I briefly discuss recent results of numerical simulations addressing the generation of a cosmological gravitational wave background produced by turbulence sources in the early universe. Contribution to the 2021 Gravitation session of the 55th Rencontres de Moriond.

1 Introduction

The field of gravitational astronomy is in a period of blossoming. Starting in 2015 with the first detected gravitational wave event GW150914, the LIGO-Virgo collaboration has recently released the first half of the third observing run O3a, with 39 new observed events, amounting to a total of 50 detected events. In the 1–100 nHz regime, pulsar-timing arrays (PTA) seek to detect GWs by monitoring the time-delay signals of an array of millisecond pulsars. Recently, it has been reported by the North American Nanohertz Observatory for Gravitational Waves (NANOGrav) after 12.5 years of observation a common-spectrum process, although the statistical significance of a quadrupolar correlation, as expected for GW signals, is still not conclusive. Even though most of the current applications focus on astrophysical events, cosmological applications are becoming more relevant, and space-based GW detectors, such as the Laser Interferometer Space Antenna (LISA), PTA and CMB anisotropies might even allow us to explore the first moments of the history of our Universe. Many sources of GWs in the early universe have been proposed, and how to disentangle the different contributions is a complicated problem that requires a lot of work from the GW community in the years to come. An interesting source of GWs in the early universe corresponds to turbulence, which can be generated by violent events, such as first-order cosmological phase transitions or by primordial magnetic fields due to the high conductivity of the primordial plasma producing magnetohydrodynamic (MHD) turbulence. Previous analytic estimates have suggested that turbulence generated at the electroweak phase transition (EWPT) can produce GWs that can be detected by LISA. Although semi-analytical models of MHD turbulence have been used to predict the GW spectral shape, the highly non-linear dynamical evolution of MHD turbulence requires the use of numerical simulations. I present an overview of the status of such simulations and a discussion
of some of the latest relevant results.

2 Magnetohydrodynamic turbulence and GW production

The stress tensor that leads to GW production is sourced by velocity and magnetic fields,

\[ T_{ij} = (p/c^2 + \rho)\gamma^2 u_i u_j - B_i B_j + \delta_{ij}B^2/2, \]  

where we have neglected electric fields due to the high conductivity of the early universe during the radiation-dominated era. The magnetohydrodynamic (MHD) conservation laws on an isotropic and homogeneous universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric tensor, with a radiation equation of state \( p = \rho/3 \), are

\[ \partial_t \ln \rho = -\frac{4}{3} (\nabla_j u_j + u_j \nabla_j \ln \rho) + \frac{1}{\rho} \left( u_j \varepsilon_{jmn} J_m B_n + \eta J^2 \right), \]  

\[ D_t u_i = \frac{1}{3} u_i (\nabla_j u_j + u_j \nabla_j \ln \rho) - \frac{u_i}{\rho} \left( u_j \varepsilon_{jmn} J_m B_n + \eta J^2 \right) - \frac{1}{4} \nabla_i \ln \rho + \frac{3}{4\rho} \varepsilon_{imn} J_m B_n + \frac{2}{\rho} \nabla_j \rho \nu S_{ij} + F_i, \]  

\[ \partial_t B_i = \varepsilon_{imn} \nabla_m (\varepsilon_{npq} u_p B_q - \eta J_n + \mathcal{E}_n), \]  

where the magnetic and density fields are expressed in units normalized by the radiation energy density at the time of turbulence generation, and are comoving with the Universe expansion, and the time is normalised to the initial conformal time of turbulence generation. We consider two main sources of turbulence in the early universe:

1. Hydrodynamic turbulence induced by a first-order phase transition.
2. Primordial magnetic fields that are produced or present at or during a phase transition.

The turbulence is characterized by the spectral peak \( k_* \) of the sourcing (velocity or magnetic) field. The scale \( k_* \) of the turbulence is comoving and normalized by the Hubble scale \( H_* \) at the time of generation. Physically motivated values of \( k_* \) are about 100 for the EWPT\(^{10}\) and 10 for the QCD phase transition.\(^{11}\) We consider the initial energy density of the turbulent source to be a fraction of the radiation energy density at the time of generation that cannot exceed 10% due to the Big Bang nucleosynthesis limit.\(^{12}\) In the case of magnetic fields, Fermi observations of blazar spectra give lower limits on the magnetic field amplitudes present in cosmic voids.\(^{13}\) Assuming that these fields have evolved from cosmological seeds allows us to impose lower bounds on primordial magnetic fields that have been generated at some early cosmological epoch. The presence of partial helicity seems to be required to explain the observed lower bounds at very large scales if the magnetic field is produced at the electroweak scale.\(^{7}\)

The tensor-mode perturbations \( h_{ij}^{\text{phys}} \) above the FLRW background are described by the GW equation. For anisotropic sources, it reads\(^{14,15}\)

\[ \left( \partial_t^2 - \nabla^2 \right) h_{ij} = 6T_{ij}^{TT}/t, \]  

for scaled strains \( h_{ij} = a h_{ij}^{\text{phys}} \), comoving coordinates and stress tensor components, and conformal time. We have used a normalization appropriate for numerical simulations\(^{8,9}\) and TT corresponds to the traceless-transverse projection.

3 Numerical results

The set of partial differential equations given in the previous section can be computed by performing numerical simulations, in which we solve the MHD equations for the stress tensor, \( T_{ij} \),
and then compute the resulting GW radiation, sourced by its TT projection. The open-source PENcil Code\textsuperscript{16} contains a module for GW computations that has been recently added\textsuperscript{8} and that is being used in recent studies of GWs sourced by early-universe turbulence.\textsuperscript{9,17,18,19,20,21} Different turbulent scenarios can be considered to model the initial magnetic or velocity fields.\textsuperscript{9}

1. Turbulent primordial magnetic fields present at the phase transition with a characteristic scale given as a fraction of the Hubble scale. When the EWPT is considered, the resulting spectrum peaks around the maximum LISA sensitivity when $k_{\ast} \sim 2\pi \times 100$, and we reach PTA sensitivities when the characteristic scale is comparable to the horizon of the QCD phase transition.

2. Magnetic or velocity fields sourced during a short period of time ($\sim 0.1 H^{-1}$) via a forcing term introduced in the induction or the momentum equations, respectively. The resulting GW production is enhanced,\textsuperscript{9,17} and hence the prospects of detectability are improved. The results show that acoustic turbulence (from velocity fields) is more efficient in producing GW than vortical turbulence (both from magnetic and velocity fields),\textsuperscript{9} as it had been previously computed numerically for sound waves.\textsuperscript{23}

In both cases, the GW is generated in a very short amount of time ($\delta t \sim 1/k_{\ast}$) after the energy of the source has reached its maximum value. The time evolution of the small wave number modes of GW energy density shows a time increase proportional to $\delta t^2$ at earlier times. Following such time evolution of the different modes, the subinertial range of the GW spectrum (with spectral peak $k_{GW} = 2k_{\ast}$, due to being sourced by the convoluted magnetic and/or velocity fields) presents $\Omega_{GW} \sim k$ spectrum below the peak down to scales of the order of the Hubble scale. Previous analytical estimates predicted $\Omega_{GW} \sim k^3$ spectrum below the peak, so this is a novel result of the numerical simulations.\textsuperscript{9} Slightly different power laws can be observed depending on the specific dynamics\textsuperscript{18,19,20,21} and the specific description is still on-going work. Longer times of forcing have been considered,\textsuperscript{17} as well as magnetic fields produced by the chiral magnetic effect.\textsuperscript{20} The effects of non-Gaussian turbulent fields (potentially produced by the dynamical MHD evolution) can also alter the slopes of the stress and the resulting GW spectra.\textsuperscript{19,24} The $k^3$ power law is observed to appear at short times\textsuperscript{9} while the GW amplitude is growing and it is expected to be kept at super-horizon scales due to causality. However, simulations that include super-horizon scales and characteristic scales require a very large dynamical range and are challenging. In general, vortical sources with a total energy density of 1-10% of the radiation energy can source GW signals detectable by LISA with a signal-to-noise ratio (SNR) of 10\textsuperscript{9}

Parity-odd violating processes during the early universe lead to circularly polarized GW signals. The degree of circular polarization can be computed from the results of the numerical simulations and it has recently been studied numerically in the case of stationary turbulence\textsuperscript{17} The detection of polarization using the dipolar response of LISA induced by our proper motion\textsuperscript{25}
or combining LISA and other space-based GW detectors, e.g., TianQin\cite{2026}, from different turbulent sources using numerical simulations is on-going work by the author and collaborators.

Following the recently reported detections by NANOGrav\cite{202}, the possibility that it corresponds to a primordial magnetic field produced at the QCD phase transition, with a characteristic scale of $k_\ast \sim 10$ has been proposed\cite{2020,2027}. Moreover, if the magnetic field is non-helical, the amplitude and coherence scales at recombination are compatible with those recently proposed to relieve the Hubble tension\cite{2028}. This is a topic currently under investigation by the author and collaborators. Note, however, that the NANOGrav observations cannot be confirmed yet to correspond to a GW signal.

4 Conclusions

The new era of gravitational astronomy is at its beginning stage, and it has already led to great discoveries, challenging our understanding on astrophysics. Future detections will keep providing us with new information of our Universe, and with new challenges, even at scales that are not accessible to traditional astronomy. One of the many cosmological sources of GWs is MHD turbulence. The numerical implementation of a GW module within the Pencil Code has been shown to be very useful as a tool to compute GW signals produced at the early universe, which combined with analytical calculations, can shed light into our understanding of the spectral shape of cosmological GW backgrounds. It is particularly useful to address the turbulence dynamics of the velocity and/or magnetic fields, which required a case-specific turbulence modelling in previous analytical calculations. The GW spectral shape highly depends on the mechanism of turbulence, and the range of possible characteristic turbulent scales and sourcing amplitudes can be explored using numerical simulations. The resulting GW signals produced by MHD turbulence at the EWPT are potentially detectable by future space-based GW detector LISA, and a novel $\Omega_{GW} \sim f$ spectrum that extends from the Hubble scale to the frequency corresponding to twice the characteristic sourcing scale, has been shown to appear while the turbulence source is acting, in time scales related to the turbulence scale. If the recent PTA observations are proven to be produced by a background of gravitational waves, this would be consistent with the presence of primordial magnetic fields at the QCD phase transition with a characteristic scale near the horizon, and could also be compatible with the presence of primordial magnetic fields at recombination. The latter have been proposed to be able to relieve the Hubble tension. The detection of such GW signals would provide clean information on the turbulence sources of the GW spectrum, e.g., primordial magnetic fields (that could explain the
current observations of magnetic fields at cosmic void scales) or velocity fields induced by the expansion of first-order phase transition bubbles, among other sources. The potential detection of the circular polarization of GWs by LISA (or by combining the results from LISA and an additional space-based GW detector, e.g., TianQin) can provide us with information of parity-violation processes in the early-universe, which can be related to, for example, the presence of helicity in magnetic fields at later times and to electroweak baryogenesis.

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