A free boundary problem for a predator-prey model with double free boundaries

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Abstract.

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1 Introduction

The expanding (migration) of a new or invasive species is one of the most important topics in mathematical ecology. A lot of mathematicians have made efforts to develop various invasion models and investigated them from a viewpoint of mathematical ecology. To describe the invasion and spreading phenomenon, there have been many interesting studies on the existence of positive traveling wave solutions connecting two different equilibria. Also, the study of asymptotic spreading speed plays an important role in invasion ecology since it can be used to predict the mean spreading rate of species. On the other hand, Du and Lin \cite{10} proposed a new mathematical model to understand the expanding of an invasive or new species. Their model is described as a free boundary problem for a logistic diffusion equation:

\begin{equation}
\begin{cases}
  u_t - du_{xx} = u(a - bu), & t > 0, \quad 0 < x < h(t), \\
  u_x(t, 0) = 0, \quad u(t, h(t)) = 0, & t > 0, \\
  h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
  h(0) = h_0, \quad u(0, x) = u_0(x), & 0 < x < h(t),
\end{cases}
\end{equation}

where \( x = h(t) \) is the moving boundary to be determined, \( a, b, d, \mu \) and \( h_0 \) are given positive constants, and \( u_0 \) is a given positive initial function. The dynamics of the free boundary is determined by Stefan-like condition \( h'(t) = -\mu u_x(t, h(t)) \). This condition means that the population pressure at the free boundary is a driving force of the free boundary. Du and Lin \cite{10} have established the existence and uniqueness of global solutions and, furthermore, derived various interesting results about the long time behavior of solution. One of very remarkable results is a spreading-vanishing dichotomy of the species, i.e., the solution \((u, h)\) of \ref{1.1} satisfies one of the following properties:

- **Spreading**: \( h(t) \to \infty, u(t, x) \to a/b \) as \( t \to \infty \);
- **Vanishing**: \( h(t) \to h_\infty \leq (\pi/2)\sqrt{d/a}, \) and \( u(t, x) \to 0 \) as \( t \to \infty \).

When the spreading occurs, it is also proved that the spreading speed approaches to a positive constant \( k_0 \), i.e., \( h(t) = (k_0 + o(1))t \) as \( t \to \infty \). See also the paper of Du and Guo \cite{6, 7}, where a free boundary problem similar to \ref{1.1} was studied in higher space dimension and the same spreading-vanishing dichotomy has been established. In \cite{18}, \ref{1.1} was discussed with \( u_x(t, 0) = 0 \) replaced by \( u(t, 0) = 0 \).

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A variety of reaction-diffusion systems are used to describe some phenomena arising in population ecology. A typical model is the following classical Lotka-Volterra type predator-prey system in a one-dimensional habitat (under the suitable rescaling)

\[
\begin{align*}
    u_t - u_{xx} &= u(1 - u + av), & t > 0, & x \in \mathbb{R}, \\
    v_t - Dv_{xx} &= v(b - v - cu), & t > 0, & x \in \mathbb{R},
\end{align*}
\]  

(1.2)

where \(u(t,x), v(t,x)\) denote, respectively, the population densities of predator and prey at the position \(x\) and time \(t\).

Understanding of spatial and temporal behaviors of interacting species in ecological systems is a central issue in population ecology. One aspect of great interest for a model with multispecies interactions is whether the species can spread successfully. Motivated by the work of Du and Lin [10], in the present paper we shall study a free boundary problem associated with [1,2] to realize the expanding mechanism of the species. In the real world, the following two kind of phenomenons often occur:

(i) At the initial state, one kind of prey species (for example, pest species) occupied the whole space or a large region. In order to control such prey species we put one kind of predator species (natural enemies) in some bounded region or a small region (initial habitat).

(ii) There is some kind of species (prey) in the whole space or a large region, and at some time (initial time) another type species (the new or invasive species, predator) enters some bounded region or a small region (initial habitat).

In general, the predator has a tendency to emigrate from the boundaries to obtain their new habitat, i.e., it will move outward along the unknown curves (free boundaries) as time increases. It is assumed that the movement speeds of free boundaries are proportional to the gradient of predator. We want to realize the dynamics/variations of predator, prey and free boundaries. According to the above arguments, the model we are concerned here is the following free boundary problem

\[
\begin{align*}
    u_t - u_{xx} &= u(1 - u + av), & t > 0, & g(t) < x < h(t), \\
    u(t,x) &= 0, & t \geq 0, & x \notin (g(t), h(t)), \\
    v_t - Dv_{xx} &= v(b - v - cu), & t > 0, & x \in \mathbb{R}, \\
    u = 0, & g'(t) = -\mu u_x, & t \geq 0, & x = g(t), \\
    u = 0, & h'(t) = -\mu u_x, & t \geq 0, & x = h(t), \\
    g(0) &= -h_0, & h(0) &= h_0, \\
    u(0,x) &= u_0(x), & x \in [-h_0, h_0]; & v(0,x) = v_0(x), & x \in \mathbb{R},
\end{align*}
\]  

(1.3)

where \(\mathbb{R} = (-\infty, \infty)\), \(x = g(t)\) and \(x = h(t)\) represent the left and right moving boundaries, respectively, which are to be determined, \(a, b, c, D, h_0\) and \(\mu\) are given positive constants. The initial functions \(u_0(x), v_0(x)\) satisfy

\[
u_0 \in C^2([-h_0, h_0]), \quad u_0(\pm h_0) = 0, \quad u_0 > 0 \quad \text{in} \ (-h_0, h_0); \quad v_0 \in C_b(\mathbb{R}), \quad v_0 > 0 \quad \text{in} \ \mathbb{R},
\]

here \(C_b(\mathbb{R})\) is the space of continuous and bounded functions in \(\mathbb{R}\). The ecological background of the free boundary conditions \(g'(t) = -\mu u_x(t, g(t))\) and \(h'(t) = -\mu u_x(t, h(t))\) can also refer to [2].

We will show that (1.3) has a unique solution \((u(t,x), v(t,x), g(t), h(t))\) defined for all \(t > 0\), with \(u(t,x) \geq 0\), \(v(t,x) > 0\), \(g'(t) < 0\) and \(h'(t) > 0\). Moreover, a spreading-vanishing dichotomy holds for (1.3), namely, as time \(t \to \infty\), either
(i) the predator $u(t, x)$ successfully establishes itself in the new environment (henceforth called spreading) in the sense that $g(t) \to -\infty$ and $h(t) \to \infty$. Moreover, both $u(t, x)$ and $v(t, x)$ go to positive constants for the weakly hunting case $b > c$ and $ac < 1$, while $u(t, x) \to 1$, $v(t, x) \to 0$ for the strongly hunting case: $b \leq c$; or

(ii) the predator $u(t, x)$ fails to establish and vanishes eventually (called vanishing), i.e., $h(t) - g(t) \to b_\infty - g_\infty \leq \pi \sqrt{1/(1 + ab)}$, $\|u(t, x)\|_{C([g(t), h(t)])} \to 0$ and $v(t, x) \to b$.

The criteria for spreading and vanishing are the following: If the initial occupying area $[-h_0, h_0]$ is beyond a critical size, namely $2h_0 \geq \pi \sqrt{1/(1 + ab)}$, then regardless of the initial population size $(u_0, v_0)$, spreading always happens. On the other hand, if $2h_0 < \pi \sqrt{1/(1 + ab)}$, then whether spreading or vanishing occurs is determined by the initial population size $(u_0, v_0)$ and the coefficient $\mu$ in the Stefan condition.

In the absence of $v$, the problem (1.3) is reduced to a one phase Stefan problem for the logistic model which has been systematically studied by many authors, see, for example [2], [6]–[10], [11], [18] [22] (including the higher dimension and heterogeneous environment case) and the references cited therein. The one phase Stefan free boundary condition in (1.3) also arises in many other applications, for instance, in the modeling of wound healing [4]. As far as population models are concerned, [20] used such a condition for a predator-prey system over a bounded interval, showing that the free boundary reaches the fixed boundary in finite time, and hence, the long-time dynamical behavior of the system is the same as the well-studied fixed boundary problem; and in [21], a two-phase Stefan condition was used for a competition system over a bounded interval, where the free boundary separates the two competitors from each other in the interval. There is a vast literature on the Stefan problems, and some important theoretical advances can be found in [8] [5] and the references therein.

The other related works concerning free boundary problems for biological models, please refer to, for instance [13] [15] [16] [17] and references cited therein.

The organization of this paper is as follows. In Section 2, we first use a contraction mapping argument to prove the local existence and uniqueness of solution to (1.3), and then show that it exists for all time $t \in (0, \infty)$. In order to estimate $(u(t, x), v(t, x))$ and $(g(t), h(t))$, in Section 3 we give some comparison principles. Section 4 is devoted to the long time behavior of $(u(t, x), v(t, x))$. Theorem 4.2 plays key roles in the following two aspects: (i) affirming the predator species disappears eventually; (ii) determining the criteria for spreading and vanishing (see the following Section 5). Moreover, its proof is very different from the single equation case (refer to the proofs of [10, Lemma 3.1] and [18, Theorem 2.10]). In Section 5 we shall provide the criteria for spreading and vanishing. The last section is a brief discussion.

Before ending this section, we should emphasize here that if $-h_0$ is replaced by another number $g_0$ with $g_0 < h_0$, and/or the free boundary conditions $g'(t) = -\mu u_x(t, g(t))$ and $h'(t) = -\mu v_x(t, h(t))$ are replaced by $g'(t) = -\mu_1 u_x(t, g(t))$ and $h'(t) = -\mu_2 v_x(t, h(t))$, respectively, and $\mu_1, \mu_2$ are positive constants, then all results of the present paper are still true.

2 Existence and uniqueness

In this section, we first prove the following local existence and uniqueness result by contraction mapping theorem, and then use suitable estimate to illustrate that the solution is defined for all $t > 0$. 

Theorem 2.1 For any given \( \alpha \in (0,1) \), there is a \( T > 0 \) such that problem \([3]\) admits a unique solution \((u,v,g,h) \in C^{1+\alpha,1+\alpha}(\overline{D_T}) \times C_T \times [C^{1+\alpha}(0,T)]^2\). And
\[
\|u\|_{C^{1+\alpha,1+\alpha}(\overline{D_T})} + \|g\|_{C^{1+\alpha}(0,T)} + \|h\|_{C^{1+\alpha}(0,T)} \leq C,
\]
where
\[
D_T = \{ 0 < t \leq T, g(t) < x < h(t) \}, \quad C_T = C_b([0,T] \times \mathbb{R}) \cap C_{loc}^{1+\alpha,2+\alpha}((0,T] \times \mathbb{R}),
\]
\( C \) and \( T \) only depend on \( h_0, \alpha, \|u_0\|_{W^2([-h_0,h_0])} \) with \( p \geq (n+2)/(1-\alpha) \) and \( \|v_0\|_{C_b(\mathbb{R})} \).

Proof. As in [3], we first straighten the free boundaries. Let \( \zeta(y) \) be a function in \( C^3(\mathbb{R}) \) satisfying
\[
\zeta(y) = 1 \text{ if } |y - h_0| < h_0/4, \quad \zeta(y) = 0 \text{ if } |y - h_0| > h_0/2, \quad |\zeta'(y)| < 6/h_0, \quad \forall \ y \in \mathbb{R},
\]
and set \( \xi(y) = -\zeta(-y) \). Consider the transformation
\[
(t,x) \rightarrow (t,y), \quad \text{where } x = y + \zeta(y)(h(t) - h_0) + \xi(y)(g(t) + h_0), \quad y \in \mathbb{R}.
\]
Notice that as long as \( |h(t) - h_0| + |g(t) + h_0| \leq h_0/16 \), the above transformation is a diffeomorphism from \( \mathbb{R} \) onto \( \mathbb{R} \). Moreover, it changes the free boundaries \( x = g(t), x = h(t) \) to the lines \( y = -h_0 \) and \( y = h_0 \) respectively. Now, direct calculations yield
\[
\frac{\partial y}{\partial x} = \frac{1}{1 + \zeta'(y)(h(t) - h_0) + \xi'(y)(g(t) + h_0)} \equiv \sqrt{\rho(g(t),h(t),y)},
\]
\[
\frac{\partial^2 y}{\partial x^2} = -\frac{\zeta''(y)(h(t) - h_0) + \xi''(y)(g(t) + h_0)}{[1 + \zeta'(y)(h(t) - h_0) + \xi'(y)(g(t) + h_0)]^2} \equiv g(g(t),h(t),y),
\]
\[
\frac{\partial y}{\partial t} = -\frac{\zeta(y)h'(t) + \xi(y)g'(t)}{1 + \zeta'(y)(h(t) - h_0) + \xi'(y)(g(t) + h_0)} \equiv \varsigma(g(t),g'(t),h(t),h'(t),y).
\]
If we set
\[
\begin{align*}
  u(t,x) &= u(t,y + \zeta(y)(h(t) - h_0) + \xi(y)(g(t) + h_0)) = w(t,y), \\
  v(t,x) &= v(t,y + \zeta(y)(h(t) - h_0) + \xi(y)(g(t) + h_0)) = z(t,y),
\end{align*}
\]
then
\[
\begin{align*}
  u_t &= w_t + \varsigma(g(t),g'(t),h(t),h'(t),y)w_y, & v_t &= z_t + \varsigma(g(t),g'(t),h(t),h'(t),y)z_y, \\
  u_x &= \sqrt{\rho(g(t),h(t),y)}w_y, & v_x &= \sqrt{\rho(g(t),h(t),y)}z_y, \\
  u_{xx} &= \rho(g(t),h(t),y)w_{yy} + g(g(t),h(t),y)w_y, & v_{xx} &= \rho(g(t),h(t),y)z_{yy} + g(g(t),h(t),y)z_y.
\end{align*}
\]
and \((w,z)\) satisfies
\[
\begin{align*}
  &\begin{cases}
    w_t - \rho w_{yy} - (\rho - \varsigma)w_y = w(1 - w + az), & t > 0, \quad |y| < h_0, \\
    w(t,y) = 0, & t \geq 0, \quad |y| \geq h_0, \\
    z_t - D\rho z_{yy} - (D\rho - \varsigma)z_y = z(b - z - cw), & t > 0, \quad y \in \mathbb{R}, \\
    w(t,-h_0) = w(t,h_0) = 0, & t \geq 0, \\
    w(0,y) = u_0(y), & y \in [-h_0,h_0]; \quad z(0,y) = v_0(y), \quad y \in \mathbb{R},
  \end{cases}
  \\
  &z(0,y) = v_0(y), \quad y \in [-h_0,h_0]; \quad w(0,y) = u_0(y), \quad y \in [-h_0,h_0];
\end{align*}
\]
where \( \rho = \rho(g(t),h(t),y), \quad g = g(g(t),h(t),y), \quad \varsigma = \varsigma(g(t),g'(t),h(t),h'(t),y). \)
Let \( g^* = -\mu u_0'(-h_0) \) and \( h^* = -\mu u_0'(h_0) \). For \( 0 < T < \frac{\ln 16}{1+\alpha} \min \{ \frac{1}{1+\alpha}, \frac{1}{1+\alpha} \} \), we define \( I_T = [0, T] \times [-h_0, h_0] \), and
\[
\begin{align*}
\mathcal{D}_T^1 &= \{ w \in C^{0,\alpha}_T(I_T) : w(t, \pm h_0) = 0, w(0, y) = u_0(y), \| w - u_0 \|_{C^{0,\alpha}_T(I_T)} \leq 1 \}, \\
\mathcal{D}_T^2 &= \{ g \in C^1([0, T]) : g(0) = -h_0, g'(0) = g^*, \| g' - g^* \|_{C([0,T])} \leq 1 \}, \\
\mathcal{D}_T^3 &= \{ h \in C^1([0, T]) : h(0) = h_0, h'(0) = h^*, \| h' - h^* \|_{C([0,T])} \leq 1 \}.
\end{align*}
\]  

It is easily seen that \( \mathcal{D}_T = \mathcal{D}_T^1 \times \mathcal{D}_T^2 \times \mathcal{D}_T^3 \) is a closed convex set in \( C^{0,\alpha}_T(I_T) \times C^1([0, T]) \times C^1([0, T]) \).

Next, we shall prove the existence and uniqueness result by using the contraction mapping theorem. First, we observe that due to our choice of \( T \), for any given \((w, g, h) \in \mathcal{D}_T\), there holds:
\[
|g(t) + h_0| \leq T \| g' \|_{C([0,T])} \leq h_0/16, |h(t) - h_0| \leq T \| h' \|_{C([0,T])} \leq h_0/16.
\]

Therefore the transformation \((t, y) \rightarrow (t, x)\) introduced at the beginning of the proof is well defined. For any \((w, g, h) \in \mathcal{D}_T\), let \( \tilde{w}(t, y) = w(t, y) \) when \( |y| \leq h_0 \), and \( \tilde{w}(t, y) = 0 \) when \( |y| > h_0 \). Since \( w \in C^{0,\alpha}_T(I_T) \), it is easy to see that \( \tilde{w} \in C^{0,\alpha}_T([0, T] \times \mathbb{R}) \). The standard partial differential equation theory guarantees that the problem
\[
\begin{cases}
& z_t - D \rho \bar{z}_{yy} - (D \varphi - \zeta) z_y = z(b - z - c \bar{w}), \quad t > 0, \quad y \in \mathbb{R}, \\
& z(0, y) = v_0(y), \quad y \in \mathbb{R}
\end{cases}
\]

admits a unique solution \( z \in C_b([0, T] \times \mathbb{R}) \cap C^{1+\frac{2}{p},2+\alpha}_{loc}(0, T] \times \mathbb{R}) \). Also, the following initial boundary value problem
\[
\begin{cases}
& \tilde{w}_t - \rho \tilde{w}_{yy} - (\varphi - \zeta) \tilde{w}_y = w(1 - w + az), \quad t > 0, \quad -h_0 < y < h_0, \\
& \tilde{w}(t, -h_0) = \tilde{w}(t, h_0) = 0, \quad t > 0, \\
& \tilde{w}(0, y) = u_0(y), \quad -h_0 < y < h_0
\end{cases}
\]

admits a unique solution \( \tilde{w} \in C^{1+\frac{2}{p},1+\alpha}_{loc}(I_T) \). Moreover, using \( L^p \) estimate for parabolic equations with \( p \geq (n + 2)/(1 - \alpha) \) and Sobolev’s inequalities, one gets
\[
\| \tilde{w} \|_{C^{1+\frac{2}{p},1+\alpha}_{loc}(I_T)} \leq C_1. \tag{2.3}
\]

where \( C_1 \) is a constant dependent on \( h_0, \alpha, \| u_0 \|_{W^{2}_{p}([-h_0, h_0])} \) and \( \| v_0 \|_{C_b(\mathbb{R})} \). Define
\[
\tilde{g}(t) = -h_0 - \int_0^t \mu \tilde{w}_y(\tau, -h_0) d\tau, \quad \tilde{h}(t) = h_0 - \int_0^t \mu \tilde{w}_y(\tau, h_0) d\tau.
\]

Then \( \tilde{g}'(t) = -\mu \tilde{w}_y(t, -h_0), \tilde{h}'(t) = -\mu \tilde{w}_y(t, h_0) \). Subsequently, \( \tilde{g}'(t), \tilde{h}'(t) \in C^{0,\alpha}_T([0, T]) \), and
\[
\| \tilde{g}'(t) \|_{C^{0,\alpha}_T([0,T])}, \| \tilde{h}'(t) \|_{C^{0,\alpha}_T([0,T])} \leq \mu C_1 := C_2. \tag{2.4}
\]

We now define \( \mathcal{F} : \mathcal{D}_T \to C^{0,\alpha}_T(I_T) \times C^1([0, T]) \times C^1([0, T]) \) by
\[
\mathcal{F}(w, g, h) = (\tilde{w}, \tilde{g}, \tilde{h}).
\]

Clearly \((w, g, h) \in \mathcal{D}_T\) is a fixed point of \( \mathcal{F} \) if and only if \((w, z, g, h)\) solves (2.2). By (2.3) and (2.4), one has
\[
\| \tilde{g}' - g^* \|_{C([0,T])} \leq \| \tilde{g}' \|_{C^{0,\alpha}_T([0,T])} T^{\frac{2}{p}} \leq \mu C_1 T^{\frac{2}{p}}, \quad \| \tilde{h}' - h^* \|_{C([0,T])} \leq \| \tilde{h}' \|_{C^{0,\alpha}_T([0,T])} T^{\frac{2}{p}} \leq \mu C_1 T^{\frac{2}{p}},
\]
and
\[
\|\tilde{w} - u_0\|_{C_T^{\frac{1}{2} \alpha, 0}(I_T)} = \|\tilde{w} - u_0\|_{C(I_T)} + \|\tilde{w} - u_0\|_{C_T^{\frac{1}{2} \alpha, 0}(I_T)} \\
\leq \|\tilde{w}\|_{C_T^{\frac{1}{2} \alpha, 0}(I_T)} T^{\frac{1}{4} \alpha} + \|\tilde{w}\|_{C_T^{\frac{1}{2} \alpha, 0}(I_T)} T^{\frac{1}{2}} + (2h_0)^{1-\alpha}\|\tilde{w}_y\|_{C_T^{\frac{1}{2} \alpha, 0}(I_T)} T^{\frac{1}{4} \alpha} \\
\leq C_1 \left( T^{\frac{1}{4} \alpha} + T^{\frac{1}{2}} + (2h_0)^{1-\alpha}T^{\frac{1}{4} \alpha} \right).
\]

Therefore, if we take \( T \leq \min \{ 1, (\mu C_1)^{-2/\alpha}, [(2 + (2h_0)^{1-\alpha})C_1]^{-2/\alpha} \} \), then \( \mathcal{F} \) maps \( \mathcal{D}_T \) into itself.

Next we attest that \( \mathcal{F} \) is a contraction mapping on \( \mathcal{D}_T \) for \( T > 0 \) sufficiently small. Indeed, let \((w_i, g_i, h_i) \in \mathcal{D}_T \) \((i = 1, 2)\) and denote \((\tilde{w}_i, \tilde{g}_i, \tilde{h}_i) = \mathcal{F}(w_i, g_i, h_i)\). Then it follows from (2.3) and (2.4) that
\[
\|\tilde{w}_i\|_{C_T^{\frac{1}{2} \alpha, 1+\alpha}(I_T)} \leq C_1, \quad \|\tilde{g}_i(t)\|_{C_T^{\frac{1}{2} \alpha}(0, T)} \leq C_2, \quad \|\tilde{h}_i(t)\|_{C_T^{\frac{1}{2} \alpha}(0, T)} \leq C_2.
\]

Setting \( \gamma = \tilde{w}_1 - \tilde{w}_2, \), \( \zeta = z_1 - z_2, \) we find that \( \gamma \) and \( \zeta \) satisfy
\[
\begin{cases}
\gamma(t) = (\rho_1 - \rho_2)w_{yy} + (\rho_1 - \rho_2)\tilde{w}_{yy} + (\rho_1 - \rho_2)\tilde{w}_{2y} + (\rho_1 - \rho_2)\tilde{w}_{2y} \\
\zeta(t) = -D(\rho_1 - \rho_2)\tilde{w}_{2y} + (\rho_1 - \rho_2)\tilde{w}_{2y} + (\rho_1 - \rho_2)\tilde{w}_{2y} + (\rho_1 - \rho_2)\tilde{w}_{2y} - c(\tilde{w}_1 - \tilde{w}_2)z_2, \quad t > 0, \quad -h_0 < y < h_0, \\
\gamma(0, y) = 0, \quad \zeta(0, y) = 0, \quad -h_0 \leq y \leq h_0
\end{cases}
\]

and
\[
\begin{cases}
\gamma(t) = D(\rho_1 - \rho_2)\zeta_{yy} + D(\rho_1 - \rho_2)\zeta_{yy} + (\rho_1 - \rho_2)\zeta_{yy} - c(\tilde{w}_1 - \tilde{w}_2)z_2, \quad t > 0, \quad y \in \mathbb{R}, \\
\zeta(0, y) = 0, \quad \xi(0, y) = 0, \quad y \in \mathbb{R},
\end{cases}
\]

respectively, where \( \rho_i = \rho_1 + \rho_2, \), \( h_i(t, y) \), \( g_i = g_1 + g_2, \) \( h_i(t, y) \) and \( \zeta_i = \zeta(g_i(t), g_i'(t), h_i(t), h_i'(t), y), \)
\( i = 1, 2. \) In view of the standard theory for parabolic partial differential equations and Sobolev’s imbedding theorem [19], we obtain that
\[
\begin{align*}
\|z_1 - z_2\|_{C(I_T)} & \leq C_3 \left( \|w_1 - w_2\|_{C(I_T)} + \|(g_1 - g_2, h_1 - h_2)\|_{C^1(0, T)} \right), \\
\|\tilde{w}_1 - \tilde{w}_2\|_{C_T^{\frac{1}{2} \alpha, 1+\alpha}(I_T)} & \leq C_3 \left( \|(w_1 - w_2, z_1 - z_2)\|_{C(I_T)} + \|(g_1 - g_2, h_1 - h_2)\|_{C^1(0, T)} \right) \\
& \leq C_4 \left( \|w_1 - w_2\|_{C(I_T)} + \|(g_1 - g_2, h_1 - h_2)\|_{C^1(0, T)} \right),
\end{align*}
\]

where \( C_3 \) and \( C_4 \) depend on \( C_1, C_2 \) and the functions \( \rho, \varrho, \zeta \). Taking the difference of equations for \( \tilde{g}_1, \tilde{h}_1, \tilde{g}_2, \tilde{h}_2 \) results in
\[
\begin{align*}
\|\tilde{g}_1 - \tilde{g}_2\|_{C_T^{\frac{1}{2} \alpha}(0, T)} & \leq \mu \|\tilde{w}_1y - \tilde{w}_2y\|_{C_T^{\frac{1}{2} \alpha}(0, T)}, \\
\|\tilde{h}_1 - \tilde{h}_2\|_{C_T^{\frac{1}{2} \alpha}(0, T)} & \leq \mu \|\tilde{w}_1y - \tilde{w}_2y\|_{C_T^{\frac{1}{2} \alpha}(0, T)}.
\end{align*}
\]

We may assume that \( T \leq 1 \). Combining (2.3) and (2.4), and applying the mean value theorem, it yields
\[
\begin{align*}
\|\tilde{w}_1 - \tilde{w}_2\|_{C_T^{\frac{1}{2} \alpha, 1+\alpha}(I_T)} & \leq C_5 \left( \|w_1 - w_2\|_{C(I_T)} + \|(g'_1 - g'_2, h'_1 - h'_2)\|_{C(0, T)} \right), \\
\end{align*}
\]

where \( C_5 \) depends on \( C_4 \) and \( \mu \). On the other hand, by direct calculations,
\[
\begin{align*}
\|\tilde{w}_1 - \tilde{w}_2\|_{C_T^{\frac{1}{2} \alpha, 1+\alpha}(I_T)} & \leq \|\tilde{w}_1 - \tilde{w}_2\|_{C_T^{\frac{1}{2} \alpha, 0}(I_T)} T^{\frac{1}{4} \alpha} + \|\tilde{w}_1 - \tilde{w}_2\|_{C_T^{\frac{1}{2} \alpha, 0}(I_T)} T^{\frac{1}{2}} \\
& \quad + (2h_0)^{1-\alpha}\|\tilde{w}_1y - \tilde{w}_2y\|_{C_T^{\frac{1}{2} \alpha, 0}(I_T)} T^{\frac{1}{4} \alpha} \\
& \leq (2 + (2h_0)^{1-\alpha}) T^{\frac{1}{4} \alpha}\|\tilde{w}_1 - \tilde{w}_2\|_{C_T^{\frac{1}{2} \alpha, 1+\alpha}(I_T)}.
\end{align*}
\]
Let $\varepsilon_1 = h_0/16$ and $\varepsilon_2 = 2 + (2h_0)^{1-\alpha}$. Then for

$$T := \min \left\{ 1, \frac{\varepsilon_1}{1 + g^*}, \frac{\varepsilon_1}{1 + h^*}, \frac{1}{(\mu C_1)^{\frac{1}{2}}} \frac{1}{(\varepsilon_2 C_1)^{2/\alpha}}, \frac{1}{(2\varepsilon_2 C_5)^{2/\alpha}} \right\},$$

it follows that

$$\|\tilde{w}_1 - \tilde{w}_2\|_{C^{T^{\frac{1}{2}},\alpha}(I_T)} + \|\tilde{g}^*_1 - \tilde{g}^*_2, \tilde{h}^*_1 - \tilde{h}^*_2\|_{C([0,T])} \leq (2 + (2h_0)^{1-\alpha}) T^{\frac{1}{2}} \|\tilde{w}_1 - \tilde{w}_2\|_{C^{T^{\frac{1}{2}},\alpha}(I_T)} + T^{\frac{1}{2}} \|\tilde{g}^*_1 - \tilde{g}^*_2, \tilde{h}^*_1 - \tilde{h}^*_2\|_{C([0,T])}$$

$$\leq (2 + (2h_0)^{1-\alpha}) C_5 T^{\frac{1}{2}} \left( \|w_1 - w_2\|_{C(I_T)} + \|g^*_1 - g^*_2, h^*_1 - h^*_2\|_{C([0,T])} \right)$$

$$\leq \frac{1}{2} \left( \|w_1 - w_2\|_{C^{T^{\frac{1}{2}},\alpha}(I_T)} + \|g^*_1 - g^*_2, h^*_1 - h^*_2\|_{C([0,T])} \right).$$

The above arguments ensure that the operator $F$ is contractive on $D_T$. It now follows from the contraction mapping theorem that $F$ has a unique fixed point $(w, g, h)$ in $D_T$. Moreover, by the $L^p$ estimates, we have additional regularity for $(w, z, g, h)$ as a solution of \((2.2)\), namely, $w \in C^{1+\alpha,1+\alpha}(I_T)$, $z \in C_b((0, T] \times \mathbb{R}) \cap C^{1+\alpha,1+\alpha}_{\text{loc}}((0, T] \times \mathbb{R})$, and $g, h \in C^{1+\alpha}([0, T])$, and \((2.3)\), \((2.4)\) hold. In other words, $(w, z, g, h)$ is the unique local classical solution of the problem \((2.2)\). Hence, $(u, v, g, h)$ is the unique classical solution of \((1.3)\). \hfill \Box

To show that the local solution obtained in Theorem \((2.1)\) can be extended to all $t > 0$, we need the following estimate.

**Lemma 2.1** The solution of the free boundary problem \((1.3)\) satisfies

$$0 < u(t, x) \leq M_1, \quad 0 < t \leq T, \quad g(t) < x < h(t),$$

$$0 < v(t, x) \leq M_2, \quad 0 < t \leq T, \quad x \in \mathbb{R},$$

$$-M_3 \leq g'(t) < 0, \quad 0 < h'(t) \leq M_3, \quad 0 < t \leq T,$$

where $M_i$ is independent of $T$ for $i = 1, 2, 3$.

**Proof.** Using the strong maximum principle, we are easy to see that $u > 0$ in $(0, T) \times (g(t), h(t))$ and $v > 0$ in $(0, T] \times \mathbb{R}$ as long as the solution exists. Since $v(t, x)$ satisfies

$$\begin{cases}
    v_t - Dv_{xx} = v(b - v - cu), & x \in \mathbb{R}, \quad t > 0, \\
    v(0, x) = v_0(x) > 0, & x \in \mathbb{R},
\end{cases}$$

it is obvious that $v \leq \max \{\|v_0\|_{\infty}, b\} := M_1$. Similarly, as $u$ satisfies

$$\begin{cases}
    u_t - u_{xx} = u(1 - u + av), & t > 0, \quad g(t) < x < h(t), \\
    u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\
    u(0, x) = u_0(x) > 0, & -h_0 < x < h_0,
\end{cases}$$

we also have $u \leq \max \{\|u_0\|_{\infty}, (1 + aM_1)\} := M_2$.

To prove $h'(t) > 0$ for $0 < t \leq T$, we use the transformation

$$y = x/h(t), \quad w(t, y) = u(t, x), \quad z(t, y) = v(t, x).$$
It is easily to see that if $f(t) = h^{-2}(t)$, $\phi(t, y) = yh'(t)/h(t)$. This is an initial-boundary value problem with fixed boundary. Since $w(t, y) > 0$ for $t > 0$ and $0 \leq y < 1$, by the Hopf boundary lemma, we have $w_y(t, 1) < 0$ for $t > 0$. This combines with the relation $u_x = h^{-1}(t)w_y$ yields $u_x(t, h(t)) < 0$, and so $h'(t) > 0$ for $t > 0$. Similarly, $g'(t) < 0$ for $t > 0$.

Now we illustrate that $g'(t) \geq -M_3$ and $h'(t) \leq M_3$ for all $t \in (0, T)$ with some $M_3$ independent of $T$. To this aim, let $M$ be a positive constant, $\Omega_M = \{0 < t < T, g(t) < x < g(t) + 1/M\}$, and construct an auxiliary function

$$w(t, x) = M_2[2M(x - g(t)) - M^2(x - g(t))^2].$$

We will choose $M$ so that $w \geq u$ in $\Omega_M$.

Direct calculations indicate that, for $(t, x) \in \Omega_M$,

$$w_t = 2M_2Mg'(t)(-1 - M(g(t) - x)) \geq 0,$$

$$-w_{xx} = 2M_2M^2, \quad u(1 - u + av) \leq M_2(1 + aM_1).$$

Therefore,

$$w_t - w_{xx} \geq 2M_2M^2 \geq M_2(1 + aM_1) \geq u(1 - u + av) \quad \text{in} \quad \Omega_M$$

provided $M^2 \geq (1 + aM_1)/2$. It is obvious that

$$w(t, g(t) + M^{-1}) = M_2 \geq u(t, g(t) + M^{-1}), \quad w(t, g(t)) = 0 = u(t, g(t)).$$

Because

$$u_0(x) = \int_{-h_0}^x u_0(y)dy \leq (x + h_0)|u_0'|_{C[-h_0, h_0]} \quad \text{on} \quad [-h_0, -h_0 + M^{-1}],$$

$$w(0, x) = M_2[2M(x + h_0) - M^2(x + h_0)^2] \geq M_2M(h_0 + x) \quad \text{on} \quad [-h_0, -h_0 + M^{-1}],$$

it is easily to see that if $MM_2 \geq ||u_0'||_{C[-h_0, h_0]}$, then

$$u_0(x) \leq (x + h_0)|u_0'|_{C[-h_0, h_0]} \leq w(0, x) \quad \text{on} \quad [-h_0, -h_0 + M^{-1}].$$

Let

$$M = \max \left\{ \sqrt{\frac{1 + aM_1}{2}}, \frac{||u_0'||_{C[-h_0, h_0]}}{M_2} \right\}.$$

We can apply the maximum principle to $w - u$ over $\Omega_M$ and deduce that $u(t, x) \leq w(t, x)$ for $(t, x) \in \Omega_M$. It would then follow that $u_x(t, g(t)) \leq w_x(t, g(t)) = 2M_2M$, and hence

$$g'(t) = -\mu u_x(t, g(t)) \geq -2\mu M_2M := -M_3.$$

Similarly, we can proved $h'(t) \leq M_3$ for $0 < t < T$. The proof is complete. \(\square\)

**Theorem 2.2** The solution of problem (1.3) exists and is unique for all $t \in (0, \infty)$. 
Theorem 2.1 that there exists a \( \delta \)

We now fix \( \varepsilon \) and as \( C > \) solution of (1.3) with initial time \( T \) \( \parallel u \) \( v \) and \( C(\bar{t}) \) in \([0, T_{\text{max}}] \times \mathbb{R} \), \( \parallel u(t, \cdot) \parallel_{W^{2}_{p}([g(t), h(t)])} \leq C \) for \( t \in [\delta, T_{\text{max}}] \). It then follows from the proof of Theorem 2.1 that there exists a \( \tau > 0 \), depending only on \( C \) and \( M_{i}(i = 1, 2, 3) \), such that the solution of (1.3) with initial time \( T_{\text{max}} - \tau/2 \) can be extended uniquely to the time \( T_{\text{max}} - \tau/2 + \tau \). But this contradicts the assumption. The proof is now complete. \( \square \)

3 Comparison principles

In this section we shall give some comparison principles which can be used to estimate the solution \((u(t, x), v(t, x))\) and the free boundaries \( x = g(t) \) and \( x = h(t) \).

Lemma 3.1 (Comparison principle) Let \( T > 0 \), \( \bar{g}, \bar{h} \in C^{1}([0, T]) \) and \( \bar{g} < \bar{h} \) in \([0, T]\). Let \( \bar{u} \in C(\mathcal{O}) \cap C^{1,2}(\bar{O}) \) with \( O = \{0 < t \leq T, \bar{g}(t) < x < \bar{h}(t)\} \), and \( \bar{v} \in C^{1,2}(G) \) with \( G = (0, T] \times \mathbb{R} \). Assume that \((\bar{u}, \bar{v}, \bar{g}, \bar{h})\) satisfies

\[
\begin{align*}
\bar{u}_{t} - \bar{u}_{xx} &\geq \bar{u}(1 - \bar{u} + \alpha \bar{v}), &0 < t \leq T, \quad \bar{g}(t) < x < \bar{h}(t), \\
\bar{v}_{t} - D \bar{v}_{xx} &\geq \bar{v}(b - \bar{v}), &0 < t \leq T, \quad x \in \mathbb{R}, \\
\bar{u}(t, \bar{g}(t)) = 0, \quad \bar{g}(t) &\leq -\mu \bar{u}_{x}(t, g(t)), &0 < t \leq T, \\
\bar{u}(t, \bar{h}(t)) = 0, \quad \bar{h}(t) &\leq -\mu \bar{u}_{x}(t, h(t)), &0 < t \leq T.
\end{align*}
\]

If \( \bar{g}(0) \leq -h_{0}, \bar{h}(0) \geq h_{0}, \bar{u}(0, x) \geq 0 \) on \([\bar{g}(0), \bar{h}(0)]\), \( u_{0}(x) \leq \bar{u}(0, x) \) on \([-h_{0}, h_{0}]\) and \( v_{0}(x) \leq \bar{v}(0, x) \) in \( \mathbb{R} \), then the solution \((u, v, g, h)\) of (1.3) satisfies

\[
g(t) \geq \bar{g}(t), \quad h(t) \leq \bar{h}(t) \quad \text{on } [0, T]; \quad u(t, x) \leq \bar{u}(t, x) \quad \text{on } \overline{D_{T}}; \quad v(t, x) \leq \bar{v}(t, x) \quad \text{on } \overline{G},
\]

where \( D_{T} \) is defined as in Theorem 2.1.

\[\text{Proof.}\] For small \( \varepsilon > 0 \), let \((u_{\varepsilon}, v_{\varepsilon}, g_{\varepsilon}, h_{\varepsilon})\) be the unique solution of (1.3) with \( h_{0}, \mu, u_{0}(x) \) and \( v_{0}(x) \) replaced by \( h_{0}^{\varepsilon} = (1 - \varepsilon)h_{0}, \mu_{\varepsilon} = (1 - \varepsilon)\mu, u_{0}(x) \in C^{2}([-h_{0}^{\varepsilon}, h_{0}^{\varepsilon}]), \) and \( v_{0}(x) \in C(\mathbb{R}) \), respectively, where \( u_{0}(x) \) and \( v_{0}(x) \) satisfy \( u_{0}^{\varepsilon}(-h_{0}^{\varepsilon}) = u_{0}^{\varepsilon}(h_{0}^{\varepsilon}) = 0 \) and

\[
0 < u_{0}^{\varepsilon}(x) \leq u_{0}(x) \quad \text{on } [-h_{0}^{\varepsilon}, h_{0}^{\varepsilon}], \quad 0 < v_{0}^{\varepsilon}(x) < v_{0}(x) \quad \text{in } \mathbb{R},
\]

and as \( \varepsilon \to 0, \)

\[
u_{0}^{\varepsilon}\left(\frac{h_{0}}{h_{0}^{\varepsilon}}x\right) \to u_{0}(x) \quad \text{in } C^{2}([-h_{0}, h_{0}]), \quad v_{0}^{\varepsilon}\left(\frac{h_{0}}{h_{0}^{\varepsilon}}x\right) \to v_{0}(x) \quad \text{in } C(\mathbb{R}).
\]

Apply the comparison principle to \( \bar{v} \) and \( v_{\varepsilon} \) we have \( v_{\varepsilon} < \bar{v} \) on \([0, T] \times \mathbb{R}\).
We claim that $g_ε(t) > \bar{g}(t)$ and $h_ε(t) < \bar{h}(t)$ for all $t \in (0, T]$. Clearly, this is true for small $t > 0$. If our claim does not hold, then we can find a first $τ \leq T$ such that $g_ε(t) > \bar{g}(t)$ and $h_ε(t) < \bar{h}(t)$ for all $t \in (0, τ)$ and at least one of $g_ε(τ) = \bar{g}(τ)$ and $h_ε(τ) = \bar{h}(τ)$ is true. Without loss of generality, it is assumed that $h_ε(τ) = \bar{h}(τ)$. Then

$$h_ε'(τ) \geq \bar{h}'(τ).$$

(3.1)

For any $0 < σ \leq T$, let

$$D_σ^ε := \{(t, x) \in \mathbb{R}^2 : 0 < t \leq σ, g_ε(t) < x < h_ε(t)\}. \quad (3.2)$$

The strong maximum principle yields $u_ε(t, x) < \bar{u}(t, x)$ in $D_σ^ε$. Obviously, $\bar{u}(t, h_ε(τ)) = u_ε(t, h_ε(τ)) = 0$ since $h_ε(τ) = \bar{h}(τ)$. Therefore, $\bar{u}_x(t, h_ε(τ)) \leq u_ε_x(t, h_ε(τ))$. Note that $u_ε_x(τ, h_ε(τ)) < 0$ and $μ_ε < μ$, it follows that $h_ε'(τ) < \bar{h}'(τ)$. This contradicts to (3.1). So, $g_ε(t) > \bar{g}(t)$ and $h_ε(t) < \bar{h}(t)$ for all $t \in (0, T]$. We may now apply the usual comparison principle over $D_T^ε$ to conclude that $u_ε < \bar{u}$ in $D_T^ε$.

Since the unique solution of (1.3) depends continuously on the parameters in (1.3), as $ε \to 0$, $(u_ε, v_ε, g_ε, h_ε)$ converges to $(u, v, g, h)$, the unique solution of (1.3). The desired result then follows by letting $ε \to 0$ in the inequalities $u_ε < \bar{u}, v_ε < \bar{v}, g_ε < \bar{g}$ and $h_ε < \bar{h}$. □

The pair $(\bar{u}, \bar{v}, \bar{g}, \bar{h})$ in Lemma 3.1 is usually called an upper solution of (1.3). In the same way as the proof of Lemma 3.1, we can prove the following two lemmas:

**Lemma 3.2 (Comparison principle)** Let $T > 0$, $\bar{h} \in C^1([0, T])$ with $\bar{h} > 0$ on $[0, T]$, $\bar{u} \in C(\bar{D}) \cap C^{1,2}(\bar{D})$ with $\bar{D} = \{0 < t \leq T, -h_0 < x < \bar{h}(t)\}$. Assume that $(\bar{u}, \bar{h})$ satisfies $\bar{h}(0) \leq h_0$, $u_0(x) \geq \bar{u}(0, x)$ on $[-h_0, \bar{h}(0)]$, and

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} \leq \bar{u}(1 - \bar{u}), & 0 < t \leq T, -h_0 < x < \bar{h}(t), \\ \bar{u}(t, -h_0) = \bar{u}(t, \bar{h}(t)) = 0, & 0 < t \leq T, \\ \bar{h}'(t) \leq -\mu \bar{u}_x(t, \bar{h}(t)), & 0 < t \leq T. \end{cases}$$

Then the solution $(u, v, g, h)$ of (1.3) satisfies $h \geq \bar{h}$ on $[0, T]$, and $u \geq \bar{u}$ on $\partial \bar{D}$.

**Lemma 3.3 (Comparison principle)** Let $T > 0$, $\bar{g} \in C^1([0, T])$ with $\bar{g} < 0$ on $[0, T]$, $\bar{u} \in C(\bar{O}) \cap C^{1,2}(\bar{O})$ with $\bar{O} = \{0 < t \leq T, \bar{g}(t) < x < h_0\}$. Suppose that $(\bar{u}, \bar{g})$ satisfies $\bar{g}(0) \geq -h_0$, $u_0(x) \geq \bar{u}(0, x)$ on $[\bar{g}(0), h_0]$, and

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} \leq \bar{u}(1 - \bar{u}), & 0 < t \leq T, \bar{g}(t) < x < h_0, \\ \bar{u}(t, \bar{g}(t)) = \bar{u}(t, h_0) = 0, & 0 < t \leq T, \\ \bar{g}'(t) \geq -\mu \bar{u}_x(t, \bar{g}(t)), & 0 < t \leq T. \end{cases}$$

Then the solution $(u, v, g, h)$ of (1.3) satisfies $g \leq \bar{g}$ on $[0, T]$, $u \geq \bar{u}$ on $\partial \bar{O}$.

4 Long time behavior of $(u, v)$

It follows from Lemma 2.1 that $x = g(t)$ is monotonic decreasing and $x = h(t)$ is monotonic increasing. Therefore, there exist $g_\infty \in [-\infty, 0)$ and $h_\infty \in (0, \infty]$ such that $\lim_{t \to \infty} g(t) = g_\infty$ and $\lim_{t \to \infty} h(t) = h_\infty$. To discuss the long time behavior of $(u, v)$, we first derive an estimate.
\textbf{Theorem 4.1} Let \((u, v, g, h)\) be the solution of (4.3). If \(h_\infty - g_\infty < \infty\), then there exists a constant \(K > 0\), such that

\[ \|u(t, \cdot)\|_{C^1([g(t), h(t)])} \leq K, \forall \ t > 1, \quad (4.1) \]

\[ \lim_{t \to \infty} g'(t) = \lim_{t \to \infty} h'(t) = 0. \quad (4.2) \]

**Proof.** Introduce new functions \(w(t, y)\) and \(z(t, y)\) by

\[ w(t, y) = u \left( t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2} \right), \quad z(t, y) = v \left( t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2} \right). \]

Clearly, \(w\) and \(z\) satisfy the following initial boundary value problem in an interval \(-1 \leq y \leq 1\) with fixed boundary \(y = \pm 1:\)

\[
\begin{aligned}
  w_t = \varphi(t)w_{yy} + \psi(t)yw_y + w(1 - w + az), \quad t > 0, \quad |y| < 1, \\
  w(t, -1) = w(t, 1) = 0, \quad t > 0, \\
  w(0, y) = u(h_0 y), \quad |y| \leq 1,
\end{aligned}
\]

where

\[ \varphi(t) = \frac{4}{(h(t) - g(t))^2}, \quad \psi(t, y) = \frac{(h'(t) - g'(t))y + h'(t) + g'(t)}{h(t) - g(t)}. \]

By Proposition [A.1] we have \(\|w\|_{C^{\frac{1}{2}+\alpha, 1+\alpha}((1, \infty) \times [-1, 1])} < K_0\) for some positive constant \(K_0\). Remember \(u_x(t, x) = \frac{2}{u(t) - g(t)}w_y(t, y)\). There exists a positive constant \(K\) such that

\[ \|u(t, \cdot)\|_{C^1([g(t), h(t)])} < K, \quad \forall \ t \geq 1. \]

We next prove \(\lim_{t \to \infty} g'(t) = 0\). Note that \(\|w_y(\cdot, -1)\|_{C^{\frac{1}{2}}((1, \infty))} < K_0\), \(-M_3 < g'(t) < 0\) and \(g'(t) = -\mu u_x(t, g(t)) = -\frac{2u}{h(t) - g(t)}w_y(t, -1)\), it yields \(\|g'|_{C^{\frac{1}{2}}((1, \infty))} < L\), where \(L\) depends on \(K_0\) and \(M_3\). In view of \(g'(t) < 0\) and \(g_\infty > -\infty\), it is easily to derive that \(\lim_{t \to \infty} g'(t) = 0\). Analogously, we can obtain \(\lim_{t \to \infty} h'(t) = 0\). The proof is complete. \(\square\)

**4.1 Vanishing case \((h_\infty - g_\infty < \infty)\)**

**Theorem 4.2** Let \((u, v, g, h)\) be any solution of (1.3). If \(h_\infty - g_\infty < \infty\), then

\[ \lim_{t \to \infty} \|u(t, \cdot)\|_{C([g(t), h(t)])} = 0, \quad (4.4) \]

\[ \lim_{t \to \infty} v(t, x) = b \text{ uniformly on the compact subset of } \mathbb{R}. \quad (4.5) \]

This result shows that if the predator can not spread into the whole space, then it will die out eventually.

From the results of Section 5 we shall see that the reason leading to the predator species disappears eventually are three aspects: (a) the initial habitat \([-h_0, h_0]\) of the predator is too narrow, (b) the initial data \(u_0(x)\) of the predator is too small, or (c) the moving parameter/coefficient \(\mu\) of free boundaries is too small.

**Proof of Theorem 4.2**

**Step 1:** Proof of (4.4). On the contrary we assume that there exist \(\varepsilon > 0\) and \(\{(t_j, x_j)\}_{j=1}^\infty\), with \(g(t_j) < x_j < h(t_j)\) and \(t_j \to \infty\) as \(j \to \infty\), such that

\[ u(t_j, x_j) \geq 3\varepsilon, \quad j = 1, 2, \ldots. \quad (4.6) \]
Since \(g_{\infty} < x_j < h_{\infty}\), there are a subsequence of \(\{x_j\}\), noted by itself, and \(x_0 \in [g_{\infty}, h_{\infty}]\), such that \(x_j \rightarrow x_0 \) as \(j \rightarrow \infty\). We claim that \(x_0 \in (g_{\infty}, h_{\infty})\). If \(x_0 = g_{\infty}\), then \(x_j - g(t_j) \rightarrow 0 \) as \(j \rightarrow \infty\). By use of the inequality \((4.6)\) firstly and the inequality \((4.1)\) secondly, it is deduced that

\[
\frac{4\varepsilon}{x_j - g(t_j)} \leq \frac{u(t_j, x_j)}{x_j - g(t_j)} = \frac{u(t_j, x_j) - u(t_j, g(t_j))}{x_j - g(t_j)} = u_x(t_j, \bar{x}_j) \leq K,
\]

where \(\bar{x}_j \in [g(t_j), x_j]\). It is a contradiction as \(x_j - g(t_j) \rightarrow 0\). Similarly, we can ensure \(x_0 < h_{\infty}\).

By use of \((4.1)\) and \((4.6)\), there exists \(\delta > 0\) such that \([x_0 - \delta, x_0 + \delta] \subset (g_{\infty}, h_{\infty})\) and

\[
u(t_j, x) \geq 2\varepsilon, \quad \forall x \in [x_0 - \delta, x_0 + \delta]
\]

for all large \(j\). As \(g(t_j) \rightarrow g_{\infty}\) and \(h(t_j) \rightarrow h_{\infty}\) as \(j \rightarrow \infty\), without loss of generality we may think that \(g(t_j) < x_0 - \delta\) and \(h(t_j) > x_0 + \delta\) for all \(j\).

Let \(l_j(t) = x_0 - \delta - (t - t_j), r_j(t) = x_0 + \delta + t - t_j\). Then \(l_j(t_j) > g(t_j)\) and \(r_j(t_j) < h(t_j)\). Set

\[
\tau_j = \inf \{t > t_j : g(t) = l_j(t), \text{ or } h(t) = r_j(t)\}.
\]

Since \(h_{\infty} < \infty\), \(g_{\infty} > -\infty\), and \(l_j(t) \rightarrow -\infty\) and \(r_j(t) \rightarrow \infty\) as \(t \rightarrow \infty\), we see that \(t_j^* < \infty\).

It is easy to obtain that \(\tau_j < t_j - \delta + (h_{\infty} - g_{\infty})/2\). Without loss of generality, we assume that \(h(\tau_j) = r_j(\tau_j)\) for all \(j\). This implies

\[
g(t) \leq l_j(t) < r_j(t) \leq h(t) \quad \text{in } [t_j, \tau_j]. \tag{4.7}
\]

Define \(y_j(t, x) = (\pi - \theta)\frac{x - x_0}{\delta + t - t_j}\) and

\[
u_j(t, x) = \varepsilon e^{-k(t-t_j)}[\cos y_j(t, x) + \cos \theta], \quad (t, x) \in \overline{\Omega}_j,
\]

where \(\theta (\theta < \pi/8)\) and \(k\) are positive constants to be chosen later, and

\[
\Omega_j = \{ (t, x) : t_j < t < \tau_j, l_j(t) < x < r_j(t) \}.
\]

It is obvious that \(u_j(t, l_j(t)) = 0 = u_j(t, r_j(t))\), and \(|y_j(t, x)| \leq \pi - \theta\) for \((t, x) \in \overline{\Omega}_j\), the latter implies \(u_j(t, x) \geq 0\) in \(\Omega_j\).

We want to compare \(u(t, x)\) and \(u_j(t, x)\) in \(\overline{\Omega}_j\). Thanks to \((4.7)\), it follows that

\[
u(t, l_j(t)) \geq 0 = u_j(t, l_j(t)), \quad u(t, r_j(t)) \geq 0 = u_j(t, r_j(t)) \quad \text{for } t \in [t_j, \tau_j].
\]

On the other hand, it is obvious that

\[
u(t_j, x) \geq 2\varepsilon \geq u_j(t_j, x) \quad \text{for } x \in [x_0 - \delta, x_0 + \delta].
\]

Thus, if the positive constants \(\theta\) and \(k\) can be chosen independent of \(j\) such that

\[
u_{jt} - u_x x - u_j(1 - u_j) \leq 0 \quad \text{in } \Omega_j, \tag{4.8}
\]

it can be deduced that \(u_j(t, x) \leq u(t, x)\) for \((t, x) \in \Omega_j\) by applying the maximum principle to \(u - u_j\) over \(\Omega_j\). Since \(u(\tau_j, h(\tau_j)) = 0 = u_j(\tau_j, r_j(\tau_j))\) and \(h(\tau_j) = r_j(\tau_j)\), it follows that \(u_x(\tau_j, h(\tau_j)) \leq u_{xx}(\tau_j, r_j(\tau_j))\). Thanks to \(\varepsilon < \pi/8\) and \(\delta + \tau_j - t_j < (h_{\infty} - g_{\infty})/2\), we derive

\[
u_{xx}(\tau_j, r_j(\tau_j)) = -\frac{\varepsilon(\pi - \theta)}{\delta + \tau_j - t_j} e^{-k(\tau_j - t_j)} \sin(\pi - \theta) \leq -\frac{7\varepsilon\pi}{4(h_{\infty} - g_{\infty})} e^{-k(h_{\infty} - g_{\infty})} \sin \theta.
\]
Note the boundary condition $-\mu u_x(\tau_j, h(\tau_j)) = h'(\tau_j)$, one has immediately

$$h'(\tau_j) \geq \frac{7\mu \pi}{4(h_\infty - \sigma_k)} e^{-k(h_\infty - \sigma_k)} \sin \theta,$$

which implies $\lim_{t \to \infty} |h'(t)| > 0$ since $\lim_{j \to \infty} \tau_j \to \infty$. This contradicts to (4.2), and (4.4) is obtained.

We claim that (4.8) holds so long as $\tau > \theta$ and $k$ satisfy

$$\theta < \frac{\pi}{8}, \sin \theta < \frac{3\delta^2 \pi}{(h_\infty - \sigma_k)^3}, \ k > \frac{\pi(h_\infty - g_\infty)}{2\delta^2(\cos \theta - \cos \theta)} + 2\varepsilon + \left(\frac{\pi}{\delta}\right)^2. \quad (4.9)$$

In fact, a series of computations indicate that, for $(t, x) \in \Omega_j$,

$$u_{jt} - u_{jxx} - u_j(1 - u_j) = -ku_j - \varepsilon e^{-k(t-j)} y_j \sin y_j + \varepsilon e^{-k(t-j)} y_{jx}^2 \cos y_j - u_j(1 - u_j) \leq (2\varepsilon + y_{jx}^2 - k) u_j - \varepsilon e^{-k(t-j)} y_{jx}^2 \cos \theta - \varepsilon e^{-k(t-j)} y_j \sin y_j \leq (2\varepsilon + \frac{\pi}{\delta} - k) u_j - 4\varepsilon e^{-k(t-j)} \left(\frac{\pi - \theta}{h_\infty - g_\infty}\right)^2 \cos \theta + \frac{\varepsilon \pi x - x_0}{\delta} e^{-k(t-j)} \sin y_j \leq I(t, x).$$

Obviously, $2\varepsilon + \frac{\pi}{\delta} - k < 0$ by the third inequality of (4.9). Since $|y_j(t, x)| \leq \pi - \theta$ in $\Omega_j$, we can decompose $\Omega_j = D_j \cup E_j$ with

$$D_j = \{(t, x) \in \Omega_j: t_j < t < \tau_j, \pi - 2\theta < |y_j(t, x)| < \pi - \theta\},$$

$$E_j = \{(t, x) \in \Omega_j: t_j < t < \tau_j, \ |y_j(t, x)| \leq \pi - 2\theta\}.$$

It is obvious that $|\sin y_j(t, x)| \leq \sin 2\theta$ in $D_j$, $\cos y_j(t, x) \geq -\cos 2\theta$ in $E_j$. Note that $u_j(t, x) \geq 0$ and $|x - x_0| \leq (h_\infty - g_\infty)/2$ in $\Omega_j$, in view of (4.9), we conclude

$$I(t, x) \leq \varepsilon e^{-k(t-j)} \left( -\frac{3\pi^2}{(h_\infty - g_\infty)^2} \cos \theta + \frac{\pi(h_\infty - g_\infty)}{2\delta^2} \sin 2\theta \right) < 0$$

when $(t, x) \in D_j$, and

$$I(t, x) \leq \varepsilon e^{-k(t-j)} \left( (2\varepsilon + \frac{\pi}{\delta} - k)(\cos \theta - \cos 2\theta) + \frac{\pi(h_\infty - g_\infty)}{2\delta^2} \right) < 0$$

when $(t, x) \in E_j$. Therefore, (4.8) holds.

Step 2: Proof of (4.5). By the comparison principle, $v(t, x) \leq \tilde{v}(t)$ for all $t \in [0, \infty)$ and $x \in \mathbb{R}$, where

$$\tilde{v}(t) = b e^{bt} \left( e^{bt} - 1 + \frac{b}{\|v_0\|_\infty} \right)^{-1},$$

which is the solution of the ODE problem

$$\tilde{v}'(t) = \tilde{v}(b - \tilde{v}), \ t > 0; \ \tilde{v}(0) = \|v_0\|_\infty.$$
Thanks to Proposition [3.1], \( \liminf_{t \to \infty} v(t, x) \geq (b - \sigma) - \varepsilon \) uniformly on \([-L, L]\). By the arbitrariness of \( \varepsilon \) and \( L \), we derive that \( \liminf_{t \to \infty} v(t, x) \geq b - \sigma \) uniformly in the compact subset of \( \mathbb{R} \). Since \( \sigma > 0 \) is arbitrary, it follows that \( \liminf_{t \to \infty} v(t, x) \geq b \) uniformly in any bounded subset of \( \mathbb{R} \). The proof is complete. \( \square \)

4.2 Spreading case \((h_\infty - g_\infty = \infty)\)

We first provide a proposition which asserts the equivalence of \( h_\infty - g_\infty = \infty \) and \( g_\infty = -\infty \), \( h_\infty = \infty \).

**Proposition 4.1** If \( h_\infty - g_\infty = \infty \), then \( g_\infty = -\infty \) and \( h_\infty = \infty \).

**Proof.** Since \( h_\infty - g_\infty = \infty \), there exists \( T > 0 \) such that \( h(T) - g(T) > \pi \). Choose a function \( \tilde{u}_0(x) \) satisfying \( \tilde{u}_0 \in C^2([g(T), h(T)]) \), \( \tilde{u}_0(x) \leq u_0(x, T) \) in \([g(T), h(T)]\), \( \tilde{u}_0(x) > 0 \) in \((g(T), h(T))\) and \( \tilde{u}_0(g(T)) = \tilde{u}_0(h(T)) = 0 \). Consider the following problem

\[
\begin{align*}
\tilde{u}_t - \tilde{u}_{xx} &= \tilde{u}(1 - \tilde{u}), & t > T, & g(T) < x < \tilde{h}(t), \\
\tilde{u}(t, g(T)) &= 0 = \tilde{u}(t, \tilde{h}(t)), & t \geq T, \\
\tilde{h}'(t) &= -\mu \tilde{u}_x(t, \tilde{h}(t)), & t \geq T, \\
\tilde{h}(T) &= h(T), & \tilde{u}(T, x) &= \tilde{u}_0(x), & g(T) \leq x \leq h(T).
\end{align*}
\]

By Theorem 2.7 of [18], this problem has a unique solution \((\tilde{u}, \tilde{h})\) and exists for all \( t \geq T \), and by Theorem 4.2 of [18], \( \lim_{t \to \infty} \tilde{h}(t) = \infty \). In view of Lemma 3.2, it concludes \( h(\infty) = \lim_{t \to \infty} h(t) \geq \lim_{t \to \infty} \tilde{h}(t) = \infty \). Similarly, by Lemma 3.3, we can obtain \( g(\infty) = -\infty \). \( \square \)

We deal with the weakly hunting case \( b > c \) and \( ac < 1 \) firstly.

**Theorem 4.3** Assume that \( g_\infty = -\infty \), \( h_\infty = \infty \). For the weakly hunting case \( b > c \) and \( ac < 1 \), we have

\[
\lim_{t \to \infty} u(t, x) = \frac{1 + ab}{1 + ac}, \quad \lim_{t \to \infty} v(t, x) = \frac{b - c}{1 + ac}
\]

uniformly in any compact subset of \( \mathbb{R} \).

**Proof.** For any given \( L > 0 \) and \( 0 < \varepsilon \ll 1 \), let \( l_\varepsilon \) be given by Proposition [3.1] with \( d = 1 \), \( \beta = 1 \) and \( \theta = 1 \). In view of \( g_\infty = -\infty \) and \( h_\infty = \infty \), there exists \( T_0 > 0 \) such that

\[
g(t) < -l_\varepsilon, \quad h(t) > l_\varepsilon, \quad \forall \ t \geq T_0.
\]

Notice that \( v > 0 \), we see that \( u \) satisfies

\[
\begin{align*}
\begin{cases}
    u_t - u_{xx} > u(1 - u), & t \geq T_0, \ x \in [-l_\varepsilon, l_\varepsilon], \\
    u(t, \pm l_\varepsilon) > 0, & t \geq T_0.
    \end{cases}
\end{align*}
\]

Since \( u(T_0, x) > 0 \) in \([-l_\varepsilon, l_\varepsilon]\), applying Proposition [3.1] it arrives at

\[
\liminf_{t \to \infty} u(t, x) \geq 1 - \varepsilon \quad \text{uniformly on } [-L, L].
\]

By the arbitrariness of \( \varepsilon \) and \( L \),

\[
\liminf_{t \to \infty} u(t, x) \geq 1 := u_1 \quad \text{uniformly on the compact subset of } \mathbb{R}.
\]
Let $M = \max\{M_1, M_2\}$, where $M_i$ is determined by Lemma 2.1, $i = 1, 2$. For any given $L > 0$, $0 < \delta \ll 1$ and $0 < \varepsilon \ll 1$, let $l_\varepsilon$ be given by Proposition 1.2 with $d = D$, $\beta = b - c(u_1 - \delta)$, $\theta = 1$ and $k = M$. In view of (4.11), there exists $T_1 > 0$ such that $u(t, x) \geq u_1 - \delta$ for all $t \geq T_1$ and $x \in [-l_\varepsilon, l_\varepsilon]$. Therefore, $v$ satisfies

$$
\begin{cases}
  v_t - Dv_{xx} \leq v[b - v - c(u_1 - \delta)], & t \geq T_1, \quad x \in [-l_\varepsilon, l_\varepsilon], \\
  v(t, \pm l_\varepsilon) \leq M, & t \geq T_1.
\end{cases}
$$

As $v(T_1, x) > 0$ in $[-l_\varepsilon, l_\varepsilon]$, in view of Proposition 1.2, it yields

$$
\limsup_{t \to \infty} v(t, x) \leq b - c(u_1 - \delta) - \varepsilon \quad \text{uniformly on } [-\varepsilon, \varepsilon].
$$

The arbitrariness of $\varepsilon$, $L$ and $\delta$ imply that

$$
\limsup_{t \to \infty} v(t, x) \leq b - c u_1 := \bar{v}_1 \quad \text{uniformly on the compact subset of } \mathbb{R}. \quad (4.12)
$$

For any given $L > 0$, $0 < \delta \ll 1$ and $0 < \varepsilon \ll 1$, let $l_\varepsilon$ be given by Proposition 1.2 with $d = 1$, $\beta = 1 + a(\bar{v}_1 + \delta)$, $\theta = 1$ and $k = M$. Taking into account (4.12), and $g_\infty = -\infty$ and $h_\infty = \infty$, there is $T_2 > 0$ such that

$$
v(t, x) \leq \bar{v}_1 + \delta, \quad g(t) < -l_\varepsilon, \quad h(t) > l_\varepsilon, \quad \forall \ t \geq T_2, \ x \in [-l_\varepsilon, l_\varepsilon].
$$

Hence, $u$ satisfies

$$
\begin{cases}
  u_t - u_{xx} \leq u[1 - u + a(\bar{v}_1 + \delta)], & t \geq T_2, \quad x \in [-l_\varepsilon, l_\varepsilon], \\
  u(t, \pm l_\varepsilon) \leq M, & t \geq T_2.
\end{cases}
$$

By the same argument as above, one gets

$$
\limsup_{t \to \infty} u(t, x) \leq 1 + a \bar{v}_1 := \bar{u}_1 \quad \text{uniformly on the compact subset of } \mathbb{R}. \quad (4.13)
$$

For any given $L > 0$, $0 < \delta \ll 1$ and $0 < \varepsilon \ll 1$, let $l_\varepsilon$ be given by Proposition 1.1 with $d = D$, $\beta = b - c(\bar{u}_1 + \delta)$ and $\theta = 1$. According to (4.13), there is $T_3 > 0$ such that $u(t, x) \leq \bar{u}_1 + \delta$ for all $t \geq T_3$ and $x \in [-l_\varepsilon, l_\varepsilon]$. Hence, $v$ satisfies

$$
\begin{cases}
  v_t - Dv_{xx} \geq v[b - v - c(\bar{u}_1 + \delta)], & t \geq T_3, \quad x \in [-l_\varepsilon, l_\varepsilon], \\
  v(t, \pm l_\varepsilon) \geq 0, & t \geq T_3.
\end{cases}
$$

Similar to the above,

$$
\liminf_{t \to \infty} v(t, x) \geq b - c \bar{u}_1 := \underline{v}_1 \quad \text{uniformly on the compact subset of } \mathbb{R}. \quad (4.14)
$$

For any given $L > 0$, $0 < \delta \ll 1$ and $0 < \varepsilon \ll 1$, let $l_\varepsilon$ be given by Proposition 1.1 with $d = 1$, $\beta = 1 + a(\underline{v}_1 - \delta)$ and $\theta = 1$. By virtue of (4.14), and $g_\infty = -\infty$ and $h_\infty = \infty$, there is $T_4 > 0$ such that

$$
v(t, x) \geq \underline{v}_1 - \delta, \quad g(t) < -l_\varepsilon, \quad h(t) > l_\varepsilon, \quad \forall \ t \geq T_4, \ x \in [-l_\varepsilon, l_\varepsilon].
$$

Thus, $u$ satisfies

$$
\begin{cases}
  u_t - u_{xx} \geq u[1 - u + a(\underline{v}_1 - \delta)], & t \geq T_4, \quad x \in [-l_\varepsilon, l_\varepsilon], \\
  u(t, \pm l_\varepsilon) \geq 0, & t \geq T_4.
\end{cases}
$$
Same as above,

\[
\liminf_{t \to \infty} u(t, x) \geq (1 + a\bar{v}_i) := \underline{u}, \quad \text{uniformly on the compact subset of } \mathbb{R}.
\]

Repeating the above procedure, we can find four sequences \( \{u_i\}, \{v_i\}, \{\bar{u}_i\} \) and \( \{\bar{v}_i\} \), such that,

\[
\begin{align*}
\liminf_{t \to \infty} u(t, x) &\leq \limsup_{t \to \infty} u(t, x) \\
\liminf_{t \to \infty} v(t, x) &\leq \limsup_{t \to \infty} v(t, x)
\end{align*}
\]

uniformly in the compact subset of \( \mathbb{R} \). Moreover, these sequences can be determined by the following iterative formulas:

\[
\begin{align*}
\bar{u}_1 &= 1, & \bar{v}_1 &= b - c\bar{u}_1, & \bar{u}_i &= 1 + a\bar{v}_i, & \bar{v}_i &= b - c\bar{u}_i, & \bar{u}_{i+1} &= 1 + a\bar{v}_i, & i &= 1, 2, \ldots
\end{align*}
\]

Denote \( A = b - c \) and \( q = ac \), then \( A > 0, 0 < q < 1 \). By the direct calculation,

\[
\begin{align*}
\bar{v}_1 &= A, & \bar{u}_1 &= 1 + aA, & \bar{v}_1 &= A(1 - q), & \bar{u}_2 &= 1 + aA(1 - q), & \bar{v}_2 &= A(1 - q + q^2).
\end{align*}
\]

Using the inductive method we have the following expressions:

\[
\begin{align*}
\bar{v}_i &= A \left(1 - q + q^2 - \cdots + q^{2i-4} - q^{2i-3} + q^{2i-2}\right), & \bar{v}_i &= \bar{v}_i - Aq^{2i-1}, & i &\geq 3.
\end{align*}
\]

Because \( 0 < q < 1 \), one has

\[
\lim_{i \to \infty} \bar{v}_i = \lim_{i \to \infty} v_i = \frac{A}{1 + q} = \frac{b - c}{1 + ac}.
\]

This fact combines with (4.16) yields that

\[
\lim_{i \to \infty} \bar{u}_i = \lim_{i \to \infty} u_i = \frac{1 + ab}{1 + ac}.
\]

The limits (4.10) are followed from (4.15), (4.17) and (4.18).

For the strongly hunting case: \( b \leq c \), similar to the above, the following theorem can be obtained.

**Theorem 4.4** Assume that \( g_\infty = -\infty \) and \( h_\infty = \infty \). For the strongly hunting case \( b \leq c \), we have

\[
\lim_{t \to \infty} u(t, x) = 1, \quad \lim_{t \to \infty} v(t, x) = 0
\]

uniformly in any compact subset of \( \mathbb{R} \).

5 The criteria governing spreading and vanishing

We first give a necessary condition for vanishing.

**Theorem 5.1** Let \((u, v, g, h)\) be any solution of (1.3). If \( h_\infty - g_\infty < \infty \), then

\[
h_\infty - g_\infty \leq \pi \sqrt{1/(1 + ab)} := \Lambda.
\]

Hence, \( h_0 \geq \Lambda/2 \) implies \( h_\infty - g_\infty = \infty \) due to \( g'(t) < 0 \) and \( h'(t) > 0 \) for \( t > 0 \).
Consequently, (5.1) holds. Hence, \( \lim \inf_{t \to \infty} v(t, x) = b \) uniformly in the bounded subset of \( \mathbb{R} \). We assume \( h_\infty - g_\infty > \Lambda \) to get a contradiction. For any small \( \varepsilon > 0 \), there exists \( \tau > 1 \) such that

\[
v(t, x) \geq b - \frac{\varepsilon}{a} := A_\varepsilon, \quad \forall \ t \geq \tau, \ x \in [g_\infty, h_\infty],
\]

\[
\frac{h(\tau) - g(\tau)}{a} > \max \{2h_0, \pi \sqrt{1/(1 + ab - \varepsilon)}\}.
\]

Set \( l_1 = g(\tau) \) and \( l_2 = h(\tau) \), then \( l_2 - l_1 > \pi \sqrt{1/(1 + ab - \varepsilon)} \). Let \( w \) be the positive solution of the following initial boundary value problem with fixed boundary:

\[
\begin{cases}
  w_t = w_{xx} + w(1 - w + aA_\varepsilon), & t > \tau, \ l_1 < x < l_2, \\
  w(t, l_1) = w(t, l_2) = 0, & t > \tau, \\
  w(\tau, x) = u(\tau, x), & l_1 < x < l_2.
\end{cases}
\]

By the comparison principle,

\[
w(t, x) \leq u(t, x) \quad \text{for} \quad t \geq \tau, \ l_1 \leq x \leq l_2.
\]

Since \( 1 + aA_\varepsilon > \pi/(l_2 - l_1)^2 \), it is well known that \( w(t, x) \to \theta(x) \) as \( t \to \infty \) uniformly in the compact subset of \( (l_1, l_2) \), where \( \theta \) is the unique positive solution of

\[
\begin{cases}
  \theta_{xx} + \theta(1 + aA_\varepsilon - \theta) = 0, & l_1 < x < l_2, \\
  \theta(l_1) = \theta(l_2) = 0.
\end{cases}
\]

Hence, \( \lim \inf_{t \to \infty} u(t, x) \geq \lim_{t \to \infty} w(t, x) = \theta(x) > 0 \) in \( (l_1, l_2) \). This is a contradiction to (4.4). Consequently, \( (5.1) \) holds.

By Theorem 5.1 and Proposition 4.1, \( h_0 \geq \Lambda/2 \) implies \( g_\infty = -\infty \) and \( h_\infty = \infty \).

Now we discuss the case \( h_0 < \Lambda/2 \).

**Lemma 5.1** Suppose that \( h_0 < \Lambda/2 \). If

\[
\mu \geq \mu^0 := \max \{1, \|u_0\|_{\infty}\} \left(\pi^2 - 4h_0^2\right) \left(2 \int_{-h_0}^{h_0} (x + h_0)u_0(x)dx\right)^{-1},
\]

then \( g_\infty = -\infty \) and \( h_\infty = \infty \).

**Proof.** Consider the following auxiliary problem

\[
\begin{cases}
  \frac{u_t - u_{xx}}{u} = u(1 - u), & t > 0, \ -h_0 < x < h(t), \\
  u(t, -h_0) = 0, \ u(t, h(t)) = 0, & t \geq 0, \\
  \frac{h'(t)}{u^2(t, h(t))} = -\mu u_t(t, h(t)), & t \geq 0, \\
  u(0) = h_0, \ u(0, x) = u_0(x), & -h_0 \leq x \leq h_0.
\end{cases}
\]

It follows from Lemma 3.2 that

\[
h(t) \leq h(t), \quad u(t, x) \leq u(t, x) \quad \text{for} \quad t > 0 \quad \text{and} \quad -h_0 < x < h(t).
\]

Recall that \( 2h_0 < \Lambda < \pi \) and \( \mu \geq \mu^0 \), by Proposition 4.8 of [18], we have \( h(\infty) = \infty \). Therefore, \( h_\infty = \infty \). Similarly, \( g_\infty = -\infty \). The proof is finished. \( \square \)
Lemma 5.2 Assume that $h_0 < \Lambda/2$. Then there exists $\mu_0 > 0$, depending also on $u_0(x)$ and $v_0(x)$, such that $h_\infty - g_\infty < \infty$ when $\mu \leq \mu_0$.

Proof. We are going to construct a suitable supper solution to (1.3) and then apply Lemma 3.1. Obviously, the function

$$\bar{v}(t) = b e^{bt} \left( e^{bt} - 1 + \frac{b}{\|v_0\|_\infty} \right)^{-1}$$

satisfies

$$\begin{cases}
\bar{v}_t - D\bar{v}_{xx} \geq \bar{v} \bar{v} - \bar{v}, & x \in \mathbb{R}, \ t > 0, \\
\bar{v}(0, x) \geq v_0(x), & x \in \mathbb{R}.
\end{cases}$$

Denote $\vartheta = \frac{1}{2}h_0 + \frac{1}{2}\Lambda$, then $h_0 < \vartheta < \Lambda/2$. Inspired by [23], we define

$$f(t) = M \exp \left\{ \int_0^t \left[ 1 + a\bar{v} - \left( \frac{\pi}{2\vartheta} \right)^2 \right] ds \right\},$$

$$\eta(t) = \left( h_0^2 (1 + \delta)^2 + \mu \pi \int_0^t f(s) ds \right)^{1/2}, \ t > 0; \ w(y) = \cos \frac{\pi y}{2}, \ -1 \leq y \leq 1,$n

$$\bar{u}(t, x) = f(t) w \left( \frac{x}{\eta(t)} \right), \ t \geq 0, \ -\eta(t) \leq x \leq \eta(t),$$

where $\delta \ll 1$ is a fixed positive constant such that $\vartheta > h_0 (1 + \delta)$ and $M$ is a positive constant to be chosen later. Clearly, $\eta'(t) > 0$ for $t > 0$ and

$$\frac{f'(t)}{f(t)} = 1 + a\bar{v}(t) - \left( \frac{\pi}{2\vartheta} \right)^2 \text{ for } t > 0. \ (5.2)$$

Remember that $\vartheta < \Lambda/2$ and $\lim_{t \to \infty} \bar{v}(t) = b$, we have $1 + a\bar{v}(t) - \left( \frac{\pi}{2\vartheta} \right)^2 < 0$ for $t$ large enough, and then $\int_0^t f(s) ds$ is uniformly bounded in $[0, \infty)$.

Let

$$\mu_0 = \frac{\vartheta^2 - h_0^2 (1 + \delta)^2}{\pi \int_0^\infty f(t) dt}.$$ 

When $0 < \mu \leq \mu_0$, it is obvious that $\vartheta \geq \eta(t)$ for all $t \geq 0$. In view of (5.2), we have that by the direct computation

$$\bar{u}_t - \bar{u}_{xx} - \bar{u}(1 - \bar{u} + a\bar{v}) = f'w - f'w - \frac{\pi}{2\eta} + f(\frac{\pi}{2\eta})^2 w - f(1 - f w + a\bar{v})$$

$$\geq f w \left[ f + \left( \frac{\pi}{2\eta} \right)^2 - 1 - a\bar{v} \right]$$

$$= \frac{\pi^2}{2} f w (\eta^{-2} - \vartheta^{-2}) \geq 0$$

for all $t > 0$ and $-\eta(t) < x < \eta(t)$. On the other hand,

$$\eta'(t) = \frac{\mu \pi}{2\eta(t)} f(t), \ \bar{u}_x(t, -\eta(t)) = \frac{\pi}{2\eta(t)} f(t), \ \bar{u}_x(t, \eta(t)) = -\frac{\pi}{2\eta(t)} f(t),$$

which imply

$$-\eta'(t) = -\mu \bar{u}_x(t, -\eta(t)), \ \eta'(t) = -\mu \bar{u}_x(t, \eta(t)).$$

Choose $M$ is so large that $u_0(x) \leq M \cos \frac{\pi x}{2h_0 (1 + \delta)}$ for $x \in [-h_0, h_0]$. Then for any $0 < \mu < \mu_0$, the pair $(\bar{u}, \bar{v})$ satisfies

$$\begin{cases}
\bar{u}_t - \bar{u}_{xx} \geq \bar{u}(1 - \bar{u} + a\bar{v}), & t > 0, \ |x| < \eta(t), \\
\bar{u}(t, \pm \eta(t)) = 0, \ \eta'(t) = \mp \mu \bar{u}_x(t, \pm \eta(t)), & t > 0, \\
\bar{u}(0, x) \geq u_0(x), & |x| \leq h_0, \\
\eta(0) > h_0.
\end{cases}$$
Take advantage of Lemma 5.1: $-\eta(t) \leq g(t)$, $\eta(t) \geq h(t)$ and $u(t,x) \leq \bar{u}(t,x)$ for $t > 0$ and $g(t) \leq x \leq h(t)$. It follows that
\[ g_\infty \geq -\lim_{t \to \infty} \eta(t) > -\vartheta > -\infty, \quad h_\infty \leq \lim_{t \to \infty} \eta(t) < \vartheta < \infty. \]
The proof is complete.

**Theorem 5.2** Suppose that $h_0 < \Lambda/2$. Then there exist $\mu^* \geq \mu_*>0$, depending on $u_0(x)$, $v_0(x)$ and $h_0$, such that $g_\infty = -\infty$ and $h_\infty = \infty$ if $\mu > \mu^*$, and $h_\infty - g_\infty \leq \Lambda$ if $\mu \leq \mu_*$ or $\mu = \mu^*$.

**Proof.** The proof is similar to that of Theorem 3.9 of [10] and Theorem 4.11 of [18]. For the convenience to reader we shall give the details. Write $(u_\mu, v_\mu, g_\mu, h_\mu)$ in place of $(u, v, g, h)$ to clarify the dependence of the solution of (1.3) on $\mu$. Define
\[ \Sigma^* = \{ \mu > 0 : h_{\mu,\infty} - g_{\mu,\infty} \leq \Lambda \}. \]
By Lemma 5.2 and Theorem 5.1 $(0, \mu_0) \subset \Sigma^*$. In view of Lemma 5.1 $\Sigma^* \cap [\mu^0, \infty) = \emptyset$. Therefore, $\mu^* := \sup \Sigma^* \in [\mu_0, \mu^0]$. By this definition and Theorem 5.1 we find that $g_{\mu,\infty} = -\infty$ and $h_{\mu,\infty} = \infty$ when $\mu > \mu^*$. Hence, $\Sigma^* \subset (0, \mu^*)$.

We will show that $\mu^* \in \Sigma^*$. Otherwise, $g_{\mu^*,\infty} = -\infty$ and $h_{\mu^*,\infty} = \infty$. There exists $T > 0$ such that $h_{\mu^*}(T) - g_{\mu^*}(T) > \Lambda$. Utilizing the continuous dependence of $(u_\mu, v_\mu, g_\mu, h_\mu)$ on $\mu$, there is $\varepsilon > 0$ such that $h_{\mu}(T) - g_{\mu}(T) > \Lambda$ for $\mu \in (\mu^* - \varepsilon, \mu^* + \varepsilon)$. It follows that for all such $\mu$,
\[ \lim_{t \to \infty} [h_{\mu}(t) - g_{\mu}(t)] \geq h_{\mu}(T) - g_{\mu}(T) > \Lambda. \]
Therefore, $[\mu^* - \varepsilon, \mu^* + \varepsilon] \cap \Sigma^* = \emptyset$, and $\sup \Sigma^* \leq \mu^* - \varepsilon$. This contradicts the definition of $\mu^*$.

Define
\[ \Sigma_* = \{ \mu : \mu \geq \mu_0 \text{ such that } h_{\mu,\infty} - g_{\mu,\infty} \leq \Lambda \text{ for all } \mu \leq \nu \}, \]
where $\mu_0$ is given by Lemma 5.2. Then $\mu_* := \sup \Sigma_* \leq \mu^*$ and $(0, \mu_*) \subset \Sigma_*$. Similar to the above, it can be obtained that $\mu_* \in \Sigma_*$. The proof is completed.

6 Discussion

In this paper, we have examined a predator-prey model with double free boundaries $x = g(t)$ and $x = h(t)$ for the predator, which describes the movement process through the two free boundaries. The dynamic behavior are discussed.

A great deal of previous mathematical investigation on the spreading of population has been based on the traveling wave fronts of the predator-prey system over the entire space $\mathbb{R}$
\[
\begin{align*}
  u_t - u_{xx} &= u(1 - u + av), & t > 0, & x \in \mathbb{R}, \\
  v_t - Dv_{xx} &= v(b - v - cu), & t > 0, & x \in \mathbb{R}.
\end{align*}
\]
(6.1)
A striking difference between (1.3) and (6.1) is that the spreading front in (1.3) is given explicitly by a function $x = h(t)$, beyond which the population density of the predator is 0, while in (6.1), the population $u(t, x)$ becomes positive for all $x$ once $t$ is positive. Second, (6.1) guarantees successful spreading of the predator species for any nontrivial initial population $u(0, x)$, regardless of its initial size and supporting area, but the dynamics of (1.3) exhibits a spreading-vanishing dichotomy. The phenomenon exhibited by this dichotomy seems closer to the reality.
The spreading-vanishing dichotomy results also indicate that:

(i) When the spreading happens, both the predator and prey will converge to positive constants for the weakly hunting case, while the predator will converge to a positive constant and the prey will vanish for the strongly hunting case. These dynamic behaviours are similar to that of solution to the Cauchy problem of (6.1).

(ii) When the vanishing occurs, the predator will vanish and the prey will converge to a positive constant.

The criteria governing spreading and vanishing (Theorems 5.1 and 5.2) tell us that whether spreading or vanishing are completely determined by sizes of both the initial habitat and initial data of the predator, and the moving parameter/coefficient $\mu$ of free boundaries.

These results tell us that in order to control the prey species (pest species) we should put predator species (natural enemies) at the initial state at least in one of three ways: (i) expand the predator’s targets, (ii) increase the moving parameter/coefficient of free boundaries, (iii) augment the initial density of the predator species.

Appendix

A Global estimate of the solution $w$ to (4.3)

Proposition A.1 Let $(u, v, g, h)$ be any solution of (4.3) and assume $h_{\infty} - g_{\infty} < \infty$, if $w(t, y)$ is the solution of (4.3), then there exists a constant $K_0$ such that

$$\|w\|_{C^{1+\alpha, 1+\alpha}([1, \infty) \times [-1, 1])} < K_0. \quad (A.1)$$

Proof. We are inspired by [1, Theorem A2]. For convenience, we denote $\varphi_n(t) = \varphi(t+n), \psi_n = \psi(t+n, y), z_n = z(t+n, y), w_n = w(t+n, y)$. Let $w(t+n, y) = a^n(t, y) + b^n(t, y)$, where $a^n$ and $b^n$ are solutions of

$$\begin{cases}
a^n = \varphi_n(t)a^n_{yy}, & t > 0, -1 < y < 1, \\
a^n(t, -1) = a^n(t, 1) = 0, & t > 0, \\
a^n(0, y) = w(n, y), & -1 \leq y \leq 1,
\end{cases}$$

and

$$\begin{cases}
b^n = \varphi_n(t)b^n_{yy} + \psi_n b^n_y + \psi_n a^n_y + w_n(1 - w_n + a z_n), & t > 0, -1 < y < 1, \\
b^n(t, -1) = b^n(t, 1) = 0, & t > 0, \\
b^n(0, y) = 0, & -1 \leq y \leq 1,
\end{cases}$$

respectively. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of the problem

$$-\phi_{yy} = \lambda \phi, \quad -1 < y < 1; \quad \phi(-1) = \phi(1) = 0,$$

and let $\phi_1, \phi_2, \cdots$ be the corresponding set of orthonormal eigenfunctions. We may express $a^n$ as

$$a^n(t, y) = \sum_{k \geq 1} \exp \left(-\lambda_k \int_0^t \varphi_n(s) ds\right) w_k^n \phi_k,$$
where \( w^n_k = (\phi_k, w(n, \cdot))_{L^2} \). In view of \( 0 < u \leq M_1 \) (cf. Lemma 2.1), it follows that
\[
\frac{\partial^2 a^n(T)}{\partial y^2} \left\| L^2(-1, 1) \right\|^2 \leq \sum_{k \geq 1} \lambda_k^{2j} \exp \left( -2\lambda_k \int_0^T \varphi_n(s)ds \right) (w^n_k)^2 \leq \sup_{t \geq 0} \left( \int_0^T \varphi_n(s)ds \right) \| w(n, x) \|_{L^2(-1, 1)}^2 \leq 2M_1^2 \left( \frac{(h_\infty - g_\infty)^2}{4T} \right)^{2j} e^{-2j}.
\]

By the \( L^p \) estimates and Sobolev’s imbedding theorem, we therefore have that \( \| a^n(t) \|_{C^2[-1, 1]} \leq K_1(1 + t^{-j}) \), where \( K_1 \) is independent of \( n \) provided that \( j \geq 2 \). From this last estimate and the differential equation satisfied by \( a^n \), we obtain \( \| a^n(t) \|_{C[-1, 1]} \leq K_1 h_0^{-2}(1 + t^{-j}) \). Hence, there exists a positive constant \( K_2 \) such that \( \| a^n \|_{C^{1+\alpha}_1, 1+\alpha(E_1)} < K_2 \), where \( E_1 = [\frac{1}{2}, 2] \times [-1, 1] \) and \( K_2 \) depends only on \( K_1 \) and \( E_1 \).

Next we estimate \( b^n \). It is obvious that the function \( c^n = e^{-\frac{1}{T}b^n} \) satisfies
\[
\begin{cases}
 c^n = \varphi_n(t)c^n_{yy} + \psi_n c^n_y + f_n(t, x), & t > 0, -1 < y < 1, \\
b^n(t, -1) = b^n(t, 1) = 0, & t > 0, \\
b^n(0, y) = 0, & -1 \leq y \leq 1,
\end{cases}
\]
where
\[
f_n(t, x) = \begin{cases}
 \frac{1}{T} e^{-s} w_n + (\psi_n + \frac{1}{T}) e^{-s} a^n + e^{-s} w_n(1 - w_n + a z_n), & t > 0, \\
0, & t = 0.
\end{cases}
\]

Note that \( \lim_{t \to 0^+} t^{-j} e^{-\frac{s}{T}} = 0 \) for any \( j > 0 \), we have \( f_n(t, x) \) is continuous in \( E = [0, 3] \times [-1, 1] \) and \( \| f_n \|_{C(E)} \leq K_3 \) where \( K_3 \) is dependent on \( K_1 \) and independent of \( n \). By using of [14, Theorem 4, p191], we can obtain that \( \| c^n \|_{C^{1+\alpha}_1, 1+\alpha(E_1)} < K_3 \) where \( K_3 \) depends on \( K_3 \). It follows that
\[
\| b^n \|_{C^{1+\alpha}_1, 1+\alpha(E_1)} < K_4.
\]
We therefore have that
\[
\| u \|_{C^{1+\alpha}_1, 1+\alpha(E_n)} \leq \| a^n \|_{C^{1+\alpha}_1, 1+\alpha(E_1)} + \| b^n \|_{C^{1+\alpha}_1, 1+\alpha(E_1)} < K_2 + K_4 = K_0,
\]
where \( E_n = [n + \frac{1}{2}, n + 2] \times [-1, 1] \). It easily to get (A.1) since the intervals \( E_n \) overlap and \( K_0 \) is independent of \( n \).

**B Estimates of solutions to parabolic partial differential inequalities**

Let \( d, \beta \) and \( \theta \) be fixed positive constants. In order to investigate the long time behavior of the solution \((u, v)\) to (1.3), we should prove the following two propositions.

**Proposition B.1** For any given \( \varepsilon > 0 \) and \( L > 0 \), there exist \( l_\varepsilon > \max \{ L, \frac{\varepsilon}{2} \sqrt{d/\beta} \} \) and \( T_\varepsilon > 0 \), such that when the continuous and non-negative function \( w(t, x) \) satisfies
\[
\begin{cases}
 w_t - dw_{xx} \geq (\leq) w(\beta - \theta w), & t > 0, \ -l_\varepsilon < x < l_\varepsilon, \\
 w(t, \pm l_\varepsilon) \geq (\leq) 0, & t \geq 0,
\end{cases}
\]
and \( w(0, x) > 0 \) in \( (-l_\varepsilon, l_\varepsilon) \), then
\[
w(t, x) > \beta/\theta - \varepsilon \ (w(t, x) < \beta/\theta + \varepsilon), \quad \forall \ t \geq T_\varepsilon, \ x \in [-L, L].
\]
Which implies
\[
\liminf_{t \to \infty} w(t, x) > \frac{\beta}{\theta} - \varepsilon \left( \limsup_{t \to \infty} w(t, x) < \frac{\beta}{\theta} + \varepsilon \right) \text{ uniformly on } [-L, L].
\]

**Proof.** Let \( l > \frac{\pi}{2} \sqrt{d/\beta} \) be a parameter. Assume that \( w_l(x) \) is the unique positive solution of
\[
\begin{cases}
-dw_{xx} = w(\beta - \theta w), & -l < x < l, \\
w(\pm l) = 0.
\end{cases}
\]
By Lemma 2.2 of [12], \( \lim_{l \to \infty} w_l(x) = \beta/\theta \) uniformly in any compact subset of \( \mathbb{R} \). So, for any given \( L > 0 \) and \( \varepsilon > 0 \), there exists \( l_\varepsilon > \max \{ L, \frac{\pi}{2} \sqrt{d/\beta} \} \), which also depends on \( d, \beta \) and \( \theta \), such that
\[
\beta/\theta - \varepsilon/2 < w_l(x) < \beta/\theta + \varepsilon/2, \quad \forall \, l \geq l_\varepsilon, \; x \in [-L, L].
\]

Let \( w_0(x) \in C([-l_\varepsilon, l_\varepsilon]) \) be a positive function and \( w_\varepsilon(t, x) \) be the unique solution of
\[
\begin{cases}
w_t - dw_{xx} = w(\beta - \theta w), & t > 0, \; -l_\varepsilon < x < l_\varepsilon, \\
w(t, \pm l_\varepsilon) = 0, & t \geq 0, \\
w(0, x) = w_0(x), & -l_\varepsilon \leq x \leq l_\varepsilon.
\end{cases}
\]
Since \( l_\varepsilon > \frac{\pi}{2} \sqrt{d/\beta} \), it is well known that \( \lim_{t \to \infty} w_\varepsilon(t, x) = w_l(x) \) uniformly in the compact subset of \((-l_\varepsilon, l_\varepsilon)\). Thanks to (B.2), there is a \( T_\varepsilon \gg 1 \) such that
\[
\beta/\theta - \varepsilon < w_\varepsilon(t, x) < \beta/\theta + \varepsilon, \quad \forall \, t \geq T_\varepsilon, \; x \in [-L, L].
\]
Our conclusion is followed from (B.3) and the comparison principle.

**Proposition B.2** Let \( k \) be a positive constant. For any given \( \varepsilon > 0 \) and \( L > 0 \), there exist \( l_\varepsilon > \max \{ L, \frac{\pi}{2} \sqrt{d/\beta} \} \) and \( T_\varepsilon > 0 \), such that when the continuous and non-negative function \( z(t, x) \) satisfies
\[
\begin{cases}
z_t - dz_{xx} \geq (\leq) z(\beta - \theta z), & t > 0, \; -l_\varepsilon < x < l_\varepsilon, \\
z(t, \pm l_\varepsilon) \geq (\leq) k, & t \geq 0,
\end{cases}
\]
and \( z(0, x) > 0 \) in \((-l_\varepsilon, l_\varepsilon)\), then we have
\[
z(t, x) > \beta/\theta - \varepsilon \left( z(t, x) < \beta/\theta + \varepsilon \right), \quad \forall \, t \geq T_\varepsilon, \; x \in [-L, L].
\]
This implies
\[
\liminf_{t \to \infty} z(t, x) \geq \beta/\theta - \varepsilon \left( \limsup_{t \to \infty} z(t, x) < \beta/\theta + \varepsilon \right) \text{ uniformly on } [-L, L].
\]

**Proof.** Let \( l > \frac{\pi}{2} \sqrt{d/\beta} \) be a parameter, and \( z_l(x) \) the unique positive solution of
\[
\begin{cases}
-dz_{xx} = z(\beta - \theta z), & -l < x < l, \\
z(\pm l) = k.
\end{cases}
\]
(refer to the proof of Lemma 2.3 in [12]). We claim that
\[
\lim_{l \to \infty} z_l(x) = \beta/\theta \quad \text{uniformly in any compact subset of } \mathbb{R}.
\]
For the case $k > \beta / \theta$. By the maximum principle we see that $\beta / \theta \leq z_l(x) \leq k$ for all $x \in [-l, l]$. Note that $z_l(x) \leq k$, by the comparison principle we have that $z_l(x)$ is decreasing in $l$. Therefore, the limit $\lim_{l \to \infty} z_l(x) = z(x)$ exists, and $z(x) \geq \beta / \theta$ and $z(x)$ satisfies

$$-d z_{xx} = z(\beta - \theta z), \quad x \in \mathbb{R}. $$

By Theorem 1.2 of [12], $z(x) \equiv \beta / \theta$. Using the interior estimate we assert that $\lim_{l \to \infty} z_l(x) = z(x)$ uniformly in any compact subset of $\mathbb{R}$. Hence (B.5) holds.

For the case $k \leq \beta / \theta$. Choose $k_0 > \beta / \theta$ and let $z_l^0$ be the unique positive solution of (B.4) with $k = k_0$. By the comparison principle we have $w_t(x) \leq z_l(x) \leq z_l^0(x)$ in $[-l, l]$, where $w_l(x)$ is the unique positive solution of (B.1) with $l > \frac{d}{\pi} \sqrt{d / \beta}$. Take into account the result we have proved in the above and Lemma 2.2 of [12], it is deduced that (B.5) holds.

In view of (B.5), for any given $L > 0$ and $\varepsilon > 0$, there is $l_\varepsilon > \max \{L, \frac{d}{\pi} \sqrt{d / \beta} \}$, which also depends on $d, \beta, \theta$ and $k$, such that

$$\frac{\beta}{\theta} - \varepsilon / 2 < z_l(x) < \frac{\beta}{\theta} + \varepsilon / 2, \quad \forall \ l \geq l_\varepsilon, \ x \in [-L, L]. \quad \text{(B.6)}$$

Let $z_0(x) \in C([-l_\varepsilon, l_\varepsilon])$ be a positive function and $z_\varepsilon(t, x)$ be the solution of

$$\begin{cases}
    z_t - d z_{xx} = z(\beta - \theta z), & t > 0, \ -l_\varepsilon < x < l_\varepsilon, \\
    z(t, \pm l_\varepsilon) = k, & t \geq 0, \\
    z(0, x) = z_0(x), & -l_\varepsilon \leq x \leq l_\varepsilon.
\end{cases}$$

Recall $l_\varepsilon > \frac{d}{\pi} \sqrt{d / \beta}$, we shall illustrate that

$$\lim_{t \to \infty} z_\varepsilon(t, x) = z_{l_\varepsilon}(x) \quad \text{uniformly in the compact subset of } (-l_\varepsilon, l_\varepsilon). \quad \text{(B.7)}$$

In fact, take a positive constant $q$ and let $\phi_q(t, x)$ be the unique solution of

$$\begin{cases}
    \phi_t - d \phi_{xx} = \phi(\beta - \theta \phi), & t > 0, \ -l_\varepsilon < x < l_\varepsilon, \\
    \phi(t, \pm l_\varepsilon) = k, & t \geq 0, \\
    \phi(0, x) = q, & -l_\varepsilon \leq x \leq l_\varepsilon.
\end{cases}$$

Let $M \gg 1$ and $0 < m \ll 1$. Then $M$ and $m$ are the ordered upper and lower solutions of (B.4) with $l = l_\varepsilon$. Therefore, $\phi_M(t, x)$ is monotone decreasing and $\phi_m(t, x)$ is monotone increasing in $t$. So, the limits $\lim_{t \to \infty} \phi_M(t, x) = \phi_M(x)$ and $\lim_{t \to \infty} \phi_m(t, x) = \phi_m(x)$ exist, and they are all positive solution of (B.4) with $l = l_\varepsilon$. Hence, $\phi_M(x) = \phi_m(x) = z_{l_\varepsilon}(x)$. Meanwhile, the comparison principle yields $\phi_m(t, x) \leq z_l(t, x) \leq \phi_M(t, x)$. Consequently, $\lim_{t \to \infty} z_\varepsilon(t, x) = z_{l_\varepsilon}(x)$. By use of the interior estimate, it can be shown that this limit is uniformly in the compact subset of $(-l_\varepsilon, l_\varepsilon)$.

Thanks to (B.6) and (B.7), there is $T_\varepsilon \gg 1$ such that

$$\beta / \theta - \varepsilon < z_\varepsilon(t, x) < \beta / \theta + \varepsilon, \quad \forall \ t \geq T_\varepsilon, \ x \in [-L, L].$$

By use of this fact and the comparison principle, the proof is immediately completed.

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