SOME TOPICS IN QUANTUM DISORDERED SYSTEMS

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ABSTRACT. We discuss some recent results connected with the properties of temperature states of quantum disordered systems. This analysis falls within the natural framework of operator algebras. Among the results quoted here, we recall some ergodic and spectral properties of KMS states, the possibility of defining the chemical potential independently of the disorder, and finally the question of the Gibbsianess for KMS states. This analysis can be considered as a step towards fully understanding the very complicated structure of the set of temperature states of quantum spin glasses, and its connection with the breakdown of the symmetry for replicas.

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1. INTRODUCTION

It is well–known that a disordered system, such as an alloy, exhibits a very complicated thermodynamical behaviour. Recently, the investigation of the structure of temperature states of spin glasses, and the questions pertaining to the breakdown of symmetry for replicas has generated considerable interest. Some interesting connections with Archimedean number theory, and with noncommutative geometry have been considered in order to understand the above mentioned problems. Unfortunately, many questions are still open. The reader is referred to [9, 22, 27, 31, 32, 33, 35] and the literature cited therein for recent results along these lines, and for further details.

Another promising approach to study the complex behaviour of the temperature states of a spin glass, is to use the standard techniques of operator algebras, the last being very fruitful for non disordered models, see e.g. [11, 12]. This approach started with the seminal paper [26]. There, besides some general results concerning the properties of equilibrium states, the phenomenon of weak Gibbsianess for quantum
disordered systems was pointed out. Recently, this strategy has been reconsidered, see [7, 8, 20].

In the present paper we discuss in an unified way such recent results about disordered systems, including spin glasses. This analysis covers states obtained by infinite volume limits of finite volume Gibbs states, that is the so called quenched disorder.

The paper is organized as follows. A preliminary section (Section 2) contains the early definitions related to the mathematical model of a quantum disordered system, and some ergodic results connected with the well-known phenomenon of self-averaging. Section 3, based on results of [7, 8], is devoted to the structure of the set of the KMS states of disordered systems, and some natural spectral properties. The model describing the Edwards–Anderson quantum spin glass is treated in some detail. Section 4, based on results in [20], concerns the algebraic description for disordered systems, of the chemical potential associated with the Gibbs grand–canonical ensemble. Following the strategy of the pivotal work [5], it is shown that one can exhibit a description of the chemical potential which does not depend on the disorder. As for non disordered systems, it is shown that the chemical potential is intrinsically connected with the Connes–Radon–Nikodym cocycles ([13, 38]) associated with the states obtained by composing the KMS state with the automorphisms carrying the localizable (abelian) charges of the disordered system under consideration. Section 5 is based on the paper [26] and is devoted to the question of the joint Gibbsianess of a KMS state of a disordered system.

The question of Gibbsianess arises when one considers infinite volume limits of states obtained from the finite volume Gibbs canonical ensemble on the skew space of joint configurations of spins and couplings. The measure on such a skew space satisfies the following properties. Its marginal distribution of the couplings is the given probability measure describing the disorder, whereas the conditional distribution of the spins, given the couplings, is some infinite volume Gibbs state almost surely. Such a field of infinite volume Gibbs states satisfies the equivariance property (2.4), that is it gives an Aizenman–Wehr metastate, see [1]. It can happen that the measure so obtained is not necessarily jointly Gibbsian. This issue was addressed in quantum setting in Proposition 5.14 of [26], and it is checked for classical disordered models in [19, 28]. In Section 5 we present a shortened discussion of these questions.

To end the introduction, we point out that a large number of problems remain open in the study of disordered systems. However, the
investigations of disordered systems by the standard techniques of operator algebras could be a significant step towards fully understanding the very complicated structure of the set of the KMS states of quantum spin glasses, and its connection with the breakdown of the symmetry for replicas.

2. Preliminaries

Our starting point for the investigation of the disordered systems is a separable $C^*$-algebra $\mathcal{A}$ with an identity $I$, describing the physical observables. Sometimes (i.e. Section 4), $\mathcal{A}$ can be obtained as the fixed point algebra $\mathcal{A} = \mathcal{F}^G$ under a strongly continuous action

$$\gamma : g \in G \mapsto \gamma_g \in \text{Aut}(\mathcal{F})$$

of a compact second countable group $G$ (the gauge group) on another separable $C^*$-algebra $\mathcal{F}$ (the field algebra), see e.g. [14, 15, 16, 17]. We suppose that the group $\{\alpha_x\}_{x \in \mathbb{Z}^d}$ of spatial translations acts on $\mathcal{F}$. The sample space for the couplings is a standard measure space $(X, \nu)$, $X$ being a compact separable space, and $\nu$ a Borel probability measure. The group $\mathbb{Z}^d$ of the spatial translations is supposed to act on the probability space $(X, \nu)$ by measure preserving transformations $\{T_x\}_{x \in \mathbb{Z}^d}$.

A one parameter random group of automorphisms

$$(t, \xi) \in \mathbb{R} \times X \mapsto \tau_t^{\xi} \in \text{Aut}(\mathcal{F})$$

is acting on $\mathcal{F}$. It is supposed to be strongly continuous in the time variable for each fixed $\xi \in X$, and jointly strongly measurable. We assume that $\tau$ acts locally, that is the strongly measurable, essentially bounded function $f_{A, t}(\xi) := \tau_t^{\xi}(A)$ belongs to the $C^*$-subalgebra $\mathcal{F} \otimes L^\infty(X, \nu)$, where the above $C^*$-tensor product is uniquely determined as any commutative $C^*$-algebra is nuclear. Furthermore, we assume that the random time evolution described above is nontrivial, if it is not otherwise specified. We assume the commutation rule

$$\tau_t^{T_x \xi} \alpha_x = \alpha_x \tau_t^{\xi}$$

(2.1)

for each $x \in \mathbb{Z}^d$, $\xi \in X$, $t \in \mathbb{R}$. If we start directly with the observable algebra, we assume analogous properties described above for $\mathcal{A}$ instead of $\mathcal{F}$.

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1To simplify the matter, we are assuming that $\mathbb{Z}^d$ is the group of the space symmetries. Some of the main results quoted below can be generalized to “continuous” disordered systems, where the group of the space symmetries is $\mathbb{R}^d$.

2This nontriviality condition simply means that the one parameter group $\{t_t\}_{t \in \mathbb{R}}$ given in (2.3) is nontrivial.
If we start with the field algebra $F$, we suppose further that

$$\alpha_x \gamma_g = \gamma_g \alpha_x, \quad \tau^\xi_t \gamma_g = \gamma_g \tau^\xi_t,$$

for each $x \in \mathbb{Z}^d$, $\xi \in X$, $t \in \mathbb{R}$, and $g \in G$. If $A$ is obtained by a principle of gauge invariance, then, by the above commutation rules, $\alpha_x$ and $\tau^\xi_t$ leave $A$ globally stable.

We address also the situation when Fermion operators are present in $F$. Namely, there exists an automorphism $\sigma$ of $F$ commuting with the all the gauge transformations, the spatial translations and the random time evolution, such that $\sigma^2 = e$. We put

$$(2.2) \quad F_+ := \frac{1}{2}(e + \sigma)(F), \quad F_- := \frac{1}{2}(e - \sigma)(F).$$

In order to achieve the disorder, it is natural to set for the field algebra,

$$\mathfrak{F} := F \otimes L^\infty(X, \nu).$$

Notice that, by identifying $\mathfrak{F}$ with a closed subspace of $L^\infty(X, \nu; F)$, each element $A \in \mathfrak{F}$ is uniquely represented by a measurable essentially bounded function $\xi \mapsto \alpha_x(A(T\gamma_x \xi))$, $t \mapsto \tau^\xi_t(A(\xi))$, $g \mapsto \gamma \otimes \text{id}_{L^\infty(X, \nu)}$, $s \mapsto \sigma \otimes \text{id}_{L^\infty(X, \nu)}$.

It is easy to verify that $\{a^x\}_{x \in \mathbb{Z}^d}$, $\{t^x_t\}_{t \in \mathbb{R}}$ and $\{g^x_g\}_{g \in G}$ define actions of $\mathbb{Z}^d$, $\mathbb{R}$ and $G$ on $\mathfrak{F}$ which are mutually commuting, and commute also with the parity automorphism $s$. The subspaces $\mathfrak{F}_+$ and $\mathfrak{F}_-$ are defined as in (2.2). Furthermore, $a^x$ and $t^x_t$ leave the disordered observable algebra $\mathfrak{A}$ globally stable. Namely, $\{a^x\}_{x \in \mathbb{Z}^d}$ and $\{t^x_t\}_{t \in \mathbb{R}}$ define by restriction, mutually commuting actions of $\mathbb{Z}^d$ and $\mathbb{R}$ on $\mathfrak{A}$, respectively.

In order to study a class of states of interest for disordered systems, we start with $\star$-weak measurable fields of states

$$\xi \in X \mapsto \varphi_\xi \in S(F).$$

We suppose that the field $\{\varphi_\xi\}_{\xi \in X}$ fulfills almost surely, the equivariance condition

$$(2.4) \quad \varphi_\xi \circ \alpha_x = \varphi_{T\gamma_x \xi}$$

w.r.t. the spatial translations, simultaneously. A state $\varphi$ on $\mathfrak{F}$ is naturally defined as

$$(2.5) \quad \varphi(A) = \int_X \varphi_\xi(A(\xi)) \nu(d\xi), \quad A \in \mathfrak{F}.$$
It is immediate to verify that \( \varphi \) defined as above is invariant w.r.t. the space translations \( a_x \). Moreover, \( \varphi[I \otimes L^\infty(X,\nu)] \) is a normal state.

Equally well, one can start with a \( a \)-invariant state \( \varphi \) on \( \mathfrak{F} \), which is normal when restricted to \( I \otimes L^\infty(X,\nu) \) (indeed, \( \varphi[I \otimes L^\infty(X,\nu)] = \int_X \cdot \nu(d\xi) \)). Then, we can recover a \( * \)-weak measurable field \( \{ \varphi_\xi \}_{\xi \in X} \subset \mathcal{F} \) fulfilling (2.4). Such a measurable field provides the direct integral decomposition of \( \varphi \) as in (2.5), see [8], Theorem 4.1, see also [26], Proposition 4.1. Similar considerations can be applied to the observable algebras \( \mathfrak{A} \) as well. In the sequel, we denote by \( \mathcal{S}_0(\mathfrak{A}), \mathcal{S}_0(\mathfrak{F}) \) the convex closed subsets of states on \( \mathfrak{A}, \mathfrak{F} \) respectively, fulfilling the properties listed above.

Now, we recall some properties of states on \( \mathfrak{F} \) or \( \mathfrak{A} \). We restrict ourselves to the field algebra, the other case being similar. Let \( C, D \in \mathfrak{F} \), and \( A, B \in \mathfrak{F}_+ \cup \mathfrak{F}_- \). Put \( \epsilon_{A,B} = -1 \) if \( A, B \in \mathfrak{F}_- \) and \( \epsilon_{A,B} = 1 \) in the three remaining possibilities. We say that the state \( \varphi \in \mathcal{S}(\mathfrak{F}) \) is asymptotically Abelian w.r.t. \( a \) if

\[
\lim_{|x| \to +\infty} \varphi\left(C(a_x(A)B - \epsilon_{A,B}Ba_x(A))D\right) = 0 ,
\]

The state \( \varphi \in \mathcal{S}(\mathfrak{F}) \) is weakly clustering w.r.t. \( a \) if

\[
\lim_{N} \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi(Aa_x(B)) = \varphi(A)\varphi(B),
\]

\( \Lambda_N \) being the box with a vertex at the origin, containing \( N^d \) points with positive coordinates. Notice that, Many of interesting states arising from quantum physics are naturally asymptotically Abelian w.r.t. the spatial translations, see e.g. [29], see also [8], Proposition 2.3 for the case of disordered systems.

It is well-know that for a \( a \)-invariant asymptotically Abelian state, the \( a \)-weak clustering property is equivalent to the \( a \)-ergodicity, see e.g. Proposition 5.4.23 of [12]. We speak, without any further mention, and if it is not otherwise specified, about asymptotic Abelianess, weak clustering, or ergodicity for states, if they satisfy these properties w.r.t. the spatial translations.

In [20], Section 3, it is shown under the ergodicity of the action \( \{T_x\}_{x \in \mathbb{Z}^d} \) on the sample space \((X,\nu)\), that \( \varphi \in \mathcal{S}_0(\mathfrak{F}) \) is weakly clustering if and only if

\[
\lim_{N} \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi_\xi(Aa_x(B(T_{-x}\xi))) = \varphi_\xi(A)\varphi(B), \quad A \in \mathcal{F}, B \in \mathfrak{F}
\]
in the weak topology of $L^1(X,\nu)$, and if
\[
\lim_{|x| \to +\infty} \varphi_\xi(A_{\alpha_x}(B)) = \varphi_\xi(A)\varphi_{T_{-x}\xi}(B), \quad A, B \in \mathcal{F}
\]
almost surely. Here, $\{\varphi_\xi\}_{\xi \in X} \subset \mathcal{F}$ is the measurable field of states associated with $\varphi$, satisfying (2.4). The last results are connected with the well–known property of self–averaging for disordered systems.

For the definition of the KMS boundary condition, the equivalent characterizations of the KMS boundary condition, the main results about KMS states, and finally the connections with Tomita theory of von Neumann algebras, we refer the reader to [12, 38] and the references cited therein. We have for the modular group $\sigma^\phi$ of a KMS state $\phi$,
\[
(2.6) \quad \sigma^\phi_t \circ \pi^\phi = \pi^\phi \circ \tau_{-\beta t},
\]
$\pi^\phi$ being the GNS representation of $\phi$.

**Remark 2.1.** It is a well–known fact that $\varphi \in \mathcal{S}_0(\mathfrak{F})$ is a KMS state if and only if $\{\varphi_\xi\}_{\xi \in X}$ are KMS states almost surely, see e.g. [20, 39].

### 3. The structure of the KMS states and their spectral properties

The present section is based on results contained in [7, 8], to which the reader is referred for further details. To simplify the matter, we suppose that the algebra of observables $\mathcal{A}$ is a quasi–local algebra (see e.g. [11], Section 2.6) whose local algebras are isomorphic to full matrix algebras, even if most of the forthcoming analysis works for more general situations. We further suppose that the action $x \mapsto T_x$ of the spatial translations on the sample space $(X,\nu)$ is ergodic.

The first result concerns the almost surely independence of the Arveson spectrum for random systems.\(^3\) This is precisely Theorem 5.3 of [7] (see also [8], Theorem 2.1). For the reader’s convenience, we report here its proof.

**Theorem 3.1.** Under the above assumptions, there exists a measurable set $F \subset X$ of full measure, and a closed set $\Sigma \subset \mathbb{R}$ such that $\xi \in F$ implies $\text{sp}(\tau^{\xi}) = \Sigma$.

**Proof.** We get by [36], Proposition 8.1.9,
\[
\text{sp}(\tau^{\xi}) = \bigcap_{f \in L^1(\mathbb{R})} \{ s \in \mathbb{R} \mid |\hat{f}(s)| \leq ||\tau_s^{\xi}|| \}
\]

\(^3\)The Arveson spectrum, known in physics as the set of the *Bohr frequencies* is made up of frequencies of the quanta which a physical system (like an atom) may emit or absorb, see [21], Theorem 5.3 for a simple example. For the definition of the Arveson spectrum, see e.g. [36].
where \( \hat{f} \) is the inverse Fourier transform of \( f \), and

\[
\tau_{f}^\xi(A) := \int_{-\infty}^{+\infty} f(t) \tau_{f}^\xi(A) \, dt.
\]

By a standard density argument, we can reduce the situation to a dense set \( \{f_k\}_{k \in \mathbb{N}} \subset L^1(\mathbb{R}) \). Define \( \Gamma_k(\xi) := \|\tau_{f_k}^\xi\| \). It was shown in [7] that the functions \( \Gamma_k \) are measurable and invariant. By ergodicity, they are constant almost everywhere. Let \( \{N_k\}_{k \in \mathbb{N}} \) be null subsets of \( X \) such that, for each \( k \in \mathbb{N} \) and \( \xi \in N_k^c \),

\[
\Gamma_k(\xi) = \|\Gamma_k\|_\infty.
\]

Consider \( F := \bigcup_{k \in \mathbb{N}} N_k^c \), and take \( \Sigma := \text{sp}(\tau^{\xi_0}) \), where \( \xi_0 \) is any element of \( F \). We have that \( F \) is a measurable set of full measure, and \( \xi \in F \) implies \( \text{sp}(\tau^{\xi}) = \Sigma \). \( \square \)

Let \( \omega \in S_0(\mathfrak{A}) \), \( \{\omega_\xi\}_{\xi \in X} \subset \mathfrak{A} \) be the measurable field of states associated to \( \omega \) as described above, and \( \pi_\omega, \pi_{\omega_\xi} \) the GNS representations of \( \omega, \omega_\xi \) respectively. Consider the subcentral decomposition

\[
M = \int_X M_\xi \nu(d\xi)
\]

of \( M := \pi_\omega'(\mathfrak{A})'' \). It is shown in Section 3 of [8], that \( M_\xi \) coincides with \( \pi_{\omega_\xi}(\mathfrak{A})'' \) almost surely.

Some of the main results in [8] (see Section 4 of that paper) are collected in the following theorem. Here, we sketch its proof for the convenience of the reader.

**Theorem 3.2.** Let \( \omega \in S_0(\mathfrak{A}) \) be a KMS state at inverse temperature \( \beta \neq 0 \). Then only type III_\lambda factors, \( \lambda \in (0, 1] \), can appear in the central decomposition of \( \pi_\omega'(\mathfrak{A})'' \).

In addition, if \( Z_{\pi_\omega} \sim L^\infty(X, \nu) \), then there exists a unique \( \lambda \in (0, 1] \) such that \( \pi_{\omega_\xi}(\mathfrak{A})'' \) are type III_\lambda factors almost surely.

**Proof.** Taking into account (2.6) and Theorem 3.1, it is shown in Proposition 4.5 of [8] that

\[
\Gamma_B(\pi_\varphi(\mathfrak{A})'') \equiv -\beta \text{sp}(t) = -\beta \text{sp}(\tau^\omega)
\]

almost surely, where \( \Gamma_B \) denotes the Borchers invariant ([10]) and \( \text{sp} \) the Arveson spectrum.\(^4\) This means that, in our situation, the infinite semifinite portion, and III_0 portion in the factor decomposition of \( \pi_{\omega_\xi}(\mathfrak{A})'' \) are avoided (cf. Theorem 4.4 of [8]). The I_fin is trivially avoided, and finally the II_1 is avoided as we are assuming that \( t_r \) is nontrivial (i.e. \( \omega \) is not a trace). This proves the first part.

\(^4\)This is a consequence of general results of [3, 23].
Now, $\mathcal{Z}_{\pi_\omega} \sim L^\infty(X, \nu)$ means that the direct integral decomposition (3.1) is the factor decomposition of $\pi_\omega(\mathfrak{A})''$. In this situation, $\Gamma_B(\pi_\omega(\mathfrak{A})'') = \Gamma(\pi_{\omega_\xi}(A)'')$ almost surely, where $\Gamma$ is Connes $\Gamma$–invariant ([13]). This implies by the previous assertion, that there exists a $\lambda \in (0, 1]$ such that $\pi_{\omega_\xi}(A)''$ is a type $\text{III}_\lambda$ factor almost surely. □

**Remark 3.3.** It is explained in [8] that states $\omega \in \mathcal{S}(\mathfrak{A})$ for which $\mathcal{Z}_{\pi_\omega} \sim L^\infty(X, \nu)$ can be thought as describing the “pure thermodynamical phase” in the case of disordered systems.\(^5\) The fact that $\pi_{\omega_\xi}$ generates a type $\text{III}_\lambda$ von Neumann algebra, $\lambda \in (0, 1]$, is in accordance with the standard fact that the physically relevant quantities do not depend on the disorder.

In order to demonstrate some possible applications, we specialize our analysis to the quantum model described by the formal random Hamiltonian

\[
H = -\frac{1}{2} \sum_{i \in \mathbb{Z}^d} \sum_{|i-j|=1} J_{i,j} \sigma(i)\sigma(j).
\]

Here, $\sigma(i)$ is the spin–operator along the $z$–axis (i.e. the Pauli matrix $\sigma_z$) acting on the $i$–th site, and the $J_{i,j}$ are independent identically distributed random variables with common distribution $p(dy)$ on the real line. To simplify, we suppose that the law $p$ is compactly supported.\(^6\)

We first suppose that for a fixed $\beta > 0$, the Ising–type model described by the Hamiltonian (3.2) admits a unique KMS state, say $\omega_\xi$, almost surely. Then the map $\xi \in X \mapsto \omega_\xi \in \mathcal{S}(\mathfrak{A})$ is automatically $\ast$–weak measurable and satisfies almost surely the equivariance condition (2.4).\(^7\) In this situation, there exists a unique state $\omega \in \mathcal{S}(\mathfrak{A})$ for the model under consideration, given by

\[
\omega(A) = \int_X \omega_\xi(A(\xi))\nu(d\xi), \quad A \in \mathfrak{A}.
\]

\(^5\)For classical disordered systems, the centre $\mathcal{Z}_{\pi_\omega}$ should be replaced by the algebra at infinity $\mathcal{Z}_{\pi_\omega}^\perp := \bigwedge_{\Lambda \text{ finite}} \pi_\omega(\mathfrak{A}_\Lambda'')''$.

\(^6\)The model with symmetric $p$ is known in literature as the (quantum) Edwards–Anderson spin glass, see [18].

\(^7\)This situation arises if the quantum model under consideration admits some critical temperature. The situation is well clarified for many classical disordered models (see e.g. [32]), contrary to the quantum situation where, in the knowledge of the authors, there are very few rigorous results concerning this fact. However, it is expected that quantum disordered systems also exhibit critical temperatures in the high temperature regime.
Furthermore, each of the other states $\omega_f \in \mathcal{S}(\mathfrak{A})$ which are normal when restricted to $I \otimes L^\infty(X, \nu)$, are in one–to–one correspondence with the positive normalized functions $f \in L^\infty(X, \nu)$ through the relation

\begin{equation}
\omega_f(A) = \int_X f(\xi) \omega_\xi(A(\xi)) \nu(d\xi), \quad A \in \mathfrak{A}. 
\end{equation}

Taking into account Theorem 3.2, it follows that KMS states in (3.3), (3.4) generate type III$\lambda$ von Neumann algebras for a fixed $\lambda \in (0, 1]$ depending on the temperature.

We end the present section by briefly describing the multiple phase regime. After taking the infinite–volume limit along various subsequences $\Lambda_{n_k} \uparrow \mathbb{Z}^d$, we will find by compactness, in general different $t$–KMS states in $\mathcal{S}_0(\mathfrak{A})$ at fixed inverse temperature $\beta$. Fix one such a state $\psi \in \mathcal{S}_0(\mathfrak{A})$, with the associated $*$–weak measurable field $\xi \in X \mapsto \psi_\xi \in \mathcal{S}(\mathfrak{A})$ of $\tau^\xi$–KMS states respectively, satisfying the equivariance property (2.4). According to Proposition 3.1 of [8], the set of the $t$–KMS states $\psi_T \in \mathcal{S}(\mathfrak{A})$, normal w.r.t. $\psi$, have the form

\begin{equation}
\psi_T(A) = \int_X \langle \pi_{\psi_\xi}(A(\xi)) T(\xi)^{1/2} \Omega_{\psi_\xi}, T(\xi)^{1/2} \Omega_{\psi_\xi} \rangle_{\mathcal{H}_{\psi_\xi}} \nu(d\xi). 
\end{equation}

Here, $(\pi_{\psi_\xi}, \mathcal{H}_{\psi_\xi}, \Omega_{\psi_\xi})$ is the GNS representation of $\psi_\xi$, $\{T(\xi)\}_{\xi \in X}$ is a measurable field of closed densely defined operators on $\mathcal{H}_{\psi_\xi}$ affiliated to the (isomorphic) centres $3_{\psi_\xi}$ respectively, satisfying $\Omega_{\psi_\xi} \in \mathcal{D}_{T(\xi)^{1/2}}$ almost surely, and $\int_X \|T(\xi)^{1/2} \Omega_{\psi_\xi}\|^2_{\mathcal{H}_{\psi_\xi}} \nu(d\xi) = 1$.

4. AN ALGEBRAIC DESCRIPTION OF THE CHEMICAL POTENTIAL

The present section is based on results contained in [20], to which the reader is referred for further details. As in the previous section, we suppose that the action $x \mapsto T_x$ of the spatial translations on the sample space $(X, \nu)$ is ergodic. Here, in order to give an algebraic description of the chemical potential, we suppose that the algebra of the observables $\mathfrak{A}$ is obtained by a principle of gauge invariance, as the fixed point subalgebra of the field algebra $\mathcal{F}$.

The technical problem which is behind the algebraic description of the chemical potential is the question of extending a KMS state on the observable algebra to a KMS state on the field algebra. This is a very delicate issue studied in a series of papers connected with non disordered models, see [5, 6, 24, 25, 30]. The strategy and the main results of the pivotal paper [5] can be effectively used to investigate this problem for disordered systems.
After proving some preliminary but crucial results such as the fact that the stabilizer and the asymmetry subgroup of a state $\varphi \in S_0(\mathcal{F})$ coincide almost surely with the corresponding objects associated with $\varphi_{\xi}$,\footnote{The stabilizer $G_\varphi$ of $\varphi \in S(\mathcal{F})$ is defined as $G_\varphi := \{ g \in G \mid \varphi \circ g = \varphi \}$. For $\varphi \in S_0(\mathcal{F})$, the stabilizers $G_{\varphi_{\xi}}$ of the $\varphi_{\xi} \in S(\mathcal{F})$ associated with $\varphi$, are defined analogously. We denote by an abuse of notation, the normalizer and the centralizer of a subgroup $H \subseteq G$ by $N(H)$ and $Z(H)$ respectively, and are defined as $N(H) := \{ g \in G \mid gHg^{-1} = H \}$, $Z(H) := \{ g \in G \mid gh = hg, h \in H \}$. The asymmetry subgroup of an invariant state is the subgroup for which the spectrum of the associated unitary implementation is one–sided, see [5], Definition II.3.} it is shown in [20] that the KMS states in $S_0(\mathcal{A})$ satisfying certain natural properties, can be extended to KMS states on $S_0(\mathcal{F})$.

We report here the main result of [20] concerning the algebraic description of the chemical potential for disordered systems.

**Theorem 4.1.** Let $\varphi \in S_0(\mathcal{F})$ be a weakly clustering asymptotically Abelian state whose restriction to $\mathcal{A}$ is $(t, \beta)$–KMS state at inverse temperature $\beta \neq 0$.

Then there exist a closed subgroup $N \subset G_\varphi$, a continuous one parameter subgroup $t \in \mathbb{R} \mapsto \varepsilon_t \in Z(G_\varphi)$, a continuous one parameter subgroup $t \in \mathbb{R} \mapsto \zeta_t \in G_\varphi$, and a measurable subset $F \subset X$ of full measure such that, for each $\xi \in F$,

1. the $N$–spectrum of $\varphi_{\xi}$ is one–sided,
2. the restriction of $\varphi_{\xi}$ to $\mathcal{F}^N := \{ A \in \mathcal{F} \mid \gamma_g(A) = A, g \in N \}$ is a $(\theta^\xi, \beta)$–KMS state for the modified time evolution $\theta^\xi_t := \tau^\xi_{t\varepsilon_t\zeta_t}$,
3. the image $[\zeta_t] := \zeta_tN$ in $G_\varphi/N$ is an element of $Z(G_\varphi/N)$.

The proof is a consequence of Theorem II.4 of [5], taking into account the above mentioned results pertaining to the stabilizers and the asymmetry subgroups, see [20], Theorem 4.7 for further details.

Theorem 4.1 can be directly applied to the case $\beta = 0$ which corresponds to the case when the restriction to $\mathcal{A}$ of a state in $S_0(\mathcal{F})$ is a trace.

**Corollary 4.2.** Under the same assumption of Theorem 4.1 but for $\beta = 0$, there exist a closed subgroup $N \subset G_\varphi$, a continuous one parameter subgroup $t \in \mathbb{R} \mapsto \zeta_t \in G_\varphi$, and a measurable subset $F \subset X$ of full measure such that the assertions of Theorem 4.1 hold true with (ii) replaced by

1. the restriction of $\varphi_{\xi}$ to $\mathcal{F}^N := \{ A \in \mathcal{F} \mid \gamma_g(A) = A, g \in N \}$ is a $(\gamma_{\zeta_t}, -1)$–KMS state.$^9$

\footnote{The value $-1$ is chosen such that the restriction of the modular group to $\pi_\varphi(\mathcal{F})$ coincides with the time evolution $\gamma_{\zeta_t}$, see (2.6).}
Proof. We start by noticing that a state $\omega$ satisfies the KMS boundary condition at inverse temperature 0 if and only if it is a trace. This means that $\omega$ is KMS at any inverse temperature $\beta \neq 0$ w.r.t. the trivial one parameter automorphism group $t \mapsto \text{id}_\mathbf{A}$. By convention, we choose $\beta = -1$.

The proof follows by the previous theorem, taking into account that, in this situation, we can choose for $t \in \mathbb{R} \mapsto \varepsilon_t \in \mathcal{Z}(G_\varphi)$ the trivial one parameter automorphism group, see [5], Theorem 2.2. □

Now, we apply the previous results to a simple case in order to understand how the chemical potential appears. To avoid technical complications, we further assume that $\mathcal{F}$ is a quasi-local algebra whose local algebras are isomorphic to full matrix algebras. We assume also that the gauge group is the unit circle $\mathbb{T}$. In this situation, the localizable charges of the model under consideration are associated with the powers $[\sigma^n] \in \text{Out}(\mathbf{A})$ of a single localized transportable outer automorphism $\sigma$, see [14].

Let $\omega \in S_0(\mathbf{A})$ be a $(t, \beta)$–KMS state at inverse temperature $\beta \neq 0$ such that $3_{\pi_\omega} \sim L^\infty(X, \nu)$. Take a weakly clustering extension $\varphi$ of $\omega$ to all of $\mathfrak{F}$ which exists by Proposition 4.2 of [8]. In order to avoid the possibility of null chemical potential, we further suppose that $\varphi$ is gauge invariant. In this situation, the asymmetry subgroup $N_\varphi$ of $\varphi$ is trivial, or equivalently $N_{\varphi_\xi}$ is trivial almost surely.\footnote{\textit{If $\omega \in S_0(\mathbf{A})$ is a trace, then $\omega_\xi$ is a trace almost surely, the proof being the same as that of Proposition 3.2 of [8].}}

Let $\rho$ be a localized automorphism of $\mathbf{A}$ carrying the charge $n$ (i.e. $\rho \in [\sigma^n]$), and $\omega \in S_0(\mathbf{A})$ as above. Then, $\rho$ extends to a measurable field $\{\rho_\xi\}_{\xi \in X}$ of normal automorphisms of the weak closure $\pi_{\omega_\xi}(\mathbf{A})''$, almost surely, see [20], Proposition 5.1.

Consider the unitary $U$ implementing $\rho$ on $\mathbf{A}$, together with the state $\varphi_U := \varphi \circ \text{Ad}(U \otimes I)$. We have for the Connes–Radon–Nikodym cocycle
([13, 38]),

\[
(D \varphi_U : D \varphi)_t = \int_X \pi_{\varphi_t}(U^*) \sigma_\varphi^t(\pi_{\varphi_t}(U)) \nu(d\xi)
\]

(4.1)

\[
= e^{i\mu \beta t} \int_X \pi_{\varphi_t}(U^* \tau_{-\beta t}^\xi(U)) \nu(d\xi),
\]

for some \( \mu \in \mathbb{R} \).

Here, we have used \( \gamma_\theta(U) = e^{i\theta}U \), and \( \sigma_\varphi^t \circ \pi_\varphi = \pi_\varphi \circ t_{-\beta t} \circ g_{\beta \mu t} \) by Theorem 4.1, taking into account (2.6).

Now, we take advantage of the fact that \( \omega \circ (\rho \otimes \text{id}) \) extends to a normal state on all of \( \pi_\omega(\mathcal{A})'' \). Denote by the same symbol \( \omega_\xi \) the normal extension of \( \omega_\xi \) itself to all of \( \pi_{\omega_\xi}(\mathcal{A})'' \). Under this identification, the equivariant measurable field of states \( \{\omega_\xi \circ \rho_\xi\}_{\xi \in X} \) provides the direct integral decomposition of the mentioned normal extension of \( \omega \circ (\rho \otimes \text{id}) \). Here, the \( \rho_\xi \) are the above mentioned normal automorphisms of \( \pi_{\varphi_\xi}(\mathcal{A})'' \) extending \( \rho \). We have by (4.1) and the fact that \( U^* \tau_{-\beta t}^\xi(U) \) is gauge invariant,

\[
(4.2)
\]

\[
(D(\omega_\xi \circ \rho_\xi) : D\omega_\xi)_t = e^{i\mu \beta t} \pi_{\omega_\xi}(U^* \tau_{-\beta t}^\xi(U))
\]

almost everywhere.

Formula (4.2) explains the occurrence of the chemical potential \( \mu \in \mathbb{R} \) as an object intrinsically associated to the observable algebra. Furthermore, according to this description, it does not depend on the disorder for KMS states \( \omega \in \mathcal{S}_0(\mathcal{A}) \) such that \( \mathfrak{Z}_{\pi_\omega} \sim L^\infty(X, \nu) \). This is in accordance with the standard fact that the physically relevant quantities should not depend on the disorder.

We end by noticing that the last analysis can be straightforwardly extended to the \( n \)-dimensional torus or, more generally, to non Abelian \( n \)-dimensional Lie groups, obtaining a \( n \)-parametric chemical potential. Furthermore, the results of the present section could be applied to continuous disordered systems such as possible disordered systems arising from quantum field theory, even if some technical gaps need to be covered in the last situation.

5. THE QUESTION OF GIBBSIANESS

The question of Gibbsianess arises when one takes infinite volume limits of finite volume Gibbs states. If one considers such an object as

\[\text{Notice that the direct integral decomposition of } (D \varphi_U : D \varphi) \text{ exists by the results in Appendix A of [8], taking into account the definition of the associated modular operator in terms of the balanced weight, see e.g. [38], Section 3.}\]
a state on the skew space of joint configurations of spins and couplings, it can happen that the state so obtained is not jointly Gibbsian.\footnote{Such a weakly Gibbsian state is known in literature as a \textit{metastate}, see [1] for the classical situation.}

The possible appearance of weakly Gibbsian non jointly Gibbsian states was firstly pointed out by Kishimoto in relation to the quantum case, see [26], see also [4]. Recently, simple examples of weakly Gibbsian states were constructed for classical disordered systems, see [19, 28]. In the present section we discuss some results of [26] connected with Gibbsianess.

We start by setting
\[ \mathcal{A} := \bigotimes_{\mathbb{Z}^d} \mathbb{M}_n(\mathbb{C}). \]

We assume that also the sample space has a local structure, and is finite for finite regions. More precisely,
\[ X := \prod_{\mathbb{Z}^d} k. \]

In this simple situation, we can take for \( \mathfrak{A} \), the separable \( C^* \)-algebra consisting of all the continuous functions from \( X \) with values in \( \mathcal{A} \):
\[ \mathfrak{A} := C(X, \mathcal{A}). \]

The algebra \( \mathfrak{A} \) of observables is equipped with a natural local structure \( \{ \mathfrak{A}_\Lambda \mid \Lambda \subset \mathbb{Z}^d, \Lambda \text{ bounded} \} \), with \( \mathfrak{A}_\Lambda = C(X_\Lambda, \mathcal{A}_\Lambda) \).

It is immediate to show that we can recover \( \mathfrak{A} \) from a principle of gauge invariance from
\[ \mathfrak{B} := \bigotimes_{\mathbb{Z}^d} \left( \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_k(\mathbb{C}) \right), \]
equipped with the local structure \( \{ \mathfrak{B}_\Lambda \mid \Lambda \subset \mathbb{Z}^d, \Lambda \text{ bounded} \} \), where
\[ \mathfrak{A}_\Lambda = \bigotimes_{\Lambda} \left( \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_k(\mathbb{C}) \right). \]

The gauge group is given by
\[ G := \prod_{\mathbb{Z}^d} \mathbb{Z}_k, \]
\( \mathbb{Z}_k = \{0,1,\ldots,k-1\} \) being the cyclic group of order \( k \). Here, the gauge action \( a_g \), \( g = \{l_r\}_{r \in \mathbb{Z}^d} \), is given by
\[ a_{\{l_r\}} := \text{id}_\mathcal{A} \bigotimes_{r \in \mathbb{Z}^d} \text{Ad}(V^{l_r}), \]
with
\[ V := \sum_{j=0}^{k-1} e^{\frac{2\pi i j}{k}} P_j, \]
\( \{P_j\}_{j=0,\ldots,k-1} \) being the set of the minimal projection of \( C^k \subset M_k(\mathbb{C}) \).

This situation covers the example described by the Hamiltonian (3.2) if the common distribution \( p(dy) \) for the couplings has finite support made of \( k \) elements, and falls within the framework of [26], even if slightly more general situations are treated there.

Let \( \{t_t\}_{t \in \mathbb{R}} \) be a strongly continuous one parameter group of automorphisms of \( \mathfrak{B} \) describing the time evolution commuting with the space translations. We suppose that
\[ t_t(\mathfrak{A}) = \mathfrak{A}, \quad t \in \mathbb{R}, \]
that is \( t_t \) leaves globally stable \( \mathfrak{A} \), and that
\[ \bigcup_{\Lambda \text{finite}} \mathfrak{B}_\Lambda \subset D(\delta), \]
\( D(\delta) \) being the domain of the infinitesimal generator of \( t_t \).

Consider, for \( \xi \in X \) the homomorphism
\[ \chi_\xi(A) := A(\xi), \quad A \in \mathfrak{A} \]
of \( \mathfrak{A} \) onto \( \mathfrak{A} \). It follows by Proposition 3.1 of [26] that a random time evolution \( \{\tau^\xi_t\}_{t \in \mathbb{R}} \) is recovered as
\[ \tau^\xi_t(\chi_\xi(A)) := \chi_\xi(t_t(A)), \quad A \in \mathfrak{A}. \]

As the time evolution and the shift on \( \mathbb{Z}^d \) are supposed to be commuting, the random time evolution \( \tau^\xi_t \) satisfies the equivariance condition (2.1), \( T_x \) being the shift on \( X \).

We report the definition of the infinite volume Gibbs condition. Let \( \omega \in S(\mathfrak{A}) \). The state \( \omega \) is a Gibbs state at inverse temperature \( \beta \neq 0 \) if
\[ (i) \ \omega \text{ is a modular state,}^16 \]

---

15 The algebra \( \mathfrak{A} \) slightly differs from the analogous object in the previous sections without affecting the forthcoming analysis. However, the principle of gauge invariance described in the present section differs significantly from that assumed in Section 4. Namely, the algebra \( \mathfrak{B} \) is non disordered (i.e. it is simple, contrary to \( \mathfrak{F} \)). In addition, \( a_\varphi \) in (5.1) acts in a different way from \( g_\varphi \) in (2.3). Finally, the gauge action (5.1) does not commute with the space translations. Thus, the theory of the chemical potential described in Section 4 cannot be directly applied to the field system (\( \mathfrak{B}, G, a_\varphi \)).

16 This means by definition, that \( \Omega_\omega \) is separating for \( \pi_\omega(\mathfrak{A})'' \).
(ii) for each bounded $\Lambda \subset \mathbb{Z}^d$, there exists a state $\psi_{\Lambda^c} \in S(\mathfrak{A}_{\Lambda^c})$ such that
\begin{equation}
\omega^{\beta H_{\Lambda}} = \text{Tr}_{\Lambda} \otimes \psi_{\Lambda^c}.
\end{equation}

Here, $H_{\Lambda} \in \mathfrak{A}$ is the selfadjoint infinitesimal generator of $t_t$ for elements in $\mathfrak{B}_{\Lambda}$ (i.e. $\delta(A) = i[H_{\Lambda}, A], A \in \mathfrak{B}_{\Lambda}$), $\omega^{\beta H_{\Lambda}}$ is the perturbation of $\omega$ by the selfadjoint element $\beta H_{\Lambda}$ (see e.g. [12] Theorem 5.4.4), and finally $\text{Tr}_{\Lambda}$ is the restriction to $\mathfrak{A}_{\Lambda}$ of the normalized trace on $\mathfrak{B}$.\footnote{The infinitesimal generator $H_{\Lambda}$ always exists for the situation considered here, taking into account Proposition 3.3 of [26], and the construction in Example 3.2.25 of [11]. This can be easily applied to models with finite range interactions such as that described by the Hamiltonian (3.2) by taking from the local infinitesimal generator $H_{\Lambda} := -\frac{1}{2} \sum_{i \in \Lambda} \sum_{|i-j|=1} J_{i,j} \sigma(i)\sigma(j)$.}

The Gibbs condition for the canonical lifting
\begin{equation}
\varphi_\omega(A) := \int_G a_g(A) \, dg, \quad A \in \mathfrak{B}
\end{equation}
is defined analogously. Denote $\nu_\omega$ the measure on $X$ obtained by restricting $\omega$ to $C(X)$, and $\nu_0 (\nu_{0,\Lambda})$ the measure on $X$ ($X_{\Lambda}$) obtained by restricting the normalized trace on $\mathfrak{B}$ to $C(X)$ ($C(X_{\Lambda})$).

Some of the main results of [26] are summarized in the following theorem. We sketch its proof for the reader’s convenience.

**Theorem 5.1.** Let $\omega \in S(\mathfrak{A})$. Then the following assertions are equivalent.

(i) $\omega$ is a Gibbs state at $\beta \neq 0$,

(ii) $\varphi_\omega \in S(\mathfrak{B})$ is a Gibbs state at $\beta \neq 0$.

If one of the above conditions holds true, then the following assertions hold true as well.

(a) $\omega$ is a $(t, \beta)$-KMS state at $\beta \neq 0$,

(b) $\varphi_\omega$ is a $(\tau^\xi, \beta)$-KMS state at $\beta \neq 0$ almost everywhere w.r.t. the measure $\nu_\omega$,

(c) there exists a positive measure $\mu$ on $X_{\Lambda^c}$ such that
\begin{equation}
\omega^{\beta \chi_\xi_{\Lambda^c}}(H_{\Lambda}) = \nu_{0,\Lambda}(d\xi) \times \mu(d\xi_{\Lambda^c}).
\end{equation}

**Proof.** The equivalence (i) $\iff$ (ii) follows from the fact that $H_{\Lambda} \in \mathfrak{A}$, provided that $\varphi_\omega$ is separating if $\omega$ is. The last fact is proved in Proposition 4.5 of [26].
It is well-known that (i)⇒(a). But (a) is equivalent to (b), see e.g. [26], or [8], Proposition 3.2.

By construction, \( \omega \) is equivalent to \( \omega^{\beta H} \), that is

\[
\omega^{\beta H} = \int_X (\omega^{\beta H})_\xi \nu_\omega (d\xi) .
\]

This means that

\[
\nu_{\omega^{\beta H}} (d\xi) = (\omega^{\beta H})_\xi (I) \nu_\omega (d\xi) .
\]

Thus, (i)⇒(c) follows by restricting (5.2) to \( C(X) \), taking into account that

\[
(\omega^{\beta H})_\xi = \omega^{\beta \chi_\xi(H)}
\]

\( \nu_\omega \)-almost everywhere. \( \square \)

**Remark 5.2.** As the \( C^* \)-algebra \( \mathcal{A} \) is simple, it is well-known that (b) of Theorem 5.1 is equivalent to

(b') \( \omega_\xi \) is a Gibbs state at \( \beta \neq 0 \) \( \nu_\omega \)-almost everywhere.

Theorem 5.1 leaves open the possibility that the Gibbs condition could be not equivalent to the KMS condition for a state \( \omega \in \mathcal{S}(\mathcal{A}) \). This is precisely the question of weak Gibbsianess, which was then addressed firstly by Kishimoto.\(^{18}\) Recently, this fact has been checked in [19, 28] for classical disordered systems even when the measure describing the disorder trivially fulfills the Dobrushin–Lanford–Ruelle (DLR for short, see e.g. [37]) condition (5.3). Even if there is no quantum examples pertaining to this point, it is expected that the quantum versions of the models in [19, 28] exhibit weakly Gibbsian non jointly Gibbsian states as well.

To end this section, we briefly mention the variational principle arising from the Gibbs condition for a state \( \omega \in \mathcal{A} \). We suppose that the local Hamiltonians \( \{ H_\Lambda \mid \Lambda \subset \mathbb{Z}^d, \Lambda \text{ bounded} \} \) are associated to a finite range interaction \( \Phi \) described by \( \{ \Phi(\Lambda) \mid \Lambda \subset \mathbb{Z}^d, \Lambda \text{ bounded} \} \), see e.g. [12]. The Gibbs condition for a translation invariant state \( \omega \in \mathcal{S}(\mathcal{A}) \) is equivalent under the above conditions to

\[
p(\Phi, \beta) = s(\nu_\omega) + \int_X p(\chi_\xi(\Phi), \beta) \nu_\omega (d\xi) ,
\]

\[
p(\chi_\xi(\Phi), \beta) = s(\omega_\xi) - \beta e(\omega_\xi) .
\]

---

\(^{18}\)The KMS boundary condition is not sensitive in classical case. This means that it gives no condition on the measure \( \nu_\omega \). However, it can happen that infinite volume limits of finite volume Gibbs states could be not Gibbsian on the skew space describing the joint configurations of spins and couplings even if \( \nu_\omega \) satisfies the DLR condition (5.3), see below.
Here, $p(\Phi, \beta)$ ($p(\chi_\xi(\Phi), \beta)$) is the pressure associated with the interaction $\Phi$ ($\chi_\xi(\Phi)$) described by the continuous field of potentials $\{\chi_\xi(\Phi(\Lambda)) | \Lambda \subset \mathbb{Z}^b, \Lambda \text{ bounded}\}$, $s(\nu_\omega)$ is the mean entropy of the translationally invariant measure $\nu_\omega$, $s(\omega_\xi)$ and $e(\omega_\xi)$ the mean entropy and the mean energy of the state $\omega_\xi$, respectively.\footnote{See e.g. [12] for the definition of the pressure. Furthermore, $s(\omega_\xi)$ and $e(\omega_\xi)$ should be defined with a little bit of care as the $\omega_\xi$ are not translationally invariant, see Proposition 7.5 and Proposition 7.6 of [26].} The functions $p(\chi_\xi(\Phi), \beta)$, $s(\omega_\xi)$, and $e(\omega_\xi)$ are measurable essentially bounded functions defined $\nu_\omega$–almost everywhere, so they depend also on the state $\omega$ under consideration. Thus, (5.5) holds true almost surely w.r.t the measure $\nu_\omega$. Notice that, if $\nu_\omega$ is ergodic w.r.t. the spatial translations, they are constant $\nu_\omega$–almost everywhere. Equation (5.5) is nothing but the variational principle associated to the Gibbs condition for the states $\omega_\xi$, which holds true $\nu_\omega$–almost everywhere, taking into account Remark 5.2.

We refer the reader to [4, 26] for the meaning of (5.4) and (5.5) as variational principles, and for further details.

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