(W_1): S(W_1 \oplus \mathbb{F}^k), C_1: A \rightarrow R(C_1)AR(C_1^{-1})$$

and $O(W_1): S(W_1) \oplus \text{Hom}_\mathbb{F}(\mathbb{F}^k, W_1) \oplus S(\mathbb{F}^k), C_1: (A_1, B, A_2) \rightarrow (C_1 A_1 C_1^{-1}, C_1 B, A_2)$ are isomorphic via the $(O(W_1))$-equivariant $\mathbb{R}$-linear isomorphism

$$\theta: S(W_1) \oplus \text{Hom}_\mathbb{F}(\mathbb{F}^k, W_1) \oplus S(\mathbb{F}^k) \rightarrow S(W_1 \oplus \mathbb{F}^k),$$

$$\theta(A_1, B, A_2): (x_1, x_2) \rightarrow (A_1 x_1 + B x_2, B^* x_1 + A_2 x_2).$$

Therefore, the representation (3.1) is isomorphic to the direct sum of some trivial real representation of the group $O(W_1)$ and the representation

$$O(W_1): (S(W_1)/(\mathbb{R} E)) \oplus \text{Hom}_\mathbb{F}(\mathbb{F}^k, W_1), C_1: (A_1 + \mathbb{R} E, B) \rightarrow (C_1 A_1 C_1^{-1} + \mathbb{R} E, C_1 B)$$

with trivial fixed point subspace. Thus, there exist a real vector space $V'$ and an isomorphism $\varphi: S(W_1 \oplus \mathbb{F}^k)/(\mathbb{R} E) \rightarrow V' \oplus (S(W_1)/(\mathbb{R} E)) \oplus \text{Hom}_\mathbb{F}(\mathbb{F}^k, W_1)$ of the representations

$$O(W_1): S(W_1 \oplus \mathbb{F}^k)/(\mathbb{R} E), C_1: A + \mathbb{R} E \rightarrow R(C_1)AR(C_1^{-1}) + \mathbb{R} E;$$

$$O(W_1): V' \oplus (S(W_1)/(\mathbb{R} E)) \oplus \text{Hom}_\mathbb{F}(\mathbb{F}^k, W_1), C_1: (v', A_1 + \mathbb{R} E, B) \rightarrow (v', C_1 A_1 C_1^{-1} + \mathbb{R} E, C_1 B).$$

In terms of the representations

$$O(W) \times O(W_1): \text{Hom}_\mathbb{F}(W_1 \oplus \mathbb{F}^k, W), (C, C_1): Z \rightarrow CZR(C_1^{-1});$$

$$O(W) \times O(W_1): (S(W)/(\mathbb{R} E)) \oplus \text{Hom}_\mathbb{F}(\mathbb{F}^k, W), (C, C_1): (A + \mathbb{R} E, B) \rightarrow (CAC^{-1} + \mathbb{R} E, CB);$$

$$O(W) \times O(W_1): V' \oplus (S(W_1)/(\mathbb{R} E)) \oplus \text{Hom}_\mathbb{F}(\mathbb{F}^k, W_1), (C, C_1): (v', A_1 + \mathbb{R} E, B) \rightarrow (v', C_1 A_1 C_1^{-1} + \mathbb{R} E, C_1 B),$$

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One can easily see that the map \( \pi_0 \) is surjective, and its fibres coincide with the \( (O(W_1) \times O(W_1)) \)-orbits. 

**Case 1.** \( k = 1 \).

Consider the representation

\[
O(W) \times O(W_1) : V' \oplus (S(W_1)/(RE)) \oplus \text{Hom}_F(F^k, W_1) \oplus F,
\]

\((C, C_1) : (v', A_1 + \mathbb{R}E, B, \mu) \rightarrow (v', C_1 A_1^{-1} + \mathbb{R}E, C_1 B, \det_F C \cdot \det_F C_1^{-1} \cdot \mu)\).

Since \( \text{dim}_F W_1 = n - 1 \), there exist a real vector space \( V_1 \) and, in terms of the representation

\[
\text{dim}_F (W_1 \oplus F^k) = (n - 1) + k = n = \text{dim}_F W, \quad \text{we can identify } W \text{ and } W_1 \oplus F^k \text{ as spaces of the class } \mathcal{P}_F. \quad \text{The map}
\]

\[
\tilde{\psi} : \text{Hom}_F(W_1 \oplus F^k, W) \rightarrow V' \oplus V_1 \oplus F^2
\]

is \( (O(W) \times O(W_1)) \)-equivariant. Also, by Corollary 2.2 the map \( \gamma \) is surjective, and its fibres coincide with the \( (SO(W)) \)-orbits. Hence, the map \( \pi_1 \circ \tilde{\psi} : \text{Hom}_F(W_1 \oplus F^k, W) \rightarrow V' \oplus V_1 \oplus F^2 \) is \( (O(W) \times O(W_1)) \)-equivariant and surjective, and its fibres coincide with the \( (SO(W) \times SO(W_1)) \)-orbits. If we consider the representation

\[
O(W) \times O(W_1) : V' \oplus V_1 \oplus \mathbb{R} \oplus F, \quad (C, C_1) : (v', v_1, t, \lambda) \rightarrow (v', v_1, t, \det_F C \cdot \lambda),
\]

then the map \( \gamma : V' \oplus V_1 \oplus F^2 \rightarrow V' \oplus V_1 \oplus \mathbb{R} \oplus F, \quad (v', v_1, \lambda, \mu) \rightarrow (v', v_1, |\lambda|^2 - |\mu|^2, \lambda \mu) \) is \( (O(W) \times O(W_1)) \)-equivariant. Denote by \( \mathbb{T} \) the multiplicative group \( \{ c \in F : |c| = 1 \} \). We have \( n > 2 - k = 1 \), \( \text{dim}_F W_1 = n - 1 > 0 \). Consequently,

\[
\{ \det_F C_1 : C_1 \in O(W_1) \} = \mathbb{T}. \quad (3.2)
\]

One can easily see that the map \( \mathbb{F}^2 \rightarrow \mathbb{R} \oplus F, \ (\lambda, \mu) \rightarrow (|\lambda|^2 - |\mu|^2, \lambda \mu) \) is surjective, and its fibres coincide with the orbits of the representation \( \mathbb{T} : \mathbb{F}^2, \ c : (\lambda, \mu) \rightarrow (c \lambda, c^{-1} \mu) \). By (3.2), the map \( \gamma \) is surjective, and its fibres coincide with the \( (O(W_1)) \)-orbits. It follows from above
that the map $\gamma \circ \pi_1 \circ \tilde{\psi}: \text{Hom}_F(W_1 \oplus \mathbb{R}^k, W) \to V' \oplus V_1 \oplus \mathbb{R} \oplus F$ is $(O(W) \times O(W_1))$-equivariant and surjective, and its fibres coincide with the $(SO(W) \times O(W_1))$-orbits.

Recall that the map $\pi_0$ is $(O(W) \times O(W_1))$-equivariant and surjective, and its fibres coincide with the $(O(W_1))$-orbits. Therefore, there exists an $(O(W))$-equivariant surjective map $\pi: (S(W')/(\mathbb{R}E)) \oplus \text{Hom}_F(\mathbb{R}^k, W) \to V' \oplus V_1 \oplus \mathbb{R} \oplus F$ satisfying $\pi \circ \pi_0 \equiv \gamma \circ \pi_1 \circ \tilde{\psi}$ whose fibres coincide with the $(SO(W))$-orbits. Since the maps $\gamma, \pi_1, \tilde{\psi}, \pi_0$ are semialgebraic, continuous, and surjective, so is the map $\pi$.

This completely proves the claim in the case $k = 1$.

**Case 2.** $k = 2$.

Since $\dim F W_1 = n - 1$, there exist a real vector space $V_1$ and a semialgebraic continuous surjective map $\pi_1: V' \oplus (S(W_1)/(\mathbb{R}E)) \oplus \text{Hom}_F(\mathbb{R}^k, W_1) \to V' \oplus V_1$ whose fibres coincide with the $(O(W_1))$-orbits. Further, $\dim F(W_1 \oplus \mathbb{R}^k) = (n-1)+k = n+1 = \dim F W + 1$, and, by Corollary 2.3, the map $\psi$ is surjective, and its fibres coincide with the $(O(W))$-orbits. Since the map $\psi$ is $(O(W) \times O(W_1))$-equivariant, the map $\pi_1 \circ \psi: \text{Hom}_F(W_1 \oplus \mathbb{R}^k, W) \to V' \oplus V_1$ is surjective, and its fibres coincide with the $(O(W) \times O(W_1))$-orbits. Recall that the map $\pi_0$ is $(O(W) \times O(W_1))$-equivariant and surjective, and its fibres coincide with the $(O(W_1))$-orbits. Hence, there exists a surjective map $\pi: (S(W')/(\mathbb{R}E)) \oplus \text{Hom}_F(\mathbb{R}^k, W) \to V' \oplus V_1$ satisfying $\pi \circ \pi_0 \equiv \pi_1 \circ \psi$ whose fibres coincide with the $(O(W))$-orbits. Since the maps $\pi_1, \psi, \pi_0$ are semialgebraic, continuous, and surjective, so is the map $\pi$.

This completely proves the claim in the case $k = 2$.

Thus, we have completely proved Theorem 1.1 (using mathematical induction on $\dim F W$ separately for each number $k \in \{1, 2\}$).
References

[1] M. A. Mikhailova, *On the quotient space modulo the action of a finite group generated by pseudoreflections*, Mathematics of the USSR-Izvestiya, 1985, vol. 24, 1, 99—119.

[2] O. G. Styrt, *On the orbit space of a compact linear Lie group with commutative connected component*, Tr. Mosk. Mat. O-va, 2009, vol. 70, 235—287 (Russian).

[3] O. G. Styrt, *On the orbit space of a three-dimensional compact linear Lie group*, Izv. RAN, Ser. math., 2011, vol. 75, 4, 165—188 (Russian).

[4] O. G. Styrt, *On the orbit space of an irreducible representation of a special unitary group*, Tr. Mosk. Mat. O-va, 2013, vol. 74, 1, 175—199 (Russian).

[5] O. G. Styrt, *On the orbit spaces of irreducible representations of simple compact Lie groups of types B, C, and D*, J. Algebra, 2014, vol. 415, 137—161.