Hölder continuity of the steepest descent direction for multiobjective optimization

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February 6, 2018

Abstract

The aim of this manuscript is to characterize the continuity properties of the multiobjective steepest descent direction for smooth objective functions. We will show that this direction is Hölder continuous with optimal exponent 1/2. In particular, this direction fails to be Lipschitz continuous even for polynomial objectives.

2000 Mathematics Subject Classification: 90C29, 90C30.

Key words: multiobjective optimization; Pareto optimality, steepest descent direction, Hölder continuity.

1 Introduction

In multiobjective optimization (MO) problems, many functions on the same argument have to be simultaneously minimized. Since in general there is no common minimizer for these functions, one shall rely in another notion of optimality. Up to now, in this settings, the most useful definition of optimality is that of Pareto [11]: A point is a Pareto optimal if its image is minimal in the image of the feasible set with respect to the componentwise (partial) order. Vector optimization in a generalization of MO where a closed convex cone is used to define the partial order for which minimal elements are to be computed. If the cone is the positive orthant one retrieves the componentwise order of MO.

Descent methods for multiobjective and vector optimization is, presently, an area of intense research ( see [5, 3, 2, 1, 13, 9, 16, 8, 14, 12, 15, 7, 11, 6] and the references therein). These are iterative methods in which all objective functions decrease along the generated sequences.

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†Partially supported by CNPq grant 306247/2015-1 and by FAPERJ grant Cientistas de Nosso Estado E-26/201.584/2014
As far as we now know, the first one of these methods was proposed by Mukai in [10]. In this work, six descent methods for constrained MO were proposed, three of which become, in the unconstrained case, the steepest descent method. In his pioneer work, Mukai neither studied the continuity properties of the MO steepest descent direction nor considered the convex case, proving that any limit point of the generated sequences satisfies a first order necessary condition for optimality.

Fliege and Svaiter, independently, reinvented the steepest descent direction and method [5] for unconstrained MO and proved continuity of this direction for smooth objective functions. They also established convergence of the generated sequences in the convex case, for stepsizes computed using an Armijo-type linesearch, under suitable assumptions (e.g. boundedness of level sets). In this manuscript we are concerned with the continuity properties of the steepest descent direction for MO. We will show that when the objective functions have Lipschitz continuous gradients this direction is locally Hölder continuous with optimal exponent 1/2. Hence, in this case, the MO steepest descend direction is not Lipschitz continuous.

2 Basic Definitions and Results

From now on $\Omega$ is a subset of $\mathbb{R}^n$, $f_1, \ldots, f_m$ are scalar function on $\Omega$, and

$$f : \Omega \to \mathbb{R}^m, \quad f(x) = (f_1(x), \ldots, f_m(x)).$$

The MO problem is

$$\min f(x), \quad x \in C$$

where $C \subseteq \Omega$ is the feasible set. A feasible point $\bar{x}$ is a Pareto optimum for this problem if

$$x \in C, \quad f(x) \leq f(\bar{x}) \Rightarrow f(x) = f(\bar{x}).$$

Henceforth we assume that

A1 $C = \Omega$ is open;

A2 each $f_i$, $i = 1, \ldots, m$, is differentiable

A vector $v \in \mathbb{R}^n$ is a multiobjective descent direction at $x$ if

$$\nabla \langle f_i(x), v \rangle < 0, \quad i = 1, \ldots, m.$$

It is trivial to verify that if $v$ is a descent direction at $x$, then

$$f_i(x + tv) < f_i(x) \quad i = 1, \ldots, m$$

for $t > 0$ small enough, whence, at Pareto optimal points in the interior of the feasible set there is no descent direction. Following [5], a point $x$ where there is no descent direction will
be called Pareto critical. Since Pareto criticality is a necessary condition for optimality (in the unconstrained case), is worth observing that

\[ x \text{ Pareto critical} \iff \text{range}(Df(x)) \cap (-\mathbb{R}_+^m) = \emptyset. \]

We believe that the condition at the right hand-side of the above implication seems to be an acceptable extension of the notion of criticality for scalar functions. A natural question is how to compute or choose a “reasonable” descent direction at a given point.

**Definition 2.1** (Mukai [10], Fliege & Svaiter [5]). *The steepest descent direction for \( f \) at \( x \in \Omega \) is \( \Lambda f(x) \),

\[
\Lambda f(x) := \arg\min_{v \in \mathbb{R}^n} \max_{i=1, \ldots, m} \langle v, \nabla f_i(x) \rangle + \frac{1}{2} \|v\|^2 .
\] (3)

It is convenient to define as in [5] the function \( \theta_f(x) \) as the optimal value of the functional whose minimizer is \( \Lambda f(x) \) in the above definition, that is,

\[
\theta_f(x) = \min_{v \in E} \max_{i=1, \ldots, m} \langle v, \nabla f_i(x) \rangle + \frac{1}{2} \|v\|^2 .
\] (4)

Both \( \Lambda f(x) \) and \( \theta_f(x) \) can be obtained minimizing a convex quadratic function under linear constraints or solving its dual by maximizing a concave quadratic function in the unit simplex.

**Proposition 2.2** ([10, 5]). *For any, \( x \in \Omega \) \( (\tau, v) = (\theta_f(x), \Lambda f(x)) \) is the solution of

\[
\min \tau + \frac{1}{2} \|v\|^2 \\
\text{s.t. } \langle v, \nabla f_i(x) \rangle \leq \tau,
\] (5)

a problem whose dual is

\[
\max -\frac{1}{2} \left\| \sum_{i=1}^{m} \alpha_i \nabla f_i(x) \right\|^2 \\
\text{s.t. } \sum_{i=1}^{m} \alpha_i = 1, \quad \alpha \geq 0.
\] (6)

**Proof.** Equivalence between (3) and (5) as well as optimality of \( (\theta_f(x), \Lambda f(x)) \) for the second problem holds trivially. The Lagrangian of (5) is

\[
\mathcal{L}((\tau, v), \alpha) = \tau + \|v\|^2 + \sum_{i=1}^{m} \alpha_i (\langle \nabla f_i(x), v \rangle - \tau),
\] (7)

which trivially implies the second part of the proposition. \( \square \)

Proposition 2.2 has two quite trivial, albeit interesting, consequences.
Corollary 2.3. For any $x \in \Omega$, $\theta_f(x) = -(1/2)\|\Lambda f(x)\|^2$ and $-\Lambda f(x)$ is the minimal norm element in the convex hull of $\{\nabla f_1(x), \ldots, \nabla f_m(x)\}$, that is, of the set

$$U = \left\{ u : u = \sum_{i=1}^{m} \alpha_i \nabla f_i(x) ; \quad \alpha_i \geq 0, i = 1, \ldots, m; \quad \sum_{i=1}^{m} \alpha_i = 1 \right\}$$

(hence $-\Lambda f(x)$ is the orthogonal projection of the origin onto $\overline{U}$).

Next we revise some useful properties of the multiobjective steepest descent direction.

Lemma 2.4 ([5, Lemma 1]). For any $x \in \Omega$,

1. if $x$ is Pareto critical, then $\theta_f(x) = 0$ and $\Lambda f(x) = 0$;
2. If $x$ is not Pareto critical then $\theta_f(x) < 0$, $\Lambda f(x) \neq 0$ and

$$\langle \Lambda f(x), \nabla f_i(x) \rangle \leq -\frac{1}{2}\|\Lambda f(x)\|^2 \quad i = 1, \ldots, m.$$  

If, additionally $\nabla f_i$, $i = 1, \ldots, m$, are continuous, then $x \mapsto \Lambda f(x)$ and $x \mapsto \theta_f(x)$ are continuous.

3 H"older continuity of the MO steepest descent direction

The main result of this work is that, under the assumption of Lipschitz continuity of the objective functions’ gradients, the MO steepest descent direction is locally Hölder continuous with optimal exponent $1/2$. We will show that in general, even for gradients with polynomial components, the MO steepest descent direction and, equivalently, the minimal norm element on the convex hull of the gradients of the objective function, fails to be Lipschitz continuous.

Theorem 3.1. Suppose that $W \subseteq \Omega$ is convex, bounded, and $\nabla f_i$, $i = 1, \ldots, m$, are $L$-Lipschitz continuous on $W$, that is,

$$\|\nabla f_i(y) - \nabla f_i(z)\| \leq L\|y - z\| \quad \forall y, z \in W, i = 1, \ldots, m. \quad (8)$$

Then

1. $x \mapsto \Lambda f(x)$ is Hölder continuous on $W$;
2. $x \mapsto \|\Lambda f(x)\|$ is Lipschitz continuous on $W$.

Proof. Define, as in [5, Section 3], for $x \in \Omega$

$$\phi_x(v) = \max\{\langle \nabla f_i(x), v \rangle : i = 1, \ldots, m\} \quad (v \in \mathbb{R}^n). \quad (9)$$
Observe that $\phi_x$ is a convex sublinear functional and for $y, z \in W$, $|\phi_y(v) - \phi_z(v)| \leq L\|y - z\|\|v\|$. Let

$$M = \max_{i=1,\ldots,m, \, x \in W} \|\nabla f_i(x)\|.$$ 

Since $W$ is bounded, $M < \infty$.

Take $y, z \in W$ and let $v_y = \Lambda f(y)$, $\alpha_y = \theta_f(y)$, $v_z = \Lambda f(z)$ and $\alpha_z = \theta_f(z)$. In view of Definition 2.1 and (9), the solution and optimal value of the problems

$$\min \, \phi_y(v) + \frac{1}{2}\|v\|^2, \quad \min \, \phi_z(v) + \frac{1}{2}\|v\|^2, \quad v \in \mathbb{R}^n$$

are, respectively, $v_y, \alpha_y$ and $v_z, \alpha_z$. Therefore,

$$\phi_y(v) + \frac{1}{2}\|v\|^2 \geq \phi_z(v) + \frac{1}{2}\|v\|^2 - L\|y - z\|\|v\|$$

$$\geq \alpha_z + \frac{1}{2}\|v - v_z\|^2 - L\|y - z\|\|v\|$$

where the first inequality follows (8) and (9) and the second inequality follows from the 1-strong convexity of $v \mapsto g_z(v) + \|v\|^2/2$ and the optimality of $v_z$ for this function. Substituting $v_y$ for $v$ in the above inequalities we conclude that

$$\alpha_y \geq \alpha_z + \frac{1}{2}\|v_y - v_z\|^2 - L\|y - z\|\|v_y\|.$$ 

By the same token,

$$\alpha_z \geq \alpha_y + \frac{1}{2}\|v_y - v_z\|^2 - L\|y - z\|\|v_z\|.$$ 

Adding the above inequalities we conclude that

$$\|v_y - v_z\|^2 \leq L\|y - z\|\left(\|v_y\| + \|v_z\|\right).$$

Since $W$ is bounded, $M = \max_{i=1,\ldots,m, \, x \in W} \|\nabla f_i(x)\| < \infty$. It follows from Corollary 2.3 that $\|v_y\| \leq M$ and $\|v_z\| \leq M$. Hence

$$\|\Lambda_f(y) - \Lambda_f(z)\| = \|v_y - v_z\| \leq \sqrt{2LM} \|y - z\|^{1/2},$$

which proves the Hölder continuity of $x \mapsto \Lambda f(x)$ on $W$ with exponent $1/2$.

To prove item 2, let $U_y$ and $U_z$ be the convex hulls of the gradients of the objective functions at $y$ and $z$, respectively. The minimal norm element of $\overline{U}_y$ is

$$-\Lambda_f(y) = \sum_{i=1}^m \alpha_i^* \nabla f_i(y).$$
for some \( \alpha^* \in \mathbb{R}^m_+ \) such that \( \sum_{i=1}^{m} \alpha^*_i = 1 \) Define \( \tilde{u} = \sum_{i=1}^{m} \alpha^*_i \nabla f_i(z) \). Then \( \tilde{u} \in \overline{U}_z \) and
\[
\| - \Lambda_f(y) - \tilde{u} \| \leq \sum_{i=1}^{m} \alpha^*_i \| \nabla f_i(y) - \nabla f_i(z) \| \leq \sum_{i=1}^{m} \alpha^*_i L \| y - z \| = L \| y - z \|
\]

Since \( -\Lambda_f(z) \) is the minimal norm element of \( \overline{U}_z \),
\[
\| \Lambda_f(z) \| \leq \| \tilde{u} \| \leq \| \Lambda_f(y) \| + \| - \Lambda_f(y) - \tilde{u} \| \leq \| \Lambda_f(y) \| + L \| y - z \|
\]

By the same token, \( \| \Lambda_f(y) \| \leq \| \Lambda_f(z) \| + L \| y - z \| \) and the conclusion follows. \( \Box \)

Finally, we establish the optimality of the Hölder exponent 1/2.

**Proposition 3.2.** Under the assumptions of Theorem 3.1, the Hölder exponent 1/2 derived there can not be improved.

**Proof.** Let
\[
\Omega = \mathbb{R}^2, \quad f_1(r, s) = \frac{r^2 + s^2}{2}, \quad f_2(r, s) = r, \quad f(x) = (f_1(x), f_2(x)) \quad \text{for} \quad x \in \mathbb{R}^2,
\]
and define, for \( 0 < t < \pi/2 \),
\[
y_t = \cos t \cos t, \quad z_t = (1, \cos t \sin t).
\]
Direct use of these definitions yields
\[
\nabla f_1(y_t) - \nabla f_2(y_t) = y_t - (1, 0) = (- \sin^2 t, \sin t \cos t) = \sin t(- \sin t, \cos t) \perp \nabla f_1(y_t)
\]
and
\[
\nabla f_1(z_t) - \nabla f_2(z_t) = z_t - (1, 0) = (0, \cos t \sin t) \perp \nabla f_2(z_t).
\]
Therefore, \( \nabla f_1(y_t) \) is the minimal norm element on the segment \( [\nabla f_1(y_t), \nabla f_2(y_t)] \), \( \nabla f_2(z_t) \) is the minimal norm element on the segment \( [\nabla f_1(z_t), \nabla f_2(z_t)] \), and it follows from Corollary 2.3 that
\[
\Lambda_f(y_t) = -\nabla f_1(y_t) = -y_t, \quad \Lambda_f(z_t) = -\nabla f_2(z_t) = -(1, 0).
\]
Combining the above equalities with (12) and the assumption \( 0 < t < \pi/2 \) we conclude that
\[
\| \Lambda_f(y_t) - \Lambda_f(z_t) \| = \| (1, 0) - y_t \| = \sin t, \quad \| y_t - z_t \| = (\sin t)^2.
\]

Let \( V \) be a neighborhood of \((1, 0)\) and \( \eta \in (0, 1) \). Since \( y_t \in V \) and \( z_t \in V \) for \( t > 0 \) small enough
\[
\sup \left\{ \frac{\| \Lambda_f(y) - \Lambda_f(y) \|}{\| y - z \|^\eta} : y, z \in V, y \neq z \right\} \geq \lim_{t \to 0^+} \sup \frac{\| \Lambda_f(y_t) - \Lambda_f(z_t) \|}{\| y_t - z_t \|^\eta} = \lim_{t \to 0^+} \frac{\sin t}{(\sin t)^{2\eta}}.
\]
To end the proof, observe that the above \( \limsup \) is \( +\infty \) for \( \eta \in (1/2, 1] \). \( \Box \)
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