Use of a physical metric for OPERA experiment

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Abstract

A physical metric is introduced as one that directly gives experimental data without further coordinate transformation. It will be shown that the geodesic equation for the Schwartzschild metric does not give the correct expression for the Shapiro time delay experiment. In the physical metric the speed of light on the surface of the earth is exactly the same as that in vacuum to first order in gravity, and this should eliminate the inconsistency of the speed of the neutrino exceeding that of light, if enough care is exerted in the discussion.

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I. INTRODUCTION

It is well known that the coordinates of the Schwartzschild metric do not correspond to observable physical coordinates\[1],[2]. The author has introduced a coordinate transformation for the Schwartzschild metric, so that the geodesic equation for the time delay of light propagation in a gravitational field gives the correct result for Shapiro’s experiment[3]. We call such a metric a physical metric. We start by constructing such a metric.

II. ASYMPTOTIC FORM FOR THE PHYSICAL METRIC

The physical metric is expressed as
\[ ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - e^{\mu(r)} r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  
(1)
for a spherically symmetric and static mass point $M$. From the fact that the transformation, $r' = re^{\mu(r)/2}$, leads to the Schwartzschild metric, one can deduce the expression for the metric,
\[ e^{\nu(r)} = 1 - (r_s/r)e^{-\mu(r)/2}, \]  
(2)
\[ e^{\lambda(r)} = (\frac{d}{dr}(re^{\mu(r)/2}))^2/(1 - (r_s/r)e^{-\mu(r)/2}), \]  
(3)
where $r_s = 2GM/c^2$ is the Schwartzschild radius. An asymptotic expansion for the metric functions can be obtained from Eq. (2) and Eq. (3), yielding
\[ e^{\nu(r)} = \sum_{n=0}^{\infty} a_n (r_s/r)^n, \]  
(4)
\[ e^{\lambda(r)} = \sum_{n=0}^{\infty} b_n (r_s/r)^n, \]  
(5)
\[ e^{\mu(r)} = \sum_{n=0}^{\infty} c_n (r_s/r)^n, \]  
(6)
where
\[ a_0 = b_0 = c_0 = 1, \]  
(7)
\[ -a_1 = b_1 = 1 \quad \text{and} \]  
(8)
\[ a_2 = c_1/2, \quad b_2 = 1 - c_1/2 + c_1^2/4 - c_2, \quad \text{etc.} \]  
(9)
It is obvious that $a_{n+1}$ and $b_n$ can be expressed as functions of $c_n, c_{n-1}, \ldots, c_1$.

III. GEODESIC EQUATIONS AND TIME DELAY

The geodesic equations can be obtained from variations of the line integral over an invariant parameter $\tau$, $\int (\frac{dx}{d\tau})^2 d\tau$, and their integrals are given by
\[
\frac{dt}{d\tau} = e^{-\nu(r)},
\]
\[
\frac{d\phi}{d\tau} = J_\phi e^{-\mu(r)} / (r \sin \theta)^2,
\]
\[
\left(\frac{d\theta}{d\tau}\right)^2 = (J_\theta^2 - J_\phi^2 / \sin^2 \theta) e^{-2\mu(r)} / r^4.
\]
Restricting the plane of motion to \(\frac{d\phi}{d\tau} = 0, \theta = \pi/2\), the radial part of the geodesic integral is given by
\[
\left(\frac{dr}{d\tau}\right)^2 = e^{-\lambda(r)} (e^{-\nu(r)} - J_\phi^2 e^{-\mu(r)} / r^2 - E)
\]
where \(J_\phi, J_\theta\) and \(E\) are constants of integration and
\[
J^2 = J_\phi^2 = J_\theta^2.
\]
The constant \(E\) is 0 for light propagation.

From Eq. (8) and Eq. (11) with Eqs. (5) and (6), it follows that
\[
\frac{dt}{dr} = \pm e^{-\nu(r)}/\sqrt{e^{-\nu(r)-\lambda(r)} - J^2 e^{-\mu(r)-\lambda(r)}/r^2}
\]
\[
= \pm \frac{rr}{\sqrt{r^2 - r_0^2}} \left(1 + \frac{(b_1 - a_1) r_s}{2r} + \frac{(c_1 - a_1) r_0 r_s}{2 (r + r_0)} + \cdots\right)
\]
for light propagation, where \(r_0\) is the impact parameter, and
\[
J^2 = r_0^2 e^{-\nu(r_0)} + \mu(r_0).
\]
Integrating from \(r_0\) to \(r\), one gets the time delay expression for light propagation,
\[
\triangle t = r_s \left(\ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right) + \frac{(c_1 + 1) r_s}{2 \sqrt{r - r_0}}} + \cdots\right.
\]
For the Schwartzschild metric, \(c_1 = 0\), the time delay is given by
\[
\triangle t = r_s \left(\ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right) + \frac{1}{2} \sqrt{r - r_0} + \cdots\right),
\]
which agrees with the calculation of Weinberg [4].

Since Shapiro’s observation [3] fits the formula
\[ \triangle t = r_s \ln \left( \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + \cdots \]  

(18)

to high accuracy (1 in 1000), we conclude that

\[ c_1 = -1. \]  

(19)

We note that the parameter values

\[ a_1 = -1, \text{ and } b_1 = 1 \]  

(20)

are coordinate independent and determined from the solution of the Einstein equation and the physical boundary condition. Thus we conclude that Eq. (19), along with Eq. (20), is the condition for the physical metric. It is important to notice that the speed of light on the surface of a sphere is identical to that in vacuo, \( c \), in the physical metric, while it is less than \( c \) in the Schwartzschild metric,

\[ \frac{c_S}{c} = (1 - r_s/r)^{1/2}, \]  

(21)

where

\[ r_s = \frac{2GM_E}{c^2} = 0.886 \text{ cm} \]  

(22)

and

\[ r_s/r = 1.39 \times 10^{-9} \]  

(23)

on the surface of the Earth.

IV. THE SPEED OF NEUTRINOS IN THE OPERA EXPERIMENT

The speed of a neutrino of 17 GeV with the assumed mass of the neutrino, \( m_\nu c^2 = 0.1 \) eV, can be calculated from

\[ E_\nu = \frac{m_\nu c^2}{\sqrt{1 - (v/c)^2}} = 17 \text{GeV} \]  

(24)

resulting in

\[ \frac{v}{c} = 1 - 1.73 \times 10^{-23}. \]  

(25)

Obviously

\[ c > v > c_S = c \left( 1 - 0.695 \times 10^{-9} \right) \]  

(26)
It is clear that if the physical metric were to be used from the beginning, the problem of the
speed of the neutrino exceeding that of the light would not occur. It exceeds the speed of
light in the Schwarzschild metric, $c_S$, but that conflicts with the time delay experiment of
Shapiro. Whenever the gravitational time delay formula is used in GPS operation, you might
be using the Schwarzschild metric unless extreme care is utilized. One way to remedy the
deficiency is to reduce distances on the surface of the earth by a factor of $(1 - 0.695 \times 10^{-9})$,
the same factor as that of the gravitational time delay on the Earth, so that the speed of
light becomes that in vacuum, $c$.

Shapiro’s time delay experiment has been successfully applied to a neutron star binary [6].
This implies that the formula in Eq. (18) is valid to higher order in gravity. Then, the
physical metric to higher order may be neccessary and important. For such a work, see the
reference of the present author [7].

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