Nilpotent Symmetries of a Modified Massive Abelian 3-Form Theory: Augmented Superfield Approach

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Abstract: We derive the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for any arbitrary D-dimensional St"uckelberg-modified massive Abelian 3-form theory within the framework of augmented version of superfield approach (AVSA) to Becchi-Rouet-Stora-Tyutin (BRST) formalism where, in addition to the horizontality condition (HC), we exploit the theoretical strength of the gauge invariant restriction (GIR) to deduce the proper transformations for the gauge, associated (anti-)ghost fields, auxiliary fields, St"uckelberg compensating field, etc. In fact, it is an elegant and delicate combination of HC and GIR (within the ambit of AVSA) that is crucial for all our discussions and derivations. One of the highlights of our present endeavor is the deduction of a new set of (anti-)BRST invariant Curci-Ferrari (CF)-type restrictions which are not found in the massless version of our present theory where only the HC plays an important role in the derivations of all the (anti-)BRST transformations and a very specific set of CF-type restrictions. The alternative ways of the derivation of the full set of the latter, from various theoretical considerations, are also interesting results of our present investigation.

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1 Introduction

The key ideas behind the (super)string theories have been at the forefront of research in the realm of theoretical high energy physics because these theories are the most promising candidates to provide a precise theoretical description of the theory of quantum gravity (see, e.g. [1-5] for details). Furthermore, as far as the idea of complete unification of all the fundamental interactions of nature is concerned, the modern developments in (super)string theories provide a glimpse of hope to accomplish this goal (which is unthinkable within the framework of quantum field theories which are not based on the ideas of (super)strings). In addition to the above positive aspects, these fundamental developments have led to many other branches of research activities in theoretical high energy physics [1-5]. For instance, the studies of noncommutative geometry and related field theories, higher p-form \((p = 2, 3, \ldots)\) gauge theories, AdS/CFT correspondence, higher spin gauge theories, specific kinds of supersymmetric Yang-Mills theories, etc., owe their origin to the key developments in the realm of (super)string theories. The central theme of our present investigation is connected with the study of the Becchi-Rouet-Stora-Tyutin (BRST) quantized version of the massive Abelian 3-form gauge theory which, it goes without saying, is inspired by the theoretical developments in (super)string theories where the higher \(p\)-form \((p = 2, 3, \ldots)\) gauge fields appear in their quantum excitations. Our present study goes beyond the realm of the standard model of particle physics (see, e.g. [6-8] and references therein) which is based on the quantum field theory of the non-Abelian 1-form Yang-Mills gauge theory.

The BRST-quantized theories are endowed with the nilpotent and absolutely anticommuting (anti-)BRST transformations corresponding to a given classical local gauge symmetry transformation for a classical gauge theory. The usual superfield approach (USFA) to BRST formalism (see, e.g. [9-11] and references therein), based on the geometrical aspects of mathematics, addresses the issues related to the BRST-quantized version of the gauge theory where the celebrated horizontality condition (HC) plays a decisive role in the derivation of \((i)\) the (anti-)BRST symmetries for the 1-form gauge and associated (anti-)ghost fields, and \((ii)\) the Curci-Ferrari (CF) condition (see, e.g. [12]) for the non-Abelian 1-form gauge theory \((without any interaction with matter fields)\). The key ideas behind USFA have been systematically augmented so as to derive the (anti-)BRST transformations for the gauge, (anti-)ghost and matter fields together along with the CF-type condition. This approach has been christened by us as the augmented version of superfield approach (AVSA) to BRST formalism (see, e.g. [13-15] for details) where, in addition to the HC, we have exploited the gauge invariant restriction (GIR). It has been found that there is no conflict between HC and GIR and they compliment each-other in a coherent manner. In fact, the results of the HC are utilized in the application of GIR that is imposed on the superfields that are defined on a suitably chosen supermanifold [see, Eqs. (20), (26) below].

In our present investigation, we apply the theoretical beauty and strength of AVSA in the context of Stückelberg-modified [16] version of the massive Abelian 3-form gauge theory in any arbitrary \(D\)-dimension of spacetime to derive \((i)\) the proper (i.e. off-shell nilpotent and absolutely anticommuting) BRST and anti-BRST symmetry transformations, \((ii)\) the precise sets of (anti-BRST invariant CF-type restrictions, and \((iii)\) the coupled (but equivalent) (anti-)BRST invariant Lagrangian densities. One of the upshots of our whole discussion is the derivation of the precise sets of (anti-)BRST invariant CF-type restrictions
from (i) the requirement of the absolute anticommutativity of the (anti-)BRST symmetry transformations, (ii) the BRST and anti-BRST invariance of the appropriate Lagrangian densities of our theory, and (iii) the Euler-Lagrange (EL) equations of motion (EoM) that emerge out from the coupled (but equivalent) (anti-)BRST invariant Lagrangian densities.

In our present endeavor, we exploit the theoretical potential and power of AVSA to obtain the full set of (anti-)BRST symmetry transformations for the St"uckelberg-modified D-dimensional massive Abelian 3-form gauge theory and show the existence of a new set of CF-type restrictions which are not found [17, 18] in the massless version of our present theory where only HC plays a decisive role. We establish the existence of all the CF-type restrictions of our theory by invoking an elegant combination of HC [cf. Eq. (8)] and a gauge invariant restriction [cf. Eq. (26)] within the framework of AVSA. We also derive the (anti-)BRST invariant Lagrangian densities in the D-dimensional ordinary space for the St"uckelberg-modified massive Abelian theory and establish their (anti-)BRST invariance in an explicit fashion. The highlight of this exercise is the derivation of all the CF-type restrictions of our theory on the beautiful idea of symmetry consideration alone.

The theoretical contents of our present paper are organized as follows. In Sec. 2, we discuss the bare essentials of the St"uckelberg-formalism [16] in the case of a massive Abelian 3-form theory in any arbitrary D-dimension of spacetime. In this section, we also capture some of the essential features of our earlier works [17, 18] where we have derived the nilpotent (anti-)BRST symmetries for the gauge, associated (anti-)ghost and auxiliary fields from the horizontality condition [i.e. $\tilde{d}\tilde{A}^{(3)} = dA^{(3)}$] [cf. Eq. (8) below] so that we can obtain the explicit form of $\tilde{A}^{(3)}_{(h)}$ [cf. Eq. (20) below] where the subscript $(h)$ denotes the explicit expression for the super Abelian 3-form after the application of HC. Our Sec. 3 deals with the derivation of the (anti-)BRST symmetry transformations for the St"uckelberg compensating field, associated (anti-)ghost fields and auxiliary fields. The subject matter of Sec. 4 concerns itself with the derivation of the coupled (but equivalent) (anti-) BRST invariant Lagrangian densities and we comment on their proper transformations under the (anti-)BRST symmetry transformations. In Sec. 5, we perform thread-bare analysis of the applications of the (anti-)BRST symmetry transformations on the coupled (but equivalent) Lagrangian densities and show the existence of the (anti-)BRST invariant CF-type restrictions. Finally, in Sec. 6, we summarize our key results, highlight new observations and point out a few future directions for further investigation(s).

In our Appendices A and B, we provide some explicit computations that have been used in the main body of our text. In particular, in Appendix A, we discuss the explicit use of HC in the derivations of the (anti-)BRST symmetry transformations and (non-)trivial CF-type restrictions which have not been performed in our earlier works [17, 18]. Our Appendix C captures, through a figure, the existence and emergence of CF-type restrictions on our theory where we concisely provide an explanation for such an existence.

**Convention and Notations:** We adopt the convention of the left-derivative w.r.t. all the fermionic fields in all our computations. Furthermore, our D-dimensional background spacetime manifold is flat with the metric tensor $\eta_{\mu\nu} = \text{diag} (+1, -1, -1,\ldots)$ so that the dot product between two non-null vectors $P_\mu$ and $Q_\mu$ (on this flat D-dimensional spacetime manifold) is defined by $P \cdot Q = \eta_{\mu\nu} P^\mu Q^\nu \equiv P_0 Q_0 - P_i Q_i$ where the Greek indices
\( \mu, \nu, \lambda, ... = 0, 1, 2, ... D - 1 \) denote the time and space directions and the Latin indices \( i, j, k, ... = 1, 2, ... D - 1 \) stand for the space directions only. We also adopt the convention of the derivatives: \( [\partial A_{\mu \nu \lambda} / \partial x^{\rho}] = \frac{1}{3!} \left[ \delta^\rho_\mu (\delta^\sigma_\nu \delta^\xi_\lambda - \delta^\sigma_\lambda \delta^\xi_\nu) + \delta^\rho_\nu (\delta^\sigma_\mu \delta^\xi_\lambda - \delta^\sigma_\lambda \delta^\xi_\mu) + \delta^\rho_\lambda (\delta^\sigma_\mu \delta^\xi_\nu - \delta^\sigma_\nu \delta^\xi_\mu) \right] \), \( (\partial B_{\mu \nu} / \partial B_{\alpha \beta}) = \frac{1}{2} [\delta^\rho_\nu \delta^\sigma_\beta - \delta^\rho_\beta \delta^\sigma_\nu] \), etc., for the basic third-ranked and second-ranked tensor fields. Throughout the whole body of our text, we denote the (anti-)BRST symmetry transformations by \( s_{(a)b} \). The generic superfield \( \Phi(x, \theta, \bar{\theta}) \) is defined on the \((D, 2)\)-dimensional supermanifold which is characterized by the super (anti-)BRST symmetry transformations by \( A_{\mu, \nu, \lambda, ...} \).

2 Preliminaries: Celebrated Stückelberg Formalism and Geometrical Horizontality Condition

This section is divided into two parts. In subsection 2.1, we discuss the key features of the Stückelberg technique of the compensating fields to derive the modified Lagrangian density [cf. Eq. (4) below] for the massive Abelian 3-form theory which respects the gauge transformations. Our subsection 2.2 is devoted to the discussion on the horizontality condition (HC) which yields the (anti-)BRST transformations \( [s_{(a)b}] \) for the massless case [17, 18] of the Abelian 3-form gauge theory in any arbitrary D-dimension of spacetime (cf. Appendix A below for details). These are essential inputs for the derivation of the complete set of off-shell nilpotent and absolutely anticommuting (anti-)BRST transformations for our entire theory which incorporates the gauge fields, associated (anti-)ghost fields, auxiliary fields and the Stückelberg-compensating field [and its associated (anti-ghost fields)].

2.1 Stückelberg Technique: Lagrangian Formulation

We begin with the Lagrangian density \( [\mathcal{L}_{(0)}] \) for the totally antisymmetric \( (A_{\mu \nu \lambda} = -A_{\nu \mu \lambda} = -A_{\mu \lambda \nu} = -A_{\lambda \nu \mu}) \) tensor field \( (A_{\mu \nu \lambda}) \) with the rest mass \( m \) (see, e.g. [16-18]) in any arbitrary D-dimension of spacetime (with \( \mu, \nu, \lambda, \zeta, ... = 0, 1, 2, ..., D - 1 \)), namely;

\[
\mathcal{L}_{(0)} = \frac{1}{24} H^{\mu \nu \lambda \zeta} H_{\mu \nu \lambda \zeta} - \frac{m^2}{6} A_{\mu \nu \lambda} A^{\mu \nu \lambda},
\]

(1)

where the field strength tensor \( H_{\mu \nu \lambda \zeta} = \partial_\mu A_{\nu \lambda \zeta} - \partial_\nu A_{\lambda \mu \zeta} + \partial_\lambda A_{\mu \nu \zeta} - \partial_\zeta A_{\mu \nu \lambda} \) is derived from the 4-form \( H^{(4)} = d A^{(3)} \) where \( A^{(3)} = [(d x^\mu \wedge d x^\nu \wedge d x^\lambda) / 3!] A_{\mu \nu \lambda} \) defines the totally antisymmetric tensor field \( A_{\mu \nu \lambda} \). Here the operator \( d = d x^\mu \partial_\mu \) (with \( d^2 = 0 \)) is the exterior derivative (see, e.g. [19-22] for details). The explicit form of \( H^{(4)} \) is as follows:

\[
H^{(4)} = d A^{(3)} = \frac{1}{4!} \left( d x^\mu \wedge d x^\nu \wedge d x^\lambda \wedge d x^\zeta \right) H_{\mu \nu \lambda \zeta}.
\]

(2)

The Euler-Lagrange (EL) equation of motion (EoM) from the Lagrangian density (1) is:

\[ \partial_\mu H^{\mu \nu \lambda \zeta} + m^2 A^{\nu \lambda \zeta} = 0 \]

which implies the subsidiary conditions: \( \partial_\nu A^{\nu \lambda \zeta} = \partial_\lambda A^{\nu \lambda \zeta} = 0 \).
\[ \partial_\xi A^{\nu\lambda} = 0 \] due to the *totally* antisymmetric nature of \( H_{\mu\nu\lambda\zeta} \) [cf. Eq. (2)] when \( m^2 \neq 0 \). With the above subsidiary conditions as inputs, we obtain the Klein-Gordon equation of motion: \( (\Box + m^2) A_{\mu\nu} = 0 \) which shows that the Lagrangian density (2) describes a relativistic field \( A_{\mu\nu} \) with the rest mass equal to \( m \). The Lagrangian density (2) is endowed with the second-class constraints in the terminology of Dirac’s prescription for the classification scheme of constraints [23, 24]. To restore the gauge symmetry transformations that are generated by the first-class constraints, we apply the following theoretical trick of the Stückelberg formalism of compensating fields [16].

\[
A_{\mu\nu} \longrightarrow A_{\mu\nu} \mp \frac{1}{m} (\partial_\mu \Phi_{\nu\lambda} + \partial_\nu \Phi_{\lambda\mu} + \partial_\lambda \Phi_{\mu\nu}), \tag{3}
\]

where the compensating antisymmetric \( (\Phi_{\mu\nu} = -\Phi_{\nu\mu}) \) tensor field \( \Phi_{\mu\nu} \) has the *same* mass dimension as \( A_{\mu\nu} \) field in any arbitrary dimension of spacetime in the natural units where \( \hbar = c = 1 \). Adopting the notation \( \Sigma_{\mu\nu\lambda} = \partial_\mu \Phi_{\nu\lambda} + \partial_\nu \Phi_{\lambda\mu} + \partial_\lambda \Phi_{\mu\nu} \), we obtain the Stückelberg-modified Lagrangian density \( \mathcal{L}_{\text{(S)}} \) (from \( \mathcal{L}_{\text{(0)}} \)) as

\[
\mathcal{L}_{\text{(0)}} \longrightarrow \mathcal{L}_{\text{(S)}} = \frac{1}{24} H^{\mu\nu\lambda\zeta} H_{\mu\nu\lambda\zeta} - \frac{m^2}{6} A^{\mu\nu\lambda} A_{\mu\nu\lambda} - \frac{1}{6} \Sigma_{\mu\nu\lambda} \Sigma_{\mu\nu\lambda} \pm \frac{m}{3} A^{\mu\nu\lambda} \Sigma_{\mu\nu\lambda}, \tag{4}
\]

which respects \( [\delta_g \mathcal{L}_{\text{(S)}} = 0] \) the following gauge symmetry transformations \( (\delta_g) \)

\[
\delta_g A_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} + \partial_\nu A_{\lambda\mu} + \partial_\lambda A_{\mu\nu}, \quad \delta_g \Phi_{\mu\nu} = \pm m \Lambda_{\mu\nu} - (\partial_\mu \Lambda_{\nu} - \partial_\nu \Lambda_{\mu}), \quad \delta_g \Sigma_{\mu\nu\lambda} = \pm m (\partial_\mu \Lambda_{\nu\lambda} + \partial_\nu \Lambda_{\lambda\mu} + \partial_\lambda \Lambda_{\mu\nu}), \quad \delta_g H_{\mu\nu\lambda\zeta} = 0, \tag{5}
\]

where \( (\Lambda_{\mu\nu} = -\Lambda_{\nu\mu}) \) is the antisymmetric gauge transformation parameter. The Lorentz vector \( (\Lambda_\mu) \) gauge transformation parameter appears in the theory due to the observation that there is a stage-one reducibility in the Stückelberg-redefinition (3) where \( \Phi_{\mu\nu} \longrightarrow \Phi_{\mu\nu} \pm (\partial_\mu \Lambda_{\nu} - \partial_\nu \Lambda_{\mu}) \) is permitted due to the antisymmetric \( (\Phi_{\mu\nu} = -\Phi_{\nu\mu}) \) nature of the compensating field \( \Phi_{\mu\nu} \) that is present in the redefinition [cf. Eq. (3)].

We end this subsection with the following remarks. First of all, we note that, in the Stückenber formalism, it is the exterior derivative \( d = d x^\mu \partial_\mu \) (with \( \mu = 0, 1, 2, \ldots, D - 1 \), and \( d^2 = 0 \)) that plays an important role because the redefinition (3) can be expressed in the differential form-language [19-22] as follows

\[
A^{(3)} \longrightarrow A^{(3)} \mp \frac{1}{m} d \Phi^{(2)}, \tag{6}
\]

where the *bosonic* 2-form \( \Phi^{(2)} = [(d x^\mu \wedge d x^\nu)/2] \Phi_{\mu\nu} \) establishes that \( \Phi_{\mu\nu} \) is an antisymmetric \( (\Phi_{\mu\nu} = -\Phi_{\nu\mu}) \) compensating field. Second, the Lorentz vector \( \Lambda_\mu(x) \) is a gauge transformations parameter in the theory due to the stage-one reducibility in the theory. Third, there is existence of stage-two reducibility, too, in our theory because the Lorentz vector \( \Lambda_\mu \) can be replaced by \( \Lambda_\mu \longrightarrow \Lambda_\mu \pm \partial_\mu \Lambda \) without spoiling the symmetry transformations where \( \Lambda(x) \) is a Lorentz scalar gauge transformation parameter. Fourth, within the framework of BRST formalism, the *bosonic* set of transformation parameters \( (\Lambda_{\mu\nu}, \Lambda_\mu, \Lambda) \) will be replaced by the *fermionic* (anti-)ghost fields. Fifth, we observe that the following
elegant and useful combination of the $A_{\mu\nu\lambda}$ and $\Phi_{\mu\nu}$ fields remains invariant under the gauge symmetry transformations ($\delta_g$):

$$\delta_g \left[ A_{\mu\nu\lambda} \mp \frac{1}{m} \Sigma_{\mu\nu\lambda} \right] = 0 \iff \delta_g \left[ A^{(3)} \mp \frac{1}{m} d \Phi^{(2)} \right] = 0. \tag{7}$$

We shall see later that this observation plays an important role in the superfield formalism (cf. Sec. 3 below). Sixth, the gauge transformations (5) are generated by the first-class constraints (see, e.g. [23-26]). The latter emerge due to the presence of the St"uckelberg compensating field ($\Phi_{\mu\nu}$) which is responsible for the conversion of the second-class constraints of the original Lagrangian density [$\mathcal{L}_{(0)}$] into the first-class constraints [25, 26]. Finally, it is straightforward to note that, due to (6), the 4-form $H^{(4)} = d A^{(3)}$ remains invariant because of the nilpotency ($d^2 = 0$) of the exterior derivative ($d$). In other words, we have the gauge invariance ($\delta_g H_{\mu\nu\lambda\zeta} = 0$) of the field-strength tensor.

### 2.2 Horizontality Condition: Superfield Approach

We have seen that the 4-form $H^{(4)} = d A^{(3)}$ (with $d^2 = 0$) defines the field strength (curvature) tensor $H_{\mu\nu\lambda\zeta} = \partial_\mu A_{\nu\lambda\zeta} - \partial_\nu A_{\lambda\zeta\mu} + \partial_\lambda A_{\mu\nu\zeta} - \partial_\zeta A_{\mu\nu\lambda}$ which remains invariant ($\delta_g H_{\mu\nu\lambda\zeta} = 0$) under the gauge symmetry transformations (5). Hence, it is a physical quantity in the theory and its generalization on the (D, 2)-dimensional supermanifold must remain independent of the Grassmannian variables ($\theta, \bar{\theta}$) that characterize this supermanifold along with the D-dimensional bosonic coordinates $x^\mu (\mu = 0, 1, 2...D - 1)$. In other words, the following horizontality condition (HC), namely;

$$\tilde{d} \tilde{A}^{(3)} = d A^{(3)}, \quad \tilde{d} = dx^\mu \partial_\mu + d\theta \partial_{\theta} + d\bar{\theta} \partial_{\bar{\theta}}, \quad d = d x^\mu \partial_\mu, \tag{8}$$

leads to the derivation of the (anti-)BRST symmetry transformations for the gauge field ($A_{\mu\nu\lambda}$), bosonic vector field $\phi_\mu$ and a set of bosonic as well as fermionic (anti-)ghost fields (see e.g. [17] for details) that are associated with the gauge transformations (5). However, before we proceed further, let us define the explicit form of the super 3-form, namely;

$$\tilde{A}^{(3)} = \frac{(d Z^M \wedge d Z^N \wedge d Z^K)}{3!} \tilde{A}_{M N K}(x, \theta, \bar{\theta}), \tag{9}$$

where the superspace coordinates $Z^M = (x^\mu, \theta, \bar{\theta})$ characterize the (D, 2)-dimensional ordinary Abelian 3-form gauge theory that is discussed within the framework of superfield approach to BRST formalism. In an explicit form, the above equation can be expressed, in terms of various kinds of bosonic as well as fermionic (D, 2)-dimensional superfields, as follows:

$$\tilde{A}^{(3)} = \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\lambda) \tilde{A}_{\mu\nu\lambda}(x, \theta, \bar{\theta}) + \frac{1}{2} (dx^\mu \wedge dx^\nu \wedge d\theta) \tilde{A}_{\mu\nu\theta}(x, \theta, \bar{\theta})$$

The Grassmannian variables ($\theta, \bar{\theta}$) are only a set of mathematical artifacts in superfield approach and they have nothing to do with the physical quantities which are gauge [i.e. (anti-)BRST] invariant.
Fmannian components in the superfields lead to the determination of the ghost numbers for the supermanifold as:

\[ 1 \] for-ghost fields with the ghost numbers (+2) and (+1), respectively. The presence of +1, respectively (see, e.g. [17] for details).

Here we identify the superfields on the r.h.s. as follows:

\[ A_{\mu
u}(x, \theta, \bar{\theta}) = A_{\mu \nu}(x, \theta, \bar{\theta}), \quad A_{\mu \theta}(x, \theta, \bar{\theta}) = F_{\mu \nu}(x, \theta, \bar{\theta}), \quad A_{\mu \bar{\theta}}(x, \theta, \bar{\theta}) = \Phi_{\mu}(x, \theta, \bar{\theta}), \]

\[ 1 \]

\[ A_{\mu \nu}(x, \theta, \bar{\theta}) = \bar{A}_{\mu \nu}(x, \theta, \bar{\theta}), \quad A_{\mu \theta}(x, \theta, \bar{\theta}) = \bar{F}_{\mu \nu}(x, \theta, \bar{\theta}), \quad A_{\mu \bar{\theta}}(x, \theta, \bar{\theta}) = \bar{\Phi}_{\mu}(x, \theta, \bar{\theta}), \]

\[ \frac{1}{3!} A_{\theta \theta \theta}(x, \theta, \bar{\theta}) = \bar{F}_2(x, \theta, \bar{\theta}), \quad \frac{1}{3!} A_{\bar{\theta} \bar{\theta} \bar{\theta}}(x, \theta, \bar{\theta}) = \bar{F}_2(x, \theta, \bar{\theta}), \]

\[ \frac{1}{2} \bar{A}_{\theta \theta}(x, \theta, \bar{\theta}) = \beta_{\mu}(x, \theta, \bar{\theta}), \quad \frac{1}{2} \bar{A}_{\bar{\theta} \bar{\theta}}(x, \theta, \bar{\theta}) = \beta_{\mu}(x, \theta, \bar{\theta}). \]

It is clear from the above identifications that the four bosonic superfields are: \( A_{\mu \nu}(x, \theta, \bar{\theta}) \), \( \Phi_{\mu}(x, \theta, \bar{\theta}) \), \( \beta_{\mu}(x, \theta, \bar{\theta}) \) and \( \bar{\beta}_{\mu}(x, \theta, \bar{\theta}) \) where the latter two superfields are the generalizations of the bosonic ghost (i.e. ghost-for-ghost) fields which carry the ghost numbers (+2) and (-2), respectively. On the contrary, the superfields \( A_{\mu \nu}(x, \theta, \bar{\theta}) \) and \( \Phi_{\mu}(x, \theta, \bar{\theta}) \) carry no ghost number (i.e. zero ghost number) and they are the generalizations of the gauge field \( A_{\mu \nu}(x) \) and a vector field \( \Phi_{\mu}(x) \) which appears in the CF-type restriction. The fermionic superfields, it is obvious, in our theory are: \( \bar{F}_{\mu \nu}(x, \theta, \bar{\theta}) \), \( \bar{F}_{\mu \nu}(x, \theta, \bar{\theta}) \) which are nothing but the generalizations of the (anti-)ghost fields \( (\bar{C}^\mu)_{\mu} \) in our theory with the ghost numbers (-1) + 1, respectively. The presence of three Grassmannian variables: \( \frac{1}{3!} A_{\theta \theta \theta}(x, \theta, \bar{\theta}) = \bar{F}_2(x, \theta, \bar{\theta}), \frac{1}{3!} A_{\bar{\theta} \bar{\theta} \bar{\theta}}(x, \theta, \bar{\theta}) = \bar{F}_2(x, \theta, \bar{\theta}) \) ensures that these fermionic superfields are the generalizations of ordinary fermionic (anti-)ghost fields \( (\bar{C}^2)_{\mu} \) which are ghost-for-ghost fields with the ghost numbers (-3) + 3 respectively. The auxiliary (anti-)ghost fields \( (\bar{C}_1)_{\mu} \) have their generalizations [cf. Eq. (12) below] on the (D, 2)-dimensional supermanifold as: \( \frac{1}{2} \bar{A}_{\theta \theta}(x, \theta, \bar{\theta}) = \bar{F}_1(x, \theta, \bar{\theta}), \frac{1}{2} \bar{A}_{\bar{\theta} \bar{\theta}}(x, \theta, \bar{\theta}) = \bar{F}_1(x, \theta, \bar{\theta}) \) with the ghost numbers (+1) − 1, respectively (see, e.g. [17] for details).

The superfields [that have been defined on the (D, 2)-dual supermanifold] can be expanded along the Grassmannian directions \( (\theta, \bar{\theta}) \) by the Taylor expansion about \( (\theta = 0, \bar{\theta} = 0) \). The resulting super expansions are (see, e.g. [9-11, 17, 18] for details)

\[ A_{\mu \nu}(x, \theta, \bar{\theta}) = A_{\mu \nu}(x) + \theta R_{\mu \nu}(x) + \bar{\theta} R_{\mu \nu}(x) + i \theta \bar{\theta} S_{\mu \nu}(x), \]

\[ \text{It is the theoretical strength of superfield approach to BRST formalism that the presence of the Grassmannian components in the superfields lead to the determination of the ghost numbers for the corresponding ordinary fields. Based on this logic, for instance, we see that the ghost-for-the-ghost-for-the-ghost fields \( (\bar{C}_2)_{\mu} \) carry the ghost numbers (-3) + 3, respectively.} \]
\[ \mathcal{F}_{\mu\nu}(x, \theta \bar{\theta}) = C_{\mu\nu} + \theta B^{(1)}_{\mu\nu}(x) + \bar{\theta} B^{(1)}_{\mu\nu}(x) + i \theta \bar{\theta} s_{\mu\nu}(x), \]
\[ \bar{\mathcal{F}}_{\mu\nu}(x, \theta \bar{\theta}) = C_{\mu\nu} + \theta B^{(2)}_{\mu\nu}(x) + \bar{\theta} B^{(2)}_{\mu\nu}(x) + i \theta \bar{\theta} s_{\mu\nu}(x), \]
\[ \vec{\beta}_{\mu}(x, \theta \bar{\theta}) = \beta_{\mu}(x) + \theta \bar{f}^{(1)}_{\mu}(x) + \bar{\theta} f^{(1)}_{\mu}(x) + i \theta \bar{\theta} b_{\mu}(x), \]
\[ \vec{\beta}_{\mu}(x, \theta \bar{\theta}) = \bar{\beta}_{\mu}(x) + \theta \bar{f}^{(2)}_{\mu}(x) + \bar{\theta} f^{(2)}_{\mu}(x) + i \theta \bar{\theta} b_{\mu}(x), \]
\[ \Phi_{\mu}(x, \theta \bar{\theta}) = \phi_{\mu}(x) + \theta \bar{f}^{(3)}_{\mu}(x) + \bar{\theta} f^{(3)}_{\mu}(x) + i \theta \bar{\theta} b^{(3)}_{\mu}(x), \]
\[ \mathcal{F}_1(x, \theta \bar{\theta}) = C_1(x) + \theta \bar{b}^{(1)}_1(x) + \bar{\theta} b^{(1)}_1(x) + i \theta \bar{\theta} s_1(x), \]
\[ \bar{\mathcal{F}}_1(x, \theta \bar{\theta}) = \bar{C}_1(x) + \theta \bar{b}^{(2)}_1(x) + \bar{\theta} b^{(2)}_1(x) + i \theta \bar{\theta} s_1(x), \]
\[ \mathcal{F}_2(x, \theta \bar{\theta}) = C_2(x) + \theta \bar{b}^{(1)}_2(x) + \bar{\theta} b^{(1)}_2(x) + i \theta \bar{\theta} s_2(x), \]
\[ \bar{\mathcal{F}}_2(x, \theta \bar{\theta}) = \bar{C}_2(x) + \theta \bar{b}^{(2)}_2(x) + \bar{\theta} b^{(2)}_2(x) + i \theta \bar{\theta} s_2(x), \]

where, because of the fermionic (\( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0 \)) nature of the Grassmannian variables (\( \theta, \bar{\theta} \)), we have the fermionic as well as bosonic secondary fields on the r.h.s. of the above super expansions. For instance, in (12), the set of secondary fields \([S_{\mu\nu\lambda}(x), B^{(1)}_{\mu\nu}(x), B^{(1)}_{\mu\nu}(x), B^{(2)}_{\mu\nu}(x), B^{(2)}_{\mu\nu}(x), b_{\mu}(x), b_{\mu}(x), b^{(3)}_{\mu}(x), \bar{b}^{(1)}(x), \bar{b}^{(1)}(x), \bar{b}^{(1)}(x), \bar{b}^{(2)}(x), \bar{b}^{(2)}(x)]\) are bosonic and the set \([R_{\mu\nu\lambda}(x), \bar{R}_{\mu\nu\lambda}(x), s_{\mu\nu}(x), \bar{s}_{\mu\nu}(x), f^{(1)}_{\mu}(x), \bar{f}^{(1)}_{\mu}(x), f^{(2)}_{\mu}(x), \bar{f}^{(2)}_{\mu}(x), f^{(3)}_{\mu}(x), \bar{f}^{(3)}_{\mu}(x), s_1(x), \bar{s}_1(x), s_2(x), \bar{s}_2(x)]\) constitutes the collection of fermionic secondary fields. These secondary fields have to be determined in terms of the basic and auxiliary fields of the theory where the HC [cf. Eq. (8)] plays an important role. The latter theoretical trick lead to the following (cf. Appendix A below for details):

\[ R_{\mu\nu\lambda} = \partial_{\mu}C_{\nu\lambda} + \partial_{\nu}C_{\lambda\mu} + \partial_{\lambda}C_{\mu\nu}, \quad \bar{R}_{\mu\nu\lambda} = \partial_{\mu}\bar{C}_{\nu\lambda} + \partial_{\nu}\bar{C}_{\lambda\mu} + \partial_{\lambda}\bar{C}_{\mu\nu}, \quad b^{(2)}_1 + b^{(2)}_2 = 0, \]
\[ S_{\mu\nu\lambda} = +i [\partial_{\mu}B^{(1)}_{\nu\lambda} + \partial_{\nu}B^{(1)}_{\mu\lambda} + \partial_{\lambda}B^{(1)}_{\mu\nu}] \equiv -i [\partial_{\mu}B^{(2)}_{\nu\lambda} + \partial_{\nu}B^{(2)}_{\mu\lambda} + \partial_{\lambda}B^{(2)}_{\mu\nu}], \quad b_{\mu} = i \partial_{\mu}\bar{b}^{(1)}_2, \]
\[ s_{\mu\nu} = +i (\partial_{\mu}\bar{f}^{(3)}_{\nu} - \partial_{\nu}\bar{f}^{(3)}_{\mu}) \equiv -i (\partial_{\mu}f^{(2)}_{\nu} - \partial_{\nu}f^{(2)}_{\mu}), \quad s_{\mu\nu} = i (\partial_{\mu}\bar{f}^{(1)}_{\nu} - \partial_{\nu}\bar{f}^{(1)}_{\mu}) \equiv -i (\partial_{\mu}f^{(3)}_{\nu} - \partial_{\nu}f^{(3)}_{\mu}), \quad B^{(2)}_{\mu\nu} = \partial_{\mu}\bar{b}_{\nu} - \partial_{\nu}b_{\mu}, \quad \bar{B}^{(2)}_{\mu\nu} = \partial_{\mu}\bar{b}_{\nu} - \partial_{\nu}b_{\mu}, \quad \bar{f}^{(2)}_{\mu} = \partial_{\mu}\bar{C}_2, \]
\[ \bar{b}_{\mu} = -i \partial_{\mu}b^{(2)}_1, \quad b^{(3)}_{\mu} = -i \partial_{\mu}b^{(2)}_1, \quad f^{(1)}_{\mu} = \partial_{\mu}C_2, \quad \bar{b}^{(1)}_1 + b^{(2)}_1 = 0, \quad f^{(1)}_{\mu} = \partial_{\mu}C_2, \quad \bar{b}^{(1)}_1 + b^{(2)}_1 = 0, \quad \bar{b}^{(2)}_1 = 0. \]

In the above equation, there are three equivalences. These turn out to be true due to the following CF-type restrictions of our theory, namely:

\[ B^{(1)}_{\mu\nu} + B^{(2)}_{\mu\nu} = \partial_{\mu}\phi_{\nu} - \partial_{\nu}\phi_{\mu}, \quad \bar{f}^{(1)}_{\mu} + f^{(3)}_{\mu} = \partial_{\mu}C_1, \quad f^{(2)}_{\mu} + \bar{f}^{(3)}_{\mu} = \partial_{\mu}\bar{C}_1. \]

The above CF-type restrictions have been obtained from the HC [cf. Eq. (8)] where we have set the coefficients of the following super 4-form differentials equal to zero, namely:

\[ (dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta}), \quad (dx^\mu \wedge d\theta \wedge d\bar{\theta} \wedge d\bar{\theta}), \quad (dx^\mu \wedge d\theta \wedge d\theta \wedge d\bar{\theta}), \quad (dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\xi), \]

which emerge out from the computation of the super 4-form \( \tilde{H}^{(4)} = \tilde{d}\tilde{A}^{(3)} \) [i.e. the l.h.s. of Eq. (8)]. It is very interesting to point out that r.h.s. and l.h.s. of Eq. (8) match because of the equality of the coefficients of the differentials \((dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\xi)\) due to the expressions for \( R_{\mu\nu\lambda}, \tilde{R}_{\mu\nu\lambda} \) and \( S_{\mu\nu\lambda} \) [cf. Eq. (13)]. Let us identify the secondary fields:
\[ \mathcal{B}^{(1)}_{\mu \nu} = \mathcal{B}_{\mu \nu}, \quad \mathcal{B}^{(2)}_{\mu \nu} = B_{\mu \nu}, \quad b_1^{(2)} = B_2, \quad b_1^{(1)} = -B_2, \quad b_1^{(1)} = -B, \quad b_1^{(2)} = -B_1, \quad f_\mu^{(3)} = f_\mu, \quad \tilde{f}_\mu^{(3)} = \tilde{f}_\mu. \]

With these identifications and their substitution in Eq. (12), we obtain the following (cf. Appendix A for details)

\[
\begin{align*}
\mathcal{A}_{\mu \lambda}^{(h)}(x, \theta, \bar{\theta}) &= A_{\mu \lambda}(x) + \theta [s_{ab}A_{\mu \lambda}(x)] + \bar{\theta} [s_{ba}\bar{A}_{\mu \lambda}(x)], \\
\mathcal{F}_{\mu \nu}^{(h)}(x, \theta, \bar{\theta}) &= C_{\mu \nu}(x) + \theta [s_{ab}C_{\mu \nu}(x)] + \bar{\theta} [s_{ba}\bar{C}_{\mu \nu}(x)], \\
\mathcal{F}_{\mu \nu}^{(2)}(x, \theta, \bar{\theta}) &= \bar{C}_{\mu \nu}(x) + \theta [s_{ab}\bar{C}_{\mu \nu}(x)] + \bar{\theta} [s_{ba}\bar{C}_{\mu \nu}(x)], \\
\mathcal{F}_{\mu}^{(h)}(x, \theta, \bar{\theta}) &= \beta_{\mu}(x) + \theta [s_{ab}\beta_{\mu}(x)] + \bar{\theta} [s_{ba}\bar{\beta}_{\mu}(x)], \\
\mathcal{F}_{\mu}^{(2)}(x, \theta, \bar{\theta}) &= \bar{\beta}_{\mu}(x) + \theta [s_{ab}\bar{\beta}_{\mu}(x)] + \bar{\theta} [s_{ba}\bar{\beta}_{\mu}(x)], \\
\Phi_{\mu}(x, \theta, \bar{\theta}) &= \phi_{\mu}(x) + \theta [s_{ab}\phi_{\mu}(x)] + \bar{\theta} [s_{ba}\phi_{\mu}(x)], \\
\mathcal{F}_{1}^{(h)}(x, \theta, \bar{\theta}) &= C_1(x) + \theta [s_{ab}C_1(x)] + \bar{\theta} [s_{ba}\bar{C}_1(x)], \\
\mathcal{F}_{2}^{(h)}(x, \theta, \bar{\theta}) &= C_2(x) + \theta [s_{ab}C_2(x)] + \bar{\theta} [s_{ba}\bar{C}_2(x)], \\
\mathcal{F}_{2}^{(2)}(x, \theta, \bar{\theta}) &= \bar{C}_2(x) + \theta [s_{ab}\bar{C}_2(x)] + \bar{\theta} [s_{ba}\bar{C}_2(x)],
\end{align*}
\]

(16)

where the superscript \((h)\) denotes that the above superfields have been obtained after the substitution of the secondary fields \((13)\) into the super expansions \((12)\) which have been obtained after the application of the HC. In the above equation \((16)\), the explicit form of the (anti-)BRST symmetry transformations \(s_{(a)b}\) are as follows:

\[
\begin{align*}
s_{ab}A_{\mu \lambda} &= \partial_\mu C_{\nu \lambda} + \partial_\nu C_{\lambda \mu} + \partial_\lambda C_{\mu \nu}, \\
s_{ab}\bar{C}_{\mu \lambda} &= \partial_\mu C_{\bar{\nu} \lambda} + \partial_{\bar{\nu}} C_{\bar{\lambda} \mu} + \partial_{\bar{\lambda}} C_{\bar{\mu} \nu}, \\
s_{ab}\beta_{\mu} &= \partial_\mu \bar{C}_2, \\
s_{ab}\bar{C}_1 &= -B_2, \\
s_{ab}B &= 0, \\
s_{ab}B_1 &= 0, \\
s_{ab}B_2 &= 0, \\
s_{ab}\bar{B}_\mu &= 0, \\
s_{ab}\bar{f}_\mu &= 0, \\
s_{ab}F_\mu &= -\partial_\mu B_2, \\
s_{ab}\phi_{\mu} &= \partial_\mu B_1, \\
s_{ab}B_{\mu \nu} &= -(\partial_\mu F_\nu - \partial_\nu F_\mu), \\
s_{ab}\bar{C}_{\mu \nu} &= B_{\mu \nu}, \\
s_{ab}\bar{B}_1 &= 0, \\
s_{ab}B_2 &= 0, \\
s_{ab}\bar{\phi}_{\mu} &= \bar{f}_\mu,
\end{align*}
\]

(17)

It is an elementary exercise to check that the above (anti-)BRST symmetry transformations \(s_{(a)b}\) are off-shell nilpotent \([s_{(a)b}^2 = 0]\) of order two. On the other hand, we note that the absolute anticommutativity property \((s_b s_{ab} + s_{ab} s_b = 0)\) of the (anti-)BRST symmetry transformations for the following fields, namely:

\[
\{ s_b, s_{ab} \} A_{\mu \nu \eta} = 0, \quad \{ s_b, s_{ab} \} C_{\mu \nu} = 0, \quad \{ s_b, s_{ab} \} \bar{C}_{\mu \nu} = 0,
\]

(19)

is satisfied if and only if the CF-type restrictions: \(B_{\mu \nu} + \bar{B}_{\mu \nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \quad f_\mu + \bar{f}_\mu = \partial_\mu C_1, \quad \bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1\) are satisfied [which are nothing but Eq. (14) but expressed in the
In this section, we apply the key idea of AVSA to BRST formalism and derive, first of all, the (anti-)BRST symmetries for the St"uckelberg invariance [cf. Eq. (7)] implies the following on the (D, 2)-dimensional supermanifold

\[ \Sigma_{\mu\nu\lambda}(x, \theta, \bar{\theta}) = 1 \]

where the expression for \( \Sigma_{\mu\nu\lambda}(x, \theta, \bar{\theta}) \) has been quoted in Eq. (16) and we have the following super expansion of \( \Sigma_{\mu\nu\lambda}(x, \theta, \bar{\theta}) \)

\[ \tilde{\Sigma}_{\mu\nu\lambda}(x, \theta, \bar{\theta}) = \Sigma_{\mu\nu\lambda}(x, \theta, \bar{\theta}) + \theta P_{\mu\nu\lambda}(x) + \bar{\theta} P_{\mu\nu\lambda}(x) + i \theta \bar{\theta} Q_{\mu\nu\lambda}(x), \] (22)

where (due to the fermionic nature of \( \theta, \bar{\theta} \)) the pair \( (P_{\mu\nu\lambda}, \bar{P}_{\mu\nu\lambda}) \) are the fermionic secondary fields and \( Q_{\mu\nu\lambda}(x) \) is the bosonic secondary field which have to be determined in terms of the basic and auxiliary fields of our D-dimensional massive Abelian 3-form gauge theory.

In fact, the substitution of the explicit expression for

\[ \tilde{A}_{\mu\nu\lambda}^{(h)}(x, \theta, \bar{\theta}) = A_{\mu\nu\lambda}(x) + \theta (\partial_\mu C_{\nu\lambda} + \partial_\nu C_{\lambda\mu} + \partial_\lambda C_{\mu\nu}) + \bar{\theta} (\partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\lambda\mu} + \partial_\lambda \bar{C}_{\mu\nu}) + \theta \bar{\theta} [\partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}], \] (23)
in Eq. (21) leads to the following:

\[ P_{\mu \nu \lambda}(x) = \pm m (\partial_\mu C_{\nu \lambda} + \partial_\nu C_{\lambda \mu} + \partial_\lambda C_{\mu \nu}) \equiv s_b \Sigma_{\mu \nu \lambda}, \]
\[ P_{\mu \nu \lambda}(x) = \pm m (\partial_\mu \bar{C}_{\nu \lambda} + \partial_\nu \bar{C}_{\lambda \mu} + \partial_\lambda \bar{C}_{\mu \nu}) \equiv s_{ab} \Sigma_{\mu \nu \lambda}, \]
\[ Q_{\mu \nu \lambda}(x) = \pm m (\partial_\mu B_{\nu \lambda} + \partial_\nu B_{\lambda \mu} + \partial_\lambda B_{\mu \nu}) \equiv s_b s_{ab} \Sigma_{\mu \nu \lambda}, \]
\[ \equiv \mp m (\partial_\mu \bar{B}_{\nu \lambda} + \partial_\nu \bar{B}_{\lambda \mu} + \partial_\lambda \bar{B}_{\mu \nu}) \equiv -s_{ab} s_b \Sigma_{\mu \nu \lambda}. \] (24)

The above forms of the secondary fields imply that we have obtained

\[ \bar{\Sigma}_{\mu \nu \lambda}^{(g)}(x, \theta, \bar{\theta}) = \Sigma_{\mu \nu \lambda}(x) + \theta (s_{ab} \Sigma_{\mu \nu \lambda}) + \bar{\theta} (s_b s_{ab} \Sigma_{\mu \nu \lambda}), \]
\[ \bar{\Sigma}_{\mu \nu \lambda}(x) + \theta (s_{ab} \Sigma_{\mu \nu \lambda}) + \bar{\theta} (s_b s_{ab} \Sigma_{\mu \nu \lambda}), \] (25)

where the superscript \((g)\) denotes the superfield [i.e. \(\bar{\Sigma}_{\mu \nu \lambda}^{(g)}(x, \theta, \bar{\theta})\)] has been obtained after the application of GIR that has been quoted in (21).

At this juncture, we are in the position to exploit the gauge invariant condition (7) for the purpose of the application of AVSA to BRST formalism. In other words, we have the following GIR in the language of differential forms, namely:

\[ \tilde{A}_{(h)}^{(3)} \equiv \frac{1}{m} \tilde{d} \tilde{\Phi}^{(2)} \equiv A^{(3)} \equiv \frac{1}{m} d \tilde{\Phi}^{(2)}, \] (26)

where the expansion for \(\tilde{A}_{(h)}^{(3)}\) has been quoted in (20) and all the superfields with the superscript \((h)\) have been written in our Appendix A. The expression for the super derivative \(\tilde{d}\) has been quoted in (8) and the explicit form of the super 2-form is as follows

\[ \tilde{\Phi}^{(2)} = \frac{1}{2!} (d Z^M \wedge d Z^N) \tilde{\Phi}_{MN} \equiv \frac{1}{2!} (d x^\mu \wedge d x^\nu) \tilde{\Phi}_{\mu \nu}(x, \theta, \bar{\theta}) + (d x^\mu \wedge d \theta) \tilde{\Phi}_{\mu \theta}(x, \theta, \bar{\theta}) \]
\[ + (d x^\mu \wedge d \bar{\theta}) \tilde{\Phi}_{\mu \bar{\theta}}(x, \theta, \bar{\theta}) + (d \theta \wedge d \bar{\theta}) \tilde{\Phi}_{\theta \bar{\theta}}(x, \theta, \bar{\theta}) + \frac{1}{2!} (d \theta \wedge d \bar{\theta}) \tilde{\Phi}_{\theta \bar{\theta}}(x, \theta, \bar{\theta}), \] (27)

where we identify the fermionic as well as bosonic superfields on the r.h.s. as\footnote{As pointed out earlier, the presence of the Grassmannian components of the superfields decides the nature of the corresponding ordinary fields and their ghost numbers.}

\[ \tilde{\Phi}_{\mu \theta} = \tilde{F}_\mu(x, \theta, \bar{\theta}), \quad \tilde{\Phi}_{\mu \bar{\theta}} = \tilde{F}_\mu(x, \theta, \bar{\theta}), \quad \tilde{\Phi}_{\theta \bar{\theta}} = \tilde{\Phi}(x, \theta, \bar{\theta}), \]
\[ \frac{1}{2!} \tilde{\Phi}_{\theta \bar{\theta}} = \tilde{\beta}(x, \theta, \bar{\theta}), \quad \frac{1}{2!} \tilde{\Phi}_{\theta \bar{\theta}} = \tilde{\bar{\beta}}(x, \theta, \bar{\theta}). \] (28)

It is evident that the \textit{fermionic} superfields \((\tilde{F}_\mu, \tilde{\Phi}_\mu)\) have the ghost numbers \((+1, -1)\) and the \textit{bosonic} superfields \((\tilde{\beta}, \tilde{\bar{\beta}})\) are characterized by the ghost numbers \((+2, -2)\), respectively. On the contrary, the scalar superfield \(\tilde{\Phi}(x, \theta, \bar{\theta})\) has the \textit{zero} ghost number. It is
straightforward to note that, we have the following
\[ + \frac{1}{m} \bar{d} \Phi^{(2)} \equiv + \frac{1}{m} \left[ \frac{1}{3!} \left( d x^\mu \wedge d x^\nu \wedge d x^\lambda \right) \{ \partial_\mu \bar{\Phi}_{\nu \lambda} + \partial_\nu \bar{\Phi}_{\lambda \mu} + \partial_\lambda \bar{\Phi}_{\mu \nu} \} ight. \]
\[ + \frac{1}{2!} \left( d x^\mu \wedge d x^\nu \wedge d \bar{\theta} \right) \{ \partial_\mu \bar{\Phi}_{\nu \bar{\theta}} + (\partial_\nu \bar{\Phi}_{\bar{\theta} \bar{\theta}} - \partial_{\bar{\theta}} \bar{\Phi}_{\nu \bar{\theta}}) \} \]
\[ + \frac{1}{2!} \left( d x^\mu \wedge d x^\nu \wedge d \theta \right) \{ \partial_\mu \bar{\Phi}_{\nu \theta} + (\partial_\nu \bar{\Phi}_{\theta \theta} - \partial_\theta \bar{\Phi}_{\nu \theta}) \} \]
\[ + (d \theta \wedge d \bar{\theta} \wedge d \theta) \partial_\theta \bar{\beta} + (d x^\mu \wedge d \theta \wedge d \bar{\theta}) \{ \partial_\mu \bar{\Phi} + \partial_{\bar{\theta}} \bar{F}_\mu + \partial_{\theta} \bar{F}_\mu \} \]
\[ + (d x^\mu \wedge d \bar{\theta} \wedge d \theta) \{ \partial_\mu \bar{\beta} + \partial_{\bar{\theta}} \bar{F}_\mu \} + (d x^\mu \wedge d \theta \wedge d \bar{\theta}) \{ \partial_\mu \bar{\beta} + \partial_{\bar{\theta}} \bar{F}_\mu \} \]
\[ + (d \bar{\theta} \wedge d \theta \wedge d \bar{\theta}) \partial_{\bar{\theta}} \bar{\beta} \]  

(29)

We would like to point out that, using the explicit expressions of (20) and (29), we can compute the l.h.s. of (28) and equate it with the r.h.s.

A close and careful look at (26) demonstrates that all the super 3-differentials with the Grassmannian variables on the l.h.s. will not have their counterparts on the r.h.s. As a consequence, their coefficients will be equal to zero. This requirement leads to the following

\[ F^{(h)}_2 \equiv \frac{1}{m} \partial_\theta \bar{\beta} = 0, \quad F^{(h)}_2 \equiv \frac{1}{m} \partial_{\bar{\theta}} \bar{\beta} = 0, \]
\[ F^{(h)}_1 \equiv \frac{1}{m} (\partial_\theta \bar{\beta} + \partial_{\bar{\theta}} \bar{\Phi}) = 0, \quad F^{(h)}_1 \equiv \frac{1}{m} (\partial_\theta \bar{\beta} + \partial_{\bar{\theta}} \bar{\Phi}) = 0, \]
\[ \bar{\beta}^{(h)} \equiv \frac{1}{m} (\partial_\mu \bar{\beta} + \partial_\nu \bar{\Phi}_\mu) = 0, \quad \bar{\beta}^{(h)} \equiv \frac{1}{m} (\partial_\mu \bar{\beta} + \partial_\nu \bar{\Phi}_\mu) = 0, \]
\[ \bar{\Phi}^{(h)}_\mu \equiv \frac{1}{m} (\partial_\mu \bar{\Phi} + \partial_{\bar{\theta}} \bar{F}_\mu + \partial_{\theta} \bar{F}_\mu) = 0, \quad \bar{F}^{(h)}_\mu \equiv \frac{1}{m} (\partial_\theta \bar{\Phi}_\mu + \partial_{\bar{\theta}} \bar{F}_\mu - \partial_\nu \bar{F}_\mu = 0), \]
\[ F^{(h)}_{\mu \nu} \equiv \frac{1}{m} (\partial_\theta \bar{\Phi}_{\mu \nu} + \partial_{\bar{\theta}} \bar{F}_{\mu \nu} - \partial_\nu \bar{F}_{\mu \nu}) = 0. \]  

(30)

It goes without saying that only the coefficient of \( (d x^\mu \wedge d x^\nu \wedge d x^\lambda) \) will be, finally, equated from the l.h.s. and r.h.s. It has been already found that both the sides are equal provided we substitute the expressions for the secondary fields\(^{3}\) in the super expansions of the following superfields which are present in (27) and (28), namely:

\[ \bar{\beta} (x, \theta, \bar{\theta}) = \beta + \theta \bar{g}_1 + \bar{\theta} g_1 + i \theta \bar{\theta} h_1, \]
\[ \bar{\beta} (x, \theta, \bar{\theta}) = \beta + \theta \bar{g}_2 + \bar{\theta} g_2 + i \theta \bar{\theta} h_2, \]
\[ \bar{\Phi} (x, \theta, \bar{\theta}) = \phi + \theta \bar{k} + \bar{\theta} k + i \theta \bar{\theta} h, \]
\[ \bar{F}_\mu (x, \theta, \bar{\theta}) = C_\mu + \theta \bar{p}_\mu^{(1)} + \bar{\theta} p_\mu^{(1)} + i \theta \bar{\theta} q_\mu, \]
\[ \bar{F}_\mu (x, \theta, \bar{\theta}) = \bar{C}_\mu + \theta \bar{p}_\mu^{(2)} + \bar{\theta} p_\mu^{(2)} + i \theta \bar{\theta} q_\mu, \]
\[ \bar{\Phi}_{\mu \nu} (x, \theta, \bar{\theta}) = \bar{\Phi}_{\mu \nu} + \theta R_{\mu \nu} + \bar{\theta} R_{\mu \nu} + i \theta \bar{\theta} S_{\mu \nu}. \]

\(^{3}\)It is gratifying to state that we have derived the exact expressions for all the secondary fields in terms of the basic and auxiliary fields of our theory and some of the (non-)trivial CF-type restrictions in equations (32), (34), (35) and (38) of our present endeavor.
where the fermionic \((\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0)\) nature of the Grassmannian variables \((\theta, \bar{\theta})\) ensures that the set of secondary fields \((k, k, g_1, g_2, q_\mu, \bar{q}_\mu, R_{\mu\nu}, \bar{R}_{\mu\nu})\) are fermionic in nature and the fields \((h, h_1, h_2, p_\mu^{(1)}, \bar{p}_\mu^{(1)}, p_\mu^{(2)}, \bar{p}_\mu^{(2)}, S_{\mu\nu})\) form a bosonic set of secondary fields which are to be determined in terms of the basic and auxiliary fields of our theory. In the above context, it can be noted that the first four entries in equation (30) lead to the following expressions for the secondary fields

\[
g_1 = \pm m C_2, \quad g_2 = \pm m \bar{C}_2, \quad h_1 = \pm i m B, \quad h_2 = \mp i m B_2
\]

\[
k + \bar{g}_1 = \pm m C_1, \quad g_2 + \bar{k} = \pm m \bar{C}_1, \quad h = \pm i m B_1,
\]

which demonstrate that we have obtained the explicit expressions for some of the secondary fields in terms of the auxiliary and basic fields of our theory. Furthermore, with identifications: \(g_2 = F, \quad \bar{k} = \bar{f}, \quad k = f, \quad \bar{g}_1 = \bar{F},\) we have also derived the new set of CF-type restrictions: \(f + \bar{F} = \pm m C_1\) and \(\bar{f} + F = \pm m \bar{C}_1\) which are over and above the CF-type restrictions (14). Ultimately, we have derived the super expansions of the three bosonic superfields [cf. Eqs (23), (28)] which are the generalization of the (anti-)ghost fields \((\beta, \bar{\beta})\) with ghost numbers \((+2, -2)\) and the scalar field \(\phi (x)\) with ghost number zero. These super expansions can be explicitly written as

\[
\bar{\beta}^{(g)}(x, \theta, \bar{\theta}) = \beta + \theta (\bar{F}) + \bar{\theta} (\pm m C_2) + \theta \bar{\theta} (\mp m B)
\]

\[
\equiv \beta + \theta (s_{ab} \beta) + \bar{\theta} (s_b \bar{\beta}) + \theta \bar{\theta} (s_b s_{ab} \bar{\beta}),
\]

\[
\bar{\beta}^{(g)}(x, \theta, \bar{\theta}) = \beta + \theta (\pm m \bar{C}_2) + \bar{\theta} (F) + \theta \bar{\theta} (\pm m B_2)
\]

\[
\equiv \beta + \theta (s_{ab} \bar{\beta}) + \bar{\theta} (s_b \bar{\beta}) + \theta \bar{\theta} (s_b s_{ab} \bar{\beta}),
\]

\[
\bar{\phi}^{(g)}(x, \theta, \bar{\theta}) = \phi + \theta (\bar{f}) + \bar{\theta} (f) + \theta \bar{\theta} (\mp m B_1)
\]

\[
\equiv \phi + \theta (s_{ab} \phi) + \bar{\theta} (s_b \phi) + \theta \bar{\theta} (s_b s_{ab} \phi),
\]

where the superscript \((g)\) on the superfields denotes the fact that the superfields on the l.h.s. have been derived after the application of the GIR in (26). It goes without saying that we have already obtained the (anti-)BRST symmetry transformations:

\[
s_b \beta = \pm m C_2, \quad s_b \phi = f, \quad s_b f = 0, \quad s_b \bar{\beta} = \bar{F}, \quad s_b \bar{F} = \mp m B_1,
\]

\[
s_{ab} \beta = \bar{F}, \quad s_{ab} \phi = \bar{f}, \quad s_{ab} \bar{f} = 0, \quad s_{ab} \bar{\beta} = \pm m \bar{C}_2,
\]

\[
s_{ab} \bar{F} = 0, \quad s_{ab} \bar{C}_2 = 0, \quad s_{ab} F = \mp m B_2, \quad s_{ab} f = \pm m B_1,
\]

where the off-shell nilpotency property has been taken into account for the (anti-)BRST symmetry transformations for some of the fields. In addition, we have derived \(s_b \bar{F} = \mp m B\) and \(s_{ab} F = \mp m B_2\) from the requirement of the (anti-)BRST invariance of the CF-type restrictions: \(f + \bar{F} = \pm m C_1\) and \(\bar{f} + F = \pm m \bar{C}_1\). As a side remarks, it is interesting to point out that the CF-type restriction: \(f + \bar{F} = \pm m C_1\) has been derived from equating the coefficient of the super differential \((d\theta \wedge d\bar{\theta} \wedge d\bar{\theta})\) equal to zero and similar exercise has been performed with the super differential \((d\theta \wedge d\theta \wedge d\bar{\theta})\) which leads to the derivation of the CF-type restriction \(\bar{f} + F = \pm m \bar{C}_1\). At this stage, we equate the coefficient of \((dx^\mu \wedge d\theta \wedge d\theta), (dx^\mu \wedge d\bar{\theta} \wedge d\bar{\theta})\) and \((dx^\mu \wedge d\theta \wedge d\bar{\theta})\) equal to zero. These amount to taking
into account the fifth, sixth and seventh entries of equation (30) which lead to the following expression for some of the secondary fields and a beautiful relationship:

\[
\begin{align*}
\tilde{p}^{(1)}_{\mu} &= \pm m \tilde{\beta}_{\mu} - \partial_{\mu} \tilde{\beta}, \\
\tilde{p}^{(2)}_{\mu} &= \pm m \phi_{\mu} - \partial_{\mu} \phi, \\
\tilde{q}_{\mu} &= -i \left[ \pm m F_{\mu} - \partial_{\mu} F \right], \\
q_{\mu} &= +i \left[ \pm m \tilde{F}_{\mu} - \partial_{\mu} \tilde{F} \right], \\
\tilde{p}^{(1)}_{\mu} + \tilde{p}^{(2)}_{\mu} &= \pm m \phi_{\mu} - \partial_{\mu} \phi \implies \tilde{B}_{\mu} + B_{\mu} = \pm m \phi_{\mu} - \partial_{\mu} \phi.
\end{align*}
\] (35)

In the above, we have identified \( p^{(1)}_{\mu} = \tilde{B}_{\mu} \) and \( p^{(2)}_{\mu} = B_{\mu} \) which lead to the derivation of a new CF-type restriction \( \tilde{B}_{\mu} + B_{\mu} = \pm m \phi_{\mu} - \partial_{\mu} \phi \). This has been obtained from the condition that the coefficient of the super differential \( (dx^\mu \wedge d\theta \wedge d\bar{\theta}) \) should be set equal to zero. Ultimately, we obtain the following super expansions

\[
\begin{align*}
\tilde{F}^{(g)}_{\mu} (x, \theta, \bar{\theta}) &= C_{\mu} + \theta \tilde{B}_{\mu} + \tilde{\theta} [\pm m \beta_{\mu} - \partial_{\mu} \beta] + \theta \tilde{\theta} [\mp m \tilde{F}_{\mu} - \partial_{\mu} \tilde{F}], \\
\tilde{F}^{(g)}_{\mu} (x, \theta, \bar{\theta}) &= \tilde{C}_{\mu} + \tilde{\theta} [\pm m \tilde{\beta}_{\mu} - \partial_{\mu} \tilde{\beta}] + \tilde{\theta} \tilde{B}_{\mu} + \theta \tilde{\theta} [\mp m \tilde{F}_{\mu} - \partial_{\mu} \tilde{F}],
\end{align*}
\] (36)

which show that we have already derived: \( s_0 C_{\mu} = \pm m \beta_{\mu} - \partial_{\mu} \beta \), \( s_0 (\pm m \beta_{\mu} - \partial_{\mu} \beta) = 0 \), \( s_0 \tilde{C}_{\mu} = B_{\mu} \), \( s_0 B_{\mu} = 0 \), \( s_{ab} C_{\mu} = \bar{B}_{\mu} \), \( s_{ab} B_{\mu} = 0 \), \( s_{ab} \tilde{C}_{\mu} = \pm m \tilde{\beta}_{\mu} - \partial_{\mu} \tilde{\beta}, s_{ab} (\pm m \tilde{\beta}_{\mu} - \partial_{\mu} \tilde{\beta}) = 0 \) where the property of the off-shell nilpotency has been taken into account.

The stage is now set to implement the last two conditions of (30). These are

\[
\begin{align*}
F_{\mu}^{(h)} &= \frac{1}{m}[\partial_{\theta} \tilde{F}_{\mu}^{(g)} + \partial_{\mu} \tilde{F}^{(g)}_{\nu} - \partial_{\nu} \tilde{F}_{\mu}^{(g)}] = 0, \\
F_{\mu}^{(h)} &= \frac{1}{m}[\partial_{\theta} \tilde{F}_{\mu}^{(g)} + \partial_{\mu} \tilde{F}^{(g)}_{\nu} - \partial_{\nu} \tilde{F}_{\mu}^{(g)}] = 0,
\end{align*}
\] (37)

where the explicit super expansions for \( F_{\mu}^{(h)} \) and \( F_{\mu}^{(h)} \) are listed in Appendix A and (36) has to be taken into account for the explicit expansions of \( \tilde{F}^{(g)}_{\mu} \) and \( \tilde{F}^{(g)}_{\mu} \). These substitutions in (37) lead to the following

\[
\begin{align*}
R_{\mu\nu} &= \pm m C_{\mu\nu} - (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} C_{\mu}), \\
S_{\mu\nu} &= -i \left[ \pm m B_{\mu\nu} - (\partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}) \right], \\
R_{\mu\nu} &= \pm m C_{\mu\nu} - (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} C_{\mu}), \\
S_{\mu\nu} &= +i \left[ \pm m \bar{B}_{\mu\nu} - (\partial_{\mu} \bar{B}_{\nu} - \partial_{\nu} \bar{B}_{\mu}) \right],
\end{align*}
\] (38)

where both the expression for \( S_{\mu\nu} \) [cf. Eq. (38)] are equal provided we take into account the CF-type restrictions: \( B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_{\mu} \phi_{\nu} - \partial_{\nu} \phi_{\mu} \) and \( B_{\mu\nu} + \bar{B}_{\mu\nu} = \pm m \phi_{\mu} - \partial_{\mu} \phi \). We are now in the position to equate the coefficients of \( (dx^\mu \wedge d\theta \wedge d\bar{\theta}) \) from the l.h.s. and r.h.s. in the GIR that is quoted in (26). This can be explicitly expressed as

\[
\begin{align*}
\tilde{A}^{(h)}_{\mu\lambda} (x, \theta, \bar{\theta}) &= \frac{1}{m} \left[ \partial_{\theta} \tilde{F}^{(g)}_{\nu\lambda} (x, \theta, \bar{\theta}) + \partial_{\nu} \tilde{F}^{(g)}_{\mu\lambda} (x, \theta, \bar{\theta}) + \partial_{\lambda} \tilde{F}^{(g)}_{\mu\nu} (x, \theta, \bar{\theta}) \right] \\
&= A_{\mu\nu\lambda} (x) \mp \frac{1}{m} \left[ \partial_{\mu} \Phi_{\nu\lambda} (x) + \partial_{\nu} \Phi_{\lambda\mu} (x) + \partial_{\lambda} \Phi_{\mu\nu} (x) \right],
\end{align*}
\] (39)

where the precise expression for \( \tilde{A}^{(h)}_{\mu\nu\lambda} (x, \theta, \bar{\theta}) \) is quoted in Appendix A [cf. Eq. (A.16)] and the explicit super expansion of \( \tilde{F}^{(g)}_{\mu\nu} (x, \theta, \bar{\theta}) \) is as follows:

\[
\begin{align*}
\tilde{F}^{(g)}_{\mu\nu} (x, \theta, \bar{\theta}) &= \Phi_{\mu\nu} (x) + \theta \left[ \pm m C_{\mu\nu} - (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} C_{\mu}) \right] \\
&+ \bar{\theta} \left[ \pm m C_{\mu\nu} - (\partial_{\mu} C_{\nu} - \partial_{\nu} \bar{C}_{\mu}) \right] + \theta \bar{\theta} \left[ \pm m B_{\mu\nu} - (\partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}) \right] \\
&\equiv \Phi_{\mu\nu} (x) + \theta \left[ \pm m C_{\mu\nu} - (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} C_{\mu}) \right] \\
&+ \bar{\theta} \left[ \pm m C_{\mu\nu} - (\partial_{\mu} C_{\nu} - \partial_{\nu} \bar{C}_{\mu}) \right] + \theta \bar{\theta} \left[ \mp m \bar{B}_{\mu\nu} - (\partial_{\mu} \bar{B}_{\nu} - \partial_{\nu} \bar{B}_{\mu}) \right].
\end{align*}
\] (40)
In the above, we have taken into account the inputs from (38). It is straightforward to note that (39) is satisfied in a precise manner. In other words, we find that the coefficient of \( \vartheta, \bar{\vartheta} \) and \( \vartheta \bar{\theta} \) of the l.h.s. are equal to zero so that only the spacetime dependent ordinary terms exist which sum up to produce the r.h.s. Thus, we have computed all the off-shell nilpotent (anti-)BRST symmetry transformations for our modified Abelian 3-form gauge theory where the mass and (anti-)BRST symmetry transformations co-exist together.

4 Coupled Lagrangian Densities: (Anti-)BRST Symmetry Transformations and CF-Type Restrictions

In this section, we explicitly write down the coupled (but equivalent) Lagrangian densities for our present massive gauge theory by exploiting the standard techniques of BRST formalism. In particular, we demonstrate that the Lagrangian density \( \mathcal{L}^{(T)}_B \) respects perfect BRST symmetry transformations as does \( \mathcal{L}^{(T)}_{\bar{B}} \) w.r.t. the anti-BRST symmetry transformations. The CF-type restrictions are also derived from the equations of motion that emerge out from \( \mathcal{L}^{(T)}_B \) and \( \mathcal{L}^{(T)}_{\bar{B}} \). For the purpose of BRST-quantization scheme, we have to incorporate the gauge-fixing and Faddeev-Popov (FP) ghost terms into the total BRST and anti-BRST invariant Lagrangian densities \( \mathcal{L}^{(T)}_B \) and \( \mathcal{L}^{(T)}_{\bar{B}} \) which are coupled but equivalent as far as the (anti-)BRST symmetry transformations are concerned [cf. Sec. 5]. The gauge-fixing and FP-ghost terms can be constructed from the basic fields that appear on the r.h.s. of (12) [i.e. \( A_{\mu \nu \lambda}, \bar{C}_{\mu \nu}, C_{\mu \nu}, \bar{\beta}_\mu, \beta_\mu, \phi_\mu, \bar{C}_2, C_2, \bar{C}_1, C_1 \)] as well as on the r.h.s. of (31) [i.e. \( \Phi_\mu, \bar{C}_\mu, C_\mu, \bar{\beta}, \beta, \phi \)]. Thus, our present section is divided into two parts. In subsection 4.1, we consider the gauge-fixing and FP-ghost terms for the massless Abelian 3-form theory where HC plays an important role. In subsection 4.2, we deal with the gauge-fixing and FP-ghost terms corresponding to the Stückelberg compensating field and associated fermionic and bosonic (anti-)ghost fields where GIR plays an important and decisive role. Finally, we write down the total gauge-fixing and FP-ghost terms for our theory. We end this section by demonstrating that \( \mathcal{L}^{(T)}_B \) and \( \mathcal{L}^{(T)}_{\bar{B}} \) (i.e. the coupled Lagrangian densities) respect perfect BRST and anti-BRST symmetry transformations, respectively.

4.1 Gauge-Fixing and FP-Ghost Terms for Massless Abelian 3-Form Theory: Basic Fields Present in HC

In our earlier work [17, 18], for the massless Abelian 3-form gauge theory, we have the following as the gauge-fixing and FP-ghost terms, namely;

\[
\begin{align*}
  s_b s_{ab} \left[ & \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_\mu C^\mu - \frac{1}{2} \phi_\mu \phi^\mu + \bar{\beta}_\mu \beta_\mu - \frac{1}{6} A_{\mu \nu \lambda} A^{\mu \nu \lambda} \right], \\
  & - s_{ab} s_b \left[ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_\mu C^\mu - \frac{1}{2} \phi_\mu \phi^\mu + \bar{\beta}_\mu \beta_\mu - \frac{1}{6} A_{\mu \nu \lambda} A^{\mu \nu \lambda} \right],
\end{align*}
\]

The above expressions lead to the Lagrangian densities [17, 18] which are a part of the perfectly BRST and anti-BRST invariant coupled (but equivalent) Lagrangian densities \( \mathcal{L}^{(HC)}_B = \mathcal{L}^{(B, H)}_{gf} + \mathcal{L}^{(B, H)}_{FP} \) and \( \mathcal{L}^{(HC)}_{\bar{B}} = \mathcal{L}^{(\bar{B}, H)}_{gf} + \mathcal{L}^{(\bar{B}, H)}_{FP} \), respectively. These Lagrangian
densities, in their explicit forms, are as follows:

\[ \mathcal{L}_{gf}^{(B,H)} + \mathcal{L}_{FP}^{(B,H)} = (\partial_\mu A^{\mu\nu\lambda})B_{\nu\lambda} + \frac{1}{2} B_{\mu\nu} B^{\mu\nu} + (\partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\lambda\mu} + \partial_\lambda \bar{C}_{\mu\nu})(\partial^\mu C^{\nu\lambda}) \]

\[ - (\partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu)(\partial^\mu \beta^\nu) - B B_2 - \frac{1}{2} B_1^2 + (\partial_\mu \bar{C}^{\mu\nu})(f_\nu - (\partial_\mu C^{\nu\lambda})(\partial^\nu C^{\mu\lambda}) - \bar{\beta}_\nu \bar{C}_{\lambda\mu} + \partial_\lambda \bar{C}_{\mu\nu})(\partial^\mu \beta^\nu) \]

\[ - \partial_\mu \bar{C}_2 \partial^\mu C_2 + f_\mu f_\nu - \bar{F}^\mu F_\mu - (\partial \cdot \beta) B_2 - (\partial \cdot \phi) B_1 + (\partial \cdot \bar{\beta}) B. \quad (42) \]

We would like to comment on the choice of the combination of the basic fields that have been incorporated into the square brackets of (41). We note that every term of the square bracket has the mass dimension \( [M]^{(D-2)} \) individually in the natural units \( (\hbar = c = 1) \). Furthermore, the ghost number for all the individual term is zero and all the individual terms are Lorentz scalars as is required by the basic tenets of the construction of the proper coupled (but equivalent) Lagrangian densities of a theory. The sign of the individual term has been chosen judiciously so that the EL-EoMs w.r.t. the auxiliary fields from (42) yield the Curci-Ferrari restrictions: \( B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\nu \phi_\nu - \partial_\nu \phi_\mu, \ f_\mu + \bar{F}_\mu = \partial_\mu C_1 \) and \( \bar{f}_\mu + F_\mu = \partial^\mu \bar{C}_1 \). In addition, we lay emphasis on the fact that the basic fields, appearing in the square bracket (41), are those that have appeared in the definition of \( \bar{A}^{(3)} \) [cf. Eqs (10) and (12)]. Using the CF-type restrictions: \( B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\nu \phi_\nu - \partial_\nu \phi_\mu, \ f_\mu + \bar{F}_\mu = \partial_\mu C_1 \), and \( \bar{f}_\mu + F_\mu = \partial^\mu \bar{C}_1 \), we write down the sum of the gauge-fixing and FP-ghost terms, in the massless sector of the Abelian 3-form theory, as:

\[ \mathcal{L}_B^{(HC)} = \mathcal{L}_{gf}^{(B,H)} + \mathcal{L}_{FP}^{(B,H)} = (\partial_\mu A^{\mu\nu\lambda})B_{\nu\lambda} - \frac{1}{2} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} \bar{B}_{\mu\nu} \bar{B}^{\mu\nu} + \frac{1}{2} \bar{B}^{\mu\nu} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) - B B_2 \]

\[ - \frac{1}{2} B_1^2 + (\partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\lambda\mu} + \partial_\lambda \bar{C}_{\mu\nu})(\partial^\mu C^{\nu\lambda}) - (\partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu)(\partial^\mu \beta^\nu) \]

\[ - \partial_\mu \bar{C}_2 \partial^\mu C_2 - (\partial \cdot \beta) B_2 - (\partial \cdot \phi) B_1 + (\partial \cdot \bar{\beta}) B - 2 F_\mu f_\nu + (\partial_\mu \bar{C}^{\mu\nu} - \partial^\mu C_1)(\partial^\nu C_1) F_\nu. \quad (43) \]

\[ \mathcal{L}_B^{(HC)} = \mathcal{L}_{gf}^{(B,H)} + \mathcal{L}_{FP}^{(B,H)} = - (\partial_\mu A^{\mu\nu\lambda})B_{\nu\lambda} - \frac{1}{2} \bar{B}_{\mu\nu} \bar{B}^{\mu\nu} + \frac{1}{2} \bar{B}^{\mu\nu} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) - B B_2 \]

\[ - \frac{1}{2} B_1^2 + (\partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\lambda\mu} + \partial_\lambda \bar{C}_{\mu\nu})(\partial^\mu C^{\nu\lambda}) - (\partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu)(\partial^\mu \beta^\nu) \]

\[ - \partial_\mu \bar{C}_2 \partial^\mu C_2 - (\partial \cdot \beta) B_2 - (\partial \cdot \phi) B_1 + (\partial \cdot \bar{\beta}) B + 2 \bar{F}^\mu \bar{f}_\mu + \bar{C}^{\mu\nu} - \partial^\mu C_1)(\partial^\nu C_1) \bar{F}_\nu. \quad (44) \]

where superscript \((HC)\) on the Lagrangian densities on the l.h.s. denotes the sum of gauge-fixing and Faddeev-Popov ghost terms corresponding to the HC that is valid for the massless Abelian 3-form gauge theory. The superscript \((H)\), on the r.h.s., carries forward
this information. It should be recalled that the terms such as \((-2F^\mu f_\mu)\) and \((+2\tilde{F}^\mu \tilde{f}_\mu)\) have ghost number equal to zero because ghost numbers for \((\tilde{F}_\mu, F_\mu)\) are equal to \((+1, -1)\), respectively. Similarly the ghost number is zero for the terms: \((\partial_\mu \tilde{C}^{\mu\nu} + \partial^\nu \tilde{C}_1) \tilde{F}_\nu\) and 
\(-[(\partial_\mu C^{\mu\nu} + \partial^\nu C_1) F_\nu].\) A close look at \(\mathcal{L}_B^{(HC)}\) and \(\mathcal{L}_B^{(HC)}\) demonstrate that \(\mathcal{L}_B^{(HC)}\) is expressed in terms of the auxiliary fields \((B_\mu, f_\mu, F_\mu)\). On the other hand, the Lagrangian density \(\mathcal{L}_B^{(HC)}\) is expressed in terms of \((\tilde{B}_\mu, \tilde{f}_\mu, \tilde{F}_\mu)\). It is straightforward to note that the following EL-EoMs emerge out from \(\mathcal{L}_B^{(HC)}\) and \(\mathcal{L}_B^{(HC)}\) w.r.t. the set of auxiliary fields \((B_\mu, f_\mu, F_\mu)\) and their counterpart set of auxiliary fields \((\tilde{B}_\mu, \tilde{f}_\mu, \tilde{F}_\mu)\); namely;
\[
B_\mu = (\partial^\lambda A_{\lambda\mu}) + \frac{1}{2}(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu), \quad 2F_\mu = \partial^\nu \tilde{C}_{\nu\mu} + \partial_\mu \tilde{C}_1, \\
\tilde{B}_\mu = -(\partial^\lambda A^{\lambda\mu}) + \frac{1}{2}(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu), \quad 2\tilde{f}_\mu = -\partial^\nu \tilde{C}_{\nu\mu} + \partial_\mu \tilde{C}_1, \\
2f_\mu = \partial^\nu C_{\nu\mu} + \partial_\mu C_1, \quad 2\tilde{F}_\mu = -\partial^\nu C_{\nu\mu} + \partial_\mu C_1.
\]

It is clear from the above equations of motion that we have obtained the CF-type restrictions: \(B_\mu + \tilde{B}_\mu = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \)
\(f_\mu + \tilde{f}_\mu = \partial_\mu C_1\) and \(\tilde{f}_\mu + F_\mu = \partial_\mu \tilde{C}_1\). It should be recalled that these (anti-)BRST invariant restrictions have emerged out in our theory because of the HC (i.e. \(\tilde{A}^{(3)} = dA^{(3)}\)) which amounts to the gauge invariance \((\delta_\mu H^{\mu\lambda\nu} = 0)\) of the field-strength tensor. Hence, the choice of the combinations of the fields in Eq. (41) are correct which lead to the derivation of the gauge-fixing and FP-ghost terms for the free massless Abelian 3-form gauge theory.

### 4.2 Gauge-Fixing and FP-Ghost Terms for the St"uckelberg-Modified Massive Abelian 3-Form Theory: GIR

In this subsection, we focus on the construction of the gauge-fixing and FP-ghost terms for the basic fields that are present in GIR [cf. Eq. (26)] which are explicitly written on the r.h.s. of (31) as the first terms (i.e. \(\Phi_\mu, \tilde{C}_\mu, C_\mu, \tilde{\beta}, \beta, \phi\)) in all the super expansions. The analogue of (41) can be written for the massive sector (i.e. the St"uckelberg antisymmetric field \((\Phi_\mu = \Phi_{\nu\mu})\) and associated gauge-fixing and FP-ghost terms) as follows
\[
\mathcal{L}_B^{(GIR)} \equiv \mathcal{L}_{gG}^{(B,G)} + \mathcal{L}_{FP}^{(B,G)} = s_b s_{ab} \left[ -\frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \tilde{\beta} \beta - \frac{1}{2} \phi^2 - \frac{1}{2} \tilde{C}_\mu C^\mu \right],
\]
\[
\mathcal{L}_B^{(GIR)} \equiv \mathcal{L}_{gG}^{(B,G)} + \mathcal{L}_{FP}^{(B,G)} = -s_{ab} s_b \left[ -\frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \tilde{\beta} \beta - \frac{1}{2} \phi^2 - \frac{1}{2} \tilde{C}_\mu C^\mu \right],
\]
where the superscript \((GIR)\) on the l.h.s. denotes the importance of GIR in our discussions. The superscript \((G)\) on the gauge-fixing and FP-ghost terms of the Lagrangian densities, on the r. h. s., encodes this information. The explicit computations of the r.h.s. in terms of the (anti-)BRST transformations \([s_{(a)b)}\) [cf. Eqs. (B.1), (B.2)] lead to the following
\[
\mathcal{L}_B^{(GIR)} = \mathcal{L}_{gG}^{(B,G)} + \mathcal{L}_{FP}^{(B,G)} \equiv -(\partial_\mu \Phi^{\mu\nu}) B_\nu \mp \frac{m}{2} B^{\mu\nu} \Phi_\mu \mp \frac{m^2}{2} \tilde{C}_\mu C^{\mu\nu} + (\partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu) \partial^\mu C_\nu \pm m (\partial_\mu \tilde{C}_{\mu\nu}) C_\nu \pm 2 m \tilde{C}_\nu (\partial^\mu C_{\mu\nu})
\]
where the top bosonic
above Lagrangian density (\(\mathcal{L}_B^{(T)}\))[cf. Appendix B].
On the contrary, its counterpart gauge-fixing and FP-ghost terms which are a part of the perfectly anti-BRST Lagrangian density \(\mathcal{L}_B^{(T)^*}\)[cf. Appendix B] are:

\[
\mathcal{L}_B^{(GIR)} = \mathcal{L}_{gf}^{(B,G)} + \mathcal{L}_{FP}^{(B,G)} \equiv (\partial_{\mu} \Phi^{\mu\nu}) B_{\nu} + \frac{m}{2} B_{\mu\nu} \Phi^{\mu\nu} + \frac{m^2}{2} C_{\mu\nu} C^{\mu\nu} \\
+ (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} C_{\mu}) \phi \phi + m \phi \phi + m \bar{C}_{\phi} (\partial_{\mu} C_{\mu}) + \frac{1}{2} \bar{B}_{\mu\nu} \dot{B}_{\mu\nu} + \frac{m}{2} \dot{m} \phi \phi \\
+ \frac{1}{2} \bar{f} (\partial_{\mu} C_{\nu}) \phi \phi + \frac{1}{2} \bar{C}_{\phi} \dot{F}_{\phi} + m \bar{C}_{\phi} \phi \phi + \bar{f} f + \bar{F} F \\
- m^2 C_{\phi} C_{\phi} + m \bar{B}_{\phi} \phi \phi + m \bar{B}_{\phi} \phi \phi.
\]

(47)

which is a part of the perfectly BRST invariant Lagrangian density \(\mathcal{L}_B^{(T)}\)[cf. Appendix B].

At this stage, we take the help of the CF-type restrictions: \(B_{\mu} + \bar{B}_{\mu} = \pm m \phi \phi - \partial_{\mu} \phi \phi, f + \bar{F} = \pm m C_{1}\) and \(\bar{f} + F = \pm m C_{1}\) to recast some of the useful and characteristic terms of the above Lagrangian density \((\mathcal{L}_{gf} + \mathcal{L}_{FP})\) as follows

\[
\frac{1}{2} B_{\mu\nu} \dot{B}_{\nu} = \frac{1}{2} B_{\mu} \left[ - \bar{B}_{\mu} \pm m \phi \phi - \partial_{\mu} \phi \phi \right] \equiv \frac{1}{2} B_{\mu} B_{\mu} + \frac{1}{2} B_{\mu} \left[ \pm m \phi \phi - \partial_{\mu} \phi \phi \right], \\
\bar{f} f - \bar{F} F = -2 F f \pm m F C_{1} \pm m \bar{C}_{1} f, \\
\frac{1}{2} B_{\mu\nu} \dot{B}_{\nu} = \frac{1}{2} \left[ - \bar{B}_{\mu} \pm m \phi \phi - \partial_{\mu} \phi \phi \right] \dot{B}_{\mu} \equiv \frac{1}{2} \bar{B}_{\mu} \dot{B}_{\mu} + \frac{1}{2} \bar{B}_{\mu} \left[ \pm m \phi \phi - \partial_{\mu} \phi \phi \right], \\
\bar{f} f - \bar{F} F = 2 \bar{F} \bar{f} \pm m C_{1} \bar{f} \pm m \bar{C}_{1} \bar{F}.
\]

(49)

where the top two entries are for \(\mathcal{L}_B^{(GIR)}\) and bottom two are for the Lagrangian density \(\mathcal{L}_B^{(GIR)}\) (cf. Sec. 5 below). It will be noted that \(\mathcal{L}_B^{(GIR)}\) has been expressed in terms of the auxiliary fields \((B_{\mu}, f, F)\) and \(\mathcal{L}_B^{(GIR)}\) has been written in terms of the set \((\bar{B}_{\mu}, \bar{f}, \bar{F})\) of auxiliary fields (cf. Sec. 5 for details).

We are in the position now to express the total gauge-fixing and FP-ghost terms which are constructed from the basic and auxiliary fields that are present in the HC and GIR. These terms are nothing but the sum of (44) and (48) as well as the sum total of (43) and (47) as (see, e.g., Appendix B for details):

\[
\mathcal{L}_{gf}^{(B)} + \mathcal{L}_{FP}^{(B)} \equiv \mathcal{L}_B^{(HC)} + \mathcal{L}_B^{(GIR)}, \\
\mathcal{L}_{gf}^{(\bar{B})} + \mathcal{L}_{FP}^{(\bar{B})} \equiv \mathcal{L}_B^{(HC)^*} + \mathcal{L}_B^{(GIR)^*}.
\]

(50)

The equations of motions w.r.t. the bosonic auxiliary fields \(B_{\mu\nu}\) and \(\bar{B}_{\mu\nu}\) that emerge out from (50) (see, e.g., Appendix B for details) are as follows:

\[
\frac{\partial}{\partial B_{\mu\nu}} \left[ \mathcal{L}_B^{(HC)} + \mathcal{L}_B^{(GIR)} \right] = (\partial^{\lambda} A_{\lambda\mu\nu}) + \frac{1}{2} (\partial_{\lambda} \phi_{\nu} - \partial_{\nu} \phi_{\lambda}) \pm \frac{m}{2} \Phi_{\mu\nu} - B_{\mu\nu} = 0, \\
\frac{\partial}{\partial B_{\mu\nu}} \left[ \mathcal{L}_B^{(HC)} + \mathcal{L}_B^{(GIR)} \right] = - (\partial^{\lambda} A_{\lambda\mu\nu}) + \frac{1}{2} (\partial_{\lambda} \phi_{\nu} - \partial_{\nu} \phi_{\lambda}) \pm \frac{m}{2} \Phi_{\mu\nu} - \bar{B}_{\mu\nu} = 0.
\]

(51)
These are modified versions of (45). However, they lead to the Curic-Ferrari type restrictions: $B_{\mu \nu} + \bar{B}_{\mu \nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$ as mentioned earlier. Similarly, we note that we have the following equations of motion w.r.t. the fermionic auxiliary fields:

\[
\frac{\partial}{\partial f_\mu} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = - (\partial_\nu \bar{C}^{\nu \mu} + \partial^{\mu} \bar{C}_1) + 2 F^\mu \pm \frac{m}{2} \bar{C}^\mu = 0 \\
\quad \implies \quad 2 F^\mu = (\partial_\nu \bar{C}^{\nu \mu} + \partial^{\mu} \bar{C}_1) \mp \frac{m}{2} \bar{C}^\mu,
\]

\[
\frac{\partial}{\partial F_\mu} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = (\partial_\nu C^{\nu \mu} - \partial^{\mu} C_1) + 2 \bar{f}^\mu \mp \frac{m}{2} \bar{C}^\mu = 0 \\
\quad \implies \quad 2 \bar{f}^\mu = - (\partial_\nu C^{\nu \mu} + \partial^{\mu} C_1) \pm \frac{m}{2} C^\mu,
\]

\[
\frac{\partial}{\partial \bar{f}_\mu} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = (\partial_\nu \bar{C}^{\nu \mu} + \partial^{\mu} \bar{C}_1) \mp \frac{m}{2} C^\mu = 0 \\
\quad \implies \quad 2 \bar{F}^\mu = - (\partial_\nu C^{\nu \mu} + \partial^{\mu} C_1) \pm \frac{m}{2} C^\mu. \tag{52}
\]

A close and careful observations of (52) imply the following:

\[
f_\mu + \bar{F}_\mu = \partial_\mu C_1, \quad \bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1. \tag{53}
\]

Thus, we observe that (i) the combination of the fields chosen in (46) and their ghost numbers as well as the mass dimension in natural units, and (ii) the numerical factors and their signs, etc., are perfectly alright as they lead to the derivation of the (anti-)BRST invariant CF-type restrictions for the EL-EoMs w.r.t. the auxiliary fields.

Finally, before we end this section, we observe the following explicit EoMs w.r.t. the fermionic auxiliary fields from the Lagrangian densities

\[
\frac{\partial}{\partial f} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = 0 \quad \implies \quad 2 F = \pm m \bar{C}_1 - \frac{1}{2} (\partial \cdot \bar{C}),
\]

\[
\frac{\partial}{\partial F} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = 0 \quad \implies \quad 2 f = \pm m C_1 - \frac{1}{2} (\partial \cdot C),
\]

\[
\frac{\partial}{\partial \bar{f}} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = 0 \quad \implies \quad 2 \bar{F} = \pm m \bar{C}_1 + \frac{1}{2} (\partial \cdot \bar{C}),
\]

\[
\frac{\partial}{\partial \bar{F}} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = 0 \quad \implies \quad 2 \bar{f} = \pm m C_1 + \frac{1}{2} (\partial \cdot C). \tag{54}
\]

It is straightforward to note that the above equations are crucial for the derivation of the CF-type restrictions: $f + \bar{F} = \pm m C_1$ and $\bar{f} + F = \pm m \bar{C}_1$. Last but not least, we are in

\[\text{The equations of motion (51), (52), (54) and (55) can be derived easily from the total Lagrangian densities that have been written in (B.5), (B.6), (B.8) and (B.9) (see Appendix B for details).}\]
position to find out the EL-EoMs w.r.t. \( B_\mu \) and \( \bar{B}_\mu \) fields as:

\[
\frac{\partial}{\partial \bar{B}_\mu} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = \left( \partial_\nu \Phi^{\nu\mu} \right) - \bar{B}^\mu \pm \frac{m^2}{2} \phi^\mu - \frac{1}{2} \partial^\mu \phi = 0,
\]

\[
\frac{\partial}{\partial B_\mu} \left[ L_B^{(HC)} + L_B^{(GIR)} \right] = - \left( \partial_\nu \Phi^{\nu\mu} \right) - B^\mu \pm \frac{m^2}{2} \phi^\mu - \frac{1}{2} \partial^\mu \phi = 0.
\]  

The above equation implies the following

\[
B_\mu + \bar{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi, \tag{56}
\]

which is nothing but the CF-type restriction. To sum up, we have derived all the CF-type restrictions of our theory by taking into account the proper forms of the gauge-fixing and FP-ghost terms for our theory. The standard techniques and tricks of the BRST formalism have been at the heart of our derivations of the (anti-)BRST invariant CF-type restrictions which are the hallmark of a BRST-quantized gauge theory [10, 11]. It is very interesting to point out that, from the mathematical structure of (41) and (46), it is clear that \( L_B^{(T)} \) and \( L_B^{(T)} \) will be BRST and anti-BRST invariant because of the off-shell nilpotency \( [s_{a(b)}^2 = 0] \) of the (anti-)BRST symmetry transformations \( [s_{a(b)}] \). This is due to the observation that \( s_{a(b)} L_{(S)} = 0 \) where \( L_{(S)} \) [cf. Eq. (4)] is the classically gauge invariant Stückelberg-modified Lagrangian density. We elaborate all these statements very clearly in the next section.

5  Symmetry Invariance and CF-Type Restrictions

Our present section is devoted to the thread-bare analysis of the BRST and anti-BRST symmetry invariance of the Lagrangian densities \( L_B^{(HC)} \) and \( L_B^{(HC)} \) (cf. Appendix B) which are coupled (but equivalent). We demonstrate that the “equivalence” of these Lagrangian densities is due to the existence of a set of CF-type restrictions on our theory [cf. Eq. (69) below]. First of all, we note that the Stückelberg-modified Lagrangian density \( L_{(S)} \), common in the coupled (but equivalent) total Lagrangian densities \( L_B^{(T)} \) and \( L_B^{(T)} \), remains (anti-)BRST invariant [i.e. \( s_{a(b)} L_{(S)} = 0 \)]. Now we focus on the gauge-fixing and FP-ghost terms of \( L_B^{(T)} \) and \( L_B^{(T)} \), respectively, and study their (anti-)BRST invariance(s). To achieve this goal, our present section is divided into three parts. In subsection 5.1, we concentrate on the Lagrangian density \( L_B^{(HC)} \) [cf. Eq. (43)] and Lagrangian density \( L_B^{(HC)} \) [cf. Eq. (44)] and focus on their (anti-)BRST invariance(s). Our subsection 5.2 is devoted to the discussion on the (anti-)BRST invariance of Lagrangian density \( L_B^{(GIR)} \) [cf. Eq. (47)] and Lagrangian density \( L_B^{(GIR)} \) [cf. Eq. (48)]. Finally, in subsection 5.3, we discuss and comment on some of the key aspects of the (anti-)BRST invariance of the total Lagrangian densities \( L_B^{(T)} \) and \( L_B^{(T)} \) of our present modified version of the massive Abelian 3-form theory.

5.1  (Anti-)BRST Invariance: \( L_B^{(HC)} \) and \( L_B^{(HC)} \)

First of all, we apply the BRST symmetry transformations \( s_b \) [cf. Eq. (B.1)] on the Lagrangian density \( L_B^{(HC)} \) [cf. Eq. (43)] which leads to following total spacetime derivative:
\[s_b \mathcal{L}^{(HC)}_B = \partial_\mu \left[ (\partial^\mu C^{\nu\lambda} + \partial^\nu C^{\lambda\mu} + \partial^\lambda C^{\mu\nu}) B_{\nu\lambda} + B^{\mu\nu} f_{\nu} - B_2 \partial^\mu C_2 \right]
- B_1 f^\mu + B F^\mu - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) F_{\nu}. \tag{57}\]

Similarly, the application of the anti-BRST symmetry transformations \(s_{ab}\) [cf. Eq. (B.2)] on the Lagrangian densities \(\mathcal{L}^{(HC)}_B\) [cf. Eq. (44)] yields the following:
\[s_{ab} \mathcal{L}^{(HC)}_B = \partial_\mu \left[ \tilde{B}^{\mu\nu} \tilde{f}_\nu - (\partial^\mu \tilde{C}^{\nu\lambda} + \partial^\nu \tilde{C}^{\lambda\mu} + \partial^\lambda \tilde{C}^{\mu\nu}) \tilde{B}_{\nu\lambda} + B \partial^\mu \bar{C}_2 \right]
- B_2 \tilde{F}^\mu - B_1 \tilde{f}^\mu - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) \tilde{F}_\nu. \tag{58}\]

It is clear from our observations in (57) and (58) that \(\mathcal{L}^{(HC)}_B\) and \(\mathcal{L}^{(HC)}_B\) respect perfect BRST and anti-BRST symmetry transformations because they transform to the total spacetime derivative where no EL-EoMs and/or CF-type restrictions are invoked for their validity.

To be precise, we have demonstrated that the Lagrangian density \(\mathcal{L}^{(HC)}_B\) is perfectly anti-BRST invariant because the action integral \(S_1 = \int d^Dx \mathcal{L}^{(HC)}_B\) remains invariant \[i.e. \ s_{ab} S_1 = 0\] due to the Gauss divergence theorem where all the physical fields vanish-off as \(x \to \pm \infty\). In exactly similar fashion, we have shown that \(\mathcal{L}^{(HC)}_B\) transforms under the BRST symmetry transformations \((s_b)\) such that the action integral \(S_2 = \int d^Dx \mathcal{L}^{(HC)}_B\) remains invariant \([s_b S_2 = 0]\) under the BRST transformations because of the Gauss divergence theorem. The latter symmetry is also a perfect symmetry because we have not imposed any kind of external conditions (e.g. EL-EoMs and/or CF-type restrictions) for its proof. Let us take this opportunity to clarify that we define a perfect symmetry is the one under which the action integral remains invariant without any kind of outside restrictions.

Within the realm of the symmetry considerations, we wish to establish the existence of the (anti-)BRST invariant CF-type restrictions: \(B_{\mu\nu} + \tilde{B}_{\nu\mu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \ f_{\nu} + \tilde{f}_\nu = \partial_\mu C_1\) and \(\bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1\) [cf. Eq. (14)]. Towards this goal in mind, first of all, we apply the continuous and nilpotent BRST symmetry transformations \((s_b)\) on the perfectly anti-BRST invariant Lagrangian density \([\mathcal{L}^{(HC)}_B]\) which yields the following:
\[s_b \mathcal{L}^{(HC)}_B = \partial_\mu \left[ -(\partial^\mu C^{\nu\lambda} + \partial^\nu C^{\lambda\mu} + \partial^\lambda C^{\mu\nu}) B_{\nu\lambda} - B^{\mu\nu} \tilde{F}_\nu - B_2 \partial^\mu C_2 + C^{\mu\nu} \partial_\nu B_1 \right]
+ 2 A^{\mu\nu} \partial_\nu \tilde{F}_\lambda - B_1 f^\mu + B F^\mu + (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \tilde{F}_\nu
- \left[ \tilde{f}^\mu + F^\mu - \partial^\mu \bar{C}_1 \right] (\partial_\mu B) + \frac{1}{2} \left[ B^{\mu\nu} + \tilde{B}^{\mu\nu} - (\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) \right] (\partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_\mu)
- (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \partial_\mu \tilde{F}_\nu + \partial_\nu \tilde{C}_1 + [f^\mu + \tilde{F}^\mu - \partial^\mu C_1] (\partial_\mu B_1)
+ (\partial^\mu C^{\nu\lambda} + \partial^\nu C^{\lambda\mu} + \partial^\lambda C^{\mu\nu}) [B_{\nu\lambda} + B_{\nu\lambda} - (\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu)]. \tag{59}\]

It is self-evident that, if we impose the (anti-)BRST invariant CF-type restrictions from outside, the perfectly anti-BRST invariant part [cf. Eq. (58)] of the Lagrangian density \([\mathcal{L}^{(HC)}_B]\) respects the BRST symmetry, too. In fact, if we invoke the validity of the CF-type
restrictions: $B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$, $f_\mu + \bar{F}_\mu = \partial_\mu C_1$ and $\bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1$, we obtain the following explicit transformation for the Lagrangian density $L_B^{(HC)}$:

$$s_b L_B^{(HC)} = \partial_\mu \left[ - (\partial^\mu C^{\nu\lambda} + \partial^\nu C^{\lambda\mu} + \partial^\lambda C^{\mu\nu}) B_{\nu\lambda} - B^{\mu\nu} \bar{F}_\nu - B_2 \partial^\mu C_2 + C^{\mu\nu} \partial_\nu B_1 \right]$$

$$+ 2 A^{\mu\nu} \partial_\nu \bar{F}_\lambda - B_1 f^\mu + B F^\mu + (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{f}_\nu - \bar{C}^{\mu\nu} \partial_\nu B_1]. \quad (60)$$

In exactly similar fashion, when we apply the anti-BRST symmetry transformations ($s_{ab}$) on the perfectly BRST invariant [cf. Eq. (57)] Lagrangian density $L_B^{(HC)}$, we obtain the following explicit transformation for the latter under $s_{ab}$ [cf. (B.2)]:

$$s_{ab} L_B^{(HC)} = \partial_\mu \left[ (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\lambda\mu} + \partial^\lambda \bar{C}^{\mu\nu}) B_{\nu\lambda} - B^{\mu\nu} F_\nu + B \partial^\mu \bar{C}_2 - C^{\mu\nu} \partial_\nu B_2 \right]$$

$$- 2 A^{\mu\nu} \partial_\nu F_\lambda - \bar{f}^\mu B_1 - \bar{F}^\mu B_2 + (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{f}_\nu - \bar{C}^{\mu\nu} \partial_\nu B_1 \right]$$

$$+ \frac{1}{2} B^{\mu\nu} \partial_\mu [f_\nu + \bar{F}_\nu - \partial_\nu C_1] + \frac{1}{2} \left[ B^{\mu\nu} + \bar{B}^{\mu\nu} - (\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) \right] \left( \partial_\mu F_\nu - \partial_\nu F_\mu \right)$$

$$- (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \partial_\mu [f_\nu + \bar{F}_\nu - \partial_\nu C_1] + [f_\mu + \bar{F}_\mu - \partial_\nu C_1] (\partial_\mu B_2)$$

$$- (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\lambda\mu} + \partial^\lambda \bar{C}^{\mu\nu}) \partial_\mu \left[ B_{\nu\lambda} + \bar{B}_{\nu\lambda} - (\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu) \right]. \quad (61)$$

A careful and close look at the above equation demonstrates that $s_{ab} L_B^{(HC)}$ is also a total spacetime derivative like Eq. (60) provided we invoke the validity of the CF-type restrictions: $B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$, $f_\mu + \bar{F}_\mu = \partial_\mu C_1$ and $\bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1$ which imply:

$$s_{ab} L_B^{(HC)} = \partial_\mu \left[ (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\lambda\mu} + \partial^\lambda \bar{C}^{\mu\nu}) B_{\nu\lambda} - B^{\mu\nu} F_\nu + B \partial^\mu \bar{C}_2 - C^{\mu\nu} \partial_\nu B_2 \right]$$

$$- 2 A^{\mu\nu} \partial_\nu F_\lambda - \bar{f}^\mu B_1 - \bar{F}^\mu B_2 + (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{f}_\nu - \bar{C}^{\mu\nu} \partial_\nu B_1 \right]. \quad (62)$$

We lay emphasis on the fact that both the Lagrangian densities $L_B^{(HC)}$ and $L_B^{(HC)}$ are equivalent w.r.t. the (anti-)BRST symmetry transformations as both of them respect both of these off-shell nilpotent symmetry transformations on a submanifold of Hilbert space of quantum fields where the CF-type restrictions (14) are satisfied. On this submanifold, the BRST and anti-BRST symmetry transformations have their own identities as they absolutely anticommute with each-other. In addition, it is worthwhile to point out that $L_B^{(HC)}$ and $L_B^{(HC)}$ respect on their own perfect BRST and anti-BRST symmetries, respectively. However, the application of the anti-BRST symmetry transformations on $L_B^{(HC)}$ and, in exactly similar fashion, the application of the BRST symmetry transformation on $L_B^{(HC)}$ lead to the existence of the (anti-)BRST invariant CF-type restrictions for their equivalence.

We end this subsection with the remark that, in the derivation of (59) and (61), we have used, at some places, the standard theoretical trick that the summation of the symmetric and antisymmetric indices turns out to be zero. This has led to the derivation of the CF-type restrictions on the r.h.s. on (59) and (61). As far as the symmetry considerations are concerned, this is an alternative way to show the existence of the (anti-)BRST invariant CF-type restrictions on our theory besides the requirement of the absolute anticommutativity (i.e. $s_b s_{ab} + s_{ab} s_b = 0$) of the (anti-)BRST symmetry transformations [cf. Appendix B].
5.2 (Anti-)BRST Invariance: $\mathcal{L}_{B}^{(GIR)}$ and $\mathcal{L}_{\bar{B}}^{(GIR)}$

In this section, first of all, we show the (anti-)BRST invariance(s) of $\mathcal{L}_{B}^{(GIR)}$ and $\mathcal{L}_{\bar{B}}^{(GIR)}$, respectively. For this purpose, by using the bottom two lines of (49), the Lagrangian density $\mathcal{L}_{B}^{(GIR)}$ [cf. Eq. (48)] can be re-expressed as follows:

$$
\mathcal{L}_{B}^{(GIR)} = \mathcal{L}_{gf}^{(B)} + \mathcal{L}_{FP}^{(B)} \equiv (\partial_{\mu} \Phi^{\mu}) \bar{B}_{\nu} \pm \frac{m}{2} \bar{B}^{\mu \nu} \Phi_{\mu \nu} + \frac{m^{2}}{2} \bar{C}_{\mu \nu} \Phi^{\mu \nu} \\
+ (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} \bar{C}_{\mu}) (\partial^{\mu} \bar{C}^{\nu}) \pm m (\partial_{\mu} \bar{C}^{\mu \nu}) C_{\nu} \pm m \bar{C}^{\nu} (\partial^{\mu} \bar{C}_{\mu \nu}) \\
- \frac{1}{2} ( \pm m \bar{\beta}^{\mu} - \partial_{\nu} \bar{\beta} ) ( \pm m \beta_{\nu} - \partial_{\mu} \beta ) - \frac{1}{2} \bar{B}^{\mu} \bar{B}_{\mu} + \frac{1}{2} \bar{B}_{\mu} ( \pm m \phi_{\mu} - \partial_{\mu} \phi ) \\
+ \frac{1}{2} \bar{f} ( \partial \cdot \bar{C} ) \pm \frac{m}{2} \bar{C}_{\mu} \bar{f}^{\mu} + \frac{1}{2} ( \partial \cdot \bar{C} ) \bar{F} \pm m B_{\mu} \phi \pm \frac{m}{2} \bar{f}^{\mu} C_{\mu} \\
- m^{2} \bar{C}_{2} C_{2} \mp m B \bar{\beta} \pm m B_{2} \beta \pm 2 \bar{F} \bar{f} \mp m \bar{f} C_{1} \mp m \bar{F} C_{1}.
$$

(63)

It should be noted, at this crucial juncture, that the above Lagrangian density has been expressed in terms of the auxiliary fields ($\bar{B}_{\mu}, \bar{F}, \bar{f}$) of (48) because the auxiliary fields ($B_{\mu}, F, f$) have been replaced by using the CF-type restrictions: $B_{\mu} + \bar{B}_{\mu} = \pm m \phi_{\mu} - \partial_{\mu} \phi, \bar{f} + \bar{F} = \pm m \bar{C}_{1}$ and $f + F = \pm m C_{1}$. We are now in the position to apply the anti-BRST symmetry transformations ($s_{ab}$) of Appendix B [cf. Eq. (B.2)] on the above Lagrangian density which leads to the following explicit expression

$$
s_{ab} \mathcal{L}_{B}^{(GIR)} = \partial_{\mu} \left[ - (\partial^{\mu} \bar{C}^{\nu} - \partial^{\nu} \bar{C}^{\mu}) \bar{B}_{\nu} \mp m \bar{B}^{\mu \nu} \bar{C}_{\nu} - \frac{1}{2} \bar{B}^{\mu} \bar{f} \\
\pm m \bar{C}^{\nu} (\pm m \bar{\beta}_{\nu} - \partial_{\mu} \bar{\beta}) \pm m (\partial^{\mu} \bar{\beta}^{\nu} - \partial^{\nu} \bar{\beta}^{\mu}) C_{\nu} \\
+ \frac{1}{2} ( \pm m \bar{\beta}^{\mu} - \partial_{\nu} \bar{\beta} ) \bar{C}_{\nu} \right].
$$

(64)

As a consequence, it is clear that the action integral corresponding to $\mathcal{L}_{B}^{(GIR)}$ remains invariant under the anti-BRST symmetry transformations [cf. Eq. (B.2)] due to Gauss’s divergence theorem because of the fact that all the physical field vanish-off as $x \to \pm \infty$.

Against the backdrop of our discussions related with $\mathcal{L}_{B}^{(GIR)}$ and its anti-BRST invariance, we concentrate on the Lagrangian density $\mathcal{L}_{\bar{B}}^{(GIR)}$ [cf. Eq. (47)]. Using the top two equations of (49) as inputs, we obtain the following form of the Lagrangian density $\mathcal{L}_{\bar{B}}^{(GIR)}$:

$$
\mathcal{L}_{\bar{B}}^{(GIR)} = \mathcal{L}_{gf}^{\bar{B}} + \mathcal{L}_{FP}^{\bar{B}} \equiv (\partial_{\mu} \Phi^{\mu}) \bar{B}_{\nu} \mp \frac{m}{2} \bar{B}^{\mu \nu} \Phi_{\mu \nu} + \frac{m^{2}}{2} \bar{C}_{\mu \nu} \Phi^{\mu \nu} \\
+ (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} \bar{C}_{\mu}) (\partial^{\mu} \bar{C}^{\nu}) \pm m (\partial_{\mu} \bar{C}^{\mu \nu}) C_{\nu} \pm m \bar{C}^{\nu} (\partial^{\mu} \bar{C}_{\mu \nu}) \\
- \frac{1}{2} ( \pm m \beta^{\mu} - \partial_{\nu} \beta ) ( \pm m \beta_{\nu} - \partial_{\mu} \beta ) - \frac{1}{2} \bar{B}^{\mu} \bar{B}_{\mu} + \frac{1}{2} \bar{B}^{\mu} ( \pm m \phi_{\mu} - \partial_{\mu} \phi ) \\
- \frac{1}{2} F ( \partial \cdot \bar{C} ) \mp \frac{m}{2} \bar{C}_{\mu} f^{\mu} - \frac{1}{2} ( \partial \cdot \bar{C} ) f \mp m B_{\mu} \phi \mp \frac{m}{2} \bar{f}^{\mu} C_{\mu} - 2 \bar{F} f \\
\pm m F C_{1} \mp m f \bar{C}_{1} - m^{2} \bar{C}_{2} C_{2} \mp m B \bar{\beta} \pm m B_{2} \beta.
$$

(65)

It is to be noted that the above Lagrangian density is a function of the set of auxiliary fields ($B_{\mu}, F, f$) because of the fact that auxiliary fields ($\bar{B}_{\mu}, \bar{F}, \bar{f}$) of (47) have been replaced
by \((B_\mu, F, f)\) using the CF-type restrictions: \(B_\mu + \bar{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi, \ f + \bar{F} = \pm m C_1\) and \(f + F = \pm m C_1\). This is the key difference between (47) and (65). The stage is now set for the application of the BRST symmetry transformations \((s_b)\) of the Appendix B [cf. Eq. (B.1)] on the Lagrangian density (65). This operation yields the following:

\[
s_b \mathcal{L}_B^{(GIR)} = \partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu \pm m B^{\mu\nu} C_\nu \mp \frac{1}{2} B_\mu f \right. \\
\left. \pm m C^{\mu\nu} (\pm m \beta_\nu - \partial_\nu \beta) \mp m (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{C}_\nu \\
\pm \frac{1}{2} (\pm m \beta^\mu - \partial^\mu \beta) F \right].
\]

(66)

As a consequence, we observe that the action integral \((S)\), corresponding to the Lagrangian density \(\mathcal{L}_B^{(GIR)}\), remains invariant \((s_b S = 0)\) under the BRST symmetry transformations \((s_b)\) that have been quoted in our Appendix B.

Within the purview of the symmetry considerations, we can show the existence of the CF-type restrictions on our theory by proving the Lagrangian densities \(\mathcal{L}_B^{(GIR)}\) and \(\mathcal{L}_B^{(GIR)}\) to be equivalent w.r.t. the nilpotent (anti-)BRST symmetry transformations. In other words, for this purpose, we have to apply \((i)\) the anti-BRST transformation on the perfectly BRST invariant [cf. Eqs (63), (64)] Lagrangian density \(\mathcal{L}_B^{(GIR)}\), and \((ii)\) the BRST transformations on the perfectly anti-BRST invariant [cf. Eqs. (65), (66)] Lagrangian density \(\mathcal{L}_B^{(GIR)}\). In this connection, first of all, we apply the BRST symmetry transformations [cf. Eq. (B.1)] on the Lagrangian density \(\mathcal{L}_B^{(GIR)}\) [cf. Eq. (65)] which leads to the following:

\[
s_b \mathcal{L}_B^{(GIR)} = \partial_\mu \left[ \pm m C^{\mu\nu} B_\nu \mp m (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{C}_\nu \mp m \bar{C}^{\mu\nu} (\pm m \beta_\nu - \partial_\nu \beta) \right. \\
\left. \pm m B^{\mu\nu} C_\nu + \Phi^{\mu\nu} (\pm m \bar{f}_\nu - \partial_\nu f) + \{ \pm m C^{\mu\nu} - (\partial^\mu C^\nu - \partial^\nu C^\mu) \} \bar{B}_\nu \\
\pm \frac{m}{2} C^\mu B_1 + \frac{1}{2} B^\mu \bar{F} - \frac{1}{2} \{ \pm m \beta^\mu - \partial^\mu \beta \} \bar{f} \mp \frac{m}{2} B \bar{C}^\mu \\
\pm \frac{m}{2} [B^{\mu\nu} + \bar{B}^{\mu\nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)] \{ (\partial^\mu C^\nu - \partial^\nu C^\mu) \mp m C^{\mu\nu} \} \\
\pm m \Phi^{\mu\nu} \partial_\mu \left( f_\nu + \bar{F}_\nu - \partial_\nu C_1 \right) \mp m B \left( \bar{f} + F \mp m \bar{C}_1 \right) \pm m B_1 \left( f + \bar{F} \mp m C_1 \right) \\
\left. - \left[ \pm m C^{\mu\nu} - (\partial^\mu C^\nu - \partial^\nu C^\mu) \right] \partial_\mu [B_\nu + \bar{B}_\nu \mp m \phi_\nu + \partial_\nu \phi] \right. \\
\left. - \frac{1}{2} \left( \pm m \beta^\mu - \partial^\mu \beta \right) \left[ \pm m (\bar{f}_\mu + F_\mu - \partial_\mu C_1) - \partial_\mu (\bar{f} + F \mp m C_1) \right] \\
\left. + \frac{1}{2} B^\mu \left[ \pm m \{ \bar{F}_\mu + f_\mu - \partial_\mu C_1 \} - \partial_\mu \{ \bar{F} + f \mp m C_1 \} \right]. \right.
\]

(67)

Thus, on the submanifold of the Hilbert space of quantum fields [defined by (anti-)BRST invariant CF-type restrictions], we note that \(\mathcal{L}_B^{(GIR)}\) also respects [cf. Eq. (67)] the BRST symmetry transformations [cf. Eq. (B.1)] thereby showing the equivalence of \(\mathcal{L}_B^{(GIR)}\) with \(\mathcal{L}_B^{(GIR)}\) w.r.t. the (anti-)BRST symmetry transformations of our theory.

To establish the equivalence of \(\mathcal{L}_B^{(GIR)}\) with \(\mathcal{L}_B^{(GIR)}\) and existence of the (anti-)BRST invariant CF-type type restrictions, we now apply the anti-BRST symmetry transformations
[cf. Eq. (B.2)] on $\mathcal{L}_B^{(GIR)}$ which leads to the following observation, namely;

\begin{align}
\mathcal{L}_B^{(GIR)} &= \partial_{\mu} \left[ \pm m (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) C_\nu \mp m \bar{C}^{\mu\nu} B_\nu \pm m C^{\mu\nu} (\pm m \bar{\beta}^\nu - \partial^\nu \bar{\beta}) \\
& \mp m \bar{B}^{\mu\nu} \bar{C}_\nu - \Phi^{\mu\nu} (\pm m \bar{f}_\nu - \partial^\nu \bar{f}) - \{ \pm m \bar{C}^{\mu\nu} - (\partial^\mu \bar{C}^{\nu} - \partial^\nu \bar{C}^{\mu}) \} B_\nu \\
& \pm \frac{m}{2} \bar{C}^{\mu} B_2 + \frac{1}{2} \bar{B}^{\mu} F - \frac{1}{2} \{ \pm m \bar{\beta}^\mu - \partial^\mu \bar{\beta} \} f \pm \frac{m}{2} \bar{C}^{\mu} B_1 \\
& \pm m B_2 [f + \bar{F} \mp m C_1] \pm m B_1 [\bar{f} + F \mp m \bar{C}_1] \\
& \pm \frac{m}{2} [ (\partial^\mu \bar{C}^{\nu} - \partial^\nu \bar{C}^{\mu}) \mp m \bar{C}^{\mu\nu} ] (B_{\mu\nu} + \bar{B}_{\mu\nu} - (\partial_{\mu} \phi_{\nu} - \partial_{\nu} \phi_{\mu} ) ] \\
& + \{ \pm m \bar{C}^{\mu\nu} - (\partial^\mu \bar{C}^{\nu} - \partial^\nu \bar{C}^{\mu}) \} \partial_{\mu} [B_{\nu} + \bar{B}_{\nu} \mp m \phi_{\nu} + \partial_{\nu} \phi ] \\
& + \frac{1}{2} \bar{B}^{\mu} [ \pm m (\bar{f}_{\mu} + F_{\mu} - \partial_{\mu} \bar{C}_1) - \partial_{\mu} (F + \bar{F} \mp m \bar{C}_1) ] \\
& - \frac{1}{2} \{ \pm m \bar{\beta}^\mu - \partial^\mu \bar{\beta} \} \pm m (f_{\mu} + \bar{F}_{\mu} - \partial_{\mu} C_1) - \partial_{\mu} (f + \bar{F} \mp m C_1) ] \\
& - \frac{1}{2} (\pm m \bar{f}^\mu - \partial^\mu \bar{f} ) [B_{\mu} + \bar{B}_{\mu} \mp m \phi_{\mu} + \partial_{\mu} \phi ] \\
& \pm m \Phi^{\mu\nu} (\partial_{\mu} [f_{\nu} + F_{\nu} - \partial_{\nu} C_1]).
\end{align}

(68)

The noteworthy point at this stage is the fact that all the six CF-type restrictions have appeared on the r.h.s. of the above transformation and (67). If we invoke the validity of these (anti-)BRST CF-type restrictions, namely;

\begin{align}
B_{\mu\nu} + \bar{B}_{\mu\nu} - (\partial_{\mu} \phi_{\nu} - \partial_{\nu} \phi_{\mu} ) = 0, & \quad f_{\mu} + \bar{F}_{\mu} - \partial_{\mu} C_1 = 0, \\
\bar{f}_{\mu} + F_{\mu} - \partial_{\mu} \bar{C}_1 = 0, & \quad f + \bar{F} \mp m C_1 = 0, \\
B_{\mu} + \bar{B}_{\mu} \mp m \phi_{\mu} + \partial_{\mu} \phi = 0, & \quad \bar{f} + F \mp m \bar{C}_1 = 0,
\end{align}

(69)

we observe that, under the anti-BRST symmetry transformations [cf. Eq. (B.2)], the perfectly BRST invariant Lagrangian density $\mathcal{L}_B^{(GIR)}$ transforms to:

\begin{align}
\mathcal{L}_B^{(GIR)} &= \partial_{\mu} \left[ \pm m (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) C_\nu \mp m \bar{C}^{\mu\nu} B_\nu \pm m C^{\mu\nu} (\pm m \bar{\beta}^\nu - \partial^\nu \bar{\beta}) \\
& \mp m \bar{B}^{\mu\nu} \bar{C}_\nu - \Phi^{\mu\nu} (\pm m \bar{f}_\nu - \partial^\nu \bar{f}) - \{ \pm m \bar{C}^{\mu\nu} - (\partial^\mu \bar{C}^{\nu} - \partial^\nu \bar{C}^{\mu}) \} B_\nu \\
& \pm \frac{m}{2} \bar{C}^{\mu} B_2 + \frac{1}{2} \bar{B}^{\mu} F - \frac{1}{2} \{ \pm m \bar{\beta}^\mu - \partial^\mu \bar{\beta} \} f \pm \frac{m}{2} \bar{C}^{\mu} B_1.
\end{align}

(70)

Thus, we note that, on a specific submanifold of the Hilbert space of quantum fields, the coupled Lagrangian density $\mathcal{L}_B^{(GIR)}$ and $\mathcal{L}_B^{(GIR)}$ are equivalent w.r.t. the (anti-)BRST symmetry transformations [cf. Eqs. (B.1), (B.2)]. The above specific submanifold is defined in terms of the field equations (67) corresponding to the (anti-)BRST invariant CF-type restrictions where the BRST and anti-BRST symmetry transformations absolutely anticommute ($s_{s_{ab}} + s_{ab} s_b = 0$) with each-other. Exactly similar kind of statement can be made for our observation in (67) where the imposition of (69) on the r.h.s. leads to the observation that $\mathcal{L}_B^{(GIR)}$ transforms to a total spacetime derivative under the off-shell nilpotent BRST symmetry transformations $s_b$ of our present massive Abelian 3-form theory.
5.3 (Anti-)BRST Invariance: $\mathcal{L}^{(T)}_B$ and $\mathcal{L}^{(T)}_\bar{B}$

In this subsection, we have to take into account the total BRST and anti-BRST symmetry transformations of Appendix B and apply them on the total Lagrangian densities of our theory which incorporate the St"uckelberg-modified Lagrangian density $\mathcal{L}_<(S)$ [cf. Eq. (4)] and total gauge-fixing and FP-ghost terms together. In this connection, we note that the application of the continuous and infinitesimal BRST symmetry transformations (B.1) and (B.2), respectively, for the Lagrangian densities (B.1) and (B.2), respectively, for the action integrals $S$ is found to be (anti-)BRST invariant (i.e. $S \xrightarrow{\text{BRST}} S$ and $S \xrightarrow{\text{anti-BRST}} S$). Thus, we have to apply the anti-BRST symmetry transformations (B.2) on $S$ to obtain the total BRST and anti-BRST symmetry transformations (B.1) for the physical fields which vanish-off as $x \to \pm \infty$. In exactly similar fashion, we observe that the application of the infinitesimal and continuous anti-BRST symmetry transformations (B.2) on $S$ leads to the following total spacetime derivative:

$$s_{ab} \mathcal{L}^{(T)}_B = \frac{\partial}{\partial t} \left[ \mathcal{B}_{ab}^T \mathcal{F}_{\nu} - (\partial^\mu \mathcal{F}^\nu + \partial^\nu \mathcal{F}^\mu + \partial^\mu \mathcal{F}^\nu) \mathcal{B}_{\nu\alpha} + B \partial^\mu \mathcal{C}_2 - B_1 \mathcal{F}^\nu - (\partial^\mu \mathcal{B}_x - \partial^\nu \mathcal{B}_x) \mathcal{B}_{\nu\alpha} - \frac{1}{2} (\pm m \mathcal{B}_x - \partial^\mu \mathcal{B}_x) \mathcal{F}_x \right] + m \mathcal{C}^{\mu\nu} (\pm m \mathcal{B}_x - \partial^\mu \mathcal{B}_x) \mathcal{C}_x \right].$$  

which demonstrates that the action integral $S_1 = \int d^D x \mathcal{L}^{(T)}_B$ remains invariant under the BRST symmetry transformations (B.1) for the physical fields which vanish-off as $x \to \pm \infty$. In exactly similar fashion, we observe that the application of the infinitesimal and continuous anti-BRST symmetry transformations (B.2) on $S_2 = \int d^D x \mathcal{L}^{(T)}_\bar{B}$ leads to the following total spacetime derivative:

$$s_{ab} \mathcal{L}^{(T)}_\bar{B} = \frac{\partial}{\partial t} \left[ \mathcal{B}_{ab}^{\bar{T}} \mathcal{F}_{\nu} - (\partial^\mu \mathcal{F}^\nu + \partial^\nu \mathcal{F}^\mu + \partial^\mu \mathcal{F}^\nu) \mathcal{B}_{\nu\alpha} + B \partial^\mu \mathcal{C}_2 - B_1 \mathcal{F}^\nu - (\partial^\mu \mathcal{B}_x - \partial^\nu \mathcal{B}_x) \mathcal{B}_{\nu\alpha} - \frac{1}{2} (\pm m \mathcal{B}_x - \partial^\mu \mathcal{B}_x) \mathcal{F}_x \right] + m \mathcal{C}^{\mu\nu} (\pm m \mathcal{B}_x - \partial^\mu \mathcal{B}_x) \mathcal{C}_x \right].$$

The above observation establishes the fact that the action integrals $S_1 = \int d^D x \mathcal{L}^{(T)}_B$ and $S_2 = \int d^D x \mathcal{L}^{(T)}_\bar{B}$ remain invariant under the BRST and anti-BRST symmetry transformations (B.1) and (B.2), respectively, for the physical fields that vanish-off as $x \to \pm \infty$.

To prove explicitly the “equivalence” of the Lagrangian densities $\mathcal{L}^{(T)}_B$ and $\mathcal{L}^{(T)}_\bar{B}$ on symmetry grounds, we have to apply the anti-BRST symmetry transformations $(s_{ab})$ on $\mathcal{L}^{(T)}_B$ and BRST symmetry transformations $(s_{ab})$ on the Lagrangian density $\mathcal{L}^{(T)}_\bar{B}$. Towards this goal in mind, first of all, we re-emphasize that the St"uckelberg-modified Lagrangian density $\mathcal{L}_<(S)$ is found to be (anti-)BRST invariant (i.e. $s_{(a)b} \mathcal{L}_<(S) = 0$). Thus, we have to focus on the gauge-fixing and FP-ghost terms to study the anti-BRST transformation of $\mathcal{L}^{(HC)}_B = \mathcal{L}^{(B)}_g + \mathcal{L}^{(B)}_FP$ [cf. Eq. (43)] and $\mathcal{L}^{(GR)}_B = \mathcal{L}^{(B)}_g + \mathcal{L}^{(B)}_FP$ [cf. Eq. (65)]. Thus, the anti-BRST symmetry transformation of $\mathcal{L}^{(T)}_B$ will lead us to obtain the sum of the r.h.s. of equations (61) and (68). In other words, we shall end up with a total spacetime derivative term plus the terms that vanish-off on the submanifold of the quantum Hilbert space of quantum fields which is defined by the CF-type restrictions (69). It is crucial to pin-point
the fact that, in (61), only the CF-type restrictions (14) appear which owe their origin to the HC. These are valid for the massless Abelian 3-form theory, too [17, 18]. On the other hand, on the r.h.s. of Eq. (68), we see the appearance of all the six CF-type restrictions of our theory. As a consequence, it is clear that \( L^{(HC)}_B + L^{(GIR)}_B \) respects the anti-BRST symmetry transformations only on the submanifold of the Hilbert space of the quantum fields where the CF-type restrictions (69) are satisfied. In exactly similar fashion, when we apply the BRST symmetry transformations on \( L^{(HC)}_B \bar{B} + L^{(GIR)}_B \bar{B} \), we obtain the sum of the r.h.s. of (59) and (67) which is the sum of a total spacetime derivative plus terms that vanish-off on the submanifold in the Hilbert space of quantum fields that is defined by the field equations corresponding to the CF-type restrictions (69).

6 Conclusions

In our present investigation, we have exploited the basic tenets of the augmented version of the superfield approach (AVSA) to BRST formalism (i) to derive all the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for all the fields of the massive Abelian 3-form theory in any arbitrary D-dimension of Minkowskian flat spacetime, and (ii) to deduce all the sets of Curci-Ferrari (CF) type restrictions which are found to be (anti-)BRST invariant and they are responsible for the absolute anticommutativity of the (anti-)BRST symmetry transformations. These CF-type restrictions lead to the existence of a coupled (but equivalent) set of Lagrangian densities that respect both the BRST and anti-BRST symmetry transformations on the submanifold (of a quantum Hilbert space of the quantum fields) which is defined by the field equations corresponding to the (anti-)BRST invariant CF-type restrictions. The CF-type restrictions are the hallmark [27, 28] of a quantum gauge theory (discussed within the ambit of BRST formalism) and they are connected with the geometrical objects called gerbes [27, 28]. Thus, the existence of the CF-type restrictions on our present theory is crucial. We have also derived these restrictions from the equations of motion that emerge out from the (anti-)BRST invariant coupled (but equivalent) Lagrangian densities (cf. Sec. 4) where the above EL-EoMs are derived w. r. t. the bosonic as well as fermionic auxiliary fields of our theory.

In the whole body of our present text, we have exploited an elegant conjunction of the HC and GIR within the realm of AVSA. Both these restrictions have been invoked on the basis of geometrically- as well as physically-backed theoretical grounds. First of all, we note that these quantities, at the classical level itself, are interesting in the sense that they are gauge invariant and, hence, these are primarily physical objects. For instance, we note that \( \delta_g H_{\mu\nu\lambda\zeta} = 0 \) and \( \delta_g [A_{\mu\nu\lambda} \mp \frac{1}{m} (\partial_{\mu} \Phi_{\nu\lambda} + \partial_{\nu} \Phi_{\lambda\mu} + \partial_{\lambda} \Phi_{\mu\nu})] = 0 \) which are nothing but the gauge [as well as the (anti-)BRST] invariance of (i) the field strength tensor of our theory, and (ii) an elegant combination of the gauge field \( A_{\mu\nu\lambda} \) and beautiful sum of a single derivative acting on the St"uckelberg field \( \Phi_{\mu\nu} \). Thus, both these quantities are invariant at the classical as well as quantum level because they respect gauge as well as (anti-)BRST symmetries. Both these invariances can be expressed in terms of the differential-form within the framework of AVSA as: \( \tilde{d} A^{(3)} = d A^{(3)} \) [cf. Eq. (8)] and \( \tilde{d} A^{(3)}_{(h)} \mp \frac{1}{m} \tilde{d} \Phi^{(2)} = d A^{(3)} \mp \frac{1}{m} d \Phi^{(2)} \) [cf. Eq. (26)]. Thus, we lay emphasis on the fact that, in these couple of restrictions, the fundamental ideas of the theoretical physics and
In our present endeavor, we have discussed and described only the complete set of nilpotent (anti-)BRST symmetry transformations and coupled (but equivalent) total Lagrangian densities that respect the above nilpotent symmetry transformations on the submanifold (in the total Hilbert space of quantum fields) that is defined by the CF-type field equations whose number is six for our D-dimensional massive model of Abelian 3-form theory. However, we have not devoted time on the derivation of the Noether currents and corresponding charges; the proof of their conservation laws by exploiting the full set of EL-EoMs that are derived from the coupled (but equivalent) Lagrangian densities; deduction of the associated BRST algebra, etc. In the immediate future, we plan to carry out research activities that will be connected with the above mentioned topics. To be precise, we shall compute the exact expressions for the conserved, off-shell nilpotent and absolutely anticommuting (anti-)BRST charges along with the ghost charge and demonstrate the existence of the standard BRST algebra in any arbitrary D-dimensions of spacetime. However, for the six (5 + 1)-dimensional (6D) massive Abelian 3-form theory, we expect the existence of the conserved, off-shell nilpotent and absolutely anticommuting (anti-)co-BRST charges along with the conserved (anti-)BRST and ghost charges. In the very near future, we wish to derive the extended BRST algebra with all the above conserved charges for the 6D massive Abelian 3-form theory and plan to show that this algebra is reminiscent of the Hodge algebra [19-22] of the de Rham cohomological operators of differential geometry.

We have worked on the BRST and superfield approaches to the massive Abelian p-form \((p = 1, 2)\) gauge theories over the years where we have shown that such theories in \(D = 2p\) (i.e. \(D = 2, 4\)) dimensions of spacetime are massive models of Hodge theory where the fields with negative kinetic term appear (absolutely on the symmetry grounds). These latter fields are found to be the possible candidates of dark matter because they obey Klein-Gordon equation with the well-defined rest mass (see, e.g. [29], [30] and references therein). In addition, such exotic fields have been christened as the “ghost” and/or “phantom” fields in the realm of modern cosmology where they play a crucial role in the cyclic, self-accelerated and bouncing models of Universe (see, e.g. [31-33] for details). We plan to extend our present work by introducing some new fields on symmetry grounds and establish that our massive Abelian 3-form gauge theory in \(D = 6\) is a massive model of Hodge theory where the axial antisymmetric tensor, axial vector and pseudo-scalar fields with negative kinetic terms (but with well-defined mass) are expected to exist. These will be, once again, exotic fields which are very popular in the domain of research activities in modern cosmology. These exotic fields will also provide a set of possible candidates of dark energy in their massless limits. It is worthwhile to mention that the Stueckelberg-modified (SUSY)QED has been discussed in an interesting set of papers [35-37] where an ultralight dark matter candidate has been purposed. It will be a nice future endeavor [38] to apply the BRST and superfield approaches to these systems [35-37].

1The massless version of the Abelian p-form \((p = 1, 2, 3)\) gauge theories have been discussed in our earlier work(s) (see, e.g. [34] and references therein) where the massless pseudo-scalar, axial-vector and axial antisymmetric tensor fields with negative kinetic term have been shown to exist on the ground of symmetries alone. These are the possible candidates for the dark energy (see, e.g. [31-33] for details).
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Appendix A: On the Horizontality Condition

The central purpose of this Appendix is to capture a few key steps in the derivation of the complete set of (i) the (anti-)BRST symmetry transformations which are off-shell nilpotent and absolutely anticommuting in nature, and (ii) the trivial as well as non-trivial (anti-) BRST invariant CF-type restrictions from the horizontality condition: \( dA^{(3)} = dA^{(3)} \).

This condition, as is obvious, implies that all the differentials in the super 4-forms on the l.h.s. will be set equal to zero so that the l.h.s. (i.e. the(super 4-forms with Grassmannian differentials) and r.h.s. can be proven to be equal. In this context, first of all, we note that when we set the coefficients of \((d\theta \wedge d\theta \wedge d\theta \wedge d\theta), (d\bar{\theta} \wedge d\bar{\theta} \wedge d\bar{\theta} \wedge d\bar{\theta}), (d\theta \wedge d\theta \wedge d\bar{\theta} \wedge d\bar{\theta}), (d\theta \wedge d\bar{\theta} \wedge d\theta \wedge d\bar{\theta}), (d\theta \wedge d\bar{\theta} \wedge d\theta \wedge d\bar{\theta})\) equal to zero, respectively, we obtain the following five restrictions on the superfields

\[
\begin{align*}
\partial_{\theta} \bar{F}_2 &= 0, & \partial_{\bar{\theta}} F_2 &= 0, & \partial_{\theta} \bar{F}_2 + \partial_{\bar{\theta}} F_1 &= 0, \\
\partial_{\bar{\theta}} \bar{F}_1 + \partial_{\theta} F_1 &= 0, & \partial_{\theta} \bar{F}_1 + \partial_{\bar{\theta}} F_2 &= 0,
\end{align*}
\]

which lead to the derivation of secondary fields [cf. Eq. (12)] as well as the trivial (anti-)BRST invariant CF-type restrictions as follows:

\[
\begin{align*}
s_1 &= \bar{s}_1 = s_2 = \bar{s}_2 = 0, & b^{(1)}_2 &= \bar{b}^{(2)}_1 = 0, \\
\bar{b}^{(1)}_1 + b^{(2)}_1 &= 0, & \bar{b}^{(2)}_1 + b^{(1)}_2 &= 0, & b^{(1)}_1 + \bar{b}^{(1)}_2 &= 0.
\end{align*}
\]

In the above, the last three entries are the trivial CF-type restrictions. We make the following (anti-)BRST invariant \([s_{(a)b} B_1 = s_{(a)b} B_2 = s_{(a)b} B = 0]\) choices

\[
\begin{align*}
\bar{b}^{(2)}_1 &= -\bar{b}^{(2)}_1 = B_2, & \bar{b}^{(1)}_1 &= -b^{(2)}_1 = B_1, & \bar{b}^{(1)}_2 &= -b^{(1)}_1 = B,
\end{align*}
\]

which satisfy the trivial CF-type restrictions of (A.2) [i.e. the last three entries]. As a consequence of the results and the choices in (A.2) and (A.3), respectively, we obtain the following super expansions of the four fermionic superfields \((F_1, \bar{F}_1, F_2, \bar{F}_2)\):

\[
\begin{align*}
F_1^{(h)}(x, \theta, \bar{\theta}) &= C_1 + \theta (B_1) + \bar{\theta} (-B) + \theta \bar{\theta} (0) \\
& \equiv C_1 + \theta (s_{ab} C_1) + \bar{\theta} (s_b C_1) + \theta \bar{\theta} (s_{ab} C_1), \\
\bar{F}_1^{(h)}(x, \theta, \bar{\theta}) &= \bar{C}_1 + \theta (-B_2) + \bar{\theta} (-B_1) + \theta \bar{\theta} (0) \\
& \equiv \bar{C}_1 + \theta (s_{ab} \bar{C}_1) + \bar{\theta} (s_b \bar{C}_1) + \theta \bar{\theta} (s_{ab} \bar{C}_1),
\end{align*}
\]

\[29\]
\[ \mathcal{F}_2^{(h)} (x, \theta, \bar{\theta}) = C_2 + \theta (B) + \bar{\theta} (0) + \theta \bar{\theta} (0) \]
\[ \equiv C_2 + \theta (s_{ab} C_2) + \bar{\theta} (s_b C_2) + \theta \bar{\theta} (s_b s_{ab} C_2), \]
\[ \mathcal{F}_2^{(h)} (x, \theta, \bar{\theta}) = \bar{C}_2 + \theta (0) + \bar{\theta} (B_2) + \theta \bar{\theta} (0) \]
\[ \equiv \bar{C}_2 + \theta (s_{ab} \bar{C}_2) + \bar{\theta} (s_b \bar{C}_2) + \theta \bar{\theta} (s_b s_{ab} \bar{C}_2). \] (A.4)

In the above, the superscript \((h)\) denotes that the above superfields and their super expansions along the Grassmannian directions \((\theta, \bar{\theta})\) of the \((D, 2)\)-dimensional supermanifold have been obtained after the application of horizontality condition (HC). Furthermore, the coefficients of \(\theta\) and \(\bar{\theta}\) are nothing but the off-shell nilpotent (anti-)BRST symmetry transformations (17) and (18) of our theory.

We observe, furthermore, that the setting the coefficients of the super 4-form differentials: \((dx^\mu \wedge \theta \wedge d\theta \wedge d\bar{\theta})\), \((dx^\mu \wedge d\theta \wedge d\theta \wedge d\bar{\theta})\), \((dx^\mu \wedge d\theta \wedge d\theta \wedge d\theta\bar{\theta})\), \((dx^\mu \wedge d\theta \wedge d\theta \wedge d\bar{\theta})\) equal to zero, respectively, lead to the following relationships
\[ \partial_\mu \mathcal{F}_2^{(h)} - \partial_\theta \bar{\beta}_\mu = 0, \quad \partial_\mu \mathcal{F}_2^{(h)} - \partial_{\bar{\theta}} \bar{\beta}_\mu = 0, \]
\[ \partial_\mu \mathcal{F}_1^{(h)} - \partial_{\bar{\theta}} \bar{\beta}_\mu - \partial_\theta \bar{\Phi}_\mu = 0, \quad \partial_\mu \mathcal{F}_1^{(h)} - \partial_\theta \bar{\beta}_\mu - \partial_{\bar{\theta}} \bar{\Phi}_\mu = 0, \] (A.5)
where the superfields with superscript \((h)\) have been illustrated in (A.4). The above relationships (A.5) lead to the following:
\[ f_\mu^{(1)} = \partial_\mu C_2, \quad b_\mu = i \partial_\mu B, \quad \bar{f}_\mu^{(2)} = \partial_\mu \bar{C}_2, \quad \bar{b}_\mu = -i \partial_\mu B_2, \]
\[ b_\mu^{(3)} = i \partial_\mu B_1, \quad \bar{f}_\mu^{(3)} + f_\mu^{(2)} = \partial_\mu C_1, \quad f_\mu^{(3)} + \bar{f}_\mu^{(1)} = \partial_\mu C_1. \] (A.6)
In the above, we have the precise expression for the secondary fields in terms of the basic and auxiliary fields of our theory and we have also derived the CF-type restrictions
\[ \bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1, \quad f_\mu + \bar{F}_\mu = \partial_\mu C_1, \] (A.7)
where we have identified \(\bar{f}_\mu^{(3)} = \bar{f}_\mu, \quad f_\mu^{(2)} = F_\mu, \quad f_\mu^{(3)} = f_\mu, \quad \bar{f}_\mu^{(1)} = \bar{F}_\mu\). It is clear that the non-trivial CF-type restrictions (A.7) emerge out when we set equal to zero the coefficients of the super differentials: \((dx^\mu \wedge d\theta \wedge d\theta \wedge d\bar{\theta})\) and \((dx^\mu \wedge d\theta \wedge d\theta \wedge d\theta\bar{\theta})\), respectively. The substitution of (A.6) leads to the following super expansions
\[ \bar{\beta}_\mu^{(h)} (x, \theta, \bar{\theta}) = \beta_\mu (x) + \theta (\bar{F}_\mu) + \bar{\theta} (\partial_\mu C_2) + \theta \bar{\theta} (-\partial_\mu B) \]
\[ \equiv \beta_\mu (x) + \theta (s_{ab} \beta_\mu) + \bar{\theta} (s_b \beta_\mu) + \theta \bar{\theta} (s_b s_{ab} \beta_\mu), \]
\[ \bar{\beta}_\mu^{(h)} (x, \theta, \bar{\theta}) = \bar{\beta}_\mu (x) + \theta (\partial_\mu \bar{C}_2) + \bar{\theta} (F_\mu) + \theta \bar{\theta} (\partial_\mu B_2) \]
\[ \equiv \bar{\beta}_\mu (x) + \theta (s_{ab} \bar{\beta}_\mu) + \bar{\theta} (s_b \bar{\beta}_\mu) + \theta \bar{\theta} (s_b s_{ab} \bar{\beta}_\mu), \]
\[ \bar{\Phi}_\mu^{(h)} (x, \theta, \bar{\theta}) = \phi_\mu (x) + \theta (\bar{f}_\mu) + \bar{\theta} (f_\mu) + \theta \bar{\theta} (-\partial_\mu B_1) \]
\[ \equiv \phi_\mu (x) + \theta (s_{ab} \phi_\mu) + \bar{\theta} (s_b \phi_\mu) + \theta \bar{\theta} (s_b s_{ab} \phi_\mu), \] (A.8)
where the superscript \((h)\) on the l.h.s., once again, denotes that the above three bosonic superfields have been obtained after the application of HC. It is worthwhile to point out a
clarification at this stage. From (A.7), it is clear that the ghost numbers for pairs $(\tilde{F}_\mu, F_\mu)$ are $(+1, -1)$, respectively, despite the notations which are a bit misleading.

At this stage, we now focus on setting the coefficients of the super 4-form differentials: $(dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta})$, $(dx^\mu \wedge dx^\nu \wedge d\bar{\theta} \wedge d\theta)$, $(dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\bar{\theta})$ equal to zero due to HC. These restrictions lead to the following relationships

$$\partial_\mu \tilde{F}_{\mu \nu} = \partial_\mu \tilde{\beta}_{\nu}^{(h)} - \partial_\nu \tilde{\beta}_{\mu}^{(h)}, \quad \partial_\mu F_{\mu \nu} = \partial_\mu \beta_{\nu}^{(h)} - \partial_\nu \tilde{\beta}_{\mu}^{(h)},$$

$$\partial_\mu \tilde{F}_{\mu \nu} + \partial_\mu \tilde{F}_{\mu \nu} = \partial_\mu \tilde{\beta}_{\nu}^{(h)} - \partial_\mu \tilde{\beta}_{\mu}^{(h)}, \quad (A.9)$$

where the super expansions of the superscript $(h)$ have been quoted in Eq. (A.8). The above relationships in (A.9) lead to

$$\tilde{B}_{\mu \nu}^{(2)} = \theta (\tilde{\beta}_\nu - \partial_\nu \tilde{\beta}_{\mu}), \quad s_{\mu \nu} = -i (\partial_\mu F_{\nu} - \partial_\nu F_{\mu}) \equiv i (\partial_\mu \tilde{f}_\nu - \partial_\nu \tilde{f}_{\mu}),$$

$$B_{\mu \nu}^{(1)} = \theta (\beta_\nu - \partial_\nu \beta_{\mu}), \quad s_{\mu \nu} = i (\partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_{\mu}) \equiv -i (\partial_\mu f_{\nu} - \partial_\nu f_{\mu}),$$

$$\tilde{B}_{\mu \nu}^{(1)} + B_{\mu \nu}^{(2)} = \theta (\tilde{\beta}_\nu - \partial_\nu \tilde{\beta}_{\mu}) \Rightarrow \tilde{B}_{\mu \nu} + B_{\mu \nu} = (\partial_\mu \beta_{\nu} - \partial_\nu \beta_{\mu}). \quad (A.10)$$

The last entry, in the above equation, is the non-trivial CF-type restriction where we have taken into account: $\tilde{B}_{\mu \nu}^{(1)} = \tilde{B}_{\mu \nu}$ and $B_{\mu \nu}^{(2)} = B_{\mu \nu}$. This CF-type restriction [i.e. $\tilde{B}_{\mu \nu} + B_{\mu \nu} = (\partial_\mu \beta_{\nu} - \partial_\nu \beta_{\mu})$] is valid if and only if the other two fermionic CF-type restrictions (i.e. $f_{\mu} + \tilde{F}_{\mu} = \partial_\mu C_{1}$, $\tilde{f}_{\mu} + F_{\mu} = \partial_\mu \tilde{C}_{1}$) are satisfied. In other words, due to the validity of the fermionic CF-type restrictions, we have the following

$$\partial_\mu \tilde{f}_\nu - \partial_\nu \tilde{f}_{\mu} = - (\partial_\mu F_{\nu} - \partial_\nu F_{\mu}), \quad \partial_\mu f_{\nu} - \partial_\nu f_{\mu} = - (\partial_\mu \tilde{F}_{\nu} - \partial_\nu \tilde{F}_{\mu}). \quad (A.11)$$

Ultimately, we have derived the following super expansions

$$\tilde{F}_{\mu \nu}^{(h)}(x, \theta, \bar{\theta}) = C_{\mu \nu} + \theta (\tilde{B}_{\mu \nu}) + \bar{\theta} (\tilde{\beta}_\nu - \partial_\nu \tilde{\beta}_{\mu}) + \theta \bar{\theta} [-(\partial_\mu \tilde{F}_{\nu} - \partial_\nu \tilde{F}_{\mu})]$$

$$\equiv C_{\mu \nu} + \theta (s_{ab} C_{\mu \nu}) + \theta \bar{\theta} (s_{ab} s_{ab} C_{\mu \nu}),$$

$$\tilde{F}_{\mu \nu}^{(h)}(x, \theta, \bar{\theta}) = \tilde{C}_{\mu \nu} + \theta (\tilde{\beta}_\nu - \partial_\nu \tilde{\beta}_{\mu}) + \bar{\theta} (B_{\mu \nu}) + \theta \bar{\theta} (\partial_\mu F_{\nu} - \partial_\nu F_{\mu})$$

$$\equiv \tilde{C}_{\mu \nu} + \theta (s_{ab} \tilde{C}_{\mu \nu}) + \theta \bar{\theta} (s_{ab} \tilde{C}_{\mu \nu}), \quad (A.12)$$

where the superscript $(h)$ on the superfields carries its standard meaning.

Before we end this Appendix, we point out that setting the coefficients of $(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\theta)$, $(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge d\bar{\theta})$ equal to zero, respectively, leads to the following conditions on the superfields:

$$\partial_\mu \tilde{F}_{\nu \lambda}^{(h)} + \partial_\nu \tilde{F}_{\lambda \mu}^{(h)} + \partial_\lambda \tilde{F}_{\mu \nu}^{(h)} = \partial_\theta \tilde{A}_{\mu \nu \lambda},$$

$$\partial_\mu F_{\nu \lambda}^{(h)} + \partial_\nu F_{\lambda \mu}^{(h)} + \partial_\lambda F_{\mu \nu}^{(h)} = \partial_\theta \tilde{A}_{\mu \nu \lambda}. \quad (A.13)$$

The above relationships imply the following

$$R_{\mu \nu \lambda} = \partial_\mu C_{\nu \lambda} + \partial_\nu C_{\lambda \mu} + \partial_\lambda C_{\mu \nu}, \quad \tilde{R}_{\mu \nu \lambda} = \partial_\mu \tilde{C}_{\nu \lambda} + \partial_\nu \tilde{C}_{\lambda \mu} + \partial_\lambda \tilde{C}_{\mu \nu},$$

$$S_{\mu \nu \lambda} = i (\partial_\mu \tilde{B}_{\nu \lambda} + \partial_\nu \tilde{B}_{\lambda \mu} + \partial_\lambda \tilde{B}_{\mu \nu}) \equiv -i (\partial_\mu B_{\nu \lambda} + \partial_\nu B_{\lambda \mu} + \partial_\lambda B_{\mu \nu}), \quad (A.14)$$
Finally, the equality of the coefficients of the 4-form differential $(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\eta)$ from the l.h.s. and r.h.s. of $d \tilde{A}^{(3)} = d A^{(3)}$ leads to

$$\partial_\mu \tilde{A}^{(h)}_{\nu \lambda \eta} - \partial_\nu \tilde{A}^{(h)}_{\lambda \mu \eta} + \partial_\lambda \tilde{A}^{(h)}_{\mu \nu \eta} - \partial_\eta \tilde{A}^{(h)}_{\mu \nu \lambda} = H_{\mu \nu \lambda \eta},$$

(A.15)

where the third-rank superfield with superscript $(h)$ is:

$$\tilde{A}^{(h)}_{\mu \nu \lambda}(x, \theta, \bar{\theta}) = A_{\mu \nu \lambda}(x) + \theta (\partial_\mu \bar{C}_{\nu \lambda} + \partial_\nu C_{\lambda \mu} + \partial_\lambda \bar{C}_{\mu \nu}) + \bar{\theta} (\partial_\mu C_{\nu \lambda} + \partial_\nu \bar{C}_{\lambda \mu} + \partial_\lambda C_{\mu \nu}) + \theta \bar{\theta} (\partial_\mu \bar{B}_{\nu \lambda} + \partial_\nu B_{\lambda \mu} + \partial_\lambda \bar{B}_{\mu \nu})$$

$$\equiv A_{\mu \nu \lambda} + \theta (s_{ab} A_{\mu \nu \lambda}) + \bar{\theta} (s_b s_{ab} A_{\mu \nu \lambda}).$$

(A.16)

A close look at the above mentioned equations establish that we have derived the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations [cf. Eqs (17), (18)] and the non-trivial (anti-)BRST invariant CF-type restrictions. We lay emphasis on the fact that (A.15) is satisfied due to the precise expression for the secondary fields in Eq. (A.14). It is interesting to mention, in passing, that the (anti-)BRST invariance of the CF-type restrictions, off-shell nilpotency and absolute anticommutativity properties lead to the complete derivations of the precise forms of the (anti-)BRST symmetry transformations (17) and (18) for the massless version of our massive Abelian 3-form theory.

Appendix B: On the Complete Set of (Anti-)BRST Symmetries and Total BRST and Anti-BRST Invariant Lagrangian Densities

The purpose of this Appendix is to collect all the off-shell nilpotent $[s_{(a)b}^2 = 0]$ and absolutely anticommuting $(s_b s_{ab} + s_{ab} s_b = 0)$ (anti-)BRST symmetry transformations $[s_{(a)b}]$ of our theory that have been derived from the HC (8) and GIR (26) in the main body of our text. We have also used the requirements of the off-shell nilpotency as well as the absolute anticommutativity properties and the invariance of the CF-type restrictions under the (anti-)BRST symmetry transformations to obtain the complete list of the (anti-)BRST symmetry transformations $[s_{(a)b}]$. The infinitesimal, continuous and off-shell nilpotent $(s_b^2 = 0)$ BRST transformations $(s_b)$ are as follows:

$s_b A_{\mu \nu \lambda} = \partial_\mu C_{\nu \lambda} + \partial_\nu C_{\lambda \mu} + \partial_\lambda C_{\mu \nu}$, $s_b C_{\mu \nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu$,

$s_b B_{\mu \nu} = - (\partial_\mu \bar{F}_\nu - \partial_\nu \bar{F}_\mu) \equiv (\partial_\mu f_\nu - \partial_\nu f_\mu)$, $s_b \bar{C}_{\mu \nu} = B_{\mu \nu}$,

$s_b \Phi_{\mu \nu} = \pm m C_{\mu \nu} - (\partial_\mu C_{\nu \mu} - \partial_\nu C_{\mu \nu})$,

$s_b C_{\mu} = \pm m \beta_\mu - \partial_\mu \beta$,

$s_b \bar{F}_\mu = - \partial_\mu B$, $s_b \bar{B}_\mu = m f_\mu - \partial_\mu f$, $s_b \bar{f}_\mu = - \partial_\mu B_1$, $s_b \beta = \pm m C_2$,

$s_b C_\mu = B_\mu$, $s_b C_2 = B_2$, $s_b C_1 = - B_1$,

$s_b \phi = f$, $s_b \bar{\phi} = \pm m B_1$, $s_b \beta = \pm m B_1$, $s_b C_1 = - B$, $s_b C = - B_1$. 

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On top of the above transformations, we have the corresponding off-shell nilpotent \( s_{ab}^2 = 0 \), infinitesimal and continuous anti-BRST symmetry transformations \( (s_{ab}) \) as:

\[
s_{ab} A_{\mu \lambda} = \partial_\mu \bar{C}_\lambda + \partial_\nu \bar{C}_{\lambda \mu} + \partial_\lambda \bar{C}_{\mu \nu}, \quad s_{ab} \bar{C}_\lambda = \partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu, \quad s_{ab} \bar{C}_{\mu \nu} = \partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu,
\]

\[
s_{ab} B_{\mu \nu} = -(\partial_\mu F_\nu - \partial_\nu F_\mu) \equiv (\partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu), \quad s_{ab} F_\mu = - \partial_\mu B_2
\]

\[
s_{ab} \Phi_\mu = \pm m \bar{C}_\mu - (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab} f_\mu = \pm m B_1
\]

\[
s_{ab} \bar{C}_\mu = \pm m \bar{f}_\mu - \partial_\mu \bar{f}_\mu, \quad s_{ab} \bar{f}_\mu = \bar{F}_\mu, \quad s_{ab} \bar{f}_\mu = \bar{f}_\mu B_1
\]

\[
s_{ab} [H_{\mu \nu \lambda}, B_{\mu \nu}, B_\mu, f_\mu, F_\mu, f, B_1, B_2, C_2] = 0. \quad (B.1)
\]

The above off-shell nilpotent \( [s_{ab}] = 0 \) symmetry transformations leave the Lagrangian densities \( L_B^{(HC,GIR)} \) and \( L_B^{(HC,GIR)} \) perfectly invariant because we observe that \( s_{ab} L_B^{(HC,GIR)} \) and \( s_{ab} L_B^{(HC,GIR)} \) are total spacetime derivatives [cf. Eqs. (57), (58), (64), (66)]. In addition, we have already mentioned that the St"uckelberg-modified Lagrangian density (4) remains invariant \( [s_{(a)b}] L_S = 0 \) under the (anti-)BRST symmetry transformations, too.

The above complete set of (anti-)BRST symmetry transformations of (B.2) and (B.1) are also anticommuting in nature. As pointed out earlier in the main body of the text [cf. Eq. (19)] that \( \{s_b, s_{ab}\} A_{\mu \lambda} = 0 \), \( \{s_b, s_{ab}\} C_{\mu \nu} = 0 \), and \( \{s_b, s_{ab}\} \bar{C}_{\mu \nu} = 0 \) are satisfied only when the CF-type conditions: \( B_{\mu \nu} + \bar{B}_{\mu \nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \), \( f_\mu + \bar{F}_\mu = \partial_\mu C_1 \), and \( \bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1 \) are imposed, respectively, from outside\(^{**}\). It is very interesting to observe that the following absolute anticommuting properties, namely:

\[
\{s_b, s_{ab}\} \Phi_\mu = 0, \quad \{s_b, s_{ab}\} C_\mu = 0, \quad \{s_b, s_{ab}\} \bar{C}_\mu = 0, \quad (B.3)
\]

are satisfied provided we take into account the following pair of (anti-)BRST invariant CF-type restrictions that have been derived using HC and GIR, namely:

\[
B_{\mu \nu} + \bar{B}_{\mu \nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \quad B_\mu + \bar{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi,
\]

\[
\bar{f}_\mu + \bar{F}_\mu = \partial_\mu C_1, \quad f + \bar{F} = \pm m C_1,
\]

\[
\bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1, \quad \bar{f} + F = \pm m \bar{C}_1. \quad (B.4)
\]

In other words, the absolute anticommutativity property for the St"uckelberg-compensating field and associated fermionic (anti-)ghost fields is satisfied if and only if we take into

\(^{**}\)It should be pointed out that (i) the proper (anti-)BRST symmetry transformations for \( A_{\mu \nu \lambda}, C_{\mu \nu} \) and \( \bar{C}_{\mu \nu} \), and (ii) the (anti-)BRST invariant CF-type restrictions: \( B_{\mu \nu} + \bar{B}_{\mu \nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \), \( f_\mu + \bar{F}_\mu = \partial_\mu C_1 \), and \( \bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1 \) have been derived from the HC (cf. Appendix A below for details).
account the pair of CF-type restrictions (B.4) together. This happens due to the fact that the GIR [cf. Eq. (26)] incorporates HC as well as gauge invariance together for its validity.

Before we end this Appendix, we write down the perfectly BRST-invariant and perfectly anti-BRST invariant coupled Lagrangian densities. First of all, we express the perfectly BRST-invariant total Lagrangian density $\mathcal{L}_B^{(T)}$ as follows

$$\mathcal{L}_B^{(T)} = \mathcal{L}_{(S)} + \mathcal{L}_{g\bar{f}}^{(B)} + \mathcal{L}_{FP}^{(B)},$$

where the above symbols stand for the following:

$$\mathcal{L}_{(S)} = \frac{1}{24} H^\mu\nu\zeta H_{\mu\nu\zeta} - \frac{m^2}{6} A^\mu A_{\mu} - \frac{1}{6} \Sigma_{\mu\nu\lambda} \Sigma_{\mu\nu\lambda} \pm \frac{m}{3} A^\mu \Sigma_{\mu\lambda},$$

$$\mathcal{L}_{g\bar{f}}^{(B)} + \mathcal{L}_{FP}^{(B)} \equiv \mathcal{L}_{B}^{(HC)} + \mathcal{L}_{B}^{(GIR)} ,$$

$$\equiv (\partial_\mu A^{\mu\nu}) B_{\nu\lambda} - \frac{1}{2} B_{\mu} B_{\nu} B^{\mu} + \frac{1}{2} B^{\mu} \left[ \pm m \phi_{\mu} - \partial_{\mu} \phi \right] + \frac{m^2}{2} \bar{C}_{\mu} C^{\mu\nu}$$

$$+ (\partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\mu\lambda} + \partial_\lambda \bar{C}_{\mu\nu} (\partial^{\mu} C^{\nu\lambda}) \pm m (\partial_\mu \bar{C}^{\nu\lambda}) C_\nu \pm m \bar{C}^{\nu} (\partial^{\mu} C_{\mu\nu})$$

$$+ (\partial_\mu \bar{C}_{\nu} - \partial_\nu \bar{C}_{\mu} ) (\partial^{\nu} C^{\mu} ) - \frac{1}{2} \left[ \pm m \beta_{\mu} - \partial^{\mu} \beta \right] \left[ \pm m \beta_{\mu} - \partial_{\mu} \beta \right]$$

$$- (\partial_\mu \beta_{\nu} - \partial_\nu \beta_{\mu} ) (\partial^{\nu} \beta^{\mu} ) - \partial_\mu \bar{C}_{2} \partial^{\mu} C_{2} - m^2 \bar{C}_{2} C_{2} + [(\partial \cdot \beta) \mp m \beta] B$$

$$- [(\partial \cdot \phi) \mp m \phi] B_{1} - [(\partial \cdot \beta) \mp m \beta] B_{2} + \left[ \partial_\nu \bar{C}^{\nu\mu} + \partial^{\mu} \bar{C}_{1} \mp \frac{m}{2} \bar{C}^{\mu} \right] f_\mu$$

$$- 2 F^\mu f_\mu - 2 F f - \left[ \partial_\nu C^{\nu\mu} + \partial^{\mu} C_{1} \mp \frac{m}{2} \bar{C}_{1} \right] F_\mu + F \left[ \pm m C_{1} - \frac{1}{2} (\partial \cdot C) \right]$$

$$- \left[ \frac{1}{2} (\partial \cdot \bar{C}) \mp m \bar{C}_{1} \right] f - B_{1} B_{2} - \frac{1}{2} B_{2}^2 .$$

(B.6) It is worth pointing out that the above perfectly BRST-invariant total Lagrangian density is function of the auxiliary fields ($B_{\mu\nu}, B_{\mu}, F_{\mu}, f_{\mu}, f, F, f$) and their counterparts ($\bar{B}_{\mu\nu}, \bar{B}_{\mu}, \bar{F}_{\mu}, \bar{f}_{\mu}, \bar{F}, \bar{f}$) do not appear in the theory due to our exercise in (49). To be precise, the total Lagrangian density (B.6) is an exact sum of (43) and the modified form of (47) and/or (63) that is re-expressed as:

$$\mathcal{L}_{B}^{(GIR)} = - (\partial_\mu \Phi^{\mu\nu}) B_{\nu} - \frac{1}{2} B^{\mu} B_{\mu} + \frac{1}{2} B^{\mu} \left( \pm m \phi_{\mu} - \partial_{\mu} \phi \right) \mp \frac{m}{2} B^{\mu\nu} \Phi_{\mu\nu}$$

$$+ \frac{m^2}{2} \bar{C}_{\mu} C^{\mu\nu} + (\partial_\mu \bar{C}_{\nu} - \partial_\nu \bar{C}_{\mu} ) \partial^{\nu} C^{\mu} \pm m (\partial_\mu \bar{C}^{\nu\lambda}) C_{\nu} \mp \frac{m}{2} F^\mu C_{\mu} \mp m \bar{C}^{\nu} (\partial^{\mu} C_{\mu\nu})$$

$$- \frac{1}{2} \left[ \pm m \beta_{\mu} - \partial^{\mu} \beta \right] \left[ \pm m \beta_{\mu} - \partial_{\mu} \beta \right] \mp \frac{m}{2} \bar{C}_{\mu} f_\mu + F \left[ \pm m C_{1} - \frac{1}{2} (\partial \cdot C) \right]$$

$$- \left[ \frac{1}{2} (\partial \cdot \bar{C}) \mp m \bar{C}_{1} \right] f \mp m B_{1} \phi - 2 F f - m^2 C_{2} C_{2} \mp m B_{2} \beta .$$

(B.7)
The analogues of (B.5) and (B.6) can be written as the perfectly anti-BRST total Lagrangian density \( \mathcal{L}_B^{(T)} \) as follows

\[
\mathcal{L}_B^{(T)} = \mathcal{L}_{(S)} + \mathcal{L}_{g\gamma}^{(B)} + \mathcal{L}_{FP}^{(B)}, \quad (B.8)
\]

where the above symbols stand for the following:

\[
\mathcal{L}_{(S)} = \frac{1}{24} H^{\mu\nu\lambda} \xi_{\mu\nu\lambda} - \frac{m^2}{6} A^{\mu\nu\lambda} A_{\mu\nu\lambda} - \frac{1}{6} \Sigma^{\mu\nu\lambda} \Sigma_{\mu\nu\lambda} \pm \frac{m}{3} A^{\mu\nu\lambda} A_{\mu\nu\lambda},
\]

\[
\mathcal{L}_{g\gamma}^{(B)} + \mathcal{L}_{FP}^{(B)} \equiv \mathcal{L}_B^{(HC)} + \mathcal{L}_B^{(GIR)},
\]

\[
\equiv - (\partial_\mu A^{\mu\nu\lambda}) \bar{B}_{\nu\lambda} - \frac{1}{2} \bar{B}^{\mu\nu} \bar{B}_{\mu\nu} + \frac{1}{2} \bar{B}^{\mu\nu} \left[ \pm m \phi_{\nu} - \partial_\mu \phi_{\nu} \pm m \Phi_{\mu\nu} \right] + \frac{m^2}{2} \bar{C}_{\mu\nu} C^{\mu\nu}
\]

\[
+ (\partial_\mu \bar{C}^{\mu\nu}) \bar{B}_{\nu} - \frac{1}{2} B^{\mu} \bar{B}_{\mu} + \frac{1}{2} \bar{B}^{\mu} \left[ \pm m \bar{\phi}_{\mu} - \partial_\mu \bar{\phi}_{\mu} + m \Phi_{\mu} \right] + \frac{m^2}{2} \bar{C}^{\mu\nu} C_{\mu\nu}
\]

\[
+ (\partial_\mu \bar{C}_{\nu} - \partial_\nu \bar{C}_{\mu}) (\partial^{\mu} C^{\nu}) - \frac{1}{2} \left[ \pm m \bar{\beta}^{\mu} - \partial_\mu \bar{\beta} \right] \left[ \pm m \beta_\mu - \partial_\mu \beta \right]
\]

\[
- (\partial_\mu \bar{\beta}_{\nu} - \partial_\nu \bar{\beta}_{\mu}) (\partial^{\mu} \beta^{\nu}) - \partial_\mu \bar{C}_{2} \partial^{\mu} C_{2} - m^2 \bar{C}_{0} C_{2} + [(\partial_{\beta} \mp m \beta) B
\]

\[
- [(\partial \cdot \phi) 
\]

\[
+ 2 \bar{F}^{\mu} \bar{f}_{\mu} + 2 \bar{F} \bar{f} - \left[ \partial_\nu \bar{C}^{\mu\nu} - \partial^{\mu} \bar{C}_{1} + \frac{m}{2} \bar{C}^{\mu} \right] \bar{F}_{\mu} + \left[ \frac{1}{2} (\partial \cdot \bar{C}) \pm m \bar{C} \right] \bar{F}
\]

\[
- \left[ \frac{1}{2} (\partial \cdot C) \pm m C \right] \bar{f} - B_{2} B_{2} - \frac{1}{2} B^2.
\] \quad (B.9)

It is worthwhile to point out the fact that, in the total perfectly anti-BRST total Lagrangian density \( \mathcal{L}_B^{(T)} \), we have the presence of the auxiliary fields \( (\bar{B}_{\mu\nu}, \bar{B}_{\mu}, \bar{F}_{\mu}, \bar{f}_{\mu}, \bar{F}, \bar{f}) \) and the set of auxiliary fields \( (B_{\mu\nu}, B_{\mu}, F_{\mu}, f_{\mu}, F, f) \) does not appear anywhere [cf. Eq. (49)]. In other words, the above equation (B.9) is the exact sum of the Lagrangian density (44) and the modified version of (48) and/or (65) that is re-expressed as:

\[
\mathcal{L}_B^{(GIR)} = (\partial_\mu \Phi^{\mu\nu}) \bar{B}_{\nu} \pm \frac{m}{2} \bar{B}^{\mu\nu} \Phi_{\mu\nu} + \frac{m^2}{2} \bar{C}_{\mu\nu} C^{\mu\nu} + (\partial_\mu \bar{C}_{\nu} - \partial_\nu \bar{C}_{\mu}) \partial^{\mu} C^{\nu}
\]

\[
\pm m (\partial_{\mu} \bar{C}^{\mu\nu}) C_{\nu} \pm m \bar{C}^{\nu} (\partial^{\nu} C_{\mu}) - \frac{1}{2} \left( \pm m \bar{\beta}^{\mu} - \partial^{\mu} \bar{\beta} \right) \left( \pm m \beta_\mu - \partial_\mu \beta \right)
\]

\[
- \frac{1}{2} B^{\mu} \bar{B}_{\mu} + \frac{1}{2} \bar{B}^{\mu} \left( \pm m \Phi_{\mu} - \partial_\mu \phi \right) + \bar{f} \left[ \frac{1}{2} \partial \cdot C \pm m C_1 \right] \pm \frac{m}{2} \bar{C}_{\mu} \bar{F}^{\mu} \pm \frac{m}{2} \bar{f}^{\mu} C_{\mu}
\]

\[
+ \left[ \frac{1}{2} \partial \cdot \bar{C} \mp m \bar{C} \right] \bar{F} \pm m B_{1} \phi + 2 \bar{F} \bar{f} - m^2 \bar{C} \bar{C} \mp m B \bar{\beta} \pm m B_2 \beta,
\] \quad (B.10)

where we have used the modifications of Eq. (49).

We end this Appendix with the concluding remarks that the total Lagrangian densities \( \mathcal{L}_B^{(T)} \) and \( \mathcal{L}_B^{(T)} \) are perfectly BRST and anti-BRST invariant [cf. Eqs. (71), (72)]. Furthermore, we observe that (i) \( \mathcal{L}_B^{(T)} \) respects anti-BRST symmetry, and (ii) \( \mathcal{L}_B^{(T)} \) remains
invariant under the BRST symmetry transformations provided we use all the CF-type restrictions [cf. Eqs. (14), (69)]. Hence, the Lagrangian density $L^T_B$ and $\bar{L}^T_B$ are coupled (but equivalent) w.r.t. the nilpotent (anti-)BRST symmetry transformations.

**Appendix C: On the Existence and Emergence of CF-Type Restrictions**

Within the framework of BRST formalism, the existence of the CF-type restriction(s) is a key and decisive feature for a BRST-quantized theory. Even in the case of a D-dimensional BRST-quantized Abelian 1-form theory, a CF-type restriction exists but it is trivial in the sense that it is the limiting case of the CF-condition [12] which is a non-trivial restriction in the case of a D-dimensional non-Abelian 1-form theory. All the higher $p$-form ($p = 2, 3, \ldots$) (non-)Abelian gauge theories are endowed with the non-trivial CF-type restriction(s) because their existence for a BRST-quantized theory is as crucial and fundamental as the existence of the first-class constraints [in the terminology of the Dirac’s prescription for the classification scheme (see, e.g. [23-26] for details) of constraints] for a given classical gauge theory. In our present Appendix, we shed some light on the existence and emergence of the six (anti-)BRST invariant CF-type restrictions on our arbitrary D-dimensional BRST-quantized modified massive Abelian 3-form gauge theory.

Towards the above goal in mind, we dwell a bit on the Fig. 1 where we have shown all the fields (with their ghost numbers) which appear in the BRST and anti-BRST symmetry transformations (B.1) and (B.2) [cf. Appendix B]. Every individual field has been shown by a circle at an appropriate ghost number that is associated with it. The BRST symmetry transformations that increase the ghost number by one (for the field with a specific ghost number on which they operate) have been shown by the red arrows. On the other hand, the anti-BRST transformations that decrease the ghost number by one (for the field with a specific ghost number on which they operate) have been displayed by the green arrows. All the BRST and anti-BRST symmetry transformations of (B.1) and (B.2) have been taken into account in our Fig. 1 and these have been displayed with appropriate colours.

Except for the fields $A_{\mu\nu\lambda}$, $B_{\mu\nu}$, $\bar{B}_{\mu\nu}$ and $\Phi_{\mu\nu}$, we have an exterior derivative $d$ that has been shown by the blue-bold arrow in the upward direction. This carries the informations about the full (anti-)BRST symmetry transformations for our massive D-dimensional Stäckelberg-modified Abelian 3-form gauge theory. For instance, the exterior derivatives $d$ that have been shown on the fields $C_{\mu\nu}$ and $\bar{C}_{\mu\nu}$ lead to the following:

\[
s_b A^{(3)} = d C^{(2)}, \quad s_{ab} A^{(3)} = d \bar{C}^{(2)}.
\]  \hspace{1cm} (C.1)

In other words, the operation of $d$ on the fermionic 2-forms $C^{(2)} = [(dx^\mu \wedge dx^\nu)/2!] C_{\mu\nu}$ and $\bar{C}^{(2)} = [(dx^\mu \wedge dx^\nu)/2!] \bar{C}_{\mu\nu}$ lifts these 2-forms fields to the layer of the 3-form fields $(A_{\mu\nu\lambda})$ in the sense that we have the following:

\[
s_b A_{\mu\nu\lambda} = \partial_\mu C_{\nu\lambda} + \partial_\nu C_{\lambda\mu} + \partial_\lambda C_{\mu\nu},
\]

\[
s_{ab} A_{\mu\nu\lambda} = \partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\lambda\mu} + \partial_\lambda \bar{C}_{\mu\nu}.
\]  \hspace{1cm} (C.2)
In exactly similar fashion, we note that

\[ s_b \Phi^{(2)} = C^{(2)} - d C^{(1)}, \quad s_{ab} \Phi^{(2)} = \bar{C}^{(2)} - d \bar{C}^{(1)}, \]  

where \( \Phi^{(2)} = [(d x^\mu \wedge d x^\nu)/2!] \Phi_{\mu\nu}, \) \( C^{(1)} = d x^\mu C_\mu, \) \( \bar{C}^{(1)} = d x^\mu \bar{C}_\mu \) are the set of a single bosonic 2-form and a couple of fermionic 1-form fields. This makes it clear that \( \Phi^{(2)} \) and \( C^{(2)} \) as well as \( \bar{C}^{(2)} \) are in the same layer but \( C^{(1)} \) and \( \bar{C}^{(1)} \) are one-step lower so that when \( d \) operates, it lifts them to the level of 2-forms. In other words, we have

\[ s_b \Phi_{\mu\nu} = \pm m C_{\mu\nu} - (\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_{ab} \Phi_{\mu\nu} = \pm m \bar{C}_{\mu\nu} - (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu). \]  

Similarly, all the other such kinds of (anti-)BRST symmetry transformations (e.g. \( s_b \bar{B}_\mu = \pm m f_\mu - \partial_\mu f, \) \( s_b C_\mu = \pm m \beta_\mu - \partial_\mu \beta_\mu, \) etc.) for our massive D-dimensional Stückelberg-modified Abelian 3-form gauge theory can be explained.

It is worthwhile to pinpoint the fact that we have two exterior derivative operators on the triplets of concentric circles at (5) and (6) in our Fig. 1 because of the following reason

\[ s_b \bar{B}_\mu = \pm m f_\mu - \partial_\mu \bar{f}, \quad f_\mu + \bar{F}_\mu = \partial_\mu C_1, \]

\[ s_{ab} B_\mu = \pm m \bar{f}_\mu - \partial_\mu \bar{f}, \quad \bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1, \]  

which can be explained in the language of differential-forms as

\[ s_b B^{(1)} = \pm m f^{(1)} - d f^{(0)}, \quad f^{(1)} + F^{(1)} = d C_1^{(0)}, \]
\[ s_{ab} B^{(1)} = \pm m \tilde{f}^{(1)} - d \bar{f}^{(0)}, \quad \tilde{f}^{(1)} + F^{(1)} = d \bar{C}^{(0)}, \quad (C.6) \]

where the 1-forms and 0-forms are: \( \tilde{B}^{(1)} = d x^\mu \tilde{B}_\mu, B^{(1)} = d x^\mu B_\mu, f^{(1)} = d x^\mu f_\mu, \bar{f}^{(1)} = d x^\mu \bar{f}_\mu, C^{(0)} = C_1, \bar{C}^{(0)} = \bar{C}_1, f^{(0)} = f, \bar{f}^{(0)} = \bar{f}, d C_1^{(0)} = d x^\mu (\partial_\mu C_1), d \bar{C}_1^{(0)} = d x^\mu (\overline{\partial}_\mu \bar{C}_1). \) In other words, we have two CF-type restrictions as well as two (anti-)BRST transformations (i.e. \( s_\mu \tilde{B}_\mu \) and \( s_{ab} B_{\mu \nu} \)) such that the exterior derivatives appear twice which lift the 0-forms \( f^{(0)}, \bar{f}^{(0)}, C_1^{(0)} \) and \( \bar{C}_1^{(0)} \) to their appropriate counterparts 1-forms.

As far as the existence and emergence of CF-type restrictions are concerned, we note that whenever there is clustering of the fields at a given ghost number, there is a relationship amongst these fields [see, e.g. the bracketed numbers (1), (2), (3), (4), (5) and (6) in our Fig. 1]. Sometime the clustering of the fields is in the same layer [e.g. (5) and (6) where we have: \( f + \tilde{F} = \pm m C_1 \) and \( \bar{f} + F = \pm m \bar{C}_1 \)]. On the other hand, at other time, the fields are in different layers but they are connected by the exterior derivative \( d = d x^\mu \partial_\mu \) (with \( d^2 = 0 \)). It is very interesting to point out that the fields: \( \phi, \phi_\mu \) and \( B_{\mu \nu} \) (and/or \( \tilde{B}_{\mu \nu} \)) are located in three different layers. The lowest layer is occupied by \( \phi \) (which is a 0-form). However, the CF-type restriction: \( B_\mu + \tilde{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi \) shows that the 0-form field \( \phi \) is lifted to the layer corresponding to the 1-forms by an exterior derivative. In other words, we have this CF-type restriction in the differential-form language as:

\[ B^{(1)} + \tilde{B}^{(1)} = \pm m \Phi^{(1)} - d \Phi^{(0)}, \quad (C.7) \]

where \( \Phi^{(0)} = \phi, \Phi^{(1)} = d x^\mu \phi_\mu, B^{(1)} = d x^\mu B_\mu \) and \( \tilde{B}^{(1)} = d x^\mu \tilde{B}_\mu \). This observation establishes that the fields \( \phi_\mu, B_\mu \) and \( \tilde{B}_\mu \) are in the same layer of the fields but \( \phi \) is one-step down of this layer. In exactly similar fashion, we note that the CF-type restriction: \( B_{\mu \nu} + \tilde{B}_{\mu \nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \) shows that the field \( \phi_\mu \) is located one-step lower than the fields \( B_{\mu \nu} \) and/or \( \tilde{B}_{\mu \nu} \). This statement becomes transparent when we express the CF-type restrictions in the form-language as:

\[ B^{(2)} + \tilde{B}^{(2)} = d \Phi^{(1)}, \quad \Phi^{(1)} = d x^\mu \phi_\mu, \quad (C.8) \]

where \( B^{(2)} = [(d x^\mu \wedge d x^\nu)/2] B_{\mu \nu} \) and \( \tilde{B}^{(2)} = [(d x^\mu \wedge d x^\nu)/2] \tilde{B}_{\mu \nu} \). All the six (anti-)BRST invariant CF-type restrictions can be explained, in exactly similar fashion, provided we express them in the form-language of differential geometry [19-22]. We end this Appendix with the final remark that the CF-type restrictions are connected with the geometrical objects called gerbes [27,28] which, primarily, provide the mathematical basis for the independent identity of the off-shell nilpotent (anti-)BRST symmetry transformations and corresponding conserved Noether charges.

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