Holograph in noncommutative geometry: 
Part 1

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ABSTRACT: In this paper, we consider the holograph principle emergent from noncommutative geometry, based on the spectral action principle. We show that under some appropriate conditions, the gravity theory on a manifold with boundary could be equivalent to a gauge theory $SU(N)$ on the boundary. Then an expression for $N$ with the geometrical quantities of the manifold is given. Based on this result, we find that the volume of the manifold and the boundary have some discrete structure. Applying the result to the black hole, we get that the radium of the Schwarzschild black hole is quantized. We also find an explanation why the extremal RN-black hole has zero temperature but with finite entropy.

KEYWORDS: Noncommutative geometry, holograph principle, spectral action principle, black hole entropy.
1. Introduction

Noncommutative geometry (NCG) [1] is a branch of mathematics that has many applications in physics, such as in string theory [2], quantum Hall effect [3], cosmology [4] and so on. It can unify the gravity theory and the gauge theory, and also can afford a starting point for quantum gravity. A good review of the noncommutative space appeared in physics is [5].

It is well believed that the holograph principle [6, 7] is a fundamental principle in quantum gravity. It roughly states that quantum gravity on a bulk space can be described by a suit theory on the boundary. This principle is motivated by the black hole physics, and strengthened by the AdS/CFT duality in M theory.

Since the NCG is an approach to quantum gravity, we can conjecture that the holograph principle will emerge from it as well. In this paper we will show that the holograph principle is a result of the spectral action principle on suit system. In this paper, we work with Planck unit system $\hbar = c = 1$. 
2. **Two systems**

Let $M$ be a $(n + 1)$ dimension smooth compact Riemannian manifold with smooth boundary $\partial M$. Then $\partial M$ has dimension $n$. Assume $n$ to be even. Let $\Delta$ be the scale Laplace operator on $M$. Define the eigenvalue problem:

$$\Delta \phi_n = \lambda_n \phi_n, \quad (2.1)$$

$$\mathcal{B}^\pm \phi = 0. \quad (2.2)$$

where $\mathcal{B}$ is called the boundary operator. The frequently used condition are the Dirichlet boundary condition $\mathcal{B}^-$ and Neumann boundary condition $\mathcal{B}^+$ respectively:

$$\mathcal{B}^- \phi = \phi|_{\partial M} \quad (2.3)$$

$$\mathcal{B}^+ \phi = (\phi, n + S\phi)|_{\partial M}, \quad (2.4)$$

where $S$ is a matrix valued function on $\partial M$. The boundary condition $\mathcal{B}^\pm$ is also called Robin condition.

For the scale Laplace operator $\Delta$ on the manifold $M$, we define its spectral action:

$$I = Tr e^{-\Delta/\Lambda^2}. \quad (2.5)$$

Then it has a heat kernel expansion [8]:

$$I = Tr e^{-\Delta/\Lambda^2} \simeq \sum_{m \geq 0} \Lambda^{(n+1)-m} a_m(\Delta). \quad (2.6)$$

where the Seeley-de Witt coefficients $a_m$ are given by [8]:

$$a_0(\Delta, \mathcal{B}) = (4\pi)^{-(n+1)/2} \int_M d^n x \sqrt{g} = (4\pi)^{-(n+1)/2} \text{vol}(M). \quad (2.7)$$

$$a_1(\Delta, \mathcal{B}) = 1/4 \chi (4\pi)^{-n/2} \int_{\partial M} d^n x \sqrt{h} = 1/4 \chi (4\pi)^{-n/2} \text{vol}(\partial M). \quad (2.8)$$

where $h$ is the induced metric on the boundary, and $\chi = \pm 1$ for boundary condition $\mathcal{B}^\pm$. This spectral action contain the cosmology term, the Hilbert-Einstein term, and other high order terms.
Then we consider another system, the gauge theory $SU(N)$ on the boundary $\partial M$, considered in paper[9]. The spectral triple for this system is $(\mathcal{A}, \mathcal{H}, D)$, where

$$\mathcal{A} = C^\infty(\partial M) \otimes M_N(C), \mathcal{H} = L^2(\partial M, S) \otimes M_N(C), D = \partial_{\partial M} \otimes 1.$$  

After inner fluctuation the Dirac operator is given by

$$D = e^\mu a^a (\partial + \omega) \otimes 1_N + 1 \otimes (-\frac{i}{2} g_0 A^i T^i).$$  

where $\omega$ is the spin connection on $\partial M$, and $T^i$ are matrices in the adjoint representation of $SU(N)$ satisfying $Tr(T^i T^j) = 2\delta^{ij}$.

Then we can calculate the spectral action for this system, and the result was given in paper [9]:

$$I' = Tr e^{-D^2/\Lambda^2} \simeq \sum_{m \geq 0} \Lambda^{m-n} \int_{\partial M} a_m(x, D^2) dv(x).$$  

The coefficients $a'_m(D^2)$ are:

$$a'_0(x, D^2) = (4\pi)^{-n/2} Tr(I),$$

$$a'_2(x, D^2) = (4\pi)^{-n/2} Tr(R/6I + E),$$

$$a'_{2m+1}(x, D^2) = 0.$$  

So we can get:

$$a'_0(D^2) = (4\pi)^{-n/2} N 2^{n/2} \int_{\partial M} d^n x \sqrt{h},$$

$$a'_2(x, D^2) = -\frac{N 2^{n/2}}{12(4\pi)^{n/2}} \int_{\partial M} d^n x \sqrt{h} R,$$

$$a'_{2m+1}(x, D^2) = 0.$$  

Now we have two systems: on the one hand, the scale Laplace operator on the manifold $M$ with boundary $\partial M$ which describes pure gravity; on the other hand, the square of the Dirac operator which contain the gauge part on the boundary $\partial M$. According to the spectral action[9], that is “the physical action only depends upon the spectrum”, if those two system have the same spectrum, hence the same spectral action, they describe the same physics. From this we get can the holograph principle. So next we will investigate under what conditions we can get the same spectral action.
3. The Correspondence

In this section, we will see under what conditions the two spectral action coincide.
First let’s introduce a length parameter \( l_c \equiv \Lambda^{-1} \). If the two spectral action coincide, we have:

\[
\begin{align*}
  a_0 l_c^{-(n+1)} + a_1 l_c^{-n} &= a'_0 l_c^{-n}, \\
  a_2 l_c^{-(n-1)} + a_3 l_c^{-n+2} &= a'_2 l_c^{-n-2}, \\
  a_n l_c^{-1} + a_{n+1} &= a'_n,
\end{align*}
\]

(3.1)

From the first equation, we get:

\[
vol(M) = vol(\partial M) \sqrt{4\pi} (N 2^{n/2} - 1/4) l_c.
\]

(3.2)

That is

\[
N = \frac{vol(M)}{vol(\partial M) \sqrt{\pi} 2^{n/2+1} l_c} + 2^{-n/2-2} \chi.
\]

(3.3)

Since \( N \) is a positive integer, the equation indicate that the volume of the manifold and the boundary must have some discrete structure.

Remark 1: We can see that the expression for the \( N \) has two terms, one term containing \( l_c \), and the other not. This splitting is essential for the holograph. In \( I' \), the odd term \( a_{2n+1} \) is zero, and \( N \) will split into two terms to correspond to the bulk and boundary terms in \( I \). This is the reason why we use the scale Laplace operator instead of the more nature Dirac operator. For Dirac operator, the \( a_1 = 0 \) due to the \( Trace(\chi) = 0 \), but other odd terms are not zero. Due to (3.3), the \( N \) only has one term, and can’t get the odd terms in \( I \).

This is our first result from our assumptions, and we want to see the consequence of it.

4. Black hole entropy

From black hole thermodynamics we know that a black hole have a temperature and an entropy \([10]\). First let’s consider the simplest one, the Schwarzschild black hole:

\[
ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]

(4.1)

The author will thank Alain Connes for explanation on this point.
which has a temperature $T_{BH} = \frac{1}{4\pi r_s} = \frac{1}{8\pi M}$, and entropy $S = \frac{A}{4\pi}$.

Consider the black hole as the region inside the horizon and the horizon as the boundary, we apply the above result. In this case, $n = 2$.

The volume inside the horizon is

$$vol(M) = \int_M d^3x \sqrt{|g|} = 4\pi \int_0^{r_s} \frac{r^2 dr}{\sqrt{r_s/r - 1}}. \quad (4.2)$$

So we can get

$$vol(M) = \frac{5}{4\pi^2} r_s^3. \quad (4.3)$$

With induced metric on the boundary, we can get $vol(\partial M) = 4\pi r_s^2$. Take those values into equation 3.2, we can get

$$\frac{5}{4\pi^2} r_s^3 = 4\pi r_s^2 \sqrt{4\pi (2N - 1/4\chi)} l_c. \quad (4.4)$$

Then we get

$$r_s = \frac{32}{5\pi^{-1/2}} (2N - 1/4\chi) l_c. \quad (4.5)$$

So the smallest radius of a black hole is $(r_s)_{min} = \frac{32}{5\pi^{-1/2}} (2 - 1/4\chi) l_c$ for different boundary condition. And the gap between nearest radius is

$$\Delta r = \frac{64}{5\pi^{-1/2} l_c}. \quad (4.6)$$

We can see that the radius is equidistant, so the area is not. This result is sharply contrary to Bekenstein’s suggestion $[11]$: $A = \alpha n l_p^2$.

Change the expression 4.3 to another way, we get

$$N = \frac{5\sqrt{\pi} r_s}{64 l_c} + \chi/8. \quad (4.7)$$

Consider the gauge theory on the horizon. We have an expression for $N$ with the geometry of the black hole, the equation 4.7. With some basic quantum statistics calculations, we can get the entropy for the free $SU(N)$ glues gases in two dimension at temperature $T$, that is

$$S = \frac{3AT^2}{4\pi} (N^2 - 1) \zeta(3). \quad (4.8)$$
With the expression for $T = T_{BH}$ and $N$, we get
\[ S = \frac{3\zeta(3)}{64\pi^2} \left( \frac{25A}{64l_c^2} + 2.5\sqrt{A/l_c} - 252 \right) \] \[ N \geq 2. \] \[ (4.9) \]
This is the entropy of free glues gas, for interacting glues gas, we have to add a factor $S' = f(g)S$, where $f(g)$ is a function depending on the coupling constant. The gauge coupling constant $g$ will appear in $a_4$ term, so we don’t know which limit of gravity theory this weak limit of gauge theory correspondence to. We know that the entropy for a black hole is expression $S = \frac{A}{4l_p^2}$, so they have the same structure, but we don’t know the exact relation between $l_c$ and $l_p$.

From equation (4.9), we can find that the entropy has a $\sqrt{A}$ term, which is absent in the usual formula. But since there maybe exist the volume correction for the black hole entropy [12], the length correction maybe also exist.

Since there exist the smallest length of black hole, we can calculate the smallest area for a Schwarzschild black hole.
\[ A_{\min} = 4\pi(r_s)^2 = (112/5l_c)^2. \] \[ (4.10) \]
We assume that the $l_c$ is a fundamental length scale in noncommutative geometry.

Next we consider the Reissner-Nordstrom(RN) black hole, which has the metric:
\[ ds^2 = -(1 - 2M/r + Q^2/r^2)dt^2 + (1 - 2M/r + Q^2/r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \]
There are two horizons with radium
\[ r_\pm = M \pm \sqrt{M^2 - Q^2}. \] \[ (4.12) \]
The extremal RN-black hole has $r_+ = r_-$. The temperature on the horizons are:
\[ T_\pm = \frac{r_+ - r_-}{4\pi r_\pm^2}. \] \[ (4.13) \]
The entropy of the RN-black hole is $S = \frac{A_\pm^2}{4l_p^2}$.

We apply our method to the RN-black hole. Consider the black hole as the region between the two horizons, and the horizons as the boundary with opposite direction. The volume between the two horizons is
\[ vol(M) = \int_\mathcal{M} d^3x \sqrt{|g|} = 4\pi \int_{r_-}^{r_+} \frac{r^3 dr}{\sqrt{(r-r_-)(r_+-r)}}. \] \[ (4.14) \]
Redefine \( r_- = a \), \( r_+ - r_- = b \). We can calculate this integral to get

\[
vol(M) = 4\pi^2(a^3 + 3/2a^2b + 9/8ab^2 + 5/16b^3).
\] (4.15)

When \( r_- = a = 0 \), then \( r_+ = b = 2M \), that is the Schwarzschild black hole, we get

\[
vol(M) = 5/4\pi^2r_+^3,
\] just the result \([4.3]\). Since the boundary have opposite direction, the boundary should be

\[
vol(\partial M) = 4\pi(A_+ - A_-) = 4\pi(r_+^2 - r_-^2) = 4\pi(2a + b)b.
\] (4.16)

Putting those expression into equation \([3.3]\) we can get

\[
N = \frac{(a^3 + 3/2a^2b + 9/8ab^2 + 5/16b^3)\sqrt{\pi}}{4(2a + b)bl_c} + \chi/8
\] (4.17)

Let’s consider the extremal RN-black, that is \( b \to 0 \), then \( T \to 0 \). On the other hand,

\[
N \approx \frac{(a^3 + 3/2a^2b)\sqrt{\pi}}{4(2a + b)bl_c} \to \infty.
\] (4.18)

The entropy of the \( SU(N) \) glues gas on the two boundary is

\[
S = \frac{3\zeta(3)}{4\pi}(A_+ T_+^2 + A_- T_-^2)N^2 \approx \frac{3\zeta(3)A^2}{4^5\pi^3}(Nb/(a + b))^2 \approx \frac{3\zeta(3)A^2}{2^{11}\pi^2l_c^2}.
\] (4.19)

From the above calculation we can see that in the extremal case, though the temperature approach zero, on the other hand, the \( N \) approach infinite, and their product is a finite number, so the entropy is finite too. To describe an extremal RN-black hole, we have to use the \( SU(\infty) \) gauge theory. Since the entropy of the glues gas is related to the black hole entropy, the black hole entropy is also finite.

For a general RN-black hole, we can also calculate the entropy. At this time,

\[
N \approx \frac{(5r_+^2 - 2r_+r_- + 5r_-^2)\sqrt{\pi}}{64(r_+ - r_-)l_c}.
\] (4.20)

So the entropy is

\[
S = \frac{3\zeta(3)}{4\pi}(A_+ T_+^2 + A_- T_-^2)N^2 \approx \frac{3\zeta(3)A^2}{2^{11}\pi^2l_c^2}(5 - 2r_-/r_+ + 5r_-^2/r_+^2)^2.
\] (4.21)

So in the general case, the entropy is a function of both \( r_+ \) and \( r_- \).
5. Conclusion

In this paper, we consider two related systems: the gravity theory on a manifold with boundary, and the gauge theory on the boundary. Under some conditions, the two system may have the same spectral action. Due to the spectral action principle, they have the same physics, so we get the holograph principle.

In this paper, we only consider the first term in the spectral action expansion, and get the relation between the geometrical quantities on the gravity side and $N$ on the gauge side. We then apply this relation to black hole entropy, and get a qualitative right expression.

In the following work, we will calculate the next few terms in the spectral action expression, and get more relations between the gravity theory and the gauge theory.

We think that the spectral action principle is a more fundamental principle in quantum gravity, and may play some essential role in the background-independent formulation of M theory. We should pay much more attention to it.

Acknowledgments

The author will thank Doctor Zhiwen Shi and Doctor Pong Song in institute of physics (Chinese Academy of Science) for many help.

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