Multivalued backward stochastic differential equations with
time delayed generators

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Abstract

Our aim is to study the following new type of multivalued backward stochastic differential equation:

\[
\begin{aligned}
- dY(t) + \partial \varphi(Y(t)) \, dt &\ni F(t,Y(t),Z(t),Y_t,Z_t) \, dt + Z(t) \, dW(t), \quad 0 \leq t \leq T, \\
Y(T) &= \xi,
\end{aligned}
\]

where \(\partial \varphi\) is the subdifferential of a convex function and \((Y_t,Z_t) := (Y(t+\theta),Z(t+\theta))_{\theta \in [-T,0]}\) represent the past values of the solution over the interval \([0,t]\). Our results are based on the existence theorem from Delong & Imkeller, Ann. Appl. Probab., 2010, concerning backward stochastic differential equations with time delayed generators.

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1 Introduction

In this paper we are interested to study a new type of multivalued backward stochastic differential equation (BSDE) formally written as

\[
\begin{aligned}
- dY(t) + \partial \varphi(Y(t)) \, dt &\ni F(t,Y(t),Z(t),Y_t,Z_t) \, dt + Z(t) \, dW(t), \quad 0 \leq t \leq T, \\
Y(T) &= \xi,
\end{aligned}
\]

where \(\partial \varphi\) is a multivalued operator of subdifferential type.

We mention that in (1) the generator \(F\) at the moment \(t \in [0,T]\) can depend, unlike the classical nonlinear BSDEs introduced in [8] and [9], on the past values \((Y_t,Z_t)\) on \([0,t]\) of the solution \((Y(t),Z(t))\), where

\[
Y_t := (Y(t+\theta))_{\theta \in [-T,0]} \quad \text{and} \quad Z_t := (Z(t+\theta))_{\theta \in [-T,0]}.
\]
For this reason we shall call (1) BSDE with time-delayed generator.

Delong and Imkeller were the first who introduced and studied in [3] and [4] the BSDE of type (1). They considered equation

$$Y(t) = \xi + \int_t^T F(s,Y_s,Z_s)ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T$$

(3)

and they obtained in [3] the existence and uniqueness of the solution for (3) if the time horizon $T$ or the Lipschitz constant for the generator $F$ are sufficiently small. Also they provide a comparison type result, existence of a measure solution and, in [4], the Malliavin differentiability of the solution of a time-delayed BSDE driven by a Levy process.

Dos Reis, Réveillac and Zhang extend in [5] the results of Delong and Imkeller by giving moment and a priori estimates in general $L^p$ spaces and by proving sufficient conditions for the existence of a solution in $L^p$. In addition it is obtained, under some appropriate regularity conditions, the relationship between the Malliavin derivatives and the classical derivatives of the solution process of a delay decoupled forward-backward stochastic equations. Delong in [2] provides some applications of the delay BSDE in real problems of financial mathematics and issues related to pricing, hedging and investment portfolio management.

Concerning the multivalued term we precise that BSDE involving a subdifferential operator (which are also called backward stochastic variational inequalities, BSVI) has been treated by Pardoux and Răşcanu in [10] where they prove the existence and the uniqueness for

$$Y(t) + \int_t^T U(s)ds = \xi + \int_t^T F(s,Y(s),Z(s))ds - \int_t^T Z(s)dW(s),$$

(4)

where $U(t)$ is an element from $\partial \varphi(Y(t))$, and they generalize the Feymann-Kac type formula in order to represent the solution of a multivalued parabolic partial differential equation (PDE). We should mention that the solution $Y$ is reflected at the boundary of the domain of $\partial \varphi$ and the role of the process $U$ is to push $Y$ in order to keep it in this domain. More recently, in [7] it is studied, in the infinite dimensional framework, a generalized version of (4) considered on a random time interval (and their applications to the stochastic PDE). Another approach is giving in [11], where, by using the Fitzpatrick function, the existence problem for the multivalued stochastic differential equations is reduced to a minimizing problem of a suitable convex lower semicontinuous function.

The above types of BSDE are connected with the reflected BSDE which were introduced (in the scalar case and with one-sided reflection) by N. El Karoui et al. in [6]. They consider BSDE such that the solution $Y$ is forced to stay above a given lower barrier. In the last years these type of equations have been intensely studied and generalized (first by considering the multidimensional BSDE with two-sided reflection) since there is a wide range of applications especially in finance, stochastic control or stochastic games. We emphasize that if the lower and upper obstacles are fixed then reflected BSDE become a particular case of BSVI of type (4), by taking $\varphi$ as a indicator function of the interval defined by obstacles.

The first connection between reflected BSDE and the time delayed equation (3) was recently made by Zhou and Ren in [12] where it is proved, under the specific assumptions of the delayed BSDE and the reflected case, that there exists a unique solution of a reflected BSDE with time-delayed generator.

The article is organized as follows: in next section we set up the notation and the assumptions. The problem is formulated and the main result is state. Section 3 is devoted to the proof of the existence of the solution of the multivalued time delayed BSDE that we consider.

2 Notations and assumptions

Let $T \in (0, \infty)$ be a finite time horizon and $\{W(t)\}_{t \in [0,T]}$ be a $d$-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\{\mathcal{F}_t\}_{t \in [0,T]}$ the natural
filtration generated by \( \{ W(t) \}_{t \in [0,T]} \) and augmented by \( \mathcal{N} \) the set of \( \mathbb{F} \)-null events of \( \mathcal{F} \), i.e.

\[
\mathcal{F}_t = \sigma \{ W(r) : 0 \leq r \leq t \} \lor \mathcal{N}.
\]

As usual, \( \mathcal{B}([-T,0]) \) stands for the Borel sets of \([-T,0]\).

Throughout the paper will be needed the following spaces:

**Definition 1** Let \( \mathcal{H}^2_{T}^{m} \) be the Hilbert space of progressively measurable stochastic processes (p.m.s.p.) \( Y : \Omega \times [0,T] \to \mathbb{R}^m \) such that

\[
||Y||^2_{\mathcal{H}^2_T} = \mathbb{E} \left[ \int_0^T |Y(s)|^2 ds \right] < \infty,
\]

and \( \mathcal{S}^2_{T}^{m} \) be the Banach space of p.m.s.p. \( Y : \Omega \times [0,T] \to \mathbb{R}^m \) such that

\[
||Y||^2_{\mathcal{S}^2_T} = \mathbb{E} \left[ \sup_{t \in [0,T]} |Y(t)|^2 \right] < \infty.
\]

**Definition 2** Let \( \mathcal{H}^2_{-T}^{m} \) be the space of measurable function \( y : [-T,0] \to \mathbb{R}^m \) such that

\[
\int_{-T}^0 |y(s)|^2 ds < \infty,
\]

and \( \mathcal{S}^2_{-T}^{m} \) be the space of measurable function \( y : [-T,0] \to \mathbb{R}^m \) such that

\[
\sup_{t \in [-T,0]} |y(t)|^2 < \infty.
\]

The aim of this section is to prove the existence and uniqueness of a solution \( (Y(t), Z(t))_{t \in [0,T]} \) for the following multivalued BSDE with time delay generator (formally written as):

\[
\begin{cases}
-dY(t) + \partial \varphi (Y(t)) dt \in F(t,Y(t),Z(t),Y_t,Z_t) dt + Z(t) dW(t), & 0 \leq t \leq T, \\
Y(T) = \xi.
\end{cases}
\]

where the generator \( F \) at time \( t \in [0,T] \) depends on the past values of the solution through \( Y_t \) and \( Z_t \) defined by (2).

We mention that we will take \( Z(t) = 0 \) and \( Y(t) = Y(0) \) for any \( t < 0 \).

The following assumptions will be needed throughout this section:

(\( A_1 \)) The function \( F : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{S}^2_{-T} \times \mathcal{H}^2_{-T} \to \mathbb{R}^m \) satisfies that there exist \( L, K > 0 \) such that, for some probability measure \( \alpha \) on \([[-T,0], \mathcal{B}([-T,0])) \) and for any \( t \in [0,T] \), \((y,z), (\bar{y}, \bar{z}) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}, (y_z, z_t), (\bar{y}_z, \bar{z}_t) \in \mathcal{S}^2_{-T} \times \mathcal{H}^2_{-T} \), \( \mathbb{P}\)-a.s.

\[
\begin{align*}
(i) & \quad F(., \cdot, y, z, y, z) \text{ is } \mathcal{F}_t\text{-progressively measurable; } \\
(ii) & \quad |F(t, y, z, y_t, z_t) - F(t, \bar{y}, \bar{z}, y_t, z_t)| \leq L (|y - \bar{y}| + |z - \bar{z}|); \\
(iii) & \quad |F(t, y, z, y_t, z_t) - F(t, y, z, y, z)|^2 \leq K \int_{-T}^0 |y(t + \theta) - \bar{y}(t + \theta)|^2 \alpha(d\theta) \\
& \quad + K \int_{-T}^0 |z(t + \theta) - \bar{z}(t + \theta)|^2 \alpha(d\theta); \\
\end{align*}
\]

and

\[
(iv) \quad \mathbb{E} \left[ \int_0^T |F(t,0,0,0,0)|^2 dt \right] < \infty; \\
(v) \quad F(t, \cdot, \cdot, \cdot, \cdot) = 0, \forall t < 0.
\]
Remark 3 The condition of the measure $\alpha$ to be of probability type is taken only for the simplicity of the calculus. If $\alpha$ will be a measure with the support in $[-T,0]$ then the constants from our results will depend in addition by $\alpha ([{-T,0}])$.

$$(A_2)$$ The function $\varphi: \mathbb{R}^m \to (-\infty, +\infty]$ is proper ($\varphi \not\equiv +\infty$), convex and lower semicontinuous (l.s.c. for short) and there is no loss of generality in assuming
$$\varphi(y) \geq \varphi(0) = 0, \forall y \in \mathbb{R}^m.$$  

$$(A_3)$$ The terminal data $\xi: \Omega \to \mathbb{R}^m$ is a $\mathcal{F}_T$-measurable random variable such that
$$\mathbb{E}[||\xi||^2 + |\varphi(\xi)|] < \infty.$$  

Remark 4 The assumption $(A_1)$ means that we can extend the solution of $(5)$ for the case of $t < 0$ by taking $(Y(t),Z(t)) := (Y(0),0)$ for $t < 0$. Concerning the value $F(t,0,0,0,0)$ from the assumption $(A_1)$, we mention that this is in fact $F(t,y,z,y_t,z_t)$ considered at $y = z = 0$, $y_t \equiv 0$ and $z_t \equiv 0$.

Remark 5 As examples we can consider the following functions as generators:

$$F_1(s,y(s),z(s),y_s,z_s) := K \int_0^s z(s) ds,$$

$$F_2(s,y(s),z(s),y_s,z_s) := Kz(s-r), \forall s \in [0,T],$$

where $r$ is a fixed time delay, 

or, more general, the linear time delayed generator

$$F(s,y(s),z(s),y_s,z_s) := \int_{-T}^0 g(s+\theta)z(s+\theta)\alpha(d\theta),$$

where $g: [0,T] \to \mathbb{R}$ is a measurable and uniformly bounded function with $g(t) = 0$ for $t < 0$.

The subdifferential operator $\partial \varphi$ is defined by

$$\partial \varphi(y) := \{ y^* \in \mathbb{R}^m : \langle y^*, v - y \rangle + \varphi(y) \leq \varphi(v), \forall v \in \mathbb{R}^m \}$$

and by $(y,y^*) \in \partial \varphi$ we understand that $y \in \text{Dom}(\partial \varphi)$ and $y^* \in \partial \varphi(y)$, where

$$\text{Dom}(\partial \varphi) := \{ y \in \mathbb{R}^m : \partial \varphi(y) \not= \emptyset \}.$$  

We know that that, if $m = 1$, then in every $y \in \text{Dom}(\varphi) := \{ y \in \mathbb{R} : \varphi(y) < +\infty \}$ we have

$$\partial \varphi(y) = [\varphi'_-(y), \varphi'_+(y)] \cap \mathbb{R},$$

where $\varphi'_-$ and $\varphi'_+$ are respectively the left and the right derivative.

Remark 6 It is known that the subdifferential operator $\partial \varphi$ is a maximal monotone operator, i.e. is maximal in the class of operators which satisfy the condition

$$\langle y^* - z^*, y - z \rangle \geq 0, \forall (y,y^*) , (z,z^*) \in \partial \varphi.$$  

Conversely, in the case $m = 1$, we recall that, if $A$ is a given maximal monotone operator on $\mathbb{R}$, then there exists a proper l.s.c. convex function $\psi$ such that $A = \partial \psi$; Hence equation $(5)$ is equivalent in this case with the study of the equation:

$$\begin{cases} -dY(t) + A(Y(t)) dt \ni F(t,Y(t),Z(t),Y_1,Z_1) dt + Z(t) dW(t), & 0 \leq t \leq T, \\ Y(T) = \xi. \end{cases}$$

We mention that in the case $m \geq 2$ the problem of the existence of a solution for $(6)$ is an open problem.
Definition 7 The triple \((Y, Z, U)\) is a solution of time-delayed multivalued BSDE (5) if

\( (i) \ (Y, Z, U) \in \mathcal{S}_{T}^{2,m} \times \mathcal{U}_{T}^{2,m×d} \times \mathcal{U}_{T}^{2,m}, \)

\( (ii) \ \mathbb{E} \left[ \int_{0}^{T} \varphi (Y(t)) \, dt \right] < \infty, \)

\( (iii) \ (Y(t), U(t)) \in \partial \varphi, \ \mathbb{P} (d\omega) \otimes dt, \) a.e. on \( \Omega \times [0, T], \)

\( (iv) \ Y(t) + \int_{t}^{T} U(s) \, ds = \xi + \int_{t}^{T} F(s, Y(s), Z(s), Y_s, Z_s) \, ds - \int_{t}^{T} Z(s) dW(s), \)

\[ \forall t \in [0, T], \ \text{a.s.} \]

Remark 8 It is easy to show that if \((Y, Z) \in \mathcal{S}_{T}^{2,m} \times \mathcal{U}_{T}^{2,m×d}\) then the generator is well defined and \(\mathbb{P}\)-integrable, since the following inequality holds true:

\[ \int_{0}^{T} |F(s, Y(s), Z(s), Y_s, Z_s)|^2 \, ds \leq 3 (2L^2 + K) T \sup_{t \in [0, T]} |Y(s)|^2 + 3 (2L^2 + K) \int_{0}^{T} |Z(s)|^2 \, ds \]

\[ + 3 \int_{0}^{T} |F(s, 0, 0, 0, 0)|^2 \, ds. \quad (8) \]

Indeed (see also Lemma 1.1 in \([5]\)), from Assumption (A1) and Fubini’s theorem

\[ \int_{0}^{T} |F(s, Y(s), Z(s), Y_s, Z_s)|^2 \, ds \leq 3 \int_{0}^{T} |F(s, Y(s), Z(s), Y_s, Z_s) - F(s, 0, 0, Y_s, Z_s)|^2 \, ds \]

\[ + 3 \int_{0}^{T} |F(s, 0, 0, Y_s, Z_s) - F(s, 0, 0, 0, 0)|^2 \, ds + 3 \int_{0}^{T} |F(s, 0, 0, 0, 0)|^2 \, ds. \quad (9) \]

\[ \leq 6L^2 \int_{0}^{T} \left( |Y(s)|^2 + |Z(s)|^2 \right) \, ds + 3K \int_{0}^{T} \int_{-T}^{T} \left( Y(s + \theta)|^2 + Z(s + \theta)|^2 \right) \alpha (d\theta) \, ds \]

\[ + 3 \int_{0}^{T} |F(s, 0, 0, 0, 0)|^2 \, ds. \]

The conclusion follows now since we have

\[ \int_{0}^{T} \int_{-T}^{0} \left( |Y(s + \theta)|^2 + |Z(s + \theta)|^2 \right) \alpha (d\theta) \, ds = \int_{-T}^{T} \int_{0}^{T} \left( |Y(s + \theta)|^2 + |Z(s + \theta)|^2 \right) d\omega (d\theta) \]

\[ = \int_{-T}^{T} \int_{0}^{T} \left( |Y(s)|^2 + |Z(s)|^2 \right) d\omega (d\theta) \leq \int_{-T}^{T} \int_{0}^{T} \left( |Y(s)|^2 + |Z(s)|^2 \right) d\omega (d\theta) \]

\[ = \int_{0}^{T} \left( |Y(s)|^2 + |Z(s)|^2 \right) \, ds. \]

Throughout this section \(C\) will designate a constant (possible depending on \(L\)) which may vary from line to line.

In order to obtain the uniqueness of the solution we will prove the next a priori estimate.

Proposition 9 Let assumptions (A1–A3) be satisfied. Let \((Y, Z, U), (\bar{Y}, \bar{Z}, \bar{U}) \in \mathcal{S}_{T}^{2,m} \times \mathcal{U}_{T}^{2,m×d} \times \mathcal{U}_{T}^{2,m}\) be the solutions of (5) corresponding to \((\xi, F)\) and \((\bar{\xi}, \bar{F})\) respectively. If time horizon \(T\) and Lipschitz constant \(K\) are small enough such that \(K e^{\beta T} < 2L^2\), then there exists some constants
$C_1 = C_1(L) > 0$ and $C_2 = C_2(L) > 0$, independent of $K$ and $T$, such that

$$
\|Y - \tilde{Y}\|_{\mathbb{S}^2_{\mathbb{H}^m}}^2 + \|Z - \tilde{Z}\|_{\mathbb{H}^2_{\mathbb{H}^m}}^2 \leq C_1 e^{C_2 T} E \left[ \xi - \bar{\xi} \right]^2 \\
+ \int_0^T |F(s, Y(s), Z(s), Y_s, Z_s) - \tilde{F}(s, Y(s), Z(s), Y_s, Z_s)|^2 ds.
$$

**Proof.** We define first, for all $t \leq T$,

$$
\Delta Y(t) := Y(t) - \tilde{Y}(t), \quad \Delta Z(t) := Z(t) - \tilde{Z}(t), \quad \Delta U(t) := U(t) - \bar{U}(t), \quad \Delta \xi := \xi - \bar{\xi}
$$

and

$$
\Delta F(t, Y(t), Z(t), Y_t, Z_t) := F(t, Y(t), Z(t), Y_t, Z_t) - \tilde{F}(t, Y(t), Z(t), Y_t, Z_t).
$$

Applying Itô’s formula we deduce that

$$
e^{\beta t} |\Delta Y(t)|^2 + \int_t^T \beta e^{\beta s} |\Delta Y(s)|^2 ds + 2 \int_t^T e^{\beta s} \langle \Delta Y(s), \Delta U(s) \rangle ds + \int_t^T e^{\beta s} |\Delta Z(s)|^2 ds
$$

$$
= e^{\beta T} |\Delta \xi|^2 + 2 \int_t^T e^{\beta s} \langle \Delta Y(s), F(s, Y(s), Z(s), Y_s, Z_s) - \tilde{F}(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \rangle ds \\
- 2 \int_t^T e^{\beta s} \langle \Delta Y(s), \Delta Z(s) dW(s) \rangle,
$$

for any $\beta > 0$ (which will be chosen later).

Since $(Y(t), U(t))$, $(\tilde{Y}(t), \bar{U}(t)) \in \partial \varphi$,

$$
(\Delta Y(s), \Delta U(s)) \geq 0.
$$

Using Young’s inequality and the assumption on $\bar{F}$, we see that

$$
2 \int_t^T e^{\beta s} \langle \Delta Y(s), F(s, Y(s), Z(s), Y_s, Z_s) - \tilde{F}(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \rangle ds
$$

$$
\leq a \int_t^T e^{\beta s} |\Delta Y(s)|^2 ds + \frac{1}{a} \int_t^T e^{\beta s} \left| F(s, Y(s), Z(s), Y_s, Z_s) - \tilde{F}(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \right|^2 ds
$$

$$
\leq a \int_t^T e^{\beta s} |\Delta Y(s)|^2 ds + \frac{a}{\bar{a}} \int_t^T e^{\beta s} |\Delta F(t, Y(t), Z(t), Y_t, Z_t)|^2 ds
$$

$$
+ \frac{\bar{a}}{a} \int_t^T e^{\beta s} \left| \tilde{F}(s, \tilde{Y}(s), \tilde{Z}(s), Y_s, Z_s) - \tilde{F}(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \right|^2 ds
$$

$$
+ \frac{\bar{a}}{a} \int_t^T e^{\beta s} \left| \tilde{F}(s, \tilde{Y}_s, \tilde{Z}_s, Y_s, Z_s) - \tilde{F}(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \right|^2 ds
$$

$$
\leq a \int_t^T e^{\beta s} |\Delta Y(s)|^2 ds + \frac{\bar{a}}{a} \int_t^T e^{\beta s} |\Delta F(t, Y(t), Z(t), Y_t, Z_t)|^2 ds
$$

$$
+ \frac{\bar{a}}{a} \int_t^T e^{\beta s} \left( |\Delta Y(s)|^2 + |\Delta Z(s)|^2 \right) ds + \frac{\bar{a}}{a} \int_t^T \int_0^T e^{\beta s} \left( |\Delta Y(s + \theta)|^2 + |\Delta Z(s + \theta)|^2 \right) \alpha(d\theta) ds,
$$

for any $a > 0$ (which will be chosen later).
But (see also the proof of (8))
\[
\int_t^T \int_{-T}^0 e^{\beta s} \left( |\Delta Y (s + \theta)|^2 + |\Delta Z (s + \theta)|^2 \right) \alpha (d\theta) \, ds \\
= \int_{-T}^0 e^{-\beta \theta} \int_t^{T+\theta} e^{\beta s} \left( |\Delta Y (s)|^2 + |\Delta Z (s)|^2 \right) \, ds \alpha (d\theta) \\
\leq \int_{-T}^0 e^{-\beta \theta} \int_0^T e^{\beta s} \left( |\Delta Y (s)|^2 + |\Delta Z (s)|^2 \right) \, ds \alpha (d\theta) \\
\leq e^{\beta T} \int_0^T e^{\beta s} \left( |\Delta Y (s)|^2 + |\Delta Z (s)|^2 \right) \, ds. 
\]

Therefore inequality (10) becomes
\[
e^{\beta t} |\Delta Y (t)|^2 + \beta \int_t^T e^{\beta s} |\Delta Y (s)|^2 \, ds + \int_t^T e^{\beta s} |\Delta Z (s)|^2 \, ds \\
\leq e^{\beta T} |\Delta \xi|^2 + \frac{3}{4} \int_t^T e^{\beta s} |\Delta F(t, Y(t), Z(t), Y_t, Z_t)|^2 \, ds \\
+ \left( a + \frac{6L^2}{a} \right) \int_t^T e^{\beta s} |\Delta Y (s)|^2 \, ds + \frac{6L^2}{a} \int_t^T e^{\beta s} |\Delta Z (s)|^2 \, ds \\
+ \frac{3K e^{\beta T}}{a} \int_0^T e^{\beta s} \left( |\Delta Y (s)|^2 + |\Delta Z (s)|^2 \right) \, ds - 2 \int_t^T e^{\beta s} \langle \Delta Y (s), \Delta Z (s) dW (s) \rangle.
\]

For \( t = 0 \) we see that
\[
\beta E \int_0^T e^{\beta s} |\Delta Y (s)|^2 \, ds + E \int_0^T e^{\beta s} |\Delta Z (s)|^2 \, ds \\
\leq E \left( e^{\beta T} |\Delta \xi|^2 \right) + \frac{3}{4} E \int_0^T e^{\beta s} |\Delta F(t, Y(t), Z(t), Y_t, Z_t)|^2 \, ds \\
+ \left( a + \frac{6L^2}{a} + \frac{3K e^{\beta T}}{a} \right) E \int_0^T e^{\beta s} |\Delta Y (s)|^2 \, ds + \left( \frac{6L^2}{a} + \frac{3K e^{\beta T}}{a} \right) E \int_0^T e^{\beta s} |\Delta Z (s)|^2 \, ds
\]

and, choosing \( a = 24L^2, \beta \geq 24L^2 + 1 \) and \( K \) and \( T \) sufficiently small such that
\[
K e^{\beta T} < 2L^2,
\]

we deduce that
\[
\frac{1}{2} E \int_0^T e^{\beta s} |\Delta Y (s)|^2 \, ds + \frac{1}{2} E \int_0^T e^{\beta s} |\Delta Z (s)|^2 \, ds \\
\leq \max \{1, \frac{1}{2e^{\beta T}}\} E \left[ e^{\beta T} |\Delta \xi|^2 + \int_0^T e^{\beta s} |\Delta F(t, Y(t), Z(t), Y_t, Z_t)|^2 \, ds \right]. 
\]

Applying Burkholder–Davis–Gundy’s inequality and once again Young’s inequality we can assert that
\[
2E \left[ \sup_{t \in [0, T]} \left| \int_t^T \langle \Delta Y (s), \Delta Z (s) dW (s) \rangle \right| \right] \leq 4E \left[ \sup_{t \in [0, T]} \int_t^T \langle \Delta Y (s), \Delta Z (s) dW (s) \rangle \right] \\
\leq 12E \left[ \int_0^T |\Delta Y (s)|^2 |\Delta Z (s)|^2 \, ds \right]^{1/2} \leq \frac{1}{2} E \left[ \sup_{t \in [0, T]} |\Delta Y (t)|^2 \right] + 72E \int_0^T |\Delta Z (s)|^2 \, ds,
\]
hence inequality (12) gives
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} e^{\beta t} |\Delta Y(t)|^2 \right] \\
\leq C \max \left\{ 1, \frac{1}{\beta} \right\} \mathbb{E} \left[ e^{\beta T} |\Delta Y|^2 + \int_0^T e^{\beta s} |\Delta F(t, Y(t), Z(t), Y_t, Z_t)|^2 \, ds \right]
\]
for some \( C > 0 \) independent of \( L, K \) and \( T \).

The main result of this section is given by

\[ \text{Theorem 10} \quad \text{Let assumptions (A1–A3) be satisfied. If time horizon } T \text{ and Lipschitz constant } K \text{ are small enough, then there exists a unique solution } (Y, Z, U) \text{ of (5).} \]

\section{Proof of the main result}

In order to prove the existence of the solution for (5) we shall consider the Yosida approximation of the multivalued operator \( \partial \varphi \). Set, for \( \epsilon > 0 \), the convex function \( \varphi_{\epsilon} \) of class \( C^1 \)

\[ \varphi_{\epsilon}(y) := \inf \left\{ \frac{1}{2\epsilon} |y - v|^2 + \varphi(v) : v \in \mathbb{R}^m \right\} \]

with the gradient being a \( 1/\epsilon \)-Lipschitz function.

If \( J_{\epsilon} y := y - \epsilon \nabla \varphi_{\epsilon}(y) \) then we can deduce the following properties (see [1]):

(i) \( \varphi_{\epsilon}(y) = \frac{1}{2\epsilon} |y - J_{\epsilon} y|^2 + \varphi(J_{\epsilon} y) \),  
(ii) \( \varphi_{\epsilon}(y) \leq \varphi(y) \), 
(iii) \( |J_{\epsilon} y - J_{\epsilon} \bar{y}| \leq |y - \bar{y}| \), 
(iv) \( \nabla \varphi_{\epsilon}(y) \in \partial \varphi(J_{\epsilon} y) \), 
(v) \( 0 \leq \varphi_{\epsilon}(y) \leq \langle y, \nabla \varphi_{\epsilon}(y) \rangle \), 
(vi) \( \langle \nabla \varphi_{\epsilon}(y) - \nabla \varphi_{\epsilon}(\bar{y}), y - \bar{y} \rangle \geq -(\epsilon + \delta) \langle \nabla \varphi_{\epsilon}(y), \nabla \varphi_{\epsilon}(\bar{y}) \rangle \),

for all \( \epsilon, \delta > 0, y, \bar{y} \in \mathbb{R}^m \).

**Proof of Theorem 10.** The uniqueness is an immediate consequence of Proposition 9.

Let now \( \epsilon > 0 \). We consider the approximating BSDE with time delayed generator:

\[ Y_{\epsilon}^t(t) + \int_t^T \nabla \varphi_{\epsilon}(Y_{\epsilon}^s(s)) \, ds = \xi + \int_t^T F(s, Y_{\epsilon}^s(s), Z_{\epsilon}^s(s), Y_{\epsilon}^s, Z_{\epsilon}^s) \, ds - \int_t^T Z_{\epsilon}^s(s) \, dW(s), \]

\[ 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \]

Since \( \nabla \varphi_{\epsilon} : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is \( 1/\epsilon \)-Lipschitz function, we can apply Theorem 2.1 from [3]. Hence there exists a unique solution \( (Y_{\epsilon}, Z_{\epsilon}) \in S_{T_2}^{2,m} \times H_T^{2,m \times d} \). We mention that the conclusion of Theorem 2.1 in [3] holds true even in the multidimensional case \( (m, d \geq 2) \). In addition, since the generator \( F \) is Lipschitz continuous in the variable \( Y(t) \) and \( Z(t) \), the proof of Theorem 2.1 can be easily change such that to allow to \( F \) to depend on the variable \( Y(t) \) and \( Z(t) \). Also, it can be see from the proof that the Lipschitz constant \( L \) can be chosen arbitrary.

The proof of the existence will be split into several steps which are adapted from the proof of Theorem 1.1 from [10].

A. **Boundedness of \( Y_{\epsilon}^t \) and \( Z_{\epsilon}^t \)**
We will first show the inequality
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} e^{\beta t} |Y^\epsilon(t)|^2 \right] + \mathbb{E} \int_0^T e^{\beta s} |Z^\epsilon(s)|^2 \, ds \leq C_1 e^{C_2 T} M_1 ,
\]
(16)
for some constants \( C_1 = C_1(L) > 0 \) and \( C_2 = C_2(L) > 0 \), independent of \( K, T \) and \( \epsilon \), and for any \( \beta > 0 \) sufficiently large, where
\[
M_1 := \mathbb{E} \left[ |\xi|^2 + \int_0^T e^{\beta s} |F(t,0,0,0)|^2 \, ds \right].
\]
Indeed, from Itô’s formula we have, for \( \beta > 0 \) arbitrarily chosen,
\[
e^{\beta t} |Y^\epsilon(t)|^2 + \int_t^T \beta e^{\beta s} |Y^\epsilon(s)|^2 \, ds + 2 \int_t^T e^{\beta s} \langle Y^\epsilon(s), \nabla \varphi_\epsilon(Y^\epsilon(s)) \rangle \, ds + \int_t^T e^{\beta s} |Z^\epsilon(s)|^2 \, ds
\]
\[
e^{\beta t} |\xi|^2 + 2 \int_t^T e^{\beta s} \langle Y^\epsilon(s), F(s,Y^\epsilon(s),Z^\epsilon(s),Y^\epsilon_s,Z^\epsilon_s) \rangle \, ds
\]
\[
-2 \int_t^T e^{\beta s} \langle Y^\epsilon(s), Z^\epsilon(s) \rangle \, dW(s) \rangle .
\]
From (14–v)
\[
\langle Y^\epsilon(s), \nabla \varphi_\epsilon(Y^\epsilon(s)) \rangle \geq 0.
\]
Using Young’s inequality and the assumption on \( F \), we obtain (see also the calculus in (11)), for \( a > 0 \) arbitrarily chosen,
\[
2 \int_t^T e^{\beta s} \langle Y^\epsilon(s), F(s,Y^\epsilon(s),Z^\epsilon(s),Y^\epsilon_s,Z^\epsilon_s) \rangle \, ds
\]
\[
\leq \left( a + \frac{6L^2}{a} \right) \int_t^T e^{\beta s} |Y^\epsilon(s)|^2 \, ds + \frac{3}{a} \int_t^T e^{\beta s} |F(t,0,0,0)|^2 \, ds
\]
\[
+ \frac{6L^2}{a} \int_t^T e^{\beta s} |Z^\epsilon(s)|^2 \, ds + \frac{3Ke^{\beta T}}{a} \int_0^T e^{\beta s} \left( |Y^\epsilon(s)|^2 + |Z^\epsilon(s)|^2 \right) \, ds.
\]
Therefore inequality (17) becomes
\[
e^{\beta t} |Y^\epsilon(t)|^2 + \int_t^T \beta e^{\beta s} |Y^\epsilon(s)|^2 \, ds + \int_t^T e^{\beta s} |Z^\epsilon(s)|^2 \, ds
\]
\[
\leq e^{\beta t} |\xi|^2 + \frac{3}{a} \int_t^T e^{\beta s} |F(t,0,0,0)|^2 \, ds + \left( a + \frac{6L^2}{a} \right) \int_t^T e^{\beta s} |Y^\epsilon(s)|^2 \, ds
\]
\[
+ \frac{6L^2}{a} \int_t^T e^{\beta s} |Z^\epsilon(s)|^2 \, ds + \frac{3Ke^{\beta T}}{a} \int_0^T e^{\beta s} \left( |Y^\epsilon(s)|^2 + |Z^\epsilon(s)|^2 \right) \, ds
\]
\[
-2 \int_t^T e^{\beta s} \langle Y^\epsilon(s), Z^\epsilon(s) \rangle \, dW(s)
\]
and for \( t = 0 \) it follows that
\[
\left( \beta - a - \frac{6L^2}{a} - \frac{3Ke^{\beta T}}{a} \right) \int_0^T e^{\beta s} |Y^\epsilon(s)|^2 \, ds + \left( 1 - \frac{6L^2}{a} - \frac{3Ke^{\beta T}}{a} \right) \int_0^T e^{\beta s} |Z^\epsilon(s)|^2 \, ds
\]
\[
\leq \mathbb{E} \left( e^{\beta T} |\xi|^2 \right) + \frac{3}{a} \mathbb{E} \int_t^T e^{\beta s} |F(t,0,0,0)|^2 \, ds.
\]
Choosing again $a = 24L^2$, $\beta \geq 24L^2 + 1$ and $K$ and $T$ sufficiently small such that 

$$Ke^{\beta T} < 6L^2,$$  \hfill (19)

we see that

$$\frac{1}{2} \mathbb{E} \int_0^T e^{\beta s} |Y^\epsilon (s)|^2 \, ds + \frac{1}{2} \mathbb{E} \int_0^T e^{\beta s} |Z^\epsilon (s)|^2 \, ds \leq \max \{1, \frac{1}{2T}\} \mathbb{E} \left[ e^{\beta T} |\xi|^2 + \int_t^T e^{\beta s} |F(t, 0, 0, 0)|^2 \, ds \right].$$  \hfill (20)

We apply Burkholder–Davis–Gundy’s inequality and we deduce the following inequality:

$$2 \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_t^T e^{\beta s} \langle Y^\epsilon (s), Z^\epsilon (s) dW(s) \rangle \right| \right] \leq 12 \mathbb{E} \left[ \int_0^T e^{\beta s} |Y^\epsilon (s)|^2 |Z^\epsilon (s)|^2 \, ds \right]^{1/2}$$

$$\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, T]} e^{\beta t} |Y^\epsilon (t)|^2 \right] + 72 \mathbb{E} \int_0^T e^{\beta s} |Z^\epsilon (s)|^2 \, ds.$$

Hence, taking sup in (18) we deduce inequality (16).

B. Boundedness of $\nabla \varphi_r (Y^\epsilon)$

We will prove that there exists some constants $C_1 = C_1 (L) > 0$ and $C_2 = C_2 (L) > 0$, independent of $K$, $T$ and $\epsilon$, such that

$$\mathbb{E} \left[ \int_0^T e^{\beta s} |\nabla \varphi_r (Y^\epsilon (s))|^2 \right] ds \leq C_1 e^{C_2 T} M_2,$$  \hfill (21a)

$$\mathbb{E} \left[ e^{\beta t} \varphi_r (J_\epsilon (Y^\epsilon (t))) \right] \leq \mathbb{E} \left[ e^{\beta s} \varphi_r (J_\epsilon (Y^\epsilon (s))) \right] ds \leq C_1 e^{C_2 T} M_2,$$  \hfill (21b)

$$\mathbb{E} \left[ e^{\beta t} |Y^\epsilon (t) - J_\epsilon (Y^\epsilon (t))|^2 \right] \leq \epsilon C_1 e^{C_2 T} M_2,$$  \hfill (21c)

for any $\beta > 0$ sufficiently large, where

$$M_2 := \mathbb{E} \left[ |\xi|^2 + \varphi (\xi) + \int_0^T |F(t, 0, 0, 0)|^2 \, ds \right].$$

Essential for the proof of this part is stochastic subdifferential inequality (2.8) from [10]:

$$e^{\beta T} \varphi_r (\xi) \geq e^{\beta t} \varphi_r (Y^\epsilon (t)) + \int_t^T e^{\beta s} \langle \nabla \varphi_r (Y^\epsilon (s)), dY^\epsilon (s) \rangle + \int_t^T \nabla \varphi_r (Y^\epsilon (s)) \, d(e^{\beta s}).$$

Therefore

$$e^{\beta t} \varphi_r (Y^\epsilon (t)) + \beta \int_t^T e^{\beta s} \varphi_r (Y^\epsilon (s)) \, ds + \int_t^T e^{\beta s} |\nabla \varphi_r (Y^\epsilon (s))|^2 \, ds \leq e^{\beta T} \varphi_r (\xi)$$

$$\quad + \int_t^T e^{\beta s} \langle \nabla \varphi_r (Y^\epsilon (s)), F(s, Y^\epsilon (s), Z^\epsilon (s), Y^\epsilon_s, Z^\epsilon_s) \rangle \, ds - \int_t^T e^{\beta s} \langle \nabla \varphi_r (Y^\epsilon (s)), Z^\epsilon (s) dW(s) \rangle.$$  \hfill (22)

To obtain (21–a) it is sufficient to use (16), inequality

$$\mathbb{E} \int_t^T e^{\beta s} \langle \nabla \varphi_r (Y^\epsilon (s)), F(s, Y^\epsilon (s), Z^\epsilon (s), Y^\epsilon_s, Z^\epsilon_s) \rangle \, ds$$

$$\leq \frac{1}{2} \mathbb{E} \int_t^T e^{\beta s} |\nabla \varphi_r (Y^\epsilon (s))|^2 \, ds + \frac{3}{2} \mathbb{E} \int_t^T e^{\beta s} |F(t, 0, 0, 0)|^2 \, ds$$

$$\quad + \frac{3}{2} (2L^2 + K e^{\beta T}) \int_t^T e^{\beta s} \left( |Y^\epsilon (s)|^2 + |Z^\epsilon (s)|^2 \right) \, ds,$$ 

10
assumption (19) and (14–ii).

Using, in addition, inequality (14–i) we see that

\[
\frac{1}{2c}e^{\beta t}|Y^\epsilon(t) - J_\epsilon(Y^\epsilon(t))| + E\left[e^{\beta s} (J_\epsilon(Y^\epsilon(t)))\right] + \beta \int_t^T e^{\beta s} \phi(J_\epsilon(Y^\epsilon(s))) \, ds \leq C_1 e^{C_2T} M_2,
\]

which is (21–b, c).

C. Cauchy sequences and convergence

The next step is to prove that there exists some constants \(C_1 = C_1(L) > 0\) and \(C_2 = C_2(L) > 0\), independent of \(K\), \(T\) and \(\epsilon\), such that

\[
E \left[ \sup_{t \in [0,T]} e^{\beta t}|Y^\epsilon(t) - Y^\delta(t)|^2 \right] + E \int_0^T e^{\beta s}|Z^\epsilon(s) - Z^\delta(s)|^2 \, ds \leq C_1 e^{C_2T} (\epsilon + \delta) M_2,
\]

for any \(\beta > 0\) sufficiently large.

Applying Itô’s formula we deduce that

\[
e^{\beta t}|Y^\epsilon(t) - Y^\delta(t)|^2 + \beta \int_t^T e^{\beta s}|Y^\epsilon(s) - Y^\delta(s)|^2 \, ds + \int_t^T e^{\beta s}|Z^\epsilon(s) - Z^\delta(s)|^2 \, ds
\]

\[
+ 2\int_t^T e^{\beta s}(Y^\epsilon(s) - Y^\delta(s), \nabla \phi_e(Y^\epsilon(s)) - \nabla \phi_\delta(Y^\delta(s))) \, ds
\]

\[
= 2\int_t^T e^{\beta s}(Y^\epsilon(s) - Y^\delta(s), F(s, Y^\epsilon(s), Z^\epsilon(s), Y^\epsilon_s, Z^\epsilon_s) - F(s, Y^\delta(s), Z^\delta(s), Y^\delta_s, Z^\delta_s)) \, ds
\]

\[
- 2\int_t^T e^{\beta s}(Y^\epsilon(s) - Y^\delta(s), (Z^\epsilon(s) - Z^\delta(s))dW(s)).
\]

Since (14–vi),

\[
\langle Y^\epsilon(s) - Y^\delta(s), \nabla \phi_e(Y^\epsilon(s)) - \nabla \phi_\delta(Y^\delta(s)) \rangle \geq - (\epsilon + \delta) |\nabla \phi_e(Y^\epsilon(s))| |\nabla \phi_\delta(Y^\delta(s))|.
\]

Using the assumption on \(F\) we obtain

\[
2\int_t^T e^{\beta s}(Y^\epsilon(s) - Y^\delta(s), F(s, Y^\epsilon(s), Z^\epsilon(s), Y^\epsilon_s, Z^\epsilon_s) - F(s, Y^\delta(s), Z^\delta(s), Y^\delta_s, Z^\delta_s)) \, ds
\]

\[
\leq (16L^2 + 4)\int_t^T e^{\beta s}|Y^\epsilon(s) - Y^\delta(s)|^2 \, ds + \frac{1}{4}\int_t^T e^{\beta s}|Z^\epsilon(s) - Z^\delta(s)|^2 \, ds
\]

\[
+ \frac{K_0 e^{\beta T}}{s} \int_0^T e^{\beta s} (|Y^\epsilon(s) - Y^\delta(s)|^2 + |Z^\epsilon(s) - Z^\delta(s)|^2) \, ds
\]

hence (24) becomes

\[
e^{\beta t}|Y^\epsilon(t) - Y^\delta(t)|^2 + \left(\beta - 16L^2 - \frac{1}{4}\right)\int_t^T e^{\beta s}|Y^\epsilon(s) - Y^\delta(s)|^2 \, ds
\]

\[
+ \frac{3}{4}\int_t^T e^{\beta s}|Z^\epsilon(s) - Z^\delta(s)|^2 \, ds
\]

\[
\leq 2(\epsilon + \delta)\int_t^T e^{\beta s} |\nabla \phi_e(Y^\epsilon(s))| |\nabla \phi_\delta(Y^\delta(s))| \, ds
\]

\[
+ \frac{K_0 e^{\beta T}}{s} \int_0^T e^{\beta s} (|Y^\epsilon(s) - Y^\delta(s)|^2 + |Z^\epsilon(s) - Z^\delta(s)|^2) \, ds
\]

\[
- 2\int_t^T e^{\beta s}(Y^\epsilon(s) - Y^\delta(s), (Z^\epsilon(s) - Z^\delta(s))dW(s)).
\]
Using (21–a) we see that
\[
\left(\beta - 16L^2 - \frac{1}{4} - \frac{K\epsilon^T}{8L^2}\right) \mathbb{E} \int_0^T e^{\beta s} |Y^\epsilon (s) - Y^\delta (s)|^2 ds + \left(\frac{3}{4} - \frac{K\epsilon^T}{8L^2}\right) \mathbb{E} \int_0^T e^{\beta s} |Z^\epsilon (s) - Z^\delta (s)|^2 ds \\
\leq (\epsilon + \delta) C_1 e^{C_2 T} M_2.
\]

Therefore, for $\beta$ large enough and for $K, T$ sufficiently small we deduce that
\[
\mathbb{E} \int_0^T e^{\beta s} |Y^\epsilon (s) - Y^\delta (s)|^2 ds + \mathbb{E} \int_0^T e^{\beta s} |Z^\epsilon (s) - Z^\delta (s)|^2 ds \leq (\epsilon + \delta) C_1 e^{C_2 T} M_2.
\]

(26)

From Burkholder–Davis–Gundy’s inequality we can infer that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \int_t^T e^{\beta s} \langle Y^\epsilon (s) - Y^\delta (s), (Z^\epsilon (s) - Z^\delta (s)) \rangle dW (s) \right] \\
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} e^{\beta t} |Y^\epsilon (t) - Y^\delta (t)|^2 \right] + 72 \mathbb{E} \int_0^T e^{\beta s} |Z^\epsilon (s) - Z^\delta (s)|^2 ds
\]
and therefore inequality (23) follows.

D. Passage to the limit

The solution will obtain as the limit of the approximating sequence $(Y^\epsilon, Z^\epsilon, \nabla \varphi_\epsilon (Y^\epsilon))$.

From Proposition 23 we see that there exist $Y \in \mathcal{S}_T^{2,m}$ and $Z \in \mathcal{H}_T^{2,m \times d}$ such that
\[
\lim_{\epsilon \searrow 0} Y^\epsilon = Y \quad \text{in} \quad \mathcal{S}_T^{2,m} \quad \text{and} \quad \lim_{\epsilon \searrow 0} Z^\epsilon = Z \quad \text{in} \quad \mathcal{H}_T^{2,m \times d}.
\]

Moreover, there exists a subsequence $\epsilon_n \searrow 0$ such that $\mathbb{P}$-a.s.
\[
\sup_{t \in [0,T]} |Y^{\epsilon_n} (t) - Y (t)| \to 0,
\]
\[
\int_0^T |Z^{\epsilon_n} (t) - Z (t)| dt \to 0.
\]

In addition, the passage to the limit in (16) gives
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y (t)|^2 \right] + \mathbb{E} \int_0^T |Z (s)|^2 ds \leq C_1 e^{C_2 T} M_1.
\]

Inequality (21–a) implies that there exists $U \in \mathcal{H}_T^{2,m}$ such that for a subsequence $\epsilon_n \searrow 0$,
\[
\nabla \varphi_{\epsilon_n} (Y^{\epsilon_n} (s)) \to U, \quad \text{weakly in Hilbert space} \ \mathcal{H}_T^{2,m}
\]
and then
\[
\mathbb{E} \int_0^T |U (s)|^2 ds \leq \liminf_{n \to \infty} \mathbb{E} \int_0^T |U^{\epsilon_n} (s)|^2 ds \leq c_1 e^{C_2 T} M_2.
\]

From (21–a, c) we see that
\[
\lim_{\epsilon \searrow 0} J_\epsilon (Y^\epsilon) = Y \quad \text{in} \quad \mathcal{H}_T^{2,m}
\]
(27)

and
\[
\lim_{\epsilon \searrow 0} \mathbb{E} \left[ |J_\epsilon (Y^\epsilon (t)) - Y (t)|^2 \right] = 0, \quad \forall t \in [0,T].
\]

(28)
Using Fatou’s Lemma, (21–b) and the lower semicontinuity of $\varphi$ we deduce that
\[ \mathbb{E} \int_0^T \varphi(Y(t)) \, dt + \mathbb{E} \left[ \varphi(Y(t)) \right] \leq c_1 e^{c_2 T} M_2. \]
Since $U^* := \nabla \varphi_c(Y^*(t)) \in \partial \varphi_c(J_c(Y^*(t)))$, for any $t$,
\[ U^*(t)(V(t) - J_c(Y^*(t))) + \varphi_c(J_c(Y^*(t))) \leq \varphi(V(t)), \] for all $V \in H_T^{2,m}$, $t \in [0, T]$,

hence for all $A \times [a, b] \subset \Omega \times [0, T]$
\[ \mathbb{E} \left( \int_a^b 1_A U^*(t)(V(t) - J_c(Y^*(t))) \, dt \right) + \mathbb{E} \left( \int_a^b 1_A \varphi_c(J_c(Y^*(t))) \, dt \right) \leq \mathbb{E} \left( \int_a^b 1_A \varphi(V(t)) \, dt \right). \]
But $\varphi$ is a proper convex l.s.c. function, hence passing to the lim inf and using (27) and (28) we deduce that
\[ \mathbb{E} \left( \int_a^b 1_A U(t)(V(t) - Y(t)) \, dt \right) + \mathbb{E} \left( \int_a^b 1_A \varphi(Y(t)) \, dt \right) \leq \mathbb{E} \left( \int_a^b 1_A \varphi(V(t)) \, dt \right), \]
for all $A \times [a, b] \subset \Omega \times [0, T]$,

which means that
\[ U(t)(V(t) - Y(t)) + \varphi(Y(t)) \leq \varphi(V(t)) \, dP \otimes dt \quad \text{a.e. on } \Omega \times [0, T]. \]

Therefore the property (7–iii) is obtained.

Finally, passing to the limit in (15) an using also the inequality
\[ \left| \int_0^T [F(s, Y^*(s), Z^*(s), Y^*_s, Z^*_s) - F(s, Y(s), Z(s), Y_s, Z_s)] \, ds \right|^2 \leq 2 (2L^2 + K) T \sup_{t \in [0, T]} |Y^*(t) - Y(t)|^2 + 2 (2L^2 + K) \int_0^T |Z^*(t) - Z(t)|^2 \, dt, \]
we deduce that the triple $(Y, Z, U)$ satisfy equation (7–iv).

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