On Difference of Operators with Different Basis Functions

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Abstract. In the recent years several researchers have studied problems concerning the difference of two linear positive operators, but all the available literature on this topic is for operators having same basis functions. In the present paper, we deal with the general quantitative estimate for the difference of operators having different basis functions. In the end we provide some examples. The estimates for the differences of two operators can be obtained also using classical result of Shisha and Mond. Using numerical examples we will show that for particular cases our result improves the classical one.

1. Introduction

Recently Acu et al. [1] and Aral et al. [2] established some interesting results for the difference of operators in order to generalize the problem posed by A. Lupas [12] on polynomial differences. Recently Gupta in [8] and [9] established some results on the difference of operators having same basis functions. Some of the results on this topic are compiled in the recent book by Gupta et al [10]. Let $I \subseteq \mathbb{R}$ be an interval and $C(I) = \{f : I \to \mathbb{R} \text{ continous} \}$, $C(I)$ containing the polynomials and $C_B(I)$ the space of all $f \in C(I)$ such that $$
\|f\| = \sup \{|f(x)| : x \in I\} < \infty.$$
Also, let $F : C(I) \to \mathbb{R}$ be a positive linear functional such that $F(e_0) = 1$. Denote
$$b_F := F(e_1), \mu_F^r = F(e_1 - b_F e_0)^r, r \in \mathbb{N}.$$
An inequality for the functional $F$ was establish in [1] as follows:

\textbf{Lemma 1.1.} [1] Let $f \in C(I)$ with $f'' \in C_B(I)$. Then
$$|F(f) - f(b_F)| \leq \frac{\mu_F^2}{2} \|f''\|.$$
Let
\[ \omega_k(f; h) := \sup \left\{ \left| \sum_{j=0}^{k} (-1)^{k-j} \left( \frac{k}{j} \right) f(x + j\delta) \right| : |\delta| \leq h, x, x + j\delta \in I \right\} \]
be the classical \( k \)th order modulus of smoothness given for a compact interval \( I \) and \( h \in \mathbb{R}_+ \).

In order to obtain one of the main results we will mention the following lemma proved by Gonska and Kovacheva in [4]:

** Lemma 1.2.** [4] If \( f \in C^0[0, 1] \), then for all \( 0 < h \leq \frac{1}{2} \) there are functions \( \nu \in C^{\alpha+2}[0, 1] \), such that

i) \( \| f^{(0)} - \nu^{(0)} \| \leq \frac{3}{4} \omega_2(f^{(0)}; h) \),

ii) \( \| f^{(\alpha+1)} \| \leq \frac{5}{\lambda} \omega_1(f^{(0)}; h) \),

iii) \( \| f^{(\alpha+2)} \| \leq \frac{3}{2\lambda^2} \omega_2(f^{(0)}; h) \).

In [1], Acu and Raşa gave an inequality for the functional \( F \) in terms of second modulus of continuity.

** Proposition 1.3.** [1] Let \( f \in C[0, 1], \lambda \geq 2 \sqrt{\mu_2} > 0 \). Then

\[ |F(f) - f(b^F)| \leq \frac{3}{\lambda} \omega_2 \left( f, \frac{\sqrt{\mu_2}}{\lambda}, \lambda^2 \right). \]

2. Difference of operators

Let \( K \) be a set of non-negative integers and \( v_{n,k}, u_{n,k} \geq 0, v_{n,k}, u_{n,k} \in C(I), k \in K \), such that \( \sum_{k \in K} u_{n,k} = \sum_{k \in K} v_{n,k} = e_0 \). Consider the positive linear functionals \( F_{n,k}, G_{n,k} : C(I) \to \mathbb{R} \) such that \( F_{n,k}(e_0) = F_{n,k}(e_0) = 1 \) and denote \( D(I) \) the set of all \( f \in C(I) \) for which

\[ \sum_{k \in K} u_{n,k} F_{n,k}(f) \in C(I) \text{ and } \sum_{k \in K} v_{n,k} G_{n,k}(f) \in C(I). \]

Let \( U_n, V_n : D(I) \to C(I) \) be two positive linear operators defined as

\[ U_n(f; x) := \sum_{k \in K} u_{n,k}(x) F_{n,k}(f), \quad V_n(f; x) := \sum_{k \in K} v_{n,k}(x) G_{n,k}(f). \]

** Theorem 2.1.** If \( f \in D(I) \) with \( f'' \in C(I) \), then

\[ |(U_n - V_n)(f; x)| \leq \alpha(x) \| f'' \| + 2 \omega_1(f; \delta_1(x)) + 2 \omega_1(f; \delta_2(x)), \]

where

\[ \alpha(x) := \frac{1}{2} \sum_{k \in K} \left( u_{n,k}(x) \mu_{l_2} + v_{n,k}(x) \mu_{l_2} \right), \]

\[ \delta_1^2(x) := \sum_{k \in K} u_{n,k}(x) \left( f^{l_2} - x \right)^2, \quad \delta_2^2(x) := \sum_{k \in K} v_{n,k}(x) \left( G^{l_2} - x \right)^2. \]
Proof. We can write

$$
|\langle U_n - V_n \rangle(f; x) \rangle \leq \sum_{k \in K} |u_{n,k}(x)| F_{n,k}(f) - v_{n,k}(x) G_{n,k}(f)|
$$

$$
\leq \sum_{k \in K} u_{n,k}(x)|F_{n,k}(f) - f(b^{n,k})| + \sum_{k \in K} v_{n,k}(x)|G_{n,k}(f) - f(b^{n,k})|
$$

$$
+ \sum_{k \in K} u_{n,k}(x)|f(b^{n,k}) - f(x)| + \sum_{k \in K} v_{n,k}(x)|f(b^{n,k}) - f(x)|
$$

$$
\leq \frac{1}{2} \sum_{k \in K} u_{n,k}(x) \mu_2^{2n,k} \|f''\| + \frac{1}{2} \sum_{k \in K} v_{n,k}(x) \mu_2^{2n,k} \|f''\|
$$

$$
+ \sum_{k \in K} u_{n,k}(x) \left(1 + \frac{(b^{n,k} - x)^2}{\delta_2^2(x)}\right) \omega_1(f, \delta_1(x)) + \sum_{k \in K} v_{n,k}(x) \left(1 + \frac{(b^{n,k} - x)^2}{\delta_2^2(x)}\right) \omega_1(f, \delta_2(x))
$$

$$
= \alpha(x) \|f''\| + 2\omega_1(f; \delta_1(x)) + 2\omega_1(f; \delta_2(x)).
$$

Remark 2.2. If we use result of Shisha and Mond [19], we can write

$$
|\langle U_n - V_n \rangle(f; x) \rangle \leq |\langle U_n(f; x) - f(x) \rangle| + |\langle V_n(f; x) - f(x) \rangle|
$$

$$
\leq 2\omega_1(f, v_1(x)) + 2\omega_1(f, v_2(x)),
$$

where

$$
v_1^2(x) = U_n \left((e_1 - x)^2; x\right) = \sum_{k \in K} u_{n,k}(x) F_{n,k} \left((e_1 - x)^2; x\right),
$$

$$
v_2^2(x) = V_n \left((e_1 - x)^2; x\right) = \sum_{k \in K} v_{n,k}(x) G_{n,k} \left((e_1 - x)^2; x\right).
$$

Since $F_{n,k}^{2n}(e_1) \leq F_{n,k}(e_1^2)$ and $G_{n,k}^{2n}(e_1) \leq G_{n,k}(e_1^2)$, it follows $\delta_i(x) \leq v_i(x), i = 1, 2$.

Remark 2.3. If $F_{n,k}^{2n}(e_1) = F_{n,k}(e_1^2)$ and $G_{n,k}^{2n}(e_1) = G_{n,k}(e_1^2)$, then the relation (1) becomes

$$
|\langle U_n - V_n \rangle(f; x) \rangle \leq 2\omega_1(f, v_1(x)) + 2\omega_1(f, v_2(x)),
$$

where $v_1, v_2$ are defined in Remark 2.2.

Theorem 2.4. Let $I = [0, 1], f \in C[0, 1], 0 < h \leq \frac{1}{2}, x \in [0, 1]$. Then

$$
|\langle U_n - V_n \rangle(f; x) \rangle \leq \frac{3}{2} \left(1 + \frac{\alpha(x)}{h^2}\right) \omega_2(f, h) + (\delta_1(x) + \delta_2(x)) \frac{10}{h} \omega_1(f, h).
$$

Proof. From relation (1) for $v \in C^2[0, 1]$, we get

$$
|\langle U_n - V_n \rangle(v; x) \rangle \leq \alpha(x) \|v''\| + 2(\delta_1(x) + \delta_2(x)) \|v'\|.
$$

We can write

$$
|\langle U_n - V_n \rangle(f; x) \rangle \leq |\langle U_n(f; x) - V_n(v; x) \rangle| + |\langle U_n(v; x) - V_n(v; x) \rangle| + |\langle V_n(v; x) - V_n(f; x) \rangle|
$$

$$
\leq 2\|f - v\| + \alpha(x) \|v''\| + 2(\delta_1(x) + \delta_2(x)) \|v'\|.
$$
Using Lemma 1.2, there exists \( \nu \in C^2[0, 1] \) such that

\[
\|f - \nu\| \leq \frac{3}{4} \omega_2(f, h); \quad \|\nu'\| \leq \frac{5}{4h} \omega_1(f, h); \quad \|\nu''\| \leq \frac{3}{2h^2} \omega_2(f, h).
\]

Therefore,

\[
|(U_n - V_n)(\nu; x)| \leq \frac{3}{2} \omega_2(f, h) + \frac{3}{2h^2} \alpha(x) \omega_2(f, h) + \frac{10}{h} (b_1(x) + \delta_2(x)) \omega_1(f, h),
\]

and the proof is complete.

\[\square\]

3. Applications

3.1. Difference of two genuine-Durrmeyer type operators

Păltănea [17] and further Păltănea and Gonska ([5, 6]) proposed a new class of Bernstein-Durrmeyer type operators \( U_n^\theta : C[0, 1] \rightarrow \Pi_{\alpha} \), defined as

\[
U_n^\theta(f; x) := \sum_{k=0}^{n} P^\theta_{n,k}(f)p_{n,k}(x)
\]

\[
:= \sum_{k=1}^{n-1} \left( \sum_{t=0}^{k-1} \frac{\alpha_{k-1}(1-t)^{(\alpha-k)-1}}{B(k, n-k)\rho} f(t) \right) p_{n,k}(x) + f(0)(1-x)^\alpha + f(1)x^n,
\]

where \( p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \). These operators constitute a link between genuine Bernstein-Durrmeyer operators \( U_n(\rho = 1) \) and the classical Bernstein operators \( B_n(\rho \to \infty) \).

Stancu [20] introduced a sequence of positive linear operators \( P_n^{\alpha,\rho} : C[0, 1] \rightarrow C[0, 1] \), depending on a parameter \( \alpha \geq 0 \) as follows

\[
P_n^{\alpha,\rho}(f; x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) p_{n,k}^{\alpha,\rho}(x), \quad x \in [0, 1],
\]

where

\[
p_{n,k}^{\alpha,\rho}(x) = \binom{n}{k} x^k (1-x)^{n-k} \rho \frac{1}{1-\alpha} \quad \text{and} \quad x^{[n,\alpha]} := x(x-\alpha) \cdots (x-n-1\alpha).
\]

Note that for \( \alpha = 0 \) are obtained Bernstein operators and for \( \alpha = \frac{1}{n} \) these operators reduce to the special case introduced by L. Lupuš and A. Lupuš [13] as follows

\[
P_n^{\frac{1}{n},\rho}(f; x) = \frac{2n!}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{k}{n} \right) (nx)(n-1x)_{n-k}.
\]

The moments of the operators (3) are given by

\[
P_n^{\frac{1}{n},\rho}(e_0; x) = 1, \quad P_n^{\frac{1}{n},\rho}(e_1; x) = x, \quad P_n^{\frac{1}{n},\rho}(e_2; x) = x^2 + \frac{2x(1-x)}{n+1}.
\]

In 2017, Neer and Agrawal [16] introduced a genuine-Durrmeyer type modification of the operators (3) as follows:

\[
U_n^\theta(f; x) := \sum_{k=0}^{n} P^\theta_{n,k}(f)p_{n,k}^{\frac{1}{n},\rho}(x), \quad \rho > 0, \quad f \in C[0, 1].
\]
Proposition 3.1. The following inequalities hold

\[ \left\| (U^p_n - \tilde{U}^p_n)(f;x) \right\| \leq \frac{x(1-x)}{n \rho + 1} \left\| f'' \right\| + 2 \omega_1 \left( f, \sqrt{\frac{x(1-x)}{n}} \right) + 2 \omega_1 \left( f, \sqrt{\frac{2x(1-x)}{n+1}} \right), f'' \in C[0, 1]; \]

\[ \left\| (U^p_n - \tilde{U}^p_n)(f;x) \right\| \leq 3 \omega_2 \left( f, \sqrt{\frac{x(1-x)}{n \rho + 1}} \right) + 10(1 + \sqrt{2}) \sqrt{\frac{n \rho + 1}{n}} \omega_1 \left( f, \sqrt{\frac{x(1-x)}{n \rho + 1}} \right), f \in C[0, 1]. \]

Proof. Denote

\[ F_{n,k}(f) = G_{n,k}(f) = \left\{ \int_0^1 \frac{\mu^{p-1}(1-t)^{(n-k)p-1}}{B(\rho, (n-k)\rho)} f(t)dt, \quad 1 \leq k \leq n-1, \right. \]

\[ \left. \quad f(0), \quad f(1), \quad k = 0, \quad k = n. \right\}

We have

\[ b_{F_{n,k}} = b_{F_{n,k}}^c = \frac{k}{n}; \]

\[ \mu_{2,F_{n,k}} = \mu_{2,F_{n,k}}^c = F_{n,k} \left( e_1 - b_{F_{n,k}}^c \right)^2 = \frac{k(n-k)}{n^2(n \rho + 1)}; \]

\[ \alpha(x) = \frac{1}{2} \sum_{k=0}^n \left( p_{n,k}(x) + p_{n,k}^c(x) \right) \mu_{2,F_{n,k}} \]

\[ = \frac{1}{2(n \rho + 1)} \left\{ B_n(e_1; x) - B_n(e_2; x) + p_{n}^{<\frac{1}{2}}(e_1; x) - p_{n}^{<\frac{1}{2}}(e_2; x) \right\} \]

\[ = \frac{x(1-x)}{2(n \rho + 1)} \left\{ 2 - \frac{1}{n} - \frac{2}{n+1} \right\} \leq \frac{x(1-x)}{n \rho + 1}, \]

where \( B_n \) is the classical Bernstein operators. Also,

\[ \delta_{2,F_{n,k}}(x) = \sum_{k=0}^n \left( p_{n,k}(x) \left( b_{F_{n,k}}^c - x \right)^2 = B_n(e_2; x) - 2xB_n(e_1; x) + x^2 = \frac{x(1-x)}{n}; \right. \]

\[ \delta_{2,F_{n,k}}^c(x) = \sum_{k=0}^n \left. \left( p_{n,k}^c(x) \left( b_{F_{n,k}}^c - x \right)^2 = P_n^{<\frac{1}{2}}(e_2; x) - 2xP_n^{<\frac{1}{2}}(e_1; x) + x^2 = \frac{2x(1-x)}{n+1}. \right\} \right. \]

Using Theorem 2.1 and Theorem 2.4 the proof is complete. \( \square \)

Example 3.2. Let \( f(x) = \frac{x}{x^2 + 1}, x \in [0, 1] \) and \( \rho = 2 \). From Proposition 3.1 we obtain the following estimate for the differences of the operators \( U^p_n \) and \( \tilde{U}^p_n \)

\[ \left\| (U^p_n - \tilde{U}^p_n)(f;x) \right\| \leq E_n(f), \]  \hspace{1cm} (5)

where

\[ E_n(f) = \frac{1}{4(n \rho + 1)} \left\| f'' \right\| + \left( \frac{1}{\sqrt{n}} + \frac{\sqrt{2}}{n \rho + 1} \right) \left\| f'' \right\|, f \in C^2[0, 1] \]

Using the result of Shisha and Mond (see Remark 2.2) and the central moments of the operators \( U^p_n \) and \( \tilde{U}^p_n \), namely

\[ U^p_n((e_1 - x)^2; x) = \frac{(\rho + 1)x(1-x)}{n \rho + 1}, \quad \tilde{U}_n^c((e_1 - x)^2; x) = \frac{(2n \rho + n + 1)x(1-x)}{n+1(n \rho + 1)}, \]
we get the next result
\[ \| (U_n^f - \bar{U}_n^f)(f; x) \| \leq E_n^{(SM)}(f), \] (6)
where
\[ E_n^{(SM)}(f) = \frac{1}{\sqrt{2n+1}} \left( \sqrt{3n+2} + \frac{2n + 1 + 1}{n+1} \right) \| f'' \|, \ f \in C^1[0, 1]. \]

Table 3.2 contains the errors for the differences of the operators \( U_n^f \) and \( \bar{U}_n^f \). Note that for this particular case the estimate (5) is better than the estimate given by Shisha and Mond’s result (6).

3.2. Difference of Durrmeyer operators and Lupas-Durrmeyer operators

The classical Durrmeyer operators were introduced by Durrmeyer [3] and, independently, by Lupas [14] and are defined as
\[ M_n(f, x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{0}^{1} p_{n,k}(t) f(t) \, dt, \ x \in [0, 1]. \] (7)
The moments of the operator (7) are given by:
\[ M_n(e_0; x) = 1, \ M_n(e_1; x) = x + \frac{1 - 2x}{n + 2}, \ M_n(e_2; x) = \frac{n(n - 1)x^2 + 4nx + 2}{(n + 2)(n + 3)}. \]
The Durrmeyer type modification of the operator (3) was introduced by Gupta et al. [11] as follows
\[ D_n^{\frac{1}{2}}(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}^{\frac{1}{2}} \int_{0}^{1} p_{n,k}(t) f(t) \, dt, \ f \in C[0, 1]. \] (8)
The moments of the operators (8) are given below (see [11])
\[ D_n^{\frac{1}{2}}(e_0; x) = 1, \ D_n^{\frac{1}{2}}(e_1; x) = \frac{nx + 1}{n + 2}, \ D_n^{\frac{1}{2}}(e_2; x) = \frac{n^2x^2 + 5nx^2 - n^2x^2 + 3nx + 2n + 2}{(n + 1)(n + 2)(n + 3)}. \]

**Proposition 3.3.** The following inequalities hold
\[ \left\| (M_n - D_n^{\frac{1}{2}})(f; x) \right\| \leq \frac{n(2n + 1)(n + 1)(n - 1)x(1 - x) + 2(n + 1)^2}{2(n + 1)(n + 3)(n + 2)^2} \| f'' \| \]
\[ + 2\omega_1 \left( f, \sqrt{\frac{nx(1 - x) + (2x - 1)^2}{(n + 2)^2}} \right) + 2\omega_1 \left( f, \sqrt{\frac{2n^2x(1 - x) + (2x - 1)^2}{(n + 2)^2(n + 1)}} \right), \ f'' \in C[0, 1]; \]
Proof. The Durrmeyer operators and the Lupas-Durrmeyer operators can be written as follows:

\[ M_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) F_{n,k}(f), \quad D_n^{ <\frac{1}{2}>}(f; x) = \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{2}>}(x) G_{n,k}(f), \]

where

\[ F_{n,k}(f) = G_{n,k}(f) = (n + 1) \int_{0}^{1} p_{n,k}(t) f(t) dt. \]

We have

\[ b_{2n,k} = b_{2n,k}^{G_{n,k}} = \frac{k + 1}{n + 2}, \]

\[ \mu_{2n,k}^{F_{n,k}} = \mu_{2n,k}^{C_{n,k}} = F_{n,k} \left( e_1 - b_{2n,k}^{F_{n,k}} \right)^2 = \frac{(k + 1)(n - k + 1)}{(n + 2)^2(n + 3)}; \]

\[ \alpha(x) = \frac{1}{2} \sum_{k=0}^{n} \left( p_{n,k}(x) + p_{n,k}^{<\frac{1}{2}>}(x) \right) \mu_{2n,k}^{C_{n,k}} \]

\[ = \frac{1}{2(n + 2)^2(n + 3)} \left\{ -n^2 B_n(e_2; x) + n^2 B_n(e_1; x) - n^2 P_n^{<\frac{1}{2}>}(e_2; x) + n^2 P_n^{<\frac{1}{2}>}(e_1; x) + 2(n + 1) \right\} \]

\[ = \frac{1}{2(n + 2)^2(n + 3)} \left\{ 2n^2 x(1 - x) - nx(1 - x) - \frac{2n^2 x(1 - x)}{n + 1} + 2(n + 1) \right\} \]

\[ = \frac{n(2n + 1)(n - 1)x(1 - x) + 2(n + 1)^2}{2(n + 1)(n + 3)(n + 2)^2}, \]

where \( B_n \) is the classical Bernstein operators. Also,

\[ \delta_{2n}^1(x) = \sum_{k=0}^{n} p_{n,k}(x) \left( b_{2n,k}^{G_{n,k}} - x \right) = \frac{nx(1 - x) + (2x - 1)^2}{(n + 2)^2}; \]

\[ \delta_{2n}^2(x) = \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{2}>}(x) \left( b_{2n,k}^{G_{n,k}} - x \right) = \frac{2n^2 x(1 - x) + (2x - 1)^2(n + 1)}{(n + 2)^2(n + 1)}. \]

Using Theorem 2.1 and Theorem 2.4 the proof is complete. \( \square \)

Example 3.4. Let \( f(x) = \sin(2\pi x), \ x \in [0, 1]. \) From Proposition 3.3 we obtain the following estimate for the differences of the operators \( M_n \) and \( D_n^{ <\frac{1}{2}>}. \)

\[ \left\| (M_n - D_n^{ <\frac{1}{2}>})(f; x) \right\| \leq E_n(f), \]

where

\[ E_n(f) = \frac{n(2n + 1)(n - 1) + 8(n + 1)^2}{8(n + 2)^2(n + 3)(n + 1)} \left\| f'' \right\| + \frac{1}{n + 2} \left( \sqrt{n + 4} + \sqrt{\frac{2(n^2 + 2n + 2)}{n + 1}} \right) \left\| f'' \right\|, \ f \in C^2[0, 1] \]
Estimate for the differences of $M_n$ and $D_n^{<1,n>}$

| $n$  | $E_n(f)$   | $E_n^{(SM)}(f)$ |
|------|------------|-----------------|
| 10   | 5.11006052700 | 5.12324449500   |
| $10^2$ | 1.59818810900 | 1.93302197800   |
| $10^3$ | 0.48908880900 | 0.62371818010   |
| $10^4$ | 0.15266270140 | 0.1976405390    |
| $10^5$ | 0.04806670559 | 0.06251225990   |
| $10^6$ | 0.01517880741 | 0.01976851709   |
| $10^7$ | 0.00479783008 | 0.00625136679   |
| $10^8$ | 0.00151699380 | 0.00197685616   |

Using the result of Shisha and Mond (see Remark 2.2) and the central moments of the operators $M_n$ and $D_n^{<1,n>}$, namely

$$M_n((e_1-x)^2; x) = \frac{2x(1-x)(n-3) + 2}{(n+2)(n+3)}; \quad D_n^{<1,n>}(e_1-x)^2; x) = \frac{x(1-x)(3n^2 - 5n - 6) + 2(n+1)}{(n+1)(n+2)(n+3)},$$

we get the next result

$$|(M_n - D_n^{<1,n>})(f; x)| \leq E_n^{(SM)}(f),$$

where

$$E_n^{(SM)}(f) = 2\left(\sqrt{\frac{n+1}{2(n+2)(n+3)}} + \sqrt{\frac{3n^2 + 3n + 2}{4(n+1)(n+2)(n+3)}}\right)\|f'\|, \quad f \in C^1[0, 1].$$

Table 3.4 contains the errors for the differences of the operators $M_n$ and $D_n^{<1,n>}$. Note that for this particular case the estimate (9) is better than the estimates given by Shisha and Mond’s result (10).

3.3. Baskakov operators and Szász operators

This section deals with the quantitative estimates for the differences of Szász operators and the Baskakov operators.

The Baskakov operators are defined as

$$V_n(f; x) = \sum_{k=0}^{\infty} v_{n,k}(x)F_{n,k}(f),$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \quad F_{n,k}(f) = f\left(\frac{k}{n}\right).$$

**Remark 3.5.** We have

$$b^{F_{n,k}} = F_{n,k}(e_1) = \frac{k}{n},$$

$$\mu_2^{F_{n,k}} := F_{n,k}(e_1 - b^{F_{n,k}} e_0)^2 = 0.$$

Some of the moments of Baskakov operators are given below

$$V_n(e_0, x) = 1, \quad V_n(e_1, x) = x, \quad V_n(e_2, x) = \frac{x^2(n+1) + x}{n},$$

$$V_n((e_1-x)^2; x) = \frac{x(x+1)}{n}.$$
The Szász operators are defined as
\[
S_n(f; x) = \sum_{k=0}^{\infty} s_{n,k}(x) G_{n,k}(f),
\]  
(12)

where
\[
s_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}, \quad G_{n,k}(f) = f\left(\frac{k}{n}\right).
\]

**Remark 3.6.** We have
\[
b^{G_{n,k}} = G_{n,k}(e_1) = \frac{k}{n}
\]
\[
\mu_2^{G_{n,k}} = G_{n,k}(e_1 - b^{G_{n,k}} e_0)^2 = 0
\]

Some of the moments of Szász operators defined by (12) are given as:
\[
S_n(e_0; x) = 1, \quad S_n(e_1; x) = x, \quad S_n(e_2; x) = x^2 + \frac{x}{n} \quad S_n((e_1 - x)^2; x) = \frac{x}{n}. \]

Using Remark 2.3 the following quantitative estimate for the difference of Baskakov operators and Szász operators is obtained.

**Proposition 3.7.** Let \( f \in C_B[0, \infty) \) and \( x \in [0, \infty) \). Then for \( n \in \mathbb{N} \), we have
\[
| (V_n - S_n)(f, x) | \leq 2 \omega_1 \left( f, \sqrt{\frac{x(1+x)}{n}} \right) + 2 \omega_1 \left( f, \sqrt{\frac{x}{n}} \right).
\]

### 3.4. Baskakov and the integral modification of the Szász-Baskakov operators

The integral modification of the Szász-Baskakov type operators considered in [18] and later improved by Gupta in [7] are defined as
\[
M_n(f; x) = \sum_{k=0}^{\infty} s_{n,k}(x) H_{n,k}(f),
\]  
(13)

where \( s_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!} \) is the Szász basis function defined in (12) and
\[
H_{n,k}(f) = (n-1) \int_0^\infty v_{n,k}(t)f(t)dt,
\]
with \( v_{n,k}(t) \) the Baskakov basis functions defined by
\[
v_{n,k}(t) = \binom{n+k-1}{k} t^k (1+t)^{n-k}.
\]

**Remark 3.8.** We have
\[
b^{H_{n,k}} = H_{n,k}(e_1) = \frac{k+1}{n-2}.
\]

Also, we have
\[
\mu_2^{H_{n,k}} := H_{n,k}(e_1 - b^{H_{n,k}} e_0)^2
\]
\[
= H_{n,k}(e_2, x) + \left( \frac{k+1}{n-2} \right)^2 - 2H_{n,k}(e_1, x) \left( \frac{k+1}{n-2} \right)
\]
\[
= \frac{(k+2)(k+1)}{(n-2)(n-3)} - \left( \frac{k+1}{n-2} \right)^2 = \frac{k^2 + nk + n - 1}{(n-2)^2(n-3)}.
\]
Following proposition provides the estimate for the difference of Baskakov and integral modification of Szász-Baskakov operators, which is application of Theorem 2.1:

**Proposition 3.9.** Let \( f^{(s)} \in C_b[0,\infty), s \in \{0,1,2\} \) and \( x \in [0,\infty) \), then for \( n \in \mathbb{N}, n > 3 \), we have

\[
| (M_n - V_n)(f;x) | \leq \frac{n^2x^2 + nx + n^2x + n - 1}{2(n-2)^2(n-3)} \| f'' \| + 2\omega_1 \left( f; \sqrt{\frac{x(1 + x)}{n}} \right) + 2\omega_1 \left( f; \sqrt{\frac{4x^2 + (4 + n)x + 1}{n - 2}} \right).
\]

**Proof.** By Remark 3.5 and Remark 3.6, we have

\[
\alpha(x) = \frac{1}{2} \left[ \sum_{k=0}^{\infty} \nu_{n,k}(x) \mu_{2}^{n,k} + \sum_{k=0}^{\infty} s_{n,k}(x) \mu_{2}^{H,n,k} \right]
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k^2 + nk + n - 1}{(n-2)^2(n-3)}
\]

\[
= \frac{n^2x^2 + nx + n^2x + n - 1}{2(n-2)^2(n-3)}.
\]

Further by using Remark 3.5, we get

\[
\delta_1^2(x) = \sum_{k=0}^{\infty} \nu_{n,k}(x) (b_L^{n,k} - x)^2
\]

\[
= \sum_{k=0}^{\infty} \nu_{n,k}(x) \left( \frac{k}{n} - x \right)^2 = \frac{x(1 + x)}{n}.
\]

Finally by Remark 3.6 and Remark 3.8, we get

\[
\delta_2^2(x) = \sum_{k=0}^{\infty} s_{n,k}(x) (b_H^{n,k} - x)^2
\]

\[
= \sum_{k=0}^{\infty} s_{n,k}(x) \left( \frac{k+1}{n-2} - x \right)^2 = \frac{4x^2 + (4 + n)x + 1}{(n-2)^2}.
\]

\[\square\]

### 3.5. Lupas and Szász-Durrmeyer operators

Lupas [12] proposed the following operators

\[
L_n(f; x) = \sum_{k=0}^{\infty} l_{n,k}(x) L_{n,k}(f), \tag{14}
\]

where \( l_{n,k}(x) = \frac{2^{-\omega_1(\alpha_k)}}{\omega_2} \) and

\[
L_{n,k}(f) = f\left( \frac{k}{n} \right).
\]

**Remark 3.10.** We have

\[
b_L^{n,k} = L_{n,k}(e_1) = \frac{k}{n}
\]

\[
\mu_2^{n,k} := L_{n,k}(e_1 - b_L^{n,k} e_0)^2 = 0.
\]
Some of the moments of Lupas operators are given as:

\[ L_n(e_0, x) = 1, \quad L_n(e_1, x) = x, \quad L_n(e_2, x) = x^2 + \frac{2x}{n} \]

\[ L_n((e_1 - x)^2; x) = \frac{2x}{n}. \]

The Szász-Durrmeyer operators [15] are defined as

\[ D_n(f; x) = \sum_{k=0}^{\infty} s_{n,k}(x)Q_{n,k}(f), \quad (15) \]

where \( s_{n,k}(x) \) is the Szász basis function defined in (12) and

\[ Q_{n,k}(f) = n \int_0^\infty s_{n,k}(t)f(t)dt. \]

**Remark 3.11.** By simple computation, we have

\[ Q_{n,k}(e_r) = \frac{(k + r)!}{k!n^r}, \]

implying

\[ b^{Q_{n,k}} = Q_{n,k}(e_1) = \frac{k + 1}{n}. \]

Next, we have

\[ \mu_2^{Q_{n,k}} := Q_{n,k}(e_1 - b^{Q_{n,k}}e_0)^2 = \frac{k + 1}{n^2}. \]

Following result provides the estimate for the difference of Lupas and Szász-Durrmeyer operators, which is application of Theorem 2.1:

**Proposition 3.12.** Let \( f^{(s)} \in C_B[0, \infty), s \in \{0, 1, 2\} \) and \( x \in [0, \infty) \), then for \( n \in \mathbb{N} \), we have

\[ \|D_n - L_n\(f; x)\| \leq \frac{nx + 1}{2n^2} \|f^{''}\| + 2\omega_1\(f; \sqrt{\frac{2x}{n}}\) + 2\omega_1\(f; \sqrt{\frac{nx + 1}{n}}\). \]

**Proof.** By Remark 3.10 and Remark 3.11, we have

\[ a(x) = \frac{1}{2} \left[ \sum_{k=0}^{\infty} l_{n,k}(x)\mu_2^{L_n} + \sum_{k=0}^{\infty} s_{n,k}(x)\mu_2^{Q_{n,k}} \right] \]

\[ = \frac{1}{2} \sum_{k=0}^{\infty} s_{n,k}(x)\frac{k + 1}{n^2} = \frac{nx + 1}{2n^2}. \]

Further by using Remark 3.10, we get

\[ \delta_1^2(x) = \sum_{k=0}^{\infty} l_{n,k}(x)(b^{Q_{n,k}} - x)^2 \]

\[ = \sum_{k=0}^{\infty} l_{n,k}(x)\left(\frac{k}{n} - x\right)^2 = \frac{2x}{n}. \]
Finally by Remark 3.6 and Lemma 3.11, we get

\[ \delta_2^2 = \sum_{k=0}^{\infty} s_{n,k}(x)((b^k-n)-x)^2 \]

\[ = \sum_{k=0}^{\infty} s_{n,k}(x)\left(\frac{k+1}{n} - x\right)^2 = \frac{nx + 1}{n^2}. \]

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Further Readings:

V. Gupta, General estimates for the difference of operators, Computational and Mathematical Methods 1(2) (2019) e1018 https://doi.org/10.1002/cmm4.1018