Overlap Fermions on a $20^4$ Lattice

K. F. Liu $^a$, S. J. Dong $^a$, F. X. Lee $^b$, and J. B. Zhang $^a$

$^a$Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA

$^b$Department of Physics, George Washington University, Washington, DC 20052, USA

$^c$Jefferson Lab, 12000 Jefferson Avenue, Newport News, VA 23606, USA

$^d$Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, P.R. China

We report results on hadron masses, fitting of the quenched chiral log, and quark masses from Neuberger’s overlap fermion on a quenched $20^4$ lattice with lattice spacing $a = 0.15$ fm. We used the improved gauge action which is shown to lower the density of small eigenvalues for $H^2$ as compared to the Wilson gauge action. This makes the calculation feasible on 64 nodes of CRAY-T3E. Also presented is the pion mass on a small volume ($6^3 \times 12$ with a Wilson gauge action at $\beta = 5.7$). We find that for configurations that the topological charge $Q \neq 0$, the pion mass tends to a constant and for configurations with trivial topology, it approaches zero possibly linearly with the quark mass.

Overlap fermion has great promise in studying chiral symmetry on the lattice [1]. Recent numerical implemenation of the overlap fermions has led to studies of chiral condensate [2,3], quark mass [4], as well as the checking of chiral symmetry and scaling [4,5]. However these studies are limited to small volumes due to the enormous numerical cost associated with approximating the matrix sign function which is about fifty times more than inverting the quark matrix of the vanilla Wilson action.

In the optimal rational approximation [6] of the matrix sign function of Neuberger’s overlap operator [7]

\[ D(m_0) = 1 + \frac{m_0}{2} + (1 - \frac{m_0}{2}) \epsilon(H), \]

where $\epsilon(H) = H/\sqrt{H^2}$ and $H$ is taken to be the hermitian Wilson-Dirac operator, i.e. $H = \gamma_5 D_w$ with $0.25 > \kappa > \kappa_c$, it is cost effective to project out a relatively few eigenmodes with very small eigenvalues in the operator $H^2$ in order to reduce the condition number and speed up the convergence in the inner do loop [8,9]. At the same time, it improves the chiral symmetry relation such as the Gell-Mann-Oakes-Renners relation [10]. However, it is shown [11] that the density of these small eigenmodes grows as $e^{c a}$ with $a$ being the lattice spacing. As a result, it is very costly and impractical to work on large volumes with the currently used lattice spacings. There are simply too many small eigenmodes to be projected out.

For this reason, we explore other options to clear this hurdle. We have tested the $O(a^2)$ improved gauge action of Morningstar and Peardon [12] and find that the density of these small eigenvalue modes is decreased to a point that it becomes feasible to go to large volumes with a size of the lattice 3 to 4 times of the Compton wavelength of the lightest pion. We further find that the anisotropic action [13] requires projection of more small eigenvalues in $H^2$ in order to achieve the same convergence in the inner loop than does the isotropic one. Thus, we use the isotropic action. We also find that using the clover action with either sign requires the projection of more small eigenvalue modes. Therefore we use the Wilson action for $H$ in the Neuberger operator with $\kappa = 0.19$. On a $20^4$ lattice with $a = 0.15$ fm as determined from the Sommer scale $r_0$, we...
project out 80 small eigenvalues. Beyond these eigenmodes, the level density becomes dense. As a result, the number of conjugate gradient steps for the inner loop is about 160 and the conjugate gradient steps is about 210 for the outer loop. These numbers are about the same as those for the Wilson gauge action on small volumes \[4\]. Therefore, other than the overhead of projecting out the small eigenvalue modes, the cost scales linearly with volume.

We shall report preliminary results on 38 gauge configurations. Given \( a = 0.15 \text{ fm} \) for the 20\textsuperscript{4} lattice, the physical length of the lattice is 3 fm. We shall study pion mass as low as \( 0.14 \text{ fm} \) for the 20\textsuperscript{4} lattice, the physical length of the lattice is 3 fm. Therefore, other than the overhead of projecting out the small eigenvalue modes, the cost scales linearly with volume.

We first show the pseudoscalar mass squared \( m^2 \) as a function of \( m_0 a \) in Figure 1. We fit them in the following form \([10]\) with and without the quenched chiral log term \( \delta \)

\[
m^2 = c + 2A m_0 a^2 \left( 1 - \delta \ln(2A m_0 / \Lambda^2) \right) + 4Bm_0^2 \delta (2)
\]

![Figure 1. Pion mass squared as a function of the bare quark mass \( m_0 a \) on the 20\textsuperscript{4} lattice with \( a = 0.15 \text{ fm} \).](image)

As shown in Table 1, the fits with and without the quenched chiral log \( \delta \) give comparable \( \chi^2/DF \) for the 6 to 10 smallest quark masses ranging from 0.014 to 0.24 (see Figure 1). The fit is insensitive to \( \Lambda_\chi \) in the range of 0.6 to 1.4 GeV. We list results with \( \Lambda_\chi = 1.0 \text{ GeV} \) in Table 1. Furthermore, the errors on \( \delta \) are much larger than the corresponding central values. We conclude that there is no signal for the quenched chiral log in the range \( m_\pi / m_\rho \sim 0.35 - 0.80 \). This is in contrast to the study of the Wilson fermion action on lattices with the same physical volume and comparable quark masses \([11]\) which supports the presence of quenched chiral logarithms with a magnitude \( \delta \sim 0.1 \). We also note that in our fit, the intercept \( c \) is consistent with zero which is to be expected.

Next we plot the nucleon, the vector and pseudoscalar masses as a function of \( m_0 a \) in Figure 2. A simple linear fit with 10 smallest quark masses gives \( m_N a = 0.534(12) \) at the chiral limit with \( \chi^2/DF = 0.05 \). Using the \( r_0 \) to set the scale, this corresponds to 712(16) MeV. Since the error in \( m_N \) is still large, we can not draw any conclusion on the \( m_N/m_\rho \) ratio yet.

![Figure 2. Nucleon, vector and pseudoscalar meson masses.](image)

We have computed the quark mass through the chiral Ward identity \( Z_A \partial_\mu A_\mu = 2Z_S^{-1} m_0 Z_P P \) where \( A_\mu = \bar{\psi} i \gamma_\mu \gamma_5 (\tau/2) \psi \) and \( P = \bar{\psi} \gamma_5 (\tau/2) \psi \). We plot \( \frac{2P(A_\mu)}{Z_A(\mu)} m_0^2 \) in Figure 2 and fit them in the form \( c + Am_0 + Bm_0 a^2 \). The fits with a few smallest quark masses are given...
Table 1
The fitted parameters in Eq. (2). $\delta = 0$ corresponds to the case without the chiral log.

| # pts. | c     | $\delta$ | A     | B     | $\chi^2$/DF |
|--------|-------|----------|-------|-------|-------------|
| 5      | 0.0016(50) | 0        | 4.51(22) | 0     | 0.008       |
| 6      | -0.0006(41) | 0        | 4.65(13) | 0     | 0.2         |
| 6      | 0.0045(70)  | 0        | 4.17(56) | 12(24) | 0.02        |
| 6      | 0.0047(53)  | 0        | 4.15(30) | 7.5(29) | 0.03        |
| 6      | 0.004(18)   | 0.015(297)| 4.13(56) | 8.2(149)| 0.035       |
| 10     | 0.0051(49)  | 0        | 4.12(22) | 7.8(20) | 0.04        |
| 10     | 0.02(15)    | 0.043(227)| 4.07(36) | 9.7(98) | 0.04        |

Table 2
The renormalized quark mass $Z_A(\mu) = Z_{\overline{\text{MS}}}^{\overline{\text{MS}}}^{\overline{\text{MS}}}(\mu) = c + A m_0 a + B m_0^2 a^2$ fitted in the form $c + A m_0 a + B m_0^2 a^2$.

| #     | c           | A            | B       | $\chi^2$/DF | $Z_A(\mu, 0)$ |
|-------|-------------|--------------|---------|-------------|---------------|
| 4     | -0.00041(34)| 0.939(24)   | 1.66(34)| 0.003       | 1.065(27)     |
| 5     | -0.00049(24)| 0.945(14)   | 1.56(16)| 0.05        | 1.058(16)     |
| 6     | -0.00063(16)| 0.956(8)    | 1.43(7) | 0.29        | 1.046(9)      |

in Table 2. We see that the intercept $c$ is consistent with zero for the case with 4 smallest masses. From this we can deduce the non-perturbative renormalization constant $Z_A(\mu, m_0)$ as a function of $m_0 a$, i.e., $Z_A^{-1}(\mu, m_0) = A + B m_0 a$. We see from Table 2 that $Z_A(\mu, m_0 = 0)$ is fairly far from the tree-level value of 1.368.

Finally we shall explore the behavior of the pseudoscalar meson mass on a small volume. Starting from the generalized Gell-Mann-Oakes-Renners relation

$$m_\pi^2 = \frac{2 m_0}{f^2} \langle \bar{\psi} \psi \rangle,$$

and assuming pion dominance, one obtains the usual Gell-Mann-Oakes-Renners relation

$$m_\pi^2 = \frac{2 m_0}{f^2} \langle \bar{\psi} \psi \rangle.$$  (4)

In the quenched approximation, the chiral condensate has the following behavior for small $m_0$

$$\langle \bar{\psi} \psi \rangle = \frac{|Q|}{m_0 V} + a + b m_0,$$

where the first term due to the fermion zero modes associated with the topological charge $Q$ is the quenched artifact which is prominent as $m_0 \to 0$ and/or at small volume $V$. This term is observed in $\langle \bar{\psi} \psi \rangle$ through a direct calculation with the overlap fermion [12]. Thus, barring additional complication due to the quenched chiral
We display in Figure 4, $m_{\pi}^2 a^2$ as a function of $m_0 a$ for the $6^3 \times 12$ lattice with 50 Wilson gauge configurations at $\beta = 5.7$. The top one shows that $m_{\pi}^2$ approaches a constant in the chiral limit as shown in Eq. (6). To further verify that the constant is indeed due to the nonvanishing topological charge in the configuration, we separate the 50 gauge configurations into 32 $Q \neq 0$ ones (the middle figure) and 18 with $Q = 0$ (the bottom figure).

We see that $m_{\pi}^2 a^2$ for the $Q \neq 0$ case approaches the same constant as $m_0 a \to 0$, while it is much smaller for the $Q = 0$ case which has a tendency to approach zero at the chiral limit. From the curvature of the dependence on $m_0 a$, one speculates that $m_0 a$ is below the Thouless energy in such a small volume so that the $a m_0$ term in Eq. (6) vanishes. As a result, $m_{\pi}^2$ is proportional to $m_0^2$ and consequently the pion mass approaches the chiral limit linearly with $m_0$. However, one needs more statistics to confirm this scenario.

To conclude, we demonstrate that it is feasible to implement the overlap fermions on large physical volumes. We do not see the quenched chiral log in the range $m_{\pi}/m_0 \sim 0.35 - 0.80$.

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