Relation between surface solitons and bulk solitons in nonlocal nonlinear media

Zhenjun Yang,1,2 Xuekai Ma,1 Daquan Lu,1 Yizhou Zheng,1 Xinghui Gao,1 and Wei Hu,1,∗

1Laboratory of Photonic Information Technology, South China Normal University, Guangzhou 510631, P. R. China
2College of Physics Science and Information Engineering, Hebei Normal University, Shijiazhuang 050016, P. R. China
huwei@scnu.edu.cn

Abstract: We find that a surface soliton in nonlocal nonlinear media can be regarded as a half of a bulk soliton with an antisymmetric amplitude distribution. The analytical solutions for the surface solitons and breathers in strongly nonlocal media are obtained, and the critical power and breather period are gotten analytically and confirmed by numerical simulations. In addition, the oscillating propagation of nonlocal surface solitons launched away from the stationary position is considered as the interaction between the soliton and its out-of-phase image beam. Its trajectory and oscillating period obtained by our model are in good agreement with the numerical simulations.

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References and links
1. M. Mitchell, M. Segev, and D. N. Christodoulides, “Observation of multihump multimode solitons,” Phys. Rev. Lett. 80, 4657-4660 (1998).
2. M. Peccianti, K. A. Brzakiewicz, and G. Assanto, “Nonlocal spatial soliton interactions in nematic liquid crystals,” Opt. Lett. 27, 1460-1462 (2002).
3. C. Conti, M. Peccianti, and G. Assanto, “Route to nonlocality and observation of accessible solitons,” Phys. Rev. Lett. 91, 073901 (2003).
4. C. Conti, M. Peccianti, and G. Assanto, “Observation of optical spatial solitons in a highly nonlocal medium,” Phys. Rev. Lett. 92, 113902 (2004).
5. W. Hu, T. Zhang, Q. Guo, X. Li, and S. Lan, “Nonlocality-controlled interaction of spatial solitons in nematic liquid crystals,” Appl. Phys. Lett. 89, 07111 (2006).
6. C. Rotschild, O. Cohen, O. Manela, and M. Segev, “Solitons in nonlinear media with an infinite range of nonlocality: first observation of coherent elliptic solitons and of vortex-ring solitons,” Phys. Rev. Lett. 95, 213904 (2005).
7. C. Rotschild, M. Segev, Z. Xu, Y. V. Kartashov, L. Torner, and O. Cohen, “Two-dimensional multipole solitons in nonlocal nonlinear media,” Opt. Lett. 31, 3312-3314 (2006).
8. S. Skupin, M. Saffman, and W. Krolikowski, “Nonlocal stabilization of nonlinear beams in a self-focusing atomic vapor,” Phys. Rev. Lett. 98, 263902 (2007).
9. P. Pedri and L. Santos, “Two-dimensional bright solitons in dipolar Bose-Einstein condensates,” Phys. Rev. Lett. 95, 200404 (2005).
10. I. Tikhonenkov, B. A. Malomed, and A. Vardi, “Anisotropic solitons in dipolar Bose-Einstein condensates,” Phys. Rev. Lett. 100, 090406 (2008).
11. Y. V. Kartashov, V. A. Vysloukh, and L. Torner, “Stability of vortex solitons in thermal nonlinear media with cylindrical symmetry,” Opt. Express 15, 9378-9384 (2007).

http://www.opticsinfobase.org/oe/abstract.cfm?URI=oe-15-15-9378
12. D. Q. Lu and W. Hu, “Multiringed breathers and rotating breathers in strongly nonlocal nonlinear media under the off-waist incident condition,” Phys. Rev. A 79, 043833 (2009).
13. Z. Xu, Y. V. Kartashov, and L. Torner, “Upper threshold for stability of multipole-mode solitons in nonlocal nonlinear media,” Opt. Lett. 30, 3171-3173 (2005).
14. L. Dong and F. Ye, “Stability of multipole-mode solitons in thermal nonlinear media,” Phys. Rev. A 81, 013815 (2010).
15. S. Ouyang, Q. Guo, and W. Hu, “Perturbative analysis of generally nonlocal spatial optical solitons,” Phys. Rev. E 74, 036622 (2006).
16. D. Buccoliero, A. S. Desyatnikov, W. Krolikowski, and Y. S. Kivshar, “Laguerre and Hermite soliton clusters in nonlocal nonlinear media,” Phys. Rev. Lett. 98, 053901 (2007).
17. D. Deng, X. Zhao, Q. Guo, and S. Lan, “Hermite-Gaussian breathers and solitons in strongly nonlocal nonlinear media,” J. Opt. Soc. Am. B 24, 2537-2544 (2007).
18. D. Deng and Q. Guo, “Propagation of Laguerre-Gaussian beams in nonlocal nonlinear media,” J. Opt. A: Pure Appl. Opt. 10, 035101 (2008).
19. D. Deng and Q. Guo, “Ince-Gaussian solitons in strongly nonlocal nonlinear media,” Opt. Lett. 32, 3206-3208 (2007).
20. Y. V. Izdebskaya, A. S. Desyatnikov, G. Assanto, and Y. S. Kivshar, “Multimode nematicon waveguides,” Opt. Lett. 36, 184-186 (2011).
21. Q. Guo, B. Luo, F. H. Yi, S. Chi, and Y. Q. Xie, “Large phase shift of nonlocal optical spatial solitons,” Phys. Rev. E 69, 016602 (2004).
22. P. D. Rasmussen, O. Bang, and W. Krolikowski, “Theory of nonlocal soliton interaction in nematic liquid crystals,” Phys. Rev. E 72, 066611 (2005).
23. A. Dreischuh, D. N. Neshev, D. E. Petersen, O. Bang, and W. Krolikowski, “Observation of attraction between dark solitons,” Phys. Rev. Lett. 96, 043901 (2006).
24. N. I. Nikolov, D. Neshev, W. Krolikowski, O. Bang, J. J. Rasmussen, and P. L. Christiansen, “Attraction of nonlocal dark optical solitons,” Opt. Lett. 29, 286-288 (2004).
25. D. Q. Lu, W. Hu, Y. J. Zheng, Y. B. Liang, L. G. Cao, S. Lan, and Q. Guo, “Self-induced fractional Fourier transform and revivable higher-order spatial solitons in strongly nonlocal nonlinear media,” Phys. Rev. A 78, 043815 (2008).
26. B. Alfassi, C. Rothschild, O. Manela, M. Segev, and D. N. Christodoulides, “Nonlocal surface-wave solitons,” Phys. Rev. Lett. 98, 213901 (2007).
27. Y. V. Kartashov, V. A. Vysloukh, and L. Torner, “Multipole surface solitons in thermal media,” Opt. Lett. 34, 283-285 (2009).
28. F. Ye, Y. V. Kartashov, and L. Torner, “Nonlocal surface dipoles and vortices,” Phys. Rev. A 77, 033829 (2008).
29. Y. V. Kartashov, V. A. Vysloukh, and L. Torner, “Ring surface waves in thermal nonlinear media,” Opt. Express 15, 16216-16221 (2007), http://www.opticsinfobase.org/oe/abstract.cfm?URI=oe-15-24-16216
30. B. Alfassi, C. Rothschild, and M. Segev, “Incoherent surface solitons in effectively instantaneous nonlocal nonlinear media,” Phys. Rev. A 80, 041808 (2009).
31. Y. V. Kartashov, V. A. Vysloukh, and L. Torner, “Optical surface waves supported and controlled by thermal waves,” Opt. Lett. 33, 506-508 (2008).
32. W. Krolikowski, O. Bang, and J. Wyller, “Modulational instability in nonlocal nonlinear Kerr media,” Phys. Rev. E 64, 016612 (2001).
33. N. Ghofraniha, C. Conti, G. Ruocco and S. Trillo, “Shocks in nonlocal media,” Phys. Rev. Lett. 99, 043903 (2007).
34. C. Conti, A. Fratalocchi, M. Peccianti, G. Ruocco, and S. Trillo, “Observation of a gradient catastrophe generating solitons,” Phys. Rev. Lett. 102, 083902 (2009).
35. N. I. Nikolov, D. Neshev, O. Bang, and W. Z. Królkowski, “Quadratic solitons as nonlocal solitons,” Phys. Rev. E 68, 036614 (2003).
36. P. V. Larsen, M. P. Søensen, O. Bang, W. Z. Królkowski, and S. Trillo, “Nonlocal description of X waves in quadratic nonlinear materials,” Phys. Rev. E 73, 036614 (2006).
37. Q. Shou, Y. B. Liang, Q. Jiang, Y. J. Zheng, S. Lan, W. Hu, and Q. Guo, “Boundary force exerted on spatial solitons in cylindrical strongly nonlocal media,” Opt. Lett. 34, 3523-3523 (2009).
38. D. Anderson, “Variational approach to nonlinear pulse propagation in optical fibers,” Phys. Rev. A 27, 3135-3145 (1983).
39. L. G. Cao, Y. Q. Zhu, D. Q. Lu, W. Hu, and Q. Guo, “Propagation of nonlocal optical solitons in lossy media with exponential-decay response,” Opt. Commun. 281, 5004-5008 (2008).
1. Introduction

The nonlocality of the nonlinear response exists in many real physical systems, such as photorefractive crystals [1], nematic liquid crystals [2-5], lead glasses [6, 7], atomic vapors [8], Bose-Einstein condensates [9, 10], etc. In nonlocal nonlinear media, various soliton solutions have been predicted theoretically, such as vortex solitons [11, 12], multi-pole solitons [7, 13-15], Laguerre-Gaussian and Hermite-Gaussian solitons [16-18], Ince-Gaussian solitons [19], and some of these have been observed experimentally [4, 5, 6, 7, 20]. For nonlocal solitons, there are many interesting properties, for instance, the large phase shift [21], attraction between two bright out-of-phase solitons [5, 22], attraction between two dark solitons [23, 24], self-induced fractional Fourier transform [25], etc.

Recently, the nonlocal surface solitons have been investigated numerically and experimentally [26, 27, 28, 29, 30, 31]. The nonlocal surface solitons occurring at the interface between a dielectric medium and a nonlocal material exhibit unique properties. Nonlocal multipole surface solitons, vortices, and bound states of vortex solitons, incoherent surface solitons and ring surface waves have been also studied recently [27, 28, 29, 30]. But the properties of nonlocal surface solitons mentioned above are all discussed by numerical simulations. In order to get a good understanding of the properties of nonlocal surface solitons, it is essential to present an analytic solution, even an approximate one.

In this paper, based on the propagation equations governing the nonlocal surface waves and assuming that all energy of the surface soliton resides in nonlocal media, we find a surface soliton in nonlocal nonlinear media can be regarded as a half of a bulk soliton with an antisymmetric amplitude distribution. The evolution regularities for the nonlocal surface waves are discussed both analytically and numerically. The analytical solutions for the surface solitons and breathers in strongly nonlocal media are obtained, and the critical power and breather period are gotten analytically and confirmed by numerical simulations. In addition, the oscillating propagation of nonlocal surface solitons launched away from the stationary position is considered as the interaction between the soliton and its out-of-phase image beam. Its trajectory and oscillating period obtained by our model are in good agreement with the numerical simulations.

2. Relation between surface solitons and bulk solitons

![Fig. 1. Sketches of a nonlocal surface soliton (a) and a nonlocal antisymmetric bulk soliton (b). Solid and dashed lines represent the distributions of the amplitude and the nonlinear refraction index, respectively.](image)

Fig. 1. Sketches of a nonlocal surface soliton (a) and a nonlocal antisymmetric bulk soliton (b). Solid and dashed lines represent the distributions of the amplitude and the nonlinear refraction index, respectively.

We consider a (1+1)-D model of an optical surface wave with an envelope $q$ propagating in $z$ direction near an interface between a nonlocal nonlinear medium and a linear medium [see Fig. 1(a)]. The propagation of surface waves is governed by the dimensionless nonlocal nonlinear Schrödinger equation (NNLSE), i.e.
(i) in nonlocal nonlinear media, $x \leq 0$,
\begin{align}
  i \frac{\partial q}{\partial z} + \frac{1}{2} \frac{\partial^2 q}{\partial x^2} + \Delta n q &= 0, \\
  \Delta n - w_m^2 \frac{\partial^2 \Delta n}{\partial x^2} &= |q|^2;
\end{align}

(ii) in linear media, $x > 0$,
\begin{align}
  i \frac{\partial q}{\partial z} + \frac{1}{2} \frac{\partial^2 q}{\partial x^2} - q n_d &= 0,
\end{align}

where $x, z$ denote, respectively, the normalized transversal and longitudinal coordinates. $\Delta n$ is the nonlinear perturbation of the refractive index, and $n_d$ denotes the normalized difference between the unperturbed refractive index of the nonlocal medium and the refractive index of the less optically dense linear medium. $w_m$ represents the characteristic length of the nonlocal material response, and $\alpha = w_m/w_0$ is the degree of nonlocality, where $w_0$ is the beam width.

The optical intensity is normalized as $I = |q|^2$. In fact, Eqs. (1) and (2) denote a NNLSE with an exponential-decay type nonlocal response. The exponential-decay type nonlocal response exists in many real physical systems, for instance, all diffusion-type nonlinearity [32, 33], orientational-type nonlinearity [2, 22], and the general quadratic nonlinearity describing parametric interaction [35, 36]. Moreover when $w_m \to \infty$, Eqs. (1) and (2) can be transformed to the forms describing the thermal nonlocal media (for example, lead glasses [6, 26]).

The boundary conditions for the surface soliton are $q(x \to -\infty) = 0, q(+0) = q(-0), \Delta n(x \to -\infty) = 0$, and $\partial \Delta n/\partial x |_{x=0} = 0$ [26, 27, 28]. References [26, 27, 28] indicate if the index difference $n_d$ is big enough, namely $n_d \gg 1$, which is easily satisfied in the actual physical system, the optical energy is almost totally confined in the nonlocal medium. Therefore one can approximately get the relation $q(-0) = q(+0) = 0$ and $q(x > 0) = 0$. Under this approximation, the propagation of surface waves can be solved totally only based on Eqs. (1) and (2) with the boundary conditions ($x \leq 0$),
\begin{align}
  q(x \to -\infty) &= q(0) = 0, \quad (4a) \\
  \left. \frac{\partial \Delta n}{\partial x} \right|_{x=0} &= 0. \quad (4b)
\end{align}

We find the solution of a surface soliton under the approximation $q(0) = 0$ is identical with the half part of an antisymmetric soliton in a bulk medium as shown in Fig. (1b). For the bulk soliton solutions with antisymmetric amplitude distribution which is also governed by Eqs. (1) and (2), one can easily obtain the following relations.
\begin{align}
  q(-0) &= q(+0) = 0, \quad (5a) \\
  q(x \to \pm \infty) &= 0, \quad (5b) \\
  \left. \frac{\partial \Delta n}{\partial x} \right|_{x=0} &= 0. \quad (5c)
\end{align}

Comparing Eq. (4) with Eq. (5), it can be found that the conditions that the surface solitons satisfy are the same as those of the left half ($x \leq 0$) of the antisymmetric bulk solitons. Therefore, the surface soliton can be regarded as a half of the bulk soliton, and all the results of antisymmetric bulk soliton can be transferred to the surface soliton. On the other hand, the right half of the bulk soliton can be regarded as an image beam of the surface soliton [37]. The interaction between the surface soliton and the interface can be regarded as the interaction between the soliton and its image beam in bulk medium.
In nonlocal bulk media, the soliton solutions can have the antisymmetric amplitude distribution [13 14 15]. Some authors have demonstrated that Hermite-Gaussian function can be applied to describe this type soliton solution in nonlinear media with several different nonlocal responses, especially for the strongly nonlocal case [15 16 17]. In addition, it has been discovered that in a nonlinear material with a finite-range nonlocal or a very long-range nonlocal response, the maximal number of peaks in stable multipole bulk solitons is four, and all higher-order soliton bound states are unstable [13 14]. Therefore, surface solitons only with less than three poles can be stable in the case of thermally nonlocal interface, which is firstly addressed by Kartashov et al [27].

In the following, we give some comparisons between the bulk solitons and the surface solitons to illustrate our conclusions, and some analytical results are given for the nonlocal surface solitons.

3. Surface solitons and breathers

In strongly nonlocal nonlinear media, there exist Hermite-Gaussian solitons or breathers [15 16 17]. Because the first-order Hermite-Gaussian beam is antisymmetric about the beam center, it can be used to describe the nonlocal surface soliton. In bulk nonlocal media, the first-order Hermite-Gaussian trial beams can be expressed as

\[ q(x,z) = a(z) x \exp \left( -\frac{x^2}{w^2(z)} + ic(z)x^2 \right) e^{i\theta(z)}, \]  

where \( a(z), w(z), c(z), \theta(z) \) represent the amplitude, beam width, phase-front curvature and phase of the beams, respectively. The input power is \( P_0 = \int |q|^2 \, dx \).

In the following, Eqs. (1) and (2) are used for the equivalent bulk case \((-\infty < x < \infty)\). Therefore Eqs. (1) and (2) can be restated as an Euler-Lagrange equation corresponding to a variational problem [16 38 39].

\[ \delta \int_{0}^{\infty} [L] \, dz = 0, \]  

where

\[ [L] = \int_{-\infty}^{\infty} L(q, q^*, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial z}, \frac{\partial q^*}{\partial x}, \frac{\partial q^*}{\partial z}) \, dx. \]  

The Lagrangian density \( L \) is given by

\[ L = \frac{i}{2} \left( q^* \frac{\partial q}{\partial z} - q \frac{\partial q^*}{\partial z} \right) - \frac{1}{2} \left| \frac{\partial q}{\partial x} \right|^2 + \frac{1}{2} |q|^2 \int_{-\infty}^{\infty} R(x-x') |q(x')|^2 \, dx', \]  

where \( R(x) = (1/2w_m) \exp(-|x|/w_m) \). Because the complexity of integration produced by \( R(x) \), the expression of \( [L] \) is too difficult to obtain. But for the strongly nonlocal case, i.e. \( \alpha = w_m/w_0 \gg 1 \), \( R(x) \) can be expanded as

\[ R(x) \approx \frac{1}{2w_m} \left( 1 - \frac{|x|}{w_m} + \frac{x^2}{2w_m^2} \right). \]  

Combining Eq. (6) and Eq. (8), \([L]\) can be obtained

\[ [L] = -\frac{3}{8} \sqrt{\pi} a^2 w \left( 1 + c^2 w^4 \right) - \frac{1}{16} \sqrt{\pi} a^2 w^3 \left( 3w^2 \frac{dc}{dz} + 4 \frac{d\theta}{dz} \right) + \frac{a^4 w^6}{512w_m^3} \left( 4\pi w_m^2 - 7 \sqrt{\pi} w_m w + 3\pi w^2 \right). \]
Following the common process of the variational approach, one can get four differential equations,

\begin{align}
2w \frac{da}{dz} + 3a \frac{dw}{dz} &= 0, \quad (12a) \\
2w \frac{da}{dz} + a \left( \frac{5dw}{dz} - 4cw \right) &= 0, \quad (12b) \\
\frac{a^2w^5}{w_m^3} \left(4\pi w_m^2 - 7\sqrt{\pi} w_m w + 3\pi w^2 \right) \\
&\quad - 48\sqrt{2} \pi \left(1 + c^2 w^4 \right) - 8\sqrt{2}\pi w^2 \left(3w^2 \frac{dc}{dz} + 4 \frac{d\theta}{dz} \right) &= 0, \quad (12c) \\
480\sqrt{2}w_m^3 c^2w^4 + a^2w^5 \left(- 24\sqrt{\pi} w_m^2 + 49w_m w - 24\sqrt{\pi} w^2 \right) \\
&\quad + 48\sqrt{2}w_m^3 \left(4w^2 \frac{d\theta}{dz} + 5w^4 \frac{dc}{dz} + 2 \right) &= 0. \quad (12d)
\end{align}

Based on the above four equations, we can investigate the propagation of the trial beams. In the following, we divide the problem into two cases to discuss, i.e. solitons and breathers.

3.1. surface solitons

For the case of solitons, in Eq. (6), \( a \) and \( w \) reduce to constants, and \( c = 0, \theta(z) = \beta z \), where \( \beta \) is propagation constant. The expression of beams can be rewritten in a simple form

\[
q(x,z) = a_0 x \exp \left(- \frac{x^2}{w_0^2}\right) e^{i\beta z}.
\]

(13)

Based on Eq. (12), we can get the input power, namely the critical power of bulk solitons

\[
P_c = \frac{48\sqrt{\pi} w_m^3}{w_0^3(7w_m - 6\sqrt{\pi}w_0)}.
\]

(14)

and the propagation constant

\[
\beta = \frac{105w_m w_0 - 6\sqrt{\pi}(8w_m^2 + 9w_0^2)}{2w_0^3(6\sqrt{\pi}w_0 - 7w_m)}.
\]

(15)

According to our conclusion in Sec. 2, the surface soliton can be expressed as

\[
q(x,z) = \begin{cases} 
  a_0 x \exp \left(- \frac{x^2}{w_0^2}\right) e^{i\beta z} & \text{for } x \leq 0, \\
  0 & \text{for } x > 0.
\end{cases}
\]

(16)

Apparently, the critical power of surface solitons is a half of bulk solitons, namely \( P_s = P_c / 2 \). When the optical beam propagates as a surface soliton, we define the position where the beam has its maximal intensity as the soliton position, i.e. the distance between the maximal intensity and the interface. According to Eq. (16), the soliton position \( x_s \) can be obtained as

\[
x_s = -\frac{\sqrt{2}w_0}{2}.
\]

(17)

Based on Eqs. (1)–(3), some numerical simulations of the propagations of surface and bulk solitons are carried out by using Eqs. (13) and (16) as incident profiles. Figure 2 shows the analytical and simulated results are in good agreement with each other in strongly nonlocal media.
Fig. 2. Comparison between the fundamental surface solitons and the first-order Hermite-Gaussian bulk solitons with the degree of nonlocality $\alpha = 10$. (a) and (c) are simulated intensity distribution for surface solitons during propagation with $w_0 = 1.0$ and 2.0, respectively; (b) and (d) are simulated intensity distribution for bulk solitons with $w_0 = 1.0$ and 2.0, respectively. (e)-(h) are, respectively, the transversal intensity distributions (solid line) and the refractive index distributions (dashed line) corresponding to row 1.

Fig. 3. Simulated propagation of surface solitons in the presence of 3% white noise at a fixed nonlocal degree $\alpha = 10$. (a) and (b) give the propagation of surface solitons with different soliton widths $w_0 = 1.0$ and 2.0, respectively. (c) and (d) give the incident intensity distribution (left) and the output intensity distribution at $z = 2000$ (right) corresponding to (a) and (b), respectively.
In fig. 2, $I = |q|^2$ is the optical intensity. To confirm the stability, we simulate the propagation of surface solitons in the presence of 3\% white noise (see fig. 3). Figure 3 shows the propagation is stable.

In general nonlocal media, the profiles of surface soliton are obtained by numerical iterative method based on Eqs. (1)-(3), and the relations between critical powers and beam width are shown in Fig. 4. For Figs. 4(a) and 4(b) which belong to the strongly nonlocal case, the analytical results are in excellent agreement with the numerical results. As the degree of nonlocality decreases to $\alpha = 6$ in Fig. 4(c), the analytical results are also accordant with the numerical results approximately. For Fig. 4(d), the analytical results begin to deviate from the exact numerical ones with the beam width increasing. Figure 4 confirms that the analytical solutions of surface solitons are very valid for the strongly nonlocal case. The relations between critical powers and propagation constants are shown in Fig. 5. It shows the analytical results from Eqs. (14) and (15) are in good agreement with the numerical results except for small $\beta$ which corresponds to a weak nonlocality.

For weakly nonlocal case, the analytical solution can not be obtained because the analytical solution for multipole bulk solitons can not be found in weakly nonlocal medium. However, in nonlocal bulk media, the diploe solitons are always existent and stable in the entire domain of...
their existence \[13\]. According to the theory in Sec. 2, the fundamental surface soliton should be existent for the weakly nonlocal case. Following the method in Ref. \[13\], we seek the dipole bulk soliton combining two out-of-phase fundamental solitons under weakly nonlocality, as a result, the surface soliton can be also found at the same time, as shown in Fig. 6.

Fig. 6. Comparison between the surface solitons and the bulk solitons for the weakly nonlocal case (\(\alpha = 0.2\)). (a) and (d) are the simulated intensity distributions for the surface soliton and the bulk soliton during propagation with \(w_0 = 1.0, w_m = 0.5\), respectively; (b) and (e) are the transversal intensity distributions (solid line) and the refractive index distributions (dashed line) corresponding to (a) and (d), respectively. (c) and (f) are the transversal amplitude distributions (dashed-dotted line) and the refractive index distributions (dashed line) corresponding to (a) and (d), respectively.

3.2. surface breathers

Because the breathers exist in nonlocal bulk media, if the input power is not equal to the soliton power, we can find the surface breathers. According to Eq. \(12\), we can get

\[
\frac{d^2w}{dz^2} = 2\sqrt{2} \frac{w^3}{w^3} - \frac{7a^2w^3}{96\sqrt{2}w_m^2} + \frac{\sqrt{\pi}a^2w^4}{16w_m^2},
\]

(18)

Above equation is equivalent to Newton’s second law in classical mechanics for the motion of an one-dimensional particle with the equivalent mass 1, thus one can get the equivalent potential

\[
V(w) = \frac{\sqrt{2}}{w^2} - \frac{a^2\sqrt{\pi}w^5}{80w_m^3} + \frac{7a^2w^4}{384w_m^4} + c_0,
\]

(19)

where \(c_0\) is a constant. Through expanding the equivalent potential \(V(w)\) into the second order at the balance position, the motion of the particle is approximated to a harmonic oscillation, and we can obtain the approximate expression of the beam width

\[
w(z) = \frac{\sqrt{2}}{2} \left[ g - \sqrt{b} + (\sqrt{2}w_0 - g + \sqrt{b}) \cos \left( \sqrt{\frac{12}{(g - \sqrt{b})^4 + \frac{1}{3w_0^2}}} \right) \right],
\]

(20)
where

\[
\begin{align*}
g &= \frac{7w_m}{12\sqrt{2\pi}} + \sqrt{t + \frac{49w_m^2}{288\pi}},
\end{align*}
\]

\[
\begin{align*}
b &= g^2 - 2t + 2\sqrt{t^2 + 3w_0p},
\end{align*}
\]

\[
\begin{align*}
t &= (h + \sqrt{p^3 + h^3})^\frac{1}{3} + (h - \sqrt{p^3 + h^3})^\frac{1}{3},
\end{align*}
\]

\[
\begin{align*}
h &= -\frac{49w_0w_m^2}{192\pi},
\end{align*}
\]

\[
\begin{align*}
p &= \frac{w_0^2P_0(6\sqrt{\pi}w_0 - 7w_m)}{18\sqrt{\pi}P_0}.
\end{align*}
\]

According to our theory, the oscillation period of surface breathers is the same as that of bulk breathers. Then, we can obtain the oscillation period of the surface breather,

\[
\Delta z = 2\pi \left[ \frac{12}{(g - \sqrt{b})} - \frac{6\sqrt{\pi}P_0}{w_0^2P_0(7w_m - 6\sqrt{\pi}w_0)} \right]^{-\frac{1}{2}},
\]

(21)

where \(P_0\) is the input power of the surface breathers. At the same time, the beam trajectory can be obtained and given by \(x_b = -\sqrt{2w(z)/2}\). Figure 7 gives the comparison between the surface breathers and the bulk breathers for \(\alpha = 10\). The analytical trajectories are in good agreement with the simulated results.

![Fig. 7. Comparison between the surface breathers and the bulk breathers with \(\alpha = 10\).](image)

(a) and (c) are, respectively, the simulated propagations of the surface breather and the bulk breather with input power being twice soliton power. (b) and (d) give the analytical trajectories of the maximal intensity corresponding to (a) and (c), respectively. Row 2 is the same as row 1 except that the input power is a half of soliton power.

4. Surface solitons launched away from the stationary position

One can learn from Ref. [26] that if the beam launched away from the soliton position, the beam will propagate oscillating about the stationary position. The oscillation induced by the interaction between the soliton and the interface is periodic, and the beam never converges to a straight line trajectory. According to our theory and introducing an out-of-phase image
beam \(^{[37]}\), the interaction between soliton and interface can be regarded as the interaction between the soliton and its image beam in nonlocal bulk media. Then the oscillating trajectory of surface solitons can be obtained from the trajectories of two out-of-phase solitons interacting in nonlocal bulk media.

For the strongly nonlocal case, the fields of two out-of-phase Gaussian solitons propagating in bulk media can be expressed as

\[
q_{\pm}(x,z) = \pm a_0 \exp \left[-\frac{[x \pm x_c(z)]^2}{w_0^2} \pm i u(z) \right] e^{i\theta(z)},
\]

where \(\theta(z)\) is the phase of whole beams, \(\pm x_c(z)\) are the mass centers of Gaussian solitons and \(x_c(0) = x_c0\). \(u(z)\) is the tilting wavefront induced by the attraction between two solitons, and we consider the normally incident case, i.e. \(u(0) = 0\). The amplitude \(a_0^2 = 32\sqrt{2}w_0^5/w_0^5(4w_m(w_0^2 + 2w_m^2) - \sqrt{\pi}w_0(w_0^2 + 4w_m^2))\) is normalized to guarantee the incident power of each soliton is equal to the critical power, so we can assume that the beam width does not change during propagation.

If the two solitons are separated over \(3w_0\) each other, their fields almost do not overlap. However because of strong nonlocality, there still exists attractive force between them. The attractive force exerted on one soliton is produced by the other soliton, specifically, by the nonlinear refractive index change induced by the other soliton. Therefore, the trajectory of one soliton (i.e. the left one \(q_+\)) can be determined by the light ray equation,

\[
\frac{d^2x_c}{dz^2} = \left. \frac{d(\Delta n_+)}{dx} \right|_{x = -x_c} \equiv \Delta n',
\]

where the refractive index change \(\Delta n_+\) is only induced by the right soliton \(q_-\), as

\[
\Delta n_+ = \int_{-\infty}^{+\infty} R(x - x')|q_-(x',z)|^2 dx'.
\]

Using the nonlinear response function \(R(x) = (1/2w_m)\exp(-|x|/w_m)\) and Eqs. \((22)-(24)\), one can obtain

\[
\frac{d^2x_c}{dz^2} = \frac{\sqrt{\pi}a^2w_0}{4\sqrt{2}w_m^2} \exp \left(\frac{w_0^2 - 16w_m w_c}{8w_m^2} \right) \times \left[ \exp \left(\frac{4x_c}{w_m} \right) \text{erfc} \left(\frac{w_0^2 + 8w_m x_c}{2\sqrt{2}w_0 w_m} \right) + \text{erf} \left(\frac{w_0^2 - 8w_m x_c}{2\sqrt{2}w_0 w_m} \right) - 1 \right],
\]

where \(\text{erf}(\cdot)\) and \(\text{erfc}(\cdot)\) denote the error and complementary error functions, respectively. Taking the conditions \(w_m \gg w_0, x_c0\), we can get a simply approximate expression

\[
\frac{d^2x_c}{dz^2} \simeq -\frac{a^2\sqrt{\pi}w_0}{2\sqrt{2}w_m^2} \text{erf} \left(\frac{4x_c}{\sqrt{2}w_0} \right),
\]

\(\Delta n'\) shown in Fig.\(^{[8]}\) represents the attractive force between two solitons. Due to the complicated form of \(\Delta n'\), Eqs. \((25)\) and \((26)\) are solved numerically using Runge-Kutta methods. The soliton trajectories and the oscillation period are shown in Figs.\(^{[8]}\)c and\(^{[8]}\)d. The trajectory of the surface soliton corresponds to the left half of trajectories of two solitons in bulk media, i.e. \(x_{sc} = -|x_c|\).

The numerical simulations based on Eqs. \((11) - (13)\) for surface soliton launched away from stationary position and for two bulk solitons are also shown in Figs.\(^{[8]}\)a and\(^{[8]}\)b. The trajectories and oscillation periods gotten from numerical simulations are compared with that from
Fig. 8. (a) Simulated propagations of a Gaussian surface wave launched $2.5w_0$ away from the surface. (b) Simulated propagations of two out-of-phase Gaussian solitons with a separated distance $5.0w_0$. (c) The trajectories of surface wave and two Gaussian solitons. (d) The oscillation period of surface wave. (e) Distribution of $\Delta n'$. The red solid line and the blue dashed-dotted line are the results based on Eqs. (25) and (26), respectively. The green dashed line and green triangles are the simulated results directly based on Eqs. (1)-(3). All cases are with the parameters $w_m = 50.0$, $w_0 = 1.0$.

Eqs. (25) and (26) in Figs. 8(c) and 8(d). These results are approximately coincident with each other. The main reason of disagreement is that Eq. (23) is valid when two beam fields do not overlap. When $x_c$ is small, the fields of two beams overlap and we can not distinguish $\Delta n_+$ and $\Delta n_-$. Then Eq. (23) is invalid, and the simulated results disagree with that obtained by solving Eqs. (25) and (26). When $x_c$ is large and the nonlocality is strong enough, the simulated results are in good agreement with the results by numerically solving Eqs. (25) and (26).

5. Conclusion

In conclusion, we have studied the nonlocal surface waves numerically and analytically. We find a surface soliton in nonlocal nonlinear media can be regarded as a half of a bulk soliton with an antisymmetric amplitude distribution. By applying the variational method and taking the first-order Hermite-Gaussian beam as an example, the analytical solutions for the surface solitons and breathers in strongly nonlocal media are obtained, and the critical power and breather period are gotten analytically and confirmed by numerical simulations. In addition, the oscillating propagation of nonlocal surface soliton launched away from the stationary position is considered as the interaction between the soliton and its out-of-phase image beam. We have discussed the oscillation period and the beam trajectory. Its trajectory and oscillating period obtained by our model are in good agreement with the numerical simulations.

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