Derivation of the Hadronic Structure Functions From Quantum Chromodynamics

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Abstract

We solve a long-standing problem in particle physics: that of deriving the Deep Inelastic structure functions of the proton from the fundamental theory of strong interactions, Quantum ChromoDynamics (QCD). In the Bjorken limit, the momenta of the constituents of the proton (the partons) can be assumed to be in a two-dimensional plane in Minkowski space: a dimensional reduction of QCD to two space-time dimensions. Two dimensional QCD is then shown to be equivalent for all energies and values of number of colors $N$ to a new theory of hadrons, Quantum HadronDynamics (QHD). The phase space of QHD is the Grassmannian (set of subspaces) of the complex Hilbert space $L^2(R)$. The natural symplectic form along with a hamiltonian define a classical dynamical system, which is equivalent to the large $N$ limit of QCD. 't Hooft’s planar limit is the linear approximation to our theory: we recover his integral equation for the meson spectrum but also all the interactions of the mesons. The Grassmannian is a union of connected components labelled by an integer (the renormalized dimension of the subspace) which has the physical meaning of baryon number. The proton is the topological soliton: the minimum of the energy in the sector with baryon number one gives the structure functions of the proton. We solve the resulting integral equations numerically; the agreement with experimental data is quite good for values of the Bjorken variable $x_B > 0.2$.
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Chapter 1

Introduction

The first indication that the atom contains a point-like nucleus came from experiments of Rutherford. He scattered alpha particles (which are positively charged) from a gold foil. Since the negative charges (electrons) were known to be much lighter than the alpha particles, they would not scatter the alpha particles very much. If the positive charges inside the atom were more or less uniformly distributed, the alpha particles would not be scattered through wide angles. Rutherford found to the contrary that the alpha particles were scattered by wide angles. In fact the probability of scattering through an angle $\theta$ is proportional to $\sin^{-4} \frac{\theta}{2}$, exactly what would happen if all the positive charge of the atom were concentrated at a point: he had discovered the atomic nucleus. Soon after, it was realized that the nucleus is of finite size, although small compared to the atom. It is composed of protons and neutrons bound together by the strong interaction.

Many years later, another celebrated series of experiments studied the structure of the proton itself. It was found that it was not an elementary particle either, in fact that it was made of point-like constituents. In such a ‘Deep Inelastic Scattering’ experiment an electron (or neutrino) beam is scattered by a proton (or a nucleus containing both protons and neutrons). The idea is to learn about the still mysterious strong interactions using the electroweak interactions as a probe.

A particle that can take part in the strong interactions is called a ‘hadron’. There are two kinds of hadrons: those of half-integer angular momentum are called ‘baryons’ and those of integer angular momentum are the ‘mesons’. The proton is the lightest baryon and the $\pi$-meson (pion) the lightest meson. There are an infinite number of baryons and mesons but the more
energetic ones are unstable against decay into the lighter hadrons. (We consider an idealized world in which the electromagnetic, weak and gravitational interactions of the hadrons with each other are ignored. Thus a particle is considered stable if its decays are purely electromagnetic or weak. This is the sense in which the pion is stable.)

The hadrons are composed of more elementary constituents: the quarks, anti-quarks and gluons. These constituents are collectively called partons[1, 2]. However, it has been found experimentaly that it is impossible to create the partons in isolation: they only exist inside hadrons, and thus, are not true particles. This is the phenomenon of ‘confinement’. More precisely, only combinations of quarks and gluons invariant under the action of the color group $SU(N)$ (see below) exist as isolation. Thus hadrons can be defined to be states that are invariant under color. Some authors use the word hadron to include the quarks and gluons. We will use the term ‘hadron’ to refer to a bound state of quarks and gluons which can exist as an isolated particle. Of course such a particle may be unstable against decay into other hadrons.

Around the same time of these developments an entirely different picture of a baryon was proposed by Skyrme [3]: that it is a topological soliton made of an infinite number of mesons. Unfortunately, this idea did not fit with the dominant view of the time and was largely ignored. In the mid-eighties Skyrme’s idea was finally revived by a group at Syracuse University (including the author) [4, 5, 6] and integrated into the modern theory of strong interactions, i.e., Quantum Chromodynamics. Witten [7] clarified why the baryons are fermions (when $N$ is odd) even though the underlying theory has only bosonic fields. (Witten had already arrived at the idea that baryons are solitons in the large $N$ limit of 't Hooft [8] by independent arguments [9].)

But then the problem remains how to reconcile it with the picture of a baryon as a bound state of point-like constituents. We will show in these lectures how the parton model can be derived from the soliton model. It fact it had never been possible to derive the distribution functions of the partons inside the hadron from fundamental principles. Our picture will solve this problem, providing for the first time a quantitative theory of the structure of a proton.

The basic idea of our approach is to find a new description of strong interactions directly in terms of hadrons rather than in terms of quarks and gluons. It should be equivalent to the color singlet sector of QCD at all ener-
gies and all values of the number of colors. We will call this theory quantum hadron dynamics. In four dimensions this is still just an idea; we havent been able to show yet that such a new paradigm for strong interactions exists. However in the two dimensional case, I constructed such an equivalent alternative formalism some years ago. Part of the motivation for studying the two dimensional case was that it would provide a solution to the problem of deriving the structure functions of four dimensional QCD. (It has been known since the early work of Feynman and Bjorken that Deep Inelastic Scattering can be explained by a two dimensional theory of strong interactions.) The first report of the idea of such an equivalent theory was in Ref. [10]. The solutions of the integral equations for the baryon wavefunction was first studied in Ref. [11]. A more detailed description of the theory appeared in Ref. [12]. In another direction, these ideas were applied to spherically symmetric situations in Ref. [13]. The methods of geometrical quantization were applied to the problem in Ref. [14]. There a version with bosonic quarks was also studied. We returned to the study of the baryon wavefunctions in Ref. [15] where the derivation of the parton model, already mentioned in [12] was given in more detail. In papers which will soon appear [16] we will extend these results to include the anti-quark and sea quark distributions. Some of the ideas that go into this work are outlined in the last chapter. The basic mathematical ideas necessary to derive the gluon structure functions have also been developed [17]. We hope to solve that problem as well in the near future.

The collinear approximation to QCD has also been studied by other papers [18]. However the combination of these ideas with the picture of a baryon as a soliton as well as the idea of a quantum hadron dynamics seem unique to our approach.

The ideas we introduce range from infinite dimensional geometry to the phenomenology of particle physics. Indeed a whole new set of tools had to be developed in order to implement the new paradigm we propose. Some of the methods are currently not associated with particle physics: they are more closely related to classical mechanics and are perhaps more familiar to mathematicians. But the new ideas of one generation become the standard lore of the next. I hope these lecture notes will prepare a new generation of theoretical physicists to pursue these ideas. Prior knowledge of quantum field theory is not essential, but will provide perspective. In general, a higher degree of mathematical maturity than specialized knowledge of particle physics is assumed.
In the next few sections of this introduction we give a quick summary of the physics background necessary. The reader who finds this boring should skip to the next chapter, after a glance at sections 1.4 and 1.5 which are essential to the argument.

1.1 Deep Inelastic Scattering

Let us consider in more detail the scattering of an electron by a proton. The electron emits a virtual photon of momentum \( q = k_f - k_i \) where \( k_i \) and \( k_f \) are the initial and final electron momenta. This photon then interacts with the hadron producing some state \( |X> \) which could contain many particles: several mesons and some excited baryons and anti-baryons. This is an inelastic scattering process. If we sum over all the states \( |X> \) so produced, we will get the ‘inclusive’ cross-section for the scattering of the electron. Upto well–known electromagnetic effects, the cross-section for this scattering is given by the matrix element[19]

\[
W_{\mu \nu}(P, q) = \frac{1}{8\pi} \sum_\sigma \sum_X <P\sigma|J_\mu(0)|X><X|J_\nu(0)|P\sigma>(2\pi)^4\delta^4(P + q - P_X)
\]

Here, \( J_\mu(x) \) is the electromagnetic current operator which is the one appropriate for electron scattering: the weak interactions of the electron are much smaller in comparison. For neutrino scattering it would instead be the charged weak current (corresponding to \( W^\pm \)) if the neutrino is converted into an electron; and the neutral current (corresponding to \( Z^0 \)) otherwise. Also, \( P \) is the momentum of the hadron and we assume that we have averaged over the spin \( \sigma \) of the hadron. (The target is usually not polarized, so that we must average over the values of the spin variables. More detailed measurements with polarized targets have been made more recently; we will not consider these for now.)

The tensor \( W_{\mu \nu} \) is symmetric and transverse: conservation of the electric current gives

\[
q^\mu W_{\mu \nu}(P, q) = 0
\]

Lorentz invariance and parity (which is a symmetry of the electromagnetic
interactions) imply that the tensor \( W_{\mu\nu}(p, q) \) has the form

\[
W_{\mu\nu}(P, q) = \left[ \eta_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right] W_1(x_B, Q^2) + \frac{1}{m^2} \left( P_\mu - q_\mu \right) \left( P_\nu - q_\nu \frac{P \cdot q}{q^2} \right) W_2(x_B, Q^2). \tag{1.2}
\]

Being Lorentz scalars, the ‘structure functions’ \( W_{1,2} \) can depend only on the Lorentz scalars \( q^2 \) and \( P \cdot q \) (\( P^2 \) is fixed to have the value \( m^2 \), where \( m \) is the mass of the proton). When an electron is scattered against a target, the momentum \( q \) of the photon it emits is space-like. It is conventional to use the positive number \( Q^2 = -q^2 \) to describe the energy of the photon. The higher the value of \( Q^2 \) the better the resolution of our measurement of the structure of the hadron. As the other independent variable we can take the dimensionless ratio (‘Bjorken variable’)

\[
x_B = \frac{Q^2}{2P \cdot q}. \tag{1.3}
\]

If the hadron we are studying is stable against strong decays (which is always the case for experimental reasons) it will be the lightest particle with its quantum numbers. Then, the mass of the intermediate state will be greater than or equal to the mass of the target:

\[
(P + q)^2 \geq m^2; \quad \Rightarrow \left[ \frac{1}{x_B} - 1 \right] \geq 0. \tag{1.4}
\]

This shows that the Bjorken variable takes values in the range

\[1 \geq x_B \geq 0. \tag{1.5}\]

It is often more convenient to use another equivalent pair of structure functions

\[
F_1(x_B, Q^2) = W_1(x, Q^2), \quad F_2(x_B, Q^2) = \frac{P \cdot q}{m^2} W_2(x_B, Q^2). \tag{1.6}
\]

In its rest frame the hadron has a certain ‘size’ \( a \) : it is of the order of the the charge radius of the proton, \( a^{-1} \sim 100\text{MeV} \). If we take the limit \( Q^2 \gg a^{-2} \) keeping \( x_B \) fixed we are looking deep inside the hadron: this is the region of Deep Inelastic Scattering. The basic idea is much like that of a
microscope: to see inside an object of size $a$ we need light of a wavelength that is small compared to $a$; or equivalently an energy for the photon that is large compared to $a^{-1}$.

When the states are normalized by the usual convention

$$< P | P' > = 2E_P(2\pi)^3\delta^3(P - P'),$$

the functions $F_{1,2}(x_B, Q^2)$ are dimensionless. It is found experimentally that these functions are approximately independent of $Q^2$: they depend essentially only on the dimensionless ratio $x_B$. The simplest explanation of this phenomenon is that the proton is made of pointlike massless constituents: the structure function would then depend only on the dimensionless ratio $x_B$.

If we multiply $P \cdot q$ and $Q^2$ by the same number, the structure functions are approximately invariant: this is the approximate symmetry of 'scale invariance'. In the first versions of this model, the partons were conceived of as free particles. Of course they must interact to bind into a hadron, but the idea was that at high enough $Q^2$ the interaction would for some reason be small. This was the germ of the idea of 'asymptotic freedom': that the constituents of the hadrons behaved like free particles at short distances, or high energies.

This 'parton' model was subsequently derived from a much deeper fundamental theory of strong interactions, Quantum Chromodynamics (QCD). This is a non–abelian gauge theory which has the unusual property that at high $Q^2$ the coupling constant (which measures the strength of the interaction) vanishes like $1/\log\frac{Q^2}{\Lambda^2}$. Here $\Lambda$ is a parameter with the dimensions of momentum which is a fundamental parameter of QCD. Hence at high energies (compared to $\Lambda$) QCD tends to a free theory, yielding asymptotic freedom and the parton model. In the limit of large $Q^2$, the structure functions are predicted to be independent of $Q^2$. At finite values of $Q^2$, the structure functions have a slow dependence on $Q^2$; as a polynomial in the coupling constant, or equivalently, $1/\log\frac{Q^2}{\Lambda^2}$. Moreover, we can calculate this dependence on $1/\log\frac{Q^2}{\Lambda^2}$ using perturbation theory: scale invariance is broken in a way that is calculable. Perturbation theory allows us to calculate the $Q^2$ dependence of the structure functions: given the value of $F_{1,2}(x_B, Q^2)$ at one (large) value of $Q^2$, we can calculate it for any other large value of $Q^2$. (We mean that $Q$ should be large compared
to $\Lambda$, which is of the order of 100 $MeV$. If $\frac{Q}{\Lambda}$ is not large, perturbative theory can no longer be used.)

It is one of the triumphs of modern particle physics that perturbative QCD accurately describes these scale violations.

However, the $x_B$ dependence of the structure functions are not as well-understood. Perturbation theory is not sufficient to calculate it: $F_2(x_B,Q^2)$ is in a certain sense the probability distribution (in momentum space) of a parton inside a hadron. Thus it describes how partons bind together to form a hadron. Perturbation theory around a free field theory can never describe such bound states. Due to the lack of any fundamental understanding of the $x_B$ dependence of the structure functions, physicists have been forced to extract them directly from experiment. A whole generation of experimentalists and phenomenologists have worked to produce a quite reliable extraction of the $x_B$ dependence of structure functions from data. It will be our goal to explain this from the fundamental theory of strong interactions, QCD.

In the next section we will summarize the definition of QCD as a quantum field theory.

### 1.2 Quantum Chromodynamics

Quantum Chromodynamics is the fundamental theory of strong interactions. It is a non-abelian gauge theory with gauge group $SU(N)$ with matter fields (quarks) which are spin half fermions transforming under ($N_f$ copies of) the fundamental representation of $SU(N)$. The natural number $N$ is called the ‘number of colors’. It has the value 3 in nature; but it will be convenient to leave it unspecified until the very end when we make comparisons with data. The ‘number of flavors’ $N_f$ is 2 for most purposes although it can be as high as 6 in principle: the heavier flavors of quarks can be ignored for most purposes. Again, it is best to leave $N_f$ as an arbitrary parameter for now.

The action principle that defines QCD is

$$S = \frac{N}{4\alpha} \int tr F_{\mu\nu} F^{\mu\nu} d^4x + \sum_{a=1}^{N_f} \int \bar{q}^a [-i\gamma \cdot \nabla + m_a] q^a d^4x$$

(1.8)

The Yang–Mills field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

(1.9)
where \( A_\mu \) are a set of four \( N \times N \) anti-hermitean matrices: \( A_\mu dx^\mu \) is a one-form taking values in the Lie algebra of the unitary group \( U(N) \). In perturbation theory, this ‘gauge field’ describes ‘gluons’: massless spin one particles analogous to the photon. \( q_a \) is a fermionic field (describing quarks) which is a Dirac spinor transforming in the fundamental representation of the ‘color’ group \( U(N) \), for each \( a \). The index takes values \( a = 1 \cdots N_f \), where \( N_f \) is the number of copies of such quarks: the number of flavors.

The parameters of the theory are (in addition to \( N \) and \( N_f \)), the quark masses \( m_a \) for \( a = 1, \cdots N_f \) and the dimensionless ‘coupling constant’ \( \alpha \). In fact the quantum theory is defined by an additional set of rules for ‘renormalization’ which replace the constant \( \alpha \) by a new parameter \( \Lambda \) which has the dimensions of mass as well. These issues are well-known and have been reviewed in several articles [23], so we will avoid them here. The basic point is that the the coupling constant acquires a dependence on the energy scale \( \alpha \sim \frac{1}{\log \frac{Q^2}{\Lambda^2}} \).

These \( N_f + 1 \) parameters should in principle determine the masses and decay rates as well as the structure functions of all the hadrons. It has turned out to be quite difficult to predict hadronic properties from this fundamental theory. We need to develop approximation methods which make the problem tractable. For the most part this problem is still not solved. In these talks we will describe how we solved a part of this puzzle: that of calculating the hadronic structure functions from QCD.

### 1.3 Structure Functions from QCD

We noted earlier that since the electromagnetic interactions are well-understood, it is sufficient to concentrate on the ‘unknown’ part of the problem, which is encoded in the structure functions \( W_{1,2}(x_B, Q^2) \). These were defined in terms of the expectation values of the product of current operators. Now, part of QCD is tractable by perturbation theory; this allows us to reduce the ‘unknown’ part of the problem further to the expectation values of simpler operators: the structure functions can be expressed in terms of ‘parton distribution functions’. The problem we will solve is that of calculating these distribution functions.

The basic result that makes this reduction possible is called the ‘factorization theorem’: the structure function can be written as a convolution of
two factors: one whose \( x_B \) and \( Q^2 \) dependence can be calculated in perturbation theory (the ‘hard’ factor); and another (the ‘soft’ factor) which has an unknown \( x_B \) dependence but which is independent of \( Q^2 \). This is a result of quantum field theory that can be proved to all orders of perturbation theory; so it is comparable in depth to the proof of renormalizability of gauge theories. We will just give an intuitive argument, pointing the reader to the literature on perturbative QCD for details [19, 24].

The photon is scattered by a quark of charge \( e_a \) inside the hadron. To leading order, the probability of the photon being scattered by the proton can be written as the product of two pieces: the probability that a quark of some momentum \( k \) will scatter the photon and the probability that there is such a quark inside the proton. (Then we of course sum over the intermediate momentum \( k \).) This is the simplest version of the factorization theorem. In the next order of perturbation theory, we will have to include the possibility that quark might emit a gluon before absorbing the photon and then re-emit it. We can imagine the scattering of the photon by the quark as a subprocess with its own structure function, except that it can be calculated within perturbation theory. The probability that a hadron contains a quark of a given momentum is of course not calculable in perturbation theory: that is the distribution function which we will attempt to understand.

In other words, the structure function of the proton can be written as the convolution of two pieces: the structure function of a parton (which is computable perturbatively) and the distribution function of the parton inside the hadron (which is non-perturbative).

In more detail,

\[
F_2(x_B, Q^2) = \sum_{i=a,\bar{a},G} \int_0^1 \frac{dx}{x} C^i(x, Q^2) \phi_i(x).
\] (1.10)

Here \( \phi_a(x_B) \) is the probability of finding a quark of flavor and momentum fraction \( x_B \) inside a proton; \( \phi_{\bar{a}} \) is the probability for an anti-quark of flavor \( a \), and \( \phi_G \) for gluons. Note that all the \( Q^2 \) dependence is in the first factor; it can be calculated as a power series in \( \frac{1}{\log \frac{Q^2}{\Lambda^2}} \) using the standard rules of perturbation theory. To leading order in perturbation theory, even the first factor is independent of \( Q^2 \):

\[
C^a_2(x_B, Q^2) = C^{\bar{a}}_2(x_B, Q^2) = e_a^2 \delta(x_B - 1), \quad C^G_2(x_B, Q^2) = 0.
\] (1.11)
If \( b_i(k), b_i^\dagger(k) \) are the creation-annihilation operators for the parton, we have

\[
\phi_i(x_B) = \sum_\sigma \int \frac{d^2k_T}{(2\pi)^2} \langle P_\sigma | b_i^\dagger(x_B P, k_T) b_i(x_B P, k_T) | P_\sigma \rangle. \tag{1.12}
\]

Here, we are averaging over all possible values of the part of the momentum orthogonal to the plane spanned by \( P \) and \( q \), (called the ‘transverse momentum’ \( k_T \)).

In terms of the field operators, we get the quark distribution functions

\[
\phi_a(x_B) = \frac{1}{2} \sum_\sigma \int dy e^{-ix_B P y} \langle P_\sigma | \bar{q}^a(y, 0_{T}) \gamma_{-} q_a(0) | P_\sigma \rangle \tag{1.13}
\]

and the anti-quark distribution functions

\[
\phi_{\bar{a}}(x_B) = -\frac{1}{2} \sum_\sigma \int dy e^{-ix_B P y} \langle P_\sigma | q^a(y, 0_{T}) [\gamma_{-}]_{\beta\alpha} \bar{q}^{\beta}(0) | P_\sigma \rangle. \tag{1.14}
\]

The position arguments of the field operators are separated by a null line in the plane spanned by \( P \) and \( q \). The averaging over transverse momenta implies that in position space, these field operators have the same transverse co-ordinates.

Since the gluons do not carry electric charge, to the leading order of perturbative QCD, the gluon distribution functions are not necessary in order to understand the structure of the proton. We will ignore them for now.

These formulae are not gauge-invariant and are to be understood in the null gauge \( A_- = 0 \). In a general gauge we should insert a parallel transport operator \( Pe^\int A-(y,0_{T}) dy \) along the null line connecting two field operators.

### 1.4 Reduction of QCD to Two Dimensions

We are probing the hadron (whose momentum is a time-like vector \( P \)) by a photon (or \( W, Z \)-boson) with a space–like momentum \( q \). These two vectors define a two-dimensional subspace of Minkowski space. There is also a space–like vector \( a \) which characterizes the ‘size’ of the hadron: the length of \( a \) is the charge radius of the hadron. In the Deep Inelastic limit, \( Q^2 = -q^2 >> |a|^{-1} \) keeping \( x_B = \frac{Q^2}{2Pq} \) fixed. This means that the hadron is of very large size in the directions transverse to the two
dimensional subspace spanned by $P$ and $q$. By the uncertainty principle, the transverse momentum of its constituents is of order $|a|^{-1}$, which is thus small compared to $Q$. A reasonable first approximation would be to let the hadron have infinite extent in the transverse directions; and to require the momenta of the constituents to lie entirely in the plane spanned by $P, q$. In other words the approximation is that the fields are independent of the transverse spatial co-ordinates.

This is analogous to the procedure of dimensional reduction popular in unified field theories of gravity. The main difference is that the dimensions that are ignored do not form a subspace of small volume: instead they are infinite in extent. The point is that the momenta in these directions are small which is the same as requiring that the fields are independent of those spatial directions: a dimensional reduction.

Thus Deep Inelastic Scattering is described by the dimensional reduction of QCD to two dimensions. That is what makes this phenomenon accessible: two dimensional gauge theories have been understood by the large $N$ method. With some further (less drastic) approximations we will be able to determine the spectrum and structure of hadrons in two dimensions. These should then be verifiable experimentally.

The dimensional reduction of QCD to two dimensions is given by the action principle

$$S = \frac{N}{4\alpha_1} \int \text{tr} F_{\mu\nu} F^{\mu\nu} d^2x + \sum_{a=1}^{N_f} \int \bar{q}_\alpha [ -i \gamma \cdot \nabla + m_a ] q_{\alpha} d^4x$$
$$+ \frac{N}{2\alpha_1} \int \text{tr} (\nabla_\mu \phi) A^2 d^2x + \frac{N}{2\alpha_1} \int \text{tr} [\phi_3, \phi_4]^2 d^2x + \int \bar{q}_\alpha ( -i ) \Gamma^A_{\alpha} \phi_A q_{\alpha} d^2x.$$

Here we allow the indices $\mu, \nu$ to take only the values $0, 1$. The gauge field now splits into a 1-form in the two dimensional space with components $A_\mu$ and a pair of scalar fields $\phi_3 = A_3, \phi_4 = A_4$ corresponding to the transverse polarization states of the gluon.

The four dimensional Dirac spinor $q$ splits into a pair of two-dimensional spinors $q^\alpha$, corresponding the two eigenvalues of $\gamma_3 \gamma_4$. Moreover $\alpha_1$ is the coupling constant of the two-dimensional theory which has dimensions of $(\text{mass})^2$. It is a combination of $\alpha$ and the size of the hadron: $\alpha_1 \sim \alpha a^{-2}$. Thus the reduction of QCD to two dimensions gives a gauge theory with scalar fields in the adjoint representation, and twice as many flavors of quarks as the original theory.
We will study for the most part two dimensional QCD, defined by the action

$$S = \frac{N}{4\alpha_1} \int \text{tr} F_{\mu\nu} F^{\mu\nu} d^2x + \sum_{a=1}^{N_f} \int \bar{q}^{a\alpha} [-i\gamma \cdot \nabla + m_a] q_{a\alpha} d^4x.$$  

This is not quite the same as the dimensional reduction of four dimensional QCD to two dimensions: there are no scalar fields. This truncated theory will be sufficient to determine the quark and anti-quark structure functions to the accuracy we need. The essential techniques required to solve the theory including the scalar fields have been developed as well. We will return to them in a later publication.

### 1.5 Two Dimensional QCD in Null Gauge

The essential simplification of two dimensional QCD is that the gauge fields can be removed completely from the problem, leaving just the quark fields $q$ and $\bar{q}$ and the scalar gluons $\phi_A$. This should not be too surprising: in $D$ dimensional space-time, a gauge field has $D-2$ polarization states. Of course, we still have the scalar fields $\phi$ which are the remnants in two dimensions of the two polarization states of the gluon in four dimensional gauge theory. But we will ignore them, as they are mostly relevant to the determination of the gluon structure functions: a problem we will postpone to a later publication.

We will change notation slightly from the last section: the indices $a, b$ etc. will denote the pairs $a\alpha, b\beta$, so they will range over $1, 2, \cdots 2N_f$. This simply reflects the fact that the two dimensional theory has twice as many flavors as the four dimensional theory: the transverse polarization label looks just just like a flavor index to the two-dimensional theory.

In the null gauge, the action of the theory becomes (see Appendix A)

$$L = \chi^{ta}(-i\partial_t)\chi_a - \chi^{ta}\left\{\frac{1}{2}[\hat{p} + \frac{m^2}{\hat{p}}] - iA_t\right\}\chi_a + \frac{N}{2\alpha_1} \text{tr}[\partial_x A_t]^2. \quad (1.15)$$

The field $A_t$ carries no dynamical degrees of freedom: it can be eliminated in terms of $\chi$ by solving its equations of motion:

$$A_{ij}(x) = \frac{i}{N} \alpha_1 \int \frac{1}{2} |x - y| : \chi^{ia}(y)\chi_{aj}(y) : dy. \quad (1.16)$$
The Coulomb potential in one-dimensional space is given by $\frac{\partial^2}{\partial x^2} \frac{1}{2|x-y|} = \delta(x-y)$. The Hamiltonian of the resulting theory is

$$H = \int dx \chi^\dagger \frac{1}{2} [\hat{p} + \frac{m^2}{\hat{p}}] \chi$$

$$-\frac{1}{2N} \alpha_1 \int \frac{1}{2} |x-y| : \chi^\dagger \chi_{aj}(x) \cdots \chi \chi_{bi}(y) : dxdy$$

Now define the operator

$$\hat{M}_\alpha^\dagger(x, y) = -\frac{2}{N} : \chi^\dagger \chi_{ai}(x) \chi_{bi}(y) :$$ (1.17)

which is gauge invariant. It describes the creation of a quark at $x$ and an anti-quark at $y$, but in a color invariant combination: in other words it describes a meson. The Hamiltonian above can be expressed entirely in terms of this operator after some reordering of factors of $\chi$.

We can rearrange the quartic operator in the $\chi$’s as a quadratic operator in $M$. First,

$$: \chi^\dagger \chi_{ai}(x) \chi_{aj}(x) \cdots \chi^\dagger \chi_{bj}(y) \chi_{bi}(y) : = : \chi^\dagger \chi_{ai}(x) \chi_{aj}(x) \chi \chi_{bj}(y) \chi_{bi}(y) :$$

$$+ : \chi^\dagger \chi_{ai}(x) \chi_{bj}(y) : < 0 | \chi_{aj}(x) \chi \chi_{bj}(y) | 0 > + : \chi^\dagger \chi_{ai}(x) : N = \frac{1}{2} \left[ \delta(x-y) + \epsilon(x-y) \right]$$

Now, (see Appendix A)

$$< 0 | \chi^\dagger \chi_{ai}(x) \chi_{bi}(y) | 0 > = N \delta^a_b \int_{-\infty}^{0} \frac{dp}{2\pi} e^{-ipx + ipy} = N \delta^a_b \frac{1}{2} \left[ \delta(x-y) + \epsilon(x-y) \right]$$

where,

$$\epsilon(x-y) = \mathcal{P} \int \frac{\text{sgn}(p)}{2\pi} e^{ip(x-y)} \frac{dp}{2\pi}.$$
On the other hand,

$$
\hat{M}_a^b(x, y) \hat{M}_a^b(y, x) = \left(\frac{2}{N}\right)^2 : \chi^{tai}(x) \chi_{bi}(y) :: \chi^{t bj}(y) \chi_{aj}(x) :
$$

$$
= \left(\frac{2}{N}\right)^2 : \chi^{t ai}(x) \chi_{bi}(y) \chi^{t bj}(y) \chi_{aj}(x) :
$$

The terms corresponding to other orderings of $\chi$ will involve $\epsilon(y, y)$ which should be interpreted as zero. So,

$$
: \chi^{tai}(x) \chi_{aj}(x) :: \chi^{t bj}(y) \chi_{bi}(y) : = -\left(\frac{N}{2}\right)^2 \hat{M}_a^a(x, y) \hat{M}_b^b(y, x)
$$

$$
+ \left(-\frac{N}{2}\right) \hat{M}_a^a(x, y) \frac{N}{2} [\delta(x - y) + \epsilon(x - y)]
$$

$$
+ \frac{N}{2} [\delta(x - y) + \epsilon(x - y)] \left(-\frac{N}{2}\right) \hat{M}_a^a(y, x)
$$

Thus,

$$
H = \left(-\frac{N}{2}\right) \int \frac{1}{2} \left[p + \frac{\tilde{\mu}^2}{p}\right] M_a^a(p, p) \frac{dp}{2\pi} +
$$

$$
- \frac{1}{2N} \alpha_1 \int \frac{1}{2} |x - y| \left\{ -\left(\frac{N}{2}\right)^2 \hat{M}_a^a(x, y) \hat{M}_b^b(y, x)
$$

$$
+ \left(-\frac{N^2}{2}\right) \hat{M}_a^a(x, y) \epsilon(x - y) \right\} dxdy
$$

We have dropped terms involving $\delta(x - y)|x - y|$ since this product is just zero; moreover the two terms involving $\epsilon(x - y)$ are equal and have been combined.

Now we use the identity

$$
\int \frac{1}{2} |x - y| \epsilon(x - y) f(x - y) dxdy = -\frac{1}{\pi} \mathcal{P} \int \frac{1}{p} \tilde{f}(p, p) \frac{dp}{2\pi}.
$$

Thus

$$
\frac{H}{N} = -\frac{1}{2} \int \frac{1}{2} \left[p + \frac{\tilde{\mu}^2}{p}\right] M_a^a(p, p) \frac{dp}{2\pi} +
$$

$$
\frac{1}{8} \alpha_1 \int \frac{1}{2} |x - y| \hat{M}_a^a(x, y) \hat{M}_b^b(y, x) dxdy
$$

where

$$
\tilde{\mu}^2 = m^2 - \frac{\alpha_1}{\pi}.
$$
The commutation relations can also be expressed entirely in terms of the variables $M$:

$$\{ \tilde{M}^a_b(p, q), \tilde{M}^c_d(r, s) \} = \frac{1}{N} \left( \delta^c_b 2\pi \delta(q - r) \left[ \delta^a_d \sgn(p - s) + \tilde{M}^a_d(p, s) \right] - \delta^a_d 2\pi \delta(s - p) \left[ \delta^c_b \sgn(r - q) + \tilde{M}^c_b(r, q) \right] \right).$$

We notice that in the limit of large $N$, these commutators become small: it is a sort of classical limit.

### 1.6 Constraint on the Variable $\hat{M}$

We have all the essential ingredients of a reformulation of two dimensional QCD in terms of the color singlet variable $M$, eliminating quarks and gluons from the picture. The commutation relations and the hamiltonian together imply the time evolution equations for $M$. However there is one more ingredient which we can miss at first: the set of allowed values of $M$. Being bilinear in the fermionic variables, $M$ behaves much like bosonic variables: they satisfy commutation relations (rather than anti-commutation relations). Also, they have a classical limit; in our case this is the large $N$ limit when there commutators become small. But their origin as fermion bilinears have a residual effect: they satisfy a quadratic constraint. This constraint is ultimately an expression of the Pauli exclusion principle; even in the large $N$ limit the fact the underlying degrees of freedom are fermions cannot be ignored.

Regard the classical variable $\rho(x, y)$ obtained by taking the large $N$ limit of $\frac{1}{N} \chi^i(x)\chi_i(y)$ as a matrix in the variables $x$ and $y$. It is in fact the density matrix of quarks. Its eigenvalues have to be between zero and one: zero when the state is completely empty and one when it is filled. For the color singlet states (hadrons), each state is either completely filled with quarks or completely empty. Hence the eigenvalue is either 0 or one, and this density matrix is a projection operator: $\rho^2 = \rho$ or,

$$\int \rho(x, y)\rho(y, z)dz = \rho(x, z). \quad (1.20)$$

This is a quadratic constraint on the classical variable.

Another (oversimplified) way to understand this constraint is that $\rho(x, y)$ corresponds to a meson state where a quark is created at $x$ and an anti-
quark at $y$. If we now create two mesons, with the position of the anti-quark of the first coinciding with the quark of the second, they will annihilate each other leaving us with one meson!.

We will usually use the normal ordered variable $M(x,y) = \frac{1}{\sqrt{N}} :\chi_i^\dagger(x)\chi_j(y) :$, which has the advantage that $M = 0$ on the vacuum. In terms of it the constraint becomes, in matrix notation

$$[\epsilon, M]^+ + M^2 = 0. \quad (1.21)$$

$M$ has to satisfy some technical conditions as well, but we will talk about them later.

It is possible to prove these constraints by a straightforward calculation on the operators, even when $N$ is held finite. (See the appendix of Ref. [13].) It is important to note that they only hold in the color singlet subspace of the fermionic operators.

This means that it is possible to understand two dimensional QCD without ever mentioning quarks, antiquarks or gluons. The operator $M$ is gauge invariant and can be the field variable of a theory that directly describes hadrons. From this point of view quarks and gluons are just mathematical artifacts. This gives us a whole new paradigm for the theory of strong interactions. Instead of a ’Quantum ChromoDynamics” in terms of unobservable quarks and gluons but a new “Quantum HadronDynamics” in terms of the directly observable particles, the hadrons.

In the next chapter we will develop just such a point of view. We will first present a classical theory which is equivalent to the large $N$ limit of the above theory. Upon quantization we will recover two-dimensional QCD.
Chapter 2

Two-Dimensional Quantum Hadron Dynamics

In this chapter we will present first the classical hadron theory and then its quantization. We will see that the quantum hadron theory is equivalent to QCD in two dimensions.

2.1 Grassmannians

Let \( H = L^2(R) \otimes C^{N_{2f}} \) be the complex Hilbert space of complex-valued functions on the real line. Define the subspace of functions whose Fourier transforms vanish for negative momenta:

\[
H_+ = \{ \psi | \psi(x) = \int_0^\infty \tilde{\psi}(p) e^{ipx} \frac{dp}{2\pi} \}
\]

and its orthogonal complement

\[
H_- = \{ \psi | \psi(x) = \int_{-\infty}^0 \tilde{\psi}(p) e^{ipx} \frac{dp}{2\pi} \}.
\]

The functions in \( H_+ \) are boundary values of analytic functions on the upper half of the complex plane in the variable \( x \). (Of course \( H_- \) is related to the lower half of the complex plane in the same way.) Define the operator

\[1\]The number \( N_{2f} \) is the number of flavors of the two dimensional theory; it is twice the number of flavors of the four-dimensional theory.
\( \epsilon \) (‘the sign of the momentum’) to be \(-1\) on \( \mathcal{H}_- \) and \(+1\) on \( \mathcal{H}_+ \). Clearly
\[
\epsilon^\dagger = \epsilon, \quad \epsilon^2 = 1.
\] (2.3)

The Grassmannian of \( \mathcal{H} \) is the set of all its subspaces. To each subspace \( W \), there is an operator \( \Phi \) which is \(-1\) on \( W \) and \(+1\) on \( W^\perp \). Again,
\[
\Phi^\dagger = \Phi, \quad \Phi^2 = 1.
\] (2.4)

Conversely any such operator corresponds to an orthogonal splitting of \( \mathcal{H} \) : it will have eigenvalues \( \pm 1 \) and \( W \) can be identified as the subspace with negative eigenvalue. Thus the set of such \( \Phi \)’s is the same as the set of subspaces of \( \mathcal{H} \). The set of subspaces of \( \mathcal{H} \) is the Grassmannian, which we can thus also regard as the set of operators \( \Phi \) satisfying the above condition.

But we don’t want to allow all such subspaces: since \( \mathcal{H} \) is infinite dimensional, that would give a Grassmannian that is too big. For example, its tangent space will not be a Hilbert space, and it wouldn’t admit a Riemann metric. We should only allow subspaces \( W \) that are ‘at a finite distance’ from \( \mathcal{H}_- \). It is convenient to introduce the variable
\[
M = \Phi - \epsilon
\] (2.5)
which measures the deviation from the standard point \( \epsilon \), which corresponds to the subspace \( \mathcal{H}_- \). We will require that \( M \) be Hilbert-Schmidt. That is, the sum of the absolute magnitudes squared of all its matrix elements is finite. (See Appendix B for a rapid summary of some functional analysis we will need.) This sum is a measure of the distance of \( W \) from \( \mathcal{H}_- \).

Thus we define the restricted Grassmannian to be,
\[
\text{Gr}(\mathcal{H}, \epsilon) = \{ M^\dagger = M, \quad [\epsilon, M]_+ + M^2 = 0, \quad \text{tr}M^2 < \infty \}.
\] (2.6)

The set of all such \( M \)’s forms an infinite dimensional Hilbert manifold. This manifold is defined by a quadratic equation in the (real) Hilbert space of self-adjoint Hilbert-Schmidt operators.

If we split the operator into \( 2 \times 2 \) blocks (as in \( \epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \)),
\[
M = \begin{pmatrix} a & b \\ b^\dagger & d \end{pmatrix}
\] (2.7)
we get (using the constraint on \( M \)),

\[
a = \frac{1}{2}(bb^\dagger + a^2), \quad d = -\frac{1}{2}(b^\dagger b + d^2), \quad ab + bd = 0. \tag{2.8}
\]

Since \( M \) as a whole is Hilbert-Schmidt, \( b : \mathcal{H}_+ \to \mathcal{H}_- \) is Hilbert-Schmidt as well. The above constraints then imply that \( a \) and \( d \) are trace-class. This means roughly speaking that the sum of their eigenvalues is absolutely convergent (see the appendix B for precise definition).

Note that \( a \) is a positive operator and \( d \) a negative operator. In particular, \( \text{tr} a \geq 0 \) and \( \text{tr} d \leq 0 \).

The tangent space at the origin is given by the special case where \( a, b, c, d \) are all infinitesimally small. The constraints then show that \( a \) and \( d \) are second order infinitesimals while \( b \) are first order. In the first order the only constraint on \( b \) is that it has to be a Hilbert-Schmidt operator. Thus, the tangent space at the origin is a complex Hilbert space, consisting of operators of the form \( M = \begin{pmatrix} 0 & b \\ b^\dagger & 0 \end{pmatrix} \). In other words we have the identification,

\[
T_0 \text{ Gr}(\mathcal{H}, \epsilon) = \mathcal{I}^2(\mathcal{H}_+ \to \mathcal{H}_-). \tag{2.9}
\]

### 2.2 The Dirac Theory of Fermions

We have introduced the Grassmannian from a purely geometric point of view above. But it has a natural physical interpretation in terms of the Dirac theory of fermions.

In Dirac’s theory, the complex Hilbert space \( \mathcal{H} \) represents the set of states of a single fermion. There is a self-adjoint operator \( h \) representing energy. In the null co-ordinates we are using, energy is

\[
p_0 = \frac{1}{2}|p + \frac{m^2}{p}|. \tag{2.10}
\]

Thus it has the same sign as the null component of momentum \( p \). The main physical obstacle to this interpretation is that the energy is not positive: all states are thus unstable with respect to decay into the negative energy states of lower and lower energy.

Dirac’s main idea was that in any physical state all except a finite number of the negative energy states are occupied. Since fermions obey the Pauli
exclusion principle, only one particle can occupy a given state: decay into states of very low energy is forbidden because those states are occupied. In particular the vacuum is not the state containing no particles. Instead the vacuum is the state where all the negative energy states are occupied and the positive energy states are empty. The filled negative energy states in the vacuum is called the ‘Dirac Sea’. All physical quantities such as energy or charge are to be measured in terms of their departure from the vacuum value: even though the vacuum contains an infinite number of particles, it is still assigned zero energy, charge etc.

An excitation from the vacuum could be a state where a finite number of positive energy states are occupied and a finite number of negative energy states are empty. Such a state will have positive energy compared to the vacuum. The empty negative energy states (also called ‘holes’) have positive energy. The holes themselves behave just like the particles except that they have the opposite value for some observables such as electric charge. They are the ‘anti-particles’. The number of particles minus the anti-particles is a conserved quantity which can take any integer value. An arbitrary physical state is a linear combination of such states containing a finite number of particles and anti-particles.

A mathematical interpretation of this situation can be given in terms of a modified exterior product of \( \mathcal{H} \). In familiar non-relativistic theories, the space of states of a multi-particle system of fermions would be the exterior power of order \( r \), where \( r \) is the number of fermions: \( \sum_{r=0}^{\infty} \Lambda^r \mathcal{H} \). Instead in Dirac’s theory we split the one particle Hilbert space into two subspaces \( \mathcal{H}_- \oplus \mathcal{H}_+ \) of negative and positive energy. The space of states of the multi-particle system is

\[
\mathcal{F} = \sum_{r,s=0}^{\infty} \Lambda^r \mathcal{H}_-^s \otimes \Lambda^r \mathcal{H}_+. \tag{2.11}
\]

This space of multi-particle states is called the Fock space.

There is a subset of states in this Fock space consisting of wedge products of single particle states. (A general state is a linear combination of such wedge products.) Suppose, \( W \) is a subspace which doesn’t differ ‘too much’ from \( \mathcal{H}_- \). That is, the intersection of \( W \) with \( \mathcal{H}_+ \) and the intersection of \( W^\perp \) with \( \mathcal{H}_- \) are both finite dimensional. Then we can form a state in \( \mathcal{F} \) in which all states of \( W \) are occupied and all those in \( W^\perp \) are empty. Suppose \( e_i \) is a basis in \( W \cap \mathcal{H}_+ \) and and \( f_j \) a basis in \( W^\perp \cap \mathcal{H}_- \).
Then the state in $F$ corresponding to occupying $W$ will contain particles in $W \cap H_+$ and holes in $W^\perp \cap H_-$:

$$f^1 \wedge f^2 \wedge \cdots \wedge e_1 \wedge e_2 \wedge e_3 \wedge \cdots. \tag{2.12}$$

(Here, $f^j$ is the dual basis in $[W^\perp \cup H_-]'$). Moreover this is independent (upto multiplication by a complex number) of the choice of basis. Thus to each such subspace $W \subset H$ there is a state (upto scalar multiple) in the Fock space $F$. In other words we have an embedding of the Grassmannian into the Projective space $PF$. This is the infinite dimensional version of the Plücker embedding familiar from algebraic geometry. [23, 26].

The Grassmannian describes this subset of states in Dirac’s theory. The condition that the operator $M = \Psi - \epsilon$ corresponding to a subspace $W$ be Hilbert-Schmidt is precisely what is required for the above construction to go through. (It is not necessary to require that the projection operator $\pi_+: W \to H_+$ be finite dimensional as we did above: the precise condition for the construction to work is that it be Hilbert-Schmidt.) What is special about this family of states is that they are coherent states: they minimize the uncertainty in the physical observables. Hence they are the states that have a sensible classical limit. The set of such coherent states is the classical phase space of the theory. That is why it is sensible to choose the Grassmannian as the classical phase space of our theory of hadrons.

Dirac’s theory was originally meant to describe electrons. We should think of it now as applied to quarks. Since the quarks are also fermions, this is reasonable. But there is one important twist: each quark comes in $N$ colors: thus each state in $H$ can be occupied by $N$ quarks, not just one. Moreover, only states that are invariant under the action of the group $SU(N)$ are allowed: these are the states that describe the hadrons. The classical limit referred to above is the large $N$ limit.

The operator $\Phi$ has eigenvalue $-1$ on states that are completely filled and $+1$ on states that are empty. The variable $M$ measures the deviation from the vacuum state, in which all the negative energy states are completely filled. A baryon is essentially a color invariant bound state of $N$ quarks. Thus the Dirac Sea of quarks can also be thought as a sea of baryons: each state is filled either by $N$ quarks or by a baryon. An infinitesimal disturbance from this can be thought of either as the creation of a meson (a quark-anti-quark pair) or as the promotion of a quark from the negative energy sea to a positive energy state. This is an infinitesimal change if the
number of colors $N$ is large. The operator $M$ thus has to be determined by a map $b^\dagger : \mathcal{H}_- \to \mathcal{H}_+$. The block-diagonal elements of $M$ are zero in this infinitesimal limit.

In addition to such small deviations from the vacuum, the theory also allows for topological solitons which are not connected to the vacuum by any continuous path. These are very important, as they describe the baryons. To see how they arise we need to understand the topological properties of the Grassmannian. This is the subject of the next section.

### 2.3 Renormalized Dimension of a Subspace

Recall that $\Phi = \epsilon + M$ has eigenvalues $\pm 1$. Hence in some formal sense the trace of $\Phi$ is an even integer: the difference between the number of positive eigenvalues and the number of negative eigenvalues. But of course this trace is not convergent. Even if we subtract the contribution of the vacuum, it is divergent: the $\text{tr}M$ can diverge, only the trace of its square needs to be convergent in general, since we only required it to be Hilbert-Schmidt. However the trace of $M$ is conditionally convergent: we can define it to be the trace with respect to the above splitting into submatrices:

$$\text{tr}_\epsilon M = \text{tra} + \text{trd}. \tag{2.13}$$

This trace exists since $a$ and $d$ are trace-class matrices. It can be shown to be an even integer, which is invariant under the continuous deformations of the operator $M$.

The interpretation of $\Phi$ in terms of subspaces will help us understand the meaning of this integer. Imagine that we take a state from $\mathcal{H}_+$ and add it to $\mathcal{H}_-$ to get a subspace $W$: then $W$ has dimension one more than $\mathcal{H}_-$. This should change the trace of $\Phi$ by $-2$: one of the eigenvalues of $\Phi$ has changed from $+1$ to $-1$. Since $\text{tr}M$ is essentially the difference between the traces of $\Phi$ and $\epsilon$, we have $\text{tr}_\epsilon M = -2$. Thus

$$B = -\frac{1}{2} \text{tr}M \tag{2.14}$$

is the ‘renormalized dimension’ of the subspace it describes: the difference between its dimension and the dimension of the standard subspace $\mathcal{H}_-$. In fact this is the only topological invariant of $M$: any two operators with the same conditional trace can be connected to each other by
a continuous path. Thus the restricted Grassmannian is a union of connected components labelled by an integer, which is called the ‘virtual rank’ or ‘renormalized dimension’ of that component. Since it is invariant under all continuous deformations, in particular it will be invariant under time evolution. This is true for any reasonable definition of time evolution, independent of the choice of hamiltonian. Thus the renormalized dimension is a ‘topologically conserved quantity’.

Each connected component of the Grassmannian by itself is an infinite dimensional manifold. Although the different components are the same (diffeomorphic) as manifolds, the component of rank zero has a special role as it contains the ‘vacuum’ \( M = 0 \). An example with renormalized dimension one is a ‘factorizable’ operator,

\[
M = -2\psi \otimes \psi^\dagger.
\] (2.15)

The constraints on \( M \) are satisfied if \( \psi \) is a positive energy state of length one:

\[
\epsilon \psi = \psi, \quad ||\psi||^2 = 1.
\] (2.16)

In the interpretation in terms of the Dirac theory, we have filled a positive energy state \( \psi \) in addition to all the negative energy states. Hence this state contains an additional fermion occupying the state \( \psi \).

More generally, let \( \psi_a \) be an orthonormal system of states satisfying

\[
\epsilon \psi = \epsilon_a \psi, \quad <\psi_a, \psi_b> = \delta_a^b.
\] (2.17)

Then

\[
M = -2 \sum_a \epsilon_a \psi_a \otimes \psi_a^\dagger
\] (2.18)

is a solution of the constraints with renormalized dimension \( \sum \epsilon_a \). In this configuration, we have filled a certain number of positive energy states (\( \psi_a \) with positive \( \mu_a \)) while creating holes in some others, \( \psi_a \), with negative \( \mu_a \). The renormalized dimension is just the difference between the number of occupied positive energy states and the number of holes in the negative energy states.

Thus we see that the renormalized dimension is just the fermion number if we interpret the Grassmannian in terms of the Dirac Sea. We will see
that the separable configuration is related to the valence quark model of the baryon.

As noted earlier, $a \geq 0$ and $d \leq 0$; thus $a$ contributes negatively to the baryon number and $d$ positively. The only configurations that have $b = 0$ are the separable ones above. To see this note that, $b = 0$ implies that $a$ and $d$ are proportional to projection operators. In order to be trace class, these have to be finite rank projections: corresponding to a certain number of baryons and anti-baryons.

The valence parton approximation of the parton model corresponds to the separable ansatz in our soliton model. In that approximation, $\phi_{a}(p)$ is zero: there are no antiquarks in the proton within the valence approximation.

### 2.4 Some Submanifolds of the Grassmannian

In principle a baryon can be in any of the configurations of renormalized dimension one: the ground state (proton) will be the one of least energy. Once we have determined the energy function (see below), the proton structure functions are determined by minimizing this energy over all possible configurations. Later on, we will describe a method to do just that numerically: the steepest descent method. However this is a computationally intensive and slow method. A much faster method will be to minimize the energy over some submanifold of the Grassmannian, which is chosen so that the minimum in this submanifold is close to the true minimum, yet is easier to find.

Physical intuition plays an important role in the choice of the ansatz of configurations explored this way. Such restricted phase spaces play an important role in our conceptual understanding as well: we will see that the theory restricted to rank one configurations of the type

$$M = -2\psi \otimes \psi^\dagger, \quad \epsilon \psi = \psi, \quad ||\psi||^2 = 1$$

(2.19)

is just the valence parton model: we will be able to derive the parton model from QHD this way.

Note that a change of phase $\psi \rightarrow e^{i\theta} \psi$ does not affect $M$. Indeed the set of rank one configurations is a submanifold of the Grassmannian diffeomorphic to $\mathcal{P}(\mathcal{H}_+)$ . If we restrict the dynamics of our theory to this submanifold, we get an approximate theory which we will show is equivalent to the valence parton model.
Allowing for a slightly larger set of configurations will give us the parton model with Sea quarks and anti-quarks. We will describe here the submanifold of the Grassmannian that describes this approximation to our theory. We generalize the above rank one ansatz to a finite rank ansatz

$$M = \mu_{\alpha\beta} \psi_\alpha \psi_\beta^\dagger.$$  

(2.20)

Here, the $\psi_\alpha$ are a finite number $r$ of eigenstates of $\epsilon$ which are orthonormal:

$$\epsilon \psi_\alpha = \epsilon_\alpha \psi_\alpha, \quad \psi_\alpha \psi_\beta^\dagger = \delta_\alpha^\beta.$$  

(2.21)

The constraints

$$M = M^\dagger, \quad (\epsilon + M)^2 = 1, \quad -\frac{1}{2} \text{tr} M = 1, \quad \text{tr} M^2 < \infty.$$  

(2.22)

become the constraints on the $r \times r$ matrix $\mu$:

$$\mu = \mu^\dagger, \quad [\tilde{\epsilon} + \mu]^2 = 1, \quad -\frac{1}{2} \text{tr} \mu = 1.$$  

(2.23)

Here, $\tilde{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & \cdots \\ 0 & \epsilon_2 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & \cdots & \epsilon_r \end{pmatrix}$ is a diagonal $r \times r$ matrix, the restriction of $\epsilon$ to the finite dimensional subspace spanned by the $\psi_\alpha$. The condition $\text{tr} M^2 < \infty$ is of course automatic since $M$ is now an operator of finite rank.

Thus, given a set of vectors $\psi_\alpha$ satisfying the above conditions, we get a point in the Grassmannian of renormalized dimension one if $\mu$ itself belongs to a finite dimensional Grassmannian. If there are $r_+$ vectors $\psi_\alpha$ with positive momentum and $r_-$ with negative momentum, we have

$$\text{tr}[\tilde{\epsilon} + \mu] = r_+ - r_- - 2.$$  

(2.24)

The simplest solution is, $r_+ = 1, r_- = 0$, so that $\mu$ is just a number: it then has to be $-2$. This is the rank one solution described earlier, which leads to the valence parton model.

If $r = r_+ + r_-$ is two, there is no solution to the above requirements (more precisely all solutions reduce to the rank one solution). The next
simplest possibility is of rank three, with \( r_+ = 2, r_- = 1 \). Then each solution to the constraints on \( \mu \) determines a one dimensional subspace of \( C^3 \); i.e., a point in \( CP^2 \). In other words, for each such \( \mu \), there is a unit vector \( \zeta = \begin{pmatrix} \zeta_- \\ \zeta_0 \\ \zeta_+ \end{pmatrix} \in C^3 \) such that
\[
\bar{\epsilon} + \mu = -1 + 2 \zeta \otimes \zeta^\dagger, \quad ||\zeta||^2 = 1.
\] (2.25)

Now, there is a \( U(1) \times U(2) \) “gauge symmetry” in the problem: we can rotate the two positive energy vectors \( \psi_0 \) and \( \psi_+ \) into each other and change the phase of the negative energy vector \( \psi_- \) without changing \( M \), provided we make the corresponding changes in \( \mu \) as well. This \( U(1) \times U(2) \) action changes the phase of \( \zeta_- \) and rotates the other two components among each other. This freedom can be used to choose
\[
\zeta = \begin{pmatrix} \zeta_- \\ 0 \\ \sqrt{1 - \zeta_-^2} \end{pmatrix}, \quad \zeta_- > 0.
\] (2.26)

In summary the submanifold of rank three configurations of renormalized dimension one is given by
\[
M = -2 \psi_0 \otimes \psi_0^\dagger + 2 \zeta_- \{ \zeta_- [\psi_- \otimes \psi_-^\dagger - \psi_+ \otimes \psi_+^\dagger] \\
+ \sqrt{1 - \zeta_-^2} [\psi_- \otimes \psi_-^\dagger + \psi_+ \otimes \psi_+^\dagger] \}
\]
where \( \psi_-, \psi_0, \psi_+ \) are three vectors in \( \mathcal{H} \) satisfying
\[
\epsilon \psi_- = -\psi_-, \quad \epsilon \psi_0 = \psi_0, \quad \epsilon \psi_+ = \psi_+,
\] (2.27)
\[
||\psi_-||^2 = ||\psi_0||^2 = ||\psi_+||^2 = 1, \quad \psi_0^\dagger \psi_+ = 0.
\] (2.28)

Also,
\[
0 \leq \zeta_- \leq 1.
\] (2.29)

The special case \( \zeta_- = 0 \) reduces to the rank one ansatz.

The physical meaning is clear if we consider the negative momentum components of \( M \). The anti-quark distribution function is
\[
\phi_a(p) = M_{aa}(-p, -p) = \zeta_-^2 |\psi_{-a}(p)|^2.
\] (2.30)

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Thus $\zeta^2$ is the probability of finding an anti-quark inside the baryon. Also, $\zeta^2|\psi_{-a}(-p)|^2$ is the distribution function of the anti-quark of flavor $a$. In the same way $|\tilde{\psi}_{0a}(p)|^2 + \zeta^2|\tilde{\psi}_{+a}(p)|^2$ is the distribution function for quarks. In a loose sense $|\tilde{\psi}_{0a}(p)|^2$ is the valence quark wavefunction and $|\tilde{\psi}_{+a}(p)|^2$ is the Sea quark wavefunction. But the splitting of the quark distribution into a valence and a Sea distribution is rather arbitrary and has no real physical meaning.

By using the ansatz above with a larger and larger rank we can get better and better approximations to the Grassmannians. But the expressions get quite complicated beyond the rank three ansatz. We will find that this rank three ansatz holds the key to understanding the anti-quark and sea quark distributions in the quark model. A variational ansatz based on these configurations will provide a derivation of the parton model with sea quarks and anti-quarks from quantum hadrodynamics.

\subsection{2.5 Integral Kernels}

We can express the operator $M$ in terms of its integral kernel in position space:\footnote{We will often suppress the indices $a, b$ etc. labelling the basis in $C^{N_2f}$}

\begin{equation}
M \psi(x) = \int M(x, y) \psi(y) dy.
\end{equation}

Alternately we can think in terms of its action on the momentum space wavefunctions:

\begin{equation}
\tilde{\psi}(p) = \int \psi(x)e^{-ipx} dx, \quad \psi(x) = \int \tilde{\psi}(p)e^{ipx} \frac{dp}{2\pi}
\end{equation}

and

\begin{equation}
[\tilde{M}\psi](p) = \int \tilde{M}(p, q) \tilde{\psi}(q) \frac{dq}{2\pi}.
\end{equation}

The two points of view are of course related by Fourier transformation:

\begin{equation}
M(x, y) = \int \tilde{M}(p, q)e^{ipx - iqu} \frac{dpdq}{(2\pi)^2}.
\end{equation}
The operator $\epsilon$ is diagonal in momentum space:

$$\tilde{\epsilon}\psi(p) = \text{sgn}(p)\tilde{\psi}(p).$$  \hspace{1cm} (2.35)

It can also be described in position space as a distribution:

$$\epsilon\psi(x) = \int \epsilon(x-y)\psi(y)dy = \frac{i}{\pi} \mathcal{P} \int \frac{1}{x-y}\psi(y)dy$$  \hspace{1cm} (2.36)

In fact this is a well-known object in complex function theory: the Hilbert transform (except for a factor of $i$). It relates the real and imaginary parts of analytic functions.

The constraints on $M$ become then,

$$\tilde{M}(p, q) = M^*(q, p)$$

$$[\text{sgn}(p) + \text{sgn}(q)]\tilde{M}(p, q) + \int \tilde{M}(p, r)\tilde{M}(r, q)\frac{dr}{2\pi} = 0$$

$$\int |M(p, q)|^2 \frac{dpdq}{2\pi} < \infty$$

or equivalently, position space,

$$M(x, y) = M^*(y, x)$$

$$\int [\epsilon(x, y)M(y, z) + M(x, y)\epsilon(y, z) + M(x, y)M(y, z)]dy = 0$$

$$\int |M(x, y)|^2 dx dy < \infty.$$

It is useful to see what the tangent space at the origin is like. Since $M$ is infinitesimally small, the constraint can be linearized:

$$[\text{sgn}(p) + \text{sgn}(q)]\tilde{M}(p, q) = 0.$$  \hspace{1cm} (2.37)

Thus $\tilde{M}(p, q)$ is only non-zero if $p$ and $q$ are of opposite signs. In fact, the case where $p > 0, q < 0$ determines the opposite one because of the hermiticity condition on $M$. Thus the tangent space is just the space of square integrable functions of one positive variable and one negative variable $\tilde{M}(p, q)$.

### 2.6 The Infinite Dimensional Unitary group

Given any self-adjoint operator $\Phi$, $g\Phi g^\dagger$ is also self-adjoint. If the transformation $g$ is unitary, $gg^\dagger = g^\dagger g = 1$, it preserves the condition
Indeed the action of the unitary group on such operators is transitive: any operator satisfying $\Phi^* = \Phi$ and $\Phi^2 = 1$ can be taken to any other by a unitary transformation. In terms of the variable $M$, the unitary transformation is

$$M \mapsto gMg^* + g[\epsilon, g^*]. \quad (2.38)$$

In the infinite dimensional case, we have required that $M$ be Hilbert-Schmidt (H-S). This means that there is a corresponding condition on the family of allowed unitary transformations: we define the restricted unitary group,

$$U(H, \epsilon) = \{g | gg^* = 1, [\epsilon, g] \in \mathcal{I}^2\}. \quad (2.39)$$

It is straightforward to verify that the H-S condition on the commutator is preserved under the multiplication and inverse operations, since $\mathcal{I}^2$ is an ideal in the algebra of bounded operators.

Indeed the restricted Grassmannian is a homogenous space of this restricted unitary group:

$$Gr(H, \epsilon) = U(H, \epsilon)/U(H_+) \times U(H_-). \quad (2.40)$$

The Lie algebra of the restricted Unitary group is

$$\mathfrak{u}(H, \epsilon) = \{u | u = -u^*, [\epsilon, u] \in \mathcal{I}^2\}. \quad (2.41)$$

The infinitesimal action on the variable $M$ is:

$$M \mapsto [u, \epsilon + M]. \quad (2.42)$$

### 2.7 Poisson Structure

We will regard the Grassmannian as the phase space of our dynamical system. The matrix elements of the operator $M$ are then a complete set of observables. We will seek a set of Poisson brackets among these variables. Fortunately there is a natural choice: there is a unique choice that is invariant under the action of the restricted unitary group. In fact the Grassmannian can be viewed as the co-adjoint orbit of the (central extension of the ) unitary group. We can regard the Poisson structure as induced by the Kirillov symplectic form on this orbit. However, the physical arguments use the Poisson
brackets rather than the symplectic form so in this paper we will not say much about the symplectic form. (See Refs. [12, 14] for an elaboration of this more geometric point of view.)

If the Poisson brackets are invariant under the action of the group, the infinitesimal action,

$$M \rightarrow [u, \epsilon + M]$$

would be a canonical transformation for any $$u$$ satisfying

$$u = -u^\dagger, [\epsilon, u] \in \mathcal{I}^2.$$  

(2.44)

The natural choice of a function that generates this canonical transformation is of the form

$$f_u = k \text{tr} uM$$

(2.45)

for some constant $$k$$. There is a technical problem: the trace may not exist. Now remember that under the splitting $$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$$,

$$u = \begin{pmatrix} \alpha & \beta \\ -\beta^\dagger & \delta \end{pmatrix} \in \left( B \begin{pmatrix} \mathcal{I}^2 & \mathcal{I}^2 \\ \mathcal{I}^2 & \mathcal{I}^1 \end{pmatrix} \right), \quad M = \begin{pmatrix} a & b \\ b^\dagger & d \end{pmatrix} \in \left( \mathcal{I}^1 \begin{pmatrix} \mathcal{I}^2 & \mathcal{I}^2 \\ \mathcal{I}^2 & \mathcal{I}^1 \end{pmatrix} \right)$$

(2.46)

so that the conditional trace

$$\text{tr}_u uM = \text{tr} a a + \text{tr} \beta b - \text{tr} \beta^\dagger b^\dagger + \text{tr} \delta d$$

(2.47)

exists. Moreover its value is some imaginary number, so that the constant $$k$$ should also be purely imaginary in order that

$$f_u = k \text{tr}_u uM$$

(2.48)

be a real-valued function.

Thus we postulate the Poisson brackets

$$\{f_u, M\} = [u, \epsilon + M].$$

(2.49)

These imply, of course that

$$\{f_u, f_v\} = k \text{tr}_u [u, \epsilon + M] = \pm f_{[u, v]} + k \text{tr}_u [u, \epsilon].$$

(2.50)

The commutation relations of the restricted unitary group are satisfied only up to a constant term: it is the central extension of the unitary group that
acts on the Grassmannian, not the group itself. This has been studied at great length in the book by Pressley and Segal [27] so we won't go too far in that direction.

We can write the Poisson brackets also in terms of the integral kernel of the operator $M$ in position space:

$$ k\{M^a_b(x, y), M^c_d(z, u)\} = \delta^c_b \delta(y - z)[\delta^a_d(x, u) + M^a_d(x, u)] - \delta^a_d \delta(x - u)[\delta^c_b(z, y) + M^c_b(z, y)], $$

or momentum space:

$$ k\{\tilde{M}^a_b(p, q), \tilde{M}^c_d(r, s)\} = \delta^c_b 2\pi \delta(q - r)[\delta^a_d \text{sgn } (p - s) + \tilde{M}^a_d(p, s)] - \delta^a_d 2\pi \delta(s - p)[\delta^c_b \text{sgn } (r - q) + \tilde{M}^c_b(r, q)]. $$

The principle of invariance under the unitary group cannot determine the constant $k$ in the Poisson brackets. In a sense it doesn't matter what value we choose for it, as long as the values of all the observables are also multiplied by $k$. However it will be convenient to choose the value $k = \frac{i}{2}$ as we will see in the next subsection.

Soon we will have to calculate the Poisson bracket of functions $f(M)$ which is not linear in $M$. The derivative of such a function can be thought of as an operator valued function $f'(M)$ of $M$:

$$ df = \text{tr} f'(M) dM = \int f''_a(M) d\tilde{M}^a_b(p, q) \frac{dpdq}{(2\pi)^2} \quad (2.51) $$

Then we can use the above Poisson brackets to get

$$ k\{f, M\} = [f'(M), \epsilon + M] \quad (2.52) $$

where the l.h.s. involves the commutator, as operators, of $f'(M)$ and $\epsilon + M$. We leave the proof as an exercise to the reader.

### 2.8 Momentum

The above Poisson brackets are invariant under translations. There must be a canonical transformation that implements this symmetry. Formally, this is given by choosing $u$ to be the derivative operator with respect to position; or, $i$ times the multiplication by momentum. (Strictly speaking this is not
in the Lie algebra of the unitary group since it is not bounded, but let us ignore this technicality for the moment.) Thus, momentum is

\[ P = ik \int p \tilde{M}(p, p) \frac{dp}{2\pi}. \tag{2.53} \]

Now, imagine calculating this quantity for the separable configuration of renormalized dimension one:

\[ M = -2\psi \otimes \psi^\dagger, \quad ||\psi||^2 = 1, \quad \epsilon \psi = \psi. \tag{2.54} \]

With the choice \( k = \frac{i}{2} \) we would have,

\[ P = \int_0^\infty p |\tilde{\psi}(p)|^2 \frac{dp}{2\pi}. \tag{2.55} \]

This has a simple physical interpretation: \( P \) is just the expectation value of momentum is a state with wavefunction \( \tilde{\psi}(p) \).

Note that \( P \) is always a positive function: the quadratic constraint becomes, for \( p = q \),

\[ 2 \text{ sgn } (p) \tilde{M}(p, p) = -\int |\tilde{M}(p, r)|^2 \frac{dr}{2\pi} \tag{2.56} \]

so that momentum can be written as

\[ P = \frac{1}{4} \int |p| |\tilde{M}(p, r)|^2 \frac{dp \, dr}{2\pi \, 2\pi} \geq 0. \tag{2.57} \]

This makes physical sense if we regard this as the null component of the momentum vector.

\section{2.9 Kinetic Energy}

For a free particle of mass \( \mu \), the time-like component of momentum (energy), \( p_0 \) is related to the spatial component \( p_1 \) by the condition

\[ p_0^2 - p_1^2 = \mu^2. \tag{2.58} \]

If we introduce the variables

\[ p = p_0 + p_1, \quad p_- = p_0 - p_1 \tag{2.59} \]
this becomes

\[(2p_0 - p)p = \mu^2, \quad pp_- = \mu^2.\]  

(2.60)

Thus the kinetic energy is related to the null component of momentum through the dispersion relation:

\[p_0 = \frac{1}{2}(p + \frac{\mu^2}{p}), \quad p_- = \frac{\mu^2}{p}.\]  

(2.61)

Notice that in this point of view the sign of energy \(p_0\) and of momentum \(p\) are the same: this will prove to be convenient when studying the structure of the ground state of relativistic fermion theories. (See the Appendix A for more details.)

Thus we will postulate the kinetic energy of our dynamical system on the Grassmannian to be

\[K = \frac{-1}{2} \int \frac{1}{2} [p + \mu^2 \frac{p}{p}] \tilde{M}(p, p) \frac{dp}{2\pi}.\]  

(2.62)

By the same argument as for momentum we can see that

\[K = \frac{1}{8} \int [\frac{|p| + \mu^2 |p|}{|p|}] |\tilde{M}(p, r)|^2 \frac{dp}{2\pi} \frac{dr}{2\pi} \geq 0\]  

(2.63)

If we add to this an appropriate potential energy \(U\) we will get the hamiltonian of the system. This potential energy must transform like \(p_-\), in order that \((p, p_- + U)\) transform like the null components of momentum.

Under Lorentz tranformations,

\[p \to \lambda p, \quad p_- \to \lambda^{-1} p_- .\]  

(2.64)

The Poisson bracket be invariant under Lorentz transformations,

\[\tilde{M}(p, q) \to \tilde{M}_\lambda(p, q) = \lambda \tilde{M}(\lambda p, \lambda q).\]  

(2.65)

In position space, \(x \to \lambda^{-1} x\) so that

\[M(x, y) \to M_\lambda(x, y) = \lambda^{-1} M(\lambda^{-1} x, \lambda^{-1} y).\]  

(2.66)

Lorentz invariance will constrain the form of the potential energy as well.
2.10 Hamiltonian

The total Hamiltonian will be a sum of kinetic energy $K$ and a potential energy $U$. While the kinetic energy is best understood in momentum space, potential energy is best written in position space. The simplest choice of $U$ will be a quadratic function of $M(x, y)$. (Anything simpler will lead to linear equations of motion.) Thus we postulate

$$U[M] = \frac{1}{4} \int M^a_b(x, y) M^b_a(y, x) v(x, y) dxdy + \frac{1}{4} \int M^a_b(x, x) M^b_a(y, y) v_1(x, y) dxdy.$$  

(2.67)

This will have the requisite invariance under the internal symmetry $U(F)$. Lorentz invariance requires that $U$ transform like $p^−$ or, equivalently, like $\frac{1}{p}$ or $x$. Thus,

$$v(\lambda^{-1}x, \lambda^{-1}y) = \lambda^{-1}v(x, y).$$  

(2.68)

Moreover, they can only depend on the difference $x − y$ due to translation invariance. Thus

$$v_1(x, y) = \frac{1}{2} \alpha_1 |x − y|, \quad v_2(x, y) = \frac{1}{2} \alpha_2 |x − y|$$  

(2.69)

for some pair of constants $\alpha_1, \alpha_2$.

Thus the Hamiltonian of our theory is

$$E[M] = -\frac{1}{4} \left[ p + \frac{\mu^2}{p} \right] \tilde{M}(p, p) \frac{dp}{2\pi} + \frac{\alpha_1}{8} \int M^a_b(x, y) M^b_a(y, x) |x − y| dxdy + \frac{\alpha_2}{8} \int M^a_b(x, x) M^b_a(y, y) |x − y| dxdy.$$  

The constants $m, \alpha_1, \alpha_2$ have to fixed later based on experimental data. They are the coupling constants of our theory.

We have already shown that the kinetic energy is positive. The potential energy is manifestly positive when $\alpha_1, \alpha_2 \geq 0$ the way we have written it in position space. Thus the minimum value of $E$ is zero and it is attained at the point $M = 0$. (This is in fact why we choose to parametrize our system by the variable $M$ and not $\Phi$.) This of course lies in the connected component with renormalized dimension zero. In the other components, energy will have a minimum again, but it is not zero. In fact we will spend much time estimating this ground state energy.
2.11 Singular Integrals

It will be convenient for later purposes to express the hamiltonian in momentum space variables. This will require the use of some singular integrals.

The basic singular integral is the Cauchy Principal value \[28\]

\[
P \int_a^b \frac{f(p)}{p-q} \frac{dp}{2\pi} = \lim_{\epsilon \to 0^+} \left[ \int_a^{q-\epsilon} \frac{f(p)}{p-q} \frac{dp}{2\pi} + \int_{q+\epsilon}^b \frac{f(p)}{p-q} \frac{dp}{2\pi} \right]
\]  

(2.70)

This exists whenever \( f \) is Hölder continuous \(^3\) of order greater than zero at the point \( p = q \). (Of course we are assuming that \( a < q < b \)). The basic idea of the Principal value integral is to cut-off the integral by removing a small interval of width \( 2\epsilon \) located symmetrically about the singular point and then take the limit as \( \epsilon \to 0 \). (If the function vanishes at the point \( p = q \) the limit might exist even if not taken symmetrically: then we have an ‘improper integral’ rather than a ‘singular integral’.)

If we had cut-off the integral asymmetrically, the contribution of the region close to the singularity would have been

\[
f(q) \left[ \int_a^{q-\epsilon_1} + \int_{q+\epsilon_1}^b \right] \frac{1}{p-q} \frac{dp}{2\pi} \sim \frac{1}{2\pi} f(q) \log \frac{\epsilon_1}{\epsilon_1}.
\]  

(2.71)

This can take any value as we let \( \epsilon_1 \) and \( \epsilon_2 \) to go to zero. Requiring that \( \epsilon_1 = \epsilon_2 \) removes this ambiguity. Indeed there are many other rules that could have been chosen: the particular one we choose must be justified by physical considerations: rather like the choice of boundary conditions in the solution of differential equations. In problems of interest to us there is a symmetry \( p \to -p \) (ultimately due to charge conjugation invariance) which selects out the Principal Value (or the Finite Part we will define soon) as the correct prescription.

We already saw a use of the Cauchy principal value in the definition of the operator \( \epsilon \) :

\[
\epsilon \psi(x) = \frac{i}{\pi} P \int \frac{\psi(y)}{x-y} dy.
\]  

(2.72)

The symmetric choice of regularization implicit in the Principal value is required by the condition that \( \epsilon \) is hermitean.

---

\(^3\) A function \( f \) is Hölder continuous of order \( \nu \) (or \( f \in C^\nu \)) if \( \lim_{p \to q} \frac{|f(p) - f(q)|}{|p-q|^\nu} \) exists for all \( q \).
We will often have to deal with integrals that have a worse singularity:

\[ \int_a^b \frac{f(p)}{(p-q)^2} \frac{dp}{2\pi}. \tag{2.73} \]

The ‘Hadamard Finite Part’ (‘part finie’) of such an integral is defined in terms of the Cauchy Principal value:

\[ \mathcal{F} \mathcal{P} \int_a^b \frac{f(p)}{(p-q)^2} \frac{dp}{2\pi} = \mathcal{P} \int_a^b \frac{f(p) - f(q)}{(p-q)^2} \frac{dp}{2\pi}. \tag{2.74} \]

As long as \( f \in C^1 \), this will make sense: the vanishing of the numerator on the r.h.s. makes the integral exist as a Cauchy principal value.

Although we wont need this yet, we note for completeness the definition of the finite part integral for a finite range of integration:

\[ \mathcal{F} \mathcal{P} \int_a^b \frac{f(p) - f(q)}{(p-q)^2} \frac{dp}{2\pi} \left[ \frac{1}{a-q} - \frac{1}{b-q} \right]. \tag{2.75} \]

We need the notion of a finite part integral because

\[ \mathcal{F} \mathcal{P} \int \frac{1}{p^2} e^{ipx} \frac{dp}{2\pi} = -\frac{1}{2} |x-y|. \tag{2.76} \]

The quantity on the right hand side is a Green’s function of the Laplace operator on the real line:

\[ \frac{d^2}{dx^2} \frac{1}{|x-y|} = \delta(x-y). \tag{2.77} \]

The choice of boundary conditions in this Green’s function corresponds to the choice of the definition of the singular integral above.

As an aside we note that

\[ \mathcal{F} \mathcal{P} \int \frac{g(p)}{p^2} \frac{dp}{2\pi} = \lim_{\epsilon \to 0^+} \left[ \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] g(p) - g(0) \frac{dp}{p^2} \geq 0 \tag{2.78} \]

if

\[ g(p) \geq g(0) \tag{2.79} \]

for all \( p \). This is useful in checking that the hamiltonian is positive.
2.12 Hamiltonian in Momentum Space

Now,

\[
\frac{1}{2} \int M^a_b(x, y) M^b_a(y, x)|x - y|dxdy = \tag{2.80}
\]

\[
-\mathcal{FP} \int \frac{1}{r^2} \tilde{M}^a_b(p, q) \tilde{M}^b_a(p', q') \frac{dpdp'dqdq'dr}{(2\pi)^5}
\]

\[
\int e^{ir(x - y) + ipx - iqy + i p'x - i q'y} dxdy
\]

\[
= -\mathcal{FP} \int \frac{1}{r^2} \tilde{M}^a_b(p, q) \tilde{M}^b_a(q + r, p + r) \frac{dpdqdr}{(2\pi)^3}
\]

Similarly,

\[
\frac{1}{2} \int M^a_b(x, x) M^b_a(y, y)|x - y|dxdy \tag{2.81}
\]

\[
-\mathcal{FP} \int \frac{1}{r^2} \tilde{M}^a_b(p, q) \tilde{M}^b_a(p', q') \frac{dpdp'dqdq'dr}{(2\pi)^5}
\]

\[
\int e^{ir(x - y) + ipx - iqy + i p'y - i q'y} dxdy
\]

\[
= -\mathcal{FP} \int \frac{1}{r^2} \tilde{M}^a_b(p, p + r) \tilde{M}^b_a(p' - r, p') \frac{dpdp'dr}{(2\pi)^3}
\]

Thus the hamiltonian becomes

\[
E[M] = -\frac{1}{4} \int [p + \mu^2/p] \tilde{M}(p, p) \frac{dp}{2\pi}
\]

\[
-\frac{\alpha_1}{4} \mathcal{FP} \int \frac{1}{r^2} \tilde{M}^a_b(p, q) \tilde{M}^b_a(q + r, p + r) \frac{dpdqdr}{(2\pi)^3}
\]

\[
-\frac{\alpha_2}{4} \mathcal{FP} \int \frac{1}{r^2} \tilde{M}^a_b(p + r, p) \tilde{M}^b_a(p' - r, p') \frac{dpdp'dr}{(2\pi)^3}
\]

In spite of the minus signs on the r.h.s., this is in fact a positive function of $\tilde{M}$. We already saw that the kinetic energy is positive whenever the constraint is satisfied.

We can rewrite the second term in a more symmetric form,

\[
E[M] = -\frac{1}{4} \int [p + \mu^2/p] \tilde{M}(p, p) \frac{dp}{2\pi}
\]

\[
-\frac{\alpha_1}{4} \mathcal{FP} \int \frac{1}{r^2} \tilde{M}^a_b(p - r/2, q - r/2) \tilde{M}^b_a(q + r/2, p + r/2) \frac{dpdqdr}{(2\pi)^3}
\]

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\[-\frac{\alpha_2}{4}\mathcal{FP}\int \frac{1}{r^2} \tilde{M}_a^a(p, p + r) \tilde{M}_b^b(p', p' - r) \frac{dpdp'dr}{(2\pi)^3}\]

Now,

\[f(r) = \int \tilde{M}_b^a(p - \frac{r}{2}, q - \frac{r}{2}) \tilde{M}_a^b(q + \frac{r}{2}, p + \frac{r}{2}) \frac{dpdq}{(2\pi)^2}\]

satisfies

\[f(r) = f^*(-r) \quad (2.82)\]

so that the second term in the hamiltonian can be written as

\[-\frac{\alpha_1}{4}\mathcal{FP}\int \frac{1}{r^2} \text{Re} f(r) \frac{dr}{2\pi}. \quad (2.83)\]

Moreover, by a simple use of the Schwarz inequality, we see that

\[\text{Re} f(r) \leq f(0). \quad (2.84)\]

Thus this term in the hamiltonian is positive. In the same way the last term can also be proved to be positive.

### 2.13 The Equations of Motion

From our Poisson brackets and the above hamiltonian we can derive the equations of motion. We can regard the derivative \( E'(M) = \frac{dE}{dM} \) of the energy with respect to \( M \) as an operator valued function of \( M \):

\[dE = \text{tr}E'(M)dM = \int \tilde{E}_a^b(M; q, p) d\tilde{M}_a^a(p, q) \frac{dpdq}{(2\pi)^2}. \quad (2.85)\]

Explicitly,

\[
dE = -\frac{1}{4} \int \left[p + \frac{\mu^2}{p}\right] d\tilde{M}_a^a(p, p) \frac{dp}{2\pi} - \frac{\alpha_1}{2}\mathcal{FP}\int \frac{1}{r^2} \tilde{M}_a^a(q + r, p + r) \frac{dr}{(2\pi)} d\tilde{M}_b^b(p, q) \frac{dpdq}{(2\pi)^2} - \frac{\alpha_2}{2}\mathcal{FP}\int \frac{1}{(q - p)^2} \tilde{M}_b^b(p', p' + p - q) \frac{dp'}{2\pi} d\tilde{M}_a^a(p, q) \frac{dpdq}{(2\pi)^2}.
\]
Thus
\[
\bar{E}_a^b(M; q, p) = -\frac{1}{4}[p + \frac{\mu^2}{p}]2\pi\delta(p - q)\delta^b_a \\
-\frac{\alpha_1}{2}\text{FP}\int \frac{1}{r^2} \tilde{M}_a^b(q + r, p + r) \frac{dr}{(2\pi)} \\
-\frac{\alpha_2}{2}\delta^b_a \text{FP} \int \frac{1}{(q - p)^2} \tilde{M}_c^c(p', p' + p - q) \frac{dp'}{2\pi}.
\]

Now the equations of motion implied by our Poisson brackets are, in operator notation,
\[
k \frac{dM}{dt} = k\{E(M), M\} = [E'(M), \epsilon + M]. \tag{2.86}
\]
(Here, \(k\) is the constant we fixed earlier to be \(\frac{i}{2}\).) Substituting the above formula for \(E'(M)\) will give a rather complicated system of integral equations as our equations of motion.

In particular, static solutions are given by
\[
[E'(M), \epsilon + M] = 0 \tag{2.87}
\]
The obvious solution to this equation is the vacuum,
\[
M = 0. \tag{2.88}
\]
This is just the point where the hamiltonian is a minimum. Actually each connected component will have a minimum for the hamiltonian; the absolute minimum is the one with renormalized dimension zero. We will return to the study of the minimum in the components of non-zero renormalized dimension.

### 2.14 Linear Approximation

If \(M\) is infinitesimally close to the vacuum value \(M = 0\) we can linearize the equations of motion:
\[
\frac{i}{2} \frac{d\tilde{M}_a^b(p, q)}{dt} = -\frac{1}{2}[\tilde{K}(p) - \tilde{K}(q)]\tilde{M}_a^b(p, q) \\
-\frac{\alpha_1}{2}\text{FP} \int \frac{1}{s^2} [\tilde{M}_a^b(p + s, r + s)2\pi\delta(r - q) \text{sgn}(q) \\
\]

\[ -2\pi \delta(p - r) \sgn(p) M^a_b(r + s, q + s) \frac{dr ds}{(2\pi)^2} \]
\[ -\frac{\alpha_2 \delta^a_b}{2} \mathcal{F} \mathcal{P} \int \left[ \frac{1}{(p - r)^2} 2\pi \delta(r - q) \sgn(q) \tilde{M}_c^c(p', p' + r - p) \right] \frac{dp' dr}{2\pi (2\pi)} \]

Simplifying,
\[
\frac{i}{2} d\tilde{M}_b^a(p, q) \frac{dt}{dt} =
\frac{-1}{2} [\tilde{K}(p) - \tilde{K}(q)] M^a_b(p, q) \\
-\frac{\alpha_1}{2} \mathcal{F} \mathcal{P} \int \frac{1}{s^2} [\tilde{M}_b^a(p + s, q + s) \sgn(q) - \sgn(p) M^a_b(p + s, q + s)] \frac{ds}{2\pi} \\
-\frac{\alpha_2}{2} \delta^a_b \mathcal{F} \mathcal{P} \int \left[ \frac{1}{(p - q)^2} \sgn(q) \tilde{M}_c^c(p', p' + q - p) \right] \frac{dp'}{2\pi} \\
-\frac{1}{(p - q)^2} \sgn(p) \tilde{M}_c^c(p', p' + q - p) \right] \frac{dp'}{2\pi} \\
\]

Here,
\[
\tilde{K}(p) = \frac{1}{2} [p + \mu^2]. \quad (2.89)
\]

Recall that the constraint on \( \tilde{M} \) becomes, in this linear approximation,
\[
[\sgn(p) + \sgn(q)] \tilde{M}(p, q) = 0; \quad (2.90)
\]
i.e., that \( p \) and \( q \) have opposite signs. We can assume that \( p > 0, q < 0 \) since the opposite case is determined by the hermiticity condition
\[
\tilde{M}(p, q) = \tilde{M}^*(q, p). \quad (2.91)
\]

Thus,
\[
\frac{i}{2} d\tilde{M}_b^a(p, q) \frac{dt}{dt} =
\frac{-1}{2} [\tilde{K}(p) - \tilde{K}(q)] M^a_b(p, q) \\
+\alpha_1 \mathcal{F} \mathcal{P} \int \frac{1}{s^2} \tilde{M}_b^a(p + s, q + s) \frac{ds}{2\pi}
\]

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Now, under translation, \( M(x, y) \to M(x + a, y + a) \) and \( \tilde{M}(p, q) \to e^{i(p-q)a} M(p, q) \). Thus the total momentum, which is a conserved quantity, is \( P = p - q \). We can use as independent variables

\[ x_B = p/P, \quad P = p - q. \]  

(2.92)

Clearly \( 0 \leq x_B \leq 1 \) and \( P \geq 0 \). For a stationary solution,

\[ \tilde{M}(p, q; t) = e^{i\omega t} \chi(x_B). \]  

(2.93)

We get thus

\[
\frac{1}{2} \omega \chi(x_B) = \frac{1}{4} \left[ 2P + \frac{\mu^2}{P x_B} + \frac{\mu^2}{P(1 - x_B)} \right] \chi(x_B) - \frac{\alpha_1}{2\pi P} \mathcal{F}\mathcal{P} \int_0^1 \frac{\chi(y)}{(x - y)^2} dy 
+ \frac{\alpha_2}{2\pi P} \int_0^1 \chi(y) dy.
\]

(2.94)

Recalling that the mass \( \mu' \) of the meson is given in light cone components by,

\[ 2\omega P - P^2 = \mu'^2 \]  

(2.95)

we get

\[
\mu'^2 \chi(x_B) = \left[ \frac{\mu^2}{x_B} + \frac{\mu^2}{(1 - x_B)} \right] \chi(x_B) - \frac{2\alpha_1}{\pi} \mathcal{F}\mathcal{P} \int_0^1 \frac{\chi(y)}{(x_B - y)^2} dy - \frac{\alpha_2}{\pi} \int_0^1 \chi(y) dy.
\]

If we set \( \alpha_2 = 0 \) this is exactly the equation that ’t Hooft obtained for the meson spectrum of two dimensional QCD.

’t Hooft obtained this result by summing over an infinite class of Feynman diagrams. He showed that in the usual perturbative series of two dimensional QCD, the diagrams of planar topology dominate in the large \( N \) limit. He then found a set of integral equations that describe the sum over these planar diagrams.
't Hooft’s approach is limited by its origins in perturbations theory. It
does not in fact reproduce the full large $N$ limit of two dimensional QCD;
only the linear approximation to the large $N$ limit is obtained this way.
The complete large $N$ limit that we have constructed, by very different
methods, is a nonlinear theory, reproduces the earlier theory of 't Hooft as
just the linear approximation.

Our theory of course contains much more information: it describes all
the interactions among the mesons, since the hamiltonian is a non-linear
function of the co-ordinates on the Grassmannian. Indeed it is not even a
polynomial which means that there are interaction vertices at every order in
perturbation theory. Some of these have been determined, but our analysis
gives all the infinite number of vertices in terms of a single constant: the
geometry of the Grassmannian fixes all these uniquely in terms of $\alpha_1$.
The first such vertex was obtained by a summation of planar diagrams as well; but
this approach will quickly get bogged down by combinatorial complications.
Before our work there was no indication that it was even possible to get a
closed formula for all the interactions among the mesons in the large $N$
limit of two dimensional QCD.

The solution of this linear integral equation will give the structure func-
tions of mesons. Experimental data exists on the structure functions of
mesons as well. It is not from Deep Inelastic Scattering, but from Drell-
Yan scattering. Preliminary results on the comparison of these structure
functions with data are encouraging. However the focus of these lectures is
the much more conceptually deep problem of obtaining the structure function
of the proton.

In fact our quantum hadrondynamics is not only a theory of mesons but
also of baryons. We will show in the next chapter that there are topological
solitons in our theory that describe baryons.

2.15 Quantization of HadronDynamics

We have so far described a classical theory, whose phase space is the Grass-
mannian. The Poisson bracket relations for the $M(x,y)$ play a role anal-
gous to the canonical commutation relations. There are two distinct ap-
proaches to quantization of this theory:
(i) Find a representation of the Lie algebra in terms of operators ona com-
plex Hilbert space (‘canonical quantization’, pursued in Ref. [12]; and, (ii)
Realize the wavefunctions as holomorphic sections of a line bundle on the Grassmannians (‘geometrical quantization’ pursued in Ref. [14].

In the first approach, we can use the complete classification of highest weight unitary representation of the (central extension of the ) infinite-dimensional unitary Lie algebra, due to Kac and Peterson [29]. These turn out to be described by Young tableaux just as in the finite dimensional case, except they may have infinite depth. The question arises which irreducible representation to choose. The quadratic constraint selects out one class of representations, whose Young Tableaux are rectangular, of width $N$ and infinite depth. Such a representation can be realized as

$$\hat{M}(x, y) = \frac{1}{N} : q^\dagger(x) q(y) :$$

(2.96)

where $q, q^\dagger$ satisfy the canonical anti-commutation relations. The representation is carried by the vector space of color invariant states in the fermionic Fock space. Thus we recover exactly the light-cone gauged fixed version of two dimensional QCD with $N$ colors as the quantum theory of hadron-dynamics. We have shown that two dimensional QCD and two dimensional hadron-dynamics are exactly equivalent at all energies and for all values of the number of colors. We have already given the details of this argument in Ref. [12].

In the geometric approach, we construct a line bundle of Chern number $N$ on the Grassmannian. There is a connection on this line bundle whose curvature is the natural symplectic form of the Grassmannian. The infinite dimensional Grassmannian is a complex manifold, just like the finite dimensional ones. Holomorphic sections of the line bundle exist when the Chern number is a positive integer. An inner product can be established on the vector space of these holomorphic sections by using ideas of Segal [30] or the measure of Pickrell [31]. The observables $M(x, y)$ can be represented as operators on this Hilbert space. Again the formulae we get are just the same as that of light-cone gauge fixed QCD. See [14] for details.

In addition it is sometimes convenient to quantize a restricted version of the theory where the phase space is some symplectic submanifold of the Grassmannian. This will give us some insight into the connection between our theory and the parton model. We will return to this in the next chapter.

To summarize, we have a theory that doesn’t just describe mesons to all orders in perturbation theory; we even have a theory of baryons. The topologically non-trivial solutions of the theory— the solitons— describe baryons.
In the next chapter we will study these solitons in more detail and obtain a theory of the structure functions of the baryon.
Chapter 3

Baryons as Solitons and Their Structure Functions

We saw that our phase space is a disconnected manifold, each connected component being labelled by an integer

\[ B = -\frac{1}{2} \text{tr}_\tau M. \]  

(3.1)

Thus each connected component will have a minimum for the energy. These will describe stable static solutions of the equations of motion. They are called topological solitons.

The minimum of the energy in each connected component with a given value of \( B \) describes a stable particle, as the only particles with lower energy would be in a sector with a different value of \( B \). It cannot decay into those states since \( B \) is a conserved quantity.

We saw that hadron dynamics describes, in the linearized approximation, the spectrum of mesons of two dimensional QCD. What does a topological soliton describe? It is an old idea of Skyrme that the topological solitons of a theory of hadrons are baryons. This idea was revived in the mid eighties \[4, 5\] and shown to be consistent with QCD in four dimensions.

With this in mind, we should expect that the topological solitons of our two-dimensional theory describe baryons in the Deep Inelastic region. In the center of mass frame, a baryon will (due to Lorentz contraction) have a thin flat shape: one that can be described within a two dimensional theory. This offers us the possibility of solving one of the long-standing problems of particle physics: to explain the structure functions of hadrons as measured
in Deep Inelastic Scattering. By solving the static equations of motion, we will be able to determine the ‘shape’ of the soliton, which will then give us the dependence of the hadronic structure function on the Bjorken $x_B$ variable. The equations we need to solve are certain singular nonlinear integral equations. They are exactly solvable, so we will have to resort to numerical techniques. Indeed some ingenuity is needed even in the numerical part of this project: ‘off the shelf’ methods do not work due to the singularities in the integrals. These singularities do not pose any new conceptual problems however: we are dealing with a finite quantum field theory. The basic definitions of these singular integrals (as dealt with an earlier section) go back to the nineteenth century work of Cauchy and Hadamard.

The static equations of motion are

$$[E'(M), \epsilon + M] = 0$$  \hspace{1cm} (3.2)

where $E'(M)$ is the derivative of the energy, computed above. It will be most convenient to view them as integral equations for $\tilde{M}(p,q)$, the integral kernel in momentum space. Of course we must solve these equations subject to the constraints

$$[\epsilon, M]_+ + M^2 = 0; \quad \text{tr}_\epsilon M = -2.$$  \hspace{1cm} (3.3)

The last condition picks out the sector with baryon number one.

### 3.1 Quark Distribution Functions from QHD

If we split the operator $M$ into submatrices according to the splitting $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$,

$$M = \begin{pmatrix} a & b \\ b^\dagger & d \end{pmatrix},$$  \hspace{1cm} (3.4)

the operator $a : \mathcal{H}_- \rightarrow \mathcal{H}_-$ is positive while $d : \mathcal{H}_+ \rightarrow \mathcal{H}_+$ is negative. The baryon number $B = -\frac{1}{2} \text{tr}_\epsilon M$ is the sum of two terms: one from positive momenta and one from negative momenta:

$$B = -\frac{1}{2} \sum_a \int_0^\infty \left[ \tilde{a}_{aa}(-p,-p) + \tilde{d}_{aa}(p,p) \right] \frac{dp}{2\pi}. $$  \hspace{1cm} (3.5)

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The first term is always negative and the second always positive. Thus, 
\( \phi_a(p) = \frac{1}{2} \tilde{d}_{aa}(-p,-p) \) can be thought of as the distribution function of anti-
quarks in a hadron while \( \phi_a(p) = -\frac{1}{2}d_{aa}(p,p) \) is the distribution function
for quarks. The quantity

\[
\phi^V_a(p) = \phi_a(p) - \bar{\phi}_a(p)
\]

(3.6)
corresponds to what is usually called the valence parton distribution. Its
integral over \( p \) and sum over \( a \) is equal to the baryon number.

So the quark distribution function can be thought of as the sum of the
‘valence’ and ‘Sea’ contributions:

\[
\phi_a(p) = \phi^V_a(p) + \phi^S_a(p).
\]

(3.7)
The ‘Sea’ quark distribution function is thus just the same as the anti-quark
distribution function:

\[
\phi^S_a(p) = \phi_a(p).
\]

(3.8)

We find this terminology a bit confusing and mention it only for purposes
of comparison. The point is that there is no physical meaning to the splitting
of the quark distribution function into a valence and a sea quark contribution:
only the sum has a measurable, physical significance. The anti-quark
distribution does in fact have a physical significance. The above definition of
a Sea quark distribution is thus completely arbitrary and is not imposed on
us by any symmetry such as charge conjugation invariance. In fact we will
find later that there is another splitting which is more natural, where the
Sea quarks have a wavefunctions are orthogonal to the valence quark wave-
function. This at least respects the Pauli principle for quarks. Even that
splitting is merely a matter of convenience of interpretation: all comparisons
with experiment should be in terms of the measurable quantities \( \phi_a(p) \) and
\( \bar{\phi}_a(p) \).

Actually the distribution functions are usually thought of as functions of
a dimensionless variable \( x_B \). If \( P \) is the total momentum (in the null
direction) of the hadron, the momentum of the quark can be measured as a
fraction of \( P \). So we should actually write

\[
\phi_a(x_B) = -\frac{1}{2} \sum_a \tilde{d}_{aa}(x_B P), \quad \phi_a(x_B) = \frac{1}{2} \sum_a \tilde{d}_{aa}(-x_B P).
\]

(3.9)
These identifications of the parton distribution functions in terms of the diagonal matrix elements of $\tilde{M}_{ab}(p,p')$ can also be seen in terms of the formula we derived in terms of QCD.

The structure functions are supposed to vanish for $x_B > 1$. Why should the r.h.s. of the above equation vanish for such $x_B$? For large $N$, the structure functions will be of order $e^{-cN^{1/2}}$ at $x_B = 1$. So even though the wavefunction doesn’t vanish at $x_B = 1$, it is exponentially small. A more proper analysis of the large $N$ limit, taking into account semi-classical corrections will in fact reproduce wavefunctions that vanish at $x_B = 1$. The difference between these two versions of the large $N$ limit is like the difference between the microcanonical ensemble and the canonical ensemble in statistical mechanics.

### 3.2 Rank One Ansatz

Before we set out to solve the above integral equations, it is useful to consider a simpler, variational approximation to them. A solution to the above constraints on $M$ is the rank one (or separable) ansatz,

$$M = -2\psi \otimes \psi^\dagger.$$  \hspace{1cm} (3.10)

We showed earlier that this is a solution to the constraints with baryon number one if $\psi$ is a positive energy wavefunction of length one:

$$\epsilon \psi = \psi, \quad \sum_a \int |\tilde{\psi}_a(p)|^2 \frac{dp}{2\pi} = 1.$$  \hspace{1cm} (3.11)

We are not claiming that the exact solution of the static equations of motion are of this form: only that this is a reasonable variational ansatz for it. The physical meaning of this ansatz will be made clear in the next section: it is equivalent to the valence quark model of the baryon in the large $N$ limit.

The technical advantage of this ansatz is that it solves the constraint; more precisely it reduces it to the much simpler constraint that $\psi$ is of length one. Geometrically, the set of states we are considering forms a projective space: changing $\psi$ by a complex number of modulus one does not change $M$. The separable ansatz is an embedding of the complex projective space $CP(H)_+$ of the positive energy states into the component of the Grassmannian with renormalized dimension one. Instead of minimizing
the energy on the whole Grassmannian, in this approximation, we minimize it on this submanifold.

The parton structure functions have a simple meaning within this approximation: the structure function \( \phi_a(x_B) \) is just the square of the wavefunction \( \tilde{\psi}(p) \) in momentum space:

\[
\phi_a(x_B) = |\tilde{\psi}_a(x_B P)|^2. \tag{3.12}
\]

The anti-quark structure function is just zero in this separable approximation.

The energy on this submanifold can be easily calculated:

\[
E_1(\psi) = \sum_a \int_0^\infty \frac{1}{2} \left[ p + \frac{\mu^2}{p} \right] |\tilde{\psi}_a(p)|^2 \frac{dp}{2\pi} + \frac{\alpha_1 + \alpha_2}{2} \sum_{ab} \int |\psi_a(x)|^2 |\psi_b(y)|^2 |x - y| dx dy \tag{3.13}
\]

where as usual,

\[
\psi_a(x) = \int_0^\infty \tilde{\psi}_a(p) \frac{e^{ipx}}{2\pi} dp. \tag{3.14}
\]

Such a ‘positive momentum’ function has an analytic continuation into the upper half plane in the \( x \) variable. A simple choice would be

\[
\psi_a(x) = \frac{C_a}{(x + ib)^2}, \tag{3.15}
\]

with the location of the double pole serving as a variational parameter. \( C_a \) is constrained by the normalization condition that the length of \( \psi \) is one. (A simple pole would have infinite kinetic energy, so we consider the next simplest possibility.) It is easy to see that there is a term in energy that scales like \( b^{-1} \) (coming from \( p|\tilde{\psi}(p)|^2 \)) and the remaining terms scale like \( b \). Thus there is a minimum. This encourages us to proceed to a more accurate determination of the wavefunction \( \psi \) that minimizes the energy.

\[\text{We will find it convenient to denote} \]

\[\alpha_1 + \alpha_2 = \tilde{g}^2, \tag{3.16}\]

\[\text{to agree with previous papers.}\]
The integral equations we have to solve are

\[
\frac{1}{2} \left[ p + \frac{\mu^2}{p} \right] \tilde{\psi}_a(p) + \mathcal{F}\mathcal{P} g^2 \int \tilde{V}(p-q) \tilde{\psi}_a(q) \frac{dq}{2\pi} = \lambda \tilde{\psi}_a(p)
\]

\[
\tilde{V}(p) = -\frac{1}{p^2} \int_0^\infty \tilde{\psi}^{*}\tilde{\psi}_a(p+q) \frac{dq}{2\pi}.
\]

subject of course to \( \tilde{\psi}(p) = 0 \) for \( p \leq 0 \) and \( \int_0^\infty |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = 1 \). The second of the above integral equations is just the momentum space version of Poisson’s equation of electrostatics:

\[
V''(x) = -\psi^{*a}(x)\psi_a(x).
\]  

(3.18)

From Gauss’ law,

\[
V(x) \sim \frac{1}{2} |x|
\]

as \( |x| \to \infty \): it is just the electrostatic potential of a unit charge located near the origin. Thus

\[
\tilde{V}(q) \sim -\frac{1}{q^2}
\]

(3.20)

as \( |q| \to 0 \). This is why the integrals are singular: the finite part prescription is just a way of imposing the boundary condition that \( V(x) \sim \frac{1}{2} |x| \) at infinity in position space. We also note that for a solution centered at the origin,

\[
\psi(x) = \psi^{*}(-x)
\]

(3.21)

and \( V(x) = V(-x) \). The boundary condition at the origin we impose is

\[
V(0) = 0
\]

(3.22)

which translates to

\[
\mathcal{F}\mathcal{P} \int \tilde{V}(q) \frac{dq}{2\pi} = 0.
\]

(3.23)
### 3.3 Approximate Analytic Solution for $\mu = 0$

It will be useful to have an approximate analytical solution; even if it only works for a physically uninteresting region it will help us to validate our numerical method.

Consider the singular integral (setting $\tilde{W}(q) = q^2 \tilde{V}(q)$):

\[
\mathcal{F}\mathcal{P} \int \tilde{V}(q) \tilde{\psi}_a(p + q) \frac{dq}{2\pi} = \mathcal{P} \int \frac{\tilde{W}(q) \tilde{\psi}_a(p + q) - \tilde{W}(0) \tilde{\psi}_a(p) \, dq}{q^2} \frac{2\pi}{2\pi} = \frac{1}{2} \mathcal{P} \int \tilde{W}(q)[\tilde{\psi}_a(p + q) + \tilde{\psi}_a(p - q)] - 2\tilde{W}(0) \tilde{\psi}_a(p) \, dq \frac{2\pi}{q^2} = \frac{1}{2} \mathcal{P} \int 2[\tilde{W}(p) - \tilde{W}(0)] \frac{dq}{q^2} \tilde{\psi}_a(p) \quad (3.25)
\]

The first integral is not singular any more. The integral in the square brackets in the last term is in fact zero by our boundary condition:

\[
\mathcal{P} \int 2[\tilde{W}(p) - \tilde{W}(0)] \frac{dq}{q^2} = \mathcal{F}\mathcal{P} \int \tilde{V}(q) \frac{dq}{2\pi} = V(0) = 0. \quad (3.24)
\]

Thus

\[
\mathcal{F}\mathcal{P} \int \tilde{V}(q) \tilde{\psi}_a(p + q) \frac{dq}{2\pi} = \frac{1}{2} \int \tilde{W}(q) \frac{\tilde{\psi}_a(p + q) + \tilde{\psi}_a(p - q) - 2\tilde{\psi}_a(p) \, dq}{q^2} \frac{2\pi}{2\pi} \quad (3.25)
\]

So far we haven’t made any approximations on this integral: just rewritten the singular integral in a better way. The main contribution to this integral ought to come from the neighbourhood of the point $q = 0$. So we should be able to approximate the quantity in the square brackets by its leading Taylor series approximation

\[
\mathcal{F}\mathcal{P} \int \tilde{V}(q) \tilde{\psi}_a(p + q) \frac{dq}{2\pi} \sim \left[ \int q^2 \tilde{V}(q) \frac{dq}{2\pi} \right] \tilde{\psi}_a''(p). \quad (3.26)
\]

Then our nonlinear singular integral equation reduces to an ordinary differential equation:

\[
- b\tilde{\psi}_a''(p) + \frac{1}{2}[p + \frac{\mu^2}{p}]\tilde{\psi}_a(p) = \lambda \tilde{\psi}_a(p) \quad (3.27)
\]
where \( b \) is determined by the self-consistency constraint

\[
b = -\int q^2 \tilde{V}(q) \frac{dq}{2\pi} = \sum_a \left| \int_0^\infty \tilde{\psi}_a(q) \frac{dq}{2\pi} \right|^2.
\] (3.28)

This equation can be thought of as the nonrelativistic Schrödinger equation for a particle of mass \( b^{-1} \) in a linear plus Coloumb potential. The only additional complications are that \( b \) is determined by the above self-consistency relation and also, we have the boundary condition \( \tilde{\psi}(p) = 0 \) for \( p \leq 0 \). The only non-linearity in the problem is in the equation determining \( b \).

Such non-relativistic models with linear plus Coloumb potentials have been used to describe mesons made of heavy quarks [34]. But the physical origin of this Schrödinger equation is completely different in our case: we get this equation in momentum space and not position space. The eigenvalue \( \lambda \) does not have the physical meaning of energy for us: energy is to be determined by substituting the solution into the formula for the hamiltonian. Moreover, we are studying the fully relativistic bound state problem. And of course we are studying baryons not mesons. Still it is encouraging that it is possible to reduce a fully relativistic bound state problem to a mathematical problem that is no more complicated than the non-relativistic case.

The special case \( \mu = 0 \) is particularly simple. In this extreme relativistic case, the solution is an Airy function:

\[
\tilde{\psi}(p) = C \text{Ai} \left( \frac{p - 2\lambda}{2b(\alpha_1 + \alpha_2)^\frac{1}{3}} \right)
\] (3.29)

\( C \) is fixed by normalization: \( \int |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = 1 \). The eigenvalue is fixed by the continuity of the wavefunction: since it must vanish for \( p \leq 0 \) we require it to vanish at \( p = 0 \) as well by continuity. Then

\[
\lambda = -\frac{1}{2} \xi_1 [2b(\alpha_1 + \alpha_2)]^\frac{1}{3}
\] (3.30)

where \( \xi = -2.33811 \) is the zero of the Airy function closest to the origin. Now we fix the constant \( b \) by putting this back into the nonlinear self-consistency condition. We get

\[
2b = \tilde{g} \frac{1}{(2\pi)^\frac{1}{2}} \frac{1}{[\int_{\xi_1}^\infty \text{Ai}(\xi)d\xi]^\frac{1}{3}} \frac{\left| \int_{\xi_1}^\infty \text{Ai}^2(\xi)d\xi \right|^\frac{1}{3}}{[\int_{\xi_1}^\infty \text{Ai}^2(\xi)d\xi]^\frac{1}{2}}
\]
The Analytic Approximation to the Wave Function with m=0

Figure 3.1: The approximate analytic solution for $\mu = 0$.

\[
\lambda = \frac{|\xi_1|}{2\sqrt{(2\pi)}} \frac{\int_{\xi_1}^{\infty} \text{Ai}(\xi) d\xi}{\int_{\xi_1}^{\infty} \text{Ai}^2(\xi) d\xi} \tilde{g} \sim 0.847589 \tilde{g}.
\]

Moreover

\[
\tilde{\psi}(p) = C \text{Ai} (\xi_1(\frac{p}{\tilde{g}\lambda} - 1)).
\] (3.31)

We plot the solution so obtained below.

### 3.4 Numerical Solution with the Rank One Ansatz

We now describe how to solve our integral equation numerically. First we replace the nonlinear integral into a recursion relation.

\[
\frac{1}{2} \left[ p + \frac{\mu^2}{p} - 2\lambda_s \right] \tilde{\psi}_{s+1}(p) + \tilde{\alpha}_1 \mathcal{P} \int_0^\infty \tilde{V}_s(p-q) \tilde{\psi}_{s+1}(q) \frac{dq}{2\pi} = 0 \quad (3.32)
\]

and

\[
\tilde{V}_s(p) = -\frac{1}{p^2} \int_0^\infty \tilde{\psi}_s^*(p+q) \tilde{\psi}(q) \frac{dq}{2\pi}.
\] (3.33)

We start with an initial guess $\tilde{V}_0(p)$, solve the linear integral equation to get a solution $\tilde{\psi}_1(p)$. Among all the solutions of this linear integral equation
we pick the one without a node; this happens to be the one with the smallest
eigenvalue $\lambda$. Then we calculate $\tilde{V}_1(p)$ as above and then again solve the
integral equation to get $\tilde{\psi}_2$ and so on till our iteration converges.

We pick the nodeless eigenfunction at each stage since we expect our final
answer for the ground state eigenfunction to have this property. It is just a
coincidence that this happens to have the smallest eigenvalue $\lambda$; in any
case $\lambda$ does not have the meaning of energy.

Of course to carry out this algorithm, we need to convert the above inte-
gral equations into matrix equations. That is done by the quadrature method
(described in the appendix C) for singular integrals. The whole procedure
can be implemented in Mathematica quite well.

We plot the numerical solution so obtained against the approximate analyt-
ical solution we got earlier for the special case $\mu = 0$.

The curves are qualitatively the same: the difference has to do with the
approximally interesting case in fact does not have $\mu = 0$: indeed we will see
that the value of $\mu^2$ is in fact negative. For example when $m = 0$, we
have, $\mu^2 = -\frac{2^2}{\pi}$. It is of much interest to see if there is a variant of the
analytic approximatin method above that applies to this more realistic case.

Figure 3.2: Comparison of Numerical and Approximate Analytic Solutions
for $\mu = 0$.
3.5 Quantization of the Rank One Ansatz

We have considered elsewhere [12, 14] the quantization of our hadron dynamics, to recover QCD. It is also instructive to study the quantization of a simpler version of our theory, corresponding to the rank one ansatz. The quantization of this truncated theory gives a sort of approximation to QHD (hence QCD) which is of some interest in itself. We will see in the next chapter that this is just the valence parton model with interactions between the partons as predicted by QCD.

Given a vector satisfying
\[ \epsilon \psi = \psi, \quad ||\psi||^2 = 1 \quad (3.34) \]
we have an element of the Grassmannian of renormalized dimension one:
\[ M = -2 \psi \otimes \psi^\dagger. \quad (3.35) \]

But this element remains the same if we change \( \psi \) by a complex number of modulus one: \( \psi \rightarrow e^{i\theta} \psi \). Thus we have an embedding of the projective space \( \mathcal{P}(\mathcal{H}_+) \) into the Grassmannian. This is a sympletic embedding: the symplectic form on the projective space induced by this embedding is the same as the natural sympletic form on the projective space. Thus we can regard the separable operators as forming a ‘reduced phase space’ describing part of the degrees of freedom of our theory.

A way to understand this is to consider \( \tilde{\psi}(p) \) as a complex-valued observable on the effective phase space. In order to reproduce the Poisson brackets of \( M \), these must satisfy
\[ \{ \tilde{\psi}_a(p), \tilde{\psi}_b(p') \} = 0 = \{ \tilde{\psi}^{*a}(p), \tilde{\psi}^{*b}(p') \}; \quad (3.36) \]
\[ \{ \tilde{\psi}_a(p), \tilde{\psi}^{*b}(p') \} = -i2\pi \delta(p - q) \delta^b_a. \quad (3.37) \]

They satisfy the constraints,
\[ \tilde{\psi}(p) = 0; \text{for } p < 0, \quad \int_0^\infty |\psi(p)|^2 dp = 1. \quad (3.38) \]

We should regard the function \( \tilde{\psi}(p) \) on momentum space to be the fundamental variable from which other observables such as
\[ \psi(x) = \int_0^\infty \tilde{\psi}(p)e^{ipx} dp \quad \frac{2\pi}{2\pi} \quad (3.39) \]
can be obtained.

Both the Poisson brackets and the constraint are simpler than the full theory, which is why we consider this case first.

The Hamiltonian of our reduced dynamical system is obtained by putting the ansatz into $E(M)$:

$$E_1(\psi) = \sum_a \int_0^\infty \frac{1}{2}[p + \frac{\mu^2}{p}]|\tilde{\psi}_a(p)|^2 \frac{dp}{2\pi}$$

$$+ \frac{\alpha_1 + \alpha_2}{2} \sum_{ab} \int |\psi_a(x)|^2 |\psi_b(y)|^2 |x - y| dx dy. \quad (3.40)$$

These Poisson brackets and hamiltonian form a perfectly well-defined dynamical system on its own right. We will now consider how to quantize this theory, and obtain the rules for the semi-classical approximation.

We can quantize the theory by looking for operators satisfying canonical commutation relations:

$$[\hat{\tilde{\psi}}_a(p), \hat{\tilde{\psi}}_b(p')] = 0 = [\hat{\tilde{\psi}}^\dagger_a(p), \hat{\tilde{\psi}}^\dagger_b(p')], \quad [\hat{\tilde{\psi}}_a(p), \hat{\tilde{\psi}}^\dagger_b(p')] = \frac{1}{N} 2\pi \delta(p - p') \delta^b_a. \quad (3.41)$$

As usual classical Poisson brackets go over to quantum commutation relations:

$$\{A, B\} \to -i\hbar [\hat{A}, \hat{B}]. \quad (3.42)$$

In our case we will denote the parameter that measures the quantum correction, analogous to $\hbar$, by $\frac{1}{N}$. In a minute we will see that this number $N$ must actually be an integer. Th elimit $N \to \infty$ is the classical limit. The constraint on the observables can be implemented by restricting attention to those states satisfying

$$\int_0^\infty \psi^* \tilde{\psi}(p) \frac{dp}{2\pi} \geq 1. \quad (3.43)$$

Now it is obvious that a representation for our commutation relations is provided by bosonic creation annihilation operators:

$$[\hat{\tilde{a}}_a(p), \hat{\tilde{a}}_b(p')] = 0 = [\hat{\tilde{a}}^\dagger_a(p), \hat{\tilde{a}}^\dagger_b(p')], \quad [\hat{\tilde{a}}_a(p), \hat{\tilde{a}}^\dagger_b(p')] = 2\pi \delta(p - q) \delta^b_a. \quad (3.44)$$

with

$$\psi_a(x) = \frac{1}{\sqrt{N}} a_a(x), \quad \psi^{\dagger a}(x) = \frac{1}{\sqrt{N}} a^{\dagger a}(x). \quad (3.45)$$
Then the constraint becomes just the condition that we restrict to states containing \( N \) particles:

\[
\int_{0}^{\infty} \tilde{a}^{\dagger a}(p)\tilde{a}_{a}(x)\frac{dp}{2\pi} = N. \tag{3.46}
\]

This is why \( N \) must be an integer!

Now we know that we are dealing with a system of \( N \) bosons interacting with each other under the hamiltonian

\[
\frac{1}{N} \hat{E}_{1}(\psi) = \sum_{a} \int_{0}^{\infty} \frac{1}{2} \left[ p + \mu^{2} \right] \tilde{a}^{\dagger a}(p)\tilde{a}_{a}(p)\frac{dp}{2\pi} + \frac{\alpha_{1} + \alpha_{2}}{2} N \int a^{\dagger a}(x)a^{\dagger b}(y)a_{b}(y)a_{a}(x)|x - y|dx dy. \tag{3.47}
\]

The classical (or large \( N \)) limit we have been discussing so far is just the mean field approximation to this many-body problem. The semi-classical approximation will give us the leading corrections in the case of finite \( N \). This basic insight is due to Witten, in a by now classic paper [9].

What are these bosons? We will see in the next chapter that these bosons are just the valence quarks of the parton model, stripped of their color!. Quarks are of course fermions. However the wavefunction of the system must be totally anti-symmetric in the color indices because of the condition that the state be invariant under \( SU(N) \). Thus in the remaining indices the wavefunction must be symmetric: if we ignore color the valence quarks behave like bosons.

Note that the momentum of the particles created by \( a^{\dagger}(p) \) is always positive. Thus the total momentum

\[
\hat{P} = \int_{0}^{\infty} p \ a^{\dagger a}(p)a_{a}(p)\frac{dp}{2\pi} \tag{3.48}
\]

is a positive operator. Indeed on a state containing \( N \) particles,

\[
|a_{1}, p_{1}; a_{2}, p_{2}; \cdots a_{N}, p_{N} > = a^{\dagger a_{1}}(p_{1})a^{\dagger a_{2}}(p_{2})\cdots a^{\dagger a_{N}}(p_{N})|0 > \tag{3.49}
\]

\( P \) is just the sum of individual momenta, each of which is positive:

\[
\hat{P}|a_{1}, p_{1}; a_{2}, p_{2}; \cdots a_{N}, p_{N} > = |p_{1} + p_{2} + \cdots p_{N}|a_{1}, p_{1}; a_{2}, p_{2}; \cdots a_{N}, p_{N} > \tag{3.50}
\]

A general state will be, in this basis described a wavefunction \( \tilde{\phi} \)

\[
|\phi > = \sum_{a_{1}, \cdots a_{N}} \int_{0}^{\infty} \frac{dp_{1}}{2\pi} \cdots \frac{dp_{N}}{2\pi} \tilde{\phi}(a_{1}, p_{1}; \cdots a_{N}, p_{N})|a_{1}, p_{1}; \cdots a_{N}, p_{N} > . \tag{3.51}
\]
An eigenstate of $\hat{P}$ with eigenvalue $P$ will satisfy

$$[p_1 + \cdots + p_N] \tilde{\phi}(a_1, p_1; \cdots; a_N, p_N) = P\tilde{\phi}(a_1, p_1; \cdots; a_N, p_N). \quad (3.52)$$

Since each of the momenta $p_i$ are positive, it follows that they must each be less than the total momentum $P$:

$$0 \leq p_i \leq P, \text{ for } i = 1, \cdots, N. \quad (3.53)$$

We will see in the next chapter that the $p_i$ are the momenta of the valence partons: the proton is a simultaneous eigensate of $\hat{H}$ and $\hat{P}$. We have just seen a very important point: our theory at finite $N$ (but within the approximation of the valence parton model) predicts that the wavefunction must vanish unless each of the parton momenta $p_i$ are less than $P$. But there is no $N$ in this inequality; so it must hold even in the large $N$ limit!

We will be finding an approximate eigenstate of $\hat{H}$, by a variational principle: a sort of mean field theory, the large $N$ limit. The naive choice is a wavefunction which is a product of single particle wavefunctions. But such a naive version of mean field theory will violate the exact inequality we just established on momentum eigenstates. We should find our variational approximation to the eigenstate of the hamiltonian, within the space of momentum eigenfunctions.

Thus we assume that the wavefunction is approximated by a wavefunction that is just a product except for the constraint that the momenta add up to $P$:

$$\tilde{\psi}(a_1, p_1; a_2, p_2; \cdots; a_N, p_N) = 2\pi\delta\left(\sum_i p_i - P\right)\tilde{\psi}(a_1, p_1)\tilde{\psi}(a_2, p_2)\cdots\tilde{\psi}(a_N, p_N). \quad (3.54)$$

Thus the fraction of the momentum carried by each particle is less than one:

$$\psi(p) = 0, \quad \text{unless } 0 \leq \frac{p}{P} \leq 1. \quad (3.55)$$

That is how we recover the fact that the quark distribution function must vanish when the Bjorken variable is greater than one. This is a sort of semiclassical correction in the $\frac{1}{N}$ approximation. We must solve the classical equations of motion for $\psi(p)$ subject to the boundary condition that it vanish outside of the interval $0 \leq p \leq P$. This requires a modification of the variational ansatz but it is possible to do that.

This variant of mean field theory is rather like the micro-canonical ensemble. The naive mean field theory where only the expectation value of momentum is required to be $P$ is like the canonical ensemble.
3.6 Valence Quark Distributions

Now we come to the point of comparison of the calculated valence parton distribution function against the data. It is not in fact necessary to make a direct comparison with the data on Deep Inelastic Scattering. A generation of phenomenologists have extracted the valence parton structure functions from the data. More precisely they have assumed a parametric form such as

\[ \phi(x_B) = A x_B^{\nu_1} (1 - x_B)^{\nu_2} [1 + a_1 x_B + a_2 x_B^2] \]  

(3.56)

for the parton distribution functions and fit to all known data points. Since the data consists of measurement of the DIS (Deep Inelastic Scattering) cross-section at different values of \( Q^2 \) this fit uses the convolution of this distribution function with the structure functions of the quarks calculated in perturbation theory. There are about 6 structure functions (corresponding to \( u, \bar{u}, d, \bar{d}, s \) quarks and the and gluons) so altogether there are about 30 parameters in addition to the fundamental parameters of perturbative QCD.

Altogether there are about a thousand data points, coming from measurement of the cross-section for \( e^- p, e^- n, \nu^- p \) or \( \nu^- n \) scattering at various energies. Thus the mere extraction of these distribution functions has itself become a subfield of particle physics [32, 19, 33]. The major groups seem to be in general agreement with each other, although a detailed analysis of the errors in their parameters is not yet available. There are recent attempts to estimate the systematic and statistical errors, but they seem incomplete.

The comparison should be made with the weighted average over the \( u \) and \( d \) quark distributions, weighted so as to get an isospin invariant combination. This is because we have not yet done a collective variable variable quantization of the isospin degrees of freedom of the proton, so the distribution function we are computing is the isospin invariant one.

Another complication in the comparison is that not all the momentum in the baryon is carried by the valence quarks: it is known that only about half of the momentum is in the valence quarks, the rest being in the anti-quarks (or ‘sea’ quarks) or gluons. This affects our momentum sum rule. Essentially what it does is that the \( N \) in the sumrule gets replaced by \( N_{\text{eff}} = \frac{N}{f} \) where \( f \) is the fraction of the momentum carried by the valence quarks. Until we do a complete calculation allowing for the contributions of the sea quarks and the gluons (i.e., without assuming the factorizable ansatz for \( M \) and without ignoring the scalar fields \( \phi_A \)) we must treat \( f \) as a parameter.
and choose the value that gives the best fit. The good news is that it is the only parameter: \( m^2 \) is fixed to be zero\(^2\) since the up and down quarks are known to have masses that are small compared to the QCD scale \( \alpha_1 \). (\( m_u \sim 5 \text{ MeV} \) and \( m_d \sim 10 \text{ MeV} \) while \( \alpha_1 \sim 100 \text{ MeV} \).)

The only other parameter of the theory is \( \tilde{g}^2 \); but since the distribution functions are dimensionless function of a dimensionless variable, the dependence on \( \alpha_1 \) cancels out.

In comparison with experimental data we should keep in mind that we are in fact ignoring some of the constituents of the baryon: the valence quarks are known to carry only about half of the momentum of the baryon. The rest is in the sea quarks and the gluons. Thus we introduce a parameter \( f \), the fraction of the momentum carried by the valence quarks. We find that for a value of \( f = 0.6 \) the wavefunction predicted by us agrees quite well with experiment. (The value of being about a half is consistent with other ways of looking at the situation.) Thus we have solved the problem of deriving the quark structure functions of the baryon from QCD.

It still remains to calculate the anti-quark and gluon structure functions.

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\(^2\)This by the way means that \( \mu^2 = m^2 - \frac{\tilde{g}^2}{4} \) is negative. There is no contradiction here, since the quark is not a true particle: the mass of the baryon is still predicted to be real.
We will turn to that in later publications. In a later chapter we will give some ideas that help in solving this problem.

In the next chapter we will see how to reconcile the soliton model with the parton model.
Chapter 4

The Parton Model

This chapter is based on the paper \[13\] which in turn was an expansion of some short statements made in Ref. \[12\].

We have been describing the idea that the baryon is a soliton, essentially made up of an infinite number of mesons. But this seems to contradict the simple and successful picture of the baryon as a bound state of quarks. At the time the Skyrme model was revived, it was shown that the static (low energy properties) of the baryon that are so well explained by the quark model (magnetic moments, mass differences, flavor multiplet structure) can all be rederived in the soliton model\[6\]. But the only explanation for the structure of the proton in deep inelastic scattering still appeared to be the quark-parton model. In these lectures, we are describing how to explain the structure functions within the soliton model as well. In fact we will be able to go beyond the simple minded parton model.

It is an old idea in the study of the structure of the proton that the constituents-the partons- can be thought of as free particles. This then raises the question of how they bind to form the proton in the first place. It is the binding mechanism that determines the wavefunction of the partons and hence the structure functions of the hadron. In this section we will present a model of interacting partons. Perhaps surprisingly, it will turn out to be equivalent to the soliton model in the factorizable ansatz (and in the large $N$ limit). Thus the parton model is merely an approximation to the soliton model. Later, by a deeper analysis of the soliton model, we will derive the sea quark and gluon distributions as well.

It is at the moment impossible to derive the particle spectrum of four dimensional QCD directly. There are many attempts to do compute the
spectrum by direct numerical simulation of QCD, but they have to yet to surmount many obstructions. For example, the asymptotic freedom of QCD implies that the energy of all physical states are exponentially small compared to the cut-off. Thus extra-ordinary accuracy is required of all direct numerical computations of the energy, a problem that is only exacerbated by the large (in principle infinite) number of degrees of freedom in the system. However, it may be possible to understand the spectrum of two dimensional QCD by diagonalizing its hamiltonian numerically. Being a finite (rather than asymptotically free) theory numerical calculations are much more reliable.

4.1 Parton Wavefunctions

We will follow largely the discussion in [15]. The valence parton model assumes that the proton is made of $N$ partons (quarks) which are fermions transforming under the fundamental representation of color. The idea of color of course was not present in the original version of this model, due to Bjorken and Feynman. Also it was not known then that the partons were spin half particles and hence fermions. But these were straightened out soon with the identification of the valence partons with the quarks. footnoe\textsuperscript{It is known that only about half of the momentum of the baryon is accounted for by these valence partons: the rest must be carried by the ‘sea quarks’ and by the gluons.

We will work think of the parton wavefunctions as functions of the null momentum $p = p_0 + p_1$. In addition, the partons carries the quantum numbers of flavor and spin (together denoted by $a$) and color $\alpha$. Thus the wavefunction of a single parton will be $\tilde{\psi}(a, \alpha, p)$. Since $p \geq 0$ for the null component of momentum, we require that $\tilde{\psi}$ vanish for negative $p$.

A baryon is made of $N$ such partons so its wavefunction is a completely antisymmetric function $\tilde{\psi}(a_1, \alpha_1, p_1; a_2, \alpha_2, p_2; \cdots ; a_N, \alpha_N, p_N)$. However, the baryon must be invariant under color: transform under the trivial representation of color $SU(N)$. This means that the wavefunction is completely antisymmetric in color alone:

$$\tilde{\psi}(a_1, \alpha_1, p_1; a_2, \alpha_2, p_2; \cdots ; a_N, \alpha_N, p_N) = \epsilon_{\alpha_1, \alpha_2, \cdots, \alpha_N} \tilde{\psi}(\alpha_1, p_1; \alpha_2, p_2; \cdots \alpha_N, p_N) \quad (4.1)$$

In other words, the wavefunction is completely symmetric in spin, flavor and momentum quantum numbers. If we factor out color from the wavefunction,
the partons behave as if they are *bosons*.

### 4.2 Hamiltonian

The kinetic energy of a single parton is

$$\frac{1}{2} [p + \frac{\mu^2}{p}] . \quad (4.2)$$

Hence the kinetic energy of the partons making up a baryon is

$$\sum_{a_1 \cdots a_N} \int_0^{\infty} \sum_{i=1}^{N} \frac{1}{2} [p_i^2 + \frac{m_{a_i}^2}{p_i}] |\bar{\psi}(a_1, p_1; \cdots; a_N, p_N)\psi(a_1, p_1; \cdots; a_N, p_N)|^2 \frac{dp_1 \cdots dp_N}{(2\pi)^N} . \quad (4.3)$$

In the early versions of the quark-parton model, they were treated as if they are free particles. That cannot be quite correct, since they must after all bind to form the baryon. The simplest idea would be to let them interact through a pairwise potential $\alpha v(x)$:

$$\frac{1}{2} \alpha_1 \sum_{a_1 \cdots a_N} \int_0^{\infty} \sum_{i \neq j} v(x_i - x_j) |\bar{\psi}(a_1, x_1; \cdots; a_N, x_N)\psi(a_1, x_1; \cdots; a_N, x_N)|^2 dx_1 \cdots dx_N . \quad (4.4)$$

Now we must allow for a self-energy term as well in the hamiltonian. The point is that the effective mass of the partons may not be the same as their bare mass. Just as the electrons in a metal have a different effective mass from the free electron, the effective mass of the parton inside a baryon may be different from that of a ‘bare’ parton: there can be a finite renormalization of the mass. We can allow for this by replacing the $\mu^2$ in the kinetic energy by an effective mass $\mu^2$.

Since the partons (quarks) cannot be separated out to infinity, one could wonder what the ‘bare’ parton mass $m$ means: it has no meaning as the mass of any particle that can exist in an asymptotic state. However, it is possible to give the bare parton mass a meaning in terms of high energy processes that do not involve asymptotic states. For example the weak interaction can be used as a probe of the quark masses: the weak decay rates of the quarks are sensitive to the bare masses. This gives a way to make sense of the bare parameter $m$. But for our purposes, what matters is the effective mass $\mu$.  

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Thus the hamiltonian of the valence quark model is

\[ \mathcal{E}_N(\tilde{\psi}) = \sum_{a_1 \cdots a_N} \int_0^\infty \left[ \sum_{i=1}^N \left( \frac{1}{2} |p_i + \frac{\mu_{a_i}}{p_i}|^2 \psi(a_1, p_1; \cdots a_N, p_N) |^2 \frac{dp_1 \cdots dp_N}{(2\pi)^N} \right) + \frac{1}{2} \alpha_1 \sum_{a_1 \cdots a_N} \int_0^\infty \sum_{i \neq j} v(x_i - x_j) |\psi(a_1, x_1; \cdots a_N, x_N)|^2 dx_1 \cdots dx_N. \]  

The ground state of this hamiltonian is the baryon of the valence quark model.

We haven’t yet decided what potential \( v(x) \) to use. There is ample evidence that the quark-quark potential is linear \[34\]. In any case as in QHD, Lorentz invariance will lead to this choice in our lightcone co-ordinates. So we choose \( v(x) = \frac{1}{2} |x| \).

With this choice we get exactly the same hamiltonian we had in the last chapter, from the quantization of the rank one ansatz with \( \frac{1}{N} \) playing the role of \( \bar{\hbar} \). Thus the rank one approximation to the topological soliton model is equivalent to this interacting valence parton model.

### 4.3 Hartree Ansatz

The ground state of a many boson system can often be described by mean field theory: each boson moves in the field created by all the others. Moreover, all the bosons can be assumed to occupy the same single particle state in this ground state. After the color is factored out, the partons in our model behave just like bosons. Hence we should be able to simplify the problem by making this mean field approximation. More specifically,

\[ \tilde{\psi}(a_1, p_1; \cdots a_N, p_N) = 2\pi \delta(\sum_i p_i - P) \prod_{i=1}^N \tilde{\psi}(a_i, p_i). \]  

(4.5)

Here \( P \) is the total momentum of the system. Since \( p \geq 0 \) we must impose

\[ \tilde{\psi}(p) = 0 \]  

(4.6)

for \( p \leq 0 \). It then follows that \( p \) is less than the total momentum \( P \).

The wavefunction satisfies the normalization condition

\[ \int_0^P |\tilde{\psi}(p)|^2 dp = 1 \]  

(4.7)

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and the momentum sum rule:

\[ N \int_0^P p |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = P. \] (4.8)

The energy is calculated by putting this ansatz into the earlier formula:

\[
E = \sum_a \int_0^\infty \frac{1}{2} \left[ p + \frac{\mu^2_a}{p} \right] |\tilde{\psi}(a, p)|^2 \frac{dp}{2\pi} + \frac{1}{2} \tilde{g}^2 \int_{-\infty}^\infty v(x - y) \sum_a |\psi(a, x)|^2 \sum_b |\psi(b, y)|^2 dxdy.
\]

At this point we notice that we have exactly the same problem as in the last chapter. The energy of the rank one ansatz for \( \tilde{M}(p, q) \) is exactly the same as that of the parton model in the Hartree approximation. Thus the mean field theory of the valence parton model is our classical theory on the projective space \( CP(\mathcal{H}_+) \). This interpretation of the rank one ansatz in terms of the parton model was already noted in Ref. [12].

What would be the leading deviation away from the large \( N \) or classical limit? We argued in the last chapter in exactly the same theory (but viewed as the quantization of the rank one ansatz) that it amounts to the ansatz

\[
\tilde{\psi}(p_1, \cdots p_N) = 2\pi \delta(p_1 + p_2 \cdots p_N - P) \prod_i \tilde{\psi}(p_i). \] (4.9)

There are now some correlations between the quarks: each momentum is less than \( P \) and also \( p_N \) can be eliminated in terms of the others. The latter is a small effect; but the inequalities on the momenta, \( 0 \leq p_i \leq P \) cannot be ignored.
Chapter 5

Beyond the Valence Parton Model

We saw that the separable ansatz in the soliton theory corresponds to the valence parton approximation. Now we see how to go beyond that and get the true minimum of the energy. We can improve on the valence parton model by adding Sea quarks and anti-quarks; this requires us to invent an improved variational ansatz. In another direction we can minimize the energy in the Grassmannian by numerical methods. We will describe ideas in both directions.

5.1 A Co-ordinate System

Recall that the equations to be solved are

\[ [E'(M), \epsilon + M] = 0, \]  
(5.1)

along with the constraints:

\[ (\epsilon + M)^2 = 1, \quad \text{tr}_2 M = -2. \]  
(5.2)

Define

\[ \epsilon_1 = \epsilon - 2\psi \otimes \psi^\dagger, \]  
(5.3)

where \( \psi \) is the minimum in the subset of separable kernels; or at least a good approximation to it. Rather than parametrize the soliton by the
deviation from the vacuum \( \epsilon \) it makes more sense to use the deviation from our approximate solution \( \epsilon_1 \) which is in the same connected component. So we define a new variable \( M_1 \) by
\[
\epsilon + M = \epsilon_1 + M_1. \tag{5.4}
\]
Then the constraints on \( M_1 \) are
\[
[\epsilon_1, M_1]_+ + M_1^2 = 0, \quad \text{tr}_r M_1 = 0. \tag{5.5}
\]
The trace of \( M_1 \) is zero since the baryon number is now already carried by \( \epsilon_1 \):
\[
-\frac{1}{2} \text{tr}[\epsilon_1 - \epsilon] = 1. \tag{5.6}
\]
Let \( W \) be the subspace where \( \epsilon_1 \) takes eigenvalue \(-1\):
\[
W = \mathcal{H}_- \oplus \langle \psi \rangle \tag{5.7}
\]
where \( \langle \psi \rangle \) denotes the linear span of \( \psi \). We have an orthogonal splitting of \( \mathcal{H} \) according to this splitting. We can now represent \( M_1 \) in terms of a co-ordinate system centered at the point \( \epsilon_1 \):
\[
M_1 = -2 \left( \frac{[1 + ZZ|^{-1} - 1}{ZZ|^{1 + ZZ|^{-1}Z]^{-1}Z} \right) \tag{5.8}
\]
Here, \( Z : W^\perp \to W \) is an arbitrary Hilbert-Schmidt operator. It is straightforward to check that this is a solution to the constraint equations. Moreover, in some finite neighborhood of \( \epsilon_1 \), all points of the Grassmannian can be represented this way. Of course this co-ordinate system will break down if we go too far away from \( \epsilon_1 \): we are assuming that the true minimum lies close enough to the approximate solution to be in this co-ordinate chart.

Given a fixed \( \psi \), the operator \( Z \) above provides a co-ordinate system on an neighborhood of the Grassmannian. This co-ordinate system only covers a part of the connected component with baryon number one. It takes a countably infinite number of such charts (corresponding to different choices of \( \psi \)) to cover the whole space of baryon number one configurations.
5.2 The Method of Steepest Descent

We need to minimize the function \( E(M) \) subject to the constraints

\[
\begin{align*}
[\epsilon, M] + M^2 &= 0, \\
-\frac{1}{2} \text{tr} M &= 1, \\
\text{tr} M^2 &< \infty.
\end{align*}
\]

A simple method for minimizing functions of several variables is steepest descent: we start at some initial point and move along the straight line opposite to the gradient of the function at that point a small distance. Then we recompute the gradient at the new point and repeat the procedure. If the topography of the constant energy surfaces is not too complicated we will eventually arrive at the minimum.

The main complication in our case is of course the constraint: the gradient vector \( E'(M) \) is not tangential to the Grassmannian so if we move opposite to it we will leave the constraint surface. Even if we project out the tangential component of the vector, we will still leave the surface if we move along the straight line in that direction. The proper geometric solution to this problem is to move a small distance along the geodesic at \( M \) tangential to the gradient vector. Then we will recompute the gradient and find the geodesic at the new point. This is feasible because the Grassmannian is a homogenous manifold and we can find the geodesic on it easily using its high degree of symmetry.

More explicitly, recall that the tangential projection of the gradient vector is

\[
T = \frac{1}{4} \left[ \epsilon + M, [\epsilon + M, E'(M)] \right].
\]

(5.10)

The vector

\[
Y = \frac{1}{2} [\epsilon + M, E'(M)]
\]

(5.11)

is at right angles to the tangential part of the gradient: it is in fact obtained by multiplying \( T \) by the complex structure of the Grassmannian. Just as in the case of the sphere, the geodesic is obtained by rotating the point about an axis orthogonal to the tangent vector. More explicitly, the geodesic starting at \( M \) tangential to \( T \) is

\[
\gamma(\tau) = e^{\tau Y} [\epsilon + M] e^{-\tau Y} - \epsilon.
\]

(5.12)
Thus the steepest descent algorithm is

1. Choose an initial configuration \( M_0 \) and small parameter \( \tau \).
2. Given the \( k^{th} \) configuration \( M_k \), calculate the gradient \( E'(M_k) \) and \( Y_k = \frac{1}{2} [\epsilon + M_k, E'(M_k)] \).
3. Set \( M_{k+1} = e^{\tau Y_k} [\epsilon + M_k] e^{-\tau Y_k} - \epsilon \) and repeat the previous step.

The value of \( \tau \) has to be chosen by some trial and error. Too small a value will produce changes in the configuration within the noise due to numerical errors. Too large a choice will not give a convergent sequence: we will bounce around all over the Grassmannian. But we found that in practice, a proper value of \( \tau \) can be found quickly.

How do we choose the initial configuration \( M_0 \)? Since we believe that the valence approximation is good, we could use as the starting point a separable configuration, minimizing the energy within that subspace.

### 5.3 Sea Quarks and Anti-quarks

We saw that if we restrict the dynamics of our theory to the rank one ansatz, \( M = -2\psi \otimes \psi^\dagger \), we get the valence parton model. To get a more general picture that includes Sea quarks and anti-quarks (but is still not the total picture) we must use the ansatz with rank three. In fact by going to ansatizes of larger and larger rank we can get better and better approximations to the baryon structure functions.

The rank three ansatz is

\[
- \frac{1}{2} M = \psi_0 \otimes \psi_0^\dagger + \zeta_- \{ \zeta_- [\psi_+ \otimes \psi_+^\dagger - \psi_- \otimes \psi_-^\dagger] \\
- \sqrt{[1 - \zeta_+] \{ \psi_- \otimes \psi_+^\dagger + \psi_+ \otimes \psi_-^\dagger \}} \}
\]

where \( \psi_-, \psi_0, \psi_+ \) are three vectors in \( \mathcal{H} \) satisfying

\[
\epsilon \psi_- = -\psi_-, \quad \epsilon \psi_0 = \psi_0, \quad \epsilon \psi_+ = \psi_+,
\]

\[
||\psi_-||^2 = ||\psi_0||^2 = ||\psi_+||^2 = 1, \quad \psi_0^\dagger \psi_+ = 0.
\]

Moreover,

\[
0 \leq \zeta_-^2 \leq 1.
\]

We saw that \( \zeta_-^2 \) is the probability of finding an anti-quark inside the baryon.
As noted earlier, the anti-quark distribution function is just $\zeta^2 |\bar{\psi}_a(-p)|^2$.

The baryon number is

$$B = \sum_a \int_0^\infty \left\{ |\bar{\psi}_{0a}(p)|^2 + \zeta^2 |\bar{\psi}_{+a}(p)|^2 - |\bar{\psi}_{-a}(p)|^2 \right\} \frac{dp}{2\pi}.$$  \hspace{1cm} (5.16)

The total momentum is, similarly,

$$P = \sum_a \int_0^\infty p \left\{ |\bar{\psi}_{0a}(p)|^2 + \zeta^2 |\bar{\psi}_{+a}(p)|^2 - |\bar{\psi}_{-a}(p)|^2 \right\} \frac{dp}{2\pi}.$$  \hspace{1cm} (5.17)

These confirm the interpretation of $\psi_0$ as the valence quark wavefunction and $\psi_+(p)$ as the Sea quark wavefunction.

Also, the kinetic energy is

$$K = \sum_a \int_0^\infty \frac{1}{2} \left\{ |\bar{\psi}_{0a}(p)|^2 + \zeta^2 |\bar{\psi}_{-a}(p)|^2 + |\bar{\psi}_{+a}(p)|^2 \right\} \frac{dp}{2\pi}.$$  \hspace{1cm} (5.18)

The potential energy is more complicated. Recall that the potential energy is simpler in position space while the kinetic energy is simpler in momentum space. Our wavefunctions now depend on a discrete variable in addition to momentum, and there is a unitary transformation in these discrete variables that is the counterpart to Fourier transformation. This is the transformation to a basis in which $\mu$ is diagonal; the wavefunctions will no longer be eigenstates of $\epsilon$.

$$-\frac{1}{2} M = \psi_0 \otimes \psi_0^\dagger + \zeta_- \left\{ \psi_1 \otimes \psi_1^\dagger - \psi_2 \otimes \psi_2^\dagger \right\}$$  \hspace{1cm} (5.19)

where

$$\psi_1 = \frac{1}{\sqrt{2}} \left\{ \sqrt{1-\zeta_-} \psi_- - \sqrt{1+\zeta_-} \psi_+ \right\}$$  \hspace{1cm} (5.20)

$$\psi_2 = \frac{1}{\sqrt{2}} \left\{ \sqrt{1+\zeta_-} \psi_- + \sqrt{1-\zeta_-} \psi_+ \right\}$$  \hspace{1cm} (5.21)

The potential energy is then,

$$U = \frac{\alpha_1}{2} \sum_a \int \left\{ |\psi_{0a}(x)|^2 V_{00}(x) +$$
\[
\begin{align*}
&\left\{ \zeta^2 |\psi_{1a}(x)|^2 V_{11}(x) + \zeta^2 |\psi_{2a}(x)|^2 V_{22}(x) \right\} dx \\
&+ \frac{\alpha_1}{2} \Re \sum_a \int \left\{ \zeta \psi_{0a}(x) \psi_{1a}^*(x) V_{10}(x) - \\
&\zeta \psi_{0a}(x) \psi_{2a}^*(x) V_{20}(x) - \zeta^2 \psi_{1a}(x) \psi_{2a}^*(x) V_{21}(x) \right\} dx.
\end{align*}
\]

The mean fields are determined by solving the differential equations

\[ V''_{\alpha\beta}(x) = \psi_{\alpha\alpha}(x) \psi_{\beta\alpha}^*(x). \tag{5.22} \]

with the boundary conditions

\[ V_{\alpha\beta}(x) \rightarrow \delta_{\alpha\beta} \frac{1}{2} |x|, \text{ for } |x| \rightarrow \infty. \tag{5.23} \]

Equivalently,

\[ V_{\alpha\alpha}(0) = V'_{\alpha\alpha}(0) = 0 \tag{5.24} \]

for the diagonal components, and,

\[ V_{\alpha\beta}(\infty) = V'_{\alpha\beta}(\infty) = 0 \text{ for } \alpha \neq \beta \tag{5.25} \]

for the off diagonal components.

By choosing appropriate variational ansatzes we can estimate the antiquark content of the proton. Or we can derive integral equations for the functions $\psi_{\pm,0}$ and solve them numerically. Both methods are being pursued. Initial results are encouraging. Detailed results will appear in a separate publication.

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Appendix A

A Null Co-ordinate System

A.1 Kinematics

It will be convenient to use a co-ordinate system that combines the advantages of the null and Cartesian co-ordinate systems.

If $x^0$ and $x^1$ are the usual Cartesian co-ordinates in Minkowski space, the metric is

$$ds^2 = [dx^0]^2 - [dx^1]^2.$$  \hspace{1cm} (A.1)

We define now

$$t = x^0 - x^1, \quad x = x^1$$ \hspace{1cm} (A.2)

so that

$$ds^2 = dt[dt + 2dx].$$ \hspace{1cm} (A.3)

The Minkowski metric is,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$ \hspace{1cm} (A.4)

Thus the vector $\frac{\partial}{\partial t}$ is time-like while $\frac{\partial}{\partial x}$ is null. The initial values of fields will be given on a surface of constant $t$, which is a null line.

Momentum $p = p_\mu dx^\mu = p_0 dt + p dx$ is a 1-form (co-vector); we will use the same letter to denote the momentum 1-form as well as its null
component, but it should be clear from the context which one we mean. To find its magnitude we must use the inverse of the above metric tensor:

\[ \eta^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \eta_{\mu\nu} = 2p_0p - p^2 . \tag{A.5} \]

Thus the mass shell condition becomes

\[ p_0 = \frac{1}{2} \left[ \vec{p} + \frac{\mu^2}{p} \right]. \tag{A.6} \]

Here, \( \mu \) is the rest mass of the particle.

We see now one of the main technical advantages of using the null-time co-ordinate system: energy and momentum have the same sign. In the usual space-time co-ordinates, \( p_0 = \pm \sqrt{p_1^2 + \mu^2} \) and therefore no such simple relationship exists. In the Dirac theory of fermions the states of negative energy are occupied; this becomes merely the condition that the negative momentum states be occupied.

### A.2 Dirac Matrices

The Dirac matrices are best thought of as matrix-valued vectors, since they appear in the form \( \gamma^\mu \nabla_\mu \) in the action. Thus the Dirac algebra \( \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \) becomes

\[ [\gamma^t]^2 = 0, \quad \gamma^t \gamma + \gamma \gamma^t = 2, \quad \gamma^2 = -1. \tag{A.7} \]

We will choose the explicit representation

\[ \gamma^t = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{A.8} \]

In any representation of Dirac matrices there is a ‘charge-conjugation’ matrix \( C \) satisfying

\[ C \gamma^\mu C^{-1} = (\gamma^\mu)^T . \tag{A.9} \]

In the usual space-time formalism this matrix is often \( \gamma^0 \) itself, but that is a representation-dependent fact. In our representation,

\[ C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{A.10} \]
Given the Dirac spinor \( q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \), the conjugate spinor is
\[
\bar{q} = q^\dagger C = \begin{pmatrix} q_2^\dagger \\ q_1^\dagger \end{pmatrix}.
\] (A.11)

### A.3 Free Fermions

The Lagrangian of a free Dirac fermion becomes
\[
L_D = \bar{q} \gamma^\mu \left[ -i \partial_\mu \right] q + m \bar{q} q \\
+ 2 \bar{q}_2 [-i \partial_x] q_2 + q_1^\dagger (-i \partial_x) q_1 - q_2^\dagger (-i \partial_x) q_2 + m [q_2^\dagger q_1 + q_1^\dagger q_2].
\]

We see that the field \( q_1 \) has no dynamical degrees of freedom: it has no time derivative in the action. Hence it can be eliminated by its equation of motion:
\[
q_1 = -\frac{m}{\hat{p}} q_2
\] (A.12)

where \( \hat{p} = -i \partial_x \). Putting this back into the action and changing variables
\[
\chi = \sqrt{2} q_2
\] (A.13)
gives us the effective action for the propagating field:
\[
L_\chi = \chi^\dagger (-i \partial_t) \chi - \chi^\dagger \frac{1}{2} [\hat{\rho} + \frac{m^2}{\hat{p}}] \chi.
\] (A.14)

### A.4 Gauge Fields

The lagrangian of two dimensional QCD is
\[
L = \frac{N}{4\alpha_1} \int \text{tr} F_{\mu \nu} F^{\mu \nu} + \sum_{a=1}^{N_f} \int \bar{q}^a \left[ -i \gamma \cdot \nabla + m_a \right] q_a.
\]

We have added in the flavor indices \( a, b \) and the color indices \( i, j \).

The freedom of gauge transformations can be utilized partially to impose the null gauge condition \( A_x = 0 \). Then, the fermionic part of the lagrangian becomes
\[
L_D = \bar{q}^a \gamma^\mu \left[ -i (\partial_\mu \delta^i_j + A^j_{\mu i}) \right] q_a + m \bar{q}^a q_a
\]
\[ + \chi^{i_1}[-i(\partial_t + A^{j}_{tt})]\chi_j + q^{j}_{1ai}(-i\partial_x)q_{1ai} - \frac{1}{2}\chi^{i_1}(-i\partial_x)\chi_{ai} + \frac{m}{\sqrt{2}}[\chi^{i_1}q_{1ai} + q^{j}_{1ai}\chi_{ai}]. \]

which becomes

\[ L_{\chi} = \chi^{i_1}(-i\partial_t)\chi_a - \chi^{i_1}\left\{\frac{1}{2}[\hat{\rho} + \frac{m^2}{\hat{p}}] - i\hat{A}_t\right\}\chi_a \tag{A.15} \]

upon eliminating \( q_1 \).

To this we must add the action of the Yang–Mills field itself, which looks quite simple in this co-ordinate system and gauge:

\[ L_{YM} = \frac{N}{4\alpha_1} \text{tr} F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma} = \frac{N}{2\alpha_1} \text{tr}[\partial_x A_t]^2. \tag{A.16} \]

The field \( A_t \) does not propagate and can be eliminated. Thus the action of two-dimensional QCD can be written entirely in terms of the field \( \chi \).

\section{A.5 The Dirac Vacuum}

Let us return to the free fermion theory with Lagrangian

\[ L = \chi^\dagger(-i\partial_t)\chi - \frac{1}{2}[\hat{\rho} + \frac{m^2}{\hat{p}}]\chi \tag{A.17} \]

Upon quantization, the field \( \chi \) becomes an operator satisfying the fermionic anti-commutation relations

\[ [\chi(x), \chi(y)]_+ = \delta(x - y), \quad [\chi(x), \chi(y)]_+ = 0. \tag{A.18} \]

If there were only a finite number of operators, such canonical anti-commutation relations would have a unique representation. In the infinite dimensional case physical ideas have to be brought in to choose the right representation. Dirac showed that the correct choice is to assume that all the negative energy states are filled even in the vacuum. Since energy and momentum have the same sign in our co-ordinate system, this condition is easy to implement. We define the vacuum by

\[ \tilde{\chi}^\dagger(p)|0> = 0 \text{ for } p < 0, \quad \tilde{\chi}(p)|0>= 0 \text{ for } p > 0. \tag{A.19} \]
We have defined the Fourier transforms

\[
\chi(x) = \int \tilde{\chi}(p)e^{ipx} \frac{dp}{2\pi}
\]  

(A.20)

e tc. Then the normal ordered product of a pair of operators is defined as

\[
:\chi^{\dagger}(p)\chi(p') : = \chi^{\dagger}(p)\chi(p')
\]  

(A.21)

unless both \( p \) and \( p' \) are negative, in which case it is

\[
:\chi^{\dagger}(p)\chi(p') : = -\chi(p')\chi^{\dagger}(p).
\]  

(A.22)

The point is that then the expectation value of normal ordered current operators are zero in the Dirac vacuum. Indeed,

\[
<0|\chi^{\dagger}(x)\chi(y)|0> = \int \frac{dp}{2\pi} \frac{dq}{2\pi} e^{-ipx+iqy} <0|\tilde{\chi}(p)\tilde{\chi}(q)|0>
\]

\[
= \int_{-\infty}^{0} \frac{dp}{2\pi} e^{ip(y-x)} = \frac{1}{2}[\delta(x-y) + \epsilon(x-y)].
\]

Here,

\[
\epsilon(x-y) = \int \text{sgn} (p)e^{ip(x-y)} \frac{dp}{2\pi} = \frac{1}{\pi i} \mathcal{P} \frac{1}{x-y}
\]  

(A.23)

is to be thought of as a distribution. It is (upto a factor of \( i \)) the kernel of a well-known integral transform, the Hilbert transform.

Thus we should regard the current operators of the fermionic theory as defined with the normal ordering. For example, the equation of motion of the \( A_t \) will be, in the quantum theory,

\[
-\partial_x^2 A_{ij}^a(x) = \frac{1}{N} \alpha_1 : \chi^{\dagger ai}(x)\chi_{aj}(x) :.
\]  

(A.24)

We can use this to eliminate the gauge field from the theory completely. Thus two dimensional QCD can be written as a theory of fermions interacting with each other through a Coulomb-like potential. The further analysis of this theory is carried out in the text, towards the end of the first chapter.
Appendix B

Operator Ideals

Here we give the basic definitions of the operator ideals we use in the text. A deeper discussion may be found in Ref. [27, 35, 36]; we give only a rough outline of the theory. It is the author’s fervent hope that experts in functional analysis will not read this appendix.

B.1 Compact and Hilbert-Schimdt Operators

The rank of an operator \( A : \mathcal{H} \rightarrow \mathcal{H} \) on a complex Hilbert space is the dimension of its range; i.e., the dimension of the subspace of all vectors that can be written as \( Au \) for some \( u \in \mathcal{H} \). When the rank of \( A \) is finite, it can be thought of as a sort of ‘rectangular matrix’ with (possibly) an infinite number of columns but only a finite number of rows; at least there is a basis in which it has this form.

For such an operator, we can define several measures of its size (norm). For example (the operator norm),

\[
|A| = \sup_u \frac{\|Au\|}{\|u\|}.
\]  

(B.1)

Another (the Hilber-Schmidt or H-S norm) is

\[
|A|_2 = \text{tr}[AA^\dagger]^\frac{1}{2}.
\]  

(B.2)

\( |A|_2 \) is also the sum of the absolute squares of all the matrix elements in any basis.
The completion of the space of finite rank operators in the operator norm is the space of compact operators. In other words, a compact operator is one that can be approximated arbitrarily closely by finite rank operators, distance between operators being measured with the operator norm. If we instead complete in the H-S norm, we get the space of Hilbert-Schmidt operators.

A bounded operator is one which has finite operator norm; i.e.,

$$|A| = \sup_{u \in \mathcal{H}} \frac{||Au||}{|u|}$$

exists. The set of bounded operators on $\mathcal{H}$ is an algebra $\mathcal{B}(\mathcal{H})$. Not all bounded operators are compact; for example the identity is bounded yet not compact.

A compact operator has the expansion, with $\mu_n > 0$,

$$A = \sum_{n=1}^\infty \mu_n \psi_n \langle \phi_n |$$

the sum being either finite (when $A$ is of finite rank) or infinite (more generally). The numbers $\mu_n$ are called singular values; if $A$ is positive they are its eigenvalues. $A$ and its adjoint $A^\dagger$ have the same singular values.

Roughly speaking, the operator norm of $A$ is the largest of its singular values, while the Hilbert-Schmidt norm is the sum of the squares of the singular values. The singular values of a compact operator form a sequence that converges to zero; in fact zero is the only limit point of the sequence. Thus, if the H-S norm is finite, the eigenvalues must be tend to zero. In fact all H-S operators are compact.

## B.2 Schatten Ideals

Many other norms can be defined in terms of the singular values. For example the trace class operators are those for which the sum of singular values is convergent. This is stronger than the requirement that the diagonal elements in some basis form a summable sequence: being trace-class requires a sort of absolute convergence. The product of two Hilbert-Schmidt operators in trace-class.

More generally, for $p \geq 1$, we have the class $\mathcal{I}_p(\mathcal{H})$ of operators for which the sum $\sum \mu_n^p$ converges. Of course $\mathcal{I}_2$ is the space of H-S operators.
and $\mathcal{I}_1$ that of trace class operators. The space of compact operators can be thought of as the limiting case $\mathcal{I}_\infty$ and that of finite rank operators as the opposite limit, $\mathcal{I}_0$. We have the inclusions

$$\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \cdots \mathcal{I}_\infty \subset \mathcal{B}. \quad (B.5)$$

It is very important for us that the $\mathcal{I}_p$ are two-sided ideals in the algebra of bounded operators; i.e., that $A \in \mathcal{B}$ and $B \in \mathcal{I}_p$ implies that both $AB \in \mathcal{I}_p$ and $BA \in \mathcal{I}_p$. These are called the Schatten ideals. We are especially interested, of course, in the cases $p = 1, 2$.

### B.3 The Restricted Unitary Group and its Grassmannian

In the text we are interested in the case of a Hilbert space $\mathcal{H}$ with a given orthogonal splitting into two infinite dimensional orthogonal subspaces: $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$. Recall that the operator $\epsilon$ is defined to have eigenvalues $\pm 1$ on $\mathcal{H}_\pm$. The restricted Unitary group is the subset of all unitaries satisfying a convergence condition:

$$U_1(\mathcal{H}, \epsilon) = \{ g | gg^\dagger = g^\dagger g = 1; [\epsilon, g] \in \mathcal{I}_2 \}. \quad (B.6)$$

It is vital for this definition to make sense that $\mathcal{I}_2$ is an ideal; that is why the product of two elements still satisfies the convergence condition. (Any unitary operator is bounded). If we split $g \in U_1(\mathcal{H}, \epsilon)$ into submatrices according to the splitting $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a : \mathcal{H}_- \to \mathcal{H}_-$ etc., the submatrices $b, c$ are H-S. The matrices $a, d$ may not be invertible in general, but they are Fredholm (see below) since $g^{-1}$ exists.

The restricted Grassmannian is the set of all self-adjoint operators of square one whose distance (in the H-S sense) from $\epsilon$ is finite.

$$Gr_1(\mathcal{H}, \epsilon) = \{ \Phi | \Phi^\dagger = \Phi; \Phi^2 = 1, \Phi - \epsilon \in \mathcal{I}_2 \}. \quad (B.7)$$

In the text we often use $M = \Phi - \epsilon$ as the variable that describes a point in the Grassmannian. We showed that each such operator corresponds to a subspace of $\mathcal{H}$ (the negative eigenspace of $\Phi$) which is at a finite distance from the standard subspace $\mathcal{H}_-$. 

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The fact that $I_2$ is an ideal ensures that the action of the restricted unitary group on this Grassmannian well-defined:

$$
\Phi \mapsto g\Phi g^\dagger, \quad M \mapsto gMg^\dagger + g[\epsilon, g^\dagger].
$$

(B.8)

## B.4 Fredholm Index

The material in this section is contained in Ref. [27] to which we refer for proofs and more precise statements. A bounded operator $A$ is **Fredholm** if it is invertible modulo a compact operator; i.e., if there is a compact operator $K$ such that $A + K$ has an inverse. The set of Fredholm operators is a topological space, with the topology induced by the operator norm. It is disconnected, each connected component being labelled by an integer called the Fredholm index.

To understand the Fredholm index, consider the kernel of an operator; i.e., the subspace of all $u$ such that $Au = 0$. Even in the finite dimensional case the kernel can change change discontinuously under a small change in $A$.; for example we may cross an eigenvalue. In the finite dimensional case, the dimensions of the kernels of $A$ and $A^\dagger$ are the same. In the case of Fredholm operators on infinite dimensional Hilbert spaces, the difference

$$
\text{index (}A\text{)} = \dim \ker A - \dim \ker A^\dagger
$$

(B.9)

is always finite but need not vanish. It however does not change under continuous changes in $A$; it is constant in each connected component of the space of Fredholm operators. In fact it is the only such function: any two Fredholm operators of the same index are connected by a continuous curve.

Now $g \in U_1(H, \epsilon)$ is of course Fredholm of index zero: it is invertible as is its adjoint. But the submatrices $a, d$ (where $a : H \to H$, etc.) defined above are not invertible in general. Yet they are invertible modulo some compact operators (products such as $bb^\dagger$), so they are Fredholm. Since $g$ as a whole invertible, they must have opposite Fredholm indexes. The index of $a$ (for example) is a topological invariant of $g$. The group $U_1$ is the union of connected components labelled by this index, which can take any integer value. The connected component of the identity, of course, has index zero.

We can understand the renormalized dimension of a subspace (called virtual rank in [27]) in terms of the Fredholm index. Every self-adjoint operator
can be diagonalized; an operator $\Phi \in Gr_1$ can be brought to the standard form $\epsilon$ by an element in $U_1: \Phi = g\epsilon g^\dagger$. The index of $a(g)$ is then the topological invariant associated to $\Phi$.

Another point of view is in terms of the subspace $W$ of $\mathcal{H}$ corresponding to $\Phi$. If $\Phi$ is at a finite distance from $\epsilon$, $W$ will not differ ‘too much’ from $\mathcal{H}_-$. More precisely, the projection operator $\pi_+: W \rightarrow \mathcal{H}_+$ will be compact and $\pi_-: W \rightarrow \mathcal{H}_-$ will be Fredholm. The index of $\pi_-$ measures the ‘difference in dimensions’ between $W$ and $\mathcal{H}_-$; this is the renormalized dimension of $W$. We saw in the text that this has the physical meaning of baryon number.

It is crucial for all this that we allow only subspaces at a finite distance from the standard one in $Gr_1$. If we had allowed for all subspaces, the Grassmannian would have been contractible. The convergence condition we must impose is required for the Poisson brackets to make sense: the symplectic form of the Grassmannian would not make sense otherwise. It is gratifying that as a consequence, we get a topological invariant which has the physical meaning of baryon number.

Although the symplectic form makes sense on all of the phase space $Gr_1$, we should not expect the Hamiltonian to make sense on all of it. There should be some dense domain in which the Hamiltonian does make sense however. We leave such questions as challenges for the analyst who is interested in solving problems of relevance to physics.
Appendix C

Quadrature of Singular Integrals

In this appendix we summarize some ideas on the numerical methods that are used in the chapter on solitons. Some originality is needed even in this part of the problem.

C.1 Quadrature Formulas

The equations we have are just too hard to be solved analytically. We have to resort to numerical methods. The basic idea is to convert the integral equation into a matrix equation by allowing \( p \) to take just a finite set of values: we must find a way to approximate the integral by a finite sum. Then we will solve the resulting nonlinear matrix equations by iteration.

Quadrature is the approximation of integrals by finite sums. There are standard methods for quadrature, (method of moments) going back to the days of Gauss. But we have to modify these methods since our integrals are singular. The basic idea is well-known in the literature on quadrature at least for the case of the Cauchy principal value [37]. Our integrals are one step harder (Hadamard Finite Part) but the idea is the same. See S. Chandrashekhar,’s classic book [38] for a clear discussion of numerical integration.

Let \( \rho(x) \) be a continuous positive function on the close interval \([a,b]\). We are interested in evaluating integrals such as

\[
\int_a^b f(x)\rho(x)dx
\]

(C.1)
by numerical approximations. Here \( f(x) \) is some continuous function.

Let \( x_j, j = 1, \ldots, \nu \) be a set of points in the interval \([a, b]\). We expect a weighted sum such as

\[
\sum_j w_j f(x_j)
\]

(C.2)

to be good approximation for the integral, provided that (i) the number points \( \nu \) is large enough and (ii) the points \( x_j \) are distributed roughly uniformly.

Every function can be approximated by a polynomial within the interval; as the order \( \nu - 1 \) of the polynomials grows the approximation gets better. We can thus approximate the integral of \( f(x) \) by that of its polynomial approximation of order \( \nu - 1 \). The weights \( w_j \) are determined (once the points \( x_j \) are chosen) by this requirement: the above formula should in particular be exact for polynomials of order \( \nu - 1 \).

This is the same as the condition

\[
\int_a^b x^k \rho(x) dx = \sum_j w_j x_j^k, \text{ for } k = 0, 1, \ldots, k - 1.
\]

(C.3)

The left hand side are the moments of the distribution \( \rho(x) dx \), and are assumed to be known. Then the above set of linear equations determine the weights in terms of \( x_j \).

If the function \( \rho(x) dx \) is not too rapidly varying, a simple choice such as equally spaced points \( x_j = a + (b - a) \frac{j-1}{\nu} \) should give reasonable approximation to the integral: certainly for polynomials up to order \( \nu - 1 \) we will get the exact answer anyway. But it is possible to do better by choosing the \( \nu \) points \( x_j \) cleverly, as pointed out by Gauss: we can ensure that the answer is exact for polynomials of order \( 2\nu - 1 \). But we wont be using this idea: we will just use equal spacing for the points, which turns out to be more convenient. This because our integrands involve terms such \( \tilde{\psi}(p + r) \), so it is convenient if the sum two points \( x_j + x_k \) is also a point at which we evaluate the integrand. The loss of precision in quadrature can be made up because the simplicity of equal spacing allows us to choose a larger number of points.
C.2 Singular Measures

Now consider integral \[ \mathcal{FP} \int_0^b f(x) \frac{dx}{x^2} \]. The symbol \( \mathcal{FP} \) indicates as before the Hadamard ‘finite part’ of the integral. For such singular integrals we can also find a numerical approximation as above. But the system of moments is no longer positive. This is because the integration measure is no longer positive: \( \mathcal{FP} \int_0^b f(x) \frac{dx}{x^2} \) can be negative even when \( f(x) \) is positive. The moments of the measure are given by

\[
\mathcal{FP} \int_0^b x^k \frac{dx}{x^2} = \frac{b^{k-1}}{k-1} \quad \text{for } k \neq 1 \quad (C.4)
\]

and

\[
\mathcal{FP} \int_0^b x^k \frac{dx}{x^2} = \log b \quad \text{for } k = 1. \quad (C.5)
\]

Note that the zeroth moment is negative. Also the second moment violates scale invariance and is negative if \( b < 1 \).

Given a system of points \( x_j \) we can approximate the singular integral by a sum

\[
\mathcal{FP} \int_0^b f(x) \frac{dx}{x^2} = \sum_{j=1}^\nu w_j f(x_j) \quad (C.6)
\]

where the weights are determined as above by solving the system

\[
\mu_k = \sum_{j=1}^\nu w_j x_j^k \quad \text{for } k = 0, \cdots \nu - 1. \quad (C.7)
\]

The choice of equally spaced points gives good answers in many cases.

These methods are used in the text (towards the end of the second chapter) to solve the integrals equations for the wavefunction of the baryon.
Bibliography

[1] J.D. Bjorken, *Partons*, Invited talk Presented at the Int. Conf. on Duality and Symmetry in Hadron Physics, Tel-Aviv, 1971

[2] R. P. Feynman, *Photon-Hadron Interactions* Benjamin, Reading (1972).

[3] T. H. R. Skyrme, Proc. Royal. Soc. Lond. A260,127(1961); Nucl. Phys. 31, 556(1961); J. Math. Phys. 12, 556(1962).

[4] A. P. Balachandran, V. P. Nair, S. G. Rajeev and A. Stern, Phys. Rev. Lett., 49,1124 (1982); Phys. Rev. D27,1153 (1983); S. G. Rajeev, Phys. Rev. D29, 2844 (1984).

[5] S. G. Rajeev, *Baryons as Solitons* Ph. D. Thesis Syracuse University (1984).

[6] A. P. Balachandran, in *Proceedings of the Yale Theoretical Advanced Study Institute, High Energy Physics 1985* ed. M. J. Bowick and F. Gursey, World Scientific, Singapore (1986).

[7] E. Witten, Nucl. Phys. B223, 422 (1983); B223, 433 (1983); G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. B228, 552 (1983).

[8] G. 't Hooft Nucl. Phys. B72, 461 (1974); B75 461 (1974).

[9] E. Witten, Nucl. Phys. B160, 57 (1979).

[10] S.G. Rajeev, ‘In 1991 Summer School on High Energy Physics and Cosmology’, ed. E. Gava, et. al. World Scientific, Singapore, (1992)

[11] P.F. Bedaque, I. Horvath, S.G. Rajeev, Mod.Phys.Lett.A7:3347-3356,1992 [hep-th/9209027]
[12] S.G. Rajeev, Int.J.Mod.Phys.A9:5583,1994;hep-th/9401115.

[13] K. S. Gupta and S. Guruswamy and S. G. Rajeev, Phys. Rev. D48, 3354 (1993); hep-ph/9301208.

[14] S.G. Rajeev and O.T. Turgut, Comm. Math. Phys. 192, 493-517, (1998); hep-th/9705103.

[15] G. Krishnaswami and S. G. Rajeev, Phys. Lett. 441, 429 (1998).

[16] V. John, G. Krishnaswami and S. G. Rajeev Derivation of the Anti-Quark Distribution function of the Proton from Quantum Chromodynamics, in preparation.

[17] C. W. H. Lee and S. G. Rajeev, Phys. Rev. Lett. 80, 2285-2288(1998); Nucl. Phys. B 529, 656,(1998); J. Math. Phys. 39, 5199 (1998); J. Math. Phys 40, 1870 (1999); Phys. Lett. B 436, 91(1998).

[18] S. J. Brodsky, hep-ph/9807212.

[19] R. Brock et al. [CTEQ Collaboration], “Handbook of perturbative QCD: Version 1.0,” Rev. Mod. Phys. 67, 157 (1995).

[20] G. 't Hooft, Unpublished lectures at the Les Houches Summer School (1973)

[21] D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973).

[22] D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

[23] D. Politzer, Phys. Rep. 14,129 (1974).

[24] A. H. Mueller, Phys. Rep. 73,237-368 (1981); J. C. Collins and D. E. Soper Nucl. Phys. B194 445 (1982)

[25] S. S. Chern, Complex Manifolds without Potential Theory, Springer-Verlag, New York (1979).

[26] J. Mickelsson, Current Algebras and Groups, Plenum (1989).

[27] A. Pressley and G. Segal, Loop Groups Clarendon Press, Oxford (1986).
[28] W. Hackbush, *Integral Equations: Theory and Numerical Treatment*
Birkhauser, Boston 1995. Verlag

[29] V. Kac and D. H. Peterson, Proc. Natl. Acad. Sci. USA 78, 3308 (1981).

[30] G. Segal, Comm. Math. Phys. 80, 301 9(1981).

[31] D. Pickrell, Journ. Funct. Anal. 70 (1987) 323.

[32] A. D. Martin, R. G. Roberts, W. J. Stirling and R. S. Thorne, hepph/9803445.

[33] M. Glück, E. Reya, Vogt, Z. Phys. C67(1995) 433.

[34] W. Kwong, J. L. Rosner and C. Quigg, Ann. Rev. Nucl. Part. Sci. 37, 325 (1987).

[35] I. C. Gohberg and M. G. Krein, *Introduction to the theory of Linear Non-Self Adjoint Operators*, Translations of the Amer. Math. Soc., 18 (1969).

[36] B. Simon, *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, (1979).

[37] P. J. Davis and P. Rabinowitz *Methods of Numerical Integration*, Academic Press, Orlando Florida (1984).

[38] S. Chandrashekhar, *Radiative Transfer* Dover, (1960).