WALL-CROSSING FOR TORIC MUTATIONS

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Abstract. This note explains how to deduce the wall-crossing formula for toric mutations established by Pascaleff-Tonkonog from the perverse schober of the corresponding local Landau-Ginzburg model. Along the way, we develop a general framework to extract a wall-crossing formula from a perverse schober on the projective line with a single critical point.

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1. Introduction

Our aim in this note is to deduce the wall-crossing formula for toric mutations established by Pascaleff-Tonkonog [6] from the perverse schober of the corresponding local Landau-Ginzburg model calculated in [5] (see also [4] for a concrete approach to the three-dimensional case). Along the way, we develop a general framework (see Theorem 1.5) to extract a wall-crossing formula from a perverse schober \( P \) on the projective line \( \mathbb{CP}^1 \) with a single critical point \( 0 \in \mathbb{CP}^1 \).

The wall-crossing formula applies to the moduli of objects of the inertia category of the clean objects (see Definition 3.8) in the generic fiber of \( P \).

1.1. Wall-crossing formula. The wall-crossing formula for toric mutations established by Pascaleff-Tonkonog [6] is the birational map

\[
y_i \mapsto y_i, \quad i = 1, \ldots, n - 1
\]
\[
y_n \mapsto y_n^{-1}(1 + y_1 + \cdots + y_{n-1})
\]
associated to the mutation $L \rightsquigarrow \mu_D(L)$ of an $n$-dimensional Lagrangian torus $L \simeq (S^1)^n$ around a singular thimble $D = \text{Cone}(T^{\text{van}})$ attached to a vanishing $(n-1)$-dimensional torus $T^{\text{van}} \simeq (S^1)^{n-1} \subset L$.

The variables $y_i$, $i = 1, \ldots, n$, are simultaneously coordinates on the moduli $T^\vee \simeq (\mathbb{G}_m)^n$ of rank one local systems on the Lagrangian torus $L \simeq (S^1)^n$ as well as its mutation $\mu_D(L) \simeq (S^1)^n$ under a natural identification. In particular, the variables $y_i$, $i = 1, \ldots, n-1$, are simultaneously coordinates on the moduli $T^\vee_0 \simeq (\mathbb{G}_m)^{n-1}$ of rank one local systems on the vanishing torus $T^{\text{van}} \simeq (S^1)^{n-1}$ as well as its mutation $\mu_D(T^{\text{van}}) \simeq (S^1)^{n-1}$.

When $n = 2$, the vanishing torus $T^{\text{van}}$ is a circle, the thimble $D = \text{Cone}(T^{\text{van}})$ is a smooth disk, and the geometry is that of a traditional “mutation configuration” [6, Def. 4.10]. In general, the geometry is given by the following local model (adapted from [6, Sect. 5.4] to suit our further considerations).

1.1.1. Local geometry. Let $z$ be a coordinate on $\mathbb{C}$, $z_1, \ldots, z_n$ coordinates on $\mathbb{C}^n$, and fix the function $W : \mathbb{C}^n \to \mathbb{C}$, $W = z_1 \cdots z_n$.

For any $\rho > 0$, $\rho \neq \epsilon$, consider the circle and corresponding Lagrangian torus

$$
\gamma_\rho = \{ |z - \epsilon| = \rho \} \simeq S^1 \subset \mathbb{C} \quad T_\rho = \{ W \in \gamma_\rho, |z_1| = \cdots = |z_n| \} \subset M
$$

In particular, for fixed $\rho_{\text{Ch}} \in (0, \epsilon)$ and $\rho_{\text{Cl}} \in (\epsilon, \infty)$, we have the Chekanov and Clifford circles and corresponding $n$-dimensional Chekanov and Clifford tori

$$
\gamma_{\text{Ch}} := \gamma_{\rho_{\text{Ch}}} \quad T_{\text{Ch}} := T_{\rho_{\text{Ch}}} \quad \gamma_{\text{Cl}} := \gamma_{\rho_{\text{Cl}}} \quad T_{\text{Cl}} := T_{\rho_{\text{Cl}}}
$$

Consider as well the closed intervals

$$
I_{\text{Ch}} = [0, \epsilon - \rho_{\text{Ch}}] \quad I_{\text{Cl}} = [\epsilon - \rho_{\text{Cl}}, 0]
$$

and the corresponding singular thimbles

$$
D_{\text{Ch}} = \{ W \in I_{\text{Ch}}, |z_1| = \cdots = |z_n| \} \simeq \text{Cone}(T^{\text{van}}_{\text{Ch}})
$$

$$
D_{\text{Cl}} = \{ W \in I_{\text{Cl}}, |z_1| = \cdots = |z_n| \} \simeq \text{Cone}(T^{\text{van}}_{\text{Cl}})
$$

Figure 1. Chekanov graph $\Gamma_{\text{Ch}}$ and Clifford graph $\Gamma_{\text{Cl}}$.
with respective boundaries the vanishing tori

\[ \partial D_{\text{Ch}} = T_{\text{Ch}}^\text{van} = \{ W = \epsilon - \rho_{\text{Ch}}, |z_1| = \cdots = |z_n| \} \]

\[ \partial D_{\text{Cl}} = T_{\text{Cl}}^\text{van} = \{ W = \epsilon - \rho_{\text{Cl}}, |z_1| = \cdots = |z_n| \} \]

It will also be useful to introduce the Chekanov and Clifford graphs

\[ \Gamma_{\text{Ch}} = \gamma_{\text{Ch}} \cup I_{\text{Ch}} \quad \Gamma_{\text{Cl}} = \gamma \cup I_{\text{Cl}} \]

which are skeleta of \( \mathbb{C} \setminus \{ \epsilon \} \cong \mathbb{C}^\times \), and the corresponding Chekanov and Clifford skeleta

\[ L_{\text{Ch}} = \{ W \in \Gamma_{\text{Ch}}, |z_1| = \cdots = |z_n| \} \quad L_{\text{Cl}} = \{ W \in \Gamma_{\text{Cl}}, |z_1| = \cdots = |z_n| \} \]

of the symplectic manifold \( M = \{ W \neq \epsilon \} \) itself.

One says the tori \( T_{\text{Ch}}, T_{\text{Cl}} \) mutate into each other around the respective thimbles \( D_{\text{Ch}}, D_{\text{Cl}} \) as the radius \( \rho \) passes through \( \epsilon \) and the circle \( \gamma_\rho \) passes through the critical value \( 0 \in \mathbb{C} \).

Remark 1.1. We could also work directly with the graph \( \Gamma_{\text{crit}} = \gamma_\epsilon \) given by the circle of critical radius \( \epsilon \), and the corresponding critical skeleton

\[ L_{\text{crit}} = \{ W \in \Gamma_{\text{crit}}, |z_1| = \cdots = |z_n| \} \]

It is the union of the two thimbles

\[ D_{\text{crit}}^\pm = \{ W \in I_{\text{crit}}^\pm, |z_1| = \cdots = |z_n| \} \cong \text{Cone}(T_{\text{crit}}^\text{van}) \]

over the two semi-circles \( I_{\text{crit}}^\pm = \{ z \in \Gamma_{\text{crit}}, \pm \text{Im}(z) \geq 0 \} \), with each thimble the cone over the the same vanishing torus

\[ \partial D_{\text{crit}}^\pm = T_{\text{crit}}^\text{van} = \{ W = 2\epsilon, |z_1| = \cdots = |z_n| \} \]

Figure 2. Critical graph \( \Gamma_{\text{crit}} \).

1.2. Landau-Ginzburg model. Let us consider the Landau-Ginzburg A-model of \( \mathbb{C}^n \) with superpotential \( W = z_1 \cdots z_n \). By results of [5], its branes are naturally organized by a perverse schober on the disk \( D = \{|z| < \epsilon \} \subset \mathbb{C}P^1 \) with a single critical point at \( 0 \in D \). (One expects any Landau-Ginzburg model to similarly give a perverse schober.)

Following Kapranov-Schechtman [3], a perverse schober on a disk \( D \) with a single critical point at \( 0 \in D \), in its minimal one-cut realization, is a spherical functor

\[ \mathcal{P}_{D,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi) \]
from a stable “vanishing” dg category $D_\Phi$ to a stable “nearby” dg category $D_\Psi$. (See Sect. 2 below for more details.)

For the Landau-Ginzburg $A$-model of $\mathbb{C}^n$ with superpotential $W = z_1 \cdots z_n$, consider the thimble and and cylinder respectively

$$L = \{ W \in \mathbb{R}_{\geq 0}, |z_1| = \cdots = |z_n| \} \subset \mathbb{C}^n$$

(1.15)

$$L^o = \{ W \in \mathbb{R}_{> 0}, |z_1| = \cdots = |z_n| \} \subset \mathbb{C}^n$$

(1.16)

A main technical result of [5] is that we obtain a perverse schober

$$P_A = (j^* : \mu Sh_L(\mathbb{C}^n) \longrightarrow \mu Sh_{L^o}(\mathbb{C}^n))$$

(1.17)

by taking dg categories of microlocal sheaves along the thimble and cylinder, with their natural restriction under the open inclusion $j : L^o \hookrightarrow L$.

Next, let us describe the mirror perverse schober. Introduce the torus $\mathbb{T}^\vee = (\mathbb{G}_m)^n$, with coordinates $x_1, \ldots, x_n$, and its quotient $\mathbb{T}_0^\vee = \mathbb{T}^\vee / \mathbb{G}_m \simeq (\mathbb{G}_m)^{n-1}$ by the diagonal, with coordinates $y_1 = x_1/x_n, \ldots, y_{n-1} = x_1/x_n$. Consider the inclusion of the pair of pants

$$i : P_{n-2} = \{ 1 + y_1 + \cdots + y_{n-1} = 0 \} \longrightarrow \mathbb{T}_0^\vee$$

Then it is elementary to check we obtain a perverse schober

$$P_B = (i_* : \text{Perf}_{\text{prop}}(P_{n-2}) \longrightarrow \text{Perf}_{\text{prop}}(\mathbb{T}_0^\vee))$$

(1.19)

by taking dg categories of perfect complexes with proper support and the usual pushforward.

A main result of [5] is the following mirror equivalence.

**Theorem 1.2** ([5]). There is an equivalence of perverse schobers $P_A \sim P_B$, i.e. a commutative diagram with vertical equivalences

$$\begin{array}{ccc}
\mu Sh_L(\mathbb{C}^n) & \xrightarrow{j^*} & \mu Sh_{L^o}(\mathbb{C}^n) \\
\sim & & \sim \\
\text{Perf}_{\text{prop}}(P_{n-2}) & \xrightarrow{i_*} & \text{Perf}_{\text{prop}}(\mathbb{T}_0^\vee)
\end{array}$$

(1.20)

**Remark 1.3.** To pin down the equivalence of the theorem, consider the vanishing torus

$$T_{\text{van}} = \{ W = 1, |z_1| = \cdots = |z_n| \} \subset \mathbb{C}^n$$

(1.21)

and note the natural diffeomorphism $L^o \simeq T_{\text{van}} \times \mathbb{R}_{> 0}$. Note we can canonically identify $\mathbb{T}_0^\vee$ with the moduli of rank one local systems on $T_{\text{van}}$, and thus obtain a canonical equivalence $\mu Sh_{L^o}(\mathbb{C}^n) \simeq \text{Perf}_{\text{prop}}(\mathbb{T}_0^\vee)$. This is the right vertical arrow in [4.1]; it fixes the left vertical arrow, since the functors $j^*, i_*$ are conservative.

1.3. **Main result.** We sketch here a general wall-crossing formula associated to any perverse schober $P_{S^2, 0}$ on the sphere $S^2 \simeq \mathbb{CP}^1$ with a single critical point $0 \in S^2$. We will regard such a perverse schober as a pair

$$P_{S^2, 0} = (S : D_\Phi \rightarrow D_\Psi, \tau : \text{id} \xrightarrow{\sim} T_{\Psi, r})$$

(1.22)

of a spherical functor $S$ and a trivialization $\tau$ of the monodromy functor $T_{\Psi, r} : D_\Psi \xrightarrow{\sim} D_\Psi$ of the nearby category.
Introduce the full subcategory $\mathcal{D}_\bullet \subset \mathcal{D}_\Psi$ of clean objects of the nearby category. We say an object $Y \in \mathcal{D}_\Psi$ is clean if we have $S^r(Y) \simeq 0$ (equivalently, $S^l(Y) \simeq 0$) for the right adjoint $S^r$ (or left adjoint $S^l$) of $S$.

**Remark 1.4.** The terminology “clean” is motivated by the following. If instead of a spherical functor $S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi$, suppose we work with a spherical pair, in the sense of a semi-orthogonal decomposition

\[
\begin{array}{ccc}
\mathcal{D}_\Phi^- & \xleftarrow{f'} & \mathcal{D}_\Phi^+ \\
\downarrow I & & \downarrow J \\
\mathcal{D}^- & \xleftarrow{f} & \mathcal{D}^+ \\
\downarrow I^* & & \downarrow J^* \\
\mathcal{D}_\Phi^- & \xleftarrow{\tau} & \mathcal{D}_\Phi^+
\end{array}
\]

Then an object $Y \in \mathcal{D}_\Phi^+$ is clean if and only if the canonical map $J_Y \to J^* Y$ is an isomorphism.

Let $\mathcal{L}D_\bullet$ denote the inertia dg category of pairs $(Y, m)$ consisting of $Y \in \mathcal{D}_\bullet$, and an automorphism $m : Y \xrightarrow{\sim} Y$. For any automorphism $m$ of the identity functor of $\mathcal{D}_\bullet$, we have the natural translated inverse functor

\[
(1.24) \quad K_m : \mathcal{L}D_\bullet \xrightarrow{\sim} \mathcal{L}D_\bullet \quad K_m(Y, m) = (Y, m^{-1} \circ m)
\]

Finally, consider the restriction

\[
(1.25) \quad \mathcal{P}_{Cyl, 0} = \mathcal{P}_{S^2, 0}|_{Cyl}
\]

of the perverse schober $\mathcal{P}_{S^2, 0}$ to the cylinder $Cyl = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{C} \mathbb{P}^1 \setminus \{\epsilon, \infty\}$, and denote by $\Gamma(Cyl, \mathcal{P}_{Cyl, 0})$ its global sections.

Here is an abstract statement of our main result.

**Theorem 1.5.** There are natural fully faithful Chekanov and Clifford embeddings fitting into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}D_\bullet & \xleftarrow{F_{Ch}} & \Gamma(Cyl, \mathcal{P}_{Cyl, 0}) \\
\downarrow K_m & & \downarrow F_{Cl} \\
\mathcal{L}D_\bullet & \xleftarrow{F_{Ch}} & \Gamma(Cyl, \mathcal{P}_{Cyl, 0})
\end{array}
\]

where $m$ is the automorphism of the identity

\[
(1.27) \quad m : id \xrightarrow{p_\Psi} T_{\Psi, \tau} \xrightarrow{\tau^{-1}} id
\]

given by the composition of the canonical map $p_\Psi$ associated to the spherical functor $S$, and the inverse of the trivialization $\tau$ of the nearby monodromy functor.

**Corollary 1.6** (Wall-crossing formula). Under the composition $F_{Cl}^{-1} \circ F_{Ch}$, objects of the inertia category $\mathcal{L}D_\bullet$ undergo the transformation

\[
(1.28) \quad (Y, m) \mapsto (Y, m^{-1} \circ m)
\]

**Example 1.7.** Recall the perverse schober

\[
(1.29) \quad \mathcal{P}_B = (i_\ast : \text{Perf}_{\text{prop}}(P_{n-2}) \xrightarrow{i_\ast} \text{Perf}_{\text{prop}}(T^*_0))
\]
on the disk $D$ with a single critical point $0 \in D$. 
The clean objects $\text{Perf}^{\text{prop}}(T_0^\vee) \subset \text{Perf}^{\text{prop}}(T_0^\vee)$ in the nearby category are those supported away from $P_{n-2}$. Thus their inertia category has a simple description
\begin{equation}
\mathcal{L}\text{Perf}^{\text{prop}}(T_0^\vee) \cong \text{Perf}^{\text{prop}}(T_0^\vee \setminus P_{n-2} \times \mathbb{G}_m)
\end{equation}

The monodromy on the nearby category $\text{Perf}^{\text{prop}}(T_0^\vee)$ is given by tensoring with $O(1)$. To smoothly extend $\mathcal{P}_B$ to a perverse schober $\tilde{\mathcal{P}}_B$ on the sphere $S^2$, we must choose a trivialization $\tau : O \to O(1)$. There are $n$ natural choices given by the homogenous coordinates $x_1, \ldots, x_n$.

Suppose we choose say $x_n$, and write $y_1 = x_1/x_n, \ldots, y_{n-1} = x_{n-1}/x_n$. Then the wall-crossing formula (1.28) yields the birational transformation
\begin{equation}
y_i \mapsto y_i, \quad i = 1, \ldots, n-1
\end{equation}
\begin{equation}
y_n \mapsto y_n^{-1}(1 + y_1 + \cdots + y_{n-1})
\end{equation}
originally appearing in (1.1).

1.3.1. Back to geometry. By the mirror equivalence of Theorem 1.2 the conclusion of Example 1.7 applies equally well to the perverse schober
\begin{equation}
\mathcal{P}_A = (j^* : \mu \text{Sh}_L(C^n) \longrightarrow \mu \text{Sh}_{L^*}(C^n))
\end{equation}

We explain here the direct geometric interpretation of the wall-crossing formula (1.31). We will resume with the constructions and notation introduced in Section 1.1.1 above. See in particular Figure 1.

Inside of $\text{Cyl} = C \setminus \{e\} \cong C^\times$, recall the Chekanov and Clifford graphs
\begin{equation}
\Gamma_{\text{ch}} = \gamma_{\text{ch}} \cup I_{\text{ch}} \quad \Gamma_{\text{Cl}} = \gamma \cup I_{\text{Cl}}
\end{equation}
and inside of the symplectic manifold $M = \{W \neq 0\} \cong (C^\times)^n$, the corresponding Chekanov and Clifford skeleta
\begin{equation}
\mathcal{L}_{\text{ch}} = \{W \in \Gamma_{\text{ch}}, |z_1| = \cdots = |z_n|\} \quad \mathcal{L}_{\text{Cl}} = \{W \in \Gamma_{\text{Cl}}, |z_1| = \cdots = |z_n|\}
\end{equation}
From Theorem 1.5 one can show there are natural equivalences for microlocal sheaves
\begin{equation}
\mu \text{Sh}_{L_{\text{ch}}}(M) \cong \Gamma(\text{Cyl}, \mathcal{P}_A) \cong \mu \text{Sh}_{L_{\text{Cl}}}(M)
\end{equation}

Note we have closed embeddings $T_{\text{ch}} \subset \mathcal{L}_{\text{ch}}, T_{\text{Cl}} \subset \mathcal{L}_{\text{Cl}}$, and hence fully faithful functors
\begin{equation}
\mathcal{L} \text{oc}(T_{\text{ch}}) \cong \mu \text{Sh}_{T_{\text{ch}}}(M) \longrightarrow \mu \text{Sh}_{L_{\text{ch}}}(M)
\end{equation}
\begin{equation}
\mathcal{L} \text{oc}(T_{\text{Cl}}) \cong \mu \text{Sh}_{T_{\text{Cl}}}(M) \longrightarrow \mu \text{Sh}_{L_{\text{Cl}}}(M)
\end{equation}
where we write $\mathcal{L} \text{oc}$ for the dg category of local systems.

Since $\gamma_{\text{ch}}$ does not wind around $0 \in C$, we have a canonical splitting $T_{\text{ch}} \cong \gamma_{\text{ch}} \cong T_{\text{ch}}$. The choice of a coordinate, say $z_n$, from among $z_1, \ldots, z_n$ provides a parallel splitting
\begin{equation}
T_{\text{ch}} \cong \gamma_{\text{ch}} \cong T_{\text{Cl}} \quad \langle (z_1, \ldots, z_n), e^{i\theta} \rangle \longrightarrow (z_1, \ldots, e^{i\theta} z_k, \ldots, z_n)
\end{equation}
The splittings in turn provide identifications for inertia stacks
\begin{equation}
\mathcal{L} \text{oc}(T_{\text{ch}}^\vee) \cong \mathcal{L} \text{oc}(T_{\text{Cl}}^\vee \times S^1) \cong \mathcal{L} \text{oc}(T_{\text{ch}})
\end{equation}
\begin{equation}
\mathcal{L} \text{oc}(T_{\text{Cl}}^\vee) \cong \mathcal{L} \text{oc}(T_{\text{Cl}}^\vee \times S^1) \cong \mathcal{L} \text{oc}(T_{\text{Cl}})
\end{equation}

More fundamentally, the choice of the coordinate $z_n$ provides an extension
\begin{equation}
\overline{W} : C^{n-1} \times CP^1 \longrightarrow CP^1 \quad \overline{W} = [W, z]
\end{equation}
where we equip $\mathbb{C}^{n-1}$ with coordinates $z_1, \ldots, z_{n-1}$, and $\mathbb{C}P^1$ with coordinates $[z_n, z]$. This gives a smooth extension of the perverse schober $\mathcal{P}_A$ to the sphere $S^2 \simeq \mathbb{C}P^1$.

The circles $\gamma_{\text{Ch}}, \gamma_{\text{Cl}} \subset \mathbb{C}$, with their counterclockwise orientations, are naturally isotopic when regarded within $\mathbb{C}P^1 \setminus \{0\} \simeq \mathbb{C}$, but with opposite orientations. We also obtain an identification $T_{\text{Ch}}^{\text{van}} \simeq T_{\text{Cl}}^{\text{van}}$ by parallel transporting each within $\mathbb{C}^{n-1} \times \mathbb{C}P^1$ above the respective intervals $[\epsilon - \rho_{\text{Ch}}, \infty]$, $[-\infty, \epsilon - \rho_{\text{Cl}}]$ to the fiber at $\infty$.

Thus altogether, we have a canonical identification $T_{\text{Ch}} \simeq T_{\text{Cl}}$ compatible with the above splittings, but notably with the inverse map on the factor $S^1$. Now with the above identifications in hand, we have the following geometric interpretation of Theorem 1.5.

**Theorem 1.8.** For the perverse schober $\mathcal{P}_A$, the Chekanov and Clifford functors of Theorem 1.5 factor into the compositions

\begin{align*}
F_{\text{Ch}} : \mathcal{D}_\bullet &\xleftarrow{\mathcal{L}} \mathcal{L}\text{Loc}(T_{\text{Ch}}^{\text{van}}) \simeq \mathcal{L}\text{Loc}(T_{\text{Ch}}) \xrightarrow{\mu} \mu\text{Sh}_{\text{Ch}}(M) \simeq \Gamma(Cyl, \mathcal{P}_A|_{\text{cyl}}) \\
F_{\text{Cl}} : \mathcal{D}_\bullet &\xleftarrow{\mathcal{L}} \mathcal{L}\text{Loc}(T_{\text{Cl}}^{\text{van}}) \simeq \mathcal{L}\text{Loc}(T_{\text{Cl}}) \xrightarrow{\mu} \mu\text{Sh}_{\text{Cl}}(M) \simeq \Gamma(Cyl, \mathcal{P}_A|_{\text{cyl}})
\end{align*}

**Corollary 1.9.** The wall-crossing formula of Corollary 1.6 is the birational map on moduli of objects for the partially defined functor $\mathcal{L}\text{oc}(T_{\text{Ch}}) \to \mathcal{L}\text{oc}(T_{\text{Cl}})$ given by comparing clean local systems on the Chekanov and Clifford tori as objects in $\Gamma(Cyl, \mathcal{P}_A)$ with coordinates related by the given extension of $\mathcal{P}_A$ to the sphere $S^2 \simeq \mathbb{C}P^1$.

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2. **Perverse schobers**

We spell out constructions with perverse schobers on some simple but useful complex curves.

2.1. **On a disk.** Following Kapranov-Schechtman \[3\], a perverse schober on the disk $D = \{ |z| < \epsilon \} \subset \mathbb{C}P^1$ with a single critical point at $0 \in D$, in its minimal one-cut realization, is a spherical functor

\begin{equation}
\mathcal{P}_{D,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Phi)
\end{equation}

between stable dg categories. By definition, a functor $S$ is spherical if it admits both a left adjoint $S^\ell$ and right adjoint $S^r$, so that we have an adjoint triple $(S^\ell, S, S^r)$ with units and counits of adjunctions denoted by

\begin{align*}
u_r : id_\Phi &\xrightarrow{\cdot} S^r S & c_r : SS^r &\xrightarrow{\cdot} id_\Phi \\
u_\ell : id_\Phi &\xrightarrow{\cdot} SS^\ell & c_\ell : S^\ell S &\xrightarrow{\cdot} id_\Phi
\end{align*}

Moreover, if we form the natural triangles

\begin{align*}
T_{\Phi,r} := \text{Cone}(u_r)[-1] &\xrightarrow{\Phi} \text{id}_\Phi \xrightarrow{\cdot} S^r S \\
S^r &\xrightarrow{c_r} \text{id}_\Phi \xrightarrow{\cdot} \text{Cone}(u_r) =: T_{\Phi,r}
\end{align*}
\[ T_{\Psi, \ell} := \text{Cone}(u_\ell)[-1] \xrightarrow{q_{\Psi}} \text{id}_\Psi \xrightarrow{u_\ell} SS^\ell \]

\[ S^\ell S \xrightarrow{c_\ell} \text{id}_\Phi \xrightarrow{P_\Psi} \text{Cone}(u_\ell) =: T_{\Phi, \ell} \]

then the following properties are required to hold (and in fact, by a theorem of Anno-Logvinenko [1], any two imply all four):

1. **(SF1)** \( T_{\Psi, r} \) is an equivalence.
2. **(SF2)** The natural composition
   \[ S^r \xrightarrow{=} S^r SS^\ell \xrightarrow{=} T_{\Psi, r} S^\ell[1] \]
   is an equivalence.
3. **(SF3)** \( T_{\Phi, r} \) is an equivalence.
4. **(SF4)** The natural composition
   \[ S^\ell T_{\Psi, r}[-1] \xrightarrow{=} S^\ell SS^r \xrightarrow{=} S^r \]
   is an equivalence.

When the above properties hold, \( T_{\Phi, \ell}, T_{\Psi, \ell} \) are respective inverses of \( T_{\Phi, r}, T_{\Psi, r} \), and we refer to them as **monodromy functors**.

**Example 2.1** (Smooth hypersurfaces). Let \( X \) be a smooth variety. Let \( \mathcal{L}_X \to X \) be a line bundle and \( \sigma : X \to \mathcal{L}_X \) a section transverse to the zero section. Let \( Y = \{ \sigma = 0 \} \) be the resulting smooth hypersurface and \( i : Y \to X \) its inclusion.

Let \( \text{Coh}(Y), \text{Coh}(X) \) denote the respective dg categories of coherent sheaves. We will regard the line bundle \( \mathcal{L}_X \) as an object of \( \text{Coh}(X) \), and its restriction \( \mathcal{L}_Y = i^* \mathcal{L}_X \) as an object of \( \text{Coh}(Y) \). We will regard the section \( \sigma \) as a morphism \( \sigma : \mathcal{O}_X \to \mathcal{L}_X \), which by duality gives a morphism \( \sigma^\vee : \mathcal{L}_X^\vee \to \mathcal{O}_X \).

We have the adjoint triple \( (i^*, i_* , i^!) \) which satisfies functorial identities

\[ i^*(-) \simeq \mathcal{O}_Y \otimes_{\mathcal{O}_X} (-) \quad i^!(*) \simeq \mathcal{L}_Y[-1] \otimes_{\mathcal{O}_X} (-) \]

Set \( D_\Psi = \text{Coh}(Y) \), \( D_\Phi = \text{Coh}(X) \) and \( S = i_* \). Then the monodromy functors satisfy

\[ T_{\Psi, r}(-) \simeq \mathcal{L}_X \otimes_{\mathcal{O}_X} (-) \quad T_{\Phi, r}(-) \simeq \mathcal{L}_Y[-2] \otimes_{\mathcal{O}_Y} (-) \]

and hence both are equivalences. Thus (SF1) and (SF3) hold, and so \( S = i_* \) is a spherical functor. We will denote this perverse schober by

\[ \mathcal{P}_{Y \subset X} = (i_* : \text{Coh}(Y) \to \text{Coh}(X)) \]

2.2. **On a sphere.** Suppose given a perverse schober on the disk \( D = \{ |z| < \epsilon \} \subset \mathbb{C}P^1 \) with a single critical point \( 0 \in D \) in its realization as a spherical functor

\[ \mathcal{P}_{D, 0} = (S : D_\Phi \to D_\Psi) \]

Recall we then have inverse monodromy functors \( T_{\Psi, r}, T_{\Psi, \ell} \) on the nearby category \( D_\Psi \).

Let us formulate what it means to extend \( \mathcal{P}_{D, 0} \) smoothly to the sphere \( S^2 \cong \mathbb{C}P^1 \).

**Definition 2.2.** (1) A **framing** for the perverse schober \( \mathcal{P}_{D, 0} = (S : D_\Phi \to D_\Psi) \) is a trivialization of the monodromy of the nearby category

\[ \tau : \text{id} \xrightarrow{\sim} T_{\Psi, r} \]
(2) A perverse schober on the sphere $S^2 \simeq \mathbb{C}P^1$ with a single critical point $0 \in S^2$ is a pair
\[
\mathcal{P}_{S^2,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, \tau : \text{id} \sim T_{\Psi,r})
\]
consisting of a spherical functor $S$ and a framing $\tau$.

**Example 2.3** (Smooth hypersurfaces). Recall from Example 2.1 the perverse schober
\[
\mathcal{P}_{S \subset X} = (i_* : \text{Coh}(Y) \to \text{Coh}(X))
\]
Recall the monodromy of the nearby category is given by
\[
T_{\Psi,r}(-) \simeq \mathcal{L}_X \otimes \mathcal{O}_X (-)
\]
Thus framings are equivalent to isomorphisms
\[
\tau : \mathcal{O}_X \sim \to \mathcal{L}_X
\]
or in other words, non-vanishing sections of $\mathcal{L}_X$.

2.3. **On a cylinder.** Suppose given perverse schober on the disk $D = \{ |z| < \epsilon \} \subset \mathbb{C}P^1$ with a single critical point $0 \in D$ in its realization as a spherical functor
\[
\mathcal{P}_{D,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi)
\]
Let us formulate what it means to extend $\mathcal{P}_{D,0}$ smoothly to the cylinder $\text{Cyl} = \mathbb{C} \setminus \{ \epsilon \} \simeq \mathbb{C}P^1 \setminus \{ \epsilon, \infty \}$.

**Definition 2.4.** (1) A perverse schober on the cylinder $\text{Cyl} = \mathbb{C} \setminus \{ \epsilon \} \simeq \mathbb{C}P^1 \setminus \{ \epsilon, \infty \}$ with a single critical point $0 \in \text{Cyl}$ is a pair
\[
\mathcal{P}_{\text{Cyl},0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, M : \mathcal{D}_\Psi \sim \to \mathcal{D}_\Psi)
\]
consisting of a spherical functor $S$ and an additional monodromy functor $M$.

(2) The *global sections* $\Gamma(\text{Cyl}, \mathcal{P}_{\text{Cyl},0})$ is the dg category of quintuples $(X, Y_1, Y_2, \Delta, m)$ consisting of $X \in \mathcal{D}_\Psi$, $Y_1, Y_2 \in \mathcal{D}_\Psi$, an exact triangle
\[
\Delta = (Y_1[-1] \xrightarrow{i} S(X) \xrightarrow{p} Y_2 \xrightarrow{\partial} Y_1)
\]
and an isomorphism
\[
m : Y_1 \sim \to M(Y_2)
\]
Morphisms are maps of diagrams: a morphism $(X, Y_1, Y_2, \Delta, m) \to (X', Y_1', Y_2', \Delta', m')$ consists of maps $f : X \to X'$, $g_1 : Y_1 \to Y_1'$, $g_2 : Y_2 \to Y_2'$, with a lift to a map of triangles
\[
\Delta = (Y_1[-1] \xrightarrow{i} S(X) \xrightarrow{p} Y_2 \xrightarrow{\partial} Y_1)
\]
and a commutativity isomorphism $M(g_2) \circ m \simeq m' \circ g_1$.

**Remark 2.5.** It is always possible to extend a perverse schober $\mathcal{P}_{D,0}$ on the disk $D$ with a single critical point $0 \in D$ to one on the cylinder $\text{Cyl}$ by taking $M$ to be the identity functor.

**Remark 2.6.** It is always possible to restrict a perverse schober $\mathcal{P}_{S^2,0}$ on the sphere $S^2$ with a single critical point $0 \in S^2$ to one on the cylinder $\text{Cyl}$ by forgetting the framing $\tau$ and taking $M$ to be the identity functor.
Remark 2.7. The notion of global sections $\Gamma(\text{Cyl}, \mathcal{P}_{\text{Cyl},0})$ given in Definition 2.4(2) follows from considering sections supported over the specific Lagrangian skeleton $\Gamma_{\text{Ch}} = \gamma_{\text{Ch}} \cup I_{\text{Ch}} \subset \text{Cyl}$. We will see immediately below an equivalent notion of global sections that follows from considering sections supported over the alternative Lagrangian skeleton $\Gamma_{\text{Cl}} = \gamma_{\text{Cl}} \cup I_{\text{Cl}} \subset \text{Cyl}$.

2.3.1. Mutated global sections. The notion of global sections $\Gamma(\text{Cyl}, \mathcal{P}_{\text{Cyl},0})$ given above in Definition 2.4(2) is one of many equivalent possibilities. We introduce here a key alternative notion and show it is equivalent to the original.

Definition 2.8. The mutated global sections $\Gamma^\#(\text{Cyl}, \mathcal{P}_{\text{Cyl},0})$ is the dg category of quintuples $(X^\#, Y_1^\#, Y_2^\#, \Delta^\#, m^\#)$ consisting of $X^\# \in D_{\Phi}$, $Y_1^\#, Y_2^\# \in D_{\Psi}$, an exact triangle

\[
\Delta^\# = (T_{\Psi,r}(Y_1^\#)[-1] \xrightarrow{i} S(X^\#) \xrightarrow{p} Y_2^\# \xrightarrow{\partial} T_{\Psi,r}(Y_1^\#))
\]

and an isomorphism

\[
m^\#: M(Y_1^\#) \xrightarrow{\sim} Y_2^\#
\]

Morphisms are maps of diagrams as in Definition 2.4(2).

Now we will define a mutation equivalence $F^\#: \Gamma(\text{Cyl}, \mathcal{P}_{\text{Cyl},0}) \rightarrow \Gamma^\#(\text{Cyl}, \mathcal{P}_{\text{Cyl},0})$ $F^\#(X,Y_1,Y_2,\Delta,m) = (X^\#, Y_1^\#, Y_2^\#, \Delta^\#, m^\#)$

First, set $Y_1^\# := Y_2$, $Y_2^\# := Y_1$, and

\[
m^\#: M(Y_1^\#) = M(Y_2) \xrightarrow{m^{-1}} Y_1 = Y_2^\#
\]

By adjunction, the map $p : S(X) \rightarrow Y_2$ provides a map $\tilde{p} : X \rightarrow S^r(Y_2)$, and we set $X^\#: = \text{Cone}(\tilde{p})$. To construct the triangle $\Delta^\#$, note that adjunction further provides a commutative diagram

\[
\begin{array}{ccc}
S(X) & \xrightarrow{S(\tilde{p})} & S(S^r(Y_2)) \\
\downarrow{p} & & \downarrow{cr} \\
Y_2 & & \\
\end{array}
\]

Recall as well the natural triangle

\[
SS^r \xrightarrow{cr} \text{id} \xrightarrow{p_{\Psi}} T_{\Psi,r} \xrightarrow{SS^r[1]} SS^r
\]

Taking the cone of each map of (2.31), we obtain another triangle

\[
\text{Cone}(S(\tilde{p})) \xrightarrow{\tilde{p}_{\Psi}} T_{\Psi,r}(Y_2) \xrightarrow{\text{Cone}(S(\tilde{p}))[1]}
\]

Now we take $\Delta^\#$ to be the rotated triangle

\[
\Delta^\# = (T_{\Psi,r}(Y_2)[-1] \xrightarrow{i} \text{Cone}(S(\tilde{p})) \xrightarrow{\tilde{p}_{\Psi}} T_{\Psi,r}(Y_2))
\]

using the canonical identification $\text{Cone}(S(\tilde{p})) \simeq S(\text{Cone}(\tilde{p}))$. 

Remark 2.9. Let us highlight a key property of the map $\tilde{p}_\Psi : Y_1 \to T_{\Psi,r}(Y_2)$ appearing in the triangle $\Delta^t$. By construction, it arises in a commutative diagram

$$
\begin{array}{ccc}
Y_2 & \xrightarrow{\partial} & Y_1 \\
\downarrow{p_\Psi} & & \downarrow{\tilde{p}_\Psi} \\
T_{\Psi,r}(Y_2) & & 
\end{array}
$$

where $\partial$ is given in the initial triangle (2.24), and $p_\Psi$ is the canonical map. In particular, if we have $Y_1 = Y_2$, and the map $\partial : Y_2 \to Y_1$ is the identity, then $\tilde{p}_\Psi = p_\Psi$.

Lemma 2.10. The mutation functor

$$\mathfrak{F} : \Gamma(Cyl, \mathcal{P}_{Cyl}) \longrightarrow \Gamma^t(Cyl, \mathcal{P}_{Cyl}) \quad \mathfrak{F}(X, Y_1, Y_2, \Delta, m) = (X^t, Y_1^t, Y_2^t, \Delta^t, m^t)$$

is an equivalence.

Proof. We can define an inverse to $\mathfrak{F}$ beginning with the evident assignments $Y_1 := Y_2^t$, $Y_1 := Y_2^t$,

$$m : Y_1 = Y_2^t \xrightarrow{m^{-1}} M(Y_1^t) = Y_2$$

For $X$ and the triangle $\Delta$, we proceed as follows. By adjunction, the map $i : T_{\Psi,r}(Y_1^t)[-1] \to S(X^t)$ provides a map $\tilde{i} : S^t(T_{\Psi,r}(Y_1^t))[-1] \to X^t$, and we take $X := \text{Cone}(\tilde{i})[-1]$.

To construct the triangle $\Delta$, note that adjunction further provides a commutative diagram

$$
\begin{array}{ccc}
S^t(T_{\Psi,r}(Y_1^t))[-1] & \xrightarrow{S^t(i)} & S(X^t) \\
\downarrow{u_i} & & \downarrow{i} \\
T_{\Psi,r}(Y_1^t)[-1] & & 
\end{array}
$$

Recall (SF4) confirms the natural composition is an equivalence.

$$S^tT_{\Psi,r}[-1] \longrightarrow S^tS^t \longrightarrow S^t$$

and thus we can write (2.33) in the form

$$
\begin{array}{ccc}
SS^t(Y_1^t) & \xrightarrow{S^t(i)} & S(X^t) \\
\downarrow{u_i} & & \downarrow{i} \\
T_{\Psi,r}(Y_1^t)[-1] & & 
\end{array}
$$

Taking the cone of each map of (2.35), we obtain another triangle

$$
\begin{array}{ccc}
Y_1^t & \longrightarrow & Y_2^t \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\text{Cone}(S(\tilde{i})) & \longrightarrow & Y_1^t[-1] \\
\text{Cone}(S(\tilde{i})) & \longrightarrow & Y_2^t \\
\end{array}
$$

We take $\Delta$ to be the rotated shifted triangle

$$\Delta = (Y_2^t[-1] \longrightarrow \text{Cone}(S(\tilde{i}))[\epsilon] \longrightarrow Y_1^t \longrightarrow Y_2^t)$$

using the canonical identification $\text{Cone}(S(\tilde{i})) \simeq S(\text{Cone}(\tilde{i}))$.

We leave it to the reader to check the constructed functor is inverse to $\mathfrak{F}$.

$\square$
In this section, we will explain how a perverse schober on the sphere \( S^2 \cong \mathbb{C}P^1 \) with a single critical point provides a wall-crossing formula.

### 3.1. Chekanov functor

Suppose given a perverse schober

\[
P_{\text{Cyl},0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, M : \mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{D}_\Psi)
\]

on the cylinder \( \text{Cyl} = \mathbb{C} \setminus \{0\} \cong \mathbb{C}P^1 \setminus \{0, \infty\} \) with a single critical point \( 0 \in \text{Cyl} \). Thus by definition, \( S \) is a spherical functor, and \( M \) is an invertible functor.

Following Definition 2.4(2), the global sections \( \Gamma(\text{Cyl}, P_{\text{Cyl},0}) \) is the dg category of quintuples \((X,Y_1,Y_2,\Delta,m)\) consisting of \( X \in \mathcal{D}_\Phi \), \( Y_1, Y_2 \in \mathcal{D}_\Psi \), an exact triangle

\[
\Delta = (Y_1[-1] \xrightarrow{i} S(X) \xrightarrow{p} Y_2 \xrightarrow{\partial} Y_1)
\]

and an isomorphism \( m : Y_1 \sim M(Y_2) \).

Introduce the smooth perverse schober

\[
P_{\text{sm}} = (0 : \{0\} \to \mathcal{D}_\Psi, M : \mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{D}_\Psi)
\]

on the cylinder with no critical points.

Following Definition 2.4(2), the global sections \( \Gamma(\text{Cyl}, P_{\text{sm}}) \) is the dg category of quintuples \((0,Y_1,Y_2,\Delta,m)\) consisting of \( Y_1, Y_2 \in \mathcal{D}_\Psi \), an exact triangle

\[
\Delta = (Y_1[-1] \xrightarrow{i} 0 \xrightarrow{p} Y_2 \xrightarrow{\partial} Y_1)
\]

and an isomorphism \( m : Y_1 \sim M(Y_2) \).

Thus \( \Gamma(\text{Cyl}, P_{\text{sm}}) \subset \Gamma(\text{Cyl}, P_{\text{Cyl}}) \) is the full subcategory of quintuples \((0,Y_1,Y_2,\Delta,m)\) with vanishing first entry.

**Remark 3.1.** There is an evident commutative diagram

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{0} & \mathcal{D}_\Psi \\
\downarrow & & \downarrow \text{id} \\
\mathcal{D}_\Phi & \xrightarrow{S} & \mathcal{D}_\Psi
\end{array}
\]

compatible with \( M : \mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{D}_\Psi \) that induces the inclusion \( \Gamma(\text{Cyl}, P_0) \subset \Gamma(\text{Cyl}, P_{\text{Cyl}}) \). In general, passing to the adjoints to the horizontal maps in (3.5) does not lead to a commutative diagram. Thus it is not clear whether to consider (3.5) as a map of perverse schobers \( P_{\text{sm}} \to P_{\text{Cyl}} \). But we will only consider (3.5) where in the top right we take the full subcategory \( \mathcal{D}_G \subset \mathcal{D}_\Psi \) of clean objects (see Definition 3.8) for which both adjoints to \( S \) vanish, and so indeed fit into a trivially commutative diagram.

Let \( L^M \mathcal{D}_\Psi \) denote the dg path category with objects pairs \((Y,m)\) consisting of \( Y \in \mathcal{D}_\Psi \), and a monodromy isomorphism \( m : Y \sim M(Y) \); morphisms \((Y,m) \to (Y',m')\) are maps \( g : Y \to Y' \) with an isomorphism \( M(g) \circ m \simeq m' \circ g \).

**Remark 3.2.** When \( M = \text{id} \), note that \( L^M \mathcal{D}_\Psi \) is the usual inertia dg category \( L \mathcal{D}_\Phi \) of pairs \((Y,m)\) consisting of \( Y \in \mathcal{D}_\Psi \), and a monodromy isomorphism \( m : Y \sim M(Y) \).

Introduce the Chekanov functor

\[
F_{\text{Ch}} : L^M \mathcal{D}_\Psi \longrightarrow \Gamma(\text{Cyl}, P_{\text{Cyl}})
\]

\[
F_{\text{Ch}}(Y,m) = (0,Y,Y,\Delta_0,m)
\]
where $\Delta_0$ is the split triangle

$$
\Delta_0 = (Y[-1] \to 0 \to Y \xrightarrow{id} Y)
$$

Lemma 3.3. $F_{Ch}$ is fully faithful with essential image $\Gamma(Cyl, P_{sm}) \subset \Gamma(Cyl, P_{Ch,0})$.

Proof. We may use the isomorphism $\partial : Y_2 \xrightarrow{\sim} Y_1$ in the triangle (3.12) to identify $Y_1, Y_2$. Thus any quintuple $(0, Y_1, Y_2, \Delta, m)$ with vanishing first entry is in the essential image of $F_{Ch}$.

Recall morphisms $(0, Y, \Delta, m) \to (0, Y', \Delta', m')$ are maps $g_1 : Y \to Y'$, $g_2 : Y \to Y'$, with a lift to a map of triangles

$$
\Delta = (Y[-1] \to 0 \to Y \xrightarrow{id} Y)
$$

and a commutativity isomorphism $M(g_2) \circ m \simeq m' \circ g_1$. The lift to triangles provides an isomorphism $g_1 \simeq g_2$, and so morphisms are simply maps $g : Y \to Y'$ such that $M(g) \circ m \simeq m' \circ g$ as in the dg path category. \qed

3.2. Clifford functor. We repeat here the constructions of the preceding section in a parallel mutated form.

Consider again the given perverse schober

$$
P_{Cyl,0} = (S : D_\Phi \to D_\Phi, M : D_\Phi \xrightarrow{\sim} D_\Phi)
$$
on the cylinder $Cyl = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{CP}^1 \setminus \{\epsilon, \infty\}$ with a single critical point $0 \in Cyl$.

Following Definition 2.8, the mutated global sections $\Gamma^\sharp(Cyl, P_{Cyl,0})$ is the dg category of quintuples $(X^1, Y_1^1, Y_2^1, \Delta^1, m^1)$ consisting of $X^1 \in D_\Phi, Y_1^1, Y_2^1 \in D_\Phi$, an exact triangle

$$
\Delta^1_0 = (T_{\psi,r}(Y_1^1)[-1] \to S(X^1) \xrightarrow{p} Y_2^1 \xrightarrow{\beta} T_{\psi,r}(Y_1^1))
$$

and an isomorphism $m^1 : M(Y_1^1) \xrightarrow{\sim} Y_2^1$.

Consider again the smooth perverse schober

$$
P_{sm} = (0 : \{0\} \to D_\Phi, M : D_\Phi \xrightarrow{\sim} D_\Phi)
$$
on the cylinder with no critical points.

Following Definition 2.8, the mutated global sections $\Gamma^\sharp(Cyl, P_{sm})$ is the dg category of quintuples $(0, Y_1^2, Y_2^2, \Delta^2, m^2)$ consisting of $Y_1^2, Y_2^2 \in D_\Phi$, an exact triangle

$$
\Delta^2_0 = (T_{\psi,r}(Y_1^2)[-1] \to 0 \xrightarrow{p} Y_2^2 \xrightarrow{\beta} T_{\psi,r}(Y_1^2))
$$

and an isomorphism $m^2 : M(Y_1^2) \xrightarrow{\sim} Y_2^2$.

Thus $\Gamma^\sharp(Cyl, P_{sm}) \subset \Gamma^\sharp(Cyl, P_{Cyl})$ is the full subcategory of quintuples $(0, Y_1^2, Y_2^2, \Delta^2, m^2)$ with vanishing first entry.

Remark 3.4. The inclusion $\Gamma^\sharp(Cyl, P_{sm}) \subset \Gamma^\sharp(Cyl, P_{Cyl,0})$ is induced by the diagram (3.9).

Let $\mathcal{L}_{M \circ T_{\psi,r}, D_\Phi}$ denote the dg path category of pairs $(Y^2, m^2)$ consisting of $Y^2 \in D_\Phi$, and a monodromy isomorphism $m^2 : M(T_{\psi,r}(Y^2)) \xrightarrow{\sim} Y^2$; morphisms $(Y^2, m^2) \to (Y^2', m^2')$ are maps $g : Y \to (Y^2')$ with an isomorphism $g \circ m^2 \simeq (m^2') \circ M(T_{\psi,r}(g))$.

Remark 3.5. When $M \circ T_{\psi,r} = id$, note that $\mathcal{L}_{M \circ T_{\psi,r}, D_\Phi}$ is the usual inertia dg category $\mathcal{L}D_\Phi$. 

Introduce the **Clifford functor**

\[ F_{\text{Cl}} : \mathcal{L}_{M=T_{\Phi,r}D_{\Psi}} \rightarrow \Gamma^t(Cyl, \mathcal{P}_{\text{Cyl},0}) \]

\[ F_{\text{Cl}}(Y^t, m^t) = (0, T_{\Psi,r}^{-1}(Y^t), Y^t, \Delta^t_0, m^t) \]

where \( \Delta^t_0 \) is the split triangle

\[ \Delta^t_0 = (Y^t[-1] \rightarrow 0 \rightarrow Y^t \rightarrow Y) \]

**Lemma 3.6.** \( F_{\text{Cl}} \) is fully faithful with essential image \( \Gamma^t(Cyl, \mathcal{P}_{\text{Cyl}}) \subset \Gamma^t(Cyl, \mathcal{P}_{\text{Cyl},0}) \).

**Proof.** We may use the isomorphism \( \partial : Y^t_2 \simeq T_{\Psi,r}(Y^t_1) \) in triangle (3.12) to identify \( T_{\Psi,r}(Y^t_1), Y^t_2 \).

Thus any quintuple \( (0, Y^t_1, Y^t_2, \Delta^t, m^t) \) with vanishing first entry is in the essential image of \( F_{\text{Cl}} \).

The rest of the proof is the same as that of Lemma 3.3. \( \square \)

### 3.3. Framing identifications.

Now suppose given a perverse schober

\[ \mathcal{P}_{S^2,0} = (S : D_{\Phi} \rightarrow D_{\Psi}, \tau : \text{id} \simeq T_{\Psi,r}) \]

on the sphere \( S^2 \simeq \mathbb{C}P^1 \) with a single critical point at \( 0 \in S^2 \).

Let us restrict to a perverse schober

\[ \mathcal{P}_{\text{Cyl},0} = \mathcal{P}_{S^2,0}|_{\text{Cyl}} = (S : D_{\Phi} \rightarrow D_{\Psi}, \text{id} : D_{\Psi} \simeq D_{\Psi}) \]

on the cylinder \( \text{Cyl} = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{C}P^1 \setminus \{\epsilon, \infty\} \) by forgetting the framing \( \tau \) and taking the additional monodromy functor \( M \) to be the identity. Consider the smooth perverse schober

\[ \mathcal{P}_{m} = (0 : \{0\} \rightarrow D_{\Psi}, \text{id} : D_{\Psi} \simeq D_{\Psi}) \]

again taking the additional monodromy functor \( M \) to be the identity.

Observe that the framing \( \tau : \text{id} \simeq T_{\Psi,r} \) provides a canonical equivalence \( \tau : \mathcal{L}D_{\Psi} \simeq \mathcal{L}T_{\Psi,r}D_{\Psi} \).

Thus the Chekanov and Clifford functors take the respective forms

\[ F_{\text{Ch}} : \mathcal{L}D_{\Psi} \rightarrow \Gamma(Cyl, \mathcal{P}_{\text{Cyl}}) \]

\[ F_{\text{Ch}}(Y, m) = (0, Y, \Delta_0, m) \]

\[ \Delta_0 = (Y[-1] \rightarrow 0 \rightarrow Y \rightarrow Y) \]

\[ F_{\text{Cl}} : \mathcal{L}D_{\Psi} \rightarrow \Gamma^t(Cyl, \mathcal{P}_{\text{Cyl}}) \]

\[ F_{\text{Cl}}(Y^t, m^t) = (0, Y^t, Y^t, \Delta^t_0, m^t) \]

\[ \Delta^t_0 = (Y^t[-1] \rightarrow 0 \rightarrow Y^t \rightarrow Y^t) \]

**Remark 3.7.** Recall the mutation equivalence \( \mathcal{F}^t : \Gamma(Cyl, \mathcal{P}_{\text{Cyl},0}) \simeq \Gamma^t(Cyl, \mathcal{P}_{\text{Cyl}}) \). We caution the reader that the following diagram is not commutative

\[ \Gamma(Cyl, \mathcal{P}_{\text{Cyl},0}) \]

\[ \mathcal{L}D_{\Psi} \]

\[ F_{\text{Ch}} \]

\[ F_{\text{Cl}} \]

\[ \mathcal{F}^t \]

\[ \Gamma^t(Cyl, \mathcal{P}_{\text{Cyl},0}) \]

We will introduce a correct intertwining relation in the next section.
3.4. **Clean objects.** Let us continue with the setup of the preceding section.

**Definition 3.8.** We say an object \( Y \in D_\Psi \) is clean if \( S^r(Y) \simeq 0 \).

We denote by \( D_\Psi \subset D_\Psi \) the full subcategory of clean objects.

**Lemma 3.9.** An object \( Y \in D_\Psi \) is clean if and only if any of the following hold:

1. \( S^r(Y) \simeq 0 \).
2. \( p_\Psi : \text{id}_\Psi \to T_{\Psi,r} \) is an isomorphism.
3. \( q_\Psi : T_{\Psi,\ell} \to \text{id}_\Psi \) is an isomorphism.

**Proof.** For the equivalence with property (1), recall (SF2): the natural composition

\[
S^r \longrightarrow S^rS^\ell \longrightarrow T_{\Phi,r}S^\ell[1]
\]

is an equivalence and (SF3): \( T_{\Phi,r} \) is an equivalence.

The equivalences with properties (2) and (3) are immediate from the triangles defining the canonical maps. \( \square \)

**Remark 3.10.** Note that the monodromy \( T_{\Psi,r} \), and hence also its inverse \( T_{\Psi,\ell} \), preserves clean objects by property (1) of the lemma and (SF4).

**Remark 3.11.** It is important to distinguish between a framing \( \tau : \text{id} \sim T_{\Psi,r} \) and the canonical map \( p_\Psi : \text{id}_\Psi \to T_{\Psi,r} \). The framing \( \tau \) is an additional structure not intrinsic to the perverse schober, and by definition, it is required to be an isomorphism. The canonical map \( p_\Psi \) is intrinsic to the perverse schober, but not necessarily an isomorphism when evaluated on some objects.

**Example 3.12** (Smooth hypersurfaces). Recall from Example 2.1 the perverse schober

\[
(3.23) \quad \mathcal{P}_{Y \subset X} = (i_* : \text{Coh}(Y) \to \text{Coh}(X))
\]

An object \( F \in \text{Coh}(X) \) is clean if and only if \( i^* F \simeq 0 \). Equivalently, an object \( F \in \text{Coh}(X) \) is clean if and only if either (hence both) of the maps

\[
(3.24) \quad F \xrightarrow{\sigma} F \otimes_{\mathcal{O}_X} \mathcal{L}_X \quad F \otimes_{\mathcal{O}_X} L_X^\vee \xrightarrow{\sigma^\vee} \text{id}
\]

is an isomorphism. From any of the above descriptions, we see that an object \( F \in \text{Coh}(X) \) is clean if and only if it is supported away from \( Y = \{ \sigma = 0 \} \).

Now introduce the perverse schober

\[
(3.25) \quad \mathcal{P}_\bullet = (0 : \{0\} \to \mathcal{D}_\bullet, \text{id} : \mathcal{D}_\bullet \sim \mathcal{D}_\bullet)
\]

on the cylinder with no critical points by taking the full dg subcategory \( \mathcal{D}_\bullet \subset \mathcal{D}_\Psi \) of clean objects and the additional monodromy functor \( M \) to be the identity.

Given any automorphism \( m \) of the identity functor of \( \mathcal{D}_\bullet \), we have a corresponding translated inverse functor on the dg inertia category

\[
(3.26) \quad K_m : \mathcal{L}\mathcal{D}_\bullet \sim \mathcal{L}\mathcal{D}_\bullet \quad K_m(Y,m) = (Y, m^{-1} \circ m)
\]

Note we have a canonical identification \( m^{-1} \circ m \simeq m \circ m^{-1} \) by functoriality.

Now we have the intertwining:
**Proposition 3.13.** The restrictions of the Chekanov and Clifford functors to clean objects fits into a canonically commutative diagram

\[
\begin{array}{ccc}
\mathcal{LD}_\bullet & \overset{F_{Ch}}\longrightarrow & \Gamma(Cyl, \mathcal{P}_{Cyl}) \\
\downarrow{\mathsf{K}_m} & & \downarrow{\mathsf{g}^\sharp} \\
\mathcal{LD}_\bullet & \overset{F_{Cl}}\longrightarrow & \Gamma^\sharp(Cyl, \mathcal{P}_{Cyl})
\end{array}
\]

where the automorphism \( m \) of the identity functor of \( \mathcal{D}_\bullet \) is the composition

\[
(3.28) \quad m : \text{id} \overset{p_\Psi}{\rightarrow} T_{\Psi, \tau} \overset{\tau^{-1}}{\rightarrow} \text{id}
\]

of the canonical morphism \( p_\Psi \) (which is invertible on clean objects) and the inverse of the framing \( \tau \).

**Proof.** For \((Y, m) \in \mathcal{LD}_\bullet\), we can apply the definitions to find:

\[
(3.29) \quad F^\sharp(F_{Ch}(Y, m)) = F^\sharp(0, Y, Y, \Delta, m) = (0, Y, Y, \Delta^\sharp, m^{-1})
\]

where \( \Delta^\sharp \) is the triangle

\[
(3.30) \quad \Delta^\sharp = (Y[-1] \rightarrow 0 \rightarrow Y \overset{m=\tau^{-1}o_\Psi}{\rightarrow} Y)
\]

as highlighted in Remark 2.9.

On the other hand, we can also apply the definitions to find:

\[
(3.31) \quad F_{Cl}(\mathsf{K}_m(Y, m)) = F_{Cl}(Y, m^{-1} \circ m) = (0, Y, Y, \Delta^\sharp_0, m^{-1} \circ m)
\]

where \( \Delta^\sharp_0 \) is the triangle

\[
(3.32) \quad \Delta^\sharp_0 = (Y[-1] \rightarrow 0 \rightarrow Y \overset{\text{id}}{\rightarrow} Y)
\]

Thus we seek an isomorphism

\[
(3.33) \quad (0, Y, Y, \Delta^\sharp, m^{-1}) \xrightarrow{\sim} (0, Y, Y, \Delta^\sharp_0, m^{-1} \circ m)
\]

or in other words, a map of triangles

\[
(3.34) \quad \begin{array}{ccc}
\Delta^\sharp = (Y[-1] & \rightarrow 0 & \rightarrow Y \overset{m=\tau^{-1}o_\Psi}{\rightarrow} Y) \\
g | & 0 | & g_1 \\
\Delta^\sharp_0 = (Y[-1] & \rightarrow 0 & \rightarrow Y \overset{\text{id}}{\rightarrow} Y)
\end{array}
\]

with a commutativity isomorphism \( g_2 \circ m^{-1} \simeq m^{-1} \circ m \circ g_1 \). We take \( g_1 = \text{id} \), \( g_2 = m \), and the canonical isomorphism \( g_2 \circ m^{-1} = m \circ m^{-1} \simeq m^{-1} \circ m \). \( \square \)

**Corollary 3.14.** \( \mathsf{K}_m \cong F^{-1}_{Cl} \circ \mathsf{g}^\sharp \circ F_{Ch} \).

**Example 3.15** (Smooth hypersurfaces). Recall from Example 2.1 the perverse schober

\[
(3.35) \quad \mathcal{P}_{Y \subset X} = (i_\ast : \text{Coh}(Y) \rightarrow \text{Coh}(X))
\]

Recall framings are equivalent to isomorphisms

\[
(3.36) \quad \tau : \mathcal{O}_X \xrightarrow{\sim} \mathcal{L}_X
\]
or in other words, non-vanishing sections of $L_X$. Recall an object $F \in \text{Coh}(X)$ is clean if and only if $F$ is supported away from $Y \subset X$. For a clean object $F \in \text{Coh}(X)$, we have the central automorphism

\[(3.37) \quad m : F \xrightarrow{\sigma} F \otimes_{\mathcal{O}_{X}} L_X \xrightarrow{\tau^{-1}} F \]

We may view $m = \sigma/\tau$ as a function vanishing on $Y = \{\sigma = 0\}$.

4. APPLICATION TO TORIC MUTATION

Now let us return to the local model of toric mutations introduced in Section 1.1.1 and further discussed in 1.3.1. We will adopt the constructions and notation established therein.

We will derived the wall-crossing formula (1.1) from the formalism developed in Section 2. Specifically Corollary 3.14.

As recalled in Section 1.2, there is a mirror equivalence of perverse schobers $\mathcal{P}_A \simeq \mathcal{P}_B$, i.e. a canonically commutative diagram with vertical equivalences

\[(4.1) \quad \mu\text{Sh}_L(\mathbb{C}^n) \xrightarrow{j^*} \mu\text{Sh}_{L^*}(\mathbb{C}^n) \xrightarrow{\sim} \text{Perf}_{\text{prop}}(P_{n-2}) \xrightarrow{i_*} \text{Perf}_{\text{prop}}(\mathbb{T}_d^1) \]

First, let us extend $\mathcal{P}_A$ to a perverse schober on the cylinder by taking the additional monodromy functor to be the identity. Let us write $\mathcal{P}_{A,\text{sm}}$ for the perverse schober on the cylinder with the same generic structure but trivial vanishing category.

Then it is an easy exercise, whose proof we sketch, to deduce the following from Theorem 1.2.

**Proposition 4.1.** For the Chekanov and Clifford skeleta $L_{Ch}, L_{Cl} \subset M$, there are natural equivalences

\[(4.2) \quad \mu\text{Sh}_{L_{Ch}}(M) \simeq \Gamma(\text{Cyl}, \mathcal{P}_A) \quad \mu\text{Sh}_{L_{Cl}}(M) \simeq \Gamma^4(\text{Cyl}, \mathcal{P}_A) \]

Moreover, for the Chekanov and Clifford tori $T_{Ch} \subset L_{Ch}, T_{Cl} \subset L_{Cl}$, the above equivalences restrict to full subcategories to give

\[\text{Loc}(T_{Ch}) \simeq \mu\text{Sh}_{T_{Ch}}(M) \simeq \Gamma(\text{Cyl}, \mathcal{P}_{A,\text{sm}}) \quad \text{Loc}(T_{Cl}) \simeq \mu\text{Sh}_{T_{Cl}}(M) \simeq \Gamma^4(\text{Cyl}, \mathcal{P}_{A,\text{sm}})\]

**Proof.** Let us focus on the first equivalence of (4.2); the second is similar.

Observe that in a small neighborhood $N \subset \mathbb{C}$ around the circle $\gamma_{Ch}$, we have an isomorphism of pairs $(N, \Gamma_{Ch} \cap N) \simeq (B^*S^1, \Lambda \cap B^*S^1)$, where $B^*S^1 \subset T^*S^1$ is a small neighborhood of the zero-section $S^1 \subset T^*S^1$, and $\Lambda \subset T^*S^1$ is the union of the zero-section $S^1 \subset T^*S^1$ and a single conormal ray $T_p^*S^1 \subset T^*S^1$ based at a point $p \in S^1$. It is a standard exercise to see

\[(4.3) \quad \mu\text{Sh}_{\Lambda \cap B^*S^1}(B^*S^1) \simeq \text{Perf}_{\text{prop}}(\mathbb{A}^1)\]

Typically one views objects of $\text{Perf}_{\text{prop}}(\mathbb{A}^1)$ as objects $Y \in \text{Perf}(pt)$ together with an endomorphism. Equivalently, we can view objects as quadruples $(Y_1, Y_2, \vartheta, m)$ consisting of objects $Y_1, Y_2 \in \text{Perf}(pt)$, a map $\vartheta : Y_2 \to Y_1$, and an isomorphism $m : Y_1 \xrightarrow{\sim} Y_2$. Namely, one uses $m$ to identify $Y_2$ and $Y_1$ so that $\vartheta$ becomes an endomorphism. Note we can normalize the equivalence (4.3) so that restriction to the ray $T_p^*S^1$ corresponds to forming the shifted cone $\text{Cone}(\vartheta)[-1]$. 
Next, observe that we also have an isomorphism of pairs \((W^{-1}(N), L_{Ch} \cap W^{-1}(N)) \cong (B^* S^1, \Lambda \cap B^* S^1) \times (T^* T^{\text{van}}_{Ch}, T^{\text{van}}_{Ch})\), and hence a natural equivalence
\[
\mu Sh_{L_{Ch}}(W^{-1}(N)) \cong \text{Perf}_{\text{prop}}(A^1) \otimes \text{Loc}(T^{\text{van}}_{Ch})
\]
Thus to see the first equivalence of (4.2), one can apply Theorem 1.2 to see that \(\mu Sh_{L_{Ch}}(M)\) classifies quadruples \((X, Y_1, Y_2, \Delta, m)\) as in the definition of \(\Gamma(\text{Cyl}, \mathcal{P}_A)\). Moreover, the asserted restricted equivalence is evident
\[
\text{Loc}(T_{Ch}) \cong \mu Sh_{T_{Ch} \cap W^{-1}(N)}(W^{-1}(N)) \cong \text{Perf}_{\text{prop}}(\mathbb{G}_m) \otimes \text{Loc}(T^{\text{van}}_{Ch}) \cong \text{Loc}(T_{Ch})
\]

**Remark 4.2.** Following Remark 4.1, we could also consider \(\mu Sh_{L_{crit}}(M)\) for the critical skeleton \(L_{crit} \subset M\). It is again straightforward to construct a natural equivalence \(\mu Sh_{L_{crit}}(M) \cong \Gamma(\text{Cyl}, \mathcal{P}_A)\) as a consequence of Theorem 1.2. Here it is less evident how to speak about the full subcategories corresponding to \(\mu Sh_{T_{Ch}}(M), \mu Sh_{T_{Cl}}(M)\).

Now let us consider framings for \(\mathcal{P}_A\).

On the one hand, the circle \(\gamma_{Ch}\) does not wind around \(0 \in \mathbb{C}\), and so we have a canonical splitting \(T^{\text{van}}_{Ch} \times \gamma_{Ch} \cong T_{Ch}\), and hence an identification for the inertia stack
\[
\mathcal{L}\text{Loc}(T^{\text{van}}_{Ch}) \cong \text{Loc}(T^{\text{van}}_{Ch} \times S^1) \cong \text{Loc}(T_{Ch})
\]

On the other hand, the circle \(\gamma_{Cl}\) winds once around \(0 \in \mathbb{C}\), and so a splitting \(T^{\text{van}}_{Cl} \times \gamma_{Cl} \cong T_{Cl}\) provides a framing \(\tau\). The choice of a coordinate, say \(z_n\), from among \(z_1, \ldots, z_n\) gives such a splitting
\[
T^{\text{van}}_{Cl} \times \gamma_{Cl} \cong T_{Cl} \quad ((z_1, \ldots, z_n), e^{i\theta}) \longrightarrow (z_1, \ldots, z_{n-1}, e^{i\theta} z_n)
\]
and thus a grading \(\tau\). The splitting in turn provides an identification for the inertia stack
\[
\mathcal{L}\text{Loc}(T^{\text{van}}_{Cl}) \cong \text{Loc}(T^{\text{van}}_{Cl} \times S^1) \cong \text{Loc}(T_{Cl})
\]

**Remark 4.3.** Under the equivalence \(\mathcal{P}_A \cong \mathcal{P}_B\), the framing given by the coordinate \(z_n\) corresponds to the homogenous coordinate section \(x_n : \mathcal{O} \to \mathcal{O}(1)\).

More fundamentally, the choice of the coordinate \(z_n\) provides an extension
\[
W : \mathbb{C}^{n-1} \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \quad W = [W, z]
\]
where we equip \(\mathbb{C}^{n-1}\) with coordinates \(z_1, \ldots, z_{n-1}\), and \(\mathbb{CP}^1\) with coordinates \([z_n, z]\).

The circles \(\gamma_{Ch}, \gamma_{Cl} \subset \mathbb{C}\), with their counterclockwise orientations, are naturally isotopic when regarded within \(\mathbb{CP}^1 \setminus \{0\} \cong \mathbb{C}\), but with opposite orientations. We also obtain an identification \(T^{\text{van}}_{Ch} \cong T^{\text{van}}_{Cl}\) by parallel transporting each within \(\mathbb{C}^{n-1} \times \mathbb{CP}^1\) above the respective intervals \([\epsilon - \rho_{Ch}, \infty], [-\infty, \epsilon - \rho_{Cl}]\) to the fiber at \(\infty \in \mathbb{CP}^1\).

Thus altogether, we have a canonical identification \(T_{Ch} \cong T_{Cl}\) compatible with the above splittings, but notably with the inverse map on the factor \(S^1\). Now with the above identifications, one can trace through the definitions to conclude the following.

**Theorem 4.4.** For the perverse schroer \(\mathcal{P}_A\), the Chekanov and Clifford functors factor into the compositions
\[
F_{Ch} : \mathcal{L}\mathcal{D} \longrightarrow \mathcal{L}\text{Loc}(T^{\text{van}}_{Ch}) \cong \text{Loc}(T_{Ch}) \longrightarrow \mu Sh_{L_{Ch}}(M) \cong \Gamma(\text{Cyl}, \mathcal{P}_A|_{\text{cycl}})
\]
\[
F_{Cl} : \mathcal{L}\mathcal{D} \longrightarrow \mathcal{L}\text{Loc}(T^{\text{van}}_{Cl}) \cong \text{Loc}(T_{Cl}) \longrightarrow \mu Sh_{L_{Cl}}(M) \cong \Gamma(\text{Cyl}, \mathcal{P}_A|_{\text{cycl}})
\]
Corollary 4.5. The wall-crossing formula of Corollary 3.14 is the birational map on moduli of objects for the partially defined functor $\text{Loc}(T_{Ch}) \to \text{Loc}(T_{Cl})$ given by comparing clean local systems on the Chekanov and Clifford tori as objects in $\Gamma(Cyl, \mathcal{P}_A)$ with coordinates related by the framing $\tau$.

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