Matroidal Approximations of Independence Systems

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June 17, 2019

Abstract [Milgrom (2017)] has proposed a heuristic for determining a maximum weight basis of an independence system $I$ given that for sets from $O \subseteq I$ we care in particular about the quality of approximation as we assume, that $O$ contains (frequently) the optimal basis. It is based on finding an ‘inner matroid’, one contained in the independence system. We show that without additional assumptions on $O$ being different from $I$ the worst-case performance of this new heuristic is no better than that of the classical greedy algorithm.

Keywords independence system · greedy algorithm · matroid.

1 Introduction

An independence system consists of a ground set $E = \{1, \ldots, n\}$ and a family $I$ of subsets of $E$. The pair $(E, I)$ is a called an independence system if $\emptyset \in I$ and for all $B \subseteq A \in I$ we have $B \in I$ as well. Elements of $I$ are called independent. For any $A \subseteq E$, a set $B \subseteq A$ is called a basis of $A$ if $B \in I$ and $B \cup \{j\} \notin I$ for all $j \in A \setminus B$. For $A \subseteq E$ denote by $B_I(A)$ (or $B(A)$ when no ambiguity) the set of bases of $A$ (with respect to $I$). The bases with respect to $E$ are denoted by $B_E$ or just $B$. Minimally dependent sets are called circuits.

If we associate weights $v_i \in \mathbb{R}_+$ for each $i \in E$, the problem of finding a maximum weight basis of $(E, I)$ can be expressed as $\max_{A \subseteq E} \sum_{i \in A} v_i$. For convenience we will write $\sum_{i \in S} v_i$ as $v(S)$ for all $S \subseteq E$. The problem is NP-hard by reduction to Hamiltonian path in a directed graph [Korte and Hausmann, 1978].

Call an independence systems $(E, I)$ with $\{i\} \in I$ for all $i \in E$ normal (for graphs or matroids, the term loopless is more prevalent). A non-normal independence system can be turned into a normal one by deleting the elements $e \in E$ with $\{e\} \notin I$ without changing the solution to the problem of finding a maximum weight basis.

Milgrom (2017) proposed a heuristic for determining a maximum weight basis of an independence system $I$ given that for sets from $O \subseteq I$ we care in particular about the quality of approximation as we assume, that $O$ contains (frequently) the optimal basis. It is based on finding an ‘inner matroid’, one contained in the independence system. We compare the worst-case performance of this heuristic with the well known greedy algorithm for the same problem. We show that the worst-case performance of this new heuristic without additional assumptions on $O$ being different from $I$ is no better than that of the classical greedy algorithm. Nevertheless, as we show by example, there are aspects of Milgrom’s proposal that bear further investigation.
We defer a discussion of these matters till the end. In the next section we describe the greedy algorithm and state its worst case performance.

2 Greedy Algorithm

We describe the greedy algorithm for finding a basis of \( E \) with possibly large weight.

**ALGORITHM 1:** Greedy Algorithm

**Given:** An independence system \((E, I)\) represented by an independence oracle. Weights \( v_i \in \mathbb{R}_+ \) for all \( i \in E \).

**Goal:** An independent set \( I_g \in I \) of "large" value.

1. Order the elements of \( E \) by non-increasing value \( v_1 \geq v_2 \geq \ldots \geq v_n \geq 0 \);
2. Set \( I \leftarrow \emptyset \);
3. for \( i \leftarrow 1 \) to \( n \) do
   4. if \( I \cup \{i\} \) independent (oracle call) then \( I \leftarrow I \cup \{i\} \);
5. end
6. Let \( I_g \leftarrow I \) and return \( I_g \).

**Remark:** If some coefficients in the objective function are equal, the order of the \( v_i \)'s is not unique and therefore the algorithm’s outcome is not necessarily unique. To avoid this assume an exogenously given tie breaking rule, so that the outcome of the greedy algorithm is unique.

A worst case bound on the quality of the greedy solution in terms of the rank quotient of an independence system can be found in Jenkyns (1976), Korte and Hausmann (1978).

**Definition 1** Let \((E, I)\) be an independence system. The rank of \( F \subseteq E \) is defined by

\[
    r(F) := \max \{|B| : B \text{ basis of } F\}.
\]

The rank-function maps \( 2^E \) into the nonnegative integers.

Similarly, for \( F \subseteq E \) we define

\[
    l(F) := \min \{|B| : B \text{ basis of } F\}
\]

to be the lower rank of \( F \). The rank quotient of \( I \) is denoted by

\[
    q(I) := \min_{F \subseteq E, r(F) > 0} \frac{l(F)}{r(F)}.
\]

Following an axiomatization of matroids by Hausmann et al. (1980), we define a matroid to be an independences system \((E, I)\) with \( q(I) = 1 \). For matroids there are several equivalent characterizations known, some of which we will use later:

Basis exchange: For every pair of bases \( B_1, B_2 \in B \) and \( i \in B_2 \setminus B_1 \) there exists a \( j \in B_1 \setminus B_2 \) such that \( (B_1 \cup \{i\}) \setminus \{j\} \) is again a basis.

Augmentation property: For every pair of independent sets \( I, J \in I \) with \( |I| < |J| \) there exists \( j \in J \setminus I \) such that \( I \cup \{j\} \) is again independent.

Rank axioms: see (R1)-(R3) on page 8.

For proofs on these and more see Oxley (1992).

If \((E, I)\) is a normal independence system, then,

\[
    q(I) \in \left\{ \frac{i}{j} : 1 \leq i, j \leq r(I) \right\}.
\]

Hence for normal independence systems, \( q(I) \geq 1/r(I) \). However, a better bound is known:

**Theorem 1** (Hausmann et al. 1980) Let \((E, I)\) be an independence system. If, for any \( A \in I \) and \( e \in E \) the set \( A \cup \{e\} \) contains at most \( p \) circuits, then \( q(I) \geq 1/p \).

If an independence system \( I \) is the intersection of \( p \) matroids, then \( q(I) \geq 1/p \).
Theorem 2 (Jenkyns, 1976, Korte and Hausmann, 1978) For an independence system $\mathcal{I}$ on $E$ with objective function $v \geq 0$, let $I_g$ be the solution returned by Algorithm (1) and $I_o$ be a maximum weight basis. Then,

$$q(\mathcal{I}) \leq \frac{v(I_g)}{v(I_o)} \leq 1.$$ 

Also, for every independence system there are weights $v \in \{0, 1\}^E$, such that the first inequality holds with equality.

The only attempt to improve upon this bound we are aware of involves incorporating a partial enumeration stage into the greedy algorithm see Hausmann et al. (1980).

3 Inner Matroid

Milgrom (2017) proposes an alternative to direct application of the greedy algorithm. It has two parts. The first is the introduction of a-priori information on where the optimal basis may lie. Formally, let $\mathcal{O} \subseteq \mathcal{I}$ be a collection of independent sets one of which is conjectured to be an optimal weight basis. $\mathcal{O}$ need not satisfy the hereditary property (so $J \subset I \in \mathcal{O}$ does not imply $J \in \mathcal{O}$) and may exclude a basis that is in fact optimal (see Example 4). Call $\mathcal{O}$ the acceptable set. The second part finds a matroid ‘inside’ the independence system (but not necessarily containing all of $\mathcal{O}$) and applies the greedy algorithm to find an optimal weight basis of that matroid. We describe this approach here.

Given an independence system $(E, \mathcal{I})$, an acceptable set $\mathcal{O}$, and a weight vector $v \in \mathbb{R}_+^E$ let

$$V_\ast(I, v) = \max_{S \in \mathcal{I}} v(S) \text{ and } V_\ast(\mathcal{O}, v) = \max_{S \in \mathcal{O}} v(S)$$

and

$$V_\ast(I, v) = \text{value of the greedy solution.}$$

Independence system $(E, \mathcal{M})$ is a matroid by previous definition involving rank quotient and Theorem 2 if and only if

$$V_\ast(\mathcal{M}, v) = V_\ast(I, v) \quad \forall v \in \mathbb{R}_+^E.$$ 

We say that the independence system $(E, \mathcal{J})$ is contained in $(E, \mathcal{I})$, an inner independence system, if $\mathcal{J} \subseteq \mathcal{I}$. If $(E, \mathcal{J})$ is a matroid it is called an inner matroid.

The approximation quality of $\mathcal{J}$ for $\mathcal{I}$ with respect to the acceptable set $\mathcal{O} \subseteq \mathcal{I}$ is given by

$$\rho(\mathcal{I}, \mathcal{O}, \mathcal{J}) := \min_{S \in \mathcal{O}\{\emptyset\}} \min_{S' \in B_{\mathcal{J}}(S)} \frac{|S'|}{|S|},$$

so this picks the element from $\mathcal{O}$ that is least well approximated by a basis of $\mathcal{J}$. If $l_{\mathcal{J}}$ denotes the lower rank function of $(E, \mathcal{J})$ then

$$\rho(\mathcal{I}, \mathcal{O}, \mathcal{J}) = \min_{S \in \mathcal{O}\{\emptyset\}} l_{\mathcal{J}}(S) \frac{|S|}{|S'|}.$$ 

If $\mathcal{M}$ is an inner matroid, then $r_\mathcal{M} = l_\mathcal{M}$ and

$$\rho(\mathcal{I}, \mathcal{O}, \mathcal{M}) = \min_{S \in \mathcal{O}\{\emptyset\}} \frac{r_\mathcal{M}(S)}{|S|} = \min_{S \in \mathcal{O}\{\emptyset\}} \max_{S' \in \mathcal{M}; S' \subseteq S} \frac{|S'|}{|S|},$$

which is the form Milgrom (2017) originally chose.

It is natural to ask when $\rho(\mathcal{I}, \mathcal{O}, \mathcal{M}) = 1$.

Theorem 3 If $(E, \mathcal{I})$ is a normal independence system with acceptable set $\mathcal{O} \subseteq \mathcal{I}$, then, there exists a matroid $\mathcal{M}$ with $\rho(\mathcal{I}, \mathcal{O}, \mathcal{M}) = 1$ if and only if $\mathcal{I}$ has an inner matroid containing $\bigcup_{F \subseteq B \in \mathcal{O}} F$.

Proof If $(\mathcal{I}, \mathcal{O}, \mathcal{M}) = 1$ then $\mathcal{O} \subseteq \mathcal{M}$. For $I \subseteq J \in \mathcal{O}$ follows $J \in \mathcal{M}$ and by hereditary property of matroid $I \in \mathcal{M}$. □

We give three examples to suggest that nothing stronger is possible.
Definition 2 Let $U_n^k$ denote the uniform matroid of all subsets of an n-element set of cardinality at most $k$, where clearly $0 \leq k \leq n$. If we want to specify the groundset $E$ of $n$ elements explicitly, we can also write $U_E^k$.

Example 1 In all cases the ground set is $E = \{1, 2, 3, 4\}$.

- Let $\mathcal{I} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}\}$. $(E, \mathcal{I})$ is not a matroid as the basis exchange axiom is violated. If $\mathcal{O} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$, then, $\rho(\mathcal{I}, \mathcal{O}, U^1_1) = 1$. This demonstrates, that the condition “$\mathcal{O}$ is matroid” is too strong. However, in this case $\mathcal{O}^I$ is a matroid.

- Again, let $\mathcal{I} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}\}$. Hence $(E, \mathcal{I})$ is not a matroid (again). If $\mathcal{O} = \{\{1, 2\}, \{3, 4\}\}$, then, $\rho(\mathcal{I}, \mathcal{O}, U^1_1) = 1/2$. In this case $\mathcal{O}^I$ is not a matroid.

- Let $\mathcal{I} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}\}$. $(E, \mathcal{I})$ is not a matroid, since $\{1, 2\}$ and $\{3, 4\}$ are bases for $E$ of different sizes. If $\mathcal{O} = \{\{2, 3\}, \{3, 4\}\}$ the smallest inner matroid of $\mathcal{I}$ containing $\mathcal{O}$ is $\mathcal{M} = U^2_{\{2,3\}}$ and $\rho(\mathcal{I}, \mathcal{O}, \mathcal{M}) = 1$. However $\mathcal{O}^I$ is not a matroid, since it lacks the basis $\{2, 4\}$ that would be required by basis exchange. So requiring “$\mathcal{O}^I$ is a matroid” is too strong.

Remark: Usually, the intersection of all matroids containing $\mathcal{O}^I$ is not a matroid, therefore it is unsuitable for a notion of hull.

Remark: Clearly

\[ \rho(\mathcal{I}, \mathcal{O}, J) \geq \rho(\mathcal{I}, \mathcal{O}^I, J), \]

since the minimum on the right hand side is determined over a larger set.

However, $\rho(\mathcal{I}, \mathcal{O}, J) > \rho(\mathcal{I}, \mathcal{O}^I, J)$ is possible, as the following example demonstrates:

Example 2 Consider $E = \{1, 2, 3, 4, 5, 6\}$ and the independence system $\mathcal{I} = 2^{\{1,2,3,4\}} \cup 2^{\{3,4,5,6\}}$. Clearly $\{1\}, \{5, 6\} \in \mathcal{I}$. The augmentation property would require, that $\{1, 5\}$ or $\{1, 6\}$ has to be independent, which is not the case. Therefore $\mathcal{I}$ is not a matroid. Let $\mathcal{O} = \{\{1, 2, 3, 4\}, \{3, 4, 5, 6\}\}$ be our benchmark set and let $\mathcal{M} = 2^{\{1,2,3,4\}}$. Now, to determine $\rho(\mathcal{I}, \mathcal{O}, \mathcal{M}) = \min_{S \in \mathcal{O} \setminus \{\emptyset\}} \frac{r_\mathcal{M}(S)}{|S|}$ we have to consider only the two elements of $\mathcal{O}$:

$\mathcal{O} = \{1, 2, 3, 4\}$ is easy, since $r_\mathcal{M}(O) = 4$;
$\mathcal{O} = \{3, 4, 5, 6\}$ is as easy, since $r_\mathcal{M}(O) = 2$.

Together, we obtain $\rho(\mathcal{I}, \mathcal{O}, \mathcal{M}) = \frac{1}{4} = \frac{1}{4}$.

Now, $\mathcal{O}^I$ contains $\mathcal{O} = \{4, 5, 6\}$ with $r_\mathcal{M}(O) = 1$. Even worse $\mathcal{O} = \{6\} \in \mathcal{O}^I$ with $r_\mathcal{M}(O) = 0$ which demonstrates $\rho(\mathcal{I}, \mathcal{O}^I, \mathcal{M}) = 0 < \frac{1}{4} = \rho(\mathcal{I}, \mathcal{O}, \mathcal{M})$.

Corollary 1 The example demonstrates, that $\rho(\mathcal{I}, \mathcal{O}^I, \mathcal{M}) > 0$ is only possible if for all $e \in \bigcup_{\mathcal{O} \in \mathcal{O}} \mathcal{O}$ holds $\{e\} \in \mathcal{M}$

Example 3 (twin-peaks) Consider two disjoint sets $E_1$ and $E_2$ and integers $k_1, k_2$ such that $k_1 < |E_1| < k_2 < |E_2|$. Let $\mathcal{M}_1 = U^k_{E_1}$ and $\mathcal{M}_2 = U^k_{E_2}$.

Define $(E, \mathcal{I})$ to be the independence system on $E := E_1 \cup E_2$ where a set $A \subseteq E$ is independent if and only if $A$ is an independent set in $\mathcal{M}_1$ or $\mathcal{M}_2$, thus $\mathcal{I} = \mathcal{M}_1 \cup \mathcal{M}_2$. Call this a ‘twin-peaks’ independence system. Choose an $F \subseteq E$ which consists of one element from $E_1$ and $k_2$ elements from $E_2$. Now $l(F) = 1$ and $r(F) = k_2$. Since larger ratios between sets in this independence system are impossible, $q(\mathcal{I}) = \frac{1}{k_2}$.

To determine the best matroid approximating $O = \mathcal{I}$ contained in $\mathcal{I}$, we must examine all matroids $\mathcal{M}'$ contained in $\mathcal{I}$.

1. Suppose first that there exists $i \in E_1, j \in E_2$ such that $\{i\}, \{j\} \in \mathcal{M}'$. We show that the rank of $\mathcal{M}'$ is 1, i.e. the largest bases of $\mathcal{M}'$ are of size one. If not, there exists a set $F \in \mathcal{M}'$ of cardinality two. By construction of $\mathcal{I}$ the set $F$ must be a subset of $E_1$ or $E_2$, wlog let $F \subseteq E_1$; since $F$ and $\{j\}$ are independent sets in $\mathcal{M}'$ there has to be an element in $F$ with which we could augment $\{j\}$ to be an independent set $F'$ in $\mathcal{M}'$. But $F'$ contains elements from $E_1$ and $E_2$ and therefore is not independent in $\mathcal{I}$ and therefore not in $\mathcal{M}'$. 

The largest matroid of this kind contained in \( \mathcal{I} \) is the one, where all singletons are independent. Hence every independent set from \( \mathcal{I} \) is approximated by an arbitrary contained singleton and we obtain the worst case bound of 
\[
\frac{1}{\max(k_1, k_2)} = \frac{1}{k_2}.
\]

2. On the other hand, if rank \( \mathcal{M}' > 1 \) then we can conclude that either all independent sets of \( \mathcal{M}' \) are contained in \( E_1 \) or they are contained in \( E_2 \) and the inclusion-wise largest matroids contained in \( E_1 \) and \( E_2 \) are \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \).

For \( \mathcal{M}' = \mathcal{M}_1 \) approximating sets \( F \in \mathcal{M}_2 \setminus \{\emptyset\} \) is only possible with the empty set which yields a quotient of \( 0/|F| \) and the same for \( \mathcal{M}' = \mathcal{M}_2 \) approximating sets approximating nonempty sets \( F \in \mathcal{M}_1 \setminus \{\emptyset\} \) is only possible with the empty set which yields a quotient of \( 0/|F| \).

In summary we obtain the best approximation in the first case and can conclude
\[
\rho(\mathcal{I}, \mathcal{I}, \mathcal{M}') = \frac{1}{k_2} = q(\mathcal{I}).
\]

However, \( \rho \) can be much better if we exploit additional knowledge. As an illustration, suppose that \( \mathcal{O} = (E_2) \), i.e., all subsets of size at most \( k_2 \) of \( E_2 \). This might be justified by a situation, where we know that the optimal solutions, that occur in our environment, have to live in \( E_2 \), maybe since \( k_1 \ll k_2 \). Unsurprisingly, the matroid \( \mathcal{M}_2 \) gives us the best conceivable guarantee of 1:
\[
\rho(\mathcal{M}_2, \mathcal{M}_2, \mathcal{I}) = 1 > \frac{1}{k_2} = q(\mathcal{I}).
\]

[Milgrom (2017)] proposes that the maximum weight basis of an inner matroid \( \mathcal{M} \) be used as a solution to the problem of finding a maximum weight basis in \( (E, \mathcal{I}) \). The objective function value of this solution will be \( V^*(\mathcal{J}, v) \). [Milgrom (2017)] gives a bound on the quality of this solution in term of \( \rho(\mathcal{I}, \mathcal{O}, \mathcal{J}) \).

**Definition 3** For a given independence system \( (E, \mathcal{I}) \) and \( e \in E \) define the new independence system \( \mathcal{I} \setminus \{e\} \) on \( E \setminus \{e\} \) by \( \mathcal{I} \setminus \{e\} := \{I \in \mathcal{I} : e \notin I\} \).

**Remark:** It is easy to see, that \( (E \setminus \{e\}, \mathcal{I} \setminus \{e\}) \) is an independence systems again. Furthermore, if \( (E, \mathcal{I}) \) is a matroid, so is \( (E \setminus \{e\}, \mathcal{I} \setminus \{e\}) \).

**Lemma 4** Given an independence system \( (E, \mathcal{I}) \) and a set \( \mathcal{O} \) the set
\[
W := \{w \in \mathbb{R}_+^E : 1^Tw = 1\} \cap \{w \in \mathbb{R}_+^E : \emptyset \notin \mathcal{O} \cap \text{arg max}_{I \in \mathcal{I}} w(I)\}
\]
is compact.

**Proof** Clearly the set \( \{w \in \mathbb{R}_+^E : 1^Tw = 1\} \) compact. Also,
\[
\{w \in \mathbb{R}_+^E : \emptyset \notin \mathcal{O} \cap \text{arg max}_{I \in \mathcal{I}} w(I)\} = \bigcup_{O \in \mathcal{O}} \{w \in \mathbb{R}_+^E : O \notin \text{arg max}_{I \in \mathcal{I}} w(I)\},
\]
where each set on the right is a closed cone. As \( \mathcal{O} \) is finite the right hand side is closed. Therefore \( W \) is compact. \( \square \)

**Theorem 5** [Milgrom (2017)] Let \( (E, \mathcal{M}) \) be an inner matroid and \( W \) be the above defined set of non-negative and non-trivial weight vectors \( v \) such that at least one member of \( \mathcal{O} \) is a maximum weight basis of \( (E, \mathcal{M}) \). Then,
\[
\rho(\mathcal{I}, \mathcal{O}, \mathcal{M}) = \min_{v \in W} \frac{V^*(\mathcal{M}, v)}{V^*(\mathcal{O}, v)}.
\]

As a convenience for the reader we provide here an independent, fuller proof of the result than [Milgrom (2017)].
Proof If the theorem is false, there must exist an independence system \((E, T), \) a set \(O\) and an inner matroid \((E, M)\) such that

\[
\rho(T, O, M) > \min_{v \in W} \frac{V^*(M, v)}{V^*(O, v)}.
\]

Among all such counterexamples choose one where \(|E|\) is minimal. Among all such counterexamples choose one, where the support of \(v\) is smallest. Denote the size of the support by \(\ell\).

Let

\[
v^* \in \arg\min_{v \in W: \text{supp}(v) \leq \ell} \frac{V^*(M, v)}{V^*(O, v)}.
\]

such that it is a vector with as few distinct values as possible. Choose \(X_O \in \arg\max_{S \in O} v^*(S)\), and \(X_M \in \arg\max_{S \in M} v^*(S)\) such that \(|X_O \cap X_M|\) is maximal. By the minimum support assumption, \(v^*_j = 0\) for all \(j \notin X_M \cup X_O\).

Claim \(v^*_j > 0\) for all \(j \in E \setminus X_O\).

Proof of claim Suppose there exists \(j \in E \setminus X_O\) with \(v^*_j = 0\). Let \(T' = T \setminus \{j\}, E' = E \setminus \{j\}, M' = M \setminus \{j\}\) and \(O' = O \setminus \{j\} := \{O \in O: j \notin O\}.\) Because \(j \notin X_O \in O'\) we have \(O' \neq \emptyset\).

For our counterexample we had:

\[
\min_{S \in O' \setminus \emptyset} \frac{r_M(S)}{|S|} = \rho(T', O', M') > \frac{V^*(M', v^*)}{V^*(O', v^*)}.
\]

Now

\[
\min_{S \in O' \setminus \emptyset} \frac{r_{M'}(S)}{|S|} \geq \min_{S \in O' \setminus \emptyset} \frac{r_M(S)}{|S|},
\]

since the minimum on the left is taken over a smaller set and \(r_{M'}(S) = r_M(S)\) for all \(S \in O'\).

On the other hand, \(V^*(M, v^*) \geq V^*(M', v^*)\) and \(V^*(O, v^*) = V^*(O', v^*) > 0\) yield

\[
\frac{V^*(M, v^*)}{V^*(O, v^*)} \geq \frac{V^*(M', v^*)}{V^*(O', v^*)}.
\]

Together

\[
\min_{S \in O' \setminus \emptyset} \frac{r_{M'}(S)}{|S|} = \rho(T', O', M') > \frac{V^*(M', v^*)}{V^*(O', v^*)}.
\]

Therefore we obtain a counterexample with \(|E'| < |E|\) in contradiction to our assumption of minimality of the counterexample. \(\square\)

Claim \(v^*_k = 0\) \(\forall k \in X_M \setminus X_O\).

Proof of claim Suppose a \(k \in X_M \setminus X_O\) with \(v^*_k > 0\) exists. Consider the valuations \(v^\alpha = v^* - \alpha e^k\) for \(0 \leq \alpha < v^*_k\) and their projection

\[
\hat{v}^\alpha = \frac{1}{\sum_{i \in E} v^\alpha_i} v^\alpha
\]

onto the unit simplex. Now

\[
\frac{\hat{v}^\alpha(S)}{\hat{v}^\alpha(T)} = \frac{v^\alpha(S)}{v^\alpha(T)} \quad \forall S, T: v(T) > 0.
\]

Since the weights on elements in \(X_O\) have not changed, while the coefficient for index \(k\) decreases, \(X_O\) is optimal for all \(\alpha\). Therefore \(\hat{v}^\alpha \in W\).

There exists \(\epsilon > 0\) such that \(X_M\) remains optimal for \(M\) with respect to \(v^\alpha\) for \(0 \leq \alpha \leq \epsilon\). If not, there is another solution \(X'\) that is optimal in \(M\) with respect to \(v^0\). By the basis exchange property of matroids there is some \(j \in X_O \setminus X_M\) such that \(X' = X_M \setminus \{k\} \cup \{j\}\). This contradicts our assumption, that \(X_M\) and \(X_O\) were chosen to have a maximal intersection.

Now, observe \(v^\alpha(M_M)\) and \(\hat{v}^\alpha(X_M)/\hat{v}^\alpha(X_O)\) are decreasing for \(\alpha\) increasing from 0 to \(\epsilon\). But this contradicts minimality of \(v^*\). \(\square\)
Together with the previous claim we obtain:

Claim $X_M \setminus X_O = \emptyset$, therefore $X_M \subseteq X_O$ and $X_O = E$.

Claim For every $i \in X_M$ there exists an $j \in X_O \setminus X_M$ such that $v_i^* = v_j^*$ and vice-versa.

Proof of claim Consider a valuation where $v_i^*$ is slightly decreased. If $X_O$ and $X_M$ do not change, then a smaller quotient for that valuation would result, which contradicts the assumption. As $X_O = E$ that cannot change either; so $X_M$ has to change for arbitrary small changes to $v_i^*$, and it can change only, by dropping $i$ and picking up some new element $j \in X_O \setminus X_M$. But this requires $v_i^* = v_j^*$.

Similarly, consider a valuation, where $v_j^*$ is slightly increased. If $X_O$ and $X_M$ do not change, then a smaller quotient for that valuation would result, which contradicts assumption. Again, $X_O = E$ ensures that $X_O$ remains optimal after these slight increases. However, it could be, that $j$ immediately enters $X_M$ but this requires, that just as immediately some $i \in X_M$ exits $X_M$. But this requires $v_i^* = v_j^*$.

Let $\bar{v} = \max_{i \in E} v_i^*$ and set

$$\hat{v}_j^\alpha := \begin{cases} \bar{v} + \alpha & \text{if } v_j^* = \bar{v}, \\ v_j^* & \text{otherwise.} \end{cases}$$

If $v^*$ were a multiple of the all ones vector, we would be done. Otherwise, for $\bar{v} := \max_{i : v_i^* \neq \bar{v}} v_i^*$ holds $\bar{v} \neq -\infty$.

Now, when $|\alpha| \leq \bar{v} - \bar{v}$, $X_O$ and $X_M$ remain optimal for $V^*(O, \hat{v}_j^\alpha)$ and $V^*(M, \hat{v}_j^\alpha)$ respectively, since $X_O = E$ and $v \geq 0$ implies, that $X_O$ is best possible. Regarding the optimality of $X_M$, notice, that the greedy algorithm can find the optimal solution, but for $|\alpha| \leq \bar{v} - \bar{v}$ the ordering of the elements did not change, hence $X_M$ remains optimal. Let

$$f(\alpha) := \frac{V^*(M, \hat{v}_j^\alpha)}{V^*(O, \hat{v}_j^\alpha)}.$$

By our analysis we deduce

$$f(\alpha) := \frac{V^*(M, \hat{v}_j^\alpha) + |\{i \in X_M : \bar{v} = v_i^*\}|}{V^*(O, \hat{v}_j^\alpha) + |\{i \in X_O : \bar{v} = v_i^*\}|}.$$

If $f'(0) < 0$, then $f(\epsilon) < f(0) = \frac{V^*(M,v^*)}{V^*(O,v^*)}$ (for small $\epsilon$) contrary to the assumption, that $v^*$ was chosen best possible among those with minimal $|E|$ and support of size $\ell$. For small $\epsilon$ neither $E$ nor $\text{supp}(v^*)$ have changed.

If $f'(0) > 0$, then $f(-\epsilon) < f(0) = \frac{V^*(M,v^*)}{V^*(O,v^*)}$ (for small $\epsilon$) contrary to the assumption, that $v^*$ was chosen best possible among those with minimal $|E|$ and support of size $\ell$. For small $\epsilon$ neither $E$ nor $\text{supp}(v^*)$ have changed.

Finally, if $f'(0) = 0$, it is straightforward to show that $f$ is a constant function on $\alpha \in [\bar{v} - \bar{v}, \bar{v} - \bar{v}]$, hence $f(\bar{v} - \bar{v}) = f(0)$ but the number of distinct values of $\bar{v} \bar{v} - \bar{v}$ has decreased by 1 contrary to the assumption that $v^*$ was chosen to have the minimum number of distinct values.

Since in all three cases we got a contradiction, we conclude that $v^*$ is just a multiple of the all-ones vector. \hfill \Box

Remark: Uniqueness of $X_O$ and $X_M$ is important for the proof. Consider a twin-peaks independence system consisting of $\{1\}$ and of $\{2,3\}$. The bases are $\{1\}$ and $\{2,3\}$. Let $2,1,1$ be the coefficients of the objective function. Optimal are both $X_1 = \{1\}$ and $X_2 = \{2,3\}$.

If we choose $X_2$ for the optimal solution and $\alpha$ increases from 0 upwards then the objective function is $\alpha |X_1|$ which does not relate to $X_2$.

If we choose $X_1$ for the optimal solution and $\alpha$ decreases from 0 downwards, then, the objective function is constantly $|X_2|$ which does not relate to $X_1$. 

$\Box$
Theorem 5 about approximating an independence system by a matroid motivates the search for a matroid that yields a best approximation. To this end, Milgrom (2017) defines the following substitutability index:

$$\rho^M(\mathcal{I}, \mathcal{O}) := \max_{\mathcal{M} \subseteq \mathcal{I}: \mathcal{M} \text{ is a matroid}} \rho(\mathcal{I}, \mathcal{O}, \mathcal{M}).$$

Milgrom (2017) gives no algorithm for determining the inner matroid \(\mathcal{M}'\). We describe an integer program for finding it.

Let \(r\) be the rank function of the independence system \((E, \mathcal{I})\). We use the fact that matroids are completely characterized by their rank functions. A function \(r : 2^E \rightarrow \mathbb{Z}_+\) is the rank function of matroid \((E, \mathcal{I})\), with \(\mathcal{I} = \{F \subseteq E : r(F) = |F|\}\) if and only if for all \(X, Y \subseteq E\):

\[
\begin{align*}
\rho^M(\mathcal{I}, \mathcal{O}) = \max_{S \subseteq E} \min_{\delta, \rho} (1 - \delta) \left( \frac{S}{r(S)} \right) \\
\text{s.t.} \quad 0 \leq \delta \leq r(S) &\quad \forall S \subseteq E \\
|r(S) - \delta| \leq |r(T) - \delta| &\quad \forall S \subseteq T \subseteq E \\
[r(S \cup j \cup k) - \delta_{S \cup j \cup k}] - [r(S \cup j) - \delta_{S \cup j}] &\leq [r(S \cup k) - \delta_{S \cup k}] \quad \forall j, k \not\in S \subseteq E \\
\delta \text{ integral} &\quad \forall S \subseteq E
\end{align*}
\]

We formulate the problem of finding the inner matroid as the problem of finding a suitable rank function, \(r' = r(S) - \delta_S\), with \(\delta_S\), integral, must be chosen to ensure that \(r'\) is a matroid rank function. Hence,

$$\rho^M(\mathcal{I}, \mathcal{O}) = \max_{\delta} \min_{S \subseteq E} (1 - \delta) \left( \frac{S}{r(S)} \right)$$

4 Comparison to Greedy

In this section we compare the outcome of the algorithm in Milgrom (2017) with greedy in the zero knowledge case of \(\mathcal{O} = \mathcal{I}\). Example 3 investigated a case, in which under zero knowledge the usual rank-quotient bound could not be improved. But that does not exclude the possibility that there exists an independence system \((E, \mathcal{I})\) where \(\rho^M(\mathcal{I}, \mathcal{I}) > q(\mathcal{I})\).

**Theorem 6** For every independence system \((E, \mathcal{I})\),

$$\rho^M(\mathcal{I}, \mathcal{I}) \leq q(\mathcal{I}).$$

**Proof** Suppose not. Then, there is an independence system \((E, \mathcal{I})\) with \(\rho^M(\mathcal{I}, \mathcal{I}) > q(\mathcal{I})\). Among all such counterexamples, choose one that minimizes \(|E|\) and among these, one that minimizes \(|\mathcal{I}|\). Then,

$$\rho^M(\mathcal{I}, \mathcal{I}) = \max_{\mathcal{M} \subseteq \mathcal{I}: \mathcal{M} \text{ is a matroid}} \min_{S \subseteq \mathcal{I} \setminus \{\emptyset\}} \frac{r_\mathcal{M}(S)}{r(S)} > \min_{F \subseteq E, r(F) > 0} \frac{l(F)}{r(F)} = q(\mathcal{I}).$$

Suppose the maximum on the left hand side of the inequality is attained by \(\mathcal{M}'\) and the minimum on the right hand side by \(F_0\), where \(F_0\) is chosen to minimize \(|F_0|\). Therefore,

$$\rho^M(\mathcal{I}, \mathcal{I}) = \min_{S \subseteq \mathcal{I} \setminus \{\emptyset\}} \frac{r_\mathcal{M'}(S)}{r(S)} > \frac{l(F_0)}{r(F_0)} = q(\mathcal{I}).$$

Now, the lower and upper rank of \(F_0\) differ (otherwise \(\mathcal{I}\) were a matroid and \(q(\mathcal{I}) = 1\)), so let \(U, L \subseteq F_0\) be bases of \(F_0\) (in \(\mathcal{I}\)) with \(|L| = l(F_0)\) and \(|U| = r(F_0)\).

\(U\) and \(L\) satisfy the following:
1. $U \cup L = F_0$ because $F_0$ was chosen to be of smallest cardinality.

2. $U \cap L = \emptyset$, otherwise $F' := (U \cup L) \setminus (U \cap L)$ satisfies $l(F')/r(F') < l(F_0)/r(F_0)$ contradicting the choice of $F_0$.

3. $F_0 = E$, otherwise the restriction of $I$ to $F_0$ yields a smaller counterexample.

Thus, our minimal counterexample has $E = U \cup L$. Furthermore, $U$ and $L$ are bases of $E$ with $|L| < |U|$.

This last inequality follows from (3). Therefore, every matroid $M$ contained in $I$ has to have a rank of at most $|U|$, i.e., $m := r_M(E) \leq |U|$.

As $U \in I$ from (3) we see that:

$$\frac{r_{M'}(U)}{r(U)} \geq \min_{S \in I \setminus \emptyset} \frac{r_{M'}(S)}{r(S)} > \frac{l(F_0)}{r(F_0)} > \frac{r_{M'}(U)}{|U|} > \frac{|L|}{|U|}$$

Consequently, $r_{M'}(U) > |L| = l(F_0)$.

Hence, for $M'$ we have $|L| < m$. Let $Z$ be a basis of $L$ in $M'$. By the augmentation property of matroid $M'$ we can augment $Z$ into a basis $Z'$ of $E$ in $M'$. If $Z \neq \emptyset$, then $L, Z' \in I$ would form a smaller counterexample on the groundset $L \cup Z'$ contrary to assumption. Therefore, all elements of $L$ are dependent in $M'$. Hence, for the matroid $M$ we have that the minimum in (2) is attained for $L$, therefore, $\frac{r_M(L)}{r(L)} = \frac{|L|}{|U|} = 0$. This contradicts the assumption that $\rho^M(I, I) > q(I) > 0$. Therefore, no minimal example $I$ with $\rho^M(I, I) > q(I)$ exists.

The zero knowledge assumption is weaker than necessary. If $v \geq 0$, we know that a maximum weight basis must be, unsurprisingly, a basis of $E$. Thus, a more reasonable knowledge assumption is $O = B_I$. Theorem 4 continues to hold under the *reasonable knowledge assumption*. This is because in the proof, $L$ and $U$ were each a basis of $E$.

Suppose $v \geq 0$ and $O \subset B_I$. In this case, prior information rules out some basis from being *optimal*. To ensure an apples with apples comparison between the greedy algorithm and the algorithm in Milgrom [2017], we must allow the greedy algorithm to make use of the information contained in $O$ as well. This may require modifying the greedy algorithm so much that it become an entirely different algorithm altogether. A straightforward way to do this is to apply the greedy algorithm to an independence system $(E, O^\uparrow)$ of sets contained in elements of $O$. The example we describe below shows that the greedy algorithm will be dominated by Milgrom’s heuristic. Under one interpretation of $O$ this is not surprising. Suppose, that $I \setminus O$ is the collection of sets that can never be optimal. Then, $O^\uparrow$ would include sets that should definitely not be selected.

**Example 4** Consider the stable set problem (problem to find a set of nonadjacent vertices) on a path of 9 vertices:

```
1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9
```

The underlying independence system has $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and the stable sets form the set-system $I$. The maximum cardinality basis of $I$ is the set $\{1, 3, 5, 7, 9\}$ of cardinality 5. An example of a small basis of $E$ would be $\{2, 5, 8\}$ of size 3. For the Korte-Hausmann bound, consider the set $\{1, 2, 3\}$ that demonstrates $q \leq \frac{1}{2}$.

Suppose $O$ is the collection of stable sets $S$ of size at most 4 with $|S \cap \{3, 6, 9\}| \leq 1$. Notice, $\{1, 3, 5, 7\} \in O$.

Consider the set $\{1, 2, 3\}$. Assuming only subsets of elements of $O$, are independent, this set has a rank quotient of $1/2$, i.e., $q(O^\uparrow) = 1/2$. Hence, the greedy algorithm applied to the independence system consisting of the elements of $O^\uparrow$ will have a worst case bound of at most $1/2$.

The following matroid $M = U^1_{\{1, 2\}} \oplus U^1_{\{4, 5\}} \oplus U^1_{\{7, 8\}}$ is contained in $I$ since every independent set in $M$ is a stable set from $I$. It is straightforward to verify that $\rho^M(I, O) = 3/4 > 1/2$. Interestingly, the largest set of $I$ is not independent in the matroid $M$.

This example highlights the essential difference between Milgrom’s heuristic and the greedy algorithm. The first is looking for a basis that has large overlap with each of the sets in $O$. The greedy algorithm seeks a basis with a large overlap with every set that is contained in $O^\uparrow$. 
5 Discussion

Our reading of Milgrom (2017) suggests that he proposed the existence of a ‘good’ inner matroid as an explanation for the success of the greedy algorithm in some settings. He writes:

“In practice, procedures based on greedy algorithms often perform very well.”

Our analysis shows that existence of a good inner matroid by itself is not good enough to explain the practical observation. It is possible for there to be no good inner matroid approximation, yet the greedy algorithm performs well. Further, there are examples where the inner matroid approximation will dominate that of the greedy algorithm. In our view the power of the inner matroid approximation comes from the prior knowledge encoded in $\mathcal{O}$.

The idea of incorporating prior information about the optimal solution into an optimization problem is not new. There are three approaches we are aware of: probabilistic, uncertainty sets and stability. The first encodes the prior information in terms of a probability distribution over the possible objective value coefficients. The second, assumes that the objective coefficients are drawn from some set given a-priori. Stability assumes that the optimal solution of the instance under examination does not change under perturbations to the objective function (see Bilu and Linial (2010)). Milgrom, instead, proposes that the prior information be encoded in a description of the set of possible optima.

Interestingly, it is unclear how to adapt the greedy algorithm (or indeed other heuristics) to incorporate prior information about the location of the optimal solution. We think this a useful and fruitful line of inquiry.

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1 This is sometimes relaxed to a class of distributions sharing common moments.