Diffraction effects in entanglement of two distant atoms

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Abstract. We investigate the process of entanglement creation between two distant atoms coupled to a single mode cavity. We consider the optical frequency regime and include a possible variation of the coupling strength of the atoms to the cavity field with the location of the atoms inside the cavity mode. We show that the degree of entanglement between the atoms is a sensitive function of the position of the atoms inside the mode. It is found that the entanglement exhibits an interesting phenomenon of diffraction when the the atoms are located between the nodes and antinodes of the cavity mode. The entanglement diffraction pattern varies with time and we explain this effect in terms of the quantum property of complementarity, which is manifested as a tradeoff between the knowledge of energy of the exchanged photon versus the evolution time of the system. Entangled states of the system are identified and the entanglement properties of the system are interpreted in terms of the coherence between the atoms.

1. Introduction
The preparation of entangled atoms is presently under enormous investigations, primarily because of their great importance and potential applications in quantum information processing and quantum computation [1, 2]. In most treatments the entanglement resulted from the direct interaction between the atoms or has been created by engineering the interaction through the coupling of the atoms to the same cavity mode [3–7]. Entanglement between atoms located (trapped) inside an optical cavity is usually studied for an idealized model of the atoms located precisely at the antinodes of the cavity field, so that the atoms experience the same coupling strengths to the cavity field. This is a relatively easy task at microwave frequencies and, in fact, detection of entangled atoms have already been performed on a beam of Rydberg atoms traversing a superconducting microwave cavity [8]. However, at optical frequencies this task may well be hard to achieve as it is rather difficult to locate atoms precisely at antinodes of a standing wave of a very short wavelength. Hence as a result, atoms located inside an optical cavity may not experience the same coupling strengths to the cavity field.

In this paper, we study the entanglement of two identical atoms in nonequivalent positions in the cavity field. We model this situation by assuming that the atoms experience different coupling strengths of the cavity field. We start by deriving the master equation for the density operator of the atoms by using the technique of adiabatic elimination of the cavity mode [9]. We work in the limit of a large detuning of the cavity field frequency from the atomic resonance. This also allows to eliminate the dynamics of the cavity mode leaving the master equation for the atoms alone. Next, we discuss in detail entangled properties of the system and the dependence of the entanglement on the position of the atoms inside the cavity mode using the well known measure, concurrence [10].
2. Master equation for the reduced density operator

We begin by a brief outline of the derivation of the master equation for the density operator of two atoms coupled to a cavity mode. The atoms are modeled as two-level system (qubits) with upper levels \( |e_i\rangle \), \( i = 1, 2 \), lower levels \( |g_i\rangle \), connected by an electric dipole transition and separated by energy \( h\omega_0 \). The atoms are coupled to a standing-wave cavity mode with the coupling constants \( g(r_j) \), having units of frequency and being proportional to the electric dipole matrix elements of the atoms. In addition, the atoms are damped at the rate \( \gamma \) by spontaneous emission to modes other than the privileged cavity mode. Moreover, we assume that the atoms are separated by many optical wavelengths and therefore ignore the direct interaction between them. The cavity mode is damped with the rate \( \kappa \) and its frequency \( \omega_c \) is significantly detuned from the atomic transition frequency \( \omega_0 \), so there is no direct exchange of photons between the atoms and the cavity mode. The behavior of the total system, the atoms plus the cavity mode, is described by the density operator \( \rho \), which in the interaction picture satisfies the master equation

\[
\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H_0, \rho] + \frac{1}{2} \sum_{j=1}^{2} \gamma_j \left( 2S_j^- \rho S_j^+ - S_j^+ S_j^- \rho - \rho S_j^+ S_j^- \right) + \frac{1}{2} \kappa \left( 2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right),
\]

where

\[
H_0 = \sum_{j=1}^{2} \left[ g(r_j) a S_j^+ e^{-i\Delta t} + \text{H.c.} \right]
\]

is the Hamiltonian describing the interaction between the cavity field and the atoms. The operators \( S_j^+ \) and \( S_j^- \) are the raising and lowering operators of the \( j \)th atom, \( S_j^\dagger \) describes its energy, \( a \) and \( a^\dagger \) are the cavity-mode annihilation and creation operators, \( \Delta = \omega_c - \omega_0 \) is the detuning of the cavity-mode frequency from the atomic transition frequency, and \( r_j \) is the position coordinate of the \( j \)th atom within the cavity mode.

To isolate the atomic dynamics, we introduce the photon number representation for the density operator with respect to the cavity mode

\[
\rho = \sum_{m,n=0}^{\infty} \rho_{mn} |m\rangle\langle n|,
\]

where \( \rho_{mn} \) are the density matrix elements in the basis of the photon number states of the cavity mode. To avoid population of higher levels we assume that the cavity mode is strongly detuned from the atomic transition frequency. In this case, we may assume that only the two lowest energy levels, the ground state \( (n = 0) \) and the one-photon state \( (n = 1) \) of the cavity mode can be populated. Thus, neglecting population in all but the two lowest levels, one finds from the master equation (1) the following equations of motion

\[
\dot{\rho}_{00} = -i \sum_{j=1}^{2} \left( g_j S_j^+ \rho_{01} - g_j^* \rho_{01} S_j^- \right) + \frac{1}{2} \gamma \sum_{j=1}^{2} \left( 2S_j^- \rho_{00} S_j^+ - S_j^+ S_j^- \rho_{00} - \rho_{00} S_j^+ S_j^- \right),
\]

\[
\dot{\rho}_{01} = i\Delta \rho_{01} - i \sum_{j=1}^{2} g_j \left( S_j^+ \rho_{11} - \rho_{01} S_j^+ \right) + \frac{1}{2} \gamma \sum_{j=1}^{2} \left( 2S_j^- \rho_{01} S_j^+ - S_j^+ S_j^- \rho_{01} - \rho_{01} S_j^+ S_j^- \right),
\]

\[
\dot{\rho}_{10} = -i\Delta \rho_{10} + i \sum_{j=1}^{2} g_j^* \left( \rho_{11} S_j^- - S_j^- \rho_{00} \right) + \frac{1}{2} \gamma \sum_{j=1}^{2} \left( 2S_j^- \rho_{10} S_j^+ - S_j^+ S_j^- \rho_{10} - \rho_{10} S_j^+ S_j^- \right),
\]

\[
\dot{\rho}_{11} = -i \sum_{j=1}^{2} \left( g_j^* S_j^+ \rho_{01} - g_j \rho_{01} S_j^+ \right) + \frac{1}{2} \gamma \sum_{j=1}^{2} \left( 2S_j^- \rho_{11} S_j^+ - S_j^+ S_j^- \rho_{11} - \rho_{11} S_j^+ S_j^- \right),
\]

(4)
where \( g_j \equiv g(r_j) \) and

\[
\hat{\rho}_{01} = \rho_{01} \exp(i\Delta t) \quad \text{and} \quad \hat{\rho}_{10} = \rho_{10} \exp(-i\Delta t)
\]  

are slowly varying parts of the coherences.

Remaining in the two-level approximation, we may apply the adiabatic approximation that for a large detuning, the coherences \( \hat{\rho}_{01} \) and \( \hat{\rho}_{10} \) vary slowly in time, so we can assume that \( \hat{\rho}_{01} \approx 0 \) and \( \hat{\rho}_{10} \approx 0 \). Hence, we find from Eq. (4) that in this approximation

\[
\hat{\rho}_{01} \approx \frac{1}{\Delta} \sum_{j=1}^{2} g_j \left( S_j^+ \rho_{11} - \rho_{00} S_j^+ \right),
\]

\[
\hat{\rho}_{10} \approx \frac{1}{\Delta} \sum_{j=1}^{2} g_j^* \left( \rho_{11} S_j^- - \rho_{00} S_j^- \right).
\]

Substituting these results into the equations of motion for the populations, \( \rho_{00} \) and \( \rho_{11} \), we obtain

\[
\dot{\rho}_{00} = \frac{i}{\Delta} \sum_{i,j=1}^{2} g_i^* g_j S_i^- S_j^+, \rho_{00} \right) + \frac{1}{2}\gamma \sum_{j=1}^{2} \left( 2S_j^- \rho_{00} S_j^+ - S_j^+ S_j^- \rho_{00} - \rho_{00} S_j^+ S_j^- \right),
\]

and

\[
\dot{\rho}_{11} = \frac{-i}{\Delta} \sum_{i,j=1}^{2} g_i^* g_j S_i^- S_j^+, \rho_{11} \right) + \frac{1}{2}\gamma \sum_{j=1}^{2} \left( 2S_j^- \rho_{11} S_j^+ - S_j^+ S_j^- \rho_{11} - \rho_{11} S_j^+ S_j^- \right).
\]

Since \( \rho_{00} + \rho_{11} = Tr_F(\rho) = \rho_A \) is the reduced density operator of the atoms, we find by adding Eqs. (7) and (8) and neglecting the population \( \rho_{11} \), as the cavity mode will never be populated, that the adiabatic elimination procedure leads to the following master equation for the density operator of the atoms

\[
\frac{d\rho_A}{dt} = i \sum_{i=1}^{2} \delta_i \left[ S_i^+ S_i^-, \rho_A \right] + i \sum_{i \neq j=1}^{2} \Omega_{ij} \left[ S_i^+ S_j^-, \rho_A \right] 
\]

\[
+ \frac{1}{2}\gamma \sum_{j=1}^{2} \left( 2S_j^- \rho_A S_j^+ - S_j^+ S_j^- \rho_A - \rho_A S_j^+ S_j^- \right),
\]

where the parameters

\[
\delta_i = \frac{|g_i|^2}{\Delta}, \quad \text{and} \quad \Omega_{ij} = \Omega_{ji} = \frac{g_i g_j^*}{\Delta}
\]

describe the dynamics of the reduced (atomic) system.

The term \( \delta_i \) describes the frequency shift of the energy levels of the \( i \)th atom. It is an analog of a dynamic Stark shift [11]. The collective term \( \Omega_{ij} \) represents the shift in energy separation of the levels of atom \( i \) due to its interaction with the atom \( j \) through the cavity mode. It is an analog of the familiar dipole-dipole interaction between the atoms [2, 12–14]. This shows that one can engineer the interaction between distant atoms by the adiabatic elimination of the cavity mode. The shift of the atomic levels can be different for different coupling constants. Thus, the system composed of two identical atoms located at different positions inside the cavity mode may behave as a system of two non-identical atoms of different transition frequencies.
3. Atomic dynamics and concurrence

One of the more interesting aspects of the atomic dynamics in a strongly detuned cavity mode, demonstrated in the previous section, is its dependence on the Stark shift of the atomic transition frequencies. We have already noted that the shift of the atomic levels is different for different locations of the atoms inside the cavity mode. It is easy to see that this difference arises from an imperfect coupling of the atoms to the cavity mode. We proceed now to present more detailed studies of the sensitivity of the atomic entanglement on position of the atoms inside a standing-wave cavity mode. Since we are interested in the effect of unequal coupling constants on the atomic entanglement, we choose the reference frame such that

$$g(\vec{r}_1) = g_0,$$

and

$$g(\vec{r}_2) = g_0 \cos (kr_{12}),$$

where $r_{12} = z_2 - z_1$ is the distance between the atoms. This choice of the reference frame corresponds to a situation where atom 1 is kept exactly at an antinode of the standing wave and the atom 2 is moved through successive nodes and antinodes of the standing wave. This choice, of course, involves no loss of generality.

In order to determine the amount of entanglement between the atoms and its dependence on the position of the atoms inside the cavity mode, we use concurrence that is the widely accepted measure of entanglement. The concurrence introduced by Wootters [10] is defined as

$$C = \max \left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right),$$

where $\{\lambda_i\}$ are the the eigenvalues of the matrix

$$R = \rho_s \tilde{\rho}_s,$$

with $\tilde{\rho}_s$ given by

$$\tilde{\rho}_s = \sigma_y \otimes \sigma_y \rho_s^* \sigma_y \otimes \sigma_y,$$

and $\sigma_y$ is the Pauli matrix. Concurrence is defined as a positive quantity varying between $C = 0$ for unentangled atoms and $C = 1$ for the maximally entangled atoms.

In terms of the density matrix elements of the two-atom system, the concurrence is given by

$$C(t) = 2 \max \left\{0, |\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}\right\},$$

where we have used the following notation for the direct-product basis

$$|1\rangle = |g_1\rangle|g_2\rangle, \quad |2\rangle = |g_1\rangle|e_2\rangle, \quad |3\rangle = |e_1\rangle|g_2\rangle, \quad |4\rangle = |e_1\rangle|e_2\rangle.$$  

(16)

It is obvious from Eq. (15) that the basic dynamical mechanism for entanglement creation in this system is the coherence $\rho_{23}(t)$. In addition, if only one photon is present in the system, $\rho_{11}(t) = \rho_{44}(t) = 0$, and then we see that the entanglement in the system is determined solely by the coherence $\rho_{23}(t)$.

In order to determine the concurrence, we use the master equation (9) and find the following equations of motion for the relevant density matrix elements

$$\dot{\rho}_{23} = - (\gamma - i \delta_{12}) \rho_{23} + i \Omega_{12} (\rho_{22} - \rho_{33}),$$

$$\dot{\rho}_{32} = - (\gamma + i \delta_{12}) \rho_{32} - i \Omega_{12} (\rho_{22} - \rho_{33}),$$

$$\dot{\rho}_{22} = - \gamma \rho_{22} + i \Omega_{12} (\rho_{23} - \rho_{32}),$$

$$\dot{\rho}_{33} = - \gamma \rho_{33} - i \Omega_{12} (\rho_{23} - \rho_{32}),$$

(17)
where we have assumed that $\rho_{11}(t) = \rho_{44}(t) = 0$ for all times.

In Eq. (17) the parameter $\delta_{12} = \delta_1 - \delta_2$ is a difference between the single-atom Stark shifts. This parameter is of central importance here as it determines the relative variation of atomic transition frequencies with position of the atoms inside the cavity mode. The parameter $\delta_{12} \neq 0$ for $g_1 \neq g_2$, and only for the special case of $g_1 = g_2$, the parameter $\delta_{12} = 0$.

The system of four equations of motion (17) can be transformed using the Laplace transform method into a system of four algebraic equations in transformed variables, which is easy to solve. We may introduce real variables, so called the Bloch vector components \[15\]

\[u = \rho_{23} + \rho_{32}, \quad v = i(\rho_{23} - \rho_{32}), \quad w = \rho_{22} - \rho_{33}\] (18)

and find the general solution of Eq. (17) in terms of the initial values of the Bloch vector components. After straightforward calculations, we find that the time evolution of the coherence $\rho_{23}(t)$ is of the form

\[\rho_{23}(t) = \frac{1}{2} \frac{e^{-\tau t}}{\alpha^2} \left[ 2\Omega_{12} A + \delta_{12} \left( v_0 \alpha \sin \alpha t + B \cos \alpha t \right) - i\alpha \left( v_0 \alpha \cos \alpha t + B \sin \alpha t \right) \right],\] (19)

where $w_0 \equiv w(0)$, $u_0 \equiv u(0)$ and $v_0 \equiv v(0)$ determine the initial population distribution and coherences in the system

\[A = 2\Omega_{12} u_0 - \delta_{12} w_0, \quad B = \delta_{12} u_0 + 2\Omega_{12} w_0,\] (20)

and $\alpha = \sqrt{4\Omega_{12}^2 + \delta_{12}^2}$ determines the frequency of the oscillations. One may notice from Eq. (19) that the parameter $\alpha$ plays a role similar to that of the detuned Rabi frequency in the atom-field interaction.

4. Entanglement diffraction pattern

To illustrate the sensitivity of the entanglement on the position of the atoms inside the cavity mode, we use the solution (19) to calculate the concurrence $C(r_{12})$ as a function of $r_{12}$ and a fixed time $t$. We will discuss the variation of the concurrence with $r_{12}$ for two cases of initial conditions. In the first, we assume that the atom 1 is initially in the lower level $|g_1\rangle$ and the atom 2 is in its upper level $|e_2\rangle$, i.e. the initial state of the system is the product state

\[|\Psi\rangle_0 = |g_1\rangle|e_2\rangle.\] (21)

In this case, the initial conditions for the Bloch vector components are $w_0 = 1$, $u_0 = v_0 = 0$.

In the second case, the atoms are initially prepared in a pure single-quantum superposition (entangled) state

\[|\Psi\rangle_0 = \frac{1}{\sqrt{2}} \left( |g_1\rangle|e_2\rangle + |e_1\rangle|g_2\rangle \right).\] (22)

In this case, the initial conditions for the Bloch vector components are $u_0 = 1$, $v_0 = w_0 = 0$.

4.1. The case $w_0 = 1$, $u_0 = v_0 = 0$

In this case, the system is initially prepared in a product (unentangled) state, and we readily find that the variation of the concurrence with position of the atoms in the standing wave is given by

\[C(r_{12}) = e^{-\Gamma \tau} \left\{ \left( \frac{\sin d}{d} \right)^2 \tau^2 + \left( \frac{\sin \frac{1}{2}d}{\frac{1}{2}d} \right)^4 \tau^4 \sin^4 k r_{12} \right\}^{\frac{1}{2}} |\cos k r_{12}|,\] (23)

where $d = (1 + \cos^2 k r_{12}) \tau / 2$ and we have introduced a scaled time variable

\[\tau = \frac{2g_0^2}{\Delta}.\] (24)
and the dimensionless spontaneous emission damping rate

\[ \Gamma = \frac{\Delta}{2g_0^2} \gamma. \]  

The concurrence \( C(r_{12}) \) depends on time, the relative distance between the atoms and decays exponentially in time with the rate \( \Gamma \). It exhibits an interesting modulation of the amplitude of the harmonic oscillation. Somewhat surprisingly our results show that the variation of the concurrence with \( r_{12} \) does not reveal the cosine form of the cavity mode function, that the concurrence is not a simple cosine function of \( r_{12} \). It is given by the product of two terms, one the absolute value of the cavity mode function \( |\cos kr_{12}| \) and the other the time- and position-dependent diffraction structure. In other words, the concurrence is in the form of position and time dependent diffraction pattern. Figure 1 shows \( C(r_{12}) \) as a function of \( r_{12}/\lambda \) for different times \( \tau \) corresponding to the evolution intervals at which the entanglement is maximal for the idealized case of \( g_1 = g_2 \). Notice that the concurrence is symmetric about an antinode of the cavity mode (\( r_{12}/\lambda = 0.25 \)). For a short time the entanglement is seen to occur over a wide range of positions centered about the antinodes of the cavity mode. The concurrence is a bell-shaped function of position without any oscillation. As time progresses, oscillations appear and consequently the region of \( r_{12} \) where the optimum entanglement occurs, becomes narrower. The evolution of \( C(r_{12}) \) tends to become increasingly oscillatory with \( r_{12} \) as time increases, and the optimum entanglement occurs in a still more restricted range of \( r_{12} \). As a result, the atom-atom entanglement oscillates with position faster than the cosine function, and the oscillations are more dramatic for larger times. Only at very early times \( (\tau \ll 1) \), the oscillations are not modulated by the diffraction pattern and the concurrence reduces to \( |\cos kr_{12}| \), but for longer times, \( C(r_{12}) \) may vary slower or faster than the cosine functions. Within the diffraction structure itself, the magnitude of the concurrence exhibits a succession of modes and of antinodes. As a consequence, the entanglement may be completely quenched even for locations of the atom close to an antinode of the cavity mode, and alternatively may achieve its optimum value even for locations of the atom close to a node of the cavity mode.

4.2. The case \( u_0 = 1, v_0 = w_0 = 0 \)

Assuming that initially the system is prepared in the maximally entangled state (22), we have \( u_0 = 1, v_0 = w_0 = 0 \), and the general solution (19) leads to the concurrence that can be written as

\[ C(r_{12}) = e^{-\Gamma \tau} \left\{ 1 - \frac{\sin^4 kr_{12}}{4} \right\} \left\{ 1 - \frac{\sin^4 kr_{12}}{4} \right\}^{\frac{1}{2}}, \]  

(26)
In this case the concurrence exhibits only the second order diffraction pattern. This may result in a weaker dependence of the concurrence on the position of the atoms than that observed with the initial unentangled state. Figure 2 shows how the concurrence evolves with the position of the atom 2 inside the cavity mode and time $\tau$, for the absence of spontaneous emission ($\gamma = 0$). Notice that the initial entanglement is reduced for positions of the atom different from the antinode of the cavity mode. That is, at some points outside the antinode the concurrence may become zero indicating the complete reduction of the initial entanglement.

![Figure 2](image)

**Figure 2.** Variation of the concurrence $C(r_{12})$ with the position of the atom 2 located between two successive nodes of the cavity field for $\gamma = 0$ and different times (a) $\tau = \pi/2$, (b) $\tau = 7\pi/2$, (c) $\tau = 31\pi/2$. The system was initially in the state $u_0 = 1, v_0 = w_0 = 0$.

The strong modulation of the concurrence in time, seen in Figures 1 and 2 can be understood as a consequence of the uncertainty relation between the evolution time and energy [16]. For the increasing time the uncertainty of the energy decreases which means that the energy (photon) becomes more localized. The increase in the localization of the energy results in a degradation of the entanglement. In other words, with increasing time, one can in principle obtain more information about the localization of the atoms inside the cavity mode.

5. Eigenstates of the system

We have seen in section 4.1 that the initially unentangled system may become entangled during the evolution, and the entanglement depends on the relative position of the atoms inside the cavity mode. Here, we will determine the explicit analytical form of the single photon entangled state of the system.

We start from the density matrix of the system, which in the basis of the product states (16) has the form

$$\rho_A(t) = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & 0 \\ 0 & \rho_{22}(t) & \rho_{23}(t) & 0 \\ 0 & \rho_{32}(t) & \rho_{33}(t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(27)

where we have assumed that only one photon is present in the system, i.e. $\rho_{44}(t) = 0$ for all times.

We see that the density matrix (27) is not diagonal in the basis of the product states (16) due to the presence of the coherences $\rho_{23}(t)$ and $\rho_{32}(t)$. Thus, the product states (16) are not the eigenstates of the system. We will find the eigenstates of the system by diagonalizing the matrix. We first find the eigenvalues of the matrix, from which we next determine eigenvectors, i.e. eigenstates of the system. Proceeding in the usual way, we calculate the determinant of the matrix $\rho_A(t)$, and find a characteristic (polynomial) equation

$$\lambda (\lambda - \rho_{11}(t)) \left[ \lambda^2 - (\rho_{22}(t) + \rho_{33}(t))\lambda + \rho_{22}(t)\rho_{33}(t) - |\rho_{23}(t)|^2 \right] = 0.$$

(28)
This is a simple fourth order polynomial whose the roots, corresponding to the eigenvalues of the matrix (27), are

\[
\begin{align*}
\lambda_1(t) &= \rho_{11}(t), \\
\lambda_2(t) &= \frac{1}{2} \left[ (\rho_{22}(t) + \rho_{33}(t)) - \sqrt{(\rho_{22}(t) - \rho_{33}(t))^2 + 4|\rho_{23}(t)|^2} \right], \\
\lambda_3(t) &= \frac{1}{2} \left[ (\rho_{22}(t) + \rho_{33}(t)) + \sqrt{(\rho_{22}(t) - \rho_{33}(t))^2 + 4|\rho_{23}(t)|^2} \right], \\
\lambda_4(t) &= 0. \\
\end{align*}
\] (29)

Since the following relations are satisfied for all times

\[
\rho_{22}(t) + \rho_{33}(t) = 1 - \rho_{11}(t), \quad (\rho_{22}(t) - \rho_{33}(t))^2 = w^2(t), \quad 4|\rho_{23}(t)|^2 = v^2(t) + u^2(t),
\] (30)

and

\[
u^2(t) + v^2(t) + w^2(t) = e^{-2\gamma t},
\] (31)

we can write the roots (29) in the form

\[
\begin{align*}
\lambda_1(t) &= \rho_{11}(t), \\
\lambda_2(t) &= \frac{1}{2} \left( 1 - \rho_{11}(t) - e^{-\gamma t} \right), \\
\lambda_3(t) &= \frac{1}{2} \left( 1 - \rho_{11}(t) + e^{-\gamma t} \right), \\
\lambda_4(t) &= 0.
\end{align*}
\] (32)

It is seen from Eq. (32) that in the presence of spontaneous emission ($\gamma \neq 0$), three of the eigenvalues are non-zero, indicating that the system is in a mixed state. However, in the absence of spontaneous emission ($\gamma = 0$) the population of the ground state $\rho_{11}(t) = 0$ for all times. Then, $\lambda_3(t) = 1$ and the remaining eigenvalues are zero indicating that in this case the system is in a pure state for all times.

Following the standard procedure of diagonalisation of matrices, we can easily find the eigenstates of the matrix (27) corresponding to the eigenvalues (32). The normalised eigenstates are of the form

\[
\begin{align*}
|\Psi_1(t)\rangle &= \rho_{11}(t)|1\rangle, \\
|\Psi_2(t)\rangle &= \frac{\rho_{23}(t)}{\sqrt{|\rho_{23}(t)|^2 + (\lambda_2(t) - \rho_{22}(t))^2}}|2\rangle + \frac{\lambda_2(t) - \rho_{22}(t)}{\sqrt{|\rho_{23}(t)|^2 + (\lambda_2(t) - \rho_{22}(t))^2}}|3\rangle, \\
|\Psi_3(t)\rangle &= \frac{\rho_{23}(t)}{\sqrt{|\rho_{23}(t)|^2 + (\lambda_3(t) - \rho_{22}(t))^2}}|2\rangle + \frac{\lambda_3(t) - \rho_{22}(t)}{\sqrt{|\rho_{23}(t)|^2 + (\lambda_3(t) - \rho_{22}(t))^2}}|3\rangle, \\
|\Psi_4(t)\rangle &= \lambda_4(t)|4\rangle.
\end{align*}
\] (33)

It is evident from Eq. (33) that the eigenstates $|\Psi_2\rangle$ and $|\Psi_3\rangle$ are superposition (entangled) states that are produced by the coherence $\rho_{23}(t)$. Thus, a non-zero value of the coherence $\rho_{23}(t)$ leads to entangled states of the atoms. In other words, whenever $\rho_{23}(t) \neq 0$, the atoms are entangled.

In Fig. 3 we plot $|\rho_{23}(t)|$ as a function of time for $\gamma = 0$ and different relative positions $r_{12}$ of the atoms inside the cavity mode. The atom 1 is located at an antinode of the cavity field and the atom 2 is located near a successive antinode of the field. The figure demonstrates that the coherence develops in time and remains different from zero for all times except some discrete times at which the population is completely in the states $|2\rangle$ or $|3\rangle$. Interestingly, when the atom 2 is not located precisely at an antinode of the cavity field, the coherence remains non-zero for longer times indicating that in this case the atoms
can be entangled on the time scale much longer than that predicted for the atoms precisely located at the antinodes of the cavity field. This surprising result can be explained in terms of the degree of localization or non-locality of the energy induced in the field by the interacting atoms [11, 17]. One can see from Eq. (17) that for equal coupling strengths the energy levels of the atoms are equally shifted ($\delta_{12} = 0$) due to the interaction with the cavity mode. In this case the induced energy by the first atom oscillates with frequency $\alpha$ such that at the particular times is fully absorbed by the second atom. Since at these times the energy is well localized in space as being completely absorbed by the localised atoms, the entanglement, which results from a non-locality of the energy, is zero. The situation changes when $\delta_{12} \neq 0$. According to Eq. (17), in this case the energy levels of the atoms are unequally shifted. Due to the frequency mismatch, the energy induced by the atom 1 is not fully absorbed by the atom 2, leading to a partial spatial delocalization of the photon at discrete times corresponding to the time at which the atom 2 is excited. Consequently, at these times an entanglement is observed.

6. Conclusions
We have analysed an experimental situation of entanglement creation between two distant atoms coupled to a single-mode optical cavity. In particular, we have investigated the effect of a variation of the coupling constant $g(\vec{r})$ with the location of the atoms in a standing-wave cavity mode on the entanglement between the atoms. We have found that the variation of the coupling constants with the position of the atoms leads to a time-dependent diffraction pattern of the concurrence. The diffraction formula also shows explicitly the effect of the initial conditions and the trend of the modification of the entanglement with the localization of the atoms when the observation time increases. For a short time the entanglement is seen to occur over a wide range of positions centered about the antinodes of the cavity mode. As time progresses, oscillations appear with an increasing frequency and consequently the spatial region where the optimum entanglement occurs, becomes narrower. This effect has been explained in terms of the quantum property of complementarity, which is manifested as a tradeoff between the knowledge of energy of the exchanged photon versus the evolution time of the system. We have also identified the entangled states of the system and interpreted the entanglement properties in terms of the coherence between the atoms and localization of the energy.

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