Viscoelastic fractures in stratified composite materials: "lenticular trumpet"

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Abstract. – We consider fractures in a stratified composite material with solid layers separated by thin slices of extremely soft matter. Viscoelastic effects associated with the soft layers are taken into account via the simplest model for weakly cross-linked polymers. We find that certain small cracks running along layers take a new "trumpet" shape quite different from previously known shapes.

Introduction. – Nacre is a composite material with solid layers (aragonite) separated by thin slices of soft organic matter (proteins) (Fig. 1). It has a spectacular toughness where a fracture propagates normal to the layers. This toughness can be explained from the absence of stress concentration in this structure. In the present note, we consider a different problem, where the fracture plane is parallel to the layers. The fracture properties have some similarity with those of a smectic liquid crystal, which have been discussed under the name of lenticular fracture. Our aim here is to consider the dynamics, when the soft layers behave like a viscoelastic fluid. But we shall start with a reminder of the statics, using a simple scaling argument.

The crucial feature is that, for a fracture cavity of size $X$ along the layers, the elastically distorted zone has a size $Y$ (perpendicular to the layers) which does not scale like $X$, but rather like

$$Y \simeq X^2/l$$

for small cracks ($X \ll l$), where $l$ is a characteristic length

$$l^2 = K_B/E_0.$$  \hspace{1cm} (2)

Here $K_B$ is a bending modulus and $E_0$ ($= \varepsilon E_s$) is an elastic modulus associated with soft layers. It is emphasized that Eq. (4) always holds at the scaling level. (For larger cracks ($X > l$) the relation (4) turns back to $X \simeq Y$; this regime will be discussed elsewhere (8)).
Fig. 1 – Narce-type structure of materials: hard layers (elastic modulus $E_h$, thickness $d_h$) are glued together by soft layers (modulus $E_s$, thickness $d_s$). Cracks in the $y-z$ plane and the $x-z$ plane are called a perpendicular and parallel fractures, respectively (the $z$-axis is perpendicular to this page).

For stratified materials, the following relation has been shown \cite{8}: 
\[
    l = d/\sqrt{\varepsilon}
\]  
(3)

where 
\[
    \varepsilon = \frac{E_s}{E_h} \cdot \frac{d_h}{d_s}
\]
(4)
in the small $\varepsilon$ limit (see Fig. 1 for notations). In the case of nacre, the small crack condition $X \ll l$ is rather severe ($l \approx 50d$). Thus, the following considerations might be more practical for some artificially synthesized layered composites where the soft part is a weak gel, $E_0$ is very small and $l \gg d$.

The small crack condition $X \ll l$ makes the situation different from smectic liquid crystals where we have $Y \gg X$ because $l$ corresponds to an atomic scale ($X/l \gg 1$ in Eq. (3)) \cite{7}. In the present case, for $X/l \ll 1$, we have $X \gg Y$; the anisotropic strain field is distributed widely in the $x$ direction compared with in the $y$ direction (see Fig. 2 below).

The potential energy (per unit length in the $z$ direction) of the crack is
\[
    F \simeq K_B \left( \frac{u}{X^2} \right)^2 XY - \sigma u X + GX
\]
(5)

Here $u$ is the displacement in the $y$-direction, $\sigma$ the pulling stress (along the $y$ axis) and $G$ the fracture energy (energy required to create a new unit area). (It has been shown that the dominant component in strain and stress tensors are indeed the $y$ components ($u$ and $\sigma$, respectively) \cite{8}). Note here the first elastic term can be equally expressed as $E_0(u/Y)^2$ due to Eq. (3); this term actually results from a local balance between these two elastic contributions and Eq. (4) originates from this balance condition \cite{8}. Minimizing $F$ with respect to $u$, then, $F(X)$ has a maximum defining the onset of fracture. At this critical of fracture we have
\[
    u \sim X, \sigma \sim X^{-1}, \sigma u \simeq G
\]
(6)
where the last equation announces that the product $\sigma u$ gives the fracture energy. These fractures are very different from conventional parabolic fractures in linear elastic fracture mechanics \cite{10}: $u \sim X^{1/2}$ and $\sigma \sim X^{-1/2}$. In a more precise analysis, we have obtained forms for $u(x, y)$ and $\sigma(x, y)$, \cite{8} which are consistent with Eqs. (5) and (4).
Eq. (1) states that, at the scaling level, a point away from the tip by a distance $X$ along the $x$ axis has the same order of strain (or stress) with that by a distance $Y$ along the $y$ axis, when $Y \approx X^2/l$; the situation can be conceptually represented as in Fig. 2.

We now proceed to the dynamics via a complex modulus of the form

$$\mu(\omega) = E_0 + (E_\infty - E_0) \frac{i\omega\tau}{1 + i\omega\tau}$$

(7)

where the ratio $E_\infty/E_0 = \lambda$ is assumed to be large as it is in a weakly cross-linked system. Here, $E_0$ is related to a small modulus associated with weak cross-links while $E_\infty$ to a large modulus originating from entanglements. Eq. (7) is a result of taking viscoelastic effects in soft layers through a complex modulus of the same form (but with $E_0$ and $E_\infty$ replaced by $E_s$ and $\lambda E_s$) [9].

Crack shape and fracture energy. — We consider a parallel crack propagating with a constant speed $V$. When $V$ is smaller than a sound velocity ($\approx \sqrt{E_0/\rho}$), the equation of motion of the density $\rho$ and the local velocity $v$ at the scaling level, $\rho Dv/Dt = -\nabla \sigma$, reduces to a static equation: $\nabla \sigma = 0$. This is even true in our linear rheological model. In addition, the stress components must also satisfy a compatibility equations, which directly result from the definitions of strain fields as derivatives of deformation fields. But these geometric conditions, again, have the same scaling structures for our linear rheological model. Thus, the scaling relation for the stress in Eq. (6), $\sigma \sim 1/X \sim 1/\sqrt{Y}$, remains unchanged even in our viscoelastic model. These observations will be more precisely addressed elsewhere [9].

Another important observation here is the scaling identification of a distance $X$ along the $x$ axis from the tip and the frequency $\omega$ via the speed $V$:

$$X \approx V/\omega;$$

(8)

small distances correspond to high frequencies while long distances to low frequencies — the farther away from the tip, the more time for relaxation. In addition, from Eq. (1), the
same magnitude of strain with the point specified by the above $X$ is developed at a point separated from the tip by a distance $Y \sim X^2/l$ along the $y$ axis; these points have the same time $(1/\omega \simeq V/X)$ to relax. Thus, a distance $Y$ from the tip sees the same frequency $\omega$ but a slower propagating speed $V_y \simeq VX/l$ ($Y \simeq V_y/\omega$). In other words, we can imagine that Fig. 2 correspond to a sequential propagation (from the center to outwards). We also note that, seen from a new coordinate $(x', y') = (x^2/l, y)$, the system returns to an isotropic system. For example, Eqs. (6) are changed into the conventional parabolic form: $\sigma \sim 1/\sqrt{X'} \sim 1/\sqrt{Y'}$ and $u \sim \sqrt{X'} \sim \sqrt{Y'}$ etc.

Our viscoelastic model in Eq. (7) has three regimes depending on frequencies: (I) at small frequencies ($\omega \tau \ll 1/\lambda$), it is like a soft solid with a small modulus $\mu(\omega) \simeq E_0$, (II) at intermediate frequencies ($1/\lambda \ll \omega \tau \ll 1$), it is like a liquid with viscosity $\mu(\omega) \simeq i\omega \eta$, and (III) at high frequencies ($1 \ll \omega \tau$), it is like a solid with a large modulus $\mu(\omega) \simeq E_\infty$. Due to the correspondence between a distance and a frequency in Eq. (8), a fracture can be thus spatially divided into three regions (Fig. 3):

(I) $\lambda V \tau \ll X$: soft solid (modulus $E_0$)
(II) $V \tau \ll X \ll \lambda V \tau$: liquid (viscosity $\eta$)
(III) $d \ll X \ll V \tau$: hard solid (modulus $E_\infty$)

Note that for a crack smaller than in this figure ($V \tau < L < \lambda V \tau$) only the regions I and II are developed while for an even smaller crack ($d < L < V \tau$) only the region I is developed; this figure corresponds to a fully developed crack ($L > \lambda V \tau$). Note also that the degree of anisotropy of the boundaries separating regions are subject to the ratio of $V \tau$ or $\lambda V \tau$ to the length $l$; the smaller the ratio, the more anisotropic.

We consider a small crack size $L$ with $L \ll \lambda V \tau \ll l$ (larger crack sizes ($L \gg l$) will be discussed elsewhere [9]); then, all the three regions (I)-(III) are fully developed. The soft-solid region (I) corresponds to low frequencies, and thus, to the static limit; in this region ($l \gg X \gg \lambda V \tau$), Eqs. (8) hold. In the liquid zone (II), the stress field scales as
\( \sigma \simeq \omega \eta u / Y \simeq \eta V lu / X^3 \) and at the same time it should scale as \( \sigma \sim Y^{-1/2} \sim 1 / X \) even in this liquid region as stated above. Thus, the strain should scale as \( u \sim X^2 \) and the product as \( \sigma u \simeq X \). The coefficients can be determined by the matching at \( X \simeq \lambda V \tau \) (the boundary between I and II):

\[
\sigma u \simeq \frac{GX}{\lambda V \tau} \quad \text{(liquid zone)}. \tag{10}
\]

In the hard-solid region of \( E_\infty \), we find \( \sigma \simeq 1 / \sqrt{Y} \) and \( u \simeq \sqrt{Y} \) as in Eqs. (6) but with \( \sigma u \simeq G_0 \) (via the same manner as in deriving Eqs. (6)). Here, \( G_0 \) is associated with the hard solid appearing near the tip. Matching this latter product \( \sigma u \) with that in Eq. (10) at \( X \simeq V \tau \), we find \( G \sim \lambda G_0 \). The overall separation energy \( G \) for a fully developed crack is enhanced from \( G_0 \) associated with local precesses near the tip. Note here that this expression for \( G \) is valid for \( d < V \tau \) in our continuum theory.

The crack shape resulting from this analysis is summarized as follows:

\[
u \sim \begin{cases} X & \text{for } \lambda V \tau < X \\ X^2 & \text{for } V \tau < X < \lambda V \tau \\ X & \text{for } X < V \tau \end{cases} \tag{11}
\]

It is just like a trumpet with a lenticular edge (Fig. 4), as has been suggested by the name of the model, but different from previously known shapes; it is not similar to the conventional parabolic form nor an isotropic parabolic trumpet predicted \([11]\) and observed \([12]\) in certain polymer systems (Fig. 4).

Fig. 4 – Lenticular and parabolic viscoelastic trumpets.

We complete our arguments by considering smaller fractures. When \( V \tau < L < \lambda V \tau \), only the hard-solid and liquid region are present; the soft solid has yet to develop. In this situation, the fracture energy is given by Eq. (10) at \( X = L: G_0 L / (V \tau) \); the toughness decreases with velocity. When \( L < V \tau \), only the hard-solid region is developed and the fracture energy is given by \( G_0 \). Thus, with increase in \( V \), the fracture energy starts from a larger plateau value \( \lambda G_0 \), and then decreases to reach a smaller plateau value \( G_0 \):

\[
G(V) \simeq \begin{cases} 
\lambda G_0 & \text{for } d / \tau < V < L / (\lambda \tau) \\
G_0 L / (V \tau) & \text{for } L / (\lambda \tau) < V < L / \tau \\
G_0 & \text{for } L / \tau < V
\end{cases} \tag{12}
\]
This behavior can be confirmed more precisely from a general formula:

\[
\frac{G(V)}{G_0} \simeq E_\infty \int \frac{d\omega}{\omega} \text{Im} \left[ \frac{1}{\mu(\omega)} \right]
\]  

(13)

which can be analytically calculated for the present model (just as in the previously known isotropic case \([13]\)). It should be emphasized here that this formula is unaltered even in our anisotropic materials, which is by no means trivial; Eq. (13) here can be shown in the following manner. We start from a relation:

\[
G(V) \simeq \int dx \int dy \dot{\sigma} e \simeq \int dXY \sigma_X \dot{e}_X.
\]  

(14)

In order to estimate \(\dot{e}_X\) we again use the dimensional identification in Eq. (8): \(\dot{e}_X \simeq \dot{e}_\omega \simeq \omega \sigma_\omega / \mu(\omega) \simeq \omega \sigma_X / \mu(\omega)\) and

\[
G(V) \simeq \int d\omega \frac{Y \sigma_X^2}{\omega \mu(\omega)} \simeq E_\infty G_0 \int d\omega \frac{1}{\omega \mu(\omega)}
\]  

(15)

Here, we have used a more precise form of Eq. (9): \(\sigma_X \simeq \sqrt{E_\infty G_0 / Y}\). Since the real and imaginary part of \(1/\mu(\omega)\) are even and odd functions, respectively, we arrive at Eq. (13).

**Conclusion.** – In this paper, we present a physical picture for fractures in nacre-type materials via scaling arguments. Viscoelastic effects for parallel fractures are taken into account via the simplest viscoelastic model for weakly cross-linked polymer. We expect that for slow crack-propagation speeds \((l \gg \lambda V \tau)\) a small crack \((L \ll l)\) takes a new trumpet shape different from previously reported shapes; in the opposite limit will be discussed in a separate paper \([9]\). The overall fracture energy \(G\) is found to decrease from a larger plateau value \(\lambda G_0\) to a smaller plateau value \(G_0\) with increase in velocity \(V\) where \(G_0\) is associated with local precesses near the tip.

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