A class of cyclic codes whose dual have five zeros

Yan Liu,∗ Chunlei Liu†

Abstract

In this paper, a family of cyclic codes over $\mathbb{F}_p$ whose duals have five zeros is presented, where $p$ is an odd prime. Furthermore, the weight distributions of these cyclic codes are determined.

Key words and phrases: cyclic code, quadratic form, weight distribution.

MSC: 94B15, 11T71.

1 INTRODUCTION

Recall that an $[n, l, d]$ linear code $C$ over $\mathbb{F}_p$ is a linear subspace of $\mathbb{F}_p^n$ with dimension $l$ and minimum Hamming distance $d$. Let $A_i$ denote the number of codewords in $C$ with Hamming weight $i$. The sequence $(A_0, A_1, A_2, \ldots, A_n)$ is called the weight distribution of the code $C$. And $C$ is called cyclic if for any $(c_0, c_1, \ldots, c_{n-1}) \in C$, also $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$. A linear code $C$ in $\mathbb{F}_p^n$ is cyclic if and only if $C$ is an ideal of the polynomial residue class ring $\mathbb{F}_p[x]/(x^n-1)$. Since $\mathbb{F}_p[x]/(x^n-1)$ is a principal ideal ring, every cyclic code corresponds to a principal ideal $(g(x))$ of the multiples of a polynomial $g(x)$ which is the monic polynomial of lowest degree in the ideal. This polynomial $g(x)$ is called the generator polynomial, and $h(x) = (x^n-1)/g(x)$ is called the parity-check polynomial of the code $C$. We also recall that a cyclic code is called irreducible if its parity-check polynomial is irreducible over $\mathbb{F}_p$, and reducible, otherwise. And a cyclic code over $\mathbb{F}_p$ is said to have $t$ zeros if all the zeros of the generator polynomial of the code form $t$ conjugate classes, or equivalently, the generator polynomial has $t$ irreducible factors over $\mathbb{F}_p$. Determining the weight distribution of a cyclic code has been the interesting subject of study for many years. For information on the weight distributions of irreducible cyclic codes, the reader is referred to [12,50]. Information on the weight distributions of reducible cyclic codes could be found in [7,10,12,14,16,18,21]. In this paper, we will determine the weight distributions of a class of reducible cyclic codes whose duals have five zeros.

Throughout this paper, let $m \geq 5$ be an odd integer and $k$ be a positive integer such that $\gcd(m,k) = 1$. Let $p$ be an odd prime and $\pi$ be a primitive element of the finite field $\mathbb{F}_{p^m}$.

∗Corresponding author, Dept. of Math., SJTU, Shanghai, 200240, liuyan0916@sjtu.edu.cn.
†Dept. of Math., Shanghai Jiaotong Univ., Shanghai, 200240, cliu@sjtu.edu.cn.
Let $\pi$ be a primitive element of the finite field $\mathbb{F}_q$. Let $h_0(x), h_{-0}(x), h_1(x), h_{-1}(x)$ and $h_2(x)$ be the minimal polynomials of $\pi^{-1}, -\pi^{-1}, \pi^{-2^{k+1}}$, $-\pi^{-2^{k+1}}$ and $\pi^{-2^{2k+1}}$ over $\mathbb{F}_p$, respectively. It is easy to check that $h_0(x), h_{-0}(x), h_1(x), h_{-1}(x)$ and $h_2(x)$ are polynomials of degree $m$ and are pairwise distinct. The cyclic codes over $\mathbb{F}_p$ with parity-check polynomial $h_0(x)h_1(x)$ have been extensively studied by [4, 14, 17, 18]. Zhengchun Zhou and Cunsheng Ding [20] proved the cyclic codes over $\mathbb{F}_p$ with parity-check polynomial $h_{-0}(x)h_1(x)$ have three nonzero weights and determined its weight distributions. In [13], the authors proved the cyclic codes over $\mathbb{F}_p$ with parity-check polynomial $h_0(x)h_{-0}(x)h_1(x)$ have six nonzero weights and determined their weight distribution. Let $C_{(p,m,k)}$ be the cyclic code with parity-check polynomial $h_0(x)h_{-0}(x)h_1(x)h_{-1}(x)h_2(x)$. In this paper, we will determine the weight distribution of the cyclic code $C_{(p,m,k)}$.

From now on, we always assume that $\lambda$ is a fixed non-square element in $\mathbb{F}_p$. Since $m$ is odd, it is also a non-square element in $\mathbb{F}_{p^m}$. Then $\lambda x$ runs through all non-square elements of $\mathbb{F}_{p^m}$ and $x$ runs through all nonzero square elements of $\mathbb{F}_{p^m}$. In addition, we have the following result.

**Lemma 1.1** ([20]) $\lambda^{(1+p^k)/2} = \lambda$ if $k$ is even, and $\lambda^{(1+p^k)/2} = -\lambda$ otherwise.

The rest of this paper is organized as follows. Some necessary results on quadratic forms will be introduced in Section 2. In Section 3, we will solve some systems of equations. A family of cyclic codes is presented and their weight distributions are determined in Section 4.

## 2 QUADRATIC FORMS OVER FINITE FIELDS

We follow the notation in Section 1. In this section, we will recall the definition of the quadratic forms over finite fields and some results about it. In particular, we present the evaluation of a specific exponential sum which is derived from the properties of the quadratic forms.

By identifying $\mathbb{F}_{p^m}$ with the $m$-dimensional $\mathbb{F}_p$-vector space $\mathbb{F}_p^m$, a function $Q$ from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ can be regarded as an $m$-variable polynomial on $\mathbb{F}_p$. Then $Q$ is called a quadratic form over $\mathbb{F}_p$ if its corresponding polynomial is a polynomial of degree two over $\mathbb{F}_p$ and can be represented as

$$Q(x_1, x_2, \ldots, x_m) = \sum_{1 \leq i \leq j \leq m} a_{ij}x_ix_j,$$

where $a_{ij} \in \mathbb{F}_p$. The rank of the quadratic form $Q(x)$ is defined as the codimension of the $\mathbb{F}_p$-vector space $V = \{x \in \mathbb{F}_{p^m} : Q(x+z) - Q(x) - Q(z) = 0 \text{ for all } z \in \mathbb{F}_{p^m}\}$.

For a quadratic form $Q(x)$, there exists a symmetric matrix $A$ of order $m$ over $\mathbb{F}_p$ such that $Q(x) = XAX^T$, where $X = (x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m$ and $X^T$ denotes the transpose of $X$. Then there exists a nonsingular matrix $H$ of order $m$ over $\mathbb{F}_p$ such that $HAH^T$ is a diagonal matrix [11]. It is easy to check that the rank of the quadratic form $Q(x)$ is exactly the rank of $A$. Under the nonsingular linear substitution $X = ZH$ with
Z = (z_1, z_2, \ldots, z_m) \in \mathbb{F}_p^m, \ Q(x) = ZHAH^T Z^T = \sum_{i=1}^r d_i z_i^2, \text{ where } r \text{ is the rank of } Q(x) \text{ and } d_i \in \mathbb{F}_p. \text{ Let } \Delta = d_1 d_2 \cdots d_r. (\text{we assume } \Delta = 0 \text{ when } r = 0). \text{ We can recall that the Legendre symbol } (\frac{a}{p}) \text{ has the value 1 if } a \text{ is a quadratic residue mod } p, \ -1 \text{ if } a \text{ is a quadratic non-residue mod } p, \text{ and zero if } p | a. \text{ Then } (\frac{a}{p}) \text{ is an invariant of } A \text{ under the action of } H \in GL_r(\mathbb{F}_p). \text{ Then we introduce the following two lemmas.}

**Lemma 2.1** (\cite{11}) With the notation as above, we have

\[
\sum_{x \in \mathbb{F}_p^m} c\beta^Q(x) = \begin{cases} 
(\frac{\Delta}{p})p^{m-\frac{r}{2}}, & p \equiv 1 \pmod{4}, \\
(\frac{\Delta}{p})(\sqrt{-1})^{p^{m-\frac{r}{2}}}, & p \equiv 3 \pmod{4}, 
\end{cases}
\]

for any quadratic form \(Q(x)\) in \(m\) variables of rank \(r\) over \(\mathbb{F}_p\), where \(\zeta_p\) is a primitive \(p\)-th root of unity.

**Lemma 2.2** (\cite{20}, Lemma 2.2) Let \(Q(x)\) be a quadratic form in \(m\) variables of rank \(r\) over \(\mathbb{F}_p\), then

\[
\sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_p^m} c\beta^Q(x) = \begin{cases} 
\pm(p-1)p^{m-\frac{r}{2}}, & r \text{ even,} \\
0, & \text{otherwise.}
\end{cases}
\]

For any fixed \((u, v, w) \in \mathbb{F}_p^2\), let \(Q_{u,v,w}(x) = \text{Tr}(ux^2 + vx^{p^k+1} + wx^{p^{2k+1}})\), where \(\text{Tr}\) is the trace mapping from \(\mathbb{F}_p^m\) to \(\mathbb{F}_p\). Moreover, we have the following result.

**Lemma 2.3** (\cite{12}) For any \((u, v, w) \in \mathbb{F}_p^2 \setminus \{(0, 0, 0)\}\), \(Q_{u,v,w}(x)\) is a quadratic form over \(\mathbb{F}_p\) with rank at least \(m - 4\).

### 3 Solutions of Some Systems of Equations

In this section, we will solve some systems of equations, which will be needed in the subsequent section. Before introducing them, for any positive integer \(k\), we define \(d_1 = p^k + 1\) and \(d_2 = p^{2k} + 1\).

**Lemma 3.1** Let \(N_2\) denote the number of solutions \((x_1, x_2, y_1, y_2) \in \mathbb{F}_p^2 \times \mathbb{F}_p^2\) of the following system of equations

\[
\begin{cases} 
y_1 x_1^2 + y_2 x_2^2 = 0 \\
y_1 x_1^{d_1} + y_2 x_2^{d_1} = 0
\end{cases}
\]

Then \(N_2 = (p-1)^2p^m\). \hspace{1cm} (1)

**Proof.**

(1) When \(y_1\) and \(y_2\) are both squares of \(\mathbb{F}_p^2\), the number of the solutions \((x_1, x_2, y_1, y_2) \in \mathbb{F}_p^2 \times \mathbb{F}_p^2\) of the system above is \(\frac{(p-1)^2}{4}\) multiplied by the number of the solutions \((x_1, x_2) \in \mathbb{F}_p^2\) of the following system of equations:

\[
\begin{cases} 
x_1^2 + x_2^2 = 0 \\
x_1^{d_1} + x_2^{d_1} = 0
\end{cases}
\]
which is equal to the number of the following system of equations:

\[
\begin{cases}
  x_1^2 + x_2^2 = 0 \\
  x_1^{d_1} + x_2^{d_1} = 0 \\
  x_1^{d_2} + x_2^{d_2} = 0.
\end{cases}
\]

(2) When \( y_1 \) is a square element but \( y_2 \) is a non-square element of \( \mathbb{F}_p^* \), the number of the solutions \((x_1, x_2, y_1, y_2) \in \mathbb{F}_{p^m}^2 \times \mathbb{F}_p^2\) of the system above is \((p-1)^2\) multiplied by the number of the solutions \((x_1, x_2) \in \mathbb{F}_{p^m}^2\) of the following system of equations:

\[
\begin{cases}
  x_1^2 + \lambda x_2^2 = 0 \\
  x_1^{d_1} + \lambda x_2^{d_1} = 0,
\end{cases}
\]

which is equal to the number of the following system of equations:

\[
\begin{cases}
  x_1^2 + \lambda x_2^2 = 0 \\
  x_1^{d_1} + x_2^{d_1} = 0 \\
  x_1^{d_2} + \lambda x_2^{d_2} = 0.
\end{cases}
\]

(3) The case of \( y_1 \) is a non-square element but \( y_2 \) is a square element of \( \mathbb{F}_p^* \) is equivalent to the above case.

(4) The case of \( y_1 \) and \( y_2 \) are both non-squares of \( \mathbb{F}_p^* \) is equivalent to case (1).

Summarizing the discussion above, the number of solutions \((x_1, x_2, y_1, y_2) \in \mathbb{F}_{p^m}^2 \times \mathbb{F}_p^2\) of the system (1) is \((p-1)^2 p^m\).

**Remark.** From the proof of Lemma 3.4 and 3.5 in [12] and the proof of Lemma 3.1 we have that the number of solutions \((x_1, x_2, y_1, y_2) \in \mathbb{F}_{p^m}^2 \times \mathbb{F}_p^2\) of (1) is equal to the number of solutions of the following system of equations

\[
\begin{cases}
  y_1 x_1^2 + y_2 x_2^2 = 0 \\
  y_1 x_1^{d_1} + y_2 x_2^{d_1} = 0 \\
  y_1 x_1^{d_2} + y_2 x_2^{d_2} = 0.
\end{cases}
\]

**Lemma 3.2** Let \( N_3 \) denote the number of solutions \((x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{F}_{p^m}^3 \times \mathbb{F}_p^3\) of the following system of equations

\[
\begin{cases}
  y_1 x_1^2 + y_2 x_2^2 + y_3 x_3^2 = 0 \\
  y_1 x_1^{d_1} + y_2 x_2^{d_1} + y_3 x_3^{d_1} = 0.
\end{cases}
\]

Then \( N_3 = (p-1)^3 (p^{m+1} + p^m - p) \).
We follow the notation fixed in the previous sections. Let \( C \) be the class of cyclic codes. Obviously, \( N \) is equal to the number of solutions of the following system of equations
\[
\begin{align*}
    y_1 x_1^2 + y_2 x_2^2 + y_3 x_3^2 &= 0 \\
    y_1 x_1^{d_1} + y_2 x_2^{d_1} + y_3 x_3^{d_1} &= 0 \\
    y_1 x_1^{d_2} + y_2 x_2^{d_2} + y_3 x_3^{d_2} &= 0.
\end{align*}
\]

**Remark.** Similarly, we also have that the number of solutions \((x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{F}_p^3 \times \mathbb{F}_p^3 \) of \( \mathcal{C} \) is equal to the number of solutions of the following system of equations
\[
\begin{align*}
    y_1 x_1^2 + y_2 x_2^2 + y_3 x_3^2 &= 0 \\
    y_1 x_1^{d_1} + y_2 x_2^{d_1} + y_3 x_3^{d_1} &= 0 \\
    y_1 x_1^{d_2} + y_2 x_2^{d_2} + y_3 x_3^{d_2} &= 0.
\end{align*}
\]

**Lemma 3.3** Let \( N_1 \) denote the number of solutions \((x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in \mathbb{F}_p^4 \) of the following system of equations
\[
\begin{align*}
    y_1 x_1^2 + y_2 x_2^2 + y_3 x_3^2 + y_4 x_4^2 &= 0 \\
    y_1 x_1^{d_1} + y_2 x_2^{d_1} + y_3 x_3^{d_1} + y_4 x_4^{d_1} &= 0.
\end{align*}
\]
Then \( N_1 = p^m (p^{m+1} + p^m - p)(p - 1)^4 \).

**Proof.** This lemma can be proved in a similar way as the proof of Lemma 3.1 by Lemmas 3.8-3.10 in [12], so we omit the details.

**Remark.** Again, we have that the number of solutions \((x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in \mathbb{F}_p^4 \times \mathbb{F}_p^4 \) of \( \mathcal{C} \) is equal to the number of solutions of the following system of equations
\[
\begin{align*}
    y_1 x_1^2 + y_2 x_2^2 + y_3 x_3^2 + y_4 x_4^2 &= 0 \\
    y_1 x_1^{d_1} + y_2 x_2^{d_1} + y_3 x_3^{d_1} + y_4 x_4^{d_1} &= 0 \\
    y_1 x_1^{d_2} + y_2 x_2^{d_2} + y_3 x_3^{d_2} + y_4 x_4^{d_2} &= 0.
\end{align*}
\]

4 THE CLASS OF REDUCIBLE CODES

We follow the notation fixed in the previous sections. Let \( C_{(p,m,k)} \) be the cyclic code defined in Section 1. In this section, we will determine the weight distribution of this class of cyclic codes. Obviously, \( C_{(p,m,k)} \) has length \( p^m - 1 \) and dimension \( 5m \). Moreover, it can be expressed as
\[
C_{(p,m,k)} = \{ c_{(a_1, a_2, b_1, b_2, c)} : a_1, a_2, b_1, b_2, c \in \mathbb{F}_p^m \},
\]
where
\[
c_{(a_1, a_2, b_1, b_2, c)} = (T r((a_1 \pi^t + a_2(-\pi)^t) + b_1 \pi^{(p^k+1)t/2} + b_2(-\pi)(p^k+1)t/2 + c \pi^{(p^k+1)t/2}))_{t=0}^{p^m-2}.
\]
In terms of exponential sums, the weight of the codeword \( c_{(a_1, a_2, b_1, b_2, c)} = (c_0, c_1, \ldots, c_{p^m-1}) \)
\(c_{p^{m-2}}\) in \(C_{(p,m,k)}\) is given by

\[
W(c_{(a_1,a_2,b_1,b_2,c)}) = \#\{0 \leq t \leq p^m - 2 : c_t \neq 0\}
\]

\[
= p^m - 1 - \frac{1}{p} \sum_{t=0}^{p^{m-2}} \sum_{y \in \mathbb{F}_p} c_y t
\]

\[
= p^m - 1 - \frac{1}{p} \sum_{t=0}^{p^{m-2}} \sum_{y \in \mathbb{F}_p} (\gamma Tr(a_1 x^t + a_2 (\pi^t + b_1 x x^{(p^k+1)/2} + b_2 (-\pi)^{(p^{k+1})/2} + c x x^{p^{2k+1}+1}))
\]

\[
= p^m - 1 - \frac{1}{2p} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_{p^m}} (\gamma Tr((a_1 + a_2)x^2 + (b_1 + b_2)x^{p^{k+1}+1} + c x x^{p^{2k+1}})
\]

It then follows from Lemma 3.1 that

\[
W(c_{(a_1,a_2,b_1,b_2,c)}) = p^m - p^{m-1} - \frac{1}{2p} S(a_1, a_2, b_1, b_2, c)
\]

when \(k\) is even, where

\[
S(a_1,a_2,b_1,b_2,c) = \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_{p^m}} (\gamma Tr((a_1 + a_2)x^2 + (b_1 + b_2)x^{p^{k+1}+1} + c x x^{p^{2k+1}})
\]

\[
+ \gamma Tr((a_1 - a_2)x^2 + (b_1 - b_2)x^{p^{k+1}+1} + c x x^{p^{2k+1}})),
\]

and

\[
W(c_{(a_1,a_2,b_1,b_2,c)}) = p^m - p^{m-1} - \frac{1}{2p} T(a_1, a_2, b_1, b_2, c)
\]

when \(k\) is odd, where

\[
T(a_1,a_2,b_1,b_2,c) = \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_{p^m}} (\gamma Tr((a_1 + a_2)x^2 + (b_1 + b_2)x^{p^{k+1}+1} + c x x^{p^{2k+1}})
\]

\[
+ \gamma Tr((a_1 - a_2)x^2 - (b_1 - b_2)x^{p^{k+1}+1} + c x x^{p^{2k+1}})),
\]

Based on the discussion above, the weight distribution of the code \(C_{(p,m,k)}\) is completely determined by the value distribution of \(S(a_1, a_2, b_1, b_2, c)\) and \(T(a_1, a_2, b_1, b_2, c)\). Before determining the value distribution of \(S(a_1, a_2, b_1, b_2, c)\) and \(T(a_1, a_2, b_1, b_2, c)\), for any \((u,v,w) \in \mathbb{F}_{p^m}^3\), we define

\[
D(u,v,w) = \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_{p^m}} \gamma Q_{u,v,w}(x) = \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_{p^m}} \gamma Tr(ax^2 + vx x^{p^{k+1}+1} + wx x^{p^{2k+1}}).
\]

From the discussion above, the value distributions of \(S(a_1, a_2, b_1, b_2, c)\) and \(T(a_1, a_2, b_1, b_2, c)\) can be deduced from the value distributions of \(D(u,v,w)\). In fact, the value distributions of \(D(u,v,0)\) can be obtained by Lemma 3.2 and 3.3 in [13] as follows.
Lemma 4.1 Let $D(u, v, w)$ be defined by (3). Then as $(u, v)$ runs through $\mathbb{F}_{p^m}^2$, the value distribution of $D(u, v, 0)$ is given by Table 1.

Table 1: Value distribution of $D(u, v, 0)$

| Value | Frequency |
|-------|-----------|
| 0     | $(p^m - 1)(p^m - p^{m-1} + 1)$ |
| $(p - 1)p^m$ | 1 |
| $(p - 1)p^{m+1}$ | $\frac{1}{2}(p^m - 1)(p^{m-1} + p^{m-1})$ |
| $-(p - 1)p^{m+1}$ | $\frac{1}{2}(p^m - 1)(p^{m-1} - p^{m-1})$ |

For the convenience, we introduce the following notation.

$$n_0 = \#\{(u, v) \in \mathbb{F}_{p^m}^2 : D(u, v, 0) = 0\}$$

and

$$n_\varepsilon = \#\{(u, v) \in \mathbb{F}_{p^m}^2 : D(u, v, 0) = \varepsilon(p - 1)p^{m+1}\},$$

where $\varepsilon = \pm 1$. That is $n_0 = (p^m - 1)(p^m - p^{m-1} + 1)$, $n_\varepsilon = \frac{1}{2}(p^m - 1)(p^{m-1} + \varepsilon p^{m-1})$.

As for any fixed $w \in \mathbb{F}_{p^m}^*$, the value distributions of $D(u, v, w)$ can be determined by the following lemma.

Lemma 4.2 Let $D(u, v, w)$ be defined by (3). Then for any fixed $w \in \mathbb{F}_{p^m}^*$, as $(u, v)$ runs through $\mathbb{F}_{p^m}^2$, the value distribution of $D(u, v, w)$ is given by Table 2.

Table 2: Value distribution of $D(u, v, w)$ for any fixed $w \in \mathbb{F}_{p^m}^*$

| Value | Frequency |
|-------|-----------|
| 0     | $p^{2m} - p^{2m-4} - p^{m-3}$ |
| $(p - 1)p^{m+1}$ | $\frac{(p^{m+2} - p^{m-1} + 1)(p^{m-1} + p^{m-1})}{2(p^2 - 1)}$ |
| $-(p - 1)p^{m+1}$ | $\frac{(p^{m+2} - p^{m-1} + 1)(p^{m-1} - p^{m-1})}{2(p^2 - 1)}$ |
| $(p - 1)p^{m+3}$ | $\frac{(p^{m-1} - 1)(p^{m-3} + p^{m-3})}{2(p^2 - 1)}$ |
| $-(p - 1)p^{m+3}$ | $\frac{(p^{m-1} - 1)(p^{m-3} - p^{m-3})}{2(p^2 - 1)}$ |

Proof. As in Eq. (3),

$$D(u, v, w) = \sum_{y \in \mathbb{F}_{p^m}^*} \sum_{x \in \mathbb{F}_{p^m}^2} \zeta_y Q_{u, v, w}(x).$$

Then for any fixed $w \in \mathbb{F}_{p^m}^*$, by Lemma 2.2 and 2.3, $D(u, v, w)$ takes on the values from the set $\{0, \pm (p - 1)p^{m+1}, \pm (p - 1)p^{m+1}\}$. To determine the distribution of $D(u, v, w)$ for any fixed $w \in \mathbb{F}_{p^m}^*$, we define

$$n_0^w = \#\{(u, v) \in \mathbb{F}_{p^m}^2 : D(u, v, w) = 0\}$$
and

\[ n^w_{e,i} = \# \{(u, v) \in \mathbb{F}_p^2 : D(u, v, w) = \varepsilon(p - 1)p^{\frac{m+i}{2}} \}, \]

where \( \varepsilon = \pm 1, i \in \{1, 3\} \). From the following discussion, the number \( n^w_{e} \) and \( n^w_{0} \) is dependent of the choice of \( w \in \mathbb{F}_p^* \). Hence in the following, for any fixed \( w \in \mathbb{F}_p^* \), we denote by \( n^w_{e} \) and \( n^w_{0} \) Instead of \( n^w_{e} \) and \( n^w_{0} \). Then we have

\[
\sum_{(u, v) \in \mathbb{F}_p^2} D(u, v) = (n_{1,1} - n_{-1,1})(p - 1)p^{\frac{m+1}{2}} + (n_{1,3} - n_{-1,3})(p - 1)p^{\frac{m}{2}}, \tag{9}
\]

\[
\sum_{(u, v) \in \mathbb{F}_p^2} D^2(u, v) = (n_{1,1} + n_{-1,1})(p - 1)^2 p^{m+1} + (n_{1,3} + n_{-1,3})(p - 1)^2 p^{m+3}, \tag{10}
\]

\[
\sum_{(u, v) \in \mathbb{F}_p^2} D(u, v) = (n_{1,1} - n_{-1,1})(p - 1)^3 p^{\frac{3(m+1)}{2}} + (n_{1,3} - n_{-1,3})(p - 1)^3 p^{\frac{3(m+3)}{2}}, \tag{11}
\]

and

\[
\sum_{(u, v) \in \mathbb{F}_p^2} D^2(u, v) = (n_{1,1} + n_{-1,1})(p - 1)^4 p^{2(m+1)} + (n_{1,3} + n_{-1,3})(p - 1)^4 p^{2(m+3)}. \tag{12}
\]

On the other hand, it follows from (8) that

\[
\sum_{u, v \in \mathbb{F}_p^m} D(u, v, w) = \sum_{u, v \in \mathbb{F}_p^m} \sum_{x, y \in \mathbb{F}_p^m} \sum_{z \in \mathbb{F}_p^m} \varepsilon y Tr(ux^2 + vxz^k + 1 + wxz^k + 1)
\]

\[
= \sum_{x, y \in \mathbb{F}_p^m} \sum_{z \in \mathbb{F}_p^m} \varepsilon y Tr(vx^k + 1) \sum_{u \in \mathbb{F}_p^m} \varepsilon y Tr(ux^2) \sum_{v \in \mathbb{F}_p^m} \varepsilon y Tr(vx^k + 1) \tag{13}
\]

\[
= (p - 1)p^{2m}.
\]

By Lemmas 5.13.3 and the remarks followed, we have the following results.

\[
\sum_{u, v \in \mathbb{F}_p^m} D^2(u, v, w)
\]

\[
= \sum_{u, v \in \mathbb{F}_p^m} \sum_{y_1, y_2 \in \mathbb{F}_p^m} \sum_{x_1, x_2 \in \mathbb{F}_p^m} \zeta_p^{y_1 Tr(u y_1 x_1^2 + y_2 x_1^k + 1 + y_2 x_1^{2k+1})} \zeta_p^{y_2 Tr(u y_2 x_2^2 + y_2 x_2^k + 1 + y_2 x_2^{2k+1})}
\]

\[
= \sum_{(y_1, y_2) \in \mathbb{F}_p^* \times \mathbb{F}_p^*} \zeta_p^{Tr(w(y_1 x_1^2 + y_2 x_2^2) + y_1 x_1^k + 1 + y_2 x_2^{2k+1})}, \tag{14}
\]

\[
= (p - 1)^2 p^{2m}.
\]
\[
\sum_{u, v, w \in \mathbb{F}_p^m} D^3(u, v, w) \\
= \sum_{(y_1, y_2, y_3) \in \mathbb{F}_p^3} \sum_{(x_1, x_2, x_3) \in \mathbb{F}_p^3} \zeta_p \text{Tr}(w(y_1x_1^{2k+1} + y_2x_2^{2k+1} + y_3x_3^{2k+1})),
\]

\[
\sum_{u \in \mathbb{F}_p^m} \zeta_p \text{Tr}(u(y_1x_1^{2k+1} + y_2x_2^{2k+1} + y_3x_3^{2k+1})),
\]

\[
\sum_{u \in \mathbb{F}_p^m} \zeta_p \text{Tr}(u(y_1x_1^{2k+1} + y_2x_2^{2k+1} + y_3x_3^{2k+1})),
\]

\[
= p^{2m} \cdot \# \{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{F}_p^3 \times \mathbb{F}_p^3 \mid y_1x_1^2 + y_2x_2^2 + y_3x_3^2 = 0,
\]

\[
y_1x_1^{d_1} + y_2x_2^{d_1} + y_3x_3^{d_1} = 0.\}
\]

\[
= p^{2m}(p - 1)^3(p^m + p^m - p).
\]

and

\[
\sum_{u, v, w \in \mathbb{F}_p^3} D^3(u, v, w) \\
= \sum_{(y_1, y_2, y_3, y_4) \in \mathbb{F}_p^4} \sum_{(x_1, x_2, x_3, x_4) \in \mathbb{F}_p^4} \zeta_p \text{Tr}(w(y_1x_1^{2k+1} + y_2x_2^{2k+1} + y_3x_3^{2k+1} + y_4x_4^{2k+1})),
\]

\[
\sum_{u \in \mathbb{F}_p^m} \zeta_p \text{Tr}(u(y_1x_1^{2k+1} + y_2x_2^{2k+1} + y_3x_3^{2k+1} + y_4x_4^{2k+1})),
\]

\[
\sum_{u \in \mathbb{F}_p^m} \zeta_p \text{Tr}(u(y_1x_1^{2k+1} + y_2x_2^{2k+1} + y_3x_3^{2k+1} + y_4x_4^{2k+1})),
\]

\[
= p^{2m} \cdot \# \{(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in \mathbb{F}_p^4 \times \mathbb{F}_p^4 \mid y_1x_1^2 + y_2x_2^2 + y_3x_3^2 + y_4x_4^2 = 0,
\]

\[
y_1x_1^{d_1} + y_2x_2^{d_1} + y_3x_3^{d_1} + y_4x_4^{d_1} = 0.\}
\]

\[
= p^{3m}(p - 1)^4(p^m + p^m - p) .
\]

Combining Eqs. (10)–(16), we have

\[
n_{1,1} = \frac{(p^{m+2} - p^m - p^{m-1} + 1)\left(p^{m-1} + p^{m+1}\right)}{2(p^2 - 1)},
\]

\[
n_{-1,1} = \frac{(p^{m+2} - p^m - p^{m-1} + 1)(p^{m-1} - p^{m-1})}{2(p^2 - 1)},
\]

\[
n_{1,3} = \frac{(p^{m-1} - 1)(p^{m-3} + p^{m-3})}{2(p^2 - 1)},
\]

\[
n_{-1,3} = \frac{(p^{m-1} - 1)(p^{m-3} - p^{m-3})}{2(p^2 - 1)}.
\]

Then we have \( n_0 = p^{2m} - n_{1,1} - n_{-1,1} - n_{1,3} - n_{-1,3} = p^{2m} - p^{m-1} + p^{2m-4} - p^{m-3} .\)

\[\blacksquare\]

The value distribution of \( S(a_1, a_2, b_1, b_2, c) \) is determined by the following theorem.

**Theorem 4.3** Let \( k \) be even and \( S(a_1, a_2, b_1, b_2, c) \) be defined by (4). Then as \((a_1, a_2, b_1, b_2, c)\) runs through \(\mathbb{F}_p^5\), the value distribution of \( S(a_1, a_2, b_1, b_2, c) \) is given by Table 5.
Similarly, we have

\[
\begin{align*}
\# \{(a_1, a_2, b_1, b_2, c) \in \mathbb{F}_p^5 : S(a_1, a_2, b_1, b_2, c) = (p - 1)p^{m+1} \} &= 2n_0n_1 + 2(p^m - 1)n_0^*n_{1,1}, \\
\# \{(a_1, a_2, b_1, b_2, c) \in \mathbb{F}_p^5 : S(a_1, a_2, b_1, b_2, c) = -(p - 1)p^{m+1} \} &= 2n_0n_{-1}, \\
\# \{(a_1, a_2, b_1, b_2, c) \in \mathbb{F}_p^5 : S(a_1, a_2, b_1, b_2, c) = 2(p - 1)p^{m+1} \} &= n_1^2 + (p^m - 1)n_1^2, \\
\# \{(a_1, a_2, b_1, b_2, c) \in \mathbb{F}_p^5 : S(a_1, a_2, b_1, b_2, c) = 2(p - 1)p^{m+1} \} &= n_2^2 + (p^m - 1)n_2^2,
\end{align*}
\]

\textbf{Proof.} By Lemma 4.1 and 4.2, }S(a_1, a_2, b_1, b_2, c)\text{ takes values from the set }\{0, (p - 1)p^m, 2(p - 1)p^m, (p - 1)(p^m \pm p^{m+1}), (p - 1)(p^m \pm p^{m+1}), \pm 2(p - 1)p^{m+1}, \pm 2(p - 1)p^{m+1}, \pm 2(p - 1)p^{m+1}, (p - 1)(p^{m+1} \pm p^{m+1}), (p - 1)(p^{m+1} \pm p^{m+1})\} . \text{ The distribution of }S(a, b, c) = (p - 1)p^m, 2(p - 1)p^m \text{ or } (p - 1)(p^m \pm p^{m+1}) \text{ can be easily obtained by Lemma 4.1.}
The weight distribution of $C$ is given by Table 4.

Following the notation above, we have $T(a_1, a_2, b_1, b_2, c) = D(a_1 + a_2, b_1 + b_2, c) + D(a_1 + a_2, -(b_1 - b_2), c)$. It can be shown that the value distribution of $T(a_1, a_2, b_1, b_2, c)$ in the case of $k$ is odd is the same as the value distribution of $S(a_1, a_2, b_1, b_2, c)$ in the case of $k$ is even.

The following is the main result of this paper.

**Theorem 4.4** Let $m$ and $k$ be any two positive integers such that $m \geq 5$ is odd, then $C_{(p, m, k)}$ is a cyclic code over $\mathbb{F}_p$ with parameters $[p^m - 1, 5m, \frac{1}{2}(p - 1)(p^{m-1} - p^{m-k})]$. Furthermore, the weight distribution of $C_{(p, m, k)}$ is given by Table 4.

**Proof.** The length and dimension of $C_{(p, m, k)}$ follow directly from its definition. The minimal weight and weight distribution of $C_{(p, m, k)}$ follow from Eqs. 10 and 17. Theorem 4.3 and the Remark above.

**Remark.** The weight distribution of $C_{(p, m, k)}$ can be determined in a similar way in a more general case where $m/gcd(m, k) \geq 5$ is odd. We omit the details in order to avoid duplication.
Table 4: Weight Distribution of $C_{(p,m,k)}$

| Weight                                   | Frequency                                    |
|------------------------------------------|----------------------------------------------|
| 0                                        | 1                                            |
| $\frac{1}{2}(p - 1)p^{m-1}$              | $2n_0$                                       |
| $\frac{1}{2}(p - 1)(p^{m-1} + p^{\frac{m+1}{2}})$ | $2n_{-1}$                                   |
| $\frac{1}{2}(p - 1)(p^{m-1} - p^{\frac{m+1}{2}})$ | $2n_1$                                       |
| $(p - 1)(p^{m-1} - \frac{1}{2}p^{\frac{m-1}{2}})$ | $2n_0n_1 + 2(p^m - 1)n_0^2n_{1,1}$          |
| $(p - 1)(p^{m-1} - \frac{1}{2}p^{\frac{m+1}{2}})$ | $2(p^m - 1)n_0^2n_{1,3}$                    |
| $(p - 1)(p^{m-1} - p^{\frac{m+1}{2}})$     | $n_1^2 + (p^m - 1)n_1^2n_{1,1}$              |
| $(p - 1)(p^{m-1} + \frac{1}{2}p^{\frac{m+1}{2}})$ | $(p^m - 1)n_1^2$                            |
| $(p - 1)(p^{m-1} + p^{\frac{m-1}{2}})$    | $2n_0n_{-1} + 2(p^m - 1)n_0^2n_{-1,1}$       |
| $(p - 1)(p^{m-1} + \frac{1}{2}p^{\frac{m-1}{2}})$ | $2(p^m - 1)n_1^2n_{-1,3}$                   |
| $(p - 1)(p^{m-1} + \frac{1}{2}p^{\frac{m+1}{2}})$ | $n_{-1}^2 + (p^m - 1)n_1^2n_{-1,1}$          |
| $(p - 1)(p^{m-1} - \frac{1}{2}p^{\frac{m-1}{2}} - \frac{1}{2}p^{\frac{m+1}{2}})$ | $(p^m - 1)n_{-1,3}$                        |
| $(p - 1)(p^{m-1} - \frac{1}{2}p^{\frac{m+1}{2}} + \frac{1}{2}p^{\frac{m-1}{2}})$ | $2(p^m - 1)n_{1,1}n_{1,3}$                 |
| $(p - 1)(p^{m-1} + \frac{1}{2}p^{\frac{m-1}{2}} - \frac{1}{2}p^{\frac{m+1}{2}})$ | $2(p^m - 1)n_{1,1}n_{-1,3}$                |
| $(p - 1)(p^{m-1} + \frac{1}{2}p^{\frac{m+1}{2}} + \frac{1}{2}p^{\frac{m+1}{2}})$ | $2(p^m - 1)n_{-1,1}n_{-1,3}$               |
| $(p - 1)p^{m-1}$                         | $n_0^2 + 2n_1n_{-1} + (p^m - 1)n_0^2n_{1,1} + n_{1,3}n_{1,3}$ |

References

[1] L.D. Baumert, R.J. McEliece, Weights of irreducible cyclic codes, *Inf. Contr.*, 20, no. 2 (1972), 158-175.

[2] L.D. Baumert, J. Mykkeltveit, Weight distribution of some irreducible cyclic codes, *DSN Progr. Rep.*, 16 (1973), 128-131.

[3] A.R. Calderbank, J.M. Goethals, Three-weight codes and association schemes, *Philips J. Res.*, 39 (1984), 143-152.

[4] C. Carlet, C. Ding, J. Yuan, Linear codes from highly nonlinear functions and their secret sharing schemes, *IEEE Trans. Inf. Theory*, 51, no. 6 (2005), 2089-2102.

[5] C. Ding, The weight distribution of some irreducible cyclic codes, *IEEE Trans. Inf. Theory*, 55, no. 3 (2009), 955-960.

[6] C. Ding, J. Yang, Hamming weights in irreducible cyclic codes, *Discrete Mathematics*, 313, no. 4 (2013), 434-446.

[7] C. Ding, Y. Liu, C. Ma, L. Zeng, The weight distributions of the duals of cyclic codes with two zeros, *IEEE Trans. Inf. Theory*, 57, no. 12 (2011), 8000-8006.
[8] K. Feng, J. Luo, Weight distribution of some reducible cyclic codes, *Finite Fields Appl.*, **14**, no. 2 (2008), 390-409.

[9] T. Feng, On cyclic codes of length $2^r - 1$ with two zeros whose dual codes have three weights, *Des. Codes Cryptogr.*, **62** (2012), 253-258.

[10] T. Feng, K. Leung, Q. Xiang, Binary cyclic codes with two primitive nonzeros, *Sci. China Math.*, **56**, no. 7 (2012), 1403-1412.

[11] R. Lidl, H. Niederreiter, Finite fields, *Addison-Wesley Publishing Inc.*, (1983).

[12] Y. Liu, H. Yan, A Class of Five-weight Cyclic Codes and Their Weight Distribution, *arXiv:1312.4638* (2013).

[13] Y. Liu, H. Yan, C. Liu, A class of six-weight cyclic codes and their weight distribution, *Des. Codes Cryptogr.*, (2014), doi: 10.1007/s10623-014-9984-y.

[14] J. Luo, K. Feng, Cyclic codes and sequences form generalized Coulter-Matthews function, *IEEE Trans. Inf. Theory*, **54**, no. 12 (2008), 5345-5353.

[15] J. Luo, K. Feng, On the weight distribution of two classes of cyclic codes, *IEEE Trans. Inf. Theory*, **54**, no. 12 (2008), 5332-5344.

[16] C. Ma, L. Zeng, Y. Liu, D. Feng, C. Ding, The weight enumerator of a class of cyclic codes, *IEEE Trans. Inf. Theory*, **57**, no. 1 (2011), 397-402.

[17] H.M. Trachtenberg, On the crosscorrelation functions of maximal linear recurring sequences, PhD dissertation, *Univ. South. Calif., Los Angels*, (1970).

[18] J. Yuan, C. Carlet, C. Ding, The weight distribution of a class of linear codes from perfect nonlinear functions, *IEEE Trans. Inf. Theory*, **52**, no. 2 (2006), 712-717.

[19] B. Wang, C. Tang, Y. Qi, Y. Yang, M. Xu, The weight distributions of cyclic codes and elliptic curves, *IEEE Trans. Inf. Theory*, **58**, no. 12 (2012), 7253-7259.

[20] Z. Zhou, C. Ding, A class of three-weight cyclic codes, *Finite Fields Appl.*, **25** (2014), 79-93.

[21] Z. Zhou, C. Ding, J. Luo, A. Zhang, A family of five-weight cyclic codes and their weight enumerators, *IEEE Trans. Inf. Theory*, **59**, no. 10 (2013), 6674-6682.