Improved extended Hamiltonian and search for local symmetries

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Abstract

We analyze a structure of the singular Lagrangian $L$ with first and second class constraints of an arbitrary stage. We show that there exist an equivalent Lagrangian (called the extended Lagrangian $\tilde{L}$) that generates all the original constraints on second stage of the Dirac-Bergmann procedure. The extended Lagrangian is obtained in closed form through the initial one. The formalism implies an extension of the original configuration space by auxiliary variables. Some of them are identified with gauge fields supplying local symmetries of $\tilde{L}$. As an application of the formalism, we found closed expression for the gauge generators of $\tilde{L}$ through the first class constraints. It turns out to be much more easy task as those for $L$. All the first class constraints of $L$ turn out to be the gauge symmetry generators of $\tilde{L}$. By this way, local symmetries of $L$ with higher order derivatives of the local parameters decompose into a sum of the gauge symmetries of $\tilde{L}$. It proves the Dirac conjecture in the Lagrangian framework.

1 Introduction

Dirac-Bergmann algorithm proves to be a principal tool for analysis of various field and particle theories with local (gauge) symmetries, and, more generally, of any theory constructed on the base of singular Lagrangian. While it has a solid mathematical ground and a well established interpretation [1-4], some problems within the formalism remain under investigation [5-18]. The aim of this work is to reveal

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one of the long standing problems, concerning the proper interpretation and treatment of so called extended Hamiltonian formulation of the singular system.

In the Hamiltonian framework, possible motions of the singular system are restricted to lie on some surface of a phase space. Algebraic equations of the surface (Dirac constraints) can be revealed in the course of the Dirac-Bergmann procedure, the latter in general case requires a number of stages. According to the order of appearance, the constraints are called primary, second-stage, ..., N-th stage constraints. All the constraints, beside the primary ones are called the higher-stage constraints and are denoted collectively $T_a$. The basic object of the Hamiltonian formulation turns out to be the complete Hamiltonian $H = H_0 + v^a \Phi_a$. Here $H_0$ is the Hamiltonian, $v^a$ represents primarily inexpressible velocities [3], and $\Phi_a$ are primary constraints. The extended Hamiltonian is constructed adding by hand the higher stage constraints with the multipliers $\lambda^a$: $H_{ext} \equiv H + \lambda^a T_a$. The Hamiltonian equations following from $H_{ext}$ involve the extra terms with derivatives of $T_a$ and hence are different from the equations obtained from $H$. Nevertheless, a detailed analysis in special basis on the phase space shows that physical sectors of the two formulations are equivalent [3].

All the constraints enter into $H_{ext}$ in the manifest form. By this reason, the extended Hamiltonian turns out to be a very useful tool for the analysis of both the general structure [3] and local symmetries [4, 5] of the singular theory. At the same time, since the higher stage constraints have been added by hand, the origin of the extended Hamiltonian and its proper interpretation in the Dirac-Bergmann framework remain somewhat mysterious. In particular, $H_{ext}$ cannot be treated as the complete Hamiltonian generated by some Lagrangian (see Sect. 2 for details). So one asks whether it is possible to construct an equivalent Lagrangian formulation that would generate the complete Hamiltonian of the same structure as $H_{ext}$. We solve this problem in the Section 3.

For the case of first class constraints, the problem has been discussed in the recent work [12]. Here we generalize this analysis to an arbitrary case, with first and second class constraints up to N-th stage presented in the original formulation $L$. We present an improvement of the extended Hamiltonian formalism according to the following scheme. Starting from the initial Lagrangian $L$ (provided
all its constraints are known), we work out an equivalent Lagrangian \( \tilde{L} \) called the extended Lagrangian. It is obtained in a closed form in terms of the quantities of initial formulation (see Eq. \( \text{(21)} \) below). Due to the equivalence of \( L \) and \( \tilde{L} \), it is matter of convenience what formulation is used to describe the theory under consideration.

By construction, all the Lagrangian counterparts of the higher-stage constraints \( T_a \) enter into \( \tilde{L} \) in the manifest form, see the last term in Eq. \( \text{(21)} \). The complete Hamiltonian \( \tilde{H} \) generated by \( \tilde{L} \) has the same structure as \( H_{\text{ext}} \). So, the improved formalism maintains all the advantages of the extended Hamiltonian formalism. Besides, since it originates from the Lagrangian, all the quantities appearing in the formalism have clear meaning in the Dirac framework.

We explore the extended Lagrangian formulation to resolve another long standing problem concerning search for constructive procedure that would give local symmetries of a given Lagrangian action [4-16]. It is well known that in a singular theory there exist the infinitesimal local symmetries with a number of local parameters \( \epsilon^a \) equal to the number of the primary first class constraints

\[
\delta q^B = \epsilon^a R^{(0)B}_a + \epsilon^a R^{(1)B}_a + \epsilon^a R^{(2)B}_a + \ldots + \epsilon^{(N-1)B} R^{(N-1)B}_a.
\]  

Here \( q^B \) is the set of configuration space variables, \( \epsilon^a \equiv \frac{d\epsilon^a}{dt} \), and the set \( R^{(k)B}_a(q, \dot{q}, \ldots) \) represents generator of the symmetry. In some particular models, the generators can be found in terms of constraints. For example, the relativistic particle Lagrangian

\[ L = \sqrt{\left(\dot{x}^\mu\right)^2} \]

implies the constraint \( T \equiv \frac{1}{2}(p^2 - 1) \), and the local symmetry \( \delta x^\mu = \epsilon \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} \). The latter can be rewritten as follows

\[
\delta x^\mu = \epsilon \{x^\mu, T\} \big|_{p^\mu \rightarrow \dot{x}^\mu},
\]  

where \( \{ \}, \) is the Poisson bracket, and the symbol \( | \) implies the indicated substitution. The equation \( \text{(2)} \) states that the gauge generator is the Lagrangian counterpart of the canonical transformation generated by the constraint on a phase space. It seems to be interesting to find a proper generalization of the recipe given by Eq. \( \text{(2)} \) on a general case. Since the Hamiltonian constraints can be found in the course of Dirac procedure, it would give a regular method for obtaining the symmetries.

General analysis of symmetry structure (classification and proof on existence of irreducible complete set of gauge generators) can be
found in [13, 14]. In the works [5] it have been observed that sym-
metries of the extended Hamiltonian with first class constraints can
be written in closed form. This observation was used in [4] to formu-
late the procedure for restoration of symmetries of the Hamiltonian
action. While the algorithm suggested is relatively simple, some of
its points remain unclarified. In particular, the completeness and
irreducibility of the symmetries of the complete Hamiltonian were
not demonstrated so far [13]. The Lagrangian symmetries have not
been discussed. Analysis of a general case (when both first and
second class constraints are present) turns out to be a much more
complicated issue (see the second model of the Section 5 for the
example). For the case, various procedures has been suggested and
discussed in the works [6-10, 14-16].

We show that namely in the extended Lagrangian formalism the
problem has a simple solution. Complete irreducible set of local
symmetries of \( \tilde{\mathcal{L}} \) will be presented in closed form through the first
class constraints of the initial formulation, see Eq. (46), (47). More-
over, all the initial variables \( q^A \) transform according to Eq. (2).

Another closely related issue is known as the Dirac conjecture [1]:
does all the higher stage constraints generate the local symmetries?
Affirmative answer on the question has been obtained by various
groups [3, 8] in the extended Hamiltonian framework. Our result
(46) can be considered as another proof of the Dirac conjecture, now
in the Lagrangian framework.

The work is organized as follows. With the aim to fix our nota-
tions, we outline in Section 2 the Hamiltonization procedure for an
arbitrary singular Lagrangian theory. In Section 3 we formulate pure
algebraic recipe for construction of the extended Lagrangian. All the
higher-stage constraints of \( \mathcal{L} \) appear as the second stage constraints
in the formulation with \( \tilde{\mathcal{L}} \). Besides, we demonstrate that \( \tilde{\mathcal{L}} \) is a the-
ory with at most third-stage constraints. Then it is proved that \( \tilde{\mathcal{L}} \)
and \( \mathcal{L} \) are equivalent. It means, that an arbitrary theory can be re-
formulated as a theory with at most third-stage constraints. Since
the original and the reconstructed formulations are equivalent, it is
matter of convenience to use one or another of them for description
of the theory under investigation. In Section 4 we demonstrate one
of advantages of the extended Lagrangian presenting its complete

\[ \text{Footnote: Popular physical theories usually do not involve more than third-stage constraints. Our}
\text{result can be considered as an explanation of this fact.} \]
irreducible set of local symmetry generators in terms of constraints. The procedure is illustrated on various examples in the Section 5.

2 Dirac-Bergmann procedure for singular Lagrangian theory

Let $L(q^A, \dot{q}^B)$ be Lagrangian of the singular theory: \( \text{rank} \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} = [i] < [A] \), defined on configuration space $q^A, A = 1, 2, \ldots, [A]$. From the beginning, it is convenient to rearrange the initial variables in such a way that the rank minor is placed in the upper left corner of the matrix $\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}$. Then one has $q^A = (q^i, q^\alpha), i = 1, 2, \ldots, [i], \alpha = 1, 2, \ldots, [\alpha] = [A] - [i]$, where $\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$.

Let us construct the Hamiltonian formulation for the theory. To fix our notations, we carry out the Hamiltonization procedure in some details. One introduces conjugate momenta according to the equations $p_i = \frac{\partial L}{\partial \dot{q}^i}, p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}$. They are considered as algebraic equations for determining velocities $\dot{q}^A$. According to the rank condition, the first $[i]$ equations can be resolved with respect to $\dot{q}^i$, let us denote the solution as

$$\dot{q}^i = v^i(q^A, p_j, \dot{q}^\alpha).$$

(3)

It can be substituted into remaining $[\alpha]$ equations for the momenta. By construction, the resulting expressions do not depend on $\dot{q}^A$ and are called primary constraints $\Phi_\alpha(q, p)$ of the Hamiltonian formulation. One finds

$$\Phi_\alpha \equiv p_\alpha - f_\alpha(q^A, p_j) = 0,$$

(4)

where

$$f_\alpha(q^A, p_j) \equiv \frac{\partial L}{\partial \dot{q}^\alpha} \bigg|_{\dot{q}^i = v^i(q^A, p_j, \dot{q}^\alpha)}.$$

(5)

The original equations for the momenta are thus equivalent to the system (3), (4). By construction, there are the identities

$$\frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \bigg|_{\dot{q}^j \rightarrow v^j(q^A, p_j, \dot{q}^\alpha)} \equiv p_i, \quad v^i(q^A, p_j, \dot{q}^\alpha) \bigg|_{p_j = \frac{\partial L}{\partial \dot{q}^j}} \equiv \dot{q}^i.$$  

(6)
Next step of the Hamiltonian procedure is to introduce an extended phase space parameterized by the coordinates $q^A, p_A, v^\alpha$, and to define the complete Hamiltonian $H$ according to the rule

$$H(q^A, p_A, v^\alpha) = H_0(q^A, p_j) + v^\alpha \Phi_\alpha(q^A, p_B),$$

(7)

where

$$H_0 = \left. (p_i \dot{q}^i - L + \dot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha}) \right|_{\dot{q}^i \rightarrow v^i(q^A, p_j, \dot{q}^\alpha)}. $$

(8)

By construction it does not contain the quantities $\dot{q}^\alpha$ and $p_\alpha$. The Hamiltonian equations

$$\dot{q}^A = \{q^A, H\}, \quad \dot{p}_A = \{p_A, H\}, \quad \Phi_\alpha(q^A, p_B) = 0,$$

(9)

are equivalent to the Lagrangian equations following from $L$, see [3]. Here $\{,\}$ denotes the Poisson bracket.

From Eq. (9) it follows that all the solutions are confined to lie on a surface of the extended phase space defined by the algebraic equations $\Phi_\alpha = 0$. It may happen, that the system (9) contains in reality more then $[\alpha]$ algebraic equations. Actually, derivative of the primary constraints with respect to time implies, as algebraic consequences of the system (9), the so called second stage equations:

$$\{\Phi_\alpha, H\} \equiv \{\Phi_\alpha, \Phi_\beta\} v^\beta + \{\Phi_\alpha, H_0\} = 0.$$  

They can be added to Eq. (9), which gives an equivalent system. Let on-shell one has $\text{rank}\{\Phi_\alpha, \Phi_\beta\} = [\alpha'] \leq [\alpha]$. Then $[\alpha']$ equations of the second-stage system can be used to represent some $v^{\alpha'}$ through other variables. It can be substituted into the remaining $[\alpha'] \equiv [\alpha] - [\alpha']$ equations, the resulting expressions do not contain $v^{\alpha}$ at all. Thus the second-stage system can be presented in the equivalent form

$$v^{\alpha'} = v^{\alpha'}(q^A, p_j, v^{\alpha''}), \quad T_{\alpha''}(q^A, p_j) = 0.$$  

(10)

Functionally independent equations among $T_{\alpha''} = 0$, if any, represent secondary Dirac constraints. Thus all the solutions of the system (9) are confined to the surface defined by $\Phi_\alpha = 0$ and by the equations (10).

The secondary constraints may imply third-stage constraints, and so on. We suppose that the theory has constraints up to $N$-th stage, $N \geq 2$. The complete set of higher stage constraints is denoted by $T_a(q^A, p_j) = 0$. Then the complete constraint system is $G_I \equiv$
All the solutions of Eq. (9) are confined to the surface defined by the equations \( \Phi_\alpha = 0 \) as well as by\(^2\)

\[
\{ G_I, H \} = 0. \quad (11)
\]

By construction, after substitution of the velocities \( v^\alpha \) determined in the course of Dirac procedure, the equations (11) vanish on the complete constraint surface \( G_J = 0 \).

Suppose that \( \{ G_I, G_J \} = \triangle_{IJ}(q^A, p_j) \), where \( \text{rank} \triangle_{IJ} \big|_{G_I=0} = [I_2] < [I] \). It means that both first and second class constraints are presented in the formulation. It will be convenient to separate them. According to the rank condition, there exist \( [I_1] = I - [I_2] \) independent null-vectors \( \vec{K}_{I_1} \) of the matrix \( \triangle \) on the surface \( G_I = 0 \), with the components \( K_{I_1J}(q^A, p_j) \). Then the bracket of constraints \( G_{I_1} \equiv K_{I_1J}G_J \) with any \( G_I \) vanishes, hence the constraints \( G_{I_1} \) represent the first class subset. One chooses the vectors \( \vec{K}_{I_2}(q^A, p_j) \) to complete \( K_{I_1} \) up to a basis of \([I]\)-dimensional vector space. By construction, the matrix

\[
K_I^J \equiv \begin{pmatrix} K_{I_1}^J \\ K_{I_2}^J \end{pmatrix},
\]

is invertible. Let us denote \( \tilde{G}_I \equiv (\tilde{G}_{I_1}, \tilde{G}_{I_2}) \), where \( \tilde{G}_{I_1} \equiv K_{I_1J}G_J, \tilde{G}_{I_2} \equiv K_{I_2J}G_J \). The system \( \tilde{G}_I \) is equivalent to the initial system of constraints \( G_I \). The constraints \( \tilde{G}_{I_2} \) form the second class subset of the complete set \( \tilde{G}_I \). In an arbitrary theory, the constraints obey the following Poisson bracket algebra:

\[
\begin{align*}
\{ \tilde{G}_{I_1}, \tilde{G}_{I_2} \} &= \triangle_{IJ}(q^A, p_B), \\
\{ \tilde{G}_{I_1}, G_J \} &= c_{I_1J}^K(q^A, p_B)G_K, \\
\{ \tilde{G}_{I_2}, G_J \} &= \triangle_{I_2J_2}(q^A, p_B),
\end{align*}
\]

where

\[
\text{rank}_I \triangle_{IJ} \big|_{G_I=0} = [I_2], \quad \det \triangle_{I_2J_2} \big|_{G_I=0} \neq 0. \quad (13)
\]

The extended Hamiltonian is defined as follows

\[
H_{\text{ext}}(q^A, p_A, v^\alpha, \lambda^\alpha) = H_0(q^A, p_j) + v^\alpha \Phi_\alpha(q^A, p_j, p_\alpha) + \lambda^\alpha T_\alpha(q^A, p_j), \quad (15)
\]

\(^2\)It is known [3], that the procedure reveals all the algebraic equations presented in the system (9). Besides, surface of solutions of Eq. (9) coincides with the surface \( \Phi_\alpha = 0, \{ G_I, H \} = 0 \).
As it was mentioned in the introduction, \( H_{\text{ext}} \) cannot be generally obtained as the complete Hamiltonian of some Lagrangian. It can be seen as follows. In the Dirac-Bergmann procedure, the total Hamiltonian is uniquely defined by Eqs. (7), (8). Consider the particular case of higher stage constraints \( T_a \) of the form \( p_a - t_a(q^A, p') \). Then it is clear that Eq. (15) does not have the desired form (7), since \( H_0 \) from (15) generally depends on \( p_a \).

3 Formalism of extended Lagrangian

Starting from the theory described above, we construct here the equivalent Lagrangian \( \tilde{\mathcal{L}}(q^A, \dot{q}^A, s^a) \) defined on the configuration space with the coordinates \( q^A, s^a \), where \( s^a \) states for auxiliary variables. By construction, it will generate the Hamiltonian of the form \( H_0 + s^a T_a \), as well as the primary constraints \( \Phi_\alpha = 0, \pi_a = 0 \), where \( \pi_a \) represent conjugate momenta\(^3\) for \( s^a \). Due to the special form of Hamiltonian, preservation in time of the primary constraints \( \pi_a = 0 \) implies that all the higher stage constraints \( T_a \) of the original formulation appear as the secondary constraints of \( \tilde{\mathcal{L}} \): \( \dot{\pi}_a = \{\pi_a, H_0 + s^a T_a\} = -T_a = 0 \).

To construct the extended Lagrangian for \( \mathcal{L} \), one introduces the following equations for the variables \( q^A, \tilde{\mathcal{p}}_j, s^a \):

\[
\ddot{q}^i - v^i(q^A, \tilde{\mathcal{p}}_j, \dot{q}^A) - s^a \frac{\partial T_a(q^A, \tilde{\mathcal{p}}_j)}{\partial \tilde{\mathcal{p}}_i} = 0.
\]

Here the functions \( v^i(q^A, \tilde{\mathcal{p}}_j, \dot{q}^A), \ T_a(q^A, \tilde{\mathcal{p}}_j) \) are taken from the initial formulation. The equations can be resolved algebraically with respect to \( \tilde{\mathcal{p}}_i \) in a vicinity of the point \( s^a = 0 \). Actually, Eq. (16) with \( s^a = 0 \) coincides with Eq. (3) of the initial formulation, the latter can be resolved, see Eq. (6). Hence \( \text{det} \left( \frac{\partial (\text{Eq. (16)})}{\partial \tilde{\mathcal{p}}_j} \right) \neq 0 \) at the point \( s^a = 0 \). Then the same is true in some vicinity of this point, and Eq. (16) thus can be resolved. Let us denote the solution as

\[
\tilde{\mathcal{p}}_i = \omega_i(q^A, \dot{q}^A, s^a).
\]

---

\(^3\)Let us stress once again, that in our formulation the variables \( s^a \) represent a part of the configuration-space variables.

\(^4\)As it will be shown below, Eq. (16) represents a solution of the equation \( \tilde{\mathcal{p}}_j \frac{\partial \mathcal{L}}{\partial q^j} \), defining the conjugate momenta \( \tilde{\mathcal{p}}_j \) of the extended formulation.
By construction, there are the identities

$$\omega_i(q, \dot{q}, s) = \dot{\tilde{p}}_i,$$

\[(18)\]

$$\left( v^i(q^A, \tilde{p}_j, \dot{q}^\alpha) + s^a \frac{\partial T_a(q^A, \tilde{p}_j)}{\partial \tilde{p}_i} \right) \equiv \dot{\tilde{q}}^i. $$

\[(19)\]

Besides, the function $\omega$ has the property

$$\omega_i(q^A, \dot{q}^A, s^a) \bigg|_{s^a = 0} = \frac{\partial L}{\partial \dot{q}^i}. $$

\[(20)\]

Now, the extended Lagrangian for $L$ is defined according to the expression

$$\tilde{L}(q^A, \dot{q}^A, s^a) = L(q^A, v^i(q^A, \omega_j, \dot{q}^\alpha), \dot{q}^\alpha) + \omega_i(\dot{\tilde{q}}^i - v^i(q^A, \omega_j, \dot{q}^\alpha)) - s^a T_a(q^A, \omega_j), $$

\[(21)\]

where the functions $v^i, \omega_i$ are given by Eqs. (3), (17). As compared with the initial Lagrangian, $\tilde{L}$ involves the new variables $s^a$, in a number equal to the number of higher stage constraints $T_a$. Let us enumerate some properties of $\tilde{L}$

$$\tilde{L}(s^a = 0) = L, $$

\[(22)\]

$$\frac{\partial \tilde{L}}{\partial \omega_i} \bigg|_{\omega(q, \dot{q}, s)} = 0, $$

\[(23)\]

$$\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}^a} = \frac{\partial L(q^A, v^i, \dot{q}^\alpha)}{\partial \dot{q}^\alpha} \bigg|_{v^i(q, \omega, \dot{q}^\alpha)} = f_a(q^A, \omega_j(q, \dot{q}, s)). $$

\[(24)\]

Eq. (22) follows from Eqs. (20), (6). Eq. (23) is a consequence of the identities (6), (18). Eq. (23) will be crucial for discussion of local symmetries in the next section. At last, Eq. (24) is a consequence of Eqs. (23), (6).

Following to the standard prescription [3, 4], let us construct the Hamiltonian formulation for $\tilde{L}$. By using of Eqs. (23), (24), one finds the conjugate momenta $\tilde{p}_A, \pi_a$ for $q^A, s^a$

$$\tilde{p}_i = \frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}^i} = \omega_i(q^A, \dot{q}^A, s^a), $$

\[(25)\]
\[ \tilde{p}_\alpha = \frac{\partial \tilde{L}}{\partial \dot{q}_\alpha} = f_\alpha(q^A, \omega_j), \]
\[ \pi_a = \frac{\partial \tilde{L}}{\partial \dot{s}^a} = 0. \]  
(26)

The equation (25) can be resolved with respect to the velocities \( \dot{q}^i \). According to the identity (18), the solution is just given by our basic equation (16). Taking this into account, the system (25), (26) is equivalent to the following one

\[ \dot{q}^i = v^i(q^A, \tilde{p}_j, \dot{q}^\alpha) + s^a \frac{\partial T_a(q^A, \tilde{p}_j)}{\partial \tilde{p}_i}, \]  
(27)

\[ \tilde{p}_\alpha - f_\alpha(q^A, \tilde{p}_j) = 0, \]  
(28)

\[ \pi_a = 0. \]  
(29)

So, in the extended formulation there are presented the primary constraints (28) of the initial formulation. Besides, there are the trivial constraints (29) in a number equal to the number of all the higher stage constraints of the initial formulation.

Using the definition (8), one obtains the Hamiltonian \( \tilde{H}_0 = H_0 + s^a T_a \), then the complete Hamiltonian for \( \tilde{L} \) is given by the expression

\[ \tilde{H} = H_0(q^A, \tilde{p}_j) + s^a T_a(q^A, \tilde{p}_j) + v^a \Phi_\alpha(q^A, \tilde{p}_j) + v^a \pi_a. \]  
(30)

Here \( v^a, v^\alpha \) are the primarily un expressible velocities of \( \tilde{L} \). Note that, if one discards the constraints \( \pi_a = 0 \), \( \tilde{H} \) coincides with the extended Hamiltonian for \( L \) after identification of the configuration space variables \( s^a \) with the Lagrangian multipliers for higher stage constraints of the original formulation.

Further, preservation in time of the primary constraints: \( \dot{\pi}_a = \{\pi_a, H_0 + s^a T_a\} = -T_a = 0 \) implies the equations \( T_a = 0 \). Hence all the higher stage constraints of the initial formulation appear now as the secondary constraints. Preservation in time of the primary constraints \( \Phi_\alpha \) leads to the equations \( \{\Phi_\alpha, \tilde{H}\} = \{\Phi_\alpha, H_0\} + \{\Phi_\alpha, \Phi_\beta\} v^\beta + \{\Phi_\alpha, T_\beta\} s^b = 0 \). In turn, preservation of the secondary constraints \( T_\alpha \) leads to the similar equations \( \{T_\alpha, \tilde{H}\} = \{T_\alpha, H_0\} + \{T_\alpha, \Phi_\beta\} v^\beta + \{T_\alpha, T_\beta\} s^b = 0 \). To continue the analysis, it is convenient to unify them as follows:

\[ \{G_I, H_0\} + \{G_I, G_J\} S^J = 0. \]  
(31)
Here $G_I$ are all the constraints of the initial formulation and it was denoted $S^J \equiv (v^\alpha, s^a)$. Using the matrix (12), the system (31) can be rewritten in the equivalent form

$$\{ \tilde{G}_{I_1}, H_0 \} + O(G_I) = 0,$$

$$\{ \tilde{G}_{I_2}, H_0 \} + \{ \tilde{G}_{I_2}, G_J \} S^J = O(G_I). \quad (32)$$

Eq. (32) does not contain any new information, since the first class constraints commute with the Hamiltonian, see Eq. (13). Let us analyze the system (33). First, one notes that due to the rank condition \( \text{rank}\{ \tilde{G}_{I_2}, G_J \}_I = [I_2] = \max \), exactly \([I_2]\) variables among \( S^I \) can be determined from the system. According to the Dirac prescription, one needs to determine the maximal number of the multipliers \( v^\alpha \). To make this, let us restore \( v \)-dependence in Eq. (33): \( \{ \tilde{G}_{I_2}, \Phi_\lambda \} v^\alpha + \{ \tilde{G}_{I_2}, H_0 \} + \{ \tilde{G}_{I_2}, T_b \} s^b = 0. \) Since the matrix \( \{ \tilde{G}_{I_2}, \Phi_\lambda \} \) is the same as in the initial formulation, from these equations one determines some group of variables \( v^{\alpha_2} \) through the remaining variables \( v^{\alpha_1} \), where \([\alpha_2]\) is the number of primary second-class constraints among \( \Phi_\alpha \). After substitution of the result into the remaining equations of the system (33), the latter acquires the form

$$v^{\alpha_2} = v^{\alpha_2}(q, \tilde{p}, s^a, v^{\alpha_1}), \quad Q_{\alpha_2 b}(q, \tilde{p}) s^b + P_{\alpha_2}(q, \tilde{p}) = 0, \quad (34)$$

where \([a_2]\) is the number of higher-stage second class constraints of the initial theory. It must be \( P \approx 0 \), since for \( s^b = 0 \) the system (33) is a subsystem of (11), but the latter vanish after substitution of the multipliers determined during the procedure, see discussion after Eq. (11). Besides, one notes that \( \text{rank}Q = [a_2] = \max \). Actually, suppose that \( \text{rank}Q = [a'] < [a_2] \). Then from Eq. (33) only \([\alpha_2] + [a'] < [I_2]\) variables among \( S^I \) can be determined, in contradiction with the conclusion made before. In resume, the system (31) for determining the second-stage and third-stage constraints and multipliers is equivalent to

$$v^{\alpha_2} = v^{\alpha_2}(q, \tilde{p}, s^{a_1}, v^{\alpha_1}), \quad (35)$$

$$s^{a_2} = \tilde{Q}^{a_2 b_1}(q, \tilde{p}) s^{b_1}, \quad (36)$$
with some matrix $\tilde{Q}$. Conservation in time of the constraints (36) leads to the equations for determining the multipliers

$$v^{a_2} = \{Q^{a_2 b_1}(q, \tilde{p}) s^{b_1}, \tilde{H}\}.$$  \hspace{1cm} (37)

Since there are no new constraints, the Dirac procedure for $\tilde{L}$ stops on this stage. All the constraints of the theory have been revealed after completing the third stage.

Now we are ready to compare the theories $\tilde{L}$ and $L$. Dynamics of the theory $\tilde{L}$ is governed by the Hamiltonian equations

$$\dot{q}^A = \{q^A, H\} + s^a \{q^A, T_a\}, \quad \dot{\tilde{p}}_A = \{\tilde{p}_A, H\} + s^a \{\tilde{p}_A, T_a\},$$

$$\dot{s}^a = v^a, \quad \dot{\pi}_a = 0,$$  \hspace{1cm} (38)

as well as by the constraints

$$\Phi_\alpha = 0, \quad T_a = 0,$$  \hspace{1cm} (39)

$$\pi_{a_1} = 0,$$  \hspace{1cm} (40)

$$\pi_{a_2} = 0, \quad s^{a_2} = Q^{a_2 b_1}(q, \tilde{p}) s^{b_1}.$$  \hspace{1cm} (41)

Here $H$ is the complete Hamiltonian of the initial theory (7), and the Poisson bracket is defined on the phase space $q^A, s^a, \tilde{p}_A, \pi_a$. The constraints $\pi_{a_1} = 0$ can be replaced by the combinations $\pi_{a_1} + \pi_{a_2} Q^{a_2 a_1}(q, \tilde{p}) = 0$, the latter represent first class subset. Let us make partial fixation of a gauge by imposing the equations $s^{a_1} = 0$ as a gauge conditions for the subset. Then $(s^a, \pi_a)$-sector of the theory disappears, whereas the equations (38), (39) coincide exactly with those of the initial theory $L$. Let us reminded that $\tilde{L}$ has been constructed in some vicinity of the point $s^a = 0$. The gauge $s^{a_1} = 0$ implies $s^a = 0$ due to the homogeneity of Eq. (36). It guarantees a self consistency of the construction. Thus $\tilde{L}$ represents one of the gauges [3] for $\tilde{L}$, which proves an equivalence of the two formulations.

\footnote{In more rigorous treatment, one writes Dirac bracket corresponding to the equations $\pi_{a_1} + \pi_{a_2} Q^{a_2 a_1}(q, \tilde{p}) = 0$, $s^{a_1} = 0$, and to the second class constraints (41). After that, the equations used in construction of the Dirac bracket can be used as strong equalities. For the case, they reduce to the equations $s^a = 0, \pi_{a_1} = 0$. For the remaining phase-space variables $q^A, \tilde{p}_A$, the Dirac bracket coincides with the Poisson one.}
Using Eqs. (18) (19), the extended Lagrangian (21) can be rewritten in the equivalent form

\[
\tilde{L}(q^A, \dot{q}^A, s^a) = L(q^A, \dot{q}^i - s^a \partial T_a(q^a, \omega_i), \dot{q}^a) + \\
s^a \left( \omega_i \frac{\partial T_a(q^a, \omega_i)}{\partial \omega_i} - T_a(q^A, \omega_i) \right)
\]

(42)

Modulo to the extra term represented by the second line in Eq. (42), \(\tilde{L}\) is obtained from \(L\) replacing the derivative \(\dot{q}^i\) by the quantity similar to the covariant derivative

\[
\partial_r q^i \longrightarrow D_r q^i = \partial_r q^i - s^a \frac{\partial T_a(q^a, \omega_i)}{\partial \omega_i}.
\]

(43)

The second line in Eq. (42) disappears when the higher stage constraints are homogeneous on momenta. For example, for the constraints of the form \(T_a = p_a\), where \(p_a\) is a part of the momenta \(p_i = (p_a, p_i')\), the extended action acquires the form

\[
\tilde{L} = L(q^A, \dot{q}^a - s^a h_a^i(q), \dot{q}^a).
\]

(44)

For the case \(T_a = h_a^i(q)p_i\) the extended Lagrangian is

\[
\tilde{L} = L(q^A, \dot{q}^i - s^a h_a^i(q), \dot{q}^a).
\]

(45)

In both cases, it can be shown that \(\tilde{L}\) is invariant under the local transformations with the transformation law for \(s^a\) being proportional to \(\dot{\epsilon}^a\). So, at least for these particular examples, \(s^a\) can be identified with a gauge field supplying the local symmetry. It leads to the suggestion that in the passage from \(L\) to \(\tilde{L}\) the local symmetries with higher order derivatives of the local parameters decompose into a sum of the gauge symmetries (with at most one derivative acting on the parameters). We confirm this statement in the next section.

4 Local symmetries of the extended Lagrangian. Dirac conjecture.

Since the initial Lagrangian is a gauge for the extended one, the physical system under consideration can be equally analyzed using

\footnote{It is known that any first class system acquires this form in special canonical variables [3].}
the extended Lagrangian. Higher stage constraints $T_a$ of $L$ turn out to be the second stage constraints of $\tilde{L}$. They enter into the expressions for $\tilde{L}$ and $\tilde{H}$ in the manifest form, see Eqs. (21), (30). Here we demonstrate one of consequences of this property: all the infinitesimal local symmetries of $\tilde{L}$ are the gauge symmetries and can be found in closed form in terms of the first class constraints.

According to the analysis made in the previous section, the primary constraints of the extended formulation are $\Phi_\alpha = 0$, $\pi_a = 0$. Among $\Phi_\alpha = 0$ there are presented first class constraints, in a number equal to the number of primary first class constraints of $L$. Among $\pi_a = 0$, we have found the first class constraints $\pi_{a_1} - \pi_{a_2} Q^{a_2} a_1 (q, p) = 0$, in a number equal to the number of all the higher-stage first class constraints of $L$. Thus the number of primary first class constraints of $\tilde{L}$ coincide with the number $[I_1]$ of all the first class constraints of $L$. Hence one expects $[I_1]$ local symmetries presented in the formulation $\tilde{L}$. Now we demonstrate that they are:

$$\delta_{I_1} q^A = \epsilon^{I_1} \left\{ q^A, \tilde{G}_{I_1} (q^B, \tilde{p}_B) \right\} \Big|_{\tilde{p}_i \rightarrow \partial \tilde{L} / \partial \dot{q}_i} ,$$  

$$\delta_{I_1} s^a = \left[ \epsilon^{I_1} K_{I_1}^a + \epsilon^{I_1} \left( b_{I_1}^a + s^b c_{I_1 b}^a + \dot{q}^{I_1, \beta} c_{I_1, \beta}^a \right) \right] \Big|_{\tilde{p}_i \rightarrow \partial \tilde{L} / \partial \dot{q}_i} .$$

Here $\epsilon^{I_1} (\tau)$, $I_1 = 1, 2, \ldots, [I_1]$ are the local parameters, and $K$ is the conversion matrix, see Eq. (12).

According to Eq. (47) variation of some $s^a$ involve derivative of parameters. Hence they can be identified with a gauge fields for the symmetry. At this point, it is instructive to discuss what happen with local symmetries on the passage from $L$ to $\tilde{L}$. Appearance of some $N$-th stage first-class constraint in the Hamiltonian formulation for $L$ implies [15], that $L$ has the local symmetry of $^{(N-1)} \epsilon$-type (1). Replacing $L$ with $\tilde{L}$, one arrives at the formulation with the secondary first class constraints and the corresponding $\dot{c}$-type symmetries (16). That is the symmetry (1) of $L$ ”decomposes” into $N$ gauge symmetries of $\tilde{L}$.

According to Eq. (16), transformations of the original variables $q^A$ are generated by all the first class constraints of initial formulation. This result can be considered as a proof of the Dirac conjecture.

We now show that variation of $\tilde{L}$ under the transformation (16) is proportional to the higher stage constraints $T_a$. So, it can be
canceled by appropriate variation of \( s^a \), the latter turns out to be given by Eq. (47). In the subsequent computations we omit all the total derivative terms. Besides, the notation \( A| \) implies the substitution indicated in Eqs. (46), (17).

To make a proof, it is convenient to represent the extended Lagrangian (21) in terms of the initial Hamiltonian \( H_0 \), instead of the initial Lagrangian \( L \). Using Eq. (8) one writes

\[
\tilde{L}(q^A, \dot{q}^A, s^a) = \omega_i \dot{q}^i + f_a(q^A, \omega_j) \dot{q}^a - H_0(q^A, \omega_j) - s^a T_a(q^A, \omega_j),
\]

where the functions \( \omega_i(q, \dot{q}, s) \), \( f_a(q, \omega) \) are defined by Eqs. (17), (5).

According to the identity (23), variation of \( \tilde{L} \) with respect to \( \omega_i \) does not give any contribution. Taking this into account, variation of Eq. (47) under the transformation (46) can be written in the form

\[
\delta \tilde{L} = -\omega_i(q, \dot{q}, s) \left( \frac{\partial \tilde{G}_I}{\partial \dot{p}_i} \right) \left| \epsilon^{I_1} - f_a(q_0, \omega(q, \dot{q}, s)) \frac{\partial \tilde{G}_I}{\partial \dot{p}_a} \right| \epsilon^{I_1}
- \left( \frac{\partial H_0(q^A, \tilde{p}_j)}{\partial q^A} + q^a \frac{\partial \Phi_a}{\partial q^A} + s^a \frac{\partial T_a(q^A, \tilde{p}_j)}{\partial q^A} \right) \{ q^A, \tilde{G}_I \} \epsilon^{I_1}
- \delta_{I_1} s^a T_a(q^A, \omega_j).
\]

To see that \( \delta \tilde{L} \) is the total derivative, we add the following zero

\[
0 \equiv \left[ \frac{\partial \tilde{L}}{\partial \omega_i} \right|_{\omega_i} \{ \tilde{p}_i, \tilde{G}_I \}
- \left( \frac{\partial H_0}{\partial \tilde{p}_j} + q^a \frac{\partial \Phi_a}{\partial \tilde{p}_j} + s^a \frac{\partial T_a}{\partial \tilde{p}_j} \right) \{ \tilde{p}_j, \tilde{G}_I \} \right| \epsilon^{I_1},
\]

to the r.h.s. of Eq. (49). It leads to the expression

\[
\delta \tilde{L} = \left[ \epsilon^{I_1} \tilde{G}_I - \epsilon^{I_1} \left( \{ H_0, \tilde{G}_I \} + q^a \{ \Phi_a, \tilde{G}_I \} + s^a \{ T_a, \tilde{G}_I \} \right) \right]
- \delta_{I_1} s^a T_a(q^A, \omega_j) = \left[ \epsilon^{I_1} \tilde{G}_I + \epsilon^{I_1} \left( b_{I_1}^a + q^a c_{I_1 a} + s^b c_{I_1 b} \right) G_I \right] - \delta_{I_1} s^a T_a(q^A, \omega_j),
\]

where \( b, c \) are coefficient functions of the constraint algebra (13). Using the equalities \( G_I = (0, T_a(q^A, \omega_j)) \), \( \tilde{G}_I = K_{I_a} T_a(q^A, \omega_j) \), one finally obtains

\[
\delta \tilde{L} = \left[ \epsilon^{I_1} K_{I_a} + \epsilon^{I_1} \left( b_{I_1}^a + q^a c_{I_1 a} + s^b c_{I_1 b} \right) - \delta_{I_1} s^a \right]_{p_i, \omega_i} T_a.
\]
Then the variation of $s^a$ given in Eq. (46) implies $\delta \tilde{L} = \text{div}$, as it has been stated.

In the absence of second class constraints, Eqs. (46), (47) acquire the form

$$
\delta I q^A = \epsilon^I \left\{ q^A, G_I(q^B, \tilde{p}_B) \right\} \bigg|_{\tilde{p}_i \to \frac{\partial \tilde{L}}{\partial \dot{q}_i}},
$$

$$
\delta I s^a = \left[ \dot{\epsilon}^a \delta a I + \epsilon^I \left( b_I^a + s^b c_I^b + \dot{q}^\beta c_I^\beta a \right) \right] \bigg|_{\tilde{p}_i \to \frac{\partial \tilde{L}}{\partial \dot{q}_i}}. \quad (53)
$$

They can be used to construct symmetries of the original Lagrangian. To this end, one notes that the extended Lagrangian coincide with the original one for $s^a = 0$: $\tilde{L}(q, 0) = L(q)$, see Eq. (22). So the initial action will be invariant under any transformation

$$
\delta q^A = \sum_{I_i} \delta_I q^A \bigg|_{s=0}, \quad (54)
$$

which obeys to the system $\delta s^a|_{s=0} = 0$, that is

$$
\dot{\epsilon}^I K_I^a + \epsilon^I \left( b_I^a + \dot{q}^\beta c_I^\beta a \right) = 0. \quad (55)
$$

One has $[a]$ equations for $[\alpha] + [a]$ variables $\epsilon^I$. Similarly to Ref. [4], the equations can be solved by pure algebraic methods, which give some $[a]$ of $\epsilon$ in terms of the remaining $\epsilon$ and their derivatives of order less than $N$. It allows one to find $[\alpha]$ local symmetries of $L$. As it was already mentioned, the problem here is to prove the completeness and the irreducibility of the set.

5 Examples

1) Model with fourth-stage constraints. Let us consider the Lagrangian

$$
L = \frac{1}{2} (\dot{x})^2 + \xi(x)^2, \quad (56)
$$

where $x^\mu(\tau), \xi(\tau)$ are configuration space variables, $\mu = 0, 1, \ldots, n$, $(x)^2 \equiv \eta_{\mu \nu} x^\mu x^\nu$, $\eta_{\mu \nu} = (-, +, \ldots, +)$.

Denoting the conjugate momenta for $x^\mu, \xi$ as $p_\mu, p_\xi$, one obtains the complete Hamiltonian

$$
H_0 = \frac{1}{2} p^2 - \xi(x)^2 + v_\xi p_\xi, \quad (57)
$$
where \( v_\xi \) is multiplier for the primary constraint \( p_\xi = 0 \). The complete system of constraints turns out to be

\[
\Phi_1 \equiv p_\xi = 0, \quad T_2 \equiv x^2 = 0, \quad T_3 \equiv xp = 0, \quad T_4 \equiv p^2 = 0.
\]  
(58)

For the case, the variable \( \xi \) plays the role of \( q^\alpha \), while \( x^\mu \) play the role of \( q^i \) of the general formalism.

The constraints are first class

\[
\{G_I, G_J\} = c_{IJK}(q^A, p_j)G_K, \quad \{G_I, H_0\} = b_I^J(q^A, p_j)G_J,
\]  
(59)

with non vanishing coefficient functions being

\[
c_{23}^2 = -c_{32}^2 = 2, \quad c_{24}^3 = -c_{42}^3 = 4, \quad c_{34}^4 = -c_{43}^4 = 2;
\]
\[
b_1^2 = 1, \quad b_2^3 = 2, \quad b_3^4 = 1, \quad b_3^3 = 2\xi, \quad b_4^3 = 4\xi.
\]  
(60)

For the present case, Eq. (16) acquires the form

\[
\dot{x}^\mu - \tilde{p}^\mu - s^3x^\mu - 2s^4\tilde{p}^\mu = 0,
\]  
(61)

so

\[
\tilde{p}^\mu = \frac{1}{1 + 2s^4}(\dot{x}^\mu - s^3x^\mu).
\]  
(62)

The r.h.s. represents the function \( \omega \) of the general formalism. Then the extended Lagrangian (12) is given by

\[
\tilde{L} = \frac{1}{2(1 + 2s^4)}(\dot{x}^\mu - s^3x^\mu)^2 + (\xi - s^2)(x^\mu)^2.
\]  
(62)

Using the equations (53), (60), its symmetries can be written immediately as follows

\[
\delta_1 \xi = \epsilon^1, \quad \delta_1 s^2 = \epsilon^1;
\]  
(63)

\[
\delta_2 s^2 = \epsilon^2 + 2\epsilon^2 s^3, \quad \delta_2 s^3 = 2\epsilon^2(1 + 2s^4);
\]  
(64)

\[
\delta_3 x^\mu = \epsilon^3 x^\mu, \quad \delta_3 s^2 = 2\epsilon^3(\xi - s^2), \quad \delta_3 s^3 = \epsilon^3, \quad \delta_3 s^4 = \epsilon^3(1 + 2s^4);
\]  
(65)

\[
\delta_4 x^\mu = 2\epsilon^4 \frac{\dot{x}^\mu - s^3x^\mu}{1 + 2s^4}, \quad \delta_4 s^3 = 4\epsilon^4(\xi - s^2), \quad \delta_4 s^4 = \epsilon^4 - 2\epsilon^4 s^3.
\]  
(66)

Since the initial Lagrangian \( L \) implies the unique chain of four first class constraints, one expects that it has one local symmetry of \( \epsilon^3 \)-type. The symmetry can be found according to the defining equations (53), for the case

\[
\begin{align*}
\epsilon^1 + \epsilon^2 + 2\epsilon^3\xi \\
2\epsilon^2 + \epsilon^3 + 4\epsilon^4\xi \\
\epsilon^3 + \epsilon^4
\end{align*}
= 0.
\]  
(67)
It allows one to find $\epsilon^1, \epsilon^2, \epsilon^3$ in terms of $\epsilon^4 \equiv \epsilon$: $\epsilon^1 = -\frac{1}{2} \epsilon + 4\dot{\epsilon} \xi + 2\epsilon\dot{\xi}$, $\epsilon^2 = \frac{1}{2} \epsilon - 2\epsilon\xi$, $\epsilon^3 = -\dot{\epsilon}$. Using Eq. (54), local symmetry of the Lagrangian (56) is given by

$$
\delta x^\mu = -\dot{\epsilon} x^\mu + 2\epsilon \dot{x}^\mu, \quad \delta \xi = -\frac{1}{2} \epsilon + 4\dot{\epsilon} \xi + 2\epsilon\dot{\xi}.
$$

(68)

2) Model with first and second class constraints. Consider a theory with configuration space variables $x^\mu$, $e$, $g$ (where $\mu = 0, 1, 2, 3$, $\eta_{\mu\nu} = (-, +, +, +)$), and with action being

$$
S = \int d\tau \left( \frac{1}{2e} (\dot{x}^\mu - g x^\mu)^2 + \frac{g^2}{2e} \right), \quad a = \text{const.}
$$

(69)

One obtains the complete Hamiltonian

$$
H = \frac{1}{2} e p^2 + g(xp) - \frac{g^2}{2e} + v_e p_e + v_g p_g,
$$

(70)

as well as the constraints

$$
\Phi_1 \equiv p_e = 0, \quad T_1 \equiv -\frac{1}{2} (p^2 + \frac{g^2}{e^2}) = 0;
$$

$$
\Phi_2 \equiv p_g = 0, \quad T_2 \equiv \frac{g}{e} - (xp) = 0.
$$

(71) (72)

They can be reorganized with the aim to separate the first class constraints

$$
\Phi_1 \equiv p_e + \frac{g}{e} p_g = 0, \quad T_1 \equiv -\frac{1}{2} (p^2 - \frac{g^2}{e^2}) - \frac{g}{e} (xp) + \frac{g^2}{e} p_g = 0;
$$

(73)

$$
p_g = 0, \quad \frac{g}{e} - (xp) = 0.
$$

(74)

The first (second) line represents the first (second) class subsets.

For the case, solution of the basic equation (16) is given by

$$
\tilde{p}^\mu = \frac{1}{e - s^2} (\dot{x}^\mu - (g - s^2) x^\mu).
$$

(75)

Using the equations (71), (72), (75) one obtains the extended Lagrangian (42)

$$
\tilde{L} = \frac{1}{2(e - s^1)} (\dot{x}^\mu - (g - s^2) x^\mu)^2 + \frac{g^2}{2e} (1 + \frac{s^1}{e}) - \frac{g}{e} s^2.
$$

(76)
Its local symmetries are obtained according to Eqs. \((46), (47)\) using the expression \((73)\) for the first class constraints

\[
\delta_1 x^\mu = -\epsilon^1 \left( \omega^\mu + \frac{g}{e} x^\mu \right), \quad \delta_1 e = 0, \quad \delta_1 g = \epsilon^1 \frac{g^2}{e}, \\
\delta_1 s^1 = \epsilon^1 - 2\epsilon^1 \left( \frac{g s^1}{e} - s^2 \right), \quad \delta_1 s^2 = \left( \epsilon^1 \frac{g}{e} \right) + \epsilon^1 \frac{g^2}{e}; \quad (77)
\]

\[
\delta_2 x^\mu = 0, \quad \delta_2 e = \epsilon^2, \quad \delta_2 g = \epsilon^2 \frac{g}{e}, \\
\delta_2 s^1 = \epsilon^2, \quad \delta_2 s^2 = \epsilon^2 \frac{g}{e}; \quad (78)
\]

Here \(\omega^\mu\) is the r.h.s. of the equation \((75)\). By tedious computations one verifies that the variation \(\delta_1 \tilde{L}\) is the total derivative \(\delta_1 \tilde{L} = \frac{1}{2} \left( \epsilon^1 (\omega^\mu)^2 + \epsilon^1 \left( \frac{g}{e} \right)^2 \right)\).

In the presence of second class constraints, local symmetries of \(L\) can not be generally restored according to the trick \((54), (55)\). The reason is that a number of equations of the system \((55)\) can be equal or more than the number of parameters \(\epsilon^a\). In particular, for the present example one obtains just two equations for two parameters 

\[
\epsilon^1 + \epsilon^2 = 0, \quad \epsilon^1 \frac{g}{e} + \epsilon^2 \frac{g^2}{e} = 0.
\]

3) Maxwell action. Consider the Maxwell action of electromagnetic field

\[
S = -\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu} = \int d^4 x \left[ \frac{1}{2} \left( \partial_0 A_i - \partial_i A_0 \right)^2 - \frac{1}{4} (F_{ij})^2 \right]. \quad (79)
\]

For the case, the functions \(v^i\) from Eq. \((6)\) are given by \(p_i + \partial_i A_0\). The action implies primary and secondary constraints

\[
p_0 = 0, \quad \partial_i p_i = 0. \quad (80)
\]

Then the basic equation \((16)\) acquires the form \(\partial_0 A_i - \omega_i - \partial_i A_0 + \partial_i s = 0\), and the extended Lagrangian action \((42)\) is

\[
\tilde{S} = \int d^4 x \left[ \frac{1}{2} \left( \partial_0 A_i - \partial_i A_0 + \partial_i s \right)^2 - \frac{1}{4} (F_{ij})^2 \right]. \quad (81)
\]

Its local symmetries can be immediately written according to Eqs. \((53)\), the nonvanishing variations are

\[
\delta_\beta A_0 = \beta, \quad \delta_\beta s = \beta;
\]

\footnote{In transition from mechanics to a field theory, derivatives are replaced by the variational derivatives. In particular, the last term in Eq. \((10)\) reads \(\frac{1}{2} \sum_{a=1}^3 \int d^4 y s^a(x) T_a(q^4(y), \omega_i(y)).\)
\[ \delta_{\alpha} A_i = -\partial_i \alpha, \quad \delta_{\alpha} s = \partial_0 \alpha. \]  
(82)

Symmetry of the initial action appears as the following combination

\[
(\delta_{\beta} + \delta_{\alpha}) A_i = -\partial_i \alpha, \\
(\delta_{\beta} + \delta_{\alpha}) A_0 = \beta,
\]
(83)

where the parameters obey to the equation \( \partial_0 \alpha + \beta = 0 \). The substitution \( \beta = -\partial_0 \alpha \) into Eq. (83) gives the standard form of \( U(1) \) gauge symmetry

\[ A'_\mu = A_\mu + \partial_\mu \alpha. \]  
(84)

6 Conclusion

In this work we have proposed an improvement of the extended Hamiltonian formalism for an arbitrary constrained system. Singular theory of a general form (with first and second class constraints of an arbitrary stage) can be reformulated as a theory that does not generate any constraints beyond the third stage. It is described by the extended Lagrangian constructed in terms of the original one according to Eq. (21). All the higher-stage constraints of \( L \) turn out to be the second-stage constraints of \( \tilde{L} \). The formalism implies an extension of the original configuration space \( q^A \) by the auxiliary variables \( s^a \). Number of them is equal to the number of all the higher stage constraints \( T_a \) of original formulation. Those of the extra variables \( s^a \) that correspond to the first class constraints, have been identified with the gauge fields supplying local symmetries of \( \tilde{L} \). Hence in the passage from \( L \) to \( \tilde{L} \), local symmetries of \( L \) with higher order derivatives of the local parameters decompose into a sum of the gauge type symmetries.

As an application of the extended Lagrangian formalism, we have presented a relatively simple way for obtaining the local symmetries of a singular Lagrangian theory. By construction, the extended Lagrangian implies only \( \dot{\epsilon} \)-type symmetries, that can be immediately written according to Eqs. (46), (47). The latter give the symmetries in terms of the first class constraints \( \tilde{G}_{I} \) of the initial formulation and the coefficient functions of the constraint algebra (13). Generators of transformations for all the original variables \( q^A \) turn out to be the Lagrangian counterparts of canonical transformations generated
by $\tilde{G}_I$. This result can be considered as a proof of the Dirac conjecture [1]. In contrast to a situation with symmetries of $L$ [14-16], the transformations (46) do not involve the second class constraints.

The extended formulation can be appropriate tool for development of a general formalism for conversion of second class constraints into the first class ones according to the ideas of the work [18]. To apply the method proposed in [18], it is desirable to have the formulation with some configuration space variables entering into the Lagrangian without derivatives. It is just what happen in the extended formulation.

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