Entanglement of electron spins in superconductors

Sangchul Oh and Jaewan Kim
School of Computational Sciences, Korea Institute for Advanced Study, Seoul 130-012, Korea

(Dated: November 5, 2021)

We investigate entanglement of two electron spins forming Cooper pairs in an s-wave superconductor. The two-electron space-spin density matrix is obtained from the BCS ground state using a two-particle Green’s function. It is demonstrated that a two spin state is not given by a spin singlet state but by a Werner state. It is found that the entanglement length, within which two spins are entangled, is not the order of the coherence length but the order of the Fermi wave length.

PACS numbers: 03.67.Mn, 03.67.-a, 03.65.Ud, 74.50.+r

*Electronic address: scoh@kias.re.kr
†Electronic address: jaewan@kias.re.kr

Entanglement, referring to the nonlocal quantum correlation between subsystems, is one of the key resources in quantum teleportation, quantum communication, and quantum computation [1]. A study of entanglement in many-body systems is of importance for not only its application to quantum information processing but also giving us new insights on its relevant physics. For example, entanglement in many spin systems has been investigated in connection with quantum phase transition [2]. For a non-interacting electron gas, the entanglement length within which two electron spins are entangled is the order of the Fermi wave length \( \lambda_F = 2\pi/k_F \) [3, 4].

A solid state entangler, analogous to the parametric down conversion producing a pair of entangled photons in quantum optics, is a device to generate a pair of entangled electrons in controlled way. It is of interest for the realization of scalable solid-state quantum computers. There have been various proposals to create entangled pairs in solid state systems; spin-entanglement via a quantum dot [5, 6] or via a magnetic impurity [7], generation of entangled electron spins by extracting a Cooper pair out of a superconductor [8, 9, 10, 11, 12, 13, 14], entangled electron-hole pairs in a degenerate electron gas [12, 14], two particle orbital entanglement [17, 18], and entangled spins in electron gases due to exchange interaction [3, 4, 19]. Usually entangling process takes two steps, generation of an entangled electron pair via some kind of interaction and separation of them from each other.

A Cooper pair of a BCS superconductor is composed of two electrons with opposite momenta and a spin singlet state, \( \uparrow \) and \( \downarrow \) [20]. Since the size of a Cooper pair is the order of the coherence length \( \xi \) (\( \sim 10^{-4} \) cm), entanglement of electron spins may survive within that scale. This means the entanglement length may be about the coherence length. If one can extract a Cooper pair and separate two electrons from each other, the superconductor may be a good natural resource of entangled spin states. In Refs. [3, 4, 10, 11, 12, 13, 14] it is implicitly assumed that the distance between two tunnel junctions attached to the superconductor should be less than the coherence length \( \xi \). Also it is unclear whether a spin state of a tunneled electron pair is a spin singlet state (one of four Bell states) or a mixed state.

In this Letter we address this problem and find the two-spin state of the BCS ground state is not given by a Bell state but by a Werner state. We investigate entanglement of the two-spin state as a function of the relative distance between two electrons. Surprisingly, we find the entanglement length is not the order of the coherence length \( \xi \) but the order of the Fermi wave length \( \lambda_F \).

Let us start with the pairing Hamiltonian of the BCS theory [20, 21]

\[
H = \sum_{k} \varepsilon_k c_{k \sigma}^{\dagger} c_{k \sigma} + \sum_{k, k'} V_{kk'} c_{k \uparrow}^{\dagger} c_{-k \downarrow}^{\dagger} c_{-k' \downarrow} c_{k' \uparrow},
\]

where \( c_{k \sigma}^{\dagger} \) is a creation operator for electrons of wave vector \( k \) and \( \sigma \)-component of spin \( s \). The normalized BCS ground state of Eq. (1) is given by

\[
|\psi_0 \rangle = \prod_k (u_k + v_k c_{k \uparrow}^{\dagger} c_{k \downarrow}^{\dagger}) |0 \rangle,
\]

where coefficients \( u_k \) and \( v_k \) are written by

\[
u_k^2 = \frac{1}{2} \left( 1 + \frac{\varepsilon_k - \mu}{E_k} \right), \quad (3a)\]

\[
u_k^2 = \frac{1}{2} \left( 1 - \frac{\varepsilon_k - \mu}{E_k} \right), \quad (3b)
\]

Here \( E_k = \sqrt{(\varepsilon_k - \mu)^2 + \Delta_k^2} \) is the excitation energy of a quasi-particle of wave vector \( k \) and \( \Delta_k \) the superconducting gap.

In order to investigate the entanglement of two-electron spins forming Cooper pairs, we introduce the two-electron space-spin density matrix in the second quantization

\[
\rho^{(2)}(x_1, x_2; x'_1, x'_2) = \frac{1}{2} \hat{\psi}^\dagger(x'_1) \hat{\psi}^\dagger(x'_2) \hat{\psi}(x_1) \hat{\psi}(x_2), \quad (4)
\]

where \( \langle \cdots \rangle = \langle \psi_0 \cdots | \psi_0 \rangle \) at zero temperature and \( x = (r, s) \). The two-particle Green’s function is defined by

\[
G(1, 2; 1', 2') = -\langle T [\hat{\psi}_H(1) \hat{\psi}_H(2) \hat{\psi}_H^\dagger(2') \hat{\psi}_H^\dagger(1')] \rangle, \quad (5)
\]
where $t^+$ denotes time infinitesimally later than $t$.

The two-particle Green’s function for the superconducting state can be factorized into single-particle Green’s functions:

$$G(1,2;1',2') = G(1,1')G(2,2') - G(1,2')G(2,1') - F(1,2)F^\dagger(1',2') .$$

The single-particle Green’s function is given by

$$G(1,1') = \frac{\pi}{2}\delta_{s,s'} G(r_1t_1, r_1't_1') ,$$

where its spin dependence for a non-magnetic system becomes a unit matrix $\delta_{s,s'}$. The anomalous Green’s function is written by

$$F^\dagger(1,2) = -i\langle [\hat{\psi}_H^\dagger(x_1t_1)\hat{\psi}_H(x_2t_2)] \rangle$$

whose spin dependence is given by an antisymmetric matrix

$$I_{ss'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i\sigma_y .$$

Also notice that $F(1,2) = I_{s,s}F(r_1t_1, r_2t_2)$.

Since the system is translationally-invariant and the Hamiltonian of the system is time-independent, the Green’s functions $G$ and $F$ depend only on the relative coordinates $r_1 - r_1'$ and the time difference $t_1 - t_1'$. In the limit that $t_1'$ goes to $t_1$, the spatial part of $G$ becomes

$$iG(r) = \langle \psi_0 | \psi_0^\dagger(r_1) \psi_0^\dagger(r_1') \psi_0 \rangle$$

where

$$\hat{\psi}_0^\dagger(r) = \frac{1}{\sqrt{\mathcal{V}}} \sum_k c_{k}\psi e^{ikr} .$$

is the field operator in Schrödinger representation. In the continuum limit one obtains

$$iG(r) = \frac{1}{\mathcal{V}} \sum_k v_k^2 \sin(kr)k \, dk .$$

Due to the similarity between $v_k^2$ of the BCS ground state and the Fermi function of an ideal electron gas at the critical temperature $T_c$, as shown in Fig. 11, Eq. 11 has a similar form of Eq. (17) in Ref. 4. The electron density $n \equiv N/V$ can be calculated by $iG(0) = n/2$. Similarly, the spatial part of $F^\dagger$ is given by

$$iF^\dagger(r_1 - r_2) = \langle \psi_0 | \psi_0^\dagger(r_1) \psi_0^\dagger(r_2) \psi_0 \rangle$$

$$= \frac{1}{\mathcal{V}} \sum_k v_k u_k e^{i(kr_1 - r_2)} .$$

Note that $F^\dagger(r_1 - r_2) = [F(r_1 - r_2)]^*$.

In the continuum limit one has

$$iF(r) = \frac{1}{\pi^2r} \int_{\mathcal{R}} \frac{\sin(kr)k}{\sqrt{\xi/\Delta}^2 + 1} \, dk$$

where $\xi_k \equiv c_k - \mu$, $\Delta_k = \Delta \theta(\omega_D - |\xi_k|)$ with a step function $\theta(x)$, and the integration over $k$ should be done on the range such that $-\hbar\omega_D \leq \xi_k \leq \hbar\omega_D$.

Here $\omega_D$ is the Debye frequency. Thus Eq. (15) becomes approximately

$$iF(r) \approx N(0)\Delta \frac{\sin(k_Fr)}{k_Fr} K_0(\frac{r}{\pi\xi_0})$$

where $N(0)$ is the density of states for one spin projection at the Fermi surface, given by

$$N(0) = \frac{1}{2\pi^2} \left[ k^2 \frac{dk}{d\xi} \bigg|_{\xi = \xi_F} \right] = \frac{\pi k_F}{2\pi^2\hbar^2} ,$$

and $K_0(y)$ is a Bessel function of order 0

$$K_0(y) = \int_0^\infty dt \frac{\cos(yt)}{\sqrt{1+t^2}} \approx \sqrt{\frac{\pi}{2y}} e^{-y} .$$

From Eqs. 9, 8 and 4, we have the two-electron space-spin density matrix

$$\rho_{s_1',s_2';s_1,s_2}(r_1, r_2; r_1', r_2')$$

$$= \frac{1}{2} \left[ \delta_{s_1,s_1'} \delta_{s_2,s_2'} G(r_1 - r_1') G(r_2 - r_2') - \delta_{s_1,s_2'} \delta_{s_2,s_1'} G(r_1 - r_2') G(r_2 - r_1') - I_{s_1,s_2} I_{s_1',s_2'} F(r_1 - r_2) F^\dagger(r_1' - r_2') \right] .$$

Eq. 17 has the same form of the two-electron space-spin density matrix for a non-interacting electron gas, Eq. (8) in Ref. 4, except the last anomalous term. In the limit that $|r_i - r_i'| \to \infty$ for $i = 1, 2$, one has

$$\rho_{s_1,s_2; s_1',s_2'}(r_1, r_2; r_1', r_2')$$

$$\to \frac{1}{2} I_{s_1,s_2} I_{s_1',s_2'} F(r_1 - r_2) F^\dagger(r_1' - r_2')$$

which shows the off-diagonal long range order of a superconductor.

Let us consider the case $r_1 = r_1'$, $r_2 = r_2'$, which is equivalent to take only the diagonal elements of the space density matrix. For a solid state entangler that produce entangled spins out of a superconductor, two leads are
attached to two tunneling points \(r_1\) and \(r_2\) on the superconductor. This implies one electron is located at \(r_1\) and the other at \(r_2\). The two spin state depending on the relative distance \(r = r_1 - r_2\) is given by

\[
\rho_{s_1, s_2; s'_1, s'_2}^{(2)}(r) = -\frac{1}{2}\left[\delta_{s_1, s'_1}\delta_{s_2, s'_2}G^2(0) - \delta_{s_1, s'_2}\delta_{s_2, s'_1}G^2(r) - I_{s_1, s_2}I_{s'_1, s'_2}|F(r)|^2\right]. \tag{19}
\]

Let us define functions \(g(r) \equiv G(0)/G(0) = 2tG(r)/\pi\) and \(f(r) \equiv F(r)/G(0)\). Eq. (19) becomes

\[
\rho_{s_1, s_2; s'_1, s'_2}^{(2)}(r) = \frac{n^2\mathcal{N}}{8}\rho_{12}, \tag{20}
\]

where the two-spin state \(\rho_{12}\) of the BCS ground state with normalization factor \(\mathcal{N} = 4 - 2g^2 + 2f^2\) is given by

\[
\rho_{12} = \frac{1}{\mathcal{N}}\begin{bmatrix}
1 - g^2 & 0 & 0 & 0 \\
0 & 1 + f^2 & -g^2 & -f^2 \\
0 & -g^2 & f^2 & 1 + f^2 \\
0 & 0 & 0 & 1 - g^2
\end{bmatrix}. \tag{21}
\]

We find that the two-spin state \(\rho_{12}\) is not a spin singlet state but a Werner state characterized by a single parameter \(p\) \((0 \leq p \leq 1)\)

\[
\rho_{12} = (1 - p)\mathbf{I} + p|\Psi^(-)\rangle\langle\Psi^(-)|, \tag{22}
\]

where \(\mathbf{I}\) is a \(4 \times 4\) unit matrix and \(|\Psi^(-)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\). The parameter \(p\) is given by a function of \(f\) and \(g\)

\[
p = \frac{f^2 + g^2}{2 + f^2 - g^2}. \tag{23}
\]

For a non-interacting electron gas, the two-spin-state is also given by a Werner state and can be obtained from Eq. (21) by putting \(f = 0\) \cite{4}.

The properties of a Werner state are well known \cite{4, 24}. According to the Peres-Horodecki separability criterion \cite{20, 27}, a Werner state is entangled for \(p > 1/3\).
one needs to extract a single Cooper pair or to confirm that two electrons are tunnelled out from a single Cooper pair.

In conclusion, we obtained the two-electron reduced density matrix of the BCS ground state based on the Green's function method. We investigated entanglement of two electron-spins forming the Cooper pair in BCS superconductors. It has been found that the two-spin density matrix for a given relative distance between two electrons is given by a Werner state not by a Bell state. Also the entanglement length is not the order of the coherence length $\xi$ but the order of the Fermi wave length $\lambda_F$.

J.K. was supported by Korean Research Foundation Grant KRF-2002-070-C00029. S.O. was partially supported by R&D Program for Fusion Strategy of Advanced Technologies of Ministry of Science and Technology of Korean Government.