Abstract

In this work we study in detail new kinds of motions of the metric tensor. The work is divided into two main parts. In the first part we study the general existence of Kerr-Schild motions—a recently introduced metric motion. We show that generically, Kerr-Schild motions give rise to finite dimensional Lie algebras and are isometrizable, i.e., they are in a one-to-one correspondence with a subset of isometries of a (usually different) spacetime. This is similar to conformal motions. There are however some exceptions that yield infinite dimensional algebras in any dimension of the manifold. We also show that Kerr-Schild motions may be interpreted as some kind of metric symmetries in the sense of having associated some geometrical invariants. In the second part, we suggest a scheme able to cope with other new candidates of metric motions from a geometrical viewpoint. We solve a set of new candidates which may be interpreted as the seeds of further developments and relate them with known methods of finding new solutions to Einstein’s field equations. The results are similar to those of Kerr-Schild motions, yet a richer algebraical structure appears. In conclusion, even though several points still remain open, the wealth of results shows that the proposed concept of generalized metric motions is meaningful and likely to have a spin-off in gravitational physics. We end by listing and analyzing some of those open points.
1 Introduction

This work deals with motions of the metric tensor of a Riemannian manifold—see e.g., [1] for a review—that, in physical terms, is to be assimilated with spacetime. In particular, we will give some examples of how one can extend the number of symmetries of the metric tensor beyond the two cases that have been mostly considered in the literature, namely isometries and conformal symmetries.

We will work along two lines. Continuous groups of Generalized Kerr-Schild (GKS) transformations, or simply Kerr-Schild motions, have been presented in [2] (hereafter referred to as I). In the first part of this work we will carry out a detailed study of this particular new metric motion (see also below). This part aims at showing how the usual study of conformal or isometric symmetries can be extended to other cases having a role in gravitation. On the other side, the aim of the second part is to pose and develop a framework to deal with some general situations, since one expects that many achievements of differential geometry on Riemannian spaces have a physical spin-off. Let us now give a more detailed account of each part.

In I some general properties and explicit examples of Kerr-Schild motions were presented. In this work we shall address the question of their general existence in any spacetime of arbitrary dimension, i.e., a detailed study of the solutions of their associated differential equations, the so-called Kerr-Schild equations (see following section). The main results are that Kerr-Schild motions are mainly divided into three families depending on the kinematical properties of the deformation direction $\vec{\ell}$. Moreover, we will show that generically they give rise to Lie algebras of finite dimensional character as happens with isometries or conformal motions, although the existence of infinite dimensional Lie algebras is possible. We also show that generically Kerr-Schild motions are “isometrizable”, i.e., they are equivalent to a subset of the isometries of a—usually different—spacetime. This solves a question raised in I.

In fact this scheme could in principle be extended to other new cases of metric motions, see below. However, besides comparing the contents of this chapter with other new proposals, it is worth relating it with current studies in other types of motions, namely collineations, see e.g., [3]–[9] for curvature collineations, [10]–[14] (and also [6]) for Ricci collineations, [15]–[20] for affine collineations and holonomy theory, [21]–[26] for projective collineations—see
also [27]—, and [28]–[30] for a general presentation and some fundamental results. We do not dare to follow this way here for the sake of brevity. With this in hand, it should be possible to place the object under study within the family of metric motions, or even within general motions, see also [1, 31, 32].

Afterwards, in section 3.1, we introduce a framework for studying “the family” of metric symmetries in general. This framework is then applied to some specific cases of physical interest. The results show that the mathematical structure of its solutions utterly surpasses the well known results for isometric and conformal motions, or those given in Sect. 2 for the Kerr-Schild case, yet a similar status as that of these two fundamental metric symmetries is still to be worked out.

Moreover, one recovers, as particular cases, isometric and conformal symmetries, as well as the Kerr-Schild case, and this consistency gives us a good control on the computations carried out, which are sometimes involved. One may ask which is the interest of this all. This question has not a simple answer. From our viewpoint the interest is twofold. On the one hand, there is the purely mathematical interest, of finding a useful way to characterize the most fundamental symmetries of a Riemannian manifold —the symmetries of the metric tensor. The scheme here developed proves to be worth towards such goal. However, for a physicist, it is impossible to detach this question from another one on the practical applications that these “generalized metric symmetries” can contribute in this domain. Thus, right from the first step, in which the example of a new metric symmetry (the “Kerr-Schild case”) is given, to the final implementation of the very general mathematical framework, it is physics which sets the pace. It is our opinion that this feedback between their mathematical and their physical features will govern the future study of new possible metric motions. Some other physical applications will be given elsewhere [33]. Finally, let us add that we have provided some examples, also with the intention of giving practical tools for solving other situations.

Section 2 deals with Kerr-Schild motions. We begin recalling a definition and two results of I which are necessary for the rest of the work. In Sect. 2.1 the case of non-geodesic Kerr-Schild motions is solved and their isometrization is proven. In Sect. 2.2 we solve the case of geodesic Kerr-Schild motions with a deformation direction that satisfies some particular conditions on its kinematical properties. This, we will show, covers almost any case of geodesic Kerr-Schild motions. We also prove a similar isotropization as in previous
case. In Sect. 2.2.1 we give the general solution of Kerr-Schild motions for Kerr-Newman spacetimes and their principal null directions. In Sect. 2.3 we address the remaining cases. We show that there is no general solution analogue to previous cases and in this case the associated Lie algebras may become infinite dimensional in any dimension of the manifold (in opposition to isometric or conformal motions, where only the latter has Lie algebras of infinite dimensional character and only in the case of a two-dimensional manifold). Moreover, we show that their existence is indeed strongly restricted and we give the main features of these cases. In Sect. 2.3.1 we study in detail the existence of this possibility under some physically interesting conditions. In Sect. 2.4 we give a brief summary of the Kerr-Schild part. In Sect. 3.1 we deal with the definition of (continuous groups of) generalized metric motions. We begin with some motivations in Sects. 3.1.1 and 3.1.2. Our choice is presented in Sect. 3.2 in Def. 3. In Sect. 3.3 we write down its differential version. In Sect. 4 the framework is implemented to the study of most relevant cases, i.e., metric motions generated by 1-dimensional subspaces. In particular, we study in detail the existence of such motions, their interrelation and, particularly, in Ex. 3 we present the general solution for the case of two covariantly constant 1-form fields. In Sect. 4.3 we address some questions regarding the addition of a conformal motion to a given metric motion. In Sect. 5 we give a list of further points that in our opinion seem worth for a deeper study in the future. We finish in Sect. 6 giving the general conclusions. In App. A a survey of useful formulae when dealing with Kerr-Schild motions is given. In App. B first steps towards the the integrability equations for a generalized metric motion under our scheme, including the Kerr-Schild case, are written down. In App. C we summarize some features of metric motions generated by a spacelike and a timelike 1-form fields. Finally, in App. D some hints towards a complete (explicit) resolution of Kerr-Schild motions in flat spacetime is given.

The conventions and notation used throughout this work are the following. \((V_n, g)\) denotes a smooth connected Hausdorff \(n\)-dimensional manifold admitting a smooth (Lorentz) metric \(g\). We will use the signature \((-1,1,\ldots,1)\), although Euclidean or other types of signatures can be considered. Greek indices run from \(0\) to \(n-1\), whereas Latin indices run from \(1\) to \(n-1\). The tensor product is denoted by \(\otimes\). 1-form fields, that is 1-covariant tensor fields, are denoted in boldface, e.g., \(\ell, u, p, \ldots\) Symmetrization and antisymmetrization on a pair of indices is defined by \(A_{(\alpha\beta)} \equiv (1/2)(A_{\alpha\beta} + A_{\beta\alpha})\),
and $A_{[\alpha\beta]} \equiv (1/2)(A_{\alpha\beta} - A_{\beta\alpha})$. The Riemann, or curvature, tensor is defined as

$$R^\alpha_{\beta\gamma\delta} \equiv \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\lambda} \Gamma^\lambda_{\beta\delta} - \Gamma^\alpha_{\delta\lambda} \Gamma^\lambda_{\beta\gamma}.$$ 

The Ricci tensor is defined by $R_{\alpha\beta} \equiv R^\lambda_{\alpha\lambda\beta}$. Other symbols used throughout this work are:

$D \equiv \ell_\lambda \nabla^\lambda$, $\theta \equiv (1/2)\nabla^\lambda \ell_\lambda$, $\overline{R} \equiv 2\Phi_{00} \equiv R_{\sigma\lambda} \ell^\sigma \ell^\lambda$, $\zeta_2 \equiv (1/2)\nabla_{\sigma} \ell_\lambda \nabla^\sigma \ell^\lambda - \theta^2$, $\varpi^2 \equiv (1/2)\nabla_{\sigma} \ell_\lambda \nabla^\sigma \ell^\lambda$. $\Psi_0 \equiv C_{\lambda\sigma\mu\nu} \ell^\lambda k^\sigma k^\mu k^\nu$, with $k$ a null complex-valued 1-form orthogonal to $\ell$. If $\ell$ is geodesic, $D\ell \equiv M\ell$, where $M$ is a function.

Throughout this work we will deal with local 1-parameter groups of local motions acting upon the metric tensor and we shall simply call them “metric motions”. A “motion” being itself a (smooth) diffeomorphism. Finally, the end of a proof or its absence is marked by $\blacksquare$.

### 2 Kerr-Schild motions

We begin by giving some basic concepts of Kerr-Schild motions (proofs and details are available in I)

**Definition 1 (Kerr-Schild vector fields)** Any solution $\vec{\xi}$ of the equations (hereafter called Kerr-Schild equations)

$$\mathcal{L}(\vec{\xi})g = 2h \ell \otimes \ell, \quad (1)$$

$$\mathcal{L}(\vec{\xi})\ell = m \ell \quad (2)$$

where $h$ and $m$ are two functions over $V_n$ and $\ell$ is a null 1-form field will be called a Kerr-Schild vector field (KSVF) with respect to $\ell$. The functions $h$ and $m$ are the gauges of the metric $g$ and of $\ell$, respectively, and $\ell$ is called the deformation direction.

It is easy to show that only the direction of $\ell$ is of relevance since $\ell$ is null, i.e., for any other $\ell' = A\ell$, with $A \neq 0$ an arbitrary function, the KSVFs with respect to $\ell'$ are the same as with respect to $\ell$. Therefore, any general result regarding Kerr-Schild equations has to include this basic property.

Other results are

**Proposition 2.1** Two metrics linked by a GKS relation, $\bar{g} = g + 2H \ell \otimes \ell$, admit the same KSVFs with respect to $\ell$. $\blacksquare$

The possibility of the infinite dimensional character of some Lie algebras is easily seen from (a slightly variant of theorem 2 in I)
Theorem 1 For any $\ell$ such that $\nabla \ell = b \ell \otimes \ell$ where $b$ is some function, $\vec{\xi} = \rho \ell$ with $d\rho \neq 0$ is a KSVF with respect to the same direction $\ell$ if and only if the functions $\rho$ are those of the ring generating $\ell$, that is to say, such that $\ell \wedge d\rho = 0$.

In the sequel, the metric tensor $g$ and the direction of the deformation $\ell$ are to be considered as data in Kerr-Schild equations. On the other hand, $h$ and $m$ are unknown $C^\infty$ functions and $\vec{\xi}$, a KSVF, is also an unknown of the problem.

Let us start with the first case.

2.1 Non-geodesic Kerr-Schild motions

We will call non-geodesic Kerr-Schild motions whenever the vector field $\vec{\ell}$ is not geodesic. We recall that a null vector field on $V_n$ is non-geodesic if and only if $a^2(\equiv g_{\mu\nu}a^\mu a^\nu) \neq 0$, where $\vec{a}$ is the four-acceleration vector associated with $\vec{\ell}$, defined by $\vec{a} \equiv D\vec{\ell}$. In this case we have,

Theorem 2 The system of non-geodesic Kerr-Schild motions is closed.

Proof: From the expression of $L(\vec{\xi}) R_{\alpha\beta}$ given in App. A and the null character of $\ell$ one gets $\ell^\lambda \ell^\mu L(\vec{\xi}) R_{\lambda\mu} = -2ha^2$. Whence $h = -(\ell^\lambda \ell^\mu / 2a^2)L(\vec{\xi}) R_{\lambda\mu}$. Thus $h$ is isolated in terms of data, and the unknowns $\xi_\alpha$, so that the system is closed.\footnote{That is, one could now rewrite Eqs. (1) in an explicit normal form for $h$, $\xi_\alpha$ and $\xi_{\alpha\beta}$, as it is done in the conformal case —see e.g., [29]. However, due to following results, this is actually secondary, because the Kerr-Schild equations will be linked with Killing equations, which have a simple and well known normal system. This result will also be applied in the following section.}

Substituting the expression for $h$ into Kerr-Schild equations one gets

Theorem 3 (Isometrization of non-geodesic Kerr-Schild motions) The KSVFs of a non-geodesic Kerr-Schild motion are the Killing vector fields of $(V_n, \gamma)$, where

$$\gamma \equiv g + (\tilde{R}/a^2)\ell \otimes \ell,$$

restricted by Eqs. (2).
Proof: First we compute $\mathcal{L}(\xi)a^2$. To do this, we first use the formula (70) of App. A and that $a_\mu\ell^\mu$ vanishes for any null vector field to obtain $\mathcal{L}(\xi)a^2 = \mathcal{L}(\xi)(\ell^\rho\nabla_\rho\ell^\rho) = (\mathcal{L}(\xi)\ell^\rho)\nabla_\rho\ell^\rho + D(\mathcal{L}(\xi)\ell^\rho)$. Recalling Eq. (71) we readily get $\mathcal{L}(\xi)a^2 = \mathcal{L}(\xi)(g_{\lambda\mu}a^\lambda a^\mu) = 2h\ell\lambda\mu a^\lambda a^\mu + 2a_\mu\mathcal{L}(\xi)a^\mu = 4ma^2$.

Now, the expression of $h$ found in the proof of theorem 2 can be rewritten as $h = \mathcal{L}(\xi)(-\bar{R}/2a^2) - m\bar{R}/a^2$, where we have put $\bar{R} \equiv R_{\mu\nu}\ell^\mu\ell^\nu$. Hence, we can write $2h\ell \otimes \ell$ as $\mathcal{L}(\xi)(-\bar{R}\ell \otimes \ell/a^2)$, which is the key result. On the other hand, the initial problem is completely characterized by $\mathcal{L}(\xi)g = 2h\ell \otimes \ell$ and $\mathcal{L}(\xi)\ell = m\ell$. Taking into account the expression obtained for $h$, for a non-geodesic $\ell$, the system is totally equivalent to $\mathcal{L}(\xi)\gamma = 0$ and $\mathcal{L}(\xi)\ell = m\ell$, with $\gamma \equiv g + (\bar{R}/a^2)\ell \otimes \ell$. Notice that the expression of $\gamma$ is independent of the parametrization of $\bar{\ell}$: if $\bar{\ell} \rightarrow A\bar{\ell}$, $\bar{a} \rightarrow A^2\bar{a} + A(DA)\bar{\ell}$ and $a^2 \rightarrow A^4a^2$. Therefore $\gamma$ remains invariant and the set of KSVFs is the same, as mentioned before.

Furthermore, since $\gamma$ is linked by a GKS relation with $g$, usual results on GKS relations —see e.g., [34]— assure that $\gamma$ is non-degenerated ($\det \gamma = \det g \neq 0$), and may be reinterpreted as another metric tensor. Besides this, we also have that $\ell$ is a non-geodesic null 1-form in $(V_n, \gamma)$ (we use that $a^2$ is invariant if two spacetimes are linked by a GKS relation). In conclusion, the Kerr-Schild motions for a non-geodesic $\ell$ are equivalent to the set of isometries of $(V_n, \gamma)$ that satisfy Eqs. (2) or equivalently that commute with the direction $\ell$ according to Eq. (70), i.e., $[\ell, \ell] = m\ell$.

Moreover, the theorem above asserts, from a structural point of view, that all non-geodesic Kerr-Schild motions are “isometrizable” under an appropriate GKS relation with the initial spacetime. We thus have that non-geodesic Kerr-Schild motions are affine motions with respect the Levi-Civitta connection of $\gamma$ and are hence linearizable [1]. This by itself constitutes an extension of the works [29, 35]–[37] to the case of Kerr-Schild motions. These authors had considered the analogous problem in the case of conformal symmetries, i.e., when a group of conformal symmetries may become a group of isometries of a certain conformally related spacetime. Our result has an extra bonus, namely, it turns out to be independent of the particular properties of the Lie algebra. Besides that, it is sometimes a useful tool for finding the set of KSVFs of a given problem. For instance, we have

Proposition 2.2 For any spacetime with $\bar{R} = 0$, with $\ell$ non-geodesic, the
solution of Kerr-Schild motions is a subset of its own isometries, restricted by Eqs. (2). In particular, this holds for any spacetime of constant curvature.

This proposition seems to be very interesting since it allows an intrinsic reduction of the whole group of isometries of a given spacetime with $\bar{R} = 0$ with the aid of the object $\ell$, which may, and often does, have a relevant physical or geometrical content.

From theorem 3 and proposition 2.2, it is evident that the knowledge of isometries—for instance, the classical book [38]—could be applied here in order to develop an extensive study of non-geodesic Kerr-Schild motions. We shall not develop here this fruitful connection in detail (see also App. D).

To end up this analysis of general properties of non-geodesic Kerr-Schild motions, let us add a pair of consequences regarding invariant quantities under a non-geodesic Kerr-Schild motion. First, due to theorem 3 a standard calculation leads to the conclusion that the Riemannian tensor associated with $\gamma$ is an invariant under a non-geodesic Kerr-Schild motion. This constitutes a result analogous to finding out that the Riemannian tensor or the Weyl tensor of a metric tensor are the invariant objects under an isometric or a conformal motion, respectively. Consequently, this allows to interpret non-geodesic Kerr-Schild motions as symmetries, i.e., transformations preserving some geometrical objects.

Moreover, this result may easily be extrapolated to spacetimes which are related by a GKS relation, in an analogous way as with the well known invariance of the Weyl tensor under a conformal relation, see e.g., [39]. The result is

**Proposition 2.3** Let two metric tensors, $\bar{g}$ and $g$, be linked by a GKS relation, that is, $\bar{g} = g + 2H\ell \otimes \ell$, with $\ell$ non-geodesic, then

$$R^{\alpha}_{\beta\gamma\delta}[\bar{g} + (R_{\lambda\mu}\ell^\lambda \ell^\mu)/a^2] \otimes \ell] = R^{\alpha}_{\beta\gamma\delta}[\bar{g} + (\bar{R}_{\lambda\mu}\ell^\lambda \ell^\mu)/\bar{a}^2] \otimes \ell].$$

Where $R_{\alpha\beta}$ is the Ricci tensor of $\bar{g}$. We have used the standard convention $\ell \equiv \ell$ and also $\bar{a}^2 = a^2$, where $\bar{a} \equiv D\bar{\ell}$ (see e.g., [40, 41]).

A final link between properties of non-geodesic Kerr-Schild motions and isometries is worth mentioning. Since this type of Kerr-Schild motions have been reduced to a restricted problem of isometries, we could also take advantage of constants of motion along geodesics of $(V_n, \gamma)$. Obviously, these will
not be in general geodesics of \((V_n, g)\). However their study could reveal new 
constants of motion for their corresponding motions in \((V_n, g)\) which could 
shed some light into non-trivial first integrals in GKS related spacetimes. 
The kind of motion in \((V_n, g)\) which corresponds to the geodesics of \(\gamma\) will be considered elsewhere.

To finish, we add an example of non-geodesic Kerr-Schild motions in 
\(n = 4\) which can be easily visualized geometrically and the solution appears 
then natural,

**Example 1** Consider Minkowski spacetime, where \(\{x^\alpha\} (\alpha = 0, 1, 2, 3)\) are the usual Cartesian coordinates, and \(x^0\) refers to the timelike coordinate. \(\ell = dx^0 + \alpha dx^1 + \beta(\cos\omega x^1 dx^2 + \sin\omega x^1 dx^3)\) where \(\beta = \sqrt{1 - \alpha^2}\), \(\alpha \in (-1, 1) - \{0\}\) and \(\omega \neq 0\) are constants, the set of Kerr-Schild motions is generated by \(\\{\partial_{x^0}, \partial_{x^2}, \partial_{x^3}, (1/\omega)\partial_{x^1} + x^2\partial_{x^3} - x^3\partial_{x^2}\}\). In this case \(m = 0\).

**Proof:** Clearly \(\ell\) is null and under the hypothesis \(\vec{a} = \alpha\beta\omega(-\sin\omega x^1 \partial_{x^2} + \cos\omega x^1 \partial_{x^3}) \neq 0\). In this case, from Prop. 2.2, Kerr-Schild motions are a set of isometries of flat spacetime. The latter are generated by \(\vec{\xi} = A^\lambda \partial_{x^\lambda} + B_i (x^i \partial_{x^0} + x^0 \partial_{x^i}) + \epsilon_{ij} x^i \partial_{x^j}\) where \(A^\lambda, B_i, \epsilon_{ij} = -\epsilon_{ji}\) are constants. On the other hand, Eqs. (2) impose

\[
\begin{align*}
m &= B_1 \alpha + \beta(B_2 \cos\omega x^1 + B_3 \sin\omega x^1), \\
m\alpha &= B_1 + \beta(\epsilon_{12} \cos\omega x^1 + \epsilon_{13} \sin\omega x^1), \\
m\beta \cos\omega x^1 &= B_2 - \epsilon_{12} \alpha + \beta \sin\omega x^1(\epsilon_{1i} x^i \omega - A^1 \omega - B_1 \omega x^0 + \epsilon_{23}), \\
m\beta \sin\omega x^1 &= B_3 - \epsilon_{13} \alpha - \beta \cos\omega x^1(\epsilon_{1i} x^i \omega - A^1 \omega - B_1 \omega x^0 + \epsilon_{23}).
\end{align*}
\]

From Eq. (3) one sees that \(m = m(x^1)\). Therefore from Eq. (4) necessarily \(B_1 = \epsilon_{1i} = 0\). Substituting Eq. (3) into Eq. (4) one gets \(B_2 = B_3 = 0\) and \(A^1 \omega = \epsilon_{23}\). The rest of equations become mutually compatible.

Finally, notice that, in general, not for any \(\ell\) a solution exists, even in flat spacetime.

### 2.2 Geodesic \(\ell\) with \(\Delta \neq 0\)

To begin with, the object \(\Delta\) is a scalar defined by (recall notation in Sect. 1)

\[
\Delta \equiv -2D\theta + 4\theta^2 - 3\bar{R} + 2\ell^\nu \nabla_\sigma \nabla^\sigma \ell_\mu + DM - 2M\theta.
\]
It can also be written in terms of the Newman-Penrose quantities and the optical scalars \cite{41}
\[
\Delta = 2(\rho^2 + \bar{\rho}^2 - \rho \bar{\rho} - \sigma \bar{\sigma} - 2\Phi_{00}) = 2(\theta^2 - \varsigma^2 - 3\bar{\varsigma}^2 - \bar{\bar{\rho}} + \bar{\rho}^2 + 2M\theta),
\]
respectively.

**Proposition 2.4** \(\Delta\) is only sensitive to changes in the parametrization of \(\vec{\ell}\) and in the form \(\Delta_A = A^2\Delta_{A=1}\), if \(\vec{\ell} \to A\vec{\ell}\).

**Proof:** Clearly, \(\Delta\) is an intrinsic scalar associated with \(\vec{\ell}\). On the other hand, under the change \(\vec{\ell} \to A\vec{\ell}\) we have
\[
\theta \to A\theta + DA/2, \quad \theta^2 \to A^2\theta^2 + A(DA)\theta + (DA)^2/4,
\]
\[
D\theta \to A^2D\theta + A(DA)\theta + (DDA)/2, \quad \bar{R} \to A^2\bar{R},
\]
\[
\ell^\alpha\nabla_\sigma\nabla^\sigma\ell_\mu \to A^2\ell^\alpha\nabla_\sigma\nabla^\sigma\ell_\mu, \quad M \to AM + DA,
\]
\[
DM \to A^2DM + A(DA)M + A(DDA).
\]
Then, a direct substitution proves the assertion. \(\square\)

For the sake of brevity, we will only display here the main results of this case. We begin with:

**Theorem 4** The system of geodesic with \(\Delta \neq 0\) Kerr-Schild motions is closed.

**Proof:** The proof is similar to that of theorem 2. Obviously the combinations leading to the isolation of \(h\) are different. In our case, we have found
\[
\mathcal{L}(\tilde{\xi})R = 2\{DDh + (4\theta + M)Dh + [2(D\theta + 2\theta^2) + 2M\theta + DM - \bar{R}]h\}, \quad (7)
\]
\[
\ell^\alpha\mathcal{L}(\tilde{\xi})R_\mu^\alpha = [DDh + 2(\theta + M)Dh + (2D\theta + \bar{R} - \ell^\lambda\nabla_\sigma\nabla^\sigma\ell_\lambda + 2M\theta)
\]
\[
+DM + M^2)h]\ell_\mu, \quad (8)
\]
\[
\ell^\sigma\ell^\mu\mathcal{L}(\tilde{\xi})R_{\sigma\beta\mu} = -[DDh + 3MDh + 2(M^2 + DM)h]\ell^\sigma\ell_\beta, \quad (9)
\]
where we have made use of some of the expressions given in App. A — recall \(\tilde{a} = M\tilde{\ell}\). Hence we have
\[
2\Delta(h\ell_\alpha\ell_\beta) = (\mathcal{L}(\tilde{\xi})R)\ell_\alpha\ell_\beta - 4(\mathcal{L}(\tilde{\xi})R_{\mu(\alpha)}\ell_\beta)\ell^\mu - 2(\mathcal{L}(\tilde{\xi})R_{\alpha\beta\mu})\ell^\sigma\ell^\mu.
\]
Clearly if \(\Delta\) is non-zero, \(h\) can be isolated only in terms of data and the unknowns \(\xi_\alpha, [\xi_\alpha]_{\beta}\). Moreover, notice that the vanishing of \(\Delta\) is a well-defined condition (see Prop. 2.4), independent of the parametrization of \(\tilde{\ell}\). \(\square\)

Again it is possible to rewrite \(h\ell \otimes \ell\) in a more compact form, namely,
Proposition 2.5  

For the case of a geodesic $\ell$ with $\Delta \neq 0$,

$$2h\ell_\alpha \ell_\beta = \mathcal{L}(\bar{\xi}) \left[ \frac{R\ell_\alpha \ell_\beta - 2(R\alpha_\beta_\mu \ell_\sigma \ell_\mu + R\alpha_\mu \ell_\mu \ell_\beta + R\beta_\mu \ell_\mu \ell_\alpha)}{\Delta} \right].$$  \hspace{1cm} (10)

**Proof:** It will suffice to prove $\mathcal{L}(\bar{\xi})\Delta = 2m\Delta$. From App. A—formulae i–vi—we have

\begin{align*}
\mathcal{L}(\bar{\xi})\theta &= m\theta + \frac{Dm}{2} \rightarrow \mathcal{L}(\bar{\xi})\theta^2 = 2m\theta^2 + (Dm)\theta, \\
\mathcal{L}(\bar{\xi})D\theta &= 2m D\theta + (Dm)\theta + D(Dm)/2, \\
\mathcal{L}(\bar{\xi})\ell_\alpha \nabla_\sigma \nabla^\sigma \ell_\mu &= 2m\ell_\mu \nabla_\sigma \nabla^\sigma \ell_\mu, \\
\mathcal{L}(\bar{\xi})M &= 2m M, \\
\mathcal{L}(\bar{\xi})D M &= 2m D M + (Dm) M + (D Dm).
\end{align*}

Consequently, $\mathcal{L}(\bar{\xi})\Delta = 2m\Delta$. Now one can follow similar steps as in the proof of theorem 3 in order to get expression (10).

Expression (10) allows us to rewrite Eq. (1) as

$$\mathcal{L}(\bar{\xi})\gamma = 0,$$

with

$$\gamma_{\alpha\beta} \equiv g_{\alpha\beta} + \frac{2(R\alpha_\sigma_\beta_\mu \ell_\sigma \ell_\mu + R\alpha_\mu \ell_\mu \ell_\beta + R\beta_\mu \ell_\mu \ell_\alpha)}{\Delta} - R\ell_\alpha \ell_\beta.$$

Despite the non-geodesic case, $\gamma$ is not always linked by a GKS relation with $g$. Moreover $\gamma$ may become degenerate—although it can only be completely degenerate for $n = 3$, as is easily seen from a study of $\gamma_{\alpha\lambda} \ell^\lambda$ and the trace of $\gamma$. Its degeneracy in $n = 4$ is controlled by the following result:

**Proposition 2.6**  

For $n = 4$, the determinant of $\gamma$, in any orthonormal cobasis, equals to $(1/\Delta^4)\{(\Delta + 2\bar{R})^2[4||\Psi_0||^2 - (\Delta + \bar{R})^2]\}$.

**Proof:** Using the Newman–Penrose formalism—see e.g., [34, 43] for definitions of each object—we write

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + \left(\frac{2}{\Delta}\right) [(\Phi_2 + \bar{\Psi}_2 - 2\Lambda - 2\Phi_{11})\ell_\alpha \ell_\beta + (\bar{\Phi}_{01} - \bar{\Psi}_1)(\ell_\alpha k_\beta + k_\alpha \ell_\beta) + (\Phi_{01} - \Psi_1)(\ell_\alpha \bar{k}_\beta + \bar{k}_\alpha \ell_\beta) + \bar{\Psi}_0 k_\alpha k_\beta + \bar{\Psi}_0 \bar{k}_\alpha \bar{k}_\beta$$

$$+ \Phi_{00}(g_{\alpha\beta} - m_\alpha \ell_\beta - m_\beta \ell_\alpha)],$$

where $\{\ell, m, k, \bar{k}\}$ is a null cobasis containing $\ell$. Indeed, any term proportional to $\ell \otimes \ell$ will not affect the value of the determinant of $\gamma$. Therefore, we only need to consider $\gamma \equiv \gamma - (2/\Delta)(\Psi_2 + \bar{\Psi}_2 - 2\Lambda - 2\Phi_{11})\ell \otimes \ell.$
In terms of an orthonormal cobasis, \( \{ \Theta^{0}, \Theta^{1}, \Theta^{2}, \Theta^{3} \} \), related with the previous one by
\[
\ell = (\Theta^{0} + \Theta^{1})/\sqrt{2}, \quad m = (\Theta^{0} - \Theta^{1})/\sqrt{2}, \quad k = (\Theta^{2} + i\Theta^{3})/\sqrt{2},
\]
we get
\[
\tilde{\gamma} = \left( 1 + \frac{4\Phi_{00}}{\Delta} \right) (-\Theta^{0} \otimes \Theta^{0} + \Theta^{1} \otimes \Theta^{1}) + \left( 1 + \frac{2\Phi_{00} + \Psi_{0} + \bar{\Psi}_{0}}{\Delta} \right) \Theta^{2} \otimes \Theta^{2}
\]
\[
+ \left( 1 + \frac{2\Phi_{00} - \Psi_{0} - \bar{\Psi}_{0}}{\Delta} \right) \Theta^{3} \otimes \Theta^{3}
\]
\[
+ \frac{(\Phi_{01} + \Psi_{01} - \Psi_{1} - \bar{\Psi}_{1})}{\Delta} (\Theta^{0} \otimes \Theta^{2} + \Theta^{2} \otimes \Theta^{0} + \Theta^{1} \otimes \Theta^{2} + \Theta^{2} \otimes \Theta^{1})
\]
\[
+ \frac{i(\Phi_{01} - \Psi_{01} + \Psi_{1} + \bar{\Psi}_{1})}{\Delta} (\Theta^{0} \otimes \Theta^{3} + \Theta^{3} \otimes \Theta^{0} + \Theta^{1} \otimes \Theta^{3} + \Theta^{3} \otimes \Theta^{1})
\]
\[
+ \frac{i(\Psi_{0} - \bar{\Psi}_{0})}{\Delta} (\Theta^{2} \otimes \Theta^{3} + \Theta^{3} \otimes \Theta^{2}).
\]
Then a standard computation gives
\[
\det \tilde{\gamma} = (1/\Delta^{4}) \{ (\Delta + 4\Phi_{00})^{2}[4||\Psi_{0}||^{2} - (\Delta + 2\Phi_{00})^{2}] \}.
\]
Recalling that \( 2\Phi_{00} \equiv \bar{R} \) and \( \det \tilde{\gamma} = \det \gamma \) we get the result claimed above.

It is worth remarking, as an example, that \( \gamma \) is a metric if \( g \) is the metric of any spacetime of constant curvature. This happens because, for a spacetime of constant curvature, one has \( R_{\alpha\beta\gamma\delta} = k(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \), where \( k \) is a constant related with the (constant) scalar curvature by:
\[
k = R/n(n-1).
\]
Whence, \( \bar{R} = g_{\lambda\mu}\ell^{\lambda\mu} = (R/n)g_{\lambda\mu}\ell^{\lambda\mu} = 0. \) Moreover, a spacetime of constant curvature is conformally flat. Therefore \( \Psi_{0} = 0 \) and \( \det \gamma = -1 \), in any orthonormal cobasis. Although \( \gamma \) will be in general non-degenerate, we do not have now a direct copy of theorem 3, but a similar result,

**Proposition 2.7 (Isometrization of geodesic \( \Delta \neq 0 \) Kerr-Schild motions)**

The KSVFs of a geodesic Kerr-Schild motion with \( \Delta \neq 0 \) are vector fields associated with the invariance of the object \( \gamma \), restricted by Eqs. (2). Moreover, when \( \det \gamma \neq 0 \), the vector fields associated with the invariance of \( \gamma \) are Killing vector fields.

Notice that \( \gamma \) is not always linked with \( g \) by a GKS relation, and, therefore, \( \mathcal{L}(\xi)\gamma = 0 \) does not imply by itself Eq. (1) in general.

We also have a similar result as in Prop. 2.2. First define \( T_{\alpha\beta} = [2(R_{\alpha\sigma\beta\mu}\ell^{\sigma\mu} + R_{\alpha\mu}\ell^{\sigma\beta} + R_{\beta\mu}\ell^{\sigma\alpha}) - R\ell_{\alpha}\ell_{\beta}]/\Delta \). Then
Proposition 2.8 For any geodesic $\ell$ with $\Delta \neq 0$ and in any spacetime with $T = 0$, the solution of Kerr-Schild motions is a subset of its own isometries, restricted by Eqs. (2).

Similar considerations as those in previous section are valid now, e.g., Prop. 2.3, for the case of non-degenerate $\gamma$, and its constants of motion. They also justify to interpret this case of Kerr-Schild motions as some kind of symmetries. These results are now more interesting because $\ell$ is clearly related with the light-cone structure of a spacetime. Inside this case most of the more used $\ell$ in General Relativity are to be found, e.g., the axially symmetric case or the spherically symmetric case. These examples are carried out in the following section.

2.2.1 KSVFs for the principal null directions of Kerr-Newman spacetimes

In this section we shall focus on the resolution of Kerr-Schild motions for a class of spacetimes which are of major astrophysical interest, i.e., Kerr-Newman spacetimes.

Formulation of the problem.

The equations to be solved are Kerr-Schild equations where $g$ is now the metric of Kerr-Newman spacetimes and $\ell$ is any of their two principal null directions —see below.

One way to solve the equations is to consider them directly, i.e., expanding them in terms of a partial derivative system. Yet this way is long. Indeed, there is an alternative path that makes use of some previous results about Kerr-Schild motions. Moreover, it may prove to be useful in other spacetimes, too.

The calculations are simplified by noticing that Kerr-Newman metrics are Kerr-Schild metrics, i.e.,

$$g_{KN} = \eta + 2H\ell \otimes \ell,$$

where $\eta$ is the metric tensor of flat spacetime, and in terms of “Cartesian-like”, or “Kerr-Schild” coordinates, $H$ and $\ell$ are expressed as [34, 42, 43]:

$$H(x, y, z) = \frac{2Mr - Q^2}{r^2 + a^2(z/r)^2}, \quad r^2(x^2 + y^2 + z^2) + a^2z^2 = r^2(r^2 + a^2),$$

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where $M$ and $Q$ represent the mass and the charge of the source, $a$ is its angular momentum per unit mass, and

$$\ell_{(\pm)} = \frac{1}{\sqrt{2}} \left( \pm dt + \frac{z}{r} \, dz + \frac{rx + ay}{r^2 + a^2} \, dx + \frac{ry - ax}{r^2 + a^2} \, dy \right)$$

(13)

are the two principal null directions of Kerr-Newman spacetimes, which are geodesic and shear-free.

First, Prop. 2.1 assures that two metrics related by a GKS relation, $\tilde{g} = g + 2H\ell \otimes \ell$, admit the same KSVFs with respect to $\ell$. The relation between the functions $\tilde{h}$, $\tilde{m}$, $h$, and $m$ is, in this case: $\tilde{h} = h + \mathcal{L}(\xi)H + 2mH$, $\tilde{m} = m$.

Thus, the problem

$$\mathcal{L}(\xi)g_{KN} = 2h_{KN} \ell \otimes \ell, \quad \mathcal{L}(\xi)\ell = m\ell$$

is equivalent to

$$\mathcal{L}(\xi)\eta = 2h_{F} \ell \otimes \ell, \quad \mathcal{L}(\xi)\ell = m\ell,$$

(14)

where “$F$” refers to flat spacetime and $\xi$ are the same set in both cases. We have also proven that the solution to a problem of Kerr-Schild motions for a geodesic $\ell$ depends on whether $\Delta$ —Eqs. (5), (6)— vanishes or not.

Thanks to Prop. 2.1, the calculation is simplified to the computation of $\Delta$ for (13) where $\{x, y, z, t\}$ are now Cartesian coordinates and where computations are much easier. Moreover, we can make use of some general results concerning the vanishing of $\Delta$ in flat spacetime. In Sect. 2.3.1 this issue will be solved when $\ell$ is a principal null direction and $R = 0$ (Clearly this includes our case). The result is that for a shear-free $\ell$, $\Delta = 0$ if in addition $\ell$ is rotation and expansion-free. However, $\ell_{\pm}$ possesses rotation unless $a = 0$. And if $a = 0$, $\ell$ is simply given by

$$\ell = \frac{1}{\sqrt{2}}(\pm dt + dr), \quad r^2 = x^2 + y^2 + z^2,$$

which satisfies $\Delta = 1/r^2 \neq 0$, as is easily computed. Summarizing, we get

\footnote{Indeed it is not difficult to show that $\Delta$ does not vanish as well in Kerr-Newman spacetimes. The reason is that $\Delta$ remains \textit{invariant} if there exists a Kerr-Schild relation between both spacetimes in which $\ell$ is geodesic. This is accomplished in Eq. (11).}
Proposition 2.9 For $\ell$ given in Eq. (13), $\Delta$ does not vanish in flat space-time.

Resolution of Kerr-Schild motions.

With this result in hand, we use Prop. 2.7 in order to conclude that Eqs. (14) reduce to

$$\mathcal{L}(\xi)\eta = 0, \quad \mathcal{L}(\xi)\ell = m\ell.$$  

The solution of the first set is simply the generators of the Poincaré group. In order to solve the second set, one only needs to consider a generic infinitesimal generator of the Poincaré group, Eq. (13) for $\ell$ and impose

$$[\xi, \ell] = m\ell.$$  

(15)

We choose the following representation of a general infinitesimal generator of the Poincaré group which is clearly adapted to our purposes:

$$\tilde{\xi} = (\alpha_1 + \beta_1 x + \beta_2 y + \beta_3 z)\partial_t + (\alpha_2 + \beta_1 t - \gamma_1 y - \gamma_2 z)\partial_x + (\alpha_3 + \beta_2 t + \gamma_1 x - \gamma_3 z)\partial_y + (\alpha_4 + \beta_3 t + \gamma_2 x + \gamma_3 y)\partial_z,$$  

(16)

where $\alpha_\lambda, \beta_i, \gamma_i$ are constants. Taking into account (13), (16) and (15), one gets that the terms with $\partial_t$ yield: $m = \beta_1 \ell + \beta_2 \ell_y + \beta_3 \ell_z$. However, combining the latter result with the terms with $\partial_x$, one obtains, $m = 0$ (and therefore $\beta_i = 0$) and $\alpha_2 = 0$. Then, from some of the terms with $\partial_y, \partial_z$, one easily gets $\alpha_3 = \alpha_4 = 0$. Finally, the remaining conditions for the $\partial_x$ terms are

$$\frac{a^2(x^2 + 2 ar y) z r}{(r^2 + a^2)^2(r^4 + a^2 z^2)}(\gamma_2 x + \gamma_3 y) - \frac{az}{r^2 + a^2} \gamma_3 + \frac{a^2 z}{r(r^2 + a^2)} \gamma_2 = 0,$$

where we have used $\partial_x r = x r^3/(r^4 + a^2 z^2), \partial_y r = y r^3/(r^4 + a^2 z^2)$ and $\partial_z r = z r(r^2 + a^2)/(r^4 + a^2 z^2)$. There are clearly two situations: $a = 0$ and $a \neq 0$. In the first situation all $\gamma_i$ remain free. The same holds for the terms with $\partial_y$ and $\partial_z$ as one can readily check. The result is therefore $\{\xi\} = \{\partial_t, x\partial_y - y\partial_x, x\partial_z - z\partial_x, y\partial_z - z\partial_y\}$ which corresponds to the (irrotational) radial case or spherically symmetric case. The other situation is $a \neq 0$. In this case, $\gamma_2$ and $\gamma_3$ must be zero necessarily. The only remaining parameter is $\gamma_1$.  

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One then calculates the remaining terms with $\partial_y$ and $\partial_z$. The $\partial_y$ terms yield the same conditions as the $\partial_x$ ones, as expected. Finally the $\partial_z$ terms are identically satisfied because $(x\partial_y - y\partial_x)r = 0$. Thus $\alpha_1$ and $\gamma_1$ remain free. This yields $\{\vec{\xi}\} = \{\partial_t, x\partial_y - y\partial_x\}$ for the rotational case.

Let us recall that the infinitesimal generators have multiple representations depending on the coordinate system being used. For the Kerr-Schild one, the result is the one displayed before. In Kerr coordinates the result is $\{\vec{\xi}\} = \{\partial_t, \partial_\phi\}$ for the rotational case, for instance, where the relation between both system of coordinates is given by

$$x + iy = (r + ia)e^{i\phi}\sin \theta, \quad z = r \cos \theta.$$ 

So far, this is the solution for flat spacetime. We have shown before that the infinitesimal generators are the same for Kerr-Newman spacetimes. Yet we can easily find their action on Kerr-Newman metrics. In our case, $m = 0$, $h_F = 0$, and we have $H = H(r, z)$ for $a \neq 0$, and $H = H(r)$ for $a = 0$. The result is, in any case $h_{KN} = m = 0$. Therefore, the Kerr-Schild motions are the isometries of Kerr-Newman spaces. Summarizing ($T_{x^\alpha}$ stands for translation along the axis $x^\alpha$)

**Proposition 2.10** There are no proper KSVFs for Kerr-Newman spacetimes and their principal null directions.

**Proposition 2.11** The Kerr-Schild motions for KN spaces associated with the principal null directions are given by $T_t \otimes T_\phi$ for rotational $\ell$, and by $T_t \otimes SO(3)$ for the irrotational case.

Here $t$ and $\phi$ are Kerr coordinates, not Boyer-Lindquist coordinates. In Kerr-Schild coordinates, cf. Eqs. above, $T_\phi$ is equivalent to $R_z$, i.e., a rotation around the $z$-axis. Another consequence is:

**Corollary 1** For any Kerr-Schild metric of the form $g = \eta + 2H\ell \otimes \ell$, where $\ell$ are the null directions given in (13), $H = H(r, z/r)$ for $a \neq 0$, and $H = H(r)$ for $a = 0$, the solution to the problem of Kerr-Schild motions is given by Prop. 2.11. In particular, this also includes flat spacetime.

---

3Of course, we could consider any $H$, yet we focus on spacetimes with the same local motions, and therefore they should share a similar action upon the metric tensor.
Indeed, this result points to an idea on physical applications of Kerr-Schild symmetries which is considered elsewhere [33]. Let us mention that among the spacetimes of this family one finds all proposed candidates to describe the macroscopic properties of non-singular quantum interiors for black holes, with a clear source origin, which are currently under study, see e.g., [59].

Another remarkable example of this section are Kerr-Schild motions for \( n \)-dimensional flat spacetime and a spherically symmetric deformation direction.

Example 2 In flat spacetime, the Kerr-Schild motions for \( \ell = d(t \pm r) \), where \( r \) is the usual radial coordinate defining the radius of the \((n-2)\) dimensional spheres, are \( SO(n-2) \times T_{x^0} \), where \( x^0 \) is the timelike coordinate. Moreover, \( m = 0 \).

Proof: This result is proven in I for \( n = 4 \); analogous steps as those prove the latter result.

We remark that in spite of the fact that for all these spacetimes the KSVFs correspond to Killing vector fields, one should not conclude that Kerr-Schild motions seem to be a simple subset of isometries. In I several examples with proper Kerr-Schild motions are solved. And even for the cases where Kerr-Schild motions reduce to isometries, they may be important because they are a subset of isometries restricted by a condition which is related with the invariance of the congruence of curves associated with \( \ell \) and, therefore, with the light-cone structure of spacetime itself —see also Exs. 3 later on.

2.3 Geodesic \( \ell \) with \( \Delta = 0 \)

The remaining case, i.e., \( \ell \) which are geodesic but satisfy \( \Delta = 0 \), is still without a complete solution. There are however some points worth to be remarked. First, the existence of such null vectors is severely restricted in general, except in flat spacetime, where the maximum number of such vectors is obtained. Indeed, the condition is very similar to that of the well known Goldberg-Sachs theorem and its generalizations (see e.g., [34]). In principle one can use the Newman-Penrose formalism to perform a complete study. In Sect. 2.3.1 we have begun with this idea for \( \ell \) which are principal null directions and satisfy \( \bar{R} = 0 \).

Secondly, one should remember that within this subcase, the system remains open. This tells us that no general solution similar to the previous
cases may be attained now for all $\ell$. In fact, one needs to classify, explicitly, the $\ell$’s which make the system open and only a particular study will reveal the characteristics of their infinite dimensional algebras. For the rest, one is faced with a closed problem. Since the existence of geodesic $\ell$ with $\Delta = 0$ completely depends on each spacetime, we do not have the confidence that a global result will be found, but rather a detailed collection of solutions. The unique exception is the two-dimensional problem. For $n = 2$ any $\ell$ belongs to this case and the general solution, containing infinite dimensional algebras, was given in I. This result is completely similar to that of conformal symmetries.

Notice, however, that the geometrical object $G$ defined by $G \equiv R_{\alpha\sigma\beta\mu}(\ell^\sigma\ell^\mu + R_{\alpha\mu}\ell^\alpha\ell_\beta + R_{\beta\mu}\ell^\mu\ell_\alpha - (R/2)\ell_\alpha\ell_\beta)$ remains conformally invariant under a geodesic, $\Delta = 0$ Kerr-Schild motion, i.e., $\mathcal{L}(\vec{\xi})G = 2mG$. This is obtained from (2) and (10). This show us that Kerr-Schild motions act in this case as a symmetry in some sense as well.

Among several possible examples we refer the reader to two examples given in I which show all the special characteristics of this case. These are the Kerr-Schild motions in an $n$-dimensional flat spacetime for an $\ell$ adapted to the cylindrical symmetry and to the parallel symmetry. Both have very peculiar features in their Lie algebras. For instance, in the parallel case the Lie algebra is of an infinite dimensional character. Although we will not discuss them here in detail, Ex. 3 —Sect. 4.1— shows the solution of the parallel case in the four dimensional case.

2.3.1 Subclassification of $\ell$ with $\Delta = 0$

In this section we would like to analyze further the classification of a geodesic $\ell$ that satisfies $\Delta = 0$ for the physical interesting case of Minkowski spacetime —where a maximum of such vectors may exist. The results are, however, easily generalized to other spacetimes as will appropriately be pointed out in the text.

The study of the existence of such $\ell$ in any spacetime is better carried out with the Newman-Penrose formalism [34]. In our case $\bar{R}(= 2\Phi_{00})$ cancels. Furthermore, without losing generality, we shall take $M$ equal to zero (recall Prop. 2.4 $\Delta = 0$ is invariant under changes in the parametrization of $\ell$). Then we have from (6)

$$\rho^2 + \bar{\rho}^2 = \rho\bar{\rho} + \sigma\bar{\sigma}.$$  (17)
The Newman-Penrose equations that are of relevance for our equation are

\[ D\rho = \rho^2 + \sigma \bar{\sigma}, \quad D\sigma = \sigma(\rho + \bar{\rho} + 4\epsilon). \]  

(18)

Deriving (17) with respect to \( \vec{\ell} \) and using (18), we get

\[ 2\rho^3 + 2\bar{\rho}^3 = (\rho + \bar{\rho})(\rho\bar{\rho} + \sigma\bar{\sigma}). \]  

(19)

Using (17) and (19) we obtain

\[ (\rho + \bar{\rho})(\rho\bar{\rho} - \sigma\bar{\sigma}) = 0. \]

Two possibilities appear. The first one is when \( \rho = -\bar{\rho} \). Substituting this condition into Eq. (17), we obtain \( 3\rho^2 = \sigma\bar{\sigma} \). Taking again the derivative with respect to \( \vec{\ell} \), we get that this condition is only fulfilled if \( \rho = \sigma = 0 \). In terms of the optical scalars, this means that the affine parametrized geodesic null vector must satisfy \( \theta = \varsigma = \eta = 0 \).

The other possibility is \( \rho\bar{\rho} = \sigma\bar{\sigma} \). Substituting it into Eq. (17), we get the condition \( \rho = \bar{\rho} \) \( (\bar{\omega} = 0) \) and therefore \( \sigma\bar{\sigma} = \rho^2 \). Provided that \( \ell \) is geodesic and \( \Phi_{00}, \Psi_0 \) are zero, one can apply the Sachs theorem, [44], to obtain that all possible such \( \ell \) must belong to one of the following sets

\[
\begin{align*}
\rho &= -\sigma = -\frac{1}{2s} \quad \text{if } \rho \neq 0, \\
\rho &= \sigma = 0 \quad \text{if } \rho = 0,
\end{align*}
\]

where \( s \) is a convenient affine parameter along the congruence generated by \( \vec{\ell} \). We remark that these conclusions are also valid for any principal null direction of a spacetime for which \( \bar{R} = 0 \) (in terms of Newman-Penrose quantities: \( \Psi_0 = 0 \) and \( \Phi_{00}(= \bar{R}/2) = 0 \)). Finally, let us recall that GKS relations used in physics have almost always \( \ell \) as a principal null direction.

Eventually, in Minkowski spacetime and with the aid of an extension of the Kerr theorem, [34], it is possible to demonstrate that the conditions \( \rho = \sigma = 0, k = 0, \epsilon + \bar{\epsilon} = 0 \) imply that \( \ell \) is covariantly constant. Whereas \( \rho = -\sigma, \bar{\rho} = \rho \), or equivalently \( \bar{\omega} = 0, \varsigma^2 = 4\theta^2 \), seems to force \( \ell \) to be the cylindrical case (see e.g., I and Ex. 3 for their explicit solutions). In that case the study of Kerr-Schild motions for geodesic \( \ell \) with \( \Delta = 0 \) in Minkowski spacetime would be complete.
2.4 Summary on Kerr-Schild motions

In previous sections, we have investigated the possibility of solving the system of differential equations of other types of metric motions than the ones of isometric or conformal motions. Specifically, we have studied in detail the existence of Kerr-Schild motions. In particular, we have shown how the problem is divided into three main different cases. In two of them, whenever $\ell$ is non-geodesic or is actually geodesic but satisfies $\Delta \neq 0$ (Sects. 2.1 and 2.2), the problem of Kerr-Schild motions yields a finite dimensional algebra. For each case several properties and some examples have been presented and discussed. Among them, we have obtained the main set of geometrical invariants for these cases. This result is important, and allows one to consider Kerr-Schild motions as Kerr-Schild symmetries.

Moreover, we have shown that Kerr-Schild symmetries can be linked with a restricted problem of isometries of another Riemannian manifold, except for very peculiar cases. This constitutes an extension of Refs. [29], [35]–[37] for Kerr-Schild motions. The reduction is far from being trivial and adds a further —geometrical— path to group isometries (see examples and App. D) —in an intrinsic way that may be helpful in the search of physical applications. Finally it gives a simple and direct procedure for solving the problem of the symmetry in some relevant cases.

In Sect. 2.2.1, we have given the solution of Kerr-Schild motions for space-times of great astrophysical interest, Kerr-Newman spacetimes, pointing to some physical applications.

The third case, Sect. 2.3, only exists under very restrictive conditions. Nevertheless, the spacetimes and null vectors satisfying such conditions are of relevance, as is the case of an $n$-dimensional flat spacetime with cylindrical or parallel direction. It is actually the most difficult one. There, the system remains open, i.e., without any further and external condition, the freedom in the unknowns is functional. Thus, infinite dimensional Lie algebras appear. This is the first time that in a metric symmetry such behavior shows up. Those facts are rather representative of what might be expected from the whole family of metric symmetries. However, everything has to be understood under a different scheme than the one used for isometric and conformal motions. Such general scheme will be dealt with in the following sections).

Finally, Sect. 2.3.1 concluded focussing on Kerr-Schild motions in space-times where $\ell$ is a principal null direction and $\bar{R} = 0$. In particular, in flat
space it was seen that the knowledge of Kerr-Schild motions is almost completed. A necessary result to finish the study of the more elaborate situation of Kerr-Schild motions. In App. A, we give some formulae for the Lie derivatives with respect a KSVF of some basic geometric objects used along previous sections.

3 Metric motions. A geometrical approach

The aim in previous sections was not focussed on a deeper study of isometric or conformal groups, nor in possible new applications of both. Rather, the scope was to try to enlarge the number of metric motions by considering a new specific example, Kerr-Schild motions. The aim of the following sections is to pose and develop a framework to deal with some general situations, since one expects that many achievements of differential geometry on Riemannian spaces have a physical spin-off. Moreover, Kerr-Schild symmetries have some general relevant features that have impelled us, along with other researchers, to consider metric motions from a more general point of view, mainly focussed on analyzing the whole set of candidates to become metric motions, which have a geometrical origin. We have decided to call them “generalized” metric motions, or, simply, metric motions when no doubt can arise.

In Sects. 3.1, 3.2 and 3.3 we elaborate this last idea. In Sect. 4, we will consider new candidates of metric groups within the proposed framework, including detailed examples in some relevant situations.

For other works that introduce some particular generalized metric motions see, for instance, the works of N.H. Ibragimov [45, 46], the ones of N. Muppinaiya [47], the ones of B.C. Xanthopoulos and K.E. Mastronikola [48]–[50], and the ones of Ll. Bel [51]. All of them can easily be included in the following framework.

3.1 Generalized metric motions

Our scope in this section is to formulate a proper definition of a generalization of isometries and conformal symmetries within a geometrical framework. Thus, we will first introduce some motivations and, eventually, will present our choice in definitions 3 and 4 —Sects. 3.2 and 3.3.
3.1.1 A remark concerning possible physical applications

An important issue in General Relativity is to find new solutions to Einstein’s field equations with a clear physical use. There are several ways to achieve that aim. An outstanding example are, e.g., Kerr-Schild relations [52] (see also I for a brief review).

Few years after the introduction of the Kerr-Schild relation, J.F. Plebański and A. Schild introduced a generalization. Their choice was [53]

\[ \tilde{g} = g + 2Fm \otimes m + G(\ell \otimes m + m \otimes \ell) + 2H\ell \otimes \ell, \]

where \( \tilde{g} \) is a rank-two symmetric tensor (in the case of having a Lorentzian signature it is to be identified with a metric tensor of a spacetime), \( g \) is a given metric tensor, \( \ell, m \) are two given null 1-forms and \( F, G, H \) are some functions. Its study has given some interesting results as e.g., regarding Kerr-Newman solutions and complex relativity (see also [54]). However, a detailed analysis still lacks, due in part to the complexity in the computations. Clearly, the GKS relation is obtained setting either \( F = G = 0 \) or \( G = H = 0 \).

On the other hand, in the early 90s, S. Bonanos [55] introduced another generalization of the Kerr-Schild relation. His choice was

\[ \check{g} = g - (p \otimes q + q \otimes p) + p^2 q \otimes q, \]

where \( \check{g} \) may always be interpreted as a metric tensor —the signature of \( g \) is that of \( g \) —, \( g \) is a given metric tensor, and \( p \) and \( q \) are two orthogonal 1-forms which may be spacelike, timelike or null. In that work some well known spacetimes were recovered and the formalism was adapted for studying vacuum solutions. However, computations are again difficult and its development still remains.

In these two examples, as well as in any further attempt, it is very important to have some knowledge on the wealth of new results. For that reason, the knowledge of the cases that give rise to an equivalent metric tensor are fundamental in order to avoid them and centre the computations in new solutions. We note that this can be posed as a problem of internal transformations, i.e., transformations of a metric tensor into itself. In this sense, some metric motions —to be defined later— become a useful tool, if not the natural one to ascertain which metric relations are redundant. Indeed, Kerr-Schild motions are a first example. That is, the knowledge of Kerr-Schild
motions of a given metric tensor informs us on those Kerr-Schild relations which do not yield in physical terms a new metric tensor. Let us now present other motivations for defining new metric motions.

3.1.2 Preliminary features on known and generalized metric motions

To begin, let us recall the two well known cases of metric motions, i.e., isometries and conformal ones. In both situations, using standard notation, one writes, respectively,

\[ \varphi^*_t(g) = g, \quad \varphi^*_t(g) = e^{2\Phi_t} g, \quad (20) \]

where \( \varphi^* \) stands for the pull-back application associated with the diffeomorphism \( \varphi(x), t \in (-a, a) \) for some appropriate \( a \in \mathbb{R} \), \( g \) is the metric tensor and \( \exp 2\Phi_t \) is the conformal factor.

Besides these two situations, it is not difficult to imagine other kind of actions over \( g \) —e.g., Kerr-Schild motions. Among possible generalizations, we are interested in those cases which admit a geometrical interpretation, i.e., those which are generated by the appearance of other geometrical objects besides \( g \) in the r.h.s. of Eqs. (20). Although this is not the only possibility, it follows a viewpoint which is close to the main line of thought used in the studies of other types of motions —e.g., in collineations. Moreover, it is an interesting choice, since it often happens that some geometrical structures have a direct connection with physical issues —recall the Kerr-Schild case and previous section.

Even though starting with Kerr-Schild motions would suffice to introduce the main features for the definition of generalized metric motions, we will present a slightly different situation which will help in noticing that a general framework for metric motions is indeed possible and that it is similar to that of the Kerr-Schild case. A possible form of a new action could be, for instance,

\[ \varphi^*_t(g) = g + \Psi_t \ u \otimes u, \quad \text{or} \quad \varphi^*_t(g) = e^{2\Phi_t} g + \Psi_t \ u \otimes u, \]

being \( u \) a 1-form field of the manifold. To fix ideas only, but without losing generality, let us consider that the manifold has a Lorentzian signature. Then \( u \) could be, for instance, a normalized timelike vector representing the four-velocity field of a fluid (obviously, in the case of a null 1-form field, one
would recover the case of Kerr-Schild motions). This new proposal is neither an isometry nor a conformal motion nor a Kerr-Schild motion. Therefore, it can be considered as a new problem of metric motions. Because this is an introduction, and for the sake of simplicity, we will just focus on the non-conformal attempt. Let us call it the “tt-like” motion in the sequel.

It is known that in any local 1-parameter group of local diffeomorphisms, it is to be satisfied that, whenever \( \varphi_t, \varphi_t', \varphi_{t+t'} \) are well defined,
\[
\varphi_t^* \circ \varphi_{t'}^* = \varphi_{t+t'}^*.
\]
Here \( \circ \) stands for the composition of applications. This property, once translated into our problem, is written as
\[
\varphi_{t'}^*(\varphi_t^*(g)) = \varphi_{t+t'}^*(g). \tag{21}
\]
Thus, one should impose in our example, that
\[
\varphi_{t'}^*(g + \Psi_t u \otimes u) = g + \Psi_{t+t'} u \otimes u. \tag{22}
\]
One may recall this basic property as the property of stability of the considered metric motion, since it ensures that a given metric transformation will belong to the same type regardless of the times it is applied. In the case of isometric and conformal motions, one readily recovers the usual result for them, namely, that there are no further consequences coming from Cond. (21).

Continuing with Eqs. (22), we write
\[
g + \Psi_{t'} u \otimes u + \Psi_t (\varphi_{t'}^*(x))[\varphi_{t'}^*(u) \otimes \varphi_{t'}^*(u)] = g + \Psi_{t+t'} u \otimes u.
\]
Whence, one deduces a necessary, and sufficient, law of transformation for \( u \), namely
\[
\varphi_t^*(u) = B_t u, \tag{23}
\]
being \( B_t \) a \( C^\infty \) function of \( t \). This is in fact analogous to the case of Kerr-Schild motions.

Thus, the \( tt \)-like motion in its initial form, i.e.,
\[
\varphi_t^*(g) = g + \Psi_t u \otimes u \tag{24}
\]
implies necessarily Eq. (23). Then, \( B_t \) may be re-interpreted as a new unknown of the complete problem. These remarks will prove to be basic in the sequel.

It is clear that in any further consideration regarding metric motions, one has to add new objects into the system besides the metric tensor itself. One possibility in order to define such general concept is to write down the expression of a particular action of a metric motion, in the general form,

\[
\varphi^* g \equiv Q,
\]

and impose now some restrictions on \( Q \) (if no restrictions on \( Q \) where considered, Eq. (25) would be nothing but an identity). For instance, one has \( Q = g \) in the case of isometries, \( Q = e^{2\Phi} g \) in the conformal case, \( Q = g + 2H \ell \otimes \ell \), for Kerr-Schild motions, and similarly for the \( tt \)-like motions — those should be viewed as guiding examples. The way one can restrict \( Q \) is varied (see [45, 47, 51]). It depends on the problem one wants to deal with. We have taken into account several situations, and peculiarities, coming from well known metric motions and collineations and their restrictions. Here we will focus only on the type of restrictions that can be expressed in terms of geometrical objects, such as \( \ell, u, g, \) etc.

### 3.2 Definition of a generalized metric motion

Bearing in mind the preliminary considerations of the previous section, we shall proceed with a suitable definition of these generalized metric motions. First, we must distinguish between different classes of metric motions.

**Definition 2 (Equivalence of metric motions)** *Two metric motions, \( Q_1 \) and \( Q_2 \), will be regarded as of the same class, or equivalent, whenever their components (in any given basis) satisfy the same restrictions, and this will be denoted by \( Q_1 \simeq Q_2 \).*

It is worth remarking that a precise definition for each possible case is meaningless, since one should list a lot of different new situations which may be described within a general framework. Our aim is half-way between the free-basis language of \( Q \), that does not give any specific equations, and the particular examples of e.g., Kerr-Schild motions, which are too specific since only one geometrical object is used. In fact, there is no single solution to
this problem. It is rather a matter of convention that depends on the degree of generality one is ready to allow. Thus, the following approach can only be evaluated from the preceding physical aims and the results of the forthcoming sections.

On the other hand, since the language of metric motions can be described in tensorial terms, one can consider a given cobasis, say \( \{ \Theta^\Omega \} (\Omega = 0, \ldots, n-1) \) in order to describe a *generalized metric motion*. It could be a natural tetrad, \( dx^\alpha \), but it is not always the most suitable choice, as it can be already noticed from Sect. 2 or from I, and Sect. 4.1 later. \(^4\) The cobasis formalism simply introduces an arbitrary cobasis in the representation of \( Q \). This, by itself, does not introduce anything new with respect to a general approach. But it will allow us to get the set of conditions of the Kerr-Schild, or \( tt \)-like examples, for a general combination of geometric objects in \( Q \).

In the case of isometric and conformal motions, this procedure is clearly unnecessary. However, for the rest, it is necessary. In most situations, one will have the bonus that the very fundamental objects will constitute part of a tetrad. For instance, Kerr-Schild motions are a very good example of this simplification, or, also, in the \( tt \) case, \( u \) may be chosen as an element of the tetrad. Consequently, in most situations, Sect. 4, the general framework will become quite natural.

We have also studied other possibilities, but they have proven to be equivalent to the formalism based on cobasis which, eventually, appears to us as the most direct one. Other approaches may be taken in problems with a different orientation (see e.g., [45]). We now introduce this formalism.

Let \( \{ \Theta^\Omega \}, \Omega = 0, \ldots, n-1 \), be any given cobasis of a \( n \)-dimensional Riemannian manifold, and let \( \varphi_t \) represent a given set of diffeomorphisms, with \( t \in (-a, a) \) with \( a \in \mathbb{R}, \varphi_{t=0} = Id \). Their action on the metric tensor, \( g \), can always be written as

\[
\varphi_t^* g \equiv Q_t = (Q_t)^{\Omega \Lambda} \Theta^\Omega \otimes \Theta^\Lambda,
\]

where \((Q_t)^{\Omega \Lambda}\) will be some \( C^\infty \) functions. However, in order to be able to compute the transformation of a transformation, one needs also to know the action of the given set of diffeomorphisms on the basis. This will be

\[
\varphi_t^* \Theta^\Omega = (M_t)^\Omega_{\Lambda} \Theta^\Lambda,
\]

\(^4\)The option of a natural cobasis will prove to be useful mainly when trying to integrate the differential equations of a particular metric group, see [45, 51] and also Ex. 3 later.
whith \((M_t)^\Omega_\Lambda\) some \(C^\infty\) functions. In fact (and this is our point) it is only necessary to give some generic restrictions on \(Q_t\). For instance, we have shown in the previous examples, including the two well-known of metric motions, that restricting \(Q_t\) to be proportional to e.g., \(g\), or to \(\ell \otimes \ell\), \ldots suffices to give a meaningful set of motions. These impositions cannot be made precise now because we want to leave an open window for other possible combinations. Nevertheless, one can actually impose that a specific type of metric motion is being considered with the aid of Def. 2. Then, we can give an expression for a set of diffeomorphisms to become a generalized metric motion, as follows.

**Definition 3 (Generalized Metric Motion, I)** A 1-parameter set of local transformations \(\{\varphi_t\}\) of \(V_n\), \(t \in (-a, a)\) for some appropriate \(a \in \mathbb{R}\), is called a generalized metric motion if and only if Eqs. (26), (27) satisfy

\[
(Q_{t+t'})_{\Omega \Lambda} := (Q_t[\varphi_t'(x)])_{\Omega \Sigma} (M_{t'})^\Pi_\Omega (M_{t'})^\Sigma_\Lambda \simeq (Q_t)_{\Omega \Lambda},
\]

where \(Q_t\) is some given class of tensors for any \(t, t', t + t' \in (-a, a)\).

Notice that in (28) Def. 2 is used.

Moreover, the functions \((M_t)^\Omega_\Lambda\) are linked with \((Q_t)_{\Omega \Lambda}\) through

\[
\varphi_t^*g = \varphi_t^*(g_{\Omega \Lambda} \Theta^\Omega \otimes \Theta^\Lambda) = (g[\varphi_t(x)])_{\Omega \Sigma} (M^\Sigma_\Omega (M^\Pi_\Lambda \Theta^\Omega \otimes \Theta^\Lambda}
\]

This gives an equation in order to obtain, partly, the expression of \((M_t)^\Omega_\Lambda\) for a given generalized metric motion. Indeed, the knowledge of \((M_t)^\Omega_\Lambda\), for all \(\Omega, \Lambda\), for a given class of metric motions is sufficient to fix the problem of a given generalized metric motion. In Sect. 4 a detailed implementation of this definition and remark will be carried out.

Since the transformations are continuous, it is possible to give the infinitesimal characterization of a generalized metric motion. This is the issue in the next section.

### 3.3 Differential version of a metric motion

The problem of a metric motion is generally much easier handled if one takes focuses on the differential equations which define the action of the whole motion.
A standard calculation, similar to the one performed in I for Kerr-Schild motions, shows that the differential version of expressions (26), (27) is

\[ L(\vec{\xi})g = q_\xi = (q_\xi)_{\Omega\Lambda} \Theta^\Omega \otimes \Theta^\Lambda, \quad L(\vec{\xi})\Theta^\Omega = m_\xi^\Omega = (m_\xi)^\Omega_{\Lambda} \Theta^\Lambda, \]

(29)

where \( \vec{\xi} \) is the differential generator associated with the differential action of the motion, and

\[ (q_\xi)_{\Omega\Lambda} \equiv \frac{d(Q_t)_{\Omega\Lambda}}{dt} \bigg|_{t=0}, \quad (m_\xi)^\Omega_{\Lambda} \equiv \frac{d(M_t)^\Omega_{\Lambda}}{dt} \bigg|_{t=0}. \]

Obviously, in order to have a vector space, the set \( \{(q_\xi)_{\Omega\Lambda}\} \) must be homogeneous, to include \( \vec{\xi} = \vec{0} \), and, moreover, their possible functional relationship among its elements must be, at most, linear. Moreover, since \( Q_t - t \in (-a, a) \) must have some restrictions, as remarked in the previous section, so must \( q_\xi \) too. That is, Eqs. (29) should be understood as meaningful for some restricted class of tensors \( q_\xi \). One thus has (Def. 2 for equivalent classes of motions is used)

**Definition 4 (Vector fields of a generalized metric motion)** Let \( \{\Theta^\Omega\} \), \( \Omega = 0, \ldots, n-1 \), be any given cobasis of \( V_n \). The solutions to the system of equations

\[ L(\vec{\xi})g = q_\xi = (q_\xi)_{\Omega\Lambda} \Theta^\Omega \otimes \Theta^\Lambda, \]

\[ L(\vec{\xi})\Theta^\Omega = m_\xi^\Omega = (m_\xi)^\Omega_{\Lambda} \Theta^\Lambda, \]

(30)

will be considered to be the vector fields of a given generalized metric motion if and only if they satisfy

\[ q_{\xi\xi} \equiv L(\vec{\xi})q_\xi = [\xi^\Lambda \partial_\Lambda (q_\xi)_{\Omega\Lambda} + 2(q_\xi)_{\Omega\Sigma}(m_\xi)^\Sigma_{\Lambda}] \Theta^\Omega \otimes \Theta^\Lambda \simeq q_\xi. \]

(32)

where \( q_\xi \) is a given class of rank-two symmetric tensor fields with \( (q_\xi)_{\Omega\Lambda} \) homogeneous functions, which are related with each other at most by linear relations.

In Eq. (32) we have made use of the fact that \( (q_\xi)_{\Omega\Lambda} = (q_\xi)_{\Lambda\Omega} \).

The previous definition could be summarized by saying that \( \vec{\xi} \) is a vector field of a metric motion if and only if \( L(\vec{\xi})g \) and \( (L(\vec{\xi}))^2g \) are of the same type.

\(^5\)The —rank-two symmetric— tensor \( q \) has not to be confused with the 1-form \( q \).
—according to Def. 2. The remarks given after Def. 3 are also valid now, when taking the differential action of the metric group. The inclusion of a cobasis is again secondary, but it gives within our aim a closer definition to any future practical problem, which departs from the isometric or conformal motions.

Let us see how the preceding definition works with some examples, although it will become apparent after having considered more general cases in the following sections. For instance, in the case of isometric motions no need for a cobasis description appears and $q_\xi = q_{\xi\xi} = 0$. Therefore, the system of equations in Def. 4 turns out to be simply Killing’s equations (see [56, 28, 29]).

In the case of conformal motions, one has $q_\xi = 2\phi_\xi g$ and $q_{\xi\xi} = 2(\mathcal{L}(\xi)\phi_\xi + 2\phi_\xi^2)g$ so that $q_{\xi\xi} \simeq q_\xi$.

Another example is that of Kerr-Schild motions. In this case $q_\xi = 2h_\xi \ell \otimes \ell$ and $q_{\xi\xi} = 2(\mathcal{L}(\xi)h_\xi + 2m_\xi h_\xi)\ell \otimes \ell$ if and only if $\mathcal{L}(\xi)\ell = m_\xi \ell$ which represent the necessary contribution of Eqs. (31). Thus, $q_{\xi\xi} \simeq q_\xi$ so that Kerr-Schild motions verify Def. 4.

On the contrary, we will show that double Kerr-Schild motions, defined by $q_\xi = 2f_\xi m \otimes m + 2h_\xi \ell \otimes \ell$, where $m$ is a null 1-form field satisfying $m \cdot \ell = -1$, do not give rise to a new metric motion in the sense of Defs. 3 or 4. This also shows that Defs. 3 or 4 are indeed half-way between a too general treatment of metric groups and a too specific treatment valid only for a few examples. The aforementioned impossibility of double Kerr-Schild motions is proven in Sect. 4.

Now we will check whether, within our generality, each vector field fulfilling Def. 4 gives rise to a local flow on the metric that belongs to Def. 3, thereby completing the general framework. This was clear in the examples of metric motion before.

**Lemma 3.1** $(\mathcal{L}(\xi))^{(n)}g \simeq \mathcal{L}(\xi)g$, for all $n \geq 1$.

**Proof:** From Def. 4, the vector field $\xi$ must satisfy:

$$(\mathcal{L}(\xi))^2g \simeq \mathcal{L}(\xi)g.$$

Whence, by induction, one has

$$(\mathcal{L}(\xi))^{(n)}g \simeq (\mathcal{L}(\xi))^{(n-1)}g$$
for all \( n \geq 2 \). Or, equivalently,

\[
(L(\xi))^{(n)} g \simeq L(\xi) g,
\]
as it was claimed.

This can also be shown by noticing, at this stage, one is not interested in a precise expression of each action. For instance, in Def. 4, we require \( L(\xi) q_\xi \) and \( q_\xi \) to be equivalent, in the sense of Def. 2. This allows to extend the equivalence to any further order.

**Proposition 3.1** Any vector field \( \vec{\xi} \) belonging to Def. 4 gives rise to a local 1-parameter group of local metric transformations belonging to Def. 3.

**Proof:** For any particular \( \vec{\xi} \) one can construct the following transformation

\[
\varphi^*_t g \equiv g + (L(\xi) g) t + \cdots + \frac{1}{n!} [(L(\xi))^n g] t^n + \cdots
\]
where \( t \in (-a, a) \) for some appropriate \( a \in \mathbb{R} \). Because \( t \) is only a parameter, from the preceding lemma, one has

\[
[(L(\xi))^{(n)} g] t^n \simeq [L(\xi) g] t.
\]
Whence, for any \( t \in (-a, a) - \{0\} \), one gets:

\[
\varphi^*_t g - g \simeq q_\xi = L(\xi) g,
\]
which tells that \( \varphi^*_t g \) is of a definite type for all \( t \in (-a, a) , t \neq 0 \).

Consequently, \( \varphi^*_{t''} g \simeq \varphi^*_t g \simeq \varphi^*_t g \) for any triad \( t'' , t' , t \), where \( \varphi^*_t , \varphi^*_t \) and \( \varphi^*_t \) are well defined. In particular, for \( t'' = t + t' \). Therefore, \( \varphi^*_t g \) constructed from the vector fields \( \vec{\xi} \) give rise to a local 1-parameter group of local metric motions belonging to Def. 3.

A typical goal when dealing with systems of differential equations is to obtain a normal system for the unknowns and for each particular metric group. Sometimes (almost always, as we will show) this will be impossible. The system of Eqs. (30)–(32) turns out to be open in most cases and infinite dimensional algebras appear, see Sects. 2, 4 later, and also I. Therefore, one can only write it as close as possible to the claimed goal. We give the expressions in App. B.
4 Metric motions defined by different substructures

Hitherto isometric, conformal and Kerr-Schild motions are the only metric motions considered. After the introduction of Kerr-Schild motions and its study, it would be logical to try to extend the number of candidates for metric motions. The first natural path to generalize the three previous cases of metric groups, in the spirit of Sects. 3.1.1, 3.1.2, is to consider some combinations of the most common geometrical objects, e.g., vector fields (or either 1-forms). Therefore, our aim here is mainly to implement Defs. 3, 4 and show how they work. Yet we will spend some time to include the most remarkable features of the candidates, specially in Sect. 4.1.

For the sake of brevity, and due to their possible physical applications, the study is centered on four-dimensional spacetimes, though many of the examples are also valid—or can easily be extended—to other dimensions and signatures of the manifold, provided these are compatible with the existence of each geometrical element used.

First, backed by the results of Kerr-Schild motions, we shall try to extend its study in order to include another (real-valued) null 1-form, say $m$ into the scheme (in this case, obviously, the metric tensor cannot have an Euclidean signature). The two 1-forms must satisfy $\ell \wedge m \neq 0$, so that each 1-form is independent from the other. Without loss of generality both null 1-forms can be taken to satisfy the orthogonality condition $\ell \cdot m = -1$ (signature $+2$). In this sense one would obtain the usual expressions for “advanced” and “retarded” null 1-forms in any spacetime. In case they are geodesic, the algebras can also be understood as the Lie algebras generated by the geometrical structure of the light cone, a point worth to be noticed. Moreover, two such null vectors can be taken as elements of any cobasis, and their study covers, in some sense, half of the cases. This is the set developed in Sect. 4.1.

Second, the other half is no longer expressible in terms of other real-valued null 1-forms. In order to complete the basis of the spacetime, one should include either complex valued null 1-forms, or choose two spacelike 1-forms, say $p, q$, satisfying $p \wedge q \wedge \ell \wedge m \neq 0$. Hence our efforts after Sects. 4.1, 4.2, are centred in the study of candidates for metric groups generated by this latter pair (this section could be adapted to Euclidean signature). The case of metric groups that are generated by the combination of a timelike and a
spacelike 1-form are indeed equivalent to Sect. 4.1. However it is worth giving
the solution, at least, in App. C. The reason is to be found in the literature.
In Sect. 3.1.1 it turned out that in aiming to find new Ansätze from 1-form
fields, different to the GKS case, two major approaches have been taken in
the literature. One of them is explicitly based in null 1-forms, whilst the other
allows to include spacelike and timelike. Let us recall that in our case they
are internal groups of transformations and our results complement, in fact,
those approaches in an analogous way as Kerr-Schild motions supplement
a new feature to the GKS Ansatz. Further work has to be done in this
direction before one could centre the study in one of the two. Therefore it
may be useful to have both expressions of the algebras in a summary, one for
each line of work (moreover, App. C will serve as well as a test of consistence
for the whole scheme). All these combinations clearly represent the seeds for
any other more general option and appear to be the simplest ones concerning
their geometrical structure, as stressed also in Sect. 5.

4.1 Metric motions generated by \( \ell - m \)

Let \( \ell \) and \( m \) be two given real null 1-form fields satisfying

\[
\ell \cdot \ell = m \cdot m = 0, \quad \ell \cdot m = -1.
\]

The differential expression of a metric motion generated by these two
ingredients would be\(^6\)

\[
q = \mathcal{L}(\xi)g = 2h\ell \otimes \ell + c(\ell \otimes m + m \otimes \ell) + 2fm \otimes m.
\]  

(33)

In this scheme \( g, \ell, \) and \( m \) may be regarded as data. In fact, since \( \ell \) and
\( m \) are null, only their direction is a necessary datum, see also Sect. 2. On
the other side, \( h, f, c \) are unknown \( C^\infty \) functions of the manifold yet to be

\(^6\)A study based upon the finite action of each motion is actually possible. However the
calculations are longer and more cumbersome, and they do not contribute, generally, to a
better understanding of the basic features of each case. In the sequel, as no confusion can
arise, we avoid writing the subindex \( \xi \) in order to clarify the notation. Moreover, in all the
cases considered in this work it is quite easy to show that the set of solutions to each type
of generalized metric motion gives rise to a vector space, and furthermore to Lie algebras.
the steps are very similar to those of Kerr-Schild motions, see I. Therefore, we omit their
proofs here.
determined by the system of equations themselves, if possible. And \( \xi \) are the differential generators of the metric motion, which constitute the main unknowns of the problem.

Taking \( \{ \ell, m \} \) as a part of the cobasis of spacetime, the previous expression tells us that the relations among the \( q_{\Omega \Lambda} \) are rather simple: free weights in the portion expanded by \( \ell \) and \( m \), and zero-valued weights for the rest of the elements that complete the cobasis.

Following Sect. 3.1, the next step is to calculate the expression of \( \mathcal{L}(\xi)(\text{cobasis}) \). Whence, one will get the internal freedoms coming from the local Lorentz character of the manifold. After that step, one should check the stability property, in order to assure that one deals with a coherent candidate for metric motion. The cobasis will be completed with the addition of two spacelike 1-forms, \( p \) and \( q \), satisfying \( p \cdot p = q \cdot q = 1, \quad p \cdot \ell = p \cdot m = q \cdot \ell = q \cdot m = p \cdot q = 0 \), but being otherwise arbitrary.

Since we are concerned with metric motions exclusively generated by the pair \( \ell - m \), the rest of the cobasis only adds an isometric action to the scheme. Therefore, the general scheme could be partially restricted to the action upon the pair \( (\ell, m) \) if one would like to focus on particular solutions. Here, we shall develop the full method in order to clarify the previous sections as much as possible. These will be the actual steps and also for the following sections. We summarize the results of this section in (A' stands for \( \mathcal{L}(\xi)A \))

**Proposition 4.1 (Metric motions generated by two null 1-forms)** The conditions in order to have a generalized metric motion generated by two given null 1-form fields, \( \ell \) and \( m \), i.e., Eqs. (33), are

\[
\begin{align*}
\{ \ell' &= \alpha_0 \ell - f m + \alpha_1 p + \alpha_2 q, \quad m' = -h \ell - (c + \alpha_0) m + \beta_1 p + \beta_2 q, \quad (34) \\
p' &= \beta_1 \ell + \alpha_1 m + \gamma_1 q, \quad q' = \beta_2 \ell + \alpha_2 m - \gamma_1 p,
\end{align*}
\]

with

\[
\begin{align*}
2h\alpha_1 + c\beta_1 &= 0, \quad 2f\beta_1 + c\alpha_1 = 0, \\
2h\alpha_2 + c\beta_2 &= 0, \quad 2f\beta_2 + c\alpha_2 = 0, \quad (35)
\end{align*}
\]

and

\[
\mathcal{L}(\xi)^2 g = 2h' \ell \otimes \ell + \tilde{c}(\ell \otimes m + m \otimes \ell) + 2\tilde{f} m \otimes m,
\]

\[
\begin{align*}
\tilde{h} &= h' + 2h\alpha_0 - hc, \\
\tilde{f} &= f' - 2f\alpha_0 - 3fc, \\
\tilde{c} &= c' - 4hf - c^2. \quad (36)
\end{align*}
\]
The volume element, \( \eta \equiv -\ell \wedge m \wedge p \wedge q \), transforms according to

\[ \eta' = -c \eta. \]  

(37)

In these expressions, \( \{ \ell, m, q, p \} \) is any semi-null cobasis of the manifold and \( \{ \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1 \} \) are \( C^\infty \) functions (we will study their freedom below).

Proof: Under any differential action of any metric motion, one can write

\[
\begin{align*}
\mathcal{L}(\xi)\ell &= A_1 \ell + A_2 m + A_3 p + A_4 q, \\
\mathcal{L}(\xi)m &= B_1 \ell + B_2 m + B_3 p + B_4 q, \\
\mathcal{L}(\xi)p &= C_1 \ell + C_2 m + C_3 p + C_4 q, \\
\mathcal{L}(\xi)q &= D_1 \ell + D_2 m + D_3 p + D_4 q,
\end{align*}
\]

where the set of functions \( A_\alpha, B_\alpha, C_\alpha, D_\alpha \) is, for the moment, arbitrary. On the other hand, \( g \) may be written as

\[ g = -(\ell \otimes m + m \otimes \ell) + p \otimes q + q \otimes p, \]

because \( \{ \ell, m, q, p \} \) is a semi-null cobasis. Then, calculating its Lie derivative and imposing Eqs. (33), one gets

\[
\begin{align*}
\mathcal{L}(\xi)\ell &= A_1 \ell - f m + A_3 p + A_4 q, \\
\mathcal{L}(\xi)m &= -h \ell - (c + A_1) m + B_3 p + B_4 q, \\
\mathcal{L}(\xi)p &= B_3 \ell + A_3 m + C_4 q, \\
\mathcal{L}(\xi)q &= B_4 \ell + A_4 m - C_4 p.
\end{align*}
\]

(38)

This result assures the fulfillment of Eqs. (30)–(31). The remaining ones, Eqs. (32), contain the last conditions to be imposed. They assure that \( g'' \approx g' = q \) for \( q = 2h \ell \otimes \ell + c(\ell \otimes m + m \otimes \ell) + 2f m \otimes m \). In operative terms, it amounts to imposing \( g'' = 2\tilde{h} \ell \otimes \ell + \tilde{c}(\ell \otimes m + m \otimes \ell) + 2\tilde{f} m \otimes m \), where the weights with a tilde represent a new set. With the aid of Expr. (38), one can then calculate \( g'' \) and after imposing Eqs. (31), one readily gets conditions (34)–(36) (where we have reordered the names of the remaining functions).

Finally, it is now straightforward to calculate \( \mathcal{L}(\xi)\eta \), being \( \eta \) the volume element of the manifold. The result is Eq. (37).

The study of Eqs. (33) should not be restricted to its general expression. Equally important are the subcases associated with further restrictions on the weights \( h, f, c \) (see table 1 and Prop. 4.2 below). Therefore, we shall develop a study for all the possibilities. In particular, the homogeneous case \( h = f = c = 0 \) lets \( \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1 \) free, which correspond to the generators of the the local Lorentz algebra, as is expected.
Table 1: Generalized metric motions generated by \((\ell, m)\). Note the presence of extra degrees of freedom in all the cases that yield a metric motion. The non-existence of double Kerr-Schild motions is a remarkable result. \(a\) is a constant under the action of the motion \(V_2\). (See the text for a detailed discussion.)

| \(q\)                                | Case | Extra freedoms |
|---------------------------------------|------|----------------|
| \(2h\ell \otimes \ell\)              | \(I_a\) | 4              |
| \(2f m \otimes m\)                   | \(I_b\) | 4              |
| \(2h\ell \otimes \ell + 2f m \otimes m\) | \(II\) | \(\frac{3}{2}\) |
| \(c(\ell \otimes m + m \otimes \ell)\) | \(III\) | 2              |
| \(2h\ell \otimes \ell + 2c(\ell \otimes m + m \otimes \ell)\) | \(IV_a\) | 2              |
| \(2f m \otimes m + 2c(\ell \otimes m + m \otimes \ell)\) | \(IV_b\) | 2              |
| \(2h\ell \otimes \ell + 2f m \otimes m + c(\ell \otimes m + m \otimes \ell)\) | \(V_1: c^2 \neq 4hf\) | \(V_1: 2\) |
| \(V_2: h = a^2f, c = \pm 2af\)       | \(V_2: 3\) |                |

Whereas the isometric motions can be completely recovered from the conformal ones, without losing generality, so that one can say that isometries are a subcase of conformal motions, it is no more valid in other general situations. Actually, one has, from Prop. 4.1 (see also the following sections)

**Proposition 4.2** Isometries and the cases I and \(V_2\) are not subcases of the general solution, case \(V_1\).

This proposition stresses the fact that the vector fields of these three subcases cannot be recovered from the solution of the most general case by imposing on the general case, \(V\), their particular conditions, e.g., \(I_{(h \neq 0)} \neq V_1(f = c = 0)\), or \(II_{f = 0} \neq I_a\). Moreover, these examples clearly show the role played by the form —class— of \(q\). Fig. 1 shows the interrelation among the different cases. Prop. 4.2 is proven in the following.

**Kerr-Schild motions.**

Kerr-Schild motions, cases \(I_a\) and \(I_b\) in table 1, correspond to setting \(c\) equal to zero and either \(h\) or \(f\) to be zero (clearly both situations are equivalent).
Figure 1: Interconnection among Lie algebras generated by two null 1-form fields. Each level represents an independent algebra. Each arrow leads to a subalgebra. $A_{B\ldots}$ means the restriction of cases $B, \ldots$ to the conditions of case $A$. The last level are isometries. (See the text and table 1 for details.)

The result is ($h \neq 0$)

$$\begin{align*}
\ell' &= \alpha_0 \ell, \\
m' &= -h \ell - \alpha_0 m + \beta_1 p + \beta_2 q, \\
p' &= \beta_1 \ell + \gamma_1 q, \\
q' &= \beta_2 \ell - \gamma_1 p, \\
\tilde{h} &= h' + 2h\alpha_0, \\
\eta' &= 0.
\end{align*}$$

There are four freedoms, namely $\alpha_0, \beta_1, \beta_2, \text{ and } \gamma_1$, which represent a freedom in the parametrization of the null curves, defined by $\ell$, two boosts, in the plane $\ell \cdot p$ and $\ell \cdot q$, and a rotation in the “orthogonal” plane $p \cdot q$. The freedom in $h$ is not constant, but is actually a functional freedom, see e.g., Sect.2.3 or Ex. 3 later on. This was a new result for metric symmetries. It implies, for instance, that $h$ cannot always be isolated in terms of the data, the metric tensor, the direction of $\ell$, and $\xi_\alpha, \xi_{\alpha\beta}, \ldots$ and tells us that the system is open (recall Prop. 1).

**Double Kerr-Schild motions.**

This case provides a very good example of how some forms of $q$ do not give rise to a new metric motion.

In this case one has to impose $c = 0$ and $fh \neq 0$. However it turns out that condition (36) imposes $hf = 0$ and, therefore, one gets a contradiction. But $c = hf = 0$ leads us to ordinary Kerr-Schild groups! We thus get the announced result

**Proposition 4.3** No proper double Kerr-Schild motions exist.

This is the first example of a $Q$ family that does not yield a new metric group. Moreover, it shows that even though it comes from the known double
Kerr-Schild relation, a direct generalization of its predecessor, i.e., Kerr-Schild relations, it completely departs from it as internal transformations. Another consequence is that any double Kerr-Schild relation gives rise to new — non-equivalent — metric tensors, in the sense described in Sect. 3.1.1.

Again, the study above can be carried out working entirely with the finite action of the motion. This study is generally more tedious. The procedure is rather similar as before. Starting from the decomposition of $g$ in the semi-null cobasis, one finds the general expressions of $\varphi_i^*(\text{cobasis})$ that fulfill the equation $\varphi_i^*(g) = g + 2H_i \ell \otimes \ell + 2F_i m \otimes m$. Then one imposes Eqs. (21) taking into account the expressions obtained for $\varphi_i^*(\ell)$ and $\varphi_i^*(m)$. Whence, one obtains a condition in order to eliminate the crossed term $\ell \otimes m + m \otimes \ell$. Eventually, one arrives at the same conclusion as before. Each of the finite steps can be translated into the infinitesimal version to see how each method works. However, for the sake of brevity, we will not write down here all these expressions.

**Semi-conformal motions.**

This new case is obtained setting $h = f = 0$. The name semi-conformal is used to mean that half of the cobasis is not used, so that one cannot talk of $c(\ell \otimes m + m \otimes \ell)$ as the usual conformal factor. Instead it should be considered as a conformal problem restricted to the subspace expanded by the pair $\ell - m$ only. This point of view also points out to how generalized metric motions allow new interesting restrictions of the typical conformal symmetry, without reducing the dimension of the manifold, yet maintaining at the same time the whole set of elements of a manifold. This feature will be considered elsewhere. Ex. 3 will show another interesting aspect connected with this motion.

In this case the conditions reduce to

\[
\begin{align*}
\ell' &= \alpha_0 \ell, \\
m' &= -(c + \alpha_0)m, \\
p' &= \gamma_1 q, \\
q' &= -\gamma_1 p, \\
\tilde{c}' &= c' - c^2, \\
\eta' &= -c \eta.
\end{align*}
\]

Only $\alpha_0$ and $\gamma_1$ are freedoms of the system. Their interpretation has been given elsewhere. Again there are functional freedoms, as will be seen in Ex. 3.

**Semi-conformal Kerr-Schild motions.**

Cases $IV_a$ and $IV_b$ are clearly equivalent. The remarks of the former section are again valid here. This case corresponds to setting either $h = 0$ or $f = 0$. 

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For the case $f = 0$, $hc \neq 0$, Eqs. (34)-(37) translate into

$$\begin{align*}
\ell' &= \alpha_0 \ell, \\
m' &= -h \ell - (c + \alpha_0) m, \\
p' &= \gamma_1 q, \\
q' &= \gamma_1 p, \\
\tilde{h} &= h' + 2h \alpha_0 - hc, \\
\tilde{c} &= c' - c^2, \\
\eta' &= -c \eta.
\end{align*}$$

The other situation, i.e., $h = 0$, $fc \neq 0$ is analogous, changing $h$ by $f$.

**Semi-conformal double Kerr-Schild motions.**

This last situation is the most general one within our scheme. Their conditions are the ones displayed in Prop. 4.1. Before proceeding with their analysis, we would like to introduce these motions from a more intuitive point of view that trivializes some of their consequences.

Since $g = -(\ell \otimes m + m \otimes \ell) + g_\perp$, where $g_\perp$ is the reduction of the metric tensor to the rest of the cobasis, Eqs. (33) can be written as $L(\vec{\xi})(\ell \otimes m + m \otimes \ell) = L(\vec{\xi})g_\perp + 2h \ell \otimes \ell + 2\tilde{c}(\ell \otimes m + m \otimes \ell) + 2\tilde{f} m \otimes m$, where $\tilde{f}$, $\tilde{c}$, $\tilde{h}$ are respectively $-f$, $-c$, $-h$. Let us further assume that $L(\vec{\xi})g_\perp$ is zero, i.e., that $\vec{\xi}$ are isometries of the complementary subspace. This last condition is not trivial, but it is not difficult to find many spacetimes fulfilling it (e.g., decomposable spacetimes). Then, the restricted problem reduces almost to an identity. Therefore, the equations are almost equivalent to a trivial problem on metric groups in the Riemannian submanifold generated by the pair $\ell$-$m$, and a great variety of solutions are expected.

Despite these conclusions, let us carefully analyze conditions (35). One must distinguish between two possibilities: $c^2 \neq 4hf$, case $V_1$, and $c^2 = 4hf$, case $V_2$.

In case $V_1$, one obtains $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$. The complete solution is

$$\begin{align*}
\ell' &= \alpha_0 \ell - f m, \\
m' &= -h \ell - (c + \alpha_0) m, \\
p' &= \gamma_1 q, \\
q' &= -\gamma_1 p,
\end{align*}$$

and, where $\tilde{h}$, $\tilde{f}$, $\tilde{c}$ and $\eta'$ are given in Eqs. (38)-(37).

The second case, $V_2$, is interesting because, apparently, it introduces a non-linear relation among the components of $q$. However, one knows that they must be linear in order to form a vector space. For instance, imposing $(c_{A+B})^2 = 4(h_{A+B})(f_{A+B})$, where $c_{A+B} = c_A + c_B$, $h_{A+B} = h_A + h_B$, $f_{A+B} = f_A + f_B$, i.e., the weights of $\vec{\xi}_{A+B} \equiv \vec{\xi}_A + \vec{\xi}_B$ —recall that, by hypothesis, $c_A^2 = 4h_A f_A$, $c_B^2 = 4h_B f_B$—, one gets the result $h_A / f_A = h_B / f_B$ for any $A$, $B$. Therefore, $h/f = \text{const.}$ If $h/f = \text{const.}$, and $c^2 = 4hf$, one gets either
where $h = a^2 f$, $c = \pm 2af$, where $a$ is a constant, or $f = b^2 h$, $c = \pm 2bh$, where $b$ is a constant. Both options are equivalent except when $a = 0$ or $b = 0$. Despite this possibility, the results of any case are analogous to each other. Therefore, we will consider in the sequel the first situation, i.e., $f$ free. Notice that “const.” stands here for constant under the action of the motion. Thus, it can be a function built upon invariants of the motion. Continuing, $q$ may be reordered to be expressed as

$$q = (\mp 4af) r_\pm \otimes r_\pm,$$

where $r_\pm \equiv (\sqrt{a/2})(\ell \pm m/a)$, with $r_\pm \cdot r_\pm = \mp 1$ and $r_\pm \cdot r_\mp = 0$. $\alpha_1$, $\beta_1$ are now free. This algebra formed by two spacelike, timelike, 1-forms will be delayed until we present the Lie algebras generated by $p-q$ and $u-n$, respectively.

We shall now give the first explicit solutions to the previous algebras. We shall deal with flat spacetime ($n = 4$) and $\ell$, $m$ covariantly constant. Despite the apparent simplicity, it will display all the properties of the algebras, as, e.g., the structure given in Fig. 1. We recall that the two independent covariantly constant null 1-forms of flat spacetime can be chosen, without losing generality, as the “advanced” and “retarded” parallel null 1-forms. Thus, the algebras can be considered as generated by the geometrical structure of the light cone of flat spacetime. In Cartesian coordinates, one has $ds^2 = -dt^2 + dz^2 + dx^2 + dy^2$, and $\ell$ and $m$ may be written as $\ell = du$, $m = dv$, where $u \equiv (1/\sqrt{2})(t - z)$, $v \equiv (1/\sqrt{2})(t + z)$.

**Example 3** In flat spacetime and for the advanced and retarded null 1-forms, the vector fields of the metric motions generated by them are-

1. For the case $V_1$,

$$\bar{\xi} = A\partial_u + B\partial_v + (c_0 y + d)\partial_x + (-c_0 x + e)\partial_y,$$

where $A$, $B$ are arbitrary $C^\infty$ functions of $u$, $v$, and $c_0$, $d$, $e$ are constants. Moreover,

$$h = -\partial_u B, \ f = -\partial_v A, \ g = -(\partial_v B + \partial_u A), \ \alpha_0 = \partial_u A, \ \gamma_1 = c_0.$$  

---

7Although the generalization to an $n$-dimensional flat metric is straightforward, it is not essential in the present discourse.
2. For cases IVa, IVb and III, the previous one restricted to each situation.

3. For the case V2,

\[ \vec{\xi} = A\partial_{s_{\pm}} + (b_1 x + b_2 y)\partial_{s_{+}} + k\partial_v + (c_0 y + b_1 s_+ + d_1)\partial_x 
+ (-c_0 x + b_2 s_+ + d_2)\partial_y, \] (41)

where \( A \) is an arbitrary \( C^\infty \) function of \( s_{\pm} \equiv v \pm au \), with \( a \neq 0 \) a constant, \( s_{\pm} \equiv u \pm av \), and \( b_1, b_2, k, c_0, d_1, d_2 \) are constants. Moreover,

\[ f = -\dot{A}, \quad \alpha_1 = b_1, \quad \alpha_2 = b_2, \] (42)

where \( \dot{}() \) means total derivation with respect to \( s_{\pm} \), and, recall, \( h = a^2 f, c = \pm 2af, \alpha_0 = \mp af \).

4. For the case Ia (Ib is analogous),

\[ \vec{\xi} = A\partial_u + (-\dot{A}v + B + x\dot{C} + y\dot{D})\partial_v + (dy + C)\partial_x + (-dx + D)\partial_y \] (43)

where \( A, B, C, D \) are arbitrary \( C^\infty \) functions of \( u \), \( \dot{}() \) means derivation with respect to \( u \), and \( d \) is a constant. Moreover,

\[ h = \dot{A}v - B - x\dot{C} - y\dot{D}, \quad \alpha_0 = \dot{A}, \quad \beta_1 = \dot{C}, \quad \beta_2 = \dot{D}, \gamma_1 = d. \] (44)

5. Notice that, in any case, setting \( h = f = c = 0 \), one obtains particular sets of restricted isometries.

All the Lie algebras, except the latter, are infinite dimensional.

**Proof:** The line element of a 4-dimensional flat spacetime can be written as

\[ ds^2 = -2 du dv + dx^2 + dy^2. \] (45)

The two parallel null directions are, without lost of generality, \( \ell = du \) and \( m = dv \). First we must solve

\[ \mathcal{L}(\vec{\xi})\eta = 2h\ell \otimes \ell + 2f m \otimes m + c(\ell \otimes m + m \otimes \ell), \] (46)

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where $\eta$ is the flat metric. After this, we will impose the conditions coming from Eqs. (31), i.e., $(\mathcal{L}(\tilde{\xi}))^2 g \simeq \mathcal{L}(\tilde{\xi}) g$, or stability conditions.

Consider a generic vector field $\tilde{\xi}$,

$$\tilde{\xi} = \xi^\lambda \partial_\lambda,$$

(47)

where $\lambda = u, v, x$ and $y$. Eqs. (46) may be written as

$$\xi^\lambda \partial_\lambda \eta_{\alpha\beta} + \eta_{\alpha\lambda} \partial_\beta \xi^\lambda + \eta_{\beta\lambda} \partial_\alpha \xi^\lambda = 2h \delta^u_\alpha \delta^u_\beta + 2f \delta^v_\alpha \delta^v_\beta + c (\delta^u_\alpha \delta^v_\beta + \delta^v_\alpha \delta^u_\beta).$$

Whence, using (45) and (47), one gets

$$h = -\partial \xi^u,$$

$$f = -\partial \xi^v,$$

$$c = -(\partial \xi^u + \partial \xi^v),$$

(48)

$$\partial \xi^x = \partial_x \xi^u,$$

$$\partial \xi^y = \partial_y \xi^v,$$

(49)

$$\partial \xi^x = \partial_x \xi^u,$$

$$\partial \xi^y = \partial_y \xi^v,$$

(50)

$$\partial_x \xi^x = \partial_y \xi^y = \partial_x \xi^u + \partial_y \xi^v = 0.$$

(52)

From Eqs. (52) one readily gets

$$\xi^x = c_0 y + F_1(u, v), \quad \xi^u = -c_0 x + F_2(u, v),$$

where $c_0$ is a constant and $F_1$, $F_2$ are $C^\infty$ functions of their arguments. Substituting this result in Eqs. (48)-(51), and using Eqs. (52), one gets

$$\xi^u = [F_3(u, v) + x \partial_u F_1(u, v) + y \partial_v F_2(u, v)],$$

$$\xi^v = [F_4(u, v) + x \partial_u F_1(u, v) + y \partial_v F_2(u, v)],$$

where $F_3$, $F_4$ are $C^\infty$ functions of their arguments. Collecting all results, we obtain the general solution for Eqs. (46)

$$\tilde{\xi} = (F_3 + x \partial_v F_1 + y \partial_v F_2) \partial_u + (F_4 + x \partial_u F_1 + y \partial_u F_2) \partial_v + (c_0 y + F_1) \partial_x + (-c_0 x + F_2) \partial_y,$$

(53)

$$h = -(\partial_u F_4 + x \partial_{uv} F_1 + y \partial_{uv} F_2),$$

(54)

$$f = -(\partial_v F_3 + x \partial_{vu} F_1 + y \partial_{vu} F_2),$$

(55)

$$c = -(\partial_u F_3 + \partial_v F_4 + 2x \partial_{uv} F_1 + 2y \partial_{uv} F_2).$$

(56)
The next step consists in imposing \((\mathcal{L}(\xi))^2 g \simeq \mathcal{L}(\xi) g\). This is collected in Conds. (34), (35) of Prop. 4.1.

In order to do this, we first calculate \(\mathcal{L}(\xi) \ell, \mathcal{L}(\xi) m, \mathcal{L}(\xi) p\) and \(\mathcal{L}(\xi) q\) for the general solution. We will take, without losing generality, \(p = dx, q = dy\). The result is,

\[
\begin{align*}
\mathcal{L}(\xi) \ell &= \alpha_0 \ell - f m + \alpha_1 p + \alpha_2 q, \\
\mathcal{L}(\xi) m &= -h \ell - (c + \alpha_0) m + \beta_1 p + \beta_2 q, \\
\mathcal{L}(\xi) p &= \beta_1 \ell + \alpha_1 m + c_0 q, \\
\mathcal{L}(\xi) q &= \beta_2 \ell + \alpha_2 m - c_0 p,
\end{align*}
\]

where

\[
\begin{align*}
\alpha_0 &\equiv \partial_u F_3 + x \partial_{uv} F_1 + y \partial_{uv} F_2, & \alpha_1 &= \partial_u F_1, \\
\alpha_2 &= \partial_u F_2, & \beta_1 &= \partial_u F_1, & \beta_2 &= \partial_u F_2,
\end{align*}
\]

and \(f, c, h\) are given in Eqs. (54)–(56). The imposition of Eqs. (35) clearly depends on each case.

1. For the case \(V_1\), where \(c^2 \neq 4hf\), \(\alpha_1, \alpha_2, \beta_1\) and \(\beta_2\) must vanish. This yields \(F_1 = d = \text{const.}\) and \(F_2 = e = \text{const.}\). Consequently, one obtains Eqs. (39)–(40).

2. For the cases \(IV_a, IV_b\) and \(III\), Eqs. (35) yield the same restrictions as before, \(\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0\). Moreover, there is one further restriction coming from \(f = 0, h = 0\) and \(f = h = 0\), respectively. Therefore the solution is, in each case, a reduction of the previous one. This result also shows that the relation \(III \subset IV_a \subset V_2\), or \(III \subset IV_b \subset V_2\), follows the same pattern as well known metric symmetries, i.e., "Killing vector fields \(\subset\) homothetic vector fields \(\subset\) special conformal vector fields \(\subset\) conformal vector fields".

3. For the case \(V_2\), one must impose

\[h = a^2 f, \quad c = \pm 2af,\]

where \(a\) a constant, and

\[\alpha_0 = \mp a f, \quad \beta_1 = \mp a \alpha_1, \quad \beta_2 = \mp a \alpha_2.\]

The last two equations yield

\[F_1 = F_1(v \mp au), \quad F_2 = F_2(v \mp au).\]
Furthermore, since \( c + \alpha_0 = \pm af \), one has, using Eqs. (55)–(56),

\[
-\partial_v F_4 \pm a \partial_v F_3 \pm 2ax\tilde{F}_1 \pm 2a\tilde{F}_2 = 0,
\]

where we have put \( \tilde{F} \equiv d^2 F/ds^2_\mp \) with \( s_\mp \equiv v \mp au \). As \( F_3, F_4 \) are functions on \( u \) and \( v \), \( \tilde{F}_2 \) and \( \tilde{F}_1 \) must vanish, that is,

\[
F_1 = b_1 s_\mp + d_1, \quad F_2 = b_2 s_\mp + d_2,
\]

with \( b_1, b_2, d_1 \) and \( d_2 \) constants. Coming back to Eq. (57), we get

\[
\partial_v F_4 = \pm a \partial_v F_3.
\]

Similarly, from \( 2\alpha_0 + c = 0, h = a^2 f \) and \( h = \mp a\alpha_0 \), one gets, respectively,

\[
\partial_u F_3 = \partial_v F_4, \quad \partial_u F_4 = a^2 \partial_v F_3, \quad \partial_u F_4 = \mp a \partial_u F_3.
\]

Combining Eqs. (58)–(59), one easily gets

\[
F_3 = F_3(s_\pm), \quad F_4 = \pm aF_3 + k,
\]

where \( k \) is a constant.

Substituting the expressions for \( F_1, F_2, F_3 \) and \( F_4 \) into (53) we get

\[
\tilde{\xi} = [F_3(s_\pm) + b_1 x + b_2 y] \partial_u + [\pm aF_3(s_\pm) + k \mp a(b_1 x + b_2 y)] \partial_v + [c_0 y + b_1 s_\mp + d_1] \partial_x + [-c_0 x + b_2 s_\mp + d_2] \partial_y.
\]

Reordering the components, and defining \( s_\pm \equiv u \pm av \), one finally gets

\[
\tilde{\xi} = F_3(s_\pm) \partial_{s_\pm} + (b_1 x + b_2 y) \partial_{s_\mp} + k \partial_v + (cy + b_1 s_\mp + d_1) \partial_x + (-cy + b_2 s_\mp + d_2) \partial_y.
\]

This is the result given in Eqs. (41), where the names of the functions have been changed for convenience.

4. For the cases \( I_a \) or \( I_b \), the solution can be read from the result given in I. The conditions are (e.g., for the case \( I_a \)): \( f = c = \alpha_1 = \alpha_2 = 0 \).

5. For the case where \( f = c = h = 0 \) is imposed from the beginning, the vector fields are the Killing vector fields of flat spacetime. However, if \( f = 0, c = 0 \) or \( h = 0 \) are imposed in the solutions of each of the preceding cases, one obtains a subset of Killing vector fields. This is worth to be remarked
since it shows how the Poincaré group can be restricted by non-isometric groups.

It is not difficult from Prop. 2.1 to enlarge Ex. 3 with several different solutions in other spacetimes, e.g., for pp-waves or Vaydia spacetimes, and for other \( \ell, m \). However, the previous example shows very clearly all the basic considerations, and subtleties, of previous sections. For instance, the interconnection among the algebras (Prop. 4.2) appears very clearly. Another remarkable fact is that the Lie algebras are, all but in the isometric case, infinite dimensional. Therefore, their differential systems are open, and their integrability equations are those of App. B (indeed, also used here in their simplest form). Nevertheless, there may be some, more or less, generic conditions under which each case may yield a closed system, and the algebra may become finite dimensional in a natural way.

As of now, one can focus on their finite dimensional subalgebras. This is accomplished setting \( f, h \) and \( c \) to be constants, say \( f_1, h_1 \) and \( c_1 \). For the case \( V_1 \), one gets (\( \lambda^1 = -f_1, \lambda^2 = -h_1, \lambda^3 = -c_1 \))

\[
\tilde{\xi} = \lambda^1 v\partial_u + \lambda^2 u\partial_v + \lambda^3 v\partial_v + \lambda^4 (u\partial_u - v\partial_v) + \lambda^5 \partial_u + \lambda^6 \partial_v + \lambda^7 (y\partial_x - x\partial_y) + \lambda^8 \partial_x + \lambda^9 \partial_y \equiv \lambda^\Omega \tilde{\xi}_\Omega, \quad \Omega = 1, \ldots, 9. \tag{61}
\]

The non-zero Lie brackets are
\[
[\tilde{\xi}_1, \tilde{\xi}_2] = -\tilde{\xi}_3, \quad [\tilde{\xi}_1, \tilde{\xi}_3] = -\tilde{\xi}_1, \quad [\tilde{\xi}_1, \tilde{\xi}_4] = 2\tilde{\xi}_1, \quad [\tilde{\xi}_1, \tilde{\xi}_6] = -\tilde{\xi}_5, \quad [\tilde{\xi}_2, \tilde{\xi}_3] = \tilde{\xi}_2,
\]
\[
[\tilde{\xi}_2, \tilde{\xi}_4] = -2\tilde{\xi}_2, \quad [\tilde{\xi}_2, \tilde{\xi}_5] = -\tilde{\xi}_6, \quad [\tilde{\xi}_3, \tilde{\xi}_6] = -\tilde{\xi}_6, \quad [\tilde{\xi}_4, \tilde{\xi}_5] = -\tilde{\xi}_5, \quad [\tilde{\xi}_4, \tilde{\xi}_6] = \tilde{\xi}_6, \quad [\tilde{\xi}_7, \tilde{\xi}_8] = \tilde{\xi}_9, \quad [\tilde{\xi}_7, \tilde{\xi}_9] = -\tilde{\xi}_8.
\]

For the cases \( IV_a, IV_0 \) or \( III \), the corresponding restriction applies. For the Kerr-Schild case, we refer the reader to I, where also different finite dimensional algebras are given. Finally, for the case \( V_2 \), one has (\( \lambda^1 = -f_1, \lambda^3 = -c_1, a \neq 0 \))

\[
\tilde{\xi} = \lambda^1 s_+ \partial_\pm + \lambda^2 (x \partial_\pm + s_+ \partial_x) + \lambda^3 (y \partial_\pm + s_\pm \partial_y) + \lambda^4 \partial_\pm + \lambda^5 \partial_\mp + \lambda^6 (y \partial_x - x \partial_y) + \lambda^7 \partial_x + \lambda^8 \partial_y \equiv \lambda^\Omega \tilde{\xi}_\Omega, \quad \Omega = 1, \ldots, 8,
\]

and
\[
[\tilde{\xi}_1, \tilde{\xi}_5] = \mp 2a\tilde{\xi}_5, \quad [\tilde{\xi}_2, \tilde{\xi}_3] = \pm 2a\tilde{\xi}_6, \quad [\tilde{\xi}_2, \tilde{\xi}_6] = -\tilde{\xi}_3, \quad [\tilde{\xi}_3, \tilde{\xi}_6] = -\tilde{\xi}_2,
\]
\[
[\tilde{\xi}_2, \tilde{\xi}_7] = -\tilde{\xi}_5, \quad [\tilde{\xi}_3, \tilde{\xi}_4] = -\tilde{\xi}_5, \quad [\tilde{\xi}_6, \tilde{\xi}_7] = \tilde{\xi}_8, \quad [\tilde{\xi}_6, \tilde{\xi}_8] = -\tilde{\xi}_7.
\]

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Notice that by choosing $c_1$ equal to zero in case $V_1$, one gets $\mathcal{L}(\xi)\eta = 2h_1 \ell \otimes \ell + 2f_1 m \otimes m$. Therefore, it seems that one has found a counterexample of our results about double Kerr-Schild. Nevertheless, calculating $(\mathcal{L}(\xi))^2\eta$, one gets $(\mathcal{L}(\xi))^2\eta = 4[\alpha_0 h_1 \ell \otimes \ell - h_1 f_1 (\ell \otimes m + m \otimes \ell) - \alpha_0 f_1 m \otimes m]$. Closer inspection shows what happens: a semi-conformal term appears, so that it is not a proper double Kerr-Schild motion, but a special case of semi-conformal double Kerr-Schild groups. (Recall that in general any metric group is defined within the two first orders).

But there is something else interesting. We have been able to start a problem without an explicit semi-conformal term at first order. It has explicitly appeared only up to the next order. This is a novelty with respect to all previously known metric symmetries, including Kerr-Schild symmetries; if a term does not appear at the first order, it does not appear ever more. One can now start conformal explicit actions at the second order level, leaving the first order as another group action. A point worth to have in consideration in case one wishes to apply metric groups to perturbative works or some processes in which a symmetry changes.

Finally, let us recall that the results obtained in this section may be helpful in the study of finite metric relations based upon two null vectors, [53] or with the conformal Kerr-Schild Ansatz, e.g., [57].

### 4.2 Metric motions generated by $p\cdot q$

In this section we will give the main results only. Let $p$ and $q$ be two given 1-forms satisfying $p \cdot p = q \cdot q = 1$, $p \cdot q = 0$. The differential expression of a metric motion generated by these two ingredients is

$$\mathcal{L}(\xi)g = 2hp \otimes p + c(p \otimes q + q \otimes p) + 2f q \otimes q.$$ 

In this scheme $g$, $p$, and $q$ may be regarded as data. On the other hand $h$, $f$, $c$ are unknown $C^\infty$ functions of the manifold, and $\xi$ are the infinitesimal generators of the group, which constitute the major unknowns of the problem.

Taking $\{p, q\}$ as part of the cabor of spacetime, it may be completed with the addition of two null 1-forms, $\ell$ and $m$, satisfying $\ell \cdot \ell = m \cdot m = 0$, $\ell \cdot p = \ell \cdot q = m \cdot p = m \cdot q = \ell \cdot m + 1 = 0$, but being otherwise arbitrary (this also enables to compare this section with the former in a direct way).

We summarize the results of this section as follows ($A'$ stands for $\mathcal{L}(\xi)A$).
Proposition 4.4 (Metric motions generated by $p-q$) The conditions in order to have a generalized metric motion generated by two given spacelike 1-form fields, $p$ and $q$, i.e., Eqs. (62), are

$$
\begin{align*}
\ell' &= \alpha_0 \ell + \alpha_1 p + \alpha_2 q, \\
m' &= -\alpha_0 m + \beta_1 p + \beta_2 q, \\
p' &= \beta_1 \ell + \alpha_1 m + h p + \gamma_1 q, \\
q' &= \beta_2 \ell + \alpha_2 m + (c - \gamma_1) p + f q,
\end{align*}
\tag{62}
$$

with

$$
\begin{align*}
2h\alpha_1 + c\alpha_2 &= 0, \\
2h\beta_1 + c\beta_2 &= 0,
\end{align*}
\tag{63}
$$

and

$$
\begin{align*}
\bar{h} &= h' + 2h^2 + c(c - \gamma_1), \\
\bar{f} &= f' + 2f^2 + c\gamma_1, \\
\bar{c} &= c' + 2(h - f)\gamma_1 + (h + 3f)c.
\end{align*}
\tag{64}
$$

The volume element, $\eta \equiv -\ell \wedge m \wedge p \wedge q$, transforms according to

$$
\eta' = (h + f) \eta.
\tag{65}
$$

In these expressions $\{\ell, m, q, p\}$ is any semi-null cobasis of the manifold and $\{\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1\}$ are $C^\infty$ functions.

**Proof:** The proof follows exactly the proof of Prop. 4.1.

Again one has, from Prop. 4.4

Proposition 4.5 Isometries and cases I, II, IV and V are not subcases of the general solution, case V.

Although the geometrical structure of both cases is similar, the Lie algebras strongly depend on their local character, see also Fig. 2. This dependence is present in conformal symmetries too, when dealing with its “isometrization”, see [35]-[37], where the signature of the metric is crucial. For the sake of brevity, we shall only show the results of the more special situations. For instance, cases I, I, II, IV, IV and V can easily be recovered from Prop. 4.4 (see also Fig. 2). For the case III, one has the following result

**Proposition 4.6** No groups proportional to $(p \otimes q + q \otimes p)$ exist.
Table 2: Generalized motions generated by \((p, q)\). The non-existence of Lie algebras for the case III is a consequence of the stability property. \(a\) is a constant under the action of the motion \(V_2\). The latter is equivalent to the case I. (See the text for a detailed discussion.)

| \(q\) | Case | Extra freedoms |
|---|---|---|
| \(2hp \otimes p\) | \(I_a\) | 3 |
| \(2fq \otimes q\) | \(I_b\) | 3 |
| \(2hp \otimes p + 2fq \otimes q\) | \(II_1: h \neq f\) \(II_2: h = f\) | \(II_1: 1\) \(II_2: 2\) |
| \(c(p \otimes q + q \otimes p)\) | III | \(\frac{2}{a}\) |
| \(2hp \otimes p + 2c(p \otimes q + q \otimes p)\) | \(IV_a\) | 1 |
| \(2fq \otimes q + 2c(p \otimes q + q \otimes p)\) | \(IV_b\) | 1 |
| \(2hp \otimes p + 2fq \otimes q + c(p \otimes q + q \otimes p)\) | \(V_1: c^2 \neq 4hf\) \(V_2: h = a^2f, c = \pm 2af\) | \(V_1: 2\) \(V_2: 3\) |

Figure 2: Interconnections among the algebras generated by two spacelike 1-forms. See Fig. 1 for its interpretation, and text and table 2 for details.
This case sets \( h = f = 0 \) and \( c \neq 0 \). However it turns out that condition (64) implies \( c = 0 \) and, therefore, one gets a contradiction.

Next, one finds case \( II_2 \), defined by \( h = f \), \( c = 0 \). In this case, contrary to the case \( II_1 \), \( \gamma_1 \) remains free. Moreover, the familial relation is linear, so that it is a well-defined case of a metric motion. In this case \( q = 2h(p \otimes p + q \otimes q) \). Therefore it may be interpreted as a semi-conformal motion, restricted to the spacelike part of the metric tensor. Its properties are

\[
\begin{align*}
\ell' &= \alpha_0 \ell, \\
m' &= -\alpha_0 m, \\
p' &= hp + \gamma_1 q, \\
q' &= hq - \gamma_1 p, \\
\bar{h}' &= h'^2 + 2h^2, \\
\eta' &= 2h\eta.
\end{align*}
\]

For the case \( V_2 \), as \( \vec{\xi} \) must form a vector space, the weights must be related with each other at most linearly. Therefore, they must satisfy \( h = a^2 f, c = \pm 2af \), where \( a \) is a constant under the action of the motion, cf. Case \( V_2 \) of Sect. 4.1.

Following similar steps as in that case, one obtains

\[
q = \left(\frac{2f}{a^2 + 1}\right)r_\pm \otimes r_\pm,
\]

with \( r_\pm \cdot r_\pm = 1, r_\pm \cdot r_\mp = 0 \). Therefore, we conclude that, in the \( p \cdot q \) algebras, the case \( V_2 \) is equivalent to the case \( I_a \), or either \( I_b \).

The Lie algebras generated by a timelike and a spacelike 1-form field, \( u \), \( n \) respectively, are given in App. C. Here we just show the summary.
### Table 3: Generalized motions generated by $(u, n)$. The non-existence of Lie algebras for the case $III$ is a consequence of the stability property. $a$ is a constant under the action of the motion $V_2$. The latter is equivalent to cases $I_a$, $I_b$ or $I$, of table 1, depending on whether $a > 1$, $a < 1$ or $a = 1$ (see the App. D).

| $q$ | Case | Extra freedoms |
|-----|------|---------------|
| $2h \ u \otimes u$ | $I_a$ | 3 |
| $2f \ n \otimes n$ | $I_b$ | 3 |
| $2h \ u \otimes u + 2f \ n \otimes n$ | $II_1 : h \neq -f$ | $II_1 : 1$ |
| | $II_2 : h = -f$ | $II_2 : 2$ |
| $c (u \otimes n + n \otimes u)$ | $III$ | $\not\exists$ |
| $2h \ u \otimes u + 2c (u \otimes n + n \otimes u)$ | $IV_a$ | 1 |
| $2f \ n \otimes n + 2c (u \otimes n + n \otimes u)$ | $IV_b$ | 1 |
| $2h \ u \otimes u + 2f \ n \otimes n$ | $V_1 : c^2 \neq 4hf$ | $V_1 : 2$ |
| $+ c (u \otimes n + n \otimes u)$ | $V_2 : h = a^2 f, c = \pm 2af$ | $V_2 : 3, 4$ |

#### 4.3 The addition of a conformal motion to a metric motion

In general, once a new metric motion is defined, one could further add the conformal group. In the case of Kerr-Schild symmetries this would mean to consider the problem

\[ \mathcal{L}(\xi)g = 2\phi g + 2h\ell \otimes \ell. \]

For instance, the algebraic classification with respect to $g$ does not change. This may suggest considering the addition of conformal motions as a completion of any metric motion. However, its differential role is completely changed by its addition, and both problems, the non-conformal and the conformal one, deserve individual attention (compare isometric and conformal motions). Nevertheless, in our case, both problems are solved separately, see e.g., [29] for the conformal symmetry. For the Kerr-Schild case, the solution is not so simple. The system is mainly closed and general solutions are to be found as shown in Sect. 2. For these “normal” situations, the conformal Kerr-Schild problem might be solved in general as well. The point is
to benefit from the linearity that gives the infinitesimal study of any metric group.

Conformal Kerr-Schild relations have been recently studied in the literature, see e.g., [34, 57]. In this last reference, the authors show that any static spherically symmetric spacetime can be related to flat spacetime locally by means of a conformal Kerr-Schild relation. They may also be useful in order to describe some field theories in General Relativity. The translation of these relations into a problem of metric symmetries, as it was the case for Kerr-Schild symmetries alone, benefits from a major simplification: only first-order terms are needed. We have started to work on this idea. Yet, at the moment, the expressions are still under study. But the main issue deserves some attention because it can also be extended to any other compound problem of symmetries.

5 Some remarks concerning possible ways of future research

It is apparent from earlier works that no single, and simple, path can solve most of current problems in the field of symmetries and in the search of useful solutions to Einstein’s field equations. Hence, in this section we point out some lines that we think will help towards a fuller comprehension of generalized metric symmetries.

The following hints try to ponder the simplicity of the problem and the reaching of the expected solutions. They follow the order of former sections. Except the last two, the rest are already under consideration with other authors, or by myself.

– First, a study of Kerr-Schild motions for the cases of non-geodesic $\ell$ and geodesic $\ell$ with $\Delta \neq 0$ through their connection with isometries. And, second, find the integrals of motion associated with these motions. (See Sects. 2.1, 2.2).

– To try to obtain a complete solution for Kerr-Schild motions with geodesic $\ell$ and $\Delta = 0$, at least in most relevant spacetimes. For this goal the Newman-Penrose formalism may be helpful. (See Sect. 2.3.1).

– To try to find a general solution of Kerr-Schild motions in flat spacetime.

A.Ya. Burinskii, private communication.
This would yield a complete knowledge of Kerr-Schild relations that do not yield new spacetimes. (See App. D).

– To investigate the relation of generalized metric motions with collineations. This path opens a different viewpoint to dealing with infinite dimensional Lie algebras in a Riemannian manifold. This can be started from the expressions of $\mathcal{L}(\xi)\Gamma^\alpha_{\beta\gamma}$ and $\mathcal{L}(\xi)R^\alpha_{\beta\gamma\delta}$ in terms of $q (\equiv \mathcal{L}(\xi)g)$ —see e.g., App. A for the Kerr-Schild case— and using the works referred to in Sect. 1.

– To address a detailed resolution of other generalized metric motions. The candidates might correspond either to a pure mathematical interest —e.g., in connection with first and second before—, to a geometrical interest, such as conformal or Kerr-Schild motions are, or to a physical aim, such as those which arise from Plebański and Schild’s work, or Bonanos’ work.

In any case, we would like to mention three other points. First, the study of the conditions for the existence of proper vector fields. Second, its complement. In particular, the issue of the set of isometries that are compatible with a generalized metric motion is perhaps of major interest (for the case of Kerr-Schild motions some results will be reported elsewhere). And, finally, the study of the conditions of maximum integrability of the chosen metric motion and the structure of the associated Lie algebras.

The case of conformal Kerr-Schild motions seems to be a good proposal fairly fulfilling previous criteria.

– To start an extension of metric motions to subspaces of a greater dimension than one. One could follow, for instance, the idea put in [58]. In that work the author considered the possibility of introducing a symmetric 2-covariant tensor, $T^\lambda$, having the algebraical properties of an electromagnetic energy momentum tensor —i.e., $T^\lambda_\lambda = 0, T_{\alpha\lambda}T^\lambda_\beta = \sigma^2 g_{\alpha\beta}$. However, no solutions were obtained. Metric motions may play an important role in analysing Ansätze with a geometrical basis. Therefore, combining the author’s approach with the ideas presented here about the role of metric motions would help towards a better knowledge of that Ansatz.

– Finally, to develop other possible lines of physical applications. (To that end, see e.g., [33]).
6 Summary and conclusions

Let us summarise very briefly the former sections. Along the preceding sections we have shown mainly two facts (more details were given in Sect. 1 and 2.4). Firstly, with regard Kerr-Schild motions, we have proven that their notion as metric motion is meaningful and that their existence and properties depend on the kinematical properties of the deformation direction \( \ell \) and also that they are generically linked with some subset of the isometries of a —generally— different spacetime.

Second, we have shown that the notion of other metric motions different to isometries or conformal motions is meaningful and contains a much richer structure than their predecessors. Despite the fact that any choice is partly subjective and that a good number of questions still remain to be solved (recall former section) the wealth of results obtained throughout former sections —both in connection with isometries and conformal motions and with new properties— prove that we were on a right track. In particular, we remark that the associated Lie algebras may be of an infinite dimensional character and this may happen in any dimension of the manifold. This result is absent in isometries and only holds for conformal motions in 2-dimensional Riemannian manifolds.

Finally, the principal role assigned to geometry in our framework will certainly help towards the decipheration of old and new physical applications of metric motions, as to finding new —physically interesting— solutions to Einstein’s equations, or in addressing some basic questions on the whole fabric of motions, i.e., including collineations.

Overall, an active issue that may stimulate the interconnection and feedback between some general mathematical issues of differential geometry and their applicability in gravitational physics.

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References

[1] Hall, G.S. (1996), Gravitation and Cosmology 2 270–276.

[2] Coll, B., Hildebrandt, S.R., and Senovilla, J.M.M. (2000). Gen. Rel. Grav. 33, 649–670.

[3] Archeburg, P.C. (1972), Gen. Rel. Grav. 3 397–400.

[4] Hall, G.S. (1983), Gen. Rel. Grav. 15 581–589.

[5] McIntosh, C.B.G., and Halford, W.D. (1982), J. Math. Phys. 23 436–441.

[6] Hall, G.S., and da Costa, J. (1991), J. Math. Phys. 32 2848–2853; ibid. 2854–2862.

[7] Carot, J., and da Costa, J. (1991), Gen. Rel. Grav. 23 1057–1069.

[8] Bokhari, A.H., Qadir, A., Ahmed, M.S., and Asghar, M. (1997), J. Math. Phys. 38 3639–3649.

[9] Hall, G.S., and Shabbir, G. (2001), Class. Quant. Grav. 18, 907.

[10] Tsamparlis, M., and Mason, D.P. (1990), J. Math. Phys. 31 1707.

[11] Carot, J., da Costa, J., and da Vaz, E. (1994), J. Math. Phys. 35 4832.

[12] Hall, G.S., Low, D.J., and Pulham, J.R. (1994), J. Math. Phys. 35 5930–5944.

[13] Carot, J., Núñez, L.A., and Percoco, U. (1997), Gen. Rel. Grav. 29 1223–1237.

[14] Qadir, A., ans Ziad, M. (1998), Nuovo Cimento Soc. Ital. Fis. B 113 773–784.

[15] Marteens, R. (1987), J. Math. Phys. 28 2051–2052.

[16] Collinson, C.D. (1988), J. Math. Phys. 29 1972–1973.

[17] Hojman, S.A., and Núñez, d. (1991), J. Math. Phys. 32 234–238.
[18] Carot, J., Mas, L., Rago, H. and da Costa, J. (1992), Gen. Rel. Grav. 24 959–972.

[19] Hall, G.S., da Costa, J. (1998), J. Math. Phys. 29 2465–2472.

[20] Hall, G.S., and Lonie, D.P. (2000), Class. Quant. Grav. 17 1369–1382.

[21] Weyl, H. (1921), Gö ttinger Nachrichten, 99–112.

[22] Eisenhart, L.P. (1927), “Non-Riemannian geometry” (Amer. Math. Soc. Coll. Publ. VIII).

[23] Hall, G.S., and Lonie, D.P. (1995), Class. Quant. Grav. 12 1007–1020.

[24] Hall, G.S. (2000), Class. Quant. Grav. 17 4637–4644.

[25] Hall, G.S. (2001), Class. Quant. Grav. 17 4637.

[26] Hall, G.S., and Khan (2001), J. Math. Phys. 42 347.

[27] Hall, G.S. (2000), Gen. Rel. Grav. 32 933–941.

[28] Schouten, J.A. (1954), “Ricci Calculus” (Springer-Verlag, Berlin).

[29] Yano, K. (1957), “The theory of Lie derivatives and its Applications” (North-Holland Publ. Co., Amsterdam).

[30] Katzin, G.H., Levine, J., and Davis, W.R. (1969), J. Math. Phys. 10 617–629.

[31] Hall, G.S. (1996), Class. Quantum Grav. 13 1479–1485.

[32] Hall, G.S. (1998), Gen. Rel. Grav. 30 1099–1110.

[33] Hildebrandt, S.R. (2002). Gen. Rel. Grav. 34 159–174.

[34] Kramer, D., Stephani, H., Herlt, E., and MacCallum, M.A.H. (1980), “Exact solutions of Einstein’s Field Equations” (Cambridge University Press, Cambridge).

[35] Bilyalov, R. (1964), Sov. Phys. 8, 878.

[36] Defrise-Carter L. (1975), Commun. Math. Phys. 40 273–282.
[37] Hall, G.S., and Steele, J.D. (1991), J. Math. Phys. 32, 1847.

[38] Petrov, A. Z. (1969), “Einstein Spaces” (Pergamon, New York).

[39] Eisenhart, L.P. (1949), “Riemannian geometry” (Princeton Univ. Press).

[40] Thompson, A.H. (1966), Tensor N.S. 17 92–95.

[41] Xanthopoulos, B.C. (1983), Ann. Phys. 149 286–295.

[42] Misner, C.W., Thorne, K.S. and Wheeler, J.A., (1973), “Gravitation” (Freeman, San Francisco).

[43] Chandrashekhar, S. (1983), ”The Mathematical Theory of Black Holes” (Clarendon Press, Oxford).

[44] Sachs, R.K. (1961), Proc. Roy. Soc. (London) A 264, 309.

[45] Ibragimov, N.H. (1985), “Transformation Groups Applied to Mathematical Physics”. Mathematics and its Applications (D. Reidel Publishing Company; Dordrecht/Boston/Lancaster).

[46] Ibragimov, N.H. (1969), Soviet. Math. Dokl. 10 780–784; Dokl. Akad. Nauk. SSSR., 187 25–28.

[47] Muppinaia, N. (1976), in Proc. of International Symposium on Relativity & Unified Field Theory (1975-1976)ed. S.N. Bose Inst. of Physical Scs., Calcuta, pp 71–82.

[48] Xanthopoulos, B.C. (1978), J. Math. Phys. 19 1607–1609.

[49] Mastronikola, K.E. (1987), Class. Quantum Grav. 4 L179–L184.

[50] Mastronikola, K.E., and Xanthopoulos, B.C. (1989), Class. Quantum Grav. 6 1613–1626.

[51] Bel, Ll. (1998), gr-qc/9812062, and references therein.

[52] Kerr, R.P., and Schild, A. (1965), in Proceedings of the Galileo Galilei Centenary Meeting on General Relativity, Problems of Energy and Gravitational Waves, (G. Barbera, ed., Comiato Nazionale per le Manifestazione Celebrative, Florence), pp. 222–233.
A Miscellaneous of Kerr-Schild-like formulae

Lie derivatives of the Levi-Civita connection, the Riemannian and Ricci tensor and of the scalar curvature.

To begin with,

\[ \mathcal{L}(\xi) g_{\alpha\beta} = 2 h_{\alpha} \ell_{\beta}, \quad \mathcal{L}(\xi) g^{\alpha\beta} = -2 h^{\alpha} \ell^{\beta}, \]

where we have used \( \mathcal{L}(\xi) g^{\alpha\beta} = -g^{\alpha\sigma} g^{\beta\mu} \mathcal{L}(\xi) g_{\sigma\mu}. \)

In any coordinate basis

\[ 2 \mathcal{L}(\xi) \Gamma^\alpha_{\beta\gamma} = g^{\alpha\lambda} [\nabla_\beta \mathcal{L}(\xi) g_{\lambda\gamma} + \nabla_\gamma \mathcal{L}(\xi) g_{\lambda\beta} - \nabla_\lambda \mathcal{L}(\xi) g_{\beta\gamma}] = \nabla_\beta (h_{\alpha} \ell_{\gamma}) + \nabla_\gamma (h_{\alpha} \ell_{\beta}) - \nabla_\alpha (h_{\beta} \ell_{\gamma}). \]

This last expression is easily generalized to any other case of metric motion. Another useful expression—which will be used in App. B—is

\[ \nabla_\beta \nabla_\gamma \xi^\alpha = R^\alpha_{\gamma\beta\lambda} \xi^\lambda + \mathcal{L}(\xi) \Gamma^\alpha_{\beta\gamma}. \]
On the other hand,
\[
\mathcal{L}(\xi) R_{\beta\gamma\delta} = 2\nabla_\gamma \mathcal{L}(\xi) \Gamma^\alpha_{\delta\beta}
\]
\[
= h(R^\alpha_{\delta\gamma\beta} \ell^\lambda_{\beta} - R^\lambda_{\delta\gamma\beta} \ell^\alpha_{\lambda}) + \nabla_\gamma \nabla_\beta (h\ell^\alpha_{\delta}) - \nabla_\delta \nabla_\beta (h\ell^\alpha_{\gamma}),
\]
(68)
where we have used Ricci identities for $h\ell^\alpha_{\beta}$.

Contracting indices $\alpha$ and $\gamma$ we get (after renaming $\beta$ and $\delta$)
\[
\mathcal{L}(\xi) R_{\alpha\beta} = (Dh_\alpha) \ell_\beta + (Dh_\beta) \ell_\alpha + h_\alpha L^\sigma_{\beta\sigma} + h_\beta L^\sigma_{\alpha\sigma}
\]
\[
+ h_\sigma (L^\alpha_{\beta\sigma} + L^\sigma_{\alpha\beta} - 2L_{\alpha\beta\sigma}) + h(L^\alpha_{\beta\sigma} + L^\sigma_{\beta\alpha} - L_{\alpha\beta\sigma})
\]
\[
- g^{\alpha\mu}(\nabla_\sigma h_\mu) \ell_\alpha \ell_\beta,
\]
(69)
where $h_\alpha \equiv \nabla_\alpha h$, $L^\alpha_{\beta\gamma} \equiv \nabla_\gamma (\ell^\alpha_{\beta})$ and $L^\alpha_{\beta\gamma\delta} \equiv \nabla_\delta \nabla_\gamma (\ell^\alpha_{\beta})$ (see notation in Sect. 1 for the rest). We have also used that $R_{\alpha\delta} \ell^\sigma_{\beta} - R^\lambda_{\beta\sigma} \ell^\alpha_{\lambda} + L^\sigma_{\beta\sigma} = L^\sigma_{\beta\delta}$ in order to express the result in a manifestly symmetric form. The quantities $L^\sigma_{\alpha\sigma}$, $L^\sigma_{\alpha\beta\sigma}$ can be expressed in terms of some kinematical quantities of $\ell$. They read:
\[
L^\sigma_{\alpha\beta} = 2\theta \ell_{\alpha} + a_{\alpha}, \quad L^\sigma_{\alpha\beta\sigma} = (2\nabla_\beta \theta + R_{\lambda\delta} \ell^\lambda) \ell_\alpha + (\nabla_\beta \ell^\sigma)(\nabla_\sigma \ell_\alpha) + 2\theta \nabla_\beta \ell_\alpha + D(\nabla_\beta \ell_\alpha).
\]

The Lie derivative of the scalar curvature with respect to a KSVF is
\[
\mathcal{L}(\xi) R = \mathcal{L}(\xi) (g^{\mu\nu} R_{\lambda\mu}) = -2h \ell^\lambda \ell^\mu R_{\lambda\mu} + g^{\alpha\mu} \mathcal{L}(\xi) R_{\lambda\mu} = -2h \bar{R} + g^{\lambda\mu} \mathcal{L}(\xi) R_{\lambda\mu}.
\]
Thus,
\[
\mathcal{L}(\xi) R = 2\{DDh + 4\theta Dh + h[2(D\theta + 2\theta^2) + \nabla_\mu a^\mu - \bar{R}] + a^\mu h_\mu\}.
\]
Whence, the Lie derivative of the Einstein’s tensor, $G_{\alpha\beta} \equiv R_{\alpha\beta} - (1/2)g_{\alpha\beta} R$, with respect to a KSVF can easily be obtained —we do not need it here.

Let us finally recall that in the case of a geodesic $\ell$ other useful expressions are already given in (7)–(9).

**Commutation of covariant and Lie derivatives.**

Let $T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$ be the components of a $p$-contravariant, $q$-covariant tensor field. A standard computation gives
\[
[\mathcal{L}(\xi), \nabla_\gamma] T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = \sum_{i=1}^{i=p} (\mathcal{L}(\xi) \Gamma^\alpha_{\gamma \lambda}) T^{\alpha_1 \ldots \alpha_{i-1} \lambda \alpha_{i+1} \ldots \alpha_p}_{\beta_1 \ldots \beta_q}
\]
\[
- \sum_{i=1}^{i=q} (\mathcal{L}(\xi) \Gamma^\lambda_{\beta_i \gamma}) T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_{i-1} \lambda \beta_{i+1} \ldots \beta_q}.
\]

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For the case of the null deformation direction \( \ell \) we readily get
\[
[\mathcal{L}(\xi), \nabla_\gamma] \ell^\alpha = -D(h \ell^\alpha \ell_\gamma), \quad [\mathcal{L}(\xi), \nabla_\gamma] \ell_\alpha = -D(h \ell_\alpha \ell_\gamma),
\]
where we have used (66).

**Lie derivatives of kinematic quantities of \( \ell \).**

First of all, from Eqs. (1), (2),
\[
\mathcal{L}(\xi) \ell = m \ell
\]

because \( \mathcal{L}(\xi) \ell^\alpha = (\mathcal{L}(\xi) g^{\alpha\sigma}) \ell_\sigma + g^{\alpha\sigma} \mathcal{L}(\xi) \ell_\sigma = m \ell^\alpha \).

In the sequel we will freely use previous results in the proofs. We then obtain

\begin{enumerate}
  \item \( \mathcal{L}(\xi) \theta = m \theta + \frac{1}{2} Dm. \)
  \item \( \mathcal{L}(\xi) D \theta = 2m D \theta + (Dm) \theta + \frac{1}{2} D^2 Dm. \)
\end{enumerate}

**Proof:** In i we have \( 2 \mathcal{L}(\xi) \theta = \mathcal{L}(\xi) \nabla_\lambda \ell^\lambda = \nabla_\lambda \mathcal{L}(\xi) \ell^\lambda = \nabla_\lambda (m \ell^\lambda) = 2m \theta + Dm. \) For ii we have \( \mathcal{L}(\xi) D \theta = \mathcal{L}(\xi)(\ell^\lambda \nabla_\lambda \theta) = (\mathcal{L}(\xi) \ell^\lambda) \nabla_\lambda \theta + \ell^\lambda \mathcal{L}(\xi) \nabla_\lambda \theta = m D \theta + \ell^\lambda \nabla_\lambda \mathcal{L}(\xi) \theta = m D \theta + D [m \theta + (1/2) Dm] = 2m D \theta + (Dm) \theta + (1/2) D^2 Dm. \)

In the case of a geodesic \( \ell - D \ell = M \ell \)— one also has

\begin{enumerate}
  \item \( \mathcal{L}(\xi) \bar{R} = 2m \bar{R} \).
  \item \( \mathcal{L}(\xi) \ell^\mu \nabla_\sigma \nabla^\sigma \ell_\mu = 2m \ell^\mu \nabla_\sigma \nabla^\sigma \ell_\mu. \)
  \item \( \mathcal{L}(\xi) M = m M + Dm. \)
  \item \( \mathcal{L}(\xi) D M = 2m D M + (Dm) M + D^2 Dm. \)
\end{enumerate}

**Proof:** In iii one has \( \mathcal{L}(\xi) (R_{\lambda \mu} \ell^\lambda \ell_\mu) = \ell^\lambda \ell_\mu \mathcal{L}(\xi) R_{\lambda \mu} + 2m \bar{R} = 2m \bar{R}, \) because \( \ell^\lambda \ell_\mu \mathcal{L}(\xi) R_{\lambda \mu} = -2ha^2 \) and vanishes for a geodesic \( \ell \). For iv it is a bit longer. We have \( \mathcal{L}(\xi)(\ell^\mu \nabla_\sigma \nabla^\sigma \ell_\mu) = m \ell^\mu \nabla_\sigma \nabla^\sigma \ell_\mu + \ell^\mu \mathcal{L}(\xi)(\nabla_\sigma \nabla^\sigma \ell_\mu) = m \ell^\mu \nabla_\sigma \nabla^\sigma \ell_\mu + \ell^\mu \nabla_\sigma (\mathcal{L}(\xi) \nabla_\sigma \ell_\mu) + \ell^\mu (\mathcal{L}(\xi) \Gamma^\sigma_{\lambda \sigma}) \nabla^\lambda \ell_\mu - \ell^\mu (\mathcal{L}(\xi) \Gamma^\lambda_{\mu \sigma}) \nabla^\sigma \ell_\lambda; \) the two latter terms cancel. On the other hand, \( \mathcal{L}(\xi) \nabla^\sigma \ell_\mu = -2M h \ell^\sigma \ell_\mu + \mathcal{L}(\xi) \nabla^\sigma \ell_\mu = -2M h \ell^\sigma \ell_\mu + \nabla^\sigma (m \ell_\mu) + (Dh) \ell^\sigma \ell_\mu + 2M h \ell^\sigma \ell_\mu = \nabla^\sigma (m \ell_\mu) + (Dh) \ell^\sigma \ell_\mu. \) Therefore \( \ell^\mu \nabla_\sigma \mathcal{L}(\xi) \nabla^\sigma \ell_\mu = m \ell^\mu \nabla_\sigma \nabla^\sigma \ell_\mu, \) and thus one gets the
claimed result. For \( v \) we have \( \mathcal{L}(\xi)(M\ell^\alpha) = \mathcal{L}(\xi)(D\ell^\alpha) = \mathcal{L}(\xi)(\ell^\mu \nabla_\mu \ell^\alpha) = m M \ell^\alpha + D(m\ell^\alpha) - \ell^\mu D(h\ell^\alpha \ell_\mu) = (2m M + Dm)\ell^\alpha \). On the other hand, \( \mathcal{L}(\xi)(M\ell^\alpha) = (\mathcal{L}(\xi)M)\ell^\alpha + mM\ell^\alpha \) and, hence, one gets \( v \). For \( v_i \) one can follow the proof of \( ii \).

This list can easily be completed for other quantities and for a non-geodesic \( \ell \). They were not used in this work and therefore we do not display them here.

B First steps towards the integrability equations of a metric motion

The general case.

First of all one needs to write the system of Eqs. (30)–(31) in a normal form for the basic unknowns \( \vec{\xi} \) and its derivatives, at least. We will assume a coordinate basis. The first result is that Eqs. (30)–(31) are equivalent to the partial differential system (\( q, m \) are used instead of \( q_\xi, m_\xi \), respectively, to clarify the notation)

\[
\begin{align*}
\nabla_\alpha \xi_\beta &= \xi_{\alpha\beta}, \\
\xi_{\alpha\beta} + \xi_{\beta\alpha} &= q_{\alpha\beta} = (q_{\Omega\Lambda})_{\alpha\beta}^{\Omega\Lambda}, \\
\xi^\lambda \nabla_\alpha \Theta^\Omega_{\alpha} + \xi^\alpha \theta^\Omega_{\alpha} &= m_{\lambda}^{\Omega\Lambda} \Theta_{\alpha}^{\Omega\Lambda}, \\
\nabla_\alpha \xi_{\beta\gamma} &= R_{\gamma\beta\alpha\lambda} \xi^\lambda + \frac{1}{2} (\nabla_\alpha q_{\gamma\beta} + \nabla_\beta q_{\gamma\alpha} - \nabla_\gamma q_{\alpha\beta}).
\end{align*}
\]

In the computation of \( \nabla_\alpha \xi^\gamma_\beta \) we have used Eq. (66) and Eq. (67). Whence, a standard —tough cumbersome— computation shows that (we put \( A_{<\alpha\gamma>} \equiv (1/2)(A_{\alpha_1\alpha_2\gamma_1\gamma_2} + A_{\gamma_1\gamma_2\alpha_1\alpha_2}) \))

**Proposition B.1** Ricci identities applied to a metric motion can be expressed as

\[
\xi^\lambda \nabla_\lambda R_{\alpha_1\alpha_2\gamma_1\gamma_2} + 4q_{[\gamma\lambda]}(\delta^\rho_{[\alpha_1} R_{\alpha_2]}^{\lambda}_{\gamma_1\gamma_2})_{<\alpha\gamma>} = -q_{\lambda[\gamma_1} R_{\gamma_2]\alpha_1\alpha_2} - 2\nabla_{\gamma_1} \nabla_{[\alpha_1} q_{\alpha_2]\gamma_2]} ,
\]

\[
\xi^\lambda \nabla_\lambda \nabla_{\alpha_3} R_{\alpha_1\alpha_2\gamma_1\gamma_2} + \xi_{[\rho\lambda]} \left[ 4 \nabla_{\alpha_3} (\delta^\rho_{[\alpha_1} R_{\alpha_2]}^{\lambda}_{\gamma_1\gamma_2})_{<\alpha\gamma>} + \delta^\rho_{\alpha_3} \nabla_{\lambda} R_{\alpha_1\alpha_2\gamma_1\gamma_2} \right] = -\frac{1}{2} q_{\alpha_3} \nabla_{\lambda} R_{\alpha_1\alpha_2\gamma_1\gamma_2} + 2 \left[ (\nabla_{\lambda} q_{\alpha_3})_{[\alpha_1} - \nabla_{[\alpha_1} q_{\alpha_3]} \right] R_{\alpha_2]}^{\lambda}_{\gamma_1\gamma_2} \right]_{<\alpha\gamma>}
\]

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\[ \nabla_{\alpha_3}(q_{\lambda[\gamma_1} R^\lambda_{\beta\gamma_2]\alpha_1 \alpha_2}) - 2 \nabla_{\alpha_3} \nabla_{[\gamma_1} q_{\alpha_2]} R^\lambda_{\beta\gamma_2]\alpha_1 \alpha_2] \]  

\[ \xi^\lambda \nabla_\lambda \Theta_\alpha + \Theta_\alpha^\lambda \xi_{[\alpha] \lambda} = -\frac{1}{2} q^\lambda_\alpha + m^\lambda_\alpha \Theta_\alpha. \]  

\[ \xi^\lambda \nabla_\lambda \nabla_\alpha \Theta_\beta^\alpha + \xi_{[\alpha \lambda]} (\delta^\lambda_\alpha \nabla^\sigma \Theta_\beta^\sigma + \delta^\lambda_\beta \nabla^\sigma \Theta_{\alpha \sigma}) = (\nabla_\alpha m^\lambda_\beta) \Theta_\beta^\lambda + \frac{1}{2} (\nabla^\lambda q_\alpha^\beta - \nabla^\beta q_\alpha^\lambda) \Theta_\gamma^\lambda + q^\lambda_\beta \nabla_\alpha \Theta_\lambda^\beta - \frac{1}{2} \nabla_\alpha q_\beta^\lambda. \]

and the Ricci identities for the functions \( m^\lambda_\alpha, q_{\alpha \beta} \) and their derivatives.

**Proof:** We will only give a sufficient sketch of it. The Ricci identities applied to \( \xi_\alpha \) are identically satisfied by virtue of Eqs. (72). The following are the Ricci identities for \( \xi_{\alpha \beta} \). A standard computation (see e.g., [29]) leads to the conclusion that these are in fact collected in the form of Eq. (68).

Using Eq. (66) one gets Eq. (68) for a generic \( q \):

\[ \mathcal{L}(\xi) R^\alpha_{\beta \gamma \delta} = R^\alpha_{\beta \gamma \delta} q^\lambda_\delta + R^\lambda_{\beta \gamma \delta} q^\alpha_\delta + (\nabla_\delta \nabla_\gamma q^\alpha_\delta - \nabla_\delta \nabla_\beta q^\gamma_\delta) + (\nabla_\delta \nabla_\alpha q^\gamma_\beta - \nabla_\gamma \nabla_\alpha q^\beta_\delta). \]

Of course, as emphasized elsewhere, some knowledge on \( q \) must be given in order to have a well-defined problem. It is worth writing them, as long as possible, in terms of the independent variables \( \xi_\alpha, \xi_{\alpha \beta} \). Substituting the constraint given by Eqs. (30) into Eqs. (77) one gets Eqs. (73). The following conditions are Ricci identities applied to \( \nabla_\nu \xi^\alpha_\beta \). Once again, a standard analysis, shows that these can be indeed written calculating \( \mathcal{L}(\xi) \nabla_\gamma R^\alpha_{\beta \gamma \delta} \).

In fact its calculation may be carried out more easily via the commutation identity

\[ [\mathcal{L}(\xi), \nabla_\nu] R^\alpha_{\beta \gamma \delta} = \mathcal{L}(\xi) \Gamma^\alpha_{\nu \lambda} R^\lambda_{\beta \gamma \delta} - \mathcal{L}(\xi) \Gamma^\lambda_{\nu \alpha} R^\alpha_{\beta \gamma \delta} - \mathcal{L}(\xi) \Gamma^\gamma_{\nu \delta} R^\delta_{\beta \gamma \lambda} - \mathcal{L}(\xi) \Gamma^\delta_{\nu \gamma} R^\gamma_{\beta \lambda \delta}. \]

It is worth writing the result in terms of \( \xi_\alpha, \xi_{\alpha \beta} \) only, and this may be performed as in the previous case. The result is (74).

This procedure has to be continued to any order (notice that only \( \xi_\alpha \) and \( \xi_{\alpha \beta} \) appear at each step). This is a usual presentation in the case of isometric
or conformal motions, of course, setting \( q \) in accordance. The difference now lies in the fact that the system may be open—except in the case of isometric or conformal motions where the system is always normal, see e.g., [29]. Thus one can only include the Ricci identities for \( m^B_\Lambda \), \( q_B \) and their derivatives in a direct way.

Secondly, Eq. (2) and its derivatives add another set of conditions for any other case which differs from isometric or conformal motions. In the previous case, Eq. (1) was directly substituted into the system, yet this is not possible for the stability conditions, Eqs. (2) or their derivatives, because of their expression (neither in the new case of Kerr-Schild motions) and they must be added directly. Again, for any derivative, one must use the commutation identity (78) (adapted to each case), in order to write them in terms of \( \xi_\alpha \) and \( \xi_{[\alpha\beta]} \) only. These are Eqs. (75), (76), ... Finally, in the expressions lower case Greek indices are raised or lowered using the tensor metric \( g_{\alpha\beta} \), and upper case ones with the aid of \( g^{B\Lambda} \equiv \Theta^B_\Omega \cdot \Theta^\Lambda_\Lambda \) and its inverse. 

The Kerr-Schild case.

This case is recovered from the general expression setting \( q = 2h\ell \otimes \ell \) although some simplifications appear. Eqs. (72) turn into

\[
\begin{align*}
\xi_{(\alpha\beta)} &= h\ell_\alpha \ell_\beta, \\
\nabla_\alpha \xi_{\beta} &= \xi_{\alpha\beta}, \\
\nabla_\alpha h &= h_\alpha, \\
\nabla_\alpha \xi_{\beta\gamma} &= R_{\gamma\beta\alpha\lambda} \xi^\lambda + h_\beta \ell_\gamma \ell_\alpha + h_\alpha \ell_\gamma \ell_\beta - h_\gamma \ell_\alpha \ell_\beta \\
&+ h[\nabla_\alpha (\ell_\beta \ell_\gamma) + \nabla_\beta (\ell_\alpha \ell_\gamma) - \nabla_\gamma (\ell_\alpha \ell_\beta)].
\end{align*}
\]

(79)

Therefore, it is clear that it suffices to consider \( \ell \) as the only relevant element of the cobasis. Whence, using the same notation as in Prop. B.1 we get:

**Proposition B.2** Ricci identities applied to a Kerr-Schild motion are

\[
\xi^\sigma \nabla_\sigma \nabla_\alpha R_{\alpha_1\alpha_2\gamma_1\gamma_2} + 4\xi_{[\rho\lambda]} (\delta^\rho_{[\alpha_1} R^\lambda_{\alpha_2]\gamma_1\gamma_2])_{<\alpha\gamma>} = -q_{\lambda[\gamma_1} R^\lambda_{\gamma_2]\alpha_1\alpha_2} - 2\nabla_{[\gamma_1} \nabla_{\alpha_1} q_{\alpha_2]\gamma_2]},
\]

(80)

\[
\xi^\sigma \nabla_\sigma \nabla_\alpha R_{\alpha_1\alpha_2\gamma_1\gamma_2} + \xi_{[\rho\lambda]} \left[ 4\nabla_{\alpha_3} (\delta^\rho_{[\alpha_1} R^\lambda_{\alpha_2]\gamma_1\gamma_2})_{(\alpha\gamma)} + \delta^\rho_{\alpha_3} \nabla^\lambda R_{\alpha_1\alpha_2\gamma_1\gamma_2} \right] = -1/2 q_{\lambda[\gamma_1} \nabla_\lambda R_{\alpha_1\alpha_2\gamma_1\gamma_2} + 2 \left[ (\nabla_\alpha q_{\alpha_3}[\gamma_1} - \nabla_{[\alpha_1} q_{\gamma_1\alpha_3} R^\lambda_{\alpha_2]\gamma_2]}_{<\alpha\gamma>} -
\]

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\[
\n\nabla_{\alpha 3}(q_{\lambda[\gamma_1} R^\lambda_{\gamma_2]\alpha_1\alpha_2]) - 4\nabla_{\alpha 3} \nabla_{[\gamma_1} \nabla_{[\alpha_1} q_{\alpha_2]_2]}, \quad (81)
\]

\[
\xim^\alpha \nabla_\sigma \ell_\alpha - \xi_{[\alpha\beta]} \ell^\sigma = m \ell_\alpha, \quad (82)
\]

\[
\xim^\alpha \nabla_\sigma \nabla_\alpha \ell_\beta + \xi_{[\alpha\beta]}(\delta^\alpha_\lambda \nabla^\sigma \ell_\beta + \delta^\beta_\lambda \nabla_\alpha \ell^\sigma) = m \nabla_\alpha \ell_\beta + \ell_\beta[\nabla_\alpha m + h a_\alpha + (Dh)\ell_\alpha], (83)
\]

and the Ricci identities for \(m, h\) and their derivatives. \[\blacksquare\]

Due to the little closure of the system in general, the above equations may be viewed as first steps towards the integrability conditions of a generalized metric motion. They may also be useful to study maximum integrability of a given generalized metric motion.

## C Metric motions generated by \(u-n\)

Let \(u\) and \(n\) be two 1-forms satisfying \(u \cdot u = -1\), \(n \cdot n = 1\), \(u \cdot n = 0\).

The differential expression of a metric motion generated by these two ingredients is

\[
\mathcal{L}(\xi)g = 2hu \otimes u + c(u \otimes n + n \otimes u) + 2fn \otimes n. \quad (84)
\]

In this scheme \(g, u,\) and \(n\) may be regarded as data. On the other hand, \(h, f, c\) are unknown \(C^\infty\) functions of the manifold yet to be determined. And \(\xi\) are the infinitesimal generators of the group. The cobasis will be completed with the addition of two spacelike 1-forms, \(p\) and \(q\), satisfying \(p \cdot p = q \cdot q = 1\), \(u \cdot p = u \cdot q = n \cdot p = n \cdot q = p \cdot q = 0\), but otherwise arbitrary.

We summarize the results of this section as follows (\(A'\) stands for \(\mathcal{L}(\xi)A\))

**Proposition C.1 (Metric motions generated by \(u-n\))** The conditions in order to have a generalized metric motion generated by \(u-n\), i.e., Eqs. (84), are

\[
\begin{align*}
  u' &= -hu + \alpha_0 n + \alpha_1 p + \alpha_2 q, \quad n' = (c + \alpha_0)u + f n + \beta_1 p + \beta_2 q, \\
  p' &= \alpha_1 u - \beta_1 m + \gamma_1 q, \quad q' = \alpha_2 u - \beta_2 n - \gamma_1 p.
\end{align*}
\]

(85)
with
\[
\begin{cases}
2h\alpha_1 + c\beta_1 = 0, \\
2h\alpha_2 + c\beta_2 = 0,
\end{cases}
\]
and
\[
\begin{cases}
\tilde{h} = h' - 2h^2 + c(c + \alpha_0), \\
\tilde{f} = f' + 2f^2 + c\alpha_0, \\
\tilde{c} = c' + 2\alpha_0(h + f) + c(3f - h).
\end{cases}
\] (86)

The volume element transforms according to
\[
\eta' = (f - h) \eta.
\] (88)

In these expressions \(\{u, n, q, p\}\) is any orthonormal cobasis of the manifold and \(\{\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1\}\) are \(C^\infty\) functions.

**Proof:** The proof follows analogous steps as those of Prop. 4.1 (now \(g = -u \otimes u + n \otimes n + p \otimes p + q \otimes q\)).

One also has

**Proposition C.2** Isometries and the cases \(I_a, I_b, II_1, IV_a, IV_b, V_2\) are disconnected from the general solution, case \(V_1\).

Again, for the sake of brevity, we shall only display some cases, see also Fig. 3 for a summary. Cases \(I_a, I_b, II_1, IV_a, IV_b, V_1\) are easily recovered from Prop. C.1. Notice, however, that now cases “a” and “b” are not equivalent because \(u\) and \(n\) are timelike and spacelike, respectively. Furthermore, case \(II_2\) is very similar to \(II_2\) of the last section. Finally, for the case \(III\) one has

**Proposition C.3** No motions proportional to \((u \otimes n + n \otimes u)\) exist.

And for the case \(V_2\), as \(\vec{\xi}\) must form a vector space, the weights must satisfy \(h = a^2 f, g = \pm 2af\), where \(a\) is a constant under the action of the group, cf. case \(V_2\) Sect. 4.1. Following similar steps as in that case, one obtains \(a^2 \neq 1\)

\[
q = \left(\frac{2f}{1-a^2}\right)r_\pm \otimes r_\pm,
\]
with \( r_\pm \cdot r_\pm = \text{sign}(1 - a^2) \), \( r_\pm \cdot r_\mp = 0 \). Therefore, we conclude that, in the \( u-n \) algebras, the case \( V_2 \) with \( a^2 \neq 1 \) is equivalent to the case \( I_a \), or either the \( I_b \). For \( a^2 = 1 \), one gets

\[
q = 2f\ell \otimes \ell,
\]

with \( r_\pm \) a null 1-form. Therefore, we conclude that the case \( V_1 \) with \( a^2 = 1 \) is equivalent to a Kerr-Schild-like problem.

We know that \( \{r_+, r_-\} \) is a combination of \( u \) and \( n \). One can check, using the expressions of \( u', n' \) given above, that after some algebra \( \{r'_+, r'_-\} \) verifies the transformation law that corresponds to each case. For \( a^2 = 1 \), \( \alpha_0 \) is no longer fixed by the relations (86). Thus, \( \alpha_0 \) adds the fourth degree of freedom which characterizes Kerr-Schild groups. This constitutes an important confirmation of the coherence of the whole scheme.

### D A procedure to find all Kerr-Schild motions in flat spacetime

Let us now begin with \( \ell \) non-geodesic and \( \ell \) geodesic with \( \Delta \neq 0 \). In both situations the Kerr-Schild problem reduces to a problem of restricted isometries in flat spacetime. The point is that one can then use a classification of all subgroups of the Poincaré group to find all the Kerr-Schild motions of this type.

One should focus on the restricting equation, i.e., \( \mathcal{L}(\xi)\ell = m\ell \), or equivalently \( \mathcal{L}(\xi)\tilde{\ell} = m\tilde{\ell} \). Commuting each Killing vector field with a general \( \tilde{\ell} \) of flat spacetime, we get explicit conditions on the functional dependence of its components. Of course, one must add the conditions that \( \ell \) be non-geodesic, or geodesic with \( \Delta \neq 0 \).

We have begun to study the non-geodesic case. We have, at the time we write this, only studied the subgroups made of either boosts, rotations or translations, and direct combinations of them. The main result is that at most there exist fourth dimensional Kerr-Schild motions (restricted isometries). For instance, the \( G_{KS,n} \) (\( G_{KS,n} \) stands for a group of Kerr-Schild motions of dimension \( n \)) are represented by Ex. 1; the \( G_{KS,3} \) are formed by three independent translations, or either by a rotation or boost with their two possible orthogonal translations. The \( G_{KS,2} \) and \( G_{KS,1} \) are still simpler.
There are no more $G_{KS}$ in this subset of all subgroups of the Poincaré group (for instance, four translations lead to a covariantly constant null vector and the same is true for three rotations or boosts).

We will not write here the functional expression of the null vectors that satisfy each of the conditions. They can be calculated assuming a general null $\ell$ and imposing the corresponding commuting restriction. This study has to be extended to all possible subgroups of the Poincaré group. In this sense holonomy theory may be helpful (see references in Sect. 1).

Finally, the same calculation should be carried out for the geodesic $\ell$ with $\Delta \neq 0$ (see Sect. 2.3). Were the second set of conditions of $\Delta = 0$ in flat spacetime to lead to the cylindrical and parallel cases (see I and Sect. 2.3.1), its solution could already be read from Ex. 3, Eqs. (43), (44). Otherwise, one only ought to solve the new $\ell$ in the same way as in that example.