Continuous dependence estimate for conservation laws with Lévy noise

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Abstract

We are concerned with multidimensional stochastic balance laws driven by Lévy processes. Using bounded variation (BV) estimates for vanishing viscosity approximations, we derive an explicit continuous dependence estimate on the nonlinearities of the entropy solutions under the assumption that Lévy noise only depends on the solution. This result is used to show the error estimate for the stochastic vanishing viscosity method. In addition, we establish fractional BV estimate for vanishing viscosity approximations in case the noise coefficient depends on both the solution and spatial variable.

Keywords: Conservation laws, stochastic forcing, Lévy noise, stochastic entropy solution, stochastic partial differential equations, Kružkov’s entropy.

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1. Introduction

The last couple of decades have witnessed remarkable advances in the studies of partial differential equations with noise/randomness. A vast literature is now available on the subject of stochastic partial differential equations (SPDEs) and the particular frontier involving hyperbolic conservation laws with noise has had its fair share of attention as well. However, this is still very much a developing story and there still a number of issues waiting to be explored. In this paper, we aim at deriving continuous dependence estimates based on nonlinearities for stochastic conservation laws driven by multiplicative Lévy noise. A formal description of our problem requires a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\) and we are interested in an \(L^p(\mathbb{R}^d)\)-valued predictable process \(u(t, \cdot)\) which satisfies the Cauchy problem

\[
\begin{aligned}
\frac{du(t, x)}{dt} + \text{div}_x F(u(t, x)) &= \int_{|z|>0} \eta(u(t, x); z) \tilde{N}(dz, dt), & x \in \Pi_T, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}^d,
\end{aligned}
\]

where \(\Pi_T = (\mathbb{R}^d \times (0, T))\) with \(T > 0\) fixed. The initial condition \(u_0(x)\) is a given function on \(\mathbb{R}^d\), and \(F : \mathbb{R} \mapsto \mathbb{R}^d\) is given (sufficiently smooth) vector valued flux function (see Section 2 for the complete list of assumptions). The right hand side of (1.1) represents the noise term and it is composed of a compensated Poisson random measure \(\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz) dt\), where \(N\) is a Poisson random measure on \(\mathbb{R} \times (0, \infty)\) with intensity measure \(\nu(dz)\), and the jump amplitude (integrand) \(\eta(u, z)\) is real valued function signifying the multiplicative nature of the noise.

Hyperbolic conservation laws are used to describe a large number of physical phenomenon from areas such as physics, economics, biology etc. The inherent uncertainty in such phenomenon prompts one to account for the same and consider random perturbation of conservation laws. As an important first step

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into the subject, a significant body of literature has grown around conservation laws that are perturbed by Brownian white noise. However, due to the complex nature of the uncertainties, it is only natural to look beyond Brownian white noise settings and consider problems with more general type of noise. We do that in this paper in the problem (1.1) by introducing Poisson noise in the right hand side. It is also mentioned the result of this paper could be extended to the general Lévy noise case.

In the case $\eta = 0$, the equation (1.1) becomes a standard conservation laws in $\mathbb{R}^d$. For the deterministic conservation laws, well-posedness analysis has a very long tradition and it goes back to the 1950s. However, we will not be able to discuss the whole literature here, but only refer to the parts that are pertinent to the current paper. The question of existence and uniqueness of solutions of conservation laws was first settled in the pioneer papers of Kružkov [15] and Vol’pert [17]. For a completely satisfactory well-posedness theory of conservation laws, we refer to the monograph of Dafermos [8]. See also [12] and references therein.

1.1. Stochastic balance laws driven by Brownian white noise

As has been mentioned, evolutionary SPDEs with Lévy noise has been the topic of interest of many authors lately, and new results are emerging faster than ever before. However, the study of stochastic balance laws driven by noise has so far been limited to equations that are driven by Brownian white noise and a satisfactory well-posedness theory is available by now.

Observe that when the noise is of additive nature, a change of variable reduces equation into a hyperbolic conservation law with random flux which could be analyzed with deterministic techniques. In fact, Kim [14] extended Kružkov’s entropy formulation to establish the well-posedness of one dimensional stochastic balance law.

However, when the noise is of multiplicative nature, one could not apply a straightforward Kružkov’s doubling method to get a $L^1$-contraction principle as in [15]. The main difficulty lies in doubling the time variable which gives rise to stochastic integrands that are anticipative and hence the stochastic integrals in the sense of Itô-Lévy would not make sense. Hence, it fails to capture a specific “noise-noise” interaction term relating two entropy solutions. This issue was first resolved by Feng & Nualart [11] with the introduction of additional condition, which captures the missing “noise-noise” interaction term, the so called strong stochastic entropy solution. They used $L^p$ framework to prove the multidimensional uniqueness result for strong stochastic entropy solution. However, existence was restricted to one space dimension since their proof of existence was based on a stochastic version of compensated compactness argument applied to vanishing viscosity approximation of the underlying problem. To overcome this problem, Debussche & Vovelle [9] introduced kinetic formulation of such problems and as a result they were able to established the wellposedness of multidimensional stochastic balance law via kinetic approach. At around the same time, Chen & Karlsen [6] also established multidimensional wellposedness of strong entropy solution in $L^p \cap BV$, via $BV$ framework. Moreover, they were able to develop continuous dependence theory for multidimensional balance laws and, as a by product, they derived an explicit convergence rate of the approximate solutions to the underlying problem. We also mention that, using the concept of measure valued solutions and Kružkov’s semi-entropy formulations, a result of existence and uniqueness of the entropy solution has been obtained by Bauzet et. al. in [1].

In the article [11], the authors used an entropy formulation which is strong in time but weak in space, which is in our view may give rise to problems where the solutions are not shown to have continuous sample paths. We refer to [3], where a few technical questions are raised and remedial measures have been proposed. We also mention that Weinert et. al. [18] published a very influential article describing the existence, uniqueness and weak convergence of invariant measures for one dimensional Burger’s equation with stochastic forcing which is periodic in $x$.

1.2. Stochastic balance laws driven by Lévy noise

Despite relatively large body of research on stochastic partial differential equations that are driven by Lévy noise, to the best of our knowledge, very little is available on the specific problem of conservation laws.
with Lévy noise. In fact, the first attempt were made to build a comprehensive theory on such problems in a very recent article by Biswas. et. al. [1]. For a detailed introduction to the SPDEs driven by Lévy processes, we refer to the monograph by Peszat. et.al. [16] and references therein. Roughly speaking, the theory developed in [16] covers semi linear parabolic equations driven by Lévy noise, which could be treated as stochastic evolution equations in some infinite dimensional Banach or Hilbert space, and typically the solutions of such equations enjoy regularizing properties. However, we can’t emulate those techniques on the specific problem of conservation laws driven by Lévy noise due to the intrinsic discontinuous nature of the solution.

In fact, independent of the smoothness of the initial data \( u_0(x) \), due to the presence of nonlinear flux term in equation (1.4), solutions to (1.1) are not necessarily smooth and weak solutions must be sought. Before introducing the concept of weak solutions, we first assume that the filtered probability space \((\Omega, P, F, \{F_t\}_{t \geq 0})\) satisfies the usual hypothesis, i.e., \(\{F_t\}_{t \geq 0}\) is a right-continuous filtration such that \(F_0\) contains all the \(P\)-null subsets of \((\Omega, F)\). Moreover, by a predictable \(\sigma\)-field on \([0, T] \times \Omega\), denoted by \(\mathcal{P}_\tau\), we mean that the \(\sigma\)-field generated by the sets of the form: \([0] \times A\) and \((s, t] \times B\) for any \(A \in \mathcal{P}_0; B \in \mathcal{F}_s\), \(0 < s, t \leq T\).

The notion of weak solution is defined as follows:

**Definition 1.1** (weak solution). An \(L^2(\mathbb{R}^d)\)-valued \(\{F_t : t \geq 0\}\)-predictable stochastic process \(u(t) = u(t, x)\) is called a stochastic weak solution of (1.1) if for all non-negative test functions \(\psi \in C^\infty_c([0, T) \times \mathbb{R}^d)\),

\[
\int_{\mathbb{R}^d} \psi(0, x)u(0, x)\, dx + \int_0^T \int_{\mathbb{R}^d} \left[ \partial_t \psi(t, x)u(t, x) + F(u(t, x)) \cdot \nabla \psi(t, x) \right]\, dx\, dt \\
+ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \eta(u(t, x); z)\psi(t, x)\, dx\, d\tilde{N}(dz, dt) = 0, \quad P - a.s. \tag{1.2}
\]

However, it is well known that weak solutions may be discontinuous and they are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. Since the notion of entropy solution is built around the so called entropy-entropy flux pairs, we begin with the definition of entropy-entropy flux pairs.

**Definition 1.2** (entropy-entropy flux pair). An ordered pair \((\beta, \zeta)\) is called an entropy-entropy flux pair if \(\beta \in C^2(\mathbb{R})\) with \(\beta \geq 0\), and \(\zeta = (\zeta_1, \zeta_2, ..., \zeta_d): \mathbb{R} \rightarrow \mathbb{R}^d\) is a vector field satisfying

\[
\zeta'(r) = \beta'(r)F'(r), \quad \text{for all } r.
\]

Moreover, an entropy-entropy flux pair \((\beta, \zeta)\) is called convex if \(\beta''(\cdot) \geq 0\).

With the help of a convex entropy-entropy flux pair \((\beta, \zeta)\), the notion of stochastic entropy solution is defined as follows:

**Definition 1.3** (stochastic entropy solution). An \(L^2(\mathbb{R}^d)\)-valued \(\{F_t : t \geq 0\}\)-predictable stochastic process \(u(t) = u(t, x)\) is called a stochastic entropy solution of (1.1) provided

1. For each \(T > 0\), \(p = 2, 3, 4, \cdots\),

\[
\sup_{0 \leq t \leq T} E\left[\|u(t, \cdot)\|_p^p\right] < \infty.
\]

2. For all test functions \(0 \leq \psi \in C^{1,2}_c([0, \infty) \times \mathbb{R}^d)\), and each convex entropy pair \((\beta, \zeta)\),

\[
\int_{\mathbb{R}^d} \psi(0, x)\beta(u(0, x))\, dx + \int_0^T \int_{\mathbb{R}^d} \left[ \partial_t \psi(t, x)\beta(u(t, x)) + \zeta(u(t, x)) \cdot \nabla \psi(t, x) \right]\, dx\, dt \\
+ \int_0^T \int_{|z| > 0} \int_{\mathbb{R}^d} \left( \beta(u(t, x) + \eta(u(t, x); z)) - \beta(u(t, x)) \right)\psi(t, x)\, dx\, d\tilde{N}(dz, dt) \\
+ \int_0^T \int_{|z| > 0} \left( \beta(u(t, x) + \eta(u(t, x); z)) - \beta(u(t, x)) - \eta(u(t, x); z)\beta'(u(t, x)) \right)\psi(t, x)\, v(dz)\, dr\, dx \\
\geq 0, \quad P - a.s.
\]
Due to the nonlocal nature of the entropy inequalities and the noise-noise interaction, the Definition 1.3 alone does not seem to give the $L^1$-contraction principle in the sense of average and hence the uniqueness is not immediate. In other words, classical “doubling of variable” technique in time variable does not work when one tries to compare directly two entropy solutions defined in the sense of Definition 1.3. To overcome this problem, the authors in [1,2] used a more direct approach by comparing one entropy solution against the solution of the regularized problem and subsequently sending the regularized parameter to zero, relying on “weak compactness” of the regularized approximations.

In order to successfully implement the direct approach, one needs to weaken the notion of stochastic entropy solution, and subsequently install the notion of so called generalized entropy solution (cf. [1,2]).

**Definition 1.4** (generalized entropy solution). An $L^2(\mathbb{R}^d \times (0,1))$-valued $\{F_t : t \geq 0\}$-predictable stochastic process $v(t) = v(t, x, \alpha)$ is called a generalized stochastic entropy solution of (1.1) provided

1. For each $T > 0$, $p = 2, 3, 4, \ldots$,
   \[
   \sup_{0 \leq t \leq T} E \left[ \|v(t, \cdot, \cdot)\|_p^p \right] < \infty.
   \]

2. For all test functions $0 \leq \psi \in C_{\text{c}}^{1,2}((0, \infty) \times \mathbb{R}^d)$, and each convex entropy pair $(\beta, \zeta)$,
   \[
   \int_{\mathbb{R}^d} \psi(0, x)\beta(v(0, x)) \, dx + \int_0^T \int_{\mathbb{R}^d} \left( \partial_t \psi(t, x)\beta(v(t, x, \alpha)) + \zeta(v(t, x, \alpha)) \cdot \nabla_x \psi(t, x) \right) \, dx \, dt
   \]
   \[
   + \int_{r=0}^T \int_{|z|>0} \int_{|\alpha|>0} \int_{|r|>0} \left( \beta(v(r, x, \alpha) + \eta(v(r, x, \alpha); z)) - \beta(v(r, x, \alpha)) \right) \psi(r, x) \, d\alpha \, d\bar{N}(dz, dr) \, dx
   \]
   \[
   + \int_{r=0}^T \int_{|z|>0} \int_{|\alpha|>0} \int_{|r|>0} \left( \beta(v(r, x, \alpha) + \eta(v(r, x, \alpha); z)) - \beta(v(r, x, \alpha)) - \eta(v(r, x, \alpha); z)\beta'(v(r, x, \alpha)) \right)
   \]
   \[
   \times \psi(r, x) \, d\alpha \, d\nu(v(z, dr) \, d\nu
   \]
   \[
   \geq 0 \quad P - \text{a.s.}
   \]

As we mentioned earlier, in a recent article [2], the authors established well-posedness along with few a priori estimates for the viscous problem with Lévy noise and proved the existence and uniqueness of generalized entropy solution for multidimensional Cauchy problem (1.1) via Young measure approach. Finally, we mention that Dong and Xu [10] established the global well-posedness of strong, weak and mild solutions for one-dimensional viscous Burger’s equation driven by Poisson process with Dirichlet boundary condition via Galerkin method. Also, they proved the existence of invariant measure of the solution.

### 1.3. Scope and outline of this paper

The above discussions clearly highlights the lack of stability estimates for the entropy solutions of stochastic balance laws driven by Lévy noise. In this paper, drawing preliminary motivation from [8], we intend to develop a continuous dependence theory for stochastic entropy solution which in turn can be used to derive an error estimate for the vanishing viscosity method. However, it seems difficult to develop such a theory without securing a BV estimate for stochastic entropy solution. As a result, we first address the question of existence, uniqueness of stochastic BV- entropy solution in $L^p(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ of the problem (1.1). Making use of the crutial BV estimate, we provide a continuous dependence estimate and error estimate for the vanishing viscosity method provided initial data lies in $u_0 \in L^p(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$.

Finally, we turn our discussions to more general stochastic balance laws driven by Lévy processes, namely when the function $\eta$ in the Lévy noise term has explicit dependency on the spatial position $x$ as well. In view of the discussions in [8], in this case we can’t expect BV estimates, but instead a fractional BV estimate is expected. However, that does not prevent us to provide an existence proof for more general class of equations in $L^p(\mathbb{R}^d)$.

The remaining part of this paper is organized as follows: we collect all the assumptions needed in the subsequent analysis, results for the regularized problem and finally state the main results in Section 2.
Section 3, we prove uniform spatial $BV$ estimate for the solution of vanishing viscosity approximation of (1.1), and thereby establishing $BV$ bounds for entropy solutions. Section 4 deals with the continuous dependence estimate, while Section 5 deals with the error estimate. Finally, in Section 6 we establish a fractional $BV$ estimate for a larger class of stochastic balance laws.

2. Preliminaries

We mention that, throughout this paper we use $C, K$ to denote a generic constants; the actual values of $C, K$ may change from one line to the next during a calculation. The Euclidean norm on any $\mathbb{R}^d$-type space is denoted by $| \cdot |$ and the norm in $BV(\mathbb{R}^d)$ is denoted by $| \cdot |_{BV(\mathbb{R}^d)}$.

Next, we collect all the basic assumptions on the data of the problem (1.1).

(A.1) The initial function $u_0(x)$ is a $\cap_{p=1,2,\ldots} L^p(\mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variable satisfying

$$
E[|u_0|^p + |u_0|^p + |u_0|_{BV}^p] < \infty \quad \text{for } p = 1, 2, \ldots .
$$

(A.2) For every $k = 1, 2, \ldots, d$, the functions $F_k(s) \in C^2(\mathbb{R})$, and $F_k(s), F_k'(s)$ and $F_k''(s)$ have at most polynomial growth in $s$.

(A.3) There exist positive constants $0 < \lambda^* < 1$ and $C > 0$, such that for all $u, v \in \mathbb{R}$; $z \in \mathbb{R}$

$$
|\eta(u; z) - \eta(v; z)| \leq \lambda^*|u - v|(|z| \wedge 1)
$$

and

$$
|\eta(u; z)| \leq C(1 + |u|)(|z| \wedge 1).
$$

(A.4) To prove existence and uniqueness of solutions, we assume that the Lévy measure $\nu(dz)$ which has a possible singularity at $z = 0$, satisfies

$$
\int_{|z|>0} (1 \wedge |z|^2) \nu(dz) < +\infty.
$$

Remark 2.1. Note that we need the assumption (A.2) as a result of the requirement that the entropy solutions satisfy $L^p$ bounds for all $p \geq 2$, which in turn forces us to choose initial data satisfying (A.1). However, it is possible to get entropy solution for initial data in $L^2(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, provided the given flux function is globally Lipschitz. The assumption (A.3) is natural in the context of Lévy noise with the exception of $\lambda^* \in (0, 1)$, which is necessary for the uniqueness. Finally, the assumptions (A.1)-(A.4) collectively ensures existence and uniqueness of stochastic entropy solution, and the continuous dependence estimate as well.

To this end, for any given fixed $\epsilon > 0$, we consider the viscous perturbation of (1.1)

$$
du_{\epsilon}(t, x) + \text{div} F_{\epsilon}(u_{\epsilon}(t, x)) dt = \int_{|z|>0} \eta_{\epsilon}(u_{\epsilon}(t, x); z) \tilde{N}(dz, dt) + \epsilon \Delta u_{\epsilon} dt, \quad t > 0, \ x \in \mathbb{R}^d, \quad u(0, x) = u_\epsilon(0, x), \ x \in \mathbb{R}^d,
$$

(2.1)

where $u_\epsilon(0, x)$ is a smooth approximation of initial data $u_0(x)$ such that

$$
E[\int_{\mathbb{R}^d} |u_\epsilon(0, x)|^p dx] \leq E[\int_{\mathbb{R}^d} |u_0(x)|^p dx].
$$

(2.2)

Moreover, if initial data $u_0(x) \in BV(\mathbb{R}^d)$, then

$$
E[\int_{\mathbb{R}^d} |\nabla u_\epsilon(0, x)| dx] \leq E[\int_{\mathbb{R}^d} |\nabla u_0(x)| dx].
$$

(2.3)
Furthermore, mainly to ease the presentation through this paper, we assume that \( F_\varepsilon, \eta_\varepsilon \) are “sufficiently smooth” approximations of \( F \) and \( \eta \) respectively. More specifically, we require that \( F_\varepsilon \) and \( \eta_\varepsilon \) satisfy the same properties as \( F \) and \( \eta \) respectively (cf. \([A.2]-[A.3]\)) and

\[
|F_\varepsilon(r) - F(r)| \leq C\varepsilon(1 + |r|^{p_0}), \quad \text{for some } p_0 \in \mathbb{N},
\]

\[
|\eta_\varepsilon(u;z) - \eta(u;z)| \leq C\varepsilon(1 + |u|(1 \wedge |z|)). \tag{2.4}
\]

Observe that, in view of \([2, \text{Subsection 3.2}]\), these properties of \( F_\varepsilon \) and \( \eta_\varepsilon \) are justified.

For the deterministic counterpart of \((2.1)\), proof of existence of global smooth solutions is classical by now. Same techniques could be used, mutatis mutandis, also for the stochastic scenario to establish the existence. More precisely, we have the following proposition from \([2]\).

**Proposition 2.1.** Let the assumptions \([A.1]-[A.3]\) and \([A.4]\) hold and \( \varepsilon > 0 \) be a given positive number. Then there exists a unique \( C^2(\mathbb{R}^d) \)-valued predictable process \( u_\varepsilon(t, \cdot) \) which solves the initial value problem \((2.1)\). Moreover,

(a) The solution \( u_\varepsilon(t, x) \) satisfies, almost surely,

\[
u_\varepsilon(t, x) = \int_{\mathbb{R}^d} G(t, x - y) u_\varepsilon(0, y) dy - \int_{t_0}^{t} \int_{\mathbb{R}^d} G(t - s, x - y) \nabla \cdot F_\varepsilon(u_\varepsilon(s, y)) dy ds
\]

\[+ \int_{t_0}^{t} \int_{\mathbb{R}^d} G(t - s, x - y) \eta(u_\varepsilon(s, y); z) dy \tilde{N}(dz, ds),
\]

where \( G(t, x) \) is the heat kernel associated with the operator \( \varepsilon \Delta_{xx} \), i.e.,

\[
G(t, x) := G_\varepsilon(t, x) = \frac{1}{(4\pi \varepsilon t)^{d/2}} e^{-\frac{x^2}{4\varepsilon t}}, \quad t > 0.
\]

(b) For positive integer \( p = 1, 2, 3, \ldots \), and \( T > 0 \),

\[
\sup_{\varepsilon > 0} \sup_{0 \leq s \leq T} E\left[ \|u_\varepsilon(t, \cdot)\|_{L_p}^p \right] < \infty. \tag{2.5}
\]

(c) For a function \( \beta \in C^2(\mathbb{R}) \) with \( \beta', \beta'', \beta''' \) having at most polynomial growth,

\[
\sup_{\varepsilon > 0} E \left[ \varepsilon \int_{t_0}^{T} \int_{\mathbb{R}^d} \beta''(u_\varepsilon(t, x)) |\nabla u_\varepsilon(t, x)|^2 dx dt \right] < \infty, \quad p = 1, 2, \ldots, T > 0.
\]

**Remark 2.2.** In view of Proposition 2.1 and assumption \([A.1]\) it follows that, for each fixed \( \varepsilon > 0 \), \( \nabla u_\varepsilon(t, x) \) is integrable. Moreover if \( E \left[ \int_{\mathbb{R}^d} |\nabla^2 u_\varepsilon(0, x)| dx \right] < +\infty \), then \( \nabla^2 u_\varepsilon(t, x) \) is also integrable for fixed \( \varepsilon > 0 \) and any finite time \( T > 0 \) (cf. \([2, \text{Section 3}]\)).

Now we are in a position to state the main results of this article.

**Main Theorem** (continuous dependence estimate). Let the assumptions \([A.1]-[A.3]\) and \([A.4]\) hold for two sets of given data \((u_0, F, \eta)\) and \((v_0, G, \sigma)\). Let \( u(t, x) \) be any entropy solution of \((1.1)\) with initial data \( u_0(x) \) and \( v(s, y) \) be another entropy solution with initial data \( v_0(y) \) and satisfies

\[
dv(s, y) + \div_y G(v(s, y)) ds = \int_{|z| > 0} \sigma(v(s, y); z) \tilde{N}(dz, ds). \tag{2.6}
\]
In addition, we assume that $F''$, $F' - G' \in L^\infty$ and define $D(\eta, \sigma) := \sup_{u \in \mathbb{R}} \int_{\mathbb{R}^d} \frac{(\eta(u; z) - \sigma(u; z))^2}{1 + |u|^2} v(dz)$. Then there exists a constant $C_T > 0$, independent of $|u_0|_{BV(\mathbb{R}^d)}$ and $|v_0|_{BV(\mathbb{R}^d)}$, such that for a.e. $t \geq 0$,

$$
E \left[ \int_{\mathbb{R}^d} |u(t, x) - v(t, x)| \phi(x) \, dx \right] 
\leq C_T \left[ (1 + E[|v_0|_{BV(\mathbb{R}^d)}]) \sqrt{D(\eta, \sigma)} |\phi(\cdot)|_{L^\infty(\mathbb{R}^d)} + E[|v_0|_{BV(\mathbb{R}^d)}] \|F' - G'\|_\infty t \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} 
+ E \left[ \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \phi(x) \, dx \right] + \sqrt{D(\eta, \sigma)} |\phi(\cdot)|_{L^1(\mathbb{R}^d)} \right],
$$

(2.7)

where $0 \leq \phi \in C^2(\mathbb{R}^d)$ such that $|\nabla \phi(x)| \leq C \phi(x)$ and $|\Delta \phi(x)| \leq C \phi(x)$ for some constant $C > 0$. Moreover, a special choice of $\phi(x)$ with the above properties

$$
\phi(x) = \begin{cases} 
1, & \text{when } |x| \leq R, \\
e^{-c(|x|-R)}, & \text{when } |x| \geq R,
\end{cases}
$$

leads to the following simplified result: For any $R > 0$, there exists a constant $C^R_T > 0$, independent of $|u_0|_{BV(\mathbb{R}^d)}$ and $|v_0|_{BV(\mathbb{R}^d)}$, such that for a.e. $t \geq 0$,

$$
E \left[ \int_{|x| \leq R} |u(t, x) - v(t, x)| \, dx \right] 
\leq C^R_T \left[ (1 + E[|v_0|_{BV(\mathbb{R}^d)}]) \sqrt{D(\eta, \sigma)} + t E[|v_0|_{BV(\mathbb{R}^d)}] \|F' - G'\|_\infty + E \left[ \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \, dx \right] \right],
$$

(2.8)

**Remark 2.3.** The condition that $F''$, $F' - G' \in L^\infty$ could be avoided if we assume that $u, v \in L^\infty((0, T) \times \mathbb{R}^d \times \Omega)$ for any time $T > 0$. In this case, an appropriate version of the main theorem would be possible. Moreover, the quantity $D(\eta, \sigma)$ is well defined in view of (A.3) and (A.4).

As a by product of the above theorem, we have the following corollary:

**Main Corollary** (error estimate). Let the assumptions (A.1) (A.2) (A.3) (A.4) hold and let $u(t, x)$ be any entropy solution of (1.1) with $E[|u(t, \cdot)|_{BV(\mathbb{R}^d)}] \leq E[|u_0|_{BV(\mathbb{R}^d)}]$ for $t > 0$. In addition, we assume that $F'' \in L^\infty$. Then, there exists a constant $C_T > 0$, independent of $|u_0|_{BV(\mathbb{R}^d)}$, such that for a.e. $t \geq 0$

$$
E \left[ \int_{\mathbb{R}^d} |u(t, x) - u(t, x)| \, dx \right] 
\leq C_T \left( 1 + E[|u_0|_{BV(\mathbb{R}^d)}] \right) (1 + t) + E \left[ \int_{\mathbb{R}^d} |u_0(0, x) - u_0(x)| \, dx \right].
$$

Moreover, if we assume that the initial error $E[\int_{\mathbb{R}^d} |u_0(0, x) - u_0(x)| \, dx] = O(\epsilon^\gamma)$, then we get

$$
E \left[ \int_{\mathbb{R}^d} |u(t, x) - u(t, x)| \, dx \right] = O(\epsilon^\gamma).
$$

Here we used the notation $O(\epsilon)$ to denote quantities that depend on $\epsilon$ and are bounded above by $Ce$, where $C$ is a constant independent of $\epsilon$.

**Remark 2.4.** We mention that, just like the deterministic case [12], we are able to show that the rate of convergence for vanishing viscosity solution is $\frac{1}{2}$. It is also worth mentioning that this rate is optimal.
We finish this section by introducing a special class of entropy functions which will play a crucial role in the analysis. Let \( \beta : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function satisfying
\[
\beta(0) = 0, \quad \beta(-r) = \beta(r), \quad \beta'(-r) = -\beta'(r), \quad \beta'' \geq 0,
\]
and
\[
\beta'(r) = \begin{cases} -1, & \text{when } r \leq -1, \\ \in [-1, 1], & \text{when } |r| < 1, \\ +1, & \text{when } r \geq 1. 
\end{cases}
\]
For any \( \xi > 0 \), define \( \beta_\xi : \mathbb{R} \to \mathbb{R} \) by
\[
\beta_\xi(r) = \xi \beta \left( \frac{r}{\xi} \right).
\]
Then
\[
|r| - M_1 \xi \leq \beta_\xi(r) \leq |r| \quad \text{and} \quad \beta_\xi''(r) \leq \frac{M_2}{\xi} 1_{|r| < \xi},
\]
where \( 1_A \) denotes the characteristic function of the set \( A \), and
\[
M_1 = \sup_{|r| \leq 1} |r| - \beta(r), \quad M_2 = \sup_{|r| \leq 1} |\beta''(r)|.
\]
Finally, by simply dropping \( \xi \), for \( \beta = \beta_\xi \) we define
\[
F^0_k(a, b) = \int_0^\alpha \beta'(\sigma - b)F'(\sigma)d(\sigma), \quad F^0(a, b) = (F^0_1(a, b), F^0_2(a, b), \ldots, F^0_d(a, b)),
\]
\[
F_k(a, b) = \text{sign}(a - b)(F_k(a) - F_k(b)), \quad F(a, b) = (F_1(a, b), F_2(a, b), \ldots, F_d(a, b)).
\]

3. A priori estimates

In this section, we derive uniform spatial BV bound for the stochastic balance laws driven by Lévy process given by (1.1) under the assumptions (A.1) (A.2) (A.3) and (A.4)

**Theorem 3.1** (spatial bounded variation). Let the assumptions (A.1) (A.2) (A.3) and (A.4) hold. Furthermore, let \( u_\epsilon(t, x) \) be a solution to the initial value problem (2.1). Then, for any time \( t > 0 \)
\[
E \left[ \int_{\mathbb{R}^d} |\nabla u_\epsilon(t, x)| \, dx \right] \leq E \left[ \int_{\mathbb{R}^d} |\nabla u_\epsilon(0, x)| \, dx \right] \leq \Delta \left[ \int_{\mathbb{R}^d} |\nabla u_\epsilon(x)| \, dx \right].
\]

**Proof.** Since \( u_\epsilon(t, x) \) is a smooth solution of the initial value problem (2.1), by differentiating (2.1) with respect to \( x_i \), we find that \( \partial_{x_i} u_\epsilon(t, x), 1 \leq i \leq d \) satisfies the stochastic partial differential equation given by
\[
d(\partial_{x_i} u_\epsilon(t, x)) + \text{div}_x(F'(u_\epsilon(t, x))\partial_{x_i} u_\epsilon(t, x)) \, dt = \int_{|z| > 0} \eta'_{\epsilon}(u_\epsilon(t, x); z) \partial_{x_i} u_\epsilon(t, x)N(dz, \, dt)
\]
\[
+ \epsilon \Delta \partial_{x_i} u_\epsilon(t, x) \, dt.
\]
To proceed further, we apply Itô-Lévy formula to \( \beta_\xi(\partial_{x_i} u_\epsilon(t, x)) \) to obtain
\[
d(\beta_\xi(\partial_{x_i} u_\epsilon(t, x))) + \text{div}_x(F'(u_\epsilon(t, x))\partial_{x_i} u_\epsilon(t, x)) \beta_\xi'(\partial_{x_i} u_\epsilon(t, x)) \, dt
\]
Since $\beta_\xi$ is convex, we conclude that

$$
\epsilon \Delta_{x_1}(\partial_{x_1}u_\xi(t, x)) \beta''_\xi(\partial_{x_1}u_\xi(t, x)) = \epsilon \left( \Delta \beta_\xi(\partial_{x_1}u_\xi(t, x)) - \beta''_\xi(\partial_{x_1}u_\xi(t, x))\|\nabla \partial_{x_1}u_\xi(t, x)\|^2 \right)
\leq \epsilon \Delta \beta_\xi(\partial_{x_1}u_\xi(t, x)),
$$

and for the martingale term, we have

$$
E \left[ \int_0^\tau \int_{|z|>0} \int_0^1 \eta''_\epsilon(u_\epsilon(s, x); z) \partial_{x_1}u_\epsilon(s, s) \beta''_\xi(\partial_{x_1}u_\epsilon(s, s) + \theta \eta''_\epsilon(u_\epsilon(s, x); z) \partial_{x_1}u_\epsilon(s, s)) d\theta \tilde{N}(dz, ds) \right] = 0.
$$

By Remark 2.2, we see that for each fixed $\epsilon > 0$ and $1 \leq i \leq d$, $\nabla \partial_{x_i}u_\epsilon(t, x)$ is integrable. Let $0 \leq \psi(x) \in C^\infty_c(\mathbb{R}^d)$. Multiply (3.1) by $\psi$ and then integrate respect to $x$ to have

$$
E \left[ \int_{\mathbb{R}^d} \beta_\xi(\partial_{x_1}u_\xi(t, x)) \psi(x) dx \right] = E \left[ \int_{\mathbb{R}^d} \beta_\xi(\partial_{x_1}u_\xi(0, x)) \psi(x) dx \right]
\leq E \left[ \int_{\mathbb{R}^d} \int_{s=0}^\tau \int_{|z|>0} \int_0^1 (1 - \theta) \beta''_\xi(\partial_{x_1}u_\epsilon(s, s) + \theta \eta''_\epsilon(u_\epsilon(s, x); z) \partial_{x_1}u_\epsilon(s, s)) \times (\eta''_\epsilon(u_\epsilon(s, x); z) \partial_{x_1}u_\epsilon(s, s)) \psi(x) d\theta v(dz) ds dx \right]
\leq E \left[ \int_{\mathbb{R}^d} \int_{s=0}^\tau \int_{|z|>0} \int_0^1 \nabla \partial_{x_1}u_\epsilon(s, s) \beta''_\xi(\partial_{x_1}u_\epsilon(s, s)) \psi(x) \nabla \partial_{x_1}u_\epsilon(s, s) + \beta_\xi(\partial_{x_1}u_\epsilon(s, s)) \nabla \psi(x) \right]
\leq E \left[ \int_{\mathbb{R}^d} \int_{s=0}^\tau \int_{|z|>0} \int_0^1 \nabla \partial_{x_1}u_\epsilon(s, s) \beta''_\xi(\partial_{x_1}u_\epsilon(s, s)) \psi(x) \nabla \partial_{x_1}u_\epsilon(s, s) \psi(x) d\theta v(dz) ds dx \right]
\leq E \left[ \int_{\mathbb{R}^d} \int_{s=0}^\tau \int_{|z|>0} \int_0^1 \nabla \partial_{x_1}u_\epsilon(s, s) \beta''_\xi(\partial_{x_1}u_\epsilon(s, s)) \psi(x) \nabla \partial_{x_1}u_\epsilon(s, s) \psi(x) d\theta v(dz) ds dx \right].
$$

To proceed further, observe that

$$
\nabla \partial_{x_1}u_\epsilon(s, s) \beta''_\xi(\partial_{x_1}u_\epsilon(s, s)) \psi(x) = \nabla \partial_{x_1}u_\epsilon(s, s) \beta''_\xi(\partial_{x_1}u_\epsilon(s, s)) \psi(x) \nabla \partial_{x_1}u_\epsilon(s, s) + \beta_\xi(\partial_{x_1}u_\epsilon(s, s)) \nabla \psi(x).
$$

Therefore, we obtain from (3.2)
Next, we move on to estimate $E_1(\epsilon, \xi)$, we proceed as follows. Note that we can rewrite $E_1(\epsilon, \xi)$ as

$$E_1(\epsilon, \xi) = E\left[ \int_{\mathbb{R}^d} \beta(y) \partial_x u_s(0, x) \psi(x) \, dx \right] + E_2(\epsilon, \xi) + E_3(\epsilon, \xi) + E_4(\epsilon, \xi).$$

To estimate $E_1(\epsilon, \xi)$, we proceed as follows. Note that we can rewrite $E_1(\epsilon, \xi)$ as

$$E_1(\epsilon, \xi) = E\left[ \int_{\mathbb{R}^d} \int_{t=0}^1 \int_{|z|>0} \int_{\theta=0}^1 (1 - \theta) h^2 \beta''(a + \theta h) \psi(x) \, d\theta \, v(\xi) \, ds \, dx \right],$$

where $a = \partial_x u_s(x, t)$ and $h = \eta_x(u_s(x, t); \xi) \partial_x u_s(x, t)$. In view of the assumption \(A.3\), it is easy to see that

$$h^2 \beta''(a + \theta h) \leq \left| \partial_x u_s(x, t) \right|^2 (1 \wedge |\xi|^2) \beta''(a + \theta h).$$

Next we move on to find a suitable upper bound on $a^2 \beta''(a + \theta h)$. Since $\beta''$ is an even function, without loss of generality we may assume that $a > 0$. Then by our assumption \(A.3\)

$$\partial_x u_s(t, x) + \theta \eta_x(u_s(t, x); \xi) \partial_x u_s(t, x) \geq (1 - \lambda') \partial_x u_s(t, x),$$

for $\theta \in [0, 1]$. In other words

$$0 \leq a \leq (1 - \lambda')^{-1}(a + \theta h).$$

Combining \(3.4\) and \(3.5\) yields

$$h^2 \beta''(a + \theta h) \leq (1 \wedge |\xi|^2) (1 - \lambda')^{-2}(a + \theta h)^2 \beta''(a + \theta h) \leq C(1 \wedge |\xi|^2) \xi.$$

Since by assumption \(A.4\) \(\int_{|z|>0} (1 \wedge |\xi|^2) \nu(\xi) < +\infty\), we infer that

$$|E_1(\epsilon, \xi)| \leq C \epsilon \xi \|\psi\|_{L^2(\mathbb{R}^d)}$$

and hence $E_1(\epsilon, \xi) \rightarrow 0$, as $\xi \downarrow 0$. \(3.6\)

Next, we move on to estimate $E_2(\epsilon, \xi)$. In fact, we have

$$|E_2(\epsilon, \xi)| \leq E\left[ \int_{\mathbb{R}^d} \int_{t=0}^1 \left| \partial_x u_s(x, t) \psi(x) \beta''(\partial_x u_s(x, t)) \right| \|\nabla \partial_x u_s(x, t)\| \|F'_s(u_s(x, t))\| \, ds \, dx \right].$$

First observe that, in view of \(2.9\), we obtain

$$|\partial_x u_s(x, t)| \beta''(\partial_x u_s(x, t)) \leq |\partial_x u_s(x, t)| M_2 \xi \frac{1}{\epsilon} \chi_{|\xi|>0} \left( |\nabla \partial_x u_s(x, t)| \right) \rightarrow 0,$$

almost surely as $\xi \downarrow 0$, and moreover we see that

$$|\partial_x u_s(x, t)| \beta''(\partial_x u_s(x, t)) \psi(x) \|\nabla \partial_x u_s(x, t)\| \|F'_s(u_s(x, t))\| \leq C \|\psi\|_{L^2} \left( \|\nabla \partial_x u_s(x, t)\|^2 + \|u_s(x, t)\|^2 p_0 \right),$$

for some $p_0 \in \mathbb{N}$.

In view of Remark 2.2 and Proposition 2.1 the right-hand side is integrable and independent of $\xi > 0$. Therefore, one can apply dominated convergence theorem to conclude that

$$E_2(\epsilon, \xi) \rightarrow 0,$$

as $\xi \downarrow 0$. \(3.7\)

Next, we consider the term $E_3(\epsilon, \xi)$. With the help of uniform estimates \(2.5\), we conclude

$$|E_3(\epsilon, \xi)| \leq E\left[ \int_{\mathbb{R}^d} \int_{t=0}^1 \left| \partial_x u_s(x, t) \right| \|\nabla \psi(x)\| |F'_s(u_s(x, t))| \, ds \, dx \right].$$
Note that since \((3.11)\) holds for which completes the proof.

To this end, we define \(0\) to obtain \((3.10)\) to obtain

In what follows, we combine all the above estimates \((3.6), (3.7), (3.8), \) and \((3.9)\) and then send \(\xi \to 0\) in \((3.10)\) to obtain

Taking advantage of \((2.9)\) in \((3.3)\) helps us to conclude

In what follows, we combine all the above estimates \((3.6), (3.7), (3.8), \) and \((3.9)\) and then send \(\xi \to 0\) in \((3.10)\) to obtain

To this end, we define \(0 \leq \psi_N(x) \in C_c^2(\mathbb{R}^d)\) such that

Note that since \((3.11)\) holds for \(\psi(x) = \psi_N(x)\), we choose \(\psi(x) = \psi_N(x)\) in \((3.11)\), and then sending \(N \to \infty\) to obtain

which completes the proof.

An important and immediate corollary of the uniform spatial BV estimate is the existence of BV bounds for the entropy solution of \((1.1)\). We have following theorem.

**Theorem 3.2 (BV entropy solution).** Suppose that the assumptions \((A.2)\) \((A.3)\) and \((A.4)\) hold. Then there exists an unique entropy solution of \((1.1)\) with initial data satisfying assumption \((A.1)\) such that

\[
E\left[ |u(t, \cdot)|_{BV(\mathbb{R}^d)} \right] \leq E\left[ |u_0|_{BV(\mathbb{R}^d)} \right], \text{ for any } t > 0.
\]  

**Proof.** We take advantage of the well-posedness results from \([A]\) and claim that the sequence \(\{u_\alpha(t, \cdot)\}\) converges, in the sense of Young measures, to the unique \(L^p(\mathbb{R}^d)\)-valued entropy solution \(u(t, \cdot)\). In view of the uniform BV estimate in Theorem \((3.1)\) by passing to the limit, we conclude \((3.12)\). In other words, the unique \(L^p\)-valued entropy solution has bounded variation if the initial condition is BV. 

\(\square\)
4. Proof of The Main Theorem

It is worth mentioning that, the average $L^1$-contraction principle [see, for example, [2]] gives the continuous dependence on the initial data in stochastic balance laws of the type (1.1). However, we intend to establish continuous dependence also on the nonlinearities, i.e., on the flux function and the noise coefficient. To achieve that, we need to consider the following regularized problem:

$$
\begin{align*}
&\frac{dv_r(s,y) + \text{div}_x G_r(v_r(s,y))}{ds} = \int_{\mathbb{R}^d} \sigma_x(v_r(s,y);z)\tilde{N}(dz, ds) + \varepsilon \Delta_y v_r(s,y) ds, \quad (s,y) \in \Pi_F, \\
&v_r(0,y) = v_0^\varepsilon(y), \quad y \in \mathbb{R}^d;
\end{align*}
$$

where $(v_0^\varepsilon, \sigma_x, G_x)$ are regularized version of $(v_0, \sigma, G)$ satisfying the conditions in (2.6). In view of Theorem 3.2 we conclude that $v_r(s,y)$ converges, as Young measures, to the unique BV-entropy solution $v(s,y)$ of (2.6) with initial data $v_0(y)$. Let $u(t, \cdot)$ be the unique BV-entropy solution of (1.1) with initial data $u_0(x)$. Moreover, we assume that the assumptions [A.1] [A.2] [A.3] and [A.4] hold for both sets of given functions $(v_0, G, \sigma)$ and $(u_0, F, \eta)$.

We estimate the $L^1$-difference between two entropy solutions $u$ and $v$. The theorem will be proved by using the “doubling of variables” technique. However, we can’t directly compare two entropy solutions $u$ and $v$, but instead we first compare the entropy solution $u(t,x)$ with the solution of the viscous approximation (4.1), i.e., $v_r(s,y)$. This approach is somewhat different from the deterministic approach, where one can directly compare two entropy solutions. For deterministic continuous dependence theory consult [4, 5, 13] and references therein.

To begin with, let $\rho$ and $\phi$ be the standard mollifiers on $\mathbb{R}$ and $\mathbb{R}^d$ respectively such that $\text{supp}(\rho) \subset [-1, 0)$ and $\text{supp}(\phi) = B_1(0)$. For $\delta > 0$ and $\delta_0 > 0$, let $\rho_{\delta_0}(r) = \frac{1}{\delta_0} \rho\left(\frac{r}{\delta_0}\right)$ and $\phi_{\delta}(x) = \frac{1}{\delta} \phi\left(\frac{x}{\delta}\right)$. For a nonnegative test function $\psi \in C_{c,1}^2([0, \infty) \times \mathbb{R}^d)$ with $|\nabla \psi(t,x)| \leq C \psi(t,x)$, $|\Delta \psi(t,x)| \leq C \psi(t,x)$ and two positive constants $\delta, \delta_0$, define

$$
\phi_{\delta,\delta_0}(t,x,s,y) = \rho_{\delta_0}(t-s)\phi_{\delta}(x-y)\psi(s,y),
$$

(4.2)

Observe that $\rho_{\delta_0}(t-s) \neq 0$ only if $s - \delta_0 \leq t \leq s$, and therefore $\phi_{\delta,\delta_0}(t,x,s,y) = 0$ outside $s - \delta_0 \leq t \leq s$.

Furthermore, let $\zeta$ be the standard symmetric nonnegative mollifier on $\mathbb{R}$ with support in $[-1,1]$ and $\zeta_l(r) = \frac{1}{l} \zeta\left(\frac{r}{l}\right)$ for $l > 0$. We now write the entropy inequality for $u(t,x)$, based on the entropy pair $(\beta(-k), F^\beta(\cdot, k))$, and then multiply by $\zeta_l(v_r(s,y) - k)$, integrate with respect to $s, y, k$ and take the expectation. The result is

$$
0 \leq E\left[ \int_{\Pi_T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta(u(t,x) - k)\phi_{\delta,\delta_0}(t,x,s,y)\zeta_l(v_r(s,y) - k) dk \, dx \, dy \, ds \right]
$$

$$
+ E\left[ \int_{\Pi_T} \int_{\Pi_T} \int_{B_R} \int_{\mathbb{R}^d} \beta(u(t,x) - k)\phi_{\delta,\delta_0}(t,x,s,y)\zeta_l(v_r(s,y) - k) \, dk \, dx \, dt \, dy \, ds \right]
$$

$$
+ E\left[ \int_{\Pi_T} \int_{B_R} \int_{\Pi_T} \int_{\mathbb{R}^d} \left( \beta(u(t,x) + \eta(u(t,x);z) - k) - \beta(u(t,x) - k) \right) \right.
$$

$$
\times \phi_{\delta,\delta_0}(t,x,s,y) \zeta_l(v_r(s,y) - k) \tilde{N}(dz, dt) \, dx \, dk \, dy \, ds
$$

$$
+ E\left[ \int_{\Pi_T} \int_{\Pi_T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta(u(t,x) + \eta(u(t,x);z) - k) - \beta(u(t,x) - k) \right.
$$

$$
\times \phi_{\delta,\delta_0}(t,x,s,y) \zeta_l(v_r(s,y) - k) \tilde{N}(dz, dt) \, dx \, dk \, dy \, ds
$$

$$
+ E\left[ \int_{\Pi_T} \int_{\Pi_T} \int_{B_R} \int_{\Pi_T} \int_{\mathbb{R}^d} \beta(u(t,x) - k) \cdot \nabla \phi_{\delta}(x-y) \psi(s,y) \rho_{\delta_0}(t-s) \right)
$$

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\[ 0 \leq E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \beta(v_\epsilon(0, y) - k)\phi_{0,\delta_0}(t, x, 0, y)\zeta_t(u(t, x) - k) \, dk \, dx \, dy \, dt \right] \\
+ E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \beta(v_\epsilon(x, y) - k)\partial_s\phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right] \\
+ E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \int_{J_{\mathcal{I}}} \left( \beta(v_\epsilon(x, y) + \sigma_\epsilon(v_\epsilon(x, y); z) - k) - \beta(v_\epsilon(x, y) - k) \right) \right. \\
\times \phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right] \\
+ E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \left( \beta(v_\epsilon(x, y) + \sigma_\epsilon(v_\epsilon(x, y); z) - k) - \beta(v_\epsilon(x, y) - k) \right) \right. \\
\times \zeta_t(u(t, x) - k) \, dk \, dy \, v(dz) \, ds \, dx \, dt \right] \\
+ E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \beta'(v_\epsilon(x, y) - k)\nabla_x\phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right] \\
- \epsilon E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \beta'(v_\epsilon(x, y) - k)\nabla_x\phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right], \quad (4.4) \]

where \( G^\theta(a, b) = \int_a^b \nu'(r) G^\theta(\theta(r)) \, dr \). It follows by direct computations that there is \( p \in \mathbb{N} \) such that

\[ \left| G^\theta(a, b) - G^\theta(a, b) \right| \leq Ce(1 + |a|^{2p} + |b|^{2p}). \]

In view of the uniform moment estimates, it follows from (4.4) that

\[ 0 \leq E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \beta(v_\epsilon(x, y) - k)\phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right] \\
+ E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \beta(v_\epsilon(x, y) - k)\partial_s\phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right] \\
+ E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \left( \beta(v_\epsilon(x, y) + \sigma_\epsilon(v_\epsilon(x, y); z) - k) - \beta(v_\epsilon(x, y) - k) \right) \right. \\
\times \phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right] \\
+ E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \left( \beta(v_\epsilon(x, y) + \sigma_\epsilon(v_\epsilon(x, y); z) - k) - \beta(v_\epsilon(x, y) - k) \right) \right. \\
\times \zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right] \\
+ E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \beta'(v_\epsilon(x, y) - k)\nabla_x\phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right] \\
- \epsilon E \left[ \int_{\Pi_t} \int_{J_{\mathcal{I}}^T} \int_{J_{\mathcal{I}}} \beta'(v_\epsilon(x, y) - k)\nabla_x\phi_{0,\delta_0}(t, x, s, y)\zeta_t(u(t, x) - k) \, dk \, dy \, ds \, dx \, dt \right], \quad (4.4) \]
Lemma 4.1. It holds that

$$\text{Lemma 4.3. It holds that}$$

$$\text{Lemma 4.4.}$$

The following hold:

To begin with, note that particular choice of test function (4.2) implies that

where $C(\beta, \psi)$ is a constant depending only on the quantities in the parentheses. Our aim is to add (4.3) and (4.5), and pass to the limits with respect to the various parameters involved. We do this by claiming a series of lemma's and proofs of these lemmas follow from [2] modulo cosmetic changes.

To begin with, note that particular choice of test function (4.2) implies that $J_1 = 0$.

**Lemma 4.1.** It holds that

$$I_1 + J_1 \xrightarrow{\delta_0 \to 0} E \left[ \int_{\Omega_T} \int_{\Omega_T} \int_{\Omega_T} \beta(u_0(x) - k)\psi(0, y)q_0(x - y)\psi'(0, y)q_0(x - y) \, dk \, dx \, dy \, ds \right]$$

$$\xrightarrow{l \to 0} E \left[ \int_{\Omega_T} \int_{\Omega_T} \beta(u_0(x) - v_0(x) - k)\psi(0, y)q_0(x - y) \, dx \, dy \right].$$

We now turn our attention to $(I_2 + J_2)$. Since $\beta, \psi$ are even functions, we see that

$$I_2 + J_2 = E \left[ \int_{\Omega_T} \int_{\Omega_T} \beta(v_0(s, y) - k)\partial_s\psi(s, y)\rho_0(t - s)q_0(x - y) \times \psi'(u(t, x) - k) \, dk \, dy \, dx \, dt \right].$$

**Lemma 4.2.** It holds that

$$I_2 + J_2 \xrightarrow{\delta_0 \to 0} E \left[ \int_{\Omega_T} \int_{\Omega_T} \int_{\Omega_T} \beta(v_0(s, y) - k)\partial_s\psi(s, y)q_0(x - y)q_0(x - y) \psi'(u(t, x) - k) \, dk \, dy \, ds \right]$$

$$\xrightarrow{l \to 0} E \left[ \int_{\Omega_T} \int_{\Omega_T} \beta(v_0(s, y) - u(s, x) - k)\partial_s\psi(s, y)q_0(x - y) \, dy \, ds \right].$$

Next, we consider the term $I_3 + J_3$ and regarding these terms we have the following lemma.

**Lemma 4.3.** The following hold:

$$\lim_{l \to 0} \lim_{\delta_0 \to 0} I_3 = E \left[ \int_{s=0}^{T} \int_{\Omega_T} \int_{\Omega_T} F_\psi(u(s, x), v_0(s, y)) \cdot \nabla_x q_0(x - y) \psi(s, y) \, dx \, dy \, ds \right]$$

and

$$\lim_{l \to 0} \lim_{\delta_0 \to 0} J_3 = E \left[ \int_{s=0}^{T} \int_{\Omega_T} \int_{\Omega_T} G_\psi(v_0(s, y), u(s, x)) \cdot \nabla_x q_0(x - y) \psi(s, y) \, dx \, dy \, ds \right]$$

**Lemma 4.4.** It holds that

$$J_6 \xrightarrow{\delta_0 \to 0} E \left[ \int_{\Omega_T} \int_{\Omega_T} \int_{\Omega_T} G_\psi(v_0(s, y), k) \cdot \nabla_x q_0(x - y)q_0(x - y) \psi'(u(t, x) - k) \, dk \, dy \, ds \right]$$

$$\xrightarrow{l \to 0} E \left[ \int_{\Omega_T} \int_{\Omega_T} \int_{\Omega_T} G_\psi(v_0(s, y), u(s, x)) \cdot \nabla_x q_0(x - y)q_0(x - y) \psi'(u(t, x) - k) \, dx \, dy \, ds \right].$$
Next, we consider the term $J_7$. Thanks to the uniform spatial $BV$ estimate for vanishing viscosity solution (cf. Theorem 5.1), we conclude that

$$|J_7| \leq \epsilon \|g\|_{\infty} E \left[ \int_{\Omega_T} \int_{\mathbb{R}^d} |\nabla_x v_k(x, y)| \nabla_x [\psi(x, y) g_0(x - y)] \, dx \, dy \, ds \right]$$

$$\leq \epsilon \|g\|_{\infty} E \left[ \int_{[0,t]} \int_{\mathbb{R}^d} |\nabla_x v_k(y, t)| \nabla_x [\psi(t, y) g_0(x - y)] \, dx \, dt \right]$$

$$\leq C \frac{\epsilon}{\delta} E[|\nabla_0 g|_{BV}] \tag{4.8}$$

Lemma 4.5. It holds that

$$\lim \lim_{\delta \to 0, \delta_0 \to 0} J_4 = E \left[ \int_{\Omega_T} \int_{\mathbb{R}^d} \int_{|\zeta| > 0} \int_{j=0}^1 (1 - \lambda) \beta''(v_k(s, y) - u(s, x) + \lambda \sigma(v_k(s, y); z)) \times |\sigma(v_k(s, y); z)|^2 \beta(s, y) g_0(x - y) \, d\lambda \, v(dz) \, dx \, dy \, ds \right]. \tag{4.9}$$

$$\lim \lim_{\delta \to 0, \delta_0 \to 0} I_4 = E \left[ \int_{\Omega_T} \int_{\mathbb{R}^d} \int_{|\zeta| > 0} \int_{j=0}^1 (1 - \lambda) \beta''(u(s, x) - v_k(s, y) + \lambda \eta(u(s, x); z)) \times |\eta(u(s, x); z)|^2 \beta(s, y) g_0(x - y) \, d\lambda \, v(dz) \, dx \, dy \, ds \right]. \tag{4.10}$$

Finally, we consider the stochastic term $I_3 + J_3$;

Lemma 4.6. It holds that $J_3 = 0$ and

$$\lim \lim_{\delta \to 0, \delta_0 \to 0} I_3 = E \left[ \int_{\Omega_T} \int_{\mathbb{R}^d} \int_{|\zeta| > 0} \left( \beta(u(r, x) + \eta(u(r, x); z) - v_k(r, y) - \sigma(v_k(r, y); z)) \right. \right.$$

$$\left. - \beta(u(r, x) - v_k(r, y) - \sigma(v_k(r, y); z)) + \beta(u(r, x) - v_k(r, y)) \right) \psi(r, y) g_0(x - y) \, v(dz) \, dx \, dy \, dr \right].$$

To proceed further, we combine Lemma 4.6 and Lemma 4.5 and conclude that

$$\lim \lim_{\delta \to 0, \delta_0 \to 0} \left( (I_3 + J_3) + (I_4 + J_4) \right)$$

$$= E \left[ \int_{\Omega_T} \int_{\mathbb{R}^d} \left( \int_{|\zeta| > 0} \int_{j=0}^1 \beta''(u(r, x) - v_k(r, y) + \rho(\eta(u(r, x); z) - \sigma(v_k(r, y); z)) \times |\beta''(u(r, x) - v_k(r, y)) - \rho(\eta(u(r, x); z) - \sigma(v_k(r, y); z))| \right) \right.$$

$$\left. \times \beta'(u(r, x) - v_k(r, y)) \right) \psi(r, y) g_0(x - y) \, d\lambda \, v(dz) \, dx \, dy \, dr \right]$$

$$= E \left[ \int_{r=0}^\tau \int_{|\zeta| > 0} \int_{\mathbb{R}^d} \int_{j=0}^1 \beta''(u(r, x) - v_k(r, y) + \rho(\eta(u(r, x); z) - \sigma(v_k(r, y); z))) \times (1 - \rho) \psi(r, y) g_0(x - y) \, d\rho \, dx \, dy \, v(dz) \, dr \right]. \tag{4.11}$$

We are now in a position to add (4.3) and (4.9) and pass to the limits $\lim \lim_{\delta \to 0, \delta_0 \to 0}$ Lemma 4.1 Lemma 4.2 Lemma 4.3 and Lemma 4.4 and the expressions (4.8) and (4.11), we arrive at

$$0 \leq E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta(u(0, x) - v_k(0, y)) \psi(0, y) g_0(x - y) \, dx \, dy \right].$$
Next, by our choice of \( \beta = \beta_\xi \), we have

\[
\left| \frac{\partial}{\partial v} \left( F^\beta(u,v) - F^\beta(v,u) \right) \right| = \left| F'(v)\beta_\xi^\prime(v-u) - F'(v)\beta_\xi^\prime(0) + \int_{x=u}^{x=v} \beta_\xi^\prime(s-v) F''(s) \, ds \right|
\]

\[
= \left| \left( F'(v) - F'(u) \right)\beta_\xi^\prime(u-v) - \int_{x=u}^{x=v} \beta_\xi^\prime(s-v) F''(s) \, ds \right|
\]

\[
= \left| \int_{x=u}^{x=v} \left( \beta_\xi^\prime(u-v) - \beta_\xi^\prime(s-v) \right) F''(s) \, ds \right| \leq M_2 \xi \| F'' \|_\infty. \tag{4.14}
\]

Also from the definition of \( F^\beta \) and \( G^\beta \), it is evident that

\[
\left| \frac{\partial}{\partial v} \left( F^\beta(v,u) - G^\beta(v,u) \right) \right| \leq |F'(v) - G'(v)| \tag{4.15}
\]

Therefore, by (4.14) and (4.15), we obtain

\[
\left| \frac{\partial}{\partial v} \left( F^\beta(u,v) - G^\beta(v,u) \right) \right| \left| F''(v) - G''(v) \right| \leq M_2 \xi \| F'' \|_\infty + |F'(v) - G'(v)| \tag{4.16}
\]

Keeping in mind the estimate (4.16), we proceed further by rewriting the term \( \mathcal{A}_3 \) as

\[
\mathcal{A}_3 = E \left[ \int_{\Omega} \int_{\mathbb{R}^d} \nabla_y v_c(s,y) \cdot \partial_y \left( F^\beta(u,v) - G^\beta(v,u) \right) \psi(s,y) \varphi_\theta(x-y) \, dy \, dx \, ds \right]
\]

Thanks to the uniform spatial BV estimate for vanishing viscosity solution (cf. Theorem 3.1), we conclude that

\[
|\mathcal{A}_3| \leq \left( M_2 \xi \| F'' \|_\infty + \| F' - G' \|_\infty \right) E \left[ \int_{j=0}^{t} \int_{\mathbb{R}^d} |\nabla_y v_c(s,y)\psi(s,y)\varphi_\theta(x-y) \, dx \, dy \, ds \right]
\]

\[
\leq E \left[ |v_0|_{BV(\mathbb{R}^d)} \right] \left( M_2 \xi \| F'' \|_\infty + \| F' - G' \|_\infty \right) \int_{j=0}^{t} \|\psi(s, \cdot)\|_{L^\infty(\mathbb{R}^d)} \, ds. \tag{4.17}
\]
Next, we recall that the function $\psi(t, x)$ satisfies $|\nabla \psi(t, x)| \leq C \psi(t, x)$ and $|F'(a, b)| \leq \|F'\|_{\infty} |a - b|$ for any $a, b \in \mathbb{R}$. Therefore, we conclude

$$|\mathcal{A}_4| \leq C \|F'\|_{L^\infty} E \left[ \int_{t=0}^T \int_{s \geq t} \int_{s \geq r} \int_{T}^1 |u(s, x) - \eta(v(s, y), y)| \psi(s, y) \rho_0(x - y) \, dx \, dy \, ds \right]$$

$$\leq C \|F'\|_{L^\infty} E \left[ \int_{t=0}^T \int_{s \geq t} \int_{s \geq r} \int_{T}^1 \beta_r(u(s, x) - \eta(v(s, y), y)) \psi(s, y) \rho_0(x - y) \, dx \, dy \, ds \right]$$

$$+ C M_1 \|F'\|_{L^\infty} \int_{t=0}^T \|\psi(s, \cdot)\|_{L^\infty(\mathbb{R})} \, ds. \quad (4.18)$$

Let us focus on the term $\mathcal{A}_5$. For this, let us define

$$a := u(r, x) - v_e(r, y), \quad \text{and} \quad b := \eta(u(r, x); z) - \sigma_r(v_e(r, y); z).$$

Then $\mathcal{A}_5$ can be rewritten in the following simplified form

$$\mathcal{A}_5 = E \left[ \int_{t=0}^T \int_{s \geq t} \int_{s \geq r} \int_{T}^1 (1 - \rho) b \beta''(a + \rho b) \psi(s, y) \rho_0(x - y) \, dp \, dx \, dy \, v(dz) \, dr \right]$$

$$\leq C E \left[ \int_{t=0}^T \int_{s \geq t} \int_{s \geq r} \int_{T}^1 \left| \eta(u(r, x); z) - \sigma(u(r, x); z) \right|^2 \beta''(a + \rho b) \right.$$ 

$$\times \psi(r, y) \rho_0(x - y) \, dp \, dx \, dy \, v(dz) \, dr \right]$$

$$+ C E \left[ \int_{t=0}^T \int_{s \geq t} \int_{s \geq r} \int_{T}^1 \left| \sigma(u(r, x); z) - \sigma(v_e(r, y); z) \right|^2 \beta''(a + \rho b) \right.$$ 

$$\times \psi(r, y) \rho_0(x - y) \, dp \, dx \, dy \, v(dz) \, dr \right]$$

$$+ C E \left[ \int_{t=0}^T \int_{s \geq t} \int_{s \geq r} \int_{T}^1 \left| \sigma(v_e(r, y); z) - \sigma_r(v_e(r, y); z) \right|^2 \beta''(a + \rho b) \right.$$ 

$$\times \psi(r, y) \rho_0(x - y) \, dp \, dx \, dy \, v(dz) \, dr \right]$$

$$:= \mathcal{A}_5^1 + \mathcal{A}_5^2 + \mathcal{A}_5^3. \quad (4.19)$$

To this end we recall that $D(\eta, \sigma) = \sup_{u \in \mathbb{R}} \int_{\mathbb{R}} \frac{|\eta(u, z) - \sigma(u, z)|^2}{1 + |u|^2} \, v(dz)$, which is well-defined in view of (A.5).

With this quantity at hand it is easy see that

$$\mathcal{A}_5^1 \leq \frac{CD(\eta, \sigma)}{\xi} E \left[ \int_{t=0}^T \int_{s \geq t} \int_{s \geq r} (1 + |u(r, x)|^2) \psi(r, y) \rho_0(x - y) \, dy \, dx \, dr \right]$$

$$\leq \frac{CD(\eta, \sigma)}{\xi} \left( \int_0^T \|\psi(s, \cdot)\|_{L^1} \, ds + \int_0^T \|\psi(s, \cdot)\|_{L^\infty} \, ds \right). \quad (4.20)$$

Next, we move on to estimate the term $\mathcal{A}_5^2$. Observe that

$$\left| \sigma(u(r, x); z) - \sigma(v_e(r, y); z) \right|^2 \beta''(a + \rho b) \leq \left| u(r, x) - v_e(r, y) \right|^2 (1 + |z|^2) \beta''(a + \rho b)$$

$$= (1 + |z|^2) a^2 \beta''(a + \rho b). \quad (4.21)$$

Therefore, it is required to find a suitable upper bound on $a^2 \beta''(a + \rho b)$. Since $\beta''$ is non-negative and symmetric around zero, without loss of generality, we may assume that $a > 0$. Then, by our assumption (A.5) we conclude that

$$\left| \eta(u(r, x); z) - \sigma(v_e(r, y); z) \right| \leq C \left| u(r, x) - v_e(r, y) \right|.$$
Finally, we make use of the estimates (4.13), (4.17), (4.18) and (4.25) in (4.12) and pass to the limit as

$$
\lim_{\epsilon \to 0} \int_{|z| > \delta} |\sigma(v_r(x, r); z) - \sigma(v_r(x, r); z)| + C\epsilon(1 + |v_r|),
$$

which implies that

$$
a + \rho b \geq -|\sigma(u(x, r); z)| - \sigma(u(x, r); z)| - C\epsilon(1 + |v_r|) + (1 - \lambda^*)a,
$$

for $\rho \in [0, 1]$. In other words

$$
0 \leq a \leq (1 - \lambda^*)^{-1}\left[a + \rho b + |\sigma(u(x, r); z)| - \sigma(u(x, r); z)| + C\epsilon(1 + |v_r|)\right].
$$

Now, we shall make use of (4.22) in (4.21), to obtain

$$
|\sigma(u(x, r); z) - \sigma(v_r(y, z); z)|^2 \beta'_\epsilon(a + \rho b)
\leq (1 - \lambda^*)^{-2}[\rho |a + \rho b|^2 + C|\sigma(u(x, r); z)|^2 + C\epsilon^2 (1 + |v_r|^2)](1 + |\epsilon|^2)\beta''_\epsilon(a + \rho b)
\leq C(\xi + \frac{\epsilon^2}{\xi})\int_\mathbb{R}^d \left|\psi(s, \cdot)\right|_{L^\infty(\mathbb{R}^d)} ds + \frac{C\mathcal{D}(\eta, \sigma)}{\xi} \left(\int_0^T \left|\psi(s, \cdot)\right|_{L^1} ds + \int_0^T \left|\psi(s, \cdot)\right|_{\infty} dr\right).
$$

This helps us to conclude

$$
\mathcal{A}_3 \leq C \mathcal{E} \left[\int_\mathbb{R}^d \left(\xi + \frac{\epsilon^2}{\xi}\right) \left(1 + |z|^2\right) \psi(r, y) \mathcal{Q}_0(x - y) dx dy m(dz) dr\right]
+ \frac{\mathcal{D}(\eta, \sigma)}{\xi} \int_\mathbb{R}^d \int_\mathbb{R}^d (1 + |u(x, r)|^2) \psi(r, y) \rho_0(x - y) dx dy dr
\leq C(\xi + \frac{\epsilon^2}{\xi}) \int_\mathbb{R}^d \left|\psi(s, \cdot)\right|_{L^\infty(\mathbb{R}^d)} ds + \frac{C\mathcal{D}(\eta, \sigma)}{\xi} \left(\int_0^T \left|\psi(s, \cdot)\right|_{L^1} ds + \int_0^T \left|\psi(s, \cdot)\right|_{\infty} dr\right).
$$

Next, we move on to estimate the term $\mathcal{A}_5$. In fact, it follows that

$$
\mathcal{E} \int_\mathbb{R}^d \int_\mathbb{R}^d \int_\mathbb{R}^d \int_\mathbb{R}^d |\sigma(v_r(y, z); z) - \sigma_r(v_r(y, r); z)|^2 \beta''(a + \rho b)
\times \psi(r, y) \mathcal{Q}_0(x - y) d\rho dy dz d\tau
\leq C\mathcal{E} \int_\mathbb{R}^d \int_\mathbb{R}^d \left(\xi + \frac{\epsilon^2}{\xi}\right) \left(1 + |z|^2\right) \psi(r, y) \mathcal{Q}_0(x - y) dx dy d\tau
\leq C\frac{\epsilon^2}{\xi} \int_\mathbb{R}^d \left|\psi(s, \cdot)\right|_{L^\infty(\mathbb{R}^d)} ds.
$$

(4.24)

We now make use of the estimates (4.20), (4.23) and (4.24). Then it is evident from (4.19) that

$$
|\mathcal{A}_5| \leq \frac{C\mathcal{D}(\eta, \sigma)}{\xi} \left(\int_0^T \left|\psi(s, \cdot)\right|_{L^1} ds + \int_0^T \left|\psi(s, \cdot)\right|_{\infty} dr\right)
+ C(\xi + \frac{\epsilon^2}{\xi}) \int_\mathbb{R}^d \left|\psi(s, \cdot)\right|_{L^\infty(\mathbb{R}^d)} ds.
$$

(4.25)

Finally, we make use of the estimates (4.13), (4.17), (4.18) and (4.25) in (4.12) and pass to the limit as $\epsilon \to 0$

(keeping $\delta$ and $\xi$ fixed) in the resulting expression to conclude that

$$
0 \leq E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v_0(y) - u(0, x)| \phi(0, y) \mathcal{Q}_0(x - y) dx dy
$$

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Now we can safely pass the limit as \( \delta \to 0 \) in (4.26) to obtain

\[
0 \leq E\left[ \int_{\mathbb{R}^d} |v_0(x) - u(0, x)| \psi(0, x) \, dx \right]
\]

\[
+ E\left[ |u_0|_{BV(\mathbb{R}^d)} \right] \left( M_2 \xi \|F''\|_{L^\infty} + \|F' - G'\|_{L^\infty} \right) \int_{s=0}^{T} \|\psi(s, \cdot)\|_{L^\infty(\mathbb{R}^d)} \, ds
\]

\[
+ C\|F'\|_{L^\infty} E\left[ \int_{s=0}^{T} \int_{\mathbb{R}^d} \beta_\xi (v(s, x) - u(s, x)) \psi(s, x) \, dx \, ds \right]
\]

\[
+ C(M_1 \|F''\|_{L^\infty} + 1) \xi \int_{s=0}^{T} \|\psi(s, \cdot)\|_{L^\infty(\mathbb{R}^d)} \, ds
\]

\[
+ \frac{C\mathcal{D} (\eta, \sigma)}{\xi} \left( \int_{0}^{T} \|\psi(s, \cdot)\|_{L^1} \, ds + \int_{0}^{T} \|\psi(r, \cdot)\|_{L^\infty} \, dr \right)
\]

\[
+ E\left[ \int_{\mathbb{R}_T^d} \beta_\xi (u(s, x) - v(s, x)) \partial_x \psi(s, x) \, dx \, ds \right].
\]  

(4.27)

To proceed further, we make a special choice for the function \( \psi(t, x) \). To this end, for each \( h > 0 \) and fixed \( t \geq 0 \), we define

\[
\psi_h(s) = \begin{cases} 
1, & \text{if } s \leq t, \\
1 - \frac{s - t}{h}, & \text{if } t \leq s \leq t + h, \\
0, & \text{if } s \geq t + h.
\end{cases}
\]

Furthermore, let \( \phi \in C_0^2(\mathbb{R}^d) \) be a cut-off function such that \( \|\nabla \phi(x)\| \leq C \phi(x) \), \( |\Delta \phi(x)| \leq C \phi(x) \). Clearly, (4.27) holds with \( \psi(s, x) = \psi_h(s) \phi(x) \). Let \( \mathcal{T} \) be the set all points \( t \in [0, \infty) \) such that \( t \) is right Lebesgue point of

\[
A(s) = E\left[ \int_{\mathbb{R}^d} \beta_\xi (v(s, x) - u(s, x)) \phi(x) \, dx \right].
\]

Clearly, \( \mathcal{T}^c \) (complement of \( \mathcal{T} \)) has zero Lebesgue measure. Fix \( t \in \mathcal{T} \). Then from (4.27), keeping in mind that we used generic \( \beta \) for the function \( \beta_\xi \), we obtain

\[
0 \leq E\left[ \int_{\mathbb{R}^d} |v_0(x) - u(0, x)| \psi(0, x) \, dx \right]
\]

\[
+ E\left[ |v_0|_{BV(\mathbb{R}^d)} \right] \left( M_2 \xi \|F''\|_{L^\infty} + \|F' - G'\|_{L^\infty} \right) \int_{s=0}^{T} \psi_h(s) \, ds
\]

\[
+ C\|F''\|_{L^\infty} E\left[ \int_{s=0}^{T} \int_{\mathbb{R}^d} \beta_\xi (v(s, x) - u(s, x)) \psi_h(s) \phi(x) \, dx \, ds \right]
\]

\[
+ \frac{C\mathcal{D} (\eta, \sigma)}{\xi} \left( \int_{0}^{T} \psi_h(s) \, ds + \int_{0}^{T} \phi(\cdot) \|\psi(r, \cdot)\|_{L^\infty} \, dr \right)
\]

\[
+ E\left[ \int_{\mathbb{R}_T^d} \beta_\xi (u(s, x) - v(s, x)) \partial_x \psi(s, x) \, dx \, ds \right].
\]

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for almost every \( t \). The proof, and second part follows from this by exploiting the specific structure of the test function \( \phi(x) \).

Since \( t \) is a right Lebesgue point of \( A(s) \), letting \( h \to 0 \) in (4.28) yields

\[
E \left[ \int_{\mathbb{R}^d} \beta_\xi(u(t, x) - v(t, x))\phi(x) \, dx \right] 
\leq E \left[ \int_{\mathbb{R}^d} |v_0(x) - u(0, x)|\phi(x) \, dx \right] + C(M_1 \|F'\|_{L^\infty} + 1) \xi \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} \, t 
+ E \left[ |v_0|_{BV(\mathbb{R}^d)} \right] \left( M_2 \xi \|F''\|_{L^\infty} + \|F' - G'\|_{L^\infty} \right) \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} \, t 
+ \frac{C_t \xi}{\xi} \left( \|\phi\|_{L^1} + \|\phi\|_{L^\infty} \right)
\]

for almost every \( t > 0 \). An weaker version of Gronwall’s inequality then yields

\[
E \left[ \int_{\mathbb{R}^d} \beta_\xi(u(t, x) - v(t, x))\phi(x) \, dx \right] 
\leq e^{C_t \|F'\|_{L^\infty}} E \left[ \int_{\mathbb{R}^d} |v_0(x) - u(0, x)|\phi(x) \, dx \right] 
+ C e^{C_t \|F'\|_{L^\infty}} \left( M_1 \|F''\|_{L^\infty} + 1) \xi \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} \, t 
+ E \left[ |v_0|_{BV(\mathbb{R}^d)} \right] \left( M_2 \xi \|F''\|_{L^\infty} + \|F' - G'\|_{L^\infty} \right) \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} \, t 
+ \frac{C_t \xi}{\xi} \left( \|\phi\|_{L^1} + \|\phi\|_{L^\infty} \right)
\]

for almost every \( t > 0 \). Next, we recall that \( |r| \leq \beta_\xi(r) + M_1 \xi \), for any \( r \in \mathbb{R} \). Using this inequality, (4.29) reduces to

\[
E \left[ \int_{\mathbb{R}^d} \beta_\xi(u(t, x) - v(t, x))\phi(x) \, dx \right] 
\leq e^{C_t \|F'\|_{L^\infty}} E \left[ \int_{\mathbb{R}^d} |v_0(x) - u(0, x)|\phi(x) \, dx \right] 
+ C e^{C_t \|F'\|_{L^\infty}} \left( M_1 \|F''\|_{L^\infty} + 1) \xi \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} \, t 
+ E \left[ |v_0|_{BV(\mathbb{R}^d)} \right] \left( M_2 \xi \|F''\|_{L^\infty} + \|F' - G'\|_{L^\infty} \right) \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} \, t 
+ \frac{C_t \xi}{\xi} \left( \|\phi\|_{L^1} + \|\phi\|_{L^\infty} \right)
\]

(4.30)

We now simply choose \( \xi = \sqrt{\xi} \) and conclude that for a.e \( t > 0 \)

\[
E \left[ \int_{\mathbb{R}^d} |u(t, x) - v(t, x)|\phi(x) \, dx \right] 
\leq C_T E \left[ \int_{\mathbb{R}^d} |u_0(x) - v_0(x)|\phi(x) \, dx \right] + E \left[ |v_0|_{BV(\mathbb{R}^d)} \right] \|F' - G'\|_{L^\infty} \, t \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} 
+ C_T \left[ \left( 1 + E[|v_0|_{BV(\mathbb{R}^d)}] \right) \sqrt{\xi} \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} + \sqrt{\xi} \|\phi(\cdot)\|_{L^\infty(\mathbb{R}^d)} \right]
\]

for some nonnegative constant \( C_T \), independent of \( |u_0|_{BV(\mathbb{R}^d)} \) and \( |v_0|_{BV(\mathbb{R}^d)} \). This completes the first part of the proof, and second part follows from this by exploiting the specific structure of the test function \( \phi(x) \).
5. Proof of The Main Corollary

It is already known that the vanishing viscosity solutions converge (in an appropriate sense) to the unique entropy solution of the stochastic conservation law. However, the nature of such convergence described by a rate of convergence is not available. As a byproduct of the Main Theorem, we explicitly obtain the rate of convergence of vanishing viscosity solutions to the unique $BV$-entropy solution of the underlying problem \( \text{(1.1)} \).

By similar arguments as in the proof of the Main Theorem (cf. Section 4), we arrive at

\[
0 \leq E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} |u_e(0, y) - u_0(0, y)| \psi_0(x - y) \, dx \, dy \right] 
\]

\[
+ E\left[ |u_0|_{BV(\mathbb{R}_t^2)} \right] M_2 \xi \|F''\|_{\infty} \int_{s=0}^{T} \|\psi(s, \cdot)\|_{L^\infty(\mathbb{R}_t)} \, ds + C \frac{\varepsilon^2}{\xi} \int_{s=0}^{T} \|\psi(s, \cdot)\|_{L^\infty(\mathbb{R}_t)} \, ds
\]

\[
+ C\|F''\|_{L^2} E\left[ \int_{s=0}^{T} \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} \beta_2(u_e(s, y) - u(s, x))\psi(s, y)\psi_0(x - y) \, dy \, dx \, ds \right]
\]

\[
+ C(M_1 \|F''\|_{L^2} + 1) \xi \int_{s=0}^{T} \|\psi(s, \cdot)\|_{L^\infty(\mathbb{R}_t)} \, ds + C \left(1 + E\left[ |u_0|_{BV(\mathbb{R}_t^2)} \right]\right) \frac{\varepsilon}{\delta}
\]

\[
+ E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} \beta_2(u_e(s, y) - u(s, x))\partial_t \psi(s, y)\psi_0(x - y) \, dx \, dy \, ds \right],
\] (5.1)

Let \( \psi(s, y) = \psi_h(s)\phi(y) \) where \( \psi_h(s) \) and \( \phi(x) \) are described previously. Let \( \mathcal{T} \) be the set all points \( t \) in \([0, \infty)\) such that \( t \) is right Lebesgue point of

\[
B(s) = E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} \beta_2(u_e(s, y) - u(s, x))\phi(y)\psi_0(x - y) \, dx \, dy \right].
\]

Clearly, \( \mathcal{T} \) has zero Lebesgue measure. Fix \( t \in \mathcal{T} \). Thus, from (5.1), we have

\[
\frac{1}{h} \int_{s=t}^{s+t+h} E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} \beta_2(u_e(s, y) - u(s, x))\phi(y)\psi_0(x - y) \, dx \, dy \right] \, ds
\]

\[
\leq C\|F''\|_{L^2} \int_{s=0}^{T} E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} \phi(y)\beta_2(u_e(s, y) - u(s, y))\psi_0(x - y) \, dx \, dy \right] \, ds
\]

\[
+ E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} |u_e(0, y) - u_0(0, y)| \phi(y)\psi_0(x - y) \, dx \, dy \right]
\]

\[
+ C E\left[ |u_0|_{BV(\mathbb{R}_t^2)} \right] M_2 \xi \|F''\|_{\infty} \int_{s=0}^{T} \psi_h(s) \, ds + C \frac{\varepsilon^2}{\xi} \|\phi(\cdot)\|_{L^\infty(\mathbb{R}_t)} \int_{s=0}^{T} \psi_h(s) \, ds
\]

\[
+ C \|F''\|_{L^2} \int_{s=0}^{T} \psi_h(s) \, ds + C \left(1 + E\left[ |u_0|_{BV(\mathbb{R}_t^2)} \right]\right) \frac{\varepsilon}{\delta}.
\]

Taking limit as \( h \to 0 \), we have

\[
E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} \beta_2(u_e(t, y) - u(t, x))\phi(y)\psi_0(x - y) \, dx \, dy \right]
\]

\[
\leq C\|F''\|_{L^2} \int_{s=0}^{T} E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} \phi(y)\beta_2(u_e(s, y) - u(s, x))\psi_0(x - y) \, dx \, dy \right] \, ds
\]

\[
+ E\left[ \int_{\mathbb{R}_t^2} \int_{\mathbb{R}_t^2} |u_e(0, y) - u_0(0, y)| \phi(y)\psi_0(x - y) \, dx \, dy \right] + C \left(1 + E\left[ |u_0|_{BV(\mathbb{R}_t^2)} \right]\right) \frac{\varepsilon}{\delta}
\]

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+ C\left(1 + E[|u_0|_{BV(R^d)}]\right)\xi\|\phi(\cdot)\|_{L^\infty(R^d)}t + C_{\xi}^2\|\phi(\cdot)\|_{L^\infty(R^d)}t

By an weaker version of Gronwall’s inequality, for a.e \( t > 0 \)

\[
E\left[\int_0^t \int_{R^d} \beta\left(u_e(t, y) - u(t, x)\right)\mathcal{Q}_\delta(x - y) \, dx \, dy\right] 
\leq e^{C\|F\|_{L^\infty}} \left\{ E\left[\int_0^t \int_{R^d} |u_e(0, y) - u_0(x)|\mathcal{Q}_\delta(x - y) \, dx \, dy\right] + C\left(1 + E[|u_0|_{BV(R^d)}]\right)\frac{\xi}{\delta}\right\} 
+ C e^{C\|F\|_{L^\infty}} \left\{ \left(1 + E[|u_0|_{BV(R^d)}]\right)\xi\|\phi(\cdot)\|_{L^\infty(R^d)}t + \frac{\epsilon^2}{\xi}\|\phi(\cdot)\|_{L^\infty(R^d)}t\right\} + C\xi\|\phi(\cdot)\|_{L^\infty(R^d)}.
\] (5.2)

First we send \( \phi \) to \( \chi_{R^d} \) in (5.2), and then choose \( \xi = \epsilon \). The resulting estimate gives

\[
E\left[\int_0^t \int_{R^d} |u_e(t, y) - u(t, x)|\mathcal{Q}_\delta(x - y) \, dx \, dy\right] 
\leq e^{C\|F\|_{L^\infty}} \left\{ E\left[\int_0^t \int_{R^d} |u_e(0, y) - u_0(x)|\mathcal{Q}_\delta(x - y) \, dx \, dy\right] + C\left(1 + E[|u_0|_{BV(R^d)}]\right)\frac{\epsilon}{\delta}\right\} 
+ C e^{C\|F\|_{L^\infty}} \left\{ \left(1 + E[|u_0|_{BV(R^d)}]\right)\epsilon t + C\epsilon\right\}.
\] (5.3)

Notice that,

\[
E\left[\int u_e(t, y) - u(t, y) \, dy\right] 
\leq E\left[\int \int |u_e(t, y) - u(t, x)|\mathcal{Q}_\delta(x - y) \, dx \, dy\right] + E\left[\int \int |u(t, x) - u(t, y)|\mathcal{Q}_\delta(x - y) \, dx \, dy\right]
\leq\left[\int \int |u_e(t, y) - u(t, x)|\mathcal{Q}_\delta(x - y) \, dx \, dy\right] + \delta E[|u_0|_{BV(R^d)}],
\] (5.4)

and

\[
E\left[\int \int |u_e(0, y) - u_0(x)|\mathcal{Q}_\delta(x - y) \, dx \, dy\right] 
\leq E\left[\int \int |u_e(0, x) - u_0(x)| \, dx\right] + \delta E[|u_0|_{BV(R^d)}].
\] (5.5)

We combine (5.4) and (5.5) in (5.3) to conclude

\[
E\left[\int \int |u_e(t, y) - u(t, y)| \, dy\right] 
\leq e^{C\|F\|_{L^\infty}} \left\{ E\left[\int \int |u_e(0, y) - u_0(y)| \, dy\right] + C\left(1 + E[|u_0|_{BV(R^d)}]\right)\frac{\epsilon}{\delta} + \delta E[|u_0|_{BV(R^d)}]\right\}
\]
+ C e^{C|F|L^1} t(1 + E[|u_0|_{BV(R^d)}]) e^t + C e + \delta E[|u_0|_{BV(R^d)}]. \tag{5.6}

We choose \( \delta = \epsilon^2 \) in (5.6), and conclude that, for a.e \( t > 0 \),

\[
E\left[ \int_{\mathbb{R}^d} |u_t(t, x) - u(t, x)| \, dx \right] \\
\leq C(T)\left( \epsilon^2 (1 + E[|u_0|_{BV(R^d)}]) (1 + t) + E\left[ \int_{\mathbb{R}^d} |u_t(0, x) - u_0(x)| \, dx \right] \right) 
\]

for some constant \( C(T) > 0 \), independent of \( E[|u_0|_{BV(R^d)}] \). This completes the proof.

6. Fractional BV Estimates

In this section, we consider a more general class of stochastic balance laws driven by Lévy noise of the type

\[
\begin{aligned}
&\left\{ du(t, x) + \text{div}_x F(u(t, x)) \, dt = \int_{|z|>0} \eta(x, u(t, x); z) \, \tilde{N}(dz, dt), \quad x \in \Pi_T, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R}^d.
\end{aligned} \tag{6.1}
\]

Observe that, the noise coefficient \( \eta(x, u; z) \) depends explicitly on the spatial position \( x \). Moreover, we assume that \( \eta(x, u; z) \) satisfies the following assumptions:

(B.1) There exist positive constants \( K > 0 \) and \( \lambda^* \in (0, 1) \) such that

\[
|\eta(x, u; z) - \eta(y, v; z)| \leq (\lambda^* |u - v| + K|x - y|)|z| \wedge 1, \quad \text{for all } u, v \in \mathbb{R}; \quad z \in \mathbb{R}; \quad x, y \in \mathbb{R}^d.
\]

(B.2) There exists a non-negative function \( g(x) \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) such that

\[
|\eta(x, u; z)| \leq g(x)(1 + |u||z| \wedge 1), \quad \text{for all } (x, u, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}.
\]

Clearly, our continuous dependence estimate is not applicable for problems of type (6.1), and primary reason for that lies in the nonavailability of BV estimate here. We refer to [3, Section 2] for discussion on this point for diffusion driven balance laws. However, it is possible to obtain a fractional BV estimate. To that context, drawing primary motivation from the discussions in [6], we intend to show that a uniform fractional BV estimate can be obtained for the solution of the regularized stochastic parabolic problem given by

\[
du_t(t, x) + \text{div}_x F(u_t(t, x)) \, dt = \int_{|z|>0} \eta_t(x, u_t(t, x); z) \tilde{N}(dz, dt) + \epsilon \Delta_x u_t(t, x) \, dt, \tag{6.2}
\]

where \( F, \eta_t \) satisfy (2.4). Regarding equation (6.2), we mention that existence and regularity of the solution to the problem (6.2) has been studied in [2]. We start with a deterministic lemma, related to the estimation of the modulus of continuity of a given integrable function, and also an useful link between Sobolev and Besov spaces. In fact, we have the following lemma, a proof of which can be found in [4, Lemma 2].

**Lemma 6.1.** Let \( h : \mathbb{R}^d \to \mathbb{R} \) be a given integrable function, \( 0 \leq \phi \in C_c^\infty(\mathbb{R}^d) \) and \( \{J_\delta\}_{\delta > 0} \) be a sequence of symmetric mollifiers, i.e., \( J_\delta(x) = \frac{1}{\delta^d} J\left(\frac{x}{\delta}\right) \), \( 0 \leq J \in C_c^\infty(\mathbb{R}) \), \( \text{supp}(J) \subset [-1, 1], J(-\cdot) = J(\cdot) \) and \( \int J = 1 \). Then

(a) For \( r, s \in (0, 1) \) with \( r < s \), there exists a finite constant \( C_1 = C_1(J, d, r, s) \) such that

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h(x + z) - h(x - z)|J_\delta(z)\phi(x) \, dz \, dx \\
\leq C_1 \delta^r \sup_{|z| \leq \delta} |z|^{-r} \int_{\mathbb{R}^d} |h(x + z) - h(x - z)|\phi(x) \, dx. \tag{6.3}
\]
Moreover, a simple calculation reveals that

\[ \sup_{|y| \leq 1} \int_{\mathbb{R}^d} |h(x + z) - h(x)| \phi(x) \, dx \leq C_2 \delta^r \sup_{0 < \delta \leq 1} \delta^{-s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h(x + z) - h(x - z)| J_\delta(z) \phi(x) \, dx \, dz + C_2 \delta^r \|h\|_{L^1(\mathbb{R}^d)}. \]  

(6.4)

Now we are in a position to state and prove a theorem regarding fractional BV estimation of solutions of (6.2).

**Theorem 6.2** (Fractional BV estimate). Let the assumptions **(A.1)**, **(A.2)**, **(B.1)**, **(B.2)** and **(A.4)** hold. Let \( u_\varepsilon \) be a solution of (6.2) with the initial data \( u_0(x) \) belongs to the Besov space \( B^\mu_{\infty, \infty}(\mathbb{R}^d) \) for some \( \mu \in (\frac{1}{2}, 1) \). Moreover, we assume that \( F''_\varepsilon \in L^\infty \). Then, for fixed \( T > 0 \) and \( R > 0 \), there exits a constant \( C(T, R) \), independent of \( \varepsilon \), such that for any \( 0 < t < T \),

\[ \sup_{|t| \leq \delta} \mathbb{E} \left[ \int_{x \in K_R} |u_\varepsilon(t, x + y) - u_\varepsilon(t, x)| \, dx \right] \leq C(T, R) \delta^r, \]

for some \( r \in (0, \frac{1}{2}) \) and \( K_R := \{x : |x| \leq R\} \).

**Proof.** Let \( 0 \leq \phi(x) \in C^\infty_0(\mathbb{R}^d) \) be any test function such that \( |\nabla \phi(x)| \leq C \phi(x) \) and \( |\Delta \phi(x)| \leq C \phi(x) \) for some constant \( C > 0 \). Let \( J_\delta \) be a sequence of mollifier in \( \mathbb{R}^d \) as mentioned in Lemma 6.1. Consider the test function

\[ \psi_\delta(x, y) := J_\delta \left( \frac{x - y}{\delta} \right) \phi \left( \frac{x + y}{\delta} \right). \]

Subtracting two solutions \( u_\varepsilon(t, x), u_\varepsilon(t, y) \) of (6.2), and applying Itô-Lévy formula to that resulting equations, we obtain

\[
\begin{align*}
\beta_\varepsilon(u_\varepsilon(t, x) - u_\varepsilon(t, y)) - \beta_\varepsilon(u_\varepsilon(0, x) - u_\varepsilon(0, y))
&= \int_{s=0}^t \beta_\varepsilon(u_\varepsilon(s, x) - u_\varepsilon(s, y)) \left( \text{div}_x F_\varepsilon(u_\varepsilon(s, y)) - \text{div}_y F_\varepsilon(u_\varepsilon(s, x)) \right) \, ds \\
&\quad + \varepsilon \int_{r=0}^t \beta_\varepsilon(u_\varepsilon(r, x) - u_\varepsilon(r, y)) \left( \Delta_x u_\varepsilon(r, x) - \Delta_y u_\varepsilon(r, y) \right) \, dr \\
&\quad + \int_{r=0}^t \int_{s>|r|>0} \int_0^1 (1 - \rho) \beta_\varepsilon(u_\varepsilon(r, x) - u_\varepsilon(r, y) + \rho(\eta_{x, y}(u_\varepsilon(r, x), \eta_{x, y}(u_\varepsilon(r, y)))) \\ &\quad \times \eta_{x, y}(u_\varepsilon(r, x), \eta_{x, y}(u_\varepsilon(r, y)))^2 \, d\rho \, v(dz) \, dr \\
&\quad + \int_{r=0}^t \int_{s>|r|>0} \beta_\varepsilon(u_\varepsilon(r, x) - u_\varepsilon(r, y) + \eta_{x, y}(u_\varepsilon(r, x), \eta_{x, y}(u_\varepsilon(r, y)))) \\ &\quad - \beta_\varepsilon(u_\varepsilon(r, x) - u_\varepsilon(r, y)) \right) \tilde{N}(dz, dr).
\end{align*}
\]

To this end, we see that

\[
\beta_\varepsilon'(u - v)(\Delta_x u - \Delta_y v) = (\Delta_x + 2 \nabla_x \cdot \nabla_y + \Delta_y) \beta_\varepsilon(u - v) - \beta_\varepsilon''(u - v)|\nabla_x u - \nabla_y v|^2. \tag{6.5}
\]

Moreover, a simple calculation reveals that

\[
\begin{align*}
(\Delta_x + 2 \nabla_x \cdot \nabla_y + \Delta_y) \phi_\delta(x, y) &= \Delta \phi \left( \frac{x + y}{2} \right) J_\delta \left( \frac{x - y}{2} \right), \\
(\nabla_x + \nabla_y) \phi_\delta(x, y) &= \nabla \phi \left( \frac{x + y}{2} \right) J_\delta \left( \frac{x - y}{2} \right).
\end{align*}
\]

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Using convexity of $\beta_\varepsilon$ and (6.5), we have

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta_\varepsilon(u(t,x) - u_t(y)) \psi_0(x,y) \, dx \, dy - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta_\varepsilon(u(0,x) - u_0(0,y)) \psi_0(x,y) \, dx \, dy \\
\leq \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_\varepsilon^0(u_t(s,x), u_t(s,y)) \cdot \nabla \phi \left( \frac{x+y}{2} \right) J_0 \left( \frac{x-y}{2} \right) \, dx \, dy \, ds \\
+ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( F_\varepsilon^0(u_t(s,y), u_t(s,x)) - F_\varepsilon^0(u_t(s,x), u_t(s,y)) \right) \cdot \nabla \psi_0(x,y) \, dx \, dy \, ds \\
+ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta_\varepsilon^\prime(u_t(s,x) - u_t(r,y)) J_0 \left( \frac{x-y}{2} \right) \Delta \phi \left( \frac{x+y}{2} \right) \, dx \, dy \, dr \\
+ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta_\varepsilon^\prime(u_t(s,x) - u_t(r,y)\psi_0(x,y) + \rho(\eta_t(x, u_t(x,t)), z) - \eta_t(y, u_t(y,t)); z) \right) \\
\times \left| \eta_t(x, u_t(x,t)); z) - \eta_t(y, u_t(y,t)); z) \right| ^2 \psi_0(x,y) \, dp \, dx \, dy \, 

Notice that since $\left| F_\varepsilon^0(u,v) - F_\varepsilon^0(v,u) \right| \leq C \left| F^\prime \right|_{\infty} \xi |u - v|$, we obtain

$$
E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta_\varepsilon(u_t(s,x) - u_t(s,y)) \psi_0(x,y) \, dx \, dy \right] - E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta_\varepsilon(u(0,x) - u_0(0,y)) \psi_0(x,y) \, dx \, dy \right] \\
\leq C \left| F^\prime \right|_{\infty} \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| u_t(s,x) - u_t(s,y) \right| \phi \left( \frac{x+y}{2} \right) J_0 \left( \frac{x-y}{2} \right) \, dx \, dy \, ds \\
+ C \left| F^\prime \right|_{\infty} \xi E \left[ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| u_t(s,x) - u_t(s,y) \right| \phi \left( \frac{x+y}{2} \right) J_0 \left( \frac{x-y}{2} \right) \, dx \, dy \, ds \right] \\
+ C \xi E \left[ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| u_t(s,x) - u_t(s,y) \right| \phi \left( \frac{x+y}{2} \right) \left| \nabla J_0 \left( \frac{x-y}{2} \right) \right| \, dx \, dy \, ds \right] \\
+ C \xi E \left[ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| u_t(s,x) - u_t(r,y) \right| \phi \left( \frac{x-y}{2} \right) \phi \left( \frac{x+y}{2} \right) \, dx \, dy \, dr \right] \\
+ E \left[ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta_\varepsilon^\prime(u_t(s,x) - u_t(r,y) + \rho(\eta_t(x, u_t(x,t)); z) - \eta_t(y, u_t(y,t)); z)) \right) \\
\times \left| \eta_t(x, u_t(x,t)); z) - \eta_t(y, u_t(y,t)); z) \right| ^2 \psi_0(x,y) \, dp \, dx \, dy \, v(dz) \, dr \right]. \quad (6.6)
$$

where we have used $|\Delta \phi(x)| \leq C \phi(x)$.

As before, one can use Cauchy-Schwartz inequality along with uniform moment estimate (2.5) to conclude

$$
C \left| F^\prime \right|_{\infty} \xi E \left[ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| u_t(s,x) - u_t(s,y) \right| \phi \left( \frac{x+y}{2} \right) J_0 \left( \frac{x-y}{2} \right) \, dx \, dy \, ds \right] \\
+ C \left| F^\prime \right|_{\infty} \xi E \left[ \int_{x=0}^{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| u_t(s,x) - u_t(s,y) \right| \phi \left( \frac{x+y}{2} \right) \left| \nabla J_0 \left( \frac{x-y}{2} \right) \right| \, dx \, dy \, ds \right] \\
\leq C \left| F^\prime \right|_{\infty} \xi \left( \frac{\xi}{\varphi} \right)^{\frac{\xi}{\varphi}} \left| \phi \right|_{L^\infty(\mathbb{R}^d)} \sqrt{V}. \quad (6.7)
$$
Next, we focus on the last term of (6.6). To estimate that term, we first let
\[
a = u_e(t, x) - u_e(t, y) \quad \text{and} \quad b = \eta_e(x, u_e(t, x); z) - \eta_e(y, u_e(t, y); z).
\]
Observe that
\[
b^2 \beta_2^\alpha(a + \rho b) = (\eta_e(x, u_e(t, x); z) - \eta_e(y, u_e(t, y); z))^2 \beta_2^\alpha(a + \rho b)
\leq \left( |u_e(t, x) - u_e(t, y)|^2 + K^2|x - y|^2 \right) (1 \wedge |z|^2) \beta_2^\alpha(a + \rho b)
= \left( a^2 + K^2|x - y|^2 \right) \beta_2^\alpha(a + \rho b) (1 \wedge |z|^2).
\]
(6.8)
As before (cf. 3.5), one can use assumption [B.1] on \( \eta(x, u; z) \) to conclude
\[
0 \leq a \leq (1 - \lambda')^{-1}(a + \rho b + K|x - y|).
\]
In view of (6.8), we have
\[
b^2 \beta_2^\alpha(a + \rho b) \leq (1 - \lambda')^{-2}(a + \rho b + K|x - y|)^2 \beta_2^\alpha(a + \rho b) (|z|^2 \wedge 1) + \frac{K|x - y|^2}{\xi} (|z|^2 \wedge 1)
\leq 2(1 - \lambda')^{-2}(a + \rho b)^2 \beta_2^\alpha(a + \rho b)(|z|^2 \wedge 1) + C(K, \lambda') \frac{|x - y|^2}{\xi} (|z|^2 \wedge 1)
\leq \left[ 2(1 - \lambda')^{-2}C \xi + C(K, \lambda') \frac{|x - y|^2}{\xi} \right] (|z|^2 \wedge 1),
\]
and hence
\[
E \left[ \int_{r=0}^{\tau} \int_{|z|>0} \int_{\mathbb{R}^d} \int_{\rho=0}^{1} b^2 \beta_2^\alpha(a + \rho b) \psi_\delta(x, y) \, d\rho \, dx \, dy \, \nu(dz) \, dr \right]
\leq E \left[ \int_{r=0}^{\tau} \int_{|z|>0} \int_{\mathbb{R}^d} \int_{\rho=0}^{1} \left[ 2(1 - \lambda')^{-2}C \xi + C(K, \lambda') \frac{|x - y|^2}{\xi} \right] (|z|^2 \wedge 1) \psi_\delta(x, y) \, dx \, dy \, \nu(dz) \, dr \right]
\leq C_1 \left( \xi + \frac{\delta^2}{\xi} \right) \| \phi(\cdot) \|_{L^\infty(\mathbb{R}^d)}.
\]
(6.9)
Now we make use of (2.12), (6.7) to (6.9) in (6.6) and conclude
\[
E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_e(t, x) - u_e(t, y)| J_0 \left( \frac{x - y}{2} \right) \phi \left( \frac{x + y}{2} \right) \, dx \, dy \right]
\leq E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_e(0, x) - u_e(0, y)| J_0 \left( \frac{x - y}{2} \right) \phi \left( \frac{x + y}{2} \right) \, dx \, dy \right]
+ C(1 + \| F' \|_{\infty}) \int_{r=0}^{\tau} E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_e(s, x) - u_e(s, y)| J_0 \left( \frac{x - y}{2} \right) \phi \left( \frac{x + y}{2} \right) \, dx \, dy \right] ds
+ C \| F' \|_{\infty} \left( \xi + \frac{\delta^2}{\xi} \right) \| \phi(\cdot) \|_{L^\infty(\mathbb{R}^d)} \sqrt{t} + C \xi \| \phi(\cdot) \|_{L^1(\mathbb{R}^d)} + C \xi \| \phi(\cdot) \|_{L^1(\mathbb{R}^d)}.
\]
A simple application of Gronwall’s inequality reveals that
\[
E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_e(t, x) - u_e(t, y)| J_0 \left( \frac{x - y}{2} \right) \phi \left( \frac{x + y}{2} \right) \, dx \, dy \right]
\leq \exp \left( t C(1 + \| F' \|_{L^\infty}) \right) E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_e(0, x) - u_e(0, y)| J_0 \left( \frac{x - y}{2} \right) \phi \left( \frac{x + y}{2} \right) \, dx \, dy \right]
\]
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Now we combine (6.12) and (6.13) to obtain
\[ \sup_{|\delta| \leq \delta_1} E\left[ \int_{|x| \leq \delta} |u_\epsilon(t,x) - u_\epsilon(t,y)|J_\delta(y) \phi(x) \, dx \right] \]
\[ \leq C(T) \left( \sup_{0 < |\delta| \leq 1} \delta^\gamma \int_{|x| \leq \delta} \left[ |u_\epsilon(t,x+y) - u_\epsilon(t,x-y)| J_\delta(y) \phi(x) \, dx \right] \right) \]
\[ + C(T) \left( \| \phi \|_{L^\infty(\mathbb{R}^d)} + \delta^\ast \| \phi \|_{L^1(\mathbb{R}^d)} \right) \]  

for some constant \( C(T) > 0 \), independent of \( \epsilon \).

Now we make use of the following change of variables
\[ \bar{x} = \frac{x-y}{2}, \quad \text{and} \quad \bar{y} = \frac{x+y}{2}, \]

to rewrite the above inequality (dropping the bar). The result is
\[ E\left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_\epsilon(t,x+y) - u_\epsilon(t,x-y)| J_\delta(y) \phi(x) \, dx \, dy \right] \]
\[ \leq C(T) \left( \sup_{0 < |\delta| \leq 1} \delta^\gamma \int_{|x| \leq \delta} \left[ |u_\epsilon(t,x+y) - u_\epsilon(t,x-y)| J_\delta(y) \phi(x) \, dx \right] \right) \]
\[ + C(T) \left( \| \phi \|_{L^\infty(\mathbb{R}^d)} + \delta^\ast \| \phi \|_{L^1(\mathbb{R}^d)} \right) \]  

In view of (6.4) of the Lemma 6.1 we obtain for \( r < \frac{1}{2} \)
\[ \sup_{|\delta| \leq \delta_1} E\left[ \int_{|x| \leq \delta} |u_\epsilon(t,x+y) - u_\epsilon(t,x)| \phi(x) \, dx \right] \]
\[ \leq C_2 \delta^r \sup_{0 < |\delta| \leq 1} \delta^\ast \left( \int_{|x| \leq \delta} \left[ |u_\epsilon(t,x+y) - u_\epsilon(t,x-y)| J_\delta(y) \phi(x) \, dx \right] \right) \]
\[ + C_2 \delta^r \left[ \| u_\epsilon(t, \cdot) \|_{L^1(\mathbb{R}^d)} \right]. \]  

Again, by (6.3) of the Lemma 6.1 and by (6.11), we see that for \( r = \frac{1}{2} \) and \( s > \frac{1}{2} \),
\[ \sup_{0 < |\delta| \leq 1} \delta^s \left( \int_{|x| \leq \delta} \left[ |u_\epsilon(t,x+y) - u_\epsilon(t,x-y)| J_\delta(y) \phi(x) \, dx \right] \right) \]
\[ \leq C(T) \sup_{0 < |\delta| \leq 1} \delta^s \left( \int_{|x| \leq \delta} \left[ |u_\epsilon(t,0,x+y) - u_\epsilon(t,0,x)| \phi(x) \, dx \right] \right) \]
\[ + C(T) \left( \| \phi \|_{L^\infty(\mathbb{R}^d)} + \| \phi \|_{L^1(\mathbb{R}^d)} \right) \]
\[ \leq C(T) \left( \| u_\epsilon \|_{L^\infty(\mathbb{R}^d)} \right) \left[ \| \phi \|_{L^\infty(\mathbb{R}^d)} + C(T) \left( \| \phi \|_{L^\infty(\mathbb{R}^d)} + \| \phi \|_{L^1(\mathbb{R}^d)} \right) \right]. \]  

Now we combine (6.12) and (6.13) to obtain
\[ \sup_{|\delta| \leq \delta_1} E\left[ \int_{|x| \leq \delta} |u_\epsilon(t,x+y) - u_\epsilon(t,x)| \phi(x) \, dx \right] \]
\[ \leq C(T) \left( \| u_\epsilon \|_{L^\infty(\mathbb{R}^d)} \right) \left[ \| \phi \|_{L^\infty(\mathbb{R}^d)} + C(T) \left( \| \phi \|_{L^\infty(\mathbb{R}^d)} + \| \phi \|_{L^1(\mathbb{R}^d)} \right) \right]. \]  

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Let $K_R = \{ x : |x| \leq R \}$. Choose $\phi \in C^\infty_c(\mathbb{R}^d)$ such that $\phi(x) = 1$ on $K_R$. Then, for $r < \frac{1}{2}$, we have

\[
\sup_{|\delta| \leq \delta} \mathbb{E} \left[ \int_{K_R} \left| u_0(t, x + y) - u_0(t, x) \right| \, dx \right] \leq C(T, R) \delta^r,
\]

which completes the proof.

In view of the well-posedness results from [2], we can finally claim the existence of entropy solutions for (6.1) that satisfies the fractional $BV$ estimate in Theorem 6.2. In other words, we have the following theorem.

**Theorem 6.3.** Suppose that the assumptions [A.2] [A.3] [A.4] [B.1] and [B.2] hold and the initial data $u_0$ belong to the Besov space $B^\mu_{1, \infty}(\mathbb{R}^d)$ for some $\mu \in (\frac{d}{2}, 1)$ and

\[
\mathbb{E} \left[ \|u_0\|^p_{L^p(\mathbb{R}^d)} + \|u_0\|^p_{L^2(\mathbb{R}^d)} \right] < \infty, \quad \text{for } p = 1, 2, \ldots. \tag{6.14}
\]

(a) Then given initial data $u_0$, there exists an entropy solution of (6.1) such that for any $t \geq 0$,

\[
\mathbb{E} \left[ \|u(t, \cdot)\|^p_{L^p(\mathbb{R}^d)} \right] < \infty, \quad \text{for } p = 1, 2, \ldots.
\]

Moreover, there exists a constant $C^p_{T, r}$ such that, for any $0 < t < T$,

\[
\sup_{|\delta| \leq \delta} \mathbb{E} \left[ \int_{K_R} |u(t, x + y) - u(t, x)| \, dx \right] \leq C^p_{T, r} \delta^r,
\]

for some $r \in (0, \frac{1}{2})$ and $K_R := \{ x : |x| \leq R \}$.

(b) Let the initial data $u_0$ only satisfies (6.14). Then there exists an entropy solution of (6.1) such that for any $t \geq 0$,

\[
\mathbb{E} \left[ \|u(t, \cdot)\|^p_{L^p(\mathbb{R}^d)} \right] < \infty, \quad \text{for } p = 1, 2, \ldots.
\]

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