Band twisting and resilience to disorder in long-range topological superconductors

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Planar topological superconductors with power-law decaying pairing display different kinds of topological phase transitions where quasiparticles dubbed non-local massive Dirac fermions emerge. These exotic particles form through long-range interactions between distant Majorana modes at the boundary of the system. We show how these propagating massive Dirac fermions neither mix with bulk states nor Anderson-localize up to large amounts of static disorder despite being finite energy. Analyzing the density of states (DOS) and the band spectrum of the long-range topological superconductor, we identify the formation of an edge gap and a surprising double-peak structure in the DOS which can be linked to a twisting of energy bands with non-trivial topology. Our findings are amenable to experimental verification in the near future using atom arrays on conventional superconductors, planar Josephson junctions on two-dimensional electron gases, and Floquet driving of topological superconductors.

I. INTRODUCTION

Symmetry-protected topological (SPT) orders are quantum phases of matter characterized by non-local order parameters (topological invariants) and protected edge states at the boundary. SPT phases with particle-hole symmetry give rise to topological superconductors with unconventional pairing and gapless edge states, dubbed Majorana zero modes (MZMs). MZMs are non-abelian anyons, which can be braided to perform topological quantum computation and are protected against thermal fluctuations by a superconducting gap. These unpaired Majorana particles were first shown to arise at the ends of a chain of fermions with p-wave superconducting pairing. However, the impracticality of p-wave pairing in nature was initially believed to be a roadblock, until proximity induced superconductivity schemes have proven to be a way to circumvent this obstacle.

In recent years, different experiments have shown Majorana physics by means of a conventional superconductor proximitized to the surface of a topological insulator, semiconductor nanowires with strong spin-orbit coupling and subject to Zeeman fields, quantum anomalous Hall insulator-superconductor structures, and atomic arrays on superconducting substrates. In particular, one-dimensional arrays of magnetic impurities, where the length of the chain is relatively small compared to the coherence length of the host superconductor, generates an effective p-wave Hamiltonian with long-range pairing. Floquet driving a p-wave superconductor and planar Josephson junctions proximitized to a 2D electron gas (2DEG) with spin-orbit coupling and Zeeman field also give rise to effective models of topological superconductivity with long-range couplings.

Inspired by these recent experimental developments, p-wave Hamiltonians with long-range couplings have been thoroughly studied. A long-range extension of the Kitaev chain with power-law decaying hopping and pairing amplitudes give rise to a combined exponential and algebraic decay of correlations, breakdown of conformal symmetry and violation of the area law of entropy. The topological nature of this new model has been also unveiled, demonstrating the existence of fractional topological numbers associated to non-local massive Dirac fermions. These particles are fermions with a highly non-local extension, as they are formed out of the long-range interaction of distant Majorana particles at the edge, and their localization properties are indeed robust to weak static disorder. Interestingly, a staircase of higher-order topological phase transitions can be induced by tuning the exponent of the power-law decaying pairing amplitude.

Generalizations of the long-range Kitaev chain to two-dimensions have been constructed, where the p-wave character of the superconductor is preserved while including power-law decaying couplings that extend over the plane. In these systems, topological phases holding propagating Majorana edge states with different chiralities get significantly enhanced by long-range couplings. In one of these topological phases, propagating Majorana fermions at each edge pair non-locally and become gapped for sufficiently long-range interactions, while remaining topological and localized at the boundary. However, the robustness of these new chiral edge states with respect to general static disorder was unclear and the effects of the long-range couplings in the band spectrum of the topological superconductor were not explored.

In this article, we study how propagating Majorana states, which become gapped by the effect of long-range interactions, are affected by the inclusion of static disorder. We show how the localization at the edge is pre-
served even for very strong disorder, demonstrating that the propagating massive Dirac fermions at the edge are not pushed to the bulk nor get delocalized. This is one of the characteristic features of all topologically protected edge states. Moreover, we study how the band spectrum of a planar p-wave topological superconductor is modified by the effect of long-range couplings. We prove how a characteristic (and previously unnoticed) double-peak structure in the density of states (DOS) of the topological superconductor is enhanced by the inclusion of power-law decaying amplitudes. Associated with that effect we find a band-twisting in the energy spectrum provided the phase is topologically non-trivial.

The paper is structured as follows. In Sec. II we introduce the 2D p-wave Hamiltonian with long-range couplings and perform a detailed study of the band structure and the density of states as function of the decaying exponents. In Sec. III we demonstrate the robustness of the non-local massive Dirac fermions due to disorder and compare it to the case with unpaired Majoranas through the non-local massive Dirac fermions due to disorder and the spatial distribution of those non-local massive Dirac fermions. Sec. IV is devoted to conclusions. In the Appendix we perform a finite-size-scaling of in-gap states and their dependence on the decaying exponent $\alpha$.

II. BAND STRUCTURE & DENSITY OF STATES

The model studied in this paper is that of a two-dimensional spinless p-wave superconductor with long-range hopping and long-range superconducting coupling. In real space the Hamiltonian can be written as

$$H = -(\mu - 4t) \sum_{r=1}^{N} (c_r^\dagger c_r - c_r^\dagger c_r^\dagger)$$

$$- \sum_{r} \sum_{r' \neq r} \frac{t}{R^{2}} \left( c_{r'}^\dagger c_{r} + c_{r}^\dagger c_{r'} \right)$$

$$- \sum_{r} \sum_{r' \neq r} \frac{\Delta}{R^{\alpha+1}} \left[ (R_x + i R_y) c_{r'}^\dagger c_{r} + (R_x - i R_y) c_{r}^\dagger c_{r'} \right],$$

(1)

where both $r$ and $r'$ run over all lattice sites labelled from 1 to $N$, where $N$ is the total number of sites. We have defined $R = (R_x, R_y)$, $|r - r'|$, and $|R| = \sqrt{R_x^2 + R_y^2} = R$. Band-width is represented by $t$ and coupling strength is represented by $\Delta$. The exponents $\alpha$ and $\beta$ control the decay of superconducting coupling range and hopping range, respectively. The chemical potential $\mu$ eventually drives the system to phase transitions, for example for short-range parameters we find a transition from a trivial superconducting phase (SC) to a topological superconducting phase characterized by Majorana fermions ($M$). Interestingly, it is known that long-range superconducting couplings give rise to new topological phases characterized by massive Dirac fermions ($D$). This phase transition happens at the critical value $\alpha = 2$ and only exists for one of the two topological phases. This differs from the semi-2D Hamiltonian where the long-range terms appear only in $x$ and $y$ directions. The phase transition then occurs at $\alpha = 1$ and is present in both topological phases. A phase diagram illustrating the former case is depicted in Fig[1]. Unless explicitly mentioned, we have used $t = 0.5$ as reference parameter, $\Delta = 0.5$ following Ref. 63, and $\beta = 10$, i.e. short-range hopping.

A. Massive Dirac fermions

The first step is to identify the differences and similarities between the Majorana phase and the massive Dirac phase. For that, the edge-state excitations will be analysed.

By exactly diagonalization $H |\psi_n\rangle = E_n |\psi_n\rangle$ we obtained the Bogoliubov excitation spectrum, $E_n$, with $n = 1, \ldots, 2N$, of a finite (squared) system with $L^2 \equiv N$ lattice sites. The results are depicted in Fig[1], in which we exemplified the two different topological phases $M$ and $D$. The parameters are indicated in the phase diagram, panel (a), by the geometric figures in diamond shape, namely we set $\alpha = 1.6$ and $\alpha = 3$, with $\mu = 1$. In both phases, the superconducting gap (stated here as bulk-gap) is easily noticed from either the excitation spectrum in panel (b) or its respective density of states (DOS) in panel (c). The topological properties are manifested as in-gap states, in particular the inset of panel (c) explicits the difference between the two topological phases. While the Majorana states manifest as a finite DOS over the entire gap, the massive Dirac states let opened a smaller gap (stated here as edge-gap since it is the energy difference between edge-state excitations).

One may also look at the localization of massive Dirac states plotting the probability of occupancy related to the $n$-th wave-vector (corresponding to energy $E_n$) inside the bulk-gap) on each site, i.e. $P_n(r) = a_n^*(r) a_n(r)$, where the amplitude $a_n(r)$ is obtained from $|\psi_n\rangle = \sum_r a_n(r) |\psi_n(r)\rangle$, and the normalization implies $\sum_r P_n(r) = 1$. Figs[1] and[1] exemplify this probability for an energy inside the bulk and for the smallest finite energy inside the bulk-gap, respectively. The probability amplitude of occupancy is better analysed if log scaled, thus for convenience we have defined a normalized logarithmic localization $\Phi = 1 - \log P_n(r) / \log P_{\text{min}}$, where $\Phi = 1$ if $P_n(r) = 1$ and $\Phi = 0$ if $P_n(r) = P_{\text{min}}$. $P_{\text{min}}$ is the global-minimum probability $P_n(r)$, i.e. among all energies $E_n$ and all sites $r$.

Equivalent to the Majorana excitations in the planar topological superconductor, the massive Dirac states are confined to the edges, see Fig[1], which form propagating modes protected by particle-hole symmetry. Technically speaking, the system still belongs to class D of topological superconductors with Z topological invariant. In Fig[1] we see the bulk energy excitations remaining spread over the sample. A thorough study of the-
bustness of the massive Dirac states is one of the main goals of this work and will be discussed in section III.

B. Twisted bands and double peak structure

We discovered that the band spectrum and the DOS of our long-range topological superconductor provide valuable informations regarding the energy distribution of the different eigenstates (see Fig 1). In addition, we may extract useful quantities such as the magnitude of the superconducting gap, the group velocity and the band dispersion.

For convenience, we consider a semi-finite system, finite in x direction and periodic in the y direction. As an example, let’s take two points in the phases $\mathcal{M}$ of the phase diagram with different chiral edge states, namely $\mu = 1$ and $\mu = 3$, with $\alpha = 3$. Fig. 2 shows the DOS of these two points, while Figs. 3a and 3b show their respective band spectrum for a semi-finite system. From these figures we highlight the following: (i) associated to the peak structures we notice an unusual band-twisting (highlighted by the arrows), and (b) there is a significant bands overlap as consequence of this band-twisting.

Next we observe that longer range superconducting couplings are responsible for the enhancement of the peak’s structure, in particular within the massive Dirac phase $\mathcal{D}$. Figs. 4-f show the results for smaller values of the superconducting coupling exponent already inside the phase $\mathcal{D}$, i.e. $\alpha = 1.6$. We clearly see a more pronounced structure of the peaks, more precisely they split into two peaks that comes along with an enlargement of the bands overlap. We further note that the two peaks structure is present in both topological phases, and that it is enhanced by decreasing $\alpha$, however they do not appear in the trivial superconducting phase (not shown in this figure).

The superconducting coupling strength is also responsible for changing the peak structure. In particular, decreasing $\Delta$ also makes the peak split into two, as shown in Fig 5. Associated with that, from the semi-finite system band spectrum shown in Figs. 4 and 5, we again notice an enlargement of the bands overlap. Indeed, we checked that lowering $\Delta$ (but finite) the two-peak structure can always be retrieved in all topological phases.

The two peak structure is not a unique long-range feature. In Figs. 5-f we show the presence of the two peaks even in the short-range limit $(\beta, \alpha \gg 1)$, although it is hard to identify. In short, both topological phases present in this work ($\mathcal{M}$ and $\mathcal{D}$) present a double peak structure in their DOS which is associated to a band twisting, which in turn leads to a band overlap. This association is highlighted by the colored arrows in Figs. 2 and 3. Surprisingly the DOS double-peak structure only appears within the topological phases. It is always achieved for finite-small values of the superconducting coupling strength and is enhanced by long-range couplings.

III. ROBUSTNESS OF THE MASSIVE EDGE STATES AGAINST DISORDER

Here we discuss the effect of static disorder in the presence of massive Dirac states. We first analyse the normalized DOS computed for a finite 2D system with different disorder strengths. The disorder is added to the Hamiltonian as

$$H_{\text{disorder}} = \nu \sum_{r=1}^{N} D_r \left( c_r^\dagger c_r - c_r c_r^\dagger \right), \tag{2}$$

where $\nu$ is the disorder strength and $|D_r| \leq 1$ is randomly distributed over the sites’ positions $r$.

Fig. 4 analyse the results for a representative point within the phase $\mathcal{D}$ (namely $\mu = 1$, $\alpha = 1.6$, and system size $N = 1681$). Fig. 4a shows the DOS for different disorder strengths. First, we clearly observe how the DOS peak decreases with this disorder. Second, we show
FIG. 2. Panel (a) shows the DOS of a finite squared system for the two phases $M$ with different chiralities, namely $\mu = 1$ and $\mu = 3$, while panels (b) and (c), respectively, show their band spectrum for a semi-finite system, i.e. periodic in the $y$ direction. The arrows indicate the two-peak structure on the DOS, and their associated band-twist in the band spectrum. Panels (d)-(f) show equivalent results for longer range couplings, in particular note that for $\mu = 1$ the system is in the phase $D$. That the bulk-gap shrinks faster than the edge-gap. In addition, the plateau formed by the massive Dirac edge states (i.e. the finite energies between the bulk-gap and edge-gap) persists quantitatively the same even for large values of disorder, which provides an indication of the robustness of the new massive edge states.

One may also look at the Anderson localization effect from the participation ratio (PR), which gives the degree of localization of each state after one disorder realization, such that

$$\text{PR} \equiv \frac{1}{N} \sum_r \langle P^2_r \rangle.$$  

(3)

For instance, for a completely delocalized state where all sites are equally probable to be occupied one finds $\text{PR} = 1$, while for a completely localized state where only one site is probable to be occupied one finds $\text{PR} = 1/N$, which goes to zero at the thermodynamic limit. Moreover, for an edge state perfectly localized at the boundary, i.e. equally distributed along the edge sites of the 2D system, one finds $\text{PR} = 4/\sqrt{N}$.

Fig.4c shows the density of participation ratio (DOPR) with respect to the energy index ($n$) for different strengths $\nu$. Note that our results average over 100 disorder realizations, and the results are an average over it. Thus, in this figure one easily notice that DOPR is concentrating near to $\text{PR} = 1$, instead of $\text{PR} \sim 10^{-3}$ for this particular system size, which signals that the bulk states are delocalized. In addition, we notice that they continue to be delocalized even for large disorder strength. From the edges states we expect a peak near to $\text{PR} \approx 0.1$ for this system size, since they are not localized at one point but spread all over the boundary. Thus the inset shows a zoom to the DOPR near $\text{PR} = 0.1$. The existing peaks are clear and they are shifting towards the left when increasing disorder strength, which reflects a trend of the edge states to be more and more localized along the edges.

The spatial localization over all the states is quantified by the mean participation ratio (MPR), namely

$$\text{MPR} \equiv \left\langle \frac{1}{2N} \sum_{n=1}^{2N} \text{PR} \right\rangle,$$  

(4)

where the average $\langle \cdots \rangle$ is over disorder realizations. Thus, Fig.4d shows the decreasing of MPR, roughly from 0.6 to 0.4 with $\nu = 0$ to $\nu = 0.5$, respectively. This shows a trend of the whole system to become more localized, besides still orders of magnitude higher than the completely localized value, typically $\text{PR} \approx 6 \times 10^{-4}$ for this system size.
FIG. 4. This picture illustrates the behavior of a non-local massive Dirac state (precisely for \( \mu = 1 \) and \( \alpha = 1.6 \)) in the presence of disorder. Panel (a) shows the DOS while the inset is a zoom in to the in-gap states. Panel (b) is the legend which holds true to all other panels. Panel (c) shows the DOPR as function of PR for different disorder strengths, in which the inset gives a zoom in to the peak coming from the edge states. Panel (d) shows the MPR for a range of disorder strength.

**A. Spatial distribution of states**

Here we analyse the spatial distribution of states subject to static disorder both for the massive Dirac and Majorana phases. Fig. 5 depicts representative states in each row associated to different energy levels, which in turn represent different behaviors. The columns of the plot stand for different disorder strengths (we have considered 100 disorder realizations). In particular, we noticed three cases where

\[ E_1 \] is the smallest finite energy inside the gap;

\[ E_2 \] illustrates the finite energies inside the gap which merge with the bulk after including strong enough disorder.

\[ E_3 \] and \[ E_4 \] represent two different bulk energies.

Remarkably, the topological robustness of the massive Dirac phase is indeed very similar to the Majorana phase. The topological energy states inside the gap display clear localization along the edges with a short tail towards the bulk. We have checked that the tail is shortened after including disorder, adding some degree of additional stability to the boundary of the system. The increase in edge localization through disorder was already noticed in the inset of Fig. 4c where the peak moves to the left (i.e. towards more localized). Moreover, Fig. 4c shows that the bulk-gap is shrinking faster than the edge-gap, which means that edge states with higher energies are merging with the bulk. This behavior is illustrated in Fig. 4a by the frames with energy \( E_2 \), in which more localised states (like clusters of probability density) are formed inside the bulk. One may notice the formation of those clusters for \( \nu \geq 0.25 \). Finally, the bulk states \( (E_3 \text{ and } E_4) \) remain fairly delocalized after incorporating disorder. However, for strong disorder we notice the formation of clusters of probability density inside the bulk.

**IV. DISCUSSIONS**

We have studied the robustness and localization properties of non-local massive Dirac fermions that appear as exotic energy quasiparticles in 2D topological superconductors with long-range interactions. Analyzing the density of states (DOS) and the energy spectrum, we identify how these topological sub-gap states at finite energy remain localized and propagating even for large static disorder. By means of the in-gap states we compute the phase diagram for different chemical potentials and long-range couplings. The propagating massive Dirac fermion is identified from a sub-gap in the superconducting phase. Looking at the probability of occupancy of the energy spectrum, we can clearly identify the localization properties of massive Dirac fermions along the edges of a 2D square lattice. The robustness of these quasiparticles is tested including chemical potential disorder. The DOS analysis indicates a strong resistance from the in-gap states to disorder, which is confirmed using a participation ratio analysis of all quantum states in the system. Finally, the stability in the probability of occupation for the edge states shows that the robustness of the massive Dirac fermions are analogous to the Majorana states.

Complementarily, for a semi-infinite periodic system, we notice that a band-twisting in the band structure is always accompanied by a double peak in the DOS. We show that this behavior also appears for purely short-range interactions, however, we notice it is an exclusive feature of topological phases and can be used as a probe to identify non-trivial topology. In addition, we show that long-range couplings and small pairing strengths strongly enhance the double-peak structure. This enhancement
can be potentially used to experimentally detect topological phases using STM measurement.\textsuperscript{33}

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For instance, in $k$-space the Hamiltonian assumes the form
\[ H = \text{even}(k)\sigma_x + \text{odd}(k)(\sigma_z + i\sigma_y), \]
where $\sigma$ acts on the Nambu basis. Thus, the particle-hole operator is $P \equiv \sigma_z K$, which satisfies the relation $H_k = -PH_{-k}P$ in real space. It also has inversion symmetry, whose operator is $I \equiv \sigma_z$ and respects the relation $H_k = IH_{-k}I$, or the relation $H_kR = IH_{-k}R$ in real space.

See Appendix III A for a detailed analysis of the excitations.

APPENDIX A: Finite size scaling of in-gap states

In the main text, Fig.1, we show the finite DOS inside the superconducting gap. Since the bulk states and the in-gap states are expected to have different finite size scalings, here we give them a detailed analysis. In Fig.1 we show the DOS for different system sizes and superconducting couplings (controlled by $\alpha$). In particular, we have used three different system sizes $N = 441, 961, 1681$, and show results for two representative points inside the phase diagram, namely $\mu = 1$ and $\mu = 3$. Each panel (b)-(d) is computed for a different $\alpha$ value, and the insets are a zoom to the in-gap states. Panel (a) shows the scaling behavior of in-gap DOS/4L value, while the inset express its dependence on $\alpha$.

FIG. 6. Here we make the finite-size-scaling analysis of the in-gap states. Panels (b)-(d) show the DOS/$N$ for three different system sizes and two representative points in the phase diagram, namely $\mu = 1$ and $\mu = 3$. Each panel (b)-(d) is computed for a different $\alpha$ value, and the insets are a zoom to the in-gap states. Panel (a) shows the scaling behavior of in-gap DOS/4L value, while the inset express its dependence on $\alpha$. 

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