Chimera and solitary states in 3D oscillator networks with inertia

V. Maistrenko\textsuperscript{1}, O. Sudakov\textsuperscript{1,2}, O. Osiv\textsuperscript{1}

\textsuperscript{1}Scientific Center for Medical and Biotechnical Research, NAS of Ukraine, 54, Volodymyrs'ka Str., Kyiv 01030, Ukraine

\textsuperscript{2}Taras Shevchenko National University of Kyiv, 60, Volodymyrs'ka Str., Kyiv 01030, Ukraine

We report the diversity of scroll wave chimeras in the three-dimensional (3D) Kuramoto model with inertia for $N^3$ identical phase oscillators placed in a unit 3D cube with periodic boundary conditions. In the considered model with inertia, we have found novel types of patterns which do not exist in a pure system without inertia. In particular, a scroll wave torus-like chimera is obtained under random initial conditions. In contrast to a pure system without inertia, where all chimera states have incoherent inner parts, these states can have partially coherent or fully coherent inner parts in a system with inertia, as exemplified by a scroll wave torus-like chimera. Solitary states exist in the considered model as separate states or can coexist with scroll wave chimeras in the oscillatory space. We also propose a method of construction of 3D images using solitary states as solutions of the 3D Kuramoto model with inertia.

PACS numbers:

I. INTRODUCTION

Chimera states\textsuperscript{1,2} as the phenomenon of the coexistence of coherence and incoherence patterns in nonlocally coupled phase oscillators have been elaborately investigated during the past decade in a wide range of systems. This new discovery has stimulated many physicists, biologists, and mathematicians to study the chimera states in different fields of science.

A number of illustrious articles are devoted to the theoretical and experimental studies of chimeras. Most of them deal with one-dimensional models.

Nonetheless, this novel approach was extended to the two-dimensional networks of oscillators. Among other factors, a new class of chimera states called the spiral wave chimeras has been introduced in [3]. This kind of space-temporal behavior is characterized by the standard wave chimeras has been introduced in [3]. This kind of factors, a new class of chimera states called the spiral two-dimensional networks of oscillators. Among other models.

As for the 3D case, we mention a few papers about chimera states in the model of coupled phase oscillators in a three-dimensional grid topology [13–18].

The first evidence of chimera states in 3D was reported in 2015 in [13] for the Kuramoto model of coupled phase oscillators in a 3D grid topology with piecewise constant oscillator’s coupling. Just this kind of coupling produces the richest variety of chimera states in the 1D Kuramoto model with respect to cosine and exponential couplings [19]. 3D oscillating chimera states, i.e., those without spiraling of the coherent region, and spirally rotating chimeras called scroll wave chimeras were obtained. It is worth to note such examples of oscillating chimeras as coherent and incoherent balls, tubes, crosses, and layers in the incoherent or coherent surrounding, and the scroll wave chimeras include incoherent rolls of different modalities in the spiraling rotating coherent surrounding.

Some time later, two new kinds of the scroll wave chimeras, Hopf link and trefoil, with linked and knotted incoherent regions were detected in [14]. The theoretical confirmation of the existence of a few kinds of 3D chimera states in the Kuramoto model with the cosine coupling of oscillators was done in 2019 in [18].

In the present paper, we will study the appearance of 3D chimera states in the Kuramoto model of coupled phase oscillators in the 3D grid topology with inertia:

\begin{equation}
\sum_{(\nu',j',k')\in B_P(i,j,k)} \sin(\varphi_{\nu'j'k'} - \varphi_{ijk} - \alpha),
\end{equation}

where $i, j, k = 1, \ldots, N$, $\varphi_{ijk}$ are phase variables, and indices $i, j, k$ are periodic mod $N$. The coupling is assumed long-ranged and isotropic: each oscillator $\varphi_{ijk}$ is coupled with equal strength $\mu$ to all its $N_P$ nearest neighbors $\varphi_{i'j'k'}$ within a ball of radius $P$, i.e., to those falling in the neighborhood

\[ B_P(i,j,k) := \{(i',j',k') : (i' - i)^2 + (j' - j)^2 + (k' - k)^2 \leq P^2 \}, \]

where the distances are calculated regarding the periodic boundary conditions of the network. The phase lag parameter $\alpha$ is assumed to belong to the attractive coupling interval from 0 to $\pi/2$.

The next control parameter, coupling radius $r = P/N$, varies from $1/N$ (local coupling) to 0.5 (close to the global coupling). The parameter $\mu$ is the oscillator coupling strength, and $\epsilon$ is the damping coefficient. The parameter $m$ is the mass. In the case $m = 0$, Eq.(1) is transformed...
FIG. 1: Screenshots of solutions of system (1). Phase distribution of $\phi_{ijk}$: (a) scroll wave torus-like chimera state, (b) cross-sections of (a) along $y = 0.5$ ($\alpha = 0.475, \mu = 0.007, r = 0.04, N = 200$); (c) scroll wave torus-like chimera state with solitary cloud ($\alpha = 0.4, \mu = 0.1, r = 0.04, N = 100$). (d) - 3D image of a real chimera generated by solitary states ($\alpha = 0.35, \mu = 0.1, r = 0.15, N = 200$); $\epsilon = 0.05$. Coordinates $x = i/N, y = j/N, z = k/N$.

In the case of the model with inertia ($m = 1$), the random initial condition can generate a stable scroll wave torus-like chimera which survives in time till $t = 10^5$, as far as we simulate with $\alpha = 0.2, r = 0.03, \mu = 0.1, \epsilon = 0.05, N = 100$.

Figure 2 illustrates the time evolution of the phase distribution $\phi_{ijk}$ for a random trajectory in the coordinates $x = i/N, y = j/N, z = k/N$. At the time $t \approx 2500$, two scroll wave tori have been generated by the random chaotic behavior of oscillators. They exist along the time interval $t \approx (2500 - 3900)$ (Fig. 2(a)), and then one torus vanishes, giving rise to the birth of a single scroll wave torus-like chimera (Fig. 2(b)) with its cross-section along $x = 0.5$ presented in Fig. 2(c) (see video 1).

FIG. 2: Generation of a scroll wave torus-like chimera state under random initial conditions. Screenshots of phase $\phi_{ijk}$: (a) $t = 2500$, (b) $t = 4000$. (c) - cross-section of (b) along $x = 0.5$. $\alpha = 0.2, r = 0.03, \mu = 0.1, \epsilon = 0.05, N = 100$. Coordinates $x = i/N, y = j/N, z = k/N$.

Changing parameters, the stability regions in the $(\alpha, \mu)$ parameter plane for scroll wave torus chimeras were built. In Fig. 3(a), the blue region corresponds to the stability of a torus chimera with incoherent or partially coherent inner part, red - torus with completely coherent inner part. Here, the stability region for solitary states, which will be considered in Sec. 4 in detail, is colored by gray. Torus chimera states exist for any infinitely small coupling strength $\mu > 0$.

Figure 3(b) illustrates the stability regions in the $(\alpha, r)$ parameter plane for scroll wave torus chimeras (blue), scroll wave coherent torus (red), sphere and ball (green), 4 scroll wave rods (magenta), and solitary states (gray).
FIG. 3: Stability regions of chimeras and solitary states in the parameter planes: (a) - $(\alpha, \mu)$ ($r = 0.03$), (b) - $(\alpha, r)$ ($\mu = 0.1$). Blue - tori with incoherent or partial coherent inner parts, red - coherent tori, sphere (green), 4 scroll wave rods (magenta), gray - solitary states. $\epsilon = 0.05$, $N = 200$.

FIG. 4: Examples of scroll wave tori. Left column - phase distributions $\phi_{i,j,k}$, next columns - cross-sections of $\phi_{i,j,k}$ and average frequencies $\bar{\omega}_{i,j,k}$ along the center of the torus, right column - double cross-sections of average frequencies $\bar{\omega}_{i,j,k}$ along the white dash line of the previous column. (a) - incoherent scroll wave torus chimera $(\alpha = 0.475, \mu = 0.007)$, (b) - scroll wave torus chimera with partial coherent inner part $(\alpha = 0.436, \mu = 0.019)$, (c) - scroll wave coherent torus $(\alpha = 0.3, \mu = 0.1)$. $r = 0.04$, $\epsilon = 0.1$, $N = 200$. Time averaging interval of frequencies $\Delta T = 1000$.

for the fixed parameters $\epsilon = 0.05$, $\mu = 0.1$, $N = 100$. The bottom boundary of the torus region touches upon to the radius coupling $P = 1$ (local coupling). Crossing the left and bottom sides of the torus stability region, all oscillators are synchronized, and tori vanish. After crossing the right side of the region, the tori are destroyed with the generation of the chaotic oscillatory behavior.

Figure 4 demonstrates the examples of scroll wave
torus chimeras with incoherent (a), partially coherent (b), and completely coherent inner parts for the parameter values indicated by black points in Fig. 3(a). Here, the left column shows the phase distributions $\phi_{i,j,k}$, next column - cross-sections of $\phi_{i,j,k}$ along the center of the torus, third column - cross-sections of average frequencies $\bar{\omega}_{i,j,k}$, right column - double cross-sections of average frequencies $\bar{\omega}_{x,y,z}$ along the white dash line of the previous column. In the case of a completely coherent torus (Fig. 4 (c)), we have a bistable behavior of oscillators. Their average frequencies $\bar{\omega}_{i,j,k}$ are equal to two average frequencies only: torus $\bar{\omega}_T$ or main synchronized cluster $\bar{\omega}_0$. So, it is a scroll wave torus pattern, but is not scroll wave torus chimeras.

The double cross-sections of the average frequencies $\bar{\omega}_{x,y,z}$ of scroll wave torus chimeras demonstrate only three possibilities for the inner part of tori. But we believe that, by increasing the dimension $N$ of system (1), the different torus structures including multiple step-wise structures similarly to the 2D case can be obtained [12].

Video 2 presents the evolution of a torus chimera with incoherent inner part for Fig. 4(a).

To our surprise, if we take a scroll wave torus with the coupling radius $r = 0.041$ and $\alpha = 0.3$ (yellow point in Fig. 3(b)) and will start to increase $r$ along the black dash line, the scroll wave torus will transform into a sphere pattern without spiral rotation.

Approaching the left and bottom boundaries of the stability regions of the sphere (Fig. 3(b), the number of incoherent oscillators on the surface decreases. Finally, the sphere vanishes crossing these boundaries.

The radius of the coherent inner part of a sphere decreases with increasing the phase lag $\alpha$. With approaching the right boundary of the stability region, the sphere becomes a incoherent ball.

Just after crossing the right boundary, the ball is transformed firstly into a cube and finally into the pattern consisting of incoherent oscillators with coherent islands which fill all the space except for a narrow coherent layer (shown in the insert in Fig. 2).

III. DIVERSITY OF SCROLL WAVE CHIMERS

The properties of scroll wave torus chimeras described in the previous sections take place also for the diversity of scroll wave chimeras. In Fig. 5, we give a few examples, as well as for other chimeras which exist in system (1), but were not mentioned here.

Some of them exist in the system without inertia, but, obviously, have incoherent inner part. As seen from the cross-sections of chimeras in Fig. 5, all presented chimeras have coherent or partially coherent inner parts.

Due to the introduction of the inertia, the scroll wave chimeras of system (1), in contrast to chimeras in a system without inertia, have a following properties.

The scroll wave chimera states in the inertial system can have partially or fully coherent inner parts, but the chimeras in a system without inertia have incoherent inner parts obviously.

Trefoil and Hopf link chimeras are fixed in the oscillatory space in contrast to their behavior in a system without inertia, where they move, by rotating around their center of mass.

Scroll wave chimera states from system (1) are very stable with respect to perturbations of the initial conditions. For example, they can still exist and save shape, even if
their phase $\varphi_{ijk}$ and frequency $\omega_{ijk}$ are perturbed by white noise with amplitude less than 0.5 for the parameter values $\alpha = 0.38, r = 0.03, \epsilon = 0.05, \mu = 0.02, N = 100$. More strong perturbations lead to changing the shape of chimeras or its destruction with complete oscillatory synchronization or creation of different types of chimera states.

IV. SOLITARY STATES

Solitary states as isolated oscillators in the oscillatory space exist in the 3D Kuramoto model with inertia.

Single (Fig. 6(a)) or multiple (Fig. 6(b)) solitary states can be easily obtained under random initial conditions. Multiple solitary states look like a “solitary cloud”.

An example of the time evolution of frequencies $\omega_{ijk}$ for a single solitary state (Fig. 6(a)) obtained under random initial conditions is shown in Fig. 6(c) with the time averaging frequency interval $\Delta T = 200$. The frequency of a solitary state $\omega_{Sol}$ becomes very soon periodic, and its average value $\bar{\omega}_{Sol}$ tends to a constant. The frequencies of synchronized oscillators $\omega_{Syn}$ and their average values $\bar{\omega}_{Syn}$ tend to constants as well. The stability regions for solitary states in the parameter spaces $(\alpha, \mu)$ and $(\alpha, r)$ are presented in Fig. 3 (gray color).

As was noted in the previous sections, the scroll wave chimera states of system (1) have strong stability with respect to perturbations of the initial conditions and the frequency by white noise in the case of the location outside of the solitary region (yellow point in Fig. 3(a)).

At the same time, if the disturbed chimera lies in the solitary region, then the perturbation of its initial conditions with amplitude in the interval $0.5 - 0.8$ can gives rise to another chimera states with solitary clouds. Examples of chimera states with solitary clouds are shown in Fig. 7. As clearly seen from these figures, the solitary clouds in the oscillatory space can appear with a perturbation of the initial conditions of a scroll torus with amplitude of 0.5 (Fig. 7(a)). Stronger perturbations with amplitudes of 0.55 (Fig. 7(b)) and 0.6 (Fig. 7(c)) save the torus chimera, but clouds becomes denser. The further perturbation of chimera’s initial conditions with amplitude more than 0.8 can destroy the original chimera.
states. We show also a multiple scroll wave chimera with solitary clouds (Fig. 7(d)) obtained under random initial conditions.

However, if the parameter values lie outside of the solitary region, the perturbation of chimeras will never give the chimera states with solitary cloud. At the same time, if we take a chimera state with solitary oscillators inside the solitary region and change the parameters to escape from this region, then the solitary clouds will vanish. On the other hand, if a chimera is located outside of the solitary region, then its entering into the solitary region will produce no solitary clouds.

So, if the scroll wave chimeras lie in the solitary region, the infinitely many different chimera states with infinitely many different solitary clouds for each fixed parameter value can be obtained with the help of perturbations of the original chimeras.

V. 3D PRINTING OF SOLUTIONS

Finally, we would like to propose a method of construction of 3D images using solitary states as the solutions of the 3D Kuramoto model with inertia (1). In addition to the image of a real chimera from the Notre-Dame Cathedral (Fig. 1(d)), we present the Eiffel Tower image in Fig. 8(a) and the Dragon Spyro from the computer game in Fig. 8(b) (see videos 3 and 4, respectively).

These images are a stable solution of the 3D Kuramoto model with inertia (1). They were built using, as a pencil, the initial conditions of the obtained solitary states.

For the distribution of solitary oscillators in the oscillatory space, we took the initial image in the STL format developed for 3D printers.

VI. CONCLUSION

The 3D Kuramoto model with inertia (1) describes the basic properties of the collective dynamics in various real physical systems. The phenomena similar to chimera and solitary states were observed experimentally in various systems. Such properties as the stability of described chimera and solitary states with respect to perturbations of the initial conditions in a wide range of parameters and the preservation of initial average oscillating frequencies for solitary states may become a key to practical applications. The practical applications of these phenomena can include (but not limited to) the creation of new information storage and transfer media, devices, architectures; development of new information encoding algorithms; clarification of the mechanisms of functioning of biological systems, etc.

ACKNOWLEDGMENT

The authors are very grateful to Yu. Maistrenko for useful discussions and valuable comments.

[1] Y. Kuramoto, D. Battogtokh. Nonlinear Phenom. Complex Syst. 5, pp. 380–385, (2002).
[2] D.M. Abrams, S.H. Strogatz. Phys. Rev. Lett., 93, 174102, (2004).
[3] Y. Kuramoto, S.I. Shima. Prog. Theor. Phys. Supp. 150, pp.115-125, (2003).
[4] O. Omel’chenko, M. Wolfrum, S. Yanchuk, Yu. Maistrenko, O. Sudakov. Phys. Rev. E 85 (3), 036210, (2012).
[5] M.J. Panaggio, D.M. Abrams. Phys. Rev. Lett., 110, 094102 (2013).
[6] J. Xie, E. Knobloch, H.-C. Kao. Phys. Rev. E, 92, 042921 (2015).
[7] C.R. Laing, SIAM J. Appl. Dyn. Syst., 16(2), pp. 974–1014 (2017).
[8] O. Omel’chenko, M. Wolfrum, E. Knobloch. SIAM J. Appl. Dyn. Syst., 17(1), pp. 97–127 (2018).
[9] J.F. Totz, J. Rode, M.R. Tinsley, et al. Nature Phys 14, pp. 282–285 (2018).
[10] O. Omel’chenko, E. Knobloch. New Journal of Physics, 21, 093034, (2019).
[11] J.F. Totz. Spiral Wave Chimera. In: Synchronization and Waves in Active Media, pp.55–97. Springer (2019).
[12] V. Maistrenko, O. Sudakov, Yu. Maistrenko. https://arxiv.org/pdf/2001.02167.pdf, (2019).
[13] Yu. Maistrenko, O. Sudakov, O. Osiv, V. Maistrenko, New Journal of Physics, 17, 073037, (2015).
[14] H.W. Lau, J. Davidsen, Phys. Rev. E, 94, 010204(R), (2016).
[15] T. Kasimatis, J. Hizanidis, A. Provata. Phys. Rev. E 97, 052213, (2018).
[16] S. Kundu, B.K. Bera, B, D. Ghosh, M. Lakshmanan. Phys. Rev. E, 99(2), (2019).
[17] O. Omel’chenko, E. Knobloch. New Journal of Physics,
21. (2019.)
[19] Yu. Maistrenko, A. Vasylenko, O. Sudakov, R. Levchenko, V. Maistrenko, Int. J. Bifurc. Chaos., 24, 1440014, (2014).
[20] P. Jaros, Yu. Maistrenko, T. Kapitaniak. Phys. Rev. E 91, 022907 (2015).
[21] P. Jaros, S. Brezetsky, R. Levchenko, D. Dudkowski, T. Kapitaniak, Yu. Maistrenko. CHAOS 28, 011103, (2018).
[22] A. Salnikov, R. Levchenko, O. Sudakov. Proc. 6th IEEE (IDAACS), pp. 198–202 (2011).
[23] O. Sudakov, A. Cherederchuk, V. Maistrenko. Proc. 9th IEEE (IDAACS), pp. 311–316 (2017).