The Sunyaev-Zel’dovich effect revisited

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March 19, 2022

Abstract

The well known Sunyaev-Zel’dovich (SZ) effect is reexamined using a Doppler shift type mechanism arising from the scattering of photons by electrons in an optically thin gas. The results are in excellent agreement with the observational data as well as with the results obtained using the diffusive pictures. A comparison of the results here obtained with other approaches is thoroughly discussed, as well as some important extensions of this method to other aspects of the SZ effect.

1 Introduction

As it is well known today, the cosmic microwave background radiation (CMBR) that fills the universe exhibits an almost perfect blackbody Planckian spectrum corresponding to a temperature of 2.726 K. A little over thirty years ago, it was predicted by the astrophysicists R.A Sunyaev and Ya. B. Zel’dovich that this spectrum could be distorted when photons from this radiation would penetrate large structures such as the hot intracluster gas now known to exist in the universe [1]. This distortion would arise from the interaction between the photons and electrons which constitute the plasma responsible for an important main contribution to the total mass contained in the cluster. Such distortion is only a very small effect changing the brightness of the spectrum by a figure of the order of 0.1 percent. This effect is now called the Sunyaev-Zel’dovich effect and its detection is at present a relatively feasible task due to the modern observational techniques available. Its main interest lies on the fact that it provides information to determine important cosmological parameters such as Hubble’s
constant and the baryonic density. Since all these and other important facts are readily available in the somewhat broad literature on the subject [1]-[6], we will not pursue any further details of its cosmological implications.

In this work we are mainly concerned with the nature of the various interpretations that have been provided to explain the effect and to propose another one which we believe, is much simpler to grasp than the others, and offers a much more direct way to account for the relativistic corrections, as well as the influence of $z$, the redshift factor, in the equations for the distorted spectrum.

The main idea behind the method we want to discuss is that photons incident on the hot electron gas of the plasma change their frequency by an absorption-emission process, so that the intensity of the spectral line is changed through the Doppler effect.

The overall effect is due to those photons that happen to be captured by an electron, an event which is in general unlikely to occur specially in an optically thin gas. Thus, an electron moving with a given thermal velocity emits (scatters) a photon with a certain incoming frequency $\nu_0$ and outgoing frequency $\nu$. The line breadth of this process is readily calculated from elementary kinetic theory taking into account that the media in which the process takes place has an optical depth directly related to the usual Compton parameter $y$. When the resulting expression is convoluted with the incoming flux of photons obtained from Planck’s distribution, one easily obtains the disturbed spectra. The results derived from the final equation are shown to be in agreement with those obtained from the observational data. To present these results we have divided the paper as follows. In section II a brief review of the previous methods developed to explain the SZ effect is given. This will allow a thorough comparison with our proposal. In Section III we develop our ideas and give the comparison between the mathematical results with both, those derived before and the observational ones. Section IV is left for some concluding remarks and future directions of this work.

2 The Diffusion-Scattering Picture of the SZ effect.

As it was clearly emphasized by the authors of this discovery in their early publications [1] [2] as well as by other authors, the distortion in the CMBR spectrum by the interaction of the photons with the electrons in the hot plasma filling the intergalactic space is due to the diffusion of the photons in the plasma which, when colliding with the isotropic distribution of a non-relativistic electron gas, generates a random walk. The kinetic equation used to describe this process was one first derived by Kompaneets back in 1956 [3]-[6]. For the particular case of interest here, when the electron temperature $T_e \sim 10^8 K$ is much larger than that of the radiation, $T \equiv T_{\text{Rad}} (2.726 K)$, the kinetic equation reads

$$\frac{dN}{dy} = \nu^2 \frac{d^2 N}{d\nu^2} + 4\nu \frac{dN}{d\nu}$$  (1)
Here \( N \) is the Bose factor, \( N = (e^x - 1)^{-1} \), \( x = \frac{h\nu}{kT} \), \( \nu \) is the frequency, \( h \) is Planck’s constant, \( k \) Boltzmann’s constant and \( y \) the “Compton parameter” given by,

\[
y = \frac{kTe}{m_e e^x c^2} \int \sigma_T n_e c \, dt = \frac{kTe}{m_e e^x c^2} \tau
\]

\( m_e \) being the electron mass, \( c \) the velocity of light, \( \sigma_T \) the Thomson’s scattering cross section and \( n_e \) the electron number density. The integral in Eq. (2) is usually referred to as the “optical depth, \( \tau \)”, measuring essentially how far in the plasma can a photon travel before being captured (scattered) by an electron. Without stressing the important consequences of Eq. (1) readily available in many books and articles [3]-[5], we only want to state here, for the sake of future comparison, that since the observed value of \( N_o(\nu) \) is almost the same as its equilibrium value \( N_{eq}(\nu) \), to a first approximation one can easily show that

\[
\delta N \equiv N_o(\nu) - N_{eq}(\nu) \approx y \left[ \frac{x^2 e^x (e^x + 1)}{(e^x - 1)^2} - \frac{4xe^x}{e^x - 1} \right]
\]

implying that, in the Rayleigh-Jeans region \( (x << 1) \),

\[
\delta N \approx -2y
\]

and in the Wien limit \( (x >> 1) \):

\[
\delta N \approx x^2 y
\]

If we now call \( I_o(\nu) \) the corresponding radiation flux for frequency \( \nu \), defined as

\[
I_o(\nu) = \frac{c}{4\pi} U_o(\nu)
\]

where \( U_o(T) = \frac{8\pi h\nu^3}{c^2} N_{eq}(\nu) \) is the energy density for frequency \( \nu \) and temperature \( T \), and noticing that

\[
\frac{\delta T}{T} = \left( \frac{\partial (\ln I(\nu))}{\partial (\ln T)} \right) \frac{\delta I}{I}
\]

we have that the change in the background brightness temperature is given, in the two limits, by

\[
\frac{\delta T}{T} \approx -2y, \ \ \ x << 1
\]

and

\[
\frac{\delta T}{T} \approx xy, \ \ \ x >> 1
\]

showing a decrease in the low frequency limit and an increase in the high frequency one. Finally, we remind the reader that the curves extracted from Eq.
Nevertheless, many authors, including Sunyaev and Zel’dovich themselves, were very reluctant in accepting a diffusive mechanism as the underlying phenomena responsible for the spectrum distortion. The difficulties of using diffusion mechanisms to study the migration of photons in turbid media, specially thin media, have been thoroughly underlined in the literature \cite{9}-\cite{11}. Several alternatives were discussed in a review article in 1980 \cite{5} and a model was set forth by Sunyaev in the same year \cite{12} based on the idea that Compton scattering between photons and electrons induce a change in their frequency through the Doppler effect. Why this line of thought has not been pursued, or at least, not widely recognized, is hard to understand. Two years ago, one of us (ASV) \cite{14} reconsidered the single scattering approach to study the SZ effect. The central idea in that paper is that in a dilute gas, the scattering law is given by what in statistical physics is known as the *dynamic structure factor*, denoted by $S(k, \nu)$ where $k = \frac{2 \pi}{\lambda}$, and $\lambda$ is the wavelength. In such a system, this turns out to be proportional to $\exp(-\nu^2 w^2)$ where $w$ is the broadening of the spectral line given by

$$w = \frac{2}{c} \left( \frac{kT}{m} \right)^{1/2} \nu$$

If one computes the distorted spectrum through the convolution integral

$$I(\nu) = \int_0^\infty I_o(\tilde{\nu}) S(k, \tilde{\nu} - (1 - ay)\nu) \, d\tilde{\nu}$$

where the corresponding frequency shift $\frac{\Delta \nu}{\nu}$ has been introduced through Eqs. (8-9) ($a = -2 + x$), one gets a good agreement with the observational data. This is exhibited in Fig. (1) for $y = 10^{-5}$. This result is interesting from, at least, two facts. One, that such a simple procedure is in agreement with the diffusive picture. This poses interesting mathematical questions which will be analyzed elsewhere, specially since the exact solution to Eq. (1) is known (see Eq. (A-8) \cite{5}). The other one arises from the fact that this is what triggered the idea of reanalyzing the SZ effect using elementary arguments of statistical mechanics and constitutes the core of this paper to be presented next.

### 3 The Doppler Effect Approach.

We begin this section by simply reminding the reader that if an atom in an ideal gas moving say with speed $u_x$ in the $x$ direction emits light of frequency $\nu_o$ at some initial speed $u_x(0)$, the intensity of the spectral line $I(\nu)$ is given by

$$\frac{I(\nu)}{I_o(\nu)} = \exp \left[ -\frac{mc^2}{2kT} \left( \frac{\nu_o - \nu}{\nu} \right)^2 \right]$$
Figure 1: Comparison of the CMBR SZ distortion as computed by Eqs. (3,6) (solid line) and Eq. (11) (dashed line) i.e using the approach of Ref. [13], with $y = 10^{-5}$ and $T = 10^8 K$. $\delta I$ is measured in $erg s^{-1} cm^{-2} ster^{-1}$, notice that the figure is scaled by a factor of $10^{18}$.

Eq. (12) follows directly from the fact the velocity distribution function for an ideal gas is Maxwellian and that $u_x$, $u_x(0)$ and $\nu$ are related through the Doppler effect. For the case of a beam of photons of intensity $I_o(\nu)$ incident on a hot electron gas regarded as an ideal gas in equilibrium at a temperature $T_e$ the full distorted spectrum may be computed from the convolution integral given by

$$I(\nu) = \frac{1}{\sqrt{\pi W(\nu)}} \int_0^\infty I_o(\bar{\nu}) \exp\left[-\left(\frac{\bar{\nu} - f(y)\nu}{W(\nu)}\right)^2\right] d\bar{\nu} \quad (13)$$

which defines the joint probability of finding an electron scattering a photon with incoming frequency $\bar{\nu}$ and outgoing frequency $\nu$ multiplied by the total number of incoming photons with frequency $\bar{\nu}$. $\frac{1}{\sqrt{\pi}} W(\nu)$ is the normalizing factor of the Gaussian function for $f(y) = 1$, $W(\nu)$ is the width of the spectral line at frequency $\nu$ and its squared value follows from Eq.(10)

$$W^2(\nu) = \frac{4kT_e}{m_e c^2} \nu^2 = 4y \nu^2 \quad (14)$$

where $\tau$ is the optical depth whose presence in Eq. (14) will be discussed later. The function $f(y)$ multiplying $\nu$ in Eq. (13) is given by $f(y) = 1 + ay$ where $a = -2$ in the Rayleigh-Jeans limit and $a = xy$ in the Wien’s limit, according to Eqs. (8-9) and the fact that $\Delta \nu = \frac{\Delta \nu}{\nu}$ for photons. Eq.(13) is the central object.
of this paper so it deserves a rather detailed examination. In the first place it is worth noticing that \( I_o(\nu) \), the incoming flux, is defined in Eq. (6). Secondly, it is important to examine the behavior of the full distorted spectrum in both the short and high frequency limits. In the low frequency limit, the Rayleigh-Jeans limit \( I_o(\nu) = \frac{2kT\nu^2}{c^2} \), so that performing the integration with \( a = -2 \) and noticing that \( y \) is a very small number, one arrives at the result

\[
\frac{\delta I}{I} \equiv \frac{I(\nu) - I_o(\nu)}{I_o(\nu)} = -2y, \quad x << 1 \quad (15)
\]

(15) is in complete agreement with the value obtained using Eq. (3), the photon diffusion equation. In the high frequency limit where \( a = xy \) and \( I_o(\nu) = \frac{2h\nu^3}{c^2} e^{-x} \), a slightly more tedious sequence of integrations leads also to a result at grips with the diffusion equation, namely

\[
\frac{\delta I}{I} = x^2y, \quad x >> 1 \quad (16)
\]

Why both asymptotic results, the ones obtained with the diffusion equation and those obtained from Eq. (13) agree so well, still puzzles us. At this moment we will simply think of them as a mathematical coincidence. Nevertheless, it should be stressed that in his 1980 paper, Sunyaev reached rather similar conclusions although with a much more sophisticated method, and less numerical accuracy for the full distortion curves. The distorted spectrum for the CMBR radiation may be easily obtained by numerical integration of Eq. (13) once the optical depth is fixed, \( y \) is determined through Eq. (2) and \( a = -2 + x \). The intergalactic gas cloud in clusters of galaxies has an optical depth \( \tau \sim 10^{-2} \) [13].

To compare our results with the curves obtained from the traditional approach, we have plotted \( \delta I(\nu) \) for several temperatures, typical of the hot intra-cluster gas, in the non-relativistic range, in Figs. (2) and (3).

From the results obtained one appreciates the rather encouraging agreement between the observational data and the theoretical results obtained with the three methods, the diffusion equation, the structure factor or scattering law approach, and the Doppler effect. This, in our opinion is rather rewarding and some efforts are in progress to prove the mathematical equivalence of the three approaches. From the physical point of view, and for reasons already given by many authors, we believe that the scattering Doppler effect picture does correspond more with reality, specially for reasons that will become clear in the last section.

4 Discussion of the results

Eq. (13), the main result of this work hardly needs a more detailed explanation, it is a direct result of elementary statistical mechanics, except of course for
Figure 2: Comparison of the CMB SZ distortion as computed by Eqs. (3-6) (dashed line) and Eq. (13) (solid line), with $\tau = 10^{-2}$, $T = 2\, K eV$ (lower curves), $T = 3\, K eV$ (middle curves) and $T = 4\, K eV$ (upper curves). Here $a = -2$ in Eq. (13). Since these are actually six curves, it is clear that the accuracy achieved by Eq. (13) is remarkable. $\delta I$ is measured in $\text{erg s}^{-1} \text{ cm}^{-2} \text{ ster}^{-1}$, the figure is again scaled by a factor of $10^{18}$.

Figure 3: The same as in figure 2, but taking $\tau = 7 \times 10^{-3}$ . The respective temperatures are 5, and 7 $K eV$. 
the factor $\tau$ which appears in the broadening of the spectral line. That this broadening must somehow depend on $\tau$ is clear. As argued by many authors in the case of clusters of galaxies, the plasma is optically thin, so most of the photons are not scattered, and how many of them are must depend on the optical depth. Now, why in particular Eq. (14) is valid remains to be rigorously shown. In our case it came as a mere accident since in ref. [14] the mass taken in $w$ to perform the calculations was that of a proton, which turns out to be of the same order of magnitude as $\frac{m_e}{\tau}$ for $y = 10^{-5}$. There is in fact one way of understanding this puzzle. In a pure diffusive (Fickian) process described by a Gaussian function one knows that the squared value of a single line width grows as $t$. Looking at the original equation of Kompaneets and thinking only in its diffusive terms, one sees that the variable time is replaced by the Compton parameter "$y$", so that one should expect that, to a first approximation, the width of the curve grows as $\sqrt{y}$. Hence, inside the gas, the effective frequency at which the photon propagates through the gas is, according to Eq. (14), proportional to $\sqrt{y} \times \nu$. This argument, which is based on ideas pertinent to the diffusion mechanism must be extended to the Doppler picture, but this has not yet been accomplished.

The two main advantages of Eq. (13) are, firstly, that its generalization to the relativistic case is straightforward and indeed, since in relativity theory one has both the transverse and parallel Doppler effects [15], one can easily study what their influence is, if any, in the full distorted spectrum. For the parallel effect the calculations carried out so far agrees reasonably well with those reported by Rephaeli and other authors [13] and will be the subject of a forthcoming publication. The case of the transverse effect is under study. Secondly, in all the work we have reported here, the cluster of galaxies, the source of the scattering processes of the CMBR and the electrons, has been considered a system at rest. One can now extend the calculations assuming that the cluster recedes to infinity according to Hubble’s law and introduce the redshift factor into the formalism. The resulting prediction could be compared with the observational data and extract valuable information on such an important cosmological parameter. Work in this direction is also in progress. Concluding, we have the strong conviction that the Doppler effect approach to study the thermal SZ effect is not only very clear from the physical point of view, it also contains ingredients which are promising for further studies more advantageous than those offered by the diffusive model.

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