GLOBAL CRYSTAL BASES FOR INTEGRABLE MODULES
OVER A QUANTUM SYMMETRIC PAIR OF TYPE AIII

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ABSTRACT. In this paper, we study basic properties of global \( j \)-crystal bases for integrable modules over a quantum symmetric pair coideal subalgebra \( U^j \) associated to the Satake diagram of type AIII without black nodes. Also, we obtain an intrinsic characterization of the \( j \)-crystal bases, whose original definition is artificial.

1. Introduction

Let \( U = U_q(\mathfrak{sl}_{2r+1}) \) be the quantum group over the field \( \mathbb{Q}(q) \) of rational functions in one variable \( q \), and \( U \) its coideal subalgebra such that \((U, U^j)\) forms a quantum symmetric pair of type AIII in the sense of [Le99]. Bao and Wang [BW18a] introduced the notion of \( j \)-canonical bases for the based \( U \)-modules. A based \( U \)-module is a \( U \)-module \( M \) with a bar-involution \( \psi_M \) and a distinguished basis \( B \) satisfying certain conditions (see [L10] for the precise definition). One of the key ingredients for the construction of the \( j \)-canonical bases is the intertwiner (also known as the quasi-\( K \)-matrix) \( \Upsilon \). Using \( \Upsilon \), Bao and Wang defined a new involution \( \psi^j_M := \Upsilon \circ \psi_M \) on \( M \) which is compatible with the bar-involution \( \psi^j \) on \( U^j \). Then, for each \( b \in B \), there exists a unique \( b^j \in M \) such that \( \psi^j_M(b^j) = b^j \) and \( b^j - b \in \bigoplus_{b' \in B} q\mathbb{Q}[q] b' \), where \( \preceq^j \) is a certain partial order on \( B \). Clearly, \( \{b^j \mid b \in B\} \) is a basis of \( M \), which is called the \( j \)-canonical basis of \((M, B)\).

The multi-parameter version of \( U^j \) was considered in [BWW18]. Thanks to the integrality of the intertwiner \( \Upsilon \), the notion of \( j \)-canonical bases can be defined analogously. The condition \( b^j - b \in \bigoplus_{b' \in B} q\mathbb{Q}[q] b' \) is replaced by \( b^j - b \in \bigoplus_{b' \preceq^j b} (p\mathbb{Q}[p,q,q^{-1}] \oplus q\mathbb{Q}[q]) b' \). The general theory of the \( j \)-canonical bases (usually called \( r \)-canonical bases) for the general quantum symmetric pairs was developed in [BW18b].

In [W17], the author classified all irreducible \( U^j \)-modules in a category \( \mathcal{O}^j_{\text{int}} \), which is an analog of the category \( \mathcal{O}_{\text{int}} \) of integrable \( U \)-modules, and proved that \( \mathcal{O}^j_{\text{int}} \) is semisimple; the isomorphism classes of irreducible modules in \( \mathcal{O}^j_{\text{int}} \) are classified by the set \( P^j \) of bipartitions of length \((r; r+1)\). When the parameters are in the asymptotic case, to each irreducible module in \( \mathcal{O}^j_{\text{int}} \), the author associated a local basis, the \( j \)-crystal basis, which is an analog of Kashiwara’s crystal basis. By the complete reducibility, every object in \( \mathcal{O}^j_{\text{int}} \) admits a \( j \)-crystal basis. In particular, each \( U \)-module in \( \mathcal{O}^j_{\text{int}} \), regarded as a \( U^j \)-module, has a \( j \)-crystal basis.

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It should be noted that the \( j \)-crystal basis of a \( U \)-module in \( O_{\text{int}} \) is the localized \( j \)-canonical basis ([W17 Section 1.3]). To be precise, let \( M \in O_{\text{int}} \) with a canonical basis (or global crystal basis) \( B \). Since \((M,B)\) is a based module, it has a \( j \)-canonical basis \( \{b^j \mid b \in B\} \). Set \( L := \text{Span}_{A_0} B \), where \( A_0 := \{ f/g \in \mathbb{Q}(p,q) \mid f,g \in p\mathbb{Q}[p,q,q^{-1}] \oplus \mathbb{Q}[q], \lim_{q \to 0}(\lim_{p \to 0} g) \neq 0 \} \). Then, \( B := \{b^j + qL \mid b \in B\} \) is a \( \mathbb{Q} \)-basis of \( L/qL \), and \((L,B)\) forms a \( j \)-crystal basis of \( M \). Hence, \( b^j + qL \) can be thought of as the localization of \( b^j \) at \( p = q = 0 \). Conversely, we may say that the \( j \)-canonical basis of a based \( U \)-module is a globalization of its \( j \)-crystal basis.

Here arises a natural question: Does a \( U^j \)-module in \( O_{\text{int}} \) that is not a based \( U \)-module admit a globalization of its \( j \)-crystal basis? One of the main results of this paper gives the affirmative answer to this question.

In our strategy, the multi-parameter \( q \)-Schur duality between \( U^j \) and the Hecke algebra of type \( B \) (BBW18), and the irreducibility of the Kazhdan-Lusztig cell representations of the asymptotic multi-parameter Hecke algebra of type \( B \) ([BI03]) play key roles. Let us recall the latter objects briefly. Kazhdan and Lusztig [KL79] gave a partition \( W = \bigsqcup_{L \subseteq W} X \) of a Coxeter group \( W \) into the left cells; here, \( L(W) \) denotes the set of left cells. To each left cell \( X \in L(W) \), they associated an \( H \)-module \( C^L_X \) which is called the left cell representation corresponding to \( X \). The left cell representation \( C^L_X \) is defined to be the quotient of a left ideal \( C_{\leq L} \) of \( H \) spanned by some Kazhdan-Lusztig basis elements by its subspace \( C_{< L} \), which is also spanned by some Kazhdan-Lusztig basis elements. Therefore, \( C^L_X \) has a basis consisting of the images of some Kazhdan-Lusztig basis elements under the canonical map \( C_{\leq L} \to C^L_X \). It is known that each left cell representation is irreducible if \( W \) is of type \( A \). When \( W \) is of type \( B \), the irreducibility of the left cell representations depends on the choice of the parameters \( p,q \). According to [BI03], the left cell representations are irreducible when the parameters are asymptotic.

By the multi-parameter \( q \)-Schur duality for type \( B \), the tensor power \( V^{\otimes d} \) of the vector representation of \( U \) is equipped with a \((U^j,H)\)-bimodule structure whose irreducible decomposition is multiplicity free, where \( H \) denotes the multi-parameter Hecke algebra of type \( B \) over the field \( \mathbb{Q}(p,q) \) of rational functions in two variables \( p,q \). Then, for each \( X \in L(W) \), the \( U^j \)-module \( V^{\otimes d} \otimes_H C^L_X \) is irreducible, where \( C^L_X := \mathbb{Q}(p,q) \otimes_{\mathbb{Z}[p^{\pm 1},q^{\pm 1}]} C^L_X \). Every irreducible \( U^j \)-module can be obtained in this way as \( d \geq 1 \) varies. The main result of this paper states that the basis of \( V^{\otimes d} \otimes_H C^L_X \) induced from the Kazhdan-Lusztig basis of \( C^L_X \) is a globalization of the \( j \)-crystal basis.

Our approach provides the following characterization of the \( j \)-crystal bases and its globalization of the finite-dimensional irreducible \( U^j \)-modules. Let \( L \in O_{\text{int}} \) be irreducible and \( v \in L \) a highest weight vector. Define two symmetric bilinear forms \((\cdot,\cdot)_1\) and \((\cdot,\cdot)_2\) on \( L \) and an involutive anti-linear automorphism \( \psi^j_L \) on \( L \) by

\[
(v,v)_1 = 1, \quad (x m, n)_1 = (m, \sigma^j(x)n)_1 \quad \text{for all } x \in U^j, \ m,n \in L,
\]

\[
(v,v)_2 = 1, \quad (x m, n)_2 = (m, \tau^j(x)n)_2 \quad \text{for all } x \in U^j, \ m,n \in L,
\]

\[
\psi^j_L(v) = v, \quad \psi^j_L(xm) = \psi^j(x)\psi^j_L(m) \quad \text{for all } x \in U^j, m \in L,
\]

where \( \sigma^j, \tau^j, \) and \( \psi^j \) are automorphisms of \( U^j \) defined in Proposition 3.1.1.

**Theorem A.** Let \( \lambda \in P^j, \ L(\lambda) \) the corresponding irreducible \( U^j \)-module, \((L(\lambda),B(\lambda))\) the \( j \)-crystal basis of \( L(\lambda) \) such that \( v + qL(\lambda) \in B(\lambda) \). Then, there exist \( G^j(b), \ b \in B(\lambda) \) satisfying the following.
(1) \( \mathcal{L}(\lambda) = \{ m \in L(\lambda) \mid (m, m)_2 \in \mathbb{A}_0 \} \).

(2) \( \mathcal{B}(\lambda) \) forms an orthonormal basis of \( \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \) with respect to the symmetric bilinear form induced from \( \langle \cdot, \cdot \rangle_2 \).

(3) Set \( L_\mathbf{A}(\lambda) := \text{Span}_\mathbf{A}\{G^j(b) \mid b \in \mathcal{B}(\lambda)\} \), where \( \mathbf{A} := \mathbb{Q}[p, p^{-1}, q, q^{-1}] \).
Then, the triple \( (\mathcal{L}(\lambda), L_\mathbf{A}(\lambda), \psi^j_\mathbf{A}(\mathcal{L}(\lambda))) \) forms a balanced triple with the global basis \( \{G^j(b) \mid b \in \mathcal{B}(\lambda)\} \).

(4) \( L(\lambda) \) has the basis dual to \( G^j(\mathcal{B}(\lambda)) \) with respect to \( \langle \cdot, \cdot \rangle_1 \).

Next, we investigate basic properties of global \( j \)-crystal basis for not necessarily irreducible \( \mathbf{U}^j \)-modules. Especially, we roughly describe the matrix coefficients of the actions of the generators of \( \mathbf{U}^j \) with respect to a given global \( j \)-crystal basis.

We end this paper by proving that the global \( j \)-crystal basis for a \( \mathbf{U}^j \)-module (not \( \mathbf{U}^j \)-module) is compatible with the filtration coming from the dominance order of the bipartitions (see subsection \( 9.3 \) for the definition of this filtration). A similar result is well-known for ordinary global crystal bases (\([K93], [L10]\)).

**Theorem B.** Let \( M \) be a \( \mathbf{U} \)-module with a global \( j \)-crystal basis \( G^j(\mathcal{B}) \). Then, for each \( \lambda \in P^j \), the subquotient \( W_\lambda(M) \) of \( M \) has \( \{ G^j(b) + W_{\succ \lambda}(M) \mid I(b) = \lambda \} \) as a global \( j \)-crystal basis. Moreover, there exists an isomorphism \( L(\lambda)^{\oplus m_\lambda} \to W_\lambda(M) \) which restricts to a bijection \( \{ G^j(b) \mid b \in \mathcal{B}(\lambda) \}^{\oplus m_\lambda} \to \{ G^j(b) + W_{\succ \lambda}(M) \mid I(b) = \lambda \} \), where \( m_\lambda \) denotes the multiplicity of \( L(\lambda) \) in \( M \).

In particular, if we take \( M \) to be an irreducible \( \mathbf{U}^j \)-module, we obtain the following.

**Corollary C.** Let \( \lambda \in P^j \). Then, \( \{ G^j(b) \mid b \in \mathcal{B}(\lambda) \} \) is a unique global \( j \)-crystal basis for \( L(\lambda) \).

Finally, we mention that the results above are valid for the quantum symmetric pair \((\mathbf{U}, \mathbf{U}^1)\) of type \( AIV \) once we define a corresponding category \( C^j_{\text{int}} \) properly.

This paper is organized as follows. In Section 2, we prepare necessary notations concerning (bi)partitions and Young (bi)tableaux. In Section 3 and 4, we give a brief review of \([W17]\). In Section 5, we introduce the notion of global \( j \)-crystal bases, and show that the \( j \)-canonical bases are examples of them. Sections 6–8 are devoted to proving the existence theorem for the global \( j \)-crystal bases of the finite-dimensional irreducible \( \mathbf{U}^j \)-modules. After studying basic properties of the global \( j \)-crystal bases in Section 9, we finally prove the compatibility of the \( j \)-crystal bases and the filtration associated to the dominance order of the bipartitions in Section 10.

### 2. Notations

Throughout this paper, we fix a positive integer \( r \). For \( n \in \frac{1}{2}\mathbb{Z} \), set \( \tilde{n} := n - \frac{1}{2} \). Note that \( -\tilde{n} = -n + \frac{1}{2} \neq -n \). We set

\[
I := \{-r, \ldots, -1, 0, 1, \ldots, r\}, \quad \mathbb{I} := \{-\underline{2}, \ldots, -1, 1, \ldots, 2\}, \quad \mathbb{V} := \{1, \ldots, r\}.
\]

A partition of \( n \in \mathbb{N} \) of length \( l \in \mathbb{N} \) is a nonincreasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_l) \) of nonnegative integers satisfying \( \sum_{i=1}^l \lambda_i = n \). Let \( |\lambda| := n \) and \( \ell(\lambda) := l \), and call them the size and the length of \( \lambda \), respectively. We denote by \( \text{Par}_l(n) \) the set of partitions of \( n \) of length \( l \).

We often identify a partition with a Young diagram in a usual way. Let \( (L, \preceq) \) be a totally ordered set. A semistandard tableau of shape \( \lambda \in \text{Par}_l(n) \) in letters
L is a filling of the Young diagram λ with elements of L, which weakly increases (with respect to the total order ≤) from left to right along the rows, and strictly increases from the top to the bottom along the columns.

A bipartition of \( n \in \mathbb{N} \) of length \((l;m) \in \mathbb{N}^2 \) is an ordered pair \( \lambda := (\lambda^-; \lambda^+) \) of partitions such that \( \ell(\lambda^-) = l, \ell(\lambda^+) = m, \) and \( |\lambda| := |\lambda^-| + |\lambda^+| = n. \) We denote by \( P_{(l;m)}(n) \) the set of bipartitions of \( n \) of length \((l;m) \). For totally ordered sets \((L^-; \lambda^-; L^+)\) and \((L^+; \lambda^+)\), a semistandard tableau of shape \( \lambda \in P_{(l;m)}(n) \) in letters \((L^-; \lambda^-; L^+)\) is an ordered pair \( T = (T^-; T^+) \), where \( T^\pm \) is a semistandard tableau of shape \( \lambda^\pm \) in letters \( L^\pm \).

For partitions \( \mu \subset \lambda \), define the skew partition \( \lambda/\mu \) in a usual way. For bipartitions \( \mu \subset \lambda \) (i.e., \( \mu^- \subset \lambda^- \) and \( \mu^+ \subset \lambda^+ \)), define the skew bipartition \( \lambda/\mu \) to be \((\lambda^-/\mu^-; \lambda^+/\mu^+) \). A skew partition \( \lambda/\mu \) is said to be a horizontal strip if each column of \( \lambda/\mu \) contains at most one box. We say that a skew bipartition \( \lambda/\mu \) is a horizontal strip if \( \lambda^\pm/\mu^\pm \) are.

Set

\begin{itemize}
  \item \( P(n) = P_r(n) := \text{Par}_{2r+1}(n) \): the set of partitions of \( n \) of length \( 2r + 1 \).
  \item \( P = P_r := \bigcup_{n \in \mathbb{N}} P(n) \): the set of partitions of length \( 2r + 1 \).
  \item \( \text{Par}_l := \bigcup_{n \in \mathbb{N}} \text{Par}_l(n) \): the set of partitions of length \( l \).
  \item \( P^l(n) = P^l_r(n) := P_{(r+1;r)}(n) \): the set of bipartitions of \( n \) of length \((r+1;r) \).
  \item \( P^l := \bigcup_{n \in \mathbb{N}} P^l(n) \): the set of bipartitions of length \((r+1;r) \).
  \item \( \text{SST}(\lambda) \): the set of semistandard tableaux of shape \( \lambda \in P(n) \) in letters \( I \).
  \item \( \text{SST}(\lambda) \): the set of semistandard tableaux of shape \( \lambda \in P^l(n) \) in letters \((I \setminus \mathbb{P}; \mathbb{P}) \) with total orders \( 0 \prec -1 \prec \cdots \prec -r \) and \( 1 \prec+ \cdots \prec+ r \).
\end{itemize}

For \( \lambda \in P^l \), we refer the \( i \)-th row of \( \lambda^- \) to as the \(-(i-1)\)-th row of \( \lambda \), and the \( j \)-th row of \( \lambda^+ \) to as the \( j \)-th row of \( \lambda \). Also, for \( i \in I \), set \( \lambda_i \) to be the length of the \( i \)-th row of \( \lambda \), i.e.,

\[
\lambda_i := \begin{cases} 
\lambda^-_{-i+1} & \text{if } i \leq 0, \\
\lambda^+_{i} & \text{if } i > 0.
\end{cases}
\]

For \( i \in \mathbb{P} \), set \( \lambda \downarrow_\lambda := (\lambda_0, \lambda_{-1}, \ldots, \lambda_{-i}; \lambda_1, \ldots, \lambda_i) \in P^l_i \).

For \( T \in \text{SST}(\lambda) \) and \( i \in \mathbb{P} \), set \( T \downarrow_\lambda \) to be the semistandard tableau obtained from \( T \) by deleting the boxes whose entries are less than \(-i\) or greater than \( i \).

For each \( \lambda \in P^l \), let \( T_\lambda \in \text{SST}(\lambda) \) be the unique semistandard tableau of shape \( \lambda \) whose entries in the \( i \)-th row are \( i \). Note that we have \( T_\lambda \downarrow_\lambda = T_{\lambda_i} \). For \( T \in \text{SST}(\lambda) \) and \( i \in I \), set \( T(i) \) to be the number of boxes of \( T \) whose entries are \( i \).

**Definition 2.0.1.**

1. \( \preceq \) is a partial order (called the dominance order) on \( \text{Par}_l \) defined as follows. For \( \lambda, \mu \in \text{Par}_l \), we have \( \lambda \preceq \mu \) if
   \begin{itemize}
     \item (a) \( |\lambda| = |\mu| \) and
     \item (b) \( \sum_{i=1}^l \lambda_i \leq \sum_{i=1}^l \mu_i \) for all \( 1 \leq j \leq l \).
   \end{itemize}
2. \( \preceq \) is a partial order (also called the dominance order) on \( P^l \) defined as follows. For \( \lambda, \mu \in P^l \), we have \( \lambda \preceq \mu \) if
   \begin{itemize}
     \item (a) \( |\lambda| = |\mu| \),
     \item (b) \( \sum_{i=0}^j \lambda_{-i} \leq \sum_{i=0}^j \mu_{-i} \) for all \( 0 \leq j \leq r \), and
     \item (c) \( |\lambda^-| + \sum_{i=1}^j \lambda_i \leq |\mu^-| + \sum_{i=1}^j \mu_i \) for all \( 1 \leq j \leq r \). 
   \end{itemize}
(3) \(\preceq\) is a partial order on \(P^j\) defined as follows. For \(\lambda, \mu \in P^j\), we have \(\lambda \preceq \mu\) if \(\lambda^- \preceq \mu^-\) (dominance order on \(\text{Par}_{r+1}\)) and \(\lambda^+ \preceq \mu^+\) (dominance order on \(\text{Par}_r\)).

Clearly, \(\lambda \preceq \mu\) implies \(\lambda \leq \mu\).

### 3. Representation theory of \(U^j\)

Let \(p\) and \(q\) be independent indeterminates.

#### 3.1. Definition of \(U^j\)

Let \(\Lambda\) be the free \(\mathbb{Z}\)-module with a free basis \(\{\epsilon_i \mid i \in I\}\), and with a symmetric bilinear form \((\cdot, \cdot)\) defined by \((\epsilon_i, \epsilon_j) = \delta_{i,j}\). For \(i \in \mathbb{I}\), set

\[
\alpha_i := \epsilon_i - \epsilon_{i+1}, \quad Q := \sum_{i \in I} \mathbb{Z}\alpha_i, \quad Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i.
\]

For \(\lambda, \mu \in \Lambda\), we write \(\mu \leq \lambda\) if \(\lambda - \mu \in Q_+\). This defines a partial order on \(\Lambda\).

The quantum group \(U = U_{2r+1} = U_q(\mathfrak{sl}_{2r+1})\) of type \(A_{2r}\) is an associative algebra over \(\mathbb{Q}(p, q)\) with generators \(E_i, F_i, K_i^{\pm 1}\), \(i \in \mathbb{I}\) subject to the following relations: For \(i, j \in \mathbb{I}\),

\[
K_iK_i^{-1} = K_i^{-1}K_i = 1, \\
K_iK_j = K_jK_i, \\
K_iE_jK_i^{-1} = q^{(\alpha_i, \alpha_j)}E_i, \\
K_iF_jK_i^{-1} = q^{-(\alpha_i, \alpha_j)}F_i, \\
E_iF_j - F_jE_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \quad \text{if } |i - j| = 1, \\
F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \quad \text{if } |i - j| = 1, \\
E_iE_j - E_jE_i = 0 \quad \text{if } |i - j| > 1, \\
F_iF_j - F_jF_i = 0 \quad \text{if } |i - j| > 1.
\]

In this paper, we use the comultiplication \(\Delta\) of \(U\) given by

\[
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i^{-1}, \\
\Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i \quad i \in \mathbb{I}.
\]

Let \((U, U^j)\) denote the quantum symmetric pair over \(\mathbb{Q}(p, q)\) of type \(A_{2r}\) without black nodes, that is, \(U^j = U^j_{r}\) is the subalgebra of \(U\) generated by

\[
k_i^{\pm 1} := (K_i^\pm K_i^\mp)^{\pm 1}, \\
e_i := E_i + p^{-\delta_{i,1}}F_{-i}K_i^{-1}, \\
f_i := E_{-i} + p^{\delta_{i,1}}K_i^{-1}F_i, \quad i \in \mathbb{I}.
\]
The $U^j$ has the following defining relations ([Le99], see also [BW18a], [BWW18]):

For $i, j \in \mathbb{I}$,

\[
\begin{align*}
    k_ik_i^{-1} &= k_i^{-1}k_i = 1, \\
    k_ik_j &= k_jk_i, \\
    k_ie_jk_i^{-1} &= q^{(\alpha_i - \alpha_i - \alpha_j)}e_j, \\
    k_if_jk_i^{-1} &= q^{-(\alpha_i - \alpha_i - \alpha_j)}f_j, \\
    e_if_j - f_je_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \quad \text{if} \ (i, j) \neq (1, 1), \\
    e_i^2e_j - (q + q^{-1})e_je_i + e_je_i^2 &= 0 \quad \text{if} \ |i - j| = 1, \\
    f_i^2f_j - (q + q^{-1})f_if_jf_i + f_jf_i^2 &= 0 \quad \text{if} \ |i - j| = 1, \\
    e_ie_j = e_je_i &= 0 \quad \text{if} \ |i - j| > 1, \\
    f_if_j - f_jf_i &= 0 \quad \text{if} \ |i - j| > 1, \\
    e_1^2f_1 - (q + q^{-1})e_1f_1e_1 + f_1e_1^2 &= -(q + q^{-1})e_1(qp_1k_1 + p^{-1}q^{-1}k_1^{-1}), \\
    f_1^2e_1 - (q + q^{-1})f_1e_1f_1 + e_1f_1^2 &= -(q + q^{-1})(pq_1k_1 + p^{-1}q^{-1}k_1^{-1})f_1.
\end{align*}
\]

**Proposition 3.1.1.**

(1) [BW18a] Lemma 6.1 (3). There exists a unique $\mathbb{Q}$-algebra automorphism $\psi^j$ of $U^j$ which maps $e_i, f_i, k_i, p, q$ to $e_i, f_i, k_i^{-1}, p^{-1}, q^{-1}$, respectively.

(2) There exists a unique $\mathbb{Q}(p, q)$-algebra anti-automorphism $\sigma^j$ of $U^j$ which maps $e_i, f_i, k_i$ to $f_i, e_i, k_i$, respectively.

(3) [BW18b] Proposition 4.6. There exists a unique $\mathbb{Q}(p, q)$-algebra anti-automorphism $\tau^j$ of $U^j$ which maps $e_i, f_i, k_i$ to $p^{-\delta_{ij}q^{-1}k_1^{-1}}f_i, p^{\delta_{ij}}qe_ik_i, k_i$, respectively.

**Proof.** It suffices to show that the images of the generators of $U^j$ satisfy the defining relations of $U^j$; it is straightforward. \qed

**Remark 3.1.2.** We have similar automorphisms on $U$:

(1) There exists a unique $\mathbb{Q}$-algebra automorphism $\psi$ of $U$ which maps $E_i, F_i, K_i, p, q$ to $E_i, F_i, K_i^{-1}, p^{-1}, q^{-1}$, respectively.

(2) There exists a unique $\mathbb{Q}(p, q)$-algebra anti-automorphism $\sigma$ of $U$ which maps $E_i, F_i, K_i$ to $F_i, E_i, K_i$, respectively.

(3) There exists a unique $\mathbb{Q}(p, q)$-algebra anti-automorphism $\tau$ of $U$ which maps $E_i, F_i, K_i$ to $qF_iK_i^{-1}, q^{-1}K_iE_i, K_i$, respectively.

Note that $\tau^j$ is the restriction of $\tau$ [BW18b] Proposition 4.6], while the others are not.

Let $U(I)$ denote the subalgebra of $U$ generated by $E_i, F_i, K_i^\pm 1$, $i \in \mathbb{I} \setminus \{1\}$, $j \in \mathbb{I}$. Note that we have $e_i, f_i, k_j \in U(I)$ for all $i \in \mathbb{I} \setminus \{1\}$, $j \in \mathbb{I}$. Note that $U(I)$ is the quantum group of type $A_r \times A_{r-1}$ with weight lattice $\Lambda$.

**3.2. Category $O'_{\text{int}}$.** Let us extend the bilinear form $(\cdot, \cdot)$ on $\Lambda$ to $\Lambda_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$. Set $\beta_i := \alpha_i - \alpha_{-i}$, $i \in \mathbb{I}$, and $J := \{\lambda \in \Lambda_{\mathbb{R}} \mid (\beta_i, \lambda) = 0 \text{ for all } i \in \mathbb{I}\}$. Then, the induced bilinear form $(\cdot, \cdot) : (\sum_{i \in \mathbb{I}} \mathbb{R}\beta_i) \times (\Lambda_{\mathbb{R}}/J) \to \mathbb{R}$ denoted by the same symbol is nondegenerate. Let $\delta_j \in \Lambda_{\mathbb{R}}/J$ be such that $(\beta_i, \delta_j) = \delta_{i,j}$ for all $i, j \in \mathbb{I}$. 
Set \( \Lambda := \sum_{i \in \mathbb{U}} \mathbb{Z} \delta_i \). Let \( \gamma_i := \alpha_i + J \in \Lambda \), \( i \in \mathbb{U} \), and \( Q'_+ := \sum_{i \in \mathbb{U}} \mathbb{Z}_{\geq 0} \gamma_i \subset \Lambda \). For \( \lambda, \mu \in \Lambda \), we write \( \mu \leq_\lambda \lambda \) if \( \lambda - \mu \in Q'_+ \). This defines a partial order on \( \Lambda \).

For a \( \mathbb{U} \)-module \( M \) and \( \lambda \in \Lambda \), we call \( M_\lambda := \{ m \in M \mid k_im = q^{(\beta, \lambda)}m \text{ for all } i \in \mathbb{U} \} \) the weight space of \( M \) of weight \( \lambda \). The category \( O^\lambda_{\text{int}} \) is the full subcategory of the category of all \( \mathbb{U} \)-modules consisting of \( \mathbb{U} \)-modules \( M \) satisfying the following:

- \( M \) has a weight space decomposition, i.e., \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \).
- Each weight space of \( M \) is finite-dimensional.
- There exist \( \mu_1, \ldots, \mu_l \in \Lambda \) such that if \( M_\lambda \neq 0 \), then \( \lambda \leq \lambda_i \mu_i \) for some \( i = 1, \ldots, l \).
- The \( f_i \)'s act on \( M \) locally nilpotently.

**Theorem 3.2.1** ([W17]). The following hold:

1. [W17] Theorem 4.4.3]. \( O^\lambda_{\text{int}} \) is semisimple.
2. [W17] Corollary 7.6.3, 7.6.4]. Each irreducible \( \mathbb{U} \)-module in \( O^\lambda_{\text{int}} \) is isomorphic to the irreducible highest weight module \( L(\lambda) \) with highest weight \( \lambda \) (in the sense of [W17]) for some \( \lambda \in P^\lambda \).
3. For \( \lambda, \mu \in P^\lambda \), we have \( L(\lambda) \simeq L(\mu) \) if and only if \( \lambda_i - \mu_i \) is constant as \( i \) runs through \(-r, \ldots, r\).

**Remark 3.2.2.** The last statement follows from the definition of \( L(\lambda) \).

For each \( \lambda \in P^\lambda \), let \( wt^\lambda(\lambda) \in \Lambda \) denote the weight of a highest weight vector of \( L(\lambda) \), namely,

\[
wt^\lambda(\lambda) := \sum_{i \in \mathbb{U}} (\lambda_{i-1} - \lambda_i + \lambda_{-(i-1)} - \lambda_{-i}) \delta_i.
\]

**Remark 3.2.3.** There is an algebra \( \mathbb{U}^\lambda \) that is closely related to \( \mathbb{U} \) (see [BW18a]). It is a coideal subalgebra of \( U_q(\mathfrak{sl}_2) \) such that the pair (\( U_q(\mathfrak{sl}_2), \mathbb{U}^\lambda \)) forms a quantum symmetric pair of type AIV. As mentioned in the introduction part, the results in this paper have counterparts for \( \mathbb{U}^\lambda \). The arguments are parallel except the definition of the category \( O^\lambda_{\text{int}} \), which is the \( \mathbb{U}^\lambda \)-analog of the category \( O^\lambda_{\text{int}} \). The main difference between \( \mathbb{U} \) and \( \mathbb{U}^\lambda \) is that \( \mathbb{U}^\lambda \) has a distinguished generator \( t \). In the definition of \( O^\lambda_{\text{int}} \), we have to add the constraint that \( t \) acts on each \( M \in O^\lambda_{\text{int}} \) diagonally with eigenvalues of the form \( \frac{q^n - q^{-n}}{q - q^{-1}} \), \( a \in \mathbb{Z} \). Then, \( O^\lambda_{\text{int}} \) becomes semisimple, and the isoclasses of irreducible modules in \( O^\lambda_{\text{int}} \) is parametrized by the set of bipartitions of length \((r; r)\).

4. **Crystal basis theory**

4.1. **Crystal bases.** The notion of crystal bases (or local bases at \( q = 0 \)) for integrable modules over quantum groups was introduced independently by Kashiwara and Lusztig in different ways ([K90], [L90a]). Although we will not review the detail, we formulate here some notations concerning the crystal bases. Let \( O_{\text{int}} \) denote the full subcategory of the BGG-category \( O \) for \( \mathbb{U} \) consisting of the integrable modules. Let \( \bar{E}_i, \bar{F}_i, i \in \mathbb{I} \) denote the Kashiwara operators. Let \( M \in O_{\text{int}}, (\mathcal{L}, \mathcal{B}) \) be its crystal basis. For \( b \in \mathcal{B} \) and \( i \in \mathbb{I} \), set

\[
\varepsilon_i(b) := \max\{n \mid \bar{E}_i^n b \neq 0\}, \quad \varphi_i(b) := \max\{n \mid \bar{F}_i^n b \neq 0\}.
\]

Also, \( \text{wt}(b) \in \Lambda \) denotes the weight of \( b \).
Recall that, for each \( \lambda \in P \), the irreducible module \( L(\lambda) \) has a unique crystal basis \((\mathcal{L}(\lambda), \mathcal{B}(\lambda))\), which is identical to \( \text{SST}(\lambda) \). For each \( M \in \mathcal{O}_{\text{int}} \) with a crystal basis \((\mathcal{L}, \mathcal{B})\), we have a unique irreducible decomposition \( \mathcal{B} = \bigsqcup_{i=1}^{l} \mathcal{B}_i \), where \( \mathcal{B}_i \simeq \mathcal{B}(\lambda_i) \) for some \( \lambda_i \in P \). By retaking \( \lambda_i \)'s if necessary, we may assume that \( |\lambda_i| - |\lambda_j| < 2r + 1 \) for all \( i, j \in \{1, \ldots, l\} \), and that there exists \( i \) such that \((\lambda_i)_{2r+1} = 0\). Then, \( \lambda_i \)'s are uniquely determined; we set \( P(M) = P_r(M) := \{\lambda_1, \ldots, \lambda_l\} \). For \( b \in \mathcal{B} \), we define \( I(b) = I_\cdot(b) \in P(M) \) to be \( \lambda_i \) if \( b \in \mathcal{B}_i \). Also let \( C(b) = C_r(b) \subset \mathcal{B} \) denote the connected component of \( \mathcal{B} \) containing \( b \). Furthermore, if we write \( b = \tilde{F}_{i_1} \cdots \tilde{F}_{i_r}b_0 \) for some \( i_1, \ldots, i_r \in \mathbb{I} \), where \( b_0 \) denotes the highest weight vector in \( C(b) \), then define \( T_b \in \text{SST}(I(b)) \) by \( T_b := \tilde{F}_{i_1} \cdots \tilde{F}_{i_r}T_{b_0} \), where \( T_0 \in \text{SST}(I(b)) \) corresponding to \( b_0 \in C(b) = \mathcal{B}(I(b)) \).

4.2. \( j \)-crystal bases. In [W17], the notion of \( j \)-crystal bases was introduced. Let us recall some properties briefly.

Set \( A := \mathbb{Q}[p, p^{-1}, q, q^{-1}] \). We denote by \( A_0 \) the subring of \( \mathbb{Q}(p, q) \) consisting of all elements of the form \( f/g \) with \( f, g \in p\mathbb{Q}[p, q, q^{-1}] \oplus \mathbb{Q}[q], \lim_{q \to 0}(\lim_{p \to 0} f/g) \neq 0 \).

Let \( \overline{\mathbb{I}} := \mathbb{I} \cup \{2', \ldots, r'\} \). The Kashiwara operators are denoted by \( \overline{e}_i \) and \( \overline{f}_i \), \( i \in \overline{\mathbb{I}} \).

The following are basic results for the crystal basis theory of \( U_\mathcal{J} \).

**Theorem 4.2.1** ([W17 Theorem 7.7.3]). Let \( \lambda \in P^j \), \( v_\lambda \in L(\lambda) \) be a highest weight vector. Set

\[
\mathcal{L}(\lambda) := \operatorname{Span}_{A_0} \{ \overline{f}_{i_1} \cdots \overline{f}_{i_l}v_\lambda \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in \overline{\mathbb{I}} \},
\]

\[
\mathcal{B}(\lambda) := \{ \overline{f}_{i_1} \cdots \overline{f}_{i_l}v_\lambda + q\mathcal{L}(\lambda) \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in \overline{\mathbb{I}} \} \setminus \{0\}.
\]

Then, \((\mathcal{L}(\lambda), \mathcal{B}(\lambda))\) is a unique \( j \)-crystal basis of \( L(\lambda) \). Moreover, \( \mathcal{B}(\lambda) \) is identical to \( \text{SST}(\lambda) \); \( v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{B}(\lambda) \) corresponds to \( T_\lambda \in \text{SST}(\lambda) \).

**Theorem 4.2.2.** Suppose that \( M \in \mathcal{O}_{\text{int}} \) has a crystal basis \((\mathcal{L}, \mathcal{B})\). Then, as a \( U_\mathcal{J} \)-module, \( M \) has a \( j \)-crystal basis whose underlying sets are equal to \((\mathcal{L}, \mathcal{B})\).

**Proof.** This is an easy consequence of [W17 Corollary 7.7.4]. \( \square \)

Let \( M \in \mathcal{O}_{\text{int}}' \) with a \( j \)-crystal basis \((\mathcal{L}, \mathcal{B})\). For each \( b \in \mathcal{B} \) and \( i \in \overline{\mathbb{I}} \), define \( \varepsilon_i(b), \varphi_i(b), \operatorname{wt}_i(b) \in \Lambda_j \), \( P^j(M) = P_r^j(M) \subset P^j \), \( P^j(b) = P_r^j(b) \in P^j(M) \), \( C_i^j(b) = C_i^j \subset \mathcal{B} \), and \( T_i^j \in \text{SST}(P^j(b)) \) in a similar way to Section 4.1.

5. Global bases

5.1. Balanced triples. Let \( \overline{\varphi} \) be the \( \mathbb{Q} \)-linear automorphism of \( \mathbb{Q}(p, q) \) sending \( p \) and \( q \) to \( p^{-1} \) and \( q^{-1} \), respectively. Set \( A_\infty := \overline{A}_0 \).

**Definition 5.1.1.** Let \( V \) be a \( \mathbb{Q}(p, q) \)-vector space and \( x \in \{0, \emptyset, \infty\} \). An \( A_x \)-lattice of \( V \) is a free \( A_x \)-submodule \( U_x \) of \( V \) of rank \( \dim_{\mathbb{Q}(p, q)} V \) such that \( \mathbb{Q}(p, q) \otimes_{A_x} U_x = V \).

**Definition 5.1.2** ([K93 Definition 2.1.2]). Let \( V \) be a \( \mathbb{Q}(p, q) \)-vector space, \( U_x \) an \( A_x \)-lattice of \( V \) for \( x \in \{0, \emptyset, \infty\} \). The triple \((U_0, U_\emptyset, U_\infty)\) is said to be balanced if the canonical map

\[
U_0 \cap U \cap U_\infty \to U_0/qU_0
\]

is an isomorphism of \( \mathbb{Q} \)-vector spaces.
Let \( V \) be a \( \mathbb{Q}(p, q) \)-vector space with a balanced triple \( (U_0, U, U_\infty) \). Take a \( \mathbb{Q} \)-basis \( B \) of \( U_0/qU_0 \). Since we have an isomorphism \( G : U_0/qU_0 \to U_0 \cap U \cup U_\infty \) of \( \mathbb{Q} \)-vector spaces, which is the inverse of the canonical map \( U_0 \cap U \cup U_\infty \to U_0/qU_0 \), we obtain an \( A_x \)-basis \( G(B) = \{ G(b) \mid b \in B \} \) of \( U_x \) for each \( x \in \{ 0, \emptyset, \infty \} \). We call \( G(B) \) the global basis of \( V \) associated to the balanced triple \( (U_0, U, U_\infty) \) and the basis \( B \).

**Lemma 5.1.3.** Let \( V, U_0, U, U_\infty, B, G \) be as above. Take a subset \( \mathcal{B}' \subset \mathcal{B} \) and set \( U'_x \) to be the \( A_x \)-span of \( G(\mathcal{B}') := \{ G(b) \mid b \in \mathcal{B}' \} \) for each \( x \in \{ 0, \emptyset, \infty \} \). Also, let \( V' \) be the \( \mathbb{Q}(p, q) \)-span of \( G(\mathcal{B}') \). Then, the following hold:

1. \( (U'_0, U'_0, U'_\infty) \) is a balanced triple with the global basis \( G(\mathcal{B}') \).
2. \( (U'_0/U'_0, U/U'_0, U_\infty/U'_\infty) \) is a balanced triple with the global basis \( \{ G(b) + V' \mid b \in \mathcal{B} \setminus \mathcal{B}' \} \).

### 5.2. Global crystal bases and global \( j \)-crystal bases

Let \( U_A \) denote the \( A \)-subalgebra of \( U \) generated by \( E_i^{(n)}, F_i^{(n)}, K_i^{\pm 1}, i \in \mathbb{I}, n \in \mathbb{Z}_{>0} \). Similarly, define \( U'_A \) to be the \( A \)-subalgebra of \( U' \) generated by \( e_i^{(n)}, f_i^{(n)}, k_i^{\pm 1}, i \in \mathbb{I}, n \in \mathbb{Z}_{>0} \).

**Lemma 5.2.1.** \( \text{(L10 1.3.5)} \). Let \( A \) be a \( \mathbb{Q}(q) \)-algebra, \( x, y \in A \) such that \( xy = q^2yx \). Then, for each \( n \in \mathbb{Z}_{>0} \), we have

\[
(x + y)^n = \sum_{t=0}^{n} q^{t(n-t)} {n \choose t} y^t x^{n-t}.
\]

**Lemma 5.2.2.** We have \( U'_A \subset U_A \).

**Proof.** It suffices to show that \( e_i^{(n)}, f_i^{(n)} \in U_A \) for all \( i \in \mathbb{I}, n \in \mathbb{Z}_{>0} \). We prove \( e_i^{(n)} \in U_A \); the proof for \( f_i^{(n)} \in U_A \) is similar. Setting \( x := E_i \) and \( y := p^{-\delta_i, 1} F_i^{-1} K_i^{-1} \), we see that

\[
e_i = x + y, \quad xy = q^2yx.
\]

Then, we can apply Lemma 5.2.1 and obtain

\[
e_i^{(n)} = \sum_{t=0}^{n} q^{t(n-t)} y^t x^{(n-t)}.
\]

It is easy to see that \( y^t = p^{-\delta_i, 1} q^{-\delta_i, 1} F_i^{-1} F_i^{(t)} K_i^{t} \in U_A \). Hence, the assertion follows. \( \square \)

Let \( V \) be a \( U \)-module in \( \mathcal{O}_{\text{int}} \) (resp., \( U' \)-module in \( \mathcal{O}'_{\text{int}} \)) with a crystal basis \( (\mathcal{L}, \mathcal{B}) \) (resp., \( j \)-crystal basis \( (\mathcal{L}, \mathcal{B}) \)). Assume that \( V \) admits a \( \mathbb{Q} \)-linear involution \( \overline{\cdot} \) satisfying the following:

\[
\overline{\psi(x)} = \psi(x), \quad \text{for all } x \in U, \quad \overline{V} = V.
\]

We call such an involution a \( \psi \)-involution (resp., \( \psi^j \)-involution) on \( V \). Since \( \mathcal{L} \) is an \( A_{\emptyset} \)-lattice of \( V \), \( \overline{\mathcal{L}} \) is an \( A_{\infty} \)-lattice of \( V \).

**Definition 5.2.3.** Let \( V, \mathcal{L}, \mathcal{B}, \overline{\cdot} \) be as above. \( V \) is said to have a global crystal basis (resp., global \( j \)-crystal basis) if there exists a \( U_A \)-submodule (resp., \( U'_A \)-submodule) \( V_A \) of \( V \) which is an \( A \)-lattice forming a balanced triple \( (\mathcal{L}, V_A, \overline{\mathcal{L}}) \). The associated global basis \( G(\mathcal{B}) \) (resp., \( G'(\mathcal{B}) \)) is called a global crystal basis (resp., global \( j \)-crystal basis) of \( V \).
Example 5.2.4. Let $\lambda \in P_1^+$ and consider the irreducible $U'_1$-module $L(\lambda)$. Recall that $L(\lambda)$ is $(\lambda_0 - \lambda_{-1} + 1)$-dimensional with a basis $G^j(\lambda) := \{ f_1^{(n)}(n) v \mid 0 \leq n \leq \lambda_0 - \lambda_{-1} \}$, where $v$ denotes a highest weight vector. Also, $L(\lambda)$ has a $j$-crystal basis $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$, where $\mathcal{L}(\lambda)$ is the $\mathbb{A}_0$-span of $G^j(\lambda)$, and $\mathcal{B}(\lambda) = \{ f_1^{(n)}(n) v + q \lambda(\lambda) \mid 0 \leq n \leq \lambda_0 - \lambda_{-1} \}$. Set $L(\lambda)_\mathbb{A}$ to be the $\mathbb{A}$-span of $G^j(\lambda)$. Note that there exists a unique $\psi_j$-involution $\psi_j^\lambda$ on $L(\lambda)$ fixing $v$. Then, $(\mathcal{L}(\lambda), L(\lambda)_\mathbb{A}, \psi_j^\lambda(\mathcal{L}(\lambda)))$ is a balanced triple, and $G^j(\lambda)$ is a global $j$-crystal basis of $L(\lambda)$.

Proposition 5.2.5. Let $M \in \mathcal{O}'_{\text{int}}$ with a global crystal $j$-crystal basis $G^j(\mathcal{B}_M)$, and $N \in \mathcal{O}_{\text{int}}$ with a global crystal basis $G(\mathcal{B}_N)$ Then, $M \otimes N$ has a global $j$-crystal basis of the form
\[
\{ G^j(b_1) \otimes G(b_2) \mid b_1 \in \mathcal{B}_M, b_2 \in \mathcal{B}_N \}.
\]
\[
G^j(b_1) \otimes G(b_2) \in G^j(b_1) \otimes G(b_2) + \sum_{b'_1 \in \mathcal{B}_M, b'_2 \in \mathcal{B}_N, \wt(b'_2) < \wt(b_2)} a_{b_1 b'_1 b_2 b'_2} G^j(b'_1) \otimes G(b'_2), \quad a_{b_1 b'_1 b_2 b'_2} \in \mathbb{A}.
\]

Proof. The fact that $\mathcal{B}_M \otimes \mathcal{B}_N$ forms a $j$-crystal basis of $M \otimes N$ is proved in [W17]. Now, one can construct a global $j$-crystal basis of $M \otimes N$ with the desired property in the same way as the proof of [BW20] Theorem 4].

5.3. $j$-canonical bases. In this subsection, we recall the notion of $j$-canonical bases, which was introduced by Bao and Wang in [BW18a], and explain that $j$-canonical bases are global $j$-crystal bases. One of the key ingredients for a construction of $j$-canonical bases is the intertwiner $\Upsilon$:

Definition 5.3.1 ([BW18a Theorem 6.4]). Let $U^-$ denote the subalgebra of $U$ generated by $F_i$, $i \in \mathbb{I}$. For each $\lambda \in Q_+$, there exists a unique $\Upsilon_\lambda \in U^-_{-\lambda}$ satisfying the following:
\begin{itemize}
  \item $\Upsilon_0 = 1$,
  \item $\Upsilon := \sum_{\lambda \in Q_+} \Upsilon_\lambda$ satisfies $\psi_j(\lambda(x)) \Upsilon = \Upsilon \psi_j(x)$ for all $x \in U^j$.
\end{itemize}

Lemma 5.3.2 ([BW18a Proposition 6.12]). Let $M \in \mathcal{O}_{\text{int}}$ with a $\psi$-involution $\psi_M$. Then, the composite $\Upsilon \circ \psi_M$ is a $\psi_j$-involution of $M$.

Theorem 5.3.3 ([BW18a Theorem 6.24]). Let $M \in \mathcal{O}_{\text{int}}$ have a global crystal basis $G(\mathcal{B})$ with a crystal basis $(\mathcal{L}, \mathcal{B})$, a $\psi$-involution $\psi_M$, and an $\mathbb{A}$-lattice $M_\mathbb{A}$. Set $\psi_M^j := \Upsilon \circ \psi_M$. Then, for each $b \in \mathcal{B}$, there exists a unique $G^j(b) \in M$ satisfying the following.
\begin{enumerate}
  \item $\psi_M^j(G^j(b)) = G^j(b)$,
  \item $G^j(b) = G(b) + \sum_{b' \in \mathcal{B}} c_{b' b} G(b')$ for some $c_{b' b} \in q \mathbb{A}_0 \cap \mathbb{A}$. Moreover, $c_{b' b} = 0$ unless $\wt(b') = \wt(b)$ and $\wt(b') < \wt(b)$.
\end{enumerate}

The new basis $G^j(\mathcal{B}) := \{ G^j(b) \mid b \in \mathcal{B} \}$ thus constructed is called the $j$-canonical basis of $(M, G(\mathcal{B}))$.

Proposition 5.3.4. We keep the notation in Theorem 5.3.3. Then, $(\mathcal{L}, \mathcal{B})$ is a $j$-crystal basis, $(\mathcal{L}, M_\mathbb{A}, \psi_M^j(\mathcal{L}))$ is a balanced triple, and $G^j(\mathcal{B})$ is the global $j$-crystal basis associated to the balanced triple $(\mathcal{L}, M_\mathbb{A}, \psi_M^j(\mathcal{L}))$ and the basis $\mathcal{B}$. 
Proof. That \((L,B)\) is a \(\gamma\)-crystal basis has already been stated in Theorem 4.2.2. Let us prove the rest. By the property (2) of Theorem 5.3.3, it is clear that \(L\) \((\text{resp., } M)\) is spanned by \(G^j(B)\) over \(A_0\) \((\text{resp., } A)\). Also, by (1) of Theorem 5.3.3, \(\psi_M^j(L)\) is spanned by \(G^j(B)\) over \(A_\infty\). Hence, the canonical homomorphism \(L \cap M \cap \psi_M^j(L) \to L/qL\) is an isomorphism, and therefore, \((L,M,A,\psi_M^j(L))\) is balanced. Finally, by Lemma 5.2.2, the \(U_A\)-module \(M_A\) is also a \(U_A^j\)-module. This proves the proposition.

\[\square\]

6. Kazhdan-Lusztig bases

The subsequent three sections are dedicated to prove the existence of a global \(\gamma\)-crystal basis and its “dual” basis for \(L(\lambda), \lambda \in P\). In this section, we formulate variants of the Kazhdan-Lusztig bases following [KL79], [Deo87], and [L03].

6.1. Hecke algebra of type \(B\). Fix \(d \in \mathbb{Z}_{>0}\). Let \(W = W_d\) be the Weyl group of type \(B_d\) with simple reflections \(S = \{s_0,s_1,\ldots,s_{d-1}\}\) such that

\[s_0s_1s_0s_1 = s_1s_0s_1s_0 = s_{i+1}s_is_{i+1} \text{ if } i \geq 1, \quad s_is_j = s_js_i \text{ if } |i-j| > 1.\]

Definition 6.1.1. The Hecke algebra \(H = H(W)\) associated to \(W\) with unequal parameters \(p,q\) is the associative algebra over \(A_{\mathbb{Z}} := \mathbb{Z}[p,p^{-1},q,q^{-1}]\) generated by \(\{H_s \mid s \in S\}\) subject to the following relations:

\[\bullet \quad (H_s - q_s^{-1})(H_s + q_s) = 0 \text{ for all } s \in S, \text{ where } q_s = p \text{ if } s = s_0 \text{ and } q_s = q \text{ otherwise.}\]

\[\bullet \quad H_{s_0}H_{s_1}H_{s_0}H_{s_1} = H_{s_1}H_{s_0}H_{s_1}H_{s_0}.\]

\[\bullet \quad H_{s_i}H_{s_{i+1}}H_{s_i} = H_{s_{i+1}}H_{s_i}H_{s_{i+1}} \text{ if } i \geq 1.\]

\[\bullet \quad H_{s_i}H_{s_j} = H_{s_j}H_{s_i} \text{ if } |i-j| > 1.\]

We often write \(H_i = H_{s_i}\). For each \(w \in W\) with a reduced expression \(w = s_{i_1}\cdots s_{i_l}\), the product \(H_{i_1}\cdots H_{i_l}\) is independent of the choice of a reduced expression of \(w\); we denote it by \(H_w\). Similarly, \(q_w := q_{s_{i_1}}\cdots q_{s_{i_l}}\) is well-defined.

Let \(U,V\) be modules over \(A_{\mathbb{Z}}\). We say a \(\mathbb{Z}\)-linear map \(f : U \to V\) is anti-linear if it satisfies \(f(gu) = \overline{f}(u)\) for all \(g \in A_{\mathbb{Z}}\) and \(u \in U\). In the sequel, we will often use the following automorphisms, all of which are involutions, of \(\mathcal{H}\).

Lemma 6.1.2.

1. There exists a unique anti-linear algebra automorphism \(\overline{\cdot}\) of \(\mathcal{H}\) such that \(\overline{H_w} = H_w^{-1}\).

2. There exists a unique anti-linear algebra automorphism \(\text{sgn}\) of \(\mathcal{H}\) such that \(\text{sgn}(H_w) = (-1)^{\ell(w)}H_w\). Here, \(\ell : W \to \mathbb{Z}_{\geq 0}\) denotes the length function on \(W\).

3. There exists a unique \(A_{\mathbb{Z}}\)-algebra anti-automorphism \((\cdot)^\flat\) of \(\mathcal{H}\) such that \(H_w^\flat = H_w^{-1}\).

Moreover, all of these automorphisms commute with each other.

For \(y, w \in W\), define \(r_{y,w} \in A_{\mathbb{Z}}\) by

\[\overline{H_w} = \sum_{y \in W} r_{y,w}H_y.\]

It is well-known and easily proved that \(r_{w,w} = 1\) for all \(w \in W\) and \(r_{y,w} = 0\) unless \(y \leq w\).
6.2. Kazhdan-Lusztig bases. Let us formulate the Kazhdan-Lusztig basis and the dual Kazhdan-Lusztig basis. Set
\[ \mathbf{A}_Z^+ := \mathbf{A}_Z \cap \mathbf{A}_0 = p\mathbb{Z}[p, q, q^{-1}] \oplus q\mathbb{Z}[q], \]
\[ \mathbf{A}_Z^- := \mathbf{A}_Z^+ = p^{-1}\mathbb{Z}[p^{-1}, q, q^{-1}] \oplus q^{-1}\mathbb{Z}[q^{-1}]. \]

**Theorem 6.2.1** ([KL79 Theorem 1.1], [Lo3 Theorem 5.2]). For each \( w \in W \), there exists a unique \( C_w \in \mathcal{H} \) such that
\begin{enumerate}
  \item \( \overline{C}_w = C_w \).
  \item \( C_w = H_w + \sum_{y < w} c_{y,w} H_y \) for some \( c_{y,w} \in \mathbf{A}_Z^+ \). Here, \( < \) denotes the Bruhat order on \( W \).
\end{enumerate}

Replacing \( \mathbf{A}_Z^+ \) with \( \mathbf{A}_Z^- \), we see the following: For each \( w \in W \), there exists a unique \( D_w \in \mathcal{H} \) such that
\begin{enumerate}
  \item \( \overline{D}_w = D_w \).
  \item \( D_w = H_w + \sum_{y < w} d_{y,w} H_y \) for some \( d_{y,w} \in \mathbf{A}_Z^- \).
\end{enumerate}

**Remark 6.2.2.** Noting that the automorphisms \( - \) and \( \text{sgn} \) commute with each other, it is easy to verify that \( D_w = (-1)^{\ell(w)} \text{sgn}(C_w) \).

It is obvious from the definitions that both \( \{ C_w \mid w \in W \} \) and \( \{ D_w \mid w \in W \} \) form \( \mathbf{A}_Z \)-bases of \( \mathcal{H} \). We call the former the Kazhdan-Lusztig basis, and the latter the dual Kazhdan-Lusztig basis of \( \mathcal{H} \).

6.3. Left cell representations. Let us recall from [KL79] the notion of left cells of \( W \) and the associated left cell representations.

**Definition 6.3.1.** Let \( y, w \in W \).
\begin{enumerate}
  \item \( y \to_L w \) if the coefficient of \( C_y \) in \( C_s C_w \) expanded in the Kazhdan-Lusztig basis is nonzero for some \( s \in S \).
  \item \( y \leq_L w \) if there exist \( y = y_0, y_1, \ldots, y_l = w \) such that \( y_{i-1} \to_L y_i \).
  \item \( y \sim_L w \) if \( y \leq_L w \) and \( w \leq_L y \).
  \item \( y <_L w \) if \( y \leq_L w \) and \( y \not\sim_L w \).
  \item Each equivalence class of \( W/\sim \) is called a left cell of \( W \). We denote by \( L(W) \) the set of left cells of \( W \).
\end{enumerate}

**Remark 6.3.2.** By Remark 6.2.2, we obtain the same equivalence relation as \( \sim \) if we replace \( C_w \)’s by \( D_w \)’s.

For each \( X \in L(W) \) and \( x \in X \), set
\[ C_{\leq_L X} = \bigoplus_{y \leq_L X} A_Z C_y, \quad C_{<_L X} = \bigoplus_{y <_L X} A_Z C_y, \quad C_X = C_{\leq_L X}/C_{<_L X}, \]
\[ D_{\leq_L X} = \bigoplus_{y \leq_L X} A_Z D_y, \quad D_{<_L X} = \bigoplus_{y <_L X} A_Z D_y, \quad D_X = D_{\leq_L X}/D_{<_L X}. \]

Note that these are independent of the choice of \( x \in X \). We denote the image of \( m \in C_{\leq_L X} \) (resp., \( m \in D_{\leq_L X} \)) under the quotient map \( C_{\leq_L X} \to C_X \) (resp., \( D_{\leq_L X} \to D_X \)) by \( [m]_X \) (resp., \( [m]_X' \)).

**Lemma 6.3.3.** Let \( X \in L(W) \). Then, \( C_{\leq_L X}, C_{<_L X}, D_{\leq_L X}, \) and \( D_{<_L X} \) are left ideals of \( \mathcal{H} \), and consequently, \( C_X \) and \( D_X \) are left \( \mathcal{H} \)-modules. Moreover, \( C_X \) has a basis \( \{ [C_x]_X \mid x \in X \} \), while \( D_X \) has a basis \( \{ [D_x]_X' \mid x \in X \} \).
Proof. The assertions are obvious from the definitions. □

We call \( C^L_X \) the left cell representation of \( \mathcal{H}(W) \) associated to \( X \in L(W) \).

6.4. Bilinear form on \( \mathcal{H} \). Let \( \mathcal{H}^* := \text{Hom}_{A_\mathbb{Z}}(\mathcal{H}, A_\mathbb{Z}) \). \( \mathcal{H}^* \) has a left \( \mathcal{H} \)-module structure given by

\[
(Hf)(H') = f(H^bH'), \quad \text{for all } f \in \mathcal{H}^*, H, H' \in \mathcal{H}.
\]

Let \( \{h_w \mid w \in W\} \subset \mathcal{H}^* \) be the dual basis of \( \{H_w \mid w \in W\} \), that is, they are characterized by \( h_y(H_w) = \delta_{y,w} \) for all \( y, w \in W \).

**Lemma 6.4.1.** For each \( w \in W \) and \( s \in S \), the following holds.

\[
H_sh_w = \begin{cases} 
    h_{sw} & \text{if } w < sw, \\
    h_{sw} + (q_s^{-1} - q_s)h_w & \text{if } sw < w.
\end{cases}
\]

**Proof.** For each \( y \in W \), we compute as

\[
(H_sh_w)(H_y) = h_w(H_sH_y) = \begin{cases} 
    h_w(H_{sy}) & \text{if } sy > y, \\
    h_w(H_{sy} + (q_s^{-1} - q_s)H_y) & \text{if } sy < y \\
    1 & \text{if } sy > y \text{ and } sy = w,
    1 & \text{if } sy < y \text{ and } sy = w,
    q_s^{-1} - q_s & \text{if } sy < y \text{ and } y = w,
    0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
    h_{sw}(H_y) & \text{if } sw > w, \\
    (h_{sw} + (q_s^{-1} - q_s)h_w)(H_y) & \text{if } sw < w.
\end{cases}
\]

This implies

\[
H_sh_w = \begin{cases} 
    h_{sw} & \text{if } sw > w, \\
    h_{sw} + (q_s^{-1} - q_s)h_w & \text{if } sw < w.
\end{cases}
\]

Thus, the proof completes. □

There exists an anti-linear automorphism \( \overline{\cdot} \) of \( \mathcal{H}^* \) defined by \( \overline{f}(H) = \overline{f(H)} \) for \( f \in \mathcal{H}^*, H \in \mathcal{H} \).

**Lemma 6.4.2.** For each \( w \in W \), we have

\[
\overline{h_w} = \sum_{y \geq w} r_{w,y}h_y.
\]

In particular, \( \overline{h_w} = h_w \), where \( w_0 \in W \) denotes the longest element.

**Proof.** Let \( y \in W \). Then, we have

\[
\overline{h_w}(H_y) = h_w(\overline{H_y}) = h_w(\sum_{z \leq y} r_{z,y}H_z) = r_{w,y}.
\]

Since \( \overline{h_w} = \sum_{y \in W} \overline{h_w}(H_y)h_y \), the assertion follows. □

Let \( \{C_w^* \mid w \in W\} \subset \mathcal{H}^* \) denote the dual basis of \( \{C_w \mid w \in W\} \).
Proposition 6.4.3. \( C^*_w \) is characterized by the following two conditions:

1. \( \overline{C^*_w} = C^*_w \).
2. \( C^*_w = h_w + \sum_{z > w} c^*_w, h_z \) for some \( c^*_w, h_z \in A^+_\mathbb{Z} \).

Proof. Thanks to Lemma 6.4.2, one can prove that there exists a unique \( C'_w \in \mathcal{H}^* \) such that \( \overline{C^*_w} = C'_w \) and \( C'_w - h_w \in \bigoplus_{y > w} A^+_\mathbb{Z} h_y \) in a similar way to Theorem 6.2.1.

Hence, it suffices to show that \( C^*_w \) satisfies the two conditions.

The first condition is verified as follows. For each \( y \in W \), we have

\[
\overline{C^*_y} (C_y) = C^*_y (C_y) = C^*_y (C_y) = \delta_{y,w} = \delta_{y,w} = C^*_w (C_y).
\]

Since \( \{C_y \mid y \in W\} \) is a basis of \( \mathcal{H} \), we obtain \( \overline{C^*_w} = C^*_w \).

Next, we prove the second condition. For each \( y \in W \), we can write \( H_y = C_y + \sum_{z < y} b_{z,y} C_z \) for some \( b_{z,y} \in A^+_\mathbb{Z} \). Then, we have

\[
C^*_w = \sum_{y \in W} C^*_w (H_y) h_y = h_w + \sum_{y > w} b_{w,y} h_y.
\]

This completes the proof. \( \square \)

Lemma 6.4.4. The linear map \( d : \mathcal{H} \to \mathcal{H}^* ; H \mapsto H \cdot h_{w_0} \) gives an isomorphism of left \( \mathcal{H} \)-modules. Moreover, we have

1. \( d(\overline{H}_y) = h_{y w_0} \) for all \( y \in W \).
2. \( d(\overline{H}) = d(\overline{H}) \) for all \( H \in \mathcal{H} \).

Proof. By Lemma 6.4.1 the linear map \( \varphi : \mathcal{H} \to \mathcal{H}^* ; H_w \mapsto h_w \) is an isomorphism of left \( \mathcal{H} \)-modules. On the other hand, the map \( \psi : \mathcal{H} \to \mathcal{H} ; H \mapsto H \cdot H_{w_0} \) is clearly an isomorphism of left \( \mathcal{H} \)-modules. Thus, the composite map \( d := \varphi \circ \psi : \mathcal{H} \to \mathcal{H}^* \) is an isomorphism of \( \mathcal{H} \)-modules satisfying

\[
d(H) = \varphi(H \cdot h_{w_0}) = H \cdot \varphi(h_{w_0}) = H \cdot h_{w_0} \quad \text{for all } H \in \mathcal{H}.
\]

Also, we have, for all \( y \in W \),

\[
d(\overline{H}_y) = \varphi(\overline{H}_y \cdot H_{w_0}) = \varphi(H^{-1}_y \cdot H y_{w_0}) = \varphi(H y_{w_0}) = h_{y w_0}.
\]

Finally, for each \( H, H' \in \mathcal{H} \), we have

\[
d(\overline{H})(H') = \overline{H} \cdot h_{w_0} (H') = h_{w_0} \left( \overline{H} H' \right),
\]

\[
d(\overline{H})(H') = h_{w_0} \left( H \overline{H} H' \right) = h_{w_0} \left( H H' \overline{H} \right).
\]

Then, the equality \( d(\overline{H}) = d(\overline{H}) \) follows from the facts that \( h_{w_0} = h_{w_0} \) and \( (\overline{H}) = \overline{H} \); the former is proved in Lemma 6.4.2 and the latter is in Lemma 6.1.2. \( \square \)

Using this isomorphism, we define a bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H} \) by

\[
\langle H, H' \rangle := d(H')(H), \quad (H, H' \in \mathcal{H}).
\]

Clearly, this bilinear form satisfies \( \langle H' \mid H H'' \rangle = \langle H H' \mid H'' \rangle \) for all \( H, H', H'' \in \mathcal{H} \).

Lemma 6.4.5. The bilinear from \( \langle \cdot, \cdot \rangle \) is symmetric.

Proof. Let \( H_1, H_2 \in \mathcal{H} \). It suffices to show that \( h_{w_0}(H_2^* H_1) = h_{w_0}(H_1^* H_2) \). Since \( H_{w_0}^* = H_{w_0} \), it holds that \( h_{w_0}(H_2^*) = h_{w_0}(H) \) for all \( H \in \mathcal{H} \). Then, the assertion follows if one notes \( (H_2^* H_1)^* = H_1^* H_2 \). \( \square \)
Proposition 6.4.6. The bases \(\{C_w \mid w \in W\}\) and \(\{D_{ww_0} \mid w \in W\}\) are dual to each other with respect to \(\langle \cdot \mid \cdot \rangle\), that is, we have \(\langle C_y \mid D_w \rangle = \delta_{yw,ww_0}\) for all \(y, w \in W\).

Proof. Recall that \(D_w = \sum_{y \leq w} d_{y,w}H_y\) with \(d_{w,w} = 1\) and \(d_{y,w} \in A_{\Sigma}^-\) for all \(y < w\). Then, we have

\[
\begin{align*}
 d(D_w) &= d(D_{ww_0}) = d(D_w), \\
 d(D_w) &= d(D_{ww_0}) = d(\sum_{y \leq w} d_{y,w}H_y) = \sum_{y \leq w} d_{y,w}h_{yw_0} = \sum_{z \geq w} d_{zw_0,w}h_z.
\end{align*}
\]

This proposition 6.4.3 show that \(d(D_w) = C^*_{ww_0}.\) Hence, it holds that \(\langle C_y \mid D_w \rangle = C^*_{ww_0}(C_y) = \delta_{yw,ww_0},\) which proves the proposition. \(\square\)

Here, we describe the duality between \(C^d_X\)'s and \(D^d_X\)'s.

Lemma 6.4.7. Let \(y, w \in W, X \in L(W).\) Then, the following hold.

1. \(y \rightarrow_L w\) if and only if \(ww_0 \rightarrow_L yw_0.\)
2. \(y \leq_L w\) if and only if \(yw_0 \leq_L w_0 yw_0.\)
3. \(Xw_0 := \{xw_0 \mid x \in X\} \in L(W).\)

Proof. We first prove part (1). Suppose that \(y \rightarrow_L w.\) Then, there exists \(s \in S\) such that \(\langle C_sC_w \mid D_{yw_0}\rangle \neq 0.\) This implies that \(\langle C_w \mid C_sD_{yw_0}\rangle \neq 0,\) and hence, we obtain \(ww_0 \rightarrow_L yw_0.\) Replacing \(y, w\) by \(yw_0, ww_0,\) we also have the opposite indication. This proves part (1). Assertion (2) is an immediate consequence of (1).

We prove part (3). Let \(x \in X.\) Then, \(X = \{y \in W \mid x \leq_L y \leq_L x\}.\) By part (2), we have \(x \leq_L y \leq_L x\) if and only if \(yw_0 \leq_L xw_0\). This implies that \(Xw_0 = \{z \in W \mid xw_0 \leq_L z \leq_L xw_0\},\) and it is a unique left cell of \(W\) containing \(xw_0.\) Thus, the proof completes. \(\square\)

Lemma 6.4.8. The bilinear from \(\langle \cdot \mid \cdot \rangle\) induces a non-degenerate bilinear form on \(C^d_X \times D^d_{Xw_0}.\) Moreover, \(\{[C_x]_X \mid x \in X\}\) and \(\{[D_{xw_0}]_{Xw_0} \mid x \in X\}\) form bases which are dual to each other.

Proof. Let \(x, y, w \in W\) be such that \(y <_L x\) and \(w <_L xw_0.\) It suffices to show that \(\langle C_y \mid D_u \rangle = 0\) for all \(u \leq_L xw_0\) and \(\langle C_y \mid D_{ww_0} \rangle = 0\) for all \(v \leq_L x.\) Both are obvious from Lemma 6.4.7 (2). \(\square\)

Proposition 6.4.9. Let \(X \in L(W).\) Then, we have an isomorphism \(D^L_{Xw_0} \simeq C^d_X\) of \(\mathcal{H}\)-modules.

Proof. It suffices to show that the characters \(\text{ch}_{D^L_{Xw_0}}\) of \(D^L_{Xw_0}\) and \(\text{ch}_{C^d_X}\) of \(C^d_X\) coincide with each other. For each \(w \in W,\) we compute as

\[
\begin{align*}
 \text{ch}_{C^d_X}(H_w) &= \sum_{x \in X} \langle C_x \mid [D_{xw_0}]_{Xw_0} \rangle \\
 &= \sum_{x \in X} \langle [C_x]_X \mid H_{w^{-1}}[D_{xw_0}]_{xw_0} \rangle \\
 &= \text{ch}_{D^L_{Xw_0}}(H_{w^{-1}}) = \text{ch}_{D^L_{Xw_0}}(H_w).
\end{align*}
\]

Thus, the proof completes. \(\square\)
6.5. **Parabolic Kazhdan-Lusztig bases.** Throughout this subsection, we fix a subset $J \subset \{0, 1, \ldots, d-1\}$ arbitrarily. Let $W_J$ denote the parabolic subgroup of $W$ generated by $\{s_j \mid j \in J\}$, $^JW$ the set of minimal length coset representatives for $W_J \backslash W$, and $w_J \in W_J$ the longest element. Also, we set

$$x_J := q_{w_J}^{-1} \sum_{w \in W_J} q_w^{-1} H_w \in \mathcal{H}.$$ 

**Lemma 6.5.1.** Let $j \in J$. Then, the following hold.

1. $x_J H_j = q_{s_j}^{-1} x_J$.
2. $x_J^\vee = x_J$.
3. $x_J = C_{w_J}$. In particular, $x_J = x_J$.

**Proof.** The assertion (1) follows from a direct calculation and the fact that $W_J = \{w \in W_J \mid w < s_j w\} \sqcup \{w \in W_J \mid s_j w < w\}$. The assertion (2) follows from the definition of $x_J$ and the facts that $W_J = \{w^{-1} \mid w \in W_J\}$, and $q_{w^{-1}} = q_w$ for all $w \in W$. The proof of (3) can be found in [X94, Proposition 1.17 (2)]. □

By Lemma 6.5.1 (1), the right ideal $x_J \mathcal{H}$ of $\mathcal{H}$ has a basis $\{x_J H_w \mid w \in ^JW\}$. Also, by Lemma 6.5.1 (3), $x_J \mathcal{H}$ is closed under the involution $\overline{\cdot}$. Hence, we can construct analogs of the Kazhdan-Lusztig basis and the dual Kazhdan-Lusztig basis of $\mathcal{H}$ in the ideal $x_J \mathcal{H}$:

**Theorem 6.5.2 ([Deo87, Proposition 3.2]).**

1. For each $w \in ^JW$, there exists a unique $^JC_w \in x_J \mathcal{H}$ such that

   a. $\overline{^JC_w} = ^J C_w$.
   b. $^JC_w = x_J (H_w + \sum_{y < w} ^J c_{y,w} H_y)$ for some $^J c_{y,w} \in A^+_2$.

2. For each $w \in ^JW$, there exists a unique $^JD_w \in x_J \mathcal{H}$ such that

   a. $\overline{^JD_w} = ^J D_w$.
   b. $^JD_w = x_J (H_w + \sum_{y < w} ^J d_{y,w} H_y)$ for some $^J d_{y,w} \in A^-_2$.

Clearly, $\{^JC_w \mid w \in ^JW\}$ and $\{^JD_w \mid w \in ^JW\}$ are linear bases of $x_J \mathcal{H}$. We call them the parabolic Kazhdan-Lusztig basis and the dual parabolic Kazhdan-Lusztig basis of $x_J \mathcal{H}$, respectively.

**Proposition 6.5.3 ([Deo87, Proposition 3.4]).** Let $w \in ^JW$. Then, $^JC_w = C_{w,jw}$.

**Proposition 6.5.4.** Let $w \in ^JW$. Then, $^JD_w = x_J D_w$. 

Proof. For each \( y \in W \), define \( y_J \in W_J \) and \( J^y \in J^W \) to be the unique elements satisfying \( y = y_J^{J^y} \) and \( \ell(y) = \ell(y_J) + \ell(J^y) \). Then, we have

\[
x_J D_w = x_J \sum_{y \leq w} d_{y,w} H_y
\]

\[
= x_J \left( H_w + \sum_{y < w} d_{y,w} H_{y,J^y} \right)
\]

\[
= x_J \left( H_w + \sum_{y < w} q_{y,J}^{-1} d_{y,w} H_{y,J^y} \right) \quad \text{(by Lemma 6.5.1 (1))}
\]

\[
= x_J \left( H_w + \sum_{y \in J^W} \sum_{x \in W_J \atop xy < w} q_{x,J}^{-1} d_{x,y,w} H_y \right).
\]

This shows that \( x_J D_w - x_J H_w \in \bigoplus_{y < w} A \sum_{x \leq y} x_J H_y \). Hence, by Theorem 6.5.2 (2), \( x_J D_w \) coincides with \( J D_w \).

For a later use, let us consider \( x_J C_y \) and \( x_J D_y \) for \( y \in W \).

Proposition 6.5.5. Let \( y \in W \). Then, we have

\[
x_J C_y = \sum_{w \in J^W} \alpha_w J C_w,
\]

for some \( \alpha_w \in A \).

Proof. Let us write

\[
x_J C_y = \sum_{w \in J^W} \alpha_w J C_w = \sum_{w \in J^W} \alpha_w C_{w,J} \quad \text{for some } \alpha_w \in A.
\]

Also, by the definition of \( \leq_L \), we can write

\[
x_J C_y = \sum_{z \leq_L y} \beta_z C_z \quad \text{for some } \beta_z \in A.
\]

This shows \( \alpha_w = 0 \) unless \( w_J w \leq_L y \). \( \square \)

Lemma 6.5.6 ([L03 Theorem 6.6 (b)]). Let \( w \in W \) and \( s \in S \) be such that \( sw < w \). Then, it holds that \( H_s D_w = -q_s D_w \).

Proposition 6.5.7. Let \( y \in W \setminus J^W \). Then, \( x_J D_y = 0 \).

Proof. Since \( y \notin J^W \), there exists \( j \in J \) such that \( s_j y < y \). For such \( j \), we have \( x_J H_j = q_j^{-1} x_J \) (Lemma 6.5.1 (1)) and \( H_j D_y = -q_j D_y \) (Lemma 6.5.6). Hence, we obtain

\[
x_J D_y = q_j x_J H_j D_y = -q_j^2 x_J D_y,
\]

which implies \( x_J D_y = 0 \), as desired. \( \square \)

Set \( P_J := q_{w_J} \sum_{x \in W_J} q_x^{-2} \in A \). Note that, by Lemma 6.5.1 (1), it holds that \( x_J^2 = P_J x_J \). Then, for each \( H, H' \in \mathcal{H} \), we have

\[
\langle x_J H \mid x_J H' \rangle = \langle x_J^2 H \mid H' \rangle = P_J \langle H \mid H' \rangle \in P_J A.
\]
here, we use Lemma 6.5.1 (2). Hence, we can define an \( A_\mathbb{Z} \)-valued bilinear form \( \langle \cdot \mid \cdot \rangle_J \) on \( x_J \mathcal{H} \) by \( \langle \cdot \mid \cdot \rangle_J := \frac{1}{P_J} \langle \cdot \mid \cdot \rangle \).

**Proposition 6.5.8.** The basis \( \{ J^w_w | w \in JW \} \) and \( \{ J^{D_w}_{w_0} | w \in JW \} \) are dual to each other with respect to \( \langle \cdot \mid \cdot \rangle_J \), that is, we have \( \langle J^w_w \mid J^{D_w}_{w_0} \rangle_J = \delta_{y,w_0}^{Jw} \) for all \( y, w \in JW \).

**Proof.** Let \( y, w \in JW \). We compute as follows:

\[
\langle J^w_w \mid J^{D_w}_{w_0} \rangle_J = \frac{1}{P_J} \langle J^w_w \mid J^{D_w}_{w_0} \rangle_J = \frac{1}{P_J} \langle C_{w_0}^{D_w} \mid x_J \rangle_D \quad (\text{by Proposition 6.5.3 and 6.5.4})
\]

\[
= \langle C_{w_0}^{D_w} \mid x_J \rangle_D \quad (\text{since } C_{w_0}^{D_w} = J^w_w \in x_J \mathcal{H})
\]

\[
= \delta_{w_0}^{Jw} = \delta_{y,w_0}^{Jw} \quad (\text{by Proposition 6.4.6}).
\]

This proves the proposition. \( \square \)

## 7. Hecke modules and their centralizers

In this section, we study the centralizer algebras of certain modules over the Hecke algebra. They are known as (generalized) \( q \)-Schur algebras. Multiparameter \( q \)-Schur algebra of type \( B \) is also studied in [LL19].

### 7.1. Fundamental properties.

We follow ideas in [DDPW08, Chapter 9.1]. Let \( \pi \) be an index set. Suppose that we are given a map \( \pi \rightarrow \{ J \mid J \subset \{ 0, 1, \ldots, d-1 \} \} \). We denote by \( I_\lambda \) the image of \( \lambda \in \pi \) under this map. For each \( \lambda \in \pi \), for simplicity, we will denote \( W_{I_\lambda}, w_{I_\lambda}, x_{I_\lambda}, \) etc. by \( W_\lambda, w_\lambda, x_\lambda, \) etc.

**Definition 7.1.1.** Associated with \( \pi \), we define a right \( \mathcal{H} \)-module

\[
\mathbb{T}(\pi) := \bigoplus_{\lambda \in \pi} x_\lambda \mathcal{H},
\]

and its centralizer algebra \( \mathbb{S}(\pi) := \text{End}_\mathcal{H}(\mathbb{T}(\pi)) \); we let \( \mathbb{S}(\pi) \) act on \( \mathbb{T}(\pi) \) from the left.

It is obvious that \( \mathbb{T}(\pi) \) has two bases \( \{ \lambda^w_w \mid \lambda \in \pi, w \in \lambda W \} \) and \( \{ \lambda^{D_w}_{w_0} \mid \lambda \in \pi, w \in \lambda W \} \); we call them the Kazhdan-Lusztig basis and dual Kazhdan-Lusztig basis, respectively.

For each \( m = \sum_{\lambda \in \pi} m_\lambda \in \mathbb{T}(\pi) \) with \( m_\lambda \in x_\lambda \mathcal{H} \), we define \( \overline{m} \in \mathbb{T}(\pi) \) to be \( \sum_{\lambda \in \pi} \overline{m_\lambda} \). Also, for each \( f \in \mathbb{S}(\pi) \), define \( \overline{f} \in \mathbb{S}(\pi) \) by \( \overline{f}(m) = \overline{f(\overline{m})} \) for all \( m \in \mathbb{T}(\pi) \). This gives anti-linear automorphisms \( \overline{\cdot} \) on \( \mathbb{T}(\pi) \) and \( \mathbb{S}(\pi) \).

For each \( \lambda \in \pi \), define \( p_\lambda \in \mathbb{S}(\pi) \) to be the composite

\[
p_\lambda: \mathbb{T}(\pi) \twoheadrightarrow x_\lambda \mathcal{H} \hookrightarrow \mathbb{T}(\pi)
\]

of the projection and the inclusion. Clearly, \( \{ p_\lambda \mid \lambda \in \pi \} \) is a family of orthogonal idempotents with \( \sum_{\lambda \in \pi} p_\lambda = \text{id}_{\mathbb{T}(\pi)} \). Hence, we have a decomposition

\[
\mathbb{S}(\pi) = \bigoplus_{\lambda, \mu \in \pi} p_\lambda \mathbb{S}(\pi) p_\mu, \quad p_\lambda \mathbb{S}(\pi) p_\mu = \text{Hom}_\mathcal{H}(x_\mu \mathcal{H}, x_\lambda \mathcal{H}).
\]
Take \( f \in \text{Hom}_H(x_\mu \mathcal{H}, x_\lambda \mathcal{H}) \) arbitrarily. Since \( x_\mu \mathcal{H} \) is generated (as a right \( \mathcal{H} \)-module) by \( x_\mu \), the \( f \) is determined by \( f(x_\mu) \in x_\lambda \mathcal{H} \). Let us write
\[
 f(x_\mu) = \sum_{w \in \lambda W} c_{\lambda,w,\mu}(f)x_\lambda H_w, \quad \text{for some } c_{\lambda,w,\mu} \in \mathbb{A}_Z.
\]

**Lemma 7.1.2.** Let \( w \in \lambda W \) and \( j \in I_\mu \) be such that \( w < ws_j \). Then, we have
\[
c_{\lambda,w,\mu}(f) = q_j c_{\lambda,ws_j,\mu}(f).
\]
Consequently, we have
\[
f(x_\mu) = \sum_{w \in \lambda W} \sum_{y \in W_\mu} q_y^{-1} c_{\lambda,w,\mu}(f)x_\lambda H_{wy},
\]
and hence, \( f \) is determined by \( (c_{\lambda,w,\mu}(f))_{w \in \lambda W_\mu} \in \mathbb{A}_Z^\lambda W_\mu \), where \( \lambda W_\mu := \lambda W \cap (\lambda W)^{-1}. \)

**Proof.** We have
\[
 q_j^{-1} f(x_\mu) = f(x_\mu H_j)
 = f(x_\mu) H_j
 = \sum_{w \in \lambda W} c_{\lambda,w,\mu}(f)x_\lambda (H_{ws_j} + (q_j^{-1} - q_j)H_w) + \sum_{w \in \lambda W} c_{\lambda,w,\mu}(f)x_\lambda H_{ws_j}
 = \sum_{w \in \lambda W} (c_{\lambda,ws_j,\mu}(f) + (q_j^{-1} - q_j)c_{\lambda,w,\mu}(f))x_\lambda H_w + \sum_{w \in \lambda W} c_{\lambda,ws_j,\mu}(f)x_\lambda H_w.
\]
Comparing the coefficients of \( x_\lambda H_w \), we obtain the assertion. \( \square \)

Conversely, given \( (c_{\lambda,w,\mu})_{w \in \lambda W_\mu} \in \mathbb{A}_Z^\lambda W_\mu \), there exists a unique
\[
g \in \text{Hom}_H(x_\mu \mathcal{H}, x_\lambda \mathcal{H})
\]
such that \( c_{\lambda,w,\mu}(g) = c_{\lambda,w,\mu} \) for all \( w \in \lambda W_\mu \). Thus, we obtain an \( \mathbb{A}_Z \)-linear isomorphism between \( \mathbb{A}_Z^\lambda W_\mu \) and \( \text{Hom}_H(x_\mu \mathcal{H}, x_\lambda \mathcal{H}) \).

**Lemma 7.1.3** ([DDP05, Theorem 4.18]). Let \( \lambda, \mu \in \pi \). For each \( x \in \lambda W_\mu \), there exists a unique \( J_x \subset \{0, 1, \ldots, d - 1\} \) such that the multiplication map
\[
 W_\lambda \times \{x\} \times J_x W_\mu \to W_\lambda x W_\mu; \quad (u, x, v) \mapsto uxv
\]
is a bijection, where \( J_x W_\mu := J_x W_\mu \). Moreover, we have \( \ell(uxv) = \ell(u) + \ell(x) + \ell(v) \) for all \( u \in W_\lambda \) and \( v \in J_x W_\mu \).

For \( \lambda, \mu \in \pi \) and \( x \in \lambda W_\mu \), define \( \xi_{\lambda,x,\mu} \in \text{Hom}_H(x_\mu \mathcal{H}, x_\lambda \mathcal{H}) \) to be the one corresponding to \( (\delta_{x,w} q_x')_{w \in \lambda W_\mu} \in \mathbb{A}_Z^\lambda W_\mu \), where \( x' \in W_\mu \) is the longest element in \( J_x W_\mu \) (\( J_x \) is as in Lemma 7.1.3). Then, the next proposition is clear.

**Proposition 7.1.4.** \( \{\xi_{\lambda,x,\mu} \mid \lambda, \mu \in \pi, \ x \in \lambda W_\mu\} \) forms a basis of \( S(\pi) \).

For each \( \lambda, \mu \in \pi \), \( x \in \lambda W_\mu \), set
\[
 \eta_{\lambda,x,\mu} = q_{w_{\lambda,x} x'} \sum_{w \in \lambda W_\mu} q_{w}^{-1} H_w.
\]
Lemma 7.1.5. Let $\lambda, \mu \in \pi$, $x \in \lambda W^\mu$.

$(1)$ $\eta^\circ_{\lambda,x,\mu} = \eta_{\mu,x^{-1},\lambda}$.

$(2)$ $\xi_{\lambda,x,\mu}(x_\mu) = \eta_{\lambda,x,\mu} \cdot x_\mu = \frac{1}{P_\mu} \eta_{\lambda,x,\mu} \cdot x_\mu = \frac{1}{P_\mu} x_\lambda \cdot \eta_{\lambda,x,\mu}$.

$(3)$ $\xi_{\lambda,e,\mu} = \xi_{\lambda,e,\mu}$, where $e$ denotes the identity element of $W$.

Proof.

$(1)$ By the definition of $x'$, we have $y := w_\lambda x x'$ is the longest element in $W_\lambda x W_\mu$. Also, it is easily checked that the map $W \to W$, $w \mapsto w^{-1}$ gives a bijection $W_\lambda x W_\mu \to W_\mu x^{-1} W_\lambda$. Since this bijection preserves the length, $y^{-1}$ is the longest element in $W_\mu x^{-1} W_\lambda$. Then, we compute $\eta^\circ_{\lambda,x,\mu}$ as follows:

$$
\eta^\circ_{\lambda,x,\mu} = q_y \sum_{w \in W_\lambda x W_\mu} q_w^{-1} H_{w^{-1}} = q_y \sum_{w \in W_\mu x^{-1} W_\lambda} q_w^{-1} H_w
= q_y^{-1} \sum_{w \in W_\mu x^{-1} W_\lambda} q_w^{-1} H_w = \eta_{\mu,x^{-1},\lambda}.
$$

$(2)$ By the definition of $\xi_{\lambda,x,\mu}$, we have

$$
\xi_{\lambda,x,\mu}(x_\mu) = \sum_{y \in W_\mu, \ y \in \lambda W} q_{x'} q_y^{-1} x_\lambda H_{xy}
= \sum_{y \in W_\mu, \ y \in \lambda W} q_{x'} q_y^{-1} q_{w_\lambda} q_z^{-1} H_{zxy} \quad \text{(by the definition of $x_\lambda$)}
= \sum_{w \in W_\lambda x W_\mu} q_{w_\lambda} q_z q_{x'} q_w^{-1} H_w = \eta_{\lambda,x,\mu}.
$$

This proves the first equation. Next, we have

$$
\eta_{\lambda,x,\mu} \cdot x_\mu = \xi_{\lambda,x,\mu}(x_\mu) = P_\mu \xi_{\lambda,x,\mu}(x_\mu) = P_\mu \eta_{\lambda,x,\mu},
$$

which implies the second equality. Finally, the third equality follows from the fact that $\eta_{\lambda,x,\mu} = \xi_{\lambda,x,\mu}(x_\mu) \in x_\lambda H$.

$(3)$ It suffices to check that $\xi_{\lambda,e,\mu}(x_\nu) = \xi_{\lambda,e,\mu}(x_\nu)$ for all $\nu \in \pi$. Only the non-trivial case is when $\nu = \mu$. Since we have

$$
\xi_{\lambda,e,\mu}(x_\mu) = \xi_{\lambda,e,\mu}(x_\mu) = \eta_{\lambda,e,\mu},
$$

the problem is reduced to proving that $\eta_{\lambda,e,\mu}$ is fixed under the involution $\overline{\cdot}$. One can write

$$
\eta_{\lambda,e,\mu} = \sum_{w \in W_\lambda W_\mu} q_{w_\lambda} q_{x'} q_w^{-1} H_w = x_\lambda \sum_{y \in \lambda W_\mu} q_{x'} q_y^{-1} H_y.
$$

On the other hand, we have

$$
x_\lambda x_\mu = x_\lambda x_{I_\lambda \cap I_\mu} \sum_{y \in \lambda W_\mu} q_{x'} q_y^{-1} H_y = P_{I_\lambda \cap I_\mu} x_\lambda \sum_{y \in \lambda W_\mu} q_{x'} q_y^{-1} H_y.
$$

Hence, we obtain

$$
\eta_{\lambda,e,\mu} = \frac{1}{P_{I_\lambda \cap I_\mu}} x_\lambda x_\mu,
$$

which is invariant under $\overline{\cdot}$. Thus, the proof completes. $\square$
**Proposition 7.1.6.** The linear map \( \flat : S(\pi) \to S(\pi); \xi_{\lambda,x,\mu} \mapsto \xi_{\mu,x^{-1},\lambda} \) defines an \( A_{\mathbb{C}} \)-algebra anti-automorphism on \( S(\pi) \).

**Proof.** We have to verify that
\[
\xi_{\lambda,x,\mu} \cdot \xi_{\mu,y,\nu} = \xi_{\nu,y^{-1},\mu} \cdot \xi_{\mu,x^{-1},\lambda}
\]
for all \( \lambda, \mu, \nu \) and \( x \in \lambda W^\mu, \ y \in \nu W^\nu \). Since the both sides are equal to zero unless \( \nu = \mu \), we may assume that \( \nu = \mu \). Let us write
\[
\xi_{\lambda,x,\mu} \cdot \xi_{\mu,y,\nu} = \sum_{z \in \lambda W^\nu} c_z \xi_{\lambda,z,\nu}
\]
for some \( c_z \in A_{\mathbb{C}} \).

Applying the both sides to \( x_\nu \in T(\pi) \), by Lemma 7.1.5 (2), we obtain
\[
\frac{1}{\mu} \eta_{\lambda,x,\mu} \eta_{\mu,y,\nu} = \sum_{z \in \lambda W^\nu} c_z \eta_{\lambda,z,\nu}.
\]

To prove the assertion, we compute as follows:
\[
\xi_{\nu,y^{-1},\mu} \cdot \xi_{\mu,x^{-1},\lambda}(x_\lambda) = \frac{1}{\mu} \eta_{\nu,y^{-1},\mu} \cdot \eta_{\mu,x^{-1},\lambda} \quad \text{(by Lemma 7.1.5 (2))}
\]
\[
= ( \frac{1}{\mu} \eta_{\lambda,x,\mu} \cdot \eta_{\mu,y,\nu} )^\flat \quad \text{(by Lemma 7.1.5 (1))}
\]
\[
= \sum_{z \in \lambda W^\nu} c_z \eta_{\lambda,z,\nu}^\flat \quad \text{(by equation (2))}
\]
\[
= \sum_{z \in \lambda W^\nu} c_z \xi_{\nu,z^{-1},\lambda}(x_\lambda) \quad \text{(by Lemma 7.1.5 (1))}
\]
\[
= \sum_{z \in \lambda W^\nu} c_z \xi_{\nu,z^{-1},\lambda}(x_\lambda) \quad \text{(by Lemma 7.1.5 (2))}
\]
\[
= ( \xi_{\lambda,x,\mu} \cdot \xi_{\mu,y,\nu} )^\flat(x_\lambda) \quad \text{(by equation (1))}.
\]

This shows that \( \xi_{\nu,y^{-1},\mu} \cdot \xi_{\mu,x^{-1},\lambda} = ( \xi_{\lambda,x,\mu} \cdot \xi_{\mu,y,\nu} )^\flat \), and hence, the proof completes. \( \square \)

Recall the bilinear form \( \langle \cdot \mid \cdot \rangle_\lambda = \langle \cdot \mid \cdot \rangle_{I_\lambda} \) on \( x_\lambda \mathcal{H} \) defined in Section 6.5.

**Proposition 7.1.7.** Let \( \lambda, \mu \in \pi, \ m \in x_\lambda \mathcal{H}, \) and \( n \in x_\mu \mathcal{H} \). Then, for each \( w \in \lambda W^\mu \), we have
\[
\langle m \mid \xi_{\lambda,w,\mu}(n) \rangle_\lambda = \langle \xi_{\lambda,w,\mu}^\flat(m) \mid n \rangle_{\mu}.
\]

**Proof.** We compute as follows:
\[
\langle m \mid \xi_{\lambda,w,\mu}(n) \rangle_\lambda = \frac{1}{\lambda} \langle m \mid \xi_{\lambda,w,\mu}(n) \rangle
\]
\[
= \frac{1}{\lambda \mu} \langle m \mid \eta_{\lambda,w,\mu} n \rangle \quad \text{(by Lemma 7.1.5 (2))}
\]
\[
= \frac{1}{\lambda \mu} \langle \eta_{\mu,w^{-1},\lambda} m \mid n \rangle \quad \text{(by Lemma 7.1.5 (1))}
\]
\[
= \frac{1}{\mu} \langle \xi_{\mu,w^{-1},\lambda}(m) \mid n \rangle \quad \text{(by Lemma 7.1.5 (2))}
\]
\[
= \langle \xi_{\lambda,w,\mu}^\flat(m) \mid n \rangle_{\mu}.
\]
This proves the proposition. \( \square \)
Define a bilinear form $\langle \cdot \mid \cdot \rangle_\pi$ on $\mathbb{T}(\pi)$ by $\langle m \mid n \rangle_\pi := \delta_{\lambda,\mu} \langle m \mid n \rangle_\pi$ for all $\lambda, \mu \in \pi$, $m \in x_\lambda \mathcal{H}$, $n \in x_\mu \mathcal{H}$.

**Corollary 7.1.8.** The two bases $\{^\lambda C_w \mid \lambda \in \pi, \ w \in \lambda W\}$ and $\{^\lambda D_{w,\lambda,\mu,w} \mid \lambda \in \pi, \ w \in \lambda W\}$ of $\mathbb{T}(\pi)$ are dual to each other with respect to the bilinear form $\langle \cdot \mid \cdot \rangle_\pi$. Moreover, for all $m,n \in \mathbb{T}(\pi)$ and $x \in S(\pi)$, we have $\langle m \mid xn \rangle_\pi = \langle x^b m \mid n \rangle_\pi$.

7.2. Cell representations. Let $X \in L(W)$ and $x \in X$. Set

- $C_{\leq L,X}(\pi) := \bigoplus_{\lambda \in \pi} \bigoplus_{w \in \lambda W} A_Z \lambda C_w$,
- $D_{\leq L}(\pi) := \bigoplus_{\lambda \in \pi} A_Z \lambda D_w$,
- $C_{< L,X}(\pi) := \bigoplus_{\lambda \in \pi} \bigoplus_{w \in \lambda W, \ w \leq L \ x} A_Z \lambda C_w$,
- $D_{< L}(\pi) := \bigoplus_{\lambda \in \pi} A_Z \lambda D_w$,
- $C_X^L(\pi) := C_{\leq L,X}(\pi)/C_{< L,X}(\pi)$,
- $D_X^L(\pi) := D_{\leq L}(\pi)/D_{< L}(\pi)$.

Note that these objects are independent of the choice of $x \in X$. We denote the image of $m \in C_{\leq L,X}(\pi)$ (resp., $D_{\leq L,X}(\pi)$) under the quotient map $C_{\leq L,X}(\pi) \to C_X^L(\pi)$ (resp., $D_{\leq L,X}(\pi) \to D_X^L(\pi)$) by $[m]_X$ (resp., $[m]_X'$).

**Proposition 7.2.1.** Let $X \in L(W)$.

1. $C_{\leq L,X}(\pi)$ is a $S(\pi)$-submodule of $\mathbb{T}(\pi)$.
2. $C_{< L,X}(\pi)$ is a $S(\pi)$-submodule of $\mathbb{T}(\pi)$.
3. $C_X^L(\pi)$ is a left $S(\pi)$-module having a basis $\{[^\lambda C_w]_X \mid \lambda \in \pi, \ w \in \lambda W \cap w_\lambda X\}$. Here, $w_\lambda X := \{w_\lambda x \mid x \in X\}$.

**Proof.** We will prove only (1) since the proof of (2) is similar to that of (1), and (3) follows from (1) and (2). Fix $x \in X$. In order to show that $C_{\leq L,X}(\pi)$ is a $S(\pi)$-submodule, it suffices to verify that $\xi_{\lambda,y,\mu} w C_w \in C_{\leq L,X}(\pi)$ for all $\lambda, \mu \in \pi$, $y \in \lambda W^\mu$, and $w \in \mu W$ such that $w_\mu w \leq L x$. By Proposition 6.5.3 and Lemma 7.1.5(2), we have

$$\xi_{\lambda,y,\mu} w C_w = \xi_{\lambda,y,\mu} C_{w_\mu w} = \frac{1}{P_\mu} \eta_{\lambda,y,\mu} C_{w_\mu w}.$$ 

Also, by Lemma 7.1.5(2), we have $\eta_{\lambda,y,\mu} = x_\lambda H$ for some $H \in \mathcal{H}$; one can write $x_\lambda H = x_\lambda H$ for some $H \in \mathcal{H}$, then $H C_w \mu w$ is a linear combination of $C_w$, $w' \leq L w \mu w' \leq L x$. Hence, by Proposition 6.5.3, $x_\lambda H C_w \mu w$ is a linear combination of $^\lambda C_w$ for $w' \in \lambda W$ with $w_\mu w' \leq L w' \leq L x$. Therefore, we have $\xi_{\lambda,y,\mu} C_w = \frac{1}{P_\mu} \eta_{\lambda,y,\mu} C_{w_\mu w} \in \frac{1}{P_\mu} C_{\leq L,X}(\pi)$. However, since $\xi_{\lambda,y,\mu} C_w \in x_\lambda H = \bigoplus_{z \in \lambda W} A_Z \lambda C_z$, we conclude that $\xi_{\lambda,y,\mu} C_w \in C_{\leq L,X}(\pi)$. This completes the proof. \qed

Similarly, one can prove the following: $D_{\leq L,X}(\pi)$ and $D_{< L,X}(\pi)$ are $S(\pi)$-submodules, and $D_X^L(\pi)$ is a left $S(\pi)$-module having a basis $\{[^\lambda D_w]_X \mid \lambda \in \pi, \ w \in \lambda W \cap X\}$.

8. Global $\mathfrak{g}$-crystal bases for the irreducible $\mathfrak{U}^-$-modules

8.1. **Surjection $\xi : \mathfrak{U}^- \to S(\pi^\prime)$.** Let $\pi^\prime = \{\lambda = (\lambda_0, \ldots, \lambda_r) \in \mathbb{Z}_{\geq 0}^{r+1} \mid \sum_{i=0}^r \lambda_i = d\}$. For $\lambda \in \pi^\prime$, set $I_\lambda = \{0,1,\ldots,d-1\} \setminus \{\lambda_0,\lambda_0,1,\ldots,\lambda_0,0\} = \{\lambda_0,\lambda_0,1,\ldots,\lambda_0,0\}$, where $\lambda_0,k = \sum_{i=0}^k \lambda_i$. 

Let $V = \bigoplus_{i=-r}^r \mathbb{Q}(p, q)v_i$ be the vector representation of $U$ with $v_r$ a highest weight vector. Then, $V \otimes^d$ has a basis $\{v_{i_1, \ldots, i_d} := v_{i_1} \otimes \cdots \otimes v_{i_d} \mid -r \leq i_1, \ldots, i_d \leq r\}$. $H := \mathbb{Q}(p, q) \otimes \mathbb{A}_\mathbb{Z} H$ acts on $V \otimes^d$ by

\[
v_{i_1, \ldots, i_d} H_0 = \begin{cases} v_{i_1, i_2, \ldots, i_d} & \text{if } i_1 > 0, \\ p^{-1}v_{i_1, \ldots, i_d} & \text{if } i_1 = 0, \\ v_{i_1, i_2, \ldots, i_d} + (p^{-1} - p)v_{i_1, \ldots, i_d} & \text{if } i_1 < 0, \end{cases}
\]

\[
v_{i_1, \ldots, i_d} H_j = \begin{cases} v_{i_1, \ldots, i_{j+1}, i_j, \ldots} & \text{if } i_j < i_{j+1}, \\ q^{-1}v_{i_1, \ldots, i_d} & \text{if } i_j = i_{j+1}, \\ v_{i_1, \ldots, i_{j+1}, i_j, \ldots} + (q^{-1} - q)v_{i_1, \ldots, i_d} & \text{if } i_j > i_{j+1}. \end{cases}
\]

Then, it is easily seen that $V \otimes^d$ is isomorphic to $T(\pi^j) := \mathbb{Q}(p, q) \otimes \mathbb{A}_\mathbb{Z} T(\pi^j)$ as a right $H$-module. Setting $S(\pi^j) := \mathbb{Q}(p, q) \otimes \mathbb{A}_\mathbb{Z} S(\pi^j)$, $V \otimes^d$ becomes a left $S(\pi^j)$-module. By the double centralizer property between $U$ and $H$ on $V \otimes^d$ ([BW18a], [BW18b]), there exists a surjective algebra homomorphism $\xi : U \rightarrow S(\pi^j)$. In particular, every $S(\pi^j)$-modules are regarded as $U^j$-modules via $\xi$. In [W17], it is proved that for each $\lambda \in P^j$, the irreducible highest weight module $L(\lambda)$ is isomorphic to $C_X(\pi^j) := \mathbb{Q}(p, q) \otimes \mathbb{A}_\mathbb{Z} C_X(\pi^j)$ for some $X \in L(W_d)$, where $d = |\lambda|$. 

For $i \in P^j$, we define two maps $\overline{e}_i, \overline{f}_i : \pi^j \rightarrow \pi^j \sqcup \{0\}$, where $0$ denotes a formal symbol, as follows. Let $\lambda = (\lambda_0, \ldots, \lambda_r) \in \pi^j$. Then, we set

\[
\overline{e}_i \lambda = \begin{cases} (\lambda_0, \ldots, \lambda_{i-2}, \lambda_{i-1} + 1, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_r) & \text{if } \lambda_i > 0, \\ 0 & \text{if } \lambda_i = 0, \end{cases}
\]

and

\[
\overline{f}_i \lambda = \begin{cases} (\lambda_0, \ldots, \lambda_{i-2}, \lambda_{i-1} - 1, \lambda_i + 1, \lambda_{i+1}, \ldots, \lambda_r) & \text{if } \lambda_{i-1} > 0, \\ 0 & \text{if } \lambda_{i-1} = 0. \end{cases}
\]

By convention, we set $\xi_{\lambda, x, \mu} = 0$ if $\lambda = 0$ or $\mu = 0$.

**Proposition 8.1.1.** For $i \in P^j$, we have

\[
\xi(e_i) = \sum_{\lambda \in \pi^j} \xi_{\overline{e}_i(\lambda), e, \lambda}, \\
\xi(f_i) = \sum_{\lambda \in \pi^j} \xi_{\overline{f}_i(\lambda), e, \lambda^*}.
\]

**Proof.** We prove only the statement for $f_1$; the proofs for $f_i$, $i \neq 1$ and for $e_i$ are similar. Recall the comultiplication $\Delta$ of $U$; we have

\[
\Delta^{(d-1)}(E_i) = \sum_{k=1}^d 1^\otimes k^{-1} \otimes E_i \otimes (K_i^{-1})^\otimes d-k, \quad \Delta^{(d-1)}(F_i) = \sum_{k=1}^d K_i^\otimes d-k \otimes F_i \otimes 1^\otimes k-1.
\]
Then, we compute as
\[
f_1v_\lambda = pq^{-1}q^{\lambda_0} \sum_{k=1}^{\lambda_0} q^{\lambda_0-k}v_{0\lambda_0-k,1,0^{k-1},1,\ldots,r^{\lambda_r}} + \sum_{k=1}^{\lambda_0} q^{\lambda_0-k}v_{0^{k-1},-1,0^{\lambda_0-k},1,\ldots,r^{\lambda_r}} = pq^{\lambda_0-1} \sum_{k=1}^{\lambda_0} q^{\lambda_0-k}v_{f_1(\lambda)}H_{\lambda_0-1} \cdots H_{\lambda_0-(k-1)} + \sum_{k=1}^{\lambda_0} q^{\lambda_0-k}v_{f_1(\lambda)}H_{\lambda_0-1} \cdots H_1H_0H_1 \cdots H_{k-1} = pq^{\lambda_0-1} \sum_{k=1}^{\lambda_0} q^{-k+1}v_{f_1(\lambda)}H_{\lambda_0-1} \cdots H_{\lambda_0-(k-1)} + pq^{\lambda_0-2} \sum_{k=1}^{\lambda_0} q^{-\lambda_0+k-2}v_{f_1(\lambda)}H_{\lambda_0-1} \cdots H_1H_0H_1 \cdots H_{k-1} = \xi_{f_1(\lambda),e(\lambda)}(v_\lambda).
\]
This proves the assertion. □

Here are immediate consequences.

**Corollary 8.1.2.** Let \( x \in U^j \). Then, \( \xi(\sigma^j(x)) = \xi(x)^b \), \( \xi(\psi^j(x)) = \overline{\xi(x)} \).

**Corollary 8.1.3.** The bilinear form \( \langle \cdot \mid \cdot \rangle_{\pi^j} \) of \( T(\pi^j) \) satisfies
\[
\langle xm \mid n \rangle_{\pi^j} = \langle m \mid \sigma^j(x)n \rangle_{\pi^j}
\]
for all \( x \in U^j \), \( m, n \in T(\pi^j) \).

### 8.2 Global \( \beta \)-crystal basis of irreducible \( U^j \)-module

Let \( X \in L(W) \). Then, \( C_X^{L}(\pi^j) \simeq L(\lambda) \) for some \( \lambda \in P^j \). Since \( L(\lambda) \) is a highest weight module, there exists a unique \( \lambda \in \pi^j \) and \( w \in \lambda W \) such that \([^\lambda C_w]_X \in C_X^{L}(\pi^j)\) is a highest weight vector.

Recall the isomorphism \( D^{L}_{X,w_0}(\pi^j) \simeq C_X^{L}(\pi^j) \) of left \( \mathcal{H} \)-modules. Set \( C_X^{L}(\pi^j) := \mathbb{Q}(p,q) \otimes_{A^j} C_X^{L}(\pi^j) \), and define \( D^{L}_{X,w_0}(\pi^j) \) similarly. Then, we have
\[
D^{L}_{X,w_0}(\pi^j) \simeq T(\pi^j) \otimes_{\mathcal{H}} D^{L}_{X,w_0}(\pi^j) \simeq T(\pi^j) \otimes_{\mathcal{H}} C_X^{L}(\pi^j)
\]
as left \( U^j \)-modules. Hence, \([^\lambda D_{w,w_0}]_{X,w_0} \in D^{L}_{X,w_0}(\pi^j)\) is also a highest weight vector. Thus, we obtain two isomorphisms
\[
\varphi_C : L(\lambda) \to C_X^{L}(\pi^j); \quad v_\lambda \mapsto [^\lambda C_w]_X,
\]
\[
\varphi_D : L(\lambda) \to D^{L}_{X,w_0}(\pi^j); \quad v_\lambda \mapsto [^\lambda D_{w,w_0}]_{X,w_0}
\]
of \( U^j \)-modules, where \( v_\lambda \in L(\lambda) \) is a fixed highest weight vector.

**Definition 8.2.1.** Let \( \lambda \in P^j \) and \( v_\lambda \in L(\lambda) \) be a highest weight vector. Define the bilinear form \( \langle \cdot \mid \cdot \rangle_1 \) on \( L(\lambda) \) by \( (v_\lambda, v_\lambda)_1 = 1 \) and \( (xm, n)_1 = (n, \sigma^j(x)n)_1 \) for all \( x \in U^j, m, n \in L(\lambda) \).

**Proposition 8.2.2.** Let \( \lambda \in P^j \). Then, the bilinear form \( \langle \cdot \mid \cdot \rangle_1 \) is nondegenerate.
Proof. For \( m, n \in L(\lambda) \), set \((m, n) := (\varphi_C(m) \mid \varphi_D(n))_{\pi_0} \). Then, we have
\[
(v_\lambda, v_\lambda) = \langle [\lambda^m C_w]_X \mid [\lambda^m D_{w_m w_0}]_{X w_0} \rangle_{\pi_0} = 1,
\]
and
\[
(x, \varphi_C(m) \mid \varphi_D(m))_{\pi_0} = \langle \varphi_C(m) \mid \sigma^2(x) \varphi_D(n) \rangle_{\pi_0} = (m, \sigma^3(x)n).
\]
Hence, we have \((\cdot, \cdot) = (\cdot, \cdot)_1 \). Then, it is clear that \( \{ \varphi_C^{-1}([\mu^\mu C_y]_X) \mid \mu \in \pi^j, \ y \in \mu W \cap w_\mu X \} \) and \( \{ \varphi_D^{-1}([\nu^\nu D_{w_\nu w_0}]_{X w_0}) \mid \nu \in \pi^j, \ y \in \mu W \cap w_\mu X \} \) form bases which are dual to each other with respect to \((\cdot, \cdot)_1 \). This proves the proposition. \( \square \)

Recall that the set \((\mu, y) \mid \mu \in \pi^j, \ y \in \mu W \cap w_\mu X \) is identical to \( B(\lambda) \). For each \( b \in B(\lambda) \), set
\[ G^b_{\text{low}} := \varphi_C^{-1}([\mu^\mu C_y]_X), \quad G^b_{\text{up}} := \varphi_D^{-1}([\nu^\nu D_{w_\nu w_0}]_{X w_0}), \]
where \( (\mu, y) \) is the pair corresponding to \( b \). Then, \( G^b_{\text{low}}(\lambda) := \{ G^b_{\text{low}}(b) \mid b \in B(\lambda) \} \) and \( G^b_{\text{up}}(\lambda) := \{ G^b_{\text{up}}(b) \mid b \in B(\lambda) \} \) are bases of \( L(\lambda) \).

Definition 8.2.3. Let \( \lambda \in P^j(d) \), and \( v_\lambda \in L(\lambda) \) be a highest weight vector. Define a bilinear form \((\cdot, \cdot)_2 \) on \( L(\lambda) \), and a \( \psi^j \)-involution \( \psi^j_\lambda \) on \( L(\lambda) \) by
\[
(v_\lambda, v_\lambda)_2 = 1, \quad (x, m, n)_2 = (m, \pi^j(x)n)_2 \quad \text{for all } x \in U^j, \ m, n \in L(\lambda),
\]
\[
\psi^j_\lambda(v_\lambda) = v_\lambda.
\]
Let \((\mathcal{L}(\lambda), B(\lambda)) \) be the unique \( j \)-crystal basis of \( L(\lambda) \) such that \( v_\lambda + q \mathcal{L}(\lambda) \in B(\lambda) \).

Theorem 8.2.4. Let \( \lambda \in P^j(d) \). Then, the following hold.

1. \( \psi^j_\lambda(G^b_{\text{low}}(b)) = G^b_{\text{low}}(b) \) for all \( b \in B(\lambda) \).
2. \( \psi^j_\lambda(G^b_{\text{up}}(b)) = G^b_{\text{up}}(b) \) for all \( b \in B(\lambda) \).
3. \( G^b_{\text{low}}(\lambda) \) and \( G^b_{\text{up}}(\lambda) \) are dual bases with respect to \((\cdot, \cdot)_1 \).
4. \( \mathcal{L}(\lambda) = \{ m \in L(\lambda) \mid (m, m)_2 \in A_0 \} \). Consequently, \((\cdot, \cdot)_2 \) induces the bilinear form \((\cdot, \cdot)_0 \) on \( \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \) defined by \((m + q\mathcal{L}(\lambda), n + q\mathcal{L}(\lambda))_0 := \lim_{n \to 0}(\lim_{m \to 0}(m, n)_2) \).
5. \( \{ G^b_{\text{low}}(b) \mid b \in B(\lambda) \} \) forms an almost orthonormal basis with respect to \((\cdot, \cdot)_2 \), i.e., we have \((G^b_{\text{low}}(b), G^{b'}_{\text{low}}(b'))_2 = \delta_{b, b'} + qA_0 \) for all \( b, b' \in B(\lambda) \).
6. \( (b, b')_0 = \delta_{b, b'} \) for all \( b, b' \in B(\lambda) \).
7. Let \( L(\lambda)_A \) be the \( A \)-span of \( G^b_{\text{low}}(\lambda) \). Then, \( (\mathcal{L}(\lambda), L(\lambda)_A, \psi^j_\lambda(L(\lambda))) \) is balanced. Moreover, the global basis associated to \( B(\lambda) \) is \( \{ G^b_{\text{low}}(b) \mid b \in B(\lambda) \} \). In particular, \( L(\lambda) \) has a global \( j \)-crystal basis.

Proof. Items (1) and (2) are obvious from the definition of \( G^b_{\text{low}}(b) \) and \( G^b_{\text{up}}(b) \).
Item (3) follows from the proof of Proposition 8.2.2
To prove the rest, observe that \( L(\lambda) \) is realized as a subquotient of \( V^{\otimes d} \) by using Kazhdan-Lusztig basis elements. To be precise, let \( X \in P^j \) be such that \( L(\lambda) \simeq C^L_\lambda(\pi^j) \) and \( x \in X \). Then,
\[
C^L_\lambda(\pi^j) = \frac{\text{Span}_{\mathbb{Q}(p, q)}[\lambda^m C_w \mid \lambda \in \pi^j, \ w \in \lambda^\lambda W, \ w_\lambda w \leq L x]}{\text{Span}_{\mathbb{Q}(p, q)}[\mu^\mu C_y \mid \mu \in \pi^j, \ y \in \mu^\mu W, \ w_\mu y \leq L x]}.
\]
Then, items (4)-(6) follows from the proof of [W17] Proposition 7.4.4. To prove item (7), it suffices to show that \( L(\lambda)_A \) is a \( U^j_A \)-module. It follows from the fact that the \( A \)-submodule of \( V^{\otimes d} \) spanned by the Kazhdan-Lusztig basis is a \( U_A \)-module, and that \( U^j_A \subseteq U_A \). \( \square \)
9. Basic properties of global crystal bases

9.1. Global crystal bases. In this subsection, we exposit some basic properties concerning global crystal bases of $U$-modules in $\mathcal{O}_{\text{int}}$. Let $M \in \mathcal{O}_{\text{int}}$, $(\mathcal{L}, \mathcal{B})$ a crystal basis of $M$, $\psi_M$ a $\psi$-involution, and $M_A$ a $U_A$-submodule of $M$. Suppose that $M$ has a global basis $G(\mathcal{B})$ with the associated balanced triple $(\mathcal{L}, M_A, \psi_M(\mathcal{L}))$.

**Proposition 9.1.1** ([K93]). Let $i \in I$, $b \in B$ and $m \in \mathbb{Z}_{\geq 0}$. Then, we have the following.

1. $\sum_{n \geq m} F_i^{(n)} M_A = \bigoplus_{b' \in B} \mathbb{A}G^j(b')$.
2. $\sum_{n \geq m} E_i^{(n)} M_A = \bigoplus_{b' \in B} \mathbb{A}G^j(b')$.
3. $F_i G^j(b) = [\varepsilon_i(b) + 1]G^j(F_i b) + \sum_{b' \in B} \varphi^{(i)}_{b',b} G^j(b')$ for some $\varphi^{(i)}_{b',b} \in q^2 - \varepsilon_i(b') \mathbb{Q}[q]$.
4. $E_i G^j(b) = [\varphi_i(b) + 1]G^j(E_i b) + \sum_{b' \in B} \varepsilon^{(i)}_{b',b} G^j(b')$ for some $\varepsilon^{(i)}_{b',b} \in q^2 - \varphi_i(b') \mathbb{Q}[q]$.

For $\lambda \in \mathcal{P}(M)$, set $I_\lambda(M)$ to be the sum of submodules of $M$ isomorphic to $L(\lambda)$. Also, we set

$$W_{\geq \lambda}(M) := \sum_{\mu \geq \lambda} I_\mu(M),$$
$$W_{\lambda}(M) := \sum_{\mu > \lambda} I_\mu(M),$$
$$W_\lambda(M) := W_{\geq \lambda}(M)/W_{\lambda}(M).$$

**Theorem 9.1.2** ([K93, L10]). Let $M, \mathcal{L}, \mathcal{B}, M_A$ be as above. Then, for each $\lambda \in \mathcal{P}(M)$, the following hold:

1. $W_{\geq \lambda}(M)$ has a global crystal basis $W_{\geq \lambda}(G(\mathcal{B})) := \{G(b) \mid I(b) \geq \lambda\}$ with the associated balanced triple $(W_{\geq \lambda}(\mathcal{L}), W_{\geq \lambda}(M_A), W_{\geq \lambda}(\psi_M(\mathcal{L})))$, where $W_{\geq \lambda}(\mathcal{L}) := W_{\geq \lambda}(M) \cap \mathcal{L}$, and so on.
2. $W_{\lambda}(M)$ has a global crystal basis $W_{\lambda}(G(\mathcal{B})) := \{G(b) \mid I(b) > \lambda\}$ with the associated balanced triple $(W_{\lambda}(\mathcal{L}), W_{\lambda}(M_A), W_{\lambda}(\psi_M(\mathcal{L})))$, where $W_{\lambda}(\mathcal{L}) := W_{\lambda}(M) \cap \mathcal{L}$, and so on.
3. $W_\lambda(M)$ has a global crystal basis $W_\lambda(M)$ with the associated balanced triple $(W_\lambda(\mathcal{L}), W_\lambda(M_A), W_\lambda(\psi_M(\mathcal{L})))$, where $W_\lambda(\mathcal{L}) := W_{\geq \lambda}(\mathcal{L})/W_{\lambda}(\mathcal{L})$, and so on.
4. There exists a $U$-module isomorphism $\xi : L(\lambda) \oplus m_\lambda \rightarrow W_\lambda(M)$ which induces an isomorphism

$$\mathcal{L}(\lambda) \oplus m_\lambda, (L(\lambda)_A) \oplus m_\lambda, \psi_\lambda(\mathcal{L}(\lambda)) \oplus m_\lambda \simeq (W_\lambda(\mathcal{L}), W_\lambda(M_A), W_\lambda(\psi_M(\mathcal{L}))),$$

where $m_\lambda := \dim \text{Hom}_U(L(\lambda), M)$ denotes the multiplicity of $L(\lambda)$ in $M$.

**Remark 9.1.3.** By replacing $P(M)$ with $P^j(M)$ and $\leq$ with $\preceq$, the same result holds for integrable modules over $U(l)$ with global crystal bases.

9.2. $j$-canonical bases. Let $M \in \mathcal{O}_{\text{int}}$ be a based $U$-module with a crystal basis $(\mathcal{L}, \mathcal{B})$, a global crystal basis $G(\mathcal{B})$, a $\psi$-involution $\psi_M$, and a balanced triple $(\mathcal{L}, M_A, \psi_M(\mathcal{L}))$. Set $\psi_M^j := \Upsilon \circ \psi_M$. We denote by $G^j(\mathcal{B})$ the associated $j$-canonical
basis. Recall that $\psi_M^j$ is a $\psi^j$-involution on $M$, and $(\mathcal{L}, M_A, \psi_M^j(\mathcal{L}))$ is a balanced triple with the associated global basis $G^j(B)$.

**Lemma 9.2.1.** Let $b \in B$. Let us write as

$$G^j(b) = G(b) + \sum_{b' \in B \setminus \{E_\gamma b\}} c_{b', b}G(b')$$

for some $c_{b', b} \in qA_0 \cap A$. Then, we have $c_{b', b} = 0$ unless

$$(3) \quad P^j(b) \leq P^j(b') \quad \text{or} \quad |P^j(b')^-| < |P^j(b)^-|.$$  

Proof. By the construction of $G^j(b)$, it suffices to show that $\psi_M^j(G(b))$ is a linear combination of $G(b')$ with $b'$ satisfying condition (3). Since $\psi_M^j(G(b)) = YG(b) \in U^{-}G(b)$, it suffices to show that for each $l \in \mathbb{Z}_{\geq 0}$ and $i_1, \ldots, i_l \in I$, we have

$$F_{i_1} \cdots F_{i_l}G(b) \in \text{Span}_{\mathbb{Q}(p, q)} \{G(b') \mid b' \text{ satisfies condition (3)}\}.$$

We prove it by induction on $l$. When $l = 0$, there is nothing to prove. So, assume that $l > 0$ and that $F_{i_{l-1}} \cdots F_{i_1}G(b) \in \text{Span}_{\mathbb{Q}(p, q)} \{G(b') \mid b' \text{ satisfies condition (3)}\}$ for all $i_1, \ldots, i_{l-1} \in I$. If $i_l \neq 1$, then, by Remark 9.1.3, we have

$$F_{i_l}G(b') \in \text{Span}_{\mathbb{Q}(p, q)} \{G(b'') \mid P^j(b') \leq P^j(b'')\}$$

for all $b'$ satisfying condition (3). Since $|P^j(b'')^-| = |P^j(b')^-|$ for all $b''$ with $P^j(b') \leq P^j(b'')$, $b''$ satisfies condition (3).

If $i_l = 1$, then $\text{wt}(F_{i_l}G(b')) = \text{wt}(G(b')) - \alpha_1$. This immediately implies that $F_{i_l}G(b') \in \text{Span}_{\mathbb{Q}(p, q)} \{G(b'') \mid |P^j(b'')^-| < |P^j(b')^-|\}$. Therefore, $F_{i_1} \cdots F_{i_l}G(b)$ is a linear combination of $G(b')$ with $|P^j(b')^-| < |P^j(b)^-|$. Thus, the proof completes.

**Proposition 9.2.2.** Let $b \in B$ and $i \in \mathbb{P}\setminus\{1\}$. Then, we have

$$e_iG^j(b) = [\varphi_{\xi_i}(b) + 1]G^j(\tilde{E}_\xi b) + \sum_{b' \in B \setminus \{E_\gamma b\}} e_{b', b}^{(i)}G^j(b'),$$

$$f_iG^j(b) = [\varphi_{-\xi_i}(b) + 1]G^j(\tilde{E}_{-\xi} b) + \sum_{b' \in B \setminus \{E_{-\gamma} b\}} f_{b', b}^{(i)}G^j(b')$$

for some $e_{b', b}^{(i)}, f_{b', b}^{(i)} \in A$. Moreover, $e_{b', b}^{(i)} = f_{b', b}^{(i)} = 0$ unless $P^j(b) \leq P^j(b')$ or $|P^j(b')^-| < |P^j(b)^-|$.

Proof. We prove the assertion only for $e_i$; the proof for $f_i$ is similar. By Lemma 9.2.1, we can write

$$G^j(b) = G(b) + \sum_{b' \in B \setminus \{b\}} c_{b', b}G(b')$$

for some $c_{b', b} \in A$ such that $c_{b', b} = 0$ unless $P^j(b) \leq P^j(b')$ or $|P^j(b')^-| < |P^j(b)^-|$.

Since $e_i \in U_q(l)$, it holds that

$$e_iG^j(b) \in \text{Span}_A \{G(b'') \mid P^j(b) \leq P^j(b') \text{ or } |P^j(b')^-| < |P^j(b)^-|\}.$$  

Hence, it suffices to show that $[e_iG^j(b) : G^j(\tilde{E}_\xi b)] = [\varphi_{\xi_i}(b) + 1]$. By the definitions of $e_i$ and $G^j(b)$, $e_iG^j(b)$ is the sum of $E_\xi G(b)$ and a linear combination of weight vectors of $M$ of weight lower than $\text{wt}(b) + \alpha_1$. We know from Proposition 9.1.1 (4) that
Proposition 9.3.3. \([E_i G^j(b) : G(\tilde{E}_i b)] = [\varphi_1(b) + 1]\). Hence, we have \([e_i G^j(b) : G^j(\tilde{E}_i b)] = [\varphi_2(b) + 1]\). This proves the assertion. 

Global \(j\)-crystal bases. Let \(M \in \mathcal{O}_\text{int}^j\), \((\mathcal{L}, \mathcal{B})\) a \(j\)-crystal basis of \(M\), \(\psi_M^j\) a \(\psi^j\)-involution, and \(M_A\) a \(U_A\)-submodule of \(M\). Suppose that \(M\) has a \(j\)-global basis \(G^j(B)\) with the associated balanced triple \((\mathcal{L}, M_A, \psi_M^j(\mathcal{L}))\).

Following [K02], let us introduce modified Kashiwara operators:

**Definition 9.3.1.** For \(n \in \mathbb{Z}\), set 
\[
\tilde{f}_i^{(n)} := \sum_{t \geq 0, -n} f_i^{(n+t)} e_i^{(t)} A_n(t; k_i), \\
\tilde{f}_1^{(n)} := \sum_{t \geq 0, -n} f_1^{(n+t)} e_1^{(t)} a_n(t; k_1),
\]
where 
\[
A_n(t; x) := (-1)^t q^{t(1-n)} x^t \prod_{s=0}^{t-1} (1 - q^{n+2s}), \\
a_n(t; x) := (-1)^t p^t q^{t(1-n)} x^t \prod_{s=0}^{t-1} q^s (1 - q^{n+2s}).
\]

**Lemma 9.3.2.** Let \(M \in \mathcal{O}_\text{int}^j\) with the \(j\)-crystal basis \((\mathcal{L}, \mathcal{B})\). For \(n \in \mathbb{Z}\), we have \(\tilde{f}_i^{(n)} \mathcal{L} \subset \mathcal{L}\), and \(\tilde{f}_i^{(n)} \mathcal{L} = \tilde{f}_1^{n} \mathcal{L}\) modulo \(q\mathcal{L}\).

**Proof.** If \(i \neq 1\), then the statement follows from [K02] Proposition 6.1. Hence, we prove the case when \(i = 1\). It suffices to prove the following: For each \(u \in \mathcal{L}\) such that \(e_1 u = 0\), \(k_1 u = q^a u\), \(e_1 f_1 u = [b] \{a - b - 1\} u\) with \(a \in \mathbb{Z}\) and \(b \in \mathbb{Z}_{\geq 0}\), we have \(\tilde{f}_1^{(n)} f_1^{(m)} u = cf_1^{(m+n)} u\) for some \(c \in 1 + qA_0 \cap A\). First of all, we have
\[
\tilde{f}_1^{(n)} f_1^{(m)} u = \sum_{t \geq 0, -n} a_n(t; q^{a-3m}) [m+n]_{m-t} [b-m+t]_t \prod_{s=0}^{t-1} (a-b-m+s) f_1^{(m+n)} u.
\]

We compute the coefficient, say \(A\), of the right-hand side as follows.
\[
A = \sum_{t \geq 0, -n} A_n(t; q^{b-2m}) [m+n]_{m-t} [b-m+t]_t \prod_{s=0}^{t-1} (1 + p^2 q^{2(a-b-m+s)})
\]
\[
= \sum_{t \geq 0, -n} B_t + p^2 \sum_{t \geq 0, -n} B_t g_t,
\]
where \(B_t := A_n(t; q^{b-2m}) [m+n]_{m-t} [b-m+t]_t\), and \(g_t \in \mathbb{Z}[p, q, q^{-1}]\) with \(\prod_{s=0}^{t-1} (1 + p^2 q^{2(a-b-m+s)}) = 1 + p^2 g_t\).

By the proof of [K02] Proposition 6.1, we have \(B_t \in 1 + q\mathbb{Z}[q]\). Also, it is clear that \(p^2 \sum_{t \geq 0, -n} B_t g_t \in p^2 \mathbb{Z}[p, q, q^{-1}]\). Thus, we have \(\tilde{f}_1^{(n)} f_1^{(m)} u \in \mathcal{L}\) and \(\tilde{f}_1^{(n)} f_1^{(m)} u = f_1^{(m+n)} u \equiv f_1^{n} f_1^{(m)} u \mod q\mathcal{L}\). This proves the lemma. 

**Proposition 9.3.3.** Let \(i \in \mathcal{D}, b \in \mathcal{B}\) and \(m \in \mathbb{Z}_{\geq 0}\). Then, we have the following.

1. \(\sum_{n \geq m} f_i^{(n)} M_A = \bigoplus_{b' \in \mathcal{B}} A G^j(b')\).
(2) \( \sum_{n \geq m} e_i^{(n)} M_A = \bigoplus_{\varphi_i(b') \geq m} AG^2(b') \) if \( i \neq 1 \).

(3) \( f_i G^2(b) = [\varepsilon_i(b) + 1] G^2(f_i b) + \sum_{\varphi_i(b') \geq \varepsilon_i(b) + 1} \varphi_{\varphi_i(b')} G^2(b') \) for some \( \varphi_i(b') \in q^{2-\varepsilon_i(b')} \mathbb{Q}[q] \).

(4) \( e_i G^2(b) = [\varphi_i(b) + 1] G^2(e_i b) + \sum_{\varphi_i(b') \geq \varphi_i(b) + 1} \varepsilon_i(b') G^2(b') \) for some \( \varepsilon_i(b') \in q^{2-\varphi_i(b')} \mathbb{Q}[q] \) if \( i \neq 1 \).

Proof. Since \((e_i, k_i, f_i), i \neq 1\) forms an \( \mathfrak{sl}_2\)-triple, most of the assertions follows from Proposition 9.1.1. What we have to prove are assertions (1) and (3) for \( i = 1 \).

First, we prove part (1) by induction on \( m \). When \( m = 0 \), the both sides of the equation to be proved are 0. Assume that assertion (1) holds for all \( m' > m \). Let \( b' \in B \) be such that \( \varepsilon_1(b') = m \). Set \( b'_0 := \bar{e}_1^m b \), and consider \( u := f_1^{(m)} G^2(b'_0) \). By the definition of \( f_1^{(m)} \) and Lemma 9.3.2, we have

\[
\sum_{n \geq m} f_1^{(n)} M_A \supseteq \sum_{n \geq m} f_1^{(n)} M_A.
\]

By our inductive hypothesis, we can write

\[
u - f_1^{(m)} G^2(b'_0) = \sum_{\varphi_i(b') \geq m} a_{\varphi_i(b')} G^2(b')
\]

for some \( a_{\varphi_i(b')} \in A \). Then, we can take \( a_{\varphi_i(b')} \in q \mathbb{Q}[q] \) in a way such that \( a_{\varphi_i(b')} = a_{\varphi_i(b')} \). Set \( v := u - \sum_{\varphi_i(b') \geq m} a_{\varphi_i(b')} G^2(b') \).

Then, we have \( v \in M_A \cap \mathcal{L} \), \( \psi_M(v) = v \), and \( v + q \mathcal{L} = u + q \mathcal{L} = b' \). These implies that \( v = G^2(b') \), and therefore, \( G^2(b') \in \sum_{n \geq m} f_1^{(n)} M_A \). Hence, we obtain

\[
\sum_{n \geq m} f_1^{(n)} M_A \supseteq \bigoplus_{\varphi_i(b') \geq m} AG^2(b').
\]

We prove the opposite inclusion. For each \( \lambda \in \Lambda \), we have

\[
(M_A)_\lambda \subseteq \sum_{b \in B_\lambda} AG^2(b)
\]

\[
= \sum_{b \in B_\lambda} AG^2(b) + \sum_{\varphi_i(b') \geq 1} AG^2(b')
\]

\[
\subseteq \sum_{b \in B_\lambda} AG^2(b) + \sum_{n \geq 1} f_1^{(n)} (M_A)_{\lambda + \gamma_1}.
\]

Hence, we obtain

\[
f_1^{(m)} (M_A)_{\lambda} \subseteq \sum_{b \in B_\lambda} AG^2(b) + \sum_{n \geq 1} f_1^{(n)} (M_A)_{\lambda + \gamma_1}
\]

\[
\subseteq \sum_{b \in B_\lambda} AG^2(b) + \sum_{n \geq 1} f_1^{(m+n)} (M_A)_{\lambda + \gamma_1}
\]

\[
= \sum_{b \in B_\lambda} AG^2(b) + \sum_{\varphi_i(b') \geq m} AG^2(b') \quad \text{(by induction hypothesis)}.
\]
Also, by the argument above, \( f_1^{(m)} G^j(b) \) with \( \varepsilon_1(b) = 0 \) is contained in
\[
\sum_{\varepsilon(b') \geq m} A G^j(b').
\]
This completes the proof of part (1).

Next, we turn to prove assertion (3) for \( i = 1 \) by descending induction on \( m := \varepsilon_1(b) \). When \( m \) is maximum among \( \{ \varepsilon_1(b') \mid b' \in B \} \), we have by (1) that
\[
f_1 G^j(b) \in \sum_{n > m} f_1^{(n)} M_{A} = \sum_{\varepsilon_1(b') > m} A G^j(b') = 0,
\]
and the equation in (3) holds. Assume that (3) is true for all \( m' > m \). As in the proof of (1), let us write
\[
G^j(b) = f_1^{(m)} G^j(\tilde{e}^m_1 b) + \sum_{b' \in B} c_{b'} G^j(b'),
\]
\[
G^j(f_1 b) = f_1^{(m+1)} G^j(\tilde{e}^m_1 b) + \sum_{b'' \in B} d_{b''} G^j(b'')
\]
for some \( c_{b'}, d_{b''} \in A \). Then, we have
\[
f_1 G^j(b) = [m + 1] f_1^{(m+1)} G^j(\tilde{e}^m_1 b) + \sum_{\varepsilon_1(b') > m} c_{b'} f_1 G^j(b')
\]
\[
= [m + 1] f_1^{(m+1)} G^j(\tilde{e}^m_1 b) + \sum_{\varepsilon_1(b') > m} c_{b'} ([\varepsilon_1(b') + 1] G^j(\tilde{f}_1 b')
\]
\[
+ \sum_{\varepsilon_1(b') > \varepsilon_1(b') + 1} \varphi_{b', b}^{(1)} G^j(b'))
\]
\[
= [m + 1] G^j(\tilde{f}_1 b) + \sum_{\varepsilon_1(b') > m} c_{b'} ([\varepsilon_1(b') + 1] G^j(\tilde{f}_1 b')
\]
\[
+ \sum_{\varepsilon_1(b') > \varepsilon_1(b') + 1} \varphi_{b', b}^{(1)} G^j(b'))
\]
\[
- \sum_{\varepsilon_1(b') > \varepsilon_1(b') + 1} (m + 1) d_{b''} G^j(b'').
\]
Thus, we obtain that
\[
f_1 G^j(b) = [\varepsilon_1(b) + 1] G^j(\tilde{f}_1 b) + \sum_{\varepsilon_1(b') > \varepsilon_1(b') + 1} \varphi_{b', b}^{(i)} G^j(b')
\]
for some \( \varphi_{b', b}^{(i)} \in A \). It remains to prove that \( \varphi_{b', b}^{(i)} \in q^{2-\varepsilon_1(b')} \mathbb{Q}[q] \). Let us write
\[
G^j(b) = \sum_{k \geq m} f_1^{(k)} u_k
\]
for some \( u_k \in \mathcal{L}_{\varepsilon_1(b') + \gamma_1} \) such that \( \epsilon_1 u_k = 0 \). Note that \( G^j(b) + q \mathcal{L} = u_m + q \mathcal{L} \).
Then, we have
\[
f_1 G^j(b) = [m + 1] f_1^{(m+1)} u_m + \sum_{k > m} [k + 1] f_1^{(k+1)} u_k,
\]
and that
\[
f_1^{(m+1)} u_m \in \mathcal{L}, f_1^{(m+1)} u_m + q \mathcal{L} = \tilde{f}_1 b.
\]
Hence, we have
\[
f_1 G^j(b) = [m + 1] G^j(\tilde{f}_1 b) + \sum_{k > m} [k + 1] f_1^{(k+1)} u_k \text{ modulo } q^{2-m} \mathcal{L}.
\]
Then, rewriting \( f_1^{(k+1)} u_k \) as a sum...
of $G^j(b'), \varepsilon_1(b') \leq k + 1$ with coefficients in $q\mathbb{A}_0$, we conclude that the coefficient of $G^j(b')$ in $f_1G^j(b)$ lies in $q^{2-\varepsilon_1(b')}\mathbb{A}_0 \cap \mathbb{A} = q^{2-\varepsilon_1(b')}\mathbb{Q}[q]$. This completes the proof. \qed

For a bipartition $\lambda \in P^j(M)$, define $I_{\lambda}(M), W_{\geq \lambda}(M), W_{> \lambda}(M), W_{\lambda}(M)$ in a similar way as $I_{\lambda}, W_{\geq \lambda}, W_{> \lambda},$ and $W_{\lambda}$ respectively.

Definition 9.3.4. We say that $M$ has the property (*) if there exists a poset $(S, \leq)$ and a map $s: B \to S$ satisfying the following:

1. The abelian group $Q := \sum_{i \in I} \mathbb{Z}\alpha_i$ acts on $S$ freely; the action is written additively.
2. $\sigma \leq \sigma + \lambda$ for all $\lambda \in Q_+, \sigma \in S$.
3. $\sigma + \lambda \leq \sigma' + \lambda$ for all $\lambda \in Q, \sigma \leq \sigma' \in S$.
4. $s(b) = s(b')$ only if $\text{wt}(b) = \text{wt}(b')$ for all $b, b' \in B$.
5. For $b \in B$ and $i \in \mathbb{P} \setminus \{1\}, s(E_i b) = s(b) + \alpha_i$ if $E_i b \neq 0$.
6. For $i \in \mathbb{P} \setminus \{1\}$, $e_i G^j(b) = [\varphi_\lambda(b) + 1]G^j(E_i b) + \sum_{b' \in B \setminus \{E_i b\}} e_{b', b}^{(i)} G^j(b'),$

$f_i G^j(b) = [\varphi_{\lambda'}(b) + 1]G^j(E_i b) + \sum_{b' \in B \setminus \{E_i b\}} f_{b', b}^{(i)} G^j(b')$

for some $e_{b', b}^{(i)}, f_{b', b}^{(i)} \in \mathbb{A}$.

Lemma 9.3.5. Let $M \in \mathcal{O}_{\text{int}},$ and $\mathcal{L}, \mathcal{B}, \psi_M, M_\mathcal{A}$ as above.

1. If $r = 1$, then $M$ has the property (*).
2. If $M \in \mathcal{O}_{\text{int}}$ and the global $j$-crystal basis is the $j$-canonical basis, then $M$ has the property (*).

Proof. Setting $S$ and $s$ to be $\Lambda$ and $\text{wt}$, respectively, part (1) is obvious, and part (2) follows from Proposition 9.2.2. \qed

The main result in this paper is the following:

Theorem 9.3.6. Suppose that $M$ has the property (*). Then, for each $\lambda \in P^j(M)$, the following hold:

1. $W_{\geq \lambda}(M)$ has a global $j$-crystal basis $W_{\geq \lambda}(G^j(\mathcal{B})) := \{G^j(b) \mid I(b) \geq \lambda\}$ with the associated balanced triple $(W_{\geq \lambda}(\mathcal{L}), W_{\geq \lambda}(M_\mathcal{A}), W_{\geq \lambda}(\psi_M(\mathcal{L})))$, where $W_{\geq \lambda}(\mathcal{L}) := W_{\geq \lambda}(\mathcal{L}) \cap \mathcal{L}$ and so on.
2. $W_{> \lambda}(M)$ has a global $j$-crystal basis $W_{> \lambda}(G^j(\mathcal{B})) := \{G^j(b) \mid I(b) > \lambda\}$ with the associated balanced triple $(W_{> \lambda}(\mathcal{L}), W_{> \lambda}(M_\mathcal{A}), W_{> \lambda}(\psi_M(\mathcal{L})))$, where $W_{> \lambda}(\mathcal{L}) := W_{> \lambda}(\mathcal{L}) \cap \mathcal{L}$ and so on.
3. $W_{\lambda}(M)$ has a global $j$-crystal basis $W_{\lambda}(G^j(\mathcal{B})) := \{G^j(b) + W_{> \lambda}(M) \mid I(b) = \lambda\}$ with the associated balanced triple $(W_{\lambda}(\mathcal{L}), W_{\lambda}(M_\mathcal{A}), W_{\lambda}(\psi_M(\mathcal{L})))$, where $W_{\lambda}(\mathcal{L}) := W_{\geq \lambda}(\mathcal{L})/W_{> \lambda}(\mathcal{L})$ and so on.
(4) There exists a \( U^j \)-module isomorphism \( \xi : L(\lambda)^{\oplus m_\lambda} \to W_\lambda(M) \) which induces an isomorphism

\[
(L(\lambda)^{\oplus m_\lambda}, (L(\lambda)_A)^{\oplus m_\lambda}, \psi^j_\lambda(L(\lambda))^{\oplus m_\lambda}) \simeq (W_\lambda(L), W_\lambda(M_A), W_\lambda(\psi^j_M(L))),
\]

where \( m_\lambda := \dim \text{Hom}_{U^j}(L(\lambda), M) \) denotes the multiplicity of \( L(\lambda) \) in \( M \).

The proof will be given in Section 10.

**Corollary 9.3.7.** Let \( \lambda \in P^j \). Then, \( G^j_{\lambda} \) is a unique global \( j \)-crystal basis of \( L(\lambda) \) satisfying the property (5).\)

9.4. **Operators \( \tilde{e}_{i^+} \) and \( \tilde{f}_{i^+} \).** The definitions of \( \tilde{e}_{i^+} \) and \( \tilde{f}_{i^+} \) given in [W17] are artificial, namely, they are defined by means of a distinguished basis \( G^j_{\lambda} \), \( \lambda \in P^j \) (in [W17], it is denoted by \( \{ b_T \mid T \in B(\lambda) \} \)). Here, we define new operators \( \tilde{e}_{i^+} \) and \( \tilde{f}_{i^+} \) for \( i \in \mathbb{P} \setminus \{ 1 \} \), and then, explain that the operators \( \tilde{e}_{i^+} \) and \( \tilde{f}_{i^+} \) on \( j \)-crystal bases are in fact intrinsic.

**Lemma 9.4.1.** Let \( r \geq 2, \lambda \in P^j \), and consider the irreducible highest weight module \( L(\lambda) \). As a \( U^j_{r-1} \)-module, \( L(\lambda) \) is multiplicity-free.

**Proof.** Let \( b \in B(\lambda) \) be a \( U^j_{r-1} \)-highest weight vector with highest weight, say, \( \mu \in P^j_{r-1} \). If we identify \( B(\lambda) \) with \( \text{SST}(\lambda) \), we have \( T^j_b \downarrow_{r-1} = T^j_\mu \). Since the entries of the boxes of \( T^j_b \) corresponding to \( \lambda/\mu \) are either \( -r \) or \( r \), it must hold that \( \lambda/\mu \) is a horizontal strip. Conversely, given \( \mu \in P^j_{r-1} \) such that \( \lambda/\mu \) is a horizontal strip, there exists a unique \( b \in B(\lambda) \) which is a \( U^j_{r-1} \)-highest weight vector with highest weight \( \mu \). This proves the lemma. \( \square \)

**Lemma 9.4.2.** Let \( r \geq 2, \lambda \in P^j \). Let \( b \in B(\lambda) \) be such that \( \tilde{e}_{r^+} b \neq 0 \). Then, there exist unique \( b' \in B(\lambda) \) and \( j \in \mathbb{P} \setminus \{ 1 \} \) satisfying the following:

- \( b' \) is a \( U^j_{r-1} \)-highest weight vector.
- There exist unique \( \varepsilon_i \in \{ 0,1 \} \) for each \( j \leq i \leq r-1 \) such that \( b = \tilde{f}_{r^+} \tilde{f}_{(r-1)^{r-1}} \cdots \tilde{f}_{j+1} b' \).

**Proof.** By the definition of \( \tilde{e}_{r^+} \), \( b \) is a \( U^j_{r-1} \)-highest weight vector with highest weight, say, \( \mu \in P^j_{r-1} \) such that \( (T^j_\mu)^- = T^j_\lambda \). Then, \( T^j_{\tilde{e}_{r^+} b} \downarrow_{r-1} \) is obtained from \( T^j_\mu \) by adding a box \( [r-1] \) to the \((j-1)\)-th row for some uniquely determined \( j \in \mathbb{P} \setminus \{ 1 \} \). Set \( b_{r-1} := \tilde{e}_{r^+} b \). Now, we have exactly one of the following: \( \tilde{e}_{r-1} b_{r-1} \neq 0 \) or \( \tilde{e}_{(r-1)^{r-1}} b_{r-1} \neq 0 \). Choose a unique \( \varepsilon_{r-1} \in \{ 0,1 \} \) in a way such that \( b_{r-2} := \tilde{e}_{(r-1)^{r-1}} b_{r-1} \neq 0 \). Then, \( T^j_{b_{r-2}} \downarrow_{r-1} \) is obtained from \( T^j_\mu \) by adding a box \([r-2]\) to the \((j-1)\)-th row. Repeating this procedure, we obtain \( \varepsilon_i \in \{ 0,1 \} \) and \( b_{r-1} \in B(\lambda) \) for \( j \leq i \leq r-1 \). By the construction, \( T^j_{b_{j-1}} \downarrow_{r-1} \) is obtained from \( T^j_\mu \) by adding a box \([j-1]\) to the \((j-1)\)-th row, which turned out to be \( T^j_{\mu'} \), where \( \mu' \in P^j_{r-1} \) such that \( \mu_k = \mu_k + \delta_{k,j-1}, k \in \{ -(r-1), \ldots, r-1 \} \). Hence, \( b_{j-1} \) is a \( U^j_{r-1} \)-highest weight vector, and we have \( b = \tilde{f}_{r^+} \tilde{f}_{(r-1)^{r-1}} \cdots \tilde{f}_{j+1} b_{j-1} \). This proves the assertion. \( \square \)

Set \( E_r(\lambda) := \{ \mu \in P^j_{r-1} \mid \mu^- = \lambda^- \downarrow_{r-1} \text{ and } \lambda^+/\mu^+ \text{ is a horizontal strip} \} \). Then, the assignment

\[
\{ b \in B(\lambda) \mid \tilde{e}_{r^+} b \neq 0 \} \rightarrow E_r(\lambda); \ b \mapsto P^j_{r-1}(b)
\]
Proposition 9.4.3. Let $\mu \in E_r(\lambda)$, we associate $b, b' \in B(\lambda), j \in I \setminus \{1\}$, and $c_i \in \{0, 1\}, j \leq i \leq r - 1$ as in Lemma 9.1.2.

Let $r \geq 2$. We define operators $\overline{e}_{t+}$ and $\overline{f}_{t+}$ on every $U^j$-modules in $O_{\text{int}}^j$ inductively for all $2 \leq l < r$. Let $\lambda \in P^j$. We define the linear operator $\overline{e}_{r+}$ on $L(\lambda)$ by

$$
\overline{e}_{r+} := \bigoplus_{\mu \in E_r(\lambda)} p_2(\mu) \circ \frac{1}{[\varphi_\lambda(b_\mu) + 1]} \epsilon_r \circ p_1(\mu),
$$

where $b_\mu \in B(\lambda)$ is the corresponding element to $\mu \in E_r(\lambda)$, $p_1(\mu)$ is the projection from $L(\lambda)$ to the one-dimensional subspace $L(\mu_{\text{wt}}(\mu))$;

$$
L(\mu_{\text{wt}}(\mu)) \subset L(\mu)_{\text{multiplicity free}} \xrightarrow{\epsilon_r} L(\lambda),
$$

and $p_2(\mu)$ is the projection from $L(\lambda)$ to the one-dimensional subspace $\overline{f}_{(r-1)^{t_{r-1}}} \cdots \overline{f}_{j} L(\lambda)$;

$$
\overline{f}_{(r-1)^{t_{r-1}}} \cdots \overline{f}_{j} L(\lambda)_{\text{multiplicity free}} \xrightarrow{\epsilon_r} L(\lambda),
$$

where $\delta_l = 0$ if $\epsilon_l = 0$, and $\delta_l = +$ if $\epsilon_l = t$ for $l = j, \ldots, r - 1$. Also, we define $\overline{f}_{r+}$ by

$$
\overline{f}_{r+} = \bigoplus_{\mu \in E_r(\lambda)} \epsilon_{r-1} \circ p_2(\mu),
$$

where $\epsilon_{r-1}$ is the inverse of the linear isomorphism $\overline{e}_{r+} : L(\mu_{\text{wt}}(\mu)) \rightarrow \overline{f}_{(r-1)^{t_{r-1}}} \cdots \overline{f}_{j} L(\lambda)$.

Finally, we extend the definitions of $\overline{e}_{r+}$ and $\overline{f}_{r+}$ to a general $U^j$-module $M \in O_{\text{int}}^j$ by the complete reducibility of $M$.

**Proposition 9.4.3.** Let $\lambda \in P^j$ and $v \in L(\lambda)$ a highest weight vector. Then, we have

$$
\mathcal{L}(\lambda) = \text{Span}_{\mathbb{A}} \{ \overline{f}_{i_1} \cdots \overline{f}_{i_l} v \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in I \setminus \{2^+, \ldots, r^+\} \},
$$

$$
\mathcal{B}(\lambda) = \{ \overline{f}_{i_1} \cdots \overline{f}_{i_l} v + q\mathcal{L}(\lambda) \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in I \setminus \{2^+, \ldots, r^+\} \} \setminus \{0\}.
$$

Moreover, on $\mathcal{B}(\lambda)$, we have $\overline{e}_{r'} = \overline{e}_{i+}$ and $\overline{f}_{r'} = \overline{f}_{i+}$ for all $i \in I \setminus \{1\}$.

**Proof.** We proceed by induction on $r$. Assume that the assertion holds for all $2 \leq l < r$ (we assume nothing when $r = 2$). Let $\mu \in E_r(\lambda)$ and $b_\mu, b', b''$ be as above. By the uniqueness of the $j$-crystal bases for $U^j_{r-1}$-modules, there exists a unique $v_\mu \in \mathcal{L}(\lambda)$ such that $U^j_{r-1} v_\mu = L(\mu), v_\mu + q\mathcal{L}(\lambda) = b_\mu$. Then, we can write

$$
v_\mu = G_{r-1}^j(b_\mu) + \sum_{b' \in \mathcal{B}(\lambda) \setminus \{b_\mu\}} a_{b'} G_{r-1}^j(b')
$$

for some $a_{b'} \in q\mathbb{A}_{0}$. Note that this equation implies that $\overline{e}_{r'}(v_\mu) \in G_{r-1}^j(\overline{e}_{r'}b_\mu) + q\mathcal{L}(\lambda)$. Also, we have

$$
\frac{1}{[\varphi_\lambda(b_\mu) + 1]} \epsilon_r v_\mu = G_{r-1}^j(\overline{e}_{r'}b_\mu) + \sum_{b' \in \mathcal{B}(\lambda)} c_{b'} G_{r-1}^j(b') \quad \text{since } \overline{e}_{r'}b_\mu = \overline{E}_\lambda b_\mu
$$

for some $c_{b'} \in \mathbb{A}$. Again, by the complete reducibility of the $U^j_{r-1}$-crystal bases, there exists a unique $v_{\mu'} \in \mathcal{L}(\lambda)$ such that $U^j_{r-1} v_{\mu'} = L(\mu'), v_{\mu'} + q\mathcal{L}(\lambda) = b'$. By
our induction hypothesis, we have \( u := \bar{f}_{(r-1)\lambda_{r-1}} \cdots \bar{f}_{j+1}(v_{\mu'}) \in \mathcal{L}(\lambda) \cap U_{r-1}v_{\mu'} \) and \( u + q\mathcal{L}(\lambda) = \bar{e}_{r+}b_{\mu} \). Then, we can write
\[
u = G^2_{\text{low}}(\bar{e}_{r+}b_{\mu}) + \sum_{b' \in B(\lambda)} d_{b'}G^2_{\text{low}}(b')
\]
for some \( d_{b'} \in qA_0 \). Hence, we have
\[
u = G^2_{\text{low}}(\bar{e}_{r+}b_{\mu}) + q\mathcal{L}(\lambda).
\]
Since we took \( \mu \in E_r(\lambda) \) arbitrarily, this equation ensures that \( \bar{e}_{r+} \) preserves \( \mathcal{L}(\lambda) \) and \( B(\lambda) \sqcup \{0\} \), and that \( \bar{e}_{r+} = \bar{e}_{r} \) on \( B(\lambda) \). By the definition of \( \bar{f}_{r+} \), it also preserves \( \mathcal{L}(\lambda) \) and \( B(\lambda) \sqcup \{0\} \), and coincides with \( \bar{f}_{r} \) on \( B(\lambda) \). Now, the assertions are clear by the definition of \((\mathcal{L}(\lambda), B(\lambda))\).

\[\square\]

**Corollary 9.4.4.** Let \( M \in \mathcal{O}^1_{\text{int}} \) be a \( U^j \)-module with a \( j \)-crystal basis \((\mathcal{L}, B)\). Then \( \bar{e}_{i'} = \bar{e}_{i+} \) and \( \bar{f}_{i'} = \bar{f}_{i+} \) on \( B \) for all \( i \in \mathbb{P} \setminus \{1\} \).

10. **Proof of Theorem 9.3.6**

For a \( U^j \)-module \( M \) with a global \( j \)-crystal basis \( G^j(B) \), and for \( m \in M, b \in B \), let \([m : G^j(b)]\) denote the coefficient of \( G^j(b) \) in \( m \).

10.1. **The \( r = 1 \) case.** In this subsection, we prove Theorem 9.3.6 for \( r = 1 \).

**Proof of Theorem 9.3.6** We proceed by descending induction on \( \lambda \) with respect to \( \preceq \). Assume that the statement holds for all \( \lambda' > \lambda \). Replacing \( M \) with \( M/W_{> \lambda}(M) \), we may assume that \( \lambda \) is maximal among \( P^j(M) \). Let \( b_1, \ldots, b_{m_\lambda} \in B \) and \( u_1, \ldots, u_{m_\lambda} \in \mathcal{L} \) be distinct highest weight vectors of type \( \lambda \) with \( u_i + q\mathcal{L} = b_i \), \( i = 1, \ldots, m_\lambda \). By retaking the \( u_i \)'s if necessary, we may assume that \([u_i : G^j(b_j)] = \delta_{i,j} \) for all \( i, j \). Fix \( i \) arbitrarily, and set \( b := b_i, u := u_i \). Then, we can write
\[
u = G^j(b) + \sum_{b' \in B \setminus \{b\}} c_{b'}G^j(b'), \quad c_{b'} \in qA_0.
\]
We first prove that \( c_{b'} = 0 \) for all \( b' \) with \( \epsilon_1(b') = 0 \). Assume contrary, and take \( b' \in B \setminus \{b\} \) such that \( c_{b'} \neq 0, \epsilon_1(b') = 0 \), and \( \varphi_1(b') \) is minimal among \( \{\varphi_1(b'') \mid c_{b''} \neq 0, \epsilon_1(b'') = 0\} \). Set \( \mu := I^j(b') \). Then, we have \( \text{wt}^j(\mu) = \text{wt}^j(\lambda) \), in particular, \( \mu_0 = \lambda_0 \). Since \( \mu \npreceq \lambda \), we have \( \varphi_1(b') = \mu_0 - \mu_{-1} > \lambda_0 - \lambda_{-1} = \varphi_1(b) \).

Hence, we have
\[
g^{d_1\epsilon_1(b)+1}G^j(b) = c_{b'}\left( G^j(\bar{f}^{d_1\epsilon_1(b)+1}b') + \sum_{\epsilon_1(b') > \varphi_1(b)+1} d_{b'',b'}G^j(b'') \right)
\]
\[
+ \sum_{b'' \neq b'} \sum_{b''' \epsilon_1(b'') \geq 1, \epsilon_1(b''') \geq \epsilon_1(b'''), \varphi_1(b)+1} d_{b''',b''}G^j(b'''),
\]
for some \( d_{b_1,b_2} \in A \). By our assumption, the coefficient of \( G^j(\bar{f}^{d_1\epsilon_1(b)+1}b') \) in the right-hand side is equal to \( c_{b'} \). On the other hand, the left-hand side is fixed by \( \psi^j_M \), and it belongs to \( M_A \). Therefore, we have \( c_{b'} \in qA_0 \cap A \) and \( c_{b'} = c_{b'} \), which implies \( c_{b'} = 0 \).

Next, we prove that \( c_{b'} = 0 \) for all \( b' \) with \( \epsilon_1(b') > 0 \). Assume contrary that \( c_{b'} \neq 0 \) for some such \( b' \). Set \( \mu := I^j(b') \). Since \( \lambda \) is maximal, we have \( \mu_0 + \mu_{-1} <
and hence, we have
\[ G_1(b) = c_{1'} \left[ \epsilon_1(b') + \varphi_1(b) + t \right] G_1(\varphi_1(b) + t b') + \text{(other terms)}. \]

This implies that \( c_{1'} \in q\mathbb{A}_0 \), \( \bar{c}_{1'} = c_{1'} \), and \( c_{1'} \left[ \epsilon_1(b') + \varphi_1(b) + t \right] \in \mathbb{A} \) for all \( t = 1, \ldots, \epsilon_1(b') + 1 \). Now, it suffices to show that \( c_{1'} \in \mathbb{A} \), which follows from next lemma.

This far, we have proved that \( G^j(b) = u \), and hence, we have \( e_1 G^j(b) = 0 \) and \( U_1 G^j(b) \cong L(\lambda) \). Then, for all \( n = 1, \ldots, \lambda_0 - \lambda_1 \), we have
\[ f_1^{(n)} G^j(b) = f_1^{(n)} u = \bar{f}_1^{n} u. \]

The left-hand side belongs to \( M_{\mathbb{A}} \), while the right-hand side belongs to \( \mathcal{L} \). Moreover, we have \( \bar{v}_d^j (f_1^{(n)} G^j(b)) = f_1^{(n)} \bar{G}^j(b) \), and \( \bar{f}_1^n u + q\mathcal{L} = \bar{f}_1^n b \). This implies that \( f_1^{(n)} G^j(b) = \bar{G}^j(\bar{f}_1^n b) \). Thus, the proof completes. \( \square \)

**Lemma 10.1.1.** Let \( A \subseteq \mathbb{Q}(p,q) \), \( m \geq n \in \mathbb{Z}_{\geq 0} \). Suppose that \( A^{[m+n]} \subseteq \mathbb{A} \) for all \( t = 1, \ldots, n+1 \). Then, we have \( A \subseteq \mathbb{A} \).

**Proof.** Let us write \( A = B/C \) for some \( B, C \subseteq \mathbb{A}_0 \cap \mathbb{A} \) that are coprime. By the hypothesis, \( C \) is a common divisor of \( [m+n] \), \( t = 1, \ldots, n+1 \). Hence, it suffices to show that the greatest common divisor of them in \( \mathbb{Z}[q] \) is equal to 1. This is equivalent to say that the greatest common divisor of \( a_t := [m+t][m+t-1] \cdots [m+t-n+1] \), \( t = 1, \ldots, n+1 \) is equal to \([n]!\). Since \([l] = q^{-l} \prod_{1 \neq d \mid l} \Phi_d \), where \( \Phi_d = \Phi_d(q^2) \) denotes the \( d \)-th cyclotomic polynomial in variable \( q^2 \), we have
\[ b_t := q^{n(m+t) - \frac{n(n-1)}{2}} a_t = \prod_{l=0}^{n-1} \prod_{1 \neq d \mid (m+t-l)} \Phi_d, \]
which is the irreducible decomposition of \( b_t \) in \( \mathbb{Z}[q^2] \). Then, we have
\[ b_t = \prod_{d \geq 2} \Phi_d^{m_{d,t}}, \text{ where } m_{d,t} := |\{0 \leq l \leq n-1 \mid d \mid (m+t-l)\}|, \]
and hence,
\[ \gcd_{1 \leq t \leq n+1} (b_t) = \prod_{d \geq 2} \Phi_d^{\min_{1 \leq t \leq n+1} (m_{d,t})}. \]
We prove that \( \min_{1 \leq t \leq n+1} (m_{d,t}) = \lfloor \frac{n}{d} \rfloor \) for all \( d \). It is clear that \( m_{d,t} \geq \lfloor \frac{n}{d} \rfloor \) for all \( t \) since \( \{m+t, m+t-1, \ldots, m+t - \lfloor \frac{n}{d} \rfloor \} \) contains exactly \( \lfloor \frac{n}{d} \rfloor \) integers divisible by \( d \). If \( \min_{1 \leq t \leq n+1} (m_{d,t}) > \lfloor \frac{n}{d} \rfloor \), then \( \{m+t - \lfloor \frac{n}{d} \rfloor, m+t - (\lfloor \frac{n}{d} \rfloor + 1), \ldots, m+t - (n-1)\} \) contains at least one multiple of \( d \) for all \( t \). Then, for \( t = 1 \), there exists \( l_1 \in \{\lfloor \frac{n}{d} \rfloor, \lfloor \frac{n}{d} \rfloor + 1, \ldots, n-1\} \) such that \( m+1 - (\lfloor \frac{n}{d} \rfloor + l_1) \in d\mathbb{Z} \).

Set \( t' := n - l_1 + 1 \), and consider the integers
\[ m + t' - \lfloor \frac{n}{d} \rfloor d, m + t' - (\lfloor \frac{n}{d} \rfloor d + 1), \ldots, m + t' - (n-1) = (m + 1 - l_1) + 1. \]
These are \( (n - \lfloor \frac{n}{d} \rfloor d) \) consecutive integers with \( (m + 1 - l_1) + 1 = 1 \) modulo \( d \). Since \( n - \lfloor \frac{n}{d} \rfloor d < d \), they have no multiples of \( d \). Hence, we have \( \min_{1 \leq t \leq n+1} (m_{d,t}) = \lfloor \frac{n}{d} \rfloor \).
for all \( d \geq 2 \). Thus, we obtain
\[
\gcd_{1 \leq t \leq n+1} (b_t) = \prod_{d \geq 2} \Phi_d^\frac{n}{d} = \prod_{d \geq 2} \Phi_d^{\frac{n}{d}} = \prod_{l=2}^{n} \left( \prod_{1 \neq d | l} \Phi_d \right) = \prod_{l=2}^{n} [l] = [n]!.
\]
This proves the lemma. \( \square \)

10.2. The \( r \geq 2 \) case. Now, we are ready to prove Theorem 9.3.6 by induction on \( r \).

When \( r = 1 \), we have already completed the proof. Let \( r \geq 2 \) and assume that the assertions hold for all \( r' < r \).

**Lemma 10.2.1.** Let \( \lambda \in P^1(M) \) be a maximal element, \( b \in B \) such that \( P(b) = \lambda \) and \( e, b = 0 \) for all \( i \in \mathbb{P} \). Suppose the following:

1. There exists a homomorphism \( \xi : L(\lambda) \to M \) of \( U^j \)-modules such that \( \xi(G^j_{\lambda}(T_{b'}^j)) = G^j(b') \) for all \( b' \in C^j(b) \) which is strongly connected to some \( b'' \in C^j(b) \) with \( wt(b') < j \cdot wt(b'') \).
2. \( \xi \) commutes with the \( \psi^j \)-involutions on \( L(\lambda) \) and \( M \).
3. \( \xi(G^j_{\lambda}(T_{b}^j)) : G^j(b) = 1 \).

Then, we have
\[
\xi(G^j_{\lambda}(T_{b}^j)) = G^j(b) + \sum_{b' \in B \setminus \{b\}, T_{b'}^j = T_{b}^j} c_{b'} G^j(b') + \sum_{b'' \in C^j(b'}, c_{b''} G^j(b'') \quad \text{for some } c_{b'}, c_{b''} \in A_0.
\]

Proof. Since \( U^j \)-module homomorphisms preserve \( j \)-crystal lattices, we have \( \xi(G^j_{\lambda}(T_{b}^j)) \in L \), and \( \xi(G^j_{\lambda}(T_{b}^j)) + qL = b \). Let us write
\[
\xi(G^j_{\lambda}(T_{b}^j)) = G^j(b) + \sum_{b' \in B \setminus \{b\}} c_{b'} G^j(b')
\]
for some \( c_{b'} \in qA_0 \). Also, since \( \xi \) commutes with \( \psi^j \)-involutions, we have \( \overline{c_b} = c_b, \overline{c_{b'}} = c_{b'} \). We claim the following: if \( b' \in B \setminus \{b\} \) satisfies

\( (\dagger) \quad c_{b'} \neq 0 \) and \( s(b') \) is maximal among \( \{s(b'') | b'' \in B \setminus \{b\} \text{ and } c_{b''} \neq 0\} \),

then \( T_{b'}^j(-i) > \lambda_{-i} \) for all \( i = 0, 1, \ldots, r \). By the case \( r = 1 \), we have \( I_1^j(b') \supset I_1^j(b) \), which implies \( T_{b'}^j(0) = T_{b}^j(0) = \lambda_0 \), and \( T_{b'}^j(-1) > T_{b}^j(-1) = \lambda_{-1} \). We proceed by induction on \( i \). Assume that \( i \geq 2 \), and that \( T_{b'}^j(-(i-1)) \geq \lambda_{-(i-1)} \) for all \( b' \) satisfying \( (\dagger) \). Suppose that there exists \( b' \) satisfying \( (\dagger) \) such that \( T_{b'}^j(-(i-1)) < \lambda_{-i} \). Let \( b'' \in B \setminus \{b\} \) be such that \( s(b'') = s(b') \) and \( \varphi_{-b''} \) is minimal among such elements. Recall that \( s(b'') = s(b') \) implies \( wt(b'') = wt(b') \), and hence, \( T_{b'}^j(-(i-1)) = T_{b'}^j(-i) < \lambda_{-i} \). Then, we have
\[
\varepsilon_{-\frac{i}{2}}(b'') = \varphi_{-\frac{i}{2}}(b') + T_{b'}^j(-(i-1)) - T_{b'}^j(-i) > T_{b'}^j(-i-1) - \lambda_{-i} + \varphi_{-\frac{i}{2}}(b'').
\]
By the minimality of \( \varphi_{-\frac{i}{2}}(b'') \), it holds that
\[
\left[ f_i^{(l)} \sum_{b' \in B \setminus \{b\}} c_{b'} G^j(b') : G^j(f_i^{(l)} b'') \right] = c_{b'} \left[ \varphi_{-\frac{i}{2}}(b'') \right] \neq 0
\]
for all \( T_{b'}^j(-(i-1)) - \lambda_{-i} + 1 \leq t \leq T_{b'}^j(-(i-1)) - \lambda_{-i} + \varphi_{-\frac{i}{2}}(b'') + 1 \). On the other hand, \( f_i^{(l)} G^j_{\lambda}(T_{b}^j) \) is the sum of \( G^j_{\lambda}(T_{f_i^{(l)} b}^j) \) and an \( A \)-linear combination of
for all \( wt \) since

\[
\xi(G^j_{\text{low}}(T^2_{b^j})) = f_i^{(t)} \xi(G^j_{\text{low}}(T^2_{b^j})) + \sum b \in A \quad G^j(\hat{b})
\]

\[
= f_i^{(t)} G^j(b) + \sum c_{b'} G^j(b') + \sum b \in A \quad G^j(\hat{b})
\]

for some \( a_\hat{b} \in A \). Here, note that we have \( \overline{e}_j \hat{f}^i b = 0 \) for all \( j = 1, \ldots, i - 1, \)

\[
[\xi(G^j_{\text{low}}(T^2_{b^j})) : G^j(\hat{f}^i b)] = 1, \quad s(\hat{f}^i b') \text{ is maximal among } \{ s(b'') | b'' \neq \hat{f}^i b \}
\]

Then, by our induction hypothesis on \( i \), we obtain that \( T^2_{b^j} b^{(i-1)} = T^2_{b^j} b^{(i-1)} = \lambda_{-i} \), which is a contradiction since \( t \geq T^2_{b^j} b^{(i-1)} - \lambda_{-i+1} \). Hence we must have \( [\xi(G^j_{\text{low}}(T^2_{b^j})) : G^j(\hat{f}^i b''')] = 0 \).

Since

\[
[\xi(G^j_{\text{low}}(T^2_{b^j})) : G^j(\hat{f}^i b''')] = c_{b'''} \left[ \frac{t}{c_{b'''}(b''')} \right] + [\xi b^{(i)} G^j(b) : G^j(\hat{f}^i b''')] + a_{\hat{f}^i b'''}
\]

and the second and the third term of the right-hand side lies in \( A \), we obtain

\[
c_{b'''} \left[ \frac{t}{c_{b'''}(b''')} \right] \in A
\]

for all \( T_{b^j} b^{(i-1)} - \lambda_{-i+1} \leq t \leq T_{b^j} b^{(i-1)} - \lambda_{-i} + c_{b'''}(b''') + 1 \). By Lemma \[10.1.1\] this implies \( c_{b'''} = 0 \).

This far, we have proved that if \( b' \in B \setminus \{ b \} \) satisfies (\dagger), then we have \( T_{b^j} b^{(i-1)} \geq \lambda_{-i} \) for all \( i \in \{0, 1, \ldots, r\} \). In particular, we have \( \mathbb{P}(b') = \lambda \) for such \( b' \) (since \( \lambda \) is maximal in \( P^j(M) \)). In this case, the condition \( T_{b^j} b^{(i-1)} \geq \lambda_{-i} \) for all \( i \) forces \( b' \) to satisfy that \( T_{b^j} b^{(i-1)} = T_{b^j} b^{(i)} \). Hence, we have

\[
\xi(G^j_{\text{low}}(T^2_{b^j})) = G^j(b) + \sum b' \in B \setminus \{ b \} \quad T_{b^j} b' = T_{b^j} b \quad c_{b'} G^j(b') + \sum b'' \in C^j(b') \quad c_{b''} G^j(b''),
\]

as desired. \( \square \)

**Lemma 10.2.2.** Let \( \lambda \in P^j(M) \) be a maximal element, \( j \in \mathbb{P} \setminus \{1\} \), \( b \in B \) such that \( P^j(b) = \lambda \), \( \epsilon_i b = 0 \) for all \( i \in \mathbb{P} \), and \( \epsilon_j b \neq 0 \). Suppose the following:

1. There exists a homomorphism \( \xi : L(\lambda) \rightarrow M \) of \( U^j \)-modules such that \( \xi(G^j_{\text{low}}(T^2_{b^j})) = G^j(b') \) for all \( b' \in C^j(b) \) which is strongly connected to some \( b'' \in C^j(b') \) with \( wt(b') \leq wt(b'') \).

2. \( \xi \) commutes with the \( \psi^j \)-involutions on \( L(\lambda) \) and \( M \).

Then, we have

\[
\xi(G^j_{\text{low}}(T^2_{b^j})) = G^j(b) + \sum b' \in B \setminus \{ b \} \quad T_{b^j} b' = T_{b^j} b \quad c_{b'} G^j(b') + \sum b'' \in C^j(b') \quad c_{b''} G^j(b''),
\]

for some \( c_{b'}, c_{b''} \in A_{0} \).

**Proof.** If we can prove that \( c_b := [\xi(G^j_{\text{low}}(T^2_{b^j})) : G^j(b)] = 1 \), then the assertion follows from the previous lemma. Hence, we aim to show \( c_b = 1 \).
By the same argument as before, we have

$$\xi(G_{\text{low}}(T^j_{b'}): G^j(\tau^j_{b''})) = c_{b''} \left[ f^{(t)} G^j(b) : G^j(\tau^j_{b''}) \right] + a_{\tau^j_{b''}}$$

for all \(b'' \in B \setminus \{b\}\) satisfying \((\dagger)\). Here, let us assume further that \(s(b') > s(b)\). Then, we have \([f^{(t)} G^j(b) : G^j(\tau^j_{b''})] = 0\) since \(f^{(t)} G^j(b)\) is a linear combination of \(G^j(b)\) with \(s(b) \leq s(b') + t^\alpha - \lambda < s(b') + t^\alpha - \lambda = s(\tau^j_{b''})\). Hence, we have \(c_{b''} \left[ \phi_j(b'') \right] \in A\), and therefore, \(c_{b''} = 0\) by Lemma 10.1.1. In particular, we obtain that \(s(b)\) is maximal. Then, we have

$$[c_j(c_b G^j(b) + \sum c_{b'} G^j(b')) : G^j(\tau^j_{b''})] = c_b [\phi_j(b) + 1].$$

On the other hand, since \([c_j G^j_{\text{low}}(T^j_b): G^j(\tau^j_{b''})] = [\phi_j(b) + 1]\), we have

$$[c_j(c_b G^j(b) + \sum c_{b'} G^j(b')) : G^j(\tau^j_{b''})] = [\phi_j(b) + 1],$$

and hence, \(c_b = 1\), as desired.

We prove Theorem 9.3.6 by descending induction (with respect to \(\preceq\)) on \(\lambda\). As in the \(r = 1\) case, we may assume that \(\lambda\) is maximal among \(P^j(M)\). Then, in order to complete the proof, we have to show the following:

1. \(I_\lambda(M)\) has a basis \(\{G^j(b) \mid P^j(b) = \lambda\}\).
2. There exists an isomorphism \(\xi : L(\lambda)^{\oplus m_\lambda} \rightarrow I_\lambda(M)\) of \(U^j\)-modules which sends the \(j\)-global basis elements of \(L(\lambda)^{\oplus m_\lambda}\) to those of \(I_\lambda(M)\), where \(m_\lambda\) denotes the multiplicity of \(L(\lambda)\) in \(M\).

Let \(b_1, \ldots, b_{m_\lambda} \in B\) and \(u_1, \ldots, u_{m_\lambda} \in L\) be distinct highest weight vectors of type \(\lambda\) with \(u_t + qL = b_t\), \(t = 1, \ldots, m_\lambda\). By retaking the \(u_t\)’s if necessary, we may assume that \([u_t : G^j(b_u)] = \delta_{t,u}\) for all \(t, u\). Let \(\xi : L(\lambda) \rightarrow M\) be the \(U^j\)-homomorphism which sends \(v_\lambda\) to \(u_t\).

**Lemma 10.2.3.** We have \(\xi_t(G^j_{\text{low}}(T^j_{b_t})) = G^j(b_t)\) for all \(t = 1, \ldots, m_\lambda\).

**Proof.** By the setting above, we can write

$$\xi_t(G^j_{\text{low}}(T^j_{b_t})) = u_t = G^j(b_t) + \sum_{b' \in P^j(\lambda) \setminus \lambda} c_{b'} G^j(b'), \quad c_{b'} \in qA_0.$$  

Then, we can apply Lemma 10.2.1 to obtain \(\xi_t(G^j_{\text{low}}(T^j_{b_t})) = G^j(b_t)\) as desired.

In order to complete the proof, it suffices to prove the following: For each \(t = 1, \ldots, m_\lambda\) and \(b \in C^j(b_t)\), we have \(\xi_t(G^j_{\text{low}}(T^j_b)) = G^j(b)\). We prove this statement by descending induction on \(wt^j(b)\) and \(P^j_{t-1}(b)\). When \(wt^j(b)\) is maximal, it must hold that \(b = b_t\), and in this case, we have already shown that \(\xi_t(G^j_{\text{low}}(T^j_{b_t})) = G^j(b_t)\). Suppose that \(wt^j(b) < \preceq wt^j(b_t)\), and the statement holds for all \(b' \in \bigcup_{t=1}^{m_\lambda} C^j(b_t)\) such that \(wt^j(b')^j > wt^j(b)\) or \(wt^j(b') = wt^j(b)\) and \(P^j_{t-1}(b') > P^j_{t-1}(b)\). In this case, since \(b\) is not a \(U^j\)-highest weight vector, the exists \(i \in I^j\) such that \(\overline{e_i}b \neq 0\).

**Lemma 10.2.4.** Suppose there exists \(i \in I^j\) such that \(\overline{e_i}b \neq 0\). Then, the statement holds.
Proof. Set $b' := \varepsilon_i(b) b$. We prove the lemma by descending induction on $\varepsilon_i(b')$. Since $\text{wt}^t(b') > \text{wt}^t(b)$, we have $G^t(b') = \xi_t(G^t_{\text{low}}(T^t_{b'})) \in U^t G^t(b)$. We know that $G^t(b)$ (resp., $G^t_{\text{low}}(T^t_{b})$) is the sum of $f_i(\varepsilon_i(b)) G^t(b)$ (resp., $\tilde{f}_i(\varepsilon_i(b)) G^t_{\text{low}}(T^t_{b})$) and a $q\mathbb{Q}[q]$-linear combination of $G^t(b')$ (resp., $G^t_{\text{low}}(T^t_{b'})$) with $\text{wt}^t(b') = \text{wt}^t(b)$ and $\varepsilon_i(b') > \varepsilon_i(b)$. By our induction hypothesis, $G^t(b') = \xi_t(G^t_{\text{low}}(b'))$ is a $q\mathbb{Q}[q]$-linear combination of $G^t(b')$'s, and is $\psi^2_M$-invariant. Such a vector must be zero, and hence, we obtain $G^t(b) = \xi_t(G^t_{\text{low}}(b))$.

Lemma 10.2.5. Suppose there exists $j \in \mathbb{P} \setminus \{1\}$ such that $\bar{\varepsilon}_j b \neq 0$ and $\varepsilon_i b = 0$ for all $i \in \mathbb{P}$. Then, the statement holds.

Proof. Apply Lemma 10.2.2 □

Now, one can complete the proof by combining Lemma 10.2.5 since each $b \in \mathcal{B}$ with $I^\lambda(b) = \lambda$ is connected to $b_t$ for some $t = 1, \ldots, m_\lambda$.

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