ON THE GEOMETRY OF CONSTANT ANGLE SURFACES IN $\text{Sol}_3$

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Abstract. In this paper we classify all surfaces in the 3-dimensional Lie group $\text{Sol}_3$ whose normals make constant angle with a left invariant vector field.

1. Preliminaries

The space $\text{Sol}_3$ is a simply connected homogeneous 3-dimensional manifold whose isometry group has dimension 3 and it is one of the eight models of geometry of Thurston [15]. As Riemannian manifold, the space $\text{Sol}_3$ can be represented by $\mathbb{R}^3$ equipped with the metric

$$\tilde{g} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$$

where $(x,y,z)$ are canonical coordinates of $\mathbb{R}^3$. The space $\text{Sol}_3$, with the group operation

$$(x, y, z) \ast (x', y', z') = (x + e^{-z}x', y + e^z y', z + z')$$

is a unimodular, solvable but not nilpotent Lie group and the metric $\tilde{g}$ is left-invariant. See e.g. [2, 15]. With respect to the metric $\tilde{g}$ an orthonormal basis of left-invariant vector fields is given by

$$e_1 = e^{-z} \frac{\partial}{\partial x}, \quad e_2 = e^z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$ 

The following transformations

$$(x, y, z) \mapsto (y, -x, -z) \quad \text{and} \quad (x, y, z) \mapsto (-x, y, z)$$

span a group of isometries of $(\text{Sol}_3, \tilde{g})$ having the origin as fixed point. This group is isomorphic to the dihedral group (with 8 elements) $D_4$. It is, in fact, the complete group of isotropy [15]. The other elements of the group are $(x, y, z) \mapsto (-x, -y, z)$, $(x, y, z) \mapsto (-y, x, -z)$, $(x, y, z) \mapsto (y, x, -z)$,
(x, y, z) ⇔ (y, x, z) and (x, y, z) ⇔ (x, −y, z). They can be unified as follows (cf. [11]):

\[ (x, y, z) \mapsto (\pm e^{-c}x + a, \pm e^{c}y + b, z + c), \]

\[ (x, y, z) \mapsto (\pm e^{c}y + a, \pm e^{c}x + b, z + c). \]

It is well known that the isometry group of \( \text{Sol}_3 \) has dimension three.

The Levi Civita connection \( \tilde{\nabla} \) of \( \text{Sol}_3 \) with respect to \( \{e_1, e_2, e_3\} \) is given by

\[
\begin{align*}
\tilde{\nabla}_{e_1}e_1 &= -e_3 & \tilde{\nabla}_{e_1}e_2 &= 0 & \tilde{\nabla}_{e_1}e_3 &= e_1 \\
\tilde{\nabla}_{e_2}e_1 &= 0 & \tilde{\nabla}_{e_2}e_2 &= e_3 & \tilde{\nabla}_{e_2}e_3 &= -e_2 \\
\tilde{\nabla}_{e_3}e_1 &= 0 & \tilde{\nabla}_{e_3}e_2 &= 0 & \tilde{\nabla}_{e_3}e_3 &= 0.
\end{align*}
\]

We recall the Gauss and Weingarten formulas

\[
\begin{align*}
(G) \quad & \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \\
(W) \quad & \tilde{\nabla}_X N = -AX
\end{align*}
\]

for every \( X \) and \( Y \) tangent to \( M \) and for any \( N \) unitary normal to \( M \). By \( A \) we denote the shape operator on \( M \).

**2. Constant angle surfaces in \( \text{Sol}_3 \) - general things**

**2.1. Motivation.** Constant angle surfaces were recently studied in product spaces \( \mathbb{Q}_\epsilon \times \mathbb{R} \), where \( \mathbb{Q}_\epsilon \) denotes the sphere \( \mathbb{S}^2 \) (when \( \epsilon = +1 \)), the Euclidean plane \( \mathbb{E}^2 \) (when \( \epsilon = 0 \)), respectively the hyperbolic plane \( \mathbb{H}^2 \) (when \( \epsilon = -1 \)). See e.g. [3, 1, 9, 4]. The angle is considered between the unit normal of the surface \( M \) and the tangent direction to \( \mathbb{R} \).

It is known, for \( \text{Sol}_3 \), that \( \mathcal{H}^1 = \{dy = 0\} \) and \( \mathcal{H}^2 = \{dx = 0\} \) are totally geodesic foliations whose leaves are the hyperbolic plane (thought as the upper half plane model).

On the other hand, for \( \mathbb{Q}_\epsilon \times \mathbb{R} \), the foliation \( \{dt = 0\} \) is totally geodesic too (\( t \) is the global parameter on \( \mathbb{R} \)). Trivial examples for constant angle surfaces in \( \mathbb{Q}_\epsilon \times \mathbb{R} \) are furnished by totally geodesic surfaces \( \mathbb{Q}_\epsilon \times \{t_0\} \).

Let us consider \( \mathcal{H}^2 \). It follows that the tangent plane to \( \mathbb{H}^2 \) (the leaf at each \( x = x_0 \)) is spanned by \( \frac{\partial}{\partial y} \) and \( \frac{\partial}{\partial x} \), while the unit normal is \( e_1 \). So, this surface corresponds to \( \mathbb{Q}_\epsilon \times \{t_0\} \), case in which the constant angle is 0. Due to these reasons we give the following definition:

An oriented surface \( M \), isometrically immersed in \( \text{Sol}_3 \), is called \textit{constant angle surface} if the angle between its normal and \( e_1 \) is constant in each point of the surface \( M \).

**2.2. First computations.** Denote by \( \theta \in [0, \pi) \) the angle between the unit normal \( N \) and \( e_1 \). Hence

\[ \tilde{g}(N, e_1) = \cos \theta. \]
Let $T$ be the projection of $e_1$ on the tangent plane $T_pM$ of $M$ in a point $p \in M$. Thus

$$e_1 = T + \cos \theta N.$$  

Case $\theta = 0$. Then $N = e_1$ and hence the surface $M$ is isometric to the hyperbolic plane $\mathcal{H}^2 = \{dx = 0\}$.

From now on we will exclude this case.

**Lemma 2.1.** If $X$ is tangent to $M$ we have

1. $\nabla_X e_1 = -\tilde{g}(X, e_1)e_3, \nabla_X e_2 = \tilde{g}(X, e_2)e_3$
2. $AT = -\tilde{g}(N, e_3)T$, hence $T$ is a principal direction on the surface
3. $g(T, T) = \sin^2 \theta$.

At this point we have to decompose also $e_2$ and $e_3$ into the tangent and the normal parts, respectively.

Let $E_1 = \frac{1}{\sin \theta} T$. Consider $E_2$ tangent to $M$, orthogonal to $E_1$ and such that the basis $\{e_1, e_2, e_3\}$ and $\{E_1, E_2, N\}$ have the same orientation. It follows that

$$\begin{aligned}
    e_1 &= \sin \theta E_1 + \cos \theta N \\
    e_2 &= \cos \alpha \cos \theta E_1 + \sin \alpha E_2 - \cos \alpha \sin \theta N \\
    e_3 &= -\sin \alpha \cos \theta E_1 + \cos \alpha E_2 + \sin \alpha \sin \theta N
\end{aligned}$$

and

$$\begin{aligned}
    E_1 &= \sin \theta e_1 + \cos \theta \cos \alpha e_2 - \cos \theta \sin \alpha e_3 \\
    E_2 &= \sin \alpha e_2 + \cos \alpha e_3 \\
    N &= \cos \theta e_1 - \sin \theta \cos \alpha e_2 + \sin \theta \sin \alpha e_3
\end{aligned}$$

where $\alpha$ a smooth function on $M$.

Case $\theta = \frac{\pi}{2}$. In this case $e_1$ is tangent to $M$ and $T = E_1$.

The metric connection on $M$ is given by

$$\nabla_{E_1} E_1 = -\cos \alpha E_2, \nabla_{E_2} E_1 = 0$$

$$\nabla_{E_1} E_2 = \cos \alpha E_1, \nabla_{E_2} E_2 = 0.$$ 

The second fundamental form is obtained from

$$h(E_1, E_1) = -\sin \alpha N, \ h(E_1, E_2) = 0, \ h(E_2, E_2) = \sigma N$$

where $\sigma$ is a smooth function on $M$.

Writing the Gauss formula (G) for $X = E_1$ and $Y = E_2$, respectively for $X = Y = E_2$ one obtains

$$E_1(\alpha) = 0 \quad \text{and} \quad E_2(\alpha) = \sin \alpha - \sigma.$$

**Remark 2.2.** The surface $M$ is minimal if and only if $\sigma = \sin \alpha$. Since $E_1$ and $E_2$ are linearly independent, it follows that $\alpha$ is constant. Moreover, $M$ is totally geodesic if and only if $\alpha = 0$, case in which $M$ coincides with $\mathcal{H}^1$.  

Due to the fact that the Lie brackets of \( E_1 \) and \( E_2 \) is \([E_1, E_2] = \cos \alpha \, E_1\), one can choose local coordinates \( u \) and \( v \) such that
\[
E_2 = \frac{\partial}{\partial u} \quad \text{and} \quad E_1 = \beta(u,v) \frac{\partial}{\partial v}.
\]
This choice implies \( \alpha \) and \( \beta \) fulfill the following PDE:
\[
\beta_u = -\beta \cos \alpha.
\]
Since \( \alpha \) depends only on \( u \), it follows
\[
\beta(u,v) = \rho(v) \, e^{-\int u \cos \alpha(\tau) d\tau}
\]
where \( \rho \) is a smooth function depending on \( v \).

Denote by
\[
F : U \subset \mathbb{R}^2 \rightarrow M \hookrightarrow \text{Sol}_3
\]
\[
(u,v) \mapsto (F_1(u,v), F_2(u,v), F_3(u,v))
\]
the immersion of the surface \( M \) in \( \text{Sol}_3 \).

We have
\[
\begin{align*}
\frac{\partial}{\partial u} &= F_u = (F_{1,u}, F_{2,u}, F_{3,u}) \\
&= E_2 = (\sin \alpha \, e_2 + \cos \alpha \, e_3)|_{F(u,v)} = \left(0, \, e^{F_3(u,v)} \sin \alpha, \, \cos \alpha\right) \\
\frac{\partial}{\partial v} &= F_v = (F_{1,v}, F_{2,v}, F_{3,v}) \\
&= \frac{1}{\beta} \, E_1 = \frac{1}{\beta} \, e_1|_{F(u,v)} = \left(\frac{1}{\beta} \, e^{-F_3(u,v)}, \, 0, \, 0\right).
\end{align*}
\]

It follows
\[
\begin{align*}
F_1 &= F_1(v) & \quad \partial_u F_1 &= \frac{1}{\rho(u,v)} \, e^{-F_3(u,v)} \\
\partial_u F_2 &= \sin \alpha(u) e^{F_3(u,v)} & \quad F_2 &= F_2(u) \\
\partial_u F_3 &= \cos \alpha(u) & \quad F_3 &= F_3(u).
\end{align*}
\]

Thus we obtain
\[
\begin{align*}
F_1(v) &= \int^v \frac{1}{\rho(\tau)} \, d\tau \\
F_2(u) &= \int^u \left(\sin \alpha(\tau) e^{\int^\tau \cos \alpha(s) ds}\right) d\tau \\
F_3(u) &= \int^u \cos \alpha(\tau) d\tau.
\end{align*}
\]

Changing the \( v \) parameter, one gets the following parametrization
\[
F(u,v) = \left(v, \, \phi(u), \, \chi(u)\right)
\]
which represents a cylinder over the plane curve \( \gamma(u) = (0, \, \phi(u), \, \chi(u)) \) where
\[
\phi(u) = \int^u \left(\sin \alpha(\tau) e^{\int^\tau \cos \alpha(s) ds}\right) d\tau \quad \text{and} \quad \chi(u) = \int^u \cos \alpha(\tau) d\tau.
\]
Notice that the surface is the group product between the curve \( v \mapsto (v, \, 0, \, 0) \) and the curve \( \gamma \).

Let us see how the curve \( \gamma \) looks like for different values of the function \( \alpha \):
ON THE GEOMETRY OF CONSTANT ANGLE SURFACES IN $Sol_3$

**a:** $\alpha$ is a constant:
\[
\gamma(u) = (0, \tan \alpha \ e^u \cos \alpha, \ u \cos \alpha)
\]

**b:** $\alpha(s) = s$
\[
\gamma(u) = \left(0, \int_u^s \sin s \ e^{\sin s} ds, \ \sin u\right)
\]

**c:** $\alpha(s) = s^2$
\[
\gamma(u) = \left(0, \int_u^s \sin^2 s \ e^{\cos \tau} d\tau ds, \ \int_u^s \cos s^2 ds\right)
\]

**d:** $\alpha(s) = \arccos(s)$, $s \in [-1, 1]$
\[
\gamma(u) = \left(0, \int_u^s \sqrt{1 - s^2} \ e^s ds, \ u\right)
\]

**e:** $\alpha(s) = 2 \arctan e^{2s}$ In this case, the expression of $\gamma$ involve hypergeometric functions. The surface $M$ is totally umbilical but not totally geodesic.

![Figure 1](image-url)  
**Figure 1.** Items: b, c, d and e

Coming back to the general case for $\theta$, we distinguish some particular situations for $\alpha$:
In order to obtain explicit embedding equations for the surface and hence the conclusion.

\[ \sin \alpha = 0. \] Then \( \cos \alpha = \pm 1 \) and the principal curvature corresponding to the principal direction \( T \) vanishes. Straightforward computations yield the case which was discussed before.

\[ \cos \alpha = 0. \] Then \( \sin \alpha = \pm 1 \) and the relations (1) and (3) may be written in an easier way, namely, for \( \sin \alpha = 1 \) we have

\[ e_1 = \sin \theta E_1 + \cos \theta N, \quad e_2 = E_2, \quad e_3 = -\cos \theta E_1 + \sin \theta N \]
\[ E_1 = \sin \theta e_1 - \cos \theta e_3, \quad E_2 = e_2, \quad N = \cos \theta e_1 + \sin \theta e_3. \]

The Levi Civita connection \( \nabla \) on the surface \( M \) is given by

\[ \nabla_{E_1}E_1 = 0, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_2}E_1 = \cos \theta E_2, \quad \nabla_{E_2}E_2 = -\cos \theta E_1. \]

**Remark 2.3.** Such surface is minimal.

**Proof.** Computing the second fundamental form, one obtains

\[ h(E_1, E_1) = -\sin \theta N, \quad h(E_1, E_2) = 0, \quad h(E_2, E_2) = \sin \theta N \]

and hence the conclusion. \( \square \)

In order to obtain explicit embedding equations for the surface \( M \) let us choose local coordinates as follows:

Let \( u \) be such that \( E_1 = \frac{\partial}{\partial u} \) and \( v \) such that \( E_2 \) and \( \frac{\partial}{\partial v} \) are collinear. This can be done due the fact that \( [E_1, E_2] = -\cos \theta E_2 \). Considering \( \frac{\partial}{\partial v} = b(u, v) E_2 \), with \( b \) a smooth function on \( M \), since \( \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] = 0 \), it follows that \( b \) satisfies \( b_u - b \cos \theta = 0 \). This PDE has the general solution \( b(u, v) = \mu(v) e^{u \cos \theta} \), with \( \mu \) a smooth function defined on certain interval in \( \mathbb{R} \).

Denote by \( F = (F_1, F_2, F_3) \) the isometric immersion of the surface \( M \) in \( Sol_3 \). We have

\[ (i) \quad \frac{\partial}{\partial u} = F_u = (\partial_u F_1, \partial_u F_2, \partial_u F_3) \]
\[ = E_1 = \sin \theta e_1|_{F(u,v)} - \cos \theta e_3|_{F(u,v)} = (\sin \theta e^{-F_3(u,v)}, 0, -\cos \theta) \]

\[ (ii) \quad \frac{\partial}{\partial v} = F_v = (\partial_v F_1, \partial_v F_2, \partial_v F_3) \]
\[ = \mu(v) e^{u \cos \theta} E_2 = \mu(v) e^{u \cos \theta} e_2|_{F(u,v)} \]
\[ = (0, \mu(v) e^{u \cos \theta + F_3(u,v)}, 0). \]

Looking at (i) we immediately get

- the third component: \( F_3(u, v) = -u \cos \theta + \zeta(v), \quad \zeta \in C^\infty(M) \)
- the second component: \( F_2(u, v) = F_2(v) \).

Replacing in (ii) we obtain

- the third component: \( \zeta(v) = \zeta_0 \in \mathbb{R} \)
- the second component: \( F_2(v) = e^{\zeta_0} \int_0^v \mu(\tau) \, d\tau \)
- the first component: \( F_1(u, v) = F_1(u) \).
Going back in (i) and taking the first component one gets
\[ F_1(u) = e^{-\zeta_0} \tan \theta e^{u \cos \theta} + \text{constant}. \]
Since the map \((x, y, z) \mapsto (x+c, y, z)\) is an isometry for \(\text{Sol}_3\), we can take the previous constant to be 0. Moreover, the map \((x, y, z) \mapsto (e^{-c}x, e^c y, z + c)\) is also an isometry of the ambient space, so \(\zeta_0\) may be assumed to be also 0. Consequently, one obtains the following parametrization for the surface \(M\)
\[ F(u, v) = \left( \tan \theta e^{u \cos \theta}, \int^v \mu(\tau) d\tau, -u \cos \theta \right). \]
Finally, we can change the parameter \(v\) such that \(\mu(v) = 1\). One can state the following

**Proposition 2.4.** The surface \(M\) given by the parametrization
\[ F(u, v) = \left( \tan \theta e^{u \cos \theta}, v, -u \cos \theta \right) \]
is a constant angle surface in \(\text{Sol}_3\).

Notice that this surface is a (group) product between the curve \(v \mapsto (0, v, 0)\) and the plane curve \(\gamma(u) = (\tan \theta e^{u \cos \theta}, 0, -u \cos \theta)\).

The angle \(\theta\) is an arbitrary constant. Moreover, the curvature of \(M\) is a negative constant \(-\cos^2 \theta\). Analogue results are obtained if \(\cos \alpha = -1\).

From now on we will deal with \(\alpha\) and \(\theta\) different from the situations above.

**Lemma 2.5.** The Levi Civita connection \(\nabla\) on \(M\) and the second fundamental form \(h\) are given by
\[ \nabla_E_1 E_1 = - \cos \alpha E_2, \quad \nabla_E_1 E_2 = \cos \alpha E_1 \]
\[ \nabla_E_2 E_1 = \sigma \cot \theta E_2, \quad \nabla_E_2 E_2 = -\sigma \cot \theta E_1 \]
\[ h(E_1, E_1) = -\sin \theta \sin \alpha N, \quad h(E_1, E_2) = 0, \quad h(E_2, E_2) = \sigma N. \]

The matrix of the Weingarten operator \(A\) with respect to the basis \(\{E_1, E_2\}\) has the following expression
\[ A = \begin{pmatrix} -\sin \alpha \sin \theta & 0 \\ 0 & \sigma \end{pmatrix} \]
for a certain function \(\sigma \in C^\infty(M)\).

Moreover, the Gauss formula yields
\[ E_1(\alpha) = 2 \cos \theta \cos \alpha \]
\[ E_2(\alpha) = \sin \alpha - \frac{\sigma}{\sin \theta} \]
and the compatibility condition
\[ (\nabla E_1 E_2 - \nabla E_2 E_1)(\alpha) = [E_1, E_2](\alpha) = E_1(E_2(\alpha)) - E_2(E_1(\alpha)) \]
gives rise to the following differential equation
\[ E_1(\sigma) + \sigma \cos \theta \sin \alpha + \sigma^2 \cot \theta = 2 \sin \theta \cos \sin^2 \alpha. \]

**Remark 2.6.** The curvature of \( M \) is equal to \( 2 \sin^2 \alpha \sin^2 \theta - \sigma \sin \alpha \sin \theta - 1 \).

We are looking for a coordinate system \((u,v)\) in order to determine the embedding equations of the surface. Let us take the coordinate \( u \) such that \( \frac{\partial}{\partial u} = E_1 \). Concerning \( v \), we will discuss later about it.

Let our attention on (7.a) which can be re-written as
\[ \partial_\alpha = 2 \cos \theta \cos \alpha. \]

Solving this PDE one gets
\[ \sin \alpha = \tanh(2u \cos \theta + \psi(v)) \]
where \( \psi \) is a smooth function on \( M \) depending on \( v \). Notice that, apparently the equation has also a second solution \( \sin \alpha = \coth(2u \cos \theta + \psi(v)) \). This is not valid because \( \coth \) takes values in \((-\infty, -1)\) or in \((1, +\infty)\).

Now, let us take \( v \) in such way that \( \frac{\partial \alpha}{\partial v} = 0 \), namely \( \psi \) is a constant, denote it by \( \psi_0 \). It follows that \( \alpha \) is given by
\[ \sin \alpha = \tanh(\bar{u}) \]
where \( \bar{u} = 2u \cos \theta + \psi_0 \).

At this point, the equation (8) becomes
\[ \sigma_u + \cot \theta (\sigma + 2 \sin \alpha \sin \theta)(\sigma - \sin \alpha \sin \theta) = 0. \]

Since \( \frac{\partial}{\partial v} \) is tangent to \( M \), it can be decomposed in the basis \( \{E_1, E_2\} \). Thus, there exist functions \( a = a(u,v) \) and \( b = b(u,v) \) such that
\[ \frac{\partial}{\partial v} = a E_1 + b E_2. \]

Due to the choice of the coordinate \( v \) we have
\[ 0 = \frac{\partial \alpha}{\partial v} = a \cdot 2 \cos \theta \cos \alpha + b \left( \sin \alpha - \frac{\sigma}{\sin \theta} \right). \]

**a.** The case \( b = 0 \) implies \( \cos \theta = 0 \) or \( \cos \alpha = 0 \). Both situations were studied separately.

**b.** Consider \( b \neq 0 \). Let us denote by \( p(u,v) = \frac{\sigma}{\sin \theta} \). Hence the equality above yields
\[ \sigma = \sin \theta \sin \alpha + p \sin 2 \theta \cos \alpha. \]

On the other hand
\[ 0 = \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] = a_u E_1 + b_u E_2 + b \left( \cos \alpha E_1 - \sigma \cot \theta E_2 \right). \]

Hence
\[ \begin{cases} a_u + b \cos \alpha = 0 \\ b_u - b \sigma \cot \theta = 0. \end{cases} \]
If we take in (11) the derivative with respect to $u$, and combining with (10), it follows

$$p_u + \cos \alpha + p \cos \theta \sin \alpha + 2p^2 \cos^2 \theta \cos \alpha = 0. \tag{13}$$

Straightforward computations yield the general solution for this equation (see the Appendix), namely

$$p(u, v) = \pm \frac{1}{\cos \theta \sinh \bar{u} + \varepsilon - \frac{\cosh \frac{1}{2} \bar{u}}{I(u) + \Lambda(v)}} \tag{14}$$

where $\varepsilon = 0, 1$ and $\Lambda$ is a certain function depending on $v$.

Let $F : U \subset \mathbb{R}^2 \rightarrow M \rightarrow \text{Sol}_3$, $(u, v) \mapsto (F_1(u, v), F_2(u, v), F_3(u, v))$ be the immersion of the surface $M$ in $\text{Sol}_3$. We have

I. $\partial_u F = (F_{1,u}, F_{2,u}, F_{3,u})$

$$= E_1 = \sin \theta \ e_{1|F(u,v)} + \cos \theta \ e_{2|F(u,v)} - \cos \theta \ e_{3|F(u,v)}$$

$$= \left(\sin \theta \ e^{-F_3(u,v)}, \ \cos \theta \ e_{F_3(u,v)}, \ -\cos \theta \ \sin \alpha \right) \tag{15.a}$$

which implies

$$\partial_u F_1 = \sin \theta \ e^{-F_3(u,v)} \tag{15.b}$$

$$\partial_u F_2 = \cos \theta \ \cos \alpha \ e^{F_3(u,v)} \tag{15.c}$$

From the last equation one immediately obtains

$$F_3(u, v) = -\frac{1}{2} \log \cosh(\bar{u}) + \zeta(v) \tag{16}$$

where $\zeta$ is a smooth function. Replacing this expression in (15.a) and (15.b), one gets

$$F_1 = \sin \theta \ e^{-\zeta(v)} \left(I(u) + f_1(v)\right) \tag{17}$$

$$F_2 = \pm \cos \theta \ e^{\zeta(v)} \left(J(u) + f_2(v)\right) \tag{18}$$

where $I(u) = \int_{0}^{u} \sqrt{\cosh(2\tau \cos \theta + \psi_0)} d\tau$, $J(u) = \int_{0}^{u} \cosh^{-\frac{1}{2}}(2\tau \cos \theta + \psi_0) d\tau$

and $f_1, f_2$ are some smooth functions which will be determined in what follows.

II. $\partial_v F = (F_{1,v}, F_{2,v}, F_{3,v})$

$$= a(u,v) E_1 + b(u,v) E_2$$

$$= a(u,v) \left(\sin \theta \ e_{1|F(u,v)} + \cos \theta \ e_{2|F(u,v)} - \cos \theta \ \sin \alpha \ e_{3|F(u,v)}\right) +$$

$$b(u,v) \left(\sin \alpha \ e_{2|F(u,v)} + \cos \alpha \ e_{3|F(u,v)}\right).$$
It follows

\[ \partial_v F_1 = a(u, v) \sin \theta \ e^{-F_3(u, v)} \]  
\[ \partial_v F_2 = \left( a(u, v) \cos \alpha \right) e^{F_3(u, v)} \]  
\[ \partial_v F_3 = -a(u, v) \cos \theta \sin \alpha + b(u, v) \cos \alpha. \]

From (16) and (19.c) we have

\[ -a(u, v) \cos \theta \sin \alpha + b(u, v) \cos \alpha = \zeta'(v) \]

and from (17) and (19.a) we obtain

\[ \zeta'(v) \left( I(u) + f_1(v) \right) - f_1'(v) + a(u, v) \sqrt{\cosh(\bar{u})} = 0. \]

Taking the derivative with respect to \( u \), one gets

\[ \zeta'(v) + a_u(u, v) + a(u, v) \cos \theta \tanh(\bar{u}) = 0. \]

The equation in \( a \) has the solution

\[ a(u, v) = \frac{-\zeta'(v)I(u) + \xi(v)}{\sqrt{\cosh(\bar{u})}}. \]

Recall that \( p(u, v) = \frac{a(u, v)}{\delta(u, v)} \). We immediately notice that the general solution given by (14) is obtained with the following identification: \( \varepsilon = 0 \iff \zeta'(v) = 0 \) and \( \varepsilon = 1 \iff \Lambda(v) = \frac{\xi(v)}{\zeta'(v)} \). It follows

\[ p(u, v) = \pm \frac{1}{\cos \theta \sinh(\bar{u}) + \frac{\zeta'(v) \cosh(\bar{u})}{\zeta'(v)I(u) + \xi(v)}}. \]

At this point we will obtain the parametrization of the surface in the following way.

1. Combining \[22\] with \[20\] one gets \( f_1'(v) - \zeta'(v)f_1(v) - \xi(v) = 0 \) which has the solution \( f_1(v) = e^{\zeta(v)} \int_{\tau}^{v} \xi(\tau)e^{-\zeta(\tau)}d\tau. \) Thus

\[ F_1(u, v) = \sin \theta \left( e^{-\zeta(v)}I(u) + \int_{\tau}^{v} \xi(\tau)e^{-\zeta(\tau)}d\tau \right). \]

2. Similarly, replace \[18\] in \[19.b\] one obtains

\[ \cos \theta(f_2'(v) + \zeta'(v)f_2(v)) + \zeta'(v)\left( \cos \theta(I(u) + J(u)) - \frac{\sinh(\bar{u})}{\sqrt{\cosh(\bar{u})}} \right) = \cos \theta \xi(v). \]
We have
\[ a(u, v) \cos \theta \cos \alpha + b(u, v) \sin \alpha = \pm \zeta'(v)\left( \sinh(u) - \cos \theta I(u) \sqrt{\cosh(u)} \right) \]
and
\[ \cos \theta (I(u) + J(u)) - \frac{\sinh(u)}{\sqrt{\cosh(u)}} = \text{constant} \]
which can be incorporated in the primitives \( I(u) \) or \( J(u) \). It follows that \( f_2 \) satisfies the following ODE
\[ f_2'(v) + \zeta'(v) f_2(v) = \xi(v) \]
which has the solution
\[ f_2(v) = e^{-\zeta(v)} \int u \xi(\tau) e^{\zeta(\tau)} d\tau. \]
Thus
\[ F_2(u, v) = \pm \cos \theta \left( e^{\zeta(v)} J(u) + \int u \xi(\tau) e^{\zeta(\tau)} d\tau \right). \]
We conclude with the following result

**Theorem 2.7.** A general constant angle surface in \( \text{Sol}_3 \) can be parameterized as
\[ (25) \quad F(u, v) = \gamma_1(v) * \gamma_2(u) \]
where
\[ (26.a) \quad \gamma_1(v) = \left( \sin \theta \int v \xi(\tau) e^{-\zeta(\tau)} d\tau, \pm \cos \theta \int v \xi(\tau) e^{\zeta(\tau)} d\tau, \zeta(v) \right) \]
\[ (26.b) \quad \gamma_2(u) = \left( \sin \theta I(u), \pm \cos \theta J(u), -\frac{1}{2} \log \cosh u \right) \]
and \( \zeta, \xi \) are arbitrary functions depending on \( v \).

The curve \( \gamma_2 \) is parametrized by arclength.

**Remark 2.8.** The only minimal constant angle surfaces in \( \text{Sol}_3 \) are: (i) the hyperbolic plane \( \mathcal{H}^2 \) (for \( \theta = 0 \)); (ii) the hyperbolic plane \( \mathcal{H}^1 \) (for \( \theta = \frac{\pi}{2} \)); (iii) surfaces furnished by Proposition 2.4.

**Proof.** In the general case when \( \theta \) is different from 0 and \( \frac{\pi}{2} \) and \( \alpha \) is such that \( \sin \alpha \) and \( \cos \alpha \) do not vanish, the minimality condition can be written as \( \sigma = \sin \alpha \sin \theta. \) But this relation is impossible due to (11) and (13).

**Final Remark.** In order to define constant angle surfaces in \( \text{Sol}_3 \) we have considered \( e_1 \) as the direction with which the normal to the surface makes constant angle. Since both \( \mathcal{H}^1 \) and \( \mathcal{H}^2 \) are totally geodesic foliations one can also propose \( e_2 \) as a candidate to the preferred direction. If this is the choice, one can define constant angle surfaces in \( \text{Sol}_3 \) to be those surfaces \( M \) whose unit normals make constant angle with \( e_2 \) in each point of \( M \).
Analogue computations give rise to similar results. Since the differences are insignificant we do not give any detail for this problem.
3. Appendix: Solution of PDE

**Problem.** Solve the equation \( p u + \cos \alpha + p \cos \theta \sin \alpha + 2p^2 \cos^2 \theta \cos \alpha = 0. \)

**Solution.** Denote by \( \bar{u} = 2u \cos \theta + \psi_0. \)

Let \( q := \frac{1}{p}; \) it follows that \( q \) satisfies

\[
q_u - q^2 \cos \alpha - q \cos \theta \sin \alpha - 2 \cos^2 \theta \cos \alpha = 0.
\]

Let \( A := q - \cos \theta \sinh \bar{u}. \) It follows that \( q_u = A_u + 2 \cos^2 \theta \cosh \bar{u}. \) Hence, \( A \) satisfies

\[
A_u - 3A \cos \theta \sinh \bar{u} - \frac{1}{\cosh \bar{u}} A^2 = 0.
\]

Let \( B := A \cosh \frac{1}{2} \bar{u}. \) It follows that \( A_u = 3B \cos \theta \sinh \bar{u} \cosh \frac{1}{2} \bar{u} + B_u \cosh \frac{1}{2} \bar{u}. \) Thus, \( B \) satisfies

\[
B_u - B^2 \cosh \frac{1}{2} \bar{u} = 0.
\]

Hence either \( B = 0 \) or \( \frac{1}{B(u,v)} = -I(u) + \Lambda(v), \) for a smooth \( \Lambda. \)

If \( B = 0 \) then \( A = 0, \) \( q = \cos \theta \sinh \bar{u}. \)

\( q \neq 0 \) if and only if \( \theta \neq \frac{\pi}{2} \) and \( \bar{u} \neq 0. \)

One gets

\[
p = \frac{1}{\cos \theta \sinh \bar{u}}.
\]

If \( B \neq 0 \) then

\[
q(u,v) = \cos \theta \sinh \bar{u} + \frac{\cosh \frac{3}{2} \bar{u}}{-I(u) + \Lambda(v)}.
\]

These solutions correspond to 1. \( \zeta' = 0 \) and 2. \( \Lambda(v) = \frac{\xi(v)}{\zeta'(v)} \)

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