ON THE POST-SYMMETRIC BRACE ALGEBRAS

IGOR MENCATTINI, ALEXANDRE QUESNEY, AND PRYSCILLA SILVA

Abstract. In this paper we identify the post-Lie analogue of the symmetric brace algebras, advocate their role in the theory of the associated \( D \)-algebras and present some relations with the so-called post-Lie Magnus expansion.

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1. Introduction
Post-Lie algebras are non-associative algebras which later on appeared to play an important role in different areas of pure and applied mathematics. In this paper we are mainly concerned with some of their properties which are more easily detected through the operadic approach. To make the present work as self-contained as possible we summarize here below the necessary background, stressing differences and similarities with the better known class of the pre-Lie algebras.

Pre-Lie algebras have been introduced by Vinberg in \[37\] in his studies about convex cones and, almost at the same time, they appeared in Gerstenhaber’s foundational work \[24\] about the deformation theory of associative algebras. Since then, pre-Lie algebras have been at the center of extensive investigations, especially because of their importance in combinatorics, mathematical physics, differential geometry, Lie theory and numerical analysis; see \[27\] and \[5\] for comprehensive

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reviews. A pre-Lie algebra is a vector space $V$ endowed with a bilinear product \( \triangleright : V \otimes V \to V \) whose associator
\[
a_v(x, y, z) := x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z, \quad \forall x, y, z \in V,
\]
satisfies
\[
(1.1) 
\quad a_v(x, y, z) = a_v(y, x, z), \quad \forall x, y, z \in V.
\]
The condition \( (1.1) \) guaranties that \([\cdot, \cdot] : V \otimes V \to V\), defined by \([x, y] := x \triangleright y - y \triangleright x\) for all \(x, y \in V\), satisfies the Jacobi identity so that \(V_{\text{Lie}} = (V, [\cdot, \cdot])\) is a Lie algebra.

The universal enveloping algebra of a Lie algebra coming from a pre-Lie algebra \((V, \triangleright)\) was analyzed in depth in the papers \[32, 33\], where it was shown that the existence of the pre-Lie product on \(V\) provides the cofree symmetric coalgebra \(S(V)\) with an associative but not commutative product \(* : S(V) \otimes S(V) \to S(V)\), defined by
\[
(1.2) \quad A \ast B = A_{(1)}(A_{(2)} \triangleright B), \quad \forall A, B \in S(V).
\]

Endowed with such a product and its usual coalgebra structure, \(S(V)\) becomes a bialgebra isomorphic to the universal enveloping algebra of \(V_{\text{Lie}}\). In Formula \(1.2\) enters both the shuffle coproduct of \(S(V)\) and a non-associative product \(\triangleright : S(V) \otimes S(V) \to S(V)\) obtained extending to \(S(V)\) the original pre-Lie product defined on \(V\). Furthermore, the restriction \(\triangleright : S(V) \otimes V \to V\) of the latter non-associative product defines a structure of symmetric brace algebra on the vector space \(V\), that is a family of operations \(B_n \in \text{End}_K(V^{\otimes n+1})\), \(n \geq 1\), symmetric in the first \(n\) variables and such that
\[
B_n(v_1, \ldots, v_n; u) = B_1(v_1; B_{n-1}(v_2, \ldots, v_n; u)) - \sum_{k=2}^{n} B_{n-1}(v_2, \ldots, B_1(v_1; v_k), \ldots, v_n; u)
\]
for all \(v_i\)'s and \(u\) in \(V\). In \[32, 25\] and \[11\] it was proved that the symmetric brace algebras form a category isomorphic to the category of the pre-Lie algebras. More precisely, in \[11\] was proposed a presentation, in terms of the set of non-planar rooted trees, of the operad \(\text{PreLie}\) controlling the pre-Lie algebras. As it turned out that the algebras over this operad are the symmetric brace algebras, this operadic approach provided a convenient way to prove the above mentioned isomorphism between the category of symmetric brace algebras and the category of pre-Lie algebras.

Post-Lie algebras were introduced by Vallette in \[36\] as being the algebras over the operad \(\text{PostLie}\), which is the Koszul dual of the operad controlling the commutative trialgebras. They were further analyzed a few years later by Munthe-Kaas and Lundervold in \[29\], in their study of the \textit{order conditions} for the Lie group integrators. Formally speaking a post-Lie algebra is a Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\) endowed with non-associative product \(\triangleright : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\) that satisfies
\[
(1.3) \quad z \triangleright [x, y] = [z \triangleright x, y] + [x, z \triangleright y]
\]
and
\[
(1.4) \quad [x, y] \triangleright z = a_v(x, y, z) - a_v(y, x, z) \quad \text{for all } x, y, z \in V.
\]
It is worth to note that, in spite the post-Lie product does not yield a Lie bracket by antisymmetrization, the bilinear product \([\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\), defined by
\[
(1.5) \quad [[x, y]] = x \triangleright y - y \triangleright x + [x, y], \quad \forall x, y \in \mathfrak{g},
\]
defines on \( g \) another Lie algebra structure, which, from now on, will be denoted by 
\[ \overline{g} = (g, [\cdot, \cdot]). \] 
Note also that (1.4) implies that a post-Lie algebra with a trivial Lie bracket is pre-Lie algebra.

After their introduction post-Lie algebras have been extensively studied, since they appeared in several different areas of mathematics like Lie theory (see [7, 8, 9, 18]), mathematical physics (see [1, 19, 21]) and numerical analysis (see [29, 14, 15, 30]). Although the post-Lie algebras were introduced to the realm of numerical analysis in [25], their relevance for the theory of numerical integration could be traced back to the fundamental work [31], where the authors performed a deep investigation of the Hopf algebra relevant for the algebraic description of the Lie group integrators. More precisely, in [31] was introduced a cocommutative but noncommutative Hopf algebra whose underlying linear space is generated by the forests of planar rooted trees (there called ordered rooted trees). This algebraic structure was then enhanced with an operation of left-grafting and was termed \( D \)-algebra.

With this concept at hand, the authors of [31] could inscribe the theory of Lie group integrators in the algebraic framework of the theory of Hopf algebras. Soon after it was understood that \( D \)-algebras played the role of universal enveloping algebras for post-Lie algebras. More precisely, in [29] it was shown that the universal enveloping algebra functor
\[ (1.6) \]
provides an adjunction between the category of post-Lie algebras and the category of \( D \)-algebras, whose adjoint is the derivation functor (see [29, Definition 3.5]).

The properties of (1.6) were further investigated in [18], where it was shown that for any post-Lie algebra \((g, \triangleright)\) there exists a unique extension \( \triangleright : U(g) \otimes U(g) \to U(g) \) of the post-Lie product which satisfies suitable compatibility conditions with its canonical coalgebra structure, see [18, Proposition 3.1]. Furthermore, it was proven that the product \( * : U(g) \otimes U(g) \to U(g) \), defined, in analogy to (1.2), by
\[ (1.7) \]
makes \( U(g) \) into a bialgebra isomorphic to \( U(\overline{g}) \). It is worth to mention that the product * corresponds to the Grossman-Larson product on the free post-Lie algebra of planar rooted trees; see [31] and also [29, 18].

Main results of the present work. The main aim of the present paper is to introduce the post-Lie analogue of the symmetric brace algebras and to study some of its properties, emphasizing its relations with other mathematical structures naturally associated to every post-Lie algebra.

First recall that a relevant point in the construction of the operad \( RRT \) in [11] is that the symmetric braces (which are nothing but iterations of the pre-Lie product) are naturally encoded in the combinatorics of non-planar rooted trees, which therefore provide a very convenient presentation of the free pre-Lie algebra. In analogy with this case, in Section [3] it is shown that the presentation of the free post-Lie algebra via planar rooted trees, see [31, 29, 18], also comes from an operadic structure. The operad in question, denoted by \( PSB \), is shown to be isomorphic to \( PostLie \) in Theorem [11].

The \( PSB \)-algebras are characterized in Section [4] where it is shown that they are post-Lie algebras endowed with brace type operations, see Proposition [12]. We term them post-symmetric brace algebras and we propose them as our candidate
to be the desired post-Lie analogue of the symmetric brace algebras. The post-
symmetric brace algebras turn out to be a convenient tool to further investigate
the properties of the product defined in (1.7). More in details, in Section 4.2, we
introduce the notion of \( D \text{-bialgebras} \) and we show that (1.6), seen as a functor
(1.8)
\[
U : \text{PostLie} \to D \text{-bialgebra},
\]
is full and faithful (see Theorem [21]), and provides an adjunction of categories (see
Proposition [22]) whose adjoint is the \textit{primitive elements} functor,
(1.9)
\[
\text{Prim} : D \text{-bialgebra} \to \text{PostLie},
\]
which associates to every \( D \text{-bialgebra} \) its primitive elements, see Definition [19]. The
faithfulness of the functor \( U \) follows, essentially, from the observation that the post-
symmetric braces on a Lie algebra \( g \) encode the left part of the \( D \text{-product} \) on \( U(g) \).

In Section 5, we analyze the so called \textit{post-Lie Magnus expansion}, introduced in
[19], see also [18, 21], from the viewpoint of the \( PSB \text{-algebras} \). In more details, in
Proposition [39] we relate the Baker-Campbell-Hausdorff formulas of the Lie algebras
\( (g, [\cdot, -]) \) and \( \bar{g} \) to the post-Lie Magnus expansion. This generalizes a result, see
Corollary [40] proven for the first time in [1], in the theoretical framework of the
pre-Lie algebras. We then provide a method for computing the coefficients of this
expansion based on a planar trees description of the universal enveloping algebra
of the free post-Lie algebra equipped with the product \( * \), which relies on tubings
on these trees.

1.1. \textbf{Conventions.} Throughout the paper \( \mathbb{K} \) will denote a field of characteristic
zero. Tensor product will be taken over \( \mathbb{K} \), and the tensor product of two
\( \mathbb{K} \text{-vector spaces} \) \( V \) and \( W \) will be denoted by \( V \otimes W \).

For a \( \mathbb{K} \text{-vector space} \) \( V \), we let \( T(V) = \oplus_{n \geq 0} V^\otimes n \), where \( V^\otimes 0 = \mathbb{K} \), be its free
tensor algebra. The map \( \Delta_{sh} : V \to T(V) \otimes T(V) \) given by \( \Delta_{sh}(x) = x \otimes 1 + 1 \otimes x \)
for all \( x \in V \) extends uniquely to \( T(V) \) as a morphism of algebras. The resulting
map \( \Delta_{sh} : T(V) \to T(V) \otimes T(V) \) is called \textit{shuffle} coproduct and makes the free
tensor algebra into a cocommutative bialgebra. It is described as follows. Let
\( X = x_1 \otimes \cdots \otimes x_n \) be in \( V^\otimes n \). For a sub order set \( I = \{i_1 < \cdots < i_k\} \) of
\( \{1 < \cdots < n\} \), we let \( X_I \) denote the element \( x_{i_1} \otimes \cdots \otimes x_{i_k} \) in \( V^\otimes |I| \). One has
\[
\Delta_{sh}(X) = \sum_{I \sqcup J} X_I \otimes X_J,
\]
where the sum runs over the ordered partitions \( I \sqcup J \) of \( \{1 < \cdots < n\} \).

In general, for a coproduct \( \Delta : W \to W \otimes W \), we will use the \textit{Sweedler} notation
in its compact form:
\[
\Delta(X) = X_{(1)} \otimes X_{(2)} \quad \text{for all } X \in W.
\]

To save notation, for a linear map \( f : V^\otimes k \to W \), we will write \( f(x_1, \ldots, x_k) \) for
\( f(x_1 \otimes \ldots \otimes x_k) \).

We will consider operads in the category of \( \mathbb{K} \text{-vector spaces} \). Our convention
follows [26] to which we refer for more details. In brief, an operad \( O \) is an \( S \text{-module} \) in the category of the \( \mathbb{K} \text{-vector spaces} \), together with partial compositions
\( \circ_i : O(m) \otimes O(n) \to O(m + n - 1) \) for \( 1 \leq i \leq m \) that satisfy associativity, equiv-
variance and unit axioms. In particular, this means that for each \( n \geq 1 \), the vector

---
space $\mathcal{O}(n)$ is acted on by the symmetric group $S_n$, and that the maps $\circ_i$ are equivariant for this action. For instance, for a vector space $V$, the $S$–module $\text{End}_V$ given by $\text{End}_V(n) = \text{Hom}_K(V^\otimes n, V)$ for $n \geq 1$, is an operad for the partial composition of linear maps:

$$f \circ_i g(x_1, ..., x_{n+m-1}) := f(x_1, ..., x_{i-1}, g(x_{i}, ..., x_{i+n-1}), x_{i+n}, ..., x_{n+m-1}),$$

for every two maps $f \in \text{End}_V(m)$ and $g \in \text{End}_V(n)$, and all $x_j \in V$. The action of the symmetric group is given by permutation of the variables. A vector space $V$ is called an $\mathcal{O}$–algebra if there is a morphism of operads $\mathcal{O} \to \text{End}_V$.

An ideal $\mathcal{I}$ of an operad $\mathcal{O}$ is a sub $S$–module of $\mathcal{O}$ such that the maps $\circ_i$ co-restrict to $\mathcal{I}$ whenever they are restricted to $\mathcal{I}$ in any of its two components i.e. they induces maps $\circ_i : \mathcal{I}(m) \otimes \mathcal{O}(n) \to \mathcal{I}(m+n-1)$ and $\circ_i : \mathcal{O}(m) \otimes \mathcal{I}(n) \to \mathcal{I}(m+n-1)$. In particular, the $S$–module quotient $(\mathcal{O}/\mathcal{I})(n) := \mathcal{O}(n)/\mathcal{I}(n)$, for $n \geq 1$, has a structure of operad.

To end this preamble, let us make a comment on decompositions. Let $\mathcal{O}$ be an operad and $T$ an element of $\mathcal{O}(n)$. Consider a decomposition of $T$, say $T = (\cdots (S_1 \circ_{i_1} S_2) \circ_{i_2} S_3) \cdots S_{k-1}) \circ_{i_k} S_k)^\sigma$. Suppose that for each $n$, the action of $S_n$ on $\mathcal{O}(n)$ is free. Because of the $S$–equivariance of the maps $\circ_i$, by applying equivariant actions of $S$ one gets other decompositions of $T$. If there is no additional decomposition of $T$, anyone of the above decompositions is called unique $S$–equivariant.

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2. An operad of planar trees

In this preliminary section, we set up notation and introduce the relevant combinatorial objects for our needs. We also define an operad of trees, which will be the starting point for defining other operads important for the present work.

**Definition 1.** A planar rooted tree is an isomorphism class of contractible graphs, embedded in the plane, and endowed with a distinguished vertex, called the root, to which is attached an adjacent half-edge, called the root-edge of the planar tree.

For a planar rooted tree $T$, we let $V(T)$ be the set of all its vertices. On it, we consider two orders:

- The level partial order $\prec$ defined by orienting the edges of $T$ towards the root, except the root-edge. For two vertices $u$ and $v$ of $V(T)$, we write $v \prec u$ if there is a string of oriented edges from $v$ to $u$. In particular, the root is maximal for this partial order.
- The canonical linear order $<$: starting from the root-edge of $T$, we run along $T$ in the clockwise direction, passing trough each edge once per direction. The order we meet the vertices for the first time gives the order $<$; see Example 2. In particular the root is the minimal element for $<$.
Pictorially, our trees are drawn with the root at the bottom, and the order on the set of the incoming edges of a vertex is given by the clockwise direction, i.e. from the left to the right.

For each vertex $v \in V(T)$, consider a small disc centered at $v$. The angles of $v$ are the connected components of the disc, when removed its intersection with the tree. The angles are ordered in the clockwise direction, starting from the component in between the outgoing edge of $v$ and its left most incoming edge (if it exists). We let $\text{Ang}_{\text{min}}(T)$ be the set of the minimal angles of all the vertices of $T$. It is in canonical bijection with the set $V(T)$, which endows it with an order; see Example 2.

**Example 2.** Let $T$ be the following planar rooted tree, together with its angles:

![Tree with angles](image)

The ordered set of the vertices of $T$ is $V(T) = \{a < c < f < e < d < b\}$. The ordered set of angles of the vertex $c$ is $\{c_1 < c_2 < c_3\}$, the ordered set of the minimal angles of $T$ is $\text{Ang}_{\text{min}}(T) = \{a_1 < c_1 < f_1 < e_1 < d_1 < b_1\}$.

From now on, when there is no ambiguity, planar rooted trees are simply called trees. We will consider trees with labelings, or more in general, with partial labelings.

**Definition 3.** Let $T$ be a tree and let $U$ be a subset of $V(T)$. A $U$–label of $T$ is a bijection $l: U \to \{1, ..., n\}$. A tree $T$ equipped with a $U$–label is called partially labeled; it is called fully labeled (or simply, labeled) if $U = V(T)$.

Let us denote by $l_d: U \to \{1 < ... < n\}$ the unique isomorphism of ordered sets, where $U$ has the induced order from $V(T)$. Since $S_n$ acts freely and transitively on the set of $U$–labels by post-composition, one can write a $U$–label in a unique form by $l_{\sigma}: U \to \{1, ..., n\}$ for some $\sigma \in S_n$ (and $l_{\sigma} := \sigma l_d$).

**Example 4.** For $(312) \in S_3$ we have the following examples of partially labeled trees:

![Examples of partially labeled trees](image)

**Definition 5.** For a partially labeled tree $T$, let $\text{Ang}_{\text{lab}}^{\text{min}}(T)$ be the set of those minimal angles of the labeled vertices of $T$.

**Definition 6.** Recall that our trees have oriented edges. For $1 \leq i \leq n$, let $\text{In}(i)$ be the order set of incoming edges of the vertex labeled by $i$. The order on $\text{In}(i)$ is the one induced by the clockwise orientation, that is, from the left to the right.
Let us define an operadic structure on the partially labeled trees. Let $\mathcal{PT}_p(n)$ be the free $K$–module generated by the set of partially trees, whose $U$–labels are with $U$ of cardinality $n$. For $m, n \geq 1$ and $1 \leq i \leq m$, let
\[(2.1) \quad \circ_i: \mathcal{PT}_p(m) \otimes \mathcal{PT}_p(n) \to \mathcal{PT}_p(m + n - 1)\]
be as follows.

For each $T \in \mathcal{PT}_p(m)$, $R \in \mathcal{PT}_p(n)$ and for a map of sets
\[\phi: \text{In}(i) \to \text{Ang}_{\text{lab}}^\text{min}(R),\]
define $T \circ_i^\phi R$ to be the tree obtained as follows: substitute the vertex labeled by $i$ by the tree $R$, and graft the incoming edges of $i$ to the labeled vertices of $R$ following the map $\phi$. The grafting is required to be performed in such a way that it respects the natural order of each fiber of $\phi$. More precisely, since for every $\alpha$ the fiber $\phi^{-1}(\alpha)$ is endowed with the induced order from $\text{In}(i)$ (given by $\text{Ang}_{\text{lab}}^\text{min}(T)$), when performing the substitution, the incoming edges of $\phi^{-1}(\alpha)$ preserves this order.

The outgoing edge of $i$ is identified with the root-edge of $R$. The tree $T \circ_i^\phi R$ has $m - 1 + n$ labeled vertices, and the partial labeling is given by classical re-indexation.

Explicitly, for a $U$–label $l_T$ of $T$ and a $U'$–label $l_R$ of $R$, the resulting label of $T \circ_i^\phi R$, say $l''$, is as follows. Suppose $v$ be the vertex of $T$ labeled by $i$ and consider the canonical inclusion of $U \{v\}$ into the set of the vertices of $T \circ_i^\phi R$, and similarly for $U'$. The $U \{v\} \cup U'$–label $l''$ is defined by
\[l''(w) = \begin{cases} l^T(w) & \text{if } w \in U \{v\} \text{ and } 1 \leq l^T(w) \leq i - 1 \\ l^R(w) + i - 1 & \text{if } w \in U' \\ l^T(w) + n & \text{if } w \in U \{v\} \text{ and } l^T(w) \geq i + 1. \end{cases}\]

For two trees $T$ and $R$ as above, their partial composition is given by
\[(2.2) \quad T \circ_i R = \sum_{\phi: \text{In}(i) \to \text{Ang}_{\text{lab}}^\text{min}(R)} T \circ_i^\phi R.\]

We extend this by linearity to get (2.1). For instance, one has
\[
\begin{array}{c}
2 \quad 3 \\
\circ_1 \\
1 \\
\end{array}
\quad =
\begin{array}{c}
2 \quad 3 \\
\circ_1 \\
1 \\
\end{array}
\]

and
\[
\begin{array}{c}
2 \quad 3 \\
\circ_1 \\
1 \\
\end{array}
\quad =
\begin{array}{c}
3 \quad 4 \\
2 \\
\circ_1 \\
1 \\
\end{array}
\quad +
\begin{array}{c}
4 \\
3 \\
\circ_1 \\
1 \\
\end{array}
\quad +
\begin{array}{c}
3 \quad 4 \\
2 \\
\circ_1 \\
1 \\
\end{array}
\quad +
\begin{array}{c}
2 \quad 4 \\
3 \\
\circ_1 \\
1 \\
\end{array}
\quad .
\]

**Proposition 7.** The $S$–module $\mathcal{PT}_p = \oplus_{n \geq 1} \mathcal{PT}_p(n)$ endowed with the partial composition maps defined in (2.1) becomes a symmetric operad.

**Proof.** This is routine check. \hfill \square

### 3. The operad $\mathcal{PSB}$

In this section we introduce the operad $\mathcal{PSB}$ defined in third author’s PhD thesis [35]. It can be seen as the post-Lie analogue of the operad of rooted trees defined in [11]. Explicitly, we show that $\mathcal{PSB}$ is isomorphic to the operad of the post-Lie algebras.
3.1. Definition of $\mathcal{P}_B$. Let $\mathcal{L} = \oplus_{n \geq 1} \mathcal{L}(n)$ be the suboperad of $\mathcal{P}_T$ generated by the fully labeled trees.

For each $n \geq 2$ let $W(n) \subset \mathcal{P}_T(n)$ be the $\mathbb{K}$-subvector space generated by those partially labeled trees $T$ that satisfy:

(a) the root of $T$ is unlabeled;
(b) if a vertex of $T$ is unlabeled, then so is its $\prec$-successor;
(c) each unlabeled vertex of $T$ has exactly two incoming edges.

We let $L(1) := L(1)$ and $L(n) := L(n) \oplus W(n)$ for $n \geq 2$.

We will prove that $L(n) := \oplus_{n \geq 1} L(n)$ carries a natural operad structure.

3.1.1. Vertex-wise action of the symmetric group. Let $R$ be a tree in $W(m)$ and let $V_{unl}(R)$ denote the set of its unlabeled vertices. For $v \in V_{unl}(R)$, let $R_v$ be the maximal subtree of $R$ with root $v$.

To any permutation $\sigma \in S_2$ one may associate the tree $R_\sigma$ that is obtained from $R$ by changing the subtree $R_v$ into

$$R_\sigma := C(v; R_{\sigma(1)}^v, R_{\sigma(2)}^v).$$

The labeling is unchanged: if a vertex of $R_v$ has a label, then its image in $R_\sigma$ has the same label.

More generally, to any tuple of permutations $\sigma \in S_2^{\times |V_{unl}(R)|}$ one may associate a tree $R_{\sigma}$ obtained by applying the above construction to each vertex $v$ of $V_{unl}(R)$. Since the order we perform the iteration does not matter, this is well defined.

For a tuple $\sigma = (\sigma_1, ..., \sigma_s) \in S_2^{\times |V_{unl}(R)|}$ we set $\epsilon(\sigma) = (-1)^{sgn(\sigma_1) + ... + sgn(\sigma_s)}$, where $sgn(\sigma_t)$ is the signature of the permutation $\sigma_t \in S_2$.

3.1.2. Contracting trees. Let $T$ be a partially labeled tree with an unlabeled vertex $v$ that is not the root, and let $v^+$ be its $\prec$-successor. Denote by $Con_v(T)$ the partially labeled tree obtained from $T$ by contracting the edge linking $v$ and $v^+$. The resulting vertex, which comes from the merging of $v$ and $v^+$, is endowed with the same label than the vertex $v^+$ of $T$, if any.

For a tree $R \in W(n)$, we denote by $Con(R)$ the tree in $\mathcal{P}_T(n)$ obtained from $R$ by contracting all the edges that are bounded by two unlabeled vertices. Thus, in $Con(R)$, the root is the unique unlabeled vertex. For instance,

$$Con\left(\begin{array}{c}
1 \\
2 \\
3 \\
5
\end{array}\right) = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array}.$$ 

3.1.3. The operad $LW$. For $m \geq 1$, $n \geq 2$ and $1 \leq i \leq m$, let

$$\delta_i : LW(m) \otimes W(n) \rightarrow LW(m + n - 1)$$

(3.1)
be linear map defined as follows. Given $T$ in $\mathcal{LW}(m)$ and $R$ in $\mathcal{W}(n)$, for $1 \leq i \leq m$, let $v$ be the vertex labeled by $i$. We set
\begin{equation}
T\tilde{\circ}_i R = \begin{cases}
\sum_{\sigma \in S_{V_{\text{unl}}(T)}} \epsilon(\sigma) \text{Con}_e(T \circ_i \text{Con} \sigma) & \text{if the } \prec \text{-successor of } v \text{ exists and is labeled;} \\
T \circ_i R & \text{otherwise},
\end{cases}
\end{equation}
where $r$ denotes the root vertex of $\text{Con} \sigma$ seen as a sub-tree of $T \circ_i \text{Con} \sigma$, and $V_{\text{unl}}(R)$ is the set of unlabeled vertices of $R$.

For instance,
\[
\begin{array}{c}
\begin{array}{c}
1 \\
\Phi_2 \\
2 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array}
+ 
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array}
+ 
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array}.
\end{array}
\]

On the other hand, note that (2.1) restricts to
\begin{equation}
o_i: \mathcal{LW}(m) \otimes \mathcal{L}(n) \rightarrow \mathcal{LW}(m+n-1).
\end{equation}

**Proposition 8.** $\mathcal{LW}$ endowed with the partial compositions defined in (3.3) and (3.4) is an operad.

**Proof.** It is a tedious but direct verification. For the reader convenience we sketch the proof of the associativity of the partial compositions, i.e. we show that for every three trees $R, S$ and $T$ in $\mathcal{W}(v), \mathcal{W}(s), \text{ and, respectively, in } \mathcal{W}(t)$
\[
R\tilde{\circ}_i(S\tilde{\circ}_jT) = (R\tilde{\circ}_iS)\tilde{\circ}_{i-1+j}T,
\]
for all $1 \leq i \leq r$ and $1 \leq j \leq s$.

Let $v$ and $w$ be the vertices of $R$ and $S$ labeled, respectively, by $i$ and $j$. If both $v$ and $w$ have no labeled $\prec$-successor, then the involved partial compositions are the ones of the operad $\mathcal{PT}_i$, see (2.1), which proves the equality. Now, let us suppose both $v$ and $w$ have a labeled $\prec$-successor (the proof for the other cases is similar). In this case, recall that one has
\begin{equation}
S\tilde{\circ}_jT = \sum_{\sigma \in S_{V_{\text{unl}}(T)}} \epsilon(\sigma) \text{Con}_e(S \circ_j \text{Con} \sigma).
\end{equation}
First, set $P\tilde{\circ}_jQ := \text{Con}_r(P \circ_j \text{Con} \sigma)$ for every couple of trees $P, Q \in \mathcal{W}$ and observe that the equality $R\tilde{\circ}_i(S\tilde{\circ}_jT) = (R\tilde{\circ}_iS)\tilde{\circ}_{i-1+j}T$ in $\mathcal{PT}_i$ implies that
\[
R\tilde{\circ}_i(S\tilde{\circ}_jT) = (R\tilde{\circ}_iS)\tilde{\circ}_{i-1+j}T.
\]
Indeed, note that the second contraction $\text{Con}_r$ in (3.4) does not affect the associativity of the $\tilde{\circ}_k$’s since the sets of the minimal labeled angles of $\text{Con}_r(S \circ_j \text{Con} \sigma)$ and of $S \circ_j \text{Con} \sigma$ are the same, and the set of inputs of $v$ and of $w$ are not concerned by these contractions. Similarly, as the contraction $\text{Con} \sigma$ involves only edges between unlabeled vertices, it has no consequence on the associativity of the $\circ_k$’s. We conclude by observing that, since $w$ has a labeled $\prec$-successor, one has $V_{\text{unl}}(\text{Con}_r(S \circ_i \text{Con} \sigma)) = V_{\text{unl}}(S)$.

Let $\mathcal{I} \subset \mathcal{LW}$ be the operadic ideal generated by
\[
\left\{ \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array} , \\
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array} , \\
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array} \right\}.
\]
Definition 9. We call \( \mathcal{PSB} \) the symmetric operad \( \mathcal{LW}/I \).

For later use, we prove the following result.

Lemma 10. For \( m \geq 2 \), every generator of the \( S_m \)-module \( \mathcal{LW}(m) \) has a unique \( S \)-equivariant decomposition of the form
\[
(3.5) \quad (\cdots (S_1 \circ_{i_1} S_2) \circ_{i_2} S_3) \cdots S_{k-1} \circ_{i_{k-1}} S_k)^\sigma,
\]
where \( S_j \in \mathcal{LW}(2) \) for \( 1 \leq j \leq k \) and \( \sigma \in S_m \).

Proof. Recall that every labeled tree \( T \in \mathcal{L}(m) \) has a unique \( S \)-equivariant decomposition into corollas, that is, there is a \( k \geq 1 \) and there are corollas \( Q_j \in \mathcal{L}(m_j) \) for \( 1 \leq j \leq k \) such that
\[
T = (\cdots ((Q_1 \circ_{i_1} Q_2) \circ_{i_2} Q_3) \cdots Q_{k-1} \circ_{i_{k-1}} Q_k).
\]
The corollas \( Q_j \in \mathcal{L}(m_j) \) and the \( i_j \)'s are unique up to the action of symmetric groups \( S_{m_j} \). Let us prove the lemma for corollas in \( \mathcal{L}(n) \). One has
\[
(3.6) \quad \begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3 \\
2
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
1 \ 2 \\
2
\end{array}
\end{array} \circ_{1} 1 \ 2 - \begin{array}{c}
\begin{array}{c}
1 \ 2 \\
1
\end{array}
\end{array} \circ_{2} 1 \ 2
\]
So the result holds for \( n = 2 \). For \( n \geq 2 \), one has
\[
\begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3 \ \cdots \ n \\
2
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
1 \ 2 \\
2
\end{array}
\end{array} - \sum_{1 \leq i \leq n-1} \begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3 \ \cdots \ n \ - \ 1 \\
2
\end{array}
\end{array} \circ_{i} 1 \ 2 \ \cdots \ n \ - \ 1
\]
where \( \sigma_i \in S_{n+1} \) is given by \( \sigma_i(1) = i, \sigma_i(i - k + 1) = i - k, \sigma_i(i + m) = i + m, 1 \leq k \leq i - 1, 1 \leq m \leq n + 1 - i \). The result follows by induction on \( n \).

Let us prove the lemma for \( \mathcal{W}(m) \). Note that the statement is true for \( \mathcal{W}(2) \). Let \( m \geq 3 \) and \( T \) be a tree of \( \mathcal{W}(m) \). By definition the root of \( T \) is unlabeled, so there are two unique trees \( T_1 \) and \( T_2 \) of \( \mathcal{LW} \) such that
\[
(3.7) \quad T = \begin{array}{c}
\begin{array}{c}
1 \ 2 \\
2
\end{array}
\end{array} \circ_{1} T_1 \circ_{2} T_2
\]
Possibly, either \( T_1 \) or \( T_2 \) is the unit tree of \( \mathcal{L}(1) \), in which case the decomposition simplifies. If both \( T_1 \) and \( T_2 \) or fully labeled (i.e. they are trees of \( \mathcal{L} \)), then the result follows from the previous case. Otherwise, one proceeds by induction, observing that for the tree(s) \( T_i \) of \( \mathcal{W}(m_i) \) one has \( m_i < m \). \( \square \)

3.2. Relating \( \mathcal{PSB} \) with other operads. In this subsection we will make explicit how the operad \( \mathcal{PSB} \) relates to a few other well known algebraic operads.

Theorem 11. The operad \( \mathcal{PSB} \) is isomorphic to the operad \( \mathcal{PostLie} \).

Proof. Recall that the operad \( \mathcal{PostLie} = \mathcal{F}(E)/(R) \) is a linear quadratic operad generated by two operations \( E = E(2) = k[S_2]/\langle [\cdot, \cdot], \triangleright \rangle \) and relations \( R = R_{\text{Lie}} \oplus R_{\text{r}} \oplus R_{t} \) where \( R_{\text{Lie}} \) are the Lie relations (antisymmetry and Jacobi) for \( [\cdot, \cdot] \) and \( R_{\text{r}} \) and \( R_{t} \) correspond to the right and left post-Lie relation \( \ominus_{\text{r}} \) and \( \ominus_{\text{t}} \) respectively:
\[
\triangleright \circ_{2} [\cdot, \cdot] - [\cdot, \cdot] \circ_{1} \triangleright - [\cdot, \cdot] \circ_{2} (\triangleright \cdot (21)) \quad \text{and} \quad \triangleright \circ_{1} [\cdot, \cdot] - a_{\circ} + a_{\circ} \cdot (213),
\]
where \( a_{\circ} = \triangleright \circ_{1} \triangleright - \triangleright \circ_{2} \triangleright \).
Let us define horizontal maps in the diagram

\[
\begin{array}{ccc}
\mathcal{F}(E) & \overset{h}{\longrightarrow} & \mathcal{LW} \\
p & \downarrow g & \downarrow \pi \\
\text{PostLie} & \underset{\bar{h}}{\longleftarrow} & \mathcal{P}SB = \mathcal{LW}/I
\end{array}
\]

Let

\[h: \mathcal{F}(E) \rightarrow \mathcal{LW}\]

be the unique morphism of operad such that

\[h([\cdot, \cdot]) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\]

and \(h(\lhd) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\).

A direct inspection shows that \(\pi h_3(R_{Jac}) = \pi h_3(R_r) = \pi h_3(R_l) = 0\), so \(\pi h\) induces a morphism \(\bar{h}: \text{PostLie} \rightarrow \mathcal{P}SB\). Let

\[g_n: \mathcal{LW}(n) \rightarrow \mathcal{F}(E)(n)\]

be as follows. Let \(g_1(\bullet) = 1 \in K = \text{PostLie}(1)\) and \(g_2 = h_2^{-1}\). For \(n \geq 3\) and a generator \(T \in \mathcal{LW}(n)\), recall the decomposition of Lemma 10 and let

\[g_n(T) := ((g_2(S_1) \circ_i g_2(S_2)) \circ_i \cdots \circ_{i_k} g_2(S_k)) \cdot \sigma.\]

We then extend it by linearity to get \(g_n\). By uniqueness of the decomposition given in Lemma 10, \(g_n\) is well-defined and is, moreover, a morphism of operads. Also, it is a section of \(h_n\) i.e. \(h_n g_n = id_n\). A direct computation shows that \(pg_3(I) = 0\).

Therefore \(pg\) induces a morphism \(\bar{g}: \mathcal{P}SB \rightarrow \text{PostLie}\).

Let us prove that \(\bar{h}_n \bar{g}_n = id_n\) and \(\bar{g}_n \bar{h}_n = id_n\) for \(n \geq 2\).

The first identity follows from \(h_n g_n = id_n\). Indeed, for \(n \geq 3\) and \([T] \in \mathcal{P}SB(n)\), one has

\[\bar{h}_n \bar{g}_n(T + I) = \bar{h}_n pg_n(T) = \bar{h}_n (g_n(T) + R) = \pi h_n (g_n(T)) = \pi(T).\]

The other identity is obtained similarly: for \(n \geq 2\), any \([T] \in \text{PostLie}(n)\) can be written as \([T] = \sum k_i T_i + R\), where each \(T_i \in \mathcal{F}(E)(n)\) is a composition of \(Q^j_i \in E(2)\). Therefore, \(\bar{h}_n([T]) = \sum k_i \bar{h}_n(T_i)\), where each \(\bar{h}_n(T_i)\) can be written as a composition, in \(\mathcal{LW}(n)\), of \(h_2(Q^j_i)\).

By unicity of the decomposition of Lemma 10 and by definition of \(g_n\), one obtains the result. \(\square\)

Remark 12. Given an operad \(O\) and a vector space \(V\), we denote by \(O(V)\) the free \(O\)-algebra generated by \(V\). It is explicitly given by \(O(V) = \bigoplus_{n \geq 0} O(n) \otimes_S V^\otimes n\).

By Theorem 11 we know that \(\mathcal{P}SB(K)\) is the free post-Lie algebra on \(K\), which is the vector space generated by trees of \(\mathcal{P}SB\), putting aside their labelings. In other words, if we let \(K = K < \bullet >\) for a generator \(\bullet\), then \(\mathcal{P}SB(K)\) is generated by the set

\[G = \{ \cdots, \begin{array}{c} \cdots \end{array}, \begin{array}{c} \cdots \end{array}, \begin{array}{c} \cdots \end{array}, \begin{array}{c} \cdots \end{array}, \begin{array}{c} \cdots \end{array}, \begin{array}{c} \cdots \end{array}, \begin{array}{c} \cdots \end{array}, \begin{array}{c} \cdots \end{array} \}.\]
The Lie product of two generators \( R \) and \( S \) is the class of the tree \( C(\bullet; R, S) \); their post-Lie product \( R \triangleright S \) is the grafting obtained from the composition in \( \mathcal{PB} \),
\[
\begin{array}{c}
1 \\
2 \\
\circ_2 S
\end{array} \circ_1 R,
\]
by putting any labeling on \( S \) and \( R \), composing and then forgetting the labeling.

In [11] Chapoton and Livernet constructed the operad \( RT \) controlling the symmetric brace algebras together with an isomorphism \( \Phi: \mathcal{PB} \to RT \). As one has a projection \( p: \mathcal{PL} \to \mathcal{PB} \) defined by killing the Lie bracket \([ - , - ]\), it is natural to ask if it is possible to recover the operad \( RT \) from \( \mathcal{PB} \).

To answer this question, recall, in short, that the operad \( RT \) is as follows (see [11, Section 1.5] for details). The underlying \( S \)–module of \( RT \) is generated by (isomorphism classes of) labeled non-planar rooted trees. The operadic composition is defined similarly to the one in (2.1) except that, since we work with non-planar trees, the map \( \phi \) in (2.2) is seen as a map from \( \text{In}(i) \) to \( V(R) \), and the condition about the order when grafting the trees is forgotten.

Said that, consider the operadic ideal \( J \) of \( \mathcal{PB} \) generated by the class of
\[
C(\bullet; 1, 2) = \begin{array}{c}
1 \\
2
\end{array}.
\]

**Proposition 13.** The operad \( \mathcal{PB}/J \) is canonically isomorphic to the operad \( RT \). Moreover, the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{PL} & \xrightarrow{\overline{h}} & \mathcal{PB} \\
p \downarrow & & \downarrow \Phi \\
\mathcal{PL} & \xrightarrow{\phi} & RT,
\end{array}
\]
here the vertical maps are the canonical projections, \( \overline{h} \) is the isomorphism of Theorem [11] and \( \Phi \) is the isomorphism from [11, Theorem 1.9].

**Proof.** We only prove the first assertion, as the commutativity of the diagram is a straightforward verification. For all \( n \) and \( 1 \leq i \leq n - 1 \), one has
\[
(3.8)
\]

Since \( J \) is an ideal, \( J(n+1) \) is closed under the action of the \( S_{n+1} \). In particular
\[
\sigma(1) \sigma(2) \cdots \sigma(i+1) \cdots \sigma(n) \quad \sigma(1) \sigma(2) \cdots \sigma(i+1) \sigma(i) \cdots \sigma(n)
\]
belongs to \( J(n+1) \) for all \( \sigma \in S_n \). Using this observations it is not difficult to see that the class in \( \mathcal{PB}/J \) of any corolla is invariant under permutation of its leaves \( i.e. \) the class of any corolla may be seen as non-planar corolla.
As every tree of \( \mathcal{PSB} \) decomposes into corollas, its class in \( \mathcal{PSB}/\mathcal{J} \) may be seen as a non-planar tree. Moreover, recall that \( \mathcal{PSB} = \mathcal{LW}/\mathcal{I} \). Since \( \mathcal{W} \) is generated, as an operad, by \( C(\bullet; 1, 2) \) (see Lemma 10), the trees of \( \mathcal{LT}/\mathcal{J} \) are (fully) labeled. From this follows the existence of a canonical bijection between \( \mathcal{LT}/\mathcal{J} \) and \( \mathcal{RT} \). It is straightforward to check that this bijection is compatible with the operadic structures. □

4. THE \( \mathcal{PSB} \)-ALGEBRAS

We start this section by characterizing the \( \mathcal{PSB} \)-algebras. In particular we show that \( \mathcal{PSB} \) naturally encodes operations that are iterations of the post-Lie product, mimicking the case of the symmetric braces which are iterations of the pre-Lie product. We term these iterated operations post-symmetric brace algebras. Then we turn our attention to the universal enveloping algebra of a post-Lie algebra and we show how its structure of \( D \)-bialgebra arises from the corresponding post-symmetric braces.

4.1. Definition of \( \mathcal{PSB} \)-algebras. Recall the notation for the shuffle coproduct of \( T(V) \), from Convention [11] \( \Delta_{sh}(X) = \sum_{I \cup J} X_I \otimes X_J \in T(V) \otimes T(V) \) for any monomial \( X \in T(V) \).

**Proposition 14.** A \( \mathcal{PSB} \)-algebra is a pair \((g, T)\) of a Lie algebra \((g, [-, -])\) together with a linear map \( T \in \text{Hom}_g(T(g) \otimes g, g) \) satisfying the following four properties.

(i) Let \( K \) be the ideal of \( T(g) \) generated by \( xy - yx - [x, y] \) for \( x, y \in g \). For all \( z \in g \) one has

(4.1) \( T(K; z) = 0 \).

(ii) \( T(1; y) = y \), for all \( y \in g \).

(iii) For any two monomials \( X \) and \( Y = y_1 \cdots y_m \) in \( T(g) \) and \( z \in g \):

(4.2) \( T(X; T(Y; z)) = \sum_{J_0 \cup \cdots \cup J_m = \{1, \ldots, m\}} T(X_{J_0}; T(X_{J_1}; y_1), \cdots, T(X_{J_m}; y_m); z) \),

where the \( J_i \)'s form an ordered partition and with the convention that \( X_{\emptyset} = 1 \in T(g) \).

(iv) For all monomial \( X \in T(g) \) and all \( x, y \in g \), one has

(4.3) \( T(X; [x, y]) = \sum_{I \cup J} [T(X_I; x), T(X_J; y)] \)

noticing that, by (ii), the two extremal terms read as \([T(X; x), y]\) and \([x, T(X; y)]\).

**Proof.** Since \( \mathcal{PSB} \) is the quotient of \( \mathcal{LW} \) by the Lie ideal \( \mathcal{I} \) (see Definition 10), it is clear that the corolla \( C(\bullet; 1, 2) \) endows \( g \) with a Lie bracket.

Let us shorten the notation \( C(n + 1; 1, \ldots, n) \) to \( C_{n+1} \). For \( n \geq 0 \), denote by \( T_n : g^{\otimes n} \otimes g \rightarrow g \) the operation corresponding to \( C_{n+1} \). They induce an operation \( T : T(g) \otimes g \rightarrow g \).

(i) is a direct consequence of (3.8).

(ii) means that \( T_0 : g \rightarrow g \) is the identity map; again this is immediate from its definition.
Let us show (iii). For \( n, m \geq 0 \), the element \( C_{n+1} \circ_{n+1} C_{m+1} \in \mathcal{L}(n + m + 1) \) is, by definition of the operadic composition, given by

\[
(4.4) \quad \sum_{J_1 \sqcup \cdots \sqcup J_{m+1} = \{1 \ldots < n\}} \sigma_{J_1} \circ_{J_1} \cdots \circ_{J_{m+1}} \sigma_{J_{m+1}}.
\]

The sum runs over all partitions of \( \{1 < \ldots < n\} \) by possibly empty order sets \( J_i \); they correspond to the fibers of the maps of sets \( \phi: \{1 < \ldots < n\} \rightarrow \operatorname{Ang}_{\min}(C_{m+1}) \) in (2.1). The leaves of the (green) corolla with root \( n + i \) are labeled by \( J_i \).

Denote by \( k_i = |J_i| \) for \( 1 \leq i \leq m \) and \( k = |J_{m+1}| \). Note that every tree of the sum (4.4) can be written as

\[
(\cdots (C_{k+m+1} \circ_{k+1} C_{k+1}) \circ_{k+2} \cdots \circ_{k+m} C_{k+m+1}) \cdot \sigma_J,
\]

where \( \sigma_J \in S_{m+n+1} \) is given by

\[
\begin{pmatrix}
1 \cdots k & k+1 \cdots k_1 + k & k_1 + k + 1 & \cdots & \kappa \cdots \kappa + k_m & \kappa + k_m + 1 & n + m + 1 \\
J_{m+1} & J_1 & n + 1 & \cdots & J_m & n + m & n + m + 1
\end{pmatrix}.
\]

Here \( \kappa = k + k_1 + 2 + k_2 + 2 + \ldots + k_{m-1} + 2 \). Since \( T_n: g^\otimes n+1 \rightarrow g \) is the operation corresponding to \( C_{n+1} \), we see that (4.5) provides the result.

The proof of (iv) is similar to that of (iii), by computing \( C_{n+1} \circ_{n+1} C(\bullet; 1, 2) \in \mathcal{LW}(n + 2) \). To end the proof, recall that every tree of \( \mathcal{W} \) decomposes into corollas, and every tree of \( \mathcal{L} \) decomposes as in (3.7). This essentially says that the relations (i)-(iv) generates any other.

\( \square \)

Remark 15. A few comments are in order.

1. As remarked in the proof of the theorem above, the morphism \( T \) defines, by restriction to the \( g^\otimes n \)'s, a family of linear maps \( \{T_n\}_{n \geq 1} \), where \( T_n \in \operatorname{Hom}_g(g^\otimes n, g) \). Obviously the properties of \( T \) expressed by the Formulas (4.1), (4.2) and (4.3) correspond to conditions on the \( T_n \)'s. For example the compatibility between \( T \) and the Lie bracket expressed in (4.1), when read on the level of the \( T_n \)'s becomes:

\[
T_n(x_1, \ldots, [x_i, x_{i+1}], \ldots; x_{n+2}) = T_{n+1}(x_1, \ldots, x_i, x_{i+1}, \ldots; x_{n+2}) - T_{n+1}(x_1, \ldots, x_{i+1}, x_i, \ldots; x_{n+2}).
\]

In Proposition 11 will be analyzed in more details how the conditions on the linear map \( T \) can be translated to the level of the family of the \( \{T_n\}_{n \geq 1} \).

2. Applied on \( X = x \in g \) and \( Y = y_1 y_2 \in g^\otimes 2 \), formula (4.2) gives

\[
T(x; T(y_1, y_2; z)) = T(x, y_1, y_2; z) + T(T(x; y_1), y_2; z) + T(y_1, T(x; y_2); z),
\]

for all \( z \in g \).

For \( X = xy \in g^\otimes 2 \) and \( u, v \in g \), equation (4.3) gives

\[
T(x, y; [u, v]) = [T(x, y; u), v] + [T(x; u), T(y; v)] + [T(y; u), T(x; v)] + [u, T(x, y; v)].
\]

Using the family of linear maps \( \{T_n\}_{n \geq 1} \), the isomorphism \( \mathcal{PSB} \cong \mathcal{PostLie} \) of Theorem 11 says that the \( \mathcal{PSB} \)-algebras can be characterized as follows.
Proposition 16. A structure of a $\mathcal{PSB}$–algebra on a vector space $V$ is the data of:

- a post-Lie algebra $(V, \triangleright, [\cdot, \cdot])$; and,
- a family $\{T_n\}_{n \geq 1}$ of linear maps $T_n \in \text{Hom}_K(V^\otimes n+1, V)$ such that for each $n \geq 2$

\begin{equation}
T_n(x_1, \ldots, x_n; y) = T_1(x_1; T_{n-1}(x_2, \ldots, x_n; y)) - \sum_{k=2}^{n} T_{n-1}(x_2, \ldots, T_1(x_k), \ldots, x_n; y),
\end{equation}

and $T_1 = \triangleright$.

Proof. Let $(\mathfrak{g}, T)$ be a $\mathcal{PSB}$–algebra and let $\{T_n\}_{n \geq 1}$ be the corresponding family of linear operators, see Remark 15. Let $X = x \in \mathfrak{g}$ and $Y = x_1 \cdots x_{n-1} \in \mathfrak{g}^\otimes n-1$. For all $y \in \mathfrak{g}$, one has

\[
T(X; T(Y; y)) = T_1(x; T_{n-1}(x_1, \ldots, x_{n-1}; y)).
\]

Since

\begin{equation}
\Delta_{sh}^{n-1}x = \sum_{i=1}^{n} x_{(i)},
\end{equation}

where $x_{(i)} = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1 \in \mathfrak{g}^\otimes n$, with $x$ in the position $i$, the term $T(X; T(Y; y))$ can be written as

\[
T_n(x, x_1, \ldots, x_{n-1}; y) + \sum_{k=1}^{n-1} T_{n-1}(x_1, \ldots, T_1(x; x_k), \ldots, x_{n-1}; y),
\]

see item (2) in Remark 16 which, up to renumbering, is Formula (4.6). Formulas (4.1) and (4.3) imply that, for all $x, y$ and $z$ in $\mathfrak{g}$

\begin{equation}
T_1([x, y]; z) = T_2(x, y; z) - T_2(y, x; z),
\end{equation}

and, respectively,

\begin{equation}
T_1(x; [y, z]) = [T_1(x; y), z] + [y, T_1(x; z)]
\end{equation}

and since $\triangleright = T_1$, one concludes that $(\mathfrak{g}, \triangleright)$ is a post-Lie algebra. On the other hand, suppose that on a post-Lie algebra $(\mathfrak{g}, \triangleright)$ is defined a family of linear operators $\{T_n\}_{n \geq 1}$, where $T_n \in \text{Hom}_K(\mathfrak{g}^\otimes n+1, \mathfrak{g})$ with $T_1 = \triangleright$, such that Formula (4.6) holds true. Let $T \in \text{Hom}_K(T(\mathfrak{g}), \mathfrak{g})$ be the unique linear map such that $T(1; y) = y$ for all $y \in \mathfrak{g}$ and such that, for each $n \geq 1$, its restriction to $\mathfrak{g}^\otimes n$ is equal to $T_n$. Equations (4.1), (4.2) and (4.3) follow from the Formulas (4.5), (4.7) and, respectively, from (4.9) using an inductive argument on the length of the monomial $X$ and $Y$. For example, since every $x \in \mathfrak{g}$ is primitive, for $X = x \in \mathfrak{g}$, (4.3) is Formula (4.9). Suppose now that (4.3) hold for every $X$ in $\mathfrak{g}^\otimes k$, for all $k \leq n-1$ and let $X' = xx$ with $X \in \mathfrak{g}^\otimes n-1$. Then $T(X'; [y, z]) = T(XX; [y, z]) = T_n(x, x_1, \ldots, x_{n-1}; [y, z])$, which, using Formula (4.9), becomes

\[
T_1(x; T_{n-1}(x_1, \ldots, x_{n-1}; [y, z])) - \sum_{k=1}^{n-1} T_{n-1}(x_1, \ldots, T_1(x; x_k), \ldots, x_{n-1}; [y, z]).
\]
To conclude the proof it suffices to apply the inductive hypothesis to the terms in the formula above and compare the result obtained with what one gets from the following computation

\[ T((xX)_{(1)}; y, z) + [y, T((xX)_{(2)}; z)] \]

recalling that for all \( x \in \mathfrak{g} \):

\[ \Delta_{sh}(xX) = (xX)_{(1)} \otimes (xX)_{(2)} = xX_{(1)} \otimes X_{(2)} + X_{(1)} \otimes xX_{(2)}. \]

One gets Formula (4.1) from Formulas (4.8) and (4.6) and Formula (4.2) from (4.6) using a similar strategy.

4.2. \( \mathcal{PSB} \)-algebras vs \( D \)-algebras. We now discuss briefly some relations between the \( \mathcal{PSB} \)-algebras and the universal enveloping algebras of the corresponding post-Lie algebras. These algebras, which were introduced in \([31]\) and there termed \( D \)-algebras, were further studied in \([29]\). In what follows we adopt the definition of \( D \)-algebra proposed in the recent preprint \([13]\).

**Definition 17** \((\ref{13})\). A \( D \)-algebra is a unital, associative algebra \( (D, \cdot, 1) \) equipped with a non-associative product \( \triangleright : V \otimes V \rightarrow V \) an exhaustive increasing filtration

\[ \mathbb{K} \cdot 1 = D^0 \subset D^1 \subset D^2 \subset \cdots \subset D^n \subset \cdots \]

and an augmentation \( \epsilon : D \rightarrow \mathbb{K} \) such that \( D^i \cdot D^j \subset D^{i+j} \) and

(i) \( 1 \triangleright X = X \) and \( X \triangleright 1 = 0 \), for all \( X \in D \).

(ii) \( D_1 = \ker(\epsilon) \cap D^1 \) is closed with respect to the antisymmetrization of the associative product and with respect to the bilinear product \( \triangleright \) and it generates \((D, \cdot)\).

(iii) \( x \triangleright (X \cdot Y) = (x \triangleright X) \cdot Y + X \cdot (x \triangleright Y) \) for all \( x \in D_1 \) and \( X, Y \in D \).

(iv) \( (x \cdot X) \triangleright Y = x \triangleright (X \triangleright Y) - (x \triangleright X) \triangleright Y \), for all \( x \in D_1 \) and \( X, Y \in D \).

**Proposition 18** \((\ref{29})\). \( D_1 \) is a post-Lie algebra.

**Proof.** By antisymmetrizing the associative product \( \cdot \) one gets a Lie bracket. Axiom (ii) says that \( D_1 \) is closed with respect to both this bracket and to the product \( \triangleright \). Now axiom (iii) implies \((\ref{13})\) and axiom (iv) implies \((\ref{14})\), proving the statement. \( \square \)

To investigate further the relations between \( D \)-algebras and \( \mathcal{PSB} \)-algebras, it is convenient to enhance the structure defining a \( D \)-algebra as follows.

**Definition 19.** A \( D \)-bialgebra is a bialgebra \( (D, \cdot, 1, \Delta, \epsilon) \) endowed with a non-associative product \( \triangleright : D \otimes D \rightarrow D \) and an exhaustive, increasing filtration

\[ \mathbb{K} \cdot 1 = D^0 \subset D^1 \subset D^2 \subset \cdots \subset D^n \subset \cdots \]

such that \( D^i : D^j \subset D^{i+j} \) and

(D1) \( 1 \triangleright X = X \) and \( X \triangleright 1 = 0 \), for all \( X \in D \).

(D2) \( D_1 = \ker(\epsilon) \cap D^1 = \text{Prim}(D) \) which generates \((D, \cdot)\).

(D3) \( \Delta(X \triangleright Y) = (X_{(1)} \triangleright Y_{(1)}) \otimes (X_{(2)} \triangleright Y_{(2)}) \).

(D4) \( X \triangleright (Y \cdot Z) = (X_{(1)} \triangleright Y) \cdot (X_{(2)} \triangleright Z) \).

(D5) \( (x \cdot X) \triangleright y = x \triangleright (X \triangleright y) - (x \triangleright X) \triangleright y \).

(D6) \( D_1 \) is closed under the antisymmetrization of the associative product.

In item \([\text{D2}]\) \( \text{Prim}(D) \) is the vector space of the primitive elements of the coalgebra \((D, \Delta, \epsilon)\) i.e. the set of all \( x \in D \) such that \( \Delta x = x \otimes 1 + 1 \otimes x. \)
First we prove that any $D$–bialgebra has an underlying structure of a $D$–algebra.

**Proposition 20.** If $(D, 1, \Delta, \epsilon, \triangleright)$ is a $D$–bialgebra, then $(D, 1, \epsilon, \triangleright)$ is a $D$–algebra.

**Proof.** Axioms \([D2]\) and \([D3]\) imply at once that $D_1$ is closed under the product $\triangleright$. To conclude the proof it suffices to prove that every $D$–bialgebra fulfills the axioms \((iii)\) and \((iv)\) of the $D$–algebras. Property \((iii)\) follows at once from \([D1]\), \([D2]\) and \([D4]\). To prove that in every $D$–bialgebra \((iv)\) holds, let $x \in D_1$ and $X, Y \in D$. If the length of $Y$ is 1, i.e. if $Y \in D_1$, then \((iv)\) is \([D5]\). Let $Y' = Y \cdot y$, where length of $Y$ is $n - 1$ and suppose that \((iv)\) holds for each $Y$ of length at least $n - 1$. Then

$$(x \cdot X) \triangleright Y' = (x \cdot X) \triangleright (Y \cdot y) = ((x \cdot X)(1) \triangleright Y) \cdot ((x \cdot X)(2) \triangleright y),$$

see \([D5]\). Since the coproduct is an algebra morphism and $x$ is primitive, the last term of the previous equality becomes

$$((x \cdot X(1) \triangleright Y) \cdot (X(2) \triangleright y) + (X(1) \triangleright Y) \cdot ((x \cdot X(2)) \triangleright y),$$

which, by the inductive hypothesis, can be written as

$$(x \triangleright (X(1) \triangleright Y) - (x \triangleright X(1)) \triangleright Y) \cdot (X(2) \triangleright y) + (X(1) \triangleright Y) \cdot (x \triangleright (X(2)) \triangleright y).$$

Since $x \in D_1$, using \([D4]\) the previous expression can be written as

$$x \triangleright (X \triangleright (Y \cdot y)) - [(x \triangleright X(1)) \triangleright Y) \cdot (X(2) \triangleright y) + (X(1) \triangleright Y) \cdot ((x \triangleright X(2)) \triangleright y)].$$

Using \([D3]\) together with the property of the coproduct of being an algebra morphism, the terms in the square bracket can be written as $(x \triangleright X) \triangleright (Y \cdot y)$, giving the proof of the proposition. □

Let

$$\mathcal{U}: \text{Lie} \to \text{Bialgebra}$$

be the classical universal enveloping algebra functor. With the following result we show that it enriches to a functor

\[(4.10) \quad \mathcal{U}: \text{PostLie} \to D \text{–bialgebra}\]

with adjoint the primitive elements functor defined in \([1.9]\).

**Theorem 21.** Let $\mathfrak{g}$ be a Lie algebra. There is a one-to-one correspondence between the structures of $\mathcal{PSB}$–algebra on $\mathfrak{g}$ and the structures of $D$–bialgebra on $\mathcal{U}(\mathfrak{g})$.

**Proof.** Suppose first that $\mathfrak{g}$ carries a structure of a $\mathcal{PSB}$-algebra. Formula \([4.1]\) implies that $T$ descends to a linear map, still called $T$, from $\mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}$ to $\mathfrak{g}$, such that $T(1; x) = x$ for all $x \in \mathfrak{g}$. Define now $\triangleright: \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ by

\[(4.11) \quad X \triangleright x = T(X; x), \forall X \in \mathcal{U}(\mathfrak{g}), x \in \mathfrak{g},\]

\[(4.12) \quad X \triangleright 1 = 0, \forall X \in \mathcal{U}(\mathfrak{g}),\]

and

\[(4.13) \quad X \triangleright Y = (X(1) \triangleright y_1) \cdots (X(n) \triangleright y_n)\]
for all $Y = y_1 \cdots y_n$ and any monomial $X \in \mathcal{U}(\mathfrak{g})$. Endowing $\mathcal{U}(\mathfrak{g})$ with its standard filtration and its standard bialgebra structure, Properties $[D2]$ and $[D6]$ are automatically fulfilled. Furthermore note that if $X = x \in \mathfrak{g}$ and $Y = y_1 \cdots y_n \in \mathcal{U}(\mathfrak{g})$, $\mathfrak{g}$ implies that

$$x \triangleright Y = \sum_{i=1}^{n} y_1 \cdots (x \triangleright y_i) \cdots y_n,$$

i.e. that $\triangleright$ extends to $\mathcal{U}(\mathfrak{g})$ as a derivation of the associative product. This observation, together with Formula (4.2), implies that $x, y, z$.

Suppose now that $\mathfrak{g}$ be the natural multiplication map. One has $\mathfrak{g}$ be defined by

$$\tau_{n-k,k} : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

be the linear map defined by

(4.14) \hspace{1cm} \nabla_n(A_1 \otimes \cdots \otimes A_n \otimes x_1 \cdots x_n) = (A_1 \triangleright x_1) \cdots (A_n \triangleright x_n)

for $A_1, \ldots, A_n \in \mathcal{U}(\mathfrak{g})$ and $x_1 \cdots x_n \in \mathcal{U}(\mathfrak{g})$. Note that $\nabla_0$ is the multiplication map of the ground field $\mathbb{K}$, that $\nabla_1(A \otimes x) = A \triangleright x$, for all $A \in \mathcal{U}(\mathfrak{g})$ and $x \in \mathfrak{g}$ and that, for all $X, Y \in \mathcal{U}(\mathfrak{g})$, with $Y = y_1 \cdots y_n$

(4.15) \hspace{1cm} \nabla_n(\Delta^{n-1}X \otimes Y) = \nabla_n(X_1 \otimes \cdots \otimes X_{(n)} \otimes Y) = X \triangleright Y,

see [4.13]. Furthermore, let

$$\tau_{n-k,k} : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

be defined by

(4.16) \hspace{1cm} \tau_{n-k,k}(A_1 \otimes \cdots \otimes A_n \otimes x_1 \cdots x_n) = (A_1 \otimes \cdots \otimes A_{n-k} \otimes x_1 \cdots x_{n-k}) \otimes (A_{n-k+1} \otimes \cdots \otimes A_n \otimes x_{n-k+1} \cdots x_n)

and let

$$m_{n-k,k} : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$$

be the natural multiplication map. One has

(4.17) \hspace{1cm} m_{n-k,k} \circ (\nabla_{n-k} \otimes \nabla_k) \circ \tau_{n-k,k} = \nabla_n.

Suppose now that $X, Y, Z \in \mathcal{U}(\mathfrak{g})$ with $Y = y_1 \cdots y_n$ and $Z = z_1 \cdots z_k$. One has

$$\mathfrak{g}(X_1 \triangleright Y) \cdot (X_2 \triangleright Z) = \nabla_n(\Delta^{n-1}X_1 \otimes Y) \cdot \nabla_k(\Delta^{k-1}X_2 \otimes Z) = m_{n-k} \circ (\nabla_n \otimes \nabla_k) \circ \tau_{n-k} \circ (\Delta^{n-1}X_1 \otimes \Delta^{k-1}X_2 \otimes (Y \cdot Z)) = \nabla_{n+k}(\Delta^{n+k-1}X \otimes (Y \cdot Z)) \hspace{1cm} \text{(1.16)}$$

$$\mathfrak{g}(X_1 \triangleright (Y \cdot Z)) = X \triangleright (Y \cdot Z),$$

where the last equality follows from the coassociativity of the coproduct. On the other hand, from

$$\nabla_{n+k}(\Delta^{n+k-1}X \otimes (Y \cdot Z)) = X \triangleright (Y \cdot Z),$$
which is what we wanted to show. It remains to prove item \( \text{[D3]} \). If \( Y = y \in \mathfrak{g} \), then \( X \triangleright y \) is primitive, which corresponds to \( \text{[D3]} \). Suppose by induction that \( \text{[D3]} \) is true for any monomial \( Y = y_1 \cdots y_k \) of length \( k \leq n \).

Let \( Y' = Y \cdot y \) for \( Y \) of length \( k \leq n \) and \( y \in \mathfrak{g} \). By \( (4.13) \), one has
\[
\Delta(X \triangleright (Y \cdot y)) = \Delta((X_1 \triangleright Y) \cdot \Delta(X_2 \triangleright y)) = \Delta((X_1 \triangleright Y) \cdot ((X_2 \triangleright y) \otimes 1 + 1 \otimes (X_2 \triangleright y))).
\]

By induction hypothesis, this gives
\[
\Delta(X \triangleright (Y \cdot y)) = ((X_1 \triangleright Y_1) \otimes (X_2 \triangleright Y_2)) \cdot ((X_3 \triangleright y) \otimes 1 + 1 \otimes (X_3 \triangleright y))
\]
\[
= ((X_1 \triangleright Y_1) \cdot (X_3 \triangleright y)) \otimes (X_2 \triangleright Y_2) + (X_3 \triangleright Y_1) \otimes ((X_2 \triangleright Y_2) \cdot (X_3 \triangleright y)).
\]

Applying again \( (4.13) \), and noticing that the coproduct is cocommutative, one obtains
\[
\Delta(X \triangleright (Y \cdot y)) = (X_1 \triangleright (Y_1 \cdot y)) \otimes (X_2 \triangleright Y_2) + (X_1 \triangleright Y_1) \otimes (X_2 \triangleright (Y_2 \cdot y))
\]
\[
= (X_1 \triangleright Y'_1) \otimes (X_2 \triangleright Y'_2).
\]

Conversely, let \( \mathcal{U}(\mathfrak{g}) \) be equipped with its standard filtration and its standard bialgebra structure and supposed it be endowed with a structure of \( D \)-bialgebra whose \( D \)-product is denoted by \( \triangleright \). \( \text{[D1]} \) implies that \( D_1 = \mathfrak{g} \) which, by \( \text{[D3]} \), has a structure of post-Lie algebra whose post-Lie product is given by the restriction of \( \triangleright \) to \( \mathfrak{g} \otimes \mathfrak{g} \). It follows from Theorem \( \text{[11]} \) that \( \mathfrak{g} \) is a \( \mathcal{PSB} \)-algebra, with higher operations given by Proposition \( \text{[16]} \).

Now let us see that the these two assignments are inverse to each other. In fact, this follows from Theorem \( \text{[11]} \) since it says that two \( \mathcal{PSB} \)-algebra structures on the Lie algebra \( \mathfrak{g} \) with the same post-Lie product are equal. \( \square \)

**Proposition 22.** The functors \( \text{Prim} \) and \( \mathcal{U} \) are adjoints.

**Proof.** Let \( \mathfrak{g} \) be a post-Lie algebra and \( D \) be a \( D \)-bialgebra. Let \( f: \mathfrak{g} \to \text{Prim}(D) \) be a morphism of post-Lie algebras. There exists a unique morphism of algebras \( \tilde{f}: \mathcal{U}(\mathfrak{g}) \to D \) that extends \( f \), that is, such that \( \text{Prim}(f) \circ \eta = f \) where \( \eta: \mathfrak{g} \to \text{Prim}(\mathcal{U}(\mathfrak{g})) \) is the canonical identification. It remains to prove that \( \tilde{f} \) is a morphism of coalgebras and that \( \tilde{f}(X \triangleright Y) = \tilde{f}(X) \triangleright \tilde{f}(Y) \) for all \( X, Y \in \mathcal{U}(\mathfrak{g}) \). The first assertion is straightforward, while the second one can be shown by an induction on \( n \) in the filtration \( \{ \mathcal{U}_n(\mathfrak{g}) \} \).

For a \( D \)-bialgebra \( (\mathcal{D}, \ast, 1, \Delta, \epsilon, \triangleright) \), let \( *: D \otimes D \to D \) be the linear map given by
\[
(4.18) \quad A \ast B := A_{(1)} \cdot (A_{(2)} \triangleright B) \text{ for all } A, B \in D.
\]

The following is a straightforward, though a bit long, verification.

**Proposition 23.** \( (\mathcal{D}, *, 1, \Delta, \epsilon) \) is a bialgebra.

We let
\[
(4.19) \quad *: D \triangleright \text{bialgebra} \to \text{Bialgebra}
\]
denote the induced functor.

**Remark 24.** If \( x \) belongs to \( \text{Prim}(D) \), then one has
\[
(4.20) \quad x \ast X = x \cdot X + x \triangleright X
\]
for all homogeneous element \( X \in D \).
Remark 25. The product $*$ defined on the universal enveloping algebra of a post-Lie algebra is known as the Grossmann-Larson product, see for example [31, 30] and references therein.

5. Post-Lie Magnus expansion

In this section we further investigate from the view-point of the $PSB$–algebras, the so called post-Le Magnus expansion, an interesting series which can be defined in a (suitable completion) of any post-Lie algebra. To this end we start by recalling a few basic properties of post-Lie algebras, see [28], then we introduce the post-Lie Magnus expansion and we give a combinatorial method to compute its coefficients.

5.1. Post-Lie Magnus expansion and BCH formula. Let $(\mathfrak{g}, [-, -], \triangleright)$ be a post-Lie algebra and let $([-,-]) : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ the Lie bracket defined in (1.5). Let $x \in \mathfrak{g}$ and define $\nabla : \mathfrak{g} \to \text{End}_K(\mathfrak{g})$ by

\[(5.1) \quad \nabla_x(y) = x \triangleright y, \forall y \in \mathfrak{g}.
\]

Lemma 26. $\nabla_x$ is a derivation of $(\mathfrak{g}, [-,-])$ and $(\nabla, \mathfrak{g})$ is a representation of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$.

Proof. The first statement is Formula (1.4), while the second follows from a simple computation which we omit. \hfill \Box

From now on $\mathfrak{g}$ and $\mathfrak{g}$ will denote the Lie algebra $(\mathfrak{g}, [-,-])$ and, with a slight abuse of notation, the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. Let $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and of $\mathfrak{g}$. Following [28], for all $x \in \mathfrak{g}$ let $\sigma_x : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ be defined by $\sigma_x(A) = x \cdot A$, where $\cdot$ denotes the associative product in $\mathcal{U}(\mathfrak{g})$ and let $M : \mathfrak{g} \to \text{End}_K(\mathcal{U}(\mathfrak{g}))$ be defined by

\[(5.2) \quad M(x) = M_x := \nabla_x + \sigma_x,
\]

for all $x \in \mathfrak{g}$. One has

Lemma 27. $(\mathcal{U}(\mathfrak{g}), M)$ is a representation of $\mathfrak{g}$, i.e.

\[M[[x,y]] = [M_x, M_y], \forall x, y \in \mathfrak{g}.
\]

Proof. It suffices to compute

\[
[M_x, M_y](z) = M_x(\nabla_y + \sigma_y)(z) - M_y(\nabla_x + \sigma_x)(z) \\
= M_x(y \triangleright z + y \cdot z) - M_y(x \triangleright z + x \cdot z) \\
= x \triangleright (y \triangleright z) + x \triangleright (y \cdot z) + x \cdot (y \triangleright z) + y \cdot (y \cdot z) \\
- y \cdot (x \triangleright z) - y \cdot (x \cdot z) - y \cdot (x \cdot z) \\
= x \triangleright (y \cdot z) - y \cdot (x \cdot z) + (x \cdot y \cdot x \cdot z) + [x, y] \cdot z,
\]

and

\[
M[[x,y]](z) = (x \triangleright y - y \triangleright y) \triangleright z + [x, y] \triangleright z + (x \triangleright y - y \triangleright x) \cdot z + [x, y] \cdot z,
\]

which, thanks to (1.4), reduces to the result of the previous computation. \hfill \Box

Using the universal property of the enveloping algebra and the previous lemma, one can extend the application $M$ above defined to an application $M : \mathcal{U}(\mathfrak{g}) \to \text{End}_K(\mathcal{U}(\mathfrak{g}))$ which defines on $\mathcal{U}(\mathfrak{g})$ a structure of (left) $\mathcal{U}(\mathfrak{g})$-module. More precisely one has the following.
Proposition 28 ([28]). The application \( \phi : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}) \), defined by
\[
\phi(A) = M_A(1)
\]
for all \( A \) monomial in \( \mathcal{U}(\mathfrak{g}) \) and then extended by linearity to \( \mathcal{U}(\mathfrak{g}) \), is both an isomorphism of (left) \( \mathcal{U}(\mathfrak{g}) \)-modules and of coalgebras.

**Proof.** First note that the universal property of the enveloping algebra implies that the application \( M : \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}_K(\mathcal{U}(\mathfrak{g})) \) obtained from the application defined in [5.2] is a morphism of associative algebras, i.e.
\[
M_{x_1 \odot \cdots \odot x_n} = M_{x_1} \circ \cdots \circ M_{x_n},
\]
for all monomials \( x_1 \odot \cdots \odot x_n \) in \( \mathcal{U}(\mathfrak{g}) \). From this observation follows that \( \phi \) restricts to the identity map from \( \mathcal{U}_{\leq 1}(\mathfrak{g}) \) to \( \mathcal{U}_{\leq 1}(\mathfrak{g}) \). Moreover, a simple induction on the length of the monomial shows that
\[
\phi(x_1 \odot \cdots \odot x_n) = x_1 \cdot x_2 \cdots x_n \in \mathcal{U}_{\leq n-1}(\mathfrak{g}),
\]
for all monomial of degree \( n \) in \( \mathcal{U}(\mathfrak{g}) \). From this one easily deduces that the restriction of \( \phi \) to \( \mathcal{U}_{\leq n}(\mathfrak{g}) \) surjects onto \( \mathcal{U}_{\leq n}(\mathfrak{g}) \), for all \( n \geq 2 \). On the other hand, if such a restriction had non trivial kernel, then there should be a monomial of length \( n \), say \( x_1 \cdot x_2 \cdots x_n \), contained in \( \mathcal{U}_{\leq n-1}(\mathfrak{g}) \), which would imply that \( x_1 \cdot x_2 \cdots x_n = 0 \). Since \( \mathcal{U}(\mathfrak{g}) \) is an integral domain, at least one \( x_i \) should be equal to zero, which, in turn would imply that \( x_1 \odot \cdots \odot x_n = 0 \), proving that, for all \( n \geq 2 \), the restriction of \( \phi \) is a linear isomorphism between \( \mathcal{U}_{\leq n}(\mathfrak{g}) \) and \( \mathcal{U}_{\leq n}(\mathfrak{g}) \). We are left to show that \( \phi \) is compatible with the coalgebra structures of the universal enveloping algebras.

The compatibility of \( \phi \) with the counits is clear. Then note that if \( A = x \odot X \) where \( X \) is a monomial of length \( n - 1 \)
\[
\phi(A) = x \cdot \phi(X) + x \triangleright \phi(X).
\]
Writing \( \Delta \) to denote the coproduct both in \( \mathcal{U}(\mathfrak{g}) \) and in \( \mathcal{U}(\mathfrak{g}) \) one has
\[
\Delta \circ \phi(x) = (\phi \otimes \phi) \circ \Delta(x), \forall x \in \mathfrak{g}.
\]
Suppose that this identity holds for every monomial \( X \in \mathcal{U}(\mathfrak{g}) \) of length \( n - 1 \), i.e.
\[
(\phi(X))(x_1) \otimes (\phi(X))(x_2) = \Delta \circ \phi(X) = (\phi \otimes \phi) \circ \Delta(X) = (\phi(X_1) \otimes \phi(X_2)),
\]
then, if \( A = x \odot X \), one can compute
\[
\Delta \circ \phi(A) = \Delta(x \odot X) = \Delta(x) \otimes \Delta(X) - (\phi \otimes \phi)(x_1) \otimes (x_2) + (x_1) \otimes (x_2) \otimes \phi(X_1) \otimes \phi(X_2) + \phi(X_1) \otimes \phi(X_2).
\]
On the other hand
\[
\Delta \circ \phi(A) \overset{\text{(1.23)}}{=} \Delta(x \cdot \phi(X) + x \triangleright \phi(X)) = \Delta(x \cdot \phi(X)) + \Delta(x \triangleright \phi(X)) = \Delta(x \cdot \phi(X)) + \Delta(x \triangleright \phi(X)) = \Delta(x \cdot \phi(X) + x \triangleright \phi(X)) = (x \cdot \phi(X))(x_1) \otimes (x \cdot \phi(X))(x_2) = (x \cdot \phi(X))(x_1) \otimes (x \cdot \phi(X))(x_2) = (x \cdot \phi(X))(x_1) \otimes (x \cdot \phi(X))(x_2).
\]

Recall the product \( * \) defined in (4.18).
Theorem 29 ([18]). \( \phi \) is an isomorphism of bialgebras between \( (\mathcal{U}(\mathfrak{g}), \circ, 1, \Delta, \epsilon) \) and \( (\mathcal{U}(\mathfrak{g}), *, 1, \Delta, \epsilon) \).

Proof. It is a direct verification. \( \square \)

Remark 30. The previous theorem was first proven in [18], even though there it was phrased in a slightly different way. The analogue if this statement for pre-Lie algebra was first shown to be true in [32, 33].

Remark 31. From a more categorical point of view one can rephrase the result of 29 in terms of the existence of an isomorphism of functors

\[
\phi: \mathcal{U} \circ \mathfrak{m} \to * \circ \mathcal{U}
\]

extending the identity

\[
\text{Prim} \circ \mathcal{U} \circ \mathfrak{m} = \text{Prim} \circ * \circ \mathcal{U}.
\]

In the previous formulas, the functor \( * \) was defined in [4,19], \( \mathcal{U} \), and Prim were defined in [4,10] and respectively in [1,9] while

\( \mathfrak{m} : \text{PostLie} \to \text{Lie} \)

is the functor induced by the bracket (1.3).

In what follows we will need to work with a suitable completion of the Hopf algebras \( \mathcal{U}(\mathfrak{g}), \mathcal{U}_c(\mathfrak{g}) \) and \( \mathcal{U}_c(\mathfrak{g}) \). We will denote these completions as \( \hat{\mathcal{U}}(\mathfrak{g}), \hat{\mathcal{U}}_c(\mathfrak{g}) \) and \( \hat{\mathcal{U}}_c(\mathfrak{g}) \), without making any notational difference between the structural operations of the original and the completed Hopf algebras (i.e. we will denote by \( \Delta \) both the coproduct on the original and of the complete Hopf algebra). The completions we will be interested in are obtained as inverse limits of quotients of the original Hopf algebras by the powers of their augmentation ideals, and in these enlarged Hopf algebras it will make sense consider infinite series like exponentials or logarithms. Furthermore, the original Lie algebras \( \mathfrak{g} \) and \( \mathfrak{g} \) inherit a completion from the ambient completed universal enveloping algebras. To save notation we will not introduce new symbols to distinguish between the original and the completed Lie algebras, hoping that it will be clear from the context which Lie algebras we are considering. For more details about the completion of Hopf algebras we refer the reader to [34], see also [26] and [21].

Note that since \( \phi(x_1 \circ \cdots \circ x_n) = x_1 * \cdots * x_n \), \( \phi \) extends to an isomorphism \( \phi: \hat{\mathcal{U}}_c(\mathfrak{g}) \to \hat{\mathcal{U}}_c(\mathfrak{g}) \) of complete Hopf algebras. Moreover, denoting with \( \circ \) any one of the products \( \cdot, * \) and \( \circ \), with \( \mathfrak{g} \) any one of the Lie algebras \( \mathfrak{g} \) or \( \mathfrak{g} \), and with \( \exp_\circ \) and \( \log_\circ \) the corresponding exponential and logarithm maps, one has that \( \xi \) is a group-like element of \( \hat{\mathcal{U}}_c(\mathfrak{g}) \) if and only if \( \xi = \exp_\circ(x) \) for a unique \( x \in \text{Prim}(\hat{\mathcal{U}}_c(\mathfrak{g})) \). In other words,

\[
\exp_\circ: \text{Prim}(\hat{\mathcal{U}}_c(\mathfrak{g})) \to \text{Group}(\hat{\mathcal{U}}_c(\mathfrak{g})),
\]

is a bijection whose inverse is

\[
\log_\circ: \text{Group}(\hat{\mathcal{U}}_c(\mathfrak{g})) \to \text{Prim}(\hat{\mathcal{U}}_c(\mathfrak{g})).
\]

Remark 32. Recall that \( \text{Prim}(\hat{\mathcal{U}}_c(\mathfrak{g})) = \mathfrak{g} \).

In particular the application \( \eta: \mathfrak{g} \to \mathfrak{g} \), defined by

\[
\eta = \log \circ \phi \circ \exp_\circ,
\]

is a bijection.
Remark 33. Since $\mathfrak{f}$ and $\mathfrak{g}$ are two Lie algebra having the same underlying vector space, $\eta$ can be thought of as a map between $\mathfrak{g}$ and itself. Furthermore, note that $\phi : \mathcal{U}_\circ(\mathfrak{f}) \to \mathcal{U}_\circ(\mathfrak{g})$ restricts to a bijection $\phi : \text{Group}(\mathcal{U}_\circ(\mathfrak{f})) \to \text{Group}(\mathcal{U}(\mathfrak{g}))$. In particular, since $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}_\circ(\mathfrak{g})$ have the same coalgebra structure, $\phi(\exp_\circ x) = \exp x$ is a group-like element of $\mathcal{U}(\mathfrak{g})$.

Let $\chi : \mathfrak{g} \to \mathfrak{g}$ be the inverse of $\eta$, i.e. $\chi$ is the application that takes every $x \in \mathfrak{g}$ to the (unique) element $\chi(x) \in \mathfrak{g}$ such that

$$\exp(x) = \exp_\circ(\chi(x)).$$

Definition 34. The map $\chi$ is called the post-Lie Magnus expansion.

Remark 35. The map $\chi$ is a very interesting mathematical object. It was introduced in [19], see also [18], to analyze iso-spectral type flow equations defined on a post-Lie algebra whose post-Lie product was coming from a solution of the modified Yang-Baxter equation. More in general, in [21], it was observed that given a post-Lie algebra $(\mathfrak{g}, \triangleright)$, for every $x \in \mathfrak{g}$, $\chi_x(x) := \chi(tx) \in \mathfrak{g}[[t]]$ satisfies the following non-linear ODE

$$\dot{\chi_x}(t) = (d\exp_\circ)^{-1}_{\chi_x(t)} \left( \exp_\circ(-\chi_x(t)) \triangleright x \right),$$

and that, the non-linear post-Lie differential equation

$$\dot{x}(t) = -x(t) \triangleright x(t),$$

for $x = x(t) \in \mathfrak{g}[[t]]$, with initial condition $x(0) = x_0 \in \mathfrak{g}$, has as a solution

$$x(t) = \exp_\circ(-\chi_{x_0}(t)) \triangleright x_0.$$ 

In the same reference $\chi$ was dubbed post-Lie Magnus expansion to stress that such a map is the analogue, for post-Lie algebras, of the so called pre-Lie Magnus expansion, see [27] and references therein. This can be defined on every completed and unital pre-Lie algebra $(\mathfrak{g}, \triangleright)$ as the map $\Omega : \mathfrak{g} \to \mathfrak{g}$ satisfying the recursive relation

$$\Omega(x) = \sum_{k=0}^{\infty} \frac{B_k}{k!} L_{\Omega(x)}^k(x),$$

where, for all $y \in \mathfrak{g}$, $L_y : \mathfrak{g} \to \mathfrak{g}$, is defined by $L_y(x) = y \triangleright x$ and the $B_k$ are the Bernoulli numbers, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, .... In particular, the first terms of the previous expansion read as

$$\Omega(x) = x - \frac{1}{2}x \triangleright x + \frac{1}{4}(x \triangleright x) \triangleright x + \frac{1}{12}x \triangleright (x \triangleright x) + \cdots$$

which is the compositional inverse of the (left) pre-Lie exponential map

$$\exp_\circ(x) := x + \frac{1}{2}x \triangleright x + \frac{1}{6}x \triangleright (x \triangleright x) + \cdots$$

In other words, the (left) pre-Lie Magnus expansion is the (left) pre-Lie logarithm, i.e. $\Omega(x) = \log_\circ(1 + x)$, for all $x \in \mathfrak{g}$. The pre-Lie Magnus expansion turned out to be an important object in several different areas of mathematics like dynamical systems, see [11], combinatorics [12], [20] and [3], quantum field theory [22] and deformation theory, see [2] and [17]. At the best of our knowledge, (5.9) was dubbed pre-Lie Magnus expansion in [20]. A very nice and comprehensive review of the classical Magnus expansion and of its many applications can be found in [6], see also [10].
Let \((g, \triangleright)\) be a post-Lie algebra and let \(\text{Exp}(\nabla_x)\) be the automorphism of (the vector space) \(g\) defined by

\[
\text{Exp}(\nabla_x)(y) = y + \sum_{n \geq 1} \frac{\nabla^n_x(y)}{n!}.
\]

**Proposition 36.** For all \(x \in g\), one has

\[
\text{Exp}(\nabla_{\chi(x)})(y) = y + \sum_{n \geq 0} \frac{1}{n!} T_n(x, \ldots, x; y), \forall y \in g.
\]

**Proof.** Recall from (4.20), that for \(a \in g\) and \(b \in U(g)\), one has \(a * b = ab + a \triangleright b\). Using \((D5)\) this gives \(a \triangleright (b \triangleright y) = (a * b) \triangleright y\). We deduce that

\[
\text{Exp}(\nabla_{\chi(x)})(y) = y + \chi(x) \triangleright y + \frac{1}{2!} (\chi(x) * \chi(x)) \triangleright y + \ldots = \exp(x) \triangleright y.
\]

The latter is, by definition of \(\chi\), equal to \(\exp(x) \triangleright y\).

\[\square\]

**Remark 37.** It is worth to mention that (5.11) was proven in [22] for the case of a pre-Lie algebra, see also [17] and [2].

We now discuss how the post-Lie Magnus expansion relates to the Hausdorff groups of the Lie algebras \(\mathfrak{g}\) and \(g\). Recall that the Hausdorff group of a complete Lie algebra \((\mathfrak{g}, [-,-])\) is the group with product defined by the Baker-Campbell-Hausdorff series \(\text{BCH}(x, y) = \log(e^x e^y)\) whose the first terms read as

\[
\text{BCH}(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots
\]

To achieve our goal, let \(\sharp : g \otimes g \to g\) be defined by

\[
x \sharp y = \log \left( \exp\right(\chi(x) * \exp\right)(y)\right), \forall x, y \in g.
\]

This operation is called *composition product* in [23] to which we refer for more informations about its relevance in the theory of geometric numerical integration.

**Lemma 38** ([23], Proposition 2.5, p.12). For all \(x, y \in g\), one has

\[
x \sharp y = \text{BCH}_{[-,-]} \left(x, \exp\right(\chi(x) \triangleright y\right).
\]

**Proof.** For completeness we recall here below the proof of this result. Since \(\exp\) is a group-like element, \((D18)\) implies that

\[
\exp(x) * \exp(y) = \exp(x) \cdot \left(\exp\right(x \triangleright \exp\right)(y)\right).
\]

A simple induction on the length of the monomials, together with \((D4)\) gives

\[
\exp\left(x \triangleright y \cdots y\right) = \underbrace{\exp\left(x \triangleright y \cdots \exp\right)_{n\text{-times}}\right)_{n\text{-times}},
\]

which implies

\[
\exp(x) * \exp(y) = \exp(x) \exp\left(\exp\left(x \triangleright y\right)\right) = \exp\left(\text{BCH}(x, \exp\left(x \triangleright y\right)\right),
\]

proving the statement. \[\square\]
Going back to the Formula (5.12) one can compute:
\[
x^\#_y = \log \left( \exp \left( \chi(x) \right) \ast \exp \left( \chi(y) \right) \right)
\]
\[
= \log \left( \phi \left( \exp_{\otimes}(x) \ast \exp_{\otimes}(y) \right) \right)
\]
\[
= \log \left( \phi \left( \exp_{\otimes} \left( \text{BCH}_{[-,-]}(\chi(x)), \chi(y) \right) \right) \right)
\]
\[
= \log \left( \exp_{\otimes} \left( \text{BCH}_{[-,-]}(\chi(x)), \chi(y) \right) \right).
\]

Therefore, one has
\[
\exp \left( x^\#_y \right) = \exp \left( \text{BCH}_{[-,-]}(\chi(x)), \chi(y) \right),
\]
which, using the definition of the map \( \chi \), becomes
\[
\exp \left( \chi(x^\#_y) \right) = \exp \left( \text{BCH}_{[-,-]}(\chi(x)), \chi(y) \right),
\]
or, equivalently,
\[
\chi(x^\#_y) = \text{BCH}_{[-,-]}(\chi(x)), \chi(y) \right).
\]

Using (5.13) and the identity
\[
\exp(x \triangleright y) = \sum_{n \geq 0} \frac{1}{n!} T_n(x, \ldots, x; y), \forall x, y \in \mathfrak{g},
\]
one obtains the following.

**Proposition 39.** For all \( x, y \in \mathfrak{g} \), one has
\[
\text{BCH}_{[-,-]}(\chi(x), \chi(y)) = \chi \left( \text{BCH}_{[-,-]}(x, \Exp(\nabla_{\chi}(x))(y)) \right).
\]

If \( (\mathfrak{g}, \triangleright) \) is a pre-Lie algebra, i.e. a post-Lie algebra such that \([-,-] \equiv 0\), then
\[
\chi(x) = \log_{\triangleright}(1 + x),
\]
see Remark [35] i.e. under this assumption the post-Lie Magnus expansion is the pre-Lie Magnus expansion, see Corollary 8 pag. 276 in [21]. From Proposition 39 one gets the following.

**Corollary 40.** If \( \mathfrak{g} \) is a pre-Lie algebra, then, for all \( x, y \in \mathfrak{g} \), one has
\[
\text{exp}_{\triangleright} \left( \text{BCH}_{[-,-]}(\chi(x), \chi(y)) \right) = 1 + x + \Exp(\nabla_{\chi}(x))(y).
\]

**Remark 41.** The identity (5.15) was first proven in [1], where it was also observed that the operation \( f : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \) defined by
\[
f(x, y) = x + \Exp(\nabla_{\chi}(x))(y), \forall x, y \in \mathfrak{g},
\]
turns \( \mathfrak{g} \) into a group, there termed the group of formal flows of \( \mathfrak{g} \), isomorphic to the Hausdorff group of the Lie algebra underlying the pre-Lie algebra \( (\mathfrak{g}, \triangleright) \), see also [27, 22, 2] and [17]. On the other hand, Lemma 38, together with Definition 34 tells us that for any (unital and complete) pre-Lie algebra \( f \) is nothing else than the composition product defined in (5.12).
5.2. Post-Lie Magnus expansion via nested tubings. We give a combinatorial method for computing the post-Lie Magnus expansion. Recall that, for a post-Lie algebra \( g \), the post-Lie Magnus expansion of \( g \) satisfies the equation \( \exp(x) = \exp_\ast(\chi(x)) \); one sees that it can be expressed as a sum \( \chi = \sum_{n \geq 1} \chi_n \), where
\[
\chi_n(x) = \frac{x^n}{n!} - \sum_{k \geq 2, p_r > 0, \sum p_r = n} \frac{1}{k!} \chi_{p_1}(x) \cdots \chi_{p_k}(x) \quad \text{for all } x \in g,
\]
see [21]. We will focus on the post-Lie Magnus expansion of the free post-Lie algebra on \( K \). We begin by a few comments on its universal enveloping algebra, equipped with the product \( \ast \). Recall from Remark 12 that the free post-Lie algebra \( PSB(K) \) is generated, as a vector space, by the set \( G \). Remark that the free associative algebra on \( PSB(K) \) is the vector space generated by the forests of \( G \), endowed with the concatenation product. The universal enveloping algebra \( (U(PSB(K)), \ast) \) is the vector space generated by the forests on \( G \), modded out by the ideal generated by \( RS - SR - C(\ast; R, S) \) for every \( R \) and \( S \) in \( G \). The product \( R \ast S \) of two trees is given by \( R \ast S = RS + R > S \) where the first term means concatenation and the second one is the grafting of Remark 12.

Now, we describe \( \chi_n : PSB(K) \to PSB(K) \subset U_\ast(PSB(K)) \) of the free post-Lie algebra on \( K = K < \mathbf{a} > \). Let \( G_a \) be the subset of \( G \) of those (class of) trees that have at least one round-shape vertex \( \mathbf{a} \); let \( For \) be the set of the forests on \( G \setminus G_a \). Note that \( \chi_n(\mathbf{a}) \) is a homogeneous Lie polynomial of degree \( n \). In particular, in \( U_\ast(PSB(K)) \) it is a sum over all forests in \( For \) with \( n \) vertices:
\[
\chi_n(\mathbf{a}) = \sum_{F \in For_n} c_F F.
\]
We propose to compute the coefficients \( c_F \)'s.

Let us introduce some vocabulary. Part of the following definitions are extracted from [16]. Recall the level partial order \( < \) and the canonical linear order \( < \) of Section 2 given for trees. For a vertex \( v \) of a tree \( T \), let \( b_v \subset V(T) \) be the subset of the \( \prec \)-predecessors of \( v \); it inherits of the order \( < \). A forest has a horizontal order \( <_h \) that is increasing as one goes from left to right: for a forest \( ST \), one has \( v <_h w \) for each vertex \( v \) of \( S \) and each vertex \( w \) of \( T \).

Definition 42. A higher set of a poset \( (P, <) \) is a subset of \( P \) that contains the \( < \)-successors of each of its elements.

Definition 43. A tube of a tree \( T \) is a connected higher set \( t \) of \( (V(T), \prec) \), such that, for each \( v \in t \), one has \( t \cap b_v \) is a higher set of \( (b_v, <) \).

Definition 44. A tube of a forest \( F \) is a higher set of \( (V(F), <_h) \) that intersects each tree of \( F \) into a tube. A tube of \( F \) is said horizontal if it intersects each tree of \( F \) into either the empty set, or the one vertex set.

Definition 45. A pre nested tubing of a forest \( F \) is a collection of tubes of \( F \) that are pairwise nested. For a pre-tubing \( t = \{t_i\}_{i \in I} \) of \( F \), the boundary of a tube \( t_i \) is \( \partial t_i = t_i \setminus \{t_j \subsetneq t_i\} \). A nested tubing is a pre nested tubing such that its horizontal tubes have boundary of cardinal at most 1.

For a forest \( F \), let \( Tub(F) \) be the poset of its nested tubings, which is partially ordered by the inclusion. The cardinal of a nested tubing is the number of its tubes.
Proposition 46. For $F \in \mathcal{F}or_n$ different from $\mathbf{a} \cdots \mathbf{a}$, one has $c_F = \sum_{t \in \text{Tub}(F)} c_t$, where $c_t = \frac{1}{|t|!} \prod_{t' \subseteq t} c_{t'}$ and $c_{\mathbf{a}} = 1$.

Proof. As $\chi_1(\mathbf{a}) = \mathbf{a}$, the first coefficient $c_\mathbf{a}$, which is the coefficient of the unique tubing of $\mathbf{a}$ is 1. Let $n \geq 2$ and $F$ be a forest in $\mathcal{F}or_n$ that is not $\mathbf{a} \cdots \mathbf{a}$. To compute $c_F$, let us remark that $F$ appears in $-\frac{1}{n} \chi_{p_1}(x) \cdots \chi_{p_k}(x)$ for some $k$–partitions $p_1 + \ldots + p_k = n$. For each of these partitions, $F$ is obtained by shuffle concatenations and/or graftings of $k$ forests, say $F_1, \ldots, F_k$, that belong to $\chi_{p_1}(x), \ldots, \chi_{p_k}(x)$ respectively. In fact, there are several operations of these types that we can exclude; the only two operations that we keep are the (simple) concatenation $A \times B = AB$ and the one vertex grafting $A \triangleright v B$ that consists in grafting of all the roots of $A$ to the vertex $v$ of $B$, for any forests $A$ and $B$ in $\mathcal{F}or$. This can be seen as follows. Note that $T * B = TB + T \triangleright B$ for all forest $B$ in $\mathcal{F}or$ and $T \in \mathcal{G}$, while this does not necessarily holds if $T$ is a forest. Since $\chi_n$ is a Lie polynomial, the forests we consider that are not trees can be grouped into trees in $\mathcal{G}_*$. Therefore, for each of the forests $F_i$, it is enough to consider the terms of the form $F_i B$ and $F_i \triangleright T$. In turns, for all $T \in \mathcal{G}$, one has $T \triangleright B = \sum_{v \in \mathcal{V}(B)} T \triangleright v B$, which allows us to consider only the terms $F_i \triangleright v B$ for forests $B$ in $\mathcal{F}or$.

Since $*$ is associative, we can restrict ourselves to applying the concatenation and the one vertex grafting operations with the left most parentheses. Let us show that there is a bijection between the set of all the possibilities of obtaining $F$ from $F_1, \ldots, F_k$ this way, and the set of all the nested tubings $t = t_1 \supset t_2 \supset \cdots \supset t_k$ of $F$ such that $|\partial t_i| = p_i$. Instead of giving a fully detailed proof of this fact, let us show how it works on a generic example: suppose

$$F = F_1 \triangleright v_2 (\cdots (\triangleright v_{k-2} (F_{k-2} \times (F_{k-1} \triangleright v_k F_k))) \cdots).$$

Since concatenation and grafting do not remove vertices nor edges, the decomposition (5.16) provides an embedding of $F_1, \ldots, F_k$ into $F$, which can be represented by a nested tubing. Explicitly, the tube $t_k$ is $F_k$ seen in $F$, the tube $t_{k-1}$ is the smallest sub forest of $F$ that contains $F_k$ and $F_{k-1}$, etc. For example, one has

A direct inspection shows that such assignment is one-to-one. In particular, the condition on $b_v$ in Definition 46 corresponds to the minimal angle condition on grafting (i.e. one grafts on the "left-side" of a vertex); the horizontal condition in Definition 46 corresponds to the absence of the forest $\mathbf{a} \cdots \mathbf{a} \in \chi_p(\mathbf{a})$ for $p \geq 2$. The coefficient $c_t$ associated to the nested tubing $t$ is given by $\frac{1}{|t|!} c_{F_1} c_{F_2} \cdots c_{F_k}$, which gives the result. 

□

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