BIDILATION OF SMALL LITTLEWOOD-RICHARDSON COEFFICIENTS

PIERRE-EMMANUEL CHAPUT AND NICOLAS RESSAYRE

Abstract. The Littlewood-Richardson coefficients $c^\nu_{\lambda,\mu}$ are the multiplicities in the tensor product decomposition of two irreducible representations of the general linear group $GL(n, \mathbb{C})$. They are parametrized by the triples of partitions $(\lambda, \mu, \nu)$ of length at most $n$. By the so-called Fulton conjecture, if $c^\nu_{\lambda,\mu} = 1$ then $c^{k\nu}_{k\lambda,k\mu} = 1$, for any $k \geq 0$. Similarly, as proved by Ikenmeyer or Sherman, if $c^\nu_{\lambda,\mu} = 2$ then $c^{k\nu}_{k\lambda,k\mu} = k + 1$, for any $k \geq 0$.

Here, given a partition $\lambda$, we set $\lambda(p, q) = (p\lambda_1, \ldots, p\lambda_1, p\lambda_2, \ldots, p\lambda_2, \ldots, p\lambda_n, \ldots, p\lambda_n)$, where each part is repeated $q$ times. Fulton’s conjecture implies that if $c^\nu_{\lambda,\mu} = 1$ then $c^{\nu(p,q)}_{\lambda(p,q),\mu(p,q)} = 1$, for any $p, q \geq 0$. Our main result is that if $c^\nu_{\lambda,\mu} = 2$ then $c^{\nu(p,q)}_{\lambda(p,q),\mu(p,q)}$ is the binomial $(p+q\choose q)$, for any $p, q \geq 0$.

1. Introduction

Fix an $n$-dimensional vector space $V$. Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ with $\lambda_i \in \mathbb{N}$, let $S^\lambda V$ be the corresponding Schur module, that is the irreducible $GL(V)$-module of highest weight $\sum \lambda_i \epsilon_i$ (notation as in [Bou02]). This paper is concerned by the Littlewood-Richardson coefficients $c^\nu_{\lambda,\mu}$, defined by

$$S^\lambda V \otimes S^\mu V \simeq \bigoplus_{\nu} C^{\nu}_{\lambda,\mu} \otimes S^\nu V,$$

where $C^{\nu}_{\lambda,\mu}$ is a multiplicity space. Given a partition $\lambda$ as above, we set $\lambda(p, q) = (p\lambda_1, \ldots, p\lambda_1, p\lambda_2, \ldots, p\lambda_2, \ldots, p\lambda_n, \ldots, p\lambda_n)$ where each part is repeated $q$ times. Fulton’s conjecture (see [KTW04, Bel07, Res11a] for various proofs) can be restated as:

**Theorem 1.** If $c^\nu_{\lambda,\mu} = 1$ then, for any positive $p$ and $q$, we have

$$c^{\nu(p,q)}_{\lambda(p,q),\mu(p,q)} = 1.$$

The ordinary formulation of Fulton’s conjecture corresponds to the case $q = 1$. The general case follows from the equality $c^{\nu'}_{\lambda',\mu'} = c^\nu_{\lambda,\mu}$, where $\lambda'$ denotes the conjugated partition of $\lambda$. This ordinary version has an extension to the case $c^\nu_{\lambda,\mu} = 2$, see [Ike16] and [She17, Theorem 1.1 and Corollary 9.4] for a generalization in the context of quivers.

This work was supported by the ANR GeoLie project, of the French Agence Nationale de la Recherche.
Theorem 2. If $c_{\lambda,\mu} = 2$ then, for any positive integers $p, q$, we have

$$c_{\lambda(p,1),\mu(p,1)} = p + 1$$ and $$c_{\lambda(1,1),\mu(1,1)} = q + 1.$$ 

Our main result is an extension of Theorem 2 in the spirit of Theorem 1:

Theorem 3. If $c_{\lambda,\mu} = 2$ then, for any positive integers $p, q$, we have

$$c_{\lambda(p,q),\mu(p,q)} = \binom{p + q}{q}.$$ 

Here, $(p+q)_q$ stands for the binomial. Ikenmeyer proved Theorem 2 using convex geometry and integral points counting, whereas we use Geometric Invariant Theory. An example of a triple of partitions $(\lambda, \mu, \nu)$ such that $c_{\lambda(p,q),\mu(p,q)} = \binom{p + q}{q}$ is given in [KTW04, Example 6.2].

The main idea for the value $c_{\lambda(p,q),\mu(p,q)} = (p+q)_q$ is the following (although the proof of the following claims is less direct than what is presented in this introduction). First, letting $G = \text{GL}_n$, we interpret the coefficient $c_{\lambda,\mu}$ as the dimension of a space of $G$-invariant sections of a line bundle $L$ on the product $X$ of three flag varieties under the group $G$. The coefficient $c_{\lambda(p,1),\mu(p,1)}$ is then simply the dimension of $H^0(X, L \otimes^p G)$. The coefficient $c_{\lambda(p,q),\mu(p,q)}$ in turn has a geometrical definition dilating the flag variety $X$. More precisely, we replace $X$ by $X(q)$ which is a product of partial flag varieties for $G(q) := \text{GL}_{nq}$, and we replace $L$ by some line bundle $L(q)$. We get:

$$c_{\lambda(p,q),\mu(p,q)} = \dim H^0(X(q), L(q) \otimes^p G(q)).$$

Using properties of the Horn cone proved in [DW11, Res11b], we observe that if $(\lambda, \mu, \nu)$ is not general, then $c_{\lambda,\mu}$ is in fact the product of two Littlewood-Richardson coefficients for smaller linear groups, and we conclude by induction.

By results of Ikenmeyer and Sherman [Ike16, She15], the polarized GIT-quotient $X^{ss}(L)//G$ is isomorphic to $(P^1, \mathcal{O}_{P^1}(1))$. The equality $c_{\lambda(p,1),\mu(p,1)} = p + 1$ is explained by the equality $\dim H^0(P^1, \mathcal{O}_{P^1}(p)) = p + 1$. We produce in [15] an inclusion of $X^q$ in $X(q)$. If $(\lambda, \mu, \nu)$ is general, then the codimension of a general $G$-orbit in $X$ has codimension 1, and we show that the codimension of a general $G(q)$-orbit in $X(q)$ will have codimension $q$, from which we deduce that the restriction induces an isomorphism

$$H^0(X(q), L(q) \otimes^p G(q)) ///G(q) \rightarrow H^0(X^q, \mathcal{O}_{X^q})^N_{G(q)}(X^q),$$

where $N_{G(q)}(X^q)$ denotes the stabilizer of $X^q$ in $G(q)$. Therefore, understanding the GIT-quotient $X(q)^{ss}(L(q))//G(q)$ comes down to understanding the GIT-quotient $(X^q)^{ss}(L(q))//N_{G(q)}(X^q)$. The action of $N_{G(q)}(X^q)$ on $X^q$ is given by the action of $G(q)$ on $X^q$ and the permutation of the $q$ factors, from which it follows that the quotient $X(q)^{ss}(L(q))//G(q)$ is isomorphic to the quotient of $(X^{ss}(L)//G)^q$ by the symmetric group $S_q$, which is $(P^1)^q//S_q$, namely $P^q$.

It follows that the polarized GIT-quotient $X(q)^{ss}(L(q))//G(q)$ is $(P^q, \mathcal{O}_{P^q}(1))$ (see Corollary 15), and taking the $p$-th power of the polarization, we obtain our binomial coefficient as the number $\dim H^0(P^q, \mathcal{O}_{P^q}(p))$. 
2. **G-ample cone of flag varieties**

2.1. **GIT-quotient.** Let $G$ be a complex connected reductive group acting on an irreducible projective variety $X$. Let $\text{Pic}^G(X)$ denote the group of $G$-linearized line bundles on $X$. For $\mathcal{L} \in \text{Pic}^G(X)$, $H^0(X, \mathcal{L})$ denotes the $G$-module of regular sections of $\mathcal{L}$ and $H^0(X, \mathcal{L}^G)$ denotes the subspace of $G$-equivariant sections. For any $\mathcal{L} \in \text{Pic}^G(X)$, we set

$$X^\text{ss}(\mathcal{L}, G) = \{ x \in X : \exists n > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^\otimes n)^G \text{ s.t. } \sigma(x) \neq 0 \}.$$ 

Note that this definition of $X^\text{ss}(\mathcal{L})$ coincides with that of [MFK94, Definition 1.7] if $\mathcal{L}$ is ample but not in general.

Assuming that $X^\text{ss}(\mathcal{L})$ is not empty, consider the following projective variety

$$X^\text{ss}(\mathcal{L})/G := \text{Proj} \left( \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^\otimes n)^G \right),$$

and the natural $G$-invariant morphism

$$\pi : X^\text{ss}(\mathcal{L}) \to X^\text{ss}(\mathcal{L})/G.$$ 

If $\mathcal{L}$ is ample then $\pi$ is a good quotient and, in particular, the points in $X^\text{ss}(\mathcal{L})/G$ correspond to the closed $G$-orbits in $X^\text{ss}(\mathcal{L})$.

2.2. **The G-ample cone.** We assume here that $\text{Pic}^G(X)$ has finite rank and we consider the rational vector space $\text{Pic}^G(X)_\mathbb{Q} := \text{Pic}^G(X) \otimes_\mathbb{Z} \mathbb{Q}$. Since $X^\text{ss}(\mathcal{L}) = X^\text{ss}(\mathcal{L}^\otimes n)$ for any positive integer $n$, $X^\text{ss}(\mathcal{L})$ can be defined for any element $\mathcal{L}$ in $\text{Pic}^G(X)_\mathbb{Q}$. The set of ample line bundles in $\text{Pic}^G(X)$ generates an open convex cone $\text{Pic}^G(X)_\mathbb{Q}^+$ in $\text{Pic}^G(X)_\mathbb{Q}$. The following cone was defined in [DH98] and is called the $G$-ample cone:

$$\mathcal{AC}^G(X) := \{ \mathcal{L} \in \text{Pic}^G(X)_\mathbb{Q}^+ : X^\text{ss}(\mathcal{L}) \neq \emptyset \}.$$ 

Accordingly, a line bundle $\mathcal{L} \in \text{Pic}^G(X)$ is said to be $G$-ample if $X^\text{ss}(\mathcal{L})$ is not empty. Since the product of two nonzero $G$-equivariant sections of two line bundles is a nonzero $G$-equivariant section of the tensor product of the two line bundles, $\mathcal{AC}^G(X)$ is convex: see [DH98] Propositions 3.1.2, 3.1.3 and Definition 3.2.1].

Let $\text{Eqd}(X, G)$ denote the minimal codimension of $G$-orbits in $X$. By [Res12, Proposition 4.1], the expected quotient dimension is the maximal dimension of the quotients:
Proof. The quotient map \( \pi : X^{ss}(\mathcal{L})\to X^{ss}(\mathcal{L})//G \) being affine, this follows from [Lun75] Theorem 2. \( \square \)

2.3. Restriction to the \( \tau \)-fixed locus. Let \( \tau : \mathbb{C}^* \to G \) be a one parameter subgroup. Let \( C \) be an irreducible component of the \( \tau \)-fixed point set \( X^\tau \). Let \( \mathcal{L} \) be an ample \( G \)-linearized line bundle.

Since the centralizer \( G^\tau \) of \( \tau \) is connected, it acts on \( C \). Moreover, by Luna (see e.g. [Res10] Proposition 8), we have

\[
C^{ss}(\mathcal{L}(C), G^\tau) = X^{ss}(\mathcal{L}, G) \cap C.
\]

Here \( \mathcal{L}(C) \) stands for the restriction of \( \mathcal{L} \) to \( C \). Thus, the following defines a morphism

\[
p : C^{ss}(\mathcal{L}(C), G^\tau)//G^\tau \to X^{ss}(\mathcal{L}, G)//G.
\]

Lemma 5. The morphism \( p \) is finite on its image.

Proof. The quotient map \( \pi : X^{ss}(\mathcal{L})\to X^{ss}(\mathcal{L}, G)//G \) being affine, this follows from [Lun75] Theorem 2. \( \square \)

2.4. Notations about flag varieties and line bundles on them. Let \( V \) be a finite dimensional complex vector space. Fix a basis \((e_1, \ldots, e_n)\) of \( V \) and identify the linear group \( \text{GL}(V) \) with \( \text{GL}_n(\mathbb{C}) \). Let \( T \subseteq \text{GL}(V) \) (resp. \( B \subseteq \text{GL}(V) \)) be the maximal torus (resp. Borel subgroup) containing all diagonal (resp. upper triangular) matrices. Let \( \epsilon_i : T \to \mathbb{C}^* \) denote the character mapping \( t \in T \) on its \( i \)-th diagonal entry. Note that \((\epsilon_i)_{1 \leq i \leq n}\) forms a \( \mathbb{Z} \)-basis of the character group \( X(T) \) of \( T \). Moreover, the set \( X(T)^+ \) of dominant characters identifies with

\[
\Lambda_n^+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \text{ with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\},
\]

by mapping \((\lambda_1, \ldots, \lambda_n) \to \sum \lambda_i \epsilon_i \). Let \( \varpi_i = \epsilon_1 + \cdots + \epsilon_i \) be the \( i \)-th fundamental weight.

Given integers \( 0 < a_1 < \cdots < a_r < n \), we let \( \text{Fl}(a_1, \ldots, a_r; V) \) denote the corresponding partial flag variety:

\[
\text{Fl}(a_1, \ldots, a_r; V) = \{ V_1 \subseteq \cdots \subseteq V_r \subseteq V : \dim(V_i) = a_i \}.
\]

This will also be denoted \( \text{Fl}(A; V) \) where \( A = \{a_1, \ldots, a_r\} \). The standard base point \( \xi_0 \) in \( \text{Fl}(A; V) \) is defined by letting \( V_i \) be the span of \( e_j \)'s for \( j \leq a_i \). The stabilizer of \( \xi_0 \) in \( \text{GL}(V) \) is denoted by \( P \). Moreover \( \varpi_i \) extends to a character of \( P \) if and only if \( i \in A \). In this case, this defines a \( G \)-linearized line bundle \( \text{GL}(V) \times^P \mathbb{C}^- \varpi_i \) on \( \text{Fl}(A; V) \), and its space of sections is \( \wedge^i V^* \), as a \( G \)-representation.

More generally, given \((\lambda_1, \ldots, \lambda_n) \in \Lambda_n^+ \), we define the line bundle

\[
\mathcal{L}_\lambda = \text{GL}(V) \times^P \mathbb{C}^- \lambda \text{ where } \lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i = \sum_{i=1}^{n} (\lambda_i - \lambda_{i+1}) \varpi_i,
\]

with the convention \( \lambda_{n+1} = 0 \). It is well-defined if and only if \( \lambda \) is a weight of \( P \), which means that

\[
0 < i < n \text{ and } \lambda_i > \lambda_{i+1} \implies i \in A.
\]

Moreover, \( \mathcal{L}_\lambda \) is ample on \( \text{Fl}(A; n) \) if and only if this is an equivalence:

\[
0 < i < n \text{ and } \lambda_i > \lambda_{i+1} \iff i \in A.
\]
Borel-Weil theorem says that
\[ H^0(\text{Fl}(A; V), L_\lambda) = S^\lambda V^*, \]
where \( S^\lambda \) is the Schur functor associated to \( \lambda \).

Finally, if \( X = \text{Fl}(A^1; V) \times \cdots \times \text{Fl}(A^k; V) \) is a product of \( k \) flag varieties, and \( \lambda^1, \ldots, \lambda^k \) are in \( \Lambda^+_n \) such that each pair \((\lambda^j, A^j)\) satisfies (7), we define the following line bundle on \( X \):
\[ L_{(\lambda^1, \ldots, \lambda^k)} = L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda^k}. \]
Thus, \( H^0(X, L_{(\lambda^1, \ldots, \lambda^k)}) = S^{\lambda^1} V^* \otimes \cdots \otimes S^{\lambda^k} V^*. \)

2.5. **The Horn cone of \( \text{GL}_n \).** Let \( k \) be an integer. The cone inside \((\mathbb{Q}^n)^k\) generated by the \( k \)-uples \((\lambda^j)\) in \( \Lambda^+_n \) such that \((S^{\lambda^1} V^* \otimes \cdots \otimes S^{\lambda^k} V^*)^{\text{GL}(V)} \neq \{0\}\) is called the **Horn cone** and has a description that we now recall.

Let \( I \subseteq \{1, \ldots, n\} \) be a subset with \( r \) elements. The linear subspace \( V_I \subseteq V \) generated by the base vectors \( e_i \) for \( i \in I \) defines a \( T \)-fixed point in the Grassmannian \( \mathbb{G}(r; V) \). The cohomology class of the closure of the \( B^- \)-orbit through this point will be denoted by \( \sigma_I \). Here \( B^- \) denotes the Borel subgroup of \( \text{GL}(V) \) consisting in lower triangular matrices.

For \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \) let \( |\lambda| = \sum_i \lambda_i \). By [Kly98] [Bel01], the \( k \)-uple \((\lambda^j)\) belongs to the Horn cone if and only if
\[ \sum_{j=1}^k |\lambda^j| = 0, \]
and the following holds for all integers \( r \in \{1, \ldots, n-1\} \) and all \( k \)-uples \((I^j)_{1 \leq j \leq k}\) of subsets of \( \{1, \ldots, n\} \) with \( r \) elements:
\[ \sigma_{I^1} \cup \cdots \cup \sigma_{I^k} = [pt] \in H^*(\mathbb{G}(r; V), \mathbb{Z}) \implies \frac{1}{n} \sum_{j=1}^k |\lambda^j_{I^j}| \leq \frac{1}{n} \sum_{j=1}^k |\lambda^j|. \]
Here, \( \lambda_I \) denotes the partition obtained by taking the parts \( \lambda_i \) for \( i \in I \). Moreover, by [Klw03] (see also [Res10]), for such \( I^1, \ldots, I^k \), each equation \( \frac{1}{n} \sum_{j=1}^k |\lambda^j_{I^j}| = \frac{1}{n} \sum_{j=1}^k |\lambda^j| \) defines a face of codimension 1 in the Horn cone.

2.6. **The G-ample cone of products of flag varieties.** We now assume that
\[ X = \text{Fl}(A^1; V) \times \cdots \times \text{Fl}(A^k; V) \]
is a product of flag varieties homogeneous under the group \( G = \text{GL}(V) \).

**Proposition 6.** A \( G \)-equivariant line bundle \( L_{(\lambda^j)} \) on \( X \) given by a \( k \)-uple \((\lambda^j)_{1 \leq j \leq k}\) in \( (\Lambda^+_n)^k \) is \( G \)-ample if and only if \( \sum_{j=1}^k |\lambda^j| = 0 \), and
\[ \lambda^j_i > \lambda^j_{i+1} \iff i \in A^j \]
\[ \sigma_{I^1} \cup \cdots \cup \sigma_{I^k} = [pt] \implies \frac{1}{n} \sum_{j=1}^k |\lambda^j_{I^j}| \leq \frac{1}{n} \sum_{j=1}^k |\lambda^j|. \]

**Proof.** The first condition is equivalent to \( L_{(\lambda^j)} \) being ample on \( X \), and the second condition is equivalent to \( H^0(X, L)^G \) being non trivial: recall respectively (8) and (11). \( \square \)
3. Geometric formulation of the main theorem

For $\nu = (\nu_1 \geq \cdots \geq \nu_n)$ in $\Lambda_n^+$, set $\nu^0 = (-\nu_n \geq \cdots \geq -\nu_1)$ such that $S^{\nu^0}V^*$ is the $\text{GL}(V)$-representation dual to $S^pV^*$. Moreover, let $c_{\lambda,\mu,\nu} = c_{\lambda,\mu,\nu}^{(1)}$. Since $\nu(p,q)^0 = \nu^0(q,p)$, our main Theorem 3 is equivalent to the implication

\[(14) \quad c_{\lambda,\mu,\nu} = 2 \implies \dim \left( H^0(X,\mathcal{L}(\lambda,\mu,\nu)) \right) = \binom{p+q}{q}.\]

Thus, let $\lambda,\mu,\nu$ be such that $c_{\lambda,\mu,\nu} = 2$. For $\eta \in \Lambda_n^+$, let $A(\eta)$ be the set $j \in \{1,\ldots,n-1\}$ such that $\eta_j > \eta_{j+1}$. We fix the product of three partial flag varieties and the ample line bundle $\mathcal{L} := \mathcal{L}(\lambda,\mu,\nu)$ on $X$, such that $H^0(X,\mathcal{L}) = S^\lambda V^* \otimes S^\mu V^* \otimes S^\nu V^*$ (see Section 2.4.

Fix a $q$-dimensional vector space $E$. If $\mathcal{F} = \text{Fl}(a_1,\ldots,a_q;V)$, set $\mathcal{F}(q) = \text{Fl}(qa_1,\ldots,qa_q;V \otimes E)$. For $\eta \in \Lambda_n^+$, let $\eta(1,q)$ denote the partition with each part $\lambda_i$ repeated $q$ times. Observe that if $\mathcal{L}_\eta$ is a line bundle (resp. ample line bundle) on $\mathcal{F}$, then $\mathcal{L}_\eta(q)$ is a line bundle (resp. ample line bundle) on $\mathcal{F}(q)$, by (11). Now, set $X(q) = \text{Fl}(A(\lambda);V)(q) \times \text{Fl}(A(\mu);V)(q) \times \text{Fl}(A(\nu);V)(q)$ and let $\mathcal{L}(q)$ be the line bundle $\mathcal{L}(\lambda(1,q),\mu(1,q),\nu(1,q))$ on $X(q)$. Then $c_{\lambda,\mu,\nu} = \dim \left( H^0(X,\mathcal{L}(\lambda,\mu,\nu))^G \right)$ and

\[(15) \quad c_{\lambda(\mu,\nu),\mu(\nu),\nu} = \dim \left( H^0(X(q),\mathcal{L}(q)^\otimes p)^G(q) \right),\]

where $G = \text{GL}(V)$ and $G(q) = \text{GL}(V \otimes E)$. Hence, our main theorem can be rephrased as the implication

\[(16) \quad \dim \left( H^0(X,\mathcal{L})^G \right) = 2 \implies \dim \left( H^0(X(q),\mathcal{L}(q))^{\otimes p}G(q) \right) = \binom{p+q}{q}.\]

4. Preparation of the proof of the main theorem

In this section, we fix $V$, $E$, $G$ and $G(q)$ as in Section 3. Fix also $k \geq 3$ and $A^1,\ldots,A^k$ subsets of $\{1,\ldots,n\}$. Consider the varieties

\[X = \text{Fl}(A^1;V) \times \cdots \times \text{Fl}(A^k;V),\]

and

\[X(q) = \text{Fl}(QA^1;V \otimes E) \times \cdots \times \text{Fl}(QA^k;V \otimes E).\]

4.1. A key construction. A key observation is that $X^q$ embeds in $X(q)$. To make this embedding explicit, fix a basis $(f_1,\ldots,f_q)$ of $E$. Let $\tau$ be a regular diagonal (with respect to the fixed basis) one-parameter subgroup of $\text{GL}(E)$.

The group $\text{GL}(E)$, and hence $\tau$, act on $V \otimes E$. A linear subspace $F \subset V \otimes E$ is $\tau$-stable if and only if there exist subspaces $(F_i)_{1 \leq i \leq q}$ of $V$ such that

\[F = F_1 \otimes \mathbb{C}f_1 + \cdots + F_q \otimes \mathbb{C}f_q.\]

Furthermore, the map

\[
\begin{align*}
\text{Gr}(a;V)^q & \quad \rightarrow \quad \text{Gr}(aq;V) \\
(F_i)_{1 \leq i \leq q} & \quad \mapsto \quad F_1 \otimes \mathbb{C}f_1 + \cdots + F_q \otimes \mathbb{C}f_q,
\end{align*}
\]
is an isomorphism onto an irreducible component of the \( \tau \)-fixed point set. Similarly, \( \text{Fl}(A; V)^q \) (resp. \( X \)) embeds in \( \text{Fl}(qA; V \otimes E)^q \) (resp. \( X(q) \)) as an irreducible component of \( \tau \)-fixed points. Denote by

\[
\tau_q : X^q \rightarrow C \subset X(q),
\]

the corresponding embedding and by \( C \) its image. It is equivariant for the action of \( G(q)^r \), that is isomorphic to \( G^q \).

4.2. Expected quotient dimension of \( X(q) \). For \( X = \text{Fl}(A^1) \times \cdots \times \text{Fl}(A^k) \) as above, we introduce some more notation:

**Notation 1.** Given \( x, y \) in \( X \), write these elements as \( x = (l^1, \ldots, l^k) \) and \( y = (m^1, \ldots, m^k) \) with \( l^i, m^i \) in \( \text{Fl}(A^j; V) \).

- Let \( \text{trans}(x, y) \) denote the subspace in \( \text{gl}(V) \) of the endomorphisms such that for any \( j \in \{1, \ldots, k\} \) and any \( i \in A^j \), \( (l^j)_i \) is sent into \( (m^j)_i \).
- Let \( \text{stab}(x) := \text{trans}(x, x) \).
- Let \( s_{\text{gen}} \) be the dimension of the vector space \( \text{stab}(x) \) for general \( x \) in \( X \).
- Let \( t_{\text{gen}} \) be the dimension of the vector space \( \text{trans}(x, y) \) for general \( (x, y) \) in \( X^2 \).

**Lemma 7.** With the above notation:

1. We always have \( t_{\text{gen}} \leq s_{\text{gen}} \);
2. If \( \text{Eqd}(X, G) > 0 \), then \( t_{\text{gen}} \leq s_{\text{gen}} - 1 \).

**Proof.** The function \( (x, y) \mapsto \dim \text{trans}(x, y) \) is upper semi-continuous on \( x \) and \( y \), hence the first point. For the second point, we assume \( t_{\text{gen}} = s_{\text{gen}} \) and we prove that \( \text{Eqd}(X, G) = 0 \). Let \( U \) be the set of \( (x, y) \in X^2 \) such that \( \dim \text{trans}(x, y) = t_{\text{gen}} \).

The theory of linear systems implies that

\[
\mathcal{E} := \{(x, y, \xi) \in U \times \text{gl}(V) : \xi \in \text{trans}(x, y)\}
\]

is a vector bundle on \( U \).

Therefore, the set \( \Sigma \) of pairs \( (x, y) \in U \) such that \( \text{trans}(x, y) \subset \{\det = 0\} \subset \text{gl}(V) \) is closed in \( U \). Since \( \text{stab}(x) = \text{trans}(x, x) \) contains the identity map of \( V \) for any \( x \in X \), \( \Sigma \) does not intersect the diagonal \( \Delta = \{(x, x) : x \in X\} \).

But, the assumption \( t_{\text{gen}} = s_{\text{gen}} \) implies that \( U \) intersects \( \Delta \). Hence \( \Sigma \) is a proper closed subset of \( U \). For any \( (x, y) \in U \setminus \Sigma \), \( \text{trans}(x, y) \) intersects \( \text{GL}(V) \), so \( x \) and \( y \) belong to the same \( \text{GL}(V) \)-orbit. Let \( p_1 : X \times X \to X \) be the first projection. For \( x \) in \( p_1(U) \) and \( y \) in the open subset \( p_1^{-1}(x) \cap U \), \( p_1^{-1}(x) \simeq X \), it follows that \( x \) and \( y \) are in the same \( G \)-orbit. Thus the \( G \)-orbit through \( x \) is dense in \( X \), and \( \text{Eqd}(X, G) = 0 \).

**Proposition 8.** With the above notation:

1. If \( \text{Eqd}(X, G) = 0 \) then \( \text{Eqd}(X(q), G(q)) = 0 \);
2. If \( \text{Eqd}(X, G) > 0 \) then \( \text{Eqd}(X(q), G(q)) \leq q^2(\text{Eqd}(X, G) - 1) + q \).

**Proof.** Let \( (x_1, \ldots, x_q) \in X^q \) be general in \( X^q \), and set \( y = \tau_q((x_1, \ldots, x_q)) \). We are interested in \( \text{stab}(y) \). The Lie algebra \( \text{gl}(V \otimes E) \) identifies with the set of \( (q \times q) \)-matrices with entries in \( \text{gl}(V) \). Accordingly, \( \text{stab}(y) \) decomposes as

\[
\text{stab}(y) = \bigoplus_{1 \leq i, j \leq q} \text{trans}(x_i, x_j) \otimes \text{Hom}(C_{f_i}, C_{f_j})
\]
which implies
\[ \dim \text{stab}(y) = \sum_{1 \leq i,j \leq q} \dim \text{trans}(x_i, x_j) = q s_{\text{gen}} + (q^2 - q) t_{\text{gen}}. \]

Assuming \( \text{Eqd}(X, G) = 0 \), we deduce from Lemma 7(1) that \( \dim \text{stab}(y) \leq q^2 s_{\text{gen}} = q^2 (\dim G - \dim X) \). It follows that the orbit \( G(q) \cdot y \) has dimension at least \( q^2 \dim G - q^2 (\dim G - \dim X) = q^2 \dim X = \dim X(q) \), so that \( X(q) \) has expected quotient dimension 0.

Assuming \( \text{Eqd}(X, G) > 0 \), set \( m = \text{Eqd}(X, G) \). We deduce from Lemma 7(2) that
\[ \dim \text{stab}(y) \leq q^2 s_{\text{gen}} - (q^2 - q) = q^2 (\dim G - \dim X + m) - (q^2 - q) = q^2 (\dim G - \dim X) + (m - 1)q^2 + q. \]
It follows that the orbit \( G(q) \cdot y \) has dimension at least \( q^2 \dim X - (m - 1)q^2 - q \), so that \( X(q) \) has expected quotient dimension at most \( (m - 1)q^2 + q \).

\[ \square \]

4.3. The stabilizer of \( C \) in \( G(q) \).

4.3.1. The statement. Recall from Section 4.1 the definition of \( C \).

**Proposition 9.** Let \( N_{\text{GL}(V \otimes E)}(C) := \{ g \in \text{GL}(V \otimes E) : g \cdot C = C \} \). We have
\[ N_{\text{GL}(V \otimes E)}(C) = \text{GL}(V)^q \ltimes S_q. \]

The proof of this proposition needs some preparation.

4.3.2. Sum of subspaces of constant dimension. The goal of this independent section is to prove some lemmas that will be useful to prove Proposition 9. We fix the following setting:

**Notation 2.** Let \( q \) be a positive integer, let \( E_1, \ldots, E_q, F \) be vector spaces, let \( \alpha_1, \ldots, \alpha_q : E_i \rightarrow F \) be linear maps, and let \( d_1, \ldots, d_q \) be integers such that \( 0 \leq d_i \leq \dim E_i \). Denote by \( S \) the sum of the subspaces \( \text{Im} \alpha_i \) for those \( i \) such that \( d_i = \dim E_i \).

We will analyse when it occurs that the dimension of \( \sum \alpha_i(U_i) \) does not depend on the vector subspaces \( U_i \subset E_i \) of dimension \( d_i \).

**Lemma 10.** Let \( \alpha : E \rightarrow F \) be a linear map between finite dimensional vector spaces, and let \( d \) be an integer between 0 and \( \dim E \). The set of all linear subspaces in \( F \) of the form \( \alpha(U) \) for \( U \subset E \) a subspace of dimension \( d \) is the set of all linear subspaces of \( \text{Im} \alpha \) of dimension between \( \max(0, d - \dim \ker \alpha) \) and \( \min(d, \text{rk} \alpha) \).

The proof of Lemma 10 will be omitted.

**Lemma 11.** Let \( q, \alpha_i : E_i \rightarrow F \) and \( d_i \) be as in Notation 2. Let \( V \subset F \) be a linear subspace. Then the set of the dimensions of the subspaces \( \sum \alpha_i(U_i) \cap V \), where \( U_i \) is any subspace in \( E_i \) of dimension \( d_i \), is an integer interval.

**Proof.** Let \( j \in \{1, \ldots, q\} \) be a fixed integer, and let \( (q - 1) \)-uple \( (U_i)_{i \neq j} \) of subspaces in the lemma be fixed. By Lemma 10 when \( U_j \) varies among the subspaces of \( E_i \) of dimension \( d_i \), the set of all subspaces of the form \( \sum_{i=1}^q \alpha_i(U_i) \) is the set of all subspaces containing \( \sum_{i \neq j} \alpha_i(U_i) \), included in \( \sum_{i \neq j} \alpha_i(U_i) + \text{Im}(\alpha_j) \), and of dimension belonging to a given integer interval.

It follows that the dimensions of the subspaces \( \sum \alpha_i(U_i) \cap V \) when \( U_j \) varies are an integer interval. Letting \( j \) vary in \( \{1, \ldots, q\} \), we deduce the lemma. \[ \square \]
Lemma 12. Let $E$ be a vector space, let $d,d'$ be integers between 0 and $\dim E$, and let $V \subset E$ be a fixed subspace of dimension $d$. Assume that the dimension of $V \cap W$, for $W \subset E$ a subspace of dimension $d'$, does not depend on $W$. Then at least one of the following occurs:

1. $d = 0$;
2. $d = \dim E$;
3. $d' = 0$;
4. $d' = \dim E$.

Proof. The minimal dimension of $V \cap W$ is $\max(d+d' - \dim E,0)$ and its maximal dimension is $\min(d,d')$. The equality of these integers implies that one of the four cases holds.

Lemma 13. Let $\alpha_i : E_i \to F$ be as in Notation 3. The dimension of $\sum \alpha_i(U_i)$ does not depend on the vector subspaces $U_i \subset E_i$ of dimension $d_i$ if and only if for all $i$, one of the following holds:

1. $d_i = 0$,
2. $0 < d_i$ and $\Im \alpha_i \subset S$,
3. $0 < d_i < \dim E_i$, $\alpha_i$ is injective, and $\Im \alpha_i \not\subset S$,

and $S$ and the subspaces $\Im \alpha_i$ for $i$ in case (iii) are in direct sum.

Proof. It is plain that the given conditions imply that the dimension of $\sum \alpha_i(U_i)$ does not depend on the $q$-uple $(U_i)$. Conversely, assume that this dimension is constant. Let $\varphi_1$ be the composition $E_1 \xrightarrow{\alpha_1} F \xrightarrow{\sum_{i\geq 2} \alpha_i(U_i)} E$. The fact that $\dim \sum \alpha_i(U_i)$ does not depend on $U_1$ implies that the dimension of $U_1 \cap \ker \varphi_1$ does not depend on $U_1$. We are thus in one of the four cases of Lemma 12. Case (α) is case (i) of our Lemma. Assume we are in case (β). This implies (ii). Moreover, letting $\varphi_i$ be the composition $E_i \xrightarrow{\alpha_i} F \xrightarrow{\Im \alpha_i} E$ for $i \geq 2$, we may assume by induction that the lemma is true for the linear maps $\varphi_2, \ldots, \varphi_k$. Since the last condition of the lemma for $\alpha_1, \ldots, \alpha_q$ is equivalent to the same condition for $\varphi_2, \ldots, \varphi_q$, the lemma is proved in this case.

Note that condition (α) or (β) holds for one $q$-uple $(U_i)$ if and only it holds for all $(U_i)$. Assume now that these conditions never hold. Then, for any $(U_i)$ we either have condition (γ), which is equivalent to $\alpha_1$ being injective and $\Im \alpha_1 \cap \sum_{i\geq 2} \alpha_i(U_i) = \{0\}$, or condition (δ), which is equivalent to $\Im \alpha_1 \subset \sum_{i\geq 2} \alpha_i(U_i)$.

If both cases (γ) and (δ) occur, we apply Lemma 11 to $V = \Im \alpha_1$ and the linear maps $\alpha_2, \ldots, \alpha_k$, and we deduce that the rank of $\alpha_1$ is at most 1. Since $\alpha_1$ is injective because case (γ) occurs, we deduce that $\dim E_1 = 0$ or $\dim E_1 = 1$, so case (α) or (β) occurs.

If only case (γ) occurs, we deduce that $\Im \alpha_1$ is in direct sum with $\sum_{i\geq 2} \Im \alpha_i$. If only case (δ) occurs, we deduce that $\Im \alpha_1 \subset S$. In each case, the conclusion of the lemma holds.

4.3.3. Proof of Proposition 4. Let $g \in N_{GL(V \otimes E)}(C)$. As in the proof of Proposition 7 we consider $g$ as a $q \times q$ matrix $g = (g_{i,j})_{1 \leq i,j \leq q}$ with coefficients $g_{i,j}$ in $gl(V)$. We choose a factor $F(A;V)$ of $X$ and we let $a \in A$.

The fact that $g$ preserves $C$ implies that given $U_1, \ldots, U_q \subset V$ of dimension $a$, there exist $V_1, \ldots, V_q \subset V$ of dimension $a$ such that $g \cdot (U_1 \otimes C f_1 \oplus \cdots \oplus U_q \otimes C f_q) = V_1 \otimes C f_1 \oplus \cdots \oplus V_q \otimes C f_q$. 


This implies that for \( j \) in \( \{1, \ldots, q\} \), we have \( \sum_i g_{i,j}(U_i) = V_j \). Let \( j \) be fixed: the dimension of \( \sum_i g_{i,j}(U_i) \) is always \( a \), and we may apply Lemma 13 to the linear maps \( g_{i,j} : V \to V \). Since we have \( d_i = a \) for all \( i \), we have \( S = \{0\} \). Case (i) does not occur, and case (ii) implies \( g_{i,j} = 0 \). If some \( g_{i,j} \) is not equal to 0, it is in case (iii) and therefore it is an isomorphism \( V \to V \). The condition that the images of the linear maps \( g_{i,j} \) are in direct sum implies that there can be at most one \( i \) in case (iii). On the other hand, there is at least one such \( i \) since \( g \) is invertible. It follows that \( g = (g_{i,j}) \) is a monomial matrix with coefficients in \( \text{GL}(V) \), proving the proposition.

\[ \square \]

5. Proof of the main theorem

In this section, we prove Theorem 3. We come back to the situation of Section 5. In particular, we have \( k = 3 \) and \( c_{\lambda, \mu, \nu} = 2 \) and \( X \) is defined by (15). We will prove that \( \dim H^0(X(q), L(q)^{\otimes p})^G(q) = \binom{p+q}{q} \).

5.1. Proof in the case of expected quotient dimension 1. In this section, we make the extra assumption that \( \text{Eqd}(X, G) = 1 \).

Step 1. Details on \( G \) acting on \( X \).

By [Tle00] Theorem 3.2 and our assumption \( c_{\lambda, \mu, \nu} = 2 \), we have

\[ (20) \quad \mathbb{C}^2 \simeq H^0(X, \mathcal{L})^G \simeq H^0(X^g(\mathcal{L}), \mathcal{L}|_{X^g(\mathcal{L})})^G. \]

By [She15] Proof of Corollary 2.4, \( X^g(\mathcal{L})/G \) is isomorphic to \( \mathbb{P}^1 \). Let \( \pi_X : X^g(\mathcal{L}) \to \mathbb{P}^1 \) be the quotient map. Observe that the stabilizer in the linear group of any point in a product of flag variety is connected, as an open subset of some vector space. By Kempf’s criterion [DNS9 Théorème 2.3], this implies that there exists a line bundle \( \mathcal{O}_{\mathbb{P}^1}(d) \) on \( \mathbb{P}^1 \) such that \( \pi_X^*(\mathcal{O}_{\mathbb{P}^1}(d)) \) is the restriction of \( \mathcal{L} \) to \( X^g(\mathcal{L}) \). Hence

\[ (21) \quad H^0(X^g(\mathcal{L}), \mathcal{L}|_{X^g(\mathcal{L})})^G \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)). \]

Combining (20) and (21), we get \( d = 1 \). Now, the same arguments imply that

\[ \begin{align*}
H^0(X, \mathcal{L}^{\otimes p})^G & \simeq H^0(X^g(\mathcal{L}), \mathcal{L}^{\otimes p})^G \\
& \simeq H^0(\mathbb{P}^1, \mathcal{O}(p)) \\
& \simeq S^p \mathbb{C}^2.
\end{align*} \]

Since the linear map \( S^p H^0(\mathbb{P}^1, \mathcal{O}(1)) \to H^0(\mathbb{P}^1, \mathcal{O}(p)) \) is an isomorphism, we get:

Lemma 14. The linear map \( S^p H^0(X, \mathcal{L})^G \to H^0(X, \mathcal{L}^{\otimes p})^G \) is an isomorphism.

Step 2. Details on \( N_{G(q)}(C) \) acting on \( C \).

It is well-known that the symmetric functions on \( q \) variables \( x_1, \ldots, x_q \) form a polynomial algebra generated by the elementary symmetric functions \( e_k \), where \( e_k \) is the coefficient of \( u^k \) in the polynomial \( \prod_{i=1}^q (x_i u + 1) \). Writing \( \mathbb{P}^1 \) as the union of two affine lines, one deduces that \( \bigoplus_{p} H^0(\mathbb{P}^1)^q, \mathcal{O}_{\mathbb{P}^1}(p))^{\otimes q} \) is a polynomial algebra generated by \( (c_k)_{0 \leq k \leq q} \), where \( c_k \) is the coefficient in \( u^k v^{q-k} \) of the product \( \prod_{i=1}^q (x_i u + y_i v) \). Here \( (x_i, y_i) \) are sections of \( \mathcal{O}_{\mathbb{P}^1}(1) \) on the \( i \)-th factor \( \mathbb{P}^1 \):
Lemma 15. The algebra $\bigoplus_p H^0((\mathbb{P}^1)^q, \mathcal{O}_{\mathbb{P}^1}(p))^G$ is freely generated by $H^0((\mathbb{P}^1)^q, \mathcal{O}_{\mathbb{P}^1}(1))^G$. Hence, the previous step implies that

$$C^s(\mathcal{L}(q) \otimes_p \mathcal{L}(q)) = X^q / \mathcal{G}(q) \cong (X^q / \mathcal{G}(q))^{\mathfrak{g}_q} = (\mathbb{P}^1)^q / \mathfrak{g}_q = \mathbb{P}^q.$$ 

Let $\pi : \mathbb{P}^q \to \mathfrak{g}_q$ and $\pi_N : C^s(\mathcal{L}(q) \otimes_p \mathcal{L}(q)) \to \mathfrak{g}_q$ be the quotient maps by $\mathfrak{g}_q$ and $\mathcal{G}(q)$, respectively. The isomorphism of Lemma 15 yields also $\pi_N^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(1)^{\mathfrak{g}_q}$. Now, Step 1 implies that $\pi_N^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{L}(q)$. Then, we have

$$H^0(C, \mathcal{L}(q) \otimes_p N_G^G(C)) \cong H^0(C^s(\mathcal{L}(q) \otimes_p \mathcal{L}(q)), \mathcal{L}(q) \otimes_p N_G^G(C)) \cong H^0((\mathbb{P}^1)^q, \mathcal{O}_{\mathbb{P}^1}(1)^{\mathfrak{g}_q}),$$

where the third isomorphism comes from Lemma 15.

Step 3. Details on $G(q)$ acting on $X(q)$.

Let $\pi : X(q)^s(\mathcal{L}(q)) \to X(q)^s(\mathcal{L}(q)) / G(q)$ be the quotient map.

Lemma 16. The map $\pi : X(q)^s(\mathcal{L}(q)) / G(q) \to X(q)^s(\mathcal{L}(q)) / G(q)$ is surjective.

Proof. The morphism $\pi$ is well defined by (1). By properness, it is sufficient to prove that it is dominant. First observe that $\tau^s_\mathcal{L}(\mathcal{L}(q) \otimes_p \mathcal{L}(q)) = \mathcal{L}^\mathfrak{g}_q$ on $X^q$. On the other hand, by (22), $\dim C^s(\mathcal{L}(q) \otimes_p \mathcal{L}(q)) = q$. On the other hand, Proposition (8) and the assumption Eqd($X, G$) = 1 imply that Eqd($X(q), G(q)$) ≤ q. Then, Proposition (7) implies $\dim X(q)^s(\mathcal{L}(q)) / G(q) = q$.

Now, Lemma 5 implies that $\pi$ is surjective being proper and finite. □

Lemma 17. For any $p \geq 0$, the restriction map

$$H^0(X(q), \mathcal{L}(q) \otimes_p G(q)) \to H^0(C, \mathcal{L}(q) \otimes_p N_G^G(C))$$

is injective. For $p = 1$, it is an isomorphism.

Proof. The last assertion follows from the first by the equality of the dimensions which follows from (16) and Theorem 2.

Let $\sigma \in H^0(X(q), \mathcal{L}(q) \otimes_p G(q))$ be such that its restriction to $C$ is zero. Let $x \in X(q)$: we show that $\sigma(x) = 0$. If $x$ is unstable, then by definition this means that any invariant section vanishes at $x$. Assume that $x$ is semistable, and set $\xi = \pi_q(x)$.

Pick $x_0$ in the closed $G(q)$-orbit in $G(q) \cdot x \cap X^s(G(q), \mathcal{L}(q))$. By semi-stability, there exists a positive integer $k$ such that the stabilizer $G(q)_{x_0}$ acts trivially on $\mathcal{L}^{\otimes k}$. It follows that the character of $G(q)_{x_0}$ which defines the $G(q)$-linearized line bundle $L^{\otimes k}_{G(q)_{x_0}}$ is the trivial character, and $L^{\otimes k}_{G(q)_{x_0}}$ is the trivial $G(q)$-linearized line bundle on $G(q) \cdot x_0$.

On the other hand, by [MFK94, Theorem 1.10], the fiber $\pi_q^{-1}(\xi)$ is affine, and by [BB63, Theorem 1], the stabilizer $G(q)_{x_0}$ is reductive. We can therefore apply [Br89, Lemma 2.1] (note that the normality assumption is not used in the proof of...
Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{SP}^0(X(q), \mathcal{L}(q))^G(q) & \to & \text{SP}^0(C, \mathcal{L}(q))^{N_G(q)}(C) \\
\downarrow \text{product} & & \downarrow \cong \\
\text{H}^0(X(q), \mathcal{L}(q)^{\otimes p})^G(q) & \to & \text{H}^0(C, \mathcal{L}(q)^{\otimes p})^{N_G(q)}(C)
\end{array}
\]

The top horizontal map is an isomorphism and the bottom one is injective by Lemma 17. The right vertical map is an isomorphism by (23). It follows that the product map is an isomorphism.

**Corollary 18.** The GIT-quotient \( X(q)^{\infty}(\mathcal{L}(q))//G(q) \) is isomorphic to \( \mathbb{P}^q \).

**Proof.** By definition, this quotient is Proj of the algebra \( \bigoplus_p H^0(X(q), \mathcal{L}(q)^{\otimes p})^G(q) \). Since the product map in (24) is an isomorphism, this algebra is the symmetric algebra on \( H^0(X(q), \mathcal{L}(q))^{G(q)} \), which is a vector space of dimension \( q + 1 \).

5.2. **Reduction to the case of expected quotient dimension** 1. Observe that \( \mathcal{L} \) is ample and has \( G \)-invariant sections, so it belongs to \( \mathcal{AC}^G(X) \). We proceed by induction on \( n \), considering two cases:

**Case 1:** \( \mathcal{L} \) belongs to the interior of \( \mathcal{AC}^G(X) \).

Then, by Proposition 4 the dimension of \( X^{\infty}(\mathcal{L})//G \) is equal to Eqd\((X, G)\). By Theorem 2 we have \( \dim H^0(X, \mathcal{L}^{\otimes p}) = p + 1 \). By definition of \( X^{\infty}(\mathcal{L})//G \), see (2), the dimension of \( X^{\infty}(\mathcal{L})//G \) is the degree of this polynomial, namely 1.

We deduce that Eqd\((X, G) = 1 \) and we are done by Section 5.1.

**Case 2:** \( \mathcal{L} \) is in the boundary of \( \mathcal{AC}^G(X) \).

By Proposition 3 and the ampleness of \( \mathcal{L} \), there exist an integer \( r \) and \( I, J, K \subset \{1, \ldots, n\} \) of cardinality \( r \) such that

\[
\sigma_I \cup \sigma_J \cup \sigma_K = [pt] \quad \text{and} \quad \frac{1}{r}(|\lambda_I| + |\mu_J| + |\nu_K|) = \frac{1}{n}(|\lambda| + |\mu| + |\nu|).
\]

Then, since the product \( \sigma_I \cup \sigma_J \cup \sigma_K \) is equal to the class of the point, by multiplicativity of Littlewood-Richardson coefficients [DW17, Res11b], we have \( 2 = c_{\lambda, \mu, \nu} = c_{\lambda_I, \mu_J, \nu_K} \cdot c_{\lambda_{I'}, \mu_{J'}, \nu_{K'}} \) where \( I' = \{1, \ldots, n\} \setminus I \) (and similarly for \( J' \) and \( K' \)).

We may thus assume the equalities \( c_{\lambda_I, \mu_J, \nu_K} = 2 \) and \( c_{\lambda_{I'}, \mu_{J'}, \nu_{K'}} = 1 \). By induction, we deduce \( c_{\lambda_I, \mu_J, \nu_K} = \binom{p+q}{p} \). By Fulton’s conjecture as stated in Theorem 1 we have \( c_{\lambda_{I'}, \mu_{J'}, \nu_{K'}} = \binom{q}{p} \). Thus, the proof in this case will be finished if we can prove that

\[
c_{\lambda, \mu, \nu} = c_{\lambda_I, \mu_J, \nu_K} \cdot c_{\lambda_{I'}, \mu_{J'}, \nu_{K'}} = \binom{p+q}{p} \binom{q}{p}.
\]

**Step 4. Conclusion.**

This Lemma, or [BH85, Corollary 6.4], and conclude that the restriction of \( \mathcal{L}^{\otimes k} \) to \( \pi_q^{-1}(\xi) \) is trivial.

Hence \( \sigma^{\otimes k} \) can be viewed as a regular constant function on \( \pi_q^{-1}(\xi) \). But Lemma 10 implies that \( C \) intersects \( \pi_q^{-1}(\xi) \). Hence \( \sigma^{\otimes k} \) vanishes on \( \pi_q^{-1}(\xi) \). Finally \( \sigma^{\otimes k} \) and \( \sigma \) vanish identically on \( \pi_q^{-1}(\xi) \). In particular \( \sigma(x) = 0 \). \( \square \)
Relation (26) is proved using multiplicativity again. First, observe that $\lambda_I(p, q)$ is equal to the partition $\lambda(p, q)_{i_q}$, where

\begin{equation}
I_q = \{(i_1 - 1)q + 1, \ldots, i_1q, (i_2 - 1)q + 1, \ldots, i_2q, \ldots, (i_r - 1)q + 1, \ldots, i_rq\}
\end{equation}

if $I = \{i_1, \ldots, i_r\}$. Note that Schubert classes in $G(r, n)$ are parametrized by subsets $I$ of $\{1, \ldots, n\}$ as we did in Section 2.5, and also by partitions whose Young diagram is included in a $r \times (n - r)$ rectangle. The correspondence maps a subset $I = \{i_1 < i_2 < \cdots < i_r\}$ to the partition $(i_r - r, \ldots, i_2 - 2, i_1 - 1)$. Therefore, the partition corresponding to $I_q$ is $q(i_r - r), \ldots, q(i_1 - 1), q(i_1 - 1)$ (with each part being repeated $q$ times).

If $\alpha$ denotes the partition corresponding to the subset $I$, then the partition corresponding to the subset $I_q$ is $\alpha(q, q)$. Thus, by Theorem 4 again, the equality

$$\sigma_I \cup \sigma_J \cup \sigma_K = [pt] \in H^*(G(r, n), \mathbb{Z})$$

implies the equality

$$\sigma_{I_q} \cup \sigma_{J_q} \cup \sigma_{K_q} = [pt] \in H^*(G(qr, qn), \mathbb{Z}).$$

By multiplicativity of Littlewood-Richardson coefficients, (26) holds.

6. About the case $c_{\lambda, \mu, \nu} > 2$

A key point in our proof is Lemma 17 showing that, under the assumption $c_{\lambda, \mu, \nu} = 2$, the restriction map

$$\rho_C : H^0(X(q), \mathcal{L}(q))^G \rightarrow H^0(C, \mathcal{L}(q))^{N_G(\mathcal{C})} = S^q H^0(X, \mathcal{L})^G$$

is injective. The following example shows that $\rho_C$ is not always injective.

**Example 1.** This example is mainly due to P. Belkale [Bel03 Example 3.7]. For $G = GL_8(\mathbb{C})$, consider $\lambda = \mu = (3, 3, 2, 2, 1, 1)$ and $\nu = (4, 4, 3, 3, 2, 2, 2)$. We have $c_{\alpha, \beta, \gamma} = 6$. Consider the Littlewood-Richardson polynomial $P_{\lambda, \mu}^\nu$ (see [DW02]) such that for any $q \in \mathbb{Z}_{\geq 0}$, $c_{\alpha, \beta, \gamma} = P(q)$. This Littlewood-Richardson coefficient is obtained as the dimension of a space of $G$-invariant sections on $X = Fl(2, 4; \mathbb{C}^8)^{\times} \times Fl(3, 5; \mathbb{C}^8)$. It is easy to check that there exists $x$ in $X$ whose isotropy group consists in the homotheties. Then $\text{Eqd}(X, G) = 6$. As a consequence the degree of $P_{\lambda, \mu}^\nu$ is at most 6. Using Buch’s calculator [Buc], one obtains that $P(0) = 1$, $P(1) = 6$, $P(2) = 22$, $P(3) = 63$, $P(4) = 154$, $P(5) = 336$ and $P(6) = 672$. Using Lagrange interpolation, one gets

$$P_{\lambda, \mu, \nu}^\nu(q) = \frac{1}{720} q^6 + \frac{1}{48} q^5 + \frac{23}{144} q^4 + \frac{35}{48} q^3 + \frac{331}{180} q^2 + \frac{9}{4} q + 1,$$

which indeed has degree 6. In particular, the map $\rho_C$ is not injective for $q$ big enough, since $\dim(S^q H^0(X, \mathcal{L})^G) = \dim(S^q \mathcal{C}_6) = \binom{q + 5}{5}$ is a polynomial function in $q$ of degree 5.

Note that similarly, one gets

$$P_{\lambda, \mu, \nu}^\nu(k) = \frac{5}{2}(k^2 + k) + 1.$$
References

[BB63] A. Bialynicki-Birula, On homogeneous affine spaces of linear algebraic groups, Am. J. Math. 85 (1963), 577–582 (English).

[Bel01] Prakash Belkale, Local systems on $\mathbb{P}^1 − S$ for $S$ a finite set, Compositio Math. 129 (2001), no. 1, 67–86. MR 1856023

[Bel03] Prakash Belkale, Irredundance in eigenvalue problems, 2003, arXiv/math/0308026.

[Bel07] Prakash Belkale, Geometric proof of a conjecture of Fulton, Adv. Math. 216 (2007), no. 1, 346–357 (English).

[BH85] H. Bass and W. Haboush, Linearizing certain reductive group actions, Trans. Am. Math. Soc. 292 (1985), 463–482 (English).

[Bou02] Nicolas Bourbaki, Lie groups and Lie algebras. Chapters 4–6, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley.

[Bri89] M. Brion, Groupe de Picard et nombres caractéristiques des variétés sphériques, Duke Math. J. 58 (1989), no. 2, 397–424.

[Buc] Anders Buch, Quantum calculator — a software maple package, Available at www.math.rutgers.edu/~asbuch/qcalc.

[DH98] Igor V. Dolgachev and Yi Hu, Variation of geometric invariant theory quotients, Inst. Hautes Études Sci. Publ. Math. 87 (1998), 5–56, With an appendix by Nicolas Ressayre.

[DN89] J.-M. Drezet and M. S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stable sur les courbes algébriques. (Picard groups of moduli varietes of semi-stable bundles on algebraic curves), Invent. Math. 97 (1989), no. 1, 53–94 (French).

[DW02] Harm Derksen and Jerzy Weyman, On the Littlewood-Richardson polynomials, J. Algebra 255 (2002), no. 2, 247–257.

[DW11] _____, The combinatorics of quiver representations, Ann. Inst. Fourier (Grenoble) 61 (2011), no. 3, 1061–1131.

[Ike16] Christian Ikenmeyer, Small Littlewood-Richardson coefficients, J. Algebr. Comb. 44 (2016), no. 1, 1–29 (English).

[Kly98] Alexander A. Klyachko, Stable bundles, representation theory and Hermitian operators, Selecta Math. (N.S.) 4 (1998), no. 3, 419–445. MR 1654578

[KTW04] Allen Knutson, Terence Tao, and Christopher Woodward, The honeycomb model of $GL_n(\mathbb{C})$ tensor products. II: Puzzles determine facets of the Littlewood-Richardson cone, J. Am. Math. Soc. 17 (2004), no. 1, 19–48 (English).

[Lun75] D. Luna, Adhérences d’orbites et invariants, Invent. Math. 29 (1975), 231–238 (French).

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, Geometric invariant theory., vol. 34, Berlin: Springer-Verlag, 1994 (English).

[Res10] Nicolas Ressayre, Geometric invariant theory and generalized eigenvalue problem, Invent. Math. 180 (2010), 389–441.

[Res11a] Nicolas Ressayre, A short geometric proof of a conjecture of Fulton, Enseign. Math. (2) 57 (2011), no. 1-2, 103–115 (English).

[Res11b] Nicolas Ressayre, Reductions for branching coefficients, J. of Lie Theory (to appear) (2011), arXiv:1102.0196.

[Res12] _____, A cohomology-free description of eigencones in types $A$, $B$, and $C$, Int. Math. Res. Not. IMRN (2012), no. 21, 4966–5005.

[She15] Cass Sherman, Geometric proof of a conjecture of King, Tollu, and Toumazet, 2015.

[She17] Cass Sherman, Quiver generalization of a conjecture of King, Tollu, and Toumazet, J. Algebra 480 (2017), 487–504 (English).

[Tel00] Constantin Teleman, The quantization conjecture revisited, Ann. Math. (2) 152 (2000), no. 1, 1–43 (English).

Université de Lorraine, CNRS, Institut Élie Cartan de Lorraine, UMR 7502, Vandœuvre-lès-Nancy, F-54506, France
Email address: pierre-emmanuel.chaput@univ-lorraine.fr

Université Claude Bernard Lyon I, Institut Camille Jordan (ICJ), UMR CNRS 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne CEDEX
Email address: ressayre@math.univ-lyon1.fr