Does the Streaming Instability Exist within the Terminal Velocity Approximation?

V. V. Zhuravlev

Sternberg Astronomical Institute, Lomonosov Moscow State University, Universitetskij pr., 13, Moscow 119234, Russia

Received 2021 December 1; revised 2022 November 7; accepted 2022 November 7; published 2022 December 13

Abstract

Terminal velocity approximation is appropriate to study the dynamics of a gas–dust mixture with solids tightly coupled to the gas. This work reconsiders its compatibility with physical processes giving rise to the resonant streaming instability in the low-dust-density limit. It is shown that the linearized equations that have been commonly used to study the streaming instability within the terminal velocity approximation actually exceed the accuracy of this approximation. For that reason, the corresponding dispersion equation recovers the long-wavelength branch of the resonant streaming instability caused by the stationary azimuthal drift of the dust. However, the latter must remain beyond the terminal velocity approximation by its physical definition. The refined equations for gas–dust dynamics in the terminal velocity approximation does not lead to the resonant streaming instability. The work additionally elucidates the physical processes responsible for the instability.

Unified Astronomy Thesaurus concepts: Protoplanetary disks (1300)

1. Introduction

Solid particles accumulated in the midplane of protoplanetary disk should drift rapidly toward the host star. Immersed in a rotating gas, this flow serves a source of free energy for the local growth of dust overdensities, which may further become the sites for planetesimal formation. Study of the corresponding linear instability was initiated by Youdin & Goodman (2005), who realized that the mutual coupling of solids with gas by the drag force may lead to amplification of small gas–dust perturbations, a process generally similar to the two-stream plasma instability known since the pioneering works of Haeff (1948) and Pierce (1948). By that analogy, Youdin & Goodman (2005) termed the new gas–dust instability as the streaming instability (hereafter SI). They showed that the free parameters determining SI are the stopping time of the particles expressed in units of the Keplerian time (the Stokes number) and the dust-to-gas density ratio. The numerical analysis of SI has revealed a rather complex pattern of its growth rate. Further analytical developments, including the most recent ones, have identified four branches of SI.

At first, Youdin & Goodman (2005) and Jacquet et al. (2011) found that the secular mode associated with the advection of the dust density perturbations grows when accounting for the dust backreaction on gas, which enhances the pressure perturbation maxima serving as dust traps. These works employed the so-called terminal velocity approximation (hereafter TVA), TVA naturally arises while considering the gas–dust dynamics in terms of the center-of-mass velocity of the gas–dust mixture and the relative velocity between the dust and the gas. According to Youdin & Goodman (2005), TVA assumes the marginal balance between the drag force and the gas pressure force, which both enter the equation for the relative velocity of the mixture. Additionally, TVA neglects the small contribution of the relative velocity to the equation for the center-of-mass velocity of the mixture. TVA is useful for the dynamics of a well-coupled mixture with a small Stokes number. Laibe & Price (2014) generalized the formulation of the gas–dust dynamics in terms of the center-of-mass velocity and relative velocity of the mixture and showed that it is optimal for use in numerical simulations. This is especially so for the well-coupled mixture, when the relative motion of two fluids is described within TVA and mathematically determined by a diffusion-like term in the equation for the dust-to-gas ratio evolution. This approach made it possible to consider the dusty gas with partially coupled solids as a single fluid with heating and cooling, and further the growth of the secular mode of SI as its overstability (see Lin & Youdin 2017). The one-fluid model, as formulated within TVA, has been employed in numerical studies of gas–dust dynamics with small solids embedded in protoplanetary disks (Lin 2019; Lovascio & Paardekooper 2019), as well as to consider some extensions of SI (see Chen & Lin 2020; Paardekooper et al. 2020; Lin 2021).

Another branch of SI is represented by the inertial wave of the gas, which also grows when accounting for the dust backreaction on gas. For that, one should retain the higher-order terms over the Stokes number, which are beyond TVA. This was shown by Jaupart & Laibe (2020), who additionally found that the terms standing beyond TVA surprisingly contribute to the growth rate of the secular mode previously considered only within TVA (see also the recent findings of Pan 2021, hereafter P21, on this issue).

The regime of strong nonlinear clumping triggered by SI has been revealed for the first time in the unstratified local simulations of Johansen & Youdin (2007). It was shown that for solids corresponding to Stokes numbers less than unity, which better meet the fragmentation barrier, the strong clumping occurs only for dust-to-gas density ratios significantly higher than unity. This feature of the nonlinear SI has manifested itself in more realistic stratified models, which included the dust settling (see, e.g., Johansen et al. 2009; Bai & Stone 2010; Carrera et al. 2015; Yang et al. 2017; Li & Youdin 2021; and Lesur et al. 2022, for a more detailed review). The analytical study of the branch of the linear SI associated with the high dust-to-gas density ratio started from Squire & Hopkins (2018) and was continued by Squire &
Hopkins (2020) and Pan (2020). It was shown that, contrary to the previously known analytic branches of SI, this one is provided by standing perturbations with a growth rate independent of the Stokes number.

Along with that, the analytic breakthrough occurred in the opposite limit of a low dust fraction. Squire & Hopkins (2018) identified the fastest-growing modes of SI on scales where the velocity of the dust radial drift matches the phase velocity of inertial waves propagating in gas. This led to understanding that, in the limit of a low dust fraction, the most prominent SI is provided by a linear resonance between the inertial wave and the dust wave, which is represented by the advection of the dust density perturbations (see Zhuravlev 2019; hereafter Z19). At the same time, it became clear that the resonant mechanism leading to instability must be subtle. In the case of a low dust fraction, Squire & Hopkins (2020) found that the force caused by the radial drift of the dust density perturbations is strictly out of phase with the oscillations of the gas as considered to leading order in a small Stokes number. The adjustment of the distribution of dust necessary to recover the instability comes from gas–dust dynamics of the next order in the Stokes number. This result appeared to be in accordance with the analysis of Z19, who showed that the mode coupling giving rise to the resonant drag instability associated with the settling of dust is absent when dust drifts only radially in a disk. As Z19 studied the mode coupling in the framework of TVA, he concluded that SI must be provided by some other resonant mechanism related to the inertia of solids, which goes beyond TVA. However, P21 claimed that the general dispersion equation constructed from the commonly known linearized two-fluid equations, also derived within TVA, do recover the resonant SI in the limit of a low dust fraction as well as the small vertical wavenumbers. The latter is in evident contradiction with the conclusions of Z19.

The purpose of this study is to resolve the discrepancy between Z19 and P21 in order to avoid possible further misunderstanding of the physical meaning of TVA. I argue that the claim of P21 is formal because it has no physical grounds. The limit of the small vertical wavenumbers considered by P21 singles out the long-wavelength branch of the resonant SI. However, this branch of the resonant SI is caused by a slow stationary azimuthal drift of the dust, which itself is an inertial effect to be neglected in TVA. In order to demonstrate this, I revisit the reduced dispersion equation, which is valid in the vicinity of resonance between the inertial wave and the dust wave in the limit of a low dust fraction (see Section 4.2 of Z19). Its derivation is intentionally performed using two different sets of state variables, in each case, at first within TVA and after that beyond TVA. I closely follow the lines of reasoning employed in Z19 while considering the resonant gas–dust dynamics leading to SI. The main conclusion of this study is drawn comparing the four cases mentioned above with each other. Finally, I suggest the refined set of general two-fluid equations, which should be used in order to study gas–dust dynamics in TVA. These equations are used to perform an additional analysis of the general dispersion equation obtained for an arbitrary value of the dust fraction. This analysis is relegated to the Appendix. It is shown that there is no more growing secular mode in the revised TVA. In contrast, the dust-rich branch of SI, which is growing for dust-to-gas ratios larger than unity, persists in the revised TVA.

In Sections 2–4 and in the Appendix, the gas–dust flow is assumed to be axisymmetric. Throughout this text, I use the original definitions and notations of Z19.

2. Resonance between Modes

Following Z19, I consider the gas–dust dynamics in the vicinity of resonance between the streaming dust wave (SDW) and the inertial wave (IW).7 IW is a wave that propagates through the interior of a rotating gas due to the presence of a Coriolis force (see, e.g., Landau & Lifshitz 1987). SDW is a trivial mode produced by the stationary drift of the dust transporting perturbations of the dust density.

In this case, perturbations taken in the form of monochromatic waves $\propto \exp(-i\omega t + ik_x x + ik_z z)$ obey the following dispersion equation

$$D_g(\omega, k) \cdot D_p(\omega, k) = \epsilon_m(\omega, k),$$

where

$$D_g(\omega, k) \equiv \omega^2 - \omega_i^2$$

independently describes IW, while

$$D_p(\omega, k) \equiv \omega - \omega_p$$

independently describes SDW with the definitions

$$\omega_p \equiv -g_s t_i k_z,$$

$$\omega_i \equiv \frac{k_z}{k},$$

with $k^2 \equiv k_x^2 + k_z^2$. Variables $\kappa$, $t_i$, and $g_s$ introduce, respectively, the local epicyclic frequency in a disk, the stopping time of the particles, and the absolute value of effective gravity along the radial direction (see Z19 for the details).

Consideration of the dynamical dust backreaction on gas leads to the appearance of the nonzero coupling term in the right-hand side (RHS) of Equation (1). As long as the dust fraction $f \ll 1$, the coupling term is small, which means that significant changes in the solution of the dispersion relation occur only close to the resonance between SDW and IW. Below, the subscript after $\epsilon$ introduces a sequence number of the model considered in this work. At the same time, it can be checked that for all models the left-hand side (LHS) of Equation (1), which can be taken to the zeroth order in $f$ near the resonance, remains the same.

The resonance between SDW and IW is determined by the condition of the mode crossing,

$$\omega_p = -\omega_i \equiv \omega_\pi,$$

and it is assumed for simplicity that $k_{x,\pi} > 0$ hereafter. All other cases can be considered in a similar way.

According to Equation (4), the length scale of mode crossing is determined by the relation

$$\frac{k_z}{k_x} = \frac{\tilde{k}}{k},$$

where

$$\tilde{k} \equiv \frac{\kappa}{g_s t_i}$$

Another representative example of such an analysis can be found in Zhuravlev (2021).
is the characteristic wavenumber of resonance between SDW and IW and thus the typical wavenumber associated with resonant SI.

The corresponding variation in the frequencies of SDW and IW at the mode crossing is estimated in the following way (see the details in Z19),

$$\Delta_m = \pm \left( \frac{\epsilon_m \omega_c}{\partial_x D \mathbf{D}_{\mathbf{m}} \cdot \partial_x D \rho_{\mathbf{m}}} \right)^{1/2} = \pm \left( \frac{\epsilon_m \omega_c}{2 \omega_c} \right)^{1/2}. \quad (7)$$

As soon as \(\Delta\) has a positive imaginary part, the corresponding model gives rise to SI.

### 3. Problem Considered with Respect to Perturbations of Gas Velocity

#### 3.1. General Equations

The local dynamics of the gas–dust mixture in a disk on scales smaller than the disk scale height can be described in the shearing sheet approximation (see Squire & Hopkins 2018 and Z19). The equation for the velocity of the gas including the dust aerodynamical backreaction on the gas reads

$$\partial_t U_g - 2\Omega_g U_g e_x + \frac{\kappa^2}{2\Omega_0} U_g e_y + \nabla \cdot (\rho_p V) = 0, \quad (8)$$

and it is assumed that the gas density, \(\rho_g\), is constant. Hence, the gas velocity is strictly free of divergence,

$$\nabla \cdot U_g = 0. \quad (9)$$

On the other side, the behavior of dust can be described by the evolution of its density distribution using the continuity equation for the gas–dust mixture,

$$\partial_t \rho_p + \nabla \cdot (\rho U) = 0, \quad (10)$$

where, by definition,

$$U = U_g + \frac{\rho_p}{\rho} V \quad (11)$$

is the center-of-mass velocity of the gas–dust mixture, and \(\rho \equiv \rho_g + \rho_p\) is the total density of mixture. The closing equation for the relative velocity of gas and dust, \(V \equiv U_p - U_g\), reads

$$\partial_t V - 2\Omega_g V e_x + \frac{\kappa^2}{2\Omega_0} V e_y + (U \nabla) V + (V \nabla) U + \frac{\rho_p}{\rho} (\rho_p V) \left( \frac{\rho_p}{\rho} V \right) - \frac{\rho_p}{\rho} (V \nabla) \left( \frac{\rho_p}{\rho} V \right) = \nabla (\rho + \rho_p) - \frac{\rho_p V}{\rho_g t_s}. \quad (12)$$

Equation (12) follows from the combination of independent equations for the velocities of the gas and dust and fully describes the local relative motion of the gas and dust in a disk (see Section 2 of Z19).

Below in this section, Equations (8)–(10), and (12) along with the definition in Equation (11) are used to derive the set of linear equations that represent the dynamics of small gas–dust perturbations in the vicinity of resonance between SDW and IW provided that the dust suffers a stationary radial drift.

Everywhere below it is assumed that

$$f \equiv \frac{\rho_p}{\rho_g} \ll 1 \quad (13)$$

as well as the Stokes number

$$\tau \equiv \Omega_0 t_s \ll 1. \quad (14)$$

#### 3.2. Stationary Drift of the Dust

##### 3.2.1. The Solution within TVA

According to Youdin \& Goodman (2005), who first used TVA, “this approximation, which ignores inertial accelerations,” “amounts to neglecting all terms on the left-hand side of Equation (here 12), both in equilibrium and in perturbation.” Assuming for simplicity

$$U_g = 0, \quad (15)$$

one obtains from Equations (8) and (12) with their LHS set to zero that, to leading order in small \(f\),

$$V = -t_s g_s e_x, \quad (16)$$

implying that \(\rho_p = \text{const}\). Solids drift exactly along the pressure gradient in a disk, which is the manifestation of TVA: “drag forces adjust quasi-statically to pressure forces” as stated by Youdin \& Goodman (2005).

##### 3.2.2. The Solution Beyond TVA

The stationary solution is extended up to the next order in \(t_s\) taking the leading inertial term on the LHS of Equation (12) into account, which is a weak relative Coriolis acceleration of the gas–dust mixture along the azimuthal direction caused by the radial drift of solids. To leading order in small \(f\), one obtains

$$U_g = f t_s g_s e_x, \quad (17)$$

whereas

$$V = -t_s g_s e_x + \frac{\kappa^2}{2\Omega_0} t_s^2 g_s e_y, \quad (18)$$

implying again that \(\rho_p = \text{const}\). Note that terms \(\sim t_s^3\) and smaller have been omitted in Equations (17)–(18). Clearly, the weak azimuthal drift of the dust found in the stationary solution goes beyond TVA, as soon as the corresponding drag force is adjusted by no pressure force.

#### 3.3. Linear Dynamics of Gas–Dust Perturbations Within TVA

The state variables, which determine the behavior of the corresponding gas–dust perturbations, are the Eulerian perturbations of the gas velocity, \(u_g\), enthalpy, \(W_g \equiv p'/\rho_g\), and relative density of dust, \(\delta \equiv \rho_p'/\rho_p\), where \(p', \rho_p', \text{ and } \rho_p\) are, respectively, the Eulerian perturbations of the pressure and density of the dust and the background density of the dust.

The equations for perturbations on the background (Equations (15–16)), which are valid in the vicinity of resonance between SDW and IW, read

$$\partial_t u_{g,x} - 2\Omega_0 u_{g,y} = -\partial_x W_g + f \frac{v_x}{t_s} - f g_x \delta, \quad (19)$$
where the terms \(\sim O(f)\) and smaller entering the equation for dust, Equation (23), have been omitted. The Eulerian perturbation of the relative velocity of the mixture, \(\mathbf{v}\), is taken from Equation (12) employing TVA to the zeroth order in \(f\). This is

\[
\mathbf{v}_i = t_i \partial_t \mathbf{W}_g,
\]

\[
\partial_t \mathbf{v}_i = \partial_t \mathbf{v}_g - \mathbf{f}_v.
\]

Further, it is appropriate to introduce the new variables for the incompressible motion of gas:

\[
\mathbf{w}_g \equiv -\partial_t u_{g,y}
\]

and

\[
\phi_g \equiv \partial_z u_{g,x},
\]

which, along with \(u_{g,z}\) and \(\phi\), obey the concise set of equations:

\[
\partial_t \phi_g = \partial_{\omega} u_{g,z} - 2 \Omega_0 \mathbf{w}_g - g_s \partial_z \delta,
\]

\[
\partial_t \mathbf{w}_g = \frac{\kappa^2}{2 \Omega_0} \phi_g,
\]

\[
\partial_{\omega} \mathbf{w}_g = -\frac{\kappa^2}{2 \Omega_0} \phi_g - \partial_z u_{g,z},
\]

\[
\partial_t \partial_z \delta = t_s g_s \partial_z \delta + 2 \tau \partial_z \mathbf{w}_g.
\]

In order to obtain Equations (26)–(28) one takes the curl of Equations (19)–(21). Equation (29) is obtained from Equations (23)–(25) after taking the divergence of Equations (19)–(21) to the zeroth order in \(f\). Note that in Equation (29) and elsewhere below the combination, \(\Omega_0 g_s\), is replaced by the Stokes number according to the definition in Equation (14).

Equations (26)–(29), written for modes of perturbations, yield the dispersion equation in the form given by Equation (1).

The coupling term emerges due to the nonzero product of the corresponding terms originating from the last term on the RHS of Equation (26) and the last term on the RHS of Equation (29):

\[
\epsilon_1 = -t_s \kappa^2 g_s k_x^2 k_z^2.
\]

According to Equation (7), the variation in the frequency at resonance between SDW and IW provided by \(\epsilon_1\) is real at \(\omega_c = \omega_p = -g_s f k_x^2 k_z^2\). Explicitly,

\[
\Delta_1 = \pm \kappa \left( \frac{f}{2} \right)^{1/2} \frac{k_z^2}{k_x^2}
\]

resulting in the absence of instability.

### 3.4. Linear Dynamics of Gas–Dust Perturbations Beyond TVA

In this case, Equations (19)–(22) for the gas change only due to the modification of the relative velocity of the mixture in the stationary solution. Indeed, revision of the LHS of Equation (12) shows that it cannot give any contributions \(\sim \delta\) to the zeroth order in \(f\), which could enter the RHS of Equations (19)–(21) via \(v\) over there. Hence, Equation (18) yields

\[
\partial_t u_{g,y} + \frac{\kappa^2}{2 \Omega_0} u_{g,x} = \frac{\kappa^2}{2 \Omega_0} t_s g_s \delta,
\]

instead of Equation (20). The new term on the RHS of this equation comes from the weak azimuthal drift of the dust.

The divergence of \(v\) on the RHS of Equation (23) should be expressed in terms of the gas variables. For that, Equations (24) and (25) are replaced by the augmented equations

\[
-2 \Omega_0 v_y + (V \nabla) u_{g,z} = \partial_z \mathbf{W}_g - \frac{v_y}{t_s},
\]

and

\[
(V \nabla) u_{g,z} = \partial_z \mathbf{W}_g - \frac{v_y}{t_s},
\]

respectively; check the LHS of Equation (12) to the zeroth order in \(f\). Note that the terms \(\partial \mathbf{v}\) and \((V \nabla) \mathbf{v}\) on the LHS of Equation (12), written for the perturbations, cancel each other near the resonance up to a small difference of \(\sim O(f)\), which is omitted here. The replacement \(\mathbf{u} \rightarrow \mathbf{u}_g\) in the advection terms on the LHS of Equations (33)–(34) has been done for the same reason.

Only the first term on the LHS of Equation (33) contributes to \(\nabla \cdot \mathbf{v}\) and thus to the contraction/rarefaction of dust. This term represents the perturbation of the relative Coriolis acceleration of a mixture due to the nonzero perturbation of the azimuthal relative velocity of the mixture. The emergence of the azimuthal relative velocity of the mixture makes one consider the dynamics of a mixture in the azimuthal direction.

It is important to note that within TVA the azimuthal motion of dust in the axisymmetric perturbed flow must be trivial: as soon as the pressure gradient has no projection onto the azimuthal direction, inertia-free solids perfectly follow the azimuthal motion of the gas, and \(v_y = 0\). However, the synchronicity of the gas and dust azimuthal motions is broken beyond TVA due to the azimuthal balance of forces coming in the next order over \(t_s\) (but still considered to the zeroth order in \(f\); see Equation (12)):

\[
t_s g_s \partial_z u_{g,y} = \frac{v_y}{t_s}.
\]

Physically,\(^3\) Equation (35) describes the azimuthal acceleration of solids by the gas drag as solids drift through the perturbed shear flow of the gas. This is a purely inertial effect, as the inertia-free particles would be instantly lifted up by the gas producing no relative velocity.

Equations (33)–(34) yield

\[
\nabla \cdot \mathbf{v} = t_s \nabla^2 \mathbf{W}_g + 2 \tau \partial_z v_y,
\]

where \(\nabla^2 \equiv \partial_{x_\omega} + \partial_{z_\omega}\). Thus, an additional compression/rarefaction of dust beyond TVA originates from the relative azimuthal motion between the gas and dust, which in turn, emerges due to the azimuthal acceleration of gas as seen for

---

\(^3\) It can be checked that the “Coriolis” term, \(\kappa^2/(2 \Omega_0) u_{g,x}\), standing also on the LHS of Equation (35) in the zeroth order in \(f\) is small compared with the term \((V \nabla) u_{g,y}\) over there.
solids penetrating the gas eddies in the course of their bulk radial drift (see Section 4 of Squire & Hopkins 2020). Equation (36) in combination with Equation (35) is used to obtain an augmented version of Equation (23).

The final equations read

$$\partial_t \phi_s = \partial_z u_{s,z} - 2\Omega_0 \omega_s - f g_s \partial_z \delta,$$  \hspace{1cm} (37)

$$\partial_t \omega_s = \frac{k^2}{2\Omega_0} \phi_s - f g_s \frac{k^2}{2\Omega_0} \partial_z \delta,$$  \hspace{1cm} (38)

$$\partial_t \omega_g = -\frac{k^2}{2\Omega_0} \partial_z u_{g,z} - f g_s \frac{k^2}{2\Omega_0} \partial_z \delta,$$  \hspace{1cm} (39)

$$\partial_t \delta = t_s g_s \partial_z \delta + 2\tau \partial_z \omega_s + 2\tau g_s t_s^2 \partial_z \omega_g.$$  \hspace{1cm} (40)

Equations (37)–(40) yield the generalized coupling term

$$\epsilon_2(\omega, k) \equiv -f \kappa^2 g_s k^2 \frac{k^2}{k^2} \left( 1 + i g_s k t_s^2 - i \omega t k^2 \frac{k^2}{k^2} \right)$$  \hspace{1cm} (41)

and the corresponding new correction to the frequency of mode crossing

$$\Delta_2 \approx \pm \kappa \frac{f}{2} \frac{k}{k} \left( 1 + i g_s k t_s^2 \frac{k^2}{k^2} \right),$$  \hspace{1cm} (42)

which fully recovers SI (see Equation (5.10) of Squire & Hopkins 2018) as well as Equation (109) of Z19.

It is important to note that the first and last terms after the imaginary unit come from the nonzero LHS of Equation (35) and the RHS of Equation (32), respectively. This means that generally resonant SI is provided by a combination of two effects:

(i) the spin-up of the gas eddies by the stationary radial drift of solids, undergoing a weak spatial redistribution due to their perturbed azimuthal motion relative to the gas as solids penetrate the gas eddies in the radial direction.

(ii) the spin-up of the gas eddies by a weak stationary azimuthal drift of solids; the subsequent growth of the pressure gradient enhances the accumulation of solids, which further enhances the azimuthal spin-up of eddies leading to instability.

Both of these effects are inertial in the sense that they become stronger with increasing mass of the particles, i.e., their stopping time (see Section 5 of Z19).

3.4.1. The Long-wavelength Limit

The long-wavelength limit of resonant SI corresponds to the limit of small $k$ as compared with the characteristic wavenumber of SI, $\tilde{k}$ (see Equations (5) and (6)). As soon as $k \ll \tilde{k}$, the resonant waves propagate almost radially, $k_c \gg k$, and $k_s \approx k$. In this case, the leading term producing SI in Equation (42) is the last one after the imaginary unit. Accordingly, one obtains

$$\Delta_2 \rightarrow \pm \frac{f}{2} \frac{k}{k} \left[ \frac{k}{k} + i \frac{g_s k t_s^2}{2} \right],$$  \hspace{1cm} (43)

which recovers the result of P21 (see his Equation (29)) taken in the same limit, $k_c \gg k_s$. The derivation of Equation (42) demonstrates that this regime of the resonant SI is associated with the stationary azimuthal drift of solids. However, the stationary azimuthal drift of solids goes beyond TVA because it is caused by the weak difference in the Coriolis force acting on solids and gas, rather than by the pressure gradient. In turn, this implies that Equations (22)–(26) from Jacquet et al. (2011), commonly known as equations derived within TVA, actually exceed the accuracy of TVA. The next section addresses this issue.

4. Problem Considered with Respect to Perturbations of the Gas–Dust Mixture Velocity

4.1. General Equations

The local dynamics of the gas–dust mixture can be equivalently described using the equation for the center-of-mass velocity of the mixture rather than the equation for the velocity of gas.

This is taken from Z19:

$$\partial_t U - 2\Omega_0 U_y e_x + \frac{\kappa^2}{2\Omega_0} U_x e_y + (U \nabla) U$$

$$+ \frac{\rho_s}{\rho} \left\{ \nabla \left( \frac{\rho_s}{\rho} \right) \right\} V + 2 \frac{\rho_p}{\rho} (V \nabla) V \right\}$$

$$= \nabla p_0 - \nabla (p + p_0).$$  \hspace{1cm} (44)

The key point is that $U$ is not free of divergence but

$$\nabla \cdot U = \nabla \cdot \left( \frac{\rho_s}{\rho} V \right).$$  \hspace{1cm} (45)

As previously, the evolution of the dust density and the relative velocity of the mixture are governed by Equations (10) and (12), respectively.

Below in this section, Equations (44)–(45) along with Equations (10) and (12) are used to reproduce the derivation of the linear equations for the dynamics of small gas–dust perturbations in the vicinity of resonance between SDW and IW in the presence of a stationary radial drift of the dust.

4.2. Stationary Drift of the Dust

4.2.1. The Solution Within TVA

Assuming for simplicity that

$$U = 0,$$  \hspace{1cm} (46)

one obtains from Equations (44) and (12) that

$$V = -t_s g_s e_x,$$  \hspace{1cm} (47)

implying that $\rho_s = \text{const}$. Note that Equations (47) and (16) are identical to each other. The solution in Equations (46)–(47) has been used by Z19 (see his Equations (22)–(24)).

4.2.2. The Solution Beyond TVA

Assuming again that

$$U = 0,$$  \hspace{1cm} (48)

one obtains to leading order in small $f$

$$V = -t_s g_s e_x + \frac{\kappa^2}{2\Omega_0} t_s^2 g_s e_y,$$  \hspace{1cm} (49)

implying again that $\rho_s = \text{const}$. Terms $\sim t_s^3$ and smaller have been omitted in Equation (49). Note that Equations (49) and (18) are identical to each other. The solution in Equations (48–
4.3. Linear Dynamics of Gas–Dust Perturbations Within TVA

In this section, I replace Equations (19)–(22) by Equations (44) and (45) with the omitted terms quadratic by \(\nu\), that are linearized on the background (Equations (46)–(47)):

\[
\begin{align*}
\partial_t u_x - 2\Omega_0 u_y & = -\partial_x W - f g_x \delta, \\
\partial_t u_y & + \frac{\nu^2}{2\Omega_0} u_x = 0, \\
\partial_t u_z & = -\partial_z W, \\
\partial_t \nu_x & + \partial_t \nu_z = f (\partial_x \nu_x + \partial_x \nu_z) - f t_x g_x \partial_x \delta, \\
\partial_t \nu_z & = t_x g_x \partial_x \delta + 2\tau \partial_z \nu_x,
\end{align*}
\]

where \(u\) and \(W \equiv \rho' / \rho\) are, respectively, the Eulerian perturbation of the center-of-mass velocity and the Eulerian perturbation of the generalized enthalpy. Equations (50)–(53) are supplemented by equations describing the dust, which are identical to Equations (23)–(25) with the replacement \(W_0 \rightarrow W\), what can be safely done to the zeroth order in \(f\).

Introducing the appropriate variables

\[\omega \equiv -\partial_t u_y\]

and

\[\phi \equiv \partial_z u_x,\]

one arrives at the equations

\[
\begin{align*}
\partial_t \phi & = \partial_x u_z - 2\Omega_0 \omega - f g_x \partial_x \delta, \\
\partial_t \omega & = \frac{\nu^2}{2\Omega_0} \phi, \\
\partial_t \nu_x & = -\frac{\nu^2}{2\Omega_0} \partial_x u_x - t_x g_x \frac{\nu^2}{2\Omega_0} \partial_x \delta, \\
\partial_t \nu_z & = t_x g_x \partial_x \delta + 2\tau \partial_z \omega.
\end{align*}
\]

Equations (54)–(57) yield the new variant of the coupling term,

\[\epsilon_3(\omega, k) \equiv -f t_x k^2 g_x k_x^2 \frac{\nu^2}{k^2} \left(1 - i\omega t_x \frac{k^2}{k_x^2}\right),\]

and the corresponding variant of correction to the frequency of mode crossing

\[\Delta_3 \approx \pm \kappa \left(\frac{f}{2}\right)^{1/2} \frac{k_x^2}{k} \left[1 + i \frac{k f}{2 k_x^2}\right],\]

which fully recovers Equation (29) of P21 and indeed recovers the long-wavelength limit of the resonant SI growth rate (see Equation (43)). It can be seen that the growth rate found in

4 Note that Equation (89) of Z19 contains a misprint.

5 In the latter case, this leads to new terms \(\sim O(f)\) in \(D_3\) and \(D_\nu\), which do not contribute to the resonant solution.

Equation (59) arises while retaining the last term on the RHS of Equation (56). In the absence of this term Equations (54)–(57) become identical to Equations (26)–(29) after the replacement \(u \rightarrow u_x\) and produce no growing modes. This is equivalent to

\[\nabla \cdot u = 0\]

as a supplementary condition for Equations (50)–(52). However, \(u\) cannot be considered free of divergence in order to derive the equation for the evolution of the dust density starting from the general Equation (10). The assumption in Equation (60) has been applied in order to derive Equations (26)–(29) in Z19; however, this is not the case for the formal solution of Equations (22)–(26) from Jacquet et al. (2011). Section 5 below is reserved for the formulation of the refined general two-fluid equations for the gas–dust dynamics within TVA.

4.4. Linear Dynamics of Gas–Dust Perturbations Beyond TVA

For completeness, this section demonstrates how the full growth rate of the resonant SI already recovered in Section 3.4 is reproduced using alternative variables.

As was shown in Section 4.2 of Z19, new terms \(\sim f^2\) appearing in the equation for \(u\) to the higher-order in \(t_x\) originate from the first gradient term \(\sim \nabla^2\) on the LHS of Equation (44). Contrary to what occurs with the equation for \(u_x\) (see Section 3.4), an extension of the stationary solution beyond TVA does not bring any terms in the equation for \(u\). As a result, Equation (50) should be replaced by the following one:

\[
\begin{align*}
\partial_t u_x - 2\Omega_0 u_y & = -f t_x g_x \frac{\nu^2}{2\Omega_0} \partial_x \delta - f t_x g_x \partial_x \delta, \\
\partial_t \omega & = \frac{\nu^2}{2\Omega_0} \phi, \\
\partial_t \nu_x & = -\frac{\nu^2}{2\Omega_0} \partial_x u_x - f t_x g_x \frac{\nu^2}{2\Omega_0} \partial_x \delta, \\
\partial_t \nu_z & = t_x g_x \partial_x \delta + 2\tau \partial_z \omega.
\end{align*}
\]

Note that, in order to derive Equation (56), the term \(\nabla \cdot \nu\), which also contributes to \(\nabla \cdot u\) via Equation (53), has been omitted as it does not contribute any term \(\sim O(f)\) to Equation (56).

It is not difficult to see that Equations (54)–(57) are not equivalent to Equations (26)–(29) neither when considering the replacement \(u \rightarrow u_x\), nor accounting for the small difference \(\sim O(f)\) between \(U\) and \(U_0\) in the stationary solutions.

Equations (54)–(57) yield the new variant of the coupling term,

\[\epsilon_4(\omega, k) \equiv -f t_x k^2 g_x k_x^2 \frac{\nu^2}{k^2} \left(1 - i\omega t_x \frac{k^2}{k_x^2}\right),\]

and the corresponding variant of correction to the frequency of mode crossing

\[\Delta_4 \approx \pm \kappa \left(\frac{f}{2}\right)^{1/2} \frac{k_x^2}{k} \left[1 + i \frac{k f}{2 k_x^2}\right],\]

which provides the dispersion equation with a correction to the frequency of the mode crossing identical to Equation (42).

\[\Delta_4 = \Delta_2.\]

5. Refinement of TVA

Additional justification of the condition in Equation (60) can be found from the ordering of terms in the set of the general Equations (44)–(45), (10), and (12). Such a procedure was
carried out in Section 2.1 of Z19. It was concluded that TVA corresponds to the limit when all terms of orders of \( \tau_s, \lambda^{-1}, \) and \( \sqrt{\lambda^{-1}} \) standing in Equations (44) and (12) should be omitted. Here

\[
\tau_s \equiv \max\{t_e l_{\text{ev}}, \tau\} \ll 1 \quad (68)
\]

and

\[
\lambda^{-1} \equiv \frac{g t_f^2}{l_{\text{ev}}} \ll 1 \quad (69)
\]

are small dimensionless parameters with \( g, l_{\text{ev}}, \) and \( t_{\text{ev}} \) being, respectively, the characteristic specific pressure gradient, length scale, and timescale of the gas–dust mixture dynamics. Note that the characteristic scale \( l_{\text{ev}} \) associated with resonant SI, \( k^{-1}, \) corresponds to \( \lambda^{-1} \sim \tau \ll 1 \) in Keplerian disks as follows from Equation (6).

However, the use of Equation (44) taken in TVA, in combination with Equation (45) leads to an excess in accuracy. Indeed, as soon as the characteristic absolute value of \( V \) is by factor of \( \sqrt{\lambda^{-1}} \) smaller than that of \( U \) (see Z19), the inertial terms on the LHS of Equation (44) rearranged with the help of Equation (45) also retain an extra term of order \( \tau_s. \) An example of such an extra term is found in Equation (56), which is the last one on its RHS. It is also clear why this term makes it possible to recover the correct resonant SI growth rate in the long-wavelength limit. The condition \( \lambda \ll \lambda^{-1} \) implies that

\[
\lambda^{-1} \ll \tau \quad (70)
\]

according to definitions of \( \lambda \) and \( \lambda^{-1}. \) So, a small term on the RHS of Equation (56) prevails upon perturbations of the gradient terms \( \sim V^2 \) on the LHS of full Equation (44).

Note that the constraint in Equation (70), which can be considered as the new constraint on the correct use of the previous formulation of TVA, is stronger than the standard constraint in Equation (69) for TVA.

5.1. The Refined General Equations for TVA in a Disk Small Shearing Sheet

Section 3 of Z19, which considers gas–dust dynamics within TVA, starts from the general Equations (13), (14), (18), and (19).\(^6\) It has become clear now that all these equations exceed the accuracy of TVA. Although Z19 derives the correct final equations being completely within TVA, I feel it would be instructive to give here the refined general equations for the local gas–dust dynamics within TVA, as well as the corresponding equations for linear perturbations. As the ordering of terms outlined above has been carried out by Z19 with the restriction \( f \lesssim 1, \) this is assumed to be the case below.\(^7\)

\[
(\partial_t - q\Omega_0 x \partial_x) U - 2\Omega_0 U_t e_x + \frac{k^2}{2\Omega_0} U_t e_y + (U \nabla) U
\]

\[
= \frac{\nabla p_0 - \nabla (p + p_0)}{\rho_b}, \quad (71)
\]

\[
\nabla \cdot U = 0, \quad (72)
\]

\(^6\) The corrections below should be made to Equations (4), (9)–(12) of Zhuravlev (2020) as well.

\(^7\) Note that this section no longer contains an assumption of axial symmetry of the flow.
interesting to find the new form of the overstability criterion for partially coupled gas and dust obtained by Lin & Youdin (2017), who based their analysis on the equations of Laibe & Price (2014) discussed just above. While considering the instabilities in dusty protoplanetary disks, Lin & Youdin (2017) recovered the simplified dispersion of Equation (27) of Jacquet et al. (2011), which partially reproduces the resonant SI as well as the nonresonant growing secular mode. The present work is focused on the former. However, it is not difficult to check that Equations (22)–(24), (26) of Jacquet et al. (2011) along with their Equation (25) with the last term omitted, which corresponds to the revised TVA, do not reproduce the growing secular mode at all (see the analysis in Appendix A below). Performing third-order expansion of the full gas–dust equations with respect to a small Stokes number, Jaupart & Laibe (2020) obtained an accurate growth rate of the secular mode and found that it reduces to the result of Youdin & Goodman (2005) and Jacquet et al. (2011) in the limit \( k_s \gg k_r \) (see Section 3.2.2 of Jaupart & Laibe (2020)). A similar reduction with respect to the resonant branch of SI as well as the corresponding explanation are shown here (see Section 3.4.1 and see also Section 5 of P21). Indeed, the previous formulation of TVA works correctly in the limit of almost radially propagating harmonics, which corresponds to long waves in the vicinity of resonance, thus leading to agreement with the constraint in Equation (70). However, this constraint can be violated well within the known condition of TVA given by Equation (69)—that is the demonstration of inconsistency of the previous formulation of TVA. Paardekooper et al. (2020) have used TVA for their study of SI extended onto polydisperse dust. They report that wavenumbers of growing modes get larger compared to SI for the monodisperse dust. Presumably, this implies that the constraint on the validity of the previous formulation of TVA becomes more essential in that case. Thus, the constraint in Equation (70) should be generalized to polydisperse dust. On the other hand, the refined equations for TVA introduced here should be generalized to polydisperse dust in order to check whether SI is absent within the revised TVA applied to polydisperse dust. Finally, it should be noted that the dust-rich branch of SI can be studied employing the refined equations for TVA, because at least its linear variant persists in the revised TVA according to Appendix B.

The work was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS” and the Program of development of Lomonosov Moscow State University. I acknowledge the support from the Ministry of Science and Higher Education of the Russian Federation (project No. 13.1902.21.0039) for the part of the study carried out beyond TVA.

**Appendix A**

**Secular SI Mode in the Revised TVA**

Equations (A5)–(A7) of Z19 yield the dispersion equation

\[
\left( \omega + \frac{g_s k_s t_s}{f + 1} \right) \left( \omega^2 - \kappa^2 \frac{k^2}{k_s^2} \right) = \frac{g_t k_t t_t}{f + 1} \kappa^2 \frac{k^2}{k_s^2} + \frac{i f}{f + 1} \omega^2 t_s (\omega^2 - \kappa^2),
\]

(A1)

which is valid for an arbitrary value of the dust fraction, \( f \). Equation (A1) is identical to Equation (27) of Jacquet et al. (2011) taking into account that their \( f_p = f/(f + 1) \) and \( f_s = 1/(f + 1) \). Note that in the revised TVA the last term on the RHS of Equation (A1) disappears, as can be checked using Equations (76)–(78).

Consider the solution of Equation (A1), which satisfies the condition \( \omega \to 0 \) for \( t_s \to 0 \). As long as \( \omega \ll \kappa \), while \( k_s \sim k_r \), the term \( \omega^2 \) in the brackets on both sides of Equation (A1) can be omitted. To leading order in \( t_s \), the corresponding approximate solution reads

\[
\omega \approx -\frac{1 - f}{f + 1} g_s k_s t_s + i t_s^2 f^2 (1 - f)^2 \frac{k_s^2 k_r^2}{(f + 1)^3 k_r^2}.
\]

(A2)

Equation (A2) represents exactly the unstable secular mode obtained by Jacquet et al. (2011); see their Equations (28)–(29)). It is clear from the above derivation that the nonzero \( \Im(\omega) \) comes from the last term on the RHS of Equation (A1). Hence, the secular mode obtained within the revised TVA is neutral.

**Appendix B**

**Dust-rich Limit of the SI in the Revised TVA**

It can be shown that Equation (A1) has another approximate solution in the high-wavenumber limit,

\[
g_t k_t t_t \gg \kappa.
\]

(B1)

As long as \( \omega \sim O(\kappa) \), \( \omega \) in the first brackets on the LHS of Equation (A1) can be omitted. Also, the condition in Equation (B1) implies that the second term on the RHS of Equation (A1) becomes small compared to the first term over there, as long as \( \kappa t_s \ll 1 \). To zeroth order in small \( \kappa/(g_s k_s t_s) \ll 1 \), this yields the solution

\[
\omega \approx \pm i \sqrt{f - 1} \frac{k_s}{k_r}.
\]

(B2)

Equation (B2) recovers the growing dust-rich mode considered by Squire & Hopkins (2018; see their Equation (A2)). Hence, the dust-rich limit of SI persists in the more restrictive approximation, which is the revised TVA.

**ORCID iDs**

V. V. Zhuravlev @ https://orcid.org/0000-0003-2346-1320

**References**

Bai, X.-N., & Stone, J. M. 2010, ApJ, 722, 1437

Carrera, D., Johansen, A., & Davies, M. B. 2015, A&A, 579, A43

Chen, K., & Lin, M.-K. 2020, ApJ, 891, 152

Haefl, A. V. 1948, PhRv, 74, 1532

Jacquet, E., Balbus, S., & Latter, H. 2011, MNRAS, 415, 3591

Jaupart, E., & Laibe, G. 2020, MNRAS, 492, 4591

Johansen, A., & Youdin, A. 2007, ApJ, 662, 627

Johansen, A., Youdin, A., & Mac Low, M.-M. 2009, ApJL, 704, L75

Laibe, G., & Price, D. J. 2014, MNRAS, 440, 2136

Landau, L. D., & Lifshitz, E. M. 1987, Fluid Mechanics (Oxford: Pergamon Press)

Lesur, G., Ercolano, B., Flock, M., et al. 2022, arXiv:2203.09821

Li, R., & Youdin, A. N. 2021, ApJ, 919, 107

Lin, M.-K. 2019, MNRAS, 485, 5221

Lin, M.-K. 2021, ApJ, 907, 64

Lin, M.-K., & Youdin, A. N. 2017, ApJ, 849, 129

Lovas, F., & Paardekooper, S.-J. 2019, MNRAS, 488, 5290

Nakagawa, Y., Sekiya, M., & Hayashi, C. 1996, Icar, 67, 375

B1

B2
Paardekooper, S.-J., McNally, C. P., & Lovasco, F. 2020, MNRAS, 499, 4223
Pan, L. 2020, ApJ, 898, 8
Pan, L. 2021, ApJ, 920, 80
Pierce, J. R. 1948, JAP, 19, 231
Squire, J., & Hopkins, P. F. 2018, MNRAS, 477, 5011
Squire, J., & Hopkins, P. F. 2020, MNRAS, 498, 1239
Yang, C.-C., Johansen, A., & Carrera, D. 2017, A&A, 606, A80
Youdin, A. N., & Goodman, J. 2005, ApJ, 620, 459
Zhuravlev, V. V. 2019, MNRAS, 489, 3850
Zhuravlev, V. V. 2020, MNRAS, 494, 1395
Zhuravlev, V. V. 2021, MNRAS, 500, 2209