EXPLICIT MERTENS’ THEOREMS FOR NUMBER FIELDS I: ASSUMING THE GENERALIZED RIEMANN HYPOTHESIS

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Abstract. Assuming the Generalized Riemann Hypothesis, we obtain effective number field analogues of Mertens’ theorems (with explicit constants).

1. Introduction

In 1874, twenty-two years before the proof of the prime number theorem by Hadamard [4] and de la Vallée Poussin [11], Mertens proved the following three results, which are collectively referred to as “Mertens’ theorems” [8]:

\[ \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1), \]
\[ \sum_{p \leq x} \frac{1}{p} = \log \log x + M + O\left(\frac{1}{\log n}\right), \]
\[ \prod_{p \leq x} \left(1 - \frac{1}{p}\right) = e^{-\gamma} \log x \left(1 + o(1)\right), \]

in which \( p \) denotes a rational prime number and

\[ M = \lim_{n \to \infty} \left( \sum_{p \leq x} \frac{1}{p} - \log \log x \right) = \gamma + \sum_p \left[ \log \left(1 - \frac{1}{p}\right) + \frac{1}{p}\right] = 0.2614972\ldots \]

is the Meissel–Mertens constant and

\[ \gamma = \lim_{x \to \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log x \right) = -\int_0^\infty e^{-t} \log t \, dt = 0.5772\ldots \]

is the Euler–Mascheroni constant. Ingham [5] Thm. 7] and Montgomery–Vaughan [9 Thm. 2.7] give modern proofs of these results. Rosser–Schoenfeld [12 (3.17) – (3.30)] provide error terms with explicit constants.

Rosen [11 Lem. 2.3, Lem. 2.4, Thm. 2] generalized Mertens’ theorems to the number field setting, but without explicit constants. We seek analogues, with completely explicit constants, of these results. We work here under the assumption that the Generalized Riemann Hypothesis is true. The unconditional setting is more complicated due to the possible presence of exceptional zeros for the corresponding Dedekind zeta function and will be addressed in the sequel.

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In what follows, let \( \mathbb{K} \) be a number field with ring of algebraic integers \( \mathcal{O}_\mathbb{K} \). Let \( n_\mathbb{K} = [\mathbb{K} : \mathbb{Q}] \) denote the degree of \( \mathbb{K} \) and \( N(a) \) the norm of an ideal \( a \subset \mathcal{O}_\mathbb{K} \). The Dedekind zeta function

\[
\zeta_\mathbb{K}(s) = \sum_{a \subseteq \mathcal{O}_\mathbb{K}} \frac{1}{N(a)^s} = \prod_p \left(1 - \frac{1}{N(p)^s}\right)^{-1},
\]

in which \( p \) runs over the prime ideals in \( \mathcal{O}_\mathbb{K} \), is analytic for \( \text{Re } s > 1 \) and enjoys a meromorphic extension to the entire complex plane with a simple pole at \( s = 1 \).

The residue \( \ell_\mathbb{K} \) of \( \zeta_\mathbb{K}(s) \) at \( s = 1 \) is

\[
\ell_\mathbb{K} = \frac{2^{r_1}(2\pi)^{r_2} h_\mathbb{K} R_\mathbb{K}}{w_\mathbb{K} \sqrt{\Delta_\mathbb{K}}},
\]

in which \( \Delta_\mathbb{K} \) denotes the absolute value of the discriminant, \( r_1 \) the number of real places of \( \mathbb{K} \), \( r_2 \) the number of complex places of \( \mathbb{K} \), \( w_\mathbb{K} \) the number of roots of unity in \( \mathbb{K} \), \( h_\mathbb{K} \) the class number of \( \mathbb{K} \), and \( R_\mathbb{K} \) the regulator of \( \mathbb{K} \) [7]. The Generalized Riemann Hypothesis (GRH) asserts that all of the zeros of \( \zeta_\mathbb{K}(s) \) with \( \text{Re } s \in (0, 1) \) satisfy \( \text{Re } s = 1/2 \).

Our main result is the following completely explicit version of Mertens’ theorems for number fields. We have made the error terms as clean and practical as possible since we hope that these results will be widely usable in this format. Although minor numerical improvements can be made, they do not substantially change the dependence upon the discriminant and degree of the number field.

**Theorem 1.** Assume GRH. For a number field \( \mathbb{K} \) and \( x \geq 2 \),

\[
\sum_{N(p) \leq x} \frac{\log N(p)}{N(p)} = \log x + A_\mathbb{K}(x), \tag{A1}
\]

\[
\sum_{N(p) \leq x} \frac{1}{N(p)} = \log \log x + M_\mathbb{K} + B_\mathbb{K}(x), \tag{B1}
\]

\[
\prod_{N(p) \leq x} \left(1 - \frac{1}{N(p)}\right) = e^{-\gamma} \frac{\ell_\mathbb{K}}{\ell_\mathbb{K}} \log x \left(1 + C_\mathbb{K}(x)\right), \tag{C1}
\]

in which

\[
M_\mathbb{K} = \gamma + \log \ell_\mathbb{K} + \sum_p \left[\frac{1}{N(p)} + \log \left(1 - \frac{1}{N(p)}\right)\right]
\]

satisfies

\[
\gamma + \log \ell_\mathbb{K} - n_\mathbb{K} \leq M_\mathbb{K} \leq \gamma + \log \ell_\mathbb{K}, \tag{M}
\]

and

\[
|A_\mathbb{K}(x)| \leq 7 \log \Delta_\mathbb{K} + 14 n_\mathbb{K}, \tag{A2}
\]

\[
|B_\mathbb{K}(x)| \leq \frac{14 \log \Delta_\mathbb{K} + (27 + 0.12 \log x) n_\mathbb{K}}{\sqrt{x}}, \tag{B2}
\]

\[
|C_\mathbb{K}(x)| \leq |E_\mathbb{K}(x)| e^{E_\mathbb{K}(x)} \text{ with } |E_\mathbb{K}(x)| \leq \frac{n_\mathbb{K}}{x - 1} + |B_\mathbb{K}(x)|. \tag{C2}
\]

In particular, \( E_\mathbb{K}(x) = o(1) \) and hence \( C_\mathbb{K}(x) = o(1) \) as \( x \to \infty \).
Our computations utilize only elementary functions, although we briefly explain (see Remark 3 below) how the constants in (A2) can be improved somewhat if numerical evaluation of nonelementary functions is permitted.

It is possible to generalize Theorem 1 further while retaining the explicit nature of the constants involved. Let \( K \subseteq L \) be a Galois extension of number fields with Galois group \( G = \text{Gal}(L/K) \). Let \( \Delta_L \) denote the absolute value of the discriminant of \( L \), \( n_L = [L : \mathbb{Q}] \), \( p \) a prime ideal of \( K \), and \( \mathfrak{p} \) a prime ideal of \( L \) lying above an unramified prime \( p \). The Artin symbol, \( [L/K]_p \) denotes the conjugacy class of Frobenius automorphisms corresponding to prime ideals \( \mathfrak{p} | p \). For each conjugacy class \( C \subset G \), the prime ideal counting function is

\[
\pi_C(x) = \# \{ \mathfrak{p} : \mathfrak{p} \text{ is unramified in } L, \ [L/K]_\mathfrak{p} = C, \ N_K(\mathfrak{p}) \leq x \},
\]

in which \( N_K(\cdot) \) denotes the norm in \( K \). Slight changes to the proof of Theorem 1 (see Remark 3) yield the following more general result.

**Theorem 2.** Assume GRH, let \( K \subseteq L \) be a Galois extension of number fields, and let \( p \) denote a typical prime ideal of \( K \) that is unramified in \( L \). For \( x \geq 2 \),

\[
\sum_{N(\mathfrak{p}) \leq x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} = \frac{\#C}{\#G} \log x + A_L(x),
\]

\[
\sum_{N(\mathfrak{p}) \leq x} \frac{1}{N(\mathfrak{p})} = \frac{\#C}{\#G} \log \log x + M_L + B_L(x),
\]

\[
\prod_{N(\mathfrak{p}) \leq x} \left( 1 - \frac{1}{N(\mathfrak{p})} \right) = \frac{e^{-\gamma}}{\ell_L(\log x)^{#C/#G}} \left( 1 + C_L(x) \right),
\]

in which \( \mathfrak{p} \) is a prime ideal in \( K \) that does not ramify in \( L \), \( \ell_L \) denotes the residue of \( \zeta_L(s) \) at \( s = 1 \), and \( A_L(x) \), \( B_L(x) \), \( C_L(x) \) and \( M_L \) are defined as \( A_K(x) \), \( B_K(x) \), \( C_K(x) \) and \( M_K \) are in Theorem 1 but with \( n_L \), \( \Delta_L \), \( \ell_L \) in place of \( n_K \), \( \Delta_K \), \( \ell_K \).

The remainder of the paper is devoted to the proof of Theorem 2.

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2. **Proof of Theorem 2**

We prove (A1), (B1), and (C1), with the associated error bounds (A2), (B2), and (C2), respectively, in separate subsections. We then prove (M), which is a consequence of the computations in the proof of (C1). We are guided largely by Rosen’s approach to Mertens-type theorems for number fields [11].

In what follows we assume the Generalized Riemann Hypothesis (GRH). Grenié–Molteni [3 Cor. 1] proved in 2019 that GRH implies

\[
\pi_K(x) = \text{Li}(x) + R_K(x) \quad \text{for } x \geq 2,
\]

in which \( K \) is a number field, \( \pi_K = \sum_{N\mathfrak{p} \leq x} 1 \) is the prime-ideal counting function,

\[
\text{Li}(x) = \int_2^x \frac{dt}{\log t}
\]
is the offset logarithmic integral, and

\[ |R_K(x)| \leq \sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta_K + \left( \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) n_K \right]. \]  \tag{1}

This strengthens a result announced by Oesterlé [10] and improves the constants Winckler obtains [13, Thm. 1.2] by using methods from Schoenfeld [13], Lagarias–Odlyzko [6] and previous work of Grenié–Molteni [2].

**Remark 3.** We briefly remark how to adjust the proof below to obtain Theorem 2. For \( x \geq 2 \), Grenié–Molteni [3, Cor. 1] proved

\[ \pi_C(x) = \frac{\#C}{\#G} \text{Li}(x) + R_{L/K}(x), \]

in which

\[ |R_{L/K}(x)| \leq \frac{\#C}{\#G} \sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta_L + \left( \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) n_L \right]. \]

Since \( \#C/\#G \leq 1 \), we can bound \( |R_{L/K}(x)| \) above by (1). The proof below goes through *mutatis mutandis* (for example, any occurrence of \( \text{Li}(x) \) should be replaced by \( (\#C/\#G) \text{Li}(x) \)).

### 2.1. Proof of (A1). If \( f \) is continuously differentiable, integration by parts yields

\[
\sum_{N(p) \leq x} f(N(p)) = f(x)\pi_K(x) - \int_2^x f'(t)\pi_K(t) \, dt \\
= f(x)\text{Li}(x) - \int_2^x f'(t)\text{Li}(t) \, dt + f(x)R_K(x) - \int_2^x f'(t)R_K(t) \, dt \\
= \int_2^x \frac{f(t)}{\log t} \, dt + f(x)\text{Li}(x) - \int_2^x f'(t)\text{Li}(t) \, dt. \tag{2}
\]

The preceding formula will be important moving forward. Let \( f(t) = \log t/t \) in (2) and obtain

\[
\sum_{N(p) \leq x} \frac{\log N(p)}{N(p)} = \log x + A_K(x),
\]

in which

\[ A_K(x) = f(x)R_K(x) - \int_2^x f'(t)R_K(t) \, dt - \log 2. \] \tag{3}

For \( x \geq 2 \),

\[ \frac{\log x}{\sqrt{x}} \] achieves its maximum value \( \frac{2}{e} \) at \( x = e^2 \)

and

\[ \frac{(\log x)^2}{\sqrt{x}} \] achieves its maximum value \( \frac{16}{e^4} \) at \( x = e^4 \).

Then (1) yields

\[
|f(x)R_K(x)| \leq \frac{\log x}{\sqrt{x}} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta_K + \left( \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) n_K \right] \\
= \frac{1}{\sqrt{x}} (3\log \Delta_K + 6n_K) + \frac{\log x}{\sqrt{x}} \left( \frac{\log \Delta_K}{2\pi} + \frac{n_K}{4\pi} \right) + \frac{(\log x)^2}{\sqrt{x}} \frac{n_K}{8\pi}.
\]
\[
\leq \frac{1}{\sqrt{2}} (3 \log \Delta_K + 6n_K) + \frac{2}{e} \left( \frac{\log \Delta_K}{2\pi} + \frac{n_K}{4\pi} \right) + \frac{2n_K}{e^2 \pi} < 2.23843 \log \Delta_K + 4.3874 n_K.
\]

Observe that
\[
f'(t) = \frac{1 - \log t}{t^2}
\]
undergoes a sign change from positive to negative at \( x = e \) and that
\[
\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x}, \quad \text{whose derivative is} \quad \frac{1}{8\pi x} - \frac{6}{x(\log x)^2},
\]
decreases until \( e^{4\sqrt{3}\pi} \approx 215,328.6 \) and then increases. Consequently,
\[
\left| \int_{\epsilon}^{e} f'(t) R_K(t) \, dt \right|
\leq \int_{\epsilon}^{e} \frac{1 - \log t}{t^2} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log t} \right) \log \Delta_K + \left( \frac{\log t}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log t} \right) n_K \right] \, dt
\leq \left( \frac{4}{\sqrt{e}} - \sqrt{2(1 + \log 2)} \right) \left[ \left( \frac{1}{2\pi} + \frac{3}{\log 2} \right) \log \Delta_K + \left( \frac{\log 2}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log 2} \right) n_K \right]
< 0.14203 \log \Delta_K + 0.27737 n_K.
\]

For \( x \geq e \), we have
\[
\left| \int_{\epsilon}^{x} f'(t) R_K(t) \, dt \right|
\leq \int_{\epsilon}^{\infty} \frac{\log t - 1}{t^2} \sqrt{t} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log t} \right) \log \Delta_K + \left( \frac{\log t}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log t} \right) n_K \right] \, dt
\leq \left( \frac{1}{2\pi} \log \Delta_K + \frac{n_K}{4\pi} \right) \int_{\epsilon}^{\infty} \frac{\log t \, dt}{t^{3/2}} + (3 \log \Delta_K + 6n_K) \int_{\epsilon}^{\infty} \frac{dt}{t^{3/2}} + \frac{n_K}{8\pi} \int_{\epsilon}^{\infty} \frac{(\log t)^2 \, dt}{t^{3/2}}
= \left( \frac{1}{2\pi} \log \Delta_K + \frac{n_K}{4\pi} \right) \left( \frac{6}{\sqrt{e}} \right) + (3 \log \Delta_K + 6n_K) \left( \frac{2}{\sqrt{e}} \right) + \frac{n_K}{8\pi} \left( \frac{26}{\sqrt{e}} \right)
= \frac{3(1 + 2\pi) \log \Delta_K}{\pi \sqrt{e}} + \frac{(19 + 48\pi) n_K}{4\pi \sqrt{e}}
< 4.21838 \log \Delta_K + 8.19543 n_K.
\]

Return to (4) and use the triangle inequality to obtain
\[
|A_K(x)| < 6.6 \log \Delta_K + 12.9 n_K + \log 2 < 7 \log \Delta_K + 14 n_K
\]
since \( n_K \geq 1 \) and \( \log 2 < 1 \). This concludes the proof of (A1). \( \square \)

**Remark 4.** Without the overestimate \( \log t - 1 \leq \log t \), one obtains
\[
\left| \int_{\epsilon}^{e} f'(t) R_K(t) \, dt \right| \leq \frac{3\pi \sqrt{e} \text{Ei}(-\frac{1}{4}) + 6\pi + 2) \log \Delta_K}{\pi \sqrt{e}} + \frac{(12\pi \sqrt{e} \text{Ei}(-\frac{1}{4}) + 24\pi + 7) n_K}{2\pi \sqrt{e}} < 2.34600 \log \Delta_K + 4.59546 n_K,
\]
where \( \text{Ei}(x) \) is the exponential integral. Thus, \( |A_K(x)| < 4.73 \log \Delta_K + 9.27 n_K \)
albeit with the numerical use of nonelementary functions.
Let $f(t) = 1/t$ in \(2\) and obtain
\[
\sum_{N(p) \leq x} \frac{1}{N(p)} = \int_2^x \frac{dt}{t \log t} + \frac{R_k(x)}{x} + \int_2^x \frac{R_k(t) \, dt}{t^2} - \log \log x - \int_2^x \frac{R_k(t) \, dt}{t^2} - \frac{R_k(x)}{x} + \int_2^x \frac{R_k(t) \, dt}{t^2}.
\]

in which the convergence of the improper integral is guaranteed by \(1\).

For \(x \geq 2\), \(1\) provides
\[
\frac{|R_k(x)|}{x} \leq \frac{1}{\sqrt{x}} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta_k + \left( \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) n_k \right]
\]
and
\[
\int_x^\infty \frac{|R_k(t)| \, dt}{t^2} \leq \int_x^\infty \frac{1}{t^{3/2}} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log t} \right) \log \Delta_k + \left( \frac{\log t}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log t} \right) n_k \right] \, dt
\]
\[
= \log \Delta_k \int_x^\infty \frac{1}{t^{3/2}} \left( \frac{1}{2\pi} + \frac{3}{\log t} \right) \, dt + n_k \int_x^\infty \frac{1}{t^{3/2}} \left( \frac{\log t}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log t} \right) \, dt
\]
\[
\leq \log \Delta_k \int_x^\infty \frac{1}{t^{3/2}} \left( \frac{1}{2\pi} + \frac{3}{\log t} \right) \, dt + n_k \int_x^\infty \frac{1}{t^{3/2}} \left( \frac{\log t}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log t} \right) \, dt
\]
\[
= \frac{1}{\sqrt{x}} \left[ \left( \frac{1}{\pi} + \frac{6}{\log x} \right) \log \Delta_k + \left( \frac{\log x}{4\pi} + \frac{1}{\pi} + \frac{12}{\log x} \right) n_k \right].
\]

For \(x \geq 2\), the triangle inequality provides
\[
|B_k(x)| \leq \frac{1}{\sqrt{x}} \left[ \left( \frac{3}{2\pi} + \frac{9}{\log x} \right) \log \Delta_k + \left( \frac{3\log x}{8\pi} + \frac{5}{4\pi} + \frac{18}{\log x} \right) n_k \right]
\]
\[
< \frac{1}{\sqrt{x}} \left[ 13.46172020 \log \Delta_k + (26.36639809 + 0.1193662 \log x) n_k \right]
\]
\[
< \frac{1}{\sqrt{x}} \left[ 14 \log \Delta_k + (27 + 0.12 \log x) n_k \right].
\]

Now we must find the constant \(M_k\); our approach follows Ingham’s \(5\). Define
\[
g(s) = \sum_p \frac{1}{N(p)^s} = \lim_{x \to \infty} \sum_{N(p) \leq x} \frac{1}{N(p)^s},
\]
which is analytic on \(\text{Re } s > 1\). For \(x \geq 2\), partial summation implies
\[
\sum_{N(p) \leq x} \frac{1}{N(p)^s} = \sum_{N(p) \leq x} \frac{1}{N(p)} N(p)^{1-s}
\]
\[
= \frac{1}{x^{s-1}} \sum_{N(p) \leq x} \frac{1}{N(p)} + (s-1) \int_2^x \left( \sum_{N(p) \leq t} \frac{1}{N(p)} \right) \frac{dt}{t^s}.
\]
Since \( \text{Re}(s - 1) > 0 \) and
\[
\sum_{N(p) \leq x} \frac{1}{N(p)} = \log \log x + O(1)
\]
by [11, Lem. 2.4], it follows that
\[
\lim_{x \to \infty} \frac{1}{x^{s-1}} \sum_{N(p) \leq x} \frac{1}{N(p)} = 0
\]
and hence
\[
g(s) = (s - 1) \int_2^\infty \left( \sum_{N(p) \leq t} \frac{1}{N(p)} \right) \frac{dt}{t^s}
\]
\[
= (s - 1) \int_2^\infty \frac{\log \log t + M_K + B_K(t)}{t^s} \frac{dt}{t^s} + (s - 1) \int_2^\infty \frac{\log \log t}{t^s} \frac{dt}{t^s}. 
\]
First observe that
\[
\lim_{s \to 1^+} I_1(s) = M_K \lim_{s \to 1^+} \left( (s - 1) \int_2^\infty \frac{dt}{t^s} \right) = M_K \lim_{s \to 1^+} 2^{1-s} = M_K.
\]
From (4), we have
\[
\frac{|B_K(t)|}{t^s} = O\left(\frac{\log t}{t^{3/2}}\right) \quad \text{and hence} \quad \int_2^\infty \frac{B_K(t)}{t^s} \frac{dt}{t} \text{ converges.}
\]
Therefore,
\[
\lim_{s \to 1^+} I_2(s) = 0.
\]
The substitution \( t^{s-1} = e^y \) and \((s - 1)t^{s-2} dt = e^y dy\) yields
\[
\log t = \frac{y}{s-1}, \quad \frac{1}{t^s} = \frac{1}{te^y}, \quad \text{and} \quad dt = \frac{t dy}{s-1}.
\]
Then
\[
I_3(s) = (s - 1) \int_2^\infty \frac{\log \log t}{t^s} \frac{dt}{t^s}
\]
\[
= (s - 1) \int_2^\infty \frac{\log\left(\frac{y}{s-1}\right)}{te^y} \frac{t dy}{s-1}
\]
\[
= \int_{\log(2^{s-1})}^\infty e^{-y} \log\left(\frac{y}{s-1}\right) dy
\]
\[
= \int_{\log(2^{s-1})}^\infty e^{-y} \log y dy - \log(s - 1) \int_{\log(2^{s-1})}^\infty e^{-y} dy
\]
\[
= \int_{\log(2^{s-1})}^\infty e^{-y} \log y dy - 2^{1-s} \log(s - 1)
\]
and hence
\[
\lim_{s \to 1^+} I_3(s) = -\gamma - \log(s - 1).
\]
Consequently, letting $s \to 1^+$, we observe
\[ g(s) = M_K - \gamma - \log(s - 1) + o(1), \]
and hence
\[ M_K = \gamma + \log(s - 1) + g(s) + o(1) \] (5)
as $s \to 1^+$. The Euler product formula for $\zeta_K(s)$ ensures that
\[
\log(s - 1) + g(s) = \log(s - 1) + \sum_p \left( \frac{1}{N(p)^s} + \log \left( 1 - \frac{1}{N(p)^s} \right) \right)
\]
in which the sum is uniformly convergent by comparison with $\sum_p N(p)^{-2}$. Since $\zeta_K(s)$ has a simple pole at $s = 1$ with residue $\ell_K$, we conclude from (5) that
\[
M_K = \gamma + \log \ell_K + \sum_p \left( \frac{1}{N(p)} + \log \left( 1 - \frac{1}{N(p)} \right) \right),
\] (6)
This concludes the proof of (B1).

2.3. Proof of (C1). Before proceeding, we require a few preliminary remarks. If $p \subset \mathcal{O}_K$ is a prime ideal, it divides exactly one rational prime $p$ and $N(p) = p^f$ for some $1 \leq f \leq n_K$ [11, Thm. 5.14c]. Moreover, $p\mathcal{O}_K$ has a unique factorization $p\mathcal{O}_K = p_1^{e_1} \cdots p_r^{e_r}$ into prime ideals $p_i$, where $e_i \in \mathbb{N}$ is the ramification index of $p_i$. The $p_i$ are the only prime ideals in $K$ with norm equal to a power of $p$. In fact, $N(p_i) = p^h_i$, in which the inertia degrees $f_i$ satisfy $f_i \leq n_K$ and $e_1 f_1 + \cdots + e_r f_r = n_K$. In particular, for each rational prime $p$ the corresponding inertia degrees satisfy
\[ \sum f_i \leq n_K. \] (7)

From (6) we deduce
\[
-\gamma - \log \ell_K + M_K = \sum_{N(p) \leq x} \left( \frac{1}{N(p)^s} + \log \left( 1 - \frac{1}{N(p)^s} \right) \right) + F_K(x), \] (8)
in which
\[ F_K(x) = \sum_{N(p) > x} \left( \frac{1}{N(p)} + \log \left( 1 - \frac{1}{N(p)} \right) \right). \] (9)
For $y \in [0, 1)$, observe that
\[ 0 \leq -y - \log(1 - y) \leq \frac{y^2}{1 - y}. \] (10)
Let $y = 1/N(p)$ and deduce
\[ |F_K(x)| = -\sum_{N(p) > x} \left( \frac{1}{N(p)} + \log \left( 1 - \frac{1}{N(p)} \right) \right) \]
\[
\leq \sum_{N(p) > x} \frac{1}{N(p)(N(p) - 1)}
\]
\[
\leq \sum_{p > x} \sum_{f_i} \frac{1}{p f_i (p f_i - 1)}
\]
\[
\leq \sum_{p > x} \left( \sum_{f_i} \frac{1}{p(p - 1)} \right)
\]
\[
< n_K \sum_{m > x} \frac{1}{m(m - 1)} \quad \text{by (7)}
\]
\[
= \frac{n_K}{[x] - 1}
\]
\[
\leq \frac{n_K}{x - 1},
\]

in which \( \sum_{f_i} \) denotes the sum over the inertia degrees \( f_i \) of the prime ideals lying over \( p \) and \([x]\) denotes the least integer greater than or equal to \( x \). In light of (B1), the right-hand side of (8) becomes
\[
\sum_{N(p) \leq x} \left[ \frac{1}{N(p)} + \log \left( 1 - \frac{1}{N(p)} \right) \right] + F_K(x)
\]
\[
= \sum_{N(p) \leq x} \frac{1}{N(p)} + \sum_{N(p) \leq x} \log \left( 1 - \frac{1}{N(p)} \right) + F_K(x)
\]
\[
= \sum_{N(p) \leq x} \log \left( 1 - \frac{1}{N(p)} \right) + \log \log x + M_K + E_K(x),
\]

in which \( E_K(x) = F_K(x) + B_K(x) \). Exponentiate (8) and use (12) to obtain
\[
e^{-\gamma} e^{M_K} \ell_K = \exp \left[ \sum_{N(p) \leq x} \log \left( 1 - \frac{1}{N(p)} \right) \right] (\log x) e^{M_K} e^{E_K(x)},
\]

which yields
\[
\prod_{N(p) \leq x} \left( 1 - \frac{1}{N(p)} \right) = \frac{e^{-\gamma}}{\ell_K \log x} e^{-E_K(x)}.
\]

Write
\[
C_K(x) = e^{-E_K(x)} - 1
\]
and use the inequality \(|e^t - 1| \leq |t|e^{|t|} \), valid for \( t \in \mathbb{R} \), to deduce that
\[
\prod_{N(p) \leq x} \left( 1 - \frac{1}{N(p)} \right) = \frac{e^{-\gamma}}{\ell_K \log x} (1 + C_K(x)),
\]
in which
\[
|C_K(x)| \leq |E_K(x)|e^{|E_K(x)|}.
\]

This concludes the proof of (C1). \( \square \)
2.4. **Proof of (M).** From (6), we have $M_K = \gamma + \log \ell_K + F_K(2-\delta)$ for $\delta \in (0,1)$, in which $F_K(x)$ is defined by (9). In particular, (10) and (11) reveal that

$$-n_K \leq \liminf_{\delta \to 0^+} F_K(2-\delta) \leq \limsup_{\delta \to 0^+} F_K(2-\delta) \leq 0.$$ 

Thus, $-n_K \leq M_K - \gamma - \log \ell_K \leq 0$, which is equivalent to (M). \qed

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