STABILITY OF NON-CLASSICAL THERMOELASTICITY MIXTURE PROBLEMS

MARGARETH S. ALVES* AND RODRIGO N. MONTEIRO

Department of Mathematics, Federal University of Viçosa
Viçosa, MG, 36570-000, Brazil

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Abstract. We discuss the stability problem for binary mixtures systems coupled with heat equations. The present manuscript covers the non-classical thermoelastic theories of Coleman-Gurtin and Gurtin-Pipkin - both theories overcome the property of infinite propagation speed (Fourier’s law property). We first state the well-posedness and our main result is related to long-time behavior. More precisely, we show, under suitable hypotheses on the physical parameters, that the corresponding solution is stabilized to zero with exponential or rational rates.

1. Introduction. Mathematical models for mixtures of elastic solids/fluids have attracted considerable attention in the past literature. We begin by mentioning the work of Truesdell in [25] that provides the balance/conservation equations for a continuum theory of mixtures -the mixture is considered as a distribution of different continuous media in the same physical space. Adkins in [1] proposed purely mechanical theory for mixtures of fluids and mixture of a fluid and an elastic solid. For an inviscid miscible mixture of fluids without chemical reactions or diffusion, we mention reference [14]. Concerning with immiscible mixtures, we cite the work of Bedford and Drumheller [7]. In connection with the present manuscript, we recall [17] where the model for mixtures of thermoviscoelastic bodies - Kelvin-Voigt material - explicitly appears. For the meaningfulness and applicability of the mixtures problems, Adkins and Craine [2] presented the constitutive theories and some applications for binary mixtures of fluids and a binary mixture of elastic solids. We also recall references [12, 18] for applications to engineering and biological problems.

1.1. Problem formulation. In this manuscript, following the theories of Bedford and Stern [8] - also Ieşan and Quintanilla [17] - we consider a portion \((0, t) \subset \mathbb{R}\) occupied by a mixture of two interacting continua with particles occupying the same position at the time \(t\). Let us consider \(u = u(x, t)\) and \(w = w(y, t)\) being the...
particles position at time \( t \) with \( x, y \in (0, \ell) \), then the mixture behavior is given by the kinematic variables 
\[
u = u(x, t) \quad \text{and} \quad w = w(x, t), \quad x \in (0, \ell).
\]

In what follows, by \( \rho_1 \) and \( \rho_2 \) we denote the mass densities of the two constituents \((u, w)\) at time \( t \). \( M \) and \( S \) are the partial stresses associated with the constituent \( u \) and \( w \), respectively, and \( P \) is the internal diffusive force. Concerning the dissipative effect, we use the symbols \((\vartheta, \vartheta, q, T_0)\) to denote the temperature deviation (difference to a fixed constant reference temperature), the entropy density, the heat flux vector and the absolute temperature in the reference configuration, respectively. Under this notation, the general form for the constitutive equations are as follows

- motion equations:
  \[
  \rho_1 u_{tt} - M_x + P = 0 \quad \text{and} \quad \rho_2 w_{tt} - S_x - P = 0. \tag{1.1}
  \]

- energy equation:
  \[
  \rho T_0 \vartheta_t - q_x = 0 \quad \text{where} \quad \rho = \rho_1 + \rho_2. \tag{1.2}
  \]

- constitutive laws:
  \[
  \begin{align*}
  M &= (a_{11}u + a_{12}w)_x + \beta_1 \vartheta, \\
  S &= (a_{12}u + a_{22}w)_x + \beta_2 \vartheta, \\
  P &= \alpha(u - w), \\
  \vartheta &= \rho_3 \rho^{-1} \vartheta - \rho^{-1}(\beta_1 u_x + \beta_2 w_x), \\
  q &= \sigma \vartheta_x + (1 - \sigma) \int_0^\infty k(s) \vartheta_x(t - s)ds,
  \end{align*}
  \tag{1.3}
  \]

with \( a_{ij}, \alpha, \beta_i, \rho_3 \ (i, j = 1, 2) \) denoting the constitutive coefficients that define the mixture, the parameter \( \sigma \in [0, 1] \) is responsible for characterization of the heat flux law and the function \( k \) is the memory kernel, whose properties are given in Section 2.

Under these notations, the constitutive equations (1.3) combined with (1.1) and (1.2), yield the following system of differential partial equations defined in \((0, \ell) \times (0, \infty)\)

\[
\begin{align*}
  \rho_1 u_{tt} - (a_{11}u + a_{12}w)_{xx} + \alpha(u - w) - \beta_1 \vartheta_x &= 0, \tag{1.4} \\
  \rho_2 w_{tt} - (a_{12}u + a_{22}w)_{xx} - \alpha(u - w) - \beta_2 \vartheta_x &= 0, \tag{1.5} \\
  \rho_3 \vartheta_t - \sigma T_0^{-1} \vartheta_{xx} - (1 - \sigma) T_0^{-1} \int_0^\infty k(s) \vartheta_{xx}(t - s)ds - (\beta_1 u_{xt} + \beta_2 w_{xt}) &= 0. \tag{1.6}
\end{align*}
\]

We also complete the PDE system (1.4)-(1.6) with boundary conditions

\[
\begin{align*}
  u(0, t) = u(\ell, t) = w(0, t) = w(\ell, t) = \vartheta_x(0, t) = \vartheta_x(\ell, t) &= 0, \quad \forall \ t \in (0, \infty), \tag{1.7} \\
  (u, w, u_t, w_t, \vartheta)|_{t=0} &= (u_0, u_1, w_0, w_1, \vartheta_0) \quad \text{and} \quad \theta(-s)|_{s>0} = \bar{\theta}_0(s) \quad \text{on} \ (0, \ell). \tag{1.8}
\end{align*}
\]

1.2. Previous results and present contribution. The mathematical studies of mixtures problems, from the point of view of the well-posedness and stability, have received considerable attention. Indeed, starting with [20], the asymptotic behavior of solutions to mixtures of thermoelastic solids was established. The energy decay rate for solutions to viscoelastic mixtures was studied in [24]. More recently, in [6], Alves et al. study the exponential decay rates for solution to thermoelastic mixture with second sound. Concerning existence of analytic solutions to viscoelastic mixtures of solids, we mention [13]. Next, for the sake of brevity, we shall concentrate
only on references for thermoelastic mixtures problems in a one-dimensional configuration. As already mentioned, the role of the parameter $\sigma \in [0, 1]$ is to characterize the heat flux law. The case when $\sigma$ assumes values in $(0, 1)$, the heat propagation law is corresponding with Coleman-Gurtin. Limit cases $\sigma = 0$ and $\sigma = 1$, the heat flux law coincides with the fully parabolic Fourier law and the fully hyperbolic Gurtin-Pipkin law, respectively. In this direction, we quote the works of Alves et al. [3, 4, 5] and Muñoz Rivera et al. [21]. Reference [3] studies the problem (1.4)-(1.6) with classical theory of thermoelasticity ($\sigma = 1$) and under suitable assumptions on the constitutive coefficients ($\alpha, a_{ij}, \beta_i, \rho_1$), the authors establish both exponential and polynomial decay rates for the corresponding solution. In [4, 5], the authors analyzed the problem (1.4)-(1.6) for Kelvin-Voigt materials with $\sigma = 1$. The first establishes decay rates for the solution and the second establishes the analyticity property of the corresponding semigroup. Both results also depend on the constitutive coefficients. In [21], the authors complete the analysis in [3] showing optimal rational decay rates for the problem with a particular boundary condition.

In this article, we complete the analysis concerning to the parameter $\sigma$. Here, we assume $\sigma \in (0, 1)$. Following the same direction as the quoted results, we study the effects of ($\alpha, a_{ij}, \beta_i, \rho_1$) on the decay rates. As an starting point, we begin introducing the stability numbers associated with (1.4)-(1.6). Let $\kappa_i$ ($i = 1, 2$) be as follows

$$\kappa_1 = a_{11}\beta_2 - a_{12}\beta_1, \quad \kappa_2 = a_{12}\beta_2 - a_{22}\beta_1.$$  

Define

$$\chi_0 = \rho_2\beta_1\kappa_1 + \rho_1\beta_2\kappa_2$$  

$$\chi_1 = \alpha(\beta_1 + \beta_2)(\rho_1\beta_2 - \rho_2\beta_1).$$  

(1.9)  

(1.10)

The critical case $\sigma = 0$ needs particular conditions. In this case, the stability numbers are defined by

$$\chi_2 = \rho_3 - \gamma_1^{-1}k(0) \quad \text{and} \quad \chi_3 = \beta_2^2\chi_2^{-1} + a_{12},$$  

(1.11)

with $\gamma_1 = (\beta_2a_{11} - \beta_1a_{12})(\rho_1\beta_2)^{-1}$.

The main results here depend only on $\chi_i$ ($i = 0, 1, 2, 3$) and (i) we first show that assumption $\chi_0 \neq 0$ and $\chi_1 \neq \chi_0(\frac{\pi n}{T})^2$ ($\forall n \in \mathbb{N}$) implies that associated semigroup is stable with exponential decay rate. The same result is valid if $\chi_0 = (\frac{\pi n}{T})^2$ for a fixed $\chi_0 \in \mathbb{N}$, $\alpha(\beta_1 + \beta_2)(\kappa_1 + \kappa_2)\chi_0^{-1} \leq 0$ and $\beta_1\beta_2 \neq 0$; (ii) we also show the lack of exponential stabilization under the following assumptions: if $\sigma \in (0, 1)$, we assume $\chi_0 = 0$ and $\chi_1 \neq 0$. If $\sigma = 0$, we additionally assume $\chi_2 \neq 0$ and $\chi_3 \neq 0$: (iii) in case of $\chi_0 = 0$ and $\chi_1 \neq 0$ rational decay rates are established. More precisely, the semigroup decays to zero as $t^{-\frac{2}{4}}$.

**Orientation:** We proceed as follows. In Section 2, we introduce the assumptions on the memory kernel ($k$) and on the physical parameters ($\alpha, a_{ij}, \beta_i, \rho_1$). The same section contains the well-posedness result for the model. In Section 3, we present the main results and also the proofs.

2. Assumptions and Well-posedness. In this section, we present the basic assumptions and we also post the well-posedness result.

**Memory kernel assumptions.** We first set the following notation for the memory kernel

$$\mu(s) = -(1 - \sigma)k'(s).$$
The assumptions here are that \( k(\infty) = 0 \), the function \( \mu(\cdot) \) is required to verify
\[
\mu \in C^1(0, \infty), \mu(s) \geq 0, \mu'(s) \leq 0 \quad \text{for every } s \in (0, \infty) \text{ and with finite total mass}
\]
\[
\int_0^\infty \mu(s) ds, \quad \int_0^\infty \mu(s) s ds > 0.
\]
We further assume that there exists a positive constant \( \nu \) such that
\[
\mu'(s) \leq -\nu \mu(s), \forall s \in (0, \infty) \text{ and } \mu(0) < \infty.
\]

**Remark 1.** Notice that assumption (2.2) implies the exponential decay of \( \mu(\cdot) \). Moreover, the assumptions on \( \mu(\cdot) \) allow us to conclude that the heat memory kernel \( k(\cdot) \) is a non-increasing function and \( k \in C^2(0, \infty) \).

**Constitutive coefficients assumptions.** We assume that \( \alpha, \rho_i \ (i = 1, 2, 3) \) are positive constants, \( \beta_i \ (i = 1, 2) \) are real numbers which do not vanish simultaneously and \( A = [a_{ij}]_{2 \times 2} \) is a positive definite symmetric matrix. The last, in particular, implies that \( a_{11}a_{22} - a_{12}^2 > 0 \).

**Semigroup Formulation.** In order to transform (1.4)-(1.6) into an equivalent autonomous problem, we introduce a new variable \( \eta = \eta^l(x, s) \) defined by
\[
\eta^l(x, s) = \int_0^t \theta(x, t - \tau) d\tau = \int_{t-s}^t \theta(x, \tau) d\tau, \text{ for } (x, t, s) \in [0, \ell] \times [0, \infty) \times [0, \infty).
\]
The variable \( \eta^l(x, s) \) satisfies
\[
\eta_t + \eta_s - \theta = 0 \quad \text{in } (0, \ell), \text{ for } (t, s) \in (0, \infty) \times (0, \infty)
\]
with
\[
\eta^l(x, 0) = 0 \text{ in } (0, \ell), \text{ for } t \in (0, \infty)
\]
and
\[
\eta^0(x, s) = \int_0^s \tilde{\theta}_0(\tau) d\tau \equiv \eta_0(x, s) \text{ in } (0, \ell), \text{ for } s \in (0, \infty).
\]
Taking these into account and assuming (without loss of generality) \( T_0 = 1 \), we find the following representation for (1.4)-(1.8)
\[
\begin{align*}
\rho_1 u_{tt} - (a_{11} u + a_{12} w)_{xx} + \alpha(u - w) - \beta_1 \theta_x &= 0, \quad (2.3) \\
\rho_2 w_{tt} - (a_{12} u + a_{22} w)_{xx} - \alpha(u - w) - \beta_2 \theta_x &= 0, \quad (2.4) \\
\rho_3 \theta_t - \sigma \theta_{xx} - \int_0^\infty \mu(s) \eta^l_{xx}(s) ds - (\beta_1 u_{xt} + \beta_2 w_{xt}) &= 0, \quad (2.5) \\
\eta_t + \eta_s - \theta &= 0, \quad (2.6)
\end{align*}
\]
with boundary conditions
\[
u(0, t) = u(0, t) = w(0, t) = w(\ell, t) = \theta_x(0, t) = \theta_x(\ell, t) = \eta^l_x(0, s) = \eta^l_x(\ell, s) = 0, \quad (2.7)
\]
and initial data
\[
(u, u_t, w, w_t, \theta, \eta^l) \big|_{t=0} = (u_0, u_1, w_0, w_1, \theta_0, \eta_0) \quad \text{on } (0, \ell). \quad (2.8)
\]
The next step is to state the Cauchy problem associated with (2.3)-(2.8). To this end, we introduce the space of the well-posedness \( \mathcal{H} \) as follows
\[
\mathcal{H} = H^1_0(0, \ell) \times L^2(0, \ell) \times H^1_0(0, \ell) \times L^2(0, \ell) \times L^2_*(0, \ell) \times \mathcal{M},
\]
where
\[
L^2_*(0, \ell) = \left\{ \theta \in L^2(0, \ell) \mid \int_0^\ell \theta(x) \, dx = 0 \right\}
\]
For \( Z \) and defined as
\[
\eta(s) = \int_0^s \mu(t) dt
\]
The space \( \mathcal{M} \) carries the inner product and corresponding norm given by
\[
\langle \eta^1, \eta^2 \rangle_{\mathcal{M}} = \int_0^1 \int_0^\infty \mu(s) \eta^1_x(s, x) \overline{\eta^2_x(s, x)} ds dx
\]
and
\[
\| \eta \|^2_{\mathcal{M}} = \int_0^1 \int_0^\infty \mu(s) |\eta_x(s, x)|^2 dx ds.
\]
We also introduce the operator \( B : D(B) \subset \mathcal{M} \to \mathcal{M} \) with domain
\[
D(B) = \left\{ \eta \in \mathcal{M} \mid B\eta \in \mathcal{M}, \eta(0) = 0 \right\}
\]
and defined as
\[
B\eta = -\eta_x,
\]
where \( \eta_x \) stands for the distributional derivative of \( \eta \) with respect to the internal variable \( s \). Following reference [15], one can show that operator \( B \) is the infinitesimal generator of a \( C_0 \)-semigroup and
\[
\text{Re} \left\{ \langle B\eta, \eta \rangle_{\mathcal{M}} \right\} = \frac{1}{2} \int_0^1 \int_0^\infty \mu'(s) \eta_x(s, x)^2 dx ds \leq 0, \forall \eta \in D(B).
\]
For \( Z(t) = (u, v, w, \omega, \theta, \eta) \), we consider \( \mathcal{H} \) endowed with energy norm
\[
\|Z\|_{\mathcal{H}}^2 = \rho_1 \|v\|_{L^2}^2 + \rho_2 \|\omega\|_{L^2}^2 + a_{11} \|u_x\|_{L^2}^2 + 2a_{12} \text{Re} \left\{ \langle u_x, w_x \rangle_{L^2} \right\}
\]
\[
+ a_{22} \|w_x\|_{L^2}^2 + \alpha \|u - w\|_{L^2}^2 + \rho_3 \|\theta\|_{L^2}^2 + \|\eta\|^2_{\mathcal{M}},
\]
where \( \| \cdot \|_{L^2} \) and \( \langle \cdot, \cdot \rangle_{L^2} \) denote the usual norm and inner product \( L^2(0, \ell) \) space, respectively.

Now, let us consider the variable \( Z(t) = (u, v, w, \omega, \theta, \eta)^\top \) with \( v = u_t \) and \( \omega = w_t \). Then, problem \( (2.3)-(2.8) \) can be expressed as a first-order equation
\[
\frac{d}{dt} Z = \mathcal{A}_\sigma Z \quad \text{and} \quad Z_0 = Z(0) = (u_0, u_1, w_0, w_1, \theta_0, \eta_0),
\]
were \( \mathcal{A}_\sigma : D(\mathcal{A}_\sigma) \subset \mathcal{H} \to \mathcal{H} \) denotes the differential operator
\[
\mathcal{A}_\sigma \begin{bmatrix} u \\
 v \\
 w \\
 \omega \\
 \theta \\
 \eta \end{bmatrix} = \begin{bmatrix} u_x & v \\
 v & \rho_1^{-1}(a_{11}u + a_{12}w)_{xx} - \rho_1^{-1}\alpha(u - w) + \rho_1^{-1}\beta_1\theta \\
 w_x & \omega \\
 \theta_x & \rho_2^{-1}(a_{12}u + a_{22}w)_{xx} + \rho_2^{-1}\alpha(u - w) + \rho_2^{-1}\beta_2\theta \\
 \eta_x & \rho_3^{-1}\sigma\theta + \rho_3^{-1}\int_0^\infty \mu(s)\eta_x(s) ds + \rho_3^{-1}(\beta_1v_x + \beta_2\omega_x) \\
 \eta & \theta \end{bmatrix}
\]
with domain
\[
D(\mathcal{A}_\sigma) = \left\{ Z \in \mathcal{H} \mid u, w \in H^2(0, \ell), \quad v, \omega \in H^1_0(0, \ell), \quad \sigma\theta + \int_0^\infty \mu(s)\eta_x(s) ds \in H^1_0(0, \ell), \quad \theta \in H^1(0, \ell) \cap L^2((0, \ell), \eta \in D(\mathcal{B})) \right\}.
\]
We end this section with the well-posedness result.
Theorem 2.1. The operator $\mathcal{A}_\sigma$ ($\sigma \in [0,1]$) generates a $C_0$-semigroup of contractions $\{S_\sigma(t)\}_{t \geq 0}$ on the finite energy space $\mathcal{H}$. Thus, for $Z_0 \in \mathcal{H}$ there exists a unique solution satisfying $Z = S_\sigma(t)Z_0 \in C([0,T];\mathcal{H})$. If $Z_0 \in D(\mathcal{A}_\sigma)$ then $Z \in C^1([0,T];\mathcal{H}) \cap C([0,T];D(\mathcal{A}_\sigma))$.

Remark 2. In the present manuscript, we are facing a combination of mixture problems with classical thermoelastic theory and memory in past history. As already mentioned the reference [3] studies the well-posedness of solutions when $\sigma = 1$. The well-posedness for problems with memory in the past history framework has been treated in [10]. Therefore, Theorem 2.1 can be proved by adapting and applying essentially the same technique as the cited works. Here, we limit to give a brief sketch.

Proof of Theorem 2.1. We first note that operator $\mathcal{A}_\sigma$ -defined in (2.10)- is dissipative. In fact, from dissipativity property (2.9), one obtains

$$\text{Re} \{(\mathcal{A}_\sigma Z, Z)_{\mathcal{H}}\} = -\sigma \| \theta_x \|^2_{L^2} + \text{Re} \{(\mathcal{B} \eta, \eta)_{\mathcal{H}}\} \leq 0, \forall Z \in D(\mathcal{A}_\sigma).$$

(2.11)

The next step is to show that $0 \in \rho(\mathcal{A}_\sigma)$, where $\rho(\mathcal{A}_\sigma)$ denotes the resolvent set of operator $\mathcal{A}_\sigma$. More precisely, we have to establish the well-posedness -in $D(\mathcal{A}_\sigma)$- of $-\mathcal{A}_\sigma Z = F$ with $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$. To this end, let us consider the definition (2.10) which gives (for $\sigma \in (0,1)$)

$$-v = f_1,$$

(2.12)

$$-(a_{11}u + a_{12}w)_{xx} + \alpha (u - w) - \beta_1 \theta_x = \rho_1 f_2,$$

(2.13)

$$-\omega = f_3,$$

(2.14)

$$-(a_{12}u + a_{22}w)_{xx} - \alpha (u - w) - \beta_2 \theta_x = \rho_2 f_4,$$

(2.15)

$$-\sigma \theta_{xx} - \int_0^\infty \mu(s)\eta_{xx}(s)ds - (\beta_1 v_x + \beta_2 \omega_x) = \rho_3 f_5,$$

(2.16)

$$\eta_x - \theta = f_6,$$

(2.17)

Identities (2.12) and (2.14) give $v, \omega \in H_0^1(0,\ell)$. From (2.17), we obtain

$$\eta = s \theta + \int_0^s f_6(\tau)d\tau.$$  

(2.18)

This promptly implies $\eta(x,0) = 0$ for all $x \in (0,\ell)$. Via equations (2.16) and (2.18), we find

$$-[\sigma + \int_0^\infty \mu(s)sds] \theta_{xx} = (\beta_1 v_x + \beta_2 \omega_x) + \rho_3 f_5 + \int_0^\infty \mu(s) f_{6,xx}(\tau)d\tau ds \in H^{-1}(0,\ell).$$

Thus by Lax-Milgram Theorem, we conclude that $\theta \in H^1(0,\ell) \cap L_2^2(0,\ell)$. This implies that $\eta, \eta_x \in \mathcal{H}$. From (2.16), we find $\sigma \theta_x + \int_0^\infty \mu(s)\eta_x(s)ds \in H_0^1(0,\ell)$. We finally back to (2.13) and (2.15) to conclude, together with Lax-Milgram Theorem and elliptic regularity, that $u, w \in H^2(0,\ell) \cap H_0^1(0,\ell)$. Moreover, we can show the existence of a positive constant $C$, such that, $\|Z\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}$. These prove that $0 \in \rho(\mathcal{A}_\sigma)$.

The proof of Theorem 2.1 is completed via Lumer-Phillips Theorem (See [22]). The proof for $\sigma = 0$ is completely similar. \hfill \Box

3. Statement of main results. The first main result is the exponential stability. To state the corresponding result, we recall the stability characterization due to Pruss [23] (See Theorem 1.3.2 in [19]). The last establishes the exponential stability
for a $C_0$-semigroup of contractions $\{S(t)\}_{t \geq 0}$, with infinitesimal generator $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$, if and only if

$$i\mathbb{R} \equiv \{i\lambda \mid \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A})$$

(3.1)

and

$$\limsup_{|\lambda| \to \infty} \| (i\lambda I - \mathcal{A})^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty,$$

(3.2)

where $\| \cdot \|_{\mathcal{L}(\mathcal{H})}$ denotes the norm in the space of continuous linear functions in $\mathcal{H}$.

Our main result reads as follows:

**Theorem 3.1.** Let the memory kernel and coefficients assumptions having validity.

(i) The semigroup $\{S_\sigma(t)\}_{t \geq 0}$ ($\sigma \in [0,1]$) associated with (2.3)-(2.8) is exponentially stable if (i.1) $\chi_0 \neq 0$ and $\chi_1 \neq \chi_0(n/R)^2$ ($\forall n \in \mathbb{N}$) or (i.2) $\frac{\Delta_1}{\chi_0} = \left(\frac{n_0R}{\ell}\right)^2$ for a fixed $n_0 \in \mathbb{N}$, $\alpha(\beta_1+\beta_2)(\kappa_1+\kappa_2)\chi_0 \leq 0$ and $\beta_1 \beta_2 \neq 0$.

(ii) In case of $\chi_0 = 0$ and $\chi_1 \neq 0$, semigroup $\{S_\sigma(t)\}_{t \geq 0}$, for $\sigma \in (0,1)$, is not exponentially stable. With additional assumption $\chi_2 \neq 0$ and $\chi_3 \neq 0$ the semigroup $\{S_0(t)\}_{t \geq 0}$ is not exponentially stable.

In case of lack of exponential stability, we obtain rational decay rates for the corresponding semigroup. Here, we make use of recent results obtained by Borichev and Tomilov (See Theorem 2.4 in [9]). In a short form, the result establishes the following: Let $\{S(t)\}_{t \geq 0}$ a bounded semigroup on a Hilbert space $\mathcal{H}$, with generator $\mathcal{A}$, such that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then, for a fixed parameter $r > 0$ the following inequalities are equivalent

$$\| (i\lambda I - \mathcal{A})^{-1} \|_{\mathcal{L}(\mathcal{H})} \leq C|\lambda|^r, \ \forall \lambda \in \mathbb{R},$$

(3.3)

and

$$\| S(t)\mathcal{A}^{-1} \|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^r},$$

(3.4)

for some positive constant $C$.

**Theorem 3.2.** Assume that $\chi_0 = 0$ and $\chi_1 \neq 0$. Then semigroup $\{S_\sigma(t)\}_{t \geq 0}$ ($\sigma \in [0,1]$) satisfies the following rational decay rate

$$\| S_\sigma(t)Z_0 \|_{\mathcal{H}} \leq \frac{C}{t^r} \| Z_0 \|_{D(\mathcal{A}_\sigma)}, \ \forall Z_0 \in D(\mathcal{A}_\sigma),$$

for some positive constant $C$ and time $t > 0$ large enough.

3.1. **Proof of the Main Results.** In the following, to further simplify notations, we will omit dependence on “$\sigma$” if no confusion arises. In this case, the symbol $\mathcal{A}$ will play the role of $\mathcal{A}_\sigma$. We first show the condition (3.1).

**Lemma 3.3.** $i\mathbb{R} \cap \sigma_p(\mathcal{A}) \neq \emptyset$ if one of the following assumptions is assumed to be true

(i) $\chi_1 = 0$ and $\chi_0 = 0$;

(ii) $\frac{\Delta_1}{\chi_0} = \left(\frac{n_0R}{\ell}\right)^2$ for some $n_0 \in \mathbb{N}$, $\alpha(\beta_1+\beta_2)(\kappa_1+\kappa_2)\chi_0 > 0$ and $\beta_1 \beta_2 \neq 0$;

(iii) $\frac{\Delta_1}{\chi_0} = \left(\frac{n_0R}{\ell}\right)^2$ for some $n_0 \in \mathbb{N}$ and $\beta_2 \beta_1 = 0$.

**Proof.** Next, we exhibit examples of purely imaginary eigenvalues.

If assumption (i) holds true and $\beta_1 + \beta_2 = 0$, we find from (1.9) and (1.10)

$$\frac{a_{11} + a_{12}}{\rho_1} = \frac{a_{22} + a_{12}}{\rho_2}.$$
In this case the complex numbers
\[ i\mathbb{R} \ni \lambda_n = i \left( \frac{a_{11} + a_{12}}{\rho_1} \right)^{\frac{1}{2}} \frac{n\pi}{\ell}, \quad \forall n \in \mathbb{N}, \]
are eigenvalues of operator \( \mathcal{A} \) with eigenvector
\[ \mathbf{Z}_n = \left( \sin \left( \frac{n\pi x}{\ell} \right), i\lambda_n \sin \left( \frac{n\pi x}{\ell} \right), \sin \left( \frac{n\pi x}{\ell} \right), i\lambda_n \sin \left( \frac{n\pi x}{\ell} \right), 0, 0 \right). \]
In case of \( \rho_2\beta_1 = \rho_1\beta_2 \), we can find, for \( n \in \mathbb{N} \) sufficiently large,
\[ i\mathbb{R} \ni i\lambda_n = i \left( \frac{\beta_2 a_{11} - a_{12}\beta_1}{\rho_1\beta_2} \left( \frac{n\pi}{\ell} \right)^2 + \frac{\alpha(\beta_1 + \beta_2)}{\rho_1\beta_2} \right)^{\frac{1}{2}}, \]
which are eigenvalues of operator \( \mathcal{A} \) with corresponding eigenvector
\[ \mathbf{Z}_n = \left( \sin \left( \frac{n\pi x}{\ell} \right), i\lambda_n \sin \left( \frac{n\pi x}{\ell} \right), -\frac{\beta_1}{\beta_2} \sin \left( \frac{n\pi x}{\ell} \right), -i\lambda_n \frac{\beta_1}{\beta_2} \sin \left( \frac{n\pi x}{\ell} \right), 0, 0 \right). \]
Now, if assumption (ii) is enforced, then the complex number
\[ i\mathbb{R} \ni i\lambda_{n_0} = i \left( \frac{\alpha(\beta_1 + \beta_2)(\kappa_1 + \kappa_2)}{\chi_1} \right)^{\frac{1}{2}} n_0 \frac{\pi}{\ell} \]
is an eigenvalue of \( \mathcal{A} \) with eigenvector \( \mathbf{Z}_{n_0} \) given by
\[ \left( \sin \left( \frac{n_0\pi x}{\ell} \right), i\lambda_{n_0} \sin \left( \frac{n_0\pi x}{\ell} \right), -\frac{\beta_1}{\beta_2} \sin \left( \frac{n_0\pi x}{\ell} \right), -i\lambda_{n_0} \frac{\beta_1}{\beta_2} \sin \left( \frac{n_0\pi x}{\ell} \right), 0, 0 \right), \]
where \( n_0 \) is a fixed natural number.
To conclude, let us assume that (iii) holds. If \( \beta_1 = 0 \), then
\[ i\mathbb{R} \ni i\lambda_{n_0} = i \left( \frac{a_{11}}{\rho_1} \left( \frac{n_0\pi}{\ell} \right)^2 + \frac{\alpha}{\rho_1} \right)^{\frac{1}{2}} \]
is an eigenvalue for the operator \( \mathcal{A} \) with
\[ \mathbf{Z}_{n_0} = \left( \sin \left( \frac{n_0\pi x}{\ell} \right), i\lambda_{n_0} \sin \left( \frac{n_0\pi x}{\ell} \right), 0, 0, 0, 0 \right) \]
being the corresponding eigenvector.
If \( \beta_2 = 0 \), then
\[ i\mathbb{R} \ni i\lambda = i \left( \frac{a_{22}}{\rho_2} \left( \frac{n_0\pi}{\ell} \right)^2 + \frac{\alpha}{\rho_2} \right)^{\frac{1}{2}}, \]
is an eigenvalue for the operator \( \mathcal{A} \) with eigenvector
\[ \mathbf{Z}_{n_0} = \left( 0, 0, \sin \left( \frac{n_0\pi x}{\ell} \right), i\lambda \sin \left( \frac{n_0\pi x}{\ell} \right), 0, 0 \right), \]
where \( n_0 \) is a fixed natural number. These conclude the proof. \( \square \)

As a consequence, we have the following result.

Lemma 3.4. \( i\mathbb{R} \subset \sigma(\mathcal{A}) \) if and only if one of the following assumptions is assumed to be true
(i) \( \chi_1 \neq \chi_0 \left( \frac{n\pi}{\ell} \right)^2 \) (\( \forall n \in \mathbb{N} \));
(ii) \( \frac{\lambda_1}{\lambda_0} = \left( \frac{n_0\pi}{\ell} \right)^2 \) for a fixed \( n_0 \in \mathbb{N} \), \( \frac{\alpha(\beta_1 + \beta_2)(\kappa_1 + \kappa_2)}{\chi_0} \leq 0 \) and \( \beta_1 \beta_2 \neq 0 \).
Proof. Via Lemma 3.3, we only need to verify the necessary condition. Here, the proof is carried out via a contradiction argument. In fact, if we suppose that $i\mathbb{R} \subset \rho(\mathcal{A})$ is not true, then by the same arguments used in [19], we obtain the existence of a real number $\lambda \neq 0$, sequences $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ with $\lambda_n \to_\infty \lambda$, $|\lambda_n| < |\lambda|$, $\{Z_n\}_{n \in \mathbb{N}} \in D(\mathcal{A})$ and $\{F_n\}_{n \in \mathbb{N}} \in \mathcal{H}$, such that

$$Z_n = (u_n, v_n, w_n, \theta_n, \eta_n), \quad \|Z_n\|_{\mathcal{H}} = 1, \quad F_n = (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}, f_{6,n})$$

and

$$(i\lambda_n I - \mathcal{A})Z_n = F_n \to_\infty 0 \text{ in } \mathcal{H}. \quad (3.5)$$

The convergence given in (3.5) is translated into the following

$$i\lambda_n u_n - v_n = f_{1,n} \to_\infty 0, \quad (3.6)$$

$$i\lambda_n \rho_1 v_n - (a_{11} u_n + a_{12} w_n)_{xx} + \alpha(u_n - w_n) - \beta_1 \theta_n x = \rho_1 f_{2,n} \to_\infty 0, \quad (3.7)$$

$$i\lambda_n w_n - \omega_n = f_{3,n} \to_\infty 0, \quad (3.8)$$

$$i\lambda_n \rho_2 \omega_n - (a_{12} u_n + a_{22} w_n)_{xx} - \alpha(u_n - w_n) - \beta_2 \theta_n x = \rho_2 f_{4,n} \to_\infty 0, \quad (3.9)$$

$$i\lambda_n \rho_3 \theta_n - \kappa \theta_n x - \int_0^\infty \mu(s) \eta_{n,x}(s) ds - (\beta_1 v_n x + \beta_2 \omega_n x) = \rho_3 f_{5,n} \to_\infty 0, \quad (3.10)$$

$$i\lambda_n \eta_n - \eta_{n,s} - \theta_n = f_{6,n} \to_\infty 0. \quad (3.11)$$

Combining convergence (3.5) with dissipative property of operator $\mathcal{A}$, allow us to conclude

$$\frac{\nu}{2} \int_0^\ell \int_0^\infty \mu(s) |\eta_{n,x}(s)|^2 ds dx + \rho \int_0^\ell |\theta_n x|^2 dx \to_\infty 0. \quad (3.12)$$

The compact embedding $H^1_0(0, \ell) \subset L^2(0, \ell)$ together with equations (3.6) and (3.8) imply

$$(u_n, v_n, w_n, \omega_n) \to_\infty (u, v, w, \omega) \text{ in } [L^2(0, \ell)]^4. \quad (3.13)$$

Now, using equation (3.10), we find

$$\int_0^\ell |\beta_1 v_n + \beta_2 \omega_n|^2 dx$$

$$= \int_0^\ell \left| i\lambda_n \rho_3 \int_0^x \theta_n ds - \sigma \theta_n x - \int_0^\infty \mu(s) \eta_{n,x}(s) ds - \rho_3 \int_0^x f_{5,n} ds \right|^2 dx. \quad (3.14)$$

Let us analyze the convergence of the integral $\int_0^\ell |i\lambda_n \rho_3 \int_0^x \theta_n ds|^2 dx$. Note that, if $\sigma \in (0, 1)$ and with convergence (3.12) at hand, one obtain

$$\int_0^\ell |i\lambda_n \rho_3 \int_0^x \theta_n ds|^2 dx \leq C|\lambda_n|^2 \int_0^\ell |\theta_n x|^2 dx \to_\infty 0. \quad (3.15)$$

Now, if one has $\sigma = 0$, we find from equation (3.11)

$$\int_0^\infty \mu(s) ds \int_0^\ell |\theta_n x|^2 dx = \left| \int_0^\infty \mu(s) \int_0^\ell (i\lambda_n \eta_n + \eta_{n,s} - f_{6,n}) \overline{\theta_{n,x}} ds dx \right|$$

$$\leq \epsilon \int_0^\ell |\theta_{n,x}|^2 dx + C\epsilon (|\lambda_n|^2 + 1) \|\eta_n\|_{\mathcal{H}}^2 + C\epsilon \|F_n\|_{\mathcal{H}}. \quad (3.16)$$

For $\epsilon > 0$ sufficiently small, the above when combined with (3.5) and (3.12) imply

$$\theta_n \to_\infty 0 \text{ in } H^1_0(0, \ell).$$
Thus, for \( \sigma = 0 \), the following convergence holds

\[
\int_0^\ell \left| i\lambda_n^3 \int_0^\sigma \theta_n ds \right|^2 dx \xrightarrow{n \to \infty} 0. \tag{3.16}
\]

Using (3.5), (3.12), (3.15) (for \( \sigma \in (0, 1) \)) and (3.16) (for \( \sigma = 0 \)), we can conclude from (3.14)

\[
\beta_1 v_n + \beta_2 \omega_n \xrightarrow{n \to \infty} 0 \text{ in } L^2(0, \ell). \tag{3.17}
\]

Moreover, using (3.6), (3.8) and (3.17), we obtain

\[
\beta_1 u_n + \beta_2 w_n \xrightarrow{n \to \infty} 0 \text{ in } L^2(0, \ell). \tag{3.18}
\]

As a consequence of (3.12), convergences (3.17) and (3.18) imply

\[
\beta_1 v + \beta_2 \omega = \beta_1 u + \beta_2 w = 0. \tag{3.19}
\]

We return to (3.7) and (3.9) to find that \( \{a_{11} u_n + a_{12} w_n\}_{n \in \mathbb{N}} \) and \( \{a_{12} u_n + a_{22} w_n\}_{n \in \mathbb{N}} \) are bounded sequences in \( H^1_0(0, \ell) \cap H^2(0, \ell) \) and the compact embedding \( H^1_0(0, \ell) \cap H^2(0, \ell) \subset H^1_0(0, \ell) \) implies the strongly convergent in \( H^1_0(0, \ell) \). These and assumption \( a_{11} a_{22} - a_{12}^2 > 0 \) yield

\[
(u_n, w_n) \xrightarrow{n \to \infty} (u, v) \text{ in } H^1_0(0, \ell). \tag{3.20}
\]

We readily obtain from (3.9), (3.20) and (3.7)

\[
i \lambda \rho_1 v - (a_{11} u + a_{12} w)_{xx} + \alpha (u - w) = 0,
\]

\[
i \lambda \rho_2 \omega - (a_{12} u + a_{22} w)_{xx} - \alpha (u - w) = 0. \tag{3.21}
\]

We begin showing the conclusion when (i) is assumed to be true. Here, we analyze two situations: (a) \( \beta_1 \beta_2 \neq 0 \) and (b) \( \beta_1 \beta_2 = 0 \). To study (a), we back to (3.19) to obtain \( w = -\frac{\beta_1}{\beta_2} u \). Inserting the last into (3.21), we find

\[
-\kappa_1 u_{xx} = \lambda^2 \rho_1 \beta_2 u - \alpha (\beta_1 + \beta_2) u, \tag{3.22}
\]

\[
-\kappa_2 u_{xx} = -\lambda^2 \rho_2 \beta_1 u + \alpha (\beta_1 + \beta_2) u. \tag{3.23}
\]

Multiplying (3.22) by \( \rho_2 \beta_1 \), (3.23) by \( \rho_1 \beta_2 \) and summing up the result, we obtain

\[
-\chi_0 u_{xx} = \chi_1 u. \tag{3.24}
\]

Assumption \( \chi_1 \neq \chi_0(\frac{\lambda^2 \rho_2}{\rho_1})^2 (\forall n \in \mathbb{N}) \) shows that problem (3.24) with zero Dirichlet boundary condition has trivial solution, that is, \( u = w = 0 \). Feeding back this information into (3.6) and (3.8), we find \( v = \omega = 0 \). These promptly imply

\[
Z_n \xrightarrow{n \to \infty} 0 \text{ in } \mathcal{H}.
\]

This contradicts \( \|Z_n\| = 1 \).

The case (b). Let us assume \( \beta_1 = 0 \). In this case, the assumption is given by \( \alpha \neq a_{12}(\frac{\pi}{2})^2 (\forall n \in \mathbb{N}) \). Now, coming back to (3.19) and (3.21), we have \( w = \omega = 0 \) and

\[
-\alpha a_{12} u_{xx} = \alpha u.
\]

As before, we find trivial solution \( u = 0 \) and this is a contradiction.

To conclude, let us assume that assumption (ii) is enforced. To this end, we back to (3.22), (3.23) and (3.24) to conclude

\[
\left( \lambda^2 - \frac{\alpha(\beta_1 + \beta_2)(\kappa_1 + \kappa_2)}{\chi_0} \right) u = 0.
\]

This implies that \( u = 0 \). As before, we arrive at a contradiction. \( \square \)
Preparation for the proof of (3.2) and (3.3): Our present goal is to establish resolvent operator estimate

\[ \|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|\lambda|^r, \]  

with \( r \geq 0 \) depending on (1.9) and (1.10).

In order to show condition (3.25) several preliminary estimates are needed. We start with the resolvent equation: let \( i\lambda \in \rho(\mathcal{A}) \), \( Z \in D(\mathcal{A}) \) and \( F \in \mathcal{H} \), such that, \((i\lambda - \mathcal{A})Z = F\). The last and the definition (2.10) imply

\[
\begin{align*}
    i\lambda u - v &= f_1, \\
    i\lambda \rho_1 v - (a_{11} u + a_{12} w)_{xx} + \alpha(u - w) &= \rho_1 f_2, \\
    i\lambda w - \omega &= f_3, \\
    i\lambda \rho_2 \omega - (a_{12} u + a_{22} w)_{xx} - \alpha(u - w) &= \rho_2 f_4, \\
    i\lambda \rho_3 \theta - \sigma \theta_{xx} - \int_0^\infty \mu(s) \eta_{xx}(s) ds - (\beta_1 v_x + \beta_2 \omega_x) &= \rho_3 f_5, \\
    i\lambda \eta + \eta_x - \theta &= f_6.
\end{align*}
\]

The first estimate is obtained combining the assumptions on the function \( \mu(\cdot) \) with (2.9) and (2.11)

\[
\sigma \|\theta_x\|^2_{L^2} + \frac{\mu}{2} \|\eta\|^2_{\mathcal{H}} \leq \sigma \|\theta_x\|^2_{L^2} - \frac{1}{2} \int_0^\ell \int_0^\infty \mu'(s) \eta_x(s) d\eta_x + \frac{\mu}{2} \int_0^\infty \mu(s) \eta_x(s) d\eta_x \leq \|Z\|_{\mathcal{H}}^2\|F\|_{\mathcal{H}}^2. \tag{3.32}
\]

Throughout this manuscript, we routinely use Young, Hölder, Poincaré inequalities and dissipative estimate (3.32), often without explicit mention. By the symbol \( C \), we denote a generic positive constant that may differ from line to line and independent on \( \sigma \in [0,1) \). We also set, for the convenience of notation, \( \mathcal{F} \) as a complex number that satisfies the estimate

\[
\text{Re}\{\mathcal{F}\} \leq C \left[ \|Z\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \|Z\|^2_{\mathcal{H}}\|F\|^2_{\mathcal{H}} + \|Z\|^2_{\mathcal{H}}\|F\|^2_{\mathcal{H}} + \|F\|^2_{\mathcal{H}} \right],
\]

for some positive constant \( C \) also independent on \( \sigma \in [0,1) \).

We observe that, when \( \sigma = 0 \), estimate (3.32) for \( \theta \) is non longer valid. Estimates for component \( \theta \) are given by the following Lemmas.

**Lemma 3.5.** Let \( \sigma = 0 \). With reference to resolvent equation (3.26)-(3.31), we have that there exists a positive constant \( C \) such that

\[ \|\theta_x\|_{L^2} \leq C \left[ |\lambda| \|Z\|^\frac{1}{2}_{\mathcal{H}}\|F\|^\frac{1}{2}_{\mathcal{H}} + \|Z\|^\frac{1}{2}_{\mathcal{H}}\|F\|^\frac{1}{2}_{\mathcal{H}} \right], \quad \text{for } |\lambda| \geq 1. \]

**Proof.** Taking the \( \mathcal{M} \) inner product of \( \theta \) with equation (3.31), we obtain

\[
\int_0^\infty \mu(s) ds \|\theta_x\|^2_{L^2} = i\lambda \int_0^\infty \mu(s) \eta_x \overline{\theta_x} ds + \int_0^\ell \int_0^\infty \mu(s) \eta_{sx} \overline{\theta_x} ds dx - \int_0^\ell \int_0^\infty \mu(s) f_{6x} \overline{\theta_x} ds dx
\leq C|\lambda|\|\theta_x\|_{L^2}\|\eta\|_{\mathcal{M}} + \text{Re} \left( \int_0^\ell \int_0^\infty \mu(s) \eta_{sx} \overline{\theta_x} ds dx \right) + C\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}}.
\]
The next step is to estimate the product defined by $I$. Via integration by parts, we find
\[
\int_0^\ell \int_0^\infty \mu(s) \eta_{s,x} \overline{\theta_x} ds dx \\
= \lim_{A \to \infty} \lim_{\epsilon \to 0} \int_0^A \frac{d}{ds} \eta_{s,x} \theta_x ds L^2 ds \\
= \lim_{A \to \infty} \mu(A) \langle \eta_{s,x}, \theta_x \rangle_{L^2} - \lim_{\epsilon \to 0} \mu(\epsilon) \langle \eta_{s,x}(\epsilon), \theta_x \rangle_{L^2} - \int_0^\infty \mu'(s) \langle \eta_{s,x}, \theta_x \rangle_{L^2} ds \\
= - \int_0^\infty \mu'(s) \langle \eta_{s,x}, \theta_x \rangle_{L^2} ds \\
\]
From Lemmas 3.3.1 and 3.3.2 in [19], we have
\[
|\mu(A)|^{\frac{1}{2}} ||\eta_{s,x}(A)||_{L^2} \xrightarrow{A \to \infty} 0 \quad \text{and} \quad |\mu(\epsilon)|^{\frac{1}{2}} ||\eta_{s,x}(\epsilon)||_{L^2} \xrightarrow{\epsilon \to 0} 0.
\]
Therefore, the above identity leads to
\[
\Re \left\{ \int_0^\ell \int_0^\infty \mu(s) \eta_{s,x} \overline{\theta_x} ds dx \right\} = \Re \left\{ \int_0^\infty -\mu'(s) \langle \eta_{s,x}, \theta_x \rangle_{L^2} ds \right\} \\
\leq \left[ \int_0^\infty -\mu'(s) ||\eta_{s,x}||_{L^2}^2 ds \right]^{\frac{1}{2}} \left[ \int_0^\infty -\mu'(s) ds \right]^{\frac{1}{2}} ||\theta_x||_{L^2}.
\]
Combining the fact that $\int_0^\infty -\mu'(s) ds = \mu(0) - \mu(\infty) > 0$ with estimate (3.32), we promptly obtain
\[
I = \Re \left\{ \int_0^\ell \int_0^\infty \mu(s) \eta_{s,x} \overline{\theta_x} ds dx \right\} \leq C ||Z||_{\mathscr{H}} \frac{1}{2} ||\eta||_{\mathscr{H}} ||F||_{\mathscr{H}} ||\theta_x||_{L^2}.
\]
Now, the conclusion follows from the above estimate and from (3.32). \qed

**Lemma 3.6.** Let $\sigma = 0$. With reference to resolvent equation (3.26)-(3.31), there exists a positive constant $C$ such that
\[
||\theta||_{L^2}^2 \leq \frac{C}{|\lambda|} \left[ ||\theta_x||_{L^2} ||Z||_{\mathscr{H}} + ||Z||_{\mathscr{H}} ||F||_{\mathscr{H}} \right].
\]

**Proof.** Taking the $L^2(0, \ell)$ inner product of $\theta$ with equation (3.30), we arrive at
\[
i \lambda \rho_3 \int_0^\ell |\theta|^2 dx + \int_0^\ell \int_0^\infty \mu(s) \theta_x \overline{\theta_x} ds dx + \int_0^\ell \left( \beta_1 v + \beta_2 w \right) \theta_x dx = \rho_3 \int_0^\ell j_x \tilde{\theta} dx.
\]
Therefore
\[
\int_0^\ell |\theta|^2 dx \leq \frac{C}{|\lambda|} \left[ ||\theta_x||_{L^2} ||\eta||_{\mathscr{H}} + ||\theta_x||_{L^2} ||Z||_{\mathscr{H}} + ||Z||_{\mathscr{H}} ||F||_{\mathscr{H}} \right].
\]
The last combined with the definition of norm in $\mathscr{H}$ implies the conclusion. \qed

**Lemma 3.7.** Let $\sigma \in [0, 1)$. With reference to resolvent equation (3.26)-(3.31), there exists a positive constant $C$ such that
\[
||\beta_1 u_x + \beta_2 w_x||_{L^2}^2 \\
\leq C \left[ (\sigma + |\lambda|^{-1}) ||\theta_x||_{L^2} ||Z||_{\mathscr{H}} + ||\eta||_{\mathscr{H}} ||Z||_{\mathscr{H}} + \sigma |\lambda|^{-1} ||\theta_x||_{L^2}^2 \right] + \Re \{\Phi\}.
\]
Proof. Using equations (3.26) and (3.28) in (3.30), we find the following identity
\[
i\lambda(\beta_1 u_x + \beta_2 w_x) = i\lambda \rho_3 \theta - a \theta x - \int_0^\infty \mu(s) \eta x x(s) ds - \rho_3 f_5 + \beta_1 f_{1,x} + \beta_2 f_{3,x}.
\]
Taking the $L^2(0, \ell)$ inner product of the above identity with $(\beta_1 u_x + \beta_2 w_x)$ and using resolvent equations (3.26), (3.28), we obtain
\[
\|\beta_1 u_x + \beta_2 w_x\|_{L^2}^2 = \frac{\rho_3}{i\lambda} \int_0^\ell \theta x (\beta_1 v + \beta_2 \omega) dx + \frac{\sigma}{i\lambda} \int_0^\ell \theta x (\beta_1 u_{xx} + \beta_2 w_{xx}) dx
+ \frac{1}{i\lambda} \int_0^\ell \int_0^\infty \mu(s) \eta x (s) (\beta_1 u_{xx} + \beta_2 w_{xx}) ds dx + \mathcal{F}. \quad (3.33)
\]
The next step is to estimate the right-hand side of (3.33). Note that
\[
\text{Re} \left\{ \frac{\rho_3}{i\lambda} \int_0^\ell \theta x (\beta_1 v + \beta_2 \omega) dx \right\} \leq C|\lambda|^{-1} \|\theta_x\|_{L^2} \|\mathbf{Z}\|_{\mathcal{F}}. \quad (3.34)
\]
Now, setting $\Delta = a_{11}a_{22} - a_{12}^2$, we can write
\[
\beta_1 u_{xx} + \beta_2 w_{xx} = -\frac{\kappa_2}{\Delta} (a_{11} u_{xx} + a_{12} w_{xx}) + \frac{\kappa_1}{\Delta} (a_{12} u_{xx} + a_{22} w_{xx}).
\]
Using equations (3.27) and (3.29), we find
\[
\beta_1 u_{xx} + \beta_2 w_{xx} = \frac{i\lambda}{\Delta} (\kappa_1 \rho_2 \omega - \kappa_2 \rho_1 v) - \frac{\alpha}{\Delta} (\kappa_1 + \kappa_2)(u - w)
- \frac{\kappa_1 \beta_2 - \kappa_2 \beta_1}{\Delta} \theta x - \frac{\kappa_1 \rho_2}{\Delta} f_4 + \frac{\kappa_2 \rho_1}{\Delta} f_2.
\]
The above implies the following estimate
\[
\text{Re} \left\{ \int_0^\ell \left[ \sigma \theta x + \int_0^\infty \mu(s) \eta x (s) ds \right] (\beta_1 u_{xx} + \beta_2 w_{xx}) dx \right\}
\leq C \left[ (\sigma + |\lambda|^{-1}) \|\theta_x\|_{L^2} \|\mathbf{Z}\|_{\mathcal{F}} + \|\eta\|_{\mathcal{H}} \|\mathbf{Z}\|_{\mathcal{F}} + \sigma |\lambda|^{-1} \|\theta_x\|_{L^2}^2 \right] + \text{Re}\{\mathcal{F}\}. \quad (3.35)
\]
Inserting (3.34) and (3.35) into (3.33), we find the desired conclusion. \hfill \square

For the subsequent Lemmas, we opt to use the notation
\[
K_1 = a_{11}\beta_1 \rho_2 + a_{12} \beta_2 \rho_1 \quad \text{and} \quad K_2 = a_{12} \beta_1 \rho_2 + a_{22} \beta_2 \rho_1.
\]

**Lemma 3.8.** With reference to resolvent equation (3.26)-(3.31), there exists a positive constant $C$ such that
\[
\|\beta_1 v + \beta_2 \omega\|_{L^2}^2 \leq C \left[ (\sigma + |\lambda|^{-1}) \|\theta_x\|_{L^2} \|\mathbf{Z}\|_{\mathcal{F}} + \|\eta\|_{\mathcal{H}} \|\mathbf{Z}\|_{\mathcal{F}} + \sigma |\lambda|^{-1} \|\theta_x\|_{L^2}^2 \right] + \text{Re}\{\mathcal{F}\}.
\]

**Proof.** Let us consider the multiplier $\xi(x) = \int_0^x (\beta_1 v + \beta_2 \omega) dy$. Then, taking the $L^2(0, \ell)$ inner product of $\xi(\cdot)$ with equation (3.30), we arrive at
\[
- \int_0^\ell |\beta_1 v + \beta_2 \omega|^2 dx = i\lambda \rho_3 \int_0^\ell \theta x \xi dx - \sigma \int_0^\ell \theta z (\beta_1 v + \beta_2 \omega) dx
- \int_0^\ell \int_0^\infty \mu(s) \eta x (s) (\beta_1 v + \beta_2 \omega) ds dx + \mathcal{F}. \quad (3.36)
\]
The next step is to estimate the right-hand side of (3.36). Firstly, we multiply resolvent equation (3.27) by $\frac{\beta_1}{\rho_1}$ and (3.29) by $\frac{\beta_2}{\rho_2}$ to find the following identity

$$i\lambda(\beta_1 v + \beta_2 \omega) = \frac{K_1}{\rho_1 \rho_2} u_{xx} + \frac{K_2}{\rho_1 \rho_2} w_{xx} - \alpha \left( \frac{\beta_1}{\rho_1} - \frac{\beta_2}{\rho_2} \right) (u - w) + \left( \frac{\beta_1^2}{\rho_1} + \frac{\beta_2^2}{\rho_2} \right) \theta_x + \beta_1 f_2 + \beta_2 f_4. \quad (3.37)$$

Secondly, we introduce the elliptic problem

$$-\psi_{xx} = \theta \text{ in } (0, \ell) \text{ and } \psi_x(0) = \psi_x(\ell) = 0.$$

Via above identities and resolvent equations (3.26) and (3.28), the real part of the first product on the right-hand side of (3.36) satisfies

$$\text{Re} \left\{ i\lambda \rho_3 \int_0^\ell \theta \xi dx \right\} = \text{Re} \left\{ i\lambda \rho_3 \int_0^\ell \psi_x(\beta_1 v + \beta_2 \omega) dx \right\} = \text{Re} \left\{ \frac{-\rho_3}{i\lambda} \int_0^\ell \theta_x \left( \frac{K_1}{\rho_1 \rho_2} v + \frac{K_2}{\rho_1 \rho_2} \omega \right) dx - \frac{\alpha \rho_3}{i\lambda} \left( \frac{\beta_1}{\rho_1} - \frac{\beta_2}{\rho_2} \right) \int_0^\ell \psi_x (v - w) dx \right\}$$

$$- \text{Re} \left\{ \rho_3 \left( \frac{\beta_1^2}{\rho_1} + \frac{\beta_2^2}{\rho_2} \right) \int_0^\ell \psi_{xx} \bar{\theta} dx + \mathcal{F} \right\} \leq C \left[ |\lambda|^{-1} \|\theta_x\|_{L^2} \|\mathbf{Z}\|_{\mathscr{H}} + \|\theta\|_{L^2}^2 \right] + \text{Re} \{ \mathcal{F} \}.$$

Here, we have used $\|\psi_x\|_{L^2} \leq C \|\theta_x\|_{L^2}$. Next, using the definition of $\mathscr{H}$ norm, we find

$$\text{Re} \left\{ - \int_0^\ell \left[ \sigma \theta_x + \int_0^\infty \mu(s) \eta_x(s) ds \right] (\beta_1 v + \beta_2 \omega) dx \right\} \leq C \left[ \sigma \|\theta_x\|_{L^2} \|\mathbf{Z}\|_{\mathscr{H}} + \|\eta\|_{\mathscr{H}} \|\mathbf{Z}\|_{\mathscr{H}} \right].$$

Combination of previous inequalities leads the desired estimate. \hfill \Box

**Lemma 3.9.** With reference to resolvent equation (3.26)-(3.31), given $\epsilon > 0$ there exists a positive constant $C_\epsilon = C(\epsilon)$ such that

$$\|K_1 u_x + K_2 w_x\|_{L^2}^2 \leq \epsilon \|\mathbf{Z}\|_{\mathscr{H}}^2 + C_\epsilon \left[ \|\beta_1 v + \beta_2 \omega\|_{L^2}^2 + |\lambda|^{-2} \|\mathbf{Z}\|_{\mathscr{H}}^2 + \|\theta\|_{\mathscr{H}}^2 \right] + \text{Re} \{ \mathcal{F} \}.$$

**Proof.** We recall (3.37) to find

$$\frac{1}{\rho_1 \rho_2} \int_0^\ell |K_1 u_x + K_2 w_x|^2 dx$$

$$= -i\lambda \int_0^\ell (\beta_1 v + \beta_2 \omega)(K_1 u + K_2 w) dx - \alpha \left( \frac{\beta_1}{\rho_1} - \frac{\beta_2}{\rho_2} \right) \int_0^\ell (u - v)(K_1 u + K_2 w) dx$$

$$- \left( \frac{\beta_1^2}{\rho_1} + \frac{\beta_2^2}{\rho_2} \right) \int_0^\ell \bar{\theta}(K_1 u_x + K_2 w_x) dx + \mathcal{F}. \quad (3.38)$$
Next, we use $i\lambda(K_1u + K_2w) = K_1v + K_2\omega + K_1f_1 + K_2f_3$ into (3.38) to obtain the estimate
\[
\int_0^\ell |K_1u_x + K_2w_x|^2 dx \leq C\|\beta_1v + \beta_2\omega\|_{L^2} \|K_1v + K_2\omega\|_{L^2} \\
+ C|\lambda|^{-1}\|\mathcal{Z}\|_{\mathcal{X}} \|K_1v + K_2\omega\|_{L^2} \\
+ C\|\theta\|_L^2 \|K_1u_x + K_2w_x\|_{L^2} + \text{Re}\{\mathcal{F}\}.
\]
Invoking now Young inequality, we conclude that for any $\epsilon > 0$ there exists a positive constant $C_\epsilon$ such that
\[
\int_0^\ell |K_1u_x + K_2w_x|^2 dx \leq \epsilon \|K_1v + K_2\omega\|_{L^2}^2 + C_\epsilon \|\beta_1v + \beta_2\omega\|_{L^2}^2 \\
+ C_\epsilon |\lambda|^{-2}\|\mathcal{Z}\|_{\mathcal{X}}^2 + C\|\theta\|_L^2 + \text{Re}\{\mathcal{F}\}.
\]
The last, when combined with inequality $\|K_1v + K_2\omega\|_{L^2} \leq C\|\mathcal{Z}\|_{\mathcal{X}}$ yields the conclusion. \hfill \Box

**Proof of the Main Result - Theorem 3.1 (i).** Under the assumptions, we already have that $i\mathbb{R} \subset \rho(\mathcal{A})$ - Lemma 3.4. To establish Theorem 3.1 completely, it suffices to show resolvent condition (3.25) with $r = 0$. We begin by taking the $L^2(0, \ell)$ inner product of equation (3.27) with $u$ and (3.29) with $w$ to obtain
\[
\rho_1 \int_0^\ell |v|^2 dx + \rho_2 \int_0^\ell |\omega|^2 dx \leq C \int_0^\ell (|u_x|^2 + |w_x|^2) dx + C\|\mathcal{Z}\|_{\mathcal{X}}\|\mathcal{F}\|_{\mathcal{X}}. \tag{3.39}
\]
We shall estimate $J$. From assumption $\chi_0 \neq 0$, we find
\[
u_x = \frac{\beta_2}{\chi_0} (K_1u_x + K_2w_x) - \frac{K_2}{\chi_0} (\beta_1u_x + \beta_2w_x)
\]
and
\[
w_x = -\frac{\beta_1}{\chi_0} (K_1u_x + K_2w_x) + \frac{K_1}{\chi_0} (\beta_1u_x + \beta_2w_x).
\]
These imply that
\[
\int_0^\ell |u_x|^2 dx + \int_0^\ell |w_x|^2 dx \leq C\|K_1u_x + K_2w_x\|_{L^2}^2 + C\|\beta_1u_x + \beta_2w_x\|_{L^2}^2. \tag{3.40}
\]
Combination of (3.39) with (3.40) implies
\[
\|\mathcal{Z}\|_{\mathcal{X}}^2 \leq C\|K_1u_x + K_2w_x\|_{L^2}^2 + C\|\beta_1u_x + \beta_2w_x\|_{L^2}^2 + C\|\mathcal{Z}\|_{\mathcal{X}}\|\mathcal{F}\|_{\mathcal{X}}.
\]
Recalling Lemmas 3.7 and 3.9, for $\epsilon > 0$ small and $|\lambda|$ large, we obtain
\[
\|\mathcal{Z}\|_{\mathcal{X}}^2 \leq C\left((\sigma^2 + |\lambda|^{-2})\|\theta_x\|_{L^2}^2 + \|\eta\|_{\mathcal{A}}^2 + |\lambda|^{-1}\|\theta_x\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) + \text{Re}\{\mathcal{F}\}.
\]
Finally, we invoke estimate (3.32) if $\sigma \in (0, 1)$ or Lemmas 3.5-3.6 for the limit case $\sigma = 0$, to conclude
\[
\|\mathcal{Z}\|_{\mathcal{X}} \leq C\|\mathcal{F}\|_{\mathcal{X}} \implies \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq C.
\]
This concludes the proof. \hfill \Box
Proof of the Lack of Exponential Decay: The proof of the present statement makes use of Lemma 5.1 in [16]. Notice that the memory kernel assumptions, in particular, imply \( \lim_{s \to 0} \sqrt{s}\mu(s) = 0 \). Then, under this condition the result in [16] states that

\[
\left| \sqrt{\lambda} \int_0^\infty \mu(s) \exp(-i\lambda s) ds \right| \to 0 \quad \text{as} \quad \lambda \to \infty.
\]

Therefore

\[
\left| \int_0^\infty \mu(s) \exp(-i\lambda s) ds \right| \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{3.41}
\]

Relation (3.41) follows directly from the Riemann-Lebesgue Lemma.

Proof of Theorem 3.1 (ii). In view of Pruss result, the proof is concluded by exhibiting the existence of a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \subset i\mathbb{R} \) with \( \lambda_n \to \infty \) and \( \{F_n\}_{n \in \mathbb{N}} \) a bounded sequence in \( \mathcal{H} \), such that, \( \{Z_n\}_{n \in \mathbb{N}} \in D(A) \) the solution of resolvent equation \( (i\lambda_n I - A)Z_n = F_n \) satisfies

\[
\|Z_n\|_\mathcal{H} = \|(i\lambda_n I - A)^{-1}F_n\|_\mathcal{H} \to \infty. \quad \tag{3.42}
\]

Let us consider \( F_n = (0, a\rho_1^{-1} \sin \left( \frac{n\pi x}{\ell} \right), 0, b\rho_2^{-1} \sin \left( \frac{n\pi x}{\ell} \right), 0, 0) \in \mathcal{H}, n \in \mathbb{N} \) and \( a, b \in \mathbb{R} \). With reference to the boundary conditions (2.7), one can assume solutions of the resolvent equation as following

\[
\begin{align*}
  u_n &= A_n \sin \left( \frac{n\pi x}{\ell} \right), & w_n &= B_n \sin \left( \frac{n\pi x}{\ell} \right), \\
  \theta_n &= C_n \cos \left( \frac{n\pi x}{\ell} \right), & \eta_n &= D_n(s) \cos \left( \frac{n\pi x}{\ell} \right).
\end{align*}
\]

These imply that

\[
\begin{align*}
  -\lambda_n^2 \rho_1 A_n + \left( \frac{n\pi}{\ell} \right)^2 (a_{11} A_n + a_{12} B_n) + \alpha (A_n - B_n) + \beta_1 \left( \frac{n\pi}{\ell} \right) C_n &= a, \quad \tag{3.43} \\
  -\lambda_n^2 \rho_2 B_n + \left( \frac{n\pi}{\ell} \right)^2 (a_{12} A_n + a_{22} B_n) - \alpha (A_n - B_n) + \beta_2 \left( \frac{n\pi}{\ell} \right) C_n &= b, \quad \tag{3.44} \\
  i\lambda_n \rho_3 C_n + \sigma \left( \frac{n\pi}{\ell} \right)^2 C_n + \left( \frac{n\pi}{\ell} \right)^2 \int_0^\infty \mu(s) \left( 1 - \exp(-i\lambda_n s) \right) ds C_n &= 0, \quad \tag{3.45} \\
  i\lambda_n D_n + D_{n,s} - C_n &= 0. \quad \tag{3.46}
\end{align*}
\]

From equation (3.46), we obtain

\[
D_n(s) = \frac{1}{i\lambda_n} (1 - \exp(-i\lambda_n s)) C_n.
\]

Replacing this into (3.45), we find

\[
\left[ i\lambda_n \rho_3 + \sigma \left( \frac{n\pi}{\ell} \right)^2 + \frac{1}{i\lambda_n} \left( \frac{n\pi}{\ell} \right)^2 \int_0^\infty \mu(s) \left( 1 - \exp(-i\lambda_n s) \right) ds \right] C_n - i\lambda_n \left( \frac{n\pi}{\ell} \right) (\beta_1 A_n + \beta_2 B_n) = 0. \quad \tag{3.47}
\]

As in Lemma 3.4, we also consider two situations: (a) \( \beta_1 \beta_2 \neq 0 \) and (b) \( \beta_1 \beta_2 = 0 \).

We start with (a). Let us consider \( a, b \in \mathbb{R} \) such that \( a \beta_2 - b \beta_1 \neq 0 \). Multiplying (3.43) by \( \frac{1}{\beta_2} \), (3.44) by \( \frac{1}{\beta_1} \), and adding up the resulting expression

\[
\begin{align*}
  -\lambda_n^2 \left( \frac{\rho_1}{\beta_1} A_n - \frac{\rho_2}{\beta_2} B_n \right) + \left( \frac{n\pi}{\ell} \right)^2 \left[ \frac{a_{11}}{\beta_1} - \frac{a_{12}}{\beta_2} \right] A_n + \left[ \frac{a_{12}}{\beta_1} - \frac{a_{22}}{\beta_2} \right] B_n + \\
  + \alpha \left[ \frac{1}{\beta_1} + \frac{1}{\beta_2} \right] (A_n - B_n) = a \frac{\beta_2 - b \beta_1}{\beta_1 \beta_2}.
\end{align*}
\]
By assumption $\chi_0 = 0$, we see
\[
\left[-\lambda_n^2 + \left(\frac{n\pi}{\ell}\right)^2 \left(\frac{\beta_2 a_{11} - \beta_1 a_{12}}{\rho_1 \beta_2}\right)\right] \left(\frac{\rho_1}{\beta_1} A_n - \frac{\rho_2}{\beta_2} B_n\right) + \alpha \left[\frac{1}{\beta_1} + \frac{1}{\beta_2}\right] (A_n - B_n) = \frac{a_2 - b_2}{\beta_1 \beta_2},
\] (3.48)

At this point, we take sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ as following
\[
\lambda_n^2 = \left(\frac{n\pi}{\ell}\right)^2 \left(\frac{\beta_2 a_{11} - \beta_1 a_{12}}{\rho_1 \beta_2}\right) \equiv \left(\frac{n\pi}{\ell}\right)^2 \gamma_1.
\] (3.49)

Firstly, note that if $\chi_0 = 0$, then $\gamma_1 > 0$ (See [3]). Secondly, inserting (3.49) into (3.48) we find
\[
A_n = B_n + \frac{a_2 - b_2}{\alpha (\beta_1 + \beta_2)} \equiv B_n + \gamma_2.
\] (3.50)

Now, we return to (3.47) with $\{\lambda_n\}_{n \in \mathbb{N}}$ replaced by (3.49) and $\{A_n\}_{n \in \mathbb{N}}$ by (3.50) to obtain
\[
C_n = i \left(\frac{n\pi}{\ell}\right)^2 \gamma_1^2 (\beta_1 + \beta_2) B_n \delta_1(n) + i \left(\frac{n\pi}{\ell}\right)^2 \gamma_1^2 \gamma_2 \beta_1 \delta_1^2(n),
\] (3.51)

where
\[
\delta_1(n) = i \left(\frac{n\pi}{\ell}\right) \gamma_1^2 \beta_2 + \sigma \left(\frac{n\pi}{\ell}\right)^2 + \frac{1}{i \gamma_1^2} \left(\frac{n\pi}{\ell}\right) \int_0^\infty \mu(s)(1 - \exp(-i\lambda_n s))ds.
\]

Inserting (3.50) and (3.51) into (3.44), we obtain
\[
B_n = -\left(\frac{n\pi}{\ell}\right)^2 a_{12} \gamma_2 + \alpha \gamma_2 + b - \delta_2(n)
\]
\[
\left(\frac{n\pi}{\ell}\right)^2 \left[-\rho_2 \gamma_1 + a_{12} + a_{22}\right] + \delta_3(n),
\]

where
\[
\delta_2(n) = i \gamma_1^2 \beta_1 \beta_2 \left(\frac{n\pi}{\ell}\right)^3 \delta_1(n) \equiv \left(\frac{n\pi}{\ell}\right)^2 \delta_2^1(n)
\]
and
\[
\delta_3(n) = i \gamma_1^2 (\beta_1 + \beta_2) \beta_2 \left(\frac{n\pi}{\ell}\right)^3 \delta_1(n) \equiv \left(\frac{n\pi}{\ell}\right)^2 \delta_3^1(n).
\]

Using (3.41) we obtain
\[
\delta_2^1(n) \rightarrow 0 \quad \text{and} \quad \delta_3^1(n) \rightarrow 0.
\]

These allow us to conclude, for $\sigma \in (0, 1)$, the following convergences
\[
B_n \rightarrow 0 \quad \text{if} \quad a_{12} \neq 0
\]

and
\[
B_n \rightarrow 0 \quad \text{if} \quad a_{12} = 0 \quad \text{(3.50)} \Rightarrow A_n \rightarrow 0 \quad \gamma_2 \neq 0.
\]

In both cases ($a_{12} = 0$ and $a_{12} \neq 0$) we find
\[
\|Z_n\|_{\mathcal{S}} \rightarrow \infty.
\] (3.52)

Let us show (b). To this end let us consider $\beta_1 = 0$. From (3.43), we obtain
\[
\left[-\lambda_n^2 \rho_1 + \left(\frac{n\pi}{\ell}\right)^2 a_{11} + \alpha\right] A_n - \alpha B_n = a.
\]

Now, we take $a \neq 0$ and $\{\lambda_n\}_{n \in \mathbb{N}}$ as
\[
\lambda_n^2 = \frac{1}{\rho_1} \left(\frac{n\pi}{\ell}\right)^2 a_{11} + \alpha.
\]
implies that

We shall estimate the right-hand side of (3.27).

Next, we multiply equation (3.27) by \( \frac{\beta_2}{\rho_1} \) and (3.29) by \( \frac{\beta_2}{\rho_2} \) to obtain

\[
\alpha \left( \frac{\beta_1 - \beta_2}{\rho_1} \right) (u - w) = -i\lambda (\beta_1 v + \beta_2 \omega) + \frac{\beta_1^2}{\rho_1^2} (K_1 u_{xx} + K_2 w_{xx}) + \frac{\beta_2^2}{\rho_2} \theta_x + \beta_1 f_2 + \beta_2 f_4
\]

(3.54)

The last combined with assumption (1.11) also implies that

\[ \|Z_n\|_{\mathcal{H}} \xrightarrow{n \to \infty} \infty. \]

This concludes the proof.

\[ \square \]

**Proof of the Polynomial Decay Rate-Theorem 3.2.** In line with Borichev and Tomilov result, we only need to verify (3.25) with \( r = 4 \).

**Step 01.** Preliminary estimates: Taking \( L^2(0, \ell) \) inner product of equation (3.27) with \( u \) and (3.29) with \( w \), we obtain

\[
\int_0^\ell (|u|^2 + |w|^2) dx \leq C \int_0^\ell (|v|^2 + |\omega|^2 + |u - w|^2) dx + C \|Z\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \quad (3.53)
\]

We shall estimate the right-hand side of (3.53). First, assumption \( \chi_0 = 0 \) promptly implies that

\[
K_1 u_{xx} + K_2 w_{xx} = \left( \frac{\rho_2 \beta_1 a_{12} + \rho_1 \beta_2 a_{22}}{\beta_2} \right) (\beta_1 u_{xx} + \beta_2 w_{xx}) = \Lambda (\beta_1 u_{xx} + \beta_2 w_{xx}). \quad (3.54)
\]

Next, we multiply equation (3.27) by \( \frac{\beta_1}{\rho_1} \) and (3.29) by \( \frac{\beta_2}{\rho_2} \) to obtain

\[
\int_0^\ell |u - w|^2 dx \leq C \text{Re} \left\{ \int_0^\ell (\beta_1 v + \beta_2 \omega)(v - \omega) dx \right\} - C \text{Re} \left\{ \int_0^\ell (\beta_1 u_x + \beta_2 w_x)(u_x - w_x) dx \right\} + C |\lambda|^{-1} \|\theta\|_{L^2} \|Z\|_{\mathcal{H}} + \text{Re} \{\mathcal{F}\}. \quad (3.55)
\]

Let us estimate the right-hand side of (3.55). To this end, we follow the same argument as in Lemmas 3.7 and 3.8 to obtain the following estimates

\[
\text{Re} \left\{ \int_0^\ell (\beta_1 v + \beta_2 \omega)(v - \omega) dx \right\} \leq C (\sigma + |\lambda|^{-1}) \|\theta\|_{L^2} \|Z\|_{\mathcal{H}} + C \|\eta\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + C \|\theta\|_{L^2}^2 + \text{Re} \{\mathcal{F}\},
\]

where \( \sigma \) is a real number.
and
\[- \text{Re} \left\{ \int_0^\ell (\beta_1 u_x + \beta_2 w_x)(u_x - w_x) \, dx \right\} \]
\[\leq C(\sigma + |\lambda|^{-1})\|\theta_x\|_{L^2} \|Z\|_{\mathcal{H}} + C\|\eta\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \text{Re}\{\mathcal{F}\}.\]

Above estimates along with (3.55) yield the following
\[\int_0^\ell |u - w|^2 \, dx \leq C[\sigma + |\lambda|^{-1}]\|\theta_x\|_{L^2} \|Z\|_{\mathcal{H}} + C\|\eta\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + |\theta|_{L^2}^2 + \text{Re}\{\mathcal{F}\}. \tag{3.56}\]

Using resolvent equation (3.26) and (3.28), we obtain
\[\int_0^\ell |v - \omega|^2 \, dx \leq C|\lambda|^2(\sigma + |\lambda|^{-1})\|\theta_x\|_{L^2} \|Z\|_{\mathcal{H}} + C|\lambda|^2 \|\eta\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + |\theta|_{L^2}^2 + |\lambda|^2 \text{Re}\{\mathcal{F}\} + C\|\mathcal{F}\|_{\mathcal{H}}^2. \tag{3.57}\]

The assumption \(\beta_1 + \beta_2 \neq 0\) implies
\[v = \frac{1}{\beta_1 + \beta_2} (\beta_1 v + \beta_2 \omega) + \frac{\beta_2}{\beta_1 + \beta_2} (v - \omega),\]
\[\omega = \frac{1}{\beta_1 + \beta_2} (\beta_1 v + \beta_2 \omega) - \frac{\beta_1}{\beta_1 + \beta_2} (v - \omega). \tag{3.58}\]

Using estimate given in Lemma 3.8 together with (3.57) and (3.58), we see
\[\int_0^\ell (|v|^2 + |\omega|^2) \, dx \leq C|\lambda|^2(\sigma + |\lambda|^{-1})\|\theta_x\|_{L^2} \|Z\|_{\mathcal{H}} + C|\lambda|^2 \|\eta\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + |\theta|_{L^2}^2 + |\lambda|^2 \text{Re}\{\mathcal{F}\} + C\|\mathcal{F}\|_{\mathcal{H}}^2. \tag{3.59}\]

**Step 02.** Completion of the proof: Recalling the definition of \(\mathcal{H}\) norm, we find
\[\|Z\|_{\mathcal{H}}^2 \leq C \int_0^\ell \left[ |v|^2 + |\omega|^2 + |u_x|^2 + |w_x|^2 + |u - w|^2 \right] \, dx + C\|Z\|_{\mathcal{H}} \|\mathcal{F}\|_{\mathcal{H}}.\]

Collecting estimates (3.53), (3.56) and (3.59)
\[\|Z\|_{\mathcal{H}}^2 \leq C|\lambda|^2 \left[ (\sigma + |\lambda|^{-1})\|\theta_x\|_{L^2} \|Z\|_{\mathcal{H}} + \|\eta\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + |\theta|_{L^2}^2 + \text{Re}\{\mathcal{F}\} \right],\]
for a positive constant \(C\) and \(|\lambda|\) large enough. Now, we recall estimate (3.32) (or Lemmas 3.5 and 3.6 for \(\sigma = 0\)) to conclude
\[\|Z\|_{\mathcal{H}}^2 \leq C|\lambda|^4 \|\mathcal{F}\|_{\mathcal{H}}.\]

Here, we have used the inequality
\[|\lambda|^2 \text{Re}\{\mathcal{F}\} \leq \epsilon \|Z\|_{\mathcal{H}}^2 + C\epsilon |\lambda|^8 \|\mathcal{F}\|_{\mathcal{H}}^2.\]

This concludes the proof. \(\square\)

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E-mail address: malves@ufv.br
E-mail address: rodrigonunesmonteiro@gmail.com