Mathematical aspect of the Marangoni effect at the interface between two immiscible fluids

Abstract

The Marangoni effect is a very important phenomenon happening at an interface between two immiscible fluids creating a source of convection. This effect is very important in two phase flow problems. Unfortunately, the Marangoni effect is neglected by many studies in two phase fluid flow and is still considered a challenging problem.

A mathematical model has been developed in this paper showing the Marangoni effect in the case of two immiscible fluids in Navier-Stokes equation. The mathematical translation of the convection term at the interface is developed in detail from the starting point of physical parameters using powerful mathematical tools.

Keywords: marangoni effect, two-phase flow, interface, navier-Stokes

Introduction

The Marangoni effect is happening at the interface between two immiscible fluids. In the absence of initial velocity, the movement of an interface is caused by a variation of interfacial tension; the displacement is the direction of positive superficial tension gradient.

The Marangoni effect is present in many domains in the instability problems in fluid mechanics,1 microstructure problems2 two phase flow problems,3 engineering flows in microfluidic devices4 and so many other domains. In the ink-jet problems,5,6 in 3D printing technology,7 these processes are complex because of physicochemical dynamics that arise from Marangoni effects, also in Surface patterning,8 interactions between suspended particles and a solid substrate.9

Mathematical model

To understand and to show the mathematical aspect of the Marangoni effect at an interface between two immiscible fluids, we are going to build the one fluid model10 of the Navier-Stokes equation for two viscous Newtonian immiscible fluids, with a variable surface tension coefficient.

Let’s consider a time dependent flow configuration of two incompressible viscous Newtonian fluids represented in Figure 1. The total domain contains two subspaces, Ω1 for the fluid 1 and Ω2 for the fluid 2. The boundary of the fluid 1 is ∂Ω1 ∩ Γ* and the boundary for the fluid 2 is ∂Ω2 ∩ Γ*. The total domain is the union of domains Ω = Ω1 ∩ Ω2, the intersection Ω1 ∩ Ω2 = ∅ and the union of all external boundaries is ∂Ω = (∂Ω1 / Γ*) ∪ (∂Ω2 / Γ*). We assume that Ω1 and Ω2 are connected but having this condition Ω1 ∩ Ω2 = ∅.

The physical properties for each domain are:

\[ P = \begin{cases} \rho_1 & \text{if } x \in \Omega_1, \\ \rho_2 & \text{if } x \in \Omega_2 \end{cases}, \quad \mu = \begin{cases} \mu_1 & \text{if } x \in \Omega_1, \\ \mu_2 & \text{if } x \in \Omega_2 \end{cases} \]  

(1)

\[ \text{For the fluid 1, we have:} \]

\[ \bm{\sigma}_1 \cdot \nabla \bm{U} \text{ and } \int \sigma \sigma_1 d\Gamma + \int \sigma \sigma_{1n} d\Gamma \]  

(2a)

\[ \text{For the fluid 2:} \]

\[ \bm{\sigma}_2 \cdot \nabla \bm{U} \text{ and } \int \sigma \sigma_2 d\Gamma + \int \sigma \sigma_{2n} d\Gamma \]  

(2b)

The addition of (2a) + (2b) gives:

![Figure 1 Two fluids model configuration space.](Image 36x797 to 57x816)

![Figure 2 Fusion of two immiscible fluids through an interface.](Image 298x32 to 212x48)
\[ \frac{D}{Dt} \int_{\Omega} (\rho \dot{U}) d\Omega = \int_{\Omega} \dot{F} d\Omega + \int_{\Omega} \nabla \sigma n d\Gamma + \int_{\Omega} \nabla \sigma_{n2} d\Gamma - \int_{\Omega} \nabla \sigma_{n1} d\Gamma \]  

With \( \Omega = \Omega_1 \cup \Omega_2 \) and \( \partial \Omega = (\partial \Omega_1 / \Gamma_1) \cup (\partial \Omega_2 / \Gamma_2) \)

At the interface we have:

\[ \lim_{\Gamma \to \Gamma^*} \left\{ \nabla \sigma n d\Gamma \right\} = \left\{ \nabla (\sigma_1 - \sigma_2) n d\Gamma \right\} \]  

\[ \lim_{\Gamma \to \Gamma^*} \left\{ \nabla \sigma_{n2} d\Gamma \right\} = \left\{ \nabla (\sigma_1 - \sigma_2) n d\Gamma \right\} \]  

With \( \overline{n} = -n_2 = \overline{n} \)

Finally, we obtain

\[ \frac{D}{Dt} \int_{\Omega} (\rho \dot{U}) d\Omega = \int_{\Omega} \dot{F} d\Omega + \int_{\Omega} \nabla \sigma n d\Gamma - \int_{\Omega} \nabla (\sigma_1 - \sigma_2) n d\Gamma \]  

Equation (7) represent the jump condition over the interface of separation and \( \Gamma \) represent the surface tension force if we multiply it with the Dirac function to have a three dimensional force. We introduce the Dirac function to express it in three dimensions. It represent the surface tension force between two fluids localized at the interface \( f_s(x,t) \).

Finally, we have the one fluid model of the Navier-Stokes equation:

\[ \frac{\partial}{\partial t} (\rho \vec{U}) + \nabla \cdot (\rho \vec{U} \vec{U}) = -\nabla \cdot \vec{P} + \nabla \cdot (\nabla \sigma_1 n \vec{U}) - \int_{\Omega} \nabla (\sigma_1 - \sigma_2) n d\Gamma \]  

\[ \]  

Equation (7) represent the jump condition over the interface of separation and it represent the surface tension force. We introduce the Dirac function to have a three dimensional force. Let’s express the jump condition (equation 7) with physical and mathematical parameters. Let’s take an interface between two immiscible fluids \( \Omega_1 \) and \( \Omega_2 \), \( S \) be a portion of this interface and \( C \) the closed contours of this portion. \( \overline{n} \), the normal vector to the interface and \( \overline{t} \) the tangential from Figure 3.

Forces acting at the interface between two immiscible fluids are composed from the force acting on the surface \( S \) and the force acting on the closed contour \( C \) of this surface. The mathematical translation of this physical phenomenon is:

\[ \vec{F} = \int_{\partial A} (\overline{\sigma}_1 - \overline{\sigma}_2) n d\Gamma + \int_{\partial C} (\overline{\gamma} \overline{n}) \times \overline{n} d\Gamma \]  

With \( \overline{\sigma}_1 \) and \( \overline{\sigma}_2 \) the stress tensors on each fluid, \( \overline{\gamma} \) the superficial tension coefficient at the interface. Note that the volumetric forces are equal to zero at the interface because the volume of an interface is equal to zero. Even in absence of equilibrium the summation of all forces is equal to zero due to the fact that the interface doesn’t have a mass.

\[ \sum \vec{F} = m \overline{a}, \text{ with } m=0. \]  

Figure 3 Forces acting on a surface of discontinuity

Equation (8) will be:

\[ \int_{\partial \Gamma} (\overline{\sigma}_1 - \overline{\sigma}_2) n d\Gamma + \int_{\partial C} (\overline{\gamma} \overline{n}) \times \overline{n} d\Gamma = 0 \]  

\[ \]  

Applying the Stokes theorem to the right-hand side of the equation (9b):

\[ \int_{\partial A} (\nabla \times \vec{F}) n dA = 0 \]

Considering \( \vec{F} \) being the product of two vectors \( \vec{F} = \vec{g} \times \vec{b} \) with \( \vec{b} \) a constant vector, we obtain:

\[ \int_{\partial A} (\nabla \times (\vec{g} \times \vec{b})) n dA = 0 \]

We have \( \nabla \times (\vec{g} \times \vec{b}) = (\nabla \vec{g}) \vec{b} - (\nabla \vec{b}) \vec{g} + \vec{b} \times \nabla \vec{g} - \vec{g} \times \nabla \vec{b} \)

But \( (\nabla \vec{g}) \vec{b} = (\nabla \vec{b}) \vec{g} = 0 \)

So, the last expression become \( (\vec{g} \times \vec{b}) \nabla \times (\vec{g} \times \vec{b}) = 0 \)

We obtain \( \int_{\partial A} (\vec{g} \times \vec{b}) n dS = \int_{\partial A} (\nabla \vec{g}) \vec{b} + \vec{b} \times \nabla \vec{g} n dA \)

With taking \( \vec{g} = \gamma \overline{n} \):

\[ \int_{\partial A} (\vec{g} \times \vec{b}) n dS = \int_{\partial A} (\nabla \vec{g}) \vec{b} + \vec{b} \times \nabla \vec{g} n dA \]

\[ \int_{\partial A} (\vec{g} \times \vec{b}) n dS = \int_{\partial A} (\nabla \vec{g}) \vec{b} + \vec{b} \times \nabla \vec{g} n dA \]

\[ \int_{\partial A} (\vec{g} \times \vec{b}) n dS = \int_{\partial A} \nabla \vec{g} \overline{n} + \vec{b} \times \nabla \vec{g} \overline{n} dA \]

\[ \int_{\partial A} (\vec{g} \times \vec{b}) n dS = \int_{\partial A} \nabla \vec{g} \overline{n} + \vec{b} \times \nabla \vec{g} \overline{n} dA \]

\[ \int_{\partial A} (\vec{g} \times \vec{b}) n dS = \int_{\partial A} \nabla \vec{g} \overline{n} + \vec{b} \times \nabla \vec{g} \overline{n} dA \]

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By definition Mansour et al.:\[ -\nabla \vec{g} \overline{n} = \kappa \]

\[ \kappa \text{ represent the curvature of the interface} \]
\[
\left( \nabla \gamma \otimes \hat{n} \right) \cdot \hat{a} = \nabla \gamma (\hat{n} \cdot \hat{a}) = \nabla \gamma
\]

Considering the divergence operator been the summation of the normal component and the tangential one, we have:

\[
\nabla \gamma = \nabla \gamma + \nabla \gamma_{\tau}
\]

\[
\nabla \gamma_{\tau} = \nabla \gamma \Delta \hat{n} \left( \nabla \gamma \right)
\]

\[
\nabla \gamma_{\tau} = (\hat{n} - \Delta \hat{n}) \left( \nabla \gamma \right)
\]

With \((\hat{n} - \Delta \hat{n})\) represent the projector of the delta operator at the interface.

Equation (10) becomes:

\[
\int_{C}^{\gamma} (\hat{n} \times \hat{n}) dS = \int_{A}^{\gamma} \hat{n} \cdot \left( \nabla \gamma + \left( \nabla \gamma \otimes \hat{n} \right) \hat{n} \right) dA \\
\int_{C}^{\gamma} (\hat{n} \times \hat{n}) dS = \int_{A}^{\gamma} \left[ \nabla \hat{n} - \left( \nabla \gamma_{\tau} \right) \right] dA \\
\int_{A}^{\gamma} \left[ \nabla \hat{n} - \left( \nabla \gamma_{\tau} \right) \right] dA = - \int_{A}^{\gamma} \left[ \nabla \hat{n} - \left( \nabla \gamma_{\tau} \right) \right] dA \\
\int_{A}^{\gamma} \left[ \nabla \hat{n} - \left( \nabla \gamma_{\tau} \right) \right] dA = 0 \\
\int_{A}^{\gamma} \left[ \nabla \hat{n} - \left( \nabla \gamma_{\tau} \right) \right] dA = 0
\]

Equation (11) represents the jump condition at an interface between two immiscible fluids with a variable surface tension coefficient.

The jump condition have a dimension of a force, it seems that the deformation of the interface is a consequence of balance forces acting on fluids or an energy balance during the evolution. The second term of this condition correspond to a surface gradient of the surface tension coefficient \( \gamma \), \( \nabla_{\tau} = (1 - \hat{n} \otimes \hat{n}) \nabla \) is a projection of \( \nabla \) on the oriented surface. This term translate the Marangoni effect.

In the case of stratified flow \( \kappa = 0 \), the fluid can’t be static until \( \nabla \gamma_{\tau} = 0 \). In other case the flow is going to be driven by the surface gradient which represent the Marangoni effect.

The Marangoni effect is only possible if the superficial tension between two points of the interface is different. It suggest the fact that in absence of initial velocity for a fluid, his motion can be driven by the Marangoni effect, in this case the flow direction will be from the point where the surface tension coefficient is low to the point of high surface tension coefficient \( \nabla \gamma = 0 \).

**Numerical results**

We implement the following numerical example of Navier-Stokes two phase flow problem. We used the XFEM for the discretization of velocity and pressure. The program was implemented in the computational FEniCS platform.

In this example we are going to consider two immiscible fluids (air/water) with an interface of separation where the superficial tension \( \gamma \) is not constant. The two immiscible fluids are without initial velocity for both. It means that the both phases are statics at \( t=0s \).

For the half of the configuration \( \gamma_{1} = 70.10^{-3} N.m^{-1} \) and for the other half \( \gamma_{2} = 72.10^{-3} N.m^{-1} \).

As we can see Figure 4, there is a displacement of the fluid from one side to another because of the difference between the coefficients of superficial tension.

![Figure 4](image)

**Conclusion**

This study gives an analytical detailed description of the Marangoni phenomenon with mathematical and physical parameters responsible for this. With this work we have the confirmation that the Marangoni effect is very important when we study two phase flow problems. The impact of this phenomenon is so important that it could be a reason for the displacement of fluids in the absence of initial velocity. In the case where we have a dynamic system, it’s a big factor to create instabilities and interfacial turbulence.

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**Conflict of interest**

Authors declare there is no conflict of interest in publishing the article.

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