NORM OPTIMAL FACTORIZATIONS OF SCALAR AND BLOCK MATRICES

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Abstract. For an $m \times n$ complex matrix $X$ of rank $r$ with Schur multiplier $S_X$ we show that there exist an $r \times m$ complex matrix $L$ and an $r \times n$ complex matrix $R$ such that $X = L^* R$ and $\|S_X\| = \|\text{diag}(L^* L)\|^\frac{1}{2} \|\text{diag}(R^* R)\|^\frac{1}{2}$, and the norm condition is optimal.

Let the completely bounded norm of the bilinear form $B_X$ induced by $X$ on $(\mathbb{C}^m, \|\cdot\|_\infty) \times (\mathbb{C}^n, \|\cdot\|_\infty)$ be denoted $\|B_X\|_{cb}$, then $X$ has a factorization $X = \Delta(\eta)^* C \Delta(\xi)$ with $\eta$ in $\mathbb{C}^m$, $\xi$ in $\mathbb{C}^n$ such that the outer factors are diagonal operators with $\|\xi\|_2 = \|\eta\|_2 = 1$ and $C$ has operator norm equal to $\|B_X\|_{cb}$, and the norm condition is optimal. A generalization to operator valued Schur block multipliers is presented too.

1. Introduction and Notation

A scalar valued $m \times n$ matrix $X$ may represent many different things in pure and applied mathematics. In this article we will focus on the interpretations of $X$ in 3 different ways

(i) As the matrix for a linear mapping $F_X$ of the $n$-dimensional C*-algebra $C(\{1, \ldots, n\}, \mathbb{C})$ into the $m$-dimensional Hilbert space $\mathbb{C}^m$. We will denote this abelian C*-algebra by $A_n$.

(ii) As the kernel for a bilinear form $B_X$ on the product $A_m \times A_n$ of C*-algebras given by

$$B_X(a, b) := \sum_{i=1}^{m} \sum_{j=1}^{n} X(i,j)a(i)b(j).$$

(iii) As a a linear mapping $S_X$ on $M_{(m \times n)}(\mathbb{C})$ induced by Schur multiplication by $X$ - or entry wise multiplication - given by

$$S_X(A)_{(i,j)} := X(i,j)A(i,j).$$

Date: November 2, 2022.

2010 Mathematics Subject Classification. Primary: 15A23, 15A60, 15A63. Secondary: 46L07, 47A30, 47L25.

Key words and phrases. Schur product, Grothendieck inequality, block matrix, minimal factorization, completely bounded, column norm.
These interpretations are very well known and have been studied in many ways for more than a century, but we think that we have discovered a connection between the classical Grothendieck inequality and the theory of completely bounded multilinear mappings which will make some of the existing results on factorization of matrices sharper. We will not define the concepts named Grothendieck inequalities or complete boundedness now, but leave that to the last part of the introduction. The short version of the content of this article is as follows; look at any of the interpretations (i) - (ii) - (iii) above of a scalar matrix. Show that the linear or multilinear operator attached to that interpretation has a completely bounded norm which only depends on it’s norm and not on the dimensions m, n. Then use our recent uniqueness result [4] for Stinespring representations of completely bounded multilinear mappings to obtain an optimal factorization of the matrix X - with respect to a certain norm property.

The estimates of the completely bounded norms come in the cases (i) and (ii) from, respectively, Grothendieck’s inequality for mappings from an abelian C*-algebra into a Hilbert space and from his inequality on bilinear forms on a pair of abelian C*-algebras [9], [18]. In the third case Smith’s work [21] shows that the norm of a Schur multiplier equals it’s completely bounded norm.

The norm optimality of the factorization results are based on the fact that the Stinespring representations for completely bounded linear or multilinear mappings contains a statement on optimality with respect to the completely bounded norm, [13]. This means - as we see it - that the optimal factorization results we obtain are all part of the theory of operator spaces and completely bounded mappings, whereas the existing versions of the results, see [18], are based on Grothendieck’s fundamental work [9]. In our setting we will say that in the case (i) Grothendieck’s "little" inequality implies that $F_X$ is completely bounded such that it’s completely bounded norm is dominated by the norm times the little Grothendieck constant $k_g$. In the case (ii) Grothendieck’s inequality implies that the completely bounded norm of the bilinear form is at most equal to the norm times Grothendieck’s complex constant $K_G$.

We need a little more notation in order to make an explicit formulation of the results we obtain in the cases (i) - (ii) - (iii) above. We will use small greek letters to denote vectors in a Hilbert space and write $\|\xi\|_2$ to denote the norm of the vector $\xi$ in some Hilbert space. For a vector $\xi$ in $\mathbb{C}^n$ we let $\Delta_n(\xi)$ denote the diagonal matrix in $M_n(\mathbb{C})$, whose diagonal equals $\xi$, and we let the expression $\xi_j$ denote the column matrix in $M_{(n,1)}(\mathbb{C})$ with entries $\xi_j$, and likewise $\xi_{-j}$ denotes the
row matrix in $M_{(1,n)}(\mathbb{C})$ with entries from $\xi$. In some instances the expressions $\xi_1$ and $\xi_\infty$ will denote the corresponding operators between the Hilbert spaces $\mathbb{C}$ and $\mathbb{C}^n$. For a matrix $T$ of scalars or operators, we let $\|T\|_\infty$ denote its operator norm, and we will use the important fact that for vector $\xi$ in $\mathbb{C}^n$ we have $\|\xi\|_2 = \|\xi\|_\infty = \|\xi_\infty\|_\infty$.

We return to the items (i) - (ii) - (iii) and let $r$ denote the rank of $X$, then the results may be written as

\begin{align*}
(i) \quad & \exists \xi \in \mathbb{C}^n, \exists T \in M_{(m,n)}(\mathbb{C}) \\
& X = T \Delta_n(\xi), \\
& \|T\|_\infty \|\xi\|_2 = \|F_X\|_cb \leq k_G \|F_X\|. \\
(ii) \quad & \exists \xi \in \mathbb{C}^n, \exists \eta \in \mathbb{C}^m, \exists T \in M_{(m,n)}(\mathbb{C}) \\
& X = \Delta_m(\eta)^* T \Delta_n(\xi), \\
& \|\eta\|_2 \|T\|_\infty \|\xi\|_2 = \|B_X\|_cb \leq K_G \|B_X\|. \\
(iii) \quad & \exists L \in M_{(r,m)}(\mathbb{C}), \exists R \in M_{(r,n)}(\mathbb{C}) \\
& X = L^* R, \text{ rank}(L) = \text{ rank}(R) = \text{ rank}(X), \\
& \|\text{diag}(L^* L)\|_1^\frac{1}{2} \|\text{diag}(R^* R)\|_1^\frac{1}{2} = \|S_X\|_cb = \|S_X\|.
\end{align*}

A first look at the items (ii) and (iii) does not show any relation between the 2 interpretations of a scalar matrix, but we will show that in some sense the two concepts may be considered as dual to each other with respect to the inner product on $M_{(m\times n)}$ given by

$$\forall X, Y \in M_{(m\times n)}(\mathbb{C}): \langle X, Y \rangle := \text{Tr}_n(Y^* X).$$

This aspect as well as quite a few of the other ones we mention are not new, see for instance equation (1.2) in [18], but some points of view and some optimality results are. The present article focusses on the use of the theory of operator spaces and completely bounded mappings, a subject which now is well described in the literature [8], [13], [17]. Pisier has made many deep and impressing contributions in this area of research and quite a few of them relate closely to some parts of this article, see [15], [16], [17], [19] to mention a few. The basic results we extend or twist go back to the famous article [9] by Grothendieck, which Pisier in the article [18] names the résumé. In the résumé Grothendieck shows a factorization result, for bilinear forms on a product of two abelian $C^*$-algebras, which now is known as the Grothendieck inequality. A reformulation of this inequality tells that
there exits a universal positive constant $K_G$ such that any complex $m \times n$ matrix with bilinear norm $\|X\|_B$ may be factored as
\begin{equation}
X = \Delta_m(\eta)^*T\Delta_n(\xi), \|\xi\|_2 = \|\eta\|_2 = 1, \|T\|_\infty \leq K_G\|B_X\|.
\end{equation}
This factorization of $X$ is analogous to the one described in (1.1) item (ii) except for the extension $\|T\|_\infty = \|B_X\|_{cb} \leq K_G\|B_X\|$, so in order to be more precise we will describe the concept named \textit{completely bounded}.

A bounded linear mapping $\varphi$ of a subspace $S$ of operators on some Hilbert space $H$ into some $B(K)$ for some Hilbert space $K$ is said to be completely bounded if there exists a positive $c$ such that for any natural number $k$ the mapping $\varphi_k := \varphi \otimes \text{id}_{M_k(\mathbb{C})} : S \otimes M_k(\mathbb{C}) \to B(H) \otimes M_k(\mathbb{C})$ has norm at most $c$. If $\varphi$ is completely bounded, it’s completely bounded norm $\|\varphi\|_{cb}$ is defined as the sup over the norms $\|\varphi_k\|$. In [5] the notion of complete boundedness was extended to multilinear mappings between spaces of bounded operators on Hilbert spaces in the following way. For a bounded bilinear mapping $\Phi : S_1 \times S_2 \to B(K)$ for a pair of operator spaces $S_1 \subseteq B(K_1)$ and $S_2 \subseteq B(K_2)$ we define $\Phi_k : (S_1 \otimes M_k(\mathbb{C})) \times (S_2 \otimes M_k(\mathbb{C})) \to B(K) \otimes M_k(\mathbb{C})$ by a formula, which is analogous to the matrix multiplication.
\begin{equation}
\forall A \in M_k(S_1) \forall B \in M_k(S_2) \forall i, j \in \{1, \ldots, k\}
\Phi_k(A, B)(i,j) := \sum_{l=1}^{k} \Phi(A_{(i,l)}, B_{(l,j)}).
\end{equation}

In the proof of Theorem 2.2 we shall see that equation (1.2) implies that $\|B_X\|_{cb} \leq K_G\|B_X\|$.

Grothendieck’s résumé [9] also shows that that the Grothendieck inequality may be used to describe those complex $m \times n$ matrices which are contractions as Schur multipliers. The following theorem is a consequence of Proposition 7 of [9].

**Theorem 1.1.** Any complex $m \times n$ matrix $X$ is contained in the closed convex hull of $m \times n$ scalar matrices $Y$ of the form $y_{(i,j)} = l_ir_j$ with $|l_i| \leq \frac{1}{2}K_G\|S_Y\|^{\frac{1}{2}}$ and $|r_j| \leq \frac{1}{2}K_G\|S_Y\|^{\frac{1}{2}}$.

This theorem is presented as Theorem 3.2 in [18] and that article contains proofs, extensions and historical notes. Based on the theorem it is possible to prove that for any complex $m \times n$ matrix $X$ there exist vectors $\{\lambda_1, \ldots, \lambda_m\}$ and $\{\rho_1, \ldots, \rho_n\}$ in the ball of radius $(K_G\|S_X\|)^{\frac{1}{2}}$ of $\ell_2(\mathbb{N}, \mathbb{C})$ such that for any pair $(i,j)$ of indices we have $X_{(i,j)} = \langle \rho_j, \lambda_i \rangle$. This result is easily reformulated such that it tells that there exist matrices $L$ in $M_{(\infty, m)}(\mathbb{C})$ and $R$ in $M_{(\infty, n)}(\mathbb{C})$ such that both...
$L$ and $R$ have column norms dominated by $(K_G\|S_X\|)^{(1/2)}$ and $X = L^*R$. The theory of completely bounded mappings later showed that the bounds could be reduced to the optimal bound $\|S_X\|^{(1/2)}$. Our improvement here is that the factorization now takes place via matrices $L, R$ in $M_{(r,n)}(\mathbb{C})$ where $r$ equals the rank of $X$. This factorization comes easily from the existing literature on completely bounded mappings see \cite{13} Theorem 8.7 (iii), but the statement on the ranks of $L$ and $R$, seems not to be noticed before.

In section 2 we will give the details in the proofs of the results on optimal factorizations as mentioned in the abstract, and we will show that the unit balls in the $m \times n$ complex matrices defined by $\mathcal{CB} := \{X \in M_{(m,n)}(\mathbb{C}) : \|BX\|_{cb} \leq 1\}$ and $\mathcal{CS} := \{Y \in M_{(m,n)}(\mathbb{C}) : \|SY\| \leq 1\}$ are polars of each other with respect to the inner product $\text{Tr}(Y^*X)$.

In Section 4 we will extend the factorization result to Schur block multipliers and study an infinite matrix $X = (x_{(i,j)})_{(i,j) \in J}$ with operator entries $x_{(i,j)}$ in the bounded operators on some Hilbert space $H$ and a *-algebra $A_0$ of bounded operators on $H$. We let $A_0^\prime$ denote the commutant of $A_0$ in $B(H)$. If the Schur block multiplier $S_X$ acts completely boundedly on the finite rank matrices in $M_J(A_0)$ then $X$ will have a factorization $X = L^*R$, such that $L$ is a matrix in $M_J(B(H))$, $R$ is a matrix in $M_J(A_0^\prime)$ and both have column norms equal to the square root of the complete Schur block multiplier norm of $S_X$ acting on the finite $J \times J$ matrices over $A_0$. This result is optimal with respect to the norm condition.

2. Factorizations

In the first place we look at a complex $m \times n$ matrix $X$ as the kernel for a linear mapping $F_X$ of $A_n$ to $\mathbb{C}^m$. The mapping $F_X$ does not map into an operator space right away, but the operator space $M_{(m,1)}(\mathbb{C})$ is isometrically isomorphic to $\mathbb{C}^m$ and hence equipped with a natural structure as an operator space. We will first show that $F_X$ is completely bounded with respect to this operator space structure and then find 2 natural Stinespring representations of this mapping, such that the factorization result drops out.

**Theorem 2.1.** For any $X$ in $M_{(m,n)}(\mathbb{C})$: $\|F_X\|_{cb} \leq k_G\|F_X\|$.

For any $X$ in $M_{(m,n)}(\mathbb{C})$ there exists a unit vector $\xi$ in $\mathbb{C}^n$ and a matrix $C$ in $M_{(m,n)}(\mathbb{C})$ such that $\|C\|_\infty = \|F_X\|_{cb}$ and $X = C\Delta_n(\xi)$.

**Proof.** It is well known that the classical little Grothendieck inequality implies that there exist a unit vector $\xi$ in $\mathbb{C}^n$ and a matrix $C$ in $M_{(m,n)}(\mathbb{C})$ such that $X = C\Delta_n(\xi)$, and $\|C\|_\infty \leq k_G\|F_X\|$. Then we
may write
\[ \forall a \in \mathcal{A}_n : \quad F_X(a) = C\Delta_n(a)\xi. \]

We recall that \(\xi\) sometimes denotes a matrix in \(M_{(m,1)}(\mathbb{C})\) and sometimes an operator in \(B(\mathbb{C},\mathbb{C}^n)\). In this equation it denotes an operator and we have obtained a Stinespring representation of \(F_X\) which proves the first statement of the theorem.

We will leave the Stinespring representation we just found and create 2 other ones now. We know that there is an optimal and also minimal Stinespring representation of \(F_X\) which we denote in the following way,
\[ \forall a \in \mathcal{A}_n : \quad F_X(a) = W^*\gamma(a)V, \]

such that \(\gamma\) is a representation of \(\mathcal{A}_n\) on a Hilbert space \(K, V\) is in \(B(\mathbb{C}, K)\) and \(W\) is in \(B(\mathbb{C}^m, K)\) and \(\|W\|\|V\| = \|F_X\|_{cb}\). This is the optimality, and the minimality means that \(K = \text{span}(\{\gamma(a)V1 : a \in \mathcal{A}_n\})\), and \(K = \text{span}(\{\gamma(a)W\eta : a \in \mathcal{A}_n, \eta \in \mathbb{C}^m\})\). We define the vector \(\Omega_n\) in \(\mathbb{C}^n\) as the vector where all entries equal 1. Then we may obtain a Stinespring representation of \(F_X\) in the following way.
\[ \forall a \in \mathcal{A}_n : \quad F_X(a) = X\Delta_n(a)(\Omega_n). \]

This Stinespring representation is minimal unless \(\text{span}(\{\Delta_n(a)X^*\eta : a \in \mathcal{A}_n, \eta \in \mathbb{C}^m\})\) is not all of \(\mathbb{C}^n\). So we have a minimal representation if and only if all columns in \(X\) are non vanishing. It is clearly no lack of generality to assume that every column in \(X\) is non trivial, the Stinespring representation from (2.2) is minimal. From [4] we know that the representations \(\gamma\) and \(\Delta_n\) of \(\mathcal{A}_n\) are unitarily equivalent, so we may as well assume that the Hilbert space \(K\) from (2.1) equals \(\mathbb{C}^n\) and that \(\gamma = \Delta_n\), so
\[ X\Delta_n(a)(\Omega_n) = W^*\Delta_n(a)V. \]

Define the vector \(\xi\) in \(\mathbb{C}^n\) as \(\xi := V1\). then \(\|\xi\|_2 = \|V\|\). and elementary algebra shows that \(X = W^*\Delta_n(\xi)\), and the theorem follows \(\square\)

We will now focus on the complex \(m \times n\) matrix as a kernel for a bilinear operator \(B_X\) on \(\mathcal{A}_m \times \mathcal{A}_n\). The result we present and it’s proof are similar to the ones we just presented.

**Theorem 2.2.** For any \(X\) in \(M_{(m,n)}(\mathbb{C})\) : \(\|B_X\|_{cb} \leq K_G\|B_X\|\).

For any \(X\) in \(M_{(m,n)}(\mathbb{C})\) there exists a unit vector \(\xi\) in \(\mathbb{C}^n\), a unit vector \(\eta\) in \(\mathbb{C}^m\) and a matrix \(C\) in \(M_{(m,n)}(\mathbb{C})\) such that \(\|C\|_\infty = \|B_X\|_{cb}\) and \(X = \Delta(\eta^*C\Delta_n(\xi))\).

**Proof.** Corollary 14.2 of [18] states that \(\|B_X\|_{cb} \leq K_G\|B_X\|\), but since we want to use [4] as the basis for our proof here, we want to use another well known approach to show this fact. It follows from the
classical Grothendieck inequality that there exist unit vectors \( \mu \in \mathbb{C}^n \), \( \nu \in \mathbb{C}^m \) and a matrix \( D \) in \( M_{(m,n)}(\mathbb{C}) \) such that \( \|D\|_{\infty} \leq K_G \|B_X\| \) and \( X = \Delta_m(\nu)^* D \Delta(\nu) \). Elementary manipulations show that

\[
(2.3) \quad \forall y \in \mathcal{A}_m \forall z \in \mathcal{A}_n : B_X(y, z) = \sum_{i=1}^{m} \sum_{j=1}^{n} (y_i \mu_i) D_{(i,j)} (\nu_j z_j) = (\mu_i)^* \Delta_m(y) D \Delta_n(z) \nu_i.
\]

So we have obtained a Stinespring representation of \( B_X \) which shows that \( \|B_X\|_{cb} \leq \|\mu\|_2 \|D\|_{\infty} \|\nu\|_2 \leq K_G \|B_X\| \).

We will take a norm optimal and minimal Stinespring representation of \( B_X \) and use the following notation. There exist Hilbert \( K, L \) representations \( \pi \) of \( \mathcal{A}_m \) on \( K \), \( \rho \) of \( \mathcal{A}_n \) on \( L \), and operators \( R \) in \( B(K, \mathbb{C}) \), \( S \) in \( B(L, K) \), \( T \) in \( B(\mathbb{C}, L) \) such that for any \( y \in \mathcal{A}_m \) for any \( z \in \mathcal{A}_n \)

\[
(2.4) \quad B_X(y, z) = R \pi(y) S \rho(z) T, \quad \text{and} \quad \|R\| \|S\| \|T\| = \|B_X\|_{cb}.
\]

As above, we let \( \Omega_m \) denote the vector in \( \mathbb{C}^m \) where all the entries are 1, and in analogy with (2.2) we define a second Stinespring representation of \( B_X \) by

\[
(2.5) \quad B_X(y, z) = (\Omega_m)^* \Delta_m(y) X \Delta_n(z) (\Omega_n^*).
\]

This second Stinespring representation is minimal if the following 4 conditions are satisfied

(i) span\(\{\Delta_n(z) \Omega_n : z \in \mathcal{A}_n\}\) = \( \mathbb{C}^n \).

(ii) span\(\{\Delta_m(y) X \Delta_n(z) \Omega_n : y \in \mathcal{A}_m, z \in \mathcal{A}_n\}\) = \( \mathbb{C}^m \).

(iii) span\(\{\Delta_m(y^*) \Omega_m : y \in \mathcal{A}_m\}\) = \( \mathbb{C}^m \).

(iv) span\(\{\Delta_n(z^*) X^* \Delta_m(y^*) \Omega_m^* : y \in \mathcal{A}_m, z \in \mathcal{A}_n\}\) = \( \mathbb{C}^n \).

Since all the entries in the vectors \( \Omega_m \) and \( \Omega_n \) are 1, it follows that the conditions (i) and (iii) are fulfilled. The conditions (ii) is fulfilled if all the rows in \( X \) are non vanishing, and, analogously, (iv) is satisfied if all the columns in \( X \) are non trivial. Since it will be no serious restriction to assume that all the columns and all the rows in \( X \) are non vanishing, we will assume so, and (2.5) gives a minimal Stinespring representation of \( B_X \). We may then apply item (ii) of Theorem 3.2 in [4] and then assume that in the Stinespring representation (2.4) we have \( K = \mathbb{C}^m \), \( L = \mathbb{C}^n \), \( \pi = \Delta_m \), and \( \rho = \Delta_n \). Then we remark that the commutant of \( \Delta_m(\mathcal{A}_m) \) equals \( \Delta_m(\mathcal{A}_m) \) and similarly for \( \mathcal{A}_n \), so when we apply item (v) of the same theorem we find that there exists a vector \( \xi \in \mathbb{C}^n \) such that \( \Delta_n(\xi) (\Omega_n) = T \). Then \( \xi = T1 \), where 1 is a unit vector in \( \mathbb{C} \), so \( \|\xi\|_2 = \|T\| \). We also get that there exists a vector \( \eta \in \mathbb{C}^m \) such that \( \Delta_m(\eta)(\Omega_m) = R^* \). Then \( \eta = R^* 1 \) and \( \|\eta\|_2 = \|R\| \). Some
elementary algebra shows that $X = \Delta_m(\eta)^*S\Delta_n(\xi)$ and from (2.4) it follows that $\|\eta\|_2\|S\|\|\xi\|_2 = \|B_X\|_{cb}$, and the theorem follows. \hfill \Box

We will now turn to the study of the norm of a Schur multiplier $S_X$ for a complex $m \times n$ matrix $X$ and the factorization result we get in this connection.

The search for estimates of the Schur multiplier norm $\|S_X\|$ has a long history and we do not intend to cover all the contributions. On the other hand several works give estimates based on norms of the diagonals of some positive matrices. The most famous estimate is of course the very first one by Schur [20], although he did not see it this way, but his result tells that for a positive matrix $X$ the Schur multiplication is a positive mapping and hence it’s norm equals the norm of the diagonal. It seems to us that the usage of the words row and column norms in connection with norms of Schur multipliers appears first in Davidson and Donsig’s article [6]. The previous researchers expressed their estimates in terms of norms of diagonals of some positive matrices. In the article [3] we constructed a concrete Stinespring representation of the Schur product, which showed that Schur multiplication is a completely bounded bilinear operator of completely bounded norm 1. This construction is sketched in Section 3, because we need an infinite version of this applied to block matrices in Section 4. Here we will just reformulate a single result from [4], which shows that the factorization of Theorem 2.4 is optimal.

**Proposition 2.3.** Let $l, m, n$ be natural numbers, $L$ and $R$ be matrices in $M_{(l,m)}(\mathbb{C})$ and $M_{(l,n)}(\mathbb{C})$ then $\|S_{(L^*R)}\| \leq \|L\|_c\|R\|_c$.

The proof is sketched in the proof of Proposition 3.2. This result gives simple proofs to some of the previous results on norms of Schur multipliers. C. Davis [7] studies the multiplier norm $\|S_X\|$ when $X$ is a self-adjoint matrix and his upper estimate on the multiplier norm is the norm of the diagonal of $|X|$ which is exactly $\|(|X|)^{\frac{1}{2}}\|^2$. When $X$ is self-adjoint with polar decomposition $X = S|X|$, then $X$ may be factored as $X = (|X|^{\frac{1}{2}})(S|X|^{\frac{1}{2}})$, and Davis result follows as an application of Proposition 2.3 to this factorization. M. Walter gets in [23] an upper bound for a general $X$ expressed in terms of the norms of some diagonals. In our language Walter’s result is $\|S_X\|_{cb} \leq \|(XX^*)(\frac{1}{2})\|_c\|\|X^*\|^\frac{1}{2}\| c$. In our setting this corresponds to the factorization of an $X$ with polar decomposition $X = V|X|$, as $X = ((XX^*)(\frac{1}{2}))(V(X^*)\frac{1}{2})$. Bożejko gives in [2] a very short proof of Walter’s result which is very close to the one we use. The difference lies mostly in his use of the existing theory on Banach spaces, whereas this
presentation is based on the Stinespring representation of the Schur product as a completely bounded bilinear operator [3], as described in Section 3. The article [6] by Davidson and Donsig contains a lot of information on problems related to our investigation, but we have not seriously tried to apply our methods to questions on which subsets $S$ of the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ that will have the property that Schur multiplication with the characteristic function of $S$ will induce a completely bounded Schur multiplier. From [1] we know that when $J = \mathbb{N}$ then the restriction of a bounded matrix to it's lower diagonal part is not a bounded mapping, but we have found no way to show that this matrix can not be factored as $L^*R$ with $\|L\|_c < \infty$ and $\|R\|_c < \infty$. We have got the idea that the remarkable results by Lust-Piquard [12] ought to be able to help us to understand some of the questions on Schur multipliers, which we have been interested in. Unfortunately we were not able do so. In [11] Livshits studies the Schur block product between block matrices with operator entries, and he obtains an inequality which in our setting is a consequence of the complete boundedness of the Schur block product.

The inclusion of $M_{(m,n)}(\mathbb{C})$ in $M_k(\mathbb{C})$ for a $k$ bigger than both $m$ and $n$ may be done in any fashion where we fix $m$ rows and $n$ columns in $M_k(\mathbb{C})$. Such an inclusion is obviously an isometry with respect to the operator norm and there is a conditional expectation of norm 1, with respect to the operator norms, from $M_k(\mathbb{C})$ onto the embedded copy of $M_{(m,n)}(\mathbb{C})$ which is given by multiplication by orthogonal projections from the left and from the right. Consequently, if we look at a complex $m \times n$ matrix $X$, the Schur multiplier norm $\|S_X\|$ is the same on both $M_{(m,n)}(\mathbb{C})$ and $M_k(\mathbb{C})$. Having this in mind we will work in the setting of square matrices until we have obtained our result in this setting and then deduce the general result at the very end.

The basic result we use in the proof of the following theorem, in replacement of the inequalities by Grothendieck used in the proofs of the theorems 2.1 and 2.2, is Smith's result [21] that the completely bounded norm $\|S_X\|_{cb}$ equals the operator norm $\|S_X\|$. We mentioned above that Grothendieck's work [9] yields a description of of the structure of a Schur multiplier of norm one, which except for a constant is analogous to the optimal one which may be obtained via the use of the theory of completely bounded mappings as described in Paulsen's book [13]. All of this gives the following theorem except for the statement on ranks. Here we will give an alternative proof based on the the results from [4], just as we have done in the former parts of this section.
Theorem 2.4. Let $X$ be in $M_{(m,n)}(\mathbb{C})$ with rank $r$, then there exist matrices $L$ in $M_{(r,m)}(\mathbb{C})$ and $R$ in $M_{(r,n)}(\mathbb{C})$ such that they both have rank $r$, and

$$L^*R = X \text{ and } \|L\|_c \|R\|_c = \|S_X\|_c = \|S_X\|.$$ 

Proof. Let us first assume that $X$ is a square matrix in $M_k(\mathbb{C})$, then by [21] there exists an optimal and minimal Stinespring representation of $S_X$ such that

$$(2.6) \quad \forall A \in M_k(\mathbb{C}) : S_X(A) = S^*\pi(A)T,$$

with $\pi$ a representation of $M_k(\mathbb{C})$ on a Hilbert space $K$, $S$ in $B(\mathbb{C}^k, K)$, $T$ in $B(\mathbb{C}^k, K)$ and $\|S^*\|\|T\| = \|S_X\|$. With the notation from [3], which also is sketched in Section 3, we get another Stinespring representation which is given as

$$(2.7) \quad \forall A \in M_k(\mathbb{C}) : S_X(A) = (V^*\lambda(X))\rho(A)V.$$

This Stinespring representation may not be minimal, but there exists by Proposition 2.2 of [4] an orthogonal projection $P$ in the commutant $\rho(M_k(\mathbb{C}))'$ such that when $\mathbb{C}^k \otimes \mathbb{C}^k$ is replaced by $\mathcal{P} := P(\mathbb{C}^k \otimes \mathbb{C}^k)$, and the representation $\rho$ and the outer operators are restricted to this subspace, then we get a minimal Stinespring representation

$$(2.8) \quad \forall A \in M_k(\mathbb{C}) : S_X(A) = (V^*(\lambda(X)|\mathcal{P}))\rho_P(A)(PV).$$

Recall that $\lambda(X) := X \otimes I_{\mathbb{C}^k}$ and $\rho(A) := I_{\mathbb{C}^k} \otimes A$, so we have the commutation relation $\rho(M_k(\mathbb{C}))' = \lambda(M_k(\mathbb{C}))$. Consequently the projection $P$ has the form $P = \lambda(p)$ for a projection $p$ in $M_k(\mathbb{C})$. By the proof of Proposition 2.2 of [4] the projection $P$ maps onto the closed linear span of the set $\{\rho(A)\lambda(X^*)V\xi : A \in M_k(\mathbb{C}), \xi \in \mathbb{C}^k\}$, and we find that $P = \lambda(p)$ is characterized by the property that $p$ is the range projection of $X^*$, i.e. the smallest orthogonal projection in $M_k(\mathbb{C})$ such that $pX^* = X^*$. By item (ii) in Theorem 3.2 of [4] we see that the representations $\rho_P$ of (2.8) and $\pi$ of (2.6) are unitarily equivalent. Hence there exists an isometry $W$ of $K$ onto $\mathcal{P}$ such that we get

$$(2.9) \quad \forall A \in M_k(\mathbb{C}) : \pi(A) = W^*\rho_P(A)W \text{ and then}$$

$$S^*W^*\rho_P(A)WT = (V^*\lambda(X)|\mathcal{P})\rho_P(A)(PV).$$

By [4] Theorem 3.2 item (v) there exists an operator $D$ on $\mathcal{P}$ which is in the commutant $\rho_P(M_k(\mathbb{C}))'$ on $\mathcal{P}$ such that $WT = DPV$. The item (iii) of the same theorem shows that there is an operator $E$ on $\mathcal{P}$ such that it commutes with $\rho_P(M_k(\mathbb{C}))$ and satisfies $WS = E\lambda(X)^*V$. The operator $DP$ is an operator on $\mathbb{C}^k \otimes \mathbb{C}^k$ which commutes with
\(\rho(M_k(\mathbb{C}))\) so there exists an \(R_0\) in \(M_k(\mathbb{C})\) such that \(\lambda(R_0) = DP\). By Proposition 3.1 we know that

\[
\|R_0\|_c = \|\lambda(R_0)V\| = \|(DP)V\| = \|WT\| = \|T\|.
\]

Analogously we find that there exists a matrix \(L_0\) in \(M_k(\mathbb{C})\) such that \(\lambda(L_0) = E\lambda(X)^*\), which satisfies

\[
\|L_0\|_c = \|\lambda(L_0)V\| = \|E\lambda(X)^*V\| = \|WS\| = \|S\|.
\]

Then we can combine the equations and we get

\[
\forall A \in M_k(\mathbb{C}) : V^*\lambda(L_0^*R_0)\rho(A)V = V^*\lambda(L_0)^*\rho(A)\lambda(R_0)V
\]

\[
= S^*W^*\rho_p(A)WT
\]

\[
= S_X(A)
\]

\[
= V^*\lambda(X)\rho(A)V.
\]

Since \(\rho(e_{(i,j)})V\delta_i = \delta_i \otimes \delta_j\), we get span\(\{\rho(A)V\xi : A \in M_k(\mathbb{C}), \xi \in \mathbb{C}^k\}\) = \(\mathbb{C}^k \otimes \mathbb{C}^k\), so we can conclude that \(L_0^*R_0 = X\) with \(\|S_X\| = \|L_0\|_c\|R_0\|_c\). With respect to the rank condition we know that the rank of \(X\) equals the dimension of it’s support projection, which is \(p\). On the other hand we have by construction that \(pR_0 = R_0\) and \(L_0^*p = L_0^*\) so both \(L_0\) and \(R_0\) have rank at most \(r\) and satisfies \(L^*R = X\), so their ranks are equal to the rank \(r\) of \(X\). There exists an isometry \(Z\) in \(M_k(\mathbb{C})\) such \(Z^*Z = p\) and \(ZZ^*\) is the projection onto the subspace of \(\mathbb{C}^k\) spanned by the first basis vectors \(\{\delta_1, \ldots, \delta_r\}\) of \(\mathbb{C}^k\). Then the matrices of the operators defined by \(L := ZL_0\) and \(R :=ZR_0\) will establish the theorem. \(\square\)

With the notation from above we will look at complex valued \(m \times n\) matrices and define 4 compact convex subsets of these matrices by

\[
B_{(m,n)} := \{X \in M_{(m,n)}(\mathbb{C}) : \|BX\| \leq 1\}
\]

\[
S_{(m,n)} := \overline{\text{conv}}\{X \in M_{(m,n)}(\mathbb{C}) : X_{(i,j)} = \overline{l_ir_j}, |l_i| = 1, |r_j| = 1\}
\]

\[
CB_{(m,n)} := \{X \in M_{(m,n)}(\mathbb{C}) : \|BX\|_{cb} \leq 1\}
\]

\[
CS_{(m,n)} := \{X \in M_{(m,n)}(\mathbb{C}) : \|SX\| \leq 1\}
\]

In the investigation of the connections between these 4 sets we need a couple of observations, which we list as propositions. A combination of Proposition 2.3 and Theorem 2.4 yield immediately the following result.

**Proposition 2.5.** Let \(m, n\) be natural numbers and \(l := \min\{m, n\}\), then

\[
CS_{(m,n)} = \{L^*R : L \in M_{(l,m)}(\mathbb{C}), \|L\|_c \leq 1, R \in M_{(l,n)}(\mathbb{C}), \|R\|_c \leq 1\}.
\]
Theorem 2.2 implies.

**Proposition 2.6.** Let $m$ and $n$ be natural numbers then

\[ \mathcal{CB}_{(m,n)} \]

\[ = \{ X \in M_{(m,n)}(\mathbb{C}) : X = \Delta_m(\xi)C\Delta_n(\eta), \text{ s.t.} \| \xi \|_2 \| C \|_{\infty} \| \eta \|_2 \leq 1 \}. \]

**Proof.** The only thing we are missing in the proof is that for an $m \times n$ matrix $X = \Delta_m(\xi)C\Delta_n(\eta)$ we have $\| B_X \|_{cb} \leq \| \xi \|_2 \| C \|_{\infty} \| \eta \|_2$, but that follows from the Stinespring representation given in the equation (2.3).

We will now look at the inner product in $M_{(m,n)}(\mathbb{C})$ defined by

\[ (2.14) \quad \forall X, Y \in M_{(m,n)}(\mathbb{C}) : \langle X, Y \rangle := \text{Tr}_n(Y^*X). \]

As usual we define the polar $\mathcal{D}^\circ$ of a subset $\mathcal{D}$ of $M_{(m,n)}(\mathbb{C})$ via the expression

\[ (2.15) \quad \mathcal{D}^\circ := \{ X \in M_{(m,n)}(\mathbb{C}) : \forall D \in \mathcal{D}, |\langle X, D \rangle| \leq 1 \}. \]

The 4 sets defined in the definitions (2.13) all have the property that they are equal to their bi-polars, so the polar circle in the statements below may be moved to the other side.

**Theorem 2.7.** For any pair of natural numbers $m, n$ we have the following relations

\[ (2.16) \quad \mathcal{B}_{(m,n)} = (\mathcal{S}_{(m,n)})^\circ \]

\[ (2.17) \quad \mathcal{CB}_{(m,n)} = (\mathcal{CS}_{(m,n)})^\circ. \]

**Proof.** The equation (2.16) is based on the identity

\[ \text{Tr}_n((\bar{l}_ir_j)^*X) = \sum_{i,j} l_i x_{(i,j)} \bar{r}_j \]

and then follows from the definition of the norm of $B_X$ as a bilinear operator on the pair of $C^*$-algebras $A_m \times A_n$, and the fact that the extreme points in the unit-ball of these algebras are the unitaries. With respect to (2.17) we will introduce the conjugation operation on $M_{(m,n)}(\mathbb{C})$ which is defined by $(\bar{X})_{(i,j)} := \overline{X_{(i,j)}}$. Since the transposition and the adjoint operation both are isometries with respect to the operator norm, it follows that the conjugation operation is a conjugate linear isometry on $M_{(m,n)}(\mathbb{C})$ equipped with the operator norm and hence we get that $\| S_X \| = \| S_{\bar{X}} \|$. An elementary calculation shows that for matrices $C, X$ in $M_{(m,n)}(\mathbb{C})$ and vectors $\eta$ in $\mathbb{C}^m$, $\xi$ in $\mathbb{C}^n$ we have

\[ (2.18) \quad \text{Tr}_n(X^*\Delta_m(\eta)^*C\Delta_n(\xi)) = \langle (\bar{X} \circ C)\xi, \eta \rangle. \]
The equality in Proposition 2.6 may be applied to the equation just above, and when \((2.18)\) is read from the left to the right we find that \((CB_{(m,n)})^\circ \subseteq CS_{(m,n)}\). On the other hand the equation also tells that for any \(X\) in \(CS_{(m,n)}(\mathbb{C})\) and any \(Y\) in \(CB_{(m,n)}(\mathbb{C})\) we have \(|\langle Y, X \rangle| \leq 1\), so \(CS_{(m,n)}(\mathbb{C}) \subseteq (CB_{(m,n)}(\mathbb{C}))^\circ\), and the theorem follows. \(\square\)

Smith [21] has proved the following proposition, see Proposition 8.11 of [13].

**Proposition 2.8.** Let \(\phi\) be a bounded linear map of an operator space into \(M_n(\mathbb{C})\), then \(\|\phi\|_{cb} = \|\phi_n\|\).

The theorem above may be combined with Proposition 2.5 to prove a similar result for bilinear forms on \(A_m \times A_n\). Smith's result may probably be used here too - but we have not pursued this possibility.

To do so, a bilinear form has to be viewed as a linear operator from \(A_{\max\{m,n\}}\) to the dual of \(A_{\min\{m,n\}}\). This point of view is the one used in [19].

**Corollary 2.9.** Let \(X\) be in \(M_{(m,n)}(\mathbb{C})\) and \(k := \min\{m, n\}\), the \(\|BX\|_{cb} = \|(BX)_k\|\).

**Proof.** By the theorem above and compactness there exists \(Y\) in \(CS_{(m,n)}(\mathbb{C})\) such that \(\|S_X\|_{cb} = \text{Tr}_n(Y^*X)\). Then by Proposition 2.5 there exists \(L\) in \(M_{(k,m)}(\mathbb{C})\) and \(R\) in \(M_{(k,n)}(\mathbb{C})\) both with column norm at most 1, such that \(Y = L^*R\). We can then construct contraction matrices \(A\) in \(M_k(A_m)\) and \(B\) in \(M_k(A_n)\), such that \(\|(BX)_k(A, B)\| = \|BX\|_{cb}\) in the following way.

For \(1 \leq s \leq k\) define elements \(a^s\) in \(A_m\) and \(b^s\) in \(A_n\) by

\[
a^s(i) := l_{(s,i)} \quad \text{and} \quad b^s(j) := \overline{r_{(s,j)}}.
\]

Since the column norms of \(L\) and \(R\) are at most 1 we have

\[
(2.19) \quad \sum_{s=1}^k a^s(a^s)^* \leq I_{A_m} \quad \text{and} \quad \sum_{s=1}^k (b^s)^*b^s \leq I_{A_n},
\]

and we can define matrices \(A\) in \(M_k(A_m)\) and \(B\) in \(M_k(A_n)\).

\[
(2.20) \quad A_{(u,v)} := \begin{cases} a^u & \text{if } u = 1, \\ 0 & \text{if } u \neq 1 \end{cases}
\]

\[
B_{(v,w)} := \begin{cases} b^w & \text{if } w = 1, \\ 0 & \text{if } w \neq 1. \end{cases}
\]

Since \(A\) is a one row operator and \(B\) a one column operator the inequalities \((2.19)\) imply that \(A\) and \(B\) are contractions. On the other
hand a simple calculation shows that \( \|(B_X)_k(A, B)\| = \|B_X\|_{cb} \), so the corollary follows.

**Remark 2.10.** The proof of the corollary actually shows that maximum is attained in the case where \( A \) is a one row matrix of length \( k \) and \( B \) is a one column matrix of length \( k \), so the completely bounded norm is given as

\[
\|B_X\|_{cb} = \sup\{ \left| \sum_{s=1}^{k} B_X(a_s, b_s) \right| : \sum_{s=1}^{k} a_s a_s^* \leq I_{A_m} \sum_{s=1}^{k} b_s^* b_s \leq I_{A_n} \}. 
\]

The finite sums above may have some implications for the Haagerup tensor product \( A_m \otimes_h A_n \), or the content of the remark may follow from well known properties of the Haagerup tensor product and the finite dimensionality of the factors in the tensor product, see [13] Ch. 17 or [18] Theorem 14.1.

We recall that Grothendieck’s inequality means that \( B_{(m,n)}(\mathbb{C}) \subseteq K_GC_B_{(m,n)}(\mathbb{C}) \). The result on polars then implies the well known result that \( C_S_{(m,n)}(\mathbb{C}) \subseteq K_GC_S_{(m,n)}(\mathbb{C}) \). As a final remark we would like to mention that the results in the Theorems 2.1, 2.2, Proposition 3.6 and Corollary 2.9 imply that Grothendieck’s constants \( k_G \) and \( K_G \) formally may be computed as

**Theorem 2.11.**

\[
k_G = \sup_{k \in \mathbb{N}} \sup_{X \in M_k(\mathbb{C})} \frac{\|(F_X)_k\|}{\|F_X\|}, 
\]

\[
K_G = \sup_{k \in \mathbb{N}} \sup_{X \in M_k(\mathbb{C})} \frac{\|(B_X)_k\|}{\|B_X\|}. 
\]

3. **On Schur Block Multipliers**

In this section we will establish some notation regarding Schur block multipliers and recall some results from the article [3]. We will study a single block matrix \( Q \) with entries in some \( B(H) \) acting from the left as a Schur block multiplier on block matrices with entries in a certain *-subalgebra \( A_0 \) contained in \( B(H) \). This left Schur multiplier is denoted \( LSQ \) and no boundedness conditions are assumed in the first place. The right Schur block multiplier \( RS \) will also occur below, but only in the cases where it is equal to the left multiplier. The theory for the left multipliers is naturally applicable to the right multipliers. We will follow Livshits in [11] and use the symbol \( \square \) to denote the Schur block product, so \( LS_Q(A) = (Q_{(i,j)}A_{(i,j)}) = Q \square A \). If we assume that
the matrix $Q$ has the property that the set $\{\|q_{(i,j)}\| : i, j \in J\}$ is bounded we will write $Q \in M^\infty_J(B(H))$. If only finitely many entries are non vanishing we will write $Q \in M^0_J(B(H))$. It is interesting to note that the unit with respect to the Schur block product is the $J \times J$ matrix with all entries equal to $I_H$. When $J$ is infinite this matrix can not have an interpretation as an unbounded operator on $\ell^2(J) \otimes H$ in any way, we think, but the mapping it induces as a Schur block multiplier is clearly completely bounded. We can now formulate the Theorem 4.1, but in a slightly soft way. The commutant $A'_0$ of $A_0$ consists of all operators in $B(H)$ which commute with each operator in $A_0$. If the mapping $LS_Q$ is completely bounded on $M^0_J(A_0)$ with completely bounded norm dominated by 1, then there exist matrices $L$ in $M^\infty_J(B(H))$ and $R$ in $M^\infty_J(A'_0)$ both with column norms at most 1 such that $Q = L^* R$. Remark that although $L$ and $R$ may not represent a pair of bounded operators on $\ell^2(J) \otimes H$, the product $L^* R$ is well defined inside $M^\infty_J(B(H))$ since $L$ and $R$ are column bounded.

We will introduce the notation required to present the Schur block product as a completely bounded bilinear operator. For a finite index set all of this is presented in the article [3], but some care has to be taken when passing to the more general setting with an infinite index set $J$ and a C*-algebra $A$ replaced by a *-algebra $A_0$.

We consider elements in $M_J(B(H))$ as formal sums of elementary tensors and we will, as in Harmonic analysis, use the symbol $\sim$ to indicate that a sum is formal rather than convergent in some topology, so we write

$$A = (a_{(i,j)}) \in M_J(B(H)), \quad A \sim \sum_{i,j \in J} e_{(i,j)} \otimes a_{(i,j)}.$$ 

It is clear that the Schur block product is well defined on the vector space $M_J(B(H))$, but the matrix product of 2 such matrices is only defined under some extra conditions on the matrices. We define 3 mappings $\lambda, \rho$ and $\sigma$ of $M_J(B(H))$ into $M_J(M_J(B(H)))$ by

$$\lambda(A) \sim \sum_{i,j,k \in J} e_{(i,j)} \otimes a_{(i,j)} \otimes e_{(k,k)},$$

$$\rho(A) \sim \sum_{i,k,l \in J} e_{(i,i)} \otimes a_{(k,l)} \otimes e_{(k,l)},$$

$$\sigma(A) \sim \sum_{i,j \in J} e_{(i,j)} \otimes a_{(i,j)} \otimes e_{(i,j)}.$$ 

When a matrix $A = (a_{(i,j)})$ represents a bounded operator, this operator will act on $\ell^2(J) \otimes H$, so we will introduce the notation that $\tilde{H}$
denotes this Hilbert space. Similarly, for an operator bounded matrix $A$ the bounded operators $\lambda(A)$, $\rho(A)$ and $\sigma(A)$ will act on the Hilbert space $\ell^2(J) \otimes H \otimes \ell^2(J)$, so we will use the short name $L$ for this Hilbert space. The standard orthonormal basis in $\ell^2(J)$ is denoted $(\delta_j)_{j \in J}$ and then we define an isometric embedding $V$ of $\tilde{H}$ into $L$ via the formula

$$V(\sum_{j \in J} \xi_j \otimes \delta_j) := \sum_{j \in J} \delta_j \otimes \xi_j \otimes \delta_j.$$  

(3.4)

The projection onto the closed range $V \tilde{H}$ in $L$ is denoted $F$ and it has the formal matrix representation $F \sim \sum_{i,j,k,l,m,n,p,q \in J} (e_{(i,i)} \otimes I_H) (e_{(k,l)} \otimes a_{(i,j)} \otimes e_{(m,m)}) (e_{(p,q)} \otimes e_{(j,j)})$

so $i = k = m = p$, $j = q = n = l$ and

(3.5)

$$F\lambda(A) \rho(B) F \sim \sum_{i,j,k,l,m,n,p,q \in J} e_{(i,j)} \otimes a_{(i,j)} b_{(i,j)} \otimes e_{(i,j)} \sim \sigma(A \Box B).$$

If $A$ and $B$ represent bounded operators on $\tilde{H}$ the equation (3.5) implies that

$$A \Box B = V^* \lambda(A) \rho(B) V,$$

so we have obtained a Stinespring representation of $\Box$ as a bilinear completely bound mapping of completely bounded norm 1. The reason why column and row norms become important when studying Schur block multipliers follow from the form of the equation (3.5) and the following proposition.

**Proposition 3.1.** Let $Y$ be a column bounded matrix in $M_J(B(H))$ then the matrices representing $\lambda(Y) F$ and $\rho(Y) F$ are matrices of bounded operators and $\|\lambda(Y) F\| = \|\rho(Y) F\| = \|Y\|_c$.

**Proof.** The proof may be found in [3], but an easy computation shows that

$$\|\lambda(Y) F\|^2 = \sup_{j \in J} \|e_{(j,j)} \otimes (\sum_{i \in J} y_{(i,j)}^* y_{(i,j)}) \otimes e_{(j,j)}\| = \|Y\|_c^2.$$

(3.7)
The proposition has the following immediate consequence, which is a generalization of a theorem of Livshits [11].

**Proposition 3.2.** Let $\mathcal{A}_0$ be an algebra of bounded operators on a Hilbert space $H$ and $Q$ an element in $M_J(B(H))$. If $Q$ may be factored as a product $Q = L^*R$ of block matrices over $J$ such that $\|L\|_c < \infty$, $\|R\|_c < \infty$ and all the operators $r(i,j)$ in the matrix $R$ are in the commutant $\mathcal{A}_0^r$, then the left Schur multiplier $LS_Q$ is completely bounded on $M_J^0(\mathcal{A}_0)$ and satisfies $\|LS_Q| M_J^0(\mathcal{A}_0)\|_{cb} \leq \|L\|_c \|R\|_c$.

**Proof.** Let $A = (a(i,j))$ be an operator in $M_J^0(\mathcal{A}_0)$, then infinite sums in the expressions below become either finite or strongly convergent sums representing the product of a row bounded block matrix and a column bounded block matrix. Further since any given $A$ is assumed to be in $M_J^0(\mathcal{A}_0)$, we are only interested in finitely many rows and finitely many columns of $Q$, when considering $Q \Box A$, so with this in mind we may assume that $Q$ is bounded and also that it is product of bounded operators, when the Schur block product is computed. Recall that by assumption and the definitions of $\lambda$ and $\rho$ the matrices $\lambda(R)$ and $\rho(A)$ commute, so

$$Q \Box A = V^*\lambda(Q)\rho(A)V = V^*\lambda(L^*)\lambda(R)\rho(A)V = (\lambda(L)V)^*\rho(A)(\lambda(R)V)\text{ and by Proposition 3.1}$$

$$\|LS_Q| M_J^0(\mathcal{A}_0)\|_{cb} \leq \|L\|_c \|R\|_c.$$

□

4. **Optimal factorization for Schur block multipliers**

In the article [4] we studied the relations between different minimal Stinespring representations of the same completely bounded operator. If we have a matrix $Q$ in $M_J(B(H))$ and a *-algebra $\mathcal{A}_0$ in $B(H)$ such that $LS_Q| M_J^0(\mathcal{A}_0)$ is completely bounded, then this mapping has an extension to the C*-algebraic closure of $M_J^0(\mathcal{A}_0)$ and therefore it has a minimal Stinespring representation $LS_Q(X) = V_1^*\gamma(X)V_2$ such that $\|V_1\|\|V_2\| = \|LS_Q| M_J^0(\mathcal{A}_0)\|_{cb}$. But the mapping wil also, by (3.6), have a sort of a Stinespring representation given as $X \rightarrow V^*\lambda(Q)\rho(X)V$. The reason why we do not call it a Stinespring representation is that $\lambda(Q)$ may not represent a densely defined closable operator in any reasonable sense. We will discuss this problem in more details in the proof of the following theorem. On the other hand $\lambda(Q)$ makes sense as a matrix over the set $(J \times J) \times (J \times J)$. This problem is the reason why we are not able to use the results from [4] to compare the 2 Stinespring like representations of $LS_Q$ we mentioned above. On the other hand the
methods from [4] may be twisted to fit the present situation such that we can get the promised factorization result.

**Theorem 4.1.** Let \( J \) be a set, \( H \) a complex Hilbert space, \( \mathcal{A}_0 \) a *-algebra on \( H \) containing the unit \( I \) of \( B(H) \) and \( Q = (q_{(i,j)})_{(i,j) \in J} \) a \( J \times J \) matrix with elements in \( B(H) \). If the Left-Schur multiplier \( LS_Q : M_j^0(\mathcal{A}_0) \to M_j(B(H)) \) is completely bounded, with completely bounded norm at most 1, then there exists a pair of matrices \( L, R \) in \( M_j^∞(B(H)) \) such that \( \|L\|_c \leq 1, \|R\|_c \leq 1, \) all the entries in \( R \) are in the commutant \( \mathcal{A}_0^\prime \) and \( L^*R = Q \).

If \( (LS_Q|M_j^0(\mathcal{A}_0)) = (RS_Q|M_j^0(\mathcal{A}_0)) \) then all the entries of \( L \) may be chosen in \( \mathcal{A}_0^\prime \) too.

**Proof.** The C*-algebra \( \mathcal{A} \) is the norm closure of the *-algebra \( \mathcal{A}_0 \) and we will let \( \mathcal{K} \otimes \mathcal{A} \) denote the norm closure of \( M_j^0(\mathcal{A}_0) \) in \( B(\bar{H}) \). Since the mapping \( LS_Q : M_j^0(\mathcal{A}_0) \to B(\bar{H}) \) is completely bounded with completely bounded norm at most 1, it extends to \( \mathcal{K} \otimes \mathcal{A} \) with the same completely bounded norm. Hence by [13] Theorem 8.4 there exist a Hilbert space \( K \), a representation \( \gamma : \mathcal{K} \otimes \mathcal{A} \to B(K) \) and contractions \( V_1, V_2 : \ell^2(J) \otimes H \to K \) such that for any \( Y \) in \( \mathcal{K} \otimes \mathcal{A} \) we have

\[
Q \square Y = LS_Q(Y) = V_1^* \gamma(Y) V_2, \quad \|V_1\| \leq 1, \|V_2\| \leq 1.
\]

When looking at (3.8) you might think that we can obtain another Stinespring representation of \( LS_Q \) simply by inserting a couple of parenthesis as follows

\[
Q \square Y = LS_Q(Y) = \left( V^* \lambda(Q) \right) \rho(Y) \left( V \right).
\]

Unfortunately the expression \( \lambda(Q) \) may not represent a densely defined closable operator. We can see how this may be so in the case where \( Q \) has the property that each \( q_{(i,j)} \) equals \( I_H \). This gives the identity mapping on the matrices, which is clearly completely bounded, but the expression \( \lambda(Q) \) makes no sense as an operator on \( L \). Each row in \( \lambda(Q) \) defines an unbounded densely defined operator, which is not closable!

It is no restriction to assume that the Stinespring representation given in (4.1) is minimal, meaning that the linear span of the vectors \( \gamma(X)V_2 \Xi \) and the linear span of the vectors \( \gamma(X)V_1 \Xi \) with \( X \) in \( \mathcal{K} \otimes \mathcal{A} \) and \( \Xi \) in \( \bar{H} \), are both dense in \( K \). The minimality condition we need with respect to the equation (3.6) is easily established since for any vector \( \xi \) in \( H \) and a pair of indices \( (i, j) \) we get

\[
\delta_i \otimes \xi \otimes \delta_j = \rho(e_{(i,j)} \otimes I_H)V(\xi \otimes \delta_i) = \lambda(e_{(i,j)} \otimes I_H)V(\xi \otimes \delta_j).
\]
In order to make use of the results from [4] we will look at submatrices of $Q$ based on a finite number of rows and just one column. We will then let $\Omega$ denote the set of all finite subsets of the index set $J$ and for an index $j$ we will also let $\{j\}$ denote the one element subset $\{j\}$ of $J$, which is an element in $\Omega$. For any $\omega$ in $\Omega$ we will define the projection $P_\omega$ in $B(\tilde{H})$ by $P_\omega := \sum_{j \in \omega} e_{(j,j)} \otimes I_H$. Then $P_\omega Q P_j$ is a one column bounded operator and we have

$$(P_\omega Q P_j) \Box X = P_\omega (Q \Box X) P_j = Q \Box (P_\omega X P_j),$$

so we get the equations

$$V^* (P_\omega Q P_j) \rho (X) V = P_\omega V_1^* \gamma (X) V_2 P_j = V_1^* \gamma (P_\omega) \gamma (X) \gamma (P_j) V_2,$$

and we have obtained $3$ bounded Stinespring representations for the same completely bounded map $LS_{(P_\omega Q P_j)}$. These Stinespring representations are not minimal so we will return to Proposition 2.5 of [4] to see how we can get some minimal representations instead, and then we can note some properties of the minimal representations. For each $j$ in $J$ we define $S_j$ as the orthogonal projection of $K$ onto the closed linear span given as

$$(4.5) \quad S_j K = \overline{\text{span}}(\gamma (X) \gamma (P_j) V_2 \Xi : X \in \mathcal{K} \otimes \mathcal{A}, \Xi \in H),$$

Then the projection $S_j^\omega$ which is the support of the minimal Stinespring representation based on $\omega, \gamma$ and $j$ above is given as the orthogonal projection onto a closed subspace of $K$ by

$$(4.6) \quad S_j^\omega K = \overline{\text{span}}(\gamma (X) S_j \gamma (P_\omega) V_1 \Xi : X \in \mathcal{K} \otimes \mathcal{A}, \Xi \in H).$$

For any family $(R_\theta)_{\theta \in \Theta}$ of orthogonal projections acting on some Hilbert space we will - as usual - let $\vee_{\theta \in \Theta} R_\theta$ denote the smallest orthogonal projection dominating all the projections $R_\theta$, and let $\wedge_{\theta \in \Theta} R_\theta$ denote the largest orthogonal projection dominated by all the projections $R_\theta$. Then the minimality of the Stinespring representation from (4.1) implies the following identities

$$(4.7) \quad I_K = \vee_{j \in J} S_j, \quad \forall j \in J : S_j = \vee_{\omega \in \Omega} S_j^\omega.$$

We will now turn to the Stinespring representation given as $X \rightarrow (V^* \lambda (P_\omega Q P_j)) \rho (X) V$, then the method from Proposition 2.5 of [4] will describe how to obtain a projection $Z_j^\omega$ in the commutant of $\rho(\mathcal{K} \otimes \mathcal{A})$ with the property that this Stinespring representation becomes minimal when $V^* \lambda (P_\omega Q P_j)$ is replaced by $V^* \lambda (P_\omega Q P_j) Z_j^\omega$, the representation $\rho$ is cut down to $Z_j^\omega L$ and $V$ is replaced by $Z_j^\omega V$. We
want to see that

\[ Z^\omega_j \leq \lambda(P_j), \]  

and in order to do so we follow the construction of \( Z^\omega_j \) as described in [4] Proposition 2.5. First we look at the closed linear span of the vectors

\[ \{ \rho(X) V \Xi : X \in \mathcal{K} \otimes \mathcal{A}, \Xi \in \tilde{H} \}, \]

and we remark that by (4.3) this space must be all of \( L \). Hence the first cut does not take place and we may obtain \( Z^\omega_j \) as the projection onto the linear span of the vectors

\[ \{ \rho(X) \lambda(P_j Q^* P_\omega) V \Xi : X \in \mathcal{K} \otimes \mathcal{A}, \Xi \in \tilde{H} \}. \]

Since \( \lambda(P_j) \) is a projection in the commutant of \( \rho(\mathcal{K} \otimes \mathcal{A}) \) we see, from the equation above, that \( Z^\omega_j \leq \lambda(P_j) \). We will need later that

\[ \forall j \in J : \bigvee_{\omega \in \Omega} Z^\omega_j = \lambda(P_j). \]  

This follows from the construction of \( Z^\omega_j \) and a new application of equation (4.3).

Now we have obtained 2 minimal Stinespring representations of the same completely bounded mapping \( S_{P_j Q P_j} \); so Theorem 3.1 of [4] applies. This means that for each pair \((j, \omega)\) there exists a densely defined closed operator \( T^\omega_j \) from \( Z^\omega_j L \) to \( S^\omega_j K \) with polar decomposition \( T^\omega_j = W^\omega_j |T^\omega_j| \) such that \( W^\omega_j \) is an isometry of \( Z^\omega_j L \) onto \( S^\omega_j K \), the positive part of the polar decomposition \( |T^\omega_j| \) is affiliated with the commutant \( \rho(\mathcal{K} \otimes \mathcal{A})' \), and

\[ \forall X \in \mathcal{K} \otimes \mathcal{A} \forall \Xi \in \text{dom}(T^\omega_j) : \rho(X) \Xi \in \text{dom}(T^\omega_j) \]  
\[ T^\omega_j \rho(X) \Xi = \gamma(X) T^\omega_j \Xi, \]

\[ \forall X \in \mathcal{K} \otimes \mathcal{A} : W^\omega_j \rho(X)(W^\omega_j)^* = \gamma(X) W^\omega_j(W^\omega_j)^*, \]

\[ \forall \Xi \in \tilde{H} : V \Xi \in \text{dom}(T^\omega_j) \text{ and } T^\omega_j V = S^\omega_j V. \]

We will focus on the equation (4.13) first and show that this equation implies that \( |T^\omega_j| \) has the form \( \lambda(t^\omega_j) \) for a contraction \( t^\omega_j \) which is a one column matrix with entries in the commutant \( \mathcal{A}' \) and supported on the \( j \)'th column. Unfortunately we do not know that \( |T^\omega_j| \) is bounded, but we may replace it by the closure of \( G|T^\omega_j| \) for a spectral projection of \( |T^\omega_j| \) such that this operator, say \( Y \), is a bounded operator in the commutant \( \rho(\mathcal{K} \otimes \mathcal{A})' \). Then \( Y \) has the form \( Y = \lambda(y) \) for some bounded operator in \( \mathcal{A}'\otimes B(\ell^2(J)) \), and the equation (4.8) implies that
The equations (3.7) and (4.13) implies that 
\[ \|y\| = \|y\|_c = \|\lambda(y)V\| \leq \|V\| \leq 1. \]
Since this is true for any spectral projection \( G \) with \( G|T_j^{\omega}| \) bounded we can conclude that
\[ \forall j \in J \forall \omega \in \Omega : \|T_j^{\omega}\| \leq 1. \]
You should keep in mind that \( T_j^{\omega} \) now is densely defined, closed and bounded, so it is domain of definition is all of \( L \). This will be used when we are going to apply the results from [4] which all are dependent on the domain of definition for \( T_j^{\omega} \).

The set \( \Omega \) of all finite subsets of \( J \) is naturally ordered by inclusion and we see that the construction of \( S_j^{\omega} \) gives an increasing net of projections \( (S_j^{\omega})_{\omega \in \Omega} \). Hence by (4.7) this net converges strongly towards \( S_j \) through \( \Omega \). Since the net \( (T_j^{\omega})_{\omega \in \Omega} \) is bounded and the operators are defined on all of \( L \), it has an ultraweakly convergent subnet \( (T_j^{\omega})_{\theta \in \Theta} \) which converges towards a contraction \( T_j \) of \( L \) to \( S_j K \), such that the support of \( T_j \) is contained in \( \lambda(P_j)L \). The equation (4.10) survives the ultraweak limit, so we get immediately
\[ \forall X \in \mathcal{K} \otimes \mathcal{A} : T_j \rho(X)\lambda(P_j)L = \gamma(X)T_j, \]
\[ T_j V = S_j V. \]

For the polar decomposition \( T_j = W_j |T_j| \) we have that \( |T_j| \) belongs to the commutant \( \rho(\mathcal{K} \otimes \mathcal{A})' \) and the equation (4.13) implies that the support of \( W_j \) is spanned by the supports of the \( W_j^{\omega} \)'s and similarly the range of \( W_j \) is spanned by the ranges of the \( W_j^{\omega} \)'s. Hence The equation (4.12) may be extended and based on (4.9) we have
\[ W_j^* W_j = \lambda(P_j), \quad W_j W_j^* = S_j, \]
\[ \forall X \in \mathcal{K} \otimes \mathcal{A} : W_j \rho(X) W_j^* = \gamma(X) W_j W_j^*. \]

These unitary equivalences between some subrepresentations of \( \rho \) and \( \gamma \) will in this case imply that \( \gamma \) is unitarily equivalent to a subrepresentation of \( \rho \). To see this we will consider families \( \{G_n : n \in N\} \) of pairwise orthogonal projections \( G_n \) in the commutant \( \gamma(\mathcal{K} \otimes \mathcal{A})' \) such that \( N \) is a subset of \( J \), \( \forall n \in N S_n = \oplus_{n \in N} G_n \) and for each \( n \) in \( N \), \( G_n \prec S_n \) in the Murray - von Neumann sense inside the von Neumann algebra commutant \( \gamma(\mathcal{K} \otimes \mathcal{A})' \). This means that inside this von Neumann algebra there exists, for each \( n \) in \( N \), a partial isometry \( U_n \) such that \( U_n U_n^* \leq S_n \) and \( U_n^* U_n = G_n \). First we remark that such families do exist, because for any \( j \) in \( J \) the one element family defined by \( N := \{j\} \) and \( G_j := S_j \), will be such a family. We order these families
by inclusion and see that any totally ordered subfamily has its union as a majorant, so Zorn’s Lemma applies and there exists a maximal element \((G_n)_{n \in N}\). Suppose that \(\bigoplus_{n \in N} G_n \neq I_k\) then there exists a \(j_0\) in \(J\) such that we can define a non-zero projection \(G_{j_0}\) in \(\gamma(\mathcal{K} \otimes \mathcal{A})'\) by

\[
G_{j_0} := (S_{j_0} \lor (\bigvee_{n \in N} S_n)) - (\bigwedge_{n \in N} S_n) \neq 0.
\]

Hence \(j_0 \notin N\) and we see immediately that \(G_{j_0} \oplus (\bigoplus_{n \in N} G_n) = S_{j_0} \lor (\bigvee_{n \in N} S_n)\). By Kaplansky’s formula, [10] Theorem 6.1.7, we see that the non-zero projection \(G_{j_0}\) is equivalent in \(\gamma(\mathcal{K} \otimes \mathcal{A})'\) to the projection

\[
S_{j_0} - (S_{j_0} \land (\bigvee_{n \in N} S_n)),
\]

so \(G_{j_0} \prec S_{j_0}\) and the maximality of the family fails, so \(\oplus G_n = I_K\). For each \(n\) in \(\tilde{N}\) there exists, by assumption, a partial isometry \(U_n\) in the commutant \(\gamma(\mathcal{K} \otimes \mathcal{A})'\) such that \(U_n^* U_n = G_n\) and \(U_n U_n^* \leq S_n\), and then the family \((W_n U_n)_{n \in N}\) consists of isometries from \(G_n K\) to a subspace of \(\lambda(P_n) L\). Since all the supports of these partial isometries are pairwise orthogonal with sum \(I_K\) and all the ranges are also pairwise orthogonal, there exists an isometry \(W^*: K \to L\) such that for each \(n\) in \(N\) we have \(W^* G_n = W_n U_n\) and based on (4.17)) we get

\[
(4.18) \quad \forall X \in \mathcal{K} \otimes \mathcal{A} : \quad W^* \gamma(X) W = \rho(X) W^* W \text{ and } \gamma(X) = W \rho(X) W^*.
\]

The equations (4.15) and (4.16) then imply

\[
(4.19) \quad \forall X \in \mathcal{K} \otimes \mathcal{A} \forall \ j \in J : \quad W^* T_j \rho(X) = \rho(X) W^* T_j;
\]

\[
(4.20) \quad W^* T_j V = W^* S_j V_2.
\]

Recall the proof of the fact that \(\|T_j\| \leq 1\), then we see that \(W^* T_j\) is a contraction in the commutant \(\rho(\mathcal{K} \otimes \mathcal{A})'\), and this algebra equals

\[
B(\ell^2(J)) \otimes \mathcal{A}' \otimes \mathbb{C} \mathcal{I}_{\ell^2(J)} = \lambda(\mathcal{A}' \otimes B(\ell^2(J))).
\]

Then we may define a one column contraction operator \(R_j\) in the von Neumann algebra on \(\tilde{H}\) given as \(\mathcal{A}' \otimes B(\ell^2(J))\) such that

\[
(4.21) \quad R_j = R_j P_j, \quad \lambda(R_j) = W^* T_j \quad \text{and} \quad \lambda(R_j) V = W^* S_j V_2.
\]

We can now define \(R\) in \(M^\infty(\mathcal{A}')\) with \(\|R\|_c \leq 1\), defining the \(j\)’th column of \(R\) to be \(R_j\). This \(R\) will be the right hand factor in the promised factorization \(Q = L^* R\). We will obtain the left hand factor \(L^*\) by constructing each of the rows \((L_i)^*\), and show that they are all contractions which satisfy the equations \((L_i)^* R_j = Q_{i,j}\). This is done via a close look into some of the equations we have obtained above. It will be a computational argument, to which we have not been able to find a more abstract replacement. We will check certain sums in order to see how \((L_i)^*\) may be picked out as a sub-matrix of \(V_i^* W \rho(P_i)\). Let
us fix the pair of indices \((i, j)\), a matrix \(X\) in \(M^0_J(\mathcal{A}_0)\) and start to compute. First we remind you that we have the following equations, which come from Section 2.

(4.22) \[ \forall X \in \mathcal{A}' \otimes B(\ell^2(J)) \forall Y \in \mathcal{A}' \otimes B(\ell^2(J)) : \lambda(X)\rho(Y) = \rho(Y)\lambda(X), \]

(4.23) \[ \forall j \in J : \lambda(P_j)V = \rho(P_j)V = \sigma(P_j)V. \]

Let \(X\) be in \(\mathcal{K} \otimes \mathcal{A}\), then we will compute, and the first equation follows from (4.22)

\[
(4.24) \quad V^*_1 W \rho(P_i)\lambda(RP_j)\rho(X)V = V^*_1 W \rho(P_i)\rho(X)\lambda(RP_j)V \\
\text{by } \lambda, \rho \text{ homomor., (4.23) and (4.22)} = V^*_1 W \rho(P_i)\rho(XP_j)\lambda(RP_j)V \\
\text{by } \rho \text{ homomor., by (4.21)} = V^*_1 W \rho(P_iXP_j)W^*S_jV_2 \\
\text{by (4.18)} = V^*_1 \gamma(P_iXP_j)S_jV_2 \\
\text{by definition of } S_j = V^*_1 \gamma(P_iXP_j)V_2 \\
\text{by (4.1)} = Q \Box (P_iXP_j) \\
\quad = (P_iQP_j) \Box X \\
\text{by (3.6)} = V^*\lambda(P_iQP_j)\rho(X)V.
\]

Here all the operators are bounded and the linear span of the set \(\{\rho(Z)V\Xi : Z \in M^0_J(\mathcal{A}), \Xi \in \ell^2(J) \otimes H\}\) is dense in \(L\) by (4.3) so

\[
V^*_1 W \rho(P_i)\lambda(RP_j) = V^*\lambda(P_iQP_j), \text{ then multiply with } V
\]

(4.25) \[ VV^*_1 W \rho(P_i)\lambda(RP_j) = VV^*\lambda(P_iQP_j) = e_{i,j} \otimes q_{i,j} \otimes e_{i,i}. \]

Let \(Z := \rho(P_i)\lambda(RP_j) = \sum_{k \in J} e_{k,j} \otimes r_{k,j} \otimes e_{i,i}\), and define

\[ Y^i := (I_{P(I)} \otimes I_H \otimes e_{i,i})VV^*_1 W (I_{P(J)} \otimes I_H \otimes e_{i,i}) \]

so the equation (4.25) implies that

(4.26) \[ Y^i Z = e_{i,j} \otimes q_{i,j} \otimes e_{i,i}. \]

\(Y^i\) must be of the form \(Y^i = \sum_{s,t \in J} e_{s,t} \otimes y_{s,t}^i \otimes e_{i,i}\). the equation (4.26) then becomes

(4.27) \[ q_{i,j} = \sum_{k \in J} y_{i,k}^i r_{k,j}, \]

The equation (4.27) only uses the \(i\)'th row of the matrix \(y^i\) so we can define a matrix \(L^*\) with row norm at most 1 in \(M^\infty_J(B(H))\) by

(4.28) \[ \forall i, j \in J : (L^*)(i,j) := y_{i,j}^i, \]

and the equation (4.27) shows that \(Q = L^*R\) and the first statement of theorem follows.
Suppose that the right Schur multiplier $RS_Q$ equals the left Schur multiplier on $M_0^0(\mathcal{A}_0)$, then we have obtained a factorization $Q = L^* R$ with $L$ and $R$ column bounded such that $\|L\|_c = \|R\|_c = \|LS_Q\|_{cb}^{\frac{1}{2}}$ with both $Q$ and $R$ in $M^\infty_0(\mathcal{A}_0')$. Each column, say $R_j$ of $R$ is bounded and has a range projection, say $G_j$, in the von Neumann tensor product $B(\ell^2(J)) \bar{\otimes} \mathcal{A}_0'$ and then the matrix of $L^* G_j$ is well defined at each entry with all the entries in $\mathcal{A}_0'$. We can then define an increasing net of projections $G_K$ indexed by finite subsets $K$ of $J$, ordered by inclusion and defined by $G_K := \vee_{j \in K} G_j$. For each index $i$ the bounded row $i L^*$ induces the bounded net of rows $(i L^* G_K)_{K \subseteq J}$, which is strongly convergent in $B(\ell^2(J)) \bar{\otimes} \mathcal{A}_0'$, and we can define the $i$'th row of a new row bounded matrix $\hat{L}^*$ as this limit. Then we get that the row norm of $\hat{L}^*$ is dominated by the row norm of $L^*$ the entries of $\hat{L}^*$ are all in $\mathcal{A}_0'$ and $\hat{L}^* R = Q$, so the theorem follows.  

\[ \Box \]

**ACKNOWLEDGEMENT**

We are happy to thank Narutaka Ozawa, Vern Paulsen and Gilles Pisier for very valuable help and comments.

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