ALTERNATING KNOTS DO NOT ADMIT COSMETIC CROSSINGS

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ABSTRACT. By examining the homology groups of a 4-manifold associated to an integral surgery on a knot \( K \) in a rational homology 3-sphere \( Y \) yielding a rational homology 3-sphere \( Y^* \) with surgery dual knot \( K^* \), we show that the subgroups generated by \([K]\) and \([K^*]\) in \( H_1(Y) \) and \( H_1(Y^*) \), respectively, have co-prime orders. We obtain an immediate corollary that, in conjunction with an argument of Lidman–Moore, proves the cosmetic crossing conjecture for knots whose branched double covers are Heegaard Floer L-spaces.

1. Introduction

The cosmetic crossing conjecture attributed to X. S. Lin (see [2, Problem 1.58]) asserts an answer to the question of when a crossing \( c \) in a planar diagram \( D \) for a knot \( K \subset S^3 \) may be changed so that the resulting diagram also presents \( K \). Following Lidman–Moore [3], we define a crossing disk at the crossing \( c \) to be a disk \( E \subset S^3 \) whose geometric intersection number with \( K \) is 2 and whose algebraic intersection number with \( K \) is 0 whose projection to the plane of the diagram \( D \) intersects \( D \) only at the crossing \( c \). A crossing \( c \) is said to be nugatory if the boundary of its crossing disk bounds a disk in \( S^3 \) that is disjoint from \( K \). Performing a crossing change at a nugatory crossing in a planar diagram for \( K \) results in a diagram presenting a knot that is isotopic to \( K \). Any non-nugatory crossing in a planar diagram for \( K \) that admits a crossing change to a diagram presenting a knot that is isotopic to \( K \) is said to be cosmetic. Any crossing that is nugatory or cosmetic is conceivably cosmetic

Conjecture 1 (Cosmetic crossing conjecture). Knots in \( S^3 \) do not admit cosmetic crossings.

Our strategy follows a standard argument inspired by the Montesinos trick, which is the same strategy as that taken up by Lidman–Moore. A crossing in a planar diagram for a knot \( K \) that does not change the isotopy class of the knot presented by the diagram gives rise to a knot \( \kappa \) in the double cover of \( S^3 \) branched along \( K, \Sigma(K) \), with a half-integral Dehn surgery to \( \Sigma(K) \). By combining Gainullin’s Dehn surgery characterization of the unknot in an L-space [1, Theorem 8.2] with the equivariant version of Dehn’s lemma of Yau and Meeks [6], Lidman–Moore show that if \( \Sigma(K) \) is an L-space and \( \kappa \) is nullhomologous, then \( K \) satisfies the cosmetic crossing conjecture. Their intended strategy for showing that \( \kappa \) is nullhomologous breaks down when the determinant of \( K \) is divisible by a square. Our strategy is less sensitive to the divisors of \( |H_1(\Sigma(K))| \) and allows us to prove the following.

Theorem 2. The knot \( \kappa \subset \Sigma(K) \) associated to a conceivably cosmetic crossing in a diagram of \( K \) is nullhomologous.

Corollary 3. If \( \Sigma(K) \) is an L-space, then \( K \) does not admit a cosmetic crossing.
Proof. Let \( \kappa \subset \Sigma(K) \) be the knot associated to a conceivably cosmetic crossing \( c \) in a planar diagram for \( K \). As argued in [3] Section 3, since \( \Sigma(K) \) is an L-space by assumption, \( \kappa \) is null-homologous by Theorem 2, and \( 1/2 \)-surgery on \( \kappa \) is homeomorphic to \( Y \), which is homeomorphic to \( 1/2 \)-surgery on the unknot in \( \Sigma(K) \), it follows from [11] Theorem 8.2 that \( \kappa \) bounds a disk in \( \Sigma(K) \). Furthermore, by an argument known to experts presented in the proof of [3] Proposition 3.3 (cf. [3]), it then follows that the crossing \( c \) is nugatory. \( \square \)

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2. The Trace of Surgery on a Homologically Essential Knot

Let \( Y \) be a rational homology 3-sphere, and let \( K \subset Y \) be a knot. Furthermore, let \( \lambda \) be a framing of the normal bundle of \( K \subset Y \). We say that \( \lambda \)-framed Dehn surgery on \( K \) is integral, for the intersection of \( \lambda \) with a meridian of \( K \) consists of a single point. Let \( Y^* \) be the result of \( \lambda \)-framed Dehn surgery on \( K \) and let \( K^* \) be the core of the surgery solid torus in \( Y^* \), or the surgery dual to \( K \). Recall that since \( \lambda \) is integral, there is some framing \( \lambda^* \) on \( K^* \) so that \( Y \) is obtained by integral \( \lambda^* \)-surgery on \( K^* \subset Y^* \). All homology groups are taken with integer coefficients unless explicitly stated otherwise. For \( K \subset Y \), denote by \( |K| \) the order of the subgroup \( \langle [K] \rangle \subset H_1(Y) \).

As in the standard treatment of framed knots in \( S^3 \), we may form the trace of \( \lambda \)-surgery on \( K \), \( W_\lambda(K) \), by gluing a 4-dimensional 2-handle \( H \) to \( Y \times [0,1] \) along \( K \times \{1\} \) with framing \( \lambda \times \{1\} \), giving us a cobordism \( Y \to Y^* \). In fact, turning this cobordism upside down gives the trace of \( \lambda^* \)-surgery on \( K^* \subset Y^* \). A straightforward computation using the Mayer-Vietoris sequence shows that \( H_2(W_\lambda(K)) \cong \mathbb{Z} \). If \( [K] = 0 \in H_1(Y) \), then we may readily furnish a generator of \( H_2(W_\lambda(K)) \). Since \( K \) is nullhomologous, there is a surface \( \Sigma_K \subset Y \) with \( \partial \Sigma_K = K \); cap this surface off with the core of \( H \), denoted \( c(H) \), to form the closed surface \( \hat{\Sigma}_K \). That \( [\hat{\Sigma}_K] \) generates \( H_2(W_\lambda(K)) \) follows from the long exact of the pair \( (Y \cup c(H), Y) \), which we delineate below:

\[
\ldots \to H_2(Y) \to H_2(Y \cup c(H)) \overset{i}{\to} H_2(Y \cup c(H), Y) \overset{j}{\to} H_1(Y) \to \ldots
\]

The cobordism \( W_\lambda(K) \) admits a deformation retraction onto \( Y \cup c(H) \) obtained by composing the obvious deformation retractions of \( Y \times [0,1] \) onto \( Y \times \{1\} \) and of \( H \) onto \( c(H) \). Since \( Y \) is a rational homology sphere, it follows that \( H_2(Y) \cong \{0\} \), and so \( i \) is an injection from \( H_2(Y \cup c(H)) \cong \mathbb{Z} \), which is generated by some class \([S]\), to \( H_2(Y \cup c(H), Y) \cong H_2(Y \cup c(H)/Y) \cong H_2(S^2) \cong \mathbb{Z} \), generated by \([c(H), \partial c(H)]\). The map \( j \) sends \([c(H), \partial c(H)]\) to \([\partial c(H)] = [K] = 0 \). It follows that \( [\hat{\Sigma}_K] \) generates \( H_2(Y \cup c(H)) \) since \( i([\hat{\Sigma}_K]) \) generates \( \text{ker}(j) \), and therefore it generates \( H_2(W_\lambda(K)) \).

If \( K \) is allowed to be homologically essential, we first appeal to the account given in [5] Section 3 to identify the generator of \( H_2(W_\lambda(K)) \), which we reproduce here. Let \( M \) be the exterior of \( K \), and note that \( H_1(M, \mathbb{Q}) \cong \mathbb{Q} \). Let \( \lambda_M \), called the rational longitude of \( K \), be a simple closed curve on \( \partial M \) whose homology class generates the kernel of the map \( H_1(\partial M, \mathbb{Q}) \to H_1(M, \mathbb{Q}) \) induced by inclusion. Then \( [\lambda_M] \) is torsion in \( H_1(M) \), and so there
is a properly embedded surface \((\Sigma_K, \partial \Sigma_K) \subset (M, \partial M)\) whose boundary is the union of \(|\lambda_M|\) copies of \(\lambda_M\). The surface \(\Sigma_K\) is called a \textit{rational Seifert surface} for \(K\).

The homology classes of the framing curve \(\lambda\) and the meridian of \(K\), \(\mu\), form a basis for \(H_1(\partial M)\), and so we may write \([\lambda_M]\) uniquely as \(a \cdot [\lambda] + b \cdot [\mu]\). Observe now that \(|K| = a \cdot |\lambda_M| = a \cdot \#(\partial \Sigma_K)\). Furthermore, we may cap off each boundary component of \(\Sigma_K\) with a surface obtained from banding together \(a\) copies of \(c(H)\) and \(b\) copies of the meridional disk bounded by \(\mu\) and contained in a regular neighborhood of \(K\) to form the closed surface \(\hat{\Sigma}_K\).

**Proposition 4.** The class \([\hat{\Sigma}_K]\) generates \(H_2(W_\lambda(K))\).

**Proof.** As before, noting that \(Y \cup c(H)\) is a deformation retraction of \(W_\lambda(K)\), we analyze the long exact sequence of the pair \((Y \cup c(H), Y)\):

\[
0 \to H_2(Y \cup c(H)) \xrightarrow{i} H_2(Y \cup c(H), Y) \xrightarrow{j} H_1(Y) \to \ldots
\]

Also as before, \(H_2(Y \cup c(H)) \cong \mathbb{Z} \cong H_2(Y \cup c(H), Y)\), and \(i\) is an injection \(\mathbb{Z} \to \mathbb{Z}\). Since \(j\) sends \([c(H), \partial c(H)]\) to \([K], \ker(j) = (\langle K | [c(H), \partial c(H)]\rangle)\). Notice that \(i([\hat{\Sigma}_K]) = (a|\lambda_M|)[c(H), \partial c(H)]\). Since \(|K| = a|\lambda_M|\), it follows that \([\hat{\Sigma}_K]\) generates \(H_2(W_\lambda(K))\). \(\Box\)

**Remark 5.** By the symmetry of the construction of \(W_\lambda(K)\), we may deduce that \([\hat{\Sigma}_K^*]\) also generates \(H_2(W_\lambda(K))\), and so there are orientations on \(\hat{\Sigma}_K\) and \(\hat{\Sigma}_K^*\) such that \([\hat{\Sigma}_K] = [\hat{\Sigma}_K^*]\).

Henceforth, we suppress notation and refer to \(W_\lambda(K)\) simply as \(W\).

In order to get a handle on the 3-dimensional surgery picture, we now seek to compute \(\iota(\hat{\Sigma}_K, \hat{\Sigma}_K)\), the oriented intersection number of \(\hat{\Sigma}_K\) and a transverse pushoff of itself. To this end, recall that \(\partial W = -Y \bigsqcup Y^*\), so the long exact sequences of the pair \((W, \partial W)\) in homology and cohomology read:

\[
\begin{align*}
(1) & \quad 0 \to H_2(W) \xrightarrow{A} H_2(W, \partial W) \xrightarrow{B} H_1(-Y) \oplus H_1(Y^*) \xrightarrow{C} H_1(W) \xrightarrow{D} H_1(W, \partial W), \\
(2) & \quad 0 \to H^2(W, \partial W) \xrightarrow{A^*} H^2(W) \xrightarrow{B^*} H^2(-Y) \oplus H^2(Y^*) \xrightarrow{C^*} H^3(W, \partial W) \xrightarrow{D^*} H^3(W).
\end{align*}
\]

Recall first that \(H_2(W) \cong \mathbb{Z}\) by Proposition 4. Note that \(H_1(Y)/\langle[K]\rangle \cong H_1(W) \cong H_1(Y^*)/\langle[K^*]\rangle\). Poincaré-Lefschetz duality gives isomorphisms \(H_k(\partial W) \cong H^{3-k}(\partial W), H_k(W) \cong H^{4-k}(W, \partial W),\) and \(H_k(W, \partial W) \cong H^{4-k}(W)\), where, by abuse of notation, in each case we denote the isomorphism by \(PD\), so that \(PD \circ A = A^* \circ PD, PD \circ B = B^* \circ PD, PD \circ C = C^* \circ PD,\) and \(PD \circ D = D^* \circ PD\). Additionally, the universal coefficient theorem allows us to identify \(H_2(W, \partial W) \cong H^2(W)\) with \(H_2(W) \oplus H_1(W)\), since in this case \(H_2(W)\) is free and \(H_1(W)\) is torsion.

Now, the map \(A^*\) sends \(PD[\hat{\Sigma}_K]\), which generates \(H^2(W, \partial W)\), to some element \((p, \alpha) \in \mathbb{Z} \oplus H_1(W)\) where \(p = A^*(PD[\hat{\Sigma}])/\langle[S]\rangle = \iota(\hat{\Sigma}, \hat{\Sigma})\). In turn, there is an identification \(H_2(W, \partial W) \cong \mathbb{Z} \oplus H_1(W)\) such that the map \(A\) is given by \([\hat{\Sigma}_K] \mapsto (p, \alpha)\).

**Proposition 6.** Let \(Y\) be a rational homology sphere, and let \(K \subset Y\) be a knot with an integral surgery slope \(\lambda\) to the rational homology sphere \(Y^*\) and whose surgery dual is \(K^*\), with surgery dual slope \(\lambda^*\). Then \(\iota(\hat{\Sigma}_K, \hat{\Sigma}_K) = |K||K^*|\). Moreover, \(|K|\) and \(|K^*|\) are co-prime.
Proof. Working backward from the right end of the sequence in (1), we will use the exactness of the sequence at each step to show that \(|p| = |K||K^*|\).

Notice first that since every homology class in \(H_1(W) \cong H_1(Y)/\langle[K]\rangle\) is represented by a 1-cycle in \(Y \subset \partial W\), the map \(D\) is identically 0. It follows from exactness that \(C\) is a surjection from \(H_1(-Y) \oplus H_1(Y^*)\) to \(H_1(W)\), and so \(\ker(C) = |K||H_1(Y^*)|\).

Now, since \(\ima(A) = \ker(B)\) and \(\ima(B) = \ker(C)\), we must have \(p \neq 0\), or else \(\ima(B)\) is not finite. Furthermore, it follows that \(B|\text{Tors}(W, \partial W)\) is injective, and moreover that \(|p| = |K||H_1(Y^*)|/|H_1(W)| = |K||K^*|\), as desired.

To see that \(|K|\) and \(|K^*|\) are co-prime, let us first recall that the symmetry of the construction of \(W_\lambda(K) \cong W \cong W_\lambda(K^*)\) yields the equality \([\hat{\Sigma}_K] = [\hat{\Sigma}_{K^*}]\).

By Poincaré-Lefschetz duality, the \(\mathbb{Z}\) summand of \(H^2(W)\) is generated by the dual of the relative homology class of a properly embedded surface \([S, \partial S] \in H_2(W, \partial W)\) such that \(\iota(S, \hat{\Sigma}_K) = 1\). Toward the end of identifying such a surface \(S\), let \(\chi\) be the properly embedded surface obtained as the union of \(K \times [0,1]\) and the core of the 2-handle \(H\) and let \(\chi^*\) be the co-core of \(H\), and note that \(\partial \chi = -K \subset Y\) and \(\partial \chi^* = K^* \subset Y^*\). Note that \(B\) sends the subgroup \(\langle[\chi, -K], [\chi^*, K^*]\rangle \subset H_2(W, \partial W)\) to \(\langle[-K]\rangle \oplus \langle[K^*]\rangle \subset H_1(-Y) \oplus H_1(Y^*)\). It follows then that \(B|\text{Tors}(H_2(W, \partial W))\) is injective, and moreover that \(|p| = |K||H_1(Y^*)|/|H_1(W)| = |K||K^*|\), as desired.

With Proposition 6 established, we are ready to prove the main theorem in support of Theorem 2.

**Theorem 7.** Let \(Y\) be a rational homology sphere, and let \(K \subset Y\) be a knot with \(|K| \equiv 1 \mod 2\). If there is some framing \(\lambda\) of \(K\) such that \(Y^*\), the result of \((\mu+2\lambda)\)-surgery on \(K\), satisfies \(|H_1(Y^*)| = |H_1(Y)|\), then \(|K| = 1\).

**Proof.** We may realize \((\mu+2\lambda)\)-surgery on \(K\) as integer surgery on the link \(L = K \cup m\), where \(m\) is a meridian of \(K\), the surgery slope on \(K\) is the framing curve \(\lambda\), and the surgery slope on \(m\) is the canonical \(-2\)-framing with respect to the Seifert framing on \(m\) induced by the disk \(m\) bounds in \(Y\). By performing \(-2\)-surgery on \(m\) first, we obtain a knot \(K' \subset Y \# \mathbb{RP}^3\) with \(|K'| = 2|K|\), since \(|K| \equiv 1 \mod 2\), that admits an integer framed surgery \(\lambda'\) to \(Y\). Let \(W\) be the trace of \(\lambda'\)-surgery on \(K'\) and let \(K^*\) denote the surgery dual of \(K'\). Note that \(|H_1(Y \# \mathbb{RP}^3)/\langle 2K \rangle\rangle = |H_1(W)| = |H_1(Y^*)/|K^*|\rangle\), and so \(|K^*| = |K|\) by the assumption that \(|H_1(Y)| = |H_1(Y^*)|\). It then follows from Proposition 6 that \(|K| = 1\).

**Proof of Theorem 2.** The Montesinos trick gives a framing \(\lambda\) on \(\kappa \subset \Sigma(K)\) such that \((\mu+2\lambda)\)-surgery on \(\kappa\) yields \(\Sigma(K)\). Since \(|H_1(\Sigma(K))|\) is equal to the determinant of \(K\), and the determinant of a knot in \(S^3\) is odd, it follows that \(|\kappa| \equiv 1 \mod 2\). By Theorem 7, it follows that \(|\kappa| = 1\).
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