THE ROPELENGTH OF SPECIAL ALTERNATING KNOTS

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ABSTRACT. A long standing open conjecture states that if a link $K$ is alternating, then its ropelength $L(K)$ is at least of the order $O(Cr(K))$. A recent result shows that the maximum braid index of a link bounds the ropelength of the link from below. Thus in the case an alternating link has a maximum braid index proportional to its minimum crossing number, such as the $T(2,2n)$ torus link, then the ropelength of the link is bounded below by a constant multiple of its minimum crossing number. However if the maximum braid index of a link is small compared to its crossing number, then there are no known results about whether its ropelength is bounded below by a constant multiple of its crossing number. For example, the $T(2,2n+1)$ torus knot has a minimum knot diagram that looks almost identical to that of the $T(2,2n)$ torus link, yet whether its ropelength is bounded below by a constant multiple of $n$ remains open to date. In this paper, we provide a first such result, and in fact for a large class of alternating knots. Specifically, we prove that if an alternating knot (namely a link with only one component) has a reduced alternating knot diagram in which the crossings are either all positive or all negative (such a knot is called a special alternating knot), then its ropelength is bounded below by a constant multiple of its crossing number.

1. Introduction

The ropelength of a link is defined (intuitively) as the minimum length of a unit thickness rope that can be used to tie the link. Let $K$ be an un-oriented link, $Cr(K)$ be the minimum crossing number of $K$ and $L(K)$ be the ropelength of $K$. A fundamental question in geometric knot theory asks how $L(K)$ is related to $Cr(K)$. For some very special cases of multi-component links, the ropelength can be precisely determined. For example, the ropelength of the link $K$ that is a simple chain of $k \geq 2$ rings (namely the connected sum of $k-1$ Hopf links) is $(4\pi + 4)k - 8 = 2(\pi + 1)Cr(K) + 4(\pi - 1)$ [4]. In general the determination of the precise ropelength of a non-trivial link is a difficult problem and no known precise formula exists for $L(K)$ even when $K$ is the trefoil, the simplest non-trivial knot. However, even if we cannot obtain a precise formula for the ropelength of a link, a good estimate of the ropelength would often suffice, much like the case of finding approximate solutions of a polynomial equation. Indeed, much progress has been made in this research direction. See for example [1] [2] [3] [4] [5] [7] [8] [9] [13] [14]. In the case of the lower bound estimates of ropelength, it has been shown in that in general $L(K) \geq 1.105(Cr(K))^{3/4}$ and that this $3/4$ power can be attained by a family of infinitely many links [3] [8]. On the other hand, not all links obey this $3/4$ power law since there exist families of infinitely many links such that the ropelength of a link from such a family grows linearly as the crossing number of the link [7] [14]. These results are based on the fact that the ropelength of these links are bounded below by either the bridge numbers or the maximum braid indices of these links which happen to be proportional to their crossing numbers because of the following recent result.

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Let $K$ be an alternating link. We will call the largest braid index among all braid indices corresponding to different orientation assignments to the components of $K$ the maximum braid index of $K$ and denote it by $B(K)$. It has been shown recently [7] that $L(K) \geq a_0 B(K)$ for some constant $a_0 > 0$ that is independent of $K$ (in fact $a_0 > 1/14$). Thus for a link $K$, if $B(K)$ is proportional to $Cr(K)$, then $L(K)$ would be bounded below by a constant multiple of $Cr(K)$. However, for a link with small maximum braid index, this result would be of little help to us. A simple example is torus knot $T(2, 2n + 1)$, which has braid index 2. As it turns out as demonstrated in this paper, this result in fact provides us the necessary tool to alternating knots that do not necessarily have braid indices proportional to their crossing numbers. More specifically, in this paper, we prove that if $K$ is a special alternating knot, namely an alternating knot that has a reduced alternating diagram in which the crossings are either all positive or are all negative, then its ropelength is bounded below by a constant multiple of its crossing number. Such alternating knots are called special alternating knots (this name was first introduced by Murasugi in [20]).

**Theorem 1.1.** Let $K$ be an alternating knot, that is, a link with only one component, that has a reduced alternating knot diagram $D$ whose crossings are either all positive or are all negative, then $L(K) \geq b_0 Cr(K)$ for some constant $b_0 > 0$ that is independent of $K$, where $Cr(K) = c(D)$ is the minimum crossing number of $K$. The statement holds if the crossings in $D$ are all negative.

By symmetry, the ropelength of a knot does not change when we change it to its mirror image, thus throughout this paper, we need only work with positive special alternating knots.

# 2. Seifert circle decomposition and Seifert graphs

## 2.1. Seifert circle decomposition

The Seifert circle decomposition of $D$ is the collection of disjoint topological circles (called Seifert circles or $s$-circles for short) obtained after all crossings in $D$ are smoothed (as shown in Figure 1). Notice that since $D$ is positive and is alternating, it can be drawn in a way that no $s$-circle can contain other $s$-circles in its interior.

![Figure 1. Left: $D$ is the diagram of a positive trefoil; Middle: The Seifert circle decomposition of $D$; Right: The Seifert graph $G_S(D)$.](image)

## 2.2. Trapped $s$-circles

Consider three $s$-circles $C_1$, $C_2$ and $C_3$ as shown in Figure 2 where $C_3$ is bounded within the topological disk created by arcs of $C_1$, $C_2$ and the two consecutive crossings as shown in the figure. We say that $C_3$ is trapped between $C_1$ and $C_2$. Similarly, $C_4$ is also trapped between $C_1$ and $C_2$. Since $D$ is a special knot diagram in our case, the interior of any $s$-circle does not contain any other $s$-circle. Thus we can always use flype moves to free any trapped $s$-circles, hence we can assume that $D$ is free of trapped $s$-circles from this point on. An immediate consequence of this is that if two $s$-circles $C_1$ and $C_2$ share crossings, then as we travel along either $C_1$ or $C_2$, we encounter all these crossings consecutively without running into crossings between them and other $s$-circles.
2.3. Seifert graphs.

Definition 2.1. Let $D$ be a reduced alternating diagram, then shrinking each $s$-circle to a vertex and each crossing between two $s$-circles to an edge connecting the two corresponding vertices, we obtain a (bipartite) plane graph. We denote this graph by $G_S(D)$ and call it the Seifert graph of $D$. See the right of Figure 1 for an example.

If two vertices are connected by one and only one edge in $G_S(D)$, then this edge is called a lone edge and it corresponds to a crossing between two $s$-circles in $S(D)$ and is the only crossing between these two $s$-circles. Such a crossing is called a lone crossing. Notice that a lone edge in $G_S(D)$ cannot be a bridge, since otherwise it would correspond to a nugatory crossing of $D$, but that is not possible since $D$ is reduced. Sometimes it is more convenient to use a different version of the Seifert graph of $D$, denoted by $\overline{G}_S(D)$. $\overline{G}_S(D)$ is similar to $G_S(D)$ except that it is a simple graph with its edges weighted. An edge of weight $k$ in $\overline{G}_S(D)$ connecting two vertices $v_1$ and $v_2$ corresponds to the case when the same vertices $v_1$ and $v_2$ in $G_S(D)$ are connected by $k$ edges. A lone edge in $G_S(D)$ is an edge of weight one in $\overline{G}_S(D)$. Since $K$ is a knot, $\overline{G}_S(D)$ cannot contain a bridge edge of even weight.

Definition 2.2. A face of a plane graph $G$ is said to be non-separating if deleting the edges on its boundary does not disconnect the graph $G$. The graph $G$ is said to be proper if every face of it is non-separating. $D$ is said to be proper if every face in $G_S(D)$ is non-separating. See Figure 2 for an example.

Definition 2.3. We call a cycle of length 2 in $G_S(D)$ a short cycle. A cycle of $G_S(D)$ that is part of the boundary of a face and is not a short cycle is called a long cycle.

Definition 2.4. A maximal path of $G_S(D)$ is said to be proper if it contains at most one lone edge (namely an edge of weight one).
Remark 2.5. Notice that if a maximal path of $G_S(D)$ contains one or more lone edges, then we can always perform flypes on $D$ so that the crossings corresponding to these lone edges occur consecutively, hence from this point on we will assume that $D$ has such a property. It follows that a maximal path of $G_S(D)$ is proper if and only if it contains no vertices of degree 2 in $G_S(D)$. Furthermore, it is not possible for a maximal path of $G_S(D)$ to contain only lone edges, with one of its two end vertices to also have degree 2 in $G_S(D)$ in our case. Since otherwise $D$ contains a connected sum component which is a two component torus link, contradicting to the fact that $D$ is a knot diagram. That is, the end vertices of a maximal path of $G_S(D)$ must have degrees higher than two in $G_S(D)$.

![Figure 4](image)

**Figure 4.** Left: Part of the diagram $D$ with two lone crossings corresponding to two edges on a path of $G_S(D)$; Right: After a proper flype move, the two lone edges are now side by side.

Lemma 2.6. Let $D$ be a special alternating knot diagram, then the following conditions are equivalent.

(i) $D$ is proper;
(ii) For any face $F$ of $G_S(D)$, there exists a spanning tree $T_F$ of $G_S(D)$ such that no edges on $T_F$ are on the boundary of $F$.
(iii) Every maximal path in $G_S(D)$ is proper.

Proof. (i) $\implies$ (ii) If $D$ is proper and $F$ is any face of $G_S(D)$, then deleting the edges on the boundary $\partial F$ of $F$ does not disconnect the graph, which means we can use find a spanning tree of $G_S(D)$ without using any edges on $\partial F$. (ii) $\implies$ (iii) If $G_S(D)$ is not proper, then it contains a maximal path with $k \geq 2$ consecutive lone edges hence at least one vertex of degree 2 connected by two lone edges. Deleting these two lone edges will then disconnect $G_S(D)$, hence $G_S(D)$, which contradicts the given condition. (iii) $\implies$ (i) Consider any face $F$ of $G_S(D)$. First consider the case that $\partial F$ contains a short cycle $\Gamma$ in $G_S(D)$ which is identified with an edge $\bar{\gamma}$ of weight at least 2 in $G_S(D)$. If $\bar{\gamma}$ is not a bridge edge of $G_S(D)$, then deleting the edges in $\gamma$ obviously will not disconnect $G_S(D)$. If $\bar{\gamma}$ is a bridge edge of $G_S(D)$, then it is necessary that its weight is an odd integer that is at least 3 since $D$ is a reduced knot diagram. Therefore deleting the edges in $\gamma$ will not disconnect $G_S(D)$ either. If $\partial F$ contains a long cycle $\gamma$, then by the given condition, $\bar{\gamma}$ contains at most one lone edge. It means that we can delete the edges in $\gamma$ without disconnecting $G_S(D)$. See Figure 3 for an illustration.

3. An outline of our approach to prove Theorem 1.1

In our approach to the proof of Theorem 1.1 we use a lattice realization of $K$ as a way to estimate $L(K)$, thus we need to have a brief discussion about links realized on the cubic lattice. Let $K$ be a link.
and $K_c$ a realization of $K$ on the cubic lattice. The length of $K_c$ is denoted by $L(K_c)$ and the minimum of $L(K_c)$ over all lattice realization $K_c$ of $K$ is denoted by $L_c(K)$. Let $B(K)$ be the largest braid index among all braid indices corresponding to all possible orientation assignments of the components of $K$ (called the maximum braid index of $K$). Of course if $K$ is a knot, then there is only one component and the two orientations of the component lead to the same braid index. The following recent result shows that $B(K)$ bounds $L(K)$ from below and $L_c(K)$ served as a bridge in between.

**Proposition 3.1.** [7] For any lattice realization $K_c$ of the link $K$, we have $B(K) < L(K_c)$. Passing to the length minimizer, we have $B(K) < L_c(K) < 14L(K)$, that is, $L(K) > (1/14)B(K)$.

The following outlines our approach in proving Theorem 1.1 using Proposition 3.1.

Let $K$ be the realization of $K$ whose projection to the $xy$-plane is the diagram $D$. Consider a thin tubular neighborhood of $K$ with $K$ as its center line. The boundary of this tube is a (knotted) torus $T$ whose longitudinal direction is parallel to $K$. Consider an arbitrary simple closed curve $K_1$ on $T$ that runs once on $T$ along its longitudinal direction, but can run around $K$ as many times in the meridian direction (positive or negative). We can isotope the twists of $K_1$ around $K$ in such a way that the projection of $K_1$ is as shown in Figure 5. $K$ and $K_1$ form a two component link $L(K, K_1)$ and the projection of $L(K, K_1)$ as shown in Figure 5 is denoted by $D_k$ if the diagram contains $|k|$ full twists, where $k > 0$ means the full twists are positive and $k < 0$ means the full twists are negative. If we assign $K$ and $K_1$ anti-parallel orientations, then the linking number between $K$ and $K_1$ is $-2c + k$. In the particular case that $k = 0$, we will use $D$ to denote the diagram. In general, if $K'$ is a realization of $K$ and $K_1'$ is a simple closed curve on a torus with $K'$ as its center line such that $K_1'$ runs once on the torus along its longitudinal direction, then $K'$ and $K_1'$ (together with the tube bounded by the torus) can be deformed to $K$ and some $K_1$ (and $T$) by an ambient isotopy. That is, there exists some $K_1$ such that $L(K', K_1')$ is topologically equivalent to $L(K, K_1)$.

![Figure 5](image)

**Figure 5.** The projection diagram of an embedded annulus whose two components are equivalent to the $5_1 = T(2, 5)$ torus knot. The top portion of the diagram contains 2 full positive twists and the linking number of the link is thus $(-10 + 4)/2 = -3$.

We have the following theorem.

**Theorem 3.2.** Let $K$ be a special alternating knot with a positive reduced alternating knot diagram $D$, and let $L(K, K_1)$ be as defined above with $K$, $K_1$ assigned anti-parallel orientations, then $B(L(K, K_1)) \geq c(K) + 2$.

We will now provide the proof of Theorem 1.1 as a consequence of Theorem 3.2. The proof of Theorem 3.2 itself is in fact the challenging part of this paper which will require us to devote the rest of the paper to it.

**Proof.** Let $K_c$ be an arbitrary realization of $K$ in the cubic lattice and let $L(K_c)$ be the length of it. The union of all cubes of side length 1 with their center points on $K_c$ is a properly defined tubular
neighborhood of $K_c$. $K'_c = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) + K_c$ is on the boundary of this neighborhood, thus we have $\mathbf{B}(L(K_c, K'_c)) \geq c(K) + 2$ by Theorem 3.2 and the discussion leading to it. If we double the cubic lattice (so every edge is doubled and a new lattice point is inserted in the middle), then we obtain a realization of $\mathcal{L}(K_c, K'_c)$ on the cubic lattice whose length is $4L(K_c)$. By Proposition 3.1, we have $c + 2 < 4L(K_c)$. Since $K_c$ is arbitrary, this leads to $c + 2 < 4L_c(K) < 56L(K)$. That is, $L(K) > a_0c(K)$, where $a_0 \geq 1/56$ is a constant which does not depend on $K$. □

4. The Murasugi-Przytycki reduction operation

Consider a link diagram $D$ with the property that none of its Seifert circles contain other Seifert circles in its interior. Here $D$ is more general, not necessarily a positive or negative alternating knot diagram. If $D$ has a lone crossing $x$ between two $s$-circles $C_1$ and $C_2$, then we can reroute the overpass at this crossing as shown in Figure 6. The effect of this rerouted strand to the Seifert circle decomposition is that $C_1$ and $C_2$ are combined into one $s$-circle $C_0$, and any $s$-circle $C'$ sharing crossings with $C_2$ becomes an $s$-circle contained within this new $s$-circle $C_0$ (we say that $C'$ is swallowed by $C_0$), while any $s$-circle sharing crossings with $C'$ will now share the same crossings with $C_0$ instead. If we ignore the $s$-circles swallowed by $C_0$, then the effect of this operation on $G_S(D)$ is that the vertices $v_1, v_2$ corresponding to $C_1, C_2$, and any vertex $v$ corresponding to an $s$-circle sharing crossings with $C_2$, is contracted to the same vertex $v_0$. This operation is known as the Murasugi-Przytycki reduction operation (we shall call it the MP operation for short) [21]. Notice that in the above described operation, we can also use $C_1$ in the place of $C_2$, which will lead to a different diagram and a different Seifert graph as it will contract the vertices connected with $v_1$ instead. A complete example of this is shown in Figure 6.

![Figure 6](image-url)

Figure 6. Top left: The overpass at the single crossing to be rerouted; Top right: The two Seifert circles merge into one after the overpass is rerouted as shown. The rerouted overpass is marked by the thick line and is above the other strands at the crossings. Notice that the operation we used here rerouted the overpass through the three $s$-circles that are adjacent to the $s$-circle on the right side of the lone crossing, so these $s$-circles are swallowed by the newly created $s$-circle; Bottom left: The Seifert graph of the diagram at the top left of Figure 6 where the arrow indicates the direction of the MP operation and the dashed line indicates the re-routed overpass; Bottom right: The Seifert graph of the diagram at the top right with the vertices swallowed treated as being contracted.
Definition 4.1. For a given link diagram \( D \), we use \( r^+(D) \) and \( r^-(D) \) to denote the maximum number of MP operations that can be performed on positive and negative lone crossings respectively.

Remark 4.2. If a lone crossing is positive (negative), then applying the MP operation to the diagram \( D \) results in a new diagram \( D' \) with \( s(D') = s(D) - 1 \) and \( w(D') = w(D) - 1 \) \((w(D') = w(D) + 1)\), since the overpass crosses an existing Seifert circle an even number of times in the re-routing process hence does not change the writhe of \( D \) except that the lone positive (negative) crossing is eliminated. Thus in general, \( D \) can be deformed via the MP operations to new diagrams \( D' \) and \( D'' \) such that \( s(D') = s(D) - r^+(D) \), \( w(D') = w(D) - r^+(D) \) and \( s(D'') = s(D) - r^-(D) \), \( w(D'') = w(D) + r^-(D) \).

5. The HOMFLY-PT Polynomial

Let us recall that the HOMFLY-PT polynomial of an oriented link is defined using any diagram \( D \) of the link with the following rules: (a) If \( D_1 \) and \( D_2 \) represent the same link, then \( H(D_1, z, a) = H(D_2, z, a) \); (b) \( H(D, z, a) = 1 \) if \( D \) is an unknot and (c) \( aH(D_+, z, a) - a^{-1}H(D_-, z, a) = zH(D_0, z, a) \) where \( D_+, D_- \) and \( D_0 \) are identical link diagrams except at one crossing as shown in Figure 7. For the sake of simplicity we shall use \( H(D) \) for \( H(D, z, a) \).

![Figure 7](image)

**Figure 7.** The sign convention at a crossing of an oriented link and the smoothing of the crossing: the crossing in \( D_+ (D_-) \) is positive (negative) and is assigned \(+1 (-1)\) in the calculation of the writhe of the link diagram.

For our purposes, we will actually be using the following two equivalent forms of the skein relation (c) above:

\[
H(D_+, z, a) = a^{-2}H(D_-, z, a) + a^{-1}zH(D_0, z, a), \\
H(D_-, z, a) = a^2H(D_+, z, a) - azH(D_0, z, a).
\]

For any Laurent polynomial \( P(z, a) \) of variables \( z \) and \( a \), we will use \( E(P(z, a)) \) and \( e(P(z, a)) \) to denote the highest and lowest power of \( a \) in \( P(z, a) \). Furthermore, if we write \( P(z, a) \) as a polynomial of \( a \) with polynomials of \( z \) as its coefficients, then the highest power term in the coefficient polynomials of the \( a^{E(P(z, a))} \) and the \( a^{e(P(z, a))} \) terms are denoted by \( p_0^E(P(a, z)) \) and \( p_0^e(P(a, z)) \) respectively (these are monomials in the variable \( z \). In the case that \( P(z, a) = H(D, z, a) \), we will abbreviate the above notations by \( E(D) \), \( e(D) \), \( p_0^E(D) \) and \( p_0^e(D) \) respectively.

**Example 5.1.** For example, if \( D \) is the positive \( T(2, 4) \) torus link with its two components assigned anti-parallel orientations, we have \( H(D) = -z^{-1}a^{-5} + (z^{-1} + z)a^{-3} + za^{-1} \). Hence \( E(D) = -1 \), \( e(D) = -5 \), \( p_0^E(D) = z \) and \( p_0^e(D) = -z^{-1} \).

Given oriented link diagram \( D \), let \( w(D) \) be the writhe of \( D \) (which is defined as the sum of the number of positive crossings minus the number of negative crossings in \( D \)) and \( s(D) \) be the number of Seifert circles in \( S(D) \). The following result is well known.
Proposition 5.2. Let $D$ be any link diagram, then $E(D) \leq s(D) - w(D) - 1$ and $e(D) \geq -s(D) - w(D) + 1$. It follows that $s(D) \geq (E(D) - e(D))/2 + 1$, hence $b(D) \geq \xi(D) = (E(D) - e(D))/2 + 1$ where $b$ is the braid index of $D$.

Corollary 5.3. Let $D$ be any link diagram, then $E(D) \leq s(D) - w(D) - 1 - 2r^-(D)$ and $e(D) \geq -s(D) - w(D) + 1 + 2r^+(D)$. 

Proof. By Remark 4.2, $D$ is equivalent to $D'$ with $s(D') = s(D) - r^+(D)$ and $w(D') = w(D) - r^-(D)$, thus we have $e(D) = e(D') \geq -s(D') - w(D') + 1 = -s(D) - w(D) + 1 + 2r^+(D)$. Similarly, $E(D) = E(D'') \leq s(D'') - w(D'') - 1 = s(D) - w(D) - 1 - 2r^-(D)$. \hfill $\square$

In the rest of this paper, we will use $c(D)$ to denote the number of crossings in the diagram $D$. If $D$ is a positive diagram (whose crossings are all positive, that is), then $c(D) = w(D)$.

The following result is known, it can be proved using the two special link trivialization operations called Operations P and N in [15]. Note that it is applicable to any link diagram whose crossings are all positive, the link diagrams do not have to be alternating.

Proposition 5.4. Let $D$ be a positive link diagram (that is, all crossings in $D$ are positive), then $E(D) = s(D) - c(D) - 1$ and $p^D_0(D) = z^{c(D) - s(D) + 1}$.

We need two more results regarding the HOMFLY-PT polynomial.

Proposition 5.5. Let $K_1$ and $K_2$ be two links and let $K_1 \# K_2$, $K_1 \sqcup K_2$ be the connected sum and disjoint sum of $K_1$ and $K_2$ respectively, then

$H(K_1 \# K_2) = H(K_1)H(K_2)$

and

$H(K_1 \sqcup K_2) = H(K_1)H(K_2) \left( \frac{a - a^{-1}}{z} \right)$.

6. The proof of Theorem 3.2

We divide the proof of the theorem into several subsections to make it easier for our reader to follow.

6.1. The Seifert graph of $D$. We shall take a systematic approach to obtain $G_S(D)$ and to gain a good understanding of it. Let us choose the parallel copy $D'$ of $D$ so that its strand is always on our right hand side as we travel on the strand of $D$ following its orientation. Figure 8 shows the details at a crossing of $D$ and how this crossing looks like in $D$. The crossing is positive and is between two Seifert circles $C_1$ and $C_2$ as marked by dashed lines in the figure. Since $D$ and $D'$ have different orientations, the two positive crossings are between strands from the same component while the negative crossings involve strands from different components. After smoothing the crossings, it is obvious that $C_1$ and $C_2$ remain s-circles, with another s-circle inserted between them and this new s-circle shares a lone positive crossing both with $C_1$ and $C_2$. Extending this to all other 4 crossing-junctions of $D$, we see that $S(D)$ consists of the original set of s-circles from $D$ (which we shall call large s-circles), the added s-circles (each corresponding to a crossing of $D$) each sharing lone crossings with two and only two large s-circles (we shall call these s-circles medium s-circles), and the remaining s-circles created by
deleting the negative crossings in $D$ (which we shall call small $s$-circles). Notice that each medium $s$-circle shares lone negative crossings with exactly two small $s$-circles.

Let us denote by $D$ the diagram that contains the large $s$-circles and the medium $s$-circles. Let us call the vertices in $G_S(D)$ corresponding to large, medium and small $s$-circles large, medium and small vertices respectively. Then $G_S(D)$ is obtained from $G_S(D)$ by inserting a medium vertex to the middle of each edge of $G_S(D)$ (which divides the original edge into two lone edges), and placing a small vertex in each face of $G_S(D)$, and adding a lone edge (corresponding to a lone negative crossing in $D$) connecting this vertex to each medium vertex on the boundary of this face. See Figure 9.

![Figure 9](image-url)

Figure 9. Left: Two vertices in $G_S(D)$ connected by three positive edges; Middle: A medium vertex (marked by small circle) is added to the middle of each edge; Right: A small vertex is added to each face and an edge is added between a small vertex and any medium vertex on the boundary of this face.

**Remark 6.1.** We make the following observations. First, the length of any cycle in $G_S(D)$ is a multiple of 4, while the length of any cycle in $G_S(D)$ is a multiple of 8. $G_S(D)$ has $s(D)$ large vertices, $c(D)$ medium vertices, and $4c(D)$ edges. The total number of small vertices of $G_S(D)$ is the same as the total number of faces of $G_S(D)$, which is $c(D) - s(D) + 2$. Thus the total number of vertices in $G_S(D)$ is $s(D) + c(D) + c(D) - s(D) + 2 = 2c(D) + 2$. Thus, $s(D) = 2c(D) + 2$ and $w(D) = 0$.

### 6.2. The determination of $e(D)$ and $p^f_0(D)$

In this subsection we prove the following result.

**Lemma 6.2.** For any reduced alternating knot diagram $D$ whose crossings are all positive, we have $e(D) = s(D) - 3c(D) - 1$ and $p^f_0(D) = (-z)^{s(D) - c(D) - 1}$.

**Proof.** Consider the class $A$ of positive alternating link diagrams that are formed in the following way. Each diagram in $A$ contains a set of $s$-circles called large $s$-circles, and a set of $s$-circles called
medium s-circles satisfying the following conditions: (1) These s-circles do not contain each other in their interiors; (2) Each medium s-circle shares lone (positive) crossings with two and only two large s-circles; (3) The diagram is reduced. Let \( \tilde{D} \) be such a link diagram with \( s_1(\tilde{D}) \) being the number of large s-circles in \( \tilde{D} \), \( s_2(\tilde{D}) \) be the number of medium s-circles in \( \tilde{D} \) (so \( s(\tilde{D}) = s_1(\tilde{D}) + s_2(\tilde{D}) \) and \( w(\tilde{D}) = c(\tilde{D}) = 2s_2(\tilde{D}) \)). We claim that \( e(\tilde{D}) = s_1(\tilde{D}) - 3s_2(\tilde{D}) - 1 \) and \( p_0^2(\tilde{D}) = (z^{s_1(\tilde{D}) - s_2(\tilde{D}) - 1}) \). Since \( D \) belongs to \( \mathcal{A} \) with \( s_1(D) = s(D), s_2(D) = c(D) \), the statement of the lemma is proved if we can prove this claim.

Use induction on \( s_2(\tilde{D}) \geq 2 \). When \( s_2(\tilde{D}) = 2 \), it is necessary that \( s_1(\tilde{D}) = 2 \) as well, so \( \tilde{D} \) is the link diagram given in Example 5.1 with \( e(\tilde{D}) = -5 \) and \( p_0^2(\tilde{D}) = -z^{-1} \), hence the statement of the claim holds.

Assume that the statement is true for \( s_1(\tilde{D}) = n \geq 2 \). Consider the case when we have \( s_1(\tilde{D}) = n + 1 \) medium s-circles. Consider a medium s-circle \( \alpha \) that shares lone crossings \( x_1 \) and \( x_2 \) with two large s-circles \( C_1 \) and \( C_2 \).

Case 1. \( \alpha \) is the only medium s-circle between its two neighboring large s-circles. In this case \( x_1 \) (and also \( x_2 \)) corresponds to a single edge in \( G_s(\tilde{D}) \) which cannot be a bridge edge. Apply (5.1) to \( x_1 \). Observe that \( \tilde{D}_- \) simplifies (by a Reidemeister II move) to a new diagram \( \tilde{D}_- \) in \( \mathcal{A} \) with \( s_1(\tilde{D}_-) = s_1(\tilde{D}) - 1, s_2(\tilde{D}_-) = s_2(\tilde{D}) - 1 \). On the other hand, the smoothing of \( x_1 \) may result in some additional bridge edge pairs in \( \tilde{D}_0 \) other than \( x_2 \). Let \( k \) be the number of such edge pairs which correspond to negligent pairs of crossings. The removing of these negligent crossings (including \( x_2 \)) yields a link diagram \( \tilde{D}_0 \) that is still in \( \mathcal{A} \) with \( s_1(\tilde{D}_0) = s_1(\tilde{D}) - 1, s_2(\tilde{D}_0) = s_2(\tilde{D}) - 1 - k \). By the induction hypothesis we now have

\[
-2 + e(\tilde{D}_-) = -2 + (s_1(\tilde{D}) - 1) - 3(s_2(\tilde{D}) - 1) - 1 = s_1(\tilde{D}) - 3s_2(\tilde{D}) - 1
\]

and

\[
-1 + e(\tilde{D}_0) = -1 + (s_1(\tilde{D}) - k) - 3(s_2(\tilde{D}) - 1 - k) - 1 = s_1(\tilde{D}) - 3s_2(\tilde{D}) + 1 + 2k.
\]

Thus \( e(\tilde{D}) = s_1(\tilde{D}) - 3s_2(\tilde{D}) - 1 \) since \( s_1(\tilde{D}) - 3s_2(\tilde{D}) + 1 + 2k > s_1(\tilde{D}) - 3s_2(\tilde{D}) - 1 \).

Case 2. \( \alpha \) is not the only medium s-circle between its two neighboring large s-circles. Say there are \( m \geq 1 \) other medium s-circles sharing crossings with \( C_1 \) and \( C_2 \). Again we apply (5.1) to \( x_1 \). In this case, \( \tilde{D}_0 \) simplifies to \( \tilde{D}_0' \in \mathcal{A} \) (after removing the negligent crossing \( x_2 \)) with \( s_1(\tilde{D}_0') = s_1(\tilde{D}), s_2(\tilde{D}_0') = s_2(\tilde{D}) - 1 \). On the other hand, \( \tilde{D}_- \) simplifies to a diagram that is the connected sum of a diagram \( \tilde{D}' \in \mathcal{A} \) with \( m \) copies of positive \( T(2,2) \), and \( s_1(\tilde{D}') = s_1(\tilde{D}) - 1, s_2(\tilde{D}') = s_2(\tilde{D}) - 1 - m \). We have \( H(T(2,2)) = -z^{-1}a^{-3} + (z^{-1} + z)a^{-1} \). Thus by the induction hypothesis and Proposition 5.5 we have

\[
-2 + e(\tilde{D}_-) = -2 - 3m + e(\tilde{D}_-)
\]

\[
= -2 - 3m + (s_1(\tilde{D}) - 1) - 3(s_2(\tilde{D}) - 1 - m) - 1 = s_1(\tilde{D}) - 3s_2(\tilde{D}) - 1
\]

\[
< -1 + e(\tilde{D}_0)
\]

\[
= -1 + s_1(\tilde{D}) - 3(s_2(\tilde{D}) - 1) - 1 = s_1(\tilde{D}) - 3s_2(\tilde{D}) + 1.
\]

Thus \( e(\tilde{D}) = s_1(\tilde{D}) - 3s_2(\tilde{D}) - 1 \).

Notice that in both cases \( e(\tilde{D}) \) comes from the term \( a^2H(\tilde{D}_-) \). Thus in the first case we have

\[
p_0^2(\tilde{D}) = p_0^2(\tilde{D}_-)
\]
\[(z - s_1(\tilde{D}) - 1)(s_2(\tilde{D}) - 1) - 1\]

and in the second case we also have
\[p^h_0(\tilde{D}) = (z - m)p^h_0(\tilde{D}')\]
\[= (z - m + (s_1(\tilde{D}) - 1) - (s_2(\tilde{D}) - 1) - m) - 1\]
\[= (z - s_1(\tilde{D}) - s_2(\tilde{D}) - 1).\]

So the statement of the claim also holds and the lemma is proved. \qed

6.3. The determination of \(E(D)\) and \(p^h_0(D)\). In this subsection we prove the following lemma.

**Lemma 6.3.** For any reduced alternating knot diagram \(D\) whose crossings are all positive, we have \(E(D) = 2s(D) - 1\) and \(p^h_0(D) = z^{2c(D)} - 2s(D) + 1\).

Let us call an edge in \(G_S(D)\) a positive (negative) edge if the crossing in \(D\) corresponding to it is positive (negative). Recall that positive edges are the ones connecting a large vertex and a medium vertex while all the negative edges are the ones connecting a medium vertex and a small vertex. Each medium vertex has two negative edges connected to it and these two edges connect to two distinct small vertices so they are uniquely associated with this medium vertex. This naturally divide the negative edges of \(G_S(D)\), hence the negative crossings of \(D\), into \(c(D)\) pairs. As illustrated in Figure 8, it is necessary that the two strands at the negative crossings belong to different components. Mark the two components of \(D\) by 1 and 2. For each pair of these negative crossings, the strand belonging to component 1 is the under strand at one and only one of the two crossings (as shown in Figure 8), and we will choose this crossing and place it in a crossing set \(C\). That is, \(C\) contains \(c(D)\) negative crossings and at these crossings, the strands belonging to component 1 are always the under strands.

Let \(C'\) be the set of negative edges in \(G_S(D)\) corresponding to the crossings in \(C\).

Furthermore, if we go around a small vertex in \(G_S(D)\) in any orientation, we encounter the edges from \(C'\) and not from \(C'\), alternately, see Figure 10 for an example. This can be observed by following the strand of a small s-circle, since the crossings we encounter along this s-circle are always negative, we either always arrive at the crossings on the overpass or always arrive at the crossings on the underpass (depending on the orientation we choose to travel), but the strands have to alternate between component 1 and component 2.

For any given face \(F\) of \(G_S(D)\), an edge \(\gamma\) on \(\partial F\), the boundary of \(F\), is said to be proper with respect to \(F\), if the negative edge connected to the medium vertex added to \(\gamma\) in \(G_S(D)\) that is contained in \(F\) belongs to \(C'\). We construct a spanning tree of \(G_S(D)\) in the following way.

Step 1. Start with any face \(F_0\) of \(G_S(D)\), and choose any proper edge \(\gamma_1\) with respect to \(F_0\), and delete \(\gamma_1\) from the graph. This eliminates the face \(F_0\).

Step 2. Let \(F_1\) be the face that shares \(\gamma_1\) with \(F_0\) on its boundary, and choose a proper edge \(\gamma_2\) with respect to \(F_1\). Notice that it is necessary that \(\gamma_2 \neq \gamma_1\). Delete \(\gamma_2\) from the graph and this eliminates the face \(F_1\).
Steps 3 to \( \beta \). We now continue this process. At each step, we choose a proper edge from the current face, which shares the proper edge chosen to be deleted from the previous edge on its boundary, to be deleted from the graph. Since there are \( c(D) - s(D) + 2 \) faces, we can do this \( \beta = \beta(D) = c(D) - s(D) + 1 \) times and at the end we reach a spanning tree of \( G_S(D) \). Notice that each \( \gamma_j \) corresponds to a unique negative edge \( \gamma_j' \) not belonging to \( C' \), see the right of Figure 10.

**Figure 10.** Left: \( G_S(D) \) of a special diagram \( D \) where the edges numbered are the proper edges by the order they are deleted in obtaining the spanning tree of \( G_S(D) \). The initial face is marked by a triangle; Middle: How the negative edges in \( C' \) are paired with the negative edges not in \( C' \) (marked by thick lines). The medium s-circles are marked by small circles and the small s-circles are marked by solid dots; Right: The negative edges not in \( C' \) that are associated with the proper edges (as indicated in the figure at left) \( G_S(D) \) are highlighted by thick lines. The deleted edge in \( C' \) is marked by dashed line, the medium vertex and the edge marked by arrows indicate where an additional MP operation can be performed.

**Remark 6.4.** We can perform \( \beta = c(D) - s(D) + 1 \) MP operations on the crossings corresponding to the \( \gamma_j' \)'s, in the sequential order of \( \gamma_0', \gamma_1', \ldots, \gamma_{\beta - 1}', \gamma_\beta' \). The reason is that at each step \( \gamma_j' \) brings with it a new medium vertex corresponding to a medium s-circle that has not been affected by the previous MP operations, hence \( \gamma_j' \) corresponds to a negative lone crossing and an MP operation can be performed on it using the direction from the medium s-circle to the small s-circle. Now consider the case when an edge \( \gamma_0 \in C' \) is deleted (meaning its corresponding negative crossing is smoothed). Let \( \gamma_0' \notin C' \) be the edge that is paired with \( \gamma_0 \) and let \( F_0 \) be the face of \( G_S(D) \) that contains \( \gamma_0' \), and use \( F_0 \) as the starting face to obtain the negative edges \( \gamma_j', 1 \leq j \leq \beta \). This time, as we perform the MP operations, the medium vertex that \( \gamma_0' \) is connected to is shielded away from being swallowed by the combined vertices (due to the MP operations) since \( \gamma_0 \) has been deleted. See the right side of Figure 10 for an illustration of this. Therefore \( \gamma_0' \) remains a lone edge after the \( \beta \) MP operations discussed above are performed, and we can perform one more MP operation on the crossing corresponds to it. This means that if we smooth a negative crossing in \( C \), then we can perform MP operations on \( \beta + 1 = c(D) - s(D) + 2 \) negative crossings, regardless whether some crossings in \( C \) have been flipped or not, since the MP operations in the above discussion only used negative crossings not in \( C \).

We are now ready to prove Lemma 6.3.

**Proof.** Choose any crossing in \( C \) and apply skein relation (5.2) to it. We have \( H(D) = H(D_\pm) = a^2H(D_\pm) - azH(D_0) \). Notice that \( s(D_0) = s(\mathbb{D}) = 2c(D) + 2 \), \( w(D_0) = w(\mathbb{D}) + 1 = 1 \). Thus by Remark 6.4 and Corollary 5.3, we have

\[
E(-azH(D_0)) \leq 1 + s(D_0) - w(D_0) - 1 - 2r^-(D_0)
\]
Thus the $-az H(D_0)$ term will not make a contribution to the $a^{2s(D)-1}$ term and will be ignored. We now consider $D_+$, which is obtained by flipping the crossing. We will choose another crossing from $S$ and repeat the above argument. Each time we can ignore the one obtained by smoothing the crossing. At the end, we arrive at a single term of the form $a^{2c(D)} H(D^f)$ where the diagram $D^f$ is obtained by flipping all crossings in $S$. We observe that $D^f$ separates into two disjoint copies of $D$ (since component 1 now sits on top of component 2 at all crossings where they intersect), hence $H(D^f) = H^2(D)(a - a^{-1})/z$ by Proposition 5.4. Hence by Propositions 5.4 we have (keep in mind that $w(D) = c(D)$):

$$E(a^{2c(D)} H(D^f)) = 2c(D) + 2s(D) - 2w(D) - 2 + 1 = 2s(D) - 1,$$

and

$$p_0^b(a^{2c(D)} H(D^f)) = z^{2c(D) - 2s(D) + 2 - 1} = z^{2c(D) - 2s(D) + 1}.$$  

This proves that $E(\mathbb{D}) = 2s(D) - 1$ and $p_0^b(\mathbb{D}) = z^{2c(D) - 2s(D) + 1}$.  

\[\square\]

6.4. The determination of $e(\mathbb{D})$ and $p_0^b(\mathbb{D})$. In this section we prove the following lemma.

**Lemma 6.5.** If $D$ is a special alternating knot diagram with positive crossings, then $e(\mathbb{D}) = 2s(D) - 2c(D) - 3$ and $p_0^b(\mathbb{D}) = -z^{2s(D) - 3}$.

**Definition 6.6.** Let $\tilde{D} \in \mathcal{A}$, where $\mathcal{A}$ is the class of alternating knot diagrams defined in the proof of Lemma 6.2 such that the graph $G'$ obtained from $G_S(\tilde{D})$ by removing the medium vertices is proper (note that $G'$ may not be bipartite anymore hence may not be the Seifert graph of a knot diagram). Consider the link diagram corresponding to a Seifert graph obtained by placing a vertex (called a small vertex) on the boundary of the face that contains the small vertex. Each such edge corresponds to a negative crossing in the corresponding diagram. If a total $k \geq 0$ such edges are added, then the corresponding link diagram is denoted by $\tilde{D}_k$.

We would like to prove the following lemma first.

**Lemma 6.7.** Let $\tilde{D}_k$ be as defined in Definition 6.6. Let $s_1$ and $s_2$ be the numbers of large and medium vertices in $G_S(\tilde{D})$, and $f = s_2 - s_1 + 2$ be the number of faces in $G_S(\tilde{D})$ (which is the same as the number of small vertices in $G_S(\tilde{D}_k)$), then $e(\tilde{D}) = k + 2s_1 - 4s_2 - 3$ and $p_0^b(\tilde{D}) = (-z)^{k+2s_1-2s_2-3}$.

**Proof.** We use induction on $k$. If $k = 0$, then $\tilde{D}$ is a disjoint union of $\tilde{D}$ and $f$ copies of trivial knots, hence the result follows from Proposition 5.5 and Lemma 6.2. Assume that this result is true for $k - 1 \geq 0$ and consider the case $k$. Choose any negative crossing in $\tilde{D}_k$ and apply (5.2) to it: $H(D_-) = a^2 H(D_+) - az H(D_0)$ with $D_- = \tilde{D}_k$ and $D_0 = \tilde{D}_{k-1}$ (to which the induction hypothesis applies). There are two cases to consider.

Case 1: the negative crossing is not a lone crossing. In this case it is necessary that $k \geq 2$ and $H(D_+)$ simplifies (via a Reidemeister II move) to $\tilde{D}_{k-2}$. Thus we have

$$e(a^2 H(D_+)) = 2 + e(\tilde{D}_{k-2}) = 2 + k - 2 + 2s_1 - 4s_2 - 3 = k + 2s_1 - 4s_2 - 3,$$
with 
\[ p_0^\ell(a^2 H(D_+)) = p_0^\ell(D_{k-2}) = (-z)^{k-2s_1+2s_2+3}. \]

On the other hand, we have 
\[ e(-azH(D_0)) = 1 + e(\hat{D}_{k-1}) = 1 + k - 1 + 2s_1 - 4s_2 - 3 = k + 2s_1 - 4s_2 - 3, \]
with 
\[ p_0^\ell((a-z)H(D_0)) = (-az)p_0^\ell(\hat{D}_{k-1}) = (-z)^{k+2s_1-2s_2+3}. \]

It follows that 
\[ e(\hat{D}_k) = k + 2s_1 - 4s_2 - 3 \]
and 
\[ p_0^\ell(\hat{D}_k) = (-z)^{k+2s_1-2s_2+3}. \]

Case 2. The negative crossing is a lone crossing, which corresponds to a (negative) lone edge connecting a small vertex \(v_s\) and a medium vertex \(v_m\) in \(G_S(\hat{D}_k)\). Again in this case the \(-azH(D_0)\) term yields the needed \(e(\hat{D}_k)\) and \(p_0^\ell(\hat{D}_k)\), so it suffices to show that \(e(a^2 H(D_+)) > k + 2s_1 - 4s_2 - 3\).

Let \(F\) be the face of \(G_S(\hat{D}_k)\) containing \(v_s\) and let \(T_F\) be a spanning tree of \(G'\) that does not contain any edge on the boundary of \(F\). Note for any edge in \(T_F\) corresponds to two large vertices each connected to a common medium vertex by a (positive) lone edge, and an MP operation can be performed using one of these lone crossings as shown in the left of Figure 11. The effect of this move is that \(C_1\) and \(C_2\) are swallowed by the newly created \(s\)-circle. Since there are \(s_1 - 1\) edges in \(T_F\), we can perform \(s_1 - 1\) such operations. Furthermore, these operations do not affect the medium vertices on the boundary of \(F\), hence the flipped crossing remains a positive lone crossing, and one more MP operation can be performed. Thus we have \(r^+(D_+) \geq s_1\). Notice that 
\[ s(D_+) = 2s_2 + 2 \] and 
\[ w(D_+) = 2s_2 + 2 - k. \] It follows from Corollary 5.3 that 
\[ e(a^2 H(D_+)) \geq 2 - s(D_+) - w(D_+) + 1 + 2r^+(D_+) \]
\[ \geq 2 - 2(s_2 + 2) - (2s_2 + 2 - k) + 1 + 2s_1 \]
\[ = k + 2s_1 - 4s_2 - 1 \]
\[ > k + 2s_1 - 4s_2 - 1. \]

\[ \square \]

**Figure 11.** Left: The effect of performing on MP operation on a positive lone crossing in \(\hat{D}_k\). The edge used here corresponds to an edge in the spanning tree \(T_F\); Middle and Right: Two examples of how the edges of the spanning tree \(T_F\) are used to perform the MP operations without affecting the boundary of the face that contains \(v_s\) and the edge corresponding to the flipped crossing.

We are now ready to prove Lemma 6.5

**Proof.** If all maximal paths in \(G_S(D)\) are proper, then the statement of the lemma follows from Lemma 6.7 by substituting \(s_1 = s(D), s_2 = c(D)\) and \(k = 2c(D)\). Let us consider the case that \(G_S(D)\) contains
maximal paths that are not proper. Remark 2.5, these maximal paths contain vertices that have degree 2 in $G_S(D)$. Furthermore, if there are multiple vertices with this property in a maximal path, they appear consecutively along the path, and the two end vertices of the path must have degree more than 2 in $G_S(D)$. Let $k_0$ be the total number of degree 2 vertices in $G_S(D)$. Each such vertex is a large vertex of $G_S(D)$ which has two lone edges connected to it. Of these two edges, we shall choose one that we encounter first by walking through the maximal path (it do not matter in which direction). Denote the set of these chosen edges by $X$. We have $|X| = k_0$. We now choose any crossing in $X$ and apply (5.1) to it: $H(D_+) = a^{-2}H(D_-) + a^{-1}zH(D_0)$. We observe that smoothing a crossing in $X$ increases the number of MP operations in the diagram by one since the we have created a positive nugatory crossing which corresponds to an isolated medium $s$-circle. That is, $s(D_0) = s(\mathbb{D}) = 2c(D) + 2$, $w(D_0) = w(\mathbb{D}) - 1 = -1$, and $r^+(D_0) \geq s(D)$. It follows that

$$e(a^{-1}zH(D_0)) \geq -1 - (2c(D) + 2) - (-1) + 1 + 2s(D) = 2s(D) - 2c(D) - 1 > 2s(D) - 2c(D) - 3.$$ 

Thus we shall ignore the $a^{-1}zH(D_0)$ term, and will continue the process with the $a^{-2}H(D_-)$ term. That is, we start with diagram obtained by flipping the first crossing, and choose another crossing in $X$, and apply (5.1) to it. Flipping the previously chosen crossings do not affect our ability to perform the $s(D) - 1$ MP operations based on any spanning tree of $G_S(D)$ as described in Case 2 of the proof of Lemma 6.7, thus the same argument always applies to the diagram obtained by smoothing the crossing. At the end, we arrive at a term of the form $a^{-2k_0}H(D^X)$, where $D^X$ is obtained by flipping every crossing in $X$. Now observe that for each flipped crossing, a Reidemeister II move can be used to simplify the diagram: the result is that a large $s$-circle and two medium $s$-circles are combined into one medium $s$-circle and the edges connecting to two medium $s$-circles to small vertices are all connected to this newly created medium vertex. Thus, $D^X$ simplifies to a diagram $\hat{D}_k$ as defined in Definition 6.6 with $s_1 = s(D) - k_0$, $s_2 = c(D) - k_0$ and $k = 2c(D)$. By Lemma 6.7 we have

$$e\left(a^{-2k_0}H(D^X)\right) = -2k_0 + e(\hat{D}_k) = -2k_0 + 2c(D) + 2(s(D) - k_0) - 4(c(D) - k_0) - 3 = 2s(D) - 2c(D) - 3,$$

and

$$p^f_0\left(a^{-2k_0}H(D^X)\right) = p^f_0\left(\hat{D}_k\right) = \left(-z\right)^{2c(D) + 2(s(D) - k_0) - 2(c(D) - k_0) - 3} = -z^{2s(D) - 3}.$$ 

This shows that $e(\mathbb{D}) = 2s(D) - 2c(D) - 3$ and $p^f_0(\mathbb{D}) = -z^{2s(D) - 3}$.\qed

6.5. The determination of $E(\mathbb{D}_k)$ and $e(\mathbb{D}_k)$. In this section we prove the following lemma.

**Lemma 6.8.** If $k \geq 0$, then $e(\mathbb{D}_k) = 2s(D) - 2c(D) - 2k - 3$ and

$$E(\mathbb{D}_k) = \begin{cases} 
2s(D) - 2k - 1 & \text{if } 0 \leq k \leq s(D); \\
-1 & \text{if } k > s(D)
\end{cases}$$

with

$$p^h_0(\mathbb{D}_k) = \begin{cases} 
z^{2c(D) - 2s(D) + 1} & \text{if } 0 \leq k \leq s(D); \\
z & \text{if } k > s(D).
\end{cases}$$
On the other hand, if $k \leq 0$, then $E(D_k) = 2s(D) - 2k - 1$, and

$$e(D_k) = \begin{cases} 
2s(D) - 2k - 2c(D) - 3 & \text{if } k \geq s(D) - c(D) - 1; \\
1 & \text{if } k < s(D) - c(D) + 1 
\end{cases}$$

with

$$p_0^k(D_k) = \begin{cases} 
-z^{2s(D)-3} & \text{if } k \geq s(D) - c(D); \\
-2z & \text{if } k = s(D) - c(D) - 1, \text{ or } s(D) = 2; \\
-z & \text{if } k < s(D) - c(D) - 1, \text{ or } k = s(D) - c(D) - 1, \text{ or } s(D) > 2.
\end{cases}$$

Proof. Use induction on $k$. The case of $k = 0$ has already been proved. Assume the lemma is true for some $k - 1 \geq 0$, consider the case $k$. For $D_k$, apply (5.1) to one of the positive crossings in the $k$ full positive twists: $H(D_k) = H(D_+) = a^{-2}H(D_-) + a^{-1}zH(D_0)$. Notice that $D_0$ deforms to the trivial knot, while $D_+$ simplifies to $D_{k-1}$ to which the induction hypothesis applies. Thus we have $e(a^{-2}H(D_-)) = -2 + 2s(D) - 2c(D) - 2(k-1) - 3 = 2s(D) - 2c(D) - 2k - 3 < e(a^{-1}zH(D_0)) = -1$ for any $k \geq 0$. On the other hand, if $k < s(D)$, then $E(a^{-2}H(D_-)) = -2 + 2s(D) - 2(k-1) - 1 = 2s(D) - 2k - 1 > E(a^{-1}zH(D_0)) = -1$, with $p_0^k(a^{-2}H(D_-)) = p_0^k(D_-) = z^{2s(D)-2s(D)+1}$. If $k = s(D)$, then $E(a^{-2}H(D_-)) = 2s(D) - 2k - 1 = -1 = E(a^{-1}zH(D_0))$, but $p_0^k(D_k) = p_0^k(a^{-2}H(D_-)) = z^{2s(D)-2s(D)+1}$ since $c(D) > s(D)$ hence $2s(D) - 2s(D) + 1 > 1$ as $p_0^k(a^{-1}zH(D_0)) = z$. If $k \geq s(D) + 1$, then we have $E(a^{-2}H(D_-)) = -2 - 1 < E(a^{-1}zH(D_0)) = -1$ hence $E(D_k) = -1$ with $p_0^k(D_k) = p_0^k(az^{-1}H(D_0)) = z$. This proves the case for $k \geq 0$.

For $D_k$ with $k < 0$, apply (5.2) to one of the negative crossings in the $k$ full negative twists: $H(D_k) = H(D_-) = a^{2}H(D_+) - azH(D_0)$. Again $D_0$ deforms to the trivial knot, while for $k \leq -1$, $D_+$ simplifies to $D_{k+1}$ to which the induction hypothesis applies. Thus we have

$$E(D_k) = E(a^{2}H(D_+)) = 2 + 2s(D) - 2(k+1) - 1 = 2s(D) - 2k - 1 > E(-azH(D_0)) = 1$$

for any $k \leq 0$. On the other hand, if $s(D) - c(D) - 1 \leq k \leq -1$, then

$$e(D_k) = e(a^{2}H(D_+)) = 2 + 2s(D) - 2(k+1) - 2c(D) - 3 = -2k - 2c(D) - s(D) + 1 - 1 < e(-azH(D_0)) = 1,$$

with

$$p_0^k(D_{-k}) = p_0^k(a^{2}H(D_+)) = p_0^k(D_{+2}) = -z^{2s(D)-3}.$$

If $k = s(D) - c(D) - 2$, and $s(D) = 2$, then

$$e(D_k) = e(a^{2}H(D_+)) = 2 + 2s(D) + 2(k-1) - 2c(D) - 3 = 2k - 2c(D) - s(D) - 3 = 1 = e(-azH(D_0)),$$

with $p_0^k(D_k) = p_0^k(a^{2}H(D_+)) = p_0^k(azH(D_0)) = -z = -2z$. If $k = s(D) - c(D) - 2$, but $s(D) > 2$, then $e(D_k) = e(a^{2}H(D_+)) = 1$, with $p_0^k(D_k) = p_0^k(a^{2}H(D_+)) = -z^{2s(D)-3}$ since $2s(D) - 3 > 1$. Finally, if $k < s(D) - c(D) - 2$, then $e(a^{2}H(D_+)) = 3 > 1 = e(-azH(D_0)) = e(D_k)$, with $p_0^k(D_k) = p_0^k(azH(D_0)) = -z$. □
Theorem 3.2 now follows trivially: the link $L(K, K_1)$ defined there corresponds to $D_k$ for some $k$. By Lemma 6.8, we have $E(D_k) - e(D_k) \geq 2c(D) + 2$ for any $k$. We end the paper with the following remark.

**Remark 6.9.** Of course, the immediate question one may ask is, can Theorem 3.2 be generalized to alternating knots that are not special? We note that the proof of Theorem 3.2 relied on the nice structure of the Seifert circle decomposition of $D$, hence that of $G_S(D)$. It is not clear to the author how to get around this, but the result in this paper is indeed very encouraging that the ropelength conjecture may hold in general for all alternating knots and links!

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