Geometric Scaling above the Saturation Scale

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Abstract

We show that the evolution equations in QCD predict geometric scaling for quark and gluon distribution functions in a large kinematical window, which extends above the saturation scale up to momenta $Q^2$ of order 100 GeV$^2$. For $Q^2 < Q_s^2$, with $Q_s$ the saturation momentum, this is the scaling predicted by the Colour Glass Condensate and by phenomenological saturation models. For $1 \lesssim \ln(Q^2/Q_s^2) \ll \ln(Q_s^2/\Lambda_{\text{QCD}}^2)$, we show that the solution to the BFKL equation shows approximate scaling, with the scale set by $Q_s$. At larger $Q^2$, this solution does not scale any longer. We argue that for the intermediate values of $Q^2$ where we find scaling, the BFKL rather than the double logarithmic approximation to the DGLAP equation properly describes the dynamics. We consider both fixed and running couplings, with the scale for running set by the saturation momentum. The anomalous dimension which characterizes the approach of the gluon distribution function towards saturation is found to be close to, but lower than, one half.
1 Introduction

At very high energies, the contribution to the hadronic wavefunction which dominates hadronic processes corresponds to a state of very high partonic (mostly gluonic) density. This high density matter is believed to reach a saturation regime, and become a Color Glass Condensate [1, 2]. (See also [3] for a recent review and more references.) Recent phenomenological analysis of the data from HERA and the RHIC show results qualitatively and semi-quantitatively consistent with this picture [4]. The results, while not entirely compelling, provide a strong motivation for the further theoretical investigation of this novel form of high density matter.

In a very important paper [5], Staśto, Golec-Biernat and Kwieciński have shown that the HERA data on deep inelastic scattering (DIS) at low $x$, which are a priori functions of two independent variables — the photon virtuality $Q^2$ and the Bjorken variable $x$ —, are consistent with scaling in terms of the variable

$$T = Q^2 R_0^2(x)$$

where $R_0^2(x) = (x/x_0)^{\lambda}/Q_0^2$ with $\lambda = 0.3 - 0.4$, $Q_0 = 1$ GeV, and $x_0 \sim 3 \times 10^{-4}$ in order to fit the data. In particular, the data for the virtual photon total cross section at $x < 0.01$ and $Q^2 < 400$ GeV$^2$ are consistent with being only a function of $T$. Note that, because $\sigma_{\gamma^* p} = 4\pi^2\alpha_{EM} F_2(x, Q^2)/Q^2$, this implies that $F_2(x, Q^2)/Q^2$ is a function only of $T$ in the indicated kinematical range.

At first sight, it is tempting to interpret this scaling as a consequence of the existence of the Color Glass Condensate: Indeed, it has been argued that in the saturated region the various parton densities are functions only of $Q^2/Q_s^2(x)$, with $Q_s$ the saturation momentum, which is proportional to the gluon density, and thus increases rapidly with $1/x$. Although $Q_s$ has not been fully determined from first principles, its estimates based on the quantum evolution of the Color Glass Condensate suggest indeed a power-like increase: $Q^2_s \sim Q^2_0 x^{-\lambda}$ [7, 8, 9, 10, 11]. (This will be further discussed in this paper.) The fundamental problem with this argument is that it is valid only for $Q^2$ less than or of the order of the saturation momentum, which is at most several GeV$^2$, while the fit of Ref. [5] extends up to $Q^2$ of the order of several hundred GeV$^2$.

The main purpose of this paper is to show that the scaling region for the various distribution functions is in fact much larger than the saturation region. We shall find geometric scaling for all momenta $Q^2$ such that the following inequality is satisfied:

$$\ln(Q^2/Q_s^2) \ll \ln(Q^2_s/\Lambda_{QCD}^2).$$

For $Q_s \sim 2$ GeV and $\Lambda_{QCD} \sim 200$ MeV, the upper scale on $Q^2$ is $Q^2_s/\Lambda_{QCD}^2 \sim 400$ GeV$^2$. While at soft momenta $Q^2 \ll Q_s^2$ this scaling is an expected consequence of saturation, at high momenta $1 < \ln(Q^2/Q_s^2) \ll \ln(Q^2_s/\Lambda_{QCD}^2)$ it rather corresponds to a regime where parton densities are small, and linear evolution equations apply. And indeed we shall see that the extended scaling at $Q^2 > Q_s^2$ arises from solutions to the BFKL equation [12], which is the appropriate limit of the general non-linear evolution equations [2,13–18] in the kinematical range of interest.

Note that, in this paper, we shall use the BFKL equation as an effective equation, valid in some range of $Q^2$ whose increase in $Q^2$ is correlated with the decrease in Bjorken $x$ (because of the
constraint $Q^2 > Q_s^2(x) \sim Q_0^2 x^{-\lambda})$. Thus, conceptual difficulties of BFKL, like the exponential increase of its solution at high energy, and its “infrared diffusion”, will be of no concern for us.

In fact, an essential step in our analysis will be to clarify the kinematical ranges in which the solution to the BFKL equation (in the saddle point approximation) is of the “genuine BFKL type” (i.e., it is close to the saddle point which governs the limit $x \to 0$ with $Q^2$ fixed), or of the “double logarithmic type” (i.e., close to the saddle point which describes the limit $Q^2 \to \infty$ with $x$ fixed). As well known, in the latter limit the BFKL solution becomes equivalent to the double logarithmic approximation (DLA) to the DGLAP equation \[ [13] \]. But this limit turns out not to be relevant for the geometric scaling: Indeed, the window for extended geometric scaling that we shall find is actually in the range controlled by the BFKL saddle point.

It is quite remarkable that even at high momenta $Q^2 > Q_s^2$, it is still the saturation momentum $Q_s(x)$ which sets the scale in the scaling variable (i.e., which plays the role of $1/R_0(x)$ in eq. \[ (1.1) \]). This is so because the solution to the BFKL equation must properly match when $Q^2 \to Q_s^2$ on the corresponding solution at saturation.

In brief, our strategy in this paper will be as follows: We start with a solution to the BFKL equation, which in general (i.e., for arbitrary $Q^2$) does not appear to scale. By extrapolating this solution down to momenta $Q^2 \sim Q_s^2(x)$, we shall estimate the saturation scale $Q_s(x)$ from the matching condition with the solution at saturation. The latter is not precisely known around $Q_s$, but for the present purposes such a precise solution is not really needed: All that we need is a saturation criterion which gives us the value of the quantity of interest at $Q^2 = Q_s^2(x)$. This criterion is simply that the gluon distribution function itself becomes of order $1/\alpha_s$. Such a criterion follows from the general analysis of the non-linear effects due to high parton densities \[ [1–3,7–11,13–18,20,21] \]. By expanding the BFKL solution around $Q^2 = Q_s^2(x)$, we shall find geometric scaling in the momentum range where the first term in this expansion is a good approximation, namely for $1 < \ln(Q^2/Q_s^2) \ll \ln(Q_s^2/\Lambda_{QCD}^2)$. This conclusion holds both for a fixed coupling and for a running coupling (with the scale for running set by the saturation momentum), although the saturation scales turn out to be different in the two cases.

Note that, although the saturation is crucial for the arguments in this paper, the emergence of an extended scaling window above the saturation scale is not an automatic consequence of the saturation. Rather, this requires some non-trivial properties of the linear evolution equations as well. To see this, consider some dimensionless functions $f(x, Q^2)$ (e.g., the gluon distribution) which at high $Q^2$ obeys a linear evolution equation (BFKL or DGLAP), but at $Q^2 \sim Q_s^2(x)$ satisfies a saturation condition of the type (after an appropriate normalization):

$$f(x, Q^2 = Q_s^2(x)) = 1.$$ \hspace{1cm} (1.3)

As a consequence of this condition, and without loss of generality, the function $f$ near $Q^2 = Q_s^2(x)$ (i.e., for $|\ln(Q^2/Q_s^2)| \ll \ln(Q_s^2/\Lambda_{QCD}^2)$), can be approximated as:

$$f(x, Q^2) \sim \left(\frac{Q^2}{Q_s^2(x)}\right)^{\lambda_s(x)},$$ \hspace{1cm} (1.4)

where $\lambda_s(x) \equiv (\partial \ln f/\partial \ln Q^2)|_{Q_s^2(x)}$. Assume that this limited expansion can be extended up to values $Q^2$ which are high enough for the non-linear effects to be subleading. Then, the exponent $\lambda_s(x)$ can be determined by solving the BFKL equation with the saturation boundary condition
A priori, the result $\lambda_s(x)$ can be a non-trivial function of $x$. If this happens, then eq. (1.4) shows no geometric scaling. However, in our subsequent analysis, we shall discover that for solutions to the BFKL equation, $\lambda_s$ is a number independent of $x$ — as a consequence of the scale-invariance of the BFKL kernel —, so that $f(x,Q^2)$ shows scaling in the window of validity of the expansion (1.4). The difference between the actual value of the exponent $\lambda_s$ and its “naïve” value expected from perturbative considerations at large $Q^2$ will be referred to as the “anomalous dimension”.

The saturation criterion that we shall actually use refers to the scattering of a small dipole off a hadronic target: $1/Q_s$ is the critical transverse size of the dipole at which the scattering amplitude becomes of order one (“black disk limit”) [22, 7, 24, 25]. This criterion, which is equivalent (at least, for our present purposes) with the more standard condition on the gluon distribution [1, 7, 9, 13, 14, 20, 21], is simpler to use in applications to deep inelastic scattering, since more directly related to the corresponding cross-section at small $x$ [22, 7, 24, 25]:

$$\sigma_{T,L}(\tau,Q^2) = \int_0^1 dz \int d^2r_\perp |\Psi_{T,L}(z,r_\perp;Q^2)|^2 \hat{\sigma}(\tau,r_\perp).$$

(1.5)

In this equation, $\sigma_{T,L}(\tau,Q^2)$ is the cross-section for the scattering of a virtual transverse ($T$) or longitudinal ($L$) photon off the hadron, at relative rapidity $\tau \equiv \ln(1/x) \sim \ln s$. (Note that, from now on, we use $\tau$ to indicate the dependence upon the total center-of-mass energy squared $s$.) Furthermore, $Q^2$ is (minus) the photon virtuality, and $\Psi_{T,L}(z,r_\perp;Q^2)$ is the light-cone wavefunction for the photon splitting into a $q\bar{q}$ pair (the “dipole”) with transverse size $r_\perp$ and a fraction $z$ of the photon’s longitudinal momentum carried by the quark. Finally, $\hat{\sigma}(\tau,r_\perp)$ is the dipole-hadron cross-section, which can be computed in the eikonal approximation as:

$$\hat{\sigma}(\tau,r_\perp) = 2 \int d^2b_\perp (1 - S_\tau(x_\perp,y_\perp)), \quad S_\tau(x_\perp,y_\perp) \equiv \frac{1}{N_c} \langle \text{tr}(V^\dagger(x_\perp)V(y_\perp)) \rangle_\tau,$$

(1.6)

with $r_\perp = x_\perp - y_\perp$ (the quark is at $x_\perp$, and the antiquark at $y_\perp$), and the impact parameter $b_\perp = (x_\perp + y_\perp)/2$. In eq. (1.6), $V^\dagger$ (or $V$) is a Wilson line along the straight line trajectory of the quark (or the antiquark), that is, a path ordered exponential of the color field created in the hadron by color source at rapidities $\tau' < \tau$. The brackets in the definition of the $S$-matrix element $S_\tau(x_\perp,y_\perp)$ refer to the average over all configurations of these colour sources [1].

A small dipole is weakly interacting: $S_\tau(r_\perp) \approx 1$ for $r_\perp \ll 1/Q_s(\tau)$. A large dipole, on the other hand, is strongly absorbed: $S_\tau(r_\perp) \ll 1$ for $r_\perp \gtrsim 1/Q_s(\tau)$. This is so because of the large density of the saturated gluons in the hadron wavefunction. At momenta $Q^2 \lesssim Q_s^2(\tau)$, there is only one intrinsic scale in the problem, the saturation momentum $Q_s(\tau)$ itself (we mean this for a fixed impact parameter and a fixed coupling). So, all physical quantities should be expressed as a dimensionless function of $Q^2/Q_s^2$ times some power of $Q_s^2$ giving the right dimension. These general properties expected for $\hat{\sigma}(\tau,r_\perp)$ have been incorporated in a simple “saturation” model by Golec-Biernat and Wüsthoff [24] $1/R_0^2(x)$ plays the role of the saturation scale):

$$\hat{\sigma}(\tau,r_\perp) = \sigma_0 \left(1 - e^{-r_\perp^2/4R_0^2(x)}\right) \equiv \sigma_0 g \left(\frac{r_\perp^2}{4R_0^2(x)}\right),$$

(1.7)

which appears to give a reasonable description of the HERA data at small $x$. By inserting this expression in eq. (1.5), one can check that the scaling property is transmitted to the DIS cross-section: $\sigma_{\gamma^*p}(\tau,Q^2) = \sigma_{\gamma^*p}(T)$ with $T$ of eq. (1.1) [3].
Written as it stands, eq. (1.7) shows exact scaling for all distances $r_\perp$, and not only in the saturation regime $r_\perp \gg R_0(x)$. In reality, however, we know this scaling to be violated at sufficiently small distances $r_\perp \ll R_0(x)$, where eq. (1.7) should be replaced by

$$\hat{\sigma}(\tau, r_\perp) = \sigma_0 \left(1 - \exp\left\{-r_\perp^2 \pi^2 \alpha_s x G(x, 1/r_\perp^2) \right\} \right), \quad (1.8)$$

where $x G(x, 1/r_\perp^2)$ is the gluon distribution function evaluated at $Q^2 = 1/r_\perp^2 \gg Q_s^2$. At such high momenta, $x G(x, Q^2)$ is a solution to some linear evolution equation, usually taken to be the DGLAP equation, which, at least for sufficiently large $Q^2$, has no scaling behaviour (see Sect. 3 below). It will be our main objective in what follows to establish up to which momenta $Q^2$ the scaling property holds for the dipole cross-section, and thus for the quark distribution in the hadron. To characterize this property also for the gluon distribution, we shall find it convenient to introduce a definition of the latter in terms of the dipole scattering amplitude (see eq. (1.8) and eqs. (2.7)–(2.8) below).

To simplify the problem, we shall assume that the dependence upon the impact parameter is not essential for the present purposes, so we can treat the hadron as being homogeneous in the transverse plane. This implies $S_\tau(x_\perp, y_\perp) = S_\tau(r_\perp)$, with $r_\perp$ the size of the dipole, and the scaling properties of $\sigma_{\gamma p}(\tau, Q^2)$ are directly related to corresponding properties of $S_\tau(r_\perp)$, that we shall study.

The function $S_\tau(r_\perp)$ will be obtained by solving an appropriate evolution equation in $\tau$. As already mentioned, we shall be mainly concerned with solutions to the BFKL equation, but this equation will be viewed as the linear limit of more general, non-linear, evolution equations. This is important, as it will allow us to keep trace of the non-linear effects via the boundary condition at $Q^2 \sim Q_s^2(\tau)$ (the “saturation condition”).

It is possible to write down a formal evolution equation for $S_\tau(x_\perp, y_\perp)$ without specifying the average on the target hadron in the definition (1.6) \[13, 14, 18\]. In general, such an equation is only the starting point of a hierarchy of coupled evolution equations for the correlators of Wilson line operators \[17\], which can be encoded in a single, functional evolution equation \[17\]. But a closed equation for $S_\tau(x_\perp, y_\perp)$ can still be obtained in the large $N_c$ limit. This has been derived by Kovchegov \[16\] within the Mueller’s dipole model \[28\], and follows also from the general evolution equations by Balitsky \[15\] by taking the large $N_c$ limit. (See also \[18\] for a different derivation.) We shall refer to this as the Balitsky-Kovchegov (BK) equation.

Alternatively, an effective theory can be constructed for the small-$x$ component of the hadron wavefunction \[1, 2, 3, 9, 20, 29, 30\], thus allowing a direct and explicit calculation of the average in eq. (1.6). In this effective theory, the high density gluon configurations at small $x$ are treated as a “Colour Glass”, i.e., as the classical colour field generated by colour sources at rapidities $\tau' < \tau = \ln(1/x)$, which are “frozen” in some random configuration. In this picture the evolution is viewed as a renormalization group operation in which layers of quantum fluctuations are successively integrated out to generate the sources \[30, 3\]. This leads to a functional renormalization group equation (RGE) for the probability distribution of the colour sources. Remarkably, this RGE is equivalent \[3\] to the Wilson line approach of Refs. \[15, 16, 17\]: it generates the same evolution
equations for the correlation functions of Wilson lines. At high transverse momenta $Q^2$, the non-linear effects are weak and can be expanded out. To lowest order in this expansion, the general RGE (and also the BK equation) reduces to the BFKL equation [12], as already mentioned.

The outline of this paper is as follows:

In the second section, we rely on the BK equation to explore the scaling properties of distribution functions in the deeply saturated region where the gluon density is of order $1/\alpha_s$. By assuming that scaling solutions exist, we determine the dependence of the saturation momentum upon $\tau$ for both the fixed coupling and the running coupling cases. Throughout this paper, the scale for the running of $\alpha_s$ is always the saturation momentum.

In the third section, we consider the intermediate $Q^2$ region where the evolution equation linearizes and reduces to the BFKL equation. We construct approximate solutions for this equation in the saddle point approximation and show that the solution which is relevant for the approach to saturation and the extended scaling is closer to the standard BFKL solution, rather than to the double-logarithmic DGLAP-like solution. We determine the saturation momentum, compute the anomalous dimension of the distribution function, and show the solution has geometric scaling for both the cases of running and fixed coupling.

The last section contains a summary of our results, and some discussion.

2 Scaling properties of the Balitsky-Kovchegov equation

In this section, we shall briefly discuss the BK equation [15,16], which is a non-linear equation for the evolution of the dipole-hadron scattering amplitude with $\tau = \ln(1/x)$. This equation will be used here just for qualitative arguments, which by themselves are valid for any $N_c$, although the BK equation holds, strictly speaking, only in the large $N_c$ limit. (The same arguments at finite $N_c$ would require the full formalism in Refs. [15, 17, 2, 3]) Specifically, the following properties of the BK equation will be useful for what follows: First, it covers both the linear and the non-linear regimes of the quantum evolution (according to the value of $Q^2$), second, it reduces to the BFKL (and eventually DLA) equation in the linear regime at high $Q^2$, and, third, it demonstrates the relation between geometric scaling and saturation at low $Q^2$.

In fact, we shall find that the structure of the BK equation is consistent with scaling solutions, although the kind of global arguments that we shall use cannot decide if such solutions actually exist or not. This will open the discussion of scaling solutions outside the saturation regime, to be pursued in the framework of linear evolution equations in Sect. 3. The scaling properties of the Balitsky-Kovchegov equation have been also investigated numerically, in Refs. [3, 11, 21].

The BK equation is most succinctly written as an equation for the $S$-matrix element $S_\tau(r_\perp)$ for dipole-hadron scattering, eq. (1.6) (below, $\tilde{\alpha}_s = N_c \alpha_s/\pi$):

$$\frac{\partial}{\partial \tau} S_\tau(x_\perp - y_\perp) = -\tilde{\alpha}_s \int \frac{d^2 z_\perp}{2\pi} \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2(y_\perp - z_\perp)^2} \times \left( S_\tau(x_\perp - y_\perp) - S_\tau(x_\perp - z_\perp)S_\tau(z_\perp - y_\perp) \right).$$

(2.1)

Given $S_\tau(r_\perp)$, the dipole-hadron scattering amplitude $N_\tau(r_\perp)$ is obtained as:

$$N_\tau(r_\perp) = 1 - S_\tau(r_\perp).$$

(2.2)
For a small dipole, \( r_\perp \ll 1/Q_s(\tau) \), \( N_\tau(r_\perp) \) is small as well, \( N_\tau(r_\perp) \ll 1 \), and one can linearize eq. (2.3) with respect to \( N_\tau(r_\perp) \); one then obtains the BFKL equation. This can be easily seen in the momentum space form of the BK equation. (The BFKL equation in coordinate space will be discussed in the next section.) Specifically, if one defines \( \varphi_\tau(k_\perp) \) by

\[
\frac{N_\tau(r_\perp)}{r_\perp^2} = \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp\cdot r_\perp} \frac{\varphi_\tau(k_\perp)}{k_\perp^2},
\]

one finds

\[
\frac{\partial}{\partial \tau} \varphi_\tau(q_\perp) = \alpha_s \int \frac{d^2k_\perp}{\pi} \frac{q_\perp^2}{k_\perp^2(q_\perp-k_\perp)^2} \left( \varphi_\tau(k_\perp) - \frac{1}{2} \varphi_\tau(q_\perp) \right) - \frac{\alpha_s}{2\pi} \frac{(\varphi_\tau(q_\perp))^2}{q_\perp^2}.
\] (2.4)

The small-dipole condition \( N_\tau(r_\perp) \ll 1 \) in coordinate space corresponds to the condition \( \varphi_\tau(q_\perp)/q_\perp^2 \ll 1 \) in momentum space, which is satisfied provided \( q_\perp^2 \) is large enough, \( q_\perp^2 \gg Q_s^2(\tau) \). In this regime, the last term, quadratic in \( \varphi_\tau(q_\perp)/q_\perp^2 \), can be neglected in the r.h.s. of eq. (2.4), which then reduces to the BFKL equation \[14\]. For momenta \( q_\perp^2 \) which are even larger, this becomes equivalent to the double-log limit of the DGLAP equation (see Sect. 3 below).

On the other hand, non-linear effects become crucial at low momenta \( q_\perp^2 \approx Q_s^2(\tau) \), that is, for a large dipole which is strongly absorbed by the hadron: \( S_\tau(r_\perp) \ll 1 \) for \( r_\perp \gg 1/Q_s(\tau) \). In this regime, the right hand side of eq. (2.3) can be simplified by neglecting the term quadratic in \( S_\tau \); this is appropriate since the dominant contribution (in the sense of the leading log) comes from \( z_\perp \) satisfying \( 1/Q_s(\tau) \ll |z_\perp - x_\perp| \ll r_\perp \) (or a similar condition on \( |z_\perp - y_\perp| \)). The equation then reduces to

\[
\frac{\partial}{\partial \tau} S_\tau(r_\perp) \simeq -\alpha_s \ln \left( r_\perp^2 Q_s^2(\tau) \right) S_\tau(r_\perp),
\] (2.5)

for which a solution will be written down shortly. (This requires the \( \tau \) dependence of the saturation scale \( Q_s^2(\tau) \).

Still in the non-linear regime, it is interesting to determine the limiting form of the function \( \varphi_\tau(q_\perp) \) at low \( q_\perp \). By solving eq. (2.4) in this regime, or, more directly, by noticing that \( N_\tau(r_\perp) \simeq 1 \) for \( r_\perp > 1/Q_s(\tau) \) in eq. (2.3), one deduces that:

\[
\varphi_\tau(q_\perp^2 \ll Q_s^2) \simeq q_\perp^2 \int_{1/q_s^2}^{1/q_\perp^2} \frac{d^2r_\perp}{r_\perp^2} = \pi q_\perp^2 \ln \frac{Q_s^2(\tau)}{q_\perp^2}.
\] (2.6)

It can be checked that this is indeed a solution to eq. (2.4) at low \( q_\perp^2 \) and to leading-log accuracy \[12\].

At this stage, one may notice that the properties of the function \( \varphi_\tau(q_\perp) \) introduced in eq. (2.3) are very similar to those expected for the “unintegrated gluon distribution”, i.e., the density of gluons in the transverse phase-space. Indeed, at large momenta \( \varphi_\tau(q_\perp) \) satisfies the BFKL equation and is truly proportional to the gluon distribution, as manifest on eq. (1.8). At low momenta, \( q_\perp^2 \ll Q_s^2 \), it has the same behaviour in \( q_\perp \), eq. (2.4), as the gluon density in the saturation regime \[8\ 9\ 3\]. It is thus quite reasonable to define the gluon phase space distribution function via the Fourier transform of the dipole scattering amplitude (with \( C_F = (N_c^2 - 1)/2N_c \)):

\[
(2\pi)^2 \frac{d^5N}{d\tau d^2k_\perp d^2b_\perp} = \frac{N_c^2 - 1}{\pi^2 \alpha_s C_F} \frac{\varphi_\tau(k_\perp, b_\perp)}{k_\perp^2},
\] (2.7)
where we have reintroduced the impact parameter dependence, for more generality (so, \(\varphi_\tau(k_\perp, b_\perp)/k_\perp^2\) is actually the Wigner transform of \(N_\tau(r_\perp, b_\perp)/r_\perp^2\), and the overall normalization follows by comparison with eq. (1.8). This gives a gluon distribution:

\[
xG(x, Q^2) \equiv \frac{dN}{dx} = \frac{2N_c}{\pi^2\alpha_s} \int d^2k_\perp \left(\frac{2\pi}{2\pi}\right)^2 \int d^2b_\perp \int d^2r_\perp e^{-ik_\perp \cdot r_\perp} \frac{N_\tau(r_\perp, b_\perp)}{r_\perp^2}.
\]

(2.8)

The definition (2.7) should be compared to the canonical definition of the gluon density, which involves the expectation value of the gluon occupation number in the infinite momentum frame and in the light-cone gauge [1, 2, 3]. In the linear regime at high \(Q^2\), these two definitions are equivalent, and provide both the standard gluon distribution which measures (Bjorken) scaling violation in \(F_2\). But in the non-linear regime at \(Q^2 \lesssim Q_s^2\), there is no simple relationship between \(F_2\) and the canonical gluon density. By contrast, the definition (2.7) has the advantage that it is actually the Wigner transform of \(N_\tau(r_\perp, b_\perp)\) and, in the light-cone gauge \(N_\tau(r_\perp, b_\perp)\), there is no simple relationship between \(F_2\) and the canonical gluon density. Encouraged by this observation, let us investigate the scaling properties of the solution to the BK equation in more generality:

1) Fixed coupling case:

We consider first the case where the coupling \(\alpha_s\) in the r.h.s. of eq. (2.1) is fixed, and search for a solution \(S_\tau(r_\perp)\) to this equation in the scaling form:

\[
S_\tau(r_\perp) \equiv 1 - \Phi(\xi), \quad \xi \equiv \ln \frac{1}{r_\perp Q_s^2(\tau)}.
\]

(2.9)

Assuming such a scaling, the \(\tau\) dependence of the saturation scale is then fixed by the equation. To see this, note first that for any function \(f(\xi)\),

\[
\frac{\partial}{\partial \tau} f(\xi) = - \left(\frac{\partial}{\partial \tau} \ln Q_s^2(\tau)\right) f'(\xi), \quad r_\perp^2 \frac{\partial}{\partial r_\perp} f(\xi) = - f'(\xi),
\]

with \(f'(\xi) = df(\xi)/d\xi\). Then, integrating the BK equation (2.1) over \(r_\perp = x_\perp - y_\perp\) (after first dividing it by \(r_\perp^2\)), one gets

\[
\int d^2r_\perp \frac{1}{r_\perp^2} \frac{\partial}{\partial \tau} S_\tau(r_\perp) = \pi \left(S_\tau(\infty) - S_\tau(0)\right) \frac{\partial}{\partial \tau} \ln Q_s^2(\tau) = - \pi \frac{\partial}{\partial \tau} \ln Q_s^2(\tau),
\]

where we have used the boundary conditions \(S_\tau(0) = 1\) and \(S_\tau(\infty) = 0\). Therefore,

\[
\frac{\partial}{\partial \tau} \ln Q_s^2(\tau) = c\bar{\alpha}_s,
\]

(2.10)
where \( c \) is given by
\[
c \equiv \int \frac{d^2 r_\perp d^2 z_\perp}{2\pi^2} \frac{1}{z_\perp^2 (r_\perp - z_\perp)^2} \left( S_\tau(r_\perp) - S_\tau(z_\perp) S_\tau(r_\perp - z_\perp) \right) .
\] (2.11)
If \( S_\tau(r_\perp) \) is a scaling solution, then the r.h.s. of eq. (2.11) is a constant independent of \( \tau \). This follows from the scale invariance of the integrand: The function (2.9) depends upon \( \tau \) only via the scale \( Q_s(\tau) \) within the scaling variable; thus, by changing variables according to \( u_i^\perp \equiv r_i^\perp Q_s(\tau) \) and \( v_i^\perp \equiv z_i^\perp Q_s(\tau) \), all the \( \tau \) dependence goes away. More explicitly, the integral in eq. (2.11) can be written only in terms of the scaling variable \( \xi = \ln \frac{1}{r_\perp^2} \):
\[
c = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi' e^{-\xi} \left[ \frac{\Phi(\xi') - \Phi(\xi)}{|e^{-\xi'} - e^{-\xi}|} + \frac{\Phi(\xi)}{\sqrt{4e^{-2\xi'} + e^{-2\xi}}} \right] - \frac{1}{2} \left( \int_{-\infty}^{\infty} d\xi \Phi(\xi) \right)^2 .
\] (2.12)
Therefore, the r.h.s. of eq. (2.10) is independent of \( \tau \), which implies that the saturation scale grows exponentially with \( \tau \):
\[
Q_s^2(\tau) = \Lambda^2 e^{c \alpha_s \tau} ,
\] (2.13)
with \( \Lambda \) fixed by the initial condition (typically, \( \Lambda \sim \Lambda_{\text{QCD}} \)). With this saturation scale, eq. (2.10) can be easily integrated, with the following result which shows how the black disk limit \( (S_\tau = 0) \) is approached when \( r_\perp \gg 1/Q_s \) [8, 9, 3]:
\[
S_\tau(r_\perp) \propto \exp \left\{ -\frac{\xi^2}{2c} \right\} .
\] (2.14)
In order to search for scaling solutions, it is preferable to rewrite the original BK equation as an equation for the scaling function \( \Phi(\xi) \). It turns out that the equation takes a simpler form when written in momentum space. That is, the scaling Ansatz (2.9) is reformulated at the level of the BK equation in momentum space (2.4). In view of eq. (2.3), it is natural to write:
\[
\Psi(\zeta) \equiv \frac{\varphi_\tau(q_\perp)}{q_\perp^2} , \quad \zeta = \ln \frac{q_\perp^2}{Q_s^2(\tau)} .
\] (2.15)
The function \( \Psi(\zeta) \) is related to \( \Phi(\xi) \) via the following relation, which follows from eq. (2.3):
\[
\Phi(\xi) = \int_{-\infty}^{\infty} \frac{d\zeta}{4\pi} e^{\xi-\zeta} J_0 (e^{(\zeta-\xi)/2}) \Psi(\zeta) ,
\] (2.16)
where \( J_0 \) is a Bessel function. By inserting the Ansatz (2.13) in eq. (2.4), and using also eq. (2.10), one obtains the following equation for the scaling function \( \Psi(\zeta) \):
\[
- c \frac{\partial}{\partial \zeta} \Psi(\zeta) = \int_{-\infty}^{\infty} d\zeta' \left\{ \frac{e^{\xi'} \Psi(\zeta') - e^{\xi} \Psi(\zeta)}{|e^{\xi'} - e^{\xi}|} + \frac{e^{\xi} \Psi(\zeta)}{\sqrt{4e^{2\xi'} + e^{2\xi}}} \right\} - \frac{1}{2\pi} \Psi^2(\zeta) .
\] (2.17)
This equation, together with the definition (2.12) of \( c \), and the relation (2.16) between \( \Psi(\zeta) \) and \( \Phi(\xi) \), form a system of a coupled equations which in principle determine the scaling solution and the associated coefficient \( c \).

2) Running coupling case:
One should stress that the running of the QCD coupling is a higher order effect, which so far has not been included in the derivation of the non-linear evolution equations from first principles. Thus, treating $\alpha_s$ in the BK equation (2.1) as a running coupling “constant” is just a phenomenological way to incorporate some (potentially large) higher order corrections, and suffers from ambiguities. Here, we shall assume the one-loop–like running

$$\bar{\alpha}_s(Q^2) = \frac{b_0}{\ln(Q^2/\Lambda_{\text{QCD}}^2)}, \quad b_0 = \frac{12N_c}{11N_c - 2N_f}, \quad (2.18)$$

with the scale $Q^2$ chosen as the saturation momentum: $Q^2 = Q_s^2(\tau)$. This is physically acceptable since $Q_s$ is the typical momentum of the gluons in the saturated regime. Moreover, numerical solutions to the fixed coupling BK equation show that the dipole scattering amplitude is peaked around the saturation scale [11]. (Other possibilities for running, e.g., $Q^2 = 1/r_\perp^2$, with $r_\perp$ the size of the dipole, will be left for a later study [32].)

Since the running coupling (2.18) involves also the QCD scale $\Lambda_{\text{QCD}}$, this is not a single scale problem any longer. It is therefore a nontrivial question in general whether a scaling solution exists with running coupling. But at least for very low $q_\perp^2$, the solution $\varphi_\tau(q_\perp)$ in eq. (2.6) is manifestly a scaling solution even for a running coupling, since independent of $\alpha_s$! (Together with eq. (2.7), this reflects the fact that the gluon density at saturation is of order $1/\alpha_s$.)

Thus, even in this case, it is legitimate to look for a scaling solution, of the form (2.9). By the same arguments as above, this leads us to the differential equation

$$\frac{\partial}{\partial \tau} \ln Q_s^2(\tau) = c\bar{\alpha}_s(Q_s^2(\tau)), \quad (2.19)$$

where $c$ is again constant, and takes the same value as in the fixed coupling case, since determined (together with the scaling function $\Phi(\xi)$) by the same coupled equations (2.12), (2.16) and (2.17).

As compared to the fixed coupling case, the growth of the saturation scale becomes somewhat milder ($\tau_0$ is an arbitrary constant)

$$Q_s^2(\tau) = \Lambda_{\text{QCD}}^2 e^{\sqrt{2b_0c(\tau+\tau_0)}}. \quad (2.20)$$

It would be interesting to clarify if the system of equations (2.12), (2.16) and (2.17) has indeed solutions, i.e., if exact scaling solutions to the BK equation really exist. But as we shall argue in the next section, the physically relevant solutions necessarily violate scaling at sufficiently small $r_\perp$. This is so since, when $r_\perp \ll 1/Q_s(\tau)$, the BK equation linearizes, so its solution should match on the corresponding solution to the BFKL (or DLA) equation, which shows scaling only in a limited range of $r_\perp$ below the saturation length $1/Q_s(\tau)$.

3 The BFKL equation in the context of saturation

We have seen in the previous section that, when the size of the dipole is small compared to the saturation scale, $r_\perp \ll 1/Q_s(\tau)$, the general evolution equation can be linearized in the scattering amplitude $N_\tau(r_\perp)$, and then it reduces to the (coordinate space form of the) BFKL equation [12, 28]:

$$\frac{\partial}{\partial \tau} N_\tau(r_\perp) = \bar{\alpha}_s \int \frac{d^2z_\perp}{\pi} \frac{r_\perp^2}{(r_\perp - z_\perp)^2z_\perp^2} \left( N_\tau(z_\perp) - \frac{1}{2} N_\tau(r_\perp) \right). \quad (3.1)$$
The solutions to this equation have been extensively studied (see, e.g., [33] for an approach similar to ours), but they will be reconsidered here in the context of saturation, which requires the function \( N_\tau(r_\perp) \) to satisfy the boundary condition \( N_\tau(r_\perp) \sim 1 \) for \( r_\perp \sim 1/Q_s(\tau) \). This is automatically satisfied by the solution to the non-linear BK equation, but in the framework of the linear BFKL equation it becomes a non-trivial condition which determines the saturation scale. (A similar strategy to determine \( Q_s \) in the context of the BFKL equation has been previously used by A. Mueller [7].)

To understand this boundary condition, recall that, at saturation, \( N_\tau(r_\perp) = 1 - S_\tau(r_\perp) \) is a scaling function (see, e.g., eq. (2.14)): \( N_\tau(r_\perp) = f(r_\perp^2/Q_s^2(\tau)) \), and thus becomes a constant \( \kappa \) when \( r_\perp Q_s(\tau) = 1 \). This constant is a number of order one (although strictly smaller: \( \kappa < 1 \)), since \( S_\tau(r_\perp) \ll 1 \) at saturation. The precise value of \( \kappa \) is a matter of convention — it defines what we mean exactly by “the saturation scale” —, but this will not matter for what follows.

### 3.1 The solution to the BFKL equation revisited

In this subsection we shall construct approximate solutions to the BFKL equation (3.1) for the case of a fixed coupling constant \( \alpha_s \). Although some of the results are quite standard (see, e.g., [33]), we prefer to go through their derivation in some detail, in order to clarify the range of validity of the various approximations. This will be important for the discussion of extended scaling in the next subsection. Moreover, the techniques that we introduce here will be also useful later.

The approximations that we shall perform rely on the following inequalities:

\[
\bar{\alpha}_s \tau \gg 1, \quad \text{and} \quad r = \ln \frac{Q^2}{\Lambda^2} \gg 1, \quad (3.2)
\]

where \( \Lambda \) is some reference scale of order \( \Lambda_{\text{QCD}} \), and \( Q^2 \equiv 1/r_\perp^2 \). The conditions (3.2) express the fact that we consider a perturbative regime at small \( x \) and large \( Q^2 \). In addition, the quantities \( r \) and \( \bar{\alpha}_s \tau \) are not free to vary independently; rather, they are constrained by \( Q^2 \gg Q_s^2(\tau) \) (which ensures that we are in a linear regime), which requires (cf. eq. (2.13)):

\[
r = \ln \frac{Q^2}{\Lambda^2} > \ln \frac{Q_s^2(\tau)}{\Lambda^2} = c\bar{\alpha}_s \tau, \quad (3.3)
\]

with the coefficient \( c \) to be determined in Sect. 3.2.

Note first that eq. (3.1) has the same structure in coordinate space as the usual BFKL equation in momentum space (i.e., the linear part of eq. (2.4)), so it can be solved via similar techniques. For the present purposes, it is convenient to use the Mellin transform with respect to the transverse coordinates:

\[
N_\tau(r_\perp) = \int_C \frac{d\lambda}{2\pi i} \left( \frac{r_\perp^2}{\ell^2} \right)^\lambda \chi_\tau(\lambda) \quad (3.4)
\]

where \( \ell^2 = 1/\Lambda^2 \) and \( r_\perp^2 \ll \ell^2 \), so that the contour is taken on the left of all the singularities of the integrand in the half plane \( \text{Re} \lambda > 0 \). Since the BFKL equation is invariant under scale transformations, the ensuing equation for \( \chi_\tau(\lambda) \) is local in \( \lambda \):

\[
\frac{\partial}{\partial \tau} \chi_\tau(\lambda) = \bar{\alpha}_s \left\{ 2\psi(1) - \psi(\lambda) - \psi(1 - \lambda) \right\} \chi_\tau(\lambda), \quad (3.5)
\]
\( \psi(\lambda) \) is the di-gamma function, and has the obvious solution
\[
\chi_\tau(\lambda) = e^{\alpha_s\tau(2\psi(1) - \psi(\lambda) - \psi(1-\lambda))} \chi_0(\lambda).
\]
(3.6)
The initial condition \( \chi_0(\lambda) \) is not important for what follows (see below), so it is left unspecified.

In order to return to coordinate space, we need to perform the integral
\[
I \equiv \int_C \frac{d\lambda}{2\pi i} e^{\alpha_s\tau(2\psi(1) - \psi(\lambda) - \psi(1-\lambda))} \chi_0(\lambda) = \int_C \frac{d\lambda}{2\pi i} e^F(\lambda, r, \tau)
\]
(3.7)
where \( r \equiv \ln(r^2/\ell^2) \) is negative and large in the range of interest. (That is, in coordinate space, the second condition (3.2) is rewritten as \( -r \gg 1 \).) The function (3.6) has essential singularities at all the positive integers \( \lambda \geq 1 \), so we can choose the contour as \( C = \{ \lambda = a + i\nu, -\infty < \nu < \infty \} \), with \( 0 < a < 1 \).

The integral (3.7) will be evaluated in the saddle point approximation, which is a good approximation when \( r \) and \( \bar{\alpha}_s\tau \) are both large. (The corrections to it are suppressed by powers of \( 1/r \) or \( 1/\bar{\alpha}_s\tau \).) To the same accuracy, we can ignore the initial condition \( \chi_0(\lambda) \), since its contribution to the function \( F(\lambda, r, \tau) \) in the exponent is not enhanced by either \( r \) or \( \bar{\alpha}_s\tau \). (We implicitly assume here that \( \chi_0(\lambda) \) is not rapidly varying in the range of \( \lambda \) of interest.)

We thus obtain:
\[
I \simeq e^{F(\lambda_0)} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} e^{-\frac{1}{2} \nu^2 F''(\lambda_0)} = e^{F(\lambda_0)} \frac{1}{\sqrt{2\pi F''(\lambda_0)}},
\]
(3.8)
where, from now on,
\[
F(\lambda) \equiv r\lambda + \bar{\alpha}_s\tau \{2\psi(1) - \psi(\lambda) - \psi(1-\lambda)\} \equiv F(\lambda, r, \tau),
\]
(3.9)
and \( \lambda_0 \) satisfies the saddle point equation:
\[
\left. \frac{\partial F(\lambda, r, \tau)}{\partial \lambda} \right|_{\lambda_0} = 0, \quad \lambda_0 = \lambda_0(r, \tau).
\]
(3.10)
In eq. (3.8), we have also included the contribution of the Gaussian fluctuations around the saddle point. That is, we have taken the contour \( C = \{ \lambda = \lambda_0 + i\nu, -\infty < \nu < \infty \} \), and we have expanded in powers of \( \nu \) to quadratic order before integrating over \( \nu \).

To visualize the solution to eq. (3.10), it is useful to keep in mind that \( 2\psi(1) - \psi(\lambda) - \psi(1-\lambda) \) is a convex function of \( \lambda \) with its minimum at \( \lambda = 1/2 \) and simple poles at \( \lambda = 0 \) and \( \lambda = 1 \). A good approximation to this function in the range \( 0 < \lambda < 1 \) is given by
\[
2\psi(1) - \psi(\lambda) - \psi(1-\lambda) \approx \frac{1}{\lambda} + \frac{1}{1-\lambda} + 4 \ln 2 - 4.
\]
(3.11)
Thus, there is an unique saddle point \( \lambda_0 \) in the region \( 0 < \lambda < 1 \), whose position moves between 0 and 1 depending upon the value of the ratio \( r/\bar{\alpha}_s\tau \). There are three limiting cases of interest:

(A) When \( r/\bar{\alpha}_s\tau \) is positive and large enough: the saddle point is close to \( \lambda = 0 \).
(B) When \( r/\bar{\alpha}_s\tau \) is small \~{} 0: the saddle point is close to \( \lambda = 1/2 \).
(C) When \( r/\bar{\alpha}_s\tau \) is strongly negative: the saddle point is close to \( \lambda = 1 \).
More precise boundaries between these various cases will be specified below.

The first case is relevant when we consider the BFKL equation in momentum space, where \( r \equiv \ln(k_{\perp}^2/\Lambda^2) \) is always positive in the range of interest. Similarly, case (C) applies only to the coordinate space, where \( r \equiv \ln(r_{\perp}^2/\ell^2) \) is negative. These two cases, (A) and (C), correspond to the double logarithmic approximation (DLA), which describes the leading behaviour of the solution at large \( k_{\perp}^2 \) (or small \( r_{\perp}^2 \)) for fixed, but large, \( \tau \); this limit is common to the BFKL and DGLAP equations. Case (B), on the other hand, applies to both the momentum space \((r > 0)\) and the coordinate space \((r < 0)\). In standard analyses of the BFKL equation, this is the case which describes the high energy limit \((\tau \to \infty \text{ at fixed } r)\) \[33\]. Here, we shall never be truly in this limit, because of the condition (3.3) which, strictly speaking, implies that \( r/\bar{\alpha}_s \tau \) is never small! If case (B) is nevertheless relevant for us here, it is because the true saddle point remains close to \( \lambda = 1/2 \) up to relatively large values of \( r/\bar{\alpha}_s \tau \), which, as we shall see, leaves enough space for the condition (3.3) to be satisfied.

(A) Assuming that \( \lambda_0 \ll 1 \), we find the saddle point

\[
\lambda_0 \simeq \sqrt{\frac{\bar{\alpha}_s \tau}{r}}, \tag{3.12}
\]

which after insertion in eq. (3.8) and performing the Gaussian integral there, leads to

\[
I \simeq e^{2\sqrt{\bar{\alpha}_s \tau}} \sqrt{\frac{\lambda_0^3}{4\pi \bar{\alpha}_s \tau}}. \tag{3.13}
\]

When \( r = \ln(k_{\perp}^2/\Lambda^2) \), this gives the well-known “DLA solution” in the momentum space, which coincides with the solution to the DGLAP equation in the double-log limit \[19\].

The condition that \( \lambda_0 \ll 1 \) should be more properly formulated as \( \lambda_0 \ll 1/4 \), since \( \lambda = 1/4 \) is the middle point between the limiting saddle points at \( \lambda = 0 \) and \( \lambda = 1/2 \). This criterion gives the following range of validity for the DLA solution (3.13) in momentum space

\[
16\bar{\alpha}_s \tau \ll \ln(k_{\perp}^2/\Lambda^2). \tag{3.14}
\]

(B) When \( r/\bar{\alpha}_s \tau \) is small, the saddle point is very close to \( \lambda = 1/2 \) :

\[
\lambda_0 \simeq \frac{1}{2} - \delta, \quad \delta \equiv \frac{r}{\beta \bar{\alpha}_s \tau}, \tag{3.15}
\]

which gives:

\[
I \simeq e^{\omega \bar{\alpha}_s \tau} e^{\frac{r}{2\beta \bar{\alpha}_s \tau}} \exp \left\{ -\frac{r^2}{2\beta \bar{\alpha}_s \tau} \right\} \left( \frac{1}{\sqrt{2\pi \beta \bar{\alpha}_s \tau}} \right), \tag{3.16}
\]

where \( \beta \equiv (-2\psi''(1/2)) = 28\zeta(3) \) and \( \omega \equiv 2\psi(1) - 2\psi(1/2) = 4\ln 2 \).

When \( r = \ln(k_{\perp}^2/\Lambda^2) \), eq. (3.16) corresponds to the usual solution to the momentum BFKL equation. On the other hand, when \( r = \ln(r_{\perp}^2/\ell^2) \), it also gives the solution of the linearized BK equation in coordinate space:

\[
N_r(r_{\perp}) \simeq \sqrt{\frac{r_{\perp}^2}{\ell^2}} \frac{e^{\omega \bar{\alpha}_s \tau}}{\sqrt{2\pi \beta \bar{\alpha}_s \tau}} \exp \left\{ -\ln^2 \left( \frac{r_{\perp}^2}{\ell^2} \right) \right\}. \tag{3.17}
\]
The saddle point in eq. (3.15) remains close to $1/2$ for all $r$ such that $\delta \ll 1/4$. This is realized when (in momentum space, for definiteness)

$$\ln\left(\frac{k^2_\perp}{\Lambda^2}\right) \ll 8\bar{\alpha_s}\tau,$$

which, as we shall soon discover, leaves a substantial window for genuine BFKL behaviour (in the sense of eq. (3.16)) above the saturation scale.

Since there is no overlapping between the conditions (3.14) and (3.18), we conclude that in the intermediate range where $8 \ll r/\bar{\alpha}_s\tau \lesssim 16$, the saddle point is relatively close to $\lambda = 1/4$. Indeed, by using the approximation (3.11), one can estimate that $\lambda_0 = 1/4$ for $r/\bar{\alpha}_s\tau \simeq 14$.

Eq. (3.16) is the solution that is usually considered in applications of the BFKL equation at very high energy [34]. It shows an exponential growth with $\tau$ and “infrared diffusion” towards small $k_\perp$ momenta, which would become problematic in the formal high energy limit $\tau \to \infty$ at fixed $r$ (which is, of course, outside the kinematical range considered here; recall eq. (3.3).) But at sufficiently high energy, $Q^2(\tau) > Q^2$, and the BFKL equation gets supplanted by the BK equation (or the more general equations in Refs. [15, 17, 2]), in which non-linear effects provide a natural solution to the difficulties of eq. (3.16) alluded to above [16, 8, 11].

(C) For $r/\bar{\alpha}_s\tau$ negative and large, the saddle point is close to $\lambda = 1$:

$$\lambda_0 \simeq 1 - \delta_1, \quad \delta_1 \equiv \frac{\bar{\alpha}_s\tau}{(-r)},$$

and the integral is estimated as

$$I \simeq \exp\left\{r + 2\sqrt{\bar{\alpha}_s\tau(-r)}\right\} \sqrt{\frac{\delta_1^3}{4\pi\bar{\alpha}_s\tau}}.$$

This saddle point is interesting only in coordinate space, where $r = \ln(r^2_\perp/\ell^2) < 0$, and gives the dominant behaviour $N_\tau(r_\perp)$ at short distances:

$$N_\tau(r_\perp) = r^2_\perp \Lambda^2 \exp\left\{2\sqrt{\bar{\alpha}_s\tau \ln\left(\frac{1}{r^2_\perp\Lambda^2}\right)}\right\} \sqrt{\frac{\delta_1^3}{4\pi\bar{\alpha}_s\tau}}, \quad \delta_1 = \sqrt{\frac{\bar{\alpha}_s\tau}{\ln(1/r^2_\perp\Lambda^2)}},$$

where we have taken $\ell^2 = 1/\Lambda^2$. As expected, this is the same as the asymptotic solution of the DLA equation in coordinate space. The saddle point (3.19) is close to 1 provided $\delta_1 \ll 1/4$, which gives

$$16\bar{\alpha}_s\tau \ll \ln(1/r^2_\perp\Lambda^2),$$

in complete agreement with the previous condition (3.14) in momentum space.

It is convenient to summarize the previous results in terms of the ratio

$$R \equiv \frac{(-r)}{\bar{\alpha}_s\tau} \equiv \frac{1}{\bar{\alpha}_s\tau} \ln\frac{1}{r^2_\perp\Lambda^2},$$

The additional factor $r^2_\perp\Lambda^2$ in front of the exponential in eq. (3.21) as compared to the DLA solution (3.13) in momentum space comes up since, in coordinate space, the DLA equation applies to $N_\tau(r_\perp)/r^2_\perp$.
(\(R\) is positive in the perturbative regime of interest) which will play a special role in the next subsection. Our analysis shows that the saddle point solution to the BFKL equation has the genuine BFKL behaviour (3.16) for \(R \ll 8\), the DLA behaviour (3.21) for \(R \gg 16\), and some intermediate behaviour in the window \(8 \lesssim R \lesssim 16\). These conclusions, based on a pure BFKL analysis, are still to be combined with the constraint (3.3) showing that we are indeed in the linear regime. This requires to determine the coefficient \(c\) in eq. (3.3), which we shall do in the next subsection.

### 3.2 Saturation scale and scaling from the BFKL equation

As already explained, it is possible to determine the saturation momentum from the solution \(N_\tau(r_\perp)\) to the BFKL equation (3.1) by using the saturation criterion:

\[
N_\tau(r_\perp = 1/Q_s(\tau)) = 1.
\]

Strictly speaking, the number in the r.h.s. of this equation is not exactly 1 (cf. the discussion at the beginning of Sect. 3), but this is irrelevant to the accuracy of the present calculation. Indeed, replacing 1 by \(\kappa < 1\) would modify the following results via subleading terms, of relative order \((1/\bar{\alpha}_s \tau) \ln \kappa\). Besides, such terms would affect the overall normalization of the saturation scale (3.29), that is, the value of the reference scale \(\Lambda^2\), which has not been fully specified anyway.

In what follows, we shall rely on the saddle point approximation in eqs. (3.8)–(3.10) to obtain an estimate for \(Q_s(\tau)\) [7], and then study the scaling properties of the BFKL solution above the saturation scale. For these purposes, it is crucial to notice from eq. (3.10), which is rewritten as

\[
\frac{\partial}{\partial \lambda} \left( \psi(\lambda) + \psi(1 - \lambda) \right) \bigg|_{\lambda_0} = \frac{r}{\bar{\alpha}_s \tau} \equiv -R,
\]

that the saddle point \(\lambda_0\) is actually a function of only one variable \(R\) (cf. eq. (3.23)) :

\[
\lambda_0(r, \tau) = \lambda_0(R).
\]

If one estimates the Mellin integral just by the saddle point, one then obtains

\[
N_\tau(r_\perp) \simeq e^{\bar{\alpha}_s \tau F(\lambda_0(R), R)},
\]

where, as compared to eqs. (3.8)–(3.10), we have changed the definition of the function \(F(\lambda, r, \tau)\) by pulling out a factor \(\bar{\alpha}_s \tau\). This is convenient since, when evaluated at the saddle point, the new function \(F(\lambda_0(R), R)\) is only a function of \(R\). Also, we have neglected the (slowly varying) factor coming from the Gaussian integral over the fluctuations. This is correct up to corrections of relative order \(\ln(\bar{\alpha}_s \tau)/\bar{\alpha}_s \tau\). The effect of this factor will be illustrated in eq. (3.43) below.

The saturation criterion (3.24) yields then the following condition on \(F\) :

\[
F(\lambda_0(R_s), R_s) = 0 \quad \text{for} \quad R_s = \frac{1}{\bar{\alpha}_s \tau} \ln \frac{Q_s^2(\tau)}{\Lambda^2}
\]

which is an equation for \(R_s\), and ultimately for \(Q_s(\tau)\).

This equation has two immediate and important consequences, which are among the main results in this paper: (i) The saturation momentum is increasing exponentially with \(\tau\), with the
slope of the exponential uniquely fixed by the saddle point solution to the BFKL equation. (ii) The “anomalous dimension” which characterizes the approach towards saturation (cf. eq. (1.3)), is constant, i.e., independent of $\tau$, which then implies geometric scaling.  

i) Indeed, the solution $R_s$ to eq. (3.28) is a pure number, $R_s \equiv c$, and not a function of $\tau$ (as it could have been if the function $F(\lambda, r, \tau)$ in eq. (3.27) was a function of two independent variables, $r$ and $\tau$, and not just of their ratio $R$). Together with the second equation (3.28), this implies

$$Q_s^2(\tau) = \Lambda^2 e^{c\bar{\alpha}_s \tau},$$

(3.29)

which is consistent with a previous result (2.13) based on scaling properties of the BK equation, but which here arises from a (BFKL) solution which has no geometric scaling (e.g., eqs. (3.16) or (3.21) are not scaling functions for generic values of $r_\perp$ and $\tau$). Moreover, the value of the slope parameter $c$ is known here, since uniquely fixed by the solution to eqs. (3.28) and (3.25).

In fact, in order to compute $R_s \equiv c$, there is no need to solve the saddle point equation (3.25) for arbitrary values of $R$ (although this could be done via numerical techniques). Rather, we notice that by combining eqs. (3.28) and (3.25) one can deduce an equation for the particular value $\lambda_s \equiv \lambda_0(R_s)$ that the saddle point takes at saturation:

$$-\lambda_s \frac{\partial}{\partial \lambda} \left( \psi(\lambda) + \psi(1-\lambda) \right) \bigg|_{\lambda_s} = \lambda_s R_s = 2\psi(1) - \psi(\lambda_s) - \psi(1-\lambda_s).$$

(3.30)

We have solved this numerically and obtained:

$$\lambda_s = 0.627549..., \quad R_s \equiv c = 4.88339....$$

(3.31)

Note that this $\lambda_s$ is not too far away from $1/2$, which is consistent with the fact that the above value of $R_s$ is in the range where we expect a “genuine” BFKL behaviour, i.e., $R_s < 8$. We shall return to this point after the discussion of geometric scaling.

ii) Let us evaluate eq. (3.27) for $R$ slightly above $R_s$, that is, for distances $r_\perp$ which, while still being much shorter than the saturation length $1/Q_s(\tau)$, are nevertheless close to it in logarithmic units. More precisely, we shall require that (with $Q^2 \equiv 1/r_\perp^2$) :

$$0 < R - R_s \ll R_s, \quad \text{or} \quad 1 < \ln \frac{Q^2}{Q_s^2(\tau)} \ll \ln \frac{Q_s^2(\tau)}{\Lambda^2}.$$  

(3.32)

The condition $R > R_s$ (i.e., $Q^2 > Q_s^2(\tau)$) ensures that we stay in the linear regime; in logarithmic units, this is effectively implemented as $Q^2 > eQ_s^2(\tau)$, with $e = 2.72...$. The condition $R - R_s \ll R_s$ allows us to study the approach of $N_\tau(r_\perp)$ towards saturation in a limited expansion in powers of $R - R_s$. To linear order in this expansion, one has

$$F(\lambda_0(R), R) \simeq F(\lambda_0(R_s), R_s) + \frac{d}{dR} F(\lambda_0(R), R) \bigg|_{R=R_s} (R - R_s) + \cdots$$

$$= -\lambda_s (R - R_s) + \cdots,$$

(3.33)

3If we had included the Gaussian fluctuations around the saddle point, this pure exponential would get multiplied by a slowly varying prefactor; see eq. (3.43) for an example.
where we have used the saturation condition \( \text{(3.28)} \), together with the fact that \( \lambda_0(R) \) is a solution of the saddle point equation, cf. eq. \( \text{(3.11)} \), so that:

\[
\frac{d}{dR} F(\lambda_0(R), R) \bigg|_{R=R_s} = -\lambda_0(R_s) + \frac{\partial F}{\partial \lambda} \bigg|_{\lambda=\lambda_0} \frac{\partial \lambda_0}{\partial R} \bigg|_{R=R_s} = -\lambda_s.
\]

Therefore, above the saturation scale, the dipole-hadron scattering amplitude is approximated as follows [recall that \( \bar{\alpha}_s \tau(R-R_s) = \ln(1/r^2_\perp Q_s^2(\tau)) \)]

\[
\mathcal{N}_\tau(r_\perp) \simeq \kappa e^{-\bar{\alpha}_s \tau \lambda_0(R-R_s)} = \kappa \left( r^2_\perp Q_s^2(\tau) \right)^{\lambda_s},
\]

where we have reintroduced the numerical factor \( \kappa < 1 \) (cf. the discussion after eq. \( \text{(3.24)} \)).

Eq. \( \text{(3.33)} \) shows geometric scaling with the anomalous dimension\(^4 \) \( \gamma = 1 - \lambda_s \simeq 0.37 \) determined by the value of the saddle point \( \lambda_s \equiv \lambda_0(R_s) \) at saturation. We summarize here the two essential ingredients in the arguments leading to this scaling: (a) the saturation condition which requires that \( \mathcal{N}_\tau(r_\perp) \to 1 \) when \( r_\perp \to 1/Q_s(\tau) \); (b) the fact that, at saturation, the saddle point \( \lambda_0(R) \) becomes independent of \( \tau \) (i.e., a pure number; cf. eq. \( \text{(3.32)} \)). In turn, point (b) is the consequence of the “scaling” property \( \text{(3.26)} \) of the saddle point, and ultimately reflects the scale invariance of the BFKL equation \( \text{(3.1)} \).

The previous arguments also shed light on potential sources of scaling violation when going beyond the present approximations. For instance, we expect \( \tau \)-dependent corrections to the anomalous dimension due to the Gaussian fluctuations, the higher-order terms in the saddle point expansion, the initial condition \( \chi_0(\lambda) \) in the Mellin function \( \text{(3.4)} \), etc. These corrections are suppressed by \( 1/\bar{\alpha}_s \tau \). Also, there are corrections to the functional form of \( \mathcal{N}_\tau(r_\perp) \), coming from higher order terms in the expansion \( \text{(3.33)} \) around the saturation scale. These corrections are controlled by the ratio \( (R-R_s)/R_s = \ln Q^2/Q_s^2(\tau) / \ln Q^2(\tau)/\Lambda^2 \). For instance, after also including the second order in this expansion, eq. \( \text{(3.33)} \) gets replaced by

\[
\mathcal{N}_\tau(r_\perp) \simeq \kappa \left( r^2_\perp Q_s^2(\tau) \right)^{\lambda_s} \exp \left\{ -\frac{\lambda_s^\prime}{2\bar{\alpha}_s \tau} \left( \ln \frac{1}{r^2_\perp Q_s^2(\tau)} \right)^2 \right\}
\]

where \( \lambda^\prime_s \equiv (d\lambda_0(R)/dR)|_{R_s} \). Clearly, the exponential term in this expression violates scaling.

The scaling behaviour \( \text{(3.35)} \) holds in a window in \( Q^2 \) specified by eq. \( \text{(3.32)} \), that is:

\[
1 < \ln \frac{Q^2}{Q^2_\tau(\tau)} \equiv \left( \ln \frac{Q^2}{\Lambda^2} - c\bar{\alpha}_s \tau \right) \ll c\bar{\alpha}_s \tau.
\]

\(^4\)We refer to \( 1 - \lambda_s \) as an “anomalous dimension” since, naively, one could expect \( \mathcal{N}_\tau(r_\perp) \) to vanish like \( r^2_\perp \) as \( r_\perp \to 0 \); this would follow from the fact that \( \mathcal{N}_\tau(0) = 0 \), together with analyticity near \( r_\perp \to 0 \). In reality, this analyticity is broken by logarithmic ultraviolet divergences in the formal expansion in powers of \( r^2_\perp \). These divergences correspond to the large logarithms \( \ln(Q^2/\Lambda^2) \) whose resummation by the DGLAP (or DLA) equation leads to the non-analytic behaviour manifest in eq. \( \text{(3.21)} \). Besides, whatever are the analytic properties of the function \( \mathcal{N}_\tau(r_\perp) \) near \( r_\perp \to 0 \) (i.e., at very high \( Q^2 \)), there is no reason why these properties should persist down to the \( Q^2 = Q_s^2(\tau) \). Compare, in this respect, eqs. \( \text{(3.35)} \) and \( \text{(3.21)} \).
This requires that both \( r = \ln(Q^2/\Lambda^2) \) and \( c\tilde{\alpha}_s\tau \) are large numbers, with \( r \) larger than \( c\tilde{\alpha}_s\tau \) (in agreement with the original assumptions (3.2) and (3.3)), but not much larger: \( r - c\tilde{\alpha}_s\tau \ll c\tilde{\alpha}_s\tau \). In practice, this requires \( \ln(Q^2/\Lambda^2) \) to be deeply inside the strip (note that \( c = 4.88... \approx 5 \))

\[
5\tilde{\alpha}_s\tau < \ln(Q^2/\Lambda^2) < 10\tilde{\alpha}_s\tau, \tag{3.38}
\]

which is indeed a large window when \( 5\tilde{\alpha}_s\tau \gg 1 \). Together with eq. (3.18), this suggests that the window for extended scaling is almost entirely located in the kinematical range controlled by the BFKL saddle point (3.13).

To verify that, let us compute directly the predictions of this saddle point and the associated solution (3.17) for the saturation scale and geometric scaling. The “BFKL saturation scale” is defined by imposing the condition (3.24) directly on eq. (3.17):

\[
\sqrt{\frac{\Lambda^2}{Q_s^2}} e^{\omega\tilde{\alpha}_s\tau} \exp \left(-\frac{\ln^2(Q_s^2/\Lambda^2)}{2\beta\tilde{\alpha}_s\tau} \right) = 1. \tag{3.39}
\]

As expected, this amounts to an equation for \( R_s \), the value of the variable \( R \) at saturation (compare to eq. (3.28)). In this case, this is a second-order equation:

\[
R_s^2 + \beta R_s - 2\beta\omega = 0, \tag{3.40}
\]

with the positive solution:

\[
R_s \bigg|_{\text{BFKL}} = \frac{1}{2} \left( -\beta + \sqrt{\beta(\beta + 8\omega)} \right) = 4.8473..., \tag{3.41}
\]

This is the value of the slope parameter \( c \) predicted by the standard BFKL solution, and is numerically very close to that in eq. (3.31). By using (3.41) and (3.15), one can compute the BFKL saddle point at saturation \( \lambda_s \) (or the anomalous dimension \( \gamma = 1 - \lambda_s \)):

\[
\lambda_s \bigg|_{\text{BFKL}} \approx \frac{1}{2} + \frac{R_s}{\beta} = 0.644..., \tag{3.42}
\]

which is indeed very close to the true saddle point (3.31). This shows that for all practical purposes one can use the explicit BFKL solution (3.17) for any \( Q^2 = 1/r_\perp^2 \) in the window for extended scaling. In fact, without any approximation, eq. (3.17) can be cast in the form of the “second-order expansion” in eq. (3.30), with \( \lambda_s \) given by eq. (3.42), and \( \lambda_s' = 1/\beta \approx 1/33.67 \). This latter is a rather small number, showing that the violations of scaling due to the exponential term are only tiny. That is, in its whole domain of applicability, the “genuine” BFKL solution (3.17) is almost an exact scaling solution.

For completeness, and also for comparison with previous analytic studies in the literature which have used the DLA approximation \[8, 9, 11\], let us finally evaluate the scaling predictions of the DLA saddle point (3.19). The saturation condition (3.24) applied to eq. (3.21) yields:

\[
Q_s^2(\tau) \bigg|_{\text{DLA}} \simeq \Lambda^2 \frac{e^{\omega\tilde{\alpha}_s\tau}}{32\pi\tilde{\alpha}_s\tau}, \tag{3.43}
\]

where the slowly varying factor multiplying the exponential is the effect of the Gaussian fluctuations around the saddle point. Eq. (3.43) implies:

\[
R_s \bigg|_{\text{DLA}} \simeq 4 - \frac{\ln(32\pi\tilde{\alpha}_s\tau)}{\tilde{\alpha}_s\tau} \simeq 4, \tag{3.44}
\]
which is slowly dependent upon $\tau$, because of the contribution of the fluctuations, but to leading order takes the constant value $c_{\text{DLA}} = 4$. This is in agreement with previous studies \[8, 9, 11\]. If inserted in eq. (3.19), it gives the DLA anomalous dimension $\gamma = 1 - \lambda_s \ [8]$:

$$\gamma \bigg|_{\text{DLA}} = \sqrt{\frac{1}{R_s}} \simeq \frac{1}{2} + \frac{\ln(32\pi\bar{\alpha}_s\tau)}{16\bar{\alpha}_s\tau} \simeq \frac{1}{2}.$$ \hspace{1cm} (3.45)

This is slightly larger than for the true saddle point, or the BFKL saddle point (3.42). The DLA window for extended geometric scaling is given by eq. (3.37) with $c = 4$.

Thus, at a first sight, the DLA predictions look rather similar to those of the BFKL approximation, both for the value of the saturation scale (which fixes also the scaling window), and for the anomalous dimension. However, one should keep in mind that these predictions are not consistent with the validity region of DLA: the saddle point (3.45) not only is not close to 1, but it even takes the typical BFKL value. Besides, the corresponding scaling window is truly within the kinematical range for BFKL.

We finally mention that numerical studies of the BK equation \[8, 10, 11\] (with a fixed coupling) have found that the saturation scale is indeed increasing exponentially with $\tau$, with a slope parameter $c \approx 4.1$ which is intermediate between the DLA and BFKL predictions obtained in this section.

### 3.3 The running coupling case: $\alpha_s(Q^2_s(\tau))$

Except for the modified $\tau$–dependence of the saturation scale, which changes according to the general expectation in eq. (2.20), all the previous discussion of the solutions to the BFKL equation and their consequences for extended scaling goes almost unchanged to the case of a running coupling in which the scale for running is set by $Q_s(\tau)$. In this subsection, we shall only indicate the few steps which involve non-trivial differences.

The extra $\tau$ dependence of the coupling $\alpha_s(Q^2_s(\tau))$ commutes with the Mellin transform (3.4), which is defined only in terms of the transverse coordinates. Thus, the previous equation for $\chi_\tau(\lambda)$ is simply replaced by

$$\frac{\partial}{\partial \tau} \chi_\tau(\lambda) = \frac{b_0}{\ln(Q^2_s(\tau)/\Lambda_{\text{QCD}}^2)} \left\{ 2\psi(1) - \psi(\lambda) - \psi(1 - \lambda) \right\} \chi_\tau(\lambda). \hspace{1cm} (3.46)$$

So far, the saturation scale is not known, but it can be absorbed into a redefinition of the “time” variable $\tau$:

$$\ln(Q^2_s(\tau)/\Lambda_{\text{QCD}}^2) \frac{\partial}{\partial \tau} \equiv \frac{\partial}{\partial \tilde{\tau}}, \hspace{0.5cm} \text{or} \hspace{0.5cm} \tilde{\tau} \equiv \int_0^\tau d\tau' \frac{1}{f(\tau')}, \hspace{1cm} (3.47)$$

where we have written $Q^2_s(\tau) = \Lambda_{\text{QCD}}^2 e^{f(\tau)}$. Then, the equation is easily solved:

$$\chi_\tau(\lambda) = e^{b_0\tilde{\tau}\left\{ 2\psi(1) - \psi(\lambda) - \psi(1 - \lambda) \right\}}, \hspace{1cm} (3.48)$$

which leads us to the following Mellin representation for the solution to the BFKL equation with running coupling:

$$N_\tau(r_\perp) = \int_C \frac{d\lambda}{2\pi i} e^{F(\lambda, r, \tilde{\tau})} \hspace{0.5cm} \text{with} \hspace{0.5cm} F(\lambda, r, \tilde{\tau}) = r\lambda + b_0\tilde{\tau}\left\{ 2\psi(1) - \psi(\lambda) - \psi(1 - \lambda) \right\} \hspace{1cm} (3.49)$$
where \( r = \ln(r^2/\ell^2) = \ln(r^2/\Lambda^2) \) as before. As anticipated, this has the same structure as in the fixed coupling case (cf. eq (3.7)). Thus, all the results in Sects. 3.1 and 3.2 can be immediately translated to the case of a running coupling by simply replacing 
\[
\bar{\alpha}_s \to b_0, \quad \text{and} \quad \tau \to \tilde{\tau},
\]
(3.50)
in the corresponding formulae. In particular, by the same arguments as before, the saddle point \( \lambda_0 \) is a function of the ratio \( R \) alone, with \( R \equiv (-r)/b_0\tilde{\tau} \). The value \( \lambda_s = \lambda_0(R_s) \) of the saddle point at saturation is again determined by eq. (3.30), so that \( \lambda_s \) and \( c \equiv R_s \) take the same values as before, cf. eq. (3.31). Thus, all the previous results on extended scaling — the value of the anomalous dimension \( \gamma = 1 - \lambda_s \) in eq. (3.35), and the momentum range (3.32) in which the scaling holds — remain unchanged, except for the expression of the saturation scale which enters these results.

To determine this scale, we use \( c = R_s = (-r_s)/b_0\tilde{\tau} \) with \( (-r_s) = \ln Q^2_s(\tau)/\Lambda^2_{\text{QCD}} = f(\tau) \), together with eq (3.47), to derive an equation for \( f(\tau) \):
\[
\frac{1}{c} f(\tau) = b_0 \int_0^\tau d\tau' \frac{1}{f(\tau')}. \tag{3.51}
\]
This has the solution \( f(\tau) = \sqrt{2b_0c(\tau + \tau_0)} \) (and therefore \( \tilde{\tau} = \sqrt{2(\tau + \tau_0)/b_0c} \)). Therefore, the saturation scale is determined as
\[
Q^2_s(\tau) = \Lambda^2_{\text{QCD}} e^{2b_0c(\tau + \tau_0)}, \tag{3.52}
\]
where \( \tau_0 \) is arbitrary and \( c \) is given by eq. (3.31). This has the same functional form as eq. (2.20) that has been obtained from the BK equation with a scaling Ansatz. In particular, this confirms the previous arguments in Sect. 2 that the coefficient \( c \) which enters the exponent of the saturation scale should be the same for fixed coupling and running coupling.

Needless to say, the BFKL and DLA predictions for the anomalous dimension, eqs. (3.42) and respectively (3.45), remain unchanged.

4 Summary and discussion

We have shown that the geometric scaling predicted at low momenta \( Q^2 \ll Q^2_s \) by the Colour Glass Condensate and phenomenological saturation models is preserved by the BFKL evolution equation up to relatively large \( Q^2 \) momenta, within the range \( 1 \ll \ln(Q^2/Q^2_s) \ll \ln(Q^2_s/\Lambda^2_{\text{QCD}}) \). By matching the solution to the BFKL equation onto the saturation condition at \( Q^2 \sim Q^2_s \), we have determined the dependence of the saturation scale \( Q_s \) upon the rapidity \( \tau = \ln(1/x) \), and the anomalous dimension of the distribution function near saturation.

The matching has been performed for the dipole scattering amplitude, which enters linearly the structure function \( F_2(x, Q^2) \), and can be also related to the gluon distribution. We have found the same kinematical window for extended scaling for both fixed and running couplings (with the scale for running set by \( Q_s(\tau) \)), although the \( \tau \)–dependence of the saturation momentum turns out to be different in the two cases. We have also found that, formally, the double logarithmic approximation to the DGLAP equation leads to qualitatively, and even quantitatively, similar
predictions for the extended scaling, but these results are inconsistent with the validity range of this approximation.

We have shown that the functional form of the \( \tau \)-dependence of the saturation scale \( Q_s \) which follows from the BFKL equation is consistent with the general scaling properties of the BK equation.

It would be extremely interesting to compare these results with the \( F_2 \) data in deep inelastic scattering, for which geometric scaling has been originally identified \( [5] \), and to explore possible implications for particle production in heavy-ion experiments, where a phenomenological scaling law has been recently reported \( [33] \). In particular, we expect these results to have consequences for the analysis of the multiplicity distributions of produced particles in heavy ion collisions at RHIC \( [4] \). Our analysis shows that while \( F_2 \) has geometric scaling, the gluon distribution function, due to an extra factor of \( 1/\alpha_s \) in eqs. (2.7)–(2.8), has logarithmic scaling violations. This is consistent with scaling violations which are seen in the RHIC data \( [4] \).

While the qualitative phenomenon of the existence of extended geometric scaling up to momenta \( Q^2 \) of order 100 GeV\(^2\) is certainly consistent with the analysis of deep inelastic scattering by Staśto, Golec-Biernat and Kwieciński \( [3] \), it is at the same time clear that the saturation scale emerging from our analysis (for either fixed or running coupling) is too rapidly increasing with \( \tau = \ln(1/x) \) to give a good description of the data for \( F_2 \).

Consider fixed coupling first. Although eq. (3.29) shows a power law increase, \( Q^2_s(x) = Q^2_0 x^{-\lambda} \), in agreement with the parametrization used in Ref. \( [1] \), the actual value of the parameter \( \lambda \) predicted by the BFKL equation, namely \( \lambda \simeq (4 - 5)\alpha_s N_c/\pi \), is sensibly larger (for \( N_c = 3 \) and realistic values of \( \alpha_s \)) than the value \( \lambda = 0.3 - 0.4 \) extracted from the fit to the data \( [5] \).

As for the corresponding prediction in the case of a running coupling, eq. (3.52), this is less rapidly increasing at very large \( \tau \): \( Q^2_s(\tau) = \Lambda^2_{\text{QCD}} \exp \{ \sqrt{C(\tau+\tau_0)} \} \). But since the number \( C = 2b_0 c \) is relatively large (of order 10; cf. eqs. (2.15) and (3.34)), this too fails to reproduce the data, unless the parameter \( \tau_0 \) is mysteriously high.

The reasons for such a failure can be several. First, there are the many approximations that we have performed in order to obtain analytic solutions to the BFKL equation. Note that we have preserved just leading order terms in the exponents, and that the subleading terms are truly small only if the inequalities (3.2) are strictly satisfied. These are strong inequalities on logarithms, which may not be well fulfilled for realistic values of \( x \) and \( Q^2 \). Besides, even small subleading terms may give a substantial effect once exponentiated. This is illustrated by eqs. (3.43)–(3.45) which, in addition to the lowest order terms, include also the corrections due to the Gaussian fluctuations of the saddle point. As obvious on eq. (3.43), these “subleading” terms may drastically change the actual value of the saturation momentum. Still, the fact that the slope parameter \( c \approx 4 - 5 \) that we have found is rather close to that obtained via numerical studies of the BK equation \( [10, 11] \) makes us confident about our control of the approximation scheme for the exponent.

The last argument also suggests that, independent of further approximations, there is a true discrepancy between the exponent \( \lambda \simeq 4\alpha_s \) predicted by the BK equation and its phenomenological value \( \lambda = 0.3 - 0.4 \) \( [3, 23] \). Recall that the BK equation is strictly valid only in the large \( N_c \) limit, and to lowest order in \( \alpha_s \) (within a leading-log approximation scheme). For finite \( N_c \), there is no simple closed non-linear equation, just an infinite hierarchy of coupled equations \( [15] \), which is equivalent to a functional Fokker-Planck equation \( [17, 2] \). This functional equation
can be studied via numerical techniques (in particular, on the basis of the associated Langevin
equation [17], or on the path integral representation [36]), and it would be very interesting to
estimate the finite $N_c$ corrections. Moreover, higher-order corrections in $\alpha_s$, which so far have
not been included in the non-linear evolution equations (see however [17]), but which are now
available for the BFKL kernel [38], may be responsible for an effective decrease in the slope
parameter $\lambda$ with respect to its lowest order BFKL value.

Note finally that the uncertainty on the $\tau$–dependence of the saturation scale should not
affect our prediction for the kinematical window in which one expects extended scaling. Indeed,
as explained in relation with eq. (1.4), this prediction relies just on a limited expansion around
$Q^2 = Q_s^2$, together with the scale-invariance of the linear evolution equation at hand (in our case,
BFKL). This argument predicts an upper limit $Q_{max}^2 \sim Q_s^4/\Lambda_{QCD}^2$ up to which geometric
scaling should be expected. To estimate this upper limit, we shall use phenomenologically reasonable
values for $Q_s$ [23, 2], and not the theoretical predictions of our analysis (which cannot give
the absolute value of $Q_s$ anyway, just its $\tau$–dependence). This gives $Q_{max}^2 \sim 100 \text{ GeV}^2$ for
$Q_s \sim 1 \text{ GeV}$, and $Q_{max}^2 \sim 400 \text{ GeV}^2$ for $Q_s \sim 2 \text{ GeV}$, values which are both of the right order
of magnitude to agree with the phenomenological analysis in Ref. [2].

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Note added

Very recently, after this work was essentially completed, a paper has been released by Kwieciński
and Staśto [39] which addresses the issue of geometric scaling at high $Q^2$ in the framework of the
DGLAP evolution equation (or its double-log approximation). The conclusions in this paper
are however different from ours: The authors of Ref. [39] have not recognized the existence of a
window for geometric scaling above $Q_s$, but rather concluded that scaling violations should be
expected for any $Q^2 > Q_s^2(x)$, even at very small values of $x$.

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