Texture zeros flavor neutrino mass matrix and triplet Higgs models

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One- and two-zero textures for the flavor neutrino mass matrix have been successful in explaining mixing in the neutrino sector. Conservatively, six cases of one-zero textures and seven cases of two-zero textures are compatible with observations. We show that one case may be the most natural in the one- and two-zero textures schemes if tiny neutrino masses are generated by the type-II seesaw mechanism in triplet Higgs models.

I. INTRODUCTION

The origin of the tiny masses and flavor mixing of neutrinos is a long-term mystery in particle physics. The seesaw mechanism is one of the leading theoretical mechanisms for generating tiny neutrino masses. There are three types of seesaw mechanisms.

1. Type I: Right-handed singlet neutrinos are introduced in the standard model.

2. Type II: A triplet scalar (triplet Higgs boson) is introduced in the standard model.

3. Type III: Triplet fermions are introduced in the standard model.

To solve the origin of flavor mixing of neutrinos, there have been various discussions on the texture zeros approach for flavor neutrino masses. In this approach, we assume that the flavor neutrino mass matrix has zero elements.

In the one-zero texture scheme, there are the following six cases for the flavor neutrino mass matrix:

\[
G_1: \begin{pmatrix} 0 & 0 & x \\ x & x & x \\ x & x & 0 \end{pmatrix}, \quad G_2: \begin{pmatrix} x & 0 & x \\ x & x & x \\ x & x & 0 \end{pmatrix},
\]

\[
G_3: \begin{pmatrix} x & x & 0 \\ x & x & x \\ x & x & 0 \end{pmatrix}, \quad G_4: \begin{pmatrix} 0 & 0 & x \\ x & x & x \\ x & x & 0 \end{pmatrix},
\]

\[
G_5: \begin{pmatrix} x & x & x \\ x & 0 & x \\ x & x & 0 \end{pmatrix}, \quad G_6: \begin{pmatrix} x & x & x \\ x & x & 0 \\ x & x & 0 \end{pmatrix}.
\]

All six cases of one-zero textures are consistent with observations.

In the two-zero texture scheme, there are 15 possible combinations of two vanishing independent elements in the 3 x 3 Majorana flavor neutrino mass matrix. The neutrino oscillation data allows only 7 out of the 15 cases to be satisfied:

\[
A_1: \begin{pmatrix} 0 & 0 & x \\ x & x & x \\ x & x & 0 \end{pmatrix}, \quad A_2: \begin{pmatrix} 0 & 0 & x \\ x & x & x \\ x & x & 0 \end{pmatrix},
\]

\[
B_1: \begin{pmatrix} x & 0 & x \\ x & x & x \\ x & x & 0 \end{pmatrix}, \quad B_2: \begin{pmatrix} x & 0 & x \\ x & x & x \\ x & x & 0 \end{pmatrix},
\]

\[
B_3: \begin{pmatrix} x & 0 & x \\ x & 0 & x \\ x & 0 & x \end{pmatrix}, \quad B_4: \begin{pmatrix} x & 0 & x \\ x & 0 & x \\ x & 0 & x \end{pmatrix},
\]

\[
C: \begin{pmatrix} 0 & x & x \\ x & 0 & x \\ x & x & 0 \end{pmatrix}.
\]

If neutrinoless double beta decay is observed in future experiments, the A1 and A2 cases should be excluded. Moreover, Singh shows only B2 and B4 are compatible with recent data at 2σ. In this paper, all seven cases of two-zero textures in Eq. (2) are included in our study in a conservative manner.

The origin of such texture zeros is discussed in Refs. [16–20]. The phenomenology of one-zero and two-zero textures is studied in, for example, Refs. [33–39] and Refs. [24–32].

In this paper, we demonstrate that all six cases of one-zero textures (G1, G2, · · · , G6) and all seven cases of two-zero textures (A1, A2, B1, · · · , B4, C) are excluded if the following two conditions are satisfied:

C1:: Neutrino masses are generated by the type-II seesaw mechanism in triplet Higgs models.

C2:: The three lepton flavor violating processes \( \mu \rightarrow eee \), \( \tau \rightarrow \mu \mu \mu \) and \( \tau \rightarrow eee \) are all explicitly forbidden either experimentally or theoretically.

Moreover, we show that the G6 case is viable only if the condition C1 as well as the following conditions is satisfied:

C3:: The three lepton flavor violating processes \( \mu \rightarrow eee \), \( \tau \rightarrow \mu \mu \mu \), and \( \tau \rightarrow eee \) are all observed experimentally or undoubtedly are predicted theoretically.

Even if part of these three lepton flavor violating processes is allowed such as BR(\( \mu \rightarrow eee \)) \( \neq 0 \), BR(\( \tau \rightarrow eee \)) \( \neq 0 \), BR(\( \tau \rightarrow \mu \mu \mu \)) \( \neq 0 \), and BR(\( \tau \rightarrow eee \)) \( \neq 0 \), then only the G6 case can be viable.
are given by the lepton flavor violating decays $\mu \rightarrow \bar{e}e\bar{e}$ and $\bar{\tau}_i \ell_j \ell_k$ directly connect with the neutrino flavor masses.

$$\text{BR}(\mu \rightarrow \bar{e}e\bar{e}) \propto |M_{\mu e}|^2 |M_{ee}|^2,$$
$$\text{BR}(\tau \rightarrow \bar{\mu}\bar{\mu}\bar{\nu}) \propto |M_{\tau \mu}|^2 |M_{\mu\mu}|^2,$$
$$\text{BR}(\tau \rightarrow \bar{e}e\bar{e}) \propto |M_{\tau e}|^2 |M_{ee}|^2,$$  
(8)

These simple relations between the branching ratios and the flavor neutrino mass matrix in Eqs. (8) and (9) are useful for testing the availability of the zero texture of the flavor neutrino mass matrix. For example, we can test the availability of a texture which has $M_{ee} = 0$ by using a branching ratio which is proportional to $M_{ee}$ such as $	ext{BR}(\mu \rightarrow \bar{e}e\bar{e}) \propto |M_{\mu e}|^2 |M_{ee}|^2$.

We would like to note again that the origin of these simple relations in Eqs. (8) and (9) is the one-to-one correspondence between $M_{ij}$ and $y_{ij}$, following the type-II seesaw mechanism. In the type-I and -III seesaw mechanisms, we obtain more complicated correspondences between $M_{ij}$ and $y_{ij}$, such as the Casas-Ibarra parametrization [50] for the type-I seesaw mechanism. This is the reason why we chose the type-II seesaw mechanism, not type I or III, to explain the neutrino mass along with texture zeros.

In the next section, we use these branching ratios of the lepton flavor violating processes to test the availability of the one- and two-zero textures. First, we will include only three branching ratios in Eq. (8) in our discussion to show our strategy. Then we include the remaining four branching ratios in Eq. (9) in our discussion to complete this paper.

### III. Texture Zeros

#### A. $\text{BR}(\mu \rightarrow 3e, \tau \rightarrow 3\mu, \tau \rightarrow 3e) = 0$ case

In this subsection, we assume that the three lepton flavor violating processes $\mu \rightarrow \bar{e}e\bar{e}$, $\tau \rightarrow \bar{\mu}\bar{\mu}\bar{\nu}$, and $\tau \rightarrow \bar{e}e\bar{e}$ are all explicitly forbidden either experimentally or theoretically.

In this case, at least the branching ratio $\text{BR}(\mu \rightarrow \bar{e}e\bar{e})$, as well as $M_{ee}$ and/or $M_{\mu\mu}$ should vanish. If we require the conditions of $M_{ee} = 0$ and/or $M_{\mu\mu} = 0$ for the $G_3$...
case in the one-zero textures scheme
\[
G_3 : \begin{pmatrix}
  \times & \times & 0 \\
  - & - & \times \\
  - & - & \times 
\end{pmatrix},
\tag{10}
\]
the following three flavor neutrino mass matrix are obtained:
\[
\begin{pmatrix}
  0 & \times & 0 \\
  - & \times & \times \\
  - & - & \times 
\end{pmatrix}, \quad \begin{pmatrix}
  0 & \times & \times \\
  - & \times & \times \\
  - & - & \times 
\end{pmatrix}, \quad \begin{pmatrix}
  0 & 0 & 0 \\
  - & \times & \times \\
  - & - & \times 
\end{pmatrix}.
\tag{11}
\]
However, the one-zero textures assumption is violated in these matrices by an additional vanishing entry. Therefore, the G_3 case in the one-zero textures scheme should be excluded if the lepton flavor violating process \( \mu \rightarrow \bar{e}ee \) is explicitly forbidden. In the same manner, we can exclude the following G_4, G_5, and G_6 cases,
\[
G_4 : \begin{pmatrix}
  \times & \times & \times \\
  - & 0 & \times \\
  - & - & \times 
\end{pmatrix}, \quad G_5 : \begin{pmatrix}
  \times & \times & \times \\
  - & - & \times 
\end{pmatrix},
\tag{12}
\]
if the lepton flavor violating process \( \mu \rightarrow \bar{e}ee \) is explicitly forbidden. Moreover, the following B_1, B_4, and C cases in the two-zero textures scheme,
\[
B_1 : \begin{pmatrix}
  \times & \times & 0 \\
  - & 0 & \times \\
  - & - & \times 
\end{pmatrix}, \quad B_4 : \begin{pmatrix}
  \times & \times & \times \\
  - & - & \times 
\end{pmatrix},
\tag{13}
\]
are also excluded if we require the conditions of \( M_{ee} = 0 \) and/or \( M_{\mu\mu} = 0 \) (the two-zero textures assumption should be violated by this requirement).

Consequently, the G_3, G_4, G_5, and G_6 cases of one-zero textures and B_1, B_4, and C cases of two-zero textures should be excluded if the lepton flavor violating process \( \mu \rightarrow \bar{e}ee \) is explicitly forbidden.

In addition to the lepton flavor violating process \( \mu \rightarrow \bar{e}ee \), we can use other two lepton flavor violating processes \( \tau \rightarrow \bar{\mu}\mu\mu \) and \( \tau \rightarrow \bar{e}ee \) to test the compatibility of the one- and two-zero textures. Table I shows the compatibility of the cases in the one- and two-zero textures schemes with the vanishing branching ratios \( \text{BR}(\mu \rightarrow \bar{e}ee) = 0 \), \( \text{BR}(\tau \rightarrow \bar{\mu}\mu\mu) = 0 \), and \( \text{BR}(\tau \rightarrow \bar{e}ee) = 0 \). The symbol \( \times \) indicates that the corresponding case should be excluded.

| Case | BR(\mu \rightarrow \bar{e}ee) = 0 | BR(\tau \rightarrow \bar{\mu}\mu\mu) = 0 | BR(\tau \rightarrow \bar{e}ee) = 0 |
|------|---------------------------------|---------------------------------|---------------------------------|
| \( G_1 \) | \times | \times | \times |
| \( G_2 \) | \times | \times | \times |
| \( G_3 \) | \times | \times | \times |
| \( G_4 \) | \times | \times | \times |
| \( G_5 \) | \times | \times | \times |
| \( G_6 \) | \times | \times | \times |
| \( A_1 \) | \times | \times | \times |
| \( A_2 \) | \times | \times | \times |
| \( A_3 \) | \times | \times | \times |
| \( B_1 \) | \times | \times | \times |
| \( B_2 \) | \times | \times | \times |
| \( B_3 \) | \times | \times | \times |
| \( B_4 \) | \times | \times | \times |
| \( C \) | \times | \times | \times |

| Case | BR(\mu \rightarrow \bar{e}ee) \neq 0 | BR(\tau \rightarrow \bar{\mu}\mu\mu) \neq 0 | BR(\tau \rightarrow \bar{e}ee) \neq 0 |
|------|---------------------------------|---------------------------------|---------------------------------|
| \( G_1 \) | \times | \times | \times |
| \( G_2 \) | \times | \times | \times |
| \( G_3 \) | \times | \times | \times |
| \( G_4 \) | \times | \times | \times |
| \( G_5 \) | \times | \times | \times |
| \( G_6 \) | \times | \times | \times |

TABLE I. Compatibility of the cases in the one- and two-zero textures scheme with the vanishing branching ratios \( \text{BR}(\mu \rightarrow \bar{e}ee) = 0 \), \( \text{BR}(\tau \rightarrow \bar{\mu}\mu\mu) = 0 \), and \( \text{BR}(\tau \rightarrow \bar{e}ee) = 0 \). The symbol \( \times \) indicates that the corresponding case should be excluded.

TABLE II. Compatibility of the cases in the one- and two-zero textures scheme with the nonvanishing branching ratios \( \text{BR}(\mu \rightarrow \bar{e}ee) \neq 0 \), \( \text{BR}(\tau \rightarrow \bar{\mu}\mu\mu) \neq 0 \), and \( \text{BR}(\tau \rightarrow \bar{e}ee) \neq 0 \). The symbol \( \times \) indicates that the corresponding case should be excluded.

B. \( \text{BR}(\mu \rightarrow \bar{e}ee, \tau \rightarrow \bar{\mu}\mu\mu, \tau \rightarrow \bar{e}ee) \neq 0 \) case

In this subsection, we assume that the three lepton flavor violating processes \( \mu \rightarrow \bar{e}ee, \tau \rightarrow \bar{\mu}\mu\mu \) and \( \tau \rightarrow \bar{e}ee \) are all observed experimentally or undoubtedly are predicted theoretically.

In this case, at least the branching ratio \( \text{BR}(\mu \rightarrow \bar{e}ee) \), as well as \( M_{ee} \) and \( M_{\mu\mu} \), cannot vanish. The nonvanishing elements \( M_{ee} \) and \( M_{\mu\mu} \) (\( M_{ee} \neq 0 \) and \( M_{\mu\mu} \neq 0 \)) are inconsistent with the \( G_1, G_2, A_1, A_2, B_2, \) and \( B_3 \) cases in
TABLE III. Allowed cases in the one- and two-zero textures scheme for \( \mu \to 3e \), \( \tau \to 3\mu \) and \( \tau \to 3e \). The abbreviation “NZ” indicates a nonzero value for the branching ratio.

| \( \text{BR}(\mu \to 3e) \) | \( \text{BR}(\tau \to 3\mu) \) | \( \text{BR}(\tau \to 3e) \) | Allowed cases |
|-----------------|-----------------|-----------------|-----------------|
| 0               | 0               | 0               | G₁, A₁, A₂      |
| 0               | NZ              | 0               | G₂, B₄         |
| 0               | 0               | NZ              | B₃             |
| NZ              | 0               | NZ              | G₄, G₅, C      |
| NZ              | NZ              | NZ              | G₆             |

Therefore, the G₁,G₂,A₁,A₂,B₂ and B₃ cases in the one- and two-zero textures scheme should be excluded if the lepton flavor violating processes \( \mu \to \bar{e}ee \) are observed experimentally or undoubtedly are predicted theoretically.

Addition to the lepton flavor violating process \( \mu \to \bar{e}ee \), the other two lepton flavor violating processes \( \tau \to \bar{\mu}\mu \mu \) and \( \tau \to \bar{e}ee \) are available for evaluation of the viability of the one- and two-zero textures. Table III shows the compatibility of the cases in the one- and two-zero textures schemes with the nonvanishing branching ratios \( \text{BR}(\mu \to \bar{e}ee) \neq 0 \), \( \text{BR}(\tau \to \bar{\mu}\mu \mu) \neq 0 \), and \( \text{BR}(\tau \to \bar{e}ee) \neq 0 \). The symbol \( \times \) indicates that the corresponding case should be excluded.

We conclude that only G₆ case is viable in one- and two-zero textures of the flavor neutrino mass matrix if the neutrino masses are generated by the type-II seesaw mechanism in triplet Higgs models and the three lepton flavor violating processes \( \mu \to \bar{e}ee \), \( \tau \to \bar{\mu}\mu \mu \) and \( \tau \to \bar{e}ee \) are all observed experimentally or undoubtedly are predicted theoretically.

C. Hybrid cases for \( \mu \to 3e, \tau \to 3\mu, \tau \to 3e \)

Based on the above discussion, it turned out that if the neutrino masses are generated by the type-II seesaw mechanism in triplet Higgs models and the three lepton flavor violating processes \( \mu \to \bar{e}ee \), \( \tau \to \bar{\mu}\mu \mu \) and \( \tau \to \bar{e}ee \) are all forbidden, there is no room for one- and two-zero textures. On the other hand, if all three processes exist, only the G₆ case is viable in one- and two-zero textures.

If parts of these three lepton flavor violating processes are allowed such as

\[
\begin{align*}
\text{BR}(\mu \to \bar{e}ee) & \neq 0, & \text{BR}(\tau \to \bar{\mu}\mu \mu) & = 0, & \text{BR}(\tau \to \bar{e}ee) & = 0, \\
\text{BR}(\tau \to \bar{e}ee) & = 0,
\end{align*}
\]

(15)

other cases of one- and two-zero textures may be allowed. For example, in the case shown in Eq. (15), the G₆ case is ruled out and only the B₁ case is allowed. Similarly, in the cases

\[
\begin{align*}
\text{BR}(\mu \to \bar{e}ee) & = 0, & \text{BR}(\tau \to \bar{\mu}\mu \mu) & \neq 0, & \text{BR}(\tau \to \bar{e}ee) & = 0, \\
\text{BR}(\tau \to \bar{e}ee) & \neq 0,
\end{align*}
\]

(16)

and

\[
\begin{align*}
\text{BR}(\mu \to \bar{e}ee) & = 0, & \text{BR}(\tau \to \bar{\mu}\mu \mu) & = 0, & \text{BR}(\tau \to \bar{e}ee) & \neq 0, \\
\text{BR}(\tau \to \bar{e}ee) & \neq 0,
\end{align*}
\]

(17)

the allowed cases of one- and two-zero textures are G₁, A₁, A₂, and B₃, respectively.

Table III shows the allowed cases in the one- and two-zero textures schemes for \( \mu \to 3e, \tau \to 3\mu \) and \( \tau \to 3e \). The abbreviation “NZ” indicates a nonzero value for the branching ratio. It is remarkable that the each of G₁, G₂, ⋱, C cases appears only once in Table III. Therefore, we can predict the allowed combination of nonvanishing branching ratios by the one- and two-zero flavor neutrino mass matrix textures.

Although whether or not the three lepton flavor violating processes \( \mu \to \bar{e}ee, \tau \to \bar{\mu}\mu \mu, \) and \( \tau \to \bar{e}ee \) are forbidden is still undetermined, we can suggest that either

\[
\begin{align*}
\text{BR}(\mu \to \bar{e}ee) = \text{BR}(\tau \to \bar{\mu}\mu \mu) = \text{BR}(\tau \to \bar{e}ee) = 0,
\end{align*}
\]

(18)

or

\[
\begin{align*}
\text{BR}(\mu \to \bar{e}ee) \neq \text{BR}(\tau \to \bar{\mu}\mu \mu) \neq \text{BR}(\tau \to \bar{e}ee) \neq 0,
\end{align*}
\]

(19)

may be the most natural case. Otherwise, the appropriate selection mechanisms for \( \ell_m \to \ell_i\ell_j\ell_k \) decay at tree level are required in the models.

We can conclude that if the tiny neutrino masses are generated by the type-II seesaw mechanism, only the G₆ case may be most natural in one- and two-zero textures schemes. This conclusion becomes more rigid in the next subsection.

D. \( \tau \to \bar{\mu}\mu \mu, \tau \to \bar{e}ep, \tau \to \bar{\mu}\mu \mu \) and \( \tau \to \bar{e}ep \)

In the last subsection, we include only three branching ratios in Eq. (19) in our discussion to show our strategy. Now we include the remaining four branching ratios in Eq. (19) in our discussion to complete this paper.
TABLE IV. Compatibility of the cases in the one- and two-zero textures schemes with the vanishing branching ratios $\text{BR}(\tau \to \bar{\mu}ee) = 0$, $\text{BR}(\tau \to \bar{\mu}e\mu) = 0$, $\text{BR}(\tau \to \bar{e}\mu\mu) = 0$ and $\text{BR}(\tau \to \bar{e}e\mu) = 0$. The symbol $\times$ indicates that the corresponding case should be excluded.

|                  | $\text{BR}(\tau \to \bar{\mu}ee) = 0$ | $\text{BR}(\tau \to \bar{\mu}e\mu) = 0$ | $\text{BR}(\tau \to \bar{e}\mu\mu) = 0$ | $\text{BR}(\tau \to \bar{e}e\mu) = 0$ |
|------------------|----------------------------------------|----------------------------------------|----------------------------------------|----------------------------------------|
| $G_1$            | $\times$                               |                                       |                                       |                                       |
| $G_2$            |                                       | $\times$                               |                                       |                                       |
| $G_3$            |                                       |                                       | $\times$                               |                                       |
| $G_4$            |                                       |                                       |                                       | $\times$                               |
| $G_5$            |                                       |                                       |                                       |                                       |
| $G_6$            |                                       |                                       |                                       |                                       |
| $A_1$            |                                       |                                       |                                       | $\times$                               |
| $A_2$            |                                       |                                       |                                       |                                       |
| $B_1$            |                                       |                                       |                                       |                                       |
| $B_2$            |                                       |                                       |                                       |                                       |
| $B_3$            |                                       |                                       |                                       |                                       |
| $B_4$            |                                       |                                       |                                       |                                       |
| $C$              |                                       |                                       |                                       | $\times$                               |

TABLE V. Compatibility of the cases in the one- and two-zero textures schemes with the vanishing branching ratios $\text{BR}(\tau \to \bar{\mu}ee) \neq 0$, $\text{BR}(\tau \to \bar{\mu}e\mu) \neq 0$, $\text{BR}(\tau \to \bar{e}\mu\mu) \neq 0$ and $\text{BR}(\tau \to \bar{e}e\mu) \neq 0$. The symbol $\times$ indicates that the corresponding case should be excluded.

|                  | $\text{BR}(\tau \to \bar{\mu}ee) \neq 0$ | $\text{BR}(\tau \to \bar{\mu}e\mu) \neq 0$ | $\text{BR}(\tau \to \bar{e}\mu\mu) \neq 0$ | $\text{BR}(\tau \to \bar{e}e\mu) \neq 0$ |
|------------------|----------------------------------------|----------------------------------------|----------------------------------------|----------------------------------------|
| $G_1$            | $\times$                               |                                       |                                       |                                       |
| $G_2$            |                                       | $\times$                               |                                       |                                       |
| $G_3$            |                                       |                                       | $\times$                               |                                       |
| $G_4$            |                                       |                                       |                                       | $\times$                               |
| $G_5$            |                                       |                                       |                                       |                                       |
| $G_6$            |                                       |                                       |                                       |                                       |
| $A_1$            |                                       |                                       |                                       | $\times$                               |
| $A_2$            |                                       |                                       |                                       |                                       |
| $B_1$            |                                       |                                       |                                       |                                       |
| $B_2$            |                                       |                                       |                                       |                                       |
| $B_3$            |                                       |                                       |                                       |                                       |
| $B_4$            |                                       |                                       |                                       |                                       |
| $C$              |                                       |                                       |                                       | $\times$                               |

According to the same method as in the last subsection, we estimate the compatibility of the cases in the one- and two-zero textures schemes with four branching ratios in Eq. (9). The results are shown in Tables IV and V.

Table IV shows the compatibility of the cases in the one- and two-zero textures schemes with the vanishing branching ratios $\text{BR}(\tau \to \bar{\mu}ee) = 0$, $\text{BR}(\tau \to \bar{\mu}e\mu) = 0$, $\text{BR}(\tau \to \bar{e}\mu\mu) = 0$ and $\text{BR}(\tau \to \bar{e}e\mu) = 0$. We see that all six cases of one-zero textures (G1, G2, · · · , G6) and all seven cases of two-zero textures (A1, A2, B1, · · · , B4, C) should be excluded if the neutrino masses are generated by the type-II seesaw mechanism in triplet Higgs models and all four lepton flavor violating processes $\tau \to \bar{\mu}ee$, $\tau \to \bar{\mu}e\mu$, $\tau \to \bar{e}\mu\mu$ and $\tau \to \bar{e}e\mu$ are explicitly forbidden.

Table V shows the compatibility of the cases in the one- and two-zero textures schemes with the nonvanishing branching ratios $\text{BR}(\tau \to \bar{\mu}ee) \neq 0$, $\text{BR}(\tau \to \bar{\mu}e\mu) \neq 0$, $\text{BR}(\tau \to \bar{e}\mu\mu) \neq 0$ and $\text{BR}(\tau \to \bar{e}e\mu) \neq 0$. We see that only the $G_6$ case is viable in one- and two-zero textures of the flavor neutrino mass matrix if the neutrino masses are generated by the type-II seesaw mechanism in triplet Higgs models and all four lepton flavor violating processes $\tau \to \bar{\mu}ee$, $\tau \to \bar{\mu}e\mu$, $\tau \to \bar{e}\mu\mu$ and $\tau \to \bar{e}e\mu$ are observed experimentally or undoubtedly are predicted theoretically.

Table V shows the allowed cases in the one- and two-zero textures schemes for $\tau \to \bar{\mu}ee$, $\tau \to \bar{\mu}e\mu$, $\tau \to \bar{e}\mu\mu$ and $\tau \to \bar{e}e\mu$. The abbreviation “NZ” indicates a nonzero value for the branching ratio. As in the last subsection, although whether or not the three lepton flavor violating processes $\tau \to \bar{\mu}ee$, $\tau \to \bar{\mu}e\mu$, $\tau \to \bar{e}\mu\mu$ and $\tau \to \bar{e}e\mu$ are forbidden is still unknown, we can suggest that either

\[
\text{BR}(\tau \to \bar{\mu}ee) = \text{BR}(\tau \to \bar{\mu}e\mu) = \text{BR}(\tau \to \bar{e}\mu\mu) = \text{BR}(\tau \to \bar{e}e\mu) = 0 \quad (20)
\]
TABLE VI. Allowed cases in the one- and two-zero textures schemes for \( \tau \rightarrow \mu e e, \tau \rightarrow \mu e \mu, \tau \rightarrow \bar{e} \mu \mu \) and \( \tau \rightarrow \bar{e} e \mu \). The abbreviation “NZ” indicates a nonzero value for the branching ratio.

| BR(\( \tau \rightarrow \mu e e \)) | BR(\( \tau \rightarrow \mu e \mu \)) | BR(\( \tau \rightarrow \bar{e} \mu \mu \)) | BR(\( \tau \rightarrow \bar{e} e \mu \)) | Allowed cases |
|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|----------------|
| 0                                 | 0                                 | 0                                 | 0                                 | \(-\) NZ\(6 \) |
| 0                                 | 0                                 | 0                                 | NZ\(3 \)                         | \(-\) A\(1 \) |
| 0                                 | 0                                 | NZ\(3 \)                         | NZ\(3 \)                         | \(-\) G\(5 \) |
| 0                                 | NZ\(3 \)                         | 0                                 | 0                                 | \(-\) A\(2 \) |
| 0                                 | NZ\(3 \)                         | 0                                 | NZ\(3 \)                         | \(-\) -.uploaded data-|
| 0                                 | NZ\(3 \)                         | 0                                 | NZ\(3 \)                         | \(-\) -Uploaded data-|
| 0                                 | NZ\(3 \)                         | NZ\(3 \)                         | 0                                 | \(-\) G\(1 \) |
| NZ\(3 \)                         | 0                                 | 0                                 | NZ\(3 \)                         | \(-\) B\(3 \) |
| NZ\(3 \)                         | 0                                 | NZ\(3 \)                         | 0                                 | \(-\) -Uploaded data-|
| NZ\(3 \)                         | NZ\(3 \)                         | 0                                 | NZ\(3 \)                         | \(-\) G\(2,B\(2 \) |
| NZ\(3 \)                         | NZ\(3 \)                         | NZ\(3 \)                         | 0                                 | \(-\) -Uploaded data-|
| NZ\(3 \)                         | NZ\(3 \)                         | NZ\(3 \)                         | NZ\(3 \)                         | \(-\) G\(6 \) |

or

\[
\begin{align*}
\text{BR}(\tau \rightarrow \mu e e) & \neq \text{BR}(\tau \rightarrow \mu e \mu) \\
\neq & \text{BR}(\tau \rightarrow \bar{e} \mu \mu) \\
\neq & \text{BR}(\tau \rightarrow \bar{e} e \mu) \\
\end{align*}
\]

(21)

may be the most natural case.

According to the combined results in the last subsection and this subsection, we conclude that if the tiny neutrino masses have been generated by type-II seesaw mechanism, only the G\(6 \) case may be the most natural in the one- and two-zero textures schemes. This is the main result of this paper.

**E. Numerical calculations**

Although the main result of this paper was already obtained in subsection 11111, an additional numerical study may be required to improve our discussions. According to the conclusion in subsection 11111, only the G\(6 \) case may be the most natural in the one- and two-zero textures schemes if the tiny neutrino masses have been generated by the type-II seesaw mechanism. In this subsection, we present the phenomenology for the G\(6 \) case.

First we give brief reviews of the neutrino mixings, useful relations for the one-zero textures, and observed data from neutrino experiments as a preparation for our numerical calculations. Then we show some predictions for the G\(6 \) case.

**Neutrino mixings:** The flavor neutrino mass matrix \( M \) is related to the diagonal neutrino mass matrix

\[
M = U \text{diag}(m_1 e^{2i\alpha_1}, m_2 e^{2i\alpha_2}, m_3) U^T, \quad (22)
\]

where \( m_i (i = 1, 2, 3) \) is a neutrino mass eigenstate and

\[
U = \begin{pmatrix}
U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\
U_{e 1} & U_{e 2} & U_{e 3}
\end{pmatrix}, \quad (23)
\]

with

\[
\begin{align*}
U_{\tau 1} & = c_{12} c_{13}, & U_{\tau 2} & = s_{12} c_{13}, & U_{\tau 3} & = s_{13} e^{-i\delta}, \\
U_{\mu 1} & = -s_{12} c_{23} - c_{12} s_{13} e^{i\delta}, & U_{\mu 2} & = c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta}, & U_{\mu 3} & = s_{23} c_{13}, \\
U_{e 1} & = s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta}, & U_{e 2} & = c_{12} s_{23} - s_{12} c_{13} s_{13} e^{i\delta}, & U_{e 3} & = c_{23} c_{13},
\end{align*}
\]

denotes the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [61–64]. We use the abbreviations \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \) \((i,j=1,2,3)\), where \( \theta_{ij} \) is a neutrino mixing angle. The Dirac CP phase is denoted by \( \delta \) and the Majorana CP phases are denoted by \( \alpha_1 \) and \( \alpha_2 \). In this paper, we assume that the mass matrix of the charged leptons is diagonal and real (some comments for this assumption will be noted in the summary).

**Useful relations for one-zero textures:** The requirement of \( M_{ij} = 0 \) for one-zero textures yields

\[
A_1 m_1 + A_2 m_2 + A_3 m_3 = 0, \quad (25)
\]

where

\[
A_1 = U_{\tau 1} U_{j 1} e^{2i\alpha_1}, \quad A_2 = U_{\tau 2} U_{j 2} e^{2i\alpha_2}, \quad A_3 = U_{\tau 3} U_{j 3}, \quad (26)
\]

This condition leads to (for examples, see Refs. [34–65])

\[
\frac{m_2}{m_1} = \frac{\text{Re}(A_1)\text{Im}(A_3) - \text{Re}(A_3)\text{Im}(A_1)}{\text{Re}(A_3)\text{Im}(A_2) - \text{Re}(A_2)\text{Im}(A_3)}. \quad (27)
\]
and
\[ m_3 = \frac{\text{Re}(A_2) \text{Im}(A_1) - \text{Re}(A_1) \text{Im}(A_2)}{\text{Re}(A_3) \text{Im}(A_2) - \text{Re}(A_2) \text{Im}(A_3)} \]
Equation (28)

The ratio of two squared mass differences is given by
\[ \frac{\Delta m^2_{21}}{|\Delta m^2_{31}|} = \frac{(m_2/m_1)^2 - 1}{(m_3/m_1)^2 - 1}, \]
where the squared mass difference is defined by \( \Delta m^2_{ij} = m^2_i - m^2_j \). Eqs. (27), (28) and (29) are useful when we search the allowed parameter sets under the requirement that \( M_{ij} = 0 \).

**Observed data:** Although the neutrino mass ordering (either the so-called normal mass ordering \( m_1 \lesssim m_2 < m_3 \) or the inverted mass ordering \( m_3 < m_1 \lesssim m_2 \)) has not been determined, a global analysis shows that the preference for normal mass ordering is due mostly to neutrino oscillation measurements [21, 68]. Upcoming experiments for neutrinos will be able to solve this problem [67]. In this paper, we assume the normal mass hierarchical spectrum for the neutrinos.

A global analysis of current data shows the following best-fit values of the squared mass differences and the mixing angles for the normal mass ordering [68]:

\[ \Delta m^2_{21}/10^{-5}\text{eV}^2 = 7.39^{+0.21}_{-0.20} \times (6.79 \rightarrow 8.01), \]
\[ \Delta m^2_{31}/10^{-3}\text{eV}^2 = 2.528^{+0.029}_{-0.031} \times (2.436 \rightarrow 2.618), \]
\[ \theta_{12}/^\circ = 33.82^{+0.78}_{-0.76} \times (31.61 \rightarrow 36.27), \]
\[ \theta_{23}/^\circ = 48.6^{+1.0}_{-1.4} \times (41.1 \rightarrow 51.3), \]
\[ \theta_{13}/^\circ = 8.60^{+0.13}_{-0.13} \times (8.22 \rightarrow 8.98), \]
\[ \delta/^\circ = 221^{+30}_{-28} \times (144 \rightarrow 357), \]
where \( \pm \) signs denote the 1\( \sigma \) region and parentheses denote the 3\( \sigma \) region. Moreover, the following constraints,
\[ \sum m_i < 0.12 - 0.69 \text{ eV}, \]
from a cosmological observation of cosmic microwave background radiation [21, 67, 72] as well as
\[ |M_{ee}| < 0.66 - 0.155 \text{ eV}, \]
from the neutrinoless double beta decay experiments [21, 22] are obtained.

**Phenomenology for G_6 case:** Now we make some predictions for the G_6 case by using numerical calculations.

In our numerical calculation, we require that the squared mass differences \( \Delta m^2_{ij} \), mixing angles \( \theta_{ij} \), and the Dirac CP violating phase \( \delta \) are varied within the 3\( \sigma \) experimental ranges, the Majorana CP violating phases \( \alpha_1 \) and \( \alpha_2 \) are varied within their full possible ranges and the lightest neutrino mass is varied within 0.01 - 0.1 eV. We also require that the constraints \( |M_{ee}| < 0.155 \text{ eV} \) and \( \sum m_i < 0.241 \text{ eV} \) (TT, TE, EE+LowE+lensing [23, 69]) are satisfied. As predictions for the one-zero textures, we estimate the ratios
\[ R_1 = \frac{\text{BR}(\tau \rightarrow \mu\mu\mu)}{\text{BR}(\tau \rightarrow \mu\mu\mu \bar{\nu})} = \frac{|M_{\mu\mu}|^2}{|M_{e\mu}|^2}, \]
\[ R_2 = \frac{\text{BR}(\mu \rightarrow e\nu\nu)}{\text{BR}(\tau \rightarrow e\nu\nu)} = \frac{|M_{e\mu}|^2}{|M_{\tau\mu}|^2}, \]
\[ R_3 = \frac{\text{BR}(\mu \rightarrow e\nu\nu)}{\text{BR}(\tau \rightarrow \mu\mu\mu \bar{\nu})} = \frac{|M_{e\mu}|^2}{|M_{\tau\mu}|^2}, \]
where \( \text{BR}(\mu \rightarrow e\nu\nu) \approx 100\% \) and \( \text{BR}(\tau \rightarrow \mu\bar{\nu}\nu) \approx 17.39\% \).

We show an example of the results of our numerical calculations for the G_6 case. A point set
\[ (\theta_{12}, \theta_{23}, \theta_{13}, \delta) = (33.82^\circ, 48.6^\circ, 8.60^\circ, 221^\circ), \]
\[ (\alpha_1, \alpha_2) = (90.03^\circ, 89.2^\circ), \]
\[ m_1 = 0.0580 \text{eV} \]
yields the following neutrino flavor masses
\[ M_{ee} = -0.0567 - 0.00125 \text{i}, \]
\[ M_{e\mu} = -0.0115 + 0.00191 \text{i}, \]
\[ M_{e\tau} = -0.00979 + 0.000647 \text{i}, \]
\[ M_{\mu\mu} = 0.0165 - 0.00176 \text{i}, \]
\[ M_{\mu\tau} = 0.0661 - 0.00120 \text{i}, \]
\[ M_{\tau\tau} = 0, \]
as well as
\[ m_2 = 0.0586 \text{ eV}, \]
\[ m_3 = 0.0768 \text{ eV}, \]
\[ \Delta m^2_{21} = 6.95 \times 10^{-5} \text{ eV}^2, \]
\[ \Delta m^2_{31} = 2.53 \times 10^{-3} \text{ eV}^2, \]
\[ \sum m_i = 0.193 \text{ eV}, \]
\[ |M_{ee}| = 0.0567 \text{ eV}. \]

These results are consistent with observations. The predicted ratios [Eq.(33)] are
\[ (R_1, R_2, R_3) = (3.85, 8.12, 2.11). \]

Figure 1 shows that the predictions for \( R_1, R_2 \) and \( R_3 \) for the lightest neutrino mass \( m_1 \) for the G_6 case. Currently, we have only the upper limits of \( \text{BR}(\tau \rightarrow \mu\mu\mu \bar{\nu}) < 2.1 \times 10^{-8}, \text{BR}(\tau \rightarrow \mu\mu\mu \bar{\nu}) < 2.7 \times 10^{-8} \) and \( \text{BR}(\mu \rightarrow e\nu\nu) < 1.0 \times 10^{-12} \) from observations [64]. If these branching ratios are determined in the future experiments,
\[ R_1 \simeq 0.6 - 3234, \]
\[ R_2 \simeq 0.07 - 542, \]
\[ R_3 \simeq 0.004 - 11, \]
support is given to the G_6 case within the type-II seesaw generation of the neutrino masses in triplet Higgs models.

Finally, we would like to mention the very recently reported tension between NOvA and T2K in the measurement of \( \delta \) and \( \sin^2 \theta_{23} \) for the normal mass ordering of
neutrinos [73, 74]. Both experiments favor the upper octant of $\theta_{23}$; however, the NOvA data show $\delta < \pi$, which is contrary to the T2K result $\delta > \pi$. In this paper, until now, we have used the data from the global analysis shown in Eq. (30) for our numerical calculations. The $3\sigma$ data in this global analysis, the upper octant of $\theta_{23}$ and $\delta > \pi$ are roughly favored.

If the mixing angle $\theta_{23}$ and the CP phase $\delta$ are varied within their full range (e.g., $0^\circ \leq \theta_{23} \leq 90^\circ$ and $0^\circ \leq \delta \leq 360^\circ$) to try to obtain insight into the tension between NOvA and T2K; unfortunately, we could not obtain a significant prediction for this tension. Figure 2 shows the allowed parameter space of $\sin^2 \theta_{23}$ and $\delta$ for the G$_6$ case with the normal mass ordering of neutrinos. The upper octant of $\theta_{23}$ is favored in the G$_6$ case. It is consistent with NOvA and T2K observations; however, the broad region is allowed for the Dirac CP phase $\delta$ in the G$_6$ case.

**IV. SUMMARY**

One- and two-zero textures for the flavor neutrino mass matrix have been successful in explaining mixing in the neutrino sector. In this paper, we have shown that all cases of one- and two-zero textures are excluded if the tiny neutrino masses are generated by the type-II seesaw mechanism in triplet Higgs models and the three lepton flavor violating processes $\mu \rightarrow \bar{e}ee$, $\tau \rightarrow \bar{\mu}\mu\mu$ and $\tau \rightarrow \bar{e}ee$ are all explicitly forbidden experimentally or theoretically. We have also shown that if all three of these lepton flavor violating processes exist, only the G$_6$ case is viable within the one- and two-zero textures.

Even if parts of these three lepton flavor violating processes are allowed, such as $\text{BR}(\mu \rightarrow 3e) \neq 0$, $\text{BR}(\tau \rightarrow 3\mu) = 0$ and $\text{BR}(\tau \rightarrow 3e) = 0$, we can suggest that the most natural case is either $\text{BR}(\mu \rightarrow 3e) = \text{BR}(\tau \rightarrow 3\mu) = \text{BR}(\tau \rightarrow 3e) = 0$ or $\text{BR}(\mu \rightarrow 3e) \neq \text{BR}(\tau \rightarrow 3\mu) \neq \text{BR}(\tau \rightarrow 3e) \neq 0$. Otherwise, the appropriate selection mechanisms for $\ell_m \rightarrow \ell_i\ell_j\ell_k$ decay at tree level are required in the models. Therefore we have concluded that if the tiny neutrino masses are generated by the type-II seesaw mechanism in the triplet Higgs models, only the G$_6$ case may be most natural in one- and two-zero textures schemes. We have also shown that this conclusion becomes more rigid by including four other lepton flavor violating processes, $\tau \rightarrow \bar{\mu}ee$, $\tau \rightarrow \bar{\mu}\mu\mu$, $\tau \rightarrow \bar{e}\mu\mu$ and $\tau \rightarrow \bar{e}e\mu$ in our discussions.
Moreover, some predictions for the $G_6$ case have been made. The ratios $R_1 = \text{BR}(\tau \rightarrow \mu \mu \mu)/\text{BR}(\tau \rightarrow e e e)$, $R_2 = \text{BR}(\mu \rightarrow e e e)/\text{BR}(\tau \rightarrow e e e)$ and $R_3 = \text{BR}(\mu \rightarrow \mu \mu \mu)/\text{BR}(\tau \rightarrow e e e)$ should be $R_1 \approx 0.6 - 3234$, $R_2 \approx 0.07 - 542$ and $R_3 \approx 0.004 - 11$, respectively, for the $G_6$ case within the type-II seesaw generation of the neutrino masses in the triplet Higgs models. In light of recent tension between NOVA and T2K in the measurement of $\delta$ and $\sin^2 \theta_{23}$ for the normal mass ordering of neutrinos, we have estimated the allowed parameter space of $\theta_{23}$ and $\delta$ for the $G_6$ case; however, we have no significant prediction for this tension.

Finally, we would like to mention the role of the charged lepton mixings. In general, the lepton mixing (PMNS) matrix $U$ is obtained as

$$U = U_\ell^T U_\nu,$$

where

$$U_\ell = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix},$$

with

$$U_{11} = c_{12} c_{13}, \quad U_{12} = s_{12} c_{13},$$

$$U_{13} = s_{13} e^{-i \delta_{13}},$$

$$U_{21} = -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta_{13}},$$

$$U_{22} = c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_{13}},$$

$$U_{23} = s_{23} c_{13},$$

$$U_{31} = s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta_{13}},$$

$$U_{32} = -c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_{13}},$$

$$U_{33} = c_{23} c_{13}. $$

We used the abbreviations $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ ($i, j = 1, 2, 3$) where $\theta_{ij}$ is a mixing angle in the charged lepton (neutrino) sector. $\theta_{ij}$ denotes the CP violating phase in the charged lepton (neutrino) sector.

As the observables, the sine and cosine of the three mixing angles of the PMNS matrix $U$ are given by

$$s_{12}^2 = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2}, \quad s_{23}^2 = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2}, \quad s_{13}^2 = |U_{\tau3}|^2,$$

$$c_{12}^2 = \frac{|U_{e1}|^2}{1 - |U_{e3}|^2}, \quad c_{23}^2 = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2}. \quad (42)$$

For example, since we obtain the following relations

$$U_{e3} = U_{e1} U_{\nu_1} + U_{e2} U_{\nu_2} + U_{e3} U_{\nu_3},$$

$$U_{\mu3} = U_{\mu1} U_{\nu_1} + U_{\mu2} U_{\nu_2} + U_{\mu3} U_{\nu_3}. \quad (43)$$

the lepton mixing angle $\theta_{23}$ depends not only on the mixing angles in the neutrino sector but also on the mixing angles in the charged lepton sector. Thus, the predicted $\sin^2 \theta_{23}$ in the Figure should be modified if the charged lepton mixing matrix $U_\ell$ is no longer the identity matrix. The detail of this modification depends on the models of the charged lepton mixings.

In this paper, we have assumed that the mass matrix of the charged leptons is diagonal and real. In this case, we obtain $U = U_\nu$ for $U_\ell = I$ where $I$ denotes the identity matrix; however, once condition $C3$ or Eq. (21) is allowed, then the charged lepton mass matrix is no longer diagonal and $U_\ell$ is no longer an identity matrix, and hence $U = U_\ell^T U_\nu$.

Our assumption, $U = U_\nu$, should be interpreted as $U \sim U_\nu$ with $U_\ell \sim I$. In this case, the contribution coming from charged lepton sector should be negligible, which is possible only if the branching ratios of the lepton flavor violating processes are far below the experimental limits; this nealy leads to the $C2$ condition. Thus, the abbreviation “NZ” in Tables III and VI indicates nonvanishing but tiny values.

If the branching ratios have measurable magnitudes for experiments in the near future, a more significant contribution of the charged leptons to the lepton mixing matrix may be necessary. We would like to discuss the details of this topic in a separate work in the future.

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