On Duality of Two-dimensional Ising Model on Finite Lattice

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Abstract

It is shown that the partition function of the 2d Ising model on the dual finite lattice with periodical boundary conditions is expressed through some specific combination of the partition functions of the model on the torus with corresponding boundary conditions. The generalization of the duality relations for the nonhomogeneous case is given. These relations are proved for the weakly nonhomogeneous distribution of the coupling constants for the finite lattice of arbitrary sizes. Using the duality relations for the nonhomogeneous Ising model, we obtain the duality relations for the two-point correlation function on the torus, the 2d Ising model with magnetic fields applied to the boundaries and the 2d Ising model with free, fixed and mixed boundary conditions.

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1. Introduction

The duality relation for the two-dimensional Ising model was discovered by Kramers and Wannier [1] in 1941 year. They established the correspondence between the partition function of the model in low-temperature phase and the partition function in high-temperature phase

\[(\sinh 2\tilde{K})^{-N/2}\tilde{Z}(\tilde{K}) = (\sinh 2K)^{-N/2}Z(K)\] (1)

\[\sinh 2K \cdot \sinh 2\tilde{K} = 1.\]

Using this self-duality property, in [1] the critical temperature in the 2d Ising model was determined before the Onsager’s exact solution [2].

In paper [3] Kramers-Wannier duality relation (1) was generalized to the nonhomogeneous case (the coupling constants are arbitrary functions of lattice site coordinates)

\[\prod_{r,i}(\sinh 2\tilde{K}_i(r))^{-1/4}\tilde{Z}[\tilde{K}_i(\tilde{r})] = \prod_{r,i}(\sinh 2K_i(r))^{-1/4}Z[K_i(r)],\] (2)

\[\sinh 2K_1(r) \cdot \sinh 2\tilde{K}_1(\tilde{r}) = 1,\]

\[\sinh 2K_2(r) \cdot \sinh 2\tilde{K}_2(\tilde{r}) = 1.\] (3)

Here \(r, \tilde{r}\) and \(K_i(r), \tilde{K}_i(\tilde{r})\) are coordinates and coupling constants on the initial and dual lattices respectively (we will define them in the following section). Since Kadanoff-Ceva anzats (2) defines connection between functionals but no functions, this relation is very useful for analysis of the thermodynamic phases of the model. So, for example, this anzats allows to correctly define disorder parameter \(\mu\), to obtain the duality relation connecting correlation functions on the initial and dual lattices, to define "mixed" correlation functions \(\langle \sigma(r_1) \ldots \sigma(r_j), \mu(r_k) \ldots \mu(r_l) \rangle\) and e.s.

As was already mentioned in [1,3] relations (1) and (2) can not be understood as literal. So, for example, using method of comparing of high- and low-temperature expansions for deriving of duality relation (1), it is hard to take into account and to compare the graphs, which include spins on the boundaries (in particularly, graphs which contain cycles wrapping up the torus). In fact (1) is correct in the thermodynamic limit (for the specific free energy). However for the nonhomogeneous case the procedure of thermodynamic limit is rather ambiguous. In any case it is usefull to have exact relations (in contrast to (1) and (2)) connecting the partition functions on the initial and dual finite lattices. This is aim of our paper.

In the section 2 we introduce denotions and define the representation of the partition function in the form of the functional Grassmann integral. In section 3 we derive exact
duality relation for the homogeneous Ising model on the finite lattice with periodical boundary
conditions. It is shown that the partition function of the model on the dual lattice is expressed
through specific combination of the partition functions on the initial lattice on the torus
with the corresponding boundary conditions. In section 4 we propose the generalization
of this relation for the nonhomogeneous case. Here we prove this relation for the weakly
nonhomogeneous distribution of the coupling constants. In section 5 the exact duality relation
between two-point order-order and disorder-disorder correlation functions on the torus is
derived. In section 6 the duality relations for the 2d Ising model with magnetic fields applied
to the boundaries and the 2d Ising model with free, fixed and mixed boundary conditions are
obtained.

2. The model

The partition function of the 2d Ising model on the torus is

\[ 2^N Z[K] = \sum_{\{\sigma\}} e^{-\beta H[\sigma]}, \]

where \( r = (x, y) \) denotes the site coordinate on the square lattice of size \( N = n \times m \), \( x = 1, \ldots, n; y = 1, \ldots, m \); \( \sigma(r) = \pm 1 \); \( K_1(r) \) and \( K_2(r) \) are coupling constants along horizontal \( X \) and vertical \( Y \) axes respectively. The one-step shift operators \( \nabla_x, \nabla_y \) are acting on \( \sigma(r) \) in the following way

\[ \nabla_x \sigma(r) = \sigma(r + \hat{x}), \quad \nabla_y \sigma(r) = \sigma(r + \hat{y}), \]

where \( \hat{x}, \hat{y} \) are unit vectors along horizontal and vertical axes. For periodical boundary
conditions along \( X \) and \( Y \) axes

\[ \left( \nabla_x^p \right)^n = 1, \quad \left( \nabla_y^p \right)^m = 1, \]

and for antiperiodical boundary conditions

\[ \left( \nabla_x^a \right)^n = -1, \quad \left( \nabla_y^a \right)^m = -1. \]

Here \( p \) and \( a \) denote periodical and antiperiodical boundary conditions respectively.

Since we have the four type of the boundary conditions on the torus let us assign the

The corresponding indeces to the partition function: \( Z^{(\alpha, \beta)}[K] \) and, for example,

\[ 2^N Z^{(a,p)}[K] = \sum_{\{\sigma\}} \exp \left( \sum_r \sigma(r) (K_1(r) \nabla_x^a + K_2(r) \nabla_y^p) \sigma(r) \right). \]
Later on we will consider \( Z^{(\alpha,\beta)}[K] \) as the four-component vector \( Z[K] \) with components \( Z_b[K], b = 1, 2, 3, 4 \)

\[
Z[K] = (Z^{(p,p)}, Z^{(p,a)}, Z^{(a,p)}, Z^{(a,a)}).
\] (6)

We denote site coordinates, functions and functionals on the dual lattice by “tilda”:

\[
\tilde{r}, \quad \tilde{\sigma}(\tilde{r}), \quad \tilde{K}_i(\tilde{r}), \quad \tilde{H}[\tilde{\sigma}], \quad \tilde{Z}[\tilde{K}], \ldots
\]

The site coordinate on the dual lattice coincides with the coordinate of the plaquet center on the initial lattice:

\[
\tilde{r} = r + (\hat{x} + \hat{y})/2.
\]

Using these denotations, we can write the dual partition function

\[
2^N \tilde{Z}[\tilde{K}] = \sum_{[\tilde{\sigma}]} e^{-\tilde{\beta}\tilde{H}[\tilde{\sigma}]},
\]

with the dual Hamiltonian

\[
-\tilde{\beta}\tilde{H}[\tilde{\sigma}] = \sum_{r} \tilde{\sigma}(\tilde{r})(\tilde{K}_1(\tilde{r})\nabla_{-x} + \tilde{K}_2(\tilde{r})\nabla_{-y})\tilde{\sigma}(\tilde{r}).
\]

The coupling constants \( K_i(r) \) and \( \tilde{K}_i(\tilde{r}) \) are connected by duality condition (3).

It is known that the partition function of the 2d Ising model can be represented as the sum of the functional integrals over the lattice real fermion field [4,5]. For our task it is important that this representation can be exactly obtained for the nonhomogeneous model [6-8] :

\[
Z^{(p,p)}[K] = \frac{1}{2}\left(-Q^{(p,p)}[K] + Q^{(p,a)}[K] + Q^{(a,p)}[K] + Q^{(a,a)}[K]\right).
\] (7)

Here \( Q^{(\alpha,\beta)}[K] \) is the Gaussian functional integral over the four-component Grassmann field :

\[
Q^{(\alpha,\beta)}[K] = \left(\prod_{r,i} \cosh K_i(r)\right) \int \mathcal{D}\psi \exp(S^{(\alpha,\beta)}[\psi]),
\] (8)

\[
\mathcal{D}\psi = \prod_r \prod_{j=1}^{4} d\psi_j(r),
\]

\( \psi_j(r) \) is Grassmann field :

\[
\{\psi_i(r), \psi_j(r')\} = 0.
\]

In (8) the action has the following form

\[
S^{(\alpha,\beta)}[\psi] = \sum_r \left( \mathcal{L}^{(0)}(\psi(r)) + \mathcal{L}^{(\alpha,\beta)}(\psi(r)) \right),
\] (9)
\[ \mathcal{L}^{(0)}(\psi(r)) = \sum_{1 \leq i < j}^4 \psi_i(r)\psi_j(r), \]
\[ \mathcal{L}^{(\alpha,\beta)}(\psi(r)) = - t_1(r)\psi_3(r)\nabla_x^\alpha \psi_1(r) + t_2(r)\psi_2(r)\nabla_y^\beta \psi_4(r), \]
\[ t_i(r) \equiv \tanh K_i(r). \]

Representation (7) was written for periodical boundary conditions. But it is obviously that we can write the similar relations for arbitrary combinations (periodical and antiperiodical) of the boundary conditions along \( X \) and \( Y \) axes. For example, \( Z^{(a,p)} \) is distinguished from \( Z^{(p,p)} \) by change of the sign of coupling constants \( K_1(r) \) in the last right lattice column:

\[ Z^{(p,p)} \rightarrow Z^{(a,p)} \quad \text{at} \quad K_1(n, y) \rightarrow -K_1(n, y), \]

and therefore

\[ t_1(n, y) \rightarrow -t_1(n, y). \quad (10) \]

But transformation (10) is equivalent to the change of the boundary conditions for the one-step operator in action (9)

\[ \nabla^p_x \leftrightarrow \nabla^a_x \]

Therefore, \( Z^{(a,p)} \) is distinguished from \( Z^{(p,p)} \) by the arrangement of signs “±” before terms in (7):

\[ Z^{(a,p)}[K] = \frac{1}{2} \left( Q^{(p,p)}[K] + Q^{(p,a)}[K] - Q^{(a,p)}[K] + Q^{(a,a)}[K] \right). \quad (11) \]

For the rest boundary coditions \( Z^{(p,a)}, Z^{(a,a)} \) we have analogous expressions. Using (6), one can write expressions for the partition functions with corresponding boundary conditions on the torus in the following compact form:

\[ Z[K] = \hat{R}Q[K]. \]

where we introduce vector

\[ Q = (Q^{(p,p)}, Q^{(p,a)}, Q^{(a,p)}, Q^{(a,a)}), \]

and matrix \( \hat{R} : \)

\[ \hat{R} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \hat{R}^2 = 1. \quad (12) \]
Functional integral (8) is expressed through the Pfaffian of the $4N \times 4N$-dimensional antysymmetric matrix $D$:

$$Q^{(\alpha,\beta)}[K] = \left( \prod_{r,i} \cosh K_i(r) \right) \text{Pf}(D^{(\alpha,\beta)}),$$  \hspace{1cm} (13)

where

$$D^{(\alpha,\beta)} = \begin{pmatrix}
0 & 1 & 1 + t_1(r - \bar{x})\nabla^\alpha_x & 1 \\
-1 & 0 & 1 & 1 + t_2(r)\nabla^\beta_y \\
-1 - t_1(r)\nabla^\alpha_x & -1 & 0 & 1 \\
-1 & -1 - t_2(r - \bar{y})\nabla^\beta_y & -1 & 0
\end{pmatrix}$$ \hspace{1cm} (14)

and its defines the quadratic form in (9)

$$S^{(\alpha,\beta)}[\psi] = \frac{1}{2}(\psi, D^{(\alpha,\beta)} \psi).$$

Similar expression one can write for the dual lattice:

$$\tilde{Z}[\tilde{K}] = \tilde{R}\tilde{Q}[\tilde{K}],$$

$$\tilde{Q}^{(\alpha,\beta)}[\tilde{K}] = \left( \prod_{\tilde{r},i} \cosh \tilde{K}_i(\tilde{r}) \right) \text{Pf}(\tilde{D}^{(\alpha,\beta)}),$$

where matrix $\tilde{D}$ has the following form

$$\tilde{D}^{(\alpha,\beta)} = \begin{pmatrix}
0 & 1 & 1 + \tilde{t}_1(\tilde{r})\nabla^\alpha_x & 1 \\
-1 & 0 & 1 & 1 + \tilde{t}_2(\tilde{r} - \bar{y})\nabla^\beta_y \\
-1 - \tilde{t}_1(\tilde{r} + \bar{x})\nabla^\alpha_x & -1 & 0 & 1 \\
-1 & -1 - \tilde{t}_2(\tilde{r})\nabla^\beta_y & -1 & 0
\end{pmatrix}$$ \hspace{1cm} (15)

$$\tilde{t}_i(\tilde{r}) \equiv \tanh \tilde{K}_i(\tilde{r})$$

In conclusion of this section we note that for the finite lattice the Pfaffian of matrix (13) or (15) is the finite power polinom of $t_i(r)$ or $\tilde{t}_i(\tilde{r})$. It is not hard to calculate $\text{Pf}(D)$ in the high-temperature limit:

$$\text{Pf}(D) = 1, \quad \text{when} \quad t_i(r) = 0,$$

and $\text{Pf}(\tilde{D})$ in the low-temperature limit:

$$\text{Pf}(\tilde{D}) = 1, \quad \text{when} \quad \tilde{t}_i(\tilde{r}) = 0.$$
3. The homogeneous case

In general case for arbitrary sizes $m, n$ and distributions of the coupling constants $K_i(r)$ we can not calculate neither the Pfaffian nor the determinant of matrix $D$. But in the homogeneous case, when

$$K_i(r) = K_i = \text{const}$$

and the matrix $D$ becomes the translation-invariant matrix the determinant of $D$ is calculated by means of the Fourier transformation:

$$\det(D) = \prod_p [(1 + t_1^2)(1 + t_2^2) - 2t_1(1 - t_2^2) \cos p_x - 2t_2(1 - t_1^2) \cos p_y],$$

where $t_i \equiv \tanh K_i$ and the momentum components $p_x, p_y \,(\mathbf{p} = (p_x, p_y))$ take the integer and half-integer values (in units of $2\pi/n$ and $2\pi/m$) for periodical and antiperiodical boundary conditions respectively. Using relation

$$(\text{Pf}(D))^2 = \det(D),$$

we obtain from (14), (18)

$$Q^2(K_1, K_2) = \prod_p (c_1 c_2 - s_1 \cos p_x - s_2 \cos p_y),$$

where

$$c_i \equiv \cosh 2K_i, \quad s_i \equiv \sinh 2K_i.$$

In similar way one gets for the dual lattice:

$$\tilde{Q}^2(\tilde{K}_1, \tilde{K}_2) = \prod_p (\tilde{c}_1 \tilde{c}_2 - \tilde{s}_1 \cos p_x - \tilde{s}_2 \cos p_y).$$

Here

$$\tilde{c}_i \equiv \cosh 2\tilde{K}_i, \quad \tilde{s}_i \equiv \sinh 2\tilde{K}_i.$$

Using (19) and (20), it is not hard to check the following equality

$$(s_1 s_2)^{-N/2}Q^2(K_1, K_2) = (\tilde{s}_1 \tilde{s}_2)^{-N/2}\tilde{Q}^2(\tilde{K}_1, \tilde{K}_2),$$

where $s_i$ and $\tilde{s}_i$ satisfy relations

$$s_1 \cdot \tilde{s}_2 = 1, \quad s_2 \cdot \tilde{s}_1 = 1.$$

Note that (21) one can consider as the duality relation for the square of the functional integrals which appear in representation (7) for the partition function. Extracting the square root from the both parts of (21), we obtain

$$(s_1 s_2)^{-N/4}Q^{(\alpha, \beta)}(K_1, K_2) = \pm(\tilde{s}_1 \tilde{s}_2)^{-N/4}\tilde{Q}^{(\alpha, \beta)}(\tilde{K}_1, \tilde{K}_2).$$
Show that sign “−” in (22) appears only for function $Q^{(p,p)}$, but for the rest components of vectors $Q$ and $\tilde{Q}$ we have the sign “+”:

$$
(\tilde{s}_1\tilde{s}_2)^{-N/4}\tilde{Q}(K_1, K_2) = (s_1s_2)^{-N/4}\tilde{g}Q(K_1, K_2),
$$

(23)

where signature matrix $\tilde{g}$ has form

$$
\tilde{g} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Since Pf($D$) is the polinom over $t_i$, then from (19), (20) it follows that det($D$) is square of this polinom. Really in product (19) the multipliers appear by pairs according to the momentum components $\pm p_x$ and $\pm p_y$. The exclusion is multipliers corresponding to the values $p_x = p_y = \pi$ and $p_x = p_y = 0$. Let us denote them by $q_\pi$ and $q_0$:

$$
q_\pi = c_1c_2 + s_1 + s_2 = (1 + t_1 + t_2 - t_1t_2)^2\cosh^2 K_1 \cdot \cosh^2 K_2
$$

$$
q_0 = c_1c_2 - s_1 - s_2 = (1 - t_1 - t_2 - t_1t_2)^2\cosh^2 K_1 \cdot \cosh^2 K_2 =
$$

$$
= (1 - s_1s_2)^2/q_\pi.
$$

Hence it is clear that in functions $Q^{(\alpha,\beta)}$ (except of $Q^{(p,p)}$) all multipliers have the constant sign in the following range of parameter values:

$$
s_1 \geq 0, \quad s_2 \geq 0,
$$

and, therefore, these functions do not change the sign. On the other hand at crossing through critical line $s_1s_2 = 1$ function $Q^{(p,p)}$ which contains multiplier $(q_0)^{1/2} \sim (1 - s_1s_2)$ changes the sign. Similar results we obtain for dual functions $\tilde{Q}^{(\alpha,\beta)}$.

Therefore, taking into account the high-temperature and low-temperature limits (16) and (17) for the Pfaffian, one gets

$$
\text{sign}(Q^{(p,p)}(K_1, K_2)) = \text{sign}(1 - s_1s_2)
$$

$$
\text{sign}(\tilde{Q}^{(p,p)}(\tilde{K}_1, \tilde{K}_2)) = \text{sign}(1 - \tilde{s}_1\tilde{s}_2) =
$$

$$
= -\text{sign}((Q^{(p,p)}(K_1, K_2)).
$$

This relation proves (23).
Multiplying the right and left parts of (23) on the matrix $\hat{R}$ (12), we obtain the duality relation for the partition functions:

$$ (\tilde{s}_1\tilde{s}_2)^{-N/4}\tilde{Z}(\tilde{K}_1, \tilde{K}_2) = (s_1s_2)^{-N/4}\hat{T}Z(K_1, K_2), $$

(24)

$$ \hat{T} = \hat{R}\hat{g}\hat{R}, \quad \hat{T}^2 = 1, $$

$$ \hat{T} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}. $$

(25)

From (24) it follows that difference between the right and left parts of Kramers-Wannier duality relation (1)

$$ (s_1s_2)^{-N/4}Z(p,p)(K_1, K_2) - (\tilde{s}_1\tilde{s}_2)^{-N/4}\tilde{Z}(p,p)(\tilde{K}_1, \tilde{K}_2) = $$

$$ = (\tilde{s}_1\tilde{s}_2)^{-N/4}\tilde{Q}(p,p)(\tilde{K}_1, \tilde{K}_2) = -(s_1s_2)^{-N/4}Q(p,p)(K_1, K_2) $$

is equal to zero only on critical line

$$ \sinh 2K_1 \cdot \sinh 2K_2 = 1. $$

Moreover outside this line the right and left parts of (26) are compared among themselves at increasing the lattice size ($m, n \to \infty$). Therefore duality relation (1) is correct only in the thermodynamic limit:

$$ \lim_{m,n \to \infty} \left( \frac{1}{N} \ln \left( \frac{\tilde{Z}(\tilde{K}_1, \tilde{K}_2)}{(\tilde{s}_1\tilde{s}_2)^{N/4}} \right) \right) = \lim_{m,n \to \infty} \left( \frac{1}{N} \ln \left( \frac{Z(K_1, K_2)}{(s_1s_2)^{N/4}} \right) \right). $$

Since for the finite lattice partition functions $Z^{(\alpha,\beta)}$ are analytic functions of $K_i$ (the finite power polinom over $e^{\pm K_i}$) the nonanalytic dependence in the right and left parts of relation (24) can appear only from multipliers $(s_1s_2)^{-N/4}$. Therefore, taking into account the circuit rules of branching points corresponding to $\sinh 2K_1 = 0$ and $\sinh 2K_2 = 0$, duality relation (24) can be continued to the rest ranges of values of the coupling constants: $K_1 \geq 0, K_2 < 0$; $K_1 < 0, K_2 \geq 0$; $K_1 < 0, K_2 < 0$. 

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4. The nonhomogeneous case

In previous section it was shown that the Kramers-Wannier duality relation (1) (having rather symbolic meaning) can be modified to the exact equality. The Kadanoff-Ceva anzats (2) also was proved in [3] for special (nonphysical) boundary conditions. Note that the thermodynamic limit for the nonhomogeneous Ising model can be defined for specific class of the functions determining distribution of the coupling constants $K_i(r)$. For illustration let us consider, for example, the sequence of functions $\{K_i^{(N)}(r)\}$ defined in the following way: the coupling constants are equal zero on the boundary $\Gamma$ of the finite-size clusters: $K_i(r \in \Gamma) = 0$. In this case at $N \to \infty$ we obtain the increasing numbers of the non-interacting finite-size clusters. Then this is rather similar on the self-averaging procedure for disordered systems than on the usual thermodynamic limit. It means that the duality relation for the nonhomogeneous system must be formulated for the finite lattice.

The covariant notation of exact duality relation (24) for the homogeneous model suggests the obvious recipit for the generalization to the nonhomogeneous case. For finite lattice on the torus Kadanoff-Ceva anzats (2) must be modified by the following way:

$$\prod_{\tilde{r},i}(\sinh 2\tilde{K}_i(\tilde{r}))^{-1/4}\tilde{Z}[\tilde{K}] = \prod_{r,i}(\sinh 2K_i(r))^{-1/4}\hat{T}Z[K].$$

(27)

Unfortunately we can not prove this relation for arbitrary lattice size and distribution of the coupling constants. But we checked duality relation (27) for lattices of small size by direct calculation on the computer. Moreover the duality relation can be proved for the weakly-nonhomogeneous case:

$$K_i(r) = K_i + \delta K_i(r), \quad K_i = \text{const}, \quad \delta K_i(r) \ll 1,$$

when $n$ and $m$ are arbitrary and finite.

For the first order over $\delta K_i(r)$ we have

$$\prod_{r,i}\left(\sinh(2K_i + \delta K_i(r))\right)^{-1/4} =$$

$$= (s_1s_2)^{-N/4}\left[1 - \frac{c_1}{2s_1}\sum_r \delta K_1(r) - \frac{c_2}{2s_2}\sum_r \delta K_2(r)\right],$$

(28)

$$Z[K] = Z(K_1, K_2)\left[1 + \sum_r (\langle \sigma(r)\sigma(r + \hat{x})\rangle\delta K_1(r) +$$

$$+ \langle \sigma(r)\sigma(r + \hat{y})\rangle\delta K_1(r))\right].$$

(29)
Note that at different $K_1$ and $K_2$ and (or) $n \neq m$ correlation functions $\langle \sigma(r)\sigma(r + \hat{x}) \rangle$ and $\langle \sigma(r)\sigma(r + \hat{y}) \rangle$ do not equal among themselves and due to translation invariance do not depend from $r$:

$$\mathbf{Z}(K_1, K_2)\langle \sigma(r)\sigma(r + \hat{x}) \rangle = \frac{1}{N} \hat{R} \frac{\partial \mathbf{Q}(K_1, K_2)}{\partial K_1},$$

$$\mathbf{Z}(K_1, K_2)\langle \sigma(r)\sigma(r + \hat{y}) \rangle = \frac{1}{N} \hat{R} \frac{\partial \mathbf{Q}(K_1, K_2)}{\partial K_2}. \tag{30}$$

Taking into account (28)-(30), in the first order over $\delta K_i(r)$ and $\delta \tilde{K}_i(\tilde{r})$ we obtain for duality relation (27):

$$(s_1 s_2)^{-1/4} \left[ \mathbf{Z}(K_1, K_2) + \hat{R} \left( \frac{1}{N} \frac{\partial \mathbf{Q}}{\partial K_1} - \frac{c_1}{2 s_1} \mathbf{Q} \right) \sum_r \delta K_1(r) + \right.$$  

$$
+ \hat{R} \left( \frac{1}{N} \frac{\partial \mathbf{Q}}{\partial K_2} - \frac{c_2}{2 s_2} \mathbf{Q} \right) \sum_r \delta K_2(r) \right] = \left.$$  

$$
= (s_1 s_2)^{-1/4} \left[ \tilde{\mathbf{Z}}(\tilde{K}_1, \tilde{K}_2) + \hat{R} \left( \frac{1}{N} \frac{\partial \tilde{\mathbf{Q}}}{\partial \tilde{K}_1} - \frac{\tilde{c}_1}{2 \tilde{s}_1} \tilde{\mathbf{Q}} \right) \sum_{\tilde{r}} \delta \tilde{K}_1(\tilde{r}) + \right.$$  

$$
+ \hat{R} \left( \frac{1}{N} \frac{\partial \tilde{\mathbf{Q}}}{\partial \tilde{K}_2} - \frac{\tilde{c}_2}{2 \tilde{s}_2} \tilde{\mathbf{Q}} \right) \sum_{\tilde{r}} \delta \tilde{K}_2(\tilde{r}) \right]. \tag{31}$$

For the order zero terms this equality is satisfied according to homogeneous duality relation (24). Show that it is satisfied for the linear terms over $\delta K$ and $\delta \tilde{K}$ terms. From (3) and (23) it follows:

$$\delta K_1(r) = -\frac{1}{s_2} \delta \tilde{K}_2(\tilde{r}), \quad \delta K_2(r) = -\frac{1}{s_1} \delta \tilde{K}_1(\tilde{r}),$$

$$\frac{\partial}{\partial K_1} = -s_2 \frac{\partial}{\partial K_2}, \quad \frac{\partial}{\partial K_2} = -s_1 \frac{\partial}{\partial K_1}, \tag{32}$$

$$\mathbf{Q}(K_1, K_2) = (s_1 s_2)^{-N/2} \tilde{\mathbf{g}} \tilde{\mathbf{Q}}(\tilde{K}_1, \tilde{K}_2). \tag{33}$$

Substituting (32) and (33) in the left part of (31) and collecting similar terms, one gets

$$(s_1 s_2)^{-N/4} \left[ \hat{T} \mathbf{Z}(\tilde{K}_1, \tilde{K}_2) + \hat{R} \tilde{\mathbf{g}} \left( \frac{1}{N} \frac{\partial \tilde{\mathbf{Q}}}{\partial \tilde{K}_2} - \frac{\tilde{c}_2}{2 \tilde{s}_2} \tilde{\mathbf{Q}} \right) \sum_{\tilde{r}} \delta \tilde{K}_2(\tilde{r}) + \right.$$  

$$
+ \hat{R} \left( \frac{1}{N} \frac{\partial \tilde{\mathbf{Q}}}{\partial \tilde{K}_1} - \frac{\tilde{c}_1}{2 \tilde{s}_1} \tilde{\mathbf{Q}} \right) \sum_{\tilde{r}} \delta \tilde{K}_1(\tilde{r}) \right]. \tag{34}$$

Using (25), (from which follows $\hat{R} \tilde{\mathbf{g}} = \hat{T} \hat{R}$) it is not hard to show that (34) coincides with the right part of (31). This proves duality relation (27) in the weakly-nonhomogeneous case for arbitrary $m$ and $n$. 

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In conclusion of this section note that for analysis of the duality relation for correlation functions it is convenient to use the other normalization in (27). Using relations
\[
\frac{\cosh^2 2K_1(r)}{\sinh 2K_1(r)} = \frac{\cosh^2 2\tilde{K}_2(\tilde{r})}{\sinh 2\tilde{K}_2(\tilde{r})},
\]
\[
\frac{\cosh^2 2K_2(r)}{\sinh 2K_2(r)} = \frac{\cosh^2 2\tilde{K}_1(\tilde{r})}{\sinh 2\tilde{K}_1(\tilde{r})},
\]
which follow from (3), and introducing denotions according to [3]
\[
Y[K] = \prod_{r,i} (\cosh 2K_i(r))^{-1/2} Z[K],
\]
\[
\tilde{Y}[\tilde{K}] = \prod_{\tilde{r},i} (\cosh 2\tilde{K}_i(\tilde{r}))^{-1/2} \tilde{Z}[\tilde{K}],
\]
(35)
we obtain instead of (27):
\[
Y[K] = \hat{T} \tilde{Y}[\tilde{K}].
\]
(36)

5. Duality relation for correlation function

The duality relation for the nonhomogeneous Ising model is useful for study the correlation function properties. For this aim it is convenient to use the magnetic dislocation representation for correlation functions [3]. This representation is based on the obvious equality
\[
e^{(K+i\pi/2)}\sigma_1\sigma_2 = i\sigma_1\sigma_2 e^{K\sigma_1\sigma_2}.
\]
Taking into account that
\[
\sigma_1\sigma_n = (\sigma_1\sigma_2)(\sigma_2\sigma_3)\ldots(\sigma_{n-1}\sigma_n),
\]
one can write
\[
\sum_{[\sigma]} e^{-\beta H[\sigma]} \sigma(r)\sigma(r') = i^{-\gamma} \sum_{[\sigma]} e^{-\beta H'[\sigma]},
\]
where Hamiltonian \(\beta H'[\sigma]\) of the Ising model with defect differs from \(\beta H[\sigma]\) by the change of the coupling constants \(K\) on \(K' = K + i\pi/2\) on the defect line \(\Gamma_\sigma\) [3] which connect sites \(r\) and \(r'\):
\[
K_i(r) = K,
\]
\[
K_i'(r) = \begin{cases} K + i\pi/2 & \text{on the links belonging to path } \Gamma_\sigma \\ K & \text{on the rest links} \end{cases}
\]
γ is the length of this path (the number of the "spoilt" bonds). Then, using functionals (35), we obtain representation for the two-point correlation function (we omit boundary condition indeces):

\[ G_\sigma(r, r') \equiv \langle \sigma(r)\sigma(r') \rangle = Y[K']/Y[K]. \] (37)

In the work [3] it was introduced the correlation function of the disorder parameter \( \mu(\tilde{r}) \). This variable characterizes the degree of disorder near the point \( \tilde{r} \) on the initial lattice and its one can consider as the result of the duality transformation for Ising spin \( \sigma(r) \). Correlation function \( \langle \mu(\tilde{r}), \mu(\tilde{r}') \rangle \) is determined by means of magnetic dislocation \( \Gamma_\mu \):

\[ G_\mu(\tilde{r}, \tilde{r}') \equiv \langle \mu(\tilde{r})\mu(\tilde{r}') \rangle = Y[K'']/Y[K], \]

where

\[ K''_i = \begin{cases} -K & \text{on the links intersecting of path } \Gamma_\mu \\ K & \text{on the rest links.} \end{cases} \]

The duality relation for correlation functions [3],

\[ \langle \tilde{\mu}(r)\tilde{\mu}(r') \rangle = \langle \sigma(r)\sigma(r') \rangle \] (38)

follows from (2) and the transformation of magnetic dislocation \( \Gamma_\sigma \) on the initial lattice to magnetic dislocation \( \tilde{\Gamma}_\mu \) on the dual lattice by means of mapping [3]

\[ K_1(r) + i\pi/2 \to \tilde{K}_2(\tilde{r}) \cdot e^{-i\pi}, \]
\[ K_2(r) + i\pi/2 \to \tilde{K}_1(\tilde{r}) \cdot e^{-i\pi}, \]

which follows from (3). Since duality relation (27) for the finite lattice on the torus differs from Kadanoff-Ceva anzats (2) the duality relation for correlation functions on the torus has more complicate form. For example, using (36), we obtain for the dual lattice with periodical boundary conditions

\[ \tilde{G}_{\mu}^{(p,p)}(r, r') = \tilde{Y}^{(p,p)}[\tilde{K''}]/\tilde{Y}^{(p,p)}[\tilde{K}] = (\tilde{T}Y[K'']/\tilde{T}Y[K])^{(p,p)} = \]

\[ = [Z^{(p,p)}G_\sigma^{(p,p)}(r, r') + Z^{(p,a)}G_\sigma^{(p,a)}(r, r') + Z^{(a,p)}G_\sigma^{(a,p)}(r, r') + Z^{(a,a)}G_\sigma^{(a,a)}(r, r')]/[Z^{(p,p)} + Z^{(p,a)} + Z^{(a,p)} + Z^{(a,a)}]. \] (39)

It is not hard to show that (39) pass to (38) only under the following condition: the correlation length is smaller of the lattice sizes (this is happen out the scaling domain and at the large \( m \) and \( n \)). Note that duality relation (39) coincides with the relation for correlation functions on the torus in the critical point obtained in the paper [9] by means of the quantum conformal field theory methods [10].
6. Duality and boundary conditions

The duality relation for the nonhomogeneous Ising model on the torus allows to obtain duality relations for the 2d Ising model with magnetic fields applied to the boundaries and for the 2d Ising model with free, fixed and mixed boundary conditions.

In order to get these relations let us consider the Ising model on the torus with defect which is defined by the following distribution of the coupling constants in Hamiltonian (5):

\[ K_1(r) = K_2(r) = K \] on all links of the lattice with the exception of the following cases – \( K_1(n-1, y) = h_1, K_1(n, y) = h_2, K_2(n, y) = h \), where \( h_1 = \beta H_1, h_2 = \beta H_2, h = \beta H, y = 1, \ldots, m \). Using (3), this defect one can define on the dual lattice by the following way:

\[ \tilde{K}_1(\tilde{r}) = \tilde{K}_2(\tilde{r}) = \tilde{K} \] on all links with the exception of the following cases – \( \tilde{K}_1(n-1, \tilde{y}) = \tilde{h}_1, \tilde{K}_1(n, \tilde{y}) = \tilde{h}_2, \tilde{K}_2(n, y) = \tilde{h} \), where \( \tilde{h}_1 = \beta \tilde{H}_1, \tilde{h}_2 = \beta \tilde{H}_2, \tilde{h} = \beta \tilde{H}, \tilde{y} = 1, \ldots, m \) and coupling constants \( h, h_1, h_2 \) and \( \tilde{h}, \tilde{h}_1, \tilde{h}_2 \) are connected by relations:

\[
\sinh 2h \cdot \sinh 2\tilde{h} = 1, \quad \sinh 2h_1 \cdot \sinh 2\tilde{h}_1 = 1, \quad \sinh 2h_2 \cdot \sinh 2\tilde{h}_2 = 1. \tag{40}
\]

Taking limit \( h \to \infty \) in partition function (2) (for simplicity we consider the ferromagnetic model), it is not hard to obtain the partition function of the 2d Ising model with magnetic fields \( H_1 \) and \( H_2 \) applied to the boundaries:

\[
2Z^p(K, h_1, h_2) = \lim_{h \to \infty} (\cosh h)^{-m} Z^{(\alpha, p)}(K, h_1, h_2, h) = \\
2 \sum_{[\sigma]} \exp(K \sum_{r,i} \sigma(r)\sigma(r+i) + h_1 \sum_{y=1}^m \sigma(n-1,y) + h_2 \sum_{y=1}^m \sigma(1,y)), \tag{41}
\]

where in order to sum over spin variables \( \{\sigma(n,y)\} \) we used the following equality:

\[
\lim_{h \to \infty} (\cosh h)^{-m} \prod_{y=1}^m \exp(h\sigma(n,y)\sigma(n,y+1)) = \prod_{y=1}^m \delta(\sigma(n,y), \sigma(n,y+1)). \tag{42}
\]

Here on the right-hand side the product of Kronecker’s \( \delta \)-functions is written. From \( \{\sigma(n,y)\} \) this product selects the two spin configurations: all spins are directed up or down.

Note, that in (41) partition function \( Z^p(h_1, h_2) \) is obtained by the limiting procedure with corresponding normalization which removes infinite constant. In other cases in consequence of the conflict between the product of Kronecker’s \( \delta \)-functions and the boundary conditions we can get the zero after taking of the limit, for example,

\[
\lim_{h \to \infty} (2 \cosh h)^{-m} Z^{(\alpha, \alpha)}(K, h_1, h_2, h) = 0, \tag{43}
\]

but this does not mean, that \( Z^{\alpha}(K, h_1, h_2) = 0 \).
In consequence of (40) for the dual lattice we have $\tilde{h} = 0$ and

$$\tilde{Z}^{\beta}(\tilde{K}, \tilde{h}_1, \tilde{h}_2) = \lim_{\tilde{h} \to 0} \tilde{Z}^{(\alpha, \beta)}(\tilde{K}, \tilde{h}_1, \tilde{h}_2, \tilde{h}) = \sum_{[\sigma]} \exp(\tilde{K} \sum_{\tilde{r}, i} \tilde{\sigma}(\tilde{r}) \tilde{\sigma}(\tilde{r} + i))$$

$$+ \tilde{h}_1 \sum_{\tilde{y}=1}^{m} \tilde{\sigma}(n, \tilde{y}) \tilde{\sigma}(n, \tilde{y} + 1) + \tilde{h}_2 \sum_{\tilde{y}=1}^{m} \tilde{\sigma}(1, \tilde{y}) \tilde{\sigma}(1, \tilde{y} + 1)$$

$$, \quad \beta = a, p \quad (44)$$

In result we obtained the partition function of the Ising model on the cylinder with the free boundary conditions and the defects on its bases.

Now, taking limits $h_i \to \infty$ and $\tilde{h} \to 0$ in (36) and using (41)-(44), it is not hard to get the duality relations for the Ising model on the square lattice wrapped on the cylinder with magnetic fields applied to its bases

$$\frac{\tilde{Z}^p(K, \tilde{h}_1, \tilde{h}_2)}{[\cosh^m 2\tilde{h}_1 \cosh^m 2\tilde{h}_2 (\cosh 2\tilde{K})^{2m(n-3)}]^{1/2}} = \frac{Z^p(K, h_1, h_2) + Z^p(K, h_1, -h_2)}{[\cosh^m 2h_1 \cosh^m 2h_2 (\cosh 2K)^{2m(n-3)}]^{1/2}} \quad (45)$$

$$\frac{\tilde{Z}^a(K, \tilde{h}_1, \tilde{h}_2)}{[\cosh^m 2\tilde{h}_1 \cosh^m 2\tilde{h}_2 (\cosh 2\tilde{K})^{2m(n-3)}]^{1/2}} = \frac{Z^p(K, h_1, h_2) - Z^p(K, h_1, -h_2)}{[\cosh^m 2h_1 \cosh^m 2h_2 (\cosh 2K)^{2m(n-3)}]^{1/2}} \quad (46)$$

Here we have lattices with sizes $(n-1) \times m$ and $n \times m$ on the right-hand and left-hand sides respectively.

In order to get the partition function of the 2d Ising model on the initial lattice with different boundary conditions on the cylinder bases it is necessary to consider in (45)-(46) different combinations of limits $h_i \to \infty$ and $h_i \to 0$, $i = 1, 2$:

1) the free boundary conditions $h_1 = h_2 = 0$, which we denote

$$Z_{(0,0)}^\alpha = 2^{-2m}Z_{(0,0)}^\alpha, \quad \alpha = a, p,$$

2) the fixed boundary conditions $h_i \to \infty$:

$$Z_{(+,+)}^\alpha = \lim(2 \cosh h_1 \cosh h_2)^{-m}Z_{(+,+)}^\alpha(h_1, h_2),$$

$$Z_{(+,-)}^\alpha = \lim(2 \cosh h_1 \cosh h_2)^{-m}Z_{(+,+)}^\alpha(h_1, -h_2),$$

3) the mixed boundary conditions $h_1 \to \infty$, $h_2 = 0$ or $h_2 \to \infty$, $h_1 = 0$:

$$Z_{(+,0)}^\alpha = \lim(4 \cosh h_1)^{-m}Z_{(+,0)}^\alpha(h_1, h_2) = \lim(4 \cosh h_2)^{-m}Z_{(+,0)}^\alpha(h_1, h_2).$$

In consequence of (40) the transition to limits $h_i \to \infty$ and $h_i \to 0$, $i = 1, 2$ on the initial lattice leads to the following results on the dual lattice:
1) \( \tilde{h}_1, \tilde{h}_2 \to 0, \)

\[
\lim Z^\alpha (\tilde{K}, \tilde{h}_1, \tilde{h}_2) = (2 \cosh \tilde{K})^{2m} \tilde{Z}^\alpha_{(0,0)}, \quad \alpha = a, p
\]  

(47)

2) \( \tilde{h}_1 \to \infty, \tilde{h}_2 \to \infty \)

\[
\lim (2 \cosh \tilde{h}_1)^{-m} (2 \cosh \tilde{h}_2)^{-m} \tilde{Z}^p(\tilde{h}_1, \tilde{h}_2) = 2 \left[ \tilde{Z}^p(\tilde{K}, \tilde{K}) + \tilde{Z}^p(\tilde{K}, -\tilde{K}) \right].
\]  

(48)

3) \( \tilde{h}_1 \to \infty, \tilde{h}_2 \to 0 \)

\[
\lim (2 \cosh \tilde{h}_1)^{-m} \tilde{Z}^a(\tilde{h}_1, \tilde{h}_2) = (2 \cosh \tilde{K})^m 2 \tilde{Z}^a(\tilde{K}, 0),
\]  

(49)

Setting in (45), (46) \( \tilde{h}_1 = \tilde{h}_2 = \tilde{K} \) and respectively \( h_1 = h_2 = K \) we obtain the first two duality relations for the 2d Ising model with the boundary conditions:

\[
\begin{align*}
\frac{\tilde{Z}^p_{(0,0)}}{(\cosh 2\tilde{K})^{m(n-3)}} &= \frac{Z^p(K, K) + Z^p(K, -K)}{(\cosh 2K)^{m(n-3)}} \quad (50) \\
\frac{\tilde{Z}^a_{(0,0)}}{(\cosh 2\tilde{K})^{m(n-3)}} &= \frac{Z^p(K, K) - Z^p(K, -K)}{(\cosh 2K)^{m(n-3)}} \quad (51)
\end{align*}
\]

Recall that here we have lattices with sizes \((n-1) \times m\) and \(n \times m\) on the right-hand and left-hand sides respectively.

Now, using (47)-(49) and taking the corresponding limits in (45), (46), it is not hard to get the following duality relations:

\[
\begin{align*}
\frac{(2 \cosh \tilde{K})^{2m} \tilde{Z}^p_{(0,0)}}{(\cosh 2\tilde{K})^{m(n-3)}} &= \frac{Z^p_{(+,+)} + Z^p_{(+,-)}}{(\cosh 2K)^{m(n-3)}}, \quad (52) \\
\frac{(2 \cosh \tilde{K})^{2m} \tilde{Z}^a_{(0,0)}}{(\cosh 2\tilde{K})^{m(n-3)}} &= \frac{Z^p_{(+,+)} - Z^p_{(+,-)}}{(\cosh 2K)^{m(n-3)}}, \quad (53) \\
\frac{\tilde{Z}^p(K, \tilde{K}) + \tilde{Z}^p(K, -\tilde{K})}{(\cosh 2\tilde{K})^{m(n-3)}} &= \frac{Z^p_{(0,0)}}{(\cosh 2K)^{m(n-3)}}, \quad (54) \\
\frac{\tilde{Z}^p(K, \tilde{K}) - \tilde{Z}^p(K, -\tilde{K})}{(\cosh 2\tilde{K})^{m(n-3)}} &= \frac{Z^a_{(0,0)}}{(\cosh 2K)^{m(n-3)}}, \quad (55) \\
\frac{(2 \cosh \tilde{K})^m \tilde{Z}^p(K, 0)}{(\cosh 2\tilde{K})^{m(n-3)}} &= \frac{Z^p_{(0,+)}}{(\cosh 2K)^{m(n-3)}}, \quad (56)
\end{align*}
\]

Note that in relations (52), (53) we have lattices with sizes \((n-1) \times m\) and \((n-2) \times m\) on the right-hand and left-hand sides respectively.

Let us make some comments about (53). Using results of exact solution [11] of the 2d Ising model with magnetic fields applied to the boundaries, it is not hard to show that
\( Z^p(K, h_1, h_2) - Z^p(K, h_1, -h_2) \) is proportional to \( \text{sign}(h_1 \cdot h_2) \) and therefore at obtaining of (46) the contradiction after taking limits \( \tilde{h}_1 \to \infty, \tilde{h}_2 \to \infty \) and \( \tilde{h}_1 \to \infty, \tilde{h}_2 \to -\infty \) is not appeared.

In the critical point relations (50)-(55) are reduced to

\[
Z^p_{(0,0)} = Z^p_{(+,+)} + Z^p_{(+,-)},
\]

which was obtained in [12] and it follows from [11].

7. Conclusion

From the duality relation for the nonhomogeneous Ising model one can be obtained some usefull consequences. Using this relation, one can correctly introduce the ”mixed” correlation function of type \( \langle \sigma \mu \sigma' \mu' \rangle \) and discover their fermionic content. In principle anzats (27) allows to construct the generating functional depending from external currents \( J(r), \tilde{J}(r) \) and \( \chi(r) \), where the first two currents are connected with fluctuations of the order and disorder variables and the last current generates the fermionic type exitation.

Undoubtedly the nonhomogeneous duality relation will be usefull for analysis of Ising model with random coupling constants.

However, unfortunately, we have not proof of this relation for arbitrary distributions of the coupling constants and sizes of the lattice. In given paper the duality relation is proved for the homogeneous case and in the first order for weakly nonhomogeneous case (it is not hard to prove one in the second order). One can prove this relation for case when the some small numbers of the coupling constants is chosen arbitrary ones on the background of the rest homogeneous coupling constants. The additional argument for correctness of duality relation (27) is the direct check of one on the small lattices with sizes \( m, n = 2, 3 \).

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