ON SUBPROJECTIVITY OF $C(K, X)$

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Abstract. We show that the Banach space $C(K, X)$ is subprojective if $K$ is scattered and $X$ is subprojective.

A Banach space $X$ is called subprojective if every closed infinite-dimensional subspace of $X$ contains an infinite-dimensional subspace that is complemented in $X$. Subprojective spaces were introduced by Whitley [14] to give conditions for the conjugate of an operator to be strictly singular. They are relevant in the study of the perturbation classes problem for semi-Fredholm operators, which has a negative answer in general [5] but a positive answer when one of the spaces is subprojective [6]. Recently Oikhberg and Spinu made a systematic study of subprojective spaces [10], widely increasing the family of known examples of spaces in this class.

Here we prove that $C(K, X)$ is subprojective whenever $K$ is a scattered compact and $X$ is a subprojective Banach space; a compact space $K$ is said to be scattered or dispersed if every non-empty closed subset of $K$ contains an isolated point. In the case where $K = [0, \lambda]$ for some ordinal $\lambda$, this result was previously obtained for countable $\lambda$ [10, Theorem 4.1] and later for arbitrary $\lambda$ [3, Theorem 2.9].

This generalises the scalar case, where it was already known that $C(K)$ is subprojective if and only if $K$ is scattered [9, Theorem 11] [11, Main Theorem], and fully characterises the subprojectivity of $C(K, X)$, as the subprojectivity of $C(K, X)$ implies that of $C(K)$ and $X$ [10, Proposition 2.1].

We will use standard notation. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in $X$, then $[x_n : n \in \mathbb{N}]$ will denote the closed linear span of $(x_n)_{n \in \mathbb{N}}$ in $X$. Given a (bounded, linear) operator $T : X \to Y$, $N(T)$ and $R(T)$ denote the kernel and the range of $T$, respectively. An operator $T : X \to Y$ is strictly singular if $T|_M$ is an isomorphism only if $M$ is finite-dimensional.

The injective tensor product of $X$ and $Y$ is denoted by $X \hat{\otimes}_\varepsilon Y$. Note that $C(K, X)$ can be identified with $C(K) \hat{\otimes}_\varepsilon X$ [12, Section 3.2], and they will be used interchangeably in the sequel. For countable ordinals $\lambda$, $\mu$ it was proved in [4] that the projective tensor product $C([0, \lambda]) \hat{\otimes}_\pi C([0, \mu])$ is subprojective.

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Let $K$, $L$ be compact spaces and let $\varphi: K \rightarrow L$ be a continuous function. It is well known that $\varphi$ induces an operator $C(L) \rightarrow C(K)$ that maps each $f \in C(L)$ to $f \circ \varphi \in C(K)$; we will denote this operator by $\hat{\varphi}: C(L) \rightarrow C(K)$.

**Definition.** Let $X$ be a Banach space and let $Z$ be a subspace of $X$. We will say that $Z$ is subprojective with respect to $X$ if every closed infinite-dimensional subspace of $Z$ contains an infinite-dimensional subspace complemented in $X$.

Note that this is a stronger notion for $Z$ than merely being subprojective, as it requires the subspace to be complemented in $X$ and not just in $Z$. Also, a space $X$ is subprojective if and only if each of its subspaces is subprojective with respect to $X$.

**Proposition 1.** Let $X$ be a Banach space, let $P: X \rightarrow X$ be a projection such that $R(P)$ is subprojective and let $Z$ be a closed subspace of $X$ such that $P(Z) \subseteq Z$ and $Z \cap N(P)$ is subprojective with respect to $X$. Then $Z$ is subprojective with respect to $X$.

**Proof.** Let $M$ be a closed infinite-dimensional subspace of $Z$. If $M \cap N(P)$ is infinite-dimensional, then it contains another infinite-dimensional subspace complemented in $X$ by hypothesis.

Otherwise, if $M \cap N(P)$ is finite-dimensional, we can assume that $M \cap N(P) = 0$ by passing to a further subspace of $M$ if necessary. If $P(M)$ is closed, then $P|_M$ is an isomorphism and, if $N$ is an infinite-dimensional subspace of $M$ such that $R(P) = P(N) \oplus H$ for some closed subspace $H$, then $X = N \oplus P^{-1}(H)$.

We are left with the case where $M \cap N(P) = 0$ and $P(M)$ is not closed. Take a normalised sequence $(x_n)_{n \in \mathbb{N}}$ in $M$ such that $\|P(x_n)\| < 2^{-n}$ for every $n \in \mathbb{N}$. Since any weak cluster point of $(x_n)_{n \in \mathbb{N}}$ must be in $M \cap N(P) = 0$, by passing to a subsequence [1 Theorem 1.5.6] we can assume that $(x_n)_{n \in \mathbb{N}}$ is a basic sequence and that there exists a bounded sequence $(x^*_n)_{n \in \mathbb{N}}$ in $X^*$ such that $x^*_i(x_j) = \delta_{ij}$ for every $i, j \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \|x^*_n\| \|P(x_n)\| < 1$. Then $K(x) = \sum_{n=1}^{\infty} x^*_n(x) P(x_n)$ defines an operator $K: X \rightarrow X$ with $\|K\| < 1$ that maps $K(x_n) = P(x_n)$ for every $n \in \mathbb{N}$, and then $I - K$ is an automorphism on $X$ such that $[(I-K)(x_n) : n \in \mathbb{N}] = [(I-P)(x_n) : n \in \mathbb{N}] \subseteq Z \cap N(P)$ and, if $N$ is an infinite-dimensional subspace of $[(I-K)(x_n) : n \in \mathbb{N}]$ complemented in $X$, then $(I-K)^{-1}(N) \subseteq M$ and is still complemented in $X$. \qed

**Proposition 2.** Let $X$ be a subprojective Banach space, let $\lambda$ be an ordinal, let $K$ be a sequentially compact space and let $\varphi: K \rightarrow [0, \lambda]$ be a continuous surjection. Then $\hat{\varphi}(C([0, \lambda])) \hat{\otimes}_\varepsilon X$ is subprojective with respect to $C(K, X)$.

Here we are using the identification $C(K, X) \equiv C(K) \hat{\otimes}_\varepsilon X$ and the fact that, if $X$ and $Y$ are Banach spaces and $M$ is a closed subspace
of $X$ and $N$ is a closed subspace of $Y$, then $M \hat{\otimes}_\varepsilon N$ can be seen as a subspace of $X \hat{\otimes}_\varepsilon Y$ [12]. Comments after Proposition 3.2).

Proof. We will proceed by induction in $\lambda$. The result is trivial for $\lambda = 0$, as then $\hat{\varphi}(C([0, \lambda])) \hat{\otimes}_\varepsilon X$ is the set of constant functions, which is complemented in $C(K, X)$ and isomorphic to $X$, which is subprojective.

Let us then assume that the result is true for every continuous surjection $K \rightarrow [0, \mu]$ with $\mu < \lambda$. Consider, for each ordinal $\mu < \lambda$, the set $K_\mu = \varphi^{-1}([0, \mu]) \subseteq K$, which is both open and closed, and the operator $P_\mu : C(K, X) \rightarrow C(K, X)$ given by $P_\mu(f) = f \chi_{K_\mu}$, which is a projection with range isometric to $C(K_\mu, X)$; also note that $P_\mu(f) \rightarrow f$ for every $f \in \hat{\varphi}(C_0([0, \lambda])) \hat{\otimes}_\varepsilon X$.

We will first prove that $\hat{\varphi}(C_0([0, \lambda])) \hat{\otimes}_\varepsilon X$ is subprojective with respect to $C(K, X)$. Let $M$ be a closed infinite-dimensional subspace of $\hat{\varphi}(C_0([0, \lambda])) \hat{\otimes}_\varepsilon X$. If there exists some $\mu < \lambda$ for which $P_M|_M$ is not strictly singular, we can assume that $P_M|_M$ is an isomorphism by passing to a further subspace of $M$ if necessary and then $P_\mu(M)$, seen as a subspace of $C(K_\mu, X)$, contains an infinite-dimensional subspace complemented in $C(K_\mu, X)$ by the induction hypothesis (with $\varphi|_{K_\mu} : K_\mu \rightarrow [0, \mu]$) so $M$ contains an infinite-dimensional subspace complemented in $C(K, X)$ [10] Proposition 2.3]. This must necessarily be the case if $\lambda$ is not a limit ordinal, as then there exists some ordinal $\mu$ such that $\lambda = \mu + 1$, for which $P_\mu$ is the identity on $M$ because functions in $M$ vanish at $\varphi^{-1}(\lambda)$.

Otherwise, if $M \subseteq \hat{\varphi}(C_0([0, \lambda])) \hat{\otimes}_\varepsilon X$ but $P_\mu$ is strictly singular for every $\mu < \lambda$, then $\lambda$ must be a limit ordinal by the previous sentence. Also, for every $\mu < \lambda$ and $\varepsilon > 0$, there exists $f \in M$ such that $\|f\| = 1$ and $\|P_\mu(f)\| < \varepsilon$, and then there is $\nu > \mu$ such that $\|P_\nu(f) - f\| < \varepsilon$. By induction, starting with an arbitrary $\mu_1 < \lambda$, there exists a strictly increasing sequence of ordinals $\mu_1 < \mu_2 < \cdots < \lambda$ and a sequence $(f_n)_{n \in \mathbb{N}}$ of normalised functions in $M$ such that $\|P_{\mu_n}(f_n)\| < 2^{-n}/32$ and $\|P_{\mu_{n+1}}(f_n) - f_n\| < 2^{-n}/32$ for every $n \in \mathbb{N}$.

Now, for each $n \in \mathbb{N}$, there exist $t_n \in K$ such that $\|f_n(t_n)\| = 1$ and then a normalised $x_n^* \in X^*$ such that $x_n^*(f_n(t_n)) = 1$; note that $t_n \in K_{\mu_{n+1}} \setminus K_{\mu_n}$ because $f_n$ cannot attain its norm outside of $K_{\mu_{n+1}} \setminus K_{\mu_n}$. As $K$ is sequentially compact, by passing to a subsequence, we may assume that $(t_n)_{n \in \mathbb{N}}$ converges to some $t_\infty \in K$. Let $Q : C(K, X) \rightarrow e_0$ and $J : e_0 \rightarrow C(K, X)$ be the operators defined as

$$Q(f) = (x_n^*(f(t_n)) - f(t_\infty))_{n \in \mathbb{N}},$$

$$J((\alpha_n)_{n \in \mathbb{N}}) = \sum_{n=1}^\infty \alpha_n(P_{\mu_{n+1}} - P_{\mu_n})(f_n) = \sum_{n=1}^\infty \alpha_n f_n \chi_{K_{\mu_{n+1}} \setminus K_{\mu_n}};$$

then $Q$ and $J$ are well defined, $\|Q\| = 2$ and $J$ is an isometry into $C(K, X)$ (into $\hat{\varphi}(C_0([0, \lambda])) \hat{\otimes}_\varepsilon X$, actually), and $QJ$ is the identity.
on $c_0$, so $JQ$ is a projection in $C(K, X)$ with range isometric to $c_0$. And, since
\[
\sum_{n=1}^{\infty} \|(P_{\mu_{n+1}} - P_{\mu_n})(f_n) - f_n\| \leq \sum_{n=1}^{\infty} (\|P_{\mu_{n+1}}(f_n) - f_n\| + \|P_{\mu_n}(f_n)\|) < \sum_{n=1}^{\infty} 2^{-n}/16 = 1/16,
\]
it follows that $[f_n : n \in \mathbb{N}]$ is also isomorphic to $c_0$ and complemented in $C(K, X)$ \[8, Proposition 1.a.9\].

Finally, let $Z = \hat{\varphi}(C([0, \lambda])) \hat{\otimes}_e X$, which is a closed subspace of $C(K, X)$, fix $t_0 \in \varphi^{-1}(\lambda) \subseteq K$ and consider the natural projection $P: C(K, X) \to C(K, X)$ defined as $P(f) = 1 \otimes f(t_0) \in C(K) \hat{\otimes}_e X$; then $R(P)$ is isomorphic to $X$, which is subprojective by hypothesis, and $R(P) \subseteq Z$. Moreover, given $f \in Z \cap N(P)$, it holds that $f(t_0) = 0$ and $f \in \hat{\varphi}(C([0, \lambda])) \hat{\otimes}_e X$, so $f$ must be constant over $\varphi^{-1}(\lambda)$ and then $f|_{\varphi^{-1}(\lambda)} = 0$. This means that $Z \cap N(P) = \hat{\varphi}(C_0([0, \lambda])) \hat{\otimes}_e X$, which is subprojective with respect to $C(K, X)$. Applying Proposition $\|Z = \hat{\varphi}(C([0, \lambda])) \hat{\otimes}_e X\|$ is subprojective with respect to $C(K, X)$.

**Lemma 3.** Let $X, Y$ be Banach spaces and let $M$ be a closed separable subspace of $X \hat{\otimes}_e Y$. Then there exist closed separable subspaces $M_X \subseteq X$ and $M_Y \subseteq Y$ such that $M \subseteq M_X \hat{\otimes}_e M_Y$ as a subspace of $X \hat{\otimes}_e Y$.

**Proof.** $M$ is separable, so there exists $(z_m)_{m \in \mathbb{N}}$ in $M$ such that $M = [z_m : m \in \mathbb{N}]$ and then, for each $m \in \mathbb{N}$, there exist $(x_{m,n})_{n \in \mathbb{N}}$ in $X$ and $(y_{m,n})_{n \in \mathbb{N}}$ in $Y$ such that $z_m \in [x_{m,n} \otimes y_{m,n} : n \in \mathbb{N}]$, so $M \subseteq [x_{m,n} \otimes y_{m,n} : m, n \in \mathbb{N}]$. Let $M_X = [x_{m,n} : m, n \in \mathbb{N}] \subseteq X$ and $M_Y = [y_{m,n} : m, n \in \mathbb{N}] \subseteq Y$; then $M_X$ and $M_Y$ are separable and $M \subseteq M_X \hat{\otimes}_e M_Y$, which can be seen as a subspace of $X \hat{\otimes}_e Y$ \[12, Proposition 3.2\].

**Theorem 4.** Let $K$ be a scattered compact space and let $X$ be a subprojective Banach space. Then $C(K, X)$ is subprojective.

**Proof.** Let $M$ be a closed infinite-dimensional subspace of $C(K, X)$, which we can assume to be separable without loss of generality. By Lemma $\|Z = \hat{\varphi}(C([0, \lambda])) \hat{\otimes}_e X\|$ there exist separable subspaces $G \subseteq C(K)$ and $Z \subseteq X$ such that $M \subseteq G \hat{\otimes}_e Z \subseteq C(K) \hat{\otimes}_e X \equiv C(K, X)$. Without loss of generality, we can replace $G$ with the least closed self-adjoint subalgebra with unit of $C(K)$ that contains it, as this is still separable, and then there exists a compact space $L$ and a continuous surjection $\varphi: K \to L$ such that $G = \varphi(C(L))$ \[2, 13, Theorem 7.5.2\], so $C(L)$ is isomorphic to $G$, which is separable, and this means in turn that $L$ is metrisable. Under these conditions, $L$ is scattered \[7, Lemma 2.5.1\] and so homeomorphic to $[0, \lambda]$ for some countable ordinal $\lambda$ \[7, Corollary 2.5.2\]. By Proposition $\|M contains an infinite-dimensional subspace complemented in C(K, X)\|$. 

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