On the Space of $C^1$ Regular Curves on Sphere with Constrained Curvature

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Abstract. In this paper, we prove that the $C^0$ and $C^1$ topologies are the same on the set of $C^1$ regular curves in the 2-sphere whose tangent vectors are Lipschitz continuous, and the a.e. existing geodesic curvatures are essentially bounded in an open interval. Besides, we study the subset consisting of curves that start and end at given points with given directions, and prove that this subset is a Banach manifold.

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1. Introduction

This paper is motivated by the investigation on the topologies of the space of $C^r$ regular curves in the unit 2-sphere $S^2$, $r \geq 1$. Here we briefly recall some results on this topic. In 1956, Smale [12] proved that the space of $C^r$ ($r \geq 1$) regular closed curves in $S^2$, has only two connected components. Each of them is homotopically equivalent to $SO_3(\mathbb{R}) \times \Omega S^3$, where $\Omega S^3$ denotes the space of all continuous closed curves in $S^3$ with the $C^0$ topology. Later in
1970, Little [4] proved that there are a total of 6 second order non-generate regular homotopy classes of $C^r$, $r \geq 2$, regular closed curves in $S^2$. In 1999, Shapiro and Khesin [3] began to study the topology of the space of all smooth regular locally convex curves (not necessarily closed) in $S^2$ which start and end at given points with given directions, and especially obtained the number of connected components of the corresponding space. During 2009–2012, in [6,7] and [8], Saldanha did further work on the higher homotopy of the space of locally convex curves on $S^2$ and gave an explicit homotopy for space of locally convex curves with prescribed initial and final Frenet frames. Recently, in 2013, Saldanha and Z¨uhlke [11] extended Little’s result to the space of $C^r$, $r \geq 2$, regular closed curves with geodesic curvature constrained in an open interval $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$. It is worth mentioning that they used $C^1$ regular curves in the proof.

In [14], we considered the space of $C^1$ regular curves in $S^2$ that start and end at given points with given directions, whose lower and upper curvatures strictly bounded in an open interval. Especially, we proved the existence of a non-trivial map from $S^{n_1}$ to this space, where the dimension $n_1$ is linked to the maximum number of arcs of angle $\pi$ for each of four types of “maximal” critical curves. We refer the interested readers to the articles [3,4,6–13,15] and the references therein for more details on this subject.

Related to our work in [14] and the one by Saldanha and Z¨uhlke in [11], in this paper we study the topological properties of the space of $C^1$ regular curves in $S^2$ with the “curvatures” constrained in an interval.

Consider the set $\mathcal{A}$ of all $C^1$ regular curves in $S^2$ whose tangent vector is Lipschitz continuous and whose geodesic curvature $\kappa(t)$ (which exists for a.e. $t$) satisfies

$$\kappa_1 < \text{ess inf}_{t \in [0,1]} \kappa(t) \leq \text{ess sup}_{t \in [0,1]} \kappa(t) < \kappa_2,$$

where $\kappa_1 < \kappa_2$ are real numbers, $\text{ess inf}_{t \in [0,1]} \kappa(t)$ and $\text{ess sup}_{t \in [0,1]} \kappa(t)$ denote the essential infimum and essential supremum of $\kappa(t)$, respectively. See the definition of $C^1$ regular curves and more details in Sect. 2.

We prove the following relation between $C^0$ and $C^1$ topologies of the space $\mathcal{A}$:

**Theorem 1.** The metric spaces $(\mathcal{A}, d^0)$ and $(\mathcal{A}, d^1)$ generate the same topology. Here $d^0$ and $d^1$ are the following metrics

$$d^0(\alpha, \beta) = \max \left\{ \left\| d(\alpha(t), \beta(t)) \right\| ; t \in [0,1] \right\},$$

where $d$ denotes the surface distance on $S^2$.

$$d^1(\alpha, \beta) = \max \left\{ d((\alpha(t), \dot{\alpha}(t)), (\beta(t), \dot{\beta}(t))) ; t \in [0,1] \right\},$$

where $d$ is the distance measured in the tangent bundle $T S^2$ with a Riemannian metric induced from $S^2$. 

By the similar proof, we may also prove that the metric spaces \((B, d^0)\) and \((B, d^1)\) have the same topology, where \(B\) denotes the set of all \(C^1\) regular curves in \(S^2\) whose tangent vector is Lipschitz continuous and whose a.e. existing geodesic curvature \(\kappa(t)\) satisfies
\[
\kappa_1 \leq \text{ess inf}_{t\in[0,1]} \kappa(t) \leq \text{ess sup}_{t\in[0,1]} \kappa(t) \leq \kappa_2,
\]
where \(\kappa_1 < \kappa_2\) are real numbers.

It is known that for a sequence of \(C^1\) functions defined on \([0,1]\), its \(C^0\) convergence implies \(C^1\) convergence if the derivatives of the functions are Lipschitz continuous and have uniform Lipschitz constant bound. However we cannot apply this property directly because we cannot obtain uniform Lipschitz constant bound for the tangent vectors of the \(C^1\) curves directly from the definition of the set \(A\). In fact, first we need to prove that the lengths of the curves in a convergent sequence must converge to the one of the limit curve, which is stated in Lemma 6.

Next, in Sect. 4, we discuss firstly the subset of \(A\) consisting of the curves with fixed initial Frenet frame, that is the set, denoted by \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot)\), consisting of all \(C^1\) curves \(\gamma\) in \(A\) with Frenet frames \(\mathfrak{F}_\gamma(0) = P \in SO_3(\mathbb{R})\). Secondly, we study the subset of \(A\) consisting of the curves that start and end at given points with given directions, denoted by \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q)\), consisting of all \(C^1\) curves \(\gamma\) in \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot)\) with Frenet frame \(\mathfrak{F}_\gamma(1) = Q \in SO_3(\mathbb{R})\) (see Definitions 4 and 5 in Sect. 2).

We will equip \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot)\) with a special norm so that it becomes a Banach space, hence a trivial Banach manifold. The idea is to write the Frenet frame of a related \(C^1\) regular curve in \(S^2\) as a weak solution of a differential equation and use the similar approach in [11]. In more detail, we first introduce the set \(\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)\) of the \((\kappa_1, \kappa_2)\)-strongly admissible parametrized curves (see Definition 8 in Sect. 4). It turns out that the set \(\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)\) is identified with \(L^\infty[0,1] \times L^\infty[0,1]\) via correspondence \(\gamma \leftrightarrow (\hat{v}, \hat{w})\) and hence it becomes a Banach space with the norm \(\|\cdot\|\) induced by \(L^\infty[0,1] \times L^\infty[0,1]\), where each pair \((\hat{v}, \hat{w})\) determines a matrix \(\Lambda(t)\) in the ODE system (10). Further we prove that \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot)\) corresponds a closed subspace of \(\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)\) consisting of the curves, each of them has the corresponding \(\hat{v}\) constant equal to the arc-length of the curve. Thus, we obtain the following result:

**Theorem 2.** \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot)\) can be identified as a closed subspace of the Banach space \((\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot), \|\cdot\|)\) and is a Banach space with the inherited norm, hence a trivial Banach manifold.

As a corollary of Theorem 2, we obtain the following result:

**Theorem 3.** \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q)\) is a Banach submanifold of \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot)\) with codimension 3 induced by the submersion \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot) \rightarrow SO_3(\mathbb{R}), \gamma \mapsto \mathfrak{F}_\gamma(1)\).

We remark that the topology of the Banach submanifold \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q)\) induced by the Banach space \((\mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot), \|\cdot\|)\) is not equivalent to the one induced by \(C^0\) and \(C^1\) metrics on \(\mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q)\). The general reason can be seen by
the following special example: Suppose that $\gamma$ is a segment of the great circle on $S^2$ and hence $\gamma$ is a curve in $\mathcal{P}^\kappa_1(\mathcal{P}, Q)$ with the corresponding $\mathcal{P}, Q$ and $\kappa_1 < 0 < \kappa_2$. Fixing a number $\kappa$ with $\kappa, -\kappa \in (\kappa_1, \kappa_2)$, we may construct a sequence of oscillating curves $\{\gamma_n(t)\}_{n \geq N_0}$ such that each $\gamma_n$ is the concatenation of $n$ arcs with the alternating geodesic curvatures $\kappa$ and $-\kappa$. Moreover, these arcs may be chosen so that they have approximately the same length and the lengths tend to 0 as $n \to \infty$. Note that as $n \to \infty$, the oscillating amplitude of the curves tends to zero. Therefore the sequence $\{\gamma_n\}$ converges to $\gamma$ in the $C^0$ metric. However $\{\gamma_n\}$ does not converge in the metric of the Banach space $(\mathcal{P}^\kappa_1(\mathcal{P}), \| \cdot \|)$ as the curvature does not converge to 0.

2. Definitions and Notations

In this section, we give some definitions and notations. Let $S^2$ denote the unit sphere in the Euclidean space $\mathbb{R}^3$. A $C^1$ regular parametrized curve in $S^2$ is a $C^1$ map $\gamma: I \to S^2$ such that the tangent vector $\dot{\gamma}(t) \neq 0$ for all $t \in I$, where $I = [a, b] \subset \mathbb{R}$. In other words, a $C^1$ regular parametrized curve in $S^2$ is a $C^1$ immersion of $I$ into $S^2$. Now we recall the definition of the space of $C^1$ regular curves.

There is an equivalence relation $\sim$ in the set of all $C^1$ regular parametrized curves in $S^2$. Any two $C^1$ regular parametrized curves in $S^2$

$$\alpha: I_\alpha \to S^2 \quad \text{and} \quad \beta: I_\beta \to S^2$$

are called equivalent if they are the same up to a reparametrization, that is, there exists a $C^1$ bijection $\beta: I_\alpha \to I_\beta$, satisfying $\frac{d\beta}{dt} > 0$ and

$$\alpha(t) = (\beta \circ \tilde{t})(t), \quad \forall t \in I_\alpha.$$  

The space of $C^1$ regular curves in $S^2$ is defined as the quotient space

$$\mathcal{I} = \{ \gamma: \gamma \text{ is a } C^1 \text{ regular parametrized curve in } S^2 \} / \sim.$$  

By abuse of notation, we will still use $\alpha$ to represent the equivalence class $[\alpha] = \{ \beta; \alpha \sim \beta \} \in \mathcal{I}$ and call $\alpha \in \mathcal{I}$ a $C^1$ regular curve in $S^2$.

For a $C^1$ regular parametrized curve $\gamma: [0, 1] \to S^2$ with parameter $t$, the arc-length $s: [0, 1] \to [0, L_\gamma]$ of $\gamma$ is given by

$$s(t) = \int_0^t |\dot{\gamma}(t)| \, dt,$$

where $L_\gamma = \int_0^1 |\dot{\gamma}(t)| \, dt$ is the length of $\gamma$. Since $|\dot{\gamma}| > 0$, $s$ is a strictly increasing function. Re-parametrizing the curve by arc-length $s$, the curve $\gamma: [0, L_\gamma] \to S^2$ satisfies $|\gamma'(s)| \equiv 1$. It is easy to check that one may re-parametrize $\gamma$ proportionally to arc-length so that $\gamma: [0, 1] \to S^2$ has constant speed $|\dot{\gamma}| \equiv L_\gamma$. In this article, unless stated otherwise, a $C^1$ regular curve will be identified with the $C^1$ regular parametrized curve using this parametrization.
Throughout this paper, we will use the notation $t_\gamma(t)$ to denote the unit tangent vector at $\gamma(t)$, that is, $t_\gamma(t) = \gamma'(s)|_{s=t}$. Derivatives with respect to $t$ and $s$ will be denoted by the symbols ' and ′, respectively. We use this convention for higher-order derivatives as well.

We may define $C^0$ and $C^1$ metrics in $\mathcal{F}$: Given any two curves $\alpha, \beta: [0,1] \to S^2$, $\alpha$ and $\beta$ are equivalent. These metrics above induce corresponding topologies in $\mathcal{F}$. Vol. 76 (2021) On the Space of $C^1$ Regular Curves on Sphere Page 5 of 17 223

$\newcommand{\gamma}{\gamma}$

Given a $C^1$ regular curve $\gamma: I \to S^2$, the unit normal vector $n_\gamma$ to $\gamma$ is

$$n_\gamma(t) = \gamma(t) \times t_\gamma(t),$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. If $\gamma$ also has the second derivative $\ddot{\gamma}(t)$ at $\gamma(t)$, the geodesic curvature $\kappa_\gamma(s)$ at $\gamma(s) = \gamma(s(t))$ is defined by

$$\kappa_\gamma(s) = \langle t_\gamma(s), n_\gamma(s) \rangle,$$

(3)

where $s$ is the arc-length of $\gamma$. However, for a $C^1$ regular curve, the geodesic curvature may not be well defined at a point of the curve.

Now we take our attention on the curves in $\mathcal{F}$ which start and end at given points with given directions. The Frenet frame of $\gamma$ is defined by:

$$\mathcal{F}_\gamma(t) = \begin{pmatrix} t_\gamma(t) \\ n_\gamma(t) \end{pmatrix} \in SO_3(\mathbb{R}).$$

(4)

The space $SO_3(\mathbb{R})$ is homeomorphic to the unit tangent bundle of sphere $UTS^2$ by mapping the matrix $M \in SO_3(\mathbb{R})$ to the vector $(M(1,0,0), M(0,1,0)) \in TM_{(1,0,0)}S^2$. We define the following space of curves:
Definition 4. Given $P, Q \in SO_3(\mathbb{R})$,

- $\mathcal{I}(P, \cdot)$ denotes the set of all $C^1$ regular curves in $S^2$ with Frenet frames $\mathfrak{F}_\gamma(0) = P$.
- $\mathcal{I}(P, Q)$ denotes the set of all $C^1$ regular curves in $S^2$ with Frenet frames $\mathfrak{F}_\gamma(0) = P$ and $\mathfrak{F}_\gamma(1) = Q$.

Let $\gamma(t), t \in [0, 1]$, be a $C^1$ regular curve in $S^2$ whose tangent vector $\dot{\gamma}$ is Lipschitz continuous. Then it is known that $\ddot{\gamma}(t)$ exists for a.e. $t$. After reparametrization by arc-length $s$, we have that $\gamma'(s)$ is Lipschitz continuous for $s$. This implies that $\ddot{\gamma}(s) = t'(s)$ exists for a.e. $s$ and

$$t'(s) = -\gamma(s) + \kappa(s)n(s), \quad \text{a.e. } s,$$

where $t(s)$ and $n(s)$ are the unit tangent vector and the unit normal vector at $\gamma(s)$ respectively, and $\kappa(s)$ is the geodesic curvature at $\gamma(s)$ (or equivalently, at $\gamma(t)$).

We have the sets $P_{\kappa_2}^\kappa(P, \cdot)$ and $P_{\kappa_2}^\kappa(P, Q)$, which are also subsets of $\mathcal{A}$, given by

Definition 5. Let $P_{\kappa_2}^\kappa(P, \cdot)$ and $P_{\kappa_2}^\kappa(P, Q)$ be the subset of all curves $\gamma$ in $\mathcal{I}(P, \cdot)$ and the subset of all curves $\gamma$ in $\mathcal{I}(P, Q)$ respectively, whose $\dot{\gamma}$ is Lipschitz continuous and whose geodesic curvature $\kappa(t)$ at a.e. $t \in [0, 1]$ satisfies

$$\kappa_1 < \text{ess inf}_{t \in [0,1]} \kappa(t) \leq \text{ess sup}_{t \in [0,1]} \kappa(t) < \kappa_2,$$

where $\kappa_1, \kappa_2 \in (-\infty, +\infty)$, with $\kappa_1 < \kappa_2$.

3. The $C^0$ and $C^1$ Topologies of the Space $\mathcal{A}$

In this section, we show that the $C^0$ and $C^1$ topologies on the set $\mathcal{A}$ are equivalent to each other. Before doing this, we prove the following result which also has independent interest.

Lemma 6. Let $\{\alpha_j\}_{j \in \mathbb{N}}$ be a sequence of $C^1$ regular curves in $\mathcal{A}$. Assume that $\{\alpha_j\}_{j \in \mathbb{N}}$ converges to a $C^1$ regular curve $\alpha$ in $d^0$ metric, where the tangent vector of $\alpha$ is Lipschitz continuous. Let $L_{\alpha_j}$ and $L_{\alpha}$ denote the lengths of $\alpha_j$ and $\alpha$ respectively. Then $\lim_{j \to \infty} L_{\alpha_j} = L_{\alpha}$.

Proof. From the equivalence of the metrics $d^0$ and $\bar{d}^0$, $\{\alpha_j\}_{k \in \mathbb{N}}$ converges to $\alpha$ in $\bar{d}^0$ metric. By contradiction, suppose that $\lim_{j \to \infty} L_{\alpha_j} \neq L_{\alpha}$. Taking a subsequence, one of cases below holds:

1. $\lim_{j \to \infty} L_{\alpha_j} = A \neq L_{\alpha}$, where $A$ is a finite number.
2. $\lim_{j \to \infty} L_{\alpha_j} = \infty$. 
For $\alpha$ and each $\alpha_j$, their tangent vectors are Lipschitz continuous on the compact set $[0, 1]$ and hence absolutely continuous. By Taylor formula with integral remainder,

$$\alpha_j(t) - \alpha_j(0) = \dot{\alpha}_j(0)t + \int_0^t \ddot{\alpha}_j(\tau)(t-\tau)d\tau$$

and

$$\alpha(t) - \alpha(0) = \dot{\alpha}(0)t + \int_0^t \ddot{\alpha}(\tau)(t-\tau)d\tau.$$ 

Note that the parameter $t \in [0, 1]$ satisfies $s = L_{\alpha_j}t$ for each $k$ and $s = L_\alpha t$, where $s$ denotes the corresponding arc-length parameter for $\alpha_j$ and $\alpha$ respectively. We have

$$\alpha_j(t) - \alpha(t) - (\alpha_j(0) - \alpha(0))$$

$$= (\dot{\alpha}_j(0) - \dot{\alpha}(0))t + \int_0^t [\ddot{\alpha}_j(\tau) - \ddot{\alpha}(\tau)](t-\tau)d\tau$$

$$= [L_{\alpha_j}t_{\alpha_j}(0) - L_\alpha t_\alpha(0)]t + \int_0^t [L_{\alpha_j}^2 t_{\alpha_j}'(\tau) - L_\alpha^2 t_\alpha'(\tau)](t-\tau)d\tau,$$ 

(6)

where $t_{\alpha_j}' = -\alpha_j + \kappa_{\alpha_j}n_{\alpha_j}$, $t_\alpha' = -\alpha + \kappa_\alpha n_\alpha$ for a.e. $s$. Without changing the value of the above integral, we may take $\kappa_{\alpha_j}$ in $(\kappa_1, \kappa_2)$ for all $t$. Also observing that $\alpha_j, \alpha \in S^2$, $\{t_{\alpha_j}'\}$ is uniformly bounded for all $j$, and $t_\alpha'$ is bounded.

**Case 1:** Since \( \lim_{j \to \infty} L_{\alpha_j} = A < \infty \), there exists a positive constant $c$ satisfying

$$|L_{\alpha_j}^2 t_{\alpha_j}'(\tau) - L_\alpha^2 t_\alpha'(\tau)| \leq c, \quad \tau \in [0, t].$$

Hence

$$|\alpha_j(t) - \alpha(t)| \geq |L_{\alpha_j}t_{\alpha_j}(0) - L_\alpha t_\alpha(0)| t - c \int_0^t (t-\tau)d\tau - |\alpha_j(0) - \alpha(0)|$$

$$\geq |L_{\alpha_j} - L_\alpha| t - \frac{c}{2} t^2 - |\alpha_j(0) - \alpha(0)|.$$ 

Since $A \neq L_\alpha$, there are a very small $\epsilon > 0$ with $\frac{\epsilon}{c} < 1$ and $N_0 \geq 0$ such that for $j \geq N_0$, $|L_j - L_\alpha| \geq \epsilon > 0$. We may choose $N_0$ so that for $j \geq N_0$, $|\alpha_j(0) - \alpha(0)| \leq \frac{\epsilon^2}{4c}$ also holds. Then, for $0 \leq t \leq \frac{\epsilon}{c}$,

$$|\alpha_j(t) - \alpha(t)| \geq t \left( \epsilon - \frac{c}{2} t \right) - \frac{\epsilon^2}{2c} \geq \frac{\epsilon}{c} t - \frac{\epsilon^2}{4c}.$$ 

Taking $t = \frac{\epsilon}{c}$, we have that

$$\left| \alpha_j \left( \frac{\epsilon}{c} \right) - \alpha \left( \frac{\epsilon}{c} \right) \right| \geq \frac{\epsilon^2}{4c} > 0.$$ 

So

$$\max_{t \in [0, 1]} |\alpha_j(t) - \alpha(t)| \geq \frac{\epsilon^2}{4c} > 0.$$
This implies that \( \lim_{j \to \infty} d^0(\alpha_j, \alpha) = \lim_{j \to \infty} \max_{t \in [0,1]} |\alpha_j(t) - \alpha(t)| \neq 0 \) which is a contradiction.

**Case 2:** Similar to Case 1, there exist positive constants \( c_1 \) and \( c_2 \) satisfying

\[
|\alpha_j(t) - \alpha(t)| \geq |L_{\alpha_j} - L_{\alpha}| t - c_1 L^2_{\alpha_j} \int_0^t (t - \tau) d\tau - c_2 L^2_{\alpha} \int_0^t (t - \tau) d\tau - |\alpha_j(0) - \alpha(0)|
\]

\[
= |L_{\alpha_j} - L_{\alpha}| t - \frac{c_1 L_{\alpha_j}^2 + c_2 L_{\alpha}^2}{2} - |\alpha_j(0) - \alpha(0)|.
\]

Since \( L_{\alpha_j} \to \infty \), there exists a number \( N_1 > 0 \) such that \( L_{\alpha_j} \geq \max\{1, 2L_{\alpha}\} \) for \( j \geq N_1 \). So for \( j \geq N_1 \),

\[
|\alpha_j(t) - \alpha(t)| \geq \frac{1}{2} \left( L_{\alpha_j} t - c L_{\alpha_j} t^2 \right) - |\alpha_j(0) - \alpha(0)|,
\]

where \( c = c_1 + \frac{c_2}{2} \). Again choose a number \( N_2 \geq N_1 \) with for \( j \geq N_2 \), \( 2c L_{\alpha_j} \leq 1 \) and \( |\alpha_j(0) - \alpha(0)| \leq \frac{1}{16c} \). For each \( j \geq N_2 \), we take \( t_j = \frac{1}{2c L_{\alpha_j}} \in [0, 1] \). Then

\[
|\alpha_j(t_j) - \alpha(t_j)| \geq \frac{1}{2} \left( \frac{1}{2c} - \frac{1}{4c} \right) - \frac{1}{16c} = \frac{1}{8c} - \frac{1}{16c} = \frac{1}{16c}.
\]

Thus

\[
\max_{t \in [0,1]} |\alpha_j(t) - \alpha(t)| \geq \frac{1}{16c}.
\]

This induces a contradiction with \( \lim_{j \to \infty} d^0(\alpha_j, \alpha) = 0 \). \( \square \)

Now we are ready to prove

**Theorem 7** (Theorem 1). The metric spaces \((\mathcal{A}, d^0)\) and \((\mathcal{A}, d^1)\) generate the same topology.

**Proof.** For a curve \( \alpha \in \mathcal{A} \), with the parameter \( t \) proportional to arc-length, \( \alpha: [0,1] \to \mathbb{S}^2 \) has constant speed \( |\dot{\alpha}| \equiv L_{\alpha} \). Since the metrics \( d^1 \) and \( d^1 \) are equivalent, it is enough to prove that the topologies induced by the metrics \( d^1 \) and \( d^0 \) are the same. Since \( d^1(\alpha, \beta) \geq d^0(\alpha, \beta) \) for any \( \alpha, \beta \in \mathcal{A} \), the topology induced by \( d^1 \) is finer than the topology induced by \( d^0 \). So it suffices to prove the converse.

Given a sequence \( \{\alpha_j\}_{j \in \mathbb{N}} \) which is convergent in \( d^0 \) to \( \alpha_0 \), we will prove that it is also convergent in \( d^1 \).

Suppose, by contrary, that \( d^1(\alpha_j, \alpha_0) \to 0 \) as \( j \to \infty \). Then there exist \( \epsilon > 0 \) and a subsequence of \( \{\alpha_j\} \), still denoted by \( \{\alpha_j\} \), so that

\[
d^1(\alpha_j, \alpha_0) \geq \epsilon > 0.
\]

Since \( d^0(\alpha_j, \alpha_0) \to 0 \), (7) implies that there exists a number \( N_0 > 0 \) such that, for \( j \geq N_0 \),

\[
\max_{t \in [0,1]} \{d(\dot{\alpha}_j(t), \dot{\alpha}_0(t))\} \geq \frac{3\epsilon}{4}.
\]

By Lemma 6, \( |\dot{\alpha}_j(t)| = L_{\alpha_j} \leq M \).
For every curve $\alpha_j \in A$, $\dot{\alpha}_j$ is Lipschitz continuous and so
\[
\dot{\alpha}_j(t_2) - \dot{\alpha}_j(t_1) = \int_{t_1}^{t_2} \ddot{\alpha}_j(\tau)d\tau = L^2_{\alpha_j} \int_{t_1}^{t_2} t'_{\alpha_j}(\tau)d\tau. \tag{8}
\]
where $\dot{\alpha}_j(t) = L_{\alpha_j} t_{\alpha_j}$ for all $t$ and $\ddot{\alpha}_j(t) = L^2_{\alpha_j} t'_{\alpha_j}$, for a.e. $t$. By Lemma 6 and (8), the family $\{\dot{\alpha}_j\}_{j \in \mathbb{N}}$ has uniform bound and uniform Lipschitz constant. This implies that the family $\{\dot{\alpha}_j\}_{j \in \mathbb{N}}$ is equicontinuous. By Azelá-Ascoli theorem, there exists a subsequence $\{\dot{\alpha}_j_l\}_{j \in \mathbb{N}}$ converges uniformly to a limit $v(t)$ which is a vector function defined at $[0, 1]$.

Claim: $\dot{\alpha}_0 = v(t)$.

Since $\dot{\alpha}_0$ and $v(t)$ are continuous, it is sufficient to prove that for any $0 \leq t_1 \leq t_2 \leq 1$, $\int_{t_1}^{t_2} (\dot{\alpha}_0(t) - v(t))dt = 0$.

\[
\int_{t_1}^{t_2} v(t)dt = \int_{t_1}^{t_2} \lim_{l \to \infty} \dot{\alpha}_j(t)dt = \lim_{l \to \infty} \int_{t_1}^{t_2} \dot{\alpha}_j(t)dt = \lim_{l \to \infty} (\alpha_j(t_2) - \alpha_j(t_1)) = \alpha_0(t_2) - \alpha_0(t_1) = \int_{t_1}^{t_2} \dot{\alpha}_0(t)dt. \tag{9}
\]

In the second equality of (9), we used the uniform convergence of $\{\dot{\alpha}_j\}_{j \in \mathbb{N}}$ and the dominated convergence theorem. This proves the claim.

By the claim, we have, for sufficiently large $l$,

\[
\max_{t \in [0, 1]} \{d(\dot{\alpha}_j(t), \dot{\alpha}_0(t))\} < \frac{3\epsilon}{4}.
\]

Thus the contradiction happens. We have proved $\bar{d}^1(\alpha_j, \alpha_0) \to 0$ as $j \to \infty$. 

4. Banach Manifold $\mathcal{P}_{\mathcal{K}_1}^{\mathcal{K}_2}(P, Q)$

We observe that the space $\mathcal{P}_{\mathcal{K}_1}^{\mathcal{K}_2}(P, Q)$, as a subspace of $A$, with $C^0$ (equivalently $C^1$) topology in Sect. 3 is not complete. In this section, we furnish $\mathcal{P}_{\mathcal{K}_1}^{\mathcal{K}_2}(P, Q)$ a complete norm such that it is a Banach manifold. In [11], Saldanha and Zählke constructed a Hilbert manifold structure on a special subspace of the space of so-called $(\kappa_1, \kappa_2)$-admissible curves. We will use an approach similar to theirs.

Denote the space $E_{\infty} = L^\infty[0, 1] \times L^\infty[0, 1]$, where $L^\infty[0, 1]$ denotes the Banach space of essentially bounded measurable functions on $[0, 1]$ together with the essential supremum norm. Let $W^{1, \infty}[0, 1]$ denote the Sobolev space
of the functions in $L^\infty[0,1]$ with their weak derivatives also in $L^\infty[0,1]$. It is well known that a function in $W^{1,\infty}[0,1]$ is Lipschitz continuous.

If $\gamma(t): [0,1] \to \mathbb{S}^2$ is a smooth regular parametrized curve, its Frenet frame $\mathfrak{F}(t)$ becomes a Banach space if there exist admissible values in the corresponding Lie group $SO_3$. If we take a pair $(\hat{v}, \hat{w}) \in E_\infty$, let $h(\kappa_1, \kappa_2) \in SO_3(\mathbb{R})$ be the smooth diffeomorphism $h(\kappa_1, \kappa_2) = (\kappa_1 - t)^{-1} + (\kappa_2 - t)^{-1}$.

If we take a pair $(\hat{v}, \hat{w}) \in E_\infty$, and $(v, w)$ given by

$$v(t) = h^{-1}(\hat{v}(t)) \quad \text{and} \quad w(t) = v(t)h_{\kappa_1, \kappa_2}^{-1}(\hat{w}(t)).$$

By the definitions of $h$ and $h_{\kappa_1, \kappa_2}$, it is straightforward to verify that $(v, w) \in E_\infty$, $v(t) > 0$ and $w(t) \in (\kappa_1, \kappa_2)$, $t \in [0,1]$. Hence we have the following definition:

**Definition 8.** A parametrized curve $\gamma: [0,1] \to \mathbb{S}^2$ is called $(\kappa_1, \kappa_2)$-strongly admissible if there exist $P \in SO_3(\mathbb{R})$ and a pair $(\hat{v}, \hat{w}) \in E_\infty$ such that $\gamma(t) = \Phi(t)(1,0,0)$ for all $t \in [0,1]$, where $\Phi$ is the unique solution in $W^{1,\infty}[0,1]$ to the initial value problem (10), with $v, w$ given by Eq. (11).

Let $\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)$ denote the set of all $(\kappa_1, \kappa_2)$-strongly admissible parametrized curves $\gamma$ such that $\Phi(0) = P$.

The set $\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)$ is identified with $E_\infty$ via correspondence $\gamma \leftrightarrow (\hat{v}, \hat{w})$ and becomes a Banach space $(\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot), \| \cdot \|)$ with the norm $\| \cdot \|$ induced by $E_\infty$. 

where

$$\Lambda(t) = \begin{pmatrix}
0 & -|\dot{\gamma}(t)| & 0 \\
|\dot{\gamma}(t)| & 0 & -|\dot{\gamma}(t)|\kappa(t) \\
0 & |\dot{\gamma}(t)|\kappa(t) & 0
\end{pmatrix} \in so_3(\mathbb{R}).$$

Now, given a $P \in SO_3(\mathbb{R})$ and a map $\Lambda: [0,1] \to so_3(\mathbb{R})$ of the form:

$$\Lambda(t) = \begin{pmatrix}
0 & -v(t) & 0 \\
v(t) & 0 & -w(t) \\
0 & w(t) & 0
\end{pmatrix},$$

where $v, w \in L^\infty[0,1]$ and $v(t) > 0$. By the ODE theory (see [5], Theorem 3.4), for the initial value problem:

$$\dot{\Phi}(t) = \Phi(t)\Lambda(t), \quad \Phi(0) = P, \quad (10)$$

there exists a unique solution $\Phi: [0,1] \to SO_3(\mathbb{R})$ which is of $W^{1,\infty}[0,1]$. The fact that $\Lambda(t)$ belongs to the Lie algebra $so_3(\mathbb{R})$ guarantees that $\Phi(t)$ assumes values in the corresponding Lie group $SO_3(\mathbb{R})$.
We also define $\mathcal{R}_{\kappa_1}^{\kappa_2}(P, Q)$ to be the subspace of $\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)$ satisfying $\Phi_\gamma(1) = Q$.

Now we prove that

**Proposition 9.** Let $-\infty < \kappa_1 < \kappa_2 < +\infty$ and $P \in \text{SO}_3(\mathbb{R})$. Any parametrized curve $\gamma \in \mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)$ can be reparametrized by arc-length $s$, such that it becomes a $C^1$ regular parametrized curve and $[\gamma(s)] \in \mathcal{P}_{\kappa_2}(P, \cdot)$.

**Proof.** The proof is divided by several steps. Let $\gamma: [0, 1] \to \mathbb{S}^2$ be a curve in $\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)$.

(i). We prove the existence of the arc-length $s$ of $\gamma$ and show some properties of $\gamma(s)$.

By the definition of $\mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot)$, the frame $\Phi_\gamma$ of $\gamma$ is of $W^{1, \infty}[0, 1]$. As a component of $\Phi_\gamma$, the function $\gamma$ is Lipschitz continuous. This implies that $\dot{\gamma}(t)$ exists a.e. $t$ and $|\dot{\gamma}(t)|$ is bounded. Further the arc-length of $\gamma$, as a curve, is well defined and is equal to

$$s(t) = \int_0^t |\dot{\gamma}(t)| dt, \quad t \in [0, 1],$$

where $s(t)$ is Lipschitz continuous. By the equation $\dot{\gamma}(t) = v(t) \cdot t(t)$ a.e. $t$,

$$s(t) = \int_0^t |\dot{\gamma}(t)| dt = \int_a^t v(t) dt, \quad t \in [0, 1].$$

Note that $v(t) > 0$ in $[0, 1]$ for a.e. $t \in [0, 1]$. $s(t)$ is strictly increasing in $[0, 1]$ and exists a strictly increasing continuous inverse function $t(s)$, $s \in [0, L_\gamma]$. Hence, $t'(s)$ exists for a.e. $s \in [0, L_\gamma]$, where $t'(s) = \frac{1}{v(t(s))}$. In addition, observe that $v(t) = h^{-1}(\hat{v}(t))$ and $\hat{v}(t) \in L^\infty[0, 1]$. Then $v(t) = \frac{\hat{v} + \sqrt{\hat{v}^2 + 4}}{2}$ is bounded below by a positive number for a.e. $t \in [0, 1]$. This implies that $v(t)$ is bounded below by a positive number for a.e. $s \in [0, L_\gamma]$ and hence $t(s)$ is Lipschitz continuous.

Now we reparametrize $\gamma$ by arc-length $s$, that is, $\gamma(s) = \gamma(t(s)), s \in [0, L_\gamma]$.

It is known that Lipschitz continuity of $\gamma(t)$ implies Lipschitz continuity of $\gamma(s)$ (see [1, Theorem 3.2]), that is $\gamma(s) \in W^{1, \infty}[0, L_\gamma]$. Moreover, $\gamma'(s)$ exists for a.e. $s$ and $|\gamma'(s)| = 1$ for a.e. $s$ (see [1, Corollary 3.7]). From the facts on $\gamma(t)$ and $\gamma(s)$ we have obtained above, it holds that

$$\gamma'(s) = \frac{\dot{\gamma}(t(s)) \cdot t'(s)}{v} = \frac{\gamma'}{v}(t(s)), \quad \text{a.e. } s.$$

By the differential system (10), we have that, for a.e. $s$,

$$\gamma'(s) = \frac{\gamma'}{v}(t(s)) = t(s),$$

where $t(s) = t(t(s))$. 
(ii). We will confirm that $\gamma'(s)$ exists for all $s$, $\gamma'(s) = t(s)$ and $\gamma(s)$ is a $C^1$ regular curve.

In fact, for any $s$, since $\gamma(s)$ is Lipschitz continuous and hence absolutely continuous,

$$\gamma'(s) = \lim_{\Delta s \to 0} \frac{\gamma(s + \Delta s) - \gamma(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\int_s^{s+\Delta s} \gamma'(u)du}{\Delta s} = \lim_{\Delta s \to 0} \frac{\int_s^{s+\Delta s} t(u)du}{\Delta s}.$$  \hspace{1cm} (12)

Note that $t(t)$, as a component of $\Phi_\gamma$, is in $W^{1,\infty}[0,1]$, that is, $t(t)$ is Lipschitz continuous. So $t(s) = t(t(s))$ is continuous for $s$. By the mean value theorem, (12) implies that

$$\gamma'(s) = t(s), \quad s \in [0,1].$$  \hspace{1cm} (13)

So we have proved that $\gamma(s)$ is a $C^1$ curve. We mention that one may reparametrize $\gamma$ with a new parameter, still denoted by $t$, such that $\gamma(t)$, $t \in [0,1]$ has the constant speed $L_\gamma$ and is in $C^1[0,1]$.

(iii). We will confirm $[\gamma(s)] \in \mathcal{P}_{\kappa_1}(P, \cdot)$.

Since the parameter $t(s)$ is Lipschitz continuous, $t(s) = t(t(s))$ is Lipschitz continuous, that is, it is of $W^{1,\infty}[0,L_\gamma]$. By (10), the following equations hold for a.e. $s$,

$$t'(s) = \left(-\gamma + \frac{w}{v}n\right)(s),$$  \hspace{1cm} (14)

where $n(s) = n(t(s))$ and $\frac{w(t)}{v(t)} \in (\kappa_1, \kappa_2)$.

Note that $\dot{w}(t) \in L^\infty[0,1]$ and $w(t) = v(t)h_{\kappa_1,\kappa_2}^{-1}(\dot{w}(t))$. It can be implied by the definition of $h_{\kappa_1,\kappa_2}$ that

$$\kappa_1 \leq \text{ess inf}_{t \in [0,1]} \frac{w(t)}{v(t)} \leq \text{ess sup}_{t \in [0,1]} \frac{w(t)}{v(t)} < \kappa_2.$$  

Taking $\gamma(s) = \gamma(t(s))$, we have proved $[\gamma(s)] \in \mathcal{P}_{\kappa_1}(P, \cdot)$. \hspace{1cm} \square

For convenience, we give the following notation

**Definition 10.** Let $\mathcal{L}_{\kappa_1}(P, \cdot)$ be the subset of $\mathcal{R}_{\kappa_1}(P, \cdot)$ satisfying $\dot{v}(t) \equiv \dot{v} \in \mathbb{R}$.

With this notation, we enunciate the following theorem.

**Theorem 11.** $\mathcal{P}_{\kappa_1}(P, \cdot) = \mathcal{L}_{\kappa_1}(P, \cdot)$ and $\mathcal{P}_{\kappa_1}(P, \cdot)$ is a Banach space with the norm, still denoted by $\| \cdot \|$, inherited from the Banach space $(\mathcal{R}_{\kappa_1}(P, \cdot), \| \cdot \|)$. 

**Proof.** First, we confirm the claim: $\mathcal{L}_{\kappa_1}(P, \cdot) = \mathcal{P}_{\kappa_1}(P, \cdot)$.

Note that $\dot{v}(t) \equiv \ddot{v}$ is constant. Proposition 9 implies that a parametrized curve $\gamma(t)$, $t \in [0,1]$, in $\mathcal{L}_{\kappa_1}(P, \cdot)$ is $C^1$ regular with its parameter $t$ proportional to the arc-length and $[\gamma(t)] \in \mathcal{P}_{\kappa_1}(P, \cdot)$. Reciprocally, let $[\gamma] \in \mathcal{P}_{\kappa_1}(P, \cdot)$, where $\gamma(t)$ is taken to be a $C^1$ regular parametrized curve with parameter
$t \in [0,1]$ proportional to the arc-length. Then it is directly verified that $\gamma(t)$ satisfies the ODE system (10) for a.e. $t$,

$$\dot{\Phi}(t) = \Phi(t) \Lambda(t) \quad \text{and} \quad \Phi(0) = P,$$

with $v(t) = |\dot{\gamma}(t)| \equiv L_\gamma$, $w(t) = L_\gamma \kappa(t)$.

Take $\dot{v}(t) = h(L_\gamma)$ and $\dot{w}(t) = h_{\kappa_1,\kappa_2}(\kappa(t))$. Then $\dot{v} \in \mathbb{R}$ and $\dot{w} \in L^\infty[0,1]$. Thus $\gamma(t) \in L^2_{\kappa_1}(P, \cdot)$. So the claim holds.

Note that $L^2_{\kappa_1}(P, \cdot)$ is a closed subspace of $\mathcal{R}^2_{\kappa_1}(P, \cdot)$. In fact, it can be obtained via correspondence $\gamma \leftrightarrow (v, \dot{w}) \in \mathbb{R} \times L^\infty[0,1]$. Hence $L^2_{\kappa_1}(P, \cdot)$ has an induced complete norm $\| \cdot \|$. By the claim, $(\mathcal{P}^2_{\kappa_1}(P, \cdot), \| \cdot \|)$ is a Banach space.

Now we prove the following Theorem 12. The proof is analogous to the one in [8], only adapting $L^2[0,1]$ to $L^\infty[0,1]$.

**Theorem 12.** $\mathcal{R}^2_{\kappa_1}(P, Q)$ is a closed submanifold of codimension 3 in $\mathcal{R}^2_{\kappa_1}(P, \cdot)$, induced by the submersion $\mathcal{R}^2_{\kappa_1}(P, \cdot) \to SO_3(\mathbb{R})$ and thus a Banach manifold of codimension 3.

**Proof.** Since the Banach space $\mathcal{R}^2_{\kappa_1}(P, \cdot)$ is identified with $E_{\infty} = L^\infty[0,1] \times L^\infty[0,1]$ via the correspondence $\gamma \leftrightarrow (v, \dot{w})$, where $\gamma(t) = \Phi(t)(1,0,0)$, it is sufficient to consider $L^\infty[0,1] \times L^\infty[0,1]$. Define the map

$$F : L^\infty[0,1] \times L^\infty[0,1] \to SO_3(\mathbb{R}), \quad (\dot{v}, \dot{w}) \mapsto \Phi(1),$$

where $\Phi : [0,1] \to SO_3(\mathbb{R})$ is given by the initial value problem (10). The smooth dependence on parameters implies that the map $F$ is smooth. Now we will confirm the following claim:

**Claim.** $F$ is a submersion.

Given a $(\dot{v}_0, \dot{w}_0) \in L^\infty[0,1] \times L^\infty[0,1]$, it determines the matrix function $\Lambda_0 = \Lambda_0(t) : [0,1] \to V^+ \subset so_3(\mathbb{R})$, thus the unique solution $\Phi_0(t)$ of the initial value problem (10) and $\gamma_0 = \Phi_0 e_1$. It suffices to prove that for any $(\dot{v}_0, \dot{w}_0)$, the differential

$$dF_{(\dot{v}_0, \dot{w}_0)} : L^\infty[0,1] \times L^\infty[0,1] \to T_{\Phi_0(1)}SO_3(\mathbb{R})$$

is surjective, where $T_{\Phi_0(1)}SO_3(\mathbb{R}) = \{ \Phi_0(1)M; M \in so_3(\mathbb{R}) \}$.

Let $V \subset so_3(\mathbb{R})$ be the plane (i.e., 2-dimensional real vector space) of matrices $M$ with the element $(M)_{31} = 0$ and $V^+ \subset V$ be the half-plane with $(M)_{21} > 0$, respectively.

Given $s \in (-\epsilon, \epsilon)$, we consider the straight line $l_s$ in the Banach space $L^\infty[0,1] \times L^\infty[0,1]$, $l_s = (\dot{v}_0, \dot{w}_0) + s(\dot{v}, \dot{w})$, where $(\dot{v}, \dot{w}) \in L^\infty[0,1] \times L^\infty[0,1]$. Then $l(s)$ generates a one-parameter family of functions $L(s) : [0,1] \to V^+$, $s \in (-\epsilon, \epsilon)$, with $L_0 = \Lambda_0$.  

For every $s$, let $G(s, \cdot) : [0, 1] \to \text{SO}_3(\mathbb{R})$ be the solution of the problem (10) for $L(s)$, i.e., $G(s, t)$ satisfies

$$\frac{\partial}{\partial t} G(s, t) = G(s, t)L(s)(t) \text{ and } G(s, 0) = P,$$

for each $s \in (-\epsilon, \epsilon)$ and $t \in [0, 1]$. (15)

Since $L(0) = \Lambda_0$, it holds that $G(0, t) = \Phi_0(t)$.

By interchanging the order of the partial derivatives $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$, a direct computation gives that the partial derivative of $G$ with respect to $s$:

$$\frac{\partial G(s, t)}{\partial s} (G(s, t))^{-1} = \int_0^t \left( G(s, \tau) \left( L'(s)(\tau) \right) (G(s, \tau))^{-1} \right) d\tau,$$

(16)

where $L'(s) = \frac{dL}{ds}$. Now we study $\frac{\partial G}{\partial s}(0, 1)$ which is just $dF_{(\hat{v}_0, \hat{w}_0)}(\hat{v}, \hat{w})$. By (16),

$$\left( G(0, 1) \right)^{-1} \frac{\partial G}{\partial s}(0, 1) = \left( G(0, 1) \right)^{-1} \left[ \int_0^1 \left( G(0, \tau) \left( L'(0)(\tau) \right) (G(0, \tau))^{-1} \right) d\tau \right] (G(0, 1)),$$

(17)

that is

$$(\Phi_0(1))^{-1} \frac{\partial G}{\partial s}(0, 1) = (\Phi_0(1))^{-1} \left[ \int_0^1 \left( \Phi_0(\tau) \left( L'(0)(\tau) \right) (\Phi_0(\tau))^{-1} \right) d\tau \right] (\Phi_0(1)).$$

(18)

Note that $L(s)(t) \in V^+ \subset \mathfrak{so}_3(\mathbb{R})$. Then $L'(s)(t) \in V \subset \mathfrak{so}_3(\mathbb{R})$ for $(s, t)$.

If we choose $(\hat{v}, \hat{w})$ so that it consists of two smooth narrow bumps around the time $t_i$, $i = 1, 2$ respectively, then

$$(\Phi_0(1))^{-1} \frac{\partial G}{\partial s}(0, 1) \approx \sum_{i=1, 2} (\Phi_0(1))^{-1} (\Phi_0(t_i))^{-1} (L'(0)(t_i))(\Phi_0(t_i))^{-1} (\Phi_0(1))$$

$$= \sum_{i=1, 2} (\Phi_0(t_i, 1))^{-1} (L'(0)(t_i)) \Phi_0(t_i, 1),$$

where we use notation $\Phi_0(t_i, 1) = (\Phi_0(t_i))^{-1} \Phi_0(1) \in \text{SO}_3(\mathbb{R})$.

We may choose $t_1, t_2 \in [0, 1]$ so that the two-dimensional spaces $(\Phi_0(t_i, 1))^{-1} V \Phi_0(t_i, 1) \subset \mathfrak{so}_3(\mathbb{R})$ are not the same for $i \in \{1, 2\}$ and hence this guarantees that

$$\sum_{i=1, 2} (\Phi_0(t_i, 1))^{-1} (L'(0)(t_i)) \Phi_0(t_i, 1)$$

may take any value in $\mathfrak{so}_3(\mathbb{R})$. Thus $(\Phi_0(1))^{-1} \frac{\partial G}{\partial s}(0, 1)$ can take any value in $\mathfrak{so}_3(\mathbb{R})$.

Therefore $dF_{(\hat{v}_0, \hat{w}_0)}$ is surjective. Thus $F^{-1}(Q)$ is the regular set and, using the fact that $\text{SO}_3(\mathbb{R})$ has dimension 3, a Banach manifold of codimension 3.
This concludes that \( \mathcal{R}_{\kappa_1}^{\kappa_2}(P, Q) \) is a Banach manifold of codimension 3, induced by the submersion \( \mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot) \to SO_3(\mathbb{R}) \).

Using Theorems 11 and 12, we prove Theorem 3, that is:

**Theorem 13.** (Theorem 3) \( \mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q) \) is a Banach submanifold of \( \mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot) \) codimension 3 which is given by the submersion \( \mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot) \to SO_3(\mathbb{R}) \).

**Proof.** Note that \( \mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q) \) is the subset of \( \mathcal{R}_{\kappa_1}^{\kappa_2}(P, Q) \) satisfying \( \hat{v}(t) = v \in \mathbb{R}, t \in [0, 1] \). We have the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{R}_{\kappa_1}^{\kappa_2}(P, Q) & \xrightarrow{i_{\mathcal{R}}} & \mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot) \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
\mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q) & \xrightarrow{i_{\mathcal{P}}} & \mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot)
\end{array}
\]

In the above, \( i_{\mathcal{R}} \) and \( i_{\mathcal{P}} \) denote the corresponding inclusions, the map \( \pi_2: \mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot) \to \mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot), \gamma \mapsto [\gamma] \) is the projection given in Proposition 9, that is, the representation of \( [\gamma] \) is the reparametrization of \( \gamma \) by the parameter \( t \in [0, 1] \) proportional to the arclength. The projecton \( \pi_1: \mathcal{R}_{\kappa_1}^{\kappa_2}(P, Q) \to \mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q) \) is the restriction of \( \pi_2 \) on \( \mathcal{R}_{\kappa_1}^{\kappa_2}(P, Q) \).

Since the projection \( \pi_2 \) preserves the fiber and thus the submersion map: \( \mathcal{R}_{\kappa_1}^{\kappa_2}(P, \cdot) \to SO_3(\mathbb{R}) \) induces a submersion map: \( \mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot) \to SO_3(\mathbb{R}) \). Therefore, \( \mathcal{P}_{\kappa_1}^{\kappa_2}(P, Q) \) is a Banach submanifold of \( \mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot) \) with codimension 3, which is given by the submersion \( \mathcal{P}_{\kappa_1}^{\kappa_2}(P, \cdot) \to SO_3(\mathbb{R}) \). This concludes the proof of the theorem.

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