Bell’s Inequality Violation (BIQV) with Non-Negative Wigner Function

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A Bell inequality violation (BIQV) allowed by the two-mode squeezed state (TMSS), whose Wigner function is nonnegative, is shown to hold only for correlations among dynamical variables (DV) that cannot be interpreted via a local hidden variable (LHV) theory.

Explicit calculations and interpretation are given for Bell’s suggestion that the EPR (Einstein, Podolsky and Rosen) state will not allow for BIQV in conjunction with its Wigner representative state being nonnegative.

It is argued that Bell’s theorem disallowing the violation of Bell’s inequality within a local hidden-variable theory depends on the DV’s having a definite value – assigned by the LHV – even when they cannot be simultaneously measured. The analysis leads us to conclude that BIQV is to be associated with endowing these definite values to the DV’s and not with their locality attributes.

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I. INTRODUCTION

In his article entitled (scented with an impish whiff) “EPR (= Einstein, Podolsky and Rosen) correlations and EPW (=Eugene Paul Wigner) Distributions”, Bell studied the possibility of underpinning quantum theory with local hidden variables (LHV’s) in the case of two spinless particles. He analyzed the correlations arising from measurements of positions of these particles in free space—a situation closer to the original one envisaged by EPR—utilizing the fact that Wigner’s distribution simulates a local “classical” model of such correlations in phase space. Bell suggested that the nonnegativity of the Wigner function for certain quantum-mechanical states would preclude Bell’s inequality violation (BIQV) with such states when one considers the correlations constructed from a dichotomous variable defined as the sign of the coordinates of the particles.

We first recall a few properties of the Wigner function. One can show that the expectation value of any operator $\hat{A}$ in a state defined by the density matrix $\hat{\rho}$ can be expressed as

$$\text{Tr}(\hat{\rho}\hat{A}) = \int d\lambda W_\rho(\lambda)W_A(\lambda),$$

where $W_Q(\lambda)$ is the Wigner representative of the quantal operator $\hat{Q}$ defined in Eq. (2.12) below, and $\lambda$ designates the appropriate phase space coordinates, i.e., $\lambda = (q, p) = (q_1, \cdots, q_n, p_1, \cdots, p_n)$, $n$ being the number of degrees of freedom. It should be noted that in Bell’s considerations of LHV’s, the values of the observables obey the so-called Bell’s factorization, which leaves the value of each observable independent of the “setting” of the other. In the expressions for two-particle correlations in terms of the Wigner representatives, when each of the DV’s depends on its own phase-space coordinates, this factorization is satisfied automatically. This is our justification for referring to the description in terms of the Wigner function as local.

We illustrate the above considerations using a two-mode squeezed state (TMSS) $|\zeta\rangle$, defined as

$$|\zeta\rangle = \exp^{\zeta (a^d b^\dagger - a b^\dagger)} |00\rangle \equiv S(|\zeta\rangle |00\rangle),$$

this equation defines the operator $S$. Here, the operators $a$, $a^\dagger$ refer to the beam of channel 1, while $b$, $b^\dagger$ refer to those of the second channel. In the limit of the squeezing parameter $\zeta$ increasing without limit, the state approaches the EPR state $|EPR\rangle = \delta(q_1 - q_2)$ (here the subscripts refer to the channels), as can be readily seen writing the state in the coordinate representation as (we use well known normal ordering formulae)

$$\langle q_1 q_2 | \zeta \rangle = \frac{1}{\cosh^2 \zeta} \sum_{n=0}^\infty \tanh^n \zeta \langle q_1 q_2 | n n \rangle \zeta \to \infty \sim \delta(q_1 - q_2).$$

Now, the Wigner function, $W_\zeta$, of the TMSS is given by

$$W_\zeta(q_1, q_2, p_1, p_2) = \frac{1}{\pi^2} \exp \left[ -\cosh(2\zeta) (q_1^2 + q_2^2 + p_1^2 + p_2^2) - 2 \sinh(2\zeta) (q_1 q_2 - p_1 p_2) \right],$$

and is clearly nonnegative for all $q$’s and $p$’s, and thus may be considered as a distribution in phase space $(q_1, q_2, p_1, p_2)$ associated with the state $|\zeta\rangle$. Thus we may refer to the variables $(q_1, q_2, p_1, p_2)$ as LHV’s and correlations weighed with $W_\zeta(q_1, q_2, p_1, p_2)$ should preclude BIQV for dynamical variables (DV’s) for which this may be a legitimate view.

As was mentioned above, Bell suggested that the nonnegativity of the Wigner function of the EPR state would preclude BIQV with this state when one considers the correlations of a dichotomous variable defined as the sign

$$\cdots$$
of the coordinates of the particles. The correlations considered in are those that are involved in the CHSH inequality, i.e., the inequality that is often studied in terms of the Bell operator. (In the present paper, Bell’s inequality and BIQV refer to this CHSH inequality.) Bell’s original argument that nonnegativity of Wigner’s function suffices to preclude BIQV was shown to be inaccurate in, where difficulties in handling normalization of the EPR state considered by Bell was shown to involve a misleading factor.

The TMSS’s were studied extensively since the early eighties in connection with BIQV in general and, in particular, for their connection to the EPR state. These studies focused on the polarization as the observable (= dynamical variable, DV). Banaszek and Wodkiewicz noted that while the Wigner function of the TMSS is non-negative, it allows for BIQV, when the dynamic variable involved in the correlations is the parity. Their study was extended by Chen et al. who showed, by using appropriately defined spin-like DV’s which together with the parity operator close an SU(2) algebra, that the TMSS, , allows the maximal possible BIQV for , i.e., when it is maximally entangled and, as stated above, it tends to the EPR state. An alternative parametrization (termed configurational) to the spin-like operators was given in. This choice of DV is more convenient for our analysis as it involves the DV considered by Bell and admits simple interpretation.

Our study aims at clarifying the relation between the non-negative Wigner function of the TMSS, , for all values of , the DV’s involved in the CHSH inequality and the possibility of BIQV. The latter, by Bell’s theorem, prohibits the underpinning of the theory with a LHV theory. Note that this attribute (non-negativity) of the Wigner function depends on variables over which it is defined.

The paper is organized as follows. In the next section we describe the properties that should be required of a QM problem in order that its translation in terms of Wigner representatives can be legitimately considered as a LHV theory. We then divide the problem indicated in the last paragraph into three levels. The first level, which the works hitherto were addressed to, is to consider BIQV for, viz that the LHV theory be such that the DV are defined simultaneously even when they cannot be measured simultaneously. This point was noted before and, indeed, such a requirement tantamounts to having the LHV endowing physical reality (in the EPR sense) to the DV’s measurable attributes.

To remain close to the formalism as discussed by Bell we shall throughout refer to variation of the DV’s as “evolution”. This retains complete generality, since to define the evolution we can choose a Hamiltonian leading to the required variation.

II. HIDDEN VARIABLES AND WIGNER’S TRANSFORM

We consider bounded QM operators , associated with DV’s for a given physical system, with eigenvalues . By a proper rescaling, we can always have

\[ |a_n| \leq 1. \tag{2.1} \]

In a HV theory we assume that we have variables , endowed with a probability distribution

\[ \rho(\lambda) \geq 0, \tag{2.2} \]

such that to every operator we associate, according to some recipe, a function – a “representative” of the DV in terms of the hidden variable – that takes on, as its possible values, the eigenvalues . When this is feasible, we say that we are dealing with a “proper” dynamical variable (PDV). Notice that this property implies that if is the representative of the operator , then is the representative of the operator , where is an
integer. We then speak of a “non-dispersive” DV. As a consequence, the $A(\lambda)$’s are bounded as

$$|A(\lambda)| \leq 1. \quad (2.3)$$

In a two-particle problem, if the DV $\hat{A}$ is associated with particle 1 and $\hat{B}$ with particle 2, the requirement that $A(\lambda)$ be independent of the setting $b$ of the instrument that measures particle 2 and $B(\lambda)$ be independent of the setting $a$ of the instrument that measures particle 1 makes the theory a LHV theory. For this two-particle problem we now introduce two other DV’s, $\hat{A}'$ and $\hat{B}'$, associated with particles 1 and 2, respectively, and not commuting, in general, with $\hat{A}$ and $\hat{B}$, respectively. To these new DV’s we associate the functions $A'(\lambda)$ and $B'(\lambda)$, respectively. Notice that the functions $A(\lambda)$ and $A'(\lambda)$ for particle 1 (and similarly $B'(\lambda)$ and $B''(\lambda)$ for particle 2) assign a definite value to the two DV’s, whether they can be measured simultaneously or not. Then one can prove the CHSH inequality

$$|\langle \mathcal{B}(\lambda) \rangle| \equiv \left| \int B(\lambda) \rho(\lambda) d\lambda \right| \leq 2, \quad (2.4)$$

where $B$ is given by

$$B = A(\lambda)B(\lambda) + A'(\lambda)B'(\lambda) + A'(\lambda)B(\lambda) - A'(\lambda)B'(\lambda). \quad (2.5)$$

As we mentioned in the Introduction, we shall call the above inequality BIQ. In other words, dealing with PDV’s implies (2.4) which, in turn, implies BIQ:

$$PDV \Rightarrow 2 \Rightarrow BIQ, \quad (2.6)$$

so that

$$PDV \Rightarrow BIQ. \quad (2.7)$$

Conversely, in a HV model in which (2.2) is fulfilled, a violation of BIQ (to be called BIQV) implies that (2.3) is not fulfilled, and hence that we are not dealing with PDV’s, i.e.

$$BIQV \Rightarrow \overline{PDV}, \quad (2.8)$$

so that

$$BIQV \Rightarrow \overline{PDV}. \quad (2.9)$$

(The bar on a proposition indicates its negation.) We mentioned these conditions with some care because of the various applications that we shall be concerned with in the following sections.

Let us mention that when we deal with dichotomous variables, i.e., with operators having only two eigenvalues ($\pm 1$), one can prove that the QM expectation value for any two-particle state $|\Psi\rangle$ of the Bell operators

$$\hat{B} = \hat{A}\hat{B} + \hat{A}'\hat{B}' + \hat{A}'\hat{B} - \hat{A}\hat{B}' \quad (2.10)$$

satisfies the Cirel’son inequality

$$|\langle \Psi| \hat{B} |\Psi\rangle| \leq 2\sqrt{2}. \quad (2.11)$$

We now discuss a specific way of implementing the above LHV program in terms of the theory of Wigner’s transforms. We define the Wigner representative $W_Q(q,p)$ of the quantal operator $Q$ (for one degree of freedom) as

$$W_Q(q,p) = \int e^{-ip\cdot y} \langle q + \frac{1}{2}\, p \mid Q \mid q - \frac{1}{2}\, p \rangle \, dy. \quad (2.12)$$

while the Wigner function for the density operator is defined with an extra factor of $\frac{1}{2\pi}$ for each degree of freedom, i.e for one degree of freedom:

$$W_\rho(q,p) = \frac{1}{2\pi} \int e^{-ip\cdot y} \langle q + \frac{1}{2}\, p \mid \hat{\rho} \mid q - \frac{1}{2}\, p \rangle \, dy. \quad (2.13)$$

Then one can prove that the expectation value of an operator $\hat{A}$ with the density matrix $\hat{\rho}$ is

$$\text{Tr}(\hat{\rho} \hat{A}) = \int W_\rho(q,p)W_{\hat{A}}(q,p) dq dp. \quad (2.14)$$

One can easily see that $W_Q(q,p)$ of Eq. (2.12) can also be expressed as

$$W_Q(q,p) = \text{Tr} \left[ \hat{Q} \hat{\Omega}(q,p) \right], \quad (2.15a)$$

$$\hat{\Omega}(q,p) = \int |q - \frac{1}{2}\, p\rangle e^{-ip\cdot y} \langle q + \frac{1}{2}\, p| \, dy, \quad (2.15b)$$

an expression that will be useful later.

It can be shown that the only wave function whose Wigner transform is non-negative is a Gaussian: in this case, the associated Wigner transform is apparently interpretable as a probability density in phase space (see Eq. (2.24)). The TMSS of Eq. (1.2) is an example where this interpretation is indeed feasible. If, in addition, the Wigner representatives of the DV’s under study are of the proper, or non-dispersive, nature required above, we have a candidate for a LHV theory, where the LHV’s are represented by the canonical variables $q$ and $p$. It seems clear from the outset that it will be rather exceptional for a DV to fall into this category. It is the purpose of the discussion that follows in the present section to identify a class of operators $\hat{A}$ that do correspond to proper DV’s. Although the analysis is certainly not exhaustive, it serves the purpose of indicating a number of sufficient conditions leading to PDV’s. For simplicity, the analysis will be restricted to systems with only one degree of freedom.

Consider a function $f(x)$, where $-\infty \leq x \leq \infty$ and the function is bounded as $|f(x)| \leq 1$.

1. We define the operator $\hat{A}_1 = f(\hat{q})$ through its spectral representation as

$$\hat{A}_1 = f(\hat{q}) = \int_{-\infty}^{\infty} |q'| f(q') \langle q' | dq'. \quad (2.16)$$
The eigenvalues of this operator are \( f(x) \), so that its spectrum lies in the interval \([-1, 1]\). For instance:

(a) \( f(x) = \tanh x \) gives a continuous spectrum in the interval \([-1, 1]\).

(b) \( f(x) = \text{sgn} x \) (where the \( \text{sgn} \) function takes on the value 1 for \( x > 0 \) and -1 for \( x < 0 \)) has a discrete spectrum, consisting of the two values 1 and -1.

One can easily show that the Wigner transform of the operator \( f(\hat{q}) \) of Eq. (2.17) is

\[
W_{f(\hat{q})}(q', p') = f(q'),
\]

(2.17)
a function which takes on, as its values, precisely the eigenvalues of the operator \( f(\hat{q}) \). According to our nomenclature, we are thus dealing with a PDV. As a result, we can write the spectral representation of the operator \( \hat{A}_3 \) of Eq. (2.19) as

\[
\hat{A}_3 = f(\hat{q}),
\]

(2.19)
where

\[
\hat{q} = a\hat{q} + b\hat{p},
\]

(2.20)

\( a \) and \( b \) being numerical constants is a linear combination of the position and momentum operators \( \hat{q} \) and \( \hat{p} \). If we add, to Eq. (2.20), the following one:

\[
\hat{p} = c\hat{q} + d\hat{p},
\]

(2.21)
c and \( d \) being numerical constants satisfying the condition

\[
ad - bc = 1,
\]

(2.22)
then the pair of equations (2.20) and (2.21) can be considered as a transformation from the canonical position and momentum operators \( \hat{q} \) and \( \hat{p} \) to the new ones \( \hat{q} \) and \( \hat{p} \). Thanks to the condition (2.22), the commutator \([\hat{q}, \hat{p}] = [\hat{q}, \hat{p}] = i\) is preserved and the transformation is canonical: it is the quantum-mechanical counterpart of the classical linear canonical transformation obtained from Eqs. (2.20) and (2.21) by removing the “hats” and considering the \( q, p, \hat{q} \) and \( \hat{p} \) as \( c \)-number canonical variables; in the classical problem it is the Poisson bracket that is preserved by the transformation.

The operators \( \hat{q}, \hat{p} \) have the same spectrum, and so do the operators \( \hat{q}, \hat{p} \); we can thus relate the two members of each pair through the unitary transformation

\[
\hat{q} = V^\dagger \hat{q} V
\]

(2.23a)
\[
\hat{p} = V^\dagger \hat{p} V.
\]

(2.23b)
The eigenstates of \( \hat{q} \) and \( \hat{p} \) to be designated by \( |q'\rangle \) and \( |p'\rangle \), respectively, i.e.,

\[
\hat{q}|q'\rangle = q'|q'\rangle \quad (2.24a)
\]

\[
\hat{p}|p'\rangle = p'|p'\rangle, \quad (2.24b)
\]

are related to the eigenstates \( |q'\rangle, |p'\rangle \) of \( \hat{q} \) and \( \hat{p} \), respectively, as

\[
|q'\rangle = V^\dagger |q\rangle \quad (2.25a)
\]

\[
|p'\rangle = V^\dagger |p\rangle. \quad (2.25b)
\]

In terms of the eigenstates \( |q'\rangle \) of \( \hat{q} \), Eq. (2.21), we can write the spectral representation of the operator \( \hat{A}_3 \) of Eq. (2.19) as

\[
\hat{A}_3 = f(\hat{q}) = \int_{-\infty}^{\infty} |q'| f(q') \langle q'| dq'.
\]

(2.26)
Using Eqs. (2.25a) and (2.21), we can write further

\[
\hat{A}_3 = f(\hat{q}) = V^\dagger \int_{-\infty}^{\infty} |q'| f(q') \langle q'| V dq'. \quad (2.27a)
\]

\[
= V^\dagger f(q)V. \quad (2.27b)
\]

From Eq. (2.20) we read off the eigenvalues of the operator \( \hat{A}_3 \) as \( f(x) \) just as for \( \hat{A}_1 \) for which the condition is preserved and the transformation is canonical: it is the quantum-mechanical counterpart of the classical linear canonical transformation obtained from Eqs. (2.20), and (2.21) by removing the “hats” and considering the \( q, p, \hat{q} \) and \( \hat{p} \) as \( c \)-number canonical variables; in the classical problem it is the Poisson bracket that is preserved by the transformation.

The operators \( \hat{q}, \hat{p} \) have the same spectrum, and so do the operators \( \hat{q}, \hat{p} \); we can thus relate the two members of each pair through the unitary transformation

\[
\hat{q} = V^\dagger \hat{q} V
\]

(2.23a)
\[
\hat{p} = V^\dagger \hat{p} V.
\]

(2.23b)
have found a class of observables, i.e., \( f(aq + bp) \) which, together with their Wigner transforms, i.e., \( f(aq' + bp') \), may be termed PDV’s.

As an application, suppose that we have a two-particle problem, with the Wigner distribution associated with the wave function being non-negative. Suppose also that we choose, as the operators \( \hat{A}, \hat{A}' \) to be associated with particle 1, any two (in general non-commuting) of the proper (\( \Rightarrow \) non-dispersive\(^\text{23} \)) DV’s discussed above, like \( A_1, A_2, \) or \( A_3, \) and similarly for the operators \( B, B' \) to be associated with particle 2. Then the CHSH inequality \(^\text{24} \) must be fulfilled, according to the discussion given right before that equation. In the presentation carried out in Sec. \( \text{IV} \) below, \( \hat{A} \) is taken as \( \text{sgn}(\hat{q}) \), i.e., as \( A_1 \) above, Eqn. \( \text{(2.19)} \), case (b); \( \hat{A}' \) is taken as \( A_3 \) above, Eqn. \( \text{(2.19)} \), again with \( f(x) = \text{sgn}(x) \), for two options for the coefficients \( a \) and \( b \). Similar choices are made for \( B \) and \( B' \). For these cases, the validity of the CHSH inequality \(^\text{24} \) is verified explicitly in Sec. \( \text{IV} \).

In contrast, it is easy to give examples of DV’s that do not fulfill the above property of having a Wigner function taking, as its values, the eigenvalues of the quantum operator. For instance, for the observable

\[
\hat{A} = \frac{1}{2} (\hat{\rho}^2 + \hat{\varphi}^2),
\]

the quantum-mechanical spectrum is \( n + 1/2 \) (\( n = 0, 1, 2, \ldots \)). (This spectrum is not bounded in the sense of \( \text{21} \); it just serves as an example to illustrate the point.) In contrast, its Wigner transform is

\[
W_{\frac{1}{2}(\hat{\rho}^2+\hat{\varphi}^2)}(q', p') = \frac{1}{2} [ (p')^2 + (q')^2 ],
\]

which takes on any value in \([0, \infty]\); the DV \( \text{23}\), together with its Wigner representative \( \text{24} \), is thus improper. Some of the observables considered in Sec. \( \text{III} \) below will, indeed, fail to be proper.

The idea of the present section has been to gain a panoramic view of the various particular cases that will be considered in the rest of this paper.

Before turning to a study of these individual situations, we mention in passing one further application of Eq. \( \text{2.28} \). Consider the variation of \( \text{Tr}(\hat{\rho} \hat{A}) \), Eq. \( \text{2.19} \), when the operator \( \hat{A} \) is subjected to the unitary transformation \( \hat{A} \Rightarrow V^\dagger AV \); obviously, the same answer is obtained if, instead, \( \hat{\rho} \) is transformed as \( \hat{\rho} \Rightarrow V \hat{\rho} V^\dagger \). We can calculate the change of the Wigner representative of \( \hat{\rho} \) from Eq. \( \text{2.23} \), valid for any Hermitian operator, replacing \( \hat{A} \) by \( \hat{\rho} \) and \( V \) by its inverse, with the result

\[
W_{V^\dagger BV}(q', p') = W_{\hat{\rho}}(dq' - bp', -cq' + ap'),
\]

which will be useful later.

III. THE EPR-EPW PROBLEM

As outlined in the Introduction, we consider the so-called EPR-EPW problem\(^\text{11} \) in successive levels. The first level is: Given a state, \( |\zeta\rangle \) in our case, whose Wigner representative function is non-negative, does such a state allow BIQV?

The answer to this was shown\(^\text{2,21} \) to be in the affirmative. The DV considered was the parity, \( S_z \) (\( N \) being the number operator),

\[
S_z \equiv \sum_{n=0}^{\infty} \left[ |2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n| \right] = (-1)^N.
\]

In\(^\text{21} \), “rotated” parity operators were introduced:

\[
S_x = \sum_{n=0}^{\infty} \left[ |2n+1\rangle\langle 2n| + |2n\rangle\langle 2n+1| \right],
\]

\[
S_y = i \sum_{n=0}^{\infty} \left[ |2n+1\rangle\langle 2n| - |2n\rangle\langle 2n+1| \right].
\]

These operators close an \( su(2) \) algebra and are viewed as 3-dimensional vector operators. We may thus consider a “rotation” in parity space by, e.g.,

\[
S_x(\vartheta) = e^{\frac{\vartheta}{2} S_z} S_x e^{-\frac{\vartheta}{2} S_z} = S_x \cos \vartheta - S_y \sin \vartheta = \mathbf{S} \cdot \mathbf{n}
\]

with \( \mathbf{n} \) a unit vector which, in this case, is in the “\( x-y \)” plane of the parity space. It will be convenient for us later to refer to the above as the “time evolution” of \( S_x \) under the “Hamiltonian” \( S_z \) in Eq. \( \text{3.3} \); in this way we refer to the “rotation” angle, \( \vartheta \), as the time, \( t \).

Sticking to the geometric notation, the Bell operator\(^\text{12} \) is (the superscript refer to the channels, \( a, a^\dagger \) being channel 1 and \( b, b^\dagger \) channel 2)

\[
\hat{B} = S^1 \cdot \mathbf{n} S^2 \cdot \mathbf{m} + S^1 \cdot \mathbf{n}' S^2 \cdot \mathbf{m}' + S^1 \cdot \mathbf{n} S^2 \cdot \mathbf{m}' - S^1 \cdot \mathbf{n}' S^2 \cdot \mathbf{m},
\]

and the Bell inequality we study is

\[
|\langle \hat{B} \rangle| \leq 2.
\]

Varying \( \mathbf{n}, \mathbf{n}' \) and \( \mathbf{m}, \mathbf{m}' \) to maximize \( |\langle \hat{B} \rangle| \) for the state \( |\zeta\rangle \) we get\(^\text{25} \)

\[
|\langle \zeta | \hat{B} | \zeta \rangle| = 2 \sqrt{1 + F^2(\zeta)};
\]

\[
F(\zeta) = |\langle \zeta | S^1_x S^2_x | \zeta \rangle| = \tanh 2 \zeta.
\]

Thus the state \( |\zeta\rangle \) allows BIQV, even though the Wigner function of the corresponding density operator may be viewed as a probability density of LHV (the phase space coordinates). However, as was stressed in the Introduction, this does not violate Bell’s theorem which prohibits BIQV for a LHV theory. Thus the correlations appearing in the Bell operator have the structure\(^\text{26} \)

\[
|\langle \zeta | S^1_x S^2_x | \zeta \rangle| = \int_{-\infty}^{\infty} dp_1 dq_1 dp_2 dq_2 W(\zeta(p_1, q_1, p_2, q_2)) \cdot W_S^1(p_1, q_1) W_S^2(p_2, q_2).
\]
Here, the factorization of the Wigner function of the two channels is automatic. As explained in detail in Sec. II for the right-hand side of Eq. (3.3) to be interpretable as a LHV theory, aside from a nonnegative Wigner function for the state, \( W_{\zeta} \), we require that the Wigner representative of the DV’s, the \( S_z \)’s in this case, give the observable values of these DV’s, viz., the eigenvalues of the quantal parity operator (for the phase point: \( q, p \)). As already indicated, we refer to a DV with this property as a proper or nondispersive DV\textsuperscript{25}. This is not the case for any of the parity operators, \( S_i \) \((i = x, y, z)\); in fact, e.g., we can easily verify that

\[
W_{S_z}(q,p) = -\pi \delta(\alpha) = -\pi \delta(q)\delta(p), \quad \alpha = q + ip. \tag{3.10}
\]

This clearly is not an eigenvalue of the parity operator (which is \( \pm 1 \)). Thus in this case this DV is improper or dispersive\textsuperscript{25}. Therefore, we are not dealing here with a LHV theory. (In addition, Eq. (3.10) makes it clear the assertion made in the Introduction that the Wigner representative of \( S_z \) violates the property of boundedness.)

We have thus completed the discussion of the first level of the EPR-EPW problem: nothing new was gained but we considered examples that will serve us below.

The second level of the EPR-EPW problem is when, in addition to having a nonnegative Wigner function for the state, we have a DV whose Wigner representative is the value of the DV - i.e. it is a proper or nondispersive DV. Would this situation allow BIQV? Would it conform to Bell’s theorem? Recently\textsuperscript{10,22} an alternative configuration was discussed for the parity operators. In this alternative configuration the operators are given in the \( q \) representation. Denoting the operators in this configuration by \( \Pi_i \) \((i = x, y, z)\), we have,

\[
\Pi_x = -\int_0^\infty dq \left[ |\mathcal{E}\rangle\langle\mathcal{E}| - |\mathcal{O}\rangle\langle\mathcal{O}| \right] = S_z; \tag{3.11}
\]

here,

\[
|\mathcal{E}\rangle = \frac{1}{\sqrt{2}}\left[ |q\rangle + |−q\rangle \right], \quad |\mathcal{O}\rangle = \frac{1}{\sqrt{2}}\left[ |q\rangle − |−q\rangle \right]. \tag{3.12}
\]

so that

\[
\Pi_x = -\int_{−\infty}^\infty dq \left[ |q\rangle⟨−q| \right]. \tag{3.13}
\]

The equality \( \langle n|\Pi_x|n'\rangle = \langle n|S_z|n'\rangle \) is easily verifiable. The natural vectorial operators that close an \( su(2) \) algebra with \( \Pi_x \) are

\[
\Pi_x = \int_0^\infty dq \left[ |\mathcal{E}\rangle\langle\mathcal{O}| + |\mathcal{O}\rangle\langle\mathcal{E}| \right], \tag{3.14}
\]

\[
\Pi_y = i\int_0^\infty dq \left[ |\mathcal{E}\rangle\langle\mathcal{O}| − |\mathcal{O}\rangle\langle\mathcal{E}| \right]. \tag{3.15}
\]

We note that \( \Pi_x \) is diagonal in \( q \), i.e.,

\[
\Pi_x = \int_0^\infty dq \left[ |q\rangle⟨q| − |−q\rangle⟨−q| \right] = sgn(q). \tag{3.16}
\]

is the spectral representation of \( \Pi_x \). Its representative Wigner function is

\[
W_{\Pi_x}(q,p) = sgn(q). \tag{3.17}
\]

i.e., it gives the eigenvalues (\( \pm 1 \)) of the operator and hence is a proper (nondispersive) DV, just as in the discussion of Eq. (2.10), case (b), of Sec. II. In this case, with \( \Pi_x \), much like in the previous case (with the \( S_i \), \( i = x, y, z \)) it is easy to get BIQV by selecting the appropriate orientational parameters. For convenience, while retaining complete generality, we consider the choice of the orientational parameters by choosing the times (for both channels) of the evolution of \( \Pi_1(t_1), \Pi_2(t_2) \) under the Hamiltonian \( H = \Pi_z \). (We note that Bell considered the same case with \( \zeta \to \infty \), i.e., the EPR state, but with the free Hamiltonian, \( H = p^2/2 \)).

Direct calculations show that by appropriate choice of the times \((t_1, t_1', t_2, t_2')\) we get, for our case,

\[
\langle \hat{B} \rangle = 2\sqrt{2}F(2\zeta); \quad \tilde{F} = \frac{2}{\pi} \arctan(\sinh 2\zeta). \tag{3.18}
\]

Thus we see that in this case, where seemingly the quantal description may be given a LHV underpinning, we get a BIQV which, we are told, is an impossibility. However, the present Bell operator involves not only the “proper” DV, \( \Pi_x \), but also \( \Pi_y \), which evolves via our Hamiltonian, \( H = \Pi_z \): the latter, i.e., \( \Pi_y \), is not a proper DV. In fact, its Wigner representative is given by

\[
W_{\Pi_y}(q,p) = −\delta(q)\mathcal{P}\frac{1}{p}. \tag{3.19}
\]

where \( \mathcal{P} \) stands for the “principal value”. Thus, once again, no LHV underpinning for the correlation involved in Eq. (3.18) is possible afterall (the boundedness condition for the Wigner representatives is violated as well).

We may attempt to consider the problem in a Schrödinger-like manner by applying the time evolution operator to the state \( |\zeta\rangle \); this, however, leads to a new state, \( |\zeta'\rangle \), whose Wigner representative function is no longer non-negative over all phase space. This can be proven most readily by considering an alternative expression for the state \( |\zeta\rangle \) obtained in\textsuperscript{25}, i.e.,

\[
|\zeta\rangle = \int_0^\infty \int_0^\infty dq dq' \left[ (g_+ + g_-)|\mathcal{E}\rangle|\mathcal{E}'\rangle + (g_+ - g_-)|\mathcal{O}\rangle|\mathcal{O}'\rangle \right], \tag{3.20}
\]

where

\[
g_{±}(q,q';\zeta) = \langle qq'|S(±\zeta)|00\rangle = \frac{1}{\sqrt{\pi}}\exp\left\{ −\frac{1}{2} \left[ q^2 + q'^2 \mp 2qq'\tanh(2\zeta) \right] \cosh(2\zeta) \right\}. \tag{3.21}
\]

Using this expression for \( |\zeta\rangle \), we have directly

\[
e^{−i\Pi_z}|\zeta\rangle = |\zeta'\rangle = \cos \gamma|\zeta\rangle + \sin \gamma|−\zeta\rangle, \tag{3.22}
\]

and the Wigner function representative of \( |\zeta'\rangle \) is no longer non-negative\textsuperscript{25}. 


IV. BILINEAR HAMILTONIANS

Level 3 of our EPR-EPW problem is the study of cases wherein: (1) The states are having non-negative Wigner representatives which, at some limit, restrict to the EPR state - our $|\zeta\rangle$ is such a state. (2) A DV (= observable) that is non-dispersive (=proper), i.e., such that the Wigner representative of its quantal version gives its eigenvalues in terms of our LHV: $p, q$ - our $\Pi_x$ is such a DV. We inquire for possible BIQV when this DV evolves via Hamiltonians which leave the Wigner representative of the state under study non-negative. Alternatively, we inquire for BIQV when our DV’s evolve by Hamiltonians which allow the initially proper DV to remain as such. In the next paragraphs we study the relationship between these two alternatives.

Since the only non-negative Wigner functions are gaussians, and as gaussians remain gaussians under linear transformations, single-channel Hamiltonians that leave the Wigner function non-negative are bilinear ones. We will consider now two such Hamiltonians:

$$H_0(i) = \frac{1}{2} (\vec{p}_i^2 + \omega_i^2 \vec{q}_i^2) \quad (4.1a)$$
$$H_f(i) = \frac{1}{2} \omega_i^2 \vec{q}_i^2 \quad (4.1b)$$

where the subscript $i = 1, 2$ denotes the channel. For simplicity we shall consider, in $H_0$, the frequency $\omega_i = 1$ for both channels. The second Hamiltonian is the one considered by Bell.

We consider the harmonic oscillator Hamiltonian $H_0$ first. Evolution of the state $|\zeta\rangle$, Eq. (4.2), under $H_0$, during a time $t_1$ for channel 1 and $t_2$ for channel 2, gives:

$$|\zeta(t_1,t_2)\rangle = |\zeta'\rangle = \exp^{-\zeta a^1 b^1 e^{-a} - a^2 e^{-a}} |00\rangle \quad (4.2)$$

where $\theta = t_1 + t_2$. The corresponding Wigner function can be obtained either directly from the state $|\zeta'\rangle$ or from Eq. (4.3), applying Eq. (4.3) with $a = \cos t_1$, $b = \sin t_1$, $c = -\sin t_1$, and $d = \cos t_1$, with the result

$$W_{\zeta(\theta)} = \frac{1}{\pi^4} \exp \left\{-\cosh(2\zeta) (q_1^2 + q_2^2 + p_1^2 + p_2^2)
-2 \sinh(2\zeta) (q_1 q_2 - p_1 p_2) \cos \theta
-(q_1 p_2 + q_2 p_1) \sin \theta\right\} \quad (4.3)$$

Direct evaluation of

$$E(t_1,t_2) = \int_{-\infty}^{\infty} dq dp W_{\zeta(\theta)}(q,p) \Pi_x^1 \Pi_x^2 \quad (4.4)$$

for this case upon the change of variables: $q_1 = q_2 \cos t_1 + p_1 \sin t_1$ and $p_{11} = -q_2 \sin t_1 + p_1 \cos t_1$. We obviously obtain the same answer at the end. Perhaps more elegantly, one can find the Wigner representative of the time evolution of $\Pi_x^1$ applying the general result $\Pi_x(\theta)$ of Sec. II with $a = \cos t_1$, $b = \sin t_1$, $c = -\sin t_1$, and $d = \cos t_1$.

It is easily shown (cf. 4.2) that, in case the time dependence occurs only in the combination $t_1 + t_2$ (which is the case in the present situation (Eq. (4.2)), the CHSH inequality implies the following inequality

$$3P_+ - P_+ (\theta) - P_- (\theta) \geq 0 \quad (4.8)$$

In the $\zeta \to \infty$ limit, i.e., when the state $|\zeta\rangle$ is maximally entangled and approaches the EPR state, $\tanh(2\zeta) \to 1$. In this limit $\chi \to \cos^{-1}(\cos \theta) = \theta$ (cf. App. A) and $P_+ (\theta) = \frac{1}{2} \theta_1$; thus the inequality is saturated. It can be shown that for finite $\zeta$ the inequality is always satisfied. Bell suggested that correlations of observables of the type of $\Pi_x^{\pm 2}$ (cf. Eq. (4.4)) for the EPR state and evolving under the free Hamiltonian would not allow for BIQV; we observe that this indeed occurs for the harmonic oscillator Hamiltonian used here.

However, his reasoning perhaps was somewhat misleading: the reason is that it is not only the nonnegativity of the relevant Wigner function that matters, but also the type of evolution induced in the observables by the Hamiltonian in question. The fulfillment of the CHSH inequality in the present case, in which the evolution is induced by the harmonic oscillator Hamiltonian, is consistent with the discussion given in Sec. II below Eq. (2.30). It is apt to notice that the free Hamiltonian is not analogous to rotation of the spins in the Bohm EPR version. The latter involves what was termed orientational variation, which leads (depending on the preferred viewpoint) either to nonproper (dispersive) DV’s even when ones starts with a proper DV, or, alternatively, to a non nonnegative Wigner function. In either case, BIQV’s do not contradict Bell’s theorem.

We now consider briefly the evolution due to the free Hamiltonian of Eq. (4.3). Again, we study the evolution of the state $|\zeta\rangle$, Eq. (4.2), under $H_f$, during a time $t_1$ for channel 1 and $t_2$ for channel 2. The corresponding Wigner function can be obtained from Eq. (4.4), applying Eq. (4.3) with $a = 1$, $b = t_1$, $c = 0$ and $d = 1$, with the result
Alternatively, just as with the previous Hamiltonian $H_a$ with solution of $\Pi$, one can find the Wigner representative of the time evolution, is, once again, consistent with the discussion of Sec. II, below Eq. (2.30).

Once the evolution is induced by the free Hamiltonian, is, once again, consistent with the discussion of the non-negative Wigner function of any dynamical variables that can be considered as accountable for by a local hidden variable theory with the phase space variables $(q, p)$ being the local hidden variables. A proper dynamical variable is one whose Wigner function representative gives the eigenvalues of the corresponding quantal dynamical variable which the local hidden variable theory aims at underpinning.

Our main conclusion is that the validity of Bell’s inequality that we have considered hinges on the assumption of having definite values for all the dynamical variables and that a proper observable (= dynamical variable) is non-dispersive. Thus only proper dynamical variables can be considered as attributable for by a local hidden variable theory with the phase space variables $(q, p)$ being the local hidden variables.

V. CONCLUSIONS AND REMARKS

In this study we took the Clauser, Horne, Shimony and Holt inequality as the representative of the so called Bell’s inequalities. Indeed this inequality is the often quoted example for underpinning Bell’s locality condition is automatically fulfilled as the Wigner function of any dynamical variables that depend on distinct phase space coordinates factorizes. Thus our discussion underscores a tacit assumption in the derivation of the Bell inequality we consider: viz the dynamical variables must all have a definite value even though they are not or even cannot be measured simultaneously. This point was noted in the past. In point of fact, two often quoted examples for underpinning non-commuting dynamical variables with LHV’s –Bell’s inequality and Wigner’s inequality – are manifestly so, although these examples are, perhaps, somewhat artificial. In the present work – which in its essence follows Bell’s suggestion – we outlined a canonical theory which automatically abides by the locality requirement (the phase space variables are local), and BIQ is abided by in cases where the DV’s are proper ones, even when they are non-commuting.

Our main conclusion is that the validity of Bell’s inequality that we have considered hinges on the assumption of having definite values for all the dynamical variables – thus endowing them with physical reality – and not the issue of locality. Of course one might ponder what would one mean by a local hidden variable theory without a definite value for all the dynamical variables; however this is a separate issue.

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**APPENDIX A: EVALUATION OF \( E(t_1, t_2) \) FOR THE HARMONIC HAMILTONIAN**

We first evaluate \( P_{++}(t_1, t_2) \), cf. Eq. (61). The integral, Eq. (59), after the integration over the \( p \)'s and letting \( q_1 \to -q_1 \), is

\[
P_{++}(t_1, t_2) = \frac{1}{\pi \cosh(2\zeta)\sqrt{(1 - \tanh^2(2\zeta)\cos^2\theta)}}
\]

\[
\cdot \int_0^\infty dq_1dq_2 \exp \left[ -\cosh(2\zeta)\Gamma(\theta, \zeta) \right.
\]

\[
\cdot \left( q_1^2 + q_2^2 - 2q_1q_2\tanh(2\zeta)\cos\theta \right).
\]

(A1)

Here \( \theta = (t_1 + t_2) \) and \( \Gamma(\theta, \zeta) = (1 - \tanh(2\zeta))/(1 - \tanh(2\zeta)\cos^2\theta) \). This integral is evaluated directly to give

\[
P_{++}(t_1, t_2) = \frac{1}{2\pi} \left[ \frac{\pi}{2} - \arctan \left( \frac{\tanh(2\zeta)\cos\theta}{\sqrt{(1 - \tanh^2(2\zeta)\cos^2\theta)}} \right) \right].
\]

Similar calculation gives for

\[
P_{--}(t_1, t_2) = \frac{1}{2\pi} \left[ \frac{\pi}{2} + \arctan \left( \frac{\tanh(2\zeta)\cos\theta}{\sqrt{(1 - \tanh^2(2\zeta)\cos^2\theta)}} \right) \right].
\]

The equality \( P_{++}(t, t') = P_{--}(t, t') \) and \( P_{-+}(t, t') = P_{+-}(t, t') \) is easily verifiable, hence we have for

\[
E(t_1, t_2) = 2P_{++}(\theta) - 2P_{--}(\theta) = \frac{\chi}{\pi}, \quad (A2)
\]

with \( \tanh(2\zeta)\cos\theta \equiv \cos \chi \), \( \theta = t_1 + t_2 \).

**APPENDIX B: THE WIGNER FUNCTION OF \( \Pi_x(t) \) FOR \( H = H_0 \)**

The Wigner function for \( \Pi_x(t) \) is given by

\[
W_{\Pi_x(t)}(x, p) = \frac{1}{2\pi} \int_0^\infty dy \int_0^\infty dq e^{-ipy} \cdot \langle x + y/2 | e^{iH_0t} \cdot \langle q | q - | -q \rangle e^{-iH_0t} | x - y/2 \rangle.
\]

(B1)

Inserting the harmonic oscillator propagators and performing the \( y \) integration gives \( \text{sgn}(x \cos t + p \sin t) \).

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