Principles of solid modelling in point calculus

E V Konopatskiy¹, A A Bezditnyi², M V Lagunova³ and A V Naidysh⁴

¹Donbas National Academy of Civil Engineering and Architecture, Ukraine
²Sevastopol branch of «Plekhanov Russian University of Economics», Russia
³Nizhny Novgorod State University of Architecture and Civil Engineering, Russia
⁴Bogdan Khmelnitsky Melitopol State Pedagogical University, Ukraine

Abstract. The article describes the principles of solid modelling in point calculus, including the definition of geometric bodies in the form of an organized set of points in space. At the same time, the choice of point calculus as a mathematical apparatus for effective modelling of geometric bodies in 3-dimensional space is substantiated, expanding the instrumental capabilities of computer graphics. By means of generalization, it has been established that the dimension of the space in which the geometric body is defined is equal to the number of current parameters. On the basis of this, a new definition of a geometric body is proposed as a geometric set of points, in which the number of its determining parameters is equal to the dimension of space. Examples of the definition of a tetrahedron body and a triangular prism body in point calculus, obtained considering the proposed definition of the term “geometric body”, are given. The obtained point equations are completely invariant with respect to the choice of the coordinate system and depend only on the coordinates of the points that define the vertices of the modelled bodies. Thus, the obtained point equations determine the entire set of bodies of tetrahedrons, bodies of triangular prisms, bodies of elliptical cylinders and cones in 3-dimensional space. The prospect of further research is the definition in point calculus of geometric bodies of curvilinear and irregular shapes, considering their relative position in space, as well as more complex composite geometric bodies in 3-dimensional space.

1. Introduction

Today, computer graphics cannot be imagined without solid modelling. This is a tool for visual representation of 3-dimensional objects on a monitor screen, which considers their relative position and the joint interaction of components in the form of geometric ones. More recently, it was somewhat exotic, and at the moment it is an integral part of most professional computer graphics packages focused on further research of a computer model using computer-aided design and calculus systems. In addition, solid modelling systems are the fundamental basis for 3D printing and an integral part of many technology cycles based on the use of numerical control. Such software products include a number of foreign (AutoCAD, Mechanical Desktop, Inventor, SolidWorks, Solid Edge, 3DS Max and etc.), and a number of domestic (COMPAS-3D, nanoCAD and etc.) developments. You can also single out a number of programs that have grown from computer graphics to information modelling, while retaining the functionality of geometric modelling and shaping of geometric bodies and surfaces (Revit, Archicad and etc.). Of course, the overwhelming majority of these software products are paid, and therefore the computational algorithms and mathematical apparatus embedded in them are a commercial secret, which developers carefully guard from competitors, which makes it difficult to assess the effectiveness of the mathematical apparatus embedded in them and computational algorithms based on it. There are a
number of works on solid modelling and computer graphics, algorithms and mathematical apparatus of computational and computer geometry [1-3]. However, it is difficult to establish a specific relationship between these works and the mathematical apparatus inherent in mass commercial software products.

The mathematical apparatus "Point calculus" (other names are BN-calculus, Balyuba-Naydysh's point calculus) was developed by a team of scientists from the Melitopol School of Applied Geometry under the leadership of Academician V.M. Naydysha [4-6]. In essence, it can be attributed to computational geometry, since any graphical operation of constructing a geometric object, point calculus associates an analytical operation, which makes it possible to represent the entire cycle of geometric constructions in the form of point equations and computational algorithms based on them. In other words, point calculus allows you to translate geometric algorithms for modelling lines, surfaces and bodies into a digital language understandable for a computer, while retaining the geometric meaning of all graphic constructions through a number of parameters that make up point equations. It should also be considered that in many cases geometric algorithms can provide a simpler solution, and tools for geometric solution of various problems have accumulated and improved over many centuries. Based on these considerations, the use of point calculus in the development of computer graphics systems, namely in solid modelling, can be considered a relevant and promising scientific direction, which has a pronounced applied nature.

2. Principles of parametrization of geometric objects in point calculus

In point calculus, any geometric object is represented by an organized set of points. A similar approach to the definition of geometric objects uses the Wurf calculus proposed by H. Staudt. In general, the point calculus in an affine space is a special case of the Wurf calculus in a projective space.

Traditionally, to determine the relative position of geometric bodies, it is necessary to coordinate each body separately, and then recalculate one coordinate system to another using rotation and translation matrices. Point calculus uses a different approach based on the definition of geometric objects in a simplex, where a simplex is understood as an n-dimensional generalization of the tetrahedron. In a particular case, a simplex can be represented as a Cartesian coordinate system. The advantage of this approach is that all geometric objects are defined in the local simplex, and the result of their joint interaction in the form of a final solid geometric model is determined in the global coordinate system. Moreover, the transition from local simplices to the global one is carried out automatically, which is a consequence of special invariant properties used in point calculus.

Point calculus is based on the invariants of affine geometry. These raises, on the one hand, additional requirements for the choice of the parameters of point equations, and on the other hand, ensures their coordinate-wise calculation. Based on this, the parameters of the point calculus can be invariants of parallel projection, presented in an explicit or implicit form. In addition, all geometric operations on geometric objects, considering their mutual position, also use the invariant properties of affine geometry. These include a simple ratio of three points of a straight line, parallelism of straight lines, intersection of straight lines, construction of a tangent, etc. All these geometric operations have their computational counterparts in point calculus and therefore can serve as an effective tool for computer modelling of complex geometric bodies, consisting of several simpler bodies.

Parameters in point calculus are of two types: fixed and movable (current). A fixed parameter defines a point in space that has specific constant coordinates. The current point calculation parameter defines the moving points, which are called the current points. Such points move in space due to a change in the current parameter and fill the space with their movement. Thus, any continuous geometric objects are represented as a set of points. To organize this set in the pointwise calculus, a special method of the moving simplex has been developed [7, 8], which is a generalization of the kinematic method of modelling geometric objects to the multidimensional space in pointwise calculus. Then the problem of solid modelling is reduced to determining such a current point, which would completely fill the space bounded by some planes and surfaces in 3-space with its motion. Of course, the simplest of these bodies are elementary bodies such as: pyramid, prism, cylinder, cone, ball and torus.
3. Definition of geometric bodies in point calculus

Modelling any solid geometric objects is inherently connected with the dimension of space and its topology, in which the sought solid model of the object is determined. So, a line is a one-parameter set that can be defined at least in 2-dimensional space. But the line, both straight and curved, is itself a one-dimensional space. And if you select a fragment of the plane bounded by lines, then we get a 2-dimensional analog of the body. Moreover, such a 2-dimensional body, as well as the space in which it is located, is a two-parameter set of points.

Accordingly, the section of the surface, and in the particular case of the plane (the surface of zero curvature), is a two-parameter set of points, which can be determined by a minimum in a 3-dimensional space and, in turn, is a 2-dimensional space. A geometrical body in 3-dimensional space as well as space itself is a three-parameter set of points. Thus, there is a complete correlation between the dimension of space and the number of current parameters that define a solid geometric object.

In accordance with [9], a body in geometric modelling is a connected set of points located on the inner side of one outer several inner shells located inside the outer shell, in combination with the points of these shells. In our opinion, this definition is too complicated for perception and requires clarification of the additional term “shell”. Based on the above reflections on the dimension of space, we propose another definition of a geometric body as a geometric set of points, in which the number of parameters defining it is equal to the dimension of space. Bodies defined in this way can also exist in spaces of higher dimensions, which can be used as an effective tool for geometric modelling of multifactor processes and phenomena by the method of multidimensional interpolation [10]. Thus, a three-parameter set of points in a 3-dimensional space will be a body, and in a 4-dimensional space – a three-parameter hypersurface.

4. Modelling of linear geometric bodies in point calculus

As already described above, elementary geometric bodies include: pyramid, prism, cylinder, cone, ball and torus. However, some of these bodies are linear, others are more complex. Based on this, we will consider the pyramid and prism to be linear geometric bodies, since they are determined by linear functional dependence. The sphere and the torus are referred to bodies of revolution. As for the set of cylinders and cones, these bodies are most widely used in computer graphics with a base in the form of a circle. Therefore, they should be identified first. Nevertheless, many other conical and cylindrical bodies with various algebraic and transcendental curves as a guide line, after defining them in point calculus, can significantly expand the tools for solid geometric modelling included in the computer graphics software packages.

4.1. Point definition of a solid model of a triangular pyramid

Let us define, as an example, in point calculus, the body of a triangular pyramid (tetrahedron) as a three-parameter set of points belonging to a 3-dimensional space. To do this, it is necessary to consider the geometric diagram shown in Figure 1. The vertices of the tetrahedron form a simplex of 3-dimensional space. To obtain the point equation of the tetrahedron body, you must first determine the set of points inside the triangle $ABC$. This means that it is necessary to determine the current point $N$ in the simplex $ABC$.

![Figure 1. Geometric scheme of the point definition of the tetrahedron body](image)
We use to determine the current point \( N \) the point equation of the plane:
\[
N = (A - C)uv + (B - C)v + C,
\]
(1)
where \( N \) – the current point that fills the space inside the triangle with its movement \( ABC \);
\( A, B \) and \( C \) – origin defining a simplex plane;
\( u \) and \( v \) – current parameters that are changing from 0 to 1;
\( \bar{v} = 1 - v \) – complement of parameter \( v \) to 1.

The set of points inside the tetrahedron is defined using the point equation of a straight-line segment:
\[
M = N\bar{w} + Dw,
\]
(2)
where \( M \) – the current point of the tetrahedron body;
\( D \) – one of the vertices of the 3-dimensional simplex \( ABCD \);
\( w \) – the current parameter, which ranges from 0 to 1.
\( \bar{w} = 1 - w \) – complement of parameter \( w \) to 1.

Substituting the point equation (1) into equation (2), after some transformations we obtain the point equation of the tetrahedron body:
\[
M = Au\bar{w} + B\bar{v}w + Cu\bar{v}w + Dw,
\]
(3)
where \( \bar{u} = 1 - u \) – complement of parameter \( u \) to 1.

Using the coordinate-wise calculation, for a 3-dimensional space we obtain the following system of parametric equations:
\[
\begin{align*}
x &= x_Auv\bar{w} + x_Bu\bar{v}w + x_Cu\bar{v}w + x_Dw \\
y &= y_Auv\bar{w} + y_Bu\bar{v}w + y_Cu\bar{v}w + y_Dw \\
z &= z_Auv\bar{w} + z_Bu\bar{v}w + z_Cu\bar{v}w + z_Dw
\end{align*}
\]

As can be seen from the obtained point equation (3), it includes only points of the simplex and linear functions of three parameters \( u, v \) and \( w \). The points of the simplex are determined using their coordinates, therefore, to determine the body of a tetrahedron, it is enough to enter its coordinates without the need to redefine the coordinate system. Another tetrahedron can be determined simply by changing the coordinates of the points of the vertices of the simplex. In this case, both tetrahedra will already be uniquely defined in the global coordinate system or in the global simplex. It should also be noted that all three current parameters varying from 0 to 1 will define the set of points of the tetrahedron body. If the values of the current parameters go beyond these limits, then we get a set of points outside the tetrahedron.

In a similar way, you can determine the point equations of pyramidal bodies with any polygons at the base.

4.2. Point determination of a solid model of a triangular prism

As a second example, consider the point definition of a triangular prism (Figure 2).

Similarly, to the definition of the body of a tetrahedron, to define the body of a triangular prism, we choose a simplex of the 3-dimensional space \( ABCD \) and define the current point \( N \) in the plane \( ABC \) using the point equation (1). Next, we determine the points \( Q \) and \( R \) using the parallel transfer rule in point calculus.
\[
Q = D + C - A;
R = N + Q - C.
\]
(4)

Substituting the point equation (1) in equation (4), we obtain the equation of the point \( R \):
\[
R = A(\bar{w}v - 1) + B\bar{v} + C\bar{u}\bar{v} + D.
\]
(5)
Next, we use the point equation of a straight-line segment similar to equation (2) and after some transformations we obtain the exact equation of the body of a triangular prism:

\[ M = N\mathbf{w} + R\mathbf{w} = A(uv - w) + B\mathbf{v} + C\mathbf{u} + Dw. \]  

(6)

**Figure 2.** Geometric scheme of point determination of the triangular prism body

In a similar way, you can determine the point equations of prismatic bodies with any polygons at the base.

The resulting point equation (6), similar to point equation (3), can be represented by a system of 3 similar parametric equations:

\[
\begin{align*}
  x &= x_A (uv - w) + x_B \mathbf{v} + x_C \mathbf{u} + x_D w \\
  y &= y_A (uv - w) + y_B \mathbf{v} + y_C \mathbf{u} + y_D w \\
  z &= z_A (uv - w) + z_B \mathbf{v} + z_C \mathbf{u} + z_D w
\end{align*}
\]

(7)

5. Modeling of cylindrical and conical bodies in point calculus

The construction of cylindrical and conical bodies is carried out using guide lines, which can have a wide variety of shapes. However, it is important to note here that if cylindrical and conical bodies are not constrained to cut planes, the original guide lines must be closed. The simplest of the closed lines are circles. But, given that a circle is a special case of an ellipse, consider, as an example, modeling cylindrical and conical bodies based on an elliptical guide line.

5.1. Point definition of a solid model of an elliptical cone

We define a solid model of an elliptic cone in a simplex of 3-dimensional space $ABCD$.

In accordance with the geometric scheme presented in Figure 3, the inner part of the cone is filled with points due to the rotation of the $CDN$ plane around the axis $CD$. This plane is a simplex of 2-dimensional space, which is filled with points with any predetermined density, using the point equation of the plane, similar to the equation (1).

\[ M = (D - C)uv + (N - C)\mathbf{v} + C. \]

(7)

The movement of the $CDN$ plane is carried out at the expense of the current point $N$, which moves along an elliptical trajectory due to the current parameter $\phi$ by the following point equation [6]:

\[ N = (A - C)\cos \phi + (B - C)\sin \phi + C, \]

(8)
where \( \varphi \in [0; 2\pi] \) – the compression (stretch) angle that defines the current point of the ellipse \( N \) when traversing the curve.

![Figure 3. Geometric scheme for modeling an elliptical solid cone](image1)

Substituting the point equation (7) into (8), we obtain the desired equation of the solid model of an elliptic cone in the form of a 3-parameter set, one of the parameters of which is angular:

\[
M = (A - C)\overline{v}\cos \varphi + (B - C)\overline{v}\sin \varphi + (D - C)\overline{\mu}\nu + C. \tag{9}
\]

Point equation (9) is, in a sense, universal for determining solid models of cones. Those, one equation describes the whole variety of solid models of elliptic and circular, straight and inclined cones, in order to select, which it is enough just to set the coordinates of the points of a simplex in 3-dimensional space \( ABCD \). It should also be noted that you can use other curves as guide lines for a tapered body in a similar way. In this case, the trajectory of motion of the movable triangle \( CDN \) can be specified not only by trigonometric, but also by any other continuous functions that determine both algebraic and transcendental curves.

5.2. Point definition of a solid model of an elliptical cylinder

By analogy with a cone, we define a solid model of an elliptic cylinder in a simplex of 3-dimensional space \( ABCD \), where a guiding elliptic curve is specified in the plane \( ABC \), and the segment \( CD \) is the axis of the cylinder (Figure 4).

![Figure 4. Geometric scheme for modeling an elliptical solid cylinder](image2)
In this case, the current point \( N \) is determined by the point equation (8) using the angular parameter \( \varphi \in [0; 2\pi] \). We define the current point \( P \) using the parameter \( u \):

\[
P = C\bar{u} + Nu = (A - C)u \cos \varphi + (B - C)u \sin \varphi + C. \tag{10}
\]

We define the point \( Q \) using parallel transfer, by analogy with (4):

\[
Q = D + P - C = D + (A - C)u \cos \varphi + (B - C)u \sin \varphi. \tag{11}
\]

Next, we define the current point \( M \) using the parameter \( v \) using equations (10) and (11):

\[
M = P\bar{v} + Qv = Au \cos \varphi + Bu \sin \varphi + C(\bar{v} - u(\cos \varphi + \sin \varphi)) + Dv. \tag{12}
\]

The principles of construction and the final point equation for circular bodies will be similar to elliptical ones. Only the coordinates of points \( A, B \) and \( C \) should be taken in such a way that \( |CA| = |CB| \).

6. Conclusion

In conclusion, it should be noted the need for further research and development of the proposed approach to solid modelling, which can take its rightful place among classical and innovative methods of computer graphics and scientific visualization. This is supported by the fact that any geometrical body, using points and parameters, can be written in point calculus in the form of just single point equation, which significantly reduces any computational algorithms with its use. The prospect of further research is modelling in point calculus of bodies with a more complex geometric shape, as well as point determination of the mutual position of geometric bodies, which is the theoretical basis for logical operations of solid modelling.

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