On Prüfer-like Properties of Leavitt Path Algebras

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Abstract

Prüfer domains and subclasses of integral domains such as Dedekind domains admit characterizations by means of the properties of their ideal lattices. Interestingly, a Leavitt path algebra \( L \), in spite of being non-commutative and possessing plenty of zero divisors, seems to have its ideal lattices possess the characterizing properties of these special domains. In [8] it was shown that the ideals of \( L \) satisfy the distributive law, a property of Prüfer domains and that \( L \) is a multiplication ring, a property of Dedekind domains. In this paper, we first show that \( L \) satisfies two more characterizing properties of Prüfer domains which are the ideal versions of two theorems in Elementary Number Theory, namely, for positive integers \( a, b, c \), \( \gcd(a, b) \cdot \lcm(a, b) = a \cdot b \) and \( a \cdot \gcd(b, c) = \gcd(ab, ac) \). We also show that \( L \) satisfies a characterizing property of almost Dedekind domains in terms of the ideals whose radicals are prime ideals. Finally, we give necessary and sufficient conditions under which \( L \) satisfies another important characterizing property of almost Dedekind domains, namely the cancellative property of its non-zero ideals.

1 Introduction

This paper is devoted to investigating some of the Prüfer-like properties of the ideals in a Leavitt path algebra \( L := L_K(E) \) of an arbitrary directed graph over a field \( K \). Recall that an integral domain \( D \) is called a valuation domain if its ideals are totally ordered by set inclusion. \( D \) is called a Prüfer domain if all its localizations at maximal ideals are valuation domains. \( D \) is called an almost Dedekind domain if all its localizations at maximal ideals are noetherian valuation domains and \( D \) is called a Dedekind domain if it is a noetherian domain and all its localizations at maximal ideals are noetherian valuation domains (see [5]). There are many equivalent characterizations of Prüfer domains that are widely studied in the literature. Some of the characterizations of Prüfer domains can be listed as given in [5, Theorem 6.6]:

\[ \gcd(a, b) \cdot \lcm(a, b) = a \cdot b \]
\[ a \cdot \gcd(b, c) = \gcd(ab, ac) \]
1. $R$ is a Prüfer domain.

2. If $AB = AC$, where $A, B, C$ are ideals of $R$ and $A$ is finitely generated and nonzero, then $B = C$.

3. $A(B \cap C) = AB \cap AC$ for all ideals $A, B, C$ of $R$.

4. $(A + B)(A \cap B) = AB$ for all ideals $A, B$ of $R$.

5. $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all ideals $A, B, C$ of $R$.

6. If $A$ and $C$ are ideals of $R$, with $C$ finitely generated, and if $A \subseteq C$, then there is an ideal $B$ of $R$ such that $A = BC$.

Although a Leavitt path algebra $L$ is non-commutative in nature and has plenty of zero divisors, it is somewhat intriguing and certainly interesting that the ideals of such a highly non-commutative algebra share many of the properties of the ideals of various types of (commutative) integral domains. To start with, the multiplication of ideals in $L$ is commutative ([1], [9, Theorem 3.4]), $L$ satisfies the property of a Bézout domain, namely, all the finitely generated ideals of $L$ are principal ([8]), every ideal of $L$ is projective, a property of Dedekind domains and the ideal lattice of $L$ is distributive ([9]) which characterizes Prüfer domains among integral domains.

In this paper, we consider characterizing properties of Prüfer domains which are ideal versions of well-known theorems in elementary number theory. Recall that if $a, b, c$ are positive integers, then $\gcd(a, b) \cdot \text{lcm}(a, b) = a \cdot b$ and $a \cdot \gcd(b, c) = \gcd(ab, ac)$ and $a \cdot \text{lcm}(b, c) = \text{lcm}(ab, ac)$. The first two results when expressed in terms of its ideals lead to a characterization of Prüfer domains: An integral domain $D$ is a Prüfer domain if and only if, for any two non-zero ideals $A, B$ of $D$, $(A \cap B)(A + B) = AB$ and, if and only if, for any three ideals $A, B, C$ in $D$, we have $A(B \cap C) = AB \cap AC$ (see [5]). We will show that every Leavitt path algebra satisfies these two characterizing properties. (Note that the ideal version of the result $a.lcm(b, c) = \text{lcm}(ab, ac)$, namely, $A(B + C) = AB + AC$ holds for ideals $A, B, C$ in any ring $R$.) We also investigate whether a Leavitt path algebra $L$ possesses any of the properties of almost Dedekind domains which form a subclass of the class of Prüfer domains. It is known (see [5]) that an integral domain $D$ is an almost Dedekind domain if and only if every ideal in $D$, whose radical is prime, is a power of a prime ideal. We show that every Leavitt path algebra also possesses this characterizing property. As a corollary, we show that two more properties of a Dedekind domain are satisfied by Leavitt path algebras. Almost Dedekind domains $D$ are also characterized among integral domains by the property that every non-zero ideal $A$ of $D$ is cancellative, that is, $AB = AC$ implies $B = C$ for any two ideals $B, C$ of $D$. While not every Leavitt path algebra satisfies this property, we give necessary and sufficient conditions on the graph $E$ under which every non-zero ideal of $L_K(E)$ is cancellative. Various graphical constructions illustrate our results.
2 Preliminaries

In this section, we will mention some of the needed basic concepts and results in Leavitt path algebras. A directed graph \( E = (E^0, E^1, r, s) \) consists of two countable sets \( E^0, E^1 \) and functions \( r, s : E^1 \to E^0 \). The elements \( E^0 \) and \( E^1 \) are called vertices and edges, respectively. For each \( e \in E^1 \), \( s(e) \) is the source of \( e \) and \( r(e) \) is the range of \( e \). If \( s(e) = v \) and \( r(e) = w \), then we say that \( v \) emits \( e \) and that \( w \) receives \( e \). A vertex which does not receive any edges is called a source, and a vertex which emits no edges is called a sink. A vertex \( v \) is called a regular vertex if it emits a non-empty finite set of edges. A vertex is called an infinite emitter if it emits infinitely many edges.

A graph is called row-finite if \( s^{-1}(v) \) is a finite set for each vertex \( v \). For a row-finite graph the edge set \( E^1 \) of \( E \) is finite if its set of vertices \( E^0 \) is finite. Thus, a row-finite graph is finite if \( E^0 \) is a finite set.

A path in a graph \( E \) is a sequence of edges \( \mu = e_1 \ldots e_n \) such that \( r(e_i) = s(e_{i+1}) \) for \( i = 1, \ldots, n - 1 \). In such a case, \( s(\mu) := s(e_1) \) is the source of \( \mu \) and \( r(\mu) := r(e_n) \) is the range of \( \mu \), and \( n \) is the length of \( \mu \), i.e., \( l(\mu) = n \). If \( s(\mu) = r(\mu) \) and \( s(e_i) \neq s(e_j) \) for every \( i \neq j \), then \( \mu \) is called a cycle.

An exit for a path \( \mu = e_1 \ldots e_n \) is an edge \( e \) such that \( s(e) = s(e_i) \) for some \( i \) and \( e \neq e_i \). A graph \( E \) is said to satisfy condition (L) if every cycle in \( E \) has an exit.

A subset \( D \) of vertices is said to be downward directed if for any \( u, v \in D \), there exists a \( w \in D \) such that \( u \geq w \) and \( v \geq w \). A subset \( H \) of \( E^0 \) is called hereditary if whenever \( v \in H \) and \( w \in E^0 \) for which \( v \geq w \), then \( w \in H \). \( H \) is saturated if whenever a regular vertex \( v \) has the property that \( \{ r(e) | e \in E^1, s(e) = v \} \subseteq H \), then \( v \in H \).

The path \( K \)-algebra over \( E \) is defined as the free \( K \)-algebra \( K[E^0 \cup E^1] \) with the relations:

\begin{enumerate}
    \item \( v_i v_j = \delta_{ij} v_i \) for every \( v_i, v_j \in E^0 \).
    \item \( e_i = e_i r(e_i) = s(e_i) e_i \) for every \( e_i \in E^1 \).
\end{enumerate}

This algebra is denoted by \( KE \). Given a graph \( E \), define the extended graph of \( E \) as the new graph \( \tilde{E} = (E^0, E^1 \cup (E^1)^*, r^*, s^*) \) where \( (E^1)^* = \{ e_i^* | e_i \in E^1 \} \) and the functions \( r^* \) and \( s^* \) are defined as

\[
    r^*|_{E^1} = r, \quad s^*|_{E^1} = s, \quad r^*(e_i^*) = s(e_i) \quad \text{and} \quad s^*(e_i^*) = r(e_i).
\]

The Leavitt path algebra of \( E \) with coefficients in \( K \) is defined as the path algebra over the extended graph \( \tilde{E} \), with relations:

\begin{enumerate}
    \item \( e_i^* e_j = \delta_{ij} r(e_j) \) for every \( e_j \in E^1 \) and \( e_i^* \in (E^1)^* \).
    \item \( v_i = \sum_{e_j \in E^1, s(e_j) = v_i} e_j e_j^* \) for every regular vertex \( v_i \in E^0 \).
\end{enumerate}

This algebra is denoted by \( L_KE \). The conditions (CK1) and (CK2) are called the Cuntz-Krieger relations.
A ring $R$ is called a ring with local units, if for every non-empty finite subset $X$ of $R$, there is a non-zero idempotent $u \in R$ such that $ux = x = xu$ for all $x \in X$. When $E^0$ is finite, $L_K(E)$ is a ring with unit element $1 = \sum_{v \in E^0} v$.

Otherwise, $L_K(E)$ is not a unital ring, but is a ring with local units consisting of sums of distinct elements of $E^0$.

A useful observation is that every element $a$ of $L_K(E)$ can be written as $a = \sum_{i=1}^{n} k_i \alpha_i \beta_i^*$, where $k_i \in K$, $\alpha_i, \beta_i$ are paths in $E$ and $n$ is a suitable integer. Moreover, $L_K(E) = \bigoplus_{v \in E^0} L_K(E)v = \bigoplus_{v \in E^0} vL_K(E)$. Another useful fact is that if $p^*q \neq 0$, where $p, q$ are paths, then either $p = qr$ or $q = ps$ where $r, s$ are suitable paths in $E$.

One of the most important properties of Leavitt path algebras is that each $L_K(E)$ is a $\mathbb{Z}$-graded $K$-algebra. That is, $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n$ induced by defining,

$$\text{deg}(v) = 0 \quad \text{for all} \quad v \in E^0$$

$$\text{deg}(e) = 1, \quad \text{deg}(e^*) = -1 \quad \text{for all} \quad e \in E^1.$$ 

Further, the homogeneous component $L_n$ for each $n \in \mathbb{Z}$ is given by

$$L_n = \{ \sum_{i=1}^{n} k_i \alpha_i \beta_i^* \in L : \text{l(}\alpha_i\text{)} - \text{l(}\beta_i\text{)} = n \} \text{ where } k_i \in K, \alpha_i, \beta_i \in \text{Path}(E).$$

An ideal $I$ of $L_K(E)$ is said to be a graded ideal if $I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n)$.

We shall be using the following concepts and results from [10]. A vertex $v$ is called a breaking vertex of a hereditary subset $H$ if $v$ belongs to the set

$$B_H := \{ v \in E^0 \setminus H \mid \text{v is an infinite emitter and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty \}.$$ 

In words, $B_H$ consists of those vertices of $E$ which are infinite emitters, which do not belong to $H$, and for which the ranges of the edges they emit are all, except for a finite (but nonzero) number, inside $H$. For $v \in B_H$, the element $v^H$ of $L_K(E)$ is defined by

$$v^H := v - \sum_{e \in s^{-1}(v) \cap r^{-1}(E^0 \setminus H)} ee^*.$$ 

We note that any such $v^H$ is homogeneous of degree 0 in the standard $\mathbb{Z}$-grading on $L_K(E)$. Given a hereditary saturated subset $H$ and a subset $S \subseteq B_H$, $(H, S)$ is called an admissible pair. The ideal generated by $H \cup \{ v^H : v \in S \}$ is denoted by $I(H, S)$ where $(H, S)$ is an admissible pair. The quotient graph $E \setminus (H, S)$ of $E$ by an admissible pair $(H, S)$ is defined as follows:

$$(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{ v^H \mid v \in B_H \setminus S \},$$
(E\(H, S))^{1} = \{ e \in E | r(e) \notin H \} \cup \{ e' | e \in E and r(e) \in B_{H}\backslash S \},

and range and source maps in E\(H, S) are defined by extending the range and source maps in E when appropriate, and in addition setting s(e') = s(e) and r(e') = r(e'). It was shown in [10] that \(L_{K}(E) / I(H, S) \cong L_{K}(E\(H, S))\).

Let \( \Lambda \) be an arbitrary non-empty set. Given a ring \( R \), \( M_{\Lambda}(R) \) denotes the ring of matrices with entries from \( R \), all but finitely many of which are non-zero and where the rows and columns are indexed by elements of \( \Lambda \). We will be using an important result about the ideals of the ring \( M_{\Lambda}(R) \). This result was proved in Theorem 3.1 of [4] when \( R \) is a ring with identity 1 and when \( \Lambda \) is finite. We need this result for rings with local units and when \( \Lambda \) is an arbitrary non-empty set. As far as we know, this generalized statement has not appeared in print and so we give the general statement and its proof. Thus Proposition [1]a below is a generalization of Theorem 3.1 of [4] to rings with local units. We thank Zak Mesyan for help in writing the proof of Proposition [1]b. For rings \( R \) with identity, this Proposition is also obtained by using the Morita equivalence of \( R \) and \( M_{\Lambda}(R) \) (see [2]).

**Proposition 1.** Suppose \( R \) is a ring with local units and \( \Lambda \) is an arbitrary non-empty set.

(a) Every ideal of \( M_{\Lambda}(R) \) is of the form \( M_{\Lambda}(A) \) for some ideal \( A \) of \( R \). The map \( A \mapsto M_{\Lambda}(A) \) defines a lattice isomorphism between the lattice of ideals of \( R \) and the lattice of ideals of \( M_{\Lambda}(R) \).

(b) For any two ideals \( A, B \) of \( R \), \( M_{\Lambda}(AB) = M_{\Lambda}(A)M_{\Lambda}(B) \).

**Proof.** If \( A \) is an ideal of \( R \), then it is easy to see that \( M_{\Lambda}(A) \) is an ideal of \( M_{\Lambda}(R) \). Also, if the ideals \( A \neq B \), then \( M_{\Lambda}(A) \neq M_{\Lambda}(B) \). Let \( I \) be a non-zero ideal of \( M_{\Lambda}(R) \). We wish to show that \( I = M_{\Lambda}(A) \) for some ideal \( A \) of \( R \). Let \( A \) be the set of all the entries at the (first row - first column) position in all the matrices belonging to \( I \). \( A \) is clearly an ideal in \( R \). We wish to show that \( I = M_{\Lambda}(A) \). Let \( U \) denote the set of all local units in \( R \). Let \( 0 \neq M = (m_{ij}) \in I \). Corresponding to all the finitely many non-zero entries \( m_{ij} \) in \( M \), choose a local unit \( u \in U \) satisfying \( um_{ij} = m_{ij} = m_{ij}u \) for all \( i, j \). For any \( k, l \), we have the identity

\[ E_{ij}uME_{kl}u = m_{jk}E_{il}u \tag{*} \]

where, for every \( i, j \), \( E_{ij}u \) denotes the \( \Lambda \times \Lambda \) matrix having \( u \) at the \( i \)th row and \( j \)th column entry and 0 at every other entry. In particular, \( E_{ij}uME_{kl}u = m_{jk}E_{ij}u \in I \) and thus \( m_{jk} \in A \) for all \( j, k \in \Lambda \). Hence \( M \in M_{\Lambda}(A) \). Conversely, let \( N = (a_{ij}) \in M_{\Lambda}(A) \). Since \( a_{il} \in A \), there is a matrix \( M = (m_{ij}) \in I \) such that \( m_{11} = a_{11} \). Let \( v \in U \) be a local unit satisfying \( va_{ij} = a_{ij} = a_{ij}v \) and also \( vm_{ij} = m_{ij} = m_{ij}v \) for all the entries \( a_{ij} \) in \( N \). It is enough to show that
$$a_{il}E_{il}v \in I$$ for all $$i, l$$. Applying the identity $$(*)$$, we have

$$a_{il}E_{il}v = m_{1l}E_{il}v = E_{il}vME_{il} \in I,$$

for all $$i, l$$.

This proves that $$I = M_{\Lambda}(A)$$.

It is straightforward to verify that, for any two ideals $$A, B$$ of $$R$$, $$M_{\Lambda}(A + B) = M_{\Lambda}(A) + M_{\Lambda}(B)$$ and $$M_{\Lambda}(A \cap B) = M_{\Lambda}(A) \cap M_{\Lambda}(B)$$. This shows that map $$A \mapsto M_{\Lambda}(A)$$ defines a lattice isomorphism.

We next show that $$M_{\Lambda}(AB) = M_{\Lambda}(A)M_{\Lambda}(B)$$ for any two ideals $$A, B$$ of $$R$$. Given $$I \in M_{\Lambda}(A)$$ and $$J \in M_{\Lambda}(B)$$, every entry of $$IJ$$ is a finite sum of the form $$\sum a_ib_j$$ for some $$a_i \in A$$ and $$b_j \in B$$ and hence an element of $$AB$$. Thus $$M_{\Lambda}(A)M_{\Lambda}(B) \subseteq M_{\Lambda}(AB)$$.

To prove the reverse inclusion, first note that every element $$M \in M_{\Lambda}(AB)$$ can be written as a finite sum of elements of the form $$abE_{ij}u$$ where $$a \in A, b \in B$$ and $$u$$ is a local unit corresponding to the finitely many non-zero entries of $$M$$ and also satisfying $$ua = a = au$$ and $$ub = b = bu$$. Since $$M_{\Lambda}(A)M_{\Lambda}(B)$$ is an ideal of $$M_{\Lambda}(R)$$, it is enough to show that $$abE_{ij}u \in M_{\Lambda}(A)M_{\Lambda}(B)$$, for all $$a \in A, b \in B$$ and $$i, j \in \Lambda$$. Now $$aE_{il}u \in M_{\Lambda}(A)$$ and $$bE_{lj}u \in M_{\Lambda}(B)$$ and so

$$abE_{ij}u = abE_{il}uE_{lj}u = (aE_{il}u)(bE_{lj}u) \in M_{\Lambda}(A)M_{\Lambda}(B).$$

Thus $$M_{\Lambda}(AB) = M_{\Lambda}(A)M_{\Lambda}(B)$$ for any two ideals $$A, B$$ of $$R$$. \hfill \Box

Throughout the following, $$L$$ will denote the Leavitt path algebra $$L_K(E)$$ of an arbitrary graph $$E$$ over a field $$K$$.

3 Prüfer-like properties satisfied in Leavitt path algebras

In this section, we shall describe how the ideals of every Leavitt path algebra $$L$$ satisfy two of the characterizing properties of a Prüfer domain mentioned in the introduction. In this connection, the graded ideals of $$L$$ seem to be well-behaved and some extra efforts are needed in dealing with the non-graded ideals of $$L$$. The following theorem consists of the results in [8] and [7] which will be used in the sequel.

**Theorem 2.** Let $$I$$ be a non-graded ideal of $$L = L_K(E)$$ with $$H = I \cap E^0$$ and $$S = \{u \in B_H : u^H \in I\}$$. Then

(i) \(\langle [8] \text{ Theorem 4}\rangle\) $$I = I(H, S) + \sum_{t \in T} \langle f_t(c_t) \rangle$$ where $$T$$ is some index set, for each $$t \in T, c_t$$ is a cycle without exits in $$E \setminus (H, S)$$, $$c_t^0 \cap c_s^0 = \emptyset$$ for $$t \neq s$$ and $$f_t(x) \in K[x]$$ is a polynomial with its constant term non-zero and is of the smallest degree such that $$f_t(c_t) \in I$$. 


(ii) ([7], Lemma 3.6) $I(H, S)$ is the largest graded ideal inside $I$.

We shall denote $I(H, S)$ by $gr(I)$ and call it the graded part of the ideal $I$.

Before proving the main theorem, we consider the case of graded ideals which are easy to handle. A useful property of graded ideals of a Leavitt path algebra $L$ (see [9], Lemma 3.1) is that if $A$ is a graded ideal of $L$, then for any ideal $B$, $AB = A \cap B$.

**Lemma 3.** Let $A, B, C$ be three ideals of a Leavitt path algebra $L$. If one of them is a graded ideal then

$$A(B \cap C) = AB \cap AC.$$  

**Proof.** Case 1: Suppose $A$ is a graded ideal. Then by [9] Lemma 3.1 (i),

$$A(B \cap C) = A \cap (B \cap C) = (A \cap B) \cap (A \cap C) = AB \cap AC.$$  

Case 2: Suppose $B$ or $C$ is a graded ideal, say, $B$ is a graded ideal. Then

$$AB \cap AC = (A \cap B) \cap AC \text{ since } B \text{ is graded} = B \cap AC \text{ since } AC \subseteq A = BAC \text{ since } B \text{ is graded} = A(B \cap C) \text{ since } B \text{ is graded}.\quad \square$$

Next, we consider the case when all ideals are non-graded in the next two lemmas. In the proofs, we shall be using Theorem 4.3 of [9], namely, $A \cap (B + C) = (A \cap B) + (A \cap C)$ for any three ideals $A, B, C$ in $L$. We shall also using the fact that for a graded ideal $I$, $IJ = I \cap J$ for any ideal $J$ in $L$.

**Lemma 4.** Let $A, B,$ and $C$ be non-graded ideals. If $A \subseteq gr(A) + gr(B \cap C)$ or $(B \subseteq gr(A) + gr(B \cap C)$ and $C \subseteq gr(A) + gr(B \cap C)$), then $A(B \cap C) = AB \cap AC$.

**Proof.** We want to show that $AB \cap AC \subseteq A(B \cap C)$ since the other inclusion is always true.

Suppose that $A \subseteq gr(A) + gr(B \cap C)$. By the Modular Law,

$$A = A \cap (gr(A) + gr(B \cap C)) = gr(A) + (A \cap gr(B \cap C)).$$

Then

$$AB = (gr(A) + (A \cap gr(B \cap C)))B = gr(A)B + (A \cap gr(B \cap C))B = gr(A)B + (A \cap gr(B \cap C)) \text{ by [9] Lemma 3.2}.$$
and similarly, \( AC = \text{gr}(A)C + (A \cap \text{gr}(B \cap C)) \). Hence
\[
AB \cap AC = (\text{gr}(A)B + (A \cap \text{gr}(B \cap C))) \cap (\text{gr}(A)C + (A \cap \text{gr}(B \cap C)))
\]
By [9, Theorem 4.3],
\[
AB \cap AC = [\text{gr}(A)B + (A \cap \text{gr}(B \cap C))] \cap \text{gr}(A)C + [(\text{gr}(A)B + (A \cap \text{gr}(B \cap C))) \cap (A \cap \text{gr}(B \cap C))]
\]
Now, by [9, Theorem 4.3, Lemma 3.1 (i)],
\[
AB \cap AC = \text{gr}(A) \cap (B \cap C) + \text{gr}(A) \cap (B \cap C) \cap C
\]
\[
= \text{gr}(A)(B \cap C) + \text{gr}(A)(B \cap C)
\]
\[
+ \text{Agr}(B \cap C)
\]
by using [9, Lemma 3.1 (i)]
\[
\subseteq A(B \cap C)
\]
Now, suppose that \( B \subseteq \text{gr}(A) + \text{gr}(B \cap C) \) and \( C \subseteq \text{gr}(A) + \text{gr}(B \cap C) \). Then, by the Modular Law, \( B = B \cap (\text{gr}(A) + \text{gr}(B \cap C)) = \text{gr}(B \cap C) + (\text{gr}(A) \cap B) \) and similarly, \( C = \text{gr}(B \cap C) + (\text{gr}(A) \cap C) \).
Hence,
\[
AB = \text{Agr}(B \cap C) + A(\text{gr}(A) \cap B)
\]
\[
= \text{Agr}(B \cap C) + (\text{gr}(A) \cap B)
\]
by [9, Lemma 3.2]
and similarly, \( AC = \text{Agr}(B \cap C) + (\text{gr}(A) \cap C) \).
Therefore, by using [9, Theorem 4.3, Lemma 3.1 (i)],
\[
AB \cap AC = [\text{Agr}(B \cap C) + (\text{gr}(A) \cap B)] \cap [\text{Agr}(B \cap C) + (\text{gr}(A) \cap C)]
\]
\[
= \text{Agr}(B \cap C) + \text{gr}(A) \cap (B \cap C) + \text{gr}(A) \cap (B \cap C) + \text{gr}(A)(B \cap C)
\]
\[
\subseteq A(B \cap C)
\]

In proving the next lemma, we shall be using the easy-to-see statement that, for any two ideals \( B, C \) of \( L \), \( \text{gr}(B \cap C) = \text{gr}(B) \cap \text{gr}(C) \).

**Lemma 5.** Let \( A, B, \) and \( C \) be non-graded ideals of \( L \). If \( A \not\subseteq gr(A) + gr(B \cap C) \) and \( B \not\subseteq gr(A) + gr(B \cap C) \) or \( C \not\subseteq gr(A) + gr(B \cap C) \), then \( A(B \cap C) = AB \cap AC \).

**Proof.** Without loss of generality, we may assume \( A \not\subseteq gr(A) + gr(B \cap C) \) and \( B \not\subseteq gr(A) + gr(B \cap C) \). Let \( I = I(H, S) = gr(A) + gr(B \cap C) \). By Theorem [2] (i),
\[
A = I(H_1, S_1) = \sum_{i \in X} \langle f_i(e_i) \rangle, \quad B = I(H_2, S_2) = \sum_{j \in Y} \langle g_j(e_j) \rangle, \quad \text{and}
\]
\[
C = I(H_3, S_3) = \sum_{k \in Z} \langle h_k(e_k) \rangle
\]
where $X, Y,$ and $Z$ are some index sets, $I(H_1, S_1) = gr(A)$, $I(H_2, S_2) = gr(B)$, $I(H_3, S_3) = gr(C)$ and for all $i \in X$, $j \in Y$, and $k \in Z$, $f_i(x), g_j(x), h_k(x) \in K[x]$ and $c_i, c_j,$ and $c_k$ are cycles without exits in $E \setminus (H_1, S_1)$, $E \setminus (H_2, S_2)$, and $E \setminus (H_3, S_3)$, respectively. In $\mathcal{T} = L/I \cong L_k(E \setminus (H, S))$, $\mathcal{A} = (A + I)/I$ is an epimorphic image of $A/gr(A)$ and let $\mathcal{B} = (B + I)/I$ and $\mathcal{C} = (C + I)/I$. Hence,

\[
\mathcal{A} = \sum_{i \in X'} (f_i(c_i)), \quad \mathcal{B} = (gr(A) + gr(B) + I)/I + \left[ \sum_{j \in Y'} (g_j(c_j)) + I \right]/I, \text{ and }
\]

\[
\mathcal{C} = (gr(A) + gr(C) + I)/I + \left[ \sum_{k \in Z'} (h_k(c_k)) + I \right]/I,
\]

where $X', Y', Z'$ are subsets of the sets $X, Y, Z$ respectively.

For the sake of convenience, we shall write

\[
\mathcal{B} = (gr(A) + gr(B) + I)/I + |\sum_2 + I|/I, \text{ where } \sum_2 = \sum_{j \in Y'} (g_j(c_j)) \text{ and }
\]

\[
\mathcal{C} = (gr(A) + gr(C) + I)/I + |\sum_3 + I|/I, \text{ where } \sum_3 = \sum_{k \in Z'} (h_k(c_k)).
\]

\[
\mathcal{B} \cap \mathcal{C} = \{(gr(A) + gr(B) + I)/I + |\sum_2 + I|/I\}
\]

\[
\cap \{(gr(A) + gr(C) + I)/I + |\sum_3 + I|/I\}
\]

\[
= (gr(A) + gr(B) + I)/I \cap |\sum_3 + I|/I
\]

\[
+ (gr(A) + gr(C) + I)/I \cap |\sum_2 + I|/I
\]

\[
+ |\sum_2 + I|/I \cap |\sum_3 + I|/I,
\]

noting that $[(gr(A) + gr(B) + I)/I] \cap [(gr(A) + gr(C) + I)/I$ simplifies to $\mathcal{B}$.

\[
\mathcal{A}(\mathcal{B} \cap \mathcal{C}) = \left[ \mathcal{A}(gr(A) + gr(B) + I)/I \right] \cap \left[ \mathcal{A}(\sum_3 + I)/I \right]
\]

\[
+ \left[ \mathcal{A}(gr(A) + gr(C) + I)/I \right] \cap \left[ \mathcal{A}(\sum_2 + I)/I \right]
\]

\[
+ \mathcal{A}(|(\sum_3 + I)/I) \cap (|\sum_2 + I)/I) \quad (1)
\]

On the other hand,

\[
\mathcal{A}B = \left[ \mathcal{A}(gr(A) + gr(B) + I)/I \right] + \left[ \mathcal{A}(\sum_2 + I)/I \right]
\]
and
\[\bar{A}C = [\bar{A}(gr(A) + gr(C) + I)/I] + [\bar{A} \sum_3 + I] / I\]

Hence, by [9, Theorem 4.3]
\[\bar{A} \bar{B} \cap \bar{A} \bar{C} = [\bar{A}(gr(A) + gr(B) + I)/I] \cap [\bar{A}(gr(A) + gr(C) + I)/I]
= 0 \text{ since } (gr(B) + gr(A) + I)/I \cap (gr(C) + gr(A) + I)/I = \bar{I}
+ [\bar{A}(gr(A) + gr(B) + I)/I \cap [\bar{A} \sum_3 + I] / I
+ [\bar{A} \sum_2 + I] / I \cap [\bar{A}(gr(A) + gr(C) + I)/I
+ [\bar{A} \sum_2 + I] / I \cap [\bar{A} \sum_3 + I] / I \quad (2)\]

We wish to show that \( \bar{A} \bar{B} \cap \bar{A} \bar{C} = \bar{A}(\bar{B} \cap \bar{C}) \). Now comparing (1) and (2), all we need is to show that
\[\bar{A}([(\sum_2 + I) / I] \cap [(\sum_3 + I) / I]) = [\bar{A}(\sum_2 + I) / I] \cap [\bar{A}(\sum_3 + I) / I].\]

Let \( G \) be the graded ideal of \( \bar{L} \) generated by the vertices on all the cycles \( c_i \) where \( i \) belongs to \( X' \). Since the \( c_i \) are cycles without exits in \( E \backslash (H, S) \), \( G \) is isomorphic to the ring direct sum \( \bigoplus_{i \in X'} M_{\Lambda_i}(K[x, x^{-1}]) \) where the \( \Lambda_i \) are suitable index sets by [9, Theorem 2.7.3]. Note that \( \bar{A} \) is contained in \( G \) and so \( \bar{A}G = \bar{A}, \) by [9, Lemma 3.2]. Then
\[\bar{A}([(\sum_2 + I) / I] \cap [(\sum_3 + I) / I]) = \bar{A}G([(\sum_2 + I) / I] \cap [(\sum_3 + I) / I])
= \bar{A}G([(\sum_2 + I) / I] \cap [G(\sum_3 + I) / I]) \text{ by Lemma 3}
= \bar{A}([(G \cap (\sum_2 + I) / I] \cap [G \cap (\sum_3 + I) / I]) \text{ as } G \text{ is graded.}\]

Now all three ideals in the preceding equation are ideals of \( G \) and \( G \) is isomorphic to the ring direct sum \( \bigoplus_{i \in X'} M_{\Lambda_i}(K[x, x^{-1}]) \). Moreover, by Proposition 1, there is an isomorphism between the ideal lattices of \( M_{\Lambda_i}(K[x, x^{-1}]) \) and \( K[x, x^{-1}] \) which preserves multiplication. Since \( K[x, x^{-1}] \) is a Prüfer domain, \( H(\Lambda \cap L) = HK \cap HL \) holds for any three ideals of \( K[x, x^{-1}] \) and consequently, any three ideals of \( G \) also satisfy this property. We observe that
\[\bar{A}([G \cap (\sum_2 + I) / I] \cap [G \cap (\sum_3 + I) / I]) = (\bar{A}[G \cap (\sum_2 + I) / I] \cap \bar{A}[G \cap (\sum_3 + I) / I])
= \bar{A}G(\sum_2 + I) / I \cap \bar{A}G(\sum_3 + I) / I \text{ as } G \text{ is graded}
= \bar{A}(\sum_2 + I) / I \cap \bar{A}(\sum_3 + I) / I, \text{ by [9] Lemma 3.2}.\]
We thus conclude that \( \bar{A} \bar{B} \cap \bar{A} \bar{C} = \bar{A}(\bar{B} \cap \bar{C}) \). Then

\[
AB \cap AC = A(B \cap C) + (gr(A) + gr(B \cap C)).
\]

Now \( AB \cap AC \) contains both \( AB \cap AC \) and \( A(B \cap C) \) and so using modular law, we have

\[
AB \cap AC = (AB \cap AC) \cap (A \cap B \cap C) + (gr(A) + gr(B \cap C)) \cap (A \cap B \cap C) \\
= A(B \cap C) + gr(A)(B \cap C) + Agr(B \cap C)
\]

\[
\subseteq A(B \cap C).
\]

Since \( A(B \cap C) \subseteq AB \cap AC \) is always true, we get \( A(B \cap C) = AB \cap AC \). \( \Box \)

Hence, we can state the main result.

**Theorem 6.** If \( A, B, C \) are any ideals of a Leavitt path algebra \( L \) of an arbitrary graph \( E \), then

\[
A(B \cap C) = AB \cap AC.
\]

**Proof.** By using Lemmas 3, 4 and 5, the result follows. \( \Box \)

In elementary number theory, it is well-known that for any two positive integers \( a, b \), we have \( \gcd(a, b) \cdot \text{lcm}(a, b) = ab \). This property can be stated for ideals as: for any ideals \( A, B \), \( (A + B)(A \cap B) = AB \). This equality holds for ideals in a Dedekind domain. If this equality holds for finitely generated ideals \( A, B \), then the integral domain is a Prüfer domain. The next theorem shows that any Leavitt path algebra satisfies this characterizing property. We shall prove this by using Theorem 6.

**Theorem 7.** For any two ideals \( A, B \) of a Leavitt path algebra \( L \),

\[
(A + B)(A \cap B) = AB.
\]

**Proof.** Now, by Theorem 6 and Theorem 3.4,

\[
(A + B)(A \cap B) = (A + B)A \cap (A + B)B \\
\supseteq BA \cap AB = AB \cap AB = AB.
\]

The converse inclusion is always true since

\[
(A + B)(A \cap B) = A(A \cap B) + B(A \cap B) \\
\subseteq AB + BA = AB + AB = AB \quad \text{by Theorem 3.4}.
\]

Thus we obtain \( (A + B)(A \cap B) = AB \). \( \Box \)
4 Almost Dedekind domains and Leavitt path algebras

As noted in the Introduction, an almost Dedekind domain $D$ is a Prüfer domain with the property that all its localizations with respect to maximal ideals are noetherian valuation domains. In this section, we investigate whether a Leavitt path algebra $L$ satisfies any of the other characterizing properties of almost Dedekind domains. Recall that the radical $\sqrt{I}$ of an ideal $I$ in a ring $R$ is the intersection of all the prime ideals of $R$ containing $I$. It is known (see [3]) that an integral domain $D$ is an almost Dedekind domain if and only if every non-zero ideal $I$ with its radical $\sqrt{I}$ a prime ideal is a power of a prime ideal. It turns out that every Leavitt path algebra $L$ satisfies this property. As a corollary, we show that if $P$ is a non-zero prime ideal in a Leavitt path algebra $L$, then all the $P$-primary ideals of $L$ form a well-ordered chain under set inclusion and that there are no ideals of $L$ strictly between $P^n$ and $P^{n+1}$, properties satisfied by Dedekind domains. We also consider the property of non-zero ideals being cancellative, an important property that characterizes almost Dedekind domains among integral domains. Recall that a non-zero ideal $A$ in a ring $R$ is cancellative if, for any two ideals $B, C$ of $R$, $AB = AC$ implies that $B = C$. Examples indicate that not all Leavitt path algebras have this property.

We show that, in a Leavitt path algebra $L := L_K(E)$, every non-zero ideal of $L$ is cancellative if and only if either (a) there is a cycle $c$ without exits in $E$ based at a vertex $v$ such that $u \geq v$ for every $u \in E^c$ and $H_E = \{H\}$ where $H$ is the hereditary saturated closure of $\{v\}$ and $B_H = \emptyset$, or (b) $E$ satisfies Condition (K), $|H_E| \leq 2$, for any two $X, Y \in H_E$ with $X \neq Y$, $X \cap Y = \emptyset$ and, for each $H \in H_E$, $B_H$ is empty and $H$ is the saturated closure of each $u \in H$. Here $H_E$ denotes the set of all non-empty proper hereditary saturated subsets of vertices in the graph $E$. Equivalent conditions on $L$ are, either (a) $L$ contains a graded ideal $M$ which contains every proper ideal of $L$ and $M \cong M_\Lambda(K[x, x^{-1}])$ where $\Lambda$ is an arbitrary finite or infinite index set or (b) $L$ has at most two non-zero ideals each of which is graded and is a principal ideal.

We begin by showing that every Leavitt path algebra satisfies the first mentioned property of almost Dedekind domains.

Theorem 8. Let $I$ be a non-zero ideal of $L$. If its radical $\sqrt{I}$ is a prime ideal, say $P$, then $I = P^n$, for some $n \geq 1$.

Proof. If $I$ is a graded ideal, then $I = \sqrt{I}$ by [3] Lemma 2.1]. So if $\sqrt{I}$ is prime, then trivially $I$ is a prime power. Suppose now that $I$ is a non-graded ideal such that its radical $\sqrt{I}$ is a prime ideal. By [3] Theorem 4], $I = I(H, S) + \sum_{i \in X} (f_i(c_i))$, where the $c_i$ are distinct cycles without exits in $E \setminus (H, S)$, $f_i(x)$ are polynomials in $K[x]$ with non-zero constant terms. Now, by [3] Lemma 5.4], $gr(\sqrt{I}) = gr(I) = I(H, S) + \langle p(c) \rangle$ where $p(x) \in K[x]$ is an irreducible polynomial with non-zero constant term, $c$ is a cycle without exits in $E \setminus (H, B_H)$ and, moreover, $E \setminus (H, B_H)$ is
downward directed. The downward directness of \( E(H, B_H) \) implies that \( c \) is the only cycle without exits in \( E(H, B_H) \). Thus \( I \) must be of the form \( I = I(H, B_H) + \langle f(c) \rangle \) where \( f(x) \in K[x] \) has its constant term non-zero. We claim \( f(x) = p^n(x) \) for some integer \( n \geq 1 \). Suppose, on the contrary, \( f(x) = q(x)g(x) \) where \( q(x) \neq p(x) \) is an irreducible polynomial with non-zero constant term. Then \( I = I(H, B_H) + \langle f(c) \rangle \subseteq I(H, B_H) + \langle q(c) \rangle \). Since \( E(H, B_H) \) is downward directed, \( I(H, B_H) + \langle q(c) \rangle \) is a prime ideal (Theorem 3.12) and so it contains the radical of \( M \) with ideal of \( \Lambda \).

Corollary 9. Let \( L \) be a Leavitt path algebra and let \( P \) be a non-zero prime ideal of \( L \). Then

(i) the set of all the \( P \)-primary ideals is totally ordered under set inclusion;

(ii) there is no ideal \( A \) of \( L \) satisfying \( P^2 \subsetneq A \subseteq P \).

Proof. (i) Note that if \( I \) is a \( P \)-primary ideal of \( L \), then \( \sqrt{I} = P \). By Theorem 8, \( I = P^k \) for some \( k \geq 1 \). Thus the set of \( P \)-primary ideals of \( L \) is the set \( \{P^n : n \geq 1\} \) which is a totally ordered (countable) set under inclusion.

(ii) We shall actually show that there is no ideal \( A \) satisfying \( P^{n+1} \subsetneq A \subseteq P^n \) for any \( n \geq 1 \). Note that if there is an ideal \( A \) of \( L \) satisfying \( P^{n+1} \subseteq A \subseteq P^n \) for some \( n \geq 1 \), then \( \sqrt{A} = P \) and so, by Theorem 8, \( A \) is a power of \( P \) and hence \( A = P^{n+1} \) or \( P^n \). \( \square \)

Next, we consider the cancellative property of the ideals of an almost Dedekind domains. This property does not seem to hold in arbitrary Leavitt path algebras as the next two examples show.

Example 10. Consider the following graph \( E \):
Given the ideals $A = \langle v_1 \rangle$, $B = \langle v_5 \rangle$ and $C = \langle v_7 \rangle$ of $\mathcal{L}_K(E)$. Clearly $B \neq C$ but $AB = 0 = AC$.

Example 11. Consider the graph

Then $H = \{ v \}$ is a hereditary saturated subset. Let $A = \langle H \rangle$, be the principal ideal generated by $H$. Clearly $c$ has no exits in $E \setminus H$. Let $B$ be the nongraded ideal $A + \langle p(c) \rangle$, where $p(x)$ is a polynomial in $K[x]$. Clearly $gr(B) = A$. Since $A$ is a graded ideal, we apply [8, Lemma 3.1(i)], to conclude that $AB = A \cap B = A = A^2 = AA$. However, $A \neq B$.

The next theorem gives necessary and sufficient conditions (both graphical and algebraic) under which non-zero finitely generated ideals of $\mathcal{L}$ is cancellative. Interestingly, in this case, every non-zero ideal of $\mathcal{L}$ also turns out to be cancellative.

We prove a useful lemma and in its proof we shall again use the result [8, Lemma 3.1] that if $A$ is a graded ideal of $\mathcal{L}$, then for any other ideal $B$, $A \cdot B = A \cap B$ and that $A^2 = AA$.

**Lemma 12.** If the cancellation property for finitely generated ideals holds in $\mathcal{L}$, then there cannot be two ideals $A, B$ with $A \subset B$ and $A$ graded and non-zero. In particular, $gr(B) = 0$.

**Proof.** By [10], $A = I(H, S)$, where $H = A \cap E^0$ and $S \subseteq B_H$. Since $A \neq 0$, $H \neq \emptyset$. Then, for a vertex $u \in H$, $C = \langle u \rangle$ is a finitely generated graded ideal. If $B$ is an ideal such that $A \subset B$ then we get $C \cdot B = C \cap B = C = C \cdot C$, but $B \neq C$, a contradiction to the cancellation property. This proves the first statement. By taking $A = gr(B)$, the second statement follows from the first. \[ \square \]

**Corollary 13.** If the cancellation property for finitely generated ideals holds in $\mathcal{L}$, then every non-empty hereditary saturated subset $H \subseteq E^0$ is the hereditary saturated closure of each single vertex $u \in H$ and, moreover, $B_H = \emptyset$. In particular, every non-zero graded ideal of $\mathcal{L}$ is a principal ideal, being generated by a single vertex.

**Proof.** If there is a vertex $u \in H$ such that the hereditary saturated closure $X$ of $\{ u \}$ is not equal to $H$, then the non-zero graded ideal $A = \langle X \rangle$ satisfies $A \subset \langle H \rangle$, contradicting Lemma 12. Likewise, if $B_H \neq \emptyset$, then again we have the non-zero graded ideal $I(H, \emptyset) \subseteq I(H, B_H)$, contradiction Lemma 12. If $A = I(H, S)$ is a non-zero graded ideal of $\mathcal{L}$, then since $B_H = \emptyset$, $S = \emptyset$ and since $H$ is the hereditary saturated closure of any vertex $u \in H$, $A = \langle H \rangle = \langle \{ u \} \rangle$ is a principal ideal. \[ \square \]
Theorem 14. Let $E$ be an arbitrary graph. The following conditions are equivalent for $L$:

(i) The cancellation property holds for all non-zero ideals in $L$;

(ii) The cancellation property holds for all non-zero finitely generated ideals of $L$;

(iii) Either (a) there is a cycle $c$ without exits in $E$ based at a vertex $v$ such that $u \geq v$ for every $u \in E^0$ and $H_E = \{H\}$ where $H$ is the hereditary saturated closure of $\{v\}$ and $B_H = \emptyset$, or (b) $E$ satisfies Condition (K), $|H_E| \leq 2$, for any two $X, Y \in H_E$ with $X \neq Y$, $X \cap Y = \emptyset$ and, for each $H \in H_E$, $B_H$ is empty and $H$ is the saturated closure of each $u \in H$.

(iv) Either (a) $L$ contains a graded ideal $M$ which contains every proper ideal of $L$ and $M \cong M_\Lambda(K[x, x^{-1}])$ where $\Lambda$ is an arbitrary finite or infinite index set or (b) $L$ has at most two non-zero graded ideals each of which is graded and is a principal ideal.

Proof. Clearly (i)$\Rightarrow$(ii).

Assume (ii). Case (a): Suppose $L$ has a non-graded prime ideal $P$ with $P \cap E^0 = H'$. By [7, Theorem 3.12], $P = I(H', B_{H'}) + \langle p(c) \rangle$, where $c$ is a cycle without exits based at a vertex $v$ in $E \setminus (H', B_{H'})$, $p(x)$ is an irreducible polynomial in $K[x, x^{-1}]$ such that $p(c) \in P$ and $u \geq v$ for all $u \in E^0 \setminus H'$. By Lemma [12] $I(H', B_{H'}) = 0$ and hence both $H'$ and $B_{H'}$ are empty. This means that $E$ contains a unique cycle $c$ without exits based at a vertex $v$ and $u \geq v$ for every vertex $u \in E^0$. Let $H$ be the hereditary saturated closure of $\{v\}$. Observe that there cannot be two members $X, Y \in H_E$ with one of them properly containing the other, say $X \subset Y$. Because, we will then have two non-zero graded ideals $A = \langle X \rangle$ and $B = \langle Y \rangle$ with $A \subset B$ and this is not possible by Lemma [12]. We claim that $H_E = \{H\}$. Suppose, on the contrary, there is another element $Z \in H_E$. As noted above, $Z \subset H$ and $H \not\subset Z$. So $Z \cap H \subset H$ is further non-empty since $v \in Z \cap H$. This then gives rise to two non-empty hereditary saturated subsets with one containing the other, a contradiction. Also $B_H = \emptyset$, by Corollary [13]. This proves (iii)(a).

Case (b): Suppose every prime ideal of $L$ is graded. Then, by [7, Corollary 3.13], $E$ satisfies Condition (K) and so every ideal of $L$ is graded. If $|H_E| = 0$, then $H_E = \emptyset$ and we are done. Assume $|H_E| \neq 0$. Suppose $X, Y \in H_E$ with $X \neq Y$. We claim $X \cap Y = \emptyset$. Because if a vertex $u \in X \cap Y$, then, by Corollary [13] both $X$ and $Y$ are saturated closures of $\{u\}$ and hence $X = Y$, a contradiction. We claim that $|H_E| \leq 2$. Indeed if there are three distinct members $X, Y, Z \in H_E$, then the three distinct non-zero graded ideals $A = \langle X \rangle$, $B = \langle Y \rangle$, $C = \langle Z \rangle$ are principal by Corollary [13] and satisfy $A \cdot B = A \cap B = 0 = A \cap C = A \cdot C$, but $B \neq C$. This contradiction shows that $|H_E| \leq 2$. By Corollary [13] each $H \in H_E$ is the hereditary saturated closure of each $u \in H$ and the corresponding $B_H = \emptyset$. This proves (iii)(b).

Assume (iii) (a). Let $M$ be the ideal of $L$ generated by the hereditary saturated closure $H$ of $\{v\}$, so $M$ is a graded ideal of $L$. Since $c$ is a cycle without
exits, by \textsuperscript{12} Lemma 2.7.1, \( M \cong M_\Lambda(K[x, x^{-1}]) \) where \( \Lambda \) is an arbitrary finite or infinite index, being the set of all paths in \( E \) that end at \( v \), but do not include the entire cycle \( c \). Since \( H_E = \{H\} \) and since \( B_H = \emptyset \), \( M \) is the only non-zero graded ideal of \( L \). We claim that every non-zero ideal \( N \) of \( L \) is contained in \( M \) (and non-graded). Suppose \( N \neq M \). Now \( N \) must be a non-graded ideal, since \( M \) is the only non-zero graded ideal of \( L \). Then, by Lemma \textsuperscript{12} \( \text{gr}(N) = 0 \) and so \( N \) does not contain any vertices. As \( E^0 \) is downward directed, \textsuperscript{12} Lemma 3.5] then implies that \( N = \langle f(c) \rangle \) where \( f(x) \in K[x] \) and \( c \) is a cycle without exits in \( E \). Since \( u \geq v \) for every \( u \in E^0 \), \( c \) is the only cycle without exits in \( E \) and so \( c = c \). Then \( N = \langle f(c) \rangle \subset M \), as \( c \in M \). This proves (iv)(a).

Assume (iii)(b). Since Condition (K) holds, every ideal of \( L \) is graded \textsuperscript{11}. If \( H_E = \emptyset \), then the only hereditary saturated subsets of \( E^0 \) are \( E^0 \) and \( \emptyset \) and so \( L \) contains no non-zero proper ideals. Suppose \( H_E = \{H\} \), with \( B_H = \emptyset \). Then \( I = \langle H, \emptyset \rangle \) is the only proper non-zero ideal of \( L \). Suppose \( H_E = \{H_1, H_2\} \). Since \( B_{H_1} = \emptyset = B_{H_2} \), \( A = \langle H_1 \rangle \) and \( B = \langle H_2 \rangle \) are the only proper non-zero ideals of \( L \) and they both are principal ideals as \( H_1, H_2 \) are saturated closures of single vertices \( v_1 \in H_1 \) and \( v_2 \in H_2 \). This proves (iv)(b).

Assume (iv)(a) so that \( L \) contains a graded ideal \( M \cong M_\Lambda(K[x, x^{-1}]) \) where \( \Lambda \) is an arbitrary finite or infinite index set and that \( M \) contains every other proper ideal of \( L \). Now, by Proposition 1, the ideals of \( M_\Lambda(K[x, x^{-1}]) \) are of the form \( M_\Lambda(I) \) where \( I \) are the ideals of \( K[x, x^{-1}] \), the map \( I \mapsto M_\Lambda(I) \) is an isomorphism of the ideal lattices of \( K[x, x^{-1}] \) and \( M_\Lambda(K[x, x^{-1}]) \) and, further, \( M_\Lambda(I \cdot J) = M_\Lambda(I) \cdot M_\Lambda(J) \) for any two ideals of \( K[x, x^{-1}] \). Since the cancellation for non-zero ideals hold in the principal ideal domain \( K[x, x^{-1}] \), we conclude that the cancellation property holds for non-zero ideals in \( M \). Now the graded ideal \( M \) possesses local units, being isomorphic to a Leavitt path algebra of a suitable graph (see \textsuperscript{11}). From this, it easy to show that the ideals of \( M \) are also the ideals of \( L \). Since \( M \) contains every other ideal \( A \) of \( L \) and is graded (so \( MA = M \cap A \)), we conclude that \( M \) is also cancellative. Thus all the non-zero ideals of \( L \) have the cancellation property. This proves (i).

Now (iv)(b)\( \Rightarrow \) (i) is immediate since \( L \) has at most two distinct non-zero proper ideals which are graded and thus the cancellation property holds trivially in \( L \).

We conclude the paper by illustrating some examples of the graphs that satisfy the conditions of part (iii) of Theorem \textsuperscript{14}.

**Example 15.** Consider the graph \( E_1 \)

\[
\begin{array}{ccc}
\bullet & - & \bullet \\
\circ & - & \circ \\
\end{array}
\]

which satisfies the conditions of part (iii) a). So, \( H = \{v\} \) is the only non-zero proper hereditary saturated subset and \( M = \langle H \rangle \cong K[x, x^{-1}] \) contains every proper ideal of \( L_K(E_1) \).
Consider the graph \( R_2 \)

\[
\begin{array}{c}
  \\
  \\
\end{array}
\]

which satisfies the conditions of part (iii) b) and there are no non-zero proper hereditary saturated subsets. Actually \( L_K(R_2) \) is the simple Leavitt algebra of type \((1,2)\).

Consider the graph \( E_3 \)

\[
\begin{array}{c}
  \\
  \\
\end{array}
\]

which satisfies the conditions of part (iii) b) and \( H = \{v\} \) is the only non-zero proper hereditary saturated subset.

Consider the graph \( F \)

\[
\begin{array}{c}
  \\
  \\
\end{array}
\]

which satisfies the conditions of part (iii) b) and \( H_1 = \{v\} \) and \( H_2 = \{w\} \) are the only two non-zero proper hereditary saturated subsets.

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