CONCORDANCE INVARIANT $\Upsilon$ FOR BALANCED SPATIAL GRAPH
USING GRID HOMOLOGY

Hajime Kubota

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Abstract

The $\Upsilon$ invariant is a concordance invariant defined by using knot Floer homology. Földvári\cite{2} gives a combinatorial restructure of it using grid homology. We define the combinatorial $\Upsilon$ invariant for balanced spatial graph using grid homology for balanced spatial graph. We show that the combinatorial $\Upsilon$ is concordance invariant for knots.

1 Introduction

There are various invariants from knot Floer homology such as the $\tau$ invariant and the $\Upsilon$ invariant defined by Ozsváth, Szabó \cite{6},\cite{7}. The $\tau$ invariant and the $\Upsilon$ invariant give homomorphisms from the (smooth) knot concordance group $\mathcal{C}$ to $\mathbb{Z}$ and lower bounds for the slice genus and the unknotting number. The $\tau$ invariant proves the Milnor conjecture $g_4(T_{p,q}) = \frac{1}{2}(p - 1)(q - 1)$. On the other hand, the $\Upsilon$ invariant is a truly stronger than the $\tau$ invariant. The $\Upsilon$ invariant is a family of concordance invariants $\Upsilon_t$ defined for every $t \in [0,2]$ and the slope of the $\Upsilon$ invariant at $t = 0$ equals the value of the $\tau$ invariant. The $\Upsilon$ invariant shows that the subgroup of $\mathcal{C}$ generated by topologically slice knots has $\mathbb{Z}^\infty$ direct summand \cite{7}.

Grid homology is a combinatorial version of knot (link) Floer homology developed by Manolescu, Ozsváth, Szabó and Thurston \cite{4}. The original definition of knot Floer homology needs gauge theory and pseudo-holomorphic curves, so its calculation is not combinatorial. In contrast, grid homology only needs planar figure called grid diagram (figure 1) and counting some rectangles on the grid diagram.

One of the main research direction of grid homology is to give a purely combinatorial proof for the known properties of knot Floer homology. For example, Sarkar \cite{10} gave a combinatorial reconstruction of the $\tau$ invariant, which we denote $\tau^{grid}$, using grid homology. Sarkar also gave purely combinatorial proof that $\tau^{grid}$ is a concordance invariant. As an application of it, he gave a combinatorial proof of the Milnor conjecture. Földvári \cite{2} defined the $\Upsilon^{grid}$ invariant using grid homology and evaluated the change of value on crossing change.

A spatial graph is an smooth embedding $f: G \to S^3$, where $G$ is a one-dimensional CW-complex. We assume that spatial graphs are always oriented. A transverse
spatial graph is a spatial graph such that for each vertex $c$, there is a small disk $D \subset S^3$ separating incoming edges and outgoing edges. In this paper, we often need one more condition: a transverse spatial graph is balanced if at each vertex, the number of incoming edges equals the number of outgoing edges.

Transverse spatial graphs are one of the extensions of oriented links and singular knots. Harvey, O’Donnol [3] extended grid homology to transverse spatial graphs. It is natural to explore generalizing of various results or invariants in knot Floer theory for transverse spatial graphs. In this direction, Vance defined $\tau^{spatial}$ which is the tau invariant for balanced spatial graph. This is an extension of Sarkar’s $\tau^{grid}$ invariant. and defined the $\tau^{spatial}$ invariant for transverse spatial graphs satisfying a certain condition which is called balanced. Vance gave the bounds for the change of the $\tau^{spatial}$ invariant between link cobordism.

The major difference between grid homology for transverse spatial graphs and the original one for knots and links is their Alexander gradings. The original grid homology for knots and links has absolutely $\mathbb{Z}$-valued Alexander grading. On the other hand, the Alexander grading of grid homology for transverse spatial graphs is relatively $\mathbb{Z}$-valued grading; it depends on the choice of graph grid diagrams. Harvey, O’Donnol [3] defined relatively Alexander grading which has values on $H_1(S^3 - f(G))$, where $g$ is a graph grid diagram representing a spatial graph $f : G \to S^3$. Vance defined the grid chain complex $CF^- (g)$ which has relatively $\mathbb{Z}$-valued Alexander filtration $\{ F^m \}_{m \in \mathbb{Z}}$ and defined the way to stabilizing the symmetrized Alexander filtration $\{ F^H_m \}_{m \in \frac{1}{2} \mathbb{Z}}$.

In this paper, we introduce the upsilon invariant $\Upsilon^{spatial}$ for balanced spatial graphs, which is an extension of Földvári’s knot invariant $\Upsilon^{grid}$ using grid homology. As in the case of Vance’s tau invariant, the definition of upsilon invariant for balanced spatial graph is not immediate due to the Alexander grading/filtration problems. We show that for knots $\Upsilon^{spatial}$ (and, also $\Upsilon^{Grid}$) is a concordance invariant. We remark that our $\Upsilon^{spatial}$ contains more information than Földvári’s $\Upsilon^{grid}$, $\Upsilon^{grid}$ is defined only for $[0, 2] \cap \mathbb{Q}$. On the other hand, our $\Upsilon^{spatial}$ is defined for all $[0, 2]$.

The grid chain complex for spatial graphs defined by Harvey and O’Donnol was defined as (Alexander) graded chain complex at first. Next, Vance [11] proved that

Figure 1: an example of balanced spatial graph and graph grid diagram
its filtered version is an invariant up to filtered quasi-isomorphism. In this paper, we extend their results: we prove that these filtered chain complexes are also filtered chain homotopy equivalent to each other (theorem 1.1).

To construct the $\Upsilon$ spatial invariant, we use $t$-modified chain complex $t\text{CF}^H(g)$ from grid chain complex and prove that the homology of $t$-modified chain complex $t\text{HF}^H(g)$ is (almost) independent from the choice of the graph grid diagram. From knot Floer homology, Ozsváth, Stipsicz, and Szabó [7] defined the $\Upsilon$ invariant using a formal construction of $t$-modified chain complex from filtered chain complex $C \rightarrow C^t$ and showed that filtered chain homotopy equivalent complexes $C \simeq D$ are lifted to chain homotopy equivalent complexes $C^t \simeq D^t$. Applying these facts and considering the connection between grid chain complexes and $t$-modified chain complexes enable us to define $\Upsilon$ spatial for all $[0, 2]$ and ensure the invariance of the $\Upsilon$ spatial.

For $t \in [0, 2]$, let $\mathcal{R}_t$ be a certain based ring (see Definition 3.1). Let $W_t$ be a two-dimensional graded vector space $W_t \cong F_0 \oplus F_{-1+t}$, where $F = \mathbb{Z}/2\mathbb{Z}$ and their indices describe the $t$-grading. For a graded $\mathcal{R}_t$-module $X$, $X[a]$ denotes a shift of graded $\mathcal{R}_t$-module so that $X[a]_d = X_{d+a}$. Then,

$$X \otimes W_t \cong X \oplus X[1-t].$$

Followings are main theorems of this paper.

**Theorem 1.1.** Let $g, g'$ be two graph grid diagrams for balanced spatial graph $f: G \rightarrow S^3$, then there is a chain homotopy equivalence

$$\text{CF}^H(g) \simeq \text{CF}^H(g').$$

**Theorem 1.2.** Let $g, g'$ be two graph grid diagrams for balanced spatial graph $f: G \rightarrow S^3$. If their grid numbers are $n, m$ ($n \geq m$), which means that $g$ has $n \times n$ squares and $g'$ has $m \times m$ squares, then as graded $\mathcal{R}$ modules,

$$t\text{HF}^H(g) \cong t\text{HF}^H(g') \otimes W_t^{\otimes(n-m)}.$$
Theorem 1.5. If $L_1, L_2$ are two links of $l_1, l_2$-components and there is a genus $g$ link cobordism from $L_1$ to $L_2$, then
\[ \Upsilon_{L_1}(t) - tg - t(l_1 - 1) - (l_1 - l_2) \leq \Upsilon_{L_2}(t) \leq \Upsilon_{L_1}(t) + tg + t(l_2 - 1) + (l_2 - l_1). \]
Especially, if $L_1, L_2$ are two knots $K_1, K_2$, then
\[ |\Upsilon_{K_1}(t) - \Upsilon_{K_2}(t)| \leq tg, \]
i.e. $\Upsilon$ is a concordance invariant for knots.

Corollary 1.6. For $t \in [0, 1]$,
\[ |\Upsilon_K(t)| \leq t \cdot g_s(K), \]
where $g_s(K)$ is the slice genus of $K$.

1.1 Some properties of $\Upsilon$

Grid homology is a combinatorial version of knot Floer homology, so it is expected that $\Upsilon$ for grid homology has the same properties as the original one. Some properties of the original $\Upsilon$ invariant are proved using algebraic techniques, and we show some of them in the same way.

Proposition 1.7. $\Upsilon_f(0) = 0$.

Proposition 1.8. If two links $L_+$ and $L_-$ differ in a crossing change, then for $t \in [0, 1]$,
\[ \Upsilon_{L_+}(t) \leq \Upsilon_{L_-}(t) \leq \Upsilon_{L_+}(t) + (2 - t) \]

Földvári proved this property when two links are knots and $t$ is rational. It is proved by constructing concrete maps between $t$-modified chain complexes (see [2]). In this paper, we show that that maps are induced from some maps on grid chain complex. Note that this proposition is weaker than the original property in proposition 1.10,[4].

For a balanced spatial graph $f: G \to S^3$ and an integer $n$, let $U(f)$ is a balanced spatial graph which is disjoint union of an unknot and $f$, regarding an unknot as a graph with one vertex and one edge.

Proposition 1.9. For a balanced spatial graph $f$,
\[ \Upsilon_{U(f)}(t) = \Upsilon_f(t). \]

For a balanced spatial graph $f: G \to S^3$ and a vertex $c$, let $f \#_c \mathcal{O}$ denote the balanced spatial graph obtained by attaching new unknotted, unlinked edge going from $c$ to $c$.

Proposition 1.10. For a balanced spatial graph $f: G \to S^3$,
\[ \Upsilon_{f \#_c \mathcal{O}}(t) = \Upsilon_f(t). \]
2 Grid homology for transverse spatial graphs

This section provides the overview of grid homology for transverse spatial graphs. See [3] for details. For grid homology for knots and links, see [4],[8].

A planar graph grid diagram $g$ (figure1) is a $n \times n$ grid of squares some of which is decorated with an $X$- or $O$- (sometimes $O^*$-) markings with following conditions.

(i) There is just one $O$ or $O^*$ on each row and column.

(ii) There is at least one $X$ on each row and column.

(iii) $O$’s (or $O^*$’s) and $X$’s do not share the same square.

We denote the set of $O$-markings by $O$ and the set of $X$-markings by $X$ and use the labeling of markings as $\{O_i\}_{i=1}^n$ and $\{X_j\}_{j=1}^m$.

Connecting the $O$ (or $O^*$) and $X$-markings by segments in each row and column and assume that vertical segments are the overpasses under horizontal segments. $O^*$-markings correspond to vertices, $O$- and $X$-markings to interior of edges.

Harvey and O’Donnol showed that any two graph grid diagrams representing same spatial graph are connected by a finite sequence of the graph grid moves. The graph grid moves are these three moves:

- **Cyclic permutation** permuting the rows and columns cyclically.
- **Commutation’** permuting two adjacent columns satisfying following condition; vertical line segments $LS_1, LS_2$ on the torus such that (1) $LS_1 \cup LS_2$ contain all the $X$’s and $O$’s in the two adjacent columns, (2) the projection of $LS_1 \cup LS_2$ to a single vertical circle $\beta_i$ is $\beta_i$, and (3) the projection of their endpoints, $\partial(LS_1) \cup \partial(LS_2)$, to a single $\beta_i$ is precisely two points. Permuting two rows are defined in the same way.
- **(de-)stabilization’** let $g$ be an $n \times n$ graph grid diagram and choose an $X$-marking. Then $g'$ is called a stabilization’ of $g$ if it is an $(n + 1) \times (n + 1)$ graph grid diagram obtained by adding one new row and column next to the $X$-marking of $g$, moving the $X$-marking to next column, and putting new one $O$-marking just above the $X$-marking and one $X$-marking just upper left of the $X$-marking. The inverse of a stabilization is called a destabilization.

A toroidal graph grid diagram is a graph grid diagram which we think it as a diagram on the torus obtained by identifying edges in a natural way. We write the horizontal circles and vertical circles which separate squares as $\alpha = \{\alpha_i\}_{i=1}^n$ and $\beta = \{\beta_j\}_{j=1}^n$

A state $x$ of $g$ is a bijection $\alpha \to \beta$. We denote by $S(g)$ the set of states of $g$. We describe a state as $n$ points on graph grid diagram (figure5).

Fix $x, y \in S(g)$, a domain $\psi$ from $x$ to $y$ is a formal sum of the closure of squares divided by $\alpha \cup \beta$ satisfying $\partial(\partial_\alpha \psi) = y - x$ and $\partial(\partial_\beta \psi) = x - y$, where $\partial_\alpha \psi$ is the
Figure 2: cyclic permutation

Figure 3: commutation’, note that gray lines are LS$_1$ and LS$_2$

Figure 4: stabilization’
Consider \( x, y \in S(g) \) that coincide with \( n - 2 \) points. An rectangle \( r \) from \( x \) to \( y \) is a domain which satisfies that \( \partial r \) is the union of four segments. A rectangle \( r \) is an empty rectangle if \( x \cap \text{Int}(r) = y \cap \text{Int}(r) = \emptyset \). Let \( \text{Rect}^\emptyset(x, y) \) be the set of empty rectangles from \( x \) to \( y \).

The grid chain complex \( (CF^-(g), \partial^-) \) is a module over \( \mathbb{F}[U_1, \ldots, U_n] \) freely generated by \( S(g) \), where \( \mathbb{F} = \mathbb{Z}/2\mathbb{Z} \) and the \( U_i \)'s are formal variables corresponding to the \( O_i \)'s in \( g \). The differential \( \partial^- \) is defined as counting empty rectangles by

\[
\partial^-(x) = \sum_{y \in S(g)} \left( \sum_{r \in \text{Rect}^\emptyset(x, y)} U_1^{O_1(r)} \cdots U_n^{O_n(r)} \right) y,
\]

where \( O_i(r) = 1 \) if \( r \) contains \( O_i \) and \( O_i(r) = 0 \) otherwise for \( i = 1, \ldots, n \).

There are two gradings for \( CF^-(g) \), the Maslov grading and the Alexander grading. A planar realization of toroidal diagram \( g \) is a planar figure obtained by cutting toroidal diagram \( g \) along \( \alpha_i \) and \( \beta_j \) for some \( i \) and \( j \), and putting on \( [0, n) \times [0, n) \in \mathbb{R}^2 \) in a natural way. For two points \( (a_1, a_2), (b_1, b_2) \subset \mathbb{R}^2 \), let \( (a_1, a_2) < (b_1, b_2) \) if \( a_1 < b_1 \) and \( a_2 < b_2 \). For two sets of finitely points \( A, B \subset \mathbb{R}^2 \), let \( \mathcal{I}(A, B) \) be the number of pairs \( a \in A, b \in B \) with \( a < b \) and let \( \mathcal{J}(A, B) = (\mathcal{I}(A, B) + \mathcal{I}(B, A))/2 \). Then for \( x \in S(g) \), the Maslov grading \( M(x) \) and the Alexander grading \( A(x) \) are defined by

\[
M(x) = \mathcal{J}(x - \emptyset, x - \emptyset) + 1, \quad (2.1)
\]

\[
A(x) = \mathcal{J}(x, x - \sum_{i=1}^n m_i O_i), \quad (2.2)
\]

where \( m_i \) is the number of \( X \)-markings on the same row as \( O_i \) \( (i = 1, \ldots, n) \). These two gradings are extended to the whole of \( CF^-(g) \) by

\[
M(U_i) = -2, A(U_i) = -m_i \quad (i = 1, \ldots, n), \quad (2.3)
\]
Note that the Alexander grading here is the same as Vance’s definition, while the original Alexander grading by Harvey and O’Dollon has values in $H_1(S^3 - f(G))$, where $f: G \to S^3$ is the transverse spatial graph represented by $g$. The Alexander grading here is given from the original one by taking the canonical homomorphism $H_1(S^3 - f(G)) \to \mathbb{Z}$ which sends the generators to 1. Also note that both of Alexander grading are not well-defined as a toroidal diagram, however relative Alexander grading $A^{rel}(x, y) = A(x) - A(y)$ is well-defined.

It is shown that the differential $\partial^-$ drops Maslov grading by 1 and preserves or drops Alexander grading, so $(CF^-(g), \partial^-)$ is Maslov graded, Alexander filtered chain complex.

Suppose the $O$-markings are labeled so that $O_1, \ldots, O_V$ are $O^*$-markings and $O_{V+1}, \ldots, O_n$ are $O$-markings. Let $U$ be the minimal subcomplex of $CF^-(g)$ containing $U_1CF^-(g) \cup \cdots \cup U_VCF^-(g)$. Then $(\hat{CF}(g), \hat{\partial})$ is also Maslov graded, Alexander filtered chain complex over $\mathbb{F}$-vector space obtained by letting $\hat{CF}(g) = CF^-(g)/U$ and $\hat{\partial}$ be the map induced by $\partial^-$. The homology of associated graded object of $CF^-(g)$ is an invariant as an absolute Maslov graded and relative Alexander graded $\mathbb{F}$-module and the homology of associated graded object of $\hat{CF}(g)$ is an invariant as an absolute Maslov graded and relative Alexander graded bigraded $\mathbb{F}$-vector space.

We denote by $\{\mathcal{F}_{m}(g)\}_{m \in \mathbb{Z}}$ (respectively $\{\hat{F}_{m}(g)\}_{m \in \mathbb{Z}}$) the Alexander filtration of $CF^-(g)$ (respectively $\hat{CF}(g)$).

**Definition 2.1** (Definition3.10,[III]). For a graph grid diagram $g$, define the symmetrized Alexander filtration $\{\hat{F}_{m}(g)\}_{m \in \frac{1}{2}\mathbb{Z}}$ to be the absolute Alexander filtration obtained by fixing the relative Alexander grading so that $m_{\max}(g) = -m_{\min}(g)$, where

$m_{\max}(g) = \max \left\{ m | H_*(\hat{F}_{m}(g)/\hat{F}_{m-1}(g)) \neq 0 \right\},$

$m_{\min}(g) = \min \left\{ m | H_*(\hat{F}_{m}(g)/\hat{F}_{m-1}(g)) \neq 0 \right\}.$

**Definition 2.2.** The symmetrized Alexander grading $A^H: S(g) \to \frac{1}{2}\mathbb{Z}$ is determined by symmetrized Alexander filtration $\{\hat{F}_{m}(g)\}_{m \in \frac{1}{2}\mathbb{Z}}$ so that for $x \in S(g)$, the value of $A^H(x)$ is the maximal filtration level which $x \in \hat{CF}(g)$ belongs.

Let $CF^{-H}(g)$ be Maslov graded, symmetrized Alexander filtered chain complex obtained from $CF^-(g)$ by using symmetrized Alexander function $A^H$ rather than Alexander grading $A$.

### 3 t-modified chain complex and the $\Upsilon$ invariant

This section provides the definition of t-modified chain complex $tCF^{-H}(g)$ and the $\Upsilon$ invariant. The t-modified chain complex here is the extension of one of Földvári.
First of all, we define the based ring \( R \) of \( tCF^{-H}(g) \).

A set \( A \subset \mathbb{R} \) is well-ordered if any subset \( A' \subset A \) has a minimal element.

**Definition 3.1** (Definition 3.1,[7]). Let \( \mathbb{R}_{\geq 0} \) denote the set of nonnegative real numbers. The ring of long power series \( \mathcal{R}_t \) defined as follows. As an abelian group, \( \mathcal{R}_t \) is the group of formal sums

\[
\left\{ \sum_{\alpha \in A} v^\alpha | A \subset \mathbb{R}_{\geq 0}, A \text{ is well - ordered} \right\},
\]

The sum in \( \mathcal{R}_t \) is given by the formula

\[
\left( \sum_{\alpha \in A} U^\alpha \right) + \left( \sum_{\beta \in B} v^\beta \right) = \sum_{\gamma \in C = A\cup B \setminus A \cap B} v^\gamma,
\]

and the product is given by the formula

\[
\left( \sum_{\alpha \in A} v^\alpha \right) \cdot \left( \sum_{\beta \in B} v^\beta \right) = \sum_{\gamma \in A+B} \#\{(\alpha, \beta) \in A \times B | \alpha + \beta = \gamma\} \cdot v^\gamma,
\]

where

\[
A + B = \{ \gamma | \gamma = \alpha + \beta \text{ for some } \alpha \in A \text{ and } \beta \in B \}.
\]

It is straightforward to check that above operations are well-defined.

For a domain \( \psi \) as a formal sum of squares, let \( O_i(\psi) \) denote the coefficient of the square containing \( O_i \), and let

\[
|O \cap p| := \sum_{i=1}^{n} O_i(p).
\]

We define \( |X \cap \psi| \) in the same manner.

**Definition 3.2.** For \( t \in [0, 2] \), t-modified grid complex \( tCF^{-H}(g) \) is a free module over \( \mathcal{R}_t \) generated by \( S(g) \) with the \( \mathcal{R}_t \)-module homomorphism \( \partial_t^- \) defined by

\[
\partial_t^-(x) = \sum_{y \in S(g)} \left( \sum_{r \in \text{Rect}^+(x,y)} v^{t|X \cap r|+2|O \cap r|-t\left( \sum_{i \in \partial r} m_i \right)} \right) y.
\]

**Definition 3.3.** For \( x \in S(g) \), let the t-grading \( \text{gr}_t \) be,

\[
\text{gr}_t(v^\alpha x) = M(x) - tA^H(x) - \alpha,
\]

where \( A^H \) is the symmetrized Alexander grading.

**Proposition 3.4.** \((tCF^{-H}(g), \partial_t^-) \) is a t-graded chain complex over \( \mathcal{R}_t \).
Proposition 3.5. $\partial_{t}^{-} \circ \partial_{t}^{-} = 0$.

Proof. For states $x$ and $z$ and a fixed domain $\psi \in \pi(x,z)$ denote by $N(\psi)$ the number of ways to decompose $\psi$ as a composite of two empty rectangles $r_1 \ast r_2$. Note that if $\psi = r_1 \ast r_2$ for some $r_1 \in \text{Rect}^\circ(x,y), r_2 \in \text{Rect}^\circ(y,z)$, then
\[
|X \cap p| = |X \cap r_1| + |X \cap r_2|,
|O \cap p| = |O \cap r_1| + |O \cap r_2|,
\sum_{O_i \in O \cap \psi} m_i = \sum_{O_i \in O \cap r_1} m_i + \sum_{O_i \in O \cap r_2} m_i.
\]

It follows that for $x \in \mathcal{S}(g)$,
\[
\partial_{t}^{-} \circ \partial_{t}^{-} (x) = \sum_{z \in \mathcal{S}(g)} \left( \sum_{\psi \in \pi(x,z)} N(\psi) v^{t|X \cap \psi|+2|O \cap \psi|-t(\sum_{O_i \in O \cap \psi} m_i)} z \right). \tag{3.3}
\]

If $\#\{x \cap (x \cap z)\} = 4$ or $\#\{x \cap (x \cap z)\} = 3$, the same argument in [2, Theorem 3.2] shows that $N(\psi)$ is even. If $x = z$, $\psi$ is an annulus and $N(\psi) = 1$. Since $r_1$ and $r_2$ are empty, this annulus has height or width equal to 1. Such an annulus is called a thin annulus. For each $x$, there are $2n$ thin annuli appearing in (3.3). We can pair annuli that contain the same $O$-marking. If $(\psi_1, \psi_2)$ is such a pair, then $|\psi_1 \cap X| = |\psi_2 \cap X|$ because $f$ is balanced spatial graph. So the all terms are canceled in pairs (we are working modulo 2). \hfill \square

Proposition 3.6. The map $\partial_{t}^{-}$ drops the $t$-grading by one.

Proof. By [4, Lemma 2.5] and [11, Lemma 3.5], for $r \in \text{Rect}^\circ(x,y)$,
\[
M(x) - M(y) = 1 - 2|r \cap O|, \tag{3.4}
A^H(x) - A^H(x) = |r \cap X| - \sum_{O_i \in r \cap O} m_i. \tag{3.5}
\]

If $v^{t|X \cap r|+2|O \cap r|-t(\sum_{O_i \in O \cap r} m_i)} y$ appears in $\partial_{t}^{-} (x)$, then
\[
gr_{t}(v^{t|X \cap r|+2|O \cap r|-t(\sum_{O_i \in O \cap r} m_i)} y) = M(y) - tA^H(y) - \left( t|X \cap r| + 2|O \cap r| - t \left( \sum_{O_i \in O \cap r} m_i \right) \right)
= M(x) - 1 + 2|r \cap O| - t \left( A^H(x) - |r \cap X| - \left( \sum_{O_i \in O} m_i \right) \right)
- \left( t|X \cap r| + 2|O \cap r| - t \left( \sum_{O_i \in O \cap r} m_i \right) \right)
= M(x) - tA^H(x) - 1.
\]
\hfill \square
proof of Proposition 3.4. From Proposition 3.5 and Proposition 3.6, we conclude that \((tCF^{-H}(g), \partial_\tau^-)\) is t-graded chain complex over.

**Definition 3.7.** Let \(tCF_d^{-H}(g) = \{\alpha \in tCF(g) | \text{gr}_t(\alpha) = d\}\), and we define \(t\)-modified graph grid homology of \(g\) to be

\[
tHF_d^{-H}(g) = \frac{\text{Ker}(\partial_\tau^-) \cap tCF_d^{-H}(g)}{\text{Im}(\partial_\tau^-) \cap tCF_d^{-H}(g)},
\]

\[
tHF^{-H}(g) = \bigoplus_d tHF_d^{-H}(g).
\]

**4 Formal construction of \(t\)-modified chain complex**

In this section, we give an alternative definition of \(t\)-modified chain complex using the original grid chain complex. See [7, Section 4] for details.

Suppose that \(C\) is a finitely generated, Maslov graded, Alexander filtered chain complex over \(\mathbb{F}[U]\). Let \(x\) be a generator of \(C\) over \(\mathbb{F}[U]\), with Maslov grading \(M(x)\). Since multiplication by \(U\) drops the Maslov grading by 2, elements of Maslov grading \(M(x) - 1\) are linear combinations of elements of the form \(U^{\frac{M(y) - M(x) + 1}{2}} y\), where \(y\) is a generator. Then the differential on \(C\) can be written as

\[
\partial(x) = \sum_y c_{x,y} \cdot U^{\frac{M(y) - M(x) + 1}{2}} y,
\]

where \(c_{x,y} \in \mathbb{F}\).

**Definition 4.1** (Definition 4.1,[7]). For \(t \in [0, 2]\), suppose that \(C\) is a finitely generated, Maslov graded, Alexander filtered chain complex over \(\mathbb{F}[U]\), and let \(R_t\) be the ring of Definition 3.1 (containing \(\mathbb{F}[U]\) by \(U = v^2\)). **formal \(t\)-modified chain complex** \(C^t\) of \(C\) is defined as follows:

- As an \(R_t\)-module, \(C^t = C \otimes_{\mathbb{F}[U]} R_t\)
- For each generator \(x\) of \(C\), define \(\text{gr}_t(v^\alpha x) = M(x) - tA(x) - \alpha\)
- Endow the graded module \(C^t\) with a differential

\[
\partial_t(x) = \sum_y c_{x,y} \cdot v^{\text{gr}_t(y) - \text{gr}_t(x) + 1} y,
\]

where \(c_{x,y}\) are determined by (4.1).

**Definition 4.2.** For a graph grid diagram \(g\), let \(CF_U^{-H}(g)\) denote the induced chain complex \(CF_U^{-H}(g) = \frac{CF^{-H}(g)}{U_1 = \cdots = U_n}\), as a Maslov graded chain complex over \(\mathbb{F}[U]\)-module.
Note that $CF_{U^{-H}}(g)$ is a Maslov graded chain complex but not Alexander filtered chain complex because the drop in Alexander grading of each $U_i$ differs by the definition (2.3).

**Proposition 4.3.** As graded chain complexes, $(CF_{U^{-H}}(g))^t$ (applying Definition 4.1) is isomorphic to $tCF_{U^{-H}}(g)$ from Definition 3.2.

**Proof.** Identifying the generators and their gradings is natural, so we only need to verify that for $x, y \in S(g)$,

$$c_{x,y} = 1 \Leftrightarrow #(\text{Rect}^\circ(x, y)) = 1,$$

and

$$\text{gr}_t(y) - \text{gr}_t(x) + 1 = t|X \cap r| + 2|\emptyset \cap r| - t\left( \sum_{\partial_i \in \partial \cap r} m_i \right).$$

If $#(\text{Rect}^\circ(x, y)) = 2$, then for two rectangles $r_1, r_2 \in \text{Rect}^\circ(x, y)$ there exists a domain with a single square $r_0$ such that $r_0 \ast r_1$ and $r_0 \ast r_2$ are thin annuli. So $\partial(x) = 0$ on $CF_{U^{-H}}(g) = CF_{U_i = \cdots = U_n}$, (4.2) is proved. We have (4.3) from (3.4) and (3.5).

Although $CF_{U^{-H}}(g)$ is not an Alexander filtered chain complex, since the definition of the Alexander grading make sense for the generators of $CF_{U^{-H}}(g)$, then Definition 4.1 also make sense for $CF_{U^{-H}}(g)$. Therefore the formal $t$-modified chain complex $(CF_{U^{-H}}(g))^t$ can be defined.

Next, we introduce the proposition playing an important role in this paper.

**Proposition 4.4** (Proposition 4.4,[7]). Let $f : C \rightarrow C'$ be a Maslov graded, Alexander filtered chain map between chain complexes over $\mathbb{F}[U]$. There is a corresponding graded chain map $f^t : C^t \rightarrow (C')^t$, with the following properties:

- If $f : C \rightarrow C'$ and $g : C' \rightarrow C''$ are two Maslov graded, Alexander filtered chain maps, then

$$(g \circ f)^t = g^t \circ f^t$$

- If $f, g : C \rightarrow C'$ are chain homotopic to each other, then $f^t$ and $g^t$ are chain homotopic to each other. In particular, filtered chain homotopy equivalent complexes are transformed by the construction $C \mapsto C^t$ into homotopy equivalent complexes.

Again note that this proposition can be applied as $CF_{U^{-H}}(g) \mapsto (CF_{U^{-H}}(g))^t$ even if $CF_{U^{-H}}(g)$ is just Maslov graded chain complex with $A^H$ grading for generators. Similar to (4.1), Maslov graded chain map $f : CF_{U^{-H}}(g) \rightarrow CF_{U^{-H}}(g')$ can be written as

$$f(x) = \sum_y c_{x,y} \cdot U^{\frac{M(y) - M(x)}{2}} y,$$

then corresponding graded chain map can be defined as

$$f^t(x) = \sum_y c_{x,y} \cdot \gamma^{\text{gr}_t(y) - \text{gr}_t(x)} y.$$

This construction gives induced chain homotopy equivalence for $t$-modified chain complexes.
5 Chain homotopy equivalences for grid chain complexes

This section provides filtered chain homotopy equivalences for grid chain complexes that are connected by a single graph grid move. Filtered chain homotopy equivalences are lifted into chain homotopy equivalences for t-modified chain complexes by applying Proposition 4.4.

5.1 cyclic permutation

Proposition 5.1. If \( g \) and \( g' \) are connected by a single cyclic permutation, then as a Maslov graded, Alexander filtered chain complexes, \( CF^{-H}(g) \) and \( CF^{-H}(g') \) are chain homotopy equivalent.

Proof. There is a natural bijection \( c: S(g) \rightarrow S(g') \). By [11, Section 3.3 and Theorem 3.15], \( c \) induces an isomorphism of Maslov graded, symmetrized Alexander filtered chain complexes \( c: CF^{-H}(g) \rightarrow CF^{-H}(g') \). As \( c: S(g) \rightarrow S(g') \) is a bijection, there is an isomorphism \( c^{-1}: CF^{-H}(g') \rightarrow CF^{-H}(g) \) induced by \( c^{-1} \) so that

\[
\begin{align*}
  c^{-1} \circ c &= id, \\
  c \circ c^{-1} &= id
\end{align*}
\]

\[\square\]

5.2 commutation’

Proposition 5.2. If \( g \) and \( g' \) are connected by a single commutation’, then as a Maslov graded, Alexander filtered chain complexes, \( CF^{-H}(g) \) and \( CF^{-H}(g') \) are chain homotopy equivalent.

Proof. By [11, Lemma 3.21] and [4, Proposition 3.2], there are chain homotopic equivalences \( \Phi_{\beta'\gamma}: CF^{-}(g) \rightarrow CF^{-}(g') \) and \( \Phi_{\alpha'\beta}: CF^{-}(g') \rightarrow CF^{-}(g) \). Since [11, Theorem 3.15], they preserve symmetrized Alexander filtration. \( \square \)

5.3 stabilization’

The case of stabilization’ is more complicated. Assume that \( g' \) is obtained from \( g \) by a single stabilization’ and \( CF^{-H}(g) \) is a Maslov graded, Alexander filtered chain complex over \( \mathbb{F}[U_2, \ldots, U_n] \) and \( CF^{-H}(g') \) is a chain complex over \( \mathbb{F}[U_1, \ldots, U_n] \). Number the O-marking so that \( O_1 \) is the new one of \( g' \), and \( O_2 \) is the O-marking in the row just below \( O_1 \). We will also assume that \( X_1 \) lies in the same row as \( O_1 \) and \( X_2 \) is the X-marking just below \( O_1 \). We denote by \( c \) the intersection point of the new horizontal and vertical circles in \( g' \) (see figure 6) and assume that the point \( c \) is on the horizontal
circle \( \alpha_i \) and on the vertical circle \( \beta_j \). Note that there may be more \( X \)-markings in the row containing \( O_2 \), and \( O_2 \) may be an \( O^* \)-marking.

For a graded, filtered chain complex \( X \), let \( X' = F_{s+b}X_{d+a} \) denote the shifted one so that \( F_sX'_d = F_{s+b}X_{d+a} \), where \( F_sX_d \) is the submodule of \( X \) whose grading is \( d \) and filtration level is \( s \).

Considering the point \( c \), we will decompose the set of states \( S(g') \) as the disjoint union \( I(g') \cup N(g') \), where \( I(g') = \{ x \in S(g') | c \in x \} \) and \( N(g') = \{ x \in S(g') | c \not\in x \} \). This decomposition gives a decomposition of \( tCF^{-}(g') = I \oplus N \), as \( I \) and \( N \) denote \( \mathcal{R} \)-modules spanned by \( I(g') \) and \( N(g') \).

There is a natural bijection between \( I(g') \) and \( S(g) \), given by

\[ e : I(g') \rightarrow S(g), \ x \cup \{ c \} \mapsto x. \]

Let \( (CF^{-H}(g)[U_1], \partial) \) be a chain complex defined by \( CF^{-H}(g)[U_1] = CF^{-H}(g) \otimes \mathbb{F}[U_2, \ldots, U_n] \) with the differential \( \partial = \partial^- \otimes \text{id} \).

Consider a chain complex \( \text{Cone}(U_1 - U_2 : CF^{-H}(g)[U_1] \rightarrow CF^{-H}(g)[U_1]) \). For the stabilization’ invariance, it is sufficient to prove the following proposition.

**Proposition 5.3.** If \( g' \) is obtained from \( g \) by stabilization’, then as Maslov graded, Alexander filtered chain complexes, \( CF^{-H}(g') \) and \( \text{Cone}(U_1 - U_2) \) are chain homotopy equivalent.

The proof is similar to the proof of stabilization invariance of grid homology for knots and links. As presented in section 13.3.2,[8], we will show this proposition by counting following types of domains.

**Definition 5.4.** For \( x \in S(g') \) and \( y \in I(g') \), a positive domain \( p \in \pi(x, y) \) is said to be of type \( \text{i}L \) if it is trivial or it satisfies the following conditions:

- At each corner point in \( x \cup y \setminus \{ c \} \), at least three of the four adjoining squares have local multiplicity zero.
- Three of four squares attaching at the corner point \( c \) have the local multiplicity \( km \) and at the southwest square meeting \( c \) the local multiplicity is \( k - 1 \).
Figure 7: Examples of $\pi^i_L(x, y)$ and $\pi^i_R(x, y)$

- $y$ has $2k + 1$ components that are not in $x$.

And, a positive domain $p \in \pi(x, y)$ is said to be of type $iR$ if it is trivial or it satisfies the following conditions:

- At each corner point in $x \cup y \setminus \{c\}$, at least three of the four adjoining squares have local multiplicity zero.
- Three of four squares attaching at the corner point $c$ have the local multiplicity $km$ and at the southeast square meeting $c$ the local multiplicity is $k + 1$.
- $y$ has $2k + 2$ components that are not in $x$.

The set of domains of type $iL$ (respectively type $iR$) from $x$ to $y$ is denoted $\pi^i_L(x, y)$ (respectively $\pi^i_R(x, y)$).

**Definition 5.5.** For $x \in S(g')$ and $y \in I(g')$, a positive domain $\phi \in \pi(x, y)$ is said to be of type $oL$ if it is trivial or it satisfies the following conditions:

- At each corner point in $x \cup y \setminus \{c\}$, at least three of the four adjoining squares have local multiplicity zero.
- Three of four squares attaching at the corner point $c$ have the local multiplicity $k$ and at the northwest square meeting $c$ the local multiplicity is $k - 1$.
- $y$ has $2k + 1$ components that are not in $x$.

And, a positive domain $p \in \pi(x, y)$ is said to be of type $oR$ if it is trivial or it satisfies the following conditions:

- At each corner point in $x \cup y \setminus \{c\}$, at least three of the four adjoining squares have local multiplicity zero.
- Three of four squares attaching at the corner point $c$ have the local multiplicity $k$ and at the northeast square meeting $c$ the local multiplicity is $k + 1$.
- $y$ has $2k + 2$ components that are not in $x$.

The set of domains of type $oL$ (respectively type $oR$) from $x$ to $y$ is denoted $\pi^o_L(x, y)$ (respectively $\pi^o_R(x, y)$).
**Figure 8:** Examples of $\pi^{oL}(\mathbf{x}, \mathbf{y})$ and $\pi^{oR}(\mathbf{x}, \mathbf{y})$

**Definition 5.6.** For $\mathbf{x}, \mathbf{y} \in \mathbf{N}(g')$, a positive domain $\phi \in \pi(\mathbf{x}, \mathbf{y})$ is said to be of type $K1$ if it satisfies the following conditions:

- $\#\{(\mathbf{x} \cup \mathbf{y}) \setminus (\mathbf{x} \cap \mathbf{y})\} \in 4\mathbb{N}$.
- One point of $(\mathbf{x} \cup \mathbf{y}) \setminus \mathbf{x} \cap \mathbf{y}$ on the vertical circle $\beta_j$ is contained in the interior of $\phi$, and at the other points, three of the four adjoining squares have local multiplicity zero.
- $\phi$ satisfies that $O_1(\phi) = X_2(\phi) = X_1(\phi) + 1$, where $X_i(\phi)$ is the coefficient of $\phi$ at the square containing $X_i$ ($i = 1, 2$).

A positive domain $\phi \in \pi(\mathbf{x}, \mathbf{y})$ is said to be of type $K2$ if it satisfies the following conditions:

- $\#\{(\mathbf{x} \cup \mathbf{y}) \setminus (\mathbf{x} \cap \mathbf{y})\} \in 4\mathbb{N}$.
- Let $m = \#\{\mathbf{x} \cup \mathbf{y} \setminus \mathbf{x} \cap \mathbf{y}\}$. Then, at $m - (m - 4)/4$ points in $(\mathbf{x} \cup \mathbf{y}) \setminus \mathbf{x} \cap \mathbf{y}$, three of the four adjoining squares have local multiplicity zero, and at $(m - 4)/4$ points are contained in the interior of $\phi$.
- $\partial(\phi)$ crosses the point $c$ vertically.
- $\phi$ satisfies that $O_1(\phi) = X_2(\phi) = X_1(\phi) + 1$

A positive domain $\phi \in \pi(\mathbf{x}, \mathbf{y})$ is said to be of type $K3$ if it satisfies the following conditions:

- $\#\{(\mathbf{x} \cup \mathbf{y}) \setminus (\mathbf{x} \cap \mathbf{y})\} \in 4\mathbb{N}$.
- Let $m = \#\{\mathbf{x} \cup \mathbf{y} \setminus \mathbf{x} \cap \mathbf{y}\}$. Then, at $m - (m - 4)/4$ points in $(\mathbf{x} \cup \mathbf{y}) \setminus \mathbf{x} \cap \mathbf{y}$, three of the four adjoining squares have local multiplicity zero, and at $(m - 4)/4$ points are contained in the interior of $\phi$.
- One of the vertical segments of $\partial(\phi)$ is on the vertical circle $\beta_{j+1}$ and it goes up.
- $\phi$ satisfies that $O_1(\phi) = X_2(\phi) = X_1(\phi)$

The set of domains of type $K1$, $K2$, and $K3$ from $\mathbf{x}$ to $\mathbf{y}$ is denoted $\pi^K(\mathbf{x}, \mathbf{y})$. 

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Note that there are some domains that are both type $K_1$ and type $K_2$, but which has no influence on our arguments.

**Definition 5.7.** The $\mathcal{R}$-module homomorphisms $D^{IL}: CF^{-H}(g') \to CF^{-H}(g)[U_1][1, 1]$ and $D^{IR}: CF^{-H}(g') \to CF^{-H}(g)[U_1]$ are given by

$$D^{IL}(x) = \sum_{y \in I(g')} \left( \sum_{\phi \in \pi^{IL}(x, y)} U_2^{O_2(\phi)} U_3^{O_3(\phi)} \cdots U_n^{O_n(\phi)} \right) \cdot e(y), \quad (5.1)$$

$$D^{IR}(x) = \sum_{y \in I(g')} \left( \sum_{\phi \in \pi^{IR}(x, y)} U_2^{O_2(\phi)} U_3^{O_3(\phi)} \cdots U_n^{O_n(\phi)} \right) \cdot e(y). \quad (5.2)$$

Then, let $D: CF^{-H}(g') \to \text{Cone}(U_1 - U_2)$ be defined by

$$D(x) = (D^{IL}(x), D^{IR}(x)).$$

**Definition 5.8.** The $\mathcal{R}$-module homomorphisms $S^{oL}: CF^{-H}(g)[U_1][1, 1] \to CF^{-H}(g')$ and $S^{oR}: CF^{-H}(g)[U_1] \to CF^{-H}(g')$ are given by

$$S^{oL}(x) = \sum_{y \in S(g')} \left( \sum_{\phi \in \pi^{oL}(e^{-1}(x), y)} U_2^{O_2(\phi)} U_3^{O_3(\phi)} \cdots U_n^{O_n(\phi)} \right) \cdot y, \quad (5.3)$$

$$S^{oR}(x) = \sum_{y \in S(g')} \left( \sum_{\phi \in \pi^{oR}(e^{-1}(x), y)} U_2^{O_2(\phi)} U_3^{O_3(\phi)} \cdots U_n^{O_n(\phi)} \right) \cdot y. \quad (5.4)$$

Then, let $S: \text{Cone}(U_1 - U_2) \to CF^{-H}(g')$ be defined by

$$S(x) = (S^{oL}(x), S^{oR}(x)).$$

**Proposition 5.9.** The maps $D, S$ are chain maps.

**Proof.** From Lemma 13.3.13, $D$ is a chain map by pairing domains of $D \circ \partial' = \partial_{\text{Cone}} \circ D$. We can check that $S$ is a chain map in the same way. \qed
**Definition 5.10.** The $R$-module homomorphisms $K: CF^{-H}(g') \to CF^{-H}(g')$ is given by

$$K(x) = \sum_{y \in S(g')} \left( \sum_{\phi \in \pi_K(x,y)} U_2^{O_2(\phi)} U_3^{O_3(\phi)} \ldots U_n^{O_n(\phi)} \right) \cdot y. \quad (5.5)$$

**Proposition 5.11.** Here $\partial^-$ denotes the differential of $CF^{-H}(g')$. The above homomorphisms satisfy that

$$D \circ S = \text{id}, \quad (5.6)$$

$$S \circ D + \partial^- \circ K + K \circ \partial^- = \text{id}. \quad (5.7)$$

**Proof.** The equation (5.6) is straightforward. To see the equation (5.7), we will count domains of left side of (5.7) and check that domains are canceled in modulo 2.

We will find it convenient to define the **complexity** of a domain to be one if it is the trivial domain and, otherwise, to be the number of horizontal segments in its boundary. Let us denote by $k(\phi)$ the complexity of $\phi$.

The map $D^{tL}$ (respectively $S^{oL}$) can be decomposed as $D^{tL} = D_1^{tL} + D_{\geq 1}^{tL}$ (respectively $S^{oL} = S_1^{oL} + S_{\geq 1}^{oL}$), where the subscript represents the restriction on the complexity of the domains. Then we can draw the following diagram, with representing $CF^{-H}(g')$ as $I \oplus N$.

$$\begin{array}{ccc}
I & \xrightarrow{D_1^{tL}} & J \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
CF^{-H}(g')[1,1] & \xrightarrow{U_1-U_2} & CF^{-H}(g')[U_1] \\
\downarrow & & \downarrow \\
S_1^{oL} & \xrightarrow{S_{\geq 1}^{oL}} & S^{oL} \\
\downarrow & & \downarrow \\
I & \xrightarrow{D^{tL}} & N \\
\end{array}$$

Note that the complexities of domains of $\pi^{tL}$ and $\pi^{oL}$ are odd, and ones of $\pi^{tL}$ and $\pi^{oL}$ are even.

Again it is convenient to write the horizontal circle crossing the point $c$ as $\alpha_i$, the vertical circle crossing the point $c$ as $\beta_j$, and the point of $x \in S(g')$ on $\alpha_i$ as $x_i$.

For $x \in S(g')$, let $\phi = d \ast s \ (d \in \pi(x,y), \ s \in \pi(y,z))$ be a composite domain appearing in $S \circ D$. Now we will observe that each $\phi$ appears also in $\partial^- \circ K + K \circ \partial^-$ or id. Consider the complexity of $d, s$. We have five cases:

(i) If $k(d) = k(s) = 1$, then the composite domain $d \ast s$ must be trivial. This domain also appears in $\text{id}_{CF^{-H}(g')}$.  

(ii) If $k(d) = 1$ or $k(s) = 1$, then we can take a small square $r$ from $\phi$ by cutting $\phi$ along $\beta_j$ and another composition $\phi = (\phi - r) \ast r$ or $\phi = r \ast (\phi - r)$ appearing in $\partial^- \circ K$ or $K \circ \partial^-$, using domains of type $K1$.  

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(iii) If $k(d) > 1$ and $k(s) > 1$ and $\phi$ does not have vertical thin annulus, then there is a 270° corner at one point $x_i$ of $(x \cup y) \setminus (x \cap y)$. Cutting $\phi$ at $x_i$ horizontally or vertically gives another way of decomposition using the domain of type $K2$.

(iv) If $k(d) > 1$ and $k(s) > 1$ and $\phi$ has vertical thin annulus $\psi$ containing $O_1$, then consider the domain $\phi - \psi$. It connects the same states as $\phi$ and can be decomposed using domains of type $K3$. If it is decomposed as $\phi - \psi = r * k$ or $\phi - \psi = k * r$, the rectangle $r$ does not contain $O_1$, so they are canceled in modulo 2.

(v) The rest terms of $\partial^{-} \circ K + K \circ \partial^{-}$ are canceled each other. When the domain $\phi$ has vertical thin annulus, then two domains $\phi$ and $\phi - \psi$ are canceled.

\[\square\]

6 Proof of the main theorem \ref{thm:main} and theorem \ref{thm:stabilization}

*Proof of Theorem \ref{thm:main}.* If $g'$ is obtained from $g$ by a single cyclic permutation or commutation', from Proposition \ref{prop:cyclic_permutation} and Proposition \ref{prop:commutation}, $\text{CF}^{-H}(g) \simeq \text{CF}^{-H}(g')$. The chain homotopy equivalent gives induced chain homotopy equivalent $\text{CF}^{-H}_U(g) \simeq \text{CF}^{-H}_U(g')$ as Maslov graded chain complexes over $\mathbb{F}[U]$-module. Proposition \ref{prop:grading} and Proposition \ref{prop:stabilization} provides $t\text{CF}^{-H}(g) \simeq t\text{CF}^{-H}(g')$.

If $g'$ is obtained from $g$ by a single stabilization', then $\text{CF}^{-H}(g') \simeq \text{Cone}(U_1 - U_2)$ by Proposition \ref{prop:stabilization}. By identifying all $U_i$'s, $\text{Cone}(U_1 - U_2)$ turns into $\text{CF}^{-H}(g) \oplus \text{CF}^{-H}(g')[1, 1]$ and we get induced chain homotopy $\text{CF}^{-H}_U(g') \simeq \text{CF}^{-H}(g) \oplus \text{CF}^{-H}(g')[1, 1]$. Then Proposition \ref{prop:grading} and Proposition \ref{prop:stabilization} provides $t\text{CF}^{-H}(g') \simeq t\text{CF}^{-H}(g) \oplus t\text{CF}^{-H}(g)[1-t]$.

*Proof of Theorem \ref{thm:stabilization}.* From Theorem \ref{thm:main}, cyclic permutation and commutation' clearly preserve the value of $\Upsilon$, so it is sufficient to check the case of stabilization'. If $t \in [0, 1]$, the grading shift $[1-t]$ does not affect the value of $\Upsilon$ because $1-t \geq 0$.

\[\square\]

7 Link cobordisms with grid homology

In this section, we observe graph grid diagrams for two links connected by a link cobordism. The basic idea of the proof is developed by Sarkar \cite{Sarkar2010}.

First we will use extended grid diagram which allows there are an $X$-marking and an $O^*$-marking sharing the same square. In this case, the square represents an unknotted, unlinked component. The homology of these grid diagrams are also invariant in the same way. A grid diagram representing a link is called \textbf{tight} if there is exactly one $O^*$-marking in each link component.

In general, the symmetrized Alexander function is not canonical because it needs calculating the homology of $\text{CF}^{-}(g)$. However, when we are thinking about links, the
Figure 10: Examples of the case (ii)

Figure 11: Examples of the case (iii)

Figure 12: Examples of the case (iv)

Figure 13: Examples of the case (v)
symmetrized Alexander function is written easily: from Section 8.2,\[8,\]

\[A_H(x) = \mathcal{J}(x, X - \emptyset) - \frac{1}{2} \mathcal{J}(X, X) + \frac{1}{2} \mathcal{J}(\emptyset, \emptyset) - \frac{n - l}{2},\]

where \(l\) is the number of link components.

According to Sarkar \[10\] and Vance \[11\], if two links are connected by a link cobordism, their tight grid diagrams are connected by finite sequence of link-grid moves. These moves are commutation’s, (de-)stabilization’s, births, \(X\)-saddles, \(O\)-saddles, and deaths.

A grid diagram \(g'\) is obtained from \(g\) by a \textbf{birth} if adding one row and column to \(g\) and putting an \(X\)-marking and \(X\)-marking in the square which is the intersection of the new row and column. A grid move \textbf{death} is the inverse move of a birth. The move birth (respectively death) represents a birth (respectively a death) on link cobordism. These moves are link-grid move (4) (respectively (7)) in \[10\].

A grid diagram \(g'\) is obtained from \(g\) by an \textbf{\(X\)-saddle} if \(g\) has a \(2 \times 2\) small squares with two \(X\)-markings at tow-left and bottom-right, and \(g'\) is obtained from \(g\) by deleting these two markings and putting new ones at top-right and bottom-left. The move \(X\)-saddle represents a saddle move on link cobordism. This moves are link-grid move (5) in \[10\].

A grid diagram \(g'\) is obtained from \(g\) by an \textbf{\(O\)-saddle} if \(g\) has a \(2 \times 2\) small squares with one \(O^*\)-marking at tow-left and one \(O\)-marking at bottom-right, and \(g'\) is obtained from \(g\) by deleting these two markings and putting new \(O^*\)-markings at top-right and bottom-left. The move \(O\)-saddle represents a split move on link cobordism. This moves are link-grid move (6) in \[10\].

In \[10\], Sarkar evaluated the maximum changes of Maslov and Alexander grading on each link-grid move. We will check that we can define the appropriate maps also on \(t\)-modified chain complexes and the changes of \(t\)-grading is the same as Sarkar’s evaluation with \(M - t \cdot A\).

In order to prove theorem \[1.5\] we evaluate the change of grading on each link-grid move. It is convenient to introduce alternative Alexander grading \(A'\):

\textbf{Definition 7.1.} For \(x \in S(g)\),

\[A'(x) = \mathcal{J}(x, X - \emptyset) - \frac{1}{2} \mathcal{J}(X, X) + \frac{1}{2} \mathcal{J}(\emptyset, \emptyset) - \frac{n - 1}{2}.\]

Note that the symmetrized Alexander grading \(A_H\) can be obtained from \(A'\) by adding \(\frac{l - 1}{2}\).

In this section, we are thinking about chain complexes \(tCF^-(g)\) with \(t\)-grading using \(A'\) rather than \(A_H\), we set \(\text{gr}_t(x) = M(x) - tA'(x)\).

For \(H(tCF^-(g)) = tHF^-(g)\), we can define \(\Upsilon'_g(t)\) in the same way as \(\Upsilon_g(t)\). It is clear that

\[\Upsilon_g(t) = \Upsilon'_g(t) - \frac{l - 1}{2} t.\] (7.1)
Figure 14: A birth in $g$ produces $g'$, a death in $g'$ produces $g$.

Figure 15: An $X$-saddle in $g$ produces $g'$

Figure 16: An $O$-saddle in $g$ produces $g'$
7.1 births and deaths

Proposition 8.3 shows that there are isomorphisms

\[ D': tHF^{-}(g') \to tHF^{-}(g) \oplus tHF^{-}(g)[1], \]
\[ S': tHF^{-}(g) \oplus tHF^{-}(g)[1] \to tHF^{-}(g'). \]

These maps preserve t-grading if we use symmetrized Alexander function \( A^H \), so \( D' \) shifts t-grading by \(-\frac{1}{2}t \) and \( S' \) by \( \frac{1}{2}t \). We see these maps as \( D' = (H((\mathcal{D}^R)^t), H((\mathcal{D}^L)^t)) \) and \( S' = (H((\mathcal{S}^R)^t), H((\mathcal{S}^L)^t)) \) using the notations in subsection 5.3 and proposition 4.4.

When we think \( H(\mathcal{D}^L) \) as the map into \( tHF^{-}(g) \), \( H(\mathcal{D}^L) \) is surjective map on homology and shifts t-grading by \( 1 - \frac{1}{2}t \). It means that for \( t \in [0, 1] \), the maximum shift of t-grading of homogeneous, non-torsion element in homology by death is \( 1 - \frac{1}{2}t \).

In contrast, the maximum change by birth is \( \frac{1}{2}t \).

7.2 X-saddles

**Proposition 7.2.** If \( g' \) is obtained by an X-saddle, there are \( \mathcal{R}_t \)-module maps

\[ \sigma: tHF^{-}(g) \to tHF^{-}(g'), \]
\[ \mu: tHF^{-}(g') \to tHF^{-}(g), \]

with the following properties:

- \( \sigma \) shifts t-grading by \(-\frac{1}{2}t \),
- \( \mu \) shifts t-grading by \(-\frac{1}{2}t \),
- \( \mu \circ \sigma = v^t \),
- \( \sigma \circ \mu = v^t \).

**Proof.** Let \( c \) be the point in the center of the \( 2 \times 2 \) squares. Define \( \sigma: tCF^{-}(g) \to tCF^{-}(g') \) and \( \mu: tCF^{-}(g) \to tCF^{-}(g') \) by

\[ \sigma(x) = \begin{cases} x & (c \in x) \\ v^t \cdot x & (c \notin x) \end{cases} \text{ and } \mu(x) = \begin{cases} v^t \cdot x & (c \in x) \\ x & (c \notin x). \end{cases} \]

Obviously, both \( \mu \circ \sigma \) and \( \sigma \circ \mu \) are multiplication by \( v^t \). It is also clear that both \( \sigma \) and \( \mu \) are chain maps. Think the maps on homology induced by them.

Let \( A \) be a \( \mathcal{R}_t \)-module. The torsion submodule of \( A \) is

\[ \text{Tors}(A) = \{ a \in A | \text{there is a non-zero } p \in \mathcal{R}_t \text{ with } p \cdot a = 0 \}. \]
Lemma 7.3. Let $A, B$ be two $\mathcal{R}_t$-module. If $\alpha: A \to B$ and $\beta: B \to A$ are two module maps with the property that $\beta \circ \alpha = v^t$, then $\alpha$ induces an injective map from $A/\text{Tors}(A)$ into $B/\text{Tors}(B)$.

Proof. If $\alpha(a) \in \text{Tors}(B)$, then there is a $r \in \mathbb{R}$ with $v^r \cdot \alpha(a) = 0$. Then $\beta(v^r \cdot \alpha(a)) = v^{r+t} \cdot a = 0$, so $a \in \text{Tors}(A)$. \hfill \Box

Using this lemma, proposition 7.2 means that the shift of t-grading of homogeneous, non-torsion element in homology by a $X$-saddle is $-\frac{1}{2}t$

7.3 $O$-saddles

Proposition 7.4. If $g'$ is obtained by an $O$-saddle, there are $\mathcal{R}_t$-module maps

$$\sigma: tHF^-(g) \to tHF^-(g'),$$
$$\mu: tHF^-(g') \to tHF^-(g).$$

with the following properties:

- $\sigma$ shifts t-grading by $-1 + \frac{1}{2}t$,
- $\mu$ shifts t-grading by $-1 + \frac{1}{2}t$,
- $\mu \circ \sigma = v^{2-t}$,
- $\sigma \circ \mu = v^{2-t}$.

Proof. We can prove this same as proposition 7.2. All we need is to define $\sigma: tCF^-(g) \to tCF^-(g')$ and $\mu: tCF^-(g) \to tCF^-(g')$ by

$$\sigma(x) = \begin{cases} x & (c \in x) \\ v^{2-t} \cdot x & (c \notin x) \end{cases} \quad \text{and} \quad \mu(x) = \begin{cases} v^{2-t} \cdot x & (c \in x) \\ x & (c \notin x) \end{cases}.$$

\hfill \Box

Again using lemma 7.3, proposition 7.4 means that the shift of t-grading of homogeneous, non-torsion element in homology by an $O$-saddle is $-1 + \frac{1}{2}t$.

8 Proof of the main theorem 1.5

Proposition 8.1 (Theorem 4.1,[10],Theorem 4.5,[11]). If $g_1, g_2$ are two tight grid diagram representing $l_1, l_2$-component links $L_1, L_2$, respectively, and if there is a link cobordism with $b$ births, $s$ saddles, and $d$ deaths, then there is a sequence of link-grid moves connecting from $g_1$ to $g_2$, such that there are exactly $b$ births, $s - d + l_1 - l_2$ $X$-saddles, $d - l_1 + l_2$ $O$-saddles, and $d$ deaths, and these happen in this order.
Proof of Theorem 1.5. Let $g_1, g_2$ be two grid diagrams represents $L_1, L_2$, respectively. Take a homogeneous, non-torsion element $\alpha \in tHF^-(g_1)$ whose t-grading is $\Upsilon'_{g_1}(t)$. Using maps in previous section, $\alpha$ is mapped into $tHF^-(g_2)$. Next, we use proposition 8.1 and consider the sum of all t-grading shifts, then we get

$$\Upsilon'_{g_1}(t) + b \cdot \frac{1}{2} t + (s - d + l_1 - l_2) \cdot \left(-\frac{1}{2} t\right) + (d - l_1 + l_2) \cdot \left(-1 + \frac{1}{2} t\right) + d \cdot \left(1 - \frac{1}{2} t\right)$$

$$\leq \Upsilon'_{g_2}(t),$$

so

$$\Upsilon'_{g_1}(t) - \frac{l_1 - 1}{2} - \left(\frac{1}{2} (s - b - d) + 1 - \frac{l_1 + l_2}{2}\right) t - (l_1 - 1)t + l_1 - l_2$$

$$\leq \Upsilon'_{g_2}(t) - \frac{l_2 - 1}{2}.t.$$

Use (7.1) and $g = \frac{1}{2}(s - b - d) + 1 - \frac{l_1 + l_2}{2}$, then

$$\Upsilon_{L_1}(t) - tg - t(l_1 - 1) - (l_1 - l_2) \leq \Upsilon_{L_2}(t)$$

We can once get the other inequality if we reverse the direction of link cobordism, so we see that

$$\Upsilon_{L_1}(t) - tg - t(l_1 - 1) - (l_1 - l_2) \leq \Upsilon_{L_2}(t) \leq \Upsilon_{L_1}(t) + tg + t(l_2 - 1) + (l_2 - l_1).$$

\[\square\]

8.1 Proof of some properties of the $\Upsilon$ invariant

8.2 Crossing change

Proposition 8.2. If two graph grid diagrams $g_+$ and $g_-$ represent two links $L_+$ and $L_-$ that differ in a crossing change, then there are $R_t$ maps $C^t_+ : tHF^{-H}(g_+) \to tHF^{-H}(g_-)$ and $C^t_- : tHF^{-H}(g_-) \to tHF^{-H}(g_+)$, with the following properties:

- $C_-$ is graded,
- $C_+$ shifts t-grading by $-2 + t$,
- $C_+ \circ C_- = v^{2-t}$,
- $C_- \circ C_+ = v^{2-t}$,

Proof. The proof is straightforward from proposition 6.1.1,8 and proposition 4.4. □

Proof of proposition 1.8. It is quickly showed from proposition 8.2 and proposition 4.4. □
8.3 The value of $\Upsilon$ at $t = 0$

Proof of Proposition 8.3. If $t = 0$, the $t$-modified chain complex $t\text{CF}^{-H}(g)$ is independent from the $X$-markings because the differential is

$$\partial_t^{-}(x) = \sum_{y \in S(g)} \left( \sum_{r \in \text{Rect}^v(x,y)} v^2|\ominus\cap r| \right) y,$$

and $t$-grading is

$$\text{gr}_t(v^\alpha x) = M(x) - \alpha.$$

Now we can ignore the difference between $O$-markings and $O^*$-markings of $g$. Remove all $X$-markings of $g$ and put some of them so that new grid diagram $g'$ represents an unknot, then $HF^{-H}(g') \cong \mathbb{F}\{U\}$ in grading zero. Since Proposition 4.4, the universal coefficient theorem, and Lemma 14.1.11, we have

$$tHF^{-H}(g') \cong tHF^{-H}(g) \oplus tHF^{-H}(g)[J_1K_0],$$

(8.1)

where $[1]$ is $t$-grading shift by 1.

8.4 Adding an unknot

Proposition 8.3. If $g$ is a graph grid diagram for balanced spatial graph $f$ and if $g'$ is obtained from $g$ by a birth, then as graded $\mathcal{R}_t$-modules,

$$tHF^{-H}(g') \cong tHF^{-H}(g) \oplus tHF^{-H}(g)[1],$$

(8.1)

where $[1]$ is $t$-grading shift by 1.

Proof. Basic idea of the proof is in Lemma 8.4.2. We assume that $CF^{-H}(g)$ is a chain complex over $\mathbb{F}\{U_2, \ldots, U_n\}$ and $CF^{-H}(g')$ is one over $\mathbb{F}\{U_1, \ldots, U_n\}$.

At first, we will show that there is a chain homotopy equivalence

$$CF^{-H}(g') \simeq CF^{-H}(g)[U_1] \oplus CF^{-H}(g)[U_1][1,0].$$
Because a birth on graph grid diagram increase its grid number by one, the argument of stabilization can be applied in the same way.

Assume that $g'$ has a $2 \times 2$ squares with both one $O$-marking and one $X$-marking at top-right and one $O$ or $O^*$-marking at bottom-left, and that $g$ does not have the square containing both one $O$-marking and one $X$-marking. We denote by $c$ the intersection point of the new horizontal and vertical circles in $g'$ (see figure 14). Under these conditions, we can use maps $D$, $S$, and $K$ in Definition 5.7, 5.8, and 5.10. Direct computation shows that the grading changes are different from the original definition: $D^{iR}$ is bigraded, $D^{iL}$ shifts Maslov grading by 1, $D^{oR}$ is bigraded, and $D^{oL}$ shifts Maslov grading by $-1$. Counting domains are independent from markings, so these maps induce chain homotopy equivalence. Using proposition 1.4 this chain homotopy equivalence induces (8.1).

Proof of 1.9. It is immediately from Proposition 8.1.

8.5 Wedge sum of an unknot

For a balanced spatial graph $f: G \to S^3$ and an vertex $c$, let $f\#_c \mathcal{O}$ denote the balanced spatial graph obtained by attaching new edge going from $c$ to $c$ which is null-homologous in $S^3 - f(G)$.

Proposition 8.4. If $g$ is a graph grid diagram for balanced spatial graph $f$ and $g'$ is for $f\#_c \mathcal{O}$, then as graded $R_t$-modules,

$$tHF^{-H}(g') \cong tHF^{-H}(g) \otimes W_t.$$  (8.2)

Proof. The proof is similar to 8.3. We can show that there is a chain homotopy equivalence

$$CF^{-H}(g') \simeq CF^{-H}(g)$$

in the same way using grid diagrams as figure 18.

Proof of 1.10. It is immediately from proposition 8.4.

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