MONOTONE SUBSEQUENCES IN LOCALLY UNIFORM RANDOM PERMUTATIONS

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Abstract. A locally uniform random permutation is generated by sampling $n$ points independently from some absolutely continuous distribution $\rho$ on the plane and interpreting them as a permutation by the rule that $i$ maps to $j$ if the $i$th point from the left is the $j$th point from below.

As $n$ tends to infinity, decreasing subsequences in the permutation will appear as curves in the plane, and by interpreting these as level curves, a union of decreasing subsequences give rise to a surface. We show that, under the correct scaling, for any $r \geq 0$, the largest union of $\lfloor r \sqrt{n} \rfloor$ decreasing subsequences approaches a limit surface as $n$ tends to infinity, and the limit surface is a solution to a specific variational problem. As a corollary, we prove the existence of a limit shape for the Young diagram associated to the random permutation under the Robinson–Schensted correspondence. In the special case where $\rho$ is the uniform distribution on the diamond $|x| + |y| < 1$ we conjecture that the limit shape is triangular, and assuming the conjecture is true we find an explicit formula for the limit surfaces of a uniformly random permutation and recover the famous limit shape of Vershik, Kerov and Logan, Shepp.

1. Introduction

It has been known since the 1970s that the longest decreasing (or increasing) subsequence of a random permutation of $\{1, 2, \ldots, n\}$ has length approximately $2\sqrt{n}$ for large $n$. More generally, the (scaled) limit of the cardinality of the largest union of $\lfloor r \sqrt{n} \rfloor$ disjoint decreasing subsequences is known for any $r \geq 0$, where $\lfloor \cdot \rfloor$ denotes the integral part. But what does this union typically look like in the permutation diagram? And what if the permutation is not sampled from the uniform distribution? The aim of this paper is to answer these questions, at least for a family of distributions called locally uniform.

Let $\sigma$ be a finite set of points in the plane, no two of which have the same $x$- or $y$-coordinate. We can interpret any such $\sigma$ as a permutation by letting $\sigma(i) = j$ if the $i$th point from the left is the $j$th point from below. If $\sigma$ consists of $n$ points that are sampled independently from some given absolutely continuous distribution $\rho$ on the plane, $\sigma$ is said to be locally uniform (with density $\rho$). In particular, if $\rho$ is the uniform distribution on the unit square $(0, 1)^2$, then, as a permutation, $\sigma$ is uniformly distributed among all permutations of order $n$.
Figure 1. The location of the largest union of 20 decreasing subsequences in a random permutation of order 100,000 drawn from the uniform distribution on the square $(0,1)^2$. The small dots are the points in the permutation, and adjacent points in the decreasing subsequences are connected by line segments. We have also sketched a local parallelogram.

In this geometric setting, decreasing subsequences of $\sigma$ appear as “decreasing subsets” of the permutation points in the plane, and we may talk about the location of a union of decreasing subsequences. Figure 1 shows an example. One could imagine a two-dimensional surface whose level curves follow the decreasing subsets, and as $n$ tends to infinity, under some rescaling one might hope to obtain a limit surface for a maximal union of $k$ decreasing subsets, where $k$ depends on $n$. (It is not hard to see that we must require that $k$ grows as $\sqrt{n}$.) This appears to be a new question already for uniform random permutations, and we will motivate below why we think it is a both natural and powerful one.

2. Background and significance

We will give a brief introduction to the history and current situation of the research area of monotone subsequences in random permutations. For a comprehensive review, we refer to Romik [24].

Let $\sigma$ be a permutation of order $n$. A subsequence of $\sigma$ is an ordered sequence $(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k))$ where $i_1 < i_2 < \cdots < i_k$. It is increasing if $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k)$ and decreasing if $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_k)$. Let $L(\sigma)$ denote the length of (any of) the longest increasing subsequence(s) of $\sigma$, that is

$$L(\sigma) := \max\{k : \text{there is an increasing subsequence of length } k\}.$$
In 1961 Ulam [28] asked the following question, sometimes known as Ulam’s problem: If \( \sigma_n \) is chosen randomly from the uniform distribution of all permutations of order \( n \), what is the expected value of \( L(\sigma_n) \)?

About ten years later, Hammersley [15] was able to prove that there exists a limit not only for the expected value \( \mathbb{E}L(\sigma_n) \) but also for \( L(\sigma_n) \) itself.

**Theorem 2.1** (Hammersley, 1972). The limit \( \Gamma = \lim_{n \to \infty} \mathbb{E}L(\sigma_n)/\sqrt{n} \) exists, and \( L(\sigma_n)/\sqrt{n} \to \Gamma \) in probability.

Though simulations suggested that \( \Gamma = 2 \), this was not proven until 1977 by Vershik and Kerov [29]. They used the fact that \( L(\sigma) \) equals the length of the first row in the Young diagram corresponding to \( \sigma \) under the Robinson–Schensted bijection, and in fact, they were able to describe a *limit shape* for the Young diagram corresponding to a random permutation as \( n \) grows to infinity. The latter result was also obtained independently by Logan and Shepp [19].

The next break-through happened in 1999, when Baik, Deift and Johansson [1] were able to describe the asymptotic behaviour of \( L(\sigma_n) \) in detail.

**Theorem 2.2** (Baik–Deift–Johansson 1999). The random variable
\[
\frac{L(\sigma_n) - 2\sqrt{n}}{n^{1/6}}
\]
converges in distribution to the Tracy–Widom distribution as \( n \to \infty \).

### 2.1. Where is the longest increasing subsequence?

Hammersley’s approach was to think about a random permutation as a set of dots randomly and independently positioned in the unit square, and that setting will be convenient also for us, so let us redefine our terminology a bit, in accordance with Section 1.

Let \( \sigma \) be a finite set of points in the plane, no two of which have the same \( x \)- or \( y \)-coordinate. We can interpret any such \( \sigma \) as a permutation by letting \( \sigma(i) = j \) if the \( i \)th point from the left is the \( j \)th point from below. A subset \( I \) of \( \sigma \) is *increasing* if, for any pair of points \( (x, y) \) and \( (x', y') \) belonging to \( I \), \( x < x' \) if and only if \( y < y' \). It is *decreasing* if \( x < x' \) if and only if \( y > y' \). This corresponds exactly to the increasing and decreasing subsequences that we defined earlier.

In this framework a natural question arises:

**Question 2.3.** For a random permutation \( \sigma \) generated by sampling \( n \) points uniformly in the unit square, where in the plane does the longest increasing subsequence typically reside?

It follows quite easily from Hammersley’s work that, with high probability, all maximal increasing subsets will be contained in a small region around the diagonal of the unit square as \( n \) tends to infinity. A much stronger result on the limit distribution of maximal increasing subsets was obtained in a recent paper by Dauvergne and Virág [4].

But the new formulation also calls for a generalization: What if the points in \( \sigma \) are sampled from some *non-uniform* distribution? Deuschel and Zeitouni [5] considered this generalization and were able to describe a
limit curve for the maximal increasing subset when $\sigma$ is a locally uniform random permutation.

We will concern ourselves with the following generalized version of Question 2.3 for locally uniform random permutations.

**Question 2.4.** Where in the plane does a maximal union of $r\sqrt{n}$ increasing subsets typically reside?

2.2. **Novelty.** Except for the above-mentioned result on the limit curve of the longest increasing subsequence by Deuschel and Zeitouni, the idea to look at not only the cardinality but also the location of monotone subsequences seems to be completely original. As we will see in Section 5 below, this approach is potentially powerful since it enables us to use analytical tools to study maximal unions of decreasing subsequences in random permutations sampled from a non-uniform distribution.

There are some results on the cardinality of increasing subsequences for non-uniform random permutations, but to the best of our knowledge those are all concerned with $q$-analogues of the uniform distribution, where a permutation $\pi$ is sampled with probability $q^{f(\pi)+f(\pi^{-1})}$ for some classical permutation statistics $f(\cdot)$ like the number of inversions (the Mallows distribution) [2, 22] or the majorant index [13]. In contrast, the family of locally uniform distributions is much larger; it is uncountably infinite-dimensional.

2.3. **Relation to limit shapes of Young diagrams.** By a theorem of Greene [14] (see Proposition 3.4 below), the cardinality of a maximal union of $k$ decreasing subsequences is encoded in the Young diagram corresponding to the permutation under the Robinson–Schensted bijection, so the asymptotic behavior of such cardinalities corresponds to a limit shape of a random Young diagram as $n$ tends to infinity. If the random permutation is drawn from the uniform distribution, the corresponding Young diagram is drawn from the Plancherel distribution, and its limit shape is the well-known result by Vershik, Kerov and Logan, Shepp that we mentioned above. For non-uniform random permutations, however, the limiting behavior of the corresponding Young diagram is an open problem.

Limit shapes of Young diagrams have gained much interest over the years, and there are results for specific probability distributions of Young diagrams, often generated by a stochastic process [17, 26]. More recent examples include [6, 7, 8, 9].

2.4. **Relation to permutation limits.** Locally uniform random permutations, our main objects of study, appear naturally as limit objects in the sense of Hoppen et al. [16]. Their main result is a definition of convergence for permutation sequences and an equivalence between such sequences and (essentially) locally uniform random permutations, and their paper is the first step towards a theory for permutations analogous to the emerging theory of limits of graphs created by Lovász and many coauthors; see [20] for an overview.
3. Terminology and results

3.1. Probabilistic setting. Our probability space will always be a simple point process in the plane viewed as a random set of points. We will define complex statements about such random point sets without worrying about the measurability of the truth-value of the statement. Typically, such statements will be parameterized by a real number $\gamma$, and we say that the statement holds asymptotically almost surely (a.a.s.) as $\gamma \to \infty$ if it is implied by an event which happens with probability tending to one. To be precise, we make the following definition.

**Definition 3.1.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{E_\gamma \subseteq \Omega\}_{\gamma > 0}$ be a collection of outcome sets indexed by a parameter $\gamma$. (Note that we do not require the sets to be elements of the $\sigma$-algebra $\mathcal{F}$.) We say that $E_\gamma$ happens asymptotically almost surely (a.a.s.) as $\gamma \to \infty$ if there is a family $\{F_\gamma \in \mathcal{F}\}_{\gamma > 0}$ such that $F_\gamma \subseteq E_\gamma$ for any $\gamma$, and $P(F_\gamma) \to 1$.

Also, if $\{X_\gamma\}_{\gamma > 0}$ is a family of functions from $\Omega$ to $\mathbb{R}$, we say that $X_\gamma \to x$ in probability if, for any $\varepsilon > 0$, the event $\{\omega \in \Omega : |X_\gamma(\omega) - x| < \varepsilon\}$ happens a.a.s. as $\gamma \to \infty$.

3.2. Increasing and decreasing sets. A set $P$ of points in $\mathbb{R}^2$ is increasing if, for any pair of points $(x,y)$ and $(x',y')$ belonging to $P$, $x < x'$ if and only if $y < y'$. It is decreasing if $x < x'$ if and only if $y > y'$. It is $k$-increasing (resp. $k$-decreasing) if it is a union of $k$ increasing (resp. decreasing) sets.

3.3. The local parallelogram. Consider a situation like that in Fig. 1, with a permutation embedded in the unit square and a chosen maximal union of $k$ decreasing subsets. Our main idea is to exploit the local uniformity of the random permutation by zooming in on a small region. Let us choose the region to have the shape of a narrow parallelogram as depicted in Fig. 1 where the long edges are parallel to the decreasing subsets passing by (the curves in the figure) and the short edges have the same slope but with a positive sign. Then, we might ask what proportion of the permutation points inside the parallelogram are “picked up” by the passing curves. Intuitively, this value is only dependent on the local density and slope of the curves together with the local density of permutation points. In fact, since the property of a set being decreasing is invariant under rescaling of the $x$- and $y$-axes, the question can be reduced to a problem about a narrow rectangle of 45-degree slope. Given a “density” of lines with slope minus one, what proportion of the permutation points inside the narrow rectangle are “picked up” by the lines? The following theorem follows directly from Proposition 4.4, proven in Section 4.

**Theorem 3.2.** Let $\Omega$ be the open rectangle (depicted in Fig. 2)

$$0 < (x + y)/\sqrt{2} < 1, \quad 0 < (y - x)/\sqrt{2} < \beta$$

for some $\beta > 0$, and let $r \geq 0$. For each $\gamma > 0$, let $\sigma_\gamma$ be a Poisson point process in the plane with homogeneous intensity $\gamma$. Define the random variable $\Lambda(\gamma)$ as the size of a maximal $\lfloor r \sqrt{\gamma} \rfloor$-decreasing subset of $\sigma_\gamma \cap \Omega$. Then, as $\gamma$ and $\beta$ tends to infinity, $\Lambda(\gamma)/\beta \gamma$ converges in $L^1$ to a constant $\Phi(r)$. 
Figure 2. The rectangle $\Omega$ in Theorem 3.2.

Figure 3. A Young diagram $\lambda$ with column lengths given by $(\lambda_1, \lambda_2, \ldots) = (4, 2, 2, 1, 0, \ldots)$.

The function $\Phi$ defined by the preceding theorem will play a main role throughout the paper.

3.4. **Limit shape under Robinson–Schensted.** While we have introduced $\Phi$ as a tool for showing our main results below, it turns out that $\Phi$ is surprisingly interesting in its own right: The derivative of $\Phi$ is a limit shape of the Young diagram associated with $\sigma$ under the Robinson–Schensted correspondence, where $\sigma$ is generated by a homogeneous Poisson point process on a diamond square.

A *Young diagram* $\lambda$ (in French notation) is a finite collection of unit cells, arranged in bottom-justified columns whose lengths are in non-increasing order from the left. The length of the $i$th column from the left is denoted by $\lambda_i$, and we let $\lambda_i = 0$ if $i$ is larger than the number of columns.\(^1\) Figure 3 shows an example.

The *Robinson–Schensted correspondence* is a bijection between permutations and pairs of *standard Young tableaux* of the same shape. We will not define this bijection or even the concept of standard Young tableaux since all we need is contained in the proposition of Greene below. For a comprehensive review we refer to [27].

**Definition 3.3.** Suppose $\sigma$ is a finite set of points in general position in the sense that no two points share the same $x$- or $y$-coordinate. Then we define the *permutation corresponding to* $\sigma$ to be the permutation of $\{1, 2, \ldots, \#\sigma\}$ defined by letting $\pi(i) = j$ if the $i$th point from the left in $\sigma$ is the $j$th point

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\(^1\)Note that this deviates from the standard notation. In the literature, usually $\lambda_i$ denotes the length of the $i$th longest row, but since we are interested in decreasing rather than increasing subsets, the columns will be most important to us.
The contours of Young diagrams corresponding to $n$ points sampled from the uniform distribution on the diamond shape $|x| + |y| < 1$.

from below. The Young diagram corresponding to $\sigma$ is the shape of the standard Young tableaux corresponding to $\pi$ under Robinson–Schensted.

In our setting, Greene’s [14] beautiful connection between the Young diagram and the decreasing (or increasing) subsequences of the permutation can be formulated as follows.

**Proposition 3.4** (Greene). Suppose $\sigma$ is a finite set of points in general position in the sense that no two points share the same $x$ or $y$ coordinate. For each $k$, let $\Lambda_k$ be the size of a maximal $k$-decreasing subset of $\sigma$. Then

$$\Lambda_k = \sum_{i=1}^{k} \lambda_i.$$

In Proposition 4.3 we will show that $\Phi$ is concave and thus differentiable almost everywhere. Our next theorem, proven in Section 10, states that $\Phi'$ is a limit shape of the Young diagram corresponding to a homogeneous Poisson point process on the diamond region $|x| + |y| < 1/\sqrt{2}$.

**Theorem 10.4.** Let $\Omega$ be the open diamond square $|x| + |y| < 1/\sqrt{2}$ and, for each $\gamma > 0$, let $\sigma_{\gamma}$ be a Poisson point processes on $\Omega$ with intensity $\gamma$. Then the Young diagram $\lambda^{(\gamma)}$ corresponding to $\sigma_{\gamma}$ approaches the limit shape $\Phi'$ in the sense that, for any $r > 0$ where $\Phi'(r)$ exists,

$$\frac{1}{\sqrt{\gamma}} \lambda^{(\gamma)}_{[r\sqrt{\gamma}]+1} \rightarrow \Phi'(r)$$

in probability as $\gamma \rightarrow \infty$.

In fact, as is evident in Fig. 4, computer simulations strongly suggest that the limit shape is an isosceles triangle! We make the following conjecture.

**Conjecture 3.5.**

$$\Phi'(r) = \begin{cases} \sqrt{2} - r & \text{if } 0 \leq r \leq \sqrt{2}, \\ 0 & \text{if } r > \sqrt{2}. \end{cases}$$
3.5. **Doubly increasing functions.** As mentioned in the introduction, we want to find some kind of limit object to a bundle of decreasing subsets like those depicted by the curves in Fig. 1, and a natural idea is to view the curves as level curves of a two-dimensional surface. Such surfaces can be described by functions of \( x \) and \( y \) that are increasing in both variables.

Define a partial order \( \leq \) on \( \mathbb{R}^2 \) by letting \( (x_1, y_1) \leq (x_2, y_2) \) if \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). For any subset \( A \) of \( \mathbb{R}^2 \), a function \( u : A \to \mathbb{R} \) is doubly increasing if \( u(x_1, y_1) \leq u(x_2, y_2) \) whenever \( (x_1, y_1) \leq (x_2, y_2) \). For \( r \geq 0 \), we let \( \mathcal{U}_r(A) \) denote the set of doubly increasing functions \( u \) on \( A \) with \( \text{diam} \ u(A) \leq r \), and we let \( \mathcal{U}(A) := \bigcup_{r \geq 0} \mathcal{U}_r(A) \) denote the set of all bounded doubly increasing functions on \( A \). Let \( \mathcal{U}_{h+r}(A) \) denote the subset of \( \mathcal{U}_r(A) \) consisting of functions with values in \([h, h + r]\).

Exactly how should a \( k \)-decreasing set be interpreted as level curves and mapped to a doubly increasing function? A simple idea is to convert each decreasing subset to a curve by joining adjacent points with line segments, but this is problematic: First, curves from different decreasing subsets might intersect, and second, there might be multiple ways of partitioning the \( k \)-decreasing set into \( k \) decreasing subsets. We can avoid both of these problems by focusing instead on the increasing subsets of the \( k \)-decreasing set, thanks to the following well-known combinatorial fact (for which we provide a proof for completeness).

**Proposition 3.6.** Let \( P \) be a finite set of points in general position in the sense that no two points share the same \( x \)- or \( y \)-coordinate. Then, \( P \) is \( k \)-decreasing if and only if it has no increasing subset of cardinality larger than \( k \).

**Proof.** Suppose \( P \) is a union of \( k \) decreasing sets. No two elements of an increasing set can belong to the same decreasing set, so, by the pigeonhole principle, there is no increasing subset of \( P \) of cardinality larger than \( k \).

For the converse, suppose \( P \) has no increasing subset with more than \( k \) elements. Let \( p_1, \ldots, p_n \) be the points in \( P \) sorted from west to east. Construct a sequence of sets \( D_1, D_2, \ldots \) by the following procedure. Initially, let \( D_1, D_2, \ldots \) be empty sets. Then, iteratively for \( i = 1, \ldots, n \), add \( p_i \) to the first of the sets \( D_1, D_2 \ldots \) that currently contains only points to the north-west of \( p_i \). After this procedure, if \( p \) is an element in \( D_{k+1} \), then \( D_k \) must contain an element south-west of \( p \); otherwise, \( p \) would have been added to \( D_k \) instead. Iterating this argument yields an increasing set of cardinality \( k + 1 \) which contradicts our assumption. Thus, \( D_{k+1} \) is empty and \( P \) is a union of the \( k \) decreasing sets \( D_1, \ldots, D_k \). \( \square \)

In accordance, our interpretation of point sets as doubly increasing functions looks as follows.

**Definition 3.7.** For any finite set \( P \) of points in the plane, define a map \( \kappa_P : \mathbb{R}^2 \to \mathbb{N} \) by letting \( \kappa_P(x, y) \) be the maximal size of an increasing subset of \( P \cap ((-\infty, x] \times (-\infty, y]) \).

See Fig. 5 for an example.

3.6. **A functional.** Now, we will take a global perspective: Instead of letting the curves be defined by a maximal union of decreasing subsets, let us
think about them just as a bunch of decreasing curves that we can bend and move freely. The goal is to position these curves so that together they pick up as many permutation points as possible. With our parameterization of the bunch of curves by a two-dimensional surface, the discrete optimization problem can be approximated and formulated as a continuous variational problem where we want to choose the two-dimensional surface that maximizes a certain functional.

Let $\mu$ denote the Lebesgue measure on $\mathbb{R}^2$. By a density domain we will mean a pair $(\Omega, \rho)$ where $\Omega$ is an open subset of $\mathbb{R}^2$ of positive finite measure and $\rho$ is a nonnegative function on $\Omega$ such that $\int_\Omega \rho \, d\mu$ is finite. We will often write $\|f\|_A$ as a shorthand for $\int_A |f| \, d\mu$.

**Definition 3.8.** For any $\eta, \theta \geq 0$, let

$$L(\eta, \theta) := \begin{cases} \eta \Phi(\sqrt{2\theta/\eta}) & \text{if } \eta > 0, \\ 0 & \text{if } \eta = 0, \end{cases}$$

and, for any density domain $(\Omega, \rho)$, let $F_\rho : \mathcal{U}(\Omega) \to \mathbb{R}$ be a (nonlinear) functional given by

$$F_\rho(u) := \int_\Omega L(\rho, u_x u_y) \, d\mu = \|L(\rho, u_x u_y)\|_\Omega,$$

where $u_x$ and $u_y$ denote partial derivatives.

We will show in Section 4 that the integrand is integrable so that $F_\rho$ is well defined.

Intuitively, the factor $\Phi(\sqrt{2u_x u_y/\rho})$ of the integrand in the definition of $F_\rho$ measures the “local efficiency” of the surface $u$, that is, the proportion of dots in the neighborhood that the curves (encoded by the surface $u$) will pick up, where we once again refer to Fig. 1. Note that it is invariant under rescaling of the $x$- and $y$-axes if the density $\rho$ is rescaled accordingly.

When the density domain $(\Omega, \rho)$ is implicitly understood, we let $F_{\max}$ be the map from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ defined by letting

$$F_{\max}(r) := \sup_{u \in \mathcal{U}_r(\Omega)} F_\rho(u).$$
3.7. Main result. In the introduction we defined a locally uniform random permutation to consist of a fixed number \( n \) of i.i.d. points with density \( \rho \). However, in many situations it is more natural to consider an (inhomogeneous) Poisson point process with density \( n\rho \) where the total number of points is a (Poisson-distributed) random variable with mean \( n \). As \( n \) tends to infinity, the difference between these random point processes becomes negligible; they approach each other in the following sense.

**Definition 3.9.** Let \( \{ \sigma_\gamma \}_{\gamma > 0} \) and \( \{ \tau_\gamma \}_{\gamma > 0} \) be two families of random point processes parameterized by \( \gamma > 0 \). Then we say that \( \tau_\gamma \) approaches \( \sigma_\gamma \) as \( \gamma \to \infty \) if \( \#(\sigma_\gamma \triangle \tau_\gamma)/\gamma \to 0 \) in probability as \( \gamma \to \infty \), where \( \triangle \) denotes symmetric difference.

Our first main theorem connects the doubly increasing functions and \( F_\rho \) with random point processes. We postpone its proof until Section 10.

**Theorem 10.2.** Let \( (\Omega, \rho) \) be a density domain and let \( \{ \tau_\gamma \}_{\gamma > 0} \) be random point processes on \( \Omega \) approaching a Poisson point process with intensity function \( \gamma \rho \) as \( \gamma \to \infty \). Then the following holds for any \( r \geq 0 \).

(a) For any \( \epsilon > 0 \), a.a.s. as \( \gamma \to \infty \), for every maximal \( \lfloor r\sqrt{\gamma} \rfloor \)-decreasing subset \( P \) of \( \tau_\gamma \) there is a \( u \in \mathcal{U}_0, r(\Omega) \) with \( F_\rho(u) = F_{\max}(r) \) such that \( \|\kappa_P/\sqrt{\gamma} - u\|_\Omega < \epsilon \).

(b) Let \( \Lambda(\gamma) \) denote the size of a maximal \( \lfloor r\sqrt{\gamma} \rfloor \)-decreasing subset of \( \tau_\gamma \). Then, \( \Lambda(\gamma)/\gamma \to F_{\max}(r) \) in probability as \( \gamma \) tends to infinity.

As a corollary, we obtain a limit shape for the Young diagram associated with a locally uniform random permutation.

**Corollary 10.3.** With the same setup as in Theorem 10.2, the Young diagram \( \lambda(\gamma) \) corresponding to \( \tau_\gamma \) approaches the limit shape \( F'_{\max} \) in the sense that, for any \( r > 0 \) where the derivative \( F'_{\max}(r) \) exists,

\[
\frac{1}{\sqrt{\gamma}} \lambda(\gamma)_{\lfloor r\sqrt{\gamma} \rfloor + 1} \to F'_{\max}(r)
\]

in probability as \( \gamma \to \infty \).

The proof will appear in Section 10.

Our second main theorem is nonprobabilistic in nature and deals with the functional \( F_\rho \) exclusively. We postpone its proof until Section 11.

**Theorem 11.3.** Let \( (\Omega, \rho) \) be a density domain. Then the following holds.

(a) For any \( r \geq 0 \), \( F_\rho \) attains its maximum on \( \mathcal{U}_r(\Omega) \).

(b) \( F_\rho \) is a concave function on \( \mathcal{U}(\Omega) \).

(c) \( F_{\max} \) is continuous, increasing and concave, and \( F_{\max}(r) \to \|\rho\|_\Omega \) as \( r \to \infty \).

In Section 5, we show how a maximizer of \( F_\rho \) can be found in practice by solving a system of partial differential equations.

Note that Theorem 11.3(c) means that there is no loss of mass in the limit shape of Corollary 10.3, that is, there is no macroscopic proportion of the boxes in the longest \( o(\sqrt{\gamma}) \) rows or columns of the Young diagram.
3.8. Consequences of Conjecture 3.5. Our final result is concerned with the consequences of Conjecture 3.5. In Section 13, provided Conjecture 3.5 holds true, we obtain a simple parameterization of the limit surface for a uniformly random permutation, and we recover the celebrated limit-shape result of Logan, Shepp and Vershik, Kerov mentioned in Section 2.

3.9. Organization of the paper. The remainder of the paper is organized as follows.

Section 4: We show that \( \Phi \) exists and that the functional \( F_\rho \) is well defined.

Section 5: We reduce the maximization problem to a PDE system.

Section 6: We study the probabilistic behavior of the local parallelogram.

Section 7: We divide a density domain into small parallelograms.

Section 8: We show that \( F_\rho \) is semicontinuous.

Section 9: We show that the set \( \mathcal{U}_{h,r}(\Omega) \) of doubly increasing functions is compact as a subset of \( L^1(\Omega) \).

Section 10: We prove our first main theorem, Theorem 10.2.

Section 11: We prove our second main theorem, Theorem 11.3.

Section 12: We show that, provided \( \Phi \) is reasonably well behaved (which it is if Conjecture 3.5 holds true), the maximizer of \( F_\rho \) is essentially unique.

Section 13: Provided Conjecture 3.5 holds true, we find the limit surfaces for a uniformly random permutation and the limit shape for the corresponding Young diagram.

Section 14: We discuss some open questions for future research.

4. Existence of \( \Phi \)

In this section we will prove Theorem 3.2 and thereby establish the existence of the function \( \Phi \). We will also show some basic properties of \( \Phi \) and \( L \) and see that the functional \( F_\rho \) defined in Section 3 is well defined.

Let \( \mathbb{N} := \{0, 1, 2, \ldots \} \) denote the set of nonnegative integers. We will use bold letters like \( \mathbf{n} \) for points in the integer lattice \( \mathbb{N}^2 \), and coordinates will be denoted by the corresponding italic letters with indices, like \( \mathbf{n} = (n_1, n_2) \). Define the operators plus, minus and the relations \( \leq \) and \( < \) coordinate-wise on \( \mathbb{N}^2 \). Also, let \( \mathbf{m} \ast \mathbf{n} = (m_1n_1, m_2n_2) \) denote coordinate-wise multiplication. Finally, we let \( \mathbf{n} \to \infty \) mean that \( \min\{n_1, n_2\} \to \infty \).

Hammersley’s proof of the existence of a limit for the size of the longest increasing subsequence (Theorem 2.1) uses Kingman’s subergodic theorem [18]. We will need a fancier version of that theorem in order to show the existence of a limiting behavior of the decreasing subsets in our two-dimensional setting. The following theorem is a special case of a result by Schürger [25, Theorem 2.1]2.

**Theorem 4.1** (Schürger). Let \( \{X_{\mathbf{m}, \mathbf{n}}\} \) be a family of real random variables, where the indices span over all \( \mathbf{m}, \mathbf{n} \in \mathbb{N}^2 \) with \( \mathbf{m} < \mathbf{n} \). Suppose the following holds.

- Translation invariance. For any \( \mathbf{k} \in \mathbb{N}^2 \), the family \( \{X_{\mathbf{m+k}, \mathbf{n+k}}\}_{\mathbf{m}<\mathbf{n}} \) has the same finite joint probability distributions as \( \{X_{\mathbf{m}, \mathbf{n}}\}_{\mathbf{m}<\mathbf{n}} \).

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2We put \( h = 0 \) in Schürger’s theorem and change the sign of the random variables.
• Superadditivity. For any $0 < m < n$, we have
  \[ X_{0,n} \geq X_{0,(m_1,n_2)} + X_{(m_1,0),n} \]
  \[ X_{0,n} \geq X_{0,(m_1,n_2)} + X_{(0,m_2),n} \]

• Integrability. The set $\{ \mathbb{E} X_{0,n}/n_1 n_2 : n > 0 \}$ is bounded.

Then, $\lim_{n \to \infty} X_{0,n}/n_1 n_2$ exists in $L^1$ and equals
  \[
  \lim_{n \to \infty} (n_1 n_2)^{-1} \lim_{m \to \infty} (m_1 m_2)^{-1} \sum_{0 < k \leq m} X_{k-n,n,k,n},
  \]
both limits existing in $L^1$.

If $I$ and $J$ are sets of points, we say that $I$ is a $k$-decreasing set compatible with $J$ if $I \cup J$ is $k$-decreasing.

**Proposition 4.2.** Fix $s > 0$ and let $\sigma$ be a Poisson point process on $\mathbb{R}^2$ with homogeneous intensity $s$.

For any pair $(m,n)$ with $m,n \in \mathbb{N}^2$ and $m < n$ we define the interval
  \[
  [m,n) := \{ x \in \mathbb{R}^2 : m \leq x < n \}.
  \]

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps $(1,0)$ to $(1,1)$ and $(0,1)$ to $(-1,1)$, and define the random variable $X_{m,n}$ as the maximal size of an $(n_1-m_1)$-decreasing subset of $\sigma \cap T[m,n)$ compatible with $T([m_1,n_1) \times \{m_2,n_2\}$, where $[m_1,n_1)$ denotes the set $\{m_1,m_1+1,\ldots,n_1-1\}$.

Then, $X_{0,n}/n_1 n_2$ converges in $L^1$ to a constant $c_s$ as $n \to \infty$.

**Proof.** We claim that the family $\{ X_{m,n} \}_{m<n}$ has the three properties defined in Theorem 4.1. Translation invariance follows immediately from the translation invariance of $\sigma$. Integrability holds since $\mathbb{E} X_{0,n}/n_1 n_2 \leq \mathbb{E} \#(\sigma \cap T[0,n))/n_1 n_2 = 2s$.

To prove superadditivity we consider any $0 < m < n$ and verify the two superadditivity inequalities.

Let $A$ be an $m_1$-decreasing subset of $\sigma \cap T[0,(m_1,n_2)]$ compatible with $T([0,m_1) \times \{n_2\})$ and let $B$ be an $(n_1 - m_1)$-decreasing subset of $\sigma \cap T([m_1,0),n)$ compatible with $T([m_1,n_1) \times \{n_2\})$, as depicted in Fig. 6a. Since the disjoint sets $A \cup T([0,m_1) \times \{n_2\})$ and $B \cup T([m_1,n_1) \times \{n_2\})$ are $m_1$- and $(n_1 - m_1)$-decreasing, respectively, their union is $n_1$-decreasing. This means that $A \cup B$ is an $n_1$-decreasing set compatible with $T([0,n_1) \times \{n_2\})$, and it follows that $X_{0,n} \geq X_{0,(m_1,n_2)} + X_{(m_1,0),n}$.

Now, instead let $A$ be an $n_1$-decreasing subset of $\sigma \cap T[0,(n_1,m_2)]$ compatible with $T([0,n_1) \times \{m_2\})$ and let $B$ be an $n_1$-decreasing subset of $\sigma \cap T([0,m_2),n)$ compatible with $T([0,n_1) \times \{m_2,n_2\})$, as depicted in Fig. 6b. The $n_1$-decreasing set $A \cup T([0,n_1) \times \{m_2\})$ is a union of $n_1$ decreasing sets $A_1,A_2,\ldots,A_{n_1}$. Since no two elements of $T([0,n_1) \times \{m_2\})$ can belong to the same decreasing set, we may assume that $T(i-1,m_2) \in A_i$ for $i = 1,2,\ldots,n_1$. Analogously, $B \cup T([0,n_1) \times \{m_2,n_2\})$ is a union of $n_1$ decreasing sets $B = B_1 \cup B_2 \cup \cdots \cup B_{n_1}$ such that $T(i-1,m_2) \in B_i$ for $i = 1,2,\ldots,n_1$. Clearly, $A_i \cup B_i$ is decreasing, and

\[ A \cup B \cup T([0,n_1) \times \{m_2,n_2\}) = \bigcup_{i=1}^{n_1} A_i \cup B_i, \]
so $A \cup B$ is an $n_1$-decreasing subset of $\sigma \cap T[0, n]$ compatible with $T([0, n_1] \times \{0, m_2, n_2\})$ and hence with $T([0, n_1] \times \{0, n_2\})$. It follows that $X_{0,n} \geq X_{0,(n_1,m_2)} + X_{(0,m_2),n}$.

We have showed that the family $\{X_{m,n}\}_{m<n}$ is translation invariant, superadditive and integrable. By Theorem 4.1, it follows that the limit $X_{\infty} := \lim_{n \to \infty} X_{0,n}/n_1n_2$ exists in $L^1$ and equals

\[
\lim_{n \to \infty} (n_1n_2)^{-1} \lim_{m \to \infty} (m_1m_2)^{-1} \sum_{0<k\leq m} X_{k,n-n,k+n}.
\]

For any fixed $m,n \in \mathbb{N}^2$,

\[
S_{m,n} := (m_1m_2)^{-1} \sum_{0<k\leq m} X_{k,n-n,k+n}
\]

is an average of $m_1m_2$ i.i.d random variables of finite expectation. By the law of large numbers, as $m \to \infty$, $S_{m,n}$ converges almost surely to $\mathbb{E} X_{0,n}$. It follows that the limit in Eq. (1) is concentrated to a constant $c_s$. □

The following proposition defines the function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

**Proposition 4.3.** Let $\Omega$ be the open rectangle

\[
0 < (x+y)/\sqrt{2} < \alpha, \quad 0 < (y-x)/\sqrt{2} < \beta
\]

and let $r \geq 0$. For each $\gamma > 0$, let $\sigma_\gamma$ be a Poisson point process in the plane with homogeneous intensity $\gamma$. Define the union of lines

\[
L_\gamma := \bigcup_{i=0}^{[ar\sqrt{2}]^{-1}} \{(x,y) \in \mathbb{R}^2 : (x+y)/\sqrt{2} = i/r\sqrt{2}\}
\]

and let $P_\gamma := \{(x,y) \in L_\gamma : y-x = 0\}$ and $Q_\gamma := \{(x,y) \in L_\gamma : (y-x)/\sqrt{2} = \beta\}$. See Fig. 7 for an illustration. Let $M_\gamma$ be the size of a maximal
Figure 7. The rectangle $\Omega$ and the sets $P_\gamma$ and $Q_\gamma$ in Proposition 4.3 and 4.4. The shaded area shows the rectangle $\Omega'$ in the proof of Proposition 4.4.

$|\alpha r\sqrt{\gamma}|$-decreasing subset of $\sigma_\gamma \cap \Omega$ compatible with $P_\gamma \cup Q_\gamma$. Then there is a constant $\Phi(r)$, independent of $\alpha$, $\beta$ and $\gamma$, such that

$M_\gamma/\alpha\beta\gamma \rightarrow \Phi(r)$

in $L^1$ as $\alpha\sqrt{\gamma}$ and $\beta\sqrt{\gamma}$ tend to infinity simultaneously in any manner.

Proof. Clearly, $\Phi(0)$ exists and is 0, so in the following we may assume that $r > 0$. First suppose $n_1 := \alpha r\sqrt{\gamma}$ and $n_2 := \beta r\sqrt{\gamma}$ are both integers. Let $s := r^{-2}/2$ and define $X_{0,(n_1,n_2)}$ as in Proposition 4.2. Then $X_{0,(n_1,n_2)}$ has the same distribution as $M_\gamma$, so $M_\gamma/\alpha\beta\gamma$ converges in $L^1$ to $\Phi(r) := r^2c_s$ as $n_1$ and $n_2$ tends to infinity under the constraint that they are integers.

Let $\alpha' := |\alpha r\sqrt{\gamma}|/r\sqrt{\gamma}$ and $\beta' := |\beta r\sqrt{\gamma}|/r\sqrt{\gamma}$, and let $\Omega'$, $L'_\gamma$, $P'_\gamma$, $Q'_\gamma$ and $M'_\gamma$ be defined as $\Omega$, $L_\gamma$, $P_\gamma$, $Q_\gamma$ and $M_\gamma$ but with $\alpha$ and $\beta$ replaced by $\alpha'$ and $\beta'$. Then, $L'_\gamma = L_\gamma$ and $P'_\gamma = P_\gamma$. Any $|\alpha' r\sqrt{\gamma}|$-decreasing subset of $\sigma_\gamma \cap \Omega'$ compatible with $P'_\gamma \cup Q'_\gamma$ is also a $|\alpha r\sqrt{\gamma}|$-decreasing subset of $\sigma_\gamma \cap \Omega$ compatible with $P_\gamma \cup Q_\gamma$, so $M_\gamma \geq M'_\gamma$ and $M_\gamma - M'_\gamma \leq \#(\sigma_\gamma \cap (\Omega \setminus \Omega'))$. Since $\alpha - \alpha'$ and $\beta - \beta'$ are both nonnegative and smaller than $1/r\sqrt{\gamma}$, we have

$\mu(\Omega \setminus \Omega') = \alpha\beta - \alpha'\beta' = (\alpha - \alpha')\beta + (\beta - \beta')\alpha - (\alpha - \alpha')(\beta - \beta') \leq (\alpha - \alpha')\beta + (\beta - \beta')\alpha \leq (\alpha + \beta)/r\sqrt{\gamma},$

where, as always, $\mu$ denotes the Lebesgue measure on $\mathbb{R}^2$.

It follows that $\#(\sigma_\gamma \cap (\Omega \setminus \Omega'))$ has a Poisson distribution with mean smaller than $(\alpha + \beta)/r\sqrt{\gamma}$, and thus $|M_\gamma - M'_\gamma|/\alpha\beta\gamma$ converges to zero almost surely as $\alpha\sqrt{\gamma}$ and $\beta\sqrt{\gamma}$ tend to infinity. We conclude that $M_\gamma/\alpha\beta\gamma$ converges to $\Phi(r)$ in $L^1$. □

From the next proposition, Theorem 3.2 follows immediately.

Proposition 4.4. With the same setup as in Proposition 4.3, define the random variable $\Lambda^{(\gamma)}$ as the size of a maximal $|\alpha r\sqrt{\gamma}|$-decreasing subset of
\(\sigma_{\gamma} \cap \Omega\). Then, as \(\alpha \sqrt{\gamma}\) and \(\beta / \alpha\) tend to infinity, we have \(\Lambda^{(\gamma)} / \alpha \beta \gamma \rightarrow \Phi(r)\) in \(L^1\). Also, for any fixed \(\alpha\) and \(\beta\), the inequality
\[
|\left(\Lambda^{(\gamma)} / \alpha \beta \gamma\right) - \Phi(r)| < 3 \alpha / \beta
\]
holds a.a.s. as \(\gamma \rightarrow \infty\).

Proof. Clearly, \(\Lambda^{(\gamma)} \geq M_{\gamma}\). Let \(\Omega'\) be the (possibly empty) rectangle given by \(0 < (x + y) / \sqrt{2} < \alpha\) and \(\alpha < (y - x) / \sqrt{2} < \beta - \alpha\); see Fig. 7 for an illustration. If \(A\) is a \([\alpha \sqrt{\gamma}]\)-decreasing subset of \(\sigma_{\gamma} \cap \Omega\), then \(A \cap \Omega'\) is a \([\alpha \sqrt{\gamma}]\)-decreasing subset of \(\sigma_{\gamma} \cap \Omega\) compatible with \(P_{\gamma} \cup Q_{\gamma}\). Thus, \(\Lambda^{(\gamma)} - M_{\gamma} \leq \#(\sigma_{\gamma} \cap (\Omega \setminus \Omega'))\). We have \(\mu(\Omega \setminus \Omega') \leq 2 \alpha^2\), so \(\#(\sigma_{\gamma} \cap (\Omega \setminus \Omega'))\) has a Poisson distribution with mean at most \(2 \alpha^2 \gamma\), and hence \((\Lambda^{(\gamma)} - M_{\gamma}) / \alpha \beta \gamma\) converges almost surely to zero as \(\alpha \sqrt{\gamma}\) and \(\beta / \alpha\) tend to infinity. It follows from Proposition 4.3 that \(\Lambda^{(\gamma)} / \alpha \beta \gamma \rightarrow \Phi(r)\) in \(L^1\). For fixed \(\alpha\) and \(\beta\), and for any \(\delta > 0\), it holds that \((\Lambda^{(\gamma)} - M_{\gamma}) / \alpha \beta \gamma < 2 (1 + \delta) / \beta\) a.a.s. as \(\gamma \rightarrow \infty\), and hence \(|(\Lambda^{(\gamma)} / \alpha \beta \gamma) - \Phi(r)| < 3 \alpha / \beta\) a.a.s.

In Section 6 we will need the following more flexible version of Proposition 4.4 that incorporates the functional \(F_\rho\) from Definition 3.8.

Proposition 4.5. Let \(\Omega\) be the open parallelogram
\[
\{(x, y) \in \mathbb{R}^2 : |ax + by| < 1, |ax - by| < \beta\}
\]
for some \(a, b, \beta > 0\), and let \(\rho\) be constant on \(\Omega\). Let \(u_{\text{linear}}(x, y) = c(ax + by)\) for some \(c \geq 0\), and let \(\{\sigma_{\gamma}\}_{\gamma > 0}\) be Poisson point processes on \(\Omega\) with intensities \(\gamma \rho\). Let \(\Lambda^{(\gamma)}\) be the size of a maximal \([2c \sqrt{\gamma}]\)-decreasing subset of \(\sigma_{\gamma}\). Then
\[
|(\Lambda^{(\gamma)} / \gamma) - F_\rho(u_{\text{linear}})| \leq 3 \rho \mu(\Omega) / \beta
\]
a.a.s. as \(\gamma \rightarrow \infty\).

Proof. If \(\rho = 0\), we have \(\Lambda^{(\gamma)} = 0\) and \(F_\rho(u_{\text{linear}}) = 0\) so the conclusion of the proposition is true.

In the following we assume that \(\rho > 0\). Rescale the \(x\)- and \(y\)-axes and generate Poisson point processes with intensities \(\gamma' = 2 \gamma \rho / ab\) on the rectangle \(|x + y| < 1 / \sqrt{2}, |x - y| < \beta / \sqrt{2}\). In Proposition 4.4 this corresponds to \(r := c \sqrt{2ab / \rho}, \alpha := 1, \beta := \beta, \gamma := \gamma'\), and the proposition yields that \(|(\Lambda^{(\gamma)} / \beta \gamma') - \Phi(r)| < 3 / \beta\) a.a.s. as \(\gamma \rightarrow \infty\). It is straightforward to check that \(\mu(\Omega) = 2 \beta / ab\) and \(F_\rho(u_{\text{linear}}) = 2 \beta \rho \Phi(r) / ab\).

Our next goal is to show some nice properties of \(\Phi\), in particular that it is increasing and concave. To this end, we need a couple of lemmas which will be used again later on when we concern ourselves with limit shapes of Young diagrams.

Let \(\partial_-\) and \(\partial_+\) denote the left and right one-sided derivative operators.

Lemma 4.6. Let \(F_1, F_2, \ldots\) be random concave functions from \(\mathbb{R}_{\geq 0}\) to \(\mathbb{R}\), and suppose there is a deterministic function \(F : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}\) such that \(F_n(x) \rightarrow F(x)\) in probability for any \(x\). Then the following holds.

(a) \(F\) is concave.

(b) \(\partial_- F_n(x) \rightarrow F'(x)\) and \(\partial_+ F_n(x) \rightarrow F'(x)\) in probability for any point \(x > 0\) where \(F'(x)\) exists.
Lemma 4.7. Let $\Lambda_k := \sum_{i=1}^k \Lambda_i^{(\gamma)}$. Let $a$ and $b$ be positive functions of $\gamma$ such that $\lim_{\gamma \to \infty} a(\gamma) = \infty$. Suppose there is a (deterministic) function $G : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that
\[
b(\gamma)\Lambda_k^{(\gamma)} \to G(r)
\]
in probability for any $r \geq 0$. Then $G$ is increasing and concave, and
\[
a(\gamma)b(\gamma)\Lambda_k^{(\gamma)} \to G'(r)
\]
in probability for any $r > 0$ where $G$ is differentiable.

**Proof.** For each $\gamma > 0$, define the function $F^{(\gamma)} : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ by

$$F^{(\gamma)}(r) := \int_0^{a(\gamma)r} \lambda_{[\gamma r + 1]} dt.$$ 

Since the integrand is a nonnegative decreasing function, $F^{(\gamma)}$ is increasing and concave, and since the integrand is piecewise constant,

$$|F^{(\gamma)}(r) - \Lambda_{[a(\gamma)r]}(\gamma)| = (a(\gamma)r - [a(\gamma)r]) \lambda_{[a(\gamma)r] + 1} \leq \frac{a(\gamma)r - [a(\gamma)r]}{a(\gamma)r} \sum_{i=1}^{a(\gamma)r} \lambda_i^{(\gamma)}$$

which equals $\Lambda_{[a(\gamma)r]}(\gamma)$ times a factor that tends to zero as $\gamma \to \infty$. It follows that

$$b(\gamma)F^{(\gamma)}(r) \to G(r)$$

in probability. Since all $F^{(\gamma)}$ are increasing, $G$ is increasing too. Furthermore, by Lemma 4.6, $G$ is concave and $b(\gamma)\partial_1 F^{(\gamma)}(r) \to G'(r)$ for any $r$ where $G$ is differentiable. Since $\partial_1 F^{(\gamma)}(r) = a(\gamma)\lambda_{[a(\gamma)r] + 1}$ the lemma follows. \hfill $\square$

**Proposition 4.8.** $\Phi$ is increasing, concave, continuous and bounded by one. Furthermore, $\Phi(0) = 0$.

**Proof.** Consider the setup of Proposition 4.4 with the specialization $\alpha = 1$ and $\beta = \gamma$. Then $\Lambda^{(\gamma)}/\gamma^2 \to \Phi(r)$ in $L^1$ as $\gamma \to \infty$. Let $\Lambda^{(\gamma)}$ be the random Young diagram corresponding to $\sigma_\gamma \cap \Omega$. By Proposition 3.4, $\Lambda^{(\gamma)} = \sum_{i=1}^{\lfloor r\sqrt{\gamma} \rfloor} \lambda_i^{(\gamma)}$, and Lemma 4.7 with $a(\gamma) = \sqrt{\gamma}$, $b(\gamma) = 1/\gamma^2$ and $G = \Phi$ yields that $\Phi$ is increasing and concave.

That $\Phi(0) = 0$ and that $\Phi$ is bounded by one follows directly from its definition together with the law of large numbers.

Since $\Phi$ is concave, it is automatically continuous on the open set $(0, \infty)$. It remains only to show that it is continuous at 0.

For any $\beta > 0$, let $\Omega_\beta$ be the open rectangle

$$0 < (x + y)/\sqrt{2} < 1, \quad 0 < (y - x)/\sqrt{2} < \beta,$$

and for any $\gamma > 0$ and $\beta > 0$, let $\sigma_{\gamma,\beta}$ be a Poisson point process on $\Omega_\beta$ with homogeneous intensity $\gamma$. Since $\Omega_\beta \subset (-\beta/\sqrt{2}, 1/\sqrt{2}) \times (0, 1 + \beta/\sqrt{2})$, by Theorem 2.1, for any $\varepsilon > 0$, the size of the largest decreasing subset of $\sigma_{\gamma,\beta}$ is smaller than $\frac{1}{\sqrt{2}} (1 + \varepsilon)(1 + \beta)\sqrt{\gamma}$ a.a.s. as $\gamma \to \infty$. (We know from the result of Vershik and Kerov [29] that $\Gamma = 2$ but we will not need that now.) Thus, for any $r > 0$, the maximum size $\Lambda^{(\gamma)}$ of a $[r\sqrt{\gamma}]$-decreasing subset of $\sigma_{\gamma,\beta}$ is smaller than $\frac{1}{\sqrt{2}} r(1 + \varepsilon)(1 + \beta)\gamma$ a.a.s. as $\gamma \to \infty$. By Proposition 4.4, $|\Lambda^{(\gamma)}/\gamma - \Phi(r)| < 3/\beta$ and thus $|\Lambda^{(\gamma)}/\gamma - (\beta\Phi(r) - 3)$ a.a.s. as $\gamma \to \infty$. So $\frac{1}{\sqrt{2}} r(1 + \varepsilon)(1 + \beta) > \Lambda^{(\gamma)}/\gamma > \beta\Phi(r) - 3$ a.a.s., and hence $\Phi(r) < \left(\frac{1}{\sqrt{2}} r(1 + \varepsilon)(1 + \beta) + 3\right)/\beta$ for any $r, \varepsilon, \beta > 0$. Letting $\beta \to \infty$ and $\varepsilon \to 0$ yields $\Phi(r) \leq \frac{r}{\sqrt{2}}$ and it follows that $\Phi$ is continuous at 0. \hfill $\square$
As a consequence of the nice properties of $\Phi$, the function $L$ from Definition 3.8 is well behaved too.

**Lemma 4.9.** $L$ is continuous and increasing in both variables. Furthermore, for any $\eta, \theta \geq 0$ it holds that $L(\eta, 0) = 0$ and $L(\eta, \theta) \leq \eta$.

**Proof.** By Proposition 4.8, $L$ is continuous at all points $(\eta, \theta)$ with $\eta > 0$. At any point $(0, \theta)$, continuity of $L$ follows from the fact that $\Phi$ is bounded.

That $L$ is increasing in the second variable follows from the fact that $\Phi$ is increasing (Proposition 4.8). That $L(\eta, 0) = 0$ and $L(\eta, \theta) \leq \eta$ for any $\eta, \theta \geq 0$ follows from the facts that $\Phi(0) = 0$ and that $\Phi$ is bounded by one (Proposition 4.8).

To show that $L$ is increasing in the first variable, take any $\theta \geq 0$ and any $\eta' > \eta > 0$. Since $\Phi$ is concave (Proposition 4.8),
\[
\Phi(\sqrt{2\theta/\eta'}) \geq \left(1 - \sqrt{\eta'/\eta}\right) \Phi(\theta) + \sqrt{\eta'/\eta} \Phi(\sqrt{2\theta/\eta}),
\]
and it follows that $L(\eta', \theta) = \eta' \Phi(\sqrt{2\theta/\eta'}) \geq \sqrt{\eta'/\eta} L(\eta, \theta) \geq L(\eta, \theta)$. □

Finally, Lemma 4.9, together with the following lemma, shows that the functional $F_\rho$ from Definition 3.8 is well defined.

**Lemma 4.10.** Let $u$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$ that is increasing in both variables. Then the following holds.

(a) $u$ is measurable.

(b) $u$ is differentiable almost everywhere.

(c) The partial derivatives of $u$ exist almost everywhere and are measurable.

**Proof.** (a) Let $a$ be a real number. We must show that $u^{-1}((-\infty, a])$ is measurable. Define a function $g$ by letting $g(x) := \sup\{y : u(x, y) \leq a\}$ whenever the supremum exists. Then the domain of $g$ is an interval and $g$ is decreasing, so it is measurable and its graph has measure zero. The inverse image $u^{-1}((-\infty, a])$ is the region below this graph (a measurable set) plus some subset of the graph itself (a null set).

(b) This is proved in [3, Sec. 6].

(c) This follows from (a), (b) and [3, Lemma 2]. □

The set of points where $u$ is differentiable is denoted by $\text{Diff}(u)$.

5. Maximizing $F_\rho$ by solving a PDE system

In this section we show that the problem of maximizing the functional $F_\rho$ can be reduced to a system of partial differential equations. This will let us compute, in terms of $\Phi$, the maximum for a parallelogram density domain with constant density. Furthermore, if Conjecture 3.5 holds, the PDE system simplifies significantly and can be solved analytically for the uniform case as we will see in Section 13.

Let us recall some standard facts from convex analysis.

Given a continuous convex function $f$ from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$, its Legendre transform $L[f]$ is a function defined by
\[
L[f](s) = \sup_{r \in I} (rs - f(r))
\]
for those \( s \) for which the supremum exists. It is well known that \( \mathcal{L}(f) \) is a continuous convex function and that its domain is an interval. Furthermore, the Fenchel–Young inequality states that
\[
f(r) + \mathcal{L}[f](s) \geq rs
\]
for any \( r \in I \) and \( s \in \text{dom} \mathcal{L}[f] \), with equality if and only if \( s \in \partial f(r) \), where the subdifferential set \( \partial f(r) \) is defined by
\[
\partial f(r) = \{ s : f(z) - f(r) \geq (z - r)s \text{ for any } z \in I \}.
\]
It is also known that \( \partial f(r) \) is nonempty for any \( r \).

**Definition 5.1.** Let \( \Phi^* \) be the real function defined on \( \mathbb{R}_{\geq 0} \) by
\[
\Phi^*(s) = \inf_{r \geq 0} (rs - \Phi(r)).
\]

Note that the infimum exists since \( \Phi \) is bounded by Proposition 4.8.

**Lemma 5.2.** \( \Phi^* \) is continuous and \(-1 \leq \Phi^*(s) \leq 0 \) for any \( s \geq 0 \). Furthermore,

1. \( \Phi^*(r) + \Phi^*(s) \leq rs \) for any \( r, s \geq 0 \), and
2. for any \( r \) there is an \( s \) such that \( \Phi^*(r) + \Phi^*(s) = rs \).

**Proof.** Once we note that \( \Phi^*(s) = -\mathcal{L}[-\Phi](-s) \), it follows that \( \Phi^* \) is continuous, and part (a) follows from the Fenchel–Young inequality while part (b) follows from the fact that all subdifferential sets are nonempty. To see that \(-1 \leq \Phi^*(s) \leq 0 \) we observe that
\[
-1 \leq -\sup_{r \geq 0} \Phi(r) \leq \inf_{r \geq 0} (rs - \Phi(r)) \leq 0 \cdot s - \Phi(0) = 0,
\]
where the first and last inequalities follow from Proposition 4.8. \( \square \)

Let us expand our terminology for doubly increasing functions to include functions decreasing in \( x \) and increasing in \( y \). Define a partial order \( \leq' \) on \( \mathbb{R}^2 \) by letting \((x_1, y_1) \leq' (x_2, y_2) \) if \( x_1 \geq x_2 \) and \( y_1 \leq y_2 \). For any subset \( A \) of \( \mathbb{R}^2 \), a function \( v : A \to \mathbb{R} \) is decreasing in \( x \) and increasing in \( y \) if \( u(x_1, y_1) \leq u(x_2, y_2) \) whenever \((x_1, y_1) \leq' (x_2, y_2) \). For \( s \geq 0 \), we let \( \mathcal{V}_s(A) \) denote the set of functions \( v \) on \( A \) that are decreasing in \( x \) and increasing in \( y \) and have \( \text{diam} v(A) \leq s \), and we let \( \mathcal{V}(A) := \bigcup_{s \geq 0} \mathcal{V}_s(A) \) denote the set of all bounded functions on \( A \) decreasing in \( x \) and increasing in \( y \). Let \( \mathcal{V}_{h,s}(A) \) denote the subset of \( \mathcal{V}_s(A) \) consisting of functions with values in \([h, h + s]\).

For any density domain \((\Omega, \rho)\), let \( F^*_\rho : \mathcal{V}(\Omega) \to \mathbb{R} \) be a (nonlinear) functional given by
\[
F^*_\rho(v) = \int_\Omega \rho \Phi^*(\sqrt{-2u_x v_y / \rho}) d\mu,
\]
where the integrand is defined to be zero at points where \( \rho = 0 \). This functional is well defined by Lemma 4.10 together with the fact that \( \Phi^* \) is continuous and bounded by Lemma 5.2.

**Lemma 5.3.** Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \), let \( u \in \mathcal{U}(\Omega) \) and \( v \in \mathcal{V}(\Omega) \), and let \( A \) be the subset of \( \Omega \) where \( u_x v_y - u_y v_x > 0 \). Then, the map \( \phi : A \to \mathbb{R}^2 \) defined by \( \phi(x, y) = (u(x, y), v(x, y)) \) is injective.
Proof. Suppose there are two distinct points \( p = (x, y) \) and \( q = (x', y') \) in \( A \) with \( \varphi(p) = \varphi(q) \). Without loss of generality we may assume that \( x \leq x' \).

Since \( \Omega \) is open, for any sufficiently small \( \varepsilon > 0 \) the point \( p_\varepsilon := (1-\varepsilon)p + \varepsilon q \) belongs to \( \Omega \). If \( x = x' \) we have \( u(p_\varepsilon) = u(p) \) and \( v(p_\varepsilon) = v(p) \) and thus \( u_y(p) = v_y(p) = 0 \). If \( y = y' \), we have \( u(p_\varepsilon) = u(p) \) and \( v(p_\varepsilon) = v(p) \) and thus \( u_x(p) = v_x(p) = 0 \). If \( x < x' \) and \( y < y' \) we have \( u(p_\varepsilon) = u(p) \) and thus \( u_x(p) = u_y(p) = 0 \). If \( x < x' \) and \( y > y' \) we have \( v(p_\varepsilon) = v(p) \) and thus \( v_x(p) = v_y(p) = 0 \). In any of the four cases above, we conclude that \( u_x v_y - u_y v_x = 0 \) in \( p \), and it follows that \( p \notin A \), a contradiction. \( \square \)

We will need the following “change of variables” theorem that appears as Theorem 263D in [12].

\( \textbf{Theorem 5.4.} \) Let \( D \subseteq \mathbb{R}^n \) be any measurable set, and \( \varphi : D \to \mathbb{R}^n \) a function differentiable relative to its domain\(^3\) at each point of \( D \). For each \( p \in D \), let \( T(p) \) be a derivative of \( \varphi \) relative to \( D \) at \( p \), and set \( J(p) := |\det T(p)| \). Then \( \mu(\varphi(D)) \leq \int_D J \, d\mu \) with equality if \( \varphi \) is injective.

Now we are ready for the main result of this section. Recall that \( \text{Diff}(u) \) denotes the set of points where \( u \) is differentiable.

\( \textbf{Theorem 5.5.} \) Let \( (\Omega, \rho) \) be a density domain.

Suppose, for some \( r, s > 0 \), there are \( u \in \mathcal{U}_r(\Omega) \) and \( v \in \mathcal{V}_s(\Omega) \) with the following properties.

(a) The set \( \{(u(x, y), v(x, y)) : (x, y) \in \text{Diff}(u) \cap \text{Diff}(v)\} \) has measure 

(b) The PDE system

\[ u_x v_y + u_y v_x = 0, \]

\[ \rho \left( \Phi(\sqrt{2u_x u_y / \rho}) + \Phi^*(\sqrt{-2v_x v_y / \rho}) \right) = 2\sqrt{-u_x u_y v_x v_y} \]

is satisfied almost everywhere in \( \Omega \), where the left-hand side of Eq. (7) is defined to be zero at points where \( \rho = 0 \).

Then, \( u \) is a maximizer of \( F_\rho \) in \( \mathcal{U}_r(\Omega) \) and \( v \) is a maximizer of \( F_\rho^* \) in \( \mathcal{V}_s(\Omega) \). Furthermore, \( s = F_{\rho_{\max}}'(r) \) if \( F_{\rho_{\max}}'(r) \) exists.

\( \textbf{Proof.} \) Let \( u \) and \( v \) be functions in \( \mathcal{U}_r(\Omega) \) and \( \mathcal{V}_s(\Omega) \), respectively. By Lemma 5.2,

\[ F_\rho(u) + F_\rho^*(v) = \int_\Omega \rho \left( \Phi(\sqrt{2u_x u_y / \rho}) + \Phi^*(\sqrt{-2v_x v_y / \rho}) \right) \, d\mu \leq 2 \int_\Omega \sqrt{-u_x u_y v_x v_y} \, d\mu \]

with equality if and only if Eq. (7) holds almost everywhere.

By the inequality of the geometric and arithmetic mean,

\[ 2 \int_\Omega \sqrt{-u_x u_y v_x v_y} \, d\mu \leq \int_\Omega (u_x v_y - u_y v_x) \, d\mu \]

---

\(^3\)We say that \( \varphi \) is differentiable relative to its domain at a point \( p \in D \) if there is a linear map \( T(p) : \mathbb{R}^n \to \mathbb{R}^n \) (called a derivative of \( \varphi \) relative to \( D \) in \( p \)) such that for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |\varphi(p) + T(p)(x - p) - \varphi(x)| \leq \varepsilon |x - p| \) for any \( x \in D \) with \( |x - p| < \delta \).
with equality if and only if Eq. (6) holds almost everywhere.

Let $D := \text{Diff}(u) \cap \text{Diff}(v)$. Let $\varphi$ be the map from $D$ to $\mathbb{R}^2$ defined by $\varphi(x, y) := (u(x, y), v(x, y))$, and let $A$ be the subset of $D$ where $u_x v_y - u_y v_x$ is positive. It follows from Lemma 5.3 that $\varphi$ is injective on $A$. The right-hand side of Eq. (8) equals

$$\int_A (u_x v_y - u_y v_x) \, d\mu,$$
and since $\varphi$ is injective on $A$, by Theorem 5.4 this equals $\mu(\varphi(A))$. By the same theorem, $\int_{\Omega}(u_x v_y - u_y v_x) \, d\mu \geq \mu(\varphi(D))$, so $\mu(\varphi(A)) \geq \mu(\varphi(D))$, which implies that $\mu(\varphi(A)) = \mu(\varphi(D))$. Thus, we obtain

$$\mu(\varphi(A)) = \mu(\varphi(D)) \leq rs$$

with equality if and only if $u$ and $v$ have property (a). (Note that $\mu(\varphi(D)) \leq rs$ holds trivially because $u \in \mathcal{U}_r(\Omega)$ and $v \in \mathcal{V}_s(\Omega)$.)

In conclusion, we have

$$F_\rho(u) + F_\rho(v) \leq rs$$
with equality if and only if $u$ and $v$ have properties (a) and (b). It follows that such $u$ and $v$ are maximizers of $F_\rho$ and $F_\rho^*$ in $\mathcal{U}_r(\Omega)$ and $\mathcal{V}_s(\Omega)$, respectively.

It remains to show that $s = F_{\max}'(r)$ if $F_{\max}'(r)$ exists. From above, it follows that

$$F_{\max}'(r) + F_\rho^*(v) \leq rs$$
for any $r, s > 0$ and any $v \in \mathcal{V}_s(\Omega)$, and that equality holds if there is a $u \in \mathcal{U}_r(\Omega)$ such that (a) and (b) hold. If equality holds for some particular $r, s, v$ and if $F_{\max}'(r)$ exists, then the partial derivative with respect to $r$ (while keeping $s$ and $v$ fixed) of the left- and right-hand sides of Eq. (9) must coincide, so $F_{\max}'(r) = s$. □

If Conjecture 3.5 holds, the PDE system of Theorem 5.5 can be written explicitly. We will exploit this fact in Section 13 where we solve the system for the uniform case.

**Proposition 5.6.** Suppose Conjecture 3.5 holds. Then Theorem 5.5 holds if $\rho > 0$ on $\Omega$ and Eq. (7) is replaced by

$$\min\{\sqrt{u_x u_y}/\rho, 1\} + \min\{\sqrt{-v_x v_y}/\rho, 1\} = 1. \tag{10}$$

**Proof.** Suppose Conjecture 3.5 holds. Then

$$\Phi(r) = \begin{cases} \sqrt{2} r - \frac{r^2}{2} & \text{if } 0 \leq r \leq \sqrt{2}, \\ 1 & \text{if } r > \sqrt{2}, \end{cases}$$

and $\Phi^*(s) = \Phi(s) - 1$ for any $s \geq 0$.

Let $p = \sqrt{u_x u_y}/\rho$ and $q = \sqrt{-v_x v_y}/\rho$. If $p > 1$, Eq. (10) implies that $q = 0$ and hence $\Phi(\sqrt{2} p) + \Phi^*(\sqrt{2} q) = 1 - 1 = 0$ so Eq. (7) is satisfied. Analogously, if $q > 1$, Eq. (10) implies that $p = 0$ and hence $\Phi(\sqrt{2} p) + \Phi^*(\sqrt{2} q) = 0 + 0 = 0$ so Eq. (7) is satisfied. If $p, q \leq 1$, Eq. (10) can be written as $p + q = 1$ which implies $2(p + q) - (p + q)^2 = 1$ and hence $2p - p^2 + 2q - q^2 - 1 = 2pq$. The last equation can be written as $\Phi(\sqrt{2} p) + \Phi^*(\sqrt{2} q) = 2pq$ which is equivalent to Eq. (7). □
Theorem 5.5 also lets us find a maximizer of $F_\rho$ on a parallelogram with constant density.

**Proposition 5.7.** Let $\Omega$ be the open parallelogram
\[
\{(x, y) \in \mathbb{R}^2 : |ax + by| < 1, |ax - by| < \beta \}
\]
for some $a, b, \beta > 0$, and let $\rho$ be constant on $\Omega$. Then, for any $c \geq 0$, in $\mathcal{U}_{2c}(\Omega)$ the functional $F_\rho$ is maximized by the function $u(x, y) = c(ax + by)$, and the maximum value is $\mu(\Omega)\rho \Phi(\sqrt{2ab}/\rho)$ if $\rho > 0$ and 0 if $\rho = 0$.

**Proof.** If $\rho = 0$, the functional $F_\rho$ is identically zero, so we may assume that $\rho > 0$. Note that $\sqrt{2u_xu_y/\rho} = c\sqrt{2ab}/\rho$ which is independent of $x$ and $y$. By Lemma 5.2(b), there is a $d \geq 0$ such that $\Phi(c\sqrt{2ab}/\rho) + \Phi^*(d\sqrt{2ab}/\rho) = 2abcd/\rho$. Let $v(x, y) := d(by - ax)$. Then, $u$ and $v$ satisfy the conditions in Theorem 5.5 with $r = 2c$ and $s = 2\beta d$, and hence $u$ is a maximizer of $F_\rho$ in $\mathcal{U}_{2c}$. The maximum value is $\mu(\Omega)\rho \Phi(\sqrt{2ab}/\rho)$.

6. Probabilistic behavior of the local parallelogram

In Definition 3.7, we defined the map $P \mapsto \kappa_P$ to convert $k$-decreasing sets into doubly increasing functions. The next definition is a device to go in the other direction.

**Definition 6.1.** For any $u$ in $\mathcal{U}(\Omega)$, we let
\[
D(u) := \{(x, y) \in \Omega : u(x, y) \in \mathbb{Z} \text{ and } u(x', y') < u(x, y) \}
\]
for any $(x', y') \in \Omega \setminus \{(x, y)\}$ such that $x' \leq x$ and $y' \leq y$.

**Lemma 6.2.** The following holds.

(a) For any $u$ in $\mathcal{U}_r(\Omega)$, $D(u)$ is $[r+1]$-decreasing, and if $u \in \mathcal{U}_{0,r}(\Omega)$, $D(u)$ is $[r]$-decreasing.

(b) If $P$ is a finite set of points in $\Omega \subset \mathbb{R}^2$ with distinct $x$-coordinates and distinct $y$-coordinates, then $D(\kappa_P) = P$.

**Proof.** (a) Let $u \in \mathcal{U}_r(\Omega)$. Suppose $(x, y)$ and $(x', y')$ are distinct points in $D(u)$ such that $u(x', y') = u(x, y)$. By definition of $D(u)$, it follows that $\{(x, y), (x', y')\}$ is a decreasing set. Hence, for each integer $k$, the fiber $u^{-1}(k) \cap D(u)$ is a decreasing set. The image of $u$ contains at most $|r| + 1$ integers, so $D(u)$ is a union of $|r| + 1$ decreasing sets. If $u \in \mathcal{U}_{0,r}(\Omega)$, the image of $u$ contains no integer outside the set $\{0, 1, \ldots, |r|\}$. Since $\Omega$ is open and $u \geq 0$, for any point $(x, y) \in \Omega$ with $u(x, y) = 0$ there is an $x' < x$ such that $u(x', y) = 0$, so the fiber $u^{-1}(0) \cap D(u)$ is empty.

(b) Let us first show that $P \subseteq D(\kappa_P)$. Take any point $(x, y) \in P$ and any point $(x', y') \in \Omega \setminus \{(x, y)\}$ such that $x' \leq x$ and $y' \leq y$. Let $Q$ be an increasing subset of $P \cap ((-\infty, x'] \times (-\infty, y])$ of cardinality $\kappa_P(x', y')$. Since no two points in $P$ have the same $x$- or $y$-coordinates, $Q \cup \{(x, y)\}$ is an increasing subset of $P \cap ((-\infty, x] \times (-\infty, y])$ of cardinality $\kappa_P(x', y') + 1$, and it follows that $\kappa_P(x', y') < \kappa_P(x, y)$. This shows that $(x, y) \in D(\kappa_P)$, and since $(x, y)$ was chosen arbitrarily in $P$, we conclude that $P \subseteq D(\kappa_P)$. 


To show that \( D(\kappa P) \subseteq P \), take any \((x, y) \in D(\kappa P)\) and let \(Q\) be an increasing subset of \(P \cap \{(-\infty, x] \times (-\infty, y]\}\) of cardinality \(\kappa P(x, y)\). Let \((x', y')\) be the point in \(Q\) with maximal coordinates. Then, \(\kappa P(x', y') \geq \kappa P(x, y)\) and, since \((x, y) \in D(\kappa P)\), it follows that \((x', y') = (x, y)\) and hence that \((x, y) \in P\). □

The next lemma is essentially a reformulation of Proposition 4.5 in terms of the \(D\) operator.

**Lemma 6.3.** Let \(\Omega\) be the open parallelogram
\[
\{(x, y) \in \mathbb{R}^2 : |ax + by| < 1, |ax - by| < \beta\}
\]
for some \(a, b, \beta > 0\), and let \(\rho\) be constant on \(\Omega\). Let \(u_{\text{linear}}(x, y) = c(ax + by)\) for some \(c \geq 0\), and let \(\{\sigma_\gamma\}_{\gamma > 0}\) be Poisson point processes on \(\Omega\) with intensities \(\gamma \rho\). Then the following two statements hold a.a.s. as \(\gamma \to \infty\).

\[
\forall w \in U_{2c}(\Omega), \#(D(w\sqrt{\gamma}) \cap \sigma_\gamma)/\gamma \leq F_{\rho}(u_{\text{linear}}) + 3(\rho + 1)\mu(\Omega)/\beta
\]
\[
\forall d \in \mathbb{R} \exists w \in U_{d,2c}(\Omega) : \#(D(w\sqrt{\gamma}) \cap \sigma_\gamma)/\gamma \geq F_{\rho}(u_{\text{linear}}) - 3(\rho + 1)\mu(\Omega)/\beta
\]

**Proof.** For any \(w \in U_{2c}(\Omega)\), by Lemma 6.2(a), \(D(w\sqrt{\gamma}) \cap \sigma_\gamma\) is a \([2c\sqrt{\gamma}]+1\)-decreasing subset of \(\sigma_\gamma\). Let \(c' = c + \frac{1}{4\sqrt{\gamma}}\) and let \(u'_{\text{linear}}(x, y) = c'(ax + by)\). Then, for any \(w \in U_{2c}(\Omega)\), \(D(w\sqrt{\gamma}) \cap \sigma_\gamma\) is a \([2c'\sqrt{\gamma}]\)-decreasing subset of \(\sigma_\gamma\), and it follows from Proposition 4.5 that the statement
\[
\forall w \in U_{2c}(\Omega), \#(D(w\sqrt{\gamma}) \cap \sigma_\gamma)/\gamma \leq F_{\rho}(u'_{\text{linear}}) + 3\rho\mu(\Omega)/\beta
\]
holds a.a.s. as \(\gamma \to \infty\).

Note that \(c' \to c\) as \(\gamma \to \infty\). By Lemma 4.9, \(L\) is continuous, so
\[
F_{\rho}(u_{\text{linear}}) - F_{\rho}(u'_{\text{linear}}) = \mu(\Omega)(L(\rho, c^2ab) - L(\rho, c'^2ab)) \to 0
\]
as \(\gamma \to \infty\), and we conclude that the statement
\[
\forall w \in U_{2c}(\Omega), \#(D(w\sqrt{\gamma}) \cap \sigma_\gamma)/\gamma \leq F_{\rho}(u_{\text{linear}}) + 3(\rho + 1)\mu(\Omega)/\beta
\]
holds a.a.s. as \(\gamma \to \infty\).

Now for the second part. This time, let \(c' = c - \frac{1}{4\sqrt{\gamma}}\) and, as before, let \(u'_{\text{linear}}(x, y) = c'(ax + by)\). We will soon let \(\gamma\) tend to infinity, so we may assume that \(c' > 0\). Let \(P_\gamma\) be any maximal \([2c'\sqrt{\gamma}]\)-decreasing subset of \(\sigma_\gamma\). It follows from Proposition 4.5 that
\[
\#P_\gamma/\gamma \geq F_{\rho}(u_{\text{linear}}) - 3\rho\mu(\Omega)/\beta
\]
a.a.s. as \(\gamma \to \infty\). Analogously to above, since \(c' \to c\) as \(\gamma \to \infty\) and since \(L\) is continuous, it follows that
\[
\#P_\gamma/\gamma \geq F_{\rho}(u_{\text{linear}}) - 3(\rho + 1)\mu(\Omega)/\beta
\]
a.a.s. as \(\gamma \to \infty\). For any \(d \in \mathbb{R}\), let \(w = (\kappa P_\gamma + [d\sqrt{\gamma}]) / \sqrt{\gamma}\). By Proposition 3.6,
\[
0 \leq \kappa P_\gamma \leq [2c'\sqrt{\gamma}] \leq 2c'\sqrt{\gamma} = 2c\sqrt{\gamma} - 1,
\]
so \(w\) belongs to \(U_{d,2c}(\Omega)\). Clearly, \(D\) is invariant under translation by an integer, so \(D(w\sqrt{\gamma}) = D(\kappa P_\gamma)\) which equals \(P_\gamma\) by Lemma 6.2(b). (Note that the points in \(P_\gamma\) are in general position almost surely.) We conclude that the statement
\[
\forall d \in \mathbb{R} \exists w \in U_{d,2c}(\Omega) : \#(D(w\sqrt{\gamma}) \cap \sigma_\gamma)/\gamma \geq F_{\rho}(u_{\text{linear}}) - 3(\rho + 1)\mu(\Omega)/\beta
\]
holds a.a.s. as $\gamma \to \infty$. \hfill \qed

7. Approximating $\Omega$ by a collection of parallelograms

Now when we have studied the behavior of a parallelogram density domain, it is time to divide a general density domain into many small local parallelograms. It is vital that the number of such parallelograms is finite since we want to infer an “in probability” result for the whole domain from similar results for each local parallelogram. To this end we will rely heavily on the theory of Vitali coverings.

First a pair of technical lemmas.

**Lemma 7.1.** Let $A$ and $B$ be disjoint closed subsets of $\mathbb{R}^2$, and suppose $A$ is compact. Then, the distance between $A$ and $B$ is positive.

*Proof.* Suppose not. Then there are sequences $a_i \in A$ and $b_i \in B$ such that $|a_i - b_i| \to 0$. Since $A$ is compact, there is a subsequence $a_{i_j}$ of $a_i$ that converges to some $a$ in $A$. By the triangle inequality, $|b_{i_j} - a| \leq |b_{i_j} - a_{i_j}| + |a_{i_j} - a|$ which tends to zero. Since $B$ is closed, this implies that $a$ belongs to $B$ which is a contradiction since $A$ and $B$ are disjoint. \hfill \Box

**Lemma 7.2.** Let $\Omega$ be an open subset of $\mathbb{R}^2$ and let $C$ be a compact subset of $\Omega$. Then there is a constant $K$ such that

$$\text{diam } w(C) \leq \text{diam } w(\Omega) + K\|w - u\|_\Omega$$

for any $u, w \in \mathcal{U}(\Omega)$.

*Proof.* By Lemma 7.1, the distance between $C$ and $\mathbb{R}^2 \setminus \Omega$ is positive, so there exists a $d > 0$ such that for each $(x, y) \in C$ we have $[x-d, x+d] \times [y-d, y+d] \subset \Omega$. It follows that, for any $u, w \in \mathcal{U}(\Omega)$,

$$\|w - u\|_\Omega \geq \sup_{(x,y) \in C} \|w - u\|_{[x-d,x+d] \times [y-d,y+d]} \geq (\sup w(C) - \sup u(\Omega))d^2$$

and

$$\|w - u\|_\Omega \geq \sup_{(x,y) \in C} \|w - u\|_{[x-d,x] \times [y-d,y]} \geq (\inf u(\Omega) - \inf w(C))d^2.$$  

Thus we can choose $K := 2/d^2$. \hfill \Box

Next, we make the idea of a local parallelogram precise.

**Definition 7.3.** Let $u \in \mathcal{U}(\Omega)$ and let $\iota > 0$. A $(u, \iota)$-parallelogram is a closed parallelogram of the form

$$P = \{(x, y) \in \mathbb{R}^2 : |\tilde{u}_x^P(x - x_P) + \tilde{u}_y^P(y - y_P)| \leq \iota c_P, |\tilde{u}_x^P(x - x_P) - \tilde{u}_y^P(y - y_P)| \leq c_P\},$$

where $(x_P, y_P)$ is a point in $\text{Diff}(u)$, $c_P > 0$, $\tilde{u}_x^P := \max\{\iota^3, u_x(x_P, y_P)\}$ and $\tilde{u}_y^P := \max\{\iota^3, u_y(x_P, y_P)\}$.

Also, for notational convenience, define $u_x^P = u_x(x_P, y_P)$ and $u_y^P = u_y(x_P, y_P)$. We say that $P$ is well behaved if $\tilde{u}_x^P = u_x^P$ and $\tilde{u}_y^P = u_y^P$. 

Lemma 7.4. For any point \((x, y)\) in a \((u, \iota)\)-parallelogram \(P\), we have

\[
|x - xp| + |y - yp| \leq c_P (1 + \iota) v^{-3}
\]

and

\[
|u^P_x(x - xp) + u^P_y(y - yp)| \leq c_P (1 + \iota).
\]

Proof. We have

\[
2\bar{u}^P_x |x - xp| \leq |\bar{u}^P_x(x - xp) + \bar{u}^P_y(y - yp)| + |\bar{u}^P_x(x - xp) - \bar{u}^P_y(y - yp)| \leq c_P (1 + \iota),
\]

where the first inequality is the triangle inequality and the second inequality follows from the definition of a \((u, \iota)\)-parallelogram. For the same reason,

\[
2\bar{u}^P_y |y - yp| \leq c_P (1 + \iota),
\]

so

\[
|x - xp| + |y - yp| \leq c_P (1 + \iota) \left( \frac{1}{2\bar{u}^P_x} + \frac{1}{2\bar{u}^P_y} \right) \leq c_P (1 + \iota) v^{-3}.
\]

Finally, by Eqs. (11) and (12),

\[
|u^P_x(x - xp) + u^P_y(y - yp)| \leq \bar{u}^P_x |x - xp| + \bar{u}^P_y |y - yp| \leq c_P (1 + \iota).
\]

Let us recall the definition of a regular Vitali covering. As always, we let \(\mu\) denote the Lebesgue measure on \(\mathbb{R}^2\).

Definition 7.5. Let \(A \subseteq \mathbb{R}^2\) and let \(\mathcal{C}\) be a collection of closed subsets of \(\mathbb{R}^2\).

- \(\mathcal{C}\) is a Vitali covering of \(A\) if, for any \(p \in A\) and any \(\delta > 0\), there is a \(C \in \mathcal{C}\) such that \(p \in C\) and \(0 < \text{diam } C < \delta\).
- \(\mathcal{C}\) is regular if there is a constant \(K\) such that \((\text{diam } C)^2 \leq K \mu(C)\) for any \(C \in \mathcal{C}\).

Lemma 7.6. Let \(u \in \mathcal{U}(\Omega)\) and let \(T\) be a subset of \(\text{Diff}(u)\) where \(u_x\) and \(u_y\) are both bounded. Then, for any \(\iota > 0\), the family of all \((u, \iota)\)-parallelograms is a regular Vitali covering of \(T\).

Proof. For any \(p \in T\), the diameter of a \((u, \iota)\)-parallelogram \(P\) centered at \((xp, yp) = p\) is bounded by

\[
2\sqrt{(x - xp)^2 + (y - yp)^2} \leq 2(|x - xp| + |y - yp|),
\]

which is at most \(2c_P (1 + \iota) v^{-3}\) by Lemma 7.4. By choosing \(c_P\) small enough we can make the diameter arbitrarily small, so the family of \((u, \iota)\)-parallelograms is a Vitali covering of \(A\). To see that it is regular, note that \(\mu(P) = 2c_P^2 / \bar{u}^P_x \bar{u}^P_y\), so the quotient

\[
\frac{\text{diam } P}{\sqrt{\mu(P)}} \leq \frac{2c_P (1 + \iota) v^{-3}}{\sqrt{\mu(P)}}
\]

is bounded since \(u_x\) and \(u_y\) are bounded on \(T\).
The following lemma divides a density domain with a doubly increasing function \( u \) into a finite number of local parallelograms such that, within each parallelogram, the situation is close to the condition of Lemma 6.3, that is, the density is nearly constant and \( u \) is nearly a linear function aligned with the parallelogram.

We will use the ordo notation \( o(1) \) to represents a function of \( \iota \) that tends to zero as \( \iota \) tends to zero.

**Lemma 7.7.** Let \( (\Omega, \rho) \) be a density domain and let \( u \in \mathcal{U}(\Omega) \) and \( \varepsilon > 0 \). Then, for any \( 0 < \iota < 1 \), there is a measurable set \( S_\iota \subseteq \text{Diff}(u) \) and a finite disjoint collection \( \mathcal{P}_\iota \) of \((u, \iota)\)-parallelograms such that the following holds.

(a) For each \( P \in \mathcal{P}_\iota \) it holds that \( P \subset \Omega \), \((x_P, y_P) \in S_\iota \) and \( c_P < 1 \). Also, \( S_\iota \subseteq \bigcup \mathcal{P}_\iota \).

(b) \( \rho \), \( u_x \) and \( u_y \) are bounded on \( \bigcup_{0 < \iota < 1} S_\iota \).

(c) \( \|\rho\|_{\Omega \setminus S_\iota} < \varepsilon + o(1) \).

(Intuition: \( S_\iota \) nearly covers the density domain.)

(d) For each \( P \in \mathcal{P}_\iota \) it holds that \( \mu(P \cap S_\iota) / \mu(P) > 1 - \iota \).

(Intuition: \( S_\iota \) nearly covers each parallelogram.)

(e) For each \( P \in \mathcal{P}_\iota \) it holds that \( |\rho(x, y) - \rho(x_P, y_P)| \leq \iota \) for any \((x, y) \in P \cap S_\iota \).

(Intuition: \( \rho \) is nearly constant on each parallelogram.)

(f) For each \( P \in \mathcal{P}_\iota \) it holds that

\[
|u(x, y) - (u(x_P, y_P) + u_x^P(x - x_P) + u_y^P(y - y_P))| \leq \iota (|x - x_P| + |y - y_P|)
\]

for any \((x, y) \in P \).

(Intuition: \( u \) is nearly linear on each parallelogram.)

(g) For each \( P \in \mathcal{P}_\iota \) it holds that

\[
\|L(\rho, u_x, u_y)\|_P / \mu(P) - L(\rho(x_P, y_P), u_x^P, u_y^P) < \iota.
\]

(Intuition: \( L(\rho, u_x, u_y) \) is nearly constant on each parallelogram.)

(h) For each \( P \in \mathcal{P}_\iota \) it holds that

\[
\inf_{(x, y) \in P} |u(x, y) - u(x_P, y_P)| < \begin{cases} c_P \iota (1 + 5\iota) & \text{if } P \text{ is well behaved,} \\ c_P (1 + 7\iota) & \text{otherwise.} \end{cases}
\]

(Intuition: \( u \) does not vary too much inside each parallelogram.)

(i) There is a function \( d : (0, 1) \to \mathbb{R}_{>0} \) such that

\[
\sup_{w \in \mathcal{U}(\Omega) : \|w - u\|_\Omega < d(\iota)} \sup_{P \in \mathcal{P}_\iota} \left( \frac{\text{diam } w(P)}{2\iota c_P} \right)^2 \|\rho\|_P < o(1).
\]

**Proof.** By Lemma 4.10, \( \mu(\Omega \setminus \text{Diff}(u)) = 0 \).

Since \( \|\rho\|_\Omega < \infty \), we can choose a subset \( T \) of \( \text{Diff}(u) \) with finite measure such that \( \rho \), \( u_x \) and \( u_y \) are all smaller than some positive constant \( C \) there, and such that

\[
\|\rho\|_\Omega < \varepsilon.
\]

For each \( 0 < \iota < 1 \), let \( T_\iota \) be the set of points in \( T \) that are Lebesgue points of \( L(\rho, u_x, u_y) \) with respect to the family of \((u, \iota)\)-parallelograms, that is, \( T_\iota \) is the set of points \((x_0, y_0) \in T \) such that for each \( \varepsilon' > 0 \) we have
\[ \|L(\rho, u_x u_y)\| \rho(1 / \rho(P) - L(\rho, u_x u_y))_{(x_0, y_0)} < \varepsilon \] for all sufficiently small \((u, \iota)\)-parallelograms \(P\) centered at \((x_0, y_0)\). By Lemma 7.6, the family of \((u, \iota)\)-parallelograms is a regular Vitali covering of \(T\), so, by Lebesgue’s differentiation theorem (see e.g. [11]), \(\mu(T_\iota) = \mu(T)\) for any \(\iota\).

For any \(0 < \iota < 1\) and any positive integer \(j\), let
\[
\tilde{S}_j^\iota = \{(x, y) \in T_\iota : (j - 1)\iota \leq \rho(x, y) < j\iota\}
\]
and let \(S_j^\iota\) be the set of points in \(\tilde{S}_j^\iota\) at which the density of \(\tilde{S}_j^\iota\) is 1. By Lebesgue’s density theorem, \(\mu(\tilde{S}_j^\iota \setminus S_j^\iota) = 0\). For any \(\iota, T_\iota\) is the union of a finite number of sets of the form \(\tilde{S}_j^\iota\), so the union \(\tilde{S}_\iota := \bigcup_j S_j^\iota\) of all \(S_j^\iota\) for a fixed \(\iota\) has the same measure as \(T_\iota\) and hence as \(T\).

For any \(0 < \iota < 1\) and any positive integer \(j\), let \(A_j^\iota\) be the family of \((u, \iota)\)-parallelograms \(P\) with center in \(S_j^\iota\) and \(c_P < 1\) such that
\begin{align*}
(\text{I}) & \quad \mu(P \cap S_j^\iota) / \mu(P) > 1 - \iota, \\
(\text{II}) & \quad |||L(\rho, u_x u_y)||| P / \mu(P) - L(\rho(x_P, y_P), u_x^P u_y^P)| | < \iota, \text{ and} \\
(\text{III}) & \quad \text{the } (u, \iota)\text{-parallelogram } P' \text{ concentric with } P \text{ but with } c_{P'} = (1 + \iota)c_P \\
& \quad \text{is contained in } \Omega, \text{ and} \\
& \quad \left| u(x, y) - (u(x_P, y_P) + u_x^P (x - x_P) + u_y^P (y - y_P)) \right| \leq \varepsilon \left( |x - x_P| + |y - y_P| \right)
\end{align*}
for any point \((x, y) \in P'\).

Let \(A_\iota = \bigcup_j A_j^\iota\).

We claim that \(A_j^\iota\) is a Vitali covering of \(S_j^\iota\). To see this, take any \((x, y) \in S_j^\iota\) and note the following:

- Since the density of \(\tilde{S}_j^\iota\) is 1 at \((x, y)\) (by the choice of \(S_j^\iota\)) and \(\mu(\tilde{S}_j^\iota \setminus S_j^\iota) = 0\), (I) holds for any sufficiently small \((u, \iota)\)-parallelogram centered at \((x, y)\).
- Since \((x, y)\) is a Lebesgue point of \(T\) (by the choice of \(T_\iota\)), (II) holds for any sufficiently small \((u, \iota)\)-parallelogram centered at \((x, y)\).
- Since \(\Omega\) is open and \(u\) is differentiable at \((x, y)\), (III) holds for any sufficiently small \((u, \iota)\)-parallelogram centered at \((x, y)\).

By Lemma 7.6, the family of all \((u, \iota)\)-parallelograms is regular, so it follows that \(A_j^\iota\) is a regular Vitali covering of \(S_j^\iota\), and hence \(A_\iota\) is a regular Vitali covering of \(\tilde{S}_\iota\). By Vitali’s cover theorem (see e.g. [10]), there is a finite disjoint subfamily \(\mathcal{P}_\iota\) of \(A_\iota\) such that \(\mu(\tilde{S}_\iota \cap \cup \mathcal{P}_\iota) \geq (1 - \iota)\mu(\tilde{S}_\iota)\) and hence
\begin{equation}
\mu(T \cap \cup \mathcal{P}_\iota) \geq (1 - \iota)\mu(T).
\end{equation}

Finally, define
\[
S_\iota := \bigcup_{j=1}^\infty \bigcup_{P \in \mathcal{P}_\iota \cap \tilde{A}_j^\iota} P \cap \tilde{S}_j^\iota.
\]

Let us check that \(S_\iota\) and \(\mathcal{P}_\iota\) have the properties claimed in the lemma.

(a) and (b) follow directly from the definitions.
(c) We have
\[ \mu(S_i) = \sum_{j=1}^{\infty} \sum_{P \in \mathcal{P}_j \cap \mathcal{A}_i^j} \mu(P \cap S_i^j) \geq \{ \text{by (I)} \} \geq \sum_{j=1}^{\infty} \sum_{P \in \mathcal{P}_j \cap \mathcal{A}_i^j} (1-\iota)\mu(P) \]
\[ = (1-\iota)\mu(\cup \mathcal{P}_i) \geq \{ \text{by Eq. (14)} \} \geq (1-\iota)^2\mu(T), \]
so (c) follows from Eq. (13) and the fact that \( \rho \) is bounded in \( T \).

(d) follows from (I).

(c) follows from the definition of the sets \( \mathcal{S}_i^j \).

(f) follows from (III).

(g) follows from (II).

(h) requires some reasoning. From (III) we know that, for each \( P \in \mathcal{P}_i \),
\[ |u(x, y) - (u(x_P, y_P) + u^P_x(x - x_P) + u^P_y(y - y_P))| \leq \iota^5(|x - x_P| + |y - y_P|) \]
for any \((x, y) \in P',\) where \( P' \subseteq \Omega \) is defined as in (III).

Consider a parallelogram \( P \in \mathcal{P}_t. \) By Lemma 7.4, inside \( P' \),
\[ |x - x_P| + |y - y_P| \leq c_{P'}(1 + \iota)\iota^{-3}. \]
Also, inside \( P' \),
\[ |u(x, y) - u(x_P, y_P)| \leq \{ \text{triangle ineq.} \} \leq \]
\[ \leq |u^P_x(x - x_P) + u^P_y(y - y_P)| \]
\[ + |u(x, y) - (u(x_P, y_P) + u^P_x(x - x_P) + u^P_y(y - y_P))| \]
\[ \leq |u^P_x(x - x_P) + u^P_y(y - y_P)| + \iota^5(|x - x_P| + |y - y_P|) \]
\[ \leq |u^P_x(x - x_P) + u^P_y(y - y_P)| + \iota^2(1 + \iota)c_{P'}, \]
where the last inequality follows from Eq. (16).

If \( P \) is well behaved, inside \( P' \) we have
\[ |u^P_x(x - x_P) + u^P_y(y - y_P)| = |\tilde{u}^P_x(x - x_P) + \tilde{u}^P_y(y - y_P)| \leq \iota c_{P'} \]
by the definition of a \((u, \iota)\)-parallelogram. Hence, by Eq. (17), inside \( P' \),
\[ |u(x, y) - u(x_P, y_P)| \leq c_{P'}(1 + \iota + \iota^2) = c_{P'}(1 + \iota)(1 + \iota + \iota^2) < c_{P'}(1 + 5\iota), \]
where the last inequality follows from the fact that \( \iota < 1. \)

If \( P \) is not well behaved, by Lemma 7.4, at least we have
\[ |u^P_x(x - x_P) + u^P_y(y - y_P)| \leq c_{P'}(1 + \iota). \]
Hence, by Eq. (17), inside \( P' \) we have
\[ |u(x, y) - u(x_P, y_P)| \leq c_{P'}(1 + \iota)(1 + \iota^2) = c_{P'}(1 + \iota)^2(1 + \iota^2) < c_{P'}(1 + 7\iota), \]
where we have used again than \( \iota < 1. \)

(i) requires some reasoning as well. Consider a parallelogram \( P \) in \( \mathcal{P}_t. \)
Let \( P' \) be the larger concentric \((u, \iota)\)-parallelogram as defined in (III). By Lemma 7.2 applied to the open set \( \text{int} P' \) and the compact set \( P \) there is a \( K_P > 0 \) such that \( \text{diam} w(P) \leq \text{diam} w(P') + K_P||w - u||_\Omega \) for any \( w \in \mathcal{U}(\Omega). \)

It follows from (h) that, if \( P \) is well behaved,
\[ \text{diam} w(P) \leq 2c_{P'}(1 + 5\iota) + K_P||w - u||_\Omega \leq 2c_{P'}(1 + 6\iota) \]
for any \( w \in \mathcal{U}(\Omega) \) with \( ||w - u||_{\Omega} \leq 2x^2 c_P/K_P \), and if \( P \) is not well behaved, \[ \text{diam } w(P) \leq 2c_P(1 + 7\epsilon) + K_P||w - u||_{\Omega} \leq 2c_P(1 + 7\epsilon + \epsilon^2) < 18c_P \]
for any \( w \in \mathcal{U}(\Omega) \) with \( ||w - u||_{\Omega} \leq 2x^2 c_P/K_P \).

Thus, if \( P \) is well behaved, for any \( w \in \mathcal{U}(\Omega) \) with \( ||w - u||_{\Omega} \leq 2x^2 c_P/K_P \) we have

\[
\left( \frac{\text{diam } w(P)}{2x c_P} \right)^2 u_x^P u_y^P - u_x^P u_y^P = \left[ \left( \frac{\text{diam } w(P)}{2x c_P} \right)^2 - 1 \right] u_x^P u_y^P \\
\leq [(1 + 6\epsilon)^2 - 1] u_x^P u_y^P = o_{1}(1)
\]

since \( u_x \) and \( u_y \) are bounded in \( \bigcup_i S_i \).

If \( P \) is not well behaved, at least one of \( u_x^P \) and \( u_y^P \) is smaller than \( \epsilon^3 \), and at least one of \( \tilde{u}_x^P \) and \( \tilde{u}_y^P \) equals \( \epsilon^3 \). It follows that, for any \( w \in \mathcal{U}(\Omega) \) with \( ||w - u||_{\Omega} \leq 2x^2 c_P/K_P \) we have

\[
\left( \frac{\text{diam } w(P)}{2x c_P} \right)^2 \tilde{u}_x^P \tilde{u}_y^P - u_x^P u_y^P \leq (9/\epsilon)^2 \tilde{u}_x^P \tilde{u}_y^P - u_x^P u_y^P = o_{1}(1).
\]

We conclude that

\[
\left( \frac{\text{diam } w(P)}{2x c_P} \right)^2 \tilde{u}_x^P \tilde{u}_y^P - u_x^P u_y^P < o_{1}(1)
\]

for any \( w \in \mathcal{U}(\Omega) \) with \( ||w - u||_{\Omega} \leq 2x^2 c_P/K_P \) whether \( P \) is well behaved or not. Thus, we can choose \( d(\epsilon) := 2x^2 \min_{P \in \mathcal{P}} (c_P/K_P) \).

\[ \Box \]

8. Semicontinuity of \( F_\rho \) and a Related Probabilistic Result

Our proof of Theorem 11.3 will rely heavily on the following result.

**Proposition 8.1.** \( F_\rho \) is upper semicontinuous in the \( L^1(\Omega) \)-norm.

**Proof.** Let \( u \in \mathcal{U}(\Omega) \). We must show that, for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( F_\rho(w) < F_\rho(u) + \epsilon \) for any \( w \in \mathcal{U}(\Omega) \) with \( ||w - u||_{\Omega} < \delta \).

Choose any \( \epsilon > 0 \) smaller than \( \epsilon' \) and apply Lemma 7.7. Let \( d : (0,1) \rightarrow \mathbb{R}_{>0} \) be the function defined in Lemma 7.7(i) and consider any family \( \{ w^{(i)} \in \mathcal{U}(\Omega) \}_{0 < i < 1} \) such that \( ||w^{(i)} - u||_{\Omega} < d(i) \). Let

\[
q^{(i)} := \sup_{w \in \mathcal{U}(\Omega): ||w - u||_{\Omega} < d(i)} \sup_{P \in \mathcal{P}_i} \left( \left( \frac{\text{diam } w(P)}{2x c_P} \right)^2 \tilde{u}_x^P \tilde{u}_y^P - u_x^P u_y^P \right).
\]

Consider any \( P \in \mathcal{P}_i \) and let \( \rho_P := \sup_{S \cap P} \rho \). By Lemma 4.9, \( L \) is increasing in the first variable, so

\[
||L(\rho, u_x^{(i)} w_y^{(i)})||_{P \cap S_i} \leq ||L(\rho_P, u_x^{(i)} w_y^{(i)})||_{P}.
\]

Let \( r_P := \text{diam } w^{(i)}(P) \). By Proposition 5.7, on \( \mathcal{U}_P(\text{int } P) \), \( F_{\rho P} \) is maximized by the function

\[
u^P_{\text{linear}}(x,y) = \frac{r_P}{2x c_P} (\tilde{u}_x^P (x - x_P) + \tilde{u}_y^P (y - y_P)) \]
\[ \|L(\rho, w_x^{(i)} w_y^{(i)})\|_P \leq F_{\rho_P}(u^P_{\text{linear}}) = \mu(P)L(\rho_P, (\frac{r_P}{2\epsilon P})^2 \tilde{u}_x^{P} \tilde{u}_y^{P}) \leq \mu(P)L(\rho_P, q^{(i)} + u^P_x u^P_y) =: \text{RHS}, \]

where the last inequality uses the fact that \( L \) is increasing in the second variable (Lemma 4.9). By Lemma 7.7(e), \( |\rho - \rho(x, y)\|_P \leq \varepsilon \), and by Lemma 7.7(i), \( q^{(i)} < o_i(1) \). By Lemma 4.9, \( L \) is continuous and hence uniformly continuous on the set \( \{(\rho(x, y), u_x(x, y)u_y(x, y)) : (x, y) \in \bigcup_{i>0} S_i\} \) which is bounded by Lemma 7.7(b). Again by Lemma 4.9, \( L \) is increasing in the second variable, so \( \text{RHS} < \mu(P)(L(\rho(x, y)P, u^P_x u^P_y) + o_i(1)) \), where \( o_i(1) \) is independent of \( P \). By Lemma 7.7(g),

\[ L(\rho(x, y)P, u^P_x u^P_y) < \frac{1}{\mu(P)}\|L(\rho, u_x u_y)\|_P + \varepsilon, \]

so

\[ \|L(\rho, w_x^{(i)} w_y^{(i)})\|_P < \|L(\rho, u_x u_y)\|_P + o_i(1)\mu(P). \]

By Lemma 7.7(a), \( S_i \subseteq \bigcup \mathcal{P}_i \), so summing over all \( P \) in \( \mathcal{P}_i \) yields

\[ \|L(\rho, w_x^{(i)} w_y^{(i)})\|_{\mathcal{P}_i} - \|L(\rho, u_x u_y)\|_{\mathcal{U}_P} < o_i(1)\mu(\mathcal{U}_P) = o_i(1). \]

By Lemma 4.9, \( L(\rho, w_x^{(i)} w_y^{(i)}) \leq \rho \), so by Lemma 7.7(c) it now follows that

\[ \|L(\rho, w_x^{(i)} w_y^{(i)})\|_\Omega \leq \|L(\rho, u_x u_y)\|_\Omega + \varepsilon + o_i(1). \]

Since \( \varepsilon < \varepsilon' \), the lemma follows. \( \square \)

In the proof of Theorem 10.2, we will need the following probabilistic analogue to Proposition 8.1.

**Lemma 8.2.** Let \( (\Omega, \rho) \) be a density domain, and let \( \{\sigma_\gamma\}_{\gamma > 0} \) be Poisson point processes on \( \Omega \) with intensity functions \( \gamma \rho \). Then, for any \( u \in \mathcal{U}(\Omega) \) and any \( \varepsilon' > 0 \) there is a \( \delta > 0 \) such that

\[ \sup_{u \in \mathcal{U}(\Omega), \|w - u\|_\Omega < \delta} \#(D(w \sqrt{\gamma}) \cap \sigma_\gamma)/\gamma < F_{\rho}(u) + \varepsilon' \]

holds a.a.s. as \( \gamma \to \infty \).

**Proof.** Choose any \( \varepsilon > 0 \) smaller than \( \varepsilon'/2 \) and apply Lemma 7.7. Let \( d : (0, 1) \to \mathbb{R}_{>0} \) be the function defined in Lemma 7.7(i). Consider any \( P \in \mathcal{P}_i \) and any \( w \in \mathcal{U}(\Omega) \) such that \( \|w - u\|_\Omega < \delta(i) \), and let \( r_P := \text{diam } w(P) \) and \( \rho_P := \sup_{S_i \cap P} \rho \). Define \( u^P_{\text{linear}} \in \mathcal{U}_{r_P}(\text{int } P) \) by

\[ u^P_{\text{linear}}(x, y) = \frac{r_P}{2\epsilon P} (\tilde{u}_x(P - x + P) + \tilde{u}_y(y - y)). \]

Since \( \sigma_\gamma \cap P \cap S_i \) is a subset of a Poisson point process on \( P \) with homogeneous intensity \( \rho_P \), Lemma 6.3 yields that, a.a.s. as \( \gamma \to \infty \),

\[ \frac{1}{\gamma} \#(D(w \sqrt{\gamma}) \cap \sigma_\gamma \cap P \cap S_i) \leq F_{\rho_P}(u^P_{\text{linear}}) + 3\mu(P)(\rho_P + 1) =: \text{RHS}. \]

As in the proof of the upper semicontinuity, we obtain

\[ \text{RHS} < \|L(\rho, u_x u_y)\|_P + \mu(P) o_i(1), \]

where \( o_i(1) \) is independent of \( P \). Summing over all \( P \) in \( \mathcal{P}_i \) yields that, a.a.s. as \( \gamma \to \infty \), for any \( w \in \mathcal{U}(\Omega) \) such that \( \|w - u\|_\Omega < \delta(i) \),

\[ \frac{1}{\gamma} \#(D(w \sqrt{\gamma}) \cap \sigma_\gamma \cap S_i) - \|L(\rho, u_x u_y)\|_{\mathcal{U}_P} < o_i(1)\mu(\mathcal{U}_P) = o_i(1). \]
By the law of large numbers, a.a.s. as $\gamma \to \infty$ we have $\frac{1}{\gamma} \#(\sigma_\gamma \setminus S_t) < \|\rho\|_{\Omega, S_t} + \varepsilon$ which is bounded by $2\varepsilon + o_t(1)$ by Lemma 7.7(c), so a.a.s.

$$\sup_{w \in \mathcal{U}(\Omega): \|w - u\|_{\Omega} < d(\iota)} \frac{1}{\gamma} \#(D(w, \sqrt{\gamma}) \cap \sigma_\gamma) - \|L(\rho, u_{x,y})\|_{\mathcal{U}, \iota} < 2\varepsilon + o_t(1).$$

Since $\varepsilon < \varepsilon'/2$, the lemma now follows. \qed

9. Compactness of the set of doubly increasing functions

In this section we put probabilistic matters aside and concern ourselves with the topology of the set of doubly increasing functions.

First, we show that doubly increasing functions can be extended to larger domains in a natural way.

**Lemma 9.1.** Let $A \subseteq B \subseteq \mathbb{R}^2$. For any $u \in \mathcal{U}(A)$ there is a $w \in \mathcal{U}(B)$ such that the restriction of $w$ to $A$ is $u$ and the images $u(A)$ and $w(B)$ have the same closure.

**Proof.** For each $(x, y) \in B$, let $P(x, y) := \{(x', y') \in A : x' \leq x, y' \leq y\}$. Define $w$ by letting $w(x, y) := \sup_{(x', y') \in P(x, y)} u$ if $P(x, y)$ is nonempty, and $w(x, y) := \inf_{A} u$ if $P(x, y)$ is empty. It is straightforward to verify that $w$ is doubly increasing.

Our proof of the existence of maximizers of $F_\rho$ (Theorem 11.3(a)) will need two key ingredients. The first one is the semicontinuity of $F_\rho$ (Proposition 8.1) and the other one is the following result about the topology of the set of doubly increasing functions, which will be essential also in the proof of Theorem 10.2.

**Proposition 9.2.** If $\Omega \subseteq \mathbb{R}^2$ is open, then $\mathcal{U}_{h,r}(\Omega)$ is a compact subset of $L^1(\Omega)$.

**Proof.** Let $\{u_n\}_{n=1}^\infty$ be a sequence of elements in $\mathcal{U}_{h,r}(\Omega)$. We need to show that there is a convergent subsequence.

Let $Q = \{q_1, q_2, \ldots\}$ be a countable dense subset of $\Omega$. We define a sequence $S_1, S_2, \ldots$ of subsequences of $\{u_n\}_{n=1}^\infty$ recursively as follows. First, let $S_1$ be the original sequence $\{u_n\}_{n=1}^\infty$. Then, for $n = 1, 2, \ldots$, let $S_{n+1}$ be a subsequence of $S_n$ that converges at the point $q_n$. This is possible since $[h, h + r]$ is a compact set. Finally, construct another subsequence $\{w_n\}_{n=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ by letting $w_n$ be the $n$th element of $S_n$. Clearly, $\{w_n\}$ converges at each point of $Q$. By Lemma 9.1, we can choose a $w \in \mathcal{U}_{h,r}(\Omega)$ such that $\lim_{n \to \infty} w_n(q) = w(q)$ for any $q \in Q$. We claim that $\lim_{n \to \infty} w_n(p) = w(p)$ for any continuity point $p$ of $w$. Take any $\varepsilon > 0$. Since $w$ is continuous at $p$, we can pick a $\delta > 0$ such that $|w(p') - w(p)| < \varepsilon/2$ for any $p'$ with $|p' - p| < \delta$. Let $A^- := \{p' \in \Omega : |p' - p| < \delta, p' \text{ strictly south-west of } p\}$ and $A^+ := \{p' \in \Omega : |p' - p| < \delta, p' \text{ strictly north-east of } p\}$. Since $\Omega$ is open, $A^-$ and $A^+$ are both open and nonempty, so there are $q^- \in Q \cap A^-$ and $q^+ \in Q \cap A^+$. For all sufficiently large $n$, we have $|w_n(q^-) - w(q^-)| < \varepsilon/2$ and $|w_n(q^+) - w(q^+)| < \varepsilon/2$, and hence

$$w(p) - \varepsilon \leq w(q^-) - \varepsilon/2 \leq w_n(q^-) \leq w_n(p) \leq w(q^+) + \varepsilon/2 \leq w(p) + \varepsilon.$$
Since \( \varepsilon \) was chosen arbitrarily, we conclude that \( \{w_n\}_{n=1}^{\infty} \) converges to \( w \) at any continuity point of \( w \).

By Lemma 4.10, \( w \) is continuous almost everywhere, so by the theorem of bounded convergence, \( w_n \) converges to \( w \) in the \( L^1(\Omega) \)-norm. \( \square \)

10. A GLUING LEMMA AND OUR FIRST MAIN THEOREM

To prove our first main theorem, Theorem 10.2, we need one final lemma. We know that we can find large unions of decreasing subsets within each local parallelogram. The following lemma glues those unions together to form a global union of decreasing subsets.

**Lemma 10.1.** Let \( \{\sigma_\gamma\}_{\gamma>0} \) be Poisson point processes on \( \Omega \) with intensity functions \( \gamma_\rho \), and let \( r \geq 0 \). For any \( u \in \mathcal{U}_{0,r}(\Omega) \) and any \( \varepsilon' > 0 \), there is a family \( \{w_\gamma \in \mathcal{U}_{0,r}\}_{\gamma>0} \) (dependent on \( \{\sigma_\gamma\}_{\gamma>0} \)) such that \( w_\gamma \rightarrow u \) and, a.a.s. as \( \gamma \rightarrow \infty \),

\[
\#(D(w_\gamma \sqrt{\gamma}) \cap \sigma_\gamma)/\gamma > F_\rho(u) - \varepsilon'.
\]

**Proof.** Choose any \( \varepsilon > 0 \) smaller than \( \varepsilon' \) and apply Lemma 7.7. We will only consider \( \varepsilon \) in the interval \((0, 1/2)\).

For each \( P \in \mathcal{P}_r \), let \( P' \) denote an open parallelogram with the same center as \( P \) but a factor \( 1 - 2\varepsilon \) as wide and high. Let \( \rho_P := \inf_{S_i \cap P'} \rho \) and let \( \bar{u}_P \in \mathcal{U}(P) \) be defined by

\[
\bar{u}_P(x, y) := u(x_P, y_P) + u_x^P(x - x_P) + u_y^P(y - y_P)
\]

if \( P \) is well behaved and \( \bar{u}_P(x, y) := u(x, y) \) otherwise. Let \( \tau_P^{(\gamma)} \) be a Poisson point process on \( P' \setminus S_i \) with homogenous intensity \( \gamma \rho_P \). Then \( (\sigma_\gamma \cap P') \cup \tau_P^{(\gamma)} \) is a superset of a Poisson point process \( \bar{\sigma}_\gamma \) on \( P' \) with homogeneous intensity \( \gamma \rho_P \).

Let \( r_P := \text{diam} \bar{u}_P(P') \). By Lemma 6.3, for any well-behaved \( P \in \mathcal{P}_r \) it holds a.a.s. as \( \gamma \) tends to infinity that there is a \( w_P^{(\gamma)} \in \mathcal{U}_{u(x_P, y_P) - r_P^{(\gamma)} P'}(P') \) such that

\[
\frac{1}{\gamma} \#(D(w_P^{(\gamma)} \sqrt{\gamma}) \cap \bar{\sigma}_\gamma) / \mu(P') \geq L(\rho_P, u_x^P u_y^P) - 3(\rho_P + 1)\varepsilon.
\]\n
Since \( \bar{\sigma}_\gamma \subseteq (\sigma_\gamma \cap P') \cup \tau_P^{(\gamma)} \), we have

\[
\#(D(w_P^{(\gamma)} \sqrt{\gamma}) \cap \sigma_\gamma) \leq \#(D(w_P^{(\gamma)} \sqrt{\gamma}) \cap ((\sigma_\gamma \cap P') \cup \tau_P^{(\gamma)})) \leq \#(D(w_P^{(\gamma)} \sqrt{\gamma}) \cap \sigma_\gamma \cap P') + \#\tau_P^{(\gamma)},
\]

and by the law of large numbers,

\[
\#\tau_P^{(\gamma)} \leq \gamma(\rho_P + 1)\mu(P' \setminus S_i)
\]

a.a.s. as \( \gamma \rightarrow \infty \). Combining Eqs. (18) to (20), we obtain

\[
\left( 1/\gamma \right) \#(D(w_P^{(\gamma)} \sqrt{\gamma}) \cap \sigma_\gamma \cap P') + (\rho_P + 1)\mu(P' \setminus S_i)] / \mu(P') \geq L(\rho_P, u_x^P u_y^P) - 3(\rho_P + 1)\varepsilon.
\]
Figure 8. The situation in the proof of Lemma 10.1. We see three disjoint well-behaved parallelograms and their slightly smaller primed counterparts. The shaded area is initially excluded from the domain of $w_\gamma$.

By Lemma 7.7(e) and the fact that $L$ is uniformly continuous (Lemma 4.9) on the bounded set \( \{(\rho(x, y), u_x(x, y)u_y(x, y)) : (x, y) \in \bigcup_i S_i\} \), we obtain

\[
\frac{1}{\gamma} \#(D(w_\gamma^{(\gamma)}) \cap \sigma_\gamma \cap P') + (p_P + 1)\mu(P' \setminus S_i)) / \mu(P') > L(\rho(x_P, y_P), u_x^{P} u_y^{P}) - o_i(1),
\]

where $o_i(1)$ is independent of $P$. But the inequality in Eq. (21) holds also (even deterministically for any $w_P^{(\gamma)}$) for any $P \in \mathcal{P}_i$ that is not well behaved by the fact that $\rho$, $u_x$ and $u_y$ are bounded on $\bigcup_i S_i$ and $u_x^{P}$ or $u_y^{P}$ is at most $i^3$ if $P$ is not well behaved together with the fact that $L$ is continuous and $L(\rho, 0) = 0$ by Lemma 4.9.

Since $w_P^{(\gamma)} \in \mathcal{U}(u_{P,y_P}) - \frac{r_P}{2}, r_P(P')$, for any well-behaved $P \in \mathcal{P}_i$ we have

\[
\sup_{(x, y) \in P'} |w_P^{(\gamma)}(x, y) - u(x_P, y_P)| \leq r_P / 2 = (1 - 2\epsilon)\epsilon P.
\]

Let $\tilde{\mathcal{P}}_i \subseteq \mathcal{P}_i$ be the set of well-behaved parallelograms in $\mathcal{P}_i$, and let $w_\gamma \in \mathcal{U}(\bigcup_{P \in \tilde{\mathcal{P}}_i} P' \cup (\Omega \setminus \text{int} \cup \tilde{\mathcal{P}}_i))$ be defined by $w_\gamma(x, y) := w_P^{(\gamma)}(x, y)$ if $(x, y) \in P'$ where $P \in \tilde{\mathcal{P}}_i$ and $w_\gamma(x, y) := u(x, y)$ if $(x, y) \in \Omega \setminus \text{int} \cup \tilde{\mathcal{P}}_i$. We claim that $w_\gamma$ is doubly increasing. To show this, it suffices to check the inequality condition for any pair of points $(x_1, y_1)$ and $(x_2, y_2)$ where

- $(x_1, y_1) \leq (x_2, y_2)$ or $(x_1, y_1) \geq (x_2, y_2)$, and
- $(x_1, y_1)$ belongs to $P'$ for some $P \in \tilde{\mathcal{P}}_i$ and $(x_2, y_2)$ lies on the boundary of $P$.

Figure 8 shows an example. We have
(23) \[ s \cdot (\bar{u}P(x_2, y_2) - u(x_P, y_P)) = \iota c_P, \]
where \( s \) is +1 if \( (x_1, y_1) \leq (x_2, y_2) \) and \(-1\) if \( (x_1, y_1) \geq (x_2, y_2) \). Also,
(24) \[ |w_\gamma(x_1, y_1) - u(x_P, y_P)| \leq (1 - 2\iota)c_P \]
by Eq. (22), and
(25) \[ |w_\gamma(x_2, y_2) - \bar{u}P(x_2, y_2)| = |u(x_2, y_2) - \bar{u}P(x_2, y_2)| \leq (|x_2 - x_P| + |y_2 - y_P|)\iota^5 \leq 2\iota^2 c_P, \]
where the first inequality follows from Lemma 7.7(f) and the last inequality follows from Lemma 7.4 together with the fact that \( \iota < 1 \). Combining Eqs. (23) to (25) and the triangle inequality, we obtain
\[ s \cdot (w_\gamma(x_2, y_2) - w_\gamma(x_1, y_1)) \geq \]
\[ s \cdot (\bar{u}P(x_2, y_2) - u(x_P, y_P)) - |w_\gamma(x_1, y_1) - u(x_P, y_P)| - |w_\gamma(x_2, y_2) - \bar{u}P(x_2, y_2)| \geq \iota c_P - (1 - 2\iota) c_P - 2\iota^2 c_P = 0, \]
and we have shown that \( w_\gamma \) is increasing.

For any \( P \in \mathcal{P}_r \), inside \( P' \) we have
\[ |w^{(\gamma)}_P(x, y) - u(x, y)| \leq |w^{(\gamma)}_P(x, y) - u(x_P, y_P)| + |u(x, y) - u(x_P, y_P)| \leq (1 - 2\iota)c_P + (1 + 5\iota)c_P, \]
where the first inequality is the triangle inequality and the second inequality follows from Eq. (22) and Lemma 7.7(h). Since \( c_P < 1 \) by Lemma 7.7(a), it follows that
(26) \[ \sup_{P \in \mathcal{P}_r} \sup_{(x,y) \in P'} |w^{(\gamma)}_P(x, y) - u(x, y)| \to 0 \]
as \( \iota \to 0 \).

By Lemma 9.1, \( w_\gamma \) can be extended to a doubly increasing function on all of \( \Omega \). Let us choose such an extension and denote it by the same symbol \( w_\gamma \). Since \( w_\gamma \) coincides with \( u \) outside the interiors of the well-behaved parallelograms, \( w_\gamma \in \mathcal{U}_{\Omega} \).

We have
\[ \|w_\gamma - u\|_\Omega = \sum_{P \in \mathcal{P}_r} \|w_\gamma - u\|_{P'} + \sum_{P \in \mathcal{P}_r} \|w_\gamma - u\|_{P \setminus P'} \]
\[ \leq \sum_{P \in \mathcal{P}_r} \mu(P') \sup_{(x,y) \in P'} |w^{(\gamma)}_P(x, y) - u(x, y)| + \sum_{P \in \mathcal{P}_r} \mu(P \setminus P') = o_c(1) \]
by Eq. (26). Also, by Eq. (21),
\[ \frac{1}{\gamma} \#(D(w_\gamma \sqrt{\gamma}) \cap \sigma_\gamma) \geq \sum_{P \in \mathcal{P}_r} \frac{1}{\gamma} \#(D(w_\gamma \sqrt{\gamma}) \cap \sigma_\gamma \cap P') \]
\[ \geq -(1 + \sup_{S_\iota} \rho) \mu(\cup \mathcal{P}_r \setminus S_\iota) + \sum_{P \in \mathcal{P}_r} (L(\rho(x_P, y_P), u^P_x u^P_y) - o_c(1)) \mu(P'), \]
By Lemma 7.7, part (d) and (g), this is greater than

\[- o_i(1) + \sum_{P \in \mathcal{P}} \left( \| L(\rho, u_x u_y) \| P \frac{\mu(P')}{\mu(P)} - o_i(1) \mu(P') \right) \geq (1 - 2\varepsilon) \| L(\rho, u_x u_y) \| \mathcal{P} - o_i(1)\]

which is greater than \( \| L(\rho, u_x u_y) \| \Omega - \varepsilon - o_i(1) \), by Lemma 7.7(c) and the fact that \( L(\rho, u_x u_y) \leq \rho \) by Lemma 4.9. Since \( \varepsilon < \varepsilon' \), we are done. \( \square \)

Finally, we are ready to prove our first main theorem.

**Theorem 10.2.** Let \((\Omega, \rho)\) be a density domain and let \(\{\tau_\gamma\}_{\gamma > 0}\) be random point processes on \(\Omega\) approaching a Poisson point process with intensity function \(\gamma \rho\) as \(\gamma \to \infty\). Then the following holds for any \(r \geq 0\).

(a) For any \(\varepsilon > 0\), a.a.s. as \(\gamma \to \infty\), for every maximal \([r \sqrt{\gamma}]\)-decreasing subset \(P\) of \(\tau_\gamma\), there is a \(u \in \mathcal{U}_{0,r}(\Omega)\) with \(F_\rho(u) = F_{\max}(r)\) such that \(\| \kappa_P / \sqrt{\gamma} - u \|_\Omega < \varepsilon\).

(b) Let \(\Lambda(\gamma)\) denote the size of a maximal \([r \sqrt{\gamma}]\)-decreasing subset of \(\tau_\gamma\). Then, \(\Lambda(\gamma) / \gamma \to F_{\max}(r)\) in probability as \(\gamma\) tends to infinity.

**Proof.** Take any \(\varepsilon > 0\) and \(r \geq 0\). Let \(M_\varepsilon\) denote the set of \(u \in \mathcal{U}_{0,r}(\Omega)\) such that \(\| u - u_{\max} : \Omega < \varepsilon\) for some \(u_{\max} \in \mathcal{U}_{0,r}(\Omega)\) with \(F_\rho(u_{\max}) = F_{\max}(r)\). Let \(M_\varepsilon' = \mathcal{U}_{0,r}(\Omega) \setminus M_\varepsilon\) denote the complementary set.

**Claim 1.** \(\sup_{u \in M_\varepsilon'} F_\rho(u) < F_{\max}(r)\).

Suppose there is a sequence \(u_1, u_2, \ldots \in M_\varepsilon'\) with \(F_\rho(u_i) \to F_{\max}(r)\). By Proposition 9.2, \(M_\varepsilon'\) is compact, so there is a convergent subsequence \(u_{i_1}, u_{i_2}, \ldots \to u \in M_\varepsilon'\). Since \(F_\rho\) is upper semicontinuous (Proposition 8.1), \(F_\rho(u) \geq \lim \sup F_\rho(u_{i_j}) = F_{\max}(r)\), which is a contradiction. This proves Claim 1.

By the claim above we can choose an \(\varepsilon' > 0\) such that

\[
\sup_{u \in M_\varepsilon'} F_\rho(u) < F_{\max}(r) - 3\varepsilon'.
\]

Without loss of generality, we assume that \(\varepsilon' < \varepsilon\). Let \(\{\sigma_\gamma\}_{\gamma > 0}\) be Poisson point processes on \(\Omega\) with intensity functions \(\gamma \rho\) such that \(\tau_\gamma\) approaches \(\sigma_\gamma\) as \(\gamma \to \infty\).

**Claim 2.** A.a.s. as \(\gamma \to \infty\), any \(w \in M_\varepsilon'\) satisfies the inequality

\[
\#(D(w \sqrt{\gamma}) \cap \sigma_\gamma) / \gamma < F_{\max}(r) - 2\varepsilon'.
\]

**Claim 3.** A.a.s. as \(\gamma \to \infty\), any \([r \sqrt{\gamma}]\)-decreasing subset \(P\) of \(\tau_\gamma\) such that \(\kappa_P / \sqrt{\gamma} \in M_\varepsilon'\) satisfies the inequality \#\(P\) / \(\gamma < F_{\max}(r) - \varepsilon'\).

**Claim 4.** A.a.s. as \(\gamma \to \infty\), there is a \([r \sqrt{\gamma}]\)-decreasing subset \(P\) of \(\tau_\gamma\) such that \#\(P\) / \(\gamma > F_{\max}(r) - \varepsilon'\).

For each \(u \in \mathcal{U}_{0,r}(\Omega)\), choose \(\delta_u\) according to Lemma 8.2. The open balls \(\{B_{\delta_u}(u) : u \in M_\varepsilon'\}\) cover \(M_\varepsilon'\), which is compact by Proposition 9.2, so there is a finite subcover \(\{B_{\delta_u}(u) : u \in A \subset M_\varepsilon'\}\), \(A\) finite. Now Claim 2 follows from Lemma 8.2 (and the choice of \(\delta_u\)) together with Eq. (27).
To show Claim 3, consider a \([r\sqrt{\gamma}]\)-decreasing subset \(P\) of \(\tau_\gamma\). We have 
\[\#P/\gamma \leq \#(P \cap \sigma_\gamma)/\gamma + \#(\tau_\gamma \setminus \sigma_\gamma)/\gamma,\]
which is smaller than \(\frac{1}{\gamma}\#(P \cap \sigma_\gamma) + \varepsilon'\) a.a.s. as \(\gamma \to \infty\). By Lemma 6.2(b), \(D(\kappa P) = P\), so we obtain the inequality 
\[\#P/\gamma < \frac{1}{\gamma}\#(D(\kappa P) \cap \sigma_\gamma) + \varepsilon'.\]
Now Claim 3 follows from Claim 2 by putting \(w = \kappa P/\sqrt{\gamma}\).

Take \(u \in U_{0,\rho}(\Omega)\) such that \(F_\rho(u) > F_{\text{max}}(r) - \frac{1}{3}\varepsilon'\). By Lemma 10.1, a.a.s. as \(\gamma \to \infty\), there is a \(w_\gamma \in U_{0,\rho}(\Omega)\) such that 
\[\#(D(w_\gamma \sqrt{\gamma}) \cap \sigma_\gamma)/\gamma > F_\rho(u) - \frac{1}{3}\varepsilon' > F_{\text{max}}(r) - \frac{2}{3}\varepsilon'\] and hence 
\[\#(D(w_\gamma \sqrt{\gamma}) \cap \tau_\gamma)/\gamma > F_{\text{max}}(r) - \varepsilon'.\]
Since \(\#(\sigma_\gamma \setminus \tau_\gamma)/\gamma\) tends to zero in probability as \(\gamma \to \infty\). By Lemma 6.2(a), 
\(D(w_\gamma \sqrt{\gamma}) \cap \tau_\gamma\) is a \([r\sqrt{\gamma}]\)-decreasing subset of \(\tau_\gamma\), and Claim 4 follows.

Now (a) follows from Claims 3 and 4.

Claim 5. A.a.s. as \(\gamma \to \infty\), any \(w \in U_{0,\rho}(\Omega)\) satisfies the inequality 
\[\#(D(w \sqrt{\gamma}) \cap \sigma_\gamma)/\gamma < F_{\text{max}}(r) + \varepsilon'.\]

Claim 6. A.a.s. as \(\gamma \to \infty\), any \([r\sqrt{\gamma}]\)-decreasing subset \(P\) of \(\tau_\gamma\) satisfies the inequality 
\[\#P/\gamma < F_{\text{max}}(r) + 2\varepsilon'.\]

There is a finite subcover \(\{B_{\delta_n}(u) : u \in A' \subset U_{0,\rho}(\Omega)\}\) of \(U_{0,\rho}(\Omega)\), so Claim 5 follows from Lemma 8.2.

By Proposition 3.6, for any \([r\sqrt{\gamma}]\)-decreasing set \(P\), we have \(\kappa P/\sqrt{\gamma} \in U_{0,\rho}(\Omega)\). With this in mind, Claim 6 follows from Claim 5 the same way as Claim 3 followed from Claim 2.

Finally, (b) follows from Claims 4 and 6 since \(\varepsilon > 0\) was chosen arbitrarily and \(\varepsilon' < \varepsilon\).

Corollary 10.3. With the same setup as in Theorem 10.2, the Young diagram \(\lambda(\gamma)\) corresponding to \(\tau_\gamma\) approaches the limit shape \(F_{\text{max}}'(r)\) in the sense that, for any \(r > 0\) where the derivative \(F_{\text{max}}'(r)\) exists,
\[\frac{1}{\sqrt{\gamma}}\lambda(\gamma)_{[r\sqrt{\gamma}] + 1} \to F_{\text{max}}'(r)\]
in probability as \(\gamma \to \infty\).

Proof. By Proposition 3.4, the maximal size \(\Lambda(\gamma)\) of a \([r\sqrt{\gamma}]\)-decreasing subset of \(\tau_\gamma\) is \(\Lambda(\gamma) = \sum_{i=1}^{\lfloor r\sqrt{\gamma} \rfloor} \lambda_i(\gamma)\), and by Theorem 10.2(b), \(\Lambda(\gamma)/\gamma \to F_{\text{max}}(r)\) in probability as \(\gamma \to \infty\). Now the corollary follows from Lemma 4.7 with \(a(\lambda) = \sqrt{\gamma}\), \(b(\gamma) = 1/\gamma\) and \(G = F_{\text{max}}\).

Theorem 10.4. Let \(\Omega\) be the open diamond square \(|x| + |y| < 1/\sqrt{2}\) and, for each \(\gamma > 0\), let \(\sigma_\gamma\) be a Poisson point process on \(\Omega\) with intensity \(\gamma\). Then the Young diagram \(\lambda(\gamma)\) corresponding to \(\sigma_\gamma\) approaches the limit shape \(\Phi'\) in the sense that, for any \(r > 0\) where \(\Phi'(r)\) exists,
\[\frac{1}{\sqrt{\gamma}}\lambda(\gamma)_{[r\sqrt{\gamma}] + 1} \to \Phi'(r)\]
in probability as \(\gamma \to \infty\).

Proof. By Proposition 5.7, for any \(r \geq 0\), the maximum value of \(F_1\) on \(U_r(\Omega)\) is \(F_{\text{max}}(r) = \Phi(r)\), so \(F_{\text{max}}(r) = \Phi'(r)\) whenever the derivative exists. Now the theorem follows from Corollary 10.3.

□
11. Concavity and existence of maximizers

Before we are ready to prove Theorem 11.3, we need another property of \( \Phi \) and a technical lemma.

The following proposition is stated in terms of the constant \( \Gamma \) from Theorem 2.1. Of course we know that \( \Gamma = 2 \) from the result of Vershik and Kerov [29], but we do not need that in the proof of Theorem 11.3. As discussed in Section 14, a proof of Conjecture 3.5 would yield a conceptually new proof of the Logan–Shepp–Vershik–Kerov limit shape, so to avoid a circular dependence we take care not to rely on that result.

**Proposition 11.1.** \( \Phi(r) = 1 \) if \( r \geq \Gamma/\sqrt{2} \), where \( \Gamma \) is the constant from Theorem 2.1.

**Proof.** By Proposition 4.8, \( \Phi \) is continuous at \( r = \Gamma/\sqrt{2} \), so it suffices to show that \( \Phi(r) = 1 \) for any \( r > \Gamma/\sqrt{2} \). For any \( \beta > 0 \), consider the density domain \( (\Omega_\beta, 1) \) where \( \Omega_\beta \) is the open rectangle \( |x + y| < 1, |x - y| < \beta \), and for any \( \gamma > 0 \) and \( \beta > 0 \), let \( \sigma_{\gamma, \beta} \) be a Poisson point process on \( \Omega_\beta \) with homogeneous intensity \( \gamma \). Since \( \Omega_\beta \subset \left(-\frac{1+\beta}{2}, \frac{1+\beta}{2}\right)^2 \), by Theorem 2.1, for any \( \varepsilon > 0 \), the size of the largest increasing subset of \( \sigma_{\gamma, \beta} \) is smaller than \( (\Gamma + \varepsilon)(1 + \beta)\sqrt{\gamma} \) a.a.s. as \( \gamma \to \infty \). Thus, by Proposition 3.6, the maximum size of a \( \lfloor (\Gamma + \varepsilon)(1 + \beta)\sqrt{\gamma} \rfloor \)-decreasing subset of \( \sigma_{\gamma, \beta} \) is \( \#(\sigma_{\gamma, \beta}) \) a.a.s. By the law of large numbers, we have \( \#(\sigma_{\gamma, \beta})/\gamma \to 2\beta \) in probability as \( \gamma \to \infty \), so, by Theorem 10.2(b), \( F_{\max}((\Gamma + \varepsilon)(1 + \beta)) = 2\beta \) for any \( \varepsilon > 0 \) and \( \beta > 0 \). On the other hand, by Proposition 5.7, \( F_{\max}((\Gamma + \varepsilon)(1 + \beta)) = 2\beta \Phi((\Gamma + \varepsilon)(1 + \beta)/\sqrt{2}) \). It follows that \( \Phi((\Gamma + \varepsilon)(1 + \beta)/\sqrt{2}) = 1 \). \( \square \)

**Lemma 11.2.** Let \( A \) be a convex subset of a vector space and let \( B \subseteq \mathbb{R} \). Let \( \psi : A \to B \) be a concave function and let \( \phi : B \to \mathbb{R} \) be an increasing concave function. Then \( \phi \circ \psi : A \to \mathbb{R} \) is concave.

**Proof.** Since \( \psi \) is concave, for any \( t \in [0, 1] \) we have
\[
\psi((1 - t)x + ty) \geq (1 - t)\psi(x) + t\psi(y)
\]
and thus, since \( \phi \) is increasing,
\[
\phi(\psi((1 - t)x + ty)) \geq \phi((1 - t)\psi(x) + t\psi(y)),
\]
which, since \( \phi \) is concave, is at least
\[
(1 - t)\phi(\psi(x)) + t\phi(\psi(y)).
\]
Thus, \( \psi \circ \phi \) is concave. \( \square \)

Finally, we have all we need to prove our second main theorem.

**Theorem 11.3.** Let \( (\Omega, \rho) \) be a density domain. Then the following holds.

(a) For any \( r \geq 0 \), \( F_\rho \) attains its maximum on \( \mathcal{U}_r(\Omega) \).
(b) \( F_\rho \) is a concave function on \( \mathcal{U}(\Omega) \).
(c) \( F_{\max} \) is continuous, increasing and concave, and \( F_{\max}(r) \to \|\rho\|_\Omega \) as \( r \to \infty \).

**Proof.** (a) By Proposition 8.1, \( F_\rho \) is upper semicontinuous in \( L^1(\Omega) \), and by Proposition 9.2, \( \mathcal{U}_{0,r}(\Omega) \) is compact, so \( F_\rho \) attains its maximum there.
(b) For any point \((x, y) \in \Omega\), let \(\mathcal{U}_{(x,y)}\) be the set of \(u\) in \(\mathcal{U}(\Omega)\) such that \(u_x(x, y)\) and \(u_y(x, y)\) exist, and define the function \(\chi_{(x,y)} : \mathcal{U}_{(x,y)} \to \mathbb{R}^2\) by letting \(\chi_{(x,y)}(u) = (u_x(x, y), u_y(x, y))\). Clearly, \(\chi_{(x,y)}\) is a linear function.

Let \(\text{supp} \rho = \{(x, y) \in \Omega : \rho(x, y) > 0\}\). For any \((x, y) \in \text{supp} \rho\), define \(\psi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}\) by \(\psi(x, y)(r, s) = \sqrt{2rs/\rho(x, y)}\). We can easily check that \(\psi(x, y)\) is concave; its Hessian determinant is negative semidefinite on \(\mathbb{R}^2_{>0}\). Also, \(\Phi\) is concave and increasing by Proposition 4.8, and hence \(\Phi \circ \psi(x, y)\) is concave by Lemma 11.2.

Since \(\chi_{(x,y)}\) is linear, \(\Phi \circ \psi(x, y) \circ \chi_{(x,y)}\) is concave. By Lemma 4.10(b), for any \(u \in \mathcal{U}(\Omega)\), \(\chi_{(x,y)}\) is defined at \(u\) for almost every \((x, y) \in \Omega\), so the integral
\[
\int_{\text{supp} \rho} \rho(x, y)(\Phi \circ \psi(x, y) \circ \chi_{(x,y)})(u) \, d\mu
\]
is well defined and concave as a function of \(u\).

(c) That \(F_{\max}\) is increasing follows from the fact that \(\mathcal{U}_{r_1}(\Omega) \subseteq \mathcal{U}_{r_2}(\Omega)\) whenever \(r_1 \leq r_2\). To show that \(F_{\max}\) is concave, we must check that \(F_{\max}((1-t)r_1 + tr_2) \geq (1-t)F_{\max}(r_1) + tF_{\max}(r_2)\) for any \(r_1, r_2 \geq 0\) and any \(t \in (0,1)\). By (a), there are \(u^{(1)} \in \mathcal{U}_{0,r_1}\) and \(u^{(2)} \in \mathcal{U}_{0,r_2}\) such that \(F_\rho(u^{(1)}) = F_{\max}(r_1)\) and \(F_\rho(u^{(2)}) = F_{\max}(r_2)\). Let \(u := (1-t)u^{(1)} + tu^{(2)}\). Clearly, \(u \in \mathcal{U}_{0,(1-t)r_1+tr_2}\). By (b), \(F_\rho\) is concave, so
\[
F_{\max}((1-t)r_1 + tr_2) \geq F_{\rho}(u) = F_\rho((1-t)u^{(1)} + tu^{(2)}) \\
\geq (1-t)F_\rho(u^{(1)}) + tF_\rho(u^{(2)}) = (1-t)F_{\max}(r_1) + tF_{\max}(r_2).
\]
This shows that \(F_{\max}\) is concave.

A concave function is automatically continuous on any open interval, so to show that \(F_{\max}\) is continuous we need only to show that it is continuous at zero. Choose any \(\varepsilon > 0\). Since \(\int_\Omega \rho \, d\mu\) is finite, there is a measurable subset \(S\) of \(\Omega\) and positive constants \(\rho_0, \rho_1\) such that \(\int_S \rho \, d\mu < \varepsilon\) and \(\rho_0 < \rho < \rho_1\) on \(S\). There is also a \(c > 0\) such that \(\int_{\Omega \setminus [-c,c]^2} \rho \, d\mu < \varepsilon\). Choose any \(r > 0\) and any \(u \in \mathcal{U}_r(\Omega)\). By Lemma 9.1, there is a \(\mathcal{U} \in \mathcal{U}_r(\mathbb{R}^2)\) that coincides with \(u\) on \(\Omega\), and by Tonelli’s theorem,
\[
\int_{\Omega \setminus [-c,c]^2} u_x \, d\mu \leq \int_{[-c,c]^2} w_x \, d\mu = \int_{-c}^c \left( \int_{-c}^c w_x(x, y) \, dx \right) \, dy.
\]
Now, by Lebesgue’s theorem for increasing functions in one dimension,
\[
\int_{-c}^c w_x(x, y) \, dx \leq w(c, y) - w(-c, y),
\]
which is at most \(r\) since \(w \in \mathcal{U}_r(\mathbb{R}^2)\). Thus, \(\int_{\Omega \setminus [-c,c]^2} u_x \, d\mu \leq \int_{-c}^c r \, dy = 2cr\) and, analogously, \(\int_{\Omega \setminus [-c,c]^2} u_y \, d\mu \leq 2cr\). Choose any \(\delta > 0\) and let \(T\) be the subset of \(S \cap [-c,c]^2\) where \(\sqrt{2u_xu_y} \geq \delta\). Since \(\rho > \rho_0\) on \(S\), on \(T\) we have \(u_xu_y \geq \delta^2 \rho/2 > \delta^2 \rho_0/2\), so \(T \subseteq T_1 \cup T_2\), where \(T_1\) is the subset of \(S \cap [-c,c]^2\) where \(u_x \geq \delta \sqrt{\rho_0}/2\) and \(T_2\) is the subset of \(S \cap [-c,c]^2\) where \(u_y \geq \delta \sqrt{\rho_0}/2\). By Markov’s inequality,
\[
\mu(T_1) \leq \frac{1}{\delta \sqrt{\rho_0}/2} \int_{S \cap [-c,c]^2} u_x \, d\mu \leq \frac{2cr}{\delta \sqrt{\rho_0}/2},
\]
and analogously for $T_2$, so $\mu(T) \leq \mu(T_1) + \mu(T_2) \leq 4cr/\delta \sqrt{\rho_0/2}$.
Combining all above, bearing in mind that $L(\rho, u_xu_y) \leq \rho$ by Lemma 4.9 and that $\Phi$ is increasing by Proposition 4.8, we obtain
\[
F_\rho(u) = \int_{\Omega} L(\rho, u_xu_y) \, d\mu
\leq \int_{\Omega \setminus S} \rho \, d\mu + \int_{\Omega \setminus [-c,c]^2} \rho \, d\mu + \int_{T \setminus (S \cap [-c,c]^2) \setminus T} \rho \, d\mu + \int_{L(\rho, u_xu_y) < \rho} \rho \Phi(\sqrt{2u_xu_y}/\rho) \, d\mu
< 2\varepsilon + \mu(T)\rho_1 + \mu((S \cap [-c,c]^2) \setminus T)\rho_1\Phi(\delta)
\leq 2\varepsilon + \frac{4cr\rho_1}{\delta \sqrt{\rho_0/2}} + \mu(\Omega)\rho_1\Phi(\delta).
\]
Since $\varepsilon$, $r$ and $\delta$ were chosen freely above, we have shown that
\[
F_{\max}(r) \leq 2\varepsilon + \frac{4cr\rho_1}{\delta \sqrt{\rho_0/2}} + \mu(\Omega)\rho_1\Phi(\delta)
\]
for any $\varepsilon, r, \delta > 0$. Here, $c$, $\rho_0$ and $\rho_1$ depend on $\varepsilon$ but not on $r$ or $\delta$. By Proposition 4.8, $\Phi$ is continuous, so for any $\varepsilon > 0$ we can choose $\delta > 0$ such that $\mu(\Omega)\rho_1\Phi(\delta) < \varepsilon$. After that, we can choose $r > 0$ such that $4cr\rho_1/(\delta \sqrt{\rho_0/2}) < \varepsilon$. It follows that $F_{\max}(r) < 4\varepsilon$ and since $F_{\max}$ is an increasing function, we conclude that $\lim_{r \to 0^+} F_{\max}(r) = 0$. Hence, $F_{\max}$ is continuous.

To show that $F_{\max}(r) \to \|\rho\|_\Omega$ as $r \to \infty$, let us again choose any $\varepsilon > 0$ and take $S$, $\rho_0$, $\rho_1$ and $c$ as above. Choose any $r \geq 0$ and let $u = u_r(S \cap [-c,c]^2)$ be defined by $u(x,y) := r(x + y)/4c$. By Lemma 9.1, $u$ can be extended to $\Omega$ without increasing the diameter of its image, so from now on we consider $u$ to be an element of $\mathcal{U}_r(\Omega)$. Since $u_x = u_y = r/4c$ almost everywhere on $S \cap [-c,c]^2$, again bearing in mind that $L(\rho, u_xu_y) \leq \rho$ and $\Phi$ is increasing we obtain
\[
0 \leq \|\rho\|_\Omega - F_\rho(u) = \int_{\Omega} \left(\rho - L(\rho, u_xu_y)\right) \, d\mu
\leq \int_{\Omega \setminus S} \rho \, d\mu + \int_{\Omega \setminus [-c,c]^2} \rho \, d\mu + \int_{S \cap [-c,c]^2} \rho \left(1 - \Phi(\sqrt{2u_xu_y}/\rho)\right) \, d\mu
< 2\varepsilon + \mu(S \cap [-c,c]^2)\rho_1(1 - \Phi(2/\rho_1 \cdot r/4c))
\leq 2\varepsilon + 4\mu(\Omega)\rho_1(1 - \Phi(\sqrt{2/\rho_1} \cdot r/4c)).
\]
Since $\varepsilon$ and $r$ were chosen freely above, we have shown that, for any $\varepsilon > 0$ and $r \geq 0$, there is a $u \in \mathcal{U}_r(\Omega)$ such that
\[
0 \leq \|\rho\|_\Omega - F_\rho(u) \leq 2\varepsilon + 4\mu(\Omega)\rho_1(1 - \Phi(\sqrt{2/\rho_1} \cdot r/4c)).
\]
Here, $c$ and $\rho_1$ depend on $\varepsilon$ but not on $r$. By Proposition 11.1, the last term tends to zero as $r$ tends to infinity, and we conclude that $\|\rho\|_\Omega - F_{\max}(r) \to 0$ as $r \to \infty$.

12. Essentially unique maximizers

In this section we show that, under reasonable assumptions, the maximizer of the functional $F_\rho$ is essentially unique.
Definition 12.1. Let \( \psi \) be a real-valued function on a convex subset \( A \) of a vector space and let \( C \) be a subset of \( A \). Then, \( \psi \) is said to be strictly concave from \( C \) if, for any \( x \in C \) and \( y \in A \) with \( x \neq y \), and any \( t \in (0,1) \), we have \( \psi((1-t)x + ty) > (1-t)\psi(x) + t\psi(y) \).

Definition 12.2. Let \( \phi \) be a function from some \( B \subseteq \mathbb{R} \) to \( \mathbb{R} \), and let \( C \) be a subset of \( B \). We say that \( \phi \) is strictly increasing from \( C \) if \( \phi(x) < \phi(y) \) for any \( C \ni x < y \in B \).

Lemma 12.3. Let \( A \) be a convex subset of a vector space and let \( B \subseteq \mathbb{R} \). Let \( \psi : A \to B \) be a concave function and let \( \phi : B \to \mathbb{R} \) be an increasing function. Suppose \( \psi \) is strictly concave on the line through any two distinct points \( x \neq y \) with \( \psi(x) = \psi(y) \). Suppose also that \( \phi \) is strictly concave from some subset \( C \) of \( B \) and strictly increasing from \( C \). Then \( \phi \circ \psi : A \to \mathbb{R} \) is strictly concave from \( \psi^{-1}(C) \).

Proof. Take any \( x \in \psi^{-1}(C) \) and \( y \in A \) with \( x \neq y \), and take any \( t \in (0,1) \). We must show that
\[
(\phi \circ \psi)((1-t)x + ty) > (1-t)(\phi \circ \psi)(x) + t(\phi \circ \psi)(y).
\]

Since \( \psi \) is concave, we have
\[
\psi((1-t)x + ty) \geq (1-t)\psi(x) + t\psi(y)
\]
and thus, since \( \phi \) is increasing,
\[
\phi(\psi((1-t)x + ty)) \geq \phi((1-t)\psi(x) + t\psi(y)).
\]
Since \( \phi \) is strictly concave from \( C \), we have
\[
\phi((1-t)\psi(x) + t\psi(y)) \geq (1-t)\phi(\psi(x)) + t\phi(\psi(y)).
\]
with equality only if \( \psi(x) = \psi(y) \). If \( \psi(x) = \psi(y) \), by the assumptions in the lemma, \( \psi \) is strictly concave on the line through \( x \) and \( y \) and hence the inequality in Eq. (28) holds strictly. Since \( \phi \) is strictly increasing from \( C \), this implies that the inequality in Eq. (29) holds strictly too.

Thus, \( \psi \circ \phi \) is strictly concave from \( \psi^{-1}(C) \).

Proposition 12.4. Suppose \( \Phi \) is strictly concave on \( [0,\sqrt{2}] \). Then, if \( u^{(1)} \) and \( u^{(2)} \) are two maximizers of the operator \( F_\rho \) in \( \mathbb{U}_r(\Omega) \), the two sets
\[
\{(x,y) : 0 < u_x^{(1)}(x,y)u_y^{(1)}(x,y) < \rho(x,y)\}
\]
and
\[
\{(x,y) : 0 < u_x^{(2)}(x,y)u_y^{(2)}(x,y) < \rho(x,y)\}
\]
are almost equal, and the partial derivatives of \( u^{(1)} \) and \( u^{(2)} \) agree almost everywhere on that set.

Proof. For \( i = 1,2 \), let \( D_i \subseteq \Omega \) be the set of points \( (x,y) \) where \( 0 < u_x^{(i)}u_y^{(i)} < \rho \). Let \( A := \mathbb{R}^2_{>0} \), \( B := \mathbb{R}_{\geq 0} \) and \( C := [0,\sqrt{2}] \). For any \( (x,y) \in \supp \rho \), let \( \chi(x,y) \) and \( \psi(x,y) \) be defined as in the proof of Theorem 11.3(b). It follows from Proposition 4.8 together with our assumption that \( \Phi \) is strictly concave on \( [0,\sqrt{2}] \) that the assumptions of Lemma 12.3 are satisfied for \( \psi_{(x,y)} \), \( \Phi \), \( A \), \( B \) and \( C \), so \( \Phi \circ \psi_{(x,y)} \) is strictly concave from \( \psi_{(x,y)}^{-1}(C) = \{(r,s) \in \mathbb{R}^2_{>0} : rs < \rho(x,y)\} \).
Let $D$ be the set of points in $D_1 \cup D_2$ where $(u_x^{(1)}, u_y^{(1)}) \neq (u_x^{(2)}, u_y^{(2)})$. We claim that $\mu(D) = 0$.

Let $w := (u^{(1)} + u^{(2)})/2$. If $(x, y) \in D$, the points $p := \chi(x,y)(u^{(1)})$ and $q := \chi(x,y)(u^{(2)})$ are distinct and at least one of them belongs to $\psi^{-1}_y(C)$. Hence, with $L_{(x,y)}$ as a shorthand for $\rho(x,y) \cdot (\Phi \circ \psi_{(x,y)} \circ \chi_{(x,y)})$, we obtain

$$L_{(x,y)}(w) = \rho(x,y)(\Phi \circ \psi_{(x,y)}((p + q)/2)$$

$$> \rho(x,y)[(\Phi \circ \psi_{(x,y)})(p) + (\Phi \circ \psi_{(x,y)})(q)]/2 = [L_{(x,y)}(u^{(1)}) + L_{(x,y)}(u^{(2)})]/2.$$ 

Suppose $\mu(D) > 0$. Then,

$$\int_D L_{(x,y)}(w) \, d\mu > \frac{1}{2} \left( \int_D L_{(x,y)}(u^{(1)}) \, d\mu + \int_D L_{(x,y)}(u^{(2)}) \, d\mu \right).$$

Also, by Theorem 11.3(b),

$$\int_{(\text{supp } \rho) \setminus D} L_{(x,y)}(w) \, d\mu \geq \frac{1}{2} \left( \int_{(\text{supp } \rho) \setminus D} L_{(x,y)}(u^{(1)}) \, d\mu + \int_{(\text{supp } \rho) \setminus D} L_{(x,y)}(u^{(2)}) \, d\mu \right),$$

so it follows that

$$\int_{\text{supp } \rho} L_{(x,y)}(w) \, d\mu > \frac{1}{2} \left( \int_{\text{supp } \rho} L_{(x,y)}(u^{(1)}) \, d\mu + \int_{\text{supp } \rho} L_{(x,y)}(u^{(2)}) \, d\mu \right).$$

This means that $F_\rho(w) > [F_\rho(u^{(1)}) + F_\rho(u^{(2)})]/2$, which is impossible since $u^{(1)}$ and $u^{(2)}$ both are maximizers of $F_\rho$. We conclude that $\mu(D) = 0$. \qed

Recall the definition of $V$ from Section 5.

**Lemma 12.5.** Let $\Omega$ be an open subset of $\mathbb{R}^2$, and let $u \in \mathcal{U}_{-c,2c}(\Omega)$ and $v \in V_{-d,2d}(\Omega)$ for some $c, d > 0$. Suppose that $u$ and $v$ are everywhere differentiable with nonzero partial derivatives and that the image of $\Omega$ under the map $\varphi_u : (x,y) \mapsto (u(x,y), v(x,y))$ is $(-c, c) \times (-d, d)$. Then, for any $w \in \mathcal{U}_{-c,2c}(\Omega)$ whose partial derivatives coincide with those of $u$ almost everywhere, it holds that $w = u$ everywhere.

**Proof.** Take any $w \in \mathcal{U}_{-c,2c}(\Omega)$ whose partial derivatives coincide with those of $u$ almost everywhere. Let $(x_0, y_0)$ be any point in $\Omega$ and let $u_0 := u(x_0, y_0)$, $v_0 := v(x_0, y_0)$ and $w_0 = w(x_0, y_0)$. We need to show that $w_0 = u_0$, but by symmetry it is enough to show that $w_0 \geq u_0$, so this will be our goal.

Since $u$ has positive partial derivatives, for any sufficiently small $\delta > 0$ there is an $x_1 < x_0$ and an $y_1 < y_0$ such that $u(x_1, y_0) = u(x_0, y_1) = u_0 - \delta$. Since $v_x < 0$ and $v_y > 0$ everywhere, we have $v_\ldots := v(x_0, y_1) < v(x_0, y_0) < v(x_1, y_0) =: v_+$. Let $S := (-c, u_0 - \delta) \times (v_-, v_+)$ and $T := \varphi_u^{-1}(S)$. We claim that every point in $T$ is south-west of $(x_0, y_0)$. To see this, first note that $v \leq v_-$ for any point south-east of $(x_0, y_1)$ and $v \geq v_+$ for any point north-west of $(x_1, y_0)$. Then note that $u \geq u_0 - \delta$ for any point north-east.

\[To be precise: Suppose $\Omega$, $u$, $v$ and $w$ satisfy the assumptions in the lemma, and define $\Omega' := \{(-x, -y) : (x, y) \in \Omega\}$, $u'(x,y) = -u(-x,-y)$, $v'(x,y) = -v(-x,-y)$ and $w'(x,y) = -w(-x,-y)$. Then, $\Omega'$, $u'$, $v'$ and $w'$ satisfy the assumptions in the lemma too, and $w(x,y) \leq u(x,y)$ if and only if $w'(x,y) \geq u'(x,y)$.\]
of \((x_0, y_1)\) or \((x_1, y_0)\). Thus, every point in \(T\) is south of \((x_1, y_0)\) and west of \((x_0, y_1)\), and we have proved the claim. Figure 9 illustrates the situation.

Let \(J := u_xv_y - u_yv_x\) be the Jacobian determinant of \(\varphi_u\). By Theorem 5.4,
\[
\int_T J d\mu \geq \mu(\varphi_u(T)) = \mu(S),
\]
where the last equality follows from the surjectivity of \(\varphi_u\) onto the set \((-c, c) \times (-d, d)\). Let \(T'\) be the subset of \(T\) where \(w\) is differentiable and the partial derivatives of \(w\) and \(u\) coincide, and define the function \(\varphi_w : T' \to [-c, c] \times [-d, d]\) by \(\varphi_w(x, y) = (w(x, y), v(x, y))\). Then \(\varphi_w\) is injective by Lemma 5.3, and by Theorem 5.4,
\[
\int_{T'} J d\mu = \mu(\varphi_w(T')).
\]
Since \(\mu(T \setminus T') = 0\), the two integrals above are equal, so
\[
\mu(\varphi_w(T')) \geq \mu(S).
\]
Recall that any point in \(T\) is south-west of \((x_0, y_0)\), so \(w(x, y) \leq w_0\) for any point \((x, y)\) in \(T\). It follows that
\[
\varphi_w(T') \subseteq [-c, w_0] \times (v_-, v_+),
\]
so \(\mu(\varphi_w(T')) \leq (w_0 + c)(v_+ - v_-)\). On the other hand, \(\mu(S) = (u_0 - \delta + c)(v_+ - v_-)\), so Eq. (30) yields that \(w_0 \geq u_0 - \delta\). Since this holds for arbitrarily small positive \(\delta\), we conclude that \(w_0 \geq u_0\).

**Lemma 12.6.** Let \(\Omega\) be the open rhombus \(a|x| + b|y| < 1\) and let \(c\) be a positive number. Let \(u \in \mathcal{U}_{-c,2c}\) and suppose that \(u_x = ca\) and \(u_y = cb\) almost everywhere in \(\Omega\). Then \(u(x, y) = c(ax + by)\) everywhere on \(\Omega\).

**Proof.** This follows from Lemma 12.5 with \(d = 1\) and \(v(x, y) = by - ax\).

Our final result in this section relies on the fact due to Vershik and Kerov [29] that the constant \(\Gamma\) in Theorem 2.1 equals 2.

**Proposition 12.7.** Suppose \(\Phi\) is strictly concave on \([0, \sqrt{2}]\). Let \(\Omega\) be the open rhombus \(a|x| + b|y| < 1\) and \(\rho > 0\) be constant, and let \(c\) be any positive number smaller than or equal to \(\sqrt{\rho/ab}\). Then, in \(\mathcal{U}_{-c,2c}(\Omega)\) the functional \(F_\rho\) is uniquely maximized by the function \(v_{\text{linear}}(x, y) = c(ax + by)\).
Proof. By Proposition 5.7, \( u_{\text{linear}} \) is a maximizer of \( F_\rho \). If \( c < \sqrt{\rho/a b} \), the uniqueness follows from Proposition 12.4 together with Lemma 12.6. Suppose \( c = \sqrt{\rho/a b} \) and suppose there is another maximizer \( u \in \mathcal{U}_{-c,2c} \). By scale invariance we can assume without loss of generality that \( a = b = \rho = 1 \). We have \( F_1(u_{\text{linear}}) = \mu(\Omega)\Phi(\sqrt{2}) \) which equals \( \mu(\Omega) \) by Proposition 11.1 together with the result from [29] that \( \Gamma = 2 \), so we must have \( \Phi(\sqrt{2}u_x u_y) = 1 \) and hence \( u_x u_y \geq 1 \) almost everywhere. It follows that

\[
\int_\Omega \sqrt{u_x u_y} \, d\mu \geq 2.
\]

On the other hand, by the inequality of the geometric and arithmetic mean,

\[
\int_\Omega \sqrt{u_x u_y} \, d\mu \leq \frac{1}{2} \int_\Omega (u_x + u_y) \, d\mu
\]

with equality if and only if \( u_x = u_y \) almost everywhere in \( \Omega \). Define a map \( \zeta : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( \zeta(\alpha, \beta) = (\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}) \). Then, \( \Omega = \zeta((-1,1)^2) \) and

\[
\frac{1}{2} \int_\Omega (u_x + u_y) \, d\mu = \frac{1}{4} \int_{(-1,1)^2} ((u_x + u_y) \circ \zeta) \, d\mu = \frac{1}{2} \int_{(-1,1)^2} \frac{\partial}{\partial \alpha} (u \circ \zeta) \, d\mu
\]

\[
= \{\text{Tonelli's theorem}\} = \frac{1}{2} \int_{-1}^{1} \left( \int_{-1}^{1} \frac{\partial}{\partial \alpha} (u \circ \zeta) \, d\alpha \right) \, d\beta
\]

\[
\leq \{\text{since } u \circ \zeta \text{ is increasing in the first variable}\} \leq \frac{1}{2} \int_{-1}^{1} 2c \, d\beta = 2c = 2.
\]

Combining Eqs. (31) to (33), we see that \( u_x = u_y = 1 \) must hold almost everywhere in \( \Omega \). By Lemma 12.6, \( u = u_{\text{linear}} \) almost everywhere. \( \square \)

13. THE UNIFORM CASE

In this section we suppose Conjecture 3.5 holds true and explore the consequences for the case of uniformly random permutations. The limit surfaces turn out to have a surprisingly simple parameterization in terms of trigonometric functions, and we are able to recover the Logan–Shepp–Vershik–Kerov limit-shape result mentioned in Section 2 and depicted in Fig. 11. Level plots of some limit surfaces are shown in Fig. 10.

Theorem 13.1. Suppose Conjecture 3.5 holds. Let \( \Omega \) be the open square \( -\frac{1}{2} < x, y < \frac{1}{2} \), let \( \rho = 1 \) and take any \( 0 < r < 2 \). Then, there is a maximizer \( u \) to \( F_\rho \) in \( \mathcal{U}_{-r/2,r}(\Omega) \) given by the following.

Let \( 0 < \alpha < \pi \) be defined by \( r = \frac{2}{\pi} (\sin \alpha - \alpha \cos \alpha) \), and define \( \phi \) and \( \psi \) by

\[
x = \frac{1}{\pi} (\phi + \frac{1}{2} \sin 2\phi), \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2},
\]

\[
y = \frac{1}{\pi} (\psi + \frac{1}{2} \sin 2\psi), \quad -\frac{\pi}{2} < \psi < \frac{\pi}{2}.
\]

Then

\[
u = \begin{cases}
\frac{1}{\pi} (\sin(\psi + \phi) - (\psi + \phi) \cos \alpha) & \text{if } |\psi + \phi| \leq \alpha, \\
-r/2 & \text{if } |\psi + \phi| < -\alpha, \\
r/2 & \text{if } |\psi + \phi| > \alpha.
\end{cases}
\]

Furthermore, every maximizer to \( F_\rho \) in \( \mathcal{U}_{-r/2,r}(\Omega) \) coincides with \( u \) in the region where \( |\psi - \phi| \leq \pi - \alpha \).
Figure 10. (a)–(c) show 11 evenly distributed level curves (solid) of $u$ in Theorem 13.1 for three different values of $\alpha$ (corresponding to the marked points in Fig. 11). The lowest and highest level curves correspond to $\psi + \phi = \pm \alpha$, and outside of these curves $u$ has a constant value of $\pm r/2$. In the region between the dashed curves $\psi - \phi = \pm(\pi - \alpha)$, any maximizer of $F_\rho$ coincides with $u$. Finally, (d) shows curves of the form $\psi + \phi = \text{const}$ (solid) and $\psi - \phi = \text{const}$ (dashed). Those are the possible level curves of $u$ and $v$, respectively, for any $0 < \alpha < \pi$.

Proof. Recall the definition of $\mathcal{V}$ from Section 5. Let $s = r + 2\cos \alpha$ and define $v \in \mathcal{V}(\Omega)$ by

$$v := \begin{cases} \frac{1}{\pi} (\sin(\psi - \phi) + (\psi - \phi) \cos \alpha) & \text{if } |\psi - \phi| \leq \pi - \alpha, \\ -s/2 & \text{if } \psi - \phi < -(\pi - \alpha), \\ s/2 & \text{if } \psi - \phi > \pi - \alpha. \end{cases}$$
We claim that \( u \) and \( v \) satisfy the assumption of Theorem 5.5. When \( \psi + \phi \) spans over \((-\alpha, \alpha)\), \( u \) spans over \((-r/2, r/2)\). Analogously, when \( \psi - \phi \) spans over \((-\pi - \alpha, \pi - \alpha)\), \( v \) spans over \((-s/2, s/2)\). From this, property (a) in Theorem 5.5 follows.

In order to verify property (b), by Proposition 5.6 it is enough to check that the equations

\[
\begin{align*}
(34) & \quad u_x v_y + u_y v_x = 0, \\
(35) & \quad \min\{\sqrt{u_x u_y / \rho}, 1\} + \min\{-\sqrt{v_x v_y / \rho}, 1\} = 1
\end{align*}
\]

hold almost everywhere in \( \Omega \). To this end, first we make the following simple calculations.

\[
\begin{align*}
& u_x = \frac{\partial u}{\partial \phi} = \frac{\cos(\psi + \phi) - \cos \alpha}{2 \cos^2 \phi} & \text{if } |\psi + \phi| \leq \alpha, \\
& u_y = \frac{\partial u}{\partial \psi} = \frac{\cos(\psi + \phi) - \cos \alpha}{2 \cos^2 \psi} & \text{if } |\psi + \phi| \leq \alpha, \\
& v_x = \frac{\partial v}{\partial \phi} = -\frac{\cos(\psi - \phi) + \cos \alpha}{2 \cos^2 \phi} & \text{if } |\psi - \phi| \leq \pi - \alpha, \\
& v_y = \frac{\partial v}{\partial \psi} = \frac{\cos(\psi - \phi) + \cos \alpha}{2 \cos^2 \psi} & \text{if } |\psi - \phi| \leq \pi - \alpha.
\end{align*}
\]

Note that

\[
(36) \quad |\psi + \phi| + |\psi - \phi| < \pi
\]

in \( \Omega \). 

Consider a point in \( \Omega \) where \( |\psi + \phi| \geq \alpha \). Then \( u_x = u_y = 0 \), so Eq. (34) holds there. Also, from Eq. (36) it follows that \( |\psi - \phi| < \pi - \alpha \) and hence

\[
\sqrt{-v_x v_y} = \frac{\cos(\psi - \phi) + \cos \alpha}{2 \cos \phi \cos \psi} = \frac{\cos(\psi - \phi) + \cos \alpha}{\cos(\psi - \phi) + \cos(\psi + \phi)} \geq 1,
\]

so Eq. (35) holds too.

Now consider a point where \( |\psi - \phi| \geq \pi - \alpha \). Then \( v_x = v_y = 0 \), so Eq. (34) holds there. Also, from Eq. (36) it follows that \( |\psi + \phi| < \alpha \) and hence

\[
\sqrt{u_x u_y} = \frac{\cos(\psi + \phi) - \cos \alpha}{2 \cos \phi \cos \psi} = \frac{\cos(\psi + \phi) - \cos \alpha}{\cos(\psi + \phi) + \cos(\psi + \phi)} \geq 1,
\]

so Eq. (35) holds too.

Finally, consider a point where \( |\psi + \phi| < \alpha \) and \( |\psi - \phi| < \pi - \alpha \). Then Eq. (34) and Eq. (35) both follow from our expressions for the partial derivatives.

We have shown that \( u \) is a maximizer of \( F_p \). It remains to be shown that every maximizer of \( F_p \) in \( U_{-r/2,r}(\Omega) \) coincides with \( u \) within the region where \( |\psi - \phi| \leq \pi - \alpha \). Let \( w \in U_{-r/2,r}(\Omega) \) be a maximizer of \( F_p \). Let \( R \) be the subregion of \( \Omega \) where \( |\psi + \phi| < \alpha \) and \( |\psi - \phi| < \pi - \alpha \). Since \( \sqrt{u_x u_y} \) and \( \sqrt{-v_x v_y} \) are both positive in \( R \), it follows from Eq. (35) that they are both smaller than one there. Then, by Proposition 12.4, the partial derivatives of \( u \) and \( w \) coincide almost everywhere in \( R \), and by Lemma 12.5, \( u \) and \( w \) coincide everywhere in \( R \). For any point \( p \) in \( \Omega \) where \( \psi + \phi \leq -\alpha \), north-east of \( p \) there are points in \( R \) with \( w \)-values arbitrarily close to \(-r/2\). Analogously, for any point \( p \) in \( \Omega \) where \( \psi + \phi \geq \alpha \), south-west of \( p \) there are points in \( R \) with \( w \)-values arbitrarily close to \( r/2 \). It follows that \( w \) coincides with \( u \) also in the region \( |\psi + \phi| \geq \alpha \). Finally, for any point \( p \) in \( \Omega \) where \( |\psi - \phi| = \pi - \alpha \), both south-west and north-east of \( p \) there are points
in $R$ with $w$-values arbitrarily close to $u(p)$, so $w$ coincides with $u$ at $p$ as well.

13.1. **Recovering the limit shape of Logan, Shepp and Vershik, Kerov.** As a consequence of Theorem 13.1, under the assumption that Conjecture 3.5 holds we are able to recover the celebrated result of Logan, Shepp [19] and Vershik, Kerov [29] on the limit shape of the Young diagram associated with a uniformly random permutation under the Robinson–Schensted correspondence. In the proof of Theorem 13.1 we showed that $u$ and $v$ satisfy the assumption of Theorem 5.5. One consequence of that theorem is that $s = s_{\text{max}}(r)$ whenever $s_{\text{max}}(r)$ exists. By Corollary 10.3, it follows that the limit shape in the $r$-$s$ plane is parameterized by

$$
\begin{align*}
    r &= \frac{2}{\pi} (\sin \alpha - \alpha \cos \alpha), \\
    s &= \frac{2}{\pi} (\sin \alpha - \alpha \cos \alpha) + 2 \cos \alpha,
\end{align*}
$$

where $0 < \alpha < \pi$; see Fig. 11 for an illustration. By the substitution $\alpha = \theta + \frac{\pi}{2}$, this is equivalent to the parameterization of the Logan–Shepp–Vershik–Kerov limit shape given in [24, Section 1.20], namely

$$
\begin{align*}
    r &= \left( \frac{2\theta}{\pi} + 1 \right) \sin \theta + \frac{2}{\pi} \cos \theta, \\
    s &= \left( \frac{2\theta}{\pi} - 1 \right) \sin \theta + \frac{2}{\pi} \cos \theta,
\end{align*}
$$

where $-\pi/2 < \theta < \pi/2$.

13.2. **Comparison with the limiting surface of a random square tableau.** Pittel and Romik [23], and more recently Maślanka and Śniady [21], studied uniform random standard Young tableaux on an $n \times n$ square shape. After rescaling the shape to a unit square in the $x$-$y$ plane and

![Figure 11. The Logan–Shepp–Vershik–Kerov limit shape of a Young diagram drawn from the Plancherel distribution. We have marked points with three specific $\alpha$-values in the parameterization given by Eq. (37). Their corresponding limit surfaces are depicted in Fig. 10.](image-url)
interpreting the entries as \( z \)-coordinates, the tableau gives rise to a two-dimensional surface. Pittel and Romik found the limit of this surface as \( n \) tends to infinity, and its level curves, shown in [23, Fig. 1(d)] and [21, Fig. 13], look very similar to the level curves of \( u \) in Theorem 13.1, shown in Fig. 10. As can be seen in Fig. 12, however, these families of curves are not the same.

14. Future research

The present work has generated plenty of open questions. The most significant one is Conjecture 3.5, of course, and since the triangular shape is arguably the simplest of all possible limit shapes, intuitively the phenomenon should have a simple explanation, even though it has evaded the author so far. As we have seen, a proof of Conjecture 3.5 would yield, as a by-product, a new proof of the Logan–Shepp–Vershik–Kerov limit shape in Fig. 11, a proof very different from the known proofs. Logan, Shepp [19] and Vershik, Kerov [29] found the limit shape independently of each other, but both proofs were based on the hook-length formula, an almost magically simple formula for computing the number of standard Young tableaux of a specific shape (see e.g. [27]).

Another category of open questions concerns the regularity of the maximizers of the functional \( F^\rho \). Is there always a continuous maximizer, or even a differentiable one? Under what conditions can we find a maximizer that satisfies the PDE system of Theorem 5.5? In the uniform case, provided Conjecture 3.5 holds true, the maximizers \( u \) of the form given by Theorem 13.1 have the following property: If \( u_1 \) and \( u_2 \) are such maximizers associated to \( r = r_1 \) and \( r = r_2 \), respectively, where \( r_1 \leq r_2 \), then every level curve of \( u_1 \) is also a level curve of \( u_2 \). Is this true in general?
Our definitions of increasing and decreasing sets and density domains can be generalized to higher dimension. For instance, we might say that a finite set of points in $\mathbb{R}^3$ is increasing if the equivalences $x < x' \iff y < y' \iff z < z'$ hold for any pair of points $(x, y, z)$ and $(x', y', z')$ in the set. Can our results be generalized to this setting?

Finally, we offer another conjecture, based on evidence from computer-generated limit shapes for various density domains.

**Conjecture 14.1.** The limit shape $F_{\text{max}}'$ that appears in Corollary 10.3 is always convex.

Somewhat surprisingly, we have the following implication, relying on the result of Vershik and Kerov [29] that the constant $\Gamma$ in Theorem 2.1 equals 2.

**Proposition 14.2.** Conjecture 14.1 implies Conjecture 3.5.

**Proof.** At the end of the proof of Proposition 4.8, we showed that $\Phi(r) \leq \frac{\Gamma}{\sqrt{2}} r$ for any $r \geq 0$, where $\Gamma = 2$ by [29]. Since $\Phi$ is concave, it follows that $\Phi'(r) \leq \sqrt{2}$ for any $r \geq 0$ where the derivative exists. On the other hand, by Proposition 11.1, $\Phi'(r) = 0$ for any $r > \sqrt{2}$.

Suppose Conjecture 14.1 holds. Since $F_{\text{max}} = \Phi$ for the diamond region $|x| + |y| < 1/\sqrt{2}$ with $\rho = 1$, it follows that $\Phi'$ is convex. Above, we saw that $\Phi'(0) \leq \sqrt{2}$ and $\Phi'(\sqrt{2} + \varepsilon) = 0$ for any small $\varepsilon > 0$, and, together with the convexity, this yields that $\Phi'(r) \leq \sqrt{2} - r$ for $0 \leq r \leq \sqrt{2}$. But

$$\int_0^{\sqrt{2}} \Phi'(r) \, dr = \Phi(\sqrt{2}) = 1 = \int_0^{\sqrt{2}} (\sqrt{2} - r) \, dr$$

by Proposition 11.1, so $\Phi'(r)$ must be equal to $\sqrt{2} - r$ in the interval $0 \leq r \leq \sqrt{2}$. $\square$

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