Twisted Covariant Noncommutative Self-dual Gravity*

S. Estrada-Jiménez
Centro de Estudios en Física y Matemáticas Básicas y Aplicadas,
Universidad Autónoma de Chiapas
Calle 4ª Oriente Norte. 1428,
Tuxtla Gutiérrez, Chiapas, México

H. García-Compeán
Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN
P.O. Box 14-740, 07000 México D.F., México and
Centro de Investigación y de Estudios Avanzados del IPN,
Unidad Monterrey, PIIT, Vía del Conocimiento 201,
Autopista nueva al Aeropuerto km 9.5,
66600, Apodaca Nuevo León, México

O. Obregón
Instituto de Física de la Universidad de Guanajuato
P.O. Box E-143, 37150, León Gto., México

C. Ramírez
Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla
P.O. Box 1364, 72000, Puebla, México

(Dated: December 4, 2008)

* Dedicated to the memory of Julius Wess.
Abstract

A twisted covariant formulation of noncommutative self-dual gravity is presented. The formulation for constructing twisted noncommutative Yang-Mills theories is used. It is shown that the noncommutative torsion is solved at any order of the $\theta$-expansion in terms of the tetrad and some extra fields of the theory. In the process the first order expansion in $\theta$ for the Plebański action is explicitly obtained.

PACS numbers: 04.70.Dy, 04.62.+v, 04.30.Nk, 11.10.Nx, 11.25.Db

†E-mail address: sestrada@unach.mx
‡E-mail address: compean@fis.cinvestav.mx
§E-mail address: octavio@fisica.ugto.mx
¶E-mail address: cramirez@fcfm.buap.mx
I. INTRODUCTION

Noncommutative structures in field theory have been studied over the years and the subject has been introduced in several forms. The idea of noncommutative space-time seems to be firstly proposed by Heisenberg [1], as a possible cut-off to cure UV divergences in quantum field theory (for some historical remarks, see [2]). This idea was further developed by H.S. Snyder and applied it to find an implementation of the Lorentz symmetry on a space-time with non-commuting coordinates [3]. Snyder’s construction was realized in (4+1)-dimensional space-time, however it allows coordinate transformations which break down the Lorentz symmetry in a (3+1)-spacetime subspace. The extension to include the gauge symmetry of the electromagnetic field was pursued in a second paper [4] (for a recent review, see [5]).

Deformation quantization and Connes approach are two of the most used formulations of noncommutative spaces. On one hand, deformation quantization was first introduced in the context of phase-space quantization [6], some few years before Snyder’s paper and finally formulated as an alternative quantization method in Ref. [7] (for a review, see [8]). On the other hand, Connes noncommutative geometry [9] is a rigorous mathematical setting containing non-trivial structures originated from von Neumann [10] and Gelfand-Naimark results [11].

An important step done recently, was the discovery that noncommutative gauge theory is obtained naturally from non-perturbative string theory (D-brane physics) via the Seiberg-Witten map [12]. Furthermore, M-theory, in its M(atrix) theory approach, was also shown to be compatible with these noncommutative structures [13]. This relation to string theory has been one of the main motivations to further explore the physics of noncommutative theories.

Non-commutative field theory can incorporate nonlocal effects in field theory at the classical and quantum levels in an interesting and subtle way. For instance it gives rise to surprising effects like the IR/UV mixing [14] (for some reviews, see [15, 16]). Recently non-perturbative studies (via Monte Carlo simulations) seem to support the existence of this mixing [17].

Noncommutative field theories can be carried over to SU(N) gauge theories through the implementation of the Universal Enveloping Algebra associated to the Lie algebra of
the gauge group (of the limiting commutative field theory). Consequently, the Standard Model or GUT’s models \[18\] can be constructed in this way by using the Seiberg-Witten map. Similarly, noncommutative versions of topological and self-dual gravity \[19\] can be constructed by using the same methods.

In fact, there are numerous proposals of noncommutative gravity theories in four dimensions \[20\]. However, they do not have a clear relation to string theory as the gauge theory counterpart do \[21\]. Moreover, the diffeomorphism invariance turns out to be broken even at the classical level. This is the problem of covariance and there is evidence that noncommutative field theories also could be non-unitary and violate causality (see for instance, \[22\]). This of course has consequences for the consistence of the theory.

More recently, proposals of a formulation of a covariant noncommutative field theory have been made \[23, 24\] (see also, \[25, 26\]). In these proposals the Lorentz symmetry transformations are deformed by a twist in order that the noncommutative theory be invariant. By this twist the Leibniz rule of the transformations on the Moyal product of two fields is consistently deformed as

\[
\delta^*_\omega (\psi \star \phi) = \delta^*_\omega \psi \star \phi + \psi \star \delta^*_\omega \phi,
\]

where \(\delta^*_\omega\) is a noncommutative variation operator to be defined below. The twist has been formulated for diffeomorphisms in \[27\], in such a way that the algebraic structure of the Lie algebra of vector fields on the manifold is deformed into a noncommutative algebra of diffeomorphisms, keeping the noncommutative parameter \(\theta\) constant. This allows to construct geometric composite covariant objects in the deformed algebra, in particular metrics, covariant derivatives, curvature and torsion. In this way in \[28\] a twisted covariant noncommutative Einstein-Hilbert action was given, which has been further explored in \[29\] (for some reviews on the subject, see \[30\]). In a similar spirit, gauge symmetries can be twisted giving rise to covariant noncommutative gauge theories described in \[26, 31, 32\] and further developed in \[33\]. An interesting point of this formulation is that the language of differential forms can be used to obtain covariant results.

In \[34\], J. Wess has given an explicit realization of the twisted co-product (1) for gauge symmetry. This formulation makes use of the functional calculus language from field theory, which allows to explicitly restrict the transformations to the fields, and to avoid the problem that the derivatives of the Moyal product are not covariant. In the present paper we follow these results, and generalize them to diffeomorphisms, in order to construct a twisted
covariant noncommutative formulation of Plebanski's self-dual gravity. This involves simultaneously local Lorentz and diffeomorphism transformations. As is well known, Plebanski's self-dual gravity is a topological constrained $SL(2, \mathbb{C})$ $BF$ theory, and self-dual variables have been the starting point to find loop variables to quantize the gravitational field. In [36] we have described $SL(2, \mathbb{C})$ noncommutative topological and self-dual gravities, respectively. Though the theories are manifestly Lorentz invariant, the diffeomorphism invariance remains broken. In this respect they are not fully symmetric with respect to the whole noncommutative symmetries. In this paper we use the twisted formalism to give a noncommutative $SL(2, \mathbb{C})$ $BF$ theory, invariant under twisted local Lorentz and diffeomorphism transformations. In order to do that, we exhibit a simple noncommutative version of the volume form, given by the product of the one-form tetrads. Then we will implement the noncommutative constraints in such a way that we get a noncommutative version of Plebanski action which is not only invariant under twisted Lorentz transformations but also under twisted diffeomorphism ones. We will show that the torsion can be solved at any order of an expansion on $\theta$. The Lagrangian, the torsion and other relevant expressions are explicitly calculated in the present paper.

This paper is organized as follows, in Section II we give an overview of twisted covariant non-commutative gauge and gravity theories. In the process we prove that the functional derivative methods introduced in Ref. [34] for gauge fields, can be carried over to the construction of noncommutative gravitational fields with twisted symmetries (a basic detailed calculation is summarized in the appendix). In Section III we give an overview of self-dual gravity. In section IV we construct the twisted covariant self-dual gravity. Section V is devoted to the final remarks.

II. NONCOMMUTATIVE GAUGE AND GRAVITY THEORIES CONSTRUCTED VIA TWISTING

In the present section we describe some important features of the twisted gauge and diffeomorphism transformations, that will be necessary in Sec. IV.

Noncommutativity is introduced ordinarily through a Moyal-Weyl space-time, with com-
mutation relations given by (for recent reviews see, \[15\])

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{2} \]

where the antisymmetric matrix \(\theta^{\mu\nu}\) has constant entries. This noncommutativity can be realized through the Wigner-Moyal correspondence, by means of the Moyal product

\[ f \ast g(x) = \mu_*(f \otimes g)(x) = \exp \left\{ \frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu} \right\} f(y)g(z) \bigg|_{x=y=z}, \tag{3} \]

where \(\mu_* = \mu \circ \mathcal{F}^{-1}\), \(\mu\) is the product map \(\mu(f \otimes g) = fg\), with \(\mathcal{F} = e^{-\frac{i}{2} \theta^{\mu\nu} \partial_{\mu} \otimes \partial_{\nu}}\). When necessary, we will write \(\mu(f \otimes g) = f \cdot g\), in order to stress that we are speaking of the commutative product. The presence of \(\theta^{\mu\nu}\) as a constant matrix and of ordinary derivatives in this definition leads to the loss of covariance. In particular it has been shown that Lorentz invariance is violated (see for instance, \[22\]) and there are problems to include in a theory representations with different charges \[18\].

If we consider a gauge group of transformations, one can construct a covariant deformed theory by introducing an appropriate twisting of the transformation law of the Moyal product of fields in some specific representation \(\mathcal{R}\). The ingredients are: (i) a Lie algebra \(\mathcal{G}\), (ii) an action of the Lie algebra \(\mathcal{G}\) on the space of functions \(\mathcal{A} = \text{Fun}(M)\) of the space-time manifold \(M\) that one wants to deform into a noncommutative algebra \(\mathcal{A}_\theta\), and (iii) a twist element \(\mathcal{F}\), constructed with the generators of the Lie algebra \(\mathcal{G}\). Then by twisting the gauge and diffeomorphism Lie algebras we can obtain a covariant noncommutative theory of gauge fields or gravitational fields, respectively.

We start by considering a gauge group \(G\), with a Lie algebra \(\mathcal{G}\) in some irreducible representation \(\mathcal{R}\). The symmetry properties of the field theory on spacetime \(M\) can be lifted in a natural way to the universal enveloping algebra

\[ U(\mathcal{G}, \mathcal{R}) \text{ of } \mathcal{G} \text{ in the irrep } \mathcal{R}. \]

This has a structure of Hopf algebra \((U(\mathcal{G}, \mathcal{R}), m, e; \Delta, \varepsilon; S)\) where \(\Delta : U(\mathcal{G}, \mathcal{R}) \to U(\mathcal{G}, \mathcal{R}) \otimes U(\mathcal{G}, \mathcal{R})\) is the co-product (for a more detailed description of the Hopf algebra structure the reader can consult for instance, \[37\]). In the commutative theory the transformation law of the product of two fields, i.e. the Leibniz rule, is given by

\[ \delta_\alpha(\phi \cdot \psi) = \mu \left[ \Delta(\delta_\alpha)(\phi \otimes \psi) \right] = (\delta_\alpha \phi) \cdot \psi + \phi \cdot (\delta_\alpha \psi), \tag{4} \]

where \(\Delta(\delta_\alpha) = \delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha\) is the co-product. In the noncommutative theory, the definition of the co-product is more elaborated. Indeed, the Moyal product of fields involves
non-covariant derivatives, hence we would expect that it will have a complicated nonlinear transformation. However the coproduct can be generalized by considering a twisting of Eq. (4).

**Twisted Gauge Transformations**

Let us write the transformation of a matter field as $\delta_\alpha \phi_j(x) = i\alpha^l(x)[T_l\phi(x)]_j = S_\alpha \phi_j(x)$, where $S_\alpha$ is given by

$$S_\alpha^0 = i \int dz \alpha^l(z)[T_l\phi(z)]_j \frac{\delta}{\delta \phi_j}.$$  \hspace{1cm} (5)

Then we define the noncommutative infinitesimal transformations by

$$\delta^*_\alpha \phi = S^*_\alpha \phi = \delta_\alpha \phi = S_\alpha \phi,$$  \hspace{1cm} (6)

which, according to [28], it can be written also as $\delta^*_\alpha \phi = -X^*_\alpha * \phi$. Here we will follow the formulation (6), which makes explicit that the transformations act only on the fields of the theory. Thus we can define a restricted Moyal product which operates only on the fields

$$\mu_*(\phi \otimes \psi) = \mu \circ \mathcal{F}^{-1}(\phi \otimes \psi) = \phi \ast \psi,$$  \hspace{1cm} (7)

where $\mathcal{F}$ is a bilinear functional operator which acts on all the fields of the theory, here denoted as $\{\phi_k\}$

$$\mathcal{F} = e^{-\frac{i}{2} \theta_{\mu \nu} \int dz \partial_\mu \phi_k(z) \frac{\delta}{\delta \phi_k(x)} \otimes \int dy \partial_\nu \phi_l(y) \frac{\delta}{\delta \phi_l(y)}}$$  \hspace{1cm} (8)

and $\phi, \psi \in \mathcal{A}_\theta$.

This restricted Moyal product does not act on functions like the parameters of the symmetry transformations, i.e., if $f$ is a function not related to the fields of the theory, then $\mu_*(f \otimes \Psi) = \mu(f \otimes \Psi) = f \Psi$.

According to the twisted noncommutative theories, the Leibniz rule is written in terms of a twisted co-product of (9)

$$\delta^*_\alpha (\phi \ast \psi) = \mu_* [\Delta \mathcal{F}(S_\alpha)(\phi \otimes \psi)]$$

$$= (\delta^*_\alpha \phi) \ast \psi + \phi \ast (\delta^*_\alpha \psi),$$  \hspace{1cm} (9)

where

$$\Delta_\theta(S_\alpha) \equiv \Delta \mathcal{F}(S_\alpha) = \mathcal{F}^{-1} \Delta(S_\alpha) \mathcal{F}$$  \hspace{1cm} (10)

and

$$\Delta_\theta(S_\alpha) \equiv \Delta \mathcal{F}(S_\alpha) = \mathcal{F}^{-1} \Delta(S_\alpha) \mathcal{F}$$  \hspace{1cm} (10)
is the Drinfeld’s twisted co-product arising in the definition of quasi-triangular Hopf algebras \cite{37,38} and $\Delta(S_\alpha) = S_\alpha \otimes 1 + 1 \otimes S_\alpha$ is the commutative co-product of \cite{5}. It can be shown in a straightforward, although somewhat cumbersome way, that the co-product \cite{9} gives the same result as in \cite{34} (an explicit derivation is worked out in the appendix).

$$\delta^*_\alpha(\phi_r \ast \phi_s) = i\alpha \cdot [(T_l \phi)_r \ast \phi_s + \phi_r \ast (T_l \phi)_s].$$  \hspace{1cm} (11)

The right hand side looks quite similar to the usual Leibniz rule, but is radically different because the Moyal product does not act on the transformation parameters, as they multiply the rest of the expression to their right with the ordinary commutative multiplication. The reason of why to use such a complicated expression like \cite{9}, is that it is part of a Hopf algebra \cite{29} which ensures its consistency, e.g. the associativity.

**Twisted Gauge Fields**

Things work quite similar for gauge fields $A^\mu_\alpha$. In this case the action of the transformations is given by \cite{34}

$$S^A_\alpha = i \int dz \left[ \partial_\mu \alpha^l(z) - \alpha^r(z) f_{rs}^l A^*_\mu(z) \right] \frac{\delta}{\delta A^*_\mu(z)}.$$ \hspace{1cm} (12)

For example, the transformation rule of the product $A_\mu \ast \phi$ is given by the expression \cite{9}

$$\delta^*_\alpha(A_\mu \ast \phi) = \mu_*[\Delta_f(S_\alpha)(A_\mu \otimes \phi)] = \mu_*[\mathcal{F}^{-1}\Delta(S_\alpha)\mathcal{F}(A_\mu \otimes \phi)] = \partial_\mu \alpha \cdot \phi + i\alpha \cdot (A_\mu \ast \phi),$$ \hspace{1cm} (13)

where it has been taken into account that the fields are in different representations of the gauge group, $\Delta(S_\alpha) = S^A_\alpha \otimes 1 + 1 \otimes S^\phi_\alpha$. Such expressions containing different fields can be handled considering, as in the case of the functional operator $\mathcal{F}$, that the transformation $S_\alpha$ must contain a sum over all the fields of the theory. Thus, the covariant derivative

$$D^*_\mu \phi = \partial_\mu \phi - iA_\mu \ast \phi,$$ \hspace{1cm} (14)

fulfils $\delta^*_\alpha D^*_\mu \phi = i\alpha \cdot D^*_\mu \phi$. The field strength is obtained as usual

$$\left(D^*_\mu \ast D^*_\nu - D^*_\nu \ast D^*_\mu\right) \phi = -i \left( \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \ast A_\nu] \right) \ast \phi = -iF^*_{\mu\nu} \ast \phi.$$ \hspace{1cm} (15)

In order to get its transformation rule, we need to compute the transformation of $A_\mu \ast A_\nu$, which turns out to be

$$\delta^*_\alpha(A_\mu \ast A_\nu) = \mu_*[\mathcal{F}^{-1}\Delta(S_\alpha)\mathcal{F}(A_\mu \otimes A_\nu)] = \partial_\mu \alpha \cdot A_\nu + \partial_\nu \alpha \cdot A_\mu + i\alpha^l \cdot [T_l, A_\mu \ast A_\nu].$$ \hspace{1cm} (16)
Thus the field strength transforms as usual, $\delta^* F^*_{\mu \nu} = i\alpha^l \cdot [T_l, F^*_{\mu \nu}]$. In order to ensure that the covariant derivative has the correct properties, it must have a co-product. Let us write the covariant derivative as

\[
D^*_\mu = \int dz \left[ \partial_\mu \phi_l(z) - i A_{\mu l}^k(z) \ast \phi_k(z) \right] \frac{\delta}{\delta \phi_l(z)}.
\]

Then we have, after straightforward but laborious computations

\[
D^*_\alpha(\phi_r \ast \phi_s) = \mu_\star [\mathcal{F}^{-1} \Delta(D^*_\mu) \mathcal{F}(\phi_r \otimes \phi_s)] = (D^*_\mu \phi)_r \ast \phi_s + \phi_r \ast (D^*_\mu \phi)_s,
\]

where $\Delta(D^*_\mu) = D^*_\mu \otimes 1 + 1 \otimes D^*_\mu$. For the adjoint representation we have similar rules.

**Twisted Diffeomorphisms**

For diffeomorphisms it is not obvious that if we use Drinfeld’s twisted coproduct, we will get similar results. Let us consider for instance a covariant vector field $U_\mu$

\[
\delta_\xi U_\mu = -\xi^\nu \partial_\nu U_\mu - \partial_\mu \xi^\nu U_\nu.
\]

As far as the second term in the r.h.s. is a matrix transformation, the considerations of the preceding section can be applied to it. Thus, in order to see if the previous formulation can be applied here, it is enough to consider scalar fields $\phi$, transforming as: $\delta_\xi \phi = -\xi^\nu \partial_\nu \phi = S^\phi_\xi \phi$. Here we define

\[
S^\phi_\xi = -\int dz \xi^\mu(z) \partial_\mu \phi(z) \frac{\delta}{\delta \phi(z)}.
\]

Thus, if we apply (9), after computations (the detailed calculation in the appendix A) we get

\[
\delta^*_\xi(\phi \ast \psi) = \mu_\star [\mathcal{F}(S^\phi_\xi)(\phi \otimes \psi)] = -\xi^\mu \cdot (\partial_\mu \phi \ast \psi + \phi \ast \partial_\mu \psi).
\]

Hence, for a covariant vector field we have

\[
S^U_\xi = -\int dz [\xi^\nu(z) \partial_\nu U_\rho(z) + \partial_\rho \xi^\nu(z) U_\nu(z)] \frac{\delta}{\delta U_\rho(z)}.
\]

Similarly, the procedure can be also carried over for a contravariant vector field $V^\mu$. Therefore, we can compute the transformations of mixed products like

\[
\delta^*_\xi(\phi \ast U_\mu) = \mu_\star [\mathcal{F}(S^\phi_\xi)(\phi \otimes U_\mu)] = \mu_\star [\mathcal{F}^{-1}(S^\phi_\xi \otimes 1 + 1 \otimes S^U_\xi) \mathcal{F}(\phi \otimes U_\mu)]
\]

\[
= -\xi^\nu \cdot (\partial_\nu \phi \ast U_\mu + \phi \ast \partial_\nu U_\mu) - \partial_\mu \xi^\nu \cdot (\phi \ast U_\nu).
\]
or
\[
\delta_\xi^*(V^\mu \star U^\nu) = \mu_*[\Delta_\mathcal{F}(S_\xi)(V^\mu \otimes U^\nu)] = \mu_*[\mathcal{F}^{-1}(S_\xi^V \otimes 1 + 1 \otimes S_\xi^U)\mathcal{F}(V^\mu \otimes U^\nu)]
\]
\[
= -\xi^\rho \cdot (\partial_\rho V^\mu \star U^\nu + V^\mu \star \partial_\rho U^\nu) + \partial_\mu \xi^\mu \cdot (V^\rho \star U^\nu) - \partial_\nu \xi^\rho \cdot (V^\mu \star U^\rho). \tag{24}
\]
Thus the Moyal product of tensor quantities lead to higher order tensors as usual, and the contraction of indices of tensor quantities lead to lower order tensors, for instance
\[
\delta_\xi^*(V^\mu \star U^\mu) = -\xi^\nu \cdot \partial_\nu (V^\mu \star U^\mu). \tag{25}
\]

**Twisted Differential Forms**

The above properties allow us to define in the usual way differential 1-forms \(U = U_\mu dx^\mu\), which can be extended to higher order differential forms if the differentials \(dx\) behave as constants under the Moyal product, i.e. for the product of two 1-differential forms \(U\) and \(V\) we have
\[
U \wedge V = U_\mu dx^\mu \wedge V_\nu dx^\nu = (U_\mu \star V_\nu)dx^\mu \wedge dx^\nu. \tag{26}
\]
The essential point is the noncommutative exterior derivative which is defined as commutation of the following diagram:

\[
f \in \mathcal{A} \xrightarrow{\mathcal{W}} f \in \mathcal{A}_\theta
\]

\[
d \downarrow \quad \downarrow d^*
\]

\[
(df) \in \Lambda^1(\mathcal{A}) \xrightarrow{\mathcal{W}} (d^* \hat{\circ} f) \in \Lambda^1(\mathcal{A}_\theta)
\]

\[
(d^* \hat{\circ} f) = (\partial_\mu \hat{\circ} f)dx^\mu = (\partial_\mu f)dx^\mu = (df). \tag{27}
\]
Here \(\Lambda^1(\mathcal{A}_\theta)\) is a left (or right) module over \(\mathcal{A}_\theta\), i.e. an \(\mathcal{A}_\theta\)-module. In the diagram \(\mathcal{W}\) is the map given by the Weyl-Wigner-Moyal correspondence \([1]\), which is an isomorphism. With this definition it easy to show that
\[
d^* \hat{\circ} d^* \hat{\circ} f = 0, \tag{28}
\]
for any \(f\). Thus \(d^* \hat{\circ} d^* = 0\). In this way, the differential of a differential form gives, as usual, higher order differential forms
\[
dU = \partial_\mu U_\nu dx^\nu \wedge dx^\mu. \tag{29}
\]
For $p$-forms

$$U_p \in \Lambda^p(A) \xrightarrow{\nabla} U_p \in \Lambda^p(A_\theta)$$

$$d_p \downarrow d_p^*$$

$$(d_p U_p) \in \Lambda^{p+1}(A) \xrightarrow{\nabla} (d_p^* U_p) \in \Lambda^{p+1}(A_\theta),$$

such that the diagram commutes, i.e.,

$$(d_p^* U_p) = (d_p U_p). \quad (30)$$

Here $\Lambda^p(A_\theta)$ is also a $A_\theta$-module.

In the general case for the wedge product of a $p$-form $U_p$ by a $q$-form $V_q$, the usual graded Leibniz rule is satisfied

$$d^* \wedge (U_p \wedge V_q) = (d^* \wedge U_p) \wedge V_q + (-1)^p U_p \wedge (d^* \wedge V_q). \quad (31)$$

Under this scheme, in which transformations $S_\xi$ act only on the fields, differential forms will not transform under diffeomorphisms as scalar fields. Usually, the transformation of the field is compensated by the coordinate transformation. In the present case, considering for instance a one-form $U$, we have $\delta_\xi U = dx^\mu S_\xi U_\mu$, where $S_\xi U_\mu$ is given by (22). However the interesting feature is that, despite of this undesirable property, four-forms continue to transform (in four dimensions) as invariant densities. Indeed, let us consider $U = dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma U_{\mu\nu\rho\sigma} = dV \varepsilon^{\mu\nu\rho\sigma} U_{\mu\nu\rho\sigma}$, where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in four dimensions. Then we have $\delta_\xi U = dV \varepsilon^{\mu\nu\rho\sigma} S_\xi U_{\mu\nu\rho\sigma}$, that is

$$\delta_\xi U = dV \varepsilon^{\mu\nu\rho\sigma} \left( -\xi^\lambda \partial_\lambda U_{\mu\nu\rho\sigma} - \partial_\mu \xi^\lambda U_{\lambda\nu\rho\sigma} - \partial_\nu \xi^\lambda U_{\mu\lambda\rho\sigma} - \partial_\rho \xi^\lambda U_{\mu\nu\lambda\sigma} - \partial_\sigma \xi^\lambda U_{\mu\nu\rho\lambda} \right)$$

$$= dV \varepsilon^{\mu\nu\rho\sigma} \left( -\xi^\lambda \partial_\lambda U_{\mu\nu\rho\sigma} - \partial_\lambda \xi^\lambda U_{\mu\nu\rho\sigma} \right) = -\partial_\lambda \left( \xi^\lambda U \right), \quad (32)$$

where we used the identity: $\varepsilon^{\mu\nu\rho\sigma} \xi^\lambda +$ cyclic permutations of $\{\mu\nu\rho\sigma\lambda\} \equiv 0$. This result can be understood from the invariance of the action under the transformations of the fields, as can be seen from

$$\delta A = \int d^4x' L[\phi'(x'), \partial'_\mu \phi'(x')] - \int d^4x L[\phi(x), \partial_\mu \phi(x)]$$

$$= \int d^4x L[\phi'(x), \partial_\mu \phi'(x)] - \int d^4x L[\phi(x), \partial_\mu \phi(x)] = 0. \quad (33)$$
In this way we can construct noncommutative invariant actions by means of four forms. For instance, if we consider the product of four one-form tetrad, we get

$$e^a \wedge e^b \wedge e^c \wedge e^d = dV \varepsilon_{\rho\sigma} e^a_\rho \star e^b_\sigma \star e^c_\rho \star e^d_\sigma,$$

which is not antisymmetric in the indices $a, b, c, d$ and consequently does not give the determinant of the tetrad. However it is still an invariant density, and we must take care only about Lorentz invariance. For example, by contracting this quantity with a suitable four tensor, we get an expression invariant under twisted Lorentz plus diffeomorphisms transformations, as follows

$$(\delta \Lambda + \delta \xi)(e^a \wedge e^b \wedge e^c \wedge e^d \wedge V_{abcd}) = -\partial \lambda \cdot (e^a \wedge e^b \wedge e^c \wedge e^d \wedge V_{abcd}).$$

In this paper we consider the Plebanski’s action of gravity [35], which is a $BF$ theory (with constraints) with the Lagrangian given by a four form. It will be twisted in the same sense as a gauge theory and will be covariant under twisted Lorentz and diffeomorphism transformations.

### III. BRIEF OVERVIEW OF THE SELF-DUAL FORMULATION OF GRAVITY

In this section we overview the self-dual formulation of gravity in four dimensions. We will follow Plebański paper [35]. Let start by considering a $SO(3)$ complex connection one-form $\Omega = \Omega_i T^i$ ($i = 1, 2, 3$), with its corresponding field strength $F = d\Omega + \Omega \wedge \Omega$ and the $BF$-action

$$I = -4i \int \text{Tr}(B \wedge F),$$

where $B$ is a Lie algebra valued two-form. Let us now write the fields into their real and imaginary parts, $\Omega = \frac{1}{2}(\omega + i\tilde{\omega})$, $B = \frac{1}{2}(\Sigma + i\tilde{\Sigma})$ and $F = \frac{1}{2}(R + i\tilde{R})$. Now let us define $\omega^{i0} = -\omega^{0i} = -\omega^i$, $\omega^{00} = 0$ and $\omega^{ij} = \varepsilon^{ijk}\tilde{\omega}^k$. Similarly $R^{0i} = -R^{i0} = R^i$, $R^{00} = 0$ and $R^{ij} = \varepsilon^{ijk}\tilde{R}^k$. In this case, putting things toghether, we get the $SO(3, 1)$ field strength,

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c^b \quad (a, b = 0, 1, 2, 3).$$

In general if $v^i$ is a complex $SO(3)$ field, its decomposition into real and imaginary parts can be rewritten as a $SO(3, 1)$

algebra valued self-dual field, $v^i = \frac{1}{2}(v^{0i} - i\varepsilon^{0i} jkv^{0j}) = v^{(+)}0i$, i.e. it fulfills $\varepsilon^{cd} v^{(+)}cd = 2iv^{(+)}ab$. Moreover $u^i v_i = -\frac{1}{4} u^{(+)}ab v^{(+)}ab$. The field strength satisfies $R^{(+)}ab(\omega^{(+)}ab) = R^{ab}(\omega^{(+)}).$
Following Plebański [35], the solution of the constraints for the $B^i$ field is given by 
$\Sigma^{ab} = e^a \wedge e^b$, where the tetrad $e^a$ are real one-forms, which are defined up to a $SO(3,1)$ transformation. Thus, the action (36) can be rewritten as

$$I = i \int \Sigma^{ab} \wedge R_{ab} = \frac{1}{2} \int \left( \frac{1}{2} \varepsilon_{abcd} e^a \wedge e^b \wedge R^{cd} + i e^a \wedge e^b \wedge R_{ab} \right). \quad (37)$$

From the tetrad we obtain the torsion two-form: $T^a = De^a = de^a + \omega^a \wedge e^b$, which satisfies the Bianchi identity $DT^a = dT^a + \omega^a \wedge T^b \equiv R^a_b \wedge e^b$. Therefore, the second term in (37) can be written in terms of the torsion. Furthermore, if we write $R^{ab} = R^{\mu \nu} e_{\mu}^a e_{\nu}^b$, the action (37) can be rewritten as

$$I = - \int \det e R_{\mu \nu}^{\mu \nu}(\omega) + \frac{i}{2} \int e^a \wedge DT_a. \quad (38)$$

The first term is the Palatini action, with the tetrad $e$ and the connection $\omega$ being independent fields. The variation of $\omega$ on this term gives

$$\delta_\omega I = i \delta_\omega \int \Sigma^{ab} \wedge R_{ab} (\omega^{(+)}) = 2i \int e^a \wedge T^b \wedge \delta \omega^{(+)}_{ab} = 0. \quad (39)$$

This is a complex equation, where the coefficient $e^a \wedge T^b$ is real. Therefore, if we set to zero the real and the imaginary parts separately, we get the equation $e^a \wedge T^b - e^b \wedge T^a = 0$, from which turns out that the torsion vanishes, $T_{\mu \nu}^a = 0$, with the well known solution given by the second Cartan structure equation

$$\omega_{\mu \nu \rho} = \frac{1}{2} \left[ e_{\mu a} (\partial_\nu e^a_{\rho} - \partial_\rho e^a_{\nu}) - e_{\nu a} (\partial_\rho e^a_{\mu} - \partial_\mu e^a_{\rho}) - e_{\rho a} (\partial_\mu e^a_{\nu} - \partial_\nu e^a_{\mu}) \right]. \quad (40)$$

If we put it back into the action (38), we get the Einstein-Hilbert action

$$I = \int \det e \cdot R_{\mu \nu}^{\mu \nu} d^4x. \quad (41)$$

IV. TWISTED COVARIANT NONCOMMUTATIVE SELF-DUAL GRAVITY

Let us consider now the noncommutative theory of the action (36). In order to take into account the form of the noncommutative field strength, we must extend the fields to the universal enveloping algebra (UEA) of $su(2)$, given in this case by $u(2)$. The formulation of twisted gauge transformations closes for arbitrary gauge groups [31, 32]. Thus, our proposal would be valid also for the Lie algebra $su(2)$. However, as it was discussed in
Ref. [32], the consistency of the classical equations of motion of noncommutative Yang-
Mills theory requires the use of the associated UEA. In Appendix B we show that these
considerations apply also for BF actions. As we precisely need to work out the equations of
motion consistently (including the torsion) from the noncommutative action corresponding
to (36), we shall use the UEA of \( su(2) \), namely \( u(2) \). Hence the action is now
\[
A = -2i \text{Tr} \left( \Sigma \wedge \widehat{R} = -4i \int \left( \Sigma^i \wedge \widehat{R}_i + \Sigma^4 \wedge \widehat{R}_4 \right) \right),
\]  
where \( \widehat{R} = d\omega + \omega \wedge \omega \) is the noncommutative field strength and \( T^A = \{ \sigma^i, \sigma^4 = 1 \} \), are the

generators of the \( u(2) \) algebra, which satisfy:
\[
[\sigma^i, \sigma^j] = 2i \varepsilon^{ij}_k T^k, \quad [\sigma^i, \sigma^4] = 0, \quad \{ \sigma^i, \sigma^j \} = 2 \delta^{ij} \sigma^4,
\]
\[
\{ \sigma^i, \sigma^4 \} = 2 \sigma^i, \quad \{ \sigma^4, \sigma^4 \} = 2 \sigma^4, \quad \text{Tr}(\sigma^A \sigma^B) = 2 \delta^{AB},
\]  
where \( A, B = 1, 2, 3, 4 \). Following Sec. II, in particular formula (35), one can see that the
action (42) is invariant under twisted gauge and diffeomorphism transformations.

The field strength is given by
\[
\widehat{R} = \widehat{R}^i T_i + \widehat{R}^4 T_4,
\]
where
\[
\widehat{R}^i = d\omega^i + i \varepsilon^i_{jk} \omega^j \wedge \omega^k + \omega^i \wedge \omega^4 + \omega^4 \wedge \omega^i,
\]
\[
\widehat{R}^4 = d\omega^4 + \omega^i \wedge \omega_i + \omega^4 \wedge \omega^4.
\]
The first term (zero-th order) in the \( \theta \)-expansion of the curvatures coincide with the com-
mutative ones.

In terms of \( SO(1,3) \) self-dual fields, by means of the relations shown in the preceding
section, the action (42) is given by
\[
A = i \int \left[ \Sigma^{(+)ab} \wedge \left( d\omega_{ab}^{(+)} - \omega_a^{(+)}c \wedge \omega_{cb}^{(+)} + \omega_{ab}^{(+)} \wedge \omega^4 + \omega^4 \wedge \omega_{ab}^{(+)} \right) \right. \\
-4 \Sigma^4 \wedge \left( d\omega^4 - \frac{1}{4} \omega_{ab}^{(+)} \wedge \omega_{ab}^{(+)} + \omega^a \wedge \omega^4 \right) \right],
\]  
where \( \Sigma^{ab} \) and \( \omega^{ab} \), which arise from \( \Sigma^i \) and \( \omega^i \), are real and antisymmetric by construction, as in the commutative case.

This action must be written explicitly in terms of the tetrad in order to be compared with
the Einstein-Hilbert action. In the commutative case, Plebański has formulated constraints
on \( \Sigma^i \), whose solution is given by \( \Sigma^{ab} = e^a \wedge e^b \). After substitution of this solution into the
commutative action, the Palatini action turns out. Here we have a noncommutative action,
with an explicit dependence on the noncommutativity parameter $\theta$. Thus the solution of the equations of motion will be given by generic fields $\phi$'s depending on $\theta$. This dependence can be made explicit by a series expansion

$$\phi(\theta) = \sum_{n=0}^{\infty} \theta^{a_1 b_1} \cdots \theta^{a_n b_n} \phi^{(n)}_{a_1 b_1 \cdots a_n b_n}. \quad (47)$$

Here we will make an ansatz on the form of the dependence of $\Sigma^{ab}$ on $\theta$, given in terms of the tetrad by

$$\Sigma^{ab} = \frac{1}{2} (e^a \hat{\wedge} e^b - e^b \hat{\wedge} e^a). \quad (48)$$

It is easy to see that the power series dependence on $\theta$ of this expression has only even powers. The next step is the variation of the action $A$ with respect to the connection $\omega_{ab}^{(+)}$, which gives us

$$\delta \omega_{ab}^{(+)} A = i \int \left\{ -d\Sigma^{ab} + [\omega \hat{\wedge} \Sigma]^{ab} - [\omega^4 \hat{\wedge} \Sigma^{0b}] - [\omega^{ab} \hat{\wedge} \Sigma^4] \right\} \wedge \delta \omega_{ab}^{(+)} = 0, \quad (49)$$

where $[\omega \hat{\wedge} \Sigma]^{ab} = [\omega^{ac} \hat{\wedge} \Sigma^d] = (\omega \hat{\wedge} \Sigma)^{ab} - (\Sigma \hat{\wedge} \omega)^{ab}$. In order to see which are the equations of motion arising from this variation, we take into account that we are dealing with complex quantities. Let us consider generic equations of the form $E^{ab} \wedge \delta \omega_{ab}^{(+)} = E^{ab(+) \wedge \delta \omega_{ab}^{(+)}} = 0$. If $E^{ab}$ are real, the real and imaginary parts of the equations give $E^{ab} = 0$. If $E^{ab}$ are complex, by the properties of the self-dual projector we can see that it is enough to set to zero their real, or their imaginary parts. Indeed, taking into account that $E^{ab(+) \wedge \delta \omega_{ab}^{(+)}} = E^{ab(-)}$, where $\overline{E}$ stands for the complex conjugated of $E$, we get for the real or the imaginary parts the same result

$$\left( E^{ab(+)} \pm E^{ab(-)} \right) \wedge \delta \omega_{ab}^{(+)} = E^{ab(+)} \wedge \delta \omega_{ab}^{(+)} = E^{ab} \wedge \delta \omega_{ab}^{(+)}. \quad (50)$$

Hence if the real part of $E$ vanishes, then the imaginary one vanishes as well. Further, if $f$ and $g$ are real functions, then $\overline{f \ast g} = g \ast f$. Thus, from the real part of the coefficient of $\delta \omega_{ab}^{(+)}$ in (49), we obtain the equations of motion

$$2d\Sigma^{ab} - [\omega \hat{\wedge} \Sigma]^{ab} + [\omega \hat{\wedge} \Sigma]^{ba} - 2i \left\{ [\eta_2 \hat{\wedge} \Sigma^{ab}] - [\omega^{ab} \hat{\wedge} \lambda_2] - \frac{1}{2} \epsilon_{cd}^{ab} \left( [\eta_1 \hat{\wedge} \Sigma^{cd}] + [\omega^{cd} \hat{\wedge} \lambda_1] \right) \right\} = 0, \quad (51)$$

where we set $\omega^4 = \eta_1 + i\eta_2$ and $\Sigma^4 = \lambda_1 + i\lambda_2$.

Considering the expansion in powers of $\theta$ of the fields (47) and of the Moyal product, expanding order by order we get for the zero-th order

$$d\Sigma^{(0)ab} - \omega^{(0)ac} \wedge \Sigma^{(0)b} + \omega^{(0)bc} \wedge \Sigma^{(0)c} = 0, \quad (52)$$
which is the second Cartan’s structure equation for \( \omega^{(0)} \), with the solution given by (40). To first order in the \( \theta \)-expansion we have that (51) yields

\[
\theta^{\alpha\beta}\left\{ -\left[\omega^{(1)}_{\alpha\beta}, \Sigma^{(0)}\right]^{ab} + \frac{1}{2} \left[ \partial_{\alpha} \eta^{(0)}_{2} \wedge \partial_{\beta} \Sigma^{(0)ab} + \partial_{\alpha} \omega^{(1)ab} \wedge \partial_{\beta} \lambda^{(0)}_{2} \right] \\
+ \frac{1}{2} \varepsilon^{ab}_{\text{cd}} \left( \partial_{\alpha} \eta^{(0)}_{1} \wedge \partial_{\beta} \Sigma^{(0)cd} + \partial_{\alpha} \omega^{(1)cd} \wedge \partial_{\beta} \lambda^{(0)}_{1} \right) \right\} = 0,
\]

(53)

where \( \Sigma^{(0)ab} = \varepsilon^{a} \wedge \varepsilon^{b} \) and \( \omega^{(1)ab} \) is given by (40).

Solving for \( \omega^{(1)} \), we have after some computations

\[
e^{d} \left( \varepsilon^{abcd} \omega^{(1)}_{\alpha\beta,dc} - \varepsilon^{abce} \omega^{(1)}_{\alpha\beta,de} \right) e_{e} = \frac{1}{2} e^{\mu\nu\rho\sigma} \left[ \partial_{\alpha} \eta^{(0)}_{2\mu} \partial_{\beta} \Sigma^{(0)ab}_{\nu\rho} + \partial_{\alpha} \omega^{(1)ab}_{\mu} \partial_{\beta} \lambda^{(0)}_{2\nu\rho} \right] \\
+ \frac{1}{2} \varepsilon^{ab}_{\text{cd}} \left( \partial_{\alpha} \eta^{(0)}_{1\mu} \partial_{\beta} \Sigma^{(0)cd}_{\nu\rho} + \partial_{\alpha} \omega^{(1)cd}_{\mu} \partial_{\beta} \lambda^{(0)}_{1\nu\rho} \right) - (\alpha \leftrightarrow \beta) \right] \equiv M_{\alpha\beta}^{\sigma\lambda} (\Phi^{(0)}),
\]

(54)

where \( \Phi^{(0)} \) are real combinations of the tetrad \( e^{a} \) and the fields \( \eta^{(k)}_{1}, \eta^{(k)}_{2}, \lambda^{(k)}_{1} \) and \( \lambda^{(k)}_{2} \) for \( k < n \). This equation can be rewritten as

\[
\omega^{(1)}_{\alpha\beta,ab} - \omega^{(1)}_{\alpha\beta,ba} = \omega^{(1)}_{\alpha\beta,da} \delta^{c}_{b} - \omega^{(1)}_{\alpha\beta,db} \delta^{c}_{a} = \varepsilon^{abde} e^{-1} M_{\alpha\beta}^{\sigma\lambda} (\Phi^{(0)}),
\]

(55)

from which we get

\[
\omega^{(1)}_{\alpha\beta,ab} - \omega^{(1)}_{\alpha\beta,ba} = \frac{1}{2} (\varepsilon^{abde} \delta^{c}_{f} + 2 \varepsilon^{abdf} \delta^{c}_{e}) M_{\alpha\beta}^{f\lambda} = M_{\alpha\beta,ab}^{c} (\Phi^{(0)}),
\]

(56)

and then

\[
\omega^{(1)}_{\alpha\beta,abc} = \frac{1}{2} \left( M_{\alpha\beta,abc}^{(1)} - M_{\alpha\beta,bca}^{(1)} + M_{\alpha\beta,cab}^{(1)} \right).
\]

(57)

Thus \( \omega^{(1)}_{\alpha\beta,abc} \) is determined by the tetrad, \( \eta^{(0)}_{1}, \eta^{(0)}_{2}, \lambda^{(0)}_{1} \) and \( \lambda^{(0)}_{2} \). Furthermore, to the \( n \)-th order we get from (51)

\[
2d \Sigma^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n}} ab = 2 \left[ \omega^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n}}, \Sigma^{(0)} \right]^{ab} + M_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n}}^{ab} (\Phi^{(0)}) = 0,
\]

(58)

where \( \Sigma^{(n)} \) vanishes if \( n \) is odd and otherwise depends on the tetrad by the ansatz (48). Moreover, by a similar computation as for the zero-th and first order cases we get

\[
\omega^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n},ab} c - \omega^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n},ba} c = M^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n},ab} c (\Phi^{(n)}),
\]

(59)

from which the \( n \)-th correction to the spin connection is given by

\[
\omega^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n},abc} = \frac{1}{2} \left( M^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n},abc} - M^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n},bca} + M^{(n)}_{\alpha_{1}\beta_{1}...\alpha_{n}\beta_{n},cab} \right).
\]

(60)
Therefore, if we substitute the spin connection obtained to all orders from these equations into the action (46), we get a noncommutative action for Einstein gravity, which depends to all orders on $e_a^\alpha$, $\Sigma^4$ and $\omega^4$, in such a way that to zero-th order it coincides with the Einstein-Hilbert action. To first order action (46) is given by

$$ A = i \int \left[ \Sigma^{(0)ab(+)} \wedge R^{(0)(+)}_{ab} - 4 \Sigma^{(0)4} \wedge d\omega^{(0)4} \right] + i \theta^{\alpha\beta} \int \left[ - 4 \Sigma^{(1)4}_{\alpha\beta} \wedge d\omega^{(0)4} - 4 \Sigma^{(0)4} \wedge d\omega^{(1)4}_{\alpha\beta} \right. $$

$$ + \Sigma^{(0)ab(+)} \wedge \left( d\omega^{(1)ab}_{\alpha\beta,c} - 2 \omega^{(0)c(+)}_a \wedge \omega^{(1)ab}_{\alpha\beta,c} - \omega^{(0)4} \wedge \omega^{(1)ab}_{\alpha\beta,ab} + \omega^{(0)ab}_{ab} \wedge \omega^{(1)4}_{\alpha\beta} \right) \right] $$

$$ - \frac{1}{2} \theta^{\alpha\beta} \int \left[ \Sigma^{(0)ab(+)} \wedge \left( - \partial_\alpha \omega^{(0)ab(+)}_a \wedge \partial_\beta \omega^{(0)4}_{ab} + 2 \partial_\alpha \omega^{(0)ab(+)}_a \wedge \partial_\beta \omega^{(0)4}_{ab} \right) \right] + O(\theta^2), $$

where $\Sigma^{(0)ab} = e_a^\alpha \wedge e_b^\beta$ and $\omega^{(0)}_a$ and $\omega^{(1)}_{\alpha\beta,ab}$ are given by (40) resp. (57).

It is worth to note that, simultaneously to the variation of $\omega^{(+)ab}$ we can vary with respect to $\omega^4$ and $\Sigma^4$, with the resulting equations of motion

$$ \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \Sigma^{(0)4}_{\nu\rho} = \varepsilon^{\mu\nu\rho\sigma} \left[ \Sigma^{(0)4}_{\mu\nu} \wedge \omega^4_{\rho} \wedge \Sigma^{(0)4}_{\nu\rho} - \frac{1}{4} \left( \omega^{(+)ab}_\mu \Sigma^{(0)ab(+)}_{\nu\rho} - \Sigma^{(0)ab(+)}_\nu \wedge \omega^{(+)ab}_{\mu} \right) \right] $$

(62)

and

$$ \partial_\mu \omega^4_\nu - \partial_\nu \omega^4_\mu = \frac{1}{4} \left( \omega^{(0)ab(+)}_\mu \wedge \omega^{(+)ab}_\nu - \omega^{(0)ab(+)}_\nu \wedge \omega^{(+)ab}_{\mu} \right) + \omega_\mu^4 \wedge \omega_\nu^4 - \omega_\nu^4 \wedge \omega_\mu^4. $$

(63)

To zero-th order we have the equations $\varepsilon^{\mu\nu\rho\sigma} \partial_\nu \Sigma^{(0)4}_{\mu\rho} = 0$ and $\partial_\mu \omega^4_\nu = \partial_\nu \omega^4_\mu = 0$, which have the solutions

$$ \Sigma^{(0)4}_{\mu\nu} = \partial_\mu S_\nu \wedge \partial_\nu S_\mu, \quad \omega^{(0)4}_\mu = \partial_\mu \phi. $$

(64)

To higher orders the equations are of the form

$$ \varepsilon^{\mu\nu\rho\sigma} \partial_\nu \Sigma^{(n)4}_{\rho\sigma,\alpha_1...\alpha_n\beta_1...\beta_n} = \text{function of } \Sigma^{(k)4}, \omega^{(k)4}_{ab}, \Sigma^{(k)}_\alpha \text{ for } (k < n), $$

(65)

$$ \partial_\mu \omega^{(n)4}_{\nu,\alpha_1...\alpha_n\beta_1...\beta_n} - \partial_\nu \omega^{(n)4}_{\mu,\alpha_1...\alpha_n\beta_1...\beta_n} = \text{function of } \omega^{(k)4}_\mu, \omega^{(k)4}_{ab} \text{ for } (k < n). $$

Thus these equations, together with the equations of $\omega^{ab}_\mu$, could be solved recursively.

V. FINAL REMARKS

In the present paper we pursue the idea of the implementation of the twisted symmetries to describe a non-commutative theory of gravity. We applied the prescription based in the
twisted gauge transformations to construct a noncommutative gauge theory of gravitation. In particular, we study noncommutative Plebański’s self-dual gravity. As well known it is a topological constrained $SL(2, \mathbb{C})$ BF theory \cite{35}. This is addressed by extending the fields to the universal enveloping algebra of $su(2)$, given by $u(2)$. This action is constructed to be invariant under twisted Lorentz and twisted diffeomorphism transformations. The constraints are implemented at the noncommutative level by the ansatz (48). This ansatz allows to solve the resulting torsion constraint to every order in the expansion of the noncommutative parameter $\theta$ (see Eq. (60)). It is shown that at any order, the solution is described in terms of the tetrad and the extra fields corresponding to the fourth components of the connection $\omega$ and of the $B$-field two-form $\Sigma$, due to the enveloping algebra. Furthermore, the noncommutative BF action is explicitly obtained to first order in $\theta$ (61). It is important to remark that, although the $BF$ theory is invariant under twisted diffeomorphisms, the invariance of the resulting noncommutative gravity theory is realized not directly through metric variables as it was described at [28], but by means of $\Sigma$ and $\omega$, through the prescription given in Refs. [31, 32] for gauge theories. Then twisted diffeomorphisms are encoded in the twisted gauge symmetry.

Finally, it is worth to mention that this procedure can be carried over to define the classical topological invariants arising in topological gravity \cite{19}, in a way invariant also under twisted diffeomorphisms. This issue was not enough clear in that paper \cite{19} and with these methods it can be clarified. Some of results on this subject will be reported elsewhere.

**Acknowledgments**

This work was supported in part by CONACyT México Grants 45713-F and 51306 and also by projects PROMEP-UGTO-CA3 and VIEP-08/EXC/07. The work of S.E.-J. was supported by a CONACyT postdoctoral fellowship.
APPENDIX A

Let us consider a linear operator $O$ acting on a set of generic fields $\phi_i$. This operator acts locally as a matrix as well as linearly in the derivatives of the field, and is defined by

$$S^\phi_O \phi = \int dz [O^\phi(z) \phi(z)]_i \frac{\delta}{\delta \phi_i(z)}, \quad (A1)$$

where

$$[O^\phi(x) \phi(x)]_i = O^{(1)}_{ij}(x) \phi_j(x) + O^{(2)}_{ij}(x) \partial_\mu \phi_j(x) = O^A(x) T^\phi_A \phi(x). \quad (A2)$$

Here the operators $T^\phi_A$ are constant and contain the matrix and the differential actions on the fields $\phi_i$.

The coproduct of this operator is given by

$$\Delta(S_O)(\phi \otimes \psi) = (S^\phi_O \otimes 1 + 1 \otimes S^\psi_O)(\phi \otimes \psi). \quad (A3)$$

Let us now define the noncommutative coproduct as

$$\delta^*(\phi \star \psi) = \mu_* [\Delta_\theta(\phi \otimes \psi)], \quad (A4)$$

where $\Delta_\theta \equiv \Delta_\mathcal{F} = \mathcal{F}^{-1} \Delta(S^\phi_O) \mathcal{F}$ with $\mathcal{F}$ given by Eq. (8).

In order to compute it, we must expand the exponentials. The action of $\mathcal{F}$ gives the Moyal product on the fields $\phi$ and $\psi$, hence

$$\delta^*(\phi_i \star \psi_k) = \sum_n \frac{1}{n!} \left( -\frac{i}{2} \right)^n \theta^{\mu_1 \nu_1} \ldots \theta^{\mu_n \nu_n}$$

$$\times \mu_* \left\{ \mathcal{F}^{-1} (S^\phi_O \otimes 1 + 1 \otimes S^\psi_O) [\partial_{\mu_1} \ldots \partial_{\mu_n} \phi_i(x) \otimes \partial_{\nu_1} \ldots \partial_{\nu_n} \psi_k(x)] \right\}. \quad (A5)$$

The action of $S_O$ on the derivatives of the fields can be computed as follows

$$S^\phi_O [\partial_{\mu_1} \ldots \partial_{\mu_n} \phi_i(x)] = \int dz [O^\phi(z) \phi(z)]_i \frac{\delta}{\delta \phi_j(z)} \partial_{\mu_1} \ldots \partial_{\mu_n} \phi_i(x)$$

$$= \partial_{\mu_1} \ldots \partial_{\mu_n} [O^\phi(x) \phi(x)]_i. \quad (A6)$$

Consequently, we have

$$\delta^*(\phi_i \star \psi_k) = \sum_n \frac{1}{n!} \left( -\frac{i}{2} \right)^n \theta^{\mu_1 \nu_1} \ldots \theta^{\mu_n \nu_n}$$

$$\times \mu_* \left\{ \mathcal{F}^{-1} (\partial_{\mu_1} \ldots \partial_{\mu_n} [O^\phi(x) \phi(x)]_i \otimes \partial_{\nu_1} \ldots \partial_{\nu_n} \psi_k(x) + \partial_{\mu_1} \ldots \partial_{\mu_n} \phi_i(x) \otimes \partial_{\nu_1} \ldots \partial_{\nu_n} [O^\psi(x) \psi(x)]_k) \right\}. \quad (A7)$$
The action of $\mathcal{F}^{-1}$ on the first term on the r.h.s. of (A7) can be written as follows

$$\sum_m \frac{1}{m!} \left( \frac{i}{2} \right)^m \theta^{\rho_1 \sigma_1} \cdots \theta^{\rho_m \sigma_m} \times \left\{ \int dz_1 \partial_{\rho_1} \phi_{j_1}(z_1) \frac{\delta}{\delta \phi_{j_1}(z_1)} \cdots \int dz_m \partial_{\rho_m} \phi_{j_m}(z_m) \frac{\delta}{\delta \phi_{j_m}(z_m)} \partial_{\mu_1}^\nu \cdots \partial_{\mu_n}^\nu [\mathcal{O}^\phi(x) \phi(x)]_i \right\}$$

and

$$\otimes \left\{ \int dy_1 \partial_{\sigma_1} \psi_{l_1}(y_1) \frac{\delta}{\delta \psi_{l_1}(y_1)} \cdots \int dy_m \partial_{\sigma_m} \psi_{l_m}(y_m) \frac{\delta}{\delta \psi_{l_m}(y_m)} \partial_{\nu_1}^\nu \cdots \partial_{\nu_n}^\nu \psi_k(x) \right\}.$$  (A8)

Furthermore, taking into account the definition (A2), the terms inside the biggest bracket in the preceding expression can be written as

$$\partial_{\mu_1}^\nu \cdots \partial_{\nu_n}^\nu \int dz_m \partial_{\rho_m} \phi_{j_m}(z_m) \frac{\delta}{\delta \phi_{j_m}(z_m)} [\mathcal{T}_A \phi(x)]_i = [\mathcal{T}_A \partial_{\rho_m} \phi(x)]_i = \partial_{\rho_m} [\mathcal{T}_A \phi(x)]_i,$$  (A10)

because $\mathcal{T}_A$ commutes with the derivatives. Therefore we have from (A10) the following

$$\sum_m \frac{1}{m!} \left( \frac{i}{2} \right)^m \theta^{\rho_1 \sigma_1} \cdots \theta^{\rho_m \sigma_m} \times \left\{ \partial_{\mu_1} \cdots \partial_{\mu_n} \{ \mathcal{O}^\phi(x) \partial_{\rho_1} \cdots \partial_{\rho_m} [\mathcal{T}_A \phi(x)]_i \} \otimes \partial_{\nu_1} \cdots \partial_{\nu_n} \partial_{\sigma_1} \cdots \partial_{\sigma_m} \psi_k(x) \ight.$$  

$$+ \partial_{\mu_1} \cdots \partial_{\mu_n} \partial_{\rho_1} \cdots \partial_{\rho_m} \phi(x)_i \otimes \partial_{\nu_1} \cdots \partial_{\nu_n} \{ \mathcal{O}^\phi(x) \partial_{\sigma_1} \cdots \partial_{\sigma_m} [\mathcal{T}_A \psi(x)]_k \}. \right\}$$ (A11)

Inserting this expression back into (A7) and considering that the sum over $n$ gives $\mathcal{F}$, which compensates $\mathcal{F}^{-1}$, we get

$$\delta^*(\phi_i \star \psi_k) = \sum_m \frac{1}{m!} \left( \frac{i}{2} \right)^m \theta^{\rho_1 \sigma_1} \cdots \theta^{\rho_m \sigma_m} \times \left\{ \mathcal{O}^\phi(x) \partial_{\rho_1} \cdots \partial_{\rho_m} [\mathcal{T}_A \phi(x)]_i \otimes \partial_{\sigma_1} \cdots \partial_{\sigma_m} \psi_k(x) + \partial_{\rho_1} \cdots \partial_{\rho_m} \phi(x)_i \otimes \mathcal{O}^\psi(x) \partial_{\sigma_1} \cdots \partial_{\sigma_m} [\mathcal{T}_A \psi(x)]_k \right\}.$$ (A12)

Then we have

$$\Delta_0(S_0) [\phi_i(x) \otimes \psi_k(x)] = [\mathcal{O}^\phi(x) \otimes 1 + 1 \otimes \mathcal{O}^\psi(x)][\phi_i(x) \otimes \psi_k(x)].$$ (A13)
From the above computation one can conclude that

$$
\delta^*(\phi_i \star \psi_k) = \mu^* \left[ \mathcal{O}^\phi(x)\phi_i(x) \otimes \psi_k + \phi_i \otimes \mathcal{O}^\psi(x)\psi_k \right] \\
= \mathcal{O}^\phi(x)\phi_i(x) \star \psi_k + \phi_i \star \mathcal{O}^\psi(x)\psi_k.
$$

(A14)

For instance, if we consider the gauge transformations (5) and translations on scalar fields (20), then we get correspondingly

$$
\delta^*_\alpha(\phi \star \psi) = \alpha^I \cdot (T_I \phi \star \psi + \phi \star T_I \psi) \\
= -\xi^\mu \cdot (\partial_\mu \phi \star \psi + \phi \star \partial_\mu \psi) = -\xi^\mu \cdot \partial_\mu (\phi \star \psi).
$$

(A15)

and

$$
\delta^*_\xi(\phi \star \psi) = -\xi^\mu \cdot (\partial_\mu \phi \star \psi + \phi \star \partial_\mu \psi) = -\xi^\mu \cdot \partial_\mu (\phi \star \psi).
$$

(A16)

**APPENDIX B**

Let us consider a noncommutative BF theory in four dimensions (without cosmological constant term) with gauge algebra $su(2)$. The action is given by $I = \int \text{Tr} B \wedge \hat{F}$, where the gauge field is $A = A^i T^i$ (with $T^i$ being the $su(2)$ generators) whose field strength is $\hat{F} = dA + A \wedge A = (dA^i + i\varepsilon_{ijk} A^j \wedge A^k) T_i + A^i \wedge A_i$ and $B = B^i T_i$ is a two-form field. This action is invariant under twisted $su(2)$ gauge transformations [31] as it has been shown in our Section 2. Moreover, due to the trace keeps only the $su(2)$ part of the field strength we get

$$
I = \int B^i \wedge (dA_i + i\varepsilon_{ijk} A^j \wedge A^k).
$$

(B1)

However, as shown in [32] for Yang-Mills theory, the consistency of the equations of motion requires the enveloping algebra. In this appendix we argue that it is also the same situation for BF actions in the case of $su(2)$.

The field equations of the action (B1) are given by

$$
\varepsilon^{\mu\nu\rho\sigma} \left( \partial_\mu B_{\nu\rho}^i \right) - \frac{i}{2} \varepsilon_{ijk} \{ A^i_\mu, B^k_{\nu\rho} \} = 0, \quad (B2)
$$

$$
\partial_\mu A^i_\nu - \partial_\nu A^i_\mu + i\varepsilon_{ijk} \{ A^j_\mu, A^k_\nu \} = 0. \quad (B3)
$$

The integrability conditions of the first equations are

$$
\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{ijk} \partial_\mu \{ A^j_\nu, B^k_{\rho\sigma} \} = \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{jk} \left( \{ \partial_\mu A^j_\nu, B^k_{\rho\sigma} \} + \{ A^j_\nu, \partial_\nu B^k_{\rho\sigma} \} \right) = 0. \quad (B4)
$$
However, if we use the equations (B2) and (B3), we have, after some manipulations
\[
\varepsilon^{\mu
u\rho\sigma} \varepsilon_{jk} \partial_{\mu} \left\{ A^{j}_{\nu} \star B^{k}_{\rho\sigma} \right\} = -\frac{i}{2} \varepsilon^{\mu
u\rho\sigma} \left[ \left\{ A^{i}_{\mu} \star \left\{ A^{j}_{\nu} \star B^{k}_{\rho\sigma} \right\} \right\} + \left\{ A^{j}_{\nu} \star \left\{ A^{i}_{\mu} \star B^{k}_{\rho\sigma} \right\} \right\} - \left\{ A^{j}_{\nu} \star B^{k}_{\rho\sigma} \right\} \right],
\]
which does not vanish identically. If instead of \( su(2) \), we had considered the enveloping algebra \( u(2) \), the corresponding equations would vanish due to the generalized Jacobi identities. Hence, even if the action is invariant under any Lie algebra, the consistency of the equations of motion requires the whole enveloping algebra.

[1] Letter of W. Heisenberg to R. Peierls (1930). W. Pauli Scientific Correspondence Vol. II, 15. Ed. Karl von Meyenn, Springer-Verlag, (1985).
[2] J. Wess, ”Nonabelian Gauge Theories on Noncommutative Spaces”, in Supersymmetry and Unification of Fundamental Interactions: Proceedings, edited by P. Nath and P.M. Zerwas (DESY, Hamburg, 2002).
[3] H.S. Snyder, Phys. Rev. 71, 38 (1947).
[4] H.S. Snyder, Phys. Rev. 72, 68 (1947).
[5] W. Bietenholz, “Cosmic Rays and the Search for a Lorentz Invariance Violation,” arXiv:0806.3713 [hep-ph].
[6] H. Weyl, Group Theory and Quantum Mechanics, Dover, New York, (1931); E.P. Wigner, Phys. Rev. 40, 749 (1932); A. Groenewold, Physica 12, 405 (1946); J.E. Moyal, Proc. Camb. Phil. Soc. 45, 99 (1949).
[7] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. 111, 61 (1978); F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. 111, 111 (1978).
[8] G. Dito and D. Sternheimer, arXiv:math/0201168; C.K. Zachos, Int. J. Mod. Phys. A 17, 297 (2002).
[9] A. Connes, Noncommutative Geometry, Academic Press, New York (1994).
[10] J. von Neumann, Math. Ann. 104, 570 (1931).
[11] I.M. Gelfand and M.A. Naimark, Mat. Sbornik 12, 197 (1943).
[12] N. Seiberg and E. Witten, JHEP 9909, 032 (1999) [arXiv:hep-th/9908142].
[13] A. Connes, M.R. Douglas and A. Schwarz, J. High Energy Phys. 02, 003 (1998).
[14] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative Perturbative Dynamics”, JHEP 0002, 020 (2000).
[15] M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. 73, 977 (2001); R.J. Szabo, Phys. Rept. 378, 207 (2003).
[16] J. M. Gracia-Bondia, J. C. Varilly and H. Figueroa, Elements of Noncommutative Geometry, Boston, USA: Birkhaeuser (2001) 685 p
[17] M. Panero, JHEP 0705, 082 (2007) [arXiv:hep-th/0608202]; SIGMA 2, 081 (2006) [arXiv:hep-th/0609205].
[18] M. Hayakawa, Phys. Lett. B 478, 394 (2000); J. Madore, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C 16, 161 (2000) [arXiv:hep-th/0001203]; B. Jurco, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C 17, 521 (2000) [arXiv:hep-th/0006246]; X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlenbenannt, Eur. Phys. J. C 23, 363 (2002); P. Aschieri, B. Jurco, P. Schupp and J. Wess, Nucl. Phys. B 651, 45 (2003) [arXiv:hep-th/0205214].
[19] H. García-Compeán, O. Obregón, C. Ramírez, M. Sabido, Phys. Rev D 68, 045010 (2003), H. García-Compeán, O. Obregón, C. Ramírez, M. Sabido, Phys. Rev. D 68, 044015 (2003).
[20] A.H. Chamseddine, G. Felder, J. Frohlich, Commun. Math. Phys. 155, 205 (1993); J. Madore, J. Mourad, Int. J. Mod. Phys. D 3, 221 (1994); A. Jevicki, S. Ramgoolam, JHEP 9904, 032 (1999); J.W. Moffat, Phys. Lett. B 491, 345 (2000); A.H. Chamseddine, Commun. Math. Phys. 218, 283 (2001); H. Nishino, S. Rajpoot, Phys. Lett. B 532, 334 (2002); A.H. Chamseddine, J. Math. Phys. 44, 2534, (2003); M.A. Cardella, D. Zanon, Class. Quant. Grav. 20, L95 (2003); A.H. Chamseddine, Phys. Rev. D 69, 024015 (2004); D.V. Vassilevich, Nucl. Phys. B 715, 695 (2005); A. H. Chamseddine, Annales Henri Poincare 482, S881 (2003) [arXiv:hep-th/0301112]; S. Marculescu and F. R. Ruiz, Phys. Rev. D 74, 105004 (2006) [arXiv:hep-th/0607201]; R. Banerjee, P. Mukherjee and S. Samanta, Phys. Rev. D 75, 125020 (2007), [arXiv:hep-th/0703128]; J. A. Nieto, “SL(2,R)-Symmetry and Noncommutative Phase Space in (2+2) Dimensions,” [arXiv:0809.3429] [hep-th].
[21] L. Alvarez-Gaume, F. Meyer and M. A. Vazquez-Mozo, Nucl. Phys. B 753, 92 (2006) [arXiv:hep-th/0605113].
[22] N. Seiberg, L. Susskind and N. Toubas, JHEP 0006, 044 (2000) [arXiv:hep-th/0005015].
[23] J. Wess, "Deformed Coordinate Spaces; Derivatives", Lecture given at BW2003 Workshop "Mathematical, Theoretical and Phenomenological Challenges Beyond Standard Model" 29 August-02 September, 2003 Vrnjacka Banja, Serbia, [arXiv:hep-th/0408080]
[24] Chaichian, P. Kulish, K. Nishijima, A. Tureanu, Phys.Lett. B 604, 98 (2004).
[25] F. Koch and E. Tsouchnika, Nucl. Phys. B 717, 387 (2005).
[26] R. Oeckl, Nucl. Phys. B 581, 559 (2000).
[27] M. Dimitrijevic and J. Wess, "Deformed bialgebra of diffeomorphisms", Talk given at 1st Vienna Central European Seminar on Particle Physics and Quantum Field Theory, 26-28 November 2004, [arXiv:hep-th/0411224]
[28] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Shupp and J. Wess, Class. Quantum Grav. 22, 3511 (2005).
[29] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, Class. Quantum Grav. 23, 1883 (2006).
[30] J. Wess, SIGMA 2, 089 (2006) [arXiv:hep-th/0611025]; P. Aschieri, J. Phys. Conf. Ser. 53, 799 (2006) [arXiv:hep-th/0608172]; P. Aschieri, Fortsch. Phys. 55, 649 (2007) [arXiv:hep-th/0703014]; R. J. Szabo, Class. Quant. Grav. 23, R199 (2006) [arXiv:hep-th/0606233].
[31] D. V. Vassilevich, “Twist to close,” Mod. Phys. Lett. A 21, 1279 (2006) [arXiv:hep-th/0602185].
[32] P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml and J. Wess, “Twisted Gauge Theories,” Lett. Math. Phys. 78, 61 (2006) [arXiv:hep-th/0603024].
[33] R. Banerjee and S. Samanta, Eur. Phys. J. C 51, 207 (2007) [arXiv:hep-th/0608214]; JHEP 0702, 046 (2007) [arXiv:hep-th/0611249]; A. Kobakhidze, Int. J. Mod. Phys. A 23, 2541 (2008) [arXiv:hep-th/0603132]; A. P. Balachandran, A. Pinzul, B. A. Qureshi, S. Vaidya and I. I. S. Bangalore, [arXiv:hep-th/0608138]; Y. Lee, Phys. Rev. D 76, 025022 (2007) [arXiv:0704.1805 [hep-th]]; A. P. Balachandran, A. Pinzul, B. A. Qureshi and S. Vaidya, Phys. Rev. D 76, 105025 (2007) [arXiv:0708.0069 [hep-th]].
[34] J. Wess, “Deformed Gauge Theories,” J. Phys. Conf. Ser. 53, 752 (2006) [arXiv:hep-th/0608135].
[35] J. Plebański, J. Math. Phys. 18, 2511 (1977).
[36] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986); Phys. Rev. D 36, 1587 (1987); Lectures on
Non-perturbative Canonical Gravity, World Scientific, Singapore (1991).

[37] V. Chari and A.N. Pressley, A Guide to Quantum Groups, Cambridge Univ. Press (1995).

[38] V.G. Drinfeld, "Quasi-Hopf algebras", Alg. Anal. 1, 114 (1989).