CLASSIFICATION OF IRREDUCIBLE BOUNDED WEIGHT MODULES OVER THE DERIVATION LIE ALGEBRAS OF QUANTUM TORI

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Abstract. Let $d > 1$ be an integer, $q = (q_{ij})_{d \times d}$ a $d \times d$ complex matrix satisfying $q_{ii} = 1$, $q_{ij} = q_{ji}^{-1}$ with all $q_{ij}$ being roots of unity. Let $C_q$ be the rational quantum torus algebra associated with $q$, and $\text{Der}(C_q)$ its derivation Lie algebra. In this paper, we give a complete classification of irreducible bounded weight modules over $\text{Der}(C_q)$. They turn out to be irreducible sub-quotients of $\text{Der}(C_q)$-module $V^\alpha(V, W)$ for a finite dimensional irreducible $\mathfrak{gl}_d$-module $V$, a finite dimensional $\Gamma$-graded-irreducible $\mathfrak{gl}_N$-module $W$, and $\alpha \in \mathbb{C}^d$ where the integer $N$ is uniquely determined by $q$.

Keywords: quantum tori; derivation algebra; weight module; irreducible module.

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1. Introduction

Let $d > 1$ be an integer, $q = (q_{ij})_{d \times d}$ be a $d \times d$ complex matrix satisfying $q_{ii} = 1$, $q_{ij} = q_{ji}^{-1}$ with all $q_{ij}$ being roots of unity. In the present paper, we consider the rational quantum torus algebra $C_q$ associated to $q$, and its derivation algebra $\text{Der}(C_q)$. The algebra $C_q$ is an important algebra, since it is the coordinate algebra of a large class of extended affine Lie algebras (See [BGK]) and shows up in the theory of noncommutative geometry (See [BVF]). When all $q_{ij} = 1$, the algebra $\text{Der}(C_q)$ is the classical Witt algebra $W_d$, i.e., the derivation algebra of the Laurent polynomial algebra $\mathcal{A} = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_d^{\pm 1}]$, see [RSS, BF1], which is also known as the Lie algebra of vector fields on a $d$-dimensional torus. It is natural to consider $\text{Der}(C_q)$ as the a quantization of the Witt algebras $W_d$.

The representation theory of Witt algebras was studied by many mathematicians and physicists for the last couple of decades, see [B, BF2, BMZ, E1, E2, GLZ, L3, L4, L5, LLZ, MZ, TZ, Z1, Z2]. In 1986, Shen defined a class of modules $F_b^\alpha(V) = V \otimes \mathcal{A}$ over the Witt algebra $W_d$ for $\alpha \in \mathbb{C}^d$ and an irreducible module $V$ over the general linear Lie algebra $\mathfrak{gl}_d$ on which the identity matrix acts as multiplication by a complex number $b$, see [Sh]. These modules were also given by Larsson...
in 1992, see [L3]. In 1996, Eswara Rao [E1] determined necessary and sufficient conditions for these modules to be irreducible when V is finite dimensional, see [GZ] for a simplified proof. Recently it was proven that the $\mathcal{W}_d$-modules $F_b^\alpha(V)$ are always irreducible when V is infinite dimensional, see [LZ2].

Very recently Billig and Futorny [BF2] gave a complete classification of all irreducible weight modules over $\mathcal{W}_d$ with finite dimensional weight spaces. More precisely, they actually showed that any irreducible bounded weight modules over $\mathcal{W}_d$ is isomorphic to some irreducible subquotient of $F_b^\alpha(V)$. To achieve this result, they introduced a new concept: the A-cover $\hat{M}$ for any bounded weight $\mathcal{W}_d$-module M. Thus they reduced the classification of irreducible bounded $\mathcal{W}_d$-modules to the classification of irreducible bounded $A\mathcal{W}_d$-modules. Using the classification of irreducible bounded $A\mathcal{W}_d$-modules in [E2, B], they classified all irreducible bounded weight modules over $\mathcal{W}_d$.

Lin and Tan defined in [LT] a class of uniformly bounded irreducible weight modules over $\text{Der}(\mathbb{C}_q)$, which generalized the construction given by Shen. These modules were clearly characterized in [LZ3]. But these modules can not exhaust all simple bounded weight modules over $\text{Der}(\mathbb{C}_q)$, since a class of more general irreducible modules $\mathcal{V}^\alpha(V, W)$ were constructed in [LZ1], see (2.5). Moreover we showed in [LZ1] that any irreducible $ZD$-weight module (similar to the notion of $A\mathcal{W}_d$-modules, see Definition 2.2) with finite dimensional weight spaces is isomorphic to some $\mathcal{V}^\alpha(V, W)$ for a finite dimensional irreducible $\mathfrak{gl}_d$-module V, a finite dimensional $\Gamma$-graded-irreducible $\mathfrak{gl}_N$-module W, and $\alpha \in \mathbb{C}^d$, where $Z := Z(\mathbb{C}_q)$ is the center of $\mathbb{C}_q$ and $D := \text{Der}(\mathbb{C}_q)$.

In the present paper, we determine irreducible bounded weight modules over $\text{Der}(\mathbb{C}_q)$, i.e., we prove a quantum version of the result by Billig and Futorny [BF2]. For an irreducible bounded weight $\text{Der}(\mathbb{C}_q)$-module M, we construct a $ZD$-module $\hat{M}$ which is called the $ZD$-cover of M. Here the $ZD$-cover $\hat{M}$ is different from the $A\mathcal{W}_d$-cover $\mathcal{W}_d \otimes M$ in [BF2], since $\mathcal{W}_d \otimes M$ is no longer a $ZD$-module in our case. The $ZD$-cover $\hat{M}$ is actually a $ZD$-quotient module of $\mathbb{C}_q' \otimes M$, see Definition 3.4. Using this technique, we prove that any irreducible bounded $\text{Der}(\mathbb{C}_q)$-weight module is isomorphic to some irreducible sub-quotient of $\mathcal{V}^\alpha(V, W)$ for a finite dimensional irreducible $\mathfrak{gl}_d$-module V, a finite dimensional $\Gamma$-graded-irreducible $\mathfrak{gl}_N$-module W, and $\alpha \in \mathbb{C}^d$. See Theorem 2.5. Actually, all such irreducible sub-quotients of $\mathcal{V}^\alpha(V, W)$ are clearly described in [E2, GZ].

Throughout this paper we denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers, rational numbers and complex numbers, respectively. We use $E_{ij}$ to denote the matrix of suitable size in the context with a 1 in the $(i, j)$ position and zeros elsewhere.
2. Notation and the main result

In this section we will collect notation and related results, then state our main theorem.

We fix a positive integer $d > 1$. Denote vector space of $d \times 1$ matrices by $\mathbb{C}^d$. Denote its standard basis by $\{e_1, e_2, \ldots, e_d\}$. Let $(\cdot | \cdot)$ be the standard symmetric bilinear form such that $(u|v) = u^T v \in \mathbb{C}$, where $u^T$ is the matrix transpose of $u$.

Let $q = (q_{ij})_{i,j=1}^d$ be a $d \times d$ matrix over $\mathbb{C}$ satisfying $q_{ii} = 1$, $q_{ij} = q_{ji}^{-1}$, where $q_{ij}$ are roots of unity for all $1 \leq i, j \leq d$. Such a matrix $q$ is called rational, see [N].

**Definition 2.1.** The rational quantum torus $\mathbb{C}_q$ is the unital associative algebra over $\mathbb{C}$ generated by $t_1^{\pm 1}, \ldots, t_d^{\pm 1}$ and subject to the defining relations $t_it_j = q_{ij}t_jt_i$, $t_it_i^{-1} = t_i^{-1}t_i = 1$ for all $1 \leq i, j \leq d$.

For convenience, denote $t^n = t_1^{n_1}t_2^{n_2} \cdots t_d^{n_d}$ for any $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. For any $n, m \in \mathbb{Z}^d$, we define the functions $\sigma(n, m)$ and $f(n, m)$ by

$$t^n t^m = \sigma(n, m) t^{n+m}, \quad t^n t^m = f(n, m) t^{m+n}. $$

It is well-known that

$$\sigma(n, m) = \prod_{1 \leq i < j \leq d} q_{ij}^{n_i m_j}, \quad f(n, m) = \prod_{i,j=1}^d q_{ij}^{n_i m_j},$$

and $f(n, m) = \sigma(n, m)\sigma(m, n)^{-1}$, see [BGK]. We also define

$$\text{Rad}(f) = \{n \in \mathbb{Z}^d \mid f(n, \mathbb{Z}^d) = 1\}, \quad \Gamma = \mathbb{Z}^d / \text{Rad}(f).$$

Clearly, the center $Z(\mathbb{C}_q)$ of $\mathbb{C}_q$ is spanned by $t^r$ for $r \in \text{Rad}(f)$.

From Theorem 4.5 in [N], up to an isomorphism of $\mathbb{C}_q$, we may assume that $q_{2i,2i-1} = q_{i}, q_{2i-1,2i} = q_{i}^{-1}$, for $1 \leq i \leq z$, and other entries of $q$ are all 1, where $z \in \mathbb{N}$ with $2z \leq d$ and with the orders $k_i$ of $q_{i}, 1 \leq i \leq z$ as roots of unity satisfy $k_{i+1}|k_{i}, 1 \leq i < z$. For an integer $l \in \{1, \ldots, d\}$, let

$$\xi_l = \begin{cases} k_i e_{2i-1}, & \text{if } l = 2i - 1 \leq 2z, \\ k_i e_{2i}, & \text{if } l = 2i \leq 2z, \\ e_l, & \text{if } l > 2z. \end{cases}$$

Then $\{\xi_1, \ldots, \xi_d\}$ is a $\mathbb{Z}$-basis of the subgroup $\text{Rad}(f)$.

Throughout the present paper, for convenience we assume that $q$ is of the above simple form. The striking advantage of this form is that $\sigma(r, n) = \sigma(n, r) = 1$ (i.e., $t^n t^r = t^{n+r}$) for all $r \in \text{Rad}(f)$ and $n \in \mathbb{Z}^d$. In this case, we know that

$$\Gamma = \bigoplus_{i=1}^z \left( (\mathbb{Z}/(k_{i}\mathbb{Z})) \oplus (\mathbb{Z}/(k_{i}\mathbb{Z})) \right).$$

Let $\text{Der}(\mathbb{C}_q)$ be the derivation Lie algebra of $\mathbb{C}_q$. Let $\text{Der}(\mathbb{C}_q)_n$ be the set of homogeneous elements of $\text{Der}(\mathbb{C}_q)$ with degree $n \in \mathbb{Z}^d$. Then
from Lemma 2.48 in [BGK], we have

\[ \text{Der}(\mathbb{C}_q) = \bigoplus_{n \in \mathbb{Z}^d} \text{Der}(\mathbb{C}_q)_n, \quad \text{Der}(\mathbb{C}_q)_n = \begin{cases} \mathbb{C}\text{ad}(t^n), & \text{if } n \not\in \text{Rad}(f), \\ \bigoplus_{i=1}^d \mathbb{C}t^n \partial_i, & \text{if } n \in \text{Rad}(f), \end{cases} \]

where \( \partial_i \) is the degree derivation defined by \( \partial_i(t^n) = n_i t^n \) for any \( n \in \mathbb{Z}^d \). We will simply denote \( \text{ad}(t^n) \) in \( \text{Der}(\mathbb{C}_q) \) by \( t^n \) for \( n \not\in \text{Rad}(f) \).

For \( n \in \text{Rad}(f), u \in \mathbb{C}^d \), we denote \( D(u, n) = t^n \sum_{i=1}^d u_i \partial_i \). The Lie bracket of \( \text{Der}(\mathbb{C}_q) \) is given by:

1. \( [t^s, t^{s'}] = (\sigma(s, s') - \sigma(s', s))t^{s+s'} \);
2. \( [D(u, r), t^s] = (u|s)t^{r+s} \);
3. \( [D(u, r), D(u', r')] = D(w, r + r') \),

where \( w = (u|r')u' - (u'|r)u, s, s' \in \mathbb{Z}^d \setminus \text{Rad}(f), r, r' \in \text{Rad}(f) \), and we have used that \( \sigma(r, s) = \sigma(r, r') = 1 \).

We can see that \( \mathfrak{h} := \text{span}\{D(u, 0) \mid u \in \mathbb{C}^d\} \) is the Cartan subalgebra (the maximal toral subalgebra) of \( \text{Der}(\mathbb{C}_q) \). Moreover the subalgebra of \( \text{Der}(\mathbb{C}_q) \) spanned by \( \{t^s \mid s \in \mathbb{Z}^d \setminus \text{Rad}(f)\} \) is isomorphic to the derived algebra \( \mathbb{C}_q' := [\mathbb{C}_q, \mathbb{C}_q] \) of \( \mathbb{C}_q \). Let

\[ W_a = \text{span}\{D(u, r) \mid r \in \text{Rad}(f), u \in \mathbb{C}^d\} \]

which is indeed isomorphic to the classical Witt algebra. Note that the algebra \( \text{Der}(\mathbb{C}_q) \) has a nature structure of \( Z(\mathbb{C}_q) \)-module, i.e.,

\[ t^r \cdot t^s = t^{s+r}, \quad t^r \cdot D(u, r') = D(u, r + r'), \]

where \( r, r' \in \text{Rad}(f), s \in \mathbb{Z}^d \setminus \text{Rad}(f), u \in \mathbb{C}^d \).

A \( \text{Der}(\mathbb{C}_q) \)-module \( V \) is called a weight module provided that the action of \( \mathfrak{h} \) on \( V \) is diagonalizable. For any weight module \( V \) we have the weight space decomposition

\[ V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \tag{2.2} \]

where \( \mathfrak{h}^* = \text{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C}) \) and

\[ V_\lambda = \{ v \in V \mid \partial v = \lambda(\partial)v \text{ for all } \partial \in \mathfrak{h} \}. \]

The space \( V_\lambda \) is called the weight space corresponding to the weight \( \lambda \).

If there is an integer \( k \in \mathbb{N} \) such that \( \dim_\mathbb{C} V_\lambda < k \) for all \( \lambda \in \mathfrak{h}^* \), the weight module \( V \) is called a bounded weight module. The following notion is important to our later arguments.

**Definition 2.2.** A \( \mathbb{Z}^d \)-module \( V \) is a module both for the Lie algebra \( \text{Der}(\mathbb{C}_q) \) and the unital commutative associative algebra \( Z(\mathbb{C}_q) \), with these two structures being compatible:

\[ (u|r')t^{r+r'}v = D(u, r)t^{r'}v - t^{r'}D(u, r)v, \tag{2.3} \]

\[ t^r t^s v = t^{r+s}v, \tag{2.4} \]

for any \( r, r' \in \text{Rad}(f), s \not\in \text{Rad}(f), v \in V \).
Clearly $C_q$ is a $ZD$-module under the adjoint action of $\text{Der}(C_q)$ and the action of $Z(C_q)$ defined as follows:

$$t^n t^m = t^{n+m}, \quad r \in \text{Rad}(f), \quad n \notin \text{Rad}(f).$$

A class of $ZD$-modules was constructed in [LZ1]. Before introducing these modules, we first recall the twisted loop algebra realization of $C_q$.

Let $\mathcal{I} = \text{span}\{t^n t^m - t^m t^n \mid n, m \in \mathbb{Z}^d, r \in \text{Rad}(f)\}$ which is an ideal of the associative algebra $C_q$. Then from Theorem 2.2 in [Z2], we know that

$$C_q/\mathcal{I} \cong \bigotimes_{n=1}^z M_{k_i}(C) \cong M_N(C)$$

as associative algebras with $N = \prod_{i=1}^z k_i$. It is well known that $\mathfrak{gl}_{k_i}, 1 \leq i \leq z$, as the associative algebra $M_{k_i}(C)$, is generated by $X_{2i-1}, X_{2i}$ with

$$X_{2i-1} = E_{11} + q_i E_{22} + \cdots + q_i^{k_i-1} E_{k_i,k_i},$$

$$X_{2i} = E_{22} + E_{23} + \cdots + E_{k_i-1,k_i} + E_{k_i,1},$$

which satisfy $X_{2i}^{k_i} = X_{2i-1}^2 = 1, X_{2i} X_{2i-1} = q_i X_{2i-1} X_{2i}$. We denote $\bigotimes_{i=1}^z X_{2i-1}^{m_i} X_{2i}^{n_i} \bigotimes_{i=1}^z N$ by $X^n$ for each $n = (m_1, \cdots, n_d)^T \in \mathbb{Z}^d$. Identifying $\mathfrak{gl}_N$ with $\bigotimes_{i=1}^z \mathfrak{gl}_{k_i}$ as associative algebras, $\mathfrak{gl}_N$ is spanned by $X^n, n \in \mathbb{Z}^d$. Note that $X^r$ equals to the identity matrix $E$ in $\mathfrak{gl}_N$ for each $r \in \text{Rad}(f)$.

**Lemma 2.3.** (See [ABFP]) As associative algebras,

$$C_q \cong \bigoplus_{n \in \mathbb{Z}^d} (C X^n \otimes x^n),$$

where the right hand side is a $\mathbb{Z}^d$-graded subalgebra of $\mathfrak{gl}_N \otimes \mathcal{A}$.

Clearly, $\mathfrak{gl}_N$ is a $\Gamma$-graded Lie algebra with the gradation

$$\mathfrak{gl}_N = \bigoplus_{\bar{n} \in \Gamma} (\mathfrak{gl}_{\bar{n}}),$$

where $(\mathfrak{gl}_{\bar{n}})_{\bar{n}} = C X^n$.

A module $W$ over the Lie algebra $\mathfrak{gl}_N$ is called a $\Gamma$-graded $\mathfrak{gl}_N$-module if $W$ has a subspace decomposition $W = \bigoplus_{\bar{n} \in \Gamma} W_{\bar{n}}$ such that $(\mathfrak{gl}_{\bar{n}})_{\bar{n}} W_{\bar{n}} \subseteq W_{\bar{m}+\bar{n}}$ for all $m, n \in \mathbb{Z}^d$. A $\Gamma$-graded $\mathfrak{gl}_N$-module $W$ is $\Gamma$-graded-irreducible if it has no nonzero proper $\Gamma$-graded submodules. We remark that all finite dimensional $\Gamma$-graded $\mathfrak{gl}_N$-modules were classified in [EK].

For any irreducible finite dimensional $\mathfrak{gl}_q$-module $V$ such that $E$ acts on $V$ by a scalar $b$, any $\Gamma$-graded-irreducible $\mathfrak{gl}_N$-module $W = \bigoplus_{\bar{n} \in \Gamma} W_{\bar{n}}$ where $E$ acts as identity, and any $\alpha \in \mathbb{C}^d$, let

$$V^n(V, W) = \bigoplus_{n \in \mathbb{Z}^d} (V \otimes W_{\bar{n}} \otimes t^n).$$

Then $V^n(V, W)$ becomes a $ZD$-module if we define the following actions

$$t^n t^m = t^{n+m}, \quad r \in \text{Rad}(f), \quad n \notin \text{Rad}(f).$$

$$t^n t^m = t^{n+m}, \quad r \in \text{Rad}(f), \quad n \notin \text{Rad}(f).$$
(1) \( t^s(v \otimes w_n \otimes t^n) = v \otimes (X^s w_n) \otimes t^{n+s} \);
(2) \( D(u,r)(v \otimes w_n \otimes t^n) = \left( (u | n + \alpha) v + (ru^T)v \right) \otimes w_n \otimes t^{n+r} \),
where \( u \in \mathbb{C}^d, v \in V, w_n \in W_n \) and \( r \in \text{Rad}(f) \), \( s \in \mathbb{Z}^d \), \( X^s \in \mathfrak{gl}_N \).

From Theorem 4.4 in [LZ1], all irreducible \( \mathbb{Z}D \)-modules with finite dimensional weight spaces are proved to be of the form \( V(V;W) \). Restricted to \( \text{Der}(\mathbb{C}q) \), \( V(V;W) \) is not necessarily irreducible. The following result easily follows from [E1] and [GZ], which gives all irreducible subquotients of the \( \text{Der}(\mathbb{C}q) \)-module \( V^a(V,W) \).

**Lemma 2.4.** The \( \text{Der}(\mathbb{C}q) \)-module \( V^a(V,W) \) is reducible if and only if \( \dim W = 1 \) and one of the following holds

(a). the highest weight of \( V \) is the fundament weight \( \omega_k \) of \( \mathfrak{sl}_d \) and \( b = k \), where \( k \in \mathbb{Z} \) with \( 1 \leq k \leq d-1 \);

(b). \( \dim V = 1, \alpha \in \mathbb{Z}^d \) and \( b \in \{0,d\} \).

We can easily see that when the \( \text{Der}(\mathbb{C}q) \)-module \( V^a(V,W) \) is reducible it has a unique maximal submodule.

In the present paper, we will obtain the classification of irreducible uniformly bounded modules over \( \text{Der}(\mathbb{C}q) \). More precisely, we will have

**Theorem 2.5.** Let \( d > 1 \) be an integer, \( q = (q_{ij})_{d \times d} \) be a \( d \times d \) complex matrix which is rational. Let \( M \) be an irreducible bounded weight \( \text{Der}(\mathbb{C}q) \)-module. Then there exist a finite dimensional irreducible \( \mathfrak{gl}_d \)-module \( V \), a finite dimensional \( \Gamma \)-graded irreducible \( \mathfrak{gl}_N \)-module \( W \), and \( \alpha \in \mathbb{C}^d \) such that \( M \) is isomorphic to some irreducible sub-quotient of \( V^a(V,W) \).

3. Proof of Theorem 2.5

In this section we will prove Theorem 2.5.

Let \( M \) be an irreducible bounded weight \( \text{Der}(\mathbb{C}q) \)-module. The irreducibility of \( M \) implies that there is an \( \alpha \in \mathbb{C}^d \) such that \( M = \bigoplus_{n \in \mathbb{Z}^d} M_{\alpha+n} \), where

\[ M_{\alpha+n} = \{ v \in M \mid \partial_i(v) = (\alpha_i + n_i) v, \ 1 \leq i \leq d \}. \]

In [BF2], in order to define the \( \mathcal{AWD} \)-cover of \( M \), they considered the the tensor product \( \mathcal{W}_d \otimes M \) of the adjoint module and \( M \). In our case, the module \( \mathcal{W}_d \otimes M \) is still an \( \mathcal{AWD} \)-module, but it is no longer a \( ZD \)-module. Now we turn to the tensor product \( \mathbb{C}'_q \otimes M \) of the \( \text{Der}(\mathbb{C}q) \)-modules \( \mathbb{C}'_q \) and \( M \), since \( \mathbb{C}'_q \) itself is a \( ZD \)-module.

**Lemma 3.1.** The \( \mathcal{W}_d \)-module \( \mathbb{C}'_q \otimes M \) is a \( ZD \) module if we define the action of \( Z(\mathbb{C}q) \) by

\[ t^r(t^n \otimes w) = t^{n+r} \otimes w, \]

where \( r \in \text{Rad}(f), n \not\in \text{Rad}(f), w \in M \).
Proof. For any \( u \in \mathbb{C}^d, m, r \in \text{Rad}(f), n, s \notin \text{Rad}(f) \) and \( w \in M \), we have that

\[
D(u, m)t^n(t^n \otimes w) - t^nD(u, m)(t^n \otimes w) = (u | r + n)t^{n+m+r} \otimes w + t^{n+r} \otimes D(u, m)w - (u | n)t^{n+m+r} \otimes w - t^{n+r} \otimes D(u, m)w = (u | r)t^{n+m+r} \otimes w,
\]

and

\[
t^s t^n(t^n \otimes w) = (\sigma(s, n + r) - \sigma(n + r, s))t^{n+s+r} \otimes w + t^{n+r} \otimes t^s w = (\sigma(s, n) - \sigma(n, s))t^{n+s+r} \otimes w + t^{n+r} \otimes t^s w = t^s t^n(t^n \otimes w).
\]

In the second equality, we have used the fact that \( \sigma(r, s) = \sigma(s, r) = 1 \). So the action of \( \text{Der}(\mathbb{C}_q) \) and \( \mathbb{Z}D \) is compatible, hence \( \mathbb{C}'_q \otimes M \) is a \( \mathbb{Z}D \)-module.

Now we define the linear map

\[
\pi : \mathbb{C}'_q \otimes M \rightarrow M
\]

by \( \pi(y \otimes w) = yw \) for \( y \in \mathbb{C}'_q, w \in M \). This map has the following nice properties.

**Lemma 3.2.** The map \( \pi \) is a \( \text{Der}(\mathbb{C}_q) \)-module homomorphism, and it is surjective if \( \mathbb{C}'_q M \neq 0 \).

**Proof.** For all \( n \notin \text{Rad}, r \in \text{Rad}, w \in M \), we have that

\[
\pi(D(u, r)(t^n \otimes w)) = [D(u, r), t^n]w + t^nD(u, r)w - D(u, r)t^n w = D(u, r)\pi(t^n \otimes w).
\]

So \( \pi \) is a \( \text{Der}(\mathbb{C}_q) \)-module homomorphism. It is easy to see that \( \mathbb{C}'_q M \) is a submodule of \( M \). Then the irreducibility of \( M \) implies that \( \pi \) is surjective. \( \square \)

Let \( J \) be the subspace of \( \mathbb{C}'_q \otimes M \) spanned by the set

\[
\{ \sum_{n \in I} t^n \otimes v_n | n \notin \text{Rad}(f), v_n \in M, \sum_{n \in I} t^{n+\gamma}v_n = 0, \text{for all } \gamma \in \text{Rad}(f) \}.
\]

Clearly, \( J \subset \ker(\pi) \).

**Lemma 3.3.** The subspace \( J \) is a \( \mathbb{Z}D \)-submodule of \( \mathbb{C}'_q \otimes M \).

**Proof.** Let \( \eta = \sum_{n \in I} t^n \otimes v_n \in J \), where \( I \subset \mathbb{Z}^d \setminus \text{Rad}(f) \) is finite. Then

\[
\sum_{n \in I} t^{n+r}v_n = 0, \text{ for all } r \in \text{Rad}(f).
\]

To show that \( J \) is a \( \mathbb{Z}D \)-submodule, we only need to show that

\[
t' \eta, D(u, r')\eta, t^s \eta \in J, \text{ for any } r' \in \text{Rad}(f), s \notin \text{Rad}(f).
\]
From $\sum_{n \in I} t^{n+r+r'} v_n = 0$, we see that

$$
\sum_{n} (u \mid n) t^{n+r+r'} v_n + \sum_{n \in I} t^{n+r} D(u, r') v_n
= \sum_{n \in I} (u \mid n) t^{n+r+r'} v_n + \sum_{n \in I} [t^{n+r}, D(u, r')] v_n + D(u, r') \sum_{n \in I} t^{n+r} v_n
= \sum_{n \in I} (u \mid n) t^{n+r+r'} v_n - \sum_{n \in I} (u \mid n + r) t^{n+r+r'} v_n
= - (u \mid r) \sum_{n \in I} t^{n+r+r'} v_n = 0.
$$

Note that

$$
t^r \eta = \sum_{n \in I} t^{n+r'} \otimes v_n,
$$

$$
D(u, r') \eta = \sum_{n \in I} (u \mid n) t^{n+r'} \otimes v_n + \sum_{n \in I} t^n \otimes D(u, r') v_n.
$$

So $t^r \eta, D(u, r') \eta \in J$.

From

$$
\sum_{n \in I} (\sigma(s, n) - \sigma(n, s)) t^{n+s+r} v_n + \sum_{n \in I} t^{n+r} t^s v_n
= \sum_{n \in I} [t^s, t^{n+r}] v_n + \sum_{n \in I} [t^{n+r}, t^s] v_n + t^s \sum_{n \in I} t^{n+r} v_n
= 0,
$$

and

$$
t^s \eta = \sum_{n \in I} (\sigma(s, n) - \sigma(n, s)) t^{n+s} \otimes v_n + \sum_{n \in I} t^n \otimes t^s v_n,
$$

we see that $t^s \eta \in J$. So $J$ is a $ZD$-submodule.

Now we introduce the concept of $ZD$-cover of a $W_d$-module.

**Definition 3.4.** The $ZD$-module $\hat{M} := (C_q \otimes M) / J$ is called the $ZD$-cover of $W_d$-module $M$.

Since $J \subset \ker(\pi)$, $\pi$ induces an epimorphism from $\hat{M}$ to $M$ which is still denoted by $\pi$. For $t^n \otimes v \in C_q \otimes M$, denote its image in $\hat{M}$ by $\psi(t^n, v)$. The next key step is to show that $\hat{M}$ is a bounded weight module. We will use the solenoidal Lie algebra (or the centerless higher rank Virasoro algebra) as an auxiliary tool.

Recall from [BF2] that a vector $u \in C^d$ is generic if $(u, r) \neq 0$ for any $r \in \mathbb{Z}^d \setminus \{0\}$. For a generic vector $u \in C^d$, let

$$
e_r = D(u, r) \text{ for } r \in \text{Rad}(f).
$$

The subalgebra $W_u$ of $\text{Der}(C_q)$ spanned by $e_r, r \in \text{Rad}(f)$ is a solenoidal Lie algebra.
From now on, we fix a generic vector \( u \in \mathbb{C}^d \). It is easy to see that the Lie bracket of \( W_u \) is given by
\[
[e_r, e_r'] = (u \mid r' - r)e_{r+r'}, \quad r, r' \in \text{Rad}(f).
\]

For \( r, h \in \text{Rad}(f), l \geq 0 \), we recall the differentiators in the universal enveloping algebra of \( \text{Der}(\mathbb{C}_q) \):
\[
\Omega_r^{(l,h)} := \sum_{i=0}^l (-1)^i \binom{l}{i} e_{r-i} e_{ih}.
\]

These operators were used in many situations, see for example [BF2, LLZ].

**Lemma 3.5.** Let \( M \) be an irreducible bounded \( \text{Der}(\mathbb{C}_q) \)-module. Then there exists an integer \( l > 1 \) such that for all \( r, h \in \text{Rad}(f) \), the differentiator \( \Omega_r^{(l,h)} \) annihilates \( M \).

**Proof.** For any \( n \in \mathbb{Z}^d \), the subspace \( M(n) := \bigoplus_{r \in \text{Rad}(f)} M_{\alpha+n+r} \) is a bounded module over \( W_u \). Clearly \( M(m) = M(n) \) for all \( m, n \in \mathbb{Z}^d \) with \( m - n \in \text{Rad}(f) \). For an \( M(n) \), by Proposition 4.6 in [BF1], there exists \( K \in \mathbb{N} \) such that for all \( r, h \in \text{Rad}(f) \) and \( l > K \), the differentiator \( \Omega_r^{(l,h)} \) annihilates \( M(n) \). Since the index of the subgroup \( \text{Rad}(f) \) in \( \mathbb{Z}^d \) is finite, \( M \) is a sum of a finite number of \( M(n) \). Thus there exists a large enough \( l \) such that for all \( r, h \in \text{Rad}(f) \), the differentiator \( \Omega_r^{(l,h)} \) annihilates \( M \). \( \square \)

**Theorem 3.6.** Let \( M \) be an irreducible bounded \( \text{Der}(\mathbb{C}_q) \)-module such that \( \mathbb{C}_q' M \neq 0 \). Then the \( \mathbb{Z} \mathbb{D} \)-cover of \( \hat{M} \) is bounded.

**Proof.** Let \( \Delta \) be a complete coset representatives of the subgroup \( \text{Rad}(f) \) in \( \mathbb{Z}^d \) with \( 0 \notin \Delta \). Clearly \( \Delta \) is a finite set. For a weight \( \lambda \in \mathbb{C}^d \), the weight space \( \hat{M}_\lambda \) is spanned by
\[
\{ \psi(t^{n+r}, M_{\lambda-n-r} : n \in \Delta, r \in \text{Rad}(f)) \}.
\]

We introduce a norm on \( \text{Rad}(f) \):
\[
\|r\| = \sum_{i=1}^d |\gamma_i|,
\]
where \( r = \sum_{i=1}^d \gamma_i \xi_i \in \text{Rad}(f) \), \( \{\xi_1, \cdots, \xi_d\} \) is the \( \mathbb{Z} \)-basis of \( \text{Rad}(f) \) defined in (2.1). By Lemma 3.5, there exists an integer \( l > 1 \) such that the differentiator \( \Omega_r^{(l,h)} \) annihilates \( M \) for all \( r \in \text{Rad}(f), i \in \{1, \ldots, d\} \).

Let \( S \) be the subspace of \( M \) spanned by
\[
\psi(t^{n+r}, M_{\lambda-n-r}) \quad n \in \Delta, r \in \text{Rad}(f) \quad \text{with} \quad \|r\| \leq \frac{ld}{2},
\]
plus \( \psi(t^{n_0+r_0}, M_0) \) if \( \lambda = n_0 + r_0 \) for some \( n_0 \in \Delta, r_0 \in \text{Rad}(f) \). Clearly \( S \) is finite dimensional.
Claim: $\overline{M}_\lambda = S$. 

In order to prove this claim, we only need to check that $t^{n+r} \otimes M_{\lambda-n-r}$ belongs to $S$ for any $n \in \Delta$, $r \in \text{Rad}(f)$. We use induction on $\|r\|$. If $|\gamma_i| \leq \frac{1}{2}$ for all $i \in \{1, \ldots, d\}$, then the claim is trivial. On the contrary, we assume that $|\gamma_j| > \frac{1}{2}$ for some $j$. Without loss of generality, we assume that $\gamma_j > \frac{1}{2}$. The case $\gamma_j < -\frac{1}{2}$ follows similarly. Clearly, the norms of $r-\xi_j, \ldots, r-k\xi_j$ are strictly smaller than $\|r\|$. Let $v \in M_{\lambda-n-r}$ with $\lambda - n - r \neq 0$. Since $e_0v = (u \mid \lambda - n - r)v$, we can write $v = e_0w$ for some $w \in M_{\lambda-n-r}$.

From $0 = \Omega_{(i, \xi_j)}t^nw = \sum_{i=0}^{l}(-1)^i i! t^{n+r-i\xi_j}e_{i\xi_j}t^n w$, we see that

$$\sum_{i=0}^{l}(-1)^i i! t^{n+r-i\xi_j}e_{i\xi_j}w = 0,$$

where we have used the fact that $(u|n) \neq 0$. Note that $e_{r-i\xi_j}t^{n+i\xi_j}w = (u \mid n + i\xi_j)t^n w + t^{n+i\xi_j}e_{r-i\xi_j}w$.

From

$$\sum_{i=0}^{l}(-1)^i \begin{pmatrix} l \cr i \end{pmatrix} = \sum_{i=0}^{l}(-1)^i i! \begin{pmatrix} l \cr i \end{pmatrix} = 0,$$

we get that

$$\sum_{i=0}^{l}(-1)^i \begin{pmatrix} l \cr i \end{pmatrix} t^{n+r-i\xi_j}e_{i\xi_j}w + \sum_{i=0}^{l}(-1)^i \begin{pmatrix} l \cr i \end{pmatrix} t^{n+i\xi_j}e_{r-i\xi_j}w = 0.$$ 

Thus

$$t^{n+r}v = -\sum_{i=1}^{l}(-1)^i \begin{pmatrix} l \cr i \end{pmatrix} t^{n+r-i\xi_j}e_{i\xi_j}w - \sum_{i=0}^{l}(-1)^i \begin{pmatrix} l \cr i \end{pmatrix} t^{n+i\xi_j}e_{r-i\xi_j}w,$$

i.e.,

$$\psi(t^{n+r}, v) = -\sum_{i=1}^{l}(-1)^i \begin{pmatrix} l \cr i \end{pmatrix} \psi(t^{n+r-i\xi_j}, e_{i\xi_j}w)$$

(3.3)

$$-\sum_{k=0}^{l}(-1)^k \begin{pmatrix} l \cr k \end{pmatrix} \psi(t^{n+k\xi_j}, e_{r-k\xi_j}w).$$

Note that $e_{i\xi_j}w \in M_{\lambda-n-(r-i\xi_j)}$, $e_{r-k\xi_j}w \in M_{\lambda-n-k\xi_j}$ and $\|r-i\xi_j\| < \|r\|$ for any $i \in \{1, \ldots, l\}$, $\|k\xi_j\| \leq \frac{1}{2}$ for any $k \in \{0, 1, \ldots, l\}$, since $d \geq 2$. By induction assumption the right hand side of (3.3) belongs to $S$. Therefore the Claim is true. Hence $\overline{M}_\lambda$ is finite dimensional. The theorem is proved. \[ \square \]

Now we are ready to prove our main theorem.
Proof of Theorem 2.5. If \( C'_q M = 0 \), the module \( M \) is an irreducible module over \( \mathcal{W}_d \). This case was proved in [BF2, Theorem 5.4]. (In the statement of this Theorem, actually \( W \) is taken as a one dimensional \( gl_N \)-module.)

Now we assume that \( C'_q M \neq 0 \). By the irreducibility of \( M \) and the fact that \( C'_q M \) is a submodule of \( M \), we see that \( C'_q M = M \). Thus the homomorphism \( \pi : \hat{M} \rightarrow M \) is surjective.

From [BF2] we know that each irreducible bounded weight \( \mathcal{W}_d \)-module has a support of the form \( \alpha + \text{Rad}(f) \) for some \( \alpha \in \mathbb{C}^d \) (possibly \( 0 \) may be removed from this coset). Since \( [Z^d : \text{Rad}(f)] < \infty \), then \( \hat{M} \) has a composition series of \( ZD \)-submodules:

\[
0 = \hat{M}_0 \subset \hat{M}_1 \subset \cdots \subset \hat{M}_s = \hat{M}.
\]

Thus each quotient \( \hat{M}_i/\hat{M}_{i-1} \) is an irreducible \( ZD \)-module. Let \( k \) be the smallest integer such that \( \pi(\hat{M}_k) \neq 0 \). By the irreducibility of \( M \), we see that \( \pi(\hat{M}_k) = M \) and \( \pi(\hat{M}_{k-1}) = 0 \). Thus we have a surjective \( \text{Der}(C_q) \)-module homomorphism from \( \hat{M}_k/\hat{M}_{k-1} \) to \( M \). By Theorem 4.4 in [LZ1], we know that \( \hat{M}_k/\hat{M}_{k-1} \) is isomorphic to \( \mathcal{V}^\alpha(V,W) \) for some finite dimensional irreducible \( gl_d \)-module \( V \), finite dimensional \( \Gamma \)-graded-irreducible \( gl_N \)-module \( W \), and \( \alpha \in \mathbb{C}^d \). This completes the proof. \( \square \)

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