A note on $f^\pm$-Zagreb indices in respect of Jaco Graphs, $J_n(1), n \in \mathbb{N}$ and the introduction of Khazamula irregularity

(Johan Kok, Vivian Mukungunugwa)

Abstract

The topological graph indices $\text{irr}(G)$ related to the first Zagreb index, $M_1(G)$ and the second Zagreb index, $M_2(G)$ are of the oldest irregularity measures researched. Alberton [3] introduced the irregularity of $G$ as $\text{irr}(G) = \sum_{e \in E(G)} \text{imb}(e)$, $\text{imb}(e) = |d(v) - d(u)|_{e=uv}$. In the paper of Fath-Tabar [7], Alberton’s indice was named the third Zagreb indice to conform with the terminology of chemical graph theory. Recently Ado et. al. [1] introduced the topological indice called total irregularity. The latter could be called the fourth Zagreb indice. We define the $\pm$Fibonacci weight, $f^\pm_i$ of a vertex $v_i$ to be $-f_{d(v_i)}$, if $d(v_i)$ is uneven and, $f_{d(v_i)}$, if $d(v_i)$ is even. From the aforesaid we define the $f^\pm$-Zagreb indices. This paper presents introductory results for the undirected underlying graphs of Jaco Graphs, $J_n(1), n \leq 12$. For more on Jaco Graphs $J_n(1)$ see [9, 10]. Finally we introduce the Khazamula irregularity as a new topological variant.

We also present five open problems.

Keywords: Total irregularity, Irregularity, Imbalance, Zagreb indices, $\pm$Fibonacci weight, Total $f$-irregularity, Fibonacci irregularity, $f^\pm$-Zagreb indices, Jaco graphs, Zeckendorf representation, Khazamula irregularity, Khazamula theorem.

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1Affiliation of author:
Johan Kok (Tshwane Metropolitan Police Department), City of Tshwane, Republic of South Africa
e-mail: kokkiek2@tshwane.gov.za

Vivian Mukungunugwa (Department of Mathematics and Applied Mathematics, University of Zimbabwe), City of Harare, Republic of Zimbabwe
e-mail: vivianm@maths.uz.ac.zw

**On advice from arXiv Moderation this paper now incorporates similar ideas and variant results of another submission which has been removed.
1 Introduction

The topological graph indices $irr(G)$ related to the first Zagreb index, $M_1(G) = \sum_{v \in V(G)} d^2(v) = \sum_{uv \in E(G)} (d(v) + d(u))$, and the second Zagreb index, $M_2(G) = \sum_{v \in V(G)} d(v)d(u)$ are of the oldest irregularity measures researched. Alberton [3] introduced the irregularity of $G$ as $irr(G) = \sum_{e \in E(G)} imb(e), imb(e) = |d(v) - d(u)|_{e=uv}$. In the paper of Fath-Tabar [7], Alberton’s indice was named the third Zagreb indice to conform with the terminology of chemical graph theory. Recently Ado et. al. [1] introduced the topological indice called total irregularity and defined it, $irr_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d(u) - d(v)|$. The latter could be called the fourth Zagreb indice.

If the vertices of a simple undirected graph $G$ on $n$ vertices are labelled $v_i, i = 1, 2, 3, \ldots, n$ then the respective definitions may be:

$M_1(G) = \sum_{i=1}^{n} d^2(v_i) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} (d(v_i) + d(v_j))_{v_i, v_j \in E(G)}$, $M_2(G) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} d(v_i)d(v_j)_{v_i, v_j \in E(G)}$,

$M_3(G) = \sum_{i=1}^{n} \sum_{j=2}^{n} |d(v_i) - d(v_j)|_{v_i, v_j \in E(G)}$ and $M_4(G) = irr_t(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} |d(v_i) - d(v_j)| = \sum_{i=1}^{n} \sum_{j=i+1}^{n} |d(v_i) - d(v_j)|$ or $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |d(v_i) - d(v_j)|$. For a simple graph on a singular vertex (1-empty graph), we define $M_1(G) = M_2(G) = M_3(G) = M_4(G) = 0$.

2 Zagreb indices in respect of $\pm$Fibonacci weights, $f^\pm$-Zagreb indices

We define the $\pm$Fibonacci weight, $f^\pm_i$ of a vertex $v_i$ to be $-f_{d(v_i)}$, if $d(v_i) = i$ is uneven and, $f_{d(v_i)}$, if $d(v_i)$ is even. The $f^\pm$-Zagreb indices can now be defined as:

$f^\pm Z_1(G) = \sum_{i=1}^{n} (f^\pm_i)^2 = \sum_{i=1}^{n-1} \sum_{j=2}^{n} (|f^\pm_i| + |f^\pm_j|)_{v_i, v_j \in E(G)}$, $f^\pm Z_2(G) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} (f^\pm_i \cdot f^\pm_j)_{v_i, v_j \in E(G)}$,

$f^\pm Z_3(G) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |f^\pm_i - f^\pm_j|_{v_i, v_j \in E(G)}$ and $f^\pm Z_4(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} |f^\pm_i - f^\pm_j| = \sum_{i=1}^{n} \sum_{j=i+1}^{n} |f^\pm_i - f^\pm_j|$.
or \( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |f_i^± - f_j^±| \). For a simple graph on a singular vertex (1-empty graph), we define \( f^±Z_1(G) = f^±Z_2(G) = f^±Z_3(G) = f^±Z_4(G) = 0 \).

2.1 Application to Jaco Graphs, \( J_n(1), n \in \mathbb{N} \)

For ease of reference some definitions in [9] are repeated. A particular family of finite directed graphs (order 1) called Jaco Graphs and denoted by \( J_n(1), n \in \mathbb{N} \) are directed graphs derived from a particular well-defined infinite directed graph (order 1), called the 1-root digraph. The 1-root digraph has four fundamental properties which are; \( V(J_\infty(1)) = \{v_i|i \in \mathbb{N}\} \) and, if \( v_j \) is the head of an edge (arc) then the tail is always a vertex \( v_i, i < j \) and, if \( v_k, \) for smallest \( k \in \mathbb{N} \) is a tail vertex then all vertices \( v_\ell, k < \ell < j \) are tails of arcs to \( v_j \) and finally, the degree of vertex \( k \) is \( d(v_k) = k \). The family of finite directed graphs are those limited to \( n \in \mathbb{N} \) vertices by lobbing off all vertices (and edges arcing to vertices) \( v_t, t > n \). Hence, trivially we have \( d(v_i) \leq i \) for \( i \in \mathbb{N} \).

Definition 2.1. The infinite Jaco Graph \( J_\infty(1) \) is defined by \( V(J_\infty(1)) = \{v_i|i \in \mathbb{N}\}, E(J_\infty(1)) \subseteq \{(v_i,v_j)|i,j \in \mathbb{N}, i < j\} \) and \( (v_i,v_j) \in E(J_\infty(1)) \) if and only if \( 2i - d^-(v_i) \geq j \), [9].

Definition 2.2. The family of finite Jaco Graphs are defined by \( \{J_n(1) \subseteq J_\infty(1)|n \in \mathbb{N}\} \). A member of the family is referred to as the Jaco Graph, \( J_n(1) \), [9].

Definition 2.3. The set of vertices attaining degree \( \Delta(J_n(1)) \) is called the Jaconian vertices of the Jaco Graph \( J_n(1) \), and denoted, \( J(J_n(1)) \) or, \( J_n(1) \) for brevity, [9].

From [9] we have Bettina’s Theorem.

Theorem 2.1. Let \( \mathbb{F} = \{f_0, f_1, f_2, f_3, \ldots\} \) be the set of Fibonacci numbers and let \( n = f_{i_1} + f_{i_2} + \ldots + f_{i_r}, n \in \mathbb{N} \) be the Zeckendorf representation of \( n \). Then

\[
d^+(v_n) = f_{i_1-1} + f_{i_2-1} + \ldots + f_{i_r-1}.
\]

Note: the degree of vertex \( v_i \), denoted \( d(v_i) \) refers to the degree in \( J_\infty(1) \) hence \( d(v_i) = i \). In the finite Jaco Graph the degree of vertex \( v_i \) is denoted \( d(v_i)_{J_n(1)} \). The degree sequence is denoted \( D_n = (d(v_1)_{J_n(1)}, d(v_2)_{J_n(1)}, \ldots, d(v_n)_{J_n(1)}) \). By convention \( D_{i+1} = D_i \cup d(v_{i+1})_{J_n(1)} \).
2.1.1 Algorithm to determine the degree sequence of a finite Jaco Graph, \( J_n(1), n \in \mathbb{N} \). 

Consider a finite Jaco Graph \( J_n(1), n \in \mathbb{N} \) and label the vertices \( v_1, v_2, v_3, \ldots, v_n \).

Step 0: Set \( n = n \). Let \( i = j = 1 \). If \( j = n = 1 \), let \( \mathbb{D}_i = (0) \) and go to Step 6, else set \( \mathbb{D}_i = \emptyset \) and go to Step 1.

Step 1: Determine the \( j \)th Zeckendorf representation say, \( j = f_{i_1} + f_{i_2} + \ldots + f_{i_r} \), and go to Step 2.

Step 2: Calculate \( d^+(v_j) = f_{i_1} - 1 + f_{i_2} - 1 + \ldots + f_{i_r} - 1 \), then go to Step 3.

Step 3: Calculate \( d^-(v_j) = j - d^+(v_j) \), and let \( d(v_j) = d^+(v_j) + d^-(v_j) \), then go to Step 4.

Step 4: If \( d(v_j) \leq n \), set \( d(v_j)_{J_n(1)} = d(v_j) \) else, set \( d(v_j)_{J_n(1)} = d^-(v_j) + (n - j) \) and set \( \mathbb{D}_j = \mathbb{D}_i \cup d(v_j)_{J_n(1)} \) and go to Step 5.

Step 5: If \( j = n \) go to Step 6 else, set \( i = i + 1 \) and \( j = i \) and go to Step 1.

Step 6: Exit.

2.1.2 Tabled values of \( F^\pm(J_n(1)) \), for finite Jaco Graphs, \( J_n(1), n \leq 12 \).

For illustration the adapted table below follows from the Fisher Algorithm [9] for \( J_n(1), n \leq 12 \). Note that the Fisher Algorithm determines \( d^+(v_i) \) on the assumption that the Jaco Graph is always sufficiently large, so at least \( J_n(1), n \geq i + d^+(v_i) \). For a smaller graph the degree of vertex \( v_i \) is given by \( d(v_i)_{J_n(1)} = d^-(v_i) + (n - i) \). In [9] Bettina’s theorem describes an arguably, closed formula to determine \( d^+(v_i) \). Since \( d^-(v_i) = n - d^+(v_i) \) it is then easy to determine \( d(v_i)_{J_n(1)} \) in a smaller graph \( J_n(1), n < i + d^+(v_i) \). The \( f_i^\pm \)-sequence of \( J_n(1) \) is denoted \( F^\pm(J_n(1)) \).
Table 1.

| $i \in \mathbb{N}$ | $d^-(v_i)$ | $d^+(v_i) = i - d^-(v_n)$ | $F^\pm(J_i(1))$ |
|-------------------|--------|----------------------|-----------------|
| 1                 | 0      | 1                    | (0)             |
| 2                 | 1      | 1                    | (-1, -1)        |
| 3                 | 1      | 2                    | (-1, 1, -1)     |
| 4                 | 1      | 3                    | (-1, 1, 1, -1)  |
| 5                 | 2      | 3                    | (-1, 1, -2, 1, 1)|
| 6                 | 2      | 4                    | (-1, 1, -2, -2, -2, 1)|
| 7                 | 3      | 4                    | (-1, 1, -2, 3, 3, -2, -2)|
| 8                 | 3      | 5                    | (-1, 1, -2, 3, -5, 3, 3, -2)|
| 9                 | 3      | 6                    | (-1, 1, -2, 3, -5, -5, 3, -2)|
| 10                | 4      | 6                    | (-1, 1, -2, 3, -5, 8, 8, -5, 3, 3)|
| 11                | 4      | 7                    | (-1, 1, -2, 3, -5, 8, -13, 8, -5, -5, 3)|
| 12                | 4      | 8                    | (-1, 1, -2, 3, -5, 8, -13, 8, 8, -5, 3)|

Since it is known that a sequence $(d_1, d_2, d_3, ..., d_n)$ of non-negative integers is a degree sequence of some graph $G$ if and only if $\sum_{i=1}^{n} d_i$ is even. It implies that a degree sequence has an even number of odd entries. Hence, we know that the $f^\pm$-sequence of $J_n(1)$ denoted, $F^\pm(J_n(1))$, $n \in \mathbb{N}$ has an even number of, $-f_{d(v_i)}$ entries. Following from Table 1 the table below depicts the values $f^\pm Z_1(J_n(1))$, $f^\pm Z_2(J_n(1))$, $f^\pm Z_3(J_n(1))$ and $f^\pm Z_4(J_n(1))$ for $J_n(1), n \leq 12$.

Table 2.

| $i \in \mathbb{N}$ | $d^-(v_i)$ | $d^+(v_i)$ | $f^\pm Z_1(J_i(1))$ | $f^\pm Z_2(J_i(1))$ | $f^\pm Z_3(J_i(1))$ | $f^\pm Z_4(J_i(1))$ |
|-------------------|--------|--------|------------------|------------------|------------------|------------------|
| 1                 | 0      | 1      | 0                | 0                | 0                | 0                |
| 2                 | 1      | 1      | 2                | 1                | 0                | 0                |
| 3                 | 1      | 2      | 3                | -2               | 4                | 4                |
| 4                 | 1      | 3      | 4                | -1               | 4                | 8                |
| 5                 | 2      | 3      | 8                | -6               | 11               | 16               |
| 6                 | 2      | 4      | 15               | 5                | 11               | 25               |
| 7                 | 3      | 4      | 32               | -26              | 35               | 56               |
| 8                 | 3      | 5      | 62               | -19              | 50               | 98               |
| 9                 | 3      | 6      | 103              | 0                | 72               | 138              |
| 10                | 4      | 6      | 211              | 38               | 119              | 251              |
| 11                | 4      | 7      | 396              | -238             | 210              | 402              |
| 12                | 4      | 8      | 604              | -158             | 273              | 566              |
3 Khazamula irregularity

Let $G \rightarrow$ be a simple directed graph on $n \geq 2$ vertices labelled $v_1, v_2, v_3, ..., v_n$. Let all vertices $v_i$ carry its $\pm$Fibonacci weight, $f_i^\pm$ related to $d(v_i) = d(v^+(v_i)) + d^-(v_i)$. Also let vertex $v_j$ be a head vertex of $v_i$ and choose any $d(v^h_i) = \max(d(v_j)_{v_j})$.

**Definition 3.1.** Let $G \rightarrow$ be a simple directed graph on $n \geq 2$ vertices with each vertex carrying its $\pm$Fibonacci weight, $f_i^\pm$. For the function $f(x) = mx + c, x \in \mathbb{R}$ and $m, c \in \mathbb{Z}$ we define the Khazamula irregularity as:

$$irr_k(G \rightarrow) = \sum_{i=1}^{n} | \int_{f_i^+}^{f_i^-} f(x)dx|.$$ 

Note: Vertices $v$ with $d^+(v) = 0$, are headless and the corresponding integral terms to the summation are defined zero. Hence, $irr_k(K_i \rightarrow) = 0$.

Let $G$ be a simple connected undirected graph on $n$ vertices which are labelled, $v_1, v_2, v_3, ..., v_n$. Also let $G$ have $\epsilon$ edges. It is known that $G$ can be oriented in $2^\epsilon$ ways, including the cases of isomorphism. Finding the relationship between the different values of $irr_k(G \rightarrow)$ and $irr_k^c(G \rightarrow)$ (to follow in subsection 3.3) in respect of the different orientations for $G$ in general is stated as an open problem. In this section we give results in respect of particular orientations of paths, cycles, wheels and complete bipartite graphs.

3.1 $irr_k$ for Paths, Cycles, Wheels and Complete Bipartite Graphs

**Proposition 3.1.** For a directed path $P_n \rightarrow$, $n \geq 2$ which is consecutively directed from left to right we have that the Khazamula irregularity, $irr_k(P_n \rightarrow) = |\frac{3}{2}(n-2)m + nc|.$

**Proof.** Label the vertices of the directed path $P_n \rightarrow$ consecutively from left to right $v_1, v_2, v_3, ..., v_n$.

From the definition $irr_k(P_n \rightarrow) = \sum_{i=1}^{n} | \int_{f_i^+}^{f_i^-} f(x)dx|$, it follows that we have:

$$\sum_{i=1}^{n} | \int_{f_i^+}^{f_i^-} f(x)dx| = | \int_{-1}^{2} f(x)dx + \int_{1}^{2} f(x)dx + \cdots + \int_{1}^{2} + \int_{1}^{1} f(x)dx|.$$

$\underline{(n-3)-terms}$
So we have, 
\[ \sum_{i=1}^{n} \left| \int_{f_{i}^{\pm}} f(x) \, dx \right| = \left| \left( \frac{1}{2} m x^2 + cx \right) \right|_1^n + (n-3) \left( \frac{1}{2} m x^2 + cx \right) \right|_1^n + 0 = \\
|2m + 2c - \frac{1}{2}m + c + (n-3)(2m + 2c - \frac{1}{2}m - c)| = \frac{3}{2}m + 3c + \frac{3}{2}(n-3)m + (n-3)c = \frac{3}{2}(n-2)m + nc. \]

**Proposition 3.2.** For a directed cycle \( C_n^+ \) which is consecutively directed clockwise we have that the Khazamula irregularity, \( irr_k(C_n^+) = n\left|\frac{3}{2}m + c\right|. \)

**Proof.** Label the vertices of the directed cycle \( C_n^+ \) consecutively clockwise \( v_1, v_2, v_3, \ldots, v_n \). So vertices carry the \( \pm \)Fibonacci weight, \( f_{v_i}^{\pm} = f_1 = 1 \). Also a head vertex is always unique with degree = 2. From the definition \( irr_k(C_n^+) = \sum_{i=1}^{n} \left| \int_{f_{i}^{\pm}} f(x) \, dx \right| \), it follows that we have:

\[ \sum_{i=1}^{n} \left| \int_{f_{i}^{\pm}} f(x) \, dx \right| = \left| \int_{1}^{2} f(x) \, dx + \int_{1}^{2} f(x) \, dx + \cdots + \int_{1}^{2} f(x) \, dx \right| = \left| n\left( \frac{1}{2} m x^2 + cx \right) \right| = \\
|n(2m + 2c - \frac{1}{2}m - c)| = |n\left( \frac{3}{2}m + c \right)| = n\left|\frac{3}{2}m + c\right|. \]

**Proposition 3.3.** For a directed Wheel graph \( W_{(1,n)}^\rightarrow \) with the axle vertex \( u_1 \) and the wheel vertices \( v_1, v_2, \ldots, v_n \) and the spokes directed \( (u_1, v_i)_v \) and the wheel vertices directed consecutively clockwise \( v_1, v_2, \ldots, v_n \), we have that:

\[ irr_k(W_{(1,n)}^\rightarrow) = \begin{cases} \\
\left| \frac{(5n-f_2^2+9)}{2} \right| m + (5n - f_n + 3)c, & \text{if } n \text{ is even}, \\
\left| \frac{(5n-f_2^2+9)}{2} \right| m + (5n + f_n + 3)c, & \text{if } n \text{ is uneven}.
\end{cases} \]

**Proof.** Consider a Wheel graph \( W_{(1,n)}^\rightarrow \) with the axle vertex \( u_1 \) and the wheel vertices \( v_1, v_2, \ldots, v_n \) and the spokes directed \( (u_1, v_i)_v \) and the wheel vertices directed consecutively clockwise \( v_1, v_2, \ldots, v_n \).

Case 1: If \( n \) is even then \( d(u_1) \) is even and carries the \( \pm \)Fibonacci weight, \( f_n \). Obviously the wheel vertices have \( d(v_i) = 3_{v_i} \), hence carry the \( \pm \)Fibonacci weight, \( f_3 = -2_{v_i} \). So from the definition of the Khazamula irregularity we have that:

\[ irr_k(W_{(1,n)}^\rightarrow) = \sum_{i=1}^{n} \left| \int_{f_{i}^{\pm}} f(x) \, dx \right| = \left| n \int_{-2}^{3} f(x) \, dx + \int_{f_n}^{3} f(x) \, dx \right| \text{ if } n \text{ is even.} \]

This results in, \( irr_k = \sum_{i=1}^{n} \left| \int_{f_{i}^{\pm}} f(x) \, dx \right| = \left| n\left( \frac{9}{2}m + 3c - 2m + 2c \right) + \left( \frac{9}{2}m + 3c - \frac{f_2^2}{2}m - f_n c \right) \right| = \\
\left| \frac{(5n-f_2^2+9)}{2} \right| m + (5n - f_n + 3)c, \]

\[ \]
\[ \left\lfloor \frac{5}{2}nm + 5nc + \frac{9}{2}m + 3c - \frac{f_n^2}{2}m - f_nc \right\rfloor = \left\lfloor \frac{5n-f_n^2+9}{2}m + (5n - f_n + 3)c \right\rfloor. \]

Case 2: If \( n \) is uneven then \( d(u_1) \) is uneven and carries the \( \pm \text{Fibonacci weight}, -f_n \). So in the Riemann integral \( \int_{-f_n}^3 f(x)dx \) we have \( (\frac{9}{2}m + 3c - \frac{f_n^2}{2}m + f_nc) \). So the result \( \text{irr}_k(W_{(1,n)}^\rightarrow) = \left\lfloor \frac{5n-f_n^2+9}{2}m + (5n + f_n + 3)c \right\rfloor \) if \( n \) is uneven, follows. \( \Box \)

Consider the complete bipartite graph \( K_{(n,m)} \) and call the \( n \) vertices the \textit{left-side vertices} and the \( m \) vertices the \textit{right-side vertices}. Orientate \( K_{(n,m)} \) strictly from \textit{left-side vertices} to \textit{right-side vertices} to obtain \( K_{(n,m)}^{l\rightarrow r} \).

**Proposition 3.4.** For the directed graph \( K_{(n,m)}^{l\rightarrow r} \) we have that:

\[
\text{irr}_k(K_{(n,m)}^{l\rightarrow r}) = \begin{cases} 
\left\lfloor \frac{(n^3-nf_m^2)}{2}m + (n^2 - nf_m)c \right\rfloor, & \text{if } m \text{ is even,} \\
\left\lfloor \frac{(n^3-nf_m^2)}{2}m + (n^2 + nf_m)c \right\rfloor, & \text{if } m \text{ is uneven.}
\end{cases}
\]

**Proof.** For the directed graph \( K_{(n,m)}^{l\rightarrow r} \) we have that all \textit{left-side vertices} say \( v_1, v_2, \ldots, v_n \) have \( d^+(v_i) = m \), whilst all \textit{right-side vertices} say \( u_1, u_2, \ldots, u_m \) have \( d^-(u_i) = n \) and \( d^+(u_i) = 0 \).

Case 1: If \( m \) is even it follows from the definition that, \( \text{irr}_k(K_{(n,m)}^{l\rightarrow r}) = n\left\lfloor \int_{-f_m}^n f(x)dx \right\rfloor \). So we have that \( \text{irr}_k(K_{(n,m)}^{l\rightarrow r}) = n\left\lfloor \left(\frac{1}{2}mx^2 + cx \right)|_{f_m}^n \right\rfloor = n\left\lfloor \frac{n^2}{2}m + nc - \left(\frac{f_n^2}{2}m + f_m c \right) \right\rfloor = \left\lfloor \frac{(n^3-nf_m^2)}{2}m + (n^2 - nf_m)c \right\rfloor. \)

Case 2: If \( m \) is uneven the \textit{left-side vertices} all carry the \( \pm \text{Fibonacci weight}, -f_m \). Hence, the result follows as in Case 1, accounting for \( -f_m \). \( \Box \)

**Example problem 1:** Let \( n = 1 \) or \( 5 \) and \( f(x) = mx \). Prove that \( \text{irr}_k(K_{(1,n)}^\rightarrow) = 0 \) or \( |12m| \) and,

\[
\text{irr}_k(K_{(1,n)}^{l\rightarrow r}) = \begin{cases} 
0, & \text{or} \\
5(\text{irr}_k(K_{(1,n)}^\rightarrow)) = 60|m|.
\end{cases}
\]
Proof. Let \( n = 1 \) and let \( f(x) = mx. \) From the definition of \( \text{irr}_k(G^+) \) it follows that
\[
\text{irr}_k(K_{(1,n)}^+) = \int_{-1}^{1} |mx.dx|_{v_1} = |\frac{1}{2}mx^2|_{-1}^{1} = 0.
\]
We also have that \( \text{irr}_k(K_{(1,n)}^-) = \int_{-1}^{1} |mx.dx|_{v_1} = |\frac{1}{2}mx^2|_{-1}^{1} = 0. \)

Let \( n = 5 \) and let \( f(x) = mx. \) Now we have that \( \text{irr}_k(K_{(1,n)}^+) = \int_{-1}^{5} |mx.dx|_{v_1} = |\frac{1}{2}mx^2|_{-1}^{5} = |12m|. \)

For \( \text{irr}_k(K_{(1,n)}^-) \) we have \( \sum_{i=1}^{5} \int_{-1}^{5} |mx.dx|_{v_1} = 5(\int_{-1}^{5} |mx.dx|) = 5|\frac{1}{2}mx^2|_{-1}^{5} = 5|12m| = 60|m|. \)

\[\Box\]

3.2 Khazamula’s Theorem

Consider two simple connected directed graphs, \( G^+ \) and \( H^+ \). Let the vertices of \( G^+ \) be labelled \( v_1, v_2, \ldots, v_n \) and the vertices of \( H^+ \) be labelled \( u_1, u_2, \ldots, u_m \). Define the directed join as \((G^+ + H^+)^+\) conventionally, with the arcs \( \{(v_i, u_j) | v_i \in V(G^+), u_j \in V(H^+)\} \).

**Theorem 3.5.** Consider two simple connected directed graphs, \( G^+ \) on \( n \) vertices and \( H^+ \) on \( m \) vertices then, \( \text{irr}_k((G^+ + H^+)^+) = |n \int_{f_1}^{\Delta(H^+)+n} f(x)dx + \sum_{i=1}^{m} |f_{d(u_i)+1}^{(v_i)} f(x)dx|. \)

Proof. Note that in the graph \( G^+ \) the maximum degree \( \Delta(G^+) = \max(d^+(v_i) + d^-(v_i)) \leq n - 1 \) for at least one vertex \( v_i \). If such a vertex \( v_i \) is indeed the head vertex of a vertex \( v_t \), then \( \sum_{i=1}^{n} |f_{d(v_i)} f(x)dx| \) will contain the term \( \int_{f_1}^{\Delta(G)} f(x)dx. \)

In \( H^+ \) the maximum degree \( \Delta(H^+) = \max(d^+(u_s) + d^-(u_s)) \geq 1 \) for some vertex \( u_s \). Hence, in the directed graph \((G^+ + H^+)^+\), all terms of \( \sum_{i=1}^{n} |f_{d(v_i)} f(x)dx| \) reduces to zero and are replaced by the terms \( \int_{f_1}^{\Delta(H^+)+n} f(x)dx, \) because \( \Delta(G^+) \leq n - 1 < \Delta(H^+) + n. \)

In respect of \( H^+ \) we have that each \( d(u_i) \) increases by exactly 1 so the value of \( f_{d(u_i)+1} \) switches between \( \pm \) and adopts the value \( f_{d(u_i)+1}. \) Similarly all head vertices’ degree increases by exactly 1. These observations result in:

\[
\text{irr}_k((G^+ + H^+)^+) = |n \int_{f_1}^{\Delta(H^+)+n} f(x)dx + \sum_{i=1}^{m} |f_{d(u_i)+1} f(x)dx|. \]

\[\Box\]
**Example problem 2:** An application of the Khazamula theorem to the graph \((C_n^+ + K_1)^+\) in respect of \(f(x) = mx\), results in \(irr_k((C_n^+ + K_1)^+) = \frac{1}{3}(n^2 - 4)irr_k(C_n^+)\) if \(f(x) = mx\).

### 3.3 Khazamula c-irregularity for orientated Paths, Cycles, Wheels and Complete Bipartite Graphs

Let \(f(x) = \sqrt{r^2 - x^2}, x \in \mathbb{R}\) and \(r = \max\{d(v_i)_{v_i.d^-(v_i) \geq 1}, or |(f_i^\pm)|_{v_i}\}\). We define Khazamula c-irregularity as \(irr_k^c(G^+) = \sum_{i=1}^n |\int_{f_i^+}^{f_i^-} f(x) dx|\). It is known that \(\int_a^b \sqrt{r^2 - x^2} dx = \left(\frac{1}{2}x\sqrt{r^2 - x^2} + \frac{r^2}{2}\arcsin\frac{x}{r}\right)|_a^b\). Also note that \(\arcsin\theta\) applies to \(\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) to ensure a singular value for the respective integral terms.

**Proposition 3.6.** For a directed path \(P_n^+, n \geq 3\) which is consecutively directed from left to right we have that the Khazamula c-irregularity, \(irr_k^c(P_n^+) = (n - 2)\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\).

**Proof.** Label the vertices of the directed path \(P_n^+, n \geq 3\) consecutively from left to right \(v_1, v_2, v_3, ..., v_n\). Note that \(r = \max\{d(v_i)_{v_i.d^-(v_i) \geq 1}, or |(f_i^\pm)|_{v_i}\} = 2\). From the definition \(irr_k^c(P_n^+) = \sum_{i=1}^n |\int_{f_i^+}^{f_i^-} f(x) dx|\), it follows that we have:

\[
\sum_{i=1}^n |\int_{f_i^+}^{f_i^-} f(x) dx| = \int_{1}^2 f(x) dx + \sum_{i=1}^{n-3} \int_{1}^2 f(x) dx + ... + \int_{1}^2 f(x) dx.
\]

So we have, \(\sum_{i=1}^n |\int_{f_i^+}^{f_i^-} f(x) dx| = \left|\left(\frac{1}{2}x\sqrt{4 - x^2} + 2\arcsin\frac{x}{2}\right)|_{1}^{2} - (n - 3)\left(\frac{1}{2}x\sqrt{4 - x^2} + 2\arcsin\frac{x}{2}\right)|_{1}^{2} + \frac{r^2}{2}\arcsin\frac{x}{r}\right|^2 = \left|\left(\arcsin\frac{x}{r}\right)^2\right| = \left|\arcsin\frac{x}{r}\right|\), and \(\arcsin\frac{x}{2}\) applies to \(\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) to ensure a singular value for the respective integral terms.

**Proposition 3.7.** For a directed cycle \(C_n^+\) which is consecutively directed clockwise we have that the Khazamula c-irregularity, \(irr_k^c(C_n^+) = n\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\).

**Proof.** Label the vertices of the directed cycle \(C_n^+\) consecutively clockwise \(v_1, v_2, v_3, ..., v_n\). So all vertices carry the \(\pm\)Fibonacci weight, \(f_{i_{vi}} = f_1 = 1\). Also a head vertex is always
unique with degree \( d(v_i) = 2 \). So \( r = \max\{d(v_i)\}_{v_i} \), or \( |f^\pm_i|_{v_i} = 2 \). From the definition
\[
irr_k(C_n^\to) = \sum_{i=1}^{n} |\int_{f^\pm_i} f(x)dx|,
\]
it follows that we have:
\[
\sum_{i=1}^{n} |\int_{f^\pm_i} f(x)dx| = |\int_{1}^{2} f(x)dx + \int_{1}^{2} f(x)dx + \cdots + \int_{1}^{2} f(x)dx| = n|(|\frac{1}{2}x\sqrt{4-x^2+2\arcsin\frac{2}{2}}|)_{i}^2| =
\]
\[
n|(0 + 2\arcsin1 - \frac{\sqrt{3}}{2} - 2\arcsin\frac{1}{2})| = n|(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})| = n(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}).
\]

**Proposition 3.8.** For a directed Wheel graph \( W_{(1,n)}^\to \) with the axle vertex \( u_1 \) and the wheel vertices \( v_1, v_2, ..., v_n \) and the spokes directed \( (u_1, v_i)_{v_i} \) and the wheel vertices directed consecutively clockwise \( v_1, v_2, ..., v_n \), we have that:

\[
irr_k(W_{(1,n)}^\to) = \begin{cases} 
4\sqrt{5} + 9\pi + 18\arcsin\frac{\pi}{3}, & \text{if } n = 3 \text{ or } 4, \\
\frac{3}{2}(n + 1)\sqrt{f_n^2 - 9} + (n + 1)\frac{\alpha_2}{2}\arcsin\frac{\beta}{f_n} - \frac{\alpha_2\pi}{4} + A, & \text{if } n \geq 6 \text{ and even}, \\
\frac{3}{2}(n + 1)\sqrt{f_n^2 - 9} + (n + 1)\frac{\alpha_2}{2}\arcsin\frac{\beta}{f_n} + \frac{\alpha_2\pi}{4} + B, & \text{if } n \geq 5 \text{ and uneven},
\end{cases}
\]

with: \( A = n(\sqrt{f_n^2 - 4} + \frac{\alpha_2}{2}\arcsin\frac{\beta}{f_n}) \) and \( B = n(\sqrt{f_n^2 - 4} - \frac{\alpha_2}{2}\arcsin\frac{\beta}{f_n}) \).

**Proof.** Consider a Wheel graph \( W_{(1,n)}^\to \) with the axle vertex \( u_1 \) and the wheel vertices \( v_1, v_2, ..., v_n \) and the spokes directed \( (u_1, v_i)_{v_i} \) and the wheel vertices directed consecutively clockwise \( v_1, v_2, ..., v_n \).

Case 1: If \( n = 3 \) we have that \( irr_k(W_{(1,3)}^\to) = |\int_{-2}^{3} \sqrt{9-x^2}dx + 3 \int_{-2}^{3} \sqrt{9-x^2}dx|, i = 1, 2, 3. \)

Therefore, \( irr_k(W_{(1,3)}^\to) = 4(|\int_{-2}^{3} \sqrt{9-x^2}dx| = 4(\frac{1}{2}x\sqrt{9-x^2+\frac{\sqrt{3}}{2}\arcsin\frac{\pi}{3}})_{3} - |\frac{\sqrt{9} - 4 - \frac{\sqrt{3}}{2}\arcsin\frac{\pi}{3}}{2})| = 4\sqrt{5} + 9\pi + 18\arcsin\frac{\pi}{3}. \)

If \( n = 4 \) then \( irr_k(W_{(1,4)}^\to) = |\int_{-2}^{3} \sqrt{9-x^2}dx + 4 \int_{-2}^{3} \sqrt{9-x^2}dx|, i = 1, 2, 3, 4. \) Hence, the result follows.
Case 2: If \( n \geq 6 \) and even we have \( \text{irr}_k^e(W_{(1,n)}^+) = \left\lfloor \int_{-2}^{n} \sqrt{f_n^2 - x^2} dx \right\rfloor, i = 1, 2, ..., n \). So we have \( \text{irr}_k^e(W_{(1,n)}^+) = \left| \left( \frac{1}{2} x \sqrt{f_n^2 - x^2} + \frac{f_n^2}{2} \arcsin \frac{x}{f_n} \right) \right|_{f_n}^{3} + n \left( \frac{1}{2} x \sqrt{f_n^2 - x^2} + \frac{f_n^2}{2} \arcsin \frac{x}{f_n} \right) \right|_{f_n}^{3} - 2 \right| = \left| \frac{3}{2} \sqrt{f_n^2 - 9} + \frac{f_n^2}{2} \arcsin \frac{3}{f_n} - \left( \frac{f_n}{2} \sqrt{f_n^2 - f_n^2} + \frac{f_n^2}{2} \arcsin 1 \right) \right| + n \left( \frac{1}{2} \sqrt{f_n^2 - 9} + \frac{f_n^2}{2} \arcsin \frac{3}{f_n} - \left( \frac{f_n}{2} \sqrt{f_n^2 - f_n^2} + \frac{f_n^2}{2} \arcsin 1 \right) \right) + n \left( \frac{3}{2} \sqrt{f_n^2 - 9} + \frac{f_n^2}{2} \arcsin \frac{3}{f_n} + \sqrt{f_n^2 - 4} + \frac{f_n^2}{2} \arcsin \frac{2}{f_n} \right) \right| = \frac{3}{2} (n + 1) \sqrt{f_n^2 - 9} + (n + 1) \frac{f_n^2}{2} \arcsin \frac{3}{f_n} + \frac{f_n^2}{4} + A \right|, \text{ with } A = n \left( \sqrt{f_n^2 - 4} + \frac{f_n^2}{2} \arcsin \frac{2}{f_n} \right).

Case 3: Similar to Case 2 and accounting for \( n \geq 5 \) and uneven.

Consider the complete bipartite graph \( K_{(n,m)} \) and call the \( n \) vertices the \textit{left-side vertices} and the \( m \) vertices the \textit{right-side vertices}. Orientate \( K_{(n,m)} \) strictly from \textit{left-side vertices} to \textit{right-side vertices} to obtain \( K_{(n,m)}^l \leftrightarrow r \).

**Proposition 3.9.** For the directed graph \( K_{(n,m)}^l \leftrightarrow r \) we have that:

\[
\text{irr}_k^e(K_{(n,m)}^l \leftrightarrow r) = \begin{cases} 
    \left| \frac{n\pi}{4} - A \right|, & \text{if } n \geq f_m \text{ and } m \text{ is even,} \\
    \left| \frac{n\pi}{4} + A \right|, & \text{if } n \geq f_m \text{ and } m \text{ is uneven,} \\
    \left| B - \frac{f_m^2}{4} \pi \right|, & \text{if } f_m > n \text{ and } m \text{ is even,} \\
    \left| B + \frac{f_m^2}{4} \pi \right|, & \text{if } f_m > n \text{ and } m \text{ is uneven,}
\end{cases}
\]

with \( A = \frac{f_m}{2} \sqrt{n^2 - f_m^2} + \frac{n^2}{2} \arcsin \frac{f_m}{n} \) and \( B = \frac{n}{2} \sqrt{f_m^2 - n^2} + \frac{f_m^2}{2} \arcsin \frac{n}{f_m} \).

**Proof.** For the directed graph \( K_{(n,m)}^l \leftrightarrow r \) we have that all \textit{left-side vertices} say \( v_1, v_2, ..., v_n \) have \( d^+(v_i) = m \), whilst all \textit{right-side vertices} say \( u_1, u_2, ..., u_m \) have \( d^-(u_i) = n \) and \( d^+(u_i) = 0 \).
Case 1: Since $d^+(u_i) = 0, \forall i$ the terms in $\sum_{i=1}^{n} | \int_{f_i^+}^{d(v_i)} f(x)dx |$, stem from vertices $v_i, \forall i$ only.

Furthermore, since $r = \max\{d(u_i)\}_{i=1}^{n}, \text{or } f_m\}$ and $n \geq f_m$, we have $r = n$.

It follows that $\text{irr}_k(K_{(n,m)}^{1 \rightarrow r}) = n | \int_{f_m}^{n} \sqrt{n^2 - x^2}dx | = |(\frac{1}{2}x \sqrt{n^2 - x^2} + \frac{n^2}{2} \arcsin \frac{x}{n})|_{f_m}^{n} = |\frac{n^2}{2} \arcsin 1 - (\frac{f_m}{2} \sqrt{n^2 - f_m^2} + \frac{n^2}{2} \arcsin \frac{f_m}{n})| = |\frac{n^2}{4} - A|$, with $A = \frac{f_m}{2} \sqrt{n^2 - f_m^2} + \frac{n^2}{2} \arcsin \frac{f_m}{n}$.

Case 2: Similar to Case 1 and accounting for $m$ is uneven.

Case 3: Similar to Case 1 and accounting for $f_m > n, m$ is even.

Case 4: Similar to Case 1 and accounting for $f_m > n, m$ is uneven.

[Open problem: If possible, generalise Khazamula’s irregularity for simple directed graphs.]

[Open problem: Find a closed or, recursive formula for $f^\pm Z_1(J_n(1)), f^\pm Z_2(J_N(1)), f^\pm Z_3(J_n(1))$, and $f^\pm Z_4(J_n(1))$.]

[Open problem: Where possible, describe the terms of the Khazamula theorem in terms of $\text{irr}_k(G^\rightarrow)$ and $\text{irr}_k(H^\rightarrow)$ for specialised classes of simple directed graphs.]

[Open problem: If possible, formulate and prove Khazamula’s $c$-Theorem related to Khazamula $c$-irregularity for simple directed graphs in general.]

[Open problem: Let $G$ be a simple connected undirected graph on $n$ vertices labelled, $v_1, v_2, v_3, ..., v_n$. Also let $G$ have $\epsilon$ edges. It is known that $G$ can be $\text{orientated}$ in $2^\epsilon$ ways, including the cases of isomorphism. Find the relationship between the different values of $\text{irr}_k(G^\rightarrow)$ in respect of the different orientations.]

[Open problem: Let $G$ be a simple connected undirected graph on $n$ vertices labelled, $v_1, v_2, v_3, ..., v_n$. Also let $G$ have $\epsilon$ edges. It is known that $G$ can be $\text{orientated}$ in $2^\epsilon$ ways, including the cases of isomorphism. Find the relationship between the different values of $\text{irr}_k(G^\rightarrow)$ in respect of the different orientations.]

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