A SYMPLECTIC LOCAL DISCONTINUOUS GALERKIN METHOD FOR STOCHASTIC SCHRÖDINGER EQUATION

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Abstract. In this paper, we propose a new symplectic local discontinuous Galerkin method to numerically solve stochastic Schrödinger equation with multiplicative noise. The method is based on an implicit midpoint scheme in time and local discontinuous Galerkin method in space. We show that our method preserves stochastic symplectic structure in temporal direction, and preserves the discrete charge conservation law which implies an \( L^2 \)-stability of our method. We derive the mean-square convergence order in \( L^2 \)-norm of the proposed method for the stochastic linear Schrödinger equation.

Key words. symplectic method, local discontinuous Galerkin method, stochastic Schrödinger equation, \( L^2 \)-stability, charge conservation law, error estimate

AMS subject classifications. 65M60, 35Q55, 37M15, 35R60

1. Introduction. The stochastic nonlinear Schrödinger (NLS) equation describes many physical phenomena and has important applications in fluid dynamics, quantum mechanics, and plasma physics, etc., see [11] and reference therein. Various kinds of numerical methods can be found for simulating solution of the stochastic NLS equation [1, 2, 3, 10, 14]. A Crank-Nickson type semi-discrete scheme for stochastic NLS equation in Stratonovich sense is proposed in [1], and authors also show the convergence of the discrete solution in probability sense in various topologies. Allowing sufficient spatial regularity, [3] proves that the numerical approximation for Itô equation has strong order \( \frac{1}{2} \) in general and order 1 if the noise is additive. Using the integral representation idea, [14] proposes a splitting temporal semi-discrete scheme to Stratonovich equation and obtains order 1 in the sense of probability. As the deterministic Schrödinger equation, multi-symplectic conservation law is shown to be preserved by stochastic case in Stratonovich sense in [10]. Authors also propose a finite-difference numerical method to preserve the corresponding discrete multi-symplectic conservation law.

As we all known, it is important to design a numerical method which preserves the properties of the original problems as much as possible. For Hamiltonian system, symplectic methods are shown to be superior to non-symplectic ones especially in long time computation, owing to their preservation of the qualitative property, the symplecticity of the underlying continuous system (see [15] for the study of stochastic symplecticity). In order to inherit the instinctive infinite-dimensional symplecticity of the stochastic Schrödinger equation in Stratonovich sense, we apply the midpoint scheme to discretize the temporal direction. It is shown that the midpoint semi-discretization not only is a symplectic method, but also possesses the discrete charge conservation law. For the spatial direction, we use local discontinuous Galerkin method to discrete the equation and obtain the full-discrete method which is called symplectic local discontinuous Galerkin method in this paper. Theoretical analysis shows that the obtained full-discrete method is \( L^2 \)-stable for the nonlinear case and preserves

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the discrete charge conservation law. To the best of our knowledge, there has been no work in the literature which studies the local discontinuous Galerkin method to the stochastic Schrödinger equation. The local discontinuous Galerkin method we discussed in this paper has several attractive properties. It can be easily designed for any order of accuracy in deterministic case. It can be used on arbitrary triangulations, thus allowing for efficient $h$ adaptivity. The method has excellent parallel efficiency. It is extremely local in data communications. Finally, it has excellent provable nonlinear stability. And there are a few numerical works in the literature for numerical simulating the deterministic Schrödinger equation based on local discontinuous Galerkin method. For more details, see [9, 12, 13, 16] and references therein.

Another goal of this work is to apply the obtained symplectic local discontinuous Galerkin method to the stochastic linear Schrödinger equation and to derive the corresponding error estimates. The key to the error estimates is the regularity properties of the numerical solution in $H^\gamma$-norm ($\gamma \geq 0$). In this paper, we obtain the regularity of the solution by first applying implicit midpoint scheme to the temporal direction of the stochastic linear Schrödinger equation. Meanwhile, we show that the semi-discrete implicit midpoint scheme is of order 1 in mean-square convergence sense. Based on the standard approximation theory of projection operator and Itô isometry, we estimate the error in spatial direction after applying the local discontinuous Galerkin method. As a result we derive the mean-square convergence order for the full-discrete symplectic local discontinuous Galerkin method in the linear case.

The rest of this paper is organized as follows. In section 2, we begin with some preliminary results about stochastic Schrödinger equation and local discontinuous Galerkin method. In section 3, we present and analyze the symplectic local discontinuous Galerkin method for the stochastic Schrödinger equation. In section 3.1, we apply implicit midpoint scheme to the equation in the temporal direction. We present a theoretical result on the symplecticity and the discrete charge conservation law. In section 3.2, we present the full discrete method and show that the discrete charge conservation law is preserved by the method, which implies the $L^2$-stability of the numerical solution. Section 4 presents the error estimate of the full-discrete local discontinuous Galerkin method for stochastic linear Schrödinger equation. Some proofs and calculations are postponed to the final appendices.

2. Preliminary results.

2.1. Stochastic nonlinear Schrödinger equation. In order to simplify the notations, in this paper we consider one-dimensional stochastic NLS equation. However, the approach and the theoretical results can be extended to the general $d$-dimensional ($d \geq 2$) problem. The one-dimensional equation with multiplicative noise is:

\begin{equation}
(2.1) \quad idu - (u_{xx} + \lambda |u|^2 u)dt = u \circ dW, \quad t \in [0,T], \quad x \in [L_f, L_r] \subset \mathbb{R},
\end{equation}

with an initial condition

\begin{equation}
(2.2) \quad u(x,0) = u_0(x)
\end{equation}

and periodic boundary conditions. Here, $u = u(x,t)$ is a complex-valued function, $W$ is a real-valued Wiener process. The $\circ$ in the last term in (2.1) means that the product is of Stratonovich type. Let $(\Omega, \mathcal{F}, P)$ be the probability space with filtration \{\mathcal{F}_t : 0 \leq t \leq T\}. Let \{\beta_k : k \in \mathbb{N}\} be a sequence of independent Brownian motions that are associated with \{\mathcal{F}_t : 0 \leq t \leq T\}. Let \{(\eta_k)_{k \in \mathbb{N}}\} be an orthonormal basis
of $L^2([L_f, L_r], \mathbb{R}), \phi \in L_2(L^2, H^\gamma)$ which is the space consisting of Hilbert-Schmidt operators from $L^2([L_f, L_r])$ into $H^\gamma([L_f, L_r])$ ($\gamma > 0$). Then

$$W(t, x, \omega) = \sum_{k=0}^{\infty} \beta_k(t, \omega) \phi e_k(x), \quad 0 \leq t \leq T, \quad x \in [L_f, L_r], \quad \omega \in \Omega,$$

is a Wiener process on the space of square integrable functions on $[L_f, L_r]$, with covariance operator $\phi \phi^*$. We will use the equivalent Itô equation. Defining the function

$$F_\phi(x) = \sum_{k=0}^{\infty} (\phi e_k(x))^2, \quad x \in [L_f, L_r]$$

which does not depend on the basis $(e_k)_{k \in \mathbb{N}}$, this equivalent Itô equation may be written as

$$idu - (u_{xx} + \lambda|u|^{2\sigma}u)dt = udW - \frac{i}{2} u F_\phi dt.$$

The well-posedness of local solution and global solution of (2.1) was studied in [2]. In particular, in the case of $\lambda = -1$ or $0 < \sigma < 2$, the solution is global. Moreover, the equation (2.1) possesses the charge conservation law which has been an important criteria of measuring whether a numerical simulation is good or not. It is given in the following theorem.

**Theorem 2.1.** [2] The stochastic NLS equation (2.1) possesses the charge conservation law almost surely

$$Q(t) = \int_{L_f}^{L_r} |u(x,t)|^2 dx = \int_{L_f}^{L_r} |u_0(x)|^2 dx = Q(0).$$

### 2.2. The local discontinuous Galerkin method.

In this subsection, we introduce some notations to be used later in the paper and also present some auxiliary results.

#### 2.2.1. Basic notations.

We denote the mesh by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], \quad \text{for } 1 \leq j \leq J$, where

$$L_f = x_{\frac{1}{2}} < x_{\frac{1}{2}} < \cdots < x_{N+\frac{1}{2}} = L_r.$$

Let

$$\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad 1 \leq j \leq J$$

with $h = \max_{1\leq j \leq J} \Delta x_j$ being the maximum mesh size.

Assume the mesh is regular, namely there is a constant $c > 0$ independent of $h$ such that

$$\Delta x_j \geq ch, \quad 1 \leq j \leq J.$$

We define a finite-element space consisting of piecewise polynomials

$$V_h^k = \{ \nu : \nu \in P^k(I_j); \quad 1 \leq j \leq J \},$$
where \( P^k(I_j) \) denotes the set of polynomials of degree up to \( k \) defined on the cell \( I_j \). Note that functions in \( V^k_h \) are allowed to have discontinuity across element interfaces. The solution of the numerical method is denoted by \( u_h \), which belongs to the finite element space \( V^k_h \). We denote by \((u_h)_{j+\frac{1}{2}}^+\) and \((u_h)_{j+\frac{1}{2}}^-\) the values of \( u_h \) at \( x_{j+\frac{1}{2}} \), from the right cell \( I_{j+1} \), and from the left cell \( I_j \), respectively.

For notational convenience we would like to introduce the following numerical flux related to the discontinuous Galerkin spatial discretization. \( f(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) \) is a given monotone numerical flux, i.e., it is Lipschitz continuous in both arguments, consistent (\( f(u,u) = f(u) \)), non-decreasing in the first argument and non-increasing in the second argument.

2.2.2. Projection. In what follows, we will consider the standard \( L^2 \)-projection of a function \( \omega \) with \( k+1 \) continuous derivatives into space \( V^k_h \), denoted by \( \mathcal{P} \), i.e., for each \( j \),

\[
\int_{I_j} (\mathcal{P}\omega(x) - \omega(x))\nu(x)dx = 0, \quad \forall \nu \in P^k(I_j),
\]

and the special projection \( \mathcal{P}^- \) into \( V^k_h \), which satisfies, for each \( j \),

\[
\int_{I_j} (\mathcal{P}^-\omega(x) - \omega(x))\nu(x)dx = 0, \quad \forall \nu \in P^{k-1}(I_j),
\]

and \( \mathcal{P}^- (\omega(x_{j+\frac{1}{2}}^-)) = \omega(x_{j+\frac{1}{2}}) \).

For the projections mentioned above, it is shown in [4], for any \( \omega \in H^{k+1}(\mathbb{R}) \)

\[
\|\tilde{\omega}(x)\|_{L^2} + h\|\tilde{\omega}(x)\|_{L^\infty} + \sqrt{h}\|\tilde{\omega}(x)\|_{\Gamma_h} \leq C h^{k+1}
\]

where \( \tilde{\omega} = \mathcal{P}\omega - \omega \) or \( \tilde{\omega} = \mathcal{P}^-\omega - \omega \). The positive constant \( C \), solely depending on the \( H^{k+1} \)-norm of \( \omega \) but independent of \( h \), \( \Gamma_h \) is the usual \( L^2 \)-norm on the cell interfaces of the mesh, which for this one-dimensional case is

\[
\|\nu\|_{\Gamma_h}^2 = \sum_{j=1}^J \left( (\nu^-_{j+\frac{1}{2}})^2 + (\nu^+_{j-\frac{1}{2}})^2 \right).
\]

3. The symplectic local discontinuous Galerkin method for the stochastic NLS equation. In this section, we will apply implicit midpoint scheme to the equation in the temporal direction, and present some properties of the semi-discrete scheme. Then, we discretize the spatial direction by local discontinuous Galerkin method and obtain the full-discrete method.

3.1. Temporal semi-discrete scheme. For actual numerical implementation, it might be more efficient to decompose the complex function \( u(x,t) \) into its real and imaginary parts by writing

\[
u(x,t) = r(x,t) + is(x,t),
\]

where \( r \) and \( s \) are real functions. Under the new notations, the problem (2.1) can be written as

\[
\begin{align*}
dr &= \left(s_{xx} + \lambda s(r^2 + s^2)^\sigma\right)dt + s \circ dW(t), \\
 ds &= \left(-r_{xx} + \lambda r(r^2 + s^2)^\sigma\right)dt - r \circ dW(t),
\end{align*}
\]
which is equivalent to the following first-order system:

\[
\begin{align*}
    dr &= \left(p_x + \lambda s(r^2 + s^2)\right) + s \circ dW(t), \\
p &= s_x, \\
ds &= -(q_x + \lambda r(r^2 + s^2)) - r \circ dW(t), \\
q &= r_x.
\end{align*}
\] (3.3)

Now, we apply the implicit midpoint method to (3.3) in temporal direction. We can get the following semi-discrete scheme

\[
\begin{align*}
r^{n+1} &= r^n + \left((p_x)^{n+\frac{1}{2}} + \lambda s^{n+\frac{1}{2}}[r^{n+\frac{1}{2}}]^2 + (s^{n+\frac{1}{2}})^2]\right) \Delta t + s^{n+\frac{1}{2}} \Delta \tilde{W}_n, \\
p^{n+\frac{1}{2}} &= (s_x)^{n+\frac{1}{2}}, \\
s^{n+1} &= s^n - \left((q_x)^{n+\frac{1}{2}} + \lambda r^{n+\frac{1}{2}}[r^{n+\frac{1}{2}}]^2 + (s^{n+\frac{1}{2}})^2]\right) \Delta t - r^{n+\frac{1}{2}} \Delta \tilde{W}_n, \\
q^{n+\frac{1}{2}} &= (r_x)^{n+\frac{1}{2}},
\end{align*}
\] (3.4)

where \(\Delta t\) is the time step size, \(v^{n+\frac{1}{2}} = \frac{1}{2}(v^{n+1} + v^n)\), and \(\Delta \tilde{W}_n = \sum_{k=0}^{\infty} \sqrt{\Delta t} \xi^k_{k,n} \phi e_k(x)\) with \(\xi^k_{k,n}\) being the truncation of a \(\mathcal{N}(0,1)\)-distribution random variable \(\xi_n\):

\[
\xi^k_{k,n} = \begin{cases} 
\kappa & \text{if } \xi_n > \kappa; \\
\xi_n & \text{if } |\xi_n| \leq \kappa; \\
-\kappa & \text{if } \xi_n < -\kappa
\end{cases}
\]

with \(\kappa := \sqrt{4|\ln(\Delta t)|}\). This choice is motivated by the fact that standard Gaussian random variables are unbounded for arbitrary values of \(\Delta t\), see [15] for more details.

For the truncated Wiener process, we have the following properties:

\[
\begin{align*}
(i) & \quad E\|\Delta \tilde{W}_n - \Delta W_n\|_{H^1}^2 \leq K \Delta t^3, \\
(ii) & \quad E\|\Delta \tilde{W}_n\|^2 - (\Delta W_n)^2 \|_{H^1}^2 \leq K \Delta t^3,
\end{align*}
\] (3.5)

where the constant \(K\) depends on \(\|\phi\|_{L^2(\Omega \times H^1)}\).

**Remark 1.** Obviously, the values of \(\kappa\) depend on \(\Delta t\). Therefore, in order to get the rates of convergence in subsection 3.2, we require a lower bound for temporal step-size to allow a uniform supremum for \(\kappa\) in the following context. I.e., we assume \(\kappa \leq \kappa_0\), with \(\kappa_0\) is a positive constant.

Rewriting (3.4) into

\[
iu^{n+1} = iu^n - \Delta t \left(u^{n+\frac{1}{2}}_{xx} + \lambda |u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}\right) + u^{n+\frac{1}{2}} \Delta \tilde{W}_n.
\] (3.6)

Based on the fact that \(W\) is real-valued, by multiplying both sides of equation (3.6) by \(u^{n+\frac{1}{2}}\), which is the conjugate of \(u^{n+\frac{1}{2}}\), and then taking the imaginary part and integrating it over the whole space domain. We can get the discrete charge conservation law as follows.

**Theorem 3.1.** Under the periodic boundary conditions, the semi-discrete scheme (3.4) of the system (3.3) has the discrete charge conservation law, i.e.,

\[
\int_{L^r} |u^{n+1}(x)|^2 dx = \int_{L^r} |u^n(x)|^2 dx, \quad n = 0, 1, ..., N.
\] (3.7)
Furthermore, the semi-discrete scheme (3.4) preserves the stochastic symplectic structure. It is stated in the following theorem.

**Theorem 3.2.** The implicit midpoint scheme (3.4) for the system (3.3) is stochastic symplectic.

**Proof.** We start with the system (3.4) by replacing $p^n + \frac{1}{2}$ and $q^n + \frac{1}{2}$ into the first and third equation, respectively.

\[
\begin{align*}
  x^{n+1} &= x^n + \left((s_{xx})^n + \frac{1}{2} + \lambda s^n + \frac{1}{2} [(r^n + \frac{1}{2})^2 + (s^n + \frac{1}{2})^2] \right) \Delta t + s^n + \frac{1}{2} \Delta \tilde{W}_n, \\
  s^{n+1} &= s^n - \left((r_{xx})^n + \frac{1}{2} + \lambda r^n + \frac{1}{2} [(r^n + \frac{1}{2})^2 + (s^n + \frac{1}{2})^2] \right) \Delta t - r^n + \frac{1}{2} \Delta \tilde{W}_n.
\end{align*}
\]

Introduce the following function $G(r,s)$:

\[
G(r,s) = \frac{\lambda \Delta t}{2(\sigma + 1)} (r^2 + s^2)^{\sigma + 1} + \frac{1}{2} (r^2 + s^2) \Delta \tilde{W}_n.
\]

It is convenient to write the midpoint scheme (3.8) as follows:

\[
\begin{align*}
  R &= r + \frac{\Delta t}{2} \left( \partial_{xx} S + \partial_{xx} s \right) + \frac{\partial G}{\partial s} \left( \frac{R + r}{2}, \frac{S + s}{2} \right), \\
  S &= s - \frac{\Delta t}{2} \left( \partial_{xx} R + \partial_{xx} r \right) - \frac{\partial G}{\partial r} \left( \frac{R + r}{2}, \frac{S + s}{2} \right),
\end{align*}
\]

where $R$ and $S$ are one-step approximation from $r$ and $s$ corresponding to (3.8), and $\frac{\partial G}{\partial r}$ means the derivative to the first component of $G$.

Then the above system (3.9) is equivalent to

\[
\begin{align*}
  \tilde{R} &= r + \frac{\Delta t}{2} \partial_{xx} \tilde{s} + \frac{1}{2} \frac{\partial G}{\partial s} (\tilde{R}, \tilde{s}), \\
  \tilde{S} &= \tilde{s} - \frac{\Delta t}{2} \partial_{xx} \tilde{r} - \frac{1}{2} \frac{\partial G}{\partial r} (\tilde{R}, \tilde{s}),
\end{align*}
\]

where $\tilde{R} = \frac{R + r}{2}$ and $\tilde{s} = \frac{s + s}{2}$.

We have (the arguments everywhere are $\tilde{R}, \tilde{s}$):

\[
\begin{align*}
  d \tilde{R} \wedge d \tilde{S} = & d \tilde{R} \wedge \left( d \tilde{s} - \frac{\Delta t}{2} d (\partial_{xx} \tilde{R}) - \frac{1}{2} \frac{\partial^2 G}{\partial r^2} d \tilde{R} - \frac{1}{2} \frac{\partial^2 G}{\partial r \partial s} d \tilde{s} \right) \\
  = & d \tilde{R} \wedge d \tilde{s} - \frac{\Delta t}{2} d \tilde{R} \wedge d (\partial_{xx} \tilde{R}) - \frac{1}{2} \frac{\partial^2 G}{\partial r \partial s} d \tilde{R} \wedge d \tilde{s}.
\end{align*}
\]

Further

\[
d \tilde{R} = dr + \frac{\Delta t}{2} d (\partial_{xx} \tilde{s}) + \frac{1}{2} \frac{\partial^2 G}{\partial r \partial s} d \tilde{R} + \frac{1}{2} \frac{\partial^2 G}{\partial s^2} d \tilde{s}.
\]

Substituting $\frac{1}{2} \frac{\partial^2 G}{\partial r \partial s} d \tilde{R}$ from here in (3.10), we obtain

\[
\begin{align*}
  d \tilde{R} \wedge d \tilde{S} = & d \tilde{R} \wedge d \tilde{s} - \frac{\Delta t}{2} d \tilde{R} \wedge d (\partial_{xx} \tilde{R}) - \left( d \tilde{R} - dr - \frac{\Delta t}{2} d (\partial_{xx} \tilde{s}) - \frac{1}{2} \frac{\partial^2 G}{\partial s^2} d \tilde{s} \right) \wedge d \tilde{s} \\
  = & dr \wedge d \tilde{s} - \frac{\Delta t}{2} \partial_{xx} \left( d \tilde{R} \wedge d (\partial_{xx} \tilde{R}) - d (\partial_{xx} \tilde{s} \wedge \tilde{s}) \right).
\end{align*}
\]
Therefore we have
\[
\int_{L_f}^{L_r} d\hat{R} \wedge dS dx = \int_{L_f}^{L_r} dr \wedge d\hat{s} dx - \int_{L_f}^{L_r} \frac{\Delta t}{2} \partial_x \left( d\hat{R} \wedge d(\partial_x \hat{R}) - d(\partial_x \hat{s} \wedge \hat{s}) \right) dx
\]
\[
= \int_{L_f}^{L_r} dr \wedge d\hat{s} dx.
\]
By the definitions of \( \hat{R} \) and \( \hat{s} \),
\[
\int_{L_f}^{L_r} d\hat{R} \wedge dS dx = \int_{L_f}^{L_r} dr \wedge ds dx.
\]
Thus the proof is finished. \( \square \)

3.2. Temporal-spatial full-discrete method. In this subsection, we consider the local discontinuous Galerkin method for the system (3.4) in the spatial direction and obtain the full-discrete method: find \( r_h, p_h, s_h, q_h \in V_h^k \), which now denotes real piecewise polynomial of degree at most \( k \), such that, for all test functions \( \nu_h, \omega_h, \alpha_h, \beta_h \in V_h^k \),

\[
\int_{L_f}^{L_r} [n+1] \nu_h dx - \int_{L_f}^{L_r} r_h^n \nu_h dx - \Delta t \left[ \left( \tilde{p}^{n+\frac{1}{2}} - \tilde{p}^{n+\frac{1}{2}} \right) - \left( \tilde{q}^{n+\frac{1}{2}} \tilde{q}_h^{n+\frac{1}{2}} \right) \right] + \Delta t \int_{L_f}^{L_r} \left( s^{n+\frac{1}{2}} \left( \nu_h \frac{1}{2} \right) + (s^{n+\frac{1}{2}}) \nu_h \right) dx - \int_{L_f}^{L_r} s^{n+\frac{1}{2}} \nu_h \Delta W_n dx = 0,
\]
\[
\int_{L_f}^{L_r} p^{n+\frac{1}{2}} \omega_h dx + \int_{L_f}^{L_r} s^{n+\frac{1}{2}} \omega_h dx - \left[ \left( \bar{s}^{n+\frac{1}{2}} \omega_h \right) - \left( \bar{s}^{n+\frac{1}{2}} \omega_h \right) \right] = 0,
\]
\[
\int_{L_f}^{L_r} s^{n+1} \alpha_h dx - \int_{L_f}^{L_r} s^n \alpha_h dx + \Delta t \left[ \left( \tilde{q}^{n+\frac{1}{2}} \tilde{q}_h^{n+\frac{1}{2}} \right) - \left( \tilde{q}^{n+\frac{1}{2}} \tilde{q}_h^{n+\frac{1}{2}} \right) \right] + \Delta t \int_{L_f}^{L_r} \left( s^{n+\frac{1}{2}} \alpha_h \Delta W_n dx = 0,
\]
\[
\int_{L_f}^{L_r} q^{n+\frac{1}{2}} \beta_h dx + \int_{L_f}^{L_r} r^{n+\frac{1}{2}} \beta_h dx - \left[ \left( \tilde{r}^{n+\frac{1}{2}} \beta_h \right) - \left( \tilde{r}^{n+\frac{1}{2}} \beta_h \right) \right] = 0.
\]
The numerical fluxes become
\[
\tilde{p} = p^+, \quad \tilde{r} = r^- , \quad \tilde{q} = q^-, \quad \tilde{s} = s^- ,
\]
where we have omitted the half-integer indices \( j + \frac{1}{2} \) as all quantities in (3.12) are computed at the same points.

**Remark 2.** The choice for the fluxes (3.12) is not unique. The important point is that \( \tilde{r} \) and \( \tilde{q} \), \( \tilde{s} \) and \( \tilde{p} \) should be chosen from different directions.

With such a choice of fluxes (3.12), we can get the first main result about discrete charge conservation law of the full-discrete method (3.11).

**Theorem 3.3.** Under the periodic boundary conditions, the full-discrete method (3.11) has the discrete charge conservation law, i.e.,

\[
\int_{L_f}^{L_r} \left| u_h^{n+1} \right|^2 dx = \int_{L_f}^{L_r} \left| u_h^n \right|^2 dx, \quad n = 0, 1, 2, ..., N.
\]
Proof. To complete the proof of the discrete charge conservation law, First, we write (3.11) as the complex form. Denote $u_h^n = r_h^n + i s_h^n, \psi_h^n = q_h^n + i p_h^n$, and take $\alpha_h = v_h, \beta_h = \omega_h$, then (3.11) become

$$i \int_{I_h} u_h^{n+1} v_h dx - i \int_{I_h} u_h^n \nu_h dx - [(\hat{\psi}^{*n+\frac{1}{2}} \nu_h^-)_{j+\frac{1}{2}} - (\hat{\psi}^{*n+\frac{1}{2}} \nu_h^+)_{j-\frac{1}{2}}] \Delta t$$

(3.14) $+ \Delta t \int_{I_h} (\psi_h^{*n+\frac{1}{2}} (\nu_h)_x - \lambda |u_h|^{n+\frac{1}{2}} |2\sigma u_h^{*n+\frac{1}{2}} \nu_h^+| dx - \int_{I_h} u_h^{*n+\frac{1}{2}} \nu_h \Delta \tilde{W}_n dx = 0,$

$$\int_{I_h} \psi_h^{*n+\frac{1}{2}} \omega_h dx + \Delta t \int_{I_h} (u_h^{*n+\frac{1}{2}} (\omega_h)_x dx - \left[ (\hat{u}^{*n+\frac{1}{2}} \omega_h^+)_{j+\frac{1}{2}} - (\hat{u}^{*n+\frac{1}{2}} \omega_h^+)_{j-\frac{1}{2}} \right] = 0.$$

where

$$\hat{u} = r_h^- + i s_h^-; \quad \hat{\psi} = q_h^+ + i p_h^+.$$

Now, we take the complex conjugate for every terms in system (3.14)

$$- i \int_{I_h} (u_h^*)^{n+1} \nu_h^* dx + i \int_{I_h} (u_h^*)^n \nu_h^* dx - \Delta t \left[ (\hat{\psi}^{*n+\frac{1}{2}} \nu_h^*)_{j+\frac{1}{2}} - (\hat{\psi}^{*n+\frac{1}{2}} \nu_h^*)_{j-\frac{1}{2}} \right]$$

(3.16) $+ \Delta t \int_{I_h} (\psi_h^{*n+\frac{1}{2}} (\nu_h^*)_x - \lambda |u_h|^n \frac{1}{2} |2\sigma u_h^{*n+\frac{1}{2}} \nu_h^+| dx - \int_{I_h} u_h^{*n+\frac{1}{2}} \nu_h^* \Delta \tilde{W}_n dx = 0,$

$$\int_{I_h} \psi_h^{*n+\frac{1}{2}} \omega_h^* dx + \int_{I_h} u_h^{*n+\frac{1}{2}} (\omega_h^*)_x dx - \left[ (\hat{u}^{*n+\frac{1}{2}} \omega_h^-)_{j+\frac{1}{2}} - (\hat{u}^{*n+\frac{1}{2}} \omega_h^-)_{j-\frac{1}{2}} \right] = 0.$$

We introduce a short-hand notation

(3.17)

$$\delta_j(u_h^n, \psi_h^n; \nu_h, \omega_h) = \int_{I_h} u_h^{n+1} \nu_h dx - \int_{I_h} u_h^n \nu_h dx - \Delta t \int_{I_h} \psi_h^{*n+\frac{1}{2}} \omega_h dx$$

$$\Delta t \int_{I_h} (\psi_h^{*n+\frac{1}{2}} (\nu_h)_x - \lambda |u_h|^n \frac{1}{2} |2\sigma u_h^{*n+\frac{1}{2}} \nu_h^+| dx - \int_{I_h} u_h^{*n+\frac{1}{2}} \nu_h^* \Delta \tilde{W}_n dx$$

$$- \Delta t \int_{I_h} u_h^{*n+\frac{1}{2}} (\omega_h)_x dx - \Delta t \left[ (\hat{u}^{*n+\frac{1}{2}} \omega_h^-)_{j+\frac{1}{2}} - (\hat{u}^{*n+\frac{1}{2}} \omega_h^-)_{j-\frac{1}{2}} \right]$$

Then from (3.16), we also have the expression of $\delta_j^*(u_h^n, \psi_h^n; \nu_h, \omega_h)$. If we take $\nu_h = u_h^{*n+\frac{1}{2}}, \omega_h = \psi_h^{*n+\frac{1}{2}}$ in both functions $\delta_j(u_h^n, \psi_h^n; \nu_h, \omega_h)$ and $\delta_j^*(u_h^n, \psi_h^n; \nu_h, \omega_h)$, both functions are zero. Hence we obtain

(3.18) $\delta_j(u_h^n, \psi_h^n; u_h^{*n+\frac{1}{2}}, \psi_h^{*n+\frac{1}{2}}) - \delta_j^*(u_h^n, \psi_h^n; u_h^{*n+\frac{1}{2}}, \psi_h^{*n+\frac{1}{2}}) = 0.$
With (3.15) of the numerical fluxes, then (3.18) becomes

\[
\int_I \left( |u_h^{n+1}|^2 - |u_h^n|^2 \right) dx + \Delta t \int_I \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} + u_h^{n+\frac{1}{2}} (\psi_{h}^{n+\frac{1}{2}})_{x} \right) dx \\
\quad - \Delta t \int_I \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} + u_h^{n+\frac{1}{2}} (\psi_{h}^{n+\frac{1}{2}})_{x} \right) dx \\
\quad - \Delta T \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} + u_h^{n+\frac{1}{2}} (\psi_{h}^{n+\frac{1}{2}})_{x} \right) \\
\quad \quad \text{for } x = x_j \\
\quad + \Delta T \left( u_h^{n+\frac{1}{2}} - \psi_{h}^{n+\frac{1}{2}} \right)_{x} \\
\quad + \Delta T \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} + u_h^{n+\frac{1}{2}} (\psi_{h}^{n+\frac{1}{2}})_{x} \right) \\
\quad - \Delta T \left( u_h^{n+\frac{1}{2}} - \psi_{h}^{n+\frac{1}{2}} \right)_{x} = 0.
\]

By the chain of rule, we can derive

\[
A = \Delta t \int_I \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right) dx = \Delta t \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j+\frac{1}{2}} - \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j-\frac{1}{2}},
\]

\[
B = \Delta t \int_I \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right) dx = \Delta t \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j+\frac{1}{2}} - \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j-\frac{1}{2}},
\]

then

\[
A - B = 2i \Delta t \left[ \text{Im} \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j+\frac{1}{2}} - \text{Im} \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j-\frac{1}{2}} \right].
\]

After some simple algebraic manipulation \( a - a^* = 2 \text{Im}(a), a \in C \), we have

\[
C = 2i \Delta t \text{Im} \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j+\frac{1}{2}}, \quad D = -2i \Delta t \text{Im} \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j+\frac{1}{2}}, \quad
\]

\[
G = 2i \Delta t \text{Im} \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j-\frac{1}{2}}, \quad H = -2i \Delta t \text{Im} \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j-\frac{1}{2}}.
\]

We combine all these equalities (3.19), (3.20) and (3.21) to obtain

\[
\int_I \left( |u_h^{n+1}|^2 - |u_h^n|^2 \right) dx + \frac{1}{2} \Delta t \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} + u_h^{n+\frac{1}{2}} (\psi_{h}^{n+\frac{1}{2}})_{x} \right) dx = 0,
\]

where the numerical entropy flux is given by

\[
\phi_{h}^{n+\frac{1}{2}} = -2 \Delta t \text{Im} \left( \psi_{h}^{n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_{x} \right)_{j-\frac{1}{2}}.
\]

Summing up over \( j \), the flux terms vanish because of the periodic boundary conditions. Thus we finish the proof. □

Corollary 3.4. The discrete charge conservation law trivially implies an \( L^2 \)-stability of the numerical solution.
4. Error estimates for the full-discrete method. In this section, we will state the error estimate of the full-discrete local discontinuous Galerkin method for the linear problem. In the sequel, $E$ denotes an expectation operator of a random variable, and $K, C$ are constants depending on the final time $T$ and the norm of $u$, but independent of $h$ and $n$. They may change from line to line.

Consider the error estimate for stochastic linear Schrödinger equation

\begin{align}
(4.1) \quad i du - (u_{xx} + \lambda u) dt = u \circ dW.
\end{align}

The full-discrete method (3.11) reads

\begin{align}
\int_{I_j} r_h^{n+1} v_h dx - \int_{I_j} r_h^n v_h dx - \Delta t \left[ \left( \hat{p}^{n+\frac{1}{2}} \nu_h^+ \right)_{j+\frac{1}{2}} - \left( \hat{p}^{n+\frac{1}{2}} \nu_h^- \right)_{j-\frac{1}{2}} \right] \\
+ \Delta t \int_{I_j} \left( \hat{p}^{n+\frac{1}{2}} (\nu_h)_{x} - \lambda s_h^{n+\frac{1}{2}} \nu_h \right) dx - \int_{I_j} s_h^{n+\frac{1}{2}} \nu_h \Delta W_n dx = 0,
\end{align}

\begin{align}
&\int_{I_j} \left[ \left( \hat{p}^{n+\frac{1}{2}} \omega_h \right)_{x} - \left( \hat{p}^{n+\frac{1}{2}} \omega_h^- \right)_{x} \right] dx - \left[ \left( \hat{p}^{n+\frac{1}{2}} \omega_h^+ \right)_{x} - \left( \hat{p}^{n+\frac{1}{2}} \omega_h^- \right)_{x} \right] = 0,
\end{align}

\begin{align}
&\int_{I_j} s_h^{n+1} \alpha_h dx - \int_{I_j} s_h^n \alpha_h dx + \Delta t \left[ \left( \hat{q}^{n+\frac{1}{2}} \alpha_h^+ \right)_{j+\frac{1}{2}} - \left( \hat{q}^{n+\frac{1}{2}} \alpha_h^- \right)_{j-\frac{1}{2}} \right] \\
+ \Delta t \int_{I_j} \left( \hat{q}^{n+\frac{1}{2}} (\alpha_h)_{x} - \lambda s_h^{n+\frac{1}{2}} \alpha_h \right) dx + \int_{I_j} s_h^{n+\frac{1}{2}} \alpha_h \Delta W_n dx = 0,
\end{align}

\begin{align}
&\int_{I_j} q_h^{n+1} \beta_h dx + \int_{I_j} r_h^{n+1} \beta_h dx - \left[ \left( \hat{r}^{n+\frac{1}{2}} \beta_h^+ \right)_{j+\frac{1}{2}} - \left( \hat{r}^{n+\frac{1}{2}} \beta_h^- \right)_{j-\frac{1}{2}} \right] = 0.
\end{align}

In order to obtain the error estimate to the full-discrete local discontinuous Galerkin method (4.2) with the fluxes (3.12), we divide the error into two parts:

\begin{align}
(4.3) \quad \|u(t_n) - u^n_h\|^2 \leq \underbrace{\|u(\cdot, t_n) - u^n\|^2}_\text{Temporal error} + \underbrace{\|u^n - u^n_h\|^2}_\text{Spatial error}.
\end{align}

4.1. Temporal error. To obtain the temporal error estimate, we need some regularity results of the numerical solution $u^n(x)$ for (3.4) in the linear case, i.e., $\sigma = 0$. We state it in the following two lemmas. The proof of these lemmas will be given in Appendix A and Appendix B, respectively.

**Lemma 4.1.** Assume that $E\|u^0\|_{H^\gamma} < \infty$, $\gamma = 0, 1, \cdots$ and $\phi \in L_2(L^2, H^\gamma)$. We have the following regularity of temporal semi-discretization, i.e., for $p \geq 1$,

\begin{align}
(4.4) \quad E\|u^n\|_{H^\gamma}^{2p} \leq K, \quad n = 1, 2, \ldots, N.
\end{align}

**Lemma 4.2.** Given $\gamma = 1, 2, \cdots$ and assume $u^0 \in L^2(\Omega, H^\gamma)$, then we have holder continuity in temporal direction, i.e., for $p \geq 1$,

\begin{align}
E\|u^{n+1} - u^n\|_{H^\gamma-1}^{2p} \leq C\Delta t^p, \quad n = 1, 2, \ldots, N.
\end{align}

Now we are in a position to establish an error estimate of the semi-discrete method (3.4) by virtue of these two lemmas.
THEOREM 4.3. Assume that \( u_0 \in L^2(\Omega, H^3) \), then it is of the mean-square order 1, i.e.,

\[
\left( \mathbb{E}\|u(t_n) - u^n\|_{L^2}^2 \right)^{1/2} \leq K\Delta t.
\]

Proof. From (A.2) (see Appendix A) and (4.1), it follows

\[
u^{n+1} = S_{\Delta t}^{n+1} u^0 - i\lambda\Delta t \sum_{\ell=1}^{n+1} S_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} - i \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{t-1},
\]

and

\[
u(t_{n+1}) = S(t_{n+1})u^0 - i\lambda \int_{t_n}^{t_{n+1}} S(t_{n+1} - \tau)u(\tau) d\tau - i \int_{t_n}^{t_{n+1}} S(t_{n+1} - \tau)u(\tau) \circ dW(\tau)
\]

\[
= S(t_{n+1})u^0 - i\lambda \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} S(t_{n+1} - \tau)u(\tau) d\tau - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} S(t_{n+1} - \tau)u(\tau) \circ dW(\tau).
\]

Subtract (4.5) from (4.6) to get

\[
u(t_{n+1}) - u^{n+1} = \left( S(t_{n+1}) - \hat{S}_{\Delta t}^{n+1} \right) u^0 - i\lambda \sum_{\ell=1}^{n+1} \left( \int_{t_{\ell-1}}^{t_{\ell}} S(t_{n+1} - \tau)u(\tau) d\tau - \Delta t \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \right)
\]

\[- i \sum_{\ell=1}^{n+1} \left( \int_{t_{\ell-1}}^{t_{\ell}} S(t_{n+1} - \tau)u(\tau) \circ dW(\tau) - \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{t-1} \right)
\]

\[=: \mathcal{A} + \mathcal{B} + \mathcal{C}.
\]

We will estimate them separately.

- **The first term \( \mathcal{A} \).**
  From [3], we know that \( \|S(t_{n+1}) - \hat{S}_{\Delta t}^{n+1}\|_{L(H^3, L^2)} \leq K\Delta t \). Thus,

\[\mathbb{E}\|\mathcal{A}\|_{L^2}^2 \leq K\mathbb{E}\|u^0\|_{H^3}^2 \Delta t^2 \leq K\Delta t^2.\]

- **The second term \( \mathcal{B} \).**
  To estimate \( \mathcal{B} \), we insert one term

\[\pm i\lambda \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} S(t_{n+1} - \tau)u_{t_{\ell-1}, u_{\ell-1}}(\tau) d\tau \]

into the expression of \( \mathcal{B} \) and we have

\[\mathcal{B} = -i\lambda \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \left( S(t_{n+1} - \tau) \left(u(\tau) - u_{t_{\ell-1}, u_{\ell-1}}(\tau)\right) d\tau - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \left( S(t_{n+1} - \tau)u_{t_{\ell-1}, u_{\ell-1}}(\tau) - \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \right) d\tau \]

\[=: \mathcal{B}^1 + \mathcal{B}^2.\]
To estimate term $B^1$, we present the estimate of $u(\tau) - u_{t_{\ell-1},u^{\ell-1}}(\tau)$ by their expression,

$$u(\tau) - u_{t_{\ell-1},u^{\ell-1}}(\tau) = S(\tau - t_{\ell-1})(u(t_{\ell-1}) - u^{\ell-1})$$

$$- i\lambda \int_{t_{\ell-1}}^{\tau} S(\tau - t_{\ell-1} - \rho)(u(\rho) - u_{t_{\ell-1},u^{\ell-1}}(\rho))d\rho$$

$$- i \int_{t_{\ell-1}}^{\tau} S(\tau - t_{\ell-1} - \rho)(u(\rho) - u_{t_{\ell-1},u^{\ell-1}}(\rho)) \circ dW(\rho).$$

Therefore, from Gronwall’s inequality, we know $E\|u(\tau) - u_{t_{\ell-1},u^{\ell-1}}(\tau)\|_{L^2}^2 \leq K\|u(t_{\ell-1}) - u^{\ell-1}\|_{L^2}^2$, and for term $B^1$

$$E\|B^1\|_{L^2}^2 \leq K\Delta t \sum_{\ell=1}^{n+1} \|u(t_{\ell-1}) - u^{\ell-1}\|_{L^2}^2.$$ 

We split term $B^2$ further as follows

$$B^2 = -i\lambda \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \left( S(t_{n+1} - r) - \hat{S}^{n+1-\ell}_\Delta T_{\Delta t} \right) u_{t_{\ell-1},u^{\ell-1}}(\tau) d\tau$$

$$- i\lambda \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}^{n+1-\ell}_\Delta T_{\Delta t} \left( u_{t_{\ell-1},u^{\ell-1}}(\tau) - u^{\ell-1} \right) d\tau$$

$$- i\lambda \Delta t \sum_{\ell=1}^{n+1} \hat{S}^{n+1-\ell}_\Delta T_{\Delta t} \left( u^\ell - u^{\ell-1} \right)$$

$$=: B^2_a + B^2_b + B^2_c.$$ 

For term $B^2_a$, based on $\|S(t_n) - \hat{S}^{n+1-\ell}_\Delta T_{\Delta t}\|_{L(H^3;L^2)} \leq K\Delta t$, $\|I - T_{\Delta t}\|_{L(H^3;L^2)} \leq K\Delta t$ and Lemma 4.1, we have

$$E\|B^2_a\|_{L^2}^2 \leq K\Delta t^2.$$ 

To estimate term $B^2_b$, we insert the expression of $u_{t_{\ell-1},u^{\ell-1}}(\tau) - u^{\ell-1}$ into it and we have

$$B^2_b = -i\lambda \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}^{n+1-\ell}_\Delta T_{\Delta t} \left( S(\tau - t_{\ell-1}) - I \right) u^{\ell-1} d\tau$$

$$- i\lambda \int_{t_{\ell-1}}^{\tau} S(\tau - \rho)(1 - \frac{i}{2}) u_{t_{\ell-1},u^{\ell-1}}(\rho) d\rho d\tau$$

$$- \lambda \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}^{n+1-\ell}_\Delta T_{\Delta t} \int_{\tau}^{\tau} S(\tau - \rho) u_{t_{\ell-1},u^{\ell-1}}(\rho) dW(\rho) d\tau.$$ 

The estimate of the first term is similar to before and is bounded by $K\Delta t^2$.

Concerning the second term, we employ Fubini’s theorem and Itô isometry and
Lemma 4.1,
\[
\mathbb{E}\| - \lambda \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} S(\tau - \rho) u_{t,\ell-1,u^{\ell-1}}(\rho) d\rho d\tau \|_{L^2}^2 \\
= \mathbb{E}\| - \lambda \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{\rho}^{\tau} S(\tau - \rho) u_{t,\ell-1,u^{\ell-1}}(\rho) d\tau d\rho \|_{L^2}^2 \\
\leq K \Delta t^2.
\]

The estimate of term $B_2^2$ is similar to that of term $B_2^2$, by replacing the expression of $u^{\ell} - u^{\ell-1}$. Combining all the above inequalities, we obtain the desired estimate of $B$

\[
\mathbb{E}\|B\|_{L^2}^2 \leq K \Delta t^2 + K \Delta t \sum_{\ell = 1}^{n+1} \|u(t_{\ell-1}) - u^{\ell-1}\|_{L^2}^2.
\]

- The third term $C$.

To estimate $C$, we change Stratonovich integral into Itô one

\[
C = - \frac{1}{2} \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} S(t_{\ell+1} - \tau) u(\tau) F_{\Delta t} d\tau \\
- i \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} S(t_{\ell+1} - \tau) u(\tau) dW(\tau) + i \sum_{\ell = 1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{\ell-1}.
\]

We split it further

\[
C = - i \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} S(t_{\ell+1} - \tau) \left( u(\tau) - u_{t,\ell-1,u^{\ell-1}}(\tau) \right) dW(\tau) \\
- i \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \left( S(t_{\ell+1} - \tau) - \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \right) u_{t,\ell-1,u^{\ell-1}}(\tau) dW(\tau) \\
- i \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \left( u_{t,\ell-1,u^{\ell-1}}(\tau) - u^{\ell-1} \right) dW(\tau) + \frac{1}{2} \sum_{\ell = 1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \left( u^{\ell} - u^{\ell-1} \right) \Delta \tilde{W}_{\ell-1} \\
- \frac{1}{2} \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} S(t_{\ell+1} - \tau) u(\tau) F_{\Delta t} d\tau + i \sum_{\ell = 1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \left( \Delta \tilde{W}_{\ell-1} - \Delta W_{\ell-1} \right).
\]

By replacing the expressions of $u_{t,\ell-1,u^{\ell-1}}(\tau) - u^{\ell-1}$ and $u^{\ell} - u^{\ell-1}$ into the above
equation, we have

(4.7)

\[
C = -i \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{\ell+1} - \tau) \left( u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\tau) \right) dW(\tau) \\
- i \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \left( S(t_{\ell+1} - \tau) - S_{\Delta t}^{n+1-\ell} T_{\Delta t} \right) u_{t_{\ell-1}, u^{\ell-1}}(\tau) dW(\tau) \\
- i \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \left( (S(\tau - t_{\ell-1}) - I) u^{\ell-1} - i \lambda \int_{t_{\ell-1}}^{\tau} S(\tau - \rho)(1 - \frac{i}{2}) u_{t_{\ell-1}, u^{\ell-1}}(\rho) d\rho \right) dW(\tau) \\
+ \frac{1}{2} \sum_{\ell = 1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \left( (\hat{S}_{\Delta t} - I) u^{\ell-1} - i \lambda \Delta t T_{\Delta t} u^{\ell-1} \right) \Delta \hat{W}_{t-1} \\
- \frac{1}{2} \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{\ell+1} - \tau) \left( u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\rho) \right) F_Q d\tau \\
- \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} S(\tau - \rho) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) dW(\tau) \\
+ \frac{1}{2} \sum_{\ell = 1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \left( \Delta \hat{W}_{t-1} \right)^2 \\
- \frac{1}{2} \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{\ell+1} - \tau) u_{t_{\ell-1}, u^{\ell-1}}(\rho) F_Q d\tau + i \sum_{\ell = 1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-1} \left( \Delta \hat{W}_{t-1} - \Delta W_{t-1} \right).
\]

We pay more attention to the last three lines, denoted by \(D\), because other terms can be estimated as before, and are bounded by \(K \Delta t^2 + K \Delta t n+1 \sum_{\ell = 1}^{n+1} \|u(t_{\ell-1}) - u^{\ell-1}\|^2_{L^2} \).

From

\[
- \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} S(\tau - \rho) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) dW(\tau) \\
- \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} \left( S(\tau - \rho) - T_{\Delta t} \right) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) dW(\tau) \\
- \sum_{\ell = 1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} T_{\Delta t} \left( u_{t_{\ell-1}, u^{\ell-1}}(\rho) - u^{\ell-1} \right) dW(\rho) dW(\tau) \\
- \sum_{\ell = 1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} dW(\rho) dW(\tau)
\]

and

\[
\int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} dW(\rho) dW(\tau) = \frac{1}{2} \left( (\Delta W_{t-1})^2 - F_Q \Delta t \right),
\]
and rearrangement of the last three lines of (4.7), it follows

\[
\mathcal{D} = -\sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S^{n+1-\ell} T_{\Delta t} \int_\tau^\tau \left( S(\tau - \rho) - \tau \right) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) dW(\tau)
\]

\[
- \frac{1}{2} \sum_{\ell=1}^{n+1} S^{n+1-\ell} T_{\Delta t} u^{\ell-1} (\Delta W_{\ell-1})^2 - (\Delta \tilde{W}_{\ell-1})^2 + \frac{1}{4} \sum_{\ell=1}^{n+1} S^{n+1-\ell} T_{\Delta t}^2 (u^{\ell} - u^{\ell-1}) \Delta \tilde{W}_{\ell-1}
\]

\[
- \frac{1}{2} \sum_{\ell=1}^{n+1} S(t_{n+1} - \tau) u_{t_{\ell-1}, u^{\ell-1}}(\rho) F_Q d\tau + \frac{1}{2} \sum_{\ell=1}^{n+1} S^{n+1-\ell} T_{\Delta t}^2 F_Q \Delta t
\]

\[
+ i \sum_{\ell=1}^{n+1} S^{n+1-\ell} T_{\Delta t} u^{\ell-1} (\Delta \tilde{W}_{\ell-1} - \Delta W_{\ell-1}).
\]

The estimates of the first two lines come from Itô isometry, and are bounded by \( K \Delta t^2 \). By the properties of the truncated Wiener process, the estimate of the last line is similar to that of \( B^2 \), and is bounded also by \( K \Delta t^2 \).

Combining all these analysis above, we obtain

\[
\mathbb{E} \left\| u(t_{n+1}) - u^{n+1} \right\|_{L^2}^2 \leq K \Delta t^2 + K \Delta t \sum_{\ell=1}^{n+1} \left\| u(t_{\ell-1}) - u^{\ell-1} \right\|_{L^2}^2.
\]

Therefore, Gronwall’s lemma leads to the assertion. \( \square \)

### 4.2. Spatial error.

We state the spatial error estimate of the full-discrete method (4.2) for the stochastic linear Schrödinger equation (4.1).

**Theorem 4.4.** Assume \( u_0 \in H^{k+2}([L_f, L_r]) \). Let \( u^n_h \) be the numerical solution of the full-discrete local discontinuous Galerkin method (4.2). Then there exists a constant \( h_0 > 0 \) such that for \( h \leq h_0 \), we have

\[
\mathbb{E} \left\| u^n - u_h^n \right\|_{L^2}^2 \leq C h^{2k+2} + C \Delta t^{-1} h^{2k+2}.
\]

**Proof.** We split the proof into two steps:

**Step 1: The error equation.**

Notice that the method (4.2) is also satisfied when the numerical solutions \( r_h, p_h, s_h, q_h \) are replaced by the exact solutions \( r, p = s_x, s, q = s_x \). For each fixed \( t_n \), we can obtain
the cell error equation

\begin{equation}
\mathcal{B}_j(r^n - r^n_h, p^n - p^n_h, s^n - s^n_h, q^n - q^n_h; \nu_h, \omega_h, \alpha_h, \beta_h) \\
= \int_{I_j} [p^{n+1} - p_h^{n+1}]} \nu_h dx - \int_{I_j} \left( p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \right) (\nu_h)_x dx \\
- \int_{I_j} \left( s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}} \right) \nu_h \Delta W_n dx - \Delta t \int_{I_j} \left( p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \right) (\omega_h)_x dx \\
- \lambda \Delta t \int_{I_j} \left( s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}} \right) \omega_h dx + \lambda \Delta t \int_{I_j} \left( p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \right) \alpha_h dx \\
- \Delta t \int_{I_j} \left( s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}} \right) (\alpha_h)_x dx + \int_{I_j} \left[ s^{n+1} - s_h^{n+1} \right] \alpha_h dx - \int_{I_j} \left[ s^n - s_h^n \right] \alpha_h dx \\
+ \int_{I_j} \left[ (r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}}) \alpha_h \Delta W_n dx - \Delta t \int_{I_j} \left( q^{n+\frac{1}{2}} - q_h^{n+\frac{1}{2}} \right) \beta_h dx - \Delta t \int_{I_j} \left( p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \right) \beta_h dx \\
- \Delta t \int_{I_j} \left( p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \right) \nu_h dx \right]_{j+\frac{1}{2}} + \Delta t \int_{I_j} \left( s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}} \right) (\omega_h)_x dx \\
- \Delta t \int_{I_j} \left( q^{n+\frac{1}{2}} - q_h^{n+\frac{1}{2}} \right) \alpha_h dx \right]_{j+\frac{1}{2}} - \Delta t \int_{I_j} \left( p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \right) \beta_h dx \right]_{j+\frac{1}{2}} \\
+ \Delta t \int_{I_j} \left( r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}} \right) \beta_h dx \right]_{j+\frac{1}{2}} - \Delta t \int_{I_j} \left( r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}} \right) \beta_h dx = 0
\end{equation}

for all \( \nu_h, \omega_h, \alpha_h, \beta_h \in V_h^k \).

Summing over \( j \), the error equation becomes

\begin{equation}
\sum_{j=1}^J \mathcal{B}_j(r^n - r^n_h, p^n - p^n_h, s^n - s^n_h, q^n - q^n_h; \nu_h, \omega_h, \alpha_h, \beta_h) = 0
\end{equation}

for all \( \nu_h, \omega_h, \alpha_h, \beta_h \in V_h^k \).

Denoting

\begin{equation}
\varepsilon^n = \mathcal{P} r^n - r_h^n, \xi^n = \mathcal{P} q^n - q_h^n, \eta^n = \mathcal{P} s^n - s_h^n, \zeta^n = p^n - \mathcal{P} p^n, \\
\varepsilon^n_e = \mathcal{P} r^n - r^n, \xi^n_e = \mathcal{P} q^n - q^n, \eta^n_e = \mathcal{P} s^n - s^n, \zeta^n_e = p^n - \mathcal{P} p^n,
\end{equation}

and taking the test functions

\begin{equation}
\nu_h = \varepsilon^{n+\frac{1}{2}}, \omega_h = \xi^{n+\frac{1}{2}}, \alpha_h = \eta^{n+\frac{1}{2}}, \beta_h = \zeta^{n+\frac{1}{2}},
\end{equation}

we obtain the important energy equality

\begin{equation}
\sum_{j=1}^J \mathcal{B}_j(\varepsilon^n - \varepsilon^n_e, \xi^n - \xi^n_e - \xi^n - \xi^n_e, \eta^n - \eta^n_e, \zeta^n - \zeta^n_e, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}) = 0.
\end{equation}

Now, we shall prove the theorem by analyzing each terms of (4.12).

**Step 2: Proof of the main result.**

We consider the left-hand side of the energy equation (4.12). Using the linearity of \( \mathcal{B}_j \) with respect to its first group of arguments, we get

\begin{equation}
\mathcal{B}_j(\varepsilon^n - \varepsilon^n_e, \xi^n - \xi^n_e, \eta^n - \eta^n_e, \zeta^n - \zeta^n_e, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}) \\
= \mathcal{B}_j(\varepsilon^n, -\xi^n_e, \eta^n_e, \xi^n_e, \zeta^n_e, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}) \\
- \mathcal{B}_j(\varepsilon^n, -\xi^n_e, \eta^n_e, \xi^n_e, \zeta^n_e, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}, \zeta_e^{n+\frac{1}{2}}).
\end{equation}
First, we consider the first term of the right-hand side in (4.13), which yields

\[
\mathcal{B}_j(e^n, -\zeta^n, \eta^n, \xi^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) \\
= \frac{1}{2} \int_{I_j} \left( (e^{n+1})^2 - (e^n)^2 \right) dx + \frac{1}{2} \int_{I_j} \left( (\eta^{n+1})^2 - (\eta^n)^2 \right) dx \\
+ \Delta t (\xi^+ e^n)^{n+\frac{1}{2}}_{j-\frac{1}{2}} - (\xi^+ e^n)^{n+\frac{1}{2}}_{j+\frac{1}{2}} + \Delta t (\eta^- e^n)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\eta^- e^n)^{n+\frac{1}{2}}_{j-\frac{1}{2}} \\
+ \Delta t (\xi^+ \eta^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\xi^+ \eta^-)^{n+\frac{1}{2}}_{j-\frac{1}{2}} + \Delta t [(e^- \xi^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (e^- \xi^-)^{n+\frac{1}{2}}_{j-\frac{1}{2}}] \\
- \Delta t \int_q [(\eta^+ \xi^+)^{n+\frac{1}{2}}_\tau + (\varepsilon^+ \zeta^+)^{n+\frac{1}{2}}_\tau] dx.
\]

(4.14)

From the integration by parts, we arrive at

\[
Q = [(\eta^- e^-)^{n+\frac{1}{2}}_{j-\frac{1}{2}} - (\eta^+ e^+)^{n+\frac{1}{2}}_{j+\frac{1}{2}}] + [(e^- \xi^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (e^+ \xi^+)^{n+\frac{1}{2}}_{j-\frac{1}{2}}].
\]

Substituting (4.15) into (4.14), we have

\[
\mathcal{B}_j(e^n, -\zeta^n, \eta^n, \xi^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) \\
= \frac{1}{2} \int_{I_j} \left( (e^{n+1})^2 - (e^n)^2 \right) dx + \frac{1}{2} \int_{I_j} \left( (\eta^{n+1})^2 - (\eta^n)^2 \right) dx + \Delta t [\Phi_j^{n+\frac{1}{2}} - \Phi_j^{n-\frac{1}{2}}],
\]

where \( \Phi = \xi^+ \eta^- + \xi^+ e^- \).

As for the second term of the right-hand side in (4.13), we have

\[
\mathcal{B}_j(e^n, -\zeta^n, \eta^n, \xi^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) = I + II + III + IV,
\]

where

\[
I = \int_{I_j} (e^{n+1} - e^n) e^{n+\frac{1}{2}} dx + \int_{I_j} (\eta^{n+1} - \eta^n) \eta^{n+\frac{1}{2}} dx,
\]

\[
II = \Delta t \int_{I_j} \left( (\xi e)^{n+\frac{1}{2}} - (\xi e)^{n-\frac{1}{2}} - (\eta e)^{n+\frac{1}{2}} - (\eta e)^{n-\frac{1}{2}} - \lambda (\xi e)^{n+\frac{1}{2}} + \lambda (\xi e)^{n+\frac{1}{2}} \right) dx,
\]

\[
III = \int_{I_j} (\xi^{n+\frac{1}{2}} \eta^{n+\frac{1}{2}} - \xi^{n+\frac{1}{2}} \eta^{n+\frac{1}{2}}) \Delta W dx,
\]

\[
IV = \Delta t \left( (\xi^+ e^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\xi^+ e^+)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\eta^+ e^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} + (\eta^+ e^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\xi^+ \eta^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} + (\xi^+ \eta^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} \right).
\]

By using the simple inequality \( ab \leq \frac{a^2}{2} + b^2 \), and the standard approximation theory (2.10) on \( e^n, \eta^n \), we have

\[
I \leq \|e^{n+1} - e^n\|_{L^2(I_j)}\|e^{n+\frac{1}{2}}\|_{L^2(I_j)} + \|\eta^{n+1} - \eta^n\|_{L^2(I_j)}\|\eta^{n+\frac{1}{2}}\|_{L^2(I_j)} \\
\leq C \Delta t^{-1}\|e^{n+1} - e^n\|^2_{L^2(I_j)} + C \Delta t\|\eta^{n+\frac{1}{2}}\|^2_{L^2(I_j)} \\
+ C \Delta t^{-1}\|\eta^{n+1} - \eta^n\|^2_{L^2(I_j)} + C \Delta t\|\eta^{n+\frac{1}{2}}\|^2_{L^2(I_j)},
\]
where \( \|e^{n+1}_e - e^n_e\|_{L^2(L_f,L_r)} = \|\mathcal{P}^{-}(r^{n+1} - r^n) - (r^{n+1} - r^n)\|_{L^2(L_f,L_r)} \) and \( \|\eta^{n+1}_e - \eta^n_e\|_{L^2(L_f,L_r)} = \|\mathcal{P}^{-}(s^{n+1} - s^n) - (s^{n+1} - s^n)\|_{L^2(L_f,L_r)} \). Summing over \( j \) and taking expectation, utilizing the property of projection and the estimate of \( \|r^{n+1} - r^n\|_{H^{k+1}} \) (see Lemma 4.2 with \( p = 1 \), we have

\[
(4.18) \quad \mathbb{E}(\sum_{j=1}^{J} I) \leq C h^{2k+2} + C \Delta t \mathbb{E}\|e^{n+1}_e\|^2_{L^2([L_f,L_r])} + C \Delta t \mathbb{E}\|\eta^{n+1}_e\|^2_{L^2([L_f,L_r])}.
\]

From the property of the projection \( \mathcal{P} \) and \( \mathcal{P}^{-} \) (see (2.8)-(2.9)), it follows that all the terms in \( II \) except the last two terms are actually zero. We can get the estimates for \( II \)

\[
(4.19) \quad \mathbb{E}(\sum_{j=1}^{J} II) \leq C h^{2k+2} + \frac{M}{4} \mathbb{E}\|e^{n+1}_e\|^2_{L^2([L_f,L_r])} + \frac{M}{4} \mathbb{E}\|\eta^{n+1}_e\|^2_{L^2([L_f,L_r])}.
\]

For the third term \( III \), we have

\[
\mathbb{E}(\sum_{j=1}^{J} III) = \frac{1}{4} \mathbb{E} \int_{L_f}^{L_r} (e^{n+1}_e - e^n_e)(\eta^{n+1}_e - \eta^n_e) \Delta W_dx + \frac{1}{2} \mathbb{E} \int_{L_f}^{L_r} (e^{n+1}_e - e^n_e) \eta^n_e \Delta W_dx + \frac{1}{2} \mathbb{E} \int_{L_f}^{L_r} (\eta^{n+1}_e - \eta^n_e) \Delta W_dx
\]

\[
=: III^a + III^b + III^c.
\]

For term \( III^a \),

\[
III^a \leq \frac{1}{4} \mathbb{E}\left( \|e^{n+1}_e - e^n_e\|_{L^2(L_f,L_r)}\|\eta^{n+1}_e - \eta^n_e\|_{L^2(L_f,L_r)}\|\Delta W_n\|_{L^\infty([L_f,L_r])} \right)
\]

\[
\leq C \Delta t \mathbb{E}\|\eta^{n+1}_e - \eta^n_e\|^2_{L^2([L_f,L_r])} + C \Delta t^{-1} \mathbb{E}\left( \|\Delta W_n\|^2_{L^\infty([L_f,L_r])}\|e^{n+1}_e - e^n_e\|^2_{L^2([L_f,L_r])} \right)
\]

\[
\leq C \Delta t \mathbb{E}\|\eta^{n+1}_e - \eta^n_e\|^2_{L^2([L_f,L_r])}
\]

\[
+ C \Delta t^{-1} \left( h^{2k+2} \mathbb{E}\|\Delta W_n\|^4_{L^\infty([L_f,L_r])} + h_{-(2k+2)}\mathbb{E}\|e^{n+1}_e - e^n_e\|^4_{L^2([L_f,L_r])} \right)
\]

\[
\leq C \Delta t \mathbb{E}\|\eta^{n+1}_e\|^2_{L^2([L_f,L_r])} + C \Delta t \mathbb{E}\|\eta^n_e\|^2_{L^2([L_f,L_r])} + C \Delta t h^{2k+2}.
\]

Similar, for term \( III^b \),

\[
III^b \leq \frac{1}{2} \mathbb{E}\left( \|e^{n+1}_e - e^n_e\|_{L^2([L_f,L_r])}\|\eta^n_e\|_{L^2([L_f,L_r])}\|\Delta W_n\|_{L^\infty([L_f,L_r])} \right)
\]

\[
\leq C \mathbb{E}\|e^{n+1}_e - e^n_e\|^2_{L^2([L_f,L_r])} + C \mathbb{E}\left( \|\eta^n_e\|^2_{L^2([L_f,L_r])}\|\Delta W_n\|^2_{L^\infty([L_f,L_r])} \right)
\]

\[
\leq C \Delta t \mathbb{E}\|\eta^n_e\|^2_{L^2([L_f,L_r])} + C \Delta t h^{2k+2},
\]

and for term \( III^c \),

\[
III^c \leq \frac{1}{2} \mathbb{E}\left( \|e^{n+1}_e\|_{L^2([L_f,L_r])}\|\eta^{n+1}_e - \eta^n_e\|_{L^2([L_f,L_r])}\|\Delta W_n\|_{L^\infty([L_f,L_r])} \right)
\]

\[
\leq C \Delta t \mathbb{E}\|\eta^{n+1}_e - \eta^n_e\|^2_{L^2([L_f,L_r])} + C \Delta t^{-1} \mathbb{E}\left( \|e^{n+1}_e\|^2_{L^2([L_f,L_r])}\|\Delta W_n\|^2_{L^\infty([L_f,L_r])} \right)
\]

\[
\leq C \Delta t \mathbb{E}\|\eta^{n+1}_e\|^2_{L^2([L_f,L_r])} + C \Delta t \mathbb{E}\|\eta^n_e\|^2_{L^2([L_f,L_r])} + C h^{2k+2}.
\]
Finally, IV only contains flux difference terms which all vanish upon a summation in \( j \). Combining these together, we know that

\[
\frac{1}{2} \mathbb{E} \left( \| e^{n+1} \|_{L^2([L_x,L_x])}^2 + \| \eta^{n+1} \|_{L^2([L_x,L_x])}^2 \right) - \frac{1}{2} \mathbb{E} \left( \| e^n \|_{L^2([L_x,L_x])}^2 + \| \eta^n \|_{L^2([L_x,L_x])}^2 \right) \\
\leq C \Delta t \mathbb{E} \| e^{n+1} \|_{L^2([L_x,L_x])}^2 + C \Delta t \mathbb{E} \| e^n \|_{L^2([L_x,L_x])}^2 \\
+ C \Delta t \mathbb{E} \| \eta^{n+1} \|_{L^2([L_x,L_x])}^2 + C \Delta t \mathbb{E} \| \eta^n \|_{L^2([L_x,L_x])}^2 + C h^{2k+2} + C \Delta t h^{2k+2}.
\]

By Gronwall’s inequality, there exists a constant \( h_0 > 0 \), for \( h \leq h_0 \), we obtain

\[
\mathbb{E} \left( \| e^n \|_{L^2([L_x,L_x])}^2 + \| \eta^n \|_{L^2([L_x,L_x])}^2 \right) \leq C h^{2k+2} + C \Delta t^{-1} h^{2k+2}, \quad \forall n.
\]

I.e.,

\[
(4.20) \quad \mathbb{E} \left( \| u^n - u_h^n \|_{L^2}^2 \right) \leq C h^{2k+2} + C \Delta t^{-1} h^{2k+2}.
\]

The proof is finished. \( \blacksquare \)

### 4.3. Main result

Combining Theorem 4.3 and Theorem 4.4, we obtain the error estimate of (4.2).

**Theorem 4.5.** Let \( u(x,t) \) be the exact solution of the problem (4.1), and assume the initial value \( u_0(x) \in H^{k+1}([L_x,L_x]) \). Let \( u_h^n \) be the numerical solution of the full-discrete local discontinuous Galerkin method (4.2). Then there exists a constant \( h_0 > 0 \) such that for \( h \leq h_0 \), we have

\[
(4.21) \quad \mathbb{E} \left( \| u(t_n) - u_h^n \|_{L^2}^2 \right) \leq C \Delta t^2 + C h^{2k+2} + C \Delta t^{-1} h^{2k+2}.
\]

**Remark 3.** Note that in the conclusion of the mean-square convergence order, the last term on the right-hand side is \( \Delta t^{-1} h^{2k+2} \). Assume that \( h/\Delta t = O(1) \), we know that \( \left( \mathbb{E} \left( \| u(t_n) - u_h^n \|_{L^2}^2 \right) \right)^{1/2} \leq C \Delta t + C h^{k+\frac{1}{2}} \). The convergence order in spatial direction is similar to the deterministic case, see ([16], Proposition 2.3). The convergence analysis for the stochastic nonlinear Schrödinger equation is our future work.

### Appendix A. Proof of lemma 4.1

**Proof.** First of all, we rewrite temporal semi-discretization system (3.4) in the linear case, i.e., \( \sigma = 0 \), into the function of \( u^n \):

\[
(\text{A.1}) \quad u^{n+1} = \hat{S}_{\Delta t} u^n - i \lambda \Delta t T_{\Delta t} u^{n+\frac{3}{2}} - i T_{\Delta t} u^{n+\frac{1}{2}} \Delta \tilde{W}_n,
\]

where \( u^n \) denotes the complex function \( v^n + i \eta^n \), operators are defined by \( \hat{S}_{\Delta t} = (I + i \frac{\Delta t}{2} \partial_{xx})^{-1} (I - i \frac{\Delta t}{2} \partial_{xx}) \) and \( T_{\Delta t} = (I + i \frac{\Delta t}{2} \partial_{xx})^{-1} \), where \( I \) is an identity operator.

It is easy to check that the operator \( \hat{S}_{\Delta t} \) is isometry in \( L^2 \), i.e., \( \| \hat{S}_{\Delta t} \|_{L^2} = 1 \). Furthermore, we know that \( \| T_{\Delta t} \|_{L^2} \leq 1 \). See reference [3] for example.

Next, we replace the function of \( u^n \) into equation (A.1) iteratively. We obtain

\[
(\text{A.2}) \quad u^n = \hat{S}_{\Delta t}^n u^0 - i \lambda \Delta t \sum_{\ell=1}^n \hat{S}_{\Delta t}^{n-\ell} T_{\Delta t} u^{\ell+\frac{3}{2}} - i \sum_{\ell=1}^n \hat{S}_{\Delta t}^{n-\ell} T_{\Delta t} u^{\ell+\frac{1}{2}} \Delta \tilde{W}_{\ell-1}.
\]
In order to bound function $u^n$, we insert the equality $u^{\ell+\frac{1}{2}} = u^{\ell} + \frac{u^{\ell} - u^{\ell-1}}{2}$ into the stochastic term and replace the expression of $u^{\ell} - u^{\ell-1}$,

$$u^n = \hat{S}_{\Delta t} T_{\Delta t} u^{\ell-1} - i\lambda \Delta t \sum_{\ell=1}^{n} \hat{S}_{\Delta t} T_{\Delta t} u^{\ell-1} \Delta \hat{W}_{\ell-1} - \frac{i}{2} \sum_{\ell=1}^{n} \hat{S}_{\Delta t} T_{\Delta t} (\hat{S}_{\Delta t} - I) u^{\ell-1} \Delta \hat{W}_{\ell-1} - \frac{1}{2} \lambda \Delta t \sum_{\ell=1}^{n} \hat{S}_{\Delta t} T_{\Delta t} u^{\ell-1} \Delta \hat{W}_{\ell-1} - \frac{1}{2} \sum_{\ell=1}^{n} \hat{S}_{\Delta t} T_{\Delta t} u^{\ell-1} (\Delta \hat{W}_{\ell-1})^2.$$

Taking $H^\gamma$-norm on both sides of equation (A.3), for the third and fourth terms on the right-hand side, using the fact that $u^{\ell-1}$ is independent of increment $\Delta \hat{W}_{\ell-1}$, and for the terms on the second line in (A.3) using $\|\Delta \hat{W}_{\ell-1}\|_{H^\gamma} \leq K \kappa_0 \Delta t^{\frac{\gamma}{2}} \leq K \Delta t^{\frac{1}{2}}$, we have

$$E\|u^n\|_{H^\gamma}^{2p} \leq K + K \Delta t \sum_{\ell=0}^{n} E\|u^{\ell}\|_{H^\gamma}^{2p},$$

where the positive constant $K$ depends on $p$, $T$, $\kappa_0$ the $L^2$-norm of operator $T_{\Delta t}$, $\|u^0\|_{H^\gamma}$, but not depends on $\Delta t$. The discrete Gronwall’s lemma leads to the assertion.

\[ \Box \]

**Appendix B. Proof of lemma 4.2.**

**Proof.** We start from equation (A.1),

$$u^{n+1} - u^n = (\hat{S}_{\Delta t} - I)u^n - i\lambda \Delta t T_{\Delta t} u^n + \hat{S}_{\Delta t} T_{\Delta t} u^n \Delta \hat{W}_n.$$

Since $\|\hat{S}_{\Delta t} - I\|_{L^1(H^\gamma, H^{\gamma-1})} \leq K \Delta t^{\frac{\gamma}{2}}$, we take $H^{\gamma-1}$-norm on both sides of the above equation and get

$$E\|u^{n+1} - u^n\|_{H^{\gamma-1}}^{2p} \leq K \Delta t^{p} E\|u^n\|_{H^\gamma}^{2p} + K \Delta t^{2p} E\left(\|u^n\|_{H^{\gamma-1}}^{2p} + \|u^{n+1}\|_{H^{\gamma-1}}^{2p}\right) + K \|u^{n+1} \Delta \hat{W}_n\|_{H^{\gamma-1}}^{2p}.$$

If $\gamma = 1$, for the last term, we use the embedding $H^1 \hookrightarrow L^\infty$, otherwise, we use the property $\|f g\|_{H^\alpha} \leq K \|f\|_{H^\alpha} \|g\|_{H^\alpha}$ ($\alpha \geq 1$). Because of $u^0 \in L^2(\Omega, H^\gamma)$, from Lemma 4.1, we know that

$$E\|u^{n+1} - u^n\|_{H^{\gamma-1}}^{2p} \leq K \Delta t^{p}.$$

This completes the proof. \[ \Box \]

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