NOTE ON THE COVERING THEOREM FOR COMPLEX POLYNOMIALS

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In this note we will give the answer to the questions posed in [D1] and [D2] concerning covering properties of complex polynomials. These questions attract our attention when we study stability properties of dynamical systems stabilized by the feedback control. It was discovered recently [DH], that these properties play the central role in the matter.

We start with the simple observation on the stable polynomials. From this observation we deduce the variant of Koebe One-Quarter Theorem for complex polynomials and from the last statement we obtain the sharp constant in the Theorem 1 from [D1] (Theorem 3.8 in [D2]).

Let \( \Delta = \{ z : |z| < 1 \} \) be the open unit disk and \( \overline{\Delta} \) its closure.

Let \( \chi(z) = a_0 + a_1 z + \ldots + a_n z^n \) be a given polynomial of degree at most \( n \). We would like to define it \( n \)-inverse by the formula

\[
\chi^*(z) = a_n + a_{n-1} z + \cdots + a_0 z^n
\]

Since \( \chi^*(z) = z^n \chi(1/z) \), polynomial \( \chi^*(z) \) "inverse" the ranges \( \chi(\Delta) \) and \( \chi(\mathbb{C} \setminus \overline{\Delta}) \).

We shall use this definition even in the case when \( a_n = 0 \). For example the 4-inverse of the polynomial \( z \) is \( z^3 \) and vice versa.

So defined \( n \)-inversion is an \textit{involution} on the set of polynomials of degree at most \( n \), i.e. \( \chi^{**}(z) = \chi(z) \).

The problem of \textit{stability} of polynomial \( \chi(z) \), which means, that all roots of \( \chi(z) \) lie in \( \Delta \), is equivalent to the problem of the description of the \textit{image} of \( \overline{\Delta} \) by the map \( \chi^* \).

Indeed, if all zeros of some polynomial \( \chi(z) \) lie in the disc \( \Delta \), then no one lies outside, which simply means, that \( 0 \notin \chi^*(\overline{\Delta}) \).

To be more precise we write the last observation like a lemma.

**Lemma 1.** Let \( \chi(z) = a_0 + a_1 z + \ldots + a_n z^n \) be a polynomial with \( a_n \neq 0 \). Then

\[
(2) \quad \text{all zeros of } \chi(z) \text{ lie in } \Delta
\]

if and only if

\[
(3) \quad 0 \notin \chi^*(\overline{\Delta})
\]

**Proof.** Let \( \chi(z) = a_n z^m (z - z_{m+1}) \ldots (z - z_n) \), where \( 0 \leq m < n \) (the case \( m = n \) is trivial). Then

\[
(4) \quad \chi^*(z) = z^n \chi(1/z) = a_n (1 - z_{m+1} z) \ldots (1 - z_n z)
\]

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Thus, from (2) we conclude that all zeros of $\chi^*(z)$ lie outside of $\bar{\Delta}$, which implies (3). On the other hand, if (3) holds, then (1) implies (2). □

The statement of lemma 1 is also holds if we change $\Delta$ and $\bar{\Delta}$. We shall state here corresponding lemma in a little different (inversion) form and without proof, which is exactly the same.

**Lemma 2.** Let $\chi(z) = a_0 + a_1z + \ldots + a_nz^n$ be a polynomial with $a_0 \neq 0$. Then,

$$0 \notin \chi(\Delta)$$

if and only if

$$\text{all zeros of } \chi^*(z) \text{ lie in } \bar{\Delta}$$

It is a well known phenomenon in Geometric Function Theory, that some restrictions on the range $f(\Delta)$ implies estimates of Taylor coefficients of $f$.

For example, for

$$f(z) = z + c_2z^2 + \ldots + c_kz^k + \ldots$$
defined in the unit disk $\Delta$: if $f$ maps $\Delta$ into the half-plane $\{z : \Re z > -1/2\}$ then $|c_k| \leq 1$ (Caratheodori) and if the function is schlicht, i.e. maps $\Delta$ in one to one way to the simply connected domain, then $|c_k| \leq k$ (de Brange).

The simplest proposition, which reflect this phenomenon ( for complex polynomials) is the following

**Lemma 3.** Let $q(z) = \hat{q}(1)z + \hat{q}(2)z^2 + \ldots + \hat{q}(n)z^n$ and

$$w \notin q(\Delta).$$

Then

$$|\hat{q}(k)| \leq \binom{n}{k}|w|$$

*Proof.* Let $\chi(z) = q(z) - w$ and $\chi^*(z) = \hat{q}(n) + \hat{q}(n-1)z + \ldots + \hat{q}(1)z^n - wz^n$ is it n-inverse. Then, $0 \notin \chi(\Delta)$ and, according to lemma 2, all roots of $\chi^*(z) = -w(z - \zeta_1)\ldots(z - \zeta_n)$ lie in $\bar{\Delta}$. Hence, by Vieta’s formulas,

$$|\hat{q}(k)| = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} (-1)^{k+1}w\zeta_{i_1}\zeta_{i_2}\ldots\zeta_{i_k} \leq |w| \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} 1 = \binom{n}{k}|w|$$

The estimates [8] are the best possible, since for the polynomial $q(z) = w - w(1 - z)^n$ point $w \notin q(\Delta)$, but $|\hat{q}(k)| = \binom{n}{k}|w|$. □

Define the *norm* of $q(z) = \hat{q}(0) + \hat{q}(1)z + \ldots + \hat{q}(n)z^n$ by the formula

$$n(q) = \max_{0 \leq k \leq n} \frac{|\hat{q}(k)|}{\binom{n}{k}}$$

Denote $\Delta_r(z_0) = \{z : |z - z_0| < r\}$ and claim, that, because $q : \mathbb{C} \to \mathbb{C}$ is an open mapping, the range $q(\Delta_r(z_0))$ always contains some disk $\Delta_r(q(z_0))$. 
It is interesting, that from lemma 3 we can obtain the (sharp) estimate of the radius of this disk and immediately conclude that \( q(z) \) is an open mapping.

It is clear, that we can suppose that \( z_0 = 0, q(z_0) = 0 \) and \( r = 1 \).

**Corollary 1.** For every polynomial \( q(z) = \hat{q}(1)z + \hat{q}(2)z^2 + \cdots + \hat{q}(n)z^n \) the range \( q(\Delta) \) contains the disk of radius \( n(q) \) with the center at the origin:

\[
\Delta_{n(q)} \subseteq q(\Delta).
\]

**Proof.** Lemma 3 implies \( |w| \geq n(q) \) for \( w \notin q(\Delta) \) and we have \( \Delta_{n(q)} \subseteq q(\Delta) \). \( \square \)

Since \( n(q) \geq 1/n \) for the polynomial \( q \) of degree \( n \), such that \( q(0) = 0 \) and \( q'(0) = 1 \) we have

**Corollary 2.** For every polynomial \( q(z) = z + \cdots q_n z^n \) the range \( q(\Delta) \) contains the disk of radius \( 1/n \) with the center at the origin:

\[
\Delta_{1/n} \subseteq q(\Delta)
\]

If we apply Corollary 2 to the polynomial

\[
\tilde{q}(z) = \frac{1}{R}q(Rz)
\]

we can get a slight general version of this assertion which gives the answer to the question posed in [D1] and [D2].

**Corollary 3.** For every polynomial \( q(z) = z + \cdots q_n z^n \) the range \( q(\Delta_R) \) contains the disk of radius \( R/n \) centered at the origin:

\[
\Delta_{R/n} \subseteq q(\Delta_R)
\]

and

\[
\Delta_{R/n} \subseteq q(\Delta_R)
\]

The second statement (12) follows from (11) by the compactness arguments. Example

\[
q(z) = \frac{R}{n}((1 + \frac{z}{R})^n - 1)
\]

shows that the size of the circle is the best possible.

Corollary 3 implies the sharp constant in Theorem 1 from [D1].

**Theorem 1.** For every polynomial \( p(z) = p_0 + p_1 z + \cdots + p_n z^n \) and any points \( z_1 \) and \( z_2 \) there exist point \( \zeta \), such that \( p(\zeta) = p(z_2) \) and

\[
|p(z_1) - p(z_2)| \geq \frac{1}{n}|p'(z_1)||z_1 - \zeta|
\]

**Proof.** Let

\[
q(z) = \frac{1}{p'(z_1)}(p(z_1) - p(z_1 - z))
\]

Thus \( q(0) = 0 \) and \( q'(0) = 1 \). For any point \( z \) there is a point \( \eta \), such that \( q(z) = q(\eta) \) and

\[
|q(z)| \geq \frac{1}{n}|
\]

\[
\eta|
\]
To prove this, denote $w = q(z)$ and define $R = n|w|$ for $w \neq 0$ (if $w = 0$ there is nothing to prove). We have $w \in \Delta_R^n$ and Corollary 3 implies that there is $\eta \in \Delta_R$, such that $w = q(z) = q(\eta)$.

Now, for the given $z_2$ define $z = z_1 - z_2$, apply (14) and choose $\zeta = z_1 - \eta$. □

References

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