# Maximum eccentricity energy of globe graph, bistar graph and some graph related to bistar graph

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## Abstract

The graphs considered in this article are undirected, finite and simple graphs. In this article we have proved that maximum eccentricity energy of bistar graph $B_{m,n}$ is $4\sqrt{n^3}+4$. Also we have investigated maximum eccentricity energy of some graphs related to bistar graph and globe graph.

## Keywords

Maximum Eccentricity Matrix, Maximum Eccentricity Eigen values, Maximum Eccentricity Energy of a Graph, Bistar Graph.

## AMS Subject Classification

05C50, 05C76.

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## 1. Introduction

The energy of a graph introduced by Gutman[6] is an important concept of spectral graph theory which links Organic Chemistry to linear algebra of Mathematics. Generally graph’s energy is summation of absolute values of eigenvalues of the adjacency matrix. Similar energies got from the eigenvalues of various graphs are considered in recent times. Several other authors also investigated energy of graph [1,2,7,10,11,13].

In a graph $G$, the distance between any two vertices $u$ and $v$ is denoted as $d(u,v)$ and it is define as the length of the minimum path between $u$ and $v$ in graph $G$, if there is no path between $u$ and $v$ in $G$ then $d(u,v)$ is defined as $\infty$. It is useful to note that in a connected graph distance between any two vertices is always finite provided graph is finite. For a vertex $v$ of $G$, the Eccentricity of a vertex $v$ is denoted as $e(v)$ and is defined as $e(v) = \max \{d(u,v) \mid u \in V(G) \}$. In [7] Naji and Soner introduced the concept of Maximum Eccentricity matrix $M_e(G)$ of connected graph $G$ and it is defined as,

$$e_{ij} = \begin{cases} \max \{e(v_i), e(v_j)\} & \text{if } v_i v_j \in E(G) \\ 0 & \text{Otherwise} \end{cases}$$

The Maximum Eccentricity eigenvalues are the eigenvalues of matrix $M_e(G)$. In this article we have considered only finite, simple and undirected graphs. To make this article self-contained, it is useful to recall some definitions from graph theory.

**Definition 1.1 ([7]):** The Maximum Eccentricity energy of graph $G$ is denoted as $EM_e(G)$ and it is defined by $EM_e(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of $M_e(G)$, Where $M_e(G)$ is the Maximum Eccentricity matrix of graph $G$.

**Definition 1.2 ([3]):** A graph $G$ is said to be bipartite if the vertex set $V$ of $G$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that $V_1 \cup V_2 = V$ and for each edge has one end vertex is in $V_1$ and other is in $V_2$.

**Definition 1.3 ([3]):** A complete bipartite graph is a bipartite graph in which all the vertices of $V_1$ are adjacent with all the vertices of $V_2$. If $|V_1| = m$ and $|V_2| = n$ respectively then the corresponding complete bipartite graph is denoted as $K_{m,n}$.

**Definition 1.4 ([3]):** A complete bipartite graph $K_{1,n}$ is known as star graph and the vertex of degree $n$ is known as the apex vertex.
Definition 1.5 ([5]). A Bistar graph is the graph obtained by joining the centre (apex) vertices of two copies of $K_1,n$ by an edge and it is denoted by $B_{n,n}$. The vertex set of $B_{n,n}$ is $V(B_{n,n}) = \{v_1,v_2,\ldots,v_n,v,u,u_1,u_2,\ldots,u_n\}$, where $v,u$ are apex vertices and $v_1,v_2,\ldots,v_n,u_1,u_2,\ldots,u_n$ are pendant vertices. The edge set of $B_{n,n}$ is $E(B_{n,n}) = \{v_1v_2,\ldots,v_nv,u_1u_2,\ldots,u_1u_n\}$.

Definition 1.6 ([11]). Let $G$ be a simple connected graph. Then the square graph of $G$ is denoted by $G^2$ and defined as the graph with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 in $G$.

Definition 1.7 ([5]). A shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$ and joining each vertex $v$ in to $G'$ the neighbours of the corresponding vertex $v'$ in $G''$.

Definition 1.8 ([9]). A globe graph $G(n)$ is a graph obtained from two isolated vertex are joined by $n$ paths of length two.

In theorem 2.1, we have shown that Maximum Eccentricity energy of Bistar graph $B_{n,n}$ is $4\sqrt{9n+1}$, for $n \in N, n \neq 1$. We also provided supportive example in Example 2.2 and in that example we have proved that $M_e(B_3,3) = 4\sqrt{46}$. We also investigate the Maximum Eccentricity of square of Bistar graph in Theorem 2.3 and we have shown that its Maximum Eccentricity energy is $1 + \sqrt{64n+1}$. In Theorem 2.5 we proved that Maximum Eccentricity energy of shadow graph of Bistar graph $D_2(B_{n,n})$ is $8\sqrt{9n+1}$ and in Theorem 2.7 we proved that the Maximum Eccentricity energy of Globe graph is $4\sqrt{2n}$.

### 2. Main Results

**Theorem 2.1.** Let $n \in N, n \neq 1$. Then $EM_e(B_{n,n}) = 4\sqrt{9n+1}$, where $EM_e(B_{n,n})$ is the Maximum Eccentricity energy of graph $B_{n,n}$.

**Proof.** Let $V(B_{n,n}) = \{v_1,v_2,\ldots,v_n,v,u,u_1,u_2,\ldots,u_n\}$. Note that $B_{n,n}$ is graph with $2n+2$ vertices and $2n+1$ edges as shown in the following Figure 1. Observe that the Maximum Eccentricity matrix $M_e(B_{n,n})$ of $B_{n,n}$ is given by

$$
\begin{bmatrix}
1 & \cdots & 1 & \cdots & 1 & \cdots & 0 \\
1 & \cdots & 1 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & 1 & \cdots & 1 & \cdots & 0 \\
1 & \cdots & 1 & \cdots & 1 & \cdots & 0 \\
1 & \cdots & 1 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
$$

Note that the characteristic polynomial of matrix $M_e(B_{n,n})$ is

$$\lambda^{2(n+1)} - 4(2n^3 + 5n^2 + 4n + 1)\lambda^{2n} - 16n^2(n + 1)^4\lambda^{2(n-1)}.$$  

So, eigenvalues of $M_e(B_{n,n})$ are $0,0,\ldots,0(2(n-1)$ times), $1 + \sqrt{9n+1}, 1 - \sqrt{9n+1}, -1 + \sqrt{9n+1}$ and $-1 - \sqrt{9n+1}$. Hence, Maximum Eccentricity energy of

$$B_{n,n} = EM_e(B_{n,n}) = 0 + \frac{1}{2}[1 + \sqrt{9n+1} + |1 - \sqrt{9n+1}| + \sqrt{9n+1} - 1 + \sqrt{9n+1} - 1 + \sqrt{9n+1} - 1 + \sqrt{9n+1} - 1]$$

$$= 4\sqrt{9n+1}$$

**Example 2.2.** Maximum Eccentricity energy of Bistar graph $B_{5,5} = 4\sqrt{46}$.

**Proof.** From the following matrix we have the characteristic polynomial of matrix $M_e(B_{5,5})$ is $\lambda^{12} - 1584\lambda^{10} - 518400\lambda^8$.

So, Maximum Eccentricity eigenvalues of $B_{5,5}$ are $0,0,\ldots,0(8$ times), $(\sqrt{46}+1), (\sqrt{46}-1), (\sqrt{46}+1)$ and $(\sqrt{46}-1)$. Hence, Maximum Eccentricity energy of $B_{5,5} = 0 + |\sqrt{46}+1| + |\sqrt{46}-1| + |\sqrt{46}+1| + |\sqrt{46}-1| = 4\sqrt{46}$. The Maximum Eccentricity matrix of $B_{5,5}$ is given by

![Image 1](image1.png)

![Image 2](image2.png)
Note that the characteristic polynomial of $B_{n,n}$ is $1 + \sqrt{64n+1}$ and $\frac{1}{2}(1 - \sqrt{64n+1})$. Therefore, Maximum Eccentricity energy of $B_{n,n}$ is given by $EM_e(B_{n,n}) = 0 + |\lambda_1| = \frac{1}{2}(1 + \sqrt{64n+1}) + \frac{1}{2}(1 - \sqrt{64n+1})$.

**Example 2.4.** Maximum Eccentricity energy of square of Bistar graph $B_{3,5}^2 = 1 + \sqrt{32}\sqrt{1}$.

**Theorem 2.3.** Let $n \in N, n \neq 1$. Then $EM_e(B_{n,n}^2) = 1 + \sqrt{64n+1}$, where $EM_e(B_{n,n}^2)$ is the Maximum Eccentricity energy of graph $B_{n,n}$.

*Proof.* Let $V(B_{n,n}^2) = \{v_1, v_2, \ldots, v_n, u, u_1, u_2, \ldots, u_n\}$.

Note that $B_{n,n}^2$ is graph with $2n + 2$ vertices and $4n + 1$ edges as shown in the following Figure 3. Observe that the Maximum Eccentricity matrix $M_e(B_{n,n}^2)$ of $B_{n,n}^2$ is given by

\[
M_e(B_{3,5}) =
\begin{pmatrix}
 v_1 & v_2 & v_3 & v_4 & v_5 & u & u_1 & u_2 & u_3 & u_4 & u_5 \\
 v_1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
v & 3 & 3 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
u & 0 & 0 & 0 & 0 & 2 & 3 & 3 & 3 & 3 & 3 \\
u_1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
u_2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
u_3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
u_4 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
u_5 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Note that the characteristic polynomial of $M_e(B_{3,5}^2)$ is $\lambda^{12} - 2541\lambda^{10} - 26620\lambda^9$. Therefore, the Maximum Eccentricity eigenvalues of $B_{3,5}^2$ are $0, 0, \ldots, 0$ (9 times), $-1, -1, -1, -1$. Hence, Maximum Eccentricity energy of $B_{3,5}^2$ is $EM_e(B_{3,5}^2) = 0 + |\lambda_1| + |\lambda_2| + \cdots + |\lambda_9| = 1 + \sqrt{32}\sqrt{1}$.

**Theorem 2.5.** Let $n \in N, n \neq 1$. Then $EM_e(D(B_{n,n})) = 8(\sqrt{9n+1})$, where $EM_e(D(B_{n,n}))$ is the Maximum Eccentricity energy of graph $D(B_{n,n})$. 
Proof. Let $V(D_2(B_{n,n})) = \{v_1, v_2, \ldots, v_n, v, u, u_1, u_2, \ldots, u_n\}$.

Note that $D_2(B_{n,n})$ is graph with $4(n+1)$ vertices and $4(2n+1)$ edges as shown in the above Figure 5.

Observe that the Maximum Eccentricity matrix $M_e(D_2(B_{n,n}))$ of $D_2(B_{n,n})$ is given by

$$
M_e(D_2(B_{n,n})) =

\begin{array}{cccccccccccc}
  & v_1 & \cdots & v_n & v_1' & \cdots & v_n' & v & v' & u & u' & u_1 & \cdots & u_n & u_1' & \cdots & u_n' \\
 v_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 3 & 3 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 v_n & 0 & \cdots & 0 & 0 & \cdots & 0 & 3 & 3 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 v_1' & 0 & \cdots & 0 & 0 & \cdots & 0 & 3 & 3 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 v_n' & 0 & \cdots & 0 & 0 & \cdots & 0 & 3 & 3 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 v & 3 & \cdots & 3 & 3 & \cdots & 3 & 0 & 0 & 2 & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 v' & 3 & \cdots & 3 & 3 & \cdots & 3 & 0 & 0 & 2 & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 u & 0 & \cdots & 0 & 0 & \cdots & 0 & 2 & 2 & 0 & 0 & 3 & \cdots & 3 & 3 & \cdots & 3 \\
 u' & 0 & \cdots & 0 & 0 & \cdots & 0 & 2 & 2 & 0 & 0 & 3 & \cdots & 3 & 3 & \cdots & 3 \\
u_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 3 & 3 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 u_1' & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 3 & 3 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 u_n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 3 & 3 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
u_n' & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 3 & 3 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 u_n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 3 & 3 & \cdots & 0 & 0 & \cdots & 0 \\
u_n' & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 3 & \cdots & 0 & 0 & \cdots & 0 \\
\end{array}
$$

Hence, Maximum Eccentricity energy of $D_2(B_{n,n})$ is given by

$$
EM_e(D_2(B_{n,n})) = 0 + |2(1 + \sqrt{9n+1})| + |2(1 - \sqrt{9n+1})| + |2(-1 + \sqrt{9n+1})| + |2(-1 - \sqrt{9n+1})| = 2(1 + \sqrt{9n+1}) - 1 + \sqrt{9n+1} - 1 + \sqrt{9n+1} + 1 + \sqrt{9n+1} = 8(\sqrt{9n+1}).
$$

Example 2.6. Maximum Eccentricity energy of shadow graph of Bistar graph $D_2(B_{5,5}) = 8(\sqrt{46})$.

Proof. The Maximum Eccentricity matrix of $D_2(B_{5,5})$ is given by

Note that the characteristic polynomial of $M_e(D_2(B_{n,n}))$ is $\lambda^{4(n+1)} - 16(2n^5 + 5n^2 + 4n + 1)\lambda^{4n+2} - 64n^2(n+1)^4\lambda^{4n}$

So, Maximum Eccentricity eigenvalues of $M_e(D_2(B_{n,n}))$ are

0, 0, \ldots, 0 [4n times], $2(1 + \sqrt{9n+1}), 2(1 - \sqrt{9n+1}), 2(-1 + \sqrt{9n+1})$ and $2(-1 - \sqrt{9n+1})$.

Example 2.6. Maximum Eccentricity energy of shadow graph of Bistar graph $D_2(B_{5,5}) = 8(\sqrt{46})$.

Proof. The Maximum Eccentricity matrix of $D_2(B_{5,5})$ is given by
The characteristic polynomial of $M(D_2(B_{5,5}))$ is

$$\lambda^{24} - 6336\lambda^{22} + 8294400\lambda^{20}.$$ 

Hence, the Maximum Eccentricity eigenvalue of $D_2(B_{5,5})$ are $0,0,\ldots,0$ (20 times), $2(1 + \sqrt{46}), 2(1 - \sqrt{46}), 2(-1 + \sqrt{46})$ and $2(-1 - \sqrt{46})$.

Therefore, Maximum Eccentricity energy

$$EM_e(D_2(B_{5,5})) = 0 + |2(1 + \sqrt{46})| + |2(1 - \sqrt{46})|$$

$$+ |2(-1 + \sqrt{46})| + |2(-1 - \sqrt{46})|$$

$$= 2(1 + \sqrt{46} - 1 + \sqrt{46} - 1 + \sqrt{46} + 1 + \sqrt{46})$$

$$= 8(\sqrt{46}).$$

Therefore, Maximum Eccentricity energy of globe graph, bistar graph and some graph related to bistar graph — 1525/1526

Theorem 2.7. Let $n \in \mathbb{N}$, $n \neq 1$. Then $EM_e(Gl(n)) = 4\sqrt{2n}$,

where $EM_e(Gl(n))$ is the Maximum Eccentricity energy of graph $Gl(n)$.

Proof. Let $V(Gl(n)) = \{v,v_1,v_2,\ldots,v_n,u\}$. Note that $Gl(n)$ is graph with $n + 2$ vertices and $2n$ edges as shown in the following Figure 7. Observe that the Maximum Eccentricity matrix $M_e(Gl(n))$ of $Gl(n)$ is given by

$$M_e(Gl(n)) = \begin{bmatrix}
0 & 2 & \cdots & v_n & u \\
v & 0 & 2 & \cdots & u \\
v_1 & 2 & 0 & \cdots & 2 \\
v_2 & \cdots & \cdots & \cdots & \cdots \\
v_n & 2 & 0 & \cdots & 2 \\
u & 0 & 2 & \cdots & 0 \\
\end{bmatrix}$$

The characteristic polynomial of $M_e(D_2(B_{5,5}))$ is $\lambda^{24} - 6336\lambda^{22} + 8294400\lambda^{20}$. Hence, the Maximum Eccentricity eigenvalue of $D_2(B_{5,5})$ are $0,0,\ldots,0$ (20 times), $2(1 + \sqrt{46}), 2(1 - \sqrt{46}), 2(-1 + \sqrt{46})$ and $2(-1 - \sqrt{46})$.

Therefore, Maximum Eccentricity energy

$$EM_e(D_2(B_{5,5})) = 0 + |2(1 + \sqrt{46})| + |2(1 - \sqrt{46})|$$

$$+ |2(-1 + \sqrt{46})| + |2(-1 - \sqrt{46})|$$

$$= 2(1 + \sqrt{46} - 1 + \sqrt{46} - 1 + \sqrt{46} + 1 + \sqrt{46})$$

$$= 8(\sqrt{46}).$$

\(\square\)
Note that the characteristic polynomial of matrix $M_e(Gl(n))$ is $\lambda^7 - 8n\lambda^5$. So, eigenvalues of $M_e(Gl(n))$ are $0, 0, \ldots, 0 (n \text{ times }), 2\sqrt{2n}$ and $-2\sqrt{2n}$.

Hence, Maximum Eccentricity energy of $Gl(n) = EM_e(Gl(n)) = 0 + |2\sqrt{2n}| + |-2\sqrt{2n}| = 4\sqrt{2n}$.

Example 2.8. Maximum Eccentricity energy of Globe graph $Gl(5) = 4(\sqrt{10})$.

Proof. The Maximum Eccentricity matrix of $Gl(5)$ is given by

$$
\begin{array}{ccccccc}
& v & v_1 & v_2 & v_3 & v_4 & v_5 & u \\
v & 0 & 2 & 2 & 2 & 2 & 2 & 0 \\
v_1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\
v_2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\
v_3 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\
v_4 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\
v_5 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\
u & 0 & 2 & 2 & 2 & 2 & 0 & 0
\end{array}
$$

Note that the characteristic polynomial of matrix $M_e(Gl(5))$ is $\lambda^7 - 40\lambda^5$. So, Maximum Eccentricity eigenvalues of $Gl(5)$ are $0, 0, \ldots, 0 (5 \text{ times }), 2\sqrt{10}$, and $-2\sqrt{10}$.

Hence, Maximum Eccentricity energy of $Gl(5) = EM_e(Gl(5)) = 0 + |2\sqrt{10}| + |-2\sqrt{10}| = 4\sqrt{10}$.

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