Asymptotically Optimal Quantum Key Distribution Protocols

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Quantum key distribution (QKD) could be the most significant application of quantum information theory. In nearly four decades, although substantial QKD protocols are developed, the BB84 protocol and its variants are still the most researched ones. It is well-known that the secure bound of qubit error rate (QBER) of BB84 protocol is about 11% while it can be increased to 12.6% by six-state protocol. It would not be surprising that employing more basis could increase the bound. However, what is the optimal one? In this paper, investigations of asymptotically optimal QKD protocols are proposed. Precisely, We present an abstraction of prepare-measure QKD protocols and investigate two special cases which are optimal among all protocols coding by the same states. Our analyses demonstrate that the asymptotically optimal QBER bounds coding by orthogonal qubits are about 22.17% for both general C-NOT attacks and memoryless attacks while the bounds coding by non-orthogonal states in two mutually unbiased basis are about 25.30% for general C-NOT attacks and 27.00% for memoryless attacks. The optimality of our protocols demonstrates the ultimate potential of the security under such attacks.

Keywords: Quantum key distribution; Optimal bound; Qubit Error Rate; C-NOT attack; Optimal Protocol.

I. INTRODUCTION

Communicating securely is always one of the most important fields in information theory. Nowadays, the only scheme whose security has been proven is coding with an one-time pad, which, however, could not be distributed to separated partners securely by classical methods. Therefore, quantum key distribution (QKD), a kind of schemes for distributing an one-time pad by quantum methods with the security only depends on physical laws, becomes significant.

The first QKD protocol, called BB84 protocol, was proposed in 1984 [1], of which the security has been proven [2]. In nearly four decades, substantial QKD protocols are developed [3–9] but BB84 protocol and its variants such as B92 protocol [10], BBM92 protocol [11], six-state protocol [12], SARG04 protocol [13] and others [2] are still the most researched ones.

Generally speaking, the security of QKD protocols comes from that if there is an eavesdropper who obtains enough information about the secret key, then she will create enough errors and detectable by the legitimate partner. However, in practically implementing a QKD protocol, errors are unavoidable due to the imperfections of channels and devices. Therefore, to analyse the security of a QKD protocol, we have to estimate the threshold (or called secure bound) of the qubit error rate (QBER) it can tolerate, namely the value that the legitimate partner can extract a secret key when the QBER is below it.

There are works analysing the security of protocols [12, 14, 21], demonstrating that the threshold of QBER for BB84 protocol is about 11% while it is increased to about 12.6% for six-state protocol, under individual attacks.

For memoryless attacks, the bound is about 15.4% for BB84 protocol, 20.4% for six-state protocol and 17.6% for SARG04 protocol [18, 20].

It would not be surprising that a protocol employing more basis for coding should be more secure. However, what is the optimal one, namely what protocol is the most secure one, theoretically, and how to analyse it? In this paper, We present an abstraction of prepare-measure QKD protocols and investigate two special cases which are optimal among all protocols coding by the same states. We calculate the secure QBER bounds for the special protocols under C-NOT attacks, demonstrating that the asymptotically optimal QBER bounds coding by orthogonal qubits are about 22.17% for both general C-NOT attacks and memoryless attacks while the bounds coding by non-orthogonal qubits in two mutually unbiased basis are increased to about 25.30% for general C-NOT attacks and 27.00% for memoryless attacks. Our investigations also reveal the meaningless of collective C-NOT attacks, namely employing C-NOT attack to every qubit. The optimality of our protocols demonstrates the ultimate potential of security under such attacks. Finally, although our protocols are idealized, they can be asymptotically realized.

II. THE ABSTRACTION OF GENERAL PREPARE-MEASURE QKD PROTOCOL

For simplicity, our scenario is under two assumptions.

1. The legitimated partner, Alice and Bob, can employ quantum memories. This assumption can be removed by basis sifting in practise as the difference between ordinary BB84 protocol [1] and BB84 protocol with Hadamard gates [2].

2. Alice and Bob only employ single photon sources
and thus the eavesdropper, Eve, will not employ attacks based on photon numbers. In practise, Alice and Bob might employ weak coherence sources and thus, Eve might employ, for example, photon-number-splitting (PNS) attacks. However, these attacks are handled by other methods such as decoy state methods.

The abstraction of a general protocol is described as follows.

**Protocol:**

**Step 1:** Alice and Bob agree to encode 0 by state $C_0|0\rangle$ and 1 by state $C_1|0\rangle$, where $C_i$, $i = 1, 2$ are unitary operators on qubits. The states can be orthogonal or non-orthogonal. Alice chooses a bit string randomly and for each bit, she chooses a unitary operator, $U$ (depending on the special protocol and can be randomly in a set), and sends $UC_i|0\rangle$ with $i = 1, 2$ chosen randomly, to Bob.

**Step 2:** After Bob receives the state, Alice publicly announces the choice of $U$. Bob measures the qubit via basis $U_B|0\rangle, U_B|1\rangle$, where $U_B$ depends on the chosen protocol (and $U$), to decode the bit.

These steps will be repeated several times until Alice and Bob share a long enough bit string.

**Step 3:** Alice and Bob discard the non-effective bits (depending on the chosen protocol) and estimate the QBER by declaring part of their bit string and public discussions. The string is aborted if the QBER is too high or not random enough.

**Step 4:** If the error rate and the randomness of the string are acceptable, they generate a raw secret key by remaining bits, followed by error correcting and privacy amplification procedures if needed.

Note that if $C_0$ is chosen to be $I, C_1$ is chosen to be the Pauli operator $X$, $U$ is chosen as $I$ or Hadamard gate $H$ randomly for each bit, and $U_B$ is chosen as $U$, then the protocol becomes BB84 protocol with Hadamard gates, while if $C_0 = I, C_1 = H, U = I$ and $U_B$ is chosen as $I$ or $H$ randomly for each bit, then the protocol becomes B92 protocol. Also if $C_0$ is chosen randomly among $I$ and $X$ for each bit, $C_1$ is chosen randomly among $H$ and $HX$ for each bit, $U = I$ and $U_B$ is chosen as $I$ or $H$ randomly for each bit, then the protocol becomes SARG04 protocol.

**III. TWO SPECIAL PROTOCOLS AND THE OPTIMALITY**

We would like to investigate two special protocols. The first one chooses $C_0 = I, C_1 = X, U_B = U$ while $U$ is chosen randomly among all unitary operators on qubits for each bit. Therefore, the protocol expands BB84 protocol. We will call it BB84 type protocol. The second one chooses $C_0 = I, C_1 = H, U_B$ be $I$ or $H$ randomly for each bit while $U$ is chosen randomly among all unitary operators on qubits for each bit. Therefore, the protocol expands the B92 protocol. We will call it B92 type protocol. Similarly, we can have SARG04 type protocol.

Our protocols are optimal among all protocols coding with the same states (namely coding 0 by state $C_0|0\rangle$ and 1 by state $C_1|0\rangle$). The optimality can be demonstrated as follows. Whatever Alice sends, since Eve is assumed to have any technology under physical laws, she can randomly operate a unitary operator $U$, followed by a normal attack, and finally operate $U^\dagger$ on the partita (in this paper, a partita always represents a subsytem) sent to Bob. Such operations of Eve make the whole procedures equal to that Alice and Bob implement our protocol while Eve implements the normal attack. In other words, Eve can transform any protocol into ours and then attack. Therefore, for Eve, the worst case could not worse than our protocol. Hence, our protocol is the worst one for Eve and thus the optimal one for Alice and Bob.

**IV. ATTACK OF EVE**

The aim of Eve is guessing the bits of Alice correctly as many as possible without resulting in the abortion of the protocol. Here we would discuss a kind of individual attack that Eve copies a qubit sent by Alice with a C-NOT gate, called C-NOT attack.

The general C-NOT attack can be described as follows. For a state sent by Alice, Eve adds an auxiliary partita (her auxiliary system) and operates a C-NOT gate under a chosen basis. Then she sends the ordinary partita to Bob while storages the auxiliary one. In general, Eve’s state is measured individually after she eavesdropped all classical communications of Alice and Bob with her measurement depending on classical messages she obtained. However, in a memoryless attack, Eve measures her state immediately.

Assume that the state sent by Alice is $U|c\rangle = aE'|0\rangle_A + bE'|1\rangle_A$. After Eve’s action, the state becomes $|x\rangle = aE'|0\rangle_a|0\rangle_E + bE'|1\rangle_a|1\rangle_E$, where $E$ denotes the partita of Eve, $E^\dagger$ is a unitary operator on qubits, $|0\rangle_E, |1\rangle_E$ are two orthogonal states in $E$ (the dimension of $E$ could be larger) and $a = \langle 0|E^\dagger U|c\rangle, b = \langle 1|E^\dagger U|c\rangle$.

**V. SECURE BOUND OF QBER**

Let us calculate the secure bound of QBER. The secure condition for the legitimated partner is the allowance to extract a secret key, which is promised by the private
information being larger than zero. The private information of the legitimated partner also provides the secret key rate [27, 28].

Denote the QBER that Alice and Bob decide to tolerate by \( r \) and the error rate of them when Eve attacks a state by \( e_B \). If Alice and Bob obtain \( N \) bits in which \( t \) bits are attacked, then \( e_B t \leq rN \) for not resulting in the abortion of the string. Therefore, the proportion of qubits Eve can attack is at most \( \frac{r}{e_B} \). An easy discussion shows that in an optimal strategy of Eve, \( \frac{r}{e_B} \leq 1 \). Now, the private information is calculated (under assumptions that \( \frac{r}{e_B} \leq 1 \) and Eve attacks \( \frac{r}{e_B} \) states) as follows, where \( e \) is the error rate of Eve (namely the probability of Eve of guessing a bit wrongly) when she launches an attack.

\[
I(A : B) - I(A : E) = \frac{r}{e_B} (I(A : B)_{att} - I(A : E)_{att}) + (1 - \frac{r}{e_B}) (I(A : B)_{non} - I(A : E)_{non})
\]

\[
= \frac{r}{e_B} (I(B : A)_{att} - I(E : A)_{att}) + (1 - \frac{r}{e_B})
\]

\[
= \frac{r}{e_B} (H(E|A)_{att} - H(B|A)_{att}) + (1 - \frac{r}{e_B})
\]

\[
= \frac{r}{e_B} ((h(e) - 1) - (h(e_B) - 1)) + (1 - \frac{r}{e_B})
\]

\[
= 1 + \frac{r}{e_B} (h(e) - h(e_B) - 1)
\]

The positivity of it implies that \( r < \frac{e_B}{1-h(e)+h(e_B)} \), where \( H(x|y) = H(x, y) - H(y) \) is the conditional entropy, \( h(x) = -x \log x - (1-x) \log (1-x) \) is the binary entropy and the subscripts denote whether Eve attacks or not. Hence, we assume that the error rate of Eve on bits 0 and bits 1 are the same, which is not surprising since Alice and Bob code with symmetric states (in our discussions below, Alice and Bob always employ symmetric states for coding) and thus if an optimal strategy of Eve obtains more errors on bits 0, then she can employ a symmetric strategy, obtaining more errors on bits 1 and she can combine the strategies (still be optimal) such that the error rate on bits 0 and bits 1 are the same.

VI. ASYMPTOTICALLY OPTIMAL QBER BOUND OF BB84 TYPE PROTOCOL

Let us investigate BB84 type protocol. In such a protocol, Alice sends state \(|0\rangle \) or \(|1\rangle \) operated by \( U \) randomly while Bob measures via basis \( U|0\rangle, U|1\rangle \). We shall calculate \( e \) and \( e_B \). As shown in section IV, after Eve’s attack, the state sent by Alice becomes \(|X_c\rangle = a|0\rangle_A|0\rangle_E + b|1\rangle_A|1\rangle_E \), where \( c = 0, 1 \). Write \( U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \) under the computational basis, and since \( U \) is unitary, \(|u_1|^2 + |u_2|^2 = 1, |u_1| = |u_4|, |u_2| = |u_3|, u_1u_3 + u_2u_4 = 0 \). Assume that Eve measures her partita by the positive operator-valued measurement (POVM) \{\(M,N\}\} (note that for general attacks, an optimal attack for Eve can contain only two measurement outcomes since she only guesses the bit of Alice and Bob be 0 or 1), depending (for general C-NOT attacks) or not depending (for memoryless attacks) on \( U \).

A. General C-NOT attack

For Eve launches a general C-NOT attack, Both Bob and Eve measure their states after knowing \( U \). Now,

\[
P_{B|A}(0|0) = \int_{U_{ave}} \langle X_0|U|0\rangle \langle 0|U^\dagger \otimes I|X_0\rangle dU = \int_{U_{ave}} \langle 0|E|U|0\rangle^2 \langle 0|E^\dagger U|0\rangle^2 + \langle 1|E|U|0\rangle^2 \langle 1|E^\dagger U|0\rangle^2 dU
\]

\[
= \int_{U_{ave}} |0|U|0\rangle^2 |0|U|0\rangle^2 dU + \int_{U_{ave}} |1|U|0\rangle^2 |1|U|0\rangle^2 dU = 2 \int_{U_{ave}} |0|U|0\rangle^4 dU = 2 \int_{U_{ave}} |u_1|^4 dU = \frac{3}{4},
\]

where \( \int_{U_{ave}} \) represents integrating over all unitary operators and taking average (namely, divided by \( \int_{U} 1dU \)). The calculation gives \( P_{B|A}(1|1) = \frac{3}{4} \) and \( e_B = \frac{1}{2} \).

To calculate \( e \), without loss generality, assume that Eve will guess the bit of Alice and Bob be 0 if her measurement outcome is \( M \) and 1 if her outcome is \( N \).
Similarly, \( P_{E|A}(0|0) = \int_{U, \text{ave}} \langle X_0 | (U|0\rangle \langle 0|U^\dagger \otimes I) (I \otimes M_U) (U|0\rangle \langle 0|U^\dagger \otimes I) X_0 \rangle dU \) is defined, an optimal strategy of Eve is making the attack of Eve, on distinguishing states \([30]\). Hence, under the optimal strategy, \( \pi_{\text{max}} \) to be \( C^2 \) with \( M_{EU} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( N_{EU} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) if \( |u_1| \geq |u_2| \) and \( M_{EU} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( N_{EU} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) otherwise, under the computational basis. It is not surprising that an optimal strategy of Eve is employing \( C^A \) as her auxiliary partita since extending dimension of the system would not give benefits on distinguishing states \([30]\). Hence, under the optimal attack of Eve, \( P_{E|A}(0|0) = \frac{1}{2} (P_{E|A}(0|0) + P_{E|A}(1|1)) = \frac{1}{2} + \frac{2}{\pi} \) and thus \( e = \frac{1}{2} - \frac{1}{\pi} \). Finally, the secure threshold of QBER is calculated as \( r < \frac{\varepsilon_B}{1 - h(\varepsilon_B) + h(e_B)} = \frac{1}{1 - h(\frac{2}{\pi}) + h(\frac{1}{2})} \approx 22.17\% \).

It is worth noting that if Eve’s measurement is restricted to projective ones, then \( P_{E|A}(0|0) \) and \( P_{E|A}(1|1) \) can be calculated directly and become maximal in the same strategy, which is coincident with the result above. Hence, an optimal strategy of Eve can only employ projective measurements for measuring.

**B. Memoryless attack**

For Eve launches a memoryless attack, Eve measures the state firstly without knowing \( U \). Now,

\[
P_{B|A}(0|0) = P_{E|A}(0|0) P_{B|A,E}(0|0, M) + P_{E|A}(N|0) P_{B|A,E}(0|0, N)
= \int_{U, \text{ave}} \langle X_0 | (U|0\rangle \langle 0|U^\dagger \otimes M) X_0 \rangle dU + \int_{U, \text{ave}} \langle X_0 | (U|0\rangle \langle 0|U^\dagger \otimes N) X_0 \rangle dU
= \int_{U, \text{ave}} \langle X_0 | (U|0\rangle \langle 0|U^\dagger \otimes I) X_0 \rangle dU + \int_{U, \text{ave}} \langle X_0 | (U|0\rangle \langle 0|U^\dagger \otimes I) X_0 \rangle dU
\]
\[ P_{E\mid A}(0|0) = \frac{1}{2} \int_{U_{\text{ave}}} \langle X_0 | I \otimes M | X_0 \rangle dU + \int_{U_{\text{ave}}} \langle X_0 | I \otimes N | X_0 \rangle dU \]
\[ = \int_{U_{\text{ave}}} \max(\langle X_0 | I \otimes M | X_0 \rangle, \langle X_0 | I \otimes N | X_0 \rangle) dU, \]
\[ P_{E\mid A}(1|1) = \frac{1}{2} \int_{U_{\text{ave}}} \langle X_1 | I \otimes N | X_1 \rangle dU + \int_{U_{\text{ave}}} \langle X_1 | I \otimes M | X_1 \rangle dU \]
\[ = \int_{U_{\text{ave}}} \max(\langle X_1 | I \otimes M | X_1 \rangle, \langle X_1 | I \otimes N | X_1 \rangle) dU. \]

where \( \langle X_0 | I \otimes M | X_0 \rangle \geq \langle X_0 | I \otimes N | X_0 \rangle \) represents the bit being more likely to be 0 when Eve obtains outcome \( M \) and 1 when she obtains outcome \( N \). An optimal strategy for Eve is making \( P_{E\mid A}(0|0) + P_{E\mid A}(1|1) \) maximal (similar to above) and can be done by choosing her partita to be \( C^2 \) with \( M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Hence, \( e = \frac{1}{2} - \frac{1}{\pi} \), \( e_B = \frac{1}{3} \) and thus the secure QBER threshold is also about 22.17\%.

It is worth noting that the investigations above also show that in such a protocol, the error rate created by Eve is always \( \frac{1}{3} \) if she attacks. Hence, if Eve employs collective attacks, then the error rate of Bob could never satisfy the secure threshold, which results in the abortion of the protocol. Therefore, discussing collective attacks would be meaningless.

\[
\begin{align*}
P_{E\mid A}(0|0) + P_{E\mid A}(1|1) &= \int_{U_{\text{ave}}} \max(\langle X_0 | I \otimes M | X_0 \rangle, \langle X_0 | I \otimes N | X_0 \rangle) dU + \int_{U_{\text{ave}}} \max(\langle X_1 | I \otimes M | X_1 \rangle, \langle X_1 | I \otimes N | X_1 \rangle) dU \\
&= \int_{U_{\text{ave}}} \max(|u_1|^2|0\rangle\langle M|0\rangle + |u_3|^2|1\rangle\langle M|1\rangle + |u_1|^2|0\rangle\langle N|0\rangle + |u_3|^2|1\rangle\langle N|1\rangle dU \\
&\leq 2 \int_{U_{\text{ave}}} \max(|u_1|^2, |u_3|^2) = 1 + \frac{2}{\pi},
\end{align*}
\]

\section{VII. Asymptotic QBER Bound of B92 Type Protocol}

Let us investigate the B92 type protocol. In such a protocol, Alice sends state \( |0\rangle \) or \( |+\rangle \) while Bob measures via basis \( \{ U |0\rangle, U |1\rangle \} \) or \( \{ U |+\rangle, U |–\rangle \} \) randomly. Others are the same as in BB84 type protocol and we will employ the same symbols. Note that our investigations also hold for SARG04 type protocol.

\subsection{A. General C-NOT attack}

Now,

\[
P_{B\mid A}(|–|0) = \int_{U_{\text{ave}}} \langle X_0 | U |–\rangle \langle – | U^\dagger \otimes I | X_0 \rangle dU
\]
\[= \frac{1}{2} \int_{U_{\text{ave}}} [|u_1|^2(1 - 2 R e(u_1 \bar{u}_2)) + |u_3|^2(1 - 2 R e(u_3 \bar{u}_4))] dU = \frac{1}{2}, \quad (8)
\]

\[
P_{B\mid A}(1|0) = \int_{U_{\text{ave}}} \langle X_0 | U |1\rangle \langle 1 | U^\dagger \otimes I | X_0 \rangle dU = 2 \int_{U_{\text{ave}}} |u_1 u_2|^2 dU = \frac{1}{4}.
\]

Therefore, \( P_{B\mid A}(0|0) = \frac{P_{B\mid A}(|–|0)}{P_{B\mid A}(|–|0) + P_{B\mid A}(1|0)} = \frac{2}{3}, \) and \( e_B = \frac{1}{3} \).
Instead of calculating $e$, we use the fact that $e \geq \frac{1}{2} - \frac{1}{2\sqrt{2}}$. Since we can view Eve’s action as a cloning procedure in which she can not do better than perfect clone. However, for perfect clone, Eve has to distinguish two non-orthogonal states $|0\rangle$ and $|+\rangle$, which can be optimal distinguished with error rate $\frac{1}{2} - \frac{1}{2\sqrt{2}}$, given in the supplied material. Hence, the secure bound of QBER is calculated as $r < \frac{e_B}{1 - h(e) + h(e_B)} = \frac{\frac{1}{2} - \frac{1}{2\sqrt{2}}}{1 - h\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)} \approx 25.30\%$. Note that even if $e = 0$, the secure bound would be about 17%.

**B. Memoryless attack**

The investigation of memoryless attacks is similar to above. $e_B$ is the same as in general attacks, since there are no difference between Bob measures firstly and Eve measures firstly. $e$ is the same as in memoryless attacks of BB84 type. Hence, $e_B = \frac{1}{3}$ while $e = \frac{1}{3} - \frac{1}{5}$. The secure QBER threshold now becomes $r < \frac{e_B}{1 - h(e) + h(e_B)} = \frac{1}{3} - h\left(\frac{1}{3} - \frac{1}{5}\right) \approx 27.00\%$.

Similar to BB84 type, note that the error rate created in a collective attack is always $\frac{1}{3}$, which could not satisfy the secure threshold. Therefore, collective attacks could always result in the abortion of the bit string. Hence, discussing collective attacks would also be meaningless.

**VIII. ASYMPTOTICALLY OPTIMAL**

Although our protocols are idealized and can not be practically implemented due to the infinite choices of $U$, the bounds can be asymptotically touched. To see this, just employing finite number of $U$ which are uniform distributed among all unitary operators on qubits. As the number of $U$ increases, the error rates and thus the QBER bounds can close to above ones.

Note that a unitary operator on qubits depends on four coefficients, $r_1, \theta_2, \theta_3, \theta_4$, where they are $|u_1|$ and arguments of $u_2, u_3, u_4$. Therefore, for example, let Alice and Bob employ $2^k$ unitary operators. The operators are chosen as $(r_1, \theta_2, \theta_3, \theta_4) = \left(\frac{1}{\sqrt{2}}, \frac{2\pi t_i}{2^k}, \frac{2\pi t_i}{2^k}, \frac{2\pi t_i}{2^k}\right)$, where $t_i \in \{0, 1, \ldots, 2^k - 1\}, i = 1, 2, 3, 4$. The limit of the QBER bounds would be the above ones as $k$ increases.

**IX. CONCLUSION**

In conclusion, we presented an abstraction of prepare-measure QKD protocols and investigated two special cases which are optimal among all protocols coding with the same states. For coding with orthogonal qubits (expanding BB84 protocol), we demonstrated that the optimal secure QBER bounds are about 22.7%, both for general and memoryless C-NOT attacks, while for coding with non-orthogonal qubits in two mutually unbiased basis (expanding B92 or SARG04 protocol), the secure bounds are increased to about 25.30% and 27.00% for general and memoryless C-NOT attacks, respectively. The optimality of the protocols demonstrates the ultimate potential of the security under such attacks.

We also demonstrated that an optimal strategy of Eve can only employ $C^2$ as her auxiliary partita and projective measurements for measuring while collective attacks are meaningless. On the other hand, although our protocols are idealized, they can be asymptotically realized. Finally, C-NOT attack might not be the most general attack, but it is one of the most normal attacks, and provides a framework for investigating such protocols.

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X. SUPPLIED MATERIAL

The optimal error rate of distinguishing states $|0\rangle$ and $|+\rangle$:

Suppose that one employs the positive operator-valued measurement (POVM) $\{M_0, M_+\}$ to distinguish the two states and judges the state be $|0\rangle$ by outcome $M_0$ and $|+\rangle$ by outcome $M_+$. Write the operators as matrices under the computational basis as $M_0 = \begin{pmatrix} m_1 & m_2 \\ \bar{m}_2 & m_4 \end{pmatrix}$, $M_+ = \begin{pmatrix} 1 - m_1 & -m_2 \\ -\bar{m}_2 & 1 - m_4 \end{pmatrix}$, where $0 \leq m_1, m_4 \leq 1$ are real and $m_1 m_4 \geq |m_2|^2$, $(1 - m_1)(1 - m_4) \geq |m_2|^2$, since $\{M_0, M_+\}$ is a POVM. Without loss generality, assume that $m_1 + m_4 \leq 1$. The correct rate is calculated as

$$P_{\text{correct}} = \frac{1}{2}(|\langle 0|M_0\rangle| + |\langle +|M_+\rangle|) = \frac{1}{2} + \frac{1}{4}(m_1 - m_4 + 2Re(m_2))$$

with conditions $0 \leq m_1, m_4 \leq 1$ are real and $m_1 m_4 \geq |m_2|^2$. It is easy to see that to make $P_{\text{correct}}$ maximal, we can choose $m_2$ be real while if $m_1 + m_4 < 1$, we can enlarge $m_1$. Therefore, when calculating maximal $P_{\text{correct}}$, we can assume that $m_1 + m_4 = 1$ and $m_2$ is real. To maximise $P_{\text{correct}}$, we should assume that $m_1 \geq m_4$. Hence, the problem becomes maximising $P_{\text{correct}} = \frac{1}{2} + \frac{1}{4}(2m_1 + 2m_2 - 1)$ in the area $\frac{1}{2} \leq m_1 \leq 1, m_1 - m_1^2 \geq m_2^2$. For every $m_1$, to maximise $P_{\text{correct}}$, we should let $m_2$ as large as possible. Therefore, $m_2 \geq 0$ and $m_1 - m_1^2 = m_2^2$. Then $P_{\text{correct}} = \frac{1}{2} + \frac{1}{4}(2m_1 + 2\sqrt{m_1 - m_1^2} - 1)$. When $m_1 = \frac{5}{8}$, $P_{\text{correct}} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ is maximal. Hence, the optimal error rate of distinguishing the two states is $1 - P_{\text{correct}} = \frac{1}{2} - \frac{1}{2\sqrt{2}}$. 

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