Riesz transforms, Hodge-Dirac operators and functional calculus for multipliers I

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Abstract
In this work, we solve the problem explicitly stated at the end of a paper of Junge, Mei and Parcet [JEMS2018]. More precisely, we prove that the Hodge-Dirac operator of the canonical “hidden” noncommutative geometry associated with a semigroup $(T_t)_{t \geq 0}$ of Markov Fourier multipliers is bisectorial and admits a bounded $H^\infty$ functional calculus on a bisector which implies a positive answer to the quoted problem. Our result can be seen as a strengthening of the dimension free estimates of Riesz transforms of the above authors. We also provide a similar result for semigroups of Markov Schur multipliers and dimension free estimates for noncommutative Riesz transforms associated with these semigroups. Our results allow us to introduce new spectral triples (i.e. noncommutative manifolds) and new quantum locally compact metric spaces in connection with the carré du champ of Meyer.

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1 Introduction

Riesz transforms associated to various geometric structures is a recurrent theme in analysis and geometry, see for example the papers [Bakl] [CCH1] [AsO1] (riemannian manifolds), [Lus1] (fermions), [Mey6] [Pis14] (gaussian spaces), [Lus2] [DoP1] (products of discrete abelian groups), [BaR1] (graphs) and [Aus1] (elliptic operators). Estimates of Riesz transforms are related to subtle geometric phenomena, see for example [Nao1] where the author uses in a decisive way some estimates of [Lus1] obtained by a noncommutative method.

Junge, Mei and Parcet obtained in [JMP2] dimension free estimates for Riesz transforms associated with semigroups ($T_t$)$_{t \geq 0}$ of Markov Fourier multipliers acting on classical $L^p$-spaces and more generally on noncommutative $L^p$-spaces $L^p(VN(G))$ associated with group von Neumann algebras $VN(G)$ where $1 < p < \infty$ and where $G$ is a (unimodular) locally compact group. Recall that if $G$ is a locally compact group then $VN(G)$ is the von Neumann algebra, whose elements act on the Hilbert space $L^2(G)$, generated by the left translation unitaries $\lambda_s : f \mapsto f(s^{-1})$, $s \in G$. If $G$ is abelian, then $VN(G)$ is $*$-isomorphic to the algebra $L^\infty(\hat{G})$ of essentially bounded functions on the dual group $\hat{G}$ of $G$. Such a semigroup ($T_t$)$_{t \geq 0}$ is characterized by a conditionally negative length $\psi : G \to \mathbb{C}$ such that the symbol of each operator $T_t$ of the semigroup is $e^{-t\psi}$. Moreover, the symbol of the (negative) generator $A_\psi$ on the noncommutative $L^p$-space $L^p(VN(G))$ of the semigroup is $\psi$. Suppose that $G$ is a discrete group. The above authors proved estimates of the form

$$\|A_\psi^p(x)\|_{L^p(VN(G))} \approx_p \|\partial_{\psi,p}(x)\|_{L^p(L^\infty(\Omega) \rtimes_\alpha G)}, \quad x \in \text{span}\{\lambda_s : s \in G\}$$

(1.1)

where $\partial_{\psi,p}$ is some kind of gradient defined on a subspace $\text{dom}\partial_{\psi,p}$ of $L^p(VN(G))$ with values in a noncommutative $L^p$-space $L^p(L^\infty(\Omega) \rtimes_\alpha G)$ associated with some crossed product $L^\infty(\Omega) \rtimes_\alpha G$. The closed subspace $\text{Ran}_{\partial_{\psi,p}}$ is equipped with a structure of bimodule and plays the role of the space of noncommutative differential 1-forms. Their approach highlights an intrinsic noncommutativity even when the locally compact group $G$ is abelian since $\text{Ran}_{\partial_{\psi,p}}$ is in general a highly noncommutative object.

In the point of view of noncommutative geometry, the underlying geometry can be resumed by the triple $(A, D_{\psi,2}, \mathcal{H})$ where $A$ is the $*$-algebra $\text{span}\{\lambda_s : s \in G\}$, $\mathcal{H}$ the Hilbert space...
\( L^2(VN(G)) \oplus_2 \operatorname{Ran} \partial_{\psi,2} \) and where the so-called Hodge-Dirac operator

\[
D_{\psi,p} \overset{\text{def}}{=} \begin{bmatrix}
0 & (\partial_{\psi,p})^*
\partial_{\psi,p} & 0
\end{bmatrix}
\]
is defined on a subspace of the Banach space \( L^p(VN(G)) \oplus_p \operatorname{Ran} \partial_{\psi,p} \). In [JMP2, Problem C.5], the authors ask for dimension free estimates for the operator \( \operatorname{sgn} D_{\psi,p} = D_{\psi,p} |D_{\psi,p}|^{-1} \).

We affirmatively answer this question by showing the following result in the spirit of [AKM1] although the approach is different.

**Theorem 1.1 (see Theorem 4.21 and Theorem 4.27)** Suppose \( 1 < p < \infty \). Let \( G \) a discrete group such that \( L^\infty(\Omega) \rtimes_\alpha G \) is QWEP. The Hodge-Dirac operator \( D_{\psi,p} \) is bisectorial on \( L^p(VN(G)) \oplus_p \operatorname{Ran} \partial_{\psi,p} \) and admits a bounded \( H^\infty(\Sigma_{\alpha}^p) \) functional calculus on a bisector.

Our result can be seen as a strengthening of the dimension free estimates (1.1) of Riesz transforms of the well-known in the commutative context, namely the carré du champ operator \( \Gamma(x) \) of Meyer associated with the generator of the semigroup (see (2.22) for the definition in the case of Schur multipliers and (2.43) for the case of Fourier multipliers). Then we obtain the following result.

**Theorem 1.2 (see Theorem 3.10)** Let \( A \) be the generator of a Markovian semigroup \( (T_t)_{t \geq 0} \) over \( B(\ell_2^\alpha) \) consisting of Schur multipliers. Suppose \( 1 < p < \infty \). For any \( x \in M_{I, \text{fin}} \), we have

\[
\|A^+_p(x)\|_{S_{\phi}^p} \approx_p \|\partial_{\alpha,p}(x)\|_{L_p(\Omega, S_{\phi}^p)}.
\]

In a third direction, we relate the norms appearing in (1.3) with a further object which is well-known in the commutative context, namely the carré du champ operator \( \Gamma(x,y) \) of Meyer associated with the generator of the semigroup (see (2.22) for the definition in the case of Schur multipliers and (2.43) for the case of Fourier multipliers). Then we obtain the following result.

**Theorem 1.3 (see Theorem 3.14)** Suppose \( 2 < p < \infty \). Let \( A \) be the generator of a Markovian semigroup over \( B(\ell_2^\alpha) \) consisting of Schur multipliers. For any \( x \in M_{I, \text{fin}} \), we have

\[
\|A^+_p(x)\|_{S_{\phi}^p} \approx_p \max \left\{ \|\Gamma(x,x)^\sharp\|_{S_{\phi}^p}, \|\Gamma(x^*,x^*)^\#\|_{S_{\phi}^p} \right\}.
\]

We also obtain a version of this theorem for the case \( 1 < p \leq 2 \), where the statement becomes more involved (see Theorem 3.14). We also discuss the question how to define the right hand side of (1.4) if \( x \) only belongs to the domain \( \operatorname{dom} A^+_p \) of the fractional power of the \( S_{\phi}^p \)-realization of \( A \) (see Lemma 3.15). We equally do this in the case where \( A \) generates a semigroup of Fourier multipliers over \( VN(G) \) (see Subsection 3.6), thus complementing the results of [JMP2]. Fourth, we extend Theorem 1.2 and (1.1) to the case where the gradient forms \( \partial_{\alpha,q,p} \) and \( \partial_{\alpha,q,p} \) take values in noncommutative \( L^p \)-spaces of \( q \)-gaussian variables \((-1 \leq q \leq 1) \) arising in second quantization which will be useful at the end of the paper to introduce some noncommutative manifolds. We refer to Propositions 3.13 and 3.25. We also give a similar result of Theorem 1.1 for semigroups of Markov Schur multipliers acting on Schatten spaces \( S_{\phi}^p \) (see Theorem 4.12).
Our estimates allow us to introduce new quantum locally compact metric spaces in the sense of \[1\] in the spirit of the quantum compact metric spaces of \[2\] (see also Subsection 5.1). Here the results read as follows. For definitions of the norms and the Gap condition \(\text{Gap}_\alpha > 0\), we refer to Section 5.

**Theorem 1.4 (see Theorems 1.4 and 5.25, and Propositions 5.20 and 5.27)** Let \(2 \leq p < \infty\). Suppose that \((T_t)_{t \geq 0}\) is a Markovian semigroup of Schur multipliers on \(B(\ell^2_1)\) associated to an injective map \(\alpha : I \to H\).

1. Let \(I\) be a finite set. Then \(\left( (S^\infty_{I,0})_{sa} \oplus \mathbb{R} \text{Id}_{\ell^2_1}, \|\cdot\|_{\Gamma,\alpha,p} \right)\) is a quantum compact metric space. Moreover, the norm \(\|\cdot\|_{\Gamma,\alpha,p}\) is a (lower semi-)continuous Leibniz norm in the sense of Latrémiölère \[3\].

2. Let \(I\) be a countable set. If \(\dim H < \infty\) and if \(\text{Gap}_\alpha > 0\) then \(\left( S^\infty, \|\cdot\|_{\Gamma,\alpha,p,\partial}, \mathcal{D}_I \right)\) is a quantum locally compact metric space. Moreover, the norm \(\|\cdot\|_{\Gamma,\alpha,p}\) is a lower semi-continuous Leibniz norm.

We also show a companion result of Theorem 1.4 in the context of discrete groups in Subsection 5.7. Finally, we introduce some spectral triples (i.e. noncommutative manifolds) strongly related to these quantum metric spaces. Our examples admits \(L^p\)-versions. So we also introduce a Banach space generalization of the notion of spectral triple satisfied by our examples.

The paper is organized as follows. Section 2 gives background and preliminary results. In Section 3, we show dimension free estimates for Riesz transforms associated to semigroups of Markov Schur multipliers. Moreover, we complement the results of \[4\] for Riesz transforms associated to semigroups of Markov Fourier multipliers. In the next Section 4, we show our main results on functional calculus of Hodge-Dirac operators. Finally, in Section 5, we introduce our new quantum locally compact metric spaces and some new spectral triples. Here we also define a Banach space generalization of the notion of spectral triple. Finally, this paper is the first part of a three part series.

### 2 Preliminaries

#### 2.1 Various

We denote by \(M_{I,\text{fin}}\) the space of matrices indexed by \(I \times I\) with a finite number of non null entries. Given a set \(I\), the set \(\mathcal{P}_f(I)\) of all finite subsets of \(I\) is directed with respect to set inclusion. For \(J \in \mathcal{P}_f(I)\) and \(A \in M_I\), we write \(T_J(A)\) for the matrix obtained from \(A\) by setting each entry to zero if its row and column index are not both in \(J\). We call \(\left( T_J(A) \right)_{J \in \mathcal{P}_f(I)}\) the net of finite submatrices of \(A\). See \[5\], page 488.

If \((T_t)_{t \geq 0}\) is a strongly continuous semigroup on a Banach space \(X\) with (negative) infinitesimal generator \(A\), we have

\[
\frac{1}{t} (T_t(x) - x) \overset{t \to 0^+}{\longrightarrow} -Ax, \quad x \in \text{dom}\ A.
\]

By \[6\], page 447, a linear operator is bisectorial if and only if

\[
i\mathbb{R} - \{0\} \subset \rho(A) \quad \text{and} \quad \sup_{t \in \mathbb{R}^+ - \{0\}} \|tR(it,A)\| < \infty.
\]
A sectorial operator $A$ is called $R$-sectorial if for some $\omega(A) < \sigma < \pi$ the family
\[(2.3) \quad \{zR(z, A) : z \notin \Sigma_{\sigma}\}\]
is $R$-bounded. For any $x \in X$, we have
\[(2.4) \quad (z - A)^{-1}x = -\int_{0}^{+\infty} e^{\omega z}T_u(x) du.\]

Recall that if $T$: dom $T \subset X \rightarrow Y$ is a densely defined unbounded operator then dom $T^*$ is equal to
\[(2.5) \quad \{y^* \in Y^* : \text{there exists } x^* \in X^* \text{ such that } \langle T(x), y^* \rangle_{Y,Y^*} = \langle x, x^* \rangle_{X,X^*} \text{ for all } x \in \text{dom } T\}.\]

If $y^* \in \text{dom } T^*$, the above $x^* \in X^*$ is determined uniquely by $y^*$ and we let $T^*(y^*) = x^*$. For any $x \in \text{dom } T$ and any $y^* \in \text{dom } T^*$, we have
\[(2.6) \quad \langle T(x), y^* \rangle_{Y,Y^*} = \langle x, T^*(y^*) \rangle_{X,X^*}.\]

If $T$: dom $T \subset X \rightarrow Y$ is closable then
\[(2.7) \quad x \in \text{dom } T \text{ iff there exists } (x_n) \subset \text{dom } T \text{ such that } x_n \rightarrow x \text{ and } T(x_n) \rightarrow y \text{ for some } y.\]

The following is a particular case of [HvNVW2, Theorem 10.6.7].

**Proposition 2.1** Suppose that $A$ is an $R$-bisectorial operator on a Banach space $X$ of finite cotype. Then $A^2$ is $R$-sectorial and for each $\omega \in (0, \frac{\pi}{2})$ the following assertions are equivalent.

1. The operator $A$ admits a bounded $H^\infty(\Sigma_{\omega}^\pm)$ functional calculus.
2. The operator $A^2$ admits a bounded $H^\infty(\Sigma_{\omega}^\pm)$ functional calculus.

Let $(M, \tau)$ be a semifinite von Neumann algebra and let $1 \leq p < \infty$. For any $a \in M$, we define a bounded operator $L_a$: $L^p(M)$ into $L^p(M)$ by letting
\[(2.8) \quad L_a(x) = ax, \quad x \in L^p(M).\]

We will call $L_a$ the left multiplication by $a$ on $L^p(M)$.

### 2.2 $q$-gaussian functors and isonormal processes

Since we will study maps between $q$-deformed algebras, we recall directly several facts about these more general algebras in the context of [BKS]. We denote by $S_n$ the symmetric group. If $\sigma$ is a permutation of $S_n$ we denote by $|\sigma|$ the number card $\{(i,j) | 1 \leq i, j \leq n, \sigma(i) > \sigma(j)\}$ of inversions of $\sigma$. Let $H$ be a real Hilbert space with complexification $H_C$. If $-1 < q < 1$ the $q$-Fock space over $H$ is
\[F_q(H) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} H_C^{\otimes_n}\]
where $\Omega$ is a unit vector, called the vacuum and where the scalar product on $H_C^{\otimes_n}$ is given by
\[\langle h_1 \otimes \cdots \otimes h_n, k_1 \otimes \cdots \otimes k_n \rangle_q = \sum_{\sigma \in S_n} q^{|\sigma|} \langle h_1, k_{\sigma(1)} \rangle_{H_C} \cdots \langle h_n, k_{\sigma(n)} \rangle_{H_C}.\]
If \( q = -1 \), we must first divide out by the null space, and we obtain the usual antisymmetric Fock space. The creation operator \( \ell(e) \) for \( e \in H \) is given by

\[
\ell(e) : \mathcal{F}_q(H) \longrightarrow \mathcal{F}_q(H) \\
h_1 \otimes \cdots \otimes h_n \longmapsto e \otimes h_1 \otimes \cdots \otimes h_n.
\]

They satisfy the \( q \)-relation

\[
\ell(f)^* \ell(e) - q \ell(e) \ell(f)^* = \langle f, e \rangle_H \text{Id}_{\mathcal{F}_q(H)}.
\]

We denote by \( s_q(e) : \mathcal{F}_q(H) \to \mathcal{F}_q(H) \) the selfadjoint operator \( \ell(e) + \ell(e)^* \). The \( q \)-von Neumann algebra \( \Gamma_q(H) \) is the von Neumann algebra generated by the operators \( s_q(e) \) where \( e \in H \). It is a finite von Neumann algebra with the trace \( \tau \) defined by \( \tau(x) = \langle \Omega, x\Omega \rangle_{\mathcal{F}_q(H)} \) where \( x \in \Gamma_q(H) \).

Let \( H \) and \( K \) be real Hilbert spaces and \( T : H \to K \) be a contraction with complexification \( T_{\mathbb{C}} : H_{\mathbb{C}} \to K_{\mathbb{C}} \). We define the following linear map

\[
\mathcal{F}_q(T) : \mathcal{F}_q(H) \longrightarrow \mathcal{F}_q(K) \\
h_1 \otimes \cdots \otimes h_n \longmapsto T_{\mathbb{C}}(h_1) \otimes \cdots \otimes T_{\mathbb{C}}(h_n).
\]

Then there exists a unique map \( \Gamma_q(T) : \Gamma_q(H) \to \Gamma_q(H) \) such that for every \( x \in \Gamma_q(H) \) we have

\[
(\Gamma_q(T)(x))\Omega = \mathcal{F}_q(T)(x\Omega).
\]

This map is normal, unital, completely positive and trace preserving. If \( T : H \to K \) is an isometry, \( \Gamma_q(T) \) is an injective \( * \)-homomorphism. If \( 1 \leq p < \infty \), it extends to a contraction \( \Gamma_p(T) : L^p(\Gamma_q(H)) \to L^p(\Gamma_q(K)) \).

Moreover, we need the following Wick formula, (see [Boz, page 2] and [EFP, Corollary 2.1]). In order to state this, we denote, if \( k \geq 1 \) is an integer, by \( \mathcal{P}_2(2k) \) the set of 2-partitions of the set \( \{1, 2, \ldots, 2k\} \). If \( V \in \mathcal{P}_2(2k) \) we let \( c(V) \) the number of crossings of \( V \), which is given, by the number of pairs of blocks of \( V \) which cross (see [EFP, page 8630] for a precise definition). Then, if \( f_1, \ldots, f_{2k} \in H \) we have

\[
\tau(s_q(f_1)s_q(f_2) \cdots s_q(f_{2k})) = \sum _{V \in \mathcal{P}_2(2k)} q^{c(V)} \prod _{(i,j) \in V} \langle f_i, f_j \rangle_H
\]

and for an odd number of factors of \( q \)-gaussians, the trace vanishes,

\[
\tau(s_q(f_1)s_q(f_2) \cdots s_q(f_{2k-1})) = 0.
\]

In particular, for any \( e, f \in H \), we have

\[
\tau(s_q(e)s_q(f)) = \langle e, f \rangle_H.
\]

Recall that if \( e \in H \) has norm 1, then the operator \( s_{-1}(e) \) satisfies

\[
s_{-1}(e)^2 = \text{Id}_{\mathcal{F}_{-1}(H)}.
\]

If \( q = 1 \), the \( q \)-gaussian functor identifies to an \( H \)-isnormal process on a probability space \((\Omega, \mu)\) [Nua1, Definition 1.1.1] [Neer1, Definition 6.5], that is a linear mapping \( W : H \to L^0(\Omega) \) with the following properties:

\[
\begin{align*}
(1.4) & \quad \text{for any } h \in H \text{ the random variable } W(h) \text{ is a centred real Gaussian;} \\
(1.5) & \quad \text{for any } h_1, h_2 \in H \text{ we have } \mathbb{E}(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_H; \\
(1.6) & \quad \text{The linear span of the products } W(h_1)W(h_2) \cdots W(h_m), \text{ with } m \geq 0 \text{ and } h_1, \ldots, h_m \in H, \text{ is dense in the real Hilbert space } L^2_{\mathbb{R}}(\Omega).
\end{align*}
\]
Here $L^0(\Omega)$ denote the space of measurable functions on $\Omega$ and we make the convention that the empty product, corresponding to $m = 0$ in (2.16), is the constant function 1. Recall that the span of elements $e^{iW(h)}$ is dense in $L^p(\Omega)$ by [Jani, Theorem 2.12] if $1 \leq p < \infty$ and weak* dense if $p = \infty$.

If $1 \leq p \leq \infty$ and if $u : H \to H$ is a contraction, we denote by $\Gamma_p(u) : L^p(\Omega) \to L^p(\Omega)$ the (symmetric) second quantization of $u$ acting on the complex Banach space $L^p(\Omega)$. Recall that the map $\Gamma_\infty(u) : L^\infty(\Omega) \to L^\infty(\Omega)$ preserves the integral\(^1\). If $u : H_1 \to H_2$ is an isometry between Hilbert spaces then $\Gamma_\infty(u) : L^\infty(\Omega_{H_1}) \to L^\infty(\Omega_{H_2})$ is a trace preserving injective normal unital *-homomorphism which is surjective if $u$ is surjective. Moreover, we have if $1 \leq p < \infty$

\begin{equation}
\Gamma_p(u)W(h) = W(u(h)), \quad h \in H.
\end{equation}

and

\begin{equation}
\Gamma_\infty(u)e^{iW(h)} = e^{iW(u(h))}, \quad h \in H.
\end{equation}

Finally, note that if $u : H \to H$ is a surjective isometry then $\Gamma_\infty(u) : L^\infty(\Omega) \to L^\infty(\Omega)$ is a *-automorphism of the von Neumann algebra $L^\infty(\Omega)$.

### 2.3 Hilbertian valued noncommutative $L^p$-spaces

We shall use various Hilbert-valued noncommutative $L^p$-spaces. We refer to [JMX, Chapter 2] for more information on these spaces. However, note that our notations are slightly different since we want that our notations remain compatible with vector-valued noncommutative $L^p$-spaces in the hyperfinite case. Suppose $1 \leq p < \infty$. Let $H$ be a Hilbert space. For any elements $\sum_{k=1}^n x_k \otimes a_k, \sum_{j=1}^m y_j \otimes b_j$ of $L^p(M) \otimes H$, we define the $L^p(M)$-valued inner product

\begin{equation}
\left\langle \sum_{k=1}^n x_k \otimes a_k, \sum_{j=1}^m y_j \otimes b_j \right\rangle_{L^p(M, H_{c,p})} \overset{\text{def}}{=} \sum_{k,j=1}^{n,m} \langle a_k, b_j \rangle_H x_k^* y_j.
\end{equation}

For any element $\sum_{k=1}^n x_k \otimes a_k$ of $L^p(M) \otimes H$, we set [JMX, (2.9)]

\begin{equation}
\left\| \sum_{k=1}^n x_k \otimes a_k \right\|_{L^p(M, H_{c,p})} \overset{\text{def}}{=} \left\| \left( \sum_{k=1}^n x_k \otimes a_k, \sum_{k=1}^n x_k \otimes a_k \right)_{L^p(M, H_{c,p})} \right\|_{L^p(M)} \overset{\text{def}}{=} \left\| \sum_{k,j=1}^n \langle a_k, a_j \rangle_H x_k^* x_j \right\|_{L^p(M)}.
\end{equation}

The space $L^p(M, H_{c,p})$ is the completion of $L^p(M) \otimes H$ for this norm. It is folklore that $L^p(M, H_{c,p})$ equipped with (2.19) is a right $L^p$-$M$-module [JuS1, Definition 3.3], see also [JuPe1, Section 1.3].

If $(e_1, \ldots, e_n)$ is an orthonormal family of $H$ and if $x_1, \ldots, x_n$ belong to $L^p(M)$, it follows from (2.20) (see [JMX, (2.10)]) that

\begin{equation}
\left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{L^p(M, H_{c,p})} = \left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M, H_{c,p})} = \left\| \sum_{k=1}^n e_k x_k \right\|_{S_p(L^p(M))}.
\end{equation}

If $T : L^p(M) \to L^p(M)$ is completely bounded then by [JMX, page 22] the map $T \otimes \text{Id}_H$ induces a bounded operator on $L^p(M, H_{c,p})$ and on $L^p(M, H_{c,p})$ and we have

\begin{equation}
\|T \otimes \text{Id}_H\|_{cb,L^p(M, H_{c,p})} \leq \|T\|_{cb,L^p(M)} \cdot \|T\|_{cb,L^p(M, H_{c,p})}.
\end{equation}

\(^1\) That means that for any $f \in L^\infty(\Omega)$ we have $\int_\Omega \Gamma_\infty(u)f \, d\mu = \int_\Omega f \, d\mu$. 

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2.4 Carré du champ $\Gamma$ and first order differential calculus for Schur multipliers

To state some of our results, we need to consider the gradient form associated to the (negative) infinitesimal generator $A$, defined by $A(e_{ij}) = a_{ij}e_{ij}$, of a weak* continuous semigroup $(T_t)_{t \geq 0}$ of selfadjoint completely positive unital Schur multipliers on $B(ell^2_I)$. For any $x, y \in M_{I, fin}$, we define the element

\[
\Gamma(x, y) = \frac{1}{2} \left[ A(x^*)y + x^*A(y) - A(x^*y) \right].
\]

of $M_{I, fin}$. For any $i, j, k, l \in I$, note that

\[
\Gamma(e_{ij}, e_{kl}) = \frac{1}{2} \left[ A(e_{ji})e_{kl} + e_{ij}A(e_{kl}) - A(e_{ij})e_{kl} \right] = \frac{1}{2} \delta_{i=k} [a_{ji} + a_{kl} - a_{jl}] e_{jl}.
\]

We refer to the books [BGL1] and [BoH1] for the general case of semigroups acting on classical $L^p$-spaces and to [Mey2]. For the semigroups acting on noncommutative $L^p$-spaces, the carré du champ and its applications were studied in the papers [CiSi1, Section 9], [CiSp1], [JM1], [Sau1], [JuZ3] and [Zen1] mainly in the $\sigma$-finite case and for $L^2$-spaces (see [DaL1] for related things). Unfortunately, these papers do not suffice for our setting. So, in the following, we describe an elementary, independent and more concrete approach. However, some results of this section could be known, at least in the case $p = 2$ and $I$ finite.

The first part of the following result shows how to construct the carré du champ directly from the semigroup $(T_t)_{t \geq 0}$.

**Lemma 2.2**

1. Suppose $1 \leq p < \infty$. For any $x, y \in M_{I, fin}$, we have

\[
\Gamma(x, y) = \lim_{t \to 0^+} \frac{1}{2t} \left( T_t(x^*)y - (T_t(x))^*T_t(y) \right)
\]

in $S^p_I$.

2. For any $x \in M_{I, fin}$, we have $\Gamma(x, x) \geq 0$.

3. For any $x, y \in M_{I, fin}$, the matrix $\begin{bmatrix} \Gamma(x, x) & \Gamma(x, y) \\ \Gamma(y, x) & \Gamma(y, y) \end{bmatrix}$ is positive.

4. For any $x, y \in M_{I, fin}$, we have

\[
\|\Gamma(x, y)\|_{S^p_I} \leq \|\Gamma(x, x)\|^\frac{1}{2} \|\Gamma(y, y)\|^\frac{1}{2}.\]

This inequality even holds for $0 < p < \infty$.

**Proof :** 1. Suppose $1 \leq p < \infty$. For any $x, y \in M_{I, fin}$, using the joint continuity of multiplication on bounded sets for the strong operator topology on $S^p_I$ [EFHN, Proposition C.19], we have

\[
\frac{1}{2t} (T_t(x^*)y - (T_t(x))^*T_t(y)) = \frac{1}{2t} (T_t(x^*y) - x^*y + x^*y - (T_t(x))^*y + (T_t(x))^*y - (T_t(x))^*T_t(y))
\]

\[
= \frac{1}{2t} (T_t(x^*y) - x^*y + (T_t(x))^*y + (T_t(x))^*(y - T_t(y)))
\]

\[
\xrightarrow{t \to 0^+} \frac{1}{2} \left[ A(x^*y) + A(x^*)y + x^*A(y) \right] = \Gamma(x, y).
\]
If $p = \infty$ we use the fact that $x, y \in M_{f, \fin}$, so all above operators belong to some finite dimensional subspace of $S_p^p \cap S_\infty^\infty$ on which convergence in $S_1^\infty$ norm follows from convergence in $S_p^p$ norm.

2. Each operator $T_t : B(\ell_2^2) \rightarrow B(\ell_2^2)$ is completely positive and unital. Hence by the Schwarz inequality [Pau, Proposition 3.3] (or [Pal2, Proposition 9.9.4]), we have $T_t(x)^* T_t(x) \leq T_t(x^* x)$. Recall that the positive cone of $B(\ell_2^2)$ is weak$^*$ closed (see e.g. [ArK1, (2.3)]). Using the point 1, we conclude that $\Gamma(x, x) \geq 0$.

3. For $x, y \in M_{f, \fin}$, using the point 1 in the first equality and the point 2 in the last inequality applied with the semigroup $(\Id_{M_2} \otimes T_t)_{t \geq 0}$, we obtain

$$\begin{bmatrix} \Gamma(x, x) & \Gamma(x, y) \\ \Gamma(y, x) & \Gamma(y, y) \end{bmatrix} = \lim_{t \rightarrow 0^+} \frac{1}{2t} \begin{bmatrix} T_t(x^* x) - (T_t(x))^* T_t(x) & T_t(x^* y) - (T_t(x))^* T_t(y) \\ T_t(x^* y) - (T_t(y))^* T_t(x) & T_t(y^* y) - (T_t(y))^* T_t(y) \end{bmatrix} \begin{bmatrix} x x^* x^* y^* \\ y x^* y^* y^* \\ T_t(x)^* T_t(y)^* 0 \\ T_t(y)^* T_t(x)^* 0 \end{bmatrix} \begin{bmatrix} T_t(x) & T_t(y) \end{bmatrix} \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_t(x^* x) - (T_t(x))^* T_t(x) & T_t(x^* y) - (T_t(x))^* T_t(y) \\ T_t(y^* y) - (T_t(y))^* T_t(y) & T_t(y^* y) - (T_t(y))^* T_t(y) \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_t(x) & T_t(y) \end{bmatrix} \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \geq 0.$$

4. By [ArK1, Lemma 2.11] or [HoJ1, Theorem 3.5.15], we have

$$\|\Gamma(x, y)\|_{S_1^p} \leq \frac{1}{2p} \left( \|\Gamma(x, x)\|_{S_p^p} + \|\Gamma(y, y)\|_{S_1^\infty} \right)^{\frac{1}{p}}.$$ 

Since we have $\Gamma(\lambda x, \frac{1}{\lambda} y) = \Gamma(x, y)$ for $\lambda > 0$, we deduce

$$\|\Gamma(x, y)\|_{S_1^p} \leq \frac{1}{2p} \inf_{\lambda > 0} \left( \lambda \|\Gamma(x, x)\|_{S_p^p} + \frac{1}{\lambda} \|\Gamma(y, y)\|_{S_1^\infty} \right)^{\frac{1}{p}}.$$ 

Ruling out beforehand the easy cases $\Gamma(x, x) = 0$ or $\Gamma(y, y) = 0$, we choose

$$\lambda = \left( \frac{\|\Gamma(y, y)\|_{S_1^\infty}^{p-1}}{\|\Gamma(x, x)\|_{S_p^p}} \right)^{\frac{1}{p}},$$

and obtain

$$\|\Gamma(x, y)\|_{S_1^p} \leq \frac{1}{2p} \left( 2 \|\Gamma(x, x)\|_{S_p^p}^{\frac{p}{p-s}} \right)^{\frac{1}{p}} \leq \|\Gamma(x, x)\|_{S_p^p}^{\frac{1}{p}} \|\Gamma(y, y)\|_{S_1^\infty}^{\frac{1}{p}} = \|\Gamma(x, x)\|_{S_1^p}^{\frac{1}{p}}.$$
The weak* infinitesimal generator $A$ acts by $A(e_{ij}) = \|\alpha_i - \alpha_j\|^2_{H} e_{ij}$. If $\Gamma_q$ is the $q$-Gaussian functor of Section 2.2 associated with the real Hilbert space $H$, we can consider the linear map $\partial_{\alpha,q}: M_{I,\text{fin}} \to \Gamma_q(H) \otimes M_{I,\text{fin}}$ (resp. $\partial_{\alpha,1}: M_{I,\text{fin}} \to L^0(\Omega) \otimes M_{I,\text{fin}}$ if $q = 1$) defined by

$$\partial_{\alpha,q}(e_{ij}) \overset{\text{def}}{=} s_q(\alpha_i - \alpha_j) \otimes e_{ij}, \quad i, j \in I.$$  

We have the following Leibniz rule.

**Lemma 2.3** Suppose $-1 \leq q \leq 1$. For any $x, y, e \in M_{I,\text{fin}}$, we have

$$\partial_{\alpha,q}(xy) = x\partial_{\alpha,q}(y) + \partial_{\alpha,q}(x)y.$$  

**Proof**: On the one hand, for any $i, j, k, l \in I$, we have

$$\partial_{\alpha,q}(e_{ij}e_{kl}) = \delta_{j=k}\partial_{\alpha,q}(e_{il}) \overset{(2.26)}{=} \delta_{j=k}s_q(\alpha_i - \alpha_l) \otimes e_{il}.$$  

On the other hand, for any $i, j, k, l \in I$, we have

$$e_{ij}\partial_{\alpha,q}(e_{kl}) + \partial_{\alpha,q}(e_{ij})e_{kl} = e_{ij}(s_q(\alpha_k - \alpha_l) \otimes e_{kl}) + (s_q(\alpha_i - \alpha_j) \otimes e_{ij})e_{kl}$$

$$= s_q(\alpha_k - \alpha_l) \otimes e_{ij}e_{kl} + s_q(\alpha_i - \alpha_j) \otimes e_{ij}e_{kl}$$

$$= s_q(\alpha_k - \alpha_l) \otimes s_q(\alpha_i - \alpha_j) \otimes e_{il}$$

$$= \delta_{j=k}s_q(\alpha_i - \alpha_l) \otimes e_{il}.$$  

The result follows by linearity. 

Now, we describe a connection between the carré du champ and the map $\partial_a$ which is analogous to the equality of [Sau1, Section 1.4] (see also [Sau2]). For that, we introduce the canonical trace preserving normal faithful conditional expectation $E_I: \Gamma_q(H) \otimes B(\ell_q^2) \to B(\ell_q^2)$.

**Proposition 2.4** Suppose $-1 \leq q \leq 1$. For any $x, e \in M_{I,\text{fin}}$, we have

$$\Gamma(x, x) = E_I\left((\partial_{\alpha,q}(x))^*\partial_{\alpha,q}(x)\right).$$  

**Proof**: On the one hand, for any $x \in M_{I,\text{fin}}$ and any $i, j \in I$, we have

$$\{\Gamma(x, x)\}_{ij} \overset{(2.22)}{=} \frac{1}{2} [A(x^*)x + x^*A(x) - A(x^*x)]_{ij}$$

$$= \frac{1}{2}\left(\sum_k [A(x^*)]_{ik}x_{kj} + \sum_k [x^*]_{ik}[A(x)]_{kj} - a_{ij}[x^*x]_{ij}\right)$$

$$= \frac{1}{2}\left(\sum_k a_{ik}x_{ki}x_{kj} + \sum_k x_{ki}a_{kj}x_{kj} - \sum_k a_{ij}x_{ki}x_{kj}\right) = \frac{1}{2}\left(\sum_k x_{ki}x_{kj}(a_{ik} + a_{kj} - a_{ij})\right).$$
On the other hand, for any \( x \in M_{I, \fin} \), we have

\[
\mathbb{E}_I \left[ (\partial_{\alpha,q}(x))^* \partial_{\alpha,q}(x) \right] = \mathbb{E}_I \left[ \left( \sum_{i,j} s_q(\alpha_i - \alpha_j) \otimes x_{ij} e_{ij} \right)^* \left( \sum_{k,l} s_q(\alpha_k - \alpha_l) \otimes x_{kl} e_{kl} \right) \right] \\
= \sum_{i,j,k,l} \mathbb{E}_I \left[ (s_q(\alpha_i - \alpha_j) \otimes x_{ij} e_{ij}) (s_q(\alpha_k - \alpha_l) \otimes x_{kl} e_{kl}) \right] \\
= \sum_{i,j,k,l} x_{ij} x_{kl} \tau (s_q(\alpha_i - \alpha_j) s_q(\alpha_k - \alpha_l)) e_{ij} e_{kl} = \sum_{k,j,l} x_{kj} x_{kl} \tau (s_q(\alpha_k - \alpha_j) s_q(\alpha_k - \alpha_l)) e_{ij} \\
(2.26) = \sum_{k,j,l} x_{kj} x_{kl} (\alpha_k - \alpha_j, \alpha_k - \alpha_l) e_{ij} = \sum_{k,j} x_{kj} x_{kl} (\alpha_k - \alpha_i, \alpha_k - \alpha_j) e_{ij} \\
= \sum_{i,j} \sum_k x_{kj} x_{kl} (\alpha_k - \alpha_i, \alpha_k - \alpha_j) e_{ij}.
\]

Thus, using

\[
\frac{1}{2} a_{ik} + a_{kj} - a_{ij} = \frac{1}{2} \left( \|\alpha_i - \alpha_k\|^2_H + \|\alpha_k - \alpha_j\|^2_H - \|\alpha_i - \alpha_j\|^2_H \right) \\
= \frac{1}{2} \left( 2 \|\alpha_k\|^2 - 2\langle \alpha_i, \alpha_k \rangle - 2\langle \alpha_j, \alpha_k \rangle + 2\langle \alpha_i, \alpha_j \rangle \right) = \langle \alpha_k - \alpha_i, \alpha_k - \alpha_j \rangle,
\]

we obtain (2.28).

**Lemma 2.5** Suppose \(-1 \leq q \leq 1\) and \(1 \leq p < \infty\).

1. For any \( x, y \in M_{I, \fin} \), we have

\[
\Gamma(x,y) = \langle \partial_{\alpha,q}(x), \partial_{\alpha,q}(y) \rangle_{S_p^\gamma(L^2(\mathcal{H}))},
\]

2. For any \( x \in M_{I, \fin} \), we have

\[
\|\Gamma(x,x)^{\frac{1}{2}}\|_{S_p^\gamma} = \|\partial_{\alpha,q}(x)\|_{S_p^\gamma(L^2(\mathcal{H}))},
\]

**Proof** 1. On the one hand, for any \( i, j, k, l \in I \) we have

\[
\langle \partial_{\alpha}(e_{ij}), \partial_{\alpha}(e_{kl}) \rangle_{S_p^\gamma(L^2(\mathcal{H}))} = \langle e_{ij} \otimes s_q(\alpha_i - \alpha_j), e_{kl} \otimes s_q(\alpha_k - \alpha_l) \rangle_{S_p^\gamma(L^2(\mathcal{H}))} \\
(2.26) = \langle s_q(\alpha_i - \alpha_j), s_q(\alpha_k - \alpha_l) \rangle_{L^2(\mathcal{H})} e_{ij} e_{kl} \\
(2.19) = \delta_{i=k} \langle \alpha_i - \alpha_j, \alpha_k - \alpha_l \rangle_{\mathcal{H}} e_{ij} e_{kl}.
\]

On the other hand, for any \( i, j, k, l \in I \), we have

\[
\Gamma(e_{ij}, e_{kl}) = \delta_{i=k} \frac{1}{2} [a_{ji} + a_{il} - a_{jl}] e_{ij}.
\]

Finally, for any \( i, j, k, l \in I \), we have

\[
\frac{1}{2} (a_{ji} + a_{il} - a_{jl}) = \frac{1}{2} \left( \|\alpha_j - \alpha_i\|^2_H + \|\alpha_i - \alpha_l\|^2_H - \|\alpha_j - \alpha_l\|^2_H \right) \\
= \frac{1}{2} \left( 2 \|\alpha_i\|^2 - 2\langle \alpha_j, \alpha_i \rangle - 2\langle \alpha_i, \alpha_l \rangle + 2\langle \alpha_j, \alpha_l \rangle \right) = \langle \alpha_i - \alpha_j, \alpha_i - \alpha_l \rangle.
\]

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2. If $x \in M_{f,\fin}$, we have
\[
\|\partial_{\alpha,q}(x)\|_{S_p^r(L^p(\Gamma_q(H)c_p))} \overset{(2.20)}{=} \left\| \langle \partial_{\alpha,q}(x), \partial_{\alpha,q}(x) \rangle \right\|_{S_p^r} \overset{(2.30)}{=} \| \Gamma(x,x) \|^\frac{1}{2} \| S_r \|.
\]

The next lemma describes the “Dirichlet form”.

Lemma 2.6 For any $x \in M_{f,\fin}$, we have
\[
\text{Tr} \left( \Gamma(x,x) \right) = \sum_{i,j \in I} |x_{ij}|^2 a_{ij}.
\]
Moreover, we have $\Gamma(x,x) = 0$ if and only if $\text{Tr}(\Gamma(x,x)) = 0$.

Proof: Using $a_{jj} = \|\alpha_j - \alpha_j\|^2_H = 0$ and $a_{kj} = \|\alpha_k - \alpha_j\|^2_H = a_{jk}$ in the third equality, we obtain
\[
\text{Tr} \left( \Gamma(x,x) \right) = \sum_j \Gamma(x,x)_{jj} \overset{(2.29)}{=} \sum_j \frac{1}{2} \left( \sum_k x_{ij} x_{kj} (a_{kj} + a_{kj} - a_{jj}) \right) = \sum_{k,j} |x_{kj}|^2 a_{kj}.
\]
The last statement is a consequence of the part 2 of Lemma 2.2 and the faithfulness of the trace. ■

2.5 Markovian semigroups of Fourier multipliers on group von Neumann algebras

In this subsection, we recall the basic theory of markovian semigroups of Fourier multipliers. Let $G$ be a discrete group and consider its representation over the Hilbert space $l^2_G$ given by left translations $\lambda_s f(t) = f(s^{-1}t)$ for $s,t \in G$ and $f \in l^2_G$. Then the group von Neumann algebra $\text{VN}(G) \subseteq B(l^2_G)$ is by definition generated by these left translations. We also recall that $C_r^*(G)$ stands for the reduced group $C^*$-algebra sitting inside $\text{VN}(G) \subseteq B(l^2_G)$ and generated by these $\lambda_s$. Let us write
\[
(2.32) \quad \mathcal{P}_G = \text{span} \{ \lambda_s : s \in G \}
\]
for the space of “trigonometric polynomials.” Then a Fourier multiplier $M_\phi$ with some symbol $\phi: G \to \mathbb{C}$ is by definition the linear mapping $M_\phi: \mathcal{P}_G \to \mathcal{P}_G$, $\lambda_s \mapsto \phi(s) \lambda_s$. The von Neumann algebra $\text{VN}(G)$ is equipped with the tracial state $\tau(\lambda_s) = \delta_{\lambda_e} = \langle \lambda_e, \delta_e \rangle$. Then one associates with it the family of non-commutative $L^p$-spaces $L^p(\text{VN}(G))$, and one can ask whether $M_\phi$ extends to a bounded operator on $L^p(\text{VN}(G))$. We will be mainly interested in the case that $M_\phi$ extends to a contraction on $L^p(\text{VN}(G))$ for all $1 \leq p \leq \infty$ (normal if $p = \infty$), more precisely we have the following.

Definition 2.7 (JMX, Chapter 5) Let $M$ be a von Neumann algebra equipped with a faithful normal semifinite trace. Let $(T_t)_t$ be a weak* continuous semigroup of normal operators on $M$. We say that $(T_t)_t$ is a markovian semigroup if
1. each $T_t$ extends to a contraction on $L^p(M)$ for any $1 \leq p \leq \infty$,
2. each $T_t$ is self-adjoint on $L^2(\text{VN}(G)),$
3. each $T_t$ is completely positive,

4. each $T_t$ maps $1 \mapsto T_t(1) = 1$.

In particular, if $M = VN(G)$ for some discrete group $G$ and each $T_t$ is a Fourier multiplier, then we say that $(T_t)_t$ is a markovian semigroup of Fourier multipliers.

We have the following characterization of such markovian semigroups of Fourier multipliers. It is known as Schoenberg’s Theorem, see [Schoe] for its classical form.

**Proposition 2.8** Let $G$ be a discrete group and $(T_t)_t$ be a semigroup on $VN(G)$. Then the following are equivalent.

1. $(T_t)_t$ is a markovian semigroup of Fourier multipliers.

2. There exists a conditionally negative definite function $\psi: G \to \mathbb{C}$ such that $T_t(\lambda_s) = \exp(-t\psi(s))\lambda_s$ for any $t \geq 0$ and $s \in G$. Conditionally negative definite means that $\psi(e) = 0$, $\psi(s) = \psi(s^{-1})$ for any $s \in G$ and $\sum_s a_s = 0 \implies \sum_{s,s'} a_s \overline{a}_{s'} \psi(s^{-1} s') \leq 0$.

3. There exists a real Hilbert space $H$ together with a mapping $b: G \to \mathbb{C}$ and a homomorphism $\alpha: G \to O(H)$, where $O(H)$ stands for the orthogonal group, such that the 1-cocycle law holds: $b(st) = b(s) + \alpha_s(b(t))$ for $s, t \in G$, and moreover, $T_t$ is a Fourier multiplier with symbol $\exp(-t\psi)$, where $\psi(s) = \|b(s)\|_H^2$.

We shall also need the following Riesz transform norm equivalence for markovian semigroup of Fourier multipliers from [JMP2, Theorem A2, Remark 1.3]. For a definition of the carré du champ operator $\Gamma$, we refer to Subsection 2.7.

**Proposition 2.9** Let $G$ be a discrete group and $(T_t)_t$ a markovian semigroup of Fourier multipliers with symbol $\psi$ of the negative generator $A$. Then for $2 \leq p < \infty$, we have the norm equivalence

$$\|A^\frac{1}{2}(x)\|_{L^p(VN(G))} \cong_p \max \left\{ \left\| \Gamma(x,x) \right\|_{L^p(VN(G))}, \left\| \Gamma(x^*,x^*) \right\|_{L^p(VN(G))} \right\}, \quad x \in \operatorname{span}\{\lambda_s : s \in G\}.$$
2.6 Fourier multipliers on crossed product von Neumann algebras: transference

We refer to [Haa2], [Haa3], [Str1], [Sun] and [Tak2]. Let $M$ be a von Neumann algebra acting on a Hilbert space $H$. Let $G$ be a locally compact group equipped with some left Haar measure $\mu_G$. Let $\alpha: G \to M$ be a representation of $G$ on $M$ which is weak*-continuous, i.e. for any $x \in M$ and any $y \in M_*$, the map $G \to M$, $s \mapsto \langle \alpha_s(x), y \rangle_{M,M_*}$ is continuous. For any $x \in M$, we define the operators $\pi(x): L^2(G,H) \to L^2(G,H)$ [Str1, (2) page 263] by

\[(2.33) \quad (\pi(x)\xi)(s) \overset{\text{def}}{=} \alpha_s^{-1}(x)\xi(s), \quad \xi \in L^2(G,H), s \in G.\]

These operators satisfy the following commutation relation [Str1, (2) page 292]:

\[(2.34) \quad (\lambda_s \otimes \text{Id}_H)\pi(x)(\lambda_s \otimes \text{Id}_H)^* = \pi(\alpha_s(x)), \quad x \in M, s \in G.\]

Recall that the crossed product of $M$ and $G$ with respect to $\alpha$ is the von Neumann algebra $M \rtimes_\alpha G = (\pi(M) \cup \{\lambda_s \otimes \text{Id}_H : s \in G\})''$ on the Hilbert space $L^2(G,H)$ generated by the operators $\pi(x)$ and $\lambda_s \otimes \text{Id}_H$ where $x \in M$ and $s \in G$. By [Str1, page 263] or [Dae1, Proposition 2.5], $\pi$ is a normal injective $\ast$-homomorphism from $M$ into $M \rtimes_\alpha G$ (hence $\sigma$-strong* continuous).

In the sequel, we suppose that $G$ is discrete. For any $s \in G$ and any $x \in M$, we let $x \rtimes \lambda_s \overset{\text{def}}{=} \pi(x)(\lambda_s \otimes \text{Id}_H)$. We recall the rules of product and adjoint:

\[(2.35) \quad (x \rtimes \lambda_s)^* = \alpha_{s^{-1}}(x^*) \rtimes \lambda_{s^{-1}}\]

and

\[(2.36) \quad (x \rtimes \lambda_s)(y \rtimes \lambda_t) = x\alpha_s(y) \rtimes \lambda_{st}, \quad s, t \in G, \xi \in H.\]

Finally, if $M$ is equipped with a normal finite faithful trace $\tau_M$ then $M \rtimes_\alpha G$ is equipped with the normal finite faithful trace defined by $\tau_{M \rtimes_\alpha G}(x \rtimes \lambda_s) = \tau_M(x)\delta_{s=e}$. Given a discrete group $G$, we can consider the “fundamental unitary” $W: \ell^2_G \otimes_2 H \otimes_2 \ell^2_G \to \ell^2_G \otimes_2 H \otimes_2 \ell^2_G$ defined by

\[(2.37) \quad W_H(\epsilon_t \otimes \xi \otimes \epsilon_r) = \epsilon_t \otimes \xi \otimes \epsilon_{tr} \quad (\text{see also [BoBr, page 8]}).\]

**Lemma 2.10** Then, for any $s \in G$ and any $x \in M$, we have

\[W_H((x \rtimes \lambda_s) \otimes \text{Id}_{\ell^2_G})W_H^{-1} = (x \rtimes \lambda_s) \otimes \lambda_s.\]

**Proof**: On the one hand, for any $s, t, r \in G$, we have

\[W_H((x \rtimes \lambda_s) \otimes \text{Id}_{\ell^2_G})(\epsilon_t \otimes \xi \otimes \epsilon_r) = W_H(x(\lambda_s \otimes \text{Id}_H) \otimes \text{Id}_{\ell^2_G})(\epsilon_t \otimes \xi \otimes \epsilon_r) = W_H(x \otimes \text{Id}_{\ell^2_G})(\epsilon_{st} \otimes \xi \otimes \epsilon_r) = W_H(\epsilon_{st} \otimes \alpha_{st}^{-1}(x)\xi \otimes \epsilon_r) = \epsilon_{st} \otimes \alpha_{st}^{-1}(x)\xi \otimes \epsilon_{str}.\]

On the other hand, we have

\[\quad (x \rtimes \lambda_s)W_H(\epsilon_t \otimes \xi \otimes \epsilon_r) = ((x(\lambda_s \otimes \text{Id}_H)) \otimes \lambda_s)(\epsilon_t \otimes \xi \otimes \epsilon_r) = (x \otimes \text{Id}_{\ell^2_G})(\epsilon_{st} \otimes \xi \otimes \epsilon_{str}) = \epsilon_{st} \otimes \alpha_{st}^{-1}(x)\xi \otimes \epsilon_{str}.\]
We conclude by linearity and density.

As a composition of an ampliation and a spatial isomorphism, the map $M \rtimes_\alpha \VN(G) \to \VN(G) \rtimes \VN(M \rtimes_\alpha G)$, $z \mapsto W_H (z \otimes \Id_G^\alpha) W_H^*$ is a unital normal injective $*$-homomorphism. This map is well-defined kind of “crossed coproduct” defined by

\[(2.38) \Delta \colon M \rtimes_\alpha \VN(G) \to \VN(G) \rtimes (M \rtimes_\alpha G), \quad x \otimes \lambda s \mapsto \lambda_s \otimes (x \otimes \lambda s)\]

This homormorphism is trace preserving since

\[
\tau_{\VN(G) \rtimes (M \rtimes_\alpha G)}(\Delta(\lambda_s \otimes (x \otimes \lambda s))) = \tau_{\VN(G) \rtimes (M \rtimes_\alpha G)}((x \otimes \lambda_s) \otimes \lambda_s) = \tau_M(x)\delta_{s=e} = \tau_{M \rtimes_\alpha \VN(G)}(x \otimes \lambda_s).
\]

By [Arh2, Lemma 4.1], it admits a completely isometric $L^p$-extension $\Delta_p \colon L^p(M \rtimes_\alpha \VN(G)) \to L^p(\VN(G) \rtimes (M \rtimes_\alpha G))$ for any $1 \leq p < \infty$.

If $x \in M$ and $s \in G$ and if $\phi \colon G \to \mathbb{C}$, we let

\[(2.39) (\Id_M \rtimes M_\alpha)(x \otimes \lambda s) \overset{\text{def}}{=} \phi(s)x \otimes \lambda s.\]

The main result of this subsection is the following transference result. The assumptions are satisfied in the case where $M$ has QWEP and where the action $\alpha \colon G \to \Aut(M)$ is amenable, see [Oza, Proposition 4.1 (vi)]. See also [Arh1, Proposition 4.8].

**Proposition 2.11** Suppose $1 \leq p \leq \infty$. Let $\phi \colon G \to \mathbb{C}$ a function which induces a completely bounded Fourier multiplier $M_\phi : L^p(\VN(G)) \to L^p(\VN(G))$. If $1 \leq p < \infty$, assume in addition that $M \rtimes_\alpha G$ has QWEP. Then (2.39) induces a completely bounded map $\Id_M \rtimes M_\phi : L^p(M \rtimes_\alpha G) \to L^p(M \rtimes_\alpha G)$ and

\[(2.40) \|\Id_M \rtimes M_\phi\|_{\text{cb}, L^p(M \rtimes_\alpha G) \to L^p(M \rtimes_\alpha G)} \leq \|M_\phi\|_{\text{cb}, L^p(\VN(G)) \to L^p(\VN(G))} \cdot \]

Moreover, we have

\[(2.41) \Delta_p(\Id_M \rtimes M_\phi)(x \otimes \lambda s) = (M_\phi \otimes \Id_{L^p(M \rtimes_\alpha G)})(\lambda_s \otimes (x \otimes \lambda s)).\]

**Proof**: On the one hand, for any $x \in M$ and any $s \in G$, we have

\[
\Delta_p(\Id_M \rtimes M_\phi)(x \otimes \lambda s) \overset{(2.39)}{=} \phi(s)\Delta_p(x \otimes \lambda s) \overset{(2.38)}{=} \phi(s)(\lambda_s \otimes (x \otimes \lambda s)).
\]

On the other hand, we have

\[
(M_\phi \otimes \Id_{L^p(M \rtimes_\alpha G)})(\lambda_s \otimes (x \otimes \lambda s)) = \phi(s)(\lambda_s \otimes (x \otimes \lambda s)).
\]

We conclude that (2.41) is true on $P_{M,G}$. Since $M \rtimes_\alpha G$ has QWEP, the von Neumann algebra $\VN(G)$ is also QWEP. So, by [Jun, (iii) page 984], the map $M_\phi \otimes \Id_{L^p(M \rtimes_\alpha G)} : L^p(\VN(G) \rtimes (M \rtimes_\alpha G)) \to L^p(\VN(G) \rtimes (M \rtimes_\alpha G))$ is completely bounded (in the case $p = \infty$, note that this assumption is useless) with

\[(2.42) \|M_\phi \otimes \Id_{L^p(M \rtimes_\alpha G)}\|_{\text{cb}, L^p(\VN(G) \rtimes (M \rtimes_\alpha G)) \to L^p(\VN(G) \rtimes (M \rtimes_\alpha G))} \leq \|M_\phi\|_{\text{cb}, L^p(\VN(G)) \to L^p(\VN(G))} \cdot \]

15
Note that \( \text{Id}_{S^p} \otimes \Delta_p : L^p(B(\ell^2)\boxtimes (M \rtimes G)) \to L^p(B(\ell^2)\boxtimes VN(G)\boxtimes (M \rtimes G)) \) is a complete isometry. If \( z \in S^p \otimes \mathcal{P}_{M,G} \), we have

\[
\begin{align*}
\| (\text{Id}_{S^p} \otimes (\text{Id}_M \rtimes M_a)) (z) \| &= \| (\text{Id}_{S^p} \otimes (\Delta_p(\text{Id}_M \rtimes M_a)) (z) \| \\
& \leq \| (\text{Id}_{S^p} \otimes (\mathcal{M}_\phi \otimes \mathcal{I}_{L^p(M \rtimes G)}) \Delta_p) (z) \| \\
& \leq \| \mathcal{M}_\phi \otimes \mathcal{I}_{L^p(M \rtimes G)} \| \| \Delta_p \| \| z \|.
\end{align*}
\]

So (2.41) is proved. ■

An operator space \( E \) has CBAP [ER, page 205] [LaS, page 2] (see also [BrO, page 365] for the particular case of C*-algebras) when there exists a net \( (T_j) \) of finite-rank linear maps \( T_j : E \to E \) satisfying the properties:

1. for any \( x \in E \), we have \( \lim_j \| T_j(x) - x \|_E = 0 \),
2. \( \sup_j \| T_j \|_{cb, E \to E} < \infty \).

In the case where \( E \) is a noncommutative \( L^p \)-space associated to a group von Neumann algebra of a discrete group, we can use an average construction of Haagerup [Han3, proof of Lemma 2.5] to replace the net \( (T_j) \) by a net of Fourier multipliers, see also [ArK1, Theorem 4.2] for a generalization. Indeed, we have the following observation which is folklore.

Lemma 2.12 Let \( G \) be a discrete group. Suppose \( 1 \leq p < \infty \). Then the operator space \( L^p(VN(G)) \) has CBAP if and only if there exists a net \( (\varphi_j) \) of functions \( \varphi_j : G \to \mathbb{C} \) with finite support which converge pointwise to \( 1 \) such that the net \( (M_{\varphi_j}) \) converges to \( \text{Id}_{L^p(VN(G))} \) in the point-norm topology with

\[
\sup_j \| M_{\varphi_j} \|_{cb, L^p(VN(G)) \to L^p(VN(G))} < \infty.
\]

Moreover, if \( L^p(VN(G)) \) has CCAP, then the above supremum can be chosen to be equal to \( 1 \).

Proof : Let \( (T_j) \) be a net of finite-rank maps on \( L^p(VN(G)) \) with \( \sup_j \| T_j \|_{cb, L^p(VN(G)) \to L^p(VN(G))} \leq C < \infty \) such that \( (T_j) \) converges to \( \text{Id}_{L^p(VN(G))} \) in the point-norm topology. Since \( \mathcal{P}_G \) from (2.32) is norm dense in \( L^p(VN(G)) \), we may suppose that the range of each \( T_j \) is a subspace of \( \mathcal{P}_G \).

Hence for any \( j \) there exist some integer \( n_j \), some \( \eta_{j,k} \in L^p(VN(G))^* \) and some \( s_{j,k} \in G \) such that \( T_j = \sum_{k=1}^{n_j} \eta_{j,k} \otimes \lambda_{s_{j,k}} \). Using the map \( P_G^p \) of [ArK1, Corollary 4.6], we let \( M_{\varphi_j} \overset{\text{def}}{=} P_G^p(T_j) \) for any \( j \). We obtain a net \( (M_{\varphi_j}) \) of completely bounded Fourier multipliers maps on \( L^p(VN(G)) \) such that \( \| M_{\varphi_j} \|_{cb, L^p(VN(G)) \to L^p(VN(G))} \leq C \) with \( \varphi_j(s) = \tau_G(T_j(\lambda_s)) \lambda_s^* \) for any \( j \) and any \( s \in G \). Then for any \( j \) and any \( s \in G \) we have

\[
\varphi_j(s) = \tau_G(T_j(\lambda_s) \lambda_s^*) = \tau_G \left( \sum_{k=1}^{n_j} \eta_{j,k}(\lambda_s) \lambda_{s_{j,k}} \lambda_{s_{j,k}}^* \right) = \sum_{k=1}^{n_j} \eta_{j,k}(\lambda_s) \tau_G(\lambda_{s_{j,k}} \lambda_{s_{j,k}}^*) = \sum_{k=1}^{n_j} \eta_{j,k}(\lambda_s) \delta_{s_{j,k},s}.
\]

This shows that each \( M_{\varphi_j} \) is a finite-rank multiplier map on \( L^p(VN(G)) \). Using the convergence of \( (T_j) \) to \( \text{Id}_{L^p(VN(G))} \) in the point-norm topology, we obtain the following pointwise convergence for any \( s \in G \)

\[
\varphi_j(s) = \tau_G(T_j(\lambda_s) \lambda_s^*) \to \tau_G(\lambda_s \lambda_s^*) = \tau_G(1_{VN(G)}) = 1.
\]

Moreover, we have the uniform bound \( \| \varphi_j \|_{L^\infty(G)} \leq \| M_{\varphi_j} \|_{cb, L^p(VN(G)) \to L^p(VN(G))} \leq C \). This implies that \( \varphi_j \) converges to \( 1 \) for the weak* topology of \( L^\infty(G) \). Using [ArK1, Lemma 6.7], we conclude that the net \( (M_{\varphi_j}) \) converges to \( \text{Id}_{L^p(VN(G))} \) in the point-norm topology.
If $L^p(\mathcal{VN}(G))$ has CCAP, then the $T_j$ above will have cb-norm less than 1. Moreover, the adaptation of the $T_j$ to have range in $P_G$ can be arranged with cb-norm less than 1. Finally, the projection $P_G^p$ from [AvK1, Corollary 4.6] has norm 1, so that the $M_{\varphi_j}$ will again have cb-norm less than 1.

2.7 Carré du champ $\Gamma$ and first order differential calculus for Fourier multipliers

Let $G$ be a discrete group. For any $1 \leq p \leq \infty$, note that we have span $\{\lambda_s : s \in G\} \subset L^p(\mathcal{VN}(G))$. To state our next result, we need to consider the gradient form associated to the infinitesimal generator $A(\lambda_s) = \psi(s)\lambda_s$ of a weak* continuous semigroup $(T_t)_{t \geq 0}$ of Markov Fourier multipliers. Namely, for $x,y \in \text{span} \{\lambda_s : s \in G\}$, we define the element

$$\Gamma(x,y) \overset{\text{def}}{=} \frac{1}{2}[A(x^*)y + x^*A(y) - A(x^*y)]$$

of span $\{\lambda_s : s \in G\}$. For any $s,t \in G$, note that

$$\Gamma(\lambda_s, \lambda_t) = \frac{1}{2}[\psi(s^{-1}) + \psi(t) - \psi(s^{-1}t)]\lambda_{s^{-1}t}.$$  

The first part of the following result shows how to construct the carré du champ directly from the semigroup $(T_t)_{t \geq 0}$. The proof is similar to the one of Lemma 2.2.

**Lemma 2.13** Let $G$ be a discrete group.

1. Suppose $1 \leq p \leq \infty$. For any $x,y \in \text{span} \{\lambda_s : s \in G\}$, we have

$$\Gamma(x,y) = \lim_{t \to 0^+} \frac{1}{2t}(T_t(x^*)y - (T_t(x))^*T_t(y))$$
in $L^p(\mathcal{VN}(G))$.

2. For any $x \in \text{span} \{\lambda_s : s \in G\}$, we have $\Gamma(x,x) \geq 0$.

3. For any $x,y \in \text{span} \{\lambda_s : s \in G\}$, the matrix $\begin{bmatrix} \Gamma(x,x) & \Gamma(x,y) \\ \Gamma(y,x) & \Gamma(y,y) \end{bmatrix}$ is positive.

4. For any $x,y \in \text{span} \{\lambda_s : s \in G\}$, we have

$$\|\Gamma(x,y)\|_{L^p(\mathcal{VN}(G))} \leq \|\Gamma(x,x)\|_{L^p(\mathcal{VN}(G))} \frac{\|x\|_{L^p(\mathcal{VN}(G))}}{\|y\|_{L^p(\mathcal{VN}(G))}} \cdot$$

This inequality even holds for $0 < p \leq \infty$.

Suppose that $(T_t)_{t \geq 0}$ is associated to the length $\psi : G \to \mathbb{R}_+$ with associated cocycle $b_\psi : G \to H$ and the orthogonal representation $\pi : G \to B(H)$, $s \mapsto \pi_s$ of $G$ on $H$. For any $s \in G$, we have

$$\psi(s) = \|b_\psi(s)\|_{H}^2.$$  

For any $s,t \in G$, we have the cocycle law

$$\pi_s(b_\psi(t)) = b_\psi(st) - b_\psi(s), \quad \text{i.e.} \quad b_\psi(st) = b_\psi(s) + \pi_s(b_\psi(t)).$$
Suppose \(-1 \leq q \leq 1\). For any \(s \in G\), we will use the second quantization \(\alpha_s \overset{\text{def}}{=} \Gamma_q(\pi_s) : \Gamma_q(H) \to \Gamma_q(H)\) which is trace preserving. We obtain an action \(\alpha : G \to \text{Aut}(\Gamma_q(H))\). So we can consider the crossed product \(\Gamma_q(H) \rtimes_\alpha \text{VN}(G)\) equipped with its canonical normal finite faithful trace \(\tau_s\).

Suppose \(-1 \leq q \leq 1\) and \(1 \leq p < \infty\). We introduce the map \(\partial_{\psi,q} : \text{span}\{\lambda_s : s \in G\} \to L^p(\Gamma_q(H) \rtimes_\alpha \text{VN}(G))\) defined by

\[
\partial_{\psi,q}(\lambda_s) = s_q(b_\psi(s)) \rtimes \lambda_s.
\]

which is a slight generalization of the map of [JMP1, page 553]. Note that \(L^p(\Gamma_q(H) \rtimes_\alpha \text{VN}(G))\) is a \(\text{VN}(G)\)-bimodule with left and right actions induced by

\[
\lambda_s(z \rtimes \lambda_t) \overset{\text{def}}{=} \alpha_s(z) \rtimes \lambda_{st} \quad \text{and} \quad (z \rtimes \lambda_t)\lambda_s \overset{\text{def}}{=} z \rtimes \lambda_{ts}, \quad z \in \Gamma_q(H), s, t \in G.
\]

The following is stated in the particular case \(q = 1\) without proof in [JMP2, page 544] and in [Zen1, page 9] without proof but with false bimodule operations. For the sake of completeness, we give a short proof.

**Lemma 2.14** Suppose \(-1 \leq q \leq 1\). Let \(G\) be a discrete group. For any \(x, y \in \text{span}\{\lambda_s : s \in G\}\), we have

\[
\partial_{\psi,q}(xy) = x\partial_{\psi,q}(y) + \partial_{\psi,q}(x)y.
\]

**Proof**: For any \(s, t \in G\), we have

\[
\lambda_s \partial_{\psi,q}(\lambda_t) + \partial_{\psi,q}(\lambda_s)\lambda_t \overset{(2.49)}{=} \lambda_s (s_q(b_\psi(t)) \rtimes \lambda_t) + (s_q(b_\psi(s)) \rtimes \lambda_s)\lambda_t
\]

\[
\overset{(2.50)}{=} \alpha_s(s_q(b_\psi(t))) \rtimes \lambda_t + s_q(b_\psi(s)) \rtimes \lambda_{st} = s_q(\pi_s(b_\psi(t))) \rtimes \lambda_{st} + s_q(b_\psi(s)) \rtimes \lambda_{st}
\]

\[
\overset{(2.49)}{=} (s_q(\pi_s(b_\psi(t))) + s_q(b_\psi(s))) \rtimes \lambda_{st} \overset{(2.48)}{=} (s_q(b_\psi(st))) \rtimes \lambda_{st} = \partial_{\psi,q}(\lambda_s \lambda_t).
\]

The following is a slight generalization of [JMP2, Remark 1.3]. The proof is not difficult.\(^2\)

Here \(E : L^p(\Gamma_q(H) \rtimes_\alpha \text{VN}(G)) \to L^p(\text{VN}(G))\) denotes the canonical conditional expectation.

---

2. On the one hand, for any \(s, t \in G\), we have

\[
\Gamma(\lambda_s, \lambda_t) \overset{(2.44)}{=} \frac{1}{2} \big| \psi(s^{-1}) + \psi(t) - \psi(s^{-1}t) \big| \lambda_{s^{-1}t} \overset{(2.47)}{=} \frac{1}{2} \left[ \left\| b_\psi(s^{-1}) \right\| ^2 + \left\| b_\psi(t) \right\| ^2 - \left\| b_\psi(s^{-1}t) \right\| ^2 \right] \lambda_{s^{-1}t}
\]

\[
\overset{(2.48)}{=} \frac{1}{2} \left[ \left\| b_\psi(s^{-1}) \right\| ^2 + \left\| b_\psi(t) \right\| ^2 - \left\| b_\psi(s^{-1}) \right\| ^2 - 2\left( b_\psi(s^{-1}), \pi_{s^{-1}}(b_\psi(t)) \right)_H - \left\| b_\psi(t) \right\| ^2 \right] \lambda_{s^{-1}t}
\]

\[
= -\left( b_\psi(s^{-1}), \pi_{s^{-1}}(b_\psi(t)) \right)_H \lambda_{s^{-1}t}.
\]

On the other hand, we have

\[
E \left[ (\partial_{\psi,q}(\lambda_s))^* \partial_{\psi,q}(\lambda_t) \right] \overset{(2.49)}{=} E \left[ (s_q(b_\psi(s)) \rtimes \lambda_s)^* (s_q(b_\psi(t)) \rtimes \lambda_t) \right]
\]

\[
\overset{(2.35)}{=} E \left[ (\alpha_{s^{-1}}(s_q(b_\psi(s))) \rtimes \lambda_{s^{-1}}) (s_q(b_\psi(t)) \rtimes \lambda_t) \right] = E \left[ (s_q(\pi_{s^{-1}}(b_\psi(s))) \rtimes \lambda_{s^{-1}}) (s_q(b_\psi(t)) \rtimes \lambda_t) \right]
\]

\[
\overset{(2.48)}{=} E \left[ (s_q(b_\psi(e) - b_\psi(s^{-1})) \rtimes \lambda_{s^{-1}}) (s_q(b_\psi(t)) \rtimes \lambda_t) \right] = -E \left[ (s_q(b_\psi(s^{-1})) \rtimes \lambda_{s^{-1}}) (s_q(b_\psi(t)) \rtimes \lambda_t) \right]
\]

\[
\overset{(2.36)}{=} -E \left[ (s_q(b_\psi(s^{-1})) \alpha_{s^{-1}}(s_q(b_\psi(t))) \rtimes \lambda_{s^{-1}}) \right] \overset{(2.48)}{=} -E \left[ (s_q(b_\psi(s^{-1})) s_q(\pi_{s^{-1}}(b_\psi(t))) \rtimes \lambda_{s^{-1}} \right]
\]

\[
\overset{(2.12)}{=} -\left( b_\psi(s^{-1}), \pi_{s^{-1}}(b_\psi(t)) \right)_H \lambda_{s^{-1}t}.
\]
Proposition 2.15 Suppose \(-1 \leq q \leq 1\). For any \(x, y \in \text{span}\{\lambda_s : s \in G\}\), we have
\begin{equation}
(2.52) \quad \Gamma(x, y) = \mathbb{E}\left[(\partial_{\psi,q}(x))^* \partial_{\psi,q}(y)\right].
\end{equation}

3 Riesz transforms associated to semigroups of Markov multipliers

3.1 Noncommutative Khintchine inequalities and Rosenthal inequalities

Recall the following notion of conditional independence, which was introduced in [DPSS1, Definition 6.1] and which is similar to the one of [JuX3, (I) page 233]. Let \(M\) be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace \(\tau\). Suppose that \((N_k)\) is a family of von Neumann subalgebras of \(M\) and that \(N\) is a common von Neumann subalgebra of the \(N_k\) such that \(\tau|N\) is semifinite. Let \(\mathbb{E}_N : M \to N\) be the canonical trace preserving faithful normal conditional expectation with respect to \(N\). We say that the family \((N_k)\) is independent\(^3\) with respect to \(N\) if for every \(k\) we have
\begin{equation}
\mathbb{E}_N(xy) = \mathbb{E}_N(x)\mathbb{E}_N(y), \quad x \in N_k, y \in W^*((N_j)_{j \neq k})
\end{equation}
where \(W^*((N_j)_{j \neq k})\) denotes the von Neumann subalgebra generated by the \(N_j\) with \(j \neq k\). A sequence \((x_k)\) of elements of \(L^p(M)\) is said to be faithfully independent with respect to a von Neumann subalgebra \(N\) if there exist a family \((N_k)\) with \(x_k \in L^p(N_k)\) for any \(k\) such that \((N_k)\) is independent with respect to \(N\).

Let \((N_k)_{k \geq 1}\) be an independent sequence of von Neumann subalgebras of \(M\) with respect to \(N\). In [JuZ2, Theorem 0.4 and (3.1)], it is shown in the case where \(M\) is finite that if \(2 \leq p < \infty\) and if \((x_k)\) is a sequence such that \(x_k \in L^p(N_k)\) and \(\mathbb{E}_N(x_k) = 0\) for all \(k\), then
\begin{equation}
(3.1) \quad \left\| \frac{1}{n} \sum_{k=1}^n x_k \right\|_{L^p(M)} \leq \max \left\{ \sqrt{p} \left( \sum_{k=1}^n \mathbb{E}_N(|x_k|^2) \right)^{\frac{1}{2}} \right\}, \quad \left\| \mathbb{E}_N\left( \frac{1}{n} \sum_{k=1}^n |x_k|^2 \right) \right\|_{L^p(M)}^{\frac{1}{2}}.
\end{equation}

We refer to [DPSS1, Theorem 6.3] for a variant of this result. Using reduction [HJX], it is easy to extend this inequality to the non-finite case.

We denote by \(\mathbb{E}_f : L^p(\Gamma_q(H) \overline{\otimes} B(\ell^2)) \to S^p_f\) the canonical conditional expectation. If \(f \in \Gamma_q(H) \overline{\otimes} B(\ell^2)\), following [Jun3], we let
\begin{equation}
(3.2) \quad \|f\|_{L^p_f(\mathbb{E}_f)} \overset{\text{def}}{=} \left\| \mathbb{E}_f(f^* f)^{\frac{1}{2}} \right\|_{S^p_f}, \quad \|f\|_{L^p_f(\mathbb{E}_f)} \overset{\text{def}}{=} \left\| \mathbb{E}_f(f f^*)^{\frac{1}{2}} \right\|_{S^p_f}.
\end{equation}
If \(1 \leq p \leq 2\), we let \(L^p_{cr}(\mathbb{E}_f) \overset{\text{def}}{=} L^p_{cr}(\mathbb{E}_f) + L^p(\mathbb{E}_f)\) and if \(2 \leq p < \infty\), we let \(L^p_{cr}(\mathbb{E}_f) \overset{\text{def}}{=} L^p(\mathbb{E}_f) \cap L^p_{cr}(\mathbb{E}_f)\). Recall that
\begin{equation}
(3.3) \quad (L^p_{cr}(\mathbb{E}_f))^* = L^{p'}_{cr}(\mathbb{E}_f).
\end{equation}

Theorem 3.1 Consider \(-1 \leq q \leq 1\).

\(^3\) The authors of [JuZ2] say fully independent.
1. Suppose $2 \leq p < \infty$. For any $f = \sum_{i,j,h} f_{i,j,h} \otimes e_{ij} \in L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^q))$ with $f_{i,j,h} \in \mathbb{C}$

$$\max \left\{ \left\| (E_I(f^* f))^{\frac{1}{2}} \right\|_{S_p^I}, \left\| (E_I(f f^*))^{\frac{1}{2}} \right\|_{S_p^I} \right\} \lesssim \| f \|_{L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^q))},$$

(3.4)

$$\lesssim \sqrt{p} \max \left\{ \left\| (E_I(f^* f))^{\frac{1}{2}} \right\|_{S_p^I}, \left\| (E_I(f f^*))^{\frac{1}{2}} \right\|_{S_p^I} \right\}.$$ 

2. Suppose $1 < p < 2$. For any $f = \sum_{i,j,h} f_{i,j,h} \otimes e_{ij} \in L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^q))$ with $f_{i,j,h} \in \mathbb{C}$, we have

$$\| f \|_{L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^q))} \approx_p \| f \|_{L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^q))},$$

(3.5)

$$\| f \|_{L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^q))} \approx_p \inf_{g + h \in H} \left\{ \left\| (E_I(g^* g))^{\frac{1}{2}} \right\|_{S_p^I}, \left\| (E_I(hh^*))^{\frac{1}{2}} \right\|_{S_p^I} \right\}.$$ 

**Proof**: Suppose $2 < p < \infty$. We begin by proving the upper estimate of (3.4). Fix $m \geq 1$. We have\footnote{If $h \in H$, we have}

$$J_m(h) \overset{\text{def}}{=} \frac{1}{\sqrt{m}} \left( \sum_{l=1}^{m} e_l \right) \otimes h.$$

Hence we can consider the associated operator $\Gamma_p(J_m) : L^p(\Gamma_q(H)) \to L^p(\Gamma_q(\ell_2^m(H)))$ of second quantization (2.9) which is an isometric completely positive map. By tensorizing with the identity $\text{Id}_{S_p^I}$, we obtain an isometric map $\pi_I \overset{\text{def}}{=} \Gamma_p(J_m) \otimes \text{Id}_{S_p^I} : L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^q)) \to L^p(\Gamma_q(\ell_2^m(H)) \overline{\otimes} B(\ell_2^q)).$ For any finite sum $f = \sum_{i,j,h} f_{i,j,h} \otimes e_{ij}$ of $L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^q))$ with
\[ f_{i,j,h} \in \mathbb{C}, \text{ we obtain} \]

\[
\| f \|_{L^p(\Gamma_q(H) \boxtimes B(\ell_2^q))} = \left\| \sum_{i,j,h} f_{i,j,h} s_q(h) \otimes e_{ij} \right\|_{L^p(\Gamma_q(H) \boxtimes B(\ell_2^q))}
\]
\[
= \left\| \pi_{\ell} \left( \sum_{i,j,h} f_{i,j,h} s_q(h) \otimes e_{ij} \right) \right\|_{L^p(\Gamma_q(\ell_2^m(H)) \boxtimes B(\ell_2^q))}
\]
\[
= \sum_{i,j,h} f_{i,j,h} \Gamma_p(J_m)(s_q(h)) \otimes e_{ij} \quad \left(2.10\right)
\]
\[
= \sum_{i,j,h} f_{i,j,h} s_{q,m}(J_m(h)) \otimes e_{ij} \quad \left(3.1\right)
\]
\[
= \sum_{i,j,h} f_{i,j,h} s_{q,m}(\frac{1}{\sqrt{m}} \left( \sum_{l=1}^{m} e_l \right) \otimes h) \otimes e_{ij} \quad \left(3.6\right)
\]
\[
= \frac{1}{\sqrt{m}} \sum_{i,j,h,l} f_{i,j,h} s_{q,m}(e_l \otimes h) \otimes e_{ij} \quad \left(3.7\right)
\]

For any \(1 \leq l \leq m\), consider the element \(f_l = \sum_{j,h} f_{i,j,h} s_{q,m}(e_l \otimes h) \otimes e_{ij}\) and the conditional expectation \(\mathbb{E}_l : L^p(\Gamma_q(\ell_2^m(H)) \boxtimes B(\ell_2^q)) \to S^p_l\). For any \(1 \leq l \leq m\), note that

\[
\mathbb{E}_l(f_l) = \mathbb{E}_l \left( \sum_{i,j,h} f_{i,j,h} s_{q,m}(e_l \otimes h) \otimes e_{ij} \right) = \sum_{i,j,h} f_{i,j,h} \mathbb{E}_l (s_{q,m}(e_l \otimes h) \otimes e_{ij})
\]
\[
= \sum_{i,j,h} f_{i,j,h} \tau(s_{q,m}(e_l \otimes h)) e_{ij} \quad \left(2.11\right) = 0.
\]

We deduce that the random variables \(f_l\) are mean-zero. Now, we prove in addition that the \(f_l\) are independent over \(B(\ell_2^q)\), so that we will be able to apply the noncommutative Rosenthal inequality (3.1) to them.

**Lemma 3.2** The \(f_l\) defined here above are independent over \(B(\ell_2^q)\). More precisely, the von Neumann algebras generated by \(f_l\) are independent with respect to \(B(\ell_2^q)\).

**Proof:** Due to normality of \(\mathbb{E}_l\), it suffices to prove

\[
\mathbb{E}_l(xy) = \mathbb{E}_l(x)\mathbb{E}_l(y)
\]

for \(x\) (resp. \(y\)) belonging to a weak* dense subset of \(W^*(f_l)\) (resp. of \(W^*(f_k)\) for \(k \neq l\)). Moreover, by bilinearity in \(x, y\) of both sides of (3.8) and self-adjointness of \(s_{q,m}(e_l \otimes h)\), it suffices to take \(x = s_{q,m}(e_l \otimes h)^n \otimes e_{ij}\) for some \(n \in \mathbb{N}\) and \(y = \prod_{t=1}^{n} s_{q,m}(e_{ki} \otimes h)^{n_t} \otimes e_{rs}\) for some \(T \in \mathbb{N}\), \(k_i \neq l\) and \(n_t \in \mathbb{N}\). We have \(\mathbb{E}_l(x)\mathbb{E}_l(y) = \tau(s_{q,m}(e_l \otimes h)^n)\tau\left( \prod_{t=1}^{n} s_{q,m}(e_{ki} \otimes h)^{n_t} \right) \otimes e_{ij} e_{rs}\) and \(\mathbb{E}_l(xy) = \tau\left( s_{q,m}(e_l \otimes h)^n \prod_{t=1}^{n} s_{q,m}(e_{ki} \otimes h)^{n_t} \right) \otimes e_{ij} e_{rs}\). We shall now apply the Wick formulae (2.10) and (2.11) to the trace term above.

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Note that if \( n + \sum_{t=1}^{T} n_t \) is odd, then according to the Wick formula (2.11) we have \( \mathbb{E}_f(xy) = 0 \). On the other hand, then either \( n \) or \( \sum_{t=1}^{T} n_t \) is odd, so according to the Wick formula (2.11), either \( \mathbb{E}_f(x) = 0 \) or \( \mathbb{E}_f(y) = 0 \). Thus, (3.8) follows in this case.

Now suppose that \( 2k \overset{\text{def}}{=} n + \sum_{t=1}^{T} n_t \) is even. Consider a 2-partition \( V \in \mathcal{P}_2(2k) \). If both \( n \) and \( \sum_{t=1}^{T} n_t \) are odd, then we must have some \((i,j) \in V\) such that one term \( \langle f_i, f_j \rangle e_{i,j}^{\circ}(H) \) in the Wick formula (2.10) equals \( \langle e_l \otimes h, e_k \otimes h \rangle = 0 \), since \( k \neq l \). Thus, \( \mathbb{E}_f(xy) = 0 \), and since \( n \) is odd, also \( \mathbb{E}_f(x) = 0 \), and therefore, (3.8) follows. If both \( n \) and \( \sum_{t=1}^{T} n_t \) are even, then in the Wick formula (2.10), we only need to consider those 2-partitions \( V \) without a mixed term as the \( \langle f_i, f_j \rangle = 0 \) above. Such a \( V \) is clearly the disjoint union \( V = V_1 \cup V_2 \) of 2-partitions corresponding to \( n \) and to \( \sum_{t=1}^{T} n_t \). Moreover, we have for the number of crossings, \( c(V) = c(V_1) + c(V_2) \). With \( (f_1, f_2, \ldots) = (e_l \otimes h, e_l \otimes h, \ldots, e_l \otimes h, e_k \otimes h, \ldots, e_k \otimes h) \), we obtain

\[
\tau \left( s_{q,m}(e_l \otimes h)^n \prod_{t=1}^{T} s_{q,m}(e_{k_t} \otimes h)^{n_t} \right) \overset{(2.10)}{=} \sum_{V \in \mathcal{P}_2(2k)} q^{c(V)} \prod_{(i,j) \in V} \langle f_i, f_j \rangle e_{i,j}^{\circ}(H) \\
= \sum_{V \in \mathcal{P}_2(2k)} q^{c(V_1) + c(V_2)} \prod_{(i,j) \in V_1} \langle f_i, f_j \rangle e_{i,j}^{\circ}(H) \prod_{(i,j) \in V_2} \langle f_i, f_j \rangle e_{i,j}^{\circ}(H) \\
= \sum_{V_1} q^{c(V_1)} \prod_{(i,j) \in V_1} \langle f_i, f_j \rangle e_{i,j}^{\circ}(H) \left( \sum_{V_2} q^{c(V_2)} \prod_{(i,j) \in V_2} \langle f_i, f_j \rangle e_{i,j}^{\circ}(H) \right) \\
= \tau \left( s_{q,m}(e_l \otimes h)^n \right) \tau \left( \prod_{t=1}^{T} s_{q,m}(e_{k_t} \otimes h)^{n_t} \right).
\]

Thus, also in this case (3.8) follows.

Now we are able to apply the noncommutative Rosenthal inequality (3.1) which yields

\[
\|f\|_{L^p(T_q(H)\otimes B(c_j^2))} = \left( \sum_{l=1}^{m} \| f_l \|_{L^p(T_q(e_{i,l}^{\circ}(H))\otimes B(c_j^2))} \right)^{(1/2)} \lesssim \frac{1}{\sqrt{m}} \left( \sum_{l=1}^{m} \| f_l \|_{L^p(T_q(H)\otimes B(c_j^2))} \right)^{(1/2)} + \sqrt{p} \left( \sum_{l=1}^{m} \mathbb{E}_f(f_l^* f_l) \right)^{(1/2)} + \sqrt{p} \left( \sum_{l=1}^{m} \mathbb{E}_f(f_l^* f_l) \right)^{(1/2)} 
\]

For any integer \( 1 \leq l \leq m \), note that

\[
\mathbb{E}_f(f_l^* f_l) = \mathbb{E}_f \left( \sum_{i,j,h} f_{i,j,h} s_{q,m}(e_l \otimes h) \otimes e_{ij} \right) \left( \sum_{r,s,k} f_{r,s,k} s_{q,m}(e_l \otimes k) \otimes e_{rs} \right) \\
= \sum_{i,j,h,r,s,k} f_{i,j,h} f_{r,s,k} \tau(s_{q,m}(e_l \otimes h)s_{q,m}(e_l \otimes k)) e_{ij} e_{rs} \\
= \sum_{i,j,h,s}^\circ f_{i,j,h} f_{r,s,k} \| e_l \otimes h \|_{e_{i,j}^{\circ}(H)}^2 s_{q,m}(e_l \otimes k) \right) \sigma_{ij} e_{rs} \\
\overset{(2.12)}{=} \sum_{i,j,h,s} f_{i,j,h} f_{r,s,k} \| e_l \otimes h \|_{e_{i,j}^{\circ}(H)}^2 e_{ij} e_{rs}.
\]

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and
\[ E_I(f^* f) = E_I \left( \left( \sum_{i,j,h} f_{i,j,h}s_q(h) \otimes e_{ij} \right)^* \left( \sum_{r,s,k} f_{r,s,k}^* s_q(k) \otimes e_{rs} \right) \right) \]
\[ = \sum_{i,j,h,r,s,k} \overline{f_{i,j,h}} f_{r,s,k} \overline{s_q(h)} s_q(k) e_{ij} e_{rs} \quad (3.12) \]
Similarly for the row terms. We conclude that
\[ (3.10) \quad E_I(f_i^* f_i) = E_I(f^* f) \quad \text{and} \quad E_I(f_i f_i^*) = E_I(f f^*). \]
Moreover, for \( 1 \leq l \leq m \), using the isometric map \( \psi_l : H \to L^2_m(H) \), \( h \to e_l \otimes h \), we can introduce the second quantization operator \( \Gamma_{\psi_l} : L^p(\Gamma_q(\ell^2_m(H))) \to L^p(\Gamma_q(\ell^2_m(H))) \). We have
\[ (3.11) \quad \|f_l\|_{L^p(\Gamma_q(\ell^2_m(H)) \otimes B(\ell^2_l))} = \left\| \sum_{i,j,k} f_{i,j,k} s_{q,m}(e_l \otimes h) \otimes e_{ij} \right\|_{L^p(\Gamma_q(\ell^2_m(H)) \otimes B(\ell^2_l))} \]
\[ = \left\| \sum_{i,j,h} f_{i,j,h} \Gamma_{\psi_l}(s_{q,m}(h) \otimes e_{ij}) \right\|_{L^p(\Gamma_q(\ell^2_m(H)) \otimes B(\ell^2_l))} \]
\[ = \left\| f \right\|_{L^p(\Gamma_q(\ell_0(H) \otimes B(\ell^2_l)))}. \]
We infer that
\[ (3.9)(3.11)(3.10) \]
\[ \left\| f \right\|_{L^p(\Gamma_q(\ell^2_m(H)) \otimes B(\ell^2_l))} \leq \frac{1}{\sqrt{m}} \left( \sum_{l=1}^m \left\| f \right\|_{L^p(\Gamma_q(\ell^2_m(H)) \otimes B(\ell^2_l))}^p \right)^{\frac{1}{p}} + \sqrt{p} \left\| \left( \sum_{l=1}^m E_I(f^* f) \right)^{\frac{1}{2}} \right\|_{S^p} \]
\[ + \sqrt{p} \left\| \left( \sum_{l=1}^m E_I(f f^*) \right)^{\frac{1}{2}} \right\|_{S^p} \]
\[ = \frac{1}{\sqrt{m}} \left[ p \left\| f \right\|_{L^p(\Gamma_q(\ell^2_m(H)) \otimes B(\ell^2_l))} + \sqrt{p} \left\| (E_I(f^* f))^{\frac{1}{2}} \right\|_{S^p} + \sqrt{p} \left\| (E_I(f f^*))^{\frac{1}{2}} \right\|_{S^p} \right] \]
\[ = p \frac{1}{\sqrt{m}} \left\| f \right\|_{L^p(\Gamma_q(\ell^2_m(H)) \otimes B(\ell^2_l))} + \sqrt{p} \left\| (E_I(f^* f))^{\frac{1}{2}} \right\|_{S^p} + \sqrt{p} \left\| (E_I(f f^*))^{\frac{1}{2}} \right\|_{S^p}. \]
Since \( p > 2 \), passing to the limit when \( m \to \infty \), we finally obtain
\[ \left\| f \right\|_{L^p(\Gamma_q(\ell^2_m(H)) \otimes B(\ell^2_l))} \leq \sqrt{p} \left[ \left\| (E_I(f^* f))^{\frac{1}{2}} \right\|_{S^p} + \left\| (E_I(f f^*))^{\frac{1}{2}} \right\|_{S^p} \right]. \]
Using the equivalence \( \ell^2_2 \approx \ell^2_{\infty} \), we obtain the upper estimate of (3.4).

The lower estimate of (3.4) holds with constant 1 from the contractivity of the conditional expectation \( E_I \) on \( L^2(\Gamma_q(H) \otimes B(\ell^2_l)) \):
\[ \max \left\{ \left\| (E_I(f^* f))^{\frac{1}{2}} \right\|_{S^p}^2, \left\| (E_I(f f^*))^{\frac{1}{2}} \right\|_{S^p}^2 \right\} = \max \left\{ \left\| E_I(f^* f) \right\|_{S^p}^{\frac{1}{2}}, \left\| E_I(f f^*) \right\|_{S^p}^{\frac{1}{2}} \right\} \]
\[ \leq \max \left\{ \left\| f^* f \right\|_{L^2(\Gamma_q(H) \otimes B(\ell^2_l))}^{\frac{1}{2}}, \left\| f f^* \right\|_{L^2(\Gamma_q(H) \otimes B(\ell^2_l))}^{\frac{1}{2}} \right\} = \left\| f \right\|_{L^2(\Gamma_q(H) \otimes B(\ell^2_l))}. \]
Let us now consider the case $1 < p < 2$. We will proceed by duality as follows. If $f \in L^p(\Gamma_q(H))$ then $f \in L^2(\Gamma_q(H))$. Hence the sequence $(\tau(f s_q(e_k)))_{k \geq 1}$ is square summable. Using Cauchy Criterion, it is easy to see that the series $\sum_{k \geq 1} \tau(f s_q(e_k))s_q(e_k)$ is convergent in $L^p(\Gamma_q(H))$. So we can define the well-known Gaussian projection by

$$P : L^p(\Gamma_q(H)) \to L^p(\Gamma_q(H))$$

$$f \mapsto \sum_{k=1}^{+\infty} \tau(f s_q(e_k))s_q(e_k)$$

which is bounded. Note that for any $f \in L^p(\Gamma_q(H))$, we have by orthogonality

$$E_t((P \otimes \text{Id}_{S^p_I^*})(f)(P \otimes \text{Id}_{S^p_I^*})(f^*)) \leq E_t((P \otimes \text{Id}_{S^p_I^*})(f)(P \otimes \text{Id}_{S^p_I^*})(f^*)) + E_t((P^\perp \otimes \text{Id}_{S^p_I^*})(f)(P^\perp \otimes \text{Id}_{S^p_I^*})(f^*))$$

$$= \tau[(P \otimes \text{Id}_{S^p_I^*})(f)(P \otimes \text{Id}_{S^p_I^*})(f^*)] + \tau[(P^\perp \otimes \text{Id}_{S^p_I^*})(f)(P^\perp \otimes \text{Id}_{S^p_I^*})(f^*)]$$

$$= \tau[(P \otimes \text{Id}_{S^p_I^*})(f) + (P^\perp \otimes \text{Id}_{S^p_I^*})(f)]((P \otimes \text{Id}_{S^p_I^*})(f^*) + (P^\perp \otimes \text{Id}_{S^p_I^*})(f^*))$$

$$= \tau(ff^*) = E_t(ff^*).$$

Similarly, we have

$$E_t((P \otimes \text{Id}_{S^p_I^*})(f^*)(P \otimes \text{Id}_{S^p_I^*})(f)) \leq E_t(f^*f).$$

We deduce that $Q = P \otimes \text{Id}_{S^p_I^*} : L^p_{\text{loc}}(E_t) \to L^p_{\text{loc}}(E_t)$ is a contraction. Using (3.3) the projection in the second equality and the upper estimate of (3.4) in the last inequality, we obtain for any $f$

$$\|f\|_{L^p_{\text{loc}}(E_t)} \leq \|\|g\|_{L^p_{\text{loc}}(E_t)} \leq 1 \leq \sup_{g \in \text{Ran} Q} \|g\|_{L^p_{\text{loc}}(E_t)} \leq \sup_{\|g\|_{L^p_{\text{loc}}(E_t)} \leq 1} \|g\|_{L^p(\Gamma_0(H) \otimes B(\ell_2^j))} \leq \|f\|_{L^p(\Gamma_0(H) \otimes B(\ell_2^j))}.$$  

The other estimate is a consequence of the continuous inclusion $L^p_{\text{loc}}(E_t) \to L^p$ for $p \leq 2$, see [JuX2, Theorem 7.1]. The case $p = 2$ is left to the reader.

For the case $p = \infty$, we can use the free gaussian functor instead [VDN1].

**Theorem 3.3** For any finite sum $f = \sum_{i,j} f_{i,j}h_0(\ell) \otimes e_{ij}$, we have

$$\|f\|_{\Gamma_0(H) \otimes B(\ell_2^j)} \approx \max \left\{ \|E_t(f^*)\|_{B(\ell_2^j)}, \|E_t(fn^*)\|_{B(\ell_2^j)} \right\}. \quad (3.13)$$

**Proof**: One of the estimates of (3.13) holds with constant 1 from the contractivity of the conditional expectation $E_t : \Gamma_0(H) \otimes B(\ell_2^j) \to B(\ell_2^j)$ since

$$\max \left\{ \|E_t(f^*)\|_{B(\ell_2^j)}, \|E_t(f^*)\|_{B(\ell_2^j)} \right\} \leq \max \left\{ \|f^*f\|_{\Gamma_0(H) \otimes B(\ell_2^j)}, \|ff^*\|_{\Gamma_0(H) \otimes B(\ell_2^j)} \right\}$$

$$= \|f\|_{\Gamma_0(H) \otimes B(\ell_2^j)}.$$
We begin to prove the other estimate of (3.13). Fix $m \geq 1$. We have an isometric embedding $J_m : H \to \ell^2_m(H)$ defined by

$$(3.14) \quad J_m(h) \defeq \frac{1}{\sqrt{m}} \left( \sum_{l=1}^{m} e_l \right) \otimes h.$$ 

Hence we can consider the associated operator $\Gamma_{\infty}(J_m) : \Gamma_0(H) \to \Gamma_0(\ell^2_m(H))$ of second quantization which is a trace preserving faithful $\ast$-homomorphism. By tensorizing with the identity $\text{Id}_{\ell^2(H)}$, we obtain an isometric map $\pi_I \defeq \Gamma_{\infty}(J_m) \otimes \text{Id}_{\ell^2(\ell^2)} : \Gamma_0(H) \otimes \ell^2 \to \Gamma_0(\ell^2_m(H)) \otimes \ell^2$. For any finite sum $f = \sum_{i,j,h} f_{i,j,h}s_0(h) \otimes e_{ij}$ of $\Gamma_0(H) \otimes \ell^2$ with $f_{i,j,h} \in \mathbb{C}$, we obtain

$$(3.15) \quad \left\| f \right\|_{\Gamma_0(H) \otimes \ell^2} = \left\| \sum_{i,j,h} f_{i,j,h}s_0(h) \otimes e_{ij} \right\|_{\Gamma_0(H) \otimes \ell^2} = \left\| \pi_I \left( \sum_{i,j,h} f_{i,j,h}s_0(h) \otimes e_{ij} \right) \right\|_{\Gamma_0(\ell^2_m(H)) \otimes \ell^2} \geq \left( \sum_{i,j,h} f_{i,j,h}s_0(h) \otimes e_{ij} \right) \otimes e_{ij} \left\|_{\Gamma_0(\ell^2_m(H)) \otimes \ell^2} \right.$$ 

For any $1 \leq l \leq m$, consider the element $f_l \defeq \sum_{i,j,h} f_{i,j,h}s_0(e_l \otimes h) \otimes e_{ij}$ and the conditional expectation $\mathbb{E}_l : \Gamma_0(\ell^2_m(H)) \otimes \ell^2 \to \ell^2$. For any $1 \leq l \leq m$, note that

$$\mathbb{E}_l(f_l) = \mathbb{E}_l \left( \sum_{i,j,h} f_{i,j,h}s_0(e_l \otimes h) \otimes e_{ij} \right) = \sum_{i,j,h} f_{i,j,h}\mathbb{E}_l \left( s_0(e_l \otimes h) \otimes e_{ij} \right) = \sum_{i,j,h} f_{i,j,h}\mathbb{E}_l \left( \tau(s_0(e_l \otimes h))e_{ij} \right) = 0.$$ 

We deduce that the random variables $f_l$ are mean-zero and it is easy to check that these random variables are freely independent over $\ell^2$. Hence, the generalization [Jun4] of Voiculescu’s inequality [Vol] yields

$$(3.16) \quad \left\| f \right\|_{\Gamma_0(H) \otimes \ell^2} \leq \left( \sum_{l=1}^{m} f_l \right) \left\|_{\Gamma_0(\ell^2_m(H)) \otimes \ell^2} \right. \leq \frac{1}{\sqrt{m}} \left[ \sum_{l=1}^{m} \left\| f_l \right\|_{\Gamma_0(\ell^2_m(H)) \otimes \ell^2} + \left\| \sum_{l=1}^{m} \mathbb{E}_l(f_l^* f_l) \right\|_{\ell^2} + \left\| \sum_{l=1}^{m} \mathbb{E}_l(f_l f_l^*) \right\|_{\ell^2} \right].$$

5. If $h \in H$, we have

$$\left\| \frac{1}{\sqrt{m}} \left( \sum_{l=1}^{m} e_l \right) \otimes h \right\|_{\ell^2_m(H)} \leq \frac{1}{\sqrt{m}} \left( \sum_{l=1}^{m} \left\| e_l \right\|_{\ell^2_m(H)} \right) \leq \left\| h \right\|_{H}.$$
Moreover, for \( \sum_{i,j,h} \int_{\ell^2_m(H)} f_{i,j,h} \circ s_0(e_l \otimes h) e_j e_r \), note that

\[
\|f\|_{\Gamma_0(H) \otimes B(\ell_1^2)} = \left\| \sum_{i,j,h} f_{i,j,h} s_0(e_l \otimes h) e_j e_r \right\|_{\Gamma_0(H) \otimes B(\ell_1^2)}.
\]

(3.17) \( \mathbb{E}_l(f^* f_l) = \mathbb{E}_l(f^* f) \) and \( \mathbb{E}_l(f_l f^*_l) = \mathbb{E}_l(ff^*) \).

Moreover, for \( 1 \leq l \leq m \), using the isometric map \( \psi_l : H \rightarrow \ell^2_m(H) \), \( h \rightarrow e_l \otimes h \), we can introduce the second quantization operator \( \Gamma_\infty(\psi_l) : \Gamma_0(H) \rightarrow \Gamma_0(\ell^2_m(H)) \). We have

\[
\| f \|_{\Gamma_0(H) \otimes B(\ell_1^2)} = \sum_{i,j,h} f_{i,j,h} e_j e_r.
\]

(3.18) We infer that

\[
\mathbb{E}_l(f^* f_l) = \mathbb{E}_l(f^* f) = \mathbb{E}_l(ff^*).
\]

(3.16)(3.17)(3.18) Passing to the limit when \( m \to \infty \), we finally obtain

\[
\| f \|_{\Gamma_0(H) \otimes B(\ell_1^2)} \lesssim \left\langle \mathbb{E}_l(f^* f) \right\rangle^{1/2} \left\| \mathbb{E}_l(f^* f) \right\|_{B(\ell_1^2)}^{1/2} + \left( \mathbb{E}_l(f^* f) \right)^{1/2} \left\| \mathbb{E}_l(f^* f) \right\|_{B(\ell_1^2)}^{1/2}.
\]

We conclude using the equivalence \( \ell_2^1 \approx \ell_2^2 \).

\[
\text{We conclude using the equivalence } \ell_2^1 \approx \ell_2^2. \]
3.2 Kato’s square root problem for semigroups of Schur multipliers

If $H$ is a Hilbert space, we denote by $H_{\text{disc}}$ the abelian group $(H, +)$ equipped with the discrete topology. We will use the trace preserving normal unital injective *-homomorphism map $J: VN(H_{\text{disc}}) \to L^\infty(\Omega) \otimes VN(H_{\text{disc}})$, $\lambda_h \mapsto 1 \otimes \lambda_h$ and the unbounded Fourier multiplier $(-\Delta)^{-\frac{1}{2}}: \mathcal{P}_{H_{\text{disc}}} \subset L^p(VN(H_{\text{disc}})) \to L^p(VN(H_{\text{disc}}))$ defined by

$$(3.19) \quad (-\Delta)^{-\frac{1}{2}}(\lambda_h) \overset{\text{def}}{=} \frac{1}{2\pi \|\lambda_h\|_H} \lambda_h.$$  

The following is inspired by [Arh7].

**Proposition 3.4** Let $H$ be a Hilbert space. There exists a unique weak* continuous group $(U_t)_{t \in \mathbb{R}}$ of *-automorphisms of $L^\infty(\Omega) \otimes VN(H_{\text{disc}})$ such that

$$(3.20) \quad U_t(f \otimes \lambda_h) = e^{\sqrt{2it\mathcal{W}(h)}} f \otimes \lambda_h, \quad t \in \mathbb{R}, f \in L^\infty(\Omega), h \in H.$$  

Moreover, each $U_t$ is trace preserving.

**Proof** : For any $t \in \mathbb{R}$, we consider the continuous function $u_t: H_{\text{disc}} \to U(L^\infty(\Omega)), h \mapsto e^{-\sqrt{2it\mathcal{W}(h)}}$. For any $t \in \mathbb{R}$ and any $h_1, h_2 \in H_{\text{disc}}$, note that

$$u_t(h_1 + h_2) = e^{-\sqrt{2it\mathcal{W}(h_1 + h_2)}} = e^{-\sqrt{2it\mathcal{W}(h_1)}} e^{-\sqrt{2it\mathcal{W}(h_2)}} = u_t(h_1) u_t(h_2).$$

By [Arh7, Proposition 2.4] applied with $M = L^\infty(\Omega)$, $G = H_{\text{disc}}$ and by considering the trivial action $\alpha$, for any $t \in \mathbb{R}$, we have a unitary $V_t: L^2(H_{\text{disc}}, L^2(\Omega)) \to L^2(H_{\text{disc}}, L^2(\Omega))$, $\xi \mapsto (h \mapsto u_t(-h)(\xi(h)))$ and a *-isomorphism $U_t: L^\infty(\Omega) \otimes VN(H_{\text{disc}}) \to L^\infty(\Omega) \otimes VN(H_{\text{disc}})$, $x \mapsto V_t x V_t^*$ satisfying (3.20). The uniqueness is clear by density.

For any $t, t' \in \mathbb{R}$ any $\xi \in L^2(H_{\text{disc}}, L^2(\Omega))$, note that almost everywhere

$$\langle V_t \mathcal{V}_{t'}(\xi) \rangle(h) = u_t(-h)(u_{t'}(-h)(\xi(h))) = e^{-\sqrt{2it\mathcal{W}(h)}} e^{-\sqrt{2it'\mathcal{W}(h)}} \mathcal{W}(h) = e^{-\sqrt{2i(t+t')\mathcal{W}(h)}} \mathcal{W}(h) = u_{t+t'}(-h)(\xi(h)) = \langle V_{t+t'}(\xi) \rangle(h).$$

We conclude that $V_t \mathcal{V}_{t'} = V_{t+t'}$. Moreover, for any $\xi, \eta \in L^2(H_{\text{disc}}, L^2(\Omega)) = L^2(H_{\text{disc}} \times \Omega)$, using dominated convergence theorem, we obtain

$$\langle V_t(\xi), \eta \rangle_{L^2(H_{\text{disc}} \times L^2(\Omega))} = \int_{H_{\text{disc}} \times \Omega} V_t(\xi)(h)(\omega) \eta(h, \omega) d\mu_{H_{\text{disc}}}(h) d\mu(\omega)$$

$$= \int_{H_{\text{disc}} \times \Omega} u_t(-h)(\xi(h))(\omega) \eta(h, \omega) d\mu_{H_{\text{disc}}}(h) d\mu(\omega)$$

$$= \int_{H_{\text{disc}} \times \Omega} e^{-\sqrt{2it\mathcal{W}(h)}}(h)(\omega) \eta(h, \omega) d\mu_{H_{\text{disc}}}(h) d\mu(\omega).$$

$$\overset{t \to 0}{\longrightarrow} \int_{H_{\text{disc}} \times \Omega} \mathcal{W}(h)(\omega) \eta(h, \omega) d\mu_{H_{\text{disc}}}(h) d\mu(\omega) = \langle \xi, \eta \rangle_{L^2(H_{\text{disc}} \times L^2(\Omega))}. $$

So $(V_t)_{t \in \mathbb{R}}$ is a weakly continuous group of unitaries hence a strongly continuous group by [Str1, Lemma 13.4] or [Tak2, page 239]. By [Tak2, page 238], we conclude that $(U_t)_{t \in \mathbb{R}}$ is a weak* continuous group of *-automorphisms.
Finally, for any $t \geq 0$, any $f \in L^\infty(\Omega)$ and any $h \in H$, we have
\[
\left( \int_\Omega \cdot \tau_{WN}(H_{disc}) \right) (U_t(f \otimes \lambda_h)) = \left( \int_\Omega \cdot \tau_{VN}(H_{disc}) \right) (e^{\sqrt{2}tW(h)} f \otimes \lambda_h) = \left( \int_\Omega e^{\sqrt{2}tW(h)} f \, d\mu \right) \tau_{VN}(H_{disc}) (\lambda_h) = \left( \int_\Omega \cdot \tau_{VN}(H_{disc}) \right) (f \otimes \lambda_h).
\]

We consider the unbounded operator $\delta$: span \{\lambda_s : s \in H_{disc}\} \subset L^p(VN(H_{disc})) \to L^p(\Omega, L^p(VN(H_{disc})))$
defined by
\[
\delta(\lambda_h) \overset{\text{def}}{=} 2\pi iW(h) \otimes \lambda_h.
\]

We consider the unbounded operator $\delta$: span \{\lambda_s : s \in H_{disc}\} \subset L^p(VN(H_{disc})) \to L^p(\Omega, L^p(VN(H_{disc})))$
defined by
\[
\delta(\lambda_h) \overset{\text{def}}{=} 2\pi iW(h) \otimes \lambda_h.
\]

Recall the classical transference principle [BGM2, Theorem 2.8]. Let $G$ be a locally compact
abelian group and $G \to B(X)$, $t \to \pi_t$ be a strongly continuous representation of $G$ on a Banach
space $X$ such that $c = \sup\{\|\pi_t\| : t \in G\} < \infty$. Let $k \in L^1(G)$ and let $T_k : X \to X$ be the
operator defined by $T_k(x) = \int_G k(t) \pi^{-1}(x) \, d\mu_G(t)$. Then
\[
\|T_k\|_{X \to X} \leq c^2 \|k * \|_{L^p(G, X) \to L^p(G, X)}.
\]

We will use the function $k_{z,R}(t) = \frac{1}{\pi^2} 1_{|t| < R} [HvNVW1, page 388].$

Recall that we say that a function $f \in L^1_{disc}(\mathbb{R}^n, X)$ admits a Cauchy principal value if the
limit $\lim_{\varepsilon \to 0+} \left( \int_{-\varepsilon}^{\varepsilon} f(t) \, dt + \int_{-\varepsilon}^{\varepsilon} f(t) \, dt \right)$ exists and we let p. v. $\int_R f(t) \, dt \overset{\text{def}}{=} \lim_{\varepsilon \to 0+} \left( \int_{-\varepsilon}^{\varepsilon} f(t) \, dt + \int_{-\varepsilon}^{\varepsilon} f(t) \, dt \right)$.

**Proposition 3.5** Suppose $1 < p < \infty$. For any Hilbert space $H$, the map
\[
H_{disc} : \quad L^p(\Omega, L^p(VN(H_{disc}))) \quad \longrightarrow \quad L^p(\Omega, L^p(VN(H_{disc})))
\]
\[
\begin{array}{c}
\text{is well-defined and completely bounded.}
\end{array}
\]

**Proof:** For any $0 < \varepsilon < R$, using the fact that the Banach space $L^p(\Omega, L^p(VN(H_{disc})), S^p_F)$ is
UMD and [AKM1, page 485] in the last inequality, we have by transference
\[
\left\| \frac{1}{\pi} \int_{\varepsilon < |t| < R} (U_t \otimes Id_{S^p_F}) \frac{dt}{t} \right\|_{L^p(\Omega, L^p(VN(H_{disc})), S^p_F) \to L^p(\Omega, L^p(VN(H_{disc})), S^p_F)}
\]
\[
= \left\| \int_R k_{z,R}(t) (U_t \otimes Id_{S^p_F}) \, dt \right\|_{L^p(\Omega, L^p(VN(H_{disc})), S^p_F) \to L^p(\Omega, L^p(VN(H_{disc})), S^p_F)}
\]
\[
\overset{(3.22)}{\leq} \left\| (k_{z,R} * \cdot) \otimes Id_{L^p(\Omega, L^p(VN(H_{disc})), S^p_F)} \right\|_{L^p(R, L^p(\Omega, L^p(VN(H_{disc})), S^p_F) \to L^p(R, L^p(\Omega, L^p(VN(H_{disc})), S^p_F))}
\]
\[
\leq_p 1.
\]

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We have the existence of the principal value if \( z = f \otimes \lambda_h \) since
\[
\int_{\frac{1}{2}}^{1} U_t(z) \frac{dt}{t} + \int_{\frac{1}{2}}^{1} U_t(z) \frac{dt}{t} = \int_{\frac{1}{2}}^{1} (U_t(z) - U_{-t}(z)) \frac{dt}{t} = \int_{\frac{1}{2}}^{1} \sin(\sqrt{2}W(h)t)(f \otimes \lambda_h) \frac{dt}{t} = 2i \left( \int_{\frac{1}{2}}^{1} \sin(\sqrt{2}W(h)t) \frac{dt}{t} \right) (f \otimes \lambda_h)
\]
admits a limit.

The following is a variation of Pisier formula.

**Proposition 3.6** For any \( h \in H \), we have
\[
(3.24) \quad \sqrt{2\pi} \delta(-\Delta)^{-\frac{1}{2}}(\lambda_h) = \left( Q \otimes \text{Id}_{L^p(VN(H_{disc}))} \right) H_{disc} J(\lambda_h).
\]

**Proof:** For any \( h \in H \), we have
\[
\sqrt{2\pi} \delta(-\Delta)^{-\frac{1}{2}}(\lambda_h) = \left( Q \otimes \text{Id}_{L^p(VN(H_{disc}))} \right) H_{disc} J(\lambda_h).
\]

Recall that for any \( \alpha \in \mathbb{R} \) we have \( \int_0^{+\infty} \frac{\sin(\alpha t)}{t} \, dt = \text{sign}(\alpha) \frac{\pi}{2} \). Consequently, we have
\[
(3.23) \quad \left( Q \otimes \text{Id}_{L^p(VN(H_{disc}))} \right) H_{disc} J(\lambda_h) = \left( Q \otimes \text{Id}_{L^p(VN(H_{disc}))} \right) \left( \text{p.v.} \frac{1}{\pi} \int_\mathbb{R} e^{i\sqrt{2}W(h)t} \frac{dt}{t} \otimes \lambda_h \right)
\]
\[
= Q \left( \text{p.v.} \frac{1}{\pi} \int_\mathbb{R} e^{i\sqrt{2}W(h)t} \frac{dt}{t} \right) \otimes \lambda_h = Q \left( \text{p.v.} \frac{1}{\pi} \int_\mathbb{R} e^{i\sqrt{2}W(h)t} \frac{dt}{t} \right) \otimes \lambda_h
\]
\[
= \frac{2i}{\pi} Q \left( \int_0^{+\infty} \sin(\sqrt{2}W(h)(t)) \frac{dt}{t} \right) \otimes \lambda_h = iQ \left( \text{sign} \circ W(h) \right) \otimes \lambda_h = \sqrt{2\pi i} W(h) \otimes \lambda_h.
\]

**Lemma 3.7** If \( 1 < p < \infty \), the operator
\[
(3.25) \quad \delta(-\Delta)^{-\frac{1}{2}} \otimes \text{Id}_{S^p_T} : L^p\left( VN(H_{disc}), S^p_T \right) \to L^p\left( \Omega, L^p(VN(H_{disc}), S^p_T) \right)
\]
is well-defined and bounded.

**Proof:** Note that the map \( J : L^p(VN(H_{disc})) \to L^p(\Omega, L^p(VN(H_{disc})) \) is completely bounded. By Proposition 3.5, the operator \( H_{disc} \otimes \text{Id}_{S^p_T} : L^p(\Omega, L^p(VN(H_{disc}), S^p_T) \to L^p(\Omega, L^p(VN(H), S^p_T)) \) is well-defined and bounded. Moreover, since the Banach space \( L^p(VN(H_{disc}), S^p_T) \) is K-convex, the map \( Q \otimes \text{Id}_{L^p(VN(H_{disc}), S^p_T) \) induces a bounded operator on \( L^p(\Omega, L^p(VN(H_{disc}), S^p_T) \). Using the equality (3.24), we obtain the result by composition.

Consider a semigroup \( (T_t)_{t \geq 0} \) of selfadjoint unital completely positive Schur multipliers on \( B(\ell^2) \). By [Arb1, Proposition 5.4], there exists a real Hilbert space \( H \) and a family \( (\alpha_t)_{t \in I} \) of elements of \( H \) such that for any \( t \geq 0 \) the Schur multiplier \( T_t : B(\ell^2) \to B(\ell^2) \) is associated with the matrix
\[
(3.26) \quad \left[ e^{-t||\alpha_i - \alpha_j||_{\ell^2}^2} \right]_{i,j \in I}.
\]
Note that \( u = \text{diag}(\lambda_i : i \in I) \) is a unitary of the von Neumann algebra \( M_I(VN(H_{\text{disc}})) = VN(H_{\text{disc}}) \otimes B(e_I^2) \). Hence we have a trace preserving normal unital injective homomorphism \( \pi: B(e_I^2) \to VN(H_{\text{disc}}) \otimes B(e_I^2) \), \( x \mapsto u(1 \otimes x)u^* \). For any \( i, j \in I \), note that
\[
\pi(e_{ij}) = \lambda_{\alpha_i} - \alpha_j \otimes e_{ij}.
\]
Moreover, if \( 1 \leq p < \infty \) and if \( y \in L^p(\Omega, S_I^p) \), we have
\[
|||\text{(Id}_{L^p(\Omega)} \otimes \pi_p)(y)|||_{L^p(\Omega, L^p(VN(H_{\text{disc}}), S_I^p))} = |||y|||_{L^p(\Omega, S_I^p)}.
\]
**Lemma 3.8 (intertwining formula)** Suppose \( 1 \leq p < \infty \). For any \( x \in M_{I, fin} \), we have
\[
i(\text{Id}_{L^p(\Omega)} \otimes \pi_p)\partial_{\alpha} A_p \frac{1}{2} = (\delta(-\Delta)^{\frac{1}{2}} \otimes \text{Id}_{S_I^p}) \pi_p.
\]
**Proof**: For any \( i, j \in I \), we have
\[
i(\text{Id}_{L^p(\Omega)} \otimes \pi_p)\partial_{\alpha} A_p \frac{1}{2} (e_{ij}) = \frac{i}{|||\alpha_i - \alpha_j|||_H} (\text{Id}_{L^p(\Omega)} \otimes \pi_p)\partial_{\alpha} (e_{ij})
\]
\[
\begin{align*}
&= \frac{i}{|||\alpha_i - \alpha_j|||_H} (\text{Id}_{L^p(\Omega)} \otimes \pi_p)(W(\alpha_i - \alpha_j) \otimes e_{ij}) = \frac{i}{|||\alpha_i - \alpha_j|||_H} W(\alpha_i - \alpha_j) \otimes \pi_p(e_{ij}) \\
&= \frac{i}{|||\alpha_i - \alpha_j|||_H} W(\alpha_i - \alpha_j) \otimes \lambda_{\alpha_i - \alpha_j} \otimes e_{ij}
\end{align*}
\]
and
\[
(\delta(-\Delta)^{\frac{1}{2}} \otimes \text{Id}_{S_I^p}) \pi_p(e_{ij}) = \frac{1}{2\pi |||\alpha_i - \alpha_j|||_H} \delta(\lambda_{\alpha_i - \alpha_j} \otimes e_{ij})
\]
\[
= \frac{1}{2\pi |||\alpha_i - \alpha_j|||_H} \delta(\lambda_{\alpha_i - \alpha_j} \otimes e_{ij}) = \frac{i}{|||\alpha_i - \alpha_j|||_H} (W(\alpha_i - \alpha_j) \otimes \lambda_{\alpha_i - \alpha_j}) \otimes e_{ij}.
\]
We conclude by linearity. 

**Lemma 3.9** For any \( x, y \in M_{I, fin} \), we have
\[
\text{Tr}_{L^\infty(\Omega) \otimes B(e_I^2)} ((\partial_{\alpha} (x))^* \partial_{\alpha} (y)) = \text{Tr} ((A^\dagger(x))^*(A^\dagger(y))).
\]
For \(-1 \leq q \leq 1 \) and any \( x, y \in M_{I, fin} \), we equally have
\[
\text{Tr}_{L^\infty(\Omega) \otimes B(e_I^2)} ((\partial_{\alpha,q} (x))^* \partial_{\alpha,q} (y)) = \text{Tr} ((A^\dagger(x))^*A^\dagger(y)).
\]
**Proof**: On the other hand, we note that
\[
\begin{align*}
\text{Tr}_{L^\infty(\Omega) \otimes B(e_I^2)} ((\partial_{\alpha} (x))^* \partial_{\alpha} (y)) &= \text{Tr}_{L^\infty(\Omega) \otimes B(e_I^2)} \left( \left( \partial_{\alpha} \left( \sum_{i,j \in I} x_{ij} e_{ij} \right) \right)^* \partial_{\alpha} \left( \sum_{k,l \in I} y_{kl} e_{kl} \right) \right) \\
&= \sum_{i,j,k,l \in I} x_{ij} y_{kl} \text{Tr}_{L^\infty(\Omega) \otimes B(e_I^2)} \left( \left( \partial_{\alpha} (e_{ij})^* \partial_{\alpha} (e_{kl}) \right) \right) \\
&= \sum_{i,j,k,l \in I} x_{ij} y_{kl} \text{Tr}_{L^\infty(\Omega) \otimes B(e_I^2)} \left( W(\alpha_i - \alpha_j) \otimes e_{ij}^* \right) \left( W(\alpha_k - \alpha_l) \otimes e_{kl} \right) \\
&= \sum_{i,j,k,l \in I} x_{ij} y_{kl} \left( \int_{\Omega} W(\alpha_i - \alpha_j)W(\alpha_k - \alpha_l) \, d\mu \right) \text{Tr}(e_{ij}^* e_{kl}) \\
&= \sum_{i,j \in I} x_{ij} y_{ij} (\alpha_i - \alpha_j, \alpha_i - \alpha_j)_H = \sum_{i,j \in I} x_{ij} y_{ij} |||\alpha_i - \alpha_j|||^2_H.
\end{align*}
\]
Moreover, we have

\[
\text{Tr} \left( (A^rac{1}{2}(x))^* (A^rac{1}{2}(y)) \right) = \text{Tr} \left( A^rac{1}{2} \left( \sum_{i,j \in I} x_{ij} e_{ij} \right) \right) A^rac{1}{2} \left( \sum_{k,l \in I} y_{kl} e_{kl} \right)
\]

\[
= \sum_{i,j,k,l \in I} x_{ij} y_{kl} \text{Tr} \left( (A^rac{1}{2}(e_{ij}))^* A^rac{1}{2}(e_{kl}) \right) = \sum_{i,j,k,l \in I} \| \alpha_i - \alpha_j \|_H \| \alpha_k - \alpha_l \|_H x_{ij} y_{kl} \text{Tr}(e_{ij}^* e_{kl})
\]

\[
= \sum_{i,j \in I} x_{ij} y_{ij} \| \alpha_i - \alpha_j \|_H^2.
\]

Finally, we have

\[
\text{Tr} \left( (\partial_{\alpha,q}(x))^* \partial_{\alpha,q}(y) \right) = \text{Tr} \left( \sum_{i,j} x_{ij} s_q(\alpha_i - \alpha_j) \otimes e_{ji} \sum_{k,l} y_{kl} s_q(\alpha_k - \alpha_l) \otimes e_{kl} \right)
\]

\[
= \sum_{i,j,k,l} x_{ij} y_{kl} \text{Tr}(e_{ji} e_{kl}) \text{Tr}(s_q(\alpha_i - \alpha_j) s_q(\alpha_k - \alpha_l))
\]

\[
\overset{(2.12)}{=} \sum_{i,j,k,l} x_{ij} y_{kl} \delta_{i=k} \delta_{j=l} (\alpha_i - \alpha_j, \alpha_k - \alpha_l) = \sum_{i,j} x_{ij} y_{ij} \| \alpha_i - \alpha_j \|_H^2.
\]

\[
\square
\]

Suppose 1 \leq p < \infty. We denote by \( A_p^\frac{1}{2} : \text{dom} A_p^\frac{1}{2} \subset S_p^p \to S_p^p \) the square root of the sectorial operator \( A_p : \text{dom} A_p \subset S_p^p \to S_p^p \).

**Theorem 3.10** Suppose 1 < p < \infty. For any \( x \in M_{I, \text{fin}} \), we have

\[
(3.32) \quad \| A_p^\frac{1}{2} (x) \|_{S_p^p} \approx_p \| \partial_{\alpha}(x) \|_{\text{Gauss}_p(S_p^p)}.
\]

**Proof**: Using Lemma 3.8, for any \( x \in M_{I, \text{fin}} \), we obtain the inequality

\[
\| \partial_{\alpha}(x) \|_{L^p(\Omega, S_p^p)} = \| \partial_{\alpha} A_p^{-\frac{1}{2}} A_p^\frac{1}{2}(x) \|_{L^p(\Omega, S_p^p)}
\]

\[
\overset{(3.28)}{=} \| (\text{Id}_{L^p(\Omega)} \otimes \pi_p) \partial_{\alpha} A_p^{-\frac{1}{2}} A_p^\frac{1}{2} (x) \|_{L^p(\Omega, L^p(\text{VN}(H_{\text{disc}}), S_p^p))}
\]

\[
\overset{(2.29)}{=} \| (\delta(-\Delta)^{-\frac{1}{2}} \otimes \text{Id}_{S_p^p}) \pi_p A_p^\frac{1}{2} (x) \|_{L^p(\Omega, L^p(\text{VN}(H_{\text{disc}}), S_p^p))}
\]

\[
\overset{(3.25)}{\leq} \| \pi_p \| \| A_p^\frac{1}{2}(x) \|_{S_p^p}
\]

The reverse estimate follows with the same constant from a duality argument. Indeed, if we fix \( x \) to be an element of \( S_p^p \), by Hahn-Banach theorem, there exists an element \( y \) of \( S_p^p \) with \( \| y \|_{S_p^p}^{-1} = 1 \) satisfying

\[
(3.33) \quad \| A_p^\frac{1}{2} (x) \|_{S_p^p} = \text{Tr} \left( y^* A_p^\frac{1}{2} (x) \right).
\]

We can suppose that \( y \in M_{I, \text{fin}} \). Note that \( A_p^{-\frac{1}{2}} \) is well-defined on \( e_{ij} \) only if \( \| \alpha_i - \alpha_j \| \neq 0 \). We consider on the set \( I \) the equivalence relation \( i \equiv j \iff \alpha_i = \alpha_j \). Then, we can write \( I = \bigcup_{k \in K} J_k \) partitioned into pairwise disjoint subsets \( J_k \) according to this equivalence.
relation. Consider the subalgebra \( M = \bigoplus_{k \in K} B(t^2_k) \otimes \delta^* \) of \( B(t^2) \) and the conditional expectation \( \mathbb{E}_I : S_p^\alpha \to L^p(M, \text{Tr} |M|) \). We obtain an element \( y_0 \) with
\[
\|y_0\|_{S_p^\alpha} = \|y - \mathbb{E}_I(y)\|_{S_p^\alpha} \leq \|y\|_{S_p^\alpha} + \|\mathbb{E}_I(y)\|_{S_p^\alpha} \leq 2.
\]
Note that \( y_{0ij} = 0 \) if \( \alpha_i = \alpha_j \), hence \( A_p^{-\frac{1}{2}}(y_0) \) is well-defined. For any \( i \in I \), we have
\[
\left( (\mathbb{E}_I(y))^* A_p^{-\frac{1}{2}}(x) \right)_{ii} = \sum_{k \in \ell} \langle \mathbb{E}_I(y)^* \rangle_{ik} \alpha_k - \alpha_i \| x_{ki} = \sum_{j=1}^{|K|} \sum_{l \in J_j} \langle \mathbb{E}_I(y)^* \rangle_{il} \alpha_l - \alpha_i \| x_{li}
\]
\[
= \sum_{j=1}^{|K|} \sum_{l \in J_j} \delta_{ij} y_{ji} \| \alpha_l - \alpha_i \| x_{li} = \sum_{j=1}^{|K|} \sum_{l \in J_j} \delta_{ij} y_{ji} \cdot 0 \cdot x_{li} = 0.
\]
Hence, \( \text{Tr} \left( (\mathbb{E}_I(y))^* A_p^{-\frac{1}{2}}(x) \right) = 0 \). Using the first part of the proof in the last estimate, we conclude that
\[
\| A_p^{-\frac{1}{2}}(x) \|_{S_p^\alpha} \leq \text{Tr} (y^* A_p^{-\frac{1}{2}}(x)) = \text{Tr} (\mathbb{E}_I(y))^* A_p^{-\frac{1}{2}}(x) - \text{Tr} \left( (\mathbb{E}_I(y))^* A_p^{-\frac{1}{2}}(x) \right) = \text{Tr} (y_0 A_p^{-\frac{1}{2}}(x))
\]
\[
\leq \| \partial_a A_p^{-\frac{1}{2}}(y_0) \|_{L^{p'}} \| \partial_a(x) \|_{L^{p}(\Omega, S_p^\alpha)} \leq_p \| \partial_a(x) \|_{L^{p}(\Omega, S_p^\alpha)}.
\]

The \( q \)-gaussian derivations equally satisfy the equivalence with \( A_p^{-\frac{1}{2}} \).

**Proposition 3.11** Suppose \(-1 \leq q \leq 1 \) and \( 2 \leq p < \infty \). For any \( x \in M_{I,\infty} \), we have
\[
(3.34) \quad \| A_p^{-\frac{1}{2}}(x) \|_{S_p^\alpha} \approx_p \| \partial_{a,q}(x) \|_{L^p(\Gamma_q(\Omega, S_p^\alpha))}.
\]
Moreover, if \( 1 < p \leq 2 \), then \( \sim_p \) still holds in (3.34) above.

**Proof**: We have
\[
\mathbb{E}_I(|\partial_{a,1}(x)|^2) = \sum_{i,j,k,l} \int_{\Omega} W(\alpha_i - \alpha_j)W(\alpha_k - \alpha_l) \cdot e_{ij} e_{kl} \delta_{i+k,j+l} (\alpha_i - \alpha_j, \alpha_k - \alpha_l) \cdot e_{ij} e_{kl} = \mathbb{E}_I(|\partial_{a,q}(x)|^2).
\]
Let first \( 2 \leq p < \infty \). Then we conclude that
\[
\| A_p^{-\frac{1}{2}}(x) \|_{S_p^\alpha} \approx_p \| \partial_{a,1}(x) \|_{L^p(\Omega, S_p^\alpha)} \approx_p \max \left\{ \left( \mathbb{E}_I(|\partial_{a,1}(x)|^2) \right)^{\frac{1}{2}}, \left( \mathbb{E}_I(|\partial_{a,1}(x)^2|) \right)^{\frac{1}{2}} \right\}_{S_p^\alpha}
\]
\[
= \max \left\{ \left( \mathbb{E}_I(|\partial_{a,q}(x)|^2) \right)^{\frac{1}{2}}, \left( \mathbb{E}_I(|\partial_{a,q}(x)^2|) \right)^{\frac{1}{2}} \right\}_{S_p^\alpha} \approx_p \| \partial_{a,q}(x) \|_{L^p(\Gamma_q(\Omega, S_p^\alpha))}.
\]
For the case \( 1 < p \leq 2 \), we use a duality argument. Namely, we have \( \| A_p^{-\frac{1}{2}}x \|_p = \text{Tr}(A_p^{-\frac{1}{2}}x y^*) \) for a certain normalized \( y \in M_{I,\infty} \cap S_p^\alpha \). As observed in the proof of Theorem 3.10, we can
replace $y$ by an element $y_0 \in M_{1,\text{fin}} \cap S_{p,0}^*$ by the price of having $\|y_0\|_p \leq 2$. Then using the case $2 \leq p < \infty$ for $p$ replaced by $p^*$ in the penultimate estimate, we obtain

\[
\left\| A\frac{1}{2}(x) \right\|_p = \text{Tr}(A\frac{1}{2}(x)y_0) = \text{Tr}(A\frac{1}{2}(x)(A^{-\frac{1}{2}}y_0)^*) \overset{(3.31)}{=} \text{Tr}(\partial_{\alpha,q}(x)(\partial_{\alpha,q}A^{-\frac{1}{2}}y_0)^*) \\
\leq \left\| \partial_{\alpha,q}(x) \right\|_p \left\| A^{-\frac{1}{2}}y_0 \right\|_p \leq 2 \left\| \partial_{\alpha,q}(x) \right\|_p .
\]

The following is a straightforward extension of [Bak1, Lemma 4.2].

**Proposition 3.12** Let $(T_t)_{t \geq 0}$ be a strongly continuous bounded semigroup acting on a Banach space $X$ with (negative) generator $A$. We have

(3.36) \[ \left\| (\text{Id}_X + A)\frac{1}{2}(x) \right\|_X \approx \|x\|_X + \|A\frac{1}{2}(x)\|_X, \quad x \in X. \]

**Proof**: It is well-known [Bak1, Lemma 2.3] that the function $f_1: t \mapsto (1 + t)^{\frac{1}{2}}(1 + t^\frac{1}{2})^{-1}$ is the Laplace transform $\mathcal{L}(\mu_1)$ of some bounded measure $\mu_1$. By [Haa1, Proposition 3.3.2] (note that $f_1 \in H^\infty(\Sigma_{a_0})$ for any $a_1 \in (0,\frac{1}{2})$ and $f_1$ has finite limits $\lim_{z \in \Sigma_{a_0}, |z| \to \infty} f_1(z)$) (see also [LM1, Lemma 2.12]), we have

\[ \int_0^\infty T_t \, d\mu_1(t) = \mathcal{L}(\mu_1)(A) = f_1(A) = (\text{Id}_X + A)\frac{1}{2}\left(\text{Id}_X + A\frac{1}{2}\right)^{-1}. \]

Hence for any $x \in \text{dom}(\text{Id}_X + A\frac{1}{2}) = \text{dom} A\frac{1}{2}$, we have

(3.37) \[ (\text{Id}_X + A)\frac{1}{2}x = \int_0^\infty T_t \left(x + A\frac{1}{2}x\right) \, d\mu_1(t). \]

By [Haa1, Remark 3.3.3], we know that $\int_0^\infty T_t \, d\mu_1(t)$ is a bounded operator on $X$ of norm $\leq \|\mu_1\|$. Using the triangular inequality in the last inequality, we conclude that

\[ \left\| (\text{Id}_X + A)\frac{1}{2}x \right\|_X \overset{(3.37)}{=} \left\| \int_0^\infty T_t \left(x + A\frac{1}{2}x\right) \, d\mu_1(t) \right\|_X \leq \|\mu_1\| \left\| x + A\frac{1}{2}x \right\|_X \leq \|\mu_1\| \left( \|x\|_X + \|A\frac{1}{2}x\|_X \right). \]

It is easy to see [Bak1, Lemma 2.3] that the functions $f_2: t \mapsto (1 + t^\frac{1}{2})(1 + t^\frac{1}{2})^{-1}$ and $f_3: t \mapsto (1 + t)^{-\frac{1}{2}}$ are the Laplace transforms $\mathcal{L}(\mu_2)$ and $\mathcal{L}(\mu_3)$ of some bounded measures $\mu_2$ and $\mu_3$. So we have

\[ \int_0^\infty T_t \, d\mu_2(t) = \left(\text{Id}_X + A\frac{1}{2}\right)(\text{Id}_X + A)^{-\frac{1}{2}} \quad \text{and} \quad \int_0^\infty T_t \, d\mu_3(t) = (\text{Id}_X + A)^{-\frac{1}{2}}. \]

Following the same argument as above, we obtain

(3.38) \[ \left\| x + A\frac{1}{2}x \right\|_X \leq \|\mu_2\| \left\| (\text{Id}_X + A)\frac{1}{2}x \right\|_X \quad \text{and} \quad \left\| x\right\|_X \leq \|\mu_3\| \left\| (\text{Id}_X + A)\frac{1}{2}x \right\|_X . \]

Note that

(3.39) \[ \left\| A\frac{1}{2}x \right\|_X = \| -x + x + A\frac{1}{2}x \|_X \leq \|x\|_X + \|x + A\frac{1}{2}x\|_X. \]

We conclude that

\[ \left\| x\right\|_X + \left\| A\frac{1}{2}x \right\|_X \overset{(3.38)}{\leq} 2 \left\| x\right\|_X + \left\| x + A\frac{1}{2}x \right\|_X \overset{(3.39)}{\leq} \left\| (\text{Id}_X + A)\frac{1}{2}x \right\|_X . \]

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We refer to [Kat1] for information on closable and closed operators acting on Banach spaces. By [Kat1, page 165], an operator $T: \text{dom} \, T \subset X \to Y$ is closed if and only if its domain $\text{dom} \, T$ is a complete space with respect to the graph norm

$$\|x\|_{\text{dom} \, T} \overset{\text{def}}{=} \|x\|_X + \|T(x)\|_Y. $$

(3.40)

For $t, s > 0$, we let $f_t(s) \overset{\text{def}}{=} \frac{t}{\sqrt{4\pi s^2}} e^{-\frac{s^2}{4t}}$. It is well-known [ABHN1, Lemma 1.6.7] that

$$ \int_0^{+\infty} e^{-s} f_t(s) \, ds = e^{-t}. $$

(3.41)

Let $B$ be the negative generator of a semigroup $(e^{-tB})_{t \geq 0}$ acting on a Banach space $X$. By [Haa1, Example 3.4.6], for any $x \in X$ we have

$$ e^{-tB} x = \int_0^{+\infty} f_t(s) e^{-sB} x \, ds. $$

(3.42)

**Proposition 3.13** Suppose $1 < p < \infty$ and $-1 \leq q \leq 1$.

1. The operator $\partial_{\alpha,q}: M_{1,\text{fin}} \subset S^p_f \to \text{L}^p(\Gamma_q(H) \hat{\otimes} B(\ell^2_f))$ is closable as a densely defined operator on $S^p_f$ into $\text{L}^p(\Gamma_q(H) \hat{\otimes} B(\ell^2_f))$. We denote by $\partial_{\alpha,q,p}$ its closure. Moreover, if $x \in \text{dom} \, \partial_{\alpha,q,p}$, we have $\partial_{\alpha,q,p} \mathcal{T}_J(x) \to \partial_{\alpha,q,p}(x)$ as $J \to I$.

2. $(\text{Id}_{S^p_f} + A_p)^{\frac{1}{2}} (M_{1,\text{fin}})$ is a dense subspace of $S^p_f$.

3. $M_{1,\text{fin}}$ is a dense subspace of $\text{dom} \, A_{\text{fin}}^{\frac{1}{2}}$ equipped with the graph norm.

4. We have $\text{dom} \, \partial_{\alpha,q,p} = \text{dom} \, A_{\text{fin}}^{\frac{1}{2}}$. Moreover, for any $x \in \text{dom} \, A_{\text{fin}}^{\frac{1}{2}}$, we have

$$ \|A^{\frac{1}{2}}(x)\|_{S^p_f} \approx_p \|\partial_{\alpha,q,p}(x)\|_{\text{L}^p(\Gamma_q(H) \hat{\otimes} B(\ell^2_f))}. $$

(3.43)

5. If $x \in \text{dom} \, \partial_{\alpha,q,p}$, we have $x^* \in \text{dom} \, \partial_{\alpha,q,p}$ and

$$ (\partial_{\alpha,q,p}(x))^* = -\partial_{\alpha,q,p}(x^*). $$

(3.44)

**Proof:** 1. Suppose that $(x_n)$ is a sequence of $M_{1,\text{fin}}$ which converges to 0 such that the sequence $(\partial_{\alpha,q}(x_n))$ converges to some $y$ belonging to $\text{L}^p(\Gamma_q(H) \hat{\otimes} B(\ell^2_f))$. For any finite subset $J$ of $I$, we have

$$ \partial_{\alpha,q} \mathcal{T}_J(x_n) = \text{Id}_{\text{L}^p(\Gamma_q(H) \hat{\otimes} B(\ell^2_f))} \otimes \mathcal{T}_J \partial_{\alpha,q}(x_n) \xrightarrow{n \to +\infty} (\text{Id}_{\text{L}^p(\Gamma_q(H) \hat{\otimes} B(\ell^2_f))} \otimes \mathcal{T}_J)(y). $$

On the other hand, for any $i, j \in I$ we have $x_{n,ij} \to 0$ as $n \to \infty$. Hence

$$ \partial_{\alpha,q} \mathcal{T}_J(x_n) = \partial_{\alpha,q} \left( \sum_{i,j \in J} x_{n,ij} e_{ij} \right) \overset{(2.26)}{=} \lim_{n \to +\infty} \sum_{i,j \in J} x_{n,ij} (s_{ij} (\alpha_i - \alpha_j) \otimes e_j) \to 0. $$

This implies by unicity of the $\text{L}^p(\Gamma_q(H) \hat{\otimes} B(\ell^2_f))$-limit that $(\text{Id}_{\text{L}^p(\Gamma_q(H) \hat{\otimes} B(\ell^2_f))} \otimes \mathcal{T}_J)(y) = 0$. Since $(\mathcal{T}_J)$ is uniformly bounded in operator norm over $S^p_f$ and converges strongly to $\text{Id}_{S^p_f}$,
we infer that \( \text{Id}_{L^p(\Gamma_u(H)\otimes B(\ell^p_q))} \otimes T_j \) converges strongly to \( \text{Id}_{L^p(\Gamma_u(H)\otimes B(\ell^p_q))} \). We deduce that 
\[ y = \lim_j \left( \text{Id}_{L^p(\Gamma_u(H)\otimes B(\ell^p_q))} \otimes T_j \right)(y) = 0. \]
We conclude with [Kat1, page 165]. The last assertion is left to the reader.

2. We consider the semigroup \((R_t)_{t \geq 0}\) associated to the negative generator \((\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}}\). Let 
\[ x \in S^p_t \] 
be orthogonal to \((\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}}(M_{f,\text{fin}})\). For an element \( y \in M_{f,\text{fin}} \), using [EnN1, Lemma 1.3 (iv)] in the second equality, we have
\[
(R_t(x), y) = \langle x, R_t(y) \rangle = \left\langle x, x + (\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}} \int_0^t R_s(y) \, ds + y \right\rangle = \left\langle x, (\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}} \int_0^t R_s(y) \, ds \right\rangle + \langle x, y \rangle = \langle x, y \rangle.
\]
By density, we deduce that \( x = R_t(x) \). We know that \((R_t(x))\) converges to 0 when \( t \) goes to \(+\infty\). Indeed, using [EnN1, page 60] in the second equality, a standard computation gives
\[
\left\| R_t(x) \right\|_{S^p_t} \left(3.42\right) = \left\| \int_0^+ \infty f_t(s)e^{-s(\text{Id}_A) (x)} \, ds \right\|_{S^p_t} = \left\| \int_0^\infty f_t(s)e^{-sT_n(x)} \, ds \right\|_{S^p_t} \lesssim \int_0^\infty f_t(s)e^{-sT_n(x)} \, ds \lesssim \left( \int_0^\infty e^{-sT_n(x)} \, ds \right) \left\| x \right\|_{S^p_t}^{\left(3.41\right)} = e^{-t} \left\| x \right\|_{S^p_t} \xrightarrow{t \to +\infty} 0.
\]
We infer that \( x = 0 \). We conclude that 0 is the only element orthogonal to \((\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}}(M_{f,\text{fin}})\) as a subset of \( S^p_t \). It follows that \((\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}}(M_{f,\text{fin}})\) is dense in \( S^p_t \).

3. Let \( x \in \text{dom} A^\frac{1}{p}_p \). By [ABH1], Proposition 3.8.2], we have \( \text{dom} A^\frac{1}{p}_p = \text{dom}(\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}} \). Hence \( x \in \text{dom}(\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}} \). From the point 2, there exists a sequence \((x_n)\) of \( M_{f,\text{fin}}\) such that \((\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}} x_n \to (\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}} x\) in \( S^p_t \). We obtain that
\[
\left\| x_n - x \right\|_{\text{dom} A^\frac{1}{p}_p} \left(3.40\right) = \left\| x_n - x \right\|_{S^p_t} + \left\| A^\frac{1}{p}_p (x_n - x) \right\|_{S^p_t} \left(3.36\right) \lesssim_p \left\| (\text{Id}_{S^p_t} + A_p)^{\frac{3}{2}} (x_n - x) \right\|_{S^p_t}.
\]
By the choice of the sequence \((x_n)\), the latter tends to 0 and consequently the sequence \((x_n)\) converges to \( x \) in \( \text{dom} A^\frac{1}{p}_p \).

4. Let \( x \in \text{dom} A^\frac{1}{p}_p \). By the point 3, \( M_{f,\text{fin}}\) is dense in \( \text{dom} A^\frac{1}{p}_p \) equipped with the graph norm. Hence we can find a sequence \((x_n)\) of \( M_{f,\text{fin}}\) converging to \( x \) in \( \text{dom} A^\frac{1}{p}_p \). For any integers \( n, m \), we obtain
\[
\left\| x_n - x_m \right\|_{S^p_t} + \left\| \partial_{\alpha,q,p}(x_n) - \partial_{\alpha,q,p}(x_m) \right\|_{L^p(\Gamma_u(H)\otimes B(\ell^p_q))} \left(3.34\right) \lesssim_p \left\| x_n - x_m \right\|_{S^p_t} + \left\| A^\frac{1}{p}_p (x_n) - A^\frac{1}{p}_p (x_m) \right\|_{S^p_t}
\]
which shows that \((x_n)\) is a Cauchy sequence in \( \text{dom} \partial_{\alpha,q,p} \). By the closedness of \( \partial_{\alpha,q,p} \), we infer that this sequence converges to some \( x' \in \text{dom} \partial_{\alpha,q,p} \) equipped with the graph norm.
Since both \( \text{dom} A^\frac{1}{p}_p \) and \( \text{dom} \partial_{\alpha,q,p} \) are continuously embedded into \( S^p_t \), we have \( x_n \to x \) and \( x_n \to x' \) in \( S^p_t \), and therefore \( x = x' \). It follows that \( x \in \text{dom} \partial_{\alpha,q,p} \). This proves the inclusion \( \text{dom} A^\frac{1}{p}_p \subset \text{dom} \partial_{\alpha,q,p} \). Moreover, for any integer \( n \), we have
\[
\left\| \partial_{\alpha,q,p}(x_n) \right\|_{L^p(\Gamma_u(H)\otimes B(\ell^p_q))} \left(3.34\right) \lesssim \left\| A^\frac{1}{p}_p (x_n) \right\|_{S^p_t}.
\]
Since \( x_n \to x \) in \( \text{dom} \partial_{\alpha,q,p} \) and in \( \text{dom} A^\frac{1}{p} \) both equipped with the graph norm, we conclude that
\[
\| \partial_{\alpha,q,p}(x) \|_{L^p(\Gamma_{s}(I)\otimes \mathbb{B}(\ell^2))} \lesssim_p \| A^\frac{1}{p}(x) \|_{S^p_I}. 
\]

The proof of the reverse inclusion and of the reverse estimate are similar. Indeed, \( \partial_{\alpha,q,p} \) being the closure of its restriction \( \partial_{\alpha,q} \) to \( M_{I,\text{fin}} \), by [Ouh1, Definition 1.40 and Proposition 1.39], we see that \( M_{I,\text{fin}} \) is a dense subspace of \( \text{dom} \partial_{\alpha,q,p} \) equipped with the graph norm.

5. For any \( i,j \in I \), we have
\[
(\partial_{\alpha,q,p}(e_{ij}^*))^{(2.26)} = (s_q(\alpha_i - \alpha_j) \otimes e_{ij}^* = s_q(\alpha_i - \alpha_j) \otimes e_{ji} = -s_q(\alpha_j - \alpha_i) \otimes e_{ji} = -\partial_{\alpha,q,p}(e_{ji}^*).
\]

The end of the proof is left to the reader. \( \blacksquare \)

3.3 Meyer’s problem for semigroups of Schur multipliers

In this section, we consider \( q = 1, \partial_\alpha = \partial_{\alpha,1} \) and \( \partial_{\alpha,p} = \partial_{\alpha,1,p} \).

**Theorem 3.14**

1. Suppose \( 1 < p \leq 2 \). For any \( x \in M_{I,\text{fin}} \) we have
\[
\| A^\frac{1}{p}(x) \|_{S^p_I} \approx_p \inf_{\partial_{\alpha,p}(x)=y+z} \left\{ \| E_I(y^*y) \|_{S^p_I} + \| E_I(zz^*) \|_{S^p_I} \right\}. 
\]

We can restrict the decomposition to the ones with \( y,z \in \text{Gauss}_p \otimes M_{I,\text{fin}} \).

2. Suppose \( 2 \leq p < \infty \). For any \( x \in M_{I,\text{fin}} \), we have
\[
\| A^\frac{1}{p}(x) \|_{S^p_I} \approx_p \max \left\{ \| \Gamma(x,x) \|_{S^p_I}, \| \Gamma(x^*,x^*) \|_{S^p_I} \right\}. 
\]

**Proof:** 2. Suppose \( 2 \leq p < \infty \). For any \( x \in \text{dom} A^\frac{1}{p} \), first note that
\[
\| A^\frac{1}{p}(x) \|_{S^p_I} \approx_p \| \partial_{\alpha,p}(x) \|_{\text{Gauss}_p(S^p_I)} \approx_p \max \left\{ \| \left( E_I((\partial_\alpha(x))^* \partial_\alpha(x)) \right) \|_{S^p_I} , \| \left( E_I((\partial_\alpha(x))(\partial_\alpha(x))^*) \right) \|_{S^p_I} \right\} \cdot 
\]
\[
\approx_p \max \left\{ \| \Gamma(x,x) \|_{S^p_I}, \| \Gamma(x^*,x^*) \|_{S^p_I} \right\}. 
\]

1. We have
\[
\| A^\frac{1}{p} x \|_{S^p_I} \approx_p \| \partial_{\alpha,p}(x) \|_{\text{Gauss}_p(S^p_I)} \approx_p \inf_{\partial_{\alpha,p}(x)=y+z} \left\{ \| (E_I y^*y) \|_{S^p_I} + \| (E_I zz^*) \|_{S^p_I} \right\}. 
\]

Moreover, replacing the space \( S^p_I \) by \( S^p_{I,\text{fin}} \), where \( x \in M_{I,\text{fin}} \) lives in \( S^p_{I,\text{fin}} \) (note that then, \( \partial_{\alpha,p}(x) \) belongs to \( \text{Gauss}_p(S^p_{I,\text{fin}}) \)), we see that in the above infimum we can restrict to \( y,z \in \text{Gauss}_p(S^p_{I,\text{fin}}) \) equipped with the graph norm. \( \blacksquare \)

With the following lemma, one can extend the definition of \( \Gamma(x,y) \) to a larger domain.
Lemma 3.15 Let $2 \leq p < \infty$. Let $x, y \in \text{dom} A^{\frac{1}{p}}$. Then the limit $\lim_{J} \Gamma(T_{J}x, T_{J}y)$ exists in $S_{t}^{p}$. We denote it by $\Gamma(x, y)$. Note that $\Gamma(x, y)$ is unambiguously defined for $x$ or $y$ belonging to $M_{L, F_{0}}$. We have $\Gamma(x, x) \geq 0$.

Proof: We recall that for an element $z \in B(\ell_{2}^{q})$, we have $z \in S_{r}^{p}$ if and only if $\sup_{a} \|T_{J}z\|_{S_{t}^{p}} < \infty$ and in this case we have $\|z\|_{S_{r}^{p}} = \sup_{a} \|T_{J}z\|_{S_{t}^{p}}$. Then, we have

$$\|\Gamma(T_{J}x, T_{J}y) - \Gamma(T_{K}x, T_{K}y)\|_{S_{t}^{p}} \leq \|\Gamma(T_{J}x, T_{J}y) - \Gamma(T_{J}x, T_{K}y)\|_{S_{t}^{p}} + \|\Gamma(T_{J}x, T_{K}y) - \Gamma(T_{K}x, T_{K}y)\|_{S_{t}^{p}}$$

where $J = \{a \in \mathbb{N} : a \leq m_{a}\}$ and $K = \{a \in \mathbb{N} : a \leq m_{a}\}$. According to the point 1 of Lemma 3.16, we infer that $\Gamma(T_{J}x, T_{J}y)$ is a Cauchy net in $S_{r}^{p}$. Reasoning in the same way for $(T_{J}A^{\frac{1}{p}}_r(x))$, we obtain that $\Gamma(T_{J}x, T_{J}y)$ is a Cauchy net in $S_{r}^{p}$. Moreover, since $\Gamma(T_{J}x, T_{J}x)$ is a positive operator by the point 2 of Lemma 2.2, its limit in $S_{r}^{p}$ is again positive.

Lemma 3.16 Suppose $2 \leq p < \infty$.

1. For any $x, y \in \text{dom} A^{\frac{1}{p}} = \text{dom} \partial_{p, \alpha}$, we have

$$\Gamma(x, y) = \langle \partial_{p, \alpha}(x), \partial_{p, \alpha}(y) \rangle_{S_{t}^{p}(L^{2}(\Omega), \alpha)}.$$  

2. For any $x \in \text{dom} A^{\frac{1}{p}} = \text{dom} \partial_{\alpha, p}$, we have

$$\|\Gamma(x, x)\|_{S_{t}^{p}} = \|\partial_{\alpha, p}(x)\|_{S_{t}^{p}(L^{2}(\Omega), \alpha)}.$$

Proof: Consider some elements $x, y \in \text{dom} A^{\frac{1}{p}}$. According to the point 1 of Proposition 3.13, we have $\partial_{p, \alpha}(T_{J}x) \rightarrow \partial_{p, \alpha}(y)$ and $\partial_{p, \alpha}(T_{J}y) \rightarrow \partial_{p, \alpha}(y)$ as $J \rightarrow I$ in $L^{p}(\Omega, S_{r}^{p})$. By [JM, Proposition 2.5 (2)], we have since $p \geq 2$ a contractive inclusion $L^{p}(\Omega, S_{r}^{p}) \hookrightarrow L^{2}(\Omega, S_{r}^{p}) \hookrightarrow S_{r}^{p}(L^{2}(\Omega), \alpha)$. We deduce that $\partial_{p, \alpha}(T_{J}x) \rightarrow \partial_{p, \alpha}(x)$ as $J \rightarrow I$ in $S_{r}^{p}(L^{2}(\Omega), \alpha)$. Thus, invoking Lemma 3.15 in the first equality, we have, with limits in $S_{r}^{p}$

$$\Gamma(x, y) = \lim_{J} \Gamma(T_{J}x, T_{J}y) = \lim_{J} \langle \partial_{p, \alpha}(T_{J}x), \partial_{p, \alpha}(T_{J}y) \rangle_{S_{t}^{p}(L^{2}(\Omega), \alpha)} = \langle \partial_{p, \alpha}(x), \partial_{p, \alpha}(y) \rangle_{S_{t}^{p}(L^{2}(\Omega), \alpha)}.$$  

2. If $x \in M_{L, F_{0}}$, we have

$$\|\partial_{\alpha}(x)\|_{S_{r}^{p}(L^{2}(\Omega), \alpha)} = \|\partial_{\alpha}(x), \partial_{\alpha}(x)\|_{S_{r}^{p}(L^{2}(\Omega), \alpha)} = \|\Gamma(x, x)\|_{S_{t}^{p}}.$$
The last statement follows in the same way from part 1 of the lemma.

Lemma 3.15 above gives an extension of $\Gamma : M_{I, \text{fin}} \times M_{I, \text{fin}} \to S_p^I$ to $\Gamma : \text{dom}(A_p^\perp) \times \text{dom}(A_p^\perp) \to S_p^I$. We show now that it is the smallest reasonable extension. To this end, suppose that $X$ is a Banach space and $a : D \times D \to Y$ is a sesquilinear mapping (linear in the second component and antilinear in the first component). Then $a$ is called a sesquilinear form. If $a(x, y)^* = a(y, x)$ for any $x, y \in D$, then $a$ is called symmetric. If $a(x, x) \geq 0$ for any $x \in D$, then $a$ is called positive. A sequence $(x_n)$ in $D$ is called $a$-convergent to $x \in X$ if $x_n \to x$ in $X$ and $a(x_n - x_m, x_n - x_m) \to 0$ for $n, m \to \infty$. Further, a positive form is called closed if for any $a$-convergent sequence $(x_n)$ in $D$, converging to $x \in X$, we have $x \in D$ and $a(x_n - x, x_n - x) \to 0$. Finally, an extension $\overline{\pi} : D \times D \to Y$ of some non-negative form $a$ is called the closure of $a$ if $\overline{\pi}$ is closed and is the smallest closed extension of $a$. If such an $\overline{\pi}$ exists, then $a$ is called closable. For all these notions, but in the case of scalar valued sesquilinear forms $Y = \mathbb{C}$, we refer to [Nes1, Section 2.1.2].

Now, let $X = S_p^I$ for some $p \geq 2$, $D = M_{I, \text{fin}}$, $Y = S_p^I \oplus S_p^I$ and we consider the form $a : M_{I, \text{fin}} \times M_{I, \text{fin}} \to S_p^I \oplus S_p^I$, $(x, y) \mapsto \Gamma(x, y) \oplus \Gamma(y^*, x^*)$. Then $a$ is a sesquilinear form. According to the point 3 of Lemma 2.2, we have $\Gamma(x, y)^* = \Gamma(y, x)$, so $a$ is symmetric. Moreover, again according to Lemma 2.2, $a$ is positive.

**Lemma 3.17** Suppose $2 \leq p < \infty$. The above form $a : M_{I, \text{fin}} \times M_{I, \text{fin}} \to S_p^I \oplus S_p^I$ is closable and its closure is given by $\overline{\pi} : \text{dom}(A_p^\perp) \times \text{dom}(A_p^\perp) \to S_p^I \oplus S_p^I$, $(x, y) \mapsto \Gamma(x, y) \oplus \Gamma(x^*, y^*)$, where in the last expression, $\Gamma(x, y)$ and $\Gamma(x^*, y^*)$ are defined by Lemma 3.15.

**Proof**: Note first that $\overline{\pi}$ is positive according to Lemma 3.15. We show that $\overline{\pi}$ is closed. To this end, let $(x_n)$ be a sequence in $\text{dom}(A_p^\perp)$ such that $x_n \to x$ in $S_p^I$ and $\overline{\pi}(x_n - x_m, x_n - x_m) \to 0$ as $n, m \to \infty$ in $S_p^I$. Then, we have

\[
\begin{align*}
\|A_p^\perp(x_n - x_m)\|_p^2 &= \lim_j \|T_j A_p^\perp(x_n - x_m)\|_p^2 = \lim_j \|A_p^\perp T_j(x_n - x_m)\|_p^2 \\
&\overset{(3.45)}{=} \lim_j \left\|\Gamma(T_j(x_n - x_m), T_j(x_n - x_m))^{1/2}\right\|_p^2 + \lim_j \left\|\Gamma(T_j(x_n^* - x_m^*), T_j(x_n^* - x_m^*))^{1/2}\right\|_p^2 \\
&= \left\|\Gamma(x_n - x_m, x_n - x_m)\right\|_p^2 + \left\|\Gamma(x_n^* - x_m^*, x_n^* - x_m^*)\right\|_p^2 \\
&\overset{x_n \to x}{\to} 0
\end{align*}
\]

as $n, m \to \infty$. Thus, $(x_n)$ is a Cauchy sequence in $\text{dom}(A_p^\perp)$ with respect to the graph norm of $A_p^\perp$. By closedness of the Schur multiplier $A_p^\perp$, $x$ belongs to $\text{dom}(A_p^\perp)$ and $A_p^\perp x_n \to A_p^\perp x$ in $S_p^I$. Then with a similar calculation as above, $\|A_p^\perp(x_n - x)\|_p^2 \overset{x_n \to x}{\to} 0$ as $n \to \infty$. Thus, $x$ is the $\overline{\pi}$-limit of $(x_n)$. We infer that $\overline{\pi}$ is closed.

Let us show that $\overline{\pi}$ is the smallest closed extension of $a$. To this end, let $x \in \text{dom}(A_p^\perp)$. We show that there exists a sequence $(x_n)$ in $M_{I, \text{fin}}$ which is $a$-converging to $x$. Simply let
\( x_n = T_{J_n} x \) for some finite subset \( J_n \) such that \( T_{J_n} x \to x \) in \( S^p_f \). Moreover,
\[
\|a(T_{J_n} x - T_m x, T_{J_n} x - T_m x)\|_{\mathcal{F}} \\
\cong \|\Gamma(T_{J_n} x - T_m x, T_{J_n} x - T_m x)\|_{\mathcal{F}} + \|\Gamma(T_{J_n} x^* - T_m x^*, T_{J_n} x^* - T_m x^*)\|_{\mathcal{F}} \\
\cong \|A^\perp_p(T_{J_n} x - T_m x)\|_p \to 0
\]
as \( n, m \to \infty \). We have shown that \((T_{J_n} x)\) is \( \alpha \)-convergent to \( x \). Thus, \( x \in \text{dom } A^\perp_p \) has to belong to the domain of any closed extension of \( a \).

**Lemma 3.18** Suppose \( 2 \leq p < \infty \). Let \( J \) be a finite subset of \( I \). If \( x \in \text{dom } A^\perp_p \), then
\[
\|\Gamma(T_J x, T_J x)\|_{S^p_f} \leq \|\Gamma(x, x)\|_{S^p_f}.
\]

**Proof**: Let \( x \in \text{dom } A^\perp_p \). By Lemma 3.15, we have \( \Gamma(x, x) = \lim_J \Gamma(T_J x, T_J x) \) in \( S^p_f \). Thus it suffices to prove that for any finite subsets \( J \subseteq K \) of \( I \), we have \( \|\Gamma(T_J x, T_J x)\|_{S^p_f} \leq \|\Gamma(T_K x, T_K x)\|_{S^p_f} \). We have
\[
(3.47) \quad \partial_\alpha T_J (x) = \sum_{i,j \in J} x_{ij} \partial_\alpha (e_{ij}) = (\text{Id}_{L^2(\Omega)} \otimes T_J) \left( \sum_{i,j \in K} x_{ij} \partial_\alpha (e_{ij}) \right) = (\text{Id}_{L^2(\Omega)} \otimes T_J) (\partial_\alpha T_K x).
\]

On the other hand, since \( T_J : S^p_f \to S^p_f \) is a complete contraction, the linear map \( T_J \otimes \text{Id}_{L^2(\Omega)} : S^p_f(L^2(\Omega)_{c,p}) \to S^p_f(L^2(\Omega)_{c,p}) \) is also a contraction according to (2.21). Hence, we have
\[
\|\Gamma(T_J x, T_J x)\|_{S^p_f} = \|\Gamma(T_J x, T_J x)\|_{S^p_f}^{(2.31)} = \|\partial_\alpha T_J (x)\|_{S^p_f(L^2(\Omega)_{c,p})} \\
\overset{(3.47)}{=} \|\Gamma(T_J \otimes \text{Id}_{L^2(\Omega)})(\partial_\alpha T_J x)\|_{S^p_f(L^2(\Omega)_{c,p})} \leq \|\partial_\alpha T_K (x)\|_{S^p_f(L^2(\Omega)_{c,p})}^{(2.31)} = \|\Gamma(T_K x, T_K x)\|_{S^p_f}^{\frac{1}{2}}.
\]

It is left to the reader to extend Proposition 2.4.

**Proposition 3.19** For any \( x \in \text{dom } A^\perp_p \), we have
\[
(3.48) \quad \Gamma(x, x) = \int_{\Omega} (\partial_\alpha (x))^\ast \partial_\alpha (x) \, d\mu = E_I ((\partial_\alpha (x))^\ast \partial_\alpha (x)).
\]

**Theorem 3.20** Suppose \( 2 \leq p < \infty \). For any \( x \in \text{dom } A^\perp_p \), we have
\[
(3.49) \quad \|A^\perp_p (x)\|_{S^p_f} \approx \max \left\{ \|\Gamma(x, x)\|_{S^p_f}, \|\Gamma(x^*, x^*)\|_{S^p_f} \right\}.
\]

**Proof**: For any \( x \in \text{dom } A^\perp_p \), first note that
\[
(3.50) \quad E_I ((\partial_\alpha (x))^\ast (\partial_\alpha (x)^*)) \overset{(3.64)}{=} E_I ((\partial_\alpha (x^*)) (\partial_\alpha (x^*))) \overset{(3.48)}{=} \Gamma(x^*, x^*).
\]
We conclude that
\[
\| A^\perp (x) \|_{S^p I} \stackrel{(3.43)}{\approx} p \| \partial_{\alpha,p} (x) \|_{\text{Gauss}_p(S^p I)} \leq \frac{1}{2} \left\{ \left\| \mathcal{E}_I (\partial_{\alpha} (x)) \right\|_{S^p I} \right\}^{\frac{1}{2}} \left\| \left( \mathcal{E}_I (\partial_{\alpha} (x)) \right)^* \right\|_{S^p I}^{\frac{1}{2}}.
\]

We let
\[
S^p_{I,0} \stackrel{\text{def}}{=} \left\{ x \in S^p_I : \lim_{t \to +\infty} T_t(x) = 0 \right\}.
\]

Since \( T_t(e_{ij}) = e^{-t||\alpha_i - \alpha_j||_H} e_{ij} \) for any \( i, j \in I \) and any \( t \geq 0 \), we have
\[
(3.51)
S^p_{I,0} = \left\{ x \in S^p_I : x_{ij} = 0 \text{ for all } i, j \text{ with } \alpha_i = \alpha_j \right\}.
\]

If \( h \in H \), we define the \( h \)-directional Riesz transform \( R_{\alpha,h} \) defined on \( M_{I,\text{fin}} \) by
\[
R_{\alpha,h}(e_{ij}) = 2\pi i \langle \alpha_i - \alpha_j, h \rangle_{H} e_{ij} \text{ if } e_{ij} \in S^p_{I,0}.
\]

and \( R_{\alpha,h}(e_{ij}) = 0 \) if \( e_{ij} \in S^p_{I,0} \). If \( (\xi_k) \) is a basis of the Hilbert space \( H \), we let
\[
R_{\alpha,k} = R_{\alpha,\xi_k}.
\]

Suppose \( 1 < p < \infty \). In the sequel, \( C_p \) stands for the subspace of \( S^p_I \) spanned by the matrices \( e_{k1} \) for \( k \in I \). For any \( k \in I \), and any finite sum \( f = \sum_{i,j} f_{ij} \otimes e_{ij} \) of \( L^\infty(\Omega) \otimes M_{I,\text{fin}} \) we define the element
\[
(3.54)
u_p(f) = \sum_k u_{pk}(f) \otimes e_{k1}
\]
of \( S^p(C_p) \) where
\[
(3.55)u_{pk}(f) = \sum_{i,j} \langle f_{ij}, W(e_k) \rangle_{L^p(\Omega),L^{p^*}(\Omega)} e_{ij}.
\]

Proposition 3.21 1. For any \( f, h \) we have
\[
(3.56)\mathcal{E}_I (f^* h) = (u_p(f))^* u_p^* (h).
\]

2. If \( x \in S^p_{I,0} \), we have
\[
(3.57)u_p \partial_{\alpha,p} A^\perp (x) = \sum_{k=1}^{+\infty} R_{\alpha,k}(x) \otimes e_{k1}.
\]
Proof : 1. We have

\[(u_p(f))^* u_p^* (h) = \left( \sum_{k \geq 1} u_{pk}(f) \otimes e_{k1} \right)^* \left( \sum_{k \geq 1} u_{pk}(h) \otimes e_{k1} \right) = \sum_{k \geq 1} (u_{pk}(f))^* u_{pk}(h) \]

\[= \sum_{k \geq 1} \left( \sum_{i,j} \langle f_{ij}, W(e_k) \rangle_{L^2(\Omega)} e_{ij} \right)^* \left( \sum_{r,s} \langle h_{rs}, W(e_k) \rangle_{L^2(\Omega)} e_{rs} \right) \]

\[= \sum_{k \geq 1} \left( \sum_{i,j} f_{ij} W(e_k) \right)_{L^2(\Omega)} \left( \sum_{r,s} h_{rs} W(e_k) \right)_{L^2(\Omega)} e_{js} = \sum_{i,j,s} \langle f_{ij}, h_{js} \rangle_{L^2(\Omega)} e_{js}. \]

On the other hand, we have

\[E_I(f^* h) = E_I \left( \left( \sum_{i,j} f_{ij} \otimes e_{ij} \right)^* \left( \sum_{r,s} h_{rs} \otimes e_{rs} \right) \right) = \sum_{i,j} \sum_{r,s} E_I(f_{ij} h_{rs} \otimes e_{ji} e_{rs}) \]

\[= \sum_{i,j,s} E_I(f_{ij} h_{rs} \otimes e_{js}) = \sum_{i,j,s} \left( \int f_{ij} h_{rs} \right) e_{js}. \]

2. We have

\[(3.58) \quad u_p \alpha_{a,p} A^{-\frac{1}{2}} (x) = u_p \left( \partial_{\alpha,p} \left( \sum_{i,j} \frac{1}{\|\alpha_i - \alpha_j\|_H} x_{ij} e_{ij} \right) \right) = \sum_{i,j} \frac{1}{\|\alpha_i - \alpha_j\|_H} x_{ij} u_p \partial_{\alpha,p} (e_{ij}) \]

\[
= \sum_{i,j} \frac{1}{\|\alpha_i - \alpha_j\|_H} x_{ij} u_p \left( W(\alpha_i - \alpha_j) \otimes e_{ij} \right) \quad (2.26) \]

\[(3.59) \quad \sum_{k=1}^{+\infty} \left( \sum_{i,j} \frac{1}{\|\alpha_i - \alpha_j\|_H} x_{ij} W(\alpha_i - \alpha_j), W(e_k) \right)_{L^2(\Omega)} e_{ij} \otimes e_{k1} \]

\[= \sum_{k=1}^{+\infty} \sum_{i,j} \langle \alpha_i - \alpha_j, e_k \rangle_H x_{ij} e_{ij} \otimes e_{k1} \quad (3.52) \]

\[= \sum_{k=1}^{+\infty} R_{a,k}(x) \otimes e_{k1}. \]

Lemma 3.22 Suppose 1 < p < \infty. Then u_p induces an isometric map from L^p_{\Omega}(E_I) into S_p^0(C_p). Moreover, a family \((a_k)_{k \in I} of elements of S_p^0 can be written as a_k = u_{pk}(f) if and only if \(\sum_k W(e_k) \otimes a_k converges in Gauss_p(S_p^0).\)

Proof : We have

\[\|f\|_{L^p_{\Omega}(E_I)} (1.2) = \|E_I(f^* f)^{1/2}\|_{S_p^0} = \|E_I(f^* f)^{1/2}\|_{S_p^0} \quad (3.56) \]

\[= \left( \|(u_p(f))^* u_p^* (f)\|_{S_p^0} \right)^{1/2} = \|u_p(f)\|_{S_p^0}. \]

The end of the proof is left to the reader with the help of [HvNVW2, Theorem 6.4.10].

Theorem 3.23 Suppose 1 < p < \infty. Let x \in S_p^0, \(\Omega, 0).\)

1. If 1 < p \leq 2, we have

\[\|x\|_{S_p^0} \approx_p \inf_{R_{a,b}(x) = a_k + b_k} \left( \sum_{k=1}^{+\infty} |a_k|^2 \right)^{1/2} + \left( \sum_{k=1}^{+\infty} |b_k|^2 \right)^{1/2} \|S_p^0. \]

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2. If $2 \leq p < \infty$, we have

$$
\|x\|_{S_p^\infty} \approx_p \max \left\{ \left\| \left( \sum_{k \geq 1} |R_{\alpha,k}(x)|^2 \right)^{\frac{1}{2}} \right\|_{S_p^\infty}, \left\| \left( \sum_{k \geq 1} |R_{\alpha,k}(x^*)|^2 \right)^{\frac{1}{2}} \right\|_{S_p^\infty} \right\}.
$$

Proof: Suppose $1 < p < 2$. For any $x \in S_p^\infty$, since $A_p^\perp(x)$ belongs to $\text{dom } A_p^\perp$, we have

$$
\|x\|_{S_p^\infty} = \|A_p^\perp A_p^{-\frac{1}{2}}(x)\|_{S_p^\infty} \overset{(3.43)}{=} \|\partial_{\alpha,p} A_p^{-\frac{1}{2}}(x)\|_{L_p(\Omega,S_p^\infty)} \overset{(3.5)}{=} \inf_{\partial_{\alpha,p} A_p^{-\frac{1}{2}}(x)=g+h} \left\| \left( E_I(g^*) \right)^{\frac{1}{2}} \right\|_{S_p^\infty} + \left\| \left( E_I(h^*) \right)^{\frac{1}{2}} \right\|_{S_p^\infty}.
$$

By the factorization of $E_I$, we infer that

$$
\left( E_I(g^*) \right)^{\frac{1}{2}} = \left( \sum_k (u_p(g))^* u_p(t) \right)^{\frac{1}{2}} \overset{(3.56)}{=} \left( \sum_k (u_p(g))^* u_p(t) \right)^{\frac{1}{2}} = \left( \sum_k (u_p(g))^* u_p(t) \right)^{\frac{1}{2}}.
$$

Similarly, we have

$$
\left( E_I(h^*) \right)^{\frac{1}{2}} = \left( \sum_k (u_p(h))^* u_p(t) \right)^{\frac{1}{2}} \overset{(3.56)}{=} \left( \sum_k (u_p(h))^* u_p(t) \right)^{\frac{1}{2}} = \left( \sum_k (u_p(h))^* u_p(t) \right)^{\frac{1}{2}}.
$$

By Lemma 3.22, the map $u_p: L_p^\infty(E_I) \rightarrow C_0 \otimes_p S_p^\infty$ is injective. So we have $\partial_{\alpha,p} A_p^{-\frac{1}{2}}(x) = g + h$ if and only if $u_p \partial_{\alpha,p} A_p^{-\frac{1}{2}}(x) = u_p(g) + u_p(h)$, that is, by (3.57), for any $k$ we have $R_{\alpha,k}(x) = u_p(g) + u_p(h)$. Hence this computation gives that $\|x\|_{S_p^\infty}$ is comparable to the norm

$$
\inf_{R_{\alpha,k}(x)=u_p(g)+u_p(h)} \left\| \sum_{k=1}^{p<\infty} [u_p(g)]^* u_p(k)(g) \right\|_{S_p^\infty} + \left\| \sum_{k=1}^{p<\infty} [u_p(h)]^* u_p(k)(h) \right\|_{S_p^\infty}.
$$

Now, note that if $h = \sum_{i,j} h_{ij} \otimes e_{ij}$ is a finite sum of $L_p^\infty(\Omega,S_p^\infty)$, since $W(e_k)$ is a real function, we have for any integer $k$

$$
[u_p(h^*)]^* = u_p \left( \sum_{i,j} (h_{ij} \otimes e_{ij})^* \right)^* = \sum_{i,j} [u_p(h_{ij} \otimes e_{ij})]^* \overset{(3.55)}{=} \sum_{i,j} [\langle h_{ij}, W(e_k) \rangle_{L_2(\Omega)} e_{ij}]^* \overset{(2.14)}{=} \sum_{i,j} \langle h_{ij}, W(e_k) \rangle_{L_2(\Omega)} e_{ij} \overset{(3.55)}{=} u_p(h).
$$
Hence the infimum of (3.60) is equal to
\[
\inf_{R_{\alpha,k}(x)=u_{pk}(g)+u_{pk}(h)} \left\| \left( \sum_{k \geq 1} \left| u_{pk}(g) \right|^p \right)^{\frac{1}{p}} \right\|_{S^p_I} + \left\| \left( \sum_{k \geq 1} \left| u_{pk}(h) \right|^p \right)^{\frac{1}{p}} \right\|_{S^p_I}.
\]

Therefore, using Lemma 3.22, this infimum is finally equal to the infimum
\[
\inf_{R_{\alpha,k}(x)=a_k+b_k} \left\| \left( \sum_{k = 1}^{\infty} |a_k|^p \right)^{\frac{1}{p}} \right\|_{S^p_I} + \left\| \left( \sum_{k = 1}^{\infty} |b_k|^p \right)^{\frac{1}{p}} \right\|_{S^p_I}.
\]

Now suppose \(2 \leq p < \infty\). For any \(x \in S^p_{I,0}\), since \(A_p^{-\frac{1}{p}}(x)\) belongs to \(A_p^{\frac{1}{p}}\), we have
\[
\|x\|_{S^p_I} = \|A_p^{\frac{1}{p}}A_p^{-\frac{1}{p}}(x)\|_{S^p_I} \overset{(3.43)}{=} \|\partial_{\alpha,p}A_p^{-\frac{1}{p}}(x)\|_{L^p(I,S^p_I)} \overset{(3.4)}{=} \max \left\{ \left\| E_I \left( \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right)^* \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right) \right\|_{S^p_I}, \left\| E_I \left( \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right)^* \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right)^* \right) \right\|_{S^p_I} \right\}.
\]

We obtain
\[
E_I \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x)^* \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right) = \left[ u_p \left( \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right)^* \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right)^* \right) \right]^* \left( \sum_{k = 1}^{\infty} R_{\alpha,k}(x) \otimes e_{k1} \right)^* \left( \sum_{l = 1}^{\infty} R_{\alpha,l}(x) \otimes e_{l1} \right)
\]
\[
\overset{(3.57)}{=} \left( \sum_{k = 1}^{\infty} R_{\alpha,k}(x) \otimes e_{k1} \right)^* \left( \sum_{l = 1}^{\infty} R_{\alpha,l}(x) \otimes e_{l1} \right) = \sum_{k = 1}^{\infty} \left( R_{\alpha,k}(x) \right)^* R_{\alpha,k}(x) = \sum_{k = 1}^{\infty} \left| R_{\alpha,k}(x) \right|^2.
\]

Similarly, using (3.61) in addition, we obtain
\[
E_I \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x)^* \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right)^* \right) \overset{(3.56)}{=} u_p \left( \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right)^* \left( \partial_{\alpha,p}A_p^{-\frac{1}{p}}(x) \right)^* \right) \left( \sum_{k = 1}^{\infty} R_{\alpha,k}(x) \otimes e_{k1} \right)^* \left( \sum_{l = 1}^{\infty} R_{\alpha,l}(x) \otimes e_{l1} \right)^* \left( \sum_{k = 1}^{\infty} R_{\alpha,k}(x) \right)^* \left( \sum_{k = 1}^{\infty} R_{\alpha,k}(x) \right)^* = \sum_{k = 1}^{\infty} \left| R_{\alpha,k}(x) \right|^2.
\]

The proof is complete. \(\blacksquare\)

### 3.4 Littlewood-Paley estimates

In this section, we give some useful square function inequalities.

**Lemma 3.24** Suppose \(2 < p < \infty\). For any sequence \((h_k)_{k \geq 1}\) of elements of \(H\) and any sequence \((x_k)\) of elements in \(S^p_{I,0}\), we have
\[
\left\| \left( \sum_{k \geq 1} \left| R_{\alpha,h_k}(x_k) \right|^2 \right)^{\frac{1}{2}} \right\|_{S^p_I} \leq_p \left( \sup_{k \geq 1} \| h_k \|_H \right) \left\| \left( \sum_{k \geq 1} |x_k|^2 \right)^{\frac{1}{2}} \right\|_{S^p_I}.
\]
Proof: Consider a basis \((\xi_i)_{i \geq 1}\) of \(H\) and the map \(\Lambda: \ell_2^2 \otimes \mathbb{C} \rightarrow \ell_2^2, \; e_{kl} \mapsto \langle h_k, \xi_i \rangle_H e_k\) where \((e_{kl})\) and \((e_k)\) denote the canonical basis of \(\ell_2^2 \otimes \mathbb{C}\) and \(\ell_2^2\). We have

\[
\left\| \sum_{k \geq 1} |R_{\alpha, \beta}(x_k)|^2 \right\|_{\ell_2^2}^{\frac{1}{2}} = \left\| \sum_{k \geq 1} R_{\alpha, \beta}(x_k) \otimes e_k \right\|_{SP(S^p)}
\]

(5.52) \[
= \left\| \sum_{k \geq 1} \left( 2\pi i \sum_{i,j} \frac{(\alpha_i - \alpha_j, h_k)_H}{||\alpha_i - \alpha_j||_H} x_{kij} e_{ij} \right) \otimes e_k \right\|_{SP(S^p)}
\]

\[
= \left\| \sum_{k,l \geq 1} \left( 2\pi i \sum_{i,j} \frac{(h_k, \xi_l)_H (\alpha_i - \alpha_j, \xi_l)_H}{||\alpha_i - \alpha_j||_H} x_{kij} e_{ij} \right) \otimes (h_k, \xi_l)_H e_k \right\|_{SP(S^p)}
\]

(5.53) \[
= \left\| \sum_{k,l \geq 1} R_{\alpha, \beta}(x_k) \otimes (h_k, \xi_l)_H e_k \right\|_{SP(S^p)}
\]

Since \(\ell_2^2\) is a homogeneous Hilbertian operator space [Pie7, page 22], the completely bounded norm of the map \(\Lambda: \ell_2^2 \otimes \mathbb{C} \rightarrow \ell_2^2\) coincides with its norm which in turn less than \(\|h\|_{\infty} = \sup_k \|h_k\|_H\):

\[
\left\| A \left( \sum_{kl} a_{kl} e_{kl} \right) \right\|_{\ell_2^2}^2 = \left\| \sum_{kl} a_{kl} A(e_{kl}) \right\|_{\ell_2^2}^2 = \left\| \sum_{kl} a_{kl} \langle h_k, \xi_l \rangle_H e_k \right\|_{\ell_2^2}^2
\]

\[
= \sum_{kl} |a_{kl} \langle h_k, \xi_l \rangle_H|^2 = \sum_{kl} |a_{kl}|^2 |\langle h_k, \xi_l \rangle_H|^2 \leq \|h\|_H^2 \sum_{kl} |a_{kl}|^2 = \|h\|_{\infty}^2 \sum_{kl} |a_{kl}|^2 e_k \right\|_{\ell_2^2}^2.
\]

If \((x_k)\) is a sequence of elements of \(S^p\) and if \(x = \sum_k x_k \otimes e_k \in S^p(S)\) we deduce that

\[
\left\| \left( \sum_{k \geq 1} |R_{\alpha, \beta}(x_k)|^2 \right)^{\frac{1}{2}} \right\|_{\ell_2^2} = \left\| \sum_{k,l \geq 1} R_{\alpha, \beta}(x_k) \otimes \Lambda(e_k) \right\|_{SP(S^p(S))} = \left\| \left( \sum_{k \geq 1} R_{\alpha, \beta}(x_k) \otimes e_k \right)^{\frac{1}{2}} \right\|_{SP(S^p(S))}
\]

\[
\leq \left\| \Lambda \right\|_{\ell_2^2 \otimes \mathbb{C} \rightarrow \ell_2^2} \left\| \sum_{k \geq 1} R_{\alpha, \beta}(x_k) \otimes e_k \right\|_{SP(S^p(S))} \leq \left( \sup_{k \geq 1} \|h_k\|_H \right) \left\| \sum_{k \geq 1} R_{\alpha, \beta}(x_k) \otimes e_k \right\|_{SP(S^p(S))}
\]

\[
= \left( \sup_{k \geq 1} \|h_k\|_H \right) \left\| \left( \sum_{k \geq 1} |R_{\alpha, \beta}(x_k)|^2 \right)^{\frac{1}{2}} \right\|_{\ell_2^2} = \left( \sup_{k \geq 1} \|h_k\|_H \right) \left\| \left( \sum_{k \geq 1} |R_{\alpha, \beta}(x_k)|^2 \right)^{\frac{1}{2}} \right\|_{\ell_2^2}
\]

\[
= \left( \sup_{k \geq 1} \|h_k\|_H \right) \left\| \left( \sum_{k \geq 1} \left( R_{\alpha, \beta}(x_k) + e_k \right)^* \otimes e_k \right) \left( \sum_{k \geq 1} \left( R_{\alpha, \beta}(x_k) + e_k \right) \otimes e_k \right) \right\|_{\ell_2^2}^{\frac{1}{2}}
\]

\[
= \left( \sup_{k \geq 1} \|h_k\|_H \right) \left\| \left( \sum_{k \geq 1} \left( R_{\alpha, \beta}(x_k) + e_k \right)^2 \right)^{\frac{1}{2}} \right\|_{\ell_2^2}^{\frac{1}{2}}
\]

\[
= \left( \sup_{k \geq 1} \|h_k\|_H \right) \left\| \left( \sum_{k \geq 1} \left( R_{\alpha, \beta}(x_k) \otimes \text{Id}_{S^p}(x) \right)^2 \right)^{\frac{1}{2}} \right\|_{\ell_2^2}^{\frac{1}{2}}
\]
Now, since Theorem 3.23 also holds for \( \text{Id}_{S^p} \otimes A_p \), the last term on the right hand side is dominated by
\[
\|x\|_{S^p(S^p_G)} = \left\| \sum_k x_k \otimes e_k \right\|_{S^p(S^p_G)} = \left\| \left( \sum_{k \geq 1} |x_k|^2 \right)^{\frac{1}{2}} \right\|_{S^p_G}.
\]

3.5 Kato’s square root problem for semigroups of Fourier multipliers

Let \( G \) be a discrete group. Suppose \( 1 < p < \infty \), \( -1 \leq q \leq 1 \). The same method as the one of [JMP2] gives the following generalization of (1.1):

\[
(3.62) \quad \left\| \frac{A_p^+}{p} (x) \right\|_{L^p(VN(G))} \approx_p \left\| \partial_{\psi,q}(x) \right\|_{L^p(\Gamma_q(H) \rtimes G)}, \quad x \in \text{span} \{ \lambda_s : s \in G \}.
\]

Proposition 3.25 Let \( G \) be a discrete group. Suppose \( 1 < p < \infty \), \( -1 \leq q \leq 1 \) and that \( L^p(VN(G)) \) has CBAP. Assume in addition that the von Neumann algebra \( L^\infty(\Omega) \rtimes \hat{G} \) has QWEP.

1. The operator \( \partial_{\psi,q} \) is closable as a densely defined operator on \( L^p(VN(G)) \) into \( L^p(L^\infty(\Omega) \rtimes \hat{G}) \). We denote by \( \partial_{\psi,q,p} \) its closure.

2. \( (\text{Id}_{L^p(VN(G))} + A_p^+) \) is a dense subspace of \( L^p(VN(G)) \).

3. \( \text{span} \{ \lambda_s : s \in G \} \) is a dense subspace of \( L^p(VN(G)) \).

4. We have dom \( \partial_{\psi,q,p} = \text{dom} A_p^+ \). Moreover, for any \( x \in \text{dom} A_p^+ \), we have

\[
(3.63) \quad \left\| \frac{A_p^+}{p} (x) \right\|_{L^p(VN(G))} \approx_p \left\| \partial_{\psi,q,p}(x) \right\|_{L^p(\Gamma_q(H) \rtimes G)}.
\]

5. If \( x \in \text{dom} \partial_{\psi,q,p} \), we have \( x^* \in \text{dom} \partial_{\psi,q,p} \) and

\[
(3.64) \quad (\partial_{\psi,q,p}(x))^* = -\partial_{\psi,q,p}(x^*).
\]

Proof: 1. Suppose that \( (x_n) \) is a sequence in \( \text{span} \{ \lambda_s : s \in G \} \) which converges to 0 such that the sequence \( (\partial_{\psi,q}(x_n)) \) converges to \( y \). Let \( (M_{\varphi_j}) \) be the sequence of Fourier multipliers approximating the identity from Lemma 2.12. For any \( j \in \mathbb{N} \), we have \( \partial_{\psi,q}(M_{\varphi_j}x_n) = (\text{Id}_{\Gamma_q(H)} \rtimes M_{\varphi_j}) \partial_{\psi,q}(x_n) \rightarrow (\text{Id}_{\Gamma_q(H)} \rtimes M_{\varphi_j})(y) \). Here we use that the mapping

\[
\text{Id}_{\Gamma_q(H)} \rtimes M_{\varphi_j} : L^p(\Gamma_q(H) \rtimes G) \rightarrow L^p(\Gamma_q(H) \rtimes G), \quad f \rtimes \lambda_s \mapsto \varphi_j(s)f \rtimes \lambda_s
\]

is continuous according to Proposition 2.11 and the fact that \( M_{\varphi_j} \) is completely bounded. On the other hand,

\[
\partial_{\psi,q}(M_{\varphi_j}x_n) = \partial_{\psi,q} \left( \sum_{s \in \text{supp}(\varphi_j)} \varphi_j(s)x_{n,s} \lambda_s \right) \overset{(2.49)}{=} \sum_{s \in \text{supp}(\varphi_j)} \varphi_j(s)x_{n,s} s_q(b_{\psi,s}(s)) \rtimes \lambda_s \rightarrow 0,
\]

since for all \( s \in \text{supp}\varphi_j \) we have \( x_{n,s} \rightarrow 0 \) as \( n \rightarrow \infty \). This implies by unicity of the \( L^p(\Gamma_q(H) \rtimes G) \) limit that \( (\text{Id}_{\Gamma_q(H)} \rtimes M_{\varphi_j})(y) = 0 \). Since the sequence \( (M_{\varphi_j}) \) is uniformly bounded in the completely bounded norm over \( L^p(VN(G)) \) and converges strongly to \( \text{Id}_{L^p(VN(G))} \), we infer by
Proposition 2.11 that \((\text{Id}_{\Gamma_p(H)} \times M_{\varphi})\) is uniformly bounded in the completely bounded norm. Moreover, \((\text{Id}_{\Gamma_p(H)} \times M_{\varphi})(x \otimes \lambda_s) = \varphi_s(s) x \otimes \lambda_s \rightarrow x \otimes \lambda_s\) in \(L^p(\Gamma_p(H) \times \alpha, G)\), so that \(\text{Id}_{\Gamma_p(H)} \times M_{\varphi}\) converges strongly to \(\text{Id}_{L_p(V^*\text{VN}(G))}\). We deduce that \(y = \lim \text{Id}_{\Gamma_p(H)} \times M_{\varphi}(y) = 0\). We conclude with [Kat1, page 165].

2. We consider the semigroup \((R_t)_{t \geq 0}\) associated to the negative generator \((\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2}\). Let \(x \in L^p(V^*\text{VN}(G))\) orthogonal to \((\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2}(\text{span}\{\lambda_s : s \in G\})\). For an element \(y \in \text{span}\{\lambda_s : s \in G\}\), using [EnN1, Lemma 1.3 (iv)] in the second equality, we have

\[
\langle R_t(x), y \rangle = \langle x, R_t(y) \rangle = \left( x, x + (\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2} \int_0^t R_s(y) \, ds + y \right) = \left( x, (\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2} \int_0^t R_s(y) \, ds \right) + \langle x, y \rangle = \langle x, y \rangle.
\]

By density, we deduce that \(x = R_t(x)\). We know that \((R_t(x))\) converges to 0 when \(t\) goes to +\(\infty\). Indeed, using [EnN1, page 60] in the second equality, a standard computation gives

\[
\|R_t(x)\|_{L_p(V^*\text{VN}(G))} \leq \int_0^t f_t(s) e^{-\|s\|} \|T_s(x)\|_{L_p(V^*\text{VN}(G))} \, ds \leq \left( \int_0^\infty e^{-\|s\|} f_t(s) \, ds \right) \|x\|_{L_p(V^*\text{VN}(G))} \overset{(3.41)}{=} e^{-t} \|x\|_{L_p(V^*\text{VN}(G))} \underset{t \to +\infty}{\to} 0.
\]

We infer that \(x = 0\). We conclude that 0 is the only element orthogonal to \((\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2}(\text{span}\{\lambda_s : s \in G\})\) as a subset of \(L^p(V^*\text{VN}(G))\). It follows that the subspace \((\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2}(\text{span}\{\lambda_s : s \in G\})\) is dense in \(L^p(V^*\text{VN}(G))\).

3. Let \(x \in \text{dom } A_p^{\frac{1}{2}}\). By [ABHN1, Proposition 3.8.2], we have \(\text{dom } A_p^{\frac{1}{2}} = \text{dom } (\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2}\). Hence \(x \in \text{dom } (\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2}\). From the point 2, there exists a sequence \((x_n)\) of \(\text{span}\{\lambda_s : s \in G\}\) such that \((\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2} x_n \rightarrow (\text{Id}_{L_p(V^*\text{VN}(G))} + A_p)^\frac{1}{2} x\) in \(L^p(V^*\text{VN}(G))\). We obtain that

\[
\|x_n - x\|_{\text{dom } A_p^{\frac{1}{2}}} = \|x_n - x\|_{L_p(V^*\text{VN}(G))} + \|A_p^{\frac{1}{2}}(x_n - x)\|_{L_p(V^*\text{VN}(G))} \leq_p \|\text{Id}_{L_p(V^*\text{VN}(G))} + A_p\|^{\frac{1}{2}}(x_n - x)\|_{L_p(V^*\text{VN}(G))}.
\]

By the choice of the sequence \((x_n)\), the latter tends to 0 and consequently the sequence \((x_n)\) converges to \(x\) in \(\text{dom } A_p^{\frac{1}{2}}\).

4. Let \(x \in \text{dom } A_p^{\frac{1}{2}}\). By the point 3, \(\text{span}\{\lambda_s : s \in G\}\) is dense in \(\text{dom } A_p^{\frac{1}{2}}\) equipped with the graph norm. Hence we can find a sequence \((x_n)\) of \(\text{span}\{\lambda_s : s \in G\}\) converging to \(x\) in \(\text{dom } A_p^{\frac{1}{2}}\). For any integers \(n, m\), we obtain

\[
\|x_n - x_m\|_{L_p(V^*\text{VN}(G))} + \|\partial_{\alpha,q,p}(x_n) - \partial_{\alpha,q,p}(x_m)\|_{L_p(\Gamma_q(H) \times \alpha, G)} \leq_p \|x_n - x_m\|_{L_p(V^*\text{VN}(G))} + \|A_p^{\frac{1}{2}}(x_n) - A_p^{\frac{1}{2}}(x_m)\|_{L_p(V^*\text{VN}(G))},
\]

which shows that \((x_n)\) is a Cauchy sequence in \(\text{dom } \partial_{\alpha,q,p}\). By the closedness of \(\partial_{\alpha,q,p}\), we infer that this sequence converges to some \(x' \in \text{dom } \partial_{\alpha,q,p}\) equipped with the graph norm. Since
both \( \text{dom} \, A^\frac{1}{2}_G \) and \( \text{dom} \, \partial_{\alpha,q,p} \) are continuously embedded into \( L^p(\text{VN}(G)) \), we have \( x_n \to x \) and \( x_n \to x' \) in \( L^p(\text{VN}(G)) \), and therefore \( x = x' \). It follows that \( x \in \text{dom} \, \partial_{\alpha,q,p} \). This proves the inclusion \( \text{dom} \, A^\frac{1}{2}_G \subset \text{dom} \, \partial_{\alpha,q,p} \). Moreover, for any integer \( n \), we have

\[
\|\partial_{\alpha,q,p}(x_n)\|_{L^p(\Gamma_x(H) \times G)} \lesssim_p \|A^\frac{1}{2}_G(x_n)\|_{L^p(\text{VN}(G))}.
\]

Since \( x_n \to x \) in \( \text{dom} \, \partial_{\alpha,q,p} \) and in \( \text{dom} \, A^\frac{1}{2}_G \) both equipped with the graph norm, we conclude that

\[
\|\partial_{\alpha,q,p}(x)\|_{L^p(\Gamma_x(H) \times G)} \lesssim_p \|A^\frac{1}{2}_G(x)\|_{L^p(\text{VN}(G))}.
\]

The proof of the reverse inclusion and of the reverse estimate are similar. Indeed, \( \partial_{\alpha,q,p} \) being the closure of its restriction \( \partial_{\alpha,q} \) to span \( \{\lambda_s : s \in G\} \), by [Ouh1, Definition 1.40 and Proposition 1.39], we see that span \( \{\lambda_s : s \in G\} \) is a dense subspace of \( \text{dom} \, \partial_{\alpha,q,p} \) equipped with the graph norm.

5. Recall that \( b_\psi(e) = 0 \). For any \( s \in G \), we have

\[
(\partial_{\psi,q,p}(\lambda_s))^* = (s_q(b_\psi(s)) \ast \lambda_s)^* = \alpha_s^{-1}(s_q(b_\psi(s))) \ast \lambda_s^{-1} = s_q(b_\psi(s) - b_\psi(s^{-1})) \ast \lambda_s^{-1} = -s_q(b_\psi(s^{-1})) \ast \lambda_s^{-1} = -\partial_{\psi,q,p}(\lambda_s^{-1}) = -\partial_{\psi,q,p}(\lambda_s^*).
\]

The end of the proof is left to the reader. ■

### 3.6 Extension of the carré du champ \( \Gamma \) for Fourier multipliers

By [JR1, Theorem 1.2], if \( G \) is a discrete group with AP and such that \( \text{VN}(G) \) has QWEP, then \( L^p(\text{VN}(G)) \) has the completely contractive approximation property. With the following lemma, one can extend the definition of \( \Gamma(x,y) \) to a larger domain.

**Lemma 3.26** Let \( 2 \leq p < \infty \). Let \( x, y \in \text{dom} \, A^\frac{1}{2}_G \). Assume that \( L^p(\text{VN}(G)) \) has CBAP. Let \( (\varphi_j)_{j \geq 1} \) be the sequence of functions \( \varphi_j : G \to \mathbb{C} \) with compact support converging pointwise to 1 such that \( \sup_j \|M_{\varphi_j}\|_{L^p(\text{VN}(G))} < \infty \). Then the limit \( \lim_{j \to +\infty} \Gamma(M_{\varphi_j}, x, M_{\varphi_j}, y) \) exists in \( L^\frac{p}{2}(\text{VN}(G)) \). We denote it by \( \Gamma(x,y) \). Note that \( \Gamma(x,y) \) is unambiguously defined for \( x \) or \( y \) belonging to span \( \{\lambda_s : s \in G\} \). Also changing the sequence \( \varphi_j \) does not alter the definition of \( \Gamma(x,y) \) either. We have \( \Gamma(x,x) \geq 0 \), so that \( \Gamma(x,x)^\frac{1}{2} \) is well-defined.

**Proof**: We recall that for an element \( z \in L^1(\text{VN}(G)) \), we have \( z \in L^p(\text{VN}(G)) \) iff \( \sup_p \|M_{\varphi_j}z\|_p < \infty \), and in this case, \( \|z\|_p \approx \sup_p \|M_{\varphi_j}z\|_p \). Then, according to [JMP2, Theorem A2] and the
proof of Proposition 5.29 below, we have
\[
\|\Gamma(M_\varphi, x, M_\varphi, y) - \Gamma(M_\varphi, x, M_\varphi, y)\|_p \leq \|\Gamma(M_\varphi, x, M_\varphi, y) - \Gamma(M_\varphi, x, M_\varphi, y)\|_p + \|\Gamma(M_\varphi, x, M_\varphi, y) - \Gamma(M_\varphi, x, M_\varphi, y)\|_p
\]
\[
= \|\Gamma(M_\varphi, x, M_\varphi, y - M_\varphi, y)\|_p + \|\Gamma(M_\varphi, x, M_\varphi, y)\|_p
\]
\[
\leq \|\Gamma(M_\varphi, x, M_\varphi, y - M_\varphi, y)\|_p + \|\Gamma(M_\varphi, x, M_\varphi, y - M_\varphi, y)\|_p
\]
\[
= \|A^{\dagger}M_\varphi, x\|_p + \|\Gamma(M_\varphi, x - M_\varphi, x)\|_p + \|\Gamma(M_\varphi, y - M_\varphi, y)\|_p
\]
\[
\leq \|A^{\dagger}x\|_p + \|\Gamma(M_\varphi, x - M_\varphi, x)\|_p + \|\Gamma(M_\varphi, y - M_\varphi, y)\|_p
\]
Note that since \(A^{\dagger}y\) belongs to \(L_p(\text{VN}(G))\), \((M_\varphi, A^{\dagger}y)\) is a Cauchy sequence in \(L_p(\text{VN}(G))\).

Reasoning in the same way for \(A^{\dagger}x\) belonging to \(L_p(\text{VN}(G))\), we obtain that \(\Gamma(M_\varphi, x, M_\varphi, y)\) is a Cauchy sequence in \(L_p(\text{VN}(G))\). Moreover, since \(\Gamma(M_\varphi, x, M_\varphi, x)\) is a positive operator, its limit in \(L_p(\text{VN}(G))\) is again positive. Note that if \(M_\varphi\) is a second sequence of such Fourier multipliers, then \(\|M_\varphi, x - M_\varphi, x\|_p \to 0\) for any \(x \in L_p(\text{VN}(G))\). Thus, by a similar calculation to the above,
\[
\|\Gamma(M_\varphi, x, M_\varphi, y) - \Gamma(M_\varphi, x, M_\varphi, y)\|_p \leq \|A^{\dagger}x\|_p + \|\Gamma(M_\varphi, x - M_\varphi, x)\|_p + \|\Gamma(M_\varphi, y - M_\varphi, y)\|_p \to 0
\]
as \(j \to \infty\). We infer that \(\lim_j \Gamma(M_\varphi, x, M_\varphi, y) = \lim_j \Gamma(M_\varphi, x, M_\varphi, y)\). By choosing now any \(x, y\) belonging to \(\text{span} \{\lambda_s: s \in G\}\) the sequence \(M_\varphi\) such that \(M_\varphi, x = x = y\) and/or \(M_\varphi, y = y\) (i.e. \(\varphi(s) = 1\) for all \(s \in \text{supp}(f)\) and/or \(s \in \text{supp}(g)\), where \(x = \hat{f}\) resp. \(y = \hat{g}\)), we finally obtain the statement of non-ambiguity.}

**Lemma 3.26** above gives an extension of \(\Gamma: \text{span} \{\lambda_s: s \in G\} \times \text{span} \{\lambda_s: s \in G\} \to L_p^2(\text{VN}(G))\) to \(\Gamma: \text{dom}(A^{\dagger}) \times \text{dom}(A^{\dagger}) \to L_p^2(\text{VN}(G))\). We show now that it is the smallest reasonable extension.

Let now \(X = L_p^2(\text{VN}(G))\) for some \(p \geq 2\), \(D = \text{span} \{\lambda_s: s \in G\} \times \text{span} \{\lambda_s: s \in G\}\) and \(a: \text{span} \{\lambda_s: s \in G\} \times \text{span} \{\lambda_s: s \in G\} \to L_p^2(\text{VN}(G))\) defined by \(a(x, y) = \Gamma(x, y)\) for \((x, y) \in \text{dom}(A^{\dagger}) \times \text{dom}(A^{\dagger})\). Then \(a\) is a sesquilinear form. According to Lemma 2.13, \(\Gamma(x, y) = \Gamma(y, x)\), so that \(a\) is symmetric. Moreover, again according to Lemma 2.13, \(a\) is non-negative.

**Lemma 3.27** Assume that \(L_p^2(\text{VN}(G))\) has CBAP. The form
\[
a: \text{span} \{\lambda_s: s \in G\} \times \text{span} \{\lambda_s: s \in G\} \to L_p^2(\text{VN}(G)) \oplus L_p^2(\text{VN}(G))
\]
above is closable and its closure is given by
\[
\pi: \text{dom}(A^{\dagger}) \times \text{dom}(A^{\dagger}) \to L_p^2(\text{VN}(G)) \oplus L_p^2(\text{VN}(G)), \quad (x, y) \mapsto \Gamma(x, y) \oplus \Gamma(y, x),
\]
where in the last expression, \(\Gamma(x, y)\) and \(\Gamma(y, x)\) are defined by Lemma 3.26.

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Proof : Note first that $\pi$ is non-negative according to Lemma 3.26. We show that $\pi$ is closed.

To this end, let $(x_n)_n$ be a sequence in $\text{dom}(A^*_p)$ such that $x_n \to x$ in $L^p(\text{VN}(G))$ and $\pi(x_n - x_m, x_n - x_m) \to 0$ as $n, m \to \infty$ in $L^2(\text{VN}(G))$. Then using [JMP2, Theorem A2] in the fifth line, we have

$$
\|\pi(x_n - x_m, x_n - x_m)\|_p = \|\Gamma(x_n - x_m, x_n - x_m) \oplus \Gamma(x_n^* - x_m^*, x_n^* - x_m^*)\|_p
$$

$$
\leq \|\Gamma(x_n - x_m, x_n - x_m)\|_p + \|\Gamma(x_n^* - x_m^*, x_n^* - x_m^*)\|_p
$$

$$
= \lim_k \|\Gamma(M_{\phi_k}(x_n - x_m), M_{\phi_k}(x_n - x_m))\|_p + \|\Gamma(M_{\phi_k}(x_n^* - x_m^*), M_{\phi_k}(x_n^* - x_m^*))\|_p
$$

$$
\leq \lim_k \|A^*_p M_{\phi_k}(x_n - x_m)\|_p^2 = \|A^*_p(x_n - x_m)\|_p^2 \to 0
$$

as $n, m \to \infty$. Thus, $(x_n)$ is a Cauchy sequence in $\text{dom}(A^*_p)$ with respect to the graph norm of $A^*_p$. By closedness of the Fourier multiplier $A^*_p$, $x$ belongs to $\text{dom}(A^*_p)$ and $A^*_p x_n \to A^*_p x$ in $L^p(\text{VN}(G))$. Then with a similar calculation as above, $\|A^*_p(x_n - x)\|_p^2 \leq \|\pi(x_n - x, x_n - x)\|_p \to 0$ as $n \to \infty$. Thus, $x$ is the $\pi$-limit of $(x_n)_n$. We infer that $\pi$ is closed.

Let us show that $\pi$ is the smallest closed extension of $a$. To this end, let $x \in \text{dom}(A^*_p)$. We show that there exists a sequence $(x_n)_n$ in span $\{\lambda_s : s \in G\}$ converging to $x$. Simply let $x_n = \phi_{\lambda_s} x$. Then $\phi_{\lambda_s} x \to x$ in $L^p(\text{VN}(G))$. Moreover,

$$
\|a(M_{\phi_s} x - M_{\phi_{n_s}} x, M_{\phi_s} x - M_{\phi_{n_s}} x)\|_p
$$

$$
\leq \|\Gamma(M_{\phi_s} x - M_{\phi_{n_s}} x, M_{\phi_s} x - M_{\phi_{n_s}} x)\|_p + \|\Gamma(M_{\phi_s} x - M_{\phi_{n_s}} x, M_{\phi_s} x - M_{\phi_{n_s}} x)\|_p
$$

$$
\leq \|A^*_p M_{\phi_s}(x_n - x_m)\|_p^2 \to 0
$$

as $n, m \to \infty$. We have shown that $(M_{\phi_s} x)_n$ is $a$-convergent to $x$. Thus, $x \in \text{dom}(A^*_p)$ has to belong to the domain of any closed extension of $a$. 

\[ \square \]

Lemma 3.28 Suppose $1 \leq p < \infty$. Let $G$ be a discrete group and assume that $L^{2p}(\text{VN}(G))$ has CCAP. Consider the completely contractive Fourier multipliers approximating the identity $M_{\phi_j} : L^{2p}(\text{VN}(G)) \to L^{2p}(\text{VN}(G))$ from Lemma 2.12. If $x \in \text{dom}(A^*_p)$, then

$$
\|\Gamma(M_{\phi_j}(x), M_{\phi_j}(x))\|_{L^p(\text{VN}(G))} \leq \|\Gamma(x, x)\|_{L^p(\text{VN}(G))}, \quad j \in \mathbb{N}.
$$

Proof : By Lemma 3.26, we have $\Gamma(x, x) = \lim_k \Gamma(M_{\phi_k} x, M_{\phi_k} x)$ in $L^p(\text{VN}(G))$, so also $\Gamma(M_{\phi_j} x, M_{\phi_j} x) = \lim_k \Gamma(M_{\phi_j} M_{\phi_k} x, M_{\phi_j} M_{\phi_k} x)$. According to Lemma 3.30, we have for $y \in \text{span} \{\lambda_s : s \in G\}$ that $\Gamma(y, y) = \int_\Omega \frac{1}{2} \partial_s(y)^T \partial_s(y) \, d\mu$, so that $\|\Gamma(M_{\phi_j} M_{\phi_k} x, M_{\phi_j} M_{\phi_k} x)\|_p = \|\hat{M}_{\phi_j} \partial_s \hat{M}_{\phi_k} x\|_{L^{2p}(\text{VN}(G), L^2(\Omega_{c,p}))}^2$ and also $\|\Gamma(M_{\phi_j} x, M_{\phi_j} x)\|_p = \|\partial_s M_{\phi_j} x\|_{L^{2p}(\text{VN}(G), L^2(\Omega_{c,p}))}^2$, where $\hat{M}_{\phi_j} : L^{2p}(\text{VN}(G), L^2(\Omega_{c,p})) \to L^{2p}(\text{VN}(G), L^2(\Omega_{c,p}))$, $x \otimes f \mapsto M_{\phi_j} x \otimes f$. Since the norm of $L^{2p}(\text{VN}(G), L^2(\Omega_{c,p}))$ is inherited from the norm of $S_{2p}^2(L^{2p}(\text{VN}(G)))$, and $M_{\phi_j}$ is a complete contraction, by [Pis5, Lemma 1.7], $\hat{M}_{\phi_j}$ is also a (complete) contraction. Thus, we deduce

$$
\|\hat{M}_{\phi_j} \partial_s \hat{M}_{\phi_k} x\|_{L^{2p}(\text{VN}(G), L^2(\Omega_{c,p}))} \leq \|\partial_s \hat{M}_{\phi_k} x\|_{L^{2p}(\text{VN}(G), L^2(\Omega_{c,p}))},
$$

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so by the above that \( \|\Gamma(M_{\varphi}, M_{\varphi} x, M_{\varphi} x)\|_p \leq \|\Gamma(M_{\varphi} x, M_{\varphi} x)\|_p \). Now we conclude letting \( k \to \infty \).

**Lemma 3.29** Let \( \pi : G \to O(H) \) be an orthogonal representation and \( \alpha : G \to L^\infty(\Omega) \), \( s \mapsto \Gamma_s \) the induced action. Then for \( 2 \leq p \leq \infty \), the natural mapping

\[
\begin{align*}
&\left\{ L^p(L^\infty(\Omega) \times_\alpha G) \to L^p(VN(G), L^2(\Omega)_{c.p}) \right. \\
&x \times \lambda_s \quad \mapsto \lambda_s \otimes x
\end{align*}
\]

is contractive.

**Proof:** We start with the case \( p = 2 \). Pick some element \( \sum_{k=1}^n x_k \times \lambda_{s_k} \in L^2(L^\infty(\Omega) \times_\alpha G) \). We can and do assume that \( s_k \neq s_l \) for \( k \neq l \). Then invoking the rules of computation (2.35) and (2.36), we have

\[
\left\| \sum_k x_k \times \lambda_{s_k} \right\|_{L^2(L^\infty(\Omega) \times_\alpha G)}^2 = \tau_{L^\infty(\Omega) \times_\alpha G} \left( \sum_{k,l} (x_k \times \lambda_{s_k})^* (x_l \times \lambda_{s_l}) \right)
\]

\[
= \tau_{L^\infty(\Omega) \times_\alpha G} \left( \sum_{k,l} (\alpha_{s_k^{-1}}(x_k^*) \times \lambda_{s_k^{-1}}) (x_l \times \lambda_{s_l}) \right) = \tau_{L^\infty(\Omega) \times_\alpha G} \left( \sum_{k,l} \alpha_{s_k^{-1}}(x_k^*) \alpha_{s_l^{-1}}(x_l) \times \lambda_{s_k^{-1}} \lambda_{s_l} \right)
\]

\[
= \tau_{L^\infty(\Omega) \times_\alpha G} \left( \sum_{k,l} \alpha_{s_k^{-1}}(x_k^* x_l) \times \lambda_{s_k^{-1}} \lambda_{s_l} \right) = \int_{\Omega} \sum_{k,l} \alpha_{s_k^{-1}}(x_k^* x_l) \delta_{k=l} \, d\mu
\]

\[
= \int_{\Omega} \sum_k \alpha_{s_k^{-1}}(x_k^* x_k) \, d\mu = \sum_k \left\| \alpha_{s_k^{-1}}(\|x_k\|_2^2) \right\|_{L^1(\Omega)} = \sum_k \|x_k^2\|_{L^1(\Omega)} = \sum_k \|x_k\|_{L^2(\Omega)}^2,
\]

where we used in the last line the fact that \( \alpha_{s_k^{-1}} \) is an isometry on the \( L^2 \) level according to \([J\text{an}1, \text{Theorem } 4.11]\). On the other hand, we calculate according to (2.20)

\[
\left\| \sum_k \lambda_{s_k} \otimes x_k \right\|_{L^2(VN(G), L^2(\Omega)_{c.p})}^2 = \left\| \left( \sum_{k,l} (x_k, x_l)_{L^2(\Omega)} \lambda_{s_k^{-1}} \lambda_{s_l^{-1}} \right) \right\|_{L^2(VN(G))}^2
\]

\[
= \tau_{VN(G)} \left( \sum_{k,l} (x_k, x_l)_{L^2(\Omega)} \lambda_{s_k^{-1}} \lambda_{s_l^{-1}} \right) = \sum_{k,l} (x_k, x_l)_{L^2(\Omega)} \delta_{k=l} = \sum_k \|x_k\|_{L^2(\Omega)}^2.
\]

In the case \( p = 2 \), we deduce that \( L^2(L^\infty(\Omega) \times_\alpha G) \) and \( L^2(VN(G), L^2(\Omega)_{c.p}) \) are isometric.

We turn to the case \( p = \infty \). On the one hand, we have, using again the rules of computation

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(2.35) and (2.36) and the embedding $L^\infty(\Omega) \times_{\alpha} G \hookrightarrow L^\infty(\Omega) \otimes B(L^2(G))$

\[
\left\| \sum_{k} x_k \times \lambda_{s_k} \right\|_{L^\infty(\Omega) \otimes_{\alpha} G} = \left\| \sum_{k,l} \alpha_{s_k}^{-1}(x_k^* x_l) \times \lambda_{s_k}^{-1} s_l \right\|_{L^\infty(\Omega) \otimes_{\alpha} G} \leq \left( \sum_{k,l} \left\| \sum_{h \in G} \alpha_{s_k}^{-1}(x_k^* x_l) \otimes e_{h, s_k^{-1} s_l \cdot h} \right\|_{L^\infty(\Omega) \otimes B(L^2(G))} \right)^{\frac{1}{2}}
\]

\[
= \sup_{\|\xi\|_{L^2(\Omega)}, \|\eta\|_{L^2(\Omega)} \leq 1} \left\| \sum_{k,l} \sum_{h \in G} \alpha_{s_k}^{-1}(x_k^* x_l) \langle e_{h, s_k^{-1} s_l \cdot h}(\xi), \eta \rangle_{L^2(G)} \right\|_{L^\infty(\Omega)} \leq \left( \sum_{k,l} \left\| \sum_{h \in G} \alpha_{s_k}^{-1}(x_k^* x_l) \xi(s_l^{-1} h) \eta(s_k^{-1} h) \right\|_{L^\infty(\Omega)} \right)^{\frac{1}{2}}
\]

On the other hand, we calculate with (2.20)

\[
\left\| \sum_{k} \lambda_{s_k} \otimes x_k \right\|_{L^\infty(\text{VN}(G), L^2(\Omega)_{c, \infty})} = \left\| \sum_{k,l} (x_k, x_l)_{L^2(\Omega)} \lambda_{s_k}^{-1} s_l \right\|_{\text{VN}(G)} \leq \left( \sum_{k,l} \left\| \sum_{h \in G} \int_{\Omega} x_k^* x_l \, d\mu \xi(s_l^{-1} h) \eta(s_k^{-1} h) \right\|_{L^\infty(\Omega)} \right)^{\frac{1}{2}}
\]

\[
= \sup_{\|\xi\|_{L^2(\Omega)}, \|\eta\|_{L^2(\Omega)} \leq 1} \left\| \sum_{k,l} \sum_{h \in G} \int_{\Omega} x_k^* x_l \, d\mu \xi(s_l^{-1} h) \eta(s_k^{-1} h) \right\|_{L^\infty(\Omega)} \leq \left( \sum_{k,l} \left\| \sum_{h \in G} \alpha_{s_k}^{-1}(x_k^* x_l) \xi(s_l^{-1} h) \eta(s_k^{-1} h) \right\|_{L^\infty(\Omega)} \right)^{\frac{1}{2}}
\]

\[
= \left\| \sum_{k,l} \sum_{h \in G} \alpha_{s_k}^{-1}(x_k^* x_l) \xi(s_l^{-1} h) \eta(s_k^{-1} h) \right\|_{L^\infty(\Omega)} \leq \left( \sum_{k,l} \left\| \sum_{h \in G} \alpha_{s_k}^{-1}(x_k^* x_l) \xi(s_l^{-1} h) \eta(s_k^{-1} h) \right\|_{L^\infty(\Omega)} \right)^{\frac{1}{2}}
\]

On the other hand, we calculate with (2.20)
where we used that $x^*_k x_l$ and $\alpha_{k-1}(x^*_k x_l)$ have same distribution, hence the same integrals. Comparing the two calculations above yields

$$\left\| \sum_k \lambda x_k \otimes x_k \right\|_{L^\infty(VN(G), L^2(\Omega), c_{\infty})} \leq \left\| \sum_k x_k \otimes \lambda x_k \right\|_{L^\infty(\Omega) \times A}.$$ 

The general case $2 \leq p \leq \infty$ follows by complex interpolation. Indeed, the spaces $L^p(L^\infty(\Omega) \times A)$ interpolate in the usual way by means of complex interpolation, since they are non-commutative $L^p$-spaces. Moreover, $L^p(VN(G), L^2(\Omega), c_{p})$ interpolate in the usual way by means of complex interpolation. Indeed, using Lemma 5.16, it is not difficult to see that they are contractively complemented in $S^p_0(L^p(VN(G)))$ via $x \otimes e_k \mapsto e_{k1} \otimes x$, where $e_k$ is an orthonormal basis of $L^2(\Omega)$ and $e_{k1}$ are elements of the standard orthonormal basis of $S^2_\Omega$. Hence we can invoke [Tri, 1.2.4 Theorem] (see also [ArK1, Lemma 4.8]).

**Lemma 3.30** Suppose $1 \leq p < \infty$.

1. For any $x, y \in \text{span} \{\lambda_s : s \in G\}$, we have

$$\Gamma(x, y) = \langle \partial_{\psi}(x), \partial_{\psi}(y) \rangle_{L^p(VN(G), L^2(\Omega), c_{p})}.$$  

Moreover, if $p \geq 2$, $L^p(VN(G))$ has the CBAP and $L^\infty(\Omega) \times A$ has QWEP, then for any $x, y \in \text{dom} A^\dagger_\psi$ we have

$$\Gamma(x, y) = \langle \partial_{\psi, p}(x), \partial_{\psi, p}(y) \rangle_{L^p(VN(G), L^2(\Omega), c_{p})}.$$ 

2. For any $x \in \text{span} \{\lambda_s : s \in G\}$, we have

$$\left\| \Gamma(x, x) \right\|_{L^p(VN(G))} = \left\| \partial_{\psi, p}(x) \right\|_{L^p(VN(G), L^2(\Omega), c_{p})}.$$ 

Moreover, if $p \geq 2$, $L^p(VN(G))$ has CBAP and $L^\infty(\Omega) \times A$ has QWEP then for any $x \in \text{dom} A^\dagger_\psi$ we have

$$\left\| \Gamma(x, x) \right\|_{L^p(VN(G))} = \left\| \partial_{\psi, p}(x) \right\|_{L^p(VN(G), L^2(\Omega), c_{p})}.$$ 

**Proof:** 1. (3.65) is in the bottom of [JRZ1, page 930]. Assume that $p \geq 2$ and let $x, y \in \text{dom} A^\dagger_\psi$. Recall that $\partial_{\psi, p} : \text{dom}(A^\dagger_\psi) \rightarrow L^p(L^\infty(\Omega) \times A)$ continuously according to Proposition 3.25. According to Lemma 3.29, since $p \geq 2$, we have $L^p(L^\infty(\Omega) \times A) \rightarrow L^p(VN(G), L^2(\Omega), c_{p})$ contractively. Now, since $L^p(VN(G))$ has CBAP, use an approximation $x_j = M_{\phi} x \in \text{span} \{\lambda_s : s \in G\}$ and $y_j = M_{\phi} y \in \text{span} \{\lambda_s : s \in G\}$ so that $x_j \rightarrow x$ and $y_j \rightarrow y$ in $\text{dom} A^\dagger_\psi$. Then according to Lemma 3.26, by the first part of the lemma, $\Gamma(x, y) = \lim_j \Gamma(x_j, y_j) = \lim_j \langle \partial_{\psi, p}(x_j), \partial_{\psi, p}(y_j) \rangle = \langle \partial_{\psi, p}(x), \partial_{\psi, p}(y) \rangle$.

2. (3.66) is in [JRZ1, page 931]. Then the second part follows from (3.66) together with the second part of 1.

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4 Boundedness of $H^\infty$ functional calculus of Hodge-Dirac operators

4.1 Hodge-Dirac operators associated with semigroups of Markov Schur multipliers

Here we consider a weak* continuous semigroup $(T_t)_{t \geq 0}$ of selfadjoint completely positive unital Schur multipliers on $B(\ell_2^p)$. If $1 < p < \infty$, we denote by $A_p$ the (negative) infinitesimal generator on $S^p_t$. By [JMX, (5.2)], we have $(A_{p'})^* = A_p$. For any $i,j \in I$, we have $A_p(e_{ij}) = \|\alpha_i - \alpha_j\|_H^2 e_{ij}$. Recall that by Proposition 3.13, we have a closed operator

$$\partial_{\alpha,q,p} : \text{dom}(\partial_{\alpha,q,p}) \subset S^p_t \to L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^p)), \quad e_{ij} \mapsto s_q(\alpha_i - \alpha_j) \otimes e_{ij}$$

and a closed operator $(\partial_{\alpha,q,p'})^* : \text{dom}(\partial_{\alpha,q,p'})^* \subset L^p(\Gamma_q(H) \overline{\otimes} B(\ell_2^p)) \to S^p_t$. The following proposition will be useful.

**Proposition 4.1** Suppose $1 < p < \infty$.

1. The space $M_{I,\fin}$ is a core of $A_p$.

2. The space $\text{dom} A_p$ is a core for $\partial_{\alpha,q,p}$.

**Proof** : 1. That is a direct consequence of [HvNVW2, Proposition G.2.4].

2. By [ABHN1, Proposition 3.8.2] and the point 4 of Proposition 3.13, dom $A_p$ is a subspace of $\text{dom} \partial_{\alpha,q,p}$. Now, it suffices to use the point 1 of Proposition 3.13 and (2.7).

Now, we give some useful formulas for the adjoint $(\partial_{\alpha,q,p'})^*$.

**Proposition 4.2** Suppose $1 < p < \infty$.

1. For any $i,j \in I$ and any $f \in L^p(\Gamma_q(H))$, we have $f \otimes e_{ij} \in \text{dom}(\partial_{\alpha,q,p'})^*$ and

$$(\partial_{\alpha,q,p'})^*(f \otimes e_{ij}) = \langle s_q(\alpha_i - \alpha_j), f \rangle_{L^{p'}(\Gamma_q(H))} e_{ij}. \quad (4.1)$$

2. As unbounded operators, we have

$$A_p = (\partial_{\alpha,q,p'})^* \partial_{\alpha,q,p}. \quad (4.2)$$

**Proof** : 1. For any $i,j,k,l \in I$ and any $f \in L^p(\Gamma_q(H))$, we have

$$\langle \partial_{\alpha,q,p'}(e_{kl}), f \otimes e_{ij} \rangle_{L^{p'}(\Gamma_q(H) \overline{\otimes} B(\ell_2^p))} = \langle s_q(\alpha_k - \alpha_l) \otimes e_{kl}, f \otimes e_{ij} \rangle_{L^{p'}(\Gamma_q(H) \overline{\otimes} B(\ell_2^p))}, \quad (2.26)$$

$$= \delta_{kl} \delta_{ij} \langle s_q(\alpha_i - \alpha_j), f \rangle_{L^{p'}(\Gamma_q(H))} e_{ij}, \quad (4.1)$$

$$= \langle e_{kl}, \langle s_q(\alpha_i - \alpha_j), f \rangle_{L^{p'}(\Gamma_q(H))} e_{ij} \rangle_{S^p_t} \equiv (\partial_{\alpha,q,p'}(e_{ij})^* f_{kl}). \quad (4.1)$$

We conclude with (2.7), the point 1 of (3.13) and (2.5).

2. Clearly by the point 1, $\partial_{\alpha,q,p}(M_{I,\fin})$ is a subspace of $\text{dom}(\partial_{\alpha,q,p'})^*$. For any $i,j \in I$ we have

$$\langle \partial_{\alpha,q,p'}(e_{ij})^* \partial_{\alpha,q,p}(e_{ij}) \rangle_{L^{p'}(\Gamma_q(H) \overline{\otimes} B(\ell_2^p))} = \langle s_q(\alpha_i - \alpha_j) \otimes e_{ij} \rangle_{L^{p'}(\Gamma_q(H))} e_{ij}, \quad (2.26)$$

$$= \tau(s_q(\alpha_i - \alpha_j) s_q(\alpha_i - \alpha_j)) e_{ij} \equiv \langle \alpha_i - \alpha_j, \alpha_i - \alpha_j \rangle_H e_{ij} = \alpha_i e_{ij} e_{ij} = A_p(e_{ij}). \quad (2.12)$$
Hence for any \( x, y \in M_{I, \text{fin}} \), by linearity we have

\[
\langle A_p(x), y \rangle_{S^p_1, S^{p^*}_1} = \langle (\partial_{\alpha, q, p})^* \partial_{\alpha, q, p}(x), y \rangle_{S^p_1, S^{p^*}_1}^{(2.6)} = \langle \partial_{\alpha, q, p}(x), \partial_{\alpha, q, p}(y) \rangle_{S^p_1, S^{p^*}_1}^{(2.6)}.
\]

Using Proposition 4.1, it is easy to check that this identity extends to all \( x \in \text{dom} A_p \) and all \( y \in \text{dom} \partial_{\alpha, q, p}^* \). By (2.5), this implies that \( \partial_{\alpha, q, p}(x) \in \text{dom}(\partial_{\alpha, q, p}^*)^* \) and that \( (\partial_{\alpha, q, p}^*)^* \partial_{\alpha, q, p}(x) = A_p(x) \). We conclude that \( A_p \subset (\partial_{\alpha, q, p})^* \partial_{\alpha, q, p} \).

To prove the other inclusion we consider some \( x \in \text{dom} \partial_{\alpha, q, p} \) such that \( \partial_{\alpha, q, p}(x) \) belongs to \( \text{dom}(\partial_{\alpha, q, p}^*)^* \). By [Kat1, Problem 5.25 and 5.26], we have \( (\partial_{\alpha, q, p})^* \partial_{\alpha, q, p} \subset (\partial_{\alpha, q, p}^*)^* \partial_{\alpha, q, p}^* \subset A_p^* \). For any \( y \in M_{I, \text{fin}} \), using \( \partial_{\alpha, q, p}(x) \in \text{dom}(\partial_{\alpha, q, p}^*)^* \) in the last equality, we deduce that

\[
\langle x, A_p^*(y) \rangle_{S^p_1, S^{p^*}_1} = \langle x, (\partial_{\alpha, q, p})^* \partial_{\alpha, q, p}(y) \rangle_{S^p_1, S^{p^*}_1}^{(2.6)} = \langle \partial_{\alpha, q, p}(x), \partial_{\alpha, q, p}(y) \rangle_{S^p_1, S^{p^*}_1}^{(2.6)} = \langle (\partial_{\alpha, q, p})^* \partial_{\alpha, q, p}(x), y \rangle_{S^p_1, S^{p^*}_1}^{(2.6)}.
\]

Since \( M_{I, \text{fin}} \) is a core for \( A_p^* \) by Proposition 4.1, this implies that \( x \in \text{dom} A_p^{**} = A_p \) ([Kat1, Th. 5.29, p. 168]) and \( A_p(x) = (\partial_{\alpha, q, p})^* \partial_{\alpha, q, p}(x) \).

**Lemma 4.3** Suppose \( 1 < p < \infty \). If \( x \in \text{dom} \partial_{\alpha, q, p} \) and \( t \geq 0 \) then \( T_{t,p}(x) \) belongs to \( \text{dom} \partial_{\alpha, q, p} \) and we have

\[
(4.3) \quad (\text{Id}_{L^p(\Gamma_q(H))} \otimes T_{t,p}) \partial_{\alpha, q, p}(x) = \partial_{\alpha, q, p} T_{t,p}(x).
\]

**Proof** : For any \( i, j \in I \), we have

\[
(\text{Id}_{L^p(\Gamma_q(H))} \otimes T_{t,p}) \partial_{\alpha, q, p}(e_{ij}) = e^{-||\alpha_i - \alpha_j||^2} \partial_{\alpha, q, p}(e_{ij}) = \partial_{\alpha, q, p} T_{t,p}(e_{ij}).
\]

So by linearity the equality (4.3) is true for elements of \( M_{I, \text{fin}} \). Now consider some \( x \in \text{dom} \partial_{\alpha, q, p} \). By [Kat1, page 166], since \( \partial_{\alpha, q, p} \) is the closure of \( \partial_{\alpha, q} \), \( M_{I, \text{fin}} \subset S^p_1 \to S^p_1 \), there exists a sequence \( (x_n) \) of elements of \( M_{I, \text{fin}} \) converging to \( x \) in \( S^p_1 \) such that the sequence \( (\partial_{\alpha, q, p}(x_n)) \) converges to \( \partial_{\alpha, q, p}(x) \). The complete boundedness of \( T_{t,p} : S^p_1 \to S^p_1 \) implies by [Jun, page 984] that we have a bounded operator \( \text{Id}_{L^p(\Gamma_q(H))} \otimes T_{t,p} : L^p(\Gamma_q(H) \boxtimes B(\ell_q^2)) \to L^p(\Gamma_q(H) \boxtimes B(\ell_q^2)) \).

We infer that in \( S^p_1 \) and \( L^p(\Gamma_q(H) \boxtimes B(\ell_q^2)) \) we have

\[
T_{t,p}(x_n) \quad \text{and} \quad \text{Id}_{L^p(\Gamma_q(H))} \otimes T_{t,p} \partial_{\alpha, q, p}(x_n) \quad \text{converges to} \quad (\text{Id}_{L^p(\Gamma_q(H))} \otimes T_{t,p}) \partial_{\alpha, q, p}(x).
\]

For any integer \( n \), we have \( (\text{Id}_{L^p(\Gamma_q(H))} \otimes T_{t,p}) \partial_{\alpha, q, p}(x_n) = \partial_{\alpha, q, p} T_{t,p}(x_n) \) by the first part of the proof. Since the left-hand side converges, we obtain that the sequence \( (\partial_{\alpha, q, p} T_{t,p}(x_n)) \) converges to \( (\text{Id}_{L^p(\Gamma_q(H))} \otimes T_{t,p}) \partial_{\alpha, q, p}(x) \) in \( L^p(\Gamma_q(H) \boxtimes B(\ell_q^2)) \). Since each \( T_{t,p}(x_n) \) belongs to \( \text{dom} \partial_{\alpha, q, p} \), the closedness of \( \partial_{\alpha, q, p} \) and [Kat1, page 166] show that \( T_{t,p}(x) \) belongs to \( \text{dom} \partial_{\alpha, q, p} \) and that \( \partial_{\alpha, q, p} T_{t,p}(x) = (\text{Id}_{L^p(\Gamma_q(H))} \otimes T_{t,p}) \partial_{\alpha, q, p}(x) \).

**Proposition 4.4** 1. For any \( t > 0 \), we have on \( \text{dom} \partial_{\alpha, q, p} \)

\[
(4.4) \quad (\text{Id}_{L^p(\Gamma_q(H))} \otimes (\text{Id}_{S^p_1} + t^2 A_p)^{-1}) \partial_{\alpha, q, p} = \partial_{\alpha, q, p} (\text{Id}_{S^p_1} + t^2 A_p)^{-1}.
\]
2. For any $t \geq 0$, we have on $\text{dom}(\partial_{\alpha,q,p})^*$

\begin{equation}
(\text{Id}_{S^p_t} + t^2 A_p)^{-1}(\partial_{\alpha,q,p})^* = (\partial_{\alpha,q,p})^*(\text{Id}_{L^p_t(\Gamma_t(H))} \otimes (\text{Id}_{S^p_t} + t^2 A_p)^{-1}).
\end{equation}

\textbf{Proof:} We will only prove the first identity since the second can be proved in a similar manner.

Note that for any $t > 0$ and any $x \in S^p_t$, the function $\mathbb{R}^+ \to S^p_t$, $u \mapsto e^{-ut} T_{u,p}(x)$ is Bochner integrable. If $t > 0$ and if $x \in \text{dom} \partial_{\alpha,q,p}$, taking Laplace transforms on both sides of (4.3) and using [HvNVW2, Theorem 1.2.4] and the closedness of $\partial_{\alpha,q,p}$ in the penultimate equality, we obtain

\begin{align}
(\text{Id}_{L^p_t(\Gamma_t(H))} \otimes (t^{-2} \text{Id}_{S^p_t} + A_p)^{-1})\partial_{\alpha,q,p}(x) &= - \left( - t^{-2} \text{Id}_{L^p_t(\Gamma_t(H))} \otimes (t^{-2} \text{Id}_{S^p_t} + A_p)^{-1} \right) \partial_{\alpha,q,p}(x) \\
&= \int_0^{+\infty} e^{-ut^2} \partial_{\alpha,q,p} \left( \int_0^{+\infty} e^{-ut^2} T_{u,p}(x) du \right) \, du
\end{align}

from which one deduces the desired identity by multiplying by $t^{-2}$.

\textbf{Proposition 4.5} The subspace $\text{dom} B_p$ is a core of the operator $(\partial_{\alpha,q,p})^* \text{Ran} \partial_{\alpha,q,p}$.

\textbf{Proof:} Let $x \in \text{dom}(\partial_{\alpha,q,p})^* \text{Ran} \partial_{\alpha,q,p}$. By [HvNVW2, Proposition G.4.2], we have

\begin{equation}
\lim_{t \to 0} (\text{Id}_{L^p_t(\Gamma_t(H))} \otimes (\text{Id}_{S^p_t} + t^2 A_p)^{-1})(x) = x
\end{equation}

in $\text{Ran} \partial_{\alpha,q,p}$. Moreover, we have

\begin{equation}
\lim_{t \to 0} (\partial_{\alpha,q,p})^*(\text{Id}_{L^p_t(\Gamma_t(H))} \otimes (\text{Id}_{S^p_t} + t^2 A_p)^{-1})(x) = \lim_{t \to 0} (\text{Id}_{S^p_t} + t^2 A_p)^{-1}(\partial_{\alpha,q,p})^* x = (\partial_{\alpha,q,p})^* x
\end{equation}

in $L^p$. We let $B_p \overset{\text{def}}{=} (\text{Id}_{L^p_t(\Gamma_t(H))} \otimes A_p)\text{Ran} \partial_{\alpha,q,p}$. The following proposition is fundamental for the sequel.

\textbf{Proposition 4.6} For any $z \in \text{dom} B_p$, we have $z \in \text{dom}(\partial_{\alpha,q,p})^*$ and $(\partial_{\alpha,q,p})^* z \in \text{dom} \partial_{\alpha,q,p}$. Moreover, we have

\begin{equation}
\partial_{\alpha,q,p}(\partial_{\alpha,q,p})^* z = B_p(z).
\end{equation}

\textbf{Proof:} Since $\text{dom} A_p$ is a core for $\partial_{\alpha,q,p}$ (Proposition 4.1), the set $\mathcal{P} \overset{\text{def}}{=} \{ \partial_{\alpha,q,p}(\text{Id} + A_p)^{-1} x : x \in \text{dom} \partial_{\alpha,q,p}\}$ is a $\text{Id} \otimes T_{t,p}$-invariant dense subspace of $\text{Ran} \partial_{\alpha,q,p}$. To see that $\mathcal{P}$ is contained in $\text{dom} A_p$, note that if $x \in \text{dom} \partial_{\alpha,q,p}$ then $f \overset{\text{def}}{=} (\text{Id} + A_p)^{-1} x \in \text{dom} A_p$ and

\begin{equation}
\partial_{\alpha,q,p}(f) = \partial_{\alpha,q,p}(\text{Id} + A_p)^{-1} x \overset{(4.4)}{=} (\text{Id} + B_p)^{-1} \partial_{\alpha,q,p}(x) \in \text{dom} B_p
\end{equation}

as claimed. It follows that $\mathcal{P}$ is a core for $\text{Id} \otimes A_p$ and hence a core for $\text{dom}(\partial_{\alpha,q,p})^* \text{Ran} \partial_{\alpha,q,p}$ by Proposition 4.5. Moreover, we have

\begin{equation}
(\text{Id} + B_p)\partial_{\alpha,q,p}(f) \overset{(4.7)}{=} \partial_{\alpha,q,p}(f) = \partial_{\alpha,q,p}(\text{Id} + A_p)(f)
\end{equation}

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and therefore \( B_p \partial_{\alpha,q,p}(f) = \partial_{\alpha,q,p}A_p(f) \).

For \( f \in \mathcal{P} \), say \( f = \partial_{\alpha,q,p}(y) \) with \( y = (\text{Id} + A_p)^{-1}(x) \) for some \( x \in \text{dom} \partial_{\alpha,q,p} \), we then have

\[
B_p(f) = B_p \partial_{\alpha,q,p}(y) \quad (4.4)
\]

\[
\text{dom}(B_p) \quad (4.8)
\]

To see that this above identity extends to arbitrary \( f \in \text{dom} B_p \), let \( f_n \rightarrow f \) for the graph norm of \( B_p \) with all \( f_n \) in \( \mathcal{P} \) (since \( \mathcal{P} \) is a core for \( B_p \)). It follows from Proposition 4.5 that \( f_n \rightarrow f \) for the graph norm of \( (\partial_{\alpha,q,p})^* \). In particular, we have in \( L^p \)

\[
(\partial_{\alpha,q,p})^* f_n \longrightarrow (\partial_{\alpha,q,p})^* f
\]

Since \( \partial_{\alpha,q,p}(\partial_{\alpha,q,p})^*(f_n) = B_p(f_n) \rightarrow B_p(f) \) in \( \text{Ran} \partial_{\alpha,q,p} \), the closedness of \( \partial_{\alpha,q,p} \) then implies that \( (\partial_{\alpha,q,p})^*(f) \in \text{dom} \partial_{\alpha,q,p} \) and \( \partial_{\alpha,q,p}(\partial_{\alpha,q,p})^*(f) = B_p(f) \). \( \blacksquare \)

We will use the restriction \( (\partial_{\alpha,q,p})^* \text{Ran} \partial_{\alpha,q,p} \) of \( (\partial_{\alpha,q,p})^* \) to \( \text{Ran} \partial_{\alpha,q,p} \). Its domain is \( \text{dom}((\partial_{\alpha,q,p})^*) \cap \text{Ran} \partial_{\alpha,q,p} \). The following proposition is left to the reader.

**Proposition 4.7** The restriction \( (\partial_{\alpha,q,p})^* \text{Ran} \partial_{\alpha,q,p} \) is closed and densely defined.

Suppose \( 1 \leq p < \infty \). We introduce the unbounded operator

\[
D_{\alpha,q,p} \overset{\text{def}}{=} \begin{bmatrix} 0 & (\partial_{\alpha,q,p})^* \\ \partial_{\alpha,q,p} & 0 \end{bmatrix}
\]

on the Banach space \( S^p_I \otimes_p \text{Ran} \partial_{\alpha,q,p} \) defined by

\[
D_{\alpha,q,p}(x,y) \overset{\text{def}}{=} ((\partial_{\alpha,q,p})^*(y), \partial_{\alpha,q,p}(x)), \quad x \in \text{dom} \partial_{\alpha,q,p}, \ y \in \text{dom}(\partial_{\alpha,q,p})^* \cap \text{Ran} \partial_{\alpha,q,p}.
\]

This operator is a closed operator and can be seen as a differential square root of the generator of the semigroup \((T_{t,p})_{t \geq 0}\). We call it the Hodge-Dirac operator of the semigroup since we have the following obvious result.

**Proposition 4.8** Suppose \( 1 \leq p < \infty \). We have

\[
D_{\alpha,q,p}^2 = \begin{bmatrix} A_p & 0 \\ 0 & (\text{Id}_{L^p(I_q(H))} \otimes A_p)|\text{Ran} \partial_{\alpha,q,p} \end{bmatrix}
\]

Now, we give a result which gives some \( R \)-boundedness.

**Proposition 4.9** Suppose \( 1 < p < \infty \). The family

\[
\left\{ t\partial_{\alpha,q,p}(\text{Id} + t^2 A_p)^{-1} : t > 0 \right\}
\]

of operators of \( B(S^p_I, L^p(I_q(H) \otimes B(I_f))) \) is \( R \)-bounded.

**Proof** : Note that the operator \( \partial_{\alpha,p}A_p^{\frac{1}{2}} : S^p_{I,0} \rightarrow L^p(I_q(H) \otimes B(I_f)) \) is bounded by (3.10). Suppose \( t > 0 \). A standard functional calculus argument gives

\[
t\partial_{\alpha,q,p}(\text{Id} + t^2 A_p)^{-1} = \partial_{\alpha,q,p}A_p^{-\frac{1}{2}} \left( (t^2 A_p)^{\frac{1}{2}}(\text{Id} + t^2 A_p)^{-1} \right).
\]

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By [Arh7], note that $A_p$ has a bounded $H^\infty(\Sigma_d)$-functional calculus with $\theta < \frac{\pi}{2}$. Moreover, by [HvNVW2, page 136] the Banach space $S^p_\theta$ has the triangular contraction property ($\Delta$). We deduce by [HvNVW2, Theorem 10.3.4] that the operator $A_p$ is $R$-sectorial. By [HvNVW2, Example 10.3.5] applied with $\alpha = \frac{1}{2}$ and $\beta = 1$, we infer that the set

$$\{(t^2 A_p)^{\frac{1}{2}} (\text{Id} + t^2 A_p)^{-1} : t > 0\}$$

of operators of $B(S^p_\theta)$ is $R$-bounded. Combining [HvNVW2, Example 8.1.7] and [HvNVW2, Proposition 8.1.19], we obtain that the set

$$\left\{ \partial_{\alpha,q,p} A_p^{\frac{1}{2}} \left( (t^2 A_p)^{\frac{1}{2}} (\text{Id} + t^2 A_p)^{-1} \right) : t > 0 \right\}$$

of operators of $B(S^p_\theta, \mathcal{L}^p(\Gamma_q(H) \otimes B(\ell^2_\theta)))$ is $R$-bounded. Hence with (4.12) we conclude that $A_p$ has a two-sided bounded inverse $S^p_\theta \oplus \text{Ran} \partial_{\alpha,q,p}$.

**Theorem 4.10** Suppose $1 < p < \infty$ and $-1 \leq q \leq 1$. The Hodge-Dirac operator $D_{\alpha,q,p}$ is $R$-bisectorial on $S^p_\theta \oplus \text{Ran} \partial_{\alpha,q,p}$.

**Proof**: We will start by showing that the set $\{it : t \in \mathbb{R}, t \neq 0\}$ is contained in the resolvent set of $D_{\alpha,q,p}$. We will do this by showing that $\text{Id} - itD_{\alpha,q,p}$ has a two-sided bounded inverse $(\text{Id} - itD_{\alpha,q,p})^{-1}$ given by

$$\begin{bmatrix}
(\text{Id} + t^2 A_p)^{-1} & \text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} \\
\text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} & (\text{Id} + t^2 A_p)^{-1}
\end{bmatrix}
$$

By Proposition 4.9 and since the operator $A_p$ satisfies the property (2.3) of $R$-sectoriality, the four entries are bounded. It only remains to check that this matrix defines a two-sided inverse of $\text{Id} - itD_{\alpha,q,p}$. We have

\begin{align*}
&\begin{bmatrix}
(\text{Id} + t^2 A_p)^{-1} & \text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} \\
\text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} & (\text{Id} + t^2 A_p)^{-1}
\end{bmatrix} \\
&= \begin{bmatrix}
(\text{Id} + t^2 A_p)^{-1} & \text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} \\
\text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} & (\text{Id} + t^2 A_p)^{-1}
\end{bmatrix} \\
&= \begin{bmatrix}
(\text{Id} + t^2 A_p)^{-1} & \text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} \\
\text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} & (\text{Id} + t^2 A_p)^{-1}
\end{bmatrix} \\
&= \begin{bmatrix}
\text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} & \text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} \\
\text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} & (\text{Id} + t^2 A_p)^{-1}
\end{bmatrix} \\
&= \begin{bmatrix}
\text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} & \text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} \\
\text{Id} \otimes (\text{Id} + t^2 A_p)^{-1} & (\text{Id} + t^2 A_p)^{-1}
\end{bmatrix}
\end{align*}
and similarly
\[
(Id - it\alpha(q,p)) \left[ (Id\alpha(q,p) + t^2A_p)^{-1} \right] = \left[ \begin{array}{ccc} \text{Id} & -it(\alpha(q,p))^{-1} & 0 \\ -it(\alpha(q,p))^{-1} & \text{Id} & 0 \\ 0 & 0 & \text{Id} \end{array} \right]
\]

It remains to show that the set \{\{it - D\alpha(q,p)^{-1} : t \neq 0\} = \{\{it - D\alpha(q,p)^{-1} : t \neq 0\} is \(R\)-bounded. For this, observe that the diagonal entries of \(\alpha\) are \(R\)-bounded by the \(R\)-sectoriality of \(A\).

Remark 4.13 The boundedness of the \(R\)-sectoriality of the other entries follows from the \(R\)-sectoriality of Proposition 4.9. Since a set of operators matrices is \(R\)-bounded precisely when each entry is \(R\)-bounded, we conclude that \(2.2\) is satisfied, i.e. that \(D\) is \(R\)-sectorial.

Note the results of [Arh7] gives the following result.

Proposition 4.11 Suppose \(1 < p < \infty\). The operators \(A\) and \(B\) have a bounded \(\text{H}^\infty(\Sigma_{\omega})\) functional calculus of angle \(\omega\) for some \(\omega < \frac{\pi}{2}\).

Now, we can state the following result.

Theorem 4.12 Suppose \(1 < p < \infty\). The Hodge-Dirac operator \(D\alpha(q,p)\) has a bounded \(\text{H}^\infty(\Sigma_{\omega}^\pm)\) functional calculus on a bisector.

Proof: By proposition 4.11, the operator \(D\alpha(q,p)\) has a \(\text{H}^\infty(\Sigma_{\omega})\) functional calculus of angle \(2\omega < \frac{\pi}{2}\). Since \(D\alpha(q,p)\) is \(R\)-sectorial by Proposition 4.10, we deduce by Proposition 2.1 that the operator \(D\alpha(q,p)\) has a bounded \(\text{H}^\infty(\Sigma_{\omega}^\pm)\) functional calculus on a bisector.

Remark 4.13 The boundedness of the \(\text{H}^\infty\) functional calculus of the operator \(D\alpha(q,p)\) implies the boundedness of the Riesz transforms and this result may be thought of as a strengthening of the equivalence (3.32). Indeed, consider the function \(\text{sgn} \in \text{H}^\infty(\Sigma_{\omega})\) defined by \(\text{sgn}(z) \triangleq 1_{\Sigma^+_{\omega}}(z) - 1_{\Sigma^-_{\omega}}(z)\). By Theorem 4.12, the operator \(D\alpha(q,p)\) has a bounded \(\text{H}^\infty(\Sigma_{\omega}^\pm)\) functional calculus on \(S^p_t \oplus \text{Ran}(\alpha(q,p))\). Hence the operator \(\text{sgn}(D\alpha(q,p))\) is bounded. This implies that
\[
(D\alpha(q,p)^2) = \text{sgn}(D\alpha(q,p))D\alpha(q,p) \quad \text{and} \quad D\alpha(q,p) = \text{sgn}(D\alpha(q,p))(D\alpha(q,p)^2).
\]

For any \(x \in \text{dom}(D\alpha(q,p))\), we deduce that
\[
\|D\alpha(q,p)(x)\|_{S^p_t \oplus \text{L}^p(\Gamma_q(H)\text{B}(t^2))} \leq_p \|(D\alpha(q,p)^2)^\star(x)\|_{S^p_t \oplus \text{L}^p(\Gamma_q(H)\text{B}(t^2))} \leq_p \|(D\alpha(q,p)^2)^\star(x)\|_{S^p_t \oplus \text{L}^p(\Gamma_q(H)\text{B}(t^2))}
\]

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Consider the mappings

\[ \text{Lemma 4.16} \]

We have

\[ \text{and} \]

\[ \|D_{\alpha,q,p}^2(x)\|_{L^p(\Gamma(H)\otimes B(\ell_j^2))} \leq \|\text{sgn}(D_{\alpha,q,p})D_{\alpha,q,p}(x)\|_{L^p(\Gamma(H)\otimes B(\ell_j^2))}. \]

Recall that on $S_1^p \oplus_p \text{Ran} \partial_{\alpha,q,p}$, we have

\[ (D_{\alpha,q,p}^2) = (4.10) \left[ \begin{array}{cc} A_p^\pm & 0 \\ 0 & \text{Id}_{L^p(\Gamma(H))} \otimes A_p^\pm \end{array} \right]. \]

By restricting to elements of the form $(y,0)$ with $y \in \text{dom} A_p^\pm$, we obtain the desired result.

**Proposition 4.14** We have \( \text{Ran} A_p = (\text{Ran}(\partial_{\alpha,q,p})^\ast) \), \( \text{Ran} B_p = \text{Ran} \partial_{\alpha,q,p}, \) \( \text{Ker} A_p = \text{Ker} \partial_{\alpha,q,p} \) and \( \text{Ker} D_{\alpha,q,p} = \text{Ker} \partial_{\alpha,q,p} \).

**Proof:** Recall that [HvNVW2, Proposition 10.6.2 (2)], we have \( \text{Ran} D_{\alpha,q,p}^2 = \text{Ran} \partial_{\alpha,q,p} \) and \( \text{Ker} D_{\alpha,q,p} = \text{Ker} \partial_{\alpha,q,p} \). It is not difficult to prove the first four equalities using (4.8) and (4.9). The last one is a consequence of the definition of \( B_p \) and of [HvNVW2, page 361].

The rest of the section is devoted to the proof of Theorem 4.19. We show several intermediate lemmas before.

**Lemma 4.15** Let \( H \) be a Hilbert space and \( i_1: H_1 \subset H \) an embedding of a sub Hilbert space \( H_1 \). Moreover, let \( I \) be an index set and \( I_1 \subset I \) a subset. Consider the mapping

\[ J_1: \begin{cases} \Gamma_q(H_1) \otimes B(\ell_1^2) & \rightarrow \Gamma_q(H) \otimes B(\ell_1^2) \\ a \otimes x & \rightarrow \Gamma_q(i_1)(a) \otimes j_1(x) \end{cases}, \]

where \( j_1: B(\ell_1^2) \rightarrow B(\ell_1^2), \) \( x \mapsto P_1^*xP_1 \) and \( P_1: \ell_1^2 \rightarrow \ell_1^2, (\xi_i)_{i \in I} \mapsto (\xi_i\delta_{i \in I_1})_{i \in I} \) is the canonical orthogonal projection. Then \( J_1 \) is a normal faithful trace preserving \(*\)-homomorphism and thus extends to a complete isometry on the \( L^p \) level, \( 1 \leq p \leq \infty \).

**Proof:** According to [BKS, Theorem 2.11], since \( i_1 \) is an isometric embedding, \( \Gamma_q(i_1) \) is a faithful \(*\)-homomorphism which preserves the traces. According to [JMX, p. 97], \( \Gamma_q(i_1) \) is normal. Moreover, it is easy to check that \( j_1 \) is also a normal faithful \(*\)-homomorphism. Thus also \( J_1 \) is a normal faithful \(*\)-homomorphism, see also [Pis7, p. 32]. Since \( \Gamma_q(i_1) \) and \( j_1 \) preserve the traces, also \( J_1 \) preserves the trace. Now \( J_1 \) is an \( L^p \) isometry according to [JMX, p 92].

In the following, we let \( (H_n)_{n \in \mathbb{N}} \) be a sequence of mutually orthogonal sub Hilbert spaces of some big Hilbert space \( H = \bigoplus_{n \in \mathbb{N}} H_n \) and let \( I = \bigsqcup_{n \in \mathbb{N}} I_n \) be a partition of a big index set \( I \) into smaller pieces \( I_n \).

**Lemma 4.16** Consider the mappings \( \Psi_n: S_{I_n}^p \rightarrow S_I^p, \) \( x \mapsto P_n^*xP_n, \) where \( P_n: \ell_I^2 \rightarrow \ell_{I_n}^2 \) is the canonical orthogonal projection,

\[ \Psi: \bigoplus_{n \in \mathbb{N}} S_{I_n}^p \rightarrow S_I^p, \ (x_n)_n \mapsto \sum_n \Psi_n(x_n) \]

and

\[ J: \bigoplus_{n \in \mathbb{N}} L^p(\Gamma_q(H_n) \otimes B(\ell_{I_n}^2)) \rightarrow L^p(\Gamma_q(H) \otimes B(\ell_I^2)), \ (y_n)_n \mapsto \sum_n J_n(y_n). \]

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1. On the $L^\infty$ level, $\text{Ran}(J_n) \cdot \text{Ran}(J_m) = \{0\}$ for $n \neq m$.

2. If the above domains of $\Psi$ and $J$ that are exterior Banach space sums, are as indicated equipped with the $\ell^p$-norms, then $\Psi$ and $J$ are isometries.

Proof : 1. By a density and normality argument, it suffices to pick $a \otimes x \in \Gamma_0(H_n) \otimes B(\ell^2_{\ell^2_{\ell^2_{\ell^2}}})$ and $b \otimes y \in \Gamma_0(H_m) \otimes B(\ell^2_{\ell^2_{\ell^2_{\ell^2}}})$ and calculate the product $J_n(a \otimes x)J_m(b \otimes y)$. We have

$$J_n(a \otimes x)J_m(b \otimes y) = \Gamma_0(i_n)(a)\Gamma_0(i_m)(b) \otimes P_n^*xP_m^*yP_mP_n = 0,$$

since $P_nP_m^*: \ell^2_{\ell^2_{\ell^2_{\ell^2}}} \to \ell^2_{\ell^2_{\ell^2_{\ell^2}}}$ equals 0.

2. According to the first point, we have for any $x_n \in \Gamma_0(H_n) \otimes B(\ell^2_{\ell^2_{\ell^2_{\ell^2}}})$,

$$\left( \sum_{n=1}^N J_n(x_n) \right)^* \left( \sum_{m=1}^N J_m(x_m) \right) = \left( \sum_{n=1}^N J_n(x_n^*) \right) \left( \sum_{m=1}^N J_m(x_m) \right) = \sum_{n=1}^N J_n(x_n^*x_n).$$

In the same way, by functional calculus of $\sum_{n=1}^N J_n(x_n)^2$ and $J_n(\|x_n\|^2)$, we obtain

$$\left( \sum_{n=1}^N J_n(x_n) \right)^p = \sum_{n=1}^N |J_n(x_n)|^p,$$

and thus, taking traces, we obtain

$$\left\| \sum_{n=1}^N J_n(x_n) \right\|_p = \left( \sum_{n=1}^N \|J_n(x_n)\|^p \right)^{1/p} = \left( \sum_{n=1}^N \|x_n\|^p \right)^{1/p}.$$

By a density argument, we infer that $J$ is an isometry. The proof for $\Psi$ is easier and left to the reader.

In the following, we let moreover $\alpha_n: I_n \to H_n$ be mappings and associate with it $\alpha: I \to H$ given by $\alpha(i) = \alpha_n(i)$ if $i \in I_n \subseteq \bigcup_{k \in N} I_k$. We thus have non-commutative gradients $\partial_{\alpha,q,p}$ and $\partial_{\alpha,q,p}^*$.

**Lemma 4.17** Recall the mappings $J_n$ and $\Psi_n$ from Lemma 4.16.

1. For any $n \in \mathbb{N}$ and $x_n \in M_{I_n,\fin}$, we have $\Psi_n(x_n) \in M_{I,\fin}$ and

$$\partial_{\alpha,q,p} \Psi_n(x_n) = J_n \partial_{\alpha_n,q,p} (x_n).$$

2. For any $n \in \mathbb{N}$, $y_n \in \Gamma_0(H_n) \otimes M_{I_n,\fin}$, we have $J_n(y_n) \in \text{dom}(\partial_{\alpha_n,q,p})$ and

$$(\partial_{\alpha,q,p}^*)^* J_n(y_n) = \Psi_n(\partial_{\alpha_n,q,p}^*)^*(y_n).$$

Proof : 1. We calculate with $x_n = \sum_{i,j \in I_n} x_{ij}e_{ij}$ and explicit embedding $k_n: I_n \hookrightarrow I$,

$$\partial_{\alpha,q,p} \Psi_n(x_n) = \partial_{\alpha,q,p}(P_n^*x_nP_n) = \partial_{\alpha,q,p} \left( P_n^* \sum_{i,j} x_{ij}e_{ij}P_n \right) = \sum_{i,j} \partial_{\alpha,q,p}(e_{k_n(i)k_n(j)}) = \sum_{i,j} x_{ij}s_q(\alpha(k_n(i)) - \alpha(k_n(j))) \otimes e_{k_n(i)k_n(j)}.$$
On the other hand, we have

\[ J_n \partial_{\alpha_n,q,p}(x_n) = \sum_{i,j} x_{ij} J_n(s_q(\alpha_n(i) - \alpha_n(j)) \otimes e_{ij}) \]

\[ = \sum_{i,j} x_{ij} \Gamma_q(i_n)(s_q(\alpha_n(i) - \alpha_n(j))) \otimes P_n^* e_{ij} P_n \]

\[ = \sum_{i,j} x_{ij} s_q(\alpha(k_n(i)) - \alpha(k_n(j))) \otimes e_{k_n(i),k_n(j)}. \]

2. First we note that \( J_n(\Gamma_q(H_n) \otimes M_{I_n,\text{fin}}) \subseteq \text{dom}(\partial_{\alpha_n,q,p})^* \) and that \((\partial_{\alpha_n,q,p})^* \) maps \( J_n(\Gamma_q(H_n) \otimes M_{I_n,\text{fin}}) \) into \( \Psi_n(S_{\text{fin},n}^p) \). Indeed for \( s_q(h_n) \otimes e_{ij} \in \Gamma_q(H_n) \otimes M_{I_n,\text{fin}} \) and \( x = \sum_{k,l} x_{kl} e_{kl} \in S_{\text{fin}}^p \), we have

\[ x \mapsto \tau_{\gamma(H)\mathbb{B}(\ell_2^n)}(s_q(h_n)) \otimes e_{k_n(i)k_n(j)} \cdot \partial_{\alpha_n,q,p}(x) \]

\[ = \sum_{k,l} x_{kl} \tau_{\gamma(H)\mathbb{B}(\ell_2^n)}(s_q(h_n)) \otimes e_{k_n(i)k_n(j)} \cdot s_q(\alpha(k) - \alpha(l)) \otimes e_{kl} \]

\[ = \tau_{\gamma(H)}(s_q(h_n)) s_q(\alpha(k_n(i)) - \alpha(k_n(j))) e_{k_n(i),k_n(j)}. \]

This defines clearly a linear form on \( S_{\text{fin}}^p \), so indeed \( J_n(\Gamma_q(H_n) \otimes M_{I_n,\text{fin}}) \subseteq \text{dom}(\partial_{\alpha_n,q,p})^* \). Moreover, we see above that if \( x = e_{kl} \) and \( k, l \) not both in \( k_n(I_n) \), then \((\partial_{\alpha_n,q,p})^*(J_n(y_n)), e_{kl} = 0 \). Therefore, \((\partial_{\alpha_n,q,p})^* \) maps \( J_n(\Gamma_q(H_n) \otimes M_{I_n,\text{fin}}) \) into \( \Psi_n(S_{\text{fin}}^p) \). Now we check the claimed equality in the statement of the lemma by applying a dual element \( x_n \) to both sides. By density of \( M_{I_n,\text{fin}} \) in \( S_{\text{fin}}^p \), we can assume \( x_n \in M_{I_n,\text{fin}} \). Moreover, by the above, we can assume \( x_n \in \Psi_n(M_{I_n,\text{fin}}) \), so that \( x_n = \Psi_n \Psi_n^*(x_n) \). Then we have according to the first point of the lemma

\[ \text{Tr}_I(\Psi_n(\partial_{\alpha_n,q,p})^*(y_n)x_n) = \text{Tr}_{I_n}(\partial_{\alpha_n,q,p})(y_n)(\Psi_n^*(x_n)) = \tau_{\gamma(H)\mathbb{B}(\ell_2^n)}(y_n, \partial_{\alpha_n,q,p}, \Psi_n^*(x_n)) \]

\[ = \tau_{\gamma(H)\mathbb{B}(\ell_2^n)}(y_n, \partial_{\alpha_n,q,p}, \Psi_n^*(x_n)) = \tau_{\gamma(H)\mathbb{B}(\ell_2^n)}(J_n(y_n), \partial_{\alpha_n,q,p}, \Psi_n^*(x_n)) \]

\[ = \tau_{\gamma(H)\mathbb{B}(\ell_2^n)}(J_n, \partial_{\alpha_n,q,p}, \Psi_n^*(x_n)) = \text{Tr}_I((\partial_{\alpha_n,q,p})^*J_n(y_n)x_n). \]

Here we have also used that \( J_n \) is trace preserving and multiplicative. The lemma is proved. 

\[ \square \]

In the following, we denote \( D_{\alpha,q,p} \) and \( D_{\alpha_n,q,p} \) the Hodge-Dirac operators for \( \alpha \) and \( \alpha_n \).

**Lemma 4.18** Recall the mappings \( \Psi \) and \( J \) from Lemma 4.16.

1. Then for any \( x = (x_n) \) with \( x_n \in M_{I_n,\text{fin}} \subseteq S_{\text{fin}}^p \) and any \( y = (y_n) \) with \( y_n \in \Gamma_q(H_n) \otimes M_{I_n,\text{fin}} \cap \text{Ran}(\partial_{\alpha_n,q,p}) \) such that only finitely many \( x_n \) and \( y_n \) are non-zero, we have

\[ D_{\alpha,q,p} \circ \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} (x, y) = \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} D_{\alpha_1,q,p} & 0 & \cdots \\ 0 & D_{\alpha_2,q,p} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} (x, y) \]

2. Let \( \omega \in (0, \frac{\pi}{2}) \) such that \( D_{\alpha,q,p} \) and all \( D_{\alpha_n,q,p} \) have an \( H^\infty(\Sigma^+_{\omega}) \) calculus, e.g. according to Theorem 4.12, \( \omega = \frac{\pi}{4} \). Then for any \( m \in H^\infty(\Sigma^+_{\omega}) \), we have

\[ m(D_{\alpha,q,p}) \circ \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} = \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} m(D_{\alpha_1,q,p}) & 0 & \cdots \\ 0 & m(D_{\alpha_2,q,p}) & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \]
as bounded operators $\bigoplus_{n \in \mathbb{N}} (S^p_{I_n} \oplus_p \text{Ran}(\partial_{\alpha,q,p})) \to S^p_I \oplus_p \text{Ran}(\partial_{\alpha,q,p})$.

Proof: 1. We have according to Lemma 4.17

$$D_{\alpha,q,p}(\Psi(x), J(y)) = D_{\alpha,q,p}\left(\sum_n \Psi_n(x_n), J_n(y_n)\right) = \left(\left(\partial_{\alpha,q,p}\right)^* \sum_n J_n(y_n), \partial_{\alpha,q,p} \sum_n \Psi_n(x_n)\right)$$

$$= \left(\sum_n \Psi_n(\partial_{\alpha,q,p})^* y_n, \sum_n J_n \partial_{\alpha,q,p} x_n\right)$$

$$= (\Psi \text{diag}(\partial_{\alpha,q,p})^* : n \in \mathbb{N}) y, J \text{diag}(\partial_{\alpha,q,p} : n \in \mathbb{N}) x).$$

This shows (4.15).

2. According to (4.15), for any $\lambda \in \rho(D_{\alpha,q,p}) \cap \bigcap_{n \in \mathbb{N}} \rho(D_{\alpha,q,p})$, we have for $(x, y)$ as in the first part of the lemma,

$$(\lambda - D_{\alpha,q,p})^{-1} \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} (\lambda - D_{\alpha,q,p}) & 0 \\ 0 & 0 \end{bmatrix} (x, y) = (\lambda - D_{\alpha,q,p})^{-1} (\lambda - D_{\alpha,q,p}) \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} (x, y)$$

$$= \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} (x, y) = \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} (\lambda - D_{\alpha,q,p})^{-1} & 0 \\ 0 & 0 \end{bmatrix} (x, y).$$

Thus, we obtain

$$(\lambda - D_{\alpha,q,p})^{-1} \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} = \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} (\lambda - D_{\alpha,q,p})^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

on a dense subspace of $\bigoplus_{n \in \mathbb{N}} (S^p_{I_n} \oplus_p \text{Ran}(\partial_{\alpha,q,p}))$. By boundedness of both sides of (4.17), we obtain equality in (4.17) as bounded operators $\bigoplus_{n \in \mathbb{N}} (S^p_{I_n} \oplus_p \text{Ran}(\partial_{\alpha,q,p})) \to S^p_I \oplus_p \text{Ran}(\partial_{\alpha,q,p})$. Now by the Cauchy integral formula, we obtain (4.16) for $m \in H^\infty(\Sigma^\omega_{\omega})$, and by the $H^\infty$ convergence lemma (see [CDMY, Lemma 2.1] for the sectorial case) applied first for fixed $(x, y)$ as in the first part of the lemma, also for $m \in H^\infty(\Sigma_\omega)$.

We can now solve the “Schur multiplier version ” of [JMP2, Problem C.5].

**Theorem 4.19** Let $1 < p < \infty$, $-1 \leq q \leq 1$, $\omega = \frac{1}{p} \frac{1}{q}$ and $D_{\alpha,q,p}$ as in Theorem 4.12. Then the $H^\infty(\Sigma^\omega_{\omega})$ calculus norm of $D_{\alpha,q,p}$ is controlled independently of $H$ and $\alpha$, that is, $\|m(D_{\alpha,q,p})\| \leq C_{q,p} \|m\|_{H^\infty(\Sigma^\omega_{\omega})}$. In particular, $\text{sgn}(D_{\alpha,q,p}) = D_{\alpha,q,p} |D_{\alpha,q,p}|^{-\frac{1}{2}}$ is bounded by a constant not depending on $H$ or its dimension nor the mapping $\alpha : I \to H$.

Proof: Suppose that the statement of the proposition is false. Then there exists a sequence of Hodge-Dirac operators $D_{\alpha,q,p}, n \in \mathbb{N}$ such that the $H^\infty(\Sigma^\omega_{\omega})$ calculus norm is bigger than $n$. We let $D_{\alpha,q,p}$ be as before the Hodge-Dirac operator of $\alpha : \bigcup_{n \in \mathbb{N}} I_n \to \bigoplus_{n \in \mathbb{N}} H_n, i \in I_n \mapsto \alpha_n(i)$. Let $m_n$ be a non-zero $H^\infty(\Sigma^\omega_{\omega})$ function such that $\|m_n\|_{\infty,\omega} = 1$ and $\|m_n(D_{\alpha,q,p})\| \geq n$. According to Theorem 4.12, $D_{\alpha,q,p}$ has an $H^\infty(\Sigma^\omega_{\omega})$ calculus. Thus, for some constant $C < \infty$ and any $m_n$ as above, according to Lemma 4.18,

$$C \geq \|m_n(D_{\alpha,q,p})\| \geq \left\| m_n(D_{\alpha,q,p}) \circ \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix}\right\| = \left\| \begin{bmatrix} \Psi & 0 \\ 0 & J \end{bmatrix} \circ \begin{bmatrix} m_n(D_{\alpha,q,p}) & 0 \\ 0 & 0 \end{bmatrix}\right\|$$

$$= \left\| m_n(D_{\alpha,q,p}) \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\| \geq \|m_n(D_{\alpha,q,p})\| \geq n.$$
Taking $n \to \infty$ yields a contradiction. The last sentence of Proposition 4.19 follows from taking $m(z) = 1_{\Sigma^+}(z) - 1_{\Sigma^-}(z)$.

4.2 Boundedness of functional calculus of Hodge-Dirac operators for Fourier multipliers

Here $G$ is a discrete group and we consider a semigroup of Markov Fourier multipliers $(T_t)_{t \geq 0}$. Suppose $1 < p < \infty$. Recall that we have a closed operator

$$\partial_{\psi,q,p} : \text{dom} \partial_{\psi,q,p} \subset L^p(VN(G)) \to L^p(\Gamma_q(H) \times G), \quad \lambda_s \mapsto s_q(\alpha_i - \alpha_j) \times \lambda_s$$

and a closed operator $(\partial_{\psi,q,p}^*)^* : \text{dom} (\partial_{\psi,q,p}^*)^* \subset L^p(\Gamma_q(H) \times G) \to L^p(VN(G))$. Suppose $1 \leq p < \infty$. We introduce the unbounded operator

$$D_{\psi,q,p} \overset{\text{def}}{=} \begin{bmatrix} 0 & (\partial_{\psi,q,p}^*)^* \\ \partial_{\psi,q,p}^* & 0 \end{bmatrix}$$

on the Banach space $L^p(VN(G)) \oplus_p \text{Ran} \partial_{\psi,q,p}$ defined by

$$D_{\psi,q,p}(x,y) \overset{\text{def}}{=} ((\partial_{\psi,q,p}^*)^*(y), \partial_{\psi,q,p}(x)), \quad x \in \text{dom} \partial_{\psi,q,p}, \quad y \in \text{dom} (\partial_{\psi,q,p}^*)^* \cap \text{Ran} \partial_{\psi,q,p}.$$

We call it the Hodge-Dirac operator of the semigroup. This operator is a closed operator and can be seen as a differential square root of the generator of the semigroup $(T_{t,p})_{t \geq 0}$ since we have the following result left to the reader.

Proposition 4.20 Suppose $1 < p < \infty$ and $-1 \leq q < 1$. As densely defined closed operators on $L^p(VN(G)) \oplus_p \text{Ran} \partial_{\psi,q,p}$, we have

$$D_{\psi,q,p}^2 = \begin{bmatrix} A_p & 0 \\ 0 & (\text{Id}_{L^p(\Gamma_q(H))} \times A_p)[\text{Ran} \partial_{\psi,q,p}] \end{bmatrix}.$$

Note that the unbounded operator $D_{\psi,q,2}$ is self-adjoint. In this section, we prove the boundedness of the functional calculus of the Hodge-Dirac operator (4.18) (which is a generalization of (1.2)). Indeed, we have the following result.

Theorem 4.21 Suppose $1 < p < \infty$ and $-1 \leq q < 1$. Let $G$ an amenable discrete group or a free group (in this case, take $q = -1$ or $q = 1$). The Hodge-Dirac operator $D_{\psi,q,p}$ is $R$-bisectorial on $L^p(VN(G)) \oplus_p \text{Ran} \partial_{\psi,q,p}$ and admits a bounded $H^\infty(\Sigma^{\pm}_q)$ functional calculus on a bisector.

We follow the same strategy that the one of the section 4.1. We only indicate the main changes. Note that we use Proposition 2.11 for the group analogue of Proposition 4.11, so we need some assumption on the group $G$.

Remark 4.22 With the same method that the one of Remark (4.13) we can see that the boundedness of the $H^\infty$ functional calculus of $D_{\psi,q,p}$ implies the boundedness of the Riesz transforms and this result may be thought of as a strengthening of the equivalence (1.1).

We have the following useful formulas for $(\partial_{\psi,q,p}^*)^*$.

Proposition 4.23 Suppose $1 < p < \infty$ and $-1 \leq q < 1$.  

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1. For any \( s \in G \) and any \( f \in L^p(\Gamma_q(H)) \), we have \( f \times \lambda_s \in \text{dom}(\partial_{\psi,q,p})^* \) and
\[
(\partial_{\psi,q,p})^*(f \times \lambda_s) = \langle s_q(b_{\psi}(s)), f \rangle_{L^p(\Gamma_q(H))}^{\text{opt}}(\partial_{\psi,q,p}) \lambda_s.
\]

2. We have
\[
A_p = (\partial_{\psi,q,p})^* \partial_{\psi,q,p}.
\]

We have the following analogue of Proposition 4.9 which can be proved in a similar manner.

**Proposition 4.24** Suppose \( 1 < p < \infty \) and \(-1 \leq q \leq 1\). The family
\[
\{ t \partial_{\psi,q,p}(\text{Id} + t^2 A_p)^{-1} : t > 0 \}
\]
of operators of \( \text{B}(L^p(\text{VN}(G)), L^p(\Gamma_q(H) \rtimes^\alpha G)) \) is \( R \)-bounded.

The following result of commutation is necessary.

**Lemma 4.25** Suppose \( 1 < p < \infty \) and \(-1 \leq q \leq 1\). If \( x \in \text{dom} \partial_{\psi,q,p} \) and \( t \geq 0 \) then \( T_{t,p}(x) \)
belongs to \( \text{dom} \partial_{\psi,q,p} \) and we have
\[
(\text{Id}_{L^p(\Gamma_q(H))} \rtimes T_{t,p}) \partial_{\psi,q,p}(x) = \partial_{\psi,q,p} T_{t,p}(x).
\]

**Proof** : For any \( s \in G \), we have
\[
(\text{Id}_{L^p(\Gamma_q(H))} \rtimes T_{t,p}) \partial_{\psi,q,p}(\lambda_s) = \left( \text{Id}_{L^p(\Gamma_q(H))} \rtimes T_{t,p} \right) (s_q(b_{\psi}(s)) \times \lambda_s)
= e^{-t||b_{\psi}(s)||^2} s_q(b_{\psi}(s)) \times \lambda_s = e^{-t||b_{\psi}(s)||^2} \partial_{\psi,q,p}(\lambda_s) = \partial_{\psi,q,p} (e^{-t||b_{\psi}(s)||^2} \lambda_s) = \partial_{\psi,q,p} T_{t,p}(\lambda_s).
\]

So by linearity the equality (4.23) is true for elements of span \( \{ \lambda_s : s \in G \} \). Then the end of
the proof is similar to the one of Lemma 4.3. \( \blacksquare \)

**Proposition 4.26** Suppose \( 1 < p < \infty \) and \(-1 \leq q \leq 1\).

1. For any \( t > 0 \), we have on \( \text{dom} \partial_{\psi,q,p} \)
\[
(\text{Id}_{L^p(\Gamma_q(H))} \rtimes (\text{Id}_{S_p^+} + t^2 A_p)^{-1}) \partial_{\psi,q,p} = \partial_{\psi,q,p} (\text{Id}_{S_p^+} + t^2 A_p)^{-1}.
\]

2. For any \( t \geq 0 \), we have on \( \text{dom}(\partial_{\psi,q,p})^* \)
\[
(\text{Id}_{S_p^+} + t^2 A_p)^{-1} (\partial_{\psi,q,p})^* = (\partial_{\psi,q,p})^* (\text{Id}_{L^p(\Gamma_q(H))} \rtimes (\text{Id}_{S_p^+} + t^2 A_p)^{-1}).
\]

**Theorem 4.27** Let \( 1 < p < \infty \), \(-1 \leq q \leq 1\), \( \omega = \frac{\pi}{2} \) and \( D_{\psi,q,p} \) as in Theorem 4.12. Then the
\( H^\infty(\Sigma_{\omega,\Gamma}) \) calculus norm of \( D_{\psi,q,p} \) is controlled independently of \( H \) and \( \alpha \), that is, \( \| \text{m}(D_{\psi,q,p}) \| \leq C_{q,p} \| m \|_{H^\infty(\Sigma_{\omega,\Gamma})} \). In particular, \( \text{sgn}(D_{\psi,q,p}) = D_{\psi,q,p} |D_{\psi,q,p}|^{-\frac{1}{2}} \) is bounded by a constant not depending on \( H \) or its dimension nor the function \( \psi \).
5 New locally compact quantum metric spaces and spectral triples

5.1 Background on quantum locally compact metric spaces

We recall definitions and characterizations of the notions that we need in this section. The main notions are that of quantum compact metric space and quantum locally compact metric space. The concept of quantum compact metric space has its origins in Connes’ paper [Con4], of 1989 in which he first proposes using Dirac operators as the vehicle for metric data in non-commutative geometry. Afterwards, Rieffel [Rie3] and Latrémolière [Lat1] among others continued and deepened the study of quantum versions of (locally) compact metric spaces. These are generalisations of \((C(X), \text{Lip}_{\text{dist}})\), where \(X\) is a (locally) compact metric space, \(\text{dist}\) is a continuous metric on \(X\) and \(\text{Lip}_{\text{dist}}\) is the Lipschitz norm (see (5.1) below). The role of the space \(C(X)\) of continuous functions is taken over by a \(C^*\)-algebra \(A\). The smallest unital \(C^*\)-algebra containing such an \(A\), i.e. either \(A\) if \(A\) is unital, or its standard unitization \(A \oplus \mathbb{C}\) otherwise, is denoted by \(uA\). If \(A\) is not unital, note that

\[(uA)_{sa} = A_{sa} \oplus \mathbb{R}1_{uA}.
\]

The following is [Lat2, Definition 2.3], [Lat6, Definition 2.2].

**Definition 5.1** A Lipschitz pair \((A, \|\cdot\|)\) is a pair where \(A\) is a \(C^*\)-algebra and where \(\|\cdot\|\) is a seminorm on a dense subspace \(\text{dom} \|\cdot\|\) of \((uA)_{sa}\) such that:

\[
\{ a \in (uA)_{sa} : \|a\| = 0 \} = \mathbb{R}1_{uA}.
\]

A unital Lipschitz pair \((A, \|\cdot\|)\) is a Lipschitz pair where \(A\) is unital.

It is natural to wonder whether the norm of some Lipschitz pair gives back the weak* topology of the “quantum metric”, that is the Monge-Kantorovich metric on states of \(A\) (see below). This desired property gives then our first central notion of this subsection. Namely, the following is [Lat2, Theorem 2.42], [Lat3, Definition 1.2] or [Lat6, Definition 2.6].

**Definition 5.2** A quantum compact metric space \((A, \|\cdot\|)\) is a unital Lipschitz pair whose Monge-Kantorovich metric

\[\text{dist}_{\text{mk},\|\cdot\|}(\varphi, \psi) \overset{\text{def}}{=} \sup \{ |\varphi(a) - \psi(a)| : a \in A_{sa}, \|a\| \leq 1 \}, \quad \varphi, \psi \in S(A)\]

metrizes the weak* topology restricted to the state space \(S(A)\) of \(A\). When a Lipschitz pair \((A, \|\cdot\|)\) is a quantum compact metric space, the seminorm \(\|\cdot\|\) is referred to as a Lip-norm.

This definition is hard to check in general. There exists the following equivalent condition to this property [AgL, Theorem 2.5], [Lat2, Theorem 2.43] and [Lat6, Theorem 2.10].

**Proposition 5.3** Let \((A, \|\cdot\|)\) be a unital Lipschitz pair. The following assertions are equivalent:

1. \((A, \|\cdot\|)\) is a quantum compact metric space,

2. there exists a state \(\mu \in S(A)\) such that the set \(\{ a \in A_{sa} : \|a\| \leq 1, \mu(a) = 0 \}\) is norm relatively compact in \(A\).
The definition of a quantum compact metric space has been extended in [Rie3, Definition 1.1], [Lat1, p. 393] to the case that \( \mathcal{A} \) is merely an order-unit space. The definition of an order-unit space below is due to Kadison [Kad] influenced by work of Stone.

**Definition 5.4**  
1. An order-unit space is an \( \mathbb{R} \)-subspace of the self-adjoint part of a unital C\(^*\)-algebra containing the unit.

2. An order-unit space \( \mathcal{A} \) together with a seminorm \( \| \cdot \| \) is called a quantum compact metric space provided that
   
   (a) \( \{ a \in \mathcal{A} : \| a \| = 0 \} = \mathbb{R}1 \),

   (b) The metric on the state space \( S(\mathcal{A}) \) of \( \mathcal{A} \) defined as in Definition 5.2 metrizes the weak\(^*\) topology restricted to \( S(\mathcal{A}) \).

The proof of [OzR, Proposition 1.3] extends without modification to the order-unit case, so that we find the following.

**Proposition 5.5** Let \( \| \cdot \| \) be a seminorm on a order-unit space \( E \) and let \( \mu \) be a state of \( E \). If \( \{ x \in E : \| x \| \leq 1 \text{ and } \mu(x) = 0 \} \) is norm relatively compact then \( \| \cdot \| \) is a Lip-norm, that is, \((E, \| \cdot \|)\) is a quantum compact metric space in the sense of Definition 5.4.

The fundamental examples of quantum compact metric spaces are given [Lat7, Example 2.9] by pairs \((C(X), \text{Lip}_{\text{dist}})\), where \( X \) is a compact metric space, \( \text{dist} \) is a continuous metric on \( X \) and, for any \( f \in C(X) \), we define:

\[
\text{Lip}_{\text{dist}}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{\text{dist}(x, y)} : x, y \in X, x \neq y \right\}.
\]

The restriction of \( \text{Lip}_{\text{dist}} \) to the space \( C(X)_{\text{sa}} \) of real-valued continuous functions on \( X \) is a Lip-norm in the above sense. The Lipschitz seminorm \( \text{Lip}_{\text{dist}} \) associated to a compact metric space \((X, m)\) enjoys a natural property with regard to the multiplication of functions in \( C(X) \), called the Leibniz property for seminorms:

\[
\text{Lip}_{\text{dist}}(fg) \leq \| f \|_{C(X)} \text{Lip}_{\text{dist}}(g) + \text{Lip}_{\text{dist}}(f) \| g \|_{C(X)}, \quad f, g \in C(X).
\]

Moreover, the Lipschitz seminorm is lower-semicontinuous with respect to the C\(^*\)-norm of \( C(X) \), i.e. the uniform convergence norm on \( X \). These two additional properties were not assumed in the above definition 5.2, yet they are quite natural. So, sometimes, some additional conditions are added as in the following Definition 5.6 and Definition 5.9.

The following is [Lat2, Definition 2.21] (see also [Lat5, Definition 1.3]).

**Definition 5.6** A Leibniz pair \((\mathcal{A}, \| \cdot \|)\) is a Lipschitz pair such that:

1. the domain \( \text{dom} \| \cdot \| \) of \( \| \cdot \| \) is a Jordan-Lie subalgebra of \( \mathcal{A}_{\text{sa}} \),

2. for all \( a, b \in \text{dom} \| \cdot \| \), we have:

\[
\| a \circ b \| \leq \| a \|_{\mathcal{A}} \| b \| + \| a \| \| b \|_{\mathcal{A}}
\]

and

\[
\| \{ a, b \} \| \leq \| a \|_{\mathcal{A}} \| b \| + \| a \| \| b \|_{\mathcal{A}}.
\]

Here and below, \( \| \cdot \|_{\mathcal{A}} \) stands for the original C\(^*\)-norm on \( \mathcal{A} \).
The following is [Lat7, Theorem 2.17], [Lat2, page 18].

**Proposition 5.7** Let $\mathcal{A}$ be a unital $C^*$-algebra and $\|\cdot\|$ be a seminorm defined on a dense $\mathbb{C}$-subspace $\text{dom}\|\cdot\| \subseteq \mathcal{A}$, such that $\text{dom}\|\cdot\|$ is closed under the adjoint operation, such that \{a \in \text{dom}\|\cdot\| : \|a\| = 0\} = C1_{\mathcal{A}}$ and, for all $a, b \in \text{dom}\|\cdot\|$, we have:

$$\|ab\| \leq \|a\|_{\mathcal{A}} \|b\| + \|b\|_{\mathcal{A}} \|a\|.$$  

If $\|\cdot\|_{sa}$ is the restriction of $\|\cdot\|$ to $A_{sa} \cap \text{dom}\|\cdot\|$, then $(\mathcal{A}, \|\cdot\|_{sa})$ is a unital Leibniz pair.

The following is [Lat2, Theorem 2.16].

**Proposition 5.8** Let $(\mathcal{A}, \|\cdot\|)$ be a Lipschitz pair. The following assertions are equivalent:

1. $\|\cdot\|$ is lower semicontinuous,
2. $\{a \in \mathcal{A}_{sa} : \|a\| \leq 1\}$ is closed in $\mathcal{A}$.

Whenever a quantum compact metric space satisfies these two additional properties, we have the following notation from [Lat2, Definition 2.45], [Lat7, Definition 2.19], [Lat6, Definition 2.19] and [Lat2, Definition 2.54].

**Definition 5.9** A unital Lipschitz pair $(\mathcal{A}, \|\cdot\|)$ is a Leibniz quantum compact metric space when it is a Leibniz pair and when $\|\cdot\|$ is a lower semicontinuous Lip-norm.

The following is [Lat1, Definition 2.15] and [Lat2, Definition 2.54].

**Definition 5.10** A topography on a $C^*$-algebra $\mathcal{A}$ is an abelian $C^*$-subalgebra $\mathcal{D}$ of $\mathcal{A}$ containing an approximate identity for $\mathcal{A}$. A topographic quantum space $(\mathcal{A}, \mathcal{D})$ is an ordered pair of a $C^*$-algebra $\mathcal{A}$ and a topography $\mathcal{D}$ on $\mathcal{A}$.

Besides quantum compact metric spaces, Latrémolière introduced in [Lat1, Lat2] also quantum locally compact metric spaces. The passage from (classical) compact metric spaces to (classical) locally compact metric spaces implicates several problems. First, the space of states over a non-unital $C^*$-algebra (such as $C_0(X)$ where $X$ is a locally compact non compact metric space) is not weak* locally compact in general. Second, the above Monge-Kantorovich construction from Definition 5.2 no longer yields a distance and can take the value $+\infty$. The solution found by Latrémolière is to “localise” the non-commutative measure space by means of a preferred commutative subfamily of the $C^*$-algebra, such as $\mathcal{D}$, the topography in Definition 5.10 above. Then for our purposes, the characterization of quantum locally compact metric spaces in Theorem 5.14 below, valid for separable $C^*$-algebras, will suffice. We shall only refer the interested reader to [Lat1, Definition 3.1] for Latrémolière’s original definition.

The following is [Sko1, Definition 5.9] (see also [Ped, 3.10.4]).

**Definition 5.11** Let $\mathcal{A}$ be a $C^*$-algebra. A positive element $x \in \mathcal{A}$ is strictly positive if for every state $\mu$ on $\mathcal{A}$ we have $\mu(x) > 0$.

Let $\mathcal{A}$ be a $C^*$-algebra. By [Sko1, Proposition 5.13] and [Ped, Proposition 3.10.5], $\mathcal{A}$ has a strictly positive element if and only if $\mathcal{A}$ has a countable $C^*$-bounded approximate identity. Moreover, the proof shows that if $(e_n)$ are the elements in the $C^*$-bounded approximate identity then $x = \sum_{n \geq 1} \frac{1}{m_n} e_n$ is a strictly positive element.

Let $(\mathcal{A}, \mathcal{D})$ be a topographic quantum space. We denote by $K(\mathcal{D})$ the set of all compact subsets of the Gelfand spectrum of the abelian $C^*$-algebra $\mathcal{D}$. If $K \in K(\mathcal{D})$ then we denote by $1_K$ the projection in $A^{**}$ defined as the indicator function of $K$ in $\mathcal{D}$. We also note that every state $\varphi$ on $\mathcal{A}$ trivially extends to a normal state on $A^{**}$, which we will again denote by $\varphi$. The following is [Lat1, Definition 2.23] and [Lat2, Definition 2.59].
Definition 5.12 Let \((\mathcal{A}, \mathcal{D})\) be a topographic quantum space. A state \(\varphi \in S(\mathcal{A})\) is local when there exists \(K \in K(\mathcal{D})\) such that \(\varphi(1_K) = 1\). The set of all local states of \((\mathcal{A}, \mathcal{D})\), denoted by \(S(\mathcal{A}|\mathcal{D})\), is called the local state space of \((\mathcal{A}, \mathcal{D})\).

The following is [Lat1, Definition 2.27] and [Lat2, Definition 2.60].

Definition 5.13 A Lipschitz triple \((\mathcal{A}, \| \cdot \|, \mathcal{D})\) is a triple where \((\mathcal{A}, \| \cdot \|)\) is a Lipschitz pair and where \(\mathcal{D}\) is a topography on \(\mathcal{A}\).

Finally, for the following characterization we refer to [Lat2, Theorem 2.74] and [Lat1, Theorem 3.10] for the third point (see also [Lat1, Theorem 3.11]).

Theorem 5.14 Let \((\mathcal{A}, \| \cdot \|, \mathcal{D})\) be a Lipschitz triple. If \(\mathcal{A}\) is separable, then the following assertions are equivalent:

1. \((\mathcal{A}, \| \cdot \|, \mathcal{D})\) is a quantum locally compact metric space,
2. there exists a strictly positive \(h \in \mathcal{D}_{sa}\) and a local state \(\mu\) of \(\mathcal{A}\) such that the set:
   \[
   \{ \text{hah} : a \in (u\mathcal{A})_{sa}, \| a \| \leq 1, \mu(\alpha) = 0 \}
   \]
   is relatively compact for \(\| \cdot \|_{\mathcal{A}}\).
3. There exists a local state \(\mu\) such that for all compactly supported \(q,p \in \mathcal{D}\) the set:
   \[
   \{ \text{gap} : a \in (u\mathcal{A})_{sa}, \| a \| \leq 1, \mu(\alpha) = 0 \}
   \]
   is relatively compact for \(\| \cdot \|_{\mathcal{A}}\).

5.2 Gaps and estimates of norms of Schur multipliers

Consider a semigroup \((T_t)_{t \geq 0}\) of selfadjoint unital completely positive Schur multipliers on \(B(\ell^2_1)\) as in (3.26). In this subsection, we suppose \(\dim H < \infty\). We define the gap of \(\alpha\) by

\[
(5.3) \quad \text{Gap}_{\alpha} \overset{\text{def}}{=} \inf_{\alpha_i, -\alpha_j \neq \alpha_k, -\alpha_l} \| (\alpha_i - \alpha_j) - (\alpha_k - \alpha_l) \|_H^2.
\]

Lemma 5.15 If \(\dim H = n\) and \(\text{Gap}_{\alpha} > 0\), for any \(k\), we have

\[
\text{card} \left\{ \alpha_i - \alpha_j : k^2 \text{Gap}_{\alpha} \leq \| \alpha_i - \alpha_j \|_H^2 \leq (k + 1)^2 \text{Gap}_{\alpha} \right\} \leq 5^n k^{n-1}.
\]

Proof : If \(B_n\) denotes the Euclidean unit ball in \(H\) and if \(\xi_1, \xi_2\) belong to \(\alpha(I) - \alpha(I)\) with \(\xi_1 \neq \xi_2\), we have

\[
(\xi_1 + \frac{\sqrt{\text{Gap}_{\alpha}}}{2} B_n) \cap (\xi_2 + \frac{\sqrt{\text{Gap}_{\alpha}}}{2} B_n) = \emptyset.
\]

This shows with a volume comparison argument of two nested sets that

\[
\left\{ \alpha_i - \alpha_j : k^2 \text{Gap}_{\alpha} \leq \| \alpha_i - \alpha_j \|_H^2 \leq (k + 1)^2 \text{Gap}_{\alpha} \right\}
\]

\[
\leq \left( \frac{\sqrt{\text{Gap}_{\alpha}}}{2} B_n \right)^{-1} \left[ \left( (k + 1) \frac{\sqrt{\text{Gap}_{\alpha}}}{2} + \frac{\sqrt{\text{Gap}_{\alpha}}}{2} \right) B_n \right] - \left[ \left( k \frac{\sqrt{\text{Gap}_{\alpha}}}{2} - \frac{\sqrt{\text{Gap}_{\alpha}}}{2} \right) B_n \right]
\]

\[
= (2k + 3)^n - (2k - 1)^n = \sum_{j=0}^{n} \binom{n}{j} (2k)^{n-j} (3^j - (-1)^j) \leq 5^n k^{n-1}.
\]
The following lemma tells that Schur multipliers with 0–1 entries of diagonal block rectangular shape are completely contractive.

**Lemma 5.16** Let \(\{I_1, \ldots, I_N\}\) and \(\{J_1, \ldots, J_N\}\) be two subpartitions of \(I\), i.e. \(I_k \subset I\) and \(I_k \cap I_l = \emptyset\) for \(k \neq l\) (and similarly for \(J_1, \ldots, J_N\)). Consider the Schur multiplier \(M_B\) associated with the matrix \(B = [b_{ij}]\), where

\[
b_{ij} = \begin{cases} 
1 & : i \in I_k, j \in J_l \text{ for the same } k \\
0 & : \text{otherwise}
\end{cases}
\]

Then \(M_B : S_1^\infty \rightarrow S_1^\infty\) is completely contractive.

**Proof:** Let \(P_i : \ell_1^2 \rightarrow \ell_1^2, (\xi_k)_{k \in I} \rightarrow (\xi_k, 1_{k \in I})_{k \in I}\) and \(Q_i : \ell_1^2 \rightarrow \ell_1^2, (\xi_k)_{k \in I} \rightarrow (\xi_k, 1_{k \in I})_{k \in I}\) be orthogonal projections for \(i \in \{1, \ldots, N\}\). Then the images \(P_i(\ell_1^2)\) and \(P_j(\ell_1^2)\) (resp. \(Q_i(\ell_1^2)\) and \(Q_j(\ell_1^2)\)) are orthogonal for \(i \neq j\). Moreover, \(M_B(C) = \sum_{i=1}^N P_i C Q_i\). Then for \(\xi, \eta \in \ell_1^2\), we have

\[
|\langle M_B(C)\xi, \eta \rangle| = \left| \sum_{i=1}^N P_i C Q_i \xi, \eta \right| \leq \sum_{i=1}^N \|P_i C Q_i \xi\|^2 \leq \|C\| S_i^\infty \sum_{i=1}^N \|P_i \xi\|^2 \leq \|C\| S_i^\infty \sum_{i=1}^N \|P_i \eta\|^2 \leq \|C\| S_i^\infty \sum_{i=1}^N \|P_i \eta\|^2 \leq \|C\| S_i^\infty \|\xi\| \|\eta\| \|\ell_1^2\|.
\]

We infer that \(\|M_B(C)\|_1 \leq \|C\| S_i^\infty\). Thus \(M_B : S_1^\infty \rightarrow S_1^\infty\) is a contraction, and since it is a Schur multiplier, even a complete contraction [Pau, Theorem 8.7].

In the following lemma, we write \(T_{m,0}\) for the Schur multiplier associated with the matrix \(m_{ij} = \tilde{m}(\alpha_i - \alpha_j)\). We also recall that \(S_{m,0}^\infty = \{x = [x_{ij}] \in S_1^\infty : x_{ij} = 0 \text{ if } A e_{ij} = 0\}\).

**Lemma 5.17** If \(\dim H = n\), \(\text{Gap}_\alpha > 0\) and \(\tilde{m} : H \rightarrow \mathbb{C}\) satisfies

\[
|\tilde{m}(\xi)| \leq c_\alpha |\xi|^{-(n+\varepsilon)} \text{ for some } \varepsilon > 0,
\]

then the Schur multiplier \(T_{m(\alpha_i - \alpha_j)} : S_1^\infty \rightarrow S_1^\infty\) is completely bounded. In particular, we have

\[
\|T_i\|_{cb, S_{m,0}^\infty \rightarrow S_1^\infty} \leq \frac{c(n)}{(\text{Gap}_\alpha t)^{2}}.
\]

**Proof:** Let \(\{I_1, \ldots, I_N\}\) be the partition of \(I\) corresponding to the equivalence relation \(i \equiv j \Leftrightarrow \alpha_i = \alpha_j\), and let \(J_1 = I_1, \ldots, J_N = I_N\). Let then \(B_0\) be the Schur multiplier symbol from Lemma 5.16 associated with these partitions. Moreover for \(\xi \in \alpha(I) - \alpha(I)\), we let \(\{I_{1\xi}, \ldots, J_{N\xi}\}\) and \(\{J^{\xi}_1, \ldots, J^{\xi}_N\}\) be the (possibly empty) subpartitions of \(I\) such that \(\alpha_i - \alpha_j = \xi \Leftrightarrow i \in I_{1\xi}\) and \(j \in J^{\xi}_k\) for the same \(k\). We let \(B_{\xi}\) be the Schur multiplier symbol from Lemma 5.16 associated with these partitions. Then, for \(x = \sum_{ij} x_{ij} \otimes e_{ij} \in S_M^\infty(S_1^\infty)\) with \(x_{ij} \in S_M^\infty\), and \(m_{ij} = \tilde{m}(\alpha_i - \alpha_j)\), using Lemma 5.15 and our growth assumption on \(\tilde{m}\) in the third inequality,
we obtain
\[
\left\| \sum_{i,j \in I} m_{ij} x_{ij} \otimes e_{ij} \right\|_{S_{M}(S_{\Gamma}^{\infty})} \leq \hat{m}(0) \left\| \sum_{i,j \in I : \alpha_i = \alpha_j} x_{ij} \otimes e_{ij} \right\|_{S_{M}(S_{\Gamma}^{\infty})} + \sum_{k \geq 1} \sum_{\xi \in \alpha(I) - \alpha(I)} |\hat{m}(\xi)| \left\| \sum_{i,j \in I : \alpha_i - \alpha_j = \xi} x_{ij} \otimes e_{ij} \right\|_{S_{M}(S_{\Gamma}^{\infty})}
\]
\[
= \|\hat{m}(0)\|_{S_{M}(S_{\Gamma}^{\infty})} \|\text{Id}_{S_{M}} \otimes M_{B_0}(x)\|_{S_{M}(S_{\Gamma}^{\infty})} + \sum_{k \geq 1} \sum_{\xi \in \alpha(I) - \alpha(I)} |\hat{m}(\xi)| \left\| \text{Id}_{S_{M}} \otimes M_{B_1}(x) \right\|_{S_{M}(S_{\Gamma}^{\infty})}
\]
\[
\leq \left( |\hat{m}(0)| + \sum_{k \geq 1} \sum_{\xi \in \alpha(I) - \alpha(I)} |\hat{m}(\xi)| \right) \left\| x \right\|_{S_{M}(S_{\Gamma}^{\infty})}
\]
\[
\leq 5^n |\hat{m}(0)| + \sum_{k \geq 1} k^{n-1} \left( \frac{\sqrt{\text{Gap}_{\alpha}}}{2} \right)^{(n+1)} \left\| x \right\|_{S_{M}(S_{\Gamma}^{\infty})} = c_n(x)(\text{Gap}_{\alpha})^{-\frac{n}{2}} .
\]

The second assertion follows similarly. Indeed, since \( x \in S_{I,0}^{\infty} \) we may ignore the term \( |\hat{m}(0)| \) above and the result follows from the inequality
\[
\sum_{k=1}^{\infty} k^{n-1}\text{e}^{-t\text{Gap}_{\alpha}k^2} \leq c(n)(\text{Gap}_{\alpha})^{-\frac{n}{2}}t^{-\frac{n}{2}} .
\]

5.3 Seminorms associated to semigroups of Schur multipliers

Suppose \( 1 \leq p \leq \infty \). Let \( I \) be an index set. Recall that \( S_{I,0}^{\infty} \) is defined by (3.51). Suppose that the map \( \alpha : I \to H \) is injective. In this case, \( S_{I,0}^{\infty} \) is the space of elements of \( S_{I}^{0} \) with null diagonal. We will also use the subspace \( M_{f,\text{fin},0} \) of matrices with a finite number of non null entries with null diagonal.

Note that we can see \( S_{I,0}^{\infty} \oplus \text{Cld}_{\ell_2} \) as the standard unitization of the C*-algebra \( S_{I}^{\infty} \). We consider the subspace \( M_{I,\text{fin},0} \oplus \text{Cld}_{\ell_2} \) of \( S_{I,0}^{\infty} \oplus \text{Cld}_{\ell_2} \).

Suppose \( 1 < p \leq \infty \). For any element \( x = x_0 + \lambda \text{ld}_{\ell_2} \) of \( M_{f,\text{fin},0} \oplus \text{Cld}_{\ell_2} \), we let
\[
\| x \|_{\Gamma,\alpha,p} \overset{\text{def}}{=} \inf_{x_0 = y + z} \left\{ \left\| \Gamma(y, y) \right\|_{S_{I}^{p}} + \left\| \Gamma(z, z)^{\frac{p}{2}} \right\|_{S_{I}^{p}} \right\} \quad \text{if} \quad 1 < p \leq 2
\]
\[
\max \left\{ \left\| \Gamma(x_0, x_0)^{\frac{p}{2}} \right\|_{S_{I}^{p}}, \left\| \Gamma(x_0, x_0) \right\|_{S_{I}^{p}} \right\} \quad \text{if} \quad p \geq 2,
\]

where the infimum is taken over all \( y, z \in M_{f,\text{fin}} \) such that \( x_0 = y + z \).

**Proposition 5.18** Suppose \( 1 < p \leq \infty \).

1. For \( 2 \leq p \leq \infty \), \( \| . \|_{\Gamma,\alpha,p} \) is a seminorm on \( M_{f,\text{fin},0} \oplus \text{Cld}_{\ell_2} \).
2. For \( 1 < p \leq 2 \), \( \| . \|_{\Gamma,\alpha,p} \) is a \( \frac{2}{p} \)-seminorm on \( M_{f,\text{fin},0} \oplus \text{Cld}_{\ell_2} \), that is, the triangle inequality holds under the form \( \| x + y \|_{\Gamma,\alpha,p} \leq \| x \|_{\Gamma,\alpha,p} + \| y \|_{\Gamma,\alpha,p} \).
3. Suppose that the map \( \alpha : I \to H \) is injective. If \( p = \infty \), we suppose that \( I \) is finite. We have
\[
\{ x \in S_{I,0}^{\infty} \oplus \text{Cld}_{\ell_2} : \| x \|_{\Gamma,\alpha,p} = 0 \} = \text{Cld}_{\ell_2}
\]
and
\[ \{ x \in (S^\infty_{\ell_0})_{\text{sa}} \oplus \mathbb{R} \text{Id}_{\ell_0} : \| x \|_{\Gamma, \alpha, p} = 0 \} = \mathbb{R} \text{Id}_{\ell_0} . \]

\textbf{Proof :} 1. If \( x = x_0 + \lambda \text{Id}_{\ell_0} \) belongs to \( \mathcal{M}_{I, \text{fin}} \oplus \mathcal{C} \text{Id}_{\ell_0} \) and \( k \in \mathbb{C} \), we have \( kx = kx_0 + k\lambda \text{Id}_{\ell_0} \). Hence
\[
\| kx \|_{\Gamma, \alpha, p} = \max \left\{ \| \Gamma(kx_0, kx_0) \|_{S_{\ell_0}^p}, \| \Gamma((kx_0)^*, (kx_0)^*) \|_{S_{\ell_0}^p} \right\}
= \max \left\{ k \| \Gamma(x_0, x_0) \|_{S_{\ell_0}^p}, k \| \Gamma(x_0^*, x_0^*) \|_{S_{\ell_0}^p} \right\}
= |k| \| \max \left\{ \| \Gamma(x_0, x_0) \|_{S_{\ell_0}^p}, \| \Gamma(x_0^*, x_0^*) \|_{S_{\ell_0}^p} \right\} = |k| \| x \|_{\Gamma, \alpha, p} .
\]

Let us turn to the triangle inequality. For any \( x_0, x_0' \in \mathcal{M}_{I, \text{fin}} \), we deduce with Lemma 2.2 that
\[
\| \Gamma(x_0, x_0') \|_{S_{\ell_0}^p} \leq \| \Gamma(x_0, x_0') \|_{S_{\ell_0}^p} \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p} ,
\]
with this inequality at hand, we can now estimate for arbitrary \( x_0, x_0' \in \mathcal{M}_{I, \text{fin}} \) (note that since \( p \geq 2 \), \( S_{\ell_0}^p \) is a normed space),
\[
\| \Gamma(x_0 + x_0', x_0 + x_0') \|_{S_{\ell_0}^p}^2 = \| \Gamma(x_0 + x_0', x_0 + x_0') \|_{S_{\ell_0}^p}^2
= \| \Gamma(x_0, x_0) + \Gamma(x_0', x_0) + \Gamma(x_0, x_0') + \Gamma(x_0', x_0') \|_{S_{\ell_0}^p}^2
\leq \| \Gamma(x_0, x_0) \|_{S_{\ell_0}^p}^2 + \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p}^2 + \| \Gamma(x_0, x_0') \|_{S_{\ell_0}^p}^2 + \| \Gamma(x_0', x_0) \|_{S_{\ell_0}^p}^2
\leq \| \Gamma(x_0, x_0) \|_{S_{\ell_0}^p}^2 + \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p}^2 + 2 \| \Gamma(x_0, x_0') \|_{S_{\ell_0}^p} \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p}
= \left( \| \Gamma(x_0, x_0) \|_{S_{\ell_0}^p} + \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p} \right)^2 .
\]

We obtain
\[
\| \Gamma(x_0 + x_0', x_0 + x_0') \|_{S_{\ell_0}^p} \leq \| \Gamma(x_0, x_0) \|_{S_{\ell_0}^p} + \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p} .
\]

Then if \( x = x_0 + \lambda \text{Id}_{\ell_0} \) and \( x' = x_0' + \lambda' \text{Id}_{\ell_0} \) belong to \( \mathcal{M}_{I, \text{fin}} \oplus \mathcal{C} \text{Id}_{\ell_0} \), we have \( x + x' = x_0 + x_0' + (\lambda + \lambda') \text{Id}_{\ell_0} \). Hence
\[
\| x + x' \|_{\Gamma, \alpha, p} \overset{(5.4)}{=} \max \left\{ \| \Gamma(x_0 + x_0', x_0 + x_0') \|_{S_{\ell_0}^p}, \| \Gamma(x_0^*, x_0^*, x_0^* + x_0^*) \|_{S_{\ell_0}^p} \right\}
\leq \max \left\{ \| \Gamma(x_0, x_0) \|_{S_{\ell_0}^p} + \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p}, \| \Gamma(x_0^*, x_0^*) \|_{S_{\ell_0}^p} + \| \Gamma(x_0^*, x_0^*) \|_{S_{\ell_0}^p} \right\}
\leq \max \left\{ \| \Gamma(x_0, x_0) \|_{S_{\ell_0}^p}, \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p} \right\} + \max \left\{ \| \Gamma(x_0', x_0') \|_{S_{\ell_0}^p}, \| \Gamma(x_0^*, x_0^*) \|_{S_{\ell_0}^p} \right\}
\overset{(5.4)}{=} \| x \|_{\Gamma, \alpha, p} + \| x' \|_{\Gamma, \alpha, p} .
\]
2. The homogeneity of $\| \cdot \|_{\Gamma, \alpha, p}$ can be shown as in the case $p \geq 2$ and is left to the reader. Let us turn to the triangle inequality. For any $x_0, x_0' \in M_{I, \text{fin}}$, we obtain as in the case $p \geq 2$ with Lemma 2.2 that

\begin{equation}
\| \Gamma(x_0, x_0') \|^2_{S_I^p} \leq \| \Gamma(x_0, x_0) \|^2_{S_I^p} \| \Gamma(x_0', x_0) \|^2_{S_I^p}.
\end{equation}

With this inequality at hand, we can now estimate for arbitrary $x_0, x_0' \in M_{I, \text{fin}}$ and for $p < 2$, so that $S_I^p$ is a $\frac{p}{2}$-normed space,

\begin{equation}
\| \Gamma(x_0 + x_0', x_0 + x_0') \|^2_{S_I^p} = \| \Gamma(x_0, x_0, x_0) \|^2_{S_I^p} \\
\leq \| \Gamma(x_0, x_0') \|^2_{S_I^p} + \| \Gamma(x_0', x_0) \|^2_{S_I^p} + \| \Gamma(x_0, x_0') \|^2_{S_I^p} + \| \Gamma(x_0', x_0) \|^2_{S_I^p}
\end{equation}

Now consider some elements $x = x_0 + \lambda \text{Id}_{C_2}$ and $x' = x_0' + \lambda' \text{Id}_{C_2}$ of $M_{I, \text{fin}, 0} \oplus \text{Cld}_{C_2}$ with some decompositions $x_0 = y + z$ and $x_0' = y' + z'$, where $y, z, y', z' \in M_{I, \text{fin}}$. We have $x_0 + x_0' = y + y' + z + z'$. We obtain

\begin{equation}
\| x + x' \|_{\Gamma, \alpha, p} ^\frac{p}{2} \leq \| \Gamma(y + y', y + y') \|_{S_I^p} ^\frac{p}{2} + \| \Gamma((z + z'), (z + z')) \|_{S_I^p} ^\frac{p}{2}
\end{equation}

Passing to the infimum, we obtain

\begin{equation}
\| x + x' \|_{\Gamma, \alpha, p} ^\frac{p}{2} \leq \| x \|_{\Gamma, \alpha, p} ^\frac{p}{2} + \| x' \|_{\Gamma, \alpha, p} ^\frac{p}{2}.
\end{equation}

3. Case $1 < p < \infty$. Let $x = x_0 + \lambda \text{Id}_{C_2}$ be an element of $M_{I, \text{fin}, 0} \oplus \text{Cld}_{C_2}$. According to Theorem 3.14, we have $\| A^\frac{p}{2}(x_0) \|_{S_I^p} \leq \| x_0 \|_{\Gamma, \alpha, p}$. Note that the expression with the infimum in this Theorem 3.14 is equivalent to the infimum in (5.4). This implies that when $\| x \|_{\Gamma, \alpha, p} = 0$, that is $\| x_0 \|_{\Gamma, \alpha, p} = 0$, we have $\| A^\frac{p}{2}(x_0) \|_{S_I^p} = 0$ and finally $A^\frac{p}{2}(x_0) = 0$. Since $\alpha: I \rightarrow H$ is injective, the symbol of $A$ and consequently that of $A^\frac{p}{2}$ has non-zero entries away from the diagonal. Thus, $x_0$ has zero entries away from the diagonal. We infer that $x_0$ is a diagonal matrix. Since $x_0 \in S_{I, 0}^{\infty}$, we conclude that $x_0 = 0$. The reverse inclusion is true by (5.4).

Case $p = \infty$ and $I$ finite. Fix some $2 < p_0 < \infty$. Let $x = x_0 + \lambda \text{Id}_{C_2}$ be an element of $M_{I, \text{fin}, 0} \oplus \text{Cld}_{C_2}$. Using the part 2 of Theorem 3.14 in the first inequality, we can write

\begin{equation}
\| A^\frac{p}{2}(x_0) \|_{S_I^{p_0}} \leq \max \left\{ \| \Gamma(x_0, x_0) \|_{S_I^{p_0}}, \| \Gamma(x_0', x_0') \|_{S_I^{p_0}} \right\} \leq \| x_0 \|_{\Gamma, \alpha, \infty}.
\end{equation}
The end of the proof is similar to the case $1 < p < \infty$.

5.4 Case where $I$ is finite: new quantum compact metric spaces

By [Rie4, page 3], the linear space of all selfadjoint elements of a unital C*-algebra is an order-unit space. So if $I$ is an index set (finite or infinite) then $(S^\infty_{I,0})_{sa} \oplus \mathbb{R} \Gamma_{I,0}$ is an order-unit space. Moreover, by [Rie3, Proposition 2.3], a subspace of an order-unit space containing the order-unit is itself also an order-unit space. We conclude that $(S^\infty_{I,0})_{sa} \oplus \mathbb{R} \Gamma_{I,0}$ is an order-unit space. Now, in the case where the set $I$ is finite, we will prove in this subsection that $(S^\infty_{I,0})_{sa} \oplus \mathbb{R} \Gamma_{I,0}$ equipped with the restriction of $\| \cdot \|_{\Gamma,\alpha,p}$ (defined by (5.4)) to this space (also denoted by $\| \cdot \|_{\Gamma,\alpha,p}$) is a quantum compact metric space [Rie3]. If $I$ is infinite and $1 < p < \infty$, we should get a quantum locally compact metric space. Unfortunately, at the present moment, the theory of quantum locally compact metric spaces from [Lat1] and [Lat2] is not developed and rich enough to take care of our setting. So we focus in this subsection to the case $I$ finite, and address the case $I$ infinite with a modified seminorm in the following subsection. Moreover, since Lipschitz pairs in the sense of Definition 5.1 are only defined for the moment for seminorms and not for quasi-seminorms, in view of Proposition 5.18 above, we restrict our family $\| \cdot \|_{\Gamma,\alpha,p}$ in the next theorem to the case $2 \leq p \leq \infty$.

**Theorem 5.19** Let $I$ be a finite set. Suppose that $(T_t)_{t \geq 0}$ is a weak* continuous semigroup of completely positive unital selfadjoint Schur multipliers on $\mathcal{B}(\ell^2_I)$ associated to an injective map $\alpha: I \to H$. Suppose $2 \leq p \leq \infty$. Then $((S^\infty_{I,0})_{sa} \oplus \mathbb{R} \Gamma_{I,0}, \| \cdot \|_{\Gamma,\alpha,p})$ is a quantum compact metric space.

**Proof:** Note that we can see $S^\infty_{I,0} \oplus \mathbb{C} \Gamma_{I,0}$ as a subspace of $S^\infty_I$. We denote by $A$ the (negative) generator of the semigroup $(T_t)_{t \geq 0}$.

Case 2 \( p \leq \infty \). By finite-dimensionality, the well-defined operator $A^{-\frac{1}{2}}: S^\infty_{I,0} \to S^\infty_{I,0}$ is bounded. This implies that the set $\{ A^{-\frac{1}{2}}(x) : x \in S^\infty_{I,0}, \| x \|_{S^\infty_I} \leq 1 \}$ is bounded in $S^\infty_I$. Hence the set

\[
\{ x \in S^\infty_{I,0} : \| A^\frac{1}{2}(x) \|_{S^\infty_I} \leq 1 \}
\]

is also bounded in $S^\infty_I$. Using Theorem 3.14, for any element $x$ of $S^\infty_{I,0} = M_{I,sa,0}$, we see that

\[
\| A^\frac{1}{2}(x) \|_{S^\infty_I} \leq \| \Gamma(x,x)^{\frac{1}{2}} \|_{S^\infty_I} \leq \| x \|_{\Gamma,\alpha,p}.
\]

This inequality shows that the set

\[
\{ x \in (S^\infty_{I,0})_{sa} \oplus \mathbb{C} \Gamma_{I,0} : \| x \|_{\Gamma,\alpha,p} \leq 1 \}
\]

is a multiple of a subset of (5.11). So the set (5.12) is also bounded in $S^\infty_I$. The desired result follows from Proposition 5.5.

Case $p = \infty$. Fix some 2 \( < p_0 < \infty \). By finite-dimensionality, the operator $A^{-\frac{1}{2}}: S^{p_0}_{I,0} \to S^\infty_{I,0}$ is bounded. This implies that the set

\[
\{ x \in S^{p_0}_{I,0} : \| A^\frac{1}{2}(x) \|_{S^{p_0}_I} \leq 1 \}
\]

is bounded in $S^\infty_I$. The inequality (5.10) says that the set

\[
\{ x \in (S^\infty_{I,0})_{sa} \oplus \mathbb{C} \Gamma_{I,0} : \| x \|_{\Gamma,\alpha,\infty} \leq 1 \}
\]

is also bounded in $S^\infty_I$. The proof is similar to the case 1 < $p < \infty$.
is bounded, i.e. relatively compact in $S^\infty_I$. The desired result follows from Proposition 5.5.

A natural additional property of the Lipschitz-seminorm over a classical compact metric space is the lower semicontinuity of the seminorm with respect to the original sup-norm. We show that the same holds in the quantum compact metric space from Theorem 5.19 above. In the following proposition, note that the proved continuity implies in particular the lower semicontinuity.

**Proposition 5.20** Let $I$ be a finite set. Suppose $1 < p \leq \infty$. Then $\|\cdot\|_{\Gamma,\alpha,p}$ is continuous on $S^\infty_{I,0} \oplus \text{CId}_{I^2}$.

**Proof**: If $2 \leq p \leq \infty$ and if $x \in M_{I,\text{fin}}$ then

$$\|\Gamma(x,x)\|_I^\frac{1}{2} \|\cdot\|_I^\frac{1}{2} \|S_I^p\|^\frac{1}{2} \lesssim_{I,\alpha} \|\cdot\|_I \|S_I^p\|^\frac{1}{2} \|x\|_I \|S^\infty_I\|^\frac{1}{2}.$$ 

In a similar manner, we have $\|\Gamma(x^*,x^*_n)\|_I^\frac{1}{2} \|\cdot\|_I^\frac{1}{2} \|S_I^p\|^\frac{1}{2} \lesssim_{I,\alpha} \|\cdot\|_I \|S^\infty_I\|^\frac{1}{2}$. Let $(x_n)$ be a sequence which converges in $S^\infty_I$ to $x$. Then by the above $\|x - x_n\|_{\Gamma,\alpha,p} \to 0$. If $1 < p \leq 2$, then $\|\cdot\|_{\Gamma,\alpha,p} \lesssim \|\Gamma(x,x)\|_I \|x\|_I \|S^\infty_I\|^\frac{1}{2}$. We conclude as in the case $2 \leq p \leq \infty$. 

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5.5 Case where \( I \) is infinite: new quantum locally compact metric spaces

We assume \( I \) countably infinite (or finite) in this subsection. For simplicity, we take hereby \( I = \mathbb{N} \) (or \( I = \{0, 1, \ldots, n\} \)). Our principal result in this subsection, Theorem 5.25, will be to describe new quantum locally compact metric spaces. As these quantum locally compact metric spaces are not yet defined for order-unit spaces, we will now create a variant \( \|\|_{\Gamma, \alpha, \rho, \omega} \) of \( \|\|_{\Gamma, \alpha, \rho} \), which in Theorem 5.25 will yield such a quantum locally compact metric space in the sense of Latremolière [Lat1], on the C*-algebra \( S^\infty \oplus \text{Cld}_{l^2} \). To this end, let \( (e_n)_{n \in \mathbb{N}} \) be the canonical orthonormal basis of \( l^2 \). Moreover, for \( x \in M_{I, \text{fin}} \oplus \text{Cld}_{l^2} \), we let

\[
\|x\|_{\Gamma, \alpha, \rho, \omega} = \sum_{n=0}^{\infty} |\langle xe_{n+1}, e_{n+1}\rangle - (xe_n, e_n)|.
\]  

Finally, for \( x \in M_{I, \text{fin}} \oplus \text{Cld}_{l^2} \), we let

\[
\|x\|_{\Gamma, \alpha, \rho} = \begin{cases} \|x\|_{\Gamma, \alpha, \rho, \omega}^{\frac{p}{2}} & \text{if } 1 < p \leq 2 \\ \|x\|_{\Gamma, \alpha, \rho, \omega} & \text{if } p > 2, \end{cases}
\]

where

\[
\|x\|_{\Gamma, \alpha, \rho} \overset{\text{def}}{=} \begin{cases} \inf_{x_0 = y + z} \left\{ \|\Gamma(y, y, y, z)^{\frac{p}{2}} + \|\Gamma(z^*, z^*)\|_{S^\infty_{l^2}}^{\frac{p}{2}}\|_{S^\infty_{l^2}} \right\} & \text{if } 1 < p \leq 2 \\ \max \left\{ \|\Gamma(x_0, x_0)\|_{S^\infty_{l^2}}, \|\Gamma(x_0, x_0)\|_{S^\infty_{l^2}} \right\} & \text{if } p > 2. \end{cases}
\]

Here, the infimum is taken over all decompositions \( x_0 = y + z \) with \( y, z \in M_{I, \text{fin}} \oplus \text{Cld}_{l^2} \). We note that if \( x = x_1 \oplus \lambda I_{l^2} \) is an element of \( M_{I, \text{fin}} \oplus \text{Cld}_{l^2} \), then \( Ax, A(x^*) \) belong to \( M_{I, \text{fin}} \), so that \( \Gamma(x, x), \Gamma(x^*, x^*) \in M_{I, \text{fin}} \) and finally, \( \Gamma(x, x)^{\frac{p}{2}}, \Gamma(x^*, x^*)^{\frac{p}{2}} \in M_{I, \text{fin}} \subseteq S^\infty_{l^2} \). We infer that \( \|x\|_{\Gamma, \alpha, \rho} \leq \infty \). Moreover, \( \|x_1\|_{\omega} < \infty \) and \( \|\lambda I_{l^2}\|_{\omega} = \infty \). Thus \( \|x\|_{\omega} < \infty \) and therefore, \( \|x\|_{\Gamma, \alpha, \rho, \omega} < \infty \). We conclude that \( \|x\|_{\Gamma, \alpha, \rho, \omega} \) is a well-defined finite expression for \( x \in M_{I, \text{fin}} \oplus \text{Cld}_{l^2} \).

**Proposition 5.21** Suppose \( 1 < p < \infty \).

1. \( \|\|_{\Gamma, \alpha, \rho, \omega} \) is a seminorm (\( \|\|_{\Gamma, \alpha, \rho} \)-seminorm if \( 1 < p \leq 2 \)) on \( M_{I, \text{fin}} \oplus \text{Cld}_{l^2} \).

2. The seminorm \( \|\|_{\Gamma, \alpha, \rho, \omega} \) (\( \|\|_{\Gamma, \alpha, \rho} \)-seminorm if \( 1 < p \leq 2 \)) is densely defined in \( S^\infty_{l^2} \oplus \text{Cld}_{l^2} \).

3. Suppose that the map \( \alpha : I \to H \) is injective. Then we have

\[
\{ x \in S^\infty_{l^2} \oplus \text{Cld}_{l^2} : \|x\|_{\Gamma, \alpha, \rho, \omega} = 0 \} = \text{Cld}_{l^2}
\]

and

\[
\{ x \in (S^\infty_{l^2} \oplus \text{Cld}_{l^2})_{\alpha} : \|x\|_{\Gamma, \alpha, \rho, \omega} = 0 \} = \text{RId}_{l^2}.
\]

**Proof** : 1. It is elementary that \( \|\|_{\omega} \) is a seminorm on \( M_{I, \text{fin}} \oplus \text{Cld}_{l^2} \). Similarly to Proposition 5.18, we prove that if \( p \geq 2 \) (resp. \( 1 < p \leq 2 \)), then the sum \( \|\|_{\Gamma, \alpha, \rho} + \|\|_{\omega} \) is a seminorm (resp. \( \left( \|\|_{\Gamma, \alpha, \rho} + \|\|_{\omega} \right)^{\frac{p}{2}} \) is a \( \frac{p}{2} \)-seminorm) on \( M_{I, \text{fin}} \oplus \text{Cld}_{l^2} \).
2. It is well-known that \( \text{M}_{I, \text{fin}} \) is dense in \( S^\infty_p \), the space of compact operators over \( \ell^2_1 \). Thus, also \( \text{M}_{I, \text{fin}} \oplus \text{Id}_{\ell^2_1} \) is dense in \( S^\infty_p \oplus \text{Id}_{\ell^2_1} \subseteq B(\ell^2_1) \).

3. If \( 2 \leq p < \infty \) (slight modification if \( 1 < p \leq 2 \) below), we clearly have

\[
\|\lambda \text{Id}_{\ell^2_1}\|_{\Gamma, \alpha, p, \partial} = |\lambda| \|\Gamma(\text{Id}_{\ell^2_1}, \text{Id}_{\ell^2_1})\|_{S^p_1} + |\lambda| \|\text{Id}_{\ell^2_1}\|_p = 0 + 0.
\]

We turn to the reciprocal inclusions. Let \( x \in \text{M}_{I, \text{fin}} \oplus \mathbb{C}1_{B(\ell^2_1)} \) such that \( \|x\|_{\Gamma, \alpha, p, \partial} = 0 \). Then

\[
\|\Gamma(x, x)\|_{S^p_1} = \|\Gamma(x^*, x^*)\|_{S^p_1} = 0,
\]

so that \( \Gamma(x, x) \) and \( \Gamma(x^*, x^*) \) are seminorms and not for quasi-seminorms, in view of Proposition 3.14, we have in case \( 2 \leq p < \infty \) (slight modification if \( 1 < p \leq 2 \) in the expression below),

\[
\left\| A^{\sharp}(x) \right\|_{S^p_1} \leq \max \left\{ \|\Gamma(x, x)\|_{S^p_1}, \|\Gamma(x^*, x^*)\|_{S^p_1} \right\} = 0,
\]

so \( A^{\sharp}(x) = 0 \). Since \( \alpha : I \rightarrow H \) is assumed to be injective, this implies that \( \langle xe_n, e_k \rangle = 0 \) for any \( n, k \in I \) with \( n \neq k \). Thus, \( x \) is diagonal in the basis \( (e_n)_{n \in I} \). We also have \( \|x\|_\partial = 0 \), so that \( \sum_{n=0}^{\infty} |\langle xe_{n+1}, e_n \rangle - \langle xe_n, e_n \rangle| = 0 \), and consequently, \( \langle xe_n, e_n \rangle = a \in \mathbb{C} \) for some \( a \in \mathbb{C} \) and all \( n \in I \). We infer that \( x \) belongs to \( \text{CId}_{\ell^2_1} \). This shows the first set identity. It also shows the second set identity, because \( (S^\infty_p \oplus \text{Id}_{\ell^2_1}) \cap \text{CId}_{\ell^2_1} = \text{RId}_{\ell^2_1} \).

Since Lipschitz pairs in the sense of Definition 5.1 are only defined for the moment for seminorms and not for quasi-seminorms, in view of Proposition 5.21 above, we restrict our family \( \|\|_{\Gamma, \alpha, p, \partial} \) to the case \( 2 \leq p < \infty \) for the rest of this subsection. In Lemma 3.15 we had extended the definition of \( \Gamma(x, y) \) to a larger domain than \( \text{M}_{I, \text{fin}} \). In the following lemma, this will give a larger domain of \( \|\|_{\Gamma, \alpha, p, \partial} \) such that all the properties of the preceding Proposition 5.21 remain true. We adopt the usual convention that if \( \|\| \) is a seminorm defined on a dense subspace \( \|\| \) of a topological vector space \( V \), and if \( x \in V \) is not in the domain of \( \|\| \), then \( \|x\| = \infty \). With this convention, we observe that:

\[
\text{dom} \|\| = \{x \in V : \|x\| < \infty \}.
\]

Note that with this convention, we do not introduce any ambiguity when talking about lower semi-continuous seminorms by exchanging the original seminorm with its extension.

**Lemma 5.22** Let \( 2 \leq p < \infty \). Assume \( \alpha : I \rightarrow H \) injective. Let \( \text{dom} \|\|_{\Gamma, \alpha, p} = \text{dom} A^{\sharp}_{I} \oplus \text{CId}_{\ell^2_1} \) and define by Lemma 3.15 for \( x = x_1 + \lambda \text{Id}_{\ell^2_1} \)

\[
\|x\|_{\Gamma, \alpha, p} = \max \left\{ \|\Gamma(x_1, x_1)\|_{p}, \|\Gamma(x_1^*, x_1^*)\|_{p} \right\}.
\]

Let also \( \text{dom} \|\|_{\partial} = \{x \in S^1 \oplus \text{Id}_{\ell^2_1} : \sum_{n=1}^{\infty} |\langle xe_{n+1}, e_n \rangle - \langle xe_n, e_n \rangle| < \infty \} \) and put \( \|x\|_\partial = \sum_{n=1}^{\infty} |\langle xe_{n+1}, e_n \rangle - \langle xe_n, e_n \rangle| \), as well as dom \( \|\|_{\Gamma, \alpha, p, \partial} = \text{dom} \|\|_{\Gamma, \alpha, p} \cap \text{dom} \|\|_{\partial} \)

Together with \( \|x\|_{\Gamma, \alpha, p, \partial} = \|x\|_{\Gamma, \alpha, p} + \|x\|_{\partial} \). Then the properties of \( \|\|_{\Gamma, \alpha, p, \partial} \) stated in Proposition 5.21 remain true.

**Proof**: We start by showing that \( \|\|_{\Gamma, \alpha, p} \) is a seminorm. It is plain that \( \|\lambda x\|_{\Gamma, \alpha, p} = |\lambda| \|x\|_{\Gamma, \alpha, p} \).

Note that if \( x, y \in \text{dom} A^{\sharp}_{I} \), then \( x + y \in \text{dom} A^{\sharp}_{I} \) and

\[
\|\Gamma(x + y, x + y)\|_{p} = \|\Gamma(x, y)\|_{p} = \lim_n \|\Gamma(T_n x + T_n y, T_n x + T_n y)\|_{p} = \lim_n \|\Gamma(T_n x + T_n y, T_n x + T_n y)\|_{p} = \|\Gamma(T_n y, T_n y)\|_{p} \leq \|\Gamma(T_n x, T_n x)\|_{p} + \|\Gamma(T_n y, T_n y)\|_{p} \leq \|\Gamma(x, x)\|_{p} + \|\Gamma(y, y)\|_{p}.
\]

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Arguing in the same way for $x^*$ and $y^*$, we obtain that $\|\cdot\|_{\Gamma, \alpha, \beta}$ is a seminorm and finally that $\|\cdot\|_{\Gamma, \alpha, \beta}$ is a seminorm on its domain defined in the lemma above. Since both $\text{dom} \|\cdot\|_{\Gamma, \alpha, \beta} = \text{dom}(A_1^2) \oplus \text{Cld} \ell_1^2$ and $\text{dom} \|\cdot\|_{\beta}$ contain $M_{\text{fin}} \oplus \text{Cld} \ell_1^2$, the domain of $\|\cdot\|_{\Gamma, \alpha, \beta, \beta}$ is clearly dense in $S_1^\infty \oplus \text{Cld} \ell_1^2$. We turn to the last statement of Proposition 5.2.1. The inclusions $\text{Cld} \ell_1^2 \subseteq \{ x \in S_1^\infty \oplus \text{Cld} \ell_1^2 : \|x\|_{\Gamma, \alpha, \beta, \beta} = 0 \}$ and $\text{Rld} \ell_1^2 \subseteq \{ x \in (S_1^\infty \oplus \text{Cld} \ell_1^2)_{\text{sa}} : \|x\|_{\Gamma, \alpha, \beta, \beta} = 0 \}$ are clear from this proposition. For the reverse inclusion, assume $x \in S_1^\infty \oplus \text{Cld} \ell_1^2$ such that $\|x\|_{\Gamma, \alpha, \beta, \beta} = 0$.

Write $x = x_1 + \lambda \text{Id} \ell_1^2$ with $x_1 \in \text{dom} A_1^2$ and $\lambda \in \mathbb{C}$. We have to show that $x_1 = 0$. Since $\|x\|_{\Gamma, \alpha, \beta} = 0$, we have $\|\Gamma(T_n x_1, T_n x_1)\|_p^2 = \|\Gamma(T_n x_1)\|_p^2 \to 0$ and $\|\Gamma(T_n x_1, T_n x_1)\|_p^2 = \|\Gamma(T_n x_1, T_n x_1)\|_p^2 \to 0$. Therefore, according to Theorem 3.20, $L_2(x, p) = \lim_\alpha_2 L_2(x, p) = 0$. We infer that $A_1^2 x = 0$. The rest of the proof is identical to that of Proposition 5.21, using that $\alpha : I \to H$ is injective.

Proposition 5.21 shows that $(S_1^\infty, \|\cdot\|_{\Gamma, \alpha, \beta})$ is a Lipschitz pair. We shall turn it into a Lipschitz triple in the sense of Definition 5.13.

Recall that $I = \{0, \ldots, n\}$ or $I = \mathbb{N}$. Let $(e_i)_{i \in I}$ be the canonical orthonormal basis of $\ell_1^2$. Let $P_i$ be the orthogonal projection onto span$\{e_0, \ldots, e_i\}$. Then $(P_i)$ is an approximate unit for $S_1^\infty$. Let $D_i$ be the $C^*$-subalgebra of $S_1^\infty$ consisting of compact operators which are diagonal for $(e_i)_{i \in I}$, i.e. the $C^*$-algebra generated by $(P_i)_{i \in I}$. Note that $D_i$ is abelian. Thus, $(S_1^\infty, \|\cdot\|_{\Gamma, \alpha, \beta, D_i})$ is a Lipschitz triple.

**Lemma 5.23** Let $(T_t)_{t \geq 0}$ be a bounded $C_0$-semigroup on a Banach space $X$. Assume that $\|T_t\|_{X \to X} \leq C_0^t$ for some $d > \frac{1}{2}$. Then $A^{-\frac{1}{2}}_t$ is initially defined on $\text{ran}(A) \cap \text{dom}(A)$, extends to a bounded operator on $\text{ran}(A) \subseteq X$. Moreover, for $x \in \text{ran}(A)$, we have

$$A^{-\frac{1}{2}}_t(x) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty t^{-\frac{1}{2}} T_t(x) \, dt.$$  

**Proof:** According to [Haa1, Proposition 3.3.5], we have for $\varepsilon > 0$ and any $x \in X$

$$\begin{align*}
(A + \varepsilon)^{-\frac{1}{2}} x &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty t^{-\frac{1}{2}} e^{-\varepsilon t} T_t(x) \, dt.
\end{align*}$$

Note that the right hand side converges to $\frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty t^{-\frac{1}{2}} T_t(x) \, dt$ as $\varepsilon \to 0$, and that this integral exists in particular. Indeed, we can apply dominated convergence together with

$$\begin{align*}
\int_0^\infty \left| t^{-\frac{1}{2}} e^{-\varepsilon t} \right| |T_t(x)| \, dt &\leq \int_0^1 \left( t^{-\frac{1}{2}} \|T_t(x)\| \right) \, dt + \int_1^\infty t^{-\frac{1}{2}} \|T_t(x)\| \, dt \\
&\leq \int_0^1 t^{-\frac{1}{2}} \|x\| \, dt + C \int_1^\infty t^{-\frac{1}{2}} t^{-d} \|x\| \, dt \\
&\leq \left( 2 + C \int_1^\infty t^{-(d+\frac{1}{2})} \right) \|x\| \|x\|.
\end{align*}$$

Moreover, according to [KuWe, 9.2 Theorem (i) and Section 15A], $A^{-\frac{1}{2}}_t$ is a well-defined closed operator and the left hand side of (5.16) converges for $x \in \text{ran}(A) \cap \text{dom}(A)$ to $A^{-\frac{1}{2}}_t x$, as $\varepsilon \to 0$. Indeed, such an $x$ can be written as $x = A(1 + A)^{-\frac{1}{2}} y$ for some $y \in X$ and $(\varepsilon + \lambda)^{-\frac{1}{2}} \lambda(1 + \lambda)^{-\frac{1}{2}}$ pointwisely and boundedly in $H^\infty(\Sigma_0)$ for any $\varepsilon \in (0, \pi)$. Then according to the estimate (5.17), $A^{-\frac{1}{2}}_t$ extends boundedly to $\text{ran}(A) \cap \text{dom}(A) = \text{ran}(A)$, where the last equality follows from the standard approximate identity $\delta_n(A)$ in [KuWe, 9.4 Proposition (b)].
Lemma 5.24 Assume that $\alpha : I \to H$ is injective where $H$ is a Hilbert space of dimension $n \in \mathbb{N}$, and that $\text{Gap}_\alpha > 0$, where we recall that $\text{Gap}_\alpha$ is defined in (5.3). Then for $2 \leq p < \infty$, $A^-_{\alpha} : S^p_{I,0} \to S^p_{I,0} \subseteq S^\infty_{I,0}$ is compact.

Proof: We begin by showing that $A^-_{\alpha} : S^2_{I,0} \to S^2_{I,0}$ is compact. Note first that since $\alpha$ is assumed to be injective, $\alpha_{ij} = \|\alpha_i - \alpha_j\|^2 > 0$ for $i \neq j$, so that $A^-_{\alpha}$ is a well-defined diagonal operator on the Hilbert space $S^2_{I,0}$. Since we have $\text{Gap}_\alpha > 0$, in particular, we cannot have infinitely many points of $\alpha(I) - \alpha(I)$ inside any ball of the finite-dimensional Hilbert space $H$. This means that the set $\{\|\alpha_i - \alpha_j\|^2 : i, j \in I, i \neq j\}$ cannot have a cluster point different from 0, so that $A^-_{\alpha} : S^2_{I,0} \to S^2_{I,0}$ is a compact operator. Next we show that $A^-_{\alpha} : S^\infty_{I,0} \to S^\infty_{I,0}$ is bounded. Indeed, note that $(T_t)_{t \geq 0}$ extends to a bounded $C_0$-semigroup on $S^\infty_{I,0}$. Moreover, Lemma 5.17 (where we inject $H$ beforehand into a 2-dimensional Hilbert space so that this lemma yields an estimate $\|T_t\|_{\infty \to \infty} \leq Ct^{-\frac{1}{2}}$ with $\frac{1}{2} > \frac{1}{2}$) together with Lemma 5.23 yield that $A^-_{\alpha}$ is bounded on $\mathcal{H}(A) = S^\infty_{I,0}$. We now interpolate the operator $A^-_{\alpha}$ between levels 2 resp. $\infty$, where it is compact resp. bounded. Thus, according to [Perss, Theorem page 217], for $2 \leq p < \infty$, $A^-_{\alpha} : S^p_{I,0} \to S^p_{I,0}$ remains to be compact. Indeed, we have $S^p_{I,0} = (S^2_{I,0}, S^\infty_{I,0})_\theta$ for the right $\theta \in (0, 1)$ and the couple $(S^2_{I,0}, S^\infty_{I,0})$ fulfills hypothesis [Perss, (H1) page 217] with the projections $P : S^\infty_{I,0} \to S^2_{I,0} \cap S^\infty_{I,0}$ given by a upper left corner, i.e $P x = Q x Q$ with $Q : \ell^2_I \to \ell^2_I$, $Q(x) = Q((\xi_i)_{i \in I}) = (\xi_{i \in I} (\xi)_{i \in I} \in I$.

Theorem 5.25 Suppose that $I$ is countable or finite. Suppose $2 \leq p < \infty$. If $\dim H < \infty$, if the map $\alpha : I \to H$ is injective and if $\text{Gap}_\alpha > 0$ then $(S^\infty_{I,0}, \|\cdot\|_{\Gamma,\alpha,p,\partial} \cap \mathcal{D}_I)$ is a quantum locally compact metric space (see Theorem 5.14 for the definition of this notion).

Proof: According to Lemma 5.24, the set
\[
\{ x \in S^\infty_{I,0} : \|A^-_{\alpha}(x)\|_{S^p_{I,0}} \leq 1 \}
\]
is relatively compact in $S^\infty_{I,0}$. Using Theorem 3.20, we see that
\[
\|A^-_{\alpha}(x)\|_{S^p_{I,0}} \leq \max \left\{ \|\Gamma(x, x)^{\frac{1}{2}}\|_{S^{p'}_{I,0}}, \|\Gamma(x^*, x)^{\frac{1}{2}}\|_{S^{p'}_{I,0}} \right\} \leq \|x\|_{\Gamma,\alpha,p,\partial}.
\]
We deduce from this inequality that
\[
\{ x \in S^\infty_{I,0} : \|x\|_{\Gamma,\alpha,p,\partial} \leq 1 \}
\]
is relatively compact in $S^\infty_{I,0}$. Subsequently, or aim will be to apply the characterization 3. of Theorem 5.14. We define the local state $\mu(x) = \langle xe_0, e_0 \rangle$. Now let $Q$ be a projection in $\mathcal{D}_I$. Then there exists some $n \in I$ such that
\[
Q \{ x \in S^\infty_{I,0} \cap \mathcal{C}Id_{I^n} : \|x\|_{\Gamma,\alpha,p,\partial} \leq 1, \mu(x) = 0 \} Q \subset P_n \{ x \in S^\infty_{I,0} \cap \mathcal{C}Id_{I^n} : \|x\|_{\Gamma,\alpha,p,\partial} \leq 1, \mu(x) = 0 \} P_n.
\]
The latter set is bounded hence (since it lives in a finite dimensional subspace) relatively compact in $S^\infty_{I,0} \cap \mathcal{C}Id_{I^n}$, provided that
\[
\{ x \in S^\infty_{I,0} \cap \mathcal{C}Id_{I^n} : \|x\|_{\Gamma,\alpha,p,\partial} \leq 1, \mu(x) = 0 \}
\]
is bounded. We show this now. Let \( x = x_1 \oplus x_2 \oplus aI_{d_2} \) belong to this set, with \( x_1 \in S_{I,0}^\infty \), \( x_2 \in D_I \) and \( a \in \mathbb{C} \), such that \( \langle x_2 e_0, e_0 \rangle = 0 \). Then \( a = 0 \) since \( \mu(x) = \mu(x_1) + \mu(x_2) + a = 0 + 0 + a = 0 \). On the other hand,

\[
\|x_2\|_\partial = \|x - x_1\|_\partial \leq \|x\|_\partial + \|x_1\|_\partial = \|x\|_\partial \leq \|x\|_{\Gamma,\alpha,p,\partial} < 1.
\]

It is easy to deduce that \( \|x_2 e_n, e_n\| \leq 1 \) for all \( n \in I \), so that \( \|x_2\|_{S_{I}^\infty} \leq 1 \). Moreover, since we multiplied with \( P_n \) from the left and the right, we can suppose that \( x_2 = P_n x_2 P_n \). Then \( \|\Gamma(x_2, x_2)\|^\frac{1}{2} \|\|S_{I}^\infty\| \leq 1 \). Similarly, \( \|x_2\|_{\Gamma,\alpha,p,\partial} \leq 1 \). Therefore, also \( \|x_1\|_{\Gamma,\alpha,p,\partial} = \|x_1\|_{\Gamma,\alpha,p,\partial} \leq 1 \). We conclude by (5.18) that \( x_1 \) varies in a relatively compact set. We conclude that \( x = x_1 \oplus x_2 \) varies in a bounded set. The desired result follows from [La1, Theorem 3.9], see also the proof of [La1, Proposition 4.7].

Now we will show that the quantum locally compact norm \( \|\|_{\Gamma,\alpha,p,\partial} \) satisfies the additional properties of lower semicontinuity and the Leibniz property discussed in Subsection 5.1. Let \( X \) be a topological space. Recall that a function \( f : X \to \mathbb{R} \) is lower semicontinuous if for any \( K \in \mathbb{R} \) the subset \( \{ x \in X : \|x\| \leq K \} \) of \( X \) is closed.

**Proposition 5.26** Let \( 2 \leq p < \infty \). Assume that \( \alpha : I \to H \) is injective, that \( \text{Gap}_\alpha > 0 \) and \( \dim H < \infty \). Then the seminorm \( \|\|_{\Gamma,\alpha,p,\partial} \) is lower semicontinuous.

**Proof**: We begin by showing that \( \|\|_{\partial} \) is lower semicontinuous. We apply the criterion [Rei, Definition 3.3] and thus show that whenever \( (x_n)_{n \in \mathbb{N}} \) is a sequence in \( S_{I}^\infty \oplus \text{Cld}_{d_2} \) converging to some \( x \in S_{I}^\infty \oplus \text{Cld}_{d_2} \) with \( \|x_n\|_{\partial} \leq 1 \), then \( x \in \text{dom} \|\|_{\partial} \) and \( \|x\|_{\partial} \leq 1 \). Since operator norm convergence implies weak operator convergence, we have \( \langle x e_k, e_k \rangle = \lim_n \langle x_n e_k, e_k \rangle \) for any \( k \in I \). Thus, using Fatou’s lemma, we obtain

\[
\|x\|_{\partial} = \sum_{k \in I} |\langle x e_k, e_k \rangle| = \sum_{k \in I} \lim_{n \to +\infty} |\langle x_n e_k, e_k \rangle| \leq \liminf_{n \to +\infty} \sum_{k \in I} |\langle x_n e_k, e_k \rangle| = \liminf_{n \to +\infty} \|x_n\|_{\partial}.
\]

We turn to the part \( \|\|_{\Gamma,\alpha,p,\partial} \). We thus have \( x_n \in \text{dom} A_{\alpha}^{\frac{1}{p}} \oplus \text{Cld}_{d_2} \) and \( \|x_n\|_{\Gamma,\alpha,p,\partial} \leq 1 \). This implies by Theorem 3.14 that \( \|\| A_{\alpha}^\frac{1}{p} T_k x_n \| \leq C \) for any \( k \in I \) and \( n \in \mathbb{N} \). Then by finite dimensionality, we have \( T_k x = \lim_n T_k x_n \) and also \( A_{\alpha}^\frac{1}{p} T_k x = \lim_n A_{\alpha}^\frac{1}{p} T_k x_n \), so that \( \|\| A_{\alpha}^\frac{1}{p} T_k x \| \leq C \). We infer that \( x \in \text{dom} A_{\alpha}^\frac{1}{p} \oplus \text{Cld}_{d_2} \). 6 We thus have \( \Gamma(x, x) = \lim_k \Gamma(T_k x, T_k x) \) in \( S_{I}^\frac{1}{2} \). Using Lemma 3.18 in the third equality, we have

\[
\|\| x \| \|_{\frac{1}{p}} = \|\| A_{\alpha}^\frac{1}{p} T_k x \| \| \|_{\frac{1}{p}} = \lim_k \Gamma(T_k x, T_k x) \| \|_{\frac{1}{p}} = \sup_k \Gamma(T_k x, T_k x) \| \|_{\frac{1}{p}} = \sup_k \|T_k x, T_k x\| \| \|_{\frac{1}{p}}.
\]

In the same way, we have \( \|\| x_n \| \|_{\frac{1}{p}} = \sup_k \|T_k x_n, T_k x_n\| \| \|_{\frac{1}{p}} \). Since on the finite dimensional edges, we have \( T_k x = \lim_n T_k x_n \) and thus \( \Gamma(T_k x, T_k x) = \lim_n \Gamma(T_k x_n, T_k x_n) \), we deduce

\[
\|\| x \| \|_{\frac{1}{p}} = \sup_k \Gamma(T_k x, T_k x) \| \|_{\frac{1}{p}} = \sup_k \Gamma(T_k x_n, T_k x_n) \| \|_{\frac{1}{p}} \leq \sup_k \Gamma(T_k x_n, T_k x_n) \| \|_{\frac{1}{p}} = \sup_k \Gamma(T_k x_n, T_k x_n) \| \|_{\frac{1}{p}} = \sup_k \Gamma(x_n, x_n) \| \|_{\frac{1}{p}} \leq 1.
\]

6. Note that if \( \alpha : I \to H \) is injective, \( \text{Gap}_\alpha > 0 \) and \( \dim H < \infty \), then according to Lemma 5.24, \( A_{\alpha}^\frac{1}{p} : S_{I,0}^p \to S_{I,0}^p \) is bounded, so that \( A_{\alpha}^\frac{1}{p} x \in S_{I,0}^p \) implies \( x \in S_{I,0}^p \oplus \text{Cld}_{d_2} \).
In the same way, we have $\| \Gamma(x^*, x^*) \|_p^\perp \leq \sup_n \| \Gamma(x_n^*, x_n^*) \|_p^\perp \leq 1$. Now we can conclude since the sum of the two lower semicontinuous functions $\| \Gamma_{\alpha, p} \|$ and $\| \cdot \|_p$ is again lower semicontinuous. ■

5.6 Leibniz Condition for seminorms associated to semigroups of Schur multipliers

Proposition 5.27 Suppose $2 \leq p < \infty$. For any $x, y \in M_{\ell, \mathbb{f}}$, we have

$$\| xy \|_{\Gamma_{\alpha, p}} \leq \| x \|_{S_p^\perp} \| y \|_{\Gamma_{\alpha, p}} + \| y \|_{S_p^\perp} \| x \|_{\Gamma_{\alpha, p}}. \tag{5.19}$$

Moreover, (5.19) also holds for $x, y \in \text{dom} A_\perp^\perp$ under the additional hypotheses from Proposition 5.26.

Proof : Suppose first that $x, y \in M_{\ell, \mathbb{f}}$. Then using the structure of bimodule of $S_p^\perp(L^2(\Omega)_{c, p})$ in the second inequality, we see that

$$\| \Gamma(x, y) \|_{S_p^\perp} \leq \| x \|_{S_p^\perp} \| y \|_{S_p^\perp} + \| y \|_{S_p^\perp} \| x \|_{\Gamma_{\alpha, p}}. \tag{2.31}$$

Similarly, we have

$$\| \Gamma(y^*, x^*) \|_{S_p^\perp} \leq \| x^* \|_{\text{B}(L^2(\Omega))} \| y^* \|_{S_p^\perp} + \| y^* \|_{\text{B}(L^2(\Omega))} \| x^* \|_{S_p^\perp}. \tag{2.34}$$

Therefore, $\| \cdot \|_{\Gamma_{\alpha, p}}$ satisfies the Leibniz condition (5.19). Now assume that the additional hypotheses from Proposition 5.26 hold. If $x, y$ belong to $\text{dom} A_\perp^\perp$, we have $x = \lim_n T_n x$ and $y = \lim_n T_n y$ where the convergence holds in $S_p^\perp$, so in particular in $S_p^\perp c, p$. Thus also $xy = \lim_n T_n x T_n y$ in $S_p^\perp$. According to the lower semicontinuity of $\| \cdot \|_{\Gamma_{\alpha, p}}$ from the proof of Proposition 5.26, we have by the first case

$$\| xy \|_{\Gamma_{\alpha, p}} \leq \liminf_n \| T_n x T_n y \|_{\Gamma_{\alpha, p}} \leq \liminf_n \| T_n x \|_{\infty} \| T_n y \|_{\Gamma_{\alpha, p}} + \| T_n y \|_{\infty} \| T_n x \|_{\Gamma_{\alpha, p}}$$

$$= \| x \|_{\infty} \| y \|_{\Gamma_{\alpha, p}} + \| y \|_{\infty} \| x \|_{\Gamma_{\alpha, p}}. \tag{5.26}$$

Here, we have also used that $A_\perp^\perp x = \lim_n A_\perp^\perp T_n x$ in $S_p^\perp$, so by Theorem 3.20, $\| \Gamma(x, x) \|_{S_p^\perp} = \lim_n \| \Gamma(T_n x, T_n y) \|_{S_p^\perp}$ (same for $x^*$ in place of $x$) and consequently, $\| x \|_{\Gamma_{\alpha, p}} = \lim_n \| T_n x \|_{\Gamma_{\alpha, p}} (\text{same for } y \text{ in place of } x).$ ■

5.7 The case of discrete groups : new compact quantum metric spaces

In this subsection, we consider a markovian semigroup of Fourier multipliers on a group von Neumann algebra as in Subsection 2.5, and introduce new compact quantum metric spaces in the spirit of [JMI]. By Lemma 3.26, the following definition is correct.
Definition 5.28 Suppose $2 \leq p \leq \infty$. Let $G$ be a discrete group such that the noncommutative $L^p$-space $L^p(VN(G))$ has CBAP. Let $A_p$ denote the $L^p$ realization of the (negative) generator of a markovian semigroup of Fourier multipliers. For any $x \in \text{dom} A_p^\frac{1}{2}$ we let

\begin{equation}
\|x\|_{\Gamma,\psi,p} \overset{\text{def}}{=} \max \left\{ \|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))}, \|\Gamma(x^*,x^*)^\frac{1}{2}\|_{L^p(VN(G))} \right\}. \tag{5.20}
\end{equation}

Proposition 5.29 Suppose $p \geq 2$ and let $G$ be a discrete group such that $L^p(VN(G))$ has CBAP. Then $\|x\|_{\Gamma,\psi,p}$ is a seminorm on the subspace $\text{dom} A_p^\frac{1}{2}$ of $L^p(VN(G))$.

**Proof:** 1. If $x$ belongs to $\text{dom} A_p^\frac{1}{2}$ and $k \in \mathbb{C}$, we have

\[
\|kx\|_{\Gamma,\psi,p} = \max \left\{ \|\Gamma(kx,kx)^\frac{1}{2}\|_{L^p(VN(G))}, \|\Gamma((kx)^*,(kx)^*)^\frac{1}{2}\|_{L^p(VN(G))} \right\} \\
= \max \left\{ \|\Gamma(kx,kx)^\frac{1}{2}\|_{L^p(VN(G))}, \|\Gamma(kx^*,kx^*)^\frac{1}{2}\|_{L^p(VN(G))} \right\} \\
= |k| \max \left\{ \|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))}, \|\Gamma(x^*,x^*)^\frac{1}{2}\|_{L^p(VN(G))} \right\} = |k| \|x\|_{\Gamma,\psi,p}.
\]

Let us turn to the triangular inequality. Assume that we have shown it for $x,y$ belonging to span $\{\lambda_s : s \in G\}$. Let $\varphi_j$ be the compactly supported multipliers from the CBAP property. Then, for general $x,y \in \text{dom} A_p^\frac{1}{2}$, we have $M_{\varphi,j}x,M_{\varphi,j}y \in \text{span} \{\lambda_s : s \in G\}$, and thus,

\[
\|\Gamma(x+y,x+y)^\frac{1}{2}\|_p = \|\Gamma(x+y,x+y)^\frac{1}{2}\|_p = \lim_j \|\Gamma(M_{\varphi,j}(x+y),M_{\varphi,j}(x+y))^\frac{1}{2}\|_p \\
= \lim_j \|\Gamma(M_{\varphi,j}(x+y),M_{\varphi,j}(x+y))^\frac{1}{2}\|_p \leq \lim_j \|\Gamma(M_{\varphi,j}(x,M_{\varphi,j}(x)^\frac{1}{2}\|_p + \lim_j \|\Gamma(M_{\varphi,j}(y,M_{\varphi,j}(y)^\frac{1}{2}\|_p \\
= \|\Gamma(x,x)^\frac{1}{2}\|_p + \|\Gamma(y,y)^\frac{1}{2}\|_p.
\]

A similar reasoning for $x,y$ replaced by $x^*,y^*$ together with (5.20) then yields the triangular inequality in the general case. So from now on assume $x,y \in \text{span} \{\lambda_s : s \in G\}$. According to Lemma 2.13 part 4., we have

\begin{equation}
\|\Gamma(x+y,x+y)^\frac{1}{2}\|_{L^p(VN(G))}^2 = \|\Gamma(x+y,x+y)^\frac{1}{2}\|_{L^p(VN(G))}^2 \\
= \|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))}^2 + \|\Gamma(y,y)^\frac{1}{2}\|_{L^p(VN(G))}^2 + \|\Gamma(x,y)^\frac{1}{2}\|_{L^p(VN(G))}^2 + \|\Gamma(y,x)^\frac{1}{2}\|_{L^p(VN(G))}^2 \\
\leq \|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))}^2 + \|\Gamma(y,y)^\frac{1}{2}\|_{L^p(VN(G))}^2 + 2\|\Gamma(x,y)^\frac{1}{2}\|_{L^p(VN(G))}^2 + 2\|\Gamma(y,x)^\frac{1}{2}\|_{L^p(VN(G))}^2 \\
\leq \|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))}^2 + \|\Gamma(y,y)^\frac{1}{2}\|_{L^p(VN(G))}^2 + 2\|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))}^2 + 2\|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))}^2 \\
= \left( \|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))} + \|\Gamma(y,y)^\frac{1}{2}\|_{L^p(VN(G))} \right)^2.
\end{equation}

We obtain

\[
\|\Gamma(x+y,x+y)^\frac{1}{2}\|_{L^p(VN(G))} \leq \|\Gamma(x,x)^\frac{1}{2}\|_{L^p(VN(G))} + \|\Gamma(y,y)^\frac{1}{2}\|_{L^p(VN(G))}.
\]

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Hence
\begin{align}
\|x + y\|_{T, \psi, p} & \overset{(5.20)}{=} \max \left\{ \|\Gamma(x + y, x + y)\|_{L_p(VN(G))}, \|\Gamma(x^*, x^*)\|_{L_p(VN(G))} \right\} \\
& \overset{(5.21)}{=} \max \left\{ \|\Gamma(x, x)\|_{L_p(VN(G))} + \|\Gamma(y, y)\|_{L_p(VN(G))}, \|\Gamma(x^*, x^*)\|_{L_p(VN(G))} \right\} \\
& \leq \max \left\{ \|\Gamma(x, x)\|_{L_p(VN(G))}, \|\Gamma(y, y)\|_{L_p(VN(G))}, \|\Gamma(x^*, x^*)\|_{L_p(VN(G))} \right\} \\
& \overset{(5.20)}{=} \|x\|_{T, \psi, p} + \|y\|_{T, \psi, p}.
\end{align}

In the next lemma, recall that if $A$ is the generator of a markovian semigroup of Fourier multipliers, then there is a Hilbert space $H$ together with a mapping $b_\psi : G \to H$ such that the symbol $\psi : G \to C$ of $A$ satisfies $\psi(s) = \|b_\psi(s)\|^2_H$. We define
\begin{align}
\text{Gap}_\psi \overset{\text{def}}{=} \inf_{b_\psi(s) \neq b_\psi(t)} \|b_\psi(s) - b_\psi(t)\|^2_H.
\end{align}

Lemma 5.30 Assume that $b_\psi : G \to H$ is injective where $H$ is a Hilbert space of finite dimension $n \in \mathbb{N}$ and that $\text{Gap}_\psi > 0$. Suppose $2 \leq p < \infty$. Then $A^{\frac{-1}{2}} : L^2_p(VN(G)) \to L^2_p(VN(G))$ is compact, where we denote $L^2_p(VN(G))$ the range of the bounded projection $P : L^p(VN(G)) \to L^p(VN(G))$, $\lambda_s \mapsto (1 - \delta_{s=\epsilon})\lambda_s = (1 - \tau(\lambda_s))\lambda_s$.

Proof: According to [JMP1, Lemma 5.8], we have $\|T_i\|_{L^1(VN(G)) \to VN(G)} \leq c\frac{1}{T^n}$, where we note that $G_0 = \{\epsilon\}$ there, since $b_\psi$ is injective. Thus, by the contractive inclusion $VN(G) \to L^1(VN(G))$ (note that $\tau_G(1) = 1$ since $G$ is discrete), we equally have $\|T_i\|_{VN(G) \to VN(G)} \leq c\frac{1}{T^n}$ and also $\|T_i\|_{L^1(VN(G)) \to L^1(VN(G))} \leq c\frac{1}{T^n}$.

Note that $A^{\frac{-1}{2}} : L^2_p(VN(G)) \to L^2_p(VN(G))$ is compact. Indeed, $A^{\frac{-1}{2}}$ is then diagonal with respect to the orthonormal basis $(\lambda_s)_{s \neq \epsilon}$ of $L^2_p(VN(G))$ and its eigenvalues $\psi(s)^{-\frac{1}{2}} = \|b_\psi(s)\|^2_H$ do not have a cluster point apart from possibly $0$, according to the gap condition (5.23), the injectivity of $b_\psi$ and the finite dimensionality of the Hilbert space $H$.

Observe that $(T_i)_{i \geq 0}$ satisfies the assumptions of Lemma 5.23 with $X = \text{span} \{\lambda_s : s \in G \}^{B(G)}$. Then according to this lemma, the map $A^{\frac{-1}{2}} : \text{span} \{\lambda_s : s \in G \} \to VN(G)_0$ is bounded. Moreover, $(T_i)_{i \geq 0}$ satisfies the hypotheses of Lemma 5.23 with $X = L^1(VN(G))$, so that also $A^{\frac{-1}{2}} : \text{Ran} A = L^1(VN(G)) \to L^1(VN(G))$ is bounded.

By positivity and self-adjointness of $T_i$, it is easy to check with (5.15) that the dual operator of $A^{\frac{-1}{2}}P : L^1(VN(G)) \to L^1(VN(G))$ is the (unique) weak* continuous extension of the map $A^{\frac{-1}{2}}P : \text{span} \{\lambda_s : s \in G \}^{B(G)} \to VN(G)$ to $A^{\frac{-1}{2}}P : VN(G) \to VN(G)$ (here $P$ denotes the projection onto the null trace subspace from the statement of the lemma). As $T_i$ maps $VN(G)_0 \to VN(G)_0$, as (5.15) also holds for $x \in VN(G)_0$ as a weak* convergent integral in $VN(G)$ and as $VN(G)_0$ is weak*-closed in $VN(G)$, we deduce that $A^{\frac{-1}{2}}$ maps $VN(G)_0 \to VN(G)_0$.

The last mapping is compatible with the compact mapping $A^{\frac{-1}{2}} : L^2_p(VN(G)) \to L^2_p(VN(G))$, so that according to [KaM, Theorem 5.5], $A^{\frac{-1}{2}} : L^2_p(VN(G)), VN(G)_0 \to L^2_p(VN(G)), VN(G)_0 \theta$.
is compact for $0 < \theta < 1$. It suffices to note that for the correct value of \( \theta \), we have $L_p^0(VN(G)) = [L_0^0(VN(G)), VN(G)]_\theta$ according to [Tri, 1.2.4 Theorem] with $S = P$ there.

In the next proposition, recall that $C^*_r(G)$ is the reduced group $C^*$ algebra associated with the discrete group $G$. We also write $C^*_r(G_0)$ for the range of the bounded projection $C^*_r(G) \to C^*_r(G) : \lambda_s \mapsto (1 - \delta_{s=e})\lambda_s$, i.e. the trace zero elements of $C^*_r(G)$.

**Proposition 5.31** Let $2 \leq p < \infty$. Let $G$ be a discrete group such that $L^p(VN(G))$ has the CBAP. Let $b_\psi : G \to H$ be an injective cocycle with values in an $n$-dimensional Hilbert space. Assume that $\text{Gap}_\psi > 0$, where $\text{Gap}_\psi$ is given in (5.23). Then $(C^*_r(G), \| \cdot \|_{\Gamma, \psi, p})$ is a quantum compact metric space.

**Proof**: First note that $\text{dom} \| \cdot \|_{\Gamma, \psi, p}$ is dense in $C^*_r(G)$. Indeed, if $x \in \text{span} \{ \lambda_s : s \in G \} \subseteq C^*_r(G)$, then $\| x \|_{\Gamma, \psi, p}$ is clearly finite. On the other hand, $\text{span} \{ \lambda_s : s \in G \}$ is by definition dense in $C^*_r(G)$. According to Proposition 5.29, $\| \cdot \|_{\Gamma, \psi, p}$ is a seminorm. It is a Lipschitz norm in the sense of Definition 5.1, i.e.

\[
\left\{ x \in C^*_r(G) : \| x \|_{\Gamma, \psi, p} = 0 \right\} = C_1.
\]

Indeed, we have $A(1) = \psi(e)1 = 0$, so that $\Gamma(1, 1) = 0$, and consequently, $\| 1 \|_{\Gamma, \psi, p} = 0$. In the other direction, if $\| x \|_{\Gamma, \psi, p} = 0$, then according to Proposition 2.9 applied for $\hat{M}_{x, \psi}$, $x \in \text{span} \{ \lambda_s : s \in G \}$ and then extended to $x \in \text{dom} \hat{A}_\psi^*$ by means of Lemma 3.26, we have $A_\psi^* x = 0$. We also infer that

\[
0 = \tau(A_\psi^*(x)\lambda_s^*) = \tau(xA_\psi^*(\lambda_s^*)) = \tau(x\psi(s)\lambda_s^*) = \psi(s)\tau(x\lambda_s^*)
\]

for any $s \in G$. Since $\psi(s) \neq 0$ for $s \neq e$, we deduce that $\tau(x\lambda_s^*) = 0$ for these $s$, so that finally, $x \in C_1$. As $C^*_r(G)_{sa} \cap C_1 = \mathbb{R}1$, we infer that

\[
\left\{ x \in C^*_r(G)_{sa} : \| x \|_{\Gamma, \psi, p} = 0 \right\} = \mathbb{R}1.
\]

Thus, $\| \cdot \|_{\Gamma, \psi, p}$ is a Lipschitz norm. It now suffices to show that

\[
\left\{ x \in C^*_r(G) : \| x \|_{\Gamma, \psi, p} \leq 1, \tau(x) = 0 \right\}
\]

is relatively compact in $C^*_r(G)$.

According to Lemma 5.30, the set

\[
\left\{ x \in C^*_r(G)_0 : \| A_\psi^*(x) \|_p \leq 1 \right\}
\]

is relatively compact in $C^*_r(G)$. Using as above Proposition 2.9, we see that

\[
\| A_\psi^*(x) \|_p \leq \max \left\{ \| \Gamma(x, x) \|_p, \| \Gamma(x^*, x^*) \|_p \right\} \leq \| x \|_{\Gamma, \psi, p}.
\]

We deduce from this inequality that

\[
\left\{ x \in C^*_r(G)_0 : \| x \|_{\Gamma, \psi, p} \leq 1 \right\}
\]

is relatively compact in $C^*_r(G)$. Since $\left\{ x \in C^*_r(G)_0 : \| x \|_{\Gamma, \psi, p} \leq 1 \right\} = \left\{ x \in C^*_r(G) : \| x \|_{\Gamma, \psi, p} \leq 1, \tau(x) = 0 \right\}$ according to (5.24), we deduce (5.25), and the proposition is proved.
Proposition 5.32 Let $2 \leq p < \infty$ and assume that $L^p(VN(G))$ has CCAP. Then the norm $\| \cdot \|_{\Gamma, \psi, p}$ is lower semicontinuous on $C^*_r(G)$.

Proof: Let $x \in C^*_r(G)$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\text{dom}(A^*_p) \cap C^*_r(G)$ such that $\|x_n - x\|_{C^*_r(G)} \to 0$ and $\|x_n\|_{\Gamma, \psi, p} \leq 1$. We have to show that $x$ belongs to $\text{dom} A^*_p$ and $\|x\|_{\Gamma, \psi, p} \leq 1$. Let $(M_{\varphi,j})_{j \geq 1}$ be a sequence of completely contractive Fourier multipliers with compactly supported symbols converging pointwise to 1. According to Proposition 2.9, we have $\|A^*_p M_{\varphi,j} x\|_p \leq C$ for any $j, n \geq 1$. Then by finite dimensionality, we have $M_{\varphi,j} x = \lim_n M_{\varphi,j} x_n$ and also $A^*_p M_{\varphi,j} x = \lim_n A^*_p M_{\varphi,j} x_n$, so that $\|A^*_p M_{\varphi,j} x\|_p = \lim_n \|A^*_p M_{\varphi,j} x_n\|_p \leq C$. We infer that $x \in \text{dom} A^*_p$. \footnote{Note that $x$ belongs to $C^*_r(G) \subset L^p(VN(G))$.} We thus have $\Gamma(x, x) = \lim_j \Gamma(M_{\varphi,j} x, M_{\varphi,j} x)$ in $L^p(VN(G))$. Using Lemma 3.28 in the third equality, we have

$$\|\Gamma(x, x)\|_p = \|\Gamma(x, x)\|_p^\frac{1}{p} = \lim_j \|\Gamma(M_{\varphi,j} x, M_{\varphi,j} x)\|_p^\frac{1}{p} = \sup_j \|\Gamma(M_{\varphi,j} x, M_{\varphi,j} x)\|_p^\frac{1}{p},$$

In the same way, we have $\|\Gamma(x_n, x_n)\|_p^\frac{1}{p} \leq \sup_j \|\Gamma(M_{\varphi,j} x_n, M_{\varphi,j} x_n)\|_p^\frac{1}{p}$. Since for any $j \geq 1$, we have $M_{\varphi,j} x = \lim_n M_{\varphi,j} x_n$ and thus $\Gamma(M_{\varphi,j} x, M_{\varphi,j} x) = \lim_n \Gamma(M_{\varphi,j} x_n, M_{\varphi,j} x_n)$, we deduce

$$\|\Gamma(x, x)\|_p \leq \sup_j \|\Gamma(M_{\varphi,j} x, M_{\varphi,j} x)\|_p = \sup_n \|\Gamma(x_n, x_n)\|_p \leq 1.$$
5.9 New spectral triples and Banach spectral triples

We refer to [CGRS1] and [CGIS1] for more information on spectral triples. Recall that \( \| \cdot \|_{\Gamma, \alpha, p} \) is defined in (5.4). If \( a \in M_{I, \fin} \), we define the bounded operator \( \pi(a) : S^p_I \oplus_p \operatorname{Ran} \partial_{\alpha, q, p} \rightarrow S^p_I \oplus_p \operatorname{Ran} \partial_{\alpha, q, p} \)

\[
\pi(a) \overset{\text{def}}{=} \begin{bmatrix} L_a & 0 \\ 0 & L_a \end{bmatrix}
\]

(5.26)

where \( L_a : S^p_I \rightarrow S^p_I \) is the left multiplication operator defined in (2.8) and where \( \tilde{L}_a : \operatorname{Ran} \partial_{\alpha, q, p} \rightarrow \operatorname{Ran} \partial_{\alpha, q, p} \) is the left action of the bimodule.

**Proposition 5.34** Suppose \( 1 < p < \infty \) and \(-1 \leq q < 1\).

1. The operator \( D_{\alpha, q, \fin} \) is selfadjoint.

2. For any \( a \in M_{I, \fin} \), we have \( \pi(a) \cdot \operatorname{dom} D_{\alpha, q, p} \subset \operatorname{dom} D_{\alpha, q, p} \).

3. For any \( a \in M_{I, \fin} \), the operator \( \left[ D_{\alpha, q, p}, \pi(a) \right] : \operatorname{dom} D_{\alpha, q, p} \subset S^p_I \oplus_p \operatorname{Ran} \partial_{\alpha, q, p} \rightarrow S^p_I \oplus_p \operatorname{Ran} \partial_{\alpha, q, p} \)

4. For any \( a \in M_{I, \fin} \), we have

\[
\left\| \left[ D_{\alpha, q, p}, \pi(a) \right] \right\|_{S^p_I \oplus_p \operatorname{Ran} \partial_{\alpha, q, p} \rightarrow S^p_I \oplus_p \operatorname{Ran} \partial_{\alpha, q, p}} = \left\| \partial_{\alpha, q, p}(a) \right\|_{\Gamma_q(H) \otimes B(I^p)}.
\]

5. Suppose \( p \geq 2 \). If \( \text{Gap}_{\alpha} > 0 \) and if \( H \) is finite-dimensional then the operator \( |D_{\alpha, q, p}|^{-1} \) is compact.

**Proof:**

1. This is a consequence of [Tre1, Proposition 2.6.3].

2. This point is obvious.

3. Let \( a \in M_{I, \fin} \). A simple computation shows that

\[
\left[ D_{\alpha, q, p}, \pi(a) \right] \overset{\text{(5.26)}}{=} \begin{bmatrix} 0 & (\partial_{\alpha, q, p})^* \\ \partial_{\alpha, q, p} L_a & 0 \end{bmatrix} \begin{bmatrix} L_a & 0 \\ 0 & L_a \end{bmatrix} - \begin{bmatrix} L_a & 0 \\ 0 & L_a \end{bmatrix} \begin{bmatrix} 0 & (\partial_{\alpha, q, p})^* \\ \partial_{\alpha, q, p} L_a & 0 \end{bmatrix} = \begin{bmatrix} L_a (\partial_{\alpha, q, p})^* - L_a (\partial_{\alpha, q, p})^* \\ (\partial_{\alpha, q, p})^* L_a - L_a (\partial_{\alpha, q, p})^* \end{bmatrix}.
\]

Hence \( \left[ D_{\alpha, q, p}, \pi(a) \right] \) extends to a bounded operator on \( S^p_I \oplus_p \operatorname{Ran} \partial_{\alpha, q, p} \).

4. If \( (x, y) \in \operatorname{dom} D_{\alpha, q, p} \), we have

\[
\left\| \left[ D_{\alpha, q, p}, \pi(a) \right] (x, y) \right\|_p = \left\| (L_{\partial_{\alpha, q}(a^*)}(y), L_{\partial_{\alpha, q}(a)}(x)) \right\|_p = \left( \left\| -L_{\partial_{\alpha, q}(a^*)}(y) \right\|_p + \left\| L_{\partial_{\alpha, q}(a)}(x) \right\|_p \right)^{\frac{1}{p}} \leq \left\| \partial_{\alpha, q}(a) \right\|_{\Gamma_q(H) \otimes B(I^p)}.
\]

The reverse inequality is obvious.

5. By Proposition 4.8, we have

\[
(D_{\alpha, q, p})^2 = \begin{bmatrix} A_p & 0 \\ 0 & (\text{Id}_{\Gamma_q(H)} \otimes A_p)(\operatorname{Ran} \partial_{\alpha, q, p}) \end{bmatrix}.
\]
So
\[ |D_{\alpha,q,p}|^{-1} = (D_{\alpha,q,p}^2)^{-\frac{1}{2}} = \begin{bmatrix} A_p^{-\frac{1}{2}} & 0 \\ 0 & B_p^{-\frac{1}{2}} \end{bmatrix}. \]

By Lemma 5.24, \( A_p^{-\frac{1}{2}} : S^p \rightarrow S^p \) is compact. In particular \( A_{2}^{-\frac{1}{2}} : S^2 \rightarrow S^2 \) is compact. By [Tha1, Corollary 5.6], since \( A_2 = (\partial_{\alpha,q,2})^* \partial_{\alpha,q,2} \) and \( B_2 = \partial_{\alpha,q,p}(\partial_{\alpha,q,2})^* \), the operator \( B_2 \) is compact. A similar argument to the one of the proof of Lemma 5.24 shows that \( B_p^{-\frac{1}{2}} \) is compact. We conclude that \( |D_{\alpha,q,p}|^{-1} \) is compact. 

In the case of the free gaussian functor, we have the following equivalence with the norm of the commutator.

**Lemma 5.35** For any \( x \in M_{f,\text{fin}}, \) we have
\[(5.28) \quad \|\partial_{\alpha,0}(x)\|_{\Gamma_0(H^{\infty}(S^p))} \approx \max \left\{ \|\Gamma(x, x)\|_{B(\ell^2_1)}, \|\Gamma(x^*, x^*)\|_{B(\ell^2_1)} \right\}. \]

**Proof**: It suffices to replace \( f \) by \( \partial_{\alpha,0}(x) \) in (3.13) and to use (2.28). \( \Box \)

**Remark 5.36** Suppose \( 2 \leq p < \infty \). If \( x \in \text{dom} A_p^{-\frac{1}{2}} \), we have seen in the proof of Corollary 3.20 that
\[(5.29) \quad \|\partial_{\alpha,q,p}(x)\|_{L^p(\Gamma_0(H^{\infty}(S^p)))} \approx \max \left\{ \|\Gamma(x, x)\|_{S^p_1}, \|\Gamma(x^*, x^*)\|_{S^p_1} \right\}. \]

**Remark 5.37** Note that the Hodge-Dirac operator \( D_{\alpha,q,p} \) anti-commutes with the involution
\[
\gamma_p = \begin{bmatrix} -\text{Id}_{S^p_1} & 0 \\ 0 & \text{Id}_{\text{Ran} \partial_{\alpha,q,p}} \end{bmatrix} : S^p_1 \oplus \text{Ran} \partial_{\alpha,q,p} \to S^p_1 \oplus \text{Ran} \partial_{\alpha,q,p}. 
\]

since
\[
D_{\alpha,q,p} \gamma_p + \gamma_p D_{\alpha,q,p} = \begin{bmatrix} 0 & (\partial_{\alpha,q,p})^* \\ -\partial_{\alpha,q,p} & 0 \end{bmatrix} \begin{bmatrix} \text{Id}_{S^p_1} & 0 \\ 0 & \text{Id}_{\text{Ran} \partial_{\alpha,q,p}} \end{bmatrix} - \begin{bmatrix} \text{Id}_{S^p_1} & 0 \\ 0 & \text{Id}_{\text{Ran} \partial_{\alpha,q,p}} \end{bmatrix} \begin{bmatrix} 0 & (\partial_{\alpha,q,p})^* \\ -\partial_{\alpha,q,p} & 0 \end{bmatrix} = \begin{bmatrix} 0 & (\partial_{\alpha,q,p})^* \\ -\partial_{\alpha,q,p} & 0 \end{bmatrix} = 0. 
\]

It is natural to state the following definition.

**Definition 5.38** A Banach spectral triple \((A, X, D)\) consists of the following data: a (reflexive) Banach space \( X \), a bisectorial operator \( D \) on \( X \) with a dense domain \( \text{dom} D \subset X \), an algebra \( A \) and a homomorphism \( \pi : A \to \text{B}(X) \) such that for all \( a \in A \) we have:

1. \( \pi(a) \cdot \text{dom} D \subset \text{dom} D \),
2. \( D \) admits a bounded functional calculus on a bisector,
3. The commutator \([D, \pi(a)] : \text{dom} D \to X \) extends to a bounded linear operator on \( X \),
4. \( \pi(a)D^{-1} \) is a compact operator on \( \text{Ran} D \).
Now, we give other examples. We forget the above Hodge-Dirac operator and we define:

\[(5.30) \quad D_{\alpha,q}(x \otimes e_{ij}) \overset{\text{def}}{=} s_q(\alpha_i - \alpha_j)x \otimes e_{ij}, \quad x \in L^p(\Gamma_q(H)), i, j \in I.\]

**Lemma 5.39** Suppose \(1 < p < \infty\). The operator \(D_{\alpha,q} : L^p(\Gamma_q(H)) \otimes M_{\mathfrak{f},\mathfrak{n}} \subset L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2)) \to L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2)) \) is closable.

**Proof**: Suppose that \((x_n)\) is a sequence of \(L^p(\Gamma_q(H)) \otimes M_{\mathfrak{f},\mathfrak{n}}\) which converges to 0 such that the sequence \((D_{\alpha,q}(x_n))\) converges to some \(y\) belonging to \(L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2))\). For any \(k \in \mathbb{N}\), we have

\[D_{\alpha,q}(\text{Id}_{L^p(\Gamma_q(H))} \otimes T_k)(x_n) = (\text{Id}_{L^p(\Gamma_q(H))} \otimes T_k)D_{\alpha,q}(x_n) \xrightarrow{n \to \infty} (\text{Id}_{L^p(\Gamma_q(H))} \otimes T_k)(y).\]

On the other hand, for any \(i, j \in I\) we have \(x_{n,ij} \to 0\) as \(n \to \infty\). Hence

\[D_{\alpha,q}(\text{Id}_{L^p(\Gamma_q(H))} \otimes T_k)(x_n) = D_{\alpha,q} \left( \sum_{i,j=1}^{k} x_{n,ij} \otimes e_{ij} \right) \xrightarrow{(5.30)} \sum_{i,j=1}^{k} s_q(\alpha_i - \alpha_j)x_{n,ij} \otimes e_{ij} \xrightarrow{n \to \infty} 0.\]

This implies by unicity of the \(L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2))\)-limit that \((\text{Id}_{L^p(\Gamma_q(H))} \otimes T_k)(y) = 0\). Since the sequence \((T_k)\) is uniformly bounded in operator norm over \(S^{1}_{\ell_1^2}\) and converges strongly to \(\text{Id}_{S^{1}_{\ell_1^2}}\), we infer that \((\text{Id}_{L^p(\Gamma_q(H))} \otimes T_k)\) converges strongly to \(\text{Id}_{L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2))}\). We deduce that \(y = \lim_k (\text{Id}_{L^p(\Gamma_q(H))} \otimes T_k)(y) = 0\). We conclude with [Kat1, page 165].

We denote by \(D_{\alpha,q,p} : \text{dom} \ D_{\alpha,q,p} \subset L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2)) \to L^p(\Gamma_q(H)) \otimes_p S^{1}_{\ell_1^2}\) its closure. It is easy to check that the operator \(D_{\alpha,2,p}\) is selfadjoint.

We define the homomorphism \(\pi : M_{\mathfrak{f},\mathfrak{n}} \to B(L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2)))\) by

\[(5.31) \quad \pi(a) \overset{\text{def}}{=} \text{Id}_{L^p(\Gamma_q(H))} \otimes L_a, \quad a \in M_{\mathfrak{f},\mathfrak{n}}.\]

**Proposition 5.40** Suppose \(1 < p < \infty\) and \(-1 \leq q < 1\). \(\text{Gap}_{\alpha} > 0\).

1. For any \(a \in M_{\mathfrak{f},\mathfrak{n}}\), we have \(\pi(a) \cdot \text{dom} \ D_{\alpha,q,p} \subset \text{dom} \ D_{\alpha,q,p}\).
2. For any \(a \in M_{\mathfrak{f},\mathfrak{n}}\), we have
\[(5.32) \quad [D_{\alpha,q,p}, \pi(a)] = L_{\partial_{\alpha,q}(a)}.\]
3. For any \(a \in M_{\mathfrak{f},\mathfrak{n}}\), we have
\[\| [D_{\alpha,q,p}, \pi(a)] \|_{L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2)) \to L^p(\Gamma_q(H) \overline{\otimes} B(\ell_1^2))} = \| \partial_{\alpha,q}(a) \|_{\Gamma_q(H) \overline{\otimes} B(\ell_1^2)},\]
4. We have \((D_{\alpha,-1,p})^2 = \text{Id}_{L^p(\Gamma_q(H))} \otimes A_p\).
5. Suppose \(p \geq 2\). If \(\text{Gap}_{\alpha} > 0\) and if \(H\) is finite-dimensional then the operator \(\| D_{\alpha,-1,p} \|^{-1}\) is compact.

**Proof**: 1. It is obvious.
2. For any \( i, j, k, l \in I \) and any \( x \in L^p(\Gamma_q(H)) \), we have
\[
\left[ \mathcal{D}_{\alpha, q, p}, \pi(e_{ij}) \right](x \otimes e_{kl}) = \mathcal{D}_{\alpha, q, p}(x \otimes e_{kl}) - \pi(e_{ij})\mathcal{D}_{\alpha, q, p}(x \otimes e_{kl})
\]
(5.30) \( \mathcal{D}_{\alpha, q, p}(x \otimes e_{ij}e_{kl}) - \pi(e_{ij})(s_q(\alpha_k - \alpha_l)x \otimes e_{kl}) \)
\( = \delta_{j=k}\mathcal{D}_{\alpha, q, p}(x \otimes e_{il}) - s_q(\alpha_k - \alpha_l)x \otimes e_{ij}e_{kl} \)
\( = \delta_{j=k}(s_q(\alpha_i - \alpha_j)x \otimes e_{il} - s_q(\alpha_k - \alpha_l)x \otimes e_{il}) = \delta_{j=k}s_q(\alpha_i - \alpha_k)x \otimes e_{il} \)
\( = s_q(\alpha_i - \alpha_j)x \otimes e_{ij}e_{kl} = (s_q(\alpha_i - \alpha_j) \otimes e_{ij})(x \otimes e_{kl}) \)
(2.26) \( \mathcal{D}_{\alpha, q, p}(x \otimes e_{kl}) \)
By linearity and density, we obtain \( \left[ \mathcal{D}_{\alpha, q, p}, \pi(e_{ij}) \right] = L_{\partial_{\alpha, q, p}} \). Finally, by linearity, we obtain (5.32).

3. For any \( a \in M_{I, fin} \), we have
\[
\left\| \left[ \mathcal{D}_{\alpha, q, p}, \pi(a) \right] \right\|_{L^p(\Gamma_q(H) \otimes B(e_j^2))} \leq \left\| L_{\partial_{\alpha, q, p}(a)} \right\|_{L^p(\Gamma_q(H) \otimes B(e_j^2))} \leq \left\| \partial_{\alpha, q, p}(a) \right\|_{\Gamma_q(H) \otimes B(e_j^2)}.
\]

4. If \( x \in L^p(\Gamma_q(H)) \) and \( i, j \in I \), note that
\[
\mathcal{D}_{\alpha, -1, p}(x \otimes e_{ij}) = \mathcal{D}_{\alpha, -1, p}(s_\alpha(\alpha_i - \alpha_j)x \otimes e_{ij}) = s_\alpha(\alpha_i - \alpha_j)^2x \otimes e_{ij}
\]
(2.13) \( x \otimes \|\alpha_i - \alpha_j\|^2 e_{ij} = (L_{\partial_{\alpha, q, p} I_p} - A_p)(x \otimes e_{ij}) \).

By [HvNVW2, Proposition G.2.4], note that \( L^p(\Gamma_q(H)) \otimes M_{I, fin} \) is a core of \( L^p(\Gamma_q(H) \otimes I_p) \). So \( \mathcal{D}_{\alpha, -1, p} \) and \( L_{\partial_{\alpha, q, p} I_p} - A_p \) coincide on a common core.

5. The proof is similar to the one of point 4 of Proposition 5.34.

Now, we generalize the construction of [JMP1, page 537] which corresponds to the case \( q = 1 \) and \( p = 2 \) below. However, note that it is written in [JMP1, page 537] that the commutator is bounded but it is not correct\(^8\). Consequently [JMP1, page 537] really does not give a spectral triple. Finally, we prove the last axiom (point 5 below) of Connes Spectral triples. Recall that \( \|\|_{\Gamma_q} \) is defined in (5.20).

If \( a \in \text{span}\{\lambda_s : s \in G\} \), we define the bounded operator \( \pi(a) : L^p(\Gamma_q(H)) \oplus_p \text{Ran} \partial_{\psi, q, p} \rightarrow L^p(\Gamma_q(H)) \oplus_p \text{Ran} \partial_{\psi, q, p} \) by
\[
\pi(a) \overset{\text{def}}{=} \begin{bmatrix} L_a & 0 \\ 0 & L_a \end{bmatrix}
\]
(5.33)
where \( L_a : L^p(\Gamma_q(H)) \rightarrow L^p(\Gamma_q(H)) \) is the left multiplication operator defined in (2.8) and where \( L_a : \text{Ran} \partial_{\psi, q, p} \rightarrow \text{Ran} \partial_{\psi, q, p} \) is the left action of the bimodule.

**Proposition 5.41** Suppose \( 1 < p < \infty \) and \(-1 \leq q < 1\).

1. The operator \( D_{\psi, q, 2} \) is selfadjoint.

2. For any \( a \in \text{span}\{\lambda_s : s \in G\} \), we have \( \pi(a) \cdot \text{dom} D_{\psi, q, p} \subset \text{dom} D_{\psi, q, p} \).

3. For any \( a \in \text{span}\{\lambda_s : s \in G\} \), the operator \( \left[ D_{\psi, q, p}, \pi(a) \right] : \text{dom} D_{\psi, q, p} \oplus_p \text{Ran} \partial_{\psi, q, p} \rightarrow L^p(\Gamma_q(H)) \oplus_p \text{Ran} \partial_{\psi, q, p} \) extends to a bounded operator.

\(^8\) The commutator \( [D_{\psi, 1, p}, \pi(a)] \) is not bounded since \( L_{\partial_{\psi, q, p}} \) is not bounded in general.
4. For any \( a \in \text{span} \{ \lambda_s : s \in G \} \), we have

\[
\left\| \left[ D_{\psi,q,p}, \pi(a) \right] \right\|_{L^p(VN(G))} = \left\| \partial_{\psi,q,p}(a) \right\|_{\Gamma_q(H) \otimes B(\ell_2^p)}.
\]

5. Suppose \( 2 \leq p < \infty \). If \( \text{Gap}_\psi > 0 \) and if \( H \) is finite-dimensional then the operator \( |D_{\psi,q,p}|^{-1} \) is compact.

**Proof:**
1. This is a consequence of [Tre1, Proposition 2.6.3].
2. This point is obvious.
3. Let \( a \in \text{span} \{ \lambda_s : s \in G \} \). A simple computation shows that

\[
[D_{\psi,q,p}, \pi(a)] = \left( \begin{array}{ccc}
0 & 0 & 0 \\
\partial_{\psi,q,p} L_a & 0 & \partial_{\psi,q,p} L_a \\
0 & \partial_{\psi,q,p} L_a & 0 \\
\end{array} \right).
\]

Hence \( [D_{\psi,q,p}, \pi(a)] \) extends to a bounded operator on \( L^p(VN(G)) \oplus_p \text{Ran} \partial_{\psi,q,p} \).

4. If \( (x,y) \in \text{dom} D_{\psi,q,p} \), we have

\[
\left\| [D_{\psi,q,p}, \pi(a)] (x,y) \right\|_p = \left\| \left( -\partial_{\psi,q,p} (a^*) \right)^* (y), L_{\partial_{\psi,q,p}}(a)(x) \right\|_p \leq \left\| L_{\partial_{\psi,q,p}}(a) \right\|_{\Gamma_q(H) \otimes VN(G)}.
\]

The reverse inequality is obvious.

5. By proposition 4.8, we have

\[
(D_{\psi,q,p})^2 = \begin{bmatrix}
A_p & 0 \\
0 & \text{Id}_{L^p(VN(G))} \otimes A_p \end{bmatrix} \text{Ran} \partial_{\psi,q,p}.
\]

So

\[
|D_{\psi,q,p}|^{-1} = (D_{\psi,q,p}^2)^{-\frac{1}{2}} = \begin{bmatrix}
A_p^{-\frac{1}{2}} & 0 \\
0 & \text{Id}_{L^p(VN(G))} \otimes A_p \end{bmatrix} \text{Ran} \partial_{\psi,q,p}^{-\frac{1}{2}}.
\]

The end is similar to the proof of 5.34.

In the case of the free gaussian functor, we have the following equivalence with the norm of the commutator.

**Lemma 5.42** For any \( x \in \text{span} \{ \lambda_s : s \in G \} \), we have

\[
\left\| \partial_{\psi,0}(x) \right\|_{\Gamma_q(H) \otimes VN(G)} \approx \max \left\{ \left\| \Gamma(x,x) \right\|_{VN(G)}, \left\| \Gamma(x^*,x^*) \right\|_{VN(G)} \right\}.
\]

**Proof:** It suffices to the ideas of [JMP1, page 589] and (2.52).

**Remark 5.43** Suppose \( 2 \leq p < \infty \). If \( x \in \text{dom} A_p^{\frac{1}{2}} \), we have essentially [JMP1] that

\[
\left\| \partial_{\psi,q}(x) \right\|_{L^p(VN(G))} \approx \max \left\{ \left\| \Gamma(x,x) \right\|_{L^p(VN(G))}, \left\| \Gamma(x^*,x^*) \right\|_{L^p(VN(G))} \right\}.
\]

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Remark 5.44 Note that the Hodge-Dirac operator $D_{\psi,q,p}$ anti-commutes with the involution

$$
\gamma_p \overset{\text{def}}{=} \begin{bmatrix}
-\text{Id}_{L^p(\text{VN}(G))} & 0 \\
0 & \text{Id}_{\text{Ran } \partial_{\psi,q,p}}
\end{bmatrix} : L^p(\text{VN}(G)) \oplus_p \text{Ran } \partial_{\psi,q,p} \rightarrow L^p(\text{VN}(G)) \oplus_p \text{Ran } \partial_{\psi,q,p}.
$$

since

$$
D_{\psi,q,p} \gamma_p + \gamma_p D_{\psi,q,p} = \begin{bmatrix}
0 & (\partial_{\psi,q,p})^* \\
\partial_{\psi,q,p} & 0
\end{bmatrix} \begin{bmatrix}
-\text{Id}_{L^p(\text{VN}(G))} & 0 \\
0 & \text{Id}_{\text{Ran } \partial_{\psi,q,p}}
\end{bmatrix} - \begin{bmatrix}
-\text{Id}_{L^p(\text{VN}(G))} & 0 \\
0 & \text{Id}_{\text{Ran } \partial_{\psi,q,p}}
\end{bmatrix} \begin{bmatrix}
0 & (\partial_{\psi,q,p})^* \\
\partial_{\psi,q,p} & 0
\end{bmatrix} = 0.
$$
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