A NOTE ON STABILITY AND CAUCHY TIME FUNCTIONS

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Abstract. Since the solution of the so-called folk problems of smoothability, there has been a special interest in the properties of classical time and volume functions of spacetimes. Here we supply some information that complements the one provided in arXiv:1108.5120v3 and arXiv:1301.2909v1, and discuss some results. As an illustration, we show that any Cauchy temporal function for a globally hyperbolic spacetime remains Cauchy temporal for close metrics—which, in particular, implies stability for global hyperbolicity.

1. Introduction

At the beginning of relativistic causality theory, the so-called folk problems of smoothability remained as open questions [11], [7], [12], Proposition 6.6.8, [20], [18], p. 1155], [9], [17, Ch. 14], [1, p.65]. In a series of paper [3, 4, 19, 5, 6], Bernal and Sánchez solved the most common of these questions by using a new approach to the problem of smoothability. This approach and the grounds of causality theory were reviewed by Minguzzi and Sánchez [15] and, since then, many papers have obtained further developments. In particular, applications to isometric embeddability in Lorentz-Minkowski were found by Müller and Sánchez [16] by refining these new smoothability techniques, a surprising new approach to time functions was introduced by Fathi and Siconolfi [10], and quite a few of properties on causality were obtained by Minguzzi and his coworkers (see for example [13, 14, 2, 8]) by using a combination of several techniques —previous classical techniques, results on smoothability and new elements which included utilities, curiously developed in Economics rather than in General Relativity. However, the connection of some results obtained with different techniques are not always clear. In this note, we discuss some connections of [2, 8] with previous references and prove a result on stability of Cauchy temporal functions.

2. Comments on stability of global hyperbolicity

The problem of stability of global hyperbolicity is discussed in [2]. There are two different versions of this article, one published in JMP and the other one posted at arxiv. In the latter, there is a discussion in relation with previous techniques, which can be summarized as follows.

In a well-known paper, Geroch [11] proved the equivalence between global hyperbolicity, and the existence of a Cauchy hypersurface, the latter obtained as a level of a (continuous) time function. He also gave an argument which showed stability of such a Cauchy hypersurface, and, thus, of global hyperbolicity. However, this argument contained a non-well justified step, as the Cauchy hypersurfaces that he could use (by using his own result) were essentially topological. However, after [3],
one knows that such a hypersurface can be taken smooth and spacelike and, after [4], that these two properties can be also retained by all the leaves of a foliation of Cauchy hypersurfaces. So, in the arxiv version of [2] (see arXiv: 1108.5120) one can read in the paragraph above its Section 4:

Geroch’s way of replacing the hypersurfaces could be amended by using the smoothability results contained in [the references [3, 4]]. One would have to make this replacement for all the Cauchy hypersurfaces appearing in Geroch’s proof showing that this step does not really affect the argument. Of course the proof would be no more self contained and would lose simplicity. Our proof is instead more topological and does not even use the notion of time function not to mention the topological splitting of globally hyperbolic spacetimes.

Our aim here is to emphasize that, at any case, the result in [4] gives directly a stronger consequence of stability, with independence of Geroch’s arguments (as these were introduced in order to handle non-spacelike Cauchy hypersurfaces). Namely, in [4] it is proven the existence of a temporal function (i.e., a smooth time function with timelike gradient) that is Cauchy (i.e., its levels are Cauchy hypersurfaces, now necessarily smooth and spacelike) in any globally hyperbolic spacetime. The existence of such a function allows to give directly the following result that, in particular, implies the stability of global hyperbolicity in [2, Th. 2.6]:

**Theorem 1.** If $t$ is a Cauchy temporal (resp. temporal) function for a globally hyperbolic (resp. stably causal) metric $g$, then there exists $g'$ with wider timecones, $g \prec g'$, such that $t$ is a Cauchy temporal (resp. temporal) function for $g'$ and, then, for any $g'' \prec g'$.

In particular, if $S$ is a spacelike Cauchy hypersurface for $g$ then it is also a spacelike Cauchy hypersurface for all such $g''$.

The last assertion does not hold if $S$ is only a topological Cauchy hypersurface, which underlies in the difficulties of Geroch’s proof. We give a proof of this result in the next section, and make further comments in the last one.

### 3. Proof of Theorem 1

For the proof of Theorem 1, we will follow the ideas in [4]. It is worth pointing out that another proof of stability has been obtained recently in [10].

The two propositions previous to our proof may have their own interest, as they show that the Cauchy temporal character of such a prescribed function $t$, remains when the timecones are slightly opened in different kind of regions of the spacetime. With this aim, the notion of $t$-perturbation is introduced below.

Consider a globally hyperbolic spacetime $(M, g)$. By [4], it can be regarded as a product $M = \mathbb{R} \times S$ and the metric can be written as $g = -\beta dt^2 + g_t$ where $g_t$ is a Riemannian metric on each slice $S_t = \{t\} \times S, t \in \mathbb{R}$ and $t$ is a Cauchy temporal function, i.e., all the slices $S_t$ are (smooth, spacelike) Cauchy hypersurfaces. As we are considering only conformally invariant properties, we can substitute $g$ with the conformal metric $g/\beta$ and redefine consistently the metrics $g_t$ so that, with no loss of generality, we can start with a globally hyperbolic spacetime type

$$(M = \mathbb{R} \times S, g = -dt^2 + g_t)$$
where \( t \) is Cauchy temporal. We will consider also a complete Riemannian metric \( g^M_0 \) on \( S \) and the corresponding Riemannian one \( g_R = dt^2 + g^M_0 \) on \( \mathbb{R} \times S \), with associated (complete) distance \( d_R \). For any two non-empty subsets \( A, B \subset M \) we denote \( J(A, B) = J^+(A) \cap J^-(B) \) and \( d_R(A, B) \) the (Hausdorff) \( d_R \)-distance between the two sets.

For any function \( \alpha \geq 0 \) on \( M \), consider the metric:
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g_\alpha = -(1 + \alpha)dt^2 + g_t.
\] Whenever \( \alpha > 0 \), \( g_\alpha \) has its timelike cones wider than the causal cones of \( g \) (\( g \prec g_\alpha \)) and \( t \) remains as a temporal function for \( g_\alpha \), as its level sets \( S_t \) are spacelike hypersurfaces (this underlies in the proof that the existence of a temporal function implies stable causality). So, our final aim is just to prove that some \( \alpha > 0 \) can be chosen small enough so that the slices \( S_t \) remain Cauchy.

We will add the subscript \( \alpha \) to denote elements computed with \( g_\alpha \) (for example, \( J^+_\alpha(p) \) is the \( g_\alpha \)-causal future of \( p \)).

**Definition 2.** Consider a globally hyperbolic spacetime with a prescribed Cauchy temporal function \( t \). Fix an open neighborhood \( U_0 \subset M \) bounded by two \( t \)-levels, i.e. \( U_0 \subset J(S_{t_-, S_{t_+}}) \), for some \( t_-, t_+ \) chosen in an optimal way (no \( t' \) \( \geq t_- \) nor \( t'_+ \leq t_+ \) satisfies \( U_0 \subset J(S_{t'_-, S_{t'_+}}) \) if one of the inequalities is strict). Let \( C_0 \subset M \) be a closed subset included in \( U_0 \), and choose \( \epsilon_0 > 0 \).

A \( t \)-perturbation of \( g \) with wider cones on \( C_0 \), support in \( U_0 \) and \( d_R \)-distance smaller than \( \epsilon_0 \) is any smooth function \( \alpha \geq 0 \) satisfying:

(i) \( \alpha > 0 \) on \( C_0 \) (\( g \prec g_\alpha \) on \( C_0 \)),

(ii) \( \alpha \equiv 0 \) outside \( U_0 \) (\( g \equiv g_\alpha \) on \( M \setminus U_0 \)),

(iii) Both, \( d_R(J(p, S_{t'_-}), J_\alpha(p, S_{t'_-})) < \epsilon_0 \) and \( d_R(J(S_{t'_-}, p), J_\alpha(S_{t'_-}, p)) < \epsilon_0 \), for all \( p \in J(S_{t_-}, S_{t_+}), t'_- \leq t_+, t'_+ \geq t_- \), and

(iv) \( t \) is Cauchy temporal for \( g_\alpha \).

Notice that if \( \alpha \) is such a perturbation then so is \( \alpha/N \) for any \( N > 1 \).

**Proposition 3.** (Existence of a \( t \) perturbation for a compact set). For any compact \( K_0 \subset M \), any neighborhood \( U_0 \) of \( K_0 \) bounded by two \( t \)-levels and any \( \epsilon_0 > 0 \), there exists a \( t \)-perturbation of \( g \) as \( g_\alpha \) in Defn. 2 with \( C_0 = K_0 \).

**Proof.** With no loss of generality, we can assume that the closure \( \overline{U_0} \) is compact. Any \( \alpha \geq 0 \) with support in \( U_0 \) and \( \alpha(K_0) > 0 \) preserves the Cauchy temporal character of \( t \). In fact, we know that \( t \) is temporal for \( g_\alpha \) and, so, no inextensible causal curve \( \gamma \) can remain partially future or past imprisoned in \( \overline{U_0} \). Therefore, \( \gamma \) will cross all the slices \( S_t \). To check that Property (iii) can be achieved by \( \alpha_m := \alpha/m, m \in \mathbb{N} \) for some large \( m \), notice that \( K := J_\alpha(\overline{U_0}, S_{t_+}) \cup J_\alpha(S_{t_-}, \overline{U_0}) \) is compact. Assuming with no loss of generality that, say, there exist \( q_m \) (necessarily in \( K \)) \( r_m \in J(q_m, S_{t_+}) \), \( r'_m \in J_\alpha(q_m, S_{t_+}) \) such that \( d_R(r_m, r'_m) = d_R(J(q_m, S_{t_+}), J_\alpha(q_m, S_{t_+})) \geq \epsilon_0 \) for all \( m \) then (up to subsequences) \( \{q_m\} \) converges to some \( q \in K \). Choosing \( q_0 \in J^-(q) \), as \( J_\alpha(q_0, S_{t_+}) \supset J(q_0, S_{t_+}) \supset J_\alpha(q_0, S_{t_+}) \supset J(q_0, S_{t_+}) \), then \( \{r_m\}, \{r'_m\} \) also converge to some \( r, r' \) with \( d_R(r, r') \geq \epsilon_0 \) and \( r' \notin J(q, S_{t_+}) \). Now, the \( g_{\alpha_m} \)-causal curves \( \gamma_m \) from \( q_m \) to \( r'_m \) will have a limit curve \( \gamma \) that connects \( q \) and \( r' \) (notice that all \( \gamma_m \) are \( \alpha \)-causal).

1 Alternatively, notice that, given a compact set (namely, \( J_\alpha(q_0, S_{t_+}) \)), the set of all its non-empty compact subsets, becomes a compact metric space with the Hausdorff distance.
But, clearly, $\gamma$ cannot leave $J^+(q)$ at, say, some first point $\gamma(s_0)$, as the causality in a normal $g$-neighborhood of $\gamma(s_0)$ is determined by its lightlike geodesics, and these vary continuously with the metric. □

**Proposition 4.** (Existence of a $t$ perturbation for a strip). For any strip $J(S_{t_1}, S_{t_2})$, $t_1 < t_2$, any neighborhood $U_0$ of the strip bounded by two $t$-levels and any $\epsilon_0 > 0$, there exists a $t$-perturbation of $g$ as in Defn. 2 with $C_0 = J(S_{t_1}, S_{t_2})$.

**Proof.** We can assume that $U_0 = (t_-, t_+)$ × $S$ (otherwise, replace the obtained $\alpha$ by a smaller one with equal value on the strip and support contained in $U_0$; this can be done by means of a partition of the unity) and $S$ is not compact (as, otherwise, the result would follow directly from Prop. 3). Choose a point $x_0 \in S$ and consider the closed $\frac{\epsilon_0}{2}$-balls $B_m \subset S$ of center $x_0$ and radius $m \in \mathbb{N}$ and the open ones $\bar{B}_m$. Put $K_m = [t_1, t_2] \times B_m$, $U_m = (t_-, t_+) \times \bar{B}_{m+1}$. Consider the function $\alpha_1$ obtained by applying Prop. 3 to the compact set $K_2$, its neighborhood $U_2$ and $\epsilon_0/2$.

Assuming inductively that $\alpha_m$ has been defined, put $\sigma(1) = 2$ and let $\sigma(m + 1)$ be the first integer greater than $2\sigma(m)$ such that $J_{\alpha_m}(\overline{U}_{\sigma(m)}, S_{t_1}) \cup J_{\alpha_m}(S_{t_1}, \overline{U}_{\sigma(m)})$ is included in $\overline{U}_{\sigma(m+1)-1}$. Now, obtain $\alpha_{m+1}$ by applying Prop. 3 to the metric $g_{\alpha_m}$, the compact set $K_{\sigma(m+1)} \setminus K_{\sigma(m)}$, its neighborhood $U_{\sigma(m+1)} \setminus \overline{U}_{\sigma(m+2)}$ and $\epsilon_0/2^m$. The sequence $\{\alpha_m\}$ converges uniformly on compact subsets (in fact, on any compact subset, all its elements are equal for large $m$) and, so, it converges to a function $\alpha$. The metric $g_{\alpha}$ satisfies all the required properties by construction—namely, when an inextensible $t$-parameterized future-directed $g_{\alpha}$-causal curve $\gamma$ comes into the strip at $\gamma(t_1)$, then all $J_{\alpha_m}(\gamma(t_1), S_{t_1})$ remains in a compact subset at a $d_{\sigma}$-distance smaller than $\epsilon_0$ from $J(\gamma(t_1), S_{t_1})$ and, thus, $J_{\alpha_m}(\gamma(t_1), S_{t_1})$ agrees with $J_\alpha(\gamma(t_1), S_{t_1})$ for large $m$ so that all $S_t$ are crossed by $\gamma$. □

**Proof of Theorem 1.** The assertions in parentheses on temporal functions follow directly from the discussion below (1). For the Cauchy temporal case, simply, apply Prop. 4 (with $U_0 = (t_-, t_+) \times S$ as above) in an inductive way. Namely, obtain some $\alpha_0$ for $t_{\pm} = \pm 1, t_1 = -1/2, t_2 = 1/2$. In order to define $\alpha_{m+1}$, apply Prop. 4 first to $g_{\alpha_m}$ with $t_{\pm} = m \pm 1, t_1 = m - 1/2, t_2 = m + 1/2$ and then to the so-obtained metric and $t_{\pm} = -m \pm 1, t_1 = -m - 1/2, t_2 = -m + 1/2$. The limit $\alpha$ (where $\{\alpha_m\} \to \alpha$ uniformly on compact subsets and on strips) is clearly well defined and satisfies the required properties—say, if an inextensible future-directed causal curve starting at some $(t_0, x_0) \in M$ failed to cross some $S_t$, with $t_0 < t$, then for large $m$ the metrics $g_{\alpha_m}$ would not admit $t$ as a Cauchy temporal function, a contradiction. The last assertion of Th. 1 can be obtained directly from the previous ones because, according to [5], $S$ can be always regarded as a level of a Cauchy temporal function. □

### 4. Further comments on time functions

In the reference [8] by Chrusciel, Minguzzi and Grant, the existence of smooth time functions by using the original approach by Seifert is revisited. This author announced the existence of such a function long time ago [20], which was used by

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2Strictly speaking, it is not necessary to choose a subsequence (i.e., one can put $\sigma(m) = m$ along the proof). Nevertheless, such a subsequence suggests that more accurate results would be also possible.

3Recall that no restriction on $\epsilon_0$ is necessary for the proof of Th. 1 (say, one can choose always $\epsilon_0 = 1$), but one may impose some additional condition on this parameter for other purposes.
many authors (for example [12, Prop. 6.4.9]). Seifert’s proof was unclear and it was regarded as an open (“folk”) problem of smoothability. However, taking into account the viewpoint of the results in the last decade, Chrusciel et al. use an approach quite close in spirit to Seifert’s original work.

In order to compare the results by these authors with previous ones, however, the following difficulty arises. Even though these authors cite some of the papers by Bernal and Sánchez, this is not the case of the article [4], which may be the most interesting for the comparison. As seen in Section 2, this article is related to the question of stability of global hyperbolicity, which is also commented in [8, Theorem 4.8]. Also the techniques in reference [4], yield the equivalence between stable causality and the existence of a time and a temporal function (see also [19]), and a fully different proof is provided in [8, Theorem 4.9]. In the reference [4], the name temporal function and Cauchy temporal function (used above), were introduced, as it is relevant that a Cauchy temporal function allows a smooth global orthogonal splitting of the manifold. In [8], the longer expression smooth time function with timelike gradient is preferred (for example, in Theorem 4.7), and it is not clear for me if some difference might appear.

Summing up, in my opinion the results in [8], which close a circle of interesting results, are highly valuable, and a better understanding of the connections and inter-relations of the different techniques developed until now would be fruitful.

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