Prediction and Control of Projectile Impact Point using Approximate Statistical Moments

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Abstract—In this paper, trajectory prediction and control design for a desired hit point of a projectile is studied. Projectiles are subject to environment noise such as wind effect and measurement noise. In addition, mathematical models of projectiles contain a large number of important states that should be taken into account for having a realistic prediction. Furthermore, dynamics of projectiles contain nonlinear functions such as monomials and sine functions. To address all these issues we formulate a stochastic model for the projectile. We showed that with a set of transformations projectile dynamics only contains nonlinearities of the form of monomials. In the next step we derived approximate moment dynamics of this system using mean-field approximation. Our method still suffers from size of the system. To address this problem we selected a subset of first- and second-order statistical moments and we showed that they give reliable approximations of the mean and standard deviation of the impact point for a real projectile. Finally we used these selected moments to derive a control law that reduces error to hit a desired point.

I. INTRODUCTION

In a broad sense, a projectile is a ranged weapon that moves through the air in the presence of external forces and endures its motion because of its own inertia. Beginning from Aristotle and later Galileo, projectile trajectory prediction and its impact points have been studied [1], [2]. The goal of these studies is to provide a realistic mathematical model in order to explain the behavior of projectiles. Mathematical modeling allows us to understand the motion of projectiles while requiring a much lower cost than experimental analysis. However, mathematical models become rapidly convoluted when all the parameters of the system are considered (e.g. atmospheric conditions, air density, fuel, etc.). Hence, forming an optimal model, that contains only critical information about a projectile, is essential. Various approximations, methods, and assumptions are used to obtain reliable models [3]–[6]. In this paper, we define a novel model to predict impact points for ballistic targets. Predicting these points is essential in order to avoid hitting constrained areas [7], and to create effective defense systems for those points [8].

In order to reach to a reasonable Impact Point Prediction (IPP), several deterministic and stochastic approaches are studied, such as various types of Kalman Filters [9], Maximum Likelihood Estimator [10], and Stochastic Model Predictive Control [7]. Recent works have also included the wind effect which is crucial to obtaining realistic IPP [11]. However, it is highly inaccessible to receive information about wind instantaneously and hastily because of limited sensor accuracy [12]. Furthermore, the deviations of wind speed and direction are random which makes it unrealizable to predict the future evolution of wind [13]. This noise also shifts the impact points of projectiles which makes classic IPP model erroneous.

In this paper, the effect of wind is modelled as a stochastic process. In the presence of such randomness, statistical moments are reliable tools that give us useful information about the mean and variance of impact points. However, it is not always possible to determine statistical moments for highly nonlinear systems because of the unclosed moment dynamics which means that higher order moments appear in lower ones. To interpret moments in this case, various approximation techniques are used which are called closure techniques (see, e.g., [14]–[18]). Unfortunately, these methods are mainly developed to deal with higher order moments of monomial form. For instance, how to approximate skewness as a function of mean and variance. However, IPP contains nonlinearities of the trigonometric form due to its nature. Here we show that by using Euler formula we can transform a system to new coordinates. The transformed system can be modelled as classic monomial form with a change of variables. In the next step, we derive moment dynamics for the transformed system. These dynamics are free of trigonometric functions, yet they are still unclosed. Hence, we apply mean-field approximation to close them [19]. Mean-field approximation gives reliable results in the limit of weak correlation between states of the system, which is the case here due to the presence of independent noise terms in different states of IPP [20].

The ultimate aim of aerospace studies is to control projectile around its desired trajectory which is named as projectile guidance. To do so, many guidance laws are developed [21], [22]. In between them, proportional navigation guidance (PNG) is the most used and well performed when the target is stationary [23]. PNG is applied to a projectile by changing forces that act on it. Such changes can be obtained by using configurations of canards, wings or tails [12]. Change of configurations is achieved through various control strategies [7], [24], [25]. In this paper, we used feedback control to move along a projectile in a desired trajectory. In the next, we start our analysis by defining projectile dynamic models.

II. EXACT DYNAMIC MODEL OF PROJECTILE

To model a projectile motion we first need to define proper coordinates. Projectile dynamics contain two frames. One
of them is the inertial frame which is often construed as the Earth coordinate frame. The other one is the projectile referenced frame which is the body frame. In these frames, position states are denoted by \((x, y, z)\), and orientation states are denoted by \((\psi, \phi, \theta)\). Note that position \((x, y, z)\) and orientation \((\phi, \theta, \psi)\) are independent parameters, and called six degrees of freedom [26], [27] which can also be seen in Figure 1.

The position states \((x, y, z)\) give information about location of the projectile in the Earth coordinate which is portrayed as

\[
\begin{align*}
\dot{x} &= \left[ C_{\phi}C_{\psi} S_{\phi}S_{\psi}C_{\theta} - S_{\phi}C_{\psi}S_{\theta} + S_{\phi}C_{\psi}C_{\theta} + S_{\phi}S_{\psi} \right] u + \left[ S_{\phi}C_{\psi}S_{\theta} - C_{\phi}C_{\psi} \right] v + \left[ -S_{\phi}S_{\psi} \right] w \\
\dot{y} &= \left[ C_{\phi}S_{\psi} S_{\phi}S_{\psi}C_{\theta} + C_{\phi}C_{\psi}S_{\theta} - C_{\phi}S_{\psi}C_{\theta} - C_{\phi}S_{\psi} \right] u + \left[ C_{\phi}S_{\psi}S_{\theta} + S_{\phi}C_{\psi} \right] v + \left[ S_{\phi}S_{\psi} \right] w \\
\dot{z} &= \left[ \right] u + \left[ \right] v + \left[ \right] w
\end{align*}
\]  

(1)

This equation shows that position states \((x, y, z)\) depend on the body translational velocities \((u, v, w)\). Furthermore \(x, y, z\) depend on angles between projectile and Earth frame origin (Figure 1). The matrix in equation (1) is known as rotation matrix. This matrix is created by the standard aerospace rotation sequence [28]. It allows us to represent the speed of projectile in all directions in the Earth coordinate system.

The orientation states of the projectile, \((\phi, \theta, \psi)\), are known as Euler’s angles, they are called specifically: roll, pitch, and yaw. The inertial angular rates \((p, q, r)\) are the time derivative of Euler angles \((\dot{\phi}, \dot{\theta}, \dot{\psi})\)

\[
\begin{pmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{pmatrix} = \begin{bmatrix} 0 & -s_{\phi}t_{\theta} & c_{\phi}t_{\theta} \\
0 & c_{\phi} & -s_{\phi} \\
0 & s_{\phi}/c_{\theta} & c_{\phi}/c_{\theta}
\end{bmatrix} \begin{pmatrix}
p \\
q \\
r
\end{pmatrix}
\]  

(2)

This transformation is a non-orthogonal transformation because the axes on which we measured Euler angle derivatives are non-orthogonal. For this reason, it is necessary to define \(L_E^B\) which is the matrix that allow us to separate Euler angle derivatives to orthonormal components [29]. Thus, from taking inverse of \(L_E^B\), we can define the dynamics of \((\dot{\phi}, \dot{\theta}, \dot{\psi})\)

\[
\begin{pmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{pmatrix} = \begin{bmatrix} 1 & s_{\phi}t_{\theta} & c_{\phi}t_{\theta} \\
0 & c_{\phi} & -s_{\phi} \\
0 & s_{\phi}/c_{\theta} & c_{\phi}/c_{\theta}
\end{bmatrix} \begin{pmatrix}
p \\
q \\
r
\end{pmatrix}
\]  

(3)

The body velocity components depend on the acting force on the projectile \((X, Y, Z)\); explicitly gravitational force, aerodynamic steady state force, aerodynamic unsteady moment, and canard lifting force [30]. Further, velocity components \((u, v, w)\) are

\[
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix} = \begin{bmatrix} \frac{X}{m} & -r & 0 \\
\frac{Y}{m} & 0 & -p \\
\frac{Z}{m} & q & 0
\end{bmatrix} \begin{pmatrix}
u \\
v \\
w
\end{pmatrix}.
\]  

(4)

Moreover, angular velocity depends on the physical moments that acting on a projectile, \((L, M, N)\). These moments are aerodynamic steady state moment, aerodynamic unsteady moment, aerodynamic Magnus moment, and the moment of the canard [30]. The connection between angular velocity and physical moment is given by

\[
\begin{pmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{pmatrix} = \begin{bmatrix} 0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{bmatrix} \begin{pmatrix}
L \\
M \\
N
\end{pmatrix}.
\]  

(5)

![Fig. 1. Model schematic of a projectile in inertial coordinate frame. A projectile is fully identified by its position in inertial coordinates \((x, y, z)\) and its orientation \((\phi, \theta, \psi)\). By knowing these 6 degrees of freedom one can predict the impact points, i.e. the point where the projectile hits.](image-url)
where $I$ is inertia matrix of a projectile

$$
I = \begin{bmatrix}
I_{XX} & I_{XY} & I_{XZ} \\
I_{YX} & I_{YY} & I_{YZ} \\
I_{ZX} & I_{ZY} & I_{ZZ}
\end{bmatrix}.
$$

The force and the moment terms in equations (4) and (5) are defined as the following: the gravitational force is the force that pulls projectile towards the center of the Earth [31]. Aerodynamic force and physical moment are the effect of pressure and shear stress on the body of projectile [32]. This force can be split into lift force which is opposite to gravitational force, and drag force which is perpendicular to gravitational force and opposite to the lateral velocity of a projectile. After these definitions, the aerodynamic moment can be constructed for any point on the body by using these forces. In addition, canard lifting force and moment are the aerodynamic force and moment over the canard surface. The last phenomena on the projectile is Magnus effect [32], and it is clear that these dynamics are highly nonlinear. Hence different variations of this model are used by considering different assumptions. In the next section, we briefly review the current approximations.

**III. Linear Model and Modified Linear Model of Projectile**

In order to decrease the complexity of projectile dynamics, researchers came up with different assumptions [33], [34]. One famous framework which is built based on series of simplifying assumptions is known as projectile linear theory. This theory allows us to define analytic solution of projectile [35], however it usually generates considerable error on impact point prediction [36]. To overcome this issue, projectile modified linear model is interpreted with relaxing some of the assumptions. Specifically,

- The pitch angle, $\theta$, is not small
  $$\sin(\theta) \neq \theta, \cos(\theta) \neq 1.$$

- Projectile roll rate and pitch angle are not assumed constant
  $$p \neq p_0, \theta \neq \theta_0.$$

After these two changes projectile modified linear model is

$$\dot{x} = \cos(\theta)D,\quad \dot{y} = \cos(\theta)D\psi + \frac{D}{V}\ddot{v},\quad \dot{z} = -D\sin(\theta) + \frac{D\cos(\theta)}{V}\ddot{w},\quad \dot{\psi} = \frac{D}{V}\ddot{\bar{\psi}},\quad \dot{\phi} = \frac{D}{V}\ddot{\bar{\phi}},$$

$$\dot{\theta} = \frac{D}{V}\ddot{\bar{\theta}},\quad \dot{\psi} = \frac{D}{V\cos(\theta)}\ddot{\bar{R}},\quad \dot{V} = -\frac{\pi p D^3 C_{NA}(\bar{u} - \bar{w})}{8m} - \frac{Dg}{V}\sin(\theta),$$

$$\ddot{w} = -\frac{\pi p D^3 C_{NA}(\bar{w} - \bar{u})}{8m} + D\ddot{\bar{q}} + \frac{Dg}{V}\cos(\theta),\quad \ddot{p} = \frac{\pi p D^4 C_{DD}D}{16I_{xx}},$$

$$\ddot{q} = \frac{\pi p D^4 R_{CM}C_{MM}(\bar{w} - \bar{u})}{16V^2 I_{yy}} + \frac{\pi p D^4 C_{MQ}\ddot{q}}{16I_{yy}},\quad \ddot{\ddot{w}} = \frac{\pi p D^4 R_{CM}C_{PA}(\bar{w} - \bar{u})}{16V^2 I_{yy}} + \frac{\pi p D^4 C_{MQ}\ddot{\bar{q}}}{16I_{yy}},$$

$$\ddot{\ddot{p}} = \frac{\pi p D^4 R_{CM}C_{PA}(\bar{w} - \bar{u})}{16V^2 I_{yy}} + \frac{\pi p D^4 C_{MQ}\ddot{\bar{q}}}{16I_{yy}}.$$
A. Statistical Moment Dynamics of PSMLM

The moment dynamics of a general SDE in the form of equation (21) can be derived by using Itô formula

\[
\frac{d\langle x^m \rangle}{dt} = \sum_{i=1}^{n} \left\{ f_i \frac{\partial \langle x^m \rangle}{\partial x_i} \right\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \left( gg^T \right)_{ij} \frac{\partial^2 \langle x^m \rangle}{\partial x_i \partial x_j} \right),
\]

where \( \langle x^m \rangle = [m_1, m_2, \ldots, m_n]^T \) [41]. The sum of \( m_j \) is the order of the moment. If \( f(x,t) \) and \( g(x,t) \) are linear, then moment dynamics can be written compactly as

\[
\frac{d\mu}{dt} = c + A\mu,
\]

where \( \mu \) contains all the moment of the system up to order \( M \)

\[
M \equiv \sum_{j=1}^{n} m_j.
\]

The vector \( c \) and the matrix \( A \) are determined by using \( f(x,t) \) and \( g(x,t) \). To solve the linear equation (27) is effortless because the desired moment order is always combination of higher or the same order moments. However, when \( f(x,t) \) or \( g(x,t) \) are nonlinear, it is not simple to determine all the moments, and (27) needs new configuration. This is

\[
\frac{d\mu}{dt} = c + A\mu + B\tilde{\mu},
\]

where \( \tilde{\mu} \) only contains the higher moments than the desired ones.

This problem is a fundamental problem of statistical moments determination when \( f(x,t) \) or \( g(x,t) \) are nonlinear. To overcome this fundamental problem, we used well-known mean-field closure technique [42], [43]. This approach is convenient for the systems that have computational complexity or high-dimension [44]. The basic idea of this moment closure technique is to define higher order moments as the product of the moments of individuals. For instance

\[
\langle x_i^{m_i} x_j^{m_j} \rangle \approx \langle x_i^{m_i} \rangle \langle x_j^{m_j} \rangle, \quad m_i, m_j \in \{0, 1, 2, \ldots\}.
\]

1) Approximation of mean of PSMLM using mean-field:

It is handy to use (26) to find moments of the system. For example for state \( x \),

\[
\frac{dx}{dt} = \frac{D}{2} \cos(\theta) dt + a_1 dW_1.
\]

This dynamic is nonlinear because of cosine term. If we use Euler's relation,

\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\]

Thus, (31) can be rewritten as

\[
\frac{dx}{dt} = \frac{D}{2} (e^{i\theta} + e^{-i\theta}) dt + dW_1.
\]

To proceed from (33), we define two new states \( \delta^+ = e^{i\theta} \) and \( \delta^- = e^{-i\theta} \)

\[
\frac{d\delta^+}{dt} = \frac{D}{V_0} \delta^+, \quad \frac{d\delta^-}{dt} = -\frac{D}{V_0} \delta^-.
\]

We start our analysis by writing the first statistical moment dynamics,

\[
\frac{d\langle \delta^+ \rangle}{dt} = \frac{D}{V_0} \langle \delta^+ \rangle, \quad \frac{d\langle \delta^- \rangle}{dt} = -\frac{D}{V_0} \langle \delta^- \rangle.
\]

These moment dynamics depend on the second order moment dynamics \( \langle \delta^+ \delta^- \rangle \). In the next step, we add dynamic of these two moment to our system which are

\[
\frac{d\langle \delta^+ \rangle}{dt} = \frac{D}{V_0} \langle \delta^+ \rangle + \pi \rho D^3 V_{MC} C_{MP} \langle \delta^+(\delta^+ - \delta^-) \rangle,
\]

\[
\frac{d\langle \delta^- \rangle}{dt} = -\frac{D}{V_0} \langle \delta^- \rangle + \pi \rho D^3 V_{MQ} C_{NA} \langle \delta^+(\delta^+ - \delta^-) \rangle.
\]

In order to calculate mean, we need to add second order moment dynamics in the next.
show up in the moment dynamics of coordinates. The rest of simulations, respectively.

blue and red ellipse are the standard deviation of x and y coordinates by using mean-field approximation and data simulations, respectively. 

Fig. 3. Our method successfully predicts the standard deviation of the impact points. Every cross represents impact point when a projectile have noise component. The impact points. Hence, here we only consider a few number of second order moments and we approximate the rest as functions of first order moments. 

Namely we add the 14 equations of the form of a sole state, for instance

\[ \frac{d\langle x^2 \rangle}{dt} = D(\langle x\delta^+ \rangle + \langle x\delta^- \rangle) + a_1. \] (41)

Moreover, we include \(\langle x\delta^+ \rangle, \langle x\delta^- \rangle\) and the moments that show up in the moment dynamics of coordinates. The rest of moments are approximated,

\[ \langle \delta^+ \hat{q} \rangle \approx \langle \delta^+ \rangle \langle \hat{q} \rangle, \quad \langle \delta^- \hat{q} \rangle \approx \langle \delta^- \rangle \langle \hat{q} \rangle, \] (42)

\[ \langle \delta^+ \hat{p}v \rangle \approx \langle \delta^+ \rangle \langle \hat{p}v \rangle, \quad \langle \delta^- \hat{p}v \rangle \approx \langle \delta^- \rangle \langle \hat{p}v \rangle, \] (43)

\[ \langle \delta^+ \hat{p}r \rangle \approx \langle \delta^+ \rangle \langle \hat{p}r \rangle, \quad \langle \delta^- \hat{p}r \rangle \approx \langle \delta^- \rangle \langle \hat{p}r \rangle. \] (44)

By applying these closures, we have closed set of first order moment dynamics and selected sub-set of second order moments of PSMLM. Solving these equations give us approximate time trend of the mean and the standard deviation.

V. SIMULATION RESULTS OF PROJECTILE PREDICTION USING MEAN-FIELD

In this paper, fin stabilized projectile is used with following initial conditions for location, \(x = 0\) ft, \(y = 0\) ft, and \(z = 0\) ft which is origin of reference frame. Speed terms are considered as \(\hat{u} = 400\) ft/s, \(\hat{v} = 0\) ft/s, and \(\hat{w} = 0\) ft/s. Angles in reference coordinates system are \(\phi = 2.9\) rad, \(\theta = 0.267\) rad, and \(\psi = -0.007\) rad. Angular velocities are \(\hat{p} = 399.7\) rad/sec, \(\hat{q} = 0.43\) rad/sec, and \(\hat{r} = -1.54\) rad/sec. Physical parameters are selected as air density of \(\rho = 0.00238\) slug/(cu ft), and gravity constant of \(g = 32.174\) ft/s². The rest of physical parameters are chosen as in [35], i.e. reference diameter is 0.343521 ft, weight of the projectile is \(m = 0.0116\) slug. Aerodynamic coefficients are \(C_{X0} = 0.279, C_{dd} = 2.672, C_{lp} = -0.042, C_{na} = 2.329, C_{qpo} = -0.295, C_{mq} = -1.800. \) Distances from center of mass for aerodynamic moment and force are \(R_{mcp} = -0.1657\) ft, and \(R_{mcem} = -0.1677\) ft. Moments of inertia are \(I_{xx} = 2.85 \times 10^{-5}\) slug/ft², \(I_{yy} = 2.72 \times 10^{-5}\) slug/ft² [35]. In addition, wind speed can be taken from weather cast agencies, here we used \(\hat{v}_w = 15\) ft/s and \(\hat{w}_w = 15\) ft/s.

After using these initial conditions and parameters, we simulated PSMLM. The result of this simulation can be seen in Figure 4. As it is clear from this figure, our approximation is able to predict the mean behavior of projectile through the time successfully. In the next, we analyzed the impact points. Figure 3 shows that our method is able to predict standard deviation of impact points around their mean with small error.

| Parameter | Mean | S.D. | Parameter | S.D. | Mean |
|-----------|------|-----|-----------|------|------|
| x         | 0    | 3   | u         | 400  | 2    |
| y         | 0    | 3   | v         | 5    | 0.01 |
| z         | 0    | 0   | w         | 5    | 0.001|
| \(\phi\)  | 2.9  | 1   | \(p\)     | 399.7| 3    |
| \(\theta\)| 0.267| 0.017| \(q\) | 0.43 | 0.01 |
| \(\psi\)  | -0.007 | 0.002 | \(r\) | -1.54 | 0.01 |

Moreover, in reality the initial condition may change because of the randomness in nature such as wind, and topology of the terrain. To address such uncertainty projectile prediction is performed for a distribution of initial conditions [7], [45], [46]. We implemented this source of uncertainty in our numerical simulations by choosing a random initial
condition drawn from a distribution introduced in Table II.

VI. DESIGN OF PROJECTILE CONTROL LAW

In this section, we focus on controlling a projectile. To apply control, four canards are used whose characteristics are exactly the same. These canards allow us to adjust forces and physical moments that act on a projectile [25]. Controlling these forces and moments are generally obtained by manipulating canard angles \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \). Here, we design a feedback control with the subject of following a desired trajectory (e.g. the trajectory with minumum flight time [12]) by using these angles. In the next, we define canard properties and feedback control law with details.

The four symmetric canards are added into projectile. The lift force of canard is [26]

\[
L_i = \frac{1}{2} \rho (u_{ci}^2 + v_{ci}^2 + w_{ci}^2) \frac{\pi D^2}{4} \alpha_i, \quad i = 1, 2, 3, 4, \tag{45}
\]

and the drag force of canard is [26]

\[
D_i = \frac{1}{2} \rho (u_{ci}^2 + v_{ci}^2 + w_{ci}^2) \frac{\pi D^2}{4} \alpha_i, \quad i = 1, 2, 3, 4, \tag{46}
\]

where \( S_i = \frac{\pi D^2}{4} \), and coefficients are defined as [25]

\[
C_L = c_{1\alpha} (M_i \alpha_i), \tag{47}
\]

\[
C_D = c_{d0} (M_i \alpha_i^2) + c_{d2} (M_i \alpha_i^2), \tag{48}
\]

Here \( M_i \) is Mach number on which aerodynamic coefficients depend [25]. Mach number is the ratio between speed of sound and the speed of air vehicles [29]. In addition, \( \alpha_i \) is the angle of attack of each canard which depends on the location of that canard and projectile angle of attack. For each canard, this angle can be defined as

\[
\alpha_i = \lambda_i \pm \tan^{-1} \left( \frac{u_{ci}}{w_{ci}} \right). \tag{49}
\]

We have 4 canards in different locations. Two of them are located parallel to missile body in y-direction. The force terms for these two are

\[
\begin{align*}
X_{ci} &= \begin{cases} 
L_i - \frac{u_{ci}}{\sqrt{u_{ci}^2 + w_{ci}^2}} - D_i - \frac{u_{ci}}{\sqrt{u_{ci}^2 + w_{ci}^2}} & , i = 1, 3. \tag{50} \\
L_i - \frac{u_{ci}}{\sqrt{u_{ci}^2 + w_{ci}^2}} - D_i - \frac{u_{ci}}{\sqrt{u_{ci}^2 + w_{ci}^2}} & , i = 2, 4.
\end{cases}
\end{align*}
\]

The other two are located parallel to missile body in z-direction. Their force terms are

\[
\begin{align*}
X_{ci} &= \begin{cases} 
- L_i - \frac{u_{ci}}{\sqrt{u_{ci}^2 + w_{ci}^2}} - D_i - \frac{u_{ci}}{\sqrt{u_{ci}^2 + w_{ci}^2}} & , i = 1, 3. \tag{51} \\
L_i - \frac{u_{ci}}{\sqrt{u_{ci}^2 + w_{ci}^2}} - D_i - \frac{u_{ci}}{\sqrt{u_{ci}^2 + w_{ci}^2}} & , i = 2, 4.
\end{cases}
\end{align*}
\]

Moreover, physical moments of these 4 canards are defined as

\[
\begin{pmatrix}
L_{ci} \\
M_{ci} \\
N_{ci}
\end{pmatrix}
= \begin{pmatrix}
0 & -r_{zi} & r_{yzi} \\
r_{zi} & 0 & -r_{zxi} \\
-r_{yzi} & r_{zxi} & 0
\end{pmatrix}
\begin{pmatrix}
X_{ci} \\
Y_{ci} \\
Z_{ci}
\end{pmatrix}, \tag{52}
\]

where \( r_{xi}, r_{yi} \) and \( r_{zi} \) are the distance from center of gravity for each canard in each direction. Then, these forces and moments are added into the system dynamic model directly.

To design a controller for a projectile by using feedback control law, a desired trajectory is required. In this paper, we assume that the desired trajectory is pre-defined. The error terms \( e_{1}, e_{2}, e_{3} \) between the desired trajectory and actual trajectory are the difference along \( x, y \) and \( z \) coordinates [25]. According to these error terms, yaw and pitch angle errors are described as

\[
\theta_E = \tan^{-1} \left( \frac{\dot{y}}{\dot{u}} \right) - \tan^{-1} \left( \frac{\hat{y}}{\hat{u}} \right), \tag{53}
\]

\[
\psi_E = - \tan^{-1} \left( \frac{\dot{z}}{\dot{u}} \right) + \tan^{-1} \left( \frac{\hat{z}}{\hat{u}} \right), \tag{54}
\]

where \( \tan^{-1} \left( \frac{\hat{y}}{\hat{u}} \right) \) and \( \tan^{-1} \left( \frac{\hat{z}}{\hat{u}} \right) \) are angle of attack (\( \alpha \)) and side slip angle (\( \beta \)). By using these errors, the control law is

\[
e_\theta = K_\theta \theta_E, \quad e_\psi = K_\psi \psi_E, \tag{55}
\]

where \( K_\theta, K_\psi, K_\theta, K_\psi \) are feedback control gains [25]. Finally, the control is applied by updating the angle of each canard [25].

\[
\lambda_1 = e_\theta - e_\phi, \quad \lambda_2 = e_\psi + e_\phi, \tag{57}
\]

\[
\lambda_3 = e_\theta + e_\phi, \quad \lambda_4 = e_\psi - e_\phi. \tag{58}
\]

The entire control design schematic is shown in Figure 5.
VII. SIMULATION RESULTS FOR IMPLEMENTING THE CONTROLLER

We assumed that the aerodynamic coefficients of canards \((c_{\alpha}, c_{d0}, c_{d2}, c_i)\) are constant. Also, the distance of each canard from center of gravity is described as the following.

| TABLE III. Distance from center of gravity for each canards |
|-----------------------------------------------------------|
| Parameter    | Value (ft) | Parameter | Value | Parameter | Value |
|------------- |------------ |----------- |------- |----------- |-------|
| \(r_{x1}\)  | 0.474      | \(r_{y1}\) | 0.102  | \(r_{z1}\) | 0     |
| \(r_{x2}\)  | 0.474      | \(r_{y2}\) | 0      | \(r_{z2}\) | 0.102 |
| \(r_{x3}\)  | 0.474      | \(r_{y3}\) | -0.102 | \(r_{z3}\) | 0     |
| \(r_{x4}\)  | 0.474      | \(r_{y4}\) | 0      | \(r_{z4}\) | -0.102|

Because of usage of identical canards, all the canard wings areas are equal to \(S_i = 0.02104 \, ft^2\). The velocities of canards are not the same because of angular rates and position of each canard [25]. However, these velocities can be calculated by considering location of canards and angular velocities in each iteration. In control process, we selected the control gain parameters as \(K_p = -2, K_d = -1.5, K_\theta = 0.01, K_\psi = 0.015\). By using these control parameters, change of standard deviation and impact points are shown in Figure 6. Our control law successfully reduced the variance of impact points. Hence, it increases the reliability and accuracy of the missile. However, the error between the simulation results and mean-field approximation is small, but not negligible. One way to reduce this error is to add more moment dynamics to our analysis. Future work will quantify bounds on error of estimation to find the optimal number of dynamics needed to be added to reach to a desired error. Such bounds on approximation error of moments is recently developed for simple dynamic systems [47], [48].

Fig. 6. By controlling a projectile, standard deviation of impact points location reduces considerably. The control law successfully rejected the contribution of the noise and made the projectile to follow the desired path. This results in lower deviation of impact point.

VIII. CONCLUSION

In this paper, we used SDEs to model projectiles under noise effect. Next, we applied Euler’s formula to deduce nonlinearity of the trigonometric to monomial form. Then, we employ mean-field approximation to obtain closed form equations describing mean and standard deviation of the system. Our approximation gives reliable results in predicting time evolution of projectile and characteristics of impact points. Finally, we proposed a control scheme to reduce the errors in impact points.

Furthermore, while the aim of a projectile is to hit the exact target point, it also evades to hit constrained areas. For this purpose, skewness, and kurtosis can be used to avoid hitting those areas by changing the shape of distribution of impact points. Further research will study the higher order moments of projectile such as skewness and kurtosis. Finally in this work we assumed that the controller is built in the projectile. Sometimes we need to give a new control law to the projectile through a transmission channel. Prospect research will merge dynamics of the projectile with random discrete transmission events modelled as renewal transitions to address this requirement [49]–[51].

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