Linked-cluster Tamm-Dancoff Field Theory

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Abstract

To solve the relativistic bound-state problem one needs to systematically and simultaneously decouple the high-energy from the low-energy modes and the many-body from the few-particle states using a consistent renormalization scheme. In a recent paper we have shown that one such approach can be a combination of the coupled cluster method as used in many-body theory and the Wilsonian exact renormalization group. Even though the method is intrinsically non-perturbative, one can easily implement a loop expansion within it. In this letter we provide further support for this aspect of our formalism by obtaining results for the two-loop renormalized $\phi^4$ theory. We show that the non-unitary representation inherent in our method leads to an economic computation and does not produce any non-hermiticity in the relevant terms.

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The Tamm-Dancoff approximation was developed in the 1950’s to describe a relativistic bound state in terms of a small number of particles. It was soon revealed that the Tamm-Dancoff truncation gives rise to a new class of non-perturbative divergences, since the truncation does not allow us to take into account all diagrams at a given order in perturbation theory. On the other hand, any naive renormalization violates Poincaré symmetry and the cluster decomposition property. Two very different remedies for this issue are the use of light-front dynamics (see, e.g. [2]) and the application of the coupled cluster method (CCM) [3]. However, both methods are too complicated to attack the issues in a self-consistent way. More recently, Glazek and Wilson [4] and independently Wegner [5] introduced an elegant approach for such problems, the so-called similarity renormalization group (SRG). There are several problems with this approach: The Hamiltonian can not be systematically diagonalized in particle number space, the computation is extremely complex and there is no efficient non-perturbative scheme.

In Ref. [6], we have proposed a new formulation for the Tamm-Dancoff renormalization in the context of the CCM by utilising the Wilsonian exact renormalization group [9]. Our method resembles the SRG approach, since we employ a similarity transformation to decouple the high-energy from the low-energy modes, leading to partial diagonalization of the Hamiltonian. Since we apply a double similarity transformation using linked-cluster amplitudes, we produce a biorthogonal representation of the problem, which is not necessarily unitary. There is a long tradition of such approaches in nuclear many-body theory [10]. One of our goals in this letter is to show that the non-hermiticity is not a problem, and leads to an economic computation. The method is intrinsically non-perturbative (it can be conceived as a topological expansion in the number of correlated excitations), but one can implement a loop expansion within it, and we shall concentrate on that aspect in this letter. By construction, we design a transformation which does not produce any fake divergences due to “small energy denominators” which plague old-fashioned perturbation theory in the Hamiltonian approach [11]. The Poincaré invariance and the cluster decomposition property can be maintained in principle at every level of truncation regardless of regularization scheme, by requiring a set of decoupling conditions [6].

In this letter we supply further evidence for the efficiency of our method by computing the 2-loop renormalization of $\phi^4$ theory. We only include a succinct discussion of the key elements of the mathematical framework; details can be found in Ref. [6]. Notice that our
formulation does not depend on the form of dynamics, i.e. on the choice of quantization (hyper-)plane.

Consider a system described by a Hamiltonian $H(\Lambda)$ which has, at the outset, a large cut-off $\Lambda$. We assume that generally the renormalized Hamiltonian $H^\text{eff}(\lambda)$ up to scale $\lambda$ can be expressed as

$$H^\text{eff}(\lambda) = H(\lambda) + H_C(\lambda),$$  \hspace{1cm} (1)

where $H_C(\lambda)$ is a “counterterm”. Our aim is to construct the renormalized Hamiltonian by obtaining this counterterm. We define two subspaces, the model-space $P$:

$$\{|L\rangle \otimes |0, b\rangle_h, L \leq \mu\}$$

and the complement-space $Q$:

$$\{|H\rangle - |0, b\rangle_h, \mu < H \leq \Lambda\}.$$  

The ket $|0, b\rangle_h$ is the bare high energy vacuum (the ground state of the free high-momentum Hamiltonian). The $P$-space contains interacting low-energy states and $Q$-space contains the orthogonal complement (the symbols $|L\rangle$ and $|H\rangle$ denote generic low- and high-energy states respectively). Our renormalization approach is based on decoupling of the complement space $Q$ from the model space $P$ by using a non-unitary transformation. The transformation of $H(\Lambda)$ is defined by

$$\mathcal{H} = e^{\hat{S}(\mu, \Lambda)} e^{-\hat{S}'(\mu, \Lambda)} H(\Lambda) e^{\hat{S}'(\mu, \Lambda)} e^{-\hat{S}(\mu, \Lambda)} \equiv H(\mu) + \delta H(\mu, \Lambda),$$  \hspace{1cm} (2)

where the operator $\hat{S}(\hat{S}')$ is a functional of certain mapping operators between $P$- and $Q$-space \cite{6}. By means of this transformation one may identify the effective interaction $\delta H(\mu, \Lambda)$ containing the physics between the scale $\Lambda$ and $\mu$. According to our prescription we expand $\hat{S}(\hat{S}')$ in terms of independent coupled cluster excitation $I$,

$$\hat{S} = \sum_{m=0} \hat{S}_m \left( \frac{\mu}{\Lambda} \right)^m, \hspace{1cm} \hat{S}_m = \sum_I \hat{s}_I C_I^\dagger,$$

$$\hat{S}' = \sum_{m=0} \hat{S}'_m \left( \frac{\mu}{\Lambda} \right)^m, \hspace{1cm} \hat{S}'_m = \sum_I \hat{s}'_I C_I^\dagger. \hspace{1cm} (3)$$

Here the primed sum means that $I \neq 0$, and momentum conservation is included in $\hat{s}_I$ and $\hat{s}'_I$. The $C_I$ and $C_I^\dagger$ are annihilation and creation operators in the high-energy Fock space for a given quantization scheme (e.g., equal time or light-cone). The indices $I$ define a subsystem, or cluster within the full system of a given configuration. Notice, that the choice of the operators $\hat{S}(\hat{S}')$ is not generally unique, due to Haag’s theorem \cite{13}. This ambiguity corresponds to the possibility of the choice of a different but equivalent representation of the canonical variables. However the physical quantities remain invariant under change of operators $\hat{S}(\hat{S}')$. 

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It is well-known in many-body applications that the parametrization Eq. (3) has the following properties: 1) it satisfies proper size-extensivity and conforms with the Goldstone linked-cluster theorem at any level of approximation 2) it is compatible with Hellmann-Feynman theorem, 3) the phase space \( \{ \hat{s}_I, \hat{s}'_I \} \) for a given \( m \) is a symplectic differentiable manifold. These features are related to each other and one can not give up one without losing the others as well \[8\]. The price we pay for these desirable features is that the representation is not longer hermitian.

One may now impose the decoupling conditions, leading to diagonalization of the transformed Hamiltonian matrix,

\[
Q\Pi P = 0 \rightarrow \langle 0 | C_I e^{-\hat{S}} H e^{\hat{S}} | 0 \rangle = 0, \\
P\Pi Q = 0 \rightarrow \langle 0 | e^{\hat{S}'} e^{-\hat{S}} H e^{\hat{S}} e^{-\hat{S}'} C^\dagger_I | 0 \rangle = 0,
\]

where \( I \neq 0 \). One can show that the right hand side of Eq. (4,5) are derivable from the dynamics of the quantum system \[6\]. Thus the effective low-energy Hamiltonian is

\[
\hat{H}^{\text{eff}} = P \hat{H} P \equiv \hbar \langle b, 0 | e^{\hat{S}'(\mu, \Lambda)} e^{-\hat{S}(\mu, \Lambda)} H(\Lambda) e^{\hat{S}(\mu, \Lambda)} e^{-\hat{S}'(\mu, \Lambda)} | 0, b \rangle. 
\]

One may prove that the effective low-energy operators Eq. (6) supplemented by decoupling conditions Eq. (4,5) indeed have the same low-energy eigenvalues as the original Hamiltonian \[6\]. The states in the full Hilbert space are constructed by adding clusters of high-energy correlation to states in the \( P \)-space, or equivalently by integrating out the high-energy modes from the Hamiltonian. It is immediately clear that states in interacting Hilbert space are normalized, due to the nature of a similarity transformation.

The decoupling property makes the \( P \) sector of the truncated Fock space independent of the rest. This means that the contribution of the excluded sector is taken into account by imposing the decoupling conditions. One can show that the energy-dependent Bloch-Feshbach formalism \[12\] is thereby made free of the small-energy denominators which spoil perturbation theory \[6\]. One can then determine the counterterm by requiring coupling coherence \[14\], namely that the transformed Hamiltonian Eq. (5) has the generic form given in Eq. (1), with \( \lambda \) replaced by \( \mu \). This requirement must be satisfied on an infinitely long

\[1\] It is of interest that according to our approach, various effective low-energy Hamiltonians can be constructed without invoking perturbation theory or hermiticity.
renormalization group trajectory and thus produces a renormalized Hamiltonian. The individual amplitudes for a given \( m \), \( \{ \hat{s}_m, \hat{s}_m' \} \equiv \{ \hat{s}_I, \hat{s}_I' \}_m \), have to be fixed by the dynamics of a quantum system incorporated the decoupling conditions Eq. (115). Equations (115) provide two sets of exact, microscopic, operatorial coupled non-linear equations for the ket and bra states which describes the flow of the coefficients in Eq. (3). One can solve the coupled equations in Eq. (4) to obtain \( \{ \hat{s}_I \}_m \) and then use them as an input in (5). The conditions given in Eq. (4) and (5) imply that all interactions of high-momentum particles should be removed from the transformed Hamiltonian \( H_{\text{eff}}(\mu) \) in Eq. (6). These are sufficient requirements to ensure partial diagonalization of the Hamiltonian in particle and momentum space.

In practice one needs to truncate both sets of coefficients \( \{ \hat{s}_I, \hat{s}_I' \}_m \) at a given order of \( m \) in the ratio of cutoffs. A consistent truncation scheme is the so-called SUB\((n, m)\) scheme, where we include up to \( n \)-body operators \( \{ \hat{S}, \hat{S}' \} \) and truncate the expansion in \((\mu/\Lambda)\) at order \( m \). The choice of \( n \) depends on the Hamiltonian interaction and needs to be fixed from the outset.

We now apply this formalism to the computation of the effective Hamiltonian for \( \phi^4 \) theory up to two-loop order in equal-time quantization. The bare \( \phi^4 \) theory Hamiltonian is

\[
H = \int d^3x \left( \frac{1}{2} \pi^2(x) + \frac{1}{2} \phi(x) ( -\nabla^2 + m^2) \phi(x) + g \phi^4(x) \right). \tag{7}
\]

According to our logic the ultraviolet-finite Hamiltonian is obtained by introducing counterterms, which depend on the UV cutoff \( \Lambda \) and some arbitrary renormalization scale. This redefines the parameters of the theory and defines the effective low-energy Hamiltonian. The renormalized Hamiltonian has the form

\[
H = \int d^3x \left( \frac{Z_\pi}{2} \pi^2(x) + \frac{1}{2} \sqrt{Z_\phi} \phi(x) \left( -\nabla^2 + Z_m m^2 \right) \sqrt{Z_\phi} \phi(x) + Z_g Z_\phi^2 g \phi^4(x) + ... \right). \tag{8}
\]

Even though the newly generated interactions are sensitive to the regularization scheme (as is well known \[16\], a sharp cutoff may lead to new non-local interaction terms), nevertheless one can ignore these if they are irrelevant in the renormalization group sense. We now split field operators into high- and low-momentum modes; \( \phi(x) = \phi_L(x) + \phi_H(x) \), where \( \phi_L(x) \) denotes modes of low-frequency with momentum \( k \leq \mu \) and \( \phi_H(x) \) denotes modes of high-frequency with momentum constrained to a shell \( \mu < k \leq \Lambda \). The field \( \phi_H(x) \) can be conceived as a background to which the \( \phi_L(x) \)-modes are coupled. Therefore, in the standard diagrammatic language, integrating out the high-frequency modes \( \phi_H(x) \) implies
that only high-frequency modes appear in internal lines. The field $\phi_H(x)$ is represented in Fock space as

$$\phi_H(x) = \sum_{\mu<k\leq\Lambda} \frac{1}{\sqrt{2\omega_k}}(a_k e^{ikx} + a_k^\dagger e^{-ikx}),$$

(9)

where $\omega_k = \sqrt{k^2 + m^2}$ and the operators $a_k$ and $a_k^\dagger$ satisfy the standard boson commutation rules. From now on all summations are implicitly over the high-frequency modes $\mu < k \leq \Lambda$. The Hamiltonian in terms of high- and low-frequency modes can be written as, after normal ordering with respect to high-frequency modes,

$$H = H_1 + H_2 + V_B + V_C + V_A,$$

(10)

where we define,

$$H_1 = \int \left( \frac{1}{2} \pi_L^2(x) + \frac{1}{2} \phi_L(x)(-\nabla^2 + m^2)\phi_L(x) + g\phi_L^4(x) \right),$$

$$H_2 = \sum \omega_k a_k^\dagger a_k,$$

$$V_B = g \sum \int \frac{e^{i(p+q+r-k)x}}{\sqrt{\omega_k\omega_p\omega_q\omega_r}} a_k^\dagger a_p a_q a_r + \frac{3e^{i(p+q-r-k)x}}{4\sqrt{\omega_k\omega_p\omega_q\omega_r}} a_k^\dagger a_p a_q a_r$$

$$+ 6\phi_L(x) \frac{e^{i(p+q-k)x}}{2\omega_k\omega_p\omega_q} a_k^\dagger a_p a_q + 3\left( \phi_L^2(x) + \frac{1}{2\omega_r} \right) \frac{e^{i(k-p)x}}{\sqrt{\omega_k\omega_p\omega_q}} a_k^\dagger a_p$$

$$+ \frac{3\phi_L^2(x)}{2\omega_r} + \text{h.c.},$$

$$V_C = g \sum \int V_C^4 a_k^\dagger a_p a_q a_r + V_C^3 a_k^\dagger a_p a_q^\dagger + V_C^2 a_k^\dagger a_p^\dagger + V_C^1 a_k^\dagger a_k^\dagger,$$

$$V_A = V_C^\dagger,$$

$$V_C^1 = \left( \frac{6\phi_L(x)}{\omega_p} + 4\phi_L^3(x) \right) \frac{e^{-ikx}}{\sqrt{2\omega_k}},$$

$$V_C^2 = 3\left( \phi_L^2(x) + \frac{1}{2\omega_r} \right) \frac{e^{i(k-p)x}}{\sqrt{\omega_k\omega_p}},$$

$$V_C^3 = 2\phi_L(x) \frac{e^{-i(k+p+q)x}}{\sqrt{2\omega_k\omega_p\omega_k}},$$

$$V_C^\dagger = \frac{e^{i(k+p+q+r)x}}{4\sqrt{\omega_k\omega_p\omega_q\omega_r}}.$$

(11)

The high-energy configurations in the Fock space are specified by $\{C_I \rightarrow \prod_{i=1} a_{k_i}\}$ and $\{C_I^\dagger \rightarrow \prod_{i=1} a_{k_i}^\dagger\}$. Up to two-loop expansion, our renormalization scheme requires to keep $S(S')$ at least to order $n = 4$, which allows us to eliminate the pure terms $V_C$ and $V_A$ at a lower level of expansion. The $\hat{S}(\hat{S})$ operators consistent with a $SUB(4,m)$ truncation scheme are,

$$\hat{S}_m = \int \sum \left( \hat{S}_m^1 a_k + \hat{S}_m^2 a_k^\dagger a_p + \hat{S}_m^3 a_k^\dagger a_p a_q + \hat{S}_m^4 a_k^\dagger a_p^\dagger a_q^\dagger a_r \right),$$

$$\hat{S}_m^\dagger = \int \sum \left( \hat{S}_m^1 a_k + \hat{S}_m^2 a_k a_p + \hat{S}_m^3 a_k a_p a_q^\dagger + \hat{S}_m^4 a_k^\dagger a_p a_q a_r \right).$$

(12)
One can expand Eqs. (4,5) in terms of $\mu/\Lambda$, leading to the introduction of a consistent hierarchy of equations in powers of $m$ [6], which can be solved for the coefficients $\hat{S}_m(\hat{S}_m')$ in Eq. (12). We split the diagonalization of the Hamiltonian matrix in an upper and lower triangle part, using the double similarity transformation. One may notice that the “most non-diagonal” terms in the Hamiltonian are $V_C$ and $V_A$ (in the light-front Hamiltonian such terms do not exist because modes with longitudinal momentum identically zero are not allowed). The potential $V_B$ is already partially diagonalized and does not change the vacuum of the high-energy states. Therefore, here we employ a minimal scheme, aiming at removal of $V_A$ and $V_C$ only.

We restrict ourselves to the elimination of the high-energy degrees of freedom up to the first order in the coupling constant $g$ and second order in the ratio of cutoffs $\mu/\Lambda$. Therefore, our truncation scheme is called $SUB(4,2)$. For $m = 0$ one finds,

$$
S_0^1 = -g\frac{V_C^1}{\omega_k}, \quad S_0^2 = -g\frac{V_C^2}{\omega_k + \omega_p},
$$

$$
S_0^3 = -g\frac{V_C^3}{\omega_k + \omega_p + \omega_q}, \quad S_0^4 = -g\frac{V_C^4}{\omega_k + \omega_p + \omega_q + \omega_r},
$$

(13)

where the $V_C^{1-4}$ are defined in Eq. (11). Here, one has $S_0' = S_0^\dagger$ [6]. At this stage the results for the one-loop renormalization can be computed. We evaluate the effective Hamiltonian by substituting $S(S')$ from Eqs. (12) and (13) into Eq. (6). In order to achieve renormalization, one should identify the potentially divergent terms (when $\Lambda \to \infty$) in the expansion of $H_{\text{eff}}(\mu)$. Such a process generally can be done by inventing a power-counting rule, using the property $S_n \simeq \frac{\mu}{\Lambda}S_{n-1}$. Here we take $\omega_k \simeq |k|$ for $\mu \gg m$ and replace $\sum_k$ by $\int \frac{d^3k}{(2\pi)^3}$. The standard tadpole one-loop mass renormalization arises from $V_B$ due to normal-ordering. We add this divergent term to $H_1$ and renormalize the bare mass

$$
\delta H^{1\text{-loop}} = \langle 0|V_B|0 \rangle = 6g \sum \int \frac{\phi^2(x)}{2\omega_k} = \frac{3g}{4\pi^2}(\Lambda^2 - \mu^2) \int d^3x \phi^2(x),
$$

$$
Z_m = 1 - \frac{3g}{2\pi^2}(\Lambda^2 - \mu^2).
$$

(14)

In this order the contribution of the terms $[V_C, S], [V_A, S']$ and $[H_1, S(S')]$ are zero, after projection on to the high-energy vacuum. The only divergent contributions come from $[V_A^{2(3)}, S_0^{2(3)}]$ due to a double and third contraction of the high-frequency fields respectively. There are two other divergent terms, $[[V_C^{2(3)}, S_0^{2(3)}]]$, however they are harmless and are
cancelled out by the divergence of $[[H_2, S_0], S_0^{(3)}]$. One thus obtains,

$$\delta H = - \frac{18g^2}{(2\pi)^6} \int \frac{\phi^2(x)\phi^2(y)}{\omega_k\omega_p(\omega_k + \omega_p)} e^{i(k+p)(x-y)}$$

$$- \frac{12g^2}{(2\pi)^6} \int \frac{\phi(x)\phi(y)}{\omega_k\omega_p\omega_q(\omega_k + \omega_p + \omega_q)} e^{i(k+p+q)(x-y)}. \tag{15}$$

In general evaluation of integrals like Eq. (15) may produce non-localities. This is due to the fact that the total momentum in integrands of Eq. (15), namely $r_1 = p+q$ and $r_2 = k+p+q$ are in the low-momentum space. To evaluate such integrations, one can firstly reduce the potential divergent integrals by a change of variable, for example for the first integrand we use $p,q \to p,r_1$, and then expand the integrand in $r_1/p$. Therefore, after expansion and evaluating the momentum integrals, one may be faced with non-analytic terms in the low-momentum space. However here these are irrelevant and will thus be ignored. We find

$$\delta H^{1\text{-loop}} = - \frac{9g^2}{2\pi^2} \ln \left( \frac{\Lambda}{\mu} \right) \int d^4x \phi^4(x) + \frac{3g^2}{2\pi^2} (2 \ln 2 - 1) \Lambda^2 \int d^4x \phi^2(x) + \frac{3g^2}{16\pi^4} \ln \left( \frac{\Lambda}{\mu} \right) \int d^4x (\nabla \phi(x))^2 + \text{finite terms.} \tag{16}$$

One can immediately deduce the renormalization factors $Z_g$ and $Z_\phi$ from above expression

$$Z_g = 1 + \frac{9g^2}{2\pi^2} \ln \left( \frac{\Lambda}{\mu} \right), \tag{17}$$

$$Z_\phi = 1 - \frac{3g^2}{8\pi^4} \ln \left( \frac{\Lambda}{\mu} \right). \tag{18}$$

The unknown coefficients in expression $S_1$ is computed by making use of Eq. (18) and solving coupled equations (14), therefore one may yield,

$$S_1^1 = \frac{6ge^{-ikx}}{\omega_k \sqrt{2\omega_k}} \left( 2\phi_L(x) - 2i\pi_L(x)\phi^2_L(x) - \frac{i\pi_L(x)}{\omega_p} \right) - \frac{g}{\omega_k} \sum_{\nu=1}^{3} \frac{1}{\nu!} V^\nu_A S_1^\nu+1,$$

$$S_1^2 = \frac{3ge^{-i(k+p)x}}{(\omega_k + \omega_p)^2 \sqrt{2\omega_k\omega_p}} \left( 1 - i2\pi_L(x)\phi_L(x) \right) - \frac{g}{\omega_k + \omega_p} \left( [V^1_C, S_1^1] + \sum_{\nu=1}^{2} \frac{1}{\nu!} V^\nu_A S_1^\nu+1 \right),$$

$$S_1^3 = - \frac{2ige^{-i(k+p+q)x}}{\omega_k + \omega_p + \omega_q} \pi_L(x) - \frac{g}{\omega_k + \omega_p + \omega_q} \left( V^1_A S_1^1 + \sum_{\nu=1}^{2} [V^\nu_C, S_1^\nu+1] \right),$$

$$S_1^4 = - \frac{g}{\omega_k + \omega_p + \omega_q + \omega_r} \sum_{\nu=1}^{3} [V^{4-\nu}_C, S_1^\nu]. \tag{19}$$

In the above expression summation over dummy momentum indices is assumed. One can find $\hat{S}_1^1$ in the same manner by exploiting Eq. (5) and using Eq. (19) as an input, which
leads to
\[ S_1^{\nu} = (S_1^{\nu})^\dagger + S_1^{\nu\alpha} \]
\[ \nu = 1, \ldots, 4, \]  
(20)

with the notations,
\[ S_{1a}^\nu = \frac{g}{\omega_k} \left( \sum_{\nu=1}^3 \frac{1}{\nu!} S_{1}^{(\nu+1)a} V_1^{\nu} - \sum_{\nu=1}^3 \frac{1}{\nu!} V_1^{\nu+1} S_1^{\nu} \right), \]
(21)

The only divergent contribution up to order \( g^2 \) arises from,
\[ \delta H = -\langle 0 | [H_1, S_1], S_0^\dagger | 0 \rangle, \]
(22)

After the evaluation of the leading divergent part, we find that
\[ \delta H = -\frac{3g^2}{16\pi^4} \ln \left( \frac{\Lambda}{\mu} \right) \int d^3x \phi^2(x), \]
(23)

which contributes to the two-loop wave-function renormalization \( Z_\pi \). By comparing Eqs. (18) and (23), one may conclude that \( Z_\pi = Z_\phi^{-1} \), as it should be. To finish the renormalization up to two-loop order, one should also take into account the contribution at order \( g^3 \). The divergent terms at this level originate from
\[ \delta H = -\langle 0 | [(V_A + 1/2V_C + V_B), S_0], S_0^\dagger | 0 \rangle. \]
(24)

After a straightforward but lengthy computation one can obtain the leading divergent parts,
\[ \delta H = \frac{27g^3}{2\pi^4} \left[ \ln \left( \frac{\Lambda}{\mu} \right) \right]^2 + \ln \left( \frac{\Lambda}{\mu} \right) \int d^3x \phi^4(x), \]
(25)

this term should be added to Eq. (15), therefore one can immediately deduce the correct total renormalization factor \( Z_g \) up to two-loop order,
\[ Z_g = 1 + \frac{9g^2}{2\pi^2} \ln \left( \frac{\Lambda}{\mu} \right) + \frac{g^3}{4\pi^4} \left( 81 \left( \ln \left( \frac{\Lambda}{\mu} \right) \right)^2 - 51 \ln \left( \frac{\Lambda}{\mu} \right) \right). \]
(26)
One can now immediately obtain the well-known two-loop $\beta$-function and anomalous dimension by making use of Eqs. (18, 26).

It is important to point out that the diagonalization at first order in the coupling constant defines a correct low-energy effective Hamiltonian which is valid up to order $g^3$. Having said that, from Eq. (20) one can observe that the non-hermiticity of the $\hat{S}$ operator appears at order $g^2$ and in a lower order of $\mu/\Lambda$. As we have shown, non-hermiticity is negligible up to two-loop order (asymmetric terms appear in irrelevant contributions). We conjecture that, for the present model, non-hermitian terms only appear in irrelevant contributions, whatever the order of truncation.

One should note that Eqs. (4, 5) are not fully consistent with the Hellmann-Feynman theorem, although this choice considerably simplifies the diagonalization of the Hamiltonian operator. This potentially leads to a different truncation scheme for the renormalization of other operators. The decoupling requirements in the exact form Eq. (15) preserve Poincaré symmetry and the cluster decomposition property regardless of regularization used. Since this is a continuous symmetry, its preservation leads to an infinite set of constraints on the phase space, which has been coded in the decoupling equations. Hence, it is of interest to consider the sensitivity of the Poincaré symmetry with respect to a given truncation scheme in the light-front dynamics.

In this letter we have employed a sharp cutoff, however this idealization should be removed since generally it may lead to pathologies in renormalization, since it induces non-locality and moreover potentially violates the gauge symmetry. We have only investigated a perturbative approach to the problem. Our method is in principle non-perturbative; this aspect remains to be exploited.

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