Entropy, symmetry, and the difficulty of self-replication

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Received: 1 January 2022 / Accepted: 30 January 2022 / Published online: 19 March 2022
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Abstract
The defining property of an artificial physical self-replicating system, such as a self-replicating robot, is that it has the ability to make copies of itself from basic parts. Three questions that immediately arises in the study of such systems are: (1) How complex is the whole robot in comparison to each basic part? (2) How disordered can the parts be while having the robot successfully replicate? (3) What design principles can enable complex self-replicating systems to function in disordered environments generation after generation? Consequently, much of this article focuses on exploring different concepts of entropy as a measure of disorder, and how symmetries can help in reliable self replication, both at the level of assembly (by reducing the number of wrong ways that parts could be assembled), and also as a parity check when replicas manufacture parts generation after generation. The mathematics underpinning these principles that quantify artificial physical self-replicating systems are articulated here by integrating ideas from information theory, statistical mechanics, ergodic theory, group theory, and integral geometry.

Keywords Entropy · Group theory · Symmetry · Self-replication · Integral geometry

1 Introduction
Artificial self-replicating systems have the potential to greatly enhance mankind’s ability to harvest resources in outer space. The basic idea is that a functional factory consisting of robots, materials processing units, and manufacturing apparatuses are formulated to harvest materials and to build infrastructure from resources found in situ before sending humans into unstructured extraterrestrial environments. This concept is by no means new, and the history can be found in many prior technical works including [1–7]. The particular presentation in this paper is an embellishment of the author’s older works [8, 9].

The basic difficulty is that to close the self-replicating loop, there must be a balance between the overall system complexity and the ability of the system to handle the simplest inputs possible. Generally speaking, the more complex the system is, the more capable it is to assemble the simplest parts, or even harvest raw materials. But when the system is very complex, it then requires more to reproduce. For example, if a robot needs microprocessors and a 3D printing system needs a high-energy laser, then producing those from in situ resources becomes an enormous challenge. One way around that problem is to focus on production of mechanical parts in situ such as structural members for robots, factories, and habitats, and to send the relatively light-weight “vitamin” components such as computers, lasers, and some chemical reagents to be considered as inputs to the system rather than as items to be replicated. In this way, a practical version of in situ resource utilization (ISRU) can be achieved by reclassifying inputs. Moreover, developing systems that are dependent on some inputs that the systems cannot manufacture from scratch is both more akin to living systems that require specific nutrients, and is also safer than developing self-replicating systems that could continue to replicate without the imposition of resource bottlenecks. Without such intentional bottlenecks, there could be fears of things running amuck for generations.

The emphasis of this article is not to produce or review any particular system architecture. This has already been done elsewhere recently [7]. Instead, this article seeks to...
enumerate mathematical modeling tools that may be useful in assessing any self-replicating system. These tools can be applied to artificial and biological systems. In particular, concepts of entropy and complexity that enter in many fields can be used here. Entropy as a measure of disorder appears in two different forms in classical thermodynamics—as discrete sums over microstates and as integrals that approximate. In classical and quantum probability and in information theory, entropy plays a prominent role. The various concepts of entropy are relevant to artificial self-replicating systems but because of the interplay between the physical and informational. (It can be argued that information is physical, but there are some aspects of information that are independent of its particular physical implementation.)

The **degree of self replication** is a concept to measure how complex a self-replicating system is [6]. Degree of self replication is additive for modular parts but can be superadditive if synergy results from adding passive components to make machines. For example, assembling gears to form a transmission gives the resulting assemblage a functionality that transcends counting the number of parts that constitute it. There are many ways to define system complexity that can be used, such as algorithmic complexity theory, and these concepts are reviewed as well. (In the example of assembling a transmission, the number of steps required to do the assembly might be counted instead of the number of input parts.)

The theory of artificial self-replicating automata has a long history, and elements were touched on by Turing, Von Neumann, Shannon, Liang, Arbib, and Freitas, as reviewed in [7]. Self replication is also related to the theory of complexity as articulated in [10–14], the thermodynamics of computation [15, 16], and physical aspects of information theory [17–19]. Many of the landmark papers in this area have been collected in the books [20–22].

Most work in the area of artificial self-replicating systems is in the form of artificial life in silico. The challenges in a-life are somewhat different than those involved in the design of systems that must replicate in a material sense in the 3D physical world governed by mechanics and thermodynamics. This is the emphasis of the current work. Other theoretical aspects related to the foundations of self-replicating machines can be found in [23, 24].

The remainder of this paper is structured as follows. Section 2 reviews the concept of ‘degree of self replication.’ Sect. 3 explains why the concept of entropy is appropriate to describe the robustness of a self-replicating system in the presence of disorder, and quantitatively compares different concepts of entropy from different fields. Section 4 explains how symmetry, and the mathematical discipline of group theory, can be used to describe how design considerations can make the process of self replication easier. Section 5 explains how a formula from the field of integral geometry, the principal kinematic formula, can be used to describe how constraining the motion of parts reduces entropy to make assembly easier. Section 6 discusses how errors propagate over generations of self replication and how symmetry can be used as an error correction mechanism.

### 2 Degree of self-replication

A question that arises naturally in the context of artificial self-replicating systems is “In comparison to a biological systems, is the artificial system really self-replicating?” In other words, a biological system such as a bacterium takes in nutrients and produces a copy of itself. The issue at hand is that the nutrients appear to be invisible to the naked eye and then the replica pops into existence, as if by magic. However, in fact, the nutrients are themselves complex molecules such as amino acids, nucleic acids, lipids, sugars, and even trace metals such as iron, sodium, and potassium. And of course this all happens in the presence of abundant water, which is another resource. Without these essential ingredients a biological system would not replicate, since biological replication is essentially a biologically process.

In an artificial self-replicating system, if the number of input parts is small, and if each part is itself a complex machine, then the effort involved in the assembly step of self-replication is relatively low compared to the effort of making the input parts. The assembly task in self-replication is then relatively unchallenging in such a scenario. In contrast, if there are many simple (easy to manufacture) parts that need to be assembled by the robot, the relative difficulty is higher. The difference between these two scenarios can be quantified using the concept of the ‘degree of self replication’. The highest degree of self replication is when raw materials are harvested by a system and used to produce a replica, since fabrication and assembly are both done by the self-replicating system in that scenario.

That is, an artificial system that picks and places a few complex macroscopic modular units and assembles them into a functional copy seems to be far less impressive than its biological counterparts. But it has been argued previously by the author and coworkers that this is a difference in degree rather than type [6]. That is, a biological replicator has a high degree of self-replication, whereas current robotic self-replicating systems have a low degree. This can be quantified quite simply. If a system is modular, each module can be assigned a complexity value by counting its constituent parts (or the number of steps required to assemble the parts), and the ratio of complexity of the overall system to the complexity of each constituent part can be computed. For example, the complexity of a bacterium may consist of millions of individual amino acids, lipids, and nucleotide bases. If these are considered as the
fundamental building blocks, then the degree of self replication might be millions.

In contrast, an artificial self-replicating system consisting of modules composed of motors, microprocessors, and sensors, each of which was built in a large factory, and meticulously assembled by a skilled worker before the robot does a simple pick and place operation seems to pale in comparison. This is reflected in the degree of self replication

\[
\text{DOSR} = \frac{\text{System Complexity}}{\text{Part Complexity}}
\]

(1)

For the biological system, this might be millions. For the artificial physical replicators to date this might be tens or hundreds. Therefore, there are several orders of magnitude difference reflected in this simple concept. Eventually, if an artificial replicator could scoop up sand and other raw materials, and cast parts in molds or 3D print, the degree of self replication for artificial systems could rival those of biological replicators.

Of course, there are subtleties regarding how to measure complexities in the simple formula above. For example, in an amino acid, one could count atomic nuclei. But for glycine, there are far fewer atoms than in the amino acids lysine or tryptophan. Then, there issues related to metabolism and synergy. Assembly of amino acids to form a polypeptide in a ribosome requires metabolic energy, and perhaps that is a better measure of the effort involved than the geometric arrangement of atomic nuclei since nature produces amino acids from nonbiological processes for use in biological systems. In contrast, in an artificial self-replicating system, any modular component containing a microprocessor represents a very high concentration of complexity (as measured either in terms of features, or manufacturing effort, or effort to build the microprocessor factory) prior to the assembly process, thereby limiting the degree of self replication.

When there is a wide range of input part complexities, it makes sense to take the ratio of the overall system complexity to the most complicated input part rather than an average complexity. Again, the issue of how to measure complexity is a whole other issue. Several options are: (1) to count the number of solid nonarticulated parts that constitute an input object or subsystem, perhaps weighted by the geometric complexity of the part; (2) to count the number of operations or total effort/energy required to produce an input object; (3) to use algorithmic complexity theory to count the length of the computer program used to produce the input parts (if they are produced by an automated fabrication system). When measuring the complexity of an assemblage of parts, there are also several ways. For example, one could simply sum up the complexities of individual parts. But this may underestimate the overall system complexity resulting from synergetic interactions of parts and the effort involved in achieving this synergy. For example, assembling gears to form a transmission might require precise operations requiring substantial effort, beyond counting the number of features of the part, such as gear teeth. And the overall functionality of the resulting assemblage may not be reflected by simply summing the complexity of the constituent parts. Therefore it might be more appropriate to consider the algorithmic complexity of the computer programs used to create parts and to assemble them. Or, one could count the total energy required to produce input parts from the stage of raw materials, and to then count the energy required to assemble the parts. Regardless of how complexity is defined, (1) can be used to measure the degree of self replication in the sense defined by each concept of complexity.

3 Entropy

Another dimension to the difficulty of self-replication is how ordered the input parts are. If they are arranged precisely in an array and the self-replicating system need only pick and place, this is less challenging than if the robot must identify, manipulate, and assemble randomly oriented parts. Moreover, if the parts have symmetries, this reduces the number of wrong ways that they can be assembled. This aspect of the difficulty of self replication can be quantified by computing entropy. Many different concepts of entropy exist from statistical mechanics, information theory, and quantum probability. Since a self-replicating system intertwines physicality and information processing, concepts from these different fields become relevant. For example, an unordered array of manufactured parts in a bowl can be described using the configurational entropy of parts. The entropy of a mixture of raw materials that must be separated, refined, and then used to produce finished parts is more related to thermodynamic and ergodic-theoretic entropy.

During a self-replication process, parts are assembled to create the replica. If \( N \) parts can exist in many different initial positions and orientations, then the set of all possible initial arrangements can be described as a probability density. When the parts are assembled properly, the disorder will decrease. Entropy is a measure of how disordered the initial and final states are.

Many different concepts of entropy exist. Two broad categories of entropy concepts are continuous and discrete. The distinction between the two can be blurred by realizing that when the problems at hand have a natural length scale, features smaller than a particular limit are not measurable. Therefore, whether one creates bins to discretize at this scale, or calls the problem continuous does not matter, because
entropy differences are essentially the same whether computed using continuous or discrete formulae.

Aside from the continuous/discrete characterization, there are distinct concepts of entropy in several different fields including information theory, statistical mechanics, ergodic/measure theory, and quantum probability. These various concepts are reviewed here with an eye towards their relevance to artificial self-replicating systems.

The various concepts of entropy examined herein share a number of common features: Entropy is reduced by environmental obstacles and containment, as well as fences, and by considering symmetry. In all cases conditioning reduces entropy. Continuous and discrete entropy differences can be equated if bin size is below feature resolution.

Both concepts (entropy and degree of self-replication) are influenced by symmetry. Symmetry reduces the entropy difference between an initial disordered ensemble and the final assemblage, making assembly tasks easier. Coset and double coset space decompositions and associated concepts of fundamental domains quantify this idea. When the order of operations is not critical, this also makes assembly tasks easier.

The remainder of this section is structured as follows. In Subsection 3.1, the basic definitions and concepts of discrete and quantum entropy are reviewed. In Sect. 3.2 continuous entropy on a measurable space is defined, and its relationship to discrete entropy is explained. In Subsection 3.3 the properties of entropy differences and relative entropy are reviewed.

### 3.1 Discrete and Quantum Entropy

An event is a discrete occurrence of some kind. For example, it can be the appearance of a printed symbol on a page, a sequence of dots and dashes in a telegraph message, a string of binary numbers stored in a computer memory, the process of pouring a discrete quantum of molten material into a mold to form a mechanical part, or the fastening of two mechanical parts in a mechanized assembly procedure. The concept of an event is extremely general.

Let \( E_i \) denote such an event and consider a finite collection of events \( C = \{E_1, E_2, ..., E_n\} \). Associated with each event assign a probability \( p_i \equiv p(E_i) \geq 0 \) and \( \sum_{i=1}^{n} p_i = 1 \). Then the self-information of each such event is

\[
I(E_i) = -\log p(E_i). \tag{2}
\]

The base of the logarithm is largely irrelevant for our purposes. The choice of base is essentially a choice of units, similar to how length can be measured in feet or meters.

The concept of \( I(E_i) \) was developed when information fed into ticker tapes or punch cards were described by binary punches in paper, wherein \( I(E_i) \) describes the amount of tape or cards to encode a message, and this concept carried over to the digital computer. In such concepts, it is, therefore, natural to use a logarithm of base 2, and to measure information in ‘bits’. In other contexts, it is more natural to use base e, and measure information in ‘nats’.

The primary purpose of the development of information theory was to maximize the amount of information that could reliably be transmitted from one time and place to another. This involves the efficient packaging of information with codes that perform self-checks to detect and correct errors. Information and its transmission has many manifestations, including the genetic code and DNA replication and transcription. In the context of self-replicating robotic systems, the spatial arrangement of parts and assembly operations required to produce a replica can be considered as messages that are transmitted. The design of physical means for error correction while efficiently replicating are then analogous to codes that protect data transmission at the highest rates possible.

The entropy of the discrete collection \( C \) is defined to be the average self-information of all events in \( C \):

\[
H(C) = \sum_{i=1}^{n} p(E_i) I(E_i) = -\sum_{i=1}^{n} p(E_i) \log p(E_i). \tag{3}
\]

Sometimes, this is denoted as a functional of the probabilities instead of the underlying events. That is, if \( p = \{p_1, p_2, ..., p_n\} \) is the collection of probabilities, then

\[
S(p) = -\sum_{i=1}^{n} p_i \log p_i. \tag{4}
\]

Obviously, \( S(p) = H(C) \). The difference in notation only reflects a difference in emphasis regarding the meaning of entropy.

The above concepts have analogs in quantum probability. In the simplest case, if a diagonal matrix \( P \) is defined with the values \( \{p_i\} \) on its diagonal, then the (von Neumann) quantum entropy

\[
S(P) = -\text{tr}(P \log P) \tag{5}
\]

Where \( \log P \) is the matrix logarithm and the terms inside of the trace is the product of two matrices. Whereas the above gives exactly the same result as the discrete entropy \( S(p) \), it is more general because \( P \) can be taken to be any symmetric real matrix (or even a complex Hermitian one) with the condition that \( \text{tr}(P) = 1 \). Hermitian matrices of this sort arise in quantum mechanics, which is the domain in which von Neumann defined quantum entropy. So far, a connection between self-replicating systems and quantum entropy has not been explored, and the concept is included here only for completeness.
3.2 Entropy of probability density functions on continuous measure spaces

A probability density function (pdf) on a measure space \((X, \mu)\) is a function \(f : X \rightarrow \mathbb{R}_{\geq 0}\) such that
\[
\int_X f(x) \, d\mu(x) = 1.
\]
The entropy of \(f\) is defined as
\[
S(f) = -\int f(x) \log f(x) \, d\mu(x). \tag{6}
\]
This is the continuous analogue of (4). Again, the choice of base of the logarithm amounts to a choice of measurement units for \(S\). Throughout this section, \(\log \triangleq \log_e = \ln\).

In the case when \(q\) parameterizes the group of rotations or the Euclidean motion group, then the entropy of a pdf \(f : G \rightarrow \mathbb{R}_{\geq 0}\) is
\[
S_G(f) = -\int_G f(g) \log f(g) \, dg \tag{7}
\]
where the natural bi-invariant measure \(dg\) takes on different appearances under changes of parametrization but the value of the integral is independent of parametrization and it is invariant under shifts of the form \(g \rightarrow g_0g\) and \(g \rightarrow g g_0\).

More than 35 years ago, A. Sanderson quantified the concept of part disorder by defining the concept of “parts entropy” \([25]\), which motivates the definition in (7).

3.2.1 Statistical mechanics, configurational entropy, and discretization

In statistical mechanics, the continuous measure space is \(2n\)-dimensional where \(n\) is the number of degrees of freedom of the physical system under consideration. This “phase space” has a volume element of the form \(d\mu(x) = dpdq|q\) where \(q\) denotes a set of generalized coordinates and \(p\) denotes the corresponding conjugate momenta. This form is invariant under changes of coordinates. In statistical mechanics, entropy is computed from Gibbs’ formula
\[
S_B = -k_B \int_Q \int_P f(p, q, t) \log f(p, q, t) \, dp \, dq \tag{8}
\]
where \(q \in Q \subseteq \mathbb{R}^n\) is a set of coordinates, and \(p \in \mathbb{R}^n\) are the corresponding conjugate momenta. The Lebesgue measure on the \(2n\)-dimensional phase space, \(dpdq|q\) is invariant under coordinate changes, though such changes do affect the bounds of integration \(Q\), unless \(Q = \mathbb{R}^n\). The formula in (8) can be considered as a special case of (6), to within the constant \(k_B\).

For a statistical mechanical system at equilibrium, the temporal dependence of \(f\) vanishes and \(f\) converges to the Boltzmann distribution
\[
f_B(q, p) = \frac{1}{Z} \exp(-\beta V(q)) \tag{9}
\]
where \(\beta = 1/k_B T\) with \(k_B\) denoting Boltzmann’s constant and \(T\) is temperature measured in degrees Kelvin. Here the Hamiltonian of a mechanical system is defined as the total system energy written in terms the conjugate momenta \(p\) and generalized coordinates \(q\):
\[
H(p, q) = \frac{1}{2}p^T M^{-1}(q) p + V(q). \tag{10}
\]
Here, \(M(q)\) is the configuration-dependent mass matrix and \(p \equiv M(q) \dot{q}\).

For recent results regarding the rate of approach to equilibrium, see [26].

But since a Gaussian integral has closed form, the marginal distribution resulting by computing the integral over \(p\) gives
\[
\int_P f_B(q, p) \, dp = C \exp(-\beta V(q)) |M(q)|^{\frac{1}{2}} \tag{11}
\]
where \(|M(q)|\) denotes the determinant of \(M(q)\) and \(C\) is the normalizing constant that makes the above expression a probability density with respect to the measure \(dq\). It is natural to re-interpret this as
\[
f_C(q) = C \exp(-\beta V(q))\]
being the pdf with respect to the measure \(|M(q)|^{\frac{1}{2}} dq\). This is the configurational Boltzmann distribution. The associated configurational entropy is defined as
\[
S_C = -\int_Q f_C(q) \log f_C(q) |M(q)|^{\frac{1}{2}} dq. \tag{12}
\]
As an example, given a rotating body with orientation parameterized by \(ZXZ\) Euler angles, \(q = [\alpha, \beta, \gamma]^T\), it can be shown that \(|M(q)|^{\frac{1}{2}} = |I|^{\frac{1}{2}} \sin \beta\) where \(I\) is the moment of inertia tensor. The measure \(\beta d\alpha d\beta d\gamma\) in fact the bi-invariant integration measure for the rotation group \(SO(3)\), and so (12) can be viewed as a specific coordinate-dependent version of (7).

3.2.2 Measure-theoretic information and entropy

Concepts used in information theory and statistical mechanics have been modified for use in the ergodic theory of deterministic dynamical systems. In this section, the concept
of measure-theoretic information, as it appears in ergodic theory, is reviewed.

Given any compact space on which a measure can be defined (for the sake of concreteness, think of a compact manifold with associated volume element), it is possible to partition that space into a finite number of disjoint subsets, the union of which is, to within a set of measure zero, the whole space. That is, given a measurable space (e.g., a compact manifold), \( M \), a partition \( \alpha = \{ A_i \} \) is defined such that

\[
A_{i_1} \cap A_{i_2} = \emptyset \quad \text{if} \quad i_1 \neq i_2 \quad \text{and} \quad \bigcup_{i \in I} A_i = M.
\]

The set indicator function,

\[
I_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \not\in A
\end{cases},
\]

(13)

together with the partition \( \alpha \) can be used to define measure-theoretic information as

\[
I_\alpha(x) = - \sum_{A \in \alpha} I_A(x) \log V(A)
\]

(14)

where \( V(A) \) is the volume of \( A \) (or, more generally, the measure of \( A \)) normalized by the volume of the whole space, \( M \). \( I_\alpha(x) \) reflects the amount of “information” that results from discovering that \( x \in A \). If \( A \) is a very large region then little information is gained by knowing that \( x \in A \). Measure-preserving actions of Lie groups such as rotations and translations do not affect this quantity. If \( \alpha \) is a very fine partition, each subset of which has roughly the same volume, then more information is obtained from \( I_\alpha(x) \) than if the partition is coarse. Sometimes, it is convenient to raise the subscript and write \( I(\alpha)(x) \) in place of \( I_\alpha(x) \).

Measure-theoretic entropy of a partition is defined as

\[
H(\alpha) = \sum_{A \in \alpha} z(V(A)).
\]

(15)

where

\[
z(\phi) = \begin{cases} 
-\phi \log \phi & \text{if } 0 < \phi \leq 1 \\
0 & \text{if } \phi = 0
\end{cases}.
\]

(16)

\( H(\alpha) \) is related to \( I(\alpha)(x) \) through the equality

\[
H(\alpha) = \int_M I(\alpha)(x) \, dx.
\]

Concepts of conditional information and entropy have been articulated previously in this measure-theoretic context, as reviewed in [30]. Even more sophisticated concepts of entropy that are used in the description of dynamical systems have been built on this concept, including the Kolmogorov-Sinai entropy [44] and the topological entropy of Adler, Konheim and McAndrew [31]. See, for example, [36] for more details. For further reading on general ergodic theory, see [32–34, 36, 39–41, 43, 46]. For works that emphasize the connection between ergodic theory and group theory, see [35, 37, 38, 42, 45, 47]

### 3.3 Entropy differences, relative entropy, and conditional entropy

Regardless of the type of entropy chosen as a descriptive tool for modeling disorder, when it comes to using entropy in the analysis of physical self-replicating systems, it is the comparison of entropy in the unassembled state and the assembled state that is important. For example, if a single planar part is located at random in a box at random orientation, its continuous entropy may simply be the log of the volume of the space of all possible motions (translations and rotations), with translations limited by the range of allowable motions of the center of the part. If in the assembly process the part is placed at a specific location, then the entropy difference is a measure of how challenging the task is. For example, if the placement has some tolerance, then it is the log of the volume of space of motions tolerated in the final part location that defines its entropy. The difference in entropy between ordered and disordered state is then the relevant measure of the difficulty of assembly, rather than the total entropy.

Discrete entropy is always nonnegative, whereas continuous entropy is not bounded from below. However, when considering entropy differences, the line between discrete and continuous entropy can be blurred. For example, discretizing the space of motions of a part at a length scale smaller than the tolerance of the part placement in the final assemblage will mean that the difference of continuous entropies and the difference of discretized versions of continuous entropies (both in the disordered and assembled state) obtained by integrating over sufficiently small bins will be negligible. That is

\[
\Delta S_{\text{cont}} \approx \Delta S_{\text{disc}} \geq 0
\]

(17)

for appropriate discretization.

A concept from information theory that is useful in proving theorems about the properties of entropy is that of relative entropy (or Kullback–Leibler divergence). This can be defined either in the discrete or continuous case. For the continuous case, it is

\[
D_{KL}(p \parallel q) = \int x \log \left( \frac{p(x)}{q(x)} \right) \, d\mu(x).
\]

(18)
In the discrete case, the integral is replaced by a sum and the integration measure disappears. Though $D_{KL}(p \parallel q) \neq S(q) - S(p)$, it should be noted that $D_{KL}(p \parallel q) \geq 0$ in both the discrete and continuous cases.

Finally, it should be mentioned that the concept of conditional entropy has a role in characterizing how sensory information can be used to reduce uncertainty during robotic assembly, including in self-replication processes.

In the context of a measure space where $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$, the conditional probability density is defined as

$$p(x_1 \mid x_2) = \frac{p(x_1, x_2)}{p(x_2)},$$

or in the event-based notation used earlier,

$$p(E_1 \mid E_2) = \frac{p(E_1 \cap E_2)}{p(E_2)}.$$  
This is a pdf in the variable $x_1$. If $p$ denotes $p(x_1, x_2)$ and $q_2$ denotes the conditional $p(x_1 \mid x_2)$, then the conditional entropy is defined as

$$S(p, q_2) = \int p(x_1, x_2) \log p(x_1 \mid x_2) dx.$$ It is related to entropy difference as

$$S(p, q_2) - S(p, q_1) = S(p_1) - S(p_2)$$

where $p_1 = p(x_1)$ is the marginal wherein all other degrees of freedom are integrated out.

Conditioning reduces entropy, and the concept of conditional entropy plays a role when information is gained by sensing, as described in [25].

### 4 Symmetry and entropy

This section focuses on the computation of entropy when there is a relationship between a probability density and a symmetry group. Such a relationship can exist when the space over which the probability density is defined is a group, or if the symmetry group acts on a space and the pdf is invariant under the action. To understand what this means, the first subsection reviews basic concepts in group theory.

#### 4.1 Symmetry of parts and assemblages

Let $A$ be an assemblage of a constituent set of parts $\{B_i\}$. For example, the assemblage could be a self-replicating robot. The assemblage $A$ always will be of the form

$$A = \bigcup_i r_i B_i$$

where $r_i \in SE(n)$ are the rigid-body transformations that describe how the parts are assembled. Each part might have discrete or continuous rotational symmetries, and likewise for the assemblage as a whole. For example, if the part or assemblage is a cube, there are 24 symmetry operations. Or if it is a cone, it has a one degree-of-freedom continuous group of rotational symmetries. In the natural world, many viruses are assemblages of protein parts that have icosahedral rotational symmetry with 60 elements, and the individual protein parts usually lack symmetry.

If $H$ denotes the symmetry group of $A$ then for every $\mathbf{x} \in A$ and $h \in H$, it is the case that $h \cdot \mathbf{x} \in A$ and as a whole we write $h \cdot A = A \forall h \in H$.

Similarly, if each $B_i$ has a symmetry group $K_i$ then

$$k \cdot B_i = B_i \forall k \in K_i.$$ Consequently, if each $r_i$ in (19) is replaced with $r_i' = hr_i k$, we see that

$$\bigcup_i r_i' B_i = h \bigcup_i r_i k B_i = h \bigcup_i r_i B_i = h \cdot A = A.$$ In other words, when there are symmetries many ways exist to form an assemblage, and this makes the job easier.

The space of possible spatial relationships of $N$ parts prior to the formation of the assemblage is the direct product group $G = SE(n)^N$. The distribution of parts can be described by the joint distribution $f(g_1, g_2, \ldots, g_N)$. When the parts are far from each other, this can be approximated as a product of independent pdfs $f_i(g_i)$. But when they are being assembled, this is not true. Regardless, what is true is that $f$ will inherit the symmetries of $A$ and each $B_i$, and this will impact the overall parts entropy. This observation is quantified in the following sections.

#### 4.2 Groups and coset spaces

Group theory is one of the foundational pillars of abstract algebra, and therefore of all of modern mathematics. The concept of a group is simple. A group is a set, $G$, together with an operation, $\circ$, such that four properties hold: (1) for every pair $g_1, g_2 \in G$, the product $g_1g_2 \in G$; (2) a special element $e \in G$ exists such that $goe = eog = g$ for every $g \in G$; (3) For each $g \in G$, there exists an element $g^{-1} \in G$ such that $gog^{-1} = g^{-1}og = e$; (4) For any three elements $g_1, g_2, g_3 \in G$ the associative property $(g_1g_2)g_3 = g_1(g_2g_3)$ holds. The group is then referred to as $(G, \circ)$, or when $\circ$ is understood,
then the group is denoted simply as $G$ and the product of $g_1$ and $g_2$ is $g_1 g_2$ rather than $g_1 \circ g_2$.

The set $G$ can be discrete like the integers, or it can be a continuous (measurable) space. If it is discrete, it can have either a finite or infinite number of elements. If it is continuous, it can have finite or infinite volume, like the group of rotations or translations of Euclidean space, respectively. The continuous groups with added conditions on the analyticity of the group product and inversion defines Lie groups.

In practice in engineering, most groups encountered are either finite-dimensional matrix Lie groups or their discrete subgroups. For example, the group of translations of the real line $(\mathbb{R}, +)$ can be described by $2 \times 2$ matrices and the group product can be described as multiplication as

\[
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + y \\ 0 & 1 \end{pmatrix}.
\]

A subgroup is a set inside of a group which contains the identity and is closed under: (1) multiplication of its elements; and (2) inversion of elements. The associative property holds automatically. For example, the integers are a subset of the group of real numbers under addition $(\mathbb{Z}, +) < (\mathbb{R}, +)$. (The notation $<$ is used to denote subgroup.)

In general, if $H < G$, then $G$ can be partitioned into left or right cosets. For any given $g \in G$, the left coset containing $G$ is defined as

\[ gH = \{ gh \mid h \in H \} \]

and a right coset is defined as

\[ Hg = \{ hg \mid h \in H \}. \]

In general, $gH \neq H g$, but equality can hold in special cases. It is possible for two different elements $g_1, g_2 \in G$ to generate the same coset, e.g., $g_1 H = g_2 H$. If all distinct cosets are collected, the result is called a coset space. The left coset space is

\[ G/H = \{ gH \mid g \in G \}. \]

If $H$ and $G$ have a finite number of elements, then the number of elements in $G/H$ is $|G|/|H|$. This is Lagrange's theorem, which has been known for more than 200 years. When $G$ is a Lie group and $H$ is a discrete subgroup, the resulting coset space will have the same dimension as $G$, otherwise the coset space will have dimension that is the difference in the dimension of $G$ and $H$.

### 4.3 Lie groups

Lie groups are groups wherein the set $G$ is a differentiable manifold and the operations of group multiplication and inversion are analytic. The group $(\mathbb{R}, +)$ is an example of a Lie group.

The group of rigid-body displacements in the Euclidean plane, $SE(2)$, can be described with elements of the form

\[
g(x, y, \theta) = \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix}.
\]  

(20)

The dimension is 3, because there are three free parameters $(x, y, \theta)$. This group is not compact as $x$ and $y$ can take values on the real line.

The group of pure rotations in 3D can be described by rotation matrices

\[
SO(3) \doteq \{ R \in \mathbb{R}^{3 \times 3} \mid R R^T = I, \, \det R = +1 \}.
\]

$SO(3)$ is a compact 3-dimensional manifold. Again, the fact that the dimension of the matrices is also 3 is coincidental.

A unimodular Lie group is defined by the property that a measure $dg$ can be constructed such that the integral over the group has the property that

\[
\int_G f(g) \, dg = \int_G f(g_0 g) \, dg = \int_G f(g g_0) \, dg
\]  

(21)

for any fixed $g_0 \in G$ and any function $f \in L^1(G)$. It can also be shown that as a consequence of (21)

\[
\int_G f(g) \, dg = \int_G f(g^{-1}) \, dg.
\]  

(22)

These properties are natural generalizations of those familiar to us for functions on Euclidean space.

As we are primarily concerned with probability density functions for which

\[
\int_G f(g) \, dg = 1,
\]

these clearly meet the condition of being in $L^1(G)$.

In the case of $SO(3)$, the bi-invariant measure expressed in terms of $Z = X - Z$ Euler angles $(\alpha, \beta, \gamma)$ is $dR = \sin \beta d\alpha d\beta d\gamma$. In the case of $SE(2)$, the bi-invariant measure is $dg = dx dy d\theta$.

The convolution of probability density functions on a unimodular Lie group is a natural operation, and is defined as

\[
(f_1 \ast f_2)(g) \doteq \int_G f_1(h) f_2(h^{-1} g) \, dh.
\]  

(23)
The convolution of two probability density functions is again a probability density.

4.4 Entropy and group-theoretic decompositions

Aside from the ability to sustain the concept of convolution, one of the fundamental ways that groups resemble Euclidean space is the way in which they can be decomposed. In analogy with the way that an integral over a vector-valued function with argument \( x \in \mathbb{R}^n \) can be decomposed into integrals over each coordinate, integrals over Lie groups can also be decomposed in natural ways. This has implications with regard to inequalities involving the entropy of pdfs on Lie groups. Analogous expressions hold for finite groups, with volume replaced by the number of group elements.

4.4.1 Decomposition of integrals over subgroups and coset spaces

Much like the way an integral over \( \mathbb{R}^2 \) can be partitioned into integrals over two copies of \( \mathbb{R} \), integrals of functions on groups can be partitioned into integrals over subgroups and corresponding coset spaces. In particular, it is possible to retain one representative element of each coset \( G/H \) to form a fundamental domain \( F_{G/H} \subset G \). Then, the integral of any measurable function can be decomposed as [27]

\[
\int f \, dg = \int_{F_{G/H}} \left( \int_{H} f(g \circ h) \, dh \right) \, d(g)
\]

where \( d(g) \) and \( dh \) are the invariant integration measures on \( F_{G/H} \) and \( H \). If \( H \) is a discrete subgroup, then the integral over \( H \) becomes a summation and \( F_{G/H} \) can be taken to be a region with the same dimension as \( G \) and \( d(g) = dg \). If \( H \) is a Lie subgroup of Lie group \( G \), then \( F_{G/H} \) is a lower dimensional subset of \( G \) and the interpretation of the integral with respect to \( d(g) \) is that it is \( dg \) restricted to a lower dimensional space, much like how the integration measure for a surface in Euclidean space is induced from the ambient Lebesgue measure.

Given a probability density function \( f_G : G \rightarrow \mathbb{R}_{\geq 0} \), which by definition satisfies the condition

\[
\int_G f_G(g) \, dg = 1,
\]

we can define

\[
f_{G/H}(gH) = \int_H f_G(g \circ h) \, dh \quad \text{and} \quad f_H(h) = \int_{F_{G/H}} f_G(g \circ h) \, dg,
\]

where \( h \in H \).

The coset space \( G/H \) is not a subset of \( G \), but there is a natural map from \( G \) to \( G/H \) defined by \( g \rightarrow gH \). The integration measure on \( G/H \) is defined by the equality

\[
\int_{G/H} f_{G/H}(gH) \, d(gH) = \int_{F_{G/H}} f_{G/H}(gH) \, dg.
\]

Then, \( F_{G/H} \) and \( f_H \) are bona fide probability density functions, because

\[
\int_{G/H} f_G(g) \, dg = \int_{F_{G/H}} \left( \int_{H} f_G(g \circ h) \, dh \right) \, d(gH) = \int_{G/H} f_{G/H}(gH) \, d(gH)
\]

and similarly

\[
\int_{G/H} f_G(g) \, dg = \int_{F_{G/H}} \left( \int_{H} f_G(g \circ h) \, dh \right) \, d(gH) = \int_{H} f_H(h) \, dh.
\]

The same construction can be made for right cosets. Moreover, if we are given a unimodular group \( G \) with unimodular subgroups \( K \) and \( H \) such that \( K \leq H \), we can use the facts that

\[
\int_{G/H} f_G(g) \, dg = \int_{G/H} \int_{H} f_G(g \circ h) \, dh \, d(gH),
\]

and

\[
\int_{H/K} f_H(h) \, dh = \int_{H/K} \int_{K} f_H(h \circ k) \, dk \, d(hK)
\]

to decompose the integral of any nice function \( f(g) \) as

\[
\int f_G(g) \, dg = \int_{G/H} \int_{H/K} \int_{K} f_G(g \circ h \circ k) \, d(hK) \, d(gH).
\]

The integral of a function on a group can also be decomposed in terms of two arbitrary subgroups and a double coset space as

\[
\int f_G(g) \, dg = \int_{K \backslash G/H} \int_{H} f_G(g \circ k) \, d(KgH) \, dk.
\]

This can be realized using the concept of fundamental domains in \( G \) constructed from one representative element of \( G \) per double coset. For concrete examples of fundamental domains of coset and double-coset spaces where \( G = SO(3) \) and \( H \) and \( K \) are finite subgroups, see [28].

4.4.2 Entropy and group-theoretic decompositions

Here some theorems related to entropy and the decomposition of integrals on groups originally derived in [29, 30] are reviewed. These will be relevant later when considering
entropy differences between disordered and assembled states of parts with symmetry.

**Theorem 4.1** The entropy of a pdf on a unimodular Lie group is no greater than the sum of the marginal entropies on a subgroup and the corresponding coset space:

\[ S(f_G) \leq S(f_{G/H}) + S(f_H). \]  

**Proof** This inequality follows immediately from the nonnegativity of the Kullback-Leibler divergence

\[ D_{KL}(f_G \parallel f_{G/H} \cdot f_H) \geq 0. \]

\[ (29) \]

For example, if \( G = SE(n) \) is a motion group of \( n \)-dimensional Euclidean space consisting of rotation-translation pairs of the form \((R, t)\) and group law

\[(R_1, t_1) \circ (R_2, t_2) = (R_1R_2, R_1t_2 + t_1)\]

and if \( H \cong \mathbb{R}^n \) is the subgroup of pure translations of the form \((l, t)\) in \( n \)-dimensional Euclidean space, then \( G/H \cong SO(n) \) consisting of all elements of the form \((R, 0)\), and an arbitrary element of \( SE(n) \) is written as a pair \((R, t) \in SO(n) \times \mathbb{R}^n\), then \( SO(n) \cong SE(n)/\mathbb{R}^n \) and we can write

\[
\int_{SE(n)} f(g) \, dg = \int_{SO(n)} \left( \int_{\mathbb{R}^n} f((l, t) \circ (R, 0)) \, dt \right) \, dR,
\]

and the marginal entropies on the right-hand-side of \((29)\) are those computed for pure rotations and pure translations.

**Theorem 4.2** The entropy of a pdf on a group is no greater than the sum of marginal entropies over any two subgroups and the corresponding double-coset space:

\[ S(f_G) \leq S(f_K) + S(f_{K \backslash G/H}) + S(f_H). \]  

**Proof** Let

\[
f_K(k) = \int_{K \backslash G/H} \int_{KgH} f_G(k_0c_{K \backslash G/H}(KgH)\circ h) \, dh \, d(KgH)
\]

\[
f_H(h) = \int_{K \backslash G/H} \int_K f_G(k_0c_{K \backslash G/H}(KgH)\circ h) \, dk \, d(KgH)
\]

and

\[
f_{K \backslash G/H}(KgH) = \int_K \int_{KgH} f_G(k_0c_{K \backslash G/H}(KgH)\circ h) \, dh \, dk,
\]

where \( c_{K \backslash G/H} : K \backslash G/H \to G \) is a function that selects an element of \( G \) from each double coset \( KgH \) to form a fundamental domain \( F_{K \backslash G/H} \subset G \). Then again using the nonnegativity of the Kullback-Leibler divergence

\[ D_{KL}(f_G \parallel f_K \cdot f_{K \backslash G/H} \cdot f_H) \geq 0 \]

gives \((30)\).

\[ 4.5 \text{ Application to self-replicating robots} \]

As stated earlier, the entropy difference between a disordered state and an assembled one is a measure of the difficulty of the assembly process. In a maximally disordered state, the entropy might simply be the log of the volume of allowable motions, which is the maximal entropy possible for an isolated part confined to a finite volume. Or, if an external potential field is applied, the probability density would become the configurational Boltzmann distribution, which has lower entropy, making an assembly process easier. Another way to reduce entropy is by reducing the size of the free space in which parts can move. This is discussed in detain in the next section.

But the relevance of the formulations earlier in this section are not related to the entropy of the disordered state. Rather, the role of part symmetry is related to the allowed entropy of the assembled state. For example, if a part is a cube that needs to be inserted into an assemblage, then the space of allowable correct orientations of the part in the final assemblage is 24 times larger than if no symmetry existed. Or, in an extreme case, if the part is spherical, and it only needs to
be inserted into a hemispherical slot, then orientation is completely irrelevant. For example, a ball on a roulette wheel easily finds its place. Another example would be loading bullets into a revolver wherein the SO(2) symmetry of the bullets and the cylindrical nature of the chambers make it much easier than if the cross-sections had no symmetry. In automated manufacturing and assembly systems, these principles are well known [48–53].

The value added by the analysis of this section is that the entropy of the assembled state can be quantified when constituent parts (and possibly the assemblage as a whole) have symmetry. If an individual part has symmetry group \( K \), then the space of motions that need to be considered during assembly is reduced from \( G \) to \( G/K \). Or, put another way, if \( K \) is a finite subgroup there are \( |K| \) times as many correct ways to assemble in comparison to when there is no symmetry. Similarly, if the overall assemblage has symmetry group \( H \), then the space is reduced from \( G \) to \( H \setminus G \). And when both the assemblage and constituent parts have symmetry then the reduction is from \( G \) to \( H \setminus G / K \). Any and all such symmetries effectively reduce the entropic burden of the assembly process.

Explicitly, if a part has symmetry then its configurational probability density function in the assemblage will inherit this symmetry as \( f(g) = f(gk) \) for all \( k \in K \), making it a left-coset function for which the analysis in the previous subsections becomes directly applicable. Or, put another way

\[
\int K f(gk) = \frac{1}{|K|} \sum_{k \in K} f(gk).
\]

In the case of a continuous symmetry, the sum is replaced by an integral, and \( |K| \) is replaced by the volume of \( K \). A general property of entropy, which follows from the theory of convex functions, is that averaging of any sort increases entropy.

5 Parts entropy and the principal kinematic formula

Physical self-replicating systems that are able to assemble basic parts to form replicas of themselves in the presence of uncertainties in the positions and orientations of feed parts are more robust than those that require perfectly palletized input parts. As discussed earlier, “parts entropy” is a statistical measure of the ensemble of all possible positions and orientations of a single part with a given probability density in position and orientation. Here a related issue is considered: if the part is confined to move uniformly at random in a finite container, what is its parts entropy?

In this section, it is shown how the “Principal Kinematic Formula” (PKF) from the field of Integral Geometry can be used to model the reduction in allowable motion imposed by the presence of an obstacle. Since entropy in the disordered state is related to the volume of allowable motion, this is relevant to the analysis of entropy change in self-replicating systems. Here, the PKF is stated without proof. References on this topic in which derivations are provided include [58].

5.1 The principal kinematic formula for collision

The indicator function on any measurable body, \( C \), is defined by:

\[
\iota(C) = \begin{cases} 
1 & \text{if } C \neq \emptyset \\
0 & \text{for } C = \emptyset
\end{cases}
\]

If \( g \in G \) is an element of a group (e.g., the group of rigid-body motions, \( SE(n) \)) that acts on \( C \) without shrinking it to the empty set, then \( g \iota(gC) = \iota(C) \) where

\[
gC = \{ g \cdot x | x \in C \}.
\]

For now let \( G = SE(n) \), the group of rigid-body motions in \( \mathbb{R}^n \). If \( g = (A, a) \) is the rigid-body motion with rotational part \( A \in SO(n) \) and translational part \( a \in \mathbb{R}^n \), then the action of \( G \) on \( \mathbb{R}^n \) is \( g \cdot x = Ax + a \), and hence \( gC \) is well defined. The indicator function is one of many functions on a body that is invariant under rigid-body motion. Others include the volume of the body, the surface area (or perimeter in the two-dimensional case).

Given two convex bodies, \( C_0 \) and \( C_1 \). Let \( C_0 \) be stationary, and let \( C_1 \) be mobile. The intersection of these two convex bodies is either a convex body or is empty. Furthermore, the rigid-body motion (or even affine deformation) of a convex body does not change the fact that it is convex. Therefore, when \( C_0 \cap gC_1 \) is not empty it will be a convex body, and

\[
f_{C_0, C_1}(g) = \iota(C_0 \cap gC_1)
\]

will be a compactly supported function on \( G \) that takes the value of 1 when \( C_0 \) and the moved version of \( C_1 \) (denoted as \( gC_1 \)) intersect, and it will be zero otherwise.

Counting up all values of \( g \) for which an intersection occurs is then equivalent to computing the integral

\[
\mathcal{I}(C_0, C_1) = \int_G \iota(C_0 \cap g \cdot C_1) \, dg.
\]

An amazing result is that the integral \( \mathcal{I} \) can be computed exactly using only elementary geometric properties of the bodies \( C_0 \) and \( C_1 \) without actually having to perform an integration over \( G \). While the general theory has been developed by mathematicians for the case of bodies in \( \mathbb{R}^n \) [55] and in manifolds on which some Lie group acts (see [58] and
references therein), we are concerned only with the cases of bodies in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In the planar case, we can write (32) explicitly as

$$I(C_0, C_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}r_1 \text{d}t_2 \text{d}\theta$$

(33)

where the rotational part of $g = (R, t)$ is described by

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

and the translational part is given by the vector $t = [t_1, t_2]^T$.

Spatial rigid-body motions can be parameterized as

$$g(t_1, t_2, t_3) = \begin{pmatrix} R(\alpha, \beta, \gamma) & t \\ 0^T & 1 \end{pmatrix},$$

where $R(\alpha, \beta, \gamma)$ is a rotation matrix expressed in terms of the ZXZ Euler-angles and $t \in \mathbb{R}^3$ is the translation vector. The bi-invariant integration measure for the group $SE(3)$ is then, to within an arbitrary scaling constant,

$$dg = \sin \beta \text{d}\sigma \text{d}\beta \text{d}\gamma \text{d}t_1 \text{d}t_2 \text{d}t_3.$$

**Theorem 5.1** (Blasche [54]) Given planar convex bodies $C_0$ and $C_1$, with $C_0$ fixed and $C_1$ free to move under the action of $SE(2)$, then the volume in the region of $SE(2)$ that places the bodies in collision is

$$I(C_0, C_1) = 2\pi [A(C_0) + A(C_1)] + L(C_0)L(C_1)$$

where $L(\cdot)$ is the perimeter of a body, and $A(\cdot)$ is the area.

For example, if the bodies are disks of radius $r_0$ and $r_1$, the above formula gives

$$I(C_0, C_1) = 2\pi [\pi r_0^2 + \pi r_1^2] + (2\pi r_0)(2\pi r_1).$$

Clearly for disks, the condition for collision is that the distance between the centers is less than or equal to $r_0 + r_1$, and so for this example

$$I(C_0, C_1) = (2\pi) \cdot \pi(r_0 + r_1)^2$$

where $2\pi$ is the volume of the space of planar rotations, $SO(2)$. The above two expressions are equal for disks.

The three-dimensional analog of this formula is given in the theorem below.

**Theorem 5.2** (Blasche [54]) Given 3D convex bodies $C_0$ and $C_1$, then when holding $C_0$ fixed and allowing $C_1$ to move, the volume in the space of motions corresponding to the bodies being in collision will be

$$I(C_0, C_1) = 8\pi^2 [V(C_0) + V(C_1)] + 2\pi[A(C_0)M(C_1) + A(C_1)M(C_0)]$$

(35)

where $V(\cdot)$ is the volume of the body and $M(\cdot)$ and $A(\cdot)$ are, respectively, the integral of mean curvature area and of the surface enclosing a body.

This result was developed by Wilhelm Blasche a century ago, and proofs can be found in [30, 54, 58].

Note that $I(C_0, C_1) = I(C_1, C_0)$, which is a consequence of the bi-invariance of integration on the Lie group $SE(3)$, which is unimodular.

As an example of the above theorem, if $C_1$ is a solid ball of radius $r_1$, then the above formula gives

$$I(C_0, C_1) = 8\pi^2 \cdot \frac{4\pi}{3} (r_0^3 + r_1^3) + 2\pi[4\pi r_2^2 \cdot 4\pi r_1 + 4\pi r_1^2 \cdot 4\pi r_0].$$

This matches the expected result of

$$I(C_0, C_1) = 8\pi^2 \cdot \frac{4\pi}{3} (r_0 + r_1)^3$$

which is the volume of $SO(3)$ multiplying the volume of the ball of radius $r_0 + r_1$ corresponding to the volume of all motions of the center of $C_1$ that would place it in collision with $C_0$.

The literature on integral geometry spanning the past century is immense. For further reading on the Principal Kinematic Formula (and Integral Geometry more generally) see [56–59]. Related work is concerned with determining when one body can be contained in another [60–63] and the characterization of free motion of one body moving inside another [64, 65]. Namely, if a part can be contained, what is the volume of its free motion? This is obviously related to the parts entropy of a final assemblage in which there are clearances. This is the subject of the following section.

### 5.2 Kinematics of containment

Instead of considering the volume within the motion group describing collision of two bodies, we can instead evaluate the volume of allowable motion of a small body within a large convex container. In this scenario a formula similar to the principal kinematic formula results under mild conditions. Namely, if all principal curvatures of the inner body are larger than every principal curvature of the container, then in the planar case [65]

$$V(C_1, C_2) = 2\pi[A(C_1) + A(C_2)] - L(C_1)L(C_2)$$

(36)

and in the spatial case

$$V(C_1, C_2) = 8\pi^2 [V(C_1) + V(C_2)] - 2\pi[A(C_1)M(C_2) + A(C_2)M(C_1)]$$

(37)

where $C_2$ is the container and $C_1$ is again the moving body.
5.3 Entropy of a convex part free to move in a container with an obstacle

Suppose that as a strategy for assembling parts in a robotic self-replication process, the original robot first either cages a part that is to be assembled, or pushes it into a bowl. These actions do not require sophisticated manipulation, and hence are appropriate for simplifying the requirements on a self-replicating system. Both a cage and bowl are examples of containers. This process physically limits the parts entropy.

If there are no preferred positions and orientations of the part within the container, then the parts entropy is simply the volume of allowable motion. In the case when there is an obstacle (such as a post or pillar) in the container that limits allowable motion, and if the obstacle and container geometries are such that the moving part never gets jammed between the container and the obstacle, the parts entropy will be

\[
S = \log \left( V(C_1, C_2) - I(C_0, C_1) \right).
\]

6 Error propagation in parts manufacturing

When considering the production of parts from raw materials, the question of how to maintain fidelity generation after generation arises. The process of reliably producing replicas by employing error-reduction techniques is related to information theoretic entropy and the theory of error-correcting codes. One way to achieve error correction is to use symmetry as a parity check. That is, original parts that are intended to be symmetrical in the first generation will lose symmetry in future generations. For example, if a mould makes a casting and that casting is used to make a subsequent mould, then with each iteration there will be some corruption of the result due to random flaws that are introduced. But if the parts have symmetries, then random flaws will affect different areas of a part in different ways, and imposing information about the symmetry of the original parts on the replicas as a post-processing step in the manufacturing process will help in maintaining tolerances and to squash the magnitude of errors in reproduction.

No matter what error correction methods are put in place, changes resulting from compounded manufacturing errors in artificial self-replicating machines would almost always result in reduction of functionality over generations. This might not be a bad thing if the purpose of a self-replicating physical system is to magnify initial human effort in harvesting in situ resources in outer space. For example, a magnification by a factor between 10 and 100 before they cease to replicate would be tremendous, whereas developing immortal (and evolvable) self-replicating robots could have unforeseen negative consequences.

Whereas living systems have various levels of feedback to ensure stability over generations from the molecular (DNA replication and transcription) to the macroscopic (competitive survival advantages and predator–prey equilibria), the same is not true in scenarios in which artificial self-replicating systems would be deployed. For example, a self-replicating factory designed to harvest materials on the moon or Mars or other inert environments in order to bootstrap mankind’s reach into the solar system need not worry about competing for resources with other entities. The goal of self-replicating systems is not evolution, but rather the reliable copying of engineered systems that have a specific mission beyond survival.

7 Conclusions

Artificial self-replicating systems that process materials, make parts, and assemble the parts to make physical copies of themselves have been imagined for more than half a century. The potential impact of such systems for the development of resources in outer space are tremendous. Indeed, this might be the only path forward for colonization of the solar system, as well as harnessing resources to mitigate the effects of global warming. Various technologies have developed in recent years that make self-replication with vitamin resources more realistic than ever. These include metal additive manufacturing (a.k.a 3D printing), and new methods for materials processing. The purpose of this paper was not to review these technological advances, but rather to expand the set of theoretical and algorithmic foundations with which to evaluate progress in this field. Several topics that interwine entropy, information, complexity, and error propagation were articulated. Advanced mathematical tools from the theory of Lie groups and Integral Geometry were introduced to the a-life community in the context of artificial physical self-replicating systems.

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