GRASSMANNIAN FRAMED SHEAVES AND GENERALIZED PARABOLIC STRUCTURES

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ABSTRACT. We build compact moduli spaces of Grassmannian framed bundles over a Riemann surface, essentially replacing a group by its bi-invariant compactification. We do this both in the algebraic and symplectic settings, and prove a Hitchin-Kobayashi correspondence between the two. The spaces are master spaces for parabolic bundles, and the reduction to parabolic bundles commutes with the correspondence. An analogous correspondence is proved for the generalized parabolic bundles of Bhosle, and the Hitchin Kobayashi correspondence is outlined.

1. Introduction

The Hitchin-Kobayashi correspondence, which establishes on a compact Kähler (or even more general) manifold a bicontinuous correspondence between on one hand unitary bundles equipped with an irreducible connection and possibly auxiliary fields satisfying a suitable curvature condition, and on the other, stable holomorphic bundles or stable holomorphic pairs, triples, etc, is by now a well established paradigm, proven over the years in increasing degrees of generality, by Narasimhan-Seshadri [NS], Mehta-Seshadri [MS], Donaldson [Do1, Do2], Uhlenbeck-Yau [UY], Simpson [Si]; good references and an overview can be found in Lübke-Teleman [LT]. This correspondence has been invaluable, both for understanding the holomorphic moduli, and in understanding the moduli of connections satisfying the curvature condition (for example, anti-self-duality in complex dimension two).

In dimension one, the curvature condition is generally one of flatness, or, for non-zero degree, of constant central curvature; the particular correspondences that concern us first are the classical ones, which served as models for the others:

- The correspondence between stable holomorphic vector bundles and flat (or constant central curvature) unitary connections. The holomorphic stable bundles then get linked to unitary representations of the fundamental group of the surface. See Narasimhan-Seshadri [NS], Donaldson [Do1].
- The correspondence between stable holomorphic vector bundles with parabolic structures at marked points, and flat unitary connections on the complement of the

2000 Mathematics Subject Classification. 14H60, 14F05.
Key words and phrases. Grassmannian framed sheaf, parabolic structure, Hitchin-Kobayashi correspondence.
marked points with fixed conjugacy classes for the holonomy around the marked points. See Mehta-Seshadri [MS], Biquard [Bi], Poritz [Po].

There are two, “master”, moduli spaces, which in some sense should contain all of the parabolic spaces, in the sense that they can be obtained as quotients of these master spaces. The first, on the algebraic or holomorphic side, is the space of framed bundles, i.e., the space of pairs of (bundles, trivializations at the marked points); the second, on the unitary or symplectic side, are the extended moduli spaces defined by Jeffrey [Je]. By rights, these spaces should also correspond under the Hitchin-Kobayashi correspondence, but so far this has not been clear.

Both of these spaces also present the difficulty of being inherently non-compact. On the algebraic side, there is a compactification, but by sheaves; on the symplectic side, as well as non-compactness, there is the problem that the symplectic form can degenerate. (One can get a compact space, at the price of considering quasi-Hamiltonian structures, as in [AMM].)

We will remedy all of these problems by replacing the framings with their graphs in a Grassmannian modelled on the Grassmannian of $n$-planes in $\mathbb{C}^{2n}$. This latter Grassmannian is a particularly nice smooth compactification of $\text{GL}(n)$; in particular it is equivariant under both the left and right actions of $\text{GL}(n)$, represented by the embeddings of $\text{GL}(n)$ into $\text{GL}(2n)$ as subgroups $\text{GL}(n) \times \{I\}, \{I\} \times \text{GL}(n)$.

It is the purpose of this paper to construct the moduli spaces of pairs (bundles, Grassmannian framing), as well as the analogous spaces on the symplectic side. We will show:

- that they are compact; on the holomorphic side, this space only involves bundles, and on the symplectic side, it is symplectic where it is reasonable to expect this (i.e., over the locus where a moment map is a submersion);
- that there is a bijective Hitchin-Kobayashi correspondence relating them;
- on both sides, one can obtain the various parabolic spaces as either holomorphic or symplectic quotients; this process commutes with the Hitchin-Kobayashi correspondence.

In a completely analogous vein, we can obtain the generalized parabolic bundles of Bhosle [Bh2]: instead of considering the Grassmannian of $n$-planes in the direct sum $E_p \oplus \mathbb{C}^n$ of the fiber $E_p$ of a vector bundle $E$ at a point $p$ and $\mathbb{C}^n$, we consider the Grassmannian of $n$-planes in $E_p \oplus E_q$; again one has both the symplectic and holomorphic points of view, with a Hitchin-Kobayashi correspondence connecting them. Again, one can construct from both points of view the moduli of bundles on a nodal curve. Indeed, it was the question of finding out what the Narasimhan-Seshadri theorem looked like for nodal curves which was the initial motivation for this work.
We note in passing that another set of master spaces for parabolic moduli, this time with trivializations parametrized by a torus instead of GL(n, C), was given by Hurtubise, Jeffrey and Sjamaar in [HJS].

There should be equivalent correspondences for arbitrary reductive groups; these should be particularly interesting, as the bi-equivariant compactifications of these groups have recently been constructed, using bundles on rational curves, by Martens and Thaddeus [MT]. We will return to these elsewhere.

Section 2 is devoted to the construction of the moduli of Grassmannian framed bundles; Section 3 recalls Jeffrey’s construction of the extended moduli spaces, and uses it to construct a Grassmannian analog. In Section 4 we prove the correspondence. In Section 5, we show how the same ideas extend to Bhosle’s generalized parabolic structures. Section 6 gives examples.

Acknowledgements. The authors would like to thank the Kerala School of Mathematics for their hospitality during the discussions which launched this project.

2. Moduli of Grassmannian-framed sheaves

2.1. Definitions and notation. Let X be an irreducible smooth complex projective curve of genus g, with ℓ marked (ordered) points p₁, · · · , pℓ. Throughout this paper, E will denote a vector bundle of rank n over X. Initially, the degree of E is δ₀ with −[(ℓn − 1)/2] ≤ δ₀ ≤ [(ℓn − 1)/2], where [t] denotes the integral part of t. (This rather odd choice of range for the degrees is due to an eventual link to bundles with parabolic structure; in the course of the moduli construction, this degree will be increased into a stable range, as the result of twisting by a line bundle of positive degree, as is usual for moduli constructions.) The Grassmannian parametrizing linear subspaces of Epᵢ ⊕ Cn of dimension n will be denoted by Grₙ(Epᵢ ⊕ Cn); if a subspace of Epᵢ ⊕ Cn is in general position, it is the graph of a trivialization of E at pᵢ, or more generally, the graph of a linear map from Cn to Epᵢ. Throughout, gi will denote an element of Grₙ(Epᵢ ⊕ Cn). Set ⃗g = (g₁, · · · , gℓ).

There are two numbers associated to an element gi of the Grassmannian Grₙ(Epᵢ ⊕ Cn):

\[ sᵢ = \dim(gᵢ \cap Eₚᵢ) \]  \hspace{1cm} (2.1)
\[ tᵢ = \dim(gᵢ \cap Cⁿ) = \dim(Eₚᵢ / Π(gᵢ)) ; \]  \hspace{1cm} (2.2)

here Π(gᵢ) is the projection of gi to Epᵢ. We note that gi is the graph of a trivialization if sᵢ = tᵢ = 0; there is the obvious bound sᵢ + tᵢ ≤ n.

We call the pair \((E, ⃗g)\) a Grassmannian framed vector bundle. For a subbundle E' ⊂ E, let gᵢ' := gi ∩ (Epᵢ' ⊕ Cn). Define

\[ sᵢ' := \dim(gᵢ \cap Eₚᵢ') \text{ and } tᵢ' := \dim(Eₚᵢ' / (Π(gᵢ) \cap Eₚᵢ')) . \]

Definition 2.1. We call \((E, ⃗g)\) semistable if the following conditions hold:
(1) the inequality
\[
\sum_{i=1}^{\ell} s_i \leq \frac{\ell n}{2} - \delta_0
\]
holds.

(2) the inequality
\[
\sum_{i=1}^{\ell} t_i \leq \frac{\ell n}{2} + \delta_0
\]
holds.

(3) for every subbundle \( E' \) of \( E \),
\[
0 \leq \frac{\delta_0}{n} - \frac{\delta_0'}{n'} + \sum_i \left[ \frac{(n' - s'_i)}{n'} - \frac{(n' - s'_i - t'_i + t_i)}{n} \right],
\]
where \( n' \) and \( \delta_0' \) are the rank and degree of \( E' \) respectively, and if in (2.5) the equality holds, then
\[
0 \leq \left[ \frac{\delta_0}{n} - \frac{\delta_0'}{n'} \right] \left[ - \frac{\delta_0}{n} - \frac{1}{2} + \sum_i \left( \frac{n' - s'_i - t'_i + t_i}{n} \right) \right].
\]

The pair \((E, \vec{g})\) is called stable if in addition one has strict inequalities in (2.3), (2.4) and, when there is equality in (2.5), in (2.6).

2.2. The moduli construction. The moduli of pairs \((E, \vec{g})\), for which the planes correspond to framings was first examined by Seshadri in [Se], and considered more extensively and in a more general context by Huybrechts and Lehn in [HL]. We adapt some of their notation and results to define a moduli space \(\mathcal{G}\mathcal{M}_{n,\delta_0} = \mathcal{G}\mathcal{M}_{n,\delta_0,p_1,\cdots,p_\ell} \) where \( n \) is the rank. We begin by an essentially linear-algebraic construction which encodes the pair \((E, \vec{g})\) with \( E \) of fixed rank \( n \) and degree \( \delta \). This follows a well-established pattern set, to name some, by Gieseker [Gi], Bhosle [Bh1], and Huybrechts and Lehn [HL].

We shall first show that there exists an integer \( e = e(n, g, \ell) \) such that for any semistable Grassmannian framed bundle \((E, \vec{g})\) of degree \( \delta \geq e \), \( E \) is generated by global sections and \( H^1(X, E) = 0 \).

If \( L \) is a line bundle on \( X \), then fixing isomorphisms \((E \otimes L)_{p_i} \cong E_{p_i}\), we get a Grassmannian framed structure \( \vec{g}_L \) on \( E \otimes L \) induced from that on \( E \). We call a Grassmannian framed bundle \((E, \vec{g})\) of rank \( n \), degree \( \delta \) pseudo semistable if for every subbundle \( E' \subset E \) of rank \( n' \) and degree \( \delta' \), we have
\[
0 \leq \frac{\delta}{n} - \frac{\delta'}{n'} + \sum_i \left[ \frac{(n' - s'_i)}{n'} - \frac{(n' - s'_i - t'_i + t_i)}{n} \right].
\]
Then the Grassmannian framed bundle \((E \otimes L, \vec{g}_L)\) is pseudo semistable if and only if \((E, \vec{g})\) is pseudo semistable. A semistable Grassmannian framed bundle is pseudo semistable, but the converse may not be true.
Lemma 2.2. Let \((E, \vec{g})\) be pseudo semistable. If \(\frac{\delta}{n} > 2g - 2 + (n-1)\ell\), then \(H^1(X, E) = 0\). If \(\frac{\delta}{n} > 2g - 1 + (n-1)\ell\), then \(E\) is generated by global sections.

Proof. Let \(B_i = \left(\frac{n'-s'}{m'} - \frac{(n'-s'-t' + t')}{n}\right), B = \sum_i B_i\). Then

\[
B_i = \frac{1}{nn'}(n'(n - n') - s'(n - n') - n'(t - t')) = \frac{(n - n')(n' - s')}{n'} - \frac{t - t'}{n} \leq 1.
\]

Hence \(B \leq \ell\).

If \(H^1(E) \neq 0\), there exists a nonzero homomorphism \(f : E \to K\). Let \(\mu(E) = \text{degree} \ E / \text{rank} \ E\).

Applying the pseudo semistability condition to the kernel of \(f\) we have

\[
\mu(E) \leq 2g - 2 + (n-1)B \leq 2g - 2 + (n-1)\ell,
\]

i.e., \(\mu(E) \leq 2g - 2 + (n-1)\ell\). This contradicts \(\frac{\delta}{n} > 2g - 2 + (n-1)\ell\).

For global generation of \(E\) it suffices to have \(H^1(X, E(-x)) = 0\) for all \(x \in X\). Since \(\mu(E(-x)) = \mu(E) - 1\), the result follows from the first part. \(\square\)

Let \(\mathcal{O}_X(1)\) denote a fixed line bundle of degree \(\ell\) over \(X\). Fix a sufficiently large positive integer \(k'\). For a vector bundle \(E_0\) on \(X\), the Hilbert polynomial of \(E = E_0(k') := E_0 \otimes \mathcal{O}_X(k')\) with \(E_0\) of rank \(n\) and degree \(\delta_0\) is

\[
P_{k'}(t) = \chi(E_0(k' + t)) = nt + \delta_0 + nk + n(1 - g) = \delta + n(1 - g),
\]

where \(k = k'\ell\). Let

\[
p = P_{k'}(0) = \delta_0 + kn + n(1 - g).
\]

Let

\[
\text{Quot} := \text{Quot}(\mathcal{O}_X^p, P_{k'}(t))
\]

be the quot scheme parametrizing all the quotients \(q : \mathcal{O}_X^p \to E\) such that \(E\) is a coherent sheaf on \(X\) with Hilbert polynomial \(P_{k'}(t)\). There exists a universal family \(\mathcal{E} \to \text{Quot} \times X\) and a (universal) quotient map

\[
\mathcal{O}_{\text{Quot} \times X}^p \to \mathcal{E},
\]

such that for any \(q \in \text{Quot}\), the restriction \(\mathcal{O}_{\text{Quot} \times X}^p \to \mathcal{E}|_{(q) \times X}\) is represented by \(q\).

Let

\[
R \subset \text{Quot}
\]

be the subset of \(\text{Quot}\) consisting of points \(q \in \text{Quot}\) corresponding to sheaves \(\mathcal{E}_q\) satisfying the following:

1. \(\mathcal{E}_q\) are vector bundles (generically) generated by sections, and
2. \(H^0(X, \mathcal{E}_q) \cong \mathbb{C}^p\) (so \(H^1(X, \mathcal{E}_q) = 0\) by the Riemann-Roch theorem).
For sufficiently large $k$, the set $R$ contains the subset of $Quot$ corresponding to all $E$ such that there is a Grassmannian framing $\vec{g}$ on $E$ satisfying the condition that the pair $(E, \vec{g})$ is semistable (Lemma 2.2). It is well known that $R$ is a Zariski open subset of $Quot$.

Let $p_R : R \times X \longrightarrow R$ be the projection. Define
$$E_{p_i} = (p_R)_*(E |_{R \times p_i}) \longrightarrow R.$$Let $C^n_R$ be the trivial vector bundle of rank $n$ on $R$. Let
$$Gr_t(E_{p_i} \oplus C^n_R) \longrightarrow R$$be the Grassmannian bundles over $R$ whose fibers at $E$ are isomorphic to the Grassmannians of $n$-planes in $E_{p_i} \oplus C^n$. Let
$$\tilde{R} := Gr_1(E_{p_1} \oplus C^n_R) \times_R \cdots \times_R Gr_t(E_{p_t} \oplus C^n_R) \longrightarrow R$$be the fiber product. A point of $\tilde{R}$ corresponds to a point $q$ of $R$, that is, a vector bundle $E$ and a point in the fiber of $Gr_t(E_{p_i} \oplus C^n_R)$ for all $p_i$.

The group $S(GL(p) \times GL(n)^t)$ acts on $Quot$ preserving $R$, and the action on $R$ lifts to $\tilde{R}$. We note that the center of $GL(p)$ acts non-trivially here on the $\beta_i$, even after projectivization; this is in contrast to many other moduli problems, such as those for parabolic bundles. Our moduli space $\mathcal{GM}_{n,\delta}$ is a GIT-quotient of $\tilde{R}$ by $GL(p)$, or, what is equivalent, by $S(GL(p) \times \mathbb{C}^*)$, where $\mathbb{C}^*$ acts by multiples of the identity on the $\mathbb{C}^n$ factors. To construct the quotient, we use an injective affine morphism of $\tilde{R}$ into a suitable projective variety; this morphism will be described now.

Set $V = \mathbb{C}^p$. Since $\dim H^0(X, E) = p$, the vector bundle $E$ is then a quotient of the trivial vector bundle $V_X = V \otimes \mathcal{O}_X$ of rank $p$ on $X$. Fixing a quotient homomorphism $V_X \rightarrow E$, we consider the determinant map on sections:
$$(2.8) \quad h : \Lambda^n(V) \longrightarrow H^0(X, \det(E)).$$

Let $P(\mathcal{U})$ be the projective Picard bundle over $Pic^d(X) =: A$, where $d = \delta_0 + nk$. We recall that $P(\mathcal{U})$ parametrizes isomorphism classes of pairs consisting of a line bundle of degree $d$ and a nonzero section of it. As our determinant bundles $\det(E)$ lie in the above component $A$ of the Picard group, the homomorphism $h$ in (2.8) gives an element $\alpha$ of the projective bundle
$$\alpha \in P := P(Hom(\Lambda^n(V), \mathcal{U})) \longrightarrow A.$$This element $\alpha$ encodes the bundle $E$ [Gi], [Bh1], [HL]. The Grassmannian framing $g_i$, in turn, defines under the evaluation map on sections of $E$ at $p_i$ a natural linear subspace in $V \oplus \mathbb{C}^n$ of codimension $n$, and so $\vec{g}$ gets encoded as an $\ell$-tuple $\beta = (\beta_1, \cdots, \beta_\ell)$ of elements $\beta_i$ of the Plücker embedding of the Grassmannian, meaning
$$\beta_i \in Gr_n(V^* \oplus \mathbb{C}^n) \subset Q = P(\Lambda^n(V^* \oplus \mathbb{C}^n)).$$
It is easy to see that associating the pair $(\alpha, \beta)$ to $(E, \vec{g})$ produces a morphism
$$f : \tilde{R} \longrightarrow P \times Q^\ell,$$
lying over the morphism $f_R : R \to P$ defined by $E \mapsto \alpha$.

The set $f_R(R)$ of elements of $P$ is described in [Gi], [Bh1], [HL]; basically, under the evaluation at any point on the curve of the elements of $U$, the element $\alpha_p \in \Lambda^n(V)^*$ that one gets must be a (non-zero) indecomposable element. Similarly, the elements $\beta_i$ must be indecomposable, meaning, they define an element of the Grassmannian of $p$-planes in $\mathbb{C}^{p+n}$ under the Plücker map. In addition, the elements $\alpha$ and $\beta_i$ must be compatible in the sense that the kernel of $\alpha_p$ must lie in the kernel of $\beta_i$.

Let $Z$ be the Zariski closure of $f(\tilde{R})$ in $P \times Q^\ell$.

As usual, we need a polarization on $Z$. As in [HL], we obtain an ample line bundle $O(1)_P$ on $P$ which is in the twist of the lift of a very ample line bundle on $A$ by a line bundle that restricts to the standard positive generating bundle on each fiber (which is a projective space). We also have the standard $O(1)_Q$ on $Q$. For a positive rational number $\eta = \nu/\mu$, where $\nu$ and $\mu$ are integers, consider the polarization $O(1)_P^\otimes \mu \boxtimes (\boxtimes_i O(1)_Q^\otimes \nu)$.

As we have made a choice of a basis of $V$, we then quotient, taking the semi-stable elements. Normally, one takes the quotient by the group $SL(V)$, as we have already projectivized our elements $(\alpha, \beta)$; we have to take a supplementary quotient here, of the scalars $\mathbb{C}^*$ acting, by a multiple of the identity map on the $\mathbb{C}^n$ factors, since this action induces isomorphisms on the level of framed bundles. We will do the quotient sequentially, and in fact will be using different polarizations.

2.3. Stability condition for points in $P \times Q^\ell$. For the polarization corresponding to $\eta$, we want to examine the stability of the element $(\alpha, \beta) = (\alpha, (\beta_1, \cdots, \beta_p))$, first under the action of $SL(V)$. We note that $(\alpha, \beta)$ is a (semi)stable point in $P \times Q^\ell$ if and only if it is a (semi)stable in $P \times Q'$ where

$$Q' = \mathbb{P}(\text{Hom}(\Lambda^n(V), H^0(\text{det}(E))^*))$$

with respect to the canonical linearization for the line bundle $O(1)_P^\otimes \mu \boxtimes (\boxtimes_i O(1)_Q^\nu)$ (see [Ma, 4.12], [HL, p.84]).

We use the Hilbert criterion, as expounded in [MFK], which involves examining the action of all one-parameter subgroups of $SL(p)$. This is equivalent to choosing a basis $v_i$ of $V^*$, and corresponding weights $a_i$ summing to zero, with $a_1 \leq a_2 \leq \cdots \leq a_n$, and taking the corresponding action. As remarked in [HL], the cone of these weights for the group $SL(V) \times \{1\}$ acting on $V$ is generated by the weights

$$((p' - p), (p' - p), \cdots, (p' - p), p', \cdots, p'),$$

where the $(p' - p)$ is repeated $p'$ times and the $p'$ is repeated $(p - p')$ times.

It suffices to consider stability for these generators. One now remarks that each of the set of choices (basis, generator of the cone of weights) corresponds to the choice of a $p'$-dimensional subspace $W$ of $V$ (the first $p'$ vectors) and a complementary space $W^\perp$ of it.
We consider the action corresponding to \((W, W^\perp)\) on \((\alpha, \beta)\). Decompose the representations in terms of weight spaces: let \(x_i\) be a local basis of weight vectors for the action on the fibers of \(P\), and set \(\alpha = \sum_i \alpha_i x_i\); similarly, put \(\beta_i = \sum_j \beta_{i,j} y_j\), for a basis of weight vectors \(y_j\) for \(Q\). Now define

\[
w_{W,\alpha} := -\min_{\alpha_i \neq 0} \text{weight}(x_i) \quad \text{and} \quad w_{W,\beta_i} := -\min_{\beta_{i,j} \neq 0} \text{weight}(y_j).
\]

Setting

\[
w_W \equiv w_{W,\alpha} + \eta \sum_i w_{W,\beta_i},
\]

for semistability (respectively, stability), one wants, as in [HL], that

\[
0 \leq w_W \quad (\text{respectively}, \; 0 < w_W)
\]

for all \(p', W\) and \(W^\perp\).

**Notation.** We will use \((\leq)\) to denote \(<\) for stability, and \(\leq\) for semi-stability.

**Remark 2.9.** We will see that the choice of \(W^\perp\) is irrelevant, and only \(W\) counts; hence the notation \(w_W\).

Let \(E_W\) be the subsheaf of \(E\) generated by \(W\). One has [HL] Lemma 1.23], for \(W\), with its accompanying weights:

**Lemma 2.3 ([HL]).** Let \(n' = \text{rank}(E_W)\). Then

\[
w_{W,\alpha} = n'(p - p') - (n - n')p' = pn' - p'n.
\]

Given \(\beta_i\), let \(g_i \in \text{Gr}_n(E_{p_i} \oplus \mathbb{C}^n)\) be the \(n\)-dimensional subspace that it defines. Let

\[
\Pi : E_{p_i} \oplus \mathbb{C}^n \to E_{p_i}
\]

be the natural projection. Let \(E_{W,p_i}\) be the image of \(E_W\) in the fiber of \(E\) at \(p_i\). We define

\[
m_i' = \dim(E_{W,p_i}), \quad s_i' = \dim(E_{W,p_i} \cap g_i), \quad t_i' = \dim(E_{W,p_i} / (E_{W,p_i} \cap \Pi(g_i))),
\]

and

\[
r_i' = m_i' - s_i' - t_i' = \dim((E_{W,p_i} \cap \Pi(g_i)) / (E_{W,p_i} \cap g_i)).
\]

Note that if \(E_W\) is a sub-bundle at \(p_i\), then \(m_i' = n'\); if the plane \(g_i\) at \(p_i\) is the graph of a framing, then \(s_i' = t_i' = 0\). Also, \(s_i' \leq s_i, t_i' \leq t_i\), and \(s_i' + t_i' \leq m_i'\). All of these quantities, when unprimed, refer to the case \(W = V^*\).

The element \(\beta_i\) can be written as

\[
\beta_i = b_1 \wedge b_2 \wedge \cdots \wedge b_{t_i} \wedge (e_{t_i+1} + b_{t_i+1}) \wedge \cdots \wedge (e_n + b_n).
\]

Here \(b_1, \cdots, b_{t_i}\) are independent elements of \(V^*\), and \(e_{t_i+1}, \cdots, e_n\) are independent elements of \((\mathbb{C}^n)^*\). Now choose a basis \(\{v_j\}\) of \(V^*\) for which the first \(p'\) vectors form a basis of \(W\), and write the components of the \(b_k\) as a matrix \(b_{k,j}\). Then row reduce (taking combinations of the \(b_k\)) and put the elements \(b_1, \cdots, b_{t_i}\) in reduced echelon form with respect to this basis, permuting if necessary; let \(c_1 < \cdots < c_{t_i}\) be the indices for which \(b_1, \cdots, b_{t_i}\) have nonzero coordinates in the basis for the first time.
Lemma 2.4. The following two hold:

\begin{align}
(2.10) & \quad t_i' = \dim\left(E_{W, p_i}/E_{W, p_i} \cap \Pi(g_i)\right) = \text{number of } c_j \leq p' \text{ with } j \leq t_i \\
(2.11) & \quad r_i' = \dim\left(E_{W, p_i} \cap \Pi(g_i)/E_{W, p_i} \cap g_i\right) = \text{number of } c_j \leq p' \text{ with } j > t_i.
\end{align}

Proof. The proof is fairly straightforward; one has \( a \in \Pi(g_i) \iff b_j(a) = 0, j = 1, \cdots, t_i \), from which the first result follows. Also, \( a \in g_i \cap E_{p_i} \iff b_j(a) = 0, j = 1, \cdots, n \), and so for all \( i \),

\[ m_i' - s_i' = \dim\left(E_{W, p_i}/E_{W, p_i} \cap g_i\right) = \text{number of } c_j \leq p', \]

from which the second result follows. \( \square \)

We note that \( SL(p) \) acts trivially on the \( e_j \), and with weight \( p' - p \) on \( v_j, j = 1, \cdots, p' \), and with weight \( p' \) on the rest. One now has the following result for (minus) the minimum weight:

Lemma 2.5. We have

\[ w_{W, \beta_i} = t_i'(p - p') - (t_i - t_i')p' + r_i'(p - p') = pt_i' - p't_i + r_i'(p - p'). \]

Putting the results for \( \alpha, \beta \) together, we have:

Proposition 2.6. The pair \( (\alpha, \beta) \) is \( SL(V) \)-semistable (respectively, stable) for the parameter \( \eta \) if and only if for all planes \( W \),

\begin{align}
(2.12) & \quad 0 \ (\leq) \ w_W = pn' - p'n + \eta \sum_i \left[ pt_i' - p't_i + r_i'(p - p')\right].
\end{align}

Lemma 2.7. Let \( W \) and \( W_1 \) be two subspaces of \( V \) such that each of \( W, W_1 \) generates the same subsheaf of \( E \) and \( W_1 \supset W \). Then \( w_{W_1} \leq w_W \) and if \( W_1 \supseteq W \) then \( w_{W_1} = w_W \).

Proof. Let \( p' = \dim W \), \( p_1' = \dim W_1 \). We have

\[ w_W - w_{W_1} = (p_1' - p')(n + \eta \sum_i (t_i + r_i') \geq 0, \]

with equality if and only if \( p' = p_1' \). \( \square \)

We now choose our stability parameter:

\begin{align}
(2.13) & \quad \eta = \frac{1}{k - g + \frac{1}{2}}.
\end{align}

Set \( \delta = \deg(E) \). For any \( W \subset V \), let \( \delta' = \deg(E_W) \), where \( E_W \) is the subsheaf of \( E \) generated by \( W \).
Suppose that \( H^1(X, E_W) = 0, h^0(X, E_W) = p' \). One then has
\[
p = \delta + (1 - g)n, \quad p' = \delta' + (1 - g)n'.
\]
Substituting into \( w_W \), and dividing by \( nn' \), one has the semistability condition for the action of \( SL(V) \):
\[
\begin{align*}
(2.14) \quad 0 \ (\leq) & \quad \frac{\delta}{n} \left( 1 + \eta \sum_i \left( \frac{r'_i + t'_i}{n'} \right) \right) - \frac{\delta'}{n'} \left( 1 + \eta \sum_i \left( \frac{r'_i + t_i}{n} \right) \right) \\
& + \eta(1 - g) \sum_i \left[ \left( \frac{r'_i + t'_i}{n'} \right) - \left( \frac{r'_i + t_i}{n} \right) \right].
\end{align*}
\]

As noted above, this vector bundle \( E \) was twisted up from an original bundle \( E_0 \); one has
\[
\delta = \delta_0 + kn, \quad p = \delta_0 + (k - g + 1)n
\]
for some \( k \). Likewise the subsheaf \( E_W \) arises from a \( E_{W,0} \) and
\[
\delta' = \delta'_0 + kn', \quad p' = \delta'_0 + (k - g + 1)n',
\]
where \( \delta'_0 \) is the degree of \( E_{W,0} \).

Substituting into our expression (2.14), we find
\[
(2.15) \quad 0 \ (\leq) \quad w_{W,k} = \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \sum_i \left[ \left( \frac{r'_i + t'_i}{n'} \right) - \left( \frac{r'_i + t_i}{n} \right) \right] \\
+ \frac{1}{k} \sum_i \left[ \frac{\delta_0}{n} \left( \frac{r'_i + t'_i}{n'} \right) - \frac{\delta'_0}{n'} \left( \frac{r'_i + t_i}{n} \right) + (1 - g) \left( \frac{r'_i + t'_i}{n'} \right) - \left( \frac{r'_i + t_i}{n} \right) \right] \\
+ \frac{1}{k} \left[ (\frac{1}{2} - g) \left( \frac{\delta_0}{n} - \frac{\delta'_0}{n'} \right) \right].
\]
Let us refer to this condition as the \( k \)-(semi-)stability condition for the \( SL(V) \)-action. Taking a limit, we have the \( \infty \)-(semi-)stability condition:
\[
(2.16) \quad 0 \ (\leq) \quad w_{W,\infty} = \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \sum_i \left[ \left( \frac{r'_i + t'_i}{n'} \right) - \left( \frac{r'_i + t_i}{n} \right) \right].
\]

Proposition 2.8. Let
\[
w_{W,A} = \left[ \frac{\delta_0}{n} - \frac{\delta'_0}{n'} \right] - \frac{\delta_0}{n} + \sum_i \left( \frac{r'_i + t_i}{n} \right) - \frac{1}{2}.\]

Then for \( k > C \), where \( C = C(n,l,g) \) is a constant, the following statements hold:

1. \( w_{W,k} \geq 0 \) if and only if \( w_{W,\infty} \geq 0 \) and in case \( w_{W,\infty} = 0 \) one has \( w_{W,A} \geq 0 \).
2. \( w_{W,k} > 0 \) if and only if \( w_{W,\infty} \geq 0 \) and in case \( w_{W,\infty} = 0 \) one has \( w_{W,A} > 0 \).
3. \( w_{W,k} = 0 \) if and only if \( w_{W,\infty} = 0 \) and then \( w_{W,A} = 0 \).
Proof. Set \( B = \sum_i \left[ \frac{r_i' + t_i'}{n'} - \frac{r_i + t_i}{n} \right] \).

We shall first show that for \( k > C_1 \), the following holds:

\[(a) \text{ if } w_{W,\infty} > 0 \text{, then } w_{W,k} > 0 \text{.}\]

Substituting for the expression for \( B \), the condition \( W_{W,\infty} > 0 \) takes the form

\[\delta_0 n' - \delta_0 n + nn'B > 0 \text{.}\]

Hence it implies that \( \delta_0 n' - \delta_0 n + nn'B \geq 1 \) so that \(-\frac{\delta_0}{n} \geq -\frac{\delta_0}{n} - B + \frac{1}{nn'}\). Substituting this in the expression for \( w_{W,k} \) one gets \( w_{W,k} \geq \frac{1}{nn'} - \frac{C_1}{knn'} \), where \( C_1 \) is a constant (i.e., dependent only on \( n, \ell, g \)). The claim \((a)\) follows for \( C = C_1 \).

Note that the statement

\[(b') \text{ if } w_{W,\infty} < 0 \text{, then } w_{W,k} < 0 \]

is equivalent to the statement

\[(b) \text{ If } w_{W,k} \geq 0 \text{, then } w_{W,\infty} \geq 0 \text{;}\]

we prove \((b')\). The condition \( w_{W,\infty} < 0 \) implies that \( \delta_0 n' - \delta_0 n + nn'B \leq -1 \). Hence \(-\frac{\delta_0}{n} \geq -\frac{\delta_0}{n} - B - \frac{1}{nn'}\). Then \( w_{W,k} \leq \frac{-1}{nn'} + \frac{C_2}{knn'} \), where \( C_2 \) is a constant. Thus for \( k \geq C_2 \), we have \( w_{W,k} \leq 0 \).

Take \( C \) to be the maximum of \( C_1 \) and \( C_2 \).

Suppose that \( w_{W,k} \geq 0 \), then by \((b)\), \( w_{W,\infty} \geq 0 \). Suppose that \( w_{W,\infty} = 0 \). Replacing \( \frac{\delta_0}{n} \) by its value from the equality \( w_{W,\infty} = 0 \), the equation \( w_{W,k} \geq 0 \) becomes

\[0 \leq \left[ \sum_i \left( \frac{r_i' + t_i'}{n'} - \frac{r_i + t_i}{n} \right) \right] \left[ \frac{\delta_0}{n} - \sum_i \left( \frac{r_i' + t_i}{n} \right) + \frac{1}{2} \right] \text{;}
\]
equivalently, \( w_{W,A} \geq 0 \).

Conversely, let \( w_{W,\infty} \geq 0 \) and in case of equality, \( w_{W,A}(\geq 0) \). If \( w_{W,\infty} > 0 \), by \((a)\) \( w_{W,k} > 0 \). If \( w_{W,\infty} = 0 \), then as seen above, \( w_{W,k} \geq 0 \) (respectively, \( w_{W,k} > 0 \)) is equivalent to \( w_{W,A} \geq 0 \) (respectively, \( w_{W,A} > 0 \)). This proves \((1)\) and \((2)\).

For \((3)\), note that \((a)\) is equivalent to

\[\text{(a') if } w_{W,k} \leq 0 \text{, then } w_{W,\infty} \leq 0 \text{.}\]

It follows from \((a')\) and \((b)\) that for \( k \geq C \), if \( w_{W,k} = 0 \), then \( w_{W,\infty} = 0 \) and then \( w_{W,A} = 0 \). The converse can be proved as in \((1)\) and \((2)\).

\[\square\]

**Corollary 2.9.** Let \( W \subset V \) be such that \( H^1(X, E_W) = 0 \). If \( w_{H^0(X,E_W),k} \geq 0 \), then \( W \) satisfies \( W_W \geq 0 \).

**Proof.** We have \( W \subset H^0(X, E_W) \) and both \( W, H^0(X, E_W) \) generate \( E_W \). By Proposition \[2.8\] applied to \( H^0(X, E_W) \subset V \) we have \( w_{H^0(X,E_W)} = w_{H^0(X,E_W),k} \geq 0 \). By Lemma \[2.7\] \( w_W \geq w_{H^0(X,E_W)} \geq 0 \).

\[\square\]
Let $K_X$ denote the canonical line bundle of $X$.

**Lemma 2.10.** There exists a subsheaf $E'_W \subseteq E_W$ such that $E'_W$ is globally generated, $H^1(X, E'_W) = 0$ and if $w_{H^0(E'_W)} \geq 0$, then $w_{H^0(E_W)} \geq 0$.

Thus, if $E_W$ destabilizes, so does $E'_W$.

**Proof.** If $H^1(X, E_W) = 0$, take $E'_W = E_W$.

If $H^1(X, E_W) \neq 0$, one has a map $E_W \rightarrow K_X$, and an exact sequence

$$0 \rightarrow F \rightarrow E_W \rightarrow K_X$$

giving a rank $n' - 1$ subbundle $F$ with $H^0(X, F) \geq (p' - g)$. Let $E^i_W$ be the subsheaf of $F$ generated by the global sections; then $H^0(X, E^i_W) \geq (p' - g)$. If $H^1(X, E^1_W) \neq 0$, we can then produce in a similar way a subsheaf $E'^i_W \subseteq E^i_W$ of rank $n' - 2$, with $H^0(X, E'^i_W) \geq (p' - 2g)$. This process eventually terminates,

(1) either at a subsheaf $E^i_W$ with $H^0(X, E^i_W) \geq (p' - ig)$ and $H^1(X, E^i_W) = 0$

for some $i$,

(2) or at a line bundle $E''_W$ with

$$H^0(X, E''_W) \geq (p' - (n' - 1)g) \quad \text{and} \quad H^1(X, E''_W) = 0$$

if $p' \geq n'g$.

Call $E'_W$ the subsheaf at which the process terminates. Let

$$W' = H^0(E_W), p' = h^0(E'_W), W'' = H^0(X, E'_W), p'' = h^0(X, E''_W).$$

Let $n', r', t'$ and $r'', t''$ be the corresponding quantities for $E_W$ and $E'_W$. The expression for $w_{W'}$ in (2.12) can be rewritten as

$$w_{W'} = p(n' + \eta \sum_i (t'_i + r'_i)) - p'(n + \eta \sum_i (r'_i + t_i)).$$

Using $p' \leq p'' + mg, m \geq 1, n' = n'' + m$, this gives

$$w_{W'} \geq p(n'' + m + \eta \sum_i (t''_i + r''_i)) - (p'' + mg)(n + \eta \sum_i (r''_i + t_i)).$$

Hence we get

$$w_{W'} - w_{W''} \geq pm - mgn + \eta \left[ \sum_i (p(t'_i + r'_i - t''_i - r''_i) - p''(r'_i - r''_i) - mg(t_i + r''_i)) \right].$$

If $w_{W''} \geq 0$, then $p'' \leq \frac{pm}{n'} + C'$, with $C'$ a constant. Substituting this in the expression for $w_{W'} - w_{W''}$, one sees that for $k$ large ($k \geq a$ constant), $\eta$ is small enough so that $w_{W'} - w_{W''} \geq 0$. \hfill \Box
Proposition 2.11. If every subsheaf $E'$ of $E$ satisfies the conditions

$$0 \leq S^1(E') := \frac{\delta_0}{n} - \frac{\delta_0}{n'} + \sum_i \left[ \frac{(r'_i + t'_i)}{n'} - \left( \frac{r_i'}{n} + \frac{t_i'}{n'} \right) \right],$$

and if one has equality,

$$0 \leq S^2(E') := \left[ \frac{\delta_0}{n} - \frac{\delta_0}{n'} \right] \left[ -\frac{\delta_0}{n} + \sum_i \left( \frac{r'_i + t'_i}{n'} - \frac{1}{2} \right) \right],$$

then $(\alpha, \beta)$ is $k$-SL($V$)-semistable.

If every subsheaf $E'$ of $E$ satisfies the conditions

$$S^1(E') \geq 0, \quad \text{and for } S^1(E') = 0 \text{ one has } S^2(E') > 0,$$

then $(\alpha, \beta)$ is $k$-SL($V$)-stable.

Proof. Suppose that every subsheaf $E$ satisfies the conditions in the statement of Proposition 2.11. Let $W \subset V$ be a subspace and $E_W$ the subsheaf of $E$ generated by $W$. By Lemma 2.10 there exists a subsheaf $E'_W \subset E_W$, $W' = H^0(X, E'_W)$ such that $E'_W$ is globally generated, $H^1(X, E'_W) = 0$ and if $w_{W''} \geq 0$, then $w_{H^0(E_W)} \geq 0$. One has $w_{W''} = S^1(E'_W), w_{W''} = S^2(E'_W)$. Since $E'_W \subset E$ satisfies the conditions in the statement of Proposition 2.11 by Proposition 2.10, $w_{H^0(E_W)} (\geq 0)$. Hence by Lemma 2.10, $w_{H^0(E_W)} (\geq 0)$. By Lemma 2.7, $w_W \geq w_{H^0(X, E_W)} (\geq 0)$. Thus $(\alpha, \beta)$ is $k$-SL($V$)-(semi)stable. \hfill \square

To prove the converse of Proposition 2.11 we need the following lemma.

Lemma 2.12. Suppose that there is a subsheaf $F$ of $E$ satisfying the conditions $S^1(F) \leq 0$ and if $S^1(F) = 0$, then $S^2(F) < 0$ (respectively, $S^2(F) = 0$). Then for degree $E$ large, there exists a subsheaf $E' \subset E$ satisfying the same respective conditions and with $H^1(X, E') = 0$.

Proof. Suppose that $E$ has a subsheaf $F$ satisfying the conditions $S^1(F) \leq 0$ and if $S^1(F) = 0$, then $S^2(F) < 0$. Take $E' \subset E$ be a subsheaf of rank $n'$ and degree $\delta'$ such that

$$S^1(E') = \min \{ S^1(F) \mid S^1(F) \leq 0 \text{ and if } S^1(F) = 0, \text{ then } S^2(F) < 0 \}. \tag{2.17}$$

Suppose that $H^0(X, E'' \otimes K_X) \neq 0$, i.e., there is a nonzero homomorphism

$$\varphi : E' \rightarrow K_X.$$

Let $E'' = \text{Ker}(\varphi)$, and let $n'', \delta''$ be its rank and degree respectively. By the choice of $E'$, $S^1(E') \leq S^1(E'')$ which gives

$$\frac{\delta''}{n''} \leq \frac{\delta'}{n'} + D, \tag{2.18}$$

where

$$D := \sum_i \left[ \left( \frac{r_i'' + t_i''}{n'} - 1 \right) - \left( \frac{r_i' + t_i'}{n'} \right) + \left( \frac{r_i' - r_i''}{n} \right) \right].$$
Hence for $\delta'' = \delta' - 2g + 2$. Hence
\begin{equation}
\frac{\delta''}{n''} \geq \frac{\delta'}{n'} + \frac{2 - 2g}{n' - 1} = \frac{\delta'}{n'(n' - 1)} + \frac{2 - 2g}{n' - 1} - D + \frac{\delta'}{n'} + D.
\end{equation}

Since $S^1(E') \leq 0$, we have
\[ \frac{\delta'}{n'} \geq \frac{\delta}{n} + \sum_i \left( \frac{r'_i + t'_i}{n} \right) - \left( \frac{r'_i + t_i}{n} \right). \]

Hence for $\delta$ larger than a constant, we have
\[ \frac{\delta'}{n'(n' - 1)} + \frac{2 - 2g}{n' - 1} - D > 0. \]

Then $\frac{\delta''}{n''} > \frac{\delta'}{n'} + D$, contradicting the inequality (2.18). This proves the lemma.

In case $S^1(E') \leq 0$, and for $S^1(E') = 0$ one has $S^2(E') = 0$, we only need to change $S^2F < 0$ to $S^2(F) = 0$ in (2.14) in the choice of $E'$. \hfill $\square$

**Proposition 2.13.** The point $(\alpha, \beta)$ is $k$-$\text{SL}(V)$-(semi)stable for $k \geq k_0(n, g, \ell)$ if and only if every subsheaf $E'$ of $E$ satisfies the conditions
\[ S^1(E') \geq 0, \quad \text{and if} \quad S^1(E) = 0, \quad \text{then} \quad S^2(E) (\geq 0). \]

**Proof.** In view of Proposition 2.11, it remains to show that if $(\alpha, \beta)$ is $\text{SL}(V)$-(semi)stable, then every subsheaf $E'$ of $E$ satisfies the conditions
\[ S^1(E') \geq 0, \quad \text{and if} \quad S^1(E) = 0, \quad \text{then} \quad S^2(E) (\geq 0). \]

Suppose that there is a subsheaf $F \subset E$ such that $S^1(F) \leq 0$ and if $S^1(F) = 0$, then $S^2(F) < 0$ (respectively, $S^2(F) = 0$). By Lemma 2.12 there is a subsheaf $E' \subset E$ satisfying the same respective conditions and with $H^1(X, E') = 0$. Then for $W = H^0(X, E')$, one has $w_{W, \infty} = S^1(E')$ and $w_{W, A} = S^2(E')$. By Proposition 2.8 this implies that $w_W < 0$ (respectively, $w_W = 0$) contradicting the $\text{SL}(V)$-semistability (respectively, stability) of $(\alpha, \beta)$. \hfill $\square$

We would like to have conditions in Proposition 2.13 to be converted into conditions for subbundles of $E$. Let $E' \subset E$ be a subsheaf and $E^c$ the minimal subbundle of $E$ containing $E'$. Define a subsheaf $\widehat{E} \subset E$ by
\begin{equation}
0 \longrightarrow \widehat{E} \longrightarrow E^c \longrightarrow \oplus_i E^c_{p_i}/(E^c_{p_i} + (E^c_{p_i} \cap g_i)) \longrightarrow 0.
\end{equation}

Then $\widehat{E}$ satisfies the following conditions:

1) $\widehat{E} = E^c$ away from $p_i$,
2) $\widehat{E}_{p_i} \cap g_i = E^c_{p_i} \cap g_i$,
and since $E'_{p_i} \cap (E^c_{p_i} \cap g_i) = E'_{p_i} \cap g_i$,
3) $E'_{p_i}/(E^c_{p_i} \cap g_i) \cong \widehat{E}_{p_i}/\widehat{E}_{p_i} \cap g_i$. 
The condition 2) says that \( s_i^c = \hat{s}_i \) and the condition 3) gives \( m'_i - s'_i = \hat{m}_i - \hat{s}_i \) or equivalently,
\[
s_i^c = \hat{s}_i, \quad r'_i + t'_i = \hat{r}_i + \hat{t}_i.
\]
Moreover, \( s_i^c = \hat{s}_i \) and \( E_{p_i}^c \cap \Pi g_i \supseteq E_{p_i}' \cap \Pi g_i \), hence \( r_i^c \geq \hat{r}_i \) with equality holding if and only if \( E_{p_i}^c \cap \Pi g_i = E_{p_i}' \cap \Pi g_i \).

We have
\[
S^1(E') - S^1(\hat{E}) = \frac{\delta_0 - \delta_0'}{n'} + \frac{\sum \hat{r}_i - r'_i}{n}.
\]
From the defining sequences of \( E' \) and \( \hat{E} \), it follows that
\[
\hat{\delta}_0 - \delta_0' = \sum_i \dim \hat{E}_{p_i}/E_{p_i} + \delta(T),
\]
where \( T \) is a torsion sheaf supported outside the \( \{p_1, \ldots, p_{\ell}\} \). Hence
\[
\hat{\delta}_0 - \delta_0' = \sum_i [(m'_i + s'_i - s_i^c) - m'_i] + \delta(T) = \sum_i (s_i^c - s_i^c') + \delta(T).
\]
Therefore,
\[
S^1(E') - S^1(\hat{E}) = \sum_i \left( \frac{s_i^c - s_i^c}{n'} \right) + \frac{\sum \hat{r}_i - r'_i}{n} + \frac{\delta(T)}{n} \geq \sum_i \left( \frac{\hat{s}_i + \hat{r}_i}{n} - \frac{s_i^c + r'_i}{n} \right) + \frac{\delta(T)}{n'}
\]
with equality holding if and only if \( n = n' \). Now,
\[
\hat{s}_i + \hat{r}_i = \dim(\hat{E}_{p_i} \cap \Pi g_i), \quad s_i^c + r'_i = \dim(E_{p_i}^c \cap \Pi g_i),
\]
and hence
\[
S^1(E') \geq S^1(\hat{E}) + \frac{\delta(T)}{n'}
\]
with equality holding if and only if \( n = n' \) and \( \hat{E}_{p_i} \cap \Pi g_i = E_{p_i}^c \cap \Pi g_i \) for all \( i \).

We now compare \( S^1(\hat{E}) \) and \( S^1(E^c) \). From the sequence \((2.20)\),
\[
\hat{\delta}_0 = \delta_0^c + \sum_i [(r_i^c + t_i^c) - (r_i^c + t_i^c)].
\]
Substituting for \( \hat{\delta}_0/n' \) in \( S^1(\hat{E}) \), we have
\[
S^1(\hat{E}) = \frac{\delta_0}{n} - \frac{\delta_0^c}{n'} + \frac{\sum_i [\frac{(r_i^c + t_i^c)}{n'} - \frac{\hat{r}_i + \hat{t}_i}{n}]}{n'} = S^1(E^c) + \sum_i \frac{r_i^c - \hat{r}_i}{n} \geq S^1(E^c),
\]
with equality holding if and only if \( r_i^c = \hat{r}_i \) for all \( i \). Hence
\[
S^1(E') \geq S^1(\hat{E}) + \frac{\delta(T)}{n'} \geq S^1(E^c) + \frac{\delta(T)}{n'}
\]
with equality holding if and only if \( n = n' \), \( r_i^c = \hat{r}_i \) and \( \hat{E}_{p_i} \cap \Pi g_i = E_{p_i}^c \cap \Pi g_i \) for all \( i \).

Thus we have proved the following lemma.

**Lemma 2.14.** We have
\[
(2.21) \quad S^1(E') \geq S^1(\hat{E}) + \frac{\delta(T)}{n'} \geq S^1(E^c) + \frac{\delta(T)}{n'}
\]
with \( S^1(E') = S^1(E^c) \) if and only if \( n = n' \), \( T = 0 \) and \( E_{p_i}^c \cap \Pi g_i = E_{p_i}^c \cap \Pi g_i = E_{p_i} \cap \Pi g_i \) for all \( i \).
Theorem 2.15. There exists $k_0 = k_0(n, g, \ell)$ such that for all $k \geq k_0$, the point $(\alpha, \beta)$ given by \((E, \bar{g})\) is \(k\)-\SL(V)-(semi)stable if and only if for all subbundles \(E'\) of \(E\),

\[
0 \leq \frac{\delta_0}{n} - \frac{\delta_0'}{n'} + \sum_i \left( \frac{n' - s_i'}{n'} - \frac{n' - s_i' - t_i' + t_i}{n} \right),
\]

and, for any \(E'\) for which one has equality,

\[
0 \leq \left( \frac{\delta_0}{n} - \frac{\delta_0'}{n'} \right) - \frac{\delta_0}{n} + \sum_i \left( \frac{n' - s_i' - t_i' + t_i}{n} - \frac{1}{2} \right),
\]

and

\[
\sum_i s_i \ (\leq \frac{n}{2} - \delta_0).
\]

Proof. Suppose that \(S^1(E^c) \geq 0\) for all subbundles \(E^c \subset E\). Let \(E'\) be a subsheaf of \(E\). By Lemma 2.14 \(S^1(E') \geq 0\) and \(S^1(E') = 0\) if and only if \(n = n', T = 0\) and \(E^c_{p_i} \cap \Xi g_i = \hat{E}_{p_i} \cap \Xi g_i = E^c_{p_i} \cap \Xi g_i \) for all \(i\). Hence if \(S^1(E') = 0\), then

\[
r_i^c = \tilde{r}_i = r_i', s_i^c = \tilde{s}_i = s_i'.
\]

For \(n = n', E^c = E, r_i^c + t_i = n - s_i\). Then

\[
\sum_i \left( \frac{r_i' + t_i}{n} \right) = \sum_i \left( \frac{n - s_i}{n} \right).
\]

Therefore

\[
S^2(E') = \left( \frac{\delta_0 - \delta_0'}{n} \right) \sum_i \left( \frac{r_i' + t_i}{n} \right) - \frac{1}{2} - \frac{\delta_0}{n} = \left( \frac{\delta_0 - \delta_0'}{n} \right) \sum_i \left( \frac{n - s_i}{n} \right) - \frac{1}{2} - \frac{\delta_0}{n}.
\]

Since \((\delta_0 - \delta_0')/n > 0\),

\[
S^2(E') \ (\geq) \ 0
\]

if and only if

\[
\sum_i \left( \frac{n - s_i}{n} \right) - \frac{1}{2} - \frac{\delta_0}{n} \ (\geq) \ 0
\]

i.e.,

\[
\sum_i s_i \ (\leq) \ \frac{n}{2} - \delta_0.
\]

The theorem now follows from Proposition 2.11 taking $k >> 0 (k \geq k_0(n, g, \ell))$ and noting that \(n' = s_i' + r_i' + t_i'\) for the subbundles. \(\square\)

We now turn our attention to the action of \(\mathbb{C}^*\). The \(\SL(V)\) and \(\mathbb{C}^*\) actions on our data commute, and we can indeed take them sequentially, and in fact independently. This is what we want to do. The action on \(V \oplus \mathbb{C}^n\) has weights \((-n, \cdots, -n, p, \cdots, p)\). The action on \(\alpha\) has a fixed weight \(-n^2\); the action on each \(\beta_i\) has lowest weight \(-t_i n - r_i n + s_i p = n(s_i - n) + s_i p\), and highest weight \(-t_i n + r_i p + s_i p = -n t_i + p(n - t_i)\). For (semi)-stability,
the highest and lowest weight must bracket the origin, and this gives the (semi-)stability condition for the action of \( \mathbb{C}^* \):

\[
\begin{align*}
(2.22) \\
0 \ (\leq) \ n^2 + \eta \sum_i (n(n - s_i) - ps_i) \\
0 \ (\leq) \ - n^2 + \eta \sum_i (-nt_i + p(n - t_i)).
\end{align*}
\]

We can use different polarizations for the \( \mathbb{C}^* \)-action, and will set

\[
(2.23) \quad \eta = \frac{\gamma}{k - g + \mu}, \quad \gamma = \frac{2n}{\ell n - 2\delta_0}, \quad \mu = \gamma \ell - 2 - \frac{\delta_0}{n}.
\]

As for \( SL(V) \) we have \( k \)-(semi)-stability:

\[
(2.24) \quad 0 \ (\leq) \ n - \gamma \sum_i s_i \\
\quad \quad + \frac{\gamma}{k} \left[ \sum_i (n - (\frac{\delta_0}{n} + (2 - g))s_i) \right] + \frac{1}{k} [n(\mu - g)]
\]

\[
(2.25) \quad 0 \ (\leq) \ - n + \gamma \sum_i (n - t_i) \\
\quad \quad + \frac{\gamma}{k} \left[ \sum_i (-n + (\frac{\delta_0}{n} + (2 - g))(n - t_i)) \right] + \frac{-1}{k} [n(\mu - g)]
\]

and \( \infty \)-(semi)-stability:

\[
0 \ (\leq) \ n - \gamma \sum_i s_i \\
0 \ (\leq) \ - n + \gamma \sum_i (n - t_i).
\]

For the latter, inserting the values of the constants:

\[
(2.26) \quad 0 \ (\leq) \ \frac{\ell n}{2} - \delta_0 - \sum_{i=1}^{\ell} s_i, \\
(2.27) \quad 0 \ (\leq) \ \frac{\ell n}{2} + \delta_0 - \sum_{i=1}^{\ell} t_i.
\]

As we have noted, if one has strict inequality, one has the \( k \)-stability; let us suppose that we have the equality. The \( k \) (semi)stability condition becomes, substituting the equality:

\[
0 \ (\leq) \ n(\gamma \ell - 2 - \mu) - \delta_0.
\]

The constraint on \( \mu \) ensures that this vanishes, so that the two notions of (semi)stability coincide.

Summarizing:
**Theorem 2.16.** Set the stability parameters \((\gamma, \mu)\) as in \((2.23)\). Let \(k\) be sufficiently large. Then \((E, \vec{g})\) is \(k\)-\(\mathbb{C}^*\) (semi)stable if and only if

\[
\sum_{i=1}^{\ell} s_i \leq \frac{\ell n}{2} - \delta_0,
\]

\[
\sum_{i=1}^{\ell} t_i \leq \frac{\ell n}{2} + \delta_0.
\]

### 2.4. The moduli space.

**Theorem 2.17.** There exists a projective scheme \(\mathcal{G}M_{n,\delta_0} = \mathcal{G}M_{n,\delta_0, p_1, \cdots, p_\ell}\) which is a coarse moduli space for semistable Grassmannian framed bundles of rank \(n\) and fixed degree with Grassmannian framed structures at \(p_1, \cdots, p_\ell\).

**Proof.** In Section 2.2, we defined an \(\text{SL}(p) \times \mathbb{C}^*\)-equivariant morphism

\[
f : \tilde{R} \longrightarrow Z \subset P \times Q^\ell.
\]

Let \(\tilde{R}^{ss}\) denote the points corresponding to semistable Grassmannian framed bundles, and let \((P \times Q^\ell)^{ss}\) denote the semistable points for \(\text{SL}(p) \times \mathbb{C}^*\)-action. From Theorem 2.15 and Theorem 2.16, it follows that \(f\) induces a morphism

\[
f^{ss} : \tilde{R}^{ss} \longrightarrow Z^{ss}.
\]

In fact,

\[
Z \subset (P \times \text{Gr}_n(V^* \oplus \mathbb{C}^n)^\ell)^{ss} \subset (P \times Q^\ell)^{ss}.
\]

Using the properness of \(\text{Gr}_n(V^* \oplus \mathbb{C}^n) \subset Q\), as in [Bh1, Proposition 3], we can prove the valuative criterion of properness for the morphism \(f^{ss}\). Thus \(f^{ss}\) is proper. It is also injective and hence affine. Therefore, the existence of the quotient of the projective scheme \(Z^{ss}\) by \(\text{SL}(p) \times \mathbb{C}^*\) implies the existence of the projective scheme \(\mathcal{G}M_{n,\delta_0} = \tilde{R}/(\text{SL}(p) \times \mathbb{C}^*)\), the GIT-quotient of \(\tilde{R}\) by \(\text{SL}(p) \times \mathbb{C}^*\).

### 2.5. Relation to parabolic structures.

The canonical basis \(e_1, \cdots, e_n\) of \(\mathbb{C}^n\) defines a natural flag of subspaces \(C^1 \subset C^2 \subset \cdots \subset C^n\). Consider a pair \((E, \vec{g})\). The direct sum \(E_{p_i} \oplus \mathbb{C}^n\) has projections \(\Pi\) and \(R\) to \(E_{p_i}\) and \(\mathbb{C}^n\) respectively. One has a flag

\[
\{0\} \subset R^{-1}(C^1) \subset R^{-1}(C^2) \subset \cdots \subset R^{-1}(C^n)
\]

in \(E_{p_i} \times \mathbb{C}^n\). The plane \(g_i\) intersects this flag, and one can project the intersections, using \(\Pi\), to \(E_{p_i}\), giving a nested sequence of subspaces

\[
0 = F_{i,-1} \subset F_{i,0} = (E_{p_i} \cap g_i) \subset F_{i,1} \subset F_{i,2} \subset \cdots \subset F_{i,n} = (E_{p_i} \cap \Pi(g_i)) \subset F_{i,n+1} = E_{p_i}.\]

Note that for a subbundle \(E'\) of \(E\), one has an induced flag \(F'_i\) in \(E'_{p_i}\).

For convenience, parabolic weights will take values in the interval \([-1/2, 1/2]\) instead of \([0, 1)\) (as we will be relating these to a moment map taking values in the interval
Now choose the weight $\alpha_{i,0} = 1/2$ for $F_{i,0}$, weight $\alpha_{i,n+1} = -1/2$ for $E_{p_{i}}/F_{i,n}$ and weights $\alpha_{i,j}$ for $F_{i,j}/F_{i,j-1}$, with $1/2 > \alpha_{i,1} \geq \alpha_{i,2} \geq \cdots \geq \alpha_{i,n} > -1/2$. We have

$$s_i = \dim(F_{i,0}) = \text{multiplicity of the weight } 1/2,$$

$$t_i = \dim(F_{i,n+1}/F_{i,n}) = \{\text{multiplicity of the weight } -1/2\}, \text{ and similarly for } E'.$$

Define as usual the parabolic degree to be

$$\text{pardeg}(E) = \delta_0(E) + \sum_{i=0}^{\ell} \sum_{j=0}^{n+1} \dim(F_{i,j}/F_{i,j-1})\alpha_{i,j}.$$ 

The usual definition of parabolic (semi)stability applies, in that one asks that for a subbundle $E'$ of rank $n' < n$,

$$0 \ (\leq) \ \frac{\text{pardeg}(E)}{n} - \frac{\text{pardeg}(E')}{n'}.$$

**Proposition 2.18.** Let $(E, \overline{g})$ satisfy the conditions for $C^*$ stability, and let it be equipped with compatible flags in $E_{p_{i}}$, as above; if the result is parabolic (semi)stable for any (one) choice $\alpha$ of weights

$$\alpha_{i,0} = 1/2 > \alpha_{i,1} \geq \alpha_{i,2} \geq \cdots \geq \alpha_{i,n} > -1/2 = \alpha_{i,n+1}$$

as above, then it is (semi)stable as a Grassmannian framed sheaf.

**Proof.** One has, for any $E'$, the condition for $\alpha$-parabolic (semi)stability,

$$0 \ (\leq) \ \frac{\delta(E)}{n} - \frac{\delta(E')}{n'} + \sum_{i=1}^{\ell} \left[ \frac{1}{n} \sum_{j=0}^{n+1} \dim(F_{i,j}/F_{i,j-1})\alpha_{i,j} - \frac{1}{n'} \sum_{j=0}^{n+1} \dim(F'_{i,j}/F'_{i,j-1})\alpha_{i,j} \right].$$

The previous inequality becomes:

$$0 \ (\leq) \ \frac{\delta(E)}{n} - \frac{\delta(E')}{n'} + \sum_{i=1}^{\ell} \left[ \frac{1}{n} \sum_{j=1}^{n} \dim(F_{i,j}/F_{i,j-1})\alpha_{i,j} - \frac{1}{n'} \sum_{j=1}^{n} \dim(F'_{i,j}/F'_{i,j-1})\alpha_{i,j} + \frac{1}{2n}(s_i - t_i) - \frac{1}{2n'}(s_i' - t_i') \right].$$

We would like this to imply Grassmann-framed semistability, for any choice of $\alpha_{i,j}$ within our simplex of weights. The inequality is an affine one in the $\alpha_{i,j}$, and so it suffices to check this for the vertices of the simplex; this corresponds to considering choices of weights of the form

$$\alpha_{i,0} = \cdots = \alpha_{i,k_i} = 1/2, \alpha_{i,k_i+1} = \cdots = \alpha_{i,n+1} = -1/2.$$

For the vector bundles $E$ and $E'$, let $m_{k_i} := \dim(F_{i,k_i}/F_{i,0})$ and $m'_{k_i} := \dim(F'_{i,k_i}/F'_{i,0})$. The inequality becomes:

$$0 \ (\leq) \ \frac{\delta(E)}{n} - \frac{\delta(E')}{n'} + \sum_{i=1}^{\ell} \left[ \frac{1}{2n}(m_{k_i} - (r_i - m_{k_i}) + s_i - t_i) - \frac{1}{2n'}(m'_{k_i} - (r_i' - m'_{k_i}) + s_i' - t_i') \right].$$
The right hand side is equal to
\[
\frac{\delta(E)}{n} - \frac{\delta(E')}{n'} + \sum_{i=1}^{\ell} \left[ \frac{(n' - s'_i)}{n'} - \frac{(n' - s'_i - t'_i + t_i)}{n} \right]
\]
\[+ \sum_{i=1}^{\ell} \left[ \frac{1}{2n}(2m_{ki} - r_i + s_i - t_i + 2r'_i + 2t_i) - \frac{1}{2n'}(2m'_{ki} - r'_i + s'_i - t'_i + 2n' - 2s'_i) \right].
\]
One then needs
\[
\sum_{i=1}^{\ell} \left[ \frac{1}{2n}(2m_{ki} - r_i + s_i - t_i + 2r'_i + 2t_i) - \frac{1}{2n'}(2m'_{ki} - r'_i + s'_i - t'_i + 2n' - 2s'_i) \right] \leq 0.
\]
To prove (2.30), note that
\[
\sum_{i=1}^{\ell} \left[ \frac{1}{2n}(2m_{ki} - r_i + s_i - t_i + 2r'_i + 2t_i) - \frac{1}{2n'}(2m'_{ki} - r'_i + s'_i - t'_i + 2n' - 2s'_i) \right]
\]
\[= \sum_{i=1}^{\ell} \left[ \frac{1}{2n}(2m_{ki} - 2r_i + n + 2r'_i) - \frac{1}{2n'}(2m'_{ki} + n') \right]
\]
\[= \sum_{i=1}^{\ell} \left[ \frac{1}{2n}(2m_{ki} - 2r_i + 2r'_i) - \frac{1}{2n'}(2m'_{ki}) \right]
\]
\[\leq \sum_{i=1}^{\ell} \left[ \frac{1}{n}(m_{ki} - r_i - m'_{ki} + r'_i) \right]
\]
which is indeed less or equal to zero, since \((r_i - m_{ki}) = \dim(F_{i,n}/F_{i,k})\) and \((r'_i - m'_{ki}) = \dim(F'_{i,n}/F'_{i,k})\).

Given a semistable parabolic vector bundle \(E\), there is no difficulty in concocting a pair \((E, \bar{g})\) to which it corresponds; this, combined with Proposition 2.18 tells us that we can obtain the parabolic moduli space as a quotient of the Grassmannian-framed moduli space:

**Theorem 2.19.** One can obtain all the parabolic moduli spaces with weights
\[
1/2 \geq \delta_{i,1} \geq \delta_{i,2} \geq \cdots \geq \delta_{i,n} \geq -1/2
\]
at \(p_i\), satisfying \(\sum_{i,j} \delta_{i,j} = -\delta_0\), by quotienting a suitable subvariety in the moduli space \(\mathcal{G}\mathcal{M}_{n,\delta_0}\) under suitable subgroups of \(\text{GL}(n)^\ell\).

**Proof.** We restrict to the subvariety of framed bundles which are semistable parabolic for our choice of weights; as noted, all semistable parabolic bundles, once one has completed the flag structure to a framing, are framed semistable, and so correspond to elements of \(\mathcal{G}\mathcal{M}_{n,\delta_0}\).

When none of the weights are \(\pm 1/2\), we are in the case \(s_i = t_i = 0\) (see subsection 2.5). This means the plane \(g_i\) in the Grassmannian is the graph of a trivialization \(f_i : E_{p_i} \to \mathbb{C}^n\). Then by the above construction of a parabolic structure from the Grassmann framing,
the flag on \( \mathbb{C}^n \) gives a flag on \( E_{p_i} \) (namely the pull back by \( f_i \)) with the same weights \( \alpha_{i,j} \) with the same multiplicities. The trivializations giving the same flag are equivalent under the action on \( \mathbb{C}^n \) of the parabolic subgroup of \( GL(n) \) which stabilizes the flag and so one quotients by the action of this group.

More generally, for an arbitrary \( s_i, t_i \), the plane \( g_i \) determines two nested subspaces of \( E_{p_i} \cap g_i \subset \Pi(g_i) \) of \( E_{p_i} \). The other spaces \( \Pi(R^{-1}(C')) \) interpolate between the two, with a full flag of subspaces; one has however \( n \) nested subspaces in \( \mathbb{C}^n \) giving \( n - s_i - t_i \) subspaces in \( E_{p_i} \), and the \( j \) for which the dimensions jump depend on the position of \( g_i \). Again, in our sequence, we should collapse together the nested subspaces with the same weight \( \alpha_{i,j} \).

Let us consider first the case \( s_i = 0 \). The plane is then the graph of a map \( \varphi_i : \mathbb{C}^n \rightarrow E_{p_i} \), and the flag is given by the images \( \varphi_i(\mathbb{C}^1), \varphi_i(\mathbb{C}^2), \ldots , \varphi_i(\mathbb{C}^n) \). If one has a \( t_i \)-dimensional subspace \( T_i = g_i \cap \mathbb{C}^n \) of \( \mathbb{C}^n \) as the kernel of \( \varphi_i \), the flag one obtains depends how \( T_i \) intersects with the standard flag in \( \mathbb{C}^n \). We choose this intersection to be maximal, i.e., \( T_i = \mathbb{C}^{t_i} \); this amounts to considering the closed orbits within each equivalence class of semi-stable orbits for the GIT construction. The flags are then the images \( \varphi_i(\mathbb{C}^{t_i+1}) \subset \varphi_i(\mathbb{C}^{t_i+2}) \subset \cdots \) and our moduli space is the quotient of the subvariety with \( s_i = 0, T_i = \text{span} \{ e_1, \ldots , e_{t_i} \} \) by the action of the parabolic subgroup of \( GL(n) \) which fixes a sub-flag of \( \mathbb{C}^{t_i} \subset \mathbb{C}^{t_i+1} \subset \mathbb{C}^{t_i+2} \subset \cdots \), the choice of sub-flag being determined by the coincidence pattern of the weights \( \delta_{i,j} \) for each \( i \).

For \( s_i \) arbitrary, one still has the graph of a map \( \varphi \) of \( \mathbb{C}^n \) into not \( E_{p_i} \); in addition \( \text{ker}(\varphi_i) = \mathbb{C}^{t_i+s_i} \). The flag in \( E_{p_i} \) is then a sub-flag of the pullback to \( E_{p_i} \) of \( \varphi_i(\mathbb{C}^{t_i+s_i+1}) \subset \varphi_i(\mathbb{C}^{t_i+s_i+2}) \subset \cdots \). Our moduli space is the quotient of the subvariety with \( s_i, t_i \) fixed, \( T_i = \mathbb{C}^{t_i}, \text{ker}(\varphi_i) = \mathbb{C}^{t_i+s_i} \), by the action of the parabolic subgroup of \( GL(n) \) which fixes a sub-flag of \( \mathbb{C}^{t_i+s_i} \subset \mathbb{C}^{t_i+s_i+1} \subset \mathbb{C}^{t_i+s_i+2} \subset \cdots \). Again, the choice of sub-flag is determined by the coincidence pattern of the weights \( \delta_{i,j} \) for each \( i \).

3. Extended moduli spaces, and their Grassmannian version

3.1. Extended moduli spaces. We now turn to the description of the extended moduli spaces of Jeffrey, as explained in [Je], and then describe their Grassmannian compactifications. Let

\[
X^* := X \setminus \{ p_1, \ldots , p_t \}
\]

be the punctured Riemann surface. Parametrize disjoint neighborhoods of the punctures as semi-infinite cylinders, with complex coordinate \( r + \sqrt{-1}\theta \), \( r \in [0, \infty) \), \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \). Choose points \( \tilde{p}_i \) given by \( (r, \theta) = (1, 0) \), thought of as close to their respective punctures. We consider the space \( EM_n \) of equivalence classes of flat unitary connections on \( X^* \), which are of the form \( \sqrt{-1}\delta d\theta \) on the semi-infinite cylinders, where \( \sqrt{-1}\delta \) is some constant skew hermitian matrix. Here the equivalence is given by gauge transformations which are the
of such forms, a closed skew form element. For smoothness, one can isolate any one of the terms \( \exp(2\pi i) \) for the \( \delta \) form near the punctures, modulo compactly supported gauge transformations; this gets diagram that they fit into, given in (3.6) of the proof of (3.1) in [Je]. The main space is

\[
\Omega (\sigma_1, \sigma_2) = -\int_{X^*} \tr (\sigma_1 \wedge \sigma_2).
\]

Jeffrey shows that the variety \( EM_n \) is smooth, and that the form \( \Omega \) is non-degenerate, for the \( \delta_i \) in a neighborhood of the origin, indeed in the neighborhood of any central element. For smoothness, one can isolate any one of the terms \( \exp(2\pi \sqrt{-1} \delta_i) \) in the defining equation (3.2), and so the variety has the form of a graph and is smooth, as long as one is at a point at which the exponential map is locally bijective. This holds for the \( \delta_i \) whose eigenvalues lie in \((-1/2, 1/2)\). Another locus at which the variety is smooth is that of irreducible representations. The form can degenerate when the stabilizer of one of the \( \exp(2\pi \sqrt{-1} \delta_i) \) differs from (and so is larger than) that of \( \delta_i \). Indeed let \( h_{1,i} \) be the stabilizer in \( u(n) \) of \( \exp(2\pi \sqrt{-1} \delta_i) \), and \( h_{2,i} \) be the stabilizer of \( \sqrt{-1} \delta_i \); Let \( s_i = h_{1,i} \cap h_{2,i} \); then the null space for \( \Omega \) is tangent to the distribution spanned by the action of \( \sum_i s_i \).

To see this, we recall from Jeffrey [Je] the various deformation spaces in play, and the diagram that they fit into, given in (3.6) of the proof of (3.1) in [Je]. The main space is the tangent space of deformations of the relevant flat connections with correct asymptotic form near the punctures, modulo compactly supported gauge transformations; this gets

\[
\prod_{j=1}^g ([a_j, b_j]) c_1 c_2 c_2^{-1} \cdots c_\ell = 1.
\]

One can integrate the connections. We note that there are implicit trivializations at each of the \( \tilde{p}_i \); these trivializations extend naturally to \( r \in [0, \infty) \), \( \theta \in (-\pi, \pi) \). In particular, the integration of the connections along each of the paths \( a, b, c \) is well defined; along the paths \( d_i \), the integral is simply \( \exp(2\pi \sqrt{-1} \delta_i) \). Our space \( EM_n \) is then the space of elements \( A_j, B_j, j = 1, \ldots, g, C_i, i = 2, \ldots, \ell \), of \( U(n) \) and \( \sqrt{-1} \delta_i, i = 1, \ldots, \ell \), of \( u(n) \) satisfying

\[
\prod_{j=1}^g ([A_j, B_j]) \exp(2\pi \sqrt{-1} \delta_1) C_2 \exp(2\pi \sqrt{-1} \delta_2) C_2^{-1} \cdots C_\ell \exp(2\pi \sqrt{-1} \delta_\ell) C_\ell^{-1} = 1.
\]

An element of \( EM_n \) can be represented either as a triple \((\tilde{E}, \nabla, f)\) consisting of a unitary bundle \( \tilde{E} \), a unitary flat connection \( \nabla \) of the form \( \sqrt{-1} \delta_i d\theta \) near the punctures, and a unitary framing \( f = (f_1, \cdots, f_\ell) \) near the punctures, alternately, as a tuple \((A_j, B_j, C_i, \delta_i)\) representing the holonomies. Under the first representation, the infinitesimal deformations of the moduli space are given by covariant constant \( u(n) \)-valued one-forms \( \sigma \) which are locally constant near the punctures, and of the form \( a_i d\theta \). One then has, for a pair \( \sigma_1, \sigma_2 \) of such forms, a closed skew form

\[
\Omega (\sigma_1, \sigma_2) = -\int_{X^*} \tr (\sigma_1 \wedge \sigma_2).
\]
expressed as a cohomology group $H^{1,0}(X^*) =: H^{1,0}$. Inside this space there is a space $H^1_c(X^*) =: H^1_c$ of compactly supported deformations; the quotient $H^{1,0}/H^1_c$ maps to $\mathfrak{u}(n)^\ell$, by taking values at the boundary. On the union $S$ of the boundary circles of $X^*$, one has the space $H^0(S)$ of covariant constant sections of the adjoint bundle, as well as the dual space $H^1(S)$; the space $H^0(S)$ contains the space $H^0(X^*)$ of covariant constant sections over the whole punctured curve. One also has the space of deformations $H^1$ of all flat connections, modulo all gauge transformations, on $X$. One has the diagram (3.6) of [2], fitting all of these spaces together:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^1_c & \longrightarrow & H^{1,0} & \longrightarrow & \mathfrak{u}(n)^\ell \\
\downarrow & & \downarrow \| & & \downarrow \tau & & \downarrow \sigma \\
H^0(X^*) & \xrightarrow{\beta} & H^0(S) & \xrightarrow{\gamma} & H^1_c & \longrightarrow & H^1 \\
\end{array}
$$

(3.3)

Note that the arrows on the bottom are indeed duals, using Poincaré duality. Let $\psi$ be a smooth function that is one on the ends of the curve, and is zero on its interior; one can find an inverse to the map $b$ over $\bigoplus_i \mathfrak{h}_{2,i}^\bot$ by associating to an element $h$ in $\mathfrak{h}_{2,i}^\bot$, the deformation $f(h) = d_A(\psi(ad(\delta_i)^{-1}h))$, where $d_A$ is the covariant derivative of the connection. These elements are supported over the ends; they are coboundaries in $H^1$, and so are mapped to zero by $\tau$. On the other hand, the elements of $\bigoplus_i \mathfrak{h}_{2,i}$ map to non-zero elements of $H^1(S)$. Finally, we note that the space $H^0(S)$ is spanned by elements $Ad(\exp(\theta\delta_i))(s), s \in \mathfrak{h}_{1,i}$; the elements corresponding to $s \in \mathfrak{h}_{2,i}$ are constant. These elements map to $-d\psi Ad(\exp(\theta\delta_i))(s)$ in $H^1_c$.

Now consider the form $\Omega(a, \cdot)$ on $H^{1,0}$; if $\tau(a)$ is non-zero, then the form is non-degenerate, as $H^1_c$ is Poincaré dual to $H^1_c$. The kernel of $\tau$ is spanned, in turn, by $f(\bigoplus_i \mathfrak{h}_{2,i}^\bot)$ and by $\gamma(H^0(S))$. From the explicit form of elements in $f(\bigoplus_i \mathfrak{h}_{2,i}^\bot)$, one can check that the form restricted to this subspace is non-degenerate. There remains the elements of $H^0(S)$. Consider those corresponding to elements of $\mathfrak{h}_{2,i}$; they are constants $s$, and map to $-sd\psi$ in $H^1_c$ under $\gamma$. On the other hand, the map $\sigma$ gives us from elements $s$ in $\mathfrak{h}_{2,i}$ elements $sd\theta$ in $H^1(S)$; if these elements come from elements of $H^1_c$, one then has a non zero pairing, as $tr(s^2)$ is non-zero. For this to be the case, $\beta^*(sd\theta)$ must vanish. Pairing with elements $\alpha$ of $(H^2_c)^* = H^0(X^*)$, this tells us that $s$ should be orthogonal to the image of $H^0(X^*)$ in $H^0(S)$. This tells us that the pairing is non degenerate on the image $\gamma(\mathfrak{h}_{2,i}) \subset H^1_c$.

There remains the subspaces of $H^0(S)$ corresponding to $s_i$; and indeed the form can degenerate on these.

Jeffrey also shows that the parabolic moduli spaces, in their symplectic description, can be obtained as symplectic quotients of $EM_n$, for weights in the open interval $(-1/2, 1/2)$ (our Grassmannian moduli space will allow us to extend this to the closed interval). For the moment, we note that associated to each puncture $p_i$, there is a natural action of $U(n)$ on the trivialization at $\tilde{p}_i$. In terms of the parametrization above, if $(g_1, \cdots, g_\ell) \in U(n)^\ell$, then...
this action is given by
\[(A_j, B_j, C_i, \delta_i) := \{(A_j), \{B_j\}, \{C_i\}, \{\delta_i\}\} \mapsto (g_1A_jg_1^{-1}, g_1B_jg_1^{-1}, g_1C_ig_1^{-1}, g_1\delta_ig_1^{-1})\].
The action is Hamiltonian, with moment map
\[\nu_{U(n)}(A_j, B_j, C_i, \delta_i) = \sqrt{-1}(\delta_1, \cdots, \delta_\ell)\].
The parabolic moduli for \(\delta_i\) such that \(\text{Stab}(\exp(2\pi\sqrt{-1}\delta_i)) = \text{Stab}(\sqrt{-1}\delta_i)\) is then the symplectic quotient \(\nu_{U(n)}^{-1}(\prod_j \mathcal{O}_{\sqrt{-1}\delta_j})/U(n)^\ell\).

We note that as a consequence of the defining constraint (3.2), we have
\[\sum_\ell \text{tr}(\delta_i) \in \mathbb{Z}\].
The actual value of the sum is minus the degree \(\delta_0\) of the eventual holomorphic bundle that we will build. The integer values of \(\delta_0\) split the moduli space into components \(EM_{n,\delta_0}\).

3.2. Grassmannian extended moduli. We have for the action of \(U(n)^\ell\) on \(EM_n\) at the punctures the moment map:
\[\nu_{U(n)}(A_j, B_j, C_i, \delta_i) = \sqrt{-1}(\delta_1, \cdots, \delta_\ell)\]
(see [Je]).

Now consider the Grassmannian \(Gr_m(m+n)\) of \(m\)-planes in \(\mathbb{C}^{m+n}\). The Kähler form on this manifold can be given either by restricting the canonical Fubini-Study form on projective space, or as the Kostant-Kirillov form on the Grassmannian thought of as the coadjoint orbit of \(\lambda = \sqrt{\frac{i}{2}}(-1, \cdots, -1, 1, \cdots, 1)\), under the action of \(U(m+n)\). The coadjoint orbit is then (writing \(\mathbb{C}^{m+n}\) as \(\mathbb{C}^m \oplus \mathbb{C}^n\)):
\[
\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{\sqrt{-1}}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \right\} = \frac{\sqrt{-1}}{2} \begin{pmatrix} -aa^* + bb^* & -ac^* + bd^* \\ -ca^* + db^* & -cc^* + dd^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(m+n)
\]
The \(U(m+n)\) moment map for this is simply the identity map, i.e., the inclusion, and so the moment map for the action of \(U(m) \times \{1\}\) is \(\sqrt{\frac{i}{2}}(-aa^* + bb^*)\), and the moment map for the action of \(I \times U(n)\) is \(\sqrt{\frac{i}{2}}(-cc^* + dd^*)\).

**Lemma 3.1.** 1) Representing the generic element of the Grassmannian \(Gr_m(m+n)\) of \(m\)-planes in \(\mathbb{C}^{m+n}\) as the graph of a linear transformation \(\gamma : \mathbb{C}^m \rightarrow \mathbb{C}^n\), the moment map for \(U(m) \times \{1\}\) acting on \(Gr_m(m+n)\) (the “right action”) in this parametrization is then
\[\mu_1(\gamma) = -\frac{\sqrt{-1}}{2}aa^* + \frac{\sqrt{-1}}{2}bb^* = \frac{\sqrt{-1}}{2}(-I + \gamma^*\gamma)(I + \gamma^*\gamma)^{-1};\]
that for \(I \times U(n)\) (the “left action”) is
\[\mu_2(\gamma) = -\frac{\sqrt{-1}}{2}cc^* + \frac{\sqrt{-1}}{2}dd^* = \frac{\sqrt{-1}}{2}(I - \gamma\gamma^*)(I + \gamma\gamma^*)^{-1}.\]
2) Representing an element of the Grassmannian as the elements annihilated by the \( n \) orthonormal rows of a matrix \((b^*,d^*)\), the moment map for the action of \( U(m) \times \{I\} \) is

\[
\mu_1(b^*,d^*) = \frac{\sqrt{-1}}{2} (-I + 2bb^*),
\]

and that of \( I \times U(n) \) is

\[
\mu_2(b^*,d^*) = \frac{\sqrt{-1}}{2} (-I + 2dd^*).
\]

**Proof.** If we represent the generic element of the Grassmannian \( \text{Gr}_m(m+n) \) as the graph of \( \gamma : \mathbb{C}^m \rightarrow \mathbb{C}^n \), the \( \ell \)-plane is spanned by the columns of

\[
\begin{pmatrix}
I \\
\gamma
\end{pmatrix}
\]

or, equivalently, by the (mutually orthogonal) columns of

\[
\begin{pmatrix}
I \\
\gamma
\end{pmatrix} (I + \gamma^*\gamma)^{-1/2} = \begin{pmatrix} a \\ c \end{pmatrix}.
\]

The orthogonal complement of this is spanned by

\[
\begin{pmatrix}
-I \\
\gamma
\end{pmatrix} (I + \gamma^*\gamma)^{-1/2} = \begin{pmatrix} b \\ d \end{pmatrix}.
\]

We then apply the formulae. In the same vein, the correspondence between the coadjoint orbit and the Grassmannian is by taking our \( m \)-plane to be the \(-\sqrt{-1}/2\) eigenspace of the matrices in the orbit under the action of \( U(n+m) \); parametrizing as above, the eigenspace is annihilated by the rows of \((b^*,d^*)\). On the other hand, as the matrix is unitary, we have \( aa^* + bb^* = I, cc^* + dd^* = I \), and so

\[
\mu_1(b^*,d^*) = \frac{\sqrt{-1}}{2} (-aa^* + bb^*) = \frac{\sqrt{-1}}{2} (-I + 2bb^*),
\]

\[
\mu_2(b^*,d^*) = \frac{\sqrt{-1}}{2} (-cc^* + dd^*) = \frac{\sqrt{-1}}{2} (-I + 2dd^*).
\]

We note that the image of both moment maps is the set of skew hermitian matrices with eigenvalues in the interval \( \sqrt{-1} \cdot [-1/2, 1/2] \).

We now want to replace framings by their graphs. We do this by acting by \( U(n)\ell \) diagonally on \( EM_{n,\delta_0} \times \text{Gr}_n(2n)\ell \), with the right action (by \((U(n) \times \{I\})\ell \)) on the Grassmannians, and reducing at zero. This means that one considers elements \((A_i, B_i, C_i, \delta_i)(b_i^*, d_i^*)\) lying in the zero locus \( M^{-1}(0) \) of the moment map, that is, satisfying

\[
\delta_i = \frac{1}{2}(I - 2b_i b_i^*),
\]

and quotients by the action of \( U(n)\ell \). 

We note that the diagonal \( S^1 \) of \((U(n) \times \{I\})\ell \) acts trivially \( EM_{n,\delta_0} \) (but not on \( \text{Gr}_n(2n)\ell \)); the quotient by \((U(n) \times \{I\})\ell \) can also be thought of as a quotient by \( S((U(n) \times \{I\})\ell \times S^1) \), where \( S^1 \) is the diagonal \( S^1 \) of \((I \times U(n))\ell \).
Proposition 3.2. The symplectic quotient at zero
\[ GM_{n,\delta_0} = (EM_{n,\delta_0} \times \text{Gr}_n(2n)^\ell) / U(n)^\ell \]
is smooth when
1) at least one of the \( \delta_i \) has all its eigenvalues in \((-1/2, 1/2)\), or
2) when the representation is irreducible, and at least one eigenvalue of at least one \( \delta_i \) is not \( \pm 1/2 \).

Over the locus where the moment map is submersive, the form is symplectic over the quotient.

The right action of the group \( U(n)^\ell \) descends to the quotient \( GM_{n,\delta_0} \), with moment map
\[ (A_j, B_j, C_i, \delta_i)(b^*_i, d^*_i) \mapsto \sqrt{-1} \frac{I - 2d_i d^*_i}{2} . \]

Proof. We first note that \( EM_{n,\delta_0} \) is smooth when either condition 1) or 2) is satisfied. The stabilizer in \( U(n)^\ell \) of an element in \( EM_{n,\delta_0} \) is a diagonal embedding of the automorphism group of the representation; on the other hand, conditions 1 or 2) then guarantee that the action of this automorphism group on the corresponding elements in the Grassmannians is free. The smoothness statement follows. If the moment map on the product is submersive at a point, one then has in the usual way that the degeneracy locus for the symplectic form restricted to \( M^{-1}(0) \) is precisely the \( U(n)^\ell \) orbit, and so the form on the quotient is indeed non degenerate, even if it is not on one of the factors. The action of \( U(n)^\ell \) on the quotient simply follows from the two commuting \( U(n) \) actions on the Grassmannian. \( \square \)

Let \( s_i \) denote the dimension of the \( 1/2 \)-eigenspace of \( \delta_i \), and \( t_i \) the dimension of the \(-1/2 \)-eigenspace. The relation (3.4) tells us that \( s_i \) is the dimension of the kernel of \( b^*_i \), and \( t_i \) is the dimension of the subspace of \( \mathbb{C}^n \) of vectors whose norm is preserved by \( b^*_i \); this dimension is then the dimension of the kernel of \( d^*_i \), as the rows of \( (b^*_i, d^*_i) \) are orthonormal. Thus, \( s_i \) is the dimension of the intersection of the plane with the first copy of \( \mathbb{C}^n \) in \( \mathbb{C}^{2n} \), and \( t_i \) is the intersection of the plane with the second copy. We have
\[
\sum_{i=1}^\ell \text{tr}(I/2 + \delta_i) = \frac{n\ell}{2} - \delta_0 .
\]
This value implies the constraints
\[
\sum_{i=1}^\ell s_i \ (\leq) \ \frac{\ell n}{2} - \delta_0 ,
\]
\[
\sum_{i=1}^\ell t_i \ (\leq) \ \frac{\ell n}{2} + \delta_0 ,
\]
as in (2.28), (2.29). We note that the relation (3.4) relates the trace of \( \delta_i \) to that of \( b_i b^*_i \) and so of \( b^*_i b_i \). The relation \( b^*_i b_i + d^*_i d_i = I \) (orthonormality of the rows of \( (b^*_i, d^*_i) \)) then ties this to the trace of \( d^*_i d_i \) and so of \( d_id^*_i \).
4. The correspondence

There is a natural map, associating to any element \((\tilde{E}, \nabla, f, \tilde{g})\) of \(GM_{n,\delta_0}\) a pair

\[ C(X) = (E, g) \]

consisting of a bundle \(E\) on \(X\) and Grassmannian framings \(g_i\) at \(p_i\). To do this, we note that the flat unitary bundle gives a holomorphic bundle \(\tilde{E}\) in a natural way on the punctured curve \(X^* = X \setminus \{p_1, \ldots, p_\ell\}\). One then must extend this to a holomorphic bundle on \(X\). This is done by first choosing a holomorphic coordinate \(z\) with \(z = 0\) corresponding to \(p_i\), taking a covariant constant unitary basis \(f_i\) for \(\tilde{E}\) on the set \(r \in [0, \infty), \theta \in (-\pi, \pi)\) for which the matrix \(\delta_i\) is diagonal with eigenvalues \(\frac{1}{2} \geq \delta_{i,1} \geq \delta_{i,2} \geq \cdots \geq \delta_{i,n} \geq -\frac{1}{2}\). One then extends the bundle to \(D\) by glueing \(\tilde{E}\) to the trivial bundle over the disk via a transition function \(\text{diag}(z^{\delta_{i,1}}, z^{\delta_{i,2}}, \ldots, z^{\delta_{i,n}})\). In other words, on the overlap, the basis \(e_{i,j}\) of the trivial bundle over the disk is identified with \(z^{-\delta_i} f_{i,j}\), where \(f_{i,j}\) is the corresponding element of the unitary basis \(f_i\). This gives us in addition a flag in the fiber at \(p_i\), whose \(k\)-th space \(F_{i,k}\) is the span of the first \(k\) vectors of the basis \(e_{i,j}, j = 1, \ldots, n\), those decaying fastest at the origin in terms of our original unitary basis. We recall that this construction of the bundle is the one used for parabolic bundles. In terms of the original trivialization, one could also say that the glueing is by the matrix \(z^{-\delta_i}\), using the matrix exponential.

This defines the bundle; we now consider the Grassmann framing. The element

\[(\tilde{E}, \nabla, f, \tilde{g})\]

contains unitary covariant constant trivializations

\[ f_i : E \longrightarrow \mathbb{C}^n \]

over the set \(r \in [0, \infty), \theta = 0\) near the \(i\)-th puncture. We transfer this (rescaling) to a trivialization \(e_i\) over the real half line in a disc surrounding the puncture \(p_i\), by composing \(f_i\) with \(z^{-\delta_i}\); the rescaling ensures that we have a well defined trivialization in the limit. This then allows us to transfer our element \(\tilde{g}_i\) in the Grassmannian \(\text{Gr}_n(\mathbb{C}^n \oplus \mathbb{C}^n)\) to an element \(g_i\) in the Grassmannian \(\text{Gr}_n(E_{p_i} \oplus \mathbb{C}^n)\); if \(\tilde{g}_i\) is the graph of a map \(\tilde{\gamma}_i : \mathbb{C}^n \longrightarrow \mathbb{C}^n\),

\[ g \text{ is the graph of } \gamma_i = \tilde{\gamma}_i \cdot z^{-\delta_i} \cdot f_i. \]

There is another useful piece of information that one can transfer to the fiber \(E_{p_i}\): a “renormalized Hermitian form” \(h\), given by taking the hermitian form \(\tilde{h}\) on \(\tilde{E}\), and conjugating it by \(z^{\delta_i}\),

\[ h = (z^{-\delta_i})^* \tilde{h} z^{-\delta_i}. \]

The framing \(e_i\) is orthonormal with respect to this new form; we also note that taking care to take the correct adjoints, if \(g\) is the graph of \(\gamma_i = \tilde{\gamma}_i \cdot z^{-\delta_i} \cdot f_i\), then

\[ \gamma_i \gamma_i^* = \tilde{\gamma}_i \tilde{\gamma}_i^*. \]
Proposition 4.1. The resulting pair \((E, g)\) is polystable. It is stable if the representation is irreducible.

Proof. When the weights \(\delta_{i,j}\) lie in the open interval \((-1/2, 1/2)\), by standard results for parabolic bundles, the bundle is parabolic polystable, and stable if irreducible. It is then, as we have seen in Proposition 2.18, Grassmann-framed polystable, and stable in case of parabolic stability. Indeed, the same arguments can apply when the weights take values at the ends of the interval: the parabolic degree of the bundle is simply the integral of the trace of the curvature over the punctured curve; any subbundle of the flat bundle \(E\) will have negative or zero curvature over \(X^*\), and so the parabolic degree of the subbundle is bounded above by zero. \(\Box\)

We have built a map

\[
C : GM_{n,\delta_0} \longrightarrow GM_{n,\delta_0}.
\]

The next step, naturally, is to show that this map is a homeomorphism. As the spaces \(GM_{n,\delta_0}\) are compact, it suffices that the map be bijective.

To obtain surjectivity, given an element of \(GM_{n,\delta_0}\), we want to consider a parabolic structure associated to the framed structure, and also choose some appropriate weights for which this is stable, and then exploit known results on parabolic bundles to build a flat connection on the complement of the punctures, with appropriate monodromies. The reason for going through the parabolic structures is that the correspondence between holomorphic bundles and flat connections involves a degenerating metric at the punctures, whose decay rate is controlled by the weights; to do the analysis, one needs the weights, but then once one has the weights, one essentially has the parabolic structure, and so one might as well use those results.

Thus one is led to the question of which weights. The left hand side of (4.1), as we saw, has a natural action of \(U(n)\), as well as a moment map, given by (3.2). On the other hand, as we saw, the resulting element of \(GM_{n,\delta_0}\) is encoded as an element \((\alpha, \beta)\) of \(P \times Q^\ell\); this is a Kähler manifold, once one has fixed a Hermitian form on \(V\), and so on \(V^* \oplus \mathbb{C}^n\). In \(Q\), in particular, one is looking at the Plücker embedding of the Grassmannian of \(n\) dimensional planes in \(V^* \oplus \mathbb{C}^n\). The group \(U(n)\) acts on the \(\mathbb{C}^n\)-factors, and so on the Grassmannians \(\text{Gr}_n(V^* \oplus \mathbb{C}^n)\). This action is by isometries.

We have some choice, that of a positive Hermitian form on \(V\). For the moment, suppose that we are at an element \((E, \vec{g})\) in the image of the map \(C\) in (4.1), and so the fibers \(E_{p_i}\) have a positive Hermitian form. We choose a form on \(V\) in a compatible way, so that

\[
ev : V \longrightarrow \bigoplus_i E_{p_i}\]

is an orthogonal projection.
Proposition 4.2. Let us consider a pair \((E, g)\) lying in the image of \(C\), corresponding to an element \((\alpha, \beta) = (\alpha, (\beta_1, \cdots, \beta_\ell))\) of \(P \times Q^\ell\). The moment map for the action of \(U(n)^\ell\) on \((\alpha, \beta)\) coincides with the one on \(GM_{n,\delta_0}\) under the map \(C\).

If the \(g_i\) are graphs of maps \(\xi_i : V \to \mathbb{C}^n\), it is given by

\[
\mu_{n,\ell}(\alpha, \beta) = \frac{-i}{2}((I - \xi_i\xi_i^*)(I + \xi_i\xi_i^*)^{-1}, \cdots, (I - \xi_\ell\xi_\ell^*)(I + \xi_\ell\xi_\ell^*)^{-1});
\]

alternately, if the \(\beta_i\) are the top exterior powers of the orthonormal rows of \(n \times (p + n)\)-matrices \((b_i^*, d_i^*)\) in a basis for \(V\), the moment map is given by:

\[
\mu_{n,\ell}(\alpha, \beta) = \frac{-i}{2}((-I + 2d_1d_1^*), \cdots, (-I + 2d_\ell d_\ell^*)).
\]

Proof. The symplectic structure for the Grassmannian moment map under the Plücker embedding is the same as for its identification as a coadjoint orbit. We can apply Lemma 3.1 This tells us that the moment map for the action of \(U(n)\) on \((\alpha, \beta)\), if the planes corresponding to \(\beta_i\) are indeed graphs of maps \(\xi_i\), is

\[
\frac{-i}{2}(I - \xi_i^*)(I + \xi_i^*)^{-1}.
\]

We note that \(\xi_i\) is the composition \(\gamma_i \cdot ev_{p_i}\), and so this then equals:

\[
\frac{-i}{2}(I - \gamma_i\gamma_i^*)(I + \gamma_i\gamma_i^*)^{-1},
\]

and we have shown that the map is indeed the one on \(GM_{n,\delta_0}\); likewise for the other representation of the element \(\beta_i\).

Fix now the Hermitian form on \(V\). Our strategy is to show that the Grassmann-framed bundle is stable, as a parabolic bundle, for the choice of weights given by the formula above; we then have the results of Biquard, Poritz et al [Bi, Po] which give us the flat connection we need, and show that our map is surjective.

To do this, we exploit the equivalence between algebraic and Kähler quotients, as expounded by Mumford, Guillemin and Sternberg, [MFK Appendix 2C], [Ki, p. 102]. We note that our previous construction of \(\mathcal{GM}_{n,\delta_0}\) can be viewed as a Kähler quotient: we have an action of \(U(p) \times S^1\) on some Kähler manifolds, with moment maps \(\mu_p, \mu_{S^1} = \sqrt{-1}\sum_i tr(\delta_i)\), and \(\mathcal{GM}_{n,\delta_0}\) can be obtained as \(\mu_p^{-1}(0) \cap \mu_{S^1}^{-1}(-\sqrt{-1}\delta_0)/(U(p) \times S^1)\). We now note, referring to the previous proposition, that the moment map \(\mu_{n,\ell}\) is invariant under \(U(p)\). Therefore, writing \(\mathcal{GM}_{n,\delta_0}\) as the \(U(p) \times S^1\) quotient, the moment map \(\mu_{n,\ell}\) descends to \(\mathcal{GM}_{n,\delta_0}\):

\[
\mu_{n,\ell} : \mathcal{GM}_{n,\delta_0} \to u(n)^\ell
\]

and \(\mathcal{GM}_{n,\delta_0}\) satisfies the \(\mu_{S^1}\) moment constraint \(\sum_i tr(\delta_i) + \delta_0 = 0\). We can fix a coadjoint orbit \(\prod_i \mathcal{O}_{\sqrt{-1}\delta_i}\) in \(u(n)^\ell\), and take the symplectic quotient

\[
(\mathcal{GM}_{n,\delta_0} \times \prod_i \mathcal{O}_{\sqrt{-1}\delta_i})/\!/PU(n)^\ell.
\]
Proposition 4.3. This symplectic quotient is a family of parabolic bundles corresponding to the choice of weights; in other words, given the parabolic bundle corresponding to an element in $\mu_{n,\ell}^{-1}(\delta_1, \ldots, \delta_\ell)$, one can reconstruct an element of $\mathcal{GM}_{n,\delta_0}$ which is unique up to the action of the stabilizer of the $\delta_i$.

Proof. On the level of the $(\alpha, \beta) \in \mu_p^{-1}(0)$ representing elements of $\mathcal{GM}_{n,\delta_0}$, we have $\mu_{n,\ell}(\alpha, \beta) = \sqrt{-1}(\delta_1, \ldots, \delta_\ell)$. One then quotients by the stabilizer of the $(\delta_1, \ldots, \delta_\ell)$, as well as by $U(p)$ . Let us just work on one puncture at a time. We take

$$\delta_i = \text{diag}(\delta_{i,1}, \ldots, \delta_{i,n}),$$

with the $\delta_{i,j}$ in decreasing order in $j$. Suppose that they have blocks of size $n_{i,0} = s_i, n_{i,1}, \ldots, n_{i,k_i}, n_{i,k_i+1} = t_i$, corresponding to the subsets of equal eigenvalues; $s_i = n_{i,0}$ is the size of the block of $1/2$ eigenvalues, and can vanish. Likewise $t_i = n_{i,k_i+1}$ is the size of the block of $-1/2$ eigenvalues, and can vanish; the other block sizes are supposed to be positive.

The essential point is being able to reconstruct the element of the Grassmannian from the flag and the weights; to do this, we will construct a normal form for the $\beta_i$. One can choose $\beta_i$ to be of norm one, and write it as the top exterior power of the orthonormal rows of an $n \times (p + n)$-matrix in a basis for $V$. Our codimension $n$ plane $g_i$ will be the kernel of this matrix. Let us choose the basis of $V$ so that $K$, the kernel of the evaluation map $V \to E_{p_i}$ is the span of the first $p - n$ vectors; we drop $K$ from now on, and work directly on $E_{p_i} \oplus \mathbb{C}^n$ so that we have a $n \times 2n$ matrix $(b^*, d^*)$ which we can think of as a map

$$\rho^*: E_{p_i} \oplus \mathbb{C}^n \to \mathbb{C}^n.$$

Our moment map is for the $U(n)$ action on the source $\mathbb{C}^n$; one is free to normalize using the $U(n)$ actions on $E_{p_i}$, and on the target $\mathbb{C}^n$. We have the orthogonality relation

$$\rho^* \rho = b^* b + d^* d = \mathbf{I},$$

as well as the moment map relation

$$\frac{1}{2}(-\mathbf{I} + 2dd^*) = \delta_i = \text{diag}(\delta_{i,1}, \ldots, \delta_{i,n}).$$

In this basis, write the source $\mathbb{C}^n$ as a sum $U \oplus U'$, with $U'$ equal to the kernel of $d^*$, corresponding to the last $t_i$ vectors of the basis. In the same way, use the action of $U(n)$ on $E_{p_i}$ to take the induced flag in $E_{p_i}$ to consist of the first $s_i + r_i$ vectors of our basis, with $g_i \cap E_{p_i}$ corresponding to the first $s_i$ vectors, and denote the space generated by the next $r_i$ vectors as $W'$; choose as a complementary subspace the space $g_i^+ \cap E_{p_i}$ of dimension $t_i$. Similarly, let the target $\mathbb{C}^n$ be decomposed as an orthogonal sum $\rho^*(g_i^+ \cap \mathbb{C}^n) \oplus W \oplus \rho^*(g_i^+ \cap E_{p_i})$ with dimensions $s_i, r_i, t_i$. We note that using $\rho^* \rho = \mathbf{I}$, one has that

1) $\rho^*(\mathbb{C}^n) \subset ((\rho^*(g_i^+ \cap \mathbb{C}^n) \oplus W),$

2) $\rho^*(E_{p_i}) \subset (W \oplus \rho^*(g_i^+ \cap E_{p_i})).$
3) In addition, as the first $s_i + r_i$ vectors $e_i$ of the basis for $E_{p_n}$ are such that there are elements $v_j$ of $\mathbb{C}^n$ with $(e_j + v_j) \in g_i = \ker(\rho^*)$, these vectors map to $W$.

Putting these together, we then have the form for $\rho^*$, mapping

$$E_{p_n} \oplus \mathbb{C}^n = (g_i \cap E_{p_n}) \oplus W' \oplus (g_i^\perp \cap E_{p_n}) \oplus U \oplus U'$$

to $\mathbb{C}^n = \rho^*(g_i^\perp \cap \mathbb{C}^n) \oplus W \oplus \rho^*(g_i^\perp \cap E_{p_n})$:

$$\rho^* = \begin{pmatrix} 0 & 0 & 0 & d_1^* & 0 \\ 0 & M & 0 & d_2^* & 0 \\ 0 & 0 & I & 0 & 0 \end{pmatrix}.$$ 

We can further normalize $M$: first use the $U(r_i)$ action (on the right on $M$) to make $M^*M$ a positive, diagonal matrix; this then means that $MD$ is unitary for a positive diagonal matrix $D$; now use the left $U(n)$ action to map $MD$ to the identity, and so $M$ to a positive diagonal matrix. Now consider the product

$$(d_1 \quad d_2) \begin{pmatrix} 0 & d_1^* \\ M & d_2^* \end{pmatrix} \begin{pmatrix} 0 & M \\ d_1 & d_2 \end{pmatrix}.$$ 

As $\rho^*\rho = I$, the product of the second and third factor is $I$. Doing the product in two different orders gives

$$(d_1 \quad d_2) = \left( \left( \frac{1}{2} + \tilde{\delta}_i \right)d_1 \quad d_2M^2 + \left( \frac{1}{2} + \tilde{\delta}_i \right)d_2 \right);$$

here $\tilde{\delta}_i$ is $\text{diag}(\delta_{i,1}, \cdots, \delta_{i,s_i+r_i})$. Therefore $d_1^* \left( \frac{1}{2} + \tilde{\delta}_i \right) = 0$, and so $d_1^*$ is of the form $(\hat{d}_1^*, 0)$, where now $\hat{d}_1^*$ is unitary, in $U(s_i)$; using the left $U(s_i)$ action, we can set it to the identity. From $\rho^*\rho = I$, one also has $d_2^*d_1 = 0$, which tells us that $d_2^*$ is of the form $(0, \hat{d}_2^*)$, with $\hat{d}_2^*$ an $r_i \times r_i$-matrix.

Now one has

$$\hat{d}_2^* = M^2\hat{d}_2^* + \hat{d}_2^*\left( \frac{1}{2} + \tilde{\delta}_i \right),$$

with $\hat{\delta}_i = \text{diag}(\delta_{i,s_i+1}, \cdots, \delta_{i,s_i+r_i})$, or $M^2\hat{d}_2^* = \hat{d}_2^*(\frac{1}{2} - \tilde{\delta}_i)$. Since $\hat{d}_2^*\hat{d}_2^*$ is an invertible matrix, $\det(\hat{d}_2^*) \neq 0$, and so there is a permutation matrix $\hat{S}$ such that $S\hat{d}_2^*$ has nonzero diagonal entries. One then has

$$(SM^2S^{-1})S\hat{d}_2^* = S\hat{d}_2^*(\frac{1}{2} - \tilde{\delta}_i).$$

Since $SM^2S^{-1}$ is again diagonal, we might as well assume $S = 1$, giving

$$M^2\hat{d}_2^* = \hat{d}_2^*(\frac{1}{2} - \tilde{\delta}_i)$$

with a $\hat{d}_2^*$ having non-zero diagonal entries. This forces $M^2 = (\frac{1}{2} - \tilde{\delta}_i)$, as well as ensuring that outside of the diagonal blocks of size $n_{i,1}, \cdots, n_{i,k_i}$, corresponding to the subsets of equal eigenvalues of $\delta_i$, the entries of $\hat{d}_2^*$ are zero. One then can use the action of the
The stability condition is that these elements are non-zero. For elements of \( \text{Gr} \) representing an element \((E, \overrightarrow{g})\) from (4.2), representing an element of the flag manifold into a product of Grassmannians, for any vector space \( V \), the subscripts denote the codimensions of the planes. One has an embedding of the symplectic quotient into a product of parabolic bundles.

From an algebraic point of view, we are quotienting the product \( \mathcal{G}M_{n, \delta_0} \times \prod_{r} \mathcal{O}_{-\sqrt{-1} \delta_i} \) by an action of \( \text{GL}(n, \mathbb{C})^\ell \); this quotient is stratified by the dimensions of intersections \( s_i, t_i \) with \( E_{p_i}, \mathbb{C}^n \). We note that from a complex point of view, the coadjoint orbit \( \mathcal{O}_{-\sqrt{-1} \delta_i} \) is a flag manifold

\[
E_{0,i} = g_i \cap E_{p_i} = \mathbb{C}^{n_i,0}, \quad E_{1,i} = \Pi(R^{-1}(\mathbb{C}^{(n_i,0+n_i,1)})) = \mathbb{C}^{(n_i,0+n_i,1)} \ldots ,
\]

in normalized position; we note that the intersections \( R^{-1}(\mathbb{C}^i) \) with the plane \( g_i \) are of generic dimension.

The normal form also tells us that given the flag in \( E_{p_i} \), and the weights \( \delta_i \), one can reconstruct the element of the Grassmannian, up to the action of the stabilizer of \( \delta_i \). In short, the symplectic quotient is indeed the family of parabolic bundles.

From an algebraic point of view, we are quotienting the product \( \mathcal{G}M_{n, \delta_0} \times \prod_{r} \mathcal{O}_{-\sqrt{-1} \delta_i} \) by an action of \( \text{GL}(n, \mathbb{C})^\ell \); this quotient is stratified by the dimensions of intersections \( s_i, t_i \) with \( E_{p_i}, \mathbb{C}^n \). We note that from a complex point of view, the coadjoint orbit \( \mathcal{O}_{-\sqrt{-1} \delta_i} \) is a flag manifold

\[
\text{Fl}_i(\mathbb{C}^n) = \text{Fl}_{(n-n_i,0),(n-n_i,0-n_i,1),\ldots,(n-n_i,0-\ldots-n_i,k_i)}(F) \subset \prod_{j=0}^{k_i} \text{Gr}_{n-n_i,0-\ldots-n_i,j}(F).
\]

Here the subscripts denote the codimensions of the planes. One has an embedding of the flag manifold into a product of Grassmannians, for any vector space \( F \):

\[
\text{Fl}_i(F) = \text{Fl}_{(n-n_i,0),(n-n_i,0-n_i,1),\ldots,(n-n_i,0-\ldots-n_i,k_i)}(F) \subset \prod_{j=0}^{k_i} \text{Gr}_{n-n_i,0-\ldots-n_i,j}(F).
\]

Representing an element \((E, \overrightarrow{g})\) by \((\alpha, \beta)\), and elements of \( \mathcal{O}_{-\sqrt{-1} \delta_i} \) (flags) by elements \( \gamma_{i,j}, j = 1, \ldots, k_i \) of \( \Lambda^{(n_i,0+\ldots+n_i,j)}(\mathbb{C}^n) \), the quotienting is achieved precisely as above:

\[
(\beta_i, \gamma_{i,j}) \mapsto \Pi(I(\gamma_{i,j})(\beta_i)) \overset{\text{def}}{=} \eta_{i,j} \in \Lambda^{n-n_i,0-\ldots-n_i,j}(V^*) .
\]

The stability condition is that these elements are non-zero. For elements of \( \text{Gr}_n(V \oplus \mathbb{C}^n) \) corresponding to maps \( f : \mathbb{C}^n \rightarrow E_{p_i} \), this map simply takes the flag \( h_i \) defined by \( \gamma_{i,j}, j = 1, \ldots, k_i \), to the flag \( f(h_i) \).

The resulting elements \((\alpha, \eta_{i,j})\) are precisely the defining elements of a bundle with quasi-parabolic structure. On the level of line bundles, the map

\[
\psi_i : [\text{Gr}_n(V \oplus \mathbb{C}^n) \times \text{Fl}_i(\mathbb{C}^n)]_s \rightarrow \text{Fl}_i(V)
\]

that we have defined (the subscript \( s \) denotes the stable locus) pulls back the standard positive line bundles \( L_j = \Lambda^{\mathcal{J}(Taut^+)} \) on the factors \( \text{Gr}_j(V) \) in (4.2) to the tensor product \( L_n \otimes L_j \) on \( \text{Gr}_n(V \oplus \mathbb{C}^n) \times \text{Fl}(\mathbb{C}^n) \), where \( L_n \) is the standard ample line bundle on \( \text{Gr}_n(V \oplus \mathbb{C}^n) \). To see this, one pulls back a divisor representing \( L_j \): the divisor of planes
$g$ meeting a fixed $j$ plane $g'$ nontrivially. We take $g'$ to correspond to a $j$-plane $\widehat{g'}$ in $E_{p'\gamma}$. Over the set of planes in $\text{Gr}_n(V \oplus \mathbb{C}^n)$ corresponding to maps $f : \mathbb{C}^n \rightarrow E_{p'\gamma}$, this pull-back divisor is given for $h \in \text{Gr}_{n-j}(\mathbb{C}^n)$ by the constraint $f(h) \cap \widehat{g'} \neq 0$.

**Proposition 4.4.** Let $(\alpha, \beta)$ correspond to $(E, \vec{g}) \in \mathcal{G}\mathcal{M}_{n,\delta}$, with $\mu_p(\alpha, \beta) = 0$, and suppose that it corresponds to a stable element. The moment map $\mu_{n,\ell}$ of Proposition 4.2 applied to $(\alpha, \beta)$ gives elements $\sqrt{-1}\delta_i$ of $u(n)^\ast$ with eigenvalues $\sqrt{-1}\delta_{i,j}$. Then the parabolic bundle defined by the $(\alpha, \eta_{i,j})$ associated to $(\alpha, \beta)$ is parabolic semistable, for the weights $\delta_{i,j}$.

**Proof.** Let us suppose that $\mu_{n,\ell}(\alpha, \beta) = \sqrt{-1}(\delta_1, \ldots, \delta_\ell)$, with $\sqrt{-1}\delta_j$ belonging to a coadjoint orbit $\mathcal{O}_{\sqrt{-1}\delta_j}$. Then

$$(\alpha, \beta, -\sqrt{-1}\delta_i) \in (\mu_p \times \mu_{n,\ell})^{-1}(0)$$

and so is semistable for the action of $S(\text{GL}(p) \times \text{GL}(n)^\ell)$; moreover, its orbit is closed. We can then quotient by the action of $S(\text{GL}(n)^\ell)$; the result is still $\text{SL}(p)$-semistable.

Before declaring that we are done, we must check that the polarizations match on both sides of $\times_{i=1}^\ell \psi_i$. Recall that $Q \supset \text{Gr}_n(V \oplus \mathbb{C}^n)$; the restriction of $\mathcal{O}_Q(1)$ to $\text{Gr}_n(V \oplus \mathbb{C}^n)$ is its standard positive line bundle (the dual of the top exterior power of the tautological bundle), we denote it again by $\mathcal{O}(1)_Q$. The map $\psi_i$, as we saw, pulls back $\otimes_k L^{\alpha_{i,k}}$ to the line bundle $T_1 \cdots T_\ell$ corresponding to the block of size $n_{i,j}$ in $\delta_i$. Note that $\delta_i^0 = 1/2, \delta_{i,k}^{k+1} = -1/2$. The standard choice of polarization on $P \times (\prod_i F_i(V))$ for parabolic bundles is the bundle

$$\mathcal{O}_P(1)^\rho \boxtimes (\Box_i (\otimes_{j=0}^{k_{i,j}} L^{(\delta_{i,j}^0-\delta_{i,j}^{k_{i,j}})})),$$

for $\rho = k-g+1/2$. (In [Bh1] [MS], one has $\rho = k-g$; however, one can check that shifting the parabolic weights from $[0, 1]$ to $[-1/2, 1/2]$ requires the change to $\rho = k-g+1/2$.)

Now pull back to $Z = \prod_i \mathcal{O}_{\sqrt{-1}\delta_i} \subset P \times \prod_i (Q \times \mathcal{O}_{\sqrt{-1}\delta_i})$. The result is

$$\mathcal{O}_P(1)^\rho \boxtimes (\Box_i \mathcal{O}_Q(1)) \boxtimes (\Box_i (\otimes_{j=0}^{k_{i,j}} L^{(\delta_{i,j}^0-\delta_{i,j}^{k_{i,j}})})),$$

The last term is the standard polarization on the coadjoint orbit $\prod_i \mathcal{O}_{\sqrt{-1}\delta_i}$; one has the correct line bundles for the $Q$ factors, and the line bundles on $P$ match also.

As our element, after projection, lies in $\mu_p^{-1}(0)$, it is semistable, with a closed orbit, and so satisfies the standard parabolic stability criterion.

One now has that the element $(\alpha, \beta)$ either yields us a parabolic stable bundle, or that it is semi-stable, but not stable. In the latter case one finds that it is polystable. This is a consequence of the orbit being closed. Indeed, let $(\alpha, \eta_{i,j})$ represent a semistable parabolic bundle $E$, with $W \subset V$ representing a destabilizing subbundle $E'$. One then has a subspace $W^\perp \subset V^\ast$; choose a complementary subspace $U$. At each point of $X$, the element $\alpha$ is represented by a product $w_1 \wedge \cdots \wedge w_k \wedge (w_{k+1} + u_{k+1}) \wedge \cdots \wedge (w_n + u_n)$, $w_i \in W^\perp, u_i \in U.$
One can act by \( \mathbb{C}^* \) so that projectively, the limit element is \( w_1 \wedge \cdots \wedge w_k \wedge u_{k+1} \wedge \cdots \wedge u_n \). For the bundle, this amounts to rescaling the extension class of \( E' \to E \to E/E' \) to zero. Similar considerations hold for the \( \eta_{i,j} \), so that the limit object is a sum of parabolic bundles. In other words, if an extension class (in the sense of parabolic bundles) is nontrivial, then the orbit is not closed.

**Theorem 4.5.** The correspondence \( C \) is bijective, and commutes with reduction to parabolic structures.

*Proof.\* We fix a Hermitian form on \( V \). Given an element \( (E, \vec{g}) \) of \( \mathcal{G} \mathcal{M}_{n, \delta_0} \), the basic problem is to find a flat unitary connection on the punctured surface with the appropriate singularities. Represent the element \( (E, \vec{g}) \) by an \( (\alpha, \beta) \) in \( \mu_{-1}(0) \); now apply the moment map \( \mu_{n,\ell} \) to find the weights \( \delta_{i,j} \). Proposition 4.4 tells us that this corresponds to a parabolic bundle for these weights. Let us first concentrate on the case when all the \( s_i \) are zero, so that there are no weights equal to \( 1/2 \). This implies that the spread of the weights is less than one and then the results for parabolic bundles (see [Bi], [Po]) give us the flat connection, with the right residues at the puncture.

Now suppose that there is a subspace \( F_0 \) of \( E_p \) with weight \( 1/2 \). We take a Hecke transform \( \tilde{E} \) as the subsheaf of sections of \( E(p) \) whose polar part lies in \( F_0 \). This Hecke transforms does not affect stability: Let \( \delta'_1, \cdots, \delta'_r \) be the distinct elements among \( \delta_1, \cdots, \delta_n \) with \( \delta'_1 > \cdots > \delta'_r \). Let

\[
0 \subset F_0 \subset F_1 \subset \cdots \subset F_r = E_p
\]

the corresponding flag with weights

\[
1/2 > \delta'_1 > \cdots > \delta'_r.
\]

Instead of this flag one has the flag

\[
0 \subset F_1/F_0 \subset \cdots \subset F_r/F_0 \subset \tilde{E}_p.
\]

with weights

\[
\delta'_1 > \cdots > \delta'_r \geq -1/2.
\]

One has that the parabolic degree of \( E \) equals the parabolic degree of \( \tilde{E} \) (with the shifted weights) and the same holds for subbundles of \( E \). Hence \( E \) is parabolic semistable if and only if \( \tilde{E} \) is. Now again the spread of the weights is less than one, and one can use the result on parabolic bundles to produce a flat connection for \( \tilde{E} \); shifting back (in the space of flat connections, taking a Schlesinger transformation) gives us the connection we want on \( E \).

This tells us that the map \( C \) in (1.1) is surjective: given an element of \( \mathcal{G} \mathcal{M}_{n, \delta_0} \), one has a flat connection corresponding to it. To see that the map is injective, suppose that a framed bundle \( (E, \vec{g}) \) is the image of two elements of \( \mathcal{G} \mathcal{M}_{n, \delta_0} \); the fact that the weights are determined by a map on \( \mathcal{G} \mathcal{M}_{n, \delta_0} \) tells us that these two elements have the same parabolic weights; but then they must correspond to the same parabolic bundle, by the injectivity of the correspondence on the level of parabolic bundles. The two elements are
then in the same orbit of the stabilizer in $U(n)$ of the element $(\delta_1, \cdots, \delta_\ell)$ corresponding to the weights. However, the map $C$ commutes with this action, and indeed maps orbits bijectively to orbits; thus the images of the two elements in question must be different. □

5. Generalized parabolic bundles

5.1. Bundles on nodal curves and generalized parabolic structures. Given a bundle $E$ on $X$ with framings at $\ell$ pairs of points $(p_i, q_i)$, there is a natural way of associating to it a bundle on the singular curve $\hat{X}$ given by identifying the points $p_i, q_i$ of each pair: the framings allow us to identify $E_{p_i}$ with $E_{q_i}$. Alternately, this identification gives, via its graph, a plane in $E_{p_i} \oplus E_{q_i}$, and more generally, one could hope for a way of associating to an element of $\mathcal{G}M_{n, \delta_0}$ a pair consisting of a bundle $E$ and a vector $\vec{g}$ of $n$-planes $g_i$ in $E_{p_i} \oplus E_{q_i}$.

Pointwise, this procedure is fairly clear: let us consider a bundle $E$ with vectors $\vec{g}^p = (g^p_1, \cdots, g^p_\ell), \vec{g}^q = (g^q_1, \cdots, g^q_\ell)$ with $g^p_i \in \text{Gr}_n(E_{p_i} \oplus \mathbb{C}^n), g^q_i \in \text{Gr}_n(E_{q_i} \oplus \mathbb{C}^n)$. Let

$$R^p_i : E_{p_i} \oplus \mathbb{C}^n \to \mathbb{C}^n \quad \text{and} \quad R^q_i : E_{q_i} \oplus \mathbb{C}^n \to \mathbb{C}^n$$

be the projections and $R_i = R^p_i \oplus R^q_i$. We suppose that:

$$R^p_i(g^p_i) + R^q_i(g^q_i) = \mathbb{C}^n$$

(5.1)

$$g^p_i \cap \mathbb{C}^n \cap g^q_i = 0.$$  

(5.2)

Let

$$g_i = \{(e_p, e_q) \in E_{p_i} \oplus E_{q_i} \mid \exists b \in \mathbb{C}^n \text{ with } (e_p, b) \in g^p_i, (e_q, b) \in g^q_i\}.$$  

In other words,

$$g_i = R_i((g^p_i \oplus g^q_i) \cap (E_{p_i} \oplus E_{q_i} \oplus \Delta)),$$

where $\Delta$ denotes the diagonal in $\mathbb{C}^n \oplus \mathbb{C}^n$. Note that (5.1) tells us that $g^p_i \oplus g^q_i$ and $E_{p_i} \oplus E_{q_i} \oplus \Delta$ span the full space $E_{p_i} \oplus E_{q_i} \oplus \mathbb{C}^n \oplus \mathbb{C}^n$; the intersection of these two spaces is then $n$-dimensional. The projection $g_j$ of this intersection to $E_{p_j} \oplus E_{q_j}$ is also $n$-dimensional, as it intersects the kernel of the projection map trivially, by (5.2). We note that $g_i$ is invariant under the diagonal action of $\text{GL}(n)$ on the $\mathbb{C}^n$s associated to $p_i, q_i$.

For example, if $g^p_i \in \text{Gr}_n(E_{p_i} \oplus \mathbb{C}^n)$ is the graph of $f^p_i \in \text{Hom}(E_{p_i}, \mathbb{C}^n)$ and $g^q_i \in \text{Gr}_n(E_{q_i} \oplus \mathbb{C}^n)$ is the graph of $f^q_i \in \text{Hom}(\mathbb{C}^n, E_{q_i})$, then $g_i \in \text{Gr}_n(E_{p_i} \oplus E_{q_i})$ is the graph of $f^p_i \circ f^q_i \in \text{Hom}(E_{p_i}, E_{q_i})$.

Algebraically, in terms of the data which defines $(E, \vec{g}^p, \vec{g}^q)$ as in Section 2, one has the natural map $\Lambda^n(V^* \oplus \mathbb{C}^n) \times \Lambda^n(V^* \oplus \mathbb{C}^n) \to \Lambda^{2n}(V^* \oplus \mathbb{C}^n)$. Since $(\Lambda^nV^* \otimes \Lambda^n\mathbb{C}^n)$ is a direct summand of $\Lambda^{2n}(V^* \oplus \mathbb{C}^n)$, we have a projection map $\Lambda^{2n}(V^* \oplus \mathbb{C}^n) \to \Lambda^nV^* \otimes \Lambda^n\mathbb{C}^n$. The restriction of the composite of these two maps gives the map described above. If $(\alpha, \beta_1^p, \cdots, \beta_\ell^p, \beta_1^q, \cdots, \beta_\ell^q)$ is the algebraic data encoding $(E, \vec{g}^p, \vec{g}^q)$, then $(\alpha, \beta) = (\beta_1, \cdots, \beta_\ell)$ encodes $(E, \vec{g})$, where

$$\beta_j = i(v)(\beta_j^p \wedge \beta_j^q) \in \Lambda^n(V^*).$$
Here \( v \) is the (co)volume element in \( \Lambda^n(\mathbb{C}^n) \). The constraint \( R_i^p(g_i^p) + R_i^q(g_i^q) = \mathbb{C}^n \) ensures that \( \beta_i^p \wedge \beta_i^q \) is non-zero, and the constraint \( g_i^p \cap \mathbb{C}^n \cap g_i^q = 0 \) then tells us that \( \beta_i \) does not vanish.

On the level of moduli spaces, there is a space, constructed in [Bh2], classifying pairs \((E, \vec{g})\). We briefly recall the construction. (In fact, the construction in [Bh2] is more general, but we restrict our attention to this particular case.) As above, fix disjoint divisors \( D_i = p_i + q_i, i = 1, \ldots , \ell \), where \((p_i, q_i)\) is a pair of distinct points of \( X \). Let \( E \) denote a vector bundle of rank \( n \), degree \( \delta_0 \) on \( X \). Let \( g_i \subset E_{p_i} \oplus E_{q_i} \) be an \( n \)-dimensional subspace and \( \vec{g} = (g_1, \ldots , g_{\ell}) \). Such a pair \((E, \vec{g})\) is called a generalized parabolic bundle (with parabolic structure over the divisors \( D_i \)); let us abbreviate to GPB.

There is a notion of (semi)stability of GPBs, analogous to that of parabolic bundles: as for parabolic bundles, there are weights, and the relevant ones here are \( \alpha_1 = 1/2, \alpha_2 = -1/2 \). For a subbundle \( E' \) of \( E \) of rank \( n' \), degree \( \delta_0' \), we set the parabolic degree to be

\[
\text{pardeg}(E') = \delta_0' + \sum_{i=1}^{\ell} \alpha_1 \dim((E'_{p_i} \oplus E'_{q_i}) \cap g_i) + \alpha_2 \dim(E'_{p_i} \oplus E'_{q_i} / ((E'_{p_i} \oplus E'_{q_i}) \cap g_i))
\]

\[
= \delta_0' + \sum_{i=1}^{\ell} (\dim((E'_{p_i} \oplus E'_{q_i}) \cap g_i) - n').
\]

(5.3)

The definition of semistability is then the usual one, using the parabolic degree to define slopes.

Let \( R, \mathcal{E}, \mathcal{E}_{p_i}, \mathcal{E}_{q_i}, \vec{R} \) be as in Section 2.2 (but with half the points \( p \) relabelled as \( q \)). For \( k \) sufficiently large, \( R \) contains the underlying bundles of all semistable GPBs. Let

\[\text{Gr}_n(\mathcal{E}_{p_i} \oplus \mathcal{E}_{q_i}) \longrightarrow R\]

be the Grassmannian bundle whose fibers are isomorphic to the Grassmannian of \( n \)-planes in the sum \( \mathcal{E}_{p_i} \oplus \mathcal{E}_{q_i} \). Let \( \text{Gr}_n(\mathcal{E}) \) be the fiber product of \( \text{Gr}_n(\mathcal{E}_{p_i} \oplus \mathcal{E}_{q_i}), i = 1, \ldots , \ell \), over \( R \). We denote the total space of \( \text{Gr}_n(\mathcal{E}) \) by \( R^{\text{par}} \). A point of \( R^{\text{par}} \) corresponds to a GPB \((E, \vec{g})\). The moduli space \( \mathcal{M}^{\text{par}}(n, \delta_0) \) is a GIT-quotient of \( R^{\text{par}} \) by \( \text{SL}(p) \).

5.2. Symplectic version. As in Section 3, one can build a symplectic version of the moduli space of GPBs. The starting point is again the space \( EM_n \) of flat connections on the complement of the points \( p_i, q_i \), framed at the punctures. We had a symplectic action of \( U(n)^{2\ell} \) on \( EM_n \), via the framings. One has \( U(n) \) acting simultaneously on the framing at \( p_i \) and on the Grassmannian \( \text{Gr}_n(\mathbb{C}^n \oplus \mathbb{C}^n) \), acting here on the first \( \mathbb{C}^n \); similarly, one has an action of \( U(n) \) on the framing at \( q_i \), and on the Grassmannian \( \text{Gr}_n(\mathbb{C}^n \oplus \mathbb{C}^n) \), now acting on the second copy of \( \mathbb{C}^n \). We take the symplectic quotient

\[\mathcal{M}^{\text{par}}(n) = (EM_n \times \text{Gr}_n(\mathbb{C}^n \oplus \mathbb{C}^n)^\ell) \big/ (U(n)^{2\ell}).\]

This gives \( \delta_{p_i} = -\delta_{q_i} \), with eigenvalues in the interval \([-1/2, 1/2]\), when all the eigenvalues are in \((-1/2, 1/2)\), the elements of the Grassmannian are graphs of maps \( \mathbb{C}^n \longrightarrow \mathbb{C}^n \).
which map eigenspaces of $\delta_{p_i}$ to eigenspaces of $\delta_{q_i}$ with the eigenvalues of the respective eigenspaces summing to zero.

As in Section 4, we can define a map 

$$C : M^{gpar}(n) \rightarrow M^{gpar}(n, 0);$$

retracing the steps of Section 4, one should be able to show that this is an isomorphism, but we will leave this discussion for elsewhere.

This map indicates what the Narasimhan-Seshadri correspondence should be for nodal curves. If $p_i, q_i$ in a desingularization $X$ of the curve are the pairs of points corresponding to the nodes, semistable vector bundles should correspond to singular unitary connections $\nabla$ to the nodes, semistable vector bundles should correspond to singular unitary connections $\nabla$ on the punctured curve $X^*$, with holonomies $\exp(2\pi \sqrt{-1}\delta_{p_i}), \exp(2\pi \sqrt{-1}\delta_{q_i})$, with $\delta_{p_i} = -\delta_{q_i}$ having eigenvalues in $(-1/2, 1/2)$, and unitary isomorphisms between the eigenspaces of $\delta_{p_i}$ and those of $\delta_{q_i}$, with the corresponding eigenvalues summing to zero.

### 5.3. Relations between $\mathcal{G}M_{n,\delta_0}$ and $M^{gpar}(n, \delta_0)$

Let us now consider the moduli space $\mathcal{G}M_{n,\delta_0}$ for the $2\ell$ marked points $p_i, q_i$; as above, we write an element of this moduli space as a triple $(E, g^p, g^q)$. The group $\text{GL}(n)^\ell$ acts on $\mathcal{G}M_{n,\delta_0}$, with the $i$-th copy of $\text{GL}(n)$ acting diagonally on the $\mathbb{C}^n$’s associated to $p_i, q_i$. We consider the quotient $\mathcal{G}M_{n,\delta_0}/\text{GL}(n)^\ell$, with the natural polarization on the product of Grassmannians $\text{Gr}_n(V \oplus \mathbb{C}^n)$.

**Lemma 5.1.** The condition (5.2) above, $g^p_i \cap \mathbb{C}^n \cap g^q_i = 0$, is a consequence of semistability for the action of $\text{GL}(n)^\ell$.

**Proof.** If $g^p_i \cap \mathbb{C}^n \cap g^q_i$ is non empty, let $e_1$ be a non-zero element of the intersection, and complete to a basis $e_i$ of $\mathbb{C}^n$. The elements $\beta_{p_i}, \beta_{q_i}$ of $\Lambda^n(V^* \oplus \mathbb{C}^n)$ describing the Grassmannian framing are of the form

$$\beta_{p_i} = b_1 \wedge (c_2 e_2^* + b_2) \wedge \cdots \wedge (c_n e_n^* + b_n), \quad \beta_{q_i} = b'_1 \wedge (c'_2 e_2^* + b'_2) \wedge \cdots \wedge (c'_n e_n^* + b'_n).$$

Here the $c_j, c'_j$ are constants, and $b_j, b'_j$ are elements of $V^*$. One can then take a 1-parameter subgroup of $S(\text{GL}(p) \times \text{GL}(n)^\ell)$ taking $\alpha, \beta_{p_i}, \beta_{q_i}$ to zero, essentially by putting positive weight on $e_1^*$, and negative weight everywhere else. \hfill $\square$

Lemma 5.1 implies that on the semi-stable locus, the projection

$$R_i((g^p_i + g^q_i) \cap (E_{p_i} \oplus E_{q_i} \oplus \Delta))$$

will always be at least $n$-dimensional, and if one defines the variety

$$Z = \{(E', \bar{g}^p, \bar{g}^q), (E, \bar{g}) \in (\mathcal{G}M_{n,\delta_0}/\text{GL}(n)^\ell) \times M^{gpar}(n, \delta_0) \mid E = E', g_i \subset R_i((g^p_i + g^q_i) \cap (E_{p_i} \oplus E_{q_i} \oplus \Delta))\},$$

(5.4)

one obtains a closed subvariety of the product.

Define

$$(\mathcal{G}M_{n,\delta_0})_{\text{gen}} = \{(E, \bar{g}^p, \bar{g}^q) \in \mathcal{G}M_{n,\delta_0} \mid R_i^p(g^p_i) + R_i^q(g^q_i) = \mathbb{C}^n\}.$$
One has that the variety $Z$ over this locus, or rather the quotient of its semi-stable locus by $S(\text{GL}(n)^t)$, is the graph of a morphism $\varphi$, provided that the image is semistable as a generalized parabolic bundle. We note that since we have already taken quotient by $\mathbb{C}^*$, only the action of $\text{GL}(n)^t/\mathbb{C}^*$ remains, hence a quotient by $S(\text{GL}(n)^t)$. Moreover, this quotient has the correct dimension $n^2(g - 1) + 1 + n^2\ell = \dim \mathcal{M}^{gpar}(n, \delta_0)$.

Specializing a bit further, let us consider the locus

$$(\mathcal{GM}_{n,\delta_0})_{\text{gen},0} = \{(E, \bar{g}^p, \bar{g}^t) \in \mathcal{GM}_{n,\delta_0} \mid s_p = s_q = t_p = t_q = 0\}$$

of framed bundles within $\mathcal{GM}_{n,\delta_0}$.

**Proposition 5.2.** Let $(E, \bar{g})$ be an element of $\mathcal{M}^{gpar}(n, \delta_0)$ for which the planes $g_i$ are the graphs of isomorphisms. If $(E, \bar{g})$ is stable, then there is a unique element $(E, \bar{g}^p, \bar{g}^t)$ of $(\mathcal{GM}_{n,\delta_0})_{\text{gen},0}/(\text{GL}(n)^t) \subset \mathcal{GM}_{n,\delta_0}/(\text{GL}(n)^t)$ corresponding to it in the variety $Z$, so that $\varphi(E, \bar{g}^p, \bar{g}^t) = (E, \bar{g})$.

If $(E, \bar{g})$ is only semistable, then the same holds, provided that $\delta_0 \leq 2\ell + (n/2)$.

**Proof.** Let $g_i$ be the $i$-th element of $\bar{g}$; it corresponds to a homomorphism $\rho_i : E_{p_i} \to E_{q_i}$. There are elements $g^p_i, g^q_i$ corresponding to linear maps $\rho_{p_i} : E_{p_i} \to \mathbb{C}^n$, $\rho_{q_i} : E_{q_i} \to \mathbb{C}^n$ with $\rho_i = \rho_{q_i}^{-1} \circ \rho_{p_i}$; these elements are unique up to the action of $\text{GL}(n)$.

If the element $(E, \bar{g})$ is stable, one has, for a subbundle of rank $n'$,

$$0 < \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \frac{\sum_{i=1}^\ell (n' - \dim((E'_{p_i} \oplus E'_{q_i}) \cap g_i))}{n'}.$$

The dimension of the intersection $\dim((E'_{p_i} \oplus E'_{q_i}) \cap g_i)$ is bounded below by $\max(0, 2n' - n)$, giving

$$0 < \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \frac{\ell(n - n')}{n'} \leq \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \frac{2\ell(n - n')}{n}$$

and so, for $2n' \geq n$,

$$0 < \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \frac{\ell(n - n')}{n'} \leq \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \frac{2\ell(n - n')}{n}$$

and for $2n' \leq n$,

$$0 < \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \ell \leq \frac{\delta_0}{n} - \frac{\delta'_0}{n'} + \frac{2\ell(n - n')}{n}.$$

These are the stability inequalities for our Grassmann framings, when $s_i, t_i = 0$.

For the semistability case, one has the same inequalities, but not strict; there is the additional condition $2.3$ which holds for $\delta_0 \leq 2\ell + (n/2)$. \hfill \□

The moduli spaces $\mathcal{GM}_{n,\delta_0}, \mathcal{M}^{gpar}(n, \delta_0)$ are compact; this then gives:

**Theorem 5.3.** There is a birational map $\varphi$ between $\mathcal{GM}_{n,\delta_0}/(\text{GL}(n)^t)$ and $\mathcal{M}^{gpar}(n, \delta_0)$.
5.4. The symplectic point of view. Let us now fix the degree $\delta_0$ of $(E, \bar{g})$ to be zero. From a symplectic point of view, one has the action of the diagonal $U(n)^{\ell}$ in $(U(n) \times U(n))^{\ell}$ on $GM_{n,\delta_0}$. Taking the symplectic quotient, the values of the moment maps $\mu_i$ at the punctures $p_i, q_i$ take opposing values; one then quotients by the diagonal action of $U(n)$ for each pair.

This quotient gives us in a natural way an element of $M^{par}(n, 0)$. Indeed, if the planes $g_{p_i}, g_{q_i}$ are cut out by matrices $(b_{p_i}^*, d_{p_i}^*), (b_{q_i}^*, d_{q_i}^*)$, as in Section 3 and Section 4, with orthonormal rows $(b_{p_i}^* b_{p_i} + d_{p_i}^* d_{p_i} = I, b_{q_i}^* b_{q_i} + d_{q_i}^* d_{q_i} = I)$ the moment map constraint gives us

$$d_{p_i} d_{p_i}^* + d_{q_i} d_{q_i}^* = I.$$  

Elements $(v_{p_i}, w_{p_i})$ of the plane $g_{p_i} \subset E_{p_i} \oplus \mathbb{C}^n$ are given by the constraint

$$(5.5) \quad (b_{p_i}^* \quad d_{p_i}^*) \begin{pmatrix} v_{p_i} \\ w_{p_i} \end{pmatrix} = 0,$$

with a similar relation for $(v_{q_i}, w_{q_i}) \in g_{q_i}$. Now let us normalize the $d_{p_i}, d_{q_i}$; one can first, by a unitary transformation $u_i$, have $u_i d_{p_i}^* u_i^{-1} = D_i$, where $D_i$ is a diagonal positive matrix, with eigenvalues in the unit interval; then $u_i d_{q_i}^* u_i^{-1} = I - D_i$. With unitary matrices $u_{p_i}, u_{q_i}$, one can set $u_i d_{p_i} u_{p_i}^{-1} = \sqrt{D_i}, u_i d_{q_i} u_{q_i}^{-1} = \sqrt{I - D_i}$. The equations in (5.5) become

$$\begin{pmatrix} u_{p_i} b_{p_i}^* \\ u_{q_i} b_{q_i}^* \end{pmatrix} \begin{pmatrix} \sqrt{D_i} \\ \sqrt{I - D_i} \end{pmatrix} = 0,$$

and so

$$\begin{pmatrix} \sqrt{I - D_i} u_{p_i} b_{p_i}^* \\ \sqrt{I - D_i} u_{q_i} b_{q_i}^* \end{pmatrix} \begin{pmatrix} v_{p_i} \\ v_{q_i} \end{pmatrix} = \sqrt{D_i} \sqrt{I - D_i} u_i (w_{q_i} - w_{p_i}).$$

If $g_{p_i}, g_{q_i}$ are graphs of isomorphisms, we can take $w_{p_i} = w_{q_i}$. Then we have

$$\begin{pmatrix} \sqrt{I - D_i} u_{p_i} b_{p_i}^* \\ \sqrt{I - D_i} u_{q_i} b_{q_i}^* \end{pmatrix} \begin{pmatrix} v_{p_i} \\ v_{q_i} \end{pmatrix} = 0.$$

Normalizing,

$$\begin{pmatrix} ((I - D_i)^2 + D_i^2)^{-1/2} \sqrt{I - D_i} u_{p_i} b_{p_i}^* - \sqrt{I - D_i} u_{q_i} b_{q_i}^* \end{pmatrix} \begin{pmatrix} v_{p_i} \\ v_{q_i} \end{pmatrix} = 0.$$
where the first copy of $U(n)$ acts on the framing at $p_i$ and on the first $\mathbb{C}^n$ in the first copy of $Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n)$, the second copy of $U(n)$ acts on the framing at $q_i$ and on the first $\mathbb{C}^n$ in the second copy of $Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n)$, and the third copy of $U(n)$ acts on the second $\mathbb{C}^n$ in both copies of $Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n)$. On the other hand,

$$M^{par}(n) = \frac{(EM_n \times Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n) \ell)}{(U(n) \ell \times U(n) \ell)},$$

where the first copy of $U(n)$ acts on the framing at $p_i$ and on the first $\mathbb{C}^n$ in $Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n)$, and the second copy of $U(n)$ acts on the framing at $q_i$ and on the second $\mathbb{C}^n$ in $Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n)$.

The relation between $S/U(n)\ell$ and $M^{par}(n)$ is thus mediated by the relation between $(Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n) \times Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n))/U(n)$ and $Gr_n(\mathbb{C}^n \oplus \mathbb{C}^n)$. These two spaces are isomorphic over the open set consisting of graphs of isomorphisms $\mathbb{C}^n \rightarrow \mathbb{C}^n$, but the quotient is not an isomorphism away from this.

6. Examples

6.1. Line bundles. We now consider line bundles over a curve of arbitrary genus; the degree $\delta_0$ will live in a range $\{-[(\ell - 1)/2], \cdots, [(\ell - 1)/2]\}$. There are no conditions on subbundles, and one just has the constraints $\sum_i s_i \leq \ell/2 - \delta_0, \sum_i t_i \leq \ell/2 + \delta_0$. (Thus, for $\ell = 1$, we have $t_1 = 0, s_1 = 0$.)

For the framed moduli, this gives us for $\ell = 1$, the Jacobian.

The other moduli spaces are fibered over the Jacobian. The fiber, over a line bundle $L$, is a quotient of $\prod_i \mathbb{P}(L_{p_i} \oplus \mathbb{C})$ by $\mathbb{C}^*$. For $\ell = 2$, the fiber is $\mathbb{P}^1$. For $\ell = 3$, there are different rational quotients depending on the degree $\delta_0$: for $\delta_0 = 1, s_i = 0,$ and $\sum t_i \leq 2$. This gives $\mathbb{P}^2$ as a quotient. Inverting the roles of $t_i$ and $s_i$, the same holds for degree minus one. For $\delta_0 = 0$, we have $\sum_i s_i \leq 1, \sum_i t_i \leq 1$, and one gets $\mathbb{P}^1 \times \mathbb{P}^1$.

From the symplectic point of view, one has elements of the moduli space $EM_1$ given by elements $A_j, B_j, j = 1, \cdots, g, C_i, i = 1, \cdots, \ell$, of $S^1$ and $\delta_i, i = 1, \cdots, \ell$, of $\mathbb{R}$. One can amalgamate the $C_i$ and the $\delta_i$ into elements $\gamma_i$ of $\mathbb{C}^*$; the space $GM_1$ is obtained by taking the cylinders $\log(|\gamma_i|) \in [-1/2, 1/2]$, and collapsing the boundaries of the cylinders to obtain spheres, then quotienting symplectically by the circle action. One fixes the sum of the $\delta_i$ to minus the degree, then quotients by $S^1$.

6.2. The genus zero, two point case. Let us consider this first from the symplectic side. In this case, the geometric data for the space $EM_n$ simplifies somewhat: one has a flat connection $\sqrt{-1} \delta d\theta = -\sqrt{-1} \delta d(-\theta)$ on a cylinder, and framings $f_1, f_2$ at each puncture. Assuming that the connection is expressed in the basis given by the framing $f_1$, the data is simply the Hermitian matrix $\delta$ and a unitary matrix $U$ expressing the second framing in terms of the first: $f_2 = U \cdot f_1$. Going to the Grassmann framed space, one has elements $g_1, g_2$ of $Gr_n(2n)$ satisfying $\mu_1(g_1) = \sqrt{-1} \delta, \mu_1(g_2) = U(-\sqrt{-1} \delta)U^{-1}$. This is to be considered modulo the action of two copies of $U(n)$, one at each puncture.
One of these copies simply undoes the action of $U$, and one then has the description of the moduli space as the set of pairs $(g_1, g_2)$, satisfying $\mu_1(g_1) = -\mu_1(g_2)$, modulo $U(n)$; in other words, one has the symplectic quotient
\[(\text{Gr}_n(2n) \times \text{Gr}_n(2n))/U(n),\]
under the diagonal action of $U(n)$.

On the open set of planes that are graphs of linear isomorphisms $\gamma_1, \gamma_2$, this gives the constraint
\[\delta = (-I + \gamma_1^* \gamma_1)(I + \gamma_1 \gamma_1^{-1}) = -(-I + \gamma_2^* \gamma_2)(I + \gamma_2 \gamma_2^{-1})^{-1}\]
with the equivalence
\[\langle \delta, \gamma_1, \gamma_2 \rangle \mapsto (U\delta U^{-1}, \gamma_1 U^{-1}, \gamma_2 U^{-1}).\]

Decomposing into a product of a positive Hermitian part and a unitary part, one has $\gamma_1 = (\gamma_1 \gamma_1^*)^{1/2} \cdot (\gamma_1 \gamma_1^*)^{-1/2} \gamma_1$, and one can use the unitary action to normalize $\gamma_1$ to a positive Hermitian matrix. One then has that $\gamma_1 = (\gamma_1 \gamma_1^*)^{1/2}$, and via the relation above, $\gamma_1$ is then computable in terms of $\gamma_2$. The open set of the moduli space is simply given by the possible choices for the matrix $\gamma_2$, and so is $GL(n, \mathbb{C})$.

From the holomorphic viewpoint, if one restricts to the open set over which the bundle is trivial, and for which the planes correspond to the graphs of framings, one again finds $GL(n, \mathbb{C})$: one has a trivial bundle $E$, and two invertible linear maps from $H^0(\mathbb{P}^1, E) = E_p = E_q$ to $\mathbb{C}^n$. One can use the automorphisms of the bundle to normalize one of the maps to the identity, with the other map giving the element of $GL(n, \mathbb{C})$.

6.3. The one point case. Let us consider the case of a framing at just one point $p$. The degree $\delta_0 \in \{-(n-1)/2, \ldots, (n-1)/2\}$. Here we have $s_1 \leq n/2 - \delta_0$, and $t_1 \leq n/2 + \delta_0$. If the base bundle $E$ is stable, the pair $(E, g)$ is also. Over the locus of stable (hence simple) bundles, the moduli space $\mathcal{G}\mathcal{M}_{n, \delta_0}$ has fiber given by the quotient of the Grassmannian by $\mathbb{C}^*$. This quotient will depend on $\delta_0$; for example, if $n$ is odd and $\delta_0 = -(n-1)/2$, then $t_1 = 0, s_1 < n$; the set of planes is then the set of graphs of non-zero maps from $\mathbb{C}^n$ to the fiber of the bundle at $p$, and, quotienting by $\mathbb{C}^*$, one simply gets the projective space $\mathbb{P}^{n^2-1}$.

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