NONTRIVIAL SOLUTIONS FOR THE FRACTIONAL LAPLACIAN PROBLEMS WITHOUT ASYMPTOTIC LIMITS NEAR BOTH INFINITY AND ZERO

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Abstract. In this paper we obtain the existence of nontrivial solutions for the fractional Laplacian equations with the nonlinearity may fail to have asymptotic limits at zero and at infinity. We make use of a combination of homotopy invariance of critical groups and the topological version of linking methods.

1. Introduction. The present paper deals with the existence of nontrivial solutions for the following nonlocal elliptic equation

\[
\begin{cases}
(-\Delta)^su = f(x, u) & x \in \Omega, \\
u = 0 & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]  

(1)

where \(s \in (0, 1)\) is fixed, \(\Omega\) is an open bounded subset of \(\mathbb{R}^N\) with Lipschitz boundary, \(N > 2s\), and \((-\Delta)^s\) is the fractional Laplace operator, which (up to normalization factors) is defined as

\[
(-\Delta)^su(x) := \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N.
\]  

(2)

We impose on the nonlinearity \(f\) in (1) the following subcritical growth condition:

\[
(f): \quad f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function satisfying } f(x, 0) \equiv 0, \quad \text{and there exist } C > 0 \text{ and } p \in [1, \frac{2N}{N-2s}) \text{ such that}
\]

\[
|f(x, t)| \leq C \left(1 + |t|^{p-1}\right) \text{ uniformly in a.e. } x \in \Omega \text{ and } t \in \mathbb{R}.
\]  

(3)

By the assumption \(f(x, 0) \equiv 0\), the problem (1) admits a trivial solution \(u = 0\), the purpose of the present paper concerns with the existence of nontrivial weak solutions for (1). A weak solution for (1) is a function \(u : \mathbb{R}^N \to \mathbb{R}\) such that

\[
\begin{cases}
\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\Omega} f(x, u(x)) \varphi(x) \, dx, \\
u \in H^s_0(\Omega),
\end{cases}
\]  

(4)

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for all $\varphi \in H_0^s(\Omega)$, where the functional space $H_0^s(\Omega)$ is a Hilbert space (see [26]) with the inner product and norm

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy, \quad \|u\| = \sqrt{\langle u, u \rangle}.$$  

Since the embedding from $H_0^s(\Omega)$ into $L^q(\Omega)$ is continuous for all $q \in \left[1, \frac{2N}{N-2s}\right]$ (see [26, 30]), it follows from (f) that (1) is variational and its energy functional $\Phi : H_0^s(\Omega) \to \mathbb{R}$ can be defined by

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \int_{\Omega} F(x, u) \, dx, \quad u \in H_0^s(\Omega)$$

(5)

where $F(x, t) = \int_0^t f(x, \zeta) \, d\zeta$. The functional $\Phi \in C^1(H_0^s(\Omega), \mathbb{R})$ and its Fréchet derivative is formulated as

$$\langle \Phi'(u), \varphi \rangle = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy - \int_{\Omega} f(x, u) \varphi \, dx$$

(6)

for all $u, \varphi \in H_0^s(\Omega)$. Therefore it follows from (4) and (6) that weak solutions of the problem (1) are exactly critical points of $\Phi$ on $H_0^s(\Omega)$.

The nonlocal equations have been experiencing impressive applications in different subjects, such as the thin obstacle problem, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes and flame propagation, conservation laws, ultrarelativistic limits of quantum mechanics, quasigeostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, elliptic problems with measure data, optimization, finance, etc. See [9] and the references therein.

In the recent years many mathematicians have made efforts to apply the minimax methods ([24]) such as the mountain pass theorem([1]), the saddle point theorem([24]) or other linking type of critical point theorems in the study of the non-local fractional Laplacian equations with different nonlinearities having sub-critical or critical growth, see [3, 14, 21, 22, 25, 26, 27, 28, 30, 31] and references therein. Both results and methods in dealing with the classical Laplace equations may be adapted to the non-local equations. One may refer to the monograph [20] for more related works. In [11, 12, 13], the saddle point theorem was applied to get the solvability for the problem likes (1) with more general nonlocal operator than $(-\Delta)^s$ with asymptotically linear nonlinearities.

It is known that the existence of nontrivial solutions of the problem (1) may be determined by the asymptotic behaviors of the nonlinearity $f$ near infinity and zero. The geometry and topology properties of the corresponding energy functional $\Phi$ are very intimately related to the position of the range of $f(x, t)/t$ with respect to the spectrum of $(-\Delta)^s$. In [6], for the completely resonant case that the asymptotically limits exist at both infinity and zero, that is, $f_1'(x, 0) = \lambda_m$ and $\lim_{|t| \to \infty} f(x, t)/t = \lambda_k$, where $\lambda_m$ and $\lambda_k$ are eigenvalues of $(-\Delta)^s$, the existence of one nontrivial weak solutions for the problem (1) was obtained via Morse theory ([5, 19]) and critical group computations. By the combinations of the cut-off technique and the mountain pass theorem the existence of two constant-sign nontrivial weak solutions for the problem (1) was obtained in [7] when $f_1'(x, t) < \lambda_1$, the first eigenvalue of $(-\Delta)^s$, and the nonlinearity having different asymptotic limits at infinity, i.e. $\lim_{|x| \to \infty} f(x, t)/|t| = \beta_+ > \lambda_1$ and $\beta_+ \neq \beta_-$. One of the novelties of [7] was that the term $f(x, t)/t$ may cross arbitrary finitely many eigenvalues of $(-\Delta)^s$ as $t$ going
from $-\infty$ to $+\infty$. In [8] one nontrivial solution was obtained for the case that
\( f_t'(x,0) = \lambda_m \) and \( \lim_{|t| \to \infty} f_t'(x,t) = \lambda_t \) via Morse theory and a penalized method. In [14] the double resonance case near infinity in the sense of \( f \) satisfying
\[
\begin{align*}
\lim_{|t| \to \infty} (f(x,t)t - 2F(x,t)) &= \infty \quad \text{uniformly in a.e. } x \in \Omega, \\
\lambda_t &\leq \liminf_{|t| \to \infty} \frac{f(x,t)}{t} \leq \limsup_{|t| \to \infty} \frac{f(x,t)}{t} \leq \lambda_{t+1}, \quad \text{uniformly in a.e. } x \in \Omega
\end{align*}
\]
has been considered. Motivated by [7], [15] and [18], the purpose of this paper
is to establish the existence of nontrivial solution for the problem (1) under some more general situations of the nonlinearity \( f \) has been considered. We extend the methods in [15] dealing with the usual Laplacian equation to the non-local setting by combining the local linking, the saddle point theorem and homotopy invariance of critical groups. We will accomplish the critical groups computations and get the existence results for the problem (1). These results will be new in the non-local settings.

Now we state the assumptions and main results in this paper. We need the following global assumption on \( f \):

\( (f) \): There are \( C > 0 \) and \( p \in (2, \frac{2N}{N-2}) \) such that
\[
|f(x,s) - f(x,t)| \leq C(|s|^{p-2}+|t|^{p-2}+1)|s-t|, \quad s, t \in \mathbb{R}, \quad \text{uniformly in a.e. } x \in \Omega. \quad (7)
\]

We note that \( (f) \) implies the subcritical growth condition \((f)\) and the functional \( \Phi \)
is of class \( C^{2,0} \) on \( H^s_0(\Omega) \) (see a proof in the next section).

Denote by \( 0 < \lambda_1 < \lambda_2 \leq \cdots < \lambda_k \leq \cdots \to \infty \) the eigenvalues of \( (-\Delta)^s \) in \( H^s_0(\Omega) \).

We impose on \( f \) the following conditions near infinity and near zero.

\( (f_{\infty}) \): There are \( \epsilon > 0 \) and \( M > 0 \) such that for two adjacent eigenvalues \( \lambda_m < \lambda_{m+1} \) of \( (-\Delta)^s \), it holds that
\[
\lambda_m + \epsilon \leq \frac{f(x,t)}{t} \leq \lambda_{m+1} - \epsilon, \quad \text{for } |t| \geq M, \quad \text{uniformly in a.e. } x \in \Omega. \quad (8)
\]

\( (f_{2}) \): There are \( \epsilon > 0 \) and \( M > 0 \) such that for two adjacent eigenvalues \( \lambda_m < \lambda_{m+1} \) of \( (-\Delta)^s \), it holds that
\[
\begin{align*}
\lambda_m + \epsilon \leq \frac{f(x,t)}{t} \leq \lambda_{m+1} - \epsilon, \quad &\text{for } |t| \geq M, \quad \text{uniformly in a.e. } x \in \Omega. \quad (9)
\end{align*}
\]

\( (f_{\infty}) \): There are \( \epsilon > 0 \) and \( M > 0 \) such that for two adjacent eigenvalues \( \lambda_m < \lambda_{m+1} \) of \( (-\Delta)^s \), it holds that
\[
\begin{align*}
\lambda_m \leq \frac{f(x,t)}{t} \leq \lambda_{m+1} - \epsilon, \quad &\text{for } |t| \geq M, \quad \text{uniformly in a.e. } x \in \Omega. \quad (10)
\end{align*}
\]

\( (f_{0}) \): There exist \( \delta > 0 \) and \( k \geq 1 \) such that for two adjacent eigenvalues \( \lambda_k < \lambda_{k+1} \) of \( (-\Delta)^s \), it holds that
\[
\lambda_k t^2 \leq 2F(x,t) \leq \lambda_{k+1} t^2, \quad \text{for } |t| \leq \delta, \quad \text{uniformly in a.e. } x \in \Omega. \quad (11)
\]

\( (f_{0}) \): There exist \( \delta > 0 \) and \( k \geq 1 \) such that for two adjacent eigenvalues \( \lambda_k < \lambda_{k+1} \) of \( (-\Delta)^s \), it holds that
\[
\lambda_k \leq \frac{f(x,t)}{t} \leq \lambda_{k+1}, \quad \text{for } 0 < |t| \leq \delta, \quad \text{uniformly in } x \in \Omega. \quad (12)
\]

The main results of the present paper are the following two theorems.
Theorem 1.1. Assume \( \hat{f} \). Then the problem \( (1) \) admits at least one nontrivial weak solution in each of the following cases: (i) \( (f_{01}) \), \( (f_{\infty 1}) \) and \( \lambda_k \neq \lambda_m \); (ii) \( (f_{01}), (f_{\infty 2}) \) and \( \lambda_k \neq \lambda_m \).

Theorem 1.2. Assume \( \hat{f} \). Then the problem \( (1) \) admits at least one nontrivial weak solution in each of the following cases: (i) \( (f_{02}) \), \( (f_{\infty 1}) \) and \( \lambda_k \neq \lambda_m \); (ii) \( (f_{02}), (f_{\infty 2}) \) and \( \lambda_k \neq \lambda_m \); (iii) \( (f_{02}), (f_{\infty 3}) \) and \( \lambda_k \neq \lambda_m \).

We give some remarks on the conditions. The condition \( (f_{\infty 1}) \) means the problem \( (1) \) is non-resonant at infinity which includes \( \lim_{|t| \rightarrow \infty} f(x,t)/t = \beta \) (see \cite{5,34}) as a special case where \( \beta \) is not an eigenvalue of \( (-\Delta)^s \). The condition \( (f_{\infty 2}) \) characterizes \( (1) \) as resonance near infinity at \( \lambda_{m+1} \) from the left side, and \( (f_{\infty 3}) \) characterizes \( (1) \) as resonance near infinity at \( \lambda_m \) from the right side. The condition \( (f_{01}) \) or \( (f_{02}) \) characterizes \( (1) \) as double resonance between two consecutive eigenvalues near zero. It is seen that \( (f_{02}) \) implies \( (f_{01}) \). The conditions \( (f_{\infty 1})-(f_{\infty 3}) \) were first introduced in \cite{15} while \( (f_{01}) \) and \( (f_{02}) \) were first introduced in \cite{18}.

The paper is organized as follows. In Section 2 we give some results about Morse theory as well as critical groups and the variational settings for the problem \( (1) \). In Section 3 we compute the critical groups at zero. In Section 4 we compute the critical groups at infinity. In Section 5 we give the proofs of the existence theorems for \( (1) \).

2. Preliminaries. In this section we will give results in Morse theory and the preliminaries for the variational structure of \( (1.1) \).

2.1. Preliminaries about Morse theory. In this subsection we collect some abstract results on Morse theory\cite{5,19} for a \( C^1 \) functional defined on a Banach space \( X \). Let \( \Phi \in C^1(X,\mathbb{R}) \) and \( K = \{ u \in X : \Phi'(u) = 0 \} \). For \( c \in \mathbb{R} \) we denote \( \Phi^c = \{ u \in X : \Phi(u) \leq c \} \) and \( K_c = K \cap \{ u \in X : \Phi(u) = c \} \).

We say that \( \Phi \) satisfies the Palais-Smale condition at the level \( c \in \mathbb{R} \) if any sequence \( \{u_n\} \subset X \) satisfying \( \Phi(u_n) \to c \) and \( \Phi'(u_n) \to 0 \) as \( n \to \infty \) has a convergent subsequence. We say that \( \Phi \) satisfies the Palais-Smale condition if \( \Phi \) satisfies the Palais-Smale condition at each \( c \in \mathbb{R} \).

Let \( u_0 \) be an isolated critical point of \( \Phi \) with \( \Phi(u_0) = c \in \mathbb{R} \), and \( U \) be a neighborhood of \( u_0 \) such that \( U \cap K_c = \{ u_0 \} \). The \( q \)-th critical group of \( \Phi \) at \( u_0 \) is defined as \( H_q(\Phi;U,K) = \{ \chi \in C_\infty^q(U,\mathbb{R}) \mid \chi|_{\partial U} = 0 \} \), \( q \in \mathbb{Z} \), is called the \( q \)-th critical group of \( \Phi \) at \( u_0 \), where \( H_q(A,B) \) denotes a singular relative homology group of the pair \( (A,B) \) with integer coefficients (see \cite{5,19}).

Assume that \( \Phi(K) \) is bounded from below by \( a \in \mathbb{R} \) and \( \Phi \) satisfies the Palais-Smale condition at all \( c \leq a \). The \( q \)-th critical group of \( \Phi \) at infinity is defined as \( H_q(\Phi,\infty) = \{ \chi \in C_\infty^q(U,\mathbb{R}) \mid \chi|_{\partial U} = 0 \) and \( \chi(\infty) = 0 \} \), \( q \in \mathbb{Z} \), is called the \( q \)-th critical group of \( \Phi \) at infinity (\cite{4}).

Assume that \( \Phi \) satisfies the Palais-Smale condition and \( K \) is a finite set containing \( 0 \). Then the critical groups of \( \Phi \) at infinity and at \( 0 \) are well-defined. The basic idea of Morse theory tells us that if \( K = \{0\} \) then \( C_q(\Phi,\infty) = C_q(\Phi,0) \) for all \( q \in \mathbb{Z} \). It follows that if \( C_q(\Phi,\infty) \neq C_q(\Phi,0) \) for some \( q \in \mathbb{Z} \) then \( \Phi \) has a nontrivial critical point. Therefore the basic method in applying Morse theory to find nontrivial critical points of \( \Phi \) is to compute critical groups of \( \Phi \) both at infinity and at \( 0 \). We first have the following partial results.
Thus for all it follows that By (14), (15) and Proposition 2.1. (Palais-Smale condition) Assume that is coercive on , i.e. as with , then \( C(\Phi, \infty) \neq 0 \) if \( \ell = \dim X_1 < \infty \).

Proposition 2.2. ([16]) Let \( \Phi \in C^1(X, \mathbb{R}) \) satisfy the Palais-Smale condition and \( 0 \in K \). Assume that has a local linking structure at 0 with respect to \( X = X_0^- \oplus X_0^+ \), i.e. there exists \( \rho > 0 \) such that

\[
\Phi(u) > 0 \quad \text{for} \quad u \in X_0^+, \quad 0 < \|u\| \leq \rho, \quad \Phi(u) \leq 0 \quad \text{for} \quad u \in X_0^-, \quad \|u\| \leq \rho. \tag{13}
\]

Then \( C_{\ell_0}(\Phi, 0) \neq 0 \) if \( \ell_0 = \dim X_0^- < \infty \).

Proposition 2.1 is a simple version of the famous Rabinowitz’s saddle point theorem [24]. It is a global linking theorem. The concept of local linking in Proposition 2.2 was introduced by Li and Liu in [17] for the existence of nontrivial critical point. We regard Propositions 2.1 and 2.2 as the topological versions of corresponding linking theorems since there are no minimax values involved.

Next we give the following homotopy invariance theorems for critical groups. We work on a Hilbert space.

Theorem 2.3. Let \( X \) be a Hilbert space and let \( \Phi_t \in C^1(X, \mathbb{R}) \) be a family of functionals, \( t \in [0, 1] \). Assume that each \( \Phi_t \) satisfies the Palais-Smale condition, \( \Phi_t' \) and \( \partial_t \Phi_t \) are locally Lipschitz continuous in \( u \). If there exists \( a \in \mathbb{R} \) and \( \delta > 0 \) such that for some \( C > 0 \)

\[
\Phi_t(u) \leq a \quad \Rightarrow \quad \|\partial_t \Phi_t(u)\| \leq C\|u\|^2, \quad \text{for all} \quad t \in [0, 1],
\]

\[
\Phi_t(u) \leq a \quad \Rightarrow \quad \|\Phi_t'(u)\| \geq \delta\|u\|, \quad \text{for all} \quad t \in [0, 1],
\]

then

\[
C_q(\Phi_0, \infty) \cong C_q(\Phi_1, \infty). \tag{16}
\]

Proof. Consider the following initial problem of ODE on \( X \)

\[
\begin{align*}
\dot{\eta}(t, u) &= -\frac{\partial_t \Phi_t(\eta(t, u))}{\|\Phi_t'(\eta(t, u))\|^2} \Phi_t'(\eta(t, u)), \\
\eta(0, u) &= u \in \Phi_0^a.
\end{align*}
\tag{17}
\]

Since

\[
\frac{d}{dt}(\Phi_t(\eta(t, u))) = \langle \Phi_t'(\eta(t, u)) , \dot{\eta}(t, u) \rangle + \partial_t \Phi_t(\eta(t, u)) = 0,
\]

it follows that

\[
\Phi_t(\eta(t, u)) = \Phi_0(\eta(0, u)) = \Phi_0(u), \quad \text{for all} \quad t \in [0, 1],
\]

and so

\[
\Phi_t(\eta(t, u)) \leq a \iff \Phi_0(u) \leq a.
\]

By (14), (15) and \( \Phi_t(\eta(t, u)) \leq a \), we have that

\[
\|\dot{\eta}(t, u)\| \leq \left\| -\frac{\partial_t \Phi_t(\eta(t, u))}{\|\Phi_t'(\eta(t, u))\|^2} \Phi_t'(\eta(t, u)) \right\| \leq C \frac{\|\eta(t, u)\|^2}{\|\Phi_t'(\eta(t, u))\|} \leq \frac{C}{\delta} \|\eta(t, u)\|.
\]

Thus for all \( u \in X \) with \( \Phi_t(\eta(t, u)) \leq a \),

\[
\|\eta(t, u)\| \leq \|u\| + \int_0^t \|\dot{\eta}(\tau, u)\| d\tau \leq \|u\| + \frac{C}{\delta} \int_0^t \|\eta(\tau, u)\| d\tau.
\]
Therefore by the Gronwall inequality, the flow \( \eta(t, u) \) generated by (17) exists for all \( t \in [0, 1] \) and for any initial value \( u \in \Phi_0^s \). It can be reversed by replacing \( \Phi_t \) with \( \Phi_{1-t} \) in (17). Thus \( \eta(1, \cdot) \) is a homeomorphism of \( \Phi_0^s \) onto \( \Phi_1^s \). It follows that
\[
C_q(\Phi_0, \infty) = H_q(X, \Phi_0^s) \cong H_q(X, \Phi_1^s) = C_q(\Phi_1, \infty) \quad \forall \ q \in \mathbb{Z}.
\]

The proof is complete. \( \square \)

Theorem 2.3 is a new modification of a result in [23](see also [15]) where the existence of the flow generated by (17) should be considered. This abstract theorem has its own meanings and can be applied to other variational problems.

**Theorem 2.4.** ([5, 19]) Let \( X \) be a Hilbert space and \( \{\Phi_\sigma \in C^{2-0}(X, \mathbb{R}) : \sigma \in [0,1]\} \) be a family of functional satisfying the Palais-Smale condition. Assume that there exists an open set \( U \) such that \( \Phi_\sigma \) has a unique critical point \( u_\sigma \in U \) for each \( \sigma \in [0,1] \) and \( \sigma \mapsto \Phi_\sigma \) is continuous in \( C^1(U) \) topology. Then \( C_q(\Phi_\sigma, u_\sigma) \) is independent of \( \sigma \in [0,1] \).

**2.2. The variational setting.** In this subsection we recall some basic facts about the eigenvalue problem associated to the variational setting for the problem (1). We first recall the functional spaces \( H^s(\mathbb{R}^N) \) and \( H_0^s(\Omega) \). See [9, 25, 26, 27, 28, 29, 30] for details.

Let \( H^s(\mathbb{R}^N) \) be the usual fractional Sobolev space endowed with the so-called Gagliardo norm
\[
\|u\|_{H^s(\mathbb{R}^N)} = \left( \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2}.
\]

Let \( H_0^s(\Omega) \) be the function space defined as
\[
H_0^s(\Omega) := \{ u \in H^s(\mathbb{R}^N) : u(x) = 0, \quad \text{a.e. } x \in \mathbb{R}^N \setminus \Omega \}.
\]

Then \( H_0^s(\Omega) \) is a Hilbert space ([26]) with the norm
\[
\|u\| = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2},
\]

and the scalar product
\[
\langle u, \varphi \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy.
\]

**Proposition 2.5.** ([26, 30]) The embedding \( H_0^s(\Omega) \hookrightarrow L^q(\mathbb{R}^N) \) is continuous for \( q \in \left[ 1, \frac{2N}{N-2s} \right] \) and it is compact for \( q \in \left[ 1, \frac{2N}{N-2s} \right) \).

Next we recall some basic facts about the eigenvalue problem associated to \((-\Delta)^s\):
\[
\begin{cases}
(-\Delta)^s u = \lambda u & \text{in } \Omega, \\
u = 0, & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]  

(18)

The number \( \lambda \in \mathbb{R} \) is an eigenvalue of (18) if there is a nontrivial function \( u \in H_0^s(\Omega) \) such that
\[
\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy = \lambda \int_{\Omega} u(x)\varphi(x) \, dx, \quad \forall \varphi \in H_0^s(\Omega).
\]

From [27, Proposition 9] we have the following conclusions.

**Proposition 2.6.** ([27]) Let \( s \in (0, 1) \) with \( N > 2s \) and let \( \Omega \) be an open bounded subset of \( \mathbb{R}^N \) with Lipschitz boundary.
(i) \( (18) \) admits an eigenvalue \( \lambda_1 \) which is positive and that can be characterized as follows

\[
\lambda_1 = \min \left\{ \| u \|^2 : u \in H^s_0(\Omega), \| u \|_{L^2(\Omega)} = 1 \right\}.
\]

\( \lambda_1 \) is simple, and there is a positive function \( \phi_1 \in H^s_0(\Omega) \) corresponding to \( \lambda_1 \), such that \( \| \phi_1 \|_{L^2(\Omega)} = 1 \) and \( \lambda_1 = \| \phi \|^2 \).

(ii) The set of the eigenvalue of \( (18) \) consists of a sequence \( \{ \lambda_k \}_{k \in \mathbb{N}} \) with

\[
0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty.
\]

For any \( k \in \mathbb{N} \), the eigenvalue \( \lambda_k \) can be characterized as follows:

\[
\lambda_k = \min \left\{ \| u \|^2 : u \in P_k, \| u \|_{L^2(\Omega)} = 1 \right\},
\]

and moreover, \( \lambda_k \) is attained at some \( \phi_k \in P_k \), that is \( \| \phi_k \|_{L^2(\Omega)} = 1 \) and \( \lambda_k = \| \phi_k \|^2 \), where \( P_k := \{ u \in H^s_0(\Omega) : \langle u, \phi_j \rangle = 0 \text{ for all } j = 1, 2, \ldots, k-1 \} \).

(iii) Each eigenvalue \( \lambda_k \) has finite multiplicity, more precisely, if \( \lambda_k \) is such that

\[
\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+v_k} < \lambda_{k+v_k+1}
\]

for some \( v_k \geq 1 \), then the set of all the eigenfunctions corresponding to \( \lambda_k \) agrees with

\[
E(\lambda_k) := \text{span} \{ \phi_k, \phi_{k+1}, \ldots, \phi_{k+v_k-1} \}, \quad \dim E(\lambda_k) = v_k.
\]

(iv) The sequence \( \{ \phi_k \}_{k \in \mathbb{N}} \) of eigenfunctions corresponding to \( \lambda_k \) is an orthonormal basis of \( L^2(\Omega) \) and an orthogonal basis of \( H^s_0(\Omega) \).

It follows from Proposition 2.6 that \( H^s_0(\Omega) \) can be split as

\[
H^s_0(\Omega) = E^- \oplus E(\lambda_1) \oplus E^+, \quad E^- = \oplus_{\lambda < \lambda_1} E(\lambda), \quad E^+ = \oplus_{\lambda > \lambda_1} E(\lambda),
\]

(19)
corresponding to the eigenvalue \( \lambda_1 \) with multiplicity \( v_1 \).

**Proposition 2.7.** ([28]) Let \( \phi \in H^s_0(\Omega) \) and \( \lambda > 0 \) be such that

\[
\int_{\mathbb{R}^N} (\phi(x) - \phi(y))(\psi(x) - \psi(y)) \frac{dx dy}{|x-y|^{N+2s}} = \lambda \int_{\Omega} \phi(x)\psi(x) dx, \quad \forall \psi \in H^s_0(\Omega).
\]

Then \( \phi \in L^{\infty}(\Omega) \) and there exists \( C > 0 \), possible depending on \( N, s \) and \( \lambda \), such that

\[
\| \phi \|_{L^{\infty}(\Omega)} \leq C \| \phi \|_{L^2(\Omega)}.
\]

**Proposition 2.8.** ([10]) Let \( \phi \in H^s_0(\Omega) \) and \( \lambda > 0 \) satisfy

\[
\int_{\mathbb{R}^N} (\phi(x) - \phi(y))(\psi(x) - \psi(y)) \frac{dx dy}{|x-y|^{N+2s}} = \lambda \int_{\Omega} \phi(x)\psi(x) dx, \quad \forall \psi \in H^s_0(\Omega).
\]

Then the nodal set \( \{ x \in \Omega : \phi(x) = 0 \} \) of the eigenfunction \( \phi \) has zero Lebesgue measure.

The \( L^{\infty} \)-regularity result for the eigenfunctions of \( (-\Delta)^s \) is from [28] and the property of the nodal set of eigenfunction of \( (-\Delta)^s \) is a corollary of [10, Theorem 1.4] and now it is known as the unique continuation property of \( (-\Delta)^s \).

**Proposition 2.9.** Assume that \( \tilde{f} \) holds. Then the functional \( \Phi \in C^{2,0}(H^s_0(\Omega), \mathbb{R}) \).
Proof. The arguments are similar to that in [2] and we sketch out for the readers' convenience. We only need to prove \( \Psi(u) = \int_{\Omega} \tilde{F}(x,u)dx \) is of \( C^{2,0} \). For any \( v, w, \phi \in H^1_0(\Omega) \) with \( ||\phi|| = 1 \), we have by \((\tilde{f})\), Proposition 2.5 and H"older inequality that
\[
||\langle \Psi'(v) - \Psi'(w), \phi \rangle || \leq \left( \int_{\Omega} |f(x,v) - f(x,w)|^{p_t} dx \right)^{\frac{1}{p_t}} \| \phi \|_{L^p} \\
\leq C \left( \int_{\Omega} (|v-w|^{p_t} (1 + |v|^{p-2} + |w|^{p-2})^{\frac{p_t}{2}} dx \right)^{\frac{p}{p-2}} (20)
\leq C \|v-w\|^{\frac{p_t}{p-2}} \left( \int_{\Omega} (1 + |v|^{p-2} + |w|^{p-2})^{\frac{p_t}{2}} dx \right)^{\frac{p}{p-2}} \|v-w\|.
\]
For \( \xi > 0 \) and \( ||v|| \leq \xi, ||w|| \leq \xi \), it follows from \((20)\) that
\[
||\Psi(v) - \Psi(w)|| = \sup_{\phi \in H^1_0(\Omega), ||\phi|| = 1} ||\langle \Psi'(v) - \Psi'(w), \phi \rangle || \leq C(\xi) ||v-w||,
\]
where \( C(\xi) \) is a constant depending on \( \xi \). Therefore \( \Psi' \) is locally Lipschitz continuous. The proof is complete. \( \square \)

3. Critical groups at infinity. This section is devoted to the computations of the critical groups of \( \Phi \) at infinity under the corresponding assumptions \((f_{\infty_1})- (f_{\infty_3})\) at infinity. We use the following notations.
\[
H^1_0(\Omega) = E^- \oplus E(\lambda_m) \oplus E^+_m, \quad E_m = \oplus_{\lambda \leq \lambda_m} E(\lambda).
\]
We will use \( C \) to denote various positive constants in the sequel.

The first result involves with the non-resonance case of the problem \((1)\) near infinity.

**Proposition 3.1.** Let \( f \) satisfy \((\tilde{f})\) and \((f_{\infty_1})\). Then \( \Phi \) satisfies the Palais-Smale condition and \( C_2(\Phi, \infty) = \delta_{q, t_m} \mathbb{F} \).

**Proof.** We will apply Theorem 2.3 to prove this proposition. Set
\[
\tilde{f}(x,t) = f(x,t) - (\lambda_{m+1} - \epsilon)t, \quad \tilde{F}(x,t) = \int_0^t \tilde{f}(x, \zeta)d\zeta.
\]
Then \( \Phi \) can be rewritten as
\[
\Phi(u) = \frac{1}{2}||u||^2 - \frac{1}{2}(\lambda_{m+1} - \epsilon) \int_{\Omega} u^2 dx - \int_{\Omega} \tilde{F}(x,u)dx.
\]
For \( u \in H^1_0(\Omega) \), we write \( u = v + w \) and set \( \tilde{u} = -v + w \) where \( v \in E_m \) and \( w \in E^+_m \). Define a family of functionals as follows:
\[
\Phi_t(u) = (1-t)\Phi(u) + \frac{t}{2} (-||v||^2 + ||w||^2), \quad t \in [0,1].
\]
By \((\tilde{f})\) and Proposition 2.9, we have that \( \Phi_t \in C^{2,0}(H^1_0(\Omega), \mathbb{R}) \) and
\[
\langle \Phi_t'(u), \phi \rangle = (1-t)\langle \Phi'(u), \phi \rangle + t\langle \tilde{u}, \phi \rangle, \quad \forall u, \varphi \in H^1_0(\Omega).
\]
By \((f_{\infty 1})\) we have that
\[
0 \leq -\frac{\tilde{f}(x, t)}{t} \leq \lambda_{m+1} - \lambda_m - 2\epsilon, \quad \forall \, |t| \geq M, \, x \in \Omega.
\]
For \(|u(x)| \geq M\) we have that
\[
\tilde{f}(x, u)\tilde{u} = -\frac{\tilde{f}(x, u)}{u} (v^2 - \omega^2) \leq \begin{cases} 
0, & u(x)\tilde{u}(x) > 0, \\
(\lambda_{m+1} - \lambda_m - 2\epsilon)v^2, & u(x)\tilde{u}(x) < 0.
\end{cases}
\]
Hence
\[
\int_{\{\|u(x)\| \geq M\}} \tilde{f}(x, u)\tilde{u}dx \leq (\lambda_{m+1} - \lambda_m - 2\epsilon) \int_{\Omega} v^2 dx.
\tag{24}
\]
By \((f)\) there is \(C > 0\) such that
\[
\int_{\{\|u(x)\| < M\}} |\tilde{f}(x, u)\tilde{u}| dx \leq C\|\tilde{u}\|.
\tag{25}
\]
Now it follows from \((23), (24)\) and \((25)\) that
\[
\langle \Phi'(u), \tilde{u} \rangle = \langle u, \tilde{u} \rangle - (\lambda_{m+1} - \epsilon) \int_{\Omega} u\tilde{u} dx - \int_{\Omega} \tilde{f}(x, u)\tilde{u} dx
\geq \frac{\epsilon}{\lambda_{m+1}} \|w\|^2 - \left[\|v\|^2 - (\lambda_{m+1} - \epsilon) \int_{\Omega} v^2 dx \right]
- \int_{\{\|u(x)\| < M\}} \tilde{f}(x, u)\tilde{u} dx - \int_{\{\|u(x)\| \geq M\}} \tilde{f}(x, u)\tilde{u} dx
\geq \frac{\epsilon}{\lambda_{m+1}} \|w\|^2 - \left[\|v\|^2 - (\lambda_m + \epsilon) \int_{\Omega} v^2 dx \right] - C\|\tilde{u}\|
\geq \frac{\epsilon}{\lambda_{m+1}} \|w\|^2 + \frac{\epsilon}{\lambda_m} \|v\|^2 - C\|\tilde{u}\|
\geq \frac{\epsilon}{\lambda_{m+1}} \|w\|^2 - C\|u\|.
\tag{26}
\]
Taking \(\varphi = \tilde{u}\) in \((23)\), we obtain from \((26)\) that
\[
\langle \Phi'(u), \tilde{u} \rangle \geq (1 - t) \left[\frac{\epsilon}{\lambda_{m+1}} \|w\|^2 - C\|u\| \right] + t\|\tilde{u}\|^2 \geq C_t \|u\|^2 - C\|u\|.
\tag{27}
\]
where \(C_t = \min \{1, \epsilon/\lambda_{m+1}\}\). By \((27)\) we see that any a Palais-Smale sequence of \(\Phi_t\) must be bounded. By Proposition 2.10 we have that \(\Phi_t\) satisfies the Palais-Smale condition for all \(t \in [0, 1]\). Moreover, it follows from \((27)\) that there are \(a \ll -1\) and \(\delta = \delta(\epsilon) > 0\) such that
\[
\Phi_t(u) \leq a \Rightarrow \|\Phi_t'(u)\| \geq \delta\|u\|.
\tag{28}
\]
For any \(a \ll -1\) being fixed, it holds that
\[
\Phi_t(u) \leq a \Rightarrow |\partial_t \Phi_t(u)| \leq C\|u\|^2.
\tag{29}
\]
From the definition we see that \(\Phi_0(u) = \Phi(u)\) and
\[
\Phi_1(u) = \frac{1}{2}(-\|v\|^2 + \|w\|^2).
\tag{30}
\]
Then \(\Phi_1\) is a \(C^2\) functional and has 0 as a unique non-degenerate critical point with Morse index \(\ell_m = \dim E_m\). It follows that
\[
C_q(\Phi_1, \infty) = C_q(\Phi_1, 0) \equiv \delta_{q, \ell_m} F, \quad \forall \, q \in \mathbb{Z}.
\tag{31}
\]
By Theorem 2.3 and (31) we have
\[ C_q(\Phi, \infty) = C_q(\Phi_0, \infty) \cong C_q(\Phi_1, \infty) = \delta_{q, \ell_m}^F. \] (32)
The proof is complete.

The next two results involve with the problem (1) being resonant near infinity at \( \lambda_{m+1} \) from the left side and at \( \lambda_m \) from the right side, respectively.

**Proposition 3.2.** Let \( f \) satisfy \((f)\) and \((f_{\infty})\). Then \( \Phi \) satisfies the Palais-Smale condition and \( C_q(\Phi, \infty) = \delta_{q, \ell_m}^F \).

**Proof.** We will also apply Theorem 2.3 to prove this proposition. Set
\[ \tilde{f}(x, t) = f(x, t) - \lambda_{m+1} t, \quad \tilde{F}(x, t) = \int_0^t \tilde{f}(x, \zeta) d\zeta. \]
Then \( \Phi \) can be rewritten as
\[ \Phi(u) = \frac{1}{2} \| u \|^2 - \frac{1}{2} \lambda_{m+1} \int_{\Omega} u^2 dx - \int_{\Omega} \tilde{F}(x, u) dx. \]

For \( u \in H^s_0(\Omega) \), we write \( u = v + z + w \) and \( \tilde{u} = -v + z + w \) where \( v \in E_m, z \in E(\lambda_{m+1}) \) and \( w \in E_{m+1}^\perp \). Define a family of functionals
\[ \Phi_t(u) = (1 - t) \Phi(u) + t \left( -\|v\|^2 + \|z\|^2 + \|w\|^2 \right), \quad t \in [0, 1]. \] (33)
By \((\tilde{f})\) and Proposition 2.9, we have that \( \Phi_t \in C^{2-0}(H^s_0(\Omega), \mathbb{R}) \) and
\[ \langle \Phi'(u), \varphi \rangle = (1 - t) \langle \Phi'(u), \varphi \rangle + t \langle \tilde{u}, \varphi \rangle. \] (34)
By \((f_{\infty})\) we have that
\[ 0 \leq -\frac{\tilde{f}(x, t)}{t} \leq \lambda_{m+1} - \lambda_m - \epsilon, \quad \forall \ |t| \geq M, \ x \in \Omega. \]
Thus for \(|u(x)| \geq M\) we have that
\[ \tilde{f}(x, u) \tilde{u} = -\frac{\tilde{f}(x, u)}{u} [u^2 - (z + w)^2] \leq \begin{cases} 0, & u(x) \tilde{u}(x) \geq 0, \\ (\lambda_{m+1} - \lambda_m - \epsilon) v^2, & u(x) \tilde{u}(x) < 0. \end{cases} \]
Hence
\[ \int_{\{|u(x)| \geq M \}} \tilde{f}(x, u) \tilde{u} dx \leq (\lambda_{m+1} - \lambda_m - \epsilon) \int_{\Omega} v^2 dx, \] (35)
and there is \( C > 0 \) such that
\[ \int_{\{|u(x)| < M \}} |\tilde{f}(x, u) \tilde{u}| dx \leq C \| \tilde{u} \|. \] (36)
Now it follows from (35) and (36) that
\[ \langle \Phi'(u), \tilde{u} \rangle = \langle u, \tilde{u} \rangle - \lambda_{m+1} \int_{\Omega} w \tilde{u} dx - \int_{\Omega} \tilde{f}(x, u) \tilde{u} dx \]
\[ \geq \left( 1 - \frac{\lambda_{m+1}}{\lambda_{m+2}} \right) \| w \|^2 - \left( \| u \|^2 - \lambda_{m+1} \int_{\Omega} v^2 dx \right) \]
Since we have by (38) that
\[
\begin{align*}
- \int_{\{|u(x)| < M\}} f(x, u) \tilde{u} dx - \int_{\{|u(x)| \geq M\}} f(x, u) \tilde{u} dx &
\geq \left( 1 - \frac{\lambda_{m+1}}{\lambda_{m+2}} \right) \|v\|^2 - \left[ \|v\|^2 - \lambda_{m+1} \int_\Omega v^2 dx \right] \\
- \int_\Omega (\lambda_{m+1} - \lambda_m - \epsilon) v^2 dx - C\|\tilde{u}\| &
\geq \left( 1 - \frac{\lambda_{m+1}}{\lambda_{m+2}} \right) \|v\|^2 + \frac{\epsilon}{\lambda_m} \|v\|^2 - C\|\tilde{u}\|.
\end{align*}
\]
Taking \( \varphi = \tilde{u} \) in (34), then we obtain from (37) that
\[
\langle \Phi'_t(u), \tilde{u} \rangle \geq (1 - t) \left[ \left( 1 - \frac{\lambda_{m+1}}{\lambda_{m+2}} \right) \|v\|^2 + \frac{\epsilon}{\lambda_m} \|v\|^2 - C\|\tilde{u}\| \right] + t \|\tilde{u}\|^2. \tag{38}
\]
We prove that there exists \( \delta > 0 \) such that for any \( a \in \mathbb{R} \) fixed
\[
\Phi_t(u) \leq a \Rightarrow \|\Phi'_t(u)\| \geq \delta \|u\|. \tag{39}
\]
We only need to prove that for each \( n \in \mathbb{N} \), there is \( u_n \in H^s_0(\Omega) \) such that if
\[
\|\Phi'_t(u_n)\| < \frac{1}{n} \|u_n\| \quad \text{and} \quad \|u_n\| \to \infty, \tag{40}
\]
then
\[
\Phi_t(u_n) \to \infty, \quad n \to \infty. \tag{41}
\]
Denote
\[
\hat{u}_n = \frac{u_n}{\|u_n\|} = \hat{v}_n + \hat{\tilde{w}}_n.
\]
Then \( \|\hat{u}_n\| \equiv 1 \). It follows from (40) that
\[
\frac{\langle \Phi'_t(u_n), \hat{u}_n \rangle}{\|u_n\|^2} \to 0, \quad n \to \infty. \tag{42}
\]
We have by (38) that
\[
\frac{\langle \Phi'_t(u_n), \hat{u}_n \rangle}{\|u_n\|^2} \geq (1 - t_n) \left[ \left( 1 - \frac{\lambda_{m+1}}{\lambda_{m+2}} \right) \|\hat{u}_n\|^2 + \frac{\epsilon}{\lambda_m} \|\hat{v}_n\|^2 - C\|\tilde{u}_n\| \right] + t_n. \tag{43}
\]
Since \( t_n \in [0, 1], \|\hat{v}_n\|^2 \leq 1 \) and \( \|\hat{v}_n\|^2 \leq 1 \), we may assume, up to a subsequence, that
\[
t_n \to t_* \in [0, 1], \quad \|\hat{v}_n\|^2 \to \alpha \in [0, 1], \quad \|\hat{w}_n\|^2 \to \beta \in [0, 1], \quad n \to \infty. \tag{44}
\]
It follows from (42), (43) and (44) that
\[
(1 - t_*) \left[ \left( 1 - \frac{\lambda_{m+1}}{\lambda_{m+2}} \right) \beta + \frac{\epsilon}{\lambda_m} \alpha \right] + t_* \leq 0. \tag{45}
\]
It must be that
\[
t_* = 0, \quad \alpha = \beta = 0.
\]
This means that
\[
\hat{v}_n \to 0, \quad \hat{w}_n \to 0, \quad n \to \infty.
\]
Since \( \|\hat{u}_n\| \equiv 1 \), it holds that
\[
\hat{z}_n \to \hat{z} \neq 0, \quad \|\hat{z}\| = 1.
\]
It follows that

\[
\Phi_{t_n}(u_n) = (1 - t_n) \left( \frac{1}{2} \|u_n\|^2 - \int_{\Omega} F(x, u_n) \, dx \right) \\
+ \frac{t_n}{2} (\|v_n\|^2 + \|z_n\|^2 + \|w_n\|^2) \\
\geq (1 - t_n) \|u_n\|^2 \left( \frac{1}{2 \lambda_{m+1}} \|\hat{z}_n\|^2 - C(\|\hat{v}_n\|^2 + \|\hat{w}_n\|^2) \right) - C \\
+ \frac{1}{2} t_n \|u_n\|^2 (-\|\hat{v}_n\|^2 + \|\hat{z}_n\|^2 + \|\hat{w}_n\|^2) \\
\to \infty, \quad n \to \infty.
\]

This proves (39). Therefore for all \( t \in [0, 1] \), \( \Phi_t \) satisfies the Palais-Smale condition. Moreover, it is easy to see that for any \( a \ll -1 \) being fixed, it also holds that

\[
\Phi_t(u) \leq a \Rightarrow |\partial_t \Phi_t(u)| \leq C\|u\|^2.
\]  
(47)

Since

\[
\Phi_1(u) = \frac{1}{2} (-\|v\|^2 + \|z\|^2 + \|w\|^2)
\]  
(48)

is a \( C^2 \) functional and has 0 as a unique non-degenerate critical point with Morse index \( \dim \ell_m = E_m \), it follows that

\[
C_q(\Phi_1, \infty) = C_q(\Phi_1, 0) \cong \delta_{q, \ell_m} F, \quad \forall q \in \mathbb{Z}.
\]  
(49)

By (39), (47), Theorem 2.3 and (49) we have

\[
C_q(\Phi, \infty) = C_q(\Phi_0, \infty) \cong C_q(\Phi_1, \infty) = \delta_{q, \ell_m} F.
\]  
(50)

The proof is complete.

**Proposition 3.3.** Let \( f \) satisfy (f) and (f∞). Then \( \Phi \) satisfies the Palais-Smale condition and \( C_{\ell_m}(\Phi, \infty) \not\equiv 0 \).

**Proof.** We will apply Proposition 2.1 to get the conclusion. We first prove that \( \Phi \) satisfies the Palais-Smale condition. Although the argument is somewhat similar to that of the previous proposition, we prefer to give the details. Assume that \( \{u_n\} \subset H^s_0(\Omega) \) satisfies

\[
|\Phi(u_n)| \leq C, \quad \Phi'(u_n) \to 0, \quad n \to \infty.
\]  
(51)

By Proposition 2.10, we only need to prove that \( \{u_n\} \) is bounded. Assume that

\[
\|u_n\| \to \infty, \quad n \to \infty.
\]

Set \( \hat{u}_n = \frac{u_n}{\|u_n\|} = \hat{v}_n + \hat{z}_n + \hat{w}_n \) where \( \hat{v}_n \in E_{m-1}, \hat{z}_n \in E(\lambda_m) \) and \( \hat{w}_n \in E^{\perp}_m \). Then \( \|\hat{u}_n\| = 1 \). Set

\[
\hat{f}(x,t) = f(x,t) - \lambda_m t, \quad \hat{F}(x,t) = \int_0^t \hat{f}(x,\zeta) d\zeta.
\]

By (f∞) we have that

\[
0 \leq \frac{\hat{f}(x,t)}{t} \leq \lambda_{m+1} - \lambda_m - \epsilon, \quad \forall |t| \geq M, \quad x \in \Omega.
\]

For \( u \in H^s_0(\Omega) \), set \( \hat{u} = -(v+z) + w \). Then for \( |u(x)| \geq M \), we have

\[
\hat{f}(x,u)\hat{u} = \frac{\hat{f}(x,u)}{u} [-\lambda_m t] \leq \begin{cases} 0, & u(x)\hat{u}(x) < 0, \\ (\lambda_{m+1} - \lambda_m - \epsilon)u^2, & u(x)\hat{u}(x) \geq 0. \end{cases}
\]
Hence
\[
\int_{\{|u(x)| \geq M\}} \hat{f}(x, u) \tilde{u} dx \leq (\lambda_{m+1} - \lambda_m - \epsilon) \int_{\Omega} w^2 dx. \tag{52}
\]
There is \( C > 0 \) such that
\[
\int_{\{|u(x)| < M\}} |\hat{f}(x, u) \tilde{u}| dx \leq C \|\tilde{u}\|. \tag{53}
\]
It follows from (52) and (53) that
\[
\left( \Phi'(u), \tilde{u} \right) = \langle u, \tilde{u} \rangle - \lambda_m \int_{\Omega} w \tilde{u} dx - \int_{\Omega} \hat{f}(x, u) \tilde{u} dx
\]
\[
= \left( \|w\|^2 - \lambda_m \int_{\Omega} w^2 dx \right) - \left( \|v\|^2 - \lambda_m \int_{\Omega} v^2 dx \right)
\]
\[
- \int_{\{|u(x)| < M\}} |\hat{f}(x, u) \tilde{u}| dx - \int_{\{|u(x)| \geq M\}} \hat{f}(x, u) \tilde{u} dx
\]
\[
\geq \left( \|w\|^2 - \lambda_m \int_{\Omega} w^2 dx \right) - \left( \|v\|^2 - \lambda_m \int_{\Omega} v^2 dx \right)
\]
\[
- \int_{\Omega} (\lambda_{m+1} - \lambda_m - \epsilon) w^2 dx - C \|\tilde{u}\|
\]
\[
\geq \left( \frac{\lambda_m}{\lambda_{m-1}} - 1 \right) \|v\|^2 + \frac{\epsilon}{\lambda_{m+1}} \|w\|^2 - C \|\tilde{u}\|. \tag{54}
\]
By (51) and (54), we obtain that
\[
o(||u_n||) = (\Phi'(u_n), \tilde{u_n}) \geq \left( \frac{\lambda_m}{\lambda_{m-1}} - 1 \right) \|v_n\|^2 + \frac{\epsilon}{\lambda_{m+1}} \|w_n\|^2 - C \|\tilde{u_n}\|. \tag{55}
\]
Therefore
\[
o(1) \geq \left( \frac{\lambda_m}{\lambda_{m-1}} - 1 \right) \|v_n\|^2 + \frac{\epsilon}{\lambda_{m+1}} \|w_n\|^2 - \frac{C}{\|u_n\|}. \tag{56}
\]
Since \( \|\tilde{w}_n\|^2 \leq 1 \) and \( \|\hat{v}_n\|^2 \leq 1 \), we may assume, up to a subsequence, that
\[
\|\hat{v}_n\|^2 \rightarrow \alpha \in [0, 1], \quad \|\tilde{w}_n\|^2 \rightarrow \beta \in [0, 1], \quad n \rightarrow \infty. \tag{57}
\]
It follows from (56) that
\[
\left( \frac{\lambda_m}{\lambda_{m-1}} - 1 \right) \alpha + \frac{\epsilon}{\lambda_{m+1}} \beta \leq 0. \tag{58}
\]
It must be that
\[
\alpha = \beta = 0.
\]
This means that
\[
\hat{v}_n \rightarrow 0, \quad \hat{w}_n \rightarrow 0, \quad n \rightarrow \infty.
\]
Since \( \|\tilde{u}_n\| \equiv 1 \), it holds that
\[
\hat{z}_n \rightarrow \hat{z} \neq 0, \quad \|\hat{z}\| = 1.
\]
Now we get
\[
\Phi(u_n) = \frac{1}{2} \|u_n\|^2 - \int_{\Omega} F(x, u_n) dx
\]
\[
\leq \|u_n\|^2 \left( -\frac{1}{2} \frac{\epsilon}{\lambda_m} \|\hat{z}_n\|^2 + C (\|\hat{v}_n\|^2 + \|\hat{w}_n\|^2) \right) + C \rightarrow -\infty
\]
as \( n \rightarrow \infty \). It is a contradiction to (51).
Next we prove that $\Phi$ satisfies the geometrical assumptions of Proposition 2.1 with respect to $H_0^1(\Omega) = X_1 \oplus X_2$ where $X_1 = E_m$ and $X_2 = E_m^{\perp}$. It follows from \((f_{\infty}^3)\) that

$$(\lambda_m + \epsilon)t^2 - C \leq 2F(x, t) \leq (\lambda_{m+1} - \epsilon)t^2 + C$$

for some $C > 0$. Then for $w \in X_2$,

$$\Phi(w) = \frac{1}{2}\|w\|^2 - \int_{\Omega} F(x, w)dx$$

$$\geq \frac{1}{2}\|w\|^2 - \frac{1}{2}\int_{\Omega} (\lambda_{m+1} - \epsilon)w^2dx - C$$

$$\geq \frac{\epsilon}{2} \int_{\Omega} |w|^2dx - C \geq -C. \quad (60)$$

For $v \in X_1$,

$$\Phi(v) = \frac{1}{2}\|v\|^2 - \int_{\Omega} F(x, v)dx$$

$$\leq \frac{1}{2}\|v\|^2 - \frac{1}{2}\int_{\Omega} (\lambda_m + \epsilon)u^2dx + C \quad (61)$$

$$\leq -\frac{\epsilon}{2\lambda_m}\|v\|^2 + C \rightarrow -\infty, \|v\| \rightarrow \infty.$$

As $\dim X_1 = \dim E_m = \ell_m < \infty$, we have by Proposition 2.1 that

$$C_{\ell_m}(\Phi, \infty) \neq 0.$$

The proof is complete. \hfill \Box

4. Critical groups at zero. In this section we compute the critical groups of $\Phi$ at zero under the assumptions \((f_{01})\) and \((f_{02})\). We make a conventional assumption that the trivial solution 0 of \((1)\) is isolated. By Proposition 2.10 the functional $\Phi$ satisfies the Palais-Smale condition over any a closed ball centered at 0. We will use the following notations.

$$H_0^1(\Omega) = E^- \oplus E(\lambda_k) \oplus E_k^+, \ E_k = \oplus_{\lambda \leq \lambda_k} E(\lambda). \quad (62)$$

**Proposition 4.1.** Let $f$ satisfy \((f)\) and \((f_{01})\). Then $C_{\ell_k}(\Phi, 0) \neq 0$.

**Proof.** We will show that $\Phi$ has a local linking structure at 0 with respect to $H_0^1(\Omega) = E_k \oplus E_k^{\perp}$.

(i) Since $E_k$ is finite dimensional, all norms on $E_k$ are equivalent, by Proposition 2.7 we have $E_k \subset L^\infty(\Omega)$. Thus there exists $\rho > 0$ such that for any $u \in E_k$,

$$\|u\| \leq \rho \Rightarrow \|u\|_{L^\infty(\Omega)} \leq \delta.$$

Therefore for $u \in E_k$ with $\|u\| \leq \rho$, we have by \((f_{01})\) that

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\Omega} \lambda_k u^2dx - \frac{1}{2}\int_{\Omega} \left(2F(x, u) - \lambda_k u^2\right)dx$$

$$\leq -\frac{1}{2}\int_{\Omega} (2F(x, u) - \lambda_k u^2)dx \leq 0. \quad (63)$$

(ii) For $u \in E_k^{\perp}$, we write $u = v + w$, where $v \in E(\lambda_{k+1})$ and $w \in E_{k+1}^{\perp}$. Then

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\Omega} \lambda_{k+1} v^2dx - \frac{1}{2}\int_{\Omega} \left(2F(x, u) - \lambda_{k+1} u^2\right)dx$$

$$\geq \frac{1}{2} \left(1 - \frac{\lambda_{k+1}}{\lambda_{k+2}}\right)\|w\|^2 - \frac{1}{2}\int_{\Omega} \left(2F(x, u) - \lambda_{k+1} u^2\right)dx. \quad (64)$$
For $|u(x)| \leq \delta$, we have by (f01) that
\[
\int_{\{|u(x)| \leq \delta\}} \left( 2F(x, u) - \lambda_{k+1} u^2 \right) dx \leq 0. \tag{65}
\]
Since $E(\lambda_{k+1})$ is finite dimensional and $E(\lambda_{k+1}) \subset L^\infty(\Omega)$, when $\rho > 0$ is small enough, we have that $\|v\| \leq \rho \Rightarrow |v(x)| < \delta/3$. Thus for $|u| \leq \rho$ and $|u(x)| > \delta$, we have
\[
|w(x)| \geq |u(x)| - |v(x)| > \frac{2}{3}|u(x)|.
\]
It follows from (f) and Proposition 2.5 that for some $q \in (2, \frac{2N}{N-2s}]$,
\[
\int_{\{|u(x)| > \delta\}} \left( 2F(x, u) - \lambda_{k+1} u^2 \right) dx \leq C \int_{\Omega} |u|^q dx \leq C \int_{\Omega} |w|^q dx \leq C \|w\|^q. \tag{66}
\]
Hence by (64)–(66) we have
\[
\Phi(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda_{k+1}}{\lambda_{k+2}} \right) \|u\|^2 - C \|w\|^q. \tag{67}
\]
Since $q > 2$, it follows from (67) that there is $\rho > 0$ small such that
\[
\Phi(u) > 0, \quad \forall \ 0 < \|u\| \leq \rho \text{ with } w \neq 0. \tag{68}
\]
We can choose $\rho > 0$ so small that $\|u\| \leq \rho \Rightarrow \|v\| \leq \rho \Rightarrow |v(x)| \leq \delta$ for all $x \in \Omega$. Then by (f01) we have
\[
2F(x, v(x)) - \lambda_{k+1} v^2(x) \leq 0 \text{ uniformly in } x \in \Omega.
\]
Thus
\[
\Phi(v) = -\frac{1}{2} \int_\Omega \left( 2F(x, v) - \lambda_{k+1} v^2 \right) dx \geq 0.
\]
Let $v_* \in E(\lambda_{k+1})$ be such that $0 < \|v_*\| \leq \rho$ and $\Phi(v_*) = 0$. Then
\[
2F(x, v_*(x)) - \lambda_{k+1} v_*^2(x) = 0 \text{ uniformly in } x \in \Omega,
\]
and so
\[
f(x, v_*(x)) = \lambda_{k+1} v_*(x) \text{ uniformly in } x \in \Omega.
\]
Going back to the problem (1), by Proposition 2.8, the unique continuation property for the eigenfunctions of $(-\Delta)^s$, we see that $v_*$ is a nontrivial solution for (1).

We conclude that there is $\rho > 0$ small such that for all $\|u\| \leq \rho$ with $w = 0$ and $v \neq 0$,
\[
\Phi(u) = \Phi(v) = -\frac{1}{2} \int_\Omega \left( 2F(x, v) - \lambda_{k+1} v^2 \right) dx > 0. \tag{69}
\]
Otherwise, for any $\tau > 0$ there exists $0 \neq v_\tau \in E(\lambda_{k+1})$ such that $\|v_\tau\| < \tau$ and $\Phi(v_\tau) = 0$. Then $v_\tau$ is a nontrivial solution of (1). It contradicts the isolation of the trivial solution. By (68) and (69) we have that
\[
\Phi(u) > 0, \quad \text{for } u \in E_k^\perp \text{ with } 0 < \|u\| \leq \rho.
\]
Therefore $\Phi$ has a local linking structure at 0 with respect to $H^s_0(\Omega) = E_k \oplus E_k^\perp$.

Since $t_k = \dim E_k < \infty$, it follows from Proposition 2.2 that $C_{t_k}(\Phi, 0) \neq 0$. The proof is complete.
Remark 4.2. (i) We remark here that the idea used in the proof of Proposition 4.1 is essentially from [18]. We would like to point out that for dealing with fractional Laplacian equation, the unique continuation property for all the eigenfunctions of $(-\Delta)^s$ is necessarily applied in the arguments. To the best of our knowledge, whether or not the arguments for proving Proposition 4.1 could be extended to the nonlocal problem with general integro-differential operator is open.

(ii) In above proposition, we verify that the $C^1$ functional $\Phi$ satisfies the so-called “local linking” condition so that the critical groups at $0$ can be computed partially. We point out that when $f$ is $C^1$ and satisfies $f_t(x,0) \equiv \lambda_k$ and $2F(x,t) \geq \lambda_k t^2$ for $|t| \leq \delta$ and $x \in \Omega$, the critical groups of $\Phi$ at $0$ can be computed completely by applying an abstract theorem built in [32, 33](see [8]). In this case $0$ is a degenerate critical point of $\Phi$.

We go on with the computations of the critical groups and a stronger result can be expected.

Proposition 4.3. Let $f$ satisfy (f) and $(f_{02})$. Then $C_q(\Phi,0) = \delta_{q,\delta} \mathbb{F}$.

Proof. For $u \in H^s_0(\Omega)$, we write $u = v + z + w$ and set $\tilde{u} = -v + z + w$ where $v \in E_k$, $z \in E(\lambda_{k+1})$ and $w \in E_{k+1}^\perp$. Define a family of functionals as follows:

$$\Phi_t(u) = (1 - t)\Phi(u) + \frac{t}{2} (-\|v\|^2 + \|z\|^2 + \|w\|^2), \quad t \in [0,1].$$  (70)

By (f) and Proposition 2.9, we have that $\Phi_t \in C^{2-0}(H^s_0(\Omega),\mathbb{R})$ and 

$$\langle \Phi'_t(u), \varphi \rangle = (1 - t)\langle \Phi'(u), \varphi \rangle + t\langle \tilde{u}, \varphi \rangle.$$ (71)

Now we show that there is $\rho > 0$ such that $u = 0$ is a unique critical point of $\Phi_t$ in the ball $B_\rho(0)$ for all $t \in [0,1]$. Denote $g(x,t) = f(x,t) - \lambda_{k+1} t$. Then by $(f_{02})$ we have that 

$$0 < -\frac{g(x,t)}{t} \leq \lambda_{k+1} - \lambda_k, \quad 0 < |t| \leq \delta, \quad x \in \Omega. $$

Then for $|u(x)| \leq \delta$, 

$$g(x,u)\tilde{u} = -\frac{g(x,u)}{u}[v^2 - (z + w)^2] \leq \begin{cases} 0, & u(x)\tilde{u}(x) \geq 0, \\ (\lambda_{k+1} - \lambda_k)v^2, & u(x)\tilde{u}(x) < 0. \end{cases}$$

Hence 

$$\int_{\{|u(x)| \leq \delta\}} g(x,u)\tilde{u}dx \leq (\lambda_{k+1} - \lambda_k) \int_{\Omega} v^2 dx.$$  (72)

Since $E_k$ and $E(\lambda_{k+1})$ are finite dimensional, there is a $\rho > 0$ such that 

$$\|v\| \leq \rho \Rightarrow |v(x)| \leq \frac{\delta}{3}, \quad \|z\| \leq \rho \Rightarrow |z(x)| \leq \frac{\delta}{3}. $$

For $\|u\| \leq \rho$ and $|u(x)| > \delta$, 

$$|u(x)| \leq |w(x)| + |z(x)| + |v(x)| \leq |w(x)| + \frac{2}{3}\delta,$$

and so 

$$|u(x)| < 3|w(x)|, \quad |\tilde{u}(x)| < 3|w(x)|.$$ 

Thus by (f) we have 

$$\int_{\{|u| > \delta\}} |g(x,u)\tilde{u}| dx \leq C \int_{\{|u| > \delta\}} |u|^{p-1}|\tilde{u}| dx \leq C \int_{\{|u| > \delta\}} |w|^p dx \leq C\|w\|^p.$$  (73)
Now for \( \|u\| \leq \rho \), it follows from (72) and (73) that
\[
\langle \Phi'(u), \hat{u} \rangle = \langle u, \hat{u} \rangle - \lambda_{k+1} \int_{\Omega} u \hat{u} dx - \int_{\Omega} g(x, u) \hat{u} dx \\
\geq \begin{cases} 
1 - \frac{\lambda_{k+1}}{\lambda_{k+2}} \|w\|^2 - \left( \|v\|^2 - \lambda_{k+1} \int_{\Omega} v^2 dx \right) \\
- \int_{\{|u(x)| \leq \delta\}} g(x, u) \hat{u} dx - \int_{\{|u(x)| > \delta\}} g(x, u) \hat{u} dx 
\end{cases} \\
\geq \begin{cases} 
1 - \frac{\lambda_{k+1}}{\lambda_{k+2}} \|w\|^2 - \left( \|v\|^2 - \lambda_k \int_{\Omega} v^2 dx \right) - C\|w\|^p. 
\end{cases} \\
(74)
\]
Therefore for \( \|u\| \leq \rho \), take \( \varphi = \hat{u} \) in (71), we obtain from (74) that
\[
\langle \Phi'(u), \hat{u} \rangle \geq \begin{cases} 
(1-t) \left[ 1 - \frac{\lambda_{k+1}}{\lambda_{k+2}} \|w\|^2 - \left( \|v\|^2 - \lambda_k \int_{\Omega} v^2 dx \right) \right] \\
-(1-t)C\|w\|^p + t\|u\|^2. 
\end{cases} \\
(75)
\]
Since \( p > 2 \), it follows that 0 is the only critical point of \( \Phi_t \) in \( \overline{B}_p(0) \) for all \( t \in [0,1] \) if \( \rho > 0 \) is sufficiently small. Since
\[
\Phi_1(u) = \frac{1}{2} \left( -\|v\|^2 + \|z\|^2 + \|w\|^2 \right) \\
(76)
\]
is a \( C^2 \) functional and has 0 as a non-degenerate critical point with Morse index \( \ell_k = \dim E_k \), it follows that
\[
C_q(\Phi_1, 0) = \delta_{q, \ell_k} F, \quad \forall q \in \mathbb{Z}. \\
(77)
\]
By Theorem 2.4 and (77) we have
\[
C_q(\Phi, 0) = C_q(\Phi_1, 0) \cong C_q(\Phi_1, 0) = \delta_{q, \ell_k} F. \\
(78)
\]
The proof is complete. \( \square \)

5. Proofs of Theorems 1.1 and 1.2. In the final section we give the proof of main theorems.

Proof of Theorem 1.1. (i) By Proposition 3.1, the functional \( \Phi \) satisfies the Palais-Smale condition and \( C_{q}(\Phi, \infty) = \delta_{q, \ell_m} F \). By Proposition 4.1, we have that \( C_{q}(\Phi, 0) \neq 0 \). Since \( \lambda_k \neq \lambda_m \) implies \( \ell_k \neq \ell_m \), it follows that \( C_{\ell_m}(\Phi, \infty) \neq C_{\ell_k}(\Phi, 0) \). Therefore \( \Phi \) has at least one nontrivial critical point. The case (ii) is proved in a similar way. \( \square \)

Proof of Theorem 1.2. (iii) By Proposition 3.3, the functional \( \Phi \) satisfies the Palais-Smale condition and \( C_{\ell_m}(\Phi, \infty) \neq 0 \). By Proposition 4.3, we have that \( C_q(\Phi, 0) = \delta_{q, \ell_k} F \). Since \( \lambda_k \neq \lambda_m \) implies \( \ell_k \neq \ell_m \), it follows that \( C_{\ell_m}(\Phi, \infty) \neq C_{\ell_k}(\Phi, 0) \). Therefore \( \Phi \) has at least one nontrivial critical point. The other cases are proved in a similar way. \( \square \)

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