Abstract

In this paper, an extended Klein-Gordon (KG) field system is introduced in $3 + 1$ dimensions. It leads to a single zero rest mass soliton solution. It is shown that this special massless soliton solution is energetically stable, i.e. any arbitrary deformation above its background leads to an increase in total energy.
Keywords: massless, zero rest mass, soliton, solitary wave, stability, extended Klein-Gordon.

I. INTRODUCTION

Any particle which moves at the speed of light, must be massless (i.e. its rest mass must be zero). It is an obvious statement from the well-known relativistic mechanic. Nevertheless, is the inverse of this statement absolutely valid? That is, does any massless particle has to move at the speed of light? In other words, is it possible to have a zero rest mass particle which be at rest or can be found at any arbitrary velocity? Mathematically, if we use the classical relativistic field theory with soliton solutions, our answer may be slightly different! Solitons in many respects behaves like real particles and satisfy the energy-momentum relation of the standard relativistic mechanic properly \([1–4]\). In the paper \([5]\), it is shown that the existence of a non-moving massless soliton solution can be possible theoretically. But, any massless particle in response to any amount of force, no matter how tiny, accelerates to the speed of light immediately. Therefore, since there is not any place in the world with zero interaction, it is not possible to find a zero rest mass particle at a velocity less than the light velocity. In other words, a non-moving massless particle just can be exist in a world with no interaction.

In the paper \([5]\), for the first time, a non-topological zero rest mass soliton solution of an extended Klein-Gordon (KG) system was introduced in 1 + 1 dimensions. The extended KG system were introduced in Refs. \([5]\) and \([6]\). Briefly, for a set of real scalar fields \(\phi_j\) \((j = 1, 2, \cdots, N)\), the extended KG systems have Lagrangian densities which are not linear in the kinetic scalars. The kinetic scalars are different contractions of the scalar field derivatives, i.e. \(S_{ij} = \partial_\mu \phi_i \partial^\mu \phi_j\). In general, such Lagrangian densities can be called non-standard Lagrangian (NSL) densities too \([7–12]\). There are many works which deal with such systems among which one can mention the works of Riazi et. al. \([13, 14]\) and El-Nabulsi \([10–12]\). Moreover, in cosmology the NSL are used for describing dark energy and dark matter \([15–18]\). Nevertheless, major works that have been done so far with relativistic solitary wave and soliton solutions, are the standard nonlinear KG systems; that is, the systems which their Lagrangian densities are linear in the kinetic scalars \([3, 6]\). For example, the systems in 1 + 1 dimensions with kink (anti-kink) \([1, 4, 19, 41]\) and Q-ball \([1, 41, 59]\) solutions, all are good examples of the standard nonlinear KG systems.
In this paper, in line with what was done in the paper \cite{5}, we are going to introduce an extended KG Lagrangian density in $3 + 1$ dimensions with a single massless (zero rest mass) soliton solution. At first step, we will show that using the same Lagrangian density which was introduced in \cite{5}, now does not lead to a single massless solitary wave solution in $3 + 1$ dimensions. Thus, for this purpose, we have to introduce a new Lagrangian density with three new additional scalar fields. However, despite the complexity of this new system in $3 + 1$ dimensions, the procedure is the same as in the previous paper \cite{5}. It will be shown that, the massless soliton solution is energetically stable; meaning that, any arbitrary deformation above its background, leads to an increase in total energy. This main property would be confirmed by this fact that all terms in the energy density functional are positive definite. It should be noted that, there are few models in $3 + 1$ dimensions with soliton solutions among which one can mention the Skyrme model \cite{60, 61} and ’t Hooft Polyakov model \cite{62, 63}. In this paper, we will introduce a new one.

The organization of this paper is as follows: In section II, similar to \cite{5}, we will consider a toy extended KG model in $3 + 1$ dimensions which does not lead to a single massless solitary wave solutions. In section III, a new extended KG system in $3 + 1$ dimensions will be introduced which yields to a single massless solitary wave solution. The last section is devoted to summary and conclusions.

II. A TOY EXTENDED KG MODEL

According to the same extended KG model in $1 + 1$ dimensions which was introduced in \cite{5} and led to a single massless soliton solution, one can think about the modified version of that in $3 + 1$ dimensions. In other words, exactly the same Lagrangian density which was introduced in $1 + 1$ dimensions (Eq. 15 in \cite{5}) for two scalar fields $\phi_1 = R$ and $\phi_2 = \theta$, now is used again here:

$$L = \sum_{i=1}^{3} K^3_i,$$

where

$$K_1 = R^2[S_{22} - 2],$$

$$K_2 = R^2[S_{22} - 2] + [S_{11} - 4R^4 + 4R^3],$$

$$K_3 = R^2[S_{22} - 2] + [S_{11} - 4R^4 + 4R^3] + 2R[S_{12}],$$
in which, $S_{11} = \partial_\mu R \partial^\mu R$, $S_{22} = \partial_\mu \theta \partial^\mu \theta$ and $S_{12} = \partial_\mu R \partial^\mu \theta$ are the allowed kinetic scalars.

Now, the main modification is that the kinetic scalars are defined in the $3+1$ dimensions; namely, $S_{11} = \partial_\mu R \partial^\mu R = (\frac{\partial R}{\partial t})^2 - (\nabla R)^2$ and so on. Thus, all the equations of motion and the energy density relations (i.e. equations (19)-(26) in [5]) would be obtained again provided one changes $R'$ and $\theta'$ (i.e. the $x$-derivative of the module and phase field) to $\nabla R$ and $\nabla \theta$, respectively. In [5], it was shown that the existence of a massless solitary wave solution is possible if for that all $K_i$’s ($i = 1, 2, 3$) to be zero simultaneously. Hence, there was just a single non-trivial solitary wave solution as follows:

$$R(x) = \frac{1}{1 + x^2}, \quad \theta(t) = \pm \sqrt{2}t. \quad (5)$$

In the $3+1$ dimensions, the required conditions $K_i = 0$’s ($i = 1, 2, 3$) lead to the following covariant PDE’s:

$$\partial_\mu \theta \partial^\mu \theta = \dot{\theta}^2 - (\nabla \theta)^2 = 2, \quad (6)$$

$$\partial_\mu R \partial^\mu R = \dot{R}^2 - (\nabla R)^2 = 4R^4 - 4R^3, \quad (7)$$

$$\partial_\mu R \partial^\mu \theta = \dot{\theta} \dot{R} - (\nabla \theta \cdot \nabla R) = 0, \quad (8)$$

where, dot indicates time derivative. In general, since there are three independent PDE’s (6)-(8) just for two scalar field $R$ and $\theta$, mathematically the existence of the common solutions is severely restricted. However, for the static massless solutions, for which $\theta(t) = \sqrt{2}t$ and $R = R(x, y, z)$, PDEs (6) and (8) are satisfied automatically, and PDE (7) reduced to

$$(\nabla R)^2 = \left(\frac{\partial R}{\partial x}\right)^2 + \left(\frac{\partial R}{\partial y}\right)^2 + \left(\frac{\partial R}{\partial z}\right)^2 = 4R^3 - 4R^4. \quad (9)$$

If we restrict ourself to the $1+1$ version of the model (1), the pervious Eq. (9) is reduced to

$$\left(\frac{dR}{dx}\right)^2 = 4R^3 - 4R^4, \quad (10)$$

in which $R = R(x)$. It is easy to show that the non-linear ordinary differential equation (10) has just a single static solution $R = 1/(1 + x^2)$, i.e. the same which was introduced in Eq. (5). But, in the $3+1$ version of the model (1), the non-linear PDE (9) has infinite solutions. For example, one can mention the followings:

$$R(r) = \frac{1}{1 + (r + \xi)^2}, \quad (11)$$

$$R = \frac{1}{1 + x^2}, \quad R = \frac{1}{1 + y^2}, \quad R = \frac{1}{1 + z^2}, \quad (12)$$

$$R = \frac{1}{1 + x^2 + y^2}, \quad R = \frac{1}{1 + x^2 + z^2}, \quad R = \frac{1}{1 + y^2 + z^2}, \quad (13)$$
where \( r = \sqrt{x^2 + y^2 + z^2} \) and \( \xi \) is any arbitrary real number. According to Eq. (11), related to different values of \( \xi \), different degenerate massless solutions can be obtained in 3 + 1 dimensions. In 1 + 1 version of this model (14), the static solution (11) is reduced to \( R = \frac{1}{1+(x+\xi)} \), but it is nothing more than a space translation in (5) and essentially can not be considered as a new special massless solution. Note that, the special solutions (12) and (13) are non-localized and can not be physically interesting.

In [5], or in the same 1 + 1 version of the model (1), the main idea which guides one to conclude the special solitary wave solution (5) is a massless soliton solution, is that three PDE’s (6)-(8) are completely independent and they have just a single solitary wave solution (5), then we ensure that (5) is a soliton solution with the minimum energy among the others. But, in 3 + 1 version of the model (1), the existence of infinite degenerate massless solution such as (11) do not let us to call them solitons. Accordingly, using two scalar fields \( R \) and \( \theta \) in the 3 + 1 version of the model (1), does not lead to a single massless solitary wave solution. In the next section we will introduce another extended KG system with three new dynamical fields \( \psi_1, \psi_2 \) and \( \psi_3 \) to overcome this problem.

III. AN EXTENDED KG SYSTEM WITH A SINGLE MASSLESS SOLITON SOLUTION IN 3 + 1 DIMENSIONS

For five real scalar fields \( \phi_1 = R, \phi_2 = \theta, \phi_3 = \psi_1, \phi_4 = \psi_2 \) and \( \phi_5 = \psi_3 \), we can propose a new extended KG system in the following form:

\[
\mathcal{L} = \sum_{i=1}^{12} K_i^3, \quad (14)
\]

where

\[
\begin{align*}
K_1 &= R^2 S_2, & K_2 &= R^2 S_2 + S_1, & K_3 &= R^2 S_2 + S_1 + 2 R S_3, \\
K_4 &= R^2 S_2 + S_4, & K_5 &= R^2 S_2 + S_5, & K_6 &= R^2 S_2 + S_6, \\
K_7 &= R^2 S_2 + S_4 + S_5 + 2 S_7, & K_8 &= R^2 S_2 + S_4 + S_6 + 2 S_8, \\
K_9 &= R^2 S_2 + S_5 + S_6 + 2 S_9, & K_{10} &= R^2 h_1 S_2 + S_1 + S_4 + 2 S_{10}, \\
K_{11} &= R^2 h_2 S_2 + S_1 + S_5 + 2 S_{11}, & K_{12} &= R^2 h_3 S_2 + S_1 + S_6 + 2 S_{12}.
\end{align*}
\]
where \( h_j = [2 + \frac{1}{2}(b_j - 1)^2] \), \( b_j = 2\psi_j(2R - 1) \) \((j = 1, 2, 3)\), and

\[
S_1 = S_{11} - 4R^4 + 4R^3, \quad S_2 = S_{22} - 2, \quad S_3 = S_{12}, \quad S_4 = S_{33} + R^2 - 4R^2\psi_1^2, \\
S_5 = S_{44} + R^2 - 4R^2\psi_2^2, \quad S_6 = S_{55} + R^2 - 4R^2\psi_3^2, \quad S_7 = S_{24} - 4R^2\psi_1\psi_2, \\
S_8 = S_{35} - 4R^2\psi_1\psi_3, \quad S_9 = S_{45} - 4R^2\psi_2\psi_3, \quad S_{10} = S_{13} - b_1R^2, \\
S_{11} = S_{14} - b_2R^2, \quad S_{12} = S_{15} - b_3R^2, 
\]

(16)

where \( S_{11} = \partial_\mu R \partial_\mu R \), \( S_{22} = \partial_\mu \theta \partial_\mu \theta \), \( S_{12} = \partial_\mu \psi_1 \partial_\mu \psi_1 \), \( S_{33} = \partial_\mu \psi_1 \partial_\mu \psi_1 \), \( S_{44} = \partial_\mu \psi_2 \partial_\mu \psi_2 \), \( S_{55} = \partial_\mu \psi_3 \partial_\mu \psi_3 \), \( S_{13} = \partial_\mu R \partial_\mu \psi_1 \), \( S_{14} = \partial_\mu R \partial_\mu \psi_2 \), \( S_{15} = \partial_\mu R \partial_\mu \psi_3 \), \( S_{34} = \partial_\mu \psi_1 \partial_\mu \psi_2 \), \( S_{35} = \partial_\mu \psi_1 \partial_\mu \psi_3 \) and \( S_{45} = \partial_\mu \psi_2 \partial_\mu \psi_3 \) are some allowed kinetic scalars which are used to introduce the new extended KG model (14).

Using the Euler-Lagrange equations, one can obtain the dynamical equations easily:

\[
\sum_{i=1}^{12} \mathcal{K}_i \left[ 2(\partial_\mu \mathcal{K}_i) \frac{\partial \mathcal{K}_i}{\partial (\partial_\mu R)} + \mathcal{K}_i \partial_\mu \left( \frac{\partial \mathcal{K}_i}{\partial (\partial_\mu R)} \right) - \mathcal{K}_i \frac{\partial \mathcal{K}_i}{\partial R} \right] = 0, 
\]

(17)

\[
\sum_{i=1}^{12} \mathcal{K}_i \left[ 2(\partial_\mu \mathcal{K}_i) \frac{\partial \mathcal{K}_i}{\partial (\partial_\mu \theta)} + \mathcal{K}_i \partial_\mu \left( \frac{\partial \mathcal{K}_i}{\partial (\partial_\mu \theta)} \right) \right] = 0. 
\]

(18)

\[
\sum_{i=1}^{12} \mathcal{K}_i \left[ 2(\partial_\mu \mathcal{K}_i) \frac{\partial \mathcal{K}_i}{\partial (\partial_\mu \psi_j)} + \mathcal{K}_i \partial_\mu \left( \frac{\partial \mathcal{K}_i}{\partial (\partial_\mu \psi_j)} \right) - \mathcal{K}_i \frac{\partial \mathcal{K}_i}{\partial \psi_j} \right] = 0, \quad (j = 1, 2, 3). 
\]

(19)

It is obvious that the sets of functions \( R, \theta \) and \( \psi_j \) \((j = 1, 2, 3)\) for which \( \mathcal{K}_i \)'s \((i = 1, 2, \ldots, 12)\) to be zero simultaneously are the special solutions (i.e. the massless solutions) of the new extended system (14). Note that, since \( \mathcal{K}_i \)'s are twelve independent linear combination of twelve independent scalars \( S_i \)'s, it is easy to understand that the conditions \( \mathcal{K}_i = 0 \) are equivalent to \( S_i = 0 \). The same procedure which was done in [5], repeats in this paper again. Hence, the energy-density belongs to the new extended Lagrangian-density \( \Pi \), would be

\[
\varepsilon(x, t) = T^{00} = \sum_{i=1}^{12} \varepsilon_i = \sum_{i=1}^{12} \mathcal{K}_i^2 [3C_i - \mathcal{K}_i], 
\]

(20)
which are divided into twelve distinct parts, in which

\[ C_i = \frac{\partial K_i}{\partial \dot{\theta}} + \frac{\partial K_i}{\partial \dot{R}} \dot{R} + \sum_{j=1}^{3} \frac{\partial K_i}{\partial \psi_j} \psi_j = \begin{cases} 
2R^2\dot{\theta}^2 & \text{i}=1 \\
2(\dot{R}^2 + R^2\dot{\theta}^2) & \text{i}=2 \\
2(\dot{R} + R\dot{\theta})^2 & \text{i}=3. \\
2(\dot{\psi}_1^2 + R^2\dot{\theta}^2) & \text{i}=4. \\
2(\dot{\psi}_2^2 + R^2\dot{\theta}^2) & \text{i}=5. \\
2(\dot{\psi}_3^2 + R^2\dot{\theta}^2) & \text{i}=6. \\
2(\dot{\psi}_1 + \dot{\psi}_2)^2 + 2R^2\dot{\theta}^2 & \text{i}=7. \\
2(\dot{\psi}_1 + \dot{\psi}_3)^2 + 2R^2\dot{\theta}^2 & \text{i}=8. \\
2(\dot{\psi}_2 + \dot{\psi}_3)^2 + 2R^2\dot{\theta}^2 & \text{i}=9. \\
(\dot{R} + \dot{\psi}_1)^2 + 2h_1R^2\dot{\theta}^2 & \text{i}=10. \\
(\dot{R} + \dot{\psi}_2)^2 + 2h_2R^2\dot{\theta}^2 & \text{i}=11. \\
(\dot{R} + \dot{\psi}_3)^2 + 2h_3R^2\dot{\theta}^2 & \text{i}=12. 
\end{cases} \] (21)

After a straightforward calculation we obtain:

\[ \varepsilon_1 = \mathcal{K}_1^2[5R^2\dot{\theta}^2 + R^2(\nabla \theta)^2 + 2R^2], \] (22)
\[ \varepsilon_2 = \mathcal{K}_2^2[5R^2\dot{\theta}^2 + 5\dot{R}^2 + R^2(\nabla \theta)^2 + (\nabla R)^2 + U(R)], \] (23)
\[ \varepsilon_3 = \mathcal{K}_3^2[5(R\dot{\theta} + \dot{R})^2 + (R\nabla \theta + \nabla R)^2 + U(R)], \] (24)
\[ \varepsilon_4 = \mathcal{K}_4^2[5R^2\dot{\theta}^2 + 5\dot{\psi}_1^2 + R^2(\nabla \theta)^2 + (\nabla \psi_1)^2 + R^2 + 4R^2\psi_1^2], \] (25)
\[ \varepsilon_5 = \mathcal{K}_4^2[5R^2\dot{\theta}^2 + 5\dot{\psi}_2^2 + R^2(\nabla \theta)^2 + (\nabla \psi_2)^2 + R^2 + 4R^2\psi_2^2], \] (26)
\[ \varepsilon_6 = \mathcal{K}_6^2[5R^2\dot{\theta}^2 + 5\dot{\psi}_3^2 + R^2(\nabla \theta)^2 + (\nabla \psi_3)^2 + R^2 + 4R^2\psi_3^2], \] (27)
\[ \varepsilon_7 = \mathcal{K}_7^2[5R^2\dot{\theta}^2 + 5(\dot{\psi}_1 + \dot{\psi}_2)^2 + R^2(\nabla \theta)^2 + (\nabla \psi_1 + \nabla \psi_2)^2 + 4R^2(\psi_1 + \psi_2)^2], \] (28)
\[ \varepsilon_8 = \mathcal{K}_8^2[5R^2\dot{\theta}^2 + 5(\dot{\psi}_1 + \dot{\psi}_3)^2 + R^2(\nabla \theta)^2 + (\nabla \psi_1 + \nabla \psi_3)^2 + 4R^2(\psi_1 + \psi_3)^2], \] (29)
\[ \varepsilon_9 = \mathcal{K}_9^2[5R^2\dot{\theta}^2 + 5(\dot{\psi}_2 + \dot{\psi}_3)^2 + R^2(\nabla \theta)^2 + (\nabla \psi_2 + \nabla \psi_3)^2 + 4R^2(\psi_2 + \psi_3)^2], \] (30)
\[ \varepsilon_{10} = \mathcal{K}_{10}^2[5h_1R^2\dot{\theta}^2 + h_1R^2(\nabla \theta)^2 + 5(\dot{R} + \dot{\psi}_1)^2 + (\nabla R + \nabla \psi_1)^2 + V(R, \psi_1)], \] (31)
\[ \varepsilon_{11} = \mathcal{K}_{11}^2[5h_2R^2\dot{\theta}^2 + h_2R^2(\nabla \theta)^2 + 5(\dot{R} + \dot{\psi}_2)^2 + (\nabla R + \nabla \psi_2)^2 + V(R, \psi_2)], \] (32)
\[ \varepsilon_{12} = \mathcal{K}_{12}^2[5h_3R^2\dot{\theta}^2 + h_3R^2(\nabla \theta)^2 + 5(\dot{R} + \dot{\psi}_3)^2 + (\nabla R + \nabla \psi_3)^2 + V(R, \psi_3)], \] (33)
where

\[ U(R) = 4R^4 - 4R^3 + 2R^2, \]  

and

\[ V(R, \varphi_j) = U(R) + 2R^2 + R^2b_j^2 + 4R^2\psi_j^2, \quad (j = 1, 2, 3). \]  

Both \( U(R) \) and \( V(R, \psi_j) \) are positive definite monotonically increasing functions and bounded from below by zero. Thus, all terms in Eqs. \((22) - (27)\) are positive definite and then the energy density function \((20)\) is also bounded from below by zero.

As we said before, a special solution with zero rest mass would be possible if \( S_i \)'s, (or equivalently \( K_i \)'s) are zero simultaneously. But, mathematically since there are twelve independent conditions \( S_i = 0 \) as twelve independent coupled PDEs just for five scalar field \( R, \theta \) and \( \psi_j \) \((j = 1, 2, 3)\), therefore, we do not expect them to be satisfied simultaneously in general. However, we build the new extended KG system \((14)\) in such a way that there is a special massless solution exceptionally, for which \( S_i = 0 \), as follows:

\[ R = \frac{1}{1 + r^2}, \quad \theta = \pm \sqrt{2}t, \quad \psi_j = \pm \frac{x^j}{1 + r^2}, \quad (j = 1, 2, 3), \]  

where \( x^1 = x, \ x^2 = y \) and \( x^3 = z \). Now, unlike the previous toy system \((11)\) with the undesirable degenerate solutions \((11)\), the following set would not be anymore a special solution of the new system \((14)\):

\[ R = \frac{1}{1 + (r + \xi)^2}, \quad \theta = \pm \sqrt{2}t, \quad \psi_j = \pm \frac{x^j}{1 + r^2}, \quad (\xi \neq 0). \]  

Moreover, one can simply check that the following sets of functions \( R, \theta \) and \( \psi_j \) \((j = 1, 2, 3)\), are not also the special solutions of the new system \((14)\); meaning that, they are not the solutions of the PDEs \( S_i = 0 \) \((i = 1, 2, \cdots, 12)\) simultaneously:

\[ R = \frac{1}{1 + (r + \xi)^2}, \quad \theta = \pm \sqrt{2}t, \quad \psi_j = \pm \frac{x^j}{1 + (r + \xi)^2}, \quad (\xi \neq 0), \]  

\[ R = \frac{1}{1 + r^2}, \quad \theta = \pm \sqrt{2}t, \quad \psi_j = \pm \frac{(1 + \xi)x^j}{1 + r^2}, \quad (\xi \neq 0), \]  

\[ R = 0, \quad \theta = \pm \sqrt{2}t, \quad \psi_j = \pm \frac{x^j}{1 + r^2}, \]  

\[ R = \frac{1}{1 + r^2}, \quad \theta = \pm \sqrt{2}t, \quad \psi_j = \psi_j = 0, \]  

\[ R = \frac{1}{1 + x^2 + y^2}, \quad \theta = \pm \sqrt{2}t, \quad \psi_j = \pm \frac{x^j}{1 + x^2 + y^2}, \]  

\[ R = \frac{1}{1 + x^2}, \quad \theta = \pm \sqrt{2}t, \quad \psi_j = \pm \frac{x^j}{1 + x^2}. \]
It should be noted that, the new system (14) does not even yield non-localized massless solutions such as (42) and (43). In general, for the static solutions, i.e. \( \theta = \sqrt{2}t, R = R(x, y, z) \) and \( \psi_j = \psi_j(x, y, z) \) \((j = 1, 2, 3)\), conditions \( S_2 = S_3 = 0 \) are satisfied simultaneously similar to the same toy model (1). But, now the static module function \( R(x, y, z) \) must participate in 10 completely different PDEs as follows:

\[
(\nabla R)^2 = 4R^3 - 4R^4, \tag{44}
\]

\[
(\nabla \psi_j)^2 = R^2 - 4R^2\psi_j^2, \quad (j = 1, 2, 3) \tag{45}
\]

\[
\nabla \psi_i \cdot \nabla \psi_j = -4R^2\psi_i\psi_j, \quad (i = 1, j = 2, 3) \quad \text{and} \quad (i = 2, j = 3) \tag{46}
\]

\[
\nabla \psi_j \cdot \nabla R = -2\psi_j(2R - 1)R^2, \quad (j = 1, 2, 3). \tag{47}
\]

These equations are built deliberately in such a way that Eq. (36) to be a special solution. Since there are ten independent PDEs (44)-(47) for four static scalar fields \( R(x, y, z) \) and \( \psi_j(x, y, z) \) \((j = 1, 2, 3)\), it seems highly unlikely that there would be any other special static solutions along with exceptional static solution (36). This argument can be generalized to all (static and dynamic) solutions of the new system (14); meaning that, as we indicated before, since there are twelve independent conditions \( S_i \)'s= 0 just for five real scalar fields \( R, \theta \) and \( \psi_j \) \((j = 1, 2, 3)\), it seems highly unlikely to have other special massless solutions along with (36).

If one is suspicious about the uniqueness of the solution (36), we can introduce more complicated systems via imposing new scalar fields with the new additional restrictive conditions \( S_i \)'s= 0 to finally be sure about the uniqueness of the massless soliton solution. However, finally we can be sure about the possibility of the existence of a system with a single zero rest mass soliton solution in 3+1 dimensions. In sum, it seems right that the special solution (36) is a single special massless solution and we use this name in the rest of the paper.

Note that \( \psi_j = \pm \frac{x^j}{\sqrt{1 + r^2}} \) \((j = 1, 2, 3)\) are not spherically symmetric functions. Therefore, since the Lagrangian density (14) is essentially Poincaré invariant, any spherically rotation of those can be used instead. In other words, instead of \( \psi_j = \pm \frac{x^j}{\sqrt{1 + r^2}} \) \((j = 1, 2, 3)\) in Eq. (36), for example, we can use \( \psi_1 = \pm \frac{\cos(\alpha)x + \sin(\alpha)y}{\sqrt{1 + r^2}}, \psi_2 = \pm \frac{-\sin(\alpha)x + \cos(\alpha)y}{\sqrt{1 + r^2}} \) and \( \psi_3 = \pm \frac{z}{\sqrt{1 + r^2}} \) (i.e. any arbitrary rotation about z-axis), where \( \alpha \) is any arbitrary angle. However, since all different spatial rotations are physically equivalent, we can just consider the same simple functions \( \psi_j = \pm \frac{x^j}{\sqrt{1 + r^2}} \) \((j = 1, 2, 3)\) as the proper candidates for all of them.
According to Eqs. (22)-(33), since all terms in the energy density functional (20) are positive definite, thus this property imposes a strong condition to ensure that the single massless solution (36) is really an energetically stable object; meaning that, any arbitrary deformation above the background of that leads to an increase in the total energy. In other words, for any arbitrary deformations above the background of the single massless soliton solution (36) (i.e. \( R = \frac{1}{1 + \gamma^2} + \delta R, \theta = \sqrt{2t} + \delta \theta \) and \( \psi_j = \pm \frac{x^j}{1 + \gamma^2} + \delta \psi_j \)), at least one of the \( K_i \)'s would be a non-zero functional which leads the energy density functional (20) to be a non-zero positive function with the total energy larger than zero.

Using a relativistic boost, one can obtain easily the moving version of the single massless soliton solution (36). For example, if it moves in the \( x \) direction, we have

\[
R = \frac{1}{1 + \gamma^2(x - vt)^2 + y^2 + z^2}, \quad \theta = k_\mu x^\mu, \quad \psi_1 = \pm \frac{\gamma(x - vt)}{1 + \gamma^2(x - vt)^2 + y^2 + z^2}, \\
\psi_2 = \pm \frac{y}{1 + \gamma^2(x - vt)^2 + y^2 + z^2}, \quad \psi_3 = \pm \frac{z}{1 + \gamma^2(x - vt)^2 + y^2 + z^2},
\]

where \( k_\mu \equiv (\gamma \omega_s, \gamma \omega_s v, 0, 0) \).

**IV. SUMMARY AND CONCLUSION**

In line with Ref. [5], we introduced an extended KG system (14) in the 3 + 1 dimensions which leads to a single massless soliton solution (36). This model (14) is based on introducing twelve independent scalar functionals \( K_i \)'s (\( i = 1, 2, \cdots, 12 \)) of five scalar field \( R, \theta \) and \( \psi_j \) (\( j = 1, 2, 3 \)). It was shown that, the solutions for which all \( K_i \)'s equal zero simultaneously, are the special massless solutions. Nevertheless, considering twelve independent conditions \( K_i = 0 \) for five scalar fields, mathematically is not possible to be satisfied simultaneously in general. However, we built this model in such a way that exceptionally there is just a single massless solution (36) for which \( K_i = 0 \). This system (14) have other solutions, but for which certainly at least one of the \( K_i \)'s is a non-zero functional, hence, this leads the total energy of the other solutions to be non-zero positive values; that is, the energy of the single massless solution (36) would be minimum among the others.

According to Eqs. (22)-(33), all terms in the related energy density functional (20) are positive definite and this property simply guarantees the energetically stability of the special massless soliton solution (36); meaning that, any arbitrary deformation above the back-
ground of the single massless solution (36) leads to increase the total energy.

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