Sinusoidal Beds as a Wave Reflector

Viska Noviantri
Department of Mathematics, Bina Nusantara University
West Jakarta, Indonesia
E-mail: viskanoviantri@yahoo.com

Abstract. In this paper, we study the relevance of sinusoidal beds as shoreline protection through the Bragg scattering mechanism. Here we take into account the presence of current and assume that the shore on the right of the sinusoidal beds can absorb wave completely. As well known, a relatively small amplitude of sinusoidal beds can reduce the amplitude of incident waves effectively, due to Bragg resonance. Bragg resonance will occur if the wavelength of the monochromatic wave is twice the wavelength of sinusoidal beds. We apply the multiple scale asymptotic expansion method to the linear Shallow Water Equation for sinusoidal beds. When there is current, two wave numbers lead to Bragg resonances. Otherwise, there will be only one wave number. A transmission and reflection coefficient are given and used to predict the amplitude reduction. The effect of sinusoidal beds to transmission and reflection waves are simulated analytically. It shows that the sinusoidal beds can reduce the amplitude of incident wave.

1. Introduction
When a wave meets a different depth, it will scatter into a transmitted and reflected wave. Let us imagine an incident wave running above a flat bottom with a finite patch of sinusoidal disturbance. During its evolution, there occur many scattering processes. Assume that the shore on the right of the sinusoidal beds can absorb wave completely, then Bragg resonance occurs when the wavelength of incident wave is twice of the wavelength of the periodic bottom disturbance [2]. We find that a larger amplitude disturbance leads to larger reflected wave amplitude. This result explains that the long shore sandbar indeed can reduce the amplitude of incident wave. This motivates the idea of artificially constructing beds to protect a beach from large amplitude of incident waves such as a tidal waves and tsunami.

In this paper, we will take into account the effect of current by analytically. We apply the asymptotic expansion method for the linear shallow water equation for sinusoidal beds in the case when there is a current. The procedure is similar to those applied by C.C. Mei et. al. in [2, 3]. Similar method applied by Philip L.-F. Liu in [4] for water waves in a channel with corrugated boundaries. We obtain there are two wave numbers that may lead to Bragg resonance. These two wave numbers reduce to the Bragg resonance wave number in the no current case. Literatures such as [5] study the numerical method for these problems.

We use the data from Heathershaw experiment [1], to simulate the wave propagating over sinusoidal beds. The larger amplitude of sinusoidal beds and current; leads smaller amplitude of transmitted wave that hit the shore. The same results will occur when the case is subcritical and supercritical detuning.
2. Bragg resonance conditions with current

The aim is to find Bragg resonance condition for surface wave above sinusoidal beds in the case there is a current. The method used is similar with the one use by C.C. Mei in [2], which is multi scale asymptotic expansion.

Consider the SWE equation with current $U$ as follows

$$\begin{align*}
\eta_t + \{(\eta + h)(U + u)\}_x &= 0 \\
u_t + (U + u)u_x + g(\eta + h)_x &= 0.
\end{align*}$$

(1)

with $\eta(\hat{x},\hat{t})$ is surface displacement at position $\hat{x}$ and time $\hat{t}$, $u(\hat{x},\hat{t})$ is horizontal component of fluid particles, $g$ is gravitation coefficient and a bottom topography

$$h(\hat{x}) = h_0(1 + \epsilon \sin K \hat{x})$$

(2)

where $h_0$ is flat depth, $h_0 \epsilon E$ and $K$ is amplitude and wave number of sinusoidal beds, respectively.

2.1. Riemann invariant form

Consider the linear equation of (1) for flat bottom case $h(x) = h_0$, which is

$$\begin{bmatrix}
\eta_t \\
\nu_t
\end{bmatrix} =
\begin{bmatrix}
-U & -h_0 \\
-g & -U
\end{bmatrix}
\begin{bmatrix}
\eta_x \\
\nu_x
\end{bmatrix}$$

(3)

In (3), equation for $\eta(\hat{x},\hat{t})$ and $u(\hat{x},\hat{t})$ are coupled. These solutions will obtain by writing them first in the following Riemann invariant form

$$\begin{bmatrix}
G_{1t} \\
G_{2t}
\end{bmatrix} =
\begin{bmatrix}
-U + \sqrt{gh_0} & 0 \\
0 & -U - \sqrt{gh_0}
\end{bmatrix}
\begin{bmatrix}
G_{1x} \\
G_{2x}
\end{bmatrix}$$

(4)

with transformation

$$[G_i \ G_j]^T = P^{-1}[\eta \ u]^T$$

(5)

where

$$P = \begin{bmatrix}
\frac{-h_0}{g} & \frac{h_0}{g} \\
\frac{-1}{g} & \frac{1}{g}
\end{bmatrix}$$

(6)

Then, will get the following system of differential equation

$$\begin{bmatrix}
G_{1t} \\
G_{2t}
\end{bmatrix} =
\begin{bmatrix}
(-U + \sqrt{gh_0})^2 & 0 \\
0 & (-U - \sqrt{gh_0})^2
\end{bmatrix}
\begin{bmatrix}
G_{1x} \\
G_{2x}
\end{bmatrix}$$

(7)

Now, in equation (7), the equation for $G_1$ and $G_2$ are separated, and they have D’Alembert solution. Assume that each solution is monochromatic wave, then

$$\begin{bmatrix}
G_1 \\
G_2
\end{bmatrix} =
\begin{bmatrix}
A e^{ik_3 x - i\omega t} + B e^{ik_3 x + i\omega t} \\
C e^{ik_3 x - i\omega t} + D e^{ik_3 x + i\omega t}
\end{bmatrix}$$

(8)
with disperse relation

\[
\frac{\omega}{k_1} = \pm c_1, \quad \frac{\omega}{k_2} = \pm c_2,
\]  

(9)

where \( \omega \) is the wave frequency, \( k_1 \) and \( k_2 \) are wave numbers, and \( c_1 = \sqrt{gh_0 - U} \) and \( c_2 = \sqrt{gh_0 + U} \) are phase velocity of each monochromatic wave.

### 2.2. Multiscale expansion method

We apply the multiple scale expansion method in order to find an analytical solution of (1), in the case of near resonance, which is valid for all \( \hat{x} \) and \( \hat{t} > 0 \). And now, we introduce fast and slow variables in space and time as belows

\[
x = \hat{x}, \bar{x} = \varepsilon \hat{x}
\]  

(10)

\[
t = \hat{t}, \bar{t} = \varepsilon \hat{t}
\]  

(11)

The relations between partial derivatives are

\[
\partial_{\bar{x}} = \partial_{\hat{x}} + \alpha \partial_{\hat{t}}, \quad \partial_{\bar{t}} = \partial_{\hat{x}} + \alpha \partial_{\hat{t}}.
\]  

(12)

Next, we expand

\[
\eta(x, \bar{x}; \bar{t}, \bar{T}) = \eta_0(x, \bar{x}; \bar{T}) + \varepsilon \eta_1(x, \bar{x}; \bar{T}) + \ldots,
\]  

\[
u(x, \bar{x}; \bar{t}, \bar{T}) = u_0(x, \bar{x}; \bar{T}) + \varepsilon u_1(x, \bar{x}; \bar{T}) + \ldots,
\]  

(13)

(14)

with \( \varepsilon > 0 \) is a small parameter.

Substituting (13) and (14) into (1) gives us the following series of equations

\[
O(1): \partial_{\bar{x}} \eta_0 - (U^2 + gh_0) \partial_{\bar{x}} \eta_0 - 2U\eta_0 \partial_{\bar{x}} u_0 = 0,
\]  

(15)

\[
O(\varepsilon): \partial_{\bar{x}} \eta_1 - (U^2 + gh_0) \partial_{\bar{x}} \eta_1 - 2U\eta_0 \partial_{\bar{x}} u_1 + 2\left\{ \partial_{\bar{x}} \eta_0 - (U^2 + gh_0) \partial_{\bar{x}} \eta_0 - 2U\eta_0 \partial_{\bar{x}} u_0 \right\} \\
+ gh_0 E \sin Kx \partial_{\bar{x}} \eta_0 + gh_0 EK \cos Kx \partial_{\bar{x}} \eta_0 \\
+ 2U\eta_0 E \sin Kx \partial_{\bar{x}} u_0 + 3U\eta_0 EK \cos Kx \partial_{\bar{x}} u_0
\]  

(16)

Note that equation (15) is equivalent with (3) and hence the solutions obtained from inverse transform of (5). There are

\[
\eta_0 = -\frac{1}{2} \left\{ h\eta e^{i(k_1 \bar{x} - \omega \bar{t})} + B(\bar{x}, \bar{T}) e^{-i(k_1 + \omega) \bar{t}} + c.c. - C(\bar{x}, \bar{T}) e^{i(k_2 - \omega) \bar{t}} - D(\bar{x}, \bar{T}) e^{-i(k_2 + \omega) \bar{t}} + c.c. \right\}
\]  

(17)

\[
u_0 = \frac{1}{2} \left\{ A(\bar{x}, \bar{T}) e^{i(k_1 \bar{x} - \omega \bar{t})} + B(\bar{x}, \bar{T}) e^{-i(k_1 + \omega) \bar{t}} + c.c. + C(\bar{x}, \bar{T}) e^{i(k_2 - \omega) \bar{t}} + D(\bar{x}, \bar{T}) e^{-i(k_2 + \omega) \bar{t}} + c.c. \right\}
\]  

(18)
with $c.c$ denotes their complex conjugate. Note that $\eta_0$ and $u_0$ are superposition of waves with two wave numbers $k_1$ and $k_2$, which now their amplitudes are functions of $\vec{x}$ and $\vec{t}$. Here, $A(\vec{x}, \vec{t})$ and $C(\vec{x}, \vec{t})$ are amplitudes of right running monochromatic wave with wave number $k_1$ and $k_2$, respectively. The others, $B(\vec{x}, \vec{t})$ and $D(\vec{x}, \vec{t})$ are amplitudes of left running monochromatic wave with wave number $k_1$ and $k_2$, respectively. The amplitudes $A(\vec{x}, \vec{t})$, $B(\vec{x}, \vec{t})$, $C(\vec{x}, \vec{t})$ and $D(\vec{x}, \vec{t})$ are complex functions to be determined.

Next, we look for solution of the order-$\varepsilon$ of equation (16) which are $\eta_1$ and $u_1$. Substituting (17) and (18) into the right hand side of (16) will yield exponent terms with wave number $\pm k_1, \pm k_2, \pm (K - k_1), \pm (K - k_2)$.

When $K = 2k_1$, the exponent terms on the r.h.s. with wave number $\pm k_1, K - k_1, -K + k_1$ have the same wave number with the natural mode $\exp(i(k_1x \pm \varepsilon t))$. To avoid unbounded resonance of $\eta_1$ and $u_1$, we can simplify equate to zero the coefficients of those terms, and get the following equations

$$\begin{cases} A_\tau + c_1 A_\tau = \beta B \\ B_\tau - c_1 B_\tau = -\beta A \end{cases}$$

with

$$\beta = \frac{c_0 E k_1}{2c_1} \left( \frac{c_0}{2} - 2U \right), \quad c_0 = \sqrt{gh_0}.$$  

(20)

where $c_0 = \sqrt{gh_0}$. As a check, we put $U = 0$ in the equations (19), we get exactly the equations by C.C. Mei in [1].

Analogously when $K = 2k_2$, we avoid unbounded resonance of $\eta_1$ and $u_1$ caused by terms with wave number $k_2$, so we get

$$\begin{cases} C_\tau + c_2 C_\tau = \beta D \\ D_\tau - c_2 D_\tau = -\beta C \end{cases}$$

(21)

3. Amplitude of transmission and reflection method

Next we study the relevance of sinusoidal bottom in reducing the amplitude of an incident monochromatic wave. Imagine that we have a sinusoidal beds patch at $0 < x < L$. A monochromatic wave coming from the left passes the sinusoidal beds and then propagates to the right until it hit the shore. In region $x < 0$, there is no interaction between the right and left propagating wave, so that the equation in this region is

$$c_1 A_\tau + A_\tau = 0$$

$$c_1^2 B_\tau + B_\tau = 0$$

(22)

When the wave propagates above sinusoidal beds, there will be many scattering processes. The waves split into a transmitted and reflected wave. If the wave number of sinusoidal beds is twice of the wave number of incident wave, then interaction between the right and left propagating wave govern by equation (19).
System of equations (19) can be separated into equations for each $A(\bar{x}, \bar{t})$ and $B(\bar{x}, \bar{t})$

$$
A_{\bar{t}} - c_1^2 A_{\bar{x}} + \beta^2 A = 0,
$$

$$
B_{\bar{t}} - c_1^2 B_{\bar{x}} + \beta^2 B = 0.

(23)
$$

Equations (23) known as the Klein-Gordon equations.

Assume that the shore on the right of the sinusoidal beds can absorb wave completely, then there is no reflected wave in $x > L$. Therefore, the right boundary condition is $B(L,t) = 0$.

Let the incident wave is

$$
\zeta = A(\bar{x}, \bar{t}) e^{ik_1x-i\omega t}
$$

where

$$
A(\bar{x}, \bar{t}) = A_0 e^{iK(\bar{x}-\bar{t})}, \bar{x} < 0
$$

(25)

In $0 < x < L$, assume that the solutions of (23) are

$$
A(\bar{x}, \bar{t}) = A_0 T(\bar{x}) e^{-iK_0\bar{x}}
$$

(26)

$$
B(\bar{x}, \bar{t}) = A_0 R(\bar{x}) e^{-iK_0\bar{x}}
$$

(27)

Here, $T(\bar{x})$ and $R(\bar{x})$ are transmission and reflection coefficient respectively. Note that the solutions must continue at $\bar{x} = 0$ and at $\bar{x} = L$.

Substituting (26) and (27) to (23) give the differential equation for $T(\bar{x})$ and $R(\bar{x})$. These solutions divide into three cases.

Case 1: Subcritical detuning
Subcritical detuning occurs when $K = 2(k_1 + \epsilon \kappa)$, where $\epsilon$ is small parameter and $\kappa$ is constant. In this case, the transmission and reflection coefficient as

$$
T(\bar{x}) = \frac{iQc_1 \cosh Q(\bar{L}-\bar{x}) + \alpha \sinh Q(\bar{L}-\bar{x})}{iQc_1 \cosh Q\bar{L} + \alpha \sinh Q\bar{L}}
$$

(28)

$$
R(\bar{x}) = \frac{-\beta \sinh Q(\bar{L}-\bar{x})}{iQc_1 \cosh Q\bar{L} + \alpha \sinh Q\bar{L}}
$$

(29)

where $Qc_1 = \sqrt{\beta^2 - \alpha^2}$ and $\alpha = \kappa c_1$

Case 2: Supercritical detuning
Supercritical detuning occurs when $K = 2(k_1 + \epsilon \kappa)$, where $\epsilon$ is small parameter and $\kappa$ is constant. In this case, the transmission and reflection coefficient as
\[
T(x) = \frac{Pc_1 \cos P(L - x) - i\alpha \sin P(L - x)}{Pc_1 \cosh QL - i\alpha \sinh PL}
\] (30)

\[
R(x) = \frac{i\beta \sin P(L - x)}{Pc_1 \cosh P L - i\alpha \sinh P L}
\] (31)

where \( Qc_1 = \sqrt{\beta^2 - \alpha^2} \) and \( \alpha = \kappa c_1 \)

Case 3: Perfect resonance

The transmission and reflection coefficient of perfect resonance \((K = 2k_1)\) are

\[
T(x) = A = \frac{\cosh \left(\frac{\beta[L - x]}{c_1}\right)}{\cosh \left(\frac{\beta L}{c_1}\right)}
\] (32)

and

\[
R(x) = \frac{B}{A_0} = \frac{i\sinh \left(\frac{\beta[L - x]}{c_1}\right)}{\cosh \left(\frac{\beta L}{c_1}\right)}
\] (33)

simulation of Bragg resonance will make using the following data: flat depth \( h_0 = 10 \) m wave number of incident wave \( k_1 = \pi /m \), the sinusoidal beds patch has wave number \( 2k_1 = 2\pi /m \), and length \( L = 10 \) m.

![Figure 1](image.png)

**Figure 1.** Left: \( R(0) \) versus \( \varepsilon E \) for some \( U \), right: \( T(L) \) versus \( \varepsilon E \) for some \( U \).

Sketch of \( R(0) \) and \( T(L) \) versus \( \varepsilon E \) for some current \( U \) draw in Figure 1. We can see from them that the larger amplitude of sinusoidal beds leads larger amplitude of reflected wave at \( x = 0 \) and leads smaller amplitude of transmitted wave at \( x = L \). Besides that, the larger current leads the larger transmitted wave amplitude at \( x = L \) and leads the smaller reflected wave amplitude at \( x = 0 \).
Figure 2 shows the reflected wave amplitude in the middle of reflected wave amplitude in subcritical and supercritical detuning case when perfect resonance occurs.

![Graph showing reflected wave amplitude](image)

**Figure 2.** Comparison of reflected wave amplitude in perfect resonance, subcritical and supercritical detuning ($\varepsilon E=0.14$ and $U=0.5$)

4. **Conclusion**

The solution of wave equation over sinusoidal-beds with current obtains by Riemann invariant form. A solution of them is superposition of monochromatic waves with two different wave numbers.

Sinusoidal beds may lead to Bragg resonance. When there is current, two wave numbers lead to Bragg resonances. Otherwise, there will be only one wave number. Bragg resonance occurs when the one of two different wavelength of incident wave is twice of the wavelength of the periodic bottom disturbance. The larger amplitude of sinusoidal beds leads larger amplitude of reflected wave at $x = 0$ and leads smaller amplitude of transmitted wave at $x = L$.

5. **References**

[1] A.D., Heathershaw 1982 Seabed-wave resonance and sandbar growth *Nature* 296 pp 343-345
[2] Chiang C. Mei 2004 Multiple Scattering by an Extended Region of Inhomogeneities *Lecture Notes* MIT
[3] Jie Yu And Chiang C. Mei 2000 Do Longshore Bars Shelter The Shore? *J. Fluid Mech.* 404 pp 251-268
[4] Philip L.-F. Liu 1987 Resonant reflection of water waves in a long channel with corrugated boundaries *J. Fluid Mech.* 179 pg. 371-381
[5] Viska Noviantri and S.R.Pudijaprasetya 2010 The Relevance of Wavy Beds as Shoreline Protection *Proceedings of the 13th Asian Congress of Fluid Mechanics* Pg. 489-492