This is a report on a joint work with Vladimir Hinich.

1. Let $X$ be a smooth algebraic variety over a field $k$ of characteristic 0. Assume that $X$ has no infinitesimal automorphisms, i.e. $H^0(X; T_X) = 0$ ($T_X$ being the sheaf of vector fields). Let $\mathcal{M} = \text{Spec}(\widehat{\mathcal{O}}_M)$ be the formal moduli space of deformations of $X$: $\widehat{\mathcal{O}}_M$ is a complete local $k$-algebra.

We have the Kodaira-Spencer isomorphism

$$T_{\mathcal{M}; X} = H^1(X; T_X) \quad (KS)$$

Here $T_{\mathcal{M}; X} = (m_{\mathcal{O}_M}/m_{\mathcal{O}_M}^2)^*$ is the tangent space of $\mathcal{M}$ at $X$. We want to consider the

**Problem.** Describe the whole algebra $\widehat{\mathcal{O}}_\mathcal{M}$ in terms of $X$.

Let

$$\mathcal{T}_X^\bullet = R\Gamma(X; T_X) : 0 \to T^0 \to T^1 \to \ldots$$

be a complex computing the sheaf cohomology of $T_X$. The last sheaf is a sheaf of Lie algebras, hence $\mathcal{T}^\bullet$ may be chosen to be a differential graded Lie algebra; it is a correctly defined object of the appropriate derived category of Homotopy Lie Algebras, [HS1].

**Theorem 1.** One has a canonical isomorphism of $k$-algebras

$$\widehat{\mathcal{O}}_\mathcal{M} = [H^0_{\text{Lie}}(\mathcal{T}_X^\bullet)]^* \quad (1)$$

The homology of a (dg) Lie algebra is a (dg) coalgebra. The dual space is an algebra. The isomorphism (1) is a generalization of the Kodaira-Spencer isomorphism.

This theorem is just an example of a quite general fact; the similar result (with the same proof) holds true for other deformation problems. For example, we may wish to describe deformations of group representations, etc. Cf. [S2], [HS1].

I know two proofs of Theorem 1. The first one works in the case when $\mathcal{M}$ is smooth, and uses the higher Kodaira-Spencer maps, cf. [HS1]. The second one works in general situation. It uses certain very natural sheaf property of Lie-Deligne functor, and is described below.
2. Deligne groupoids. Let \( \mathfrak{g} = \oplus_{i \geq 0} \mathfrak{g}^i \) be a nilpotent dg Lie algebra. Recall that groupoid is a category with all morphisms being isomorphisms. The Deligne groupoid \( \mathcal{G}(\mathfrak{g}^\bullet) \) is defined as follows. Its objects are Maurer-Cartan elements

\[
MC(\mathfrak{g}^\bullet) := \{ y \in \mathfrak{g}^1 | dy + \frac{1}{2}[y, y] = 0 \}
\]

Let \( \mathcal{G}(\mathfrak{g}^0) \) be the Lie group corresponding to the nilpotent Lie algebra \( \mathfrak{g}^0 \). The algebra \( \mathfrak{g}^0 \) acts on \( MC(\mathfrak{g}^\bullet) \) by the rule

\[
x \circ y = dx + [x, y], \quad x \in \mathfrak{g}^0, y \in MC(\mathfrak{g}^\bullet),
\]

hence the group \( \mathcal{G}(\mathfrak{g}^0) \) acts on \( MC(\mathfrak{g}^\bullet) \). By definition,

\[
\text{Hom}_{\mathcal{G}(\mathfrak{g}^\bullet)}(y, y') = \{ g \in \mathcal{G}(\mathfrak{g}^0) | y' = gy \}
\]

Morphisms are composed in the obvious way. Of course, this is a generalization of the Lie functor from Lie algebras to Lie groups.

3. Let us return to our deformation situation. The variety \( X \) defines a functor

\[
\text{Def}_X : \text{Art}_k \longrightarrow \text{Groupoids}
\]

where \( \text{Art}_k \) is the category of artinian \( k \)-algebras with residue field \( k \). Namely, \( \text{Def}_X(A) \) is the groupoid whose objects are flat deformations of \( X \) over \( A \), and morphisms are isomorphisms identical on \( X \).

On the other hand, if \( \mathfrak{g}^\bullet \) is a dg Lie algebra over \( k \), it defines a functor

\[
\mathcal{G}_{\mathfrak{g}^\bullet} : \text{Art}_X \longrightarrow \text{Groupoids},
\]

by

\[
\mathcal{G}_{\mathfrak{g}^\bullet}(A) = \mathcal{G}(\mathfrak{m}_A \otimes \mathfrak{g}^\bullet)
\]

where \( \mathfrak{m}_A \) is the maximal ideal of \( A \).

Example. Assume that \( X = \text{Spec}(R) \) is affine. Then one sees immediately from the definitions (Grothendieck) that one has an isomorphism of functors

\[
\text{Def}_X = \mathcal{G}_{\mathfrak{g}^\bullet}
\]

(2)

where \( \mathfrak{g}^\bullet = T_X = H^0(X; T_X) = \text{Der}_k(R) \) considered as a dg Lie algebra concentrated in dimension 0.

Sometimes when (2) holds, the people say that the dg Lie algebra \( \mathfrak{g}^\bullet \) governs the deformations of \( X \).

Theorem 2. Let \( X \) be arbitrary, and (2) holds for some \( \mathfrak{g}^\bullet \). If \( H^0(\mathfrak{g}^\bullet) = 0 \) then

\[
\hat{\mathcal{O}}_\mathcal{M} = (H^0_{\mathcal{M}}(\mathfrak{g}^\bullet))^*
\]
Proof. For an arbitrary $A \in \text{Art}_K$, we have

$$\text{Hom}_{\text{Art}_K}(\hat{\mathcal{O}}_M, A) = \pi_0(\text{Def}_X(A)) = \pi_0(\mathcal{G}(\mathfrak{m}_A \otimes \mathfrak{g}^*)) =$$

$$= \text{Hom}_{\text{alg}}((H^0_{\text{Lie}}(\mathfrak{g}^*))^*, A)$$

4. Now, we know the Lie algebra $\mathfrak{g}^*$ for affine varieties, and we want to know it for arbitrary ones, i.e. we want to glue them.

Let $\mathfrak{g}^*$ be a sheaf of nilpotent Lie algebras on a topological space $X$. It defines a presheaf of groupoids $\mathcal{G}(\mathfrak{g}^*)$,

$$U \mapsto \mathcal{G}(\Gamma(U; \mathfrak{g}^*))$$

**Theorem 3.** [H]. Assume that $\mathfrak{g}^*$ is a sheaf in the homotopy sense, i.e. for an open covering $\mathcal{U} = \{U_i\}$, $U = \bigcup U_i$, the natural map

$$\Gamma(U; \mathfrak{g}^*) \rightarrow \check{\mathcal{C}}(\mathcal{U}; \mathfrak{g}^*)$$

is quasiisomorphism. Then $\mathcal{G}(\mathfrak{g}^*)$ is a sheaf (i.e. stack).

Here $\check{\mathcal{C}}(\mathcal{U}; \mathfrak{g}^*)$ is the Čech complex of the covering $\mathcal{U}$.

Theorem 1 follows immediately from Theorems 2 and 3, applied to a homotopy sheaf resolution of $T_X$.

5. It seems that the similar statements hold true in characteristic $p$, or in mixed characteristics, cf. [S2]. One should work with algebras of distributions instead of Lie algebras, and with simplicial objects instead of dg objects.

**References**

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