Least squares estimator for non-ergodic Ornstein-Uhlenbeck processes driven by Gaussian processes

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Abstract: The statistical analysis for equations driven by fractional Gaussian process (fGp) is relatively recent. The development of stochastic calculus with respect to the fGp allowed to study such models. In the present paper we consider the drift parameter estimation problem for the non-ergodic Ornstein-Uhlenbeck process defined as \(dX_t = \theta X_t dt + dG_t, \ t \geq 0\) with an unknown parameter \(\theta > 0\), where \(G\) is a Gaussian process. We provide sufficient conditions, based on the properties of \(G\), ensuring the strong consistency and the asymptotic distribution of our estimator \(\tilde{\theta}_t\) of \(\theta\) based on the observation \(\{X_s, \ s \in [0, t]\}\) as \(t \to \infty\). Our approach offers an elementary, unifying proof of [4], and it allows to extend the result of [4] to the case when \(G\) is a fractional Brownian motion with Hurst parameter \(H \in (0, 1)\). We also discuss the cases of subfractional Brownian motion and bifractional Brownian motion.

Key words: Parameter estimation, Non-ergodic Gaussian Ornstein-Uhlenbeck process.

1 Introduction

While the statistical inference of Ito’s type diffusions has a long history, the statistical analysis for equations driven by fractional Gaussian process is relatively recent. The development of stochastic calculus with respect to the fGp has allowed to study such models. We will recall several approaches to estimate the parameters in fractional models but we mention that the list below is not exhaustive:

- The MLE approach in [12], [16]: In general the techniques used to construct maximum likelihood estimators (MLE) for the drift parameter are based on Girsanov’s transforms for fBm and depend on the properties of the deterministic fractional operators (determined by the Hurst parameter) related to the fBm. In this case, the MLE is not easily computable.

- A least squares approach has been proposed in [10]: The study of the asymptotic properties of the estimator is based on certain criteria formulated in terms of the Malliavin calculus. In the ergodic case, the statistical inference for several fractional Ornstein-Uhlenbeck (fOU) models has been recently developed in the papers [10], [1], [2], [11], [3], [5]. The case of non-ergodic fOU process of the first kind and of the second kind can be found in [4] and [7] respectively.

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Method of moments: A new idea has been provided in [9], to develop the statistical inference for stochastic differential equations related to stationary Gaussian processes by proposing a suitable criteria. This approach is based on the Malliavin calculus, and it makes in principle the estimators easier to be simulated. Moreover, as an application, the models discussed in [10], [1], [2], [6] have been studied in [9] by using this approach.

In this paper, we consider the non-ergodic Ornstein-Uhlenbeck process \( X_t \) given by

\[
X_0 = 0; \quad dX_t = \theta X_t dt + dG_t, \quad t \geq 0,
\]

where \( G \) is a Gaussian process and \( \theta > 0 \) is an unknown parameter. A problem here is to estimate the parameter \( \theta \) when one observes the whole trajectory of \( X \). In the case when the process \( X \) has Hölder continuous paths of order \( \delta \in (\frac{1}{2}, 1] \) we can consider the following least squares estimator (LSE)

\[
\hat{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}, \quad t \geq 0,
\]

as estimator of \( \theta \), where the integral with respect to \( X \) is a Young integral (see Appendix). The estimator \( \hat{\theta}_t \) is obtained by the least squares technique, that is, \( \hat{\theta}_t \) (formally) minimizes

\[
\theta \mapsto \int_0^t \left| X_s - \theta X_s \right|^2 ds.
\]

Moreover, using the formula (A-1) we can rewrite \( \hat{\theta}_t \) as follows,

\[
\hat{\theta}_t = \frac{X_t^2}{2 \int_0^t X_s^2 ds}, \quad t \geq 0.
\]

Motivated by (1.3) we propose to use, in the general case, the right hand of (1.3) as a statistic to estimate the drift coefficient \( \theta \) of the equation (1.1). More precisely, we define

\[
\tilde{\theta}_t = \frac{X_t^2}{2 \int_0^t X_s^2 ds}, \quad t \geq 0.
\]

This estimator \( \tilde{\theta}_t \) may exist even if \( X \) does not have Hölder continuous paths of order \( \delta \in (\frac{1}{2}, 1] \).

We shall provide sufficient conditions, based on the properties of \( G \), under which the estimator \( \tilde{\theta}_t \) is consistent (see Theorem 2.1), and the limit distribution of \( \tilde{\theta}_t \) is a standard Cauchy distribution (see Theorem 2.2).

Examples of the Gaussian process \( G \).

Fractional Brownian motion:
Suppose that the process \( G \) given in (1.1) is a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \). By assuming that \( H > \frac{1}{2} \), [4] studied the LSE \( \hat{\theta}_t \) which coincides, in this case, with \( \tilde{\theta}_t \) by Remark 2.1. In this paper, we extend the result of [4] to the case \( H \in (0, 1) \). Moreover, we offer an elementary proof (see Section 3.1).
Sub-fractional Brownian motion:
Assume that the process $G$ given in (1.1) is a subfractional Brownian motion with parameter $H \in (0, 1)$. For $H > \frac{1}{2}$, using an idea of [4], [14] studied the LSE $\hat{\theta}_t$ which also coincides with $\tilde{\theta}_t$. But the proof of Lemma 4.3 in [14] relies on a possibly awed technique because the passage from line -7 to -6 on page 671 does not allow to obtain the convergence of $E \left[ \left( e^{-\theta t} \int_0^t e^{\theta s} dS_s^H \right)^2 \right]$ as $t \to \infty$. In the present paper, we give a solution of this problem and we extend the result to $H \in (0, 1)$ (see Section 3.2).

Bifractional Brownian motion:
To the best of our knowledge there is no study of the problem of estimating the drift of (1.1) in the case when $G$ is a bifractional Brownian motion with parameters $(H, K) \in (0, 1)^2$. Section 3.3 is devoted to this question.

2 Asymptotic behavior of the estimator

Let $G = (G_t, t \geq 0)$ be a continuous centered Gaussian process defined on some probability space $(\Omega, \mathcal{F}, P)$ (Here, and throughout the text, we assume that $\mathcal{F}$ is the sigma-field generated by $G$). The following assumptions are required.

$(\mathcal{H}1)$ The process $G$ has Hölder continuous paths of order $\delta \in (0, 1]$.

$(\mathcal{H}2)$ For every $t \geq 0$, $E \left( G_t^2 \right) \leq ct^{2\gamma}$ for some positive constants $c$ and $\gamma$.

2.1 Strong consistency

We will prove that the estimator $\hat{\theta}_t$ given by (1.4) is strongly consistent. It is clear that the linear equation (1.1) has the following explicit solution

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} dG_s, \quad t \geq 0,$$

where the integral is interpreted in the Young sense (see Appendix). Suppose $(\mathcal{H}1)$ holds. Applying the formula (A-1) we can write

$$X_t = G_t + \theta e^{\theta t} Z_t, \quad t \geq 0,$$

where

$$Z_t := \int_0^t e^{-\theta s} G_s ds, \quad t \geq 0.$$  

Let us introduce the following process

$$\xi_t := \int_0^t e^{-\theta s} dG_s, \quad t \geq 0.$$  

Thus, we can also write

$$\xi_t = e^{-\theta t} G_t + \theta Z_t, \quad t \geq 0.$$  

Remark 2.1. Suppose that $G$ has Hölder continuous paths of order $\delta \in (\frac{1}{2}, 1]$. Then the process $X$ has $\delta$-Hölder continuous paths which implies that the estimator $\tilde{\theta}_t$ coincides with the LSE $\hat{\theta}_t$ by using (A-7). This property is satisfied in the cases of fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, sub-fractional Brownian motion with parameter $H > \frac{1}{2}$ and bifractional Brownian motion with parameters $(H, K) \in (0, 1)$ such that $HK > \frac{1}{2}$ (see Section 3).

Indeed, let us prove that $X$ has $\delta$-Hölder continuous paths. From (2.6), it suffices to prove that the process $Z$ given in (2.7) has $\delta$-Hölder continuous paths. Furthermore, Mean Value Theorem and the continuity of $G$ entail that $Z$ has Hölder continuous paths of order $\frac{1}{2}$. Thus, the result is obtained.

The following theorem gives the strong consistency of the estimator $\tilde{\theta}_t$.

Theorem 2.1. Assume that $(H1)$ and $(H2)$ hold and let $\tilde{\theta}_t$ be given by (1.4). Then

$$\tilde{\theta}_t \to \theta \text{ almost surely as } t \to \infty. \quad (2.9)$$

In order to prove this theorem we make use of the following technical lemmas. We will analyze separately the numerator and the denominator in the right hand side of the estimator (1.4). The proofs of the following lemmas are given in Appendix.

**Lemma 2.1.** Assume that $(H1)$ and $(H2)$ hold. Let $Z$ be the process defined in (2.7). Then,

$$Z_t \to Z_\infty \text{ almost surely and in } L^2(\Omega). \quad (2.10)$$

Thus, as $t \to \infty$

$$\xi_t \to \xi_\infty := \theta Z_\infty \text{ almost surely and in } L^2(\Omega). \quad (2.11)$$

**Lemma 2.2.** Assume that $(H1)$ and $(H2)$ hold. Then, as $t \to \infty$,

$$e^{-2\theta t} \int_0^t X_s^2 ds = e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds \to \frac{\xi_\infty^2}{2\theta} = \frac{\theta}{2} Z_\infty^2 \text{ almost surely.}$$

Proof of Theorem 2.1. By (2.6), we can write

$$\tilde{\theta}_t = \frac{(G_t + \theta e^{\theta t} Z_t)^2}{2 \int_0^t e^{2\theta s} \xi_s^2 ds} = \frac{\xi_t^2}{2e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds}. \quad (2.12)$$

The convergence (2.9) follows from Lemmas 2.1 and 2.2.

**2.2 Asymptotic distribution**

This section is devoted to the investigation of asymptotic distribution of the estimator $\tilde{\theta}_t$ of $\theta$. Then, the following assumptions are required.

$(H3)$ The limiting variance of $e^{-\theta t} \int_0^t e^{\theta s} dG_s$ exists as $t \to \infty$, i.e., there exists a constant $\sigma_G > 0$ such that

$$\lim_{t \to \infty} E \left[ \left( e^{-\theta t} \int_0^t e^{\theta s} dG_s \right)^2 \right] \to \sigma_G^2.$$
For all fixed $s \geq 0$

$$\lim_{t \to \infty} E \left( G_s e^{-\theta t} \int_0^t e^{\theta r} dG_r \right) = 0.$$

The next theorem proves that $\tilde{\theta}_t$ is asymptotically Cauchy.

**Theorem 2.2.** Assume that $(H1)$-$(H4)$ hold. Then, as $t \to \infty$,

$$e^{\theta t} \left( \tilde{\theta}_t - \theta \right) \xrightarrow{law} \frac{2\sigma_G}{\sqrt{E(Z_\infty^2)}} C(1),$$

(2.13)

with $C(1)$ is the standard Cauchy distribution with the probability density function $\frac{1}{\pi(1+x^2)}$: $x \in \mathbb{R}$.

In order to prove this theorem we need the following lemmas. The proofs of these lemmas are given in Appendix.

**Lemma 2.3.** Assume that $(H1)$ holds. Then, for every $t \geq 0$, we have

$$\frac{1}{2} X_t^2 = \theta \int_0^t X_s^2 ds + \theta Z_t \int_0^t e^{\theta s} dG_s + R_t,$$

where

$$R_t := \frac{1}{2} G_t^2 - \theta \int_0^t G_s^2 ds + \theta^2 \int_0^t \int_0^s dG_s G_r e^{-\theta(s-r)}.$$

**Lemma 2.4.** Assume that $(H1)$, $(H3)$ and $(H4)$ hold. Let $F$ be any $\sigma\{G\}$-measurable random variable such that $P(F < \infty) = 1$. Then, as $t \to \infty$,

$$\left( F, e^{-\theta t} \int_0^t e^{\theta s} dG_s \right) \xrightarrow{law} (F, \sigma_G N),$$

where $N \sim N(0,1)$ is independent of $G$.

**Proof of Theorem 2.2.** Using Lemma 2.3 we can write

$$e^{\theta t} \left( \tilde{\theta}_t - \theta \right) = \frac{e^{-\theta t} \int_0^t e^{\theta s} dG_s}{Z_\infty} \times \frac{\theta Z_t Z_\infty}{e^{-2\theta t} \int_0^t X_s^2 ds} + \frac{e^{-\theta t} R_t}{e^{-2\theta t} \int_0^t X_s^2 ds}$$

$$:= a_t \times b_t + c_t.$$ 

Lemma 2.4 yields, as $t \to \infty$,

$$a_t \xrightarrow{law} \frac{\sigma_G}{Z_\infty} N,'$$

where $N' \sim N(0,1)$ is independent of $G$, whereas Lemmas 2.1 and 2.2 imply that $b_t \to 2$ almost surely as $t \to \infty$. On the other hand, $e^{-\theta t} R_t \to 0$ in $L^1(\Omega)$ as $t \to \infty$ because, as $t \to \infty$,

$$e^{-\theta t} E \left( G_t^2 \right) \leq c \ell^{2\gamma} e^{-\theta t} \to 0,$$

$$e^{-\theta t} \int_0^t E \left( G_s^2 \right) ds \leq c \ell^{2\gamma+1} 2^{1/\gamma} e^{-\theta t} \to 0,$$
Lemma 3.1. Let \( g : [0, \infty) \times [0, \infty) \to \mathbb{R} \) be a symmetric function such that \( \frac{\partial}{\partial s} g(s, r) \) and \( \frac{\partial^2 g}{\partial s \partial r}(s, r) \) exist on \((0, \infty) \times [0, \infty)\). Then, for every \( t \geq 0 \),
\[
\Delta_g(t) := g(t, t) - 2\theta e^{-\theta t}\int_0^t g(s, t)e^{\theta s}ds + \theta^2 e^{-2\theta t}\int_0^t \int_0^t g(s, r)e^{\theta(s+r)}drds
\]
\[= 2e^{-2\theta t}\int_0^t e^{\theta s}\frac{\partial g}{\partial s}(s, 0)ds + 2e^{-2\theta t}\int_0^t e^{\theta s}\int_0^s \int_0^r \frac{\partial^2 g}{\partial s \partial r}(s, r)e^{\theta r}drds. \tag{3.14} \]

Proof. Set \( h(s) := \int_0^s g(s, r)e^{\theta r}dr \). Combining the integration by parts formula together with
\[\frac{\partial h}{\partial s}(s) = e^{\theta s}g(s, s) + \int_0^s \frac{\partial g}{\partial s}(s, r)e^{\theta r}dr, \]
we obtain
\[
\Delta_g(t) = g(t, t) - 2\theta e^{-2\theta t}\int_0^t g(s, t)e^{2\theta s}ds - 2\theta e^{-2\theta t}\int_0^t e^{\theta s}\int_0^s \frac{\partial g}{\partial s}(s, r)e^{\theta r}drds
\]
\[= e^{-2\theta t}\int_0^t \frac{\partial g(s, s)}{\partial s}e^{2\theta s}ds - 2\theta e^{-2\theta t}\int_0^t e^{\theta s}\int_0^s \frac{\partial g}{\partial s}(s, r)e^{\theta r}drds. \]

Since \( g \) is symmetric, \( 2\frac{\partial g}{\partial s}(s, r)1_{(r=s)} = \frac{\partial g(s, s)}{\partial s}(s) \) where \( \frac{\partial g(s, s)}{\partial s} \) is the derivative of the function \( s \to g(s, s) \). Thus by using again the integration by parts formula, the claim [3.14] is obtained.

Lemma 3.2. Let \( \lambda > -1 \). Define
\[J_\lambda(t) := e^{-2\theta t}\int_0^t \int_0^t e^{\theta s}e^{\theta r}|s-r|^{\lambda}drds; \quad I_\lambda(t) := e^{-\theta t}\int_0^t e^{\theta r}(t-r)^{\lambda}dr. \]

Then
\[
\lim_{t \to \infty} J_\lambda(t) = \lim_{t \to \infty} \left( \frac{1}{\theta} I_\lambda(t) \right) = \frac{\Gamma(\lambda+1)}{\theta^{\lambda+2}}. \tag{3.15} \]
Proof. Let \( t \geq 0 \). We have

\[
J_\lambda(t) = 2e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} (s - r)^\lambda
\]

\[
= 2e^{-2\theta t} \int_0^t ds e^{2\theta s} \int_0^s dr e^{-\theta u} u^\lambda
\]

\[
= 2e^{-2\theta t} \int_0^t ds e^{-\theta u} u^\lambda \int_u^t dr e^{2\theta s}
\]

\[
= \frac{1}{\theta} \left( \int_0^t u^\lambda e^{-\theta u} du - e^{-2\theta t} \int_0^t u^\lambda e^{\theta u} du \right)
\]

\[
\to \frac{\Gamma(\lambda + 1)}{\theta^{\lambda+2}} \quad \text{as} \quad t \to \infty,
\]

which proves (3.15).

\[\square\]

3.1 Fractional Brownian motion

The fractional Brownian motion (fBm) \( B^H = (B^H_t, t \geq 0) \) with Hurst parameter \( H \in (0, 1) \) is defined as a centered Gaussian process starting from zero with covariance

\[
E(B^H_t B^H_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).
\]

Note that, when \( H = \frac{1}{2} \), \( B^{\frac{1}{2}} \) is a standard Brownian motion.

Proposition 3.1. Suppose that, in (1.1), the process \( G \) is the fBm \( B^H \). Then for all fixed \( H \in (0, 1) \) the convergences (2.9) and (2.13) hold.

Proof. By Kolmogorov’s continuity criterion and the fact

\[
E(B^H_t - B^H_s)^2 = |s - t|^{2H}; \quad s, t \geq 0,
\]

we deduce that \( B^H \) has Hölder continuous paths of order \( H - \varepsilon \) for all \( \varepsilon \in (0, H) \). So, the process \( B^H \) satisfies the assumptions (H1) and (H2). Thus the strong consistence (2.9) is obtained in the case where \( G = B^H \).

For the convergence (2.13), it suffices to check (H3) and (H4). Let us first compute the limiting variance of \( e^{-\theta t} \int_0^t e^{\theta s} dB^H_s \) as \( t \to \infty \). We have

\[
E \left[ \left( e^{-\theta t} \int_0^t e^{\theta s} dB^H_s \right)^2 \right] = E \left[ \left( e^{-\theta t} \left( e^{\theta t} B^H_t - \theta \int_0^t e^{\theta s} B^H_s ds \right) \right)^2 \right]
\]

\[
= t^{2H} - 2\theta e^{-\theta t} \int_0^t e^{\theta s} E(B^H_s B^H_t) ds + \theta^2 e^{-2\theta t} \int_0^t \int_0^t e^{\theta s} e^{\theta r} E(B^H_s B^H_r) ds dr
\]

\[
= t^{2H} - \theta e^{-\theta t} \int_0^t e^{\theta s} \left( s^{2H} + t^{2H} - (t - s)^{2H} \right) ds
\]

\[
+ \frac{1}{2} \theta^2 e^{-2\theta t} \int_0^t \int_0^t e^{\theta s} e^{\theta r} \left( s^{2H} + r^{2H} - |r - s|^{2H} \right) ds dr
\]

\[
= \Delta_g B^H(t) + \theta I_{2H}(t) - \frac{\theta^2}{2} J_{2H}(t),
\]

(3.16)
where \( g_{BH}(s, r) = \frac{1}{2} (s^{2H} + t^{2H}) \).

On the other hand, (3.14) implies that
\[
\Delta g_{BH}(t) = 2He^{-\theta t} \int_0^t s^{2H-1}e^{\theta s} ds \to 0 \text{ as } t \to \infty.
\] (3.17)

Combining (3.13) and (3.17), we get for every \( H \in (0, 1) \)
\[
E \left[ \left( e^{-\theta t} \int_0^t e^{\theta s} dB^H_s \right)^2 \right] \to \frac{H \Gamma(2H)}{\theta^{2H}} \text{ as } t \to \infty.
\]

Hence, to finish the proof it remains to check that, for all fixed \( s \geq 0 \)
\[
\lim_{t \to \infty} E \left( B^H_s e^{-\theta t} \int_0^t e^{\theta r} dB^H_r \right) = 0.
\]

Let us consider \( s < t \). Setting \( f_{BH}(s, r) = E(B^H_s B^H_r) \), it follows from (A-1) that
\[
E \left( B^H_s e^{-\theta t} \int_0^t e^{\theta r} dB^H_r \right) = f_{BH}(s, t) - \theta e^{-\theta t} \int_0^t e^{\theta r} f_{BH}(s, r) dr
\]
\[
= f_{BH}(s, t) - \theta e^{-\theta t} \int_0^t e^{\theta r} f_{BH}(s, r) dr - \theta e^{-\theta t} \int_0^s e^{\theta r} f_{BH}(s, r) dr
\]
\[
= -\theta (t-s) f_{BH}(s, s) - \theta e^{-\theta t} \int_0^s e^{\theta r} \frac{\partial f_{BH}}{\partial r}(s, r) dr - \theta e^{-\theta t} \int_0^s e^{\theta r} f_{BH}(s, r) dr.
\]

It is clear that \( e^{-\theta (t-s)} f_{BH}(s, s) = 0 \) for every \( r > s \). Then, for \( H = \frac{1}{2} \)
\[
e^{-\theta t} \int_0^t e^{\theta r} \frac{\partial f_{BH}}{\partial r}(s, r) dr = 0.
\]

Now, suppose that \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \). Since
\[
\int_s^t e^{\theta r} |r^{2H-1} - (r-s)^{2H-1}| dr \geq 0
\]
\[
|2H-1|s \int_s^t e^{\theta r} r^{2H-2} dr
\]
\[
\geq 2H-1|st^{2H-2} \int_s^t e^{\theta r} dr
\]
\[
\to \infty \text{ as } t \to \infty,
\]

we can apply L’Hospital’s rule to obtain
\[
\lim_{t \to \infty} \left| e^{-\theta t} \int_0^t e^{\theta r} \frac{\partial f_{BH}}{\partial r}(s, r) dr \right| = \lim_{t \to \infty} H \left| e^{-\theta t} \int_s^t e^{\theta r} (r^{2H-1} - (r-s)^{2H-1}) dr \right|
\]
\[
\leq \lim_{t \to \infty} \left( H \left| e^{-\theta t} \int_s^t e^{\theta r} (r^{2H-1} - (r-s)^{2H-1}) dr \right| \right)
\]
\[
= \lim_{t \to \infty} \left( \frac{H}{\theta} |t^{2H-1} - (t-s)^{2H-1}| \right)
\]
\[
\leq \lim_{t \to \infty} \left( \frac{sH}{\theta} \frac{\Gamma(2H-1)}{(t-s)^{2H-2}} \right)
\]
\[
\to 0 \text{ as } t \to \infty,
\]

which finishes the proof of Proposition 3.1.\( \square \)
3.2 Sub-fractional Brownian motion

The sub-fractional Brownian motion (sfBm) \( S^H \) with parameter \( H \in (0, 1) \) is a centred Gaussian process with covariance function

\[
E(S^H_t S^H_s) = t^{2H} + s^{2H} - \frac{1}{2}((t + s)^{2H} + |t - s|^{2H}).
\]

Note that, when \( H = \frac{1}{2} \), \( S^{1/2} \) is a standard Brownian motion.

**Proposition 3.2.** Suppose that, in (1.2), the process \( G \) is the sfBm \( S^H \). Then for all fixed \( H \in (0, 1) \) the convergences (2.9) and (2.13) hold.

**Proof.** By Kolmogorov’s continuity criterion and the fact

\[
E(S^H_t - S^H_s)^2 \leq (2 - 2^{2H-1})|s - t|^{2H}; \quad s, t \geq 0,
\]

we deduce that \( S^H \) has H"older continuous paths of order \( H - \varepsilon \) for all \( \varepsilon \in (0, H) \). So, the process \( S^H \) satisfies the assumptions \((\mathcal{H}1)\) and \((\mathcal{H}2)\). Thus, by Theorem 2.1 the convergence (2.9) is obtained.

To prove (2.13), it suffices to check \((\mathcal{H}3)\) and \((\mathcal{H}4)\). The case \( H = \frac{1}{2} \) has already been established above. Suppose now that \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \). Using the same argument as in (3.16), we get

\[
E\left( e^{-\theta t} \int_0^t e^{\theta s} dS^H_s \right)^2 = \Delta_{g_{SH}}(t) + \theta I_{2H}(t) - \frac{\theta^2}{2} J_{2H}(t), \quad (3.19)
\]

where \( g_{SH}(s, r) = s^{2H} + t^{2H} - \frac{1}{2}(s + t)^{2H} \).

Moreover, we have

\[
\Delta_{g_{SH}}(t) = 2He^{-2\theta t} \int_0^t s^{2H-1} e^{\theta s} ds - 2H(2H - 1)e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} (s + r)^{2H-2}.
\]

It is easy to see that \( 2He^{-2\theta t} \int_0^t s^{2H-1} e^{\theta s} ds \to 0 \) as \( t \to \infty \).

Furthermore, using the fact that

\[
\int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} (s + r)^{2H-2} \geq \frac{(2t)^{2H-2}}{2} \left( \int_0^t e^{2\theta s} ds \right)^2 \to \infty \quad \text{as} \quad t \to \infty,
\]

L’Hôpital’s rule entails

\[
\lim_{t \to \infty} \left( e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} (s + r)^{2H-2} \right) = \lim_{t \to \infty} \left( \frac{1}{2\theta} e^{-\theta t} \int_0^t e^{\theta r} (t + r)^{2H-2} dr \right)
\]

\[
\leq \lim_{t \to \infty} \left( \frac{t^{2H-2}}{2\theta} e^{-\theta t} \int_0^t e^{\theta r} dr \right) \to 0 \quad \text{as} \quad t \to \infty.
\]
Thus, we deduce that
\[ \Delta g_{SH}(t) \to 0 \text{ as } t \to \infty. \] (3.20)

Combining (3.19), (3.20) and (3.15) we get
\[ E \left( e^{-\theta t} \int_0^t e^{\theta r} dS^H_r \right)^2 \to \frac{H \Gamma(2H)}{\theta^{2H}} \text{ as } t \to \infty. \]

Hence, to finish the proof it remains to check that, for all fixed \( s \geq 0 \)
\[ \lim_{t \to \infty} E \left( S^H_s e^{-\theta t} \int_0^t e^{\theta r} dS^H_r \right) = 0. \]

Let us consider \( s < t \) and let \( f_{S^H}(s, r) = E(S^H_s S^H_r) \). Then, as in the fBm case, we can write
\[ e^{-\theta(t-s)} f_{S^H}(s, s) + e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{S^H}}{\partial r}(s, r) dr - \theta e^{-\theta t} \int_0^s e^{\theta r} f_{S^H}(s, r) dr. \]
It is clear that \( e^{-\theta(t-s)} f_{S^H}(s, s) - \theta e^{-\theta t} \int_0^s e^{\theta r} f_{S^H}(s, r) dr \to 0 \) as \( t \to \infty. \)

On the other hand, since
\[ e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{S^H}}{\partial r}(s, r) dr = \frac{H}{2} e^{-\theta t} \int_s^t e^{\theta r} \left( 2(2H-1) - (r+s)^{2H-1} - (r-s)^{2H-1} \right) dr, \]
the same argument as in (3.18) leads to
\[ e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{S^H}}{\partial r}(s, r) dr \to 0 \text{ as } t \to \infty, \]
which finishes the proof. \( \square \)

### 3.3 Bifractional Brownian motion

Let \( B^{H,K} = \left( B^{H,K}_t, t \geq 0 \right) \) be a bifractional Brownian motion (bifBm) with parameters \( H \in (0, 1) \)
and \( K \in (0, 1] \). This means that \( B^{H,K} \) is a centered Gaussian process with the covariance function
\[ E(B^{H,K}_s B^{H,K}_t) = \frac{1}{2K} \left( (t^{2H} + s^{2H})^K - |t-s|^{2HK} \right). \]

The case \( K = 1 \) corresponds to the fBm with Hurst parameter \( H \). The process \( B^{H,K} \) verifies
\[ E \left( \left| B^{H,K}_t - B^{H,K}_s \right|^2 \right) \leq 2^{1-K} |t-s|^{2HK}, \]
so \( B^{H,K} \) has \( (HK - \varepsilon) \)-Hölder continuous paths for any \( \varepsilon \in (0, HK) \) thanks to Kolmogorov’s continuity criterion. The bifBm \( B^{H,K} \) can be extended for \( 1 < K < 2 \) with \( H \in (0, 1) \) and \( HK \in (0, 1) \) (see [4] and [13]).

**Proposition 3.3.** Suppose that, in (1.1), the process \( G \) is the bifBm \( B^{H,K} \). Then the convergences (2.9) and (2.13) hold true for all fixed \((H, K) \in (0, 1)^2\).
Proof. Since $B^{H,K}$ has Hölder continuous paths of order $HK - \varepsilon$ for all $\varepsilon \in (0, HK)$, it satisfies the assumptions (H1) and (H2). Thus the convergence (2.9) is satisfied. To prove (2.13), it suffices to check (H3) and (H4). Using the same argument as in (3.16), we have

$$
E \left[ e^{-\theta t} \int_0^t e^{\theta s} dB^H_s, K \right]^2 = \Delta g_{B^{H,K}}(t) + 2^{1-K} \theta I_{2HK}(t) - 2^{-K} \theta^2 J_{2HK}(t),
$$

(3.21)

where $g_{B^{H,K}}(s, r) = \frac{1}{2K} (s^{2H} + r^{2H})^K$.

On the other hand,

$$
\Delta g_{B^{H,K}}(t) = 2^{2-K} HK e^{-2\theta t} \int_0^t s^{2HK-1} e^{\theta s} ds
- 2^{2-K} H^2 K (K - 1) e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} (sr)^{2H-1} (s^{2H} + r^{2H})^{K-2}.
$$

The convergence $2^{2-K} HK e^{-2\theta t} \int_0^t s^{2HK-1} e^{\theta s} ds \to 0$ as $t \to \infty$ is immediate.

Also, it is straightforward to check that there exists a constant $C_{H,K}$ depending on $H, K$ such that

$$
\int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} (sr)^{2H-1} (s^{2H} + r^{2H})^{K-2} \geq C_{H,K} t^{2HK-2} \int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} \\
\geq \frac{C_{H,K}}{2} t^{2HK-2} \int_0^t s e^{\theta s} ds \\
\to \infty \quad \text{as } t \to \infty.
$$

So, we can apply L'Hôpital’s rule to obtain

$$
\lim_{t \to \infty} \left( e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} (sr)^{2H-1} (s^{2H} + r^{2H})^{K-2} \right)
= \lim_{t \to \infty} \left( e^{-\theta t} \frac{1}{2\theta} \int_0^t e^{\theta r} (tr)^{2H-1} (t^{2H} + r^{2H})^{K-2} dr \right)
\leq \lim_{t \to \infty} \frac{a^{K-3} e^{-\theta t}}{\theta} \int_0^t e^{\theta r} (tr)^{H-1} dr
\leq \lim_{t \to \infty} \frac{a^{K-3} 2H - 2}{\theta} e^{-\theta t} \int_0^t e^{\theta r} dr
\to 0 \quad \text{as } t \to \infty.
$$

Hence, for every $(H, K) \in (0, 1)^2$

$$
\Delta g_{B^{H,K}}(t) \to 0 \quad \text{as } t \to \infty.
$$

(3.22)

Consequently, (3.21), (3.22) and (3.15) imply

$$
E \left[ \left( e^{-\theta t} \int_0^t e^{\theta s} dB^H_s, K \right)^2 \right] \to \frac{HK \Gamma(2HK)}{\theta^{2HK}} \quad \text{as } t \to \infty.
$$

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Hence, to finish the proof it remains to check that, for all fixed $s \geq 0$

$$\lim_{t \to \infty} E \left( B_s^{H,K} e^{-\theta t} \int_0^t e^{\theta r} dB_{r}^{H,K} \right) = 0.$$ 

Let us consider $s < t$ and let $f_{B,H}(s,r) = E(B_s^{H,K} D_r^{H,K})$. Then, as in the fBm case, we can write

$$E \left( B_{H,K}^s e^{-\theta t} \int_0^t e^{\theta r} dB_{r}^{H,K} \right) = e^{-\theta(t-s)} f_{B,H,K}(s,s) + e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B,H,K}}{\partial r}(s,r) dr - \theta e^{-\theta t} \int_0^s e^{\theta r} f_{B,H,K}(s,r) dr.$$ 

We have $e^{-\theta(t-s)} f_{B,H,K}(s,s) - \theta e^{-\theta t} \int_0^s e^{\theta r} f_{B,H,K}(s,r) dr \to 0$ as $t \to \infty$.

Also,

$$e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B,H,K}}{\partial r}(s,r) dr = 2^{1-K} H K e^{-\theta t} \int_s^t e^{\theta r} \left( r^{2H-1} \left( s^{2H} + r^{2H} \right)^{K-1} - (r-s)^{2HK-1} \right) dr.$$ 

Hence, if $HK < \frac{1}{2}$, L’Hôpital’s rule leads to

$$\left| e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B,H,K}}{\partial r}(s,r) dr \right| \leq 2^{1-K} H K e^{-\theta t} \int_s^t e^{\theta r} \left( r^{2HK-1} + (r-s)^{2HK-1} \right) dr \leq 2^{2-K} H K e^{-\theta t} \int_s^t e^{\theta r} (r-s)^{2HK-1} dr \to \lim_{t \to \infty} \frac{2^{2-K} H K}{\theta} (t-s)^{2HK-1} = 0 \quad \text{as } t \to \infty.$$ 

If $HK = \frac{1}{2}$,

$$\left| e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B,H,K}}{\partial r}(s,r) dr \right| = 2^{-K} e^{-\theta t} \int_s^t e^{\theta r} \left( 1 - \left( 1 + \left( \frac{s}{r} \right)^{2H} \right)^{K-1} \right) dr \to 0 \quad \text{as } t \to \infty.$$ 

The last convergence comes from the fact that for $r$ large,

$$1 - \left( 1 + \left( \frac{s}{r} \right)^{2H} \right)^{K-1} \leq 1 - \left( 1 + \frac{s}{r} \right)^{K-1} \sim (1 - K) \frac{s}{r},$$

and L’Hôpital’s rule. Similarly, if $HK > \frac{1}{2}$,

$$\left| e^{-\theta t} \int_s^t e^{\theta r} \frac{\partial f_{B,H,K}}{\partial r}(s,r) dr \right| \leq 2^{1-K} H K e^{-\theta t} \int_s^t e^{\theta r} r^{2HK-1} \left( 1 + \left( \frac{s}{r} \right)^{2H} \right)^{K-1} - \left( 1 + \frac{s}{r} \right)^{2HK-1} dr \to 0 \quad \text{as } t \to \infty,$$

which completes the proof.

\[\square\]
We first notice that the integral

\[ \int_0^T f(t) \, dt. \]

Proof of Lemma 2.1. We deduce that, for any \( \alpha \), it can be shown (see, e.g., [15, Section 3.1]) that, for any \( \beta \), let

\[ f \in H((0, \infty) \times [0, T]) \]

We also set \( |f|_\alpha := \sup_{0 \leq s \leq T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty. \) We set \( |f|_\infty = \sup_{t \in [0, T]} |f(t)| \), and we equip \( H([0, T]) \) with the norm \( \|f\|_\alpha := |f|_\alpha + |f|_\infty \). Let \( f \in H([0, T]) \), and consider the operator \( T_f : C^1([0, T]) \to C^0([0, T]) \) defined as

\[ T_f(g)(t) = \int_0^t f(u)g'(u) \, du, \quad t \in [0, T]. \]

It can be shown (see, e.g., [15, Section 3.1]) that, for any \( \beta \in (1 - \alpha, 1) \), there exists a constant \( C_{\alpha, \beta, T} > 0 \) depending only on \( \alpha, \beta, T \) such that, for any \( g \in H^\beta([0, T]) \),

\[ \left\| \int_0^t f(u)g'(u) \, du \right\|_\beta \leq C_{\alpha, \beta, T} \|f\|_\alpha \|g\|_\beta. \]

We deduce that, for any \( \alpha \in (0, 1) \), any \( f \in H^\alpha([0, T]) \) and any \( \beta \in (1 - \alpha, 1) \), the linear operator

\( T_f : C^1([0, T]) \subset H^\beta([0, T]) \to H^\beta([0, T]) \), defined as \( T_f(g) = \int_0^t f(u)g'(u) \, du \), is continuous with respect to the norm \( \| \cdot \|_\beta \). By density, it extends (in an unique way) to an operator defined on \( H^\beta \). As consequence, if \( f \in H^\alpha([0, T]) \), if \( g \in H^\beta([0, T]) \) and if \( \alpha + \beta > 1 \), then the (so-called) Young integral \( \int_0^t f(u) \, dg(u) \) is well-defined as being \( T_f(g) \).

The Young integral obeys the following formula. Let \( f \in H^\alpha([0, T]) \) with \( \alpha \in (0, 1) \) and \( g \in H^\beta([0, T]) \) for all \( \beta \in (0, 1) \). Then \( \int_0^t g(u) \, df(u) \) and \( \int_0^t f(u) \, dg(u) \) are well-defined as the Young integrals. Moreover, for all \( t \in [0, T] \),

\[ f(t)g(t) = f_0g_0 + \int_0^t g(u) \, df(u) + \int_0^t f(u) \, dg(u). \]  

(A-1)

Proof of Lemma 2.7 We first notice that the integral \( Z_\infty = \int_0^\infty e^{-\theta s} G_s \, ds \) is well-defined because

\[ \int_0^\infty e^{-\theta s} E(|G_s|) \, ds \leq \sqrt{\pi} \int_0^\infty s^{\gamma} e^{-\theta s} \, ds < \infty. \]

Now, we prove (2.10). By using Borel-Cantelli’s lemma, it is sufficient to prove that, for any \( \varepsilon > 0 \),

\[ \sum_{n \geq 0} P \left( \sup_{n \leq t \leq n+1} \left| \int_t^{t+n} e^{-\theta s} G_s \, ds \right| > \varepsilon \right) < \infty. \]
Notice that for every $\varepsilon > 0$,
\[
E \left( \sup_{n \leq t \leq n+1} \int_t^\infty e^{-\theta s} G_s ds \right) \leq E \left( \int_n^\infty e^{-\theta s} |G_s| ds \right) \\
\leq \sqrt{c} \int_n^\infty e^{-\theta s} \gamma s ds \\
\leq \sqrt{ce^{-\frac{\theta}{2} n}} \int_0^\infty e^{-\frac{\theta}{2} s} \gamma s ds \\
= \sqrt{c} \Gamma (1 + \gamma) \left( \frac{2}{\theta} \right)^{1+\gamma} e^{-\frac{\theta}{2} n}.
\]
Consequently,
\[
\sum_{n \geq 0} P \left( \sup_{n \leq t \leq n+1} \int_t^\infty e^{-\theta s} dG_s > \varepsilon \right) \leq \varepsilon^{-1} \sum_{n \geq 0} E \left( \sup_{n \leq t \leq n+1} \int_t^\infty e^{-\theta s} dG_s \right) \\
\leq \varepsilon^{-1} \sqrt{c} \Gamma (1 + \gamma) \left( \frac{2}{\theta} \right)^{1+\gamma} \sum_{n \geq 0} e^{-\frac{\theta}{2} n} < \infty,
\]
which implies that $Z_t \to Z_\infty$ almost surely as $t \to \infty$. Moreover, since
\[
E \left[ (Z_t - Z_\infty)^2 \right] = \int_t^\infty \int_t^\infty e^{-\theta r} e^{-\theta s} E (G_r G_s) dr ds \\
\leq c \int_t^\infty \int_t^\infty e^{-\theta r} e^{-\theta s} (rs)^\gamma dr ds \\
= c \left( \int_t^\infty e^{-\theta s} \gamma ds \right)^2 \\
\to 0 \text{ as } t \to \infty,
\]
the proof of the claim (2.10) is finished. The convergence (2.11) is a direct consequence of (2.10) and (2.8). Thus the proof of Lemma 2.1 is done.

**Proof of Lemma 2.2** It follows from (2.11) that $\xi_\infty \sim \mathcal{N} \left( 0, E \left[ \xi_\infty^2 \right] \right)$, where
\[
E \left[ \xi_\infty^4 \right] = \theta^2 E \left[ Z_\infty^2 \right] = \theta^2 \int_0^\infty \int_0^\infty e^{-\theta r} e^{-\theta s} E (G_r G_s) dr ds \\
\leq c \theta^2 \int_0^\infty \int_0^\infty e^{-\theta r} e^{-\theta s} (rs)^\gamma dr ds \\
= c \left( \frac{\Gamma (\gamma + 1)}{\theta^\gamma} \right)^2 < \infty.
\]
This implies that
\[
P(\xi_\infty = 0) = 0. \quad (A-2)
\]
The continuity of $\xi$ entails that, for every $t \geq 0$
\[
\int_0^t e^{2\theta s} \xi_s^2 ds \geq \int_0^t e^{2\theta s} \xi_s^2 ds \geq \frac{t}{2} \theta t \left( \inf_{\frac{t}{2} \leq s \leq t} \xi_s^2 \right) \text{ almost surely.} \quad (A-3)
\]
Furthermore, the continuity of $\xi$ and (2.11) yield
\[
\lim_{t \to \infty} \left( \inf_{\frac{t}{2} \leq s \leq t} \xi_s^2 \right) = \xi_\infty^2 \text{ almost surely.}
\]
Combining this last convergence with (A-2) and (A-3), we deduce that
\[
\lim_{t \to \infty} \int_0^t e^{2\theta s} \xi_s^2 ds = \infty \text{ almost surely.}
\]
Hence, we can use L'Hospital's rule to obtain
\[
\lim_{t \to \infty} \int_0^t e^{2\theta s} \xi_s^2 ds = \infty \text{ almost surely,}
\]
which completes the proof of Lemma 2.2.

**Proof of Lemma 2.3** Let $t \geq 0$. Setting $\eta_t = \int_0^t X_s ds$, the equation (1.1) leads to
\[
\frac{1}{2} X_t^2 = \frac{1}{2} \theta^2 \eta_t^2 + \frac{1}{2} G_t^2 + \theta \eta_t G_t.
\]
Moreover, (A-1) and (1.1) entail
\[
\frac{1}{2} \eta_t^2 = \int_0^t \eta_s d\eta_s = \int_0^t \eta_s X_s ds = \theta^{-1} \left( \int_0^t X_s^2 ds - \int_0^t G_s X_s ds \right).
\]
Define $Y_t := \int_0^t e^{\theta s} G_s ds$. Then, by (2.6) and (A-1)
\[
\int_0^t G_s X_s ds = \int_0^t G_s \left( G_s + \theta e^{\theta s} Z_s \right) ds
\]
\[
= \int_0^t G_s^2 ds + \theta \int_0^t e^{\theta s} G_s Z_s ds
\]
\[
= \int_0^t G_s^2 ds + \theta \int_0^t Z_s dY_s
\]
\[
= \int_0^t G_s^2 ds + \theta Z_t Y_t - \theta \int_0^t Y_s dZ_s
\]
\[
= \int_0^t G_s^2 ds + \theta Z_t Y_t - \theta \int_0^t ds \int_0^s dr G_s G_r e^{-\theta(s-r)}.
\]
Thus, we deduce that
\[
\frac{1}{2} X_t^2 = \theta \int_0^t X_s^2 ds - \theta^2 Z_t Y_t + \theta \eta_t G_t + R_t. \tag{A-4}
\]
On the other hand, by (1.1) and (2.6) we get
\[
\theta \eta_t G_t = G_t (X_t - G_t) = -\theta e^{\theta t} G_t Z_t.
\]
This implies that
\[
-\theta^2 Z_t Y_t + \theta \eta_t G_t = -\theta Z_t (\theta Y_t - e^{\theta t} G_t) = \theta Z_t \int_0^t e^{\theta s} dG_s.
\]
Combining this with (A-4) the proof of Lemma 2.3 is done.
Proof of Lemma 2.4. For any $d \geq 1$, $s_1 \ldots s_d \in [0, \infty)$, we shall prove that, as $t \to \infty$,

$$
\left( B_{s_1}, \ldots, B_{s_d}, e^{-\theta t} \int_0^t e^{\theta s} dG_s \right) \xrightarrow{\text{law}} \left( B_{s_1}, \ldots, B_{s_d}, \sigma N \right),
$$

(A-5)

which is enough to lead to the desired conclusion. Because the left-hand side in the previous convergence is a Gaussian vector (see proof of Lemma 7 in [8]), to get (A-5) it is sufficient to check the convergence of its covariance matrix. Thus, the assumptions \((H3)\) and \((H4)\) complete the proof.

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