Generalized Ernst equations for plane symmetric spacetimes and compatible lattice structures

Anastasios Tongas
Department of Mathematics, University of Patras, 26 500 Patras, Greece
E-mail: tasos@math.upatras.gr

Abstract. It is shown that the symmetries of a partial differential equation which incorporates the hyperbolic Ernst equation in general relativity, generate the complete hierarchy of the Korteweg-de Vries soliton equations.

1. Introduction
The fundamental field theories have been formulated in such a way that a single, or a system of partial differential equations (PDEs) univocally indicates the corresponding field theory. The prime examples are the linear field theories, such as the wave equation of the massless scalar field and Maxwell’s equations of electromagnetism, which are the prototypes of the nonlinear field theories of gauge theory and relativity theory.

The d’Alembertian on Minkowski spacetime $M^4$ with standard coordinates $x^i$, is the second order differential operator

$$\Box = \partial^2_{x^0} - \partial^2_{x^1} - \partial^2_{x^2} - \partial^2_{x^3}.$$ (1)

A massless scalar field is a real-valued function $\varphi$ on $M^4$ satisfying the wave equation $\Box \varphi = 0$. Plane symmetric solutions of the wave equation satisfy in addition, the compatible constraints $\partial_{x^2} \varphi = \partial_{x^3} \varphi = 0$. Thus, plane symmetric massless fields satisfy the reduced 2D-wave equation

$$\varphi_{uv} = 0,$$ (2)

where $u, v$ are null coordinates introduced by $u = x^0 + x^1$ and $v = x^0 - x^1$. A remarkable property of the solutions of the wave equation (2) is given by the identity

$$\varphi(O) + \varphi(B) = \varphi(A) + \varphi(\Gamma).$$ (3)

It relates the values of the field $\varphi$ assigned on the four vertices $O, A, B, \Gamma$ of any rectangular quadrilateral whose edges are formed by the characteristic “curves” of equation (2). The significance of the above identity is revealed immediately when we turn to the characteristic boundary value problem of the wave equation (figure 1). Another interpretation of the identity (3) is obtained by rewriting it in the form

$$\varphi + \varphi_{12} = \varphi_1 + \varphi_2,$$ (4)

which indicates a rather trivial superposition principle of solutions, stemming from the linearity and homogeneity of equation (2). Nonlinear superposition principles connecting solutions of
certain nonlinear PDEs appeared long time ago in the transformation theory of surfaces in ordinary space. These investigations were initiated in the late nineteenth century by Bäcklund, who introduced an important class of surface transformations. In virtue of a commutativity property, repeated applications of Bäcklund transformations can be performed in a purely algebraic manner resulting a nonlinear superposition principle, see [1] for recent advances on these topics. The prototypical example is given by the equation

\[(u - v) \tan \left( \frac{\phi_{12} - \phi}{4} \right) = (u + v) \tan \left( \frac{\phi_2 - \phi_1}{4} \right).\]  \hspace{1cm} (5)

It relates a solution \(\phi_{12}\) of the sine-Gordon equation

\[\phi_{xy} = \sin \phi,\]  \hspace{1cm} (6)

with an arbitrary seed solution \(\phi\) and two solutions \(\phi_1\) and \(\phi_2\) obtained from \(\phi\) via the Bäcklund transformations specified by the parameters values \(u\) and \(v\), respectively. On the other hand, equation (5) may be interpreted as a partial difference equation. This interpretation is obtained by simply identifying \(\phi_{1}\) and \(\phi_{2}\), respectively, with the values attained by the dependent variable \(\phi\) when two discrete independent variables \(n_1\) and \(n_2\) change by a unit step. Nonlinear PDEs, such as the sine-Gordon equation, and their discrete counterparts have been the subject of intensive investigations over the past century, leading to the vast developing soliton theory or theory of integrable systems.

The significance of Bäcklund transformations in general relativity, as invaluable methods for generating exact solutions of the Einstein field equations for spacetimes admitting two commuting Killing vectors, was established in the pioneering studies [2, 3, 4] on the celebrated Ernst equation [5, 6]. Here, we report some results obtained recently by investigating compatible partial difference and partial differential equations [7, 8, 9, 10]. In particular, in section 2 we give a representation of the infinite symmetries of a PDE which incorporates the hyperbolic Ernst equation, in terms of the complete hierarchy of the KdV soliton equations.

2. The KdV hierarchy and the infinite symmetries of the Ernst equation

The simplest discrete systems in two dimensions are one-field equations on quadrilaterals, i.e. equations of the form

\[\mathcal{H}(F, F_{[1]}, F_{[2]}, F_{[12]}; u, v) = 0.\]  \hspace{1cm} (7)

They may be regarded as the discrete analogues of hyperbolic type PDEs involving two independent variables. The fields take values in the complex numbers and are assigned on

\[\text{See also the contribution of G. Neugebauer in this volume.}\]
the vertices at sites \((n, m)\), which vary by unit steps only. The continuous lattice parameters \(u, v \in \mathbb{C}\) are assigned on the edges of an elementary quadrilateral as shown in figure 2. The updates of a field \(F \in \mathbb{C}\), along a shift in the \(n\) and \(m\) direction of the lattice are denoted by \(F_{[1]}, F_{[2]}\) respectively, i.e.

\[
F_{[1]} = F(n + 1, m), \quad F_{[2]} = F(n, m + 1), \quad F_{[12]} = F(n + 1, m + 1).
\]  

A specific nonlinear equation of the type (7) is the discrete KdV equation [11]

\[
(f_{[12]} - f)(f_{[1]} - f_{[2]}) = u - v.
\]

Due to a property that satisfies equation (9), consisting of its three-dimensional consistency, it is possible to construct from equation (9), an associated auxiliary linear system (discrete Lax pair). Moreover, interchanging the role of the lattice variables \((n, m)\) with that of the continuous lattice parameters \((u, v)\) and using the aforementioned property, it is also possible to derive a compatible PDE. The latter is the fourth order equation obtained from the Euler-Lagrange equation for the variational problem associated with the Lagrangian density [7]

\[
L = (u - v)\left(\frac{1}{f_{u}f_{v}}\right)^{2} + \frac{1}{u - v}\left(m^{2}\frac{f_{u}}{f_{v}} + n^{2}\frac{f_{v}}{f_{u}}\right).
\]  

The physical significance of the above Lagrangian was shown in [8], where the associated Euler-Lagrange equation is identified as a generalization of the hyperbolic Ernst equation for an Einstein-Weyl field, an equation which is very familiar to relativists.

The Euler-Lagrange equation associated with \(L\), admits a linear representation. This means that the field equation is a compatibility condition for the existence of the solution of an auxiliary, linear, overdetermined system of equations of the form

\[
(u - \lambda)\psi_{u} = H_{u}\psi, \quad (v - \lambda)\psi_{v} = H_{v}\psi.
\]

Here, \(\psi\) is a complex 2-vector, \(H\) is a \(2 \times 2\)-matrix valued function depending on \(u, v\), and \(\lambda\) is a complex parameter, called the spectral parameter. The explicit form of the matrices in (11) is

\[
H_{u} = \left(\begin{array}{c}
\frac{n - af_{u}}{a(n - af_{u})} & \frac{f_{u}}{af_{u}} \\
\frac{m - bf_{v}}{b(m - bf_{v})} & \frac{f_{v}}{bf_{v}}
\end{array}\right), \quad H_{v} = \left(\begin{array}{c}
\frac{m - bf_{v}}{b(m - bf_{v})} & \frac{f_{v}}{bf_{v}} \\
\frac{n - af_{u}}{a(n - af_{u})} & \frac{f_{u}}{af_{u}}
\end{array}\right),
\]

where the complex scalars \(a, b\) serve as auxiliary potentials for the field equation, while on the discrete level are identified with the shifted values of the dependent variable \(f\), when the discrete independent variables \(n\) and \(m\) change by a unit step, i.e. \(a = f_{[1]}, b = f_{[2]}\). The compatibility condition \(\psi_{uv} = \psi_{vu}\) on equations (11) is satisfied for every fixed value of \(\lambda\), whenever the Euler-Lagrange equation associated with \(L\) holds. The latter PDE acquires the matrix form

\[
(v - u)H_{uv} + [H_{u}, H_{v}] = 0.
\]

The symmetries of equation (13) are generated by the vector fields \(X_{Q} = Q\partial_{H}\), where the matrix-valued characteristic \(Q\) satisfies the linearized field equation

\[
(v - u)Q_{uv} + [Q_{u}, H_{v}] + [H_{u}, Q_{v}] = 0,
\]

2 By generalization here we mean that all solutions of the Ernst equation are embedded in the solution space of the Euler-Lagrange equation associated with \(L\), while the converse does not hold in general.
Using induction, one easily proves that the compatibility conditions
\[ \partial_u (uQ_u + [Q, H_u]) = \partial_u (vQ_v + [Q, H_v]), \]
which implies the existence of the matrix-valued potential \( \tilde{Q} \) introduced by
\[ \tilde{Q}_u = uQ_u + [Q, H_u], \quad \tilde{Q}_v = vQ_v + [Q, H_v]. \]
Using the Jacobi identity, one finds that the potential \( \tilde{Q} \) is also a symmetry characteristic of
equations (13), whenever \( Q \) and \( H \) satisfy (14) and (13), respectively. Thus, starting with a
known symmetry characteristic \( Q^{(0)} \), such as \( Q^{(0)} = [J, H] + K \) where \( J \) and \( K \) are complex
constant \((2 \times 2)\)-matrices, we may generate an infinite number of symmetries of equation (13),
using recursively equations (16). Explicitly we have
\[ Q^{(p+1)} = \mathcal{R} Q^{(p)}, \quad Q^{(0)} = [J, H] + K, \quad p \in \mathbb{N}, \]
where the recursion operator \( \mathcal{R} \) can be written formally as follows
\[ \mathcal{R} = d^{-1} \cdot (\sigma d - \varrho \ast d + ad_{dH}). \]
Here \( \ast \) is the two dimensional Hodge duality operator acting by \( \ast du = du, \ast dv = -dv \) on the
basis of one forms, \( ad_{A}(B) = [B, A] \), and \( \sigma = \frac{1}{2}(v + u), \varrho = \frac{1}{2}(v - u) \).
We now give a representation of the infinite symmetries, in terms of a compatible infinite set
of PDEs involving the field variable \( H \) and its derivatives with respect to higher time variables,
which correspond to the infinitesimal group parameters of the symmetry transformations. Let
us denote the latter by \( t^p = (t^0, t^1, \ldots), p \in \mathbb{N}, \) i.e.
\[ \partial_{t^p} H = Q^{(p)}, \quad p \in \mathbb{N}. \]
Since for every fixed \( p \), the characteristic \( Q^{(p)} \) defines a symmetry of equation (13), the pair
of PDEs which is formed by taking each of equations (19) together with equation (13), is a
compatible system of PDEs for \( H(u, v, t^p) \) (\( p \) fixed). By virtue of this fact, every such pair of
PDEs, admits also a linear representation. The latter is given by the linear equations (11),
together with each of the following linear equations
\[ \psi_{t^0} = J \psi, \quad \psi_{t^p+1} = -\lambda \psi_{t^p} - Q^{(p)} \psi, \quad p \in \mathbb{N}. \]
Using induction, one easily proves that the compatibility conditions \( \partial_{t^p} \partial_{u} \psi = \partial_{u} \partial_{t^p} \psi \) and
\( \partial_{t^p} \partial_{v} \psi = \partial_{v} \partial_{t^p} \psi \) are satisfied for every \( p \in \mathbb{N}, \) whenever \( H \) satisfies equation (13) and \( Q^{(p)} \)
are the preceding symmetry characteristics. Thus, it suffices to restrict our considerations on
the linear system (20) and generate the compatible set of PDEs for \( H \) with respect to the
variables \( t^p \), by taking the relevant compatibility conditions. However, in order to give an
explicit description and for purposes of calculation we proceed as follows. Firstly we remove the
leading equation from the linear system (20). Next we fix \( J \) and \( K \) as follows
\[ (J_{ij}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (K_{ij}) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \]
\[ i = 1, 2 \] and finally we introduce the matrix \( \Lambda = -\lambda J - K \). From this starting point we have
\[ \psi_{t^1} = (\Lambda + [\Lambda', H]) \psi, \quad \psi_{t^{p+1}} = -\lambda \psi_{t^p} - H_{t^p} \psi, \quad p \in \mathbb{N} \setminus \{0\}. \]
where prime denotes differentiation with respect to \( \lambda \). The first compatibility condition
\( \partial_{t^0} \partial_{u} \psi = \partial_{u} \partial_{t^0} \psi \) on equations (22), yields a set of four, first order PDEs for the components
of \( H \). Using (12) to identify the main field variable \( f \), and eliminating all variables in favour of
\( f \), we arrive at the following PDE
\[ 4 q_x + 12 q q_x - q_{xxx} = 0, \]
where \( q = f_x \) and \( x = t^1, t = t^2 \). Equation (23) is the celebrated KdV equation. Higher members
of the KdV hierarchy are obtained from system (22), in a similar manner.
3. Concluding remarks

Considering the discrete Boussinesq system, i.e. the three field quadrilateral system

\[ W_{[1]} = UU_{[1]} - V, \quad W_{[2]} = UU_{[2]} - V, \quad W = UU_{[12]} - V_{[12]} + \frac{u - v}{U_{[2]} - U_{[1]}}, \tag{24} \]

it was also possible to derive a compatible system of PDEs, \([9, 10]\). The latter equations are the Euler-Lagrange equations for the variational problem associated with the Lagrangian density

\[ \mathcal{L} = (u - v) \frac{\mathcal{F} \mathcal{E}}{\mathcal{J}^2} + m \frac{\mathcal{F}}{\mathcal{J}} + n \frac{\mathcal{E}}{\mathcal{J}}. \tag{25} \]

The scalars quantities \(\mathcal{F}, \mathcal{E}\) and \(\mathcal{J}\) are given by

\[ \mathcal{F} = f_{uv} \land f_u, \quad \mathcal{E} = f_{uv} \land f_v, \quad \mathcal{J} = f_u \land f_v, \tag{26} \]

where \((f_1, f_2) = (W, -U)\). The physical significance of the above Lagrangian stems from the fact that the Euler-Lagrange equations incorporate the Ernst equations for plane symmetric spacetimes in the presence of a source free Maxwell field and a Weyl neutrino field. In connection with the preceding discussion, the infinite symmetries of the Euler-Lagrange equations associated with \(\mathcal{L}\), build the complete hierarchy of the Boussinesq soliton equations. Moreover, by exploiting the compatibility between the discrete and continuous equations, an auto-Bäcklund transformation for the associated PDEs was constructed in an algorithmic manner. Thus, it would be interesting to consider to what extent, solutions of the larger continuous PDEs and their discrete compatible systems, could provide new exact solutions of the Einstein equations which govern the head-on collision of plane fronted gravitational waves, coupled with electromagnetic and neutrino waves, see \([12]\) and references therein.

Acknowledgments

This work was supported by the grant Pythagoras B-365-015 of the European Social Fund (ESF), Operational Program for Educational and Vocational Training II (EPEAEK II).

References

[1] Rogers C and Schief W K 2002 Bäcklund and Darboux transformations, Geometry and modern applications in soliton theory (Cambridge: Cambridge University Press)
[2] Harrison B.K. 1978 Bäcklund transformation for the Ernst equation of general relativity Phys. Rev. Let. 41 1197–1200
[3] Neugebauer G 1979 Bäcklund transformations of axially symmetric stationary gravitational fields J. Phys. A. 12 L67–L70
[4] Kramer D and Neugebauer G 1981 Prolongation structure and linear eigenvalue equations for Einstein-Maxwell fields J. Phys. A. 14 L333–L338
[5] Ernst F J 1968 New formulation of the axially symmetric gravitational field problem Phys. Rev. 167 1175–1178
[6] Ernst F J 1968 New formulation of the axially symmetric gravitational field problem II Phys. Rev. 168 1415–1417
[7] Nijhoff F, Hone A and Joshi N 2000 On a Schwarzian PDE associated with the KdV hierarchy Phys. Lett. A 267 147–156
[8] Tongas A, Tsoubelis D and Xenitidis P 2001 A family of integrable nonlinear equations of hyperbolic type J. Math. Phys. 42 5762–5784
[9] Tongas A and Nijhoff F 2005 Generalized hyperbolic Ernst equations of an Einstein-Maxwell-Weyl field J. Phys. A: Math. Gen. to appear
[10] Tongas A and Nijhoff F 2005 The Boussinesq integrable system: Compatible lattice and continuum structures Glasgow Math. J. to appear
[11] Nijhoff F W and Capel H W 1995 The discrete Korteweg-de Vries equation Acta Appl. Math. 133-158
[12] Griffiths J B 1991 Colliding plane waves in general relativity (Oxford: Oxford University Press)