Solutions for Functional Fully Coupled Forward-Backward
Stochastic Differential Equations

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Abstract: In this paper, we study a functional fully coupled forward-backward stochastic differential equations (FBSDEs). Under a new type of integral Lipschitz and monotonicity conditions, the existence and uniqueness of solutions for functional fully coupled FBSDEs is proved. We also investigate the relationship between the solution of functional fully coupled FBSDE and the classical solution of the path-dependent partial differential equation (P-PDE). When the solution of the P-PDE has some smooth and regular properties, we solve the related functional fully coupled FBSDE and prove the P-PDE has a unique solution.

Keywords: forward-backward stochastic differential equations (FBSDEs); monotonicity conditions; functional Itô calculus; path-dependent partial differential equation (P-PDE)

1 Introduction

Linear backward stochastic differential equations (in short BSDEs) was introduced by Bismut [2]. Pardoux and Peng [18] established the existence and uniqueness theorem for nonlinear BSDEs under a standard Lipschitz condition. Since then, backward stochastic differential equations and forward-backward stochastic differential equations (FBSDEs) have been widely recognized that they provide useful tools in many fields, especially mathematical finance and the stochastic control theory (see [3], [7], [13], [24], [25] and the references therein).

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A state dependent fully coupled FBSDE is formulated as:

\[ X(t) = x + \int_0^t b(X(s), Y(s), Z(s))ds + \int_0^t \sigma(X(s), Y(s), Z(s))dW(s), \]
\[ Y(t) = g(X(T)) - \int_t^T h(X(s), Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T]. \]

(1.1)

There have been three main methods to solve FBSDE (1.1), i.e., the Method of Contracting Mapping (see [1] and [16]), the Four Step Scheme (see [14]) and the Method of Continuation (see [12], [19] and [23]). In [15], Ma et al. studied the wellposedness of the FBSDEs in a general non-Markovian framework. They find a unified scheme which combines all existing methodology in the literature, and overcome some fundamental difficulties that have been long-standing problems for non-Markovian FBSDEs.

In this paper, we study the following functional fully coupled FBSDE:

\[ X(t) = x + \int_0^t b(X_s, Y(s), Z(s))ds + \int_0^t \sigma(X_s, Y(s), Z(s))dW(s), \]
\[ Y(t) = g(X(T)) - \int_t^T h(X_s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T], \]

(1.2)

where \( X_s := X(t)_{0 \leq t \leq s} \).

As mentioned above, Hu and Peng [12] initiated the continuation method in which the key issue is a certain monotonicity condition. But unfortunately, the Lipschitz and monotonicity conditions in [12] and [19] do not work for equation (1.2). Here the main difficulty is that the coefficients of (1.2) depend on the path of the solution \( X(t)_{0 \leq t \leq T} \). In this paper, we revise the continuation method and propose a new set of Lipschitz and monotonicity conditions. These new conditions involve an integral term with respect to the path of \( X(t)_{0 \leq t \leq T} \). Thus, we call them the integral Lipschitz and monotonicity conditions. The readers may see Assumption 2.3 and 2.4 for more details. In particular, we present two examples to illustrate that our assumptions are not restrictive. Under the integral Lipschitz and monotonicity conditions, the continuation method can go through and it leads to the existence and uniqueness of the solution to equation (1.2).

It is well known that quasilinear parabolic partial differential equations are related to Markovian FBSDEs (see [20], [17] and [16]), which generalizes the classical Feynman-Kac formula. Recently, a new framework of functional Itô calculus was introduced by Dupire [7] and later developed by Cont and Fourni [4], [5], [6]. Inspired by Dupire’s work, Peng and Wang [22] obtained a so-called functional Feynman-Kac formula for classical solutions of path-dependent partial differential equation (P-PDE) in terms of non-Markovian BSDEs. Furthermore, under a special condition, Peng [21] proved that the viscosity solution of the second order fully nonlinear P-PDE is unique. Ekren, Touzi, and Zhang [9], [10], [11] gave another definition of the viscosity solution of the fully nonlinear P-PDE and obtained the uniqueness result of viscosity solutions.
In this paper, we explore the relationship between the solution of functional fully coupled FBSDE \((1.2)\) and the classical solution of the following related P-PDE:

\[
D_t u(\gamma_t) + \mathcal{L} u(\gamma_t) - h(\gamma_t, u(\gamma_t), v(\gamma_t)) = 0,
\]

\[
v(\gamma_t) = D_x u(\gamma_t) \tilde{\sigma}(\gamma_t, u(\gamma_t), v(\gamma_t)),
\]

\[
u(\gamma_t) = g(\gamma^1_t, \gamma^2(T)), \quad \gamma^1_t \in \Lambda^d, \quad \gamma^2(T) \in \mathbb{R}^n,
\]

where

\[
\mathcal{L} u = (\mathcal{L} u_1, \ldots, \mathcal{L} u_n), \quad \mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{n+d} (\tilde{\sigma} \tilde{\sigma}^T)_{i,j}(\gamma_t, u, v) D_{x_{ij}} + b_i(\gamma_t, u, v) D_{x_i}.
\]

We prove that if the solution \(u\) of the above P-PDE has some smooth and regular properties, then we can solve the related equation \((1.2)\) and consequently, the P-PDE has a unique solution.

The paper is organized as follows. In section 2, we formulate the problem and give the integral Lipschitz and monotonicity conditions. The existence and uniqueness of the solution for \((1.2)\) are proved in the first part of section 3. In the second part of section 3, we show the relationship between functional FBSDEs and P-PDEs.

## 2 Formulation of the problem

Let \(\Omega = C([0, T]; \mathbb{R}^d)\) and \(P\) the Wiener measure on \((\Omega, \mathcal{B}(\Omega))\). We denote by \(W = (W(t)_{t \in [0, T]}\) the canonical Wiener process, with \(W(t, \omega) = \omega(t), \quad t \in [0, T], \quad \omega \in \Omega\). For any \(t \in [0, T]\), we denote by \(\mathcal{F}_t\) the \(P\)-completion of \(\sigma(W(s), s \in [0, t])\).

For \(n \in \mathbb{N}\), set

\[
C^n_t = C([0, t]; \mathbb{R}^n) \text{ and } C^n = \bigcup_{t \in [0, T]} C^n_t.
\]

Consider the following functional fully coupled FBSDE:

\[
X(t) = x + \int_0^t b(X_s, Y_s, Z_s) ds + \int_0^t \sigma(X_s, Y_s, Z_s) dW(s), \quad \text{(2.1)}
\]

\[
Y(t) = g(X(T)) - \int_t^T h(X_s, Y_s, Z_s) ds - \int_t^T Z(s) dW(s), \quad t \in [0, T], \quad \text{(2.2)}
\]

where the processes \(X, Y, Z\) take values in \(\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m \times d}\), \(X_s = X(r)_{0 \leq r \leq s}\) and

\[
b : C^m \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \Omega \rightarrow \mathbb{R}^n;
\]

\[
\sigma : C^m \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \Omega \rightarrow \mathbb{R}^{n \times d};
\]

\[
h : C^m \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \Omega \rightarrow \mathbb{R}^m;
\]

\[
g : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^m.
\]
For $z \in \mathbb{R}^{m \times d}$, define $|z| = \{\text{tr}(zz^T)\}^{1/2}$, where “$T$” means transpose. For $z^1 \in \mathbb{R}^{m \times d}, z^2 \in \mathbb{R}^{m \times d}$,

\[
((z^1, z^2)) = \text{tr}(z^1(z^2)^T).
\]

We use the notations

\[
u^1 = (y^1, z^1) \in \mathbb{R}^m \times \mathbb{R}^{m \times d};
\]

\[
u^2 = (y^2, z^2) \in \mathbb{R}^m \times \mathbb{R}^{m \times d};
\]

\[[\nu^1, \nu^2] = \langle y^1, y^2 \rangle + ((z^1, z^2)).
\]

Given an $m \times n$ full-rank matrix $G$, for $(x, u) \in C^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, define

\[
f(x, u) = (G^T h(x, u), Gb(x, u), G\sigma(x, u)),
\]

where $G\sigma = (G\sigma_1, \cdots, G\sigma_d)$.

**Definition 2.1** We denote by $M^2(0, T; \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\vartheta(\cdot)$ such that

\[
E \int_0^T |\vartheta(s)|^2 \, ds < +\infty.
\]

**Definition 2.2** A triple $(X, Y, Z) : [0, T] \times \Omega \rightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ is called an adapted solution of the equations (2.1) and (2.2), if $(X, Y, Z) \in M^2(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$, and it satisfies (2.1) and (2.2) $P$-a.s.

We rewrite (2.1) and (2.2) in a differential form:

\[
\begin{align*}
&dX(t) = b(X_t, Y(t), Z(t)) \, dt + \sigma(X_t, Y(t), Z(t)) \, dW(t), \\
&dY(t) = h(X_t, Y(t), Z(t)) \, dt + Z(t) \, dW(t), \\
&X(0) = x, \quad Y(T) = g(X(T)).
\end{align*}
\]

Now we give the assumptions:

**Assumption 2.3** For each $(x_T, u_T) \in C^n \times C^m \times C^{m \times d}$, $f(x, u(\cdot)) \in M^2(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$. There exists a constant $c_1 > 0$, such that

\[
\int_0^T |f(x^1_t, u^1(t)) - f(x^2_t, u^1(t))|^2 \, dt \leq c_1 \int_0^T |x^1(t) - x^2(t)|^2 \, dt,
\]

\[
|f(x^1_s, u^1(s)) - f(x^2_s, u^2(s))| \leq c_1 |u^1(s) - u^2(s)|, \quad 0 \leq s \leq T,
\]

$\forall (x^1_T, u^1_T), (x^2_T, u^2_T) \in C^n \times C^m \times C^{m \times d}$, and

\[
|g(x^1) - g(x^2)| \leq c_1 |x^1 - x^2|, \forall (x^1, x^2) \in \mathbb{R}^n \times \mathbb{R}^n, \quad P$-a.s..
Assumption 2.4 There exists constants $\beta_1, \beta_2, \mu_1$ such that

$$\int_0^T [f(x_1(t), u_1(t)) - f(x_1^2, u_2(t)), (x_1(t) - x_2(t), u_1(t) - u_2(t))]dt$$

$$\geq \int_0^T \beta_1 |Gx_1(t) - Gx_2(t)|^2 + \beta_2 (|G^Ty_1(t) - G^Ty_2(t)|^2 + |G^T z_1(t) - G^T z_2(t)|^2) dt, \quad P-a.s.,$$

$\forall (x_1^1, u_1^1), (x_2^1, u_2^1) \in C^n \times C^m \times C^{m \times d}$, and

$$\langle g(x_1^1) - g(x_2^1), G(x_1^1 - x_2^1) \rangle \leq -\mu_1 |Gx_1 - Gx_2|^2, \quad P-a.s.,$$

$\forall (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\beta_1, \beta_2$ and $\mu_1$ are given nonnegative constants with $\beta_1 + \beta_2 > 0, \mu_1 + \beta_2 > 0$. Moreover we have $\beta_1 > 0, \mu_1 > 0$ (resp., $\beta_2 > 0, \mu_1 > 0$) when $m > n$ (resp., $n > m$).

The coefficients of the following two examples satisfy Assumptions 2.4 and 2.1

Example 2.5 Set $n = m = 1, T = 1$, and the functional FBSDE is

$$dX(t) = \left( \int_0^t X(s)ds + 2Y(t) \right)dt + \left( \int_0^t X(s)ds + 2Z(t) \right)dW(t),$$

$$dY(t) = \left( \int_0^t X(s)ds + 3X(t) \right)dt + Z(t)dW(t),$$

$$X(0) = x, \quad Y(1) = -X(1).$$

We first show that the coefficients of (2.3) satisfy Assumption 2.4. For any $(x_1^1, y_1^1, z_1^1), (x_2^1, y_2^1, z_2^1) \in C \times C \times C$, it is easy to check that $g$ satisfies the Lipschitz condition. For $f = (h, b, \sigma)$, we have

$$\int_0^1 |f(x_1^1, y_1^1(t), z_1^1(t)) - f(x_2^1, y_1^1(t), z_1^1(t))|^2 dt$$

$$\leq \int_0^1 [4\sigma^2 \int_0^1 |x_1^1(s) - x_2^1(s)|^2 ds + 18 |x_1^1(t) - x_2^1(t)|^2] dt$$

$$\leq 22 \int_0^1 |x_1^1(t) - x_2^1(t)|^2 dt.$$

and

$$|f(x_1^1, y_1^1(t), z_1^1(t)) - f(x_1^1, y_2^1(t), z_2^1(t))|$$

$$\leq 2 |y_1^1(t) - y_2^1(t)| + 2 |z_1^1(t) - z_2^1(t)|.$$

Then we check Assumption 2.4. For any $(x_1^1, y_1^1, z_1^1), (x_2^1, y_2^1, z_2^1) \in C \times C \times C, g$ satisfies the monotonicity condition:

$$(g(x_1^1)) - g(x_2^1)) \cdot (x_1^1 - x_2^1) \leq - |x_1^1(t) - x_2^1(t)|^2.$$
For \( f = (h, b, \sigma) \), we have

\[
\int_0^t [f(x(s), u(s)) - f(x(s), u(s)), (x(t) - x(s), u(t) - u(s))] dt
\]

\[
= \int_0^t \left[ \int_0^s (x(s) - x_2(s)) ds \right] (x(t) - x(s), u(t) - u(s)) + 3 |x(t) - x(s)|^2
\]

\[
+ \left( \int_0^1 x(t) - x_2(s) ds \right) \cdot (y(t) - y_2(t)) + 2 |y(t) - y_2(t)|^2
\]

\[
+ \left( \int_0^1 x(t) - x_2(s) ds \right) \cdot (z(t) - z_2(s)) + 2 |z(t) - z_2(s)|^2 dt
\]

\[
\geq \int_0^t \left[ |x(t) - x_2(s)|^2 + |y(t) - y_2(s)|^2 + |z(t) - z_2(s)|^2 \right] dt.
\]

**Example 2.6** We consider a nonlinear functional fully coupled FBSDE. Set \( n = m = 1, T = 1 \).

Suppose that \( \hat{f} = (\hat{h}, \hat{b}, \hat{\sigma}) \) satisfies the next Lipschitz and monotonicity conditions:

there exists a constant \( c_1 > 0 \), such that

\[
|\hat{f}(x^1, y^1, z^1) - \hat{f}(x^2, y^2, z^2)| \leq c_1( |x^1 - x^2| + |y^1 - y^2| + |z^1 - z^2| ), \quad P - \text{a.s.},
\]

and

\[
|\hat{f}(x^1, y^1, z^1) - \hat{f}(x^2, y^2, z^2), (x^1 - x^2, y^1 - y^2, z^1 - z^2)|
\]

\[
\geq c_1( |x^1 - x^2| + |y^1 - y^2| + |z^1 - z^2| ), \quad P - \text{a.s.},
\]

\( \forall (x^1, y^1, z^1), (x^2, y^2, z^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \). Then we set

\[
h(x, y, z) = \hat{h}(\int_0^t x(s) ds, y, z);
\]

\[
b(x, y, z) = \hat{b}(\int_0^t x(s) ds, y, z);
\]

\[
\sigma(x, y, z) = \hat{\sigma}(\int_0^t x(s) ds, y, z);
\]

\[
g(x(1)) = -x(1),
\]

\( \forall (x, y, z) \in C \times \mathbb{R} \times \mathbb{R} \). The corresponding functional FBSDE is

\[
dX(t) = \hat{b}(\int_0^t X(s) ds, Y(t), Z(t)) dt + \hat{\sigma}(\int_0^t X(s) ds, Y(t), Z(t)) dW(t),
\]

\[
dY(t) = \hat{h}(\int_0^t X(s) ds, Y(t), Z(t)) dt + Z(t) dW(t),
\]

\[
X(0) = x, \quad Y(1) = -X(1).
\]

Applying similar analysis as in Example 2.5, we can check that \( f = (h, b, \sigma) \) and \( g \) satisfy Assumptions 2.3 and 2.4.
3 Functional Fully Coupled FBSDEs

3.1 Existence and uniqueness for functional fully coupled FBSDEs

In this section, we combine the continuation method in [19] and the integral Lipschitz and monotonicity conditions to prove our main result.

Theorem 3.1 Let Assumptions 2.3 and 2.4 hold. Then there exist a unique adapted solution \((X,Y,Z)\) for equations (2.1) and (2.2).

Firstly, we prove the uniqueness.

Proof. Let \(U^1(\cdot) \triangleq (Y^1(\cdot),Z^1(\cdot)), U^2(\cdot) \triangleq (Y^2(\cdot),Z^2(\cdot))\) be two adapted solutions of (2.1) and (2.2). We set

\[
(X,\hat{Y},\hat{Z}) = (X^1 - X^2, Y^1 - Y^2, Z^1 - Z^2);
\]

\[
\hat{b}(t) = b(X^1_1, U^1(t)) - b(X^2_1, U^2(t));
\]

\[
\hat{\sigma}(t) = \sigma(X^1_1, U^1(t)) - \sigma(X^2_1, U^2(t));
\]

\[
\hat{h}(t) = h(X^1_1, U^1(t)) - h(X^2_1, U^2(t)).
\]

From Assumption 2.3, it follows that \(\hat{X}(\cdot)\) and \(\hat{Y}(\cdot)\) are continuous, and

\[
E[\int_0^T (|\hat{X}(t)|^2 + |\hat{Y}(t)|^2) dt] < +\infty.
\]

Applying the Itô formula to \(\hat{Y}(t), G\hat{X}(t)\),

\[
d(\hat{Y}(t), G\hat{X}(t)) = [f(X^1_1, U^1(t)) - f(X^2_1, U^2(t)), (X^1(t) - X^2(t), U^1(t) - U^2(t))] dt
\]

\[
+ (\hat{X}(t) G\hat{T} \hat{Z}(t) + \hat{Y}(t) G\hat{\sigma}(t)) dW(t).
\]

It yields that

\[
E\langle g(X^1(T)) - g(X^2(T)), GX^1(T) - GX^2(T) \rangle
\]

\[
= E \int_0^T [f(X^1_1, U^1(t)) - f(X^2_1, U^2(t)), (X^1(t) - X^2(t), U^1(t) - U^2(t))] dt.
\]

By Assumptions 2.3 and 2.4, we obtain

\[
-\mu_1 E |GX^1(T) - GX^2(T)|^2
\]

\[
\geq E\langle g(X^1(T)) - g(X^2(T)), GX^1(T) - GX^2(T) \rangle
\]

\[
\geq E \int_0^T (\beta_1 |Gx^1(t) - Gx^2(t)|^2 + \beta_2(|G^TY^1(t) - G^TY^2(t)| + |G^TZ^1(t) - G^TZ^2(t)|)) dt.
\]
In the case \( m > n \), we have \( \beta_1 > 0 \) and \( \mu_1 > 0 \). Then we have \( X^1 = X^2 \). From the uniqueness of BSDE, it follows that \( U^1 = U^2 \). In the case \( m < n \), we have \( \beta_2 > 0 \) and \( \mu_1 > 0 \). It yields that \( U^1 = U^2 \). Thus, by the uniqueness of SDE, it yields that \( X^1 = X^2 \). \( \Box \)

The following lemma is quoted from [19].

**Lemma 3.2** Suppose that \((b_{0}(\cdot), \sigma_{0}(\cdot), h_{0}(\cdot)) \in M^{2}(0,T; \mathbb{R}^{n} \times \mathbb{R}^{n \times d} \times \mathbb{R}^{m})\), \( g_{0} \in L^{2}(\Omega, \mathcal{F}_{T}; \mathbb{R}^{m})\). Then the following linear forward-backward stochastic differential equation

\[
X(t) = x + \int_{0}^{t} (-\beta_{2}G^{T}Y(s) + b_{0}(s))ds + \int_{0}^{t} (-\beta_{2}G^{T}Z(s) + \sigma_{0}(s))dW(s),
\]

\[
Y(t) = \lambda G X(T) + g_{0} - \int_{t}^{T} (-\beta_{1}G X(s) + h_{0}(s))ds - \int_{t}^{T} Z(s)dW(s)
\]

have a unique adapted solution \((X, Y, Z)\), where \( \lambda \) is a nonnegative constant.

For any given \( \alpha \in \mathbb{R} \), we define

\[
b^{\alpha}(x_{t}, y, z) = \alpha b(x_{t}, y, z) + (1 - \alpha)\beta_{2}(-G^{T}y);
\]

\[
\sigma^{\alpha}(x_{t}, y, z) = \alpha \sigma(x_{t}, y, z) + (1 - \alpha)\beta_{2}(-G^{T}z);
\]

\[
h^{\alpha}(x_{t}, y, z) = \alpha h(x_{t}, y, z) + (1 - \alpha)\beta_{1}(-Gx);
\]

\[
g^{\alpha}(x) = \alpha g(x) + (1 - \alpha)\beta_{1}(Gx),
\]

and consider the following equations:

\[
X(t) = x + \int_{0}^{t} (b^{\alpha}(X_{s}, U(s)) + b_{0}(s))ds + \int_{0}^{t} (\sigma^{\alpha}(X_{s}, U(s)) + \sigma_{0}(s))dW(s),
\]

\[
Y(t) = (g^{\alpha}(X(T)) + g_{0}) - \int_{t}^{T} (h^{\alpha}(X_{s}, U(s)) + h_{0}(s))ds - \int_{t}^{T} Z(s)dW(s),
\]

where \( U = (Y, Z) \).

**Lemma 3.3** We assume that, a priori, for a given \( \alpha_{0} \in [0,1] \) and for any \((b_{0}(\cdot), \sigma_{0}(\cdot), h_{0}(\cdot)) \in M^{2}(0,T; \mathbb{R}^{n} \times \mathbb{R}^{n \times d} \times \mathbb{R}^{m})\), \( g_{0} \in L^{2}(\Omega, \mathcal{F}_{T}; \mathbb{R}^{m})\), there exists a solution of (3.1) and (3.4). Then there exists a \( \delta_{0} \in (0,1) \) which depends only on \( \mu_{1}, \beta_{1}, \beta_{2} \) and \( T \), such that for all \( \alpha \in [\alpha_{0}, \alpha_{0} + \delta_{0}] \), and for any \((b_{0}(\cdot), \sigma_{0}(\cdot), h_{0}(\cdot)) \in M^{2}(0,T; \mathbb{R}^{n} \times \mathbb{R}^{n \times d} \times \mathbb{R}^{m})\), \( g_{0} \in L^{2}(\Omega, \mathcal{F}_{T}; \mathbb{R}^{m})\), (3.3) and (3.4) have an adapted solution.

**Proof.** Note that

\[
b^{\alpha+\delta}(x_{t}, y, z) = b^{\alpha}(x_{t}, y, z) + \delta(\beta_{2}G^{T}y + b(x_{t}, y, z));
\]

\[
\sigma^{\alpha+\delta}(x_{t}, y, z) = \sigma^{\alpha}(x_{t}, y, z) + \delta(\beta_{2}G^{T}z + \sigma(x_{t}, y, z));
\]

\[
h^{\alpha+\delta}(x_{t}, y, z) = h^{\alpha}(x_{t}, y, z) + \delta(\beta_{1}Gx + h(x_{t}, y, z));
\]

\[
g^{\alpha+\delta}(x) = g^{\alpha}(x) + \delta(-\beta_{1}Gx + g(x)).
\]
Let \((X^0, U^0) = (X^0, Y^0, Z^0) = 0\). We solve the following equations iteratively:

\[ X^{i+1}(t) = x + \int_0^t [b^{\alpha_0}(X^{i+1}_s, U^{i+1}(s)) + \delta(\beta_2 G T Y^i(s) + b(X^i_s, U^i(s)))] ds + \int_0^t [\sigma^{\alpha_0}(X^{i+1}_s, U^{i+1}(s)) + \delta(\beta_2 G T Z^i(s) + \sigma(X^i_s, U^i(s)))] dW(s), \tag{3.5} \]

\[ Y^{i+1}(t) = [g^{\alpha_0}(X^{i+1}(T)) + \delta(-\beta_1 G X^i(T) + g(X^i(T))) + g_0(s)] - \int_t^T Z^{i+1}(s) dW(s) + \int_t^T [h^{\alpha_0}(X^{i+1}_s, U^{i+1}(s)) + \delta(\beta_1 G X^i(s) + h(X^i_s, U^i(s)))] ds, \tag{3.6} \]

where \(U^i = (Y^i, Z^i)\).

Applying Itô formula to \((\hat{Y}^{i+1}(t), G \hat{X}^{i+1}(t))\),

\[ E\{g^{\alpha_0}(X^{i+1}(T)) - g^{\alpha_0}(X^i(T)), G \hat{X}^{i+1}(T)\} \]

\[ = \delta E(\beta_1 G \hat{X}^i(T) - (g(X^i(T)) - g(X^{i-1}(T))), G \hat{X}^{i+1}(T)) + E \int_0^T [f^{\alpha_0}(X^{i+1}_t, U^{i+1}(t)) - f^{\alpha_0}(X^i_t, U^i(t)), (\hat{X}^{i+1}(t), \hat{U}^{i+1}(t))] dt + \delta E \int_0^T [(\beta_1 G T G \hat{X}^i(t), \beta_2 G G T \hat{U}^i(t)) + f(X^i_t, U^i(t)) - f(X^{i-1}_t, U^{i-1}(t)), (\hat{X}^{i+1}(t), \hat{U}^{i+1}(t))]. \]

From Assumptions \(3.1\) and \(3.2\) there exists \(K_1 > 0\) such that

\[ (\alpha_0 \mu_1 + (1 - \alpha_0)) E | G \hat{X}^{i+1}(T) |^2 + \beta_1 E \int_0^T |G \hat{X}^{i+1}(t)|^2 dt + \beta_2 E \int_0^T |G^T \hat{U}^{i+1}(t)|^2 dt \]

\[ \leq \delta K_1 (E | \hat{X}^i(T) |^2 + E \int_0^T |\hat{X}^i(t)|^2 dt + E \int_0^T |\hat{U}^i(t)|^2 dt + E \int_0^T |\hat{X}^{i+1}(t)|^2 dt + E \int_0^T |\hat{U}^{i+1}(t)|^2 dt). \tag{3.7} \]

We also have \(\forall i \geq 1\),

\[ \hat{X}^{i+1}(s) = \int_0^s [b^{\alpha_0}(X^{i+1}_t, U^{i+1}(t)) - b^{\alpha_0}(X^i_t, U^i(t)) + \delta(\hat{Y}^i(t) + b(X^i_t, U^i(t)) - b(X^{i-1}_t, U^{i-1}(t)))] dt \]

\[ + \int_0^s [\sigma^{\alpha_0}(X^{i+1}_t, U^{i+1}(t)) - \sigma^{\alpha_0}(X^i_t, U^i(t)) + \delta(\hat{Z}^i(t) + \sigma(X^i_t, U^i(t)) - \sigma(X^{i-1}_t, U^{i-1}(t)))] dW(t). \]

By the usual technique of estimation, we can derive that

\[ \sup_{0 \leq s \leq T} E | \hat{X}^{i+1}(s) |^2 \]

\[ \leq K_1 \delta (E \int_0^T |\hat{U}^i(t)|^2 dt + E \int_0^T |\hat{X}^i(t)|^2 dt) + K_1 E \int_0^T |\hat{U}^{i+1}(t)|^2 dt, \quad \forall i \geq 1. \tag{3.8} \]
\[ E \int_0^T |\dot{X}^{i+1}(t)|^2 dt \leq K_1 T \delta (E \int_0^T |\dot{U}^i(t)|^2 dt + E \int_0^T |\dot{X}^i(t)|^2 dt) + K_1 T E \int_0^T |\dot{U}^{i+1}(t)|^2 dt, \quad \forall i \geq 1. \]  

(3.9)

For equation (3.6), by the standard estimation of BSDE, we have

\[ E \int_0^T |\dot{U}^{i+1}(t)|^2 dt \leq K_1 \delta (E \int_0^T |\dot{U}^i(t)|^2 dt + E \int_0^T |\dot{X}^i(t)|^2 dt) + K_1 E \int_0^T |\dot{X}^{i+1}(t)|^2 dt, \]

(3.10)

where the constant \( K_1 \) depends on the Lipschitz constants \( c_1, \beta_1, \beta_2 \) and \( T \).

Due to the above estimations, we conclude that

\[ E \int_0^T |\dot{U}^{i+1}(t)|^2 dt + E \int_0^T |\dot{X}^{i+1}(T)|^2 dt + E \int_0^T |\dot{X}^{i+1}(t)|^2 dt \]

\[ \leq K \delta (E \int_0^T |\dot{U}^i(t)|^2 dt + E \int_0^T |\dot{X}^i(t)|^2 dt) + E \int_0^T |\dot{X}^{i+1}(T)|^2 dt, \]

where the constant \( K \) depends on \( \beta_1, \beta_2, \mu, K_1 \) and \( T \). Taking \( \delta_0 = \frac{1}{2K} \), for each \( 0 < \delta \leq \delta_0 \),

\[ E \int_0^T |\dot{U}^{i+1}(t)|^2 dt + E \int_0^T |\dot{X}^{i+1}(T)|^2 dt + E \int_0^T |\dot{X}^{i+1}(t)|^2 dt \]

\[ \leq \frac{1}{2} (E \int_0^T |\dot{U}^i(t)|^2 dt + E \int_0^T |\dot{X}^i(t)|^2 dt) + E \int_0^T |\dot{X}^{i+1}(T)|^2 dt. \]

Thus, the limit of \( (U^i, X^i) \) exists. We denote its limit by \( (X, U) = (X, Y, Z) \). Passing to the limit in equations (3.5) and (3.6), when \( 0 < \delta \leq \delta_0 \), \( (X, U) = (X, Y, Z) \) solves equations (3.3) and (3.4) for \( \alpha = \alpha_0 + \delta \). This completes the proof. \( \square \)

In the following, we give the proof of the existence.

**Proof.** By Lemma 3.1, when \( \alpha = 0 \), for any \( (b_0(\cdot), \sigma_0(\cdot), b_0(\cdot)) \in M^2(0, T; \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m) \), \( g_0 \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^m) \) equations (3.3) and (3.4) have an adapted solution. Then from Lemma 3.3, there exists a constant \( \delta_0 \) which depends only on \( \mu_1, \beta_1, \beta_2 \) and \( T \), such that for all \( \alpha \in [\alpha_0, \alpha_0 + \delta_0] \), (3.3) and (3.4) have an adapted solution. Thus, we can solve equations (3.3) and (3.4) successively for the case \( \alpha \in [0, \delta_0], [\delta_0, 2\delta_0], \ldots \) which leads to that, when \( \alpha = 1 \), there exists an adapted solution of equations (2.1) and (2.2). \( \square \)
3.2 Relationship between functional fully coupled FBSDEs (1.2) and related P-PDEs

In this section, we first review some basic notions and results of functional Itô calculus. Then we investigate the relationship between the solution of functional fully coupled FBSDE and the classical solution of the related P-PDE.

Let $T > 0$ be fixed. For each $t \in [0, T]$, we denote by $\Lambda^n_t$ the set of càdlàg $\mathbb{R}^n$-valued functions on $[0, t]$.

For each $\gamma \in \Lambda^n_t$, the value of $\gamma$ at time $s \in [0, T]$ is denoted by $\gamma(s)$. Thus $\gamma = \gamma(s)_{0 \leq s \leq T}$ is a càdlàg process on $[0, T]$ and its value at time $s$ is $\gamma(s)$. The path of $\gamma$ up to time $t$ is denoted by $\gamma_t$, i.e., $\gamma_t = \gamma(s)_{0 \leq s \leq t} \in \Lambda^n_t$. We denote $\Lambda^n = \bigcup_{t \in [0, T]} \Lambda^n_t$. For each $\gamma_t \in \Lambda^n$ and $x \in \mathbb{R}^n$ we denote by $\gamma_t(s)$ the value of $\gamma_t$ at $s \in [0, t]$ and $\gamma_t^r := (\gamma_t(s)_{0 \leq s < t}, \gamma_t(t) + x)$ which is also an element in $\Lambda^n_t$.

We now introduce a distance on $\Lambda^n$. Let $\langle \cdot , \cdot \rangle$ and $| \cdot |$ denote the inner product and norm in $\mathbb{R}^n$. For each $0 \leq t, \bar{t} \leq T$ and $\gamma_t, \bar{\gamma}_t \in \Lambda^n$, we denote

$$
\| \gamma_t \| := \sup_{s \in [0, t]} | \gamma_t(s) |,
$$

$$
\| \gamma_t - \bar{\gamma}_t \| := \sup_{s \in [0, t \land \bar{t}]} | \gamma_t(s \land \bar{t}) - \bar{\gamma}_t(s \land \bar{t}) |,
$$

$$
d_\infty(\gamma_t, \bar{\gamma}_t) := \sup_{s \in [0, t \land \bar{t}]} | \gamma_t(s \land \bar{t}) - \bar{\gamma}_t(s \land \bar{t}) | + | t - \bar{t} |.
$$

It is obvious that $\Lambda^n$ is a Banach space with respect to $\| \cdot \|$. Since $\Lambda^n$ is not a linear space, $d_\infty$ is not a norm.

**Definition 3.4** A function $u : \Lambda^n \mapsto \mathbb{R}$ is said to be $\Lambda^n$–continuous at $\gamma_t \in \Lambda^n$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\bar{\gamma}_t \in \Lambda^n$ with $d_\infty(\gamma_t, \bar{\gamma}_t) < \delta$, we have $| u(\gamma_t) - u(\bar{\gamma}_t) | < \varepsilon$.

$u$ is said to be $\Lambda^n$–continuous if it is $\Lambda^n$–continuous at each $\gamma_t \in \Lambda^n$.

**Definition 3.5** Let $v : \Lambda^n \mapsto \mathbb{R}$ and $\gamma_t \in \Lambda^n$ be given. If there exists $p \in \mathbb{R}^n$, such that

$$
v(\gamma_t^p) = v(\gamma_t) + \langle p, x \rangle + o(|x|) \text{ as } x \to 0, \ x \in \mathbb{R}^n.
$$

Then we say that $v$ is (vertically) differentiable at $\gamma_t$ and denote the gradient of $D_x v(\gamma_t) = p$. $v$ is said to be vertically differentiable in $\Lambda^n$ if $D_x v(\gamma_t)$ exists for each $\gamma_t \in \Lambda^n$. We can similarly define the Hessian $D^2_{xx} v(\gamma_t)$. It is an $\mathbb{S}(n)$-valued function defined on $\Lambda^n$, where $\mathbb{S}(n)$ is the space of all $n \times n$ symmetric matrices.
For each $\gamma_t \in \Lambda^n$ we denote

$$\gamma_{t,s}(r) = \gamma_t(r)1_{[0,t)}(r) + \gamma_t(t)1_{[t,s]}(r), \; r \in [0,s].$$

It is clear that $\gamma_{t,s} \in \Lambda^n$.

**Definition 3.6** For a given $\gamma_t \in \Lambda^n$ if we have

$$v(\gamma_{t,s}) = v(\gamma_t) + a(s-t) + o(|s-t|) \; \text{as} \; s \to t, \; s \geq t,$$

then we say that $v(\gamma_t)$ is (horizontally) differentiable in $t$ at $\gamma_t$ and denote $D_tv(\gamma_t) = a$. $u$ is said to be horizontally differentiable in $\Lambda^n$ if $D_tv(\gamma_t)$ exists for each $\gamma_t \in \Lambda^n$.

**Definition 3.7** Define $C^{j,k}(\Lambda^n)$ as the set of function $v := (v(\gamma_t))_{\gamma_t \in \Lambda^n}$ defined on $\Lambda^n$ which are $j$ times horizontally and $k$ times vertically differentiable in $\Lambda^n$ such that all these derivatives are $\Lambda^n$–continuous.

The following Itô formula was firstly obtained by Dupire [7] and then by Cont and Fournié [4] for a more general formulation.

**Theorem 3.8 (Functional Itô’s formula)** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be a probability space, if $X$ is a continuous semi-martingale and $u$ is in $C^{1,2}(\Lambda^n)$, then for any $t \in [0,T)$,

$$v(X_t) - v(X_0) = \int_0^t D_s v(X_s) \, ds + \int_0^t D_x v(X_s) \, dX(s) + \frac{1}{2} \int_0^t D_{xx} v(X_s) \, d\langle X \rangle(s), \quad P \text{-a.s.}$$

Note that the coefficients of (2.1) and (2.2) are permitted to be random. For a given $t \in [0,T]$ and $\gamma_t = (\gamma_t^1, \gamma_t^2) \in C^d \times C^n$, consider the following functional fully coupled FBSDE:

$$W^{\gamma_t}(s) = \gamma_t^1(t) + W(s) - W(t), \; s \in [t,T],$$

$$X^{\gamma_t}(s) = \gamma_t^2(t) + \int_t^s b(W^{\gamma_t}, X^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) \, dr + \int_t^s \sigma(W^{\gamma_t}, X^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) \, dW(r), \; s \in [t,T],$$

$$Y^{\gamma_t}(s) = g(W^{\gamma_t}_T, X^{\gamma_t}(T)) - \int_s^T h(W^{\gamma_t}_r, X^{\gamma_t}_r, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) \, dr - \int_s^T Z^{\gamma_t}(r) \, dW(r), \; s \in [t,T],$$

and

$$W^{\gamma_t}_s = \gamma_t^1(s), \; 0 \leq s \leq t,$$

$$X^{\gamma_t}_s = \gamma_t^2(s), \; 0 \leq s \leq t.$$
In order to apply Theorem 3.1, the above equations (3.11) and (3.12) can be rewritten as:

\[ \dot{X}_t^\gamma(s) = \gamma_t(t) + \int_t^s \dot{b}(\dot{X}_r^\gamma, Y^\gamma(r), Z^\gamma(r))dr + \int_t^s \dot{\sigma}(\dot{X}_r^\gamma, Y^\gamma(r), Z^\gamma(r))dW(r), \]
\[ \dot{X}_t^\gamma = \gamma_t, \quad X_t^\gamma = (W_t^\gamma, X_s^\gamma), \quad t \leq s \leq T, \]

(3.13)

\[ Y_t^\gamma(s) = g(W_t^\gamma, X_t^\gamma) - \int_s^T h(\dot{X}_r^\gamma, Y_t^\gamma, Z^\gamma)dr - \int_s^T Z^\gamma ds + \int_s^T W(r)dW(r), \quad s \in [t, T], \]

(3.14)

where \( \tilde{b} \) and \( \tilde{\sigma} \) are defined as

\[ \tilde{b}(\tilde{x}_t^1, u^1(t)) = ((0, \ldots, 0, b(\tilde{x}_t^1, u^1(t))))^T, \quad P - a.s., \]
\[ \tilde{\sigma}(\tilde{x}_t^1, u^1(t)) = (I_{d \times d}, \sigma(\tilde{x}_t^1, u^1(t)))^T, \quad P - a.s., \]

\( \forall (\tilde{x}_T^1, u_T^1), (\tilde{x}_T^2, u_T^2) \in C^{d+n} \times C^m \times C^{m \times d} \) and \( (0, \ldots, 0) \in \mathbb{R}^d \), \( I_{d \times d} \) is the \( d \)-dimensional identity matrix.

Now we consider the related P-PDE:

\[ D_t u(\gamma_t) + L u(\gamma_t) - h(\gamma_t, u(\gamma_t), v(\gamma_t)) = 0, \]
\[ v(\gamma_t) = D_x u(\gamma_t) \tilde{\sigma}(\gamma_t, u(\gamma_t), v(\gamma_t)), \]
\[ u(\gamma_T) = g(\gamma_T^1, \gamma_T^2(T)), \quad \gamma_T^1 \in \Lambda^d, \quad \gamma_T^2(T) \in \mathbb{R}^n, \]

where

\[ L u = (L u_1, \ldots, L u_n), \quad L = \frac{1}{2} \sum_{i,j=1}^{n+d} (\tilde{\sigma} \tilde{\sigma}^T)_{ij}(\gamma_t, u, v) D_x_{ix} + \sum_{i=1}^{n+d} b_i(\gamma_t, u, v) D_{x_i}. \]

**Theorem 3.9** Suppose that Assumptions (2.3) and (2.4) hold. If \( u^1 \in C^{1,2}(\Lambda^{d+n}) \) is the solution of equation (3.13) and \( (u^1, v^1) \) is uniformly Lipschitz continuous and has linear growth, then we have \( u^1(\gamma_t) = Y^\gamma(t) \), for each \( \gamma_t = (\gamma_t^1, \gamma_t^2) \in \Lambda^d \times C^n \), where \( (Y^\gamma(s), Z^\gamma(s))_{s \leq t} \) is the unique solution of equations (3.13) and (3.14). Consequently, the P-PDE (3.14) has a unique \( C^{1,2}(\Lambda^{d+n}) \)-solution.

**Proof.** Note that \((u^1, v^1)\) is uniformly Lipschitz continuous and has linear growth. Then, under Assumptions (2.3) and (2.4), the equation

\[ d\dot{X}_t^\gamma(s) = \tilde{b}(\dot{X}_t^\gamma, u^1(\dot{X}_s^\gamma), v^1(\dot{X}_s^\gamma))ds + \tilde{\sigma}(\dot{X}_s^\gamma, u^1(\dot{X}_s^\gamma), v^1(\dot{X}_s^\gamma))dW(s), \]
\[ \dot{X}_s^\gamma = \gamma_t, \quad s \in [t, T], \]
has a unique solution. It is easy to see that we can denote $\tilde{X}_s^{\gamma}$ by $(W_s^{\gamma}, X_s^{\gamma})$.

Set $(Y^1(s), Z^1(s)) = (u^1(\tilde{X}_s^{\gamma}), v^1(\tilde{X}_s^{\gamma}))$, $t \leq s \leq T$. Applying the functional Itô formula to $u^1(\tilde{X}_s^{\gamma})$, we have

$$dY^1(s) = h(\tilde{X}_s^{\gamma}, Y^1(s), Z^1(s))dr + Z^1(s)dW(s),$$

$$Y^1(T) = g(W_T^{\gamma}, X_T^{\gamma}), \quad s \in [t, T].$$

Thus, by Theorem 3.1, we have $(Y^1(s), Z^1(s)) = (Y^\gamma(s), Z^\gamma(s))$. In particular, $u^1(\gamma_t) = Y^\gamma(t)$. This completes the proof. □

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