FUJITA-TYPE FREENESS FOR QUASI-LOG CANONICAL THREEFOLDS

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ABSTRACT. In this paper, we show that Fujita’s basepoint-freeness conjecture for projective quasi-log canonical singularities holds true in dimension three. Immediately, we prove Fujita-type basepoint-freeness for projective semi-log canonical threefolds.

CONTENTS

1. Introduction 1
2. Quasi-log schemes 3
3. Deficit at a point 7
4. Global index one cover 9
5. Freeness for terminal singularities 10
6. Freeness for qlc singularities 15
7. Appendix 17
References 19

1. INTRODUCTION

This paper is a continuous study of [FL2] to treat Fujita’s basepoint-freeness conjecture for quasi-log canonical singularities (see Section 2 for a quick review of the theory of quasi-log schemes). Note that we have posted Fujita’s conjecture for quasi-log canonical pairs in [FL2]. Before we recall it in this paper, we first agree on a convention for reader’s convenience.

Notation. Let $N$ be a $\mathbb{R}$-Cartier divisor on a scheme $X$ of dimension $n$. We call that $N$ satisfies Fujita’s condition with respect to $n$ if

(1) $N^{\dim X_i} \cdot X_i > (\dim X_i)^{\dim X_i}$ for every positive-dimensional irreducible component $X_i$ of $X$, and

(2) for every positive $k$-dimensional irreducible subvariety $Z$ which is not an irreducible component of $X$, we put

$$n_Z = \min_i \{\dim X_i \mid X_i \text{ is an irreducible component of } X \text{ with } Z \subset X_i\}$$

and assume that $N^k \cdot Z \geq n_Z^k$.

When $X$ is equidimensional, our notation here is the same as that in [F5] for semi-log canonical pairs, which is slightly stronger than that in Fujita’s original basepoint-freeness conjecture in [Fu].

We restate [FL2, Conjecture 1.2] by using this notation.

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Conjecture 1.1 (Fujita-type freeness for quasi-log canonical pairs). Let \([X, \omega]\) be a projective quasi-log canonical pair of dimension \(n\). Let \(M\) be a Cartier divisor on \(X\). We put \(N = M - \omega\) and assume that \(N\) satisfies Fujita’s condition. Then the complete linear system \(|M|\) is basepoint-free.

Osamu Fujino and the author have proved that Conjecture 1.1 holds true for \(n \leq 2\) in [FL2]. In this paper, we continue their work to prove that Conjecture 1.1 holds true for \(n = 3\). That is:

Theorem 1.2 (Theorem 4.3). Let \([X, \omega]\) be a projective quasi-log canonical pair of dimension three. Let \(M\) be a Cartier divisor on \(X\). We put \(N = M - \omega\) and assume that \(N\) satisfies Fujita’s condition. Then the complete linear system \(|M|\) is basepoint-free.

In particular, since semi-log canonical pairs contain natural quasi-log canonical structures by [F3], we have the following corollary immediately by Theorem 1.2:

Corollary 1.3. Let \((X, \Delta)\) be a projective semi-log canonical pair of dimension three. Let \(M\) be a Cartier divisor on \(X\). We put \(N = M - K_X - \Delta\) and assume that \(N\) satisfies Fujita’s condition. Then the complete linear system \(|M|\) is basepoint-free.

The basic idea to tackle basepoint-freeness conjecture for a qlc pair \([X, \omega]\) is to use Riemann-Roch theorem and Fujita’s condition to create an effective divisor \(L_1\) such that \(L_1\) is very singular at point \(x\) but as smooth as possible elsewhere. Then there is a maximal number \(r_1 < 1\) such that \([X, \omega + r_1 L_1]\) is qlc at around \(x\) and a minimal qlc center \(Z\) passing through \(x\). Since \(\dim Z < \dim X\) and \([Z, (\omega + r_1 L_1)]_Z\) is an induced quasi-log pair which is qlc at around \(x\), we can use Riemann-Roch theorem and Fujita’s condition repeatedly and finally get a sequence of numbers \(c = \Sigma c_i\) and a sequence of divisors \(L = \Sigma r_i L_i\) such that \(x\) is the minimal qlc center of \([X, \omega + L]\). If \(c < 1\), then Fujita’s conjecture is true by the vanishing theorem. We call above operation inductive procedure.

Let us quickly explain the strategy to prove Theorem 1.2 based on inductive procedure. Note that in the content of this paper, we will show this strategy from the bottom up. We take an arbitrary closed point \(x\) of \(X\).

- If \(x \in \text{Nqklt}(X, \omega)\), then there exists an irreducible minimal qlc center \(W\) passing through \(x\) such that \(\dim W < \dim X\). By adjunction (see Theorem 2.7 (i)), \(|W, \omega|_W\) is a quasi-log canonical pair. By the vanishing theorem (see Theorem 2.7 (ii)), the natural restriction map \(H^0(X, O_X(M)) \to H^0(W, O_W(M))\) is surjective. Therefore, we can replace \(X\) with \(W\) and use induction on the dimension.

- If \(x \not\in \text{Nqklt}(X, \omega)\), then \(X\) is normal at \(x\). Let \(\nu : \tilde{X} \to X\) be the normalization. Then by Theorem 2.9, \([\tilde{X}, \nu^* \omega]\) is a qlc pair and isomorphic to \([X, \omega]\) in a neighborhood of \(x\). To prove that \(|M|\) is basepoint-free at \(x\), we try to prove that \(|\nu^* M|\) is basepoint-free at \(\tilde{x} := \nu^{-1}(x)\). To descend the obtained section back to \(X\), we need more freeness for \(|\nu^* M|\). That is, we need that \(\nu^* M\) can separated \(\tilde{x}\) and \(\text{Nqklt}(\tilde{X}, \nu^* \omega)\).

- We have deduced to show the (stronger) freeness under the assumption that \(X\) is normal. By using Theorem 2.8, we can take a boundary \(\mathbb{R}\)-divisor \(\Delta\) on \(X\) such that \(K_X + \Delta \sim_{\mathbb{R}} \omega + \varepsilon N\) for \(0 < \varepsilon \ll 1\) and \((X, \Delta)\) is klt in a neighborhood of \(x\). That is, by a small perturbation, we can “almost” view \([X, \omega]\) as a klt pair from now on, although such a perturbation will weaken Fujita’s condition a little bit. Thanks to the first condition in Fujita’s condition, we can set the volume of \(N\) bigger to offset the negative effect of this perturbation. Note that if [F1, Conjecture 1.5] is true, then we don’t need such a perturbation.

- Next we assume that \(x\) is not a terminal point on \(X\). Let \(h : \tilde{X} \to X\) be the terminalization by Lemma 2.10 and \([\tilde{X}, h^* \omega]\) be the induced qlc pair. Then \(\dim h^{-1}(x) \geq 1\).
and thus we can choose a general point $\bar{x} \in h^{-1}(x)$ smooth on $\bar{X}$. To prove that $|M|$ is basepoint-free at $x$ (and separate $\text{Nqklt}(X, \omega)$), we try to prove that $|h^*M|$ is basepoint-free at $\bar{x}$ (and separate $\text{Nqklt}(\bar{X}, h^*\omega)$). Note that at this point, we get a qlc pair $[\bar{X}, h^*\omega]$ whose singularities are better, but we lose Fujita’s condition partially and ampleness on it. Therefore, we need to go up to $\bar{X}$ to search for high multiplicity (such that $c = \Sigma c_i < 1$ in the inductive procedure), and go down to $X$ to use ampleness and the vanishing theorem.

- Finally we turn to consider that $x$ is a terminal point on $X$. Let $p : X' \to X$ be the global index one cover defined in Section 4. At this point, we only have a neighborhood $X'_0$ of point $x' = p^{-1}(x)_{\text{red}}$ such that $(X'_0, \Delta_0)$ is a klt pair (where $\Delta_0 = p^*\Delta|_{X'_0}$). But Fujita’s condition is kept for those possible qlc minimal centers passing through $x'$ coming from the inductive procedure. Again, we go up to $X'$ to search for high multiplicity, and go down to $X$ to use ampleness and the vanishing theorem.

We strongly recommend those interested readers to read [Lin] and [FL2] as a warm-up on basepoint-freeness for quasi-log canonical singularities. Note also that Angehrn–Siu type effective freeness for quasi-log canonical pairs can be proved by above strategy without using inversion of adjunction for quasi-log canonical pairs in [Lin] Theorem 2.10.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. A scheme means a separated scheme of finite type over $\mathbb{C}$. A variety means a reduced scheme, that is, a reduced separated scheme of finite type over $\mathbb{C}$. We sometimes assume that a variety is irreducible without mentioning it explicitly if there is no risk of confusion. We will freely use the standard notation of the minimal model program and the theory of quasi-log schemes as in [F2] and [F6]. For the details of semi-log canonical pairs, see [F3].

2. Quasi-log schemes

In this section, we collect some basic definitions and explain some results on quasi-log schemes.

Definition 2.1 ($\mathbb{R}$-divisors). Let $X$ be an equidimensional variety, which is not necessarily regular in codimension one. Let $D$ be an $\mathbb{R}$-divisor, that is, $D$ is a finite formal sum $\sum_i d_i D_i$, where $D_i$ is an irreducible reduced closed subscheme of $X$ of pure codimension one and $d_i$ is a real number for every $i$ such that $D_i \neq D_j$ for $i \neq j$. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad D^{>1} = \sum_{d_i > 1} d_i D_i, \quad \text{and} \quad D^{=1} = \sum_{d_i = 1} D_i.$$  

We also put

$$[D] = \sum_i [d_i] D_i \quad \text{and} \quad [D] = -[-D],$$

where $[d_i]$ is the integer defined by $d_i \leq [d_i] < d_i + 1$. When $D = D^{\leq 1}$ holds, we usually say that $D$ is a subboundary $\mathbb{R}$-divisor.

Let $B_1$ and $B_2$ be $\mathbb{R}$-Cartier divisors on $X$. Then $B_1 \sim_\mathbb{R} B_2$ means that $B_1$ is $\mathbb{R}$-linearly equivalent to $B_2$.

Let us quickly recall singularities of pairs for the reader’s convenience. We recommend the reader to see [F6] Section 2.3] for the details.
Definition 2.2 (Singularities of pairs). Let $X$ be a normal variety and let $\Delta$ be an $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a projective birational morphism from a smooth variety $Y$. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E,$$

where $a(E, X, \Delta) \in \mathbb{R}$ and $E$ is a prime divisor on $Y$. By taking $f : Y \to X$ suitably, we can define $a(E, X, \Delta)$ for any prime divisor $E$ over $X$ and call it the \textit{discrepancy} of $E$ with respect to $(X, \Delta)$. If $a(E, X, \Delta) > -1$ (resp. $a(E, X, \Delta) \geq -1$) holds for any prime divisor $E$ over $X$, then we say that $(X, \Delta)$ is \textit{sub klt} (resp. \textit{sub log canonical}). If $(X, \Delta)$ is sub klt (resp. sub log canonical) and $\Delta$ is effective, then we say that $(X, \Delta)$ is \textit{klt} (resp. \textit{log canonical}). If $(X, \Delta)$ is log canonical and $a(E, X, \Delta) > -1$ for any prime divisor $E$ that is exceptional over $X$, then we say that $(X, \Delta)$ is \textit{plt}.

If there exist a projective birational morphism $f : Y \to X$ from a smooth variety $Y$ and a prime divisor $E$ on $Y$ such that $a(E, X, \Delta) = -1$ and $(X, \Delta)$ is log canonical in a neighborhood of the generic point of $f(E)$, then $f(E)$ is called a \textit{log canonical center} of $(X, \Delta)$.

Definition 2.3 (Multiplier ideal sheaves). Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a projective birational morphism from a smooth variety such that

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

and $\text{Supp} \Delta_Y$ is a simple normal crossing divisor on $Y$. We put

$$\mathcal{J}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor)$$

and call it the \textit{multiplier ideal sheaf} of $(X, \Delta)$. We can easily check that $\mathcal{J}(X, \Delta)$ is a well-defined ideal sheaf on $X$. The closed subscheme defined by $\mathcal{J}(X, \Delta)$ is denoted by $\text{Nklt}(X, \Delta)$.

The notion of \textit{globally embedded simple normal crossing pairs} plays a crucial role in the theory of quasi-log schemes described in [F6, Chapter 6].

Definition 2.4 (Globally embedded simple normal crossing pairs). Let $Y$ be a simple normal crossing divisor on a smooth variety $M$ and let $B$ be an $\mathbb{R}$-divisor on $M$ such that $Y$ and $B$ have no common irreducible components and that the support of $Y + B$ is a simple normal crossing divisor on $M$. In this situation, $(Y, B_Y)$, where $B_Y := B|_Y$, is called a \textit{globally embedded simple normal crossing pair}. A \textit{stratum} of $(Y, B_Y)$ means a log canonical center of $(M, Y + B)$ included in $Y$.

Let us recall the notion of \textit{quasi-log schemes}, which was first introduced by Florin Ambro (see [A]). The following definition is slightly different from the original one. For the details, see [F4, Appendix A]. In this paper, we will use the framework of quasi-log schemes established in [F6, Chapter 6].

Definition 2.5 (Quasi-log schemes). A \textit{quasi-log scheme} is a scheme $X$ endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) $\omega$ on $X$, a closed subscheme $\text{Nqlc}(X, \omega) \subseteq X$, and a finite collection \{C\} of reduced and irreducible subschemes of $X$ such that there exists a proper morphism $f : (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair $(Y, B_Y)$ satisfying the following properties:

1. $f^* \omega \sim_\mathbb{R} K_Y + B_Y$.
2. The natural map $\mathcal{O}_X \to f_* \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ induces an isomorphism

$$\mathcal{I}_{\text{Nqlc}(X, \omega)} \xrightarrow{\sim} f_* \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{<1} \rfloor),$$

where $\mathcal{I}_{\text{Nqlc}(X, \omega)}$ is the defining ideal sheaf of $\text{Nqlc}(X, \omega)$. 


(3) The collection of subvarieties \{C\} coincides with the images of \((Y, B_Y)\)-strata that are not included in \(\text{Nqlc}(X, \omega)\). We simply write \([X, \omega]\) to denote the above data
\[
(X, \omega, f : (Y, B_Y) \to X)
\]
if there is no risk of confusion. We note that the subvarieties \(C\) are called the \textit{qlc strata} of \((X, \omega, f : (Y, B_Y) \to X)\) or simply of \([X, \omega]\). If \(C\) is a qlc stratum of \([X, \omega]\) but is not an irreducible component of \(X\), then \(C\) is called a \textit{qlc center} of \([X, \omega]\). The union of all qlc centers of \([X, \omega]\) is denoted by \(\text{Nqklt}(X, \omega)\).

If \(B_Y\) is a subboundary \(\mathbb{R}\)-divisor, then \([X, \omega]\) in Definition \ref{def:qlc} is called a \textit{quasi-log canonical pair}.

\begin{definition}[Quasi-log canonical pairs]
Let \((X, \omega, f : (Y, B_Y) \to X)\) be a quasi-log scheme as in Definition \ref{def:qlc}. We say that \((X, \omega, f : (Y, B_Y) \to X)\) or simply \([X, \omega]\) is a \textit{quasi-log canonical pair} (qlc pair, for short) if \(\text{Nqlc}(X, \omega) = \emptyset\). Note that the condition \(\text{Nqlc}(X, \omega) = \emptyset\) is equivalent to \(B^+_{Y} = 0\), that is, \(B_Y = B^\leq_{Y} = B^\geq_{Y}\).
\end{definition}

One of the most important results in the theory of quasi-log schemes is the following theorem.

\begin{theorem}
Let \([X, \omega]\) be a quasi-log scheme and let \(X'\) be the union of \(\text{Nqlc}(X, \omega)\) with a (possibly empty) union of some qlc strata of \([X, \omega]\). Then we have the following properties.
\begin{enumerate}
\item[(i)] (Adjunction). Assume that \(X' \neq \text{Nqlc}(X, \omega)\). Then \([X', \omega']\) is a quasi-log scheme with \(\omega' = \omega|_{X'}\) and \(\text{Nqlc}(X', \omega') = \text{Nqlc}(X, \omega)\). Moreover, the qlc strata of \([X', \omega']\) are exactly the qlc strata of \([X, \omega]\) that are included in \(X'\).
\item[(ii)] (Vanishing theorem). Assume that \(\pi : X \to S\) is a proper morphism between schemes. Let \(L\) be a Cartier divisor on \(X\) such that \(L - \omega\) is nef and log big over \(S\) with respect to \([X, \omega]\), that is, \(L - \omega\) is \(\pi\)-nef and \((L - \omega)|_C\) is \(\pi\)-big for every qlc stratum \(C\) of \([X, \omega]\). Then \(R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0\) for every \(i > 0\), where \(\mathcal{I}_{X'}\) is the defining ideal sheaf of \(X'\) on \(X\).
\end{enumerate}
\end{theorem}

For the proof of Theorem \ref{thm:adjunction}, see, for example, \cite[Theorem 6.3.5]{Fujita}. We note that we generalized Kollár’s torsion-free and vanishing theorems in \cite[Chapter 5]{Fujita} by using the theory of mixed Hodge structures on cohomology with compact support in order to establish Theorem \ref{thm:adjunction}.

The following theorem is a special case of \cite[Theorem 1.5]{Fujita}. It is a deep result based on the theory of variations of mixed Hodge structures on cohomology with compact support.

\begin{theorem}[\cite[Theorem 1.5]{Fujita}]
Let \([X, \omega]\) be a quasi-log canonical pair such that \(X\) is a normal projective irreducible variety. Then there exists a projective birational morphism \(p : X' \to X\) from a smooth projective variety \(X'\) such that
\[
K_{X'} + B_{X'} + M_{X'} = p^*\omega,
\]
where \(B_{X'}\) is a subboundary \(\mathbb{R}\)-divisor, that is, \(B_{X'} = B^\geq_{X'}\), such that \(\text{Supp}B_{X'}\) is a simple normal crossing divisor and that \(p_*B_{X'}\) is effective, and \(M_{X'}\) is a nef \(\mathbb{R}\)-divisor on \(X'\). Furthermore, we can make \(B_{X'}\) satisfy \(p(B^+_{X'}) = \text{Nqklt}(X, \omega)\).
\end{theorem}

Next theorem will play an important role as we explained in the strategy of introduction.

\begin{theorem}[\cite[Theorem 1.1]{FL}]
Let \([X, \omega]\) be a quasi-log canonical pair such that \(X\) is irreducible. Let \(\nu : \tilde{X} \to X\) be the normalization. Then \([\tilde{X}, \nu^*\omega]\) naturally becomes a quasi-log canonical pair with the following properties:
\end{theorem}
(i) if \( C \) is a qlc center of \([\widetilde{X}, \nu^*\omega]\), then \( \nu(C) \) is a qlc center of \([X, \omega]\), and
(ii) \( \text{Nqklt}(\widetilde{X}, \nu^*\omega) = \nu^{-1}(\text{Nqklt}(X, \omega)) \). More precisely, the equality
\[
\nu_* \mathcal{I}_{\text{Nqklt}(\widetilde{X}, \nu^*\omega)} = \mathcal{I}_{\text{Nqklt}(X, \omega)}
\]
holds, where \( \mathcal{I}_{\text{Nqklt}(X, \omega)} \) and \( \mathcal{I}_{\text{Nqklt}(\widetilde{X}, \nu^*\omega)} \) are the defining ideal sheaves of \( \text{Nqklt}(X, \omega) \) and \( \text{Nqklt}(\widetilde{X}, \nu^*\omega) \) respectively.

We prepare one more useful lemma for our paper. It is a kind of terminalization, which is well known to experts in the framework of log canonical pairs.

**Lemma 2.10** (terminalization). Let \((X, \omega, f : (Y, B_Y) \to X)\) be a quasi-log canonical pair such that \( X \) is a normal irreducible variety. Then there is a morphism \( h : \widetilde{X} \to X \) where \( \widetilde{X} \) is a normal \( \mathbb{Q} \)-factorial terminal variety and an induced quasi-log canonical pair \((\widetilde{X}, \widetilde{\omega}, \widetilde{f} : (\widetilde{Y}, B_{\widetilde{Y}}) \to \widetilde{X})\) where \( \widetilde{\omega} = h^*\omega \).

**Proof.** By a modification of \( f \) (cf. [FL1]), we can assume that every stratum of \( Y \) is dominant onto \( X \). By [F7, Lemma 11.1], we can decompose \((X, \omega, f : (Y, B_Y) \to X)\) into a combination of qlc \( \mathbb{Q} \)-structures and prove our lemma for each qlc \( \mathbb{Q} \)-structure and then for the final \( \mathbb{R} \)-structure by combination as the proof of [F7, Theorem 1.7]. Therefore, we can assume that \((X, \omega, f : (Y, B_Y) \to X)\) has a \( \mathbb{Q} \)-structure and thus is a so-called basic slc-trivial fibration [F7, Definition 4.1]. By Theorem 2.8 and further blowing ups of \((Y, B_Y)\), there is a commutative diagram as follows:

\[
\begin{array}{ccc}
(Y, B_Y) & \xrightarrow{g} & (Y, B_Y) \\
\downarrow{f} & & \downarrow{f} \\
X' & \xrightarrow{p} & X
\end{array}
\]

such that \( K_{X'} + B_{X'}^+ + M_{X'} \sim_{p, \mathbb{Q}} B_{X'}^- \), where \( B_{X'} = B_{X'}^+ - B_{X'}^- \), \( B_{X'}^+ \), and \( B_{X'}^- \) are effective \( \mathbb{Q} \)-divisors. Note that \((X', B_{X'})\) is log smooth and \( M_{X'} \) is a nef \( \mathbb{Q} \)-divisor on \( X' \). We can run the relative \((K_{X'} + B_{X'}^+ + M_{X'})\)-MMP over \( X \) and terminate at a map \( h : \widetilde{X} \to X \) such that \( B_{X'}^+ \) is contracted. Then we get the following commutative diagram:

\[
\begin{array}{ccc}
(Y, B_Y) & \xrightarrow{g} & (Y, B_Y) \\
\downarrow{f} & & \downarrow{f} \\
X' & \xrightarrow{\pi} & \widetilde{X} \xrightarrow{h} X
\end{array}
\]

where \( \widetilde{X} \) is \( \mathbb{Q} \)-factorial. Let \( \widetilde{K}_{\widetilde{X}} = \pi_* K_{X'} \), \( \widetilde{B}_{\widetilde{X}} = \pi_* B_{X'}^+ \), and \( \widetilde{M}_{\widetilde{X}} = \pi_* M_{X'} \). Then \( \widetilde{\omega} = K_{\widetilde{X}} + \widetilde{B}_{\widetilde{X}} + \widetilde{M}_{\widetilde{X}} = h^*\omega \) and \( X' \to \widetilde{X} \to X \) is a generalized polarized dlt pair such that \( B_{\widetilde{X}} \) is the boundary part and \( M_{\widetilde{X}} \) the nef part (cf. [BZ, Definition 1.4]). Replacing \( \widetilde{X} \) with its terminalization, we can further assume that \( \widetilde{X} \) is terminal. By Definition 2.5, to prove that \((\widetilde{X}, \widetilde{\omega}, \widetilde{f} : (Y, B_Y) \to \widetilde{X})\) is a quasi-log canonical pair, we only need to prove that
\[
\alpha : \mathcal{O}_{\widetilde{X}} \to \widetilde{f}_*\mathcal{O}_Y([-B_Y^{\leq 1}])
\]
is an isomorphism. Since \( h : \widetilde{X} \to X \) is an isomorphism at the generic point of any prime divisor on \( X \), we only need to check that \( \alpha \) is isomorphic at every generic point of exceptional locus. Let \( P \) be a prime divisor in the exceptional locus and \( t_P = \text{mult}_P B_{\widetilde{X}} \). Since \( B_{\widetilde{X}} \) is a boundary, \( 0 \leq t_P \leq 1 \). Note that \((\widetilde{X}, \widetilde{\omega}, \widetilde{f} : (Y, B_Y) \to \widetilde{X})\) is an induced basic slc-trivial fibration. In particular, \( K_Y + B_Y + (1 - t_P)\widetilde{f}^* P \) is sub slc over the generic
point of $P$ by [F7 (4.5)]. That is, there is a prime divisor $F$ on $Y$ dominant onto $P$, $a := \text{mult}_F B_Y$ and $b := \text{mult}_F f^* P$ such that $a + (1 - t_P)b = 1$. Since $\tilde{X}$ is smooth in codimension two, the generic point of $P$ is Cartier. That is, $b$ is an integer. Then

$$[-a] = [(1 - t_P)b - 1] \leq b - 1.$$ 

This is equivalent to say that $[-B_Y^{\leq 1}] \notin f^* P$. Therefore,

$$\mathcal{O}_{\tilde{X}} \hookrightarrow \tilde{f}_* \mathcal{O}_Y([-B_Y^{\leq 1}]) \subset \mathcal{O}_{\tilde{X}}(P)$$

at the generic point of $P$. That is, $\mathcal{O}_{\tilde{X}} \simeq \tilde{f}_* \mathcal{O}_Y([-B_Y^{\leq 1}])$ and this is what we want. □

**Corollary 2.11.** Let $[X, \omega]$ be a qlc pair such that $X$ is normal and $[\tilde{X}, \tilde{\omega}]$ be the terminalization of $[X, \omega]$ induced by $h: \tilde{X} \to X$ as in Lemma 2.10. Let $L$ be an effective $\mathbb{R}$-Cartier divisor on $X$ and $\tilde{L} = h^* L$. Then $[X, \omega + L]$ is qlc at a point $x$ if and only if $[\tilde{X}, \tilde{\omega} + \tilde{L}]$ is qlc at every point of $h^{-1}(x)$. In particular, let $W$ be a connected union of qlc centers of $[X, \omega + L]$ and $\tilde{W}$ be the union of all qlc centers of $[\tilde{X}, \tilde{\omega} + \tilde{L}]$ mapping into $W$, then $\tilde{W}$ is also connected.

**Proof.** By the construction of the induced qlc pair $[\tilde{X}, \tilde{\omega}]$ as in Lemma 2.10 $[\tilde{X}, \tilde{\omega}]$ and $[X, \omega]$ are dominated by the same globally embedded simple normal crossing pair $(Y, B_Y)$. Therefore, the first part is a direct implication of Definition 2.6. By adjunction theorem, $W$ and $\tilde{W}$ are also dominated by the same union of strata $Y'$ of $(Y, B_Y)$. Since $W$ is connected, $Y'$ is also connected and thus so is $\tilde{W}$. □

### 3. Deficit at a point

In this section, we collect some definitions and explanations to reach to a definition of deficit at a point of a klt pair at a closed point, which is a number to measure how far this pair away from having this point as its minimal lc center. Most of them are due to Ein’s [E], Ein-Lazarsfeld’s [EL], Lee’s [Lee] and Helmke’s [H1], [H2], and the references therein.

**Definition 3.1.** Let $X$ be a variety with dim $X = n$ and $x$ be a closed point on $X$. Let $\varphi: X' \to X$ be the blowing up of point $x$ (blowing up with respect to $m_x$ where $m_x$ is the maximal ideal sheaf of point $x$) and $E = \varphi^{-1}(x)$ be the Cartier exceptional divisor. Note that $E$ is not necessarily reduced or irreducible. Let $G$ be an effective $\mathbb{R}$-Cartier divisor on $X$. We define the order of $G$ at point $x$, $\text{ord}_x G$ for short, to be the coefficient of $E$ in $\varphi^* G$.

**Remark 3.2.** When $G$ is Cartier, we can define

$$\text{ord}_x G = \text{ord}_x(f) = \max\{n \in \mathbb{N}; f \in m_x^n\}$$

where $f$ is a local equation of $G$ and $m_x$ is the maximal ideal sheaf of point $x$. We can also generalize it to effective $\mathbb{R}$-Cartier divisors by $\mathbb{R}$-linear combination. Note that these two definitions of $\text{ord}_x G$ are the same. Note also that if $x$ is smooth and $G$ is prime, then $\text{ord}_x G = \text{mult}_x G$ where $\text{mult}_x G$ is denoted as the multiplicity of a variety at $x$. In general, $\text{ord}_x G$ and $\text{mult}_x G$ are not the same, which will make some confusions. Therefore, we will use $\text{ord}_x G$ (order) for divisors and $\text{mult}_x G$ (multiplicity) for varieties.

Let $(X, \Delta)$ be a log pair which is klt at around a closed point $x$. Let $G$ be an effective $\mathbb{R}$-Cartier divisor. Let $f: Y \to X$ be a log resolution of $(X, \Delta + G)$ that factors through the blowing up $\varphi$ by a morphism $g: Y \to X'$ such that $f = \varphi \circ g$. Let $K_Y + B_Y = f^*(K_X + \Delta)$.

**Definition 3.3 (Deficit).** Let $B_Y = \sum a_i F_i$, $f^* G = \sum b_i F_i$ and $g^* E = \sum e_i F_i$ where $F_i$ are simple normal crossing divisors. Assume that $(X, \Delta + G)$ is log canonical at around $x$. Then:
Remark 3.4. In Definition 3.3 (2), the minimal lc center of \((X, \Delta + G)\) passing through \(x\) is always called the \textit{critical variety} of \(G\). We also call that \(G\) is \textit{critical at} \(x\) if not necessary to mention \(W\). See \cite{E} Definition 2.4] or \cite{Lee} Definition 2.5]. But we don’t need these definitions in our paper since we don’t need the tie-breaking trick (cf. \cite{E} Remark 2.5] or \cite{Lee} Remark 2.6].

Remark 3.5. It is equivalent to define \textit{deficit of} \(G\) as the smallest \(c \in \mathbb{R}_{\geq 0}\) such that for any effective \(\mathbb{R}\)-Cartier divisor \(D\) with \(\operatorname{ord}_x D \geq c\), we will have \(x \in \text{Nklt}(X, \Delta + G + D)\). This is the approach used in \cite{Lee} Definition 2.9] when \(x\) is a singular point on \(X\) based on \cite{E} Section 4]. When \(x\) is smooth, our two definitions of deficit coincide with the so-called \textit{local discrepancy} defined by Helmke \cite{H1} \cite{H2}.

Remark 3.6. We could also define \textit{deficit} of a qlc pair \([X, \omega]\) at a closed point \(x\) where \(x \notin \text{Nqklt}(X, \omega)\). But thanks to Theorem 2.8 we could “almost” turn a qlc pair into a klt pair by a small perturbation. See the Appendix in this paper.

Definition 3.7 (nice lifting). Notations are as in Definition 3.3 (2). Let \(W\) be the minimal lc center of \((X, \Delta + G)\) passing through \(x\). Let \(D\) be an effective \(\mathbb{R}\)-Cartier divisor on \(W\). An effective \(\mathbb{R}\)-Cartier divisor \(B\) on \(X\) is said to be a \textit{nice lifting} of \(D\), if \(B\) satisfies the following two properties:

(1) \(B|_W = D\);

(2) \(\text{Nklt}(X - W, (\Delta + G + B)|_{X - W}) = \text{Nklt}(X - W, (\Delta + G)|_{X - W})\).

Remark 3.8. Roughly speaking, a lifting \(B\) of \(D\) is \textit{nice}, means that \(B\) keeps the required singularities as \(D\) on \(W\) but as \textit{smooth} as possible outside of \(W\). Up to quasi-log canonical singularities, a nice lifting of \(D\) will always exist and if there are two nice liftings of \(D\), they will contribute nothing difference in our settings. Therefore, we can choose any nice lifting as we want. See \cite{E} Definition 4.5], \cite{Lee} Proposition 2.7], \cite{Ko} Claim 6.8.4], \cite{F1} Lemma 2.9] and \cite{Lin} Proposition 3.4] up to quasi-log canonical singularities for more details of nice lifting.

By above definitions, we immediately have the following propositions:

Proposition 3.9. Let \((X, \Delta)\) be a klt pair. Let \(G\) be an effective \(\mathbb{R}\)-Cartier divisor and \(x\) be a closed point on \(X\). Let \(W\) be the minimal lc stratum of \((X, \Delta + G)\) passing through \(x\) with \(\dim W = k\). Let \(D\) be an effective \(\mathbb{R}\)-divisor on \(W\). Then:

(1) \(d_x(G) \leq k\). If \(W\) is singular at \(x\), then \(d_x(G) \leq k - 1\);

(2) we can choose a nice lifting \(B\) such that \(\operatorname{ord}_x B \geq \operatorname{ord}_x D\);

(3) \(d_x((1 - t)G + B) \leq d_x((1 - t)G) - \operatorname{ord}_x B\) for sufficiently small \(t\).

Proof. See \cite{E} Proposition 4.2 and Lemma 4.6] or \cite{Lee} Proposition 2.9]. See also \cite{H2} Equation (2.6)].
4. Global index one cover

Recall that an index one cover (cf. [CKM (6.8)] or [KM Definition 5.19]) is constructed locally on an affine variety. In this section, we construct index one cover globally. Let $X$ be a projective $n$-dimensional normal $\mathbb{Q}$-Gorenstein variety. We assume that $x := \text{Sing } X$ is a unique isolated point. Let $r = \text{Index}_x X$. That is, $r$ is the smallest positive integer such that $rK_X$ is Cartier. Take a sufficiently ample line bundle $\mathcal{A}$ on $X$ such that $\mathcal{A}^r \otimes \mathcal{O}_X(rK_X)$ is generated by global sections. Let $D = (s = 0)$ be a general member of $|\mathcal{A}^r \otimes \mathcal{O}_X(rK_X)|$ which is smooth and does not pass through point $x$. By general ramified cyclic cover in [KM Definition 2.52], there is a cyclic cover

$$p : X' = \text{Spec}_X \oplus_{i=0}^{r-1} (\mathcal{A} \otimes \mathcal{O}_X(K_X))^{[-i]} \to X$$

ramified along $x \cup D$ with $\deg p = r$. In particular, let $U$ be an open neighborhood of point $x$ and $U' = p^{-1}(U)$. Then

$$p|_{U'} : U' = \text{Spec}_U \oplus_{i=0}^{r-1} (\mathcal{A} \otimes \mathcal{O}_X(K_X)|_U)^{[-i]} \to U$$

is the restriction of $p$ and the multiplication is given by

$$s|_U : \mathcal{O}_U \to (\mathcal{A} \otimes \mathcal{O}_X(K_X)|_U)^{[r]} \cong \mathcal{O}_U$$

by shrinking $U$ suitably. Note that this is exactly the definition of local index one cover defined by $s|_U$.

**Definition 4.1** (global index one cover). Such a cyclic cover $p : X' \to X$ is called global index one cover at point $x$. Note that $p$ is heavily depended on $D$.

**Proposition 4.2.** Let $p : X' \to X$ be a global index one cover at point $x$ with $r = \text{Index}_x X$. Then:

1. $X'$ is normal and Gorenstein, i.e., $K_{X'}$ is Cartier,
2. $x' := p^{-1}(x)_{\text{red}}$ is the unique possible singular point,
3. $p$ is étale in codimension one over $X \setminus D$,
4. the extension of the function fields $\mathbb{C}(X')/\mathbb{C}(X)$ is Galois and the Galois group $G \cong \mathbb{Z}/(r)$ acts on $X'$ over $X$, and
5. $g \cdot m_{x'} \simeq m_x$, where $g \in G$ is an action and $m_{x'}$ is the ideal sheaf of point $x'$.

**Proof.** Note that outside the singular point $x$, $p$ coincides with the ramified cyclic cover for line bundle case as in [KM Definition 2.50]. Since $D$ is smooth, $X'\setminus p^{-1}(x)$ is smooth by [KM Lemma 2.51]. Note also that outside the ramified locus $D$, $p$ coincides with the local index one cover determined by the nowhere vanishing local section of $D$. Then it is easy to see that (1)–(4) are direct conclusions of [Mi 4-5-1]. For (5), we write down the local expression of $m_{x'}$ as in [Mi 4-5-1]:

$$m_{x'} = m_x \oplus \{ \oplus_{i=1}^{r-1} R_i \cdot (\sqrt[r]{f})^i \}.$$ 

Assume the action of $g$ is presented by a fixed primitive $r$-th root of unity $\zeta$, then

$$g \cdot m_{x'} = \zeta \cdot (m_x \oplus \{ \oplus_{i=1}^{r-1} R_i \cdot (\sqrt[r]{f})^i \}) = m_x \oplus \{ \oplus_{i=1}^{r-1} R_i \cdot (\zeta \cdot \sqrt[r]{f})^i \}).$$

It is easy to see that $g \cdot m_{x'} \subset m_{x'}$. Using a converse action $g^{-1}$, we get what we want. □

Let $(X, \Delta)$ be a projective klt pair such that $X$ is an $n$-dimensional normal $\mathbb{Q}$-Gorenstein variety. Assume that $x := \text{Sing } X$ is a closed point. Let $r = \text{Index}_x X$. Let $N$ be an ample $\mathbb{R}$-Cartier divisor on $X$ such that $N^n > \frac{n^n}{r}$. Let $p : X' \to X$ be the global index one cover at point $x$ determined by $D$ and $X_0 = X \setminus D$. Let $X_0' = p^{-1}(X_0)$. Let $\Delta'$ be the $\mathbb{R}$-divisor such that $K_{X'} + \Delta' = p^*(K_X + \Delta)$ and $\Delta'_0 = \Delta'|_{X_0'}$. Then $(X_0', \Delta'_0)$ is also klt by Proposition 4.2 (3) and [KM Proposition 5.20]. Note that $(X', \Delta')$ is sub klt since $\Delta'$ may contain negative component supported on $p^{-1}(D)$. Let $N' = p^*N$ be the ample $\mathbb{R}$-Cartier
divisor on $X'$. Then $(N')^n = r \cdot N^n > n^n$. By [Ko] Theorem 6.7.1 (where we only need to assume that $(X', \Delta')$ is klt in a neighborhood of $x'$, see also [LM] Proposition 3.3), there is an effective $\mathbb{R}$-Cartier divisor $L' \sim_\mathbb{R} N'$ on $X'$ such that $(X', \Delta' + L')$ is not log canonical at $x'$. Let $d_{x'}$ be the deficit of 0 with respect to $(X', \Delta')$.

**Lemma 4.3.** Notations are as above. If we further assume that $\text{ord}_{x'} L' > d_{x'}$, then there is an effective $\mathbb{R}$-Cartier divisor $L \sim_\mathbb{R} N$, a positive number $0 < c < 1$, and an open neighborhood $x \in U \subset X$ such that:

1. $(U, (\Delta + c L)|_U)$ is log canonical, and
2. there is a minimal lc center $W$ of $(U, (\Delta + c L)|_U)$ passing through $x$ with dim $W < \text{dim } X$.

**Proof.** Let $L'_g$ be the Galois conjugates of $L'$ for $g \in G$. Then $\Sigma_g L'_g$ is $G$-invariant and thus, there is an effective $\mathbb{R}$-Cartier divisor $\tilde{L} \sim_\mathbb{R} r N$ on $X$ such that $\Sigma_g L'_g = p^* \tilde{L}$. Note that

$$\text{ord}_{x'} \Sigma_g L'_g = r \cdot \text{ord}_{x'} L' > r \cdot d_{x'}$$

by Proposition 4.2 (5) and assumption. Therefore, there is a maximal number

$$c' \leq \frac{d_{x'}}{\text{ord}_{x'} \Sigma_g L'_g} < \frac{1}{r}$$

such that $(X'_0, \Delta'_0 + c' \Sigma_g L'_g)$ is log canonical at around point $x'$ by Remark 3.5. By shrinking $X_0$ and [KM] Proposition 5.20, $(U, (\Delta + c' \tilde{L})|_U)$ is log canonical in a suitable open neighborhood $x \in U$. That is, there is an effective $\mathbb{R}$-Cartier divisor $L = \frac{1}{r} \tilde{L} \sim_\mathbb{R} N$ and a number $c = c' \cdot r < 1$ such that $(U, (\Delta + c L)|_U)$ is log canonical. This is (1). By our construction, (2) is trivial.

Without assuming that $\text{ord}_{x'} L' > d_{x'}$, it may happen that $x' \notin \text{Nklt}(X', \Delta' + 1/r(\Sigma_g L'_g))$ by replacing $L'$ with $1/r(\Sigma_g L'_g)$, and thus the inductive procedure stops. Note also that $\text{ord}_x L$ may be smaller than $d_x$ on $X$. It will be interesting to ask the relationship between $d_x$ and $d_{x'}$.

### 5. FREENESS FOR TERMINAL SINGULARITIES

It is known that Fujita-type basepoint-freeness in dimension three has been proved up to Gorenstein terminal singularities by Lee in [Lee] and Kakimi in [K1] separately. When the threefold $X$ is not Gorenstein, the result is still not known. This is because that, for a given terminal point $x$, mult$_x X$ can go to infinity when the index $r$ is increasing (cf. [K2] Theorem 2.1). This makes the low bound of $\text{ord}_x L$ where $L \sim_\mathbb{R} N$ constructed by Riemann-Roch theorem (as in the proof of [H1] Proposition 3.2) so small that the inductive procedure stops. In this section, we will overcome this problem by using global index one cover. Note that Sing $X$ is a union of isolated points since $X$ has only terminal singularities. For simplicity, we assume that $x := \text{Sing } X$ is a unique point.

**Theorem 5.1.** Let $[X, \omega]$ be a projective quasi-log canonical pair such that $X$ is a normal $\mathbb{Q}$-factorial terminal threefold. Assume that $x := \text{Sing } X$ is a point and $x \notin \text{Nklt}(X, \omega)$. Let $r = \text{Index}_x X$. Let $M$ be a Cartier divisor on $X$. We put $N = M - \omega$ and assume that $N^3 > \frac{27}{r}$ and that $N^k \cdot Z \geq 3^k$ for every subvariety $Z$ with $0 < \text{dim } Z = k < 3$. Then the complete linear system $|M|$ is basepoint-free at point $x$.

**Proof.** By using Theorem 2.8 (see the proof of [FL2] Theorem 3.2), we can take a boundary $\mathbb{R}$-divisor $\Delta_x$ on $X$ such that $K_X + \Delta_x \sim_\mathbb{R} \omega + \varepsilon N$ for $0 < \varepsilon \ll 1$ and $\mathcal{J}(X, \Delta_x) = \mathcal{I}_{\text{Nklt}(X, \omega)}$ where $\mathcal{J}(X, \Delta_x)$ is the multiplier ideal sheaf of $(X, \Delta_x)$. Since $\mathcal{J}(X, \Delta_x) = \mathcal{I}_{\text{Nklt}(X, \omega)}$, $(X, \Delta_x)$ is klt in a neighborhood of $x$. Let $p : X' \to X$ be the global index one cover at $x$. Let $x' \in X'$ is a point.
By Riemann-Roch theorem, there is an effective $R$. Therefore, there is a maximal real number $0 < c < \varepsilon$ such that $\text{ord}_x L'_3 \geq \sigma_3 > 2$ for any irreducible surface $S'$. By Proposition 3.9 (2), there is an effective $R$ that $\text{ord}_x L'_3 \geq \sigma_3 > 2$ for any irreducible curve $C'$. Moreover, $\text{ord}_x L'_3 \geq \sigma_3 > 2 - 3\varepsilon$.

Since $X'$ has at most Gorenstein terminal singularity, $m_1 = \text{mult}_{x'} X' \leq 2$ (cf. [CKM]). By Riemann-Roch theorem, there is an effective $R$-Cartier divisor $L'_3 \sim_R N'_e$ on $X'$ such that $\text{ord}_{x'} L'_3 \geq \frac{\sigma_3}{\sqrt{\varepsilon}}$. Let $d_{x'}$ be the deficit of pair $(X'_0, \Delta'_e | X'_0)$. By Proposition 3.9 (1),

$$\text{ord}_{x'} L'_3 \geq \sigma_3 > 3 \geq d_{x'} \text{ if } x' \text{ is smooth, or}$$

$$\text{ord}_{x'} L'_3 \geq \frac{\sigma_3}{\sqrt{2}} > 2 \geq d_{x'} \text{ if } x' \text{ is singular.}$$

Therefore, there is a maximal real number $0 < c_3 < 1$ such that $(X'_0, (\Delta'_e + c_3 L'_3)| X'_0)$ is log canonical at around point $x'$. The same as Lemma 4.3, we can assume that $L'_3$ is $G$-invariant by (5.1) and thus $(X'_0, (\Delta'_e + c_3 L'_3)| X'_0)$ is log canonical at around $x$. Let $Z'$ be the closure of the minimal lc center of $(X'_0, (\Delta'_e + c_3 L'_3)| X'_0)$ in $X'$ and $Z$ be the minimal lc center of $(X, \Delta_e + c_3 L_3)$ passing through $x$. We discuss various cases according to the dimension of $Z'$.

**Case 1.** Assume that $\dim Z' = 0$, that is, $x' = Z'$. Then $(X, \Delta_e + c_3 L_3)$ is log canonical at around $x$ and $x = p(Z')$ is the minimal lc center of $(X, \Delta_e + c_3 L_3)$ passing through $x$. Let $W = \text{Nl}(X, K_X + \Delta_e + c_3 L_3) \cup x$. Since $M - (K_X + \Delta_e + c_3 L_3) \sim_R (1 - c_3)(1 - \varepsilon)N$ is ample, the natural restriction map

$$H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(W, \mathcal{O}_W(M))$$

is surjective by the vanishing theorem. Since $x$ is an isolated point in $W$, it is obviously that $|M|$ is basepoint-free at point $x$.

**Case 2.** Assume that $\dim Z' = 1$, that is, $Z'$ is an irreducible curve smooth at $x'$. By condition (3), there is an effective $R$-Cartier divisor $L'_{Z'} \sim_R N'_e|_{Z'}$ on $Z'$ such that $\text{ord}_{x'} L'_{Z'} \geq \sigma_1$. By Proposition 3.9 (2), there is an effective $R$-Cartier divisor $L'_1 \sim_R N'_e$ on $X'$ such that $\text{ord}_{x'} L'_1 \geq \text{ord}_{x'} L'_{Z'} \geq \sigma_1$. By Proposition 3.9 (1),

$$\text{ord}_{x'} L'_1 \geq \sigma_1 = 3(1 - \varepsilon) > 1 \geq d_{x'}(c_3 L'_3).$$

Therefore, there is a maximal real number $0 < c_1 < 1$ such that $(X'_0, (\Delta'_e + c_3 L'_3 + c_1 L'_1)| X'_0)$ is log canonical at around point $x'$. By (5.2) and Lemma 4.3, we assume that $L'_1$ is $G$-invariant and thus $L'_1 = p^*L_1$ where $(X, \Delta_e + c_3 L_3 + c_1 L_1)$ is log canonical at around $x$. In particular, $x$ is the minimal lc center of $(X, \Delta_e + c_3 L_3 + c_1 L_1)$. We need to show that $c_3 + c_1 < 1$. If so, then

$$M - (K_X + \Delta_e + c_3 L_3 + c_1 L_1) \sim_R (1 - c_1 - c_3)(1 - \varepsilon)N$$

is ample, and the natural restriction map

$$H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(W, \mathcal{O}_W(M))$$
is surjective by the vanishing theorem where \( W = \text{Nqlc}(X, K_X + \Delta + c_3 L_3 + c_1 L_1) \cup x \).

Since \( x \) is an isolated point in \( W \), it is obviously that \(|M|\) is basepoint-free at point \( x \).

Therefore, we prove that \( c_3 + c_1 < 1 \) in the rest of this case. Let \( b_3 = \text{ord}_{x'} L'_3 \geq \frac{\sigma_3}{\sqrt{m}} \) and \( b_1 = \text{ord}_{x'} L'_1 \geq \sigma_1 \). Let \( d_0 = d_{x'} \) and \( d_1 = d_{x'}(c_3 L'_3) \). When \( x' \) is smooth, we have the following relationship by Proposition 3.9:

(1) \( d_0 \leq 3 \),
(2) \( c_3 b_3 \leq d_0 \),
(3) \( d_1 \leq d_0 - c_3 b_3 \),
(4) \( c_1 b_1 \leq d_1 \),

and thus

\[
\frac{c_3 + c_1}{c_3 + \frac{d_0 - c_3 b_3}{b_1}} \leq \frac{3 - c_3 \sigma_3}{\sigma_1} = \frac{3}{\sigma_1} + (1 - \frac{\sigma_3}{\sigma_1})c_3 = \frac{1 - 4\varepsilon c_3}{1 - \varepsilon}.
\]

Note that we can always assume that \( c_3 \geq \frac{2}{3} \). Otherwise, we can easily check that on \((X, \Delta_\varepsilon + c_3 L_3), M|_Z - (K_X + \Delta_\varepsilon + c_3 L_3)|_Z \) satisfies Fujita’s condition with respect to \( \dim Z \leq 1 \). Then we can use induction on dimension and prove that \(|M|_Z| \) is base-point free at point \( x \) by FL2, Theorem 1.3 and thus \(|M| \) is base-point free at point \( x \) by the vanishing theorem. Then

\[
c_3 + c_1 \leq \frac{1 - 4\varepsilon c_3}{1 - \varepsilon} \leq \frac{3 - 8\varepsilon}{3 - 3\varepsilon} < 1.
\]

When \( x' \) is singular, we have the same relationship except that \( d_0 \leq 2 \). Then

\[
c_3 + c_1 \leq c_3 + \frac{d_0 - c_3 b_3}{b_1} \leq c_3 + \frac{3 - c_3 \sigma_3}{\sigma_1} \leq \frac{2\sqrt{2} - c_3 \sigma_3}{\sigma_1 \sqrt{2}}
\]

\[
= \frac{2}{\sigma_1} + (1 - \frac{\sigma_3}{\sigma_1 \sqrt{2}})c_3 < \frac{2}{3 - 3\varepsilon} + \frac{1}{4} < 1
\]

and this is what we want.

**Case 3.** Assume that \( \dim Z' = 2 \), that is, \( Z' \) is an irreducible surface normal at \( x' \). Since \( X' \) is Gorenstein and \( Z' \) is \( \mathbb{Q} \)-Cartier by assumption, \( Z' \) is Cartier by FL2, Lemma 5.1. Since \((X', Z')\) is plt at around \( x' \), \( K_{Z'} = (K_{X'} + Z'_3)|_{Z'} \) by adjunction and thus \( Z' \) is also Gorenstein. In particular, \( Z' \) has at most rational double point. Let \( m_2 = \text{mult}_{x'} Z' \). Then \( m_2 \leq 2 \). By condition (2), there is an effective \( \mathbb{R} \)-Cartier divisor \( L'_2 \sim_{\mathbb{R}} N'_\varepsilon|_{Z'} \) on \( Z' \) such that \( \text{ord}_{x'} L'_2 \geq \frac{\sigma_2}{\sqrt{m_2}} \). By a nice lifting, there is an effective \( \mathbb{R} \)-Cartier divisor \( L'_2 \sim_{\mathbb{R}} N'_\varepsilon \) on \( X' \) such that \( b_2 := \text{ord}_{x'} L'_2 \geq \text{ord}_{x'} K_{Z'} \geq \frac{\sigma_2}{\sqrt{m_2}} \). Let \( d_0 = d_{x'} \) and \( d_1 = d_{x'}(c_3 L'_3) \). Let \( b_3 = \text{ord}_{x'} L'_3 \geq \frac{\sigma_3}{\sqrt{m_3}} \). The same as Case 2, we can assume that \( c_3 \geq \frac{1}{3} \). By Proposition 3.9 and \cite{Lee} Lemma 3.3 (or 

\[
\text{ord}_{x'} L'_2 \geq \frac{3 - \varepsilon}{\sqrt{2}} \geq 2 \geq d_0 - c_3 b_3 \geq d_1 \quad \text{if } m_1 = 2 \text{ and } m_2 = 1
\]

\[
(5.3)\text{ord}_{x'} L'_2 \geq \frac{3 - \varepsilon}{\sqrt{2}} > \frac{3 + \sqrt{2}}{3} \geq d_0 - c_3 b_3 \geq d_1 \quad \text{if } m_1 = 1 \text{ and } m_2 = 2
\]

Therefore, there is a maximal real number \( 0 < c_2 < 1 \) such that \((X'_0, (\Delta'_\varepsilon + c_3 L'_3 + c_2 L'_2)|_{X'_0})\) is log canonical at around point \( x' \). By (5.3) and Lemma 1.3, we assume that \( L'_2 \) is \( G \)-invariant and thus \( L'_2 = p^* L_2 \) where \((X, \Delta_\varepsilon + c_3 L_3 + c_2 L_2)\) is log canonical at around \( x \).
Let $S'$ be the closure of the minimal lc center of $(X'_0, (\Delta'_\varepsilon + c_3 L'_3 + c_2 L'_2)|_{X'_0})$. If $\dim S' = 0$, then the same as Case 2 we only need to show that $c_3 + c_2 < 1$.

If $m_1 = 1$ and $m_2 = 1$, then
\[
c_3 + c_2 \leq c_3 + \frac{d_0 - c_3 b_3}{b_2} \leq c_3 + \frac{3 - c_3 \sigma_3}{\sigma_2} = \frac{3}{\sigma_2} + (1 - \frac{\sigma_3}{\sigma_2}) c_3 = \frac{1 - 4\varepsilon c_3}{1 - \varepsilon} \leq \frac{3 - 4\varepsilon}{3 - 3\varepsilon} < 1.
\]

If $m_1 = 2$ and $m_2 = 1$, then
\[
c_3 + c_2 \leq c_3 + \frac{2\sqrt{2} - c_3 \sigma_3}{\sigma_2 \sqrt{2}} = \frac{2}{\sigma_2} + (1 - \frac{\sigma_3}{\sigma_2 \sqrt{2}}) c_3 < \frac{2}{3 - 3\varepsilon} + \frac{1}{4} < 1.
\]

If $m_1 = 2$ and $m_2 = 2$, then
\[
c_3 + c_2 \leq c_3 + \frac{2\sqrt{2} - c_3 \sigma_3}{\sigma_2 \sqrt{2}} = \frac{2\sqrt{2}}{\sigma_2} + (1 - \frac{\sqrt{2} \sigma_3}{\sigma_2 \sqrt{2}}) c_3 < \frac{2\sqrt{2}}{3 - 3\varepsilon} < 1.
\]

Finally we consider the case $m_1 = 1$ and $m_2 = 2$. By choosing $\varepsilon$ small enough and [Lee Lemma 3.3], we have
\[
\frac{b_3}{b_3 - 1} < \frac{\sqrt{2} b_3}{\sqrt{2}(3 - 1)} < \frac{\sigma_2}{\sqrt{2}} \leq b_2.
\]

If $d_0 - c_3 b_3 \geq 1$, then
\[
c_3 + c_2 \leq c_3 + \frac{d_0 - c_3 b_3}{b_2} = \frac{d_0}{b_2} + (1 - \frac{b_3}{b_2}) c_3 \leq \frac{1}{b_2} + \frac{d_0 - 1}{b_3} \leq \frac{b_3 - 2}{b_3} + \frac{d_0 - 1}{b_3} \leq 1;
\]
and if $d_0 - c_3 b_3 < 1$, then by (5.4) and simple calculation, we have that:
\[
d_0 - c_3 b_3 < (1 - c_3) b_2.
\]
Therefore,
\[
c_3 + c_2 \leq c_3 + \frac{d_0 - c_3 b_3}{b_2} < c_3 + (1 - c_3) < 1.
\]

Case 4. We continue to discuss Case 3. Assume that $\dim S' = 1$. That is, $S'$ is an irreducible curve smooth at $x'$. By condition (3), there is an effective $\mathbb{R}$-Cartier divisor $L'_x \sim_{\mathbb{R}} N'_1|_{S'}$ on $S'$ such that $\ord_{x'} L'_1 \geq \sigma_1$. By a nice lifting, there is an effective $\mathbb{R}$-Cartier divisor $L'_1 \sim_{\mathbb{R}} N'_1$ on $X'$ such that $b_1 := \ord_{x'} L'_1 \geq \ord_{x'} L'_x \geq \sigma_1$. Let $d_2 = d_{x'}(c_3 L'_3 + c_2 L'_2)$ and $b_2 = \ord_{x'} L'_2$. By Proposition 3.9
\[
\ord_{x'} L'_1 \geq \sigma_1 = 3(1 - \varepsilon) > 1 \geq d_2.
\]
Therefore, there is a maximal real number $0 < c_1 < 1$ such that $(X'_0, (\Delta'_\varepsilon + c_3 L'_3 + c_2 L'_2 + c_1 L'_1)|_{X'_0})$ is log canonical at around point $x'$. By (5.4) and Lemma 1.3 we assume that $L'_1$ is $G$-invariant and thus $L'_1 = p^* L_1$ where $(X, \Delta_\varepsilon + c_3 L_3 + c_2 L_2 + c_1 L_1)$ is log canonical at around $x$. In particular, $x$ is the minimal lc center of $(X, \Delta_\varepsilon + c_3 L_3 + c_2 L_2 + c_1 L_1)$. Again, we only need to show that $c_3 + c_2 + c_1 < 1$.

If $m_1 = 1$ and $m_2 = 1$, then
\[
c_3 + c_2 + c_1 \leq c_3 + c_2 + \frac{d_2}{b_1} \leq c_3 + c_2 + \frac{d_0 - c_3 b_3 - c_2 b_2}{b_1} \leq \frac{3}{\sigma_1} + (1 - \frac{\sigma_3}{\sigma_1}) c_3 = \frac{1 - 4\varepsilon c_3}{1 - \varepsilon} \leq \frac{3 - 4\varepsilon}{3 - 3\varepsilon} < 1.
\]

If $m_1 = 2$ and $m_2 = 1$, then
\[
c_3 + c_2 + c_1 \leq c_3 + c_2 + \frac{d_0 - c_3 b_3 - c_2 b_2}{b_1} \leq \frac{2}{\sigma_1} + (1 - \frac{\sigma_3}{\sigma_1 \sqrt{2}}) c_3 < \frac{2}{\sigma_1} + \frac{1}{4} < 1.
\]
Proof.

In the proof of Theorem 5.1, we in fact get an induced quasi-log structure

where $W$

Note that

$14$ HAIDONG LIU

and

condition. Then there exists a section $M$

Let $\mathcal{L}$ be a Cartier divisor on $X, K_X + \Delta X, \omega$ and assume that $s(x) \neq 0$, where $\mathcal{L}_{N_{\text{qlc}}(X, \omega)}$ is the defining ideal sheaf of $N_{\text{qlc}}(X, \omega)$ on $X$.

Proof. In the proof of Theorem 5.1, we in fact get an induced quasi-log structure

such that $x$ is the minimal qlc center of this quasi-log structure and $c_3 + c_2 + c_1 < 1$ where $c_i$ may be zero for some $i$. Let $V$ be the union of all irreducible qlc centers of $[X, K_X + \Delta_x + c_3 L_3 + c_2 L_2 + c_1 L_1]$ passing through $x$. Let $W$ be the closure of

$\text{Nqkl}(X, K_X + \Delta_x + c_3 L_3 + c_2 L_2 + c_1 L_1) \setminus V$.

Note that $W$ is a union of some qlc centers and $N_{\text{qlc}}(X, K_X + \Delta_x + c_3 L_3 + c_2 L_2 + c_1 L_1)$. Then:

$M - (K_X + \Delta_x + c_3 L_3 + c_2 L_2 + c_1 L_1) \sim (1 - c_1 - c_2 - c_3)(1 - \varepsilon)N$

is ample, and the natural restriction map

$H^0(X, \mathcal{O}_X(M)) \to H^0(W \cup x, \mathcal{O}_{W \cup x}(M))$
is surjective by the vanishing theorem. Since \( x \) is an isolated point in \( W \cup x \), there exists a section \( s \in H^0(X, \mathcal{O}_X(M)) \) such that \( s(x) \neq 0 \) and \( s(W) = 0 \). Since every \( L_i \) is effective, we have that

\[
\text{Nqklt}(X, \omega) = \text{Nqklt}(X, K_X + \Delta_\varepsilon) \subset W
\]

by the construction of \( W \). Therefore, \( s \in H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M)) \) and \( s(x) \neq 0 \). \( \square \)

6. Freeness for qlc singularities

First we deal with the normal case.

**Theorem 6.1.** Let \([X, \omega]\) be a projective quasi-log canonical pair such that \( X \) is a normal threefold. Let \( x \) be a closed point on \( X \). Let \( M \) be a Cartier divisor on \( X \). We put \( N = M - \omega \) and assume that \( N \) satisfies Fujita’s condition. Then the complete linear system \( |M| \) is basepoint-free at point \( x \).

**Proof.** Assume that \( x \notin \text{Nqklt}(X, \omega) \). Let \( h : \widetilde{X} \to X \) be the \( \mathbb{Q} \)-factorial terminalization and \([\widetilde{X}, \widetilde{\omega}]\) be the induced qlc pair by Lemma 2.10. In particular, \( \widetilde{X} \) is \( \mathbb{Q} \)-factorial and terminal. By Corollary 5.3, we can assume that \( \dim h^{-1}(x) \geq 1 \). Since \( \dim \text{Sing} \widetilde{X} = 0 \), we can choose a general point \( \widetilde{x} \in h^{-1}(x) \) which is smooth on \( \widetilde{X} \). Let \( \widetilde{M} = h^*M \) and \( \widetilde{N} = h^*N \).

By using Theorem 2.8, we can take a boundary \( \mathbb{R} \)-divisor \( \Delta_\varepsilon \) on \( X \) with \( K_X + \Delta_\varepsilon \sim_{\mathbb{R}} \omega + \varepsilon N \) for \( 0 < \varepsilon \ll 1 \) such that \( (X, \Delta_\varepsilon) \) is klt in a neighborhood of \( x \), and a boundary \( \mathbb{R} \)-divisor \( \widetilde{\Delta}_\varepsilon \) on \( \widetilde{X} \) with \( K_{\widetilde{X}} + \widetilde{\Delta}_\varepsilon = h^*(K_X + \Delta_\varepsilon) \sim_{\mathbb{R}} \widetilde{\omega} + \varepsilon \widetilde{N} \) such that \( (\widetilde{X}, \widetilde{\Delta}_\varepsilon) \) is klt in a neighborhood of \( \widetilde{x} \). Let \( N_\varepsilon = M - (K_X + \Delta_\varepsilon) \sim_{\mathbb{R}} (1 - \varepsilon)N \) and \( \widetilde{N}_\varepsilon = h^*N_\varepsilon \sim_{\mathbb{R}} (1 - \varepsilon)\widetilde{N} \). Let \( \sigma_3 = 3(1 + \varepsilon) \), \( \sigma_2 = \sigma_1 = 3(1 - \varepsilon) \). Then by choosing \( \varepsilon \) small enough, we have that:

1. \( \widetilde{N}_\varepsilon = N_\varepsilon > \sigma_3^2 > 27 \),
2. \( \widetilde{N}_\varepsilon \cdot S = N_\varepsilon \cdot h(S) \geq \sigma_2^2 \) for any irreducible surface \( S \) such that \( \dim h(S) = 2 \),
3. \( \widetilde{N}_\varepsilon \cdot C = N_\varepsilon \cdot h(C) \geq \sigma_1 \) for any irreducible curve \( C \) such that \( \dim h(C) = 1 \).

By condition (1), we can find an effective \( \mathbb{R} \)-Cartier divisor \( L_3 \sim_\mathbb{R} N_\varepsilon, \widetilde{L}_3 = h^*L_3 \sim_\mathbb{R} \widetilde{N}_\varepsilon \), and a maximal positive number \( 0 < c_3 < 1 \) such that \( b_3 := \text{ord}_x \widetilde{L}_3 > \sigma_3, (\widetilde{X}, \widetilde{\Delta}_\varepsilon + \widetilde{L}_3) \) is not lc at point \( \widetilde{x} \) but \( (\widetilde{X}, \widetilde{\Delta}_\varepsilon + c_3\widetilde{L}_3) \) is lc at point \( \widetilde{x} \). Let \( \widetilde{Z} \) be the irreducible minimal lc center passing through \( \widetilde{x} \). Then \( \dim \widetilde{Z} \leq 2 \) and \( \widetilde{Z} \) is normal at around point \( \widetilde{x} \). Note that \( \widetilde{Z} \) may be contained in the exceptional locus of \( h \). We discuss various cases according to the dimension of \( \widetilde{Z} \).

**Case 1.** Assume that \( \dim \widetilde{Z} > \dim h(\widetilde{Z}) \). If \( \dim h(\widetilde{Z}) = 0 \), that is, \( x = h(\widetilde{Z}) \) since \( \widetilde{Z} \) passing through \( \widetilde{x} \in h^{-1}(x) \), then by Corollary 2.11 \( x \) is the minimal lc center of \( (X, \Delta_\varepsilon + c_3L_3) \). Note that we get a natural quasi-log structure on \([X, K_X + \Delta_\varepsilon + c_3L_3]\). That is, \( x \) is the minimal qlc center of \([X, K_X + \Delta_\varepsilon + c_3L_3]\) since \( x \) is the minimal lc center of \( (X, \Delta_\varepsilon + c_3L_3) \). Let \( W = \text{Nqlc}(X, K_X + \Delta_\varepsilon + c_3L_3) \cup x \). Since \( M - (K_X + \Delta_\varepsilon + c_3L_3) \sim_{\mathbb{R}} (1 - c_3)(1 - \varepsilon)N \) is ample, the natural restriction map

\[
H^0(X, \mathcal{O}_X(M)) \to H^0(W, \mathcal{O}_W(M))
\]
is surjective by the vanishing theorem. Since $x$ is an isolated point in $W$, it is obviously that $|M|$ is basepoint-free at point $x$.

The left case for $\dim \tilde{Z} > \dim h(\tilde{Z})$ is that $\dim \tilde{Z} = 2$ and $\dim h(\tilde{Z}) = 1$. Let $C = h(\tilde{Z})$ be the curve on $X$. Note that $C$ is the minimal lc center of $(X, \Delta_x + c_3 L_3)$ by Corollary 2.11. By condition (3), there is an effective $\mathbb{R}$-Cartier divisor $L_C \sim_{\mathbb{R}} N_\varepsilon|_C$ on $C$ such that $\ord_x L_C \geq \sigma_1 > 1$ and $[C,(K_X + \Delta_x + c_3 L_3)]|_C + L_C|$ is not qlc at point $x$. By a nice lifting, there is an effective $\mathbb{R}$-Cartier divisor $L_1 \sim_{\mathbb{R}} N_\varepsilon$ on $X$ such that $\ord_x L_1 \geq \sigma_1$ and $(X, \Delta_x + c_3 L_3 + L_1)$ is not lc at point $x$. Therefore, there is a maximal number $0 < c_1 < 1$ such that $(X, \Delta_x + c_3 L_3 + c_1 L_1)$ is lc at point $x$ and $x$ is exactly the minimal lc center of $(X, \Delta_x + c_3 L_3 + c_1 L_1)$. We need to show that $c_3 + c_1 < 1$. Then

$$M - (K_X + \Delta_x + c_3 L_3 + c_1 L_1) \sim_{\mathbb{R}} (1 - c_3 - c_1)(1 - \varepsilon)N$$

is ample, and the natural restriction map

$$H^0(X, \mathcal{O}_X(M)) \to H^0(W, \mathcal{O}_W(M))$$

is surjective by the vanishing theorem where $W = \text{Nqlc}(X, K_X + \Delta_x + c_3 L_3 + c_1 L_1) \cup x$. Since $x$ is an isolated point in $W$, it is obviously that $|M|$ is basepoint-free at point $x$.

Therefore, we prove that $c_3 + c_1 < 1$ in the rest of this case. Let $\tilde{L}_1 = h^* L_1$. Then by Corollary 2.11 $(\tilde{X}, \tilde{\Delta}_x + c_3 \tilde{L}_3 + c_1 \tilde{L}_1)$ is lc at point $\tilde{x}$. Note that $\ord_{h^{-1}(x)} \tilde{L}_1 = \ord_x L_1 \geq \sigma_1$. Since $T^k_{h^{-1}(x)} \to T^k_{\tilde{x}}$ is injective for any positive integer $k$,

$$\ord_{\tilde{x}} \tilde{L}_1 \geq \ord_{h^{-1}(x)} \tilde{L}_1 \geq \sigma_1.$$ 

Let $b_3 = \ord_{\tilde{x}} \tilde{L}_3 > \sigma_3$ and $b_1 = \ord_{\tilde{x}} \tilde{L}_1 \geq \sigma_1$. Let $d_0 = d_{\tilde{x}}$ and $d_1 = d_{\tilde{x}}(c_3 L_3)$. Then by Proposition 3.9

1. $d_0 \leq 3$,
2. $c_3 b_3 \leq d_0$,
3. $d_1 \leq d_0 - c_3 b_3$, and
4. $c_1 b_1 \leq d_1$,

and thus

$$c_3 + c_1 \leq c_3 + \frac{d_0 - c_3 b_3}{b_1} \leq \frac{3}{\sigma_3} + (1 - \frac{\sigma_3}{\sigma_1})c_3 = \frac{1 - 2\varepsilon c_3}{1 - \varepsilon} \leq \frac{3 - 4\varepsilon}{3 - 3\varepsilon} < 1$$

by adding assumption that $c_3 \geq \frac{2}{3}$ as in Case 2 of Theorem 5.1.

Case 2. Assume that $\dim Z = \dim h(Z) = 2$. Then by condition (2) we can find an effective $\mathbb{R}$-Cartier divisor $L_2 \sim_{\mathbb{R}} N_\varepsilon$, $\tilde{L}_2 = h^* L_2 \sim_{\mathbb{R}} \tilde{N}_\varepsilon$, and a maximal positive number $0 < c_2 < 1$ such that $b_2 := \ord_{\tilde{x}} \tilde{L}_2 \geq \sigma_2$, $(\tilde{X}, \tilde{\Delta}_x + c_3 \tilde{L}_3 + \tilde{L}_2)$ is not lc at point $\tilde{x}$ but $(\tilde{X}, \tilde{\Delta}_x + c_3 \tilde{L}_3 + c_2 \tilde{L}_2)$ is lc at point $\tilde{x}$. Let $\tilde{T}$ be the irreducible minimal lc center passing through $\tilde{x}$. Then $\dim \tilde{T} \leq 1$ and $\tilde{T}$ is normal at around point $\tilde{x}$. We further assume in this case that $\dim \tilde{T} > \dim h(\tilde{T})$. That is, $x = h(\tilde{T})$.

Under these assumptions, we only need to prove that $c_3 + c_2 < 1$ and the rest are the same as Case 1. By [Kal] Theorem 2.2 or [Lee] Lemma 3.3, we can further assume that $b_3 = \ord_{\tilde{x}} \tilde{L}_3 > 3 + \sqrt{2}$. Then we can get $c_3 + c_2 < 1$ exactly the same as Case 3 of Theorem 5.1.

Case 3. Besides above cases, we have that $\dim \tilde{Z} = \dim h(\tilde{Z})$ and $\dim \tilde{T} = \dim h(\tilde{T})$ (when $\dim \tilde{Z} = \dim h(\tilde{Z}) = 2$). Then condition (2) and (3) make sure that we could create the inductive procedure exactly the same as the proof of Theorem 5.1.

Anyway, we finish our proof of Fujita-type freeness for normal qlc threefolds. □
Remark 6.2. As we saw in above proof, to create the desired inductive procedure we only need the Fujita’s condition holds true for those possible lc (or qlc) minimal centers.

Finally, we finish our proof of Fujita-type basepoint-freeness for general quasi-log canonical threefolds.

Theorem 6.3. Let \([X, \omega]\) be a projective quasi-log canonical threefold. Let \(x\) be a closed point on \(X\). Let \(M\) be a Cartier divisor on \(X\). We put \(N = M - \omega\) and assume that \(N\) satisfies Fujita’s condition. Then the complete linear system \(|M|\) is basepoint-free at \(x\).

Proof. Let \(x\) be an arbitrary closed point of \(X\) and let \(W\) be the irreducible minimal qlc stratum of \([X, \omega]\) passing through \(x\). By adjunction theorem, \([W, \omega]|_W\) is a quasi-log canonical pair. By the vanishing theorem, the natural restriction map

\[ H^0(X, \mathcal{O}_X(M)) \to H^0(W, \mathcal{O}_W(M)) \]

is surjective. From now on, we will see that \(|M|\) is basepoint-free in a neighborhood of \(x\). If \(W = x\), that is, \(x\) is a qlc center of \([X, \omega]\), then the complete linear system \(|M|\) is obviously basepoint-free in a neighborhood of \(x\) by the surjection (6.1). Let us consider the case where \(0 < m = \dim W < 3\). Let \(M_W = |M||_W\) and \(N_W = |N||_W = (M - \omega)|_W\). Then \(N_m^W \cdot W = N_m^W \geq 3m > m^m\) and \(N_k^W \cdot Z = N_k^W \cdot Z \geq 3^k \geq m^k\) for every subvariety \(Z \subset W\) with \(0 < \dim Z = k < m\). That is, \(N_W\) also satisfies Fujita’s condition. Using induction of dimension, \(|M_W|\) is basepoint-free at \(x\) by (6.1). Combining (6.1), we see that \(|M|\) is basepoint-free at \(x\). Thus we may assume that \(\dim W = \dim X = n\). It is also easy to check that \(N_W = |N||_W\) satisfies Fujita’s condition as above. Therefore, by replacing \(X\) with \(W\), we can assume that \(X\) is irreducible and \(x \notin \text{Nqklt}(X, \omega)\). In particular, \(X\) is normal near point \(x\).

Let \(\nu : \tilde{X} \to X\) be the normalization. Note that \([\tilde{X}, \nu^*\omega]\) is a qlc pair by Lemma 2.9. We put \(\tilde{M} = \nu^*M\) and \(\tilde{N} = \nu^*N = \tilde{M} - \nu^*\omega\). It is obvious that \(\tilde{M}\) is Cartier. Moreover, \((\tilde{N})^3 \cdot \tilde{X} = \tilde{N}^3 \cdot X \geq 27\) and \((\tilde{N})^k \cdot Z \geq \tilde{N}^k \cdot \nu(Z) \geq 3^k\) for every subvariety \(Z \subset \tilde{X}\) with \(0 < \dim Z = k < 3\). Note that \(\dim \nu(Z) = \dim Z = k\) since normalization \(\nu\) is finite. We also note that, \(\tilde{x} := \nu^{-1}(x)\) is a point since \(\nu : \tilde{X} \to X\) is an isomorphism over some open neighborhood of the normal point \(x\), and that the non-normal part of \(X\) is contained in \(\text{Nqklt}(X, \omega)\) by Lemma 2.9. The same as Corollary 5.3 there is a section

\[ \tilde{s} \in H^0(\tilde{X}, \mathcal{I}_{\text{Nqklt}(\tilde{X}, \nu^*\omega)} \otimes O_{\tilde{X}}(\tilde{M})). \]

such that \(\tilde{s}(\tilde{x}) \neq 0\) by the proof of Theorem 6.1. By \(\nu_*\mathcal{I}_{\text{Nqklt}(\tilde{X}, \nu^*\omega)} = \mathcal{I}_{\text{Nqklt}(X, \omega)}\) in Lemma 2.9, we have that:

\[ H^0(\tilde{X}, \mathcal{I}_{\text{Nqklt}(\tilde{X}, \nu^*\omega)} \otimes \mathcal{O}_{\tilde{X}}(\tilde{M})) \cong H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M)). \]

Thus we can descend the section \(\tilde{s}\) to a section \(s \in H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M))\) and \(s(x) \neq 0\). This \(s \in H^0(X, \mathcal{O}_X(M))\) is what we want.

\[ \square \]

7. Appendix

Let \((X, \omega, f : (Y, B_Y) \to X)\) be a quasi-log canonical pair such that \(X\) is an \(n\)-dimensional normal variety and \(x\) be a closed point such that \(x \notin \text{Nqklt}(X, \omega)\). By the universal property of blowing up, we can assume that \(f\) factors through the blowing up \(\varphi\) defined in Definition 3.1 by morphism \(g : Y \to X'\) such that \(f = \varphi \circ g\). That is, there is a commutative diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X' \\
\downarrow & & \downarrow \varphi \\
X' & \xrightarrow{f} & X
\end{array}
\]
Let $G$ be an effective $\mathbb{R}$-Cartier divisor and $[X, \omega + G]$ be the induced quasi-log structure by [F5, Lemma 4.6].

**Definition 7.1.** Let $B_Y = \sum a_i F_i, f^* G = \sum b_i F_i$ and $g^* E = \sum e_i F_i$ where $F_i$ are simple normal crossing divisors. If $x \notin \text{Nqklt}(X, \omega + G)$, then the **deficit of $G$ at $x$**, is defined as

$$d_x[X, \omega + G] = \inf \frac{1 - a_i - b_i}{e_i},$$

where $f$ varies among all quasi-log resolutions factoring through $\varphi$. If $x \in \text{Nqklt}(X, \omega + G)$ but $x \notin \text{Nqklt}(X, \omega + (1 - t)G)$ for any $0 < t < 1$, then the **deficit of $G$ at $x$**, is defined as

$$d_x[X, \omega + G] = \lim_{t \to 0^+} d_x[X, \omega + (1 - t)G].$$

**Lemma 7.2.** It is equivalent to define deficit of $G$ as the smallest $c \in \mathbb{R}_{\geq 0}$ such that for any effective $\mathbb{R}$-Cartier divisor $D$ with $\text{ord}_x D \geq c$, we have $x \in \text{Nqklt}(X, \omega + G + D)$.

**Proof.** Let $d = d_x[X, \omega + G]$. Let $0 < \varepsilon \ll 1$ be a sufficiently small number. Let $D$ be any given effective $\mathbb{R}$-Cartier divisor with $\text{ord}_x D \geq d$. Then

$$g_i := \text{mult}_F f^* D = \text{ord}_x D \cdot e_i \geq d \cdot e_i.$$

By Definition 7.1, there is a prime divisor $F$ on $Y$ such that $f(F) = x$ and

$$1 - a - b < (d + \varepsilon) \cdot e$$

where $a = \text{mult}_F B_Y, b = \text{mult}_F f^* G$ and $e = \text{mult}_F g^* E$. Let

$$f : (Y, B_Y + f^* G + f^* D) \to [X, \omega + G + D]$$

be the induced quasi-log structure. Then (7.1) and (7.2) imply that

$$\text{mult}_F (B_Y + f^* G + f^* D) = a + b + g \geq a + b + d \cdot e > 1 - \varepsilon \cdot e.$$

where $g = \text{mult}_F f^* D$. Note that the numbers $a, b, g$ and $e$ will not change anymore if we further blow up $(Y, B_Y)$ and consider the strict transform of $F$. Therefore, let $\varepsilon \to 0$, we have that (after replacing $F$ with its strict tranform):

$$\text{mult}_F (B_Y + f^* G + f^* D) \geq 1.$$

That is, $x \in \text{Nqklt}(X, \omega + G + D)$ which implies that $c \leq d$ by assumption.

Conversely, let $D$ be an effective $\mathbb{R}$-Cartier divisor with $\varepsilon + c > \text{ord}_x D \geq c$. Then by assumption, there is a prime divisor $F$ on $Y$ such that $f(F) = x$ and

$$a + b + (\varepsilon + c) \cdot e > \text{mult}_F (B_Y + f^* G + f^* D) = a + b + g \geq 1.$$

Therefore, $\varepsilon + c > d$ by definition of $d$. Let $\varepsilon \to 0$, we have that $c \geq d$. We get what we want. \hfill \square

Let $N$ be an ample $\mathbb{R}$-Cartier divisor. By using Theorem 2.8 (see the proof of [FL2 Theorem 3.2]), we can take a boundary $\mathbb{R}$-divisor $\Delta_z$ on $X$ such that $K_X + \Delta_z \sim_{\mathbb{R}} \omega + \varepsilon N$ for $0 < \varepsilon \ll 1$ and $J(X, \Delta_z) = I_{\text{Nqklt}(X, \omega)}$ where $J(X, \Delta_z)$ is the multiplier ideal sheaf of $(X, \Delta_z)$. Since $J(X, \Delta_z) = I_{\text{Nqklt}(X, \omega)}$, $(X, \Delta_z)$ is klt in a neighborhood of $x$. Let $D_z$ be an effective $\mathbb{R}$-Cartier divisor. Note that we get a natural quasi-log structure on $[X, \omega_z + D_z]$ with $\omega_z := K_X + \Delta_z$. $W_z$ is the minimal qlc center of $[X, \omega_z + D_z]$ passing through $x$, is equivalent to say that, $W_z$ is the minimal log canonical center of $(X, \Delta_z + D_z)$ passing through $x$. Let $d_z$ be the deficit of $0$ with respect to $(X, \Delta_z)$ defined in Definition 3.3 Then:

**Lemma 7.3.** $d_z[X, \omega] = \lim_{\varepsilon \to 0^+} d_x$.
Proof. By blowing up \((Y, B_Y)\) further and the universal property of blowing up, there is a commutative diagram as follows:

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow h & & \downarrow f \\
X' & \xrightarrow{\varphi} & X
\end{array}
\]

where \(\varphi\) is the blowing up of point \(x\) as in Definition 3.3 and Definition 7.1 and \(p\) is a sufficiently high log resolution as in Theorem 2.8. Let \(D_\varepsilon\) be an effective \(\mathbb{R}\)-Cartier divisor such that \(\text{ord}_x D_\varepsilon \geq d_\varepsilon\). By Lemma 7.2, \(x \in \text{Nqklt}(X, \omega_x + D_\varepsilon)\). By our construction, \(\text{Nqklt}(X, \omega_x + D_\varepsilon)\) is dominated by \((B_Y + \varepsilon f^*N + f^*D_\varepsilon)_{\geq 1}\). Since

\[
(B_Y + f^*D_\varepsilon)_{\geq 1} \leq (B_Y + \varepsilon f^*N + f^*D_\varepsilon)_{\geq 1},
\]

\(x\) is not necessarily contained in \(\text{Nqklt}(X, \omega + D_\varepsilon)\). This means that \(d_x[X, \omega] \geq d_\varepsilon\). By limiting, \(d_x[X, \omega] \geq \lim_{\varepsilon \to 0^+} d_\varepsilon\).

Conversely, let \(t\) be a sufficiently small number. Let \(D_t\) be an effective \(\mathbb{R}\)-Cartier divisor such that \(\text{ord}_x D_t = d_x[X, \omega] - t\) and \(x \notin \text{Nqklt}(\omega + D_t)\). We can choose a small number \(0 < \varepsilon \ll 1\) such that \(\varepsilon \cdot \text{ord}_x N < t\) and \(x \notin \text{Nqklt}(\omega + \varepsilon N + D_t)\). That is, \(d_\varepsilon \geq d_x[X, \omega] - t\). Let \(\varepsilon \to 0\), we have that \(\lim_{\varepsilon \to 0^+} d_\varepsilon \geq d_x[X, \omega] - t\). Let \(t \to 0\), we have that \(\lim_{\varepsilon \to 0^+} d_\varepsilon \geq d_x[X, \omega]\). We get what we want. 

By Lemma 7.3, any property belonging to deficit of klt pair is also belonging to deficit of qlc pair. In particular, since a klt pair \((X, \Delta)\) has a natural qlc structure, the deficits of both cases are the same. Note that if [FL1, Conjecture 1.5] is true, then Lemma 7.3 is tedious.

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