Complexity of oscillatory integrals on the real line

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Abstract We analyze univariate oscillatory integrals defined on the real line for functions from the standard Sobolev space $H^s(\mathbb{R})$ and from the space $C^s(\mathbb{R})$ with an arbitrary integer $s \geq 1$. We find tight upper and lower bounds for the worst case error of optimal algorithms that use $n$ function values. More specifically, we study integrals of the form

$$I^\varrho_k(f) = \int_{\mathbb{R}} f(x) e^{-i k x \varrho(x)} \, dx \quad \text{for} \quad f \in H^s(\mathbb{R}) \text{ or } f \in C^s(\mathbb{R})$$

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with \( k \in \mathbb{R} \) and a smooth density function \( \varrho \) such as \( \varrho(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \). The optimal error bounds are \( \Theta((n + \max(1, |k|))^{-s}) \) with the factors in the \( \Theta \) notation dependent only on \( s \) and \( \varrho \).

**Keywords** Oscillatory integrals · Complexity · Sobolev space

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1 Introduction

In the last decades, many papers have been published on the approximate computation of highly oscillatory univariate integrals over finite intervals, see the two surveys of Huybrechs and Olver [3], Milovanović and Stanić [4] and papers cited there. Our paper [5] belongs to this group of papers. We studied the integration interval [0, 1] and found tight lower and upper error bounds for algorithms that use \( n \) function values for periodic and nonperiodic functions from the standard Sobolev spaces \( H^s([0, 1]) \) with an integer \( s \geq 1 \). These results can be strengthened in the case \( s = 1 \), see [7].

For the case when the integration interval is unbounded, the literature is not so rich. We refer the readers to Blakemore, Evans and Hyslop [1], Chen [2] and Xu, Milovanović and Xiang [13] for pointers to the literature. However, we could not find any paper where tight lower error bounds were found.

The aim of this paper is to generalize results of [5] for oscillatory integrals of the form Eq. 1 defined over the real line for functions from the space \( H^s(\mathbb{R}) \) with smooth density functions such as the normal one. The main, and possibly surprising, result of this paper is that for the real line and the space \( H^s(\mathbb{R}) \), sharp error bounds for algorithms that use \( n \) function values are roughly the same as for the interval [0, 1] and the periodic space \( H^s([0, 1]) \). More precisely, they are of order \( (n + \max(1, |k|))^{-s} \).

We add in passing that sharp error bounds are higher for the density \( \varrho = l_{[0,1]} \) if we consider the whole class \( H^s([0, 1]) \) without additional conditions on the boundary.

To approximate the univariate oscillatory integrals (1), we use a smooth partition of unity, and reduce the integration problem over the whole real line to the case of the integration problem over finite intervals. The last problem could be solved by the change of variables and the use of the results from [5] for the integration domain [0, 1]. However this approach has some drawbacks. First of all, we assume in [5] that \( k \) is an integer, which is not required in this paper. We also used a slightly different norm of the space \( H^s([a,b]) \) than the more standard norm which is now used. Furthermore, the change of variables yields to larger factors in the upper error bounds. Finally, and more importantly, we present a new proof technique which is based on Poisson’s summation formula as the basic tool to obtain upper error bounds. That is why we decided to use this new approach and not to use the results from [5].

Sharp error bounds allow us to find sharp estimates on the information complexity which is defined as the minimal number of function values needed to find an algorithm with an error \( \varepsilon \cdot \text{CRI} \). Here, \( \varepsilon \) is a presumably small error threshold and \( \text{CRI} = 1 \).
when the absolute error criterion is used, and CRI is the initial error obtained by the zero algorithm when the normalized error criterion is used.

Consider first the absolute error criterion. The information complexity is then roughly $c_{s,\varrho} e^{-1/s} - \max(1, |k|)$ for some positive $c_{s,\varrho}$. Hence, large $|k|$ helps for not too small $\varepsilon$ and is irrelevant if $\varepsilon$ goes to zero.

Consider now the normalized error criterion. Then the information complexity is roughly $\max(1, |k|)(c_{s,\varrho} e^{-1/s} - 1)$ again for some positive $c_{s,\varrho}$. In this case, the information complexity is proportional to $\max(1, |k|)$ so that large $|k|$ hurts for all $\varepsilon < 1$.

The paper is organized as follows. In Section 2 some definitions and preliminaries are given. In Section 3 we study integration of functions with compact support, whereas in Section 4 we consider integration of functions defined over the real line. In both cases, we find matching lower and upper error bounds for algorithms that use $n$ function values.

2 Preliminaries

In this paper we consider real or complex valued functions defined on the whole real line $\Omega = \mathbb{R}$ or on an interval $\Omega = [a, b]$ with $-\infty < a < b < \infty$. Let $\langle \cdot, \cdot \rangle_{0,\Omega}$ be the usual inner product in $L^2(\Omega)$, i.e., $\langle f, g \rangle_{0,\Omega} := \int_{\Omega} f(x)g(x)\,dx$. We consider the standard Sobolev space $H^s(\Omega) = \{ f \in L^2(\Omega) \mid f^{(s-1)} \text{ is abs. cont.}, f^{(\ell)} \in L^2(\Omega) \text{ for } \ell = 0, 1, \ldots, s \}$ for $s \in \mathbb{N}$, which is equipped with the inner product

$$\langle f, g \rangle_{s,\Omega} = \sum_{\ell=0}^{s} \langle f^{(\ell)}, g^{(\ell)} \rangle_{0,\Omega} \quad \text{for } f, g \in H^s(\Omega).$$

The norm in $H^s(\Omega)$ is given by $\| f \|_{H^s(\Omega)} = \langle f, f \rangle_{s,\Omega}^{1/2}$.

We also consider the space $C^s(\Omega)$ of $s$ times continuously differentiable functions with the norm

$$\| f \|_{C^s(\Omega)} := \max_{\ell=0,1,\ldots,s} \sup_{x \in \Omega} |f^{(\ell)}(x)|.$$

Moreover, we consider functions with compact support and define the respective classes $H^s_0([a, b])$ and $C^s_0([a, b])$. More exactly, for functions $f \in H^s_0([a, b])$ we assume that $f \in H^s([a, b])$ and

$$f^{(\ell)}(a) = f^{(\ell)}(b) = 0 \quad \text{for } \ell = 0, 1, \ldots, s - 1,$$

and for functions $f \in C^s_0([a, b])$ we assume that $f \in C^s([a, b])$ and

$$f^{(\ell)}(a) = f^{(\ell)}(b) = 0 \quad \text{for } \ell = 0, 1, \ldots, s.$$

Given a nonzero and non-negative integrable function $\varrho: \Omega \to [0, \infty)$, we consider the approximation of oscillatory integrals of the form Eq. 1, i.e.,

$$I_k^\varrho(f) = \int_{\Omega} f(x) e^{-ikx} \varrho(x)\,dx, \quad i = \sqrt{-1},$$
where $k \in \mathbb{R}$ and $f \in F$, where $F \in \{H^s(\Omega), H_0^s(\Omega), C^s(\Omega), C_0^s(\Omega)\}$. Specific smoothness assumptions on $\varrho$ are given in the corresponding theorems. For $\Omega = \mathbb{R}$, these assumptions are satisfied for the normal density

$$\varrho(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \quad \text{for} \quad x \in \mathbb{R},$$

whereas for $\Omega = [a, b]$, we study $\varrho = 1_{[a,b]}$ which was already considered in [5] for $[a, b] = [0, 1]$ and an integer $k$.

For the approximation of $I_k^\varrho$ we consider algorithms that use $n$ function values. It is well known that linear algorithms $A_n$ are optimal in our setting, see e.g. [11] or [6], hence there is no need to study more general algorithms such as nonlinear or adaptive algorithms. The linear algorithms, or quadrature formulas, are of the form

$$A_n(f) = \sum_{j=1}^{n} a_j f(x_j),$$

where the coefficients $a_j$ and the nodes $x_j$ of course may depend on $\Omega, s, k, \varrho$ and $n$.

The aim of this paper is to prove upper and lower bounds on the $n$th minimal (worst case) errors

$$e(n, I_k^\varrho, F) := \inf_{A_n} \sup_{f \in F: \|f\|_F \leq 1} |I_k^\varrho(f) - A_n(f)|.$$

This number is the worst case error on the unit ball of $F$ of an optimal algorithm $A_n$ that uses at most $n$ function values for the approximation of the functional $I_k^\varrho$. The initial error is given for $n = 0$ when we do not sample the functions. In this case the best we can do is to take the zero algorithm $A_0(f) = 0$ and

$$e(0, I_k^\varrho, F) := \sup_{f \in F: \|f\|_F \leq 1} |I_k^\varrho(f)| = \|I_k^\varrho\|_F.$$

We are ready to define the information complexity, which is the minimal number $n$ of function values for which the $n$th minimal error of at most $\varepsilon$ CRI. Here, CRI = 1 if we consider the absolute error criterion, and CRI = $e(0, I_k^\varrho, F)$ if we consider the normalized error criterion. Hence, for the absolute error criterion the information complexity is defined as

$$n_{\text{abs}}(\varepsilon, I_k^\varrho, F) := \min\{n : e(n, I_k^\varrho, F) \leq \varepsilon\}, \quad (2)$$

while for the normalized error criterion the information complexity is defined as

$$n_{\text{nor}}(\varepsilon, I_k^\varrho, F) := \min\{n : e(n, I_k^\varrho, F) \leq \varepsilon e(0, I_k^\varrho, F)\}. \quad (3)$$

As already mentioned, our basic tool to derive upper error bounds will be the Poisson summation formula. We now remind the reader of this formula. For integrable functions $f$ on the whole real line, the Fourier transform of $f$ is defined by

$$[\mathcal{F} f](z) = \int_{\mathbb{R}} f(y) e^{-2\pi i z y} dy \quad \text{for} \quad z \in \mathbb{R}.$$

The study of quadrature rules with equidistant nodes can be done by Poisson’s summation formula, see e.g. [10, Thm. VII.2.4]. We state here only the univariate version.
Lemma 1 Let \( f \in L_1(\mathbb{R}) \) be continuous. Then its periodization
\[
g(x) := \sum_{m \in \mathbb{Z}} f(x + m) \quad \text{for } x \in \mathbb{R}
\]
converges in the norm of \( L_1([0, 1]) \). The resulting (1-periodic) function has the Fourier expansion
\[
g(x) = \sum_{z \in \mathbb{Z}} [\mathcal{F}f](z) e^{2\pi i zx} \quad \text{for } x \in \mathbb{R}.
\]
A consequence of Lemma 1 applied to the function \( h(x) = cf(cx)e^{-ikcx} \) for an integrable and continuous \( f \), real \( c \neq 0 \) and \( k \in \mathbb{R} \), and then taking \( g(0) \), yields
\[
c \sum_{x \in \mathbb{Z}} f(x) e^{-ikx} = c \sum_{m \in \mathbb{Z}} f(cm) e^{-ikm} = \sum_{m \in \mathbb{Z}} h(m) = \sum_{z \in \mathbb{Z}} [\mathcal{F}h](z) = \sum_{z \in \mathbb{Z}} \left[ \mathcal{F}f \right](\frac{z}{c} + \frac{k}{2\pi}),
\]
by a change of variable, see e.g. [12, Lemma 12]. In fact, to use Lemma 1 pointwise, we need that the periodization of \( f \), say \( g \), has a pointwise convergent Fourier series, e.g. we could assume that \( g \) is in the Wiener algebra. This assumption will be fulfilled in all cases considered.

Furthermore, we need the following lemma.

Lemma 2 Let \( s \geq 1 \). For every \( \Omega \subset \mathbb{R} \) we have
\[
(i) \quad \|fg\|_{H^s(\Omega)} \leq 2^s \|f\|_{H^s(\Omega)} \|g\|_{C^s(\Omega)} \quad \text{for } f \in H^s(\Omega) \text{ and } g \in C^s(\Omega),
\]
\[
(ii) \quad \|fg\|_{C^s(\Omega)} \leq 2^s \|f\|_{C^s(\Omega)} \|g\|_{C^s(\Omega)} \quad \text{for } f \in C^s(\Omega) \text{ and } g \in C^s(\Omega).
\]

Proof (i) Using the product rule and the Cauchy-Schwarz inequality we obtain
\[
\|fg\|^2_{H^s(\Omega)} = \sum_{\ell=0}^{s} \|f g(\ell)\|^2_{L_2(\Omega)} \leq \sum_{\ell=0}^{s} \left( \sum_{m=0}^{\ell} \left( \begin{array}{c} \ell \\ m \end{array} \right) \|f^{(m)}\|_{L_2(\Omega)} \|g^{(\ell-m)}\|_{L_2(\Omega)} \right)^2 \\
\leq \sum_{\ell=0}^{s} 2^\ell \left( \sum_{m=0}^{\ell} \left( \begin{array}{c} \ell \\ m \end{array} \right) \|f^{(m)}\|_{C^s(\Omega)} \|g^{(\ell-m)}\|_{C^s(\Omega)} \right) \\
\leq 2^s \|g\|^2_{C^s(\Omega)} \|f\|^2_{H^s(\Omega)} \left( \max_{m=0,\ldots,s} \sum_{\ell=m}^{s} \left( \begin{array}{c} \ell \\ m \end{array} \right) \right).
\]
This proves the bound since \( \sum_{\ell=m}^{s} \left( \begin{array}{c} \ell \\ m \end{array} \right) = \left( \begin{array}{c} s+1 \\ m+1 \end{array} \right) \leq 2^s \).

(ii) For \( \ell = 0, 1, \ldots, s \) we have
\[
(f g)^{(\ell)}(x) = \sum_{m=0}^{\ell} \left( \begin{array}{c} \ell \\ m \end{array} \right) f^{(m)}(x) g^{(\ell-m)}(x).
\]
Therefore
\[
\|fg\|_{C^s(\Omega)} \leq \|f\|_{C^s(\Omega)} \|g\|_{C^s(\Omega)} \sum_{m=0}^{\ell} \left( \begin{array}{c} \ell \\ m \end{array} \right) = \|f\|_{C^s(\Omega)} \|g\|_{C^s(\Omega)} 2^\ell,
\]

\( \square \) Springer
and
\[ \|fg\|_{C^s(\Omega)} \leq 2^s \|f\|_{C^s(\Omega)} \|g\|_{C^s(\Omega)}. \]

\[ \square \]

3 Functions with compact support

In this section we study the approximation of \( I_\varrho^s_k \) for functions from \( H_0^s(\Omega) \) and \( C_0^s(\Omega) \) with a bounded \( \Omega = [a, b] \) and \( \varrho = 1 \). In particular, we determine the dependence of the optimal error bounds on the length \( |\Omega| = b - a \) of the interval.

For \( k \in \mathbb{R} \), we now study the functional
\[ I_k(f) := \int_{\Omega} f(x) e^{-ikx} \, dx = \int_{\Omega} f(x) e^{-ikx} \, dx \quad \text{for} \quad f \in H_0^s(\Omega) \quad \text{or} \quad f \in C_0^s(\Omega). \]

First we find upper error bounds for the initial error and for a specific algorithm that uses \( n \) function values and whose error will be almost minimal. Then we provide matching lower bounds. Similar to [5, Prop. 3] we prove the following assertion.

**Proposition 3** The initial error of \( I_k \) satisfies
\[ e(0, I_k, H_0^s(\Omega)) \leq \frac{|\Omega|^{1/2}}{v_s(k)} \leq \frac{|\Omega|^{1/2}}{\bar{k}s} \]
and
\[ e(0, I_k, C_0^s(\Omega)) \leq \frac{|\Omega|}{\bar{k}s}, \]
with \( v_s(k) := \sqrt{1 + \sum_{\ell=1}^{s} k^{2\ell}} \) and \( \bar{k} = \max(1, |k|) \).

**Proof** Consider the function \( e_k(x) = e^{ikx} \). Then \( I_k(f) = \langle f, e_k^0,\Omega \rangle \) for every \( f \in L_1(\Omega) \). Integration by parts yields
\[ |k|^\ell \|\langle f, e_k^\ell,\Omega \rangle = \left| \left\langle f, e_k^\ell \right\rangle_{0,\Omega} \right| = \left| \left\langle f^\ell, e_k \right\rangle_{0,\Omega} \right| \]
for each \( \ell = 0, 1, \ldots, s \). Hence
\[ v_s(k)^2 |I_k(f)|^2 = \sum_{\ell=0}^{s} \left| \left\langle f^\ell, e_k \right\rangle_{0,\Omega} \right|^2 \leq \sum_{\ell=0}^{s} \| f^\ell \|_{L_2(\Omega)}^2 \| e_k \|_{L_2(\Omega)}^2 = |\Omega| \| f \|_{H^s(\Omega)}^2 \]
and
\[ \bar{k}s |I_k(f)| = \max_{\ell=0,\ldots,s} \left| \left\langle f^\ell, e_k \right\rangle_{0,\Omega} \right| \leq \max_{\ell=0,\ldots,s} \| f^\ell \|_{L_\infty(\Omega)} \| e_k \|_{L_1(\Omega)} = |\Omega| \| f \|_{C^s(\Omega)}. \]

Here we used that \( \bar{k}s = \max_{\ell=0,\ldots,s} |k|^\ell \), where by convention \( 0^0 = 1 \). Additionally, note that \( \bar{k}s \leq (1 + \sum_{\ell=1}^{s} k^{2\ell})^{1/2} \). This completes the proof. \[ \square \]
Remark 4 The upper bounds for the initial error can be proved analogously for the more general Sobolev spaces $W^{s,p}_0(\Omega)$ which are normed by

$$
\|f\|_{W^{s,p}_0(\Omega)} := \left( \sum_{\ell=0}^{s} \|f^{(\ell)}\|_{L^p(\Omega)}^p \right)^{1/p}.
$$

The upper bound would be $\bar{k}^{-s} |\Omega|^{1-1/p}$.

We now turn to the definition of an algorithm which uses $n$ function values and whose error is, as we prove it later, almost minimal. For $n \geq 1$, define $c_n := |\Omega|/n$ and the algorithm

$$
A_n^\Omega(f) = c_n \sum_{x \in (c_n \mathbb{Z}) \cap \Omega} f(x) e^{-ikx} \quad \text{for all } f \in H^s_0(\Omega). \quad (5)
$$

Note that $x \in (c_n \mathbb{Z}) \cap \Omega$ means that $x = c_n j \in [a, b]$ for some integer $j$. Or equivalently,

$$
\frac{j}{n} \in \left[ \frac{a}{b-a} \frac{b}{b-a} \right].
$$

The number of such $j$ is clearly at most $n + 1$. In fact, it can be $n + 1$ only if $a/(b-a) = m/n$ for some integer $m$. In this case, $A_n^\Omega(f)$ uses one function value on the left and one on the right boundary of $\Omega$. Since functions from $f \in H^s_0(\Omega)$ are zero at these points, they can be omitted from the summation. Hence, the number of function values used by the algorithm $A_n^\Omega$ is at most $n$. We now prove the following error bound for $A_n^\Omega$ for a relatively large $n$, whereas the case of small $n$ will be considered later.

Theorem 5 Let $s \in \mathbb{N}$, $k \in \mathbb{R}$ with $\bar{k} = \max(1, |k|)$, and $\Omega = [a, b] \subset \mathbb{R}$. The algorithm $A_n^\Omega(f)$ from Eq. 5 satisfies

(i) for each $f \in H^s_0(\Omega)$ and $n \geq (2\pi)^{-1}(1 + |k|) |\Omega|$:

$$
|I_k(f) - A_n^\Omega(f)| \leq \frac{2}{(2\pi)^s} \frac{|\Omega|^{1/2}}{(n/|\Omega| - |k|/2\pi)^s} \|f\|_{H^s(\Omega)}.
$$

(ii) for each $f \in H^s_0(\Omega)$, $\alpha \in [1/3, 1)$ and $n \geq \left[(1 + \alpha)/(1 - \alpha)\right] (2\pi)^{-1} \bar{k} |\Omega|$:

$$
|I_k(f) - A_n^\Omega(f)| \leq \frac{2}{(2\pi \alpha)^s} \frac{|\Omega|^{1/2}}{(n/|\Omega| + \bar{k}/2\pi)^s} \|f\|_{H^s(\Omega)}.
$$

(iii) for each $f \in C^s_0(\Omega)$, $\alpha \in [1/3, 1)$ and $n \geq \left[(1 + \alpha)/(1 - \alpha)\right] (2\pi)^{-1} \bar{k} |\Omega|$:

$$
|I_k(f) - A_n^\Omega(f)| \leq \frac{2}{(\sqrt{2\pi \alpha})^s} \frac{|\Omega|}{(n/|\Omega| + \bar{k}/2\pi)^s} \|f\|_{C^s(\Omega)}.
$$

Proof Let $f \in H^s_0(\Omega)$. Then we can rewrite (5) as

$$
A_n^\Omega(f) = c_n \sum_{x \in c_n \mathbb{Z}} f(x)e^{-ikx}.
$$
Using Eq. 4 we have
\[ A_n^{\Omega}(f) = \sum_{z \in (1/c_n)\mathbb{Z}} [\mathcal{F}f](z + \frac{k}{2\pi}) = \sum_{z \in \mathbb{Z}} [\mathcal{F}f]\left(\frac{zn}{|\Omega|} + \frac{k}{2\pi}\right). \]
Noting that \( I_k(f) = \mathcal{F}f(k/2\pi) \) we have
\[ |I_k(f) - A_n^{\Omega}(f)| = \left| \sum_{z \in \mathbb{Z}\setminus\{0\}} [\mathcal{F}f]\left(\frac{zn}{|\Omega|} + \frac{k}{2\pi}\right) \right|. \]
Define
\[ v_{k,n}^{s}(j) := \left(1 + \sum_{\ell=1}^{s} \left|2\pi \frac{jn}{|\Omega|} + k\right|^{1/2}\right). \] (6)
We bound the error by
\[ |I_k(f) - A_n^{\Omega}(f)| = \left| \sum_{j \in \mathbb{Z}\setminus\{0\}} [v_{k,n}^{s}(j)]^{-1} v_{k,n}^{s}(j) [\mathcal{F}f]\left(\frac{jn}{|\Omega|} + \frac{k}{2\pi}\right) \right| \leq \left( \sum_{j \in \mathbb{Z}\setminus\{0\}} [v_{k,n}^{s}(j)]^{-2} \right)^{1/2} \left( \sum_{j \in \mathbb{Z}\setminus\{0\}} [v_{k,n}^{s}(j)]^2 \right)^{1/2} \left|\mathcal{F}f\left(\frac{jn}{|\Omega|} + \frac{k}{2\pi}\right)\right|^2 \].
We first bound the second factor. Integrating by parts yields
\[ \left|\mathcal{F}f\left(\frac{jn}{|\Omega|} + \frac{k}{2\pi}\right)\right|^2 = \left(2\pi \frac{jn}{|\Omega|} + k\right)^{-2\ell} \left|\mathcal{F}[D^{\ell}f]\left(\frac{jn}{|\Omega|} + \frac{k}{2\pi}\right)\right|^2 \]
for all \( \ell = 0, 1, \ldots, s \). Summing up with respect to \( \ell \), we obtain
\[ [v_{k,n}^{s}(j)]^2 \left|\mathcal{F}f\left(\frac{jn}{|\Omega|} + \frac{k}{2\pi}\right)\right|^2 = \sum_{\ell=0}^{s} \left|\mathcal{F}[D^{\ell}f]\left(\frac{jn}{|\Omega|} + \frac{k}{2\pi}\right)\right|^2. \] (7)
Since \( c_n = |\Omega|/n \), from Eq. 7 we obtain
\[ \sum_{j \in \mathbb{Z}\setminus\{0\}} [v_{k,n}^{s}(j)]^2 \left|\mathcal{F}f\left(\frac{jn}{|\Omega|} + \frac{k}{2\pi}\right)\right|^2 = \sum_{\ell=0}^{s} \left|\mathcal{F}[D^{\ell}f]\left(\frac{j}{c_n} + \frac{k}{2\pi}\right)\right|^2 \]
\[ = \sum_{\ell=0}^{s} \sum_{j \in \mathbb{Z}\setminus\{0\}} \left|\int_{\mathbb{R}} D^{\ell}f(y) e^{-ik\gamma} e^{-2\pi i j y} dy\right|^2 \]
\[ = \sum_{\ell=0}^{s} \sum_{j \in \mathbb{Z}\setminus\{0\}} \left|c_n \int_{\mathbb{R}} D^{\ell}f(c_n x) e^{-ikc_n x} e^{-2\pi ij x} dx\right|^2 \]
\[ = \sum_{\ell=0}^{s} \sum_{j \in \mathbb{Z}\setminus\{0\}} \left|c_n \sum_{m \in \mathbb{Z}} \int_{0}^{1} D^{\ell}f(c_n(x + m)) e^{-ikc_n(x+m)} e^{-2\pi ij x} dx\right|^2. \]
Define the function
\[ g_{\ell,n}(x) = c_n \sum_{m \in \mathbb{Z}} D^{\ell}f(c_n(x + m)) e^{-ikc_n(x+m)} \quad \text{for } x \in [0, 1], \]
and note that for each fixed \( x \) the number of non-zero terms in the last series is
\[ |\{m \in \mathbb{Z} : c_n(x + m) \in \Omega\}| = \left|\mathbb{Z} \cap \left(n\frac{\Omega}{|\Omega|} - x\right)\right| \leq n. \]
We now show that \( g_{\ell,n} \in L_2([0, 1]). \) Indeed,
\[ |g_{\ell,n}(x)|^2 \leq c_n^2 n \sum_{m \in \mathbb{Z}} |D^{\ell}f(c_n(x + m))|^2, \]
and
\[
\int_0^1 |g_{\ell,n}(x)|^2 \, dx \leq c_n^2 n \int_\mathbb{R} |D^\ell_f(c_n x)|^2 \, dx = c_n n \int_\mathbb{R} |D^\ell f(x)|^2 \, dx < \infty,
\]
since \( f \in H_0^s(\Omega) \) implies that \( D^\ell f \in L_2(\mathbb{R}) \) for all \( \ell = 0, 1, \ldots, s \). Hence, \( g_{\ell,n} \in L_2([0,1]) \), as claimed.

By Parseval’s theorem and the Cauchy-Schwarz inequality we have
\[
\sum_{\nu \in \mathbb{Z}} v_{k,n}(\nu)^2 \left| (\mathcal{F} f)\left( \frac{j_n}{2\pi} + \frac{k}{2\pi} \right) \right|^2 = \sum_{\nu \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left| f_0^{(1)} g_{\ell,n}(x) e^{-2\pi ij x} \right|^2 \\
\leq \sum_{\nu = 0}^\infty \sum_{j = 0}^{\infty} \left| f_0^{(1)} g_{\ell,n}(x) \right|^2 \, dx \\
\leq \sum_{\nu = 0}^\infty c_n n \int_\mathbb{R} |D^\ell f(x)|^2 \, dx = |\Omega| \| f \|^2_{H^s(\Omega)}.
\]

We now bound the first factor in the estimate of \( |I_k(f) - A_n^\Omega(f)| \). For this assume that \( n \geq (2\pi)^{-1} (1+|k|) |\Omega| \). Then \( 2\pi n / |\Omega| \geq 1 + |k| \). Since \( v_{k,n}(j) \geq 2\pi j n / |\Omega| + |k|^s \), we have
\[
\sum_{j \in \mathbb{Z}} v_{k,n}(j)^{-2} \leq \sum_{j \in \mathbb{Z}} 2 \sum_{j = 1}^\infty \left( 2 \pi j n / |\Omega| - |k| \right)^{-2s} \\
= 2 \left( 2 \pi n / |\Omega| - |k| \right)^{-2s} + 2 \sum_{j = 1}^\infty \left( 2 \pi j n / |\Omega| - |k| \right)^{-2s} \\
\leq 4 \left( 2 \pi n / |\Omega| - |k| \right)^{-2s}.
\]

This proves (i).

Let \( n \geq [(1+\alpha)/(1-\alpha)] (2\pi)^{-1} \bar{k} |\Omega| \) for \( \alpha \in [1/3, 1) \). Then we have \( n / |\Omega| - |k| \geq \alpha(n / |\Omega| + \bar{k}) \) and \( \bar{k} (1+\alpha)/(1-\alpha) \geq 1 + |k| \). Since now \( n \geq (1 + |k|) |\Omega| / (2\pi) \), (i) easily yields (ii). For (iii) we simply use \( \| f \|^2_{H^s(\Omega)} \leq (s+1)|\Omega| \| f \|^2_{C^s(\Omega)} \) and \( \sqrt{s+1} \leq 2^{s/2} \). This completes the proof.

**Remark 6** It should be noted that we could prove the same upper bounds also for the case of “periodic” functions. This means for all functions \( f \in H^s([a,b]), \) or \( C^s([a,b]) \), such that \( f(x) e^{-ikx} \) is periodic with period \( |\Omega| \). However, we omit it since this leads to some technicalities and is not needed later.

Before we state the final result on the \( n \)th minimal errors, including matching lower bounds, we present a modification of the algorithm \( A_n^\Omega \) that satisfies good error bounds also for small \( n \). For small \( n \), this algorithm, which we denote by \( A_n^\Omega \), simply uses no information of the function \( f \) and outputs zero. Although this seems artificial, it is known, at least in special cases, that for small \( n \) the zero algorithm outperforms \( A_n^\Omega \), see [5, Thm. 4(ii)]. More precisely, we define
\[
A_n^\Omega(f) = \begin{cases} 
0 & \text{if } n < \frac{1}{\bar{k}} |\Omega|, \\
A_n^\Omega(f) & \text{if } n \geq \frac{1}{\bar{k}} |\Omega|.
\end{cases}
\]

Theorem 5 immediately implies the following error bound on \( A_n^\Omega \).
Corollary 7 For all \( n, s \in \mathbb{N} \) and \( k \in \mathbb{R} \) with \( \tilde{k} = \max(1, |k|) \), the algorithm \( A_n^\Omega \) satisfies

\[
|I_k(f) - A_n^\Omega(f)| \leq \frac{2}{2^s} \frac{|\Omega|^{1/2}}{(n/|\Omega| + \tilde{k}/2\pi)^s} \|f\|_{H^s(\Omega)} \quad \text{for } f \in H_0^s(\Omega),
\]

\[
|I_k(f) - A_n^\Omega(f)| \leq \frac{2}{2^{s/2}} \frac{|\Omega|^{1/2}}{(n/|\Omega| + \tilde{k}/2\pi)^{s/2}} \|f\|_{C^s(\Omega)} \quad \text{for } f \in C_0^s(\Omega).
\]

Proof Assume first that \( n \geq \frac{1}{\pi} \tilde{k} |\Omega| \). In this case the upper bounds follow from Theorem 5 (ii) and (iii) with \( \alpha = 1/3 \). It remains to consider the case \( n < \frac{1}{\pi} \tilde{k} |\Omega| \).

Then we have \( \tilde{k} > 2(n/|\Omega| + \tilde{k}/2\pi) \), and therefore for \( f \in H_0^s(\Omega) \) with \( \|f\|_{H^s(\Omega)} \leq 1 \),

\[
|I_k(f) - A_n^\Omega(f)| \leq e(0, I_k, H_0^s(\Omega)) \leq \frac{|\Omega|^{1/2}}{\tilde{k}^s} \leq \frac{1}{2^s} \frac{|\Omega|^{1/2}}{(n/|\Omega| + \tilde{k}/2\pi)^s},
\]

as claimed. Again, we use \( \|f\|_{H^s(\Omega)}^2 \leq 2^s |\Omega| \|f\|_{C^s(\Omega)}^2 \) for the second bound. This completes the proof.

This enables us to give sharp bounds on the \( n \)th minimal error.

Theorem 8 Let \( k \in \mathbb{R} \) with \( \tilde{k} = \max(1, |k|) \). Consider the integration problem \( I_k \) defined for functions from the spaces \( H_0^s(\Omega) \) or \( C_0^s(\Omega) \) with \( \Omega = [a, b] \) and \( s \in \mathbb{N} \). Then there exist numbers \( c_s \in (0, 1/2^{s-1}] \) and \( \tilde{c}_s \in (0, 1/2^{(s-2)/2}] \) such that for every \( n \in \mathbb{N}_0 \) there are numbers \( d_s = d_s(n, k) \) and \( \tilde{d}_s = \tilde{d}_s(n, k) \) such that

\[
d_s \in [c_s, 1/2^{s-1}] \quad \text{and} \quad \tilde{d}_s \in [\tilde{c}_s, 1/2^{(s-2)/2}]
\]

and

\[
e(n, I_k, H_0^s(\Omega)) = d_s \frac{|\Omega|^{1/2}}{(n/|\Omega| + \tilde{k}/2\pi)^s}, \quad \text{(i)}
\]

\[
e(n, I_k, C_0^s(\Omega)) = \tilde{d}_s \frac{|\Omega|}{(n/|\Omega| + \tilde{k}/2\pi)^s}. \quad \text{(ii)}
\]

Moreover, the lower bounds hold for all algorithms that use at most \( n \) function or derivative (up to order \( s - 1 \)) values.

Proof The upper bound follows from Corollary 7. The proof of the lower bound is the same as the proof of Theorem 9 in [5] with two minor modifications. First, there are now about \( \frac{1}{\pi} |k||\Omega| \) roots of the oscillatory weight \( \cos(kx) \) in \( \Omega \) and, secondly, the fooling function \( f \) that is constructed there satisfies \( \|f\|_{H^s(\Omega)} = \Theta(|\Omega|^{1/2}) \) or \( \|f\|_{C^s(\Omega)} = \Theta(1) \). \( \square \)

We stress that the last bounds are sharp with respect to \( n, \tilde{k} \) and \( |\Omega| \) as well as with respect to the convergence rate. The only part which is not sharp involves factors which depend on \( s \). However, even the upper bounds on \( d_s \) and \( \tilde{d}_s \) are exponentially small in \( s \).
From Theorem 8 we easily obtain sharp estimates on the information complexities defined by Eqs. 2 and 3.

**Corollary 9** Let $k \in \mathbb{R}$ with $\bar{k} = \max(1, |k|)$. Consider the integration problem $I_k$ defined for the spaces $H_0^s(\Omega)$ or $C_0^s(\Omega)$ with $\Omega = [a, b]$ and $s \in \mathbb{N}$. Then for any positive $\varepsilon$

- there exist $\alpha_s = \alpha_s(\varepsilon, k) \in [c_s, 2^{-(s-1)}]$ and $\beta_s = \beta_s(\varepsilon, k) \in [c_s(2\pi)^s, 1/(c_s 2^{s-1})]$ with $c_s$ given in Theorem 8 such that
  \[
  n_{abs}(\varepsilon, I_k, H_0^s(\Omega)) = \left|\Omega\right| \left(\frac{\left|\alpha_s\right|\left|\Omega\right|^{1/2}}{\varepsilon} - \frac{\bar{k}}{2\pi}\right)^{1/s} + 1,
  \]
  \[
  n_{nor}(\varepsilon, I_k, H_0^s(\Omega)) = \left|\Omega\right| \frac{\bar{k}}{2\pi} \left(\left(\frac{\beta_s}{\varepsilon}\right)^{1/s} - 1\right) + 1
  \]
- there exist $\tilde{\alpha}_s = \tilde{\alpha}_s(\varepsilon, k) \in [\tilde{c}_s, 2^{-(s-2)/2}]$ and $\tilde{\beta}_s = \tilde{\beta}_s(\varepsilon, k) \in [\tilde{c}_s(2\pi)^s, 1/(\tilde{c}_s 2^{(s-2)/2})]$ with $\tilde{c}_s$ given in Theorem 8 such that
  \[
  n_{abs}(\varepsilon, I_k, C_0^s(\Omega)) = \left|\Omega\right| \left(\frac{\left|\tilde{\alpha}_s\right|\left|\Omega\right|^{1/2}}{\varepsilon} - \frac{\bar{k}}{2\pi}\right)^{1/s} + 1,
  \]
  \[
  n_{nor}(\varepsilon, I_k, C_0^s(\Omega)) = \left|\Omega\right| \frac{\bar{k}}{2\pi} \left(\left(\frac{\tilde{\beta}_s}{\varepsilon}\right)^{1/s} - 1\right) + 1
  \]

**Proof** The results for the absolute error criterion, i.e. the bounds on $n_{abs}$, are obvious from Theorem 8. In view of Eq. 3 the information complexity for the normalized error criterion is given by $n_{nor}(\varepsilon, I_k, F) = n_{abs}(\varepsilon \cdot e(0, I_k, F), I_k, F)$ for $F \in \{H_0^s(\Omega), C_0^s(\Omega)\}$. From Theorem 8 (for $n = 0$) and Proposition 3 we know that $e(0, I_k, H_0^s(\Omega)) \in \left[c_s \left|\Omega\right|^{1/2}(\bar{k}/2\pi)^{-s}, \left|\Omega\right|^{1/2}\bar{k}^{-s}\right]$ and $e(0, I_k, C_0^s(\Omega)) \in \left[\tilde{c}_s \left|\Omega\right|^{1/2}(\bar{k}/2\pi)^{-s}, \left|\Omega\right|^{1/2}\bar{k}^{-s}\right]$. Putting this in the bounds on $n_{abs}$ this shows $(2\pi)^s \alpha_s \leq \beta_s \leq c_s^{-1} \alpha_s$ and $(2\pi)^s \tilde{\alpha}_s \leq \tilde{\beta}_s \leq c_s^{-1} \tilde{\alpha}_s$ and proves the claim.

The formulas in Corollary 9 can be simplified when $\varepsilon$ goes to zero. Then we have

\[
\begin{align*}
  n_{abs}(\varepsilon, I_k, H_0^s(\Omega)) &= \Theta\left(\frac{\left|\Omega\right|^{1+1/(2s)}}{\varepsilon^{1/s}}\right), \\
  n_{nor}(\varepsilon, I_k, H_0^s(\Omega)) &= \Theta\left(\frac{\left|\Omega\right| \bar{k}}{\varepsilon^{1/s}}\right), \\
  n_{abs}(\varepsilon, I_k, C_0^s(\Omega)) &= \Theta\left(\frac{\left|\Omega\right|^{1+1/s}}{\varepsilon^{1/s}}\right), \\
  n_{nor}(\varepsilon, I_k, C_0^s(\Omega)) &= \Theta\left(\frac{\left|\Omega\right| \bar{k}}{\varepsilon^{1/s}}\right),
\end{align*}
\]

where the factors in the $\Theta$ notations depend only on $s$.
We stress that we now have sharp dependence on \( \varepsilon, |\Omega| \) and \( \bar{k} \). In all cases the dependence on \( \varepsilon \) is through \( \varepsilon^{-1/s} \), whereas the dependence on \( |\Omega| \) and \( \bar{k} \) varies and is different for the spaces \( H^s_0(\Omega) \) and \( C^s_0(\Omega) \) as well as it depends on the error criterion. For the absolute error criterion there is no asymptotic dependence on \( \bar{k} \), however, for large \( \bar{k} \) we have to wait longer to see this asymptotic dependence. For the normalized error criterion, the information complexity of the integration problem \( I_k \) is roughly the same for \( H^s_0(\Omega) \) and \( C^s_0(\Omega) \) and the dependence on \( |\Omega| \) and \( \bar{k} \) is through \( |\Omega| \bar{k} \). In this case, large \( \bar{k} \) hurts.

Observe that the dependence on the size of \( |\Omega| \) with \( |\Omega| > 1 \) is more severe for the absolute than for the normalized error criterion, however, for large \( s \) this difference disappears. For small \( |\Omega| < 1 \), the opposite holds and the absolute error criterion is easier than the normalized error criterion.

### 4 Functions on the real line

We now consider the approximation of integrals of the form

\[
I_k^\rho(f) = \int_{\mathbb{R}} f(x) e^{-i k x \rho(x)} \, dx \quad \text{for} \quad f \in H^s(\mathbb{R}) \text{ or } f \in C^s(\mathbb{R})
\]

with a sufficiently smooth density function \( \rho \). The primal example of such \( \rho \) is the normal density

\[
\rho(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \quad \text{for} \quad x \in \mathbb{R}.
\]

We establish conditions on \( \rho \) such that the optimal error bounds are of the order \( (n + \bar{k})^{-s} \), just as in the case for a bounded interval with functions of compact support, see Theorem 8.

We need the notion of a smooth partition of unity. We call a family \( \{g_m\}_{m \in \mathbb{Z}} \) of functions a smooth partition of unity if \( g_m \in C^\infty(\mathbb{R}) \) for all \( m \in \mathbb{Z} \) and

\[
\sum_{m \in \mathbb{Z}} g_m(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}.
\]

In this section we use a partition of unity with a specific structure. Namely, we choose a (fixed) nonnegative function \( g \in C^\infty(\mathbb{R}) \) such that

\[
\text{supp}(g) = [-1, 1] \quad \text{and} \quad g(x) + g(x - 1) = 1 \quad \text{for} \quad x \in [0, 1],
\]

and define the functions

\[
g_m(x) = g(x - m) \quad \text{for} \quad m \in \mathbb{Z} \quad \text{and} \quad x \in \mathbb{R}.
\]

Such functions \( g \) obviously exist; consider for example the up-function defined in [8]. This function is defined by

\[
\frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} \prod_{k=1}^\infty \sin \left( \frac{t2^{-k}}{t2^{-k}} \right) dt.
\]
it is a solution of the equation
\[ y'(x) = 2y(2x + 1) - 2y(2x - 1) \]
with compact support.

Based on such a partition of unity \( \{ g_m \}_{m \in \mathbb{Z}} \), we define the partition of the density \( \varrho \) by \( \{ \varrho_m \}_{m \in \mathbb{Z}} \) with \( \varrho_m = g_m \cdot \varrho \). Clearly, \( \text{supp}(\varrho_m) = [m - 1, m + 1] \), \( \sum_{m \in \mathbb{Z}} \varrho_m = \varrho \), and therefore
\[ I_k^0(f) = \sum_{m \in \mathbb{Z}} I_k^{0m}(f) = \sum_{m \in \mathbb{Z}} I_k(f \varrho_m). \tag{9} \]
Note that \( f \varrho_m \) is now a function with support \( \Omega_{1m} \), \( \varrho \) belongs to \( H_{s}((\Omega_{1m})) \), whereas for the space \( C_{s}((\Omega_{1m})) \) we assume that \( \varrho \) is \( s \) times continuously differentiable and its derivatives decay exponentially fast to zero if its argument goes to infinity. Therefore Eqs. 11 and 12 hold.

The algorithm for the approximation of \( I_k^0(f) \) we are going to analyze is based on a suitable distribution of the \( n \) function evaluations over the real line. For this, let
\[ p = \{ p_m \}_{m \in \mathbb{Z}} \]
be a family of real numbers, to be specified in a moment, such that
\[ p_m \in [0, 1] \quad \text{for all} \quad m \quad \text{and} \quad \sum_{m \in \mathbb{Z}} p_m = 1. \tag{10} \]
For the space \( H^{s}(\mathbb{R}) \) we assume that \( \varrho \in C^{s}(\mathbb{R}) \) and that the sequence \( \{ \| \varrho \|_{C^{s}(\Omega_{m})} \}_{m \in \mathbb{Z}} \) belongs to \( \ell_{1/(s+1/2)} \), i.e.,
\[ \sum_{m \in \mathbb{Z}} \| \varrho \|_{C^{s}(\Omega_{m})}^{1/(s+1/2)} < \infty, \tag{11} \]
whereas for the space \( C^{s}(\mathbb{R}) \) we assume that \( \varrho \in H^{s}(\mathbb{R}) \) and \( \{ \| \varrho \|_{H^{s}(\Omega_{m})} \}_{m \in \mathbb{Z}} \) is in every \( \ell_{p} \), \( p > 0 \), due to its exponential decay. Therefore Eqs. 11 and 12 hold.

In particular, this is true for the normal density \( \varrho(x) = 1/\sqrt{2\pi \sigma^2} \exp(-x^2/(2\sigma^2)) \) with the standard deviation \( \sigma > 0 \). In this case, we can estimate the norms \( \| \varrho \|_{C^{s}(\Omega_{m})} \) for \( m \in \mathbb{Z} \) by Cramer’s bound which states that
\[ \| \varrho \|_{C^{s}(\Omega_{m})} \leq (2\pi)^{-1/4} \sigma^{-1} \sqrt{\lambda} e^{-(\tilde{m} - 1)^2/(4\sigma^2)}, \]
with \( \tilde{m} = \max(1, |m|) \), see e.g. [9, p. 324]. Clearly, the sequence \( \{ \| \varrho \|_{C^{s}(\Omega_{m})} \}_{m \in \mathbb{Z}} \) (and hence the sequence \( \{ \| \varrho \|_{H^{s}(\Omega_{m})} \}_{m \in \mathbb{Z}} \)) is in every \( \ell_{p} \), \( p > 0 \), due to its exponential decay. Therefore Eqs. 11 and 12 hold.

Note that due to Lemma 2 (ii), (11) implies that
\[ \| \varrho \|_{C^{s}(\Omega_{m})} \leq 2^{1/(s+1/2)} \| \varrho \|_{C^{s}(\mathbb{R})}^{1/(s+1/2)} \sum_{m \in \mathbb{Z}} \| \varrho \|_{C^{s}(\Omega_{m})}^{1/(r+1/2)} < \infty, \]
whereas due to Lemma 2 (i), (12) implies that
\[ \rho_{Hs} := \sum_{m \in \mathbb{Z}} \| \rho_m \|_{H^s(\Omega_m)}^{1/(s+1)} \| \varrho \|_{C^s(\mathbb{R})} \sum_{m \in \mathbb{Z}} \| \varrho_m \|_{H^s(\Omega_m)} < \infty. \]

Then
\[ p_m^s = \frac{\| \rho_m \|_{C^s(\Omega_m)}^{1/(s+1/2)}}{\rho_{C^s}} \quad \text{for } H^s(\mathbb{R}), \tag{13} \]
\[ p_m^s = \frac{\| \rho_m \|_{H^s(\Omega_m)}^{1/(s+1)}}{\rho_{H^s}} \quad \text{for } C^s(\mathbb{R}), \tag{14} \]
are well defined and satisfy (10).

We are ready to define the algorithm for approximating \( I_{\varrho k} \) by
\[ A_{n,p}(f) = \sum_{m \in \mathbb{Z}} A_{n_m}^{\varrho_m}(f \cdot \varrho_m) \quad \text{with } n_m := \lfloor p_m n \rfloor, \tag{15} \]
where \( n \in \mathbb{N}_0, f \in H^s(\mathbb{R}) \) or \( f \in C^s(\mathbb{R}) \), and \( A_{n_m}^{\varrho_m} \) is given by Eq. 8. Note that \( f \in H^s(\mathbb{R}) \) implies that \( f \varrho_m \in H^s(\Omega_m) \) and \( f \in C^s(\mathbb{R}) \) implies that \( f \varrho_m \in C^s(\Omega_m) \). Hence \( A_{n_m}^{\varrho_m}(f \cdot \varrho_m) \) is well defined. Note also that only finitely many \( n_m \) are nonzero. Therefore almost all \( n_m = 0 \), and since \( A_{n}^{\varrho_m} = 0 \) the series in Eq. 15 has only finitely many nonzero terms. Hence, \( A_{n,p} \) is well defined. We now estimate the error of \( A_{n,p} \).

**Theorem 10** Assume that Eq. 11 holds if we consider the space \( H^s(\mathbb{R}) \), and Eq. 12 if we consider the space \( C^s(\mathbb{R}) \). Let \( A_{n,p} \) be given by Eq. 15 for \( n \in \mathbb{N}_0 \) with \( p = \{ p_m \}_{m \in \mathbb{Z}} \) from Eqs. 13 and 14, respectively, and let \( k = \max(1, |k|) \). Then
\[ |I_{\varrho k}(f) - A_{n,p}(f)| \leq \sum_{m \in \mathbb{Z}} |I_k(f \cdot \varrho_m) - A_{n_m}^{\varrho_m}(f \cdot \varrho_m)|, \]
where we need to express the right hand side of this inequality in terms of \( p_m n \) instead of \( n_m \). For \( n_m = \lfloor p_m n \rfloor < \frac{2k}{\pi} \) we have \( A_{n_m}^{\varrho_m} = 0 \), see Eq. 8. Note that now \( p_m n + \bar{k} < \frac{2k}{\pi} + 1 + \bar{k} \leq \bar{k}(2/\pi + 2) \leq 3\bar{k} \). Since \( |\Omega_m| = 2 \), we obtain from Proposition 3 that
\[ \frac{|I_k(f \cdot \varrho_m) - A_{n_m}^{\varrho_m}(f \cdot \varrho_m)|}{\| f \cdot \varrho_m \|_{H^s(\Omega_m)}} \leq \frac{\sqrt{2}}{\bar{k}^{s}} \leq \frac{2^{1/2}3^{s}}{(p_m n + \bar{k})^{s}}. \tag{16} \]
For $n_m = \lfloor p_m n \rfloor \geq 2\hat{k}/\pi$, we cannot have $n_m = 0$. Therefore $n_m \geq 1$. We also have $n_m \geq p_m n/\pi$. Indeed, it is true if $p_m n \leq \pi$ and for $p_m n > \pi$ we have $n_m \geq p_m n - 1 \geq p_m n/\pi$. Hence $n_m + \hat{k}/\pi \geq \frac{1}{\pi}(p_m n + \hat{k})$. This and Corollary 7 yield

\[
I_k^\theta(f \cdot \varphi_m) - A_{n_m}^{\Omega_m}(f \cdot \varphi_m) \leq \frac{2^{3/2} \pi_s}{(p_m n + \hat{k})^s}.
\] (17)

Due to Eq. 16, the last inequality (17) holds for all $n$. Since $(p_m n + \hat{k})^s \geq p_m^s (n + \hat{k})^s$, we have

\[
|I_k^\theta(f) - A_n.p(f)| \leq \sum_{m \in \mathbb{Z}} \frac{2^{3/2} \pi_s}{(p_m n + \hat{k})^s} \|f \cdot \varphi_m\|_{H^s(\Omega_m)} \leq \frac{2^{3/2} \pi_s}{(n + \hat{k})^s} \sum_{m \in \mathbb{Z}} \|f \cdot \varphi_m\|_{H^s(\Omega_m)}. \tag{18}
\]

It remains to bound the last sum. We first prove the result for $f \in H^s(\mathbb{R})$. In this case, $p_m = \varphi^{-1}_C \varphi_m \|\varphi_m\|_{C^s(\Omega_m)}$ as in Eq. 13. Clearly,

\[
\sum_{m \in \mathbb{Z}} \|f\|_{H^s(\Omega_m)}^2 = 2 \|f\|_{H^s(\mathbb{R})}^2.
\]

By Lemma 2(i), we obtain

\[
\sum_{m \in \mathbb{Z}} \frac{\|f \cdot \varphi_m\|_{H^s(\Omega_m)}}{p_m^s} \leq 2^s \sum_{m \in \mathbb{Z}} \frac{1}{p_m} \|\varphi_m\|_{C^s(\Omega_m)} \|f\|_{H^s(\Omega_m)} \leq 2^s \left( \sum_{m \in \mathbb{Z}} \frac{\|\varphi_m\|_{C^s(\Omega_m)}^2}{p_m^s} \right)^{1/2} \left( \sum_{m \in \mathbb{Z}} \|f\|_{H^s(\Omega_m)}^2 \right)^{1/2} = 2^{s+1/2} \|f\|_{H^s(\mathbb{R})} \left( \sum_{m \in \mathbb{Z}} \frac{\|\varphi_m\|_{C^s(\Omega_m)}^2}{p_m^s} \right)^{1/2} = 2^{s+1/2} \varphi^{1/s}_C \|f\|_{H^s} \left( \sum_{m \in \mathbb{Z}} \|\varphi_m\|_{C^s(\Omega_m)}^{1/(s+1)} \right)^{1/2}.
\]

With Eq. 18 this leads to

\[
|I_k^\theta(f) - A_n.p(f)| \leq \frac{\tilde{C}_{s,\ell} \|f\|_{H^s(\mathbb{R})}}{(n + \hat{k})^s},
\]

where $\tilde{C}_{s,\ell} = 4(2\pi)^s \varphi^{1/s+1}_C$, and proves the first estimate.

For $f \in C^s(\mathbb{R})$, we have $p_m = \varphi^{-1}_H \|\varphi_m\|_{H^s(\Omega_m)}$ as in Eq. 14, and Lemma 2 (i) yields

\[
\sum_{m \in \mathbb{Z}} \frac{\|f \cdot \varphi_m\|_{H^s(\Omega_m)}}{p_m^s} \leq 2^s \sum_{m \in \mathbb{Z}} \frac{1}{p_m} \|\varphi_m\|_{C^s(\Omega_m)} \|f\|_{H^s(\Omega_m)} \leq 2^s \|f\|_{C^s(\mathbb{R})} \sum_{m \in \mathbb{Z}} \frac{\|\varphi_m\|_{H^s(\Omega_m)}^2}{p_m^s} = (2\varphi_H)^s \|f\|_{C^s} \sum_{m \in \mathbb{Z}} \|\varphi_m\|_{H^s}^{1/(s+1)} = 2^s \varphi^{1+1/s}_H \|f\|_{C^s(\mathbb{R})}.
\]

Hence

\[
|I_k^\theta(f) - A_n.p(f)| \leq \frac{C_{s,\ell} \|f\|_{C^s(\mathbb{R})}}{(n + \hat{k})^s}
\]

with $C_{s,\ell} = 2^{3/2}(2\pi)^s \varphi^{1+1/s}_H$. This proves the second estimate and completes the proof. \hfill $\square$
We are ready to present sharp bounds on the $n$th minimal errors.

**Theorem 11** Consider the integration problem $I_k^\rho$ defined over the spaces $H^s(\mathbb{R})$ or $C^s(\mathbb{R})$ with $s \in \mathbb{N}$ and a nonzero density $\rho$. Then for every $n \in \mathbb{N}_0$ and $k \in \mathbb{R}$ we have

\[
e(n, I_k^\rho, H^s(\mathbb{R})) = \Theta((n + \bar{k})^{-s}) \quad \text{if} \quad \{\|\rho\|_{C^s(\Omega_m)}\}_{m \in \mathbb{Z}} \in \ell^1(1/(s+1/2)),
\]

\[
e(n, I_k^\rho, C^s(\mathbb{R})) = \Theta((n + \bar{k})^{-s}) \quad \text{if} \quad \{\|\rho\|_{H^s(\Omega_m)}\}_{m \in \mathbb{Z}} \in \ell^1(1/(s+1)),
\]

where the factors in the $\Theta$ notation depend only on $s$ and $\rho$. As always, $\bar{k} = \max(1, |k|)$.

**Proof** The proof of lower bounds can be done as in the proof of Theorem 9 in [5].

We only use the fact that $\rho$ is continuous and different than zero and conclude that

\[
e(n, I_k^\rho, F) \ge c_{\rho,s}(n + \bar{k})^{-s} \quad \text{for} \quad F = H^s(\mathbb{R}) \text{ and } F = C^s(\mathbb{R}).
\]

Note that for $n = 0$ we have a lower bound on the initial error. The upper bounds are attained by the algorithm $A_{n,p}$ whose error is bounded in Theorem 10. □

The assumptions on $\rho$ in Theorem 11 for $H^s(\mathbb{R})$ and $C^s(\mathbb{R})$ differ in the conditions on the decay of $\rho$ at infinity. One reason for this difference is that the space $C^s(\mathbb{R})$ does not guarantee any integrability property. We did not try to find optimal conditions on $\rho$.

The essence of Theorem 11 is that if $\rho$ is smooth enough and decays fast enough at infinity, we see that the $n$th minimal errors for the integration problem $I_k^\rho$ for the spaces $H^s(\mathbb{R})$ and $C^s(\mathbb{R})$ are of the same order and may be different only in the dependence on $s$ in the factors in the $\Theta$ notation.

We now rewrite Theorem 11 in terms of the information complexities similarly as it was done in Corollary 9.

**Corollary 12** Consider the integration problem $I_k$ for $k \in \mathbb{R}$ defined for the space $F \in \{H^s(\mathbb{R}), C^s(\mathbb{R})\}$ with $s \in \mathbb{N}$. Then

\[
n_{abs}(\varepsilon, I_k, F) = \left\lceil \Theta\left(\left(\frac{1}{\varepsilon}\right)^{1/s} - \bar{k}\right) \right\rceil,
\]

\[
n_{nor}(\varepsilon, I_k, F) = \left\lceil \bar{k}\left(\Theta\left(\left(\frac{1}{\varepsilon}\right)^{1/s} - 1\right) \right) \right\rceil,
\]

where the factors in the $\Theta$ notations depend only on $s$ and $\rho$. □

Hence, modulo the dependence on $s$ in the factors in $\Theta$ notations, the information complexities for the spaces $H^s(\mathbb{R})$ and $C^s(\mathbb{R})$ are the same. The parameter $\bar{k}$ helps for the absolute error criterion although it does not change the asymptotic behaviour of the information complexity when $\varepsilon$ goes to zero. The parameter $\bar{k}$ plays a different role for the normalized error criterion since the information complexity is now proportional to $\bar{k}$.
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