Updated Estimate of Running Quark Masses

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Abstract

Stimulated by recent development of the calculation methods of the running quark masses $m_q(\mu)$ and renewal of the input data, for the purpose of making a standard table of $m_q(\mu)$ for convenience of particle physicists, the values of $m_q(\mu)$ at various energy scales $\mu$ ($\mu = 1 \text{ GeV}$, $\mu = m_c$, $\mu = m_b$, $\mu = m_t$ and so on), especially at $\mu = m_Z$, are systematically evaluated by using the mass renormalization equations and by taking into consideration a matching condition at the quark threshold.
I. INTRODUCTION

It is very important to know reliable values of quark masses $m_q$ not only for hadron physicists who intend to evaluate observable quantities on the basis of an effective theory, but also for quark-lepton physicists who intend to build a model for quark and lepton unification. For such a purpose, for example, a review article [1] of 1982 by Gasser and Leutwyler has offered useful information on the running quark masses $m_q(\mu)$ to us. However, during the fifteen years after the Gasser and Leutwyler’s review article, there have been some developments in the input data and calculation methods: the QCD parameter $\Lambda^{(n)}_{\overline{MS}}$ has been revised [2]; top-quark mass $m_t$ has been observed [3–5]; the three-loop diagrams have been evaluated for the pole mass $M_{q \text{pole}}$ [6] and for the running quark mass $m_q(\mu)$ [7]; a new treatment of the matching condition at the quark threshold has been proposed [8]. On the other hand, so far, there are few articles which review masses of all quarks systematically, although there have been some re-estimates [9–18] for specific quark masses. For recent one of such few works in systematical study of all quark masses, for example, see Ref. [19] by Rodrigo. We will give further systematical studies on the basis of recent data and obtain a renewed table of the running quark mass values.

The purpose of the present paper is to offer a useful table of the running quark masses $m_q(\mu)$ to hadron physicists and quark-lepton physicists. In Sec. IV, by using the mass renormalization equation (4.1), we will evaluate the value of $m_q(\mu)$ at various energy scales $\mu$, e.g., $\mu = 1$ GeV, $\mu = m_q (q = c, b, t)$, $\mu = M_{q \text{pole}}$, $\mu = m_Z$, $\mu = \Lambda_W$, and so on, where $M_{q \text{pole}}$ is a “pole” mass of the quark $q$, and $\Lambda_W$ is the symmetry breaking energy scale of the electroweak gauge symmetry $SU(2)_L \times U(1)_Y$:

$$\Lambda_W \equiv \langle \phi^0 \rangle = (\sqrt{2}G_F)^{-\frac{1}{2}}/\sqrt{2} = 174.1 \text{ GeV}. \quad (1.1)$$

In the next section, we review the light quark masses $m_u(\mu)$, $m_d(\mu)$ and $m_s(\mu)$ at $\mu = 1$ GeV. In Sec. II, we review pole mass values of the heavy quark masses $M_{c \text{pole}}$, $M_{b \text{pole}}$ and $M_{t \text{pole}}$. In Sec. IV, running quark masses $m_q(\mu)$ are evaluated for various energy scales $\mu$ below $\mu = \Lambda_W = 174.1$ GeV. In Sec. V, we comment on the reliability of the perturbative calculations of the running quark masses $m_q(\mu)$ ($\mu \leq \Lambda_W$). In Sec. VI, we summarize our numerical results of the running quark mass values $m_q(\mu)$, the charged lepton masses $m_{\ell}(\mu)$, the Cabibbo-Kobayashi-Maskawa (CKM) [20] matrix $V_{CKM}(\mu)$, and the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge coupling constants $g_i(\mu)$ ($i = 1, 2, 3$) at $\mu = m_Z$. In Sec. VII, for reference, the evolution of the Yukawa coupling constants is estimated at energy scales higher than $\mu = \Lambda_W$ for the cases of (A) the standard model with one Higgs boson and (B) the minimal SUSY model. Finally, Sec. VIII is devoted to summary and discussions.

II. LIGHT QUARK MASSES AT $\mu = 1$ GeV

Gasser and Leutwyler [1] have concluded in their review article of 1982 that the light quark masses $m_u(\mu)$, $m_d(\mu)$ and $m_s(\mu)$ at $\mu = 1$ GeV are
\[ m_u(1 \text{ GeV}) = 5.1 \pm 1.5 \text{ MeV} , \]
\[ m_d(1 \text{ GeV}) = 8.9 \pm 2.6 \text{ MeV} , \]
\[ m_s(1 \text{ GeV}) = 175 \pm 55 \text{ MeV} , \]

from QCD sum rules.

On 1987, Dominguez and Rafael have re-estimated those values from QCD finite energy sum rules. They have obtained the same ratios of the light quark masses with those estimated by Gasser and Leutwyler, but they have used a new value of \((m_u + m_d)\) at \(\mu = 1\) GeV

\[ (m_u + m_d)_{\mu=1 \text{ GeV}} = (15.5 \pm 2.0) \text{ MeV} , \]

instead of Gasser–Leutwyler’s value \((m_u + m_d)_{\mu=1 \text{ GeV}} = (14 \pm 3) \text{ MeV}\). Therefore, Dominguez and Rafael have concluded as

\[ m_u(1 \text{ GeV}) = 5.6 \pm 1.1 \text{ MeV} , \]
\[ m_d(1 \text{ GeV}) = 9.9 \pm 1.1 \text{ MeV} , \]
\[ m_s(1 \text{ GeV}) = 199 \pm 33 \text{ MeV} . \]

Recently, by simulating \(\tau\)-like inclusive processes for the old Das-Mathur-Okubo sum rule relating the \(e^+e^-\) into \(I = 0\) and \(I = 1\) hadron total cross-section data, Narison (1995) has obtained the following values:

\[ m_u(1 \text{ GeV}) = 4 \pm 1 \text{ MeV} , \]
\[ m_d(1 \text{ GeV}) = 10 \pm 1 \text{ MeV} , \]
\[ m_s(1 \text{ GeV}) = 197 \pm 29 \text{ MeV} , \]

which are roughly in agreement with (2.3).

On the other hand, by combining various pieces of the information on the quark mass ratios, Leutwyler (1996) has recently re-estimated the ratios

\[ m_u/m_d = 0.553 \pm 0.043 , \]
\[ m_s/m_d = 18.9 \pm 0.8 , \]

and has obtained

\[ m_u(1 \text{ GeV}) = 5.1 \pm 0.9 \text{ MeV} , \]
\[ m_d(1 \text{ GeV}) = 9.3 \pm 1.4 \text{ MeV} , \]
\[ m_s(1 \text{ GeV}) = 175 \pm 25 \text{ MeV} , \]

The values (2.6) are in agreement with (2.1), (2.3) and (2.4).

There is not so large discrepancy among these estimates as far as \(m_u\) and \(m_d\) are concerned, except for estimates by Donoghue, Holstein and Wyler (1992), who have obtained

\[ m_d/m_u = 3.49, \quad m_s/m_d = 20.7 , \]

(2.7)
from the constraints of chiral symmetry treated to next-to-leading order. Eletsky and Ioffe (1993) [13], and Adami, Drukarev and Ioffe (1993) [14] have obtained

\[(m_d - m_u)_{\mu=0.5 \text{ GeV}} = 3 \pm 1 \text{ MeV },\]  

(2.8)

from the QCD sum rules on the isospin-violating effects for \(D\) and \(D^*\) and for \(N, \Sigma\) and \(\Xi\), respectively. The value (2.8) is consistent with (2.3) and (2.6). The value

\[(m_u + m_d)_{\mu=1 \text{ GeV}} = (12 \pm 2.5) \text{ MeV },\]  

(2.9)

obtained from QCD finite energy sum rules and Laplace sum rules by Bijnens, Prades and Rafael (1995) [15] is consistent with (2.2).

On the contrary, for the strange quark mass \(m_s\), two different values, \(m_s \approx 175 \text{ MeV}, [(2.1) and (2.6)]\), and \(m_s \approx 200 \text{ MeV}, [(2.3) and (2.4)],\) have been reported. Recently, Chetyrkin et al. (1997) [16] have estimated

\[m_s(1 \text{ GeV}) = 205.5 \pm 19.1 \text{ MeV },\]  

(2.10)

by an order-\(\alpha_s^3\) determination from the QCD sum rules. The value (2.10) is consistent with (2.3). (Of course, if we take their errors into consideration, these values are consistent.) Hereafter, as the light quark masses at \(\mu = 1 \text{ GeV}\), we will use the following values which are weighted averages of the values (2.3), (2.4), (2.6) and (2.10).

\[
\begin{align*}
m_u(1 \text{ GeV}) &= 4.88 \pm 0.57 \text{ MeV }, \\
m_d(1 \text{ GeV}) &= 9.81 \pm 0.65 \text{ MeV }, \\
m_s(1 \text{ GeV}) &= 195.4 \pm 12.5 \text{ MeV }.
\end{align*}
\]

(2.11)

III. HEAVY QUARK MASSES

A. Charm and bottom quark masses

Gasser and Leutwyler (1982) [1] have estimated charm and bottom quark masses \(m_c\) and \(m_b\) from the QCD sum rules as

\[
\begin{align*}
m_c(m_c) &= 1.27 \pm 0.05 \text{ GeV }, \\
m_b(m_b) &= 4.25 \pm 0.10 \text{ GeV }.
\end{align*}
\]

(3.1)

Titard and Ynduráin (1994) [17] have re-estimated charm and bottom quark masses by using the three-level QCD and the full one-loop potential. They have concluded that

\[
\begin{align*}
M_c^{\text{pole}} &= 1.570 \pm 0.019 \pm 0.007 \text{ GeV }, \\
M_b^{\text{pole}} &= 4.906^{+0.009}_{-0.004} \pm 0.004^{+0.011}_{-0.0000} \text{ GeV },
\end{align*}
\]

(3.2)

\[
\begin{align*}
m_c(m_c) &= 1.306^{+0.021}_{-0.034} \pm 0.006 \text{ GeV }, \\
m_b(m_b) &= 4.397^{+0.007}_{-0.004} \pm 0.004^{+0.016}_{-0.002} \text{ GeV }.
\end{align*}
\]

(3.3)
where the first- and second-errors come from the use of the QCD parameter $\Lambda_{MS}^{(4)} = 0.20^{+0.08}_{-0.06}$ GeV and the gluon condensate value $\langle \alpha_s G^2 \rangle = 0.042 \pm 0.020$ GeV$^4$, respectively, and the third error denotes a systematic error.

On the other hand, from the QCD spectral sum rules to two-loops for $\psi$ and $\Upsilon$, Narison (1995) \cite{18} has estimated the running quark masses

$$m_c(M_{PT}^{c}) = 1.23^{+0.02}_{-0.04} \pm 0.03\text{ GeV},$$
$$m_b(M_{PT}^{b}) = 4.23^{+0.03}_{-0.04} \pm 0.02\text{ GeV},$$

(3.4)

corresponding to the short-distance perturbative pole masses to two-loops

$$M_{PT}^{c} = 1.42 \pm 0.03\text{ GeV},$$
$$M_{PT}^{b} = 4.62 \pm 0.02\text{ GeV},$$

(3.5)

and the three-loop values of the short-distance pole masses

$$M_{PT}^{c} = 1.64^{+0.10}_{-0.07} \pm 0.03\text{ GeV},$$
$$M_{PT}^{b} = 4.87 \pm 0.05 \pm 0.02\text{ GeV}.$$  

(3.6)

The values (3.6) are in agreement with the values (3.2) estimated by Titard and Ynduráin while the values (3.5) are not so. Narison asserts that one should not use $M_{PT}^{c}$ because the hadronic correlators are only known to two-loop accuracy.

Although we must keep the Narison’s statement in mind, since we use the three-loop formula (4.5) for the running quark masses $m_q(\mu)$ for all quarks $q = u, d, \ldots, t$, hereafter, we adopt the following weighted averages of (3.2) and (3.6),

$$M_{pole}^{c} = 1.59 \pm 0.02\text{ GeV},$$
$$M_{pole}^{b} = 4.89 \pm 0.05\text{ GeV},$$

(3.7)

as the pole mass values.

B. Top quark mass

The explicit value of the top quark mass was first reported by the CDF collaboration (1994) \cite{3} from the data of $p\bar{p}$ collisions at $\sqrt{s} = 1.8$ TeV:

$$m_t = 174 \pm 10^{+13}_{-12}\text{ GeV}.$$  

(3.8)

They (1995) \cite{4} have also reported an updated value

$$m_t = 176 \pm 8 \pm 10\text{ GeV}.$$  

(3.9)

On the other hand, the D0 collaboration \cite{5} has reported the value

$$m_t = 199^{+19}_{-21} \pm 22\text{ GeV}.$$  

(3.10)

The particle data group (PDG96) \cite{21} has quoted the value

$$m_t = 180 \pm 12\text{ GeV},$$  

(3.11)
as the top quark mass from direct observations of top quark events.

Hereafter, we use the value \(3.11\) as the pole mass of the top quark.

C. Mass values \(m_q(\mu)\) at \(\mu = M_q^{\text{pole}}\)

The relation between the pole mass \(M_q^{\text{pole}}\) and the running quark mass \(m_q(M_q^{\text{pole}})\) at \(\mu = M_q^{\text{pole}}\) has been calculated by Gray et al.\[6\]:

\[
m_q(M_q^{\text{pole}}) = M_q^{\text{pole}} \left[ 1 + \frac{4 \alpha_s(M_q^{\text{pole}})}{3 \pi} + K_q \left( \frac{\alpha_s(M_q^{\text{pole}})}{\pi} \right)^2 + O(\alpha_s^3) \right]^{-1},
\]

(3.12)

where \(K_c = 14.5\), \(K_b = 12.9\) and \(K_t = 11.0\). The definition of \(K_q\) and their estimates are given in Appendix A. The values of \(\alpha_s(\mu)\) at various values of \(\mu\) and errors are given in Table VII in Appendix B. By using (3.12), from (3.7) and (3.11), we obtain

\[
m_c(M_c^{\text{pole}}) = 1.213 \pm 0.018_{-0.034}^{+0.040} \text{ GeV},
\]

\[
m_b(M_b^{\text{pole}}) = 4.248 \pm 0.046_{-0.036}^{+0.040} \text{ GeV},
\]

\[
m_t(M_t^{\text{pole}}) = 170.1 \pm 11.4 \pm 0.3 \text{ GeV},
\]

(3.13)

where the first and second errors come from \(\pm \Delta M_q^{\text{pole}}\) and \(\pm \Delta \Lambda^{(5)}_{\text{MS}}\), respectively.

IV. BEHAVIORS OF \(m_q(\mu)\) AT THE QUARK THRESHOLDS

The scale dependence of a running quark mass \(m_q(\mu)\) is governed by the equation \[7\]

\[
\mu \frac{d}{d\mu} m_q(\mu) = -\gamma(\alpha_s)m_q(\mu),
\]

(4.1)

where

\[
\gamma(\alpha_s) = \gamma_0 \frac{\alpha_s}{\pi} + \gamma_1 \left( \frac{\alpha_s}{\pi} \right)^2 + \gamma_2 \left( \frac{\alpha_s}{\pi} \right)^3 + O(\alpha_s^4).
\]

(4.2)

\[
\gamma_0 = 2, \quad \gamma_1 = \frac{101}{12} - \frac{5}{18} n_q,
\]

\[
\gamma_2 = \frac{1}{32} \left[ 1249 - \left( \frac{2216}{27} + \frac{160}{3} \zeta(3) \right) n_q - \frac{140}{81} n_q^2 \right],
\]

(4.3)

Then, \(m_q(\mu)\) is given by

\[
m_q(\mu) = R(\alpha_s(\mu)) \tilde{m}_q,
\]

(4.4)

\[
R(\alpha_s) = \left( \frac{\beta_0 \alpha_s}{2 \pi} \right)^{2 \gamma_0 / \beta_0} \left\{ 1 + \left( \frac{\gamma_1}{\beta_0} - \frac{\beta_1 \gamma_0}{\beta_0^2} \right) \frac{\alpha_s}{\pi} \right\}
\]

6
\[ + \frac{1}{2} \left( \frac{2 \gamma_1 - \beta_1 \gamma_0}{\beta_0^2} \right)^2 + \frac{2 \gamma_2}{\beta_0^2} - \frac{\beta_1 \gamma_1 - \beta_2 \gamma_0}{\beta_0^2} + \frac{\beta_2^2 \gamma_0}{2 \beta_0^2} \right) \left( \frac{\alpha_s}{\pi} \right)^2 + O(\alpha_s^3) \]  

where \( \bar{m}_q \) is the renormalization group invariant mass which is independent of \( \ln(\mu^2/\Lambda^2) \), \( \alpha_s(\mu) \) is given by (B4) in Appendix B and \( \beta_i \) \( (i=0,1,2) \) are also defined by (B3). By using (4.5) and \( \Lambda_{\overline{\text{MS}}}^{(n)} \) obtained in Appendix B, we can evaluate \( R^{(n)}(\mu) \) for \( \mu < \mu_{n+1} \), where \( \mu_n \) is the \( n \)th quark flavor threshold and we take \( \mu_n = m_{qn}(m_{qn}) \).

Quite recently, the four-loop beta function and quark mass anomalous dimension have been obtained by several authors [22]. In this paper, we evaluate the running quark masses by using the three-loop results (4.1)-(4.5). The effects of the four-loop results to the three-loop results will be discussed in the next section.

We can evaluate the values of \( m_q(m_q) \) \( (q = c, b, t) \) by using the values of \( M_{\overline{\text{MS}}}^{\text{pole}} \) given in Sec. III and the relation

\[ m_{qn}(\mu) = \left[ \frac{R^{(n)}(\mu)/R^{(n)}(M_{\overline{\text{MS}}}^{\text{pole}})}{m_{qn}(M_{\overline{\text{MS}}}^{\text{pole}})} \right] m_{qn}(M_{\overline{\text{MS}}}^{\text{pole}}) \quad (\mu < \mu_{n+1}) \]  

Similarly, we evaluate the light quark masses \( m_q(m_q) \) \( (q = u, d, s) \) by using the relation

\[ m_q(\mu) = \left[ \frac{R^{(3)}(\mu)/R^{(3)}(1 \text{ GeV})}{m_q(1 \text{ GeV})} \right] m_q(1 \text{ GeV}) \quad (\mu < \mu_4) \]  

and the values \( m_q(1 \text{ GeV}) \) given in (2.11). The results are summarized in Table 1. The values of \( m_c(m_c) \), \( m_d(m_d) \) and \( m_s(m_s) \) should not be taken rigidly, because the perturbative calculation is not reliable for such a region in which \( \alpha_s(\mu) \) takes a large value (see the next section).

Exactly speaking, the estimates of \( \Lambda_{\overline{\text{MS}}}^{(n)} \) in Table 11 in Appendix I are dependent on the choices of the quark threshold \( \mu_n = m_{qn}(m_{qn}) \). The values in Table 11 and Table 1 have been obtained by iterating the evaluation of \( \Lambda_{\overline{\text{MS}}}^{(n)} \) and \( m_q(m_q) \).

Running quark mass values \( m_{qn}(\mu) \) at \( \mu \geq \mu_{n+1} \) cannot be evaluated by the formula (4.4) straightforwardly, because of the quark threshold effects. As seen in Fig. 4, the behavior of \( R(\mu) \) is discontinuous at \( \mu = \mu_n \equiv m_{qn}(m_{qn}) \).

The behavior of the \( n \)th quark mass \( m_{qn}^{(N)}(\mu) \) \( (n < N) \) at \( \mu_N \leq \mu < \mu_{N+1} \) are given by the matching condition

\[ m_{qn}^{(N)}(\mu) = m_{qn}^{(N-1)}(\mu) \left[ 1 + \frac{1}{12} \left( x_N^2 + \frac{5}{3} x_N + \frac{89}{36} \right) \left( \frac{\alpha_s^{(N)}(\mu)}{\pi} \right)^2 \right]^{-1} \]  

where

\[ x_N = \ln \left[ \left( m_{qN}^{(N)}(\mu) \right)^2 / \mu^2 \right] \]  

(4.9)

For example, the behavior of \( m_c(\mu) \) at \( \mu < \mu_5 \), \( m_{t}^{(4)}(\mu) \), can be evaluated by using (4.6), while those at \( \mu_5 \leq \mu < \mu_6 \) and \( \mu_6 \leq \mu \) must be evaluated by using (4.8) with \( m_{t}^{(4)}(\mu) \) and \( x_5 = \ln \left[ \left( m_{t}^{(5)}(\mu) \right)^2 / \mu^2 \right] \) and with \( m_{c}^{(5)}(\mu) \) and \( x_6 = \ln \left[ \left( m_{c}^{(6)}(\mu) \right)^2 / \mu^2 \right] \), respectively. In Fig. 2, we illustrate the \( \mu \)-dependency of the light quark masses \( m_q(\mu) \) \( (q = u, d, s) \), where
we have taken the matching condition (4.8) into account. We can see that the discontinuity which was seen in Fig. 1 disappears in Fig. 2.

We also illustrate the behavior of the heavy quark masses \( m_q(\mu) \) (\( q = c, b, t \)) in Fig. 3. Exactly speaking, the word “the running mass value \( m_Q(\mu) \)” of a heavy quark \( Q \) at a lower energy scale \( \mu \) than \( \mu = m_Q(m_Q) \) loses the meaning. For example, the effective quark flavor number \( n_q \) is three at \( \mu = 1 \text{ GeV} \), so that the value of \( m_t(\mu) \) at \( \mu = 1 \text{ GeV} \) has not the meaning. However, for reference, in Fig. 3, we have calculated the value of \( m_Q(\mu) = \hat{m}_Q R^{(n)}(\mu) \) [not \( m_Q(\mu) = \hat{m}_Q R^{(n)}(\mu) \)].

The numerical results are summarized in Table II. As stressed by Vermaseren et al. [22], the invariant mass \( \hat{m}_q \) is good reference mass for the accurate evolution of the \( \overline{\text{MS}} \) quark masses to the necessary scale \( \mu \) in phenomenological applications. The values of \( \hat{m}_q \) are also listed in Table II.

V. RELIABILITY OF THE PERTURBATIVE CALCULATION BELOW \( \mu \sim 1 \text{ GeV} \)

As we noted already, the values of the light quark masses \( m_q(m_q) \) (\( q = u, d, s \)) should not be taken rigidly, because the perturbative calculation below \( \mu \sim 1 \text{ GeV} \) seems to be not reliable.

In order to see the reliability of the calculation of \( \alpha_s(\mu) \), in Fig. 4 we illustrate the values of the second and third terms in \{ \} of (B4) in Appendix 1 separately. The values of the second and third terms exceed one at \( \mu \simeq 0.42 \text{ GeV} \) and \( \mu \simeq 0.47 \text{ GeV} \), respectively. Also, in Fig. 5, we illustrate the values of the second and third terms in \{ \} of (4.5) separately. The values of the second and third terms exceed one at \( \mu \simeq 0.58 \text{ GeV} \) and \( \mu \simeq 0.53 \text{ GeV} \), respectively. These means that the perturbative calculation is not reliable below \( \mu \simeq 0.6 \text{ GeV} \). Therefore, the values with asterisk in Tables I, II and VI should not be taken strictly.

These situations are not improved even if we take the four-loop correction into consideration. For example, for \( n_q = 3 \), \( d(\alpha_s/\pi)/d \ln \mu \) is given by [22]

\[
\frac{d(\alpha_s/\pi)}{d \ln \mu} = -\frac{9}{2} \left( \frac{\alpha_s}{\pi} \right)^2 \left[ 1 + 1.79 \frac{\alpha_s}{\pi} + 4.47 \left( \frac{\alpha_s}{\pi} \right)^2 + 21.0 \left( \frac{\alpha_s}{\pi} \right)^3 + \cdots \right].
\] (5.1)

Since the value of \( \alpha_s/\pi \) is \( \alpha_s/\pi \simeq 0.16 \) at \( \mu \simeq 1 \text{ GeV} \), the numerical values of the right-hand side of (5.1) becomes

\[
\frac{d(\alpha_s/\pi)}{d \ln \mu} = -\frac{9}{2} \left( \frac{\alpha_s}{\pi} \right)^2 \left[ 1 + 0.28 + 0.11 + 0.085 + \cdots \right],
\] (5.2)

so that the fourth term is not negligible compared with the third term. This suggests that the fifth term which is of the order of \( (\alpha_s/\pi)^6 \) will also not be negligible below \( \mu \sim 1 \text{ GeV} \).

However, we consider that the evolution of \( m_q(\mu) \) above \( \mu \sim 1 \text{ GeV} \) (from \( \mu \simeq 1 \text{ GeV} \) to \( \mu \sim m_Z \)) is reliable in spite of the large error of \( \alpha_s(\mu) \) at \( \mu \sim 1 \text{ GeV} \).
VI. OBSERVABLE QUANTITIES $m_q(\mu)$, $V_{CKM}(\mu)$ AND $\alpha_i(\mu)$ AT $\mu = m_Z$

For quark mass matrix phenomenology, values of $m_q(\mu)$ at $\mu = m_Z$ are useful, because the observed CKM matrix parameters $|V_{ij}|$ are given at $\mu = m_Z$. We summarize quark and charged lepton masses at $\mu = m_Z$

$$m_u = 2.33^{+0.42}_{-0.45} \text{ MeV} , \quad m_c = 677^{+56}_{-61} \text{ MeV} , \quad m_t = 181 \pm 13 \text{ GeV} ,$$
$$m_d = 4.69^{+0.66}_{-0.66} \text{ MeV} , \quad m_s = 93.4^{+11.8}_{-13.6} \text{ MeV} , \quad m_b = 3.00 \pm 0.11 \text{ GeV} ,$$

(6.1)

where the running charged lepton masses $m_\ell(\mu)$ have been evaluated from the relation for the physical (pole) masses $M_\ell$

$$m_\ell(\mu) = M_\ell \left[ 1 - \frac{\alpha(\mu)}{\pi} \left( 1 + \frac{3}{4} \ln \frac{\mu^2}{m_\ell^2} \right) \right].$$

(6.2)

The value of $m_b(m_Z)$ in (6.1) is in good agreement with the value $[24]$

$$m_b(m_Z) = 2.67 \pm 0.25 \pm 0.27 \pm 0.34 \text{ GeV} ,$$

(6.3)

which has recently been extracted from CERN LEP data.

On the other hand, the standard expression [25] of the CKM matrix $V$ is given by

$$V = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}s_{13}e^{i\delta_{13}} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}s_{13}
\end{pmatrix}.$$  

(6.4)

The observed values $|V_{us}|$, $|V_{ub}|$ [21] and $|V_{cb}|$ [26,27] are

$$|V_{us}| = 0.2205 \pm 0.0018 ,$$
$$|V_{cb}| = 0.0373 \pm 0.0018 ,$$
$$|V_{ub}/V_{cb}| = 0.08 \pm 0.02 ,$$

(6.5)

where the value of $|V_{cb}|$ has been obtained by combining the OPAL97 value [26] $|V_{cb}| = 0.0360 \pm 0.0021 \pm 0.0024 \pm 0.0012$ and the ALEPH97 value [27] $|V_{cb}| = 0.0344 \pm 0.0016 \pm 0.0023 \pm 0.0014$ with the PDG96 value $|V_{cb}| = 0.041 \pm 0.003$. Because of the hierarchical structure $|V_{us}|^2 \gg |V_{cb}|^2 \gg |V_{ub}|^2$, the following expression of $V$ will also be useful:

$$V \simeq \begin{pmatrix}
    1 - \frac{1}{2} \lambda^2 & \frac{\lambda}{2} & \sigma e^{-i\delta} \\
    -\frac{\lambda}{2} & 1 - \frac{1}{2} \lambda^2 & \rho \\
    \lambda \rho - \sigma e^{i\delta} & -\rho & 1 - \frac{1}{2} \rho^2
\end{pmatrix},$$

(6.6)

where $\lambda = |V_{us}|$, $\rho = |V_{cb}|$ and $\sigma = |V_{ub}|$. Hereafter, we will use the observed values (6.5) as the values of $|V_{ij}(\mu)|$ at $\mu = m_Z$. Then, from the expression (6.4) (not the approximate expression (6.6)), we obtain the numerical expression of $V(\mu)$ at $\mu = m_Z$,
Numerically, by using (6.7), but without using the approximate expression (6.11), we obtain

\[ V(m_Z) = \begin{pmatrix} 0.9754 & 0.2205 & 0.0030e^{-i\delta} \\ -0.2203 - 0.0001e^{i\delta} & 0.9747 & 0.0373 \\ 0.0082 - 0.0029e^{i\delta} & -0.0364 - 0.0007e^{i\delta} & 0.9993 \end{pmatrix} . \] (6.7)

Since we have already known the numerical values of \( D_u = \text{diag}(m_u, m_c, m_t) \), \( D_d = \text{diag}(m_d, m_s, m_b) \) and \( V_{ij} \) (except for the parameter \( \delta \)) at \( \mu = m_Z \), by using the relations

\[ U^U_L^M_uU^R_R^{\dagger} = D_u \quad U^U_L^dM_dU^R_R^{\dagger} = D_d \quad V = U^U_L^U_L^{\dagger} , \] (6.8)

we can determine the numerical structures of the squared mass matrices \( H_u \) and \( H_d \) which are defined by

\[ H_u = M_uM_u^{\dagger} , \quad H_d = M_dM_d^{\dagger} . \] (6.9)

Especially, at a special quark-family basis on which the up-quark mass matrix takes a diagonal form \( D_u \), we can readily obtain the matrix form \( H_u \) and \( H_d \):

\[ H_u = D_u^2 = m_t^2 \begin{pmatrix} m_u^2/m_t^2 & 0 & 0 \\ 0 & m_c^2/m_t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \] (6.10)

\[ H_d = VD_d^2V^{\dagger} \simeq m_b^2 \begin{pmatrix} \sigma^2(1 + x^2) & \rho \sigma(y + e^{-i\delta}) & \rho \sigma^{-i\delta} \\ \rho \sigma(y + e^{i\delta}) & \rho^2(1 + y^2/x^2) & \rho \\ \rho e^{i\delta} & \rho & 1 \end{pmatrix} , \] (6.11)

where

\[ x = \frac{\lambda m_s}{\sigma m_b} , \quad y = \frac{\lambda}{\rho \sigma} \left( \frac{m_s}{m_b} \right)^2 . \] (6.12)

Numerically, by using (6.7), but without using the approximate expression (6.11), we obtain

\[ H_u(m_Z) = m_t^2(m_Z) \begin{pmatrix} 1.66 \times 10^{-10} & 0 & 0 \\ 0 & 1.40 \times 10^{-5} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \] (6.13)

\[ H_d(m_Z) = m_b^2(m_Z) \begin{pmatrix} 5.84 \times 10^{-5} & (2.08 + 1.11e^{-i\delta}) \times 10^{-4} & 2.98 \times 10^{-3}e^{-i\delta} \\ (2.08 + 1.11e^{i\delta}) \times 10^{-4} & 2.31 \times 10^{-3} & 3.72 \times 10^{-2} \\ 2.98 \times 10^{-3}e^{i\delta} & 3.72 \times 10^{-2} & 0.9986 \end{pmatrix} , \] (6.14)

where \( m_t^2(m_Z) = 3.24 \times 10^4 \text{ GeV}^2 \) and \( m_b^2(m_Z) = 9.00 \text{ GeV}^2 \).

In the standard model [not \( SU(2)_L \times SU(2)_R \times U(1)_Y \), but \( SU(2)_L \times U(1)_Y \)], by a suitable transformation of the right-handed fields, we can always make quark mass matrices \( (M_u, M_d) \) Hermitian. Furthermore, in the quark-family basis where \( M_u = D_u \), the quark mass matrices are given by
\[ M_u = D_u = m_t \begin{pmatrix} m_u/m_t & 0 & 0 \\ 0 & m_c/m_t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.15) \]

\[ M_d = V D_d V^\dagger \simeq m_b \begin{pmatrix} m_s/m_b \left( \frac{m_d}{m_s} + \lambda^2 \right) \lambda m_s/m_b \sigma e^{-i\delta} \\ \frac{\lambda m_s}{m_b} & m_s/m_b & \rho \\ \sigma e^{i\delta} & \rho & 1 \end{pmatrix}. \quad (6.16) \]

It is well known that if we assume \((M_d)_{11} = 0\), we obtain the relation
\[ \lambda \equiv |V_{us}| \simeq \sqrt{-m_d/m_s}. \quad (6.17) \]

Then, we obtain a simpler expression of \(M_d\)
\[ M_d \simeq m_b \begin{pmatrix} 0 & \sqrt{-m_d m_s/m_b^2} \sigma e^{-i\delta} \\ \sqrt{-m_d m_s/m_b^2} & m_s/m_b & \rho \\ \sigma e^{i\delta} & \rho & 1 \end{pmatrix}. \quad (6.18) \]

Numerically, by using Eq. (6.7), we obtain
\[ M_u(m_Z) = m_t(m_Z) \begin{pmatrix} 1.29 \times 10^{-5} & 0 & 0 \\ 0 & 3.75 \times 10^{-3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.19) \]

\[ M_d(m_Z) = m_b(m_Z) \times \begin{pmatrix} 3.01 \times 10^{-3} & (6.36 + 0.11e^{-i\delta}) \times 10^{-3} & (0.24 + 2.97e^{-i\delta}) \times 10^{-3} \\ (6.36 + 0.11e^{i\delta}) \times 10^{-3} & 0.0310 & 0.0362 \\ (-0.24 + 2.97e^{i\delta}) \times 10^{-3} & 0.0362 & 0.9986 \end{pmatrix}, \quad (6.20) \]

where \(m_t(m_Z) = 180\) GeV and \(m_b(m_Z) = 3.00\) GeV. For the case of \(m_s < 0\), instead of (6.20), we obtain
\[ M_d(m_Z) = m_b(m_Z) \times \begin{pmatrix} -1.8 \times 10^{-5} & (-7.03 + 0.11e^{-i\delta}) \times 10^{-3} & (0.26 + 2.98e^{-i\delta}) \times 10^{-3} \\ (-7.03 + 0.11e^{i\delta}) \times 10^{-3} & -0.0281 & 0.0384 \\ (0.26 + 2.98e^{i\delta}) \times 10^{-3} & 0.0384 & 0.9986 \end{pmatrix}. \quad (6.21) \]
We can obtain quark mass matrix forms on arbitrary quark-family basis by the unitary transformation $H'_u = U H_u U^\dagger$ and $H'_d = U H_d U^\dagger$ for (6.10) and (6.11), respectively [and also $M'_u = U M_u U^\dagger$ and $M'_d = U M_d U^\dagger$ for (6.15) and (6.16), respectively]. Explicit mass matrix forms on other special quark-family basis are, for example, given in Refs. [29,30].

By starting from the numerical expressions of the mass matrices $H_u$ and $H_d$ at $\mu = m_Z$, (6.10) and (6.11), we can also obtain the mass matrix form $M_q (q = u, d)$ (in other words, the Yukawa coupling constants) at arbitrary energy scale $\mu$ which is larger than the electroweak scale $\Lambda_W$. In the next section, we discuss the evolution of the Yukawa coupling constants. Then, we will use the following values of the SU(3)$_c \times$SU(2)$_L \times$U(1)$_Y$ gauge coupling constants at $\mu = m_Z$:

$$\alpha_1 (m_Z) = 0.016829 \pm 0.000017,$$
$$\alpha_2 (m_Z) = 0.033493^{+0.000060}_{-0.000058},$$
$$\alpha_3 (m_Z) = 0.118 \pm 0.003.$$

which are derived from [31]

$$\alpha (m_Z) = (128.89 \pm 0.09)^{-1},$$
$$\sin^2 \theta_W = 0.23165 \pm 0.000024,$$

and $\Lambda_{\text{MS}}^{(5)} = 209^{+39}_{-33}$ MeV [2]. Here, the coupling constants of U(1)$_Y$, SU(2)$_L$, and SU(3)$_c$ gauge bosons, $g_1$, $g_2$ and $g_3$, are defined as they satisfy the relation

$$\frac{1}{e^2} = \frac{5}{3}\frac{1}{g_1^2} + \frac{1}{g_2^2},$$

and the relation in the SU(5)-GUT [32] limit

$$g_1 = g_2 = g_3.$$

VII. EVOLUTION OF YUKAWA COUPLING CONSTANTS

So far, we have evaluated values of the running quark masses $m_q (\mu)$ at energy scales which are below the electroweak symmetry breaking energy scale $\Lambda_W$ by using the formula (4.1). However, for the quark masses at an extremely high energy scale far from $\Lambda_W$, we must use “evolution” equations of Yukawa coupling constants $y_{ij}^a (a = u, d : i, j = 1, 2, 3)$. The numerical results of the Yukawa coupling constants have already been given in many literatures. Since our interest in the present paper is in the updated values of the quark masses $m_q (\mu)$ (i.e., the Yukawa coupling constants $y_q$), we give only a short review of the evolution of the Yukawa coupling constants, and do not give a systematical study of the numerical results.

We define the Yukawa coupling constants $y_{ij}^a$ as follows:

$$H_{\text{mass}} = \sum_{a=u,d} \sum_{i=1}^3 \sum_{j=1}^3 y_{ij}^a \overline{\psi}_{Lai} \psi_{Raj} \phi_a^0 + h.c.,$$

(7.1)
where \( \phi_a^0 \) are the vacuum expectation values of the neutral components of the Higgs bosons \( \phi_a \) which couple with fermions \( \psi_a \), and they mean \( \phi_u^0 \) and \( \phi_d^0 \) for the minimal SUSY model (Model B), while they mean a single Higgs boson \( \phi^0 = \phi_u^0 = \phi_d^0 \) for the standard model with one Higgs boson (Model A). The quark mass matrices \( M_u \) and \( M_d \) at \( \mu = \Lambda_W \) are given by

\[
M_a(\mu) = \frac{1}{\sqrt{2}} Y_a(\mu) v_a ,
\]

where \( Y_a \) denotes a matrix \((Y_a)_{ij} = y_{ij}^a\), and \( v_a \) are the vacuum expectation values of \( \phi_a^0 \), \( v_u = \sqrt{2} \langle \phi_u^0 \rangle \), and \( v_d = \sqrt{2} \Lambda_W \) for Model A and \( \sqrt{v_u^2 + v_d^2} = \sqrt{2} \Lambda_W \) for Model B.

The evolution of the coupling constants \( Y_a(\mu) \) from \( Y_a(\Lambda_W) \) is given by the following equations [33]:

\[
dY_a \quad dt = \left[ \frac{1}{16\pi^2} \beta_a^{(1)} + \frac{1}{(16\pi^2)^2} \beta_a^{(2)} \right] Y_a , \quad (a = u, d, e)
\]

\[
t = \ln(\mu/\Lambda_W) ,
\]

\[
\beta_a^{(1)} = c_a^{(1)} 1 + \sum b_a^b H_b ,
\]

\[
\beta_a^{(2)} = c_a^{(2)} 1 + \sum b_a^b H_b + \sum b_a^{bc} H_b H_c ,
\]

\[
H_a = Y_a Y_a^\dagger ,
\]

where, for convenience, we have changed the definition of the Hermitian matrix \( H_a \) from (6.8) in the previous section to (7.7). The coefficients \( c_a^{(1)} \) and \( b_a^b \) in the one-loop contributions \( \beta_a^{(1)} \) are given in Table [III] according to Models A and B, where

\[
c_a^{(1)} = T_a - G_a .
\]

The coefficients \( c_a^{(2)} \), \( b_a^b \) and \( b_a^{bc} \) in the two-loop contributions \( \beta_a^{(2)} \) are given in Appendix C, because they have too long expressions. The evolution of the gauge coupling constants \( g_i(\mu) \) is given in Appendix D.

By using the information of \( V_{ij}(\mu) \) at \( \mu = m_Z \) in the previous section, we can obtain not the knowledge of \( M_q(m_Z) \), but that of \( H_q(m_Z) \), i.e., \( H_u = D_u^2 \) and \( H_d = V D_d^2 V^\dagger \), where \( D_q \) (\( q = u, d \)) are the diagonalized matrices of \( Y_q \). Then, the expression for \( H_a(\mu) \)

\[
\frac{d}{dt} H_a = \left[ \frac{1}{16\pi^2} \beta_a^{(1)} + \frac{1}{(16\pi^2)^2} \beta_a^{(2)} \right] H_a + H_a \left[ \frac{1}{16\pi^2} \beta_a^{(1)\dagger} + \frac{1}{(16\pi^2)^2} \beta_a^{(2)\dagger} \right] ,
\]

is useful rather than (7.3) which is the expression for \( Y_a \). Hereafter, for simplicity, we calculate the evolution not from \( \mu = \Lambda_W \), but from \( \mu = m_Z \) because most of the input values at \( \mu = m_Z \) have already given in Sec. [VI]. Since the numerical results are insensitive to the value of the phase parameter \( \delta_{13} (\pi/3 < \delta_{13} < 2\pi/3) \) in the CKM matrix \( V \), (6.4),
we will use the value \( \delta \equiv \delta_{13} = \pi/2 \) below. For Model A (Standard model with one Higgs boson), we must assume the value of the Higgs boson mass \( m_H \). We will take a typical value \( m_H = \sqrt{2} \Lambda_W = 246.2 \) GeV (see later discussion). For Model B (Minimal SUSY model), we must assume the value of \( \tan \beta = v_u/v_d \). We will take a typical value \( \tan \beta = 10 \). The numerical results of \( y_q \) are given below. Here, the values \( y_{ai} \) are obtained by diagonalizing the matrix \( H_a \), and it does not mean \( \sqrt{(H_a)_{ii}} \).

(A) Standard model with one Higgs boson

As seen in Appendix A in the calculation of the two-loop contributions, the evolution of the Yukawa coupling constants \( y_q \) depends on the coupling constant \( \lambda_H \) of the Higgs boson \( \phi \), which is related to the Higgs boson mass \( m_H \) as

\[
\lambda_H = m_H^2/v^2. \tag{7.10}
\]

We find [34] that the input value of \( m_H(m_Z) \) which is less than \( 2.2 \times 10^2 \) GeV leads to a negative \( \lambda_H \) at a unification scale \( \mu = M_X \), while that which is larger than \( 2.6 \times 10^2 \) GeV leads to the burst of \( \lambda_H \) at the unification scale. Therefore, if we put an ansatz that Nature accepts only the parameter regions in which the perturbative calculations are valid, we can conclude that the Higgs boson mass \( m_H \) in the standard model must be in

\[
220 \text{ GeV} < m_H(m_Z) < 260 \text{ GeV}. \tag{7.11}
\]

In Table IV, we list the numerical results of \( m_q(\mu) = y_q(\mu)v/\sqrt{2} \) at the typical energy scales \( \mu = m_Z, \mu = 10^9 \) GeV and \( \mu = M_X \). For the comparison with the SUSY model (Model B) later, the values \( m_q(\mu) \) at \( \mu = M_X \) are listed, where \( M_X \) is a unification scale of SUSY, \( M_X = 2 \times 10^{16} \) GeV. Here, we have tentatively taken a value \( m_H = \sqrt{2} \Lambda_W = 246.2 \) GeV (i.e., \( \lambda_H = 1 \)) as the input value of \( m_H(m_Z) \).

We also obtain the numerical expression of the CKM matrix \( V(\mu) \) at \( \mu = M_X \)

\[
V(M_X) = \begin{pmatrix}
0.9754 & 0.2206 & -0.0035i \\
-0.2203 & 0.9745 & 0.0433 \\
0.0101e^{-19\pi i} & -0.0422e^{+1.0\pi i} & 0.9991
\end{pmatrix}, \tag{7.12}
\]

correspondingly to (6.7) at \( \mu = m_Z \), where we have taken \( \delta = 90^\circ \) tentatively. We also obtain the numerical result of \( (M_u, M_d) \) at \( \mu = M_X \) correspondingly to (6.19), (6.20) and (6.21):

\[
M_u(M_X) = m_t(M_X) \begin{pmatrix}
1.11 \times 10^{-5} & 0 & 0 \\
0 & 3.23 \times 10^{-3} & 0 \\
0 & 0 & 1
\end{pmatrix}, \tag{7.13}
\]

\[
M_d(M_X) = m_b(M_X) \begin{pmatrix}
0.0035 & 0.0074e^{-1.2\pi i} & 0.0035e^{-95.3\pi i} \\
0.0074e^{+1.2\pi i} & 0.0363 & 0.0418e^{+0.03\pi i} \\
0.0035e^{+95.3\pi i} & 0.0418e^{-0.03\pi i} & 0.9982
\end{pmatrix}, \tag{7.14}
\]

and
where $m_t(M_X) = 84.2$ GeV and $m_b(M_X) = 1.071$ GeV.

(B) Minimal SUSY model

The scale of the SUSY symmetry breaking $m_{SUSY}$ is usually taken as $m_{SUSY} \simeq m_t$ or $m_{SUSY} \simeq 1$ TeV. For simplicity, we take $m_{SUSY} = m_Z$ in the present numerical study, because the numerical results of $y_b(\mu)$ are not sensitive to the value of $m_{SUSY}$.

The values of $m_q(\mu) = y_q(\mu)v/\sqrt{2} \ (q = u, d)$ are sensitive to the value of $\tan \beta = v_u/v_d$. A large value of $\tan \beta$, $\tan \beta \simeq 60$, leads to the burst of $m_b(\mu)$ at the unification scale $\mu = M_X \simeq 2 \times 10^{16}$ GeV. On the other hand, a small value of $\tan \beta$, $\tan \beta \simeq 1.5$, leads to the burst of $m_t(\mu)$ at the unification scale. The values of $m_q(\mu)$ are insensitive to the value of $\tan \beta$ in the region from $\tan \beta \simeq 5$ to $\tan \beta \simeq 30$ [35]. In Table [V] we list the numerical results of $m_q(\mu)$ at the typical energy scales, $\mu = m_Z$, $\mu = 10^9$ GeV and $\mu = M_X$. Here, we have tentatively taken a value $\tan \beta = 10$ as the input value of $\tan \beta$.

In Fig. 5 for reference, we illustrate the behavior of $m_t(\mu)$, $m_b(\mu)$ and $m_\tau(\mu)$. The value of $m_t(M_X)$ is highly dependent on the input value of $m_t(m_Z)$. Therefore, the value of $m_t(M_X)$ in Table [V] should not be taken strictly. Also, the energy scale $\mu_X$ at which $m_b(\mu_X) = m_\tau(\mu_X)$ is highly dependent on the input value of $m_b(m_Z)$. Therefore, the value $\mu_X$ should also not be taken strictly.

As seen in Fig. 5, it is very interesting that the observed top quark mass value is given by almost the upper value which gives $m_q(\Lambda_W) \leq m(M_X)$. However, since the purpose of the present paper is not to investigate the evolution of the Yukawa coupling constants in the SUSY model under some postulation [e.g., $m_b(\mu) = m_\tau(\mu)$ at $\mu = M_X$], we do not go further more. Some of such studies will be found in Refs. [35,36].

We also obtain the numerical expression of the CKM matrix $V(\mu)$ at $\mu = M_X$

$$V(M_X) = \begin{pmatrix}
0.9754 & 0.2205 & -0.0026i \\
-0.2203e^{+0.03i} & 0.9749 & 0.0318i \\
0.0075e^{-19i} & -0.0311e^{+10i} & 0.9995
\end{pmatrix}, \quad (7.16)$$

correspondingly to (6.7) at $\mu = m_Z$, where we have taken $\delta = 90^\circ$ tentatively. We also obtain the numerical result of $(M_u, M_d)$ at $\mu = M_X$ correspondingly to (6.19), (6.20) and (6.21):

$$M_u(M_X) = m_t(M_X) \begin{pmatrix}
8.0 \times 10^{-6} & 0 & 0 \\
0 & 2.33 \times 10^{-3} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (7.17)$$

$$M_d(M_X) = m_b(M_X) \begin{pmatrix}
0.0026 & 0.0054e^{-0.9i} & 0.0025e^{-93.9i} \\
0.0054e^{+0.9i} & 0.0263 & 0.0310e^{+0.03i} \\
0.0025e^{+93.9i} & 0.0310e^{-0.03i} & 0.9990
\end{pmatrix}, \quad (7.18)$$

and
\[
M_q(M_X) = m_b(M_X) \left( \begin{array}{ccc}
-1.6 \times 10^{-5} & -0.0060e^{+0.8^\circ i} & 0.0026e^{-85.8^\circ i} \\
-0.0060e^{-0.8^\circ i} & -0.0241 & 0.0326e^{-0.03^\circ i} \\
0.0026e^{+85.8^\circ i} & 0.0326e^{+0.03^\circ i} & 0.9990
\end{array} \right),
\]
(7.19)

where \( m_t(M_X) = 129.3 \) GeV and \( m_b(M_X) = 0.997 \) GeV.

VIII. SUMMARY

In conclusion, we have evaluated the running quark mass values \( m_q(\mu) \) (\( q = u, d, s, c, b, t \)) at various energy scales \( \mu \) (\( \mu = 1 \) GeV, \( \mu = m_q, \mu = m_Z, \) and so on). The values of \( m_q(m_q) \) given in Table II in Sec. [V] will be convenient for hadron physicists who want to calculate hadronic matrix elements on the bases of quark-parton model, heavy-quark effective theory, and so on. Also, the values of \( m_q(\mu), m_\ell(\mu), |V_{ij}(\mu)| \) and \( \alpha_i(\mu) \) at \( \mu = m_Z \) given in Sec. [VI] will be convenient for quark and lepton mass-matrix model-builders. In quark mass matrix phenomenology, the values of \( m_q(\mu) \) at \( \mu = 1 \) GeV have conventionally been used. However, we recommend the use of the values \( m_q(m_Z) \) rather than \( m_q(1 \) GeV), because we can use the observed values of \( |V_{ij}| \) as the values \( |V_{ij}(m_Z)| \) straightforwardly, and, exactly speaking, the value of \( m_t(1 \) GeV) does not have the meaning.

Although, in Sec. [VII], we have given the values of \( m_q(\mu) \) at \( \mu = M_X \), i.e., the evolution of the Yukawa coupling constants \( y_q(\mu) \), the study was not systematical in contrast to the study for \( \mu \leq \Lambda_W \). The values of \( y_q(\mu) \) in the standard model with one Higgs depend on the input value of the boson mass \( m_H(m_Z) \). The values of \( y_q(\mu) \) in the minimal SUSY model depend on the values of the parameters \( m_{SUSY} \) and \( \tan \beta \). Therefore, the values \( m_q(M_X) \) given in Table IV and Table V in Sec. [VII] should be taken only for reference.

We hope that the most of the present results, Table II in Sec. [IV] and (6.1), (6.7), (6.13) and (6.14) in Sec. [VI] are usefully made by particle physicists.

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APPENDIX A: RELATION BETWEEN $m_q(m_q)$ AND $M_q^{\text{pole}}$

The pole mass, $M_q^{\text{pole}}(p^2 = m_q^2)$, is a gauge-invariant, infrared-finite, renormalization-scheme-independent quantity. Generally, mass function $M(p^2)$, which is defined by

$$S(p) = Z(p^2)/\left(M(p^2) - p^2\right),$$  \hspace{1cm} (A1)

$$Z(p^2) = 1 - \frac{\alpha_s}{3\pi} \left(a - 3b + \frac{2}{3}\right) \lambda_H + O(\alpha_s^2),$$  \hspace{1cm} (A2)

is related to

$$M(p^2) = m(\mu) \left[1 + \frac{\alpha_s}{\pi} (a + \lambda b) + O(\alpha_s^2)\right],$$  \hspace{1cm} (A3)

$$a = \frac{4}{3} - \ln \frac{m^2}{\mu^2} + \frac{m^2 - p^2}{p^2} \ln \frac{m^2 - p^2}{m^2},$$  \hspace{1cm} (A4)

$$b = -\frac{m^2 - p^2}{3p^2} \left(1 + \frac{m^2}{p^2} \ln \frac{m^2 - p^2}{m^2}\right),$$  \hspace{1cm} (A5)

where $\lambda_H$ is given by $\lambda_H = 0$ in the Landau gauge and by $\lambda_H = 1$ in the Feynman gauge. For $p^2 = m^2$, we obtain $a = 4/3$ and $b = 0$, so that we obtain the relation

$$M_q^{\text{pole}}(p^2 = m_q^2) = m_q(m_q) \left(1 + \frac{4}{3} \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right).$$  \hspace{1cm} (A6)

Similarly, for the spacelike value of $p^2$, $p^2 = -m_q^2$, we obtain $a = 4/3 - 2\ln 2$ and $b = (2/3)(1 - \ln 2)$, so that we obtain the gauge-dependent “Euclidean” masses

$$M_q^{\text{pole}}(p^2 = -m_q^2) = m_q(m_q) \left[1 + \frac{\alpha_s}{\pi} \left(\frac{4}{3} - 2\ln 2\right) + O(\alpha_s^2)\right].$$  \hspace{1cm} (A7)

The estimate of the pole mass has been given by Gray et al [6] (also see [38]):

$$m_q(M_q^{\text{pole}}) = M_q^{\text{pole}} / \left[1 + \frac{4}{3} \frac{\alpha_s(M_q^{\text{pole}})}{\pi} + K_q \left(\frac{\alpha_s(M_q^{\text{pole}})}{\pi}\right)^2 + O(\alpha_s^2)\right],$$  \hspace{1cm} (A8)

$$K_q = K_0 + \frac{4}{3} \sum_{i=1}^{n-1} \Delta(M_i^{\text{pole}}/M_n^{\text{pole}}).$$  \hspace{1cm} (A9)
\[K_0 = \frac{1}{9} \pi^2 \ln 2 + \frac{7}{18} \pi^2 - \frac{1}{6} \zeta(3) + \frac{3673}{288} \left( \frac{1}{18} \pi^2 + \frac{71}{144} \right) n, \quad (A10)\]

\[\Delta(r) = \frac{1}{4} \left[ \ln^2 r + \frac{1}{6} \pi^2 - \left( \ln r + \frac{3}{2} \right) r^2 \right.\]
\[- \left. (1 + r)(1 + r^3) L_+(r) - (1 - r)(1 - r^3) L_-(r) \right], \quad (A12)\]

\[L_\pm(r) = \int_0^{1/r} dx \frac{\ln x}{x \pm 1}, \quad (A13)\]

Here the sum in (A9) is taken over \(n - 1\) light quarks with masses \(M_{\text{pole}}^i (M_{\text{pole}}^i < M_{\text{pole}}^n \equiv M_{\text{pole}}^q)\). The numerical results are summarized in Table VI.

In Table VI, the values of \(M_{\text{pole}}^q\) and \(m_q\) for the light quarks \(q = u, d, s\) have been obtained by solving the relations (A8) with the help of (A7) with the inputs (2.11). These values for the light quarks should not be taken rigidly, because the perturbative calculation is unreliable for the region at which \(\alpha_s(\mu)\) takes a large value. Fortunately, the values of \(K_q\) are not sensitive to the values of \(M_{\text{pole}}^q\) for the light quarks \(q = u, d, s\). Therefore, the values of \(K_q\) in Table VI are valid not only for the heavy quarks \(q = c, b, t\) but also for the light quarks \(q = u, d, s\).

**APPENDIX B: ESTIMATE OF \(\Lambda_{\overline{MS}}^{(n)}\)**

The effective QCD coupling \(\alpha_s = g_s^2 / 4\pi\) is governed by the \(\beta\)-function:

\[\frac{\partial \alpha_s}{\partial \mu} = \beta(\alpha_s), \quad (B1)\]

where

\[\beta(\alpha_s) = -\frac{\beta_0}{2\pi} \alpha_s^2 - \frac{\beta_1}{4\pi^2} \alpha_s^3 - \frac{\beta_2}{64\pi^3} \alpha_s^4 + O(\alpha_s^5), \quad (B2)\]

\[\beta_0 = 11 - \frac{2}{3} n_q, \quad \beta_1 = 51 - \frac{19}{3} n_q, \quad \beta_2 = 2857 - \frac{5033}{9} n_q + \frac{325}{27} n_q^2, \quad (B3)\]

and \(n_q\) is the effective number of quark flavors [39]. The solution \(\alpha_s(\mu)\) of (B1) is given by [2]

\[\alpha_s(\mu) = \frac{4\pi 1}{\beta_0 L} \left\{ 1 - \frac{2\beta_1}{\beta_0^2} \ln L + \frac{4\beta_2}{\beta_0^3 L^2} \left[ \left( \ln L - \frac{1}{2} \right)^2 + \frac{\beta_2 \beta_0}{8\beta_1^2} - \frac{5}{4} \right] \right\} + O \left( \frac{\ln^2 L}{L^3} \right), \quad (B4)\]

where
\[ L = \ln(\mu^2 / \Lambda^2) \, . \]  

(B5)

The value of \( \alpha_s(\mu) \) is not continuous at \( n \)th quark threshold \( \mu_n \) (at which the \( n \)th quark flavor channel is opened), because the coefficients \( \beta_0, \beta_1 \) and \( \beta_2 \) in (B2) depend on the effective quark flavor number \( n_q \). Therefore, we use the expression \( \alpha_s(n)(\mu) \) (B3) with a different \( \Lambda_{\text{MS}}^{(n)} \) for each energy scale range \( \mu_n \leq \mu < \mu_{n+1} \). The relationship between \( \Lambda_{\text{MS}}^{(n-1)} \) and \( \Lambda_{\text{MS}}^{(n)} \) is fixed at \( \mu = m_q^{(n)} \), where \( m_q^{(n)} \) is the value of the \( n \)th running quark mass \( m_q^{(n)} = m_q(m_q) \), and is given as follows [40]:

\[
2 \beta_0^{(n-1)} \ln \left( \frac{\Lambda_{\text{MS}}^{(n)}}{\Lambda_{\text{MS}}^{(n-1)}} \right) = \left( \beta_0^{(n)} - \beta_0^{(n-1)} \right) L_{\text{MS}}^{(n)}
+ 2 \left( \frac{\beta_1^{(n)}}{\beta_0^{(n)}} - \frac{\beta_1^{(n-1)}}{\beta_0^{(n-1)}} \right) \ln \left( \frac{L_{\text{MS}}^{(n)}}{\Lambda_{\text{MS}}^{(n)}} \right) - \frac{2 \beta_1^{(n-1)}}{\beta_0^{(n-1)}} \ln \left( \frac{\beta_0^{(n)}}{\beta_0^{(n-1)}} \right)
+ \frac{4 \beta_1^{(n)}}{\beta_0^{(n)}} \left( \frac{\beta_1^{(n)}}{\beta_0^{(n)}} - \frac{\beta_1^{(n-1)}}{\beta_0^{(n-1)}} \right) \ln \left( \frac{L_{\text{MS}}^{(n)}}{\Lambda_{\text{MS}}^{(n)}} \right)
+ \frac{1}{\beta_0^{(n)}} \left[ \left( \frac{2 \beta_1^{(n)}}{\beta_0^{(n)}} \right)^2 - \left( \frac{2 \beta_1^{(n-1)}}{\beta_0^{(n-1)}} \right)^2 \right] - \frac{\beta_2^{(n)}}{2 \beta_0^{(n)}} + \frac{\beta_2^{(n-1)}}{2 \beta_0^{(n-1)}} - \frac{22}{9} \frac{1}{L_{\text{MS}}^{(n)}} ,
\]

(B6)

where

\[
L_{\text{MS}}^{(n)} = \ln \left( m_q^{(n)} / \Lambda_{\text{MS}}^{(n)} \right)^2 .
\]

Particle data group (PDG96) [2] has concluded that the world average of \( \Lambda_{\text{MS}}^{(5)} \) is

\[
\Lambda_{\text{MS}}^{(5)} = 209^{+20}_{-33}\text{MeV} .
\]

(B8)

Starting from \( \Lambda_{\text{MS}}^{(5)} = 0.209 \text{ GeV} \), by using the relation (B6), at \( \mu_5 = m_b(m_b) = 4.352 \text{ GeV} \), \( \mu_4 = m_c(m_c) = 1.302 \text{ GeV} \), and \( \mu_6 = m_t(m_t) = 170.8 \text{ GeV} \), we evaluate the values of \( \Lambda_{\text{MS}}^{(n)} \) for \( n = 3, 4, 6 \). The results are summarized in Table VII.

We show the threshold behaviors of \( \alpha_s(n)(\mu) \) in Fig. 7. We can see that \( \alpha_s^{(n-1)}(\mu) \) in \( \mu_{n-1} \leq \mu < \mu_n \) connects with \( \alpha_s^{(n)}(\mu) \) in \( \mu_n \leq \mu < \mu_{n+1} \) continuously.

APPENDIX C: EVOLUTION OF THE YUKAWA COUPLING CONSTANTS

The coefficients \( c_a^{(2)} \), \( b_{ab}^{(2)} \) and \( b_{ab}^{(bc)} \) in the two-loop contributions \( \beta^{(2)} \) are given as follows. Here, \( T_a \) (\( a = u, d, e \)) are given in Table II in Sec. VII and \( n_q \) is the number of generations. (A) Standard model with one Higgs scalar

\[
b_u^{uu} = b_d^{dd} = \frac{3}{2} , \quad b_u^{dd} = b_d^{uu} = \frac{11}{4} , \quad b_e^{ee} = \frac{3}{2} ,
\]

\[
b_u^{ud} = b_d^{du} = -\frac{1}{4} , \quad b_u^{du} = b_d^{ud} = -1 ,
\]

(C1)
\[ b_u^u = -\frac{9}{4} T_u - 6\lambda_H + \frac{223}{80} g_1^2 + \frac{135}{16} g_2^2 + 16g_3^2, \]
\[ b_d^d = -\frac{9}{4} T_d - 6\lambda_H + \frac{187}{80} g_1^2 + \frac{135}{16} g_2^2 + 16g_3^2, \]
\[ b_e^e = -\frac{9}{4} T_e - 6\lambda_H + \frac{387}{80} g_1^2 + \frac{135}{16} g_2^2, \quad (C2) \]
\[ b_u^u = 5\frac{T_u}{4} - 2\lambda_H - \left( \frac{43}{80} g_1^2 - \frac{9}{16} g_2^2 + 16g_3^2 \right), \]
\[ b_d^d = 5\frac{T_d}{4} - 2\lambda_H - \left( \frac{79}{80} g_1^2 - \frac{9}{16} g_2^2 + 16g_3^2 \right) , \quad (C3) \]
\[ c^{(2)}_u = -X_4 + \frac{5}{2} Y_4 + 3\frac{\lambda_H^2}{2} + \left( \frac{9}{200} + \frac{29}{45} n_g \right) g_1^4 - \frac{9}{20} g_1^2 g_2 \]
\[ + \frac{19}{15} g_1^2 g_3^2 - \left( \frac{35}{4} - n_g \right) g_2^4 + 9g_2^2 g_3^2 - \left( \frac{404}{3} - \frac{80}{9} n_g \right) g_3^4 , \]
\[ c^{(2)}_d = -X_4 + \frac{5}{2} Y_4 + 3\frac{\lambda_H^2}{2} - \left( \frac{29}{200} + \frac{1}{45} n_g \right) g_1^4 + \frac{27}{20} g_1^2 g_2 \]
\[ + \frac{31}{15} g_1^2 g_3^2 - \left( \frac{35}{4} - n_g \right) g_2^4 + 9g_2^2 g_3^2 - \left( \frac{404}{3} - \frac{80}{9} n_g \right) g_3^4 , \]
\[ c^{(2)}_e = -X_4 + \frac{5}{2} Y_4 + 3\frac{\lambda_H^2}{2} + \left( \frac{51}{200} + \frac{11}{5} n_g \right) g_1^4 + \frac{27}{20} g_1^2 g_2 - \left( \frac{35}{4} - n_g \right) g_2^4, \quad (C4) \]

where
\[ X_4 = \frac{9}{4} \text{Tr} \left( 3H_u^2 - \frac{2}{3} H_u H_d + 3H_d^2 + H_e^2 \right) , \quad (C5) \]
\[ Y_4 = \left( \frac{17}{20} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2 \right) \text{Tr} H_u + \left( \frac{1}{4} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2 \right) \text{Tr} H_d \]
\[ + \left( \frac{3}{4} g_1^2 + \frac{3}{4} g_2^2 \right) \text{Tr} H_e, \quad (C6) \]
\[ \lambda_H = \frac{m_H^2}{v^2} , \quad (C7) \]
and the evolutions of $g_i$ ($i = 1, 2, 3$) and $\lambda_H$ are given in Sec. [3]

(B) Minimal SUSY model

$$b_u^{uu} = b_d^{dd} = -4 , \quad b_u^{uu} = b_d^{dd} = 2 , \quad b_e^e = -4 ,$$

$$b_u^{ud} = b_d^{du} = 0 , \quad b_u^{du} = b_d^{ud} = -2 , \quad (C8)$$

$$b_u^{u} = -3T_u + \frac{2}{5} g_1^2 + 6g_2^2 , \quad b_d^{d} = -3T_d + \frac{4}{5} g_1^2 + 6g_2^2 , \quad b_e^e = -3T_e + 6g_2^2 ,$$

$$b_u^{d} = -T_u + \frac{2}{5} g_1^2 , \quad b_d^{u} = -T_d + \frac{4}{5} g_1^2 , \quad (C9)$$

$$c_u^{(2)} = -3\text{Tr} \left( 3H_u^2 + H_uH_d \right) + \left( \frac{4}{5} g_1^2 + 16g_3^2 \right) \text{Tr} H_u$$

$$+ \left( \frac{403}{450} + \frac{26}{15} n_g \right) g_1^4 + g_1^2 g_2^2 + \frac{136}{45} g_1^2 g_3^2$$

$$- \left( \frac{21}{2} - 6n_g \right) g_2^4 + 8g_2^2 g_3^2 - \left( \frac{304}{9} - \frac{32}{3} n_g \right) g_3^4 ,$$

$$c_d^{(2)} = -3\text{Tr} \left( 3H_d^2 + H_uH_d + H_e^2 \right) + \left( \frac{-2}{5} g_1^2 + 16g_3^2 \right) \text{Tr} H_d + \frac{6}{5} g_1^2 \text{Tr} H_e + \left( \frac{7}{18} + \frac{14}{15} n_g \right) g_1^4$$

$$+ g_1^2 g_2^2 + \frac{8}{9} g_1^2 g_3 - \left( \frac{21}{2} - 6n_g \right) g_2^4 + 8g_2^2 g_3^2 - \left( \frac{304}{9} - \frac{32}{3} n_g \right) g_3^4 ,$$

$$c_e^{(2)} = -3\text{Tr} \left( 3H_e^2 + H_uH_d + H_e^2 \right) + \left( \frac{-2}{5} g_1^2 + 16g_3^2 \right) \text{Tr} H_d + \frac{6}{5} g_1^2 \text{Tr} H_e$$

$$+ \left( \frac{27}{10} + \frac{18}{5} n_g \right) g_1^4 + \frac{9}{5} g_1^2 g_2^2 - \left( \frac{21}{2} - 6n_g \right) g_2^4 . \quad (C10)$$
APPENDIX D: EVOLUTION OF THE GAUGE COUPLING CONSTANTS

Evolution of gauge coupling constants is given by

\[
\frac{dg_i}{dt} = -b_ig_i^3 \frac{3}{16\pi^2} - \sum_k b_{ik} \frac{g_i^3 g_k^2}{(16\pi^2)^2} - \frac{g_i^3}{(16\pi^2)^2} \sum_a c_{ia} \text{Tr} H_a , \tag{D1}
\]

where the coefficients \( b_i, b_{ik} \) and \( c_{ia} \) are given in Table VIII.

The evolution of the coupling constants \( \lambda_H \) given in (C7) is given by

\[
\frac{d\lambda_H}{dt} = \frac{1}{16\pi^2} \beta^{(1)}_{\lambda} + \frac{1}{(16\pi^2)^2} \beta^{(2)}_{\lambda} , \tag{D2}
\]

\[
\beta^{(1)}_{\lambda} = 12\lambda_H^2 - \left( \frac{9}{5} g_1^2 + 9g_2^2 \right) \lambda_H + \frac{9}{4} \left( \frac{3}{25} g_1^4 + \frac{2}{5} g_1^2 g_2^2 + g_2^4 \right) + 4\lambda_H \text{Tr}(3H_u + 3H_d + H_e) - 4\text{Tr}(3H_u^2 + 3H_d^2 + H_e^2) , \tag{D3}
\]

\[
\beta^{(2)}_{\lambda} = -78\lambda_H^3 + 18 \left( \frac{3}{5} g_1^2 + 3g_2^2 \right) \lambda_H^2
\]

\[
- \left[ \left( \frac{313}{8} - 10n_g \right) g_2^4 - \frac{117}{20} g_1^2 g_2^2 + \frac{9}{25} \left( \frac{229}{24} + \frac{50}{9} n_g \right) g_1^4 \right] \lambda_H
\]

\[
+ \left( \frac{497}{8} - 8n_g \right) g_2^6 - \frac{3}{5} \left( \frac{97}{24} + \frac{8}{3} n_g \right) g_1^2 g_2^4 - \frac{9}{25} \left( \frac{239}{24} + \frac{40}{9} n_g \right) g_1^4 g_2^2 - \frac{27}{125} \left( \frac{59}{24} + \frac{40}{9} n_g \right) g_1^6
\]

\[
- 64g_3^2 \text{Tr} \left( H_u^2 + H_d^2 \right) - \frac{8}{5} g_1^2 \text{Tr} \left( 2H_u^2 - H_d^2 + 3H_e^2 \right) - \frac{3}{2} g_4^2 \text{Tr}(3H_u + 3H_d + H_e)
\]

\[
+ 10\lambda_H \left[ \left( \frac{17}{20} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2 \right) \text{Tr} H_u + \left( \frac{1}{4} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2 \right) \text{Tr} H_d + \frac{3}{4} (g_1^2 + g_2^2) \text{Tr} H_e \right]
\]

\[
+ \frac{3}{5} g_1^2 \left[ \left( -\frac{57}{10} g_1^2 + 21g_2^2 \right) \text{Tr} H_u + \left( \frac{3}{2} g_1^2 + 9g_2^2 \right) \text{Tr} H_d + \left( -\frac{15}{2} g_1^2 + 11g_2^2 \right) \text{Tr} H_e \right]
\]

\[
- 24\lambda_H^2 \text{Tr}(3H_u + 3H_d + H_e) - \lambda_H \text{Tr} \left[ 3(H_u - H_d)^2 + H_e^2 \right]
\]

\[
+ 20\text{Tr}(3H_u^3 + 3H_d^3 + H_e^3) - 12\text{Tr}[H_u H_d(H_u + H_d)] . \tag{D4}
\]
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FIGURES

FIG. 1. Threshold behavior of $R^{(n)}(\mu)$ versus $\mu$.

FIG. 2. Light quark masses $m_q(\mu)$ ($q = u, d, s$) versus $\mu$. 
FIG. 3. Heavy quark masses $m_q(\mu)$ ($q = c, b, t$) versus $\mu$.

FIG. 4. Reliability of the perturbative calculation of $\alpha_s^{(n)}(\mu)$. The curves show the behaviors of the second and third terms in $\{ \}$ in (B4).
FIG. 5. Reliability of the perturbative calculation of $m_q(\mu)$. The curves show the behaviors of the second and third terms in $\{\}$ in (4.5).

FIG. 6. Behavior of the Yukawa coupling constants $y_t(\mu)$, $y_b(\mu)$ and $y_\tau(\mu)$ of in the minimal SUSY model. For convenience, the values are illustrated by the form $m_t(\mu) = y_t(\mu)v \sin \beta/\sqrt{2}$, $m_b(\mu) = y_b(\mu)v \cos \beta/\sqrt{2}$ and $m_\tau(\mu) = y_\tau(\mu)v \cos \beta/\sqrt{2}$. 
FIG. 7. Threshold behavior of $\alpha_s^{(n)}(\mu)$ versus $\mu$. 

\[ \mu_5 = m_b(m_b) \]

\[ \mu_4 = m_c(m_c) \]

\[ \mu_6 = m_t(m_t) \]
TABLES

TABLE I. Running quark mass values \( m_q(\mu) \) at \( \mu = m_q \). Input values \( m_q(1 \text{ GeV}) \) for \( q = u, d, s \) and \( m_q(M_q^{\text{pole}}) \) for \( q = c, b, t \) are used. The first and second errors come from \( \pm \Delta m_q \) (or \( \pm \Delta m_q^{\text{pole}} \)) and \( \pm \Delta M_q^{(5)} \), respectively. The values with asterisk should not be taken rigidly, because these values have been calculated in the region with a large \( \alpha_s(\mu) \).

| Input : \( m_q(1 \text{ GeV}) \) or \( m_q(M_q^{\text{pole}}) \) | Output : \( m_q(\mu) \) |
|-----------------|------------------|
| \( u \) | 4.88 ± 0.57 MeV | *0.436 ±0.001 , 0.052 GeV |
| \( d \) | 9.81 ± 0.65 MeV | *0.448 ± 0.001+0.059 , 0.053 GeV |
| \( s \) | 195.4 ± 12.5 MeV | *0.553 ± 0.005+0.052 GeV |
| \( c \) | 1.213 ± 0.018−0.036 GeV | 1.302 ± 0.0180.019 GeV |
| \( b \) | 4.248 ± 0.046+0.036 GeV | 4.339 ± 0.046+0.027 GeV |
| \( t \) | 170.1 ± 11.4 ± 0.3 GeV | 170.8 ± 11.5 ± 0.2 GeV |

TABLE II. Running quark masses \( m_q(\mu) \) and invariant masses \( \bar{m}_q \) (in unit of GeV). The values with asterisk should not be taken strictly, because the perturbative calculation is not reliable in the region with a large \( \alpha_s(\mu) \).

| \( q \) | \( u \) | \( d \) | \( s \) | \( c \) | \( b \) | \( t \) |
|-------|-----|-----|-----|-----|-----|-----|
| \( M_q^{\text{pole}} \) | *0.501 +0.068 | *0.517 +0.068 | *0.687 +0.074 | 1.59 | 4.89 | 180 |
| \( m_q(M_q^{\text{pole}}) \) | *0.0307 +0.0026 | *0.0445 +0.026 | *0.283 +0.016 | 1.213 | 4.248 | 170 |
| \( m_q(m_q) \) | *0.436 +0.059 | *0.448 +0.054 | *0.553 +0.054 | 1.302 | 4.339 | 171 |
| \( m_q(1\text{GeV}) \) | 0.00488 ±0.00057 | 0.00981 ±0.00065 | 0.1954 ±0.0125 | 1.506 | 7.18 | 475 |
| \( m_q(m_c) \) | 0.00418 +0.00056 | 0.00840 +0.00071 | 0.1672 +0.0137 | 1.302 | 6.12 | 399 |
| \( m_c = 1.302 \) | 0.00418 +0.00056 (Λ) | 0.00840 +0.00071 (Λ) | 0.1672 +0.0137 (Λ) | 1.302 | 6.12 | 399 |
| \( m_q(m_b) \) | 0.00317 +0.00052 | 0.00637 +0.00073 | 0.1268 +0.0142 | 0.949 | 4.34 | 272 |
| \( m_b = 4.339 \) | 0.00317 +0.00052 (Λ) | 0.00637 +0.00073 (Λ) | 0.1268 +0.0142 (Λ) | 0.949 | 4.34 | 272 |
| \( m_q(m_W) \) | 0.00235 +0.00042 | 0.00473 +0.00061 | 0.0942 +0.0119 | 0.684 | 3.03 | 183 |
| \( m_W = 80.33 \) | 0.00235 +0.00042 (Λ) | 0.00473 +0.00061 (Λ) | 0.0942 +0.0119 (Λ) | 0.684 | 3.03 | 183 |
| \( m_q(m_Z) \) | 0.00233 +0.00045 | 0.00469 +0.00067 | 0.0934 +0.0131 | 0.677 | 3.00 | 181 |
| \( m_Z = 91.187 \) | 0.00233 +0.00045 (Λ) | 0.00469 +0.00067 (Λ) | 0.0934 +0.0131 (Λ) | 0.677 | 3.00 | 181 |
| \( m_q(m_t) \) | 0.00223 +0.00054 | 0.00449 +0.00064 | 0.0894 +0.0114 | 0.646 | 2.85 | 171 |
| \( m_t = 170.8 \) | 0.00223 +0.00054 (Λ) | 0.00449 +0.00064 (Λ) | 0.0894 +0.0114 (Λ) | 0.646 | 2.85 | 171 |
| \( m_q(\Lambda_W) \) | 0.00223 +0.00049 | 0.00448 +0.00064 | 0.0893 +0.0114 | 0.645 | 2.84 | 171 |
| \( \Lambda_W = 174.1 \) | 0.00223 +0.00049 (Λ) | 0.00448 +0.00064 (Λ) | 0.0893 +0.0114 (Λ) | 0.645 | 2.84 | 171 |
| \( \bar{m}_q \) | 0.00496 +0.00101 | 0.00998 +0.00153 | 0.1999 +0.030 | 1.59 | 7.87 | 546 |
| \( \bar{m}_q \) | 0.00496 +0.00101 (Λ) | 0.00998 +0.00153 (Λ) | 0.1999 +0.030 (Λ) | 1.59 | 7.87 | 546 |
TABLE III. Coefficients $\beta^{(1)}_a$ in the evolution equations of Yukawa coupling constants $Y_a$.

|                | Model A StandardsingleHiggs | Model B SUSY |
|----------------|------------------------------|--------------|
| $G_u$          | $\frac{13}{12} g_1^2 + \frac{9}{16} g_2^2 + 8 g_3^2$ | $\frac{13}{12} g_1^2 + 3 g_2^2 + \frac{10}{3} g_3^2$ |
| $G_d$          | $\frac{14}{19} g_1^2 + \frac{9}{16} g_2^2 + 8 g_3^2$ | $\frac{14}{19} g_1^2 + 3 g_2^2 + \frac{10}{3} g_3^2$ |
| $G_e$          | $\frac{1}{2} g_1^2 + \frac{9}{16} g_2^2 + 8 g_3^2$ | $\frac{1}{2} g_1^2 + 3 g_2^2 + \frac{10}{3} g_3^2$ |
| $T_u = T_d = T_e$ | $= 3 \text{Tr}(H_u + H_d) + \text{Tr}H_e$ | $T_u = 3 \text{Tr}H_u$ |
| $T_d = T_e$    | $= 3 \text{Tr}H_d + \text{Tr}H_e$ | $T_d = T_e$ |
| $a_u^u = a_d^d$ | $= +3/2$ | $a_u^u = a_d^d = +3$ |
| $a_u^d = a_d^u$ | $= -3/2$ | $a_u^d = a_d^u = +1$ |
| $a_e^e$        | $= +3/2$ | $a_e^e = +3$ |

TABLE IV. Evolution of the Yukawa coupling constants $y_a$ in the standard model with one Higgs boson (Model A). For convenience, instead of $y_a(\mu)$, the values of $m_a(\mu) = y_a(\mu)v/\sqrt{2}$ are listed, where $v = \sqrt{2} \Lambda_W = 246.2$ GeV. The errors $\pm \Delta m$ at $\mu = 10^9$ GeV and $\mu = M_X$ denote only those from $\pm \Delta m$ at $\mu = m_Z$.

|                | $\mu = m_Z$                                     | $\mu = 10^9$ GeV                               | $\mu = M_X$        |
|----------------|-----------------------------------------------|-----------------------------------------------|-------------------|
| $m_u(\mu)$    | $2.33^{+0.42}_{-0.45}$ MeV                     | $1.28^{+0.23}_{-0.25}$ MeV                     | $0.94^{+0.17}_{-0.18}$ MeV |
| $m_c(\mu)$    | $677^{+16}_{-61}$ MeV                          | $371^{+31}_{-33}$ MeV                         | $272^{+24}_{-24}$ MeV |
| $m_t(\mu)$    | $181 \pm 13$ GeV                              | $109^{+19}_{-13}$ GeV                         | $84^{+18}_{-13}$ GeV |
| $m_d(\mu)$    | $4.69^{+0.60}_{-0.66}$ MeV                     | $2.60^{+0.32}_{-0.37}$ MeV                    | $1.94^{+0.25}_{-0.28}$ MeV |
| $m_s(\mu)$    | $93.4^{+11.8}_{-13.0}$ MeV                     | $51.9^{+11.7}_{-7.2}$ MeV                     | $38.7^{+19}_{-5.4}$ MeV |
| $m_b(\mu)$    | $3.00 \pm 0.11$ GeV                           | $1.51^{+0.35}_{-0.06}$ GeV                    | $1.07 \pm 0.04$ GeV |
| $m_e(\mu)$    | $0.48684727$ MeV                              | $0.51541746$ MeV                              | $0.49348567$ MeV |
|               | $\pm 0.00000014$                               | $\pm 0.00000015$                               | $\pm 0.00000014$   |
| $m_\mu(\mu)$  | $102.75138$ MeV                                | $108.78126$ MeV                               | $104.15246$ MeV    |
|               | $\pm 0.00033$                                  | $\pm 0.00035$                                 | $\pm 0.00033$      |
| $m_\tau(\mu)$ | $1746.7 \pm 0.3$ MeV                           | $1849.2 \pm 0.3$ MeV                          | $1770.6 \pm 0.3$ MeV |
TABLE V. Evolution of the Yukawa coupling constants $y_a$ in the minimal SUSY model (Model B). For convenience, instead of $y_a(\mu)$, the values of $m_a(\mu) = y_a(\mu)v \sin \beta/\sqrt{2}$ for up-quark sector and $m_a(\mu) = y_a(\mu)v \cos \beta/\sqrt{2}$ for down-quark sector are listed, where $v = \sqrt{2}\Lambda_W$. The errors $\pm \Delta m$ at $\mu = 10^9$ GeV and $\mu = M_X$ denote only those from $\pm \Delta m$ at $\mu = m_Z$.

|       | $\mu = m_Z$ | $\mu = 10^9$ GeV | $\mu = M_X$ |
|-------|-------------|------------------|-------------|
| $m_u(\mu)$ | $2.33^{+0.42}_{-0.45}$ MeV | $1.47^{+0.20}_{-0.28}$ MeV | $1.04^{+0.19}_{-0.20}$ MeV |
| $m_c(\mu)$ | $677^{+100}_{-61}$ MeV | $427^{+33}_{-38}$ MeV | $302^{+20}_{-27}$ MeV |
| $m_t(\mu)$ | $181 \pm 13$ GeV | $149^{+30}_{-26}$ GeV | $129^{+190}_{-40}$ GeV |
| $m_d(\mu)$ | $4.69^{+0.60}_{-0.66}$ MeV | $2.28^{+0.29}_{-0.32}$ MeV | $1.33^{+0.17}_{-0.19}$ MeV |
| $m_s(\mu)$ | $93.4^{+11.8}_{-13.0}$ MeV | $45.3^{+3.3}_{-3.2}$ MeV | $26.5^{+3.5}_{-3.7}$ MeV |
| $m_b(\mu)$ | $3.00 \pm 0.11$ GeV | $1.60 \pm 0.06$ GeV | $1.00 \pm 0.04$ GeV |
| $m_e(\mu)$ | $0.48684727$ MeV | $0.40850306$ MeV | $0.32502032$ MeV |
| $m_\mu(\mu)$ | $102.75138$ MeV | $86.21727$ MeV | $68.59813$ MeV |
| $m_\tau(\mu)$ | $1746.7 \pm 0.3$ MeV | $1469.5^{+0.3}_{-0.2}$ MeV | $1171.4 \pm 0.2$ MeV |

TABLE VI. Pole masses $M^\text{pole}_q$ and the related quantities. The values with asterisk should not be taken rigidly, because these values have been calculated in the region with a large $\alpha_s(\mu)$.

| $K_0$ | $\Delta(M_i/M_n)$ | $K$ | $M^\text{pole}_q$ | $m_q(M^\text{pole}_q)$ |
|-------|------------------|-----|------------------|------------------|
| $u$    | 16.11            | 0   | *16.11*0.501 MeV | *0.0307 MeV      |
| $d$    | 15.07            | *0.838 | *16.19*0.517 MeV | *0.0445 MeV      |
| $s$    | 14.03            | *1.364 | *15.85*0.687 MeV | *0.283 MeV       |
| $c$    | 12.99            | 1.114 | 14.47            | 1.59 GeV         |
| $b$    | 11.94            | 0.746 | 12.94            | 4.89 GeV         |
| $t$    | 10.90            | 0.0555 | 10.98          | 180 GeV          |

TABLE VII. The values of $\Lambda^{(n)}_\overline{MS}$ in unit of GeV and $\alpha_s(\mu_n)$. The underlined values are input values.

| $n$ | $\Lambda^{(n)}_\overline{MS}$ | $\alpha_s^{(n)}(\mu_n)$ | $\mu_n$ |
|-----|-------------------------------|--------------------------|---------|
| 3   | $0.333^{+0.030}_{-0.042}$     | $1.69^{+0.038}_{-0.033}$ | $\mu_3 = 0.553$ GeV |
| 4   | $0.291^{+0.035}_{-0.041}$     | $0.379^{+0.034}_{-0.039}$ | $\mu_4 = 1.302$ GeV |
| 5   | $0.209^{+0.039}_{-0.033}$     | $0.222^{+0.013}_{-0.012}$ | $\mu_5 = 4.339$ GeV |
| 6   | $0.088^{+0.019}_{-0.0156}$    | $0.107^{+0.0036}_{-0.0055}$ | $\mu_6 = 170.8$ GeV |
TABLE VIII. Coefficients in the evolution equations of gauge coupling constants.

| Model(A) | Model(B) |
|----------|----------|
| $b_1 = -\left(\frac{1}{10} + \frac{1}{3}n_g\right)$ | $b_1 = -\left(\frac{3}{5} + 2n_g\right)$ |
| $b_2 = \frac{43}{6} - \frac{4}{3}n_g$ | $b_2 = 5 - 2n_g$ |
| $b_3 = 11 - \frac{4}{3}n_g$ | $b_3 = 9 - 2n_g$ |
| $(b_{ik}) = \begin{pmatrix} -9 & -\frac{10}{3} & 0 \\ -\frac{9}{10} & -\frac{25}{6} & 0 \\ 0 & 0 & 102 \end{pmatrix}$ | $(b_{ik}) = \begin{pmatrix} -9 & -\frac{9}{5} & 0 \\ -\frac{9}{5} & 17 & 0 \\ 0 & 0 & 54 \end{pmatrix}$ |
| $-n_g \begin{pmatrix} 19 & 3 & 44 \\ \frac{19}{3} & 41 & 4 \\ \frac{1}{3} & \frac{17}{2} & 26 \end{pmatrix}$ | $-n_g \begin{pmatrix} 38 & 6 & 88 \\ \frac{38}{3} & \frac{14}{5} & \frac{18}{5} \\ \frac{15}{2} & \frac{11}{5} & \frac{68}{3} \end{pmatrix}$ |
| $(c_{ia}) = \begin{pmatrix} 17 & 1 & 3 \\ \frac{17}{3} & \frac{3}{4} & 1 \\ 2 & 2 & 0 \end{pmatrix}$ | $(c_{ia}) = \begin{pmatrix} 26 & 14 & 18 \\ \frac{5}{3} & \frac{5}{3} & \frac{3}{3} \\ 6 & 6 & 2 \end{pmatrix}$ |