Research Article

General Forms of Solutions for Linear Impulsive Fuzzy Dynamic Equations on Time Scales

Shihuang Hong, Xiaoyu Cao, Ji Chen, Haiyang Hou, and Xinggang Luo

Science College, Hangzhou Dianzi University, Hangzhou 310018, China

Correspondence should be addressed to Shihuang Hong; hongshh@hotmail.com

Received 18 December 2019; Accepted 3 February 2020; Published 12 March 2020

Abstract

A class of linear impulsive fuzzy dynamic equations on time scales is considered by using the generalized differentiability concept on time scales. Some novel criteria and general forms of solutions are established for such models whose significance lies in proposing the possibility to get unifying forms of solutions for discrete and continuous dynamical systems under uncertainty and to unify corresponding problems in the framework of fuzzy dynamic equations on time scales. Finally, some examples show the applicability of our results.

1. Introduction

In the real world, some processes vary continuously, while others vary discretely. These processes can be modeled by differential and difference equations, respectively. There are also some processes that vary both continuously and discretely. Usage of fuzzy differential and difference equations is a natural way to model dynamical systems under possibilistic uncertainty [1, 2]. First-order linear fuzzy differential (difference) equations are one of the simplest fuzzy equations which are very basic, important, and may appear in many applications. Thus, it is reasonable to seek conditions under which the resulting fuzzy systems would have a solution with a general form. Much progress has been seen in the fuzzy differential (difference) equation direction, and many criteria are established based on different approaches (for instance, fuzzy differential equations [1, 3–17] and fuzzy difference equations [18–23]). Careful investigation reveals that it is similar to explore the existence of solutions for fuzzy differential equations and their discrete analogue in the approaches, methods, and the main results. For example, extensive research shows that many results concerning the existence of fuzzy differential equations can be carried over to their discrete analogues [24–26]. However, other results seem to be completely different [3]. It is natural to ask whether we can explore such an existence problem in a unified way and offer more general conclusions.

For the certainty system, the theory of time scale calculus and dynamic equations on time scales provides us with a powerful tool for attacking such mixed processes [27]. The calculus on time scales (see [28–31]) was initiated by Hilger in [28] in order to unify continuous and discrete analysis under the certainty system, and it has a tremendous potential for applications and has recently received much attention.

The $H$-derivative of a fuzzy-number-valued function was introduced in [32], and it has its starting point in the Hukuhara derivative of set-valued functions. The first approach to modeling the uncertainty of dynamical systems uses the $H$-derivative or its generalized, and mainly the existence and uniqueness of the solution of a fuzzy differential equation are studied under this setting (see for example [11, 14, 33–35]). Fuzzy differential equations have been studied under other approaches (see [12, 36]). Furthermore, there are several works that have dealt with fuzzy-number-valued functions on time scales and focused on a class of new derivative of such fuzzy functions, see [37–40], as well as the Hukuhara derivative of set-valued functions has been extended onto the time scales by Hong in [41–43] and fuzzy or set dynamic equations have afterward been discussed in cited.
above references and [44, 45]. The aim of this paper is to establish a general form of solutions for linear fuzzy impulsive dynamic equations whose significance lies in proposing the possibility to get unifying forms of solutions for discrete and continuous dynamical systems under uncertainty and to build a unifying framework for the study of corresponding problems. As mentioned above, the notion of the $H$-derivative plays a fundamental role in the theory of fuzzy differential equations and the calculus on time scales has the features of unification and extension. In order to achieve our purpose, a derivative of fuzzy-number-valued functions on time scales, which is similar to the one in [37] and called the $\Delta_{H}$-derivative in this paper, will be developed to study our fuzzy dynamic equations. The proposed approach forms the appropriate environment within which the study of fuzzy dynamic systems on time scales can be developed.

This paper contains four sections. In Section 2, we recall several basic definitions and properties of time scales and generalized differentiability of fuzzy-number-valued functions on time scales proposed by [38] which is the extension of that on the real axis $\mathbb{R}$ introduced in [24]. Moreover, it contains the $\Delta_{H}$-derivative introduced in [41]. In addition, some corresponding properties of the $\Delta_{H}$-derivative are explored which provide the necessary background for our further consideration. Subsequently, in Section 3, we consider first order linear fuzzy dynamic equations on account of $\Delta_{H}$-differentiability. The idea of the present section originates from the study of an analogous problem examined by Khastan et al. [3] for a variation of constant formula for the first-order linear fuzzy differential equations in $\mathbb{R}$. As distinct from [3], we consider the impulsive problem on an infinite time scale interval instead of the initial value problem on a finite real number interval and present the solutions with general expressions in this setting. This study reveals that, when we deal with the existence of solutions with general expressions for linear fuzzy differential equations and the difference counterparts, it is unnecessary to prove results for fuzzy differential equations and separately again for their discrete analogues. In other words, one can get a unifying expression of solutions for such continuous and discrete uncertainty systems. Finally, several examples are given to illustrate the applicability of our results in Section 4.

2. Preliminaries

In this section, we first recall a notion of the time scale built by Hilger and Bernd Aulbach. For more details, we refer the reader to [28, 29].

A closed nonempty subset $\mathbb{T}$ of real axis $\mathbb{R}$ is called a time scale or measure chain. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \inf\{\tau \in \mathbb{T}: \tau > t\}$, while the backward jump operator $\rho: \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) = \sup\{\tau \in \mathbb{T}: \tau < t\}$. The function $\mu: \mathbb{T} \rightarrow [0, \infty)$ called the graininess function is defined by $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. In this definition, we put $\inf\emptyset = \sup\mathbb{T}(i.e. \sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup\emptyset = \inf\mathbb{T}(i.e., \rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. t is said to be right (left) scattered if $\sigma(t) > t (\rho(t) < t)$, and t is said to be right (left) dense if $\sigma(t) = t (\rho(t) = t)$. A point is said to be isolated (dense) if it is right-scattered (right-dense) and left-scattered (left-sense) at the same time. In this paper, we stipulate that the time scale $\mathbb{T}$ is $\mathbb{T} - \{M\}$ if $\mathbb{T}$ has a left-scattered maximum $M$.

A function $f$ is right-dense continuous (rd-continuous, for short) if it is continuous at each right-dense point in $\mathbb{T}$ and its left-sided limits exist at each left-dense points in $\mathbb{T}$. For a function $f: \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}$, S. Hilger defined the $\Delta$-derivative of $f$ at $t$, $f^{\Delta}(t)$, to be the number (when it exists), with the property that, for each $\epsilon > 0$, there exists a neighborhood $U_{\delta}$ of $t$ (i.e. $U_{\delta} = (t - \delta, t + \delta) \cap \mathbb{T}$) such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| < \epsilon|\sigma(t) - s|,$$

(1)

for all $t \in U_{\delta}$. A function $f$ is said to be $\Delta$-differentiable at $t$ if its $\Delta$-derivative exists at $t$ and $\Delta$-differentiable at $T$ if its $\Delta$-derivative exists at each $t \in \mathbb{T}$.

We also recall the concept of the matrix-valued functions introduced by [29]. An $m \times n$-matrix-valued function $A: \mathbb{T} \rightarrow \mathbb{R}^{mn}$ (a collection of all $m \times n$-real matrices) is said to be $\Delta$-differentiable on $\mathbb{T}$ provided each entry of $A$ is $\Delta$-differentiable on $\mathbb{T}$. In this case, we put

$$A^{\Delta} = (a^{\Delta}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$$

where $A = (a_{ij})$. (2)

An $n \times n$-matrix-valued function $A$ on $\mathbb{T}$ is called regressive provided

$$I + \mu(t)A(t) \quad \text{is invertible for all} \quad t \in \mathbb{T}.$$  (3)

Here, $I$ stands for an $n \times n$-identity matrix (and so is it in what follows). Let

$$\mathcal{R} = \{A | A: \mathbb{T} \rightarrow \mathbb{R}^{mn} \text{is a regressive and rd-continuous} \times n \text{ matrix - valued function}\},$$

$$\mathcal{R}_{I}^{1} = \{p | p: \mathbb{T} \rightarrow \mathbb{R} \text{ is a rd-continuous function and} 1 + \mu(t)p(t) > 0 \text{ for } t \in \mathbb{T}\}.$$  (4)

From now on, unless otherwise mentioned, the matrix-valued functions under consideration are always assumed to belong to $\mathcal{R}$. For $A, B \in \mathcal{R}$, the “circle plus” and “circle minus” of matrix-valued functions are referred to as, respectively,

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t),$$

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t),$$

with $(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) = -A(t)[I + \mu(t)A(t)]^{-1}.$  (5)

A matrix exponential function $e_{A}(t, t_{0})$ is defined as a unique matrix-valued solution of the following initial value problem:

$$Y^{\Delta} = A(t)Y,$$

$$Y(t_{0}) = I,$$  (6)

where $A \in \mathcal{R}$ and $t_{0} \in \mathbb{T}$. In [29], matrix exponential functions have been proved to possess the following properties:
(a1) \( e_0(t, s) \equiv 1, e_d(t, t) \equiv I, e_A(\sigma(t), s) = (I + \mu(t)A(t)) e_A(t, s) \)

(a2) \( e_A(s, t) = e_{A^*}(t, s) = e_{A_d}(t, s) \), where \( A^* \) stands for the conjugate transpose of the matrix \( A \)

(a3) \( e_A(t, s) e_A(s, r) = e_A(t, r) \)

For an interval \( J \subset T \), if a function \( g: J \rightarrow \mathbb{R} \) is \( \Delta \)-differentiable and \( g^\Delta(t) = f(t) \); then, in [29], the authors defined the Cauchy integral by

\[
\int_a^f g(s) \Delta s = g(t) - g(a). \tag{7}
\]

In this case, \( f \) is said to be \( \Delta \)-integrable on \( J \). In particular, by \( \int_a^f g(t) \Delta t := \lim_{p \rightarrow \infty} \int_a^f g(t) \Delta t \), we mean that \( f \) is \( \Delta \)-integrable on \( J = [a, \infty) \cap T \) provided this limit exists.

In the following, we introduce the necessary definitions and notation for fuzzy numbers on time scales which are the extension of the corresponding concepts in \( \mathbb{R} \) (see, for example, [46]). Let us denote by \( \mathbb{T}_f \) the class of fuzzy subsets of \( \mathbb{T} \) satisfying the following properties, that is, \( u \in \mathbb{T}_f \), i.e. \( u: \mathbb{T} \rightarrow [0, 1] \) and

(f1) \( u \) is normal, i.e., there exists \( s_0 \in \mathbb{T} \) such that \( u(s_0) = 1 \)

(f2) \( u \) is fuzzy convex on \( \mathbb{T} \), i.e., \( u(ta + (1 - t)b) \geq \min(u(a), u(b)) \) for all \( t \in [0, 1] \) with \( ta + (1 - t)b \in \mathbb{T} \), where \( a, b \in \mathbb{T} \)

(f3) \( u \) is upper semicontinuous on \( \mathbb{T} \)

(f4) \( [u]^0 = \{ s \in \mathbb{T} : u(s) > 0 \} \cap \mathbb{T} \) is compact, where \( \mathbb{A} \) denotes the closure of \( A \) in \( \mathbb{R} \).

Then, \( \mathbb{T}_f \) is called the space of fuzzy numbers. Obviously, \( \mathbb{T} \subset \mathbb{T}_f \). Here, \( \mathbb{T} \subset \mathbb{T}_f \) is understood as \( T = \{ x | \chi_{[\alpha]}(x) \text{ is an element in the time scale} \} \). For \( 0 < \alpha \leq 1 \), set \( [u]^a = \{ s \in \mathbb{T} : u(s) \geq \alpha \} \) and \( [u]^a = \{ s \in \mathbb{T} : |u(s)| \cap \mathbb{T} = \emptyset \} \). From (f1)–(f4), it follows that the \( \alpha \)-level set \( [u]^a \) is a nonempty compact interval of \( \mathbb{T} \) for all \( 0 \leq \alpha \leq 1 \) if \( u \) belongs to \( \mathbb{T}_f \) (i.e., \( [u]^a \) is an intersection of a nonempty compact interval of \( \mathbb{T} \) and \( \mathbb{T} \)). The notation

\[
[u]^a = [u^a, u^b] \cap \mathbb{T} = [u^a, u^b], \quad \text{with } u^a, u^b \in \mathbb{T}, \tag{8}
\]

denotes explicitly the \( \alpha \)-level set of \( u \) on \( \mathbb{T} \). We refer to \( u^a \) and \( u^b \) as the lower and upper branches of \( u \), respectively. For \( u \in \mathbb{T}_f \), we can define the length of \( u \) as

\[
\text{diam}(u) = u^b - u^a. \tag{9}
\]

For \( u, v \in \mathbb{T}_f \) and \( \lambda \in \mathbb{R} \), the sum \( u + v \) and the product \( \lambda u \) are defined by \( [u + v]^a = [u]^a + [v]^a \), \( [\lambda u]^a = \lambda [u]^a \) for any \( \alpha \in [0, 1] \), where \( [u]^a + [v]^a \) is defined as the same as the usual addition of two intervals (subsets) of \( \mathbb{T} \) and \( \lambda [u]^a \) means the usual product between a scalar and a subset of \( \mathbb{T} \). The metric structure is given by the Hausdorff distance \( D: \mathbb{T}_f \times \mathbb{T}_f \rightarrow \mathbb{R}_+ = (0, \infty) \subset \mathbb{R} \).

\[
D(u, v) = \sup_{\alpha \in [0, 1]} \max \left\{ \left| [u]^a - [v]^a \right|, \left| u^a - v^a \right| \right\}. \tag{10}
\]

\((\mathbb{T}, D)\) is a complete metric space [42, 46], and the following properties are well known:

\[
D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in \mathbb{T}_f,
\]

\[
D(ku, kv) = |k|D(u, v), \quad \forall k \in \mathbb{R}, u, v \in \mathbb{T}_f,
\]

\[
D(u + v, z + w) \leq D(u, z) + D(v, w), \quad \forall u, v, z, w \in \mathbb{T}_f. \tag{11}
\]

**Definition 1.** Let \( x, y \in \mathbb{T}_f \). If there exists \( z \in \mathbb{T}_f \) such that \( x = y + z \), then \( z \) is called the \( H \)-difference of \( x \) and \( y \) and it is denoted by \( x -_H y \).

Let us remark that, in general, \( x -_H y \neq x + (-1)y \). Usually, we denote \( x + (-1)y \) by \( x - y \), while \( x -_H y \) stands for the \( H \)-difference. Similar to the analysis in [24, 37, 38], we have the following remark:

(1) We denote \( \bar{0} \in \mathbb{T}_f \) as a neutral element with respect to \( + \); if \( u + \bar{0} = \bar{0} + u = u \) for all \( u \in \mathbb{T}_f \).

(2) \( (\lambda + \mu)u = (\lambda u) + (\mu u) \), with \( \lambda \mu > 0 \)

(3) \( \lambda(u + v) = (\lambda u) + (\lambda v) \)

The following lemma appears in references [37, 38, 47].

**Lemma 1.** If \( u -_H v \) exists, it is unique and one has

(i) \( u -_H u = 0, u -_H \bar{0} = u, \bar{0} -_H u = -u \) and \( (u + v) -_H (u + v) = u \)

(ii) \( v -_H u = -(u -_H v) = (-u) -_H (-v) \) provided \( v -_H u \) exists.

In the sequel, we fix \( \bar{0} = \mathbb{T}_f \cap \mathbb{R}_+ \). The strongly generalized differentiability on the real axis \( \mathbb{R} \) was introduced in [24] and studied in [1, 26]. Motivated by these works, we introduce generalized differentiability on a time scale \( \mathbb{T} \) which appears to [38] and can be regarded as a generalization of \( \Delta_H \)-differentiability introduced in [41].

**Definition 2.** Let \( F: \mathbb{T} \rightarrow \mathbb{T}_f \) and \( t \in \mathbb{T} \) and a neighborhood \( U_T \) of \( t \) be defined by \( U_T = (t - \delta, t + \delta) \cap \mathbb{T} \) for some \( \delta > 0 \). Then,

(1) \( F \) is said to be \( \Delta \)-right differentiable at \( t \) if there exists an element \( \Delta_F(t) \) of \( \mathbb{T}_f \) and, for any given \( \varepsilon > 0 \), there exists a neighborhood \( U_T \) of \( t \) such that either the \( H \)-differences \( F(t + h) -_H F(\sigma(t)) \) exists and

\[
(c_1) D[F(t + h) -_H F(\sigma(t)), \Delta_F(t)(h - \mu(t))] \leq \varepsilon(h - \mu(t))
\]

or \( F(\sigma(t)) -_H F(t + h) \) exists and

\[
(c_2) D[F(\sigma(t)) -_H F(t + h), \Delta_F(t)(\mu(t) - h)] \leq \varepsilon(h - \mu(t))
\]

for all \( t + h \in U_T \) with \( 0 \leq h < \delta \). Moreover, \( \Delta_F(t) \) is called the \( \Delta \)-right derivative of \( F \) at \( t \).
(2) $F$ is said to be $\Delta$-left differentiable at $t$ if there exists an element $\Delta F(t)$ of $\mathbb{T}$ and, for any given $\varepsilon > 0$, there exists a neighborhood $U_{\varepsilon}$ of $t$ such that either the $H$-differences $F(t - h) - hF(\sigma(t))$ exists and
\[
(c_1) \ D[F(t - h) - hF(\sigma(t)), \Delta F(t)(-h - \mu(t))] \\
\leq \varepsilon (h + \mu(t))
\]
or $F(\sigma(t)) - hF(t - h)$ exists and
\[
(c_2) \ D[F(\sigma(t)) - hF(t - h), \Delta F(t)(h + \mu(t))] \\
\leq \varepsilon (\mu(t) + h),
\]
(13)
for all $t - h \in U_{\varepsilon}$ with $0 \leq h < \delta$. Moreover, $\Delta F(t)$ is called the $\Delta$-left derivative of $F$ at $t$.

(3) $F$ is said to be $\Delta$-differentiable at $t$ if $F$ is both $\Delta$-left and $\Delta$-right differentiable at $t$ and $\Delta F(t) = \Delta F(t)$. The element $\Delta F(t)$ or $\Delta F(t)$ is said to be the $\Delta$-derivative of $F$ at $t$ and is denoted by $\Delta F(t)$. We say that $F$ is $\Delta$-differentiable at $t$ if its $\Delta$-derivative exists at $t$. Moreover, we say $F$ is $\Delta$-differentiable on $\mathbb{T}$ if its $\Delta$-derivative exists at each $t \in \mathbb{T}$. The fuzzy set-valued function $\Delta F: \mathbb{T} \rightarrow \mathbb{T}$ is then called the $\Delta$-derivative of $F$ on $\mathbb{T}$.

The principal properties of the $\Delta$-derivative in the sense of Definition 2 have been proposed in [37–39, 41]. Next, we shall write some properties whose a majority of proofs are similar to the above mentioned references.

**Proposition 1** (see [38]). Let $F: \mathbb{T} \rightarrow \mathbb{T}$. For $t \in \mathbb{T}$, we have the following results:

(I) If $F$ is $\Delta$-differentiable at $t$, then $F$ is continuous at $t$.

(II) If $F$ is continuous at $t$ and $t$ is right scattered, then $F$ is $\Delta$-differentiable at $t$ with
\[
\Delta F(t) = \frac{F(\sigma(t)) - hF(t)}{\mu(t)}
\]
or $F(\sigma(t)) = F(t) + \mu(t)\Delta F(t)$.

(III) If $t$ is right-dense, then $F$ is $\Delta$-differentiable at $t$ if and only if
\[
\lim_{h \rightarrow 0^+} \frac{F(t + h) - H F(t)}{h} = \frac{F(t) - h F(t + h)}{H} \\
\lim_{h \rightarrow 0^+} \frac{F(t + h) - H F(t + h)}{H} = \frac{F(t) - h F(t + h)}{H} \\
\lim_{h \rightarrow 0^+} \frac{F(t - h) - H F(t)}{h} = \frac{F(t) - h F(t + h)}{H} \\
\lim_{h \rightarrow 0^+} \frac{F(t - h) - H F(t)}{h} = \frac{F(t) - h F(t + h)}{H}
\]
exist as a finite number and satisfy any one of the following equations:
\[
\lim_{h \rightarrow 0^+} \frac{F(t + h) - H F(t)}{h} = \frac{F(t) - h F(t + h)}{H} = \Delta F(t) \\
\lim_{h \rightarrow 0^+} \frac{F(t + h) - H F(t)}{h} = \frac{F(t) - h F(t + h)}{H} = \Delta F(t) \\
\lim_{h \rightarrow 0^+} \frac{F(t - h) - H F(t)}{h} = \frac{F(t) - h F(t + h)}{H} = \Delta F(t) \\
\lim_{h \rightarrow 0^+} \frac{F(t - h) - H F(t)}{h} = \frac{F(t) - h F(t + h)}{H} = \Delta F(t)
\]

Remark 1. Proposition 1 implies that, under the hypothesis that $\mathbb{T}$ is a discrete system, the fuzzy number-valued function $F: \mathbb{T} \rightarrow \mathbb{T}$ is $\Delta$-differentiable if and only if $F$ is continuous and the corresponding $H$-differences exist. However, if $\mathbb{T}$ is a continuous system, $\Delta$-derivative of $F$ does not always exist even if the corresponding $H$-differences exist.

Remark 2. If $\mathbb{T} = \mathbb{R}$, equations (16)–(19) are identical to the relevant provisions in Definition 5 in [24] in which four cases for derivatives were considered in $\mathbb{R}$. In addition, equation (16) is identical to (16)-differentiability and equation (19) to (17)-differentiability in [3].

If $\mathbb{T} = \mathbb{Z}$, then the previous definition expresses some generalized difference operators, for example, corresponding to the difference operator $\Delta F_n = F_{n + 1} - F_n$ given in [21].

For functions $F, G: \mathbb{T} \rightarrow \mathbb{T}$, we define the sum $F + G$ by $(F + G)(t) = F(t) + G(t)$ and the $H$-difference $F - H G$ by $(F - H G)(t) = F(t) - H G(t)$ for each $t \in \mathbb{T}$. We have the following.

**Proposition 2.** Assume that $F, G: \mathbb{T} \rightarrow \mathbb{T}$ are $\Delta$-differentiable at $t \in \mathbb{T}$ and $\lambda$ is any constant. Then,

(i) The sum is $\Delta$-differentiable at $t \in \mathbb{T}$. Moreover,
\[
\Delta F(t) + \Delta G(t) = \Delta F(t) + \Delta G(t)
\]

(ii) $\lambda F$ is $\Delta$-differentiable at $t$ and $\lambda \Delta F(t) = \lambda \Delta F(t)$.

In the sequel, we say that a fuzzy function is $\Delta$-differentiable meaning that it is in two cases of $(c_1)$ and $(c_2)$-differentiability (denoted by (i)-differentiable) or $(c_2)$ and $(c_1)$-differentiability (denoted by (ii)-differentiable) on $\mathbb{T}$.

**Proposition 3.** Let $F, G: \mathbb{T} \rightarrow \mathbb{T}$ be $\Delta$-differentiable such that $F$ is (i)-differentiable and $G$ is (ii)-differentiable or $F$ is (ii)-differentiable and $G$ is (i)-differentiable on $\mathbb{T}$. If the $H$-difference $F(t) - H G(t)$ exists for $t \in \mathbb{T}$, then $F - H G$ is $\Delta$-differentiable and
\[
\Delta_H (F - \mu G)(t) = \Delta_H F(t) + (-1) \cdot \Delta_H G(t). 
\]

(21)

**Proof.** For \( t \in \mathbb{T} \), if \( t \) is a right-dense point, it is the same as the proof of Theorem 4 in [1]. If \( t \) is a right-scattered point, by means of Proposition 2-(II) and Lemma 1-(i) and (ii), we have
\[
\Delta_H (F - G)(t) = \frac{(F - \mu G)(\sigma(t)) - \mu(F - \mu G)(t)}{\mu(t)} \\
= \frac{F(\sigma(t)) - \mu F(\sigma(t)) - \mu(F - \mu G)(t)}{\mu(t)} \\
= \frac{F(\sigma(t)) - \mu F(t) + G(t) - \mu G(\sigma(t))}{\mu(t)} \\
= \Delta_H F(t) + (-1) \cdot \frac{G(\sigma(t)) - \mu G(t)}{\mu(t)} = \Delta_H F(t) \\
+ (-1) \cdot \Delta_H G(t).
\]

(22)

This proof is complete.

The following lemma roots in Theorem 5 of [1]. \( \square \)

**Lemma 2.** Let \( a: \mathbb{T} \rightarrow \mathbb{R} \) be \( \Delta \)-differentiable and \( G: \mathbb{T} \rightarrow \mathbb{R} \) \( \Delta_H \)-differentiable.

(a) If \( a(\sigma(t))a^\Delta(t) > 0 \) and \( G \) is (i)-differentiable, then \( aG \) is (i)-differentiable and we have
\[
\Delta_H (aG)(t) = a^\Delta(t)G(t) + a(\sigma(t))\Delta_H G(t). 
\]

(23)

(b) If \( a(\sigma(t))a^\Delta(t) < 0 \) and \( G \) is (ii)-differentiable, then \( aG \) is (ii)-differentiable and we have
\[
\Delta_H (aG)(t) = a^\Delta(t)G(t) + a(\sigma(t))\Delta_H G(t). 
\]

(24)

(c) If \( a(\sigma(t))a^\Delta(t) > 0 \), \( G \) is (ii)-differentiable and the \( H \)-differences \((aG)(\sigma(t)) - a_G(t + h) + a_G(t - h) - a(\sigma(t))a^\Delta(t) \) exist, then \( aG \) is (ii)-differentiable, and we have
\[
\Delta_H (aG)(t) = a(\sigma(t))\Delta_H G(t) - i_H(-a^\Delta(t))G(t). 
\]

(25)

Proof. We get into details regarding the discussion of the cases (b) and (c), while the proof of (a) is similar to (b).

(b) For any \( t \in \mathbb{T} \) and \( \varepsilon > 0 \), there exists a neighborhood \( U_\varepsilon \) of \( t \) for some \( \delta > 0 \) such that
\[
D[G(\sigma(t)) - \mu G(t + h), \Delta_H G(t)(\mu(t) - h)] \leq \varepsilon(h - \mu(t)),
\]

(26)

\[
\left|a(\sigma(t)) - a(t + h) - a^\Delta(t)(\mu(t) - h)\right| \leq \varepsilon(h - \mu(t)),
\]

(27) for \( 0 \leq h < \delta \) with \( t + h \in U_\varepsilon \). On the contrary, we have
\[
D\left[a(\sigma(t))G(\sigma(t)) - \mu a(t + h)G(t + h), (a^\Delta(t)G(t)
+ a(\sigma(t))\Delta_H G(t))\mu(t) - h\right]
\leq D[a(\sigma(t))G(\sigma(t)) - \mu a(\sigma(t))G(t + h), a(\sigma(t))\Delta_H
\cdot G(t)(\mu(t) - h)]
+ D\left[a(\sigma(t))G(t + h) - \mu a(t + h)G(t + h), a^\Delta(t)
\cdot G(t)(\mu(t) - h)\right]
\leq D[a(\sigma(t))G(t + h), \Delta_H G(t)(\mu(t) - h)]
\cdot |a(\sigma(t))|
+ D\left[a(\sigma(t))G(t + h) - \mu a(t + h)G(t + h), a^\Delta(t)G(t)(\mu(t) - h)\right]
\leq D\left[a(\sigma(t))G(t + h), \Delta_H G(t)(\mu(t) - h)\right]
\cdot |a(\sigma(t))|
+ D\left[a(\sigma(t))G(t + h), a(t + h)G(t + h) + a^\Delta(t)G(t + h)
\cdot G(t)(\mu(t) - h)\right]
\cdot (\mu(t) - h).
\]

(28)

Note that \( a(\sigma(t)) \) has the same sign as \( a(t + h) \) for sufficiently small \( h > 0 \). In addition, \( a(\sigma(t))a^\Delta(t) < 0 \) and \( \mu(t) - h \leq 0 \) imply that \( a^\Delta(t)(\mu(t) - h) \) has the same sign as \( a(\sigma(t)) \). Hence, \( a(t + h)G(t + h) + a^\Delta(t)G(t + h)(\mu(t) - h) = [a(t + h) + a^\Delta(t)(\mu(t) - h)]G(t + h) \) and
\[
D\left[a(\sigma(t))G(t + h), a(t + h)G(t + h) + a^\Delta(t)G(t + h)
\cdot G(t)(\mu(t) - h)\right] = [a(\sigma(t)) - (a(t + h) + a^\Delta(t)(\mu(t) - h))]G(t + h)
\cdot \|G(t + h)\|.
\]

(29)

It follows that
\[
D\left[a(\sigma(t))G(\sigma(t)) - \mu a(t + h)G(t + h), (a^\Delta(t)G(t)
+ a(\sigma(t))\Delta_H G(t))\mu(t) - h\right]
\leq D\left[a(\sigma(t))G(t + h), \Delta_H G(t)(\mu(t) - h)\right]|a(\sigma(t))|
+ \left[a(\sigma(t)) - (a(t + h) + a^\Delta(t)(\mu(t) - h))\right]\|G(t + h)\|
+ D\left[G(t), G(t + h)\right]|a^\Delta(t)(\mu(t) - h)|
\]

(30)

In view of this and the continuity of \( G \), together with the inequalities (26) and (27), we see that \( aG \) satisfies
the first inequality of Definition 2-(c²). We can similarly check the second inequality of Definition 2-(c₁). Consequently, the desired conclusion arrives.

\[ D \left[ a(\sigma(t))G(\sigma(t)) - h_1 a(t + h)G(t + h), \left( a(\sigma(t))\Delta_H G(t) - h_1 (-a^t(t))G(t) \right) (\mu(t) - h) \right] \]

\[ \leq D \left[ a(\sigma(t))G(\sigma(t)) - h_1 a(\sigma(t))G(t + h), a(\sigma(t))\Delta_H G(t) (\mu(t) - h) \right] \]

\[ + D \left[ a(\sigma(t))G(t + h), a(t + h)G(t + h) + (-a^t(t))G(t + h)(\mu(t) - h) \right] \]

\[ \leq D \left[ a(\sigma(t)) - h_1 a(\sigma(t))G(t + h), a(t + h)G(t + h) (\mu(t) - h) \right] \]

\[ + D \left[ a(\sigma(t)) - a(t + h) + a^t(t)(\mu(t) - h) \right] \| G(t + h) \|

As in case (b), we arrive at the desired result. This proof is complete. \(\square\)

**Proposition 4.** Let \( F : \mathbb{T} \rightarrow \mathbb{T}_f \) and put \([F(t)]^a = [F^a(t), F^\Delta(t)]\) for each \( a \in [0, 1] \).

(i) If \( F \) is (i)-differentiable, then \( F^a \) and \( F^\Delta \) are \( \Delta \)-differentiable functions and

\[ [\Delta_H F(t)]^a = \left[ \left( \frac{a^t}{\Delta} \right)(t), \left( \frac{a^\Delta}{\Delta} \right)(t) \right]. \]  

(ii) If \( F \) is (ii)-differentiable, then \( F^a \) and \( F^\Delta \) are \( \Delta \)-differentiable functions, and we have

\[ [\Delta_H F(t)]^a = \left[ \left( \frac{a^\Delta}{\Delta} \right)(t), \left( \frac{a^\Delta}{\Delta} \right)(t) \right]. \]  

Proposition 4 for its proof is similar to Theorem 5 in [11].

A fuzzy-number-valued function \( F : J \subset \mathbb{T} \rightarrow \mathbb{T}_f \) is called regulated if it has right-sided limit exists at any right-dense point in \( \mathbb{T} \) and left-sided limit exists at any left-dense point in \( \mathbb{T} \).

\( F \) is called right dense continuous, denoted \( r \)-dense-continuous, provided \( F \) is continuous at each right dense point in \( \mathbb{T} \), its left-sided limits exist at each left dense points in \( \mathbb{T} \). Similarly, we can define \( l \)-density. The sets of all \( r \)-dense-continuous fuzzy-number-valued functions, and all such functions \( F : J \rightarrow \mathbb{T}_f \) whose \( r \)-dense-continuous \( \Delta_H \)-derivative exist are denoted, respectively, by

\[ \mathcal{C}_{rd} = \mathcal{C}_{rd}(J, \mathbb{T}_f), \]

\[ \mathcal{C}_{rd}^1 = \mathcal{C}_{rd}^1(J, \mathbb{T}_f). \]

A function \( f : J \subset \mathbb{T} \rightarrow \mathbb{R} \) is called an integrable selector of the fuzzy-number-valued function \( F : J \rightarrow \mathbb{T}_f \) if \( f \) is \( \Delta \)-integrable and \( f(t) \in [F(t)]^a \) for all \( t \in J \) with \( a \in [0, 1] \). \( F \) is called integrable on \( I \) if it has at least an integrable selector. The integral of \( F \), denoted by \( \int_J F(s) \Delta s \), is defined levelwise by

\[ \left[ \int_J F(s) \Delta s \right]^a = \int_J [F(s)]^a \Delta s = \left\{ \int_J f(s) \Delta s \colon f \right\}. \]  

For the fuzzy version of the fundamental properties of calculus in the sense of (i)-differentiability, we refer to the analogue of set-valued functions in [41, 43], and in the sense of (ii)-differentiability we present the following results which are similar to those proposed in [3, 11]:

The integral \( \int J F(s) \Delta s = (i)-differentiable \) and \( \Delta_H F(t) = F(t) \). If \( F \in \mathcal{C}_{rd}(J, \mathbb{T}_f) \) and the function \( F \) is (ii)-differentiable, then \( \Delta_H F(t) = F(t) \).

As the authors pointed out in [3], in general, the function \( F(t) \) is not (ii)-differentiable. Indeed, suppose that is (ii)-differentiable, then the length of the support decreases in \( t \), but the function \( F(t) \), if \( F \) is fuzzy non-real-valued, has increasing length of the support. A (ii)-differentiable function needs to have decreasing length of support which is a contradiction.

**Lemma 3** (see [42]). Let \( J = [t_0, T] \cap \mathbb{T} \) with \( t_0, T \in \mathbb{T} \). Then, we have

(i) Let \( F \in \mathcal{C}_{rd}(J, \mathbb{T}_f) \) and define \( U(t) = \gamma^{-H} \int_{t_0}^t -H F(s) \Delta s \) for \( t \in J \), where \( \gamma \in \mathbb{T}_f \) such that the previous \( H \)-derivative exists for \( t \in J \). Then, \( U \) is (ii)-differentiable and \( \Delta_H U(t) = F(t) \);

(ii) Let \( F \) be (ii)-differentiable and \( \Delta_H F \) be integrable on \( \mathbb{T}_+ \). Then, for each \( t \in \mathbb{T}_+ \) with \( t \geq t_0 \in \mathbb{T}_+ \), we have
To explore the existence of solutions to LIFDE (39), we need the following essential preliminaries. In virtue of Theorem 5.24 in [29], the problem

$$\left\{ \begin{array}{l} u^A(t) = A(t)u(t) + f(t), \quad t \in T, \\ u(0) = v_0, \end{array} \right.$$  

has a unique solution \( v: T \rightarrow \mathbb{R}^n \) given by

$$v(t) = e_A(t)v_0 + \int_0^t e_A(t, \sigma(s))f(s)\Delta s,$$  

where \( A \in \mathcal{R}, e_A(t) = e_A(t, 0) \) and \( w: T \rightarrow \mathbb{R}^n \) is \( rd \)-continuous.

Let \( A \in \mathcal{R}, f: T_+ \rightarrow \mathbb{R}^n \) be an \( rd \)-continuous function and \( \lim_{t \rightarrow T} f(t) = f(T) \) exist for \( k = 1, 2, \ldots \), and let \( L_k \) be a continuous linear operator acting in \( \mathbb{R}^n \) for \( k = 1, 2, \ldots \). By an analogue of the proof of the above result, we can prove that the linear impulsive dynamic equation:

$$\left\{ \begin{array}{l} u^A(t) = A(t)u(t) + f(t), \quad t \in J_k, \\ u_{r_k} = \Phi_k = L_k(u(t_k)), \\ u(t_k) = u_{k-1}(t_k), \end{array} \right.$$  

for each \( t_k \in J \) has a unique solution

$$\left\{ \begin{array}{l} u_k(t) = e_A(t, t_{r_k})\Phi_k + \int_{t_k}^t e_A(t, \sigma(\tau))f(\tau)\Delta \tau, \quad t \in J_k, \\ u_k(t_k) = u_{k-1}(t_k), \end{array} \right.$$  

for \( k = 0, 1, 2, \ldots \), where \( u_{0-1} = v(t_0) \). Thus, we obtain that the linear impulsive dynamic equation:

$$\left\{ \begin{array}{l} u^A(t) = A(t)u(t) + f(t), \quad t \in T_+ \setminus J, \\ u_{r_k} = \Phi_k = L_k(u(t_k)), \quad t_k \in J(k = 0, 1, 2, \ldots), \\ u(0) = v_0, \end{array} \right.$$  

has a unique solution \( w = w(v_0) \) on \( T_+ \) which is left continuous on \( T_+ \) and defined by

$$w(t) = \left\{ \begin{array}{l} \nu(t), \quad t \in J, \\ u_0(t), \quad t \in J_0, \\ \ldots, \ldots, \\ u_k(t), \quad t \in J_k, \\ \ldots, \ldots. \end{array} \right.$$  

Similar to the formulation in [3], we study LIFDE (39) in three cases \( r(t) < 0, r(t) > 0 \) and \( r(t) = 0 \) for \( t \in T_+ \), where \( r \) is a function given in LIFDE (39). We first observe that the hyperbolic functions proposed by Bohner and Peterson [31] can be extended to
Let $E_r(t, s) = e_r(t, s)e_{-r}(t, s)$. Obviously, for all $t, s \in \mathbb{T}$, the hyperbolic functions possess the following properties:

(p1) $\cosh_r(s, s) = 1$, $\sinh_r(s, s) = 0$

(p2) $\cosh^{-1}_r(t, s) = r \sinh_r(t, s), \sinh^{-1}_r(t, s) = r \cosh_r(t, s)$, where $a^{-1}_r(t, s)$ means the $\Delta$-derivative of $a$ with respect to the variable $t$

(p3) $\cosh_r(s, t) = (1/E_r(t, s))\cosh_r(t, s), \sinh_r(s, t) = -(1/E_r(t, s))\sinh_r(t, s)$



\[
\begin{align*}
\cosh_r(t, s) &= \frac{e_r(t, s) + e_{-r}(t, s)}{2}, \\
\sinh_r(t, s) &= \frac{e_r(t, s) - e_{-r}(t, s)}{2}. \\
\end{align*}
\] (46)

We are now in a position to state and verify our main results.

**Theorem 1.** If $r \in \mathbb{R}^+_0$ satisfies $r(t) < 0$ for all $t \in \mathbb{T}_+$, then

(i) LIFDE (39) has a (i)-solution on $\mathbb{T}_+$ given by

\[
U(t) = \begin{cases} V(t), & t \in J_-; \\
U_k(t), & t \in J_k(k = 0, 1, 2, \ldots), \end{cases}
\] (47)

where

\[
\begin{align*}
V(t) &= \cosh_r(t) \left\{ U_0 + \int_0^t \left[ F(r) \frac{\cosh_r(\sigma(r), 0)}{E_r(\sigma(r), 0)} - hF(r) \frac{\sinh_r(\sigma(r), 0)}{E_r(\sigma(r), 0)} \right] \Delta r \right\}, \\
+ \sinh_r(t) \left\{ U_0 + \int_0^t \left[ F(r) \frac{\cosh_r(\sigma(r), t_k)}{E_r(\sigma(r), t_k)} - hF(r) \frac{\sinh_r(\sigma(r), t_k)}{E_r(\sigma(r), t_k)} \right] \Delta r \right\}, \\
U_k(t) &= \cosh_r(t, t_k^*) \left\{ L_k U_k(t_k) + \int_{t_k}^t \left[ F(r) \frac{\cosh_r(\sigma(r), t_k)}{E_r(\sigma(r), t_k)} - hF(r) \frac{\sinh_r(\sigma(r), t_k)}{E_r(\sigma(r), t_k)} \right] \Delta r \right\}, \\
+ \sinh_r(t, t_k^*) \left\{ L_k U_k(t_k) + \int_{t_k}^t \left[ F(r) \frac{\cosh_r(\sigma(r), t_k)}{E_r(\sigma(r), t_k)} - hF(r) \frac{\sinh_r(\sigma(r), t_k)}{E_r(\sigma(r), t_k)} \right] \Delta r \right\}, \\
\end{align*}
\] (48)

with $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ for $k = 0, 1, 2, \ldots$, and

\[
\begin{align*}
U(t) &= \begin{cases} V(t) = e_r(t) \left\{ U_0 - H \int_0^t F(r) e_{or}(\sigma(r)) \Delta r \right\}, & t \in J_-; \\
U_k(t) = e_r(t, t_k^*) \left\{ L_k U_k(t_k) - H \int_{t_k}^t F(r) e_{or}(\sigma(r), t_k) \Delta r \right\}, & t \in J_k, \end{cases}
\] (49)

(ii) The (ii)-solution of the LIFDE (39) on $\mathbb{T}_+$ is given by

\[
\begin{align*}
(U^{as})^\Delta (t) &= r(t) U^{as}(t) + \mathbb{F}^a(t), & t \in \mathbb{T}_+ \setminus J; \\
(\mathbb{U}^a)^\Delta (t) &= r(t) \mathbb{U}^a(t) + \mathbb{F}^a(t), & t \in \mathbb{T}_+ \setminus J; \\
(U^{as})_t &= L_k U^{as}(t_k), \\
(\mathbb{U}^a)_t &= L_k \mathbb{U}^a(t_k), \\
t_k &\in J(k = 0, 1, 2, \ldots), \\
U^{as}(0) = (U^{as})_0, \\
\mathbb{U}^a(0) = (\mathbb{U}^a)_0.
\] \] (50)

**Proof.** Proposition 4 shows us how to translate the LIFDE (39) into a system of ordinary dynamic equations (ODEs), that is, if $r(t) < 0$ with $t \in \mathbb{T}_+$ and $U$ is (i)-differentiable, then $[\Delta_r U(t)]^a = [U^{as}]^\Delta(t), (U^{as})_t^\Delta(t]$ with $[U]^a = [U^{as}, \mathbb{U}^a]$ for all $a \in [0, 1]$, and (39) is translated into the following impulsive system of ODEs:
where \( [F(t)]^a = [F^a(t), F^a(t)] \) for all \( a \in [0, 1] \). For solving system (50), we translate it into system (44) with

\[
\begin{aligned}
u(t) &= \left( \frac{U^a(t)}{U^a(t)} \right), \\
u_{t_k} &= \left( \frac{U^a_{t_k}}{U^a_{t_k}} \right), \\
f(t) &= \left( \frac{F^a(t)}{F^a(t)} \right), \\
v_0 &= \left( \frac{U^a_0}{U^a_0} \right),
\end{aligned}
\]

\( (a \in [0, 1]), \quad A(t) = \begin{pmatrix} 0 & r(t) \\ r(t) & 0 \end{pmatrix}. \tag{51} \)

Similarly, if \( U \) is (ii)-differentiable then \( \Delta_H U(t)^a = [(U^a)^2(t), (U^a)^3(t)] \) and (39) is translated into (44) with \( u, u_{t_k}, v_0 \) given as in (51) and

\[
\begin{aligned}
A(t) &= \begin{pmatrix} r(t) & 0 \\ 0 & r(t) \end{pmatrix}, \\
f(t) &= \left( \frac{F^a(t)}{F^a(t)} \right).
\end{aligned}
\]

(i) Under the case of the (i)-differential, we check that \( A \) given by (51) belongs to \( \mathcal{R} \). From \( r \in \mathcal{R}^+_1 \), it follows that the matrix

\[
I + \mu(t)A(t) = \begin{pmatrix} 1 & \mu(t)r(t) \\ \mu(t)r(t) & 1 \end{pmatrix}, \tag{53}
\]

is invertible for each \( t \in T_+ \), that is, \( A \in \mathcal{R} \). Moreover, we easily check that

\[
e_A(t, s) = \begin{pmatrix} \cosh(t, s) & \sinh(t, s) \\ \sinh(t, s) & \cosh(t, s) \end{pmatrix}, \tag{54}
\]

with \( A \) given in (51).

Now, by substituting this matrix exponential function for \( e_A(t, t_k^+) \) and \( e_A(t, \sigma(t)) \) of (43), we have

\[
\begin{aligned}
\Phi_k &= \left( \begin{array}{c}
\cosh(t_k, t_k^+) \\
\sinh(t_k, t_k^+)
\end{array} \right), \\
\Delta_f(t, t_k^+) &= \int_{t_k}^{t} \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) f(\tau) d\tau, \\
\Phi_k &= \left( \begin{array}{c}
\cosh(t_k, t_k^+) \\
\sinh(t_k, t_k^+)
\end{array} \right) \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) \Delta_f(t, t_k^+).
\end{aligned}
\]

Therefore, for \( t \in I_k \) with \( k = 0, 1, 2, \ldots \), we have

\[
\begin{aligned}
\left( \frac{U^a(t)}{U^a(t)} \right) &= \left( \frac{U_k^a(t)}{U_k^a(t)} \right) = \left( \begin{array}{c}
\cosh(t_k, t_k^+) \\
\sinh(t_k, t_k^+)
\end{array} \right), \\
\left( \frac{L_k U_k^a(t_k)}{L_k U_k^a(t_k)} \right) &= \left( \begin{array}{c}
\cosh(t_k, t_k^+) \\
\sinh(t_k, t_k^+)
\end{array} \right)
\times \left[ \frac{F^a(t)}{F^a(t)} \right] + \int_{t_k}^{t} \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) f(\tau) d\tau, \\
\left( \frac{L_k U_k^a(t_k)}{L_k U_k^a(t_k)} \right) &= \left( \begin{array}{c}
\cosh(t_k, t_k^+) \\
\sinh(t_k, t_k^+)
\end{array} \right)
\times \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) \left( \begin{array}{c}
\cosh(t_k, \sigma(t)) \\
\sinh(t_k, \sigma(t))
\end{array} \right) \Delta_f(t, t_k^+).
\end{aligned}
\]

(57)
Then, by the property (p3), the solution of the linear dynamic equation system is

\[
\begin{align*}
U^a(t) &= U^a_k(t) \\
&= \cosh_r(t, t^*_k) \left\{ \sum_{k=0}^{\infty} \int_{t_k}^{t} \left[ \frac{F^a(\tau)}{E_r(\sigma(\tau), t_k)} \cosh_r(\sigma(\tau), t_k) - \frac{H^a(\tau)}{E_r(\sigma(\tau), t_k)} \sinh_r(\sigma(\tau), t_k) \right] \Delta \tau \right\} \\
&\quad + \sinh_r(t, t^*_k) \left\{ \sum_{k=0}^{\infty} \int_{t_k}^{t} \left[ -\frac{F^a(\tau)}{E_r(\sigma(\tau), t_k)} \sinh_r(\sigma(\tau), t_k) + \frac{H^a(\tau)}{E_r(\sigma(\tau), t_k)} \cosh_r(\sigma(\tau), t_k) \right] \Delta \tau \right\},
\end{align*}
\]

(58)

Thus, we obtain the (i)-solution of LIFDE (39) on \( J_k \) for \( k = 0, 1, 2, \ldots \) when \( r(t) < 0 \) with \( t \in \mathbb{U}_+ \) as follows:

\[
U(t) = U_k(t) \\
= \cosh_r(t, t^*_k) \left\{ \sum_{k=0}^{\infty} \int_{t_k}^{t} \left[ \frac{F(\tau)}{E_r(\sigma(\tau), t_k)} \cosh_r(\sigma(\tau), t_k) - \frac{H(\tau)}{E_r(\sigma(\tau), t_k)} \sinh_r(\sigma(\tau), t_k) \right] \Delta \tau \right\} \\
&\quad + \sinh_r(t, t^*_k) \left\{ \sum_{k=0}^{\infty} \int_{t_k}^{t} \left[ -\frac{F(\tau)}{E_r(\sigma(\tau), t_k)} \sinh_r(\sigma(\tau), t_k) + \frac{H(\tau)}{E_r(\sigma(\tau), t_k)} \cosh_r(\sigma(\tau), t_k) \right] \Delta \tau \right\},
\]

(59)

Similarly, if \( t \in J_- \), we have

\[
U(t) = V(t) = \cosh_r(t) \left\{ U_0 + \int_{0}^{t} \left[ \frac{F(\tau)}{E_r(\sigma(\tau), 0)} \cosh_r(\sigma(\tau), 0) - \frac{H(\tau)}{E_r(\sigma(\tau), 0)} \sinh_r(\sigma(\tau), 0) \right] \Delta \tau \right\} \\
&\quad + \sinh_r(t) \left\{ U_0 + \int_{0}^{t} \left[ -\frac{F(\tau)}{E_r(\sigma(\tau), 0)} \sinh_r(\sigma(\tau), 0) + \frac{H(\tau)}{E_r(\sigma(\tau), 0)} \cosh_r(\sigma(\tau), 0) \right] \Delta \tau \right\}.
\]

(60)

Let us remark that the \( H \)-difference

\[
F(\tau) \frac{\cosh_r(\sigma(\tau), s)}{E_r(\sigma(\tau), s)} - H F(\tau) \frac{\sinh_r(\sigma(\tau), s)}{E_r(\sigma(\tau), s)}
\]

always exists for \( s \in [0, \sigma(\tau)] \) and \( r(t) < 0 \). As explained in [3], the diameters of the \( a \)-level sets of \( F(\tau)\cosh_r(\sigma(\tau), s)/E_r(\sigma(\tau), s) \) and \( F(\tau)\sinh_r(\sigma(\tau), s)/E_r(\sigma(\tau), s) \) are, respectively,

\[
\text{diam}(\lfloor F(\tau) \rfloor) \frac{\cosh_r(\sigma(\tau), s)}{E_r(\sigma(\tau), s)},
\]

(61)

\[
\text{diam}(\lfloor F(\tau) \rfloor) \frac{-\sinh_r(\sigma(\tau), s)}{E_r(\sigma(\tau), s)}.
\]

(62)

While the former is greater than the latter since we have the inequality:
Finally, note that \( \cosh_1(t, s) \cdot \cosh_2(t, s) = r \cosh_1(t, s) \cdot \sinh_1(t, s) + \sinh_2(t, s) \cdot \cosh_1(t, s) = r(t) \sinh_1(t, s) \cdot \cosh_1(t, s) > 0 \) for \( r(t) < 0 \), we see that \( U(t) \) is (i)-differentiable on \( \mathbb{T}_+ \). Consequently, LIFDE (39) has a (ii)-solution and (47) holds.

(ii) For \( t \in J_k \), under the hypothesis of (ii)-differentiability, LIFDE (39) is translated into the corresponding system (42) with \( A \) and \( f \) given as (52). Obviously, \( A \in \mathcal{R} \) and

\[
e_A(t, s) = e_r(t, s). (64)
\]

By means of (43), we have

\[
\begin{align*}
\| u_k(t) &= e_r(t, t_k) \| \mathcal{M}_k + \int_{t_k}^t e_{or}(\sigma(t), t_k) F(t)\Delta t, \quad r \in J_k, \\
\| u_k(t_k) &= u_{k-1}(t_k). (65)
\end{align*}
\]

Repeating the arguments of (i), we obtain that the solution of the corresponding ODE system is

\[
\begin{align*}
U_k^a(t) &= e_r(t, t_k^*) \left[ L_k U_k^a(t_k) + \int_{t_k}^t F^a(t) e_{or}(\sigma(t), t_k) \Delta t \right], \\
U_k^a(t) &= e_r(t, t_k^*) \left[ L_k U_k^a(t_k) + \int_{t_k}^t F^a(t) e_{or}(\sigma(t), t_k) \Delta t \right]. (66)
\end{align*}
\]

\[
\begin{align*}
V(t) &= \cosh_1(t) \left[ U_0 - H \int_0^t F(\tau) \sinh_1(\sigma(\tau), 0) - H F(\tau) \cosh_1(\sigma(\tau), 0) \right] \Delta t - H[-\sinh_1(t)] \\
&\quad \cdot \left[ U_0 - H \int_0^t F(\tau) \sinh_1(\sigma(\tau), 0) - H F(\tau) \cosh_1(\sigma(\tau), 0) \right] \Delta t, \quad r \in J_-, \\
U_k(t) &= \cosh_1(t, t_k^*) \left[ L_k U_k(t_k) - H \int_{t_k}^t F(\tau) \sinh_1(\sigma(\tau), t_k) - H F(\tau) \cosh_1(\sigma(\tau), t_k) \right] \Delta t - H[-\sinh_1(t, t_k^*]) \\
&\quad \cdot \left[ L_k U_k(t_k) - H \int_{t_k}^t F(\tau) \sinh_1(\sigma(\tau), t_k) - H F(\tau) \cosh_1(\sigma(\tau), t_k) \right] \Delta t, \quad r \in J_k.
\end{align*}
\]

for all \( \alpha \in [0, 1] \). We assert that the (ii)-solution of LIFDE (39) on \( J_k \) is

\[
U(t) = U_k(t) = e_r(t, t_k^*) \left[ L_k U_k(t_k) - H \int_{t_k}^t -F(\tau) e_{or}(\sigma(\tau), t_k) \Delta t \right],
\]

where \( k = 0, 1, 2, \ldots \) and \( r(t) < 0 \). In fact, we observe that \( e_r(\sigma(t), t_k) e^\sigma(t, t_k) = r(t) e_r(\sigma(t), t_k) e_r(t, t_k) < 0 \). If we denote \( G_k(t) = L_k U_k(t_k) - H \int_{t_k}^t -F(\tau) e_{or}(\sigma(\tau), t_k) \Delta t \), then Lemma 3-(i) guarantees that \( G_k \) is (ii)-differentiable and

\[
\Delta G_k(t) = F(t) e_{or}(\sigma(t), t_k). (68)
\]

Now, the conditions in Lemma 2-(b) are met, so

\[
\Delta U_k(t) = r(t) e_r(t, t_k^*) G_k(t) + e_r(\sigma(t), t_k) \Delta G_k(t) = r(t) U_k(t) + F(t), (69)
\]

for \( k = 0, 1, 2, \ldots \). Similarly, for \( t \in J_-, \) we have

\[
U(t) = V(t) = e_r(t, 0) \left[ U_0 - g \int_0^t -g F(\tau) e_{or}(\sigma(\tau), 0) \Delta t \right]. (70)
\]

We obtain that LIFDE (39) has a (ii)-solution satisfying (49). The proof is complete. 

**Theorem 2.** If \(-r \in \mathcal{R}_+^1\) and \( r(t) > 0 \) with \( t \in \mathbb{T}_+ \), then

(i) LIFDE (39) has a (ii)-solution on \( \mathbb{T}_+ \) given as in (47), where
provided that the above $H$-differences exist and $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ for $k = 0, 1, 2, \ldots$

(ii) The $(i)$-solution of LIFDE (39) on $\mathbb{T}_+$ is given by

\[ U(t) = \begin{cases} 
V(t) = e_r(t) \left[ U_0 + \int_{0}^{t} F(r)e_{\sigma(r)}(\sigma(r))\Delta r \right], & t \in J_-, \\
U_k(t) = e_r(t, t_k^*) \left[ L_kU_k(t_k) + \int_{t_k}^{t} F(r)e_{\sigma(r)}(\sigma(r), t_k)\Delta r \right], & t \in J_k,
\end{cases} \tag{72} \]

where $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ for $k = 0, 1, 2, \ldots$

Proof. (i) LIFDE (39) with $(ii)$-differentiability is transformed into the linear impulsive dynamic equation system (44) with $u(t), u_\nu, A(t), v_0$ given in (51) and $f(t)$ as in (52). By (44), we obtain

\[
\begin{aligned}
&u_k(t) = A(t, t_k^*) \left[ \Phi_k + \int_{t_k}^{t} A(\tau, \sigma(\tau))f(\tau)\Delta \tau \right], & t \in J_k, \\
&u_k(t_k) = u_{k-1}(t_k),
\end{aligned}
\tag{73} \]

\[ U^a(t) = U_k^a(t) \]

\[ = \cosh_r(t, t_k^*) \left[ L_kU_k^a(t_k) + \int_{t_k}^{t} \left\{ T^a(\tau) \cosh_r(\sigma(\tau), t_k) E_r(\sigma(\tau), t_k) - E^a(\tau) \sinh_r(\sigma(\tau), t_k) E_r(\sigma(\tau), t_k) \right\} \Delta \tau \right]
\] 

\[ + \sinh_r(t, t_k^*) \left[ L_kU_k^a(t_k) + \int_{t_k}^{t} \left\{ -T^a(\tau) \sinh_r(\sigma(\tau), t_k) E_r(\sigma(\tau), t_k) + E^a(\tau) \cosh_r(\sigma(\tau), t_k) E_r(\sigma(\tau), t_k) \right\} \Delta \tau \right], \tag{74} \]

\[ U_k(t) = \cosh_r(t, t_k^*)G_k(t) - H[\sinh_r(t, t_k^*)]G_k(t), \tag{75} \]

\[ U_k(t) = \cosh_r(t, t_k^*)G_k(t) - H[\sinh_r(t, t_k^*)]G_k(t), \tag{75} \]

\[ \text{with} \]

\[ G_k(t) = L_kU_k(t_k) - H[\int_{t_k}^{t} F(\tau) \sinh_r(\sigma(\tau), t_k) E_r(\sigma(\tau), t_k) - H[\int_{t_k}^{t} F(\tau) \cosh_r(\sigma(\tau), t_k) E_r(\sigma(\tau), t_k) \Delta \tau, \tag{76} \]

is $(ii)$-differentiable and $U_k$ is a $(ii)$-solution of LIFDE (39) on $J_k$.

The following argument is due to the proof of Theorem 3.3 in [3]. First, by our hypothesis $G_k$ is well defined. Second, Lemma 3-(i) guarantees that $G_k$ is $(ii)$-differentiable and the diameter of $G_k$ is nonincreasing in the variable $t$ for fixed $\alpha \in [0, 1]$. Note that $\cosh, (t, t_k^*) - \sinh, (t, t_k^*) = e_{-t} (t, t_k^*)$ is nonnegative and decreasing in $t$. Thus, $\text{diam}[U_k(t)]^a$ is nonincreasing in $t$ for fixed $\alpha \in [0, 1]$. Therefore, the $H$-differences $U_k(\sigma(t))-U_k(t+h)$ and $U_k(t-h)-U_k(\sigma(t))$ exist. Third, we check that the $(ii)$-derivative of $U_k$ is $r(t)U_k(t) + F(t)$ for $t \in E_k(k = 0, 1, 2, \ldots)$. If $t$ is a
right-dense point, in view of an analogous argument of Theorem 3.3 in [3], we can check that
\[
\lim_{h \to 0^-} \frac{U_k(t) - hU_k(t + h)}{-h} = \lim_{h \to 0^-} \frac{U_k(t - h) - hU_k(t)}{-h} = r(t)U_k(t) + F(t).
\]

(77)

In the light of Proposition 1-(III), we have
\[
\Delta H_k(t) = r(t)U_k(t) + F(t). \quad \text{If } t \text{ is a right-scattered point, we denote}
\]

\[
\xi_k(t) = \int_{t_k}^{t} \left[ \Phi^a(\tau) \cosh_h(\sigma(\tau), t_k) - \phi^a(\tau) \sinh_h(\sigma(\tau), t_k) \right] \Delta \tau,
\]

\[
\eta_k(t) = \int_{t_k}^{t} \left[ -\Phi^a(\tau) \sinh_h(\sigma(\tau), t_k) + \phi^a(\tau) \cosh_h(\sigma(\tau), t_k) \right] \Delta \tau.
\]

(78)

Then,

\[
\xi_k(t) = F^a(t) \cosh_h(\sigma(t), t_k) - F^a(t) \sinh_h(\sigma(t), t_k),
\]

\[
\eta_k(t) = -F^a(t) \sinh_h(\sigma(t), t_k) + F^a(t) \cosh_h(\sigma(t), t_k),
\]

\[
\xi_k(\sigma(t)) - \xi_k(t) = \int_{t_k}^{t} \left[ \Phi^a(\tau) \cosh_h(\sigma(\tau), t_k) - \phi^a(\tau) \sinh_h(\sigma(\tau), t_k) \right] \Delta \tau
\]

(79)

\[
= \mu(t)F^a(t) \cosh_h(\sigma(t), t_k) - \mu(t)F^a(t) \sinh_h(\sigma(t), t_k).
\]

\[
\eta_k(\sigma(t)) - \eta_k(t) = -\mu(t)F^a(t) \sinh_h(\sigma(t), t_k) + \mu(t)F^a(t) \cosh_h(\sigma(t), t_k).
\]

(80)

From Proposition 1-(II) and the property (p2) it follows
\[
sinh_h(\sigma(t), t_k) - sinh_h(t + h, t_k) = r(t) \cosh_h(t, t_k) \mu(t),
\]
\[
cosh_h(\sigma(t), t_k) - cosh_h(t + h, t_k) = r(t) \sinh_h(t, t_k) \mu(t).
\]

(81)

Due to the above results, for each \( a \in [0, 1] \), we uniformly have

\[
U_k^a(\sigma(t)) - U_k^a(t) - \left[ r(t)U_k^a(t_k) + F^a(t) \right] \mu(t)
\]

\[
= \cosh_h(\sigma(t), t_k) \left[ L_ku_k^a(t_k) + \xi_k(\sigma(t)) \right] + \sinh_h(\sigma(t), t_k) \left[ L_ku_k^a(t_k) + \eta_k(\sigma(t)) \right]
\]

\[
- \cosh_h(t, t_k) \left[ L_ku_k^a(t_k) + \xi_k(t) \right] - \sinh_h(t, t_k) \left[ L_ku_k^a(t_k) + \eta_k(t) \right]
\]

\[
- r(t)\mu(t) \left[ \sinh_h(t, t_k) \left[ L_ku_k^a(t_k) + \xi_k(t) \right] + \cosh_h(t, t_k) \left[ L_ku_k^a(t_k) + \eta_k(t) \right] \right] - F^a(t) \mu(t)
\]

\[
= \left[ \cosh_h(\sigma(t), t_k) - \cosh_h(t, t_k) - r(t)\mu(t) \sinh_h(t, t_k) \right] \left[ L_ku_k^a(t_k) + \xi_k(\sigma(t)) \right]
\]

\[
+ \left[ \sinh_h(\sigma(t), t_k) - \sinh_h(t, t_k) - r(t)\mu(t) \cosh_h(t, t_k) \right] \left[ L_ku_k^a(t_k) + \eta_k(\sigma(t)) \right]
\]

\[
+ \cosh_h(t, t_k) \left( \xi_k(\sigma(t)) - \xi_k(t) \right) + r(t)\mu(t) \sinh_h(t, t_k) \left[ \xi_k(\sigma(t)) - \xi_k(t) \right]
\]

\[
+ \sinh_h(t, t_k) \left( \eta_k(\sigma(t)) - \eta_k(t) \right) + r(t)\mu(t) \cosh_h(t, t_k) \left[ \eta_k(\sigma(t)) - \eta_k(t) \right] - F^a(t) \mu(t)
\]
Now, (71) holds, and (i) is proved. We now check that the (i)-solution of LIFDE (39) on $J_k$.

(ii) For $k \leq (0, \alpha)$ and $\sigma$ given in (51). Load $\mu (t, t, \tau, \sigma (t), t_k) = \sigma (t, t_k) \Delta \tau$ for all right-scattered point $t \in J_k$. In consequence, $U_k$ is a (ii)-solution of LIFDE (39) on $J_k$ for $k = 0, 1, 2, \ldots$ Similarly, for $t \in J_0$, we have

$$U_k (t) = U (t) = \mu (t) \left( \psi (t) + r (t) \sinh (t) \right) - e (t, \tau, \sigma (t)) \left( L_k U_k (t) + \int_{t_k}^t F (r) e (r, \sigma (r), t_k) \Delta \tau \right).$$

This shows that (71) holds, and LIFDE (39) has a (i)-solution on $\mathbb{T}_+$. This proof is complete.

**Theorem 3.** If $r \equiv 0$, then LIFDE (39) has a (i)-solution
provided the above H-differences exist and $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ for $k = 0, 1, 2, \ldots$.

**Remark 3.** If $\mathbb{T} = \mathbb{R}$, then $E_r(t, s) \equiv 1$, $\sinh_r(t, s) = \sinh \left\{ \int_r^s r(\tau) d\tau \right\}$, and $\cosh_r(t, s) = \cosh \left\{ \int_r^s r(\tau) d\tau \right\}$. Thus, our results are the extension and improvement of the corresponding results in [3]. In particular, the present results restricted to $(t, s)$ in between, for instance, $t(1/2)$, the solution of LIFDE (39) is not unique in general.

In addition, we also extend these “classical cases” to cases “in between”, for instance, $\mathbb{T} = \{ \sum_{k=1}^{n} (1/k) : n \in \mathbb{N} \}$. In this case,

$$E_r(t, s) = \begin{pmatrix} n-s+r \\ n-s \end{pmatrix} \begin{pmatrix} n-s-r \\ n-s \end{pmatrix},$$

$$\sinh_r(t, s) = \frac{1}{2} \left( \begin{pmatrix} n-s+r \\ n-s \end{pmatrix} - \begin{pmatrix} n-s-r \\ n-s \end{pmatrix} \right), \quad \cosh_r(t, s) = \frac{1}{2} \left( \begin{pmatrix} n-s+r \\ n-s \end{pmatrix} + \begin{pmatrix} n-s-r \\ n-s \end{pmatrix} \right),$$

where $t = \sum_{k=1}^{n} (1/k)$. We can also consider the so-called q-difference problems.

**Remark 4.** Although problem (44) has a unique solution on $\mathbb{T}_+$, the solution of LIFDE (39) is not unique in general.

**4. Examples**

In this section, we present several examples to further illustrate the applicability of the results involved in the above sections.

**Example 1.** Consider the impulsive fuzzy dynamic equation:

$$\Delta_H U(t) + aU(t) + f(t), \quad t \in \mathbb{T} \setminus \mathcal{J},$$

$$U(t_k^+) = L_k U(t_k), \quad k = 0, 1, 2, \ldots, \quad U(t_0) = U_0,$$

where $a < 0, a \in \mathbb{R}_1^+$, and $L_k$ is linear bounded. It is not difficult to infer the existence of the H-differences $U_0 \bigcup \mathbb{R} \int_0^t r e_{\mathbb{R} a}(\alpha(\tau)) d\tau$ and $L_k U(t_k) - \mathbb{R} \int_{t_k}^t r e_{\mathbb{R} a}(\alpha(\tau), t_k) d\tau$. Thus, by Theorem 1, (ii)-solution of LIFDE (92) on $\mathbb{T}_+$ is given by

$$V(t) = e^a t \left[ U_0 \bigcup \mathbb{R} \int_0^t -r e^{-\alpha r} d\tau \right], \quad t \in \mathbb{J}_-,$$

$$U_k(t) = e^a \left[ L_k U_k(t_k) - \mathbb{R} \int_{t_k}^t r e^{-\alpha r} d\tau \right], \quad t \in \mathbb{J}_+(k = 0, 1, 2 \ldots).$$

**Remark 5.** If $\mathbb{T} = \mathbb{R}$, then we get
We obtain by calculating
\[ U_k(t) = \frac{1}{a}(t + \frac{1}{a})T + \left[ L_kU_k(t_k) + \left( \frac{t}{a} + \frac{1}{a^2} \right) \right] e^{a(t-t_k)}, \]
(95)
with \( T = \chi_{[1]} \), \( t \in J_k \). We observe that in this case
\[ D[U(t), (1/a)(t + (1/a)T)] \leq \|L_kU_k(t_k) + (t/a)\| e^{\alpha}, \]
and this implies that \( \lim_{t \to \infty} D[U(t), (1/a)(t + (1/a)T)] = 0 \). An interpretation in the light of [1] is that the uncertainty asymptotically disappears on the fuzzy system.
If \( T = \mathbb{Z} \), then
\[ V(t) = (1 + a)^{\left[ U_0 - \sum_{r=0}^{t-1} (-r) (1 + a)^{-r+1} \right] \}, \quad t \in J_- \]
\[ U_k(t) = (1 + a)^{t-t_k} \left[ L_kU_k(t_k) - \sum_{r=t_k}^{t-1} (-r) (1 + a)^{-r+1} \right], \quad t \in J_k \] \( \{k = 0, 1, 2, \ldots\} \).
(96)
For \( t \in J_k \), we have
\[ U(t) = U_k(t) = L_kU_k(t_k) (1 + a)^{t-t_k} - \sum_{r=t_k}^{t-1} (-r) (1 + a)^{-r+1} \]
\[ = \frac{(1 + a)^{t-t_k}}{a \cdot 1} + \left[ L_kU_k(t_k) - \frac{1 + a}{a} \right] (1 + a)^{t-t_k}. \]
(97)
Therefore, in the discrete case, the phenomenon that the uncertainty asymptotically disappears on the fuzzy system arises only \( a > -1 \).

**Example 2.** Let us consider the LIFDE
\[ U(t) = \begin{cases} 
V(t) = \frac{1}{2} \left( 3e^t - 1 \right) \gamma, & t \in J_- = [0, \sqrt{\ln 2}] \cap \mathbb{T}_\gamma, \\
U_k(t) = 4U_k(\sqrt{(k+1)\ln 2}) e^{r-(k+1)\ln 2} + \frac{1}{2} \left( e^{r-(k+1)\ln 2} - 1 \right) \gamma, & t \in J_k \{k = 0, 1, 2, \ldots\}. \end{cases} \]
(99)
To seek the (ii)-solution, we note that \( \cosh(t, t_k) = \cosh(t^2 - t_k^2) \) and \( \sinh(t, t_k) = \sinh(t^2 - t_k^2) \) when \( r(t) = 2t \). The solution of the ODEs system corresponding to (43) is
\[ \frac{U_k^a(t)}{U_k^{a-1}(t_k)\cosh(t^2-t_k^2) + 4U_k^{-a}(t_k)\sinh(t^2-t_k^2)} \]
\[ - \frac{1}{2} \left[ \cosh(t^2-t_k^2) + \sinh(t^2-t_k^2) \right] - 1 \left[ \cosh(t^2-t_k^2) - \sinh(t^2-t_k^2) \right], \]
\[ \frac{U_k^{a-1}(t)}{U_k^a(t_k)\sinh(t^2-t_k^2) + 4U_k^{-a}(t_k)\cosh(t^2-t_k^2)} \]
\[ - \frac{1}{2} \left[ \cosh(t^2-t_k^2) + \sinh(t^2-t_k^2) \right] - 1 \left[ \cosh(t^2-t_k^2) - \sinh(t^2-t_k^2) \right]. \]
(100)
It is easy to see that the \( H \)-difference
\[ y - H \int_0^t \left[ ry \sinh(r^3) \right] dr, \]
(101)
exists on \([0, \sqrt{\ln 2}]\), and the (ii)-solution \( V(t) \) of (98) on \([0, \sqrt{\ln 2}]\) can be written as
\[ V(t) = \frac{1}{2} \left( 3e^t - 1 \right) \gamma. \]
(102)
In particular, $V(\sqrt{\ln 2}) = (1/4)y$. From $L_0 U_0(t_0) = 4 \times (1/4)y = y$, it follows that the $H$-difference 

$$L_0 U_0(t_0) - H \int_{t_0}^{t} \left[ \gamma y \sinh(r^2 - r^2_0) \gamma y \cosh(r^2 - r^2_0) \right] dr,$$

$$= y - H \int_{\sqrt{\ln 2}}^{t} \left[ \gamma y \sinh(r^2 - \ln 2) \gamma y \cosh(r^2 - \ln 2) \right] dr,$$

(103)

exists on $(\sqrt{\ln 2}, \sqrt{2 \ln 2})$, and the (ii)-solution $U_0(t)$ of (98) on $(\sqrt{\ln 2}, \sqrt{2 \ln 2})$ can be written as

$$U_0(t) = \cosh(t^2 - \ln 2) \left[ y - H \int_{\sqrt{\ln 2}}^{t} \left[ \gamma y \sinh(r^2 - \ln 2) \gamma y \cosh(r^2 - \ln 2) \right] dr \right]$$

$$- H \left[ - \sinh(t^2 - \ln 2) \left[ y - H \int_{\sqrt{\ln 2}}^{t} \left[ \gamma y \sinh(r^2 - \ln 2) \gamma y \cosh(r^2 - \ln 2) \right] dr \right] \right]$$

$$= (1 - \frac{1}{2} (\sinh(t^2 - \ln 2) + \cosh(t^2 - \ln 2) - 1))(\cosh(t^2 - \ln 2) - \sinh(t^2 - \ln 2)) y$$

$$= \frac{1}{2} (3e^{-((t^2 - \ln 2))} - 1).$$

In particular, $U_0(\sqrt{2 \ln 2}) = (1/4)y$. On the analogy of this process, we obtain that the (ii)-solution of (98) on $J_k = (\sqrt{(k+1) \ln 2}, \sqrt{(k+2) \ln 2})$ can be expressed by

$$U_k(t) = \frac{1}{2} \left( 3e^{-((t^2 - (k+1) \ln 2))} - 1 \right), \quad \text{for } t \in J_k,$$

$$U_k(\sqrt{(k+2) \ln 2}) = \frac{1}{4} y, \quad k = 1, 2, \ldots$$

(105)

$$U(t) = \begin{cases} V(t) = y, \\ U_k(t) = \prod_{r=2k+1}^{r-1} (1 + 2r) \cdot \left[ 4U_{k-1}(t_k) + \sum_{r=2k+1}^{r-1} \frac{r}{(1 + 2r)^2} \right], \quad t \in J_k(k = 0, 1, 2, \ldots) \end{cases}$$

(106)

Here, $J_0 = [0, 1] \cap Z$, $J_k = (2k + 1, 2k + 3) \cap Z$, $U_0(t_0) = y$, and $U_0(t_1) = 3 \cdot 5 (4y) + (1/5)y + 2y, \ldots$

Similarly, we can present the expression of the (ii)-solutions of (98) in the discrete case.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Acknowledgments**

This study was supported by the National Natural Science Foundation of China (grant nos. 71771068, 71831006, and 71471051).

**References**

[1] B. Bede, I. J. Rudas, and A. L. Bencsik, “First order linear fuzzy differential equations under generalized differentiability,” *Information Sciences*, vol. 177, no. 7, pp. 1648–1662, 2007.

[2] L. A. Zadeh, “Toward a generalized theory of uncertainty (GTU)--an outline,” *Information Sciences*, vol. 172, no. 1-2, pp. 1-40, 2005.

[3] A. Khastan, J. J. Nieto, and R. Rodrigo-Lopez, “Variation of constant formula for first order fuzzy differential equations,” *Fuzzy Sets and Systems*, vol. 177, no. 1, pp. 20–33, 2011.
[4] A. Khastan and R. Rodríguez-López, "On the solutions to first order linear fuzzy differential equations," Fuzzy Sets and Systems, vol. 295, pp. 114–135, 2016.

[5] M. Zeinali and S. Shahmorad, "An equivalence lemma for a class of fuzzy implicit integro-differential equations," Journal of Computational and Applied Mathematics, vol. 327, pp. 388–399, 2018.

[6] Y. Shen, W. Chen, and J. Wang, "Fuzzy Laplace transform method for the Ulam stability of linear fuzzy differential equations of first order with constant coefficients," Journal of Intelligent & Fuzzy Systems, vol. 32, no. 1, pp. 671–680, 2017.

[7] J. J. Nieto, A. Khastan, and K. Ivaz, "Numerical solution of fuzzy differential equations under generalized differentiability," Nonlinear Analysis: Hybrid Systems, vol. 3, no. 4, pp. 700–707, 2009.

[8] J. J. Nieto, R. Rodríguez-lópez, and D. Franco, "Linear first-order fuzzy differential equations," International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, vol. 14, no. 6, pp. 687–709, 2006.

[9] J. J. Nieto, R. Rodríguez-López, and M. Villanueva-Pesquera, "Exact solution to the periodic boundary value problem for a first-order linear fuzzy differential equation with impulses," Fuzzy Optimization and Decision Making, vol. 10, no. 4, pp. 323–339, 2011.

[10] S. Cai and Q. Zhang, "Existence and stability of periodic solutions for impulsive fuzzy BAM Cohen-Grossberg neural networks on time scales," Advances in Difference Equations, vol. 2016, no. 1, p. 64, 2016.

[11] Y. Chalco-Cano and H. Román-Flores, "On new solutions of fuzzy differential equations," Chaos, Solitons & Fractals, vol. 38, no. 1, pp. 112–119, 2008.

[12] Y. Chalco-Cano and H. Román-Flores, "Comparison between some approaches to solve fuzzy differential equations," Fuzzy Sets and Systems, vol. 160, no. 11, pp. 1517–1527, 2009.

[13] O. S. Fard and N. Ghal-Eh, "Numerical solutions for linear system of first-order fuzzy differential equations with fuzzy constant coefficients," Information Sciences, vol. 181, no. 20, pp. 4765–4779, 2011.

[14] E. J. Villamizar-Roa, V. Angulo-Castillo, and Y. Chalco-Cano, "Existence of solutions to fuzzy differential equations with generalized Hukuhara derivative via contractive-like mapping principles," Fuzzy Sets and Systems, vol. 265, pp. 24–38, 2015.

[15] M. Chehlabi and T. Allahviranloo, "Concreted solutions to fuzzy linear fractional differential equations," Applied Soft Computing, vol. 44, pp. 108–116, 2016.

[16] J. E. Macias-Díaz and S. Tomasiello, "A differential quadrature-based approach à la Picard for systems of partial differential equations associated with fuzzy differential equations," Journal of Computational and Applied Mathematics, vol. 299, no. 1, pp. 15–23, 2016.

[17] A. M. Bertone, R. M. Jafelice, L. C. de Barros, and R. C. Bassanezi, "On fuzzy solutions for partial differential equations," Fuzzy Sets and Systems, vol. 219, pp. 68–80, 2013.

[18] J. J. Buckley, T. Feuring, and Y. Hayashi, "Fuzzy differential equations: the initial value problem," Journal of Advanced Computational Intelligence and Intelligent Informatics, vol. 5, no. 6, pp. 315–325, 2001.

[19] K. A. Chrysafis, B. K. Papadopoulos, and G. Papaschinopoulos, "On the fuzzy differential equations of finance," Fuzzy Sets and Systems, vol. 159, no. 24, pp. 3259–3270, 2008.

[20] E. Y. Deeba, A. D. Korvin, and E. L. Koh, "A fuzzy difference equation with an application," Journal of Difference Equations and Applications, vol. 2, no. 4, pp. 365–374, 1996.

[21] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fuzzy differential equations," Journal of Difference Equations and Applications, vol. 8, no. 11, pp. 957–968, 2002.

[22] B. K. Papadopoulos and G. Papaschinopoulos, "On the fuzzy differential equation $x_{n+1} = A + (x_{n-m}),$" Fuzzy Sets and Systems, vol. 129, pp. 73–81, 2002.

[23] G. Stefanidou, G. Papaschinopoulos, and C. J. Schinas, "On an exponential-type fuzzy difference equation," Advances in Difference Equations, vol. 2010, Article ID 196920, 2010.

[24] B. Bede and S. G. Gal, "Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations," Fuzzy Sets and Systems, vol. 151, no. 3, pp. 581–599, 2005.

[25] V. Lakshmikatham, T. Gnana Bhaskar, and J. V. Devi, Theory of Set Differential Equations in Metric Spaces, Cambridge Scientific Publishers, Cambridge, UK, 2006.

[26] J. Li, A. Zhao, and J. Yan, "The Cauchy problem of fuzzy differential equations under generalized differentiability, Fuzzy Sets and Systems, vol. 200, pp. 1–24, 2012.

[27] V. Spedding, "Taming nature’s numbers," New Science, vol. 174, pp. 28–31, 2003.

[28] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," Results in Mathematics, vol. 18, no. 1-2, pp. 18–56, 1990.

[29] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, an Introduction with Applications, Birkhäuser, Basel, Switzerland, 2001.

[30] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Basel, Switzerland, 2003.

[31] M. Bohner and A. Peterson, "First and second order linear dynamic equations on time scales," Journal of Difference Equations and Applications, vol. 7, no. 6, pp. 767–792, 2001.

[32] M. L. Puri and D. A. Ralescu, "Differentials of fuzzy functions," Journal of Mathematical Analysis and Applications, vol. 91, no. 2, pp. 552–558, 1983.

[33] O. Kaleva, "Fuzzy differential equations," Fuzzy Sets and Systems, vol. 24, no. 3, pp. 301–317, 1987.

[34] R. Alkhami, F. Bahrami, and A. Jabbari, "Existence of global solutions to nonlinear fuzzy Volterra integro-differential equations," Nonlinear Analysis: Theory, Methods & Applications, vol. 75, no. 4, pp. 1810–1821, 2012.

[35] J. J. Buckley and T. Feuring, "Fuzzy differential equations," Fuzzy Sets and Systems, vol. 110, no. 1, pp. 43–54, 2000.

[36] P. E. Kloeden and T. Lorenz, "Fuzzy differential equations without fuzzy convexity," Fuzzy Sets and Systems, vol. 230, pp. 65–81, 2013.

[37] O. S. Fard and T. A. Bidgoli, "Calculus of fuzzy functions on time scales (I)," Soft Computing, vol. 19, no. 2, pp. 293–305, 2015.

[38] C. Vasavi, G. Suresh Kumar, and M. S. N. Murty, "Generalized differentiability and integrability for fuzzy set-valued functions on time scales," Fuzzy Sets and Systems, vol. 20, no. 3, pp. 297–305, 2010.

[39] C. Vasavi, G. Suresh Kumar, and M. S. N. Murty, "Fuzzy dynamic equations on time scales under second type Hukuhara delta derivative," International Journal of Chemical Sciences, vol. 14, no. 1, pp. 49–66, 2016.

[40] C. Vasavi, G. Suresh Kumar, and M. S. N. Murty, "Fuzzy Hukuhara delta dierential and applications to Fuzzy Dynamic equations on time scales," Journal of Uncertain Systems, vol. 10, no. 3, pp. 163–180, 2016.

[41] S. Hong, "Differentiability of multivalued functions on time scales and applications to multivalued dynamic equations," Nonlinear Analysis: Theory, Methods & Applications, vol. 71, no. 9, pp. 3622–3637, 2009.
[42] S. Hong and Y. Peng, “Almost periodicity of set-valued functions and set dynamic equations on time scales,” Information Sciences, vol. 330, pp. 157–174, 2016.

[43] S. H. Hong, J. Gao, and Y. Peng, “Solvability and stability of impulsive set dynamic equations on time scales,” Abstract and Applied Analysis, vol. 2014, Article ID 610365, 19 pages, 2014.

[44] S. Hong, “Stability criteria for set dynamic equations on time scales,” Computers & Mathematics with Applications, vol. 59, no. 11, pp. 3444–3457, 2010.

[45] S. Hong and J. Liu, “Phase spaces and periodic solutions of set functional dynamic equations with infinite delay,” Nonlinear Analysis: Theory, Methods & Applications, vol. 74, no. 9, pp. 2966–2984, 2011.

[46] P. Diamond and P. Kloeden, Metric Spaces of Fuzzy Sets, World Scientific, Singapore, 1994.

[47] L. Stefanini, “A generalization of Hukuhara difference and division for interval and fuzzy arithmetic,” Fuzzy Sets and Systems, vol. 161, no. 11, pp. 1564–1584, 2010.