The Schwinger and Thirring models at finite chemical potential and temperature

Ramón F. Alvarez-Estrada
Departamento de Física Teórica, Universidad Complutense, 28040, Madrid, Spain

Angel Gómez Nicola
Theoretical Physics Group, Imperial College, Prince Consort Road, London SW7 2BZ, United Kingdom
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The imaginary time generating functional $Z$ for the massless Schwinger model at nonzero chemical potential $\mu$ and temperature $T$ is studied in a torus with spatial length $L$. The lack of hermiticity of the Dirac operator gives rise to a non-trivial $\mu$ and $T$ dependent phase $J$ in the effective action. When the Dirac operator has no zero modes (trivial sector), we evaluate $J$, which is a topological contribution, and we find exactly $Z$, the thermodynamical partition function, the boson propagator and the thermally averaged Polyakov loop. The $\mu$-dependent contribution of the free partition function cancels exactly the nonperturbative one from $J$, for $L \to \infty$, yielding a zero charge density for the system, which bosonizes at nonzero $\mu$. The boson mass is $e/\sqrt{\pi}$, independent on $T$ and $\mu$, which is also the inverse correlation length between two opposite charges. Both the boson propagator and the Polyakov loop acquire a new $T$ and $\mu$ dependent term at $L \to \infty$. The imaginary time generating functional for the massless Thirring model at nonzero $T$ and $\mu$ is obtained exactly in terms of the above solution of the Schwinger model in the trivial sector. For this model, the $\mu$ dependences of the thermodynamical partition function, the total fermion number density and the fermion two-point correlation function are obtained. The phase $J$ displayed here leads to our new results and allows to complement non trivially previous studies on those models.

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I. INTRODUCTION

The Schwinger model is QED in 1+1 space-time dimensions. Although it is a toy model, it shares many interesting physical properties with more realistic theories such as QCD or the electroweak theory. It is perhaps the simplest example in which gauge invariance does not necessarily imply a massless gauge boson, analogously to the Higgs phenomenon. Other interesting properties of the model are dynamical mass generation, chiral symmetry breaking and confinement. The model with massless fermions was shown to be exactly solvable in vacuum (that is, without thermal effects) long time ago. It is equivalent to a theory describing a free boson with mass $e/\sqrt{\pi}$ (bosonization), which is physically a fermion-antifermion bound state (confinement). The finite mass implies a finite correlation length, which physically corresponds to charge screening, long range forces being absent. On the other hand, chiral symmetry is broken through the chiral anomaly rather than spontaneously, since Coleman’s theorem prevents any continuous symmetry to be spontaneously broken in two dimensions. When a mass parameter for the fermions is included, the model is no longer solvable but it is still possible to analyse exactly some of the above properties, like fermion confinement.

In the last ten years, there has been a renewed interest in the study of the Schwinger model including statistical mechanics features. The model was also solved and, in particular, the two point correlation functions and the partition function were obtained at finite temperature $T$ and zero chemical potential $\mu$ in when the Dirac operator has no zero modes. Its thermodynamics can be expressed in terms of those of a free boson of mass $e/\sqrt{\pi}$ and free fermions, i.e., bosonization also takes place at finite temperature. A more complete study of the Schwinger model on a torus, which naturally incorporates temperature effects, still at $\mu = 0$, has been performed in . More specifically, in the model was treated with an arbitrary number of zero modes and the two-point fermion correlation function was calculated, whereas in higher correlation functions were obtained. In , the correlation functions have also been studied for nonzero $\mu$. In a recent work, the problem of charge screening at finite temperature (with $\mu = 0$) in the Schwinger model is analysed, in connection with the spontaneous breaking of the discrete $Z$ symmetry, which corresponds to the freedom of choosing gauge fields in the Euclidean time direction with any winding number around $S^1$. These nontrivial gauge transformations will play an essential role in our analysis with a nonzero chemical potential.

On the other hand, we remind that the Thirring model, which describes massless fermions in 1+1 dimensions with a quartic self-interaction, can also be explicitly solved in vacuum ($T = \mu = 0$) and free fermions, i.e., bosonization also takes place at finite temperature. A more complete study of the Schwinger model on a torus, which naturally incorporates temperature effects, still at $\mu = 0$, has been performed in . More specifically, in the model was treated with an arbitrary number of zero modes and the two-point fermion correlation function was calculated, whereas in higher correlation functions were obtained. In , the correlation functions have also been studied for nonzero $\mu$. In a recent work, the problem of charge screening at finite temperature (with $\mu = 0$) in the Schwinger model is analysed, in connection with the spontaneous breaking of the discrete $Z$ symmetry, which corresponds to the freedom of choosing gauge fields in the Euclidean time direction with any winding number around $S^1$. These nontrivial gauge transformations will play an essential role in our analysis with a nonzero chemical potential.

In this paper, we shall study, first, the Schwinger model in a medium at thermodynamical equilibrium, by introducing both the temperature and the fermion chemical potential $\mu$. By considering a nonzero $\mu$, we are able to study the system when there is a finite net fermion charge density (in the free case, $\mu$ is just the Fermi energy). Some of the questions that naturally arise are: i) whether one can provide simple solutions for the Schwinger model with a non zero $\mu$, which extend previous studies non trivially, ii) whether bosonization takes place at finite fermion charge density and, if so, which is the boson mass, iii) which is the net fermionic charge of the resulting system, that is, whether the fermions are still confined into neutral mesons and iv) how does the chemical potential affect charge screening. We shall try to give answers to all of these questions. The second aim of this work is to provide an exact solution for the Thirring model at nonvanishing $T$ and $\mu$ has been obtained in terms of the fermionic one (with an external electromagnetic source) for the Schwinger model . The Thirring model at nonvanishing $T$ and $\mu$ has been analysed in for real time and in in the torus.

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The plan of this paper is as follows. In section , we shall deal with the generating functional $Z$ of the Schwinger model at nonzero $T$ and $\mu$, analysing several important items: the fermionic generating functional $Z_f$ with an external electromagnetic field, the role of the zero modes, the determinant of the Dirac operator, etc. By following steps similar to those in , the lack of hermiticity of the Dirac operator and a non-trivial phase factor $J$ will be genuine and crucial features of the $\mu \neq 0$ case. They both will make necessary to extend the methods developed in . From section onwards, we shall restrict ourselves to the trivial sector, which is the only relevant one in order to study the thermodynamics of the system. We shall get $Z$, $Z_f$ and $J$ by using functional methods, generalising what was done for $\mu = 0$ in and deriving the proper extension of the point-splitting regularization when $\mu \neq 0$. Section is devoted to: a) several physical results for the Schwinger model: the fermion charge density, the thermodynamical partition function, the boson propagator in the trivial sector, the Polyakov loop (the order parameter of the confining symmetry) and the screening length, b) the consistency of our methods. The tasks of obtaining an explicit solution and new results for the Thirring model at nonzero $T$ and $\mu$ are undertaken in section . Section contains the conclusions and some discussions. Several results pertaining to the zero modes in the Schwinger model at nonzero $T$ and $\mu$ are collected in an Appendix.
II. THE GENERATING FUNCTIONAL AT FINITE TEMPERATURE AND DENSITY

Our starting point will be the generating functional for the Schwinger model in the imaginary time formalism of Thermal Field Theory \[\mathbb{Z}\mathbb{Z}\]. We shall work in Euclidean two-dimensional space-(imaginary) time. In principle, we shall keep the length of the system \(L\) as finite, by imposing suitable boundary conditions in the spatial direction (see below). Thus, one properly defines the spectrum of the Dirac operator and avoids infrared divergences \[\mathbb{Z}\mathbb{Z}\mathbb{Z}\mathbb{Z}\].

At the end of the calculations we shall take the \(L \rightarrow \infty\) limit. The finite density effects will be implemented by including a chemical potential \(\mu\) associated to the conservation of the total electric charge (or the number of electrons minus that of positrons). Let \(A = (A^\mu) = (A^0, A^1)\) be the electromagnetic potential. Then, the generating functional reads

\[
Z[J, \xi, \bar{\xi}] = N(\beta) \int DA \exp \left[ \int_T d^2x \left( \Gamma[A] + JA \right) \right] Z_f[A, \xi, \bar{\xi}]
\]

where the fermionic generating functional is

\[
Z_f[A, \xi, \bar{\xi}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ \int_T d^2x \left( -\bar{\psi} \mathcal{D}(A; \mu)\psi + \xi \psi + \bar{\xi} \psi \right) \right]
\]

and the Dirac operator is given by

\[
\mathcal{D}(A; \mu) = \partial - ie A - \mu \gamma^0
\]

In the above equations, \(N(\beta)\) is a temperature dependent normalisation constant, \(\beta = 1/T\), \(T\) being the temperature, \(\int_T\) is the integral over the Euclidean two-dimensional torus \([0, \beta] \times [0, L]\) and \(e\) is the electric charge, which has dimensions of energy. The fermionic and bosonic external sources are \(\xi, \bar{\xi}\) and \(J\) respectively. The electric field \(E = F_{01} = \partial_0 A_1 - \partial_1 A_0\) and \(\alpha\) is the covariant gauge-fixing parameter. It is important to remark here that the above covariant gauge fixing does not fix the gauge completely on the torus. There is still some residual gauge arbitrariness related to global gauge transformations, which we shall deal with later. The Faddeev-Popov determinant has been absorbed in the measure in \([\mathbb{Z}\mathbb{Z}]\), as it plays no dynamical role. Our conventions for the Euclidean Dirac matrices \((\gamma_\mu, \gamma_\nu) = \delta_{\mu\nu}\) are: \(\gamma^0 = \gamma_0, \gamma^1 = \gamma_1\) and \(\gamma^2 = -i\gamma^0\gamma^1\) are the Pauli matrices.

The electromagnetic field and the bosonic external source are periodic in the Euclidean time with period \(\beta\) whereas the fermionic fields and sources are anti-periodic. An alternative approach, which we shall not follow here, would have been to take \(\mathcal{D}(A; 0)\) in \([\mathbb{Z}\mathbb{Z}]\), with fermions satisfying the boundary condition \(\psi(x^0 + \beta, x^1) = -\exp(\beta\mu)\psi(x^0, x^1)\). Concerning the spatial boundary conditions, they cannot be chosen as periodic, in general (after the above choice for the temporal ones), as the Dirac operator may have zero modes on the torus (to avoid duplications, we refer to \([\mathbb{Z}\mathbb{Z}]\) for a justification). Without loss of generality, we shall choose \(A_\mu\) so that \(A_\mu(x^0, x^1 + L) = A_\mu(x^0, x^1) = \partial_\mu(-\Phi x^0/e\beta)\) and hence

\[
\psi(x^0, x^1 + L) = \exp \left( -i \frac{\Phi}{\beta} x^0 \right) \psi(x^0, x^1)
\]

\[
\overline{\psi}(x^0, x^1 + L) = \overline{\psi}(x^0, x^1) \exp \left( i \frac{\Phi}{\beta} x^0 \right),
\]

with \(\Phi\) the total flux of the electric field over the torus

\[
\Phi = e \int_T d^2x E(x) = 2\pi(n_+ - n_-)
\]

where \(n_\pm\) are the number of zero modes with positive and negative chirality. The relation \([\mathbb{Z}\mathbb{Z}]\) follows directly from the axial anomaly \([\mathbb{Z}\mathbb{Z}\mathbb{Z}\mathbb{Z}]\). We shall introduce \(k = n_+ + n_-\), the total number of zero modes. The gauge sector with \(k = 0\) will be referred to as the trivial sector. For later use, we recall the following factorisation property of the Dirac operator

\[
\mathcal{D}(A; \mu) = \exp(x^0 \mu) \mathcal{D}(A; 0) \exp(-x^0 \mu)
\]
A. General structure of $Z_f[A, \xi, \bar{\xi}]$ with zero modes

The contribution of the zero modes to the generating functional has to be analysed carefully, in order to properly define the functional determinant of the Dirac operator. For that purpose, we shall follow the same steps as in \cite{footnote}. However, there is an important distinctive feature of the $\mu \neq 0$ case, namely, that $i\slashed{D}[A; \mu]$ is non-hermitian. Hence, the set of eigenfunctions of $i\slashed{D}$ is no longer an orthonormal basis in which the spinor fields could be expanded. To avoid this difficulty we shall expand the spinors in the basis of the hermitian operators $\slashed{D}^\dagger \slashed{D}$ and $\slashed{D} \slashed{D}^\dagger$. This will allow to separate the zero mode contribution up to a phase factor. We shall discuss below this factor and its relevance to the calculation. First, let us consider the set of eigenfunctions of the hermitian operators

$$H(A; \mu) \phi_n = \left[ \slashed{D}^\dagger (A; \mu) \slashed{D} (A; \mu) \right] \phi_n = \mu_n \phi_n \quad ; \quad \overline{\slashed{D}}(A; \mu) \varphi_n = \left[ \slashed{D}(A; \mu) \slashed{D}^\dagger (A; \mu) \right] \varphi_n = \mu_n \varphi_n$$ \hspace{1cm} (7)

The operators $H$ and $\overline{\slashed{D}}$ have the same eigenvalues $\mu_n \geq 0$ (for $\mu_n > 0$, $\varphi_n$ is an eigenstate of $\overline{\slashed{D}}$) and the zero modes of $H$ ($\overline{\slashed{D}}$) are the same as those of $\slashed{D}$ ($\slashed{D}^\dagger$). In addition, since the anomaly \cite{footnote} is $\mu$-independent (for general results on the independence of anomalies on thermal effects, see \cite{footnote}), $n_+ - n_-$ is the same for both $\slashed{D}(A; \mu)$ and $\overline{\slashed{D}}(A; 0)$. As we shall see in section \ref{section}, all zero modes have always the same chirality. Therefore, the number $k$ of zero modes is the same for $H$, $\overline{\slashed{D}}$ and $\slashed{D}(A; 0)$.

At this point let us expand the spinor fields $\psi$ and $\overline{\psi}$ as

$$\psi(x) = \sum_{p=1}^{k} \alpha_p \phi_p + \sum_{q=k+1}^{\infty} \beta_q \phi_q$$

$$\overline{\psi}(x) = \sum_{p=1}^{k} \overline{\alpha}_p \phi_p^\dagger + \sum_{q=k+1}^{\infty} \beta_q \phi_q^\dagger$$ \hspace{1cm} (8)

with

$$\phi_q = \frac{1}{\sqrt{\mu_q}} \delta_{\mu_q} \quad q = k + 1, \ldots, \infty$$ \hspace{1cm} (9)

$\phi_p$ ($\phi_p^\dagger$) being the zero modes of $H$ ($\overline{\slashed{D}}$). In this basis, we have $(\phi_q, \varphi_q^\dagger) = (\phi_q, \phi_q^\dagger) = \delta_{qq'}$, where $(\chi, \psi) = \int_T d^2 x \chi^\dagger \psi$ is the scalar product on the torus. We get for the fermionic action

$$\int_T d^2 x \overline{\psi} \slashed{D}(A; \mu) \psi = \sum_{q=k+1}^{\infty} \sqrt{\mu_q} \beta_q$$ \hspace{1cm} (10)

Then, the action is diagonal in this basis and, by doing the integration over the grassmanian variables $\alpha_p$, the contribution of the zero modes can be factorized. A crucial point should be noticed here. As the spinors $\psi$ and $\overline{\psi}$ are expanded in different basis, the Jacobian of the change of basis from $\slashed{D}\psi$ to $\prod_{p,q} d\alpha_p d\beta_q$ is not the inverse of that from $\overline{\slashed{D}}\overline{\psi}$ to $\prod_{p,q} d\overline{\alpha}_p d\overline{\beta}_q$. This fact was already noted by Fujikawa \cite{footnote} in the context of anomalies with non-hermitian Dirac operators. Since both changes of variables are formally unitary, when doing them simultaneously we are left with some phase factor $\exp[iJ(A; \mu)]$. Notice that $J(A; 0) = 0$ since then $H = \overline{\slashed{D}} = -\slashed{D}^2(2; 0)$. Also, in principle, the phase factor is different for every $k$ sector, a feature to be reminded by means of a superscript $(k)$. Thus, performing the Gaussian Grassman integrals over $d\alpha d\beta$ we get

$$Z_f[A, \xi, \bar{\xi}] = \exp[iJ^{(k)}(A; \mu)] \exp \left[ -i \int_T d^2 x d^2 y \xi(x)G(x, y, eA; \mu)\xi(y) \right]$$

$$\times \prod_{p=1}^{k} \int_T d^2 x d^2 y \xi(x)\phi_p(x)\varphi_p(y)\xi(y) \sqrt{\det' H(A; \mu)}$$ \hspace{1cm} (11)

where $\det'$ is the functional determinant when the zero modes are omitted (or factored out) and $G(x, y, eA) = \sum_{q=k+1}^{\infty} \frac{1}{\sqrt{\mu_q}} \phi_q(x)\varphi_q^\dagger(y)$ is the exact fermionic two-point function, satisfying the differential equation

$$\slashed{D}(A; \mu)G(x, y, eA; \mu) = \delta^{(2)}(x - y) - \sum_{p=1}^{k} \varphi_p(x)\varphi_p^\dagger(y)$$ \hspace{1cm} (12)
The second term on the right hand side of the above equation is the projector onto the $\overline{H}$ zero mode subspace. For simplicity, we have omitted a superscript $(k)$ in both $G(x,y,eA;\mu)$ and det$' H(A;\mu)$. From (11), we see that the zero mode contribution can be factorized in this basis, in which we obtain the contribution of $|\text{det}' D| = (\text{det}' H)^{1/2}$. However, we have still to clarify which is the role of the phase factor $\exp(i\mathcal{J})$: this will be carried out in the next sections. Let us now remind how to obtain different quantities of physical interest from (11). If we are interested in the thermodynamics of the Schwinger model, the relevant quantity is the partition function $Z = Z(0,0,0)$, so that, from (11), only the trivial sector contributes:

$$Z(0,0,0) = N \int_{\text{periodic,0}} DA \exp \left[ i\mathcal{J}^{(0)}(A;\mu) + \int_T d^2 x \Gamma[A] \right] \sqrt{\text{det} H(A;\mu)} \tag{13}$$

We can obtain thermodynamic observables such as the free energy and the particle charge density, by differentiating $Z$ with respect to temperature and the chemical potential respectively, thereby generalising for finite charge density the study carried out in [6]. The subscript '0' in the functional integral above indicates that only the trivial sector contributes. Then, in the trivial sector $\mathcal{J}$ can be identified with the phase of the fermionic determinant. We can also calculate the average fermion charge density $\rho \equiv L^{-1} \int_0^L dx \bar{1} (\overline{\psi} \gamma^0 \psi)$ in terms of the two point Green function.

A remark is in order here: the equations of motion for the $A$ field imply $\partial_1 E(x) = -i e \overline{\psi} \gamma^0 \psi$, which is Gauss law. If the latter is imposed as a quantum constraint on physical states [23], then it would imply $\rho = 0$ with the boundary conditions chosen. As it is customary [3], one may consider that an external compensating (say, ion) charge $\rho_{ex}$ is present to ensure charge neutrality and hence Gauss law holds for the total charge $\rho_{tot} = \rho + \rho_{ex} = 0$. Alternatively, one may consider an open system that exchanges particles with a reservoir ensuring charge neutrality. With this in mind, we shall make no further reference to $\rho_{ex}$ and concentrate only in the fermion charge density $\rho$ for the electron-photon system. Therefore, (11) yields:

$$\rho = \frac{1}{L} \int_0^L dx \left( \overline{\psi} \gamma^0 \psi \right) = \frac{1}{\beta L} \frac{\partial}{\partial \mu} \log Z = \frac{1}{L} \int_0^L dx \left( \frac{1}{Z} \frac{\delta}{\delta \xi} \int_0^\infty \frac{\delta}{\delta \xi} Z(0,\xi,\xi) \right)_{\xi = \xi = 0}$$

$$= i \frac{1}{Z} \int_0^L dx \left\{ \int_{\text{periodic,0}} DA \exp \left[ i\mathcal{J}^{(0)}(A;\mu) + \int_T d^2 x \Gamma[A] \right] \sqrt{\text{det} H(A;\mu)} \right. \right. $$

and the $k \geq 1$ contributions vanish. We shall show in section II C that the zero modes of $H$ and $\overline{H}$ have all the same chirality, given by $\text{sgn} (\Phi)$. Hence, $\varphi_1$ and $\phi_1$ are both eigenstates of $\gamma^5$ with the same eigenvalue and therefore the second piece in the above equation vanishes. Then, only the trivial sector contributes to the fermion number density.

We also remark that the property $\{ \gamma_5, G(x,y) \} = 0$, which is not difficult to prove with the above definitions, implies, as in the case of finite $T$ but vanishing $\mu$ [7], that the chiral condensate $\langle \overline{\psi} P_k \psi \rangle$, with $P_k = (1 \pm \gamma^5)/2$, does not depend on $G(x,y)$. However, from (11) we see that it will contain the phase factor $\exp(i\mathcal{J}^{(1)}(A;\mu))$ (see Appendix).

**B. The imaginary part of the effective action**

As $i\mathcal{D} \neq (i\mathcal{D})^\dagger$, we have found the extra factor $\mathcal{J}(A;\mu)$, which is the source-independent piece of the phase of the generating functional. We shall analyse here its physical interpretation, at least in the trivial sector. The general form of $\mathcal{J}(A;\mu)$ can be inferred from the symmetry transformation properties of the phase of the different quantities obtained from $Z_f$ after switching off the external sources. For instance, in the trivial sector, the object of interest is the effective action $Z_f[A,0,0]$. Now, recall that the $\mu$-dependent term in the Dirac action is odd under the operation of charge conjugation $C$, since it is the number of particles minus the number of antiparticles operator. The rest of the Dirac action is even under $C$, so that $C$ acts on $Z_f$ by replacing $\mathcal{D} \rightarrow -\mathcal{D}'$ or, in other words, $Z_f[A^C,0,0] = Z_f[A,0,0]$ and, therefore, the phase of the effective action is odd under $C$, while the modulus is even. This is analogous to the case of the QCD effective chiral lagrangian, when the symmetry under consideration is spatial parity ($P$), the phase of the effective action being, then, the Wess-Zumino-Witten term [22]. In our case, $\mathcal{J}^{(0)}(A;\mu)$ should contain only $C$-odd combinations of the gauge field. As $P$ is a symmetry of the effective action, $\mathcal{J}$ should be $CP$ odd. In addition, it is not difficult to check that the $\mu$ term does not generate any anomaly in the gauge current, so that imposing local gauge invariance (see comments below), the only term which fulfills such symmetry requirements is of the form:
\[ \mathcal{J}^{(0)}(A; \mu) = \tilde{F}(T, \mu, L) \int_T d^2 x A_0(x) \]

where \( \tilde{F}(T, \mu, L) \) is a function, undetermined so far (to be found explicitly later) such that \( \tilde{F}(T, \mu, L) = -\tilde{F}(T, -\mu, L) \) since, by changing simultaneously \( \mu \rightarrow -\mu \) and particle by antiparticle, the theory remains unchanged. There is another point that is worth noticing here. Recall that in the torus the gauge transformations \( g(x^0, x^1) : S^1 \times S^1 \rightarrow U(1) \) are parametrised by \( \mathbb{Z} \times \mathbb{Z} \), corresponding to the two winding numbers \( (n, m) \) around the two circles. For any \( \Phi \), the most general gauge transformation \( A_\mu \rightarrow A_\mu + \partial_\nu \alpha \), which keeps \( \partial_\mu A^\mu \) fixed (so that \( \partial_\mu \partial^\mu \alpha = 0 \)) and leaves unchanged the boundary conditions (in space and time) for both fermion and gauge fields is: \( \alpha(x_0, x_1) = (2\pi nx^0)/\beta + (2\pi mx^1)/L \), up to an additive constant. Different choices of \( (n, m) \) correspond to nontrivial, homotopically disconnected, gauge transformations. But then, we note that the integral in (15) is precisely equal to \( n \). In the next section, we shall get \( \tilde{F} \) explicitly, through another method, thereby justifying our assumption.

C. Instanton decomposition

Following [7], we shall decompose the gauge field into an instanton part \( \tilde{A}_\mu \) and a local fluctuation \( \phi \) as

\[ A_\mu = \tilde{A}_\mu - \epsilon_{\mu\nu} \partial_\nu \phi \]

with \( \phi(x) \) periodic in both space-time directions. Note that \( \tilde{A}_\mu \) yields a constant electric field \( \tilde{E} = \Phi/(eL\beta) \) and, hence, \( E = \tilde{E} + \Delta \phi \). By following steps similar to those in [7], we separate first the contribution of \( \tilde{A} \) and \( \phi \) in \( \sqrt{\det H} \). Using that \( \psi(A; 0) = \exp(\epsilon^5 \phi) \psi(\tilde{A}; 0) \exp(\epsilon^5 \phi) \) and (6), it follows immediately that

\[ \psi(A; \mu) = \exp(\epsilon^5 \phi) \psi(\tilde{A}; \mu) \exp(\epsilon^5 \phi) \]

Notice that the operator \( H(A; \mu) \) in (15) can be cast as

\[ H(A; \mu) = -\nabla_\nu \nabla_\nu (A_\mu) - eE\gamma^5 \nabla_\nu = \partial_\nu - ieA_\nu + i\mu\gamma^5 \delta_{\nu1} \]

and the operator \( \overline{H} \) is obtained from \( H \) by changing \( \mu \rightarrow -\mu \). Hence, \( H(\tilde{A}; \mu) = -\nabla_\nu \nabla_\nu (\tilde{A}_\mu) - \gamma^5 \frac{\Phi}{L^2} \), so that, as \( -\nabla_\nu \nabla_\nu \) in (18) is a positive operator, all the zero modes of \( H(\tilde{A}; \mu) \) have the same chirality, equal to the sign of \( \Phi \) (recall that \( [\gamma^5, H] = 0 \)). On the other hand, from (17), we get a zero mode of \( \psi(A; \mu) \) by multiplying a zero mode of \( \psi(\tilde{A}; \mu) \) by \( \exp(-\epsilon^5 \phi) = \exp(-\epsilon sgn(\Phi) \phi) \), which is in turn a zero mode of \( H(A; \mu) \). We can apply exactly the same argument to \( \overline{H} \). Then, the zero modes of \( H \) and \( \overline{H} \) in (15) have both the same chirality, which is equal to \( sgn(\Phi) \). This was already used in section [4], in order to omit the one zero mode contribution to the particle density.

It is possible to separate the contribution of \( \det H(A; \mu) \) in \( \det H(A; \mu) \), for arbitrary \( k \). We have sketched the derivation in the Appendix, the general formula, for any \( k \), being given in (15). From that expression, we read the usual induced mass term for the boson field, with mass \( m = e/\sqrt{\pi} \), which is independent on both the temperature and the chemical potential, thereby generalising the result for \( \mu = 0 \) previously derived in [7]. Let us quote here the result for the trivial sector \( k = 0 \):

\[ \det H(A; \mu) = \det H(\tilde{A}; \mu) \exp \frac{e^2}{\pi} \int_T d^2 x \phi(x) \Delta \phi(x) \]

D. The determinant of the instanton operator

In order to complete the analysis in the previous section, one should still study the spectrum of \( H(\tilde{A}; \mu) \), which will be the purpose of the present section. First, by following [6], we shall decompose the field \( \tilde{A} \) as:

6
\[
\begin{align*}
\hat{A}_0 &= -\frac{\Phi}{eL}\beta x_1 + \frac{2\pi}{\beta}h_0 + \partial_0 \lambda \\
\hat{A}_1 &= \frac{2\pi}{L}h_1 + \partial_1 \lambda
\end{align*}
\] (20)

which is the Hodge decomposition of the gauge field in the torus. The contributions proportional to \(h_0\) and \(h_1\) are the so-called harmonic parts and are essential to correctly quantise the model \([13][14]\). Notice that under a non-trivial gauge transformation \((n, m)\) of the type commented in section \([13]\), the \(h\) fields above are the only ones changing and they do so as \(h_0 \to h_0 + n\) and \(h_1 \to h_1 + m\), even for \(\Phi = 0\). The \(\lambda\)-dependent terms in the last two equations are pure gauge contributions with \(\lambda\) periodic in \(x^0\) and \(x_1\), which will not play any physical role. For instance, with the covariant choice \(\partial_\mu A^\mu = 0\), \(\lambda\) is just a constant and that term does not appear in \((24)\). Let us consider first the case \(\Phi = 0\), which is the only relevant one for the partition function.

Since \(\gamma^5\) commutes with \(\hat{H}\), we choose the eigenfunctions of \(H(\hat{A}, \mu)\) as states of definite chirality, that is:

\[
\Psi^\pm = \begin{pmatrix} \phi^+ \\ 0 \end{pmatrix} \quad \Psi^- = \begin{pmatrix} 0 \\ \phi^- \end{pmatrix}
\] (21)

with \(\gamma^5\Psi^\pm = \pm \Psi^\pm\), since \(\gamma^5 = \text{diag}(1, -1)\). Then, for \(\Phi = 0\) we have to solve \(H^\pm(\hat{A}; \mu)\phi^\pm = \lambda^\pm \phi^\pm\) with

\[
H^\pm(\hat{A}; \mu) = - (\partial_0 - i\bar{h}_0)^2 - (\partial_1 - i\bar{h}_1 \pm i\mu)^2
\] (22)

and (anti-)periodic boundary conditions in the (time) space direction. In the above equation we have introduced \(\bar{h}_0 = 2\pi e h_0/\beta\) and \(\bar{h}_1 = 2\pi e h_1/L\). The eigenfunctions are plane waves and the corresponding eigenvalues are:

\[
\lambda^\pm_{n,k} = \left(\frac{2\pi}{\beta}\right)^2 \left[ n + \frac{1}{2} - e\bar{h}_0 \right]^2 + \left[ \frac{2\pi}{L}(k - e\bar{h}_1) \pm \mu \right]^2
\] (23)

with \(n, k\) integers. Notice that the chemical potential breaks the chiral degeneracy which was originally present in the \(\mu = 0\) case. Now, by using \(\log \det H = \text{Tr} \log H\) and \((24)\), we get:

\[
\log \det H(\hat{A}; \mu) = \frac{1}{2} \sum_{n,k=-\infty}^{\infty} \sum_{\pm} \log \left[ (\omega_n)^2 + (\omega_k - \bar{h}_1 \pm \mu \pm i\bar{h}_0)^2 \right] \tag{24}
\]

with \(\omega_n = (2n + 1)\pi/\beta\) and \(\omega_k = 2\pi k/L\). Now, let us add and subtract \(\log \beta^2 \sum_{n,k} \) to the right-hand-side of the above expression. This procedure will give rise to a \(T\)-dependent infinite constant, which, in turn, will be absorbed, as customarily, in the normalisation constant \(N(\beta)\) \([13][14]\). We can perform the summation over \(n\) in the above equation with the help of the two formulae \([13][22]\):

\[
\log \left[ (2n + 1)^2 \pi^2 + \beta^2 (\omega_k \pm \alpha)^2 \right] = \int_1^{\frac{\beta^2 (\omega \pm \alpha)^2}{\theta^2 + (2n + 1)^2 \pi^2}} \frac{d\theta}{\theta^2 + (2n + 1)^2 \pi^2} + \log \left[ 1 + (2n + 1)^2 \pi^2 \right] = \frac{1}{\beta} \left( \frac{1}{2} - \frac{1}{e^\theta + 1} \right)
\] (25)

In so doing, we obtain, finally:

\[
\log \det H(\hat{A}; \mu) = \sum_{k=-\infty}^{\infty} \left\{ 2\beta \left( \omega_k - \bar{h}_1 \right) + \sum_{\pm} \log \left[ 1 + \exp -\beta \left( \omega_k - \bar{h}_1 \pm \mu \pm \bar{h}_0 \right) \right] \right\}
\] (26)

up to an irrelevant \(T\) and \(\mu\) independent constant. In order to obtain the full partition function, we have to multiply the above expression (which for \(e = 0\) reproduces the partition function for free fermions at finite density) by \(\exp[iJ(0)]\) in \([13]\) and by the \(\phi\)-dependent one in \([13]\) and, then, integrate over the gauge fields \((\phi, h_0, h_1)\). As a consequence of the decomposition of the gauge field chosen here, the phase factor only depends on \(h_0\), so that:

\[
\int_T d^2x A_0(x) = -\frac{\Phi L}{2e} + 2\pi e h_0 L
\] (27)
since φ is periodic in the space direction. Then, the φ contribution to the partition function in (19) gives the partition function of a free massive boson [6]. All the dependence on the chemical potential is included in the (h₀, h₁) part, as given in (26) and (27). However, we have still to determine the value of \( F(T, \mu, L) \) in (15). Before undertaking that task, and for completeness, let us analyse the spectrum of \( H(\tilde{A}; \mu) \) when \( \Phi \neq 0 \). In this case, we have

\[
H^\pm(\tilde{A}; \mu) = -(\partial_0 - i\tilde{h}_0 + i\frac{\Phi}{\beta L} \gamma_1)^2 - (\partial_1 - i\tilde{h}_1 \pm i\mu)^2 \mp e\tilde{E}
\]

(28)
together with the boundary conditions in the spatial direction given in (1). This eigenvalue problem is solved in the Appendix. From the result found there, we remark here that the \( \mu \) dependence, when \( \Phi \neq 0 \), appears only in the states, while the determinant in (A7) depends on the temperature \( T \) but not on \( \mu \). As the norm of the zero modes in (A8) is also \( \mu \)-independent, then, if we go to (A3) and (A1), we realize that the dependence of the fermionic generating functional on \( \mu \), when \( \Phi \neq 0 \), is encoded in the determinants of the matrices in (A3), that follow immediately from the spectrum found for \( \alpha = 0 \). Besides, there are \( \mu \) dependences in both \( G(x, y, e\tilde{A}; \mu) \) and \( J^{(k)}(A; \mu) \).

III. THE GENERATING FUNCTIONAL IN THE TRIVIAL SECTOR

In this section, we shall use functional methods in order to calculate the generating functional in the trivial sector. By employing these methods, we shall also obtain the phase factor \( J^{(0)}(A; \mu) \). This will allow to get the full fermion charge density and the partition function, as well as to establish the consistency with the results of the previous section. We remind that the covariant gauge-fixing is not complete on the torus. We still have the freedom of performing a nontrivial gauge transformation \((h_0, h_1) \to (h_0 + n, h_1 + m)\), with the \( h \) fields in (21) and \( n, m \) integers, corresponding to loops that wind \( n \) times around the temporal direction and \( m \) times around the spatial one. It is clear that \( n \) and \( m \) are not fixed by \( \partial^\mu A_\mu = 0 \). Throughout this section, though, we shall work with the covariant gauge-fixing, ignoring this residual gauge arbitrariness. The latter has also been treated in a situation related to the one analysed here, but not quite identical with it: specifically, for (real time) QED on a spatial circle, at zero temperature and chemical potential [24]. We have to bear in mind that in (21), we are integrating the gauge field over all possible values of the fields \( h_\mu \), that is, \( h_\mu \in \mathbb{R} \). Fixing the gauge for those fields would consist in restricting them to a range \([0, 1] \times \mathbb{Z} \), since they change by an integer under a global gauge transformation. Then, if the effective action is globally gauge invariant, and unaffected by the residual gauge arbitrariness, the difference between integrating over all \( h \) or restricting them to a \([0, 1]\) interval, is an infinite constant independent on \( T \) and \( \mu \). So that arbitrariness cannot affect physical observables such as the free energy or the particle density. For \( e = 0 \), the action depends on derivatives of the electromagnetic field, and the covariant gauge fixing, even if not complete, suffices to get a well-defined propagator, unaffected by that arbitrariness. In general, when \( e \) is non-vanishing, \( Z[J, \xi, \bar{\xi}] \) is also unaffected by the arbitrariness, after having integrated over all fields. However, for a given \( A_\mu \), both \( \det H(\tilde{A}; \mu) \) and the fermionic generating functional \( Z_F \) may be subject to it, even if \( \mu = 0 \). In particular, when \( \mu \neq 0 \), we have seen in section IIIB that \( Z_F \) is not globally gauge invariant, due to the induced topological term in the phase \( J \), which changes when \( h_0 \to h_0 + n \). Thus, if we restrict \( h_0 \) to a \([0, 1]\) interval, the result for the observables would depend on our choice and then it is consistent to let \( h_0 \in \mathbb{R} \). We shall come again to this point at the end of section IV.A, where we shall perform the integration over the \( h \) fields explicitly, using the results derived in section II. It is not difficult to check that all the formal functional manipulations that we shall carry out in this section, except those related to \( L[A] \), are also unaffected by the residual gauge arbitrariness.

Thus, as a first step, let us rewrite the generating functional in (1) for the trivial sector, with the aid of standard functional techniques, as:

\[
Z[J, \xi, \bar{\xi}] = Z_{EM} Z_F \exp \left[ -ie \int_T d^2x \frac{\delta}{\delta \xi(x)} \gamma^\nu \frac{\delta}{\delta J^\nu(x)} \delta(x) \right] \
\times \exp \int_T d^2x d^2y \left[ \frac{1}{2} J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) - i\bar{\xi}(x) S(x, y; \mu) \xi(y) \right]
\]

(29)

where \( Z_{EM} \) and \( Z_F \) (\( Z_F(T, \mu, L) = Z_F \) for short) are the free boson and fermion partition functions and \( D^{\mu\nu}(x-y) \) and \( S(x, y; \mu) \) are the free gauge boson and fermion propagators respectively:

\[
D_{\mu\nu}(x-y) = \frac{1}{\beta L} \sum_{n,k=-\infty}^{\infty} e^{i\omega(x-y)} \frac{1}{\omega^2} \left[ \delta_{\mu\nu} + \left( \alpha - 1 \right) \frac{\omega_j \omega_k}{\omega^2} \right]
\]

\[
S(x, y; \mu) = -\frac{i}{\beta L} \sum_{n,k=-\infty}^{\infty} e^{i\omega(x-y)} \frac{1}{\gamma^{\beta}(\omega_n + i\mu) + \gamma^{\mu}}
\]

(30)
where $\omega = (\omega_n, \omega_b)$, with $\omega_b = 2\pi k/L$, $\omega_n = 2\pi n/\beta$ in the bosonic propagator and $\omega_n = (2n + 1)\pi/\beta$ in the fermionic one. Now, we shall make use of some known functional differentiation formulae \[12\], and, in particular of:

$$\exp \left[ -i \int d^2x d^2y \frac{\delta}{\delta \xi(x)} A(x, y) B(x, y) \xi(y) \right] \times \exp i \int d^2x d^2y \bar{\xi}(x) B(x, y) \xi(y)$$

is gauge invariant. However, if we replace (36) into (34), calculate the divergence of the current

$$\delta \int d^2x d^2y \bar{\xi}(x) B(x, y) \xi(y) + L$$

Here, $A(x, y)$ and $B(x, y)$ are arbitrary functions, to be regarded as the kernels of the operators $A$ and $B$, respectively, $\bar{B} = B(1 + AB)^{-1}$ and $L = -\text{Tr} \log[1 + AB]^{-1}$, $\text{Tr}$ indicating the trace over functional and Dirac spaces. Thus, one finds:

$$Z[J, \xi, \bar{\xi}] = Z_{EM} Z_F \exp \left[\frac{1}{2} \int_T d^2x d^2y J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) \right]$$

$$\times \exp \left[ -\frac{1}{2} \int_T d^2x d^2y \delta A_\mu(x) D^{\mu\nu}(x-y) \delta A_\nu(y) \right]$$

$$\times \exp \left\{ -i \int_T d^2x d^2y \bar{\xi}(x) G(x, y, i\epsilon A; \mu) \xi(y) + L[A] \right\}$$

with $A_\mu(x) \equiv -i \int_T d^2y D_{\mu\nu}(x-y) J_\nu(y)$, after having performed the functional differentiations, which appears to leave no trace of the residual gauge arbitrariness in $Z[J, \xi, \bar{\xi}]$. The so-called closed fermion loop functional $L[A]$ can be written formally as:

$$L[A] = \text{tr}_D \int_0^\infty de' \int_T d^2x A(x) G(x, x, i\epsilon A; \mu)$$

where $\text{tr}_D$ denotes the Dirac trace. We remind that $G(x, y, eA; \mu)$ is the two-point function, which, in the trivial sector, fulfils Eq. (33) with $k = 0$, that is, with its second term on the right hand side omitted.

In order to get a well defined expression for the generating functional, we need, first, to regularise $L[A]$ in (33). For that purpose, we shall appeal to the point-splitting regularization \[12\]. Therefore, we should deal with the following limit: $\lim_{x\rightarrow y} G(x, y, eA; \mu)$. In Minkowski space-time, the limit should be taken by keeping the points $x$ and $y$ relatively space-like, in order to maintain causality \[12\]. As we are working in Euclidean space-time, we shall not impose, in principle, such a restriction. We shall comment below on the different ways of taking the limit. Before that, and generalising \[12\], we shall derive the point-splitting regularization prescription in our present case with nonzero $\mu$. We start with the formal definition of the gauge current in the presence of an external background field $A_\mu$:

$$\langle j_\mu(x) \rangle_f [eA] = \langle \bar{\psi}(x) \gamma_\mu \psi(x) \rangle_f [eA] = i \lim_{x\rightarrow y} \text{tr}_D \gamma_\mu G(x, y, eA; \mu)$$

where $\langle O \rangle_f = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp[-\int \bar{\psi} \mathcal{D}\psi]$. We shall obtain the regularised version of the right-hand-side of the above equation as follows: we shall demand that such a regularised gauge current be conserved and gauge invariant. Notice that, under a gauge transformation $A_\mu \rightarrow A_\mu - \partial_\mu \Lambda$, $G(x, y, eA; \mu)$ changes as:

$$G(x, y, eA; \mu) \rightarrow G(x, y, eA; \mu) \exp i\epsilon[A(x) - A(y)]$$

Based upon this, it is easy to show that the product:

$$G(x, y, eA; \mu) \exp[-i\epsilon \int_x^y d\xi A_\sigma(\xi)]$$

is gauge invariant. However, if we replace (36) into (34), calculate the divergence of the current $j_\mu$ so defined, and use $\mathcal{D}(A; \mu) G = \delta^{(2)}(x-y)$, we find that such a divergence does not vanish for $\mu \neq 0$. To ensure that the current is divergenceless, we have to add an extra $\mu$-dependent term, which leads to the regularised gauge current:

$$\langle j_\mu(x) \rangle_f^{reg} [eA] = i \lim_{x\rightarrow y} \text{tr}_D \gamma_\mu G(x, y, eA; \mu) \exp[-i\epsilon \int_x^y d\xi A_\sigma(\xi)] \exp[-\mu(x^0 - y^0)]$$

which is, indeed, gauge invariant and divergenceless. Note that Euclidean covariance is broken since the system is in a thermal bath. We are now ready to define the regularised fermion closed loop as:
\[
L^{reg}[A] = -it_{\mathcal{D}} \int_0^e \int_T d^2 x A_\mu(x) \langle j_\mu(x) \rangle^{reg}_T [ie'A] \\
= t_{\mathcal{D}} \int_0^e \int_T d^2 x A_\mu(x) \lim_{x \to y} t_{\mathcal{D}} \gamma_\mu G(x, y, ie'A; \mu) \\
\times \exp[ie \int_x^y d\xi A_\sigma(\xi)] \exp[-\mu(x^0 - y^0)] \tag{38}
\]

The limit \( x \to y \) has to be taken in a symmetric way, regarding \((x, y)\) \([12]\). In order to calculate the fermion closed loop in \((38)\), we shall consider an ansatz for the exact Green function similar to that in \([1]\):

\[
G(x, y, eA; \mu) = \exp[-ie[\chi(x) - \chi(y)]S(x, y; \mu)] \tag{39}
\]

It is not difficult to check that, with the above ansatz, \(G(x, y)\) is a solution of \(\mathcal{D}G(x, y) = \delta^{(2)}(x - y)\), provided that \(\chi(x)\) be a solution of \(\partial_x \chi(x) = -\mathcal{A}(x)\). In turn, the solution is:

\[
\chi(x) = -\int_T d^2 y \Delta(x - y) \partial_y \mathcal{A}(y)
\]

\[
\Delta(x - y) = -\frac{1}{\beta L} \sum_{n, k = -\infty}^{\infty} e^{i\omega_n(x - y)} \frac{1}{\omega_n^2 + \omega_k^2} \tag{40}
\]

where \(\omega = (\omega_n, \omega_k)\) with \(\omega_n = 2\pi n/\beta\), \(\omega_k = 2\pi k/L\). So \(G(x, y, eA; \mu)\) appears to be unaffected by the residual gauge arbitrariness.

Hence, from the above equations, we have to find the behaviour of \(S(x, y; \mu)\) when \(x \to y\). For that purpose, we apply the standard formulae:

\[
\frac{1}{\beta} \sum_{n = -\infty}^{\infty} f(\omega_n + i\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega f(\omega) + \oint_C d\omega f(\omega)
\]

\[
- \frac{1}{2\pi} \sum_{\epsilon} \int_{-\infty + i\epsilon}^{\infty + i\epsilon} d\omega f(\omega) \frac{1}{e^{\pm i\beta \omega} + 1}
\]

\[
\frac{1}{\beta} \sum_{k = -\infty}^{\infty} f(\omega_k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega f(\omega)
\]

\[
+ \frac{1}{2\pi} \sum_{\epsilon} \int_{-\infty + i\epsilon}^{\infty + i\epsilon} d\omega f(\omega) \frac{1}{e^{\pm i\beta \omega} + 1} \tag{41}
\]

where \(\epsilon \to 0^+\), \(\omega_n = (2n + 1)\pi i/\beta\), \(\omega_k = 2\pi k/L\) and \(C\) is the rectangular contour in the complex \(\omega\) plane running through the points \((+\infty, -\infty, -\infty + i\mu, +\infty + i\mu)\). We apply \([11]\) to \(S(x, y; \mu)\) in \((38)\) and retain only the dominant contributions in \(x \to y\). The complex plane integrals along the lines \((-\infty + i\epsilon, +\infty + i\epsilon)\) and \((-\infty + i\epsilon + i\mu, +\infty + i\epsilon + i\mu)\) are performed by forming a closed contour, by means of an infinite arc above (below), corresponding to the (+/-) sign in \((41)\), and applying the Residue Theorem. We get:

\[
\lim_{x \to y} S(x, y; \mu) = e^{(x^0 - y^0)/\mu} \left[ \frac{1}{2\pi} \frac{(x - y)\gamma_\mu}{(x - y)^2} + i\gamma_0 F(T, \mu, L) + \mathcal{O}(x - y)^2 \right] \tag{42}
\]

where

\[
F(T, \mu, L) = \frac{1}{2L} \sum_{k = -\infty}^{\pm\infty} \left[ \frac{1}{e^{\beta(\omega_k + \mu)} + 1} - \frac{1}{e^{\beta(\omega_k - \mu)} + 1} \right] \tag{43}
\]

with \(\omega_k = 2\pi k/L\). We recall that \([12]\) reproduces the \(T = \mu = 0\) result given in \([12]\) and the \(\mu = 0, T \neq 0\) one \((F(\mu = 0) = 0)\) in \([1]\).

Next, we shall replace both \([10]\) and the limit \((42)\) into \((39)\) and \((38)\). We have taken the \(x \to y\) limit in two different ways and established that the same result is arrived at. We have taken, firstly, \(x^0 - y^0 \to 0, x^1 - y^1 \to 0\) with \((x^1 - y^1)/(x^0 - y^0) = 1\) and, secondly, the Minkowski causal choice (see \([12]\)). \(x^0 = y^0\) and \((x^1 - y^1) \to 0\). Anyway, what it is important to note here is that the exponential \(\mu\) dependence in \((43)\) is exactly cancelled with that in the
regulator in $\{33\}$. Then, no matter how we take the $x \to y$ limit, we always get a term $\text{tr} \int A(x) \gamma^0 F$ in $L^{reg}$. The possible divergence in $L^{reg}$ arising from the first piece in $\{42\}$ is absent since we have taken the limit symmetrically. Thus, finally we arrive at

$$L^{reg}[A] = \frac{1}{2} \int_T d^2 x d^2 y A_\alpha(x) \Pi^{\alpha\beta}(x-y) A_\beta(y) + 2\epsilon F(T, \mu, L) \int_T d^2 x A_0(x)$$

with

$$\Pi^{\alpha\beta}(x-y) = \frac{1}{\beta L \pi} e^2 \sum_{n,k=-\infty}^{+\infty} e^{i\omega(x-y)} \left[ \delta_{\alpha\beta} - \frac{\omega_\alpha \omega_\beta}{\omega^2} \right]$$

where $\omega_n = 2\pi n/\beta$ and $\omega_k = 2\pi k/L$. So, $L^{reg}[A]$ is, in principle, affected by the residual gauge arbitrariness. The functional differentiations with respect to $A(x)$ in $\{32\}$ can be performed by employing the following formula, which is valid for any linear operators $P$ and $Q$ $\{12\}$

$$\exp \left[ -i \int \frac{\delta}{\delta A} P \frac{\delta}{\delta A} \right] \exp \left[ i \int AQA + i \int f \cdot A \right] = \exp \left[ i \int A\overline{P} A \right] + i \int A(1-QP)^{-1} \cdot f + \frac{1}{2} \text{tr} \log(1-QP)^{-1} + i \int f Q(1-QP)^{-1} f$$

where $\overline{P} = P(1-QP)^{-1}$ and we have omitted, for simplicity, all the space-time dependences. We shall concentrate on the bosonic generating functional $Z[J, 0, 0]$, as we have already analysed the dependence on the fermionic sources in the previous section, up to the phase factor (to be calculated below). Upon applying $\{40\}$ in $\{32\}$, $L[A]$ being given in $\{44\}$, we get:

$$Z[J, 0, 0] = Z_{EM} Z_F \exp \left[ \frac{1}{2} \int_T d^2 x d^2 y (J-iG)_\mu(x) \mathbb{D}_{\mu\nu}^{(0)}(x-y) (J-iG)_\nu(y) \right] \times \exp \left[ \frac{1}{2} \text{Tr} \log(1+\Pi D)^{-1} \right]$$

where $G_0 = 2\epsilon F(T, \mu, L)$, $G_1 = 0$. In turn, $\mathbb{D}_{\mu\nu}^{(0)}$ is the exact boson propagator at $\mu = 0$, which can be expressed, formally, as: $\mathbb{D}^{(0)} = D(1+\Pi D)^{-1}$, $D$ and $\Pi$ being given in $\{30\}$ and $\{45\}$, respectively. Its explicit representation is:

$$\mathbb{D}_{\mu\nu}^{(0)}(x-y) = \frac{1}{\beta L} \sum_{n,k=-\infty}^{\infty} e^{i\omega(x-y)} \left[ \frac{1}{\omega^2 + m^2} \left( \delta_{\mu\nu} - \frac{\omega_\mu \omega_\nu}{\omega^2} \right) + \frac{\omega_\mu \omega_\nu}{(\omega^2)^2} \right]$$

where $m^2 = e^2/\pi$ is the induced boson mass.

In the following section, we shall obtain, from $\{44\}$ and $\{45\}$, the complete form of the fermionic generating functional in the trivial sector, including the phase factor, the exact boson propagator at $\mu \neq 0$ and the partition function and check the results with the method used in the previous section.

### IV. PHYSICAL RESULTS

**A. The charge density and the partition function**

By setting $J = 0$ in the expression $\{47\}$ we get the partition function

$$Z(T, \mu) = \frac{Z_F(T, \mu)}{Z_F(T, \mu = 0)} Z(T, \mu = 0) \exp -\frac{1}{2} [2\epsilon F(T, \mu, L)]^2 \int_T d^2 x d^2 y \mathbb{D}_{\mu\nu}^{(0)}(x-y)$$

However, the above integrals of $\mathbb{D}_{\mu\nu}$ are ambiguous, in the sense that changes in the order in which such integrals are done yield different results. For instance, let us perform, firstly, the spatial integrals: that would force us to set $k = 0$ in $\{48\}$, which would give rise, after doing the temporal integral, to an infinite and gauge dependent result. Conversely, by changing the order of the integrals, let us perform, first, the one over the (imaginary) time. In so doing, we arrive at a finite and gauge-independent answer. The latter prescription seems to be a natural and physically
reasonable choice. In addition, as we shall see, it is consistent with the results obtained in the previous section. If we adopt this prescription in (49), we get:

\[
Z(T, \mu) = \frac{Z_F(T, \mu)}{Z_F(T, \mu = 0)} Z(T, \mu = 0) \exp[-2\beta L \pi F^2(T, \mu, L)]
\] (50)

This is a genuine nonperturbative result, since the argument of the exponential in the term that corrects the free fermionic partition function is independent of the electric charge. As we shall see, such a nonperturbative behaviour comes directly from the topological structure discussed in section I, namely, from the phase \( \mathcal{J}[A; \mu] \) of the fermionic generating functional. At this point, we recall the result obtained in [6] for \( Z(T, \mu = 0) \). The latter partition function was proven to factorise into the product of that for a free fermionic field times that for a free massive boson field with mass \( m \), divided by the one for a free massless field (finite temperature bosonization). In our present case, with \( \mu \neq 0 \), we may wonder if such a factorisation actually takes place as well and, if so, whether the whole system may have a net fermionic charge or not. Let us consider, first, the free fermionic partition function \( Z_F(T, \mu) \):

\[
\log Z_F(T, \mu, L) = \sum_{k=-\infty}^{+\infty} \left\{ \log \left[ 1 + e^{-\beta(\omega_k - \mu)} \right] + \log \left[ 1 + e^{-\beta(\omega_k + \mu)} \right] \right\}
\] (51)

The net free fermion charge density is

\[
\frac{1}{\beta L} \frac{\partial}{\partial \mu} \log Z_F(T, \mu, L) = -2F(T, \mu, L)
\] (52)

\( F \) being given in (49). Thus, from (52) and (50) we have

\[
Z(T, \mu) = Z(T, \mu = 0, L) \exp \left\{ -2\beta L \left[ \pi F^2(T, \mu, L) + \int_0^\mu d\mu' F(T, \mu', L) \right] \right\}
\] (53)

which is our final result for the partition function at \( \mu \neq 0 \) for finite \( L \). Now, let us analyse the behaviour of the function \( F \) in the limit \( L \to \infty \). In such a limit, the sum over \( k \) becomes a trivial integral, which yields:

\[
F(T, \mu, L \to \infty) = -\frac{\mu}{2\pi}
\] (54)

Thus, in the \( L \to \infty \) limit, the \( \mu \) dependence in (53) exactly cancels out. We get for the full partition function:

\[
Z(T, \mu, L \to \infty) = Z(T, \mu = 0, L \to \infty) = Z_{EM} Z_F(T, \mu = 0, L \to \infty)
\times \exp \left\{ \frac{1}{2} \text{Tr} \log(1 + \Pi D)^{-1} \right\}
\] (55)

which is only \( T \) dependent. Its explicit expression can be found in section IV of [6]. That is, in the \( L \to \infty \) limit the system bosonizes as well, and the only effective degrees of freedom are those of a massive boson field, the net fermionic charge of the system being \( \rho = (\beta L)^{-1} \partial \log Z/\partial \mu = 0 \). We remark that: i) This is a nonperturbative effect (which has been established due to the fact that the Schwinger model at finite chemical potential and temperature can, still, be solved exactly); ii) It is characteristic of two dimensions. Recall for instance that in perturbative four-dimensional QED in the infinite volume limit, the free energy log \( Z \) is \( \mu \)-dependent and hence \( \rho \neq 0 \) [2]; iii) The result (53) holds in the \( L \to \infty \) limit: if we keep \( L \) finite, \( F \) is no longer given by (54) but it acquires further corrections and the study of the counterpart of (55) will not be attempted here.

We now turn to the issue of the \( C \)-violating phase factor, addressed previously in section I. Thus, we consider the fermionic generating functional \( Z_f[A, \xi, \bar{\xi}] \), treating now \( A \) as an external background field. We apply (53) and we arrive at an expression analogous to (52), but now taking \( J = 0 \), omitting the derivatives with respect to \( A \) and replacing \( e \to -ie \). By recalling the regularised fermion closed loop functional obtained in (58), we get:

\[
Z_f[A, \xi, \bar{\xi}] = Z_F Z_{EM} \exp \left[ -i \int_T d^2 x d^2 y \bar{\xi}(x) G(x, y, eA; \mu) \xi(y) \right]
\times \exp \left[ -\frac{1}{2} \int_T d^2 x d^2 y A_{\alpha}(x) \Pi^\alpha \beta(x - y) A_{\beta}(y) \right]
\times \exp \left[ -2ieF(T, \mu, L) \int_T d^2 x A_0(x) \right]
\] (56)
The comparison with (11) in the trivial sector leads to identify:

\[ \mathcal{J}^{(0)}(A, \mu) = -2eF(T, \mu, L) \int_T d^2xA_0(x) \]

(57)

that has the form that we had anticipated in (15), based upon the \( C \) symmetry properties of the phase of the fermionic determinant, and, so, \( \bar{F} = -2eF \).

We shall now provide an interesting check of consistency. On the basis of (13), (19), (26) and (27), we can calculate the partition function through the method developed in section II. We shall do so in the \( L \rightarrow \infty \) limit. Let us first differentiate with respect to \( \mu \) in (26). Using again (58) we get in the \( L \rightarrow \infty \) limit by replacing \( \omega_k \) by continuous \( \omega \) and \( \sum_k \) by \( (L/2\pi) \int d\omega \). The resulting integral can be done using

\[ \int_{-\infty}^{+\infty} d\omega \left[ \frac{1}{1 + e^{\beta (\omega - a + b)}} - \frac{1}{1 + e^{\beta (\omega - a - b)}} \right] = -2b \]

(58)

which is convergent when both pieces of the integrand are added together. Then, in the \( L \rightarrow \infty \) limit we obtain:

\[ \frac{\partial}{\partial \mu} \log \det H(\bar{A}; \mu) = \frac{2\beta L\mu}{\pi} \]

(59)

and, hence:

\[ \sqrt{\det H(\bar{A}; \mu, T)} = \sqrt{\det H(\bar{A}; \mu = 0, T)} \exp \left( \frac{L^2 \mu^2}{2\pi} \right) \]

(60)

We now insert the above result in (19) and, then, in (11), for the trivial sector, and we compare to (56). We see that, before integrating out the photon field, the results obtained with both methods are consistent with each other in \( L \rightarrow \infty \), since we get the same \( \mu \) dependence, namely, the one coming from the phase factor (the explicit form of which has been obtained with the second method) times that of the free fermionic partition function, as it appears in (55) when \( F \rightarrow -\mu/2\pi \). To complete the check, we shall see that the integration over the gauge boson fields also gives the same result with both methods in the \( L \rightarrow \infty \) limit, namely, that in (54). From our previous discussion, it follows that the only part that depends on \( \mu \) in the full partition function is the integral over the \( (h_0, h_1) \) fields, the integrand of which is the exponential of (27) for \( \Phi = 0 \) times \( -2ieF \), multiplied by \( \sqrt{\det H(\bar{A})} \). From (60), it follows that all the \( h \) dependence of \( \det H \) is contained in the \( \mu = 0 \) part. In order to extract the dependence on \( h_0 \), we differentiate again in (26). Using again (58) we get in the \( L \rightarrow \infty \) limit:

\[ \frac{\partial}{\partial h_0} \log \det H(\bar{A}; \mu = 0; h_0, h_1) = 2ie \sum_{s = \pm} \int_{-\infty}^{+\infty} d\omega \frac{s}{1 + e^{\beta (\omega - h_1 - is h_0)}} \]

(61)

which turns out to be independent on \( h_1 \). Thus, from (61) and (11), we derive:

\[ \sqrt{\det H(\bar{A}; \mu, T; h_0, h_1)} = \sqrt{\det H(\bar{A}; \mu = 0, T; 0, h_1)} \]

\[ \times \exp \left( \frac{L^2 \mu^2}{2\pi} - \frac{2\pi L^2 h_0^2}{\beta} \right) \]

(62)

At this stage, we have to integrate over the fields \((h_0, h_1)\). As commented at the beginning of section II, it is necessary to integrate over \( h_0 \in \mathbb{R} \), to achieve consistency. This is reinforced by the topological term (52) in the phase of the fermionic generating functional. Thus, from (27) and (28), it follows that the relevant factor carrying the \( \mu \) dependence is:

\[ e \int_{-\infty}^{+\infty} dh_0 \exp \left( -2i \mu h_0 L - \frac{2\pi L h_0^2}{\beta} \right) = e \sqrt{\frac{\beta}{2L}} \exp \left( \frac{\mu^2 L^2}{2\pi} \right) \]

(63)

It turns out that the exponential \( \mu \) dependence in (63) cancels that in (53), and, hence, we arrive, again at the result (53), obtained with our previous method.
B. The boson propagator, the screening length and the Polyakov loop

The exact boson propagator, that results after the integration of both fermion and gauge fields, can be obtained, in the trivial sector, just by differentiating (17) with respect to the external sources and setting $J = 0$. We find:

$$D^{(\mu)}_{\alpha\beta}(x - y) = D^{(0)}_{\alpha\beta}(x - y) - (2eF)^2 \int_T d^2u d^2z D^{(0)}_{\alpha0}(x - u) D^{(0)}_{\beta0}(y - z)$$

(64)

$D^{(0)}_{\alpha\beta}(x - y)$ being given in (58). The above expression is, again, formal, in the sense that we have to specify the order in which the spatial and temporal integrals should be done in the second piece on the right-hand-side. If we adopt the same prescription as that in section IV A (that is, by doing the integrals in the order that gives a finite answer) we arrive in momentum space at:

$$D^{(\mu)}_{\alpha\beta}(\omega_n, \omega_k) = D^{(0)}_{\alpha\beta}(\omega_n, \omega_k) - \delta_{\alpha0} \delta_{\beta0} \frac{4\pi^4 F^2 (T, \mu, L)}{e^2}$$

(65)

$D^{(0)}_{\alpha\beta}(\omega_n, \omega_k)$ being given by the expression between square brackets in (58). In the $L \to \infty$ limit, $\omega_k$ is replaced by a continuous variable $\omega$ and $L \delta_{0\beta} F^2$ goes to $(\mu^2/4\pi^2) \delta(\omega)$, in (53). As a check of consistency, we notice that the propagator (53) (which is $\mu$ dependent) satisfies the gauge invariance condition $\omega^\mu \omega_{\mu} D_{\mu\nu} = \alpha$, with $\alpha$ the gauge-fixing parameter. For finite $L$, the only Euclidean pole of the propagator (53) remains at $\omega^2 = -m^2 = -e^2/\pi$, which defines the mass of the boson and is independent on $\mu$. However, in the $L \to \infty$ limit, a new pole appears at $\omega^2 = 0$ due to the contribution of $\delta(\omega)$ in the second ($\mu$ dependent) term. To clarify what this pole means physically, let us calculate first the inverse correlation length squared $M^2$ in the $L \to \infty$ limit, through the usual definition

$$M^2 = D_{00}(\omega_n = 0, \omega \to 0)$$

(66)

where $D_{\alpha\beta}(\omega_n, \omega)$ is the boson self-energy, which is defined in momentum space through

$$D_{\alpha\beta}(\omega_n, \omega) = [D^{-1}]_{\alpha\beta}(\omega_n, \omega) = \frac{1}{[D^{-1}]}_{\alpha\beta}(\omega_n, \omega)$$

(67)

Accordingly, from (56) and the propagators in (58) and (52), we obtain

$$M^2(T, \mu) = 0$$

(68)

That is, the mass $M^2$ vanishes in the $L \to \infty$ limit. This is indeed a consistent result if we recall the connection between the screening length and the equation of state of the system given in (16):

$$M^2(T, \mu) = e^2 \frac{\partial}{\partial \mu} \rho(T, \mu)$$

(69)

$\rho$ being the total fermion charge density. Hence (68) and (53) are consistent in the $L \to \infty$ limit: we got a total zero charge density, while $M^2$ tends to zero. However, were $M^2$ be interpreted as a vanishing screening mass, then the system would be in a confined phase and the $Z$ symmetry would be restored, which does not occur for $\mu = 0$ and $T \neq 0$ in the $L \to \infty$ limit (11). We shall outline below a correct understanding of the screening mass. The order parameter of this symmetry is the thermal average of the Polyakov loop

$$P e(x^1) \equiv \exp \left[ i \tilde{e} \int_0^\beta d\tau A_0(\tau, x^1) \right]$$

(70)

for an arbitrary charge $\tilde{e}$ and a spatial point $x^1$. In addition, the correlator of two Polyakov loops with charges $\tilde{e}$ and $-\tilde{e}$ and spatial points $x^1$ and $u^1$ measures, at large spatial separation, the effective potential between both charges. Thus, in order to clarify the role of the two different masses $m$ and $M$, let us calculate the Polyakov loop and the correlator for $\mu \neq 0$ in the $L \to \infty$ limit. First, we shall use the functional methods developed in section II. After the obvious relabelling of variables, the insertion of $P e(x^1)$ in the functional integral (1) amounts formally to replace $J(y) \to iP(y)$ with $P_0(y) = \tilde{e}\delta(y^1 - x^1)$ and $P_1(y) = 0$. Then, recalling our result (17), we get
\[ \langle P_\xi(x^1) \rangle = \exp \left[ -\frac{1}{2} \int d^2y d^2z [\Pi^{(0)}_{00}(y - z) P_0(y) (P_0(z) - 2G_0(z))] \right] = \exp \left[ -\frac{\beta \xi^2}{2m} - \beta \mu \tilde{\epsilon} \right] \]  

(71)

where in the last step we have first followed our convention of performing the time integration before the spatial one and then we have taken the limit \( L \to \infty \), so as to transform spatial sums into integrals. The result (71) can also be obtained using the decomposition (16) and (20) for the gauge field. As we saw in previous sections, all the \( \mu \) dependence is included in the \( h_0 \)-dependent part. Then, since

\[ \int_0^\beta d\tau A_0(\tau, x^1) = 2\pi h_0 - \partial_x^2 \int_0^\beta d\tau \phi(\tau, x^1), \]  

(72)

all we have to do is to insert a piece \( \exp(2\pi i h_0 \tilde{\epsilon}/e) \) in the integrand in (63), in the \( L \to \infty \) limit. It is straightforward to get again the \( \mu \) dependence of the Polyakov loop in (71) consistently, once we integrate over \( h_0 \in \mathbb{R} \).

We see that \( \langle P_\xi(x^1) \rangle \) never vanishes for any value of \( T \) and \( \mu \), so that the \( Z \) symmetry is never restored for \( L \to \infty \). Our result extends that obtained in [10] for \( \mu = 0 \). Then, there should exist a nonvanishing screening mass, which, by consistency with our result for the propagator, should be \( m = e/\sqrt{\pi} \), independent on \( \mu \). This is confirmed by calculating the correlator of two Polyakov loops. Following the same steps as for \( \langle P \rangle \) we obtain with both methods that \( \langle P_\xi(x^1)P_{-\xi}(u^1) \rangle \) is independent on \( \mu \) and

\[ \lim_{x^1 - u^1 \to \infty} \langle P_\xi(x^1)P_{-\xi}(u^1) \rangle = \exp \left[ -\frac{\beta \xi^2}{2m} \right], \]  

(73)

that is, \( m \) is the screening mass between the two opposite charges. It seems clear that the definition (66), leading to (68), does not give the right answer for the screening mass between two opposite charges, nor can be used to infer any conclusion about the breaking of the confinement or \( Z \) symmetry. However, it is the consistent equation to use if we want to get the equation of state of the system through (68).

Notice that we have restricted ourselves to the trivial sector. The boson propagator could receive additional contributions from sectors with \( k \geq 1 \), apart from that in (64). However, we have seen that \( \rho \) only receives contributions from the trivial sector, so that, (68) is valid for any sector if (69) holds. In addition, it is enough to restrict ourselves to the trivial sector to calculate correlators of Polyakov loops [10]. Hence, we expect the above results for the screening mass to remain valid for \( \Phi \neq 0 \).

If the length of the system \( L \) is kept finite, the definition (68) for \( M^2 \) is no longer valid. Using (68) instead would give rise to a nonzero and \( \mu \)-dependent \( M^2 \), directly from (68). Remember that for finite \( L \) there is no pole of the propagator at \( \omega^2 = 0 \). However, it is not clear whether (69) remains valid for finite \( L \). On the other hand, for finite \( L \) and \( \mu = 0 \), the screening mass is only different from zero if \( \tilde{\epsilon}/e \in \mathbb{Z} \) [10]. A more rigorous analysis of the finite length corrections for \( \mu \neq 0 \) is beyond the scope of this work.

V. THE THIRRING MODEL AT FINITE \( T \) AND \( \mu \)

A. The generating functional

We shall consider a system of many massless fermions (and antifermions) in one dimension (inside a finite interval of length \( L \)) at equilibrium at absolute temperature \( T \) and chemical potential \( \mu \). By assumption, their dynamics is described by the Thirring quartic lagrangian. Let \( \xi, \bar{\xi} \) be fermionic external sources. Then, in the imaginary time formalism, the generating functional of the system reads now

\[ Z[\xi, \bar{\xi}] = N(\beta, \mu) \int_{\text{antiperiodic}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left[ \int_T d^2x \left( -\bar{\psi}(\partial - \mu \gamma^0)\psi + \bar{\xi}\psi + \bar{\psi}\xi \right) \right. \]

\[ - \left[ \frac{g^2}{2} \left( \bar{\psi}\gamma^\nu\psi \right) \left( \bar{\psi}\gamma^\nu\psi \right) \right] \]  

(74)

where \( g \) is the coupling constant. The partition function is \( Z[0, 0] \). One can also cast \( Z[\xi, \bar{\xi}] \) as follows

\[ Z[\xi, \bar{\xi}] = \exp \left[ -g \int_T d^2x \frac{\delta}{\delta \bar{\psi}(x) \gamma^\nu} \frac{\delta}{\delta \psi(x) \gamma^\nu} \right] \times Z[\xi, \bar{\xi}; J]_{J=0} \]  

(75)
\[
Z_1[\xi, \xi; J] = N Z_F \exp \left[ \frac{1}{2} \int \frac{d^2 x' d^2 y' J_\alpha(x') K^{\alpha\beta}(x' - y') J_\beta(y') \right] \\
\times \exp \left[ \int_T d^2 x' d^2 y' \xi(x') S(x', y''; \mu) \xi(y'') \right] \\
K^{\alpha\beta}(x' - y') = \frac{1}{\beta L} \sum_{n,k=0}^{+\infty} e^{i\omega(x'-y')} \left[ f_{\alpha\beta} \omega^{\alpha\beta} \right] (76)
\]

with \( S \) in (30), \( Z_F \) the free fermionic partition function and \( J = (J_0, J_1) \) a boson source, to be set equal to zero after all functional differentiations with respect to it have been performed in (23) and \( \omega \) in (77) is the same as in the gauge boson free propagator in (30). We have introduced an arbitrary function \( f(x^2) \), by virtue of the fact that the current \( \bar{\psi}\gamma^\nu\psi \) in the actual Thirring model is conserved. A proof of equations (75) and (76) follows readily through steps similar to those in [10]. At this stage, using standard techniques [12], one can rewrite eqs. (75) and (76) as

\[
Z[\xi, \xi] = \exp \left[ -\frac{1}{2} \int_T d^2 x' d^2 y' \frac{\delta}{\delta A_{\alpha}^*(x')} K^{\alpha\beta}(x' - y') \frac{\delta}{\delta A_{\beta}^*(y')} \right] \\
\times Z_J[A_{\alpha}^*, \xi, \xi]|_{A^*=0} (78)
\]

\[
Z_J[A_{\alpha}^*, \xi, \xi] = N Z_F \exp \left[ -ig \int_T d^2 x \frac{\delta}{\delta \xi(x')} A_{\alpha}^*(x) \frac{\delta}{\delta \xi(x)} \right] \\
\times \exp \left[ \int_T d^2 x' d^2 y' \xi(x') S(x', y''; \mu) \xi(y'') \right] (79)
\]

\[
A_{\alpha}^*(x) = -i \int_T d^2 z K_{\alpha\beta}(x - z) J^\beta(z) (80)
\]

Again, one sets \( A^* = 0 \) in the above equations, after having carried out all functional differentiations. Standard functional techniques allow us now to establish that \( Z_J[A_{\alpha}^*, \xi, \xi] \) as given in (79) also coincides with the right-hand-side of (2) (when due care is taken of the normalisation factor \( \tilde{N}(\beta, \mu) \)) provided that, in the latter, one replaces \( e \) by \( g \) and \( A \) by \( A^* \). Let us concentrate on \( A^* \) belonging to the trivial sector (see comments below). Then, by recalling the developments in section [11], one finds immediately

\[
Z_J[A_{\alpha}^*, \xi, \xi] = Z_F \exp \left[ -i \int_T d^2 x \int_T d^2 y \xi(x') G(x, y, gA^*; \mu) \xi(y) + L[A^*] \right] (81)
\]

where \( G(x, y, gA^*; \mu) \) and \( L[A^*] \) are now given by the right-hand-sides of (39), (40) and (44) (with the same \( S, \Delta, \Pi \) and \( F \), when one replaces \( e, A \) by \( g, A^* \), respectively. Thus, we have provided the solution for the Thirring model at finite \( T, \mu \) in terms of the fermionic generating functional for the Schwinger model, without zero modes (\( \Phi = 0 \)).

The formal use of eqs. (75)-(77) and (78)-(80) (which can be checked upon comparing the corresponding expansions into powers of \( g \)) is justified if we restrict \( Z_J[A_{\alpha}^*, \xi, \xi] \) to the trivial sector of the Schwinger model, as implemented through (81). In this regard, it is interesting to note that, upon using (80) in (4) and taking into account that the propagator \( K_{\alpha\beta} \) in (77) satisfies periodic boundary conditions in both \( x \) and \( y \) directions, one immediately finds \( \Phi[A^*] = 0 \), which confirms that \( Z_J[A^*, \xi, \xi] \) should be restricted to the trivial sector when use is made of (78)-(80) and, hence, the consistency of (81). On the other hand, this appears also to be consistent with the idea that, in the end, we are going to set \( A^* = 0 \) and then we can take the vector field \( A_{\alpha}^* \) as a configuration in the trivial sector, that is, topologically connected with \( A^* = 0 \). It is unknown to us whether the Thirring model may have other solutions (besides that given in (78) and (81)).

**B. The thermodynamical partition function and fermion correlation function**

The thermodynamical partition function becomes, upon applying (16) to (78) and (81)

\[
Z[0, 0] = Z_F \exp \left\{ \frac{1}{2} \text{Tr} \log(1 + \Pi K)^{-1} \right\} \\
\left\{ -\frac{1}{2} (2gF(T, \mu, L))^2 \int_T d^2 x d^2 y \left[ K(1 + \Pi K)^{-1} \right]^{100}(x - y) \right\} (82)
\]

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where in momentum space we have

\[
[(1 + \Pi K)^{-1}]^{\alpha\beta}(x - y) = \frac{1}{\beta L} \sum_{n, k = -\infty}^{+\infty} e^{i\omega(x - y)} \left[ \frac{\delta^{\alpha\beta} - \frac{g^2/\pi}{1 + g^2/\pi} \left( \delta^{\alpha\beta} - \frac{\omega^\alpha\omega^\beta}{\omega^2} \right)}{1 + \frac{2}{\pi}} \right]
\]

\[
[K(1 + \Pi K)^{-1}]^{\alpha\beta}(x - y) = \frac{1}{\beta L} \sum_{n, k = -\infty}^{+\infty} e^{i\omega(x - y)} \left[ \frac{\delta^{\alpha\beta}}{1 + \frac{2}{\pi}} - \frac{\omega^\alpha\omega^\beta}{\omega^2} \left( f(\omega^2) - \frac{g^2/\pi}{1 + g^2/\pi} \right) \right]
\]

(83)

Using (84), (82) yields readily for \(L \to \infty\)

\[
Z[0, 0] = Z_F(T, \mu) \exp \left\{ L \alpha \log \frac{1}{1 + \frac{2}{\pi}} - \beta L \frac{g^2\mu^2}{2\pi^2} \left[ \frac{1}{1 + \frac{2}{\pi}} \right] \right\} - b \left( f(0) - \frac{\frac{2}{\pi}}{1 + \frac{2}{\pi}} \right)
\]

(84)

where

\[
a = \frac{1}{2} \sum_{n = -\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dk}{2\pi}
\]

\[
b = \lim_{L \to \infty} \frac{1}{(\beta L)^2} \int_T d^2x d^2y \sum_{n, k = -\infty}^{+\infty} e^{i\omega(x - y)} \frac{\omega_0(0)}{\omega^2}
\]

(85)

Notice that the first exponential on the right-hand-side of (84) is independent on \(\beta, \mu\) and then it is irrelevant as far as the thermodynamics of the model is concerned. On the other hand, the constant \(b\) (which is independent on both \(T\) and \(\mu\)) could in principle give rise to a dependence on \(f(0)\). Again, the very definition of \(b\) is ambiguous. If we agree to evaluate the (imaginary time) integrals over \(x^0\) and \(y^0\) in (83) before the spatial ones, then \(b = 0\) and, hence, the independence of the partition function on \(f(\omega^2)\) follows, which is a welcomed result. Another independent reason to favour this prescription is that it is the same as that leading from (48) to (49). Now, by recalling the expression for the free charge density in (82), we have from (84) that the total fermion number density of the system reads in the \(L \to \infty\) limit

\[
\rho = \frac{\mu}{\pi + g^2}
\]

(86)

Hence we obtain that the Thirring model at finite \(T\) and \(\mu\) is no longer a free fermion gas, but the fermion density acquires a correction in \(g^2\), as it stands in (83). It differs from the result in (13) in which only the free contribution to the fermion density remains. It is clear from our analysis starting from the Schwinger model that the correction to the free gas comes entirely from the topological contribution depending on the \(F\) function. This contribution only depends on the harmonic field \(h_0\) in the decomposition of the gauge field. The calculation in (13) was done in real time formalism, in which this term is not present, whereas in (13) the model is solved in the torus. As it is emphasised in (13), the toroidal compactification is very useful to deal with infrared divergences and the harmonic parts of the gauge field are essential to correctly quantise the model. It is the most natural choice when using the imaginary time formalism, as we have done in this work. On the other hand, if we evaluate the pressure of the system, which follows directly by taking the logarithm of the partition function in (84), our result (not quoted for brevity) agrees with (13), which provides a check of consistency between our methods and those used in that work.

Finally, we shall give the exact fermion correlation function for the Thirring model at nonzero \(T\) and \(\mu\)

\[
G(x, y) = \frac{\delta^2 \log Z[\xi, \bar{\xi}]}{\delta \xi(x) \delta \xi(y)} \bigg|_{\xi = \bar{\xi} = 0} = \Theta(x, y) S(x, y)
\]

\[
\Theta(x, y) = \exp \left\{ -g^2 \frac{1}{\beta L} \sum_{n, k = -\infty}^{+\infty} \frac{1}{\omega^2} \left[ 1 - e^{i\omega(x - y)} \right] \right\}
\]

\[
\times \left[ f(\omega^2) - \frac{g^2/\pi}{1 + g^2/\pi} \right] + 2g F(T, \mu, L) \cdot c
\]

(87)
where we have used again (46) into (78) and (81) and performed the functional differentiation. In turn, c is given by the formal expression

$$c = \sum_{n,k=-\infty}^{+\infty} \left[ e^{-\omega x} - e^{-\omega y} \right] \frac{\delta \omega_n \delta \omega_k}{\omega^2} \left[ -\frac{\omega \gamma^0}{1 + g^2/\pi} + \omega_0 \left( f(\omega^2) - \frac{g^2/\pi}{1 + g^2/\pi} \right) \right]$$

which, again, turns out to be ambiguous. Like we did with the same ambiguities before, let us evaluate the summation over n in (88) (which is reminiscent of imaginary-time integrations) before the spatial summation and let $L \to \infty$. Then one gets

$$c = \frac{i}{1 + g^2/\pi} (x_1 - y_1) \gamma^1 \gamma^0$$

**VI. CONCLUSIONS AND DISCUSSION**

The main new results obtained in this work are the following:

1) In the imaginary time formalism, the fermionic generating functional $Z_f$ with an external electromagnetic field and the full generating functional $Z$ for the Schwinger model have been explicitly obtained for any spatial length $L$ in the trivial sector (in which the Dirac operator has no zero modes).

2) The work previously done in [7] at finite $T$ but $\mu = 0$, in which the model was formulated in a two-dimensional torus for an arbitrary number of zero modes, can be extended when both $T$ and $\mu$ are nonzero. Such an extension has to be worked out carefully due to some non trivial peculiarities of the $\mu \neq 0$ case. Technically, the main distinctive feature is the lack of hermiticity of the Dirac operator. This implies a non-vanishing phase factor $\mathcal{J}^{(k)}$ for the fermion determinant in the sector with $k$ zero modes. Using functional methods we have evaluated this term for $k = 0$ (the trivial sector), which plays a crucial role in the solutions for the Schwinger and Thirring models presented here and in the physical features thereof. That phase depends on $T$ and $\mu$, is linear in the zeroth component of the electromagnetic potential $A_0$ (in agreement with charge conjugation symmetry arguments) and vanishes if $\mu = 0$ for any $T, A_0$. Furthermore, this term is topological, in the sense that it changes only under nontrivial gauge transformations, with nonzero winding number around the $S^1$ parametrising the Euclidean time. In terms of the Hodge decomposition of the gauge field in the torus, it only depends on the harmonic part. The existence of topological $\mu$-induced effective actions seems to be a common feature of different models.

3) For the Schwinger model we have calculated the thermodynamical partition function. The topological phase factor in the effective action gives rise to a nonperturbative contribution that in the $L \to \infty$ limit exactly cancels the $\mu$ dependence contained in the free fermionic partition function. Then, at $L \to \infty$, the partition function is independent on $\mu$ and hence the total charge density of the system is zero. In other words, the system bosonizes even though it could have a net fermionic charge density at nonzero $\mu$. The partition function factorises, as in [6] (into that of free fermions at $\mu = 0$ times a factor, which is the ratio of that of massive bosons, divided by that of free massless bosons), the mass of the boson being $m = e/\sqrt{\pi}$, independent of $T$ and $\mu$. The exact boson propagator has an additional $\mu$-dependent piece for any $L$. At $L \to \infty$ this new piece gives rise to a vanishing inverse correlation length squared $M^2$, which is interpreted through the relationship between $M^2$ and the first derivative of the charge density given in [14]. However, by calculating for $\mu \neq 0$ the thermal average of the Polyakov loop (which is $\mu$-dependent) and its correlator ($\mu$-independent), we have shown that the $Z$ symmetry is broken for any $T$ and $\mu$ (deconfinement) and that the screening mass between two opposite charges is the mass $m$ of the boson. A study of what happens regarding the above mentioned cancellation in the thermodynamical partition function, when $L$ is kept finite, lies outside the scope of this work. Our computations of the thermodynamical partition function through two different methods yielding the same result establish the consistency of our approach.

4) Several important features of the solution of the Schwinger model for $\mu \neq 0$ in the sectors with zero modes are summarised, as they are closely related to the analysis in the trivial sector. Namely, we have given the general structure of the fermion determinant, solving the spectrum of the Dirac operator for an instanton configuration when $\mu \neq 0$. The chemical potential breaks the chiral degeneracy of the spectrum. We remark that the correlation functions for $\mu \neq 0$ have been analysed in [6], although in that paper a different approach based on bosonization is used and the harmonic part of the gauge field (and hence the contribution of the phase factors) is not considered. The analysis of the phase factors and the fermionic two point function when there are zero modes lies beyond the scope of this work.

5) In the imaginary time formalism as well, the generating functional for the massless Thirring model at finite $T$, $\mu$ is constructed in terms of the fermionic generating functional $Z_f$ for the Schwinger model, previously found in this work. We have justified that it is enough to restrict ourselves to the trivial sector for $Z_f$. The thermodynamical
partition function, the total fermion number density and the fermion correlation function have been computed for non-vanishing \( \mu \) and \( T \). A distinctive feature is that all of them depend nontrivially on \( \mu \), as a consequence of the non-trivial phase \( J^{(0)} \) of the Schwinger model. Our result for the pressure agrees with [14], which shows that our different approach is consistent. Our total fermion density differs from [13], where it was obtained, using real time formalism, that the model is equivalent to that of free fermions. The origin of that discrepancy is that in [13] the harmonic pieces of the vector field are not considered.

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APPENDIX A: THE FERMION FUNCTIONAL DETERMINANT FOR \( \Phi \neq 0 \)

We shall outline here the calculation of \( \text{det}' H(A; \mu) \) in section [1C], following steps similar to those in [1], with suitable generalisations for our present case. Firstly, we shall relate \( \text{det}' H(A; \mu) \) with \( \text{det}' H(\tilde{A}; \mu) \). For that purpose, we define \( \Phi_\alpha \), replacing \( e \rightarrow e\alpha \) in (7). Then, the corresponding \( H_\alpha = \mathcal{D}_\alpha \Phi_\alpha \) interpolates between \( H(A) \) and \( H(\tilde{A}) \) when \( \alpha \) varies from 0 to 1, and so on for \( \mathcal{D}_\alpha = \mathcal{D}_\nu \mathcal{D}_\nu' \). The operator \( H_\alpha \) can be cast as in (18), but now with \( A_\mu = \tilde{A}_\mu - \alpha e\mu_\nu \partial_\nu \phi \) and the operator \( \mathcal{D}_\alpha \) is obtained from \( H_\alpha \) by changing \( \mu \rightarrow -\mu \). By using \( \zeta \)-regularization [27], we have:

\[
\log \text{det}' H_\alpha = - \frac{d}{ds} \zeta_H(s; \alpha) \bigg|_{s=0}
\]

\[
\zeta_H(s; \alpha) = \sum_{q=k+1}^{\infty} \mu_q^{-s}(\alpha)
\]

\( \mu_q(\alpha) \) denoting, generically, the non-vanishing eigenvalues of \( H_\alpha \). As in section [1A], we choose the eigenstates of \( \mathcal{D}_\alpha \) as \( \phi_q^{(\alpha)} = (\mathcal{D}_\alpha \phi_q^{(\alpha)})/\sqrt{\mu_q(\alpha)} \), where \( \phi_q^{(\alpha)} \) are the eigenstates of \( H_\alpha \) for \( \mu_q(\alpha) \neq 0 \). Now we use the Feynman-Hellmann formula \( \mu_q(\alpha) = \left( \phi_q^{(\alpha) \dagger} H_\alpha \phi_q^{(\alpha)} \right) \), where the dot indicates derivation with respect to \( \alpha \), and the Seeley-de Witt expansion [27] for \( H_\alpha \). Then, following similar steps as in [1] we can write the derivative of \( \log \text{det}' H_\alpha \) in (A1) with respect to \( \alpha \), in terms of \( E_\alpha = \tilde{E} + \alpha \Delta \phi, \phi(x) \) and the zero modes \( \phi_p^{(\alpha)} \) and \( \phi_p^{(\alpha)} \) of \( H_\alpha \) and \( \mathcal{D}_\alpha \). The latter are related to the zero modes of \( H \) and \( \mathcal{D} \) simply by multiplying by \( \exp(-e\alpha\gamma_5) \). Then, the integral in \( \alpha \) can be done and we obtain [27], [1] for through:

\[
\text{det}' H(A; \mu) = \text{det}' H(\tilde{A}; \mu) \det \left\{ N(1) \left[ N(0) \right]^{-1} \right\} \det \left\{ M(1) \left[ M(0) \right]^{-1} \right\} \times \exp \frac{2 \varepsilon^2}{\pi} \int_T d^2 x \phi(x) \left[ \tilde{E} + \frac{1}{2} \Delta \phi(x) \right]
\]

where the elements of the matrices \( N(\alpha) \) and \( M(\alpha) \) are \( N_{pp'}^{(\alpha)} = \int_T d^2 x \phi_p^{(\alpha) \dagger} \phi_{p'}^{(\alpha)} \) and \( M_{pp'}^{(\alpha)} = \int_T d^2 x \phi_p^{(\alpha) \dagger} \phi_p^{(\alpha)} \).

Secondly, we are going to calculate the spectrum of \( H(A; \mu) \) in [28], with the boundary conditions discussed in the text. Like in the \( \mu = 0 \) case [1], we shall try eigenfunctions bearing the form:

\[
\phi_{n,m}^{\pm}(x) = e^{(2n+1)\pi i/\beta} e^{i \hat{n}_1 \pi \mu x_1} \phi_{n,m}^{\pm}(x_1)
\]

By plugging (A3) into the eigenvalue equation, we arrive at an harmonic oscillator eigenvalue problem, which can be solved in the standard fashion. However, the functions \( \phi_{n,m}^{\pm} \) in (A3) do not satisfy the right boundary conditions. It is not difficult to see that:
\[ \hat{\phi}^{\pm}_{n,m}(x^0, x^1) = \sum_{j=-\infty}^{\infty} e^{(2\pi i\epsilon_1 \pm i\mu L)j} \hat{\phi}^{\pm}_{n+j,k',m}(x^0, x^1) \]  
(A4)

with \( k' \equiv n_+ - n_- = \Phi/2\pi \), are the correct eigenfunctions, which do satisfy the right boundary conditions in (1). We have used that the \( \xi(x_1) \) functions in (A3) depend on \( x^1 \) and \( n \) only through the combination \( y = x^1 + 2\pi L(n + 1/2 - \epsilon h_0)/\Phi \).

The usual harmonic oscillator quantisation condition for the \( \xi^{\pm}_{n,m} \) states reads in this case:

\[ \frac{\lambda^{\pm} L\beta}{|\Phi|} \pm \text{sgn}\Phi = 2m + 1 \]  
(A5)

\( m \) being an integer, with \( m \geq 0 \). Then, the eigenvalues for \( \Phi \neq 0 \) are independent on \( \mu \), and they are: \( \lambda = 0 \) with degeneracy \( k \), and \( \lambda_m = 2m|\Phi|/L\beta \) with degeneracy \( 2k \). As we had anticipated, the zero modes appear with only one chirality, equal to the sign of \( \Phi \). The eigenfunctions are those in (A4) and (A3) with

\[ \xi^{\pm}_{n,m} = H_m \left[ \frac{|\Phi|}{L\beta} y \right] \exp \left[ -\frac{|\Phi|^2}{2L\beta y^2} \right] \]  
(A6)

where \( H_m \) are the Hermite polynomials. Once we know the eigenvalues, we can calculate the determinant using again \( \zeta \)-function regularization and we get:

\[ \text{det}' H(\hat{A}; \mu) = \exp \left[ -\frac{d}{ds} \xi_H(s) \bigg|_{s=0} \right] = \left( \frac{\pi L\beta}{|\Phi|} \right)^k \]  
(A7)

independent on \( \mu \). If we concentrate only in the zero modes, that is, the \((n,m)\) eigenstates in (A4) with \( m = 0 \), it turns out that they are already orthogonal and that their norms are independent on \( \mu \):

\[ ||\phi_{n,0}||^2 = \left( \frac{\pi L\beta^3}{|\Phi|} \right)^{1/2} \]  
(A8)

As it was commented in the text, once we know the spectrum of the Dirac operator for \( \Phi \neq 0 \) we could calculate the chiral condensates, which do not depend on the Green function \( G(x, y, eA; \mu) \), up to the phase factor. In particular, we have

\[ \langle \overline{\psi}(x) P_{\pm} \psi(x) \rangle_f = \exp \left[ i \mathcal{J}^{(1)}(A; \mu) \right] \sqrt{\text{det}' H(A; \mu) \varphi_1^+(x) P_{\pm} \varphi_1(x)} \]  
(A9)

where the superscript 1 indicates the sector with only one zero mode.

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