D-type Vortex and N-type Vortex in Two-gap Superconductor

Y. M. Cho and Pengming Zhang
Center for Theoretical Physics and School of Physics
College of Natural Sciences,
Seoul National University, Seoul 151-742, Korea

We show that the two-gap superconductor has two types of non-Abrikosov magnetic vortex, the D-type which has no concentration of the condensate at the core and the N-type which has a non-vanishing concentration of the condensate at the core, which may carry $4\pi$-flux, $2\pi$-flux, or a fractional flux, depending on the parameters of the potential. Furthermore, we show that the non-Abrikosov vortex is described by two types of topology. The integer flux vortex has the non-Abelian topology $\pi_2(S^2)$, but the fractional flux vortex has the Abelian topology $\pi_1(S^2)$. We show that the inclusion of the Josephson interaction does not affect the existence of the magnetic vortex, but alter the shape of the vortex drastically.

PACS numbers: 74.20.-z, 74.20.De, 74.60.Ge, 74.60.Jg, 74.90.+n
Keywords: non-Abrikosov vortex, fractional magnetic flux in two-gap superconductor

The Abrikosov vortex in one-gap superconductors and similar ones in Bose-Einstein condensates and superfluids have played a very important role in condensed matter physics [1]. But the recent advent of two-component Bose-Einstein condensates and two-gap superconductors [2, 3] has opened up an exciting new possibility for us to construct far more interesting topological objects in laboratories [1, 3, 4, 5, 6, 7, 8]. It has been shown that the two-gap superconductor can admit a non-Abrikosov vortex which has the non-Abelian $\pi_2(S^2)$ topology [4, 5, 6]. The purpose of this Letter is to show that the two-gap superconductor actually allows two types of non-Abrikosov vortex, D-type and N-type, whose magnetic flux can be anywhere between $4\pi/2$ and zero. And this holds true even when the interband Josephson interaction is present. The reason for this is that the magnetic vortex in two-gap superconductor has two types of boundary condition at the core.

In mean field approximation the free energy of the two-gap superconductor could be expressed by [3, 5, 6]

$$H = \frac{\hbar^2}{2m_1} |(\nabla + igA)\tilde{\phi}_1|^2 + \frac{\hbar^2}{2m_2} |(\nabla + igA)\tilde{\phi}_2|^2 + V(\tilde{\phi}_1, \tilde{\phi}_2) + \frac{1}{2} |(\nabla \times A)|^2,$$

(1)

where $V$ is the effective potential. We choose the potential to be the most general quartic potential which has the $U(1) \times U(1)$ symmetry,

$$V = \frac{\tilde{\lambda}_{11}}{2} |\tilde{\phi}_1|^4 + \frac{\tilde{\lambda}_{12}}{2} |\tilde{\phi}_1|^2 |\tilde{\phi}_2|^2 + \frac{\tilde{\lambda}_{22}}{2} |\tilde{\phi}_2|^4 - \tilde{\mu}_1 |\tilde{\phi}_1|^2 - \tilde{\mu}_2 |\tilde{\phi}_2|^2,$$

(2)

where $\tilde{\lambda}_{ij}$ are the quartic coupling constants and $\tilde{\mu}_i$ are the chemical potentials of $\tilde{\phi}_i$ ($i = 1, 2$). The Josephson interaction which breaks the $U(1) \times U(1)$ symmetry down to $U(1)$ will be discussed separately in the following.

With the normalization of $\tilde{\phi}_1$ and $\tilde{\phi}_2$ to $\phi_1$ and $\phi_2$,

$$\phi_1 = \frac{\hbar}{\sqrt{2m_1}} \tilde{\phi}_1, \quad \phi_2 = \frac{\hbar}{\sqrt{2m_2}} \tilde{\phi}_2.$$

(3)

one can simplify the above Hamiltonian,

$$H = |(\nabla + igA)\phi|^2 + V(\phi_1, \phi_2) + \frac{1}{2} |(\nabla \times A)|^2,$$

(4)

where $V$ is the normalized potential,

$$V = \frac{\lambda_{11}}{2} |\phi_1|^4 + \frac{\lambda_{12}}{2} |\phi_1|^2 |\phi_2|^2 + \frac{\lambda_{22}}{2} |\phi_2|^4 - \mu_1 |\phi_1|^2 - \mu_2 |\phi_2|^2.$$

(5)

The potential has the following types of vacuum:

A. Type I: Integer flux vacuum

$$\left( \begin{array}{c} |\phi_1| > \\ |\phi_2| > \end{array} \right) = \left( \begin{array}{c} \sqrt{\mu_1/\lambda_{11}} \\ 0 \end{array} \right).$$

(6)

This is possible when we have one of the following three cases,

(a) \quad 0 < \lambda_{12}, \quad \frac{\lambda_{12}}{\lambda_{22}} < \frac{\lambda_{11}}{\lambda_{12}} \leq \frac{\mu_1}{\mu_2},

(b) \quad 0 < \lambda_{12}, \quad \frac{\lambda_{11}}{\lambda_{12}} < \sqrt{\frac{\lambda_{11}}{\lambda_{22}}} < \frac{\mu_1}{\mu_2},

(c) \quad 0 < \lambda_{12}, \quad \frac{\lambda_{11}}{\lambda_{12}} = \frac{\lambda_{12}}{\lambda_{22}} \leq \frac{\mu_1}{\mu_2}.$$

(7)

We call this integer flux vacuum because, as we will see, the magnetic vortex with this type of vacuum has an
integer flux.

B. Type II: Fractional flux vacuum

\[
\begin{pmatrix}
\langle |\phi_1| \rangle \\
\langle |\phi_2| \rangle 
\end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},
\]

\[
\hat{\phi}^2 = \frac{\mu_1 \lambda_{12} - \mu_2 \lambda_{12}}{\lambda_{11} \lambda_{22} - \lambda_{12}^2}, \quad \hat{\phi}_2 = \frac{\mu_2 \lambda_{11} - \mu_1 \lambda_{12}}{\lambda_{11} \lambda_{22} - \lambda_{12}^2}.
\]

This is possible when we have one of the following three cases,

(a) \( \lambda_{12} < 0, \quad \frac{|\lambda_{12}|}{\lambda_{22}} < \frac{\lambda_{11}}{|\lambda_{12}|} \),

(b) \( 0 < \lambda_{12}, \quad \frac{\lambda_{12}}{\lambda_{22}} < \frac{\mu_1}{\mu_2} < \frac{\lambda_{11}}{\lambda_{12}} \),

(c) \( \lambda_{12} = 0 \).

We call this fractional flux vacuum because the magnetic vortex with this type of vacuum has a fractional flux.

C. Type III: Degenerate vacuum

\[ \mu_1 < |\phi_1|^2 + \mu_2 < |\phi_2|^2 = \frac{\mu_1 \mu_2}{\lambda_{11}}. \]

This is what we have when

\[ \frac{\mu_1}{\mu_2} = \frac{\lambda_{12}}{\lambda_{11}} = \frac{\lambda_{22}}{\lambda_{12}}. \]

This includes the special (and familiar) case

\[ \lambda_{11} = \lambda_{12} = \lambda_{22} = \lambda, \quad \mu_1 = \mu_2 = \mu. \]

In this case the potential has the full \( SU(2) \) symmetry. Notice that the potential has no vacuum when

\[ \lambda_{12} < 0, \quad \frac{\lambda_{11}}{|\lambda_{12}|} \leq \frac{|\lambda_{12}|}{\lambda_{22}}. \]

All other cases can be reduced to one of the above cases by re-labelling \( \phi_1 \) and \( \phi_2 \).

With

\[ \phi = \frac{1}{\sqrt{2}} \rho \xi, \quad (\xi^1 \xi = 1), \quad \hat{n} = \xi^1 \partial \xi, \]

we can obtain the following equation of motion from the Hamiltonian

\[ \partial^2 \rho - \left( \frac{1}{4} (\partial \hat{n})^2 + g^2 (A_{\mu} - \frac{1}{2} C_{\mu})^2 \right) \rho = \left[ \frac{\lambda}{2} (\rho^2 - \rho^2) + \left( \frac{\alpha}{2} \rho^2 - \gamma \right) n_3 + \frac{\beta}{2} \rho^2 n_3^2 \right] \rho, \]

\[ \hat{n} \times \partial^2 \hat{n} + 2 \frac{\partial \rho}{\rho} \hat{n} \times \partial \hat{n} - 2 \frac{g \rho^2}{g \rho^2} \partial \mu F_{\mu \nu} \partial \nu \hat{n} = (2 \gamma - (\frac{\alpha}{2} + \beta n_3) \rho^2) \hat{k} \times \hat{n}, \]

\[ \partial \mu F_{\mu \nu} = j_\nu = g^2 \rho^2 \left( A_\nu - \frac{1}{2} C_\nu \right), \]

where

\[ C_\mu = \frac{2i}{g} \xi^1 \partial \rho \xi, \quad \rho^2 = \frac{2 \mu}{\lambda}, \]

\[ \lambda = \frac{\lambda_{11} + \lambda_{22} + 2 \lambda_{12}}{4}, \quad \alpha = \frac{\lambda_{11} - \lambda_{12}}{2}, \quad \beta = \frac{\lambda_{11} + \lambda_{22} - 2 \lambda_{12}}{4}, \]

\[ \mu = \frac{\lambda_{11} + \lambda_{22}}{2}, \quad \gamma = \frac{\lambda_{11} - \lambda_{12}}{2}, \]

and \( \hat{k} = (0, 0, 1) \). This is the equation for two-gap superconductor which allows a large class of interesting topological objects, straight magnetic vortex, helical magnetic vortex, and magnetic knot, all with \( 4\pi / g \)-flux, \( 2\pi / g \)-flux, or a fractional flux.

To discuss the vortex solution let \( (\rho, \varphi, z) \) be the cylindrical coordinates and choose the ansatz

\[ \rho = \rho(\varphi), \quad \xi = \left( \cos \frac{f(\varphi)}{2} \exp(-in\varphi) \right), \]

\[ A_\mu = \frac{n}{g} A(\varphi) \partial_\mu \varphi. \]

With this one can obtain the vortex solution by solving \( \Box \). To do this, we have to fix the boundary conditions. At the core the smoothness allows (for \( n = 1 \)) two types of boundary condition:

A. Dirichlet boundary condition

\[ \rho(0) = 0, \quad \hat{\rho}(0) = 0, \quad A(0) = -1. \]

B. Neumann boundary condition

\[ \rho(0) \neq 0, \quad \hat{\rho}(0) = 0, \quad A(0) = 0. \]

This is a new feature of two-gap superconductor, which we do not have in ordinary superconductor.

At the infinity all fields must assume the vacuum values. In particular, the electromagnetic current must vanish. This means that we must have

\[ \rho(\infty) = \sqrt{2} (|\phi_1|^2 + |\phi_2|^2), \]

\[ \cos f(\infty) = \frac{|\phi_1|^2}{|\phi_1|^2} - \frac{|\phi_2|^2}{|\phi_2|^2}, \]

\[ A(\infty) = \frac{|\phi_1|^2}{|\phi_1|^2} - \frac{|\phi_2|^2}{|\phi_2|^2}. \]

Notice that for the integer flux vacuum we have \( A(\infty) = 1 \), but for the fractional flux vacuum \( A(\infty) \) becomes fractional.

The existence of two types of boundary conditions in two-gap superconductor has an important impact. To understand this notice that the magnetic flux of vortex is given by

\[ \Phi = \int A_\mu dx^\mu = \frac{2\pi n}{g} (A(\infty) - A(0)). \]
shown with different $\alpha, \beta, \gamma$. In (C) and (D) the N-type vortex with $2\pi/g$ becomes $4\pi/g$ fractional. As importantly, when $A$ is an ordinary superconductor.

With the ansatz $\Phi = \int F_{\alpha\beta}d^2x = \int \partial_\alpha A_\beta d^2x = \frac{2\pi n}{g}$, we find the solutions with the different parameters shown in Fig. 1C. We call this a N-type vortex. In this case $\phi_1$ behavior is the same as before. But notice that $\phi_2$ has a maximum concentration at the core, and approaches zero at the infinity. This is a generic feature of a N-type vortex. The magnetic flux of the vortex is given by

$$f(0) = \pi, \quad f(\infty) = 0,$$

$$A(0) = 0, \quad A(\infty) = 1,$$  \hspace{1cm} (23)

which is same as that of Abrikosov vortex. But the topology of the $\CP^1$ field $\hat{n}$ is the same as the $4\pi/g$-flux vortex, $\pi_2(S^2) = n$. The reason why there exist two vortices which have different magnetic flux but have the same topology is because the magnetic flux is determined by the boundary condition $A(\infty) - A(0)$, not by the topology. The topology assures the quantization of the flux, but does not determine the magnitude of the flux.

C. Fractional flux vortex: This is possible when we have the fractional flux vacuum at the infinity

$$\rho(\infty) = 2 \sqrt{\frac{2\mu - \alpha}{4\lambda - \alpha^2}}, \quad \cos f(\infty) = \frac{2\gamma\lambda - \alpha\mu}{2\beta\mu - \alpha\gamma},$$

$$A(\infty) = \frac{2(\gamma\lambda + \beta\mu - \alpha(\mu + \gamma)}{2\beta\mu - \alpha\gamma}.$$ \hspace{1cm} (25)

At the core we can impose either the Dirichlet condition or the Neumann condition, and obtain the D-type vortex shown in Fig. 1B or the N-type vortex shown in Fig. 1D. The fractional vortex is also topological, but the topology of the fractional vortex is different from that of integer flux vortex. For the fractional flux vortices the $\pi_2(S^2)$ topology of $\hat{n}$ becomes trivial, but in this case we still have a $U(1)$ topology $\pi_1(S^1)$, the topology of the $U(1)$ symmetry which leaves $\hat{n}$ invariant. And this Abelian topology describes the topology of the fractional flux vortex. So the topology of the fractional flux vortices is the same as that of the Abrikosov vortex. An important feature of the fractional flux vortex is that the energy per unit length of the vortex is logarithmically divergent. This, however, does not make the fractional flux vortex unphysical. In laboratory setting one can observe such vortex because one has a natural cutoff parameter $\Lambda$ fixed by the size of the superconductor, which can effectively make the energy of the fractional flux vortex finite.

It must be emphasized, however, that the actual magnetic flux of vortex depends on the two-gap superconductor at hand because it is fixed by the parameters of the potential which characterizes the superconductor. Independent of this all two-gap superconductors have two types of vortex, D-type and N-type. On the other hand one must keep the followings in mind. First, the D-type vortex has more energy than the N-type vortex, because it carries $2\pi/g$ more flux. Secondly, the energy (per unit...
the magnetic vortex in the presence of Josephson interaction.

FIG. 2: The density profile of $|\phi_1|^2$ in (A) and $|\phi_2|^2$ in (B) of the magnetic vortex in the presence of Josephson interaction. Here we have put $\bar{\rho} = 1$, $\gamma = 0.05$, $\eta = 0.25$, and $\lambda/g^2 = 2$.

length) of the fractional flux vortex is logarithmically divergent, so that they can exist only when a cutoff parameter makes the energy finite.

It is well-known that two-gap superconductor may allow the interband Josephson interaction [10]. To discuss the impact of the Josephson interaction in two-gap superconductor, we let $\lambda_{11} = \lambda_{22} = \lambda_{12} = \lambda$ and adopt the potential which has the Josephson interaction

$$V = \frac{\lambda}{2} (|\phi_1|^2 + |\phi_2|^2)^2 - \mu_1 |\phi_1|^2 - \mu_2 |\phi_2|^2 + \eta (\phi_1 \phi_2 + \phi_1^* \phi_2^*).$$

Now, we choose the following ansatz for the magnetic vortex

$$\rho = \rho(g),$$

$$\xi = \left( \begin{array}{c} \cos \frac{f}{2} \cos \frac{\omega}{2} \exp(-i\varphi) + \sin \frac{f}{2} \sin \frac{\omega}{2} \\ \cos \frac{f}{2} \sin \frac{\omega}{2} \exp(-i\varphi) + \sin \frac{f}{2} \cos \frac{\omega}{2} \end{array} \right),$$

$$A_\mu = \frac{n}{g} A(g) \partial_\mu \varphi,$$

$$\tan \omega = \frac{2\eta}{\mu_1 - \mu_2}.$$  (26)

Notice that the ansatz is not axially symmetric. With this we obtain two types of magnetic vortex. The density profile of the N-type $2\pi/g$-flux vortex is plotted in Fig. 2. Notice that the vortex can be viewed as a “bound state” of two vortices made of $\phi_1$ and $\phi_2$. This confirms that the Josephson interaction does not prevent the existence of two types of magnetic vortex. It makes the vortex more interesting.

Finally we emphasize that all these non-Abrikosov vortices can be twisted to form a helical vortex which is periodic in $z$-coordinate. In particular, with the Josephson interaction, we can construct a “braided” helical vortex from the above bound state of $\phi_1$ and $\phi_2$ vortices by twisting and making it periodic in $z$-coordinate. Perhaps more importantly, we can construct a twisted magnetic vortex ring from the helical vortex by smoothly bending it and connecting two periodic ends together. And the vortex ring becomes a stable magnetic knot whose knot topology $\pi_3(S^2)$ is fixed by the Chern-Simons index of the electromagnetic potential $\mathcal{A}_\mu$. Because of the helical structure of the magnetic flux the knot has two magnetic flux linked together, one around the knot tube and one along the knot, whose linking number is given by the knot quantum number. And since the flux trapped inside the vortex ring can not be squeezed out, it makes the knot stable against collapse by providing a stabilizing repulsive force. This makes the knot dynamical (as well as topologically) stable.

It is really remarkable that the two-gap superconductor allows such diverse topological objects. This is because it has a non-Abelian structure in which the dou- blet $\phi$ can be treated as an $SU(2)$ doublet. The fact that the Hamiltonian [4] has the full $SU(2)$ symmetry when $\alpha = \beta = \gamma = 0$ tells that the $SU(2)$ symmetry survives as an approximate symmetry of two-gap superconductor, which is broken by the $\alpha$, $\beta$, and $\gamma$ interactions [4, 5].

There are other topological objects which have not been discussed in this paper. The detailed discussion on the topological objects in two-gap superconductor will be presented in a separate paper [11].

ACKNOWLEDGEMENT

The work is supported in part by the ABRL Program of Korea Science and Engineering Foundation (Grant R02-2003-000-10043-0).

[1] A. Abrikosov, Sov. Phys. JETP 5, 1174 (1957).
[2] C. Myatt et al., Phys. Rev. Lett. 78, 586 (1997); D. Stamper-Kurn, et al., Phys. Rev. Lett. 80, 2027 (1998).
[3] J. Nagamatsu et al., Nature 410, 63 (2001); C. U. Jung et al., Appl. Phys. Lett. 78, 4157 (2001).
[4] Y. M. Cho, cond-mat/0112325, Int. J. Pure Appl. Phys. 1, 246 (2005); cond-mat/0308182, submitted to Int. J. Pure Appl. Phys.
[5] Y. M. Cho, cond-mat/0112498, Phys. Rev. B72, 212516 (2005).
[6] J. Ruostekoski and J. Anglin, Phys. Rev. Lett. 87, 120407 (2001).
[7] Y. M. Cho, Phys. Rev. A72, 063603 (2005).
[8] E. Babaev, Phys. Rev. Lett. 89, 067001 (2002); E. Babaev, L. Faddeev, and A. Niemi, Phys. Rev. B65, 100512 (2002); M. Zhitomirsky and V. Dao, Phys. Rev. B69, 054508 (2004).
[9] Y. M. Cho and Pengming Zhang, cond-mat/0601347, Phys. Rev. B, submitted.
[10] A. Leggett, Prog. Theo. Phys. 36, 901 (1966).
[11] Y. M. Cho and Pengming Zhang, to be published.