ON PPT SQUARE CONJECTURE.

WLADYSŁAW ADAM MAJEWSKI

Abstract. A detailed analysis of the PPT square conjecture is given.
1. Introduction

The notion of positive partial transpose maps, PPT maps for short, is playing an important role in the quantum information theory. We remind the reader that linear maps \( T : B(H) \rightarrow B(H) \) that are both completely positive and completely copositive are called PPT maps. The important point to note here is an observation that various classes of positive maps are used to describe quantum and useful maps in bipartite systems. In particular, quantum maps which are useless for preserving quantum entanglement are called entanglement breaking maps. 

\[ T(a) = \sum_k s_k(a)P_k, \]

where \( s_k(a) = (aw_k, w_k), a \in B(H), P_k = |v_k><v_k| \) for some sets of vectors \( w_k, v_k \) in \( H \); see [1], [2].

Recently, Christandl, [3], raised the following conjecture, see also [4], [5], [6]:

**Conjecture 1.1. (PPT square conjecture)** The composition \( T_1 \circ T_2 \) of a pair of \( T_1 \) and \( T_2 \) PPT maps on \( B(H) \) is always entanglement breaking.

It was argued that PPT square conjecture 1.1 is equivalent to

**Conjecture 1.2.** [7] If \( T_1 \) is positive and \( T_2 \) is PPT map then \( T_1 \circ T_2 \) is decomposable.

In Section 2 we give definitions of the concepts just introduced. Furthermore, we will define and discuss the associated classes of positive maps. Section 3 is devoted to a study of PPT square conjecture. In particular, we will present the general approach to the problem. Keeping in mind that entanglement breaking maps are so important in Quantum Information, in Section 4 we will give a detailed analysis of the relevant tensor cones and associated classes of maps. This will be done for finite dimensional cases.

2. Preliminaries

To facilitate access to the considered maps, here, we will make the following assumptions: let \( \{e_i\}_{1}^{d<\infty} \) be a basis in \( H \), where \( d = \dim H \). The conjugation \( J : H \rightarrow H \) is defined as

\[ Jf = J \left( \sum_i (f, e_i)e_i \right) = \sum_i (f, e_i)e_i, \]

where \( f \in H \). In particular, a vector \( f \in H \) is called real if \( Jf = f \). The transposition \( t(\cdot) \) of \( a \in B(H) \) with respect to the basis \( \{e_i\} \) is given by

\[ t(a) = Ja^*J. \]

Our next observation is that any \( V \in B(H) \) can be written as

\[ Vf = V \left( \sum_k (f, e_k)e_k \right) = \sum_k (f, e_k)V e_k = \sum_k (V e_k \otimes \bar{w}) f, \]

where \( f \in H \), and \( v \otimes \bar{w} \cdot z = (z, w)v; \) for details see [8].
In the very special case, if \( \text{Ran}V = 1 \) one has \( Ve_i = \lambda_i x \), where \( x \) is a certain vector in \( \mathcal{H} \) and \( \lambda_i \in \mathbb{F} \). Moreover, in this case,
\[
V = \sum_i \lambda_i x \otimes e_i = x \otimes \sum_i \lambda_i e_i = x \otimes y.
\]

Thus, (\(1\)) can be rewritten as
\[
T(a) = \sum_k v_k \otimes w_k \cdot a \cdot w_k \otimes \overline{w_k} \equiv \sum_k V_k \cdot a \cdot V_k^*,
\]
and, obviously, \( \text{Ran}v_k \otimes \overline{w_k} \equiv \text{Ran}V_k = 1 \).

However, as a quantum system demands infinite dimensional Hilbert space, from now on we make the assumption that Hilbert spaces \( \mathcal{H} \) is, in general, infinite dimensional. We need to distinguish the following families of maps \( T : B(\mathcal{H}) \to B(\mathcal{H}) \): positive, decomposable, completely positive, PPT, and superpositive. To present the concise and rigorous description we will use the selection rule which gives the required characterization in terms of tensor cones, see [9] for all details.

- positive maps; they are determined by the tensor cone \( \text{conv} \{ B(\mathcal{H})_+ \otimes \mathcal{F}_T(\mathcal{H})_+ \} = C_p \). Here and subsequently we are using the identification \( \mathcal{F}_T(\mathcal{H}) = B(\mathcal{H}_*) \).
- decomposable maps; they are determined by the tensor cone \( \text{conv} \{ (B(\mathcal{H}) \otimes \mathcal{F}_T(\mathcal{H}))_+ \cap (\text{id} \otimes t) (B(\mathcal{H}) \otimes \mathcal{F}_T(\mathcal{H}))_+ \} \equiv C_d \).
- completely positive maps; they are determined by the tensor cone \( (B(\mathcal{H}) \otimes \mathcal{F}_T(\mathcal{H}))_+ \equiv C_{cp} \).
- PPT maps; they are determined by the tensor cone \( \text{conv} \{ (B(\mathcal{H}) \otimes \mathcal{F}_T(\mathcal{H}))_+ \cup (\text{id} \otimes t) (B(\mathcal{H}) \otimes \mathcal{F}_T(\mathcal{H}))_+ \} \equiv C_{PPT} \).
- super positive maps; they are determined by the largest tensor cone \( C_1 \). Equivalently, \( T(a) = \sum_i V_i^* a V_i \) is superpositive if there exists a Krauss decomposition such that for all \( V_i \) one has the \( \text{Ran}V_i = 1 \), cf Stømer’s book [11].

We remind the reader that the selection rule is designed to determine certain classes of positive maps with the prescribed properties, see [9]. It reads as follows: the class of positive maps \( P_\alpha \) is defined as
\[
P_\alpha = \{ T \in L(B(\mathcal{H}), B(\mathcal{H})); T(\sum a_i \otimes b_i) = \sum TrT(a_i) t(b_i) \geq 0 \text{ for any } \sum a_i \otimes b_i \in C_\alpha \}
\]
where \( C_\alpha \) is the specified tensor cone. In other words, properties of a positive map \( T \) are specified by the associated functional \( \bar{T} \), for all details see [9].

3. PPT MAPS AND PPT SQUARE CONJECTURE

To get some intuition we begin this section with an example.

**Example 3.1.** Let \( V_1 = v \otimes e_1 + Jv \otimes e_2 \), \( V_2 = Jv \otimes e_1 + v \otimes e_2 \) where \( v \) is a not real vector in \( \mathcal{H} \). \{\( e_i \)\} a basis in \( \mathcal{H} \). Define
\[
T(a) = \sum_{i=1}^2 V_i a V_i^*, \quad a \in B(\mathcal{H}).
\]
Then
\[ T(a) = (ae_1, e_1)v \otimes \tau + (ae_2, e_1)v \otimes Jv + (ae_1, e_2)v \otimes \tau + (ae_2, e_2)v \otimes Jv \]
\[ + (ae_1, e_1)v \otimes Jv + (ae_2, e_1)v \otimes \tau + (ae_1, e_2)v \otimes Jv + (ae_2, e_2)v \otimes \tau, \]
and
\[ t \circ T(a) = T(a). \]

Further, we note
\[ T \circ T(a) = \sum_{ij} V_i V_j^* a V_i^* V_j^*, \]
with
\[ V_i^2 = (v \otimes \tau_1 + Jv \otimes \tau_2) (v \otimes \tau_1 + Jv \otimes \tau_2) \]
\[ = (v, e_1)v \otimes \tau_1 + (e_1, v)v \otimes \tau_2 + (v, e_2)Jv \otimes \tau_1 + (e_2, v)Jv \otimes \tau_2, \]
etc.

It is clear that \( \text{Ran} V_i = 2 \) while \( \text{Ran} V_i^2 \) is also equal to 2 if \( v \) is not orthogonal to \( \{e_1, e_2\} \), \( (v \in \mathcal{H} \) is not a real vector). Obviously, \( T \) is a PPT map. In general, \( T \) is not unital. However, there do exist such vectors \( \{v \} \) determined by vectors \( a, b \)
respectively), where \( a, b \in \mathcal{H} \) are not orthogonal. Therefore, to show that \( T \) is superpositive we apply the characterization of these maps given in terms of the injective cone, see the previous section. To this end it would be enough to observe that, in 2-dimensional case, the tensor cone \( \text{conv} \{ (B(\mathcal{H}) \otimes \mathcal{F}_T(\mathcal{H}))_+ \cup (\text{id} \otimes t)(B(\mathcal{H}) \otimes \mathcal{F}_T(\mathcal{H}))_+ \} \) is equal to the injective cone \( C_1 \).
We will see that this claim is dual to the equivalence, in two dimensional case, of the tensor cone \( C_d \) determining decomposable maps with the tensor cone determining positive maps \( C_p \). Thus to proceed with the examination of superpositivity, we should introduce the notion of duality and develop its description. Then, the superpositivity of \( T \) will follow.

To define the dual tensor cone, firstly, we will define the notion of duality, cf [10]. To this end we begin with an observation that the pair \( < B(\mathcal{H}), \mathcal{F}_T(\mathcal{H}) > \) of vector spaces \( B(\mathcal{H}) \) and \( \mathcal{F}_T(\mathcal{H}) \) is separating dual, i.e.
\[ a \times \varrho \mapsto < a, \varrho > = \text{Tr} \varrho \in \Phi; \ a \in B(\mathcal{H}), \ \varrho \in \mathcal{F}_T(\mathcal{H}), \]
and for any \( a \neq 0 (\varrho \neq 0) \) there is \( \varrho \neq 0 (b \neq 0) \) with \( \text{Tr} \varrho \neq 0 (\text{Tr} b \varrho \neq 0 \) respectively), where \( a, b \in B(\mathcal{H}) \) and \( \varrho, \sigma \in \mathcal{F}_T(\mathcal{H}). \)

The separating dual pairing is defined as, cf Section 15 in [10].
\[ < \sum_n a_n \otimes \varrho_n, \sum_m b_m \otimes \sigma_m > \to \Phi. \]
\[ < \sum_n a_n \otimes \varrho_n, \sum_m b_m \otimes \sigma_m > = \sum_n \sum_m \text{Tr} a_n \sigma_m \cdot \text{Tr} \varrho_n b_m. \]

Now, we are in position to define

**Definition 3.2.** The dual cone \( C_d^\alpha \) is defined as
\[ C_d^\alpha = \{ \sum_n a_n \otimes \varrho_n; \sum_m \sum_n \text{Tr} a_n \sigma_m \cdot \text{Tr} b_m \varrho_n \geq 0 \text{ for all } \sum_m b_m \otimes \sigma_m \in C_\alpha \}. \]
In particular, it is an easy observation that
\[ C_p^d = \{ \sum_{n} a_n \otimes q_n ; \sum_{m} \sum_{n} \text{Tr}a_n \sigma_m \cdot \text{Tr}b_m \varrho_n \geq 0, \text{ for all } b_m \geq 0, \sigma_m \geq 0 \} = C_i. \]

In addition, we note:

**Remark 3.3.**

1. As tensor cones are defined for the projective tensor product, the above tensor products \( \otimes \) denote the projective tensor product \( \otimes_p \).
2. Now it is an easy exercise to prove the final claim of the above example.
3. Finally, the above notion of duality suggests that two conjectures \([1.1]\) and \([1.2]\) are dual each other. We return to this issue at the end of this section.

We now turn to an explanation of what is going on with the composition of a positive map with a PPT map. This will be done using the structure of positive maps, and in particular, the selection rule; for details see [9].

Our first observation is that Conjecture \([1.2]\) can be rephrased as follows: for any positive map \( T_1 \) the functional \( T_0 \) associated with the restriction \( T_0 \equiv T_1|_{T_2 C_d} \), where \( T_2 \) is a PPT map, should be positive on the tensor cone \((B(H) \otimes F_T(H))_+ \cap (id \otimes t) (B(H) \otimes F_T(H))_+ \equiv C_d \).

The associated functional \( T_0 \) to the map \( T_0 \) has the form, cf [9]
\[ T_0(\sum_i a_i \otimes b_i) = \sum_i \text{Tr}T_0(a_i)b_i^i, \]
where \( b_i \equiv t(b) \) and \( \sum_i a_i \otimes b_i \) is taken from the specified tensor cone.

As Conjecture \([1.2]\) claims the decomposability of the composition \( T_1 \circ T_2 \) the specified cone should be \( C_d \).

To proceed with the study of Conjecture \([1.2]\) we will look more closely at the dual relations between the distinguished tensor cones. It easy to check that
\[ C_d = \{ \sum_n A_n \otimes \varrho_n ; \sum_{m,n} \text{Tr}A_n \sigma_m \cdot \text{Tr}B_m \varrho_n \geq 0 \text{ for all } B_m \otimes \sigma_m \in C_d \} = C_{\text{PPT}}. \]

As for the preimage \( R^{-1}(B) \) of a mapping \( R \) one has \( R \circ R^{-1}(B) \subseteq B \), it is easy to see
\[ 0 \leq \langle T_2 C_d, (T_2^d)^{-1} C_{\text{PPT}} \rangle > \]
where \( T_i \equiv T_i \otimes id, i = 1, 2 \), and \( (T_2^d) \) stands for the dual map of \( T_2 \). ( We emphasize that, in general, the map \( T_2^d \) has not the inverse map. Therefore, we employed the inverse image technique.)

**Proposition 3.4.** Let \( T_2 \) be such that \( T_2 C_d \subseteq C_p \). Then, the Conjecture \([1.2]\) holds.

**Proof.** Under the assumption, the composition \( T_1 \circ T_2 \) is well defined on \( C_d \) for any positive map \( T_1 \). Consequently, the associated functional \( T_1 \circ T_2 \) is positive defined on the cone \( C_d \). Thus, the selection rule implies that \( T_1 \circ T_2 \) is a decomposable map.

Further we note \( \langle C_p, C_i \rangle \geq 0 \) as the corresponding cones are dual each other. Hence \( \langle T_2 C_d, C_i \rangle \geq 0 \) and \( \langle C_d, T_2^d C_i \rangle \geq 0 \). Thus
\[ T_2 C_d \subseteq C_p, \text{ implies } T_2^d C_i \subseteq C_{\text{PPT}} \]
Consider $T_1 \circ T_2$ where now $T_1$ is a PPT map. It is obvious that $T_1 \circ T_2$ is well defined on the cone $C_i$. Consequently, $T_1 \circ T_2$ is well defined on $C_i$, and the selection rule implies that $T_1 \circ T_2$ is a superpositivity map. And this shows that the examined Conjectures 1.1 and 1.2 are dual each other in the above sense.

What is left is to examine the condition $T_2 C_d \subseteq C_p$. This will be done in the next Section, for finite dimensional case.

4. Finite dimensional case

In this section we indicate how techniques described in the previous section may be used to examine the relations (20).

As finite dimensional models are prevailing in Quantum Information, from now on, again, we make the assumption that the considered Hilbert space $\mathcal{H}$ is finite dimensional. Needless to say the finite dimension of $\mathcal{H}$ will be assumed only for simplicity. It is possible to proceed the study of the condition $T_2 C_d \subseteq C_p$ in more general setting. It would require the more careful analysis of trace class operators, continuity of corresponding functionals and maps. But, we will not develop this point here. The advantage of finite dimensional approach lies in the fact that it sheds some new light on the structure of the set of positive maps, in particular, on the origin of non-decomposable maps.

We have seen that the condition (20) is crucial and we wish to examine in details this condition. To this end we will look more closely at the image $T_2(C_d)$ of $C_d$ where $T_2$ is a PPT map. We remind that any $a \in B(\mathcal{H}) \otimes B(\mathcal{H})$ can be uniquely represented as

$$a = \sum_{k,l=1}^{\dim \mathcal{H}} A_{kl} \otimes e_{kl} = \sum_{k,l=1}^{\dim \mathcal{H}} e_{ij} \otimes B_{ij},$$

where $A_{kl} = \text{Tr}_2(a \cdot 1 \otimes e_{lk}) \in B(\mathcal{H})$ and the system $\{e_{kl}\}_{k,l=1}^{\dim \mathcal{H}} \in B(\mathcal{H})$ consists of matrix units (so $e_{kl} = e_{lk}$, $e_{kl}e_{mn} = \delta_{lm} e_{kn}$, $\sum_i e_{ii} = 1$). Obviously, similar characterization is valid for $B_{kl}$. Furthermore, as $\dim \mathcal{H} < \infty$, we have used the identification : $F_T(\mathcal{H}) = B(\mathcal{H})$. $\text{Tr}_2$ stands for the partial trace with respect to the second term.

So

$$(B(\mathcal{H}) \otimes B(\mathcal{H}))_+ \ni a^* a = \sum_{kln} A_{kl}^* A_{kn} \otimes e_{ln},$$

and

$$(\text{id} \otimes t) (a^* a) = \sum_{kln} A_{kl}^* A_{kn} \otimes e_{nl}.$$ (23)

Consequently, $C_d$ consists of those elements $\sum_{kln} A_{kl}^* A_{kn} \otimes e_{ln}$ for which

$$\sum_{k} A_{kl}^* A_{kn} \otimes e_{ln} \geq 0,$$ (24)

and

$$\sum_{k} A_{kl}^* A_{kl} \otimes e_{ln} \geq 0.$$ (24)

However, the matrix $\sum_{k} A_{kl}^* A_{kn} \otimes e_{ln} \geq 0$ if and only if there exist $B_{kn} \in B(\mathcal{H})$ such that $\sum_{k} A_{kl}^* A_{kn} \otimes e_{ln} = \sum_{k} B_{kl}^* B_{kn} \otimes e_{ln}$. Thus $C_d$ consists of the following elements $\sum_{kln} A_{kl}^* A_{kl} \otimes e_{ln} = \sum_{kln} B_{kl}^* B_{kn} \otimes e_{ln}$.
However, to examine the image of $C_d$ under a PPT map $\varphi$ we need more informations about PPT maps. As the first step we recall Evans result on $n+1$ positive maps \[12\]. For $n+1$ positive map $\varphi$ the matrix inequality

\[
\|\varphi\|((\varphi(a_i^*a_j))_{i,j=1}^n \geq (\varphi(a_i^*)\varphi(a_j))_{i,j=1}^n
\]
is valid for all $a_1, a_2, \ldots, a_n \in \mathfrak{A}$, where $\mathfrak{A}$ is a $C^*$ algebra. Furthermore, if a bounded linear map $\varphi : \mathfrak{A} \to B(H)$ satisfies these inequalities for all $n$, then $\varphi$ is completely positive.

Applying (25) to elements of $C_d$ we obtain

\[
(\varphi(A_{kl}^*A_{km}))_{l,m=1}^n \geq (\varphi(A_{kl}^*)\varphi(A_{km}))_{l,m=1}^n
\]
as we assumed $||\varphi|| = 1$ (what follows from the assumption that $\varphi$ is unital). If $\varphi$ is PPT map then $\varphi \circ t$ is also CP map. Thus

\[
(\varphi \circ t)(A_{kl}^*A_{km}))_{l,m=1}^n \geq (\varphi \circ t(A_{kl}^*) \varphi \circ t(A_{km}))_{l,m=1}^n
\]
Hence

\[
(\varphi(A_{km}^*(A_{kl})))_{m,l=1}^n \geq (\varphi \circ (A_{kl}^*) \varphi \circ t(A_{km}))_{l,m=1}^n
\]
as for the matrix $[A_{kl}^*A_{km}]_{m \leq 0}$ its transposition is $[A_{km}^*A_{kl}]_{ml}$. Consequently, the PPT property of $\varphi$ ensures that

\[
(\varphi \otimes id)(C_d) \subseteq C_d.
\]

One can say even more: namely for a PPT map $\varphi$ one has

\[
(\varphi \otimes id)(C_{cp}) \subseteq C_d \quad \text{and} \quad (\varphi \circ t \otimes id)(C_{cp}) \subseteq C_d,
\]
what clearly indicates the big difference between CP map and PPT map.

The task is now to examine the structure of $\varphi(C_d) \subseteq C_d$. To answer this question, we begin by noting that $C_d$ is a convex closed set, thus also $C_d^B = \{A \in C_d ; \|A\| \leq B\}$. In particular, $C_d^B$ is a convex compact set. As $\varphi$ is a continuous map, $\varphi(C_d^B)$ is a compact convex set. This clearly forces, by Krein-Milman theorem, that $\varphi(C_d^B)$ is a convex hull of extreme points.

To proceed we need:

**Lemma 4.1.** If $\varphi$ is a PPT map then $\varphi \otimes t$ is a positive map on $B(H) \otimes B(H)$.

Prove. It is easily seen that

\[
(\varphi \otimes id) \circ (id \otimes t) = \varphi \otimes t = \varphi \circ t \otimes t = (\varphi \circ t \otimes id) \circ t \otimes t.
\]

As $t \otimes t$ is a positive map, $t \circ \varphi$ as well as $\varphi \circ t$ are CP maps (cf \[2\]) and the composition of two positive maps is a positive map the claim is proved. 

We next follow the deep Woronowicz \[13\] and Størmer \[11\] idea of looking for a specific vector $v$ in the range of $\varphi(a_0)$, $a \in C_d$.

**Theorem 4.2.** Let for any $a \in (\varphi \otimes id)(C_d)$ there exists a simple tensor $x \otimes y$ in $Ran(a)$, $x, y \in H$. Then, for PPT map $\varphi$ one has $(\varphi \otimes id)(C_d) \subseteq C_p$.

Prove. Our proof starts with an observation that the assumption $x \otimes y \in Ran(a)$ and the assumption of finite dimensionality of $H$ implies existence of $\epsilon > 0$ such that $a \geq \epsilon |x > x| \otimes |y > y|$. Obviously, cf. Lemma 4.1

\[
(id \otimes t) \circ (\varphi \otimes id)(a_0) = (\varphi \otimes id) \circ (id \otimes t)(a_0) \in (\varphi \otimes id)(C_d),
\]
for \( a_0 \in C_d \). In particular, \((\varphi \otimes id) - \epsilon|x| > x > \otimes |y| > y \in (\varphi \otimes id)(C_d)\) where \( \bar{y} = Jy \) and \( J \) is a conjugation on \( \mathcal{H} \) associated with the transposition \( t \). Consequently, any \( a - \epsilon|x| > x > \otimes |y| > y \in (\varphi \otimes id)(C_d) \) is majorized. In particular, as it is true for any \( b_0 \in Ext(\varphi \otimes id)(C_d) \) \( (Ext \) stands for extremal points) we infer that for \( b_0 \) there exists \( \lambda_0 > 0 \) such that \( b_0 = \lambda_0|x| > x|\otimes|y| > y|\).

Hence, it follows easily that under given assumptions

\begin{equation}
\label{eq:31}
(\varphi \otimes id)(C_d) \subseteq C_p,
\end{equation}

as required. \hfill \square

We note that the existence of a product vector \( x \otimes y \) in the range of \((\varphi \otimes id)(a), a \in C_d \) is a nontrivial condition. Namely, there should be \( \mathcal{H} \otimes \mathcal{H} \ni f = \sum_{p} f_p \otimes e_p, f_p \in \mathcal{H} \) such that

\[
\sum_{i} \lambda_i x \otimes e_i = x \otimes y = \sum_{ijp} \varphi(a_{ij}) \otimes e_{ij} \cdot f_p \otimes e_p = \sum_{ij} \varphi(a_{ij})f_j \otimes e_i,
\]

where \( \lambda_i = (y, e_i) \). Hence, for each \( i \) and \( a = \sum_{ij} a_{ij} \otimes e_{ij} \in C_d \) one should have

\[
\lambda_i x = \sum_{j} \varphi(a_{ij})f_j.
\]

Let us now turn to an examination of conditions

\begin{equation}
\label{eq:32}
x \otimes y \in Ran(a); \quad a \in (\varphi \otimes id)(C_d),
\end{equation}

The question of existence of product vectors in a subspace of the tensor product of two Hilbert spaces can be formulated in terms of algebraic geometry. In particular, properties of projective spaces as well as the Segre variety appeared to be crucial. For more information the reader may consult [14]. The following Proposition stated in [13] is a special case of basic theorem given in [14].

**Proposition 4.3** ([13, 16]). Suppose the subspace \( V \subseteq \mathcal{F}^m \otimes \mathcal{F}^n \). If \( Dim(V) > (m - 1)(n - 1) \) then \( V \) contains at least one product vector. Furthermore, if \( Dim(V) > (m - 1)(n - 1) + 1 \) then \( V \) has infinitely many product vectors.

As an application of the above result we consider the following example:

**Example 4.4.** (1) 3D-case. Let \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) where \( \dim \mathcal{H}_1 = 3 = \dim \mathcal{H}_2 \). Consider \( a \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) \) where \( a \geq 0 \). Note that \( \mathcal{H}_1 \otimes \mathcal{H}_2 = \ker a \oplus \text{Ran} a \) (\( \ker \) stands for a kernel while \( \text{Ran} \) for a range). Thus, if \( \ker a < 5 \) then there exists at least one product vector in the range of \( a \). If \( \dim \ker a = 5 \) then \( a \in C_d \) would be defined on 4 dimensional Hilbert space. So, one can suppose that \( a \) is separable and consequently has a product vector in its range.

(2) 4D-case. The condition given in Proposition 4.3 says that subspaces with dimension larger than 9 have a product vector. But there are subspaces with smaller dimension than 9 and larger than 4 where the studied question still is unanswered.

**Remark 4.5.** It is worth pointing out that the conditions given in Theorem 1.1, [17], indicate the optimality of the criterion given in Proposition 4.3.

To formulate next proposition we need the idea of strict positive map, see [18], [19], cf also [20]. We say that \( \varphi : \mathfrak{A} \rightarrow \mathfrak{A} \), where \( \mathfrak{A} \) is a \( C^* \)-algebra, is strictly positive if \( \varphi(a) > 0 \) for all non-zero \( a \in \mathfrak{A}_+ \).
Proposition 4.6. Let $\phi$ be a strictly positive map and $a \in C_{cp}$. Then $\ker a = \{0\}$. In particular, $f \otimes g \notin \ker(\phi \otimes \text{id})(a)$.

Proof. For any $f = \sum_k f_k \otimes e_k$ one has

$$0 < \sum_{k,l,i,j} (f_k \otimes e_k, \phi(a_{ij} \otimes e_{ij} f_l \otimes e_l) = \sum ij (f_i, \phi(a_{ij}) f_j).$$

Hence $f = \sum f_i \otimes e_i \notin \ker a$.

Therefore, as $\mathcal{H} = \ker a \oplus \text{Ran } a (a \geq 0)$ one can find a product vector in the range of $a$. It is easily seen from Theorem 4.2 that $(\phi \otimes \text{id})(C_d) \subseteq C_p$.

Finally, we consider two specific types of positive mappings. The first type we will consider is an extreme CP mapping and we will examine its action on the cone $C_d$. The significance of such analysis stems from the following observation. Any element of $C_p$ can be written as

$$\sum_i a_i \otimes b_i = \sum_{kl} (\sum_i |\lambda_{kl}^i|a_i) \otimes e_{kl} = \sum_{kl} (\sum_i |\lambda_{kl}^i|a_i \otimes e_{kl}),$$

where $a_i, b_i$ are positive operators in $B(\mathcal{H})$ and, for each $i$, $|\lambda_{kl}^i|$ is a semipositive defined complex matrix. Consequently, $C_p$ is a conical hull of specific subcones.

We note that an extreme CP map $\phi$ has the form $\phi(a) = V^*aV$ where $V \in B(\mathcal{H})$ see [21]. Assume that the map $\phi$ is also a PPT map. Then, as $C_d \ni a = \sum_n \lambda_n x_n \otimes x_n^*$, where $\lambda_n \geq 0$ and $\{x_n\}$ form an orthonormal system, one gets

$$(\phi \otimes \text{id})(a) = \sum_n \lambda_n (\phi \otimes \text{id})(|x_n><x_n|).$$

However, $|x_n><x_n| = \sum_{ij} |x_i^n><x_j^n| \otimes e_{ij}$ as $x_n = \sum_i x_i^n \otimes e_i$, $x_i^n \in \mathcal{H}$. Hence

$$(\phi \otimes \text{id})(a) = \sum_n \lambda_n \sum_{ij} |V^* x_i^n><x_j^n| \otimes e_{ij}.$$ But, if $\phi$ is PPT map then the matrices $[V^* x_i^n><x_j^n]_{ij}$ and $[V^* x_i^n><x_j^n]_{ij}$ should be semipositive. By Woronowicz arguments [13], we can infer that $\{x_i^n\}_i$ are simple vectors. Thus $x_i^n = c_i^n y^n$ where $c_i^n \in \mathfrak{C}$. Consequently

$$(\phi \otimes \text{id})(a) = \sum_n \sum_{ij} \lambda_n c_i^n c_j^n |y^n><y^n| \otimes e_{ij}.$$ As $\sum_{ij} \lambda_n c_i^n c_j^n |y^n><y^n| \otimes e_{ij} = |y^n><y^n| \otimes d^n$ and $0 \leq d^n \in B(\mathcal{H})$ we infer that $(\phi \otimes \text{id})(a) \in C_p$.

The second type of positive mapping we will consider is rank-1 nonincreasing positive map $\varphi$, i.e. a map such that $\varphi(P) \leq 1$ for every 1-dimensional projection $P$, see [22], [23], where it is shown that such a mapping has a form analogous to extreme CP map. Consequently, analogous arguments to those given above show that the action of $\varphi \otimes \text{id}$ on the $C_d$ cone is similar to the action of the extreme CP mapping $\phi$.

5. ACKNOWLEDGMENTS

The author wishes to express his thanks to Marcin Marciniak for drawing the author’s attention on non-uniqueness of the Kraus decomposition, to Louis Labuschagne for his helpful comments and to A. Müller-Hermes for his critical comments.
References

[1] M. Horodecki, P. W. Shor, M. B. Ruskai, General entanglement breaking channels, Rev. Math. Phys. 15, 629-641 (2003)
[2] M. Kennedy, N. A. Manor, V. I. Paulsen, Composition of PPT maps, Quantum Information & Computation 18 Issue 5-6 (2018)
[3] M. B. Ruskai, M. Junge, D. Kribs, P. Hayden, A. Winter, Operator structures in quantum information theory, Final Report, Banff International Research Station (2012)
[4] S. Bäuml, On the bound key and the use of bound entanglement, Diploma Thesis, Maximilians Universität München (2010)
[5] S. Bäuml, M. Christandl, K. Horodecki, A. Winter, Limitations of key repeaters, Nat. Comm. 6, 6908 (2015)
[6] S–H. Kye, E. Størmer, Duality between tensor products and mapping spaces through ampliation, arXiv 2002.09614 [math.OA]
[7] M. Christandl, A. Müller-Hermes, M. M. Wolf, When do composed maps become entanglement breaking? Ann. Henri Poincaré, 20 2295-2322 (2019)
[8] R. Schatten, Norm ideals of completely continuous operators, Springer Verlag, 1970
[9] W. A. Majewski, On the structure of positive maps, Positivity, DOI 10.1007/s11117-019-00708-x (2019), the preliminary version is given in arXiv 1812.01607, v.2[math.OA]
[10] A. Defant, K. Floret, Tensor norms and operator ideals, North-Holland, 1993
[11] E. Størmer, Positive linear maps of operator algebras, Springer, 2013
[12] D. E. Evans, Positive linear maps on operator algebras, Commun. math. Phys., 48, 15-22 (1976)
[13] S. L. Woronowicz, Positive maps of low dimensional matrix algebras, Rep. Math. Phys. 10, 165 (1976)
[14] R. Hartshorne, Algebraic Geometry, Springer Verlaag, New York (1977)
[15] L. Chen, D.Z. Dokovic, Properties and construction of extreme bipartite states having positive partial transpose, Commun. Math. Phys., 323, 241-284 (2013)
[16] Y. Liang, J. Yan, D. Si, L. Chen "Inertia of partial transpose of positive semidefinite matrices, arXiv:2310.16567 [quant-ph] 25 Oct 2023
[17] Y-H Kiem, S-H Kye, J. Na, Product vectors in the ranges of multi-partite states with positive partial transposes and permanents matrices, Commun. Math. Phys., 338, 6211-639 (2015)
[18] D. E. Evans, R. Høegh-Krohn, Spectral properties of positive maps on $\mathcal{C}^*$-algebras, J. London Math. Soc. 2s2-17 345-355 (1978)
[19] W.A. Majewski, D. W. Robinson, Strictly positive and strongly positive semigroups, J. Austral. Math. Soc. (Series A) 34 36-48 (1983)
[20] J. Agredo, F. Fagnola, D. Poletti, The Kossakowski matrix and strict positivity of Markovian Quantum Dynamics, Open Systems & Inf. Dyn. 29 (2022)
[21] W. Arverson, Subalgebras of $\mathcal{C}^*$-algebras, Acta Math. 123, 141-224 (1969)
[22] M. Marciniak, On extremal positive maps acting between type I factors, Banach Center Publications 89, 201-221 (2010)
[23] E. B. Davies, Quantum stochastic processes, Commun. math. Phys. 15 277-304 (1969)

DSI-NRF CoE in Math. and Stat. Scl., Focus Area for Pure and Applied Analytics, Internal Box 209, School of Comp., Stat., & Math. Scl., NWU, PVT. BAG X6001, 2520 Potschefstroom, South Africa

Email address: fizwan@gmail.com