PHASE STRUCTURE OF FOUR FLAVOUR QCD IN THE $T - \mu$ PLANE FROM A NEW METHOD FOR SIMULATIONS OF LATTICE GAUGE THEORIES AT NON-ZERO BARYON DENSITY

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We review a method for numerical simulations of lattice gauge theories at non-zero baryonic chemical potential we recently proposed. We first report on a test of the method using a solvable model and then present results for the phase structure of four flavour QCD. For the first time the region of chemical potential up to $1.4 T_C$ is explored, finding a first order transition line.

1. Introduction

The existence and nature of deconfined phase(s) of QCD at high baryon density is one of the open key issues of the theory of Strong Interaction. Progresses in this field are heavily delayed by the absence of a direct method of numerical simulation based on an importance sampling Monte Carlo scheme. In order to, at least partially, overcome this impasse, analytical approaches in the strong coupling limit using the Lagrangian or Hamiltonian approaches, as well as other indirect methods, have been proposed in the last years, which in principle allow to extract physical informations of the behaviour of the vacuum of QCD in the portion of the space parameter that is not accessible to direct numerical simulations. Between the previously denoted indirect methods it is worthwhile to cite the Imaginary Chemical Potential approach and the Doubly Reweighting method.

Following this line we have recently proposed a different approach to the prob-
lem, which we hope can greatly help to improve the knowledge in the field\textsuperscript{11}. In this proceeding we present a concise description of the method, some results of the test carried out in an exactly solvable model (3-dimensional Gross-Neveu at non zero baryon density in the large $N$ limit) and finally discuss the new results we obtained concerning the phase structure of QCD with four flavour of light quarks at finite temperature and baryon density\textsuperscript{12}.

2. The Method

Starting from the usual discretization of QCD with staggered fermions with the chemical potential term introduced \textit{a l\'a} Hasenfratz and Karsch

$$ S = S_{PG} + \frac{1}{2} \sum_n \sum_{i=1}^{3} \bar{\psi}_n \eta_i(n) \left(U_{n,i} \psi_{n+i} - U_{n-i,i}^\dagger \psi_{n-i}\right) $$

$$ + \frac{1}{2} \sum_n \bar{\psi}_n \eta_0(n) \left(e^{i\mu} U_{n,0} \psi_{n+0} - e^{-i\mu} U_{n-0,0}^\dagger \psi_{n-0}\right) + m \sum_n \bar{\psi}_n \psi_n $$ \hspace{1cm} (1)

we generalize the action introducing two independent parameters $x$ and $y$: the coefficient of the antihermitian and hermitian terms in the temporal part of the fermionic action. The usual action is recovered when $x = \cosh(\mu a)$ and $y = \sinh(\mu a)$

$$ S = S_{PG} + m \sum_n \bar{\psi}_n \psi_n + \frac{1}{2} \sum_n \sum_{i=1}^{3} \bar{\psi}_n \eta_i(n) \left(U_{n,i} \psi_{n+i} - U_{n-i,i}^\dagger \psi_{n-i}\right) $$

$$ + \frac{1}{2} x \sum_n \bar{\psi}_n \eta_0(n) \left(U_{n,0} \psi_{n+0} - U_{n-0,0}^\dagger \psi_{n-0}\right) $$

$$ + \frac{1}{2} y \sum_n \bar{\psi}_n \eta_0(n) \left(U_{n,0} \psi_{n+0} + U_{n-0,0}^\dagger \psi_{n-0}\right) $$ \hspace{1cm} (2)

As in the case of \textsuperscript{11} this action suffers, in general, from the sign problem; only when $x$ is real and $y = i\tilde{y}$ is imaginary the model defined through \textsuperscript{2} can be simulated with standard Monte Carlo methods. Note that the case of imaginary chemical potential is a particular line in this subspace, defined by $x = \cos(\mu a)$ and $\tilde{y} = \sin(\mu a)$. Studying the phase structure in the parameter space defined from $\beta$, $x$ and $\tilde{y}$ and performing an analytical continuation to real values of $y$ we can recover the phase structure of the original model.

In Ref. \textsuperscript{11} the reader can find an accurate discussion of what we expect for the phase structure in the $(x, y)$ and $(x, \tilde{y})$ planes and the motivations for such guess; it is of little interest, in our opinion, to repeat that discussion here, because we want to concentrate on the results concerning the critical line in the $(\mu, T)$ plane. Assuming this attitude, let us sketch in some detail the procedure used to extract the critical line in the cases under examination in the rest of the paper, \textit{i.e.} Gross-Neveu model in three dimension and four flavours QCD.
Not to work in a three parameter space we can look at the critical line in two dimensional sections; in particular we can choose to work at fixed $\beta$, in the $(x,\bar{y})$ plane or at fixed $x$, in the $(\beta,\bar{y})$ plane.

In the first case we work at $\beta$ smaller than the critical value of the zero density case and look for a critical line that starts from the $\bar{y} = 0$ axis moving towards larger values of $x$ and $\bar{y}$. This can be accomplished performing simulations at fixed $\bar{y}$ for several values of $x$ starting from $\bar{y} = 0$ and repeating the procedure for several times at different values of $\bar{y}$. Having determined the critical line in the $(x,\bar{y})$ plane we can fit it with an even function of $\bar{y}$ and analytically continue the line in the $(x,y)$ plane. At this point the intersection of this line with the $x^2 - y^2 = 1$ line, that corresponds to real values of the chemical potential, determines the critical value of $\mu a$ for the given $\beta$. Repeating the whole procedure for several values of $\beta$ will determine several points on the critical line in the $(\beta,\mu a)$ plane; values in physical units can be obtained using the perturbative two-loop beta function.

The second alternative is simply described by exchanging the role of $\beta$ and $x$. We work at fixed $x = \cosh(\mu a)$ and look for lines in the $(\beta,\bar{y})$ plane, analytically continue to the $(\beta,y)$ plane and intersect them with $y = \sinh(\mu a)$ to get the critical values in the $(\beta,\mu a)$ plane, as before.

3. Results in the Gross-Neveu model

The Gross-Neveu model (i.e. four-Fermi model in 2+1 dimensions) is an ideal test bench for our approach: in the chiral limit it possesses a phase structure in the $(\mu,T)$ plane that is similar to the one expected for four flavour QCD and it can be analytically solved in the large $N$ limit; the exact solution allows us to determine the critical line in the $(x,\bar{y})$ plane for a fixed $\beta$. In Fig. 1 the whole procedure is illustrated (take in mind that the variable in horizontal axis is $\bar{y}$ for the upper curve and $y$ for the two lower curves).

Repeating this procedure for others values of $\beta$ produces the critical structure of the model in the $(\mu,\beta)$ plane depicted in Fig. 2. In this figure the symbols stand for the critical points obtained with our method, the continuous line represents the exact results for $N_t = 4$ and the dashed line is what can be obtained in the same situation performing an analytical extension starting from results obtained working in the Imaginary Chemical Potential approach (in all cases the analytical extrapolations rely on quadratic fits). It is evident a clear advantage in using our new method for increasing chemical potential values.

4. Four Flavour QCD

While in the Gross-Neveu model we start from data (the critical points at imaginary values of $y$) obtained solving seminumerically the gap equation, for QCD we have to rely uniquely on Monte Carlo simulation. For this purpose we choose to work with a Hybrid Monte Carlo code for staggered fermions. The lattices we have used, due to the fact that our method is quite expensive from a computational point of view...
1. Analytical continuation of the critical line from imaginary to real $y$ and determination of the physical critical $\mu$; Gross-Neveu model at $\beta = 0.7$ and $N_t = 4$. 

![Graph](image1.png)

Fig. 1. Analytical continuation of the critical line from imaginary to real $y$ and determination of the physical critical $\mu$; Gross-Neveu model at $\beta = 0.7$ and $N_t = 4$. 

2. Critical structure of Gross-Neveu model at $N_t = 4$ in the $\mu$, $\beta$ plane. 

![Graph](image2.png)

Fig. 2. Critical structure of Gross-Neveu model at $N_t = 4$ in the $\mu$, $\beta$ plane.

(for any point in the ($\mu$, $T$) plane we have to run the Monte Carlo to span a two dimensional space), are limited to $8^3 \times 4$ volumes at most. We choose a quark mass of $ma = 0.05$ in order to compare with existing results obtained with Imaginary Chemical Potential and Doubly Reweighting methods.

For each value of $x$, $\bar{y}$ and $\beta$ we run around 40000 molecular dynamics trajectories of unit time, measuring plaquette, chiral condensate and Polyakov loop for each trajectory. The simulations have been performed on Linux Clusters for a total of 100000 CPU hours.
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We have worked, here, in the fixed $x$ scheme, choosing 5 values of $x$ corresponding to $\mu a$ of 0.2, 0.25, 0.3, 0.4 and 0.5. For each $x$ value we perform a search of the critical point in $\beta$ for a number of $\tilde{y}$ values varying from 6 to 9, using a Ferrenberg-Swendsen procedure to precisely determine the critical values. In all cases the signal from the susceptibilities of the three observables we used gave identical results for the estimation of the critical points. The transition line was found to be a first order one, with a latent heat essentially independent of the $\tilde{y}$ value.

For the following stage we fitted the critical line in the $(\tilde{y}, \beta)$ plane, for each $x$, with a second order even polynomial:

$$\beta(\tilde{y}) = \beta_0(x) + b_2(x)\tilde{y}^2 \quad (3)$$

It is straightforward to continue the critical line to real values of $y$ and intersect with the $y = \sinh(\mu a)$ vertical line to obtain the corresponding critical coupling $\beta_c(\mu a)$. As the latent heat was constant along the line in the $(\tilde{y}, \beta)$ plane, we assume his analytical extension shares the same behaviour; hence we can affirm that the transitions at different $\mu a$ are still first order, as the one at $\mu = 0$. This procedure is sketched in Fig. 3 for a representative value of $x$ (as before the horizontal axis is $\tilde{y}$ for the upper curve and $y$ for the lower one).

Repeating this procedure for the five values of $x$ (or $\mu a$) we have chosen we obtain the critical line in the $(\mu a, \beta)$ plane. At this point we can rewrite our results in physical units, as reported in Fig. 4, setting the scale with the critical temperature at $\mu = 0$ and using the two loop beta function. In this figure there are also shown results obtained by D’Elia and Lombardo using the Imaginary
Chemical Potential approach as well as a fit of our results with a power law:

\[
\frac{T}{T_C} = \left[1 - c \left(\frac{\mu}{T_C}\right)^2\right]^p
\]

(4)

Note that, if we take seriously this ansatz for the critical line in the \((\mu/T_c, T/T_c)\) plane, this leads to a prediction of the critical chemical potential at zero temperature of \(1.5T_c \simeq 250\text{ MeV}\).

Fig. 4. Phase diagram in physical units.

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