Dissipation in Non-equilibrium Spacetime Thermodynamics

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Abstract. It has been recently realized [1, 2] that the thermodynamical derivation of the Einstein equation introduced by Jacobson [3] needs a generalization to a non-equilibrium thermodynamical setting. Here we show that the non-equilibrium character of spacetime thermodynamics — both in GR as well as in $f(R)$ gravity — is generally related to non-local heat fluxes associated with the purely gravitational/internal degrees of freedom of the theory. In particular, we show that the allowed gravitational degrees of freedom can be fixed by the kinematics of the local spacetime causal structure, through the specific Equivalence Principle formulation. In this sense, the thermodynamical description seems to go beyond Einstein’s theory as an intrinsic property of gravitation.

1. Introduction
The basic idea of a spacetime thermodynamics [3] bases on the relation between the thermal character of quantum fields vacuum, as perceived by an accelerated Rindler observer, and the stationarity properties of the respective Killing horizon. By making use of the Equivalence Principle [4], the notion of Rindler frame can be used at a local level as an experimental setting for studying the local spacetime dynamics in terms of the way the variation of the spacetime geometry follows from the energy variation of the matter fields [1, 2, 3, 5].

Actually, for a given general spacetime, an approximate local Rindler frame can always be introduced by invoking local flatness (Local Lorentz Invariance) around a point $p$. By further assuming that the ground state of the fields living in the original spacetime is locally approximated by the Minkowski vacuum, the vacuum state associated with the set of Rindler wedge observers, can be interpreted as an approximate thermal state, with a temperature $T \approx T_{\text{un}} \sqrt{-g_{00}}$, where $T_{\text{un}} = \hbar/2\pi \chi$ is the the Unruh temperature [6], while $\sqrt{-g_{00}} = \kappa \chi$ is the gravitational Doppler factor defined with the Rindler metric $ds^2 = -\kappa^2 \chi^2 d\eta^2 + d\chi^2 + dy^2 + dz^2$. The temperature $T$ stays constant throughout the Rindler wedge and it is well defined on the horizon. Therefore, the thermal character of the Rindler vacuum is effectively extended from the single Rindler observer to the whole wedge.

Moreover, since the accelerated observer in the wedge can only access information on spacelike slices bounded by the bifurcation plane and the vacuum fluctuations between the inside and the outside of the wedge are correlated, she/he will perceive an entanglement entropy. This entropy will scale with the area of the local boundary and diverge with the density of field states in the UV limit. However, with the introduction of an UV cut-off one can make this entropy proportional to the area, that is $S = \alpha A$, where the proportionality factor $\alpha$ can \textit{a priori} depend...
on the nature of the quantum fields as well as be some complicate function of the position in spacetime \cite{7, 8}. Together with the temperature $T$, this notion of entropy allows to consider the local Rindler wedge with its Killing horizon as an analogue of an equilibrium canonical ensemble (Gibbs state) bounded by a diathermic wall.

2. Spacetime Thermodynamics

The notion thermal equilibrium for the wedge is ultimately associated to the stationarity of the Killing horizon. In this sense the equilibrium is realized at the horizon bifurcation surface at $p$, where $dS = \delta(\alpha A) = 0$ by definition.

Moving away from the bifurcation surface along the approximate Killing horizon, the spacetime becomes dynamical: the presence of matter distorts the local Rindler causal structure and perturbs the thermal state. Thereby, one can describe the slight perturbation of the causal structure by looking at the variation of the affine quantities of the null geodesic comprising the horizon, provided that some condition to relate the spacetime geometry deformation to the variation of the fields energy content can be found.

This condition can be borrowed from classical equilibrium thermodynamics. For a slightly perturbed Gibbs state at temperature $T$, variations of entropy and internal energy are related by the Clausius relation, $dS = -\delta Q/T$, where the change in the mean energy $\delta Q$ is due to the fluxes into/from an unobservable region of spacetime. For a system undergoing a quasi-static process of energy exchange with the surroundings, the Clausius relation is nothing but a local thermodynamical equilibrium condition.

Now, since the thermal Rindler wedge behaves like a Gibbs state, one can actually use the Clausius relation to relate the horizon entropy and the boost energy across the Rindler horizon, where $T$ is the wedge temperature \cite{3}. As a second fundamental postulate, one can assume a universal entropy density $\alpha$ per unit horizon area $\delta A$, such that

$$dS = \alpha \delta A,$$  \hspace{1cm} (1)

then implicitly considering a constant UV cut-off for the fields. With this choice, the changes of the entanglement entropy of the fields in the wedge can be effectively described in terms of geometrical variations of the horizon cross-section\cite{3}.

Let us highlight here that the assumption $\alpha = \text{constant}$, made in the GR derivation of \cite{1, 3}, can be indeed recast as an explicit choice of a specific formulation of the Equivalence principle. As said, generally the UV cut-off $\alpha$ is fixed at the quantum gravity scale. This can be identified as the scale at which the gravitational action is of the order of the quantum of action $\hbar$. For GR this is the standard Planck length $l_p = \sqrt{G\hbar/c^3}$ and hence it is directly related to the Newton constant. However, for a general scalar-tensor theory, i.e. a theory compatible just with the Einstein Equivalence principle (EEP) \cite{4}, $G$ is promoted to a spacetime field. As a consequence of this, one should expect that the cut-off will be generically position dependent. In this sense, assuming $dS = \alpha \delta A$ is equivalent to assume the strong formulation of the Equivalence Principle (SEP) \cite{4}, hence effectively to allow for the only two SEP-compatible gravity theories: GR and Nordström gravity \cite{9}.

Given the assumption in (1), a quantitative expression for the system entropy variation is obtained just by applying the definition for the change of the horizon area in terms of the expansion rate of the null geodesics comprising it, that is

$$\delta A = \int_{H} \tilde{\epsilon} \theta \, d\lambda,$$  \hspace{1cm} (2)

with $\tilde{\epsilon}$ indicating the 2-surface area element of the horizon cross-section, while $\lambda$ is the affine parameter for the null geodesic congruence comprising the horizon.
Moving away from the equilibrium surface at $\lambda = 0$, along the null congruence, the infinitesimal evolution of $\theta$ is given by a linear expansion around its equilibrium value at $p$, up to the first order in $\lambda$, say $\theta \approx \theta_p + \lambda \frac{d\theta}{d\lambda} p + O(\lambda^2)$. This first order coefficient will be determined as usual by the Raychaudhuri equation, $\frac{d\theta}{d\lambda} = -1/2 \theta^2 - \|\sigma\|^2 - R_{ab} \ell^a \ell^b$, where $\|\sigma\|^2$ stands for the squared congruence shear $\sigma^{ab} \sigma_{ab}$, while $\ell^a$ is the null vector tangent to the null congruence comprising the horizon. In this way, the entropy variation, is directly related to the purely geometrical degrees of freedom of the horizon congruence, up to $O(\lambda^2)$

$$dS = \alpha \int_H \bar{\epsilon} d\lambda \left[ \theta - \lambda \left( \frac{1}{2} \theta^2 + \|\sigma\|^2 + R_{ab} \ell^a \ell^b \right) \right]_p.$$  \hspace{1cm} (3)

On the other hand, the mean energy variation of the thermal system, given by the boosted energy current flux of matter, is then described by the heat flux across the horizon as

$$\delta Q = \int_H T_{ab} \chi^a d\Sigma^b = \int_H \bar{\epsilon} d\lambda \left( -\lambda \kappa \right) T_{ab} \ell^a \ell^b,$$  \hspace{1cm} (4)

where $T_{ab}$ is the matter stress energy tensor, while the volume element is given by $d\Sigma^b = \bar{\epsilon} d\lambda \ell^b$. In the last passage the heat is expressed with respect to the null vector $\ell^a$, given the relation

$$\lambda = -e^{-\nu r}$$

between the $\lambda$ affine and the $v$ Killing parameter along the approximated Killing horizon, from which $\chi^a = (d\lambda/dv) \ell^a$.

At this stage, asking for the Clausius relation to hold for all null vectors $\ell^a$, one can equate the $O(\lambda)$ integrands in (3) and (4). At the zeroth order in $\lambda$, the value of heat flux at $p$ is zero, hence one necessarily gets $\theta_p = 0$. Then, to the first order, one finds the constitutive relation

$$\frac{2\pi}{\hbar} T_{ab} \ell^a \ell^b = \left( \|\sigma\|^2 + R_{ab} \ell^a \ell^b \right)_p.$$  \hspace{1cm} (5)

Now, if one further postulates that $\sigma_p = 0$, as originally done in [3], one is left with

$$2\pi/\hbar \alpha T_{ab} = R_{ab} + \Phi g_{ab},$$

where $\Phi$ is an undetermined integration function. Then, by assuming the local energy conservation, that is $\nabla^b T_{ab} = 0$, applying the divergence operator on both sides of (5), and using the contracted Bianchi identity $\nabla^b R_{ab} = \frac{1}{2} \nabla_a R$, one eventually gets

$$\Phi = -\frac{1}{2} R - \Lambda,$$

hence $2\pi/\hbar \alpha T_{ab} = R_{ab} - \frac{1}{2} R g_{ab} - \Lambda g_{ab}$, where $\Lambda$ is some arbitrary integration constant. Once the condition $\alpha = 1/4\hbar G$ is imposed, one can easily recognize the familiar Einstein equations and, noticeably, the latter condition implies that the entropy density of the local Rindler horizon is the same as the one of a black hole.

On the other hand, by allowing for some shear at $p$, that is for $\sigma_p \neq 0$, the Einstein equation is no more recovered [1, 2]. This argument forces the thermodynamical derivation to be recasted in a non-equilibrium setting, where $dS > \delta Q/T$. In this new context, the Clausius law is replaced by the entropy balance law, $dS = \delta Q/T + d_i S$ and the extra shear term in (5) is then associated with the internal entropy production $d_i S$, generated by the system out of equilibrium. The internal entropy contribution, $O(\lambda)$, has the form

$$d_i S = -\frac{4\pi \eta}{\hbar} \int_H \bar{\epsilon} \lambda d\lambda \|\sigma\|^2_p$$  \hspace{1cm} (6)

and it is interpreted as an internal entropy production term [10] due to some internal spacetime viscosity, with $\eta = \hbar \alpha/4\pi$. 


3. Some Remarks

The geometrical properties of the horizon null congruence and consequently the value of affine expansion and shear at \( p \) are given with the local spacetime curvature in that point. In order to define an equilibrium surface one just need to require a suitably smooth curvature for the spacetime patch around \( p \), without fixing \textit{a priori} the values of affine expansion and shear. This means that the local Rindler frame is not sensitive to the exact value of the affine expansion and shear at \( p \). Therefore, setting \( \sigma_p = 0 \) is actually an unjustified arbitrary choice in [3].

Thereby, the assumption that the affine congruence orthogonal to the local spacetime patch around \( p \) has zero expansion and shear is equivalent to require that the chosen local patch (and hence its associated null congruence) is less general than the one allowed by the assumed entropy-area relation (or alternatively by the SEP).

On the one hand, allowing for non zero affine shear at \( p \) give rise to further interesting clues. In some way, the shear contribution in (5) brings into the entropy balance process a new degree of freedom, which is not fixed by the Ricci tensor and so has nothing to do with the local matter energy sources. Actually, the surface shear is generally related to the Weyl tensor and usually associated with the distortion on the geodesics congruence due to a gravitational perturbation.

In fact, this argument opens an issue about the absence so far of any role for gravitational fluxes in the system energy perturbation mechanism. Due to their non-local nature, the gravitational energy fluxes cannot be taken into account with a proper stress-energy tensor (SET). However, allowing for non-local terms, as the one in \( ||\sigma||^2 \) in (5), seems at odds with neglecting the role of these non-local energy contributions.

In this sense, such a spacetime viscosity seems naturally related to the distorsive effect of a gravitational flux, to be intended as a local curvature perturbation which is independent from the Einstein equation. This suggest that gravitational energy fluxes can possibly play a role into the total entropy balance of the system without entering into the Einstein equilibrium relation.

4. Internal entropy in non-equilibrium thermodynamics

In classical non-equilibrium thermodynamics, the rate of change of the entropy is generally written as the sum of two contributions: \( dS = d_eS + d_iS \), where \( d_eS \) is the rate of entropy exchange with the surroundings, while \( d_iS \) comes from the process occurring inside the system and is a non-negative quantity, accordingly to the second law of thermodynamics. In particular, \( d_iS \) is zero for reversible (quasi-static) processes and positive for irreversible processes.

However, as some irreversible processes come into play, also the Clausius relation must be generalized to the expression

\[
    dS = \frac{\delta Q}{T} + \delta N, \tag{7}
\]

where \( \delta Q \) is generally referred to as \textit{compensated heat}, that is the heat transferred between the system and its surroundings, while \( \delta N \), the so called \textit{uncompensated heat}, indicates the amount of entropy associated with the heat which is intrinsic to the system when it undergoes an irreversible process. The above definition generalizes the notion of local equilibrium by extending the entropy balance to the unbalanced contributions related to the irreversible processes, like dissipation (see e.g. [11]).

The generalized Clausius relation (7) is helpful in order to clarify the nature of the equilibrium and non-equilibrium contributions defining our system entropy. In fact, by using the definition of the non-equilibrium entropy given above, we can write \( d_eS + d_iS = \delta Q/T + \delta N \) and identify the external and internal entropy contributions in terms of the compensated and uncompensated heat, respectively \( d_eS = \delta Q/T \), at the reversible level, and \( d_iS = \delta N \), at the irreversible level.

With the notion of generalized Clausius relation, the argument described in Sec.2 acquires a clearer interpretation. Indeed, the extra contribution (6) introduced by the non vanishing horizon shear is an internal entropy production term allowed by the most general choice of the
null congruence associated with the starting local spacetime patch compatible with the area-entropy relation for GR (that we linked to the choice of the EP formulation). Therefore, it has to be seen as a by-product of the presence of internal/purely gravitational degrees of freedom of the theory which can be responsible for irreversible dissipative processes.

A possible way to capture non-equilibrium features of the thermal system is to use the analogy between the congruence bundle comprising the horizon and a classical fluid [2]. Indeed, given the Price and Thorne membrane approach [12, 13], the local Rindler horizon can be effectively approximated by a timelike hypersurface living just inside the true Rindler wedge, i.e. a timelike stretched horizon.

With a 2+1 decomposition, the timelike congruence comprising the stretched horizon is formally equivalent to a 2+1 continuous medium (fluid) living on the spacelike two dimensional cross section of the hypersurface and moving with velocity $\mathbf{v}^i$, defined by the unit timelike vector tangent to the hypersurface [2, 12].

This analogy can be used to get an insight on the possible expression for the horizon internal entropy. Indeed, for a slightly anisotropic fluid (viscous medium), the internal entropy is defined by

$$d_s = -\frac{1}{T} P_{ij}^{\text{vis}} \mathbf{V}^{ij} = -\frac{1}{T} (P^{\text{vis}} \nabla^\rho \mathbf{v}_p + P_{ij}^{\text{vis}} \mathbf{V}^{ij}) = \frac{\zeta}{T} (\nabla^\rho \mathbf{v}_p)^2 + \frac{2\eta}{T} \| \mathbf{V} \|^2,$$

where $P_{ij}^{\text{vis}} = p^{\text{vis}} \delta_{ij} + \bar{P}_{ij}^{\text{vis}}$ is the viscous pressure acting on the medium, respectively decomposed in its bulk and trace-less components and the linear constitutive equation $P^{\text{vis}} = -\zeta \nabla^\rho \mathbf{v}_p$ and $\bar{P}_{ij}^{\text{vis}} = -2\eta \mathbf{V}_{ij}$ have been used, with $\zeta$ and $\eta$ respectively the medium bulk and shear viscosity [11]. Now, from a purely kinematical point of view, we can consider the velocity gradient of the medium, $\nabla_{\mathbf{v}_i}$, as the equivalent of the deformation tensor field of the horizon congruence (extrinsic curvature), in the Killing frame, $\hat{B}_{ab}$. In both cases, the tensor can be decomposed into trace and symmetric trace-free part, $\mathbf{V}_{ij} = 1/3 (\nabla^\rho \mathbf{v}_p) \delta_{ij} + \tilde{V}_{ij}$, where $\tilde{V}_{ij}$ represents the symmetric component of $\nabla_{\mathbf{v}_i}$, $\delta_{ij}$ is the identity tensor and $\tilde{V}_{ij} = \frac{1}{2} (\nabla_{\mathbf{v}_j} + \nabla_{\mathbf{v}_i})$ is the deviatoric traceless tensor. Similarly, $\hat{B}_{ab} = 1/2 \hat{\theta} h_{ab} + \hat{\sigma}_{ab}$, where $\hat{\theta}$ and $\hat{\sigma}_{ab}$ are the usual scalar expansion and shear of the null congruence.

Therefore, given the stretched horizon approximation and by associating the bulk term ($\nabla^\rho \mathbf{v}_p$) to $\hat{\theta}$ and the deviatoric traceless tensor $\tilde{V}_{ij}$ to $\hat{\sigma}_{ab}$, one can expect a dissipative internal entropy production term for the horizon congruence, of the form

$$(d_s S)^{\text{vis}} = \frac{1}{T} \int_{\mathcal{H}} \tilde{c} \, dv \, \zeta \, \hat{\theta}^2 + 2\eta \| \hat{\sigma} \|^2.$$  

The above expression identifies the congruence kinematical quantities which are responsible for the irreversible viscous transfer of momentum into the system, and for the consequent production of internal entropy.

5. Non-equilibrium Spacetime Thermodynamics: GR

We can now reproduce the thermodynamical derivation of the Einstein equations with a non-equilibrium irreversible thermodynamics setting in a quite general way, by starting from a generic spacelike 2-surface patch comprising $p$, with non vanishing $\theta_p$ and $\sigma_p$.

As for the previous derivation, the strong equivalence principle will allow us to use the entropy area relation as in (1). However, in the new setting, the Clausius relation will be generalized to the expression (7), where $\delta N$ will now encode all the information about both microscopic properties and irreversible perturbations of the system.

The new argument starts from the same definition of entropy, given in (3). Since we are now dealing with a non-equilibrium setting, we expect that the entropy can be expressed as a sum
of two different contributions $dS = d_e S + d_i S$. Moreover, we can separate the reversible and irreversible levels, as previously argued,

$$d_e S = \alpha \int_H \bar{\epsilon} d\lambda (\theta - \lambda R_{ab} \ell^a \ell^b)_p = \delta Q/T \quad \text{at the reversible level,} 
$$

$$d_i S = -\alpha \int_H \bar{\epsilon} d\lambda \lambda \left( \frac{1}{2} \theta^2 + \|\sigma\|^2 \right)_p = \delta N \quad \text{at the irreversible level.}$$

From the first expression above, one can see that the heat flux is still defined by the expression in (4). Even for the non-equilibrium setting the reversible heat will vanish at $\lambda = 0$. Thereby, at the zero order in $\lambda$, one deduces again $\theta_p = 0$, while the relation $R_{ab} + \Phi g_{ab} = (2\pi/h\alpha)T_{ab}$, is recovered for all null vectors $\ell^a$ at the first order. Following the previous discussion this leads to the Einstein equations if $\alpha = (4hG)^{-1}$.

On the other hand, for the irreversible level, we identify

$$\delta N = d_i S = -\alpha \int_H \bar{\epsilon} d\lambda \lambda \|\sigma\|^2_p.$$ 

(12)

This again identifies the shear contribution as an internal entropy term, associating it to some irreversible dissipative process occurring in the thermal Rindler wedge, in accordance with (6).

To get a physical interpretation of $\delta N$ with respect to the thermal properties of the Rindler wedge, it is helpful to express equation (12) in terms of the Killing horizon parameters. In the new frame,

$$\delta N = d_i S = \frac{\alpha}{\kappa} \int_H \bar{\epsilon} dv \|\hat{\sigma}\|^2_p \geq 0,$$ 

(13)

in accordance with the second law of thermodynamics.

By a comparison with expression (9), one can actually interpret the expression in (13) as the standard entropy production term for a fluid with shear viscosity $\eta$, defined by $2\eta/T = \alpha/\kappa$, that is $\eta = h\alpha/4\pi$, in agreement with the universal relation for the viscosity to entropy density ratio found in the AdS/CFT context [14].

While the previous discussion shows that the spacetime thermodynamics nicely fits into a non-equilibrium setting, we now want to take this arguments a step further and ask whether the expression in (13) can be effectively related to some gravitational energy flux.

The expression for the uncompensated heat given in (13) quantifies the energy of the system which is effectively dissipated by the viscous process,

$$T \delta N = \frac{\alpha}{\kappa} \int_H \bar{\epsilon} dv \|\hat{\sigma}\|^2_p = \frac{1}{8\pi G} \int_H \bar{\epsilon} dv \|\hat{\sigma}\|^2_p,$$ 

(14)

which, by substituting $\alpha = (4hG)^{-1}$, from the reversible sector of the thermodynamical approach, coincides with the Hartle-Hawking formula for the tidal heating of a classical black hole [15, 16, 17, 18, 19].

This is a striking result as it defines the internal entropy production as a purely gravitational effect. Indeed, it can be associated with the work done on the horizon by the perturbative tidal field which is described by the electric part of the Weyl curvature tensor. The horizon viscosity implies that such a work will be converted into internal heat. Hence, the presence of the internal entropy term can be directly related to the process of dissipation via gravitational/internal degrees of freedom. In this sense, the irreversible sector contains the information about the possible activation/propagation of such degrees of freedom of the theory.
6. Non-equilibrium Spacetime Thermodynamics: $f(R)$

In the previous derivation, a crucial assumption lies in the validity of the SEP which allowed to consider the entropy density $\alpha$ as a constant. One might wonder what are the consequences of relaxing such an assumption in favor of the less restrictive Einstein Equivalence Principle (EEP). In this case, one might generically expect that the entropy density is promoted to a spacetime function (basically because the EEP implies a spacetime dependent Newton constant). However, in the definition for the entanglement entropy of the Rindler wedge, this implies a possibly very complicated spacetime dependence for the UV cut-off.

Following the treatment of [1], we consider here the specific case of $f(R)$ gravity, which is known to be equivalent to a single field scalar-tensor theory (more precisely a Brans-Dicke theory with $\omega = 0$ and a specific potential for the scalar field [20]). In this case, the UV cut-off is known to be proportional to some function of the curvature $f'(R) \equiv f'_R$ (where the prime indicates the derivative with respect to $R$), playing the role of the inverse of the gravitational coupling. In this case, the area entropy relation is known to be given by $S = \alpha f(R)$, where $\alpha$ is still a constant (albeit a priori different from the one considered in the previous section).

It is easy to see that in this case the entropy variation along the null congruence will be $dS/\lambda = \alpha (d\Phi/d\lambda + \Phi d\tilde{\epsilon}/d\lambda)$, where, by definition $\tilde{\epsilon}^{-1} d\tilde{\epsilon}/d\lambda = \theta$. Consequently, the entropy change along the horizon will read [1]

$$dS = \alpha \int d\lambda (\tilde{\epsilon} + \epsilon \theta), \quad (15)$$

then acquiring, with respect to the previous argument, an extra contribution $\dot{f}$ coupled to the dynamics of the scalar function $f$. (Here the dot stays for differentiation with respect to $\lambda$.)

For this reason, in order to set the instantaneous stationarity condition at $p$, that is $dS = 0$, the affine expansion is no more a good dynamical variable. In this sense, it is helpful to define the quantity $\tilde{\theta} \equiv (\theta f + \dot{f})$ as a sort of effective expansion for the congruence. Consequently, the equilibrium surface for the system will be fixed by the condition $\tilde{\theta}_p = 0$, that is $\tilde{\theta}_p = -\dot{f}/f$,

where $\tilde{\epsilon}^a = f'(R) \ell^a R_{ab}$ is generally nonzero. In particular, this actually provides an example of local Rindler frame equilibrium surface, for which $\theta_p$ is always non-vanishing, apart from the trivial case where $f$ is constant, for which the theory will be equivalent to GR. From the fluid analogy, we could already expect that the presence of the non-vanishing affine expansion would produce a non-equilibrium contribution to the system entropy.

In order to get a quantitative expression for the entropy change in the neighborhood of $p$, one again can consider an infinitesimal deviation of the entropy from its equilibrium value. Let us then Taylor expand the integrand in (15) around $p$ up to the first order in $\lambda$, that is $\tilde{\theta} = \tilde{\theta}_p + \lambda (\dot{f} - f^{-1} \dot{f}^2 + \Theta \tilde{\theta}) + \mathcal{O}(\lambda^2)$. Therefore, one can use the Raychaudhuri equation and the geodesic equation $\ell^a \ell^b_{\ ;a} = 0$, to obtain the $\mathcal{O}(\lambda)$ expression for the entropy change

$$dS = \alpha \int_H \tilde{\epsilon} d\lambda \left[(f_{ab} - f R_{ab}) \ell^a \ell^b - 3/2 f \theta^2 - f \|\sigma\|^2\right]_p, \quad (16)$$

where relation $\tilde{\theta}_p = 0$ is used to substitute $f^{-2} \dot{f}^2 = \theta^2$ at $p$. Now, keeping the expression in (4) for the heat flux, one can finally reproduce the same approach used in Sec.5.

At the reversible level, the generalized Clausius relation gives the constitutive equation $f R_{ab} - f_{ab} + \Psi_{gab} = (2\pi/\hbar \alpha) T_{ab}$ where $\Psi$ is an undetermined function. With the same argument given in Sec.2, one then requires the conservation of the matter stress-energy tensor and use the contracted Bianchi identity to write the commutator of the covariant derivative as $2v^{[c}_{;a}] R_{ab} = R_{abcd} v^d$. In this way, one finds

$$(f R_{ab} - f_{ab})^{\alpha} = \left(1/2 f - \Box f\right), \quad \text{and thereby} \quad \Psi = \left(\Box f - 1/2 f\right). \quad (17)$$
Eventually, equation (17), together with the constitutive equation found leads to the field equations of \( f(R) \) gravity

\[
 f R_{ab} - f_{ab} + \left( \Box f - \frac{1}{2} f \right) g_{ab} = \frac{2\pi}{\hbar\alpha} T_{ab}. \tag{18}
\]

with the identification \( \alpha = (4\hbar G)^{-1} \). In [1], the same result was obtained starting from the entropy balance relation, assuming \( \sigma_p = 0 \), and then identifying the extra entropy term in \( \theta \) in the second line of (16) with a suitable internal entropy term.

Indeed, following the previous discussion, the above term is expected (together with a shear dependent term) as an unavoidable contribution related to the irreversible sector of the generalized Clausius relation (7)

\[
\delta N = - \int_H \tilde{\epsilon} \, d\lambda \, \lambda \left( \alpha f \right) \left[ \frac{3}{2} \theta^2 + \| \sigma \|^2 \right], \tag{19}
\]

which, as explained in Sec.4, identifies the internal entropy production terms of the system.

As expected, the internal entropy in (19) now shows contributions both from scalar and tensorial degrees of freedom. Indeed, by using the same argument as in the GR case, we again have a natural interpretation for the expression in (19) as the dissipative function of the system.

The shear squared contribution is equivalent to the one found for GR, with a shear viscosity coefficient which now takes a factor \( f \), \( \eta = \frac{\hbar\alpha}{4\pi} \), as a consequence of the UV cut-off chosen for the area entropy relation.

On the other hand, the internal entropy contribution due to the scalar degree of freedom is now given by

\[
d_i S_\theta = - \int_H \tilde{\epsilon} \, d\lambda \, \lambda \left( \alpha f \right) \left[ \frac{3}{2} \theta^2 - \| \sigma \|^2 \right]. \tag{20}
\]

By making use of the kinematical analogy described in Sec.4 and by expressing the above equation in the Killing frame, one is naturally led to define the bulk viscosity \( \zeta \) as \( \zeta/T = \frac{3}{2} \hbar\alpha f/2\kappa \), that is \( \zeta = \frac{3}{4} \hbar\alpha f/4\pi \), as already found in [1].

Furthermore, in order to give a physical interpretation to (20), one can use the equivalence between \( f(R) \) and scalar-tensor gravity, thereby interpreting \( f \) as an effective massive dilaton.

The action for \( f(R) \) gravity is given by

\[
S = \frac{\hbar\alpha}{4\pi} \int d^4x \sqrt{-g} \, f(R) + S_{\text{mat}}. \tag{21}
\]

By introducing an auxiliary field \( \varphi \equiv f(R) \) and assuming \( f''(R) \neq 0 \) for all \( R \), one can take \( V(\varphi) \) as the Legendre transform of \( f(R) \) so that \( R = V'(\varphi) \), thereby rewriting the expression in (21) as

\[
S = \frac{\hbar\alpha}{4\pi} \int d^4x \sqrt{-g} \left[ \varphi R + V(\varphi) \right] + S_{\text{mat}}. \tag{22}
\]

The Euler-Lagrange equations, in the Jordan frame, take the form

\[
\varphi \left( R_{ab} - \frac{1}{2} g_{ab} R \right) + (g_{ab} \nabla^c - \nabla_a \nabla_b) \varphi + \frac{1}{2} g_{ab} V(\varphi) = \frac{2\pi}{\hbar\alpha} T_{ab}, \tag{23}
\]

equivalent to field equations given in (18).

In this frame, by using the relation \( \tilde{\theta}_p = 0 \), one can express the dissipated energy coupled to the bulk and shear viscosity in (20), in terms of the auxiliary scalar field \( \varphi \)

\[
T \delta N = - \int_H \tilde{\epsilon} \, d\lambda \, \lambda (\alpha \varphi) T \left[ \frac{3}{2} \varphi^{-2} \varphi^2 + \| \sigma \|^2 \right]. \tag{24}
\]
The interpretation of the term related to the shear is straightforward as it is clearly the generalization to a scalar-tensor theory of the tidal heating already obtained for the GR case. More problematic is the interpretation of the bulk viscosity (purely scalar) contribution. In this direction, as a first step, one can look for the effective field source terms which drive the local deformation of the null horizon congruence. Again, we consider the effective expansion $\tilde{\theta}$ as the suitable quantity to describe the scalar perturbations of the horizon given that $\tilde{\theta} = 0$ is the condition for equilibrium.

Let us then use $\tilde{\theta}$ as the effective expansion rate for the horizon bundle at $p$. Starting from this equation, where now $f$ is substituted by $\varphi$, then using the Raychaudhuri equation and the Jordan frame field equation given in (23), one gets

$$\dot{\tilde{\theta}}_p = -\frac{\theta_p^2}{2} - ||\sigma||_p^2 - (2\pi/h\alpha) \varphi^{-1} T_{ab} \ell^a \ell^b - \varphi^{-2} \nabla_a \varphi \nabla_b \varphi \ell^a \ell^b.$$  

(25)

Now, since we are interested only in the scalar contributions to $\dot{\tilde{\theta}}_p$, we can set $T_{ab} = 0$ and $||\sigma||_p^2 = 0$. Then, by using the equilibrium relation in the new frame $\theta = -\varphi^{-1}\dot{\varphi}$, we are left with $\dot{\tilde{\theta}}_p = -3/2 \varphi^{-2} \nabla_a \varphi \nabla_b \varphi \ell^a \ell^b = -3/2 \varphi^{-2} \dot{\varphi}^2$. This equation identifies the quantity $(3/2) \varphi^{-2} \dot{\varphi}^2$ as what one might define as the gravitational energy flux associated with the solely scalar field degrees of freedom.

In conclusion, we can now provide a clean interpretation of the viscous terms in (24) as those representing the thermal system internal energy loss due to both scalar and tensor gravitational energy fluxes through the horizon. In particular, by moving to the Killing frame, (24) can be rewritten as

$$T \delta N = \frac{1}{8\pi G} \int_H \bar{\xi} dV \left[ \frac{3}{2} \varphi^{-1} \dot{\varphi}^2 + \varphi ||\sigma||^2 \right]_p,$$

(26)

where we have set $\alpha = (4\hbar G)^{-1}$ as required by the equations of motion (18). The striking similarity of the above expression with the energy loss rate due to the gravitational radiation in scalar-tensor gravity (see e.g. equation (10.135) of [21]) further reinforces the above suggested interpretation. Furthermore, we now see that also for the case of $f(R)$ gravity the relation in (6), the expression for $\zeta$ and the above interpretation of (26) provide a correlation between the transmission coefficient of the gravitational energy through the horizon and the horizon congruence viscosity.

7. Discussion

In non-equilibrium spacetime thermodynamics the viscous dissipative effects appear to be naturally associated with purely gravitational energy fluxes. Their association to the irreversible/dissipative sector of the theory strongly suggests an interpretation of their nature as, non-local, internal heat flows associated with the internal spacetime degrees of freedom and clarifies why in GR a local, background independent, description of gravitational waves is precluded.

However, a different issue is the interpretation of the internal entropy production terms with respect to a particular spacetime solution. While the association between internal entropy and allowed form of gravitational fluxes seems quite clear, it might seem however puzzling that the arbitrariness in the choice of initial spacetime patch around $p$ allows for non-zero shear and expansion of the null congruence (and hence for internal entropy production terms) even, for example, if one imagine to have performed the local Rindler wedge construction in a Minkowski spacetime. In fact, the thermodynamical approach is providing us just with the constitutive equations of the thermal system associated with local Rindler wedge, not of the spacetime in...
which the latter is constructed. The arbitrariness of the choice of the starting local spacetime patch implies that such equations will at most characterize the structure of the gravitational theory selected by the entropy-area relation (the EP formulation). In this sense they will not be associated with physical fluxes or curvatures of the spacetime as a whole.

Another important aspect of this approach highlighted by the work done has to do with the role that the different Equivalence Principle formulations play in selecting the possible gravitational dynamics. The Strong Equivalence Principle implies, for a generic choice of the local spacetime patch, $\theta_p = 0$, thereby leading to the equations of motion of GR (with irreversible level only corresponding to tensorial gravitational fluxes). The Einstein Equivalence Principle, does not fix either $\theta$ nor $\sigma$, thereby allowing for a generalized theory of gravity. For the simple case of $f(R)$, we showed how the scalar degree of freedom in fact produces a dissipative contribution which is actually associated with some purely gravitational scalar energy flux through the system boundary (and which is not part of the SET of the scalar field).

The presence of an unavoidable purely scalar gravitational energy flux seems to indicate that GR and scalar-tensor are truly separated theories. However, the very same thermodynamical approach, might also suggest to look at such gravitational theories as different regimes of some more general effective description of gravity. This seems an intriguing possibility worth further investigation as it might lead to a unified framework which associates different gravitational theories with different hydrodynamical regimes of the analogue flow associated with the horizon null congruence.

For what regards the spacetime viscosity, the quantities $\eta$ and $\zeta$ are found to be always related to the UV cut-off scale of the theory through the entropy density. This might suggest an underlying microscopic interpretation of gravity\footnote{P. Candelas, D. W. Sciama, Phys. Rev. Lett. 38, 1372 (1977).} along the ideas of induced gravity (see e.g. [23] for a related discussion).

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