The condition for entanglement enhanced information transmission of Pauli memory channel

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Abstract

The sufficient condition of entanglement enhanced classical capacity is given for Pauli memory channel with arbitrary channel parameters. In some special case the condition is also necessary but fail to be necessary in general. The theory of majorization and perturbation are used in the proving.

1 Introduction

Calculating the capacities of quantum channels is an important task of quantum information theory. Besides the uncorrelated channel of qubit system, aiming at finding the entanglement enhanced channel capacity, the memory channel has been widely discussed [1] [2] [3] [4] [5] [6] [7] [8] [9] [10]. Among which the is Markov channel. The simplest situation is Pauli memory channel, which transform the state in the fashion of

$$\rho \rightarrow \sum_{i,j=0}^3 p_{ij} \sigma_i \otimes \sigma_j \rho \sigma_i \otimes \sigma_j$$

where $p_{ij} = (1-\mu)q_i q_j + \mu q_i \delta_{ij}$, $\sigma_0 = I_2$, $\sigma_i$ ($i = 1, 2, 3$) are Pauli matrices. Some of the special channels such as the channel with $q_1 = q_2 = q_3 = (1-x)/3$, $q_0 = x$ and the channel with $q_0 = q_1 = x$, $q_2 = q_3 = \frac{1}{2} - x$ have been studied throughoutly, and the conditions for entangled states maximizing the channel capacity have been obtained. While for the general single qubit channel parameters $q_i$, the problems that if the channel capacity can be maximized by entangled state and at what condition it can be maximized are not known.

For a general quantum channel $E$, the output state is $E(\rho)$ where $\rho$ is the input state. Holevo quantity is defined as $\chi(E) = S(E(\sum_i p_i \rho_i)) - \sum_i p_i S(E(\rho_i))$, where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy of the density operator $\rho$, the input ensemble is $\{p_i, \rho_i\}$ where $\rho_i$ are the input state on which classical information is encoded, and are transmitted with prior probabilities $p_i$. The channel capacity is the maximization of Holevo quantity over all input ensembles $\{p_i, \rho_i\}$

$$C = \max_{\{p_i, \rho_i\}} \chi(E).$$

In the memory Pauli channel [3] (we hereafter denote it as $E$), the $\rho_i$ describe state of two qubits, thus the capacity is the two-qubit capacity of the channel. Due to the symmetry of the channel, the capacity was simplified to

$$C = 2 - \min_{\rho} S(E(\rho)).$$

Also it is proven by the concavity of the von Neumann entropy that the input state $\rho$ achieving the capacity should be pure state.

2 Bell basis representation and strictly solvable channel

The extremal input state that achieving the capacity can be expressed in Bell basis:

$$|\psi\rangle = a_0 \Phi^+ + a_1 \Psi^+ + a_2 \Psi^- + a_3 \Phi^-,$$

where $\Phi^+$, $\Psi^+$, $\Psi^-$, $\Phi^-$ are the Bell states.
then in Bell basis representation, one has $E(\bra{\psi}\ket{\psi}) = B$. The $4 \times 4$ Hermite matrix $B$ has its elements

$$B_{ii} = (A_0, A_1, A_2, A_3) U_i (|a_0|^2, |a_1|^2, |a_2|^2, |a_3|^2)^T,$$

$$B_{ij} = a_i a_j^* A_{ij} + a_j^* a_j A_{ij}, \quad i \neq j$$

where $U_0 = \sigma_0 \otimes \sigma_0, U_1 = \sigma_0 \otimes \sigma_1, U_2 = \sigma_1 \otimes \sigma_0, U_3 = \sigma_1 \otimes \sigma_1, A_{ij} = A_{ji}$ (if $i \neq j$). Denote $A_4 = A_{01}, A_6 = A_{02}, A_8 = A_{03}; A_5 = A_{01}, A_7 = A_{02}, A_9 = A_{03}.$

With $A_0 = \sum_i p_i, A_{1.5} = 2(p_{01} \pm p_{23}), A_{2.7} = 2(p_{02} \pm p_{13}), A_{3.9} = 2(p_{03} \pm p_{12}), A_4 = p_{00} + p_{11} - p_{22} - p_{33}, A_6 = p_{00} - p_{11} + p_{22} - p_{33}, A_8 = p_{00} - p_{11} - p_{22} + p_{33}.$

The completely solvable channel is the channel that has been considered by Macchiavello [3]: the channel $B_{1}$ can be obtained by substituting $a_0$ with $a_2$ and $a_1$ with $a_3$. Let $a_0 = k \cos \theta, a_1 = k \sin \theta e^{i \varphi}, (k \leq 1)$ without lose of generality we set $k \geq 1/2$. The two eigenvalues of $B_1$ will be

$$\lambda_{0,1} = \frac{1}{2} \sqrt{2A_2 + k(A_0 + A_1 - 2A_2) \pm k\sqrt{(A_0 - A_1)^2 \cos^2 2\theta + \sin^2 2\theta(A_2^2 + A_3^2 + 2A_4A_5 \cos 2\varphi)}},$$

A similar expression for the two eigenvalues $\lambda_{2,3}$ of $B_2$ is immediately obtained. To derive the extremal input state, we need the following lemma.

**Lemma:** for any quantum state, if the difference of two of its eigenvalues increases while keeping all the other eigenvalues invariant, the entropy of the state will decrease.

**Proof:** suppose the two eigenvalues are $\lambda_i, \lambda_j$ with $\lambda_i > \lambda_j$, the two eigenvalues contribute to the entropy as $S_{ij} = -\lambda_i \log_2 \lambda_i - \lambda_j \log_2 \lambda_j$, denote $x = \lambda_i / (\lambda_i + \lambda_j)$, then $x > 1/2$. One has $S_{ij} = (\lambda_i + \lambda_j)[H_2(z) - \log_2(\lambda_i + \lambda_j)]$. The binary entropy function $H_2(z) = z \log_2 z - (1 - z) \log_2 (1 - z)$ is a monotonic decreasing function of $z$ for $z > 1/2$. Thus $S_{ij}$ decreases as the bigger eigenvalue $\lambda_i$ increasing. The entropy of the state decreases when only two the eigenvalues become more apart while the other eigenvalues are kept invariant.

The eigenvalues of $B_1$ can be made apart as possible while keeping that of $B_2$. Then we deal with $B_2$ in the same manner. When $A_0 - A_1 > |A_4| + |A_5|$, which can be simplified as

$$\mu > |4x - 1|,$$

we have

$$\lambda_{0,1} = kA_{0,1} + (1 - k)A_2,$$

$$\lambda_{2,3} = (1 - k)A_{0,1} + kA_2.$$
While for the situation of $A_0 - A_1 < |A_4| + |A_5|$, the eigenvalues of $B$ matrix are

$$
\begin{align*}
\lambda_{0,1} &= kA'_{0,1} + (1-k)A_2, \\
\lambda_{2,3} &= (1-k)A'_{0,1} + kA_2.
\end{align*}
$$

(12) (13)

Where $A'_{0,1} = \frac{1}{2}(|A_0 + A_1| \pm (|A_4| + |A_5|))$. For $k \geq \frac{1}{2}$, we also have $\lambda_0 > \lambda_1 \geq \lambda_3, \lambda_0 \geq \lambda_2 > \lambda_3$. Suppose the extremal input states are again the states with $k = 1$, the eigenvalues in descending order are $A' = (A'_{0,1}, A'_1, A_2, A_3)^\downarrow$. When $A'_1 > A_2$, we have $\lambda_0 - A'_{0,1} = (1-k)(A_2 - A'_{0,1}) < 0$, $\lambda_0 + \lambda_1 - A'_0 - A'_1 = (1-k)(2A_2 - A_0 - A_1) < 0$, $\lambda_0 + \lambda_1 + \lambda_2 - A'_0 - A'_1 - A'_2 = \lambda_1 - A'_1 = (1-k)(A_2 - A'_1) < 0$ for the case of $\lambda_1 \geq \lambda_2$; and $\lambda_0 + \lambda_2 - A'_0 - A'_1 = A_2 - A'_1 < 0$ for the case of $\lambda_1 < \lambda_2$.

When $A'_1 < A_2$, we have $\lambda_0 + \lambda_1 - A'_0 - A_2 = \lambda_0 + \lambda_1 - A'_0 - A'_1 - (A_2 - A'_1) < 0, \lambda_0 + \lambda_1 + \lambda_2 - A'_0 - 2A_2 = k(A'_1 - A_2) < 0$ for the case of $\lambda_1 \geq \lambda_2$; and $\lambda_0 + \lambda_2 - A'_0 - A_2 = 0$ for the case of $\lambda_1 < \lambda_2$.

The entanglement enhanced capacity condition [3] is proved.

### 3 The general Pauli channel

Usually, for general Pauli channels, the eigenvalues as well as the entropy of output state $E(|\psi\rangle \langle \psi|)$ can not be analytically obtained. With the numerical steepest descending method, the extremal states can be found to be either with only one Bell state, or the superposition of two Bell states with the same probabilities (separable state). We will prove that such input states are extremal by perturbation theory in the next section.

To compare the maximal of entanglement input and that of separable state input in order to obtain the capacity, we first need to regularize the channel. The channel $E$ possesses the property that

$$E(\sigma_1 \otimes \sigma_3 \rho \sigma_1 \otimes \sigma_3) = \sigma_1 \otimes \sigma_3 E(\rho) \sigma_1 \otimes \sigma_3. \tag{14}$$

As entropy is invariant under unitary transformation, the input states $\rho$ and $\sigma_1 \otimes \sigma_3 \rho \sigma_1 \otimes \sigma_3$ have the same output entropy. If $q_0$ is not the maximal among $q_0, q_1, q_2, q_3$, see $q_1$ is the maximal, we use $\rho' = \sigma_1 \otimes \sigma_1 \rho \sigma_1 \otimes \sigma_1$ as the input state, the action of the channel $E$ on $\rho'$ will be equivalent to a channel $E'$ applying on $\rho$ in the sense of output entropy. The channel $E'$ is produced from $E$ by exchanges of $q_0 \leftrightarrow q_1, q_2 \leftrightarrow q_3$. The two channels are equivalent for our problem of classical capacity. If $q_2$ or $q_3$ is the maximal, we can use $\sigma_2 \otimes \sigma_2 \rho \sigma_2 \otimes \sigma_2$ or $\sigma_3 \otimes \sigma_3 \rho \sigma_3 \otimes \sigma_3$ as the input state. Thus the channel can be regularized to channel with $q_0$ being the maximal among $q_0, q_1, q_2, q_3$. Without lose of generality, we can suppose $q_1 \geq q_0, q_3$. The Pauli channel is regularized to

$$q_0 \geq q_1 \geq q_2, q_3. \tag{15}$$

It follows immediately that $A_0 \geq A_1 \geq A_2, A_3$ and $A_4 \geq 0, A_5 \geq 0$. The numerical steepest descending calculation exhibits that the extremal input pure state will be with the form of $a_0 = a_1 = 0$ or $a_2 = a_3 = 0$. Thus the matrix $B$ is reduced to a $2 \times 2$ matrix and two diagonal elements. When $a_2 = a_3 = 0$,

$$B = \begin{bmatrix}
A_0 a_0^2 + A_1 |a_1|^2, & A_4 a_0^2 a_1^2 + A_5 a_0^2 a_1^2, \\
A_4 a_0^2 a_1 + A_5 a_0 a_1^2, & A_1 |a_0|^2 + A_0 |a_1|^2
\end{bmatrix} \oplus \lambda_2 \oplus \lambda_3. \tag{16}
$$

with $\lambda_2 = A_2 |a_0|^2 + A_3 |a_1|^2$, $\lambda_3 = A_3 |a_0|^2 + A_2 |a_1|^2$. Let $a_0 = \cos \theta, a_1 = \sin \theta e^{i\varphi}$, and using the above lemma to eliminate $\varphi$, the eigenvalues of $B$ are

$$
\begin{align*}
\lambda_{0,1} &= \frac{1}{2}[A_0 + A_1 \pm \sqrt{(A_0 - A_1)^2 \cos^2 2\theta + \sin^2 2\theta (A_4 + A_5)^2}], \\
\lambda_{2,3} &= \frac{1}{2}[A_2 + A_3 \pm (A_2 - A_3) \cos 2\theta].
\end{align*}
$$

(17) (18)

If $A_0 - A_1 > A_4 + A_5$, we have $\lambda_1 > A_1 \geq \max(A_2, A_3) \geq \lambda_2, \lambda_3$. The descending order of the eigenvalues are $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, where we have denoted $\lambda'_{2,3} = \frac{1}{2}[A_2 + A_3 \pm (A_2 - A_3) \cos 2\theta]$ without lose of generality.
Suppose the extremal state is with $\sin^2 2\theta = 0$, the descending order of the extremal eigenvalues are $\lambda^e = (\lambda_0, \lambda_1, \lambda_2')$, with $\lambda_2' = \max(A_2, A_3), A_3' = \min(A_2, A_3)$. It is not difficult to see that $\lambda < \lambda^e$.

If $A_0 - A_1 < A_4 + A_5$, the descending order of the eigenvalues are $\lambda = (\lambda_0, \lambda_1, \lambda_2', \lambda_3')$. When $\lambda_1 > \lambda_2', \lambda = (\lambda_0, \lambda_1, \lambda_2', \lambda_3')$. Suppose the extremal state is with $\cos 2\theta = 0$, the descending order of the eigenvalues are $\lambda^e = (A_0', A_1', \frac{1}{2}(A_2 + A_3), \frac{1}{2}(A_2 + A_3))$ with $A_0' = \frac{1}{2} (|A_0 + A_1 + (A_4 + A_5)|)$. We have $\lambda_0 < A_0', \lambda_0 + \lambda_1 = A_0' + A_1'$, but $\lambda_2' > \frac{1}{2}(A_2 + A_3)$. When $\lambda_1 < \lambda_2'$, the situation may even be worse. Thus we can not conclude that $\lambda^e$ is extremal by majorization. Only in the situation of $A_2 = A_3$, that is $q_0 = q_1$ or $q_2 = q_3$, we can conclude that $\lambda^e$ is extremal by majorization. For most of channels, our numerical result exhibits that the extremal state is with $\cos 2\theta = 0$, thus the input state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ or $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. There exist the situation that Bell state maximize the capacity even when $A_0 - A_1 < A_4 + A_5$, although it takes place scarcely in our numerical calculation.

Thus $A_0 - A_1 > A_4 + A_5$ is only the sufficient condition (not necessary) of entanglement enhanced capacity. It can be written as

$$\mu > \frac{2q_0q_1 - q_2^2 - q_3^2}{q_2 + q_3 + 2q_0q_1 - q_2^2 - q_3^2}. \quad (19)$$

### 4 Perturbation Verification

The numerical calculation give rise to the results that the extremal input pure state is either Bell state or superposition of two Bell states (with equal probabilities, a separable state). In the Bell state basis, they are $[1, 0, 0, 0]$ are $\frac{1}{\sqrt{2}}[1, 1, 0, 0]$ or the similar. For a perturbation to Bell state input $\Phi^+$, suppose the perturbed input is $[a_0, a_1, a_2, a_3]$ with $||1 - a_0, a_1, a_2, a_3|| \rightarrow 0$, we have The matrix $B$ now can be written as $B_0 + B'$, with

$$B_0 = \text{diag}\{A_0, A_1, A_2, A_3\}, \quad (20)$$

The first order perturbations to the eigenvalues of $B_0$ are $B'_{ii}$ ($i=0, 1, 2, 3$).

$$B'_{ii} = (A_0, A_1, A_2, A_3) U_i (|a_0|^2, |a_1|^2, |a_2|^2, |a_3|^2)^T - A_i. \quad (21)$$

Keep in mind that $A_0 \geq A_1 \geq A_2, A_3$, we have $B'_{00} = (|a_0|^2 - 1) A_0 + |a_1|^2 A_1 + |a_2|^2 A_2 + |a_3|^2 A_3 = |a_1|^2 (A_1 - A_0) + |a_2|^2 (A_2 - A_0) + |a_2|^2 (A_3 - A_0) \leq 0$; $B'_{01} = (|a_0|^2 - 1) (A_0 + A_1) + |a_1|^2 (A_0 + A_1) + |a_2|^2 (A_2 + A_3) + |a_3|^2 (A_2 + A_3) = -(|a_2|^2 + |a_3|^2) (A_0 + A_1 - A_2 + A_3) \leq 0$. Suppose $A_3 < A_2$, $B'_{22} = -B'_{33} = A_3 - (A_3 |a_0|^2 + A_2 |a_1|^2 + A_3 |a_2|^2 + A_3 |a_3|^2) \leq 0$. If $A_3 > A_2$, we get $B'_{00} + B'_{11} + B'_{33} = 0$. In either cases, the descending order eigenvalues of $B_0$ majorizes that of $B_0 + B'$. So that the perturbation will increase the entropy of the output state.

For a perturbation to the product input state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(\Phi^+ + \Phi^+)$, suppose the perturbed input is $[a_0, a_1, a_2, a_3]$ with $||1/\sqrt{2} - a_0, 1/\sqrt{2} - a_1, a_2, a_3|| \rightarrow 0$, the unperturbed matrix is

$$B_0 = \frac{1}{2} \left[ \begin{array}{cccc} A_0 + A_1, & A_4 + A_5, & A_4 + A_5, & A_1 + A_0 \\ A_4 + A_5, & A_1 + A_0, & A_0 + A_1, & A_4 + A_5 \end{array} \right] \oplus \frac{1}{2} (A_2 + A_3) + \frac{1}{2} (A_2 + A_3).$$

with eigenvalues $\lambda_{0,1} = \frac{1}{2} [A_0 + A_1 \pm (A_4 + A_5)], \lambda_{2,3} = \frac{1}{2} (A_2 + A_3)$. The first order perturbations to the the first two eigenvalues of $B_0$ are

$$\lambda'_{0,1} = \frac{1}{2} [-(1 - |a_0|^2 - |a_1|^2) (A_0 + A_1 - A_2 - A_3) \pm (a_0a_1^* + a_0a_1^* - 1)(A_4 + A_5)] \quad (22)$$

The last two eigenvalues of $B_0$ are degenerate, thus the degenerate perturbation should be applied. The perturbation to the eigenvalues are

$$\lambda'_{2,3} = \frac{1}{2} [(1 - |a_0|^2 - |a_1|^2) (A_0 + A_1 - A_2 - A_3) \pm \sqrt{C}], \quad (24)$$

4
where $C = [(|a_0|^2 - |a_1|^2)(a_2 - A_3) + (|a_2|^2 - |a_3|^2)(A_0 - A_1)^2 + 4||a_2||^2|a_3|^2(A_4^2 + A_5) + (a_2^2a_3^2 + a_2^2a_3^2)a_4A_5].$

We can easily see that $\lambda_0' \leq 0$. If $A_0 + A_1 - A_4 - A_5 > A_2 + A_3$, that is

$$\mu > 1 - \frac{1}{2(q_0 + q_1)},$$

(25)

the descending order of eigenvalues of $B_0$ is $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$. We need to verify that $\lambda_0' + \lambda_1' \leq 0$ and $\lambda_0' + \lambda_2' + \lambda_3' = -\lambda_5' \leq 0$. The former is evident. The condition $\lambda_5' \geq 0$ should be true for all possible perturbations. As far as $A_2 \neq A_3$, we can choose the situation of $|a_2| = |a_3| = 0$ while $|a_0| \neq |a_1|$, so that $\lambda_3' < 0$. When $A_2 = A_3$, the condition for $\lambda_3' \geq 0$ is

$$4(A_0 - A_2)(A_1 - A_2) > (A_4 + A_5)^2.$$  

(26)

If $A_0 + A_1 - A_4 - A_5 < A_2 + A_3$, the descending order of eigenvalues of $B_0$ is $(\lambda_0, \lambda_2, \lambda_3, \lambda_1)$. We need to verify that $\lambda_0' + \lambda_2' \leq 0$ and $\lambda_0' + \lambda_2' + \lambda_3' = -\lambda_1' \leq 0$. The later is evident by the fact that $A_4 + A_5 > A_0 + A_1 - A_4 - A_5$ and $a_0a_1^* + aa_1^* \leq |a_0|^2 + |a_1|^2$. The condition $\lambda_0' + \lambda_2' \leq 0$ requires

$$(1 - a_0a_1^*-a_0a_1^*)(A_4 + A_5) \geq \sqrt{C}.$$  

(27)

When $A_2 \neq A_3$, we can choose the situation of $|a_2| = |a_3| = 0$ to verify that $(27)$ can not be true in general. For the situation of $A_2 = A_3$, the condition $(27)$ will reduce to $A_4 + A_5 > A_0 - A_1$ which is just the requirement of separable state achieving the capacity.

Hence, when $A_2 \neq A_3$, even in the sense of perturbation, majorization can not be used to prove the extremal property of the product state as an input to the channel.

5 Conclusion

The classical capacity of Pauli channel is investigated with the representation of Bell states. A new proof is given for the capacity of some strictly solvable symmetric Pauli channel ($q_0 = q_1$ and $q_2 = q_3$), the full expression of the entanglement enhanced classical condition is $(22)$. For the most general Pauli channel, the condition for entanglement enhanced classical capacity is given. The condition is a sufficient condition but is not necessary. When in the situation of $A_2 = A_3$, that is, the two big $q_i$ are equal or the two small $q_i$ are equal, the condition is also necessary under a group of additional inequalities which are obtained by perturbation theory. This comprise the well studied channel of $q_1 = q_2 = q_3 = (1 - x)/3, q_0 = x$. For this channel, it should be mentioned that the full expression of the entanglement enhanced classical capacity condition is $\mu > \frac{14x - 11}{14x - 11/3}$.

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