SUBSPACES OF INTERVAL MAPS RELATED TO THE TOPOLOGICAL ENTROPY

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Abstract. For $a \in [0, +\infty)$, the function space $E_{\geq a}$ ($E_{> a}; E_{\leq a}; E_{< a}$) of all continuous maps from $[0, 1]$ to itself whose topological entropies are larger than or equal to $a$ (larger than $a$; smaller than or equal to $a$; smaller than $a$) with the supremum metric is investigated. It is shown that the spaces $E_{\geq a}$ and $E_{> a}$ are homeomorphic to the Hilbert space $l_2$ and the spaces $E_{\leq a}$ and $E_{< a}$ are contractible. Moreover, the subspaces of $E_{\leq a}$ and $E_{< a}$ consisting of all piecewise monotone maps are homotopy dense in them, respectively.

1. Introduction

One of the central topics in the study of infinite-dimensional topology is that which kinds of function spaces are homeomorphic to the separable infinite dimensional Hilbert space $l_2$ or its well-behaved subspaces. The well-known Anderson-Kadec’s theorem states that the countable infinite product $\mathbb{R}^N$ of lines is homeomorphic to $l_2$, see [1, 10]. Using this result, it was proved that the function space of real valued maps of an infinite compact metric space with the supremum metric is homeomorphic to $l_2$. See [4, 12, 15] for more on this topic. Moreover, in [5], the authors proved that the function space of real valued maps of an infinite countable metric space with the topology of pointwise convergence is homeomorphic to the subspace $c_0 = \{(x_n) \in \mathbb{R}^N : \lim_{n \to \infty} x_n = 0\}$ of $\mathbb{R}^N$. In a series of papers, the fourth named author of the present paper and his coauthors gave a condition for the continuous functions from a k-space to $I = [0, 1]$ with the Fell hypergraph topology being homeomorphic to $c_0$, see [16, 17, 18, 19].

In the study of dynamical systems, some function spaces naturally appear. The group of measure preserving transformations of the unit interval equipped with the weak topology is homeomorphic to $l_2$ (see [6] and [13]). Recently, in [11], Kolyada et al. proposed the study of dynamical topology: investigating the topological properties of spaces of maps that can be described in dynamical terms. They showed in [11] that the space of transitive interval maps is contractible and uniformly locally arcwise connected, see also [12] for more detailed results. In [8], Grine et al. discussed some topological properties of subspaces of interval maps related to the periods of periodic points.

In this paper, we will follow the idea in [11] to study subspaces of interval maps related to the topological entropy. Let $I = [0, 1]$ and $C(I)$ be the collection of continuous maps on $I$.

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with the supremum metric $d$. For each $f \in C(I)$, denote by $h_{\text{top}}(f)$ the topological entropy of $f$. For any $a \in [0, +\infty)$, let

$$
E_{\geq a} = \{ f \in C(I) : h_{\text{top}}(f) \geq a \};
$$

$$
E_{\geq a} = \{ f \in C(I) : h_{\text{top}}(f) > a \};
$$

$$
E_{\leq a} = \{ f \in C(I) : h_{\text{top}}(f) \leq a \};
$$

$$
E_{< a} = \{ f \in C(I) : h_{\text{top}}(f) < a \}.
$$

A map $f \in C(I)$ is said to be **piecewise monotone** if there exist $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $f|_{[t_{i-1}, t_i]}$ is monotone for every $i = 1, 2, \ldots, n$. Similarly, we can define a map to be **piecewise linear**. We use $C_{\text{PM}}(I)$ to denote the set of all piecewise monotone continuous maps on $I$ and

$$
E_{PM} = E_{\leq a} \cap C_{\text{PM}}(I).
$$

The main results of this paper are as follows:

**Theorem 1.1.** For every $a \in [0, +\infty)$, both $E_{\geq a}$ and $E_{\geq a}$ are homeomorphic to $l_2$.

**Theorem 1.2.** There exists a homotopy $H : C(I) \times I \to C(I)$ such that

1. $H_0 = \text{id}_{C(I)}$;
2. $h_{\text{top}}(H_t(f)) \leq h_{\text{top}}(f)$ and $H_t(f) \in C_{\text{PM}}(I)$ for any $t \in (0, 1)$ and $f \in C(I)$;
3. $H_1(f) \equiv 0$ for any $f \in C(I)$.

Restricting the homotopy in Theorem 1.2 to $E_{\geq a}$ and $E_{> a}$ respectively, we can obtain the following corollary:

**Corollary 1.3.** For every $a \in [0, +\infty)$, $E_{\geq a}$ ($E_{< a}$, respectively) is contractible and $E_{PM}$ ($E_{PM}$, respectively) is homotopy dense in $E_{\geq a}$ ($E_{< a}$, respectively).

The paper is organized as follows. In Section 2, we recall some basic notions which we will use in the paper. Theorems 1.1 and 1.2 are proved in Sections 3 and 4 respectively.

2. Preliminaries

In this section, we recall some notions and aspects of infinite-dimensional topology and topological entropy which will be used later.

2.1. Infinite-dimensional topology. In this subsection, we give some concepts and facts on general topology and infinite-dimensional topology. For more information, we refer the reader to [7][4][14][15].

Let $(X, d)$ be a metric space. We say that

- $X$ is **nowhere locally compact** if no non-empty open set in $X$ is locally compact;
- $X$ is an **absolute (neighborhood) retract** (A(N)R, briefly) if for every metric space $Y$ which contains $X$ as a closed subspace, there exists a continuous map $r : Y \to X$ ($r : U \to X$ from a neighborhood $U$ of $X$) such that $r|_X = \text{id}$;
- $X$ has the **strong discrete approximation property** (SDAP, briefly) if for every continuous map $\varepsilon : X \to (0, 1)$, every compact metric space $K$ and every continuous map $f : K \times \mathbb{N} \to X$, there exists a continuous map $g : K \times \mathbb{N} \to X$ such that $\{g(K \times \{n\}) : n \in \mathbb{N}\}$ is discrete in $X$ and $d(f(k, n), g(k, n)) < \varepsilon(f(k, n))$ for every $(k, n) \in K \times \mathbb{N}$.
A homotopy on $X$ is a continuous map $H : X \times I \to X$, $(x, t) \mapsto H_t(x)$. The space $X$ is said to be contractible if there exists a homotopy $H : X \times I \to X$ such that $H_0 = \text{id}_X$ and $H_1$ is a constant map. A subset $A$ of $X$ is called homotopy dense if there exists a homotopy $H : X \times I \to X$ such that $H_0 = \text{id}_X$ and $H_t(x) \in A$ for every $x \in X$ and $t \in (0, 1]$.

We will need the following important results in infinite-dimensional topology.

Proposition 2.1. ([14] Theorem 5.2.15]) A metric space is an AR if and only if it is a contractible ANR.

Theorem 2.2. ([2] 1.2.1 Proposition and Exercise 1.3.4]) Let $Y$ be a homotopy dense subspace of $X$. If $X$ is an ANR (with SDAP) then $Y$ is also an ANR (with SDAP).

Theorem 2.3. ([2] 1.1.14 (Characterization Theorem)]) A separable topologically complete metric space is homeomorphic to $l_2$ if and only if it is an AR with SDAP.

Theorem 2.4. ([2] 5.5.2 Corollary]) A convex subspace $X$ of a separable Banach space is homeomorphic to $l_2$ if and only if $X$ is topologically complete and nowhere locally compact.

The following result must be “folklore”, but we can not find a proper reference and therefore we provide a proof for the completeness.

Proposition 2.5. The function space $C(I)$ is homeomorphic to $l_2$.

Proof. Let $C(I, \mathbb{R})$ be the collection of all continuous maps from $I$ to $\mathbb{R}$ with the standard linear structure and the supremum norm. Then $C(I, \mathbb{R})$ is a separable Banach space. The space $C(I)$ is a closed and convex subspace of $C(I, \mathbb{R})$. It is not hard to verify that $C(I)$ is nowhere locally compact. It follows from Theorem 2.4 that $C(I)$ is homeomorphic to $l_2$. □

Combining the above results, we have the following useful criterion when a subspace of $C(I)$ is homeomorphic to $l_2$.

Corollary 2.6. A homotopy dense subspace $A$ of $C(I)$ is homeomorphic to $l_2$ if and only if it is topologically complete and contractible.

Proof. The necessity is clear and we only need to prove the sufficiency. By Proposition 2.5, $C(I)$ is homeomorphic to $l_2$. So by Theorem 2.3, $C(I)$ is an ANR with SDAP. Since $A$ is homotopy dense in $C(I)$, it follows from Theorem 2.2 that $A$ is also an ANR with SDAP. By the assumption we have $A$ is contractible, then by Proposition 2.1 $A$ is an AR. Finally by Theorem 2.3 again, $A$ is homeomorphic to $l_2$. □

2.2. Topological entropy. Let $X$ be a compact metric space. Denote by $Cov(X)$ the family of all open covers of $X$. For $\alpha, \beta \in Cov(X)$ and $f \in C(X)$, let

$$N(\alpha) = \min \{ n \in \mathbb{N} : \text{there exist } U_1, U_2, \cdots, U_n \in \alpha \text{ such that } \bigcup_{i=1}^n U_i = X \};$$

$$\alpha \vee \beta = \{ U \cap V : U \in \alpha, V \in \beta \}, \quad f^{-1}(\alpha) = \{ f^{-1}(U) : U \in \alpha \};$$

and

$$h_{\text{top}}(f, \alpha) = \lim_{n \to \infty} \frac{\log N(\alpha \vee f^{-1}(\alpha) \vee \cdots \vee f^{-n+1}(\alpha))}{n}.$$
The topological entropy of a continuous map \( f : X \to X \) is defined as

\[
h_{\text{top}}(f) = \sup \{ h_{\text{top}}(f, \alpha) : \alpha \in \text{Cov}(X) \}.
\]

Let \( f \in C(I) \). A family \( \{ J_1, J_2, \cdots, J_n \} \) of non-degenerate closed intervals is called an \( n \)-horseshoe if

1. \( \text{int}(J_i) \cap \text{int}(J_j) = \emptyset \) for all \( 1 \leq i < j \leq n \), where \( \text{int}(J_i) \) is the interior of \( J_i \) in \( I \);
2. \( J_i \subset f(J_j) \) for all \( 1 \leq i, j \leq n \).

The following result can be easily obtained, see e.g. [3] Proposition VIII.8.

**Lemma 2.7.** If \( f \in C(I) \) has an \( n \)-horseshoe, then \( h(f) \geq \log n \).

The following result was first proved by Misirewicz, see e.g. [3] Proposition VIII.30.

**Theorem 2.8.** The entropy function \( h_{\text{top}} : C(I) \to [0, +\infty], f \mapsto h_{\text{top}}(f) \) is lower-semicontinuous.

**Corollary 2.9.** For every \( a \in [0, +\infty) \), \( E_{\geq a} \) is open and \( E_{\leq a} \) is a \( G_{\delta} \)-set in \( C(I) \).

The convexity of \( C(I) \) in the Banach space \( C(I, \mathbb{R}) \) plays a key role in the proof of Proposition 2.5. The following examples show that neither \( E_{\leq a} \) nor \( E_{\geq a} \) is convex in \( C(I, \mathbb{R}) \).

**Example 2.10.** Note that for every \( f \in C(I) \), if \( f(\frac{1}{2} - x) = f(\frac{1}{2} + x) \) for all \( x \in [0, \frac{1}{2}] \), then \( f \) and \( 1 - f \) are topologically conjugate and thus \( h_{\text{top}}(1 - f) = h_{\text{top}}(f) \). But \( h_{\text{top}}(\frac{1}{2}f + \frac{1}{2}(1 - f)) = h_{\text{top}}(\frac{1}{2}) = 0 \). It follows that \( E_{\geq a} \) is not convex for any \( a \in [0, +\infty) \).

**Example 2.11.** It is well-known that for every \( f \in C(I) \), \( h_{\text{top}}(f) = 0 \) if and only if all periods of \( f \) are of the form \( 2^n \) (see e.g. Proposition VIII.34 and Theorem II.14 in [3]). Let \( f \) and \( g \) be the broken line maps through the points \((0, 1), (\frac{1}{2}, 0), (1, 0)\) and the points \((0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2})\), respectively. Then it is not hard to verify that \( n \) is a period for \( f \) or \( g \) if and only if \( n = 1 \) or \( n = 2 \). It follows that \( h_{\text{top}}(f) = h_{\text{top}}(g) = 0 \). For the convex combination \( \varphi = \frac{1}{2}f + \frac{1}{2}g \), we have \( \varphi(0) = \frac{3}{4}, \varphi(\frac{1}{2}) = \frac{1}{4} \) and \( \varphi(\frac{1}{4}) = 0 \). It follows that \( 0 \) is a periodic point with period \( 3 \) for \( \varphi \), which implies \( h_{\text{top}}(\varphi) > 0 \). This shows that \( E_{\leq 0} \) is not convex.

### 3. Proof Theorem [11]

In this section, we will prove Theorem 1.1. At first, we introduce the box maps defined in [11]. Define a subset \( \Lambda \) of \( \mathbb{R}^5 \) as follows

\[
\Lambda = \{(a_t, a_r, a_b, a_t, a_r) \in \mathbb{R}^5 : a_b < a_t, a_t, a_r \in [a_b, a_t], a_b \geq 20\}.
\]

For every non-degenerate closed interval \( K = [a_0, a_1] \) and \( \lambda = (a_t, a_r, a_b, a_t, a_r) \in \Lambda \), the authors in [11] defined a continuous surjection \( \xi_\lambda : K \to [a_b, a_t] \), which was called a box map, such that \( \xi_\lambda \) is piecewise linear with constant slope \( \frac{a_t(a_t - a_b)}{a_t - a_r} \), \( \xi_\lambda(a_0) = a_t \) and \( \xi_\lambda(a_1) = a_r \). We make this construction both from left and right, \( \xi_\lambda \) is increasing on the leftmost lap unless \( a_t = a_r \) and decreasing on the rightmost one unless \( a_t = a_r \). We choose the meeting point \( m \) to be on the fifth decreasing lap from the left (see Figure 1 for example). If the left and right graphs coincide, then there is no well-defined meeting point, but the graph of \( \xi_\lambda \) is clear.
Remark 3.1. Let $\xi_4$ be a box map on $K$. If $a_b = a_0$ and $a_t = a_1$, then there exist closed subintervals $J_1$, $J_2$, $\ldots$, $J_{[a_s-4]}$ of $K$ with disjoint interiors such that $f(J_j) = K$ for $j = 1, 2, \ldots, [a_s - 4]$, where $[x]$ is the greatest integer less than or equal to $x$. Hence, $J_1$, $J_2$, $\ldots$, $J_{[a_s-4]}$ form an $[a_s - 4]$-horseshoe of $\xi_4$. By Lemma 2.7, $h_{\text{top}}(\xi_4) \geq \log([a_s - 4])$.

Following the idea in [11], for every $\alpha \geq 20$ we first construct a homotopy $\tilde{H}^\alpha : C(I) \times I \to C(I)$ as follows. Fix a function $f \in C(I)$. First let $\tilde{H}^0_0(f) = f$. For $t \in (0, 1]$, let $s$ be the largest non-negative integer such that $st < 1$. We obtain $s + 1$ closed intervals:

$$I_i = [(i-1)t, it], \ i = 1, 2, \ldots, s, \ I_{s+1} = [st, 1].$$

In particular, if $t = 1$, then $s = 0$ and we have only one closed interval $I_1 = [0, 1]$. For $i = 1, 2, \ldots, s + 1$, let $\alpha_i = \max([I_i], |f(I_i)|)$, where $|J|$ is the length of a closed interval $J$, and

\[
\begin{align*}
  a_i^+ &= \max\{0, \min(f(I_i)) - 4\alpha_i\}; \\
  a_i^- &= \min\{1, \max(f(I_i)) + 4\alpha_i\}; \\
  a_i^1 &= f(\min I_i); \\
  a_i^2 &= f(\max I_i).
\end{align*}
\]

It is not hard to verify that if $I_i \cap f(I_i) \neq \emptyset$ then

\begin{equation}
I_i \subset [a_i^+, a_i^-].
\end{equation}

It is clear that $\lambda_i^\alpha = (a_i^+, a_i^-, a_i^1, a_i^2, \alpha) \in \Lambda$ and then we define $\tilde{H}^\alpha_i(f)$ on $I_i$ as the box map $\xi_{\lambda_i^\alpha} \in C(I_i, I)$. So $H^\alpha_i(f)$ is well-defined for $t \in (0, 1]$. By Lemma 2.2 of [11], $\tilde{H}^\alpha : C(I) \times I \to C(I)$ is a homotopy. Note that $\tilde{H}^0_0 = \text{id}_{C(I)}$ and for every $f \in C(I)$, $\tilde{H}^0_1(f)$ is the box map on $I$ with the parameter $(f(0), f(1), 0, 1, \alpha)$. Now we construct another homotopy $\tilde{H}^\alpha : \tilde{H}^0_1(C(I)) \times I \to C(I)$. For every $f \in C(I)$ and $t \in [0, 1]$ we define $\tilde{H}^\alpha_1(f)$ to be the box map on $I$ with the parameter $((1-t)f(0), (1-t)f(1), 0, 1, \alpha)$. By Lemma 2.1 of [11], $\tilde{H}^\alpha$ is continuous then it is a homotopy. It should be noticed that for every $f \in C(I)$, $H^\alpha(f)$ is the box map on $I$ with the parameter $(0, 0, 1, \alpha)$. Finally, we define a homotopy $H^\alpha : C(I) \times I \to C(I)$ by joining $\tilde{H}^\alpha$ and $H^\alpha$, that is, for every $f \in C(I)$, $H^\alpha(f) = \tilde{H}^\alpha_1(f)$ for $t \in [0, \frac{1}{2}]$ and $H^\alpha(f) = \tilde{H}^\alpha_{2t-1}(f)$ for $t \in (\frac{1}{2}, 1]$.

We have the following estimation of the topological entropy of $H^\alpha(f)$.

Lemma 3.2. For every $t \in (0, 1]$, $\alpha \geq 20$ and $f \in C(I)$, we have

$$h_{\text{top}}(H^\alpha_t(f)) \geq \log([\alpha - 4]).$$
Proposition 3.6. An important fact is that there is no continuous selection of fixed points.

Proof. Suppose that \( t \in [0, \frac{1}{2}] \). By the construction of \( H^t \), there exists an interval \( I_t \) and \( x_0 \in I_t \) such that \( f(x_0) = x_0 \). By the formula (5.7), we have \( I_1 \subset [a_j', a'_j] \). Now by the construction of the box map on \( J_l \), there exist closed subintervals \( J_1, J_2, \ldots, J_{[\alpha-4]} \) of \( I_t \) with disjoint interiors such that \( H^t(f)(J_j) = [a_j', a'_j] \) for \( j = 1, 2, \ldots, [\alpha-4] \). Then \( J_1, J_2, \ldots, \)

We summarize the above results as follows.

Proposition 3.3. For every \( \alpha \geq 20 \), there exists a homotopy \( H^\alpha : C(I) \times I \to C(I) \) such that

1. \( H^\alpha_0 = \text{id}_{C(I)} \);
2. \( h_{\text{top}}(H^\alpha_t(f)) \geq \log([\alpha - 4]) \) for \( t \in (0, 1) \) and for every \( f \in C(I) \);
3. \( H^\alpha_t(f) \) is the box map on \( I \) with the parameter \( (0, 0, 0, 1, \alpha) \).

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Fix \( a \in [0, +\infty) \) and choose \( \alpha \in [20, +\infty) \) such that \( \log([\alpha - 4]) > a \). Let \( H^\alpha \) as in the Proposition 3.3. Then both \( E_{\leq a} \) and \( E_{> a} \) are homotopy dense in \( C(I) \). Using the homotopies \( H^\alpha|_{E_{\leq a} \times 4} \) and \( H^\alpha|_{E_{> a} \times 4} \), both \( E_{\leq a} \) and \( E_{> a} \) are contractible. By Corollary 2.9, both \( E_{\leq a} \) and \( E_{> a} \) are topologically complete. Now using Corollary 2.6, \( E_{\leq a} \) and \( E_{> a} \) are homeomorphic to \( I_2 \).

Corollary 3.4. For every \( a \in [0, +\infty) \), \( E_{\leq a} \) and \( E_{> a} \) are homotopy dense in \( C(I) \). Moreover, \( E_{> a} \cap E_{< +\infty} \) is homotopy dense and open in \( E_{< +\infty} \).

Proof. The former was shown in the proof of Theorem 1.1. To show the latter, we only note that the topological entropy of a piecewise monotone map is finite (see e.g. [3] Proposition VIII.18).

By Theorem 2.8 and Corollary 3.4, we know that the subspace \( E_{> a} = \{ f \in C(I) : h_{\text{top}}(f) = +\infty \} \) is a dense \( G_\delta \) set in \( C(I) \). But the following question remains open.

Problem 3.5. Is \( E_{> a} \) homeomorphic to \( I_2 \)?

In Proposition 3.3, for every \( f \in C(I) \) and \( t \in (0, 1] \), \( H^\alpha_t(f) \) is piecewise monotone and then it has finite topological entropy. So we can not use the method in the beginning of this section to construct a proper homotopy to show that \( E_{> a} \) is contractible. Another important fact is that there is no continuous selection of fixed points.

Proposition 3.6. There does not exist a continuous map \( \phi : C(I) \to I \) such that \( \phi(f) \) is a fixed point of \( f \) for every \( f \in C(I) \).

Proof. Suppose that \( \phi : C(I) \to I \) is such a map. Choose \( x_0 \in I \setminus \{ 0, 1, \phi(\text{id}_I) \} \). Let \( l_n : I \to I \) be the broken line map through the points \( (0, \frac{1}{n}), (x_0, x_0) \) and \( (1, 1 - \frac{1}{n}) \). Then \( l_n \to \text{id}_I \) in \( C(I) \) as \( n \to \infty \). Since \( l_n \) has a unique fixed point \( x_0, \phi(l_n) = x_0 \neq \phi(\text{id}_I) \) as \( n \to \infty \). So \( \phi \) is not continuous, which is a contradiction.
4. Proof of Theorem 1.2

In this section we construct the homotopy in Theorem 1.2 which is done by connecting three homotopies.

Inspired by [8] and [9], we introduce the following concept. Let $f$ and $\tilde{f} \in C([a, b], \mathbb{I})$. We say that $\tilde{f}$ is made from $f$ by procedure of making constant pieces (PMCP, briefly) if there exists a sequence of open intervals $\{U_n\}_{n=1}^\infty$ of $[a, b]$ in the relative topology such that $\tilde{f}|_{\bigcup_{n=1}^\infty U_n} = f|_{\bigcup_{n=1}^\infty U_n}$ and $\tilde{f}|_{U_n}$ is constant for every $n \in \mathbb{N}$. It should be noticed that our definition here is more general than the one in [8]. We will need the following result which was proved in [9] Lemma 5.

Lemma 4.1. Let $f \in C(\mathbb{I})$. If $\tilde{f}$ is made from $f$ by PMCP, then $h_{\text{top}}(\tilde{f}) \leq h_{\text{top}}(f)$.

For every $c \in \mathbb{I}$, the map $\max\{f(x), c\}$ can be thought to be made from $f$ by PMCP. For every $f \in C([a, b], \mathbb{I})$, let

\[
M(f) = \max\{f(x) : x \in [a, b]\};
\]
\[
c_1(f) = \min\{x \in [a, b] : f(x) = M(f)\};
\]
\[
c_2(f) = \max\{x \in [a, b] : f(x) = M(f)\}.
\]

Now we define $\tilde{f} : [a, b] \to \mathbb{I}$ as follows

\[
\tilde{f}(x) = \begin{cases} 
\max\{f(t) : a \leq t \leq x\}, & x \in [a, c_1(f)], \\
M(f), & x \in [c_1(f), c_2(f)], \\
\max\{f(t) : x \leq t \leq b\}, & x \in [c_2(f), b]. 
\end{cases}
\]

First we have the following lemma.

Lemma 4.2. For any $f, g \in C([a, b], \mathbb{I})$, we have

1. $\tilde{f}$ is made from $f$ by PMCP and it is in $C^{PM}([a, b], \mathbb{I})$;
2. $\tilde{f}(a) = f(a), f(b) = f(b)$ and $\tilde{f}([a, b]) \subset f([a, b])$;
3. $d(\tilde{f}, g) \leq d(f, g)$;
4. if $c \in (a, b)$ and $\varepsilon > 0$ satisfy either

\[
\max f|_{[a, c]} - \min f|_{[a, c]} < \varepsilon \quad \text{or} \quad \max f|_{[c, b]} - \min f|_{[c, b]} < \varepsilon,
\]

that is, the amplitude of $f$ on $[a, c]$ or on $[c, b]$ is smaller than $\varepsilon$, then

\[
d(\tilde{f}, \overline{f|_{[a, c]}} \cup \overline{f|_{[c, b]}}) < \varepsilon.
\]

Proof. (1) and (2) are obvious. We only need to show (3) and (4).

(3) We note that for any maps $h, k : J \to \mathbb{I}$,

\[
\sup\{h(x) : x \in J\} - \sup\{k(x) : x \in J\} \leq \sup\{|h(x) - k(x)| : x \in J\}.
\]

It follows that (3) holds in the case $[c_1(f), c_2(f)] \cap [c_1(g), c_2(g)] \neq \emptyset$. For the case $[c_1(f), c_2(f)] \cap [c_1(g), c_2(g)] = \emptyset$, without loss of generality, we assume that $c_2(f) < c_1(g)$. For $x \in [a, b] \setminus [c_2(f), c_1(g)]$ using the formula (4.1), we have that

\[
|\tilde{f}(x) - g(x)| \leq d(f, g).
\]
If $x \in (c_2(f), c_1(g))$ and $\tilde{f}(x) \geq \tilde{g}(x)$, then
$$0 \leq \tilde{f}(x) - \tilde{g}(x) \leq f(c_2(f)) - g(c_2(f)) \leq d(f, g).$$

If $x \in (c_2(f), c_1(g))$ and $\tilde{g}(x) > \tilde{f}(x)$, then
$$0 < \tilde{g}(x) - \tilde{f}(x) \leq g(c_1(g)) - f(c_1(g)) \leq d(f, g).$$

Hence (3) holds in the case $M(4.2)$.

Therefore, follows that

Moreover, for every $h \in [c_1(f), c_2(g)]$, by the assumption in (4),
$$h \in [1, n] \cup [c_1(f), c_2(g)] \subseteq C([a, b], I).$$

**Case A:** $c \in [c_1(f), c_2(f)]$. By the assumption, we have $M(f) - \varepsilon < f(c) \leq M(f)$. It follows that
$$M(f) - \varepsilon < f(c) \leq h(x) \leq M(f) = \tilde{f}(x), \quad x \in [c_1(f), c_2(f)].$$

Hence
$$|\tilde{f}(x) - h(x)| < \varepsilon, \quad x \in [c_1(f), c_2(f)].$$

Moreover, it is trivial that $\tilde{f}(x) = h(x)$ for $x \in [a, b] \setminus [c_1(f), c_2(f)]$. Hence
$$d(\tilde{f}, h) < \varepsilon.$$

**Case B:** $c \in [a, c_1(f)]$. In this case,
$$\tilde{f}(x) = h(x), \quad x \in [c_1(f), b].$$

Moreover, by the assumption in (4),
$$|\tilde{f}(x) - h(x)| < \varepsilon, \quad x \in [a, c].$$

Furthermore, for every $x \in [c, c_1(f)],$
$$h(x) = \tilde{f}_{|_{[c, b]}}(x) \leq \tilde{f}(x) < \tilde{f}_{|_{[c, b]}}(x) + \varepsilon = h(x) + \varepsilon.$$

Therefore,
$$d(\tilde{f}, h) < \varepsilon.$$

**Case C:** $c \in [c_2(f), b]$. By the assumption, $M(f) - \varepsilon < f(c) \leq M(f)$. It follows that
$$M(f) - \varepsilon < f(c) \leq f(c_2(f)_{|_{[c, b]}}) \leq M(f).$$

Note that
$$\tilde{f}(x) = f_{|_{[c, b]}}(x) \leq f_{|_{[c, b]}}(x), \quad x \in [c_1(f), c_2(f)], b].$$

Moreover, using this and the formula (4.2), we have
$$|\tilde{f}(x) - h(x)| < \varepsilon, \quad x \in [a, c_2(f)_{|_{[c, b]}}].$$

So in this case we also have $d(\tilde{f}, h) < \varepsilon.$

Using the above, we can give the first homotopy.

**Lemma 4.3.** There exists a homotopy $H^1 : C(I) \times I \rightarrow C(I)$ such that

1. $H^1_0 = \text{id}_{C(I)}$;
2. $h_{\text{top}}(H^1_t(f)) \leq h_{\text{top}}(f)$ and $H^1_t(f) \in C^{PM}(I)$ for $t \in (0, 1]$ and $f \in C(I)$. 

\[\square\]
Proof. In the same way as in the construction of the homotopy $H^a$ in Section 3, let $H^l_0 = \text{id}_{C(I)}$, and for $t \in (0,1]$, let $s$ be the largest non-negative integer such that $st < 1$. We can obtain $s + 1$ closed intervals:

$$I_i = [(i-1)t, it], i = 1, 2, \ldots, s, I_{s+1} = [st, 1].$$

The integer $s$ and the interval $I_i$ are also denoted by $s(t)$ and $I_i$ if necessary. We define $H^l_i$ such that, for every $f \in C(I)$ and $i = 1, 2, \ldots, s + 1$,

$$H^l_i(f)|_{I_i} = \tilde{f}|_{I_i}.$$  

Using Lemma 4.2, $H^l : C(I) \times I \to C(I)$ is well-defined. Trivially, it satisfies (1). From Lemmas 4.1 and 4.2, it follows that it satisfies (2). It remains to verify that $H^l : C(I) \times I \to C(I)$ is continuous.

At first, we show that $H^l(f, \cdot)$ is continuous for every fixed $f \in C(I)$. For every $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$(4.3) \quad |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \frac{\varepsilon}{2}.$$  

Now, for every $t_0 \in I$, we verify that there exists $\delta(t_0) \in (0, \delta]$ such that

$$|t - t_0| < \delta(t_0) \implies d(H^l(f, t), H^l(f, t_0)) < \varepsilon,$$

which shows that $H^l(f, \cdot)$ is continuous.

If $t_0 = 0$, we let $\delta(t_0) = \delta$. For every $t \in (0, \delta)$ and $i$, from Lemma 4.2 it follows that

$$\tilde{f}|_{I_i}(I_i) \subset f(I_i).$$

Since $|I_i| \leq \varepsilon < \delta$, using the formula (4.3), we have $|f(I_i)| < \varepsilon$. Thus the formula (4.4) holds.

If $t_0 \in (0, 1]$, choose $\delta(t_0) \in (0, \delta)$ small enough such that for every $t \in I \cap (t_0 - \delta(t_0), t_0 + \delta(t_0))$, we have $|s(t_0) - s(t)| < 2$ and $(s(t_0) + 2)\delta(t_0) < \delta$. Then all points $\{it, j|t_0\}$ divide $I$ into closed intervals $\{J_j\}$. Let

$$G = \bigcup \tilde{f}|_{I_i} \subset C(I).$$

Then, for every $i$, $I_i$ is either a union of the two closed intervals in $\{J_j\}$ or just a closed interval in $\{J_j\}$. If the former holds, then by the choice of $\delta(t_0)$ and the formula (4.3), the amplitude of $f$ in one of the two closed intervals is smaller than $\frac{\varepsilon}{2}$. Using Lemma 4.2, we have that

$$d(H^l(f, t_0)|_{I_i}, G|_{I_i}) < \frac{\varepsilon}{2}.$$  

If the later holds, then $H^l(f, t_0)|_{I_i} = G|_{I_i}$ and hence the above formula also holds. Thus,

$$d(H^l(f, t_0), G) < \frac{\varepsilon}{2}.$$  

Similarly, we have that

$$d(H^l(f, t), G) < \frac{\varepsilon}{2}.$$  

Hence the formula (4.4) holds.

By Lemma 4.2, we can obtain that $d(H^l(f, t), H^l(g, t)) \leq d(f, g)$. In combination with the continuity of $H^l$ on $t$, we have that $H^l : C(I) \times I \to C(I)$ is jointly continuous. □

The second homotopy we need is the following.
Lemma 4.4. There exists a homotopy $H^2 : C(I) \times I \rightarrow C(I)$ satisfying

1. $H^2_0 = \text{id}_{C(I)}$;
2. $h_{\text{top}}(H^2_t(f)) \leq h_{\text{top}}(f)$ and $H^2_t(C^{PM}(I)) \subset C^{PM}(I)$ for any $t \in (0, 1]$ and $f \in C(I)$;
3. $H^2_t$ is a constant map for any $f \in C(I)$.

Proof. For every $f \in C(I)$, let

$$M(f) = \max\{f(x) : x \in I\}, \quad m(f) = \min\{f(x) : x \in I\}.$$ 

Then $M, m : C(I) \rightarrow I$ are continuous. Using them, we can define our homotopy as follows

$$H^2(f, t)(x) = \max\{f(x), (M(f) - m(f))t + m(f)\}.$$

Then it is not hard to verify that $H^2 : C(I) \times I \rightarrow C(I)$ is continuous and it satisfies (1) and (3). Moreover, $H^2(f, t)$ is made from $f$ by PMCP. It follows from Lemma 4.1 that $H^2$ also satisfies (2).

The third homotopy $H^3 : I^2 \rightarrow I$ is defined as

$$H^3(s, t) = (1 - t)s,$$

which is a homotopy between the identical map and the constant map 0 in $I$. Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Define $H : C(I) \times I \rightarrow C(I)$ as

$$H(f, t) = \begin{cases} H^1(f, 3t), & t \in [0, \frac{1}{2}), \\ H^2(H^1(f, 1), 3t - 1), & t \in [\frac{1}{2}, \frac{2}{3}), \\ H^3(H^2(H^1(f, 1), 1), 3t - 2), & t \in [\frac{2}{3}, 1]. \end{cases}$$

Since $H^2(H^1(f, 1), 1)$ is a constant map, the homotopy $H$ is well-defined. Note that $h_{\text{top}}(c) = 0$ for every constant map $c$. By Lemmas 4.3 and 4.4 it is easy to see that $H : C(I) \times I \rightarrow C(I)$ is the homotopy as required.

It follows from Corollary 3.4 and Theorem 2.8 that $E_{\leq a}$ is nowhere dense and closed in the space $E_{< + \infty}$. Hence

$$E_{< + \infty} = \bigcup_{n \in \mathbb{N}} E_{\leq n}$$

is not topologically complete. Therefore, $E_{< + \infty}$ is not homeomorphic to $l_2$. It is natural to put the following problem:

Problem 4.5. Does there exist $a \in (0, + \infty)$ such that $E_{< a}$ is homeomorphic to $l_2$?

For every $a \in [0, + \infty)$, by Corollary 1.3, we know that $E_{\leq a}$ is contractible. By Theorem 2.8, $E_{\leq a}$ is a closed subset of $C(I)$ and hence it is topologically complete. But the following problem is still open.

Problem 4.6. Is $E_{\leq a}$ homeomorphic to $l_2$ for every $a \in [0, + \infty)$? In particular, is $E_{\leq 0}$ homeomorphic to $l_2$?
By Anderson-Kadec’s theorem, $l_2$ is homeomorphic to $\mathbb{R}^\mathbb{N}$, then it is also homeomorphic to $s = (-1, 1)^\mathbb{N}$. Let

\[ Q = [-1, 1]^{\mathbb{N}}, \]
\[ \Sigma = \{(x_n) \in Q : \sup |x_n| < 1\}, \]
\[ P_{<2^n} = \{ f \in C(I) : \text{there exists } n \in \mathbb{N} \text{ such that} \}
\]
\[ \text{the set of periods of } f \text{ is } \{2^i : 0 \leq i \leq n\}. \]

Using these symbols, we have the following problem:

**Problem 4.7.** For every $a \in (0, +\infty)$, does there exist a homeomorphism $h : E_{<a} \to s \times Q$ such that $h(E_{<a}) = s \times \Sigma$? Does there exist a homeomorphism $h : E_{\leq 0} \to s \times Q$ such that $h(P_{<2^n}) = s \times \Sigma$?

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