A VERTEX ALGEBRA ATTACHED TO THE FLAG MANIFOLD AND
LIE ALGEBRA COHOMOLOGY

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Abstract. We compute the cohomology vertex algebra
\( H^\bullet(G/B, D^\text{ch}_{G/B}) \) to the effect that it is isomorphic to
\( H^\bullet(\mathfrak{n}, \mathbb{C}) \otimes \mathbb{V}(\mathfrak{g})_{\chi=0} \).

We then find the Friedan-Martinec-Shenker-Borisov bosonization of
\( H^\bullet(\mathbb{P}^1, D^\text{ch}_{G/\mathbb{P}^1}) \) and verify that the latter algebra vanishes nonperturbatively.

1. Introduction

If \( X \) is a smooth algebraic variety and \( V \) a sheaf of vertex algebras over it,
then \( H^\bullet(X, V) \) is naturally a vertex algebra. Although various examples of
such sheaves, most notably sheaves of algebras of chiral differential operators (CDO),
[3, 11, 14] and references therein, have been studied, almost no cohomology
vertex algebras have been explicitly described, except for the case where \( X = \mathbb{P}^n \), [13].

In the present paper we fill this gap by determining the vertex algebra structure of
\( H^\bullet(G/B, D^\text{ch}_{G/B}) \), where \( G/B \) is a flag manifold and \( D^\text{ch}_{G/B} \) is the CDO
over it, [11, 14]. The CDO \( D^\text{ch}_{G/B} \) is a sheaf of \( \hat{\mathfrak{g}} \)-modules at the critical level,
and the \( \hat{\mathfrak{g}} \)-module structure of \( H^\bullet(G/B, D^\text{ch}_{G/B}) \) was computed in [2]. It is clear
from [7] that \( H^\bullet(G/B, D^\text{ch}_{G/B}) \) is a direct sum of several copies of \( \mathbb{V}(\mathfrak{g})_{\chi=0} \),
the vacuum \( \hat{\mathfrak{g}} \)-module at the critical level and zero central character. The computation carried out in [2] establishes a natural 1-1 correspondence between
these copies and Schubert cells of \( G/B \). This suggests that as a vertex algebra
\( H^\bullet(G/B, D^\text{ch}_{G/B}) \) is isomorphic to \( H^\bullet(G/B, \mathbb{C}) \otimes \mathbb{V}(\mathfrak{g})_{\chi=0} \), a proposition im-
mediately seen to be wrong because it distorts the cohomological degree. What
happens in reality is rather a manifestation of Bott’s “strange equality”, [4]:
\[
\dim H^i(\mathfrak{n}, \mathbb{C}) = \dim H^{2i}(G/B, \mathbb{C})
\]
where \( \mathfrak{n} \subset \mathfrak{g} \) is a maximal nilpotent subalgebra. We show that the associa-
tive commutative algebra \( H^\bullet(\mathfrak{n}, \mathbb{C}) \) carries a derivation – thus becoming
a commutative vertex algebra – so that \( H^\bullet(G/B, D^\text{ch}_{G/B}) \) is isomorphic to
\( H^\bullet(\mathfrak{n}, \mathbb{C}) \otimes \mathbb{V}(\mathfrak{g})_{\chi=0} \) as a vertex algebra. Furthermore, \( H^\bullet(\mathfrak{n}, \mathbb{C}) \) thus embedded
in \( H^\bullet(G/B, D^\text{ch}_{G/B}) \) is the center of \( H^\bullet(G/B, D^\text{ch}_{G/B}) \).

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Our result is not without omission for we do not compute the derivation, see Theorem 2.5.1. We show, however, that this derivation is 0 when \( \mathfrak{g} \) is either \( \mathfrak{sl}_2 \) or \( \mathfrak{sl}_3 \) and expect it to be always 0.

Next, we focus on the case of \( \mathfrak{sl}_2 \) and establish what can be called a Friedan-Martinec-Shenker-Borisov bosonization of \( H^\bullet(\mathbb{P}^1, \mathcal{D}_C^{ch}) \). This means a differential graded vertex algebra \((V_0 L, D)\) such that the cohomology vertex algebra \( H_D(V_0) \) is \( H^\bullet(\mathbb{P}^1, \mathcal{D}_C^{ch}) \). The DGVA \((V_0 L, D)\) can be included in a flat family \((V_t L, D), t \in \mathbb{C}\), where \( V_t \) is a certain lattice vertex algebra if \( t \neq 0 \). We show that generically the cohomology vanishes: \( H_D(V_t) = 0 \) if \( t \neq 0 \). This result should be contrasted with an analogous result of [13], where a similar computation gives that \( H_D(V_0) \) is a full-fledged infinite dimensional vertex algebra, the cohomology of the chiral de Rham complex over a projective space, while generically, if \( t \neq 0 \), \( H_D(V_t) \) is the quantum cohomology of the same projective space. If we are to believe that the passage from \((V_0 L, D)\) to \((V_t L, D)\) is a way to take into account the instanton corrections (arguments in [8] seem to be in favor of this), then our result is a confirmation of Witten’s prediction [16] that nonperturbatively the algebra \( H^\bullet(X, \mathcal{D}_C^{ch}) \) vanishes.

2. Cohomology vertex algebra

2.1. Multiplicative structures on cohomology. Let \( X \) be a smooth algebraic variety over \( \mathbb{C} \), \( \mathcal{A} \) a sheaf of \( \mathbb{C} \)-vector spaces over \( X \). We would like to show that any multiplication on \( \mathcal{A} \), i.e., a sheaf morphism

\[
m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}
\]

induces a canonical multiplication on cohomology

\[
h(m) : H^\bullet(X, \mathcal{A}) \otimes \mathcal{A} \to H^\bullet(X, \mathcal{A}).
\]

This and the discussion to follow should be termed well-known, but we failed to find a reference with the necessary results. The earliest paper we were able to locate where such issues are analyzed is [10]. Our thinking about the subject was greatly influenced by the Beilinson-Drinfeld approach to chiral algebras [3]; more on this in Remark 2.3.2.

Consider the diagonal \( \Delta : X \hookrightarrow X \times X \), the 2 projections \( p_i : X \times X \to X \), \( i = 1, 2 \), and the exterior tensor square \( \mathcal{A} \boxtimes \mathcal{A} = p_1^*(\mathcal{A}) \otimes_C p_2^*(\mathcal{A}) \), where \( \Delta^\bullet \) is the inverse image in the category of sheaves of \( \mathbb{C} \)-vector spaces. Since \( \mathcal{A} \otimes_C \mathcal{A} = \Delta^\bullet(\mathcal{A} \boxtimes \mathcal{A}) \) and \( \Delta^\bullet \) is the right adjoint of \( \Delta_\bullet \), the direct image in the category of sheaves of \( \mathbb{C} \)-vector spaces, we obtain

\[
\text{Hom}(\mathcal{A} \otimes_C \mathcal{A}, \mathcal{A}) \cong \text{Hom}(\mathcal{A} \boxtimes \mathcal{A}, \Delta_\bullet(\mathcal{A})).
\]

Hence (2.1) gives

\[
\tilde{m} : \mathcal{A} \boxtimes \mathcal{A} \to \Delta_\bullet(\mathcal{A}).
\]

Let us compute the cohomology of the sheaves involved. If \( \mathcal{I} \) is an injective module over \( X \), then \( \Delta_\bullet(\mathcal{I}) \) is an injective module over \( X \times X \) with \( \Gamma(X \times X \times \mathcal{I}) \).
$X, \Delta_\bullet(I) = \Gamma(X, I)$. Therefore, if $A \to I^\bullet$ is an injective resolution, then $\Delta_\bullet(A) \to \Delta_\bullet(I^\bullet)$ is also, and we obtain
\begin{equation}
(2.4) \quad H^\bullet(X \times X, \Delta_\bullet(A)) = H^\bullet(X, A).
\end{equation}
Even if $I$ and $J$ are injective, $I \boxtimes J$ does not have to be, but its higher cohomology vanish. Indeed, denoting by $\pi$ the projection to a point and factoring out $\pi = \pi \circ p_1$, we obtain
\begin{equation}
(2.5) \quad R\pi(I \boxtimes J) = R\pi \circ Rp_1(I \boxtimes J) = \Gamma(X, I) \otimes_C \Gamma(X, J).
\end{equation}
It follows that the resolution $A \boxtimes A \to I^\bullet \boxtimes I^\bullet$ is adapted to the computation of the cohomology $H^\bullet(X \times X, A \boxtimes A); (2.5)$ implies
\begin{equation}
(2.6) \quad H^\bullet(X \times X, A \boxtimes A) = \Gamma(X, I^\bullet \boxtimes I^\bullet) = H^\bullet(X, A) \otimes_C H^\bullet(X, A).
\end{equation}
Multiplication $(2.3)$ gives a multiplication on the resolution, well-defined in the derived category,
\begin{equation}
(2.7) \quad I^\bullet \boxtimes I^\bullet \to \Delta_\bullet(I^\bullet).
\end{equation}
Taking the spaces of global sections of both sides of $(2.7)$ and then using $(2.4)$ and $(2.6)$, we obtain the desired multiplication
\begin{equation}
(2.8) \quad h(m) : H^\bullet(X, A) \otimes_C H^\bullet(X, A) \to H^\bullet(X, A).
\end{equation}

2.2. The Čech complex and cup-product. Suppose now each $x \in X$ has a neighborhood $U \subset X$ so that $H^i(U, A|_U) = 0$ if $i > 0$. Then for each fine enough open cover $\mathcal{U}$, the (sheaf version of) Čech complex $C^\bullet = C^\bullet(\mathcal{U}, A)$ is a resolution of $A$. Hence $A \boxtimes A \to C^\bullet \boxtimes C^\bullet$.

Multiplication $(2.3)$ allows us to define a cup-product on Čech cochains in a familiar manner
\begin{equation}
(2.10) \quad C^p \boxtimes C^q \ni f \boxtimes g \mapsto f \ast g, \quad (f \ast g)(i_0, \ldots, i_{p+q}) = m(f(i_0, \ldots, i_p), g(i_p, \ldots, i_q)).
\end{equation}
We have obtained the following commutative diagram
\[
\begin{array}{ccc}
A \boxtimes A & \longrightarrow & C^\bullet \boxtimes C^\bullet \\
\downarrow & & \downarrow \\
\Delta_\bullet(A) & \longrightarrow & \Delta_\bullet(C^\bullet)
\end{array}
\]
Taking cohomology we obtain yet another, Čech, multiplication
\begin{equation}
(2.11) \quad \hat{h}(m) : H^\bullet(X, A) \otimes_C H^\bullet(X, A) \to H^\bullet(X, A).
\end{equation}
We wish to show that this multiplication coincides with the canonical one $(2.8)$. In fact, this is true at the cochain level. Note that $I^\bullet$ being injective we obtain quasi-isomorphisms, $C^\bullet \to I^\bullet, \Delta_\bullet(C^\bullet) \to \Delta_\bullet(I^\bullet), C^\bullet \boxtimes C^\bullet \to I^\bullet \boxtimes I^\bullet$, canonical at the level of the derived category.
Lemma 2.2.1. Multiplications (2.7) and (2.10) coincide in the derived category. In particular,

$$\hat{h}(m) = h(m).$$

Proof. Collect all the morphisms in sight in the following diagram:

![Diagram](image)

By construction, commutative in this diagram are the top and bottom triangles, the left hand trapezoid, and the square. The right hand trapezoid commutes only at the level of derived categories – as indeed asserted by the lemma. To see this, one only needs to precompose the 2 of its paths leading from $C^* \otimes C^*$ to $\Delta_* (I^*)$ and then use the listed commutativity properties. Applying the cohomology functor to this trapezoid, we obtain the commutative diagram

$$
\begin{align*}
H^*(X, A) \otimes H^*(X, A) &\longrightarrow H^*(X, A) \otimes H^*(X, A) \\
\hat{h}(m) &\downarrow \quad h(m) \\
H^*(X, A) &\longrightarrow H^*(X, A)
\end{align*}
$$

The proof is complete. □

2.3. Identities. Assume that the algebra $A$ satisfies an order $n$ multilinearized identity. This means there is a function $f : S_n \rightarrow \mathbb{C}$, $S_n$ the symmetric group, such that for each $U \subset X$ and each $n$-tuple $a_1, ..., a_n \subset A(U)

$$
(2.13) \quad \sum_{\sigma \in S_n, ()} f(\sigma)((x_{\sigma_1} \ast x_{\sigma_2} \ast \cdots \ast x_{\sigma_n})) = 0.
$$

We are not assuming associativity, and so $((x_{\sigma_1} \ast x_{\sigma_2} \ast \cdots \ast x_{\sigma_n}))$ stands for the product of the $x$’s taken in the indicated order and with the brackets inserted in some way that is somewhat schematically denoted by $()$.

Each operation $A^{\otimes n} \rightarrow A$, $x_1 \otimes \cdots \otimes x_n \mapsto (x_1 \ast x_2 \cdots \ast x_n)$ extends to an operation on the resolution $(I^*)^{\otimes n} \rightarrow \Delta^{(n)} (I^*)$, where $\Delta^{(n)} : X \rightarrow X \times X$ is the diagonal.

Each permutation $A^{\otimes n} \rightarrow A^{\otimes n}$ also extends to a permutation on $(I^*)^{\otimes n}$, but in order to obtain a morphism of complexes a standard rule of sign must be enforced.
If a map such as one defined by (2.13) equals 0, then the corresponding map on $(\mathcal{I}^\bullet)^{\otimes n}$ is homotopic to 0 – because a map such as (2.7) is unique up to homotopy. The passage to the cohomology proves the following.

**Lemma 2.3.1.** If $\mathcal{A}$ satisfies identity (2.13), then $H^\bullet(X, \mathcal{A})$ satisfies its super-version

$$f^{super} : \sum_{\sigma \in S_n} (-1)^{\sigma(x)} f(\sigma)((x_{\sigma_1} \ast x_{\sigma_2} \ast \cdots x_{\sigma_n})) = 0,$$

where $\sigma(x)$ is the number of inversions of odd cohomological degree $x$’s in $x_{\sigma_1} \ast x_{\sigma_2} \ast \cdots x_{\sigma_n}$.

**Remark 2.3.2.** All of this can be rephrased as follows. The category with objects sheaves of vector spaces over $X$, morphisms various $\text{Hom}(\mathcal{A}_1 \boxtimes \cdots \boxtimes \mathcal{A}_n, \Delta^{(n)}(\mathcal{A}))$ is a pseudo-tensor category [3]. An algebra $\mathcal{A}$ defines an algebra object in this category, the multiplication being a morphism $\mathcal{A} \boxtimes \mathcal{A} \to \Delta_{\bullet}(\mathcal{A})$. If $\mathcal{A}$ satisfies an order $n$ multilinearized identity, so will the corresponding object in the pseudo-tensor category. The cohomology functor can be applied to the multiplication $\mathcal{A} \boxtimes \mathcal{A} \to \Delta_{\bullet}(\mathcal{A})$ so as to endow $H^\bullet(X, \mathcal{A})$ with a $C$-algebra structure satisfying a super-version of the identity.

### 2.4. Vertex algebras

A vertex algebra is a collection $(V, 1_{(\cdot)}; n \in \mathbb{Z})$, where $V$ is a $C$-supervector space, $(\cdot)_{(n)} : V \otimes V \to V, a \otimes b \mapsto a_{(n)}b$

$1 \in V$ is a distinguished element known as the vacuum vector; the following axioms hold:

$$a_{(n)}b = 0 \text{ if } n \gg 0,$$

$$1_{(n)} = \delta_{n,-1}1, a_{(-1)}1 = a, a_{(n)}1 = 0 \text{ if } n \geq 0$$

$$\sum_{j \geq 0} \binom{m}{j} a_{(n+j)}b_{(m+k-j)}c = \sum_{j \geq 0} (-1)^j \binom{n}{j} \{a_{(m+n-j)}b_{(k+j)}c - (-1)^n(-1)^{\delta b_{(n+k-j)}(a_{(m+j)}c)}\}$$

If $\mathcal{V}$ is a sheaf of vertex algebras, then each multiplication is extended to an injective resolution of $\mathcal{V}$ via (2.7) or to a Čech resolution via the cup-product (2.10). These multiplications define ones on the cohomology $H^\bullet(X, \mathcal{V})$ via (2.8) or (2.11). The validity of the nilpotency condition (2.15) follows at once from the cup-product formula (2.10). The multiplications on the cochains satisfy the Borcherds identity (2.17), an example of a trilinearized identity, up to homotopy, the ones on the cohomology satisfy (2.17) on the nose; this follows from sect. [2.3] especially Lemma 2.3.1 or rather its obvious multioperational version. We have established the following:

**Lemma 2.4.1.** If $\mathcal{V}$ is a sheaf of vertex algebras, then $H^\bullet(X, \mathcal{V})$ carries a canonical vertex algebra structure with $1 \in \Gamma(X, \mathcal{V})$ the vacuum vector and multiplications $(\cdot)_{(n)}$, $n \in \mathbb{Z}$, defined via (2.8) or (2.11).
Here is our main application of the above discussion.

2.5. A CDO over the flag manifold. Let $G$ be a simple complex Lie group, $B \subset G$ the Borel subgroup. The flag manifold $G/B$ carries an algebra of chiral differential operators (CDO), an example of a sheaf of vertex algebras introduced initially in [14]; see also [11] for a different point of view.

The vertex algebra $\mathcal{V}(\mathfrak{g})_{-h^+}$ attached to the Lie algebra $\mathfrak{g} = \text{Lie} G$ at the critical level contains the Feigin-Frenkel center $\mathfrak{z}(\mathfrak{g}) \subset \mathcal{V}(\mathfrak{g})_{-h^+}$. Denote by $\mathcal{V}(\mathfrak{g})_{X=0}$ the quotient of $\mathcal{V}(\mathfrak{g})_{-h^+}$ by the ideal generated by $\mathfrak{z}(\mathfrak{g})$. There is an embedding

$$
(2.20) \quad \mathcal{V}(\mathfrak{g})_{X=0} \hookrightarrow \Gamma(X, \mathcal{D}_{G/B}^{ch}),
$$

which gives the cohomology $H^\bullet(G/B, \mathcal{D}_{G/B}^{ch})$ a $\mathcal{V}(\mathfrak{g})_{X=0}$--structure. The latter was computed in [2] to the effect that

$$
(2.19) \quad H^i(G/B, \mathcal{D}_{G/B}^{ch}) = \bigoplus_{w \in W, l(w)=i} \mathcal{V}(\mathfrak{g})_{X=0},
$$

where $W$ is the Weyl group, $l(.)$ is the length function. Our task is to compute the vertex algebra structure on $H^\bullet(G/B, \mathcal{D}_{G/B}^{ch})$, see Lemma 2.4.1

Denote by $\text{Sing}$ the commutator (i.e., the space of singular vectors) of $\mathcal{V}(\mathfrak{g})_{X=0}$ in $H^\bullet(G/B, \mathcal{D}_{G/B}^{ch})$, cf. (2.18). More explicitly,

$$
(2.20) \quad \text{Sing} = \{ a \in H^\bullet(G/B, \mathcal{D}_{G/B}^{ch}) \text{ s.t. } g_{(n)} a = 0 \forall g \in \mathcal{V}(\mathfrak{g})_{X=0}, n \geq 0 \}.
$$

It is an easy consequence of (2.17) that $\text{Sing}$ is a vertex subalgebra of $H^\bullet(G/B, \mathcal{D}_{G/B}^{ch})$. Furthermore, the irreducibility of $\mathcal{V}(\mathfrak{g})_{X=0}$, [7], implies that $\text{Sing}$ is a $\mathbb{C}$-span of several copies of the vacua, one for each copy of $\mathcal{V}(\mathfrak{g})_{X=0}$ appearing in the decomposition (2.19). It follows from decomposition (2.19) that the two embeddings $\text{Sing} \hookrightarrow H^\bullet(G/B, \mathcal{D}_{G/B}^{ch}) \hookrightarrow \mathcal{V}(\mathfrak{g})_{X=0}$ give a vertex algebra isomorphism

$$
(2.21) \quad \text{Sing} \otimes_{\mathbb{C}} \mathcal{V}(\mathfrak{g})_{X=0} \sim H^\bullet(G/B, \mathcal{D}_{G/B}^{ch}).
$$

To determine a vertex algebra structure on $H^\bullet(G/B, \mathcal{D}_{G/B}^{ch})$, we need that on $\text{Sing}$. Now let us recall that a vertex algebra $V$ is called commutative if multiplications $(a) = 0$ if $n \geq 0$. In this case, the triple $(V_i(-1), 1)$ is a unital associative commutative algebra with derivation

$$
(2.22) \quad T : V \to V, \quad T(a) = a_{(-2)} 1.
$$

In fact, the assignment $V \mapsto (V_i(-1), 1, T)$ establishes an equivalence between the category of commutative vertex algebras and associative commutative unital algebras with derivation, see e.g. [12]. In particular, any choice of a derivation makes a commutative associative unital algebra into a vertex algebra.

An associative commutative algebra we wish to focus on is $H^\bullet(\mathfrak{n}, \mathbb{C})$, the cohomology of the maximal nilpotent $\mathfrak{n} \subset \mathfrak{g}$ with trivial coefficients.

**Theorem 2.5.1.** (i) The vertex algebra $\text{Sing}$ is commutative and, for some derivation of $H^\bullet(\mathfrak{n}, \mathbb{C})$, there is a vertex algebra isomorphism

$$
(2.23) \quad \text{Sing} \sim H^\bullet(\mathfrak{n}, \mathbb{C}).
$$
(ii) $H^\bullet(n, \mathbb{C})$ being equipped with a vertex algebra structure as in (2.23), there is a vertex algebra isomorphism

$$ (2.24) \quad H^\bullet(n, \mathbb{C}) \otimes \mathcal{V}(g)_{\chi=0} \xrightarrow{\sim} H^\bullet(G/B, \mathcal{D}^{ch}_{G/B}). $$

(iii) If $g$ is either $sl_2$ or $sl_3$, then the derivation of $H^\bullet(n, \mathbb{C})$ chosen in assertion (i) is 0.

**Corollary 2.5.2.** The center of $H^\bullet(G/B, \mathcal{D}^{ch}_{G/B})$ is isomorphic to $H^\bullet(n, \mathbb{C})$ as an associative commutative algebra.

**Proof of Corollary 2.5.2.** Recall that the center of a vertex algebra $V$ is $\{v \in V : v_{(n)}V = 0 \forall n \geq 0\}$. We have $\text{Sing}_{(n)}(\mathcal{V}(g)_{\chi=0}, n \geq 0$ by definition, $\text{Sing}_{(n)} \text{Sing} = 0$ by virtue of Theorem 2.5.1(i), hence $\text{Sing}_{(n)}(\mathcal{V}(g)_{\chi=0} \otimes \mathcal{V}(g)_{\chi=0}) = 0$; it remains to use (2.23) and (2.24). □

**Proof of Theorem 2.5.1.**

**Proof of assertion (i).**

A. Let us establish a semi-infinite cohomology interpretation of $\text{Sing}$; this will serve as a bridge to the Lie algebra cohomology $H^\bullet(n, \mathbb{C})$. Denote by $\mathcal{C}^{\infty/2+\bullet}(Ln, \mathbb{C})$ the semi-infinite cochain complex for the loop algebra $Ln$, i.e., the vertex algebra whose various differentials lead to various cohomology theories of $Ln$. For $V$ a vertex algebra and $V(n) \rightarrow V$ a vertex algebra morphism, $\chi : n \rightarrow \mathbb{C}$ the principal character, let

$$ \mathcal{C}^{\infty/2+\bullet}(Ln; V)_{DS} \overset{\text{def}}{=} (\mathcal{C}^{\infty/2+\bullet}(Ln, \mathbb{C}) \otimes V, d_{DS}) $$

be the Drinfeld-Sokolov reduction complex. It is a differential graded vertex algebra with differential $d_{DS}$; its cohomology will be denoted by $H^{\infty/2+\bullet}(Ln; V)_{DS}$.

Suppose now $V = H^\bullet(G/B, \mathcal{D}^{ch}_{G/B})$. A singular vector $v \in \text{Sing}$ determines a cohomology class:

$$ (2.25) \quad \text{Sing} \rightarrow H^{\infty/2+0}(Ln; H^\bullet(G/B, \mathcal{D}^{ch}_{G/B})_{DS}, v \mapsto \text{class of } 1 \otimes v. $$

Frenkel and Gaitsgory [7] have computed the cohomology $H^{\infty/2+\bullet}(Ln; \mathcal{V}(g)_{\chi=0})_{DS}$. Since $H^\bullet(G/B, \mathcal{D}^{ch}_{G/B})_{DS}$ is a direct sum of several copies of the module $\mathcal{V}(g)_{\chi=0}$, (2.19), their result applies and implies that map (2.25) defines a vertex algebra isomorphism

$$ (2.26) \quad \text{Sing} \xrightarrow{\sim} H^{\infty/2+\bullet}(Ln; H^\bullet(G/B, \mathcal{D}^{ch}_{G/B})_{DS}, v \mapsto \text{class of } 1 \otimes v. $$

It is convenient to recast this result as follows. Consider a sheaf of differential graded vertex algebras $\mathcal{C}^{\infty/2+\bullet}(Ln; \mathcal{D}^{ch}_{G/B})_{DS}$. The *hypercohomology* $H^\bullet(G/B, \mathcal{C}^{\infty/2+\bullet}(Ln; \mathcal{D}^{ch}_{G/B})_{DS})$ arises; this is a vertex algebra by virtue of (an obvious version of) Lemma 2.4.1 Isomorphism (2.26) implies

$$ (2.27) \quad \text{Sing} \xrightarrow{\sim} H^\bullet(G/B, \mathcal{C}^{\infty/2+\bullet}(Ln; \mathcal{D}^{ch}_{G/B})_{DS}). $$

Let us now relate this to the ordinary Lie algebra cohomology $H^\bullet(n, \mathbb{C})$. 

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**Note:**

- $\mathcal{D}^{ch}$ denotes the sheaf of differential graded vertex algebras.
- $\chi$ is the principal character.
- $\mathcal{V}(g)_{\chi=0}$ is a module over the vertex algebra $\mathcal{V}(g)$.
- $d_{DS}$ is the differential associated with the Drinfeld-Sokolov reduction complex.
- $H^{\infty/2+\bullet}(Ln; V)_{DS}$ is the cohomology of the Drinfeld-Sokolov reduction complex.
- $\text{Sing}$ denotes the set of singular vectors in $V$. 

B. For an $n$-module $V$, denote by $\mathcal{C}^\bullet(n, V)$ the standard cochain complex with values in $V$. Since $n \subset g$ operates on $G/B$, a sheaf of commutative differential graded algebras $\mathcal{C}^\bullet(n, \mathcal{O}_{G/B})$ arises. We have an obvious embedding $\mathcal{O}_{G/B} \hookrightarrow \mathcal{D}^e_{G/B}$, $f \mapsto f1$. We would like to assert that it is a vertex algebra morphism, but it is not because $\mathcal{O}_{G/B}$ does not carry an appropriate derivation because if $f \in \mathcal{O}_{G/B} \subset \mathcal{D}^e_{G/B}$, then $f(-2)1 \notin \mathcal{O}_{G/B}$, cf. (2.22). We can, however, equip $\mathcal{O}_{G/B}$ with multiplications $(n)$, $n \geq -1$, by declaring that $(-1)$ is the standard multiplication on $\mathcal{O}_{G/B}$ and $(n) = 0$ if $n \geq 0$. Let us for the purpose of the remainder of this proof call a vector space space with multiplications $(n)$, $n \geq -1$, an \textit{algebra}. Thus the morphism $\mathcal{O}_{G/B} \hookrightarrow \mathcal{D}^e_{G/B}$ becomes an \textit{algebra} morphism.

It naturally extends to a differential graded algebra morphism

(2.28) $\mathcal{C}^\bullet(n, \mathcal{O}_{G/B}) \to \mathcal{C}^{\infty/2+\bullet}(L_n; \mathcal{D}^e_{G/B})_{DS}$.

Since $H^\bullet(G/B, \mathcal{O}_{G/B}) = H^0(G/B, \mathcal{O}_{G/B}) = \mathbb{C}$, $H^\bullet(G/B, \mathcal{C}^\bullet(n, \mathcal{O}_{G/B})) = H^\bullet(n, \mathbb{C})$. Therefore, (2.28) gives

(2.29) $H^\bullet(n, \mathbb{C}) \to H^\bullet(G/B, \mathcal{C}^{\infty/2+\bullet}(L_n; \mathcal{D}^e_{G/B})_{DS})$.

Lemma 2.5.3. \textit{Map (2.29) is an algebra isomorphism.}

This lemma implies Theorem 2.5.1 (i) at once because assertion (i) is the lemma combined with (2.27); in particular, the derivation of $H^\bullet(n, \mathbb{C})$, the missing ingredient, is one pulled back from Sing.

\textit{Proof of Lemma 2.5.3} We shall use the approach of [2]. Starting with the BGG-resolution of the trivial $g$-module

(2.30) $0 \to \mathbb{C} \to M^*$

$M^* : M^c_0 \to \bigoplus_{w \in W; l(w) = 1} M^c_{w0} \to \cdots \to \bigoplus_{w \in W; l(w) = 1} M^c_{w0} \to \cdots$,

we pull the bootstraps to obtain, first, the Cousin-Grothendieck resolution of $\mathcal{O}_{G/B}$, via the Beilinson-Bernstein localization functor $\Delta$,

(2.31) $0 \to \mathcal{O}_{G/B} \to \Delta(M^*)$;

second, a resolution of $\mathcal{D}^e_{G/B}$, using a version of the Zhu functor [1]

(2.32) $0 \to \mathcal{D}^e_{G/B} \to \mathcal{Zhu} \circ \Delta(M^*)$.

In the derived category, the map $\mathcal{O}_X \to \mathcal{D}^e_{G/B}$ is nothing but the natural embedding

$\Delta(M^*) \hookrightarrow \mathcal{Zhu} \circ \Delta(M^*)$,

hence

(2.33) $\mathcal{C}^\bullet(n, \Delta(M^*)) \to \mathcal{C}^{\infty/2+\bullet}(L_n; \mathcal{Zhu} \circ \Delta(M^*))_{DS}$.

The map of hypercohomology induced by the latter is precisely (2.29), but now we can compute it in a different way. An application of $R\Gamma(G/B, \cdot)$ to both sides of (2.33) gives

$\mathcal{C}^\bullet(n, M^*) \to \mathcal{C}^{\infty/2+\bullet}(L_n; \Gamma(G/B, \mathcal{Zhu} \circ \Delta(M^*)))_{DS}$.
Since $M^\bullet$ is a co-free resolution of $\mathbb{C}$, the cohomology of the left hand side is $H^\bullet(n, \mathbb{C})$. Furthermore, the cohomology classes are 0-cochains defined by highest weight vectors of the contragredient Verma modules $M^\bullet_{G/B}$, cf. (2.30).

Similarly, it is the essence of the proof of (2.19) proposed in [2] that the cohomology of the right hand side is $H^\bullet(G/B, C^\infty/2+*)(Ln; D^ch_{G/B})_{DS}$ and that the cohomology classes are precisely 0-cochains defined by the highest weight vectors of $\Gamma(G/B, Zhu \circ \Delta(M^\bullet))$.

The map $M^\bullet \to \Gamma(G/B, Zhu \circ \Delta(M^\bullet))$, by its definition, sends highest weight vectors to highest weight vectors. Hence the resulting map

$$H^\bullet(n, \mathbb{C}) \to H^\bullet(G/B, C^\infty/2+*)(Ln; D^ch_{G/B})_{DS}$$

is an isomorphism.

Assertion (ii) is decomposition (3.21) and assertion (i) combined.

Proof of assertion (iii). As follows from (3.22), we need to show that $a(\pi \cdot 1) = 0$ for $\forall a \in \text{Sing}$. Definition (2.20) implies that both $a$ and $a(-1)1$ are highest weight vectors w.r.t. $\hat{\mathfrak{g}}$ of conformal weights that differ from each other by 1. On the other hand, if $a \in H^\bullet(G/B, D^ch_{G/B})$, then $a(-2)1 \in H^\bullet(G/B, D^ch_{G/B})$ too. However, Theorem 1.1 (2) of [2] shows at once that if $\mathfrak{g}$ is either $sl_2$ or $sl_3$, then all highest weight vectors of the same cohomological degree have the same conformal weight; hence $a(-2)1$ must be 0.

This concludes the proof of Theorem 2.5.1. □

3. Bosonization over the Projective Line

3.1. The CDO. The CDO $D^ch_{P^1}$ can be defined very explicitly. Let $\{\mathbb{C}_0, \mathbb{C}_\infty\}$ be the standard atlas of $\mathbb{P}^1$ with $x, y = 1/x$ the respective coordinates. The space $D^ch_{P^1}(\mathbb{C}_0)$ is what is known as the $\beta\gamma$-system, a vertex algebra generated by 2 even elements, $x, \partial_x$, and relations

$$\partial_{x(x)}x = -x(x)\partial_x = \delta_{n0}1 \text{ if } n \geq 0.$$  

$D^ch_{P^1}(\mathbb{C}_\infty)$ is defined similarly with $x$ replaced with $y$. That $x$ and $y$ play a 2-faceted role, a coordinate and a vertex algebra element, should not lead to too much confusion for their transformation laws, which we are about to introduce, are the same in each of their capacities.

One can localize $D^ch_{P^1}(\mathbb{C}_0)$ ($D^ch_{P^1}(\mathbb{C}_\infty)$) over $\mathbb{C}_0$ ($\mathbb{C}_\infty$ resp.) and then glue over $\mathbb{C}^* = \mathbb{C}_0 \cap \mathbb{C}_\infty$ sending

$$x \mapsto 1/y, \partial_x \mapsto -\partial(y(-1)(y(-1)y) - 2y(-2)1.$$  

The result is the $D^ch_{P^1}$.

3.2. The Friedan-Martinec-Shenker bosonization. Consider the lattice $L = \mathbb{Z}p \oplus \mathbb{Z}q$ with $(p, p) = (q, q) = 1, (p, q) = 0$. There arises a lattice vertex algebra, $V_L$, see an exposition in [12], sect.5.4,5.5. Its main ingredients are the Lie algebra $\mathfrak{h}$ with basis $\{p_n, q_n, n \in \mathbb{Z}\}$ and commutation relations

$$[p_n, p_m] = -[q_n, q_m] = n\delta_{n,-m}, \ [p_n, q_m] = 0,$$  

$$\partial p_n = -q_n - np_{n+1}, \ \partial q_n = 0.$$  

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$$x \mapsto 1/y, \partial_x \mapsto -\partial(y(-1)(y(-1)y) - 2y(-2)1.$$  

The result is the $D^ch_{P^1}$.
S, the vacuum $\hat{h}$-module, and $\mathbb{C}[L]_\epsilon$, a twisted group algebra of $L$ with multiplication

$$e^\alpha e^\beta = e(\alpha, \beta)e^{\alpha+\beta},$$

see [12], sect.5.5, for an explicit formula for the cocycle $\epsilon$.

As a vector space, $V_L = \mathbb{C}[L]_\epsilon \otimes S$. The formulas for products $(n)$ are classic and can be found in *loc. cit.*; we shall only record the generating function for the operators $e_{(n)}^\alpha$ given by the celebrated vertex operators:

$$\sum_{n \in \mathbb{Z}} e_{(n)}^\alpha z^{-n-1} = e^\alpha z^{(\alpha,.)} \exp \sum_{n<0} \frac{\alpha_n}{-n} z^{-n} \exp \sum_{n>0} \frac{\alpha_n}{-n} z^{-n},$$

where $(\alpha,.)$ is understood as a function that sends $e^\beta \otimes s$ to $(\alpha, \beta)e^\beta \otimes s$.

Define the grading $V_L = \oplus_{m \in \mathbb{Z}} V_L[m]$, where $V_L[m]$ is the eigenspace of $(p + q, .)$ with eigenvalue $m$. We let

$$V_{L_\geq} = \oplus_{m \geq 0} V_L[m], \quad V_{L_\leq} = \oplus_{m \leq 0} V_L[m].$$

If we let $D_{\geq} = e_{(0)}^p$, $D_{\leq} = e_{(0)}^{-p}$, then we obtain 2 differential graded vertex algebras

$$(V_{L_{\geq}}, D_{\geq}), \quad (V_{L_{\leq}}, D_{\leq})$$

and the cohomology vertex algebras, $H^\bullet(V_{L_{\geq}})$ and $H^\bullet(V_{L_{\leq}})$.

Now consider $D_{\rho_{\geq}}(\mathbb{C}_0)$ as a differential graded vertex algebra with trivial grading and zero differential.

**Lemma 3.2.1.** The assignment

$$x \mapsto e^{p+q}, \quad \partial_x \mapsto p(-1)e^{-p-q}$$

defines a quasiisomorphism of differential graded vertex algebras

$$\rho_{\geq} : D_{\rho_{\geq}}(\mathbb{C}_0) \to (V_{L_{\geq}}, D_{\geq}).$$

In particular,

$$D_{\rho_{\geq}}(\mathbb{C}_0) \sim H^0(V_{L_{\geq}}).$$

This is known as Friedan-Martinec-Shenker bosonization, [9]. The assertion about $H^0(V_{L_{\geq}})$ was proved in [6]. The vanishing of the higher cohomology (undoubtedly known to Feigin and Frenkel) is easy and follows from the fact that $e_{(-1)}^{-p}$ is a contracting homotopy; indeed $[e_{(0)}^p, e_{(-1)}^{-p}] = 1$.

The advantage of passing from $D_{\rho_{\geq}}(\mathbb{C}_0)$ to $(V_{L_{\geq}}, D_{\geq})$ is that in the latter the element $x$ is invertible: just let $x^{-1} = e^{-p-q}$. Upon localization, we obtain a vertex algebra morphism

$$D_{\rho_{\geq}}^h(\mathbb{C}^*) \to V_L[0],$$

which is readily seen to be an isomorphism. Therefore, an embedding $D_{\rho_{\geq}}^h(\mathbb{C}_\infty) \hookrightarrow V_L[0], \text{ compatible with } D_{\rho_{\geq}}^h(\mathbb{C}_0) \to V_L[0]$ that appears in Lemma 3.2.1 must arise. It does and its construction is natural: the map $x \mapsto x^{-1}$ lifts to an automorphism of $D_{\rho_{\geq}}^h(\mathbb{C}^*)$. In terms of $V_L[0]$, the latter is nothing but the automorphism engendered by the lattice automorphism $p \mapsto -p$, $q \mapsto -q$. Hence the following assertion, a companion of Lemma 3.2.1
Corollary 3.2.2. The assignment
\[ y \mapsto e^{-p-q}, \ \partial y \mapsto -p(-1)e^{p+q} \]
defines a quasiisomorphism of differential graded vertex algebras
\[ \rho_\leq : D^{ch}_{p1}(\mathbb{C}_\infty) \to (V_L, D_\leq) \]
so that
\[ \rho_\geq \mid_{D^{ch}_{p1}(\mathbb{C}_0) \cap D^{ch}_{p1}(\mathbb{C}_\infty)} = \rho_\leq \mid_{D^{ch}_{p1}(\mathbb{C}_0) \cap D^{ch}_{p1}(\mathbb{C}_\infty)}, \]
where the intersection \( D^{ch}_{p1}(\mathbb{C}_0) \cap D^{ch}_{p1}(\mathbb{C}_\infty) \) is that of the images w.r.t. the embeddings \( D^{ch}_{p1}(\mathbb{C}_0) \hookrightarrow D^{ch}_{p1}(\mathbb{C}_*) \hookrightarrow D^{ch}_{p1}(\mathbb{C}_\infty) \) In particular,
\[ D^{ch}_{p1}(\mathbb{C}_\infty) \cong H^0(V_L). \]

3.3. A bosonization à la Borisov. Consider a complex
\[ 0 \to V_L[0] \to V_L[1] \oplus V_L[-1] \to 0. \]
It follows at once from Lemma 3.2.1 and Corollary 3.2.2 that its 0-th cohomology is \( H^0(\mathbb{P}^1, D^{ch}_{p1}) \). We shall now extend it to a complex that computes the entire \( H^*(\mathbb{P}^1, D^{ch}_{p1}) \). Our argument is a straightforward purely even version of the construction that Borisov proposed on the case of the chiral de Rham complex [5].

Consider the Čech complex \( \check{\mathcal{C}}^\bullet = \check{\mathcal{C}}^\bullet(\mathcal{U}, D^{ch}_{p1}) \) attached to the open cover \( \mathcal{U} = \{ \mathbb{C}_0, \mathbb{C}_\infty \} \):

\begin{equation}
\begin{array}{ccc}
D^{ch}_{p1}(\mathbb{C}_*) & \to & D^{ch}_{p1}(\mathbb{C}_0) \oplus D^{ch}_{p1}(\mathbb{C}_\infty) \\
D^{ch}_{p1}(\mathbb{C}_0) & \to & D^{ch}_{p1}(\mathbb{C}_\infty) \\
\end{array}
\end{equation}

Lemma 3.2.1 and Corollary 3.2.2 give us a resolution of \( \check{\mathcal{C}}^\bullet \), to be denoted \( R^\bullet(\check{\mathcal{C}}^\bullet) \), that is a double complex as follows:

\begin{equation}
\begin{array}{ccc}
V_L[0] & \to & 0 \\
V_L[0] \oplus V_L[0] & \to & V_L[1] \oplus V_L[-1] \\
V_L[1] \oplus V_L[0] & \to & V_L[2] \oplus V_L[-2] \\
\end{array}
\end{equation}

Recall that the cup-product formula (2.10) makes \( \check{\mathcal{C}}^\bullet \) into a differential graded algebra with products \( (n) \), \( n \in \mathbb{Z} \), and it is this algebra structure that defines a vertex algebra structure on \( H^*(\mathbb{P}^1, D^{ch}_{p1}) \). Lemma 2.4.1. The cup-product is easy to extend from \( \check{\mathcal{C}}^\bullet \) to \( R^\bullet(\check{\mathcal{C}}^\bullet) \): in order to do that simply write \( R^\bullet(\check{\mathcal{C}}^0) = R^\bullet(\check{\mathcal{C}}^0_{01}) \), \( R^\bullet(\check{\mathcal{C}}^1)_{01} = R^\bullet(\check{\mathcal{C}}^1_{01}) \) and then use the same (2.10). The quasiisomorphism
\[ \check{\mathcal{C}}^\bullet \to R^\bullet(\check{\mathcal{C}}^\bullet) \]
becomes a differential graded algebra quasiisomorphism, the quasiisomorphism assertion being the result of computing the cohomology of \( \mathcal{R}^\bullet(\mathcal{C}^\bullet) \) w.r.t. the horizontal differential in (3.4).

A computation starting with the vertical differential will again display a collapse of the spectral sequence and show that \( H^\bullet(\mathbb{P}^1, \mathcal{D}_{P1}^{ch}) \) equals the cohomology of the complex

\[
V_L[0] \xrightarrow{(D_\geq, D_\leq)} V_L[+1] \oplus V_L[-1] \xrightarrow{(D_\geq, D_\leq)} V_L[+2] \oplus V_L[-2] \rightarrow \cdots
\]

This complex is also a differential graded algebra with a family of products. It is a bit unexpected that not only this algebra structure defines a vertex algebra structure on the cohomology, but it itself is a vertex algebra on the nose. To see this, note that the cup-product on 0-chains being defined by \((f \circ g)(i) = f(i)g(i)\), cf. (2.10), the products on (3.5) are

\[
u_{(n)}v = \begin{cases} u_{(n)}v & \text{if } u \in V_L[m], v \in V_L[n], m \cdot n \geq 0 \\ 0 & \text{if } u \in V_L[m], v \in V_L[n], m \cdot n < 0 \end{cases}
\]

It is easy to show directly that (3.6) is a vertex algebra structure; it is even more useful to obtain it as a degeneration of one on \( V_L \). Indeed upon rescaling

\[ V_L[m] \ni v \mapsto t^{|m|}v, \]

the product \((n)\) is replaced with \((n), t\) so that

\[
u_{(n), t}v = \begin{cases} u_{(n)}v & \text{if } \deg u \cdot \deg v \geq 0 \\ t^{\deg u + |\deg v| - |\deg u + \deg v|}u_{(n)}v & \text{if } \deg u \cdot \deg v < 0 \end{cases}
\]

This gives a family of vertex algebras, \( \{V_L^t, t \in \mathbb{C}^*\} \), isomorphic to each other and with a well-defined limit as \( t \to 0 \). Furthermore, \( V_L^0 \overset{\text{def}}{=} \lim_{t \to 0} V_L^t \) is precisely (3.6); hence the latter also defines a vertex algebra structure. Since \( D_\geq + D_\leq \) is a derivation of the latter, our discussion proves the following.

**Lemma 3.3.1.** There is a vertex algebra isomorphism

\[ H^\bullet(\mathbb{P}^1, \mathcal{D}_{P1}^{ch}) \sim H_{D_\geq + D_\leq}^\bullet(V_L^0). \]

Since \( D_\geq + D_\leq \) remains a derivation in the deformed algebra, we can consider \( H_{D_\geq + D_\leq}^\bullet(V_L^t) \).

**Theorem 3.3.2.**

\[ H_{D_\geq + D_\leq}^\bullet(V_L^t) = \begin{cases} H^\bullet(\mathbb{P}^1, \mathcal{D}_{P1}^{ch}) & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** The argument is similar to that in [13], and we shall be brief. The differential on \( V_L^t \) can be written as \( D = D_0 + tD_1 \), where \( D_0 \) coincides with the differential on the degenerated \( V_L^0 \), \( D_1 \) commutes with \( D_0 \) and maps in the opposite direction. Indeed, while \( D_0 \) is as in (3.5), the complex \( (V_L^t, D_0 + tD_1) \) is as follows:

\[
V_L[0] \xrightarrow{(D_\geq, D_\leq)} V_L[+1] \oplus V_L[-1] \xrightarrow{(D_\geq, D_\leq)} V_L[+2] \oplus V_L[-2] \rightarrow \cdots
\]
The case \( t = 0 \) of the theorem is but Lemma 3.3.1. If \( t \neq 0 \), then a spectral sequence arises with \( E_1 \), the \( D_0 \)-cohomology. The latter is known, [2, 14]: \( E_1 = H^0(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) \oplus H^1(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}), H^0(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) = H^1(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) = \mathbb{V}(\mathfrak{sl}_2)_{x=0}, \) cf. (2.19). Therefore, \((E_1, d_1)\) is

\[
tD_1 : H^1(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) \to H^0(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}).
\]

\( tD_1 \) is a \( \mathfrak{g} \)-module morphism. Let us see how it acts on the highest weight vector.

The highest weight vector of \( H^1(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) \) was computed in [14]; it equals the Čech cochain \( C^* \mapsto x_{(-2)} x^{-1} \). According to Lemma 3.3.1, upon bosonization it is identified with

\[
e^{-p-q}((p+q)_{(-1)} e^{p+q}) = (p+q)_{(-1)} 1 \in V_L[0].
\]

This places our cocycle in the uppermost-leftmost corner of (3.4). In order to identify it with a cocycle in \( V_L^0 \), see (3.5), we have to do a bit of a diagram chase.

It is clear that the cocycle belongs to the image of \((p+q)_{(-1)} 1, 0\) under the action of the vertical differential in (3.4), hence cohomologous to

\[
(D_{ge}(p+q)_{(-1)} 1, 0) = (e^{p}_{(0)} (p+q)_{(-1)} 1, 0) = (-e^p, 0) \in V_L^0[1].
\]

Therefore, its image under the action of \( tD_1 \) is, see (3.7),

\[
tD_1((-e^p, 0)) = t(-D_{ge} e^p, 0) = t e^{p}_{(0)} e^p = t 1,
\]

a highest weight vector of \( H^0(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) \) if \( t \neq 0 \). Since \( H^0(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) \) and \( H^1(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) \) are irreducible, morphism (3.8) is an isomorphism. Therefore, \( E_2 = 0 \), and the assertion of Theorem 3.3.2 follows. □

Remark 3.3.3. Witten explains [16] that \( H^\bullet(\mathbb{P}^1, D_{\mathbb{P}^1}^{eh}) = H^\bullet_{D_2 + D_{ge}}(V_L^0) \) is Witten’s half-twisted algebra of a certain \( \sigma \)-model with \( (0,2) \)-supersymmetry in perturbative regime. If we are to believe that deforming \( H^\bullet_{D_2 + D_{ge}}(V_L^0) \) to \( H^\bullet_{D_2 + D_{ge}}(V_L^1) \) means taking into account instanton corrections – such apparently was original Borisov’s intention [5], and indeed this construction gives quantum cohomology in case of \( \mathbb{P}^n \) and the chiral de Rham complex [13], see also [8] – then the vanishing \( H^\bullet_{D_2 + D_{ge}}(V_L^1) = 0 \) if \( t \neq 0 \) confirms Witten’s assertion [16] that nonperturbatively half-twisted algebra vanishes.

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REFERENCES

[1] T. Arakawa, D. Chebotarov, F. Malikov, Algebras of twisted chiral differential operators and affine localization of \( g \)-modules, [arXiv:0810.4964]
[2] T. Arakawa, F. Malikov, A chiral Borel-Weil-Bott theorem, posted on [arXiv:0903.1281]
[3] A. Beilinson, V. Drinfeld, Chiral algebras, (Colloquium Publications, AMS Providence, Rhode Island, 2004).
[4] R. Bott, Homogeneous vector bundles, Ann. Math. 66 203–248 (1957)

13
[5] L. Borisov, Vertex algebras and mirror symmetry, *Comm. Math. Phys.* **215**, no. 3, 517-557 (2001).
[6] B. Feigin, E. Frenkel, Semi-infinite Weil complex and the Virasoro algebra, *Comm. in Math. Phys.* **137**, 617-639 (1991).
[7] E. Frenkel, D. Gaitsgory, Local Geometric Langlands Correspondence: the Spherical Case, posted on arXiv:0711.1132 (2007).
[8] E. Frenkel, A. Losev, Mirror symmetry in two steps: A-I-B. *Comm. Math. Phys.* **269** (2007), no. 1, 39–86.
[9] D. Friedan, E. Martinec, S. Shenker, Conformal invariance, supersymmetry and string theory, Nuclear Physics B271, 93-165 (1986).
[10] J. Gamst, K. Hoechsmann, Products in sheaf cohomology, *Tôhoku Math. Journ.* **22**, 143–162 (1970).
[11] V. Gorbounov, F. Malikov, V. Schechtman, On chiral differential operators over homogeneous spaces. *Int. J. Math. Math. Sci.* 26, no.2, 83–106 (2001).
[12] V. Kac, *Vertex algebras for beginners*, University Lecture Series, **10. American Mathematical Society, Providence, RI, 1997**.
[13] F. Malikov, V. Schechtman, Deformations of vertex algebras, quantum cohomology of toric varieties, and elliptic genus. *Comm. Math. Phys.* 234 (2003), no. 1, 77–100.
[14] F. Malikov, V. Schechtman, A. Vaintrob, Comm. in Math. Phys. **204**, 439-473 (1999).
[15] Meng-Chwan Tan, Junya Yagi, Chiral algebras of (0, 2) models: beyond perturbation theory. *Lett. Math. Phys.* **84** (2008), no. 2-3, 257–273.
[16] E. Witten, Two-dimensional models with (0,2) supersymmetry: perturbative aspects, *Adv. Theor. Math. Phys.* **11**, Number 1 (2007), 1-63.

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