GROUP SUPERSCHEMES

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Abstract. We develop a general theory of algebraic group superschemes, which are not necessarily affine. Our key result is a category equivalence between those group superschemes and Harish-Chandra pairs, which generalizes the result known for affine algebraic group superschemes. Then we present the applications, including the Barsotti-Chevalley Theorem in the super context, and an explicit construction of the quotient superscheme \( G/H \) of an algebraic group superscheme \( G \) by a group super-subscheme \( H \).

Introduction

The purpose of this paper is to generalize a description of algebraic supergroups, which uses Harish-Chandra pairs, to the category of locally algebraic group superschemes. The first results of this kind were developed in [11], where it has been shown, using Hopf superalgebra technique, that any algebraic supergroup is the product of its largest even super-subgroup and a purely-odd super-subscheme. An explicit construction of such a decomposition was first proposed in the article [5], and then significantly revised and generalized in the papers [12, 13]. The main result of these papers can be formulated as a fundamental equivalence between the category of (affine) algebraic supergroups and specific Harish-Chandra pairs.

In the broadest sense, a Harish-Chandra pair is the couple \((G, V)\), where \( G \) is a group scheme, and \( V \) is a finite-dimensional \( G \)-module, equipped with a bilinear symmetric \( G \)-equivariant map from \( V \times V \) to the Lie algebra \( g \) of \( G \) (see Section 12.1 for more details). We call a Harish-Chandra pair affine, algebraic, or locally algebraic, provided \( G \) is an affine, algebraic, or locally algebraic group scheme, respectively.

Throughout this article we freely use the equivalence between the category of geometric superschemes \( \mathcal{SV} \) and the category of superschemes \( \mathcal{SF} \), regarded as \( k \)-functors. The objects and morphisms of \( \mathcal{SF} \) are denoted by \( X, Y, \ldots \), and \( f, g, \ldots \) respectively. The corresponding, by virtue of this equivalence, objects and morphisms of \( \mathcal{SV} \) are denoted by \( X, Y, \ldots \), and \( f, g, \ldots \), and vice versa.

To every affine algebraic group superscheme \( G \), we can associate the affine algebraic Harish-Chandra pair \((G_{ev}, g_1)\), where \( G_{ev} \) is the largest purely-even group super-subscheme of \( G \) and \( g \) is the Lie superalgebra of \( G \). The action of \( G_{ev} \) on \( g_1 \) is induced by the adjoint action of \( G \) on \( g \). The corresponding bilinear map is the restriction of Lie super-bracket to \( g_1 \). The functor \( G \mapsto (G_{ev}, g_1) \) is called the Harish-Chandra functor.

There is a functor that, given an affine algebraic Harish-Chandra pair \((G, V)\), constructs an algebraic supergroup as a superscheme product \( G \times E \), where \( E \) is a purely-odd superscheme isomorphic to \( \text{SSp}(\Lambda(V^*)) \). Moreover, this functor is a quasi-inverse of the Harish-Chandra functor.
The Harish-Chandra functor can be extended to the larger category of locally algebraic group superschemes. We prove that this functor has a quasi-inverse as well. It means that every locally algebraic group supervariety $G$ is isomorphic to a supervariety product $G_{ev} \times E$ as above, where $G_{ev}$ is regarded as a group scheme. Although the proof mostly follows the papers [12, 13], we introduce a new object that plays a crucial role for non-affine group superschemes. We show that every group superscheme $G$ contains a normal group subfunctor $\mathcal{N}(G)$, which can be regarded as a formal supergroup (see [7, Remark 3.6]). We call $\mathcal{N}(G)$ a formal neighborhood of the identity. The superscheme $E$ is contained in $\mathcal{N}(G)$. In particular, $G = \mathcal{G}_{ev}\mathcal{N}(G)$, i.e., $\mathcal{N}(G)$ is sufficiently large as a group subfunctor.

Moreover, $\mathcal{N}(G)$ is "quasi-affine" in the sense that a complete Hopf superalgebra $\mathcal{O}_e$ can represent it (see Section 9). Note that $\mathcal{N}(G)$ is "negligible" from the topological point of view. That is, for every open super-subscheme $U$ of $G$, we have $UN(G) \subseteq U$. The reason for introducing $\mathcal{N}(G)$ is that $E$ is not a group subfunctor, but every product $xy$ of elements of $E$ can be uniquely expressed as $fz$, where $f \in \mathcal{N}(G)_{ev}$ and $z \in E$.

In our proof, we also exploit a Hopf superalgebra pairing between the Hopf superalgebra $\mathcal{O}_e$ and the hyperalgebra $hyp(G)$ of $G$. For example, each $\mathcal{N}(G)/(R)$ can be identified with the subgroup of $(hyp(G) \otimes R)^\times$ consisting of all group-like elements.

Finally, to show that $G$ coincides with $\mathcal{G}_{ev}E$, we use the endofunctor $X \mapsto \text{gr}(X) = \text{gr}_{H}(X)$ of the category of geometric superschemes, more precisely, its functor-of-points counterpart.

The functor $\text{gr}$ is interesting on its own. Indeed, it preserves immersions and induces an endofunctor of the category of group superschemes. Moreover, a morphism $f : X \to Y$ of superschemes (locally of finite type) is an isomorphism if and only if $\text{gr}(f)$ is. The group superscheme $\text{gr}(G)$ is a semi-direct product of $\text{gr}(G)_{ev} \simeq G_{ev}$ and a normal purely-odd group super-subscheme $G_{odd}$. In this case, $G_{odd} = E$. Then Proposition [11.3] implies that the graded companion of the natural embedding $G_{ev} E \to G$ is an isomorphism, hence $G_{ev} E = G$.

The remainder of the paper is devoted to applications of the fundamental equivalence. First, we prove that a slightly modified (super)version of the well-known Barsotti-Chevalley theorem occurs in the category of algebraic group superschemes. More precisely, a connected algebraic group superscheme $G$ has normal group super-subschemas $G_1 \leq G_2$ such that $G_1$ is affine, $G_2/G_1$ is an abelian group variety, and $G/G_2$ is again affine.

Further, we describe abelian supervarieties and anti-affine algebraic group superschemes. Contrary to the purely-even case, we show that the class of pseudoabelian group superschemes is extensive. Regardless of whether the ground field is perfect or not, we construct pseudoabelian group superschemes, which are neither abelian supervarieties nor even solvable group functors (compare with [10], chapter 8).

In the last section, we prove that for any algebraic group superscheme $G$ and its group super-subscheme $H$, the sheaf quotient $G/H$ is a superscheme of finite type. Using [15] Remark 9.11, and Theorem 1.1 from [2], we reduce the general case to the case when $G_{ev} = M \times A(G)$, where $M$ is an affine group subscheme, $A(G)$ is an abelian variety, and $H_{ev} \leq M$. This immediately implies that $H$ is affine. Note also that $A(G)$ is a central group subscheme of $G_{ev}$.
Under these conditions, $G$ has an open covering by (finitely many) affine $\mathbb{H}$-saturated super-subschemas $\mathcal{U} \in \mathcal{E}$, where $\mathcal{U} = \mathcal{U}_{aff} \times \mathcal{U}_{ab}$, $\mathcal{U}_{aff}$ and $\mathcal{U}_{ab}$ form open affine coverings of $M$ and $A(G)$, respectively, and $\mathcal{U}/\mathcal{H} \simeq \mathcal{U}_{aff}/\mathcal{H} \times \mathcal{U}_{ab}$ is an affine superscheme. Modifying the proof of the main theorem from [14], we show that $\mathcal{U}$ is isomorphic to a homogeneous fiber quotient $\mathcal{X} \times^H \mathbb{H}$, where $\mathcal{X}$ is an affine superscheme on which $H$ acts on the right, and such that $(\mathcal{X} \times^H \mathbb{H})/\mathbb{H}$ is affine. The morphism $\mathcal{X} \times^H \mathbb{H} \to \mathcal{U}$ is constructed in a more straightforward way than in [14]. To prove that it is an isomorphism, we use the above functor $gr$. Each $\mathcal{U}/\mathbb{H}$ is affine, and the standard arguments imply that they form an open covering of $G/\mathbb{H}$.

The article is organized as follows. In the first section, we have collected all necessary notations, definitions, and results on superschemes. Throughout the article, we distinguish between superschemes as functors and geometric superschemes, although the categories of superschemes and geometric superschemes are equivalent to each other by the Comparison Theorem. For example, to introduce the notions of separated and proper morphisms in the second section, the language of geometric superschemes seems to be more natural and convenient. However, the group superschemes are introduced in two incarnations, as group functors and as group objects in the category of geometric superschemes (see Section 4). In the third section we recall the notions of flat and faithfully flat morphisms of superschemes. Lemma 3.3 plays crucial role in the proof of Theorem 14.1, the main result of the final section.

In the fifth section, we prove that every group superscheme has the largest affine quotient, again using the language of geometric superschemes. In the sixth section, for every $k$-functor $\mathcal{X}$, we define its tangent functor at a $k$-point and investigate its properties in the case when $\mathcal{X}$ is a superscheme of locally finite type. In the seventh section, the Lie superalgebra of a group superscheme is introduced as functor and a superalgebra. In the eighth section, we develop a fragment of the differential calculus on locally algebraic group superschemes. In the ninth section we introduce the normal group subfunctor $N(G)$ and discuss how it relates to the Hopf superalgebra $\text{hyp}(G)$.

The tenth section is devoted to studying the functor $gr$. We prove that it commutes with direct products, thus it induces an endofunctor of the category of geometric group superschemes. Moreover, a superscheme $gr(X)$ is always "split" in the sense that there are two superscheme morphisms $i_X : X_{ev} \to gr(X)$ and $q_X : gr(X) \to X_{ev}$ such that $q_X i_X = \text{id}_{X_{ev}}$, and $i_X$ induces an isomorphism onto $gr(X)_{ev}$. In particular, if $G$ is a locally algebraic group superscheme, then $gr(G) \simeq G_{ev} \times G_{odd}$, where $G_{odd} = \ker q_G$ is a purely-odd group superscheme. In the eleventh section, we show that $G_{odd} \simeq SSp(\Lambda(g_1^*))$ and for any morphism $G \to \mathbb{H}$ of group superschemes, the induced morphism $G_{odd} \to \mathbb{H}_{odd}$ is uniquely defined by the corresponding linear map $g_1 \to h_1$.

In the twelfth section, using the techniques developed previously, we prove that the category of locally algebraic group superschemes is equivalent to locally algebraic Harish-Chandra pairs. The content of the last two sections has already been discussed in detail above.

1. Superschemes

For the content of this section, we refer to [3, 8, 9, 13, 14]. Throughout this article, $k$ is the ground field of odd or zero characteristic.
1.1. **Superschemes as ℓ-functors.** Let $\mathbf{SAlg}_k$ denote the category of super-commutative $k$-superalgebras. If $A \in \mathbf{SAlg}_k$, then let $I_A$ and $\overline{A}$ denote the super-ideal $AA_1$ and the factor-algebra $A/I_A$, respectively.

A $k$-functor $X$ is a functor from the category $\mathbf{SAlg}_k$ to the category of sets $\mathbf{Sets}$. The category of $k$-functors is denoted by $\mathcal{F}$. The morphisms in $\mathcal{F}$ are denoted by bold letters $f, g, ...$

A $k$-functor $X$ is called an **affine superscheme** if it is representable by a superalgebra $A$, that is,

$$X(B) = \text{Hom}_{\mathbf{SAlg}_k}(A, B)$$

for $B \in \mathbf{SAlg}_k$.

In this case, $X$ is denoted by $\mathbf{SSp}(A)$. A closed affine super-subscheme of $X = \mathbf{SSp}(A)$ is defined as

$$\forall(I)(B) = \{ \phi \in X(B) \mid \phi(I) = 0 \}$$

for $B \in \mathbf{SAlg}_k$,

where $I$ is a super-ideal of $A$. It is easy to see that $\forall(I) \simeq \mathbf{SSp}(A/I)$.

Similarly, a super-ideal $I$ of $A$ defines an open $k$-subfunctor of $X = \mathbf{SSp}(A)$ as

$$\mathcal{D}(I)(B) = \{ \phi \in X(B) \mid \phi(I)B = B \}$$

for $B \in \mathbf{SAlg}_k$.

In general, a $k$-subfunctor $Y$ of a $k$-functor $X$ is called closed (respectively, open) if for any morphism of functors $f : \mathbf{SSp}(A) \to X$ the inverse image $f^{-1}(Y)$ is closed (respectively, open) in $\mathbf{SSp}(A)$.

A collection of open $k$-subfunctors $\{X_i\}_{i \in I}$ is said to be an *open covering* of a $k$-functor $X$, provided $X(F) = \bigcup_{i \in I} X_i(F)$ for any field extension $k \subseteq F$. A $k$-functor $X$ is called local if for any $k$-functor $Y$ and any open covering $\{Y_i\}_{i \in I}$ of $Y$ the diagram

$$\text{Mor}_\mathcal{F}(Y, X) \to \prod_{i \in I} \text{Mor}_\mathcal{F}(Y_i, X) \to \prod_{i,j \in I} \text{Mor}_\mathcal{F}(Y_i \cap Y_j, X),$$

$$f \mapsto \prod_{i \in I} f_{|Y_i}, \prod_{i \in I} f_i \mapsto \prod_{i,j \in I} f_{i|Y_i \cap Y_j}, \prod_{i \in I} f_i \mapsto \prod_{i,j \in I} f_{i|Y_i \cap Y_j},$$

is exact.

A local $k$-functor $X$ is called a *superscheme* whenever $X$ has an open covering by affine super-subschemas. Superschemes form a full subcategory of $\mathcal{F}$, denoted by $\mathcal{S}\mathcal{F}$.

Let $X$ be a $k$-functor. If $Y$ is a subfunctor of $X$, such that for any $A \in \mathbf{SAlg}_k$, the set $Y(A)$ contains exactly one element, then $Y$ is called a one-point subfunctor of $X$.

**Lemma 1.1.** Assume that $X$ is a superscheme. Then every one-point subfunctor of $X$ is closed.

**Proof.** Let $Y$ be a one-point subfunctor of $X$. For any superalgebra $A$, we have $Y(A) = \{ y_A \}$. Choose a covering of $X$ by open affine super-subschemas $X_i \simeq \mathbf{SSp}(A_i)$ for $i \in I$. A verbatim superization of [15, I.1.7(6)], implies that $X_i$ meets $Y$ if and only if $y_i$ belongs to $X_i(k)$. In the latter case, $y_i$ corresponds to a superalgebra morphism $\phi : A_i \to k$ and $Y \cap X_i = \mathcal{V}(\ker \phi)$. Then [15, Lemma 9.1] concludes the proof. □
1.2. Geometric superschemes. Recall that a geometric superspace \( X \) consists of a topological space \( X^e \) and a sheaf of super-commutative superalgebras \( \mathcal{O}_X \) such that all stalks \( \mathcal{O}_{X,x} \) for \( x \in X^e \) are local superalgebras. A morphism of superspaces \( f : X \to Y \) is a pair \((f^e, f^s)\), where \( f^e : X^e \to Y^e \) is a morphism of topological spaces and \( f^s : \mathcal{O}_Y \to f_* \mathcal{O}_X \) is a morphism of sheaves such that \( f_x^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) is a local morphism for any \( x \in X^e \). Let \( \mathcal{V} \) denote the category of geometric superspaces.

To simplify the notations, we use \( \mathcal{O}(X) \) instead of \( \mathcal{O}_X(X^e) \). For any open subsets \( V \subseteq U \subseteq X^e \) the image of \( f \in \mathcal{O}_X(U) \) in \( \mathcal{O}_X(V) \) is denoted by \( f|_V \).

Let \( X \) be a geometric superspace. If \( U \) is an open subset of \( X^e \), then \( (U, \mathcal{O}_X|_U) \) is again a geometric superspace, called an open super-subspace of \( X \). In what follows \( (U, \mathcal{O}_X|_U) \) is denoted by \( U \).

Let \( R \) be a superalgebra. An affine superscheme \( \text{SSpec}(R) \) can be defined as follows. The underlying topological space of \( \text{SSpec}(R) \) coincides with the prime spectrum of \( R \), endowed with the Zariski topology. For any open subset \( U \subseteq \text{SSpec}(R)^e \), the super-ring \( \mathcal{O}_{\text{SSpec}(R)}(U) \) consists of all locally constant functions \( h : U \to \bigcup_{p \in U} R_p \) such that \( h(p) \in R_p, p \in U \).

A superspace \( X \) is called a (geometric) superscheme if there is an open covering \( X^e = \bigcup_{i \in I} U_i \), such that each open super-subspace \( U_i \) is isomorphic to an affine superscheme \( \text{SSpec}(R_i) \). Superschemes form a full subcategory of \( \mathcal{V} \), denoted by \( \mathcal{S} \).

If \( X \) is a superscheme, then every open super-subspaces of \( X \) is a superscheme, called an open super-subscheme of \( X \). A superscheme \( Z \) is a closed super-subscheme of \( X \) if there is a closed embedding \( \iota : Z^e \to X^e \) such that the sheaf \( \iota_* \mathcal{O}_Z \) is an epimorphic image of the sheaf \( \mathcal{O}_X \).

**Example 1.2.** The sheafification of the presheaf \( U \mapsto I_{\mathcal{O}_X(U)} = \mathcal{O}_X(U) / \mathcal{O}_X(U)_1 \) is a sheaf of \( \mathcal{O}_X \)-super-ideals, which is denoted by \( \mathcal{I}_X \). Then \((X^e, \mathcal{O}_X / \mathcal{I}_X) \) is the largest purely-even closed super-subscheme of \( X \), denoted by \( X_{ev} \). Regarded as a geometric scheme, \( X_{ev} \) is denoted by \( X_{res} \). Moreover, every morphism \( f : X \to Y \) induces the morphism \( f_{ev} : X_{ev} \to Y_{ev} \) so that \( X \to X_{ev} \) is an endofunctor of \( \mathcal{S} \).

1.3. Comparison Theorem. (see [3, Theorem 10.3.7], or [15, Theorem 5.14], and [4, I, §1, 4.4])

**Theorem 1.3.** The functor \( X \mapsto \mathbb{X}, \) where \( \mathbb{X}(A) = \text{Mor}_{\mathcal{S}}(\text{SSpec}(A), X) \) for \( A \in \text{SAlg}_{\mathbb{S}} \), defines an equivalence \( \mathcal{S} \mathcal{V} \simeq \mathcal{S} \mathcal{F} \).

If \( f : X \to Y \) is a morphism in the category \( \mathcal{S} \mathcal{V} \), then the corresponding morphism \( \mathbb{X} \to \mathbb{Y} \) in the category \( \mathcal{S} \mathcal{F} \) is denoted by \( f \) and vice versa.

1.4. Immersions. A morphism \( f : X \to Y \) in the category \( \mathcal{S} \mathcal{V} \) is called an immersion if there is an open subset \( U \) in \( Y^e \) such that \( f^e \) induces a homeomorphism from \( X^e \) onto the closed subset \( f^e(X^e) \subseteq U \) and the induced morphism of sheaves \( f^* : \mathcal{O}_U \to f_* \mathcal{O}_X \) is surjective. The latter is equivalent to the condition that for every \( x \in X^e \), the induced superalgebra morphism \( \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) is surjective. If \( f^e(X^e) = U \), then \( f \) is called an open immersion. Finally, if \( U = Y^e \), then \( f \) is called a closed immersion.

The proofs of the following lemmas are standard, and we leave them for the reader.
Lemma 1.4. A morphism \( f : X \to Y \) in the category \( \mathcal{SV} \) is closed, open or immersion if and only if for any open covering \( \{ V_i \}_{i \in I} \) of \( Y \), each morphism \( f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i \) is closed, open or immersion, correspondingly.

Lemma 1.5. Let \( f : X \to Y \) be a morphism in the category \( \mathcal{SV} \) such that \( f^c \) is injective. Then \( f \) is closed, or an immersion, respectively if and only if for any finite open covering \( \{ U_j \}_{j \in J} \) of \( X \), each morphism \( f|_{U_j} : U_j \to Y \) is closed, or an immersion, respectively. Finally, \( f \) is an open immersion if and only if for any open covering \( \{ U_j \}_{j \in J} \) of \( X \), each morphism \( f|_{U_j} : U_j \to Y \) is an open immersion.

Proposition 1.6. (cf. [10, 1.1]) Let \( f : Z \to X \) be a morphism of geometric superschemes. Then \( f \) is a closed immersion if and only if \( f \) is an isomorphism of \( Z \) onto a closed super-subscheme of \( X \).

Proof. Let \( \{ U_j \}_{j \in J} \) be an open covering of \( X \). Then \( \{ Z_j = i^{-1}(U_j) \}_{j \in J} \) is an open covering of \( Z \). By [15, Lemma 5.2(2,3)], \( \{ U_j \}_{j \in J} \) is an open covering of \( X \). Moreover, each \( U_j \) is affine whenever \( U_j \) is.

Observe that, for every superalgebra \( A \), and every open subset \( U \subseteq X^c \), the subset \( \mathbb{U}(A) \subseteq \mathbb{X}(A) \) consists of all superscheme morphism \( h : \text{SSpec}(A) \to X \) such that \( h^c(\text{SSpec}(A)^c) \subseteq U \). Thus \( f^{-1}(\mathbb{U}) = \mathbb{Z} \), for any \( j \in J \).

Assume now that \( f \) is a closed immersion. Then \( f \) is an embedding of \( \mathbb{X} \)-functors. The above remarks, combined with [15, Lemma 9.1 (1)], reduce the general case to \( X = \text{SSpec}(A) \) for \( A \in \text{SAlg}_{\mathbb{K}} \). There is a super-ideal \( I \) of \( A \) such that \( f \) can be identified with the closed immersion \( \text{SSpec}(A/I) \to \text{SSpec}(A) \). Then [15, Lemma 4.1] implies that \( f \) is the canonical isomorphism of \( \text{SSp}(A/I) \) onto the closed super-subscheme \( \mathbb{V}(I) \) of \( \text{SSp}(A) \).

For the reverse statement, we use [15, Lemma 9.1 (1)] again, and the general case can be reduced to \( X = \text{SSpec}(A) \). There is a super-ideal \( I \) of \( A \) and an isomorphism \( h : Z \to \text{SSp}(A/I) \) such that \( f = \text{SSp}(\pi)h \), where \( \pi \) is the canonical epimorphism \( A \to A/I \). By Comparison Theorem, there is an isomorphism \( h : Z \to \text{SSpec}(A/I) \) such that \( f = \text{SSpec}(\pi)h \). In other words, for every open affine covering \( \{ U_j \}_{j \in J} \) of \( X \), each morphism \( f|_{Z_j} : Z_j \to U_j \) is a closed immersion. Therefore, \( f \) is a closed immersion, proving the proposition.

Example 1.7. The closed immersion \( \mathbb{X}_{ev} \to X \) corresponds to the embedding \( \mathbb{X}_{ev} \to \mathbb{X} \), where the subfunctor \( \mathbb{X}_{ev} \) is defined as
\[
\mathbb{X}_{ev}(A) = \mathbb{X}(i)(\mathbb{X}(A_0)) \cong \mathbb{X}(A_0) \quad \text{for} \ A \in \text{SAlg}_{\mathbb{K}},
\]
and \( i : A_0 \to A \) is the natural embedding. In particular, \( \mathbb{X}_{ev} \) is a closed super-subscheme of \( X \) (see also [13, Proposition 9.2]). As above, \( \mathbb{X} \to \mathbb{X}_{ev} \) is an endofunctor of the category \( \mathcal{SF} \). Note that if \( \mathbb{X} \) is not a superscheme, then the above map \( \mathbb{X}(i) \) is no longer injective. For example, consider the functor \( A \to \overline{A} \) for \( A \in \text{SAlg}_{\mathbb{K}} \).

Let \( f : X \to Y \) be an immersion. Assume that \( f \) factors through an open super-subscheme \( U \) of \( Y \). Let \( \mathcal{J} \) denote \( \ker(f_*\mathcal{O}_U \to \mathcal{O}_X) \). For any non-negative integer \( n \), one can define the \( n \)-th neighborhood of \( f \) as a closed super-subscheme of \( U \), given by the super-ideal sheaf \( \mathcal{J}^{n+1} \), denoted by \( Y^n_f \).

Lemma 1.8. The definition of \( Y^n_f \) does not depend on \( U \).

Proof. If \( f \) factors through another open super-subscheme \( V \) of \( Y \), then \( f \) factors through \( U \cap V \). Without loss a generality, one can assume that \( V \subseteq U \). Then
for every \( x \in X^e \subseteq V^e \) there is a natural isomorphism \( O_{Y^e,x} \simeq O_{U,x}/J_{x}^{n+1} \approx O_{V,x}/(J|_V)^{n+1} \).

For example, let \( B \) be a local superalgebra with nilpotent maximal super-ideal \( \mathfrak{m} \) such that \( B/\mathfrak{m} = k \). Every superscheme morphism \( \text{SSpec}(B) \to X \) is uniquely defined by a \( k \)-point \( x \in X^e \) and by a local morphism of superalgebras \( O_x \to B \). In particular, such morphism factors through any open neighborhood of \( x \), and it is a closed immersion if and only if \( O_x \to B \) is surjective. In particular, we have a canonical closed immersion \( \text{SSpec}(O_x/\mathfrak{m}_x^{n+1}) \to X \), which is just the \( n \)-th neighborhood of the closed immersion \( i_x : \text{SSpec}(O_x/\mathfrak{m}_x) \to X \).

1.5. **Morphisms (locally) of finite type.** Recall that a morphism \( f : X \to Y \) of geometric superschemes is said to be **locally of finite type** if there is an open covering of \( Y \) by affine super-subschemes \( V_i \simeq \text{SSpec}(B_i) \) such that for every \( i \), the open super-subscheme \( f^{-1}(V_i) \) is covered by open super-subschemes \( U_{ij} \simeq \text{SSpec}(A_{ij}) \), where each \( A_{ij} \) is a finitely generated \( B_i \)-superalgebra.

If each \( f^{-1}(V_i) \) can be covered by a finite number of \( U_{ij} \), then \( f \) is said to be a morphism of **finite type** (cf. [6, II.3]).

**Lemma 1.9.** Let \( \phi : B \to A \) be a superalgebra morphism. Then \( A \), regarded as a \( B \)-superalgebra via \( \phi \), is finitely generated if and only if there are \( b_1, \ldots, b_s \in B_0 \) such that \( \sum_{1 \leq i \leq s} B_0 b_i = B_0 \) and \( A_{\phi(b_i)} \) is a finitely generated \( B_0 \)-superalgebra for each \( 1 \leq i \leq s \). Symmetrically, if there are \( a_1, \ldots, a_s \in A_0 \) such that \( \sum_{1 \leq i \leq s} A_0 a_i = A_0 \) and \( A_{a_i} \) is a finitely generated \( B \)-superalgebra for each \( 1 \leq i \leq s \), then \( A \) is a finitely generated \( B \)-superalgebra.

**Proof.** There is a super-subalgebra \( C \) of \( A \), finitely generated over \( B[\phi(b_1), \ldots, \phi(b_s)] \) (respectively, finitely generated over \( B[a_1, \ldots, a_s] \)), such that for every \( i \), there is \( A_{\phi(b_i)} = C_{\phi(b_i)} \) (respectively, \( A_{a_i} = C_{a_i} \)). Then [19] Lemma 1.2 concludes the proof.

Combining Lemma 1.9 with [15] Lemma 3.5, one can easily derive the following characterization of morphisms locally of finite type, and morphisms of finite type as well (cf. [6, Exercise II.3.3(b)]).

**Lemma 1.10.** A morphism \( f : X \to Y \) in \( \mathcal{SV} \) is **locally of finite (respectively, of finite) type** if and only if for every open super-subscheme \( V \simeq \text{SSpec}(B) \) of \( Y \), the open super-subscheme \( U = f^{-1}(V) \) has a (finite) open covering by super-subschemes \( U_i \simeq \text{SSpec}(A_i) \) such that each \( A_i \) is a finitely generated \( B \)-superalgebra.

In particular, if both \( X \) and \( Y \) are affine, then \( f \) is of finite type if and only if \( \mathcal{O}(X) \) is a finitely generated \( \mathcal{O}(Y) \)-superalgebra.

A superscheme \( X \) is said to be (locally) of finite type if the canonical morphism \( X \to \text{SSpec}(k) \) is (locally) of finite type.

2. **Separated and proper morphisms of superschemes**

Let \( \mathcal{P} \) be a property of a class of geometric superschemes or a class of morphisms in \( \mathcal{SV} \). We say that this property is **even reducible**, provided \( X \) satisfies \( \mathcal{P} \) if and only if \( X_{ev} \) does (respectively, \( f \) satisfies \( \mathcal{P} \) if and only if \( f_{ev} \) does). For example, the property of a Noetherian (geometric) superscheme to be affine is even reducible (see [18, Theorem 3.1]). The equivalence \( \mathcal{SV} \cong \mathcal{SF} \) naturally translates this definition.
to the category $\mathcal{SF}$. That is, $\mathcal{P}$ is even reducible in $\mathcal{SV}$ if and only if it is even reducible in $\mathcal{SF}$.

Since the categories $\mathcal{SV}$ and $\mathcal{SF}$ are equivalent and $\mathcal{SF}$ has fibered products, $\mathcal{SV}$ has as well. Let $p_X$ and $p_Y$ denote the canonical projection morphisms $X \times_S Y \to X$ and $X \times_S Y \to Y$, respectively. The fibered product $X \times_{\text{SSpec}(k)} Y$, denoted by $X \times Y$, is a direct product of superschemes $X$ and $Y$ in $\mathcal{SV}$.

The following lemma will be used later.

**Lemma 2.1.** Let $X$ and $Y$ be superschemes over a superscheme $S$. Let $\{U_i\}_{i \in I}$ be an open covering of $X$. Then the superschemes $U_i \times_S Y$ form an open covering of $X \times_S Y$.

**Proof.** By Theorem 5.14 and remark after [15 Proposition 5.12], one can work in the category $\mathcal{SF}$. The easy superization of both [8 I.1.7(3)] and [8 I.1.7(4)] implies the statement. $\square$

A morphism $f : X \to Y$ in $\mathcal{SV}$ is called *separated* if the diagonal morphism $\delta_f : X \to X \times_Y X$ is a closed immersion. We also say that $X$ is *separated* over $Y$. In particular, a superscheme $X$ is called *separated*, provided $X$ is separated over $\text{SSpec}(k)$ (cf. [6 II, §4]). For example, any morphism of affine superschemes is separated (see [6 Proposition II.4.1]). In particular, any affine superscheme is separated. The following lemma superizes [6 Corollary II.4.2].

**Lemma 2.2.** A morphism $f : X \to Y$ is separated if and only $\delta_f^e(X^e)$ is a closed subset of $(X \times_Y X)^e$.

**Proof.** The proof of [6 Corollary II.4.2] can be copied verbatim, provided we prove the following. If $V$ is an affine super-subscheme of $Y$ and $U$ is an affine super-subscheme of $X$ such that $f(U) \subseteq V$, then the natural morphism $U \times_Y U \to X \times_Y X$ is an open immersion. By the remark after [15 Proposition 5.12], all we need is to check the analogous statement in the category $\mathcal{SF}$ which is evident (see [8 I.1.7(3)]). $\square$

**Lemma 2.3.** For any morphisms $f : X \to S$ and $g : Y \to S$ in $\mathcal{SV}$, the fibred product $X_{ev} \times_{S_{ev}} Y_{ev}$ is isomorphic to $(X \times_S Y)_{ev}$.

**Proof.** For every $L \in \mathcal{SV}$, let $L_0$ denote a purely-even superscheme $(L^e, (\mathcal{O}_L)_0)$. The morphisms $L \to Z_{ev}$ are in one-to-one correspondence with the morphisms $L_0 \to Z$, which factor through $L_0 \to Z_{ev}$. Therefore, by the universality of a fiber product, the morphisms $L \to X_{ev} \times_{S_{ev}} Y_{ev}$ are in one-to-one correspondence with the morphisms $L_0 \to X \times_S Y$ hence with the morphisms $L \to (X \times_S Y)_{ev}$, proving the lemma. $\square$

**Proposition 2.4.** A morphism $f : X \to Y$ is separated if and only if $f_{ev}$ is separated, if and only if $f_{res}$ is separated (the latter is regarded as a morphism of geometric schemes).

**Proof.** Lemma 2.3 implies that $\delta_f|_{X_{ev}}$ can be identified with $\delta_{f|_{X_{ev}}}$. The last equivalence is now apparent. $\square$

**Corollary 2.5.** The property of a superscheme morphism to be separated is even reducible. In particular, all standard properties of separated morphisms of schemes, formulated in [6 Corollary II.4.6], are valid for superschemes.
A morphism of geometric superschemes $f : X \to Y$ is called closed if $f^* \text{ takes}$ closed subsets of $X^e$ to closed subsets of $Y^e$. It is called universally closed if for any morphism $Y' \to Y$ the projection $X \times_Y Y' \to Y'$ is closed. Obviously, the first property is even reducible. Hence, by Lemma 2.3 the second one is, too.

A morphism of geometric superschemes $f : X \to Y$ is called proper if it is separated, universally closed, and of finite type.

**Lemma 2.6.** If $X$ is Noetherian, then $f : X \to Y$ is of finite type if and only if $f_{ev}$ is of finite type, if and only if $f_{res}$ is of finite type.

**Proof.** By Lemma 1.10, it is enough to consider the case $X = \text{SSpec}(A)$ and $Y = \text{SSpec}(B)$, where $A$ is a Noetherian superalgebra, and $\overline{A}$ is finitely generated over $\overline{B}$. Since $A^n_1/A^{n+1}_1$ is a finitely generated $\overline{A}$-module for every $n \geq 1$ and $A^N_1 = 0$ for sufficiently large $N$, $A$ is finitely generated over $B_0$, hence over $B$. □

**Corollary 2.7.** In the full subcategory consisting of Noetherian superschemes, [6, Corollary II.4.8], can be superized verbatim.

A Noetherian superscheme $X$ is called complete if it is separated and proper over $\text{SSpec}(k)$, i.e., the morphism $X \to \text{SSpec}(k)$ is separated and proper simultaneously. By the above, this property is even reducible.

Recall that a superalgebra $A$ is called reduced, provided the algebra $\overline{A}$ is. A geometric superscheme $X$ is called reduced if $X_{\text{res}}$ is a reduced scheme (cf. [17, 2.3]). This property is local, i.e., $X$ is reduced if and only if the superalgebra $\mathcal{O}_x$ is reduced for every $x \in X^e$. In particular, an affine superscheme $\text{SSpec}(A)$ is reduced if and only if the superalgebra $A$ is reduced. Thus a superscheme $X$ is reduced if and only if every open affine super-subscheme of $X$ is reduced.

Following [10], we call a geometric superscheme $X$ (algebraic) supervariety if $X$ is of finite type, separated and geometrically reduced. The latter means that the superscheme $X_k = X \times \text{SSpec}(k)$ is reduced for an algebraically closed field $k$ of $k$.

A superalgebra $A$ is called Grassman-like if $\overline{A} = k$ or, equivalently, if odd elements generate $A$.

**Lemma 2.8.** A connected supervariety $\text{SSpec}(A)$ is complete if and only if $A$ is a Grassman-like superalgebra.

**Proof.** Use even reducibility of this property and [10, A.114(f)]. □

Let $f : X \to Y$ be a morphism in $SV$ and let $X$ be a Noetherian superscheme. Then there is a closed super-subscheme $Z$ of $Y$ such that $f^e(X^e) \subseteq Z^e$, the morphism $f$ factors through the closed immersion $Z \to Y$, and for any closed super-subscheme $Z'$ of $Y$ such that $f$ factors through the closed immersion $Z' \to Y$, $Z \to Y$ factors through $Z' \to Y$. Since $X$ is a Noetherian superscheme, the third remark after [15, Proposition 3.1] implies that $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-supermodules. Then, by [15, Corollary 3.2] and [17, Proposition 2.5], the superideal sheaf $\ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is a coherent sheaf of $\mathcal{O}_Y$-supermodules, which defines the superscheme $Z$, called a superscheme-theoretic image of $f$ and denoted by $f(X)$. Observe also that if $X$ is (geometrically) reduced, then $f(X)$ is also (geometrically) reduced.

**Proposition 2.9.** The following statements hold:
(1) If $Z$ is a closed super-subscheme of a complete Noetherian superscheme $X$, then $Z$ is complete.

(2) Let $f : X \to Y$ be a morphism of algebraic supervarieties. If $X$ is complete, then $f^*(X^c)$ is closed, and $f(X)$ is complete.

(3) If $X$ is a complete connected supervariety, then $\mathcal{O}_X(X^c)$ is a Grassman-like superalgebra.

Proof. The first statement follows by superized [3] Corollary II.4.8(a, b).

It is easy to see that $f_{res}(X_{res})$ coincides with $f(X)_{res}$. Then the second statement follows by [10] A.114(d).

Let $A$ denote $\mathcal{O}(X)$. The induced morphism $f : X \to S\text{Spec}(A)$ factors through its superscheme-theoretic image $f(X)$. Since $f(X)$ is closed, $f(X) \simeq S\text{Spec}(A/I)$ for a superideal $I$ of $A$. On the other hand, the composition of superalgebra morphisms $A \to A/I \to \mathcal{O}(X)$ is an identity map. Thus $I = 0$, hence $f(X) = S\text{Spec}(A)$ is a complete connected supervariety. Lemma 2.8 concludes the proof.

□

3. Flat and faithfully flat morphisms

Let $A$ be a superalgebra and $M$ be an $A$-supermodule. Recall that $M$ is said to be flat, if the functor $N \mapsto M \otimes_A N$ takes any exact sequence of $A$-supermodules to the exact sequence of superspaces. If this functor is faithfully exact, then $M$ is called faithfully flat. By [11] Lemma 5.1(1), $A$-supermodule $M$ is (faithfully) flat if and only if $M$ is (faithfully) flat as the (left and right) $A$-module. Besides, $M$ is faithfully flat as an $A$-supermodule if and only if $M$ is flat and for any maximal super-ideal $m$ of $A$ there is $M \neq mM$ (combine [19] Lemma 1.1(iii]) and [11] Proposition 1, 1, §3.1).

A morphism of geometric supervarieties $f : X \to Y$ is said to be flat if for any $x \in X^c$ the induced superalgebra morphism $O_Y.f^*(x) \to O_{X,x}$ is flat. If, additionally, $f^*$ is surjective, then $f$ is called faithfully flat.

Lemma 3.1. A morphism $f : X \to Y$ is flat if and only if for any open affine super-subspaces $U \subseteq X$ and $V \subseteq Y$, such that $f^*(U^c) \subseteq V^c$, the restriction of $f$ to $U$ is flat if and only if $\mathcal{O}(U)$ is a flat $\mathcal{O}(V)$-supermodule via superalgebra morphism $f^*$.

Proof. The first equivalence is obvious. The part "only if" follows by [19] Proposition 1.1 (iii)]. Conversely, let $\mathcal{O}(U)$ be a flat $\mathcal{O}(V)$-supermodule. For any prime superideal $p$ of $\mathcal{O}(U)$ set $q = p \cap \mathcal{O}(V)$. Then $\mathcal{O}(U)_q \simeq \mathcal{O}(V)_q \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$ is flat over $\mathcal{O}(V)_q$. By [19] Lemma 1.2 (i)], $\mathcal{O}(U)_p$ is flat over $\mathcal{O}(U)_q$. Lemma is proven.

□

Corollary 3.2. If $X$ and $Y$ are affine supervarieties, then $f$ is faithfully flat if and only if $\mathcal{O}(X)$ is faithfully flat over $\mathcal{O}(Y)$.

Proof. The map $f^c$ is surjective if and only if $\mathcal{O}(X)n \neq \mathcal{O}(X)$ for any maximal super-ideal $n$ of $\mathcal{O}(Y)$.

Recall that a morphism of geometric supervarieties $f : X \to Y$ is called affine, if for any affine superscheme $Z$ and any morphism $Z \to Y$ the fibered product $X \times_Y Z$ is an affine superscheme (cf. [8, 15]). In particular, if $f$ is affine and $Z$ is an affine super-subscheme of $Y$, then $f^{-1}(Z)$ is an affine super-subscheme of $X$. 

□
Lemma 3.3. Let
\[
\begin{array}{ccc}
Z & \rightarrow & W \\
\downarrow & & \downarrow \\
U & \rightarrow & V
\end{array}
\]
be a cartesian square of geometric superschemes. Assume that the morphisms \( p_U : Z \rightarrow U, \ g : U \rightarrow V \) and \( q : W \rightarrow V \) are affine, and \( g \) is faithfully flat. If the superscheme \( Z \) is of finite type and \( g \) factors through \( q \), then \( W \) is of finite type as well.

Proof. Choose a covering of \( V \) by open affine super-subschemes \( V_i \simeq \text{SSpec}(A_i) \), \( 1 \leq i \leq l \). The open super-subschemes \( W_i = q^{-1}(V_i) \) are affine and form a covering of \( W \).

Since \( p_U^{-1}(V_i) = p_W^{-1}(W_i) \) for each \( 1 \leq i \leq l \), we have cartesian squares in the subcategory of affine superschemes
\[
\begin{array}{ccc}
(gp_U)^{-1}(V_i) & \rightarrow & W_i \\
\downarrow & & \downarrow \\
g^{-1}(V_i) & \rightarrow & V_i
\end{array}
\]
each of which satisfies the conditions of our lemma. Therefore, without loss of generality one can assume that all superschemes \( V, U, W, Z \) are affine, say \( V \simeq \text{SSpec}(A), W \simeq \text{SSpec}(B), U \simeq \text{SSpec}(C) \) and \( Z \simeq \text{SSpec}(C \otimes_A B) \). Since \( C \otimes_A B \) is a finitely generated \( k \)-superalgebra, there is a finitely generated \( k \)-super-subalgebra \( L \) of \( B \) such that \( C \otimes_A L = C \otimes_A B \). Since \( C \) is a faithfully flat \( A \)-supermodule, we obtain \( L = B \). Lemma is proven. \( \square \)

4. Group superschemes

A group object in the category \( SF \) is called a group superscheme. Similarly, a group object in the category \( SV \) is called a geometric group superscheme. These objects form subcategories in \( SF \) and \( SV \), respectively, with morphisms preserving their group structures. They are denoted by \( SFG \) and \( SVG \), correspondingly. Theorem 1.3 implies that \( SFG \simeq SVG \).

A geometric group superscheme is called locally algebraic if it is of locally finite type as a superscheme. Symmetrically, a group superscheme \( G \) is called locally algebraic if its geometric counterpart \( G \) is locally algebraic.

Locally algebraic group superschemes form a full subcategory of \( SFG \), denoted by \( SFG_{la} \). The category \( SFG_{la} \) is equivalent to the full subcategory \( SVG_{la} \) of \( SVG \), consisting of all locally algebraic geometric group superschemes.

A geometric group superscheme is called algebraic if it is of finite type as a superscheme. As above, one can define algebraic group superschemes as two equivalent categories \( SVG_a \) and \( SFG_a \).

Let \( E \) denote the trivial group superscheme. That is, for every \( A \in SAlg_k \), the group \( E(A) = \{ e_A \} \) is trivial. Its geometric counterpart is isomorphic to \( e \simeq \text{SSpec}(k) \) with the trivial group structure. The unique point of \( (\text{SSpec}(k))^e \) is also denoted by \( e \).

A sequence
\[
e \rightarrow H \rightarrow G \xrightarrow{f} R \rightarrow e
\]
in the category \( SVG \) is called exact if the corresponding sequence
\[
E \rightarrow H \rightarrow G \xrightarrow{f} R \rightarrow E
\]
in the category $\mathcal{SF}_G$ is exact, that is, for any superalgebra $A$ we have $\mathbb{H}(A) = \ker f(A)$, and the sheafification of the naive quotient functor
\[
A \mapsto (G/\mathbb{H})(A) = G(A)/\mathbb{H}(A) \subseteq R(A),
\]
with respect to the Grothendieck topology of fpqc coverings, coincides with $R$. The latter is equivalent to the following condition. For any superalgebra $A$ and any $g \in R(A)$, there is a finitely presented $A$-superalgebra $A'$, which is a faithfully flat $A$-supersubmodule such that $R(\iota_A)(g) \in f(A')(G(A'))$, where $\iota_A : A \to A'$ is the corresponding monomorphism of superalgebras (cf. [19], page 144, and Proposition 5.15 therein).

A geometric group superscheme $G$ (or the corresponding group superscheme $\mathbb{G}$) is called an abelian supervariety if $G$ is a complete supervariety. Since both properties are even reducible, we obtain that $G$ is an abelian supervariety if and only if $G_{ev}$ is an abelian variety (cf. [19], Definition 10.13).

Let $R$ be a group superscheme that acts on a superscheme $X$ on the right, and on a superscheme $Y$ on the left. Then $R$ acts on $X \times Y$ as $(x, y) \cdot r = (x \cdot r, r^{-1} \cdot y)$ for $x \in X(R), y \in Y(R), r \in R(B)$ and $B \in SAlg_k$. The sheafification of the naive quotient $B \mapsto (X(B) \times Y(B))/R(B)$ is called the homogeneous fiber quotient, and it is denoted by $X \times^R Y$. This construction is functorial in both $X$ and $Y$.

Assume that all the previous superschemes are affine, say $\mathbb{X} \approx SSp(A), \mathbb{Y} \approx SSp(B)$, and $R \approx SSp(D)$. In particular, $A$ and $B$ are affine and $D$-coideal superalgebras (cf. [19]).

Let $V$ and $W$ be the right and left $D$-supersubmodules respectively. Let $\tau_V$ and $\tau_W$ denote the corresponding supercomodule maps $V \to V \otimes D$ and $W \to D \otimes W$. Then the cotensor product $V \square_D W$ is defined as $\ker(id_V \otimes \tau_W - \tau_V \otimes id_W)$.

**Lemma 4.1.** If $\mathbb{X} \times^R \mathbb{Y}$ is an affine superscheme, it is canonically isomorphic to $SSp(A \square_D B)$. Moreover, if there are $R$-equivariant morphisms $\mathbb{X}^' \to \mathbb{X}$ and $\mathbb{Y}^' \to \mathbb{Y}$, where $\mathbb{X}^' \approx SSp(A')$, $\mathbb{Y}^' \approx SSp(B')$, and $\mathbb{X}^' \times^R \mathbb{Y}^'$ are affine, then the induced morphism $\mathbb{X}^' \times \mathbb{Y}^' \to \mathbb{X} \times \mathbb{Y}$ is dual to the natural superalgebra morphism $A \square_D B' \to A \square_D B$.

**Proof.** Use [19], Proposition 4.1. $\square$

5. The Affinization of Group Superschemes

Let $X$ be a geometric superscheme, and $U$ be its open super-subscheme. As it has been already observed, a (geometric) superscheme morphism $f : Z \to X$ factors through $U$ if and only if $f^*(Z^c) \subseteq U$.

In particular, if $X \times_Z Y$ is a fibred product in $\mathcal{S}V$, then for every open supersubchemes $U$ and $V$ of $X$ and $Y$, respectively, the open super-subscheme $p_X^{-1}(U) \cap p_Y^{-1}(V)$ of $X \times_Z Y$ is naturally isomorphic to $U \times_Z V$. Here $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$ are the canonical projections. Note that if $X(\mathbb{k}) \neq \emptyset$ (respectively, $Y(\mathbb{k}) \neq \emptyset$), then $p_X^*$ (respectively, $p_Y^*$) is injective.

Note that the sheaf morphisms $p_X^*$ and $p_Y^*$ induce a superalgebra morphism $\mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y) \to \mathcal{O}(X \times_Z Y)$.

Let $G$ be a group superscheme, and let $m, i$ and $\epsilon$ denote the multiplication morphism $G \times G \to G$, the inversion morphism $G \to G$, and the closed immersion $e \to G$, respectively. We also denote by $p_1$ and $p_2$ the corresponding projections from $G \times G$ to $G$. 
Recall that if $G$ is affine, say $G \simeq \text{SSpec}(A)$, then $A$ is a Hopf superalgebra with the coproduct $\Delta_A = m^*$, counit $\epsilon_A = \epsilon^*$ and antipode $S_A = \iota^*$.

**Lemma 5.1.** Let $H \simeq \text{SSpec}(B)$ be an affine geometric group superscheme. Then every morphism $G \to H$ in $\text{SV}_G$ is uniquely defined by a superalgebra morphism $\phi : B \to \mathcal{O}(G)$, that satisfies the following conditions:

1. The superalgebra morphism $B \otimes B \xrightarrow{\phi \otimes \phi} \mathcal{O}(G) \otimes \mathcal{O}(G) \xrightarrow{p_1^* \otimes p_2^*} \mathcal{O}(G \times G)$ makes the diagram

   \[
   \begin{array}{ccc}
   B \otimes B & \xrightarrow{\phi \otimes \phi} & \mathcal{O}(G) \otimes \mathcal{O}(G) \\
   \uparrow & & \uparrow \\
   B & \xrightarrow{\phi} & \mathcal{O}(G)
   \end{array}
   \]

   where the vertical arrows are $\Delta_B$ and $m^*$, respectively, to be commutative.

2. $\epsilon^* \phi = \epsilon_B$.

**Proof.** Use [15, Lemma 4.1].

It is clear that there is the largest super-subalgebra $C$ of $\mathcal{O}(G)$, such that the superalgebra morphism $\Delta_C : C \to \mathcal{O}(G) \xrightarrow{m^*} \mathcal{O}(G \times G)$ maps $C$ to $(p_1^* \otimes p_2^*)(C \otimes C)$.

**Proposition 5.2.** $C$ is a Hopf superalgebra with the coproduct $\Delta_C = m^*|_C$, the antipode $\iota^*|_C$ and the counit $\epsilon^*|_C$.

**Proof.** Let $q_{12}$ and $q_1$ denote the canonical projections $(G \times G) \times G \to G \times G$ and $(G \times G) \times G \to G$, respectively. Symmetrically, let $q_2$ and $q_{23}$ denote the canonical projections $G \times (G \times G) \to G$ and $G \times (G \times G) \to G \times G$, respectively. The natural isomorphism $\alpha : (G \times G) \times G \to G \times (G \times G)$ is uniquely defined by the identities:

\[ q_1 \alpha = p_1 p_{12}, \quad p_1 q_{23} \alpha = p_2 q_{12} \quad \text{and} \quad q_3 = p_2 q_{23} \alpha. \]

Moreover, the following identities are satisfied:

\[ m q_{12} = p_1 (m \times \text{id}_G), \quad q_3 = p_2 (m \times \text{id}_G), \quad m q_{23} = p_2 (\text{id}_G \times m), \quad \text{and} \quad q_1 = p_1 (\text{id}_G \times m). \]

Then there is a commutative diagram

\[
\begin{array}{cccccc}
C^{\otimes 3} & \xrightarrow{\Delta_C \otimes \text{id}_{\mathcal{O}(G)}} & \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) & \xrightarrow{p_1^* \otimes p_2^* \otimes \text{id}_{\mathcal{O}(G)}} & ((G \times G) \times G) & \xrightarrow{q_{12} \otimes q_3} \\
\uparrow & \quad & \uparrow & \quad & \uparrow \\
C^{\otimes 2} & \xrightarrow{\Delta_C} & \mathcal{O}(G) \otimes \mathcal{O}(G) & \xrightarrow{p_1^* \otimes p_2^*} & \mathcal{O}(G \times G) & \xrightarrow{\text{id}_{\mathcal{O}(G)} \otimes \text{id}_{\mathcal{O}(G)}} \\
\uparrow & \quad & \uparrow & \quad & \uparrow \\
C & \xrightarrow{\Delta_C} & \mathcal{O}(G)
\end{array}
\]

where the upper vertical arrows are $\Delta_C \otimes \text{id}_{\mathcal{O}(G)}, m^* \otimes \text{id}_{\mathcal{O}(G)}$, and $(m \times \text{id}_G)^*$, respectively. The lower vertical arrows are $\Delta_C$ and $m^*$, respectively. Symmetrically, there is a commutative diagram

\[
\begin{array}{cccccc}
\mathcal{O}(G \times (G \times G)) & \xleftarrow{q_{23}^*} & \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) & \xleftarrow{\text{id}_{\mathcal{O}(G)} \otimes p_1^* \otimes p_2^*} & \mathcal{O}(G)^{\otimes 3} & \xleftarrow{C^{\otimes 3}} \\
\uparrow & \quad & \uparrow & \quad & \uparrow \\
\mathcal{O}(G \times G) & \xleftarrow{p_1^* \otimes p_2^*} & \mathcal{O}(G)^{\otimes 2} & \xleftarrow{C^{\otimes 2}} & \mathcal{O}(G) & \xleftarrow{C}
\end{array}
\]
where the upper vertical arrows are \((\text{id}_C \times m)^*\), \(\text{id}_{\mathcal{O}(G)} \otimes m^*\), and \(\text{id}_C \otimes \Delta_C\), respectively. Using the isomorphism \(\alpha\) and the axiom of associativity, we obtain that \(\Delta_C\) is a coproduct. Therefore, \(C\) is a superbialgebra. We leave it for the reader to verify the identity
\[
\Delta_C t^* = t_{C,C} (t^* \otimes t^*) \Delta_C,
\]
where \(t_{C,C}\) is the braiding \(c_1 \otimes c_2 \mapsto (-1)^{|c_1||c_2|} c_2 \otimes c_1\) for \(c_1, c_2 \in C\). Therefore, \(\Delta_C\) maps \(t^*(C)\) to \(t^*(C)^{\otimes 2}\), hence \(t^*(C) \subseteq C\). The proposition is proven. \(\square\)

**Corollary 5.3.** The geometric group superscheme \(\text{SSpec}(C)\) is the largest affine quotient of \(G\). This means that every homomorphism from \(G\) to an affine geometric group superscheme uniquely factors through \(G \to \text{SSpec}(C)\). Similarly, \(\text{SSp}(C)\) is the largest affine quotient of \(G\). We denote the superschemes \(\text{SSpec}(C)\) and \(\text{SSp}(C)\) by \(G^{aff}\) and \(\mathbb{G}^{aff}\), respectively.

In general, \(C\) is a proper super-subalgebra of \(\mathcal{O}(G)\). But if \(G\) is algebraic, then \(C = \mathcal{O}(G)\). The proof is given in two lemmas below.

**Lemma 5.4.** (see [4] Lemma 1.8, I, §2) If \(X\) and \(Y\) are geometric superschemes such that \(Y\) is of finite type and \(X\) is affine, then \(\mathcal{P}_X \otimes \mathcal{P}_Y^*\) induces an isomorphism \(\mathcal{O}(X) \otimes \mathcal{O}(Y) \cong \mathcal{O}(X \times Y)\), which is functorial in both \(X\) and \(Y\).

**Proof.** Choose a covering of \(Y\) by open affine super-subschemes \(V_i \cong \text{SSpec}(A_i), 1 \leq i \leq t\). By Lemma \(2.1\), \(X \times V_i\) form a covering of \(X \times Y\) by open affine super-subschemes. We have a commutative diagram
\[
\begin{array}{c}
\mathcal{O}(X \times Y) \\
\uparrow \\
\mathcal{O}(X) \otimes \mathcal{O}(Y)
\end{array}
\begin{array}{c}
\prod_{1 \leq i \leq t} \mathcal{O}(X \times V_i) \\
\uparrow \\
\prod_{1 \leq i \leq j \leq t} \mathcal{O}(X \times (V_i \cap V_j))
\end{array}
\begin{array}{c}
\prod_{1 \leq i \leq t} \mathcal{O}(X) \otimes \mathcal{O}(V_i) \\
\uparrow \\
\prod_{1 \leq i \neq j \leq t} \mathcal{O}(X) \otimes \mathcal{O}(V_i \cap V_j)
\end{array}
\]
with the exact top and bottom rows. Besides, the left vertical arrows are \(\mathcal{P}_X \otimes \mathcal{P}_Y, \mathcal{P}_X \otimes \mathcal{P}_Y^*\), and \(\mathcal{P}_X \otimes \mathcal{P}_Y^*_{V_i \cap V_j}\), respectively. Since all of them, but \(\mathcal{P}_X \otimes \mathcal{P}_Y^*\), are the isomorphisms, \(\mathcal{P}_X \otimes \mathcal{P}_Y^*\) is an isomorphism as well. \(\square\)

**Lemma 5.5.** If \(X\) and \(Y\) are superschemes of finite type, then \(\mathcal{P}_X \otimes \mathcal{P}_Y^*\) is an isomorphism, which is functorial in both \(X\) and \(Y\).

**Proof.** Just consider the similar diagram for any finite open affine covering of \(X\) and apply Lemma \(5.4\). \(\square\)

### 6. THE TANGENT FUNCTOR AT SPECIFIC POINTS

The content of this section extends [4] [19].

Let \(\mathcal{X}\) be a \(\mathbb{k}\)-functor. For every point \(x \in \mathcal{X}(\mathbb{k})\), one can define a \(\mathbb{k}\)-functor \(T_x(\mathcal{X})\), called a tangent \(\mathbb{k}\)-functor at point \(x\), as follows. For each superalgebra \(R\), set \(x_R = \mathcal{X}(i_R^R(x))\), where \(i_R^R : \mathbb{k} \to R\) is the canonical embedding. Observe that \(x = x_{x}\).

Let \(R[\varepsilon_0, \varepsilon_1]\) denote the superalgebra of dual super-numbers, that is
\[
R[\varepsilon_0, \varepsilon_1] = R[x][y]/\langle x^2, y^2, xy, yx \rangle,
\]
and \(\varepsilon_0\) and \(\varepsilon_1\) denote the residue classes of \(x\) and \(y\), respectively. We have two \(R\)-superalgebra morphisms \(p_R : R[\varepsilon_0, \varepsilon_1] \to R\) and \(i_R : R \to R[\varepsilon_0, \varepsilon_1]\) such that \(p_R(r) = r, p_R(\varepsilon_0) = 0\) for \(i = 0, 1\), and \(i_R(r) = r\) for \(r \in R\). Since, \(p_R i_R = \text{id}_R\), the map \(\mathcal{X}(p_R)\) is surjective.

Set \(T_x(\mathcal{X})(R) = \mathcal{X}(p_R)^{-1}(x_R)\). It is clear that \(T_x(\mathcal{X})\) is a \(\mathbb{k}\)-functor.
Assume that $X$ is a superscheme. Let $U \simeq \text{SSp}(A)$ be an open affine supersubscheme of $X$. As it has been already observed, $x_R$ belongs to $U(R)$ if and only if $x$ belongs to $U(k)$. Besides, $x$ can be identified with a superalgebra morphism $\epsilon_A : A \to k$ and $x_R$ can be identified with $i^R_k \epsilon_A$.

Let $x \in X$ be a point. If $x$ is a closed point, then $x = (i^R_k \epsilon_A)$, so $x_R$ is the $R$-point of $X$. If $x$ is an open point, then $x_R$ is the $R$-point of $X$.

**Proof.** Let $U$ be an open affine supersubscheme of $X$ that corresponds to $U$. Then the statement of the lemma is equivalent to the following. If $f : \text{SSp}(R) \to X$ satisfies $(f_\circ \text{SSp}(p))''(\text{SSp}(R))'' \subseteq U^c$, then $f''(\text{SSp}(R))'' \subseteq U^c$.

**Lemma 6.1.** For every open super-subscheme $U$ of $X$ and every $R \in \text{SAAlg}_k$, there is $X(p_R)^{-1}(U(R)) \subseteq U(R[\epsilon_0, \epsilon_1])$. In particular, $T_x(X)(R) = T_x(U)(R)$, provided $x \in U(k)$.

**Proof.** Let $U$ be an open affine super-subscheme of $X$ that corresponds to $U$. Then the statement of the lemma is equivalent to the following. If $f : \text{SSp}(R[\epsilon_0, \epsilon_1]) \to X$ satisfies $(f_\circ \text{SSp}(p))''(\text{SSp}(R))'' \subseteq U^c$, then $f''(\text{SSp}(R[\epsilon_0, \epsilon_1]))'' \subseteq U^c$.

**Lemma 6.2.** The functor $T_x(X)$ is isomorphic to $\mathcal{V}_a$, where $V = (m_x/m_x^2)^*$.  

**Proof.** By Lemma 6.1, we have a commutative diagram

| $T_x(X)(R)$ | $X(R[\epsilon_0, \epsilon_1])$ | $X(R)$ |
|-------------|-------------------------------|--------|
| $\downarrow$ | $\uparrow$                    | $\uparrow$ |
| $T_x(U)(R)$ | $U(R[\epsilon_0, \epsilon_1])$ | $U(R)$ |
| $\downarrow$ | $\uparrow$                    | $\uparrow$ |
| $T_x(\text{SSp}(O_x))(R)$ | $\text{SSp}(O_x)(R[\epsilon_0, \epsilon_1])$ | $\text{SSp}(O_x)(R)$ |

where $U$ is an open affine super-subscheme of $X$ with $x \in U(k)$. The lower middle and rightmost vertical arrows are $i_x(R[\epsilon_0, \epsilon_1])$ and $i_x(R)$, respectively.

In particular, if $U \simeq \text{SSp}(A)$, then $(m_A/m_A^2)^* \otimes R \simeq T_x(U)(R)$ via $(\phi \circ r)(a) = \epsilon_A(a) + (-1)^{|r|\epsilon_k}(\epsilon_k \phi(a)r)$, where $r \in R, \phi \in (m_A/m_A^2)^*, a \in A, \bar{a} = a - \epsilon_A(a)$ and $k \equiv |r| + |\phi|$ (mod 2).

Similarly, we have $(m_x/m_x^2)^* \otimes R \simeq T_x(\text{SSp}(O_x))(R)$, and both isomorphisms are functorial in $R$. Since $m_A/m_A^2 \simeq m_x/m_x^2$, the lemma follows.

To an homogeneous element $r \in R$, we associate an endomorphism $\hat{r}$ of the superalgebra $R[\epsilon_0, \epsilon_1]$ defined by

\[ \hat{r}(a + \epsilon_0 b + \epsilon_1 c) = a + (-1)^{|a|\epsilon_0}(\epsilon_0 b + \epsilon_1 c). \]

In fact, we have

\[ \hat{r}((a + \epsilon_0 b + \epsilon_1 c)(a' + \epsilon_0 b' + \epsilon_1 c')) =
\]
\[ a a' + (-1)^{|a|\epsilon_0}(\epsilon_0 b + \epsilon_1 c) =
\]
\[ a a' + (-1)^{|a|\epsilon_0}(\epsilon_0 b + \epsilon_1 c) =
\]
\[ (a + (-1)^{|a|\epsilon_0}(\epsilon_0 b + \epsilon_1 c))(a' + (-1)^{|a'|\epsilon_0}(\epsilon_0 b + \epsilon_1 c)) =
\]
\[ \hat{r}(a + \epsilon_0 b + \epsilon_1 c) = a + (-1)^{|a|\epsilon_0}(\epsilon_0 b + \epsilon_1 c). \]
Moreover, if $r'$ is another homogeneous element, then $r^r = r^r$. This proves the following lemma.

**Lemma 6.3.** An endomorphism $\hat{r}$ satisfies $p_R \hat{r} = p_R$ and $r \hat{r} = r$. Therefore, $X(\hat{r})$ maps $T_x(X)(R)$ to itself. Moreover, if we identify $T_x(X)(R)$ with $(m_x/m_x^2)^* \otimes R$, then $X(\hat{r})$ coincides with the map $\hat{r} \otimes r' \mapsto \phi \otimes r^r$. In particular, $T_x(X)(R)$ has a natural structure of the right $R$-supermodule.

Let $f : X \to Y$ be a morphism of $k$-functors. Choose points $x \in X(K)$ and $y \in Y(k)$ such that $f(k)(x) = y$. For every superalgebra $R$, the map $f(R)$ takes $x_R$ to $y_R$. Therefore, there is a commutative diagram

$$
\begin{array}{ccc}
T_x(X)(R) & \to & X(R[\epsilon_0, \epsilon_1]) \to X(R) \\
\downarrow & & \downarrow \\
T_y(Y)(R) & \to & Y(R[\epsilon_0, \epsilon_1]) \to Y(R)
\end{array}
$$

The induced map $T_x(X)(R) \to T_y(Y)(R)$ is functorial in $R$. That is, $f$ induces a $k$-functor morphism $d_x f : T_x(X) \to T_y(Y)$, called the differential of $f$ at the point $x$.

Let $f : X \to Y$ be a morphism of locally algebraic geometric superschemes. If $f^*(x) = y$, then $f^*_x$ induces a morphism of superspaces $d_x f : (m_x/m_x^2)^* \to (n_y/n_y^2)^*$, where $n_y$ is the maximal superideal of $O_{Y,y}$.

**Lemma 6.4.** If we identify $T_x(X)$ and $T_y(Y)$ with $V_a$ and $W_a$, respectively, where $V = (m_x/m_x^2)^*$ and $W = (n_y/n_y^2)^*$, then $d_x f$ is naturally induced by $d_x f$.

**Proof.** Replace $X$ and $Y$ by open affine neighborhoods $U$ and $V$ of $x$ and $y$, respectively, such that $U \subseteq f^{-1}(V)$, and use the diagram from Lemma 6.3. \hfill \Box

### 7. Lie superalgebra of a group superscheme

Let $G$ be a locally algebraic group superscheme, $e$ be the identity element of $G(k)$, and $\text{Lie}(G)$ be the tangent functor $T_e(G)$. Then $\text{Lie}(G)$ is called the Lie superalgebra functor of $G$.

Arguing as in Lemma 6.3, we obtain the exact sequence

$$
1 \to \text{Lie}(G)(R) \to G(R[\epsilon_0, \epsilon_1]) \to G(R) \to 1
$$

$$
0 \to (m_x/m_x^2)^* \otimes R \to \text{SSp}(O_e)(R[\epsilon_0, \epsilon_1]) \to \text{SSp}(O_e)(R)
$$

Denote the superspace $(m_x/m_x^2)^*$ by $g$ and call it the Lie superalgebra of $G$.

Since $G(p_R)$ and $G(i_R)$ are group homomorphisms, there is an exact split sequence of groups:

$$
1 \to \text{Lie}(G)(R) \to G(R[\epsilon_0, \epsilon_1]) \xrightarrow{G(p_R)} G(R) \to 1.
$$

In other words, $G(R[\epsilon_0, \epsilon_1]) \simeq G(R) \times \text{Lie}(G)(R)$, where $G(R)$ is identified with the subgroup $G(i_R)(G(R)) \leq G(R[\epsilon_0, \epsilon_1])$. The group $G(R)$ acts on $\text{Lie}(G)(R)$ as

$$
\text{Ad}(g)(z) = G(i_R)(g) \circ G(i_R)(g)^{-1} \text{ for } g \in G(R) \text{ and } z \in \text{Lie}(G)(R).
$$

This action is called adjoint.

**Lemma 7.1.** The adjoint action of $G(R)$ on $\text{Lie}(G)(R)$ is $R$-linear and functorial in $R$. Moreover, $G(R)$ acts on $\text{Lie}(G)(R)$ by parity preserving operators.
Proof. The first statement can be easily derived from \( \hat{r} i_R = i_R \). The second is obvious. To prove the last statement, we define an endomorphism \( \iota \) of superalgebra \( R[e_0, e_1] \) by

\[
a + e_0 b + e_1 c \mapsto a + e_0 b - e_1 \text{ for } a, b, c \in R.
\]

It satisfies \( p_R \iota = p_R, \iota i_R = i_R \) and commutes with each \( \hat{r} \). Thus \( G(\iota) \) induces an automorphism of \( R \)-supermodule \( \text{Lie}(G)(R) \) that commutes with the adjoint action of \( G(R) \). On the other hand, if we identify \( \text{Lie}(G)(R) \) with \( (m_r/m_r^{\infty}) \otimes R \), then one immediately sees that \( G(\iota) \) sends a homogeneous element \( x \in \text{Lie}(G)(R) \) to \((-1)^{|x|} x\). \( \square \)

8. A FRAGMENT OF DIFFERENTIAL CALCULUS

In this section, we superize a fragment of differential calculus from [4, II, §4], for locally algebraic group superschemes (see also [3, 19]).

Let \( V \) be a finite-dimensional superspace. Recall that the affine (algebraic) group superscheme \( \text{GL}(V) \) is defined as

\[
\text{GL}(V)(R) = \text{End}_R(V \otimes R)^{\times} \text{ for } R \in \text{SAlg}_k.
\]

In other words, each \( \text{GL}(V)(R) \) consists of all graded (parity-preserving) \( R \)-linear automorphisms of \( R \)-supermodule \( V \otimes R \). The group superscheme \( \text{GL}(V) \) is called the general linear supergroup.

Let \( G \) be a group superscheme. Then a (finite-dimensional) superspace \( V \) is called a (left) \( G \)-supermodule, provided there is a group superscheme morphism \( f : G \to \text{GL}(V) \).

From now on, all \( G \)-supermodules are assumed to be finite-dimensional unless stated otherwise. Note that \( G \)-supermodules form an abelian category (with graded morphisms).

For example, \( V \) is a \( \text{GL}(V) \)-supermodule. If \( G \) is a locally algebraic group superscheme, then Lemma 7.1 implies that the Lie superalgebra \( g \) of \( G \) is a \( G \)-supermodule via \( \text{Ad} : G \to \text{GL}(g) \).

Corollary 5.3 immediately implies the following lemma.

Lemma 8.1. Let \( G \) be a group superscheme. Then the category of \( G \)-supermodules is naturally isomorphic to the category of \( G^{aff} \)-supermodules. It is also isomorphic to the category of (finite-dimensional) right \( \mathcal{O}(G^{aff}) \)-supercomodules (cf. [19]).

Lemma 8.2. If

\[
\begin{array}{cccc}
E & \to & H & \to \ G & \to \ P & \to \ R & \to \ E
\end{array}
\]

is an exact sequence of locally algebraic group superschemes, then we have an exact sequence of \( k \)-functors

\[
\begin{array}{cccc}
E & \to & \text{Lie}(H) & \xrightarrow{d_{\iota}(A)} \text{Lie}(G) & \xrightarrow{d_p(A)} \text{Lie}(R),
\end{array}
\]

that is, the sequence of \( A \)-supermodules

\[
0 \to \text{Lie}(H)(A) \xrightarrow{d_{\iota}(A)} \text{Lie}(G)(A) \xrightarrow{d_p(A)} \text{Lie}(R)(A)
\]

is exact for every \( A \in \text{SAlg}_k \).

Proof. The standard diagram chasing implies the first statement. The second statement follows by Lemma 6.4. \( \square \)
From now on, all group superschemes are locally algebraic. Following [3, 4, 19], we denote the image of $x \otimes r \in g \otimes R$ in $G(R[e_0, e_1])$ by $e^{[x] + [r]|x \otimes r}$. If $f : G \to \mathbb{H}$ is a morphism of group superschemes, then
\[ f(R[e_0, e_1])(e^{[x] + [r]|x \otimes r}) = e^{[x] + [r]|x \otimes r} = e^{[x] + [r]|x \otimes r}. \]
In particular, if $V$ is a $G$-supermodule with respect to a homomorphism $f : G \to \text{GL}(V)$, then we have
\[ f(R[e_0, e_1])(e^{[x] + [r]|x \otimes r}) = \text{id}_V + \epsilon_{[x] + [r]} d_e f(k)(x) \otimes r. \]
If we denote
\[ (f(R[e_0, e_1])(e^{[x] + [r]|x \otimes r}))(v \otimes 1) \text{ and } (d_e f(R)(x \otimes r))(v \otimes 1) \]
by
\[ e^{[x] + [r]|x \otimes r} \cdot (v \otimes 1) \text{ and } (x \otimes r) \cdot (v \otimes 1), \]
respectively, where $v \in V$, then the above formula can be recorded as
\[ e^{[x] + [r]|x \otimes r} \cdot (v \otimes 1) = v \otimes 1 + (-1)^{|r||v|} \epsilon_{[x]}(x \cdot v) \otimes r. \]

Consider the category of pairs $(G, V)$, where $G$ is a group superscheme, and $V$ is a $G$-supermodule. The morphisms in this category are couples $(f, h)$, where $f : G \to \mathbb{H}$ is a group superscheme morphism and $h : V \to W$ is a linear map of superspaces, such that for every $R \in \mathcal{SAlg}_K$, the diagram
\[ \begin{array}{ccc}
G(R) \times V_a(R) & \rightarrow & V_a(R) \\
\downarrow & & \downarrow \\
\mathbb{H} \times W_a(R) & \rightarrow & W_a(R)
\end{array} \]
is commutative. Here the horizontal maps correspond to the actions of $G(R)$ and $\mathbb{H}(R)$ on $V_a(R)$ and $W_a(R)$, respectively, the right vertical map is $h_a(R) = h \otimes \text{id}_R$, and the left vertical map is $f(R) \times h_a(R)$.

**Lemma 8.3.** If $(f, h)$ is a morphism of pairs $(G, V) \to (\mathbb{H}, W)$, then for every $x \in g, r \in R$, and $v \in V$ there is
\[ h_a(R)((x \otimes r) \cdot (v \otimes 1)) = (-1)^{|r||v|} d_e f(k)(x) \cdot h(v)) \otimes r. \]

**Proof.** We have
\[ h_a(R)(e^{[x] + [r]|x \otimes r} \cdot v) = h(v) \otimes 1 + \epsilon_{[x] + [r]} h_a(R)((x \otimes r) \cdot (v \otimes 1)) =
\]
\[ e^{[x] + [r]|d_e f(k)(x) \otimes r} \cdot (h(v) \otimes 1) = h(v) \otimes 1 + (-1)^{|r||v|} \epsilon_{[x] + [r]}(d_e f(k)(x) \cdot h(v)) \otimes r, \]
proving the formula. \qed

Let $\text{ad}$ denote $d_e \text{Ad} : \text{Lie}(g) \rightarrow \text{Lie}(\text{GL}(g)) = \mathfrak{gl}(g)$. For every $x, y \in g = \text{Lie}(g)(k)$ and $r, r' \in R$ we set $[x \otimes r, y \otimes r'] = (\text{ad}(R)(x \otimes r))(y \otimes r')$.

**Example 8.4.** The Lie superalgebra of $\text{GL}(V)$ is canonically isomorphic to $\mathfrak{gl}(V) = \text{End}_k(V)$, regarded as a superspace with $\mathfrak{gl}(V)_i = \{ \phi \mid \phi(V_j) \subseteq V_{i+j \mod 2} \}$ for $i = 0, 1$. Moreover, if $X, Y \in \mathfrak{gl}(V)$ and $r, r' \in R$, then
\[ [X \otimes r, Y \otimes r'] = (-1)^{|r||Y|} [X, Y] \otimes r r' = (-1)^{|r||Y|} (XY - (-1)^{|X||Y|} YX) \otimes r r'. \]
Indeed, we have
\[ (e^{[X] + [r]|X \otimes r}) \cdot (Y \otimes r') = (\text{id}_V \otimes 1 + \epsilon_{[X] + [r]} X \otimes r)(Y \otimes r') \text{(id}_V \otimes 1 - \epsilon_{[X] + [r]} X \otimes r) =
\]
\[ Y \otimes r' + \epsilon_{[X] + [r]} ((-1)^{|X||Y|} XY \otimes r r' - (-1)^{|X|+|Y|+|r'|||X|} YX \otimes r r') =
\]


\[ Y \otimes r' + (-1)^{|Y||r|} \epsilon_{|X|+|r|}(XY - (-1)^{|X||Y|}YX) \otimes rr', \]

proving the formula.

**Lemma 8.5.** Let \( f : G \to H \) be a morphism of locally algebraic group superschemes. Let \( h \) denote the Lie superalgebra of \( H \). Then for every \( x, y \in g \) and \( r, r' \in R \) there is

\[ d_* f(R)([x \otimes r, y \otimes r']) = (-1)^{|r||y|}[d_* f(k)(x), d_* f(k)(y)] \otimes rr'. \]

In particular, \( g \otimes R \) and \( h \otimes R \) are Lie superalgebras for the operation \([ , |\) and \(d_* f \) is a morphism of Lie superalgebra functors.

**Proof.** We have

\[ d_* f(R) \text{Ad}(R)(g) = \text{Ad}(R)(f(R)(g)) d_* f(R) \]

for every superalgebra \( R \) and \( g \in G(R) \). That is, \( (f, d_* f(k)) \) is a morphism of pairs \((G, g) \to (H, h)\), where \( g \) and \( h \) are regarded as \(G\)-supermodule and \( H\)-supermodule with respect to the adjoint actions. Lemma 8.3 implies the first statement.

To prove the second statement, one needs to show that the operation \([ , |\) on \( g \) satisfies the identities (B2), (B3), and (B4) from [12] (recall that \( \text{char}(k) \neq 2 \)). Applying the first statement to \( \text{Ad} : G \to \text{GL}(g) \), one obtains

\[ [[x, y], z] = (XY - (-1)^{|x||y|}YX)(z), \]

where \( X = \text{ad}(k)(x) \), \( Y = \text{ad}(k)(y) \). Therefore, (B4) (or the super Jacobi identity)

\[ [[x, y], z] = [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]] \]

follows.

Next, let \( R \) denote \( k[\epsilon_0, \epsilon_1] \). Then the group commutator

\[ e^{\epsilon_{|x||x|} x}, e^{\epsilon_{|y||y|} y} \]

is equal to

\[ e^{\text{Ad}(R[\epsilon_0, \epsilon_1])(e^{\epsilon_{|x||x|} x})(e^{\epsilon_{|y||y|} y})} e^{-\epsilon_{|x||x|} x} e^{-\epsilon_{|y||y|} y} = e^{\epsilon_{|y||y|} y + (-1)^{|x||y|} \epsilon_{|x||x|} x} \text{ad}(k)(x)(y) e^{-\epsilon_{|y||y|} y} \]

The group identity \([g, h] = [h, g]^{-1} \) implies \( \epsilon_{|x||x|} x \) \([x, y] = -\epsilon_{|y||y|} y \) \([y, x] \), hence (B3) (or the super skew-symmetry) \([x, y] = -(-1)^{|x||y|}[y, x] \) follows.

The identity (B2) states \([x, x], x] = 0 \) for every \( x \in g_1 \). If \( \text{char}(k) \neq 3 \), then (B2) follows from (B4). If \( \text{char}(k) = 3 \), then (B4) does not imply (B2). However, in Lemma 8.4 we prove the identity (B2) in general. This finishes the proof. \( \square \)

**Lemma 8.6.** The adjoint action of \( G \) on \( \text{Lie}(G) \) commutes with the super-bracket functor.

**Proof.** For every \( g \in G(R) \) we have

\[ \text{Ad}(R)(g) e^{(-1)^{|x||y|} \epsilon_{|x||x|} x} e^{\epsilon_{|y||y|} y} = e^{(-1)^{|x||y|} \epsilon_{|x||x|} x} \text{Ad}(R)(g)[x, y] = \]

\[ \text{Ad}(R)(g)\{ e^{\epsilon_{|x||x|} x}, e^{\epsilon_{|y||y|} y} \} = [\text{Ad}(R)(g)e^{\epsilon_{|x||x|} x}, \text{Ad}(R)(g)e^{\epsilon_{|y||y|} y}] = \]

\[ \{ e^{\epsilon_{|x||x|} x}, \text{Ad}(R)(g)[x, y] = e^{(-1)^{|x||y|} \epsilon_{|x||x|} x} [\text{Ad}(R)(g)x, \text{Ad}(R)(g)y], \]

proving our statement. \( \square \)
Let $G$ be a geometric group superscheme. Let $e$ denote the identity morphism $e \to G$. We have a commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{m} & G \\
\uparrow & & \uparrow \\
G \times G & \xrightarrow{\pi_1} & e \\
\xrightarrow{\pi_2} & & \xrightarrow{\epsilon} \\
G & \xleftarrow{\epsilon} & e \\
\end{array}
\]

where the closed immersion $e \to G \times G$ is induced by the universality of the direct product. For every affine open neighborhood $U$ of $e$, the immersion $e \to G \times G$ factors through $U \times U \subseteq G \times G$. If we denote the image of the point $e$ in $(G \times G)^s$ by $e \times e$, then $\mathcal{O}_{e \times e}$ is naturally isomorphic to $(\mathcal{O}_e \otimes \mathcal{O}_e)_{n_{e \times e}}$, where $n_{e \times e} = \mathcal{O}_e \otimes m_e + m_e \otimes \mathcal{O}_e$. Moreover, $m_{e \times e}$ is a local superalgebra morphism from $\mathcal{O}_e$ to $(\mathcal{O}_e \otimes \mathcal{O}_e)_{n_{e \times e}}$.

For any non-negative integer $k$, let $N_k(G) \simeq \text{SSpec}(\mathcal{O}_e/m_{e}^{k+1})$ denote the $k$-th neighborhood of $e$.

**Lemma 9.1.** For any non-negative integers $k$ and $t$, there are commutative diagrams

\[
\begin{array}{ccc}
G \times G & \xrightarrow{m} & G \\
\uparrow & & \uparrow \\
N_k(G) \times N_t(G) & \rightarrow & N_{k+t}(G) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
G & \xrightarrow{\iota} & G \\
\uparrow & & \uparrow \\
N_k(G) & \rightarrow & N_k(G) \\
\end{array}
\]

**Proof.** The local superalgebra $\mathcal{O}_e/m_e^{k+1} \otimes \mathcal{O}_e/m_e^{t+1}$ is canonically isomorphic to $(\mathcal{O}_e \otimes \mathcal{O}_e)_{n_{e \times e}}/(\mathcal{O}_e \otimes m_e^{k+1} + m_e^{t+1} \otimes \mathcal{O}_e)_{n_{e \times e}}$. Moreover, $m_{e \times e}$ induces a local superalgebra morphism

\[
\mathcal{O}_e/m_e^{k+t+1} \rightarrow (\mathcal{O}_e \otimes \mathcal{O}_e)_{n_{e \times e}}/(m_e^{k+1} \otimes \mathcal{O}_e + \mathcal{O}_e \otimes m_e^{t+1})_{n_{e \times e}}
\]

that makes the diagram

\[
\begin{array}{ccc}
\mathcal{O}_e & \xleftarrow{m_{e \times e}} & \mathcal{O}_e \\
\downarrow & & \downarrow \\
\mathcal{O}_e/m_e^{k+1} \otimes \mathcal{O}_e/m_e^{t+1} & \leftarrow & \mathcal{O}_e/m_e^{k+t+1}
\end{array}
\]

commutative. The proof of the second statement is analogous. \qed

Translating to the category $\mathcal{SF}_G$, one sees that the group superscheme $G$ contains an ascending chain of closed super-subschemes $N_0(G) \subseteq N_1(G) \subseteq \ldots$, such that each $N_k(G)$ is isomorphic to $\text{SSpec}(\mathcal{O}_e/m_e^{k+1})$. Moreover, the terms satisfy $N_k(G)^{-1} \subseteq N_k(G)$ and $N_k(G)N_t(G) \subseteq N_{k+t}(G)$ for any non-negative integers $k$ and $t$. Therefore, $\mathbb{N}(G) = \cup_{k \geq 0} N_k(G)$ is a group subfunctor of $G$ such that for every $R \in \mathcal{SAlg}_k$, the group $\mathbb{N}(G)(R)$ consists of all superalgebra morphisms $\mathcal{O}_e \to R$ vanishing on some power of $m_e$. We call $\mathbb{N}(G)$ the *formal neighborhood* of the identity in $G$.

The group functor $\mathbb{N}(G)$ is "quasi-affine" in the following sense. The complete local superalgebra $\widehat{\mathcal{O}_e} = \mathop{\text{lim}}_{n} \mathcal{O}_e/m_e^{n+1}$ is a complete Hopf superalgebra for the
comultiplication $\Delta : \hat{\mathcal{O}}_e \to \hat{\mathcal{O}}_e \otimes \hat{\mathcal{O}}_e$ and the antipode $S : \hat{\mathcal{O}}_e \to \hat{\mathcal{O}}_e$, induced by $m_{e \times e}^*$ and $i_e^*$, respectively (see [7], Definition 3.3). Moreover, there is

$$\Delta(m_e) \subseteq \hat{\mathcal{O}}_e \otimes m_e + m_e \otimes \hat{\mathcal{O}}_e \text{ and } S(m_e) \subseteq m_e,$$

that is, $m_e$ is a (closed) Hopf superideal of $\hat{\mathcal{O}}_e$.

For every $R \in SAlg_k$, the group $\mathbb{N}(G)(R)$ consists of all continuous superalgebra morphisms $\hat{\mathcal{O}}_e \to R$, where $R$ is regarded as a discrete superalgebra. The group operations of $\mathbb{N}(G)(R)$ are defined by $(gh)(a) = (g \otimes h)(\Delta(a))$ and $g^{-1}(a) = g(S(a))$ for $g, h \in \mathbb{N}(G)(R)$ and $a \in \hat{\mathcal{O}}_e$. We use the "Sweedler notation" and write $\Delta(a) = a_{(0)} \otimes a_{(1)}$, by omitting the summation symbol. The sum on the left is convergent in the $n_{\times e}$-adic topology. Then $\phi(a) = g(a_{(0)})h(a_{(1)})$, where the sum on the right contains only finitely many non-zero summands.

Assume again that $G$ is locally algebraic. For a non-negative integer $n$, let $\text{hyp}_{\mathbb{N}}(G)$ denote the superspace $(\mathcal{O}_e / m_{e}^{n+1})^*$. Set $\text{hyp}(G) = \bigcup_{n \geq 0} \text{hyp}_{\mathbb{N}}(G) \subseteq (\mathcal{O}_e)^*$. The superspace $\text{hyp}(G)$ has a natural Hopf superalgebra structure with respect to the product

$$(\phi \psi)(a) = (-1)^{|\phi||a_{(0)}|} \phi(a_{(0)}) \psi(a_{(1)}),$$

the coproduct $\Delta^*(\phi) = \phi(0) \otimes \phi(1)$, uniquely defined by the identity

$$\phi(ab) = (-1)^{|\phi(a_{(1)}||a_{(0)}|} \phi(a_{(0)}) \phi(1)(b) \text{ for } \phi, \psi \in \text{hyp}(G) \text{ and } a, b \in \hat{\mathcal{O}}_e,$$

and the antipode $S^*$, such that $(S^*(\phi))(a) = \phi(S(a))$. Furthermore, for every $R \in SAlg_k$, $\mathcal{O}_e \otimes R = \varprojlim_{n \to -\infty} (\mathcal{O}_e \otimes R)/(m_e \otimes R)^{n+1}$ is a complete Hopf $R$-superalgebra with respect to the comultiplication $\Delta \otimes \text{id}_R$ and the antipode $S \otimes \text{id}_R$, and $\text{hyp}(G) \otimes R$ has the unique structure of Hopf $R$-superalgebra for every $R \in SAlg_k$ such that the pairing $(\text{hyp}(G)) \otimes R \times (\mathcal{O}_e \otimes R) \to R$ given by $< \phi \otimes r, a \otimes r' > = (-1)^{|r||a|} \phi(a) r r'$, is a Hopf pairing in the sense of [12].

The following lemma is folklore.

**Lemma 9.2.** For every non-negative integers $k$ and $t$, there is:

1. $\text{hyp}_k(G) \text{hyp}_t(G) \subseteq \text{hyp}_{k+t}(G)$,
2. $\Delta^*(\text{hyp}_k(G)) \subseteq \oplus_{0 \leq s \leq k} \text{hyp}_s(G) \otimes \text{hyp}_{k-s}(G)$,
3. $S^*(\text{hyp}_k(G)) \subseteq \text{hyp}_k(G)$.

For a Hopf superalgebra $H$, let $\text{Gpl}(H)$ denote the subgroup of $H^*$ consisting of all (even) group-like elements of $H$. Let $n_G$ denote the Lie superalgebra of $\text{N}(G)$.

**Lemma 9.3.** The group functor $\mathbb{N}(G)$ is canonically identified with the group functor $R \mapsto \text{Gpl}(\text{hyp}(G) \otimes R)$. Additionally, we have $n_G \simeq \text{hyp}_1(G)^+ = \{ \phi \in \text{hyp}_1(G) | \phi(1) = 0 \}$ and the Lie super-bracket on $n_G$ is defined by $[\phi, \psi] = \phi \psi - (-1)^{|\phi||\psi|}\psi \phi$. In particular, this operation satisfies the identity (B2).

**Proof.** We identify the superspace $\text{hyp}(G) \otimes R$ with $\varprojlim_{n} \text{Hom}_K(\mathcal{O}_e / m_{e}^{n+1}, R)$ via the above pairing. Then the first statement follows.

Furthermore, the previous dual super-numbers technique shows that the Lie superalgebra $n_G$ can be identified with the super-subspace $P$ of $\text{hyp}(G)^+$ consisting of all primitive elements. Also, $\dim n_G = \dim g$ (see Lemma 9.2 below) and $\text{hyp}_1(G)^+ \subseteq P$. 

Since $N(G)$ is "represented" by the complete Hopf superalgebra $\widehat{O}_c$, to prove the last statement, one can mimic the calculation from [19, Lemma 3.4]. Finally, $[[\phi, \phi], \phi] = [2\phi^2, \phi] = 0$ for every $\phi \in \mathfrak{s}_1$.

\noindent \textbf{Lemma 9.4.} \textit{Let $G$ be a locally algebraic group superscheme. Then the operation $[,]$ on $g$ satisfies the identity (B2).}

\textbf{Proof.} Since each immersion $N_k(G) \to G$ factors through any open affine neighborhood $U$ of $e = e_k \in G(k)$, we have the following commutative diagram

\[
\begin{array}{ccccccc}
1 & \to & \text{Lie}(G)(R) & \to & G(R[\epsilon_0, \epsilon_1]) & \to & G(R) & \to & 1 \\
\| & & \| & & \| & & \| & & \\
(\mathfrak{m}_c/\mathfrak{m}_c^2)^* \otimes R & \to & N_k(G)(R[\epsilon_0, \epsilon_1]) & \to & N_k(G)(R) & \to & N_k(G)(R) & \to & 1 \\
\| & & \| & & \| & & \| & & \\
(\mathfrak{m}_c/\mathfrak{m}_c^2)^* \otimes R & \to & N_k(G)(R[\epsilon_0, \epsilon_1]) & \to & N_k(G)(R) & \to & N_k(G)(R) & \to & 1 \\
\end{array}
\]

where $t \leq k$. Using Lemma 9.1 we obtain a commutative diagram of groups

\[
\begin{array}{ccccccc}
1 & \to & \text{Lie}(G)(R) & \to & G(R[\epsilon_0, \epsilon_1]) & \to & G(R) & \to & 1 \\
\| & & \| & & \| & & \| & & \\
1 & \to & (\mathfrak{m}_c/\mathfrak{m}_c^2)^* \otimes R & \to & N_k(G)(R[\epsilon_0, \epsilon_1]) & \to & N_k(G)(R) & \to & 1 \\
\end{array}
\]

Thus $g = n_c$ and using the same trick with group commutators as in Lemma 8.3 one can derive that the Lie super-bracket on $g$ coincides with the Lie super-bracket on $n_c$. Lemma 9.3 concludes the proof. \hfill \Box

Let the $k$-functor morphism $c : G \times G \to G$ defines the conjugation action of $G$ on itself. In other words, for any $R \in \text{SAAlg}_k$ and $g, h \in G(R)$ there is $c(R)(g, h) = ghg^{-1}$.

\noindent \textbf{Lemma 9.5.} \textit{Each $N_k(G)$ is invariant with respect to the conjugation action of $G$ on itself. Therefore, $\overline{N}(G)$ is a normal group subfunctor of $G$.}

\textbf{Proof.} We use the equivalent language of geometric superschemes. The conjugation action is defined by a superscheme morphism $c : G \times G \to G$. Let $U \simeq \text{SSpec}(A)$ be an affine neighborhood of $e$. For every affine open super-subscheme $V \simeq \text{SSpec}(B)$, there is a commutative diagram

\[
\begin{array}{ccccccc}
V \times U & \overset{c|_{V \times U}}{\to} & G \\
\| & & \| & & \\
V \times e & \overset{c|_{V \times e}}{\to} & e \\
\end{array}
\]

Then $c^{-1}|_{V \times U}(U) = (V \times U) \cap c^{-1}(U)$ can be covered by open affine supersubschemes $\text{SSpec}(B \otimes A)_g$. There is a dual commutative diagram

\[
\begin{array}{ccccccc}
(B \otimes A)_g & \overset{c^*}{\to} & A \\
\| & & \| & & \\
B_{\overline{g}} & \leftarrow & k \\
\end{array}
\]

where the vertical arrows are $\text{id}_{B \otimes A}$ and $\epsilon_A$, respectively, and $\overline{g} = (\text{id}_{B \otimes A})(g)$. Since localization is a faithful functor, we obtain $c^*(\mathfrak{m}_A) \subseteq (B \otimes \mathfrak{m}_A)_g$. Since $c^*(\mathfrak{m}_A^{k+1}) \subseteq (B \otimes \mathfrak{m}_A)^{k+1} = (B \otimes \mathfrak{m}_A^{k+1})_g$, $c^*$ induces a superalgebra morphism $(B \otimes A/\mathfrak{m}_A^{k+1})_g \leftarrow A/\mathfrak{m}_A^{k+1}$. Considering all $g$ and $V$, one sees that $c$ sends $G \times N_k(G)$ to $N_k(G)$, which proves the lemma. \hfill \Box
**Lemma 9.6.** Let $X$ be a geometric superscheme. For every finite-dimensional superalgebra $A$, we have $\mathcal{O}(X \times \text{SSpec}(A)) \simeq \mathcal{O}(X) \otimes A$.

**Proof.** Let $Y$ denote $\text{SSpec}(A)$. Choose a covering of $X$ by affine open super-subschemas $U_i$. Then $V_i = U_i \times Y$ is an open covering of $X \times Y$ by affine super-subschemas. If the elements $a_1, \ldots, a_k$ form a basis of $A$, then each element $f \in \mathcal{O}(V_i) \simeq \mathcal{O}(U_i) \otimes A$ can be uniquely expressed as $\sum_{1 \leq j \leq k} f_j \otimes a_j$. Moreover, if for some $f_j \in \mathcal{O}(V_i)$ we have $f_j|_{V_i \cap V_i} = f_j'|_{V_i \cap V_i}$, then $f_j|_W = f_j'|_W$ for any affine open super-subscheme $W$ of $U_i \cap U_i$ and $1 \leq j \leq k$, that is, $f_j|_{V_i \cap V_i} = f_j'|_{V_i \cap V_i}$ for every $1 \leq j \leq k$. Thus, our lemma follows. \hfill $\square$

**Lemma 9.7.** The conjugation action of $G$ on $\mathbb{N}(G)$ factors through the action of $\mathbb{G}^{aff}$ on $\mathbb{G}(G)$. 

**Proof.** For simplicity, we use $N_k$ instead of $N_k(G)$ and $N$ instead of $\mathbb{N}(G)$. Let $c_k$ denote $c|_{G \times N_k}$. By Lemma 9.6, $c_k$ maps $\mathcal{O}(N_k)$ to $\mathcal{O}(G) \otimes \mathcal{O}(N_k)$, that is, in Sweedler’s notation, $c_k^* (a) = a_0 \otimes a_{(1)}$ for $a \in \mathcal{O}(N_k)$. The superspace generated by the coefficients $\hat{a}_{(0)}$, where $a$ ranges over $\mathcal{O}(N_k)$, is called the coefficient superspace of $\mathcal{O}(N_k)$, and it is denoted by $\text{cf}(\mathcal{O}(N_k))$.

The fact that $c_k$ defines an action implies the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{O}(G) \otimes \mathcal{O}(N_k) & \stackrel{m^* \otimes \text{id}_{\mathcal{O}(N_k)}}{\longrightarrow} & \mathcal{O}(G \times G) \otimes \mathcal{O}(N_k) \\
\uparrow \quad & & \uparrow \quad \\
\mathcal{O}(N_k) & \stackrel{c_k^*}{\longrightarrow} & \mathcal{O}(G) \otimes \mathcal{O}(N_k)
\end{array}
$$

where the vertical arrows are $c_k^*$ and $(\text{id}_G \times c_k)^*$, respectively, and $\beta$ is the canonical isomorphism $(G \times G) \times N_k \rightarrow G \times (G \times N_k)$. Arguing as in Proposition 5.2, one sees that $(\text{id}_G \times c_k)^*$ factors through $\text{id}_{\mathcal{O}(G)} \otimes c_k^*$. Moreover, the composition of the morphism

$$
\mathcal{O}(G)^{\otimes 2} \otimes \mathcal{O}(N_k) \simeq \mathcal{O}(G) \otimes \mathcal{O}(G \times N_k) \rightarrow \mathcal{O}(G \times (G \times N_k))
$$

with $\beta^*$ can be identified with $p_1^* \otimes p_2^* \otimes \text{id}_{\mathcal{O}(N_k)}$. Therefore, $\text{cf}(\mathcal{O}(N_k)) \subseteq \mathcal{O}(G^{aff})$, and $\mathcal{O}(N_k) = \mathcal{O}_e / m_e^{t+1}$ is a left $\mathcal{O}(G^{aff})$-coideal superalgebra (cf. [19]). Furthermore, for any $t \geq k$, there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_e / m_e^{t+1} & \stackrel{c^*}{\longrightarrow} & \mathcal{O}(G^{aff}) \otimes \mathcal{O}_e / m_e^{t+1} \\
\downarrow \quad & & \downarrow \\
\mathcal{O}_e / m_e^{k+1} & \stackrel{c^*}{\longrightarrow} & \mathcal{O}(G^{aff}) \otimes \mathcal{O}_e / m_e^{k+1},
\end{array}
$$

hence $c^* = \lim_{t \rightarrow k} c^*_k$ defines a structure of left $\mathcal{O}(G^{aff})$-coideal superalgebra on $\mathcal{O}_e$. In other words, the conjugation action of $G$ on $\mathbb{N}$ is defined by

$$(g \cdot h)(a) = (\overline{g} \cdot h)(a) = \overline{g}(a_{(0)}) h(a_{(1)}),$$

where $g \in \mathbb{G}(R), h \in \mathbb{N}(R), a \in \mathcal{O}_e, c^*(a) = a_{(0)} \otimes a_{(1)}, R \in \text{SAlg}_{k}$, and $\overline{g}$ is the image of $g$ under the group homomorphism $\mathbb{G}(R) \rightarrow \mathbb{G}^{aff}(R)$. \hfill $\square$

Since $\mathcal{O}_e$ is a left $\mathcal{O}(G^{aff})$-coideal superalgebra, $\mathbb{G}(R)$ acts on $\mathcal{O}_e \otimes R$ on the right by $R$-superalgebra automorphisms

$$(a \otimes r) \cdot g = \overline{g}(a_{(0)}) a_{(1)} \otimes r$$

for $g \in \mathbb{G}(R)$ and $r \in R$, where $\overline{g}(a_{(0)}) a_{(1)} \otimes r$ for $g \in \mathbb{G}(R)$ and $r \in R$. 


and this action is functorial in \( R \). We also have
\[
< g \cdot h, x > = < h, x \cdot g > \text{ for } x \in \widehat{O}_c \otimes R,
\]
where \( h \in N(R) \) is regarded as a group-like element of \( \text{hyp}(\mathbb{G}) \otimes R \).

**Lemma 9.8.** The group superscheme \( \mathbb{G} \) acts on \( \widehat{O}_c \) by Hopf superalgebra automorphisms. Moreover, if we define its action on \( \text{hyp}(\mathbb{G}) \) by the above formula
\[
< g \cdot \phi, a > = < \phi, a \cdot g > \text{ for } g \in \mathbb{G}(R), \phi \in \text{hyp}(\mathbb{G}) \otimes R, \text{ and } a \in \widehat{O}_c \otimes R,
\]
then this action also preserves the Hopf superalgebra structure of \( \text{hyp}(\mathbb{G}) \otimes R \) and is functorial in \( R \).

**Proof.** Since \( < , > \) is a Hopf duality, the second statement is obvious.

We have a commutative diagram
\[
\begin{array}{cccc}
\mathbb{G}^{aff} \times \mathbb{N} & \xrightarrow{\mathcal{E}} & \mathbb{N} \\
\uparrow & & \uparrow \\
\mathbb{G}^{aff} \times \mathbb{N} \times \mathbb{N} & \rightarrow & (\mathbb{G}^{aff} \times \mathbb{N}) \times (\mathbb{G}^{aff} \times \mathbb{N}),
\end{array}
\]
where the lower horizontal arrow is defined as \((g, s, s') \mapsto (g, g, s, s')\), the left vertical arrow is \( \text{id}_{\mathbb{G}^{aff}} \times \mathfrak{m} \), and the right vertical arrows are \( \mathfrak{c} \times \mathfrak{c} \) and \( \mathfrak{m} \), read from the bottom to the top, respectively. The commutativity of the corresponding diagram of superalgebras
\[
\begin{array}{cccc}
\mathcal{O}(\mathbb{G}^{aff}) \otimes \widehat{O}_c & \xrightarrow{\mathfrak{c}} & \widehat{O}_c \\
\downarrow & & \downarrow \\
\mathcal{O}(\mathbb{G}^{aff}) \otimes \widehat{O}_c \otimes \widehat{O}_c & \leftarrow (\mathcal{O}(\mathbb{G}^{aff}) \otimes \widehat{O}_c) \otimes (\mathcal{O}(\mathbb{G}^{aff}) \otimes \widehat{O}_c),
\end{array}
\]
is equivalent to the identity
\[
(*) \quad a_{(0)} \otimes (a'_{(1)})_{(0)} \otimes (a'_{(1)})_{(1)} = (-1)^{(\ell(a_{(1)}))_{(0)} \cdot \ell(a'_{(0)}))_{(1)} \cdot \ell(a'_{(0)}))_{(0)} \otimes (a'_{(1)})_{(0)} \otimes (a'_{(1)})_{(1)},
\]
for any \( a \in \widehat{O}_c \), where
\[
\Delta(a) = a_{(0)} \otimes a'_{(1)}, \quad a^*(a) = a_{(0)} \otimes a'_{(1)}, \quad \Delta(a_{(1)}) = (a_{(1)})_{(0)} \otimes (a_{(1)})_{(1)}, \\
\quad c^*(a_{(0)}) = (a_{(0)})_{(0)} \otimes (a_{(0)})_{(1)}, \quad c^*(a'_{(1)}) = (a'_{(1)})_{(0)} \otimes (a'_{(1)})_{(1)}.
\]
Applying \( g \otimes \text{id}_{\widehat{O}_c}^{\otimes 2} \) to both parts of (*), we obtain \( \Delta(a \cdot g) = a_{(0)} \cdot g \otimes a'_{(1)} \cdot g. \)

**Corollary 9.9.** The above action of \( \mathbb{G} \) on \( \text{hyp}(\mathbb{G}) \) coincides with \( \text{Ad} \).

**Lemma 9.10.** If \( U \) is an open super-subscheme of \( \mathbb{G} \), then \( NU \subseteq U \), and \( UN \subseteq U \).

**Proof.** All one needs to show is that for any \( k \geq 0 \), the morphism \( U \times N_k \rightarrow \mathbb{G} \) factors through \( U \). That is, \( m^k \) maps \( (U \times N_k)^c \) to \( U^c \). Without losing generality, one can assume that \( U \) is affine, say \( U \simeq \text{SSpec}(A) \). Then every prime super-ideal of \( A \otimes \mathcal{O}_c/m^{k+1}_c \) has a form \( p \otimes \mathcal{O}_c/m^{k+1}_c + A \otimes \mathfrak{m}_c/m^{k+1}_c \), where \( p \in (\text{SSpec}(A))^c \).

Thus \((U \times N_k)^c = (U \times N_0)^c = (U \times e)^c \) and our statement follows by one of the group axioms.
10. The functor $\text{gr}$

10.1. Filtered superalgebras and their graded companions. A \textit{(downward)} filtered superalgebra is a couple $(A, I)$, where $A$ is a superalgebra, and $I$ is a superideal filtration

$$A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$$

of $A$ such that $I_k I_l \subseteq I_{k+l}$ for every $k, l \geq 0$. We associate with any filtered superalgebra $(A, I)$ a graded superalgebra $\text{gr}_I(A) = \oplus_{k \geq 0} I_k/I_k^{+1}$.

If there is a superideal $I$ such that $I_k = I^k$ for each $k \geq 0$, then this filtration is called $I$-adic, and the corresponding graded superalgebra is denoted by $\text{gr}_I(A)$.

If $(B, J)$ is another filtered superalgebra, then the tensor product $A \otimes B$ is a filtered superalgebra with respect to the filtration

$$T_k = I_0 \otimes J_k + I_1 \otimes J_{k-1} + \ldots + I_k \otimes J_0 \text{ for } k \geq 0.$$

We call the filtration $T$ a \textit{tensor product} of filtrations $I$ and $J$.

A morphism of filtered superalgebras from $(A, I)$ to $(B, J)$, is a superalgebra morphism $\phi : A \rightarrow B$ such that $\phi(I_k) \subseteq J_k$ for every $k \geq 0$. Any such morphism induces a superalgebra morphism $\text{gr}_I(A) \rightarrow \text{gr}_J(B)$.

If $(A, I)$ is a filtered superalgebra, then for every $n \geq 0$ the superalgebra $A/I_{n+1}$ has a finite filtration

$$A/I_{n+1} = I_0/I_{n+1} \supseteq I_1/I_{n+1} \supseteq \ldots \supseteq I_n/I_{n+1} \supseteq 0,$$

which is denoted by $I \leq n$.

\textbf{Proposition 10.1.} \textit{There is a natural isomorphism}

$$\text{gr}_I(A) \otimes \text{gr}_J(B) \simeq \text{gr}_T(A \otimes B),$$

\textit{induced by the canonical embeddings $A \rightarrow A \otimes B$ and $B \rightarrow A \otimes B$, which is functorial in both $A$ and $B$.}

\textbf{Proof.} Both $A \rightarrow A \otimes B$ and $B \rightarrow A \otimes B$ are morphisms of filtered superalgebras. Therefore, they induce the canonical morphisms of graded superalgebras

$$\text{gr}_I(A) \rightarrow \text{gr}_T(A \otimes B) \text{ and } \text{gr}_J(B) \rightarrow \text{gr}_T(A \otimes B)$$

given by

$$a + I_{k+1} \mapsto a \otimes 1 + T_{k+1} \text{ and } b + J_{l+1} \mapsto 1 \otimes b + T_{l+1}.$$

Thus, there is a superalgebra morphism

$$\text{gr}_I(A) \otimes \text{gr}_J(B) \rightarrow \text{gr}_T(A \otimes B)$$

which takes $(a + I_{k+1}) \otimes (b + J_{l+1})$ to $a \otimes b + T_{k+l+1}$ for every $a \in I_k, b \in J_l$.

Moreover, if we define a graded superalgebra structure on $\text{gr}_I(A) \otimes \text{gr}_J(B)$ as

$$(\text{gr}_I(A) \otimes \text{gr}_J(B))(n) = \oplus_{0 \leq k \leq n} I_k/I_k^{+1} \otimes J_{n-k}/J_{n-k}^{+1} \text{ for } n \geq 0,$$

then the above map is graded superalgebra morphism.

For every $k \geq 0$, choose elements $a_{i,k} \in I_k$ and $b_{j,k} \in J_k$, where $i$ and $j$ run over index sets $S_k$ and $L_k$, respectively, such that they form bases of $I_k$ modulo $I_{k+1}$, and $J_k$ modulo $J_{k+1}$, respectively. Then the homogeneous component $(\text{gr}_I(A) \otimes \text{gr}_J(B))_n$ has a basis consisting of all elements

$$(a_{i,k} + I_{k+1}) \otimes (b_{j,n-k} + J_{n-k+1}) \text{ for } 0 \leq k \leq n, i \in S_k \text{ and } j \in L_{n-k}.$$
Moreover, the elements
\[ a_{i,k} \otimes b_{j,n-k} \] for \( 0 \leq k \leq n, \ i \in S_k \) and \( j \in L_{n-k} \) generate \( T_n \) modulo \( T_{n+1} \). Since our morphism induces a one-to-one correspondence between the basis of \( (\operatorname{gr}(A) \otimes \operatorname{gr}(B))_n \) and the generating set of \( T_n/T_{n+1} \), all we need is to show that the elements \( a_{i,k} \otimes b_{j,n-k} \) are linearly independent over \( T_{n+1} \).

Consider the filtered superalgebra \((A/I_n, 1 \leq n)\) and \((B/J_n, 1 \leq n)\). The elements \( \overline{a_{i,k}} = a_{i,k} + I_{n+1} \) and \( \overline{b_{j,l}} = b_{j,l} + I_{n+1} \), where \( 0 \leq k, l \leq n, i \in S_k \), and \( j \in L_l \), form bases of vector (super)spaces \( A/I_{n+1} \) and \( B/J_{n+1} \), respectively. The superalgebra
\[ A/I_{n+1} \otimes B/J_{n+1} \cong (A \otimes B)/(I_{n+1} \otimes B + A \otimes J_{n+1}) \]
has a basis \( \overline{a_{i,k}} \otimes \overline{b_{j,l}} \) for \( 0 \leq k, l \leq n, i \in S_k \) and \( j \in L_l \).

Let \( T' \) denote the tensor product of filtrations \( I \subseteq n \) and \( J \subseteq n \). Then \( T_{2n+1}' = 0 \) and for every \( 0 \leq s \leq 2n \) the superspace \( T_{s}'/T_{s+1}' \) is spanned by the part of the above basis consisting of the elements \( \overline{a_{i,k}} \otimes \overline{b_{j,s-k}} \), where \( k, s - k \leq n \). In particular, the latter elements are linearly independent over \( T_{s+1}' \). If we apply this remark to \( s = n \), and observe that \( T_n'/T_{n+1}' \) is naturally isomorphic to \( T_n/T_{n+1} \), the proof concludes.

**10.2. Graded companions of geometric superschemes.** Recall that if \( \mathcal{F} \) is a presheaf of abelian groups on a topological space \( X \), then its sheafification is denoted by \( \mathcal{F}^+ \).

If the index set \( I \) is infinite, then a direct sum of sheaves \( \mathcal{F} = \bigoplus_{i \in I} \mathcal{F}_i \) is a presheaf but not necessary a sheaf (see [8, Exercise II.1.10]).

The direct product \( \mathcal{F} = \prod_{i \in I} \mathcal{F}_i \) is a sheaf on \( X \). Let \( \mathcal{F} \) denote a sub-presheaf of \( \mathcal{F} \) such that for every open subset \( U \subseteq X \) a section \( f \in \mathcal{F}(U) \) belongs to \( \mathcal{F}(U) \) if and only if there is an open covering \( \{U_j\}_{j \in J} \) of \( U \) with \( f|_{U_j} \in \mathcal{F}(U_j) \) for each \( j \in J \). The following lemma can be easily derived from [6, Proposition-Definition II.1.2].

**Lemma 10.2.** \( \mathcal{F}^+ \) is a sheaf isomorphic to \( \mathcal{F}^+ \).

Let \( X \) be a geometric superscheme. A quasi-coherent super-ideal sheaf \( \mathcal{J} \) on \( X \) is called locally nilpotent if for every \( x \in X^e \) the stalk \( \mathcal{J}_x \) is a locally nilpotent super-ideal of \( \mathcal{O}_{X,x} \). Let \( \operatorname{gr}_{\mathcal{J}}(\mathcal{O}_X) \) denote the sheaf of superalgebras \( (\oplus_{n \geq 0} \mathcal{J}^n/\mathcal{J}^{n+1})^+ \). Recall that \( \mathcal{I}_X \) is the super-ideal sheaf generated by \( \mathcal{O}_X^1 \).

**Proposition 10.3.** The following statements hold:

1. A geometric superspace \( \operatorname{gr}_{\mathcal{J}}(X) = (X^e, \operatorname{gr}_{\mathcal{J}}(\mathcal{O}_X)) \) is a superscheme;
2. \( X \to \operatorname{gr}_{\mathcal{I}_X}(X) \) is an endofunctor of the category \( \mathcal{SV} \) that takes immersions to immersions;
3. A morphism \( f : X \to Y \) of superschemes of locally finite type is an isomorphism if and only if \( \operatorname{gr}(f) : \operatorname{gr}_{\mathcal{I}_X}(X) \to \operatorname{gr}_{\mathcal{I}_Y}(Y) \) is.

**Proof.** Without loss of generality, one can assume that \( X \) is affine, say \( X = \operatorname{SSpec}(A) \). Then \( \mathcal{J} = \mathcal{J}_p \), where \( J \) is a super-ideal of \( A \) such that \( J_p \) is locally nilpotent for every \( p \in (\operatorname{SSpec}(A))^e \). Therefore, \( J \) is locally nilpotent.
Since \((\text{SSpec}(\mathfrak{gr}_J(A)))^e = (\text{SSpec}(A/J))^e = (\text{SSpec}(A))^e\), [18] Proposition 2.1 (3) (see also [6] Proposition II.5.2 (c))] implies
\[
\mathfrak{gr}_J(\mathcal{O}_X) \simeq (\oplus_{n \geq 0} J^n/J^{n+1})^+ \simeq \mathcal{O}_{\text{SSpec}(\mathfrak{gr}_J(A))},
\]
hence \(\mathfrak{gr}_J(X) \simeq \text{SSpec}(\mathfrak{gr}_J(A))\).

If \(f : X \to Y\) is a morphism in \(SV\), then \(f^*(\mathcal{I}_Y) \subseteq f^*_e \mathcal{I}_X\), and \(f^*\) induces the required morphism of sheaves \(\mathfrak{gr}_{f*} \mathcal{O}_Y \to f^*_e \mathfrak{gr}_{f*} \mathcal{O}_X\).

Since the functor \(A \to \mathfrak{gr}_{I_A}(A)\) commutes with localizations, for every \(x \in X^e\) there is
\[
\mathcal{O}_{\mathfrak{gr}_{I_A}(x),x} \simeq \mathfrak{gr}_{I_{O_{x}}}(\mathcal{O}_x),
\]
and this isomorphism is functorial in \(X\). Thus, the second statement follows.

Similarly, if \(\mathfrak{gr}(f)\) is an isomorphism, then \(f^e\) is a homeomorphism of topological spaces and for every \(x \in X^e\) the local morphism \(f^e_x\) induces the isomorphism
\[
\mathfrak{gr}_{I_{O_{y}}}(\mathcal{O}_{Y,y}) \simeq \mathfrak{gr}_{I_{O_{X,x}}}(\mathcal{O}_{X,x}),
\]
where \(y = f^e(x)\). By [17] Proposition 1.10, the local superalgebras \(O_{X,x}\), and \(O_{Y,y}\) are Hausdorff spaces with respect to their \(I_{O_{X,x}}\)-adic, and \(I_{O_{Y,y}}\)-adic topologies, respectively. Hence \(O_{Y,y} \simeq O_{X,x}\) and the proposition is proven. \(\Box\)

**Proposition 10.4.** For every geometric superschemes \(X\) and \(Y\), there is an isomorphism
\[
\mathfrak{gr}_{I_{X \times Y}}(X \times Y) \simeq \mathfrak{gr}_{I_X}(X) \times \mathfrak{gr}_{I_Y}(Y).
\]
Moreover, for every morphisms \(X \to Z\) and \(Y \to T\) in \(SV\), the diagram
\[
\begin{array}{ccc}
\mathfrak{gr}_{I_{X \times Y}}(X \times Y) & \simeq & \mathfrak{gr}_{I_X}(X) \times \mathfrak{gr}_{I_Y}(Y) \\
\downarrow & & \downarrow \\
\mathfrak{gr}_{I_{Z \times T}}(Z \times T) & \simeq & \mathfrak{gr}_{I_Z}(Z) \times \mathfrak{gr}_{I_T}(T)
\end{array}
\]
is commutative.

**Proof.** Let \(U \simeq \text{SSpec}(A)\) and \(V \simeq \text{SSpec}(B)\) be open super-subschemes of \(X\) and \(Y\), respectively. Let \(p_X\) and \(p_Y\) denote the canonical projections \(X \times Y \to X\) and \(X \times Y \to Y\) respectively. As we have already observed, \(p_X^{-1}(U) \cap p_Y^{-1}(V) \simeq U \times V \simeq \text{SSpec}(A \otimes B)\) is an open super-subscheme of \(X \times Y\). Proposition [10,3] and Proposition [11,1] imply that there is the unique open immersion
\[
\phi_{UV} : \mathfrak{gr}_{I_{U \times V}}(U \times V) \to \mathfrak{gr}_{I_X}(X) \times \mathfrak{gr}_{I_Y}(Y)
\]
such that
\[
p_{\mathfrak{gr}_{I_X}(X)} \phi_{UV} = \text{gr}(p_U) \quad \text{and} \quad p_{\mathfrak{gr}_{I_Y}(Y)} \phi_{UV} = \text{gr}(p_V).
\]

Besides, \(\phi_{UV}\) induces an isomorphism onto the open affine super-subscheme \(\mathfrak{gr}_{I_U}(U) \times \mathfrak{gr}_{I_V}(V)\) of \(\mathfrak{gr}_{I_X}(X) \times \mathfrak{gr}_{I_Y}(Y)\). Furthermore, for every open affine super-subschemes \(U' \subseteq U\) and \(V' \subseteq V\), the composition of the natural open immersion
\[
\mathfrak{gr}_{I_{U \times V}}(U' \times V') \to \mathfrak{gr}_{I_{U \times V}}(U \times V)
\]
and \(\phi_{U,V}\) coincides with \(\phi_{U',V'}\).

Using these remarks, one can construct a collection of open immersions
\[
\phi_{UV} : \mathfrak{gr}_{I_{U \times V}}(U \times V) \to \mathfrak{gr}_{I_X}(X) \times \mathfrak{gr}_{I_Y}(Y)
\]
compatible with each other, where \(U\) and \(V\) run over open affine coverings of \(X\) and \(Y\), respectively. Thus the first statement follows.
The same reduction to affine open coverings can be used to prove the second statement. We leave the details for the reader. □

From now on, the graded companion \( \text{gr}_{\mathcal{X}}(X) \) of a geometric superscheme \( X \) is denoted just by \( \text{gr}(X) \). Similarly, \( \text{gr}_{\mathcal{A}}(A) \) is denoted by \( \text{gr}(A) \).

**Proposition 10.5.** \( G \mapsto \text{gr}(G) \) is an endofunctor of the category \( \mathcal{SVG} \). Moreover, if \( G \) is (locally) algebraic, then \( \text{gr}(G) \) is.

**Proof.** Let \( m, \iota, \) and \( \epsilon \) denote the multiplication, the inverse, and the identity morphisms of \( G \), respectively.

Composition of \( \text{gr}(m) \) with the above isomorphism \( \text{gr}(G \times G) \simeq \text{gr}(G) \times \text{gr}(G) \) defines the multiplication morphism \( m' : \text{gr}(G) \times \text{gr}(G) \to \text{gr}(G) \). Moreover, if \( f : G \to H \) is a morphism of (geometric) group superschemes, then \( \text{gr}(f) \) is a morphism of (geometric) groupoid superschemes.

Similarly, \( \text{gr}_(\iota) \) and \( \text{gr}(\epsilon) \) are the inverse and the unit morphisms of \( \text{gr}(G) \), respectively.

Using Proposition 10.4 one obtains the following commutative diagram

\[
\begin{array}{ccc}
\text{gr}(G \times G \times G) & \simeq & \text{gr}(G) \times \text{gr}(G \times G) \\
\downarrow & & \downarrow \\
\text{gr}(G \times G) & \simeq & \text{gr}(G) \times \text{gr}(G) \\
\downarrow & & \downarrow \\
\text{gr}(G) & = & \text{gr}(G)
\end{array}
\]

where the top vertical arrows are \( \text{gr}(\text{id}_G \times m) \), \( \text{id}_{\text{gr}(G)} \times \text{gr}(m) \) and \( \text{id}_{\text{gr}(G)} \times m' \), respectively, and the bottom vertical arrows are \( \text{gr}(m) \) and twice \( m' \), respectively.

Constructing the symmetric diagram for

\[
G \times G \times G \xrightarrow{m \times \text{id}_G} G \times G \xrightarrow{m} G,
\]

one can derive that \( m' \) satisfies the axiom of associativity. We leave for the reader the routine checking of all remaining group axioms. □

**Lemma 10.6.** Let \( X \) be a geometric superscheme. There are natural morphisms \( i_X : X_{ev} \to \text{gr}(X) \) and \( q_X : \text{gr}(X) \to X_{ev} \), both functorial in \( X \), such that \( q_X i_X = \text{id}_{X_{ev}} \). Moreover, \( i_X \) is a closed immersion that maps \( X_{ev} \) isomorphically onto \( \text{gr}(X)_{ev} \).

**Proof.** Since the underlying topological spaces of all the appearing superschemes are the same, all we need is to define morphisms of their superalgebra sheaves. We have two morphisms of superalgebra presheaves

\[
\mathcal{O}_{X_{ev}} = \mathcal{O}_X / \mathcal{I}_X \to \oplus_{n \geq 0} \mathcal{I}_X^n / \mathcal{I}_X^{n+1}
\]

and

\[
\oplus_{n \geq 0} \mathcal{I}_X^n / \mathcal{I}_X^{n+1} \to \mathcal{O}_X / \mathcal{I}_X = \mathcal{O}_{X_{ev}},
\]

where the first morphism is the isomorphism onto the 0th component of \( \oplus_{n \geq 0} \mathcal{I}_X^n / \mathcal{I}_X^{n+1} \) and the second morphism is the projection onto its 0th component. These morphisms uniquely extend to morphisms of superalgebra sheaves

\[
q_X^* : \mathcal{O}_{X_{ev}} \to (\oplus_{n \geq 0} \mathcal{I}_X^n / \mathcal{I}_X^{n+1})^+ = \mathcal{O}_{\text{gr}(X)}
\]

and

\[
i_X^* : \mathcal{O}_{\text{gr}(X)} = (\oplus_{n \geq 0} \mathcal{I}_X^n / \mathcal{I}_X^{n+1})^+ \to \mathcal{O}_{X_{ev}}.
\]
Since the composition of morphisms of presheaves is an identity map, we have \( i_X q_X = \text{id}_{X_{\text{ev}}} \). The functoriality follows.

Finally, since the last statement is local, one can assume that \( X \) is affine, say \( X = \text{SSpec}(A) \). Then \( i_X \) is induced by the projection \( \text{gr} A \to \overline{A} \) (and \( q_X \) is induced by the embedding \( \overline{A} \to \text{gr}(A) \)). The lemma is proven.

**Lemma 10.7.** For every geometric superschemes \( X \) and \( Y \), there are the following commutative diagrams

\[
\begin{align*}
(X \times Y)_{\text{ev}} & \simeq X_{\text{ev}} \times Y_{\text{ev}} & \text{gr}(X \times Y) & \simeq \text{gr}(X) \times \text{gr}(Y) \\
\downarrow & & \downarrow & \downarrow \\
\text{gr}(X \times Y) & \simeq \text{gr}(X) \times \text{gr}(Y) & (X \times Y)_{\text{ev}} & \simeq X_{\text{ev}} \times Y_{\text{ev}} ,
\end{align*}
\]

where the vertical arrows in the first diagram are \( i_{X \times Y} \) and \( i_X \circ i_Y \), while the vertical arrows in the second diagram are \( q_{X \times Y} \) and \( q_X \times q_Y \).

**Proof.** Arguing as in Proposition 10.4, one can reduce the general case to \( X = \text{SSpec}(A) \) and \( Y = \text{SSpec}(B) \). Then the commutativity of the above diagrams is equivalent to the commutativity of the diagrams

\[
\begin{align*}
A \otimes B & \simeq \overline{A} \otimes \overline{B} & \text{gr}(A \otimes B) & \simeq \text{gr}(A) \otimes \text{gr}(B) \\
\uparrow & & \uparrow & \uparrow \\
\text{gr}(A \otimes B) & \simeq \text{gr}(A) \otimes \text{gr}(B) & A \otimes B & \simeq \overline{A} \otimes \overline{B} ,
\end{align*}
\]

where the vertical arrows are the corresponding projections and embeddings. \( \square \)

**Corollary 10.8.** If \( G \) is a group superscheme, then \( i_G \) and \( q_G \) are morphisms of group superschemes.

**Proof.** For example, let us consider \( i_G \). We have two commutative diagrams

\[
\begin{align*}
(G \times G)_{\text{ev}} & \simeq G_{\text{ev}} \times G_{\text{ev}} & (G \times G)_{\text{ev}} & \overset{m_{\text{ev}}}{\longrightarrow} G_{\text{ev}} \\
\downarrow & & \downarrow & \downarrow \\
\text{gr}(G \times G) & \simeq \text{gr}(G) \times \text{gr}(G) & \text{gr}(G \times G) & \overset{\text{gr}(m)}{\longrightarrow} \text{gr}(G) ,
\end{align*}
\]

where \( m : G \times G \to G \) is the multiplication morphism. Since the multiplication morphisms of \( G_{\text{ev}} \) and \( \text{gr}(G) \) are factored through the isomorphisms (horizontal arrows) of the first diagram, our statement follows. The case of \( q_G \) is similar. \( \square \)

11. **An Interpretation of the Functor of Points and Some Useful Consequences**

Due to the category equivalence \( \mathcal{SV}G \simeq \mathcal{SF}G \), the results of the previous section can be reformulated as follows. There is an endofunctor of the category \( \mathcal{SF}G \) that assigns to each group superscheme \( G \) a group superscheme \( \text{gr}(G) \) such that there are natural morphisms (in \( \mathcal{SF}G \))

\[
i_G : G_{\text{ev}} \to \text{gr}(G), \quad q_G : \text{gr}(G) \to G_{\text{ev}},
\]

where \( i_G \) is an isomorphism onto \( \text{gr}(G)_{\text{ev}} \) and \( q_G i_G = \text{id}_{G_{\text{ev}}} \).

Let \( G_{\text{odd}} \) denote \( \ker q_G \). Lemma 1.1 implies that \( G_{\text{odd}} \) is a closed normal group super-subscheme of \( \text{gr}(G) \). Moreover, we have

\[
\text{gr}(G) = \text{gr}(G)_{\text{ev}} \times G_{\text{odd}} \simeq G_{\text{ev}} \times G_{\text{odd}}.
\]

If \( f : G \to H \) is a morphism of group superschemes, then Lemma 10.4 implies \( \text{gr}(f)(G_{\text{odd}}) \leq H_{\text{odd}}. \) Thus, \( G \to G_{\text{odd}} \) is an endofunctor of the category \( \mathcal{SF}G \).
**Proposition 11.1.** Let $\mathcal{G}$ be a locally algebraic group superscheme. The group superscheme $\mathcal{G}_{\text{odd}}$ is affine, and represented by a local graded Hopf superalgebra $\Lambda(\mathfrak{g}_e^*)$, where the elements of $\mathfrak{g}_e^*$ are assumed to be primitive.

Moreover, if $f : \mathcal{G} \to \mathbb{H}$ is a morphism of locally algebraic group superschemes, then $\mathcal{G}_{\text{odd}} \to \mathbb{H}_{\text{odd}}$ is induced by the (dual) linear map $(d_e f)^* : \mathfrak{h}_1^* \to \mathfrak{g}_1^*$.

**Proof.** We have the exact sequence

$$e \to G_{\text{odd}} \to \text{gr}(G) \xrightarrow{q_{\mathcal{G}}} G_{ev} \to e,$$

that is the geometric counterpart of the exact sequence

$$\mathcal{E} \to \mathcal{G}_{\text{odd}} \to \text{gr}(\mathcal{G}) \xrightarrow{\mathfrak{q}_{\mathcal{G}}} \mathcal{G}_{ev} \to \mathcal{E}.$$ 

Since for every field extension $k \subset F$ and every group superscheme $\mathbb{H}$ there is $\mathbb{H}(F) = \mathbb{H}_{ev}(F)$, the underlying topological space of $G_{\text{odd}}$ consists of the unit element $e$ only (see [13 Lemma 5.5]). Therefore, $G_{\text{odd}}$ is affine, and represented by a local Hopf superalgebra $A$ with the nilpotent maximal superideal $A^+ = \ker \epsilon_A$.

Since $q_{\mathcal{G}} \in \mathbb{I}_A$, Lemma 5.2 implies that the Lie superalgebra of $G_{\text{odd}}$ is purely-odd. Hence, $A$ is generated by odd elements. In particular, $I_A = A^+$.

Furthermore, $\text{gr}(G)$ is naturally isomorphic to $G_{ev} \times G_{\text{odd}}$ so that the projection onto $G_{ev}$ is identified with $q_{\mathcal{G}}$. Thus

$$O_{\text{gr}(G),e} \simeq O_{G_{ev},e} \otimes O_{G_{odd},e} = O_{G_{ev},e} \otimes A$$

and the maximal superideal $n$ of $O_{\text{gr}(G),e}$ is identified with $\mathfrak{m}_e \otimes A + O_{G_{ev},e} \otimes A^+$, where $\mathfrak{m}_e = m_e/I_{O_{G_{ev}}}$.

Since the superideal $A^+$ is nilpotent, the $n$-adic topology on $O_{\text{gr}(G),e}$ coincides with its $\mathfrak{m}_e$-adic topology. Besides, the grading of $O_{\text{gr}(G),e}$ is given by

$$O_{\text{gr}(G),e}(k) \simeq O_{G_{ev},e} \otimes (A^+)^k/(A^+)^{k+1} \text{ for } k \geq 0.$$

In particular, there is an isomorphism of complete graded superalgebras $\widehat{O_{\text{gr}(G),e}} \simeq \widehat{O_{G_{ev},e}} \otimes A$, such that

$$\widehat{O_{\text{gr}(G),e}}(k) \simeq \widehat{O_{G_{ev},e}} \otimes (A^+)^k/(A^+)^{k+1} \text{ for } k \geq 0.$$

Moreover, the epimorphism of complete Hopf superalgebras $\widehat{O_{\text{gr}(G),e}} \to A$, induced by the closed immersion $G_{\text{odd}} \to \text{gr}(G)$, is identified with

$$\widehat{O_{\text{gr}(G),e}} \to \widehat{O_{\text{gr}(G),e}}/\widehat{O_{\text{gr}(G),e}} O_{G_{ev},e}^+.$$

Thus $A \simeq \text{gr}(A)$ is a graded Hopf superalgebra, generated by component $A(1) = A^+/A^{+2}$, whose elements are primitive. Let $J$ denote the kernel of the natural epimorphism $\Lambda(A(1)) \to A$ of graded Hopf superalgebras. Choose a basis $a_1, \ldots, a_t$ of $A(1)$. The induction on $t$ shows that for every $1 \leq i \leq t$, the induced epimorphism $\Lambda(A(1))/\Lambda(A(1))a_i \to A/aa_i$ is an isomorphism. Hence $J \subseteq \cap_{1 \leq i \leq t} \Lambda(A(1))a_i = \langle a_1 \ldots a_t \rangle$. Since $a_1 \ldots a_t$ is not Hopf superideal, $J = 0$.

Finally, $A(1)$ is the odd component of $n/n^2 \simeq m_e/m_e^2$. That is, $A(1) \simeq \mathfrak{g}_e^*$, and Lemma 5.4 concludes the proof. □

**Lemma 11.2.** The induced morphism $\text{gr}(N_k) \to \text{gr}(G)$ coincides with the $k$th neighborhood of the closed embedding $e \to \text{gr}(G)$.
Proof. It is enough to show that the kernel of the induced epimorphism
\[ \text{gr}(\mathcal{O}_e) \rightarrow \text{gr}(\mathcal{O}_e/\mathfrak{m}_e^{k+1}) \]
c coinides with \( \mathfrak{n}^{k+1} \), where \( \mathfrak{n} \) is the maximal superideal of \( \text{gr}(\mathcal{O}_e) \simeq \mathcal{O}_{\text{gr}(\mathcal{G}),e} \).

Let \( V \) denote the superspace \( \mathcal{I}_{\mathcal{O}_e}/\mathcal{I}_{\mathcal{O}_e}^2 \). As it was already shown in the previous lemma, the graded superalgebra \( \text{gr}(\mathcal{O}_e) \) is naturally isomorphic to \( \mathcal{O}_e \otimes \Lambda(V) \).

Besides, \( \mathfrak{n} \) is identified with the graded superideal \( \mathfrak{m}_e \otimes 1 + \mathcal{O}_e \otimes \Lambda(V)^+ \), where \( \mathfrak{m}_e = \mathfrak{m}_e/\mathcal{I}_{\mathcal{O}_e} \). For every \( k \geq 0 \), there is
\[
\mathfrak{n}^{k+1} = \sum_{k+1-t \leq i \leq k+1} \mathfrak{m}_e^i \otimes (\oplus_{k+1-t \leq i \leq t} \Lambda^i(V)) = \oplus_{0 \leq i \leq t} \mathfrak{m}_e^{k+1-i} \otimes \Lambda^i(V),
\]
where \( t = \dim V \). In particular, the graded superalgebra \( \text{gr}(\mathcal{O}_e)/\mathfrak{n}^{k+1} \) is isomorphic to \( \oplus_{0 \leq i \leq t} \mathcal{O}_e/\mathfrak{m}_e^{k+1-i} \otimes \Lambda^i(V) \).

On the other hand, the isomorphism \( \text{gr}(\mathcal{O}_e) \simeq \mathcal{O}_e \otimes \Lambda(V) \) implies that each element \( f \in \mathcal{O}_e \) has the "canonical" form
\[
f = \sum_{0 \leq s \leq t, 1 \leq i_1 < \ldots < i_s \leq t} f_{i_1, \ldots, i_s} v_{i_1} \ldots v_{i_s},
\]
where the elements \( v_1, \ldots, v_t \in (\mathcal{O}_e)_0 \) form a basis of \( V \), and the coefficients \( f_{i_1, \ldots, i_s} \in (\mathcal{O}_e)_0 \) are uniquely defined modulo \( \mathcal{I}_{\mathcal{O}_e} \), or equivalently, modulo \( (\mathcal{O}_e)_1^2 \).

In particular, \( f \) belongs to \( \mathfrak{m}_e \) if and only if its "free" coefficient (corresponding to \( s = 0 \)) does. Moreover, \( f \) belongs to \( \mathfrak{m}_e^{k+1} \) if and only if each \( f_{i_1, \ldots, i_s} \) belongs to \( \mathfrak{m}_e^{k+1-s} \) modulo \( \mathcal{I}_{\mathcal{O}_e} \). Therefore, every element \( f \) of superalgebra \( \mathcal{O}_e/\mathfrak{m}_e^{k+1} \) has a form
\[
\frac{f_{i_1, \ldots, i_s} v_{i_1} \ldots v_{i_s}}{I_{\mathcal{O}_e}}, \text{ where } f_{i_1, \ldots, i_s} \in (\mathcal{O}_e/\mathfrak{m}_e^{k+1})_0,
\]
and each \( f_{i_1, \ldots, i_s} \) is uniquely defined modulo \( \mathfrak{m}_e^{k+1-s} + \mathcal{I}_{\mathcal{O}_e} \). The lemma follows. \( \square \)

Corollary 11.3. The formal neighborhood of the identity in \( \text{gr}(\mathcal{G}) \) coincides with \( \cup_{k \geq 0} \text{gr}(\mathbb{N}_k) \).

If we denote the formal neighborhood of the identity in \( \text{gr}(\mathcal{G}) \) by \( \text{gr}(\mathbb{N}) \), then the above corollary states that \( \text{gr}(\mathbb{N}) \) is "represented" by complete Hopf superalgebra \( \text{gr}(\hat{\mathcal{G}}) \simeq \text{gr}(... \otimes \Lambda(V) \ldots) \), whose comultiplication and antipode are \( \Delta \) and \( S \), respectively. Moreover, arguing as in Lemma 11.2 one sees that each element \( f \in \mathcal{O}_e \) has a "canonical" form
\[
\sum_{0 \leq s \leq t, 1 \leq i_1 < \ldots < i_s \leq t} f_{i_1, \ldots, i_s} v_{i_1} \ldots v_{i_s}, \text{ where } f_{i_1, \ldots, i_s} \in (\mathcal{O}_e)_0,
\]
and the coefficients \( f_{i_1, \ldots, i_s} \) are uniquely defined modulo \( (\hat{\mathcal{O}}_e)_1^2 \).

It is clear that \( \mathbb{N}_{ev} = \mathbb{N} \cap \mathcal{G}_{ev} \) is the formal neighborhood of the identity in \( \mathcal{G}_{ev} \). Moreover, \( \text{hyp}(... \otimes \Lambda(V) \ldots) \) can be naturally identified with the purely-even subalgebra of \( \text{hyp}(\mathcal{G}) \), consisting of all \( \phi \in \text{hyp}(\mathcal{G}) \) such that \( \phi(I_{\hat{\mathcal{G}}}) = 0 \). More generally, let \( \text{hyp}(... \otimes \Lambda(V) \ldots) \) denote the super-subspace \( \{ \phi \in \text{hyp}(\mathcal{G}) \mid \phi(I_{\hat{\mathcal{G}}}) = 0 \} \). In particular, \( \text{hyp}(... \otimes \Lambda(V) \ldots) = \text{hyp}(0)(\mathcal{G}) \) and \( \text{hyp}(\mathcal{G}) = \text{hyp}(\mathcal{G}) \).

The following lemma is similar to Lemma 11.2.

Lemma 11.4. For every nonnegative integers \( 0 \leq k, l \leq t \) there is:

(1) \( \text{hyp}(k)(\mathcal{G}) \subseteq \text{hyp}(k+l)(\mathcal{G}) \),

(2) \( \text{hyp}(k)(\mathcal{G}) \subseteq \text{hyp}(k+l)(\mathcal{G}) \),

(3) \( \text{hyp}(k)(\mathcal{G}) \subseteq \text{hyp}(k+l)(\mathcal{G}) \),

(4) \( \text{hyp}(k)(\mathcal{G}) \subseteq \text{hyp}(k+l)(\mathcal{G}) \),

(5) \( \text{hyp}(k)(\mathcal{G}) \subseteq \text{hyp}(k+l)(\mathcal{G}) \).
(2) $\Delta^*(\text{hyp}^{(k)}(G)) \subseteq \sum_{0 \leq s \leq k} \text{hyp}^{(s)}(G) \otimes \text{hyp}^{(k-s)}(G)$,
(3) $S^*(\text{hyp}^{(k)}(G)) \subseteq \text{hyp}^{(k)}(G)$.

Lemma 11.3 immediately implies that
$$\text{gr}(\text{hyp}(G)) = \oplus_{0 \leq k \leq t} \text{hyp}^{(k)}(G) / \text{hyp}^{(k-1)}(G)$$
has a natural structure of (finitely) graded Hopf superalgebra.

**Proposition 11.5.** There is a canonical isomorphism $\text{gr}(\text{hyp}(G)) \simeq \text{hyp}(\text{gr}(G))$ of graded Hopf superalgebras.

**Proof.** First of all, the superspace $\text{hyp}(\text{gr}(G))$ is isomorphic to $\bigoplus_{0 \leq k \leq t} (I^k_{\hat{O}_e} / I^{k+1}_{\hat{O}_e})^{\bullet}$, where $(I^k_{\hat{O}_e} / I^{k+1}_{\hat{O}_e})^{\bullet}$ consists of all continuous linear maps $I^k_{\hat{O}_e} / I^{k+1}_{\hat{O}_e} \to k$, and $k$ is regarded as a discrete vector space. In fact, arguing as in Lemma 11.2, one sees that the $\hat{\bullet}$-adic topology of $\text{gr}((\hat{G}_e))$ (respectively, the factor-topology of $I^k_{\hat{O}_e} / I^{k+1}_{\hat{O}_e}$) is equivalent to its $\hat{\bullet}$-adic topology. Thus, $\text{gr}((\hat{G}_e))$ is isomorphic to $\bigoplus_{0 \leq k \leq t} I^k_{\hat{O}_e} / I^{k+1}_{\hat{O}_e}$ as a (linear) topological superspace.

Next, we have a natural superspace morphism
$$\text{hyp}^{(k)}(G) / \text{hyp}^{(k-1)}(G) \to (I^{k-1}_{\hat{O}_e} / I^k_{\hat{O}_e})^{\bullet},$$
induced by the Hopf pairing $\langle , \rangle$, which is obviously injective. On the other hand, every $\phi \in (I^{k-1}_{\hat{O}_e} / I^k_{\hat{O}_e})^{\bullet}$ is some $\psi \in \text{gr}((\hat{G}_e))^{\bullet}$ restricted on $I^{k-1}_{\hat{O}_e} / I^k_{\hat{O}_e}$. Since $\psi$ is continuous, there is $l \geq 0$ such that $\psi(\hat{n}^{l+1}) = 0$.

Choose a basis of $\text{gr}((\hat{G}_e))$ consisting of elements $\overline{f}v_{i_1} \ldots v_{i_s}$, where $1 \leq i_1 < \ldots < i_s \leq t, 0 \leq s \leq t$, and $f \in (\hat{O}_e)_0$. We also assume that for every $k \geq 0$, the elements $\overline{f}$, that belong to $\hat{\mathbb{m}}^k / \hat{\mathbb{m}}^{k+1}$, form a basis of $\hat{\mathbb{m}}^k / \hat{\mathbb{m}}^{k+1}$. Then the elements $fv_{i_1} \ldots v_{i_s}$ form a basis of $\hat{\mathbb{O}}_e$. Moreover, the superideal $\hat{\mathbb{m}}^{k+1}$ has a basis consisting of all $fv_{i_1} \ldots v_{i_s}$ such that $f \in \hat{\mathbb{m}}^{k+1} + I_{\hat{O}_e}$ (or equivalently, $f \in \hat{\mathbb{m}}^{k+1-s}$). Similarly, the superideal $I^k_{\hat{O}_e}$ has a basis consisting of all $fv_{i_1} \ldots v_{i_s}$ with $s \geq k$.

Set $\psi'(fv_{i_1} \ldots v_{i_s}) = \psi(\overline{f}v_{i_1} \ldots v_{i_s})$ for each basic element $fv_{i_1} \ldots v_{i_s}$. Then $\psi'$ belongs to $\text{hyp}^{(k)}(G)$ and its image in $(I^{k-1}_{\hat{O}_e} / I^k_{\hat{O}_e})^{\bullet}$ coincides with $\psi$. In other words, we have a natural graded pairing
$$\text{gr}(\text{hyp}(G)) \times \text{gr}((\hat{G}_e)) \to k$$
that induces an isomorphism $\text{gr}(\text{hyp}(G)) \simeq \text{hyp}(\text{gr}(G))$ of superspaces. It is a Hopf pairing by the definition of Hopf superalgebra structure on $\text{gr}((\hat{G}_e))$. The proposition is proven. \qed

Let $\gamma_1, \ldots, \gamma_t$ be a basis of $\mathfrak{g}_1 \simeq V^*$, dual to the basis $v_1, \ldots, v_t$ of $V$.

**Proposition 11.6.** Every element $\phi \in \text{hyp}_k(G)$ can be uniquely expressed as
$$\phi = \sum_{0 \leq s \leq t, 1 \leq i_1 < \ldots < i_s \leq t} \phi_{i_1, \ldots, i_s} \gamma_{i_1} \ldots \gamma_{i_s}, \text{ where } \phi_{i_1, \ldots, i_s} \in \text{hyp}_{k-s}(G_{cev}).$$

**Proof.** It is sufficient to prove this statement for a (locally algebraic) group super-scheme $G \simeq \text{gr}(G)$. In this case, we have $\hat{G}_e \simeq A \otimes \Lambda(V)$, where $A \simeq \overline{\hat{O}_e}$ is a
closed Hopf subalgebra of \( \hat{\mathcal{O}}_c \). Since \( \hat{\mathcal{O}}_c \) is also a graded Hopf superalgebra with \( \hat{\mathcal{O}}_c(1) = A \otimes V \), then each \( v \in V \) is skew primitive, i.e. \( \Delta'(v) = v \otimes a + b \otimes v \), where \( a \) and \( b \) are group-like elements from \( A \), and \( \Delta' = \text{gr}(\Delta) \).

Further, the Hopf pairing \( \text{hyp}(G) \times \hat{\mathcal{O}}_c \to k \) is graded. Therefore, it is enough to show that \( < \phi \gamma_i \cdots \gamma_i, f_v \gamma_j \cdots \gamma_j > \neq 0 \) implies \( s = l, i_1 = j_1, \ldots, i_s = j_s \). Moreover, in this case

\[
< \phi \gamma_i \cdots \gamma_i, f_v \gamma_j \cdots \gamma_j > = < \phi, f_b \gamma_i \cdots \gamma_i >,
\]

where \( \Delta'(v) = a_i \otimes u_i + v_i \otimes b_i, 1 \leq i \leq t \). We use an induction on \( s \).

If \( s = 0 \), then the statement is obvious. Let \( s \geq 1 \). We have

\[
\Delta'(f_v \gamma_j \cdots \gamma_j) = (f_0(0) \otimes f_{(1)}) \prod_{1 \leq p \leq t} (v_{jp} \otimes a_{jp} + b_{jp} \otimes v_{jp}) =
\sum_{K \subseteq \{j_1, \ldots, j_t\}} \pm f_0(0)^b_K \otimes f_{(1)}a_K V_K,
\]

where \( \Delta'(f) = f_0(0) \otimes f_{(1)} \) and \( V_K \) (respectively, \( a_K \) or \( b_K \)) denotes a product of all \( v_{jp} \) (respectively, \( a_{jp} \) or \( b_{jp} \)) for \( j_p \in K \) (in an arbitrary order), and \( K = \{j_1, \ldots, j_t\} \setminus K \) is the complement of \( K \). Then

\[
< (\phi \gamma_i \cdots \gamma_i), f_v \gamma_j \cdots \gamma_j > =
\sum_{K \subseteq \{j_1, \ldots, j_t\}} \pm < \phi \gamma_i \cdots \gamma_i, f_0(0)^b_K V_K > < \gamma_i, f_{(1)}a_K V_K >
\]

contains nonzero summands for \( K = \{i_1, \ldots, i_s-1\} \) only. Moreover, \( K \) should be a singleton, say \( K = \{j_p\} \). Thus \( s = l \) and \( < \gamma_i, f_{(1)}a_K V_{j_p} > = \epsilon_A(f_{(1)}) < \gamma_i, v_{j_p} > \neq 0 \) implies \( i_s = j_p \). Besides, \( < (\phi \gamma_i \cdots \gamma_i), f_v \gamma_j \cdots \gamma_j > \) is equal to

\[
< \phi \gamma_i \cdots \gamma_i, f_0(0)b_i v_1 \cdots v_{i_s-1} > \epsilon_A(f_{(1)}) = < \phi, f_b \gamma_i \cdots \gamma_i >.
\]

The proposition is proven.

\[ \square \]

12. The fundamental equivalence

12.1. The category of Harish-Chandra pairs. In this subsection, we follow [12].

A pair \((G, V)\), where \( G \) is a locally algebraic group scheme, and \( V \) is a \( G \)-module, is a Harish-Chandra pair if the following conditions hold:

(a) There is a functor \( V_a \times V_a \to g_a \), denoted by \([\cdot, \cdot]\), where \( g \) is the Lie algebra of \( G \), such that for every \( R \in \text{Alg}_k \), the map \( V_a(R) \times V_a(R) \to g_a(R) \) is \( R \)-bilinear and symmetric.

(b) The functor \( V_a \times V_a \to g_a \) is \( G \)-equivariant for the diagonal action of \( G \) on \( V_a \times V_a \) and the adjoint action of \( G \) on \( g_a \).

(c) The induced action of \( g \) on \( V \) satisfies \([v, v] \cdot v = 0 \) for \( v \in V \).

The morphism of Harish-Chandra pairs \((G, V) \to (H, W)\) is a morphism of pairs \((f, u)\) as in Section 7, such that the diagram

\[
\begin{array}{ccc}
V_a \times V_a & \to & g_a \\
\downarrow & & \downarrow \\
W_a \times W_a & \to & h_a
\end{array}
\]

is commutative, where the first vertical map is \( u_a \times u_a \) and the second vertical map is \( d_e f \). The category of Harish-Chandra pairs is denoted by HCP.
To every algebraic group superscheme $G$, we associate a Harish-Chandra pair $(G, V)$ with $G = G_v$ (regarded as a group scheme), and $V = g_1$. Besides, the action of $G$ on $V$ is induced by the adjoint action of $G$, and $[\ , \ ]$ is the restriction of Lie super-bracket of $g$ on its odd component.

**Lemma 12.1.** The correspondence $\Phi: G \mapsto (G, V)$ is a functor from $\mathcal{SF}G_{\text{la}}$ to HCP.

We call $\Phi$ the Harish-Chandra functor.

Recall that each group $\mathbb{N}(R)$ is identified with $\text{Gpl}(\text{hyp}(G) \otimes R)$. For every $v \in V = g_1, x \in g = g_0$ and $a \in R_1, b \in R_0$ such that $b^2 = 0$, define the group-like elements

$$f(b, x) = e^* \otimes 1 + x \otimes b, e(a, v) = e^* \otimes 1 + v \otimes a.$$  

These elements are images of the group-like elements $e^{a(x \otimes 1)}$ and $e^{-c_1(v \otimes 1)}$ from $\mathbb{N}(R[\epsilon_0, \epsilon_1])$ under a homomorphism of groups, induced by $\epsilon_0 \mapsto b, \epsilon_1 \mapsto a$. Using the identity involving group commutators from Lemma 8.5, we obtain the first three relations in the list:

1. $[e(a, v), e(a', v')] = f(-aa', [v, v'])$,
2. $[f(b, x), e(a, v)] = e(ba, [x, v])$,
3. $[f(b, x), f(b', x')] = f(bb', [x, x'])$,
4. $e(a, v)e(a', v) = f(-aa', \frac{1}{2}[v, v])e(a + a', v)$.

A direct computation can derive the fourth relation. The relations (1), (2), (3) are the relations (i), (iii), (iv) from [12] Lemma 4.2, and the relation (4) is (ii) therein.

Also, for every $g \in G(R)$ we have

5. $e(a, v)^g = e(a, \text{Ad}(g)v))$ and $f(b, x)^g = f(b, \text{Ad}(g)x))$.

Define the group subfunctors $\Sigma$ and $F$ of $\mathbb{N}$ and $\mathbb{N} = \mathbb{N}_v$ respectively, such that $\Sigma(R)$ is generated by all the elements $f(b, x)$ and $e(a, v)$, but $F(R)$ is generated only by $f(b, x)$, where $a, b \in R$ and $R \in \text{SAlg}_k$.

**Lemma 12.2.** The following statements hold:

1. $F(R) = \Sigma(R) \cap \mathbb{N}(R)$.
2. If $v_1, \ldots, v_t$ is a basis of $V$, then each element of $\Sigma(R)$ is uniquely expressed in the form
   $$fe(a_1, v_1) \ldots e(a_t, v_t),$$
   where $f \in F(R)$ and $a_i \in R_1$ for $1 \leq i \leq t$.
3. Both group subfunctors $F$ and $\Sigma$ are invariant for the conjugation action of $G$.

**Proof.** To prove the first two statements, use Proposition 11.6 and copy the proof of [12] Proposition 4.3. The third statement follows from the relation (5) above. \Box

**Remark 12.3.** The symmetric version of Lemma 12.2 (2), where $f$ appears on the right-hand side, is also valid.

Now we can define a group subfunctor $G'$ of $G$, such that $G'(R) = G(R)\Sigma(R)$ for $R \in \text{SAlg}_k$.

**Corollary 12.4.** Each element of $G'(R)$ is uniquely expressed in the form

$$ge(a_1, v_1) \ldots e(a_t, v_t),$$

where $g \in G(R)$, and $a_i \in R_1$ for $1 \leq i \leq t$. 

Let $\mathbf{E}$ denote a subfunctor of $\Sigma$ such that $\mathbf{E}(R)$ consists of all products

$$e(a_1, v_1) \ldots e(a_i, v_i),$$

where $a_i \in R$ for $1 \leq i \leq t$.

It is clear that $\mathbf{E} \simeq \text{SSp}(\Lambda(V^*))$ is a purely-odd affine superscheme. Then $G'$ is isomorphic to $G \times \mathbf{E}$ as a superscheme. In particular, $G'$ is a (locally algebraic) group superscheme.

**Theorem 12.5.** We have $G = G'$.

*Proof.* By Proposition [10,3] one has to prove that the induced morphism $\text{gr}(G') \to \text{gr}(G)$ of group superschemes is an isomorphism.

Since $G_{ev} = G'_{ev}$, Lemma [10,6] implies that $\text{gr}(G')_{ev}$ is mapped isomorphically onto $\text{gr}(G')_{ev}$. By Proposition [11,1] it is sufficient to prove that the embedding $G' \to G$ induces an isomorphism of odd components of their Lie superalgebras. But as it has already been observed, in both $G$ and $G'$, the elements $e^{t_1 + 1}(v_i \otimes 1)$ coincide with $e(-e_1, v_i)$. The theorem is proven. \qed

**Corollary 12.6.** Let $N$ denote $N_{ev}$. Then $N = NE = EN$.

*Proof.* If $s \in N(R)$ is expressed as $gx$, where $g \in G(R)$ and $x \in \mathbf{E}(R)$, then $g = sx^{-1} \in G_{ev}(R) \cap N = N_{ev}(R)$, since $E \subseteq N$. It is clear that $EN \subseteq N$. Moreover, for any $s \in N(R)$ and $x \in \mathbf{E}(R)$, we have $x^s = sx^{-1} = x'f$ for $f \in F(R)$ and $x' \in E(R)$, by the symmetric version of Lemma [12,2] (2). Thus, $sx = x's = x'(fs)$. \qed

**12.2. A quasi-inverse of the Harish-Chandra functor.** Let $(G, V)$ be a Harish-Chandra pair. Note that $g \otimes V$ has a natural Lie superalgebra structure, induced by the bilinear map $V \times V \to g$, and by the differential of the action of $G$ on $V$. Let $g$ denote this Lie superalgebra. Fix a basis $x_1, \ldots, x_t$ of $g_0 = g$.

Following [12], one can define a group functor $\Sigma'$ that associates with each superalgebra $R$ a group generated by symbols $e'(a, v), f'(b, x)$ for $a \in R_1, b \in R_0$ such that $b^2 = 0, v \in V$ and $x \in g$, subject to the relations (1) – (4). Note that if either one of the parameters $a, b, v, x$ is zero, then the corresponding generator is the unit. Let $F'$ be any group subfunctor of $\Sigma'$, such that $F'(R)$ is generated only by the symbols $f'(b, x)$.

Since the proof of [12, Lemma 4.4] uses only the above relations, one can conclude again that any element of $\Sigma'(R)$ has a form $f' e(a_1, v_1) \ldots e'(a_i, v_i)$, where $f' \in F'(R)$. On the other hand, there is a natural morphism of the group functor $\Sigma'$ to the group functor $R \mapsto \text{Gp}(U(g) \otimes R)$ that sends $e'(a, v)$ to $1 \otimes 1 + v \otimes a$ and $f'(b, x)$ to $1 \otimes 1 + x \otimes b$, respectively. Then [12, Proposition 4.3] implies that the above form is unique.

Further, the action of $G$ on $V$ induces the action on the generators by

$$g \cdot e'(a, v) = e'(r_1 a, v_1) \ldots e'(r_t a, v_t) \text{ and } g \cdot f'(b, x) = f'(s_1 b, x_1) \ldots f'(s_t b, x_t),$$

where $g \cdot v = \sum_{1 \leq i \leq t} v_i \otimes r_i$ and $\text{Ad}(g)(x) = \sum_{1 \leq j \leq t} x_j \otimes s_j$.

**Lemma 12.7.** The above action preserves the relations (1) – (4). Hence, it defines a homomorphism $\alpha(R) : G(R) \to \text{Aut}(\Sigma'(R))$.

*Proof.* Consider the relation (1). Suppose that $g \cdot v' = \sum_{1 \leq i \leq t} v_i \otimes r_i$. Then for every $x, y, z \in \{e'(r_1 a, v_1), e'(r_2 a, v_2) \mid 1 \leq i, k \leq t\}$ we have $[[x, y], z] = 1$. Thus, the
well-known commutator identity $[xy, z] = [x, z][y, z]$ implies $[xy, z] = [x, z][y, z]$. Therefore,

g \cdot [e'(a, v), e'(a', v')] = [g \cdot e'(a, v), g \cdot e'(a', v')] = \prod_{1 \leq i, k \leq t} [e'(r_i a, v_i), e'(r_k a', v_k)] =

\prod_{1 \leq i, k \leq t} f'(-r_i a r_k a', [v_i, v_k]) = f'(-a a', g \cdot [v, v']) = g \cdot f'(-a a', [v, v']).

The identities (3) and (4) can be derived similarly. The identity (2) follows from $g \cdot [x, v] = [\text{Ad}(g)(x), g \cdot v]$, which is obtained by using the identity at the beginning of the proof of Lemma 5.3. The lemma is proven.

The map $f'(b, x) \rightarrow f(b, x)$ induces a morphism of group functors $i : F' \rightarrow F$. In fact, it is the composition of the above morphism from $F'$ to the group functor $R \mapsto \text{Gp}(U(g) \otimes R_0)$ and the natural morphism from the latter to $F$, induced by the Hopf algebra morphism $U(g) \rightarrow \text{hyl}(G_{ev})$.

It is clear that $(\Sigma'(R), F'(R), G(R), i(R), \alpha(R))$ is a quintuple in the sense of [12], with the only difference that $\alpha(R)$ is a group homomorphism. We define a group $\Gamma(R)$ as a factor-group of $G(R) \rtimes \Sigma'(R)$ by a normal subgroup $\Xi(R) = \{(i(R)(f), f^{-1}) \mid f \in F'(R)\}$.

**Remark 12.8.** $\Gamma(R)$ is isomorphic to the factor-group of the amalgamated free product $G(R) \ast_{ev(R)} \Sigma'(R)$ modulo the relations $s^g = (\alpha(R)(g))(s)$ for $g \in G(R)$ and $s \in \Sigma'(R)$.

By [12] Lemma 2.1 every element of $\Gamma(R)$ can be uniquely expressed as

$$ge'(a_1, v_1) \ldots e'(a_t, v_t),$$

where $g \in G(R)$ and $a_i \in R_1$ for $1 \leq i \leq t$. In other words, the superscheme $\Gamma$ is isomorphic to $G \times E$, hence locally algebraic.

The following lemma is now apparent.

**Lemma 12.9.** $R \mapsto \Gamma(R)$ is a group functor, hence a locally algebraic group superscheme. Moreover, $\Psi : (G, V) \rightarrow \Gamma$ is a functor from HCP to $SF\hat{G}_{la}$.

**Theorem 12.10.** The functors $\Phi$ and $\Psi$ are quasi-inverse to each other.

**Proof:** Since the group structure of $\Gamma(R)$ is defined by the relations similar to (1) – (5), Theorem [12, 5] implies $\Psi \circ \Phi \simeq \text{id}_{SF\hat{G}_{la}}$. Further, the odd component of the Lie superalgebra of $\Gamma$ can be identified with the (commutative) subgroup of $\Gamma[K[\epsilon_0, \epsilon_1]]$ consisting of all elements $e'(c_1 \epsilon_1, v_1) \ldots e'(c_t \epsilon_1, v_t)$, where $c_i \in k$ for $1 \leq i \leq t$, which can be identified with $V$. Moreover, the adjoint action of $G$ is identified with the original action of $G$. Thus $\Phi \circ \Psi \simeq \text{id}_{\text{HCP}}$. The theorem is proven.

In what follows, let $E$ denote $E_{ev}$.

Let $G$ be a locally algebraic group superscheme. If $\Phi(G) \simeq (G, V)$ or $\Psi((G, V)) \simeq G$, then we say that $G$ is represented by $(G, V)$.

A sequence $(R, W) \rightarrow (G, V) \rightarrow (H, U)$ in HCP is called exact whenever the following conditions hold:

1. The sequences $0 \rightarrow W \rightarrow V \rightarrow U \rightarrow 0$ and $E \rightarrow R \rightarrow G \rightarrow H \rightarrow E$ are exact in the categories of superspaces and group schemes, respectively.
2. $W$ is a $G$-submodule of $V$. 


(2b) \( R \) acts trivially on \( V/W \).

(2c) \([V, W] \subseteq \text{Lie}(R)\).

**Theorem 12.11.** A sequence of group superschemes \( R \to G \to H \) is exact if and only if the sequence \( \Phi(R) \to \Phi(G) \to \Phi(H) \) is exact.

**Proof.** Suppose that \( R, G \) and \( H \) are represented by the pairs \((R, W), (G, V), \) and \((H, U), \) respectively. Without loss of a generality, one can replace \( R \to G \to H \) by the sequence \( \Psi((R, W)) \to \Psi((G, V)) \to \Psi((H, U)). \) Then \( R \) coincides with \( \ker(G \to H) \) if and only if \( R = \ker(G \to H), W = \ker(V \to U) \) and \( W \) satisfies the conditions (2a)–(2c). For example, let \( A \in \text{SAlg}_{2}. \) Since \( e'(a, v)h e'(a, v)^{-1} \in R(A) \) for every \( h \in R(A), a \in A_{1} \) and \( v \in V, \) then

\[
e'(a, v)h e'(a, v)^{-1}h^{-1} = e'(a, v)e'(a, -h \cdot v) = e'(a, v - h \cdot v) \in R(A)
\]

implies \( v - h \cdot v \in W, \) hence (2b). The converse statement is obvious.

Suppose that \( G \to H \) is surjective in the Grothendieck topology. Consider a couple \( h \in H(A_{0}) = H(A_{0}) \) and \( e'(a, u) \in H(A) \). There is an fpfp covering \( A' \) of \( A, \) such that \( e'(\iota(a), u) = e'(a', \pi) \), where \( v \in V, \) and \( \pi \) is the image of \( v \) in \( U. \) That is, \( V \to U \) is a surjective map. Finally, let \( \iota: A_{0} \to B \) be an fpfp covering of \( A_{0}, \) such that \( H(\iota)(h) \) belongs to the image of \( G_{ev}(B) = G(B_{0}) \). Then \( B_{0} \) is an fpfp covering of \( A_{0}. \) Conversely, if \( h \in H(A_{0}) \) belongs to the image of \( G \) up to an fpfp covering \( B \) of \( A_{0} \) (in the category of algebras), then \( h \) belongs to the image of \( G \) up to an fpfp covering \( B \otimes_{A_{0}} A \) of \( A. \) The theorem is proven. \( \square \)

**Corollary 12.12.** If \( R \) is a normal group super-subscheme of an algebraic group scheme \( G, \) then the sheaf quotient morphism \( G \to G/R \) is faithfully flat.

**Proof.** The superschemes \( G \) and \( G/R \) can be identified with \( G \times \text{SSp}(\Lambda(V^{*})) \) and \( G/R \times \text{SSp}(\Lambda((V/W)^{*})). \) Then the quotient morphism is induced by the quotient morphism \( G \to G/R, \) which is faithfully flat by [10] Theorem 7.35, and by the canonical embedding \( \Lambda((V/W)^{*}) \to \Lambda(V^{*}), \) so that \( \Lambda(V^{*}) \) is a free \( \Lambda((V/W)^{*}) \)-supermodule. \( \square \)

# 13. Applications

13.1. **Radicals.** Let \( G \) be an algebraic group superscheme, represented by a Harish-Chandra pair \((G, V), \) where \( V = g_{1}, \) and \( G = G_{ev} \) is regarded as an algebraic group scheme. In what follows \( H^{0} \) denotes the connected component of an algebraic group scheme \( H. \)

Let \( R \) be a normal group subscheme of \( G. \) The largest normal group super-subscheme \( H \) of \( G \) such that \( H = H_{ev} \subset R, \) is called the \( R \)-**radical** of \( G. \) If the \( R \)-radical of \( G \) is trivial, then \( G \) is called \( R \)-**semisimple.**

Let \( W \) be a \( G \)-submodule of \( V \) such that \( [W, V] \subseteq \text{Lie}(R) \) and \( [W, V] = [[W, V], V] \subseteq W. \) Such a submodule is called \( R \)-subordinated. The sum of two \( R \)-subordinated submodules is again \( R \)-subordinated. Thus, there is the largest \( R \)-subordinated submodule, denoted by \( W_{R}. \)

Set \( H_{R} = \ker(R \to GL(V/W_{R})). \) Since \( W_{R} \) is a \( G \)-submodule of \( V, \) we have \( H_{R} \subseteq G. \)

**Lemma 13.1.** The Harish-Chandra sub-pair \((H_{R}, W_{R})\) represents the \( R \)-radical of \( G. \)
Proof. Let \((H,W)\) be a Harish-Chandra sub-pair of \((G,V)\). Then \((H,W)\) represents a normal group super-subscheme of \(G\) if and only if it satisfies the conditions \((2a)-(2c)\) from Theorem 12.11. Assume that \(H \leq R\). Since \(H\) acts trivially on \(V/W\), we have

\[ [W, V, V] \subseteq [\text{Lie}(H), V] \subseteq W, \]

that is, \(W\) is \(R\)-subordinated. Thus \(W \subseteq W_R\) and \(H \leq H_R\). To complete the proof, it remains to show that \([W_R, V] \subseteq \text{Lie}(H_R)\). Recall that

\[ \text{Lie}(H_R) = \{ x \in \text{Lie}(R) \mid [x, V] \subseteq W_R \}, \]

hence \([W_R, V] \subseteq \text{Lie}(H_R)\) if and only if \([W_R, V, V] \subseteq W_R\). The lemma is proven. \(\Box\)

Remark 13.2. The submodule \(W_R\) can be also defined as

\[ W_R = \{ w \in V \mid \text{for any odd positive integer } n, \text{ there is } [\ldots[w, V], \ldots V] \subseteq \text{Lie}(R) \}. \]

Corollary 13.3. Every algebraic group superscheme \(G\) contains the largest connected normal affine group super-subscheme, denoted by \(G_{aff}\). The group superschemes \(G_{aff}\) can be characterized by the property that \(G/G_{aff}\) does not contain non-trivial normal connected affine group super-subschemes.

Proof. Let \(G_{aff}\) denote the largest connected normal affine group subscheme of \(G\) (cf. [10, Proposition 8.1]). Then the sub-pair \((H_0^G, W_{G_{aff}})\) represents \(G_{aff}\), i.e., \(G_{aff}\) is nothing else but the connected component of the \(G_{aff}\)-radical of \(G\). The second statement is now apparent. \(\Box\)

Let \(G\) be an algebraic group superscheme, represented by a Harish-Chandra pair \((G,V)\). For every group subscheme \(S\) of \(G\), let \(V_S\) denote the smallest \(G\)-submodule of \(V\) such that \(S \leq \ker(G \to GL(V/V_S))\).

Lemma 13.4. Let \(R\) be a normal group subscheme of \(G = G_{ev}\). Then \(G\) is \(R\)-semisimple if and only if at least one of the following conditions hold:

(1) For any non-trivial normal group subscheme \(S\) of \(G\) such that \(S \leq R\), there is \([V, V_S] \subseteq \text{Lie}(S)\);
(2) There is no non-zero \(G\)-submodule \(W\) of \(V\) such that \([V, W] = 0\).

Proof. If on the contrary, \([V, V_S] \subseteq \text{Lie}(S)\) or there is non-zero \(G\)-submodule \(W\) of \(V\) such that \([V, W] = 0\), then \((S, V_S)\) or \((E, W)\) represent a non-trivial normal group super-subscheme of \(G\). Conversely, suppose that \((S, W)\) represents a normal group super-subscheme of \(G\). Since \([V, V_S] \subseteq [V, W] \subseteq \text{Lie}(S)\), \((1)\) implies \(S = E\), and \((2)\) concludes the proof. \(\Box\)

A connected algebraic group superscheme \(G\) is called pseudoabelian if \(G_{aff} = E\).

Let \(G\) be a pseudoabelian group superscheme and \(G \neq E\). Then \(G_{ev} = G\) is not affine. Recall that \(G_{aff}^G\) denotes the largest affine quotient of \(G\).

Set \(A(G) = \ker(G \to G_{aff}^G)\). Then \(A(G)\) is a nontrivial anti-affine algebraic group, hence smooth and connected normal group subscheme of positive dimension (cf. [10 Corollary 8.14 and Proposition 8.37]). Since the natural morphism \(G \to GL(V)\) factors through \(G \to G_{aff}^G\), \(A(G)\) acts trivially on \(V\). Besides, by [10 Corollary 8.13], \(A(G)\) is central in \(G\).

Lemma 13.5. The group scheme \(A(G)\) is an abelian group variety such that \(G = G_{aff}A(G)\).
Proof: The sub-pair \((A(G)_\text{aff}, 0)\) represents a connected normal affine group supersubscheme of \(G\). Thus, \(A(G)_\text{aff} = E\), and by [10, Theorem 8.28], \(A(G)\) is an abelian group variety. The group scheme \(G/G_{\text{aff}}A(G)\) is a quotient of both \(G/G_{\text{aff}}\) and \(G/A(G)\). Recall that \(G/G_{\text{aff}}\) is an abelian variety by [10, Theorem 8.28]. Therefore, \(G/G_{\text{aff}}A(G)\) is a complete and affine connected scheme simultaneously, hence trivial (cf. [10, A.75(g)]).

The following example shows that the class of pseudoabelian group superschemes is extensive.

**Example 13.6.** For every connected algebraic group scheme \(G\) such that \(A(G)\) is an abelian variety, and \(G = G_{\text{aff}}A(G)\), there exists a pseudoabelian group superscheme \(G\) with \(G_{\text{ev}} = G\). Let \(W\) be a faithful \(G_{\text{aff}}/(G_{\text{aff}} \cap A(G))\)-module and \(V = W \oplus W^*\). Then \(V\) has the natural structure of \(G\)-module for the diagonal action of \(G_{\text{aff}}\) and the trivial action of \(A(G)\). Furthermore, we have a bilinear symmetric \(G\)-equivariant map \(V \times V \to \text{Lie}(G)\) defined by the rule

\[
[v, w] = [\phi, \psi] = 0, [\phi, v] = \phi(v)x \text{ for } v, w \in W \text{ and } \phi, \psi \in W^*,
\]

where \(x \in \text{Lie}(A) \setminus 0\).

Then \((G, V)\) is a Harish-Chandra pair representing a group superscheme \(G\). By Lemma [13.7] the group superscheme \(G\) is pseudoabelian.

The following theorem is a super-version of well-known Barsotti-Chevalley theorem (cf. [10, Theorem 8.27]).

**Theorem 13.7.** Let \(G\) be a connected algebraic group superscheme. Then there are normal group super-subschemas \(G_1\) and \(G_2\) of \(G\) such that \(G_1\) is affine and connected, \(G_2/G_1\) is a (purely-even) abelian group variety, and \(G/G_2\) is affine.

Proof. Set \(G_1 = G_{\text{aff}}\). Then \(H = G/G_1\) is a pseudoabelian group superscheme represented by a Harish-Chandra pair \((H, V)\). As above, \((A(H), 0)\) represents a normal (purely-even) group subvariety \(A\) of \(H\). Moreover, Lemma [13.5] implies that \(H/A\) is an affine group superscheme. We define \(G_2\) as \(\pi^{-1}(A)\), where \(\pi : G \to G/G_1\) is the quotient morphism.

**Corollary 13.8.** Let \(G\) be an abelian group supervariety. Then

1. \(G_{\text{ev}} \subseteq G\), and \(G_{\text{ev}}\) is an abelian group supervariety;
2. \(G_{\text{aff}} \cap G_{\text{ev}} = E\), hence \(G_{\text{aff}}\) is a (purely-odd) affine unipotent group superscheme;
3. \(G/(G_{\text{aff}} \times G_{\text{ev}})\) is a (purely-odd) affine unipotent group superscheme.

Proof. We have seen that \(G\) is an abelian group supervariety if and only if \(G = G_{\text{res}}\) is an abelian group variety. Thus \(G\) acts trivially on \(V = \text{Lie}(G)_1\) and \((G_1)_{\text{ev}}\) is trivial, which implies the first and second statements. The third statement is clear.

**Remark 13.9.** If \(G\) is an abelian group supervariety, then the group super-subscheme \(G_{\text{aff}}\) is represented by the pair \((E, W)\), where \(W = \{w \in V \mid [w, V] = 0\}\). Respectively, \(G_{\text{ev}} \times G_{\text{aff}}\) is represented by the pair \((G, W)\).
13.2. **Anti-affine group superschemes.** An algebraic group superscheme $\mathcal{G}$ is called *anti-affine* whenever $\mathcal{O}(\mathcal{G})$ is a Grassman-like superalgebra. Set $\Phi(\mathcal{G}) = (\mathcal{G}, V)$. Theorem 12.5 implies that $\mathcal{G}$ is anti-affine if and only if the algebraic group scheme $\mathcal{G}$ is (cf. [10], page 39). Moreover, $(A(\mathcal{G}), 0)$ represents a central anti-affine group super-subscheme $\mathcal{A}$ of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{A} \simeq \mathcal{G}^{aff}$. But an anti-affine group super-subscheme of $\mathcal{G}$ is not necessarily central in $\mathcal{G}$, contrary to the purely-even case (cf. [10 Corollary 8.13]). For example, if $\mathcal{G}$ is a pseudoabelian group superscheme from Example 13.6 then its group super-subscheme, represented by the pair $(A(\mathcal{G}), V)$, is anti-affine but not central.

Similarly, an anti-affine group superscheme is no longer commutative, but it is always nilpotent. If $\mathcal{G}$ is anti-affine, then $\mathcal{G}_{ev}$ is central, and $\mathcal{G}/\mathcal{G}_{ev}$ is a purely-odd unipotent group superscheme.

14. **Sheaf quotients are superschemes**

Let $\mathcal{G}$ be an algebraic group superscheme, and $\mathfrak{H}$ its group super-subscheme. They are represented by Harish-Chandra pairs $(\mathcal{G}, V)$ and $(H, W)$, respectively.

For more details about sheaves, sheaf quotient and sheafification (completions) in the Grothendieck topology of fppf coverings, we refer to [11, 13, 20]. In this section, we prove the following theorem.

**Theorem 14.1.** The sheaf quotient $\mathcal{G}/\mathfrak{H}$ is a superscheme of finite type and the quotient morphism $\mathcal{G} \to \mathcal{G}/\mathfrak{H}$ is faithfully flat.

The proof will be given in the series of lemmas.

14.1. **A reduction.** As above, let $A(\mathcal{G})$ denote $\text{ker}(\mathcal{G} \to \mathcal{G}^{aff})$.

Let us start with two elementary observations. First, suppose $L$ is a group subscheme of $A(\mathcal{G})$. In that case, the pair $(L, 0)$ represents a central group super-subscheme $L$ of $\mathcal{G}$. Besides, the group superscheme $\mathcal{G}/L$ is represented by the pair $(\mathcal{G}/L, V)$ such that $\text{ker}(\mathcal{G}/L \to (\mathcal{G}/L)^{aff}) = A(\mathcal{G})/L$, i.e. $(\mathcal{G}/L)^{aff}$ is naturally isomorphic to $\mathcal{G}^{aff}$.

Second, if $R$ is an affine normal group subscheme of $\mathcal{G}$, then $(\mathcal{G}/R)_aff = \mathcal{G}^{aff}R/R$. Every normal affine connected group subscheme of $\mathcal{G}/R$ has the form $L/R$, where $L$ is a normal affine group subscheme of $\mathcal{G}$ (use [10 Proposition 8.1]). By [10 Proposition 1.52], $L^{0}$ is normal, hence $L^{0} \leq \mathcal{G}^{aff}$. Since $(L/R)/(L^{0}R/R) \simeq L/L^{0}R$ is etale and $L^{0}R/R$ is connected, we have $L/R = L^{0}R/R \leq \mathcal{G}^{aff}R/R$.

**Lemma 14.2.** If Theorem 14.1 holds for affine group super-subschemes $\mathfrak{H}$, then it holds for arbitrary $\mathfrak{H}$.

**Proof.** By the first observation, $(\mathcal{H} \cap A(\mathcal{G}), 0)$ represents a central group super-subscheme $L$ of $\mathcal{G}$ such that $L \leq \mathfrak{H}$. By Proposition 4.2 from [20], we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{G} & \rightarrow & \mathcal{G}/L \\
\downarrow & & \downarrow \\
\mathcal{G}/\mathfrak{H} & \simeq & (\mathcal{G}/L)/(\mathfrak{H}/L)
\end{array}
$$

where $\mathcal{H}/L \cap A(\mathcal{G})/L = \mathcal{E}$, hence $\mathcal{H}/L$ is affine and, in its turn, $\mathfrak{H}/L$ is affine as well. It remains to note that $\mathcal{G} \to \mathcal{G}/L$ and $\mathcal{G}/L \to (\mathcal{G}/L)/(\mathfrak{H}/L)$ are faithfully flat morphisms of superschemes of finite type by Corollary 12.12 and by the assumption of lemma, respectively. $\square$
Lemma 14.3. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be (affine) group super-sub-schemes of $G$ such that $\mathbb{H}_1 \leq \mathbb{H}_2$. If Theorem 14.1 holds for the couple $\mathbb{H}_2 \leq G$ and $\mathbb{H}_2/\mathbb{H}_1$ is an affine superscheme of finite type, then it holds for the couple $\mathbb{H}_1 \leq G$.

Proof. We have a cartesian square (see [19, Remark 9.11]):

$$
\begin{array}{ccc}
G \times \mathbb{H}_2/\mathbb{H}_1 & \rightarrow & G/\mathbb{H}_1 \\
\downarrow & & \downarrow \\
G & \rightarrow & G/\mathbb{H}_2,
\end{array}
$$

which satisfies the conditions of Lemma 3.3 (use [19, Proposition 9.8 and Remark 9.11]). Therefore, $G/\mathbb{H}_1$ is a superscheme of finite type. □

Lemma 14.4. Theorem 14.1 holds for any couple $\mathbb{H} \leq G$, whenever it holds for each couple that satisfies:

1. $G = M \times A(G)$;
2. $M$ is an affine group subscheme of $G$ with $H \leq M$.

Proof. Let $R$ denote the group scheme $G_{aff}H \cap A(G)$. Set $\mathbb{R} = \Psi(R, 0)$. The second observation implies that $(G/R)_{aff} = G_{aff}R/R$ and $(G_{aff}H)/(R \cap A(G))/R = E$. Since $\mathbb{H}/\mathbb{H} \simeq \mathbb{R}$ is affine and of finite type, $G/\mathbb{H}$ is a superscheme of finite type provided $G/\mathbb{H} \simeq (G/\mathbb{R})/(HR/R)$ is. Therefore, one can assume that $G_{aff}H \cap A(G) = E$. Arguing as in Lemma 13.5, one can show that $A(G)$ is an abelian variety and $G_{0} = G_{aff}A(G)$. In particular, $A(G)$ can be identified with $(G/G_{aff}H)^{0}$. By [2, Theorem 1.1], there is a finite, hence affine, group subscheme $F$ of $G/G_{aff}H$ such that $FA(G) = G/G_{aff}H$. The inverse image of $F$ in $G$ is an affine group subscheme $M$ such that $G = MA(G)$. Repeating the above arguments with $R = M \cap A(G)$, one can reduce the general case to $M \cap A(G) = E$. The proposition is proven. □

14.2. Sheaf quotient. Let $G$ be an algebraic group superscheme, represented by a Harish-Chandra pair $(G, V)$. Let $\mathbb{H}$ be its closed (affine) group super-sub-scheme, represented by a sub-pair $(H, W)$. Without loss of generality, one can assume that $G$ and $\mathbb{H}$ satisfy the conditions of Lemma 14.4.

Since $X = G/\mathbb{H}$ is a scheme (of finite type), any open affine subscheme $U'$ of $G/\mathbb{H}$ has a form $U/\mathbb{H}$, where $U$ is an open affine $H$-saturated subscheme of $G$ (cf. [8, I.5.7.1(1)]).

Set $Sp(A) = U$ and $Sp(D) = H$. Then $A' = O(U') \simeq A^{D} = A \square pk$.

Recall that the superschemes $G$ and $\mathbb{H}$ can be represented as $GE$ and $HE'$, respectively.

Lemma 14.5. The subfunctor $U = UE = EU$ is an open affine $\mathbb{H}$-saturated super-sub-scheme of $G$.

Proof. It is clear that $UE$ is open in $G$. Further, we have $U = UE = UNE = UN$, where $N = N(G)$ and $N = N_{eV}$. Since $N$ is a normal group subfunctor, we also have $U = NU = ENU = EU$. Finally, for every $hx \in \mathbb{H}$, where $h \in H$ and $x \in E'$, there is $U(hx) = UN(hx) = (Uh)(N^{h^{-1}}x) \subseteq UN = U$.

The lemma is proven. □
Set $SSp(A) = \mathbb{U}$ and $SSp(D) = \mathbb{H}$. Let $\tau : A \to A \otimes D$ be the corresponding $D$-coideal superalgebra map.

Recall that $G = M \times A(\mathbb{G})$, where $H \leq M$ and $M$ is affine. Thus $U$ can be chosen so that $U = U_{aff} \times U_{ab}$, where $U_{aff}$ is an open affine subscheme of $M$ and $U_{ab}$ is an open affine subscheme of $A(\mathbb{G})$. Moreover, the algebra $A$ is isomorphic to $A_{aff} \otimes A_{ab}$ as a $D$-comodule, where $A_{aff} \simeq k[U_{aff}], A_{ab} \simeq k[U_{ab}]$, and $A_{ab}$ is a trivial $D$-comodule.

Let $V$ be a right (super)comodule over a (super)coalgebra $C$. Let $C \to B$ be a morphism of (super)coalgebras. Then $V$ and $C$ has the natural structures of right and left (super)comodules over $B$. Moreover, the (super)comodule map $\sigma : V \to V \otimes C$ factors through $V \square_B C \subseteq V \otimes C$ and we denote $V \to V \square_B C$ by $\sigma_B$.

Following notations from [13], one can define a morphism of $D$-coideal superalgebras

$$\omega \theta : A \to (\wedge(V^*) \otimes A) \square_D D \wedge (\theta') \square_D (A \otimes \wedge(Z)) \square_D D,$$

where $Z = \ker(V^* \to W^*)$. The proofs of Lemma 4.1, Proposition 4.2, and Lemma 4.4 in [13] can be repeated verbatim. Only the definition of $\theta'$ and the proof of Lemma 4.6 in [13] needs a commentary. The structure of the right $G$-module of $V^*$ is completely defined by the structure of $M$-module because $A(\mathbb{G})$ acts trivially on $V^*$. Therefore, one can define the map

$$\kappa = \kappa_A : A \otimes V^* \to V^* \otimes A$$

by

$$a_1 \otimes a_2 \otimes v^* \mapsto v^*_{(0)} \otimes a_1 j(v^*_{(1)}) \otimes a_2,$$

where $a_1 \in A_{aff}, a_2 \in A_{ab}, v^* \mapsto v^*_{(0)} \otimes v^*_{(1)}$ is the corresponding comodule map $V^* \to V^* \otimes k[M]$, and $j : k[M] \to A_{aff}$ is dual to the open immersion $U_{aff} \to M$.

We will show that $\omega \theta$ is an isomorphism.

**Remark 14.6.** The definition of $\omega \theta$ is consistent with a base change, and once needed, $k$ can be replaced by a suitable field extension $L \supseteq k$.

It has been proven that $gr(\mathbb{G}) \simeq G \ltimes G_{odd}$ and $gr(\mathbb{H}) \simeq H \ltimes H_{odd}$, where $G_{odd} \simeq SSp(A(V^*))$, $H_{odd} \simeq SSp(A(W^*))$ and $V^*$ and $W^*$ consist of primitive elements.

Using Proposition 10.3 and arguing as in Proposition 10.5 one sees that $gr(U) \simeq SSp(gr(A))$ is an open $gr(\mathbb{H})$-saturated super-subscheme of $gr(\mathbb{G})$.

**Proposition 14.7.** The following statements hold:

1. $gr(U) \simeq U_{G_{odd}} = G_{odd} U$.
2. As the right and left $D$-coideal superalgebras, $gr(A)$ and $gr(D)$ are naturally isomorphic to $A$ and $D$, respectively.
3. $gr(\tau_B) = gr(\tau)_D$.

**Proof.** The geometric counterpart of $U_{G_{odd}}$ is the open super-subscheme $q^{-1}_{G}(U_{ev}) \simeq U_{ev} \times G_{odd}$, where $U$ is the geometric counterpart of $U$ in $G$. Since $gr(U)^e = U_{ev} = q^{-1}_{G}(U_{ev})^e$, the condition (1) follows.

Further, the right $gr(D)$-coideal superalgebra structure on $gr(A)$ is defined by a superalgebra morphism $gr(A)^{gr(\tau)} gr(A \otimes D) \simeq gr(A) \otimes gr(D)$. More precisely, for an element $a \in I^n_A$, let $\tau(a) = \sum_{0 \leq k \leq n} a(0) \otimes a(1), n-k$ be adapted to the $I_{A \otimes D}$-adic
filtration. That is, \( a_{(0),k} \in I_A^k \) and \( a_{(1),k} \in I_D^{n-k} \) for every \( 0 \leq k \leq n \). Then \( \text{gr}(\tau) \) is defined by

\[
a + I_A^{n+1} \mapsto \sum_{0 \leq k \leq n} (a_{(0),k} + I_A^{k+1}) \otimes (a_{(1),n-k} + I_D^{n-k+1}).
\]

It follows that the right D-coideal superalgebra structure of \( \text{gr}(A) \) is defined as

\[
a + I_A^{n+1} \mapsto (a_{(0),n} + I_A^{n+1}) \otimes (a_{(1),0} + I_D),
\]

where \( a \mapsto a_{(0),n} \otimes (a_{(1),0} + I_D) \) is the right D-coideal superalgebra map of \( A \).

On the other hand, if we identify \( A \) with \( \Lambda(V^*) \otimes A \), then the latter map is \( x \otimes a \mapsto x \otimes a_{(0)} \otimes a_{(1)} \), where \( x \in \Lambda(V^*) \), \( a \in A \) and \( a \mapsto a_{(0)} \otimes a_{(1)} \) is the D-coideal algebra map of \( A \). Moreover, we have \( I_A^k = (\oplus_{x \geq k} A^x(V^*)) \otimes A \) for \( 0 \leq k \leq \dim V \). Then \( A \) and \( \text{gr}(A) \) are isomorphic as D-coideal superalgebras. The case of \( D \) is similar.

Furthermore, \( A \otimes_D D \) is isomorphic to \( \Lambda(V^*) \otimes B \otimes \Lambda(W^*) \), where

\[
B = \{ a_{(0)} \otimes a_{(1)} \mid a \in A \} \subseteq A \otimes D.
\]

In particular, \( I_A \otimes_D A \cap I_A \otimes_D D \) for \( 0 \leq k \leq \dim V + \dim W \), and we obtain a natural isomorphism \( \text{gr}(A \otimes_D D) \simeq \text{gr}(A) \otimes_D \text{gr}(D) \) that makes the diagram

\[
\begin{array}{ccc}
\text{gr}(A) & \xrightarrow{\text{gr}(\tau_0)} & \text{gr}(A \otimes_D D) \\
\| & & \downarrow \\
\text{gr}(A) & \xrightarrow{\text{gr}(\tau)} & \text{gr}(A) \otimes_D \text{gr}(D)
\end{array}
\]

commutative. The proposition is proven. \( \square \)

**Lemma 14.8.** The morphism \( \text{gr}(\Lambda(\tau) \boxtimes_D \text{id}_D) \) is identified with \( \Lambda(\tau) \boxtimes_D \text{id}_{\text{gr}(D)} \).

**Proof.** Recall that the morphism \( \Lambda(\tau') \) is defined as

\[
v_1^* \wedge \ldots \wedge v_k^* \otimes a \mapsto \sum_{i_1, \ldots, i_k} a_{i_1} \ldots a_{i_k} a \otimes z_{i_1} \wedge \ldots \wedge z_{i_k},
\]

where \( \tau'(v_k^*) = \sum_{i_s} a_{i_s} \otimes z_{i_s} \) for \( 1 \leq s \leq k \). Moreover, the morphism \( \Lambda(\tau') \boxtimes_D \text{id}_D \) is defined as

\[
v_1^* \wedge \ldots \wedge v_k^* \otimes a_{(0)} \otimes a_{(1)} \wedge w_1^* \wedge \ldots \wedge w_i^* \mapsto \sum_{i_1, \ldots, i_k} a_{i_1} \ldots a_{i_k} a_{(0)} \otimes z_{i_1} \ldots \wedge z_{i_k} \otimes a_{(1)} \wedge w_1^* \wedge \ldots \wedge w_i^*,
\]

and is compatible with the corresponding filtrations of the superalgebras \( \Lambda(V^*) \otimes A \otimes_D D \) and \( (A \otimes \Lambda(Z)) \otimes_D D \). \( \square \)

The following lemma is now obvious.

**Lemma 14.9.** The morphism \( \text{gr}(\omega_0) \) can be naturally identified with

\[
\text{gr}(A) \xrightarrow{\text{gr}(\tau)_0} (\Lambda(V^*) \otimes A) \boxtimes_{\text{gr}(D)} (\Lambda(\tau') \boxtimes_D \text{id}_{\text{gr}(D)}) \xrightarrow{\text{id}_{\text{gr}(D)}} (A \otimes \Lambda(Z)) \boxtimes_{\text{id}_D} \text{gr}(D).
\]

Let \( \theta \) be a retract of the inclusion \( A \otimes Z \to A \otimes V^* \) from [14] Lemma 4.4]. Let \( V^* = Z \oplus T \), and elements \( t_j \) form a basis of \( T \) for \( 1 \leq j \leq s = \dim W^* \). The free \( A \)-module \( \ker \theta \) has a basis consisting of the elements \( t_j' \) such that \( t_j' = t_j \) (mod \( A \otimes Z \)).

Choose a closed point \( x \) in \( U \). By Remark 14.6, one can assume that \( x \in U(k) \).

That is, \( x \) is an algebra morphism \( A \to k \).
The natural embedding $\mathbb{H}_{\text{odd}} \to U \times G_{\text{odd}} \simeq \text{gr}(U)$, defined as $h \mapsto (x, h)$, is dual to the superalgebra morphism induced by $x$ and $V^* \to W^*$. The natural (right-hand side) action of group $\mathbb{H}_{\text{odd}}$ on $\text{gr}(U)$ is defined by a coideal superalgebra map

$$z \mapsto z \otimes 1 \text{ and } t'_j \mapsto t'_j \otimes 1 + 1 \otimes t_j \text{ for } z \in Z,$$

where $t_j$ is the image of $t_j$ in $W^*$. Finally, the $A$-superalgebra morphism $\land(\theta) : A \otimes \Lambda(V^*) \to A \otimes \Lambda(Z)$ induces the embedding $s : U \times \text{SSp}(\Lambda(Z)) \to \text{gr}(U)$.

**Lemma 14.10.** There is a natural isomorphism

$$(U \times \text{SSp}(\Lambda(Z))) \times \mathbb{H}_{\text{odd}} \simeq \text{gr}(U),$$

induced by the above two embeddings and the multiplication map.

**Proof.** Choose a basis $z_1, \ldots, z_{t-n}$ of $Z$. We use notations from Proposition 11.6. Any couple $(g, h) \in (U \times \text{SSp}(\Lambda(Z))) \times \mathbb{H}_{\text{odd}}$ is taken to a superalgebra morphism

$$(g, h)(azt'_j) = \sum_{S \subseteq J} g(\land(\theta)(azt'_jS))h(\overline{t}_{jS}) = g(azt_j)h(\overline{t}_j),$$

where $I \subseteq \{1, \ldots, t-s\}, J \subseteq \{1, \ldots, s\}$. The proof concludes by the observation that $A \otimes \Lambda(V^*) \simeq (A \otimes \Lambda(Z)) \otimes \Lambda(T')$, where $T'$ is the $k$-span of $t'_j$ for $1 \leq j \leq s$. ⊓⊔

**Lemma 14.11.** The superalgebra morphism $\text{gr}(\omega^0)$ is an isomorphism, and therefore $\omega^0$ is as well.

**Proof.** Consider the following sequence of superscheme morphisms

$$(\ast) \quad \text{gr}(U) \overset{a}{\leftarrow} \text{gr}(U) \times^H \text{gr}(\mathbb{H}) \overset{b}{\rightarrow} (U \times G_{\text{odd}}) \times^H \text{gr}(\mathbb{H}) \overset{d}{\longleftarrow} (U \times \text{SSp}(\Lambda(Z))) \times^H \text{gr}(\mathbb{H}) \simeq \text{SSp}((A \otimes \Lambda(Z)) \boxtimes \text{gr}(D)).$$

The morphism $a$ is induced by $(u, h) \mapsto uh$ for $u \in \text{gr}(U)$ and $h \in \text{gr}(\mathbb{H})$. Since $\text{gr}(\mathbb{H}) \simeq H \times \mathbb{H}_{\text{odd}}$, the superscheme $\text{gr}(U) \times^H \text{gr}(\mathbb{H})$ is canonically isomorphic to $\text{gr}(U) \times \mathbb{H}_{\text{odd}}$, hence it is affine. Lemma 4.4 implies that $\text{gr}(U) \times^H \text{gr}(\mathbb{H}) \simeq \text{SSp}((A \otimes \Lambda(Z)) \boxtimes \text{gr}(D))$ and the morphism $a$ is dual to $\text{gr}(\tau_0)$.

Similarly, we have

$$(U \times G_{\text{odd}}) \times^H \text{gr}(\mathbb{H}) \simeq (U \times G_{\text{odd}}) \times \mathbb{H}_{\text{odd}} \simeq \text{SSp}((\Lambda(V^*) \otimes A) \boxtimes \text{gr}(D)),$$

and the isomorphism $b$ is dual to the superalgebra isomorphism induced by $\kappa^{-1}$,

$$(U \times \text{SSp}(\Lambda(Z))) \times^H \text{gr}(\mathbb{H}) \simeq (G_{\text{odd}} \times U) \times \mathbb{H}_{\text{odd}} \simeq \text{SSp}((A \otimes \Lambda(Z)) \boxtimes \text{gr}(D))$$

and the morphism $d = s \times^H \text{id}_{\text{gr}(\mathbb{H})}$ is just $\text{SSp}(\land(\theta) \boxtimes \text{id}_{\text{gr}(D)})$. Thus the composition $abcd$ coincides with $\text{SSp}(\text{gr}(\omega^0))$. The above isomorphisms allow us to identify $abcd$ with the isomorphism from Lemma 14.10. Thus, the lemma follows. ⊓⊔

**Lemma 14.12.** The sheaf quotient $U/\mathbb{H}$ is an affine superscheme of finite type.

**Proof.** By Theorem 3.1 (1) and Corollary 4.14(2)], we need to show that the superalgebra morphism $\alpha : A \otimes A \to A \otimes D$ given by $a \otimes a' \mapsto \sum aa'_{(0)} \otimes a'_{(1)}$, is surjective. Equivalently, the functor morphism $\text{SSp}(\alpha) : U \times \mathbb{H} \to U \times U$ given by $(u, h) \mapsto (u, uh)$, maps $U \times \mathbb{H}$ isomorphically onto a closed super-subscheme of $U \times U$. Define a functor morphism $p : G \times G \to G$ by $(g, g') \mapsto g^{-1}g'$. Then $\text{SSp}(\alpha)(U \times \mathbb{H}) = (U \times U) \cap p^{-1}(\mathbb{H})$, hence it is closed in $U \times U$. The lemma follows. ⊓⊔
Let $V'$ be an open affine subscheme of $U'$. As above, $V' = V/H$, where $V \simeq \text{Sp}(B)$ is an affine subscheme of $U$ and $B' = O(V') \simeq B^D$. Set $V = V/E$. Then $V$ is an open affine $\mathbb{H}$-saturated super-subscheme of $U$.

**Lemma 14.13.** $V/H$ is an open affine super-subscheme of $U/H$.

**Proof.** The inclusion $V' \subseteq U'$ is defined by a morphism $\phi : A \to B$ of $D$-coideal algebras. By [[15, Lemma 3.5]], there are (even) elements $a_1, \ldots, a_k \in A'$ such that $\sum_{1 \leq i \leq k} B'\phi(a_i) = B'$, and for each $i$, the induced morphism $A'_i \to B'_i = B'_{\phi(a_i)}$ is an isomorphism. Thus, $B = \sum_{1 \leq i \leq k} B\phi(a_i)$ and $A_i \simeq B_{\phi(a_i)}$ for $1 \leq i \leq k$. There is $B = O(V) \simeq (B \otimes \Lambda(Z)) \sqcup D$, and the morphism of $D$-coideal superalgebras $A \to B$, dual to the inclusion $V \subseteq U$, is $(\phi \otimes \text{id}_{\Lambda(Z)}) \sqcup \text{id}_D$. Moreover, by [[14, Proposition 4.2(1)]], $V/H \simeq \text{SSp}(B \otimes \Lambda(Z)^D)$ and $U/H \simeq \text{SSp}(A \otimes \Lambda(Z)^D)$. Therefore, the inclusion $V/H \subseteq U/H$ is dual to $\phi \otimes \text{id}_{\Lambda(Z)}$, and the superalgebras $O(U/H)$ and $O(V/H)$ satisfy the conditions of [[15, Lemma 3.5]], for the elements $a_i \otimes 1$, where $1 \leq i \leq k$. The lemma is proven. $\Box$

**Corollary 14.14.** The statement of Lemma 14.13 remains valid for an arbitrary open subscheme $V'$ of $U'$.

**Proof.** Indeed, $V'$ is a finite union of open affine subschemes. $\Box$

Fix a finite open covering of $G/H$ by affine subschemes $U'_i$ for $1 \leq i \leq l$. Then there is a collection of affine superschemes $U_i = U_i/H$ and quotient morphisms $U_i \to U'_i$ such that for every pair of indices $1 \leq i \neq j \leq l$ we have $(U_i \cap U_j)/H = U'_i \cap U'_j$.

Consider the collection of corresponding geometric superschemes $U'_i$. Since the underlying topological space of each $U'_i$ coincides with $(U'_i)^c$ (see [[14, Proposition 4.2]]), one can construct a geometric superscheme $Z$ with $Z^c = X^c$ as follows. For any open subset $W \subseteq Z^c$, we define

$$
O_Z(W) = \ker(\prod_{1 \leq i \leq l} O_{U'_i}(W \cap (U'_i)^c) \to \prod_{1 \leq i \neq j \leq l} O_{U'_i \cap U'_j}(W \cap (U'_i)^c \cap (U'_j)^c)),
$$

where $U'_i \cap U'_j$ denotes the geometric counterpart of $U_i \cap U_j$. It is clear that $O_Z$ is a superalgebra sheaf such that $O_Z|_{(U'_i)^c} \simeq O_{U'_i}$ for $1 \leq i \leq l$.

The following lemma concludes the proof of the main theorem.

**Lemma 14.15.** There is the unique morphism $G \to Z$, the restriction of which on each $U_i$ coincides with $U'_i \subseteq Z$. In particular, $Z \simeq G/H$.

**Proof.** The open immersions $U'_i \to Z$ induce the open embeddings $U'_i \to Z$. Thus, $U'_i$ form an open covering of $Z$. Since $Z$ is a local functor, the collection of morphisms $U_i \to U'_i$ uniquely extends to a morphism $G \to Z$, which is constant on $\mathbb{H}$-orbits. Moreover, any morphism from $G$ to a faisceau $\mathcal{Y}$, which is constant on $\mathbb{H}$-orbits, is uniquely defined by its restrictions on $U_i$ for $1 \leq i \leq l$. Since such morphisms $U_i \to \mathcal{Y}$ factor through the morphisms $U'_i \to \mathcal{Y}$, $G \to \mathcal{Y}$ factors through the unique morphism $Z \to \mathcal{Y}$. Lemma is proven. $\Box$

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