Critical and Subcritical Anisotropic Trudinger–Moser Inequalities on the Entire Euclidean Spaces

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We investigate the subcritical anisotropic Trudinger–Moser inequality in the entire space \( \mathbb{R}^N \), obtain the asymptotic behavior of the supremum for the subcritical anisotropic Trudinger–Moser inequalities on the entire Euclidean spaces, and provide a precise relationship between the supremums for the critical and subcritical anisotropic Trudinger–Moser inequalities. Furthermore, we can prove critical anisotropic Trudinger–Moser inequalities under the nonhomogenous norm restriction and obtain a similar relationship with the supremums of subcritical anisotropic Trudinger–Moser inequalities.

1. Introduction

Let \( \Omega \) be a domain with finite measure in \( \mathbb{R}^N \) and \( W^{1,p}_0 (\Omega) \) represent the completion of \( C_\infty (\Omega) \) in the norm

\[
\| u \|_{W^{1,p}_0 (\Omega)} = \left( \int_{\Omega} (|\nabla u| + |u|^p) \, dx \right)^{1/p}.
\]  

(1)

The classical Sobolev embedding theorem tells us that \( W^{1,p}_0 (\Omega) \subseteq L^p (\Omega) \), where \( p > 1 \), but \( W^{1,p}_0 (\Omega) \subseteq L^{\infty} (\Omega) \). One knows from the studies of Yudovich [1], Peetre [2], Pohozzaev [3], and Trudinger [4] that \( W^{1,p}_0 (\Omega) \) embeds into the Orlicz space \( L_\phi (\Omega) \), with the N-function

\[
\phi (t) = \exp (|t|^{N/N-1} - 1).
\]

This embedding was made more precise by Moser [5] and obtained the following inequality:

\[
\sup_{\| u \|_{W^{1,p}_0 (\Omega)} \leq 1} \int_{\Omega} \{ \exp (\alpha |u|^{N/N-1}) - 1 \} \, dx < +\infty \quad \text{if} \quad \alpha \leq \alpha_N,
\]

\[
= +\infty \quad \text{if} \quad \alpha > \alpha_N,
\]

(2)

where \( \alpha_N = N \omega_{N-1}^{1/N-1} \) and \( \omega_{N-1} \) is the measure of the unit sphere in \( \mathbb{R}^N \).

In 2000, Adachi-Tanaka [6] obtained a sharp Trudinger–Moser inequality on \( \mathbb{R}^N \):

\[
\sup_{u \in W^{1,N} (\mathbb{R}^N)} \int_{\mathbb{R}^N} \Phi_N (\alpha |u|^{N/N-1}) \, dx \leq C (\alpha, N) \| u \|_N^N, \quad \text{iff} \quad 0 < \alpha < \alpha_N,
\]

\[
\int_{\mathbb{R}^N} |\nabla u|^N \, dx \leq 1
\]

(3)
where \( \Phi_N(t) = e^t - \sum_{i=0}^{N-2} \frac{t^i}{i!} \). Note that inequality (3) has the subcritical form, that is, \( \alpha < \alpha_N \). Later, in [7, 8], Li and Ruf showed that the best exponent \( \alpha_N \) becomes admissible if the Dirichlet norm \( \int_{\mathbb{R}^N} |u|^N \, dx \) is replaced by Sobolev norm \( \int_{\mathbb{R}^N} (|u|^N + |\nabla u|^N) \, dx \). More precisely, they proved that

\[
\sup_{u \in W^{1,N} (\mathbb{R}^N)} \int_{\mathbb{R}^N} \Phi_N \left( \frac{N}{\alpha |u|^N - 1} \right) \, dx < +\infty, \quad \text{iff} \quad \alpha \leq \alpha_N.
\]

The proofs of both critical and subcritical Trudinger–Moser inequalities (3) and (4) rely on the Pólya–Szegő inequality and the symmetrization argument. Lam and Lu [9, 10] developed a symmetrization-free method to establish the critical Trudinger–Moser inequality (see also Li, Lu, and Zhu [11]) in settings such as the Heisenberg group where the Pólya–Szegő inequality fails. Such an argument also provides an alternative proof of both critical and subcritical Trudinger–Moser inequalities (3) and (4) in the Euclidean space. In fact, the equivalence and relationship between the supremums of critical and subcritical Trudinger–Moser inequalities have been established by Lam, Lu, and Zhang [12].

**Theorem 1.** Let \( N \geq 2 \) and \( \alpha_N = \alpha^{1/N-1}_N \), where \( 0 \leq \beta < N \) and \( 0 \leq \alpha < \alpha_N \). Then, \( AT(\alpha, \beta) \) is defined as

\[
AT(\alpha, \beta) \triangleq \sup_{||u||_{L^N} \leq 1} \frac{1}{||u||_{N/\beta}^N} \int_{\mathbb{R}^N} \Phi_N \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|^{N/N-1} \right) \frac{dx}{|x|^\beta}.
\]

Then, there exist positive constants \( c(N, \beta) \) and \( C(N, \beta) \) when \( \alpha \) is close enough to \( \alpha_N \); one has

\[
\frac{c(N, \beta)}{(1 - (\alpha/\alpha_N)^{N-1})^{N-\beta/N}} \leq AT(\alpha, \beta) \leq \frac{C(N, \beta)}{(1 - (\alpha/\alpha_N)^{N-1})^{N-\beta/N}}.
\]

and the constant \( \alpha_N \) is sharp, that is, \( AT(\alpha_N, \beta) = \infty \).

**Theorem 2.** Let \( AT(\alpha_N, \beta) = \infty \), where \( 0 \leq \beta < N \), and \( 0 < a, b \), and define

\[
MT_{a,b}(\beta) \triangleq \sup_{|\nabla u|, \|u\|_N \leq 1} \int_{\mathbb{R}^N} \Phi_N \left( \alpha_N \left( 1 - \frac{\beta}{N} \right) |u|^{N/N-1} \right) \frac{dx}{|x|^\beta}.
\]

Then, \( MT_{a,b}(\beta) < \infty \) if and only if \( b \leq N \), the constant \( \alpha_N \) is sharp, and

\[
MT_{a,b}(\beta) = \sup_{a \in (0, \alpha_N)} \left( \frac{1 - (\alpha/\alpha_N)^{(N-1)/a}}{(\alpha/\alpha_N)^{(N-1)/b}} \right)^{(N-\beta)/b} AT(\alpha, \beta),
\]

in particular,
\[ MT_{a,b}(\beta) = \sup_{x \in (0,a \alpha_0)} \left( \frac{1 - (a/\alpha_0)^{(N-1)}}{(a/\alpha_0)^{(N-1)}} \right)^{(N-\beta)/N} AT (\alpha, \beta). \]

(9)

Trudinger–Moser inequalities have many other related extensions: concentration-compactness principle and Adimurthi–Druet-type inequalities on unbounded domains (see Do et al. [13] and Lu-Zhu [14]); extensions of the Trudinger–Moser inequality to higher-order Sobolev spaces (see Adams [15], Ruf-Sani [16], Lam-Lu [10], and Chen-Lu-Zhu [17]); Trudinger–Moser inequalities on manifolds or Heisenberg groups (see Li [18, 19]); Trudinger–Moser inequalities on Heisenberg groups (see the work of Cohn, Lam, Lu, and Zhu [9, 11, 20]).

In 2012, Wang and Xia [21] investigated a sharp Trudinger–Moser inequality involving the anisotropic Dirichlet norm \( (\int_{\Omega} F^N(\nabla u))^{1/N} \) on \( W^{1,N}_0(\Omega) \) for \( N \geq 2 \):

\[ \sup_{u \in W^{1,N}_0(\Omega)} \left( \int_{\Omega} F^N(\nabla u) dx \right)^{1/N} \leq C \left( \int_{\Omega} |u|^{N(\kappa - 1)/N} dx \right)^{1/N} \]

for \( \lambda \leq \lambda_N \) and the integral above will tend to infinity for any \( \lambda > \lambda_N \).

In this paper, we will establish the Adachi–Tanaka-type subcritical Trudinger–Moser inequality and the equivalence relationship between the suprema of critical and subcritical Trudinger–Moser inequalities involving the anisotropic norm restriction similar as in [12].

Our main results can be stated as follows.

**Theorem 3.** Let \( N \geq 2, \lambda_N = N^{N/2}/\kappa^{N-1} \), where \( 0 \leq \beta < N \) and \( 0 \leq \lambda < \lambda_N \). \( AAT (\lambda, \beta) \) is defined as

\[ AAT (\lambda, \beta) \equiv \sup_{|u|_{W^{1,N}_0} \leq 1} \frac{1}{|u|^N_{W^{1,N}_0}} \int_{\mathbb{R}^N} \Phi_N \left( \frac{1}{\lambda} \Phi (|u|^{(N-\beta)/N}) \right) \frac{dx}{F^\beta (x)} \]

If \( \lambda \) is close enough to \( \lambda_N \), then there exist constants \( c(N, \beta) \) and \( C(N, \beta) \) such that

\[ \frac{c(N, \beta)}{(1 - \lambda/\lambda_N)^{N\beta/N}} \leq AAT (\lambda, \beta) \leq \frac{C(N, \beta)}{(1 - \lambda/\lambda_N)^{N\beta/N}}, \]

where \( \lambda_N \) is sharp, that is, \( AAT (\lambda_N, \beta) = \infty \).

**Theorem 4.** Let \( N \geq 2, 0 \leq \beta < N, \) and \( 0 < a, b \). Define
\[
\text{AMT}_{a,b}(\beta) \doteq \sup_{|F(\nabla u)|_{L^1} \leq \beta, |u|_{L^a} \leq 1} \int_{\mathbb{R}^N} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u|^{N/N-1} \right) \frac{dx}{F^o(x)^{N/N}},
\]
(14)

Then, \( \text{AMT}_{a,b}(\beta) < \infty \) if and only if \( b \leq N \), and \( \lambda_N \) is sharp, and we also have
\[
\text{AMT}_{a,b}(\beta) = \sup_{\lambda \in (a,\lambda_N]} \left( 1 - \frac{\lambda}{\lambda_N} \right)^{(N-1)/(N-1)a} N^{-\beta/b} AAT(\lambda,\beta),
\]
(15)
in particular,
\[
\text{AMT}(\beta) = \sup_{\lambda \in (0,\lambda_N]} \left( 1 - \frac{\lambda}{\lambda_N} \right)^{(N-1)/(N-1)} N^{-\beta/N} AAT(\lambda,\beta).
\]
(16)

2. Finsler Metric and Some Useful Lemmas

Before giving the proof, for the convenience of the readers, we provide some notations and basic facts about the Finsler metric. Let \( F: \mathbb{R}^N \to \mathbb{R} \) be a nonnegative convex function of class \( C^2(\mathbb{R}^N \setminus \{0\}) \) which is even and positively homogeneous of degree 1, for any \( \xi \in \mathbb{R}^N \) and \( t \in \mathbb{R} \) so that
\[
F(t\xi) = |t| F(\xi).
\]
(17)

A typical example is \( F(\xi) = (\sum |\xi|^q)^{1/q} \) for \( q \in [1,\infty) \).

For any \( \xi \neq 0 \), we then assume \( F(\xi) > 0 \).

Because of homogeneity of \( F \), there exist two constants \( 0 < a \leq b < \infty \) such that
\[
a|\xi| \leq F(\xi) \leq b|\xi|.
\]
(18)

If we consider the map
\[
\phi: S^{N-1} \to \mathbb{R}^N, \phi(\xi) = F_\xi(\xi).
\]
(19)

Its image \( \phi(S^{N-1}) \) is a smooth, convex hypersurface in \( \mathbb{R}^N \) which is called Wulff shape of \( F \). Let \( F^o \) be the support function of
\[
\text{AAT}(\lambda,\beta) = \sup_{|F(\nabla u)|_{L^1} \leq \beta, |u|_{L^a} \leq 1} \int_{\mathbb{R}^N} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u|^{N/(N-1)} \right) \frac{dx}{F^o(x)^{N/N}},
\]
(23)

Proof. For any \( u \in W^{1,N}(\mathbb{R}^N) \) satisfying \( \|F(\nabla u)\|_{L^N} \leq 1 \), we can define \( \nu(x) \) as
\[
\nu(x) = u(\lambda x), \quad \lambda = |u|_{L^N},
\]
(24)
and then we have
\[
\nabla \nu(x) = \lambda \nabla u(\lambda x).
\]
(25)

Due to the homogeneity of \( F(x) \), we obtain
\[
\|F(\nabla \nu)\|_{L^N} = \left( \int_{\mathbb{R}^N} F^o(\nabla \nu(\lambda x)) d\lambda x \right)^{1/N} = \|F(\nabla u)\|_{L^N} \leq 1,
\]
(26)
and so we have

\[
\int_{\mathbb{R}^N} \Phi_N \left( \frac{1 - \frac{\beta}{N}}{N} |v(x)|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} = \int_{\mathbb{R}^N} \Phi_N \left( \frac{1 - \frac{\beta}{N}}{N} |\mu(\lambda x)|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} = \lambda^{\beta-N} \int_{\mathbb{R}^N} \Phi_N \left( \frac{1 - \frac{\beta}{N}}{N} |\mu(\lambda x)|^{N(N-1)} \right) \frac{d\lambda x}{F^\beta(\lambda x)^\beta} = \|u\|_{L^N}^{\beta-N} \int_{\mathbb{R}^N} \Phi_N \left( \frac{1 - \frac{\beta}{N}}{N} |u|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta}
\]

Therefore,

\[
AAT(\lambda, \beta) \leq \sup_{\|F(v)\|_{L^N} \leq 1, \|v\|_{L^N} = 1} \int_{\mathbb{R}^N} \Phi_N \left( \frac{1 - \frac{\beta}{N}}{N} |v|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} \leq \sup_{\|F(v)\|_{L^N} \leq 1} \|u\|_{L^N}^{\beta-N} \int_{\mathbb{R}^N} \Phi_N \left( \frac{1 - \frac{\beta}{N}}{N} |u|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} = AAT(\lambda, \beta),
\]

and the proof is finished. \(\square\)

By Lemma 2, when we consider the sharp Trudinger–Moser inequality, we can always assume \(\|u\|_{L^N} = 1\).

Lemma 2. The sharp subcritical Trudinger–Moser inequality is the sequence of the sharp critical Trudinger–Moser inequality. More precisely, if \(AAT_{a,b}(\beta)\) is bounded, then \(AAT(\lambda, \beta)\) is also bounded and

\[
AAT(\lambda, \beta) \leq \left( \frac{\lambda(\lambda N)^{(N-1)/N}}{1 - (\lambda(\lambda N)^{(N-1)/N})} \right)^{(N-\beta)/b} AAT_{a,b}(\beta),
\]

in particular,

\[
\|F(v)\|_{L^N}^a = \left( \frac{\lambda}{\lambda N} \right)^{(N-1)/N} \|F(v)\|_{L^N}^a = \left( \frac{\lambda}{\lambda N} \right)^{(N-1)/N} \|F(v)\|_{L^N}^a \leq \left( \frac{\lambda}{\lambda N} \right)^{(N-1)/N} \|F(v)\|_{L^N}^a \leq \left( \frac{\lambda}{\lambda N} \right)^{(N-1)/N} \|F(v)\|_{L^N}^a,
\]

where

\[
\lambda = \left( \frac{\lambda(\lambda N)^{(N-1)/N}}{1 - (\lambda(\lambda N)^{(N-1)/N})} \right)^{1/b}.
\]

Then,

\[
\|v\|_{L^N}^b = \left( \frac{\lambda}{\lambda N} \right)^{(N-1)/b} \|v\|_{L^N}^b \leq 1 - \left( \frac{\lambda}{\lambda N} \right)^{(N-1)/N}.
\]

Because \(\|F(v)\|_{L^N}^a + \|v\|_{L^N}^b \leq 1\), we have
\[ \int_{\mathbb{R}^N} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} = \int_{\mathbb{R}^N} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u(\tilde{x})|^{N/(N-1)} \right) \frac{d\tilde{x}}{F^\rho(\tilde{x})^\beta} = \tilde{\lambda}^{-N} \int_{\mathbb{R}^N} \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |v(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \leq \left( \frac{(\lambda/\lambda_N)^{(N-1)/N}b}{1 - (\lambda/\lambda_N)^{(N-1)/N}a} \right)^{(N-\beta)/b} \text{AMT}_{a,b}(\beta). \]

(34)

3. Equivalence between the Critical and Subcritical Anisotropic Trudinger–Moser Inequalities under the Homogeneous Norm Restriction

In this section, we give the asymptotic behavior of the supremum for the subcritical anisotropic Trudinger–Moser inequalities, show the equivalence between the critical and subcritical anisotropic Trudinger–Moser inequalities under the homogeneous norm restriction, and finish the Proof of Theorem 3.

**Proof of Theorem 3.** Assume \( u \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \|F(\nabla u)\|_{L^N} \leq 1, \) and \( \|u\|_{L^N} = 1 \). Set

\[ \Omega_u = \left\{ x : u(x) > \left( 1 - \left( \frac{\lambda}{\lambda_N} \right)^{N-1} \right)^{1/N} \right\}. \]

(35)

Now, we estimate the volume of \( \Omega_u \):

\[ |\Omega_u| = \int_{\Omega_u} 1 \, dx \leq \frac{1}{\int_{\Omega_u} 1 - (\lambda/\lambda_N)^{N-1} \, dx} \leq \frac{1}{1 - (\lambda/\lambda_N)^{N-1}}. \]

(36)

We rewrite

\[ \int_{\mathbb{R}^N} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \]

as

\[ \left( \int_{\mathbb{R}^N} + \int_{\Omega_u} \right) \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \triangleq I_1 + I_2. \]

(38)

Then,

\[ I_1 = \int_{\mathbb{R}^N \setminus \Omega_u} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \leq \int_{\{u \leq 1\}} \Phi_N \left( |u(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \leq e^d \int_{(u\leq1)F^\alpha(x)^\beta} u^N \, dx \leq e^d \left( \int_{(x \in P(x) \geq 1)F^\alpha(x)^\beta} u^N \, dx + \int_{(x \in P(x) < 1)F^\alpha(x)^\beta} u^N \, dx \right) \leq C(N,\beta) \frac{1}{1 - (\lambda/\lambda_N)^{(N-1)/(N-\beta)/N}}. \]

(39)

By calculation,

\[ I_2 = \int_{\Omega_u} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \leq \int_{\Omega_u} \exp \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \]

(40)

In the area \( \Omega_u \), we assume

\[ v(x) = u(x) - \left( 1 - \left( \frac{\lambda}{\lambda_N} \right)^{N-1} \right)^{1/N} \]

(41)

where \( v \in W_0^{1,N}(\Omega_u) \), and we can easily have

\[ \|F(\nabla v)\|_{L^N} \leq 1. \]

(42)

Set \( \varepsilon = \lambda_N/\lambda - 1 \), for any \( a, b, \varepsilon > 0 \) and \( \rho > 1 \). We have that
\[
|u(x)|^{N/(N-1)} \leq \left( |v| + \left( 1 - \left( \frac{\lambda}{\lambda_N} \right)^{N-1} \right)^{1/N} \right)^{N/(N-1)}
\]

\[
\leq (1 + \varepsilon)|v|^{N/(N-1)} + \left( 1 - \left( \frac{1}{1 + \varepsilon} \right)^{N-1} \right)^{1/(1-N)} \left( 1 - \left( \frac{\lambda}{\lambda_N} \right)^{N-1} \right)^{1/(N-1)}
\]

\[
= \frac{\lambda_N}{\lambda} |v|^{N/(N-1)} + \left( 1 - \left( \frac{\lambda}{\lambda_N} \right)^{N-1} \right)^{1/(1-N)} \left( 1 - \left( \frac{\lambda}{\lambda_N} \right)^{N-1} \right)^{1/(N-1)}
\]

by using the following elementary inequality:

\[
(a + b)^p \leq \varepsilon b^p + (1 + \varepsilon)^{(1-p)/(1-1)} (1 - \varepsilon)^{1-p} a^p.
\]

Using the singular Trudinger–Moser inequality under the anisotropic norm in the bounded domain [23], we have that

\[
I_2 \leq \int_{\Omega_a} \exp \left( \lambda \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha (x)^{1/\rho}}
\]

\[
\leq \int_{\Omega_a} \exp \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |v(x)|^{N/(N-1)} + \lambda \left( 1 - \frac{\beta}{N} \right) \right) \frac{dx}{F^\alpha (x)^{1/\rho}}
\]

\[
\leq C(N, \beta) |\Omega_a|^{1-(\beta/N)}
\]

\[
\leq \frac{C(N, \beta)}{(1 - (\lambda/\lambda_N)^{N-1})^{N-\beta/N}}.
\]

Therefore,

\[
\text{AAT}(\lambda, \beta) \leq \frac{C(N, \beta)}{(1 - (\lambda/\lambda_N)^{N-1})^{N-\beta/N}}.
\]

Next, we show that AAT(\lambda_N, \beta) = \infty. Set \{u_k(x)\}:

\[
u_k (x) = \begin{cases} 
0, & \text{if } F^\alpha (x) \geq 1, \\
\left( \frac{N - \beta}{N^N} \right)^{1/N} \log \left( \frac{1}{F^\alpha (x)} \right), & \text{if } e^{-k/(N-\beta)} < F^\alpha (x) < 1, \\
\left( \frac{1}{N^N} \right)^{1/N} \left( \frac{k}{N - \beta} \right)^{1/N}, & \text{if } 0 \leq F^\alpha (x) \leq e^{-k/(N-\beta)}. 
\end{cases}
\]

By calculation, we have
\[
\|\mathbf{u}_k\|_{L^N} = \int_0^1 \left( N\kappa_N \right)^{-1} \left( \frac{k}{N - \beta} \right)^{N - 1} r^{N - 1} \, dr
\]
\[
+ \int_0^1 e^{-\lambda(N - \beta)} \left( \frac{N - \beta}{N\kappa_N k} \right) \ln \left( \frac{1}{r} \right)^N r^{N - 1} \, dr
\]
\[
= Ck^{N - 1} \int_0^1 e^{-\lambda(N - \beta)} \left( \frac{N - \beta}{N\kappa_N k} \right) \ln \left( \frac{1}{r} \right)^N r^{N - 1} \, dr
\]
\[
+ C \frac{1}{k} \int_0^{\lambda(N - \beta)} y^N e^{-Ny} \, dy \sim k^N e^{-Nk/(N - \beta)} + \frac{1}{k} \sim \frac{1}{k}
\]
\[
\|F(\nabla \mathbf{u}_k)\|_{L^N} = 1, \|\mathbf{u}_k\|_{L^N} = O\left( \frac{1}{k} \right).
\]

Therefore,
\[
\int_{\mathbb{R}^N} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) |\mathbf{u}_k(x)|^{N/(N - 1)} \right) \frac{dx}{F^\beta(x)^\beta}
\]
\[
\geq C \int_0^{e^{-\lambda(N - \beta)}} \Phi_N \left( \lambda \left( 1 - \frac{\beta}{N} \right) \left( \frac{1}{N\kappa_N} \right)^{1/(N - 1)} \left( \frac{k}{N - \beta} \right) \right) r^{N - 1 - \beta} \, dr
\]
\[
\geq C \int_0^{e^{-\lambda(N - \beta)}} \Phi_N \left( \frac{\lambda}{N\kappa_N} k \right) r^{N - 1 - \beta} \, dr \sim \Phi_N \left( \frac{\lambda}{N\kappa_N} k \right) e^{-k}.
\]
When \( \lambda/\lambda_N \geq (1/2) \), there exists a very large constant \( M_2 \) which is independent of \( \lambda \); when \( k \geq M_2 \), we have that
\[
\Phi_N\left( \frac{\lambda}{\lambda_N}k \right) \approx e^{\left( \frac{\lambda}{\lambda_N}k \right)}. \tag{55}
\]
Then,
\[
\int_{\mathbb{R}^N} \Phi_N\left( \lambda \left( 1 - \frac{\beta}{N} \right) |u_k(x)|^{N/(N-1)} \right) \frac{dx}{F'(x)^{\beta}} \sim e^{\left( \frac{\lambda}{\lambda_N}k \right)}e^{-k} = e^{-\left( 1 - \frac{\lambda}{\lambda_N} \right)k}. \tag{56}
\]
When \( \lambda \) is close enough to \( \lambda_N \), we can always find a suitable \( k \) satisfying \( 1 \leq (1 - (\lambda/\lambda_N))k \leq 2 \) and
\[
\lambda = \left( 1 - \frac{1}{k} \right) \lambda_N \geq \left( 1 - \frac{1}{\max(M_1, M_2)} \right) \lambda_N, \tag{57}
\]
or
\[
\max(M_1, M_2) \leq k \approx \frac{1}{1 - (\lambda/\lambda_N)}. \tag{58}
\]
Then,
\[
\frac{1}{\|u_k\|_{L^N}} \int_{\mathbb{R}^N} \Phi_N\left( \lambda \left( 1 - \frac{\beta}{N} \right) |u_k(x)|^{N/(N-1)} \right) \frac{dx}{F'(x)^{\beta}} \sim k^{(N-\beta)/N}e^{-2} \sim \left( \frac{1}{1 - (\lambda/\lambda_N)} \right)^{(N-\beta)/N}
\]
\[
\sim \frac{1}{\left( 1 - (\lambda/\lambda_N)^{N-1} \right)^{(N-\beta)/N}}. \tag{59}
\]
Since \( \lambda \) is close enough to \( \lambda_N \), we have
\[
\frac{1 - (\lambda/\lambda_N)^{N-1}}{1 - (\lambda/\lambda_N)} \approx 1, \tag{60}
\]
which is
\[
AAT(\lambda, \beta) \geq \frac{c(N, \beta)}{\left( 1 - (\lambda/\lambda_N)^{N-1} \right)^{(N-\beta)/N}}. \tag{61}
\]

4. Critical Anisotropic Trudinger–Moser Inequalities under the Nonhomogenous Norm Restriction and the Relationship with the Subcritical Anisotropic Trudinger–Moser Inequalities

In this section, we prove critical anisotropic Trudinger–Moser inequalities under the nonhomogenous norm restriction and give a precise relationship between the suprema for the critical and subcritical anisotropic Trudinger–Moser inequalities under the nonhomogenous norm restriction.

Proof of Theorem 4. Let \( b \leq N \) and \( u \in W^{1, N} (\mathbb{R}^N) \setminus \{0\} \), with \( \|F(\nabla u)\|_{L^N} + \|u\|_{L^N}^b \leq 1 \). Assume
\[
\|F(\nabla u)\|_{L^N} = \theta \in (0, 1); \|u\|_{L^N}^b \leq 1 - \theta^b. \tag{62}
\]
If \( 1/2 < \theta < 1 \), let
\[
\nu(x) = \frac{u(\tilde{\lambda}x)}{\tilde{\lambda}}, \tag{63}
\]
where
\[
\tilde{\lambda} = \frac{(1 - \theta^b)^{1/b}}{\theta} > 0. \tag{64}
\]
By the simple calculation,
\[
\|F(\nabla \nu)\|_{L^N} = \left( \int_{\mathbb{R}^N} \left( F\left( \nabla \frac{\nu(\tilde{\lambda}x)}{\tilde{\lambda}} \right) \right)^N dx \right)^{1/N} = \frac{\|F(\nabla u)\|_{L^N}}{\theta} = 1, \tag{65}
\]
\[
\|\nu\|_{L^N}^b = \int_{\mathbb{R}^N} |\nu|^N dx = \frac{1}{\theta^b} \int_{\mathbb{R}^N} |\nu(\tilde{\lambda}x)|^N dx = \frac{1}{\theta^N \tilde{\lambda}^N} \|u\|_{L^N}^b \leq \frac{(1 - \theta^b)^{N/b}}{\theta^{N-1} \tilde{\lambda}^N} = 1.
\]

By Theorem 3, we can obtain
that is finite, since the last but one term and then we can obtain

\[ \int_{\mathbb{R}^n} \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} \]

\[ = \int_{\mathbb{R}^n} \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |\lambda_x(x)|^{N(N-1)} \right) \frac{d\lambda}{F^\beta(\lambda_N)^\beta} \]

\[ \leq \lambda^{N-\beta} \int_{\mathbb{R}^n} \Phi_N \left( \theta^{N(N-1)} \lambda_N \left( 1 - \frac{\beta}{N} \right) |v(x)|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} \]

\[ \leq \lambda^{N-\beta} \text{AAT} \left( \theta^{N(N-1)} \lambda_N, \beta \right) \leq \left( \frac{1 - (\theta)N/(b^N)}{\theta^N} \right)^{1 - (\beta/N)} \frac{C(N, \beta)}{\left( 1 - \left( (\theta)^{N(N-1)} \lambda_N / \lambda_N \right)^{(N-1)/(N-\beta)} \right)^{N/(N-\beta)}} \]

\[ \leq \left( \frac{1 - (\theta)N/(b^N)}{1 - \theta^N} \right)^{1 - (\beta/N)} c(N, \beta) \leq c. \]

When \( \theta \rightarrow 1 \), we can use L'Hospital's rule to estimate the last but one term \( ((1 - \theta)^N/b^N)/(1 - \theta^N)^{1 - (\beta/N)}c(N, \beta) \) that is finite, since \( b \leq N \).

If \( 0 < \theta \leq 1/2 \), we define

\[ v(x) = 2u(2x), \]

and then we can obtain

\[ \|F(vv)\|_{L^N} = 2\|F(vu)\|_{L^N} \leq 1, \]

\[ \|v\|_{L^N} = \|u\|_{L^N} \leq 1. \]

By Theorem 3, we can obtain

\[ \int_{\mathbb{R}^n} \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |u(x)|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} \]

\[ = \int_{\mathbb{R}^n} \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |u(2x)|^{N(N-1)} \right) \frac{d2x}{F^\beta(2x)^\beta} \]

\[ \leq 2^{N-\beta} \int_{\mathbb{R}^n} \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) 2^{-N(N-1)} |v(x)|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} \]

\[ \leq 2^{N-\beta} \text{AAT} \left( 2^{-N(N-1)} \lambda_N, \beta \right) \]

\[ \int_{\mathbb{R}^n} \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |w_k(x)|^{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} \]

\[ \geq \int_{\{0 \leq F^\beta(x) \leq e^{-k/(N-\beta)}\}} \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) \left( \frac{1}{\lambda_N} \right)^\frac{1}{N(N-1)} \left( \frac{1}{N(N-\beta)} \right)^\frac{1}{N(N-1)} \right) \frac{dx}{F^\beta(x)^\beta} \]

\[ \sim \Phi_N \left( \lambda_N \left( 1 - \frac{\beta}{N} \right) \left( \frac{1}{\lambda_N} \right)^{1/(N-1)} \left( \frac{1}{N(N-\beta)} \right)^{1/(N-1)} \lambda_k \right) \int_{\{0 \leq F^\beta(x) \leq e^{-k/(N-\beta)}\}} \frac{dx}{F^\beta(x)^\beta} \]

\[ = \Phi_N \left( \lambda_N \lambda_k \right) \int_{\{0 \leq F^\beta(x) \leq e^{-k/(N-\beta)}\}} \frac{dx}{F^\beta(x)^\beta} \]

\[ \sim \exp \left[ k \left( \frac{1}{\lambda_N \lambda_k} - 1 \right) \right]. \]
We define \( \lambda \), i.e., when AMT be the maximizing sequence of AMT
\[ AMT_{ab}(\beta) = \sup_{\lambda \in (0, \lambda_N)} \left( 1 - \frac{(\lambda/\lambda_N)^{(N-1)/a}}{(\lambda/\lambda_N)^{(N-1)/b}} \right)^{(N-\beta)/b} \cdot AAT(\lambda, \beta), \] when AMT_{ab}(\beta) < \infty. By Lemma 2, we have
\[ \sup_{\lambda \in (0, \lambda_N)} \left( 1 - \frac{(\lambda/\lambda_N)^{(N-1)/a}}{(\lambda/\lambda_N)^{(N-1)/b}} \right)^{(N-\beta)/b} \cdot AAT(\lambda, \beta) \leq AMT_{ab}(\beta). \] Let \( \{u_k(x)\} \) be the maximizing sequence of AMT_{ab}(\beta), i.e.,
\[ u_k \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}: \|F(Vu_k)\|_{L^N}^a + \|u_k\|_{L^N}^b \leq 1, \]
\[ \int_{\mathbb{R}^N} \Phi_N\left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |u_k(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \rightarrow AMT_{ab}(\beta)(k \rightarrow \infty). \] We define \( v_k(x) = u_k(\bar{\lambda}_k, x)/\|F(Vu_k)\|_{L^N} \), where
\[ \bar{\lambda}_k = \left( \frac{1 - \|F(Vu_k(x))\|_{L^N}^a}{\|F(Vu_k(x))\|_{L^N}^b} \right)^{1/b} > 0. \] Thus, we have
\[ \|F(Vu_k)\|_{L^N} = 1; \|v_k\|_{L^N} \leq 1, \]
\[ \int_{\mathbb{R}^N} \Phi_N\left( \lambda_N \left( 1 - \frac{\beta}{N} \right) |u_k(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \]
\[ = \bar{\lambda}_k^{N-\beta} \int_{\mathbb{R}^N} \Phi_N\left( \|F(Vu_k(x))\|_{L^N}^{N/(N-1)} \lambda_N \left( 1 - \frac{\beta}{N} \right) |v_k(x)|^{N/(N-1)} \right) \frac{dx}{F^\alpha(x)^\beta} \]
\[ \leq \bar{\lambda}_k^{N-\beta} \cdot AAT\left( \|F(Vu_k(x))\|_{L^N}^{N/(N-1)} \lambda_N, \beta \right) \]
\[ = \left( \frac{1 - \|F(Vu_k(x))\|_{L^N}^a}{\|F(Vu_k(x))\|_{L^N}^b} \right)^{(N-\beta)/b} \cdot AAT\left( \|F(Vu_k(x))\|_{L^N}^{N/(N-1)} \lambda_N, \beta \right) \]
\[ \leq \sup_{\lambda \in (0, \lambda_N)} \left( 1 - \frac{(\lambda/\lambda_N)^{(N-1)/a}}{(\lambda/\lambda_N)^{(N-1)/b}} \right)^{(N-\beta)/b} \cdot AAT(\lambda, \beta). \]
Therefore, we have proved

\[ AMT_{ab}(\beta) = \sup_{\lambda \in (0, \lambda_N)} \left( 1 - \frac{\lambda A N}{\lambda^{N-1}(N-1)\beta b} \right)^{(N-1)/N} \leq \frac{1}{AAT(\lambda, \beta)}, \]

(81)

when \( AMT_{ab}(\beta) < \infty \)

At last, we will prove that \( AMT_{ab}(\beta) < \infty \) if and only if \( b \leq N \). Recall that \( AMT_{ab}(\beta) < \infty \) if \( b \leq N \). Here, if there exist some \( b > N \) such that \( AMT_{ab}(\beta) < \infty \), then we have

\[ \limsup_{\lambda \to \lambda_N} \left( 1 - \frac{\lambda A N}{\lambda^{N-1}(N-1)\beta b} \right)^{(N-1)/N} \leq \frac{1}{AAT(\lambda, \beta)} < \infty. \]

(82)

Since \( AMT_{ab}(\beta) < \infty \),

\[ \limsup_{\lambda \to \lambda_N} \left( 1 - \frac{\lambda A N}{\lambda^{N-1}(N-1)\beta b} \right)^{(N-1)/N} \leq \frac{1}{AAT(\lambda, \beta)} < \infty. \]

(83)

From Theorem 3, we can obtain

\[ \liminf_{\lambda \to \lambda_N} \left( 1 - \frac{\lambda A N}{\lambda^{N-1}(N-1)\beta b} \right)^{(N-1)/N} > 0. \]

(84)

Thus, when \( \lambda \) is close enough to \( \lambda_N \), we have that \( AAT(\lambda, \beta) \sim (1 - (\lambda/\lambda_N)^{N-1}(\beta b N)\beta b). \) Thus,

\[ \liminf_{\lambda \to \lambda_N} \left( 1 - \frac{\lambda A N}{\lambda^{N-1}(N-1)\beta b} \right)^{(N-1)/N} < \infty, \]

(85)

which is impossible because \( b > N \). Hence, we complete the proof. \( \square \)

**Data Availability**

Some or all data, models, or codes that support the findings of this study are available from the corresponding author upon reasonable request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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