A general geometric construction for affine surface area

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Abstract

Let $K$ be a convex body in $\mathbb{R}^n$ and $B$ be the Euclidean unit ball in $\mathbb{R}^n$. We show that

$$\lim_{t \to 0} \frac{|K| - |K_t|}{|B| - |B_t|} = \frac{as(K)}{as(B)},$$

where $as(K)$ respectively $as(B)$ is the affine surface area of $K$ respectively $B$ and $\{K_t\}_{t \geq 0}$, $\{B_t\}_{t \geq 0}$ are general families of convex bodies constructed from $K$, $B$ satisfying certain conditions. As a corollary we get results obtained in [M-W], [Schm], [S-W] and [W].

The affine surface area $as(K)$ was introduced by Blaschke [B] for convex bodies in $\mathbb{R}^3$ with sufficiently smooth boundary and by Leichtweiss [L1] for convex bodies in $\mathbb{R}^n$ with sufficiently smooth boundary as follows

$$as(K) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x),$$

where $\kappa(x)$ is the Gaussian curvature in $x \in \partial K$ and $\mu$ is the surface measure on $\partial K$. As it occurs naturally in many important questions, so for example in the approximation of convex bodies by polytopes (see the survey article of Gruber [Gr] and the paper by Schütt [S]) or in a priori estimates for PDEs [Lu-O], one wanted to have extensions of the affine surface area to arbitrary convex bodies in $\mathbb{R}^n$ without any smoothness assumptions of the boundary.

Such extensions were given in recent years by Leichtweiss [L2], Lutwak [Lu], Meyer and Werner [M-W], Schmuckenschläger [Schm], Schütt and Werner [S-W] and Werner [W].

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The extensions of affine surface area to an arbitrary convex body $K$ in $R^n$ in [L2], [M-W], [Schm], [S-W] and [W] have a common feature:

first a specific family $\{K_t\}_{t \geq 0}$ of convex bodies is constructed. This family is different in each of the extensions [L2], [M-W], [Schm], [S-W] and [W] but of course related to the given convex body $K$.

Typically the families $\{K_t\}_{t \geq 0}$ are obtained from $K$ through a “geometric” construction. In [L2] respectively [S-W] this geometric construction gives as $\{K_t\}_{t \geq 0}$ the family of the floating bodies respectively the convex floating bodies.

In [M-W] the geometric construction gives the family of the Santaló-regions, in [Schm] the convolution bodies and in [W] the family of the illumination bodies.

The affine surface area is then obtained by using expressions involving volume differences $|K| - |K_t|$ respectively $|K_t| - |K|$.

Therefore it seemed natural to ask whether there are completely general conditions on a family $\{K_t\}_{t \geq 0}$ of convex bodies in $R^n$ that (in connection with volume difference expressions) will give us affine surface area. We give a positive answer to this question which was asked - among others - by A. Pełczyński.

Throughout the paper we shall use the following notations.

$B(a, r) = B^n(a, r)$ is the $n$-dimensional Euclidean ball with radius $r$ centered at $a$. We put $B = B(0, 1)$. By $\|\|$ we denote the standard Euclidean norm on $R^n$, by $\langle \cdot, \cdot \rangle$ the standard inner product on $R^n$. For two points $x$ and $y$ in $R^n$ $[x, y] = \{\alpha x + (1 - \alpha)y : 0 \leq \alpha \leq 1\}$ denotes the line segment from $x$ to $y$. For a convex set $C$ in $R^n$ and a point $x \in R^n \setminus C$, co$[x, C]$ is the convex hull of $x$ and $C$.

$\mathcal{K}$ denotes the set of convex bodies in $R^n$. For $K \in \mathcal{K}$, int$(K)$ is the interior of $K$ and $\partial K$ is the boundary of $K$. For $x \in \partial K$, $N(x)$ is the outer unit normal vector to $\partial K$ in $x$. We denote the $n$-dimensional volume of $K$ by $\text{vol}_n(K) = |K|$. Let $K \in \mathcal{K}$ and $x \in \partial K$ with unique outer unit normal vector $N(x)$. We say that $\partial K$ is approximated in $x$ by a ball from the inside (respectively from the outside) if there exists a hyperplane $H$ orthogonal to $N(x)$ such that $H \cap \text{int}(K) \neq \emptyset$ and a Euclidean ball $B(r) = B(x - rN(x), r)$ (respectively $B(R) = B(x - RN(x), R)$) such that

$$B(r) \cap H^+ \subseteq K \cap H^+$$

respectively

$$K \cap H^+ \subseteq B(R) \cap H^+.$$

Here $H^+$ is one of the two halfspaces determined by $H$. 
Definition 1

For $t \geq 0$, let $F_t : K \to K$, $K \mapsto F_t(K) = K_t$, be a map with the following properties

(i) $K_0 = K$ and either $K_t \subseteq K$ for all $t \geq 0$ and $F_t$ is decreasing in $t$ (that is $K_{t_1} \subseteq K_{t_2}$ if $t_1 \geq t_2$) or $K \subseteq K_t$ for all $t \geq 0$ and $F_t$ is increasing in $t$.

(ii) For all affine transformations $A$ with $\det A \neq 0$, for all $t$

\[(A(K))_{\mid \det A \mid t} = A(K_t)\]

(iii) For all $t \geq 0$, $B_t$ is a Euclidean ball with center 0 and radius $f_1(t)$ and

\[\lim_{t \to 0} \frac{|B| - |B_t|}{t^{n+1}} = c,\]

where $c$ is a constant (depending on $n$ only).

(iv) Let $x \in \partial K$ be approximated from the inside by a ball $B(r)$. If $H^+ \cap \partial(K_t) \cap \partial(B(r)) \neq \emptyset$ for some $s$ and $t$, then $s \leq Ct$ where $C$ is a constant (depending only on $n$).

(v) Let $\epsilon > 0$ be given and $x \in \partial K$ be such that it is approximated from the inside by a ball $B(\rho - \epsilon)$ and from the outside by a ball $B(\rho + \epsilon)$. There exists a hyperplane $H$ orthogonal to $N(x)$ and $t_0$ such that whenever

$H^+ \cap \partial(K_t) \cap \partial(B(\rho - \epsilon)) \neq \emptyset$, for $t \leq t_0$, $s = s(t),$

respectively

$H^+ \cap \partial(K_t) \cap \partial(B(\rho + \epsilon)) \neq \emptyset$, for $t \leq t_0$, $s = s(t),$

then

$s \leq (1 + \epsilon)t$

respectively

$s \geq (1 - \epsilon)t.$
Remarks 2

(i) Note that the maps $F_t$ are essentially determined by the invariance property 1 (ii) and by their behaviour with respect to Euclidean balls.

(ii) Let $f_r(t)$ be the radius of $B(0, r)_t$. Then it follows immediately from Definition 1 (ii), (iii) that

$$\lim_{t \to 0} \frac{r - f_r(t)}{1 - f_1(t)} = r^{\frac{n-1}{n+1}}.$$

(iii) For some examples the following Definition 1’ is easier to check than Definition 1.

Definition 1’

(i) - (iii) as in Definition 1.

(iv)’ If $s < t$, then $K_t \subseteq \text{int}(K_s)$.

(v)’ If $K \subseteq L$ where $L$ is a convex body in $R^n$, then $K_t \subseteq L_t$ for all $t \geq 0$.

However not all the examples mentioned below satisfy (iv)’ and (v)’. For instance the illumination bodies (defined below) do not satisfy (v)’.

Examples for Definitions 1 and 1’

1. The (convex) floating bodies [S-W]

Let $K$ be a convex body in $R^n$ and $t \geq 0$. $F_t$ is a (convex) floating body if it is the intersection of all half-spaces whose defining hyperplanes cut off a set of volume $t$ of $K$. More precisely, for $u \in S^{n-1}$ let $a^u_t$ be defined by

$$t = |\{x \in K : <u, x> \geq a^u_t\}|.$$

Then

$$F_t = \bigcap_{u \in S^{n-1}} \{x \in K : <u, x> \geq a^u_t\}$$

is a (convex) floating body.

The family $\{F_t\}_{t \geq 0}$ satisfies Definitions 1 and 1’.

2. The Convolution bodies [K], [Schm]

Let $K$ be a symmetric convex body in $R^n$ and $t \geq 0$. Let

$$C(t) = \{x \in R^n : |K \cap (K + x)| \geq 2t\}$$

is
and 
\[ C_t = \frac{1}{2} C(t). \]

Then \( \{C_t\}_{t \geq 0} \) satisfies Definitions 1 and 1'.

3. The Santaló-regions [M-W]

For \( t \in R \) and a convex body \( K \) in \( R^n \) the Santaló-region \( S(K, t) \) of \( K \) is defined as
\[ S(K, t) = \{ x \in K : \frac{|K||K^x|}{|B|^2} \leq t \}, \]
where \( K^x = (K - x)^0 = \{ z \in R^n : < z, y - x > \leq 1 \text{ for all } y \in K \} \) is the polar of \( K \) with respect to \( x \). (We consider only these \( t \) for which \( S(K, t) \neq \emptyset \)).

Put 
\[ S_t = S(K, \frac{|K|}{t|B|^2}) = \{ x \in K : |K^x| \leq \frac{1}{t} \}. \]

Then the family \( \{S_t\}_{t \geq 0} \) satisfies Definitions 1 and 1'.

4. The Illumination bodies [W]

Let \( K \) be a convex body in \( R^n \) and \( t \geq 0 \). The illumination body \( I_t \) is the convex body defined as
\[ I_t = \{ x \in R^n : |\text{co}[x, K]\backslash K| \leq t \}. \]

Then the family \( \{I_t\}_{t \geq 0} \) satisfies Definition 1.

**Theorem 3**

Let \( K \) be a convex body in \( R^n \). For all \( t \geq 0 \) let \( K_t \) respectively \( B_t \) be convex bodies obtained from \( K \) respectively \( B \) by Definition 1 or 1'. Then
\[ \lim_{t \to 0} \frac{|K| - |K_t|}{|B| - |B_t|} = \frac{as(K)}{as(B)}. \]

**Remark**

Note that 
\[ as(B) = vol_{n-1}(\partial B) = n|B|. \]
Corollary 4

(i) [S-W]
Let $K$ be a convex body in $\mathbb{R}^n$ and for $t \geq 0$ let $F_t$ be a floating body. Then
\[
\lim_{t \to 0} c_n \frac{|K| - |F_t|}{t^{n+1}} = \text{as}(K).
\]
where $c_n = 2 \left( \frac{|B_n|}{n+1} \right)^{\frac{2}{n+1}}$.

(ii) [Schm]
Let $K$ be a symmetric convex body in $\mathbb{R}^n$ and for $t \geq 0$ let $C_t$ be a convolution body. Then
\[
\lim_{t \to 0} c_n \frac{|K| - |C_t|}{t^{n+1}} = \text{as}(K).
\]
where $c_n$ is as in (i).

(iii) [M-W]
Let $K$ be a convex body in $\mathbb{R}^n$ and for $t \geq 0$ let $S_t$ be a Santaló-region. Then
\[
\lim_{t \to 0} e_n \frac{|K| - |S_t|}{t^{n+1}} = \text{as}(K).
\]
where $e_n = \frac{2 |B|}{|B|^{\frac{n+1}{n+2}}}$.

(iii) [W]
Let $K$ be a convex body in $\mathbb{R}^n$ and for $t \geq 0$ let $I_t$ be an illumination body. Then
\[
\lim_{t \to 0} d_n \frac{|I_t| - |K|}{t^{\frac{n}{n+1}}} = \text{as}(K).
\]
where $d_n = 2 \left( \frac{|B_n|}{n(n+1)} \right)^{\frac{2}{n+1}}$.

For the proof of Theorem 3 we need several Lemmas. The basic idea of the proof is as in [S-W].
Lemma 5

Let $K$ and $L$ be two convex bodies in $\mathbb{R}^n$ such that $0 \in \text{int}(L)$ and $L \subseteq K$. Then

(i)  
$$|K| - |L| = \frac{1}{n} \int_{\partial K} <x, N(x)> \left(1 - \left(\frac{\|x_L\|}{\|x\|}\right)^n\right) d\mu(x),$$

where $x_L = [0, x] \cap \partial L$ and $\mu$ is the usual surface measure on $\partial K$.

(ii)  
$$|K| - |L| = \frac{1}{n} \int_{\partial L} <x, N(x)> \left((\frac{\|x_K\|}{\|x\|})^n - 1\right) d\mu(x),$$

where $x_K$ is the intersection of the half-line from 0 through $x$ with $\partial K$ and $\mu$ is the usual surface measure on $\partial L$.

The proof of Lemma 5 is standard.

For $x \in \partial K$ denote by $r(x)$ the radius of the biggest Euclidean ball contained in $K$ that touches $\partial K$ at $x$. More precisely

$$r(x) = \max\{r : x \in B(y, r) \subseteq K \text{ for some } y \in K\}.$$

Remark

It was shown in [S-W] that

(i)  If $B \subseteq K$, then

$$\mu\{x \in \partial K : r(x) \geq \beta\} \geq (1 - \beta)^{n-1} \text{vol}_{n-1}(\partial K)$$

(ii)  
$$\int_{\partial K} r(x)^{-\alpha} d\mu(x) < \infty \quad \text{for all } \alpha, \ 0 \leq \alpha < 1$$

Lemma 6

Suppose $0$ is in the interior of $K$. Then we have for all $x$ with $r(x) > 0$ and for all $t \geq 0$

$$0 \leq \frac{x, N(x)}{n(|B| - |B_t|)} \leq g(x),$$

where $\int_{\partial K} g(x) d\mu(x) < \infty$.

$x_t = [0, x] \cap \partial K$ if $K_t \subseteq K$. $x_t$ is the intersection of the half-line from 0 through $x$ with $\partial K_t$ if $K \subseteq K_t$. 

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Lemma 7 Let \( x_t \) be as in Lemma 6. Then

\[
\lim_{t \to 0} \frac{< x, N(x) > \left(1 - (\|x_t\|/\|x\|)^n \right)}{n(|B| - |B_t|)} \text{ exists a.e.}
\]

and is equal to

(i) \( \rho(x) \frac{n-1}{n|B|} \) if the indicatrix of Dupin at \( x \in \partial K \) is an \((n-1)\)-dimensional sphere with radius \( \sqrt{\rho(x)} \).

(ii) 0, if the indicatrix of Dupin at \( x \) is an elliptic cylinder.

Remark

(i) \( r(x) > 0 \) a.e. [S-W] and the indicatrix of Dupin exists a.e. [L2] and is an ellipsoid or an elliptic cylinder.

(ii) If the indicatrix is an ellipsoid, we can reduce this case to the case of a sphere by an affine transformation with determinant 1 (see [S-W]).

Proof of Theorem 3

We may assume that 0 is in the interior of \( K \). By Lemma 5 and with the notations of Lemma 6 we have

\[
\frac{|K| - |K_t|}{|B| - |B_t|} = \frac{1}{n} \int_{\partial K} \frac{< x, N(x) > \left(1 - (\|x_t\|/\|x\|)^n \right)}{|B| - |B_t|} d\mu(x)
\]

By Lemma 6 and the Remark preceding it, the functions under the integral sign are bounded uniformly in \( t \) by an \( L^1 \)-function and by Lemma 7 they are converging pointwise a.e. We apply Lebesgue’s convergence theorem.

Proof of Lemma 6

Let \( x \in \partial K \) such that \( r(x) > 0 \). We consider the proof in the case of Definition 1' and of Definition 1 in the case where \( K_t \subseteq K \) for all \( t \geq 0 \). The case of Definition 1 where \( K \subseteq K_t \) for all \( t \geq 0 \) is treated in a similar way.

As \( \|x\| \geq \|x_t\| \), we have for all \( t \)

\[
\frac{1}{n} < x, N(x) > \left(1 - (\|x_t\|/\|x\|)^n \right) \leq < x/\|x\|, N(x) > \|x - x_t\|
\]

(1)

Put \( r(x) = r, x - r(x)N(x) = z \) and \( < x/\|x\|, N(x) >= \cos \theta \).

We can assume that there is an \( \alpha > 0 \) such that

\[
B(0, \alpha) \subseteq K \subseteq B(0, \frac{1}{\alpha})
\]

(2)
and hence
\[ \cos \theta \| x - x_t \| \leq \frac{2}{\alpha}. \]

Let \( \epsilon > 0 \) be given. By Remark 2 (ii) there exists \( t_1 \) such that for all \( t \leq t_1 \)
\[ r(1 - \frac{1 - f_1(t)}{r^{\frac{n}{n+1}}}(1 + \epsilon)) \leq f_r(t) \leq r(1 - \frac{1 - f_1(t)}{r^{\frac{n}{n+1}}}(1 - \epsilon)). \] (3)

Let \( t_0 \) be such that \( Ct_0 < t_1 \). By Definition 1, (i) \( f_1(t) \) is decreasing in \( t \), hence we have for all \( t \geq t_0 \)
\[ f_1(t) \leq f_1(t_0) \]
and thus for all \( t \geq t_0 \) with (1) and (2)
\[ \frac{\langle x, N(x) \rangle}{n(|B| - |B_t|)} \leq \frac{2}{\alpha|B|(1 - (f_1(t_0))^n)}. \]

Therefore the expression in question is bounded by a constant in this case and hence integrable. It remains to consider the case when \( t < t_0 \).

a) Suppose first that
\[ \| x - x_t \| < r \cos \theta. \]

For \( B(z,r) \) we construct the corresponding inner body \( (B(z,r))_s \) such that \( x_t \) is a boundary point of \( (B(z,r))_s \). By Definition 1 (iii) \( (B(z,r))_s \) is a Euclidean ball with center \( z \) and radius \( f_r(s) \). As \( x_t \) is a boundary point of \( (B(z,r))_s \),
\[ f_r(s) = r(1 - \frac{2\| x - x_t \| \cos \theta}{r^{\frac{n}{n+1}}} + \frac{\| x - x_t \|^2}{r^2})^{1/2} \leq r(1 - \frac{\| x - x_t \| \cos \theta}{2r}) \] (4)
The last inequality holds by assumption a).

So far the arguments are the same for Definition 1 and Definition 1’. From now on they differ slightly.

By Definition 1 (iv) \( s \leq Ct \), hence by monotonicity \( f_r(s) \geq f_r(Ct) \) and thus, as \( Ct < t_1 \), with (3)
\[ f_r(Ct) \geq r(1 - (1 + \epsilon) \frac{1 - f_1(Ct)}{r^{\frac{n}{n+1}}}), \]
which, using Definition 1 (iii) can be shown to be
\[ \geq r(1 - (1 + \epsilon)(C(\frac{2}{r^{\frac{n}{n+1}}}} + \epsilon) \frac{1 - f_1(t)}{r^{\frac{n}{n+1}}}). \] (5)

We get from (4) and (5)
\[ 1 - f_1(t) \geq \frac{\| x - x_t \| \cos \theta}{2(1 + \epsilon)(C(\frac{2}{r^{\frac{n}{n+1}}}} + \epsilon). \] (6)
Observe also that

\[ |B| - |B_t| = |B|(1 - f^n_t(t)) \geq |B|(1 - f_1(t)). \]

This inequality together with (1) and (6) shows that

\[ < x, N(x) > \left(1 - \left(\frac{|x_t|}{||x||}\right)^n\right) \leq \frac{2(1 + \epsilon)(C + 1 + \epsilon) - \frac{1}{R^n}}{|B|}. \]

And the latter is integrable by the Remark preceding Lemma 6.

In the case of Definition 1' it follows from (iv)' and (v)' that \( s \leq t \). For if not, then \( s > t \), therefore by (iv)' \( (B(z, r))_s \subset \text{int}(B(z, r))_t \) and by (v)' \( \text{int}(B(z, r))_s \subset \text{int}(K_t) \), which contradicts that \( x_t \in \partial K_t \cap \partial(B(z, r))_s \). Therefore \( f_r(s) \geq f_r(t) \) and thus, as \( t < t_1 \), with (3)

\[ f_r(t) \geq r(1 - (1 + \epsilon)\frac{1 - f_1(t)}{R^{\frac{n}{n+1}}}). \]

We then conclude as above.

b) Now we consider the case when

\[ ||x - x_t|| \geq r \cos \theta. \]

We choose \( \alpha \) so small that \( x_t \notin B(0, \alpha) \). Let \( H \) be the hyperplane through 0 orthogonal to \( x \). Then the spherical cone \( C = [x, H \cap B(0, \alpha)] \) is contained in \( K \) and \( x_t \in C \). Let \( d = \text{distance}(x_t, C) \). Then

\[ d = ||x - x_t|| \leq \frac{\alpha}{(\alpha^2 + ||x||^2)^{\frac{n}{2}}}. \] (7)

Let \( w \in [0, x_t] \) such that \( ||x_t - w|| = \frac{d}{R} \). Let \( B(w, R) \subseteq K \) be the biggest Euclidean ball with center \( w \) such that \( B(w, R) \subseteq K \). Then \( \partial B(w, R) \cap \partial K \neq \emptyset \). Moreover \( R \geq d \), which implies that \( x_t \in B(w, R) \). Let \( (B(w, R))_s \) be the corresponding inner ball such that \( x_t \in \partial(B(w, R))_s \).

Now we have to distinguish between Definition 1 and 1'.

By Definition 1, (iv) \( s \leq Ct \). By monotonicity \( f_R(s) \geq f_R(Ct) \) which, as above, is

\[ \geq R(1 - (1 + \epsilon)\left(C + \frac{1}{R}\right)\frac{1 - f_1(t)}{R^{\frac{n}{n+1}}}). \]

As \( R \geq d \), the latter is

\[ \geq d(1 - (1 + \epsilon)\left(C + \frac{1}{d}\right)\frac{1 - f_1(t)}{d^{\frac{n}{n+1}}}). \]
On the other hand by construction $f_R(s) = \frac{d}{2}$. Therefore

$$1 - f_1(t) \geq \frac{d^{\frac{2n-1}{2n}}}{2(1 + \epsilon)(C^{\frac{2n-1}{2n}} + \epsilon)}.$$ 

Note also that (2) implies that $\cos \theta \geq \alpha^2$. Hence with (1), (2), (7) and assumption b) we get that

$$\frac{< x, N(x) >}{n(|B| - |B_t|)} \leq \frac{2(1 + \alpha^4)\frac{\alpha}{C} (1 + \epsilon)(C^{\frac{2n-1}{2n}} + \epsilon)}{|B| \alpha^{\frac{n-2}{n-1}}} r^{-\frac{n-1}{n-1}}.$$

The case of Definition 1' is treated similarly and the above inequalities hold true with $C = 1$ and $C^{\frac{2n-1}{2n}} + \epsilon = 1$.

**Proof of Lemma 7**

We again consider the case when $K_t \subseteq K$ for all $t \geq 0$ for Definition 1. The case $K \subseteq K_t$ for all $t \geq 0$ for Definition 1 and the case of Definition 1' are done in a similar way (compare the proof of Lemma 6).

As in the proof of Lemma 6 we can choose $\alpha > 0$ such that

$$B(0, \alpha) \subseteq K \subseteq B(0, \frac{1}{\alpha}).$$

Therefore

$$1 \geq \frac{x}{||x||} \cdot N(x) \geq \alpha^2. \quad (8)$$

We put again $\cos \theta = \frac{x}{||x||} \cdot N(x) >$. (1) holds, that is

$$\frac{1}{n} < x, N(x) > \left( 1 - \left( \frac{||x_t||}{||x||} \right)^n \right) \leq \frac{x}{||x||} \cdot N(x) > ||x - x_t||$$

Since $x$ and $x_t$ are colinear,

$$||x|| = ||x_t|| + ||x - x_t||$$

and hence

$$\frac{1}{n} < x, N(x) > \left( 1 - \left( \frac{||x_t||}{||x||} \right)^n \right) = \frac{1}{n} < x, N(x) > \left( 1 - \left( 1 - \frac{||x - x_t||}{||x||} \right)^n \right) \geq \frac{x}{||x||} \cdot N(x) > ||x - x_t|| \left( 1 - k_1 \cdot \frac{||x - x_t||}{||x||} \right). \quad (9)$$
for some constant $k_1$, if we choose $t$ sufficiently large.

(i) Case where the indicatrix is an ellipsoid

We have seen that then we can assume that the indicatrix is a Euclidean sphere. Let $\sqrt{\rho(x)}$ be the radius of this sphere. We put $\rho(x) = \rho$ and we introduce a coordinate system such that $x = 0$ and $N(x) = (0, \ldots, 0, -1)$. $H_0$ is the tangent hyperplane to $\partial K$ in $x = 0$ and $\{H_\alpha : \alpha \geq 0\}$ is the family of hyperplanes parallel to $H_0$ that have non-empty intersection with $K$ and are of distance $\alpha$ from $H_0$. For $\alpha > 0$, $H_\alpha^+$ is the half-space generated by $H_\alpha$ that contains $x = 0$. For $a \in R$, let $z_a = (0, \ldots, 0, a)$ and $B_a = B(z_a, a)$ be the Euclidean ball with center $z_a$ and radius $a$. As in [W], for $\varepsilon > 0$ we can choose $\alpha_0$ so small that for all $\alpha \leq \alpha_0$

$$B_{\rho - \varepsilon} \cap H_\alpha^+ \subseteq K \cap H_\alpha^+ \subseteq B_{\rho + \varepsilon} \cap H_\alpha^+.$$ 

We choose $t$ so small that $x_t \in \text{int}(B_{\rho - \varepsilon} \cap H_\alpha^+)(\subseteq \text{int}(B_{\rho + \varepsilon} \cap H_\alpha^+))$. For $B_{\rho + \varepsilon}$ we construct the corresponding inner body $(B_{\rho + \varepsilon})_s$ such that $x_t$ is a boundary point of $(B_{\rho + \varepsilon})_s$. $(B_{\rho + \varepsilon})_s$ is a Euclidean ball with center $z_{\rho + \varepsilon}$ and radius $f_{\rho + \varepsilon}(s)$. We have

$$f_{\rho + \varepsilon}(s) = ((\rho + \varepsilon)^2 + \|x - x_t\|^2 - 2(\rho + \varepsilon)\|x - x_t\| \cos \theta)^{\frac{1}{2}},$$

$$\geq (\rho + \varepsilon)(1 - \frac{\|x - x_t\| \cos \theta}{\rho + \varepsilon}).$$

Definition 1, (v) implies that $s \geq (1 - \varepsilon)t$, hence by monotonicity $f_{\rho + \varepsilon}(s) \leq f_{\rho + \varepsilon}((1 - \varepsilon)t)$, which, for $t$ small enough is (compare with the proof of Lemma 6)

$$\leq (\rho + \varepsilon)(1 - (1 - k_2\varepsilon) \frac{1 - f_1(t)}{(\rho + \varepsilon)^{n-1}}),$$

where $k_2$ is a constant. Thus

$$1 - f_1(t) \leq \frac{\|x - x_t\| \cos \theta (\rho + \varepsilon)^{\frac{n-1}{2}}}{1 - k_2\varepsilon}.$$ 

Note that

$$|B| - |B_t| = |B|(1 - f_1^n(t)) \leq n|B|(1 - f_1(t)).$$

Therefore with (9)

$$< x, N(x) > \left(1 - \left(\frac{\|x_t\|}{\|x\|}\right)^n\right) \geq (1 - k_2\varepsilon)(1 - k_1 \frac{\|x - x_t\|}{\|x\|} (\rho + \varepsilon)^{\frac{n-1}{2n}} n|B|).$$

This is the lower bound for the expression in question.
To get an upper bound we proceed similarly. For $B_{\rho-\varepsilon}$ we construct the corresponding inner body $(B_{\rho-\varepsilon})_s$ such that $x_t$ is a boundary point of $(B_{\rho-\varepsilon})_s$. $(B_{\rho-\varepsilon})_s$ is a Euclidean ball with center $z_{\rho-\varepsilon}$ and radius $f_{\rho-\varepsilon}(s)$. We have

$$f_{\rho-\varepsilon}(s) = ((\rho-\varepsilon)^2 + \|x - x_t\|^2 - 2(\rho-\varepsilon)\|x - x_t\|\cos\theta)^{\frac{1}{2}},$$

$$\leq (\rho-\varepsilon)(1 - \frac{\|x - x_t\|\cos\theta}{\rho - \varepsilon} - \frac{\|x - x_t\|}{2(\rho - \varepsilon)\cos\theta})(1 + k_3 \frac{\|x - x_t\|\cos\theta}{\rho - \varepsilon} (1 - \frac{\|x - x_t\|}{2(\rho - \varepsilon)\cos\theta})), $$

for some constant $k_3$, if $t$ is small enough. Again by Definition 1 (v) $s \leq (1 + \varepsilon)t$ and therefore $f_{\rho-\varepsilon}(s) \geq f_{\rho-\varepsilon}((1 + \varepsilon)t)$ which with arguments similar as before is

$$\geq (\rho-\varepsilon)(1 - (1 + k_4)\frac{1 - f_1(t)}{(\rho-\varepsilon)^{\frac{n+1}{n}}})$$

with a suitable constant $k_4$. Thus

$$1 - f_1(t) \geq$$

$$\frac{\|x - x_t\|\cos\theta}{1 + k_4 \varepsilon} (1 - \frac{\|x - x_t\|}{2(\rho - \varepsilon)\cos\theta}) (1 + k_3 \frac{\|x - x_t\|\cos\theta}{\rho - \varepsilon} (1 - \frac{\|x - x_t\|}{2(\rho - \varepsilon)\cos\theta}))(\rho-\varepsilon)^{\frac{n+1}{n}}. $$

Observe now that

$$|B| - |B_t| = |B|(1 - f_1(t)) \geq n|B|(1 - f_1(t))(1 - \frac{n-1}{2}(1 - f_1(t))).$$

We choose $t$ so small that $1 - f_1(t) < \frac{2\varepsilon}{n-1}$. This together with (1), (10) and (11) implies that

$$\frac{x, N(x)}{n(|B| - |B_t|)} \leq$$

$$\frac{1 + k_4 \varepsilon}{(1 - \varepsilon)(1 - \frac{\|x - x_t\|}{2(\rho - \varepsilon)\cos\theta}) (1 + k_3 \frac{\|x - x_t\|\cos\theta}{\rho - \varepsilon} (1 - \frac{\|x - x_t\|}{2(\rho - \varepsilon)\cos\theta}))(\rho-\varepsilon)^{\frac{n+1}{n}} |B|.$$ 

Note that $\cos\theta \geq \alpha^2$ by (8).

This finishes the proof of Lemma 7 in the case where the indicatrix is an ellipsoid.
(ii) Case where the indicatrix of Dupin is an elliptic cylinder

Recall that then we have to show that

$$\lim_{t \to 0} \frac{\langle x, N(x) \rangle}{n(|B| - |B_t|)} = 0.$$ 

We can again assume (see [S-W]) that the indicatrix is a spherical cylinder i.e. the product of a $k$-dimensional plane and a $n - k - 1$ dimensional Euclidean sphere of radius $\rho$. We can moreover assume that $\rho$ is arbitrarily large (see also [S-W]).

By Lemma 9 of [S-W] we then have for sufficiently small $\alpha$ and some $\varepsilon > 0$

$$B_{\rho-\varepsilon} \cap H^+_\alpha \subseteq K \cap H^+_\alpha.$$ 

Using similar methods, this implies that

$$\lim_{t \to 0} \frac{\langle x, N(x) \rangle}{n(|B| - |B_t|)} = 0.$$
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