HOMOTOPY TYPES OF GAUGE GROUPS OVER RIEMANN SURFACES

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Abstract. Let $G$ be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$. We study the homotopy types of gauge groups of principal $G$-bundles over Riemann surfaces. This can be applied to an explicit computation of the homotopy groups of the moduli spaces of stable vector bundles over Riemann surfaces.

1. Introduction

Let $G$ be a compact connected Lie group, and let $P$ be a principal $G$-bundle over a finite complex $X$. The gauge group of $P$ is defined to be the topological group of $G$-equivariant self-maps of $P$ which fix $X$. There may be infinitely many distinct principal $G$-bundles over $X$. For example, there are infinitely many bundles when $X$ is an orientable 4-manifold. Each bundle has a gauge group, so there may be potentially certainly infinitely many gauge groups. However, Crabb and Sutherland [7] showed that these gauge groups have only finitely many homotopy types. Then it has been intensely studied the precise number of homotopy types of gauge groups for specific $G$ and $X$. The study began with simply-connected Lie groups [6, 10, 12, 15, 16, 18, 20, 28, 29], and recently, non-simply-connected cases are also studied as in [11, 14, 17, 24].

In this paper, we study the homotopy types of gauge groups of principal $G$-bundles over a compact connected Riemann surface, where $\pi_1(G) \cong \mathbb{Z}$. This includes an important case, gauge groups of principal $U(n)$-bundles over a Riemann surface, whose topology was first studied by Atiyah and Bott [2]. To state the results, we need to define an intrinsic structure of $G$. Suppose $\pi_1(G) \cong \mathbb{Z}$. Then there are a compact connected simply-connected Lie group $H$ and a subgroup $C$ of the center of $S^1 \times H$ such that

(1.1) \[ G \cong (S^1 \times H)/C \]

Note that $H$ is uniquely determined by $G$, but $C$ is not. For example, if $G = S^1 \times H$, then $C$ can be any finite subgroup of $S^1 \times 1 \subseteq S^1 \times H$. We define

\[ s(G) = |p_2(C)| \]
where \( p_2 : S^1 \times H \to H \). By Theorem 1.4 below, we can see that \( s(G) \) is independent from the choice of \( C \).

**Example 1.1.** Since \( U(n) \) is the quotient of \( S^1 \times SU(n) \) by the diagonal central subgroup isomorphic to \( \mathbb{Z}/n \), we have \( s(U(n)) = n \).

Let \( X \) be a compact connected Riemann surface. Then there is a one-to-one correspondence between principal \( G \)-bundles over \( X \) and \( \pi_2(BG) \cong \mathbb{Z} \). Let \( G_k(X, G) \) denote the gauge group of a principal \( G \)-bundle over \( X \) corresponding to \( k \in \mathbb{Z} \). Now we state our results.

**Theorem 1.2.** Let \( G \) be a compact connected Lie group with \( \pi_1(G) \cong \mathbb{Z} \), and let \( X \) be a compact connected Riemann surface. If \( (k, s(G)) = (l, s(G)) \), then \( G_k(X, G) \) and \( G_l(X, G) \) are homotopy equivalent after localizing at any prime or zero.

We remark that the \( p \)-localization of a disconnected space will mean the disjoint union of the \( p \)-localization of path-connected components. For a prime \( p \), Theriault \[27\] gave a \( p \)-local homotopy decomposition of \( G_k(X, U(p)) \), which implies the converse implication of Theorem 1.2 holds for \( G = U(p) \). We will prove the converse implication of Theorem 1.2 holds for other Lie groups.

**Theorem 1.3.** Let \( G \) be a compact connected Lie group with \( \pi_1(G) \cong \mathbb{Z} \), and let \( X \) be a compact connected Riemann surface. If \( G \) is locally isomorphic to \( S^1 \times SU(n)^r \) or \( S^1 \times SU(4n - 2)^s \times Sp(2n - 1)^t \), then the following statements are equivalent:

1. \( (k, s(G)) = (l, s(G)) \);
2. \( G_k(X, G) \) and \( G_l(X, G) \) are homotopy equivalent after localizing at any prime or zero.

The homotopy type of a gauge group \( G_k(X, G) \) is closely related with a Samelson product in \( G \), as we will see in Section 2. In our context, the Samelson product of a generator of \( \pi_1(G) \cong \mathbb{Z} \) and the identity map of \( G \) is of particular importance. Then we will prove the following theorem, which is of independent interest.

**Theorem 1.4.** Let \( G \) be a compact connected Lie group with \( \pi_1(G) \cong \mathbb{Z} \), and let \( \epsilon \) denote a generator of \( \pi_1(G) \). Then the Samelson product \( \langle \epsilon, 1_G \rangle \) in \( G \) is of order \( s(G) \).

Now we consider an application. Gauge groups over a Riemann surface are closely related to the moduli spaces of stable vector bundles over a Riemann surface as follows. Let \( X \) be a Riemann surface of genus \( g \), and let \( M(n, k) \) denote the moduli space of stable vector bundles over \( X \) of rank \( n \) and degree \( k \). Daskalopoulos and Uhlenbeck \[8\] showed that there is an isomorphism

\[
\pi_i(M(n, k)) \cong \pi_{i-1}(G_k(X, U(n)))
\]

for \( 2 < i \leq 2(g-1)(n-1)-2 \) and \( (n, k) \neq (2, 2) \). There is a polystable Higgs bundle analog due to Bradlow, Garcia-Prada and Gothen \[5\]. Then we can compute the homotopy groups of these moduli spaces through the following homotopy decomposition.
Theorem 1.5. Let $G$ be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$, and let $X$ be a compact connected Riemann surface of genus $g$. If $s(G)$ divides $k$, then

$$\mathcal{G}_k(X, G) \cong G \times (\Omega G)^{2g} \times \Omega^2 G.$$ 

Moreover, the above homotopy equivalence also holds after localizing at $p$ whenever $p$ does not divide $s(G)$.

The paper is structured as follows: Section 2 recalls a connection of gauge groups to Samelson products, and then proves Theorems 1.2 and 1.5 by assuming Theorem 1.4 holds. Section 3 shows some general results on Samelson products in a Lie group, which will be used for a practical computation. Sections 4 and 5 compute the Samelson products in $G$ when $H$ is simple. Finally, Section 6 collects all results so far together to prove Theorems 1.4 and Theorem 1.3.

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2. Gauge groups and Samelson products

This section recalls a connection of gauge groups to Samelson products, and then Theorems 1.2 and 1.5 are proved by assuming Theorem 1.4 holds. First, we recall a connection of gauge groups to mapping spaces. Let $G$ be a topological group, and let $P$ be a principal $G$-bundle over a base $X$, which is classified by a map $\alpha : X \to BG$. Recall that the gauge group of $P$, denoted by $G(P)$, is a topological group of $G$-equivariant self-maps of $P$ which fix $X$. Gottlieb [9] proved that there is a natural homotopy equivalence

$$B\mathcal{G}(P) \cong \text{map}(X, BG; \alpha)$$

where $\text{map}(A, B; f)$ denotes the path component of the space of maps $\text{map}(A, B)$ containing a map $f : A \to B$. Then evaluating at the basepoint of $X$ yields a homotopy fibration

$$\text{map}_*(X, BG; \alpha) \to B\mathcal{G}(P) \to BG$$

where $\text{map}_*(X, BG; \alpha)$ is a subspace of $\text{map}(X, BG; \alpha)$ consisting of basepoint preserving maps. So the gauge group $\mathcal{G}(P)$ is homotopy equivalent to the homotopy fiber of the connecting map

$$\partial_\alpha : G \to \text{map}_*(X, BG; \alpha)$$

of the above homotopy fibration.

Next, we assume $X = S^n$ for $n \geq 1$ and describe the connecting map $\partial_\alpha$. Clearly, there is a homotopy equivalence $\text{map}_*(S^n, BG; \alpha) \cong \Omega_0^{n-1}G$, where $\Omega_0^{n-1}G$ denotes the path component of $\Omega^{n-1}G$ containing the constant map. Then by adjoining, the connecting map $\partial_\alpha$ corresponds to a map

$$d_\alpha : S^{n-1} \wedge G \to G$$
The original definition of Whitehead products in [30] and adjointness of Whitehead products
and Samelson products prove the following.

Lemma 2.1. The map $d_\alpha$ is the Samelson product $\langle \bar{\alpha}, 1_G \rangle$ in $G$, where $\bar{\alpha}: S^{n-1} \to G$ is the adjoint of $\alpha: S^n \to BG$.

The following lemma due to Theriault [26] shows how to identify the homotopy type of a
gauge group $G(P)$ from the order of a Samelson product $\langle \bar{\alpha}, 1_G \rangle$.

Lemma 2.2. Suppose that a map $f: X \to Y$ into an $H$-space $Y$ is of order $n < \infty$. Then $(n, k) = (n, l)$ implies $F_k(p) \simeq F_l(p)$ for any prime $p$, where $F_k$ denotes the homotopy fiber of
a map $k \circ f: X \to Y$.

Finally, we recall a homotopy decomposition of a gauge group. Theriault [25] showed a
homotopy decomposition of a gauge group over principal $U(n)$-bundle over a Riemann surface.
We can easily see that his proof works in verbatim for any compact connected Lie group $G$ with $\pi_1(G) \cong \mathbb{Z}$. Then we get:

Proposition 2.3. Let $G$ be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$, and let $X$ be a compact connected Riemann surface of genus $g$. Then there is a homotopy equivalence
\[ G_k(X, G) \simeq (\Omega G)^{2g} \times G_k(S^2, G). \]

Now we prove Theorems 1.2 and 1.5 by assuming Theorem 1.4 holds.

Proof of Theorem 1.2. Combine Lemmas 2.1, 2.2, Proposition 2.3 and Theorem 1.4.

Proof of Theorem 1.5. By Lemma 2.1 and Theorem 1.4, if $k$ is divisible by $s(G)$, then $G_k(S^2, G)$ is homotopy equivalent to the homotopy fiber of the constant map $G \to \Omega_0G$. So since $\pi_2(G) = 0$, $G_k(S^2, G) \simeq G \times \Omega^2G$. Thus by Proposition 2.3, the proof is done.

3. Samelson Products in Lie Groups

This section shows some criteria for computing Samelson products in a Lie group. For the
rest of the paper, we will use the following notation.

- Let $G$ be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$.
- Let $\epsilon_G$ denote a generator of $\pi_1(G) \cong \mathbb{Z}$.
- Let $H$ and $C$ be as in the decomposition (1.1).
- Let $j_H: \Sigma H \to BH$ denote the natural map.
- Let $p_C: S^1 \times H \to G$ denote the projection.
- Let $p_1: S^1 \times H \to S^1$ and $p_2: S^1 \times H \to H$ denote projections.
- Let $K = H/p_2(C)$.
- Let $q_G: G \to K$ and $\bar{q}_K: H \to K$ denote projections.
We will abbreviate $\epsilon_G, j_H, p_G, q_G, q_K$ to $\epsilon, j, p, q, q$, respectively, if $G, H, K$ are clear from the context. First, we show two properties of the group $C$.

**Lemma 3.1.** The abelian group $p_2(C)$ is cyclic.

*Proof.* There is a fibration
\begin{equation}
S^1 \to G \xrightarrow{2} K
\end{equation}
and so by the homotopy exact sequence, we can see that $\pi_1(K) \cong p_2(C)$ is a quotient of $\pi_1(G) \cong \mathbb{Z}$. Then $p_2(C)$ is a cyclic group, as stated. $\square$

**Lemma 3.2.** We may choose a group $C$ such that $|p_1(C)| = s(G)$.

*Proof.* Note that $p_2(C)$ is a cyclic group by Lemma 3.1. We prove an inequality $|p_1(C)| \geq s(G)$ always holds. If $|p_1(C)| < s(G)$, then $C_1 = |p_1(C)|C$ is a non-trivial subgroup of the center of $1 \times H \subset S^1 \times H$. In particular, there is a covering

$$C/C_1 \to (S^1 \times H)/C_1 \to G.$$ 

Then $\pi_1(G) \cong \mathbb{Z}$ includes a non-trivial finite abelian group $C_1$, which is a contradiction. Thus $|p_1(C)| \geq s(G)$.

Suppose that $|p_1(C)| > s(G)$. Then $C_2 = s(G)C$ is a finite subgroup of $S^1 \times 1 \subset S^1 \times H$. Then $(S^1 \times H)/C_2 \cong S^1 \times H$, implying

$$G \cong (S^1 \times H)/C \cong ((S^1 \times H)/C_2)/(C/C_2) \cong (S^1 \times H)/(C/C_2).$$

By definition, $|p_1(C/C_2)| = |p_2(C/C_1)| = |p_2(C)| = s(G)$, and thus the proof is finished. $\square$

By Lemma 3.1, $\pi_1(K) \cong p_2(C)$ is a cyclic group of order $s(G)$. For the rest of this section, we will also use the following notation.

- Let $\bar{\epsilon}_K$ denote a generator of $\pi_1(K)$.

We will abbreviate it by $\bar{\epsilon}$ if $K$ is clear from the context.

Next, we show an upper bound and a lower bound for the order of $\langle \epsilon, 1_G \rangle$.

**Lemma 3.3.** The order of $\langle \epsilon, 1_G \rangle$, hence $\langle \epsilon, p \rangle$, divides $s(G)$.

*Proof.* The proof of Lemma 3.1 implies $q \circ \epsilon = \bar{\epsilon}$. Then since $q$ is a homomorphism, we get

$$q_*(s(G)\langle \epsilon, 1_G \rangle) = s(G)\langle q \circ \epsilon, q \rangle = s(G)\bar{\epsilon}, q = 0.$$

So since there is a fibration (3.1), $s(G)\langle \epsilon, 1_G \rangle$ lifts to a map $S^1 \wedge G \to S^1$. Since $S^1 \wedge G$ is simply-connected, this lift is trivial, and thus $s(G)\langle \epsilon, 1_G \rangle$ itself is trivial, completing the proof. $\square$

**Lemma 3.4.** The order of $\langle \bar{\epsilon}, q \rangle$ divides the order of $\langle \epsilon, p \rangle$.
Proof. Let \( i: H \to S^1 \times H \) denote the inclusion. By definition, \( q \circ p \circ i = \bar{q} \), and the proof of Lemma 3.2 implies that \( q \circ \epsilon = \bar{\epsilon} \). Then
\[
(1 \wedge i) \circ q \ast (\langle \epsilon, p \rangle) = q \ast (\langle \epsilon, p \circ i \rangle) = (q \circ \epsilon, q \circ p \circ i) = (\bar{\epsilon}, \bar{q})
\]
and so the proof is done. \( \square \)

Finally, we give a cohomological criterion for the Samelson product \( \langle \bar{\epsilon}, \bar{q} \rangle \) being non-trivial. For an algebra \( A \), let \( QA \) denote the module of indecomposables.

**Lemma 3.5.** Suppose there are \( x, y, z \in QH^\ast(BK; \mathbb{Z}/p) \) and a Steenrod operation \( \theta \) satisfying the following conditions:

1. \( |y| = 2 \) and \( QH^n(BK; \mathbb{Z}/p) = \langle z \rangle \) for \( n > 2 \);
2. \( \theta(x) \) is decomposable and includes the term \( y \otimes z \);
3. \( (\bar{q} \circ j)^* (z) \) is non-trivial and not included in any element of \( \theta(H^\ast(\Sigma H; \mathbb{Z}/p)) \).

Then the Samelson product \( \langle \bar{\epsilon}, \bar{q} \rangle \) is non-trivial.

Proof. Suppose that \( \langle \bar{\epsilon}, \bar{q} \rangle \) is trivial. Let \( \hat{\epsilon}: S^2 \to BK \) and \( \hat{\mu}: \Sigma H \to BK \) denote the adjoint of \( \bar{\epsilon} \) and \( \bar{q} \), respectively. Then by adjointness of Samelson products and Whitehead products, the Whitehead product \( [\hat{\epsilon}, \hat{\mu}] \) is trivial, so that there is a homotopy commutative diagram

\[
\begin{array}{ccc}
S^2 \vee \Sigma H & \xrightarrow{\hat{\epsilon} \circ \hat{\mu}} & BK \\
\downarrow & & \downarrow \\
S^2 \times \Sigma H & \xrightarrow{\mu} & BK.
\end{array}
\]

By the first condition, \( \hat{\epsilon}^*(y) = u \), where \( u \) is a generator of \( H^2(S^2; \mathbb{Z}/p) \cong \mathbb{Z}/p \). Then by the first and the second conditions, \( \mu^*(\theta(x)) \) includes the term \( u \otimes \hat{\mu}^*(z) \). Since \( \hat{q} = \bar{q} \circ j \), the third condition implies \( u \otimes \hat{q}^*(z) \neq 0 \). On the other hand, by the third condition, \( \theta(\mu^*(x)) \) cannot include the term \( u \otimes \hat{q}^*(z) \). Thus since \( \mu^*(\theta(x)) = \theta(\mu^*(x)) \), we obtain a contradiction. Therefore \( \langle \bar{\epsilon}, \bar{q} \rangle \) is non-trivial, completing the proof. \( \square \)

Recall that compact simply-connected simple Lie groups with non-trivial center are

\[ SU(n), \ Sp(n), \ Spin(n) \ (n \geq 7), \ E_6, \ E_7. \]

Then in the following two sections, we will compute the Samelson product \( \langle \epsilon, p \rangle \) for \( H \) being one of the above Lie groups.

### 4. Classical case

This section determines the order of the Samelson product \( \langle \epsilon, p \rangle \) for \( H = SU(n), Sp(n), Spin(n) \).
4.1. The case $H = SU(n)$. First we consider the case $H = SU(n)$.

**Proposition 4.1.** If $H = SU(n)$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.

**Proof.** By Lemma 3.3, it suffices to show that the order of $\langle \epsilon, p \rangle$ is a non-zero multiple of $s(G)$. The center of $SU(n)$ is isomorphic to $\mathbb{Z}/n$. Then since $U(n) = S^1 \times_{\mathbb{Z}/n} SU(n)$, it follows from Lemma 3.2 that there is a homomorphism $\rho: G \to U(n)$ which is a $\frac{n}{s(G)}$-sheeted covering. Let $\alpha_{2i-1}$ denote a generator of $\pi_{2i-1}(U(n)) \cong \mathbb{Z}$ for $i = 1, 2, \ldots, n$. Then

$$\rho_* (\epsilon) = \frac{n}{s(G)} \alpha_1.$$

On the other hand, it is shown in [4] that the order of $\langle \alpha_1, \alpha_{2n-1} \rangle$ is a non-zero multiple of $n$. Since $\rho_*: \pi_{2n-1}(G) \to \pi_{2n-1}(U(n))$ is an isomorphism, there is $\tilde{\alpha} \in \pi_{2n-1}(G)$ such that $\rho_*(\tilde{\alpha}) = \alpha_{2n-1}$. Then since

$$\rho_* (\langle \epsilon, \tilde{\alpha} \rangle) = \langle \rho_*(\epsilon), \rho_*(\tilde{\alpha}) \rangle = \langle \frac{n}{s(G)} \alpha_1, \alpha_{2n-1} \rangle = \frac{n}{s(G)} \langle \alpha_1, \alpha_{2n-1} \rangle,$$

the order of $\rho_* (\langle \epsilon, \tilde{\alpha} \rangle)$ is a non-zero multiple of $s(G)$.

Thus since the map $\rho_*: \pi_{2n}(G) \to \pi_{2n}(U(n))$ is an isomorphism, the order of $\langle \epsilon, \tilde{\alpha} \rangle$ is a non-zero multiple of $s(G)$ too. Since $p_*: \pi_{2n-1}(S^1 \times SU(n)) \to \pi_{2n-1}(G)$ is an isomorphism, there is $\beta \in \pi_{2n-1}(S^1 \times SU(n))$ such that $p \circ \beta = \tilde{\alpha}$. Thus since $(1 \wedge \beta)^* (\langle \epsilon, p \rangle) = \langle \epsilon, \tilde{\alpha} \rangle$, the order of $\langle \epsilon, p \rangle$ is a non-zero multiple of $s(G)$, completing the proof. \(\square\)

4.2. The case $H = Sp(n)$. Next, we consider the case $H = Sp(n)$. Recall that the center of $Sp(n)$ is isomorphic to $\mathbb{Z}/2$, and the quotient of $Sp(n)$ by its center is denoted by $PSp(n)$. We apply Lemma 3.5 to the case $H = Sp(n)$. To this end, we compute the mod 2 cohomology of $BPSp(2n)$ in low dimensions.

**Lemma 4.2.** Let $\Delta = \{ \pm(1, \ldots, 1) \in Sp(2)^n \}$. Then for $* \leq 7$

$$H^*(B(Sp(2)^n/\Delta); \mathbb{Z}/2) = \mathbb{Z}/2[x_2, x_3, x_5] \otimes_{\mathbb{Z}/2} \bigotimes_{k=1}^{n} \mathbb{Z}/2[x_{4k}], \quad Sq^2 x_{4k} = x_2 x_{4k}$$

where $|x_i| = i$ and $|x_{4k}| = 4$.

**Proof.** Consider the Serre spectral sequence for a homotopy fibration $\mathbb{R}P^\infty \to BSp(2)^n \to B(Sp(2)^n/\Delta)$. Since $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[w]$ with $|w| = 1$, $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \Delta(w, Sq^1 w, Sq^2 Sq^1 w)$.

for $* \leq 7$, where $\Delta(a_1, \ldots, a_k)$ denotes the simple system of generators in $a_1, \ldots, a_k$. Clearly, $\tau(w) = x_2$ for a generator $x_2$ of $H^2(B(Sp(2)^n/\Delta); \mathbb{Z}/2) \cong \mathbb{Z}/2$, where $\tau$ denotes the transgression. Then we get $H^*(B(Sp(2)^n/\Delta); \mathbb{Z}/2)$ for $* \leq 7$ as stated. It remains to show
By considering a homotopy fibration implies that
\[ 4.2 \]
where \( w_i \) is the \( i \)-th Stiefel-Whitney class. Then since \( PSp(2) \cong SO(5), \)
\[ H^*(BPSp(2); \mathbb{Z}/2) = \mathbb{Z}/2[y_2, y_3, y_4, y_5], \quad Sq^2 y_4 = y_2 y_4 \]
where \( |y_i| = i \). Let \( q_k : Sp(2)^n \to Sp(2) \) denote the \( k \)-th projection for \( k = 1, 2, \ldots, n \). Then \( q_k^*(y_2) = x_2 \) and \( q_k^*(y_4) = x_{4,k} \). Thus we obtain \( Sq^2 x_{4,k} = x_2 x_{4,k} \), completing the proof.

\[ \triangleq \]

**Proposition 4.3.** For \( * \leq 7, \)
\[ H^*(BPSp(n); \mathbb{Z}/2) = \mathbb{Z}/2[x_2, x_3, x_4, x_5], \quad Sq^2 x_4 = x_4 x_2, \quad |x_i| = i. \]

**Proof.** We can compute the mod 2 cohomology of \( BPSp(2n) \) in the same way as in the proof of Lemma 4.2 by considering a homotopy fibration \( \mathbb{R}P^{\infty} \to BSp(2n) \to BPSp(2n) \). Then it remains to show \( Sq^2 x_4 = x_4 x_2 \). Let \( \Delta \) be as in Lemma 4.2. Then there is an inclusion \( i : Sp(2)^n/\Delta \to PSp(2n) \). Clearly, \( i^*(x_2) = x_2 \) and \( i^*(x_4) = x_{4,1} + \cdots + x_{4,n} \). Then we obtain \( Sq^2 x_4 = x_4 x_2 \) by Lemma 4.2.

Now we prove:

**Proposition 4.4.** If \( H = Sp(n) \), then \( \langle \epsilon, p \rangle \) is of order \( s(G) \).

**Proof.** Since the center of \( Sp(n) \) is isomorphic to \( \mathbb{Z}/2 \), we only consider \( G = S^1 \times_{\mathbb{Z}/2} Sp(n) \). In this case, \( s(G) = 2 \), so by Lemmas 3.3, it suffices to show \( \langle \epsilon, p \rangle \) is non-trivial. First, we consider the case \( G = S^1 \times_{\mathbb{Z}/2} Sp(2n - 1) \). The natural inclusion \( Sp(2n - 1) \to SU(4n - 2) \) sends the center of \( Sp(2n - 1) \) injectively into the center of \( SU(4n - 2) \). Then we get a homomorphism \( G \to S^1 \times_{\mathbb{Z}/2} SU(4n - 2) \) which is an isomorphism in \( \pi_1 \). It is well known that the induced map \( \pi_{8n-5}(Sp(2n - 1)) \to \pi_{8n-5}(SU(4n - 2)) \) is an isomorphism, hence so is \( \pi_{8n-5}(G) \to \pi_{8n-5}(S^1 \times_{\mathbb{Z}/2} SU(4n - 2)) \). Then the proof of Proposition 4.1 implies that the Samelson product \( \langle \epsilon, p \rangle \) is non-trivial.

Next, we consider \( G = S^1 \times_{\mathbb{Z}/2} Sp(2n) \). We apply Lemma 3.5 to \( K = PSp(2n) \) by setting \( x = z = x_4, y = x_2 \) and \( \theta = Sq^2 \). By Proposition 4.3, the first and the second conditions of Lemma 3.5 are satisfied. The proof of Proposition 4.3 implies \( q^*(x_4) \) is non-trivial, where \( H^4(BSp(2n); \mathbb{Z}/2) \cong QH^4(BSp(2n); \mathbb{Z}/2) \cong \mathbb{Z}/2 \). Since the map
\[ j^* : QH^4(BSp(2n); \mathbb{Z}/2) \to \Sigma QH^3(Sp(2n); \mathbb{Z}/2) \]
is an isomorphism, we have \( (q \circ j)^*(x_4) \neq 0 \). Moreover, for degree reasons, \( (q \circ j)^*(x_4) \) is not included in any element of \( \theta(H^4(\Sigma Sp(2n); \mathbb{Z}/2)) \). Then the third condition of Lemma 3.5 is also satisfied. Thus \( \langle \epsilon, q \rangle \) is non-trivial, and so by Lemma 3.4, \( \langle \epsilon, p \rangle \) is non-trivial too.
4.3. The case $H = \text{Spin}(n)$. Finally, we consider the case $H = \text{Spin}(n)$. We show some properties of the mod 2 cohomology of $B\text{Spin}(n)$ that we are going to use. Recall that the mod 2 cohomology of $B\text{SO}(n)$ is given as in (4.1).

**Lemma 4.5.** (1) The mod 2 cohomology of $B\text{Spin}(n)$ is given by

$$H^* (B\text{Spin}(n); \mathbb{Z}/2) = \mathbb{Z}/2 [u_2, u_3, \ldots, u_n, z]/(u_2, Sq^2, Sq^3, \ldots, Sq^k u_2 | k \geq 0)$$

where $\bar{q}_{\text{SO}(n)}(w_j) = u_j$, $|z| = 2^h$ for some $h > 0$ and $Sq^i u_j$ is computed by replacing $w_j$ with $u_j$ in the formula (4.1).

(2) For $2 \leq i \leq n$ with $i \neq 2^k + 1$, $j_{\text{Spin}(n)}^* (u_i) \neq 0$.

**Proof.** (1) is a result of Quillen [23]. We prove the statement (2). It is well known that $(j')^* (w_i) \neq 0$ for $i = 2, 3, \ldots, n$, where $j' : \Sigma \text{SO}(n) \to B\text{SO}(n)$ is the natural map. On the other hand, it is shown in [13] that $(\Sigma \bar{q}_{\text{SO}(n)})^* \circ (j')^*(w_i) \neq 0$. Then for $2 \leq i \leq n$ with $i \neq 2^k + 1$,

$$0 \neq (\Sigma \bar{q}_{\text{SO}(n)})^* \circ (j')^*(w_i) = j^* \circ \bar{q}_{\text{SO}(n)}(w_i) = j^* (u_i).$$

Thus the statement (2) is proved. \qed

The following lemma is easily deduced from the formula (4.1).

**Lemma 4.6.** In $H^* (B\text{SO}(n); \mathbb{Z}/2)$, we have:

(1) If $n \equiv 0, 1 \mod 4$, then $Sq^2 w_i$ for $i = n - 3, n - 1$ are decomposable and $Sq^2 w_{n-1}$ includes the term $w_2 w_{n-1}$;

(2) if $n \equiv 2 \mod 8$, then $Sq^5 w_i$ for $i = n - 4, n - 9$ are decomposable and $Sq^5 w_{n-4}$ includes the term $w_2 w_{n-1}$;

(3) if $n \equiv 6 \mod 8$, then $Sq^5 w_i$ for $i = n - 2, n - 4$ are decomposable and $Sq^5 w_{n-2}$ includes the term $w_2 w_{n-1}$;

(4) if $n \equiv 3 \mod 4$, then $Sq^2 w_i$ for $i = n - 2, n$ are decomposable and $Sq^2 w_n$ includes the term $w_2 w_n$.

Let $C_n$ denote the center of $\text{Spin}(n)$. Then we have:

(1) $C_{2n+1} \cong \mathbb{Z}/2$ and $\text{Spin}(2n + 1)/C_{2n+1} \cong \text{SO}(2n + 1)$.

(2) $C_{4n+2} \cong \mathbb{Z}/4$ and $\text{Spin}(4n + 2)/(\mathbb{Z}/2) \cong \text{SO}(4n + 2)$.

(3) $C_{4n} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, $\text{Spin}(4n)/\mathbb{Z}/2 \cong \text{SO}(4n)$ and $\text{Spin}(4n)/(1 \times \mathbb{Z}/2) \cong Ss(4n)$.

**Proposition 4.7.** If $H = \text{Spin}(n)$ and $K = \text{SO}(n)$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.

**Proof.** We only give a proof for $n$ odd because the case $n$ even is quite similarly proved. We apply Lemma 3.5 by setting $x = z = w_{n-1}, y = w_2$ and $\theta = Sq^2$. By Lemma 4.6, the first and the second conditions of Lemma 3.5 are satisfied. By Lemmas 4.5 and 4.6, $(\bar{q} \circ j)^* (w_{n-1})$ is non-trivial and not included in any element of $Sq^2 (H^* (\Sigma \text{Spin}(n); \mathbb{Z}/2))$. Then the third
condition of Lemma 3.5 is also satisfied, so $\langle \bar{\epsilon}, \bar{q} \rangle \neq 0$. Thus since $s(G) = 2$, Lemmas 3.3 and 3.4 complete the proof. \hfill \Box

Let $PO(n) = Spin(n)/C_n$. Then we have:

**Corollary 4.8.** If $H = Spin(4n + 2)$ and $K = PO(4n + 2)$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.  

**Proof.** Let $\bar{\rho}: SO(4n + 2) \to PO(4n + 2)$ denote the projection. Then $\bar{\rho}_* (\epsilon_{SO(4n+2)}) = 2\bar{\epsilon}_{PO(4n+2)}$. Since $S^1 \wedge Spin(4n + 2)$ is simply-connected, the map

$$\bar{\rho}_*: [S^1 \wedge Spin(4n + 2), SO(4n + 2)] \to [S^1 \wedge Spin(4n + 2), PO(4n + 2)]$$

is an isomorphism. By definition, $\bar{q}_{PO(4n+2)} = \bar{\rho} \circ \bar{q}_{SO(4n+2)}$. So by Proposition 4.7,

$$2\langle \bar{\epsilon}_{PO(4n+2)}, \bar{q}_{PO(4n+2)} \rangle = \bar{\rho}_* (\langle \epsilon_{SO(4n+2)}, \bar{q}_{SO(4n+2)} \rangle) \neq 0.$$  

Then by Lemma 3.3, the order of $\langle \bar{\epsilon}_{PO(4n+2)}, \bar{q}_{PO(4n+2)} \rangle$ is a non-zero multiple of $s(G) = 4$. Thus the proof is complete by Lemmas 3.3 and 3.4. \hfill \Box

Let $\Delta$ denote the diagonal subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$.

**Proposition 4.9.** If $H = Spin(4n)$ and $p_2(C) = 1 \times \mathbb{Z}/2, \Delta$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.  

**Proof.** By triality of $Spin(8)$, the case $H = Spin(8)$ is proved by Proposition 4.7. Then we assume $n > 2$. The mod 2 cohomology of $PO(4n)$ was determined by Baum and Browder [3] such that

$$H^*(PO(4n); \mathbb{Z}/2) = \mathbb{Z}/2[v]/(v^{2^r}) \otimes \Delta(u_1, \ldots, u_{2^{r-1}}, \ldots, u_{n-1}), \quad \bar{\rho}_*(u_i) = w_i$$

where $4n = 2^r(2m + 1)$, $|v| = 1$ and $|u_i| = i$. The elements $v$ and $u_1$ correspond respectively to generators of subgroups $1 \times \mathbb{Z}/2$ and $\mathbb{Z}/2 \times 1$ of $C_{4n} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. The Hopf algebra structure of $H^*(PO(4n); \mathbb{Z}/2)$ was also determined such that

$$\tilde{\phi}(v) = 0 \quad \text{and} \quad \tilde{\phi}(u_i) = \sum_{j=1}^{i-1} \binom{i}{j} u_j \otimes v^{i-j}$$

where $\tilde{\phi}$ is the reduced diagonal map. Let $\gamma: PO(4n)^2 \to PO(4n)$ denote the commutator map. Since $\bar{\epsilon}(v) \neq 0$, it suffices to show $\gamma^*(x)$ includes the term $v \otimes y$ such that $\rho^*(y) \neq 0$, where $\rho: Spin(4n) \to PO(4n)$ denotes the projection. Let $\mu: PO(4n)^2 \to PO(4n)$ and $\Delta: PO(4n) \to PO(4n)^2$ denote the multiplication and the diagonal map, respectively. Let $\iota: PO(4n) \to PO(4n)$ be a map given by $\iota(x) = x^{-1}$, and let $T: PO(4n)^2 \to PO(4n)^2$ be the switching map. Then

$$\gamma = \mu \circ (\mu \times \mu) \circ (1 \times 1 \times \iota \times \iota) \circ (1 \times T \times 1) \circ (\Delta \times \Delta).$$
Let \( I_k = \tilde{H}^*(PO(n)^\infty; \mathbb{Z}/2) \). Now we compute \( \gamma^*(u_i) \):

\[
\begin{align*}
\mu^* &\mapsto u_i \otimes 1 + 1 \otimes u_i + i u_{i-1} \otimes v \mod I_2^3 \\
(\mu \times \mu)^* &\mapsto i(u_{i-1} \otimes v \otimes 1 \otimes 1 + \otimes u_{i-1} \otimes v + u_{i-1} \otimes 1 \otimes 1 \otimes v + 1 \otimes u_{i-1} \otimes v \otimes 1) \\
&\mod I_1 \otimes I_1 \otimes 1 + 1 \otimes I_1 \otimes 1 \otimes I_1 + I_4^3 \\
(1 \times 1 \times 1 \times 1)^* &\mapsto i(u_{i-1} \otimes v \otimes 1 \otimes 1 + \otimes u_{i-1} \otimes v - u_{i-1} \otimes 1 \otimes 1 \otimes v - 1 \otimes u_{i-1} \otimes v \otimes 1) \\
&\mod I_1 \otimes I_1 \otimes 1 + 1 \otimes I_1 \otimes 1 \otimes I_1 + I_4^3 \\
(1 \times T \times 1)^* &\mapsto i(u_{i-1} \otimes 1 \otimes v \otimes 1 + \otimes u_{i-1} \otimes 1 \otimes v - u_{i-1} \otimes 1 \otimes 1 \otimes v - 1 \otimes v \otimes u_{i-1} \otimes 1) \\
&\mod I_1 \otimes I_1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes I_1 \otimes I_1 + I_4^3 \\
(\Delta \times \Delta)^* &\mapsto i(u_{i-1} \otimes y - y \otimes u_{i-1}) \mod I_1 \otimes 1 + 1 \otimes I_1 + I_2^3.
\end{align*}
\]

Then for \( n \) odd, \( \gamma^*(u_7) \) includes the term \( v \otimes u_6 \), where \( \rho^*(u_6) \neq 0 \) by Lemma 4.5, and for \( n \) even, \( \gamma^*(u_{11}) \) includes the term \( v \otimes u_{10} \), where \( \rho^*(u_{10}) \neq 0 \) by Lemma 4.5. Thus the Samelson product \( \langle \epsilon, q \rangle \) is non-trivial, completing the proof by Lemmas 3.3 and 3.4 because \( s(G) = 2 \).

\[\square\]

5. Exceptional case

First, we consider the case \( H = E_6 \).

**Proposition 5.1.** If \( H = E_6 \), then \( \langle \epsilon, p \rangle \) is of order \( s(G) \).

**Proof.** Since the center of \( E_6 \) is isomorphic to \( \mathbb{Z}/3 \), we only need to consider the case \( G = S^1 \times \mathbb{Z}/3 \times E_6 \). The mod 3 cohomology of \( Ad(E_6) \), which is the quotient of \( E_6 \) by its center, was determined by Kono [19] such that

\[
H^*(Ad(E_6); \mathbb{Z}/3) = \mathbb{Z}/3[x_2, x_8]/(x_2^6, x_8^3) \otimes \Lambda(x_1, x_3, x_7, x_9, x_{11}, x_{16})
\]

such that

\[
\tilde{\phi}(x_9) = x_8 \otimes x_1 + x_2 \otimes x_7 - x_3^3 \otimes x_3 + x_2^4 \otimes x_1 \text{ and } \tilde{q}^*(x_3) \neq 0
\]

where \( |x_i| = i \). Then by the same computation as in the proof of Proposition 4.9, we can see that \( \langle \epsilon, q \rangle \) is non-trivial. Thus by Lemmas 3.3 and 3.4, \( \langle \epsilon, 1_G \rangle \) is of order \( s(G) = 3 \). \[\square\]

Next, we consider the case \( H = E_7 \). Because the center of \( E_7 \) is isomorphic to \( \mathbb{Z}/2 \), we only need to consider the case \( G = S^1 \times \mathbb{Z}/2 \times E_7 \). The Hopf algebra structure of \( H^*(Ad(E_7); \mathbb{Z}/2) \) was determined by Ishitoya, Kono and Toda [13], from which we can see that the same computation as \( Ad(E_6) \) does not apply to \( Ad(E_7) \). So we apply Lemma 3.5. Kono and Mimura [21] showed that the mod 2 cohomology of \( BAd(E_7) \) is generated by elements \( x_i \) for \( i \in \{2, 3, 6, 7, 10, 11, 18, 19, 34, 35, 64, 66, 67, 96, 112\} \), where \( |x_i| = i \). We determine \( Sq^2 x_6 \).
Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Elements of the spin group $\text{Spin}(n)$ are expressed by using $e_1, e_2, \ldots, e_n$. See [1, Chapter 3]. Recall from [1, Proposition 4.2] that there are two representations

$$\Delta_{2n}^+, \Delta_{2n}^- : \text{Spin}(2n) \to SU(2^{n-1})$$

such that $\Delta_{2n}^+$ has weights $\frac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_n)$ with even numbers of minus signs and $\Delta_{2n}^-$ has weights $\frac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_n)$ with odd numbers of minus signs.

**Proposition 5.2.** There is a natural isomorphism

$$\text{Spin}(4) \cong \text{Ker} \Delta_4^+ \times \text{Ker} \Delta_4^-.$$  

**Proof.** There is a product decomposition $\text{Spin}(4) \cong SU(2) \times SU(2)$ such that $\Delta_4^+ : \text{Spin}(4) \to SU(2)$ are identified with projections $SU(2) \times SU(2) \to SU(2)$. Then the statement is proved. \(\square\)

As in [1, Theorem 6.1], there is a homomorphism

$$\theta : \text{Spin}(16) \to E_8$$

whose kernel is \{1, $e_1e_2 \cdots e_{16}$\}. Let $\mu : \text{Spin}(4) \times \text{Spin}(12) \to \text{Spin}(16)$ denote the homomorphism covering the inclusion

$$SO(4) \times SO(12) \to SO(16), \quad (A, B) \mapsto \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$  

Define $\bar{\mu} = \theta \circ \mu : \text{Spin}(4) \times \text{Spin}(12) \to E_8$. Then

$$\text{Ker} \bar{\mu} = \{(1, 1), (-1, -1), (e_1e_2e_3e_4, e_5e_6 \cdots e_{16}), (-e_1e_2e_3e_4, -e_5e_6 \cdots e_{16})\}.$$  

Recall from [1, Chapter 8] that $E_7$ is defined as the centralizer of $\bar{\mu}(\text{Ker} \Delta_4^+ \times 1)$ in $E_8$. Then by Lemma 5.2, there is a homomorphism

$$\hat{\mu} : \text{Ker} \Delta_4^+ \times \text{Spin}(12) \to E_7.$$  

Since $-e_1e_2e_3e_4 \in \text{Ker} \Delta_4^+$, $\bar{\mu}(-e_1e_2e_3e_4, 1)$ commutes with every element of $E_7$ in $E_8$. Moreover, $\bar{\mu}(-e_1e_2e_3e_4, 1) = \bar{\mu}(e_1e_2e_3e_4, -1) = \bar{\mu}(e_1e_2e_3e_4, -1)$, which belongs to $E_7$ and is not the unit of $E_7$. Then we obtain:

**Proposition 5.3.** The center of $E_7$ is \{1, $\hat{\mu}(e_1e_2e_3e_4, -1)$\}.

Let $L = (\text{Ker} \Delta_4^- \times \text{Spin}(12))/\{(1, 1), (e_1e_2e_3e_4, -1)\}$. Then by Proposition 5.3, there is a map

$$\rho : L \to \text{Ad}(E_7)$$

which is an isomorphism in the second mod 2 cohomology.

**Lemma 5.4.** In $H^*(B\text{Ad}(E_7); \mathbb{Z}/2)$, $Sq^2x_6$ is decomposable and includes the term $x_2x_6$. 
Proof. By [21, 22], \((\bar{\mu} \circ (1 \times \bar{q}))^*(x_6)\) includes the term \(1 \otimes u_6\), where \(u_i\) is as in Lemma 4.5. Note that the composition
\[ Spin(12) \to Ker \Delta_4^{-} \times Spin(12) \to L \overset{\underline{2}}{\longrightarrow} SO(12) \]
is the natural projection, where \(q_2\) is the second projection. Then by degree reasons,
\[ \rho^*(x_6) + a \rho^*(x_2)^3 + b \rho^*(x_3)^2 = q_2^*(w_6) \]
for some \(a, b \in \mathbb{Z}/2\). On the other hand, \(q_2^* : H^2(BO(12); \mathbb{Z}/2) \to H^2(BL; \mathbb{Z}/2)\) is an isomorphism, implying \(\rho^*(x_2) = q_2^*(w_2)\). Then since \(Sq^2 w_6 = w_2 w_6\) by (4.1) and \(Sq^2 x_6\) is decomposable by degree reasons, \(Sq^2 x_6\) is decomposable and includes the term \(x_2 x_6\), as stated.

We are ready to prove:

**Proposition 5.5.** If \(H = E_7\), then \(\langle \epsilon, p \rangle\) is of order \(s(G)\).

**Proof.** As mentioned above, we only need to consider \(G = S^1 \times_{\mathbb{Z}/2} E_7\). We apply Lemma 3.5 by setting \(x = z = x_6, y = x_2\) and \(\theta = Sq^2\). By Lemma 5.4, the first and the second conditions of Lemma 3.5 are satisfied. As in [22], \(\bar{q}^*(x_6)\) is a generator of \(H^6(BE_7; \mathbb{Z}/2)\) such that \((\bar{q} \circ j)^*(x_6)\) is non-trivial. Then by degree reasons, the third condition of Lemma 3.5 is also satisfied, implying \(\langle \bar{\epsilon}, \bar{q} \rangle\) is non-trivial. Since \(s(G) = 2\), the proof is complete by Lemmas 3.3 and 3.4. \(\square\)

### 6. Proofs of Theorems 1.3 and 1.4

This section proves Theorems 1.3 and 1.4. First, we prove Theorem 1.4.

**Proof of Theorem 1.4.** Suppose \(H \cong H_1 \times \cdots \times H_k\), where each \(H_i\) is a simple Lie group. Let \(r_i : S^1 \times H \to S^1 \times H_i\) be the projection, and let \(G_i = (S^1 \times H_i)/(r_i(C))\) for \(i = 1, 2, \ldots, k\). By definition, \(s(G)\) is the least common multiple of \(s(G_1), \ldots, s(G_k)\).

Let \(\bar{r}_i : G \to G_i\) and \(\iota_i : S^1 \times H_i \to S^1 \times H\) denote the projection and the inclusion, respectively. Then \(\bar{r}_i \circ \epsilon_G = \epsilon_{G_i}\) and \(\bar{r}_i \circ p_G \circ \iota_i = p_{G_i}\), so that
\[ (1 \wedge \iota_i)^* \circ (\bar{r}_i)_* (\langle \epsilon_G, p_G \rangle) = \langle \bar{r}_i \circ \epsilon_G, \bar{r}_i \circ p_G \circ \iota_i \rangle = \langle \epsilon_{G_i}, p_{G_i} \rangle. \]
Thus the order of \(\langle \epsilon_G, p_G \rangle\) is a non-zero multiple of the order of \(\langle \epsilon_{G_i}, p_{G_i} \rangle\). So by Propositions 4.1, 4.4, 4.7, 5.1 and 5.5, the order of \(\langle \epsilon_G, p_G \rangle\) is a non-zero multiple of \(s(G_i)\) for \(i = 1, 2, \ldots, k\), hence so is \(\langle \epsilon_G, 1_G \rangle\). Therefore by Lemma 3.3, the proof is complete. \(\square\)

Next, we prove Theorem 1.3.
Proof of Theorem 1.3. We only prove the case $H = SU(n)^r$ because the case $H = SU(4n - 2)^s \times Sp(2n - 1)^t$ is proved analogously. The implication (1) $\Rightarrow$ (2) follows from Theorem 1.2. We prove the implication (2) $\Rightarrow$ (1). Let $\partial_0 : G \to \map_*(S^2, BG; k) \cong \Omega_0 G$ be as in Section 2, and let $q_i : H \to SU(n)$ be the projection onto the $i$-th $SU(n)$. Then by Lemma 2.1, the proof of Proposition 4.1 implies that the image of the map
\[
(\partial_k)_* : \pi_{2n-1}(G) \to \pi_{2n-1}(\Omega_0 G)
\]
is isomorphic to $\prod_{i=1}^r \mathbb{Z}/\frac{n!}{(k_i|q_i(C))}$, where $\pi_{2n-1}(\Omega_0 G) \cong (\mathbb{Z}/n!)^r$. By (2.1), there is an exact sequence
\[
0 \to \prod_{i=1}^r \mathbb{Z}/\frac{n!}{(k_i|q_i(C))} \to (\mathbb{Z}/n!)^r \to \pi_{2n-1}(\mathcal{G}_k(S^2, G)) \to \pi_{2n-1}(BG) \cong \pi_{2n-1}(BSU(n)^r) = 0.
\]
Then since $\pi_{2n-1}(\mathcal{G}_k(S^2, G)) \cong \pi_{2n-2}(\mathcal{G}_k(S^2, G))$, $\pi_{2n-2}(\mathcal{G}_k(S^2, G)) \cong \prod_{i=1}^r \mathbb{Z}/(k_i|q_i(C))$.

So if $\mathcal{G}_k(X, G) \cong \mathcal{G}_l(X, G)$, then $\pi_{2n-2}(\mathcal{G}_k(S^2, G)) \cong \pi_{2n-2}(\mathcal{G}_l(S^2, G))$, implying
\[
(k, |q_1(C)|) \cdots (k, |q_r(C)|) = (l, |q_1(C)|) \cdots (l, |q_r(C)|).
\]
As in the proof of Theorem 1.3, $s(G)$ is the least common multiple of $|q_1(C)|, \ldots, |q_r(C)|$. Then it is easy to see that the above equality implies $(k, s(G)) = (l, s(G))$, completing the proof.

References

[1] J.F. Adams, Lectures on exceptional Lie groups, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996. With a foreword by J. Peter May; Edited by Zafer Mahmud and Mamoru Mimura.

[2] M.F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523-615.

[3] P.F. Baum and W. Browder, The cohomology of quotients of classical groups, Topology 3 (1965), 305-336.

[4] R. Bott, A note on the Samelson products in the classical groups, Comment. Math. Helv. 34 (1960), 249-256.

[5] S. B. Bradlow, O. Garcia-Prada, and P. B. Gothen, Homotopy groups of moduli spaces of representations, Topology 47 (2008) 203-224.

[6] T. Cutler, The homotopy types of $Sp(3)$-gauge groups, Topology Appl. 236 (2018), 44-58.

[7] M.C. Crabb and W.A. Sutherland, Counting homotopy types of gauge groups, Proc. London Math. Soc. 83 (2000), 747-768.

[8] G.D. Daskalopoulos and K.K. Uhlenbeck, An application of transversality to the topology of the moduli space of stable bundles, Topology 34 (1995) 203-215.

[9] D.H. Gottlieb, Applications of bundle map theory, Trans. Amer. Math. Soc. 171 (1972) 23-50.

[10] H. Hamanaka and A. Kono, Unstable $K^1$-group and homotopy type of certain gauge groups, Proc. Roy. Soc. Edinburgh Sect. A 136 (2006), 149-155.

[11] S. Hasui, D. Kishimoto, A. Kono, and T. Sato, The homotopy types of $PU(3)$- and $PSp(2)$-gauge groups, Algebr. Geom. Topol. 16 (2016), no. 3, 1813-1825.

[12] S. Hasui, D. Kishimoto, T. So, and S. Theriault, Odd primary homotopy types of the gauge groups of exceptional Lie groups, Proc. Amer. Math. Soc. 147 (2019), no. 4, 1751-1762.
[13] K. Ishitoya, A. Kono, and H. Toda, Hopf algebra structure of mod 2 cohomology of simple Lie groups, Publ. RIMS 12 (1976), 141-167.

[14] Y. Kamiyama, D. Kishimoto, A. Kono, and S. Tsukuda, Samelson products of SO(3) and applications, Glasg. Math. J. 49 (2007), no. 2, 405-409.

[15] D. Kishimoto and A. Kono, On the homotopy types of Sp(n) gauge groups, Algebr. Geom. Topol. 19 (2019), no. 1, 491-502.

[16] D. Kishimoto, A. Kono, and M. Tsutaya, On p-local homotopy types of gauge groups, Proc. Roy. Soc. Edinburgh Sect. A 144 (2014), no. 1, 149-160.

[17] D. Kishimoto, I. Membrillo-Solis, and S. Theriault, The homotopy types of SO(4)-gauge groups, accepted by Eur. J. Math.

[18] D. Kishimoto, S. Theriault, and M. Tsutaya, The homotopy types of G2-gauge groups, Topology Appl. 228 (2017), 92-107.

[19] A. Kono, Hopf algebra structure of simple Lie groups, J. Math. Kyoto Univ. 17 (1977), no. 2, 259-298.

[20] A. Kono, A note on the homotopy type of certain gauge groups, Proc. Roy. Soc. Edinburgh Sect. A 117 (1991), 295-297.

[21] A. Kono and M. Mimura, On the cohomology mod 2 of the classifying space of AdE7, J. Math. Kyoto Univ. 18 (1978), no. 3, 535-541.

[22] A. Kono, M. Mimura, and N. Shimada, On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie group E7, J. Pure Appl. Alg. 8 (1976), 267-283.

[23] D. Quillen, The mod 2 cohomology rings of extra-special 2-groups and the spinor groups, Math. Ann. 194 (1971) 197-212.

[24] S. Rea, Homotopy types of gauge groups of PU(p)-bundles over spheres, J. Homotopy Rel. Str. 16 (2021), 61-74.

[25] S.D. Theriault, Odd primary homotopy decompositions of gauge groups, Algebr. Geom. Topol. 10 (2010) 535-564.

[26] S.D. Theriault, The homotopy types of Sp(2)-gauge groups, Kyoto J. Math. 50 (2010) 591-605.

[27] S.D. Theriault, Homotopy decompositions of gauge groups over Riemann surfaces and applications to moduli spaces, Int. J. Math. 22 (2011), no. 12, 1711-1719.

[28] S.D. Theriault, The homotopy types of SU(5)-gauge groups, Osaka J. Math. 52 (2015), 15-29.

[29] S. Theriault, Odd primary homotopy types of SU(n)-gauge groups, Algebr. Geom. Topol. 17 (2017), no. 2, 1131-1150.

[30] G.W. Whitehead, On products in homotopy groups, Ann. Math. 47 (1946), no. 3, 460-475.