THE GALOIS ACTION ON THE LOWER CENTRAL SERIES OF THE FUNDAMENTAL GROUP OF THE FERMAT CURVE

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Abstract. Information about the absolute Galois group $G_K$ of a number field $K$ is encoded in how it acts on the étale fundamental group $\pi$ of a curve $X$ defined over $K$. In the case that $K = \mathbb{Q}(\zeta_n)$ is the cyclotomic field and $X$ is the Fermat curve of degree $n \geq 3$, Anderson determined the action of $G_K$ on the étale homology with coefficients in $\mathbb{Z}/n\mathbb{Z}$. The étale homology is the first quotient in the lower central series of the étale fundamental group. In this paper, we determine the Galois module structure of the graded Lie algebra for $\pi$. As a consequence, this determines the action of $G_K$ on all degrees of the associated graded quotient of the lower central series of the étale fundamental group of the Fermat curve of degree $n$, with coefficients in $\mathbb{Z}/n\mathbb{Z}$.

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1. Introduction

Let $X$ be the Fermat curve of degree $n$, where $n \geq 3$. Consider the cyclotomic field $K = \mathbb{Q}(\zeta_n)$; let $\mathcal{K}$ be its algebraic closure, and let $G_K$ be its absolute Galois group. Anderson described the action of $G_K$ on the étale homology $H_1(X; \mathbb{Z}/n\mathbb{Z})$ with coefficients in $\mathbb{Z}/n\mathbb{Z}$ of the base change $X_{\mathcal{K}}$ of $X$ to $\mathcal{K}$ (the base change is suppressed in the notation $H_1(X; \mathbb{Z}/n\mathbb{Z})$); more precisely, he analyzed the $G_K$-action on the relative homology $H_1(U, Y; \mathbb{Z}/n\mathbb{Z})$ of the open affine Fermat curve $U = \{(x, y) : x^n + y^n = 1\}$ relative to the set $Y$ of the $2n$ cusps with $xy = 0$.

The main result of [DPSW16, Sections 4 & 5] is that Anderson’s description uniquely determines the action of $G_K$ on $H_1(U, Y; \mathbb{Z}/n\mathbb{Z})$ when $n$ is prime. In [DPSW18, Theorem 1.1], the authors find an explicit formula for the action of each $\sigma \in G_K$ on $H_1(U, Y; \mathbb{Z}/n\mathbb{Z})$ when $n$ is a prime satisfying Vandiver’s conjecture.

In this paper, we describe the action of $G_K$ on each of the higher graded quotients $[\pi]/[\pi]_m \otimes \mathbb{Z}/n\mathbb{Z}$ in the lower central series filtration of $\pi$, with coefficients in $\mathbb{Z}/n\mathbb{Z}$; when $n$ is prime, this description determines the action uniquely. One motivation for this work is that it sheds light on the 2-nilpotent quotient of the étale fundamental group of the Fermat curve, because of the exact sequence:

$$1 \to [\pi]/[\pi]_2 \otimes \mathbb{Z}/n\mathbb{Z} \to [\pi]/[\pi]_3 \otimes \mathbb{Z}/n\mathbb{Z} \to [\pi]/[\pi]_2 \otimes \mathbb{Z}/n\mathbb{Z} \to 1.$$  

(1.a)

To state the results more precisely, consider the graded Lie algebra $\text{gr}(\pi) = \oplus_{m \geq 1} [\pi]/[\pi]_{m+1}$ associated with the lower central series for $\pi$, ([Laz54, Ser65]), which is equipped with its $G_K$-action. The group $\mu_n \times \mu_n$ acts on $X$ by multiplying $x$ and $y$ by $n$th roots of unity, and therefore acts $G_K$-equivariantly on $\pi$. Let $F$ be the free profinite group on $2g$ generators, and consider its graded Lie algebra $\text{gr}(F) = \oplus_{m \geq 1} \text{gr}_m(F)$. It

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follows from work of Labute in [Lab70, Theorem, page 17] that there is an element \( \rho \) of weight 2 such that 
\[
\text{gr}(\pi) \cong \text{gr}(F)/\langle \rho \rangle.
\]

The right hand side may be equipped with a Galois action by identifying \( \text{gr}(F) \) with the étale fundamental group of the open complement in \( X_{\overline{K}} \) of a \( K \)-rational point, and the isomorphism may be chosen to be the one induced from the inclusion of the open subscheme. It thus respects the Galois actions on both sides. This Galois action is determined by the action on \( \text{gr}_1(F) \cong \text{gr}_1(\pi) \cong H_1(X;\overline{\mathbb{Z}}) \) by [MKS04, Section 5.7, Corollary 5.12 (v)].

In Theorem 1.1, we determine the isomorphism class of \( \text{gr}(\pi) \) as a graded Lie group with action of \( \mu_n \times \mu_n \).

Since \( \text{gr}(F) \) is generated in degree 1, it suffices to obtain a complete description of the ideal \( \langle \rho \rangle \subseteq \text{gr}(F) \) and the action of \( \mu_n \times \mu_n \) on it.

Furthermore, when \( n \) is prime, we determine the isomorphism class of \( \text{gr}(\pi) \otimes \mathbb{Z}/n\mathbb{Z} \) as a graded Lie algebra with \( \mu_n \times \mu_n \times G_K \)-action. This gives the stated application of determining the action of \( G_K \) on each of the higher graded quotients \( [\pi]_m/[\pi]_{m+1} \otimes \mathbb{Z}/n\mathbb{Z} \). For this, it suffices to use the description of the ideal \( \langle \rho \rangle \subseteq \text{gr}(F) \) from Theorem 1.1 together with the action of \( G_K \) on \( [\pi]_1/[\pi]_2 \otimes \mathbb{Z}/n\mathbb{Z} \) from our earlier result in [DPSW18, Theorem 1.1].

To find the ideal \( \langle \rho \rangle \subseteq \text{gr}(F) \), we use the isomorphism of \( G_K \)-modules [Hai97, Corollary 8.3]
\[
[\pi]_2/[\pi]_3 \cong (H_1(X) \wedge H_1(X))/\text{Im}(\mathcal{C}),
\]
where
\[
\mathcal{C} : H_2(X) \to H_1(X) \wedge H_1(X)
\]
is the dual map to the cup product \( H^1(X) \wedge H^1(X) \to H^2(X) \).

The image \( \text{Im}(\mathcal{C}) \) is cyclic since \( H_2(X) \cong \mathbb{Z}(1) \). We use the basis of \( H_1(X) \) as a \( \mathbb{Z} \)-module from [Ejd19, Theorem 1.2], see (4.1), which interacts well with the \( \mu_n \times \mu_n \)-action. This basis gives an isomorphism \( \text{gr}_1(F) \cong H_1(X) \), which in turn induces an isomorphism \( \text{gr}_2(F) \cong H_1(X) \wedge H_1(X) \). We may therefore compute \( \rho \) as an element of \( H_1(X) \wedge H_1(X) \), and any generator of \( \text{Im}(\mathcal{C}) \) is a valid choice for \( \rho \).

We note that \( H_1(X) \) is a quotient of \( H_1(U) \), which is a subspace of the relative homology \( H_1(U,Y) \). For all integers \( n \geq 3 \), we determine \( \rho \) as the image of an element \( \Delta \) in \( H_1(U) \wedge H_1(U) \), with the result expressed in terms of a basis \( \{[E_{i,j}]\} \) for \( H_1(U) \) defined in Section 4.1, Lemma 4.1. This basis is convenient because we know the action of \( \mu_n \times \mu_n \) on \( G_K \) on its elements.

**Theorem 1.1.** For \( n \geq 3 \), a generator \( \rho \) for \( \text{Im}(\mathcal{C}) \) is given by the image in \( H_1(X) \wedge H_1(X) \) of the following element \( \Delta \) of \( H_1(U) \wedge H_1(U) \):
\[
\Delta = \sum_{\substack{1 \leq i_1 \leq i_2 \leq n-1 \leq j_1 \leq j_2 \leq n-1 \leq i_1,j_1 \neq (i_2,j_2)}} \epsilon(i_1,j_1,i_2,j_2)[E_{i_1,j_1}] \wedge [E_{i_2,j_2}],
\]
where
\[
\epsilon(i_1,j_1,i_2,j_2) = \begin{cases} 
1 & \text{if } j_2 - j_1 \equiv i_2 - i_1 \neq 0 \pmod{n-1} \\
-1 & \text{if } j_2 - j_1 + 1 \equiv i_2 - i_1 \neq 0 \pmod{n-1} \\
0 & \text{otherwise.}
\end{cases}
\]

The action of the absolute Galois group \( G_K \) on the homology of the Fermat curve is the subject of several foundational papers, including [Iha86], [And87], [AI88], [And89], and [Col89]. Let \( n = p \) be an odd prime. Let \( L \) be the splitting field of \( 1 - (1 - x^p)^p \). In [And87, Section 10.5], Anderson proved that the action of \( G_K \) on the relative homology \( H_1(U,Y;\mathbb{Z}/p\mathbb{Z}) \) factors through the finite Galois extension \( L/K \) and gave a theoretical formulation for the action of \( q \in Q = \text{Gal}(L/K) \). From [And87, Section 10.5] and the result of Labute quoted above, it follows that the action of \( G_K \) on \( [\pi]_m/[\pi]_{m+1} \otimes \mathbb{Z}/p\mathbb{Z} \) factors through \( Q = \text{Gal}(L/K) \), for any \( m \geq 2 \).
In [DPSW18, Theorem 1.1] and [DPSW16, Theorem 1.1], we made a completely explicit calculation of the $Q$-action on $H_1(U, Y; \mathbb{Z}/p\mathbb{Z})$ when $p$ is an odd prime satisfying Vandiver’s conjecture.* Our main motivation for Theorem 1.1 is that the $G_K$-module $[\pi_2/\pi_3]$ occurs as the coefficient group in a map that measures an obstruction for rational points: $\delta_2 : H^1(G_K, H_1(X)) \rightarrow H^2(G_K, [\pi_2/\pi_3])$. For this reason, we highlight the following result.

**Corollary 1.2.** Combining [DPSW18, Theorem 1.1] with Theorem 1.1 yields an explicit computation of $[\pi_2/\pi_3] \otimes \mathbb{Z}/p\mathbb{Z}$ as a $G_K$-module when $p$ is an odd prime satisfying Vandiver’s conjecture.

Section 7 contains several applications. In Section 7.1, we give an independent verification for the formula for $p = 5$, using the fact that $p$ satisfies certain invariance properties under the action of $\text{Aut}(X)$ and $\text{Gal}(L/\mathbb{Q})$. Using these invariance properties, if $p = 5$, we also compute that the dimension of the $G_Q$-invariant subspace of $H_1(X; \mathbb{Z}/5\mathbb{Z})$ is 2, see Example 7.7. This provides a new proof of a result of Tzermias [Tze97, Corollary 2].

In Section 7.3, we consider the short exact sequence

$$0 \rightarrow (\mathbb{Z}/p\mathbb{Z})_\rho \rightarrow H_1(X; \mathbb{Z}/p\mathbb{Z}) \cap H_1(X; \mathbb{Z}/p\mathbb{Z}) \rightarrow [\pi_2/\pi_3] \otimes \mathbb{Z}/p\mathbb{Z} \rightarrow 0.$$  

Since $Q$ fixes $\rho$, this yields a long exact sequence

$$(1.c) \quad 0 \rightarrow (\mathbb{Z}/p\mathbb{Z})_\rho \rightarrow H^0(Q; H_1(X; \mathbb{Z}/p\mathbb{Z}) \cap H_1(X; \mathbb{Z}/p\mathbb{Z})) \rightarrow H^0(Q; [\pi_2/\pi_3] \otimes \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} H^1(Q; (\mathbb{Z}/p\mathbb{Z})_\rho).$$

If $p = 5$, as an application of Corollary 1.2, we compute that the dimension of the $G_K$-invariant subspace of $H_1(X; \mathbb{Z}/5\mathbb{Z}) \cap H_1(X; \mathbb{Z}/5\mathbb{Z})$ (resp. $[\pi_2/\pi_3] \otimes \mathbb{Z}/5\mathbb{Z}$) is 35 (resp. 34). This shows that the coboundary map $\delta$ in (1.c) is trivial if $p = 5$, see Example 7.8; this is a non-trivial fact since $p \mid |Q|$.

# 2. The Fundamental Group of the Fermat Curve

Let $\zeta = \zeta_n = e^{2\pi i/n}$ (resp. $\epsilon = e^{\pi i/n}$) be a primitive $n$th (resp. $2n$th) root of unity.

Consider the Fermat curve $X$ of exponent $n$ with equation $x^n + y^n = z^n$. Let $Z_0$ be the set of $n$ points where $z = 0$. Consider the open affine subset $U = X - Z_0$. In Sections 2-4, the field of definition is the complex numbers; let $X := X(\mathbb{C})$ and $U := U(\mathbb{C}) = \{(x, y) \in \mathbb{C}^2 \mid x^n + y^n = 1\}$.

**2.1. The fundamental group and the definition of $\Delta$.** Note that $U$ is a real surface of genus $g = \binom{n-1}{2}$ with $n$ punctures. We choose the base point $b = (0, 1)$. There exist loops $a_i, z_i$ for $1 \leq i \leq g$ and $c_j$ for $0 \leq j \leq n-1$, with base point $b$, such that $\pi_1(U)$ has a presentation

$$(2.d) \quad \pi_1(U) = \langle a_i, z_i, c_j : i = 1, \ldots, g, \quad j = 0, \ldots, n-1 \rangle / \prod_{i=1}^{g} [a_i, z_i] \prod_{j=0}^{n-1} c_j.$$  

Let $\bar{a}_i, \bar{z}_i, \bar{c}_j$ denote the images of $a_i, z_i, c_j$ in $H_1(U)$. $H_1(U)$ is equipped with an intersection pairing, which may be defined using Poincaré duality $H_1(U) \cong H^1_c(U)$ and the cup product on compactly supported cohomology. We can suppose that

1. the loop $c_j$ circles the puncture $[c_j : \epsilon : 0] \in Z_0$;
2. each $c_j$ pairs trivially with $\bar{a}_i, \bar{z}_i$; each $\bar{c}_j$ pairs trivially with $\bar{c}_i$ for $i \neq j$;
3. the images of $\bar{a}_i, \bar{z}_i$ in $H_1(X)$ form a standard symplectic basis.

Item (2) may be arranged by choosing loops $c_j$ whose images have no set-theoretic intersection with each other or with the loops $a_i, z_i$ for $1 \leq i \leq g$. This in turn can be arranged using a standard gluing of an $n$-punctured polygon with $4g$ sides, with the sides labeled consecutively by $a_1, z_1, a_1^{-1}, z_1^{-1}, \ldots, a_g, z_g, a_g^{-1}, z_g^{-1}$.

The set $\{\bar{c}_j\}$ generates the kernel of $H_1(U) \rightarrow H_1(X)$. Define

$$(2.e) \quad \Delta = \sum_{i=1}^{g} \bar{a}_i \wedge \bar{z}_i \in H_1(U) \wedge H_1(U).$$

---

*Vandiver’s Conjecture states that $p$ does not divide the order of the class group of $Q(\zeta_p + \zeta_p^{-1})$. It is true for all regular primes $p$ and all primes less than 163 million.
Let \([\pi_1(U)]_2 = [\pi_1(U), \pi_1(U)]\) and \([\pi_1(U)]_3 = [\pi_1(U), [\pi_1(U)]_2]\). Consider the map

\[ C : H_1(U) \land H_1(U) \to [\pi_1(U)]_2/[\pi_1(U)]_3, \]

which takes the simple wedge \(r \land s\) to the (equivalence class of the) commutator of a lift of \(r\) and a lift of \(s\) to elements of \(\pi_1(U)\). Since \(U\) is not proper, \(C\) is an isomorphism. Recall the definition of \(\mathcal{C}\) from (1.b).

**Lemma 2.1.** The image of \(\Delta\) under the map \(\wedge^2(H_1(U) \to H_1(X))\) is a generator of \(\text{Im}(\mathcal{C})\).

**Proof.** By definition, \(\mathcal{C}\) is the dual of the cup product pairing. Since the images of \(\bar{a}_i, \bar{z}_i\) form a standard symplectic basis, the image of \(\mathcal{C}\) is generated by the image of \(\sum_{i=1}^g \bar{a}_i \land \bar{z}_i\) under the map \(\wedge^2(H_1(U) \to H_1(X))\), which is the image of \(\Delta\) by definition. \(\square\)

Lemma 2.1 shows that the image of \(\Delta\) under the map \(\wedge^2(H_1(U) \to H_1(X))\) is a valid choice for \(\rho\), and we henceforth let \(\rho\) denote this image.

### 2.2. The second graded quotient in the lower central series.

By (2.e), \(\Delta = \sum_{i=1}^g \bar{a}_i \land \bar{z}_i\). Our goal is to determine \(\Delta\) in terms of a basis of \(H_1(U) \land H_1(U)\) for which we know the action of the absolute Galois group. In order to do this, we investigate the element \(T := \prod_{i=1}^g [a_i, z_i]\) in \(\pi_1(U)\). Note that \(T = (\epsilon_0 \circ \epsilon_1 \circ \cdots \circ \epsilon_{n-1})^{-1}\).

The next lemma shows that \(\Delta\) does not depend on the representation as a product of commutators.

**Lemma 2.2.** Suppose \(r_1, \ldots, r_N, s_1, \ldots, s_N\) are loops in \(U\), with images \(\bar{r}_i, \bar{s}_i\) in \(H_1(U)\). If

\[ T \text{ is homotopic to } [r_1, s_1] \circ \cdots \circ [r_N, s_N], \]

then \(\sum_{i=1}^g \bar{a}_i \land \bar{z}_i = \sum_{i=1}^N \bar{r}_i \land \bar{s}_i\) in \(H_1(U) \land H_1(U)\).

**Proof.** By hypothesis, in \(\pi_1(U)\),

\[ (2.f) \quad [a_1, z_1] \circ \cdots \circ [a_g, z_g] = [r_1, s_1] \circ \cdots \circ [r_N, s_N]. \]

Note that both sides of the equation are elements of \([\pi_1(U)]_2\). Therefore (2.f) holds in \([\pi_1(U)]_2/[\pi_1(U)]_3\). Under the inverse of the isomorphism \(C\), (2.f) becomes \(\sum_{i=1}^g \bar{a}_i \land \bar{z}_i = \sum_{i=1}^N \bar{r}_i \land \bar{s}_i\) in \(H_1(U) \land H_1(U)\). \(\square\)

The next lemma is key for simplifying later calculations.

**Lemma 2.3.** Suppose \(\alpha, \beta, \gamma \in \pi_1(U)\).

1. If \(\alpha \gamma \in [\pi_1(U)]_2\), then \(\gamma \alpha \in [\pi_1(U)]_2\), and \(\alpha \gamma\) and \(\gamma \alpha\) have the same image in \([\pi_1(U)]_2/[\pi_1(U)]_3\).
2. If \(\gamma^{-1} \alpha \beta \in [\pi_1(U)]_2\), then \(\alpha \beta \in [\pi_1(U)]_2\), and the difference between the images of \(\gamma^{-1} \alpha \beta\) and \(\alpha \beta\) in \([\pi_1(U)]_2/[\pi_1(U)]_3\) is \(\gamma \land (-\alpha)\).

**Proof.** (1) Note that \(\gamma \alpha = \alpha^{-1}(\alpha \gamma)\alpha\). If \(\alpha \gamma \in [\pi_1(U)]_2\), then \(\gamma \alpha \in [\pi_1(U)]_2\) because the commutator subgroup is normal. Also \(\alpha \gamma\) and \(\gamma \alpha\) have the same image in \([\pi_1(U)]_2/[\pi_1(U)]_3\) because conjugation acts trivially on \([\pi_1(U)]_2/[\pi_1(U)]_3\).

(2) In \(\pi_1(U)\),

\[ \gamma^{-1} \alpha \gamma, \gamma = (\gamma^{-1} \alpha \gamma)(\gamma^{-1} \alpha^{-1} \gamma)\gamma^{-1} = \gamma^{-1} \alpha \gamma \alpha^{-1} = [\gamma^{-1}, \alpha]. \]

So \([\gamma^{-1} \alpha \gamma, \gamma] \alpha \beta = \gamma^{-1} \alpha \beta\). In particular, if \(\gamma^{-1} \alpha \beta\) is in \([\pi_1(U)]_2\), then so is \(\alpha \beta\).

Since \(U\) is affine, \([\pi_1(U)]_2/[\pi_1(U)]_3 \cong H_1(U) \land H_1(U)\). In \([\pi_1(U)]_2/[\pi_1(U)]_3\), the difference between the images of \(\gamma^{-1} \alpha \beta\) and \(\alpha \beta\) is the image of \([\gamma^{-1} \alpha \gamma, \gamma]\). Since \([\gamma^{-1} \alpha \gamma, \gamma] = [\gamma^{-1}, \alpha]\), this image is \(-\gamma \land \alpha\), which equals \(\gamma \land (-\alpha)\). \(\square\)
2.3. Elements of the fundamental groupoid. Recall that \( U := U(\mathbb{C}) = \{(x, y) \mid x^n + y^n = 1\} \) and \( Y \) is the set of \( 2n \) points such that \( xy = 0 \). We compute in the fundamental groupoid \( \pi_1(U, Y) \) of \( U \) with respect to the base points in \( Y \). Let \( \beta \) be the path in \( U \), which begins at the base point \( b_0 = b = (0, 1) \) and ends at \( d_0 = (1, 0) \), given by
\[
(2.\text{g}) \quad \beta = (\sqrt[n]{t}, \sqrt[n]{1-t}) \quad \text{for} \quad t \in [0, 1].
\]
Throughout this section, let \( 0 \leq i \leq n-1 \) and \( 0 \leq j \leq n-1 \). It is sometimes convenient to think of \( i \) and \( j \) as elements of \( \mathbb{Z}/n\mathbb{Z} \).

Let \( b_j = (0, \zeta^j) \) and \( d_i = (\zeta^i, 0) \). Consider the automorphisms \( \epsilon_0, \epsilon_1 \in \text{Aut}(U) \) defined by \( \epsilon_0(x, y) = (\zeta x, y) \) and \( \epsilon_1(x, y) = (x, \zeta y) \). Consider the path in \( U \), which begins at \( b_j \) and ends at \( d_i \), given by
\[
(2.\text{h}) \quad \epsilon_{i,j} = \epsilon_0 \epsilon_1 \beta.
\]
Consider the loop \( E_{i,j} \) in \( U \), formed by the composition of four paths, where path composition is written from left to right:
\[
E_{i,j} = \epsilon_{0,0} \circ (\epsilon_{0,j})^{-1} \circ \epsilon_{i,j} \circ (\epsilon_{i,0})^{-1}.
\]
Then \( E_{i,j} \) proceeds through the following points:
\[
b_0 \mapsto d_0 \mapsto b_j \mapsto d_i \mapsto b_0.
\]
If \( ij = 0 \), then \( E_{i,j} \) is trivial in the fundamental groupoid. The converse is also true; see Lemma 4.1 below.

3. A formula for the classifying element

The main goal of this section is to find a formula for \((\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1})^{-1}\) in terms of the elements \( E_{i,j} \) in the fundamental groupoid \( \pi_1(U, Y) \). The formula is stated in Section 3.4 and proved to be correct in Proposition 3.6. The reason for finding this formula is that

\[
\text{Proposition 3.6.}
\]

The equation for the curve is \( y^n = f(x) \) where \( f(x) = -\prod_{i=0}^{n-1} (x - \zeta^i)^1 \) is separable. This implies that the inertia type of \( \varphi \) is the \( n \)-tuple \((1, 1, \ldots, 1)\). This means that the canonical generator of inertia at each ramification point is \( \zeta \). In other words, the chosen generator of the Galois group of \( \varphi \) acts on a uniformizer at each ramification point by the same root of unity \( \zeta \).

Let \( w_i \) be the path in \( V \) given by \( x = \zeta^i \sqrt[n]{t} \) for \( t \in [0, 1] \). Let \( V^\circ = V \setminus \{w_i \mid 0 \leq i \leq n-1\} \). Let \( U^\circ = \varphi^{-1}(V^\circ) \). The restriction of \( \varphi \) to \( U^\circ \) is unramified. This is because the monodromy around each root of unity is multiplication by \( \zeta \), so a loop going around all \( n \) of the roots of unity is multiplication by \( \zeta^n \), which is trivial. Therefore, the monodromy action of \( \pi_1(V^\circ) \) is trivial on \( U^\circ \), proving that the restriction of \( \varphi \) to \( U^\circ \) is unramified. Thus \( U^\circ \) is a disjoint union of \( n \) sheets.

In order to label these sheets, we define the following notation, for \( 0 \leq i \leq n-1 \). Let \( r_i \) be the ray \( x = \zeta^i \sqrt[n]{t} \) for \( t \in [0, \infty) \). We define the angular segment, or sector, \( R_i \), to be the intersection of a small neighborhood of \( x = 0 \) in \( V^\circ \) with the segment of \( V^\circ \) bounded by \( r_{i-1} \mod n \) and \( r_i \). In other words, for some small \( \epsilon_n \in \mathbb{R} > 0 \),
\[
R_i = \{x = re^{i\theta} \in V^\circ \mid (i-1) \frac{2\pi}{n} < \theta < i \frac{2\pi}{n}, \quad 0 < r < \epsilon_n \}.
\]
For $0 \leq i, j \leq n - 1$, we denote by $\tilde{R}_{i,j}$ the intersection of a small neighborhood of $b_j = (0, \zeta^j)$ with $U^\circ \cap \varphi^{-1}(R_i)$. We label by $U_k$ the sheet containing $\tilde{R}_{k,0}$ for $0 \leq k \leq n - 1$.

Thus our small neighborhood of $b_j = (0, \zeta^j)$ intersected with $U^\circ$ is the disjoint union of the $n$ neighborhoods $\tilde{R}_{i,j}$ for $0 \leq i \leq n - 1$. Since we may choose to base fundamental groups at a point $b_j$ or at a simply connected neighborhood of $b_j$, we can think of $b_j$ as a base point divided into $n$ pieces, one piece on each sheet $U_k$, or as $n$ different tangential basepoints.

In Lemma 3.2, we determine the value of $k$ such that the sheet $U_k$ contains $\tilde{R}_{i,j}$ when $j \neq 0$.

### 3.2. The cusps with $z = 0$.

Recall that $Z_0$ is the set of $n$ points of $X$ where $z = 0$. Let $\epsilon = e^{\pi t/n}$ be a primitive $n$th root of $-1$. The points of $Z_0$ have projective coordinates $z_k = [\zeta^k : \epsilon : 0]$ for $0 \leq k \leq n - 1$. The unramified cover $U^\circ \to V^\circ$ extends to an unramified cover $U^\circ \cup Z_0 \to V^\circ \cup \{\infty\}$. The next result shows that the boundary of the sheet $U_k$ contains exactly one point of $Z_0$.

**Lemma 3.1.** The point $z_k$ is contained in the boundary of the sheet $U_k$.

**Proof.** Consider the ray $Q_k$ in $U^\circ$ given by $(x, y) = (e^{-1} \zeta^k \sqrt{t}, \sqrt{1 + t})$ for $t \in (0, \infty)$. As $t \to 0$, it approaches $b = (0, 1)$ and is contained in the sheet $U_k$. As $t \to \infty$, the value of $x/y$ on $Q_k$ approaches
\[
\lim_{t \to \infty} \epsilon^{-1} \zeta^k \sqrt{t}/\sqrt{1 + t} = \epsilon^{-1} \zeta^k.
\]
The point $z_k$ is at the end of the ray $Q_k$ and is thus contained in the sheet $U_k$. \hfill \Box

### 3.3. Loops in the cover.

The information from the inertia type indicates how to glue the sheets $U_k$ together along the paths $e_{i,j}$ to obtain a ramified cover of Riemann surfaces. Recall that the canonical generator of inertia at each ramification point is $\zeta$.

Consider a loop $L_i$ in $V$, with starting point $x = 0$, which makes a counterclockwise circle around the path $w_i$, starting in the region $R_i$ and ending in $R_{i+1}$. Note that $L_i$ depends only on the value of $i$ modulo $n$. Define $L_{i,j}$ to be the lift of $L_i$ to a path in $U^\circ$ which starts at the point $b_j = (0, \zeta^j)$. By the definition of the inertia type, $L_{i,j}$ ends at the point $b_{j+1} = (0, \zeta^{j+1})$. Note that $L_{i,j}$ is homotopic to the composition of paths
\[
L_{i,j} = e_{i,j} \circ (e_{i,j+1})^{-1} : (0, \zeta^j) \mapsto (\zeta^j, 0) \mapsto (0, \zeta^{j+1}).
\]
The next result determines which of the lifts $\tilde{R}_{i,j}$ of the sector $R_i$ are on each sheet $U_k$.

**Lemma 3.2.** With notation as above, $\tilde{R}_{i,j} \subset U_k$ if and only if $i - j = k$.

**Proof.** By definition, $\tilde{R}_{k,0} \subset U_k$. The path $L_{i,j}$ is contained in a unique sheet. Since $L_{i,j}$ starts in $\tilde{R}_{i,j}$ and ends in $\tilde{R}_{i+1,j+1}$, then these neighborhoods are contained in the same sheet. \hfill \Box
Note that \( U^\circ = U - \{ e_{i,j} \mid 0 \leq i,j \leq n - 1 \} \). In order to reconstruct the ramified cover of Riemann surfaces \( U \to V \), we need to glue the sheets \( \{ U_k \}_{0 \leq k \leq n-1} \) together along the missing segments; specifically, we glue \( R_{i,j} \) and \( R_{i+1,j} \) together along the edge \( e_{i,j} \) for \( 0 \leq i,j \leq n - 1 \).

3.4. Lifting of a star shape. Recall that path composition is written from left to right. For \( 0 \leq \ell \leq n - 1 \), define a loop in \( V \) by:

\[
S_\ell = L_{n-\ell} \circ L_{n-\ell+1} \circ \cdots \circ L_{n-\ell+(n-1)}.
\]

Each loop \( S_\ell \) traces in a counterclockwise direction along the outside of the slits \( \{ w_i \} \) and forms an \( n \)-pointed star shape. Each is homotopic to a large circle in \( V^\circ \) traced in a counterclockwise direction.

**Definition 3.3.** For \( 0 \leq \ell \leq n - 1 \), let \( \tilde{S}_\ell \) denote the unique lift under \( \varphi \) of \( S_\ell \) to a loop in \( U \) with starting point \( b_0 = (0,1) \). Let \( \tilde{S} := \tilde{S}_0 \circ \ldots \circ \tilde{S}_{n-1} \).

By the proof of Lemma 3.2, \( \tilde{S}_\ell \) is contained in \( U_{n-\ell} \). Later, we will see that \( \tilde{S} \in [\pi_1(U)]_2 \); see Remark 4.5.

**Lemma 3.4.** For \( 0 \leq \ell \leq n - 1 \), the loop \( \tilde{S}_\ell \) is homotopic to

\[
\tilde{S}_\ell = e_{-\ell,0} \circ (\varphi_{-\ell,1})^{-1} \circ e_{-\ell+1,1} \circ (\varphi_{-\ell+2,1})^{-1} \circ \cdots \circ e_{-\ell+n-1,1} \circ (\varphi_{-\ell+n-1,0})^{-1},
\]

or, equivalently, \( L_{n-\ell,0} \circ L_{n-\ell+1,0} \circ \cdots \circ L_{n-\ell+(n-1),0} \).

**Proof.** The loop \( \tilde{S}_\ell \) is homotopic to the composition of \( 2n \) of the edges \( e_{i,j} \) and \( (e_{i,j})^{-1} \). Because of the inertia type of \( \varphi \), this composition involves loops of the form \( L_{i,j} = e_{i,j} \circ (e_{i,j+1})^{-1} \). The condition that \( \tilde{S}_\ell \) is contained in \( U_{n-\ell} \) implies that its initial edge is the path \( e_{-\ell,0} \) from \( (0,1) \) to \( (\zeta^\ell,0) \). Thus the initial loop is \( L_{n-\ell,0} = e_{-\ell,0} \circ (e_{-\ell,1})^{-1} \). Consider the loop \( L_{j',j'} \) following the loop \( L_{i,j} \). Then \( j' = i + 1 \) because \( \tilde{S}_\ell \) circles counterclockwise around the point \( (\zeta^i,0) \). Also \( j' = j + 1 \) because the starting point \( (0,\zeta^j) \) of \( L_{j',j'} \) is the same as the ending point \( (0,\zeta^{j+1}) \) of \( L_{i,j} \).

For example, \( \tilde{S}_0 \) passes through the points in this order:

\[
(0,1) \to (1,0) \to (0,\zeta) \to (\zeta,0) \to (0,\zeta^2) \to \cdots \to (\zeta^{n-1},0) \to (0,1),
\]

and

\[
\tilde{S}_0 = e_{0,0} \circ (e_{0,1})^{-1} \circ e_{1,1} \circ (e_{1,2})^{-1} \circ \cdots \circ e_{n-1,0} \circ (e_{n-1,0})^{-1}.
\]

3.5. Comparison with loops around cusps with \( z = 0 \). We prove that \( \tilde{S} \) is path homotopic to the boundary of a disk containing the \( n \) points of \( Z_0 \) and that \( \tilde{S}_\ell \) can be taken to be the loop \( (c_{n-\ell})^{-1} \) in a standard presentation of \( \pi_1(U) \) (a presentation of the form \( (2,d) \)). For \( 0 \leq j \leq n - 1 \), let \( z_j = [\zeta^j : \epsilon : 0] \).

**Proposition 3.5.** The loop \( \tilde{S}_\ell \) is homotopic to the clockwise loop bounding a disk containing \( z_{n-\ell} \).

**Proof.** Note that \( V^\circ \) is homeomorphic to \( \mathbb{A}^1 - \{ 0 \} \) and \( S_\ell \) is homotopic to a counterclockwise loop around \( 0 \). Thus \( S_\ell \) is homotopic to a clockwise loop around \( \infty \). By definition, the lift \( \tilde{S}_\ell \) of \( S_\ell \) is a loop in \( U_{n-\ell} \). The restriction of \( \varphi \) to \( U_{n-\ell} \) yields a homeomorphism \( U_{n-\ell} \to V^\circ \). Thus \( \tilde{S}_\ell \) is a clockwise loop around the point missing from \( U_{n-\ell} \). By Lemma 3.1, this point is \( z_{n-\ell} \).

**Proposition 3.6.** The loop \( \tilde{S} \) is homotopic to the boundary of a disk in the Fermat curve which contains the \( n \) points where \( z = 0 \); it follows that we may take a presentation of the form \( (2,d) \) where \( \tilde{S}_\ell = (c_{n-\ell})^{-1} \) and \( \tilde{S} = (c_1 \circ \cdots \circ c_{n-1} \circ c_0)^{-1} \).

In the following proof, it is convenient to use a small ball around \( b_0 \) as our basepoint \( b_0 \) (which we may do because balls are simply connected). The intersection of this ball with \( R_{i,0} \) will then be called the fractional point of \( b_0 \) in the region \( R_{i,0} \).
Applying Proposition 3.5, it follows that we can take a presentation of the form (2.d) where \( \tilde{S}_0 \) is illustrated in Figure 2.

The loop \( \tilde{S}_0 \) in \( U_0 \) starts and ends at the fractional point \( b_0 \) in the region \( \tilde{R}_{0,0} \). With a small homotopy adjustment, the end of \( \tilde{S}_0 \) can cross the path \( e_{n-1,0} \) rather than return to the point \( b_0 \). Since the sheet \( U_0 \) is glued to the sheet \( U_{n-1} \) along the edge \( e_{n-1,0} \), this yields an ending point in the region \( \tilde{R}_{n-1,0} \) near the fractional point \( b_0 \) in the sheet \( U_{n-1} \).

The loop \( \tilde{S}_1 \) in \( U_{n-1} \) starts and ends at the fractional point \( b_0 \) in the region \( \tilde{R}_{n-1,0} \). With a small homotopy adjustment, the end of \( \tilde{S}_1 \) can cross the path \( e_{n-2,0} \) rather than return to the point \( b_0 \). Since the sheet \( U_{n-1} \) is glued to the sheet \( U_{n-2} \) along the edge \( e_{n-2,0} \), this yields an ending point in the region \( \tilde{R}_{n-2,0} \) near the fractional point \( b_0 \) in the sheet \( U_{n-2} \).

We continue in this way through all the loops \( \tilde{S}_0 \circ \tilde{S}_1 \circ \cdots \circ \tilde{S}_{n-1} \). Finally, with a small homotopy adjustment, the end of \( \tilde{S}_{n-1} \) crosses the path \( e_{0,0} \) and returns to the region \( \tilde{R}_{0,0} \) in \( U_0 \). Thus \( \tilde{S} \) is path homotopic to a loop in the Fermat curve \( X \) composed of \( n \) paths each contained on a single \( U_k \) of the form shown in Figure 2 for \( n = 3 \).

This path divides \( X \) into an external and internal piece, where the internal piece contains the ramification points \( \{ d_i \mid 0 \leq i \leq n-1 \} \), and the external piece contains \( Z_0 \) and is homeomorphic to a disk. (To see that the external piece is homeomorphic to a disk, note the following. The external piece is the union of \( n \) pieces, each homotopic to a wedge, which are glued together along the edges of the wedges. The picture for \( n = 3 \) is illustrated in Figure 2.)

Applying Proposition 3.5, it follows that we can take a presentation of the form (2.d) where \( \tilde{S}_0 = (e_{n-0})^{-1} \).

Thus \( \tilde{S} = (c_0)^{-1} \circ (c_{n-1})^{-1} \circ \cdots \circ (c_1)^{-1} = (c_1 \circ \cdots \circ c_{n-1} \circ c_0)^{-1} \). \( \square \)

---

**Figure 2.**

The difference between \( (c_0 \circ c_1 \circ \cdots \circ c_{n-1})^{-1} \) and \( (c_1 \circ \cdots \circ c_{n-1} \circ c_0)^{-1} \) is not significant by Lemma 2.3(1).

---

**Lemma 3.7.** Let \( S^* \) (resp. \( S^*_i \)) be the element of \( \pi_1(U) \) obtained by substituting \( E_{i,j} \) for the path \( e_{i,j} \) in \( \tilde{S} \) (resp. \( \tilde{S}_i \)). Then \( S^* \) and \( S^i \) (resp. \( S^*_i \) and \( \tilde{S}_i \)) are homotopic in \( \pi_1(U) \).

**Proof.** We fix the path \( e_{0,0} \circ e_{0,j}^{-1} \) from \( b_0 \) to \( b_j \), the initial point of \( e_{i,j} \). We fix the path \( e_{i,0} \) from \( b_0 \) to \( d_i \), the final point of \( e_{i,j} \). In the composition of paths, after completing each path \( e_{i,j} \), we return to the base point \( b_0 \) and then return to the initial point of the next path. This does not change the homotopy class. The statement can also be proven using algebraic cancelation. \( \square \)
3.6. **Combinatorial analysis.** By Proposition 3.6, the loop $\tilde{S}$ is homotopic to the boundary of a disk in the Fermat curve which contains the $n$ points where $z = 0$. By Lemma 3.7, the loop $S^*$ also has this property. So the goal is to find the image of $S^*$ in $[\pi_1(U)]_2/\pi_1(U)_3$ in terms of the elements $E_{i,j}$. By Lemma 2.3(2), this is possible if we have a complete understanding of the edges in between $E_{i,j}^{-1}$ and $E_{i,j}$ in $S^*$. We describe this combinatorially in this section.

**Lemma 3.8.** The loop $\tilde{S}$ is the composition of $2n^2$ paths, with each path $e_{i,j}$ and each path $(e_{i,j})^{-1}$ occurring exactly once.

**Proof.** Immediate from Lemma 3.4. □

We begin a combinatorial analysis of the ordering of the elements $E_{i,j}$ and $E_{i,j}^{-1}$ in $S^*$, viewed as a cycle, rather than a word. For $1 \leq j \leq n - 1$ and $0 \leq a \leq n - 2$, let $j + \overline{a}$ be the unique value in $\{1, \ldots, n - 1\}$ congruent to $j + a$ modulo $n - 1$.

**Proposition 3.9.** The ordering of the elements $E_{i,j}$ and $E_{i,j}^{-1}$ in the cycle $S^*$ satisfies the following:

1. The edges between $E_{i,1}^{-1}$ and $E_{1,1}$ are:
   \[ f_i := E_{2,1} \circ (E_{2,2})^{-1} \circ \cdots \circ E_{n-1,n-2} \circ (E_{n-1,n-1})^{-1}. \]

2. The edges between $E_{i,j}$ and $E_{i,j}$ are:
   \[ f_j := E_{2,j} \circ (E_{2,j+1})^{-1} \circ \cdots \circ E_{n-1,j+n-3} \circ (E_{n-1,j+n-2})^{-1}. \]

3. For $1 \leq i, j \leq n - 1$, the edges $E_{i',j'}$ between $E_{i,j}^{-1}$ and $E_{i,j}$ with $i' \geq i$ are:
   \[ E_{i',j'} \text{ with } i < i' \leq n - 1 \text{ and } i' - j' \equiv i - j + 1 \mod n - 1; \text{ and } (E_{i',j'})^{-1} \text{ with } i < i' \leq n - 1 \text{ and } i' - j' \equiv i - j \mod n - 1. \]

**Proof.** Item (1) is a special case of (2), which is a special case of (3), which follows from Lemma 3.4. □

## 4. The homology of the Fermat curve

The homology of the Fermat curve has been studied from many perspectives, see e.g., [Gro78, appendix] and [Lim91, Section 4]. By [Gro78, Theorem 1, appendix], $H_1(X)$ is a cyclic $\Lambda$-module, where $\Lambda = \mathbb{Z}[\mu_n \times \mu_n]$, and the annihilator of $H_1(X)$ in $\Lambda$ can be found in [Gro78, page 210] and [Lim91, Proposition 4.1]. The facts about the structure of $H_1(U)$ and $H_1(X)$ in this section will be familiar to the experts.

In Sections 4 and 5, we consider homology with coefficients in $\mathbb{Z}$; however, we follow an approach which is compatible with the results in [And87], [DPSW16], and [DPSW18] about the étale homology with coefficients in $\mathbb{Z}/n\mathbb{Z}$ and the action of the absolute Galois group upon it; see Section 6.

Consider the relative homology $H_1(U,Y)$, viewed as a groupoid with one object. To study $H_1(U)$, we use the homomorphism of groupoids from $\pi_1(U,Y)$ to $H_1(U,Y)$, which sends composition to addition; we denote this homomorphism by $- \mapsto [-]$.

### 4.1. A basis for the homology of $X$

Let $A_1 = \mathbb{Z}[\mu_n \times \mu_n]$ and let $[e]_0$ and $[e]_1$ denote the generators of $\mu_n \times \mu_n$. Let $A_1 = \langle ([e]_0 - 1)([e]_1 - 1) \rangle \subset A_1$ denote the augmentation ideal.

Let $[e_{i,j}]$ (resp. $[E_{i,j}]$) denote the class of $e_{i,j}$ (resp. $E_{i,j}$) in the relative homology $H_1(U,Y)$. Let $\beta$ denote the class of $[e_{0,0}]$ and recall that $[e_{i,j}] = [e]_0 [e]_1 \beta$. Also

\[ [E_{i,j}] = [e_{0,0}] - [e_{i,0}] - [e_{0,j}] + [e_{i,j}]. \]

Using modular symbols, Ejder proves in [Ejd19, Theorem 1.2] that a basis for $H_1(X)$ is given by

\[ \{ [e]_0 [e]_1 (1 - [e]_0)(1 - [e]_1) \beta \mid 1 \leq i \leq n - 2, 0 \leq j \leq n - 2 \}. \]
In our notation, that means that a basis for \( H_1(X) \) is given by
\[
\{ [e_{ij}^1] | 1 \leq i \leq n-2, \quad 0 \leq j \leq n-2 \}.
\]

4.2. Facts about the homology of the affine curve. Next we find a basis for \( H_1(U) \).

Lemma 4.1. The elements \( [E_{i,j}] \) from (4.1) are in \( H_1(U) \) and the set \( \{ [E_{i,j}] | 1 \leq i, j \leq n-1 \} \) is a basis for \( H_1(U) \) as a \( \mathbb{Z} \)-module.

Proof. The first claim is true because \( [E_{i,j}] \) is the image of a path in the fundamental groupoid starting and ending at the same point.

We first show that \( \{ [E_{i,j}] | 1 \leq i, j \leq n-1 \} \) is a basis for \( H_1(U; \mathbb{Z}/n\mathbb{Z}) \). There is an isomorphism \( H_1(U, Y; \mathbb{Z}/n\mathbb{Z}) \cong \Lambda_1 \otimes (\mathbb{Z}/n\mathbb{Z}) \), taking \( \beta \mapsto 1 \), [And87, Theorem 6]. Thus \( \{ [E_{i,j}] | 0 \leq i, j \leq n-1 \} \) is a basis for \( H_1(U, Y; \mathbb{Z}/n\mathbb{Z}) \).

Consider the augmentation ideal \( A_1 \otimes \mathbb{Z}/n\mathbb{Z} \subset \Lambda_1 \otimes \mathbb{Z}/n\mathbb{Z} \). If \( \alpha \in \Lambda_1 \otimes \mathbb{Z}/n\mathbb{Z} \), write \( \alpha = \sum_{i,j} a_{ij} [e_{ij}^1] \).

One can check that \( \alpha \in A_1 \otimes \mathbb{Z}/n\mathbb{Z} \) if and only if the rows and columns of the matrix \( [a_{ij}] \) sum to \( 0 \) modulo \( n \). By [DPSW16, Proposition 6.2], \( H_1(U; \mathbb{Z}/n\mathbb{Z}) \cong A_1 \otimes \mathbb{Z}/n\mathbb{Z} \).

The element \( [e_{i,j}] \) appears in \( [E_{i',j'}] \) if and only if \( i' = i \) and \( j' = j \). It follows that \( \{ [E_{i,j}] | 1 \leq i, j \leq n-1 \} \) is a set of linearly independent elements in \( H_1(U, Y; \mathbb{Z}/n\mathbb{Z}) \), and thus also in \( H_1(U; \mathbb{Z}/n\mathbb{Z}) \). Their span contains \( n(n-1)^2 \) elements of \( H_1(U; \mathbb{Z}/n\mathbb{Z}) \). Since \( H_1(U; \mathbb{Z}/n\mathbb{Z}) \) has rank \( (n-1)^2 \), this span is the entirety of \( H_1(U; \mathbb{Z}/n\mathbb{Z}) \). This completes the proof that \( \{ [E_{i,j}] | 1 \leq i, j \leq n-1 \} \) is a basis for \( H_1(U; \mathbb{Z}/n\mathbb{Z}) \).

It follows that \( \{ [E_{i,j}] | 1 \leq i, j \leq n-1 \} \) is a set of linearly independent elements in \( H_1(U) \). Consider the span of the image of this set in \( H_1(X) \). This span contains \( [E_{1,1}] \) and is a \( \Lambda_1 \)-module, thus contains \( [e_{ij}^1]_{i,j}^1 \). By (4.1), the image of this set spans \( H_1(X) \).

To complete the proof, we need to show that \( \{ [E_{i,j}] | 1 \leq i, j \leq n-1 \} \) spans the kernel of \( H_1(U) \rightarrow H_1(X) \). A basis for the kernel is \( \{ e_j | 0 \leq j \leq n-1 \} \). By Proposition 3.5, the loop \( \tilde{S}_i \) is homotopic to the clockwise loop bounding a disk containing \( z_{n-i} \). Thus a basis for the kernel is the set of images of \( \tilde{S}_i \) in \( H_1(U) \). By Lemma 3.4, \( \tilde{S}_i \) is homotopic to a loop with a formula written in terms of \( e_{i,j} \). By Lemma 3.7, the same is true after replacing \( e_{i,j} \) by \( E_{i,j} \). Also \( E_{i,j} = 0 \) if \( i = j \). Thus the set of images of \( \tilde{S}_i \) in \( H_1(U) \) is generated by \( \{ [E_{i,j}] | 1 \leq i, j \leq n-1 \} \). This completes the proof. \( \square \)

By Lemma 4.1, there is an injection \( H_1(U) \wedge H_1(U) 

\[ \cong \Lambda_1 \wedge \Lambda_1 \] and an isomorphism

\[ H_1(U) \wedge H_1(U) \rightarrow A_1 \wedge A_1. \]

Lemma 4.2. Consider the index set
\[
I = \{ (i_1, j_1, i_2, j_2) | 1 \leq i_1, i_2, j_1, j_2 \leq n-1, \quad i_1 \leq i_2, \text{ and if } i_1 = i_2 \text{ then } j_1 < j_2 \}.
\]

Then \( H_1(U) \wedge H_1(U) \) is a free \( \mathbb{Z} \)-module with basis \( \{ [E_{i_1,j_1}] \wedge [E_{i_2,j_2}] \mid (i_1, j_1, i_2, j_2) \in I \} \).

Proof. By Lemma 4.1, \( H_1(U) \) is a free \( \mathbb{Z} \)-module of rank \( m := (n-1)^2 \) with basis \( \{ [E_{i,j}] | 1 \leq i, j \leq n-1 \} \). Then \( H_1(U) \wedge H_1(U) \) is a free \( \mathbb{Z} \)-module of rank \( \binom{m}{2} \). Because \( z \wedge w = -w \wedge z \) and \( z \wedge z = 0 \), a basis is given by the set of simple wedges \( [E_{i_1,j_1}] \wedge [E_{i_2,j_2}] \) with \( i_1 \leq i_2 \) and \( (i_1, j_1) \neq (i_2, j_2) \), which is indexed by \( I \). \( \square \)

4.3. Facts about the homology of the projective curve. We characterize \( H_1(X) \wedge H_1(X) \) as a quotient of \( H_1(U) \wedge H_1(U) \) both for theoretical reasons and for the computational applications in Sections 7.1-7.3.

Lemma 4.3. Let \( S \) be the kernel of \( H_1(U) \rightarrow H_1(X) \). Then the kernel of \( H_1(U) \wedge H_1(U) \rightarrow H_1(X) \wedge H_1(X) \) equals \( S \wedge H_1(U) \).

Proof. Since \( H_1(X) \) is a free module, the quotient map \( H_1(U) \rightarrow H_1(X) \) splits, giving a direct sum decomposition \( H_1(U) \cong H_1(X) \oplus S \), where \( S, H_1(X) \) and \( H_1(U) \) are all free modules. The wedge of the direct sum decomposes as \( H_1(U) \wedge H_1(U) \cong (H_1(X) \wedge H_1(X)) \oplus (H_1(X) \wedge S) \oplus (S \wedge S) \), showing the claim. \( \square \)
We need an explicit description of $H_1(X)$ as a quotient of $H_1(U)$ for the computational applications in Sections 7.1-7.3. Define $\gamma_i \in \Lambda_1$ by the formula

\[(4.n) \quad \gamma_i = [\epsilon]_0^{-1}(1 - [\epsilon]_1)(1 + [\epsilon]_0[\epsilon]_1 + \cdots + [\epsilon]_0^{-1}[\epsilon]_1^{n-1}).\]

**Lemma 4.4.** The set $\{\gamma_i\beta \mid 1 \leq i \leq n - 1\}$ is a basis for $S = \text{Ker}(H_1(U) \to H_1(X))$.

**Proof.** By Proposition 3.5, a basis for the kernel of $H_1(U) \to H_1(X)$ is the set of images of $\tilde{S}_\ell$ in $H_1(U)$. Using Lemma 3.4, one can check that $\gamma_i\beta$ is the image of $\tilde{S}_\ell$ under the map $\pi_1(U) \to H_1(U)$. Thus $\{\gamma_i\beta \mid 1 \leq i \leq n - 1\}$ is a basis for the kernel of $H_1(U) \to H_1(X)$. \(\square\)

**Remark 4.5.** Since $\gamma_i\beta$ is the image of $\tilde{S}_\ell$, one can see that the image of $\tilde{S}$ is $\sum_{\ell=0}^{n-1} \gamma_i\beta = 0$. Thus $\tilde{S} \in [\pi_1(U)]_2$.

4.4. **Properties of the classifying element $\rho$.** The classifying element $\rho \in H_1(X) \wedge H_1(X)$ satisfies the following invariance property for the geometric action of automorphisms in $\text{Aut}(X)$. See Proposition 6.2 for an invariance property for the arithmetic action of automorphisms in the absolute Galois group $G_\mathbb{Q}$.

**Proposition 4.6.** Let $\rho$ be a generator for the image of $H_2(X) \to H_1(X) \wedge H_1(X)$. If $\phi \in \text{Aut}(X)$, then $\phi(\rho) = \rho$.

**Proof.** Every algebraic automorphism $\phi$ of $X$ is orientation-preserving and preserves the fundamental class in $H_2(X)$. It follows that $\phi$ preserves the fundamental class in $H_2(X)$. \(\square\)

By [Tze95], or [Leo96], if $n \geq 4$, then $|\text{Aut}(X)| = 6n^2$. We will apply Proposition 4.6 to:

1. the automorphisms $\phi_0([x : y : z]) = [\zeta x : y : z]$ and $\phi_1([x : y : z]) = [x : \zeta y : z]$ which act on $H_1(U,Y)$ via multiplication by $[\epsilon]_0$ and $[\epsilon]_1$ respectively. So $\rho$ is invariant under the action of $\Lambda_1$.

2. the transposition $\tau([x : y : z]) = [y : x : z]$; after using $\beta$ to fix an isomorphism between $\Lambda_1$ and $H_1(U,Y)$, then $\beta$ acts on $H_1(U,Y)$ by the ring automorphism of $\Lambda_1$ that switches $[\epsilon]_0$ and $[\epsilon]_1$. So $\rho$ is symmetric.

3. the 3-cycle $\omega([x : y : z]) = [z : -x : y]$; this automorphism does not stabilize $U$ and $Y$.

5. **Main result**

In this section, we complete the analysis of the structure of $\text{gr}(\pi)$ as a graded Lie algebra, by finding a formula for the element $\Delta \in H_1(U) \wedge H_1(U)$ that maps to $\rho \in H_1(X) \wedge H_1(X)$. Then we give some examples for $n = 3, 4, 5$.

5.1. **Proof of the main result.** By Lemma 4.2, with $I$ defined as in (4.m), $H_1(U) \wedge H_1(U)$ is a free $\mathbb{Z}$-module with basis $\{[E_{i_1,j_1}] \wedge [E_{i_2,j_2}] \mid (i_1,j_1,i_2,j_2) \in I\}$. Thus there exist $\epsilon(i_1,j_1,i_2,j_2) \in \mathbb{Z}$, such that $\Delta \in H_1(U) \wedge H_1(U)$ can be uniquely represented as the linear combination

$$\Delta = \sum_{(i_1,j_1,i_2,j_2) \in I} \epsilon(i_1,j_1,i_2,j_2)[E_{i_1,j_1}] \wedge [E_{i_2,j_2}].$$

Theorem 1.1 follows immediately from the next result.

**Theorem 5.1.** In $H_1(U) \wedge H_1(U)$, the coefficient $\epsilon(i_1,j_1,i_2,j_2)$ of the basis element $[E_{i_1,j_1}] \wedge [E_{i_2,j_2}]$ in $\Delta$ is

$$\epsilon(i_1,j_1,i_2,j_2) = \begin{cases} 1 & \text{if } j_2 - j_1 \equiv i_2 - i_1 \not\equiv 0 \mod n - 1, \\ -1 & \text{if } j_2 - j_1 + 1 \equiv i_2 - i_1 \not\equiv 0 \mod n - 1, \\ 0 & \text{otherwise.} \end{cases}$$
Proof. Recall that \( T = \prod_{i=1}^n \beta_i \), by Lemma 2.2, if \( r_1, \ldots, r_N, s_1, \ldots, s_N \in \pi_1(U) \) are such that \( T = [r_1, s_1] \cdots [r_N, s_N] \), then \( \Delta = \sum_{i=1}^N \tilde{r}_i \land \tilde{s}_i \) in \( H_1(U) \land H_1(U) \). By Proposition 3.6, \( \tilde{S} \) is homotopic to \(((c_1 \cdots c_{n-1} \circ c_0)^{-1})\). The difference between \(((c_1 \cdots c_{n-1} \circ c_0)^{-1})\) and \(((c_1 \cdots c_{n-1} \circ c_0)^{-1})\) is not significant by Lemma 2.3(1); thus \( \tilde{S} \) and \( T \) have the same image in \( H_1(U) \land H_1(U) \).

By Lemma 3.7, \( S^* \) is homotopic to \( \tilde{S} \). It thus suffices to express \( S^* \) as a product of commutators. By Lemma 2.3(2), the image of \( S^* \) in \([\pi_1(U)\]//[\pi_1(U)]\) depends only on the ordering of the edges in between \( E_{i,j}^{-1} \) and \( E_{i,j} \) in \( S^* \). We may view \( S^* \) as a cycle rather than a word, meaning that the last element precedes the first one. By Proposition 3.9, the ordering of the elements \( E_{i,j} \) and \( E_{i,j}^{-1} \) in \( S^* \) is:

\[
(E_{1,1}^{-1} \circ f_1 \circ E_{1,1}) \circ (E_{1,2}^{-1} \circ f_2 \circ E_{1,2}) \circ \cdots \circ (E_{1,n-1}^{-1} \circ f_{n-1} \circ E_{1,n-1}),
\]

where \( f_j \) is defined in Proposition 3.9.

By Lemma 2.3(2), \( \Delta \) is the sum of

\[(E_{1,1} \land (-f_1) + \cdots + (E_{1,n-1} \land (-f_{n-1}), \)

and the image of \( \tilde{f} := f_1 \cdots f_{n-1} \) in \( H_1(U) \land H_1(U) \).

Note that \( E_{1,j} \) and \( E_{1,j}^{-1} \) do not appear in \( \tilde{f} \) for any \( j \). Thus the coefficient of \( [E_{1,j}] \land [E_{1,j}] \) is zero unless \( E_{1,j}^{-1} \) or its inverse appears in \( f_j \). In particular, it is zero if \( i_2 = 1 \). For \( i_2 \neq 1 \), by the definition of \( f_j \), the coefficient of \( [E_{1,j}] \land [E_{1,j}] \) is +1 if \( j_2 - j_2 = j - 1 \) and is -1 if \( j_2 - j_2 = j - 2 \). This is equivalent to the coefficient being +1 if \( j_2 - j_2 = i_2 - 1 \neq 0 \) mod \( n - 1 \) and being -1 if \( j_2 - j_2 = i_2 - 1 \neq 0 \) mod \( n - 1 \), which is the claimed statement for \( i = 1 \).

Furthermore, the ordering of the edges in the cycle \( \tilde{f} \) is the same as for the cycle \( S^* \), except the edges \( e_{1,j} \) and \( e_{1,j}^{-1} \) do not appear. Using Proposition 3.9(3) and repeating the argument shows that the statement is true for \( i = 2 \). The result follows by induction. \( \square \)

Remark 5.2. For \( p = 5 \), we were able to independently verify using Magma that the image of \( \Delta \) generates \( \langle \rho \rangle \) in \( H_1(X) \land H_1(X) \); see Section 7.1. This uses the invariance properties from Propositions 4.6 and 6.2 and the explicit formulas for the Galois action from [DPSW18, Theorem 1.1], [DPSW16, Theorem 1.1].

Remark 5.3. The combinatorial description of \( \Delta \in H_1(U) \land H_1(U) \) can be related with the ring of cliques as follows. Consider the graph whose vertices are indexed by the \((n-1)^2\) elements \([E_{i,j}]\) of the basis of \( H_1(U) \). Place these vertices on \( n - 1 \) levels indexed by the value of \( j - i \) mod \( n - 1 \in \{0, \ldots, n-2\} \). Elements \([E_{i,j}] \land [E_{i,j}]\) of \( H_1(U) \land H_1(U) \) can be indexed by a subset of edges in this graph. The elements in \( \Delta \) yield the complete graph \( K_{n-1} \) on each level; also each vertex on level \( i \) is connected to \( n - 2 \) vertices from levels \( i - 1 \) mod \( n - 1 \) and \( i + 1 \) mod \( n - 1 \).

5.2. Examples. In Sections 5.2.1-5.2.3, we illustrate the process of finding \( \Delta \) when \( n = 3, 4, 5 \); of course, the results match the formula for \( \Delta \) found in Theorem 1.1.

5.2.1. The case \( n = 3 \). Let \( \zeta = e^{2\pi i/3} \). By Lemma 3.4:

\[
\begin{align*}
\tilde{S}_0 &= (0, 1) \mapsto (1, 0) \mapsto (0, \zeta) \mapsto (\zeta, 0) \mapsto (0, \zeta^2) \mapsto (\zeta^2, 0) \mapsto (0, 1)\\
&= e_{0,0} \circ e_{0,1}^{-1} \circ e_{1,1} \circ e_{1,2}^{-1} \circ e_{2,2} \circ e_{2,0}^{-1}; \\
\tilde{S}_1 &= (0, 1) \mapsto (\zeta^2, 0) \mapsto (0, \zeta) \mapsto (1, 0) \mapsto (0, \zeta^2) \mapsto (\zeta, 0) \mapsto (0, 1)\\
&= e_{2,0} \circ e_{2,1}^{-1} \circ e_{0,1} \circ e_{0,2}^{-1} \circ e_{1,2} \circ e_{1,0}^{-1}, \text{ and} \\
\tilde{S}_2 &= (0, 1) \mapsto (\zeta, 0) \mapsto (0, \zeta) \mapsto (\zeta^2, 0) \mapsto (0, \zeta^2) \mapsto (1, 0) \mapsto (0, 1)\\
&= e_{1,0} \circ e_{1,1}^{-1} \circ e_{2,1} \circ e_{2,2}^{-1} \circ e_{0,2} \circ e_{0,0}^{-1}.
\end{align*}
\]
By Lemma 3.7, \( S^* = E_{0,0} \circ E_{0,1}^{-1} \circ E_{1,1}^{-1} \circ E_{2,2}^{-1} \circ E_{2,0}^{-1} \circ E_{2,0} \circ E_{2,1}^{-1} \circ E_{0,1}^{-1} \circ E_{0,2} \circ E_{1,2}^{-1} \circ E_{1,0}^{-1} \circ E_{1,1} \circ E_{2,1} \circ E_{2,2} \circ E_{0,2} \circ E_{0,0} \).

By Lemma 2.3(1), in \([\pi_1(U)]_2/\pi_1(U)\), the image of \(S^*\) is the same as the image of \((E_{1,1}^{-1} \circ E_{2,1} \circ E_{2,2}^{-1} \circ E_{1,1}) \circ (E_{1,2}^{-1} \circ E_{2,2} \circ E_{2,1}^{-1} \circ E_{1,2})\).

By Lemma 2.3(2),

\[
\Delta = E_{1,1} \wedge (E_{2,2} - E_{2,2}) + E_{1,2} \wedge (E_{2,1} - E_{2,2})
\]

\[
= E_{1,1} \wedge E_{2,2} - E_{1,1} \wedge E_{2,1} + E_{1,2} \wedge E_{2,1} - E_{1,2} \wedge E_{2,2}.
\]

5.2.2. The case \(n = 4\). Let \(\zeta = e^{2\pi i/4}\). By Lemma 3.4:

\[
\tilde{S}_0 = (0,1) \mapsto (1,0) \mapsto (0,\zeta) \mapsto (\zeta,0) \mapsto (0,\zeta^2) \mapsto (\zeta^2,0) \mapsto (0,\zeta^3) \mapsto (\zeta^3,0) \mapsto (0,1)
\]

\[
= e_{0,0} \circ e_{0,1}^{-1} \circ e_{1,1} \circ e_{1,2}^{-1} \circ e_{2,2} \circ e_{3,3} \circ e_{3,0}^{-1}.
\]

\[
\tilde{S}_1 = (0,1) \mapsto (\zeta^3,0) \mapsto (0,\zeta) \mapsto (1,0) \mapsto (0,\zeta^2) \mapsto (\zeta,0) \mapsto (0,\zeta^3) \mapsto (\zeta^2,0) \mapsto (0,1)
\]

\[
= e_{3,0} \circ e_{3,1} \circ e_{0,1} \circ e_{0,2}^{-1} \circ e_{1,2} \circ e_{1,3} \circ e_{2,3} \circ e_{2,0}^{-1}.
\]

\[
\tilde{S}_2 = (0,1) \mapsto (\zeta^2,0) \mapsto (0,\zeta) \mapsto (\zeta^3,0) \mapsto (0,\zeta^2) \mapsto (1,0) \mapsto (0,\zeta^3) \mapsto (\zeta,0) \mapsto (0,1)
\]

\[
= e_{2,0} \circ e_{2,1}^{-1} \circ e_{3,1} \circ e_{3,2} \circ e_{0,3} \circ e_{1,3} \circ e_{1,0}^{-1} \text{ and}
\]

\[
\tilde{S}_3 = (0,1) \mapsto (\zeta,0) \mapsto (0,\zeta) \mapsto (\zeta^2,0) \mapsto (0,\zeta^2) \mapsto (\zeta^3,0) \mapsto (0,\zeta^3) \mapsto (1,0) \mapsto (0,1)
\]

\[
= e_{1,0} \circ e_{1,1}^{-1} \circ e_{2,1} \circ e_{2,2}^{-1} \circ e_{3,2} \circ e_{3,3} \circ e_{0,3} \circ e_{0,0}^{-1}.
\]

By Lemma 3.7, \( S^* = E_{0,1} \circ E_{1,2}^{-1} \circ E_{2,2} \circ E_{2,3}^{-1} \circ E_{3,3} \circ E_{3,1}^{-1} \circ E_{1,2} \circ E_{1,3}^{-1} \circ E_{2,3} \circ E_{2,1} \circ E_{3,1} \circ E_{1,3} \circ E_{2,1} \circ E_{2,2} \circ E_{3,2} \circ E_{3,3}^{-1} \).

By Lemma 2.3(1), in \([\pi_1(U)]_2/\pi_1(U)\), the image of \(S^*\) is the same as the image of

\[
(E_{1,1}^{-1} \circ E_{2,1} \circ E_{2,2}^{-1} \circ E_{3,2} \circ E_{3,3}^{-1} \circ E_{1,1}) \circ \]

\[
(E_{1,2}^{-1} \circ E_{2,2} \circ E_{2,3}^{-1} \circ E_{3,3} \circ E_{3,1}^{-1} \circ E_{1,2}) \circ \]

\[
(E_{1,3}^{-1} \circ E_{2,3} \circ E_{2,1} \circ E_{3,1} \circ E_{3,2}^{-1} \circ E_{1,3})
\]

By Lemma 2.3(2),

\[
\Delta = E_{1,1} \wedge (E_{2,2} - E_{2,1} + E_{3,3} - E_{3,2}) + E_{1,2} \wedge (E_{2,3} - E_{2,2} + E_{3,1} - E_{3,3}) + E_{1,3} \wedge (E_{2,1} - E_{2,3} + E_{3,2} - E_{3,1}) + E_{2,1} \wedge (E_{3,2} - E_{3,1}) + E_{2,2} \wedge (E_{3,3} - E_{3,2}) + E_{2,3} \wedge (E_{3,1} - E_{3,3}).
\]
5.2.3. The case \(n = 5\). Let \(\zeta = e^{2\pi i/5}\). By Lemma 3.4:

\[
\begin{align*}
\tilde{S}_0 &= (0, 1) \mapsto (1, 0) \mapsto (0, \zeta) \mapsto (\zeta, 0) \mapsto (0, \zeta^2) \mapsto (\zeta^2, 0) \mapsto (0, \zeta^3) \mapsto (\zeta^3, 0) \mapsto (0, \zeta^4) \mapsto (\zeta^4, 0) \mapsto (0, 1) \\
&= e_{0,0} \circ e_{0,1} \circ e_{1,1} \circ e_{1,2} \circ e_{2,2} \circ e_{2,3} \circ e_{3,3} \circ e_{3,4} \circ e_{4,4} \circ e_{4,0}; \\
\tilde{S}_1 &= (0, 1) \mapsto (\zeta^4, 0) \mapsto (0, \zeta) \mapsto (1, 0) \mapsto (0, \zeta^2) \mapsto (\zeta, 0) \mapsto (0, \zeta^3) \mapsto (\zeta^2, 0) \mapsto (0, \zeta^4) \mapsto (\zeta^3, 0) \mapsto (0, 1) \\
&= e_{4,0} \circ e_{4,1} \circ e_{0,1} \circ e_{0,2} \circ e_{1,1} \circ e_{1,2} \circ e_{2,3} \circ e_{2,4} \circ e_{3,4} \circ e_{3,0}; \\
\tilde{S}_2 &= (0, 1) \mapsto (\zeta^3, 0) \mapsto (0, \zeta) \mapsto (\zeta^2, 0) \mapsto (0, \zeta^2) \mapsto (1, 0) \mapsto (0, \zeta^3) \mapsto (\zeta, 0) \mapsto (0, \zeta^4) \mapsto (\zeta^2, 0) \mapsto (0, 1) \\
&= e_{3,0} \circ e_{3,1} \circ e_{4,1} \circ e_{1,2} \circ e_{0,2} \circ e_{0,1} \circ e_{1,3} \circ e_{2,4} \circ e_{2,4} \circ e_{3,4} \circ e_{3,0}; \\
\tilde{S}_3 &= (0, 1) \mapsto (\zeta^2, 0) \mapsto (0, \zeta) \mapsto (\zeta^3, 0) \mapsto (0, \zeta^2) \mapsto (\zeta^4, 0) \mapsto (0, \zeta^3) \mapsto (1, 0) \mapsto (0, \zeta^4) \mapsto (\zeta, 0) \mapsto (0, 1) \\
&= e_{2,0} \circ e_{2,1} \circ e_{3,1} \circ e_{3,2} \circ e_{2,2} \circ e_{4,1} \circ e_{1,3} \circ e_{0,3} \circ e_{0,1} \circ e_{1,4} \circ e_{1,0}; \text{ and} \\
\tilde{S}_4 &= (0, 1) \mapsto (\zeta, 0) \mapsto (0, \zeta) \mapsto (\zeta^2, 0) \mapsto (0, \zeta^2) \mapsto (\zeta^3, 0) \mapsto (0, \zeta^3) \mapsto (\zeta^4, 0) \mapsto (0, \zeta^4) \mapsto (1, 0) \mapsto (0, 1) \\
&= e_{1,0} \circ e_{1,1} \circ e_{2,1} \circ e_{2,2} \circ e_{3,2} \circ e_{2,3} \circ e_{3,3} \circ e_{4,3} \circ e_{4,4} \circ e_{0,4} \circ e_{0,1}.
\end{align*}
\]

By Lemma 3.7,

\[
S^* = E_{1,1} \circ E_{2,2} \circ E_{3,3} \circ E_{4,4} \circ E_{4,0} = E_{1,2} \circ E_{2,3} \circ E_{3,4} \circ E_{4,4} \circ E_{4,0}.
\]

By Lemma 2.3(1)-(2), in \([\pi_1(U)]_2/[[\pi_1(U)]_3\), the image of \(S^*\) is

\[
\Delta = E_{1,1} \wedge (-E_{2,1} + E_{2,2} - E_{3,2} + E_{3,3} - E_{4,3} + E_{4,4}) \\
+ E_{1,2} \wedge (-E_{2,2} + E_{2,3} - E_{3,3} + E_{4,3} - E_{4,4} + E_{4,1}) \\
+ E_{1,3} \wedge (-E_{2,3} + E_{2,4} - E_{3,4} + E_{3,1} - E_{4,1} + E_{2,2}) \\
+ E_{1,4} \wedge (-E_{2,4} + E_{2,1} - E_{3,1} + E_{3,2} - E_{4,2} + E_{4,3}) \\
+ E_{2,1} \wedge (-E_{3,1} + E_{3,2} - E_{4,2} + E_{4,3}) \\
+ E_{2,2} \wedge (-E_{3,2} + E_{3,3} - E_{4,3} + E_{4,4}) \\
+ E_{2,3} \wedge (-E_{3,3} + E_{4,4} - E_{4,1}) \\
+ E_{2,4} \wedge (-E_{4,4} + E_{3,1} - E_{4,1} + E_{4,2}) \\
+ E_{3,1} \wedge (-E_{4,1} + E_{4,2}) \\
+ E_{3,2} \wedge (-E_{4,2} + E_{4,3}) \\
+ E_{3,3} \wedge (-E_{4,3} + E_{4,4}) \\
+ E_{4,4} \wedge (E_{4,1} - E_{4,4}).
\]

6. The étale homology and action of the absolute Galois group

Let \(K = \mathbb{Q}(\zeta_n)\). We consider \(X\) and \(U\) as curves over \(K\). Let \(Y \subset U\) be the set of \(2n\) points where \(xy = 0\). In this section, we denote the étale fundamental group by \(\pi_1(U)\), the étale homology by \(H_1(U)\), and the relative étale homology by \(H_1(U, Y)\).

Remark 6.1. In previous sections, the homology has coefficients in \(\mathbb{Z}\); the étale homology has coefficients in a finite or \(\ell\)-adic ring. After choosing an embedding \(K \subset \mathbb{C}\) and applying Riemann’s Existence Theorem, we may identify the profinite completion of \(H_1(U(\mathbb{C}))\) with the étale homology \(H_1(U)\). Similarly, we may identify the profinite completion of \(\pi_1(U(\mathbb{C}))\) with the étale fundamental group \(\pi_1(U)\).
We therefore can consider the elements \(a_i, z_i, c_j, T, E_{i,j}\) to be in \(\pi_1(U)\) and \(a_i, \bar{z}_i, \bar{c}_j, [E_{i,j}]\) to be in \(H_1(U)\). Similarly, we may consider \(\beta, \epsilon_{i,j}\) to be in the étale fundamental groupoid and \([\epsilon_{i,j}]\) to be in \(H_1(U,Y)\). Likewise, we can consider \(\Delta\) to be an element of \(H_1(U) \land H_1(U)\) and its image \(\rho\) to be an element of \(H_1(X) \land H_1(X)\). The results in Sections 2–5 about these elements remain true in this context as well. In particular, Theorem 1.1 is true in the context of the étale homology.

Beginning in Section 6.2, we use the coefficients \(\mathbb{Z}/n\mathbb{Z}\) for the étale homology, where \(n\) is the degree of the Fermat curve \(X\). As in Remark 6.1, we may identify \(H_1(U(\mathbb{C}); \mathbb{Z}/n\mathbb{Z})\) with \(H_1(U; \mathbb{Z}/n\mathbb{Z})\).

### 6.1. An arithmetic property of the action.

**Proposition 6.2.** If \(\sigma \in G_Q\), then \(\sigma\) acts on \(\rho\) via the cyclotomic character: if \(\sigma(\zeta) = \zeta^i\), then \(\sigma(\rho) = \zeta^i\rho\). In particular, if \(\sigma \in G_K\), then \(\sigma\) acts trivially on \(\rho\).

**Proof.** Recall that \(\rho\) is a generator for the image of \(H_2(X) \rightarrow H_1(X) \land H_1(X)\). The map \(H_2(X) \rightarrow H_1(X) \land H_1(X)\) is \(G_Q\)-equivariant. By Poincaré duality, \(\sigma \in G_Q\) acts on \(H_2(X)\) via the cyclotomic character. The mod \(n\) cyclotomic character is trivial when restricted to \(G_K\). \(\square\)

### 6.2. Partial information about the \(G_Q\)-action. Let \(n = p\) be a prime satisfying Vandiver’s conjecture. In this section, we collect some information about the action of \(G_Q\) on \(H_1(U,Y; \mathbb{Z}/p\mathbb{Z})\).

By [And87, Section 10.5], the action of \(\sigma \in G_K\) on the generator \(\beta\) for \(H_1(U,Y; \mathbb{Z}/p\mathbb{Z})\) factors through \(Q = \text{Gal}(L/K)\). For \(q \in Q\), in [DPSW18, Theorem 1.1], the authors provide a completely explicit formula for the element \(B_q \in \Lambda_1 \otimes \mathbb{Z}/p\mathbb{Z}\) such that \(q \circ \beta = B_q \beta\).

Here is some partial information about how \(\text{Gal}(K/Q)\) acts on \(\beta\). Let \(a\) be a primitive root modulo \(p\). Let \(\xi_a \in \text{Gal}(K/Q)\) be the automorphism such that \(\xi_a(\zeta_p) = \zeta_p^a\). It generates \(\text{Gal}(K/Q) \cong (\mathbb{Z}/p\mathbb{Z})^*\). By [DPSW18, Lemma 2.2], \(\text{Gal}(L/Q)\) is a semi-direct product of the form \(Q \rtimes (\mathbb{Z}/p\mathbb{Z})^*\). We fix a lifting \((1, \xi_a)\) of \(\xi_a\) in \(\text{Gal}(L/Q)\) and denote it also by \(\xi_a\).

Since \(H_1(U,Y)\) is stabilized by \(G_Q\), there exists \(R_a \in \Lambda_1\) such that \(\xi_a(\beta) = R_a \beta\). Modifying the lifting of \(\xi_a\) by \(q \in Q\) changes \(R_a\) by multiplication by the element \(B_q \in \Lambda_1\) from [DPSW18, Theorem 1.1]. By [And87, Theorem 7], \(R_a\) is symmetric, meaning invariant when \(\epsilon_0\) and \(\epsilon_1\) are switched. By [And87, Section 9.6], \(R_a - 1\) is in the augmentation ideal \(\langle y_0 y_1 \rangle\). This is because \(\xi_a(\beta)\) and \(\beta\) have the same endpoints and so \(R_a \beta - \beta\) is in \(H_1(U) = \langle y_0 y_1 \rangle\). Also \(R_a R_b = R_{ab}\).

Proposition 6.2 implies that \(\xi_a(\rho) = a \rho\). To state one more property of \(R_a\), we consider the permutation action on \(\Lambda_1\) given by \(\text{perm}_a(\epsilon_0^i \epsilon_1^j) = \epsilon_0^a i \epsilon_1^a j\).

**Lemma 6.3.** Let \(p\) be an odd prime and let \(a\) be a generator of \((\mathbb{Z}/p\mathbb{Z})^*\). Then \(\prod_{i=0}^{(p-1)/2-1} \text{perm}_a^i(R_a) = 1\).

**Proof.** The automorphism \(\xi_a^{(p-1)/2}\) is the restriction of complex conjugation to \(K\). This fixes \(\beta\), since \(\beta\) is defined over \(\mathbb{R}\). By induction, we check that \(\xi_a^i(\beta) = (\prod_{i=0}^{(p-1)/2-1} \text{perm}_a^i(R_a))^i(\beta)\). Thus

\[
\beta = \xi_a^{(p-1)/2}(\beta) = \left(\prod_{i=0}^{(p-1)/2-1} \text{perm}_a^i(R_a)\right)^{(p-1)/2}(\beta).
\]

\(\square\)

If \(p = 5\), the properties above determine the action of \(\text{Gal}(K/Q)\) on \(H_1(X; \mathbb{Z}/p\mathbb{Z})\); see Section 7.1.
7. Examples

In Section 7.1, if \( n = 5 \), we verify the formula in Theorem 5.1 through an independent method using invariance properties. This method provides additional information that allows us to determine the action of \( G_Q \) on \( H_1(X) \) if \( n = 5 \); see Section 7.2. In Section 7.3, as a final application of the formula if \( n = 5 \), we compute the dimension of the \( G_K \)-invariant subspace of \( [\pi]/[\pi] \otimes \mathbb{Z}/5\mathbb{Z} \) and use it to show a coboundary map is trivial. For the calculations in this section, we use Magma [BCP97]; the code for our calculations is available here [Dav].

7.1. An independent verification of the formula for \( \rho \) if \( n = 5 \). Recall that \( \rho \) is a generator of the image of \( H_2(X) \to H_1(X) \wedge H_1(X) \). We study the subspace \( \mathcal{A} \) of \( H_1(X) \wedge H_1(X) \) of elements that are invariant under \( \text{Aut}(X) \) and \( G_K \). By Propositions 6.1 and 6.2, \( \rho \) is contained in \( \mathcal{A} \). Using the material in Section 6.2, we determine which elements of \( \mathcal{A} \) may be compatible with the action of \( \text{Gal}(K/Q) \). In Proposition 7.5, if \( n = 5 \), we verify that there is a unique 1-dimensional subspace of \( H_1(X) \wedge H_1(X) \) determined by the requirements from the actions of \( \text{Aut}(X) \), \( G_K \), and \( \text{Gal}(K/Q) \), and we verify that this subspace is the same as the one given by the formula in Theorem 1.1.

Definition 7.1. Let \( \mathcal{A} \) be the subset of \( \alpha \in H_1(X) \wedge H_1(X) \) that satisfy these properties:

1. \( \alpha \) is invariant under the automorphisms \( \phi_0, \phi_1, \tau, \omega \) of \( X \); and
2. \( \alpha \) is invariant under the action of \( \sigma \in G_K \).

Lemma 7.2. If \( n = 5 \), then \( \mathcal{A} \) is a 2-dimensional subspace of \( H_1(X) \wedge H_1(X) \).

Proof. To find \( \mathcal{A} \), we first compute the actions of \( \epsilon_0, \epsilon_1, \tau, \omega \) on \( H_1(U) \). Using the exterior wedge product, we then compute their actions on \( H_1(U) \wedge H_1(U) \). Lemma 4.1 provides a basis for the kernel \( S \) of \( H_1(U) \to H_1(X) \). By Lemma 4.3, \( S \wedge H_1(U) \) is the kernel of \( H_1(U) \wedge H_1(U) \to H_1(X) \wedge H_1(X) \). We find the image in \( H_1(X) \wedge H_1(X) \) of all \( D \in H_1(U) \wedge H_1(U) \) that satisfy these properties: \( [\epsilon_0 D - D, [\epsilon_1 D - D, \tau D - D \) are in \( S \wedge H_1(U) \); if \( \sigma \in G_K \), then \( (\sigma - 1) D \in S \wedge H_1(U) \).

For the 3-cycle \( \omega \in \text{Aut}(X) \), it is more complicated to determine the action of \( \omega \) on \( H_1(X) \) since \( \omega \) does not stabilize \( H_1(U) \). To check invariance under \( \omega \), we use a basis for \( H_1(X) \) found in [Ejd19, Theorem 1.2], together with information about how \( \omega \) acts on \( H_1(X) \) found in [Ejd19, Section 4.3 and Proposition 5.1].

If \( n \) is a prime \( p \) satisfying Vandiver’s conjecture, the action of \( \sigma \in G_K \) on \( H_1(U) \) can be calculated. As explained in the introduction, the reason is that the action of \( \sigma \) factors through \( Q = \text{Gal}(L/K) \). In [DPSW18, Theorem 1.1 and Example 3.8], we gave an explicit formula for the action of each \( q \in Q \) on \( H_1(U) \). This yields an explicit formula for the action of \( q \in Q \) on \( H_1(U) \wedge H_1(U) \); for \( n = 5 \), we implemented this formula in Magma [Dav]. See Example 7.7 for more details about this.

If \( n = 5 \), we explicitly find all of the actions above and compute in Magma that \( \mathcal{A} \) is a 2-dimensional subspace of \( H_1(X) \wedge H_1(X) \).

By Proposition 6.2, the action of \( \text{Gal}(K/Q) \) on \( \rho \) is compatible with the cyclotomic character. We consider which \( \alpha \in \mathcal{A} \) have this compatibility property.

Let \( a \) be a primitive root modulo \( n = p \). Let \( \xi_a \) denote the automorphism \( (1, \xi_a) \in \text{Gal}(L/Q) \) from Section 6.2. As seen in Section 6.2, the choice of lifting does not matter when working with \( G_K \)-invariant elements. Recall that \( \xi_a(\zeta) = \zeta^a \). The element \( \alpha \in \mathcal{A} \) is compatible with the cyclotomic character if it is the image of an element \( D \in H_1(U) \wedge H_1(U) \) such that

\[
\xi_a(D) - aD \in S \wedge H_1(U).
\]

As seen in Section 6.2, \( \xi_a(\beta) = R_a \beta \) for some \( R_a \in \Lambda_1 \) such that:

1. \( R_a - 1 \) is in the augmentation ideal \( \langle y_0y_1 \rangle \);
2. \( R_a \) is symmetric; and
3. \( \prod_{i=0}^{(p-1)/2-1} \text{perm}_i(R_a) = 1 \) (Lemma 6.3).

For \( p > 3 \), properties (i)-(iii) do not determine \( R_a \) but they do give partial information.
Definition 7.3. Let \( \mathcal{R} \) be the set of \( R_a \in \Lambda_1 \) satisfying conditions (i)-(iii).

We compute the following in Magma.

Lemma 7.4. If \( n = 5 \), then \( \mathcal{R} \) is a set of size 125.

If \( n = 5 \) and \( a = 2 \), the next result shows that we can uniquely determine \( \langle \rho \rangle \) from these restrictions, despite the ambiguity for \( R_a \).

Proposition 7.5. Let \( n = 5 \) and \( a = 2 \). There are exactly 5 elements \( \alpha \in \mathcal{A} \) lying under some \( D \in H_1(U) \setminus H_1(U) \) for which there is an \( R_a \in \mathcal{R} \) such that the pair \( (D, R_a) \) satisfies (7.5). These \( \alpha \) are exactly the multiples of the image in \( H_1(X) \setminus H_1(X) \) of the element \( \Delta \in H_1(U) \setminus H_1(U) \) found in Theorem 1.1.

Proof. This follows from a Magma computation. We consider all \( D \in H_1(U) \setminus H_1(U) \) lying above \( \mathcal{A} \). To compute \( \zeta_a \) on \( D \), we write \( D \) as a sum of simple tensors \( D = \sum_{i \in \mathcal{T}} D_i \beta \), where \( \mathcal{T} \) is a finite index set and \( D_i, D_i' \in \Lambda_1 \). We compute \( \zeta_a(D) = \sum_{i \in \mathcal{T}} \operatorname{perm}_a(D_i') R_a \beta \). \( \square \)

We do not know if the analogue of Proposition 7.5 is true for a prime \( n > 5 \).

7.2. The action of \( G_Q \). Furthermore, if \( n = 5 \) and \( a = 2 \), we have enough information about \( R_a \in \mathcal{R} \) to determine the action of \( \operatorname{Gal}(K/Q) \) on \( H_1(X) \).

Proposition 7.6. Let \( n = 5 \) and \( a = 2 \). There are 25 possibilities for \( R_a \in \mathcal{R} \) from the calculation in Proposition 7.5. Each of the 25 elements \( R_a - 1 \) has the same action on \( H_1(X) \).

Proof. Magma calculation. \( \square \)

Here is one of the possibilities for \( R_a \):

\[
R_{a,0} = 4 \epsilon_0^4 \epsilon_1^3 + 4 \epsilon_0^3 \epsilon_1^2 + 2 \epsilon_1^4 \epsilon_1 + 3 \epsilon_0^3 \epsilon_1^2 + 4 \epsilon_0^2 \epsilon_1 + 4 \epsilon_0^3 \epsilon_1 + 4 \epsilon_0^3 \epsilon_1^2 + 4 \epsilon_0^2 \epsilon_1 + 4 \epsilon_0^3 \epsilon_1 + 4 \epsilon_1^2 + 4 \epsilon_1^2.
\]

Write \( y_0 = [\epsilon_0]_0 - 1 \) and \( y_1 = [\epsilon_1]_1 - 1 \). Then

\[
R_{a,0} = 4y_0^4 y_1^3 + 4y_0^3 y_1^2 + 2y_0^4 y_1 + y_0^3 y_1^2 + 4y_0^3 y_1 + 4y_0^2 y_1 + 4y_0^3 y_1 + 4y_0^3 y_1^2 + 2y_0^3 y_1 + 2y_0 y_1 + 1.
\]

The set of 25 possibilities for \( R_a \) in Proposition 7.6 is \( \{R_{a,0} + iv_1 + jv_2 | i, j \in \{0, \ldots, 4\}\} \), where:

\[
v_1 = 2y_0^4 y_1 + 3y_0^3 y_1^2 + 3y_0^2 y_1^3 + y_0^4 y_1^4.
\]

\[
v_2 = 2y_0^4 y_1 + 3y_0^3 y_1^2 + 2y_0^3 y_1^3 + 3y_0^2 y_1^4.
\]

Recall from Section 4.4 that \( R_a \) is well-defined after making a choice of automorphism \( \zeta_a \) in \( \operatorname{Gal}(L/Q) \) lifting the automorphism \( \zeta_a \in \operatorname{Gal}(K/Q) \). Changing \( \zeta_a \) by \( q \in Q \) changes \( R_a \) by multiplication by the element \( B_q \in \Lambda_1 \) found in [DPSW18, Theorem 1.1].

Suppose \( \delta \in H_1(X)^{G_K} \). Then \( \delta \) is fixed by any automorphism \( q \in \operatorname{Gal}(L/K) \). By definition, the action of \( q \) on \( \delta \) is given by multiplication by \( B_q \). Then \( B_q R_a \delta = R_a B_q \delta = R_a \delta \) for any \( q \in \operatorname{Gal}(L/K) \). This means that the action of \( \operatorname{Gal}(L/Q) \) on \( H_1(X)^{G_K} \) does not depend on the choice involved in the definition for \( R_a \).

Let \( J_5 \) be the Jacobian of the Fermat curve of degree 5. Since \( H_1(X, \mathbb{Z}/5\mathbb{Z})^{G_L} \cong J_5(L)[5] \) for a number field \( L \), the next example can be deduced from earlier work of Rohlich and Zerms. Let \( J_5^{\infty} \) be the subgroup of \( J_5 \) of divisors of degree 0 supported at the points where \( xyz = 0 \). By [Roh77, Theorem 1], \( \dim_{\mathbb{Z}/5\mathbb{Z}}(J_5^{\infty}) = 8 \). By [Tze97, Proposition, Corollary 2, page 663], \( J_5(\mathbb{Q}(\zeta_5)) = J_5^{\infty} \) and \( \dim_{\mathbb{Z}/5\mathbb{Z}}(J_5(\mathbb{Q}(\zeta_5))) = 2 \).

Example 7.7. If \( n = 5 \), the \( \mathcal{G}_K \)-invariant subspace of \( H_1(X; \mathbb{Z}/5\mathbb{Z}) \) has dimension 8, and the \( \mathcal{G}_Q \)-invariant subspace of \( H_1(X; \mathbb{Z}/5\mathbb{Z}) \) has dimension 2.
Proof. Write \( n = p \). The action of \( G_K \) on \( \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) \) factors through the field extension \( L/K \) where \( L \) is the splitting field of \( 1 - (1 - x^p)^p \). If \( p = 5 \), there are 3 generators \( \tau_0, \tau_1, \tau_2 \) for \( Q = \text{Gal}(L/K) \). The formula for the action of each of these on \( \text{H}_1(U; \mathbb{Z}/5\mathbb{Z}) \) can be found in [DPSW18, Example 3.8].

Let \( \text{Fix}([e]_0[e]_1) = \{ \alpha \in \text{H}_1(U; \mathbb{Z}/5\mathbb{Z}) \mid [e]_0[e]_1\alpha = \alpha \} \).

By [DPSW16, Proposition 6.3], letting \( S = \text{Fix}([e]_0[e]_1) \),

(7.1) \[ \text{H}_1(U; \mathbb{Z}/5\mathbb{Z}) = \text{H}_1(U; \mathbb{Z}/5\mathbb{Z})/S. \]

In Magma, we computed the action of \( \tau_0, \tau_1, \tau_2 \) on \( \text{H}_1(X; \mathbb{Z}/5\mathbb{Z}) \). To determine the \( G_K \)-invariant subspace \( I \) of \( \text{H}_1(X; \mathbb{Z}/5\mathbb{Z}) \), we computed the intersection of the kernels of the 3 operators \( \tau_i - 1 \) for \( i = 0, 1, 2 \). For the \( G_Q \)-invariant subspace, we computed the subspace of \( I \) which is fixed by multiplication by \( \text{perm}_a(R_a) \).

7.3. An application about coboundaries. Let \( p \) be a prime satisfying Vandiver’s Conjecture and let \( K = \mathbb{Q}((\zeta_p)) \). In [DPSW18, Theorem 1.1], we gave an explicit formula for the action of \( G_K \) on \( \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) = \pi/|\pi|_2 \otimes \mathbb{Z}/p\mathbb{Z} \). From the results in this paper, we obtain an explicit action of \( G_K \) on the higher quotients \( |\pi|_m/|\pi|_{m+1} \otimes \mathbb{Z}/p\mathbb{Z} \) as well.

We would like to thank the referee for bringing this idea to our attention. Consider the short exact sequence

\[ 0 \to (\mathbb{Z}/p\mathbb{Z})^3 \to \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) \wedge \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) \to |\pi|_2/|\pi|_3 \otimes \mathbb{Z}/p\mathbb{Z} \to 0. \]

Since \( Q \) fixes \( \rho \), this yields a long exact sequence

(7.2) \[ 0 \to (\mathbb{Z}/p\mathbb{Z})^3 \to \text{H}^0(Q; \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) \wedge \text{H}_1(X; \mathbb{Z}/p\mathbb{Z})) \to \text{H}^0(Q; |\pi|_2/|\pi|_3 \otimes \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} \text{H}^1(Q; (\mathbb{Z}/p\mathbb{Z})^3). \]

The fact that \( Q \) fixes \( \rho \) also implies that \( \text{H}^1(Q; (\mathbb{Z}/p\mathbb{Z})^3) \cong \text{Hom}(Q, \mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{(p+1)/2} \). Given a \( Q \)-invariant element \( \alpha \) of \( |\pi|_2/|\pi|_3 \otimes \mathbb{Z}/p\mathbb{Z} \), consider a lift of \( \alpha \) to \( \tilde{\alpha} \in \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) \wedge \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) \). If \( q \in Q \), then \( q(\tilde{\alpha}) = \tilde{\alpha} + s_q \rho \) for some \( s_q \in \mathbb{Z}/p\mathbb{Z} \). Then \( \delta(\alpha) \) can be identified with the homomorphism \( Q \to \mathbb{Z}/p\mathbb{Z} \) given by \( q \mapsto s_q \).

Recall that \( X \) has genus \( g = (p - 1)(p - 2)/2 \) and so \( \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) \wedge \text{H}_1(X; \mathbb{Z}/p\mathbb{Z}) \) has dimension \((p^2 - 1)/2\). Thus \(|\pi|_2/|\pi|_3 \otimes \mathbb{Z}/p\mathbb{Z} \) has dimension \((p^2 - 1)/2 - 1\). If \( p = 5 \), then \( g = 6 \) and \(|\pi|_2/|\pi|_3 \otimes \mathbb{Z}/5\mathbb{Z} \) has dimension 65.

Example 7.8. If \( p = 5 \), then the \( G_K \)-invariant subspace of \( \text{H}_1(X; \mathbb{Z}/5\mathbb{Z}) \wedge \text{H}_1(X; \mathbb{Z}/5\mathbb{Z}) \) has dimension 35; the \( G_K \)-invariant subspace of \(|\pi|_2/|\pi|_3 \otimes \mathbb{Z}/5\mathbb{Z} \) has dimension 34; and thus the coboundary map \( \delta \) in (7.1) is trivial.

Proof. From the computation in Example 7.7, we know the action of \( \tau_i \) on \( \text{H}_1(X; \mathbb{Z}/5\mathbb{Z}) \) for \( i = 0, 1, 2 \). From this, we computed the action of \( \tau_i \) on \( \text{H}_1(X; \mathbb{Z}/5\mathbb{Z}) \wedge \text{H}_1(X; \mathbb{Z}/5\mathbb{Z}) \) (resp. on the quotient of this by \( \rho \)). We then computed the dimension of the intersection of the kernels of the 3 operators \( \tau_i - 1 \) for \( i = 0, 1, 2 \). This dimension, which is 35 (resp. 34), is the dimension of the \( G_K \)-invariant subspace.

The fact that the coboundary map is trivial follows from the exact sequence in (7.1). As an additional check (not included here), we computed the cocycle computationally and verified that it is trivial.

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