Conservative Currents of Boundary Charges in $AdS_{2+1}$ Gravity

Sze-Shiang Feng #, 1, 2, 3, Bin Wang 1, 4, Xin-He Meng 5
1. High Energy Section, ICTP, Trieste, 34100, Italy
e-mail:fengss@ictp.trieste.it
2. CCAST(World Lab.), P.O. Box 8730, Beijing 100080
3. Department of Modern Physics, University of Science and Technology of China, 230026, Hefei, China
e-mail:zhdp@ustc.edu.cn
4. Physics Department, Shanghai Normal University, 200234, Shanghai, China
5. Theoretical Physics Division, Department of Physics, Nankai University, Tianjin, China

Abstract

The boundary charges which constitute the Virasoro algebra in 2+1 dimensional anti-de Sitter gravity are derived by way of Noether theorem and diffeomorphic invariance. It shows that the boundary charges under discussion recently exhaust all the independent nontrivial charges available. Therefore, the state counting via the Virasoro algebra is complete.

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1 Introduction

General relativity is a highly non-linear field theory which is very complicated at both classical and quantum level. In fact, the concept of quantum gravity has not been established in
3+1 dimensions, let alone the whole theory of it. This renders the gravitational interaction be the only one in nature still to quantized. While paying efforts to the quantization of gravitation in 3+1 dimensions, physicists have also been working with the problem in lower dimensions in the past decades in order to get some hints. A remarkable observation was made by Brown and Henneaux [1] that the asymptotic symmetry group of $\text{AdS}_{2+1}$ is generated by (two copies of) the Virasoro algebra, and that therefore any consistent quantum theory of gravity on $\text{AdS}_{2+1}$ is a conformal field theory. They further computed the value of the central charge as $c = \frac{3}{2G\sqrt{-\lambda}}$, where $G$ is the Newton’s gravitational constant and $\lambda$ is the cosmological constant. Witten first showed that 2+1 dimensions Einstein gravity (with or without cosmological term) can be formulated as a Chern-Simons theory[2][3] and this renders the theory exactly soluable at the classical and quantum levels. As in 3+1 dimensions, black hole aspects in lower dimensions have also been of great interests, and the discovery of BTZ black hole [4][5]has proven to be a milestone. Based on the discovery in [1] and Cardy’s formula for state counting in conformal field theories[6], Strominger was able to compute microscopically the black hole entropy from the asymptotic growth of states[7], and this helped a lot about the understanding of the origin of Bekenstein-Hawking entropy. Now it is understood that boundary behavior of the spacetime is very important to the understanding of both the classical dynamics and the quantum aspects[8] - [10]. The highlights is like this. Firstly, one is to find Hilbert space consisting of all the solutions. Secondly, one is to find as many as possible charges $Q$ in order to classify the solutions accoding to whether one can be generated by the charges from another. It is obvious that once we can find a complete set of charges (like the complete set of operators in quantum mechanics) $Q$, i.e. no more, the structure of the Hilbert space can be completely determined. Then two problems arises immediately. One is whether the charges are physical observables, the other is how we could find as many as possible the charges. To the first one, we take the point of view that the charges in [8] - [10] are observables because even if they do not commute with the constraints, they could still be according to the argument in [11]. The second question is to be answered here.

In this paper, we make use of the diffeomorphism invariance and the Noether theorem to obtain the conservative charges corresponding to each arbitrary diffeormorphism trans-
transform one solution to another different one. So the collection of charges there exhausts all the conservative charges corresponding to diffeomorphisms. The layout of this paper is like this. Section 2 is a presentation of the general approach to conservation laws in general relativity. Section 3 applies this approach to 2+1 dimensional gravity. Section 4 is devoted to final discussions.

2 General Scheme for Conservation Laws in General Relativity

As in 1+3 Einstein gravity, conservation laws are also the consequence of the invariance of the action corresponding to some transforms. In order to study the covariant energy-momentum of more complicated systems, it is beneficial to discuss conservation laws by Noether theorem in general [12][13]. Suppose that the spacetime is of dimension $D = 1 + d$ and the Lagrangian is in the first order formalism, i.e.

$$I = \int_G \mathcal{L}(\phi^A, \partial_\mu \phi^A) d^D x$$

(1)

where $\phi^A$ denotes the generic fields. If the action is invariant under the infinitesimal transforms

$$x'^\mu = x^\mu + \delta x^\mu \quad \phi'^A(x') = \phi^A(x) + \delta \phi^A(x)$$

(2)

(it is not required that $\delta \phi^A |_{\partial \mathcal{L}} = 0$), then the following relation holds [8]-[10] (see the proof in the appendix).

$$\partial_\mu (\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A} \delta_0 \phi^A) + [\mathcal{L}]_{\phi^A} \delta_0 \phi^A = 0$$

(3)

where

$$[\mathcal{L}]_{\phi^A} = \frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A}$$

(4)

and $\delta_0 \phi^A$ is the Lie variation of $\phi^A$

$$\delta_0 \phi^A = \phi'^A(x) - \phi^A(x) = \delta \phi^A(x) - \partial_\mu \phi^A \delta x^\mu$$

(5)

If $\mathcal{L}$ is the total Lagrangian of the system, the field equations of $\phi^A$ is just $[\mathcal{L}]_{\phi^A} = 0$. Hence from eq.(3), we can obtain the conservation equation corresponding to transform eq.(2)

$$\partial_\mu (\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A} \delta_0 \phi^A) = 0$$

(6)
It is important to recognize that if $L$ is not the total Lagrangian, e.g. the gravitational part $L_g$, then so long as the action of $L_g$ remains invariant under transform eq.(2), eq.(3) is still valid yet eq.(6) is no longer admissible because of $[L_g]_{\phi^A} \neq 0$.

Suppose that $\phi^A$ denotes the Riemann tensors $\phi^A_\mu$ and Riemann scalars $\psi^A$ (for the model considered in this paper, they are $A^{(\pm)a}$ and there are no Riemann scalar fields). Eq.(3) reads

$$\partial_\mu (L_g \delta x^\mu + \frac{\partial L_g}{\partial \phi^A_\nu} \delta_0 \phi^A_\nu) + [L_g]_{\phi^A} \delta_0 \phi^A_\mu = 0$$  \(7\)

Under transforms eq.(2), the Lie variations are

$$\delta_0 \phi^A_\nu = -\delta x^\alpha,\nu \phi^A_\alpha - \phi^A_{\nu,\alpha} \delta x^\alpha$$  \(8\)

where the dot "," denotes partial derivative. So eq.(7) reads

$$\partial_\mu [L_g \delta x^\mu - \frac{\partial L_g}{\partial \phi^A_\lambda} (\delta x^\nu, \phi^A_\nu + \phi^A_{\lambda,\nu} \delta x^\nu)] - [L_g]_{\phi^A} \delta_0 \phi^A_\mu \delta x^\nu = 0$$  \(9\)

Comparing the coefficients of $\delta x^\nu, \delta x^\nu, \lambda$ and $\delta x^\nu, \mu, \lambda$, we may obtain an identity

$$\partial_\lambda ([L_g]_{\phi^A_\nu} \phi^A_\mu) = [L_g]_{\phi^A_\nu} \phi^A_\lambda$$  \(10\)

Then eq.(9) can be written as

$$\partial_\mu [L_g \delta x^\mu - \frac{\partial L_g}{\partial \phi^A_\lambda} (\delta x^\nu, \phi^A_\nu + \phi^A_{\lambda,\nu} \delta x^\nu)] - [L_g]_{\phi^A} \phi^A_\nu \delta x^\nu = 0$$  \(11\)

or

$$\partial_\mu [(L_g \delta^\mu_\nu - \frac{\partial L_g}{\partial \phi^A_\lambda} \phi^A_{\lambda,\nu} - [L_g]_{\phi^A_\nu} \phi^A_\lambda) \delta x^\nu - \frac{\partial L_g}{\partial \phi^A_{\lambda,\nu}} \phi^A_{\lambda,\nu} \delta x^\nu] = 0$$  \(12\)

By definition, we introduce

$$\tilde{I}^\mu_\nu = -(L_g \delta^\mu_\nu - \frac{\partial L_g}{\partial \phi^A_\lambda} \phi^A_{\lambda,\nu} - [L_g]_{\phi^A_\nu} \phi^A_\lambda)$$  \(13\)

$$\tilde{Z}^{\lambda\mu}_\nu = \frac{\partial L_g}{\partial \phi^A_{\lambda,\mu}} \phi^A_\nu$$  \(14\)

Then eq.(12) gives

$$\partial_\mu (\tilde{I}^\mu_\nu \delta x^\nu + \tilde{Z}^{\lambda\mu}_\nu \delta x^\nu, \lambda) = 0$$  \(15\)

So by comparing the coefficients of $\delta x^\nu, \delta x^\nu, \mu$ and $\delta x^\nu, \mu, \lambda$, we have the following from eq.(15)

$$\partial_\mu \tilde{I}^\mu_\nu = 0$$  \(16\)
\[ I_{\nu} = -\partial_{\mu} \bar{Z}_{\nu}^{\lambda \mu} \quad \bar{Z}_{\nu}^{\mu \lambda} = -\bar{Z}_{\nu}^{\lambda \mu} \]  

Eq.(16)-(17) are fundamental to the establishing of conservation law of energy-momentum in [12] and [13].

Now suppose that \( \delta x^{\mu} = \epsilon \xi^{\mu}(x) \), with \( \epsilon \) is an infinitesimal constant parameter and \( \xi^{\mu}(x) \) is an arbitrary vector. Then it follows that from eq.(15) - eq.(17)

\[ \partial_{\mu} \bar{j}^{\mu}(\xi) = 0 \]  

(18)

where

\[ \bar{j}^{\mu}(\xi) = \partial_{\nu} \bar{Z}^{\nu \mu} \]  

(19)

and

\[ \bar{Z}^{\nu \mu} = \bar{Z}^{\nu \mu}_{\alpha} \xi^{\alpha} \]  

(20)

Accordingly, we have the conserved charge associated with \( xi \)

\[ Q[\xi] = \int_{\Sigma} \bar{j}^{0} d^{2}x = \int_{\partial \Sigma} \bar{Z}^{0 \mu} \epsilon_{ij} d\xi^{j} \]  

(21)

It is obvious that if we choose \( \xi^{\mu} = \epsilon^{\mu}_{a} e^{a}, \epsilon^{a} = const. \), we can obtain the energy-momentum immediately.

3 Diffeomorphic Charges in \( AdS_{2+1} \) Gravity

Einstein gravity with a negative cosmological constant can be re-formulated as a Chern-Simons theory for the group \( SL(2, R) \times SL(2, R) \) [4], with connection one-forms

\[ A^{(\pm)}{a} = \omega^{a} \pm \frac{1}{\ell} e^{a} \]  

(22)

where \( e^{a} = e^{a}_{\mu} dx^{\mu} \) is the triad and the \( \omega^{a} = \frac{1}{2} e^{abc} \omega_{\mu bc} dx^{\mu} \) is the spin connection. The Einstein-Hilbert action becomes

\[ I = I_{CS}[A^{(-)}] - I_{CS}[A^{(+)\,}] \]  

(23)

where

\[ I_{CS}[A] = \frac{k}{4\pi} \int_{M} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \]  

(24)

is the Chern-Simons action. \( A^{(\pm)} = A^{(\pm){a}} J^{(\pm)}{a} \). The value of the coupling constant \( k \) depends on the choice of the representation and the definition of the trace. With the choice \( J^{(\pm)}{a} = \)
\[ -i \sigma_2, J_1^{(\pm)} = \frac{1}{2} \sigma_3, J_2^{(\pm)} = \frac{1}{2} \sigma_1, \text{ where } \sigma_{1,2,3} \text{ are the ordinary Pauli matrices}, \]

\[ k \text{ should take the value } k = \frac{i}{4G} \text{ and } \text{Tr} J^a J^b = -\frac{1}{2} \eta^{ab}, \text{Tr} J^a J^b J^c = \frac{1}{4} \epsilon^{abc}, \epsilon^{012} = 1, \eta^{ab} = \text{diag}(1, -1, -1). \]

So the action is explicitly expressed as

\[ I_{CS}[A] = -\frac{k}{8\pi} \int_M (\eta_{ab} A^a \wedge dA^b - \frac{1}{3} \epsilon_{abc} A^a \wedge A^b \wedge A^c) \]

\[ = -\frac{k}{8\pi} \int_M \epsilon^{\alpha\beta\nu} (\eta_{ab} A^a_{\alpha} \partial_\beta A^b_\nu - \frac{1}{3} \epsilon_{abc} A^a_{\alpha} A^b_{\beta} A^c_\nu) \]  
(25)

The field equations are

\[ F = 0 \]  
(26)

where \( F = dA + A \wedge A \).

It should be mentined that it is quite novel that the Chern-Simons formulation not only reproduce the Enstein-Hilbert part of the Lagrangian, but also can reproduce the Chern-Simons of the spin connection, this is can accomplished by modifying the quardratic forms \[12\] \[13\].

Now from eq.(14), (19) and eq.(25), we have

\[ \tilde{Z}^{\mu\nu}(\xi) = \frac{k}{8\pi} \epsilon^{\mu\nu\alpha} \eta_{ab} A^a_\alpha A^b_\beta \xi^\beta \]  
(27)

Or, if we define \( \lambda^a = A^a_{\mu} \xi^\mu \) as in \[8\],we have up to a constant factor

\[ Q(\lambda) = \frac{k}{4\pi} \int \eta_{ab} \lambda^a A^b_\phi dx^k \]  
(28)

Treating the parameter \( \lambda^a \) as field independent, this is just the charge generating gauge transformations. How to obtain the charge generating diffeomorphism transformations?

Note that to an arbitrary vector whose zero component is zero \( \xi^\mu = (0, \xi^\rho, \xi^\phi) \), there exists a charge corresponging to it, now consider another vector \( \xi'^\mu = (0, \xi^\rho, 0) \). The sum of \( Q(\xi) \) and \( Q(\xi') \) then realizes the charge required which is re-termed as \( Q(\xi) \) (in the special gauge \( A^a_{\rho} = \alpha^a \), it is just the eq.(2.18) in \[9\])

\[ Q(\xi) = \frac{k}{4\pi} \int_{\partial\Sigma} \eta_{ab} (\xi^\rho A^a_\rho A^b_\phi + \xi^i A^a_\rho A^b_\phi) d\phi \]  
(29)

Note that in both cases, the charges is differentiable with respect to the connection \( A \). One can naturally go even further: what it will generate if the \( \xi \) is made more field-dependent?
Suppose that $\lambda^a$ is a functional of $A_i^a$ and the functional derivative is well-defined everywhere. Then consider the functional

$$G(\lambda) = \frac{k}{8\pi} \int_\Sigma \lambda_a \epsilon^{ij} F_{ij}^a d^2 x - Q(\lambda)$$

(30)

Under any variation $\delta A_i^a$ we have

$$\delta Q = \frac{k}{8\pi} \int_\Sigma (\delta \lambda_a \epsilon^{ij} F_{ij}^b + 2 \lambda_a \epsilon^{ij} \partial_i \delta A_j^b + 2 \lambda_b \epsilon^{ij} \epsilon_{bcd} A_i^c \delta A_j^d) d^2 x - 2 \int_{\partial \Sigma} (\delta_b \lambda A_k^b + \lambda_b \delta A_k^b) dx^k$$

(31)

So if we impose the condition that $\delta \lambda^a |_{\partial \Sigma} = 0$, then functional derivative of $Q$ with respect to $A$ is well-defined.

$$\frac{\delta G}{\delta A_j^a} = \frac{k}{8\pi} (2 \epsilon^{ji} D_i \lambda_d + \frac{\delta \lambda_a}{\delta A_j^a} \epsilon^{mn} F_{mn}^a)$$

(32)

Using the canonical Poisson brackets $\{ A_i^a(x), A_j^b(y) \} = -\frac{8\pi}{k} \epsilon_{ij} \eta^{ab} \delta^2(x-y)$, we have

$$\delta A_k^c = \{ G(\lambda), A_k^c \} = -2 D_k \lambda^c - 2 \frac{\delta \lambda_a}{\delta A_k^c} F_{jk}^a \approx -2 D_k \lambda^c$$

(33)

where the $\approx$ means modulo the constraints. That is, $G(\lambda(A))$ generates still a gauge transform.

Since there are two copies of vector fields, i.e. $A^{(\pm)}$, for each copy, there exists a conservative charge associated to every diffeomorphism. For any specific spacetime, the set of charges is unique as far as the numerical quantity of the charges are concerned. Especially, the energy-momentum and angular-momentum are also among them (it should not be strange that the angular-momentum can be also obtain since gauge transformation of $e_i^a$ and $\omega_i^a$ can be reproduced by diffeomorphism transformations in Chern-Simons formulation of gravity in 2+1 dimensions) since they can be acquired by linear combinations from $Q^{(\pm)}$ associated to constant $\lambda$.

4 Discussions

In this paper, we have obtained the conservative charges associated with diffeomorphisms in Chern-Simons formulation of 2+1 gravity and shown that the boundary charges generating Virasoro algebra exhaust all the independent charges. Therefore the state counting based on the representation of Virasoro algebra is complete.
It seems non-trivial to ask the question that why the boundary dynamics is so important to the understanding of the quantum feature of gravity in 2+1 dimensions. To our understanding, observables are at the kernel in any physical theory. In other non-gravitational physical theories, the observables such as energy-momentum and angular-momentum are localized while for gravitation, they are both determined by the dynamics of the field at spatial infinity [12]-[17]. Therefore, the boundary behaviour should play an important role in both classical and quantum aspects.

If the quantum theory of 2+1 dimensional gravity can be extended to the more realistic 3+1 Einstein gravity, it soon becomes clear that the quantum feature of gravity will be determined to a great extent by the boundary behavior while that of other interactions (such as QED and QCD) is determined mainly by local behaviour (of course some global properties are also important). Does it mean that the future unification of quantum gravity and other quantum field theories is an unification of local quantum theory with global quantum theory? This is any way the gravity an important conceptual problem.

Appendix: Proof of equation (3)

The action is supposed to be of the first order form eq.(1). The transformation of $\phi^A$ in eq.(2) contains two parts: the induced variation due to the coordinate transformation and the variation by its own. Under the coordinate transform, the integration domain $G$ is transformed to $G'$, so the variation of the action is

$$
\delta I = \int_{G'} \mathcal{L}(\phi^A(x'), \partial'_\mu \phi^A(x'))d^4x' - \int_G \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))d^4x
$$

$$
= \int_{G'} [\mathcal{L}(\phi^A(x'), \partial'_\mu \phi^A(x')) - \mathcal{L}(\phi^A(x'), \partial'_\mu \phi^A(x'))] + \mathcal{L}(\phi^A(x'), \partial'_\mu \phi^A(x'))d^4x' - \int_G \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))d^4x
$$

$$
= \int_{G'} \left[ \frac{\partial \mathcal{L}}{\partial \phi^A(x')} (\phi^A(x') - \phi^A(x')) + \frac{\partial \mathcal{L}}{\partial \partial'_\mu \phi^A(x')} (\partial'_\mu \phi^A(x') - \partial'_\mu \phi^A(x')) \right] + \int_{G'} \mathcal{L}(\phi^A(x'), \partial'_\mu \phi^A(x'))d^4x' - \int_G \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))d^4x
$$

In the part underlined, it makes no difference if one uses $\frac{\partial \mathcal{L}}{\partial \phi^A(x')}$ and $\frac{\partial \mathcal{L}}{\partial \partial'_\mu \phi^A(x')}$ because they coincide to the first order of infinitesimals. Using the definition of $\delta^0 \phi^A(x)$ in eq.(5), we have

$$
\delta I = \int_{G'} \left[ \frac{\partial \mathcal{L}}{\partial \phi^A(x')} \delta^0 \phi^A(x') + \frac{\partial \mathcal{L}}{\partial \partial'_\mu \phi^A(x')} \delta_0 (\partial'_\mu \phi^A(x')) \right]d^4x'
$$
In the one-dimensional case, the underlined part is simply

\[
\int_{x_1+\delta x_1}^{x_2+\delta x_2} \mathcal{L}(\phi^A(x'), \partial_\mu \phi^A(x')) dx' - \int_{x_1}^{x_2} \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x)) dx
\]

\[
= (\int_{x_1+\delta x_1}^{x_2+\delta x_2} - \int_{x_1}^{x_2+\delta x_2} + \int_{x_1+\delta x_1}^{x_2}) \mathcal{L}(\phi^A(x'), \partial_\mu \phi^A(x')) dx' - \int_{x_1}^{x_2} \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x)) dx
\]

\[
= \delta x_2 \mathcal{L}(\phi^A(x_2), \partial_\mu \phi^A(x_2)) - \delta x_1 \mathcal{L}(\phi^A(x_1), \partial_\mu \phi^A(x_1)) = \int_{x_1}^{x_2} \frac{d(\delta x \mathcal{L})}{dx} dx
\]

(36)

In general, it is \( f_G \partial_\mu (\delta x^\mu \mathcal{L}) d^4 x \). Therefore, it follows that

\[
\delta I = \int_G \left[ \partial_\mu (\delta x^\mu \mathcal{L}) + \frac{\partial \mathcal{L}}{\partial \phi} \delta_0 \phi^A + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A} \delta_0 (\partial_\mu \phi^A) \right] d^4 x
\]

(37)

Note that the Lie derivative operator \( \delta_0 \) commutes with the ordinary partial differential operator \( \partial_\mu \), so we have

\[
\delta I = \int_G \left[ \partial_\mu (\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A} \delta_0 \phi^A) + [\mathcal{L} \phi^A] \delta_0 \phi^A \right] d^4 x
\]

(38)

In general relativity, the action \( I \) is invariant under the transformation eq.(2) due to the invariance of the theory, accordingly we have eq.(3). Note that we do not require that \( \delta \phi^A |_{\partial G} = 0 \).

\textit{Note added in proof} Just at the finishing of this paper, the authors are awared of the works [17] and [18] which have some overlap with this paper.

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