Dirac versus reduced quantization and operator ordering

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Abstract

We show an equivalence between Dirac quantization and the reduced phase space quantization. The equivalence of the both quantization methods determines the operator ordering of the Hamiltonian. Some examples of the operator ordering are shown in simple models.

I. INTRODUCTION

In recent years some authors [1] are discussing on Dirac quantization and the reduced phase space quantization. Their arguments are that the reduced phase space quantization and Dirac quantization may be different in the constraint system with a non-trivial metric. In order to clarify the problem, let us consider the simplest model, as an example.

Lagrangian is given by

\[ L = \frac{1}{2} \dot{x}^2 + \frac{f(x)}{2} (\dot{y} - \lambda)^2 \]

where \( \lambda \) is a Lagrange multiplier. There is a non-trivial metric \( f(x) \). This is not a field theory but a quantum mechanics. The Hamiltonian of this system is

\[ H = \frac{1}{2} p_x^2 + \frac{1}{2f(x)} p_y^2 + \lambda p_y \]

and there are two constraints

\[ p_\lambda \equiv \pi \approx 0, \quad (3) \]

\[ p_y \approx 0. \quad (4) \]

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These are first-class constrains. We set \( p_y = 0 \) in the Hamiltonian before the quantization. Then the Hamiltonian reduces to

\[
H = \frac{1}{2} p_x^2
\]

and the Hamiltonian operator is

\[
\hat{H} = \frac{1}{2} \partial_x^2. \tag{6}
\]

This is the reduced phase space quantization. The procedure of the reduced phase space quantization is to reduce first and then quantize.

In the case of Dirac quantization, its procedure is to quantize first and then reduce. The Hamiltonian in this model is defined on the two-dimensional space of \( x \) and \( y \) without a constraint term. To ensure the invariance under the coordinate transformation, the Hamiltonian operator is written by

\[
\hat{H} = \frac{1}{2 \sqrt{f}} \partial_x \sqrt{f} \partial_x + \frac{1}{2 \sqrt{f}} \partial_y \sqrt{f} \frac{1}{f} \partial_y, \tag{7}
\]

where \( \sqrt{f} \) is \( \sqrt{\text{det} g_{\mu\nu}} \). The metric \( g_{\mu\nu} \) is the two-dimensional metric of \( x-y \) space. Since \( \hat{p}_y = \partial_y \approx 0 \), \( y \) derivatives in the Hamiltonian operator are eliminated. Then the Hamiltonian operator in Dirac quantization is

\[
\hat{H} = \frac{1}{2 \sqrt{f}} \partial_x \sqrt{f} \partial_x. \tag{8}
\]

This is not the same with the result of the reduced phase space quantization. This is the problem of an inconsistency of the reduced phase space quantization and Dirac quantization.

In section 2 we show the equivalence of the both quantization methods. It is shown that the Hamiltonian operator of Dirac quantization should include the constraint term and be invariant under the three-dimensional coordinate transformation of \( x, y \), and a configuration variable conjugate to the Lagrange multiplier.

In section 3 we discuss a problem of the operator ordering. If the Hamiltonian has a non-trivial metric:

\[
H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu, \tag{9}
\]

the Hamiltonian operator may have a function of scalars like \( R, R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}, \cdots \) from the invariance of the coordinate transformation, in addition to the Laplacian

\[
\hat{H} = \frac{1}{2} \Delta + F(R, R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}, \cdots). \tag{10}
\]

The Laplacian is indispensable from the invariance and derived from the discretized path integral \[2\]. The additional function is called the quantum mechanical potential. The problem of operator ordering is to determine the quantum mechanical potential. Using the equivalence between the reduced phase quantization and Dirac quantization, we determine this potential in simple models.

Section 4 is devoted to the conclusions and discussions.
II. THE REDUCED PHASE SPACE QUANTIZATION AND DIRAC QUANTIZATION

Let us reconsider the Hamiltonian (2). We take a gauge condition $\dot{\lambda} = 0$ to quantize this system in the path integral formalism. The Hamiltonian form path integral \[Z = \int d\mu \exp[iS],\]

$$d\mu = [dx dp_x dy dp_y d\pi d\lambda],$$

$$S = \int dt p_x \dot{x} + p_y \dot{y} - \pi \dot{\lambda} - \frac{1}{2} p_x^2 - \frac{1}{2f(x)} p_y^2 - \lambda p_y.$$  \hspace{1cm} (11)

$\lambda$ is a momentum variable so that the sign of gauge fixing term is negative. After the partial integration, it becomes usual one. Since this gauge is an Abelian, we need not to introduce any ghost. After the integration of $\pi, \lambda, p_y, \text{ and } y$, the partition function becomes

$$Z = \int d\mu' \exp[iS'],$$

$$d\mu' = [dx dp_x],$$

$$S' = \int dt p_x \dot{x} - \frac{1}{2} p_x^2.$$  \hspace{1cm} (12)

This is nothing but the partition function of a free particle. Then the Hamiltonian operator is equation (6). This means the operator formalism corresponding to the stage of the path integral (12) is the reduced phase quantization. On the other hand, Dirac quantization is the operator formalism corresponding to the stage of the path integral (11). In equation (11) any variable is not integrated and constraint variables are still alive. The symmetry of this path integral is the coordinate transformation of the whole configuration space including $\pi$ which is a configuration variable conjugate to $\lambda$. Therefore, the Hamiltonian operator should be made invariant under the three-dimensional coordinate transformation, not the two-dimensional one. Then the Hamiltonian operator is

$$\hat{H} = \frac{1}{2\sqrt{g}} \partial_\mu \sqrt{gg^{\mu\nu}} \partial_\nu$$

$$= \frac{1}{2} \partial_x^2 + \frac{1}{2} \partial_y f \partial_y + \frac{1}{2} \partial_y \partial_x + \frac{1}{2} \partial_x \partial_y$$  \hspace{1cm} (13)

where $g_{\mu\nu}$ is a inverse of $g^{\mu\nu}$ of the Hamiltonian. The original Lagrangian (1) has a singular metric. However, the gauge fixed Lagrangian which is made by the integration of momentum variables in equation (11) has a regular metric and it coincides with the inverse of $g^{\mu\nu}$.

Using the constraint $\rho_y = \partial_y \approx 0$, we get the Hamiltonian

$$\hat{H} = \frac{1}{2} \partial_x^2.$$  \hspace{1cm} (14)

This is Dirac quantization and we obtain the same Hamiltonian operator with the reduced phase space quantization. This is natural because we start from the same path integral (11). This simplest example indicates that Dirac quantization and the reduced phase space quantization should be coincide.
A naive Dirac quantization showed in the introduction is made by the requirement that the Hamiltonian operator should be invariant under the coordinate transformation of \( x \) and \( y \). In that case the constraint term is treated separately. However, under some coordinate transformation, the net Hamiltonian and the constraint term are mixed. The naive Dirac quantization does not represent the symmetry correctly. This is the reason why the naive Dirac quantization is different from the reduced phase space quantization.

In general case with many variables, we can propose Dirac quantization and the reduced phase space quantization are equivalent because both quantizations are the operator versions of the different forms of the same path integral as before. We can determine the quantum potentials with this property.

III. THE OPERATOR ORDERING

Let us now consider the Lagrangian

\[
L = \frac{1}{2} \hbar(x) \dot{x}^2 + \frac{g(x)^2}{2f(x)}(y - \frac{\lambda}{g(x)})^2. \tag{15}
\]

This Lagrangian leads the Hamiltonian

\[
H = \frac{1}{2\hbar(x)} p_x^2 + \frac{f(x)}{2g(x)^2} p_y^2 + \frac{1}{g(x)} \lambda p_y, \tag{16}
\]

and constraints

\[
p_y \approx 0, \quad p_\lambda \equiv \pi \approx 0, \tag{17}
\]

as before. The reduced phase space quantization makes the Hamiltonian

\[
H = \frac{1}{2\hbar(x)} p_x^2 \tag{18}
\]

by the constraints. Then the Hamiltonian operator is

\[
\hat{H} = \frac{1}{2\sqrt{\hbar}} \partial_x \frac{1}{\sqrt{\hbar}} \partial_x. \tag{19}
\]

While in Dirac quantization, we consider the Hamiltonian in three-dimension at first. The invariance of the three-dimensional coordinate transformation allows the Hamiltonian operator of the form

\[
\hat{H} = \frac{1}{2} \triangle + F(R, R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}, \cdots)
= \frac{1}{2\sqrt{\hbar}} \partial_x \frac{g}{\sqrt{\hbar}} \partial_x + \frac{1}{2\sqrt{\hbar}} \partial_y \frac{f\sqrt{\hbar}}{g} \partial_y + \frac{1}{2\sqrt{\hbar}} \partial_y \sqrt{\hbar} \partial_\pi
+ \frac{1}{2\sqrt{\hbar}} \partial_\pi \sqrt{\hbar} \partial_y + F \tag{20}
\]
because in this model $R, \cdots$ are not zeros. $g_{\mu\nu}$ is an inverse matrix of $g^{\mu\nu}$ of the Hamiltonian and is same with that of the gauge fixed Lagrangian as before. The constraint $\hat{p}_y = \partial_y \approx 0$ makes the Hamiltonian simple form;

$$\hat{H} = \frac{1}{2g\sqrt{h}}\partial_x - \frac{g}{\sqrt{h}}\partial_x + F.$$  \hspace{1cm} (21)

The inner product for the reduced phase quantization is defined as

$$\int \sqrt{h} dx \psi^*_r \psi_r$$ \hspace{1cm} (22)

where $\psi_r$ is a wave function of the reduced phase quantization. On the other hand, for Dirac quantization, it is written by

$$\int \sqrt{h} dx dy d\pi \psi^*_D \psi_D$$

where $\psi_D$ is a Dirac quantized wave function. Since the constraint $\pi \approx 0$ means $\int d\pi \psi^*_D \psi_D = 0$, $\psi^*_D \psi_D$ is proportional to $\delta(\pi)$ and could be written as $\psi^*_D \psi_D \delta(\pi)$. We rewrite $\psi_D^*$ as $\psi_D$ again and the inner product reads

$$\int dy \int \sqrt{h} dx \psi^*_D \psi_D.$$ \hspace{1cm} (23)

$\int dy$ is a gauge volume and it should be ignored. For the both inner products to agree with each other,

$$\psi_D = \frac{1}{\sqrt{g}} \psi_r$$ \hspace{1cm} (24)

must be satisfied.

An expectation value of the energy for the reduced phase space quantization is

$$< E >_r = \int \sqrt{h} dx \psi^*_r \hat{H} \psi_r$$

$$= \int \sqrt{h} dx \psi^*_r \frac{1}{2g\sqrt{h}}\partial_x - \frac{g}{\sqrt{h}}\partial_x \psi_r.$$ \hspace{1cm} (25)

While in Dirac quantization it is given by

$$< E >_D = \int \sqrt{h} dx \psi^*_D \hat{H} \psi_D$$

$$= \int \sqrt{h} dx \frac{1}{\sqrt{g}} \psi^*_r (\frac{1}{2g\sqrt{h}}\partial_x - \frac{g}{\sqrt{h}}\partial_x + F) \frac{1}{\sqrt{g}} \psi_r$$

$$= < E >_r + \int \sqrt{h} dx \psi^*_r (\frac{g''}{4hg} + \frac{g'^2}{8h^2g} + \frac{g'''}{8gh^2} + F) \psi_r,$$ \hspace{1cm} (26)

where $'$ is a $x$ derivative. To be consistent with each other, the second term should be zero in the last equation. In other words, the function $F$ is determined so that the both quantization methods coincide.
In this space, $R$ and $R^\mu_\nu R^\mu_\nu$ are written as

$$R = \frac{1}{h} \left( -\frac{g''}{g} + \frac{g'^2}{g^2} + \frac{g'h'}{2gh} \right) - \frac{gh'}{gh},$$  \hspace{1cm} (27)$$

$$R^\mu_\nu R^\mu_\nu = \left( -\frac{g''}{g} + \frac{g'^2}{g^2} + \frac{g'h'}{2gh} \right)^2 \frac{1}{h^2} + \frac{1}{2} \left( \frac{gh'}{gh} \right)^2.$$  \hspace{1cm} (28)$$

If we define

$$A \equiv \frac{1}{h} \left( -\frac{g''}{g} + \frac{g'^2}{g^2} + \frac{g'h'}{2gh} \right),$$ \hspace{1cm} (29)$$

$$B \equiv \frac{g''}{gh};$$ \hspace{1cm} (30)$$

$R$ and $R^\mu_\nu R^\mu_\nu$ are rewritten as

$$R = A - B,$$ \hspace{1cm} (31)$$

$$R^\mu_\nu R^\mu_\nu = A^2 + \frac{1}{2} B^2.$$ \hspace{1cm} (32)$$

From these equations, we get

$$A = \frac{R \pm \sqrt{6R^\mu_\nu R^\mu_\nu - 2R^2}}{3}.$$ \hspace{1cm} (33)$$

Then if we take

$$F = -R + \frac{\sqrt{6R^\mu_\nu R^\mu_\nu - 2R^2}}{12},$$ \hspace{1cm} (34)$$

$<E>_r$ coincides with $<E>_D$. Here we take a positive sign of root. We discuss the reason later. The operator ordering for the Hamiltonian (16) is, then,

$$\hat{H} = \frac{1}{2} \triangle + \frac{-R + \sqrt{6R^\mu_\nu R^\mu_\nu - 2R^2}}{12}.$$ \hspace{1cm} (35)$$

Let us consider the next example. The Hamiltonian is

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

$$= \frac{1}{2} \gamma^{ij} p_i p_j - \frac{N^i}{M} p_i p_y + \frac{N^2}{2M^2} p_y^2 + \frac{1}{M} p_y \lambda,$$ \hspace{1cm} (36)$$

where $p_i$ means $p_{x_i}$ and $i$ runs from 1 to n. $\lambda$ is a Lagrange multiplier. The metric $g^{\mu\nu}$ of the Hamiltonian and its inverse which accords with the metric $g_{\mu\nu}$ of the gauge fixed Lagrangian are
\[ g^{\mu\nu} = \begin{pmatrix} \gamma_{ij} & -N_j^i & 0 \\ -N^i_j & N^2_j & M \\ 0 & M & 0 \end{pmatrix}, \] (37)

\[ g_{\mu\nu} = \begin{pmatrix} \gamma_{ij} & 0 & N_j \\ 0 & 0 & M \\ N_i & M & -N^2 + N_iN_j \end{pmatrix}. \] (38)

The metric \( g^{\mu\nu} \) depends on only \( x \). Constraints are \( p_\lambda \equiv \pi \approx 0 \) and \( p_y \approx 0 \) as before.

Since the Hamiltonian of the reduced phase space quantization;

\[ H = \frac{1}{2} \gamma^{ij} p_ip_j \] (39)

has a non-trivial metric in this case, the Hamiltonian operator is

\[ \hat{H} = \frac{1}{2\sqrt{\gamma}} \partial_i \sqrt{\gamma} \gamma^{ij} \partial_j + F(R, \cdots). \] (40)

Here \( F \) is a function of \( R, \cdots \) of \( \gamma_{ij} \). In this model the reduced phase space quantization may have an additional function \( F \), too.

While the Hamiltonian operator of Dirac quantization is

\[ \hat{H} = \frac{1}{2\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu + G(R, \cdots) \]

\[ = \frac{1}{2M\sqrt{\gamma}} \partial_i \sqrt{\gamma} M \gamma^{ij} \partial_j - \frac{1}{2M\sqrt{\gamma}} \partial_i \sqrt{\gamma} N^i \partial_y - \frac{1}{2M\sqrt{\gamma}} \partial_y \sqrt{\gamma} N^j \partial_j \]

\[ + \frac{1}{2M\sqrt{\gamma}} \partial_y \sqrt{\gamma} \frac{N^2}{M} \partial_y + \frac{1}{2M\sqrt{\gamma}} \partial_y \sqrt{\gamma} \partial_\pi + \frac{1}{2M\sqrt{\gamma}} \partial_\pi \sqrt{\gamma} \partial_y \]

\[ + G(R, \cdots). \] (41)

Here \( G \) is a function of \( R, \cdots \) of \( g_{\mu\nu} \). The constraint \( \hat{p}_y = \partial_y \approx 0 \) makes the Hamiltonian

\[ \hat{H} = \frac{1}{2M\sqrt{\gamma}} \partial_i \sqrt{\gamma} M \gamma^{ij} \partial_j + G. \] (42)

The inner product for the reduced phase quantization is defined as

\[ \int \sqrt{\gamma} dx \Psi_r^* \Psi_r. \] (43)

On the other hand, for Dirac quantization, it is written by

\[ \int \sqrt{\gamma} M dx dy d\pi \Psi_D^* \Psi_D \]

\[ = \int dy \int \sqrt{\gamma} M dx \Psi_D^* \Psi_D \] (44)
as before. For the both inner products to agree with each other

\[
\Psi_D = \frac{1}{\sqrt{\gamma}} \Psi_r
\]  

(45)

must be satisfied in this case.

The expectation value of the energy for the reduced phase space quantization is

\[
\langle E \rangle_r = \int \sqrt{\gamma} dx \Psi^*_r \left( \frac{1}{2\gamma} \partial_i \sqrt{\gamma} \gamma^{ij} \partial_j + F \right) \Psi_r.
\]  

(46)

While in Dirac quantization it is given by

\[
\langle E \rangle_D = \int \sqrt{\gamma} dx \Psi^*_r \left( \frac{1}{2\gamma} \partial_i \sqrt{\gamma} \gamma^{ij} \partial_j + G \right) \frac{1}{\sqrt{M}} \Psi_r
\]

\[
= \langle E \rangle_r + \int \sqrt{\gamma} dx \Psi^*_r \left( -\frac{1}{2} \gamma^{ij} \nabla_i \nabla_j \sqrt{M} + G - F \right) \Psi_r
\]  

(47)

where \(\nabla_i\) is a covariant derivative with respect to \(\gamma_{ij}\). For the both quantization to be equivalent, the second term should be zero in the second equation.

To simplify the problem, suppose that \(\gamma_{ij}\) is a two-dimensional metric. The dimension of the space on which Dirac quantization is performed is four. Four-dimensional \(R, R^{\mu\nu}R_{\mu\nu},\) and \(R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}\) of the metric(37) and (38) are related with two-dimensional ones of the metric \(\gamma_{ij}\) as

\[
R = R^{(2)} - 4\gamma^{\bar{z}z} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}} - 2\gamma^{\bar{z}z} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} M}{M},
\]  

(48)

\[
R^{\mu\nu}R_{\mu\nu} = R^{(2)ij} R_{ij}^{(2)} - 4R^{\bar{z}z} \frac{2\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}} + 2(2\gamma^{\bar{z}z} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}})^2
\]

\[
+ 8(\gamma^{\bar{z}z})^2 \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}} + \frac{1}{2} (2\gamma^{\bar{z}z} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} M}{M})^2,
\]  

(49)

\[
R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = R^{(2)ijkl} R_{ijkl}^{(2)} + 4(2\gamma^{\bar{z}z} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}})^2
\]

\[
+ 16(\gamma^{\bar{z}z})^2 \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}}
\]

\[
+ (4\gamma^{\bar{z}z} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} \sqrt{M}}{\sqrt{M}} - \gamma^{\bar{z}z} \frac{\nabla_{\bar{z}} \nabla_{\bar{z}} M}{M})^2,
\]  

(50)

where \(R^{(2)}, R_{ij}^{(2)},\) and \(R_{ijkl}^{(2)}\) are two-dimensional ones. We use a complex coordinate in two-dimension where \(\gamma_{z\bar{z}} \neq 0\) and \(\gamma_{zz} = \gamma_{\bar{z}\bar{z}} = 0\). Using relations \(R^{\bar{z}z} R_{\bar{z}z} = \frac{1}{2} R^2, R^{ijkl} R_{ijkl} = R^2,\) and \(R^{\bar{z}z} = \frac{1}{2} R \gamma^{\bar{z}z}\) in two-dimension and defining
\[ a \equiv 2\gamma^{zz} \frac{\nabla_z \nabla \sqrt{M}}{\sqrt{M}}, \quad (51) \]

\[ b \equiv (\gamma^{zz})^2 \frac{\nabla_z \nabla \sqrt{M} \nabla \nabla \sqrt{M}}{\sqrt{M}}, \quad (52) \]

\[ B \equiv 2\gamma^{zz} \frac{\nabla_z \sqrt{M}}{M}, \quad (53) \]

we can rewrite equations (48), (49), and (50) as

\[ R = R^{(2)} - 2a - B, \quad (54) \]

\[ R^{\mu\nu} R_{\mu\nu} = \frac{1}{2} R^{(2)^2} - 2R^{(2)} a + 2a^2 + 8b + \frac{B^2}{2}, \quad (55) \]

\[ R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} = (R^{(2)^2} + 4a^2 + 16b + (2a - B)^2). \quad (56) \]

From these equations, we get

\[ \frac{a}{2} = \frac{-R \pm \sqrt{R^2 - 6R^{\mu\nu} R_{\mu\nu} + 3R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}}}{12}. \quad (57) \]

Therefore, if we take this quantity as G in equation (47), the reduced phase space quantization coincide with Dirac quantization. Since two-dimensional quantity does not appear in the right hand side of equation (57), F in equation (47) is zero in two-dimension.

Now we get two operator orderings. The Hamiltonian operator for the equation (36) in four-dimension is

\[ \hat{H} = \frac{1}{2} \Delta + \frac{-R \pm \sqrt{R^2 - 6R^{\mu\nu} R_{\mu\nu} + 3R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}}}{12}, \quad (58) \]

where we take positive sign of root as before. And for the two-dimensional Hamiltonian of equation (39), the Hamiltonian operator is

\[ \hat{H} = \frac{1}{2} \Delta. \quad (59) \]

In two-dimension there does not appear any function of R.

So far we get three operator orderings. These operator orderings have the relations each other. In the three-dimensional constraint system of the metric of equations (37) and (38), \[ R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} \] is written as

\[ R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} = 4R^{\mu\nu} R_{\mu\nu} - R^2. \quad (60) \]

Substituting this equation into equation (58), we get equation (35). In two-dimension \[ R^{\mu\nu} R_{\mu\nu} = \frac{1}{2} R^2 \] is satisfied. If we substitute this relation into equation (35), we get two-dimensional trivial Hamiltonian operator (33). This is the reason why we take the positive sign of root.
IV. CONCLUSION AND DISCUSSION

We showed the equivalence of the reduced phase space quantization and Dirac quantization. Both methods are the different operator formalism out of the same path integral. Using this equivalence and the reparametrization invariance, we determined operator orderings in three examples. However, these expressions are not unique. Because scalars are expressible by other scalars. We can derive many equivalent forms.

In general the n-dimensional Hamiltonian operator is determined by the equivalence with the artificially extended (n+2)-dimensional constraint system and the (n+2)-dimensional Hamiltonian operator of the constraint system is determined at the same time. However, it is difficult to determine the concrete form of the quantum potential.

In the case of the quantum gravity, the Hamiltonian operator is not positive definite. However, this method is applicable to the quantum gravity. For example, in the minisuperspace model with scale factor and scalar matter, Weeler-DeWitt equation reduces $\Box \Psi = 0$. Because in the case of two-dimension there does not appeare the quantum potential.

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