REMARKS ON THE REGULARITY CRITERIA OF
GENERALIZED MHD AND NAVIER-STOKES SYSTEMS

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Abstract. We study the regularity criteria of the three dimensional generalized MHD and Navier-Stokes systems. In particular, we show that the regularity criteria of the generalized MHD system may be reduced to depend only on two diagonal entries of the Jacobian matrix of the velocity vector field or one vorticity component and one entry of the Jacobian matrix of the velocity vector field.

Keywords: MHD system, Navier-Stokes system, regularity criteria

1. Introduction and statement of results

We study the generalized magnetohydrodynamics (MHD) and Navier-Stokes (NSE) systems in $\mathbb{R}^3$:

$$
\begin{aligned}
\frac{\partial}{\partial t} u + (u \cdot \nabla) u - (b \cdot \nabla) b + \nabla p + \nu \Lambda^{2\alpha} u &= 0 \\
\frac{\partial}{\partial t} b + (u \cdot \nabla) b - (b \cdot \nabla) u + \eta \Lambda^{2\beta} b &= 0 \\
\nabla \cdot u &= \nabla \cdot b = 0, \quad u(x,0) = u_0(x), \quad b(x,0) = b_0(x)
\end{aligned}
$$

(1)

$$
\begin{aligned}
\frac{\partial}{\partial t} u + (u \cdot \nabla) u + \nabla p + \nu \Lambda^{2\alpha} u &= 0 \\
\nabla \cdot u &= 0, \quad u(x,0) = u_0(x)
\end{aligned}
$$

(2)

where $u : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$ represents velocity vector field, $b : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$ the magnetic vector field, $p : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ the pressure scalar field and $\nu, \eta > 0$ the kinematic viscosity and diffusivity constants respectively. The operator $\Lambda = (-\Delta)^{\frac{\alpha}{2}}$ is a fractional Laplacian with power $\alpha, \beta > 0$ as parameters. Without loss of generality, we set $\nu = \eta = 1$ throughout the rest of the paper.

The global regularity issue of these systems remain one of the most challenging outstanding open problems in mathematical analysis. In two dimensional case, both MHD and NSE admit a unique global strong solution respectively; however, in three dimensional case, such results hold only locally in time (e.g. [21]).

Starting from the pioneering works of Serrin in [18] and [19] on NSE, much effort was devoted to provide sufficient conditions for a strong solution to exist globally in time and similarly for MHD (cf. e.g. [1], [2], [3], [10], [14], [15], [20], [24], [28])

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in case of NSE and \([7], [9], [11], [22]\) in case of MHD). In particular, recently in \([4]\) it was shown that the global regularity issue of the solution to NSE may depend only on one entry of the Jacobian matrix of the velocity vector field while in \([5]\) the global regularity issue of the solution to MHD only on a partial derivative of \(u\) in \(x_3\)-direction. In relevance to such component reduction type results of regularity criteria, we mention that recently we have seen developments in the case of active scalars as well (cf. \([23]\)).

However, to the best of our knowledge, it is not known whether the regularity criteria of MHD system may be reduced to rely on only the entries of the Jacobian matrix of velocity vector field with number less than three. In fact, there is no regularity criteria for the system (1) even in terms of \(\nabla u_3\), although it is known to exist for the NSE system (cf. e.g. \([13]\)). In \([12]\) the authors obtained partial results toward direction.

Moreover, it is not clear whether the regularity criteria in terms of one entry in the Jacobian matrix for the classical NSE may be generalized to the case with a fractional Laplacian. Because the system (2) with \(\alpha \geq \frac{5}{4}\) admits a unique global solution (cf. \([21]\)), it is of interest if we may generalize such a result for the case \(\alpha \in (1, \frac{5}{4})\). We answer these questions:

**Theorem 1.1.** Let \(\alpha, \beta \in (1, \frac{5}{4})\). Suppose the solution \((u, b)(x, t)\) solves (1) in \([0, T]\) and

\[
\int_0^T \|\partial_2 u_2\|_{L^p}^r + \|\partial_3 u_3\|_{L^p}^r dt < \infty
\]

for \(3 \leq p < \infty \) and

\[
\frac{3}{p} + \frac{2\alpha}{r} \leq \min \left\{ \frac{3}{p} + \frac{[5\alpha(1 - \frac{1}{p}) + 4\alpha^2(1 - \frac{1}{p}) - 10 + 4\alpha]}{2(5 - 2\alpha)}, \right. \\
\frac{3}{p} + \frac{\alpha}{\beta} \frac{[5\beta(1 - \frac{1}{p}) + 4\alpha\beta(1 - \frac{1}{p}) - 10 + 4\alpha]}{2(5 - 2\alpha)}, \right. \\
\frac{3}{p} + \frac{[5\alpha(1 - \frac{1}{p}) + 4\alpha\beta(1 - \frac{1}{p}) - 10 + 4\beta]}{2(5 - 2\beta)}, \right. \\
\left. \frac{3}{p} + \frac{\alpha}{\beta} \frac{[5\beta(1 - \frac{1}{p}) + 4\beta^2(1 - \frac{1}{p}) - 10 + 4\beta]}{2(5 - 2\beta)} \right\},
\]

then there is no singularity up to time \(T\).

The modification of the proof of Theorem 1.1 for the classical MHD system is possible:

**Theorem 1.2.** Let \(\alpha, \beta = 1\). Suppose the solution \((u, b)(x, t)\) solves (1) in \([0, T]\) and

\[
\int_0^T \|\partial_2 u_2\|_{L^p}^r + \|\partial_3 u_3\|_{L^p}^r dt < \infty
\]

for \(3 < p < \infty \) and

\[
\frac{3}{p} + \frac{2}{r} \leq \frac{3}{2p} + \frac{1}{2}.
\]
then there is no singularity up to time $T$.

We state an immediate interesting corollary of Theorem 1.2 which does not seem to follow from the work of [5] or [12]:

**Corollary 1.3.** Let $\alpha, \beta = 1$. Suppose the solution $(u, b)(x, t)$ solves (1) in $[0, T]$ and

$$\int_0^T \|w_3\|_{L^p}^r + \|\partial_3 u_3\|_{L^p}^r dt < \infty$$

for $3 < p < \infty$ and

$$\frac{3}{p} + \frac{2}{r} < \frac{3}{2p} + \frac{1}{2}$$

where $w_3 = \partial_1 u_2 - \partial_2 u_1$, then there is no singularity up to time $T$.

**Theorem 1.4.** Let $\alpha \in \left(1, \frac{5}{4}\right)$. Suppose the solution $u(x, t)$ solves (2) in $[0, T]$ and

$$\int_0^T \|\partial_3 u_3\|_{L^p}^r dt < \infty$$

where $2 < p < \infty$ and

$$\frac{3}{p} + \frac{2\alpha}{r} \leq \frac{3}{p} + \left(\frac{p - 2}{p}\right) \frac{\alpha(5 + 4\alpha)}{4(5 - 2\alpha)}$$

then there is no singularity up to time $T$.

**Remark 1.1.**  
(1) The key to the proof of theorems of this type is an appropriate decomposition of nonlinear terms. It is not clear whether a direct extension of the proof in [4] is possible due to the complex structure of the four nonlinear terms of (1), as discussed in [12]. Our approach is based on an observation that upon $\|\nabla h u\|_{L^2}^2 + \|\nabla h b\|_{L^2}^2$ estimate, every nonlinear term has $u$ involved. Hence, making use of the incompressibility of both $u$ and $b$, we may separate $u_1, u_2$ and $u_3$. Our second observation is that Lemma 2.2 below due to [4], of which originally $i = 3$, may be used for $i$ any direction. Thus, we can use this lemma to concentrate the regularity dependence on $\partial_2 u_2$ and $\partial_3 u_3$.

(2) From the proof, it becomes clear that in fact we could have selected any one of the three partial derivatives of $u_1, u_2$ and $u_3$. Thus, for Theorem 1.2, we also proved the criteria in terms of $\partial_3 u$ which is the result from [5]; hence, our results are more general. Moreover, Theorem 1.1 may be seen as a component reduction type result of the work of [8], [25], and [27]. Moreover, our proof may be extended to a regularity criteria of a component and a partial derivative, e.g. $u_3$ and $\partial_3 u_3$ in the case of Theorem 1.4 as done in [26]; we chose to state the case of only partial derivatives for simplicity.

(3) The lower bound of $p$ in the Theorem 1.1 may be optimized furthermore in terms of $\alpha$ and $\beta$; we chose to state so for simplicity.

(4) Concerning Corollary 1.3, we refer readers to [6] for similar result in the case of the NSE.

In the next sections, we list a few lemmas and thereafter prove our theorems.
2. Preliminaries

We denote by $\nabla_p$ the horizontal gradient while $\Delta_p$ the horizontal Laplacian, i.e.

$$\nabla_p = (\partial_1, \partial_2), \quad \Delta_p \partial_1^2 + \partial_2^2$$

Moreover, we denote for simplicity

$$X(t) = \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2,$$

$$Y(t) = \|\nabla_h u(t)\|_{L^2}^2 + \|\nabla_h b(t)\|_{L^2}^2$$

and

$$Z(t) = \|\partial_2 u_2(t)\|_{L^2}^{\gamma - 1} + \|\partial_3 u_3(t)\|_{L^2}^{\gamma - 1}.$$

**Lemma 2.1.** (cf. [14]) Let $u \in H^2(\mathbb{R}^3)$ be smooth and divergence free. Then

$$\sum_{i,j=1}^2 \int_{\mathbb{R}^3} u_i \partial_i u_j \Delta_p u_j$$

$$= \frac{1}{2} \sum_{i,j=1}^2 \int_{\mathbb{R}^3} \partial_i u_j \partial_i u_j \partial_3 u_3 - \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 + \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_1 \partial_3 u_3$$

**Lemma 2.2.** (cf. [4], [26]) For $f, g, \in C_c^\infty(\mathbb{R}^3)$, we have

$$|\int_{\mathbb{R}^3} f g h| \leq c \|f\|_{L^q}^{\frac{\gamma - 1}{\gamma - 2}} \|\partial_i f\|_{L^2}^{\frac{1}{\gamma - 2}} \|g\|_{L^2}^{\frac{\gamma - 2}{\gamma - 2}} \|\partial_j g\|_{L^2}^{\frac{1}{\gamma - 2}} \|h\|_{L^2}^{\frac{1}{\gamma - 2}}$$

where

$$2 < \gamma, \quad 1 < q, \quad s \leq \infty, \quad \frac{\gamma - 1}{q} + \frac{1}{s} = 1$$

and $i, j$ and $k$ are any combinations of 1, 2 and 3.

The proof of the following elementary inequality is simple and we omit it:

**Lemma 2.3.** For $0 \leq p < \infty$ and $a, b \geq 0$,

$$(a + b)^p \leq 2^p (a^p + b^p)$$

3. Proofs

3.1. **Proof of Theorem 1.1.** We start by taking an inner product of the first equation in (1) with $u$ and the second with $b$ and integrating in time to obtain

$$\sup_{t \in [0, T]} \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2 \leq c(u_0, b_0) \quad (6)$$
3.1.1. Estimate of $\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2$. Local well-posedness is shown in [21]. We devote our effort to obtain $H^1$ estimate below. We take an inner product of the first equation in (1) with $-\Delta_h u$ and the second with $-\Delta_h b$ to obtain

\[
\frac{1}{2} \partial_t Y + \|\Lambda^\alpha \nabla_h u\|_{L^2}^2 + \|\Lambda^\beta \nabla_h b\|_{L^2}^2
= \int (u \cdot \nabla)u \cdot \Delta_h u - (b \cdot \nabla)b \cdot \Delta_h u + (u \cdot \nabla) b \cdot \Delta_h b - (b \cdot \nabla)u \cdot \Delta_h b = \sum_{i=1}^4 J_i
\]

For $J_1$, we notice that applying Lemma 2.1 and integrating by parts implies

\[
J_1 \leq c \int |u_3||\nabla u||\nabla \nabla_h u|
\]

For $J_2, J_3, J_4$, we decompose them as follows:

\[
J_2 + J_4 = -\int (b \cdot \nabla) b \cdot \Delta_h u + (b \cdot \nabla)u \cdot \Delta_h b
= \sum_{i,j=1}^3 \sum_{k=1}^2 \int \partial_k b_i \partial_i b_j \partial_k u_j + \partial_k b_i \partial_i u_j \partial_k b_j
\]

due to the incompressibility of $b$. We integrate by parts once more to obtain

\[
J_2 + J_4 \leq c \int |\nabla \nabla_h b||\nabla b|(|u_1| + |u_2| + |u_3|)
\]

Similarly, integrating by parts and using incompressibility of $u$, we obtain

\[
J_3 \leq c \int |\nabla \nabla_h b||\nabla b|(|u_1| + |u_2| + |u_3|)
\]

Now we apply Lemma 2.2 with

\[
f = |u_3|, g = |\nabla u|, h = |\nabla \nabla_h u|, i = 3, j = 1, k = 2, q = 2
\]

to bound using (6)

\[
J_1 \leq c \|\partial_3 u_3\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{2-\alpha}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h u\|_{L^2}^{\frac{2}{2}}
\leq c \|\partial_3 u_3\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{2-\alpha}{2}} \|\Lambda^\alpha \nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h u\|_{L^2}^{\frac{2}{2}}
\leq c \|\Lambda^\alpha \nabla_h u\|_{L^2}^{2} + \frac{1}{4} \|\nabla \nabla_h u\|_{L^2}^{2} + c \|\partial_3 u_3\|_{L^2}^{\frac{2(2\gamma - \theta - 1)}{\gamma(1 - \gamma)}} \|\nabla u\|_{L^2}^{\frac{2(2\gamma - \theta - 1)}{\gamma(1 - \gamma)}} \|\Lambda^\alpha \nabla u\|_{L^2}^{\frac{2}{2}}
\]

where we used Gagliardo-Nirenberg inequalities and Young’s inequalities. Similarly, applying Lemma 2.2 and using (6) we obtain
\[ J_2 + J_3 + J_4 \leq c(\|\partial_1 u_1\|_{L^2} \|\nabla b\|_{L^2} \|\nabla \nabla_h b\|_{L^2} \|\Delta b\|_{L^2} \]
\[ + \|\partial_2 u_2\|_{L^2} \|\nabla b\|_{L^2} \|\nabla \nabla_h b\|_{L^2} \|\Delta b\|_{L^2} \]
\[ + \|\partial_3 u_3\|_{L^2} \|\nabla b\|_{L^2} \|\nabla \nabla_h b\|_{L^2} \|\Delta b\|_{L^2} \]

Now we use divergence-free condition of \( u \) and apply Gagliardo-Nirenberg inequalities and Young’s inequalities to bound by

\[ c(\|\partial_2 u_2\|_{L^2} \|\nabla b\|_{L^2} \|\nabla \nabla_h b\|_{L^2} \|\Delta b\|_{L^2} \]
\[ + c(\|\partial_3 u_3\|_{L^2} \|\nabla b\|_{L^2} \|\nabla \nabla_h b\|_{L^2} \|\Delta b\|_{L^2} \]
\[ \leq c(\|\partial_2 u_2\|_{L^2} \|\nabla b\|_{L^2} \|\nabla \nabla_h b\|_{L^2} \|\Delta b\|_{L^2} \]
\[ + c(\|\partial_3 u_3\|_{L^2} \|\nabla b\|_{L^2} \|\nabla \nabla_h b\|_{L^2} \|\Delta b\|_{L^2} \]
\[ \leq c Z(\|\nabla u\|_{L^2} \|\nabla b\|_{L^2} \sum_{\alpha=0}^{a} \|\Lambda^\alpha \nabla \Delta_h u\|_{L^2} \|\nabla b\|_{L^2} \sum_{\beta=0}^{a} \|\Lambda^\beta \nabla b\|_{L^2} )
\]

That is,

\[
\sup_{t \in [0, T]} Y(t) + \int_0^T \|\Lambda^\alpha \nabla \Delta_h u\|_{L^2}^2 + \|\Lambda^\beta \nabla b\|_{L^2}^2 dt 
\leq c(T) + c \int_0^T Z(\|\nabla u\|_{L^2} \sum_{\alpha=0}^{a} \|\Lambda^\alpha \nabla \Delta_h u\|_{L^2} \|\nabla b\|_{L^2} \sum_{\beta=0}^{a} \|\Lambda^\beta \nabla b\|_{L^2} ) dt 
\]

3.1.2. Estimate of \( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \). Next, we take inner products of the first equation in (1) with \( -\Delta u \) and the second with \( -\Delta b \) to obtain

\[ \frac{1}{2} \partial_t X + \|\Lambda^\alpha \nabla u\|_{L^2}^2 + \|\Lambda^\beta \nabla b\|_{L^2}^2 = \int (u \cdot \nabla) u \cdot \Delta_h u + (u \cdot \nabla) u \cdot \partial_{33}^2 u - (b \cdot \nabla) b \cdot \Delta_h u - (b \cdot \nabla) b \cdot \partial_{33}^2 u \]
\[ + (u \cdot \nabla) b \cdot \Delta_h b + (u \cdot \nabla) b \cdot \partial_{33}^2 b - (b \cdot \nabla) u \cdot \Delta_h b - (b \cdot \nabla) u \cdot \partial_{33}^2 b \]

We see that

\[
\int (u \cdot \nabla) \partial_{33}^2 u = \int u \cdot \nabla_h u \cdot \partial_{33}^2 u - \frac{1}{2} (\partial_3 u_3)(\partial_3 u)^2 
\leq - \int \partial_3 u \cdot \nabla_h u \cdot \partial_3 u - \frac{1}{2} (\nabla_h \cdot u)(\partial_3 u)^2 - \frac{1}{2} (\partial_1 u_1 + \partial_2 u_2)(\partial_3 u)^2 
\leq c \int |\partial_3 u|^2 |\nabla_h u|
\]

Similarly
\[
\int (u \cdot \nabla) b \cdot \partial^2_{33} b = \int (u \cdot \nabla h) b \cdot \partial^2_{33} b - \frac{1}{2} \partial_3 u_3 (\partial_3 b)^2 \\
= - \int \partial_3 u \cdot \nabla h \cdot \partial_3 b - \frac{1}{2} (\nabla_h \cdot u)(\partial_3 b)^2 - \frac{1}{2} (\partial_1 u_1 + \partial_2 u_2)(\partial_3 b)^2 \\
\leq c \int |\partial_3 u| |\nabla h b| |\partial_3 b| + |\nabla h u||\partial_3 b|^2
\]

Next, we combine two other terms:

\[
\int (b \cdot \nabla) b \cdot \partial^2_{33} u = \int (b \cdot \nabla h) b \cdot \partial^2_{33} u + (b \cdot \nabla) b \cdot \partial^2_{33} b \\
= - \int (\partial_3 b \cdot \nabla h) b \cdot \partial_3 u + (\partial_3 b \cdot \nabla h) u \cdot \partial_3 b \\
- (\nabla_h \cdot b)(\partial_3 b \cdot \partial_3 u) - (\partial_1 b_1 + \partial_2 b_2)(\partial_3 b \cdot \partial_3 u) \\
\leq c \int |\partial_3 b||\nabla h b||\partial_3 u| + |\partial_3 b|^2 |\nabla h u|
\]

Thus, we have shown

\[
\int (u \cdot \nabla) u \cdot \partial^2_{33} u - (b \cdot \nabla) b \cdot \partial^2_{33} u + (u \cdot \nabla) b \cdot \partial^2_{33} b - (b \cdot \nabla) u \cdot \partial^2_{33} b \\
\leq c (\|\nabla h u\|_{L^2} \|\nabla u\|_{L^4}^2 + \|\nabla u\|_{L^4}^2 \|\nabla h b\|_{L^2} + \|\nabla h b\|_{L^2} \|\nabla b\|_{L^4}^2 + \|\nabla h u\|_{L^2} \|\nabla b\|_{L^4}^2) \\
\leq c (\|\nabla h u\|_{L^2} \|\Lambda^\alpha u\|_{L^2}^2 \|\Lambda^\beta b\|_{L^6}^4 + \|\Lambda^\alpha u\|_{L^2} \|\Lambda^\beta b\|_{L^6}^4 \|\Lambda^\alpha u\|_{L^2} \|\Lambda^\beta b\|_{L^6}^4) \\
\]

by Hölder’s inequalities, Gagliardo-Nirenberg inequalities and the Sobolev embeddings. Next, we use the well-known inequality of

\[
\|f\|_{L^6} \leq c \|\nabla h f\|_{L^2}^\frac{2}{3} \|\partial_3 f\|_{L^2}^\frac{1}{3}
\] (cf. [5]) to bound by a constant multiples of

\[
\|\nabla h u\|_{L^2} \|\Lambda^\alpha u\|_{L^2}^{\frac{4\alpha-3}{\alpha}} \|\Lambda^\alpha \nabla h u\|_{L^2}^{\frac{7-4\alpha}{\alpha}} \|\Lambda^\alpha \nabla u\|_{L^2}^{\frac{7-4\alpha}{\alpha}} \\
+ \|\nabla h b\|_{L^2} \|\Lambda^\beta b\|_{L^2}^{\frac{4\beta-3}{\beta}} \|\Lambda^\beta \nabla h b\|_{L^2}^{\frac{7-4\beta}{\beta}} \|\Lambda^\beta \nabla b\|_{L^2}^{\frac{7-4\beta}{\beta}} \\
+ \|\nabla h u\|_{L^2} \|\Lambda^\beta b\|_{L^2}^{\frac{4\beta-3}{\beta}} \|\Lambda^\beta \nabla h b\|_{L^2}^{\frac{7-4\beta}{\beta}} \|\Lambda^\beta \nabla b\|_{L^2}^{\frac{7-4\beta}{\beta}} \\
+ \|\nabla h u\|_{L^2} \|\Lambda^\beta b\|_{L^2}^{\frac{4\beta-3}{\beta}} \|\Lambda^\beta \nabla h b\|_{L^2}^{\frac{7-4\beta}{\beta}} \|\Lambda^\beta \nabla b\|_{L^2}^{\frac{7-4\beta}{\beta}}
\]

We combine our previous estimate and integrate in time \([0, T]\) to obtain
Now we use Hölder’s inequalities and (6) to bound by a constant multiples of

\[
\begin{align*}
X(T) - X(0) &+ \int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^2 + \|\Lambda^\beta \nabla b\|_{L^2}^2 \, dt \\
&\leq c \int_0^T Z(\|\nabla u\|_{L^2}^{2(\alpha-\alpha-1)}) \|\Lambda^\alpha \nabla u\|_{L^2}^{-\frac{\alpha(\alpha-1)}{\alpha-1}} + \|\nabla b\|_{L^2}^{2(\beta-\beta-1)} \|\Lambda^\beta \nabla b\|_{L^2}^{-\frac{\beta(\beta-1)}{\beta-1}} \, dt \\
&+ \int_0^T X \, dt \\
&+ c \int_0^T (\|\nabla_h u\|_{L^2} \|\Lambda^\alpha u\|_{L^2}^{4\alpha-3} + \|\Lambda^\alpha \nabla u\|_{L^2}^{2} + \|\Lambda^\beta \nabla b\|_{L^2}^{2}) \, dt \\
&+ \|\nabla_h b\|_{L^2} \|\Lambda^\alpha u\|_{L^2}^{4\alpha-3} + \|\Lambda^\alpha \nabla u\|_{L^2}^{2} + \|\Lambda^\beta \nabla b\|_{L^2}^{2} \\
&+ \|\nabla_h u\|_{L^2} \|\Lambda^\beta \nabla b\|_{L^2}^{4\beta-3} + \|\Lambda^\beta \nabla b\|_{L^2}^{2} \, dt \\
&+ \sup_{t \in [0,T]} \|\nabla_h u(t)\|_{L^2} \left( \int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^{2} \, dt \right)^{\frac{\alpha-1}{\alpha}} \\
&+ \sup_{t \in [0,T]} \|\nabla_h b(t)\|_{L^2} \left( \int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^{2} \, dt \right)^{\frac{\beta-1}{\beta}} \\
&+ \sup_{t \in [0,T]} \|\nabla_h b(t)\|_{L^2} \left( \int_0^T \|\Lambda^\beta \nabla b\|_{L^2}^{2} \, dt \right)^{\frac{\alpha-1}{\alpha}} \\
&+ \sup_{t \in [0,T]} \|\nabla_h u(t)\|_{L^2} \left( \int_0^T \|\Lambda^\beta \nabla b\|_{L^2}^{2} \, dt \right)^{\frac{\beta-1}{\beta}} \, dt
\end{align*}
\]
\[
\left( \int_0^T Z^{\frac{\alpha(\gamma-1)\mu}{\alpha(\gamma-1)-1}} \| \nabla u \|_{L^2}^2 dt \right)^{\frac{\alpha(\gamma-1)-1}{\alpha(\gamma-1)}} \left( \int_0^T \| \Lambda^\alpha \nabla u \|_{L^2}^2 dt \right)^{\frac{1}{\alpha(\gamma-1)}}
+ \left( \int_0^T Z^{\frac{\beta(\gamma-1)\mu}{\beta(\gamma-1)-1}} \| \nabla b \|_{L^2}^2 dt \right)^{\frac{\beta(\gamma-1)-1}{\beta(\gamma-1)}} \left( \int_0^T \| \Lambda^\beta \nabla b \|_{L^2}^2 dt \right)^{\frac{1}{\beta(\gamma-1)}} + \int_0^T X dt
+ \left( \int_0^T Z^{\frac{2(\gamma-1)\mu}{\gamma(\gamma-1)-1}} \| \Lambda^\alpha \nabla u \|_{L^2}^2 dt \right)^{\frac{2(\gamma-1)-1}{\gamma(\gamma-1)}} \left( \int_0^T \| \Lambda^\beta \nabla b \|_{L^2}^2 dt \right)^{\frac{2(\gamma-1)-1}{\gamma(\gamma-1)}}
\times \left( \int_0^T \| \Lambda^\beta \nabla b \|_{L^2}^2 dt \right)^{\frac{2(\gamma-1)-1}{\gamma(\gamma-1)}}
\]

On the first two terms, we use Young’s inequalities to bound by
\[
c \int_0^T Z^{\frac{\alpha(\gamma-1)\mu}{\alpha(\gamma-1)-1}} \| \nabla u \|_{L^2}^2 dt + c \int_0^T Z^{\frac{\beta(\gamma-1)\mu}{\beta(\gamma-1)-1}} \| \nabla b \|_{L^2}^2 dt
+ \frac{1}{2} \left( \int_0^T \| \Lambda^\alpha \nabla u \|_{L^2}^2 dt + \int_0^T \| \Lambda^\beta \nabla b \|_{L^2}^2 dt \right)
\]

On the last two terms, by Hölder’s inequalities we bound by
\[
c \left( \int_0^T Z^{\frac{\alpha(\gamma-1)\mu}{\alpha(\gamma-1)-1}} \| \nabla u \|_{L^2}^2 dt \right)^{\frac{\alpha(\gamma-1)-1}{\alpha(\gamma-1)}} \left( \int_0^T \| \Lambda^\alpha \nabla u \|_{L^2}^2 dt \right)^{\frac{1}{\alpha(\gamma-1)}}
+ \left( \int_0^T Z^{\frac{\beta(\gamma-1)\mu}{\beta(\gamma-1)-1}} \| \nabla b \|_{L^2}^2 dt \right)^{\frac{\beta(\gamma-1)-1}{\beta(\gamma-1)}} \left( \int_0^T \| \Lambda^\beta \nabla b \|_{L^2}^2 dt \right)^{\frac{1}{\beta(\gamma-1)}}
\times \left( \int_0^T \| \Lambda^\beta \nabla b \|_{L^2}^2 dt \right)^{\frac{2(\gamma-1)-1}{\gamma(\gamma-1)}}
+ c \left( \int_0^T Z^{\frac{\alpha(\gamma-1)\mu}{\alpha(\gamma-1)-1}} \| \nabla u \|_{L^2}^2 dt \right)^{\frac{\alpha(\gamma-1)-1}{\alpha(\gamma-1)}} \left( \int_0^T \| \Lambda^\alpha \nabla u \|_{L^2}^2 dt \right)^{\frac{1}{\alpha(\gamma-1)}}
+ \left( \int_0^T Z^{\frac{\beta(\gamma-1)\mu}{\beta(\gamma-1)-1}} \| \nabla b \|_{L^2}^2 dt \right)^{\frac{\beta(\gamma-1)-1}{\beta(\gamma-1)}} \left( \int_0^T \| \Lambda^\beta \nabla b \|_{L^2}^2 dt \right)^{\frac{1}{\beta(\gamma-1)}}
\times \left( \int_0^T \| \Lambda^\beta \nabla b \|_{L^2}^2 dt \right)^{\frac{2(\gamma-1)-1}{\gamma(\gamma-1)}}
\]

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Next, by Lemma 2.3 we bound by a constant multiples of

\[
\left(\int_0^T Z^{\alpha/(\gamma-1)-T} \left\| \nabla u \right\|_{L^2}^2 dt \right)^{\frac{\alpha(\gamma-1)-1}{\alpha(\gamma-1)-5-2\alpha}} \left(\int_0^T \left\| \Lambda^\alpha \nabla u \right\|_{L^2}^2 dt \right)^{\frac{5-2\alpha}{5-2\alpha}} \left(\int_0^T \left\| \Lambda^\beta \nabla \beta \right\|_{L^2}^2 dt \right)^{\frac{7-4\alpha}{7-4\alpha}} + \int_0^T \left\| \Lambda^\alpha \nabla u \right\|_{L^2}^2 dt + \left\| \Lambda^\beta \nabla \beta \right\|_{L^2}^2 dt
\]

Now we use Young’s inequalities to bound by a constant multiples of

\[
\left(\int_0^T Z^{\alpha/(\gamma-1)-T} \left\| \nabla u \right\|_{L^2}^2 dt \right)^{\frac{\alpha(\gamma-1)-1}{\alpha(\gamma-1)-5-2\alpha}} \left(\int_0^T \left\| \Lambda^\alpha \nabla u \right\|_{L^2}^2 dt \right)^{\frac{3\alpha(\gamma-1)}{3\alpha(\gamma-1)-5+2\alpha}} + \left(\int_0^T Z^{\beta/(\gamma-1)-T} \left\| \nabla \beta \right\|_{L^2}^2 dt \right)^{\frac{\beta(\gamma-1)-1}{\beta(\gamma-1)-5-2\beta}} \left(\int_0^T \left\| \Lambda^\beta \nabla \beta \right\|_{L^2}^2 dt \right)^{\frac{7-4\beta}{7-4\beta}}
\]

Next we will use the following Young’s inequalities with

\[
\begin{align*}
\left(\frac{7-4\alpha}{4}\right) \frac{\alpha(\gamma-1)}{3\alpha(\gamma-1)-5+2\alpha} + \frac{5\alpha(\gamma-1)+4\alpha^2(\gamma-1)-20+8\alpha}{4[3\alpha(\gamma-1)-5+2\alpha]} & = 1 \\
\left(\frac{7-4\beta}{4}\right) \frac{\alpha(\gamma-1)}{3\alpha(\gamma-1)-5+2\alpha} + \frac{5\beta(\gamma-1)+4\beta^2(\gamma-1)-20+8\beta}{4[3\beta(\gamma-1)-5+2\alpha]} & = 1 \\
\left(\frac{7-4\beta}{4}\right) \frac{\alpha(\gamma-1)}{3\alpha(\gamma-1)-5+2\alpha} + \frac{5\beta(\gamma-1)+4\beta^2(\gamma-1)-20+8\beta}{4[3\beta(\gamma-1)-5+2\alpha]} & = 1
\end{align*}
\]

Justification of these Young’s inequalities require in addition to the hypothesis of Lemma 2.2,
\[ \gamma > \max\{20 - 3\alpha + 4\alpha^2, 20 + 5\beta - 8\alpha + 4\alpha\beta, 20 + 5\alpha - 8\beta + 4\alpha\beta, 20 - 3\beta + 4\beta^2\} \]

Thus, letting

\[
\begin{align*}
\frac{1}{p_1} &= \frac{4(\alpha(\gamma-1)-1)[5-2\alpha]}{5\alpha\gamma-1+4\alpha\beta(\gamma-1)-20+8\alpha} \\
\frac{1}{p_2} &= \frac{5\beta(\gamma-1)+4\alpha\beta(\gamma-1)-20+8\alpha}{4[\alpha(\gamma-1)-1][5-2\beta]} \\
\frac{1}{p_3} &= \frac{5\beta(\gamma-1)+4\alpha\beta(\gamma-1)-20+8\beta}{4[\beta(\gamma-1)-1][5-2\beta]} \\
\frac{1}{p_4} &= \frac{5\beta(\gamma-1)+4\alpha\beta(\gamma-1)-20+8\beta}{4[\beta(\gamma-1)-1][5-2\beta]}
\end{align*}
\]

we have the bound by a constant multiples of

\[
\left( \int_0^T Z^{\alpha(\gamma-1)-1} \|\nabla u\|_{L^2}^{2(p_1)} \|\nabla u\|_{L^2}^{2(1-p_1)} dt \right)^{\frac{1}{p_1}} + \left( \int_0^T Z^{\beta(\gamma-1)-1} \|\nabla b\|_{L^2}^{2(p_2)} \|\nabla b\|_{L^2}^{2(1-p_2)} dt \right)^{\frac{1}{p_2}} + \left( \int_0^T Z^{\alpha(\gamma-1)-1} \|\nabla u\|_{L^2}^{2(p_3)} \|\nabla u\|_{L^2}^{2(1-p_3)} dt \right)^{\frac{1}{p_3}} + \left( \int_0^T Z^{\beta(\gamma-1)-1} \|\nabla b\|_{L^2}^{2(p_4)} \|\nabla b\|_{L^2}^{2(1-p_4)} dt \right)^{\frac{1}{p_4}}
\]

By Hölder’s inequalities we further bound by a constant multiples of

\[
\left( \int_0^T Z^{\alpha(\gamma-1)-1} \|\nabla u\|_{L^2}^{2} dt \right)^{\frac{1}{p_1}} + \left( \int_0^T Z^{\beta(\gamma-1)-1} \|\nabla b\|_{L^2}^{2} dt \right)^{\frac{1}{p_2}} + \left( \int_0^T Z^{\alpha(\gamma-1)-1} \|\nabla u\|_{L^2}^{2} dt \right)^{\frac{1}{p_3}} + \left( \int_0^T Z^{\beta(\gamma-1)-1} \|\nabla b\|_{L^2}^{2} dt \right)^{\frac{1}{p_4}}
\]

By Gagliardo-Nirenberg inequalities and (6), we have obtained
range of
take an inner product of the first equation in (1) with

$\sum_{i \in \mathbb{I}} J_i$

\[ X(T) + \int_0^T \| A^\alpha \nabla u \|_{L^2}^2 + \| A^\beta \nabla b \|_{L^2}^2 dt \]

\[ \leq X(0) + c \int_0^T Z^{\alpha(\gamma-1)-1} \| \nabla u \|_{L^2}^2 + Z^{\beta(\gamma-1)-1} \| \nabla b \|_{L^2}^2 + X \]

\[ + Z^{\alpha(\gamma-1)} \| \nabla u \|_{L^2}^2 + Z^{\beta(\gamma-1)} \| \nabla b \|_{L^2}^2 \]

By Lemma 2.3

\[ Z^{\alpha(\gamma-1)-1} \| \nabla u \|_{L^2}^2 + Z^{\beta(\gamma-1)-1} \| \nabla b \|_{L^2}^2 \]

\[ \leq c(\partial_2 u_2)_{L^2}^{\alpha(\gamma-1)+4\alpha^2(\gamma-1)-20+8\alpha} + \| \partial_3 u_3 \|_{L^2}^{\alpha(\gamma-1)+4\alpha^2(\gamma-1)-20+8\alpha} \]

\[ + \| \partial_2 u_2 \|_{L^2}^{\alpha(\gamma-1)+4\beta^{(\gamma-1)}-20+8\beta} + \| \partial_3 u_3 \|_{L^2}^{\alpha(\gamma-1)+4\beta^{(\gamma-1)}-20+8\beta} \]

\[ + \| \partial_2 u_2 \|_{L^2}^{\alpha(\gamma-1)+4\beta^{(\gamma-1)}-20+8\beta} + \| \partial_3 u_3 \|_{L^2}^{\alpha(\gamma-1)+4\beta^{(\gamma-1)}-20+8\beta} \]

Gronwall’s inequality and (3) imply the desired result. Lastly, considering the range of $\alpha, \beta$, all the conditions of

\[ 2 \leq 2 - 3\beta + 4\beta^2 \]

may be simplified to say that $\gamma > \frac{7}{3}$ suffices.

3.2. Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1 and hence we only sketch it. We start by taking an inner product of the first equation in (1) with $u$ and the second with $b$ and integrating in time to obtain

\[ \sup_{t \in [0,T]} \| u(t) \|_{L^2}^2 + \| b(t) \|_{L^2}^2 + \int_0^T X dt \leq c(u_0, b_0) \]  

(8)

3.2.1. Estimate of $\| \nabla h u \|_{L^2}^2 + \| \nabla h b \|_{L^2}^2$. Local well-posedness is shown in [17]. We take an inner product of the first equation in (1) with $-\Delta h u$ and the second with $-\Delta h b$ to obtain

\[ \frac{1}{2} \partial_t Y + \| \nabla h \nabla u \|_{L^2}^2 + \| \nabla h \nabla b \|_{L^2}^2 \]

\[ = \int (u \cdot \nabla) u \cdot \Delta h u - (b \cdot \nabla) b \cdot \Delta h u + (u \cdot \nabla) b \cdot \Delta h b - (b \cdot \nabla) u \cdot \Delta h b = \sum_{i=1}^4 J_i \]

Similarly as before, Lemma 2.2, (8) and Young’s inequality give

\[ J_1 \leq c \| \nabla h u \|_{L^2}^2 + c \| \partial_2 u \|_{L^2}^{\frac{5}{4}} \| \nabla u \|_{L^2}^2 \]
and

\[ J_2 + J_3 + J_4 \leq \epsilon \| \nabla \nabla_h b \|_{L^2}^2 + c Z \| \nabla b \|_{L^2}^{2(\frac{2}{3} - \frac{1}{2})} \| \Delta b \|_{L^2}^{\frac{2}{3}} \]

Thus, for \( \epsilon > 0 \) sufficiently small, due to the divergence-free property of \( u \),

\[
\sup_{t \in [0,T]} Y(t) + \int_0^T \| \nabla \nabla_h u \|_{L^2}^2 + \| \nabla \nabla_h b \|_{L^2}^2 dt \\
\leq Y(0) + c \int_0^T \| \partial_3 u_3 \|_{L^2}^{\frac{2}{3}} X + c Z \| \nabla b \|_{L^2}^{2(\frac{2}{3} - \frac{1}{2})} \| \Delta b \|_{L^2}^{\frac{2}{3}} dt
\]

3.2.2. Estimate of \( \| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 \). Next, we take inner products of the first equation in (1) with \( -\Delta u \) and the second with \( -\Delta b \) to obtain

\[
\frac{1}{2} \partial_t X + \| \Delta u \|_{L^2}^2 + \| \Delta b \|_{L^2}^2 \\
= \int (u \cdot \nabla) u \cdot \Delta u + (u \cdot \nabla) u \cdot \partial_{33}^2 u - (b \cdot \nabla) b \cdot \Delta u - (b \cdot \nabla) b \cdot \partial_{33}^2 u \\
+ (u \cdot \nabla) b \cdot \Delta b + (u \cdot \nabla) b \cdot \partial_{33}^2 b - (b \cdot \nabla) u \cdot \Delta b - (b \cdot \nabla) u \cdot \partial_{33}^2 b
\]

Similarly as before, we can obtain

\[
\int (u \cdot \nabla) u \cdot \partial_{33}^2 u - (b \cdot \nabla) b \cdot \partial_{33}^2 u + (u \cdot \nabla) b \cdot \partial_{33}^2 b - (b \cdot \nabla) u \cdot \partial_{33}^2 b \\
\leq c \int |\partial_3 u|^2 |\nabla_h u| + |\partial_3 u||\nabla_h b||\partial_3 b| + |\nabla_h u||\partial_3 b|^2
\]

Then, Hölder’s inequality, (7) and the previous estimate give after integrating in time

\[
X(T) + \int_0^T \| \Delta u \|_{L^2}^2 + \| \Delta b \|_{L^2}^2 dt \\
\leq X(0) + c \int_0^T \| \partial_3 u_3 \|_{L^2}^{\frac{2}{3}} X dt + c \int_0^T Z \| \nabla b \|_{L^2}^{2(\frac{2}{3} - \frac{1}{2})} \| \Delta b \|_{L^2}^{\frac{2}{3}} dt \\
+ c \int_0^T \| \nabla_h u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla \nabla_h u \|_{L^2} \| \Delta u \|_{L^2}^{\frac{1}{2}} + \| \nabla_h b \|_{L^2} \| \nabla u \|_{L^2} \| \nabla \nabla_h u \|_{L^2} \| \Delta u \|_{L^2}^{\frac{1}{2}} \\
+ \| \nabla_h b \|_{L^2} \| \nabla b \|_{L^2} \| \nabla \nabla_h b \|_{L^2} \| \Delta b \|_{L^2}^{\frac{1}{2}} + \| \nabla_h u \|_{L^2} \| \nabla b \|_{L^2} \| \nabla \nabla_h b \|_{L^2} \| \Delta b \|_{L^2}^{\frac{1}{2}} dt
\]

Another Hölder’s inequalities imply
Next, (8), Young’s inequality and previous estimates imply

\[ X(T) + \int_0^T \| \Delta u \|_{L^2}^2 + \| \Delta b \|_{L^2}^2 dt \]

\[ \leq X(0) + c \int_0^T \| \partial_3 u_3 \|_{L^2}^{2\gamma} X dt + c \int_0^T Z^{\frac{2\gamma}{2\gamma+1}} \| \nabla b \|_{L^2}^2 dt + \epsilon \int_0^T \| \Delta b \|_{L^2}^2 dt \]

\[ + c \left[ \sup_t \| \nabla h u(t) \|_{L^2} \left( \int_0^T \| \nabla u \|_{L^2}^2 dt \right) \right]^{\frac{3}{4}} \left( \int_0^T \| \nabla \nabla_h u \|_{L^2}^2 dt \right)^{\frac{1}{4}} \left( \int_0^T \| \Delta u \|_{L^2}^2 dt \right)^{\frac{1}{4}} \]

\[ + \sup_t \| \nabla h b(t) \|_{L^2} \left( \int_0^T \| \nabla u \|_{L^2}^2 dt \right) \left( \int_0^T \| \nabla \nabla_h u \|_{L^2}^2 dt \right) \left( \int_0^T \| \Delta u \|_{L^2}^2 dt \right) \]

\[ + \sup_t \| \nabla h b(t) \|_{L^2} \left( \int_0^T \| \nabla b \|_{L^2}^2 dt \right) \left( \int_0^T \| \nabla \nabla_h b \|_{L^2}^2 dt \right) \left( \int_0^T \| \Delta b \|_{L^2}^2 dt \right) \]

Hölder’s inequality after absorbing the dissipative term gives us the bound by
\[ X(0) + c \int_0^T \| \partial_3 u_3 \|_{L^2}^{\frac{2}{\gamma-2}} X dt + c \int_0^T Z^{\frac{2}{\gamma-2}} \| \nabla b \|_{L^2}^2 dt \]

\[ + c \int_0^T \| \partial_3 u_3 \|_{L^2}^{\frac{2}{\gamma-2}} X dt \left( \left( \int_0^T \| \Delta u \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} + \left( \int_0^T \| \Delta b \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} \right) \]

\[ + c \left( \int_0^T Z^{\frac{2}{\gamma-2}} X dt \right)^{\frac{2}{\gamma-2}} \left( \int_0^T \| \Delta b \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} \left( \int_0^T \| \Delta u \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} + \left( \int_0^T \| \Delta b \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} \]

We expand and bound the last term using Young's inequalities by

\[ c \left( \int_0^T Z^{\frac{2}{\gamma-2}} X dt \right)^{\frac{2}{\gamma-2}} \left( \int_0^T \| \Delta b \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} \left( \int_0^T \| \Delta u \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} \]

\[ + c \left( \int_0^T Z^{\frac{2}{\gamma-2}} X dt \right)^{\frac{2}{\gamma-2}} \left( \int_0^T \| \Delta b \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} \left( \int_0^T \| \Delta u \|_{L^2}^2 dt \right)^{\frac{1}{\gamma}} \]

\[ \leq c \left( \int_0^T Z^{\frac{2}{\gamma-2}} X dt \right)^{\frac{4(\gamma-2)}{3\gamma-4}} + \epsilon \int_0^T (\| \Delta u \|_{L^2}^2 + \| \Delta b \|_{L^2}^2) dt \]

Thus, after absorbing the dissipative term and applying Young's inequality on the middle term we obtain

\[ X(T) + \int_0^T \| \Delta u \|_{L^2}^2 + \| \Delta b \|_{L^2}^2 dt \]

\[ \leq X(0) + c \int_0^T \| \partial_3 u_3 \|_{L^2}^{\frac{2}{\gamma-2}} X dt + c \int_0^T Z^{\frac{2}{\gamma-2}} \| \nabla b \|_{L^2}^2 dt \]

\[ + c \left( \int_0^T \| \partial_3 u_3 \|_{L^2}^{\frac{2}{\gamma-2}} X dt \right)^{\frac{4}{\gamma}} + c \int_0^T \| \Delta u \|_{L^2}^2 + \| \Delta b \|_{L^2}^2 dt + c \left( \int_0^T Z^{\frac{2}{\gamma-2}} X dt \right)^{\frac{4(\gamma-2)}{3\gamma-4}} \]

On the fourth and sixth terms, we use Hölder's inequalities and (8) to finally obtain

\[ X(T) + \int_0^T \| \Delta u \|_{L^2}^2 + \| \Delta b \|_{L^2}^2 dt \leq X(0) + c \int_0^T \| \partial_3 u_3 \|_{L^2}^{\frac{2}{\gamma-2}} X dt + c \int_0^T Z^{\frac{2}{\gamma-2}} \| \nabla b \|_{L^2}^2 dt \]

\[ + c \int_0^T \| \partial_3 u_3 \|_{L^2}^{\frac{2}{\gamma-2}} X dt + c \int_0^T Z^{\frac{2}{\gamma-2}} X dt \]

Lemma 2.3 implies that if we take \( \gamma > 3 \),

\[ \sum_{i=2}^3 \int_0^T \| \partial_i u_i \|_{L^3}^8 dt < \infty \]

suffices to complete the proof with Gronwall's inequality.
3.3. **Proof of Corollary 1.3.** Corollary 1.3 is immediate from the following special case of the lemma due to [16]:

**Lemma 3.1.** Let \( u \) be a divergence-free sufficiently smooth vector field in \( \mathbb{R}^3 \). Then there exists a constant \( C = C(q) \) such that

\[
\|\partial_i u_j\|_{L^q} \leq C(\|w_3\|_{L^q} + \|\partial_i u_3\|_{L^q})
\]

for \( 1 < q < \infty, i,j = 1,2 \).

Thus,

\[
\|\partial_2 u_2\|_{L^p} \leq c(\|w_3\|_{L^p} + \|\partial_3 u_3\|_{L^p})^T \leq c(\|w_3\|_{L^p} + \|\partial_3 u_3\|_{L^p})
\]

by Lemma 2.3 as \( r \geq 0 \).

3.4. **Proof of Theorem 1.4.** Taking an \( L^2 \)-inner product with \( u \) on (2), integrating in time we see that

\[
\frac{1}{2} \partial_t \|\nabla h u\|_{L^2}^2 + c \|\Lambda^\alpha \nabla u\|_{L^2}^2 \\
\leq c \|\partial_3 u_3\|_{L^2} \|\nabla u\|_{L^2} \|\nabla h u\|_{L^2} \|\Lambda^\alpha \nabla h u\|_{L^2}
\]

due to Gagliardo-Nirenberg inequality. Young’s and Gronwall’s inequalities give us similarly as before

\[
\sup_{t \in [0,T]} \|\nabla h u(t)\|_{L^2}^2 + c \int_0^T \|\Lambda^\alpha \nabla h u\|_{L^2}^2 dt \leq c(T) + c(T) \int_0^T \|\partial_3 u_3\|_{L^2} \|\nabla u\|_{L^2}^2 dt
\]

3.4.1. **Estimate of \( \|\nabla_h u\|_{L^2}^2 \).** Taking an inner product with \( -\Delta h u \) on (2), applying Lemma 2.1 and integration by parts just like \( J_1 \) estimate in the previous proofs, Lemma 2.2 give us

\[
\frac{1}{2} \partial_t \|\nabla u\|_{L^2}^2 + c \|\Lambda^\alpha \nabla u\|_{L^2}^2 \\
\leq c \|\partial_3 u_3\|_{L^2} \|\nabla u\|_{L^2} \|\nabla h u\|_{L^2} \|\Lambda^\alpha \nabla h u\|_{L^2}
\]

where

\[
J_2 = c \int |\nabla h u| |\partial_3 u_3|^2 \leq c \|\nabla h u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla h u\|_{L^4} \|\nabla u\|_{L^2} \|\Lambda^\alpha u\|_{L^6} \|\nabla^\alpha u\|_{L^6}^{\frac{7-4\alpha}{5-2\alpha}}
\]

by H"{o}lder’s and Gagliardo-Nirenberg inequality. Now Sobolev embedding and (7) combined with previous estimate give

\[
\frac{1}{2} \partial_t \|\nabla u\|_{L^2}^2 + \|\Lambda^\alpha \nabla u\|_{L^2}^2 \\
\leq c \|\partial_3 u_3\|_{L^2} \|\nabla u\|_{L^2}^2 + \|\nabla h u\|_{L^2}^2 \\
+ c \|\nabla h u\|_{L^2} \|\Lambda^\alpha u\|_{L^2} \|\nabla^\alpha u\|_{L^2} \|\nabla^\alpha u\|_{L^2}^{\frac{7-4\alpha}{5-2\alpha}}
\]

We integrate in time to obtain
\[ \| \nabla u(T) \|_{L^2}^2 + \int_0^T \| \Lambda^\alpha \nabla u \|_{L^2}^2 dt \]
\[ \leq c \int_0^T \| \partial_3 u_3 \|_{L^2} \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 dt \]
\[ + c \sup_{t \in [0,T]} \| \nabla_h u \|_{L^2} \left( \int_0^T \| \Lambda^\alpha u \|_{L^2}^{(4\alpha/3)} 2^{(4\alpha/3)} dt \right)^{4\alpha/3} \]
\[ \times \left( \int_0^T \| \nabla_h \Lambda^\alpha u \|_{L^2}^{4(5-4\alpha)/3} \left( \frac{2}{5-4\alpha} \right) dt \right)^{4(5-4\alpha)/3} \left( \int_0^T \| \nabla \Lambda^\alpha u \|_{L^2}^{2(7-4\alpha)/12} \left( \frac{4}{7-4\alpha} \right) dt \right)^{7-4\alpha/12} \]

by Hölder’s inequality. By the previous estimate and (9) we have the second term bounded by

\[ c \left( \int_0^T \| \partial_3 u_3 \|_{L^2} \| \nabla u \|_{L^2} dt \right)^{5-2\alpha} \left( \int_0^T \| \nabla \Lambda^\alpha u \|_{L^2}^{2\alpha} dt \right)^{7-4\alpha} \]
\[ \leq c \left( \int_0^T \| \partial_3 u_3 \|_{L^2} \| \nabla u \|_{L^2} dt \right)^{4(5-2\alpha)/3} + \left( \int_0^T \| \nabla \Lambda^\alpha u \|_{L^2}^{2\alpha} dt \right)^{7-4\alpha/12} \]

by Young’s inequality. Absorbing the last term, we have the bound of

\[ c \int_0^T \| \partial_3 u_3 \|_{L^2} \| \nabla u \|_{L^2} + \| \nabla u \|_{L^2}^{15-12\alpha} \]
\[ + c \left( \int_0^T \| \partial_3 u_3 \|_{L^2} \| \nabla u \|_{L^2} dt \right)^{15-12\alpha} \left( \int_0^T \| \nabla u \|_{L^2}^{2\alpha} dt \right)^{7-4\alpha/12} \]

by Hölder’s inequality. Therefore,

\[ \| \nabla u(T) \|_{L^2}^2 + \int_0^T \| \Lambda^\alpha \nabla u \|_{L^2}^2 dt \]
\[ \leq c \int_0^T \| \partial_3 u_3 \|_{L^2} \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 dt + c \left( \int_0^T \| \partial_3 u_3 \|_{L^2} \| \nabla u \|_{L^2}^{2\alpha} \left( \frac{5-2\alpha}{5-4\alpha} \right) \| \nabla u \|_{L^2}^{2\alpha} dt \right) \]

by the Gagliardo-Nirenberg inequality and Hölder’s inequality and (9). Gronwall’s inequality and (5) complete the proof of Theorem 1.4.

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