ON A GENERALIZED 1-HARMONIC EQUATION AND THE INVERSE MEAN CURVATURE FLOW

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ABSTRACT. We introduce and study generalized 1-harmonic equations (1.1). Using some ideas and techniques in studying 1-harmonic functions from [W1] (2007), and in studying nonhomogeneous 1-harmonic functions on a cocompact set from [W2, (9.1)] (2008), we find an analytic quantity \( w \) in the generalized 1-harmonic equations (1.1) on a domain in a Riemannian \( n \)-manifold that affects the behavior of weak solutions of (1.1), and establish its link with the geometry of the domain. We obtain, as applications, some gradient bounds and nonexistence results for the inverse mean curvature flow, Liouville theorems for \( p \)-subharmonic functions of constant \( p \)-tension field, \( p \geq n \), and nonexistence results for solutions of the initial value problem of inverse mean curvature flow.

1. Introduction

Some ideas and techniques in studying 1-harmonic functions from [W1], and in studying nonhomogeneous 1-harmonic functions on a cocompact set from [W2, (9.1)], can be carried over to a more general setting, e.g., to a large class of the following partial differential equations:

\[
\text{div} \left( \frac{\nabla f}{|\nabla f|} \right) = A(x,f,\nabla f) \quad \text{on} \quad \Omega,
\]

where \( A(x,f(x),\nabla f(x)) \) is a continuous real-valued function of \( x \in \Omega \), or more generally \( A(x,f(x),\nabla f(x)) \) is a nonnegative-valued or nonpositive-valued function, and \( \Omega \) is a domain in a complete Riemannian \( n \)-manifold \( M \). These equations, in particular, include the ones for 1-harmonic functions when \( A(x,f,\nabla f) \equiv 0 \), functions of constant 1-tension field when \( A(x,f,\nabla f) \equiv \text{const} \), nonhomogeneous 1-harmonic functions when \( A(x,f,\nabla f) = A(x) \), almost 1-harmonic functions when \( A(x,f,\nabla f) = A(x,f) \), the mean curvature flow when \( A(x,f,\nabla f) = -\frac{1}{|\nabla f|} \), and

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the inverse mean curvature flow when \( A(x, f, \nabla f) = |\nabla f| \) (in the level set formulation, where the evolving hypersurfaces \( X_t \) are given as level sets of \( f \), via \( X_t = \partial \{ x : f(x) < t \} \)). Whereas 1-harmonic functions have been applied to solve Plateau’s problem in Euclidean space ([BDG]) and the Bernstein conjecture in hyperbolic geometry ([WW]), the inverse mean curvature flow (i.e., a solution of the parabolic evolution equation \( \frac{\partial X_t}{\partial t} = \nu H \), where \( H \), assumed to be positive, is the mean curvature of \( X_t \), \( \nu \) is the outward unit normal, and \( \frac{\partial X_t}{\partial t} \) denotes the normal velocity field along \( X_t \)) has been applied to solve fundamental problems in general relativity, such as proving the Penrose inequality ([HI]), and to compute the Yamabe invariant of three-dimensional real projective space ([BN]). Thus, in view of numerous relations among \( p \)-harmonic maps, geometric flows, and other areas of mathematics and science (e.g. cf. [HI, M, KN, BN, BDG, WW, W1, W2, D, KS, ES, CGG]), we would like to call equations of the form (1.1) generalized 1-harmonic equations, and initiate their study in this paper.

A \( W^{1,1}_{\text{loc}}(\Omega) \) function \( f : \Omega \to \mathbb{R} \) is said to be a weak solution of (1.1) in the distribution sense, or a generalized 1-harmonic function, if for every \( \psi(x) \in C_0^\infty(\Omega) \), we have

\[
\int_\Omega \frac{\nabla f}{|\nabla f|} \cdot \nabla \psi \, dv = - \int_\Omega A(x, f(x), \nabla f(x)) \psi(x) \, dv.
\]

We note that in the case of the inverse mean curvature flow, our definition of a weak solution \( f \in W^{1,1}_{\text{loc}}(\Omega) \) is a critical point of the functional \( J^K_f(g) = \int_K |\nabla g| + g|\nabla f| \, dv \) with fixed \( |\nabla f| \) (where \( g \in W^{1,1}_{\text{loc}}(\Omega) \), and \( K \subset \subset \Omega \)), in the distribution sense. This is in contrast to that used in G. Huisken and T. Ilmanen [HI]. According to the definition used in [HI], \( f \in C^{0,1}_{\text{loc}}(\Omega) (= W^{1,\infty}_{\text{loc}}(\Omega)) \) is a weak solution of (1.7) if \( f \) is an absolute minimum of the functional \( J^K_f(g) = \int_K |\nabla g| + g|\nabla f| \, dv \) (where \( g \in C^{0,1}_{\text{loc}}(\Omega) \), and \( K \subset \subset \Omega \)). By this definition, a constant function \( f \) would be a weak solution of (1.7), whereas constant functions are not admissible as weak solutions of (1.1) by our definition. However, in both definitions, weak solutions are not in the space of functions of bounded variation for problems of this type, and in general they are closely related by the inclusions \( C^{0,1}_{\text{loc}}(\Omega) \subset W^{1,1}_{\text{loc}}(\Omega) \).

The purpose of this paper is to find an analytic quantity \( w \) (cf. (1.3)) in a generalized 1-harmonic equation (1.1) on a domain \( \Omega \) in a complete Riemannian manifold \( M \) that affects the behavior of weak solutions of (1.1), and establish its link with the geometry of the domain \( \Omega \), in terms of its doubling constant \( D(\Omega) \), Cheeger constant \( I_\infty(\Omega) \), Sobolev constant \( S_\infty(\Omega) \) and the first eigenvalue \( \lambda_1(\Omega) \) of the Laplacian. In particular, we have:

**Theorem 2.1**  Let \( \Omega \subset M \) have the doubling property (2.1). Assume \( f \in W^{1,1}_{\text{loc}}(\Omega) \) is a weak solution of (1.1). Let

\[
w = \text{ess inf}_{x \in \Omega} |A(x, f(x), \nabla f(x))|.
\]
Then for every $B(x_0, r) \subset \Omega$, the essential infimum $w$ of $|A|$ over $\Omega$ satisfies

$$0 \leq w \leq \frac{C_1 D(\Omega)}{r},$$

where constant $C_1$ is as in [W1] Lemma 1, and $D(\Omega)$ is a doubling constant of $\Omega$ as in (2.1).

**Theorem 2.2** Let $\Omega \subset M$, the Cheeger constant $I_\infty(\Omega)$ be as defined in (2.3), and $w$ be as in (1.3). Then

$$0 \leq w \leq I_\infty(\Omega),$$

and for all $x_0 \in \Omega$ and $B(x_0, r) \subset \Omega$, $w$ satisfies

$$w \text{Vol}(B(x_0, r)) \leq \frac{d}{dr} \text{Vol}(B(x_0, r))$$

for almost all $r > 0$. In particular, if $I_\infty(\Omega) = 0$, then $w = 0$.

Examples include that on a complete manifold with nonnegative Ricci curvature, the essential infimum $w$ of $A$ over $\Omega$ satisfies (1.4), on a complete manifold with negative sectional curvature $\text{Sec}^M \leq -a^2 < 0$, $w$ satisfies (1.5) and (1.6), and in Euclidean space $\mathbb{R}^n$, $w = 0$.

As applications of our methods, we obtain some gradient bounds and nonexistence results for the inverse mean curvature flow:

**Corollary 3.1** Let $\Omega$ have a doubling constant and $f : \Omega \to \mathbb{R}$ be a $W^{1,1}_{\text{loc}}$ weak solution of the level set formulation of the inverse mean curvature flow

$$(1.7) \quad \text{div}\left(\frac{\nabla f}{|\nabla f|}\right) = |\nabla f| \quad \text{on} \quad \Omega,$$

Then

$$\text{ess inf}_{x \in B(x_0, r)} |\nabla f|(x) \leq \frac{C_1 D(\Omega)}{r},$$

for any $x_0 \in \Omega$, and $B(x_0, r) \subset \Omega$.

**Corollary 3.2** Let $\Omega \subset M$, the Sobolev constant $S_\infty(\Omega)$ be as defined in (2.4), and $f \in W^{1,1}_{\text{loc}}(\Omega)$ be a weak solution of (1.7); then

$$\text{ess inf}_{x \in \Omega} |\nabla f|(x) \leq I_\infty(\Omega)$$

and

$$\text{ess inf}_{x \in \Omega} |\nabla f|(x) \leq S_\infty(\Omega).$$

and

$$\text{ess inf}_{x \in \Omega} |\nabla f|(x) \leq 2\sqrt{\lambda_1(\Omega)}.$$

**Proposition 3.2** Let $M$ be a complete noncompact manifold with the global doubling property. Then there does not exist a $W^{1,1}_{\text{loc}}$ weak solution of the level set formulation of the inverse mean curvature flow (1.7) on $M$ that satisfies

$$(1.11) \quad \lim_{r \to \infty} \text{ess inf}_{x \in B(x_0, r)} |\nabla f|(x) > 0,$$

for some $x_0 \in M$. 
By the same method, we also obtain Liouville theorems for \( p \)-subharmonic functions of constant \( p \)-tension field:

**Theorem 4.1** Let \( n \leq p \), and \( f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) be bounded below. If \( f \) is a weak subsolution of the \( p \)-Laplace equation with constant \( p \)-tension field, i.e.,

\[
\text{div}(|\nabla f|^{p-2} \nabla f) = c,
\]

and with bounded weak derivative (i.e., \( |\nabla f| \leq C_2 \) for some constant \( C_2 > 0 \)), then \( f \) is constant.

One can drop the assumption on the bounded gradient if the function is in \( W^{1,p}(\mathbb{R}^n) \):

**Theorem 4.2** Let \( n \leq p \). If \( f \in W^{1,p}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) is bounded below, and is a weak subsolution of the \( p \)-Laplace equation with constant \( p \)-tension field, then \( f \) is constant.

These augment Theorem 1.1 in [BD] (cf. Theorem 4.3) in which a weak supersolution of the \( p \)-Laplace equation was considered. It should also be mentioned that a positive \( p \)-subharmonic (resp. \( p \)-superharmonic) function \( f \) on a complete noncompact Riemannian manifold with one of the following: \( p \)-finite, \( p \)-mild, \( p \)-obtuse, \( p \)-moderate, and \( p \)-small growth, in which \( 1 < p < \infty \), and the growth exponent \( q > p-1 \) (resp. \( q < p-1 \)), is constant (WLW). As a further application, we obtain in particular, for \( p = 1 \), that if a complete noncompact Riemannian manifold \( M \) has the global doubling property, and \( f \in W^{1,1}_{\text{loc}}(M) \) is a weak subsolution of the 1-harmonic equation on \( M \) of a constant 1-tension field, then \( f \) is constant (cf. Proposition 4.1). This generalizes the case \( M = \mathbb{R}^n \) in [W1].

Being motivated by the work in [W2] (cf. Theorem 5.1), we study generalized 1-harmonic functions on cocompact domains in Section 5, and obtain the nonexistence of solutions of the initial value problem for inverse mean curvature flow in Section 6 (cf. Theorems 5.2 and 6.2).

Our methods developed in [W1, W2] and in this paper can be employed and carried over to other settings, such as generalized constant mean curvature type equations for differential forms in Euclidean space and on manifolds [DW], and nonhomogeneous \( A \)-harmonic equations [L].

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### 2. Generalized 1-harmonic equations

We recall that a domain \( \Omega \subset M \) is said to have the doubling property if \( \exists D(\Omega) > 0 \) s.t. \( \forall r > 0, \forall x \in M \), with \( B(x, 2r) \subset \Omega \), the volumes of the geodesic balls centered at \( x \) of radii \( 2r \) and \( r \) satisfy

\[
\text{Vol}(B(x, 2r)) \leq D(\Omega) \ \text{Vol}(B(x, r)).
\]
Let $\Omega \subset M$ have the doubling property (2.1). Assume that $f \in W_{loc}^{1,1}(\Omega)$ is a weak solution of (1.1). Let the essential infimum $w$ of $|A|$ over $\Omega$ be defined as in (1.3). Then for every $B(x_0, r) \subset \Omega$, $w$ satisfies (1.4).

Proof. We consider two cases:

Case 1. $A(x, f(x), \nabla f(x))$ is continuous on $\Omega$ and assumes both positive and negative values: By the intermediate value theorem, $A(x, f(x), \nabla f(x))$ assumes value 0 at some point, and thus $w = \inf_{x \in \Omega} |A(x, f(x), \nabla f(x))| = 0$.

Case 2. $A(x, f(x), \nabla f(x))$ is nonpositive-valued or nonnegative-valued: Let $\psi \geq 0$ be as in [W1] Lemma 1, in which $t = r, s = \frac{r}{2}$. Substituting $\psi$ into (1.2), applying (1.3) and the Cauchy-Schwarz inequality, we have

$$0 \leq w \text{Vol}(B(x_0, \frac{r}{2}))$$

$$= \int_{B(x_0, \frac{r}{2})} w\psi(x)dx$$

$$\leq \int_{B(x_0, \frac{r}{2})} A(x, f(x), \nabla f(x))\psi(x)dv$$

$$\leq \int_{B(x_0, r)} A(x, f(x), \nabla f(x))\psi(x)dv$$

$$= \int_{B(x_0, r)} \frac{\nabla f}{|\nabla f|} \cdot \nabla \psi dv$$

$$\leq \int_{B(x_0, r)} |\nabla \psi| dv$$

$$\leq \frac{C_1}{r} \text{Vol}(B(x_0, r))$$

$$\leq \frac{C_1 D(\Omega)}{r} \text{Vol}(B(x_0, \frac{r}{2})).$$

By letting $r \to \infty$ in the above expression, we obtain immediately the following

Corollary 2.1. Let $M$ be a complete noncompact manifold with a doubling constant. Then there does not exist a $W_{loc}^{1,1}$ weak subsolution $f : M \to \mathbb{R}$ of equation (1.1) on $M$ such that for some $x_0 \in M$,

$$\lim_{r \to \infty} \inf_{x \in B(x_0, r)} |A(x, f(x), \nabla f(x))| > 0.$$
Define the Cheeger constant $I_\infty(\Omega)$ by
\begin{equation}
I_\infty(\Omega) = \sup \{ C : \text{Area}(\partial \Omega') \geq C \text{Vol}(\Omega') \text{ for any domain } \Omega' \subset \subset \Omega \}
\end{equation}

**Theorem 2.2.** Let $\Omega \subset M$, the Cheeger constant $I_\infty(\Omega)$ be as defined in (2.3), and $w$ be as in (1.3). Then $w$ satisfies (1.5), and for all $x_0 \in \Omega$ and $B(x_0, r) \subset \subset \Omega$, $w$ satisfies (1.6) for almost all $r > 0$. In particular, if $I_\infty(\Omega) = 0$, then $w = 0$.

**Proof.** By (1.1), (1.3), Stokes' theorem, and the Cauchy-Schwarz inequality, we have, for any domain $\Omega' \subset \subset \Omega$
\begin{align*}
w \text{Vol}(\Omega') &\leq \left| \int_{\Omega'} A(x, f(x), \nabla f(x)) \, dx \right|
= \left| \int_{\partial \Omega'} \nabla f(x) \cdot v \, dS \right|
\leq \int_{\partial \Omega'} 1 \, dS = \text{Vol}(\partial \Omega'),
\end{align*}
This yields (1.5) immediately, and (1.6) by the coarea formula and letting $\Omega' = B(x_0, r)$.

Define a Sobolev constant $S_\infty(\Omega)$ by
\begin{equation}
S_\infty(\Omega) = \sup \{ C : C \int_\Omega |f| \, dv \leq \int_\Omega |\nabla f| \, dv \text{ for any } f \in C_0^\infty(\Omega) \}
\end{equation}
Then one always has:

**Corollary 2.2.**
$w \leq S_\infty(\Omega)$.

In particular, if $S_\infty(\Omega) = 0$, then $w = 0$.

**Proof.** This follows from a theorem of Federer and Fleming (FF) that $S_\infty(\Omega) = I_\infty(\Omega)$, and Theorem 2.2.

**Corollary 2.3.** Under the assumption of Theorem 2.2 or Corollary 2.2, we have
$$w \leq 2 \sqrt{\lambda_1(\Omega)}$$
where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian on $\Omega$.

**Proof.** This follows from the inequality $\lambda_1(\Omega) \geq \frac{1}{4} I_\infty(\Omega)^2$ (cf. [C]), and Theorem 2.2.

**Proposition 2.1.** Let $M$ be a complete noncompact Riemannian manifold, and $w$ be as in (1.3) in which $\Omega = M$. Then, for every $x_0 \in M$, there exists a constant $K$ such that $w$ satisfies
\begin{equation}
\liminf_{r \to \infty} e^{-wr} \text{Vol}(B(x_0, r)) \geq K > 0,
\end{equation}
In particular, if $M$ has pth-power volume growth, for some $p \geq 0$ (i.e., $\text{Vol}(B(x_0, r)) = O(r^p)$ as $r \to \infty$ for some $x_0 \in M$), then $w = 0$. 

Let $r_0$ and $r_1$ be any two numbers with $r_1 > r_0 > 0$. Integrating (1.6) with respect to $r$ from $r_0$ to $r_1$, we have
\[
wr_1 - wr_0 = \int_{r_0}^{r_1} wr \, dr \leq \int_{r_0}^{r_1} \frac{4}{\text{Vol}(B(x_0, r))} \text{Vol}(B(x_0, r)) \, dr
\]
\[
= \log \text{Vol}(B(x_0, r_1)) - \log \text{Vol}(B(x_0, r_0)),
\]
for any $r_1 > r_0$. This implies, by the monotonicity of the exp function, $e^{wr_1} \cdot e^{-wr_0} \leq \text{Vol}(B(x_0, r_1)) \cdot \text{Vol}(B(x_0, r_0))$, i.e., $e^{-wr_1} \text{Vol}(B(x_0, r_1)) \geq K > 0$, for any $r_1 > r_0$, where $K = e^{-er_0} \text{Vol}(B(x_0, r_0))$. Now taking $r_1$ to $\infty$ gives the desired (2.5). Suppose the contrary for the last assertion; then (2.5) would lead to $0 = \liminf_{r \to \infty} e^{-wr}r^p > 0$, a contradiction. \hfill \Box

3. THE INVERSE MEAN CURVATURE FLOW

For completeness and the readers’ reference, we collect the consequences of our observation on the inverse mean curvature flow in this section.

**Corollary 3.1.** Let $\Omega$ have a doubling constant, and $f : \Omega \to \mathbb{R}$ be a $W_{1,1}^{1,1}$ weak solution of the level set formulation of the inverse mean curvature flow (1.7). Then $\text{ess inf}_{x \in B(x_0, r)} |\nabla f|(x)$ satisfies (1.8), for any $x_0 \in \Omega$, and $B(x_0, r) \subset \Omega$.

**Proof.** This follows at once from Theorem 2.1 in which $A(x, f(x), \nabla f(x)) = |\nabla f(x)|$. \hfill \Box

**Proposition 3.1.** Let $M$ be a complete noncompact manifold with a doubling constant. Let $f : M \to \mathbb{R}$ be a $W_{1,1}^{1,1}$ weak subsolution of the level set formulation of the inverse mean curvature flow (1.7) on $M$. Then
\[
\lim_{r \to \infty} \text{ess inf}_{x \in B(x_0, r)} |\nabla f|(x) = 0
\]
for any $x_0 \in M$.

**Proof.** In view of Corollary 3.1 we have $\text{ess inf}_{x \in B(x_0, r)} |\nabla f|(x) \leq \frac{C_1 D(M)}{r}$, for any $x_0 \in M$. Letting $r \to \infty$ gives the desired. \hfill \Box

**Example 3.1.** ([III]) The expanding sphere $\partial B_t(t)$ of radius $r(t)$ in $\mathbb{R}^n$, where
\[
r(t) = e^{\frac{t}{n-1}}
\]
satisfies the inverse mean curvature flow ($\frac{\partial X}{\partial t} = \nabla f$), and its level curve formulation satisfies $\lim_{r \to \infty} \text{ess inf}_{x \in B(x_0, r)} |\nabla f|(x) = 0$ for any $x_0 \in \mathbb{R}^n$.

**Proposition 3.2.** Let $M$ be a complete noncompact manifold with the global doubling property. Then there does not exist a $W_{1,1}^{1,1}$ weak solution of the level set formulation of the inverse mean curvature flow (1.7) on $M$ such that (1.11) holds for some $x \in M$. 
Proof Suppose the contrary. Letting $r \to \infty$ in Corollary 3.1 would lead to a contradiction.

Corollary 3.2. Let $\Omega \subset M$, and $f : \Omega \to \mathbb{R}$ be a $W^{1,1}_{loc}$ weak solution of (1.7). Then (1.9) and (1.10) hold.

Proof. This is an immediate consequence of Theorem 2.2 in which $A(x, f(x), \nabla f(x)) = |\nabla f(x)|$, and $S_\infty(\Omega) = I_\infty(\Omega)$.

Theorem 3.1. Let $\Omega$ be a domain in a Cartan-Hadamard manifold $M$ with sectional curvature $\text{Sec}_M \leq -a^2$, where $a > 0$. Then there does not exist a $W^{1,1}_{loc}$ weak subsolution $f : \Omega \to \mathbb{R}$ of (1.7) with $w = \text{ess inf}_{x \in \Omega} |\nabla f(x)| > I_\infty(\Omega) > 0$.

Proposition 3.3. Let $M$ be a complete noncompact manifold with $p$th-power volume growth, $p \geq 0$. Let $f : M \to \mathbb{R}$ be a $W^{1,1}_{loc}$ weak subsolution of the level set formulation of the inverse mean curvature flow (1.7) on $M$. Then (3.1) holds.

In particular, if $M$ is a complete manifold of finite volume, or of nonnegative Ricci curvature, then (3.1) holds.

Proof. This follows at once from the last assertion of Proposition 2.1 and the Bishop-Gromov volume comparison theorem.

4. Liouville Theorem for $p$-subharmonic functions of constant $p$-tension field

Theorem 4.1. Let $n \leq p$, and $f \in W^{1,p}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be bounded below. If $f$ is a weak subsolution of the $p$-Laplace equation with constant $p$-tension field, i.e., $\text{div}(|\nabla f|^{p-2} \nabla f) = c$, and with bounded weak derivative (i.e., $|\nabla f| \leq C_2$ for some constant $C_2 > 0$), then $f$ is constant.

Proof. Let $\psi \geq 0$ be as in [W1] Lemma 1 in which $M = \mathbb{R}^n, t = r, s = \frac{r}{2}$. Choose $\psi$ to be a test function in the distribution sense of the constant $p$-tension field equation. Then via the Cauchy-Schwarz inequality we have

$$\int_{B(x_0, \frac{r}{2})} |c| \psi(x) dx \leq \int_{B(x_0, r)} |c| \psi(x) dx = \left| \int_{B(x_0, r)} |\nabla f|^{p-2} \nabla f \cdot \nabla \psi dx \right| \leq C_2^{p-1} \int_{B(x_0, r)} |\nabla \psi| dx.$$

Hence,

$$|c| \text{Vol} \left( B \left( x_0, \frac{r}{2} \right) \right) \leq \frac{C_1 C_2^{p-1}}{r} \text{Vol}(B(x_0, r))$$

implies that $c = 0$ on letting $r \to \infty$. Now the assertion follows from Theorem 1.1 in [BD].
Theorem 4.2. Let $n \leq p$. If $f \in W^{1,p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is bounded below, and is a weak subsolution of the $p$-Laplace equation with constant $p$-tension field, then $f$ is constant.

Proof. Proceeding as in the proof of Theorem 4.1, we have via the H"older inequality

$$\int_{B(x_0, \frac{r}{2})} |c| \psi(x) dx \leq \left| \int_{B(x_0, r)} |\nabla f|^{p-2} \nabla f : \nabla \psi dx \right|$$
$$\leq \left( \int_{\mathbb{R}^n} |\nabla f|^p dx \right)^{\frac{p-1}{p}} \cdot \left( \int_{B(x_0, r)} |\nabla \psi|^p dx \right)^{\frac{1}{p}}.$$

Hence,

$$|c| \text{Vol} \left( B \left( x_0, \frac{r}{2} \right) \right) \leq \frac{C_1 C_4}{r} \text{Vol}(B(x_0, r))$$

where $C_3 \geq \left( \int_{\mathbb{R}^n} |\nabla f|^p dx \right)^{\frac{p-1}{p}}$ is a constant independent of $r$ (here we use the assumption $f \in W^{1,p}(\mathbb{R}^n)$). Hence $c = 0$ on letting $r \to \infty$, and the assertion follows. □

Theorems 4.1 and 4.2 also augment Theorem 1.1 in [BD]:

Theorem 4.3. Let $n \leq p$. If $u \in W^{1,p}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is a weak supersolution of

$$\text{div}(|\nabla f|^{p-2} \nabla f) = 0 \quad \text{in} \quad \mathbb{R}^n$$

and is bounded below then $u$ is constant.

As a further application of Theorem 2.1 for $p$-subharmonic functions of constant $p$-tension field for $p = 1$, we have

Proposition 4.1. Let $M$ be a complete noncompact Riemannian manifold with a doubling constant $D(M)$, and $f : M \to \mathbb{R}$ be a $W^{1,1}_{loc}$ weak subsolution of the 1-harmonic equation

$$\text{div} \left( \frac{\nabla g}{|\nabla g|} \right) = 0 \quad \text{on} \quad M,$$

with constant 1-tension field $c$, i.e.,

$$\text{div} \left( \frac{\nabla f}{|\nabla f|} \right) = c$$

in the distribution sense. Then $f$ is a 1-harmonic function.

Proof. This follows at once from Theorem 2.1 in which $A(x, f, \nabla f) \equiv c$. □

This generalizes the case $M = \mathbb{R}^n$ of [W1].
5. Generalized 1-harmonic Functions on cocompact domains

In [W2], we study \( p \)-harmonic geometry and related topics, and obtain, in particular:

**Theorem 5.1.** ([W2, Theorem 9.6]) Let a complete manifold \( M \) have the global doubling property \(^{2}\), and \( K \) be a compact subset of \( M \). If \( g : M \setminus K \rightarrow \mathbb{R} \) is a continuous function with \( \inf_{x \in M \setminus K} |g(x)| > 0 \), then there does not exist a \( C^2 \) solution \( f \) of the equation \( \text{div} \left( \frac{\nabla f}{|\nabla f|} \right) = g(x) \) on \( M \setminus K \).

These ideas and techniques in studying nonhomogeneous 1-harmonic functions in [W2, (9.1)] on the complement of a compact set on manifolds can provide a unified method in the following more general setting:

**Theorem 5.2.** Let \( M \) and \( K \) satisfy the assumptions in Theorem 5.1. Let \( f \in W^{1,1}_{\text{loc}}(M) \) be a weak solution of
\[
\text{div} \left( \frac{\nabla f}{|\nabla f|} \right) = A(x, f, \nabla f) \text{ on } M \setminus K
\]
where \( A \) is a continuous real-valued function on \( M \), or \( A \) is either nonnegative or nonpositive valued, with
\[
w_{K^c} := \text{ess} \inf_{x \in M \setminus K} |A(x, f, \nabla f)|
\]
Then \( w_{K^c} = 0 \).

**Lemma 5.1.** [W3] Every complete manifold \( M \) with the global doubling property \(^{2}\) has infinite volume.

For completeness, we provide the following:

**Proof of Lemma 5.1** Suppose on the contrary \( \text{Vol}(M) = V_0 \) for some \( 0 < V_0 < \infty \). Since \( M \) is complete, \( M = \bigcup_{i=0}^{\infty} B(x_0; 2^i r) \), there would exist \( n \) such that \( \text{Vol}(B(x_0; 2^n r)) > \frac{D(M)}{D(M) + 1} V_0 \). Hence, \( \text{Vol}(M \setminus B(x_0; 2^n r)) < \frac{1}{2} \frac{D(M)}{D(M) + 1} V_0 \). Choose \( x_1 \in M \) such that the distance between \( x_0 \) and \( x_1 \) is \( 3 \cdot 2^n r \), then \( B(x_1; 2 \cdot 2^n r) \subset M \setminus B(x_0; 2^n r) \) and \( B(x_0; 2^n r) \subset B(x_1; 4 \cdot 2^n r) \). By the global doubling property, this would lead the following contradiction:
\[
\frac{D(M)}{D(M) + 1} V_0 < \text{Vol}(B(x_1; 4 \cdot 2^n r)) \leq D(M) \text{Vol}(B(x_1; 2 \cdot 2^n r)) < \frac{D(M)}{D(M) + 1} V_0.
\]

As an immediate consequence of Lemma 5.1 one obtains the following result due to Calabi and Yau by different methods:

**Corollary 5.1 ([Ca, Y]).** Every complete manifold of nonnegative Ricci curvature has infinite volume.
Proof of Theorem 5.2. We consider two cases:

Case 1. \( A(x, f(x), \nabla f(x)) \) is a continuous function on \( M \setminus K \) : Then either it assumes both positive and negative values, in which case the assertion follows from the intermediate value theorem, or we go to Case 2.

Case 2. \( A(x, f(x), \nabla f(x)) \) is nonpositive or nonnegative valued: We proceed as in the proof of [W2, Theorem 9.6]. Since \( K \subset M \) is compact, choose a sufficiently large \( r_0 < r \) such that \( K \subset B(x_0, r_0) \). Let \( 0 \leq \psi \leq 1 \) be the cut-off function as in Lemma 1 in which \( t = r, s = 2r \) (i.e. \( \psi \equiv 1 \) on the closure \( B(x_0, r) \), \( \psi \equiv 0 \) off \( B(x_0, 2r) \), and \( |\nabla \psi| \leq \frac{C}{r} \)). Multiplying both sides of (5.1) by \( \psi \), integrating over \( B(x_0, 2r) \setminus B(x_0, r) \), and applying Stokes’ theorem, we have

\[
(5.3) \quad \int_{\partial B(x_0, r_0)} \frac{\nabla f}{|\nabla f|} \cdot \nabla \psi \, dS + \int_{B(x_0, 2r) \setminus B(x_0, r_0)} \frac{\nabla f}{|\nabla f|} \cdot \nabla \psi \, dx \leq \text{Vol}(B(x_0, r_0)) + \frac{C_1 \text{Vol}(B(x_0, 2r))}{r},
\]

where \( \nu \) is the unit normal to \( \partial B(x_0, r) \), \( dS \) is the area element of \( \partial B(x_0, r) \), and \( C_1 > 0 \) is the constant as above. Hence, dividing (5.3) by \( \text{Vol}(B(x_0, r)) \), and using (5.3) and (2.2), one has

\[
w_{k^*} \left( 1 - \frac{\text{Vol}(B(x_0, r_0))}{\text{Vol}(B(x_0, r))} \right) \leq \frac{\text{Vol}(\partial B(x_0, r_0))}{\text{Vol}(B(x_0, r))} + \frac{C_1 D(M)}{r} \to 0
\]
as \( r \to \infty \), since by Lemma 5.1 \( M \) has infinite volume.

Remark 5.1. From the above proof, it is clear that if we weaken the assumption that \( K \) is compact in Theorem 5.2 to \( K \) being precompact or bounded, the conclusion in Theorem 5.2 remains true. Similarly, the following Proposition 5.1 and Corollary 5.2 also hold on the complement of any precompact or bounded set \( K \).

As an immediate consequence of Theorem 5.2, we have:

**Proposition 5.1.** Let \( K, M \), and \( w \) be as in Theorem 5.2. Then there does not exist a generalized 1-harmonic function on the complement of a compact set \( K \) with \( w_{k^*} > 0 \).

**Proof.** This follows immediately from Theorem 5.2.

**Corollary 5.2.** On the complement of any compact subset \( K \) in a complete manifold \( M \) of nonnegative Ricci curvature, there does not exist a weak solution of the level set formulation of the inverse mean curvature flow (2.1), with constant 1-tension field (4.2).
Proof. Suppose, on the contrary, that there were such a weak solution. Then by Theorem 5.2, (1.7) and (4.2), we would have $w_{K_c} = 0 = c = |\nabla f|$, i.e., $f = \text{constant}$. This contradicts our remark in the introduction that a constant is not a weak solution of (1.1).

6. Nonexistence of solutions of the Initial value problem for Inverse Mean Curvature Flow

In the section, we begin by combining the definition of a weak solution (in the distribution sense; cf. (1.2)) with an initial condition consisting of a bounded open set $E_0$ with a boundary at least $C^1$. We say that a function $f$ is a weak solution of the level set formulation of the inverse mean curvature flow

\begin{equation}
\text{div} \left( \frac{\nabla f}{|\nabla f|} \right) = |\nabla f|
\end{equation}

with initial condition $E_0$ if

$f \in W^{1,1}_{\text{loc}}(M), E_0 = \{x : f(x) < 0\},$ and $f$ is a weak solution of (6.1) in $M \setminus E_0$.

All the local estimates (1.4), (1.6), and (1.8) (cf. Theorems 2.1 and 2.2 and Corollary 3.1) hold for $\Omega = M \setminus E_0$. On the other hand, we have the following global result for the initial value problem for inverse mean curvature flow, which is in contrast to Proposition 3.1 in which $\Omega = M$.

**Theorem 6.1.** Let $M$ be a complete manifold with the global doubling property. Let $f$ be a weak solution of the level set formulation of the inverse mean curvature flow (6.1) with initial condition $E_0$. Then

$$\text{ess inf}_{x \in M \setminus E_0} |\nabla f|(x) = 0.$$  

or equivalently,

**Theorem 6.2.** [Nonexistence of Weak Solutions] Let $M$ be a complete manifold with the global doubling property. If

$$\text{ess inf}_{x \in M \setminus E_0} |\nabla f|(x) > 0,$$

then there does not exist a weak solution of the level set formulation of the inverse mean curvature flow (6.1) with initial condition $E_0$.

**Proof of Theorems 6.1 and 6.2** This follows immediately from Theorem 5.2 and Remark 5.1.

**Corollary 6.1.** There does not exist a weak solution of (6.1) with initial condition $E_0$, and with constant 1-tension field on a complete manifold $M$ of nonnegative Ricci curvature.
Proof of Corollary 6.1. This follows at once from Proposition 6.1, Remark 6.1, and the fact that a manifold of nonnegative Ricci curvature has the global doubling property.

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