Some Continuation results in Uniquely Geodesic Spaces

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Abstract. In this paper, we prove some continuation results based on essential maps in $\Gamma$–uniquely geodesic spaces, which are generalizations of the existing results in Banach spaces. Also, we obtain Leray Schauder principle in $\Gamma$–uniquely geodesic spaces which generalizes Leray Schauder principle in the setting of Hyperbolic spaces.

1. Introduction
This paper is based on the notion of essential maps defined by Granas [13] in 1976. Many studies and extensions of this concept in variety of settings have been done by several authors [2, 12, 26, 25, 27]. Essential map techniques are one of the best tools to prove continuation results for compact maps [1].

Compact maps play a vital role in proving existence and uniqueness of solutions of differential and integral equations. The famous Schauder fixed point theorem proved by Juliusz Schauder in 1930 in the setting of Banach spaces as a generalization of the celebrated Brouwer's fixed point theorem and the Leray-Schauder principle are milestones in the theory of fixed points and are based on compact maps. For many years, researchers were trying to extend the concepts in normed spaces to more general spaces. In that direction, several authors have extended the famous Schauder fixed point theorem to various general spaces [3, 23] and the geodesic spaces grasped the attention of many fixed point theorists after the publication of the papers [16, 17] due to Kirk. A version of Schauder fixed point theorem has been proved by Niculescu and Roventa in $CAT(0)$ spaces [23] by assuming the compactness of convex hull of finite number of elements. Followed by this, in [3], Ariza, Li and Lopez have proved Schauder Fixed point theorem in the setting of $\Gamma$–uniquely geodesic spaces, which includes Busemann spaces, Linear spaces, $CAT(\kappa)$ spaces with diameter less than $D_k/2$, Hyperbolic spaces [28] etc.

The Leray Schauder principle was first proved for compact mappings in Banach spaces and this has been broadly used to obtain fixed points of variety of mappings under different settings and many interesting contributions can be found in the literature [22, 4, 8, 10, 21].

Granas introduced essential maps in order to prove continuation results for compact maps in Banach spaces [13]. In this paper he has proved topological transversality principle and the Leray Schauder principle. Apart from this, he has also proved several fixed point results for compact fields using essential mappings and homotopical methods. Followed by the results established by Granas, Agarwal and O'Regan [2] have extended the concept of essential maps to a large class of mappings and established several fixed point theorems in 2000. The concept...
of essential maps has been further extended to $d-$essential maps and $d-L$-essential maps \cite{27}.

In this paper we prove some continuation results in $\Gamma-$uniquely geodesic spaces using essential maps and generalize the Leray Schauder Principle in hyperbolic spaces \cite{3} to $\Gamma-$uniquely geodesic spaces.

2. Preliminaries

We start with some basic notions.

**Definition 2.1.** \cite{3} Let $\phi$ be a metric space. A geodesic segment (or geodesic) from $x$ to $y$ in $X$ is a map $\gamma$ from a closed interval $[0, l] \subseteq \mathbb{R}$ to $X$ such that $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2$ in $[0, l]$. $(X, d)$ is said to be a geodesic space if every two points in $X$ are joined by a geodesic and $(X, d)$ is uniquely geodesic if there is exactly one geodesic joining $x$ to $y$, for all $x, y \in X$.

We denote the set of all geodesic segments in $X$ by $\Omega$ and for $A \subseteq X$, closure and boundary of $A$ in $X$ is denoted by $\overline{A}$ and $\partial A$ respectively.

**Definition 2.2.** \cite{3} Let $(X, d)$ be a metric space and $\Gamma \subseteq \Omega$ be a family of geodesic segments. We say that $(X, d)$ is a $\Gamma-$uniquely geodesic space if for every $x, y \in X$, there exists a unique geodesic in $\Gamma$ passes through $x$ and $y$. We will denote unique geodesic segment in $\Gamma$ joining $x$ and $y$ by $\gamma_{x,y}$.

**Remark 2.1.** \cite{3} Let $(X, d)$ be a $\Gamma-$uniquely geodesic space. Then, the family $\Gamma$ induces a unique mapping $\bigoplus_{\Gamma} : X^2 \times [0, 1] \to X$ such that $\bigoplus_{\Gamma}(x, y, t) \in \gamma_{x,y}$ and the following properties hold for each $x, y \in X$:

(i) $d(\bigoplus_{\Gamma}(x, y, t), \bigoplus_{\Gamma}(x, y, s)) = |t - s|d(x, y)$ for all $t, s \in [0, 1]$

(ii) $\bigoplus_{\Gamma}(x, y, 0) = x$ and $\bigoplus_{\Gamma}(x, y, 1) = y$

Also, it is enough to consider $\bigoplus_{\Gamma}(x, y, t) = \gamma_{x,y}(td(x, y))$, i.e., it is a point on $\gamma$ at a distance $td(x, y)$ from $x$ and we denote it by $(1 - t)x \oplus ty$

**Definition 2.3.** \cite{1} Let $X$ and $Y$ be two metric spaces. A map $F : X \to Y$ is called compact if $F(X)$ is contained in a compact subset of $Y$.

**Definition 2.4.** \cite{1} Let $C$ be a closed convex subset of a Banach space $X$ and $U$ be an open subset of $C$. Denote set of all continuous, compact maps $\phi : \overline{U} \to C$ by $\mathcal{K}(\overline{U}, C)$ and set of all maps $\phi \in \mathcal{K}(\overline{U}, D)$ with $x \neq \phi(x)$ by $\mathcal{K}_{\partial U}(\overline{U}, C)$ for $x \in \partial U$.

A map $\phi \in \mathcal{K}_{\partial U}(\overline{U}, C)$ is essential in $\mathcal{K}_{\partial U}(\overline{U}, C)$ if for every map $\psi \in \mathcal{K}_{\partial U}(\overline{U}, C)$ with $\phi|_{\partial U} = \psi|_{\partial U}$ there exists $x \in U$ with $x = \psi(x)$. Otherwise $\phi$ is inessential in $\mathcal{K}_{\partial U}(\overline{U}, C)$, that is, there exists a fixed point free $\psi \in \mathcal{K}_{\partial U}(\overline{U}, C)$ with $\phi|_{\partial U} = \psi|_{\partial U}$.

**Definition 2.5.** \cite{1} Two maps $\phi, \psi \in \mathcal{K}_{\partial U}(\overline{U}, C)$ are homotopic in $\mathcal{K}_{\partial U}(\overline{U}, C)$, written $\phi \simeq \psi$ in $\mathcal{K}_{\partial U}(\overline{U}, C)$, if there exists a continuous, compact mapping $H : \overline{U} \times [0, 1] \to C$ such that $H_0(x) := H(\cdot, 0) : \overline{U} \to C$ belongs to $\mathcal{K}_{\partial U}(\overline{U}, C)$ for each $t \in [0, 1]$ with $H_0 = \phi$ and $H_1 = \psi$.

**Definition 2.6.** \cite{1} A $\Gamma-$uniquely geodesic space is said to have property $(P)$ if

$$\limsup_{\varepsilon \to 0} \{ d((1 - t)x \oplus ty, (1 - t)x \oplus tz) : t \in [0, 1], x, y, z \in X, d(y, z) \leq \varepsilon \} = 0$$

**Definition 2.7.** \cite{3} Let $C$ be a nonempty subset of a $\Gamma-$uniquely geodesic space. We say that $C$ is $\Gamma-$convex if $\gamma_{x,y} \subseteq C$ for all $x, y \in C$.

**Remark 2.2.** \cite{3} Let $(X, \|\cdot\|)$ be a normed linear space, $\Gamma_L$ the family of linear segments and let $d$ denote the metric induced by the norm $\|\cdot\|$. Then $(X, d)$ is a $\Gamma_L$-uniquely geodesic space with property $P$. 

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Definition 2.8. [3] A metric space $(X, d)$ is a hyperbolic space (in the sense of Reich-Shafir [25]) if $X$ is $\Gamma$– uniquely geodesic and the following inequality holds

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \leq \frac{1}{2}d(y, z).$$

Various extensions and restrictions of Schauder’s fixed point theorem have been done by many authors [14, 29]. A Schauder type fixed point theorem in the setting of geodesic spaces having the property (P) is proved in [3] which is stated below. We use this theorem for our work in this paper.

Theorem 2.1. [3] Let $(X, d)$ be a $\Gamma$–geodesic space with property (P) and all balls are $\Gamma$-convex. Let $K$ be a nonempty, closed, $\Gamma$–convex subset of $(X, d)$. Then, any continuous mapping $T : K \rightarrow K$ with compact range $T(K)$ has at least one fixed point in $K$.

Throughout this paper, we denote a closed $\Gamma$–convex subset of the geodesic space $X$ by $D$ and interior of $C$ by $\text{int} C$, where $C$ is a closed subset of $D$. We denote the set of all continuous, compact maps $\phi : C \rightarrow D$ by $\mathcal{X}(C, D)$ and set of all maps $\phi \in \mathcal{X}(C, D)$ with $x \neq \phi(x)$ by $\mathcal{X}_{\partial C}(C, D)$ for $x \in \partial C$.

3. Main Results

This section deals with main theorems and the corresponding corollaries.

Theorem 3.1. Let $(X, d)$ be a $\Gamma$–uniquely geodesic space satisfying property (P) and $\Phi, \Psi \in \mathcal{X}_{\partial C}(C, D)$. Suppose that for all $(x, t) \in \partial C \times [0, 1]$,

$$x \neq (1 - t)\Phi(x) \oplus t\Psi(x)$$

i.e., geodesic segment joining $\Phi(x)$ and $\Psi(x)$ does not contain $x$. Then $\Phi \simeq \Psi$ in $\mathcal{X}_{\partial C}(C, D)$.

Proof Let $H(x, t) = \bigoplus_t (\Phi(x), \Psi(x), t) = \gamma_{\Phi(x), \Psi(x)}(td(\Phi(x), \Psi(x)))$. Clearly $H$ is continuous. We show that $H : C \times [0, 1] \rightarrow D$ is a compact map.

Let $\{x_n\}$ be a sequence in $C$. Since $\Phi, \Psi : C \rightarrow D$ are compact maps, $\Phi(x_n) \rightarrow u$ and $\Psi(x_n) \rightarrow v$ as $n \rightarrow \infty$ for some subsequence $S$ of natural numbers and $u, v \in D$. Let $t \in [0, 1]$ be such that $t$ is the limit of some sequence $t_n \in [0, 1]$.

Now using Remark 2.1

$$d(H(x_n, t_n), (1 - t)u \oplus tv) \leq |t_n - t|d(\Phi(x_n), \Psi(x_n)) + d((1 - t)\Phi(x_n) \oplus t\Psi(x_n), (1 - t)\Phi(x_n) \oplus tv) + d((1 - t)\Phi(x_n) \oplus tv, (1 - t)u \oplus tv)$$

Now by using property (P) and continuity of $\Phi$ and $\Psi$, it is easy to show that $H(x_n, t_n) \rightarrow (1 - t)u \oplus tv$ for $t_n \in [0, 1]$. Since $D$ is $\Gamma$–convex, $(1 - t)u \oplus tv \in D$. Hence $H$ is compact.

But it is given that $x \neq (1 - t)\Phi(x) \oplus t\Psi(x)$ for $(x, t) \in \partial C \times [0, 1]$. Hence $H_t(x) \in \mathcal{X}_{\partial C}(C, D)$, $H(x, 0) = \Phi(x)$ and $H(x, 1) = \Psi(x)$. Therefore $\Phi \simeq \Psi$ in $\mathcal{X}_{\partial C}(C, D)$.

Following result proved in [1] is a corollary to our theorem.

Corollary 3.1. Let $X$ be a Banach space, $C$ a closed, convex subset of $X$, $U$ an open subset of $C$ and $\Phi, \Psi \in \mathcal{X}_{\partial U}(U, C)$. Suppose that for all $(u, \lambda) \in \partial U \times [0, 1],

$$u \neq (1 - \lambda)\Phi(u) + \lambda\Psi(u)$$

Then $\Phi \simeq \Psi$ in $\mathcal{X}_{\partial U}(U, C)$. 

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Suppose Φ is inessential in Proof essential in Φ ∈ Let Theorem 3.3. follows from Theorem 3.2. □

Proof We have H M Define ζ with K ξ with M H K ζ ζ ξ = Φ with Φ |∂C = Ψ |∂C. Hence the inessentiality of Φ in K x. Therefore Φ has a fixed point, which is a contradiction. Thus x ≠ ζ(x). Now it follows from above facts that Φ is inessential in K x C,D. Hence the proof.

As a consequence of Theorem 3.2 we obtain the following corollary.

Corollary 3.2. Let X be a Hyperbolic space (in the sense of [28]), C a closed, convex subset of X, U an open subset of C and F ∈ K∂U(U, C). Then the following are equivalent:

(i) Φ is inessential in K∂U(U, C).
(ii) There exists Ψ ∈ K∂U(U, C) with Ψ(x) ≠ x for all x ∈ U and Φ ≃ Ψ in K∂U(U, C).

Proof Every Hyperbolic space is a Γ-uniquely geodesic space with property P. Hence the result follows from Theorem 3.2. □

Theorem 3.3. Let (X, d) be a Γ-uniquely geodesic space satisfying property (P) and let Φ ∈ K∂C(C, D). Suppose that Φ, Ψ ∈ K∂C(C, D) with Φ ≃ Ψ in K∂C(C, D). Then Φ is essential in K∂C(C, D) if and only if Ψ is essential in K∂C(C, D).

Proof Suppose Φ is inessential in K∂C(C, D). Then from Theorem 3.2 there exists T ∈ K∂C(C, D) with Φ ≃ T in K∂C(C, D) such that T(x) ≠ x for all x ∈ C. Hence, Ψ ≃ T in K∂C(C, D). Therefore by Theorem 3.2 Ψ is inessential in K∂C(C, D). Hence the proof. □
Corollary 3.3. Let $X$ be a Hyperbolic space (in the sense of [28]), $C$ a closed, convex subset of $X$, $U$ an open subset of $C$ and $\Phi, \Psi \in \mathcal{KH}(\mathcal{F}, C)$ with $\Phi \simeq \Psi$ in $\mathcal{KH}(\mathcal{F}, C)$. Then $\Phi$ is essential in $\mathcal{KH}(\mathcal{F}, C)$ if and only if $\Psi$ is essential in $\mathcal{KH}(\mathcal{F}, C)$.

Above corollary is a special case of Theorem 3.3.

Theorem 3.4. Let $(X, d)$ be a $\Gamma$-uniquely geodesic space with all balls are $\Gamma$-convex and satisfies property (P). Let $u \in \text{int } C$. Then the map $\Phi(C) = u$ is essential in $\mathcal{KH}(\mathcal{F}, C, D)$.

Proof Consider the continuous compact map $\Psi : C \to D$ which agrees with $\Phi$ on $\partial C$. It is enough to show that $\Psi(x) = x$ for some $x \in \text{int } C$. Let $\zeta : D \to D$ given by

$$\zeta(x) = \begin{cases} 
\Psi(x), & x \in C; \\
u, & x \in D \setminus C.
\end{cases}$$

Then $\zeta$ is continuous and compact. Thus by Theorem 2.1, $\zeta(x) = x$ for some $x \in D$. Clearly $x \in \text{int } C$, since $u \in \text{int } C$. Hence $x$ is a fixed point of $\Psi$. Therefore $\Phi$ is essential in $\mathcal{KH}(\mathcal{F}, C, D)$.

Following result proved in [1] is a corollary to our theorem.

Corollary 3.4. Let $X$ be a Banach space, $C$ a closed, convex subset of $X$, $U$ an open subset of $C$ and $u \in U$. Then the constant map $\Phi(C) = u$ is essential in $\mathcal{KH}(\mathcal{F}, C)$.

Proof We know that all balls in Banach spaces are convex. Hence the result follows from theorem 3.3.

As a consequence of above theorems, we obtain the Leray Schauder principle in $\Gamma$–uniquely geodesic spaces which generalizes the Leray Schauder principle in Hyperbolic spaces proved in [3].

Theorem 3.5. Let $(X, d)$ be a $\Gamma$-uniquely geodesic space satisfying property (P) and all balls are $\Gamma$–convex. Suppose that $\Phi : C \to D$ is a continuous compact map. Then either

(i) $\Phi$ has a fixed point in $C$, or
(ii) There exists $(x_0, \lambda) \in \partial C \times (0, 1)$ such that $x_0 = (1 - \lambda)u \oplus \lambda \Phi(x_0)$.

Proof Suppose that (ii) does not hold and $\Phi(x) \neq x$ for all $x \in \partial C$. Define $\Psi : C \to D$ by $\Psi(x) = u$ for all $x \in C$. Consider the map $H : C \times [0, 1] \to D$ defined by

$$H(x, t) := (1 - t)u \oplus t\Phi(x),$$

which is continuous and compact. Also, for all $x \in \partial C$, $H_t(x) \neq x$ for a fixed $t$. Then by Theorem 3.3 and Theorem 3.4, $\Phi$ is essential in $\mathcal{KH}(\mathcal{F}, C, D)$ and hence $\Phi(x) = x$ for some $x \in \text{int } C$.

Following corollary is a consequence of Theorem 3.5. Which is proved in [3].

Corollary 3.5. Let $X$ be a Hyperbolic space (in the sense of [28]), $x_0 \in X$ and $r > 0$. Suppose that $T : B[x_0, r] \to X$ be a continuous mapping with $T[B[x_0, r]]$ compact. Then either

(i) $T$ has at least one fixed point in $B[x_0, r]$, or
(ii) There exists $(x, \lambda) \in \partial B[x_0, r] \times (0, 1)$ with $x = (1 - \lambda)x_0 \oplus \lambda T(x)$.

Proof Hyperbolic spaces are geodesic spaces with property (P) and balls are $\Gamma$-convex. Hence the result follows from above theorem.
Conclusion
In this paper, we have proved some continuation results in \( \Gamma \)-uniquely geodesic spaces. We have also generalized the Leray-Schauder principle established in Hyperbolic spaces established by Ariza, Li and Lopez [3] to \( \Gamma \)-uniquely geodesic spaces. Busemann spaces, Linear spaces, \( CAT(\kappa) \) spaces with diameter less than \( D_k/2 \), Hyperbolic spaces (in the sense of [28]) etc. are geodesic spaces with property (P) and balls are \( \Gamma \)-convex. Therefore, all the results in this paper holds true in these spaces too. One can try to establish similar results for multivalued mappings in geodesic spaces and prove similar results for \( d \)-essential maps, \( d \)-L essential maps and other general class of maps [2][27].

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