Abstract. It is shown that the fractional integral operators with the parameter $\alpha$, $0 < \alpha < 1$, are not bounded between the generalized grand Lebesgue spaces $L^{p,\theta_1}$ and $L^{q,\theta_2}$ for $\theta_2 < (1 + \alpha q)\theta_1$, where $1 < p < 1/\alpha$ and $q = \frac{p}{1-\alpha p}$. Besides this, it is proved that the one–weight inequality

$$\|I_\alpha(f w^\alpha)\|_{L^{q,\theta_1}(1 + \alpha q)} \leq c\|f\|_{L^{p,\theta_1}},$$

where $I_\alpha$ is the Riesz potential operator on the interval $[0,1]$, holds if and only if $w \in A_{1+q/p'}$.

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Introduction

In this paper we show that potential operators with the parameter $\alpha$, $0 < \alpha < 1$, are not bounded from $L^p$ to $L^q$, where $1 < p < \infty$ and $q$ is the Hardy–Littlewood–Sobolev exponent of $p$: $q = \frac{p}{1-\alpha p}$. This phenomena motivates us to investigate the boundedness problem for the Riesz potential operator $I_\alpha$ in the generalized grand Lebesgue spaces. In particular, we study this problem in $L^{p,\theta}_w$ spaces and prove that the one–weight inequality

$$\|I_\alpha(f w^\alpha)\|_{L^{q,\theta_1}(1 + \alpha q)} \leq c\|f\|_{L^{p,\theta}_w([0,1])}$$

holds if and only if $w$ belongs to the Muckenhoupt’s class $A_{1+q/p'}$. 

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The unweight spaces $L^p,\theta$ (i.e. $L^p_w,\theta$ for $w \equiv \text{const}$) were introduced by E. Greco, T. Iwaniec and C. Sbordone [6] when they studied existence and uniqueness of the nonhomogeneous $n$–harmonic equation $\text{div} A(x, \nabla u) = \mu$.

The grand Lebesgue spaces $L^p) = L^p,1$ first appeared in the paper by T. Iwaniec and C. Sbordone [7]. In that paper the authors showed that if $f = (f_1, \cdots, f_n) : \Omega \to \mathbb{R}^n$ belongs to the Sobolev class $W^{1,1}$, where $\Omega$ is an open subset in $\mathbb{R}^n$, $n \geq 2$, then the Jacobian determinant $J = J(f, x) = \det Df(x)$ ($J(x, f) \geq 0 \ a.e.$) of $f$ belongs to the class $L^1_{\text{loc}}(\Omega)$ provided that $g \in L^n_1), where $g(x) := |Df(x)| = \{\sup_{y \in S^{n-1}} |Df(y)| : y \in S^{n-1}\}$.

Recently necessary and sufficient conditions guaranteeing the one–weight inequality for the Hardy–Littlewood maximal operator in $L^p_w(\Omega)$, where $I = [0,1]$, were established by A. Fiorenza, B. Gupra and P. Jain [4], while the same problem for the Hilbert transform was studied in the paper [8]. In particular, it turned out that the Hardy–Littlewood maximal operator (resp. the Hilbert transform) is bounded in $L^p_w(\Omega)$ if and only if the weight $w$ belongs to the Muckenhoupt class $A_p(I)$.

1 Preliminaries

Let $\Omega$ be a bounded subset of $\mathbb{R}^n$ and let $w$ be an a.e. positive, integrable function on $\Omega$ (i.e. a weight). The weighted generalized grand Lebesgue space $L^p,\theta(\Omega)$ ($1 < p < \infty$) is the class of those $f : \Omega \to \mathbb{R}$ for which the norm

$$
\|f\|_{L^p,\theta(\Omega)} = \sup_{0 < \varepsilon < p-1} \left( \frac{\varepsilon^\theta}{|\Omega|} \int_{\Omega} |f(t)|^{p-\varepsilon} w(t) dt \right)^{1/(p-\varepsilon)}
$$

is finite.

If $w \equiv 1$, then we denote $L^p,\theta(\Omega) := L^p,\theta(\Omega)$. The space $L^p,\theta(\Omega)$ is not rearrangement invariant unless $w \equiv \text{const}$.

Hölder’s inequality and simple estimates yield the following embeddings (see also [6], [4]):

$$
L^p_w(\Omega) \subset L^p_w,\theta_1(\Omega) \subset L^p_w,\theta_2(\Omega) \subset L^{p-\varepsilon}_w(\Omega),
$$

where $0 < \varepsilon < p - 1$ and $\theta_1 < \theta_2$.

In the classical weighted Lebesgue spaces $L^p_w$ the equality

$$
\|f\|_{L^p_w} = \|w^{1/p} f\|_{L^p}
$$

holds but this property fails in the case of grand Lebesgue spaces. In particular, there is $f \in L^p_w$ such that $w^{1/p} f \notin L^p$ (see also [4] for the details).
Let \( \varphi \) be positive increasing function on \((0, p - 1)\) satisfying the condition \( \varphi(0^+) = 0 \), where \( 1 < p < \infty \). We will also need the following auxiliary class of functions defined on \( \Omega \) and associated with \( \varphi \):

\[
L^p_{w, \varphi}(\Omega) := \left\{ f : \sup_{0 < \varepsilon \leq p - 1} \left( \frac{\varphi(\varepsilon)}{\varepsilon} \right)^{\frac{1}{p - 1}} \| f \|_{L^{p - \varepsilon}_w} < \infty \right\}.
\]

The space \( L^p_{w, \varphi}(\Omega), \theta > 0 \), is the special case of \( L^p_{w, \varphi}(\Omega) \) taking \( \varphi(x) = x^\theta |\Omega| \).

Throughout the paper the symbol \( \varphi(t) \approx \psi(t) \) means that there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \varphi(t) \leq \psi(t) \leq c_2 \psi(t).
\]

Constants (often different constants in the same series of inequalities) will generally be denoted by \( c \) or \( C \). By the symbol \( p' \) we denote the conjugate number of \( p \), i.e. \( p' := \frac{p}{p - 1}, 1 < p < \infty \).

### 2 Fractional Integrals and Fractional Maximal Functions in Unweighted Grand Lebesgue Spaces

Let

\[
(I_\alpha f)(x) = \int_0^1 \frac{f(y)}{|x - y|^{1 - \alpha}} dy, \quad 0 < \alpha < 1
\]

be the Riesz potential operator defined on \([0, 1]\). We begin with the following result:

**Theorem 2.1.** Let \( 0 < \alpha < 1, 1 < p < \frac{1}{\alpha} \), \( \theta_1 \) and \( \theta_2 \) be positive numbers such that \( \theta_2 < \theta_1(1 + \alpha q) \), where \( q = \frac{p}{1 - \alpha} \). Then the operator \( I_\alpha \) is not bounded from \( L^p_{\theta_1} \) to \( L^q_{\theta_2} \).

**Proof.** Suppose the contrary: \( I_\alpha \) is bounded from \( L^p_{\theta_1} \) to \( L^q_{\theta_2} \). i.e. the inequality

\[
\|I_\alpha f\|_{L^q_{\theta_2}} \leq c \|f\|_{L^p_{\theta_1}} \tag{2.1}
\]

holds, where the positive constant \( c \) does not depend on \( f \). Taking \( f = \chi_J \) in (2.1), where \( J \) is an interval in \([0,1]\), we have

\[
(I_\alpha f)(x) = \int_J \frac{dy}{|x - y|^{1 - \alpha}} \geq |J|^\alpha, \quad x \in J.
\]

Consequently,

\[
\|I_\alpha f\|_{L^q_{\theta_2}} \geq |J|^\alpha \|\chi_J\|_{L^q_{\theta_2}}.
\]

Taking inequality (2.1) into account we have that

\[
|J|^\alpha \|\chi_J\|_{L^q_{\theta_2}} \leq c \|\chi_J\|_{L^p_{\theta_1}}, \tag{2.2}
\]

where the positive constant \( c \) does not depend on \( J \).
Let us define the number \( \varepsilon_J \) which is between 0 and \( p - 1 \) and satisfies the condition
\[
\sup_{0 < \varepsilon \leq p - 1} \left( \varepsilon \theta |J| \right)^{1/p - 1} = \left( \varepsilon \theta_J |J| \right)^{1/p - 1}.
\] (2.3)

Now we claim that \( \lim_{|J| \to 0} \varepsilon_J = 0 \). Indeed, suppose the contrary: that there is a sequence of intervals \( J_n \) and a positive number \( \lambda \) such that \( |J_n| \to 0 \) and \( \varepsilon_{J_n} \geq \lambda > 0 \) for all \( n \in N \).

It is obvious that we can choose \( J_{n_0} \) so that
\[
\frac{|J_{n_0}|^{\frac{1}{p-1}} (p-1)}{e} < \varepsilon < \frac{|J_{n_0}|^{\frac{1}{p-1}} (p-1)}{e} \leq e^{-\frac{\lambda}{p}}.
\]

Now we claim that \( f'(x) < 0 \) for all \( x \in [\lambda/2, p - 1] \), where \( f(x) = (x^{\theta_1} |J_{n_0}|)^{1/p - 1} \). Indeed, it is easy to see that for \( \lambda/2 \leq x \leq p - 1 \), the inequalities
\[
\frac{|J_{n_0}|^{\frac{1}{p-1}} x}{e} \leq \frac{|J_{n_0}|^{\frac{1}{p-1}} (p-1)}{e} < \frac{x}{e^{\lambda/p}} \leq \frac{e^{-p}}{e}.
\]

hold. Hence, using the formula
\[
f'(x) = f(x) \cdot \frac{1}{p-x} \left[ \frac{\ln (x^{\theta_1} |J_{n_0}|)}{p-x} + \frac{\theta_1}{x} \right]
\]
and the fact that
\[
f'(x) < 0 \iff \frac{x |J_{n_0}|^{\frac{1}{p-1}}}{e} < e^{-\frac{p}{x}}
\]
we conclude that \( f'(x) < 0 \).

This observation together with the equality \( \lim_{x \to 0} f(x) = 0 \) gives that \( \varepsilon_{J_{n_0}} < \lambda \), where \( \varepsilon_{J_{n_0}} \) is defined by
\[
\sup_{0 < \varepsilon \leq p - 1} \left( \varepsilon \theta |J_{n_0}| \right)^{1/p - 1} = \left( \varepsilon \theta_{J_{n_0}} |J_{n_0}| \right)^{1/(p-\varepsilon_{J_{n_0}})}.
\]

This contradicts the assumption that \( \varepsilon_{J_n} \geq \lambda > 0 \) for all \( n \). Further, we choose \( \eta_J \) so that
\[
\alpha = \frac{1}{p} - \frac{1}{q} = \frac{1}{p - \varepsilon_J} - \frac{1}{q - \eta_J}.
\]
This is equivalent to say that
\[
\eta_J = q - \frac{p - \varepsilon_J}{1 - \alpha (p - \varepsilon_J)}.
\] (2.4)

By (2.2) and (2.3) we have that
\[
|J|^\alpha \cdot \eta_J^{-\theta_{\eta_J}} |J|^{-\frac{1}{q-\eta_J}} \leq \varepsilon \theta_{J} |J|^{-\frac{1}{p-\varepsilon_J}}. \] (2.5)
(here we used the fact that if \( \varepsilon_J \) is small, then \( 0 < \eta_J < q - 1 \)). Now (2.5) yield:
\[
\eta_J^{-\frac{q}{q-\eta_J}} \varepsilon_J^{-\frac{\eta_J}{p-\varepsilon_J}} \leq c. \tag{2.6}
\]

Further, (2.4) and (2.6) imply
\[
\left( q - \frac{p-\varepsilon_J}{1-\alpha(p-\varepsilon_J)} \right) \varepsilon_J^{-\theta_J - \frac{\theta_J}{p-\varepsilon_J} - \alpha \theta_2} \leq c. \tag{2.7}
\]

Passing now to the limit as \(|J| \to 0\) we see that the left-hand side of (2.7) tends to \(+\infty\) because the limit of the first factor is
\[
\left[ \frac{1}{(1-\alpha p)^2} \right]^{\frac{\theta_J}{p} - \alpha \theta_2},
\]
and
\[
\lim_{|J| \to 0} \varepsilon_J^{-\theta_J - \frac{\theta_J}{p-\varepsilon_J} - \alpha \theta_2} = \lim_{|J| \to 0} \varepsilon_J^{-\theta_J - \frac{\theta_J}{p} - \alpha \theta_2} = \infty.
\]

(Here we used the observation \( \frac{\theta_J}{\theta_1} < 1 + \alpha q \iff \frac{\theta_J - \theta_1}{p} - \alpha \theta_2 < 0 \)).

Analysing the proof of Theorem 2.1 we have the result similar to that of the previous statement for the fractional maximal operator
\[
M_\alpha f(x) = \sup_{J \ni x} \frac{1}{|J|^{1-\alpha}} \int_J |f|, \quad x \in [0, 1].
\]

**Theorem 2.2.** Let the conditions of Theorem 2.1 be satisfied. Then the operator \( M_\alpha \) is not bounded from \( L^{p,\theta_1} \) to \( L^{q,\theta_2} \).

**Proof.** Proof is the same as in the case of Theorem 2.1. We only need to observe that the inequality
\[
M_\alpha f(x) \geq \frac{1}{|J|^{1-\alpha}} \int_J dx = |J|^\alpha, \quad x \in J,
\]
holds for \( f(x) = \chi_J(x) \), where \( J \) is a subinterval of \([0, 1]\). Details are omitted.

### 3 Sobolev’s Embedding in Weighted Generalized Grand Lebesgue Spaces

This section is devoted to the investigation of the one–weight inequality for the operator \( I_\alpha \) in \( L^{p,\theta}_w \) spaces.

First we introduce the function
\[
\varphi(u) = \left[ \frac{u - q}{1 - \alpha(u - q) + p} \right]^{1-(u-q)\alpha} \tag{3.1}
\]
where $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, \( q = \frac{p}{1-\alpha p} \).

To prove the main results we need some auxiliary statements.

**Lemma 3.1.** \( \varphi(x) \approx x^{1+\alpha q} \) near 0.

The proof is straightforward and therefore is omitted.

**Lemma 3.2.** Let \( 1 < q < \infty \) and let \( w \) be a weight. Then
\[
\| f \|_{L^q_w[0,1], \varphi(x)} \approx \| f \|_{L^q_w[0,1], 1+\alpha q},
\]
where \( \varphi \) is defined by (3.1).

*Proof.* Follows immediately from Lemma 3.1.

**Lemma 3.3.** Let \( 1 < q < \infty \) and let \( \theta > 0 \). Then
\[
\| f \|_{L^q_w[0,1], \varphi(x)} \approx \| f \|_{L^q_w[0,1], \theta^{1+\alpha q}},
\]
where \( \varphi \) is defined by (3.1).

The proof follows immediately from Lemma 3.1.

**Lemma 3.4.** Let \( 1 < p < \infty \) and let \( \varphi \) be as above. Then there is a positive constant \( c \) such that for all intervals \( J \subset [0,1] \) and \( f \in L^p_w, \varphi(x) \) the inequality
\[
\| f \|_{L^p_w[0,1], \varphi(x)} \leq c(\varphi(J))^{-\frac{1}{p-\alpha}} \left( \int_J |f(x)|^{p-\alpha} w(x) dx \right)^{\frac{1}{p-\alpha}} \| \chi_J \|_{L^p_w, \varphi(x)}
\]
holds.

*Proof.* We have
\[
\| f \|_{L^p_w[0,1], \varphi(x)} = \sup_{0 < \varepsilon \leq p-1} \left( \varphi(\varepsilon) \int_J |f(x)|^{p-\varepsilon} w(x) dx \right)^{\frac{1}{p-\varepsilon}}
\]
\[
= \sup_{0 < \varepsilon \leq p-1} \left( \varphi(\varepsilon) \int_J |f(x)|^{p-\varepsilon} w(x)^{\frac{p-\varepsilon}{p}} w(x) dx \right)^{\frac{1}{p-\varepsilon}}
\]
\[
\leq \sup_{0 < \varepsilon \leq p-1} \varphi(\varepsilon)^{-\frac{1}{p-\varepsilon}} \left( \int_J (|f(x)|^{p-\varepsilon} w(x)^{\frac{p-\varepsilon}{p}} w(x) dx \right)^{\frac{p}{p-\varepsilon}} \left( \int_J [\frac{\varphi(x)}{\varphi(\varepsilon)}]^{\frac{p-\varepsilon}{p-\varepsilon}} dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

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\[
\varphi(\varepsilon)^{\frac{1}{p}} \left( \int_{J} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}} \rho \sup_{0<\varepsilon \leq p-1} \varphi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

\[
= \left( \int_{J} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left( \int_{J} w(x) dx \right)^{\frac{1}{p}} \sup_{0<\varepsilon \leq p-1} \varphi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

\[
= \left( \int_{J} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left( \int_{J} w(x) dx \right)^{\frac{1}{p}} \varphi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

\[
\lambda = \varphi(\varepsilon)^{\frac{1}{p}} \left( \int_{J} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}} \rho \sup_{0<\varepsilon \leq p-1} \varphi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

\[
\exp \left( \frac{1}{p} \int_{J} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left( \int_{J} w(x) dx \right)^{\frac{1}{p}} \sup_{0<\varepsilon \leq p-1} \varphi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

\[
\phi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

\[
\lambda = \exp \left( \frac{1}{p} \int_{J} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left( \int_{J} w(x) dx \right)^{\frac{1}{p}} \sup_{0<\varepsilon \leq p-1} \varphi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

\[
\lambda = \exp \left( \frac{1}{p} \int_{J} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left( \int_{J} w(x) dx \right)^{\frac{1}{p}} \sup_{0<\varepsilon \leq p-1} \varphi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

\[
\phi(\varepsilon) \left( \int_{J} w(x) dx \right)^{\frac{\varepsilon}{p-\varepsilon}}
\]

**Lemma 3.5.** Let \( \theta > 0, 1 < p < \infty, 0 < \alpha < 1/p \) and let \( q = \frac{p}{1-\alpha p} \). Suppose that the inequality

\[
\| I_{\alpha}(f w^\alpha) \|_{L^q_w([0,1])} \leq c \| f \|_{L^p_w([0,1])}
\]

holds. Then

\[
\int_0^1 w^{-\nu'/q}(x) dx < \infty.
\]

**Proof.** Suppose the contrary: \( \int_0^1 w^{-\nu'/q}(x) dx = \| w^{\alpha-1} \|_{L^p_w} = \infty \). This means that there is a function \( g \in L^p_w \) such that \( \int_0^1 g w^\alpha = \infty \).

On the other hand,

\[
I_{\alpha}(g w^\alpha)(x) = \int_0^1 \frac{g(t) w^\alpha(t)}{|x-t|^{1-\alpha}} dt \geq \int_0^1 g(t) w^\alpha(t) dt = \infty, \quad x \in [0,1].
\]

Further, Lemma 3.4 with \( \varphi(x) = x^\theta \) implies that \( g \in L^p_w([0,1]) \). But \( I_{\alpha}(g w^\alpha)(x) = \infty \) for \( x \in [0,1] \). This contradicts inequality (3.2). \( \square \)

**Definition 3.1.** Let \( 1 < r < \infty \). We say that a weight function \( w \) belongs to the Muckenhoupt’s class \( A_r([0,1]) \) \((w \in A_r([0,1]))\) if

\[
A_r(w) := \sup_{J \subset [0,1]} \left( \frac{1}{|J|} \int_J w \right)^{1/r} \left( \frac{1}{|J|} \int_J w^{1-r'/r} \right)^{1/r'} < \infty,
\]
where the supremum is taken over all subintervals \( J \) of \([0, 1]\).

**Lemma 3.6.** Let \( 0 < \alpha < 1 \), \( 1 < p < 1/\alpha \). We set \( q = \frac{p}{1-\alpha p} \). Suppose that \( w \in A_{1+q/p'}([0, 1]) \), i.e.,

\[
\sup_{J \subset [0, 1]} \left( \frac{1}{|J|} \int_J w \right)^{1/q} \left( \frac{1}{|J|} \int_J w^{-p'/q} \right)^{1/p'} < \infty.
\]

Then there are positive constants \( \sigma_1, \sigma_2 \) and \( L \) satisfying the conditions:

\[
\frac{1}{p - \sigma_2} - \frac{1}{q - \sigma_1} = \alpha, \quad w \in A_{1+q/(\sigma_2(\sigma_2 - \sigma_1))},
\]

\[
\|K_\alpha\|_{L_w^p \to L_w^q} \leq L
\]

for all \( 0 \leq \varepsilon \leq \sigma_1 \), \( 0 \leq \eta \leq \sigma_2 \) with \( \frac{1}{p - \eta} - \frac{1}{q - \varepsilon} = \alpha \), where \( K_\alpha \) is the operator defined as follows.

**Proof.** Since \( w \in A_{1+q/p'} \) by the openness property of Muckenhoupt’s classes (see [9]) we have that there are small positive numbers \( \sigma_1 \) and \( \sigma_2 \) such that \( \frac{1}{p - \sigma_2} - \frac{1}{q - \sigma_1} = \alpha \) and \( w \in A_{1+q/(\sigma_2(\sigma_2 - \sigma_1))} \).

By the result of B. Muckenhoupt and R. L. Wheeden [10] we have that the operator \( K_\alpha \) is bounded from \( L_w^p \) to \( L_w^q \) and from \( L_w^{p - \sigma_2} \) to \( L_w^{q - \sigma_1} \). Let \( 0 < t < 1 \) and let us define positive numbers \( \eta \) and \( \varepsilon \) so that

\[
\frac{1}{p - \eta} = \frac{t}{p} + \frac{1 - t}{p - \sigma_2}, \quad \frac{1}{q - \varepsilon} = \frac{t}{q} + \frac{1 - t}{q - \sigma_1}.
\]

Then by applying the Riesz–Thorin theorem (see e.g. [2], p. 16) we have that \( K_\alpha \) is bounded from \( L_w^{p - \eta} \) to \( L_w^{q - \varepsilon} \) and moreover,

\[
\|K_\alpha\|_{L_w^{p - \eta} \to L_w^{q - \varepsilon}} \leq \|K_\alpha\|_{L_w^p \to L_w^q} \|K_\alpha\|_{L_w^{p - \sigma_2} \to L_w^{q - \sigma_1}}^{1 - t}.
\]

Observe now that

\[
\frac{1}{p - \eta} - \frac{1}{q - \varepsilon} = \frac{t}{p} - \frac{1 - t}{q} + \frac{1 - t}{p - \sigma_2} - \frac{1 - t}{q - \sigma_1}
\]

\[
= t\left(\frac{1}{p} - \frac{1}{q}\right) + (1 - t)\left(\frac{1}{p - \sigma_2} - \frac{1}{q - \sigma_1}\right) = t\alpha + (1 - t)\alpha = \alpha.
\]

The lemma is proved since we can take \( L = \|K_\alpha\|_{L_w^p \to L_w^q} \|K_\alpha\|_{L_w^{p - \sigma_2} \to L_w^{q - \sigma_1}} \) (since without loss of generality we can assume that each term is greater or equal to 1).

**Theorem 3.1.** Let \( 1 < p < \infty \) and let \( 0 < \alpha < 1/p \). Suppose that \( \theta > 0 \). We set \( q = \frac{p}{1-\alpha p} \). Then the inequality

\[
\|I_{\alpha}(f w^\sigma)\|_{L_w^{q, \theta(1+\alpha q)}} \leq c\|f\|_{L_w^{p, \theta}}
\]

holds if and only if \( w \in A_{1+q/p'}([0, 1]) \).
Proof. By Lemma 3.1 we have that (3.3) is equivalent to the inequality

\[ \|I_\alpha(f w^\alpha)\|_{L^q_w,\psi(x)([0,1])} \leq c \|f\|_{L^p_w,\varphi(x)([0,1])}, \]  

where

\[ \psi(x) = \varphi(x^q), \quad \varphi(x) = \left[ \frac{x - q}{1 - \alpha(x - q)} + p \right]^{1-(x-1)\alpha}. \]  

Necessity. Let (3.3) and hence (3.4) hold. By Lemma 3.5 we have that \( \frac{1}{0} \int w^{-p'/q} < \infty \). Let us take \( f = \chi_J w^{-\alpha - p'/q} \). Then for \( x \in J \), we get that

\[ I_\alpha(w^\alpha f)(x) \geq \frac{1}{|J|^{1-\alpha}} \int_J w^\alpha f = \frac{1}{|J|^{1-\alpha}} \int_J w^{-p'/q}. \]

Hence,

\[ \|I_\alpha(w^\alpha f)\|_{L^q_w,\psi(x)([0,1])} \geq |J|^{\alpha-1} \left( \int_J w^{-p'/q} \right) \|\chi_J\|_{L^p_w,\varphi(x)([0,1])}. \]

Further, by Lemma 3.4 we find that

\[ |J|^{\alpha-1} \left( \int_J w^{-p'/q} \right) \|\chi_J\|_{L^p_w,\varphi(x)([0,1])} \leq c \|f\|_{L^p_{w,\varphi(x)}} \leq c(w(J))^{-\frac{1}{p}} \left( \int_J |f(t)|^p w(t) dt \right)^{\frac{1}{p}} \|\chi_J\|_{L^p_{w,\varphi(x)}} \]

\[ = c w(J)^{-\frac{1}{p}} \left( \int_J w^{-p'/q} \right)^{1/p} \|\chi_J\|_{L^p_{w,\varphi(x)}}. \]

Further, it is easy to see that there is a number \( \eta_J \) depending on \( J \) such that \( 0 < \eta_J \leq p - 1 \) and

\[ |J|^{\alpha-1} w(J)^{\frac{1}{p}} \left( \int_J w^{-p'/q} \right)^{\frac{1}{p'}} \|\chi_J\|_{L^q_w,\psi(x)([0,1])} \leq c(J w(J))^{-\frac{1}{p-\eta_J}}. \]

For such \( \eta_J \) we choose \( \varepsilon_J \) so that

\[ \frac{1}{p - \eta_J} - \frac{1}{q - \varepsilon_J} = \alpha. \]
Then \(0 < \varepsilon_J \leq q - 1\) and
\[
|J|^\alpha w(J)\left(\int_J w^{-p'/q} \right)^{1/p'} \leq c.
\]

Observe that by Lemma 3.1 we have that
\[
\frac{\eta - \theta}{p - \eta} \varphi(\varepsilon_J) J^{1/q - \varepsilon} = \left(\frac{\eta - \theta}{p - \eta} \varphi(\varepsilon_J) J^{1/q - \varepsilon}\right) \approx \left(\frac{\eta - \theta}{p - \eta} \varphi(\varepsilon_J) J^{1/q - \varepsilon}\right)^{\theta}
\]
and also,
\[
\frac{1}{p} - \frac{1}{p - \eta} + \frac{1}{q - \varepsilon} = \frac{1}{p - \alpha} = \frac{1}{q}.
\]

Finally, we have that
\[
|J|^\alpha w(J) \left(\int_J w^{-p'/q} \right)^{1/p'} \leq c.
\]

Necessity is proved.

Sufficiency. Using Lemma 3.6 we have that there are positive constants \(\sigma_1, \sigma_2\) and \(L\) satisfying the conditions: \(\frac{1}{p - \sigma_2} - \frac{1}{q - \sigma_1} = \alpha\), \(w \in A_{1+\frac{q - \sigma_1}{(p - \sigma_2)\sigma_2}}\), \(\|K_\alpha\|_{L^{q-\varepsilon}_w \to L^q_w} \leq L\) for all \(0 \leq \varepsilon \leq \sigma_1, 0 \leq \eta \leq \sigma_2\) with \(\frac{1}{p - \eta} - \frac{1}{q - \varepsilon} = \alpha\), where \(K_\alpha\) is the operator defined by \(K_\alpha f = I_\alpha (f w^\alpha)\).

Let \(\sigma\) be a small positive number such that \(\sigma < \sigma_1 < q - 1\) and let us fix \(\varepsilon \in (\sigma, q - 1]\). Then \(\frac{q - \sigma}{q - \varepsilon} > 1\). By Hölder’s inequality we have that
\[
\|I_\alpha (f w^\alpha)\|_{L^{q-\varepsilon}_w([0,1])} \leq \left(\int_0^1 |I_\alpha (f w^\alpha)(x)|^{q - \sigma} w(x) dx \right)^{\frac{1}{q - \sigma}} w([0,1])^{\frac{\varepsilon - \sigma}{(q - \sigma)(q - \varepsilon)}}
\]
because \(\frac{q - \sigma}{q - \varepsilon} = \frac{q - \sigma}{\varepsilon - \sigma}\).

Further, the conditions \(\sigma < q - 1\) and \(\sigma < \varepsilon < q - 1\) yield
\[
0 < \frac{\varepsilon - \sigma}{(q - \sigma)(q - \varepsilon)} < \frac{q - \sigma}{q - \sigma}, \quad (q - 1)\sigma - \frac{1}{1 - \alpha} > 1.
\]
Consequently, using the well-known result by B. Muckenhoupt and R. L. Wheeden [10] for the classical weighted Lebesgue spaces:
\[
\|I_\alpha (f w^\alpha)\|_{L^q_w([0,1])} \leq c\|f\|_{L^q_w([0,1])} \iff w \in A_{1+\frac{q}{p'}}([0,1]), \quad q = \frac{p}{1 - \alpha p},
\]
we find that
\[ \|I_\alpha f\|_{L^q_w,\psi([0,1])} = \max \left\{ \sup_{0<\varepsilon \leq \sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}, \right. \]
\[ \left. \sup_{\sigma<\varepsilon \leq q-1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])} \right\} \]
\[ \leq \max \left\{ \sup_{0<\varepsilon \leq \sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}, \right. \]
\[ \left. \sup_{\sigma<\varepsilon \leq q-1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])} \right\} \]
\[ \leq \max \left\{ 1, \sup_{\sigma<\varepsilon \leq q-1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \psi(\sigma)^{\frac{1}{q-\varepsilon}} w([0,1])^{\frac{\varepsilon-\sigma}{q-\sigma}} \right\} \sup_{0<\varepsilon \leq \sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])} \]
\[ \leq c \max \left\{ 1, \left[ \sup_{\sigma<\varepsilon \leq q-1} \left( \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \right) \right] \varphi(\sigma)^{\frac{1}{q-\varepsilon}} \left( 1 + w([0,1])^{\frac{q-1-\sigma}{q-\sigma}} \right) \right\} \sup_{0<\eta \leq \sigma_0} \eta^{\frac{a}{p-a}} \|f\|_{L^{p-a}_w([0,1])} \]
\[ \leq c \left( \sup_{\sigma<\varepsilon \leq q-1} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \varphi(\sigma)^{\frac{1}{q-\varepsilon}} \left( 1 + w([0,1])^{\frac{q-1-\sigma}{q-\sigma}} \right) \right)^{\frac{q-1-a}{q-\sigma}} \|f\|_{L^{p-a}_w([0,1])} \].

Here \( \sigma_0 \) is a small positive number such that when \( 0 < \varepsilon \leq \sigma \), then \( 0 < \eta \leq \sigma_0 < \sigma_1 < p-1 \). Also, we used the estimates:
\[ \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \approx \varepsilon^{\frac{\theta(1+\alpha q)}{q-\varepsilon}} \approx \varphi(\varepsilon)^{\frac{\theta}{q-\varepsilon}} = \eta^{\frac{a}{p-a}}, \text{ as } \varepsilon \to 0, \]
where \( \frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha. \]

**Corollary 3.1.** Let \( \theta > 0 \) and let \( 1 < p < \infty \). Suppose that \( 0 < \alpha < 1/p \). We set \( q = \frac{p}{1-\alpha p} \). Then \( I_\alpha \) is bounded from \( L^p,\theta_1([0,1]) \) to \( L^{q,\theta_2}([0,1]) \) provided that \( \theta_2 > (1+\alpha q)\theta_1 \).

**Proof** follows immediately from Theorem 3.1 (in the unweighted case \( w(x) \equiv \text{const} \)) and (1.1). \( \square \)

## 4 One-sided potentials

In this section we show that the unboundedness result in grand Lebesgue spaces is also true for the one–sided potentials:
\[(R_{\alpha}f)(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x \in (0,1);\]

and

\[(W_{\alpha}f)(x) = \int_x^1 \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad x \in (0,1),\]

where \(0 < \alpha < 1\). In particular, we claim that \(R_{\alpha}\) and \(W_{\alpha}\) are not bounded from \(L^{p,\theta_1}\) to \(L^{q,\theta_2}\), where \(q = \frac{p}{1-\alpha p}, 1 < p < \infty, \theta_1, \theta_2 > 0, \theta_2 < \frac{\theta_1 q}{p}\). Indeed, let us show the result first for \(R_{\alpha}\).

Suppose the contrary:

\[\|R_{\alpha}f\|_{L^{q,\theta_2}(\[0,1\])} \leq c \|f\|_{L^{p,\theta_1}(\[0,1\])}, \quad \theta_2 < \frac{\theta_1 q}{p}, \quad (4.1)\]

where \(c\) does not depend on \(f\). Let \(f_n(x) = \chi(0,1/2n)(x)\) in (4.1). Then taking the following inequality

\[(R_{\alpha}f_n)(x) \geq \int_0^{1/2n} \frac{1}{(x-t)^{1-\alpha}} dt \geq \left(\frac{1}{2n}\right)^{\alpha}, \quad x \in \left(\frac{1}{2n}, \frac{1}{n}\right), \quad (4.2)\]

into account, (4.1) yields that

\[(2n)^{-\alpha} \left\| \chi(\frac{1}{2n}, \frac{1}{n}) \right\|_{L^{p,\theta_1}(\[0,1\])} \leq c \left\| \chi(0,1/2n) \right\|_{L^{p,\theta_1}(\[0,1\])}, \quad (4.3)\]

Now we choose \(\varepsilon_n\) positive number so that

\[\sup_{0 < \varepsilon \leq \frac{p}{p-1}} \left(\varepsilon^{\theta_1} \frac{1}{2n}\right)^{\frac{1}{p-\varepsilon_n}} = \left(\varepsilon_n^{\theta_1} \frac{1}{2n}\right)^{\frac{1}{p-\varepsilon_n}}. \quad (4.4)\]

We now observe that \(\lim_{n \to 0} \varepsilon_n = 0\) (see the proof of Theorem 2.1 for the similar arguments). Choose now \(\eta_n\) so that

\[\alpha = \frac{1}{p} - \frac{1}{q} = \frac{1}{p - \varepsilon_n} - \frac{1}{q - \eta_n}. \quad (4.5)\]

Hence,

\[\eta_n = q - \frac{p - \varepsilon_n}{1 - \alpha(p - \varepsilon_n)}. \quad (4.5)\]

By (4.3)-(4.5) we conclude that

\[(2n)^{-\alpha} \left[\frac{\theta_1}{q - \eta_n} \left(\frac{1}{2n}\right)^{\frac{1}{q - \eta_n}}\right] \leq c \varepsilon_n^{\frac{\theta_1}{p - \varepsilon_n}} (2n)^{-1/(p - \varepsilon_n)}, \quad (4.6)\]
From (4.6) we have that
\[ \eta_n^{1-p} \varepsilon_n^{-p} \leq c_p, \]  
for all \( n \in N \) (4.7) because
\[ \frac{1}{2} \leq \left( \frac{1}{2} \right)^{\frac{1}{p}} \leq \left( \frac{1}{2} \right)^{\frac{1}{q}}, \]
\[ \frac{1}{2} \leq \left( \frac{1}{2} \right)^{\frac{1}{q-n}} \leq \left( \frac{1}{2} \right)^{\frac{1}{p}}. \]

Now (4.5) yields
\[ \left[ q \frac{1}{1-\alpha(p-\varepsilon_n)} - \frac{\theta_2}{p-\varepsilon_n} - \frac{\theta_1}{p-\varepsilon_n} - \alpha \theta_2 \right] \frac{\theta_2}{p-\varepsilon_n} \leq c_p. \]

Hence,
\[ \left[ q \frac{1}{1-\alpha(p-\varepsilon_n)} - \frac{\theta_2}{p-\varepsilon_n} - \frac{\theta_1}{p-\varepsilon_n} - \alpha \theta_2 \right] \frac{\theta_2}{p-\varepsilon_n} \leq c_p, \]
which is impossible, because \( \lim_{n \to \infty} \varepsilon_n^{-p} = \infty \) (recall that \( \frac{\theta_2-\theta_1}{p} - \alpha \theta_2 = \frac{\theta_2}{q} - \frac{\theta_1}{p} < 0 \)).

Analogously, we have that \( W_\alpha \) is not bounded from \( L^p)_{\theta_1} \) to \( L^q)_{\theta_2} \). This follows from the inequalities
\[ (W_\alpha)(x) \geq \int_x^{1-\frac{1}{3n}} \frac{f(t)}{(t-x)^{1-\alpha}} dt \geq \left( \frac{2}{3n} \right)^{\alpha-1} \cdot \frac{1}{6n} = c_\alpha n^{-\alpha}, \quad x \in \left( 1 - \frac{1}{n}, 1 - \frac{1}{2n} \right), \]
where \( f(t) = \chi_{(1-\frac{1}{2n}, 1-\frac{1}{3n})}(t) \). Hence,
\[ c_\alpha n^{-\alpha} \left\| \chi_{(1-\frac{1}{2n}, 1-\frac{1}{3n})} \right\|_{L^q)_{\theta_2}([0,1])} \leq c \left\| \chi_{(1-\frac{1}{2n}, 1-\frac{1}{3n})} \right\|_{L^p)_{\theta_1}([0,1])}. \]

Choosing now \( \varepsilon_n \) so that
\[ \left[ \frac{\theta_2}{p-\varepsilon_n} \right] \frac{1}{6n} = \sup_{0 < \varepsilon_n \leq p-1} \left[ \frac{\theta_2}{p-\varepsilon_n} \right] \frac{1}{6n} , \quad 0 < \varepsilon_n \leq p-1, \]
and observing that \( \lim_{n \to \infty} \varepsilon_n = 0 \) (see the proof of Theorem 2.1 for the similar arguments) we find that the conclusion similar to the case of \( R_\alpha \) is valid.
4.1 Conclusions and Remarks

Let $0 < \alpha < 1$ and let $I_\alpha, R_\alpha, W_\alpha$ be potential operators defined above. In the sequel we denote by $T_\alpha$ one of these operators.

**Corollary 5.1.** Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. We set $q = \frac{p}{1-\alpha p}$. Suppose that $\theta_1$ and $\theta_2$ be positive numbers. Then:

(i) If $\theta_2 < (1 + \alpha q) \theta_1$, then $T_\alpha$ is not bounded from $L^p, \theta_1$ to $L^q, \theta_2$.
(ii) If $\theta_2 \geq (1 + \alpha q) \theta_1$, then $T_\alpha$ is bounded from $L^p, \theta_1$ to $L^q, \theta_2$.

**Remark 5.1.** There is a function $f$ from $L^p \setminus L^p$ such that $T_\alpha f \in L^q \setminus L^q$. Indeed, let $f(t) = t^{-\frac{1}{\alpha}}$, $t \in (0, 1)$. Then $f \in L^p \setminus L^p$. On the other hand, (see e. g. [11]), $T_\alpha f \approx t^{-\frac{1}{\alpha}}$. Hence $T_\alpha f \in L^q \setminus L^q$.

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