SCHUR Q-POLYNOMIALS, MULTIPLE HYPERGEOMETRIC SERIES
AND ENUMERATION OF MARKED SHIFTED TABLEAUX

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Abstract. We study Schur $Q$-polynomials evaluated on a geometric progression, or equivalently $q$-enumeration of marked shifted tableaux, seeking explicit formulas that remain regular at $q = 1$. We obtain several such expressions as multiple basic hypergeometric series, and as determinants and pfaffians of continuous $q$-ultraspherical or continuous $q$-Jacobi polynomials. As special cases, we obtain simple closed formulas for staircase-type partitions.

1. Introduction

The Schur $Q$-polynomials originate in the work of Schur [Sc] on projective representations of the symmetric group. Nowadays, they are recognized as the case $t = -1$ of the Hall–Littlewood polynomials, which are in turn a special case of the Macdonald polynomials [Ma].

In the present paper we report on some investigations on Schur $Q$-polynomials that were motivated by applications to sums of squares [R3]. In [R1], we used elliptic pfaffian evaluations to derive, and generalize, certain triangular number identities conjectured by Kac and Wakimoto [KW] and proved first by Milne [M2, M3, M4] and later by Zagier [Z]. Extending this work to the case of square numbers leads naturally to quantities related to Schur $Q$-polynomials. To be precise, we note that, although one usually assumes that the polynomial $Q_\lambda$ is indexed by a partition $\lambda$, the definition (2.1) makes sense for general $\lambda \in \mathbb{Z}^m$. The quantities alluded to may then be written $Q_{(\lambda, -\lambda)}(1, \ldots, 1)$, where $\lambda$ is a partition. In the present paper, however, we investigate $Q_\lambda(1^n)$ for general $\lambda$.

In many ways, Schur $Q$-polynomials and Schur polynomials have analogous properties. Therefore, it may be instructive to compare with the much simpler situation for the latter. The standard definition

$$s_\lambda(x) = \frac{\det_{1 \leq i, j \leq n}(x_i^{\lambda_j + n - j})}{\det_{1 \leq i, j \leq n}(x_i^{n - j})}$$

has a removable singularity at $x_1 = \cdots = x_n = 1$. A well-known way to compute $s_\lambda(1^n)$ is by passing to the case when the variables are in geometric progression.
Namely, by the Vandermonde determinant evaluation,
\[ s_\lambda(1, q, \ldots, q^{n-1}) = \frac{\prod_{1 \leq i < j \leq n}(q^{\lambda_j + n-j} - q^{\lambda_i + n-i})}{\prod_{1 \leq i < j \leq n}(q^{n-j} - q^{n-i})}, \tag{1.1} \]
which in the limit \( q \to 1 \) gives
\[ s_\lambda(1^n) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \]

Our method for computing \( Q_\lambda(1^n) \) is similar. In that case, \( Q_\lambda(1, q, \ldots, q^{n-1}) \) is a multiple basic hypergeometric sum, see (2.5), again with a removable singularity at \( q = 1 \). Since it does not simplify in general, the best one can hope for is to find a transformation formula, leading to a different sum without the apparent singularity. In the limit \( q \to 1 \) one would then obtain a formula for \( Q_\lambda(1^n) \) as a multiple classical hypergeometric sum. Finding such identities is our main goal.

It should be remarked that \( Q_\lambda(1, q, \ldots, q^{n-1}) \) has a combinatorial meaning as a generating function (or \( q \)-enumeration) on marked shifted tableaux, see (2.3). Thus, we can equivalently formulate our goal as finding explicit solutions to the \( q \)-enumeration problem that remain regular in the limit \( q \to 1 \), corresponding to classical enumeration.

The key fact for finding transformation formulas with the desired property is Lemma 5.1, which allows us to identify the two-row \( Q \)-polynomial
\[ Q_{(\lambda_1, \lambda_2)}(1, q, \ldots, q^n) \]
with the Christoffel–Darboux kernel of certain continuous \( q \)-Jacobi polynomials. Like most of our results, it is formulated in terms of multivariable polynomials \( P_n \) that are related to \( Q_\lambda \) through
\[ Q_\lambda(1, q, \ldots, q^n) = 2^n P_n(q^{\lambda_1}, \ldots, q^{\lambda_m}). \tag{1.2} \]

The multi-row \( Q \)-polynomial similarly corresponds to a multivariable Christoffel–Darboux kernel. This fact implies several pfaffian and determinantal formulas for the polynomials \( P_n \). Besides the pfaffian formula (2.27), which is essentially Schur’s definition of the \( Q \)-polynomial, we have the transformed pfaffian formula of Corollary 5.7 and the determinant formula of Corollary 5.8. In the limit \( q \to 1 \), the latter leads to an elegant determinant formula for \( Q_\lambda(1^n) \), see Corollary 5.10.

We then use our new pfaffian and determinantal formulas to obtain hypergeometric formulas. Corollary 5.7 allows us to express \( P_n \) as a pfaffian of double sums. Series manipulation then leads to the multiple hypergeometric sum of Theorem 6.1. Similarly, Corollary 5.8 expresses \( P_n \) as a determinant of single sums, which leads to the Schlosser-type hypergeometric sums of Theorems 7.1 and 7.3. In the limit \( q \to 1 \), Theorems 6.1 and 7.3 yield hypergeometric formulas for \( Q_\lambda(1^n) \), thus fulfilling our main goal. (Theorem 7.1 retains the singularity at \( q = 1 \).)

In some special cases, our expressions for \( Q_\lambda(1, q, \ldots, q^{n-1}) \) simplify, giving completely factored expressions similarly as for Schur polynomials.
• $n = \infty$, see Section 3 and Remark 5.2. The explicit evaluation of
  
  \[ Q_\lambda(1, q, q^2, \ldots) \]

  follows from an identity of Kawanaka [K], who does not, however, mention
  the relation to $Q$-polynomials.

• $\lambda$ staircase, see Remark 7.4. If $\lambda = (m, m - 1, \ldots, 1)$, then $Q_\lambda(1, q, q^2, \ldots) = 2^m s_\lambda$, so
  
  \[ Q_\lambda(1, q, q^2, \ldots, q^{n-1}) \]

  factors by (1.1).

• $\lambda$ odd staircase, see Section 4, Remark 6.3 and Remark 7.4. If $\lambda = (2m - 1, 2m - 3, \ldots, 1)$, then $Q_\lambda(1, q, q^2, \ldots, q^{n-1})$ can be evaluated using a discrete Selberg integral due to Milne [M1], or in the form needed here to Krattenthaler [Kr]. Again, the connection to $Q$-polynomials seems not to have been noticed before.

• $\lambda$ even staircase, see Corollary 7.6. If $\lambda = (2m, 2m - 2, \ldots, 2)$, then $Q_\lambda(1, q, q^2, \ldots, q^{n-1})$ factors. In contrast to the previously mentioned cases, we have not been able to reduce this to previously known results.

The plan of the paper is as follows. Section 2 contains preliminaries. In Section 3 we consider the limit $n \to \infty$ and in Section 4 the odd staircase, describing the connection to previously known results. After this preliminary material, we lay the foundation for our main results in Theorem 5.4, where we relate $Q_\lambda(1, q, q^2, \ldots, q^n)$ to multivariable Christoffel–Darboux kernels, obtaining as a consequence new Pfaffian and determinantal identities. These are then used to derive the Pfaffian hypergeometric identity of Theorem 6.1 and the determinantal hypergeometric identities in Section 7. Finally, in the Appendix we give an alternative proof of Theorem 5.4.

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2. Preliminaries

We will use the terminology and notation of [Ma] for partitions and symmetric functions, and that of [GR] for classical and basic hypergeometric functions.

2.1. Schur $Q$-polynomials and marked shifted tableaux. When $m \leq n$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition of length $m$, that is,

\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0, \]

the Schur $Q$-polynomial $Q_\lambda$ is defined by

\[ Q_\lambda(x_1, \ldots, x_n) = 2^m \sum_{\sigma \in S_n/S_{n-m}} \sigma \left( x_1^{\lambda_1} \cdots x_m^{\lambda_m} \prod_{1 \leq i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \right). \]  (2.1)

Here, $S_n$ acts by permuting the variables $x_1, \ldots, x_n$ and $S_{n-m}$ is the subgroup acting on $x_{m+1}, \ldots, x_n$. Note that $Q_\lambda = 0$ unless $\lambda$ is strict, that is,

\[ \lambda_1 > \lambda_2 > \cdots > \lambda_m > 0. \]

For strict $\lambda$, $Q_\lambda$ has a combinatorial interpretation in terms of marked shifted tableaux. We briefly recall the relevant definitions.
Let $\lambda$ be the diagram of a strict partition, and let $S(\lambda)$ denote the diagram obtained by shifting the $i$-th row $(i - 1)$ steps to the right, for each $i$. A marked shifted tableau of shape $S(\lambda)$ is a labelling of the boxes of $S(\lambda)$ with symbols from the ordered alphabet $1' < 1 < 2' < 2 < \cdots$ such that:

(1) The labels increase weakly along rows and down columns.
(2) Each unmarked symbol occurs at most once in each column.
(3) Each marked symbol occurs at most once in each row.

Let $a_k$ be the number of boxes labelled either $k$ or $k'$, and let $x^T$ denote the monomial $\prod_{k \geq 1} x_k^{a_k}$. Then, [Ma] (III.8.16')

\[ Q_\lambda(x) = \sum_T x^T, \tag{2.2} \]

where the sum is over all marked shifted tableaux of shape $S(\lambda)$. In (2.2) the number of variables is infinite; restricting the alphabet to $1' < 1 < \cdots < n' < n$ gives a formula for $Q_\lambda(x_1, \ldots, x_n)$.

When $T$ is a marked shifted tableau, we let $|T| = \sum_{k \geq 1} (k - 1) a_k$, with $a_k$ as above. Then,

\[ Q_\lambda(1, q, \ldots, q^{n-1}) = \sum_T q^{|T|} \tag{2.3} \]

is the generating function for $|T|$ on marked shifted tableaux of shape $\lambda$ and alphabet $1' < 1 < \cdots < n' < n$. In particular, the cardinality of this set is $Q_\lambda(1^n)$.

### 2.2. Schur $Q$-polynomials and basic hypergeometric series

Representing $\sigma$ by the $m$-tuple $(k_1, \ldots, k_m) = (\sigma(1), \ldots, \sigma(m))$, we may rewrite (2.1) as

\[ Q_\lambda(x_1, \ldots, x_n) = 2^m \sum_{k_1, \ldots, k_m = 1}^{\text{distinct}} \prod_{i=1}^{m} x_{k_i}^{\lambda_i} \prod_{1 \leq i < j \leq m} \frac{x_{k_i} + x_{k_j}}{x_{k_i} - x_{k_j}} \prod_{i=1}^{m} \prod_{j \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_m\}} x_{k_i} \sum_{k_1, \ldots, k_m = 1}^{\text{distinct}} \prod_{i=1}^{m} x_{k_i}^{\lambda_i} \prod_{1 \leq i < j \leq m} \frac{x_{k_i} - x_{k_j}}{x_{k_i} + x_{k_j}} \prod_{i=1}^{m} \prod_{j \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_m\}} x_{k_i} - x_{j}. \tag{2.4} \]

Consider the case when $x_1, \ldots, x_n$ are in geometric progression. Then, (2.4) is a multiple basic hypergeometric sum. Indeed, replacing $n$ by $n + 1$ and $k_i$ by $k_i + 1$, we obtain after simplification

\[ Q_\lambda(1, q, \ldots, q^n) = \frac{(-1; q)_{n+1}}{(q; q)_n^m} \sum_{k_1, \ldots, k_m = 0}^{n} \prod_{1 \leq i < j \leq m} \frac{q^{k_j} - q^{k_i}}{q^{k_j} + q^{k_i}} \prod_{i=1}^{m} (-q, q^{-n}; q)_{k_i} q^{\lambda_i k_i}, \tag{2.5} \]

where we use the standard notation [GR]

\[ (a; q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1}), \]
\[(a_1, \ldots, a_m; q)_k = (a_1; q)_k \cdots (a_m; q)_k.\]

It will be convenient to view \(q^{\lambda_i}\) as free variables. Thus, we introduce the polynomials
\[P_n(x_1, \ldots, x_m) = \frac{(-q; q)_n}{(q; q)_n} \sum_{k_1, \ldots, k_m=0}^n \prod_{1 \leq i < j \leq m} \frac{q^{k_j} - q^{k_i}}{q^{k_j} + q^{k_i}} \prod_{i=1}^m \frac{(-q, q^{-n}; q)_k}{(q, -q^{-n}; q)_k} x_i^{k_i}, \quad (2.6)\]
so that \(1.2\) holds. By anti-symmetry, we may equivalently write
\[P_n(x_1, \ldots, x_m) = \frac{(-q; q)_n}{(q; q)_n} \sum_{0 \leq k_m \cdots k_1 \leq n, 1 \leq i < j \leq m} \prod_{i=1}^m \frac{(-q, q^{-n}; q)_k}{(q, -q^{-n}; q)_k} \det (x_i^{k_j}). \quad (2.7)\]
This can be viewed as a Schur polynomial expansion, see \(3.2\). One easily verifies that
\[P_n(x_1, \ldots, x_m, 0) = (-1)^m \frac{(-q; q)_n}{(q; q)_n} x_1 \cdots x_m P_{n-1}(x_1, \ldots, x_m). \quad (2.8)\]

For \(m = 1\), \(P_n\) is a terminating well-poised \(2\phi_1\) series, and can thus be expressed in terms of continuous \(q\)-ultraspherical polynomials \([GR]\) as
\[P_n(x) = \frac{(-q; q)_n}{(q; q)_n} 2\phi_1 \left[-q; q^{-n}, -q^{-n} : q, x \right] = e^{i n \theta} C_n(\cos \theta; -q|q), \quad x = -e^{2i \theta}. \quad (2.9)\]

It will be convenient to introduce the monic polynomials
\[c_n(x) = \frac{(q; q)_n}{(-q; q)_n} C_n(x/2; -q|q). \quad (2.10)\]

Then, assuming a fixed choice of \(\sqrt{-x}\),
\[P_n(x) = \frac{(-q; q)_n}{(q; q)_n} \left(\sqrt{-x}\right)^n c_n \left(\sqrt{-x} + (\sqrt{-x})^{-1}\right). \quad (2.11)\]
For \(0 < q < 1\), the polynomials \((c_k)_{k=0}^\infty\) form an orthogonal system with respect to a unique positive measure (see \(A.11\)), which we can normalize so that
\[\|c_k\|^2 = \frac{(q; q)_k (q; q)_{k+1}}{(-q; q)_k (-q; q)_{k+1}}. \quad (2.12)\]

### 2.3. Orthogonal polynomials

Our main interest is in the case \(q = 1\). Unfortunately, the right-hand side of \(2.5\) is then singular. However, when \(m = 1\) one can use transformation formulas for basic hypergeometric series to derive a plethora of other expressions, many of which remain regular when \(q \to 1\).

The transformations that we will use correspond to different identifications of \(q\)-ultraspherical polynomials with continuous \(q\)-Jacobi polynomials, which in turn
form a sub-class of the Askey–Wilson polynomials. Recall that the latter polynomials are defined by

\[ p_n(\cos \theta; a, b, c, d | q) = \frac{(ab, ac, ad; q)_n}{a^n} 4 \phi_3 \left[ q^{-n}, acdq^{n-1}, ae^{i\theta}, ae^{-i\theta}; q, q \right]; \]

they are symmetric in the parameters \( a, b, c, d \) and satisfy

\[ p_n(-x; a, b, c, d | q) = (-1)^n p_n(x; -a, -b, -c, -d | q). \]  

(2.13)

By [GR, Eq. (7.5.33–36)],

\[ C_n(x; a | q) = \frac{(a; q)_n}{(q, a^2 q^n; q)_n} p_n(x; \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} | q) \]  

(2.14a)

\[ = \frac{(a; q^{1/2})_n}{(q, aq^{1/2}; q)_n} p_n(x; \sqrt{a}, -\sqrt{a}, q^{1/4}, -q^{1/4} | q^{1/2}), \]  

(2.14b)

\[ C_{2n}(x; a | q) = \frac{(a; q)_n}{(q, a^2 q_{2n}; q)_{2n}} p_n(2x^2 - 1; a, -1, \sqrt{q}, -\sqrt{q} | q) \]  

(2.14c)

\[ = \frac{(a^2; q^2)_{2n}}{(q, -a; q)_{2n}} p_n(2x^2 - 1; a, -q, -1, -q | q^2), \]  

(2.14d)

\[ C_{2n+1}(x; a | q) = \frac{(a; q)_{n+1}}{(q, q_{2n+1}; q)_{2n+1}} 2x p_n(2x^2 - 1; a, -q, \sqrt{q}, -\sqrt{q} | q) \]  

(2.14e)

\[ = \frac{(a^2; q^2)_{n+1}}{(q, -a; q)_{2n+1}} 2x p_n(2x^2 - 1; a, -q, -q^2 | q^2). \]  

(2.14f)

It is easy to prove this by verifying that the orthogonality measures for the various polynomials agree.

In view of (2.9), we are particularly interested in the case \( a = -q \). Using also (2.13), we may then write (2.14) as

\[ P_n(-e^{2i\theta}) = \frac{(-q; q)_n}{(q, q^{n+2}; q)_n} e^{in\theta} p_n(\cos \theta; iq^{1/2}, -iq^{1/2}, iq, -iq | q) \]  

(2.15a)

\[ = \frac{(-q^{1/2}; q)_n}{(q, -q^{3/2}; q)_n} e^{in\theta} p_n(\cos \theta; iq^{1/2}, -iq^{1/2}, q^{1/4}, -q^{1/4} | q^{1/2}), \]  

(2.15b)

\[ P_{2n}(x) = \frac{(-q; q)_n}{(q, q_{2n}; q)_{2n}} x^n p_n(\frac{1}{2}(x + x^{-1}); 1, q, q, -\sqrt{q} | q) \]  

(2.16a)

\[ = \frac{(q^2; q^2)_n}{(q, q^2 q_{2n}; q)_{2n}} x^n p_n(\frac{1}{2}(x + x^{-1}); 1, q, q^2 | q^2), \]  

(2.16b)
These identities yield many useful expressions for $P_n(x)$. For instance, \((2.15a)\) implies
\[
P_n(x) = \frac{1 - q^{n+1}}{1 - q} \left( \frac{\sqrt{x/q}}{\sqrt{q}} \right)^n \left[ q^{-n}, q^{n+2}, \sqrt{q/x}, -\sqrt{q}x; q, q \right].
\]  
(2.17)

In contrast to \((2.9)\), this expression is regular in the limit
\[
Q_{(\lambda)}(1^{n+1}) = \lim_{q \to 1} 2P_n(q^\lambda) = (2n + 2) \, _3F_2 \left[ \frac{-n, n + 2, (1 - \lambda)/2}{1, 3/2}; 1 \right].
\]  
(2.18)

This will be generalized to general $m$ in Theorem 6.1 and Corollary 6.2.

Similarly, \((2.15b)\) gives
\[
P_n(x) = \frac{1 - q^{n+1}}{1 - q} \left( \frac{\sqrt{x/q}}{\sqrt{q}} \right)^n \left[ q^{-n/2}, -q^{(n+2)/2}, \sqrt{q/x}, -\sqrt{q}x; q^{1/2}, q^{1/2} \right],
\]
which implies the simple formula
\[
Q_{(\lambda)}(1^{n+1}) = \lim_{q \to 1} 2P_n(q^\lambda) = (2n + 2) \, _2F_1 \left[ \frac{-n, 1 - \lambda}{2}; 2 \right].
\]  
(2.19)

In Section 7 we will need the following eight expressions, which are all consequences of \((2.16)\):
\[
P_{2n}(x) = \frac{1 - q^{2n+1}}{1 - q} q^{-n} x^n \, _4\Phi_3 \left[ q^{-n}, -q^{n+1}, qx, q/x; q^2, q^{3/2}, -q^{3/2}; q, q \right],
\]  
(2.20a)

\[
= \frac{(q^{3/2}, -q; q)_n}{(q, -q^{1/2}; q)_n} q^{-n/2} x^n \, _4\Phi_3 \left[ q^{-n/2}, -q^{(n+1)/2}, \sqrt{q/x}, -\sqrt{q}x; q^{1/2}, q^{1/2}; q, q \right],
\]  
(2.20b)

\[
= \frac{1 - q^{2n+1}}{1 - q} q^{-n} x^n \, _4\Phi_3 \left[ q^{-2n}, q^{2n+2}, qx, q/x; q, q^2, q^{2}; q^2, q^2 \right],
\]  
(2.20c)

\[
= \left( \frac{1 - q^{2n+1}}{1 - q} \right)^2 q^{-2n} x^n \, _4\Phi_3 \left[ q^{-2n}, q^{2n+2}, q^2 x, q^2 x; q^2, q^2, q^2 \right],
\]  
(2.20d)
Let us elaborate on this point, introducing the notation continuous Hahn polynomials. Explicitly, in the notation of \([KS]\),

\[
P_k(x) = \frac{i^k k!}{2^k} \lim_{q \to 1} P_k(q^{-ix}),
\]

where the constant is chosen so as to make \(p_k\) monic. By \((2.18)\) and \((2.19)\),

\[
p_k(x) = \frac{i^k (k + 1)!}{2^k} 2F_1 \left[\begin{array}{c}
-k, 1 + i x \\
2
\end{array} ; 2\right] = \frac{i^k (k + 1)!}{2^k} 3F_2 \left[\begin{array}{c}
-n, n + 2, (1 + i x)/2 \\
1, 3/2
\end{array} ; 1\right].
\]

It follows that \(p_k\) may be identified with Meixner-Pollaczek polynomials and with continuous Hahn polynomials. Explicitly, in the notation of \([KS]\),

\[
p_k(x) = \frac{k!}{2^k} P_k^{(1)}(x; \pi/2) = \frac{2^k(k + 1)!}{(2k + 1)!} P_k(x/2; 1/2, 1, 1/2, 1, 1). \tag{2.20f}
\]

In particular, \(\langle p_k(x) \rangle_{k=0}^{\infty}\) are the monic orthogonal polynomials corresponding to the measure

\[
\int_{-\infty}^{\infty} \frac{x}{\sinh(\pi x)} f(x) \, dx,
\]

with norms

\[
\|p_k\|^2 = \frac{k!(k + 1)!}{2^{2k+1}}.
\]

Let

\[
p_{2k}(x) = p_k^{(0)}(x^2), \quad p_{2k+1}(x) = x p_k^{(1)}(x^2), \tag{2.21}
\]

so that \(\langle p_k^{(\varepsilon)}(x) \rangle_{k=0}^{\infty}\) are the monic orthogonal polynomials corresponding to

\[
\int_{0}^{\infty} \frac{x^\varepsilon f(x)}{\sinh(\pi \sqrt{x})} \, dx, \quad \varepsilon = 0, 1,
\]

with norms

\[
\|p_k^{(0)}\|^2 = \frac{(2k)! (2k + 1)!}{2^{4k+1}}, \quad \|p_k^{(1)}\|^2 = \frac{(2k + 1)!(2k + 2)!}{2^{4k+3}}. \tag{2.22}
\]
Then, $p_k^{(c)}$ are continuous dual Hahn and Wilson polynomials; explicitly,

$$p_k^{(0)}(x) = (-1)^k S_k(x; 0, 1/2, 1) = (-1)^k \frac{4^k k!}{(2k)!} W_k(x/4; 0, 1/2, 1/2, 1),$$

$$p_k^{(1)}(x) = (-1)^k S_k(x; 1/2, 1, 1) = (-1)^k \frac{4^k (k + 1)!}{(2k + 1)!} W_k(x/4; 1/2, 1/2, 1, 1).$$

This may be seen by letting $q \to 1$ in (2.20), or by checking that the orthogonality measures agree.

**Remark 2.1.** It may seem strange that the variables $\lambda_i$, which are originally integers, have to be taken as imaginary at the support of the orthogonality measure of $p_k$. Actually, if we let

$$f_k(x) = \frac{p_k(ix)}{i^k} = \frac{(k + 1)!}{2^k} {}_2F_1 \left[ \begin{array}{c} -k, 1 - x \\ 2 \end{array} ; 2 \right], \quad (2.23)$$

then $f_k$ are “almost” orthogonal on the positive integers, in view of the Abel sum

$$\lim_{t \to 1} \sum_{k=1}^{\infty} (-1)^{k+1} t^k f_m(k) f_n(k) = \frac{(-1)^n (n + 1)!}{4^{n+1}} \delta_{mn}. \quad (2.24)$$

This type of orthogonality plays an important role in [R3]. It is a limit case of the orthogonality of Meixner polynomials $M_n(x; \beta, c)$ [KS] with $\beta = 2, c \to -1$. When $q \neq 1$, a similar formula can be obtained as a limit case of the discrete orthogonality for $q$-ultraspherical polynomials considered in [AK].

**2.4. Schur $Q$-polynomials and pfaffians.** The pfaffian of a skew-symmetric even-dimensional matrix is defined by

$$\text{pfaff} \left( a_{ij} \right) = \sum_{\sigma \in S_{2m}/G} \text{sgn}(\sigma) \prod_{i=1}^{m} a_{\sigma(2i-1), \sigma(2i)},$$

where $G$ is the subgroup of order $2^m m!$ consisting of permutations preserving the set of pairs $\{1, 2\}, \{3, 4\}, \ldots, \{2m - 1, 2m\}$. Pfaffians and Schur $Q$-polynomials are intimately related. Indeed, Schur’s original definition of the latter [Sc] is based on the identity

$$Q(\lambda_1, \ldots, \lambda_{2m}) = \text{pfaff}_{1 \leq i,j \leq 2m} \left( Q(\lambda_i, \lambda_j) \right). \quad (2.24)$$

In the same work, Schur obtained the pfaffian evaluation

$$\text{pfaff}_{1 \leq i,j \leq 2m} \left( \frac{x_j - x_i}{x_j + x_i} \right) = \prod_{1 \leq i < j \leq 2m} \frac{x_j - x_i}{x_j + x_i}, \quad (2.25)$$

which we recall here together with its companion [Stë, Proposition 2.3]

$$\text{pfaff}_{1 \leq i,j \leq 2m} \left( \frac{x_j - x_i}{1 - tx_i x_j} \right) = t^{m(m-1)} \prod_{1 \leq i < j \leq 2m} \frac{x_j - x_i}{1 - tx_i x_j}. \quad (2.26)$$
We need an elementary property of pfaffians, which is stated below as Corollary 2.3. Incidentally, it can be used to prove (2.24), see Remark 2.5. Before formulating the result, we find it instructive to give a generalization.

**Lemma 2.2.** Let \((X_i, \mu_i)_{1 \leq i \leq 2m}\) be a collection of measure spaces, and \((b_{ij})_{1 \leq i, j \leq 2m}\) a collection of integrable functions, \(b_{ij} : X_i \times X_j \to \mathbb{R}\), such that \(b_{ij}(x, y) = -b_{ji}(y, x)\). Then,

\[
\text{pfaff}_{1 \leq i, j \leq 2m} \left( \iint b_{ij}(x, y) \, d\mu_i(x) d\mu_j(y) \right) = \int \cdots \int \text{pfaff}_{1 \leq i, j \leq 2m} (b_{ij}(x_i, x_j)) \, d\mu_1(x_1) \cdots d\mu_{2m}(x_{2m}).
\]

**Proof.** The left-hand side equals

\[
\frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \prod_{i=1}^{m} \iint b_{\sigma(2i-1), \sigma(2i)}(x_i, y_i) \, d\mu_{\sigma(2i-1)}(x_i) d\mu_{\sigma(2i)}(y_i).
\]

Introducing new integration variables by \(x_i \mapsto x_{\sigma(2i-1)}\), \(y_i \mapsto x_{\sigma(2i)}\) and interchanging the finite sum and the integral, we obtain the desired result. \(\Box\)

Consider the special case of Lemma 2.2 when each \(X_i\) is equal to the same finite space \(X\). Writing \(X = \{1, \ldots, n\}\), \(\mu_i(j) = A_{ij}\), \(b_{ij}(k, l) = B_{kl}^{ij}\), it takes the following form.

**Corollary 2.3.** Let \((A_{ij})_{1 \leq i, j \leq 2m}\) and \((B_{kl}^{ij})_{1 \leq i, j \leq n, 1 \leq k, l \leq 2m}\) be matrices such that \(B_{kl}^{ij} = -B_{lk}^{ji}\). Then,

\[
\text{pfaff}_{1 \leq i, j \leq 2m} \left( \sum_{x, y = 1}^{n} A_{ix} A_{jy} B_{xy}^{ij} \right) = \sum_{k_1, \ldots, k_{2m}} A_{ik_1} \text{pfaff}_{1 \leq i, j \leq 2m} (B_{k_ik_j}^{ij}).
\]

**Remark 2.4.** We will only need Corollary 2.3 in the case when \(B_{ij} = B_{kl}^{ij}\) is independent of \(k\) and \(l\). It can then equivalently be written

\[
\text{pfaff}(ABA^t) = \sum_{1 \leq k_{2m} < \cdots < k_1 \leq n} \det (A_{i,k_j}) \text{pfaff}_{1 \leq i, j \leq 2m} (B_{k_ik_j}),
\]

which is a version of the “minor summation formula” of Ishikawa and Wakayama [IW1]. The corresponding special case of Lemma 2.2 is due to de Bruijn [B].

**Remark 2.5.** To prove (2.24), one may apply Corollary 2.3 in the case when

\[
A_{ik} = x_k^{\lambda_i} \prod_{j=1, j \neq k}^{n} \frac{x_k + x_j}{x_k - x_j}, \quad B_{kl}^{ij} = B_{ij} = \frac{x_j - x_i}{x_j + x_i}.
\]

Using (2.25) to compute \(\text{pfaff}(B_{k_ik_j})\) and comparing with (2.4) gives indeed (2.24).
Similarly, choosing
\[ A_{ik} = x_i^{k-1}(-q, q^{-n}; q)_{k-1}, \quad B_{ij} = B_{ij} = \frac{q^j - q^i}{q^j + q^i}. \]
it follows from (2.6) that, for \( m \) even,
\[ P_n(x_1, \ldots, x_m) = \text{pfaff}_{1\leq i,j\leq m}(P_n(x_i, x_j)). \tag{2.27} \]
Together with (2.8), this reduces the study of the polynomials \( P_n(x_1, \ldots, x_m) \) to the case \( m = 2 \).

2.5. Multivariable Christoffel–Darboux kernels. Let
\[ f \mapsto \int f(x) \, d\mu(x) \tag{2.28} \]
be a positive moment functional and \((p_k(x))_{k=0}^{\infty}\) the corresponding family of monic orthogonal polynomials. We may then introduce the Christoffel–Darboux kernel
\[ K_n(x, y) = \frac{1}{\|p_n\|^2} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y}. \tag{2.29} \]
In [R2], we studied more general multivariable kernels
\[ K^n_m(x) = K^n_m(x_1, \ldots, x_{2m}). \]
For present purposes, the crucial fact is that the following three expressions define the same kernel. As is explained in [R2], this can be deduced from results of Ishikawa and Wakayama [IW2], Lascoux [L1, L2], Okada [O] and Strahov and Fyodorov [SF]. A self-contained proof is given in [R2].

Lemma 2.6. If
\[ K^n_m(x) = \frac{1}{\prod_{i=1}^m \|p_{n-i}\|^2} \frac{\det_{1\leq i,j\leq 2m}(p_{n-m+j-1}(x_i))}{\prod_{1\leq i,j\leq 2m}(x_j - x_i)}, \tag{2.30} \]
then, for any choice of square roots \( \sqrt{x_i} \),
\[ K^n_m(x) = \frac{1}{\prod_{1\leq i,j\leq 2m}(\sqrt{x_j} - \sqrt{x_i})} \text{pfaff}_{1\leq i,j\leq 2m}((\sqrt{x_j} - \sqrt{x_i})K(x_i, x_j)) \]
and, for any choice of \( \xi_i \) such that \( \xi_i + \xi_i^{-1} = x_i + 2 \),
\[ K^n_m(x) = \frac{1}{\prod_{1\leq i,j\leq 2m}(\xi_j - \xi_i)} \text{pfaff}_{1\leq i,j\leq 2m}((\xi_j - \xi_i)K(x_i, x_j)). \]
We note in passing the less symmetric identities

\[ K_m^n(x_1, \ldots, x_m, y_1, \ldots, y_m) = \frac{\det_{1 \leq i, j \leq m}(K_m^n(x_i, y_j))}{\prod_{1 \leq i < j \leq m}(x_j - x_i)(y_j - y_i)} \]

\[ = \frac{1}{\prod_{1 \leq i < j \leq m}(x_j - x_i)(y_j - y_i)} \sum_{0 \leq k_m < \cdots < k_1 \leq n-1} \prod_{i=1}^{m} \frac{1}{\|p_k\|^2} \det_{1 \leq i, j \leq m}(p_k(x_i)) \det_{1 \leq i, j \leq m}(p_k(y_j)). \]  (2.31)

Thus, we have two determinant and two pfaffian formulas for the multiple sum in (2.31); these give four different multivariable extensions of the Christoffel–Darboux formula (2.29).

We will apply Lemma 2.6 in slightly modified form. To this end, assume that all odd moments of the functional (2.28) vanish. Let \((q_k)_{k=0}^\infty\) and \((r_k)_{k=0}^\infty\) be the monic orthogonal polynomials associated with the moment functionals \[ f \mapsto \int f(-x^2) \, d\mu(x), \quad f \mapsto \int f(-x^2)x^2 \, d\mu(x), \]
respectively; the minus signs will be convenient. Then, \[ p_{2k}(x) = (-1)^k q_k(-x^2), \quad p_{2k+1}(x) = (-1)^k x r_k(-x^2). \]

Later, we will choose \(p_k = c_k\) as the continuous \(q\)-ultraspherical polynomials in (2.10). Then, one may use (2.16) to identify \(q_k\) and \(r_k\) with continuous \(q\)-Jacobi polynomials.

Let \[ \tilde{K}_m^n(x, y) = \sum_{0 \leq k \leq n-1, k \equiv \bar{n}-1 \mod 2} \frac{p_k(x)p_k(y)}{\|p_k\|^2}. \]  (2.32)

Then, applying (2.29) to the systems \((q_k)_{k=0}^\infty\) and \((r_k)_{k=0}^\infty\) gives

\[ \tilde{K}_m^n(x, y) = \frac{1}{\|p_{n-1}\|^2} \frac{p_{n+1}(x)p_{n-1}(y) - p_{n+1}(x)p_{n-1}(y)}{x^2 - y^2}. \]  (2.33)

We note in passing that (2.32) implies

\[ \tilde{K}_m^n(x, y) = K_m^n(x, y) - \tilde{K}_m^{n-1}(x, y), \]

which, upon iteration, yields

\[ \tilde{K}_m^n(x, y) = \sum_{j=0}^{n-1} (-1)^{n+j+1} K^{j+1}_m(x, y). \]  (2.34)

More generally, let us temporarily write \(Q_m^n\) and \(R_m^n\) for the kernels obtained by replacing \((p_k)_{k=0}^\infty\) in \(K_m^n\) with \((q_k)_{k=0}^\infty\) and \((r_k)_{k=0}^\infty\), respectively. Moreover, let

\[ \tilde{K}_m^{2n-1}(x_1, \ldots, x_{2m}) = Q_m^n(-x_1^2, \ldots, -x_{2m}^2), \]
\[ \tilde{K}^n_m(x_1, \ldots, x_{2m}) = x_1 \ldots x_{2m} R^m_n(-x_1^2, \ldots, -x_{2m}^2). \]

Then, applying Lemma 2.6 to the kernels \( Q \) and \( R \), the result may be expressed in unified form as follows.

**Corollary 2.7.** Under the assumption of vanishing odd moments, the following identities hold:

\[
\tilde{K}^n_m(x) = \frac{1}{\prod_{i=1}^{m} \|p_{n+1-2i}\|^2} \frac{\det_{1 \leq i,j \leq 2m}(p_{n-2m+2j-1}(x_i))}{\prod_{1 \leq i,j \leq m}(x_j^2 - x_i^2)} = \frac{1}{\prod_{1 \leq i,j \leq 2m}(w_j - w_i)} \text{pfaff} \left( (w_j - w_i) \tilde{K}^n(x_i, x_j) \right) \quad (2.35)
\]

\[
= \frac{1}{\prod_{1 \leq i,j \leq 2m}(\xi_j - \xi_i)} \text{pfaff} \left( (\xi_j - \xi_i) \tilde{K}^n(x_i, x_j) \right), \quad (2.36)
\]

where \( w_i^2 = -x_i^2 \) and \( \xi_i + \xi_i^{-1} = 2 - x_i^2 \), \( 1 \leq i \leq 2m \).

The right-hand side of (2.30) is an analogue of a rectangular Schur polynomial; in fact, it can be identified with a Schur function over an abstract alphabet \([1, \infty]\). These polynomials make sense also for odd-dimensional matrices. Let

\[
P_n^m(x_1, \ldots, x_m) = \frac{\det_{1 \leq i,j \leq m}(p_{n+j-1}(x_i))}{\det_{1 \leq i,j \leq m}(p_{n+j-1}(x_i))} \frac{\det_{1 \leq i,j \leq m}(p_{n+j-1}(x_i))}{\prod_{1 \leq i,j \leq m}(x_j - x_i)}, \quad (2.37)
\]

so that

\[
\tilde{K}^n_m(x) = \frac{1}{\prod_{i=1}^{m} \|p_{n-i}\|^2} P_{(n-m)2m}^m(x).
\]

Under the assumption of vanishing odd moments, we will also write

\[
\tilde{P}_n^m(x_1, \ldots, x_m) = \frac{\det_{1 \leq i,j \leq m}(p_{n+2j-2}(x_i))}{\prod_{1 \leq i,j \leq m}(x_j^2 - x_i^2)}, \quad (2.38)
\]

so that

\[
\tilde{K}^n_m(x) = \frac{1}{\prod_{i=1}^{m} \|p_{n+1-2i}\|^2} \tilde{P}_{(n+1-2m)2m}^m(x).
\]

Then, if \( Q \) and \( R \) denote the kernels obtained by replacing \( p_k \) in (2.37) with \( q_k \) and \( r_k \), respectively, we have

\[
\tilde{P}_{(2n)m}^m(x_1, \ldots, x_m) = (-1)^mn Q_n^m(-x_1^2, \ldots, -x_m^2),
\]

\[
\tilde{P}_{(2n+1)m}^m(x_1, \ldots, x_m) = (-1)^mn x_1 \cdots x_m R_n^m(-x_1^2, \ldots, -x_m^2).
\]

It will be useful to note that

\[
\lim_{t \to \infty} t^{-n} \tilde{P}_{n+1}^m(x_1, \ldots, x_m, t) = \tilde{P}_n(x_1, \ldots, x_m), \quad (2.39)
\]

this is easily derived from (2.38).
Finally, we mention the integral formula
\[
\mathbb{P}_{n,m}(x_1, \ldots, x_m) = \frac{1}{n! \prod_{i=1}^n \|p_{i-1}\|^2} \int \prod_{1 \leq j \leq m \atop 1 \leq k \leq n} (x_j - y_k) \prod_{1 \leq i < j \leq n} (y_j - y_i)^2 \, d\mu(y_1) \cdots d\mu(y_n),
\]
which is at least implicitly due to Christoffel, see also [BH, R2]. Applying this to the polynomials \( q_k \) and \( r_k \) gives
\[
\tilde{\mathbb{P}}_{(2n)^m}(x_1, \ldots, x_m) = \frac{1}{n! \prod_{i=1}^n \|p_{2i-2}\|^2} \int \prod_{1 \leq j \leq m \atop 1 \leq k \leq n} (x_j^2 - y_k^2) \prod_{1 \leq i < j \leq n} (y_j^2 - y_i^2)^2 \, d\mu(y_1) \cdots d\mu(y_n), \tag{2.40a}
\]
\[
\tilde{\mathbb{P}}_{(2n+1)^m}(x_1, \ldots, x_m) = \frac{x_1 \cdots x_m}{n! \prod_{i=1}^n \|p_{2i-1}\|^2} \int \prod_{1 \leq j \leq m \atop 1 \leq k \leq n} (x_j^2 - y_k^2) \prod_{1 \leq i < j \leq n} (y_j^2 - y_i^2)^2 \prod_{k=1}^n y_k^2 \, d\mu(y_1) \cdots d\mu(y_n). \tag{2.40b}
\]
These formulas will be used in the Appendix.

3. Relation to Kawanaka’s identity

Kawanaka [K, Theorem 1.1] obtained the Schur polynomial identity
\[
\sum_{\mu} \prod_{\alpha \in \mu} \frac{1 + q^{h(\alpha)}}{1 - q^{h(\alpha)}} s_\mu(x_1, \ldots, x_m) = \prod_{i=1}^m \frac{(-x_i; q)_\infty}{(x_i; q)_\infty} \prod_{1 \leq i < j \leq m} \frac{1}{1 - x_i x_j}. \tag{3.1}
\]
Here, the sum is over all partitions \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0 \) and \( h(\alpha) \) denotes the hook-length at the box \( \alpha \) of the diagram of \( \mu \), see [Ma]. We will recover a proof of (3.1) below, see Remark 5.2.

Although it is not mentioned by Kawanaka, (3.1) can be viewed as evaluating a Schur \( Q \)-function at an infinte geometric progression. To see this we observe that, writing \( k_i = \mu_i + m - i \), the determinant in (2.7) equals
\[
\prod_{1 \leq i < j \leq m} (x_i - x_j) s_\mu(x_1, \ldots, x_m).
\]
Moreover, it is easy to check that (cf. [Ma] Examples I.1.1 and I.1.3)
\[
\prod_{i=1}^m \frac{(-q; q)_{k_i}}{(q; q)_{k_i}} \prod_{1 \leq i < j \leq m} \frac{q^{k_j} - q^{k_i}}{q^{k_j} + q^{k_i}} = \prod_{\alpha \in \mu} \frac{1 + q^{h(\alpha)}}{1 - q^{h(\alpha)}},
\]
where \( c \) denotes content. Plugging all this into (2.7) gives

\[
P_n(x_1, \ldots, x_m) = \prod_{i=1}^{m} \left( \frac{(-q; q)_{n+1-i}}{(q; q)_{n+1-i}} \right) \prod_{1 \leq i < j \leq m} (x_j - x_i)
\]

\[
\times \sum_{0 \leq \mu_1 \leq \cdots \leq \mu_1 \leq n+1-m} \prod_{\alpha \in \mu} \frac{1 + q^{|\alpha|}}{1 - q^{c(\alpha) + m - n - 1}} s_{\mu}(x). \quad (3.2)
\]

Let us now take the limit \( n \to \infty \), which can be justified analytically if \( |q| < 1 \) and \( |x_i| < 1 \) for all \( i \). Note that

\[
\lim_{n \to \infty} \prod_{\alpha \in \mu} \frac{1 - q^{c(\alpha) + m - n - 1}}{1 + q^{c(\alpha) + m - n - 1}} = (-1)^{\sum_i \mu_i},
\]

which may be absorbed into \( s_{\mu} \) by homogeneity. Thus, we obtain

\[
\lim_{n \to \infty} P_n(x_1, \ldots, x_m) = \left( \frac{-q; q}{q; q} \right)^m \prod_{1 \leq i < j \leq m} (x_j - x_i) \sum_{\mu} \prod_{\alpha \in \mu} \frac{1 + q^{|\alpha|}}{1 - q^{-c(\alpha) - m + n - 1}} s_{\mu}(x), \quad (3.3)
\]

where the sum is computed by (3.1). Although the corresponding Schur \( Q \)-function identity is equivalent to (3.1), it seems not to have appeared explicitly in the literature.

**Proposition 3.1.** For \( \lambda \) a partition of length \( m \),

\[
Q_{\lambda}(1, q, q^2, \ldots) = \prod_{i=1}^{m} \frac{(-1; q)_{\lambda_i}}{(q; q)_{\lambda_i}} \prod_{1 \leq i < j \leq m} \frac{q^{\lambda_j} - q^{\lambda_i}}{1 - q^{\lambda_j + \lambda_i}},
\]

where the left-hand side is interpreted as

\[
\lim_{n \to \infty} Q_{\lambda}(1, q, \ldots, q^n), \quad |q| < 1.
\]

In terms of tableaux, the result may be written as follows.

**Corollary 3.2.** For \( \lambda \) a strict partition of length \( m \),

\[
\sum_T q^{|T|} = \prod_{i=1}^{m} \frac{(-1; q)_{\lambda_i}}{(q; q)_{\lambda_i}} \prod_{1 \leq i < j \leq m} \frac{q^{\lambda_j} - q^{\lambda_i}}{1 - q^{\lambda_j + \lambda_i}}, \quad |q| < 1,
\]

where the sum is over all marked shifted tableaux of shape \( S(\lambda) \).

This should be compared with the identity

\[
\sum_T q^{|T|} = \prod_{i=1}^{m} \frac{1}{(q; q)_{\lambda_i}} \prod_{1 \leq i < j \leq m} \frac{q^{\lambda_j} - q^{\lambda_i}}{1 - q^{\lambda_i + \lambda_j}}, \quad (3.4)
\]
where the sum is over column-strict shifted tableaux of shape $S(\lambda)$, that is, only unmarked symbols appear, so that conditions (1)–(3) in Section 2.1 may be summarized as

(4) The labels increase weakly along rows and strictly down columns.

As was pointed out by Stembridge [St], (3.4) is equivalent to a result conjectured by Stanley [St] and first proved by Gansner [G]. Comparison with Corollary 3.2 suggests the following problem.

**Problem 3.3.** Prove directly (that is, without summing the series) that

$$\sum_{T \text{ marked}} q^{|T|} = \prod_{i=1}^{m} (-1; q)_{\lambda_i} \sum_{T \text{ column-strict}} q^{|T|},$$

where $\lambda$ is a strict partition of length $m$ and the sums are over shifted tableaux of shape $S(\lambda)$. For instance, writing $M_k$ and $C_k$ for the set of marked, respectively column-strict, shifted tableaux $T$ of shape $S(\lambda)$ such that $|T| = k$, construct a map $\phi : M_k \to \bigcup_{j \leq k} C_j$ such that

$$\sum_{k} |M_{k+j} \cap \phi^{-1}(x)| q^k = \prod_{i=1}^{m} (-1; q)_{\lambda_i}, \quad x \in C_j.$$  

This would lead to a new proof of the Stanley–Gansner identity as a consequence of Kawanaka’s identity (and vice versa).

## 4. Relation to a discrete Selberg integral

Krattenthaler [Kr, Theorem 6] gave the following multiple extension of the $q$-Chu–Vandermonde summation, which was applied to enumeration problems for perfect matchings:

$$\sum_{0 \leq k_0 < \cdots < k_1 \leq n} \prod_{1 \leq i < j \leq m} (q^{k_j} - q^{k_i}) \prod_{i=1}^{m} \frac{(x; q)_{k_i} (y; q)_{n-k_i}}{(q; q)_{k_i} (q; q)_{n-k_i}} y^{k_i}$$

$$= q^{2 \binom{m}{3} y \binom{m}{2}} \prod_{i=1}^{m} \frac{(x, y, q; q)_{i-1} (xyq^{i+m-2}; q)_{n+1-m}}{(q; q)_{n+i-m}}. \quad (4.1)$$

An equivalent identity was previously obtained by Milne [M1, Theorem 5.3]. Moreover, as noted in [Kr], another equivalent formulation is a special case of a discrete Selberg integral conjectured by Askey [A] and proved by Evans [E]. However, the form given by Krattenthaler is more useful for our purposes.

In the special case $x = y = -q$, [Kr1] evaluates the Schur $Q$-polynomial $Q_{\lambda}(1, q, \ldots, q^n)$, where $\lambda$ is the odd staircase partition $(2m-1, 2m-3, \ldots, 3, 1)$. To see this, we let $x_i \equiv q^{2i-1}$ in (2.7). The resulting sum contains the Vandermonde determinant

$$\det_{1 \leq i \leq j \leq m} (q^{2i-1} k_j) = \prod_{i=1}^{m} q^{2k_i} \prod_{1 \leq i < j \leq m} (q^{2k_j} - q^{2k_i})$$
and is thus evaluated by (4.1) as

\[ P_n(q, q^3, \ldots, q^{2m-1}) \]

\[ = \sum_{0 \leq k_m < \cdots < k_1 \leq n} \prod_{1 \leq i < j \leq m} (q^{k_j} - q^{k_i})^2 \prod_{i=1}^{m} \frac{(-q; q)_{k_i}(-q; q)_{n-k_i}}{(q; q)_{k_i} (q; q)_{n-k_i}} (-q)^{k_i} \]

\[ = (-1)^{m} q^\frac{1}{2} \binom{2m}{3} \prod_{i=1}^{m} \frac{(-q, -q; q)_{i-1} (q^{i+m}; q)_{n+1-m}}{(q; q)_{n+1-m}} \] (4.2)

\[ = (-1)^{m} q^\frac{1}{2} \binom{2m}{3} \prod_{i=1}^{m} \frac{(q^{n+1-m+i}; q)_{m}}{(q; q^2)_{i-1}(q; q^2)_{i}} \]

To check the equality of the last two expressions, the elementary identity

\[ \prod_{i=1}^{m} (aq^i; q)_{m} = \prod_{i=1}^{m} (aq^i; q)_{i}(aq^i; q)_{i-1} \] (4.3)

with \( a = 1 \), is useful.

Since, by anti-symmetry,

\[ P_n(q, q^3, \ldots, q^{2m-1}) = (-1)^{m} P_n(q^{2m-1}, \ldots, q^3, q) \]

we obtain the following Schur \( Q \)-polynomial identity. Although it is equivalent to a special case of (4.1), we have not found it explicitly in the literature.

**Proposition 4.1.** When \( \lambda = (2m-1, 2m-3, \ldots, 3, 1) \), then

\[ Q_{\lambda}(1, q, \ldots, q^n) = 2^m q^\frac{1}{2} \binom{2m}{3} \prod_{i=1}^{m} \frac{(q^{n+1-m+i}; q)_{m}}{(q; q^2)_{i-1}(q; q^2)_{i}}. \]

We will recover Proposition 4.1 twice below, see Remark 6.3 and Remark 7.4.

A very similar result holds for the even staircase, see Corollary 7.6.

We write down the limit case \( q \to 1 \) explicitly, starting from (4.2) with \( n \) replaced by \( n-1 \).

**Corollary 4.2.** When \( \lambda = (2m-1, 2m-3, \ldots, 3, 1) \), then \( Q_{\lambda}(1^n) \), that is, the number of marked shifted tableaux of shape \( S(\lambda) \), with labels from the finite alphabet \( 1' < 1 < 2' < 2 < \cdots < n' < n \), equals

\[ 2^{m^2} \prod_{i=1}^{m} \frac{(n+i-1)!(i-1)!}{(n+i-m-1)!(i+m-1)!}. \]

**Problem 4.3.** Give a combinatorial proof of Corollary 4.2 or more generally of Proposition 4.1.
5. Relation to Christoffel–Darboux Kernels

The following fundamental result identifies $P_n(x, y)$ with a Christoffel–Darboux kernel for continuous $q$-Jacobi polynomials.

**Lemma 5.1.** One has

$$P_n(x, y) = \frac{1 - q^{n+1} y P_{n+1}(x) P_{n-1}(y) - x P_{n-1}(x) P_{n+1}(y)}{1 - xy}. \quad (5.1)$$

Equivalently, fixing square roots $\sqrt{-x}$ and $\sqrt{-y}$ and writing $\xi = \sqrt{-x} + (\sqrt{-x})^{-1}$, $\eta = \sqrt{-y} + (\sqrt{-y})^{-1}$, one has

$$P_n(x, y) = (\sqrt{-x})^{n-1} (\sqrt{-y})^{n-1} (y - x) \tilde{K}_n(\xi, \eta), \quad (5.2)$$

where $\tilde{K}_n$ is as in (2.33), with $p_n = c_n$ defined in (2.10).

**Proof.** By definition, $(1 - xy) P_n(x, y)$ equals

$$(1 - xy) \frac{(-q; q)_n^2}{(q; q)_n^2} \sum_{k, l=0}^{n} q^k q^l \frac{(-q, q^{-n}; q)_k (-q, q^{-n}; q)_l}{(-q, q^{-n}; q)_k (-q, q^{-n}; q)_l} x^k y^l$$

$$= \frac{(-q; q)_n^2}{(q; q)_n^2} \sum_{k, l=0}^{n+1} \frac{q^k - q^l (-q, q^{-n}; q)_{k-1} (-q, q^{-n}; q)_{l-1}}{q^k + q^l (-q, q^{-n}; q)_k (-q, q^{-n}; q)_l} x^k y^l$$

$$\times \left\{ (1 + q^k)(1 - q^{k-1})(1 + q^l)(1 - q^{l-1}) \right. \right.$$}

$$- (1 - q^k)(1 + q^{k-1})(1 - q^l)(1 + q^{l-1}) \}.$$

We observe that the quantity within brackets factors as

$$2(q^k + q^l)(1 - q^{-n-1})(1 - q^{k+l-n-1}),$$

and then split the summand in a different way, writing

$$2(q^k + q^l)(1 + q^{-n-1})(1 - q^{k+l-n-1})$$

$$= (1 - q^k)(1 - q^{k-1})(1 + q^l)(1 + q^{l-1}) - (1 + q^k)(1 + q^{k-1})(1 - q^l)(1 - q^{l-1}).$$

This gives

$$(1 - xy) P_n(x, y)$$

$$= \frac{(-q; q)_n^2}{(q; q)_n^2} \left( \sum_{k=1}^{n} \frac{(-q; q)_{k-1}(q^{-n}; q)_k}{(q; q)_{k-1}(-q^{-n}; q)_k} x^k \sum_{l=0}^{n+1} \frac{(-q, q^{-n-1}; q)_l}{(q, -q^{-n-1}; q)_l} y^l \right.$$

$$- \sum_{k=0}^{n+1} \frac{(-q, q^{-n-1}; q)_k}{(q, -q^{-n-1}; q)_k} x^k \sum_{l=1}^{n} \frac{(-q; q)_{l-1}(q^{-n}; q)_l}{(q; q)_{l-1}(-q^{-n}; q)_l} y^l \right).$$
Replacing $k$ by $k+1$ in the first term and $l$ by $l+1$ in the second term yields (5.1). It is then straightforward to derive (5.2), using (2.11), (2.12) and the elementary identity

$$\xi^2 - \eta^2 = \frac{1}{xy}(x - y)(1 - xy).$$

□

Remark 5.2. By the $q$-binomial theorem [GR (II.3)],

$$\lim_{n \to \infty} P_n(x) = \left(- q; q \right)_\infty \left( q, -q; q \right)_\infty \sum_{k=0}^\infty \frac{(-q; q)_k}{(q; q)_k} (-x)^k,$$

for $|q|, |x| < 1$. Lemma 5.1 then gives

$$\lim_{n \to \infty} P_n(x, y) = \left(- q; q \right)_\infty \left( q, -q; q \right)_\infty \sum_{j=0}^{m-1} \frac{(-q; q)_j}{(q; q)_j} \left( qx, qy; q \right)_\infty \left( q, -q; q \right)_\infty \frac{x_j - x_i}{1 - x_i x_j},$$

for $m$ even. By (2.8), this holds also for odd $m$. Comparing (5.3) and (3.3) yields Kawanaka’s identity (3.1), which thus follows from Lemma 5.1. In fact, our proof of Lemma 5.1 generalizes the proof of Kawanaka’s identity given by Ishikawa and Wakayama [IW2].

We rewrite (2.34) in terms of the polynomials $P_n$, assuming for convenience that we have chosen $\sqrt{x}$ and $\sqrt{y}$ so that $\sqrt{x}/\sqrt{y} = \sqrt{-x}/\sqrt{-y}$. The resulting identity will be used in the proof of Theorem 6.1.

**Corollary 5.3.** One has

$$P_n(x, y) = \sqrt{x + y} \sum_{j=0}^{n-1} \left( \sqrt{x/y} \right)^{n-1-j} \left( \sqrt{y/x} \right)^{n-1-j} \left( \sqrt{y} P_{j+1}(x) P_j(y) - \sqrt{x} P_j(x) P_{j+1}(y) \right).$$

Next, combining (5.2) and (2.27) gives

$$P_n(x_1, \ldots, x_{2m}) = \prod_{i=1}^{2m} (\sqrt{-x_i})^{n-1-j} \text{pfaff}_{1 \leq i,j \leq 2m} \left( (x_j - x_i) K^n(\xi_i, \xi_j) \right),$$

where $\xi_i = \sqrt{-x_i} + (\sqrt{-x_i})^{-1}$. We see that the pfaffian is of the form (2.36), with $\zeta = x_1$, $z_i = \xi_i$. This shows the following result when $m$ is even. The case of odd $m$ follows as a limit case, using (2.8) and (2.39), and the expression (2.12) for the norms.
Theorem 5.4. For any fixed choice of square roots $\sqrt{-x_i}$,

$$P_n(x_1, \ldots, x_m) = \prod_{i=1}^{m} \frac{(-q; q)_{n+1-i}((\sqrt{-x_i})^{n+1-m}}{(q; q)_{n+1-i} \prod_{1 \leq i < j \leq m} (x_j - x_i)^\tilde{P}_{n+1-m}(\xi_1, \ldots, \xi_m)},$$

where $\xi_i = \sqrt{-x_i} + (\sqrt{-x_i})^{-1}$, and where $\tilde{P}$ is defined by choosing $p_k = c_k$ in (2.38). When $m$ is even, we equivalently have

$$P_n(x_1, \ldots, x_{2m}) = \prod_{i=1}^{2m} ((\sqrt{-x_i})^{n+1-2m} \prod_{1 \leq i < j \leq 2m} (x_j - x_i)^{\tilde{K}_m}(\xi_1, \ldots, \xi_{2m}),$$

where $\tilde{K}$ is defined by choosing $p_k = c_k$ in Corollary 2.7.

We will give an alternative proof of Theorem 5.4 in the Appendix.

Note that $\xi_i$ is invariant under the transformations $\sqrt{-x_i} \mapsto (\sqrt{-x_i})^{-1}$ or, equivalently, $\sqrt{-x_i} \mapsto -\sqrt{-x_i}$. This yields the following hyperoctahedral symmetry, which is quite non-obvious from the definition (2.16).

Corollary 5.5. The function

$$\prod_{i=1}^{m} ((\sqrt{-x_i})^{m-n-1} \frac{P_n(x_1, \ldots, x_m)}{\prod_{1 \leq i < j \leq m} (x_j - x_i)}$$

is invariant under the action of the hyperoctahedral group $B_m$ generated by permutation of the variables together with inversions $\sqrt{-x_i} \mapsto -(\sqrt{-x_i})^{-1}$.

It should be noted that, although phrased in terms of square roots, this is a polynomial statement. For instance, the symmetry under $\sqrt{-x_1} \mapsto -(\sqrt{-x_1})^{-1}$ can be rephrased as

$$\prod_{i=2}^{m} (1 - x_1 x_i) P_n(x_1, x_2, \ldots, x_m) = ((-x_1)^n \prod_{i=2}^{m} (x_i - x_1) P_n(x_1^{-1}, x_2, \ldots, x_m).$$

We note in passing the formulas for $P_n$ obtained from (2.31). These will not be used in the remainder of this paper.

Corollary 5.6. One has

$$P_n(x_1, \ldots, x_m, y_1, \ldots, y_m) = \prod_{1 \leq i < j \leq m} (1 - x_i x_j)(1 - y_i y_j) \frac{\det (P_n(x_i, y_j))}{\det (y_j - x_i)} \prod_{1 \leq i < j \leq m} (1 - q^{k_i+1})$$

$$\times \sqrt{\prod_{1 \leq i,j \leq m} (x_j^{(n-1-k_i)/2} P_k(x_j)) \prod_{1 \leq i,j \leq m} (y_j^{(n-1-k_i)/2} P_k(y_j))},$$

where $k_i = \sum_{0 \leq k_m < \ldots < k_1 \leq n-1 \mod 2} 1 \leq k_i \leq n-1$. 

\[ \]
Turning to (2.35), we have
\[
P_n(x_1, \ldots, x_{2m}) = \prod_{1 \leq i < j \leq 2m} (x_j - x_i) \prod_{1 \leq i \leq j \leq 2m} (w_j - w_i) \frac{\text{pfaff} \left( \frac{w_j - w_i}{x_j - x_i} P_n(x_i, x_j) \right)}{\prod_{1 \leq i \leq j \leq 2m} (w_j - w_i) \prod_{1 \leq i \leq j \leq 2m} (w_i - w_j) \prod_{1 \leq i,j \leq 2m} (w_j - w_i) \prod_{1 \leq i,j \leq 2m} (1 + \sqrt{x_i x_j}) P_n(x_i, x_j)} ,
\]
where \( w_i = \sqrt{x_i} - (\sqrt{x_i})^{-1} \), so that
\[
w_j - w_i = \frac{1}{\sqrt{x_i x_j}}(\sqrt{x_j} - \sqrt{x_i})(1 + \sqrt{x_i x_j}).
\]
This gives the following pfaffian formula, which forms the basis for Theorem 6.1.

**Corollary 5.7.** For \( m \) even,
\[
P_n(x_1, \ldots, x_m) = \prod_{1 \leq i < j \leq m} \frac{\sqrt{x_i} + \sqrt{x_j}}{1 + \sqrt{x_i x_j}} \text{pfaff} \left( \frac{1 + \sqrt{x_i x_j}}{\sqrt{x_i} + \sqrt{x_j}} P_n(x_i, x_j) \right).
\]

Rewriting the determinant formula (2.38) in terms of the one-variable polynomials \( P_n(x) \) and reversing the order of the columns, we obtain the following identity.

**Corollary 5.8.** One has
\[
\prod_{1 \leq i < j \leq m} (1 - x_i x_j) P_n(x_1, \ldots, x_m) = \prod_{i=1}^m \frac{(q^{n+1-m+i}; q)_{i-1}}{(-q^{n+1-m+i}; q)_{i-1}} \det_{1 \leq i,j \leq m} \left( x_i^{j-1} P_{n+m+1-2j}(x_i) \right). \quad (5.4)
\]

Corollary 5.8 seems interesting enough to write down also in standard notation for Schur \( Q \)-polynomials.

**Corollary 5.9.** For \( \lambda \) a partition of length \( m \),
\[
Q_\lambda(1, q, \ldots, q^n) = \prod_{i=1}^m \frac{(q^{n+1-m+i}; q)_{i-1}}{(-q^{n+1-m+i}; q)_{i-1}} \prod_{1 \leq i < j \leq m} \frac{1}{1 - q^{\lambda_i + \lambda_j}} \times \det_{1 \leq i,j \leq m} \left( q^{(j-1)\lambda_i} Q_{\langle \lambda_i \rangle}(1, q, \ldots, q^{n+m+1-2j}) \right).
\]

Replacing \( n \) by \( n - 1 \) and letting \( q \to 1 \) gives the following simple determinant formula for \( Q_\lambda(1^n) \).

**Corollary 5.10.** For \( \lambda \) a partition of length \( m \),
\[
Q_\lambda(1^n) = 2^{\frac{m}{2}(2n+1-m)} \prod_{i=1}^m \frac{1}{(n-m+i-1)!} \prod_{1 \leq i < j \leq m} \frac{1}{\lambda_i + \lambda_j} \det_{1 \leq i,j \leq m} \left( f_{n+m-2j}(\lambda_i) \right),
\]
where \( f_k \) are the hypergeometric polynomials defined in (2.23).
For future reference, we rewrite Corollary 5.10 in terms of generalized Schur polynomials and multivariable Christoffel–Darboux kernels corresponding to the continuous dual Hahn polynomials \( p_k^{(0)} \) and \( p_k^{(1)} \) introduced in (2.21). Let \( \mathbb{P}_n^{(\varepsilon)} \) be the polynomial obtained by choosing \( p_k = p_k^{(\varepsilon)} \) in (2.37), and let \( K_m^{n,\varepsilon} \) be the kernel similarly obtained from Lemma 2.6. Using the expression (2.22) for the norms, we arrive at the following result, which can also be obtained directly from Theorem 5.4 as a limit case.

**Corollary 5.11.** Let \( n - m = 2k + \varepsilon \), with \( k \) an integer and \( \varepsilon \in \{0, 1\} \). Then,

\[
Q(\lambda_1, \ldots, \lambda_m)(1^n) = 2^{\frac{1}{2}m(2n+1-m)} (1)_{km} \prod_{i=1}^{m} \lambda_i^\varepsilon \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j) \mathbb{P}_n^{(\varepsilon)}(-\lambda_1^2, \ldots, -\lambda_m^2).
\]

In particular,

\[
Q(\lambda_1, \ldots, \lambda_{2m})(1^{2n}) = 4^m \prod_{1 \leq i < j \leq 2m} (\lambda_i - \lambda_j) K_m^{n,0}(-\lambda_1^2, \ldots, -\lambda_{2m}^2),
\]

\[
Q(\lambda_1, \ldots, \lambda_{2m})(1^{2n+1}) = 4^m \prod_{i=1}^{2m} \lambda_i \prod_{1 \leq i < j \leq 2m} (\lambda_i - \lambda_j) K_m^{n,1}(-\lambda_1^2, \ldots, -\lambda_{2m}^2).
\]

6. **A Pfaffian Hypergeometric Identity**

Using the pfaffian formula of Corollary 5.7, we derive the following hypergeometric identity. Note that it displays the hyperoctahedral symmetry of Corollary 5.7.

**Theorem 6.1.** For any fixed choice of \( \sqrt{x_i} \),

\[
P_n(x_1, \ldots, x_m) = q^{m(m-1-2n)} \prod_{i=1}^{m} \frac{1 - q^{n+1}}{1 - q} \prod_{1 \leq i < j \leq m} \frac{\sqrt{x_i} + \sqrt{x_j}}{1 + \sqrt{x_i x_j}} \times \sum_{1 \leq i < j \leq m} \prod_{k_i, k_j = 0, i \neq j}^{n} \prod_{1 \leq i \leq m} \frac{q^{k_j} - q^{k_i}}{1 - q^{k_i + k_j + 1}} \prod_{i=1}^{m} \frac{(q^{n}, q^{n+2}, \sqrt{q/x_i}, -\sqrt{q/x_i}; q)_{k_i}}{(q, q^{3/2}, -q^{3/2}; q)_{k_i}}.
\]
Proof. We first consider the case $m = 2$. Using (2.17) in Corollary 5.3 and changing the order of summation gives

$$P_n(x, y) = \frac{q^{-1/2}}{(1-q)^2} \left( \sqrt{xy} + \frac{q^{-1/2}}{1 + \sqrt{xy}} \right)^n \times \sum_{k,l=0}^n \frac{(\sqrt{q/x}, -\sqrt{q/x}; q)_k}{(q, q^{3/2}, -q^{3/2}/q; q)_l} q^{k+l} \times \sum_{j=\max(k-1,l-1)}^{n-1} \frac{(1 - q^{j+1})(1 - q^{j+2}) q^{-j}}{((q^{-j-1}, q^{j+3}; q)_k(q^{-j}, q^{j+2}; q)_l - (q^{-j}, q^{j+2}; q)_k(q^{-j-1}, q^{j+3}; q)_l)} \times \left\{ (q^l - q^k)(1 - q^{-2j+3}) \right\}. \quad (6.2)$$

Let $S$ denote the inner sum in (6.2). It can be rewritten as

$$S = \sum_{j=\max(k-1,l-1)}^{n-1} (1 - q^{j+1})(1 - q^{j+2}) q^{-j} \times \left\{ (q^l - q^k)(1 - q^{-2j+3}) \right\}$$

where the expression within brackets factors as

$$(q^l - q^k)(1 - q^{-2j+3}).$$

Replacing $j$ by $n - 1 - j$, elementary manipulations give

$$S = q^{1-n}(q^l - q^k)(1 - q^{2n+1})(q^{-n}, q^{n+1}; q)_k(q^{-n}, q^{n+1}; q)_l \times \sum_{j=0}^{\min(n-k,n-l)} \frac{1 - q^{-2n-1+2j}}{1 - q^{-2n-1}} \frac{(q^{k-n}, q^{-n}; q)_j}{(q^{-k-n}, q^{-n}; q)_j} q^{-(k+l+1)j}.$$ 

Here, the sum is a very-well-poised \(6_5\phi_5\), which by [GR Eq. (II.21)] equals

$$\frac{(1 - q^{n+k+1})(1 - q^{n+l+1})}{(1 - q^{2n+1})(1 - q^{k+l+1})},$$

so that

$$S = q^{1-n}(1 - q^{n+1})^2 \frac{q^l - q^k}{1 - q^{k+l+1}} \frac{(q^{-n}, q^{n+2}; q)_k(q^{-n}, q^{n+2}; q)_l}{(q^{-n}, q^{n+2}; q)_k(q^{n+2}, q^{n+2}; q)_l}.$$
This gives

$$P_n(x, y) = q^{\frac{1}{2}-n} \left( 1 - q^{n+1} \right)^2 \left( \frac{1 - q}{1 - q} \right)^n \left( \sqrt{x} + \sqrt{y} \right) \left( \frac{1}{1 + \sqrt{xy}} \right)$$

$$\times \sum_{k,l=0}^{n} \left( \frac{q^l - q^k}{1 - q^{k+1}} \left( q^{-n}, q^{n+2}, \sqrt{q/x}, -\sqrt{q/y}; q \right)_k q^k \right)$$

$$\times \left( \frac{q^{-n}, q^{n+2}, \sqrt{q/y}, -\sqrt{q/x}; q \right)_l q^l) \right), \quad (6.3)$$

which is the case \( m = 2 \) of (6.1).

Suppose now that \( m \) is even. We plug (6.3) into Corollary 5.7, thus expressing \( P_n(x_1, \ldots, x_m) \) as a pfaffian of two-dimensional sums. Applying Corollary 2.3 leads to an \( m \)-dimensional sum, each term containing a pfaffian of the form (2.26). This proves (6.1) in the case when \( m \) is even.

Finally, if \( m \) is odd, we use (2.8) to write

$$P_n(x_1, \ldots, x_m) = \lim_{t \to 0} \frac{(q; q)_{n+1}}{(-q; q)_{n+1}} \frac{(-1)^m}{x_1 \cdots x_m} P_{n+1}(x_1, \ldots, x_m, t).$$

Applying (6.1) to the right-hand side, only the term with \( k_{m+1} = n + 1 \) survives in the limit. After simplification, one arrives at the desired expression. \( \square \)

In contrast to (2.6), (6.1) is regular in the limit

$$Q_{\lambda}(1^{n+1}) = 2^n \lim_{q \to 1} P_n(q^{\lambda_1}, \ldots, q^{\lambda_m}). \quad (6.4)$$

**Corollary 6.2.** Let \( \lambda \) be a partition of length \( m \). Then,

$$Q_{\lambda}(1^{n+1}) = (2n + 2)^m \sum_{k_1, \ldots, k_m=0}^{n} \prod_{1 \leq i < j \leq m} \frac{k_i - k_j}{k_i + k_j + 1} \prod_{i=1}^{m} \frac{(-n, n + 2, (1 - \lambda_i)/2)_{k_i}}{(1, 1, 3/2)_{k_i}}.$$ 

**Remark 6.3.** If \( x_i \equiv q^{2i-1} \), corresponding to the odd staircase partition, the sum in (6.1) reduces to the term with \( k_i \equiv i - 1 \). After simplification, one recovers Proposition 4.1.

### 7. Determinantal hypergeometric identities

By inserting hypergeometric expressions for \( P_n(x) \) into the determinant in Corollary 5.8, one may obtain further formulas for \( P_n(x_1, \ldots, x_m) \). We are particularly interested in cases when the result simplifies using a determinant evaluation such
as \([S]\) Lemma A.1]

\[
\det_{1 \leq i, j \leq m} \left( \frac{(AX_i, AC/X_i; q)_{j-1}}{(BX_i, BC/X_i; q)_{j-1}} \right) = q^{\binom{m}{2}}(AC)^{\binom{m}{2}} \times \prod_{1 \leq i < j \leq m} (X_j - X_i)(1 - X_i X_j/C) \prod_{i=1}^{m} (B/A, ABCq^{2m-2i}; q)_{i-1} X_{i}^{m-1}(BX_i, BC/X_i; q)_{m-1}
\]

(7.1)

or the degenerate case

\[
\det_{1 \leq i, j \leq m} ((AX_i; q)_{j-1}(BX_i; q)_{m-j}) = q^{\binom{m}{3}}A^{\binom{m}{2}} \prod_{1 \leq i < j \leq m} (X_i - X_j) \prod_{i=1}^{m} (q^{i-m}B/A; q)_{i-1}.
\]

(7.2)

Multiple hypergeometric sums obtained in this way have been called “Schlosser-type”, see \([GK, S]\).

First of all, using our original definition of \(P_n(x)\), we find the following identity, which has a very similar structure to sums found in \([RS]\).

**Theorem 7.1.** One has

\[
\prod_{1 \leq i < j \leq m} (1 - x_i x_j) P_n(x_1, \ldots, x_m)
\]

\[
= (-1)^{\binom{m}{2}} q^{(n+1)(\binom{m}{2})+2\binom{m}{3}} \frac{(-q; q)_{m}^{m}}{(q; q)_{m+n-1}^{m}} \prod_{i=1}^{m} \frac{(q^{n+1-m+i}; q)_{i-1}}{(-q^{2-m}; q)_{i-1}}
\]

\[
\times \sum_{k_1, \ldots, k_m = 0}^{n+m-1} \left( \prod_{1 \leq i < j \leq m} (q^{k_j} - q^{k_i})(1 - q^{k_i + k_j + 1-m-n})
\]

\[
\times \prod_{i=1}^{m} \frac{(q^{1-m-n}; -q^{2-m}; q)_{k_i}}{(q, -q^{-n}; q)_{k_i}} x_i^{k_i} \right).
\]

(7.3)

**Proof.** By \((2.9)\), the right-hand side of \((5.4)\) equals

\[
\prod_{i=1}^{m} \frac{(-q; q)_{n+1-i}}{(q; q)_{n+1-i}^{m+i}} \sum_{\sigma \in S_m} \text{sgn}(\sigma) \sum_{k_1, \ldots, k_m \geq 0} \prod_{i=1}^{m} \frac{(-q; q^{2\sigma(i)-n-m-1}; q)_k}{(q, -q^{2\sigma(i)-n-m-1}; q)_k} x_i^{k_i+\sigma(i)-1}.
\]

Replacing \(k_i\) by \(k_i + 1 - \sigma(i)\) and interchanging summations gives after simplification

\[
(-1)^{\binom{m}{2}} \frac{(-q; q)_{m+n-1}^{m}}{(q; q)_{m+n-1}^{m+n-1}} \prod_{i=1}^{m} \frac{(q^{n+1-m+i}; q)_{i-1}}{(-q^{n+1-m+i}; q)_{i-1}}
\]

\[
\times \sum_{k_1, \ldots, k_m = 0}^{m+n-1} \det_{1 \leq i, j \leq m} \left( \frac{(q^{k_i+1-m-n}; q^{-k_i}; q)_{j-1}}{(-q^{k_i+1-m-n}; -q^{-k_i}; q)_{j-1}} \right) \prod_{i=1}^{m} \frac{(-q; q^{1-m-n}; q)_k}{(q, -q^{1-m-n}; q)_k} x_i^{k_i}.
\]
By (7.1), the determinant equals
\[
q^{\binom{m}{3}} \prod_{1 \leq i < j \leq m} (q^{k_j} - q^{k_i})(1 - q^{k_i+k_j+1-m-n}) \prod_{i=1}^{m} \frac{(-1, -q^{1+m-n-2i}; q)_{i-1}}{(-q^{k_i+1-m-n}, -q^{-k_i}; q)_{m-1}} q^{(1-m)k_i}.
\]

After further simplification, one arrives at the right-hand side of (7.3).

**Remark 7.2.** Writing Theorem 7.1 as
\[
\prod_{1 \leq i < j \leq m} (1 - x_i x_j) P_n(x_1, \ldots, x_m) = \sum_{k_1, \ldots, k_m = 0}^{n+m-1} \chi(k_1, \ldots, k_m) \prod_{i=1}^{m} x_i^{k_i},
\]
one easily checks that \((-1)^{|k|} \chi(k)\) is antisymmetric and invariant under the reflections \(k_i \mapsto n + m - 1 - k_i\). This fact is equivalent to Corollary 5.5. In analogy with the Schur polynomial expansion (3.2), one may exploit this symmetry to rewrite Theorem 7.1 in terms of characters of the classical groups SO(2m) and SO(2m + 1), according to whether \(n + m\) is odd or even, respectively.

Inserting different expressions for \(P_n(x)\) in Corollary 5.8, one may obtain further hypergeometric formulas. We will not exploit all possibilities, but merely give a few examples, corresponding to the eight expressions in (2.20). The resulting identities share the nice properties of Theorem 6.1, that is, they are regular in the limit \(q \to 1\), have completely factored terms, display hyperoctahedral symmetry, and have nice specializations at staircase-type partitions.

For instance, assuming that \(n + m\) is odd, we insert (2.20) into (5.4). After simplification, we obtain
\[
\prod_{1 \leq i < j \leq m} (1 - x_i x_j) P_n(x_1, \ldots, x_m)
\]
\[
= q^{-\frac{1}{2}nm} (1 - q)^n \prod_{i=1}^{m} \frac{(q^{n+1-m+i}; q)_i}{(-q^{n+1-m+i}, q)_{i-1}(q^{1-n-m}/2, -q^{3+n-m}/2; q)_{i-1}} x_i^{(n+m-1)/2}
\]
\[
\times \sum_{k_1, \ldots, k_m = 0}^{(n+m-1)/2} \prod_{i=1}^{m} \frac{(q^{1-n-m}/2, -q^{3+n-m}/2, q x_i, q/x_i; q)_{k_i} q^{k_i}}{(q, q, q^{3/2}, -q^{3/2}; q)_{k_i}}
\]
\[
\times \det_{1 \leq i, j \leq m} \left( (q^{k_i+(1-n-m)/2}; q)_{j-1}(-q^{k_i+(3+n-m)/2}; q)_{m-j} \right),
\]

where, by (7.2), the determinant equals
\[
q^{-\frac{1}{2} \binom{n}{3} - \frac{n+1}{2} \binom{m}{2}} \prod_{1 \leq i \leq j \leq m} (q^{k_i} - q^{k_j}) \prod_{i=1}^{m} (-q^{n+1-m+i}; q)_{i-1}.
\]

Repeating this for all eight expressions (2.20) yields the following result. In the case of (7.4c) and (7.5d) we have also used (4.3).
Theorem 7.3. If \( n + m \) is odd, then \( \prod_{1 \leq i < j \leq m} (1 - x_i x_j) P_n(x_1, \ldots, x_m) \) is equal to each of the four expressions

\[
\frac{(-1)^{m/2} q^{\frac{1}{2} \binom{m+1}{3} - \frac{1}{2} \binom{m+1}{2}}}{(1 - q)^m} \prod_{i=1}^{m} \frac{\left( q^{n+1-m+i}; q \right)_i}{\left( q^{(1-n-m)/2}, -q^{(3+n-m)/2}; q \right)_{i-1}} x_i^{(n-m-1)/2} \\
\times \sum_{k_1, \ldots, k_m = 0}^{(n+m-1)/2} \left( \prod_{1 \leq i < j \leq m} \left( q^{k_j} - q^{k_i} \right) \right) \\
\times \prod_{i=1}^{m} \frac{\left( q^{1-n-m}/2, -q^{3+n-m}/2, q x_i, q/x_i; q \right)_{k_i} q^{k_i}}{(q, q^{1/2}, q^{3/2}, -q^{3/2}; q)_{k_i}}, \quad (7.4a)
\]

\[
\frac{q^{-\frac{1}{4}nm} (-q; q)_{(n+1-m)/2}^m}{(q; q)_{(n+m-1)/2}^m} \prod_{i=1}^{m} \frac{\left( q^{n+1-m+i}; q \right)_{i-1} \left( q^{3/2}; q \right)_{(n+m+1-2i)/2} x_i^{(n-m-1)/2}}{\left( -q^{1/2}; q \right)_{(n+m+1-2i)/2}} \\
\times \sum_{k_1, \ldots, k_m = 0}^{(n+m-1)/2} \left( \prod_{1 \leq i < j \leq m} \left( q^{k_j} - q^{k_i} \right) \right) \\
\times \prod_{i=1}^{m} \frac{\left( q^{1-n-m}/2, -q^{3+n-m}/2, q^{1/2} x_i, q^{1/2}/x_i; q \right)_{k_i} q^{k_i}}{(q, q^{1/2}, q^{3/2}, -q; q)_{k_i}}, \quad (7.4b)
\]

\[
\frac{q^{-\frac{1}{4}nm}}{(1 - q)^m (q^{3+n-m}; q^2)_{m-1}^m} \prod_{i=1}^{m} \frac{\left( q^{n+1-m+i}; q \right)_m x_i^{(n+m-1)/2}}{\left( q^{3+n-m}, q x_i, q/x_i; q^2 \right)_{k_i} q^{2k_i}} \\
\times \sum_{k_1, \ldots, k_m = 0}^{(n+m-1)/2} \left( \prod_{1 \leq i < j \leq m} \left( q^{2k_j} - q^{2k_i} \right) \right) \\
\times \prod_{i=1}^{m} \frac{\left( q^{1-n-m}, q^{3+n-m}, q^{1/2} x_i, q^{1/2}/x_i; q^2 \right)_{k_i} q^{2k_i}}{(q^2, q^2, q^3, q^3; q^2)_{k_i}}, \quad (7.4c)
\]

\[
\frac{q^{-nm}}{(1 - q)^{2m} (q^{3+n-m}; q^2)_{m-1}^m} \prod_{i=1}^{m} \frac{\left( q^{n+1-m+i}; q \right)_i^2 x_i^{(n+m-1)/2}}{\left( q^{2k_j} - q^{2k_i} \right) \left( q^2, q^2, q^3, q^3; q^2 \right)_{k_i} q^{2k_i}} \\
\times \sum_{k_1, \ldots, k_m = 0}^{(n+m-1)/2} \left( \prod_{1 \leq i < j \leq m} \left( q^{2k_j} - q^{2k_i} \right) \right) \\
\times \prod_{i=1}^{m} \frac{\left( q^{1-n-m}, q^{3+n-m}, q^{2} x_i, q^{2}/x_i; q^2 \right)_{k_i} q^{2k_i}}{(q^2, q^2, q^3, q^3; q^2)_{k_i}}, \quad (7.4d)
\]
whereas if $n + m$ is even, \[ \prod_{1 \leq i < j \leq m} (1 - x_i x_j) P_n(x_1, \ldots, x_m) \] equals
\[
\frac{(-1)^m q^{\frac{1}{2} (m + 1)} - 1}{(1 - q)^{2m}} \prod_{i=1}^{m} \frac{(q^{n+1-m+i}; q)^i}{(q^{(2-n-m)/2}, -q^{(4+n-m)/2}; q)_{i-1}} \times \prod_{i=1}^{m} (1 - x_i) x_i^{(n+m-2)/2} \sum_{k_1, \ldots, k_m=0}^{n-1} \left( \prod_{1 \leq i < j \leq m} (q^{k_j} - q^{k_i}) \right) \times \prod_{i=1}^{m} \frac{(q^{2-n-m}/2, -q^{(4+n-m)/2}, q x_i, q/x_i; q)_{k_i}}{(q, q^2, q^3/2, -q^3/2; q)_{k_i}} ,
\]
\[ (7.5a) \]
\[
\frac{q^{-\frac{1}{2} (n-1)m}}{(1 - q)^m (q^{4+n-m}; q^2)_{m-2}} \prod_{i=1}^{m} (q^{n+1-m+i}; q)^2 \times \prod_{i=1}^{m} (1 - x_i) x_i^{(n+m-2)/2} \sum_{k_1, \ldots, k_m=0}^{n-1} \left( \prod_{1 \leq i < j \leq m} (q^{2k_j} - q^{2k_i}) \right) \times \prod_{i=1}^{m} \frac{(q^{2-n-m}, q^{4+n-m}, q x_i, q/x_i; q^2)_{k_i}}{(q^2, q^2, q^3, q^3; q^2)_{k_i}} ,
\]
\[ (7.5b) \]
\[
\frac{q^{- (n-1)m}}{(1 - q)^{2m} (1 - q^2) (q^{4+n-m}; q^2)_{m-2}} \prod_{i=1}^{m} (q^{n+1-m+i}; q) \times \prod_{i=1}^{m} (1 - x_i) x_i^{(n+m-2)/2} \sum_{k_1, \ldots, k_m=0}^{n-1} \left( \prod_{1 \leq i < j \leq m} (q^{2k_j} - q^{2k_i}) \right) \times \prod_{i=1}^{m} \frac{(q^{2-n-m}, q^{4+n-m}, q^2 x_i, q^2/x_i; q^2)_{k_i}}{(q^2, q^2, q^3, q^3; q^2)_{k_i}} ,
\]
\[ (7.5c) \]
\[
\frac{q^{- (n-1)m}}{(1 - q)^{2m} (1 - q^2) (q^{4+n-m}; q^2)_{m-2}} \prod_{i=1}^{m} (q^{n+1-m+i}; q) \times \prod_{i=1}^{m} (1 - x_i) x_i^{(n+m-2)/2} \sum_{k_1, \ldots, k_m=0}^{n-1} \left( \prod_{1 \leq i < j \leq m} (q^{2k_j} - q^{2k_i}) \right) \times \prod_{i=1}^{m} \frac{(q^{2-n-m}, q^{4+n-m}, q^2 x_i, q^2/x_i; q^2)_{k_i}}{(q^2, q^2, q^3, q^3; q^2)_{k_i}} ,
\]
\[ (7.5d) \]

Note that all eight expressions are regular in the limit (6.4). We do not write down the resulting identities for $Q_{\lambda}(1^n)$ explicitly.
Remark 7.4. By specializing the variables $x_i$, each of the eight sums in Theorem (7.3) may be reduced to the term with $k_i = i - 1$. In the case of (7.4a) and (7.5a), this happens if $x_i \equiv q^j$, leading to the identity

$$Q_{(m,m-1,\ldots,1)}(1, q, \ldots, q^n) = 2^m(-1)^{(m)} P_n(q, q^2, \ldots, q^m)$$

$$= 2^m q^{\binom{m}{3}} \prod_{i=1}^{m} \frac{(q^{n+1-m+i}; q)_i}{(q^2)_i},$$

which follows more easily from [Ma, Ex. III.8.3]

$$Q_{(m,m-1,\ldots,1)} = 2^m s_{(m,m-1,\ldots,1)}.$$

In the case of (7.4b) and (7.5b), one should let $x_i \equiv q^{-\frac{j}{2}}$. We write down the result in Corollary 7.5. Similarly, when $x_i \equiv q^{2i-1}$, (7.4c) and (7.5c) reduce to Proposition 4.1 and when $x_i \equiv q^{2i}$, (7.4d) and (7.5d) yield Corollary 7.6.

Corollary 7.5. If $x_i \equiv q^{-\frac{j}{2}}$, then $P_n(x_1, \ldots, x_m)$ equals

$$(-1)^{(m)} q^{\binom{m}{3}} \prod_{i=1}^{m} \frac{(q, q^{n+1-m+i}; q)_{i-1}(q^{3/2}, -q; q)_{(n+m+1-2i)/2}}{(-q, q^{1/2}, q^{3/2}; q)_{i-1}(q, -q^{1/2}; q)_{(n+m+1-2i)/2}}$$

if $n + m$ is odd and

$$(-1)^{(m)} q^{\binom{m}{3}} \prod_{i=1}^{m} \frac{(q, q^{n+1-m+i}; q)_{i-1}(q^{3/2}; q)_{(n-m-2i)/2}(-q, q)_{(n+m+2-2i)/2}}{(-q, q^{3/2}, q^{3/2}; q)_{i-1}(-q^{3/2}; q)_{(n+m-2i)/2}}$$

if $n + m$ is even.

Corollary 7.6. When $\lambda = (2m, 2m - 2, \ldots, 2)$, then

$$Q_\lambda(1, q, \ldots, q^n) = 2^m q^{\binom{m}{3}} \prod_{i=1}^{m} \frac{(q^{n+1-m+i}; q)_i^2}{(q^2)_i^2}.$$
is the unique measure such that the polynomials (2.10) are orthogonal with norm (2.12). By (2.40), Theorem 5.4 is then equivalent to the integral formulas

\[
P_n(x_1, \ldots, x_m) = \frac{1}{l!} \prod_{i=1}^{n} \left( \frac{-q; q}{q; q} \right)_i \prod_{i=1}^{m} x_i^l \prod_{1 \leq i < j \leq m} (x_j - x_i)
\]

\[
\times \int_{0 \leq \theta_1, \ldots, \theta_l \leq \pi} \prod_{1 \leq j \leq m, 1 \leq k \leq l} (x_j + x_j^{-1} + e^{2i\theta_k} + e^{-2i\theta_k})
\]

\[
\times \prod_{1 \leq j < k \leq l} (e^{2i\theta_j} + e^{-2i\theta_j} - e^{2i\theta_j} - e^{-2i\theta_j})^2 \prod_{k=1}^{l} w(\theta_k) d\theta_k, \quad (A.2a)
\]

when \( n - m = 2l - 1 \) and

\[
P_n(x_1, \ldots, x_m) = \frac{1}{l!} \prod_{i=1}^{n} \left( \frac{-q; q}{q; q} \right)_i \prod_{i=1}^{m} x_i^l (1 - x_i) \prod_{1 \leq i < j \leq m} (x_j - x_i)
\]

\[
\times \int_{0 \leq \theta_1, \ldots, \theta_l \leq \pi} \prod_{1 \leq j \leq m, 1 \leq k \leq l} (x_j + x_j^{-1} + e^{2i\theta_k} + e^{-2i\theta_k})
\]

\[
\times \prod_{1 \leq j < k \leq l} (e^{2i\theta_j} + e^{-2i\theta_j} - e^{2i\theta_j} - e^{-2i\theta_j})^2 \prod_{k=1}^{l} (e^{i\theta_k} + e^{-i\theta_k})^2 w(\theta_k) d\theta_k, \quad (A.2b)
\]

when \( n - m = 2l \).

To prove this we will use the following alternative expression for \( Q_\lambda \). An equivalent identity is due to Nimmo [N, Eq. (A12)], see also [Ma, Ex. III.8.13].

**Lemma A.1.** Let \( l \) be the integral part of \((n - m)/2\). Then,

\[
Q_{(\lambda_1, \ldots, \lambda_m)}(x_1, \ldots, x_n)
\]

\[
= \frac{2^{m-l}}{l!} \prod_{1 \leq i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{m} x_{\lambda_i}^{\sigma(i)} \prod_{i=1}^{l} x_{\sigma(m+2i-1)} \frac{x_{\sigma(m+2i)} - x_{\sigma(m+2i+1)}}{x_{\sigma(m+2i)} + x_{\sigma(m+2i+1)}}.
\]

In particular, if \( l \) is the integral part of \((n + 1 - m)/2\), then

\[
P_n(x_1, \ldots, x_m)
\]

\[
= \frac{1}{2^{m-l}} \prod_{j=1}^{n} \left( \frac{-q; q}{q; q} \right)_j \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{i=1}^{m} x_{\sigma(i)-1}^{\sigma(i)} \prod_{i=1}^{l} \frac{1 - q^{\sigma(m+2i)-\sigma(m+2i+1)}}{1 + q^{\sigma(m+2i)-\sigma(m+2i+1)}}. \quad (A.3)
\]

**Proof.** By definition,

\[
Q_\lambda(x) = \frac{2^{m}}{(n - m)!} \sum_{\sigma \in S_n} \prod_{i=1}^{m} x_{\sigma(i)}^{\lambda_i} \prod_{1 \leq i < j \leq n} \frac{x_{\sigma(i)} + x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}}.
\]
We write
\[
\prod_{1 \leq i \leq m} \frac{x_{\sigma(i)} + x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} = AB,
\]
where
\[
A = \prod_{1 \leq i < j \leq n} \frac{x_{\sigma(i)} + x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} = \text{sgn}(\sigma) \prod_{1 \leq i < j \leq n} \frac{x_i + x_j}{x_i - x_j}.
\]
\[
B = \prod_{m+1 \leq i < j \leq n} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_{\sigma(i)} + x_{\sigma(j)}}.
\]
Next, we note that
\[
\prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{x_i + x_j} = \frac{1}{2^l!} \sum_{\sigma \in S_{n-m}} \text{sgn}(\sigma) \prod_{i=1}^{l} \frac{x_{\sigma(2j-1)} - x_{\sigma(2j)}}{x_{\sigma(2j-1)} + x_{\sigma(2j)}}.
\]
Indeed, if \(n - m\) is even this is Schur’s pfaffian evaluation (2.25), while if \(n - m\) is odd it is the case \(x_{2m} = 1\) of (2.25). It follows that the factor \(B\) can be traded for
\[
\frac{(n - m)!}{2^l!} \prod_{i=1}^{l} \frac{x_{\sigma(m+2i-1)} - x_{\sigma(m+2i)}}{x_{\sigma(m+2i-1)} + x_{\sigma(m+2i)}}
\]
since the result is antisymmetrized over \(S_n\) we need not perform the antisymmetrization over the smaller group \(S_{n-m}\). This completes the proof. \(\square\)

It will be convenient to introduce the functional
\[
\int f(t) \, d\lambda(t) = \int_{\pi}^{\pi} \frac{f(-e^{2i\theta}) - f(-e^{-2i\theta})}{e^{2i\theta} - e^{-2i\theta}} \, w(\theta) \, d\theta,
\]
defined on Laurent polynomials.

**Lemma A.2.** One has
\[
\int t^k \, d\lambda(t) = \frac{1 - q^k}{1 + q^k}, \quad k \in \mathbb{Z}.
\]

**Proof.** Since
\[
\int f(t) \, d\lambda(t) = -\int f(t^{-1}) \, d\lambda(t),
\]
we may assume that \(k\) is positive. We may then write
\[
\int t^k \, d\lambda(t) = \frac{(-1)^{k+1}}{2} \int_{-\pi}^{\pi} \frac{e^{2i\theta k} - e^{-2i\theta k}}{e^{2i\theta} - e^{-2i\theta}} \, w(\theta) \, d\theta
\]
\[
= \frac{(-1)^{k+1}}{2} \sum_{j=0}^{k-1} \int_{-\pi}^{\pi} e^{2i\theta(2j+1-k)} \, w(\theta) \, d\theta.
\]
By [GR Eq. (7.4.18)],
\[
\int_{-\pi}^{\pi} e^{2ik\theta} w(\theta) d\theta = 2(-1)^k q^k \frac{q^{-1} - q}{(1 + q^{k-1})(1 + q^{k+1})} = 2(-1)^k (a_{k-1} - a_{k+1}),
\]
where \(a_k = q^k/(1 + q^k)\). Thus, the above sum telescopes, giving
\[
\int t^k \, d\lambda(t) = a_{-k} - a_k,
\]
which simplifies to the desired expression. \(\square\)

We now use Lemma A.2 to represent the numbers
\[
\frac{1 - q^{\sigma(m+2) - \sigma(m+2i-1)}}{1 + q^{\sigma(m+2) - \sigma(m+2i-1)}},
\]
appearing in (A.3) as integrals. This gives
\[
P_n(x_1, \ldots, x_m)
= \frac{(-1)^l}{2^l l!} \prod_{j=1}^{n} \frac{(-q; q)_j}{(q; q)_j} \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{i=1}^{m} x_i^{\sigma(i) - 1} \prod_{i=1}^{l} \int (-t)^{\sigma(m+2) - \sigma(m+2i-1)} \, d\lambda(t).
\]
Interchanging summation and integration, the sum becomes a Vandermonde determinant, for which we introduce the notation
\[
\Delta(x) = \det(x_j^{i-1}) = \prod_{i<j} (x_j - x_i).
\]

If \(n - m = 2l - 1\), we thus obtain
\[
P_n(x_1, \ldots, x_m) = \frac{1}{2^l l!} \prod_{j=1}^{n} \frac{(-q; q)_j}{(q; q)_j} \times \int \Delta(x_1, \ldots, x_m, t_1^{-1}, t_1, \ldots, t_l^{-1}, t_l) \, d\lambda(t_1) \cdots d\lambda(t_l),
\]
whereas if \(n - m = 2l\),
\[
P_n(x_1, \ldots, x_m) = \frac{1}{2^l l!} \prod_{j=1}^{n} \frac{(-q; q)_j}{(q; q)_j} \times \int \Delta(x_1, \ldots, x_m, t_1^{-1}, t_1, \ldots, t_l^{-1}, t_l, 1) \, d\lambda(t_1) \cdots d\lambda(t_l).
\]
Next, we apply the following elementary identities.
Lemma A.3. One has
\[
\Delta(x_1, \ldots, x_m, t_1, t_1^{-1}, \ldots, t_l, t_l^{-1})
= \prod_{i=1}^{m} x_i^l \prod_{j=1}^{l} (t_j - t_j^{-1}) \Delta(x) \Delta(\tau)^2 \prod_{1 \leq j \leq m, 1 \leq k \leq l} (\xi_i - \tau_j),
\]
\[
\Delta(x_1, \ldots, x_m, t_1, t_1^{-1}, \ldots, t_l, t_l^{-1}, 1)
= \prod_{i=1}^{m} x_i^l (1 - x_i) \prod_{j=1}^{l} (t_j - t_j^{-1})(2 - \tau_j) \Delta(x) \Delta(\tau)^2 \prod_{1 \leq j \leq m, 1 \leq k \leq l} (\xi_i - \tau_j),
\]
where \(\xi_i = x_i + x_i^{-1}, \tau_i = t_i + t_i^{-1}\).

Writing
\[
\int (t_j - t_j^{-1}) f(\tau_j) d\lambda(t_j) = 2 \int_0^\pi f(-e^{2i\theta_j} - e^{-2i\theta_j}) w(\theta_j) d\theta_j,
\]
we recover (A.2). This completes our alternative proof of Theorem 5.4.

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