Torsion theories of simplicial groups with truncated Moore complex

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Abstract
We introduce a linearly ordered lattice $\mu(Grp)$ of torsion theories in simplicial groups. The torsion theories are defined where the torsion/torsion-free subcategories are given by the simplicial groups with bounded above/below Moore complex, respectively. These torsion theories extend naturally the torsion theories in internal groupoids in groups. Connections of this lattice with the homotopy groups are established since the homotopy groups of a simplicial group can be calculated as the quotients of torsion subobjects.

1 Introduction

The notion of semi-abelian category [16] allows a categorical and unified treatment of the categories of groups, rings, Lie-algebras and other non-abelian categories in a similar way as abelian categories generalise abelian groups and categories of modules. Torsion theories were originally introduced for abelian categories by Dickson, and have been generalized by several authors to different non-abelian categories as for example in [4], [7] and [18].

For a torsion theory in a semi-abelian category $X$ we mean a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that:

1. any morphism $f : T \to F$ with $T$ in $\mathcal{T}$ and $F$ in $\mathcal{F}$ is the zero morphism;
2. for any object $X$ in $X$ there is a short exact sequence

$$0 \longrightarrow T_X \longrightarrow X \longrightarrow F_X \longrightarrow 0$$

with $T_X$ in $\mathcal{T}$ and $F_X$ in $\mathcal{F}$.

An internal groupoid $X$ in $X$ is a diagram:

$$X_2 \xrightarrow{p_2} X_1 \xrightarrow{d_1} X_0$$

$$\xleftarrow{p_0} \xleftarrow{d_0}$$

where

$$X_2 \xrightarrow{p_1} X_1$$

$$X_1 \xrightarrow{d_0} X_0$$

is a pullback square. The objects $X_0$ and $X_1$ are called the ‘objects of objects’ and the ‘object of arrows’ of $X$, the morphisms $d_0, d_1$ are called ‘domain’ and ‘codomain’ and the morphisms $p_0, p_1, p_2, d_0, d_1, s_0$ satisfy the usual equations that determine a category. The category $Grpd(\mathcal{X})$ of internal groupoids in a semi-abelian category $\mathcal{X}$, which is itself semi-abelian, exhibits two examples of non-abelian torsion theories. The first is given by the pair $(Ab(\mathcal{X}), Eq(\mathcal{X}))$ where $Ab(\mathcal{X})$ is the
category of internal abelian objects in $X$ and $Eq(X)$ is the category of equivalence relations, i.e. internal groupoids where the induced morphism $(d_0, d_1) : X_1 \to X_0^2$ is monic \[^4\]. The second example is given by $(\text{Conn}(\text{Grpd}(X)), \text{Dis}(X))$, where $\text{Conn}(\text{Grpd}(X))$ is the category of connected internal groupoids and $\text{Dis}(X)$ is the category of discrete groupoids. Since for an internal groupoid $X$ the nerve $N(X)$ is a simplicial object in $X$ it is natural to ask if there are torsion theories in simplicial objects such that they expand or generalize those of internal groupoids.

In section 2, we recall the basics theory of torsion theories in semi-abelian categories. Section 3 and 4 introduce two different families of torsion theories, $COK_n$ and $KER_n$, in the category of proper chain complexes and we exhibits some connections with the homological aspects of chain complexes. Section 5 and 6 introduce the torsion theories $\mu_{n \geq}$, $\mu_{\geq n}$ in simplicial groups whose associated Moore complex behave as those in proper chains. Section 7 studies the homotopy groups of the simplicial groups defined by the torsion theories of the lattice $\mu(\text{Grp})$. In particular, the homotopy groups of a simplicial group $X$ can be studied using torsion subobjects.

2 Torsion theories in semi-abelian categories

2.1 Notation. By a regular category $X$ we mean a finitely complete category with coequalizers of kernel pairs with the property that any morphism $f : X \to Y$ in $X$ factors as a regular epimorphism $e_f$ followed by a monomorphism $m_f$:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{e_f} & & \downarrow{m_f} \\
\text{f}(X) & & \\
\end{array}
$$

and these factorizations are pullback stable. As usual, we will call the subobject represented by $m_f$ the image of $f$. A category $X$ is pointed if it has a zero object 0, i.e. an object which is both initial and terminal. For any pair of objects $X, Y$ in $X$ the unique morphism $X \to Y$ that factors through the zero object will be denoted by 0.

A regular category $X$ is called (Barr-)exact if any equivalence relation is a kernel pair $Eq(f)$ for some morphism $f$ in $X$ \[^1\].

Definition 2.2. \[^{16}\] A category $X$ is called semi-abelian if it is pointed, (Barr-)exact, protomodular in the sense of Bourn \(^{[3]}\) and has binary coproducts.

In a semi-abelian category, a short exact sequence is a pair of composable morphisms $(k, p)$, as in the diagram

$$
0 \longrightarrow K \xrightarrow{k} X \xrightarrow{p} Y \longrightarrow 0
$$

such that $k = \ker(p)$ is the kernel of $p$ and $p = \cok(k)$ is the cokernel of $k$. In such a short exact sequence the object $Y$ will be denoted as $X/K$. Recall that in a semi-abelian category $X$ regular epimorphisms are normal epimorphisms, that is cokernels of some morphisms in $X$.

We will need the following results.

Lemma 2.3. \[^{2}\] Let $X$ be a semi-abelian category. Given two normal subobjects $k : K \to A$ and $l : L \to A$ such that $k \leq l$, i.e. $k$ factors through $l$, then there is a short exact sequence:

$$
0 \longrightarrow L/K \longrightarrow A/K \longrightarrow A/L \longrightarrow 0.
$$

Proposition 2.4. \[^{16}\] Let $X$ be a semi-abelian category then it satisfies the following property: Given a commutative diagram in $X$:

$$
\begin{array}{ccc}
A & \xrightarrow{m} & B \\
\downarrow{p} & & \downarrow{q} \\
C & \xrightarrow{n} & D
\end{array}
$$

2
with \(p\) and \(q\) normal epimorphisms, \(m\) a normal monomorphism, and \(n\) a monomorphism; then \(n\) is a normal monomorphism.

Torsion theories can be defined in a more general context, but in this article we will restrict to semi-abelian categories.

**Definition 2.5.** Let \(\mathcal{X}\) be a semi-abelian category. A *torsion theory* in \(\mathcal{X}\) is a pair \((\mathcal{T}, \mathcal{F})\) of full and replete subcategories of \(\mathcal{X}\) such that:

1. A morphism \(F : T \to F\) with \(T\) in \(\mathcal{T}\) and \(F\) in \(\mathcal{F}\) is a zero morphism.
2. For any object \(X\) in \(\mathcal{X}\) there is a short exact sequence:
   \[
   0 \longrightarrow T_X \overset{\epsilon_X}{\longrightarrow} X \overset{n_X}{\longrightarrow} F_X \longrightarrow 0
   \]
   with \(T_X\) in \(\mathcal{T}\) and \(F_X\) in \(\mathcal{F}\) (which is necessarily unique up to isomorphism).

In a torsion theory \((\mathcal{T}, \mathcal{F})\), \(\mathcal{T}\) is the torsion category whose objects are called torsion objects, and similarly \(\mathcal{F}\) is the torsion-free category of the torsion theory. Torsion subcategories are normal mono-coreflective subcategories of \(\mathcal{X}\), i.e. coreflective subcategories such that each component \(\epsilon_X\) of the counit \(\epsilon\) is a normal monomorphism, while torsion-free subcategories are normal epi-reflective subcategories, so that each component \(\eta_X\) is a normal epimorphism:

\[
\mathcal{T} \overset{J}{\leftarrow} \mathcal{X} \overset{F}{\leftarrow} \mathcal{F} .
\]

The \(X\)-component of the counit \(\epsilon\) of \(J \dashv T\) and of the unit \(\eta\) of \(F \dashv I\) both appear in the short exact sequence \([1]\). A subcategory \(\mathcal{A}\) of \(\mathcal{X}\) is closed under extensions in \(\mathcal{X}\) if every time we have a short exact sequence

\[
0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0
\]

with \(A\) and \(B\) in \(\mathcal{A}\) then \(X\) belongs to \(\mathcal{A}\). In a torsion theory both \(\mathcal{T}\) and \(\mathcal{F}\) are closed under extensions in \(\mathcal{X}\) \([2]\).

**Definition 2.6.** Let \(\mathcal{X}\) be a semi-abelian category. A *preradical* in \(\mathcal{X}\) is a normal subfunctor \(\sigma : r \to \text{Id}_\mathcal{X}\) of the identity functor of \(\mathcal{X}\), i.e. for all object \(X\) we have a normal monomorphism \(\sigma_X : r(X) \to X\) and for every morphism \(f : X \to Y\) a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\sigma_X \downarrow & & \sigma_Y \downarrow \\
r(X) & \xrightarrow{r(f)} & r(Y).
\end{array}
\]

Moreover, a preradical \(r\) is called:

- *idempotent* if \(r(r(X)) = r(X)\) for all objects \(X\);
- *a radical* if \(r(X/r(X)) = 0\) for all objects \(X\);
- *hereditary* if for every monomorphism \(f : X \to Y\) diagram \([3]\) is a pullback.

Given a preradical \(r\) we can consider the \(r\)-torsion subcategory \(\mathcal{T}_r\) and the \(r\)-torsion-free subcategory \(\mathcal{F}_r\) of \(\mathcal{X}\):

\[
\mathcal{T}_r = \{ X \in \mathcal{X} \mid r(X) \cong X \} \quad \text{and} \quad \mathcal{F}_r = \{ X \in \mathcal{X} \mid r(X) \cong 0 \}.
\]
In general, the pair \((T_r, F_r)\) only satisfies axiom TT1 of a torsion theory. Conversely, a torsion theory \((T, F)\) defines an idempotent radical
\[
t = JT : X \longrightarrow X.
\]
In fact, there is a bijection:
\[
\{\text{torsion theories in } X\} \leftrightarrow \{\text{idempotent radicals in } X\}.
\]
It is easy to see that a hereditary preradical is always idempotent. Conversely, an idempotent preradical is hereditary if and only if the category \(T_r\) is closed under subobjects in \(X\), i.e. for every monomorphism \(m : X \rightarrow Y\) with \(Y\) in \(T_r\) then \(X\) is in \(T_r\). Furthermore, the previous bijection is restricted to a bijection:
\[
\{\text{hereditary torsion theories in } X\} \leftrightarrow \{\text{hereditary radicals in } X\}.
\]
A torsion theory \((T, F)\) is called\(\) \textit{hereditary} if \(T\) is closed under subobjects. Similarly, \((T, F)\) is called \textit{cohereditary} if \(F\) is closed under quotients, i.e. for every normal epimorphism \(p : X \rightarrow Y\) with \(X\) in \(F\) then so is \(Y\) in \(F\) (see [7]). It is also useful to recall that in any torsion theory \((T, F)\), \(T\) is always closed under quotients and \(F\) is closed under subobjects.

In order to characterize torsion-free subcategories among normal epi-reflective subcategories, it is useful to recall the following result, which was first proved in [4] for homological categories:

\textbf{Theorem 2.7.} ([4], [11], [8]) Let \(X\) be a semi-abelian category and \(F \dashv I : X \rightarrow \mathbb{A}\) a normal epi-reflective subcategory of \(X\) with unit \(\eta\), then the following are equivalent:

1. \(F\) is a torsion-free subcategory of \(X\);
2. the induced radical of \(F \dashv I\) is idempotent;
3. the reflector \(F : X \rightarrow \mathbb{A}\) is semi-left-exact;
4. the reflector \(F : X \rightarrow \mathbb{A}\) is normal, i.e. \(F(\ker(\eta_X)) = 0\) for every object \(X\) in \(X\).

Under these conditions the corresponding torsion category of \(F\) is given by the full subcategory \(T = \text{Ker}(F) = \{X \mid F(X) \cong 0\}\). So, \((\text{Ker}(F), F)\) is a torsion theory in \(X\).

\textit{Proof.} Equivalences (1) \(\Leftrightarrow\) (2) \(\Leftrightarrow\) (3) are proved in [11] and (3) \(\Leftrightarrow\) (4) is proved in [4].

\textbf{2.8.} Given torsion theories \((T, F)\) and \((S, G)\) in \(X\) with associated idempotent radicals \(\tau\) and \(\sigma\) we have that \(T \subseteq S\) if and only if \(G \subseteq F\), this allows us to define a partial order in the (possibly big) lattice \(\mathbb{X}^{\text{tors}}\) of torsion theories in \(X\):

\[
(T, F) \leq (S, G) \quad \text{if and only if} \quad T \subseteq S.
\]

In this case we have that \(\tau \leq \sigma\), so for an object \(X\) in \(X\) we have \(\tau(X) \leq \sigma(X)\). The lattice \(\mathbb{X}^{\text{tors}}\) has as bottom and top element the trivial torsion theories denoted as:

\[
0 := (0, X) \quad \text{and} \quad \mathbb{X} := (X, 0).
\]

Given preradicals \(\tau \leq \sigma\) we can define the quotient endofunctor as:
\[
\sigma/\tau : X \longrightarrow X, \quad \sigma/\tau(X) = \sigma(X)/\tau(X).
\]
and as consequence of Lemma [2,3] for each object \(X\) we have a short exact sequence:
\[
0 \longrightarrow \sigma(X)/\tau(X) \longrightarrow X/\tau(X) \longrightarrow X/\sigma(X) \longrightarrow 0.
\]
For abelian categories the next result is due to P. Gabriel ([24]). A localization of a category \( \mathbb{X} \) is a reflective subcategory \( L \dashv I : \mathbb{X} \to \mathbb{A} \) such that \( L \) preserves finite limits.

**Theorem 2.9.** Let \( L \dashv I : \mathbb{X} \to \mathbb{A} \) be a localization of a semi-abelian category \( \mathbb{X} \) with unit \( \eta \). The subcategories of \( \mathbb{X} \)

\[
\mathcal{T}_L = \text{ker}(L) = \{X \mid L(X) \cong 0\} = \{X \mid \eta_X = 0\}
\]

and

\[
\mathcal{F}_L = \{X \mid \eta_X : X \to IL(X) \text{ is monic}\}
\]

define a torsion theory \((\mathcal{T}_L, \mathcal{F}_L)\) in \( \mathbb{X} \).

**Proof.** First notice that for any object \( X \) the morphism \( X \to 0 \) factors through \( \eta_X \). Then, if \( \eta_X = 0 \) the morphism \( \eta_X \) factors through \( X \to 0 \) and, so \( L(X) \cong 0 \) since \( L \) is a reflection. Hence, for any object \( X \) we have that \( L(X) = 0 \) if and only if \( \eta_X = 0 \). TT1) For a morphism \( f : X \to Y \) consider the diagram given by the naturality of \( \eta \):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\eta_X \downarrow & & \downarrow \eta_Y \\
IL(X) & \xrightarrow{IL(f)} & IL(Y).
\end{array}
\]

Now, if \( \eta_X = 0 \) and \( \eta_Y \) is monic it is clear that \( f \) is the zero morphism.

TT2) Consider for an object \( X \) the normal epi-mono factorization \((p, m)\) of \( \eta_X \) and the short exact sequence

\[
0 \longrightarrow \text{ker}(\eta_X) \xrightarrow{k} X \xrightarrow{p} \eta_X(X) \longrightarrow 0
\]

(5)

To see that \( \text{ker}(\eta_X) \) is torsion, consider the commutative diagram:

\[
\begin{array}{ccc}
\text{ker}(\eta_X) & \xrightarrow{k} & X \\
\eta_{\text{ker}(\eta_X)} \downarrow & & \downarrow \eta_X \\
IL(\text{ker}(\eta_X)) & \xrightarrow{IL(k)} & IL(X).
\end{array}
\]

Since \( L \) preserves finite limits then \( IL(k) \) is monic and since \( \eta_X k = 0 \) this implies that \( \eta_{\text{ker}(\eta_X)} = 0 \).

To see that \( \eta_X(X) \) is torsion-free consider the diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & IL(X) \\
\eta_X \downarrow & & \downarrow \eta_{IL(X)} \\
IL(X) & \xrightarrow{IL(\eta_X)} & IL(IL(X)).
\end{array}
\]

Notice that since \( \mathbb{A} \) is a reflective subcategory then \( IL(\eta_X) \) and \( \eta_{IL(X)} \) are isomorphisms. Finally, \( \eta_{\eta_X(X)} \) is also a monomorphism. 

\[\square\]
It is also worth mentioning that, under the assumptions from above, a localization \( L : \mathcal{X} \to \mathcal{A} \) induces a preradical on \( \mathcal{X} \) as \( r = \ker(\eta) \), so we have \( T_L = T_r \).

**Corollary 2.10.** The torsion theory \((T_L, F_L)\) induced by a localization \( L \dashv i \) of a semi-abelian category \( \mathcal{X} \) is hereditary.

**Proof.** Since \( L \) preserves finite limits it preserves monomorphisms so if \( m : S \to X \) with \( L(X) = 0 \) then \( L(S) = 0 \).

**Lemma 2.11.** Let \((T, F)\) be a hereditary torsion theory in a semi-abelian category \( \mathcal{X} \). Then \( T \) is closed under finite limits in \( \mathcal{X} \). In particular, \( T \) is closed under kernel pairs of morphisms in \( \mathcal{X} \).

**Proof.** Since \( T \) is closed under kernels of arrows in \( T \), we only need to prove that \( T \) is closed under pullbacks of morphisms in \( T \). Consider the commutative diagram

\[
\begin{array}{ccc}
\ker(p_1) & \longrightarrow & P \\
\downarrow p_0 & & \downarrow p_0 \\
\ker(f) & \longrightarrow & A
\end{array}
\]

with \( f \) and \( g \) morphisms in \( T \). Since \( P \) is a pullback then \( \ker(p_1) \cong \ker(f) \). Now, since \( T \) is closed under subobjects, if \( A \) is torsion then \( \ker(f) \) is also torsion. Consider \( e, m \) the normal epi/mono factorization of \( p_1 \) and the short exact sequence:

\[
0 \longrightarrow \ker(p_1) \longrightarrow P \rightarrow p_1(P) \rightarrow 0.
\]

Then \( p_1(P) \) is a subobject of \( C \) so it is torsion, and finally, since \( T \) is closed under extension \( P \) is torsion.

A *quasi-hereditary* torsion theory \((T, F)\) in \( \mathcal{X} \) is a torsion theory such that \( T \) is closed under regular subobjects, i.e. if \( e : X \to T \) is an equalizer with \( T \) torsion then \( X \) is torsion. In [14] quasi-hereditary torsion theories are studied in homological categories.

**Theorem 2.12.** [14] Let \((T, F)\) be a torsion theory in a homological category \( \mathcal{X} \). The following are equivalent:

1. \((T, F)\) is quasi-hereditary.
2. The associated idempotent radical \( t : \mathcal{X} \to \mathcal{X} \) preserves finite limits.
3. The associated idempotent radical \( t : \mathcal{X} \to \mathcal{X} \) preserves equalizers.
4. For every regular subobject \( e : E \to A \) in \( \mathcal{X} \) then \( F(e) \) is a monomorphism in \( \mathcal{F} \).

**Lemma 2.13.** Let \((T, F)\) be a torsion theory in a semi-abelian category \( \mathcal{X} \). Then \( T \) is an exact category.

**Proof.** We will first prove that an arrow \( q \) in \( T \) is a regular epimorphism in \( \mathcal{X} \) if and only if it is a regular epimorphism in \( T \). Clearly, if \( q \) is a regular epimorphism in \( T \) and the inclusion \( J : T \to \mathcal{X} \) preserves colimits then \( q \) is a regular epimorphism in \( \mathcal{X} \). Now, if \( q \) is a regular epimorphism in \( \mathcal{X} \) it is a coequalizer of its kernel pair \( K[q] \) in \( \mathcal{X} \), and since \( T \) is closed under kernel pairs in \( \mathcal{X} \) we have the isomorphism \( Eq(q) \cong t(Eq(q)) \), so \( q \) is a regular epimorphism in \( T \).

To prove pullback stability of regular epimorphism consider the pullback diagram in \( T \):

\[
\begin{array}{ccc}
P & \longrightarrow & X \\
\downarrow p' & & \downarrow p \\
A & \longrightarrow & B
\end{array}
\]
with $p$ a regular epimorphism in $\mathcal{T}$. Since the inclusion $i: \mathcal{T} \to \mathcal{X}$ preserves pullbacks and quotients, we have that $p$ is a regular epimorphism in $\mathcal{X}$, and so is $p'$ in $\mathcal{T}$.

Finally, since $\mathcal{T}$ is closed under quotients and $\mathcal{X}$ is an exact category, any equivalence relation in $\mathcal{T}$ must be effective and $\mathcal{T}$ is an exact category.

**Theorem 2.14.** Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in a semi-abelian category $\mathcal{X}$. Then $\mathcal{T}$ is a semi-abelian category.

**Proof.** First since $\mathcal{T}$ is coreflective in $\mathcal{X}$ it is complete and cocomplete as well as pointed. By Lemma 2.13 $\mathcal{T}$ is exact. Finally, by Theorem 2.12 the full inclusion $J: \mathcal{T} \to \mathcal{X}$ preserve finite limits and hence short split exact sequences, so if $\mathcal{X}$ is protomodular then so is $\mathcal{T}$.

This theorem admits a dual version. A Birkhoff subcategory $\mathcal{A}$ of a regular category $\mathcal{X}$, is a full regular epi-reflective subcategory that is closed under subobjects and quotients in $\mathcal{X}$. It is known that if $\mathcal{X}$ semi-abelian then so is any Birkhoff subcategory $\mathcal{A}$.

**Corollary 2.15.** Let $(\mathcal{T}, \mathcal{F})$ be a cohereditary torsion theory in a semi-abelian category $\mathcal{X}$. Then $\mathcal{F}$ is a semi-abelian category.

**Proof.** If $(\mathcal{T}, \mathcal{F})$ is cohereditary, $\mathcal{F}$ is closed under quotients in $\mathcal{X}$, so $\mathcal{F}$ is a Birkhoff subcategory of $\mathcal{X}$.

### 3 Torsion theories in chain complexes

Throughout this section $\mathcal{X}$ will denote a semi-abelian category.

A chain complex $M$ in $\mathcal{X}$ is a family of morphisms $\{\delta_i : M_i \to M_{i-1}\}_{i \in \mathbb{Z}}$ with the condition $\delta_i \delta_{i+1} = 0$ for all $i$. A morphism of chain complexes $f: M \to N$ is a family of morphisms $f_i: M_i \to N_i$ such that $f_{i-1} \delta_i = \delta_i f_i$ for all $i$. For a chain complex $M$ and for each $i$ we will write $e_i$ and $m_i$ for the normal epi/mono factorization of each $\delta_i$:

$$
\begin{array}{ccc}
M_i & \xrightarrow{\delta_i} & M_{i-1} \\
e_i & \downarrow{m_i} & \\
\delta_i(M_i) & & \\
\end{array}
$$

And we call a chain complex $M$ proper if each $\delta_i$ is a proper morphism i.e. each $m_i$ is a normal monomorphism for each $i$. We will write $ch(\mathcal{X})$ for the category of chain complexes in $\mathcal{X}$ and $pch(\mathcal{X})$ for the subcategory of proper chain complexes. In [12] it is noticed that since $\mathcal{X}$ is a semi-abelian category then $ch(\mathcal{X})$ is also semi-abelian, but this is not the case for $pch(\mathcal{X})$, since it may not have kernels. However, $pch(\mathcal{X})$ does have cokernels as follows.

**Lemma 3.1.** The category $pch(\mathcal{X})$ has cokernels and they are computed as in $ch(\mathcal{X})$.

**Proof.** For a morphism $f: M \to N$ of proper chain complexes $(M, d), (N, \delta)$ consider the commutative diagram for each $i$:

$$
\begin{array}{ccc}
M_i & \xrightarrow{f_i} & N_i \\
d_i & \downarrow{\delta_i} & \\
M_{i-1} & \xrightarrow{f_{i-1}} & N_{i-1} \\
\end{array}
$$

$$
\begin{array}{ccc}
M_i & \xrightarrow{p} & \text{cok}(f_i) \\
\downarrow{e_i} & & \downarrow{e_i'} \\
\delta_i(N_i) & \xrightarrow{q} & \delta_i'(\text{cok}(f_i)) \\
\downarrow{m_i} & & \downarrow{m_i'} \\
\text{cok}(f_{i-1}) & & \\
\end{array}
$$
where $\delta'_{i}$ is induced by universal property of the cokernel $p$ and $m_{i}$, $e_{i}$ and $m'_{i}$, $e'_{i}$ are the normal epi-mono image factorizations of $\delta_{i}$ and $\delta'_{i}$ respectively. Now, since taking images is functorial we have $q'$ such that $q'e_{i} = e'_{i}p$ and $m'_{i}q' = qm_{i}$, then $q'$ is a normal epimorphism since $p$ and $e'_{i}$ are also normal epimorphisms. Finally, by Proposition 2.4 if $m_{i}$ is a normal monomorphism and $m'_{i}$ is a monomorphism, then $m'_{i}$ is a normal monomorphism. So, $\delta'_{i}$ is a proper morphism and it is the cokernel of $f$ in $pch(\mathcal{X})$.

By a short exact sequence in $pch(\mathcal{X})$ we mean a short exact sequence in $ch(\mathcal{X})$ such that every object is a proper chain complex. We will introduce well-known functors in very different settings of algebraic topology that still makes sense in our context, we will follow the terminology of [5].

3.2. Let $ch(\mathcal{X})_{n}\geq$ be the category of $n$-truncated (above) chain complexes, i.e. chain complexes defined for degrees $n \geq i$ for a fixed $n \in \mathbb{Z}$. We can identify $ch(\mathcal{X})_{n}\geq$ with the full subcategory of $ch(\mathcal{X})$ of chain complexes with $M_{i} = 0$ for $i > n$. Actually, we have the functors:

- $\text{tr}_{n} : ch(\mathcal{X}) \longrightarrow ch(\mathcal{X})_{n}\geq$ is the canonical (above) truncation:
  \[
  \text{tr}_{n}(M) = M_{n} \longrightarrow M_{n-1} \longrightarrow M_{n-2} \longrightarrow \ldots
  \]

- $\text{sk}_{n} : ch(\mathcal{X})_{n}\geq \longrightarrow ch(\mathcal{X})$ is the canonical inclusion or skeleton functor:
  \[
  \text{sk}_{n}(M) = \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow M_{n} \longrightarrow M_{n-1} \longrightarrow \ldots
  \]

- $\text{cosk}_{n} : ch(\mathcal{X})_{n}\geq \longrightarrow ch(\mathcal{X})$ the coskeleton functor is given by:
  \[
  \text{cosk}_{n}(M) = \ldots \longrightarrow 0 \longrightarrow \ker(\delta_{n}) \overset{k(\delta_{n})}{\longrightarrow} M_{n} \overset{\delta_{n}}{\longrightarrow} M_{n-1} \longrightarrow \ldots
  \]

- $\text{cot}_{n} : ch(\mathcal{X}) \longrightarrow ch(\mathcal{X})_{n}\geq$ the (above) cotruncation functor:
  \[
  \text{cot}_{n}(M) = \text{Cok}(\delta_{n+1}) \overset{\delta'_{n}}{\longrightarrow} M_{n-1} \longrightarrow M_{n-2} \longrightarrow \ldots,
  \]

where $\delta'_{n}$ is induced by $\delta_{n} : M_{n} \rightarrow M_{n-1}$ and the universal property of $\text{cok}(\delta_{n+1})$.

This functors give a string of adjunctions:

\[
\begin{array}{cccc}
\text{cot}_{n} & \dashv & \text{sk}_{n} & \dashv \text{tr}_{n} & \dashv \text{cosk}_{n} \quad \begin{array}{c}
\begin{pmatrix}
\text{cot}_{n} \\
\text{sk}_{n} \\
\text{tr}_{n} \\
\text{cosk}_{n}
\end{pmatrix} \\
\begin{pmatrix}
\begin{pmatrix}
\text{cot}_{n} \\
\text{sk}_{n} \\
\text{tr}_{n} \\
\text{cosk}_{n}
\end{pmatrix}
\end{pmatrix}
\end{array}
\end{array}
\]

We will write $\text{Sk}_{n} = \text{sk}_{n}\text{tr}_{n}$, $\text{Cosk}_{n} = \text{cosk}_{n}\text{tr}_{n}$ and $\text{Cot}_{n} = \text{sk}_{n}\text{cot}_{n}$.

Notice that for abelian categories the cotruncation functor above is exactly L. Illusie’s truncation functor in [15].

Lemma 3.3. The category $ch(\mathcal{X})_{n-1}\geq$ is a normal epireflective subcategory of $ch(\mathcal{X})$ with the adjunction $\text{cot}_{n-1} \dashv \text{sk}_{n-1}$. Moreover, the category $ch(\mathcal{X})_{n-1}\geq$ is closed under extensions in $ch(\mathcal{X})$. 8
Proof. For a chain complex \( M \) the unit \( \eta_M \) of \( \text{cot}_{n-1} \dashv \text{sk}_{n-1} \) is given by:

\[
\begin{array}{c}
M = \\
\eta_M \\
\text{Cot}_{n-1}(M)
\end{array}
\begin{array}{ccc}
\rightarrow & M_n & \rightarrow & M_{n-1} & \rightarrow & M_{n-2} & \rightarrow & \ldots \\
\delta_n & \delta_{n-1} & \delta_{n-2} & \ldots \\
\rightarrow & \text{cok}(\delta_n) & \rightarrow & \text{Cok}(\delta_n) & \rightarrow & \ldots \\
\rightarrow & 1 & \rightarrow & \ldots \\
\rightarrow & 0 & \rightarrow & \ldots \\
\rightarrow & \delta'_{n-1} & \rightarrow & \delta'_{n-2} & \rightarrow & \ldots \\
\end{array}
\]

which is a component-wise normal epimorphism, and hence, \( \eta_M \) is a normal epimorphism in \( \text{ch}(\mathcal{X}) \).

Since a short exact sequence in \( \text{ch}(\mathcal{X}) \) is a component-wise short exact sequence then it is clear that \( \text{ch}(\mathcal{X})_{n-1} \geq \) is closed under extension in \( \text{ch}(\mathcal{X}) \).

3.4. In [18] conditions are given for a normal epireflective subcategory closed under extensions to be a torsion-free category. Here, \( \text{ch}(\mathcal{X})_{n-1} \geq \) provides a counter-example of this situation, in the sense that \( \text{ch}(\mathcal{X})_{n-1} \geq \) is a normal epireflective subcategory closed under extensions of \( \text{ch}(\mathcal{X}) \) that is not a torsion-free subcategory. Indeed, the functor \( \text{cot}_{n-1} \) is not normal.

For example, we can consider the truncated case of the category \( \text{Arr}(\mathcal{X}) \) of arrows in \( \mathcal{X} \) and \( \text{cot}_0 = \text{cok} : \text{Arr}(\mathcal{X}) \rightarrow \mathcal{X} \) and the dihedral group \( D_4 = \langle a, b \mid a^2 = b^4 = 0, aba = b^{-1} \rangle \). Let \( X = \langle a \rangle \rightarrow D_4 \) be the inclusion. Then \( \eta_X \) is given by the vertical morphisms in the diagram

\[
\begin{array}{c}
x \\
\eta_X, 1 \\
0
\end{array}
\begin{array}{c}
D_4 \\
\eta_X, 0 \\
D_4/\langle a, b^2 \rangle
\end{array}
\]

where \( \text{ker}(\eta_X) \) is the inclusion \( \langle a \rangle \rightarrow \langle a, b^2 \rangle \) which does not have trivial kernel, so \( \text{cot}_0(\text{ker}(\eta_X)) \) is not trivial.

However, we obtain a torsion theory when restricted to the case of proper chains and also \( \text{cot}_{n-1} \) will be a normal functor.

Lemma 3.5. For each \( n \in \mathbb{Z} \) the adjunction \( \text{cot}_{n-1} \dashv \text{sk}_{n-1} \) can be restricted to proper chains:

\[
\text{cot}_{n-1} \dashv \text{sk}_{n-1} : \text{pch}(\mathcal{X}) \perp \text{pch}(\mathcal{X})_{n-1} \geq .
\]

Proof. It suffices to prove that the \((n-1)\)-cotruncation of a proper chain complex \( M \) is again proper. Indeed, consider the diagram

\[
\begin{array}{c}
\ldots \\
\rightarrow & M_n & \rightarrow & M_{n-1} & \rightarrow & M_{n-2} & \rightarrow & \ldots \\
\delta_n & \delta_{n-1} & \delta_{n-2} & \ldots \\
\rightarrow & \text{cok}(\delta_n) & \rightarrow & \text{Cok}(\delta_n) & \rightarrow & \ldots \\
\rightarrow & 1 & \rightarrow & \ldots \\
\rightarrow & 0 & \rightarrow & \ldots \\
\rightarrow & \delta'_{n-1} & \rightarrow & \delta'_{n-2} & \rightarrow & \ldots \\
\end{array}
\]

where \( \delta'_{n-1} \) is induced by the cokernel \( q_n = \text{cok}(\delta_n) \) and consider \((e_{n-1}, m_{n-1})\) and \((e'_{n-1}, m'_{n-1})\) the image factorizations of \( \delta_{n-1} \) and \( \delta'_{n-1} \). Since \( m_{n-1} = m'_{n-1}q'_n \) is a monomorphism then \( q'_n \) is both a monomorphism and a normal epimorphism, hence \( q'_n \) is an isomorphism. Then \( m'_{n-1} \) is a normal monomorphism. \( \Box \)
Definition 3.6. We define the full subcategories in $\text{pch}(\mathcal{X})$ for each $n \in \mathbb{Z}$:

$$\mathcal{EP}_n = \{ M \mid \delta_n \text{ is a normal epi and } M_i = 0 \text{ for } n-1 > i \}.$$ 

And, similarly,

$$\mathcal{MN}_n = \{ M \mid \delta_n \text{ is a normal mono and } M_i = 0 \text{ for } i > n \}.$$ 

For instance, a proper chain complex $M$ in $\mathcal{EP}_n$ looks like this:

$$\ldots \rightarrow M_{n+1} \xrightarrow{\delta_{n+1}} M_n \xrightarrow{\delta_n} M_{n-1} \rightarrow 0 \rightarrow 0 \rightarrow \ldots$$

with $\delta_n$ a normal epimorphism. Similarly, a proper chain complex $M$ in $\mathcal{MN}_n$ looks like this:

$$\ldots \rightarrow 0 \rightarrow 0 \rightarrow M_n \xleftarrow{\delta_n} M_{n-1} \xleftarrow{\delta_{n-1}} M_{n-2} \rightarrow \ldots$$

where $\delta_n$ is a normal monomorphism.

In addition, we will also consider the full subcategory of $\text{ch}(\mathcal{X})$:

$$\text{Ker}(\text{cot}_n) = \{ M \mid \text{cot}_n(M) \cong 0 \}.$$ 

Lemma 3.7. Let $f : A \rightarrow B$ a morphism in $\mathcal{X}$. If $f$ is proper and has trivial cokernel, then $f$ is a normal epimorphism.

Proof. Consider $e, m$ the normal epi/mono factorization of $f$ and the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{e} & & \downarrow{q} \\
\text{Ker}(q) & \xleftarrow{m'} & \text{Cok}(f)
\end{array}
$$

where $m'$ is induced by the kernel $\text{Ker}(q)$. Then if $f$ is a proper morphism then $m'$ is an isomorphism since the normal monomorphism $m$ is the kernel of its cokernel $q = \text{cok}(f)$. Also, if $\text{Cok}(f) = 0$ then $k$ is also an isomorphism. Finally, $m$ is an isomorphism and $f$ is a normal epimorphism. \hfill $\square$

Lemma 3.8. The restriction of the subcategory $\text{Ker}(\text{cot}_n)$ to proper chains is equivalent to $\mathcal{EP}_n$:

$$\text{Ker}(\text{cot}_n) \cap \text{pch}(\mathcal{X}) = \mathcal{EP}_n.$$ 

Proof. A chain complex $M$ in $\text{Ker}(\text{cot}_n)$ has all $M_i = 0$ for $n-1 > i$ and with the differential $\delta_n$ with trivial cokernel, so it follows from lemma 3.7. \hfill $\square$

Theorem 3.9. For each $n \in \mathbb{Z}$ we have:

1. The pair $(\text{Ker}(\text{cot}_n), \text{ch}(\mathcal{X})_{n-1})$ of subcategories in $\text{ch}(\mathcal{X})$ satisfy axiom TT1 of a torsion theory.

2. The pair $(\mathcal{EP}_n, \text{pch}(\mathcal{X})_{n-1})$ of subcategories in $\text{pch}(\mathcal{X})$ is a torsion theory in $\text{pch}(\mathcal{X})$.

Proof. 1) Let $f : M \rightarrow N$ a morphism in $\text{ch}(\mathcal{X})$ with $M$ in $\text{Ker}(\text{cot}_n)$ and $N$ in $\text{ch}(\mathcal{X})_{n-1}$:

$$
\begin{array}{ccccccc}
\ldots & \rightarrow & M_{n+1} & \xrightarrow{\delta_{n+1}} & M_n & \xrightarrow{\delta_n} & M_{n-1} & \rightarrow & 0 & \rightarrow & \ldots \\
\downarrow{f_{n+1}} & & \downarrow{f_n} & & \downarrow{f_{n-1}} & & \downarrow{f_{n-2}} \\
\ldots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & N_{n-1} & \xrightarrow{\delta'_{n-1}} & N_{n-2} & \rightarrow & \ldots
\end{array}
$$

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Since $\delta_n : M_n \to M_{n-1}$ has trivial cokernel then $f_{n-1} = 0$, hence $f = 0$.

From 1), we only need to prove axiom TT2 of a torsion theory. For a proper chain complex $M$ the short exact sequence is given by:

$$
\begin{array}{cccccccc}
\ldots & \to & M_{n+1} & \delta_{n+1} & M_n & \delta_n & M_{n-1} & \delta_{n-1} & M_{n-2} & \to & \ldots \\
\downarrow{id} & & \downarrow{id} & & \downarrow{m_n} & & \downarrow{cok(\delta_n)} & & \downarrow{id} & \\
\ldots & \to & 0 & & 0 & & Cok(\delta_n) & & M_{n-2} & \to & \ldots
\end{array}
$$

(6)

Notice that since $M$ is a proper chain complex $Cok(\delta_n) \cong M_{n-1}/\delta_n(M_n)$.

3.10. By duality, let $ch(\mathbb{X}) \geq n$ be the category of $n$-truncated below chain complexes. It is straightforward to define the duals of the functors of 3.2:

- $tr'_n(M) = \ldots \to M_{n+2} \to M_{n+1} \to M_n$
- $sk'_n(M) = \ldots \to M_{n+1} \to M_n \to 0 \to 0 \to \ldots$
- $cosk'_n(M) = \ldots \to M_{n+1} \to M_n \to Cok(\delta_{n+1}) \to 0 \ldots$
- $cot'_n(M) = \ldots \to M_{n+2} \to M_{n+1} \to ker(\delta_n)$

and indeed these also give a string of adjunctions:

$$
\begin{array}{cccc}
\cosk'_n & \dashv & \tr'_n & \dashv & \sk'_n & \dashv & \cot'_n
\end{array}
$$

$ch(\mathbb{X}) \geq n$

$ch(\mathbb{X})$

We consider $ch(\mathbb{X}) \geq n$ as a subcategory of $ch(\mathbb{X})$ through $sk'_n$. However, unlike in the case of $ch(\mathbb{X}) \geq n$, the category $ch(\mathbb{X}) \geq n$ is a torsion category without the need of the restriction to proper chain complexes with $sk'_n \dashv cot'_n$ as the coreflection.

**Theorem 3.11.** For each $n \in \mathbb{Z}$ we have:

1. The adjunction $tr_{n-1} \dashv \cosk_{n-1} : ch(\mathbb{X}) \to ch(\mathbb{X})_{\leq n}$ is a localization and thus by theorem 2.9 it induces a hereditary torsion theory $(\tilde{T}_{tr_{n-1}}, \tilde{F}_{tr_{n-1}})$ in $ch(\mathbb{X})$.

2. The category $\tilde{T}_{tr_{n-1}}$ is equivalent to $ch(\mathbb{X})_{\geq n}$.

3. The reflector of $ch(\mathbb{X})_{\geq n}$ is given by $cot'_n$.

4. The restriction of $(ch(\mathbb{X})_{\geq n}, \tilde{F}_{tr_{n-1}})$ to $pch(\mathbb{X})$ is the torsion theory $(pch(\mathbb{X})_{\geq n}, MN_n)$ in $pch(\mathbb{X})$.

**Proof.** 1) The functor $tr_{n-1}$ preserves finite limits since it admits a left adjoint, namely $sk_{n-1}$. 2) is trivial since by definition $\tilde{T}_{tr_{n-1}} = Ker(cot'_n)$. 3) and 4) follow immediately from the associated short exact sequence of $(\tilde{T}_{tr_{n-1}}, \tilde{F}_{tr_{n-1}})$, which is given by Theorem 2.9. 

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To be precise, the unit $\eta$ of $tr_{n-1} \downarrow \text{cosk}_{n-1}$ for a chain complex $M$ is:

$$M = \ldots \xrightarrow{} M_{n+1} \xrightarrow{\delta_{n+1}} M_n \xrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_{n-1}} M_{n-2} \xrightarrow{} \ldots$$

$$\text{Cosk}_{n-1}(M) = \ldots \xrightarrow{} 0 \xrightarrow{k}\text{er}(\delta_{n-1}) \xrightarrow{1} M_{n-1} \xrightarrow{1} M_{n-2} \xrightarrow{} \ldots$$

The normal epi-mono factorization of $\delta'_n$ is given by $(e_n, m'_n)$ where $m'_n$ is given by $m_n$ and the universal property of $\text{ker}(\delta_{n-1})$:

$$
\begin{array}{ccc}
M_n & \xrightarrow{\delta_n} & M_{n-1} \\
| & \downarrow & | \\
e_n & \xrightarrow{m_n} & \text{ker}(\delta_{n-1}) \\
\delta_n(M_n) & \xrightarrow{m'_n} & \text{Ker}(\delta_{n-1}) \\
\end{array}
$$

So the associated short exact sequence for a chain complex $M$ is:

$$
\begin{array}{ccc}
\ldots & \xrightarrow{} & M_{n+1} \\
\downarrow & \quad & \downarrow \\
\ldots & \xrightarrow{} & M_{n+1} \\
\downarrow & \quad & \downarrow \\
\ldots & \xrightarrow{} & 0 \\
\downarrow & \quad & \downarrow \\
\ldots & \xrightarrow{} & \delta(M_n) \\
\downarrow & \quad & \downarrow \\
\ldots & \xrightarrow{} & M_{n-1} \\
\downarrow & \quad & \downarrow \\
\ldots & \xrightarrow{} & M_{n-2} \\
\end{array}
$$

(7)

Now, we will now restrict ourselves to the case of proper chains.

**Definition 3.12.** For each $n \in \mathbb{Z}$, the torsion theories in $\text{pch}(\mathcal{X})$ from theorems 3.9 and 3.11 will be denoted as

$$\text{COK}_n = (\mathcal{EP}_n, \text{pch}(\mathcal{X})_{n-1} \geq)$$

and

$$\text{KER}_n = (\text{pch}(\mathcal{X}) \geq n, \text{MN}_n)$$

with the preradicals $\text{ker}_n, \text{cok}_n : \text{pch}(\mathcal{X}) \to \text{pch}(\mathcal{X})$, respectively.

We will write $\text{COT}(\mathcal{X})$ for the set of torsion theories $\text{COK}_n, \text{KER}_n$ given by the cotruncation functors:

$$\begin{array}{ccc}
\text{COK}_n = & \mathcal{EP}_n & \text{pch}(\mathcal{X}) \downarrow \text{pch}(\mathcal{X})_{n-1} \geq \\
& \downarrow & \downarrow \\
& \text{sk}_{n-1} & \text{sk}_{n-1} \\
\text{KER}_n = & \text{pch}(\mathcal{X})_{n-1} \geq & \text{MN}_n \\
& \downarrow & \downarrow \\
& \text{cot}_{n-1} & \text{cot}_{n-1} \\
\end{array}$$

The next result asserts that $\text{COT}(\mathcal{X})$ is a linearly order sublattice of $(\text{pch}(\mathcal{X}))_{\text{tors}}$.

**Proposition 3.13.** In $\text{pch}(\mathcal{X})$ for each $n \in \mathbb{Z}$ we have the embeddings of full subcategories

$$\ldots \leq \text{pch}(\mathcal{X})_{n+1} \leq \mathcal{EP}_{n+1} \leq \text{pch}(\mathcal{X})_{n} \leq \mathcal{EP}_{n} \leq \text{pch}(\mathcal{X})_{n-1} \leq \ldots$$
Equivalently,
\[ \ldots \geq \mathcal{M}n_{+1} \geq \text{pch}(X)_{n+1} \geq \mathcal{M}n \geq \text{pch}(X)n \geq \mathcal{M}n_{-1} \geq \ldots . \]

Moreover, there is a linearly ordered lattice of torsion theories in \( \text{pch}(X) \):
\[ O \leq \ldots \leq \mathcal{E}R_{n+1} \leq \mathcal{C}K_{n+1} \leq \mathcal{E}R_n \leq \mathcal{C}K_n \leq \ldots \leq \text{pch}(X) \]

**Proof.** By definition we have \( \mathcal{E}P_n \leq \text{pch}(X)_{n+1} \) and since a morphism \( M_{n+1} \to 0 \) is a normal epimorphism we have \( \text{pch}(X)_{n+1} \leq \mathcal{E}P_n \). Recall that the order is reverse for the torsion-free subcategories.

This construction works with truncated or bounded chain complexes, in particular we will be interested in the case for \( \text{pch}(X) \geq 0 \), and \( \text{pch}(X)n \geq 0 \), the category of chain complexes bounded above \( n \) and below 0 for a fixed \( n \).

**Corollary 3.14.** In \( \text{pch}(X) \geq 0 \) there is a linearly ordered lattice of torsion theories given by:
\[ O \leq \ldots \leq \mathcal{E}R_n \leq \mathcal{C}K_n \leq \ldots \leq \mathcal{E}R_1 \leq \mathcal{C}K_1 \leq \text{pch}(X) \geq 0 \]

**Corollary 3.15.** In \( \text{pch}(X)n \geq 0 \) there is a linearly ordered lattice of torsion theories given by:
\[ O \leq \mathcal{E}R_n \leq \mathcal{C}K_n \leq \ldots \leq \mathcal{C}K_2 \leq \mathcal{E}R_1 \leq \mathcal{C}K_1 \leq \text{pch}(X)n \geq 0 \]

We will write \( \mathcal{C}O\mathcal{T}(\text{pch}(X) \geq 0) \) and \( \mathcal{C}O\mathcal{T}(\text{pch}(X)n \geq 0) \) for the corresponding lattice of torsion theories in the bounded cases of \( \text{pch}(X) \geq 0 \) and of \( \text{pch}(X)n \geq 0 \).

**3.16. Example.** For the bounded case of \( \text{pch}(X) \geq 2 \geq 0 \) the lattice \( \mathcal{C}O\mathcal{T}(\text{pch}(X)2 \geq 0) \) is:
\[ O \leq \mathcal{E}R_2 \leq \mathcal{C}K_2 \leq \mathcal{E}R_1 \leq \mathcal{C}K_1 \leq \text{pch}(X)2 \geq 0 . \]

This lattice induces a lattice of preradicals \( \text{pch}(X)2 \geq 0 \) and hence, for a fixed proper chain complex \( M \), a lattice of torsion subobjects of \( M \):
\[
M = \begin{align*}
M_2 & \xrightarrow{\delta_2} M_1 & \xrightarrow{\delta_1} M_0 \\
cok_1(M) &= M_2 & \xrightarrow{\delta_2} M_1 & \xrightarrow{\epsilon_1} \delta_1(M_1) \\
ker_1(M) &= M_2 & \xrightarrow{\delta_2} ker(\delta_1) & \xrightarrow{0} \\
cok_2(M) &= M_2 & \xrightarrow{\epsilon_2} \delta_2(M_2) & \xrightarrow{0} \\
ker_2(M) &= ker(\delta_2) & \xrightarrow{0} & \xrightarrow{0} \\
0 &= 0 & \xrightarrow{0} & \xrightarrow{0} .
\end{align*}
\]

**4 Homology**

In abelian categories, the \( n \)th-homology objects of a chain complex \( M \) is usually defined as \( H_n(M) = ker(\delta_n)/\delta_{n+1}(M_{n+1}) \). We can also consider the dual homology object \( K_n \). In other words, consider the commutative diagram:

\[
\begin{array}{cccccc}
M_{n+1} & \xrightarrow{\delta_{n+1}} & M_n & \xrightarrow{\delta_n} & M_{n-1} \\
\downarrow{\epsilon_{n+1}} & & \downarrow{k_n} & & \downarrow{m_n} \\
\delta_{n+1}(M_{n+1}) & \xrightarrow{m_{n+1}} & Ker(\delta_n) & \xrightarrow{q_{n+1}} & Cok(\delta_{n+1}) \\
\end{array}
\]

\[
\begin{array}{cccccc}
\delta_{n+1}(M_{n+1}) & \xrightarrow{q_{n+1}} & Cok(\delta_{n+1}) & \xrightarrow{e_n} & \delta_n(M_n) \\
\end{array}
\]
where \( \delta_n = m_n e_n \), \( \delta_{n+1} = m_{n+1} e_{n+1} \) are the normal epi/mono factorizations and \( m'_{n+1} \) and \( e'_n \) are induced by \( \text{Ker}(\delta_n) \) and \( \text{Cok}(\delta_{n+1}) \), respectively. Then we have
\[
H_n(M) = \text{Cok}(M_{n+1} \to \text{Ker}(\delta_n)) = \text{Cok}(m'_{n+1})
\]
and
\[
K_n(M) = \text{Ker}(\text{Cok}(\delta_{n+1} \to M_{n-1})) = \text{Ker}(e'_n).
\]
It is well-known that in abelian categories the objects \( H_n(M) \) and \( K_n(M) \) are naturally isomorphic, and in [12] this was proved to be the case also for exact homological categories provided that the chain complex \( M \) is proper. The following result provides an alternative proof of this fact as well as showing the connection of the objects \( H_n(M), K_n(M) \) with the preradicals in \( \text{COT} \).

**Lemma 4.1.** Let \( M \) be a proper chain complex then the objects \( H_n(M), K_n(M) \) are isomorphic and are given by
\[
H_n(M) \cong K_n(M) \cong \ker_n(M)/\text{cok}_n(M)
\]
where \( \text{cok}_{n+1}(M), \ker_n(M) \) are the torsion subobjects of \( M \) given by the torsion theories as in Definition 3.12 and where \( H_n(M), K_n(M) \) are considered as trivial chain complexes except at the order \( n \) that have the objects \( H_n(M), K_n(M) \) respectively.

**Proof.** Since \( \text{ch}(\mathcal{X}) \) is semi-abelian it follows from Theorem (2.3) when we consider the normal subobjects \( \text{cok}_{n+1}(M) \leq \ker_n(M) \) of \( M \). So we have a short exact sequence in \( \text{ch}(\mathcal{X}) \):
\[
0 \longrightarrow \ker_n(M)/\text{cok}_{n+1}(M) \longrightarrow M/\text{cok}_{n+1}(M) \longrightarrow M/\ker_n(M) \longrightarrow 0.
\]
To be more precise observe the short exact sequences that define \( H_n(M), K_n(M) \). Then \( H_n(M) \) is the cokernel of the inclusion \( \text{cok}_{n+1}(M) \leq \ker_n(M) \):

\[
\begin{array}{cccccccc}
0 & \downarrow & \\
\text{cok}_{n+1}(M) & \longrightarrow & \ldots & \longrightarrow & M_{n+1} & \longrightarrow & \delta_{n+1}(M_{n+1}) & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ker_n(M) & \longrightarrow & \ldots & \longrightarrow & M_{n+1} & \longrightarrow & \ker(\delta_n) & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & & & \downarrow & & \\
H_n(M) & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & H_n(X) & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow & & & & & \downarrow & & & & & \\
0 & \downarrow & & & & & \downarrow & & & & &
\end{array}
\]
so \( H_n(M) \cong \ker_n(M)/\text{cok}_{n+1}(M) \); and on the other hand

\[
\begin{array}{cccccccc}
0 & \downarrow & \\
\text{K}_n(M) & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & \text{K}_n(M) & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M/\text{cok}_{n+1}(M) & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & \text{cok}(\delta_{n+1}) & \longrightarrow & M_{n-1} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M/\ker_n(M) & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & \delta_n(M) & \longrightarrow & M_{n-1} & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & & & \downarrow & & \\
0 & & & & & & & & & &
\end{array}
\]
so $K_n(M) \cong \ker(M/cok_{n+1}(M) \to M/\ker_n(M))$. Theorem \[2.3\] yields the isomorphism $H_n(M) \cong K_n(M)$.

**Proposition 4.2.** For a proper chain complex $M$ the following are equivalent:

1. $H_n(M) = 0$
2. $M/\ker_{n+1}(M) \cong \text{Cosk}_n(M)$.

Similarly, the following are equivalent:

1. $\ker_n(M) = 0$
2. $\text{Cok}_{n+1}(M) \cong \text{Cosk}'_n(M)$.

where $\ker_{n+1}(M)$ and $cok_n(M)$ are the torsion subobjects of $M$ given by the torsion theories in \[3.12\]

**Proof.** First, recall from Theorem \[3.11\] that the unit $M \to \text{Cosk}_n(M)$ factors through the reflection of $\mathcal{M} \mathcal{N}_{n+1}$:

\[
M = \cdots \to M_{n+1} \xrightarrow{\delta_{n+1}} M_n \xrightarrow{\delta_n} \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
M/\ker_{n+1}(M) = \cdots \to 0 \xrightarrow{\delta_{n+1} \to M_n} \delta_n \to \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Cosk}_n(M) = \cdots \to 0 \xrightarrow{\delta_n \to M_n} \delta_n \to \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\ker_n(M) = \cdots \to 0 \xrightarrow{\delta_n \to M_n} \delta_n \to \cdots
\]

And by definition, $\delta_{n+1}(M_{n+1}) \cong \ker(\delta_n)$ if and only if $H_n(M) = 0$. The second part is similar, since $cok(\delta_{n+1}) \cong \delta_n(M)$ if and only if $H_n(M) = 0$. \[\square\]

From \[13\], it is known that the truncation functor $\text{cot}_n$, $\text{cot}'_n$ give truncations in the homology objects. The following lemma generalises these facts.

**Lemma 4.3.** Let $\ker_n$, $cok_n$ be the preradicals of the torsion theories in Definition \[3.12\]. For a proper chain complex $M$ we have:

1. For all $n > 0$
   
   $$H_i(cok_n(M)) = H_i(ker_n(M)) = \begin{cases} H_i(M) & i \geq n \\ 0 & n > i \end{cases}$$

2. For all $n > 0$
   
   $$H_i\left(\frac{M}{cok_n(M)}\right) = H_i\left(\frac{M}{ker_n(M)}\right) = \begin{cases} 0 & i \geq n \\ H_i(M) & n > i \end{cases}$$

3. For all $n > 0$
   
   $$H_i\left(\frac{cok_n(M)}{ker_n(M)}\right) = 0 \text{ for all } i.$$

4. For $m > n$
   
   $$H_i\left(\frac{cok_n(M)}{ker_m(M)}\right) = H_i\left(\frac{cok_n(M)}{cok_m(M)}\right) = \begin{cases} H_i(M) & m > i \geq n \\ 0 & \text{otherwise} \end{cases}.$$

5. Moreover, for $m > n$
   
   $$H_i\left(\frac{cok_n(M)}{cok_m(M)}\right) = H_i\left(\frac{ker_n(M)}{ker_m(M)}\right) = H_i\left(\frac{ker_n(M)}{ker_m(M)}\right) \cdot$$
6. In particular, for $m = n + 1$

$$H_i \left( \frac{\text{cok}_n(M)}{\text{ker}_{n+1}(M)} \right) = \begin{cases} H_i(M) & i = n \\ 0 & i \neq n. \end{cases}$$

**Proof.** It is straightforward to calculate the homology of each chain complex. For 1) consider:

$$\text{cok}_n(M) = \ldots \to M_{n+1} \xrightarrow{\delta_{n+1}} M_n \xrightarrow{\epsilon_n} \delta_n(M_n) \xrightarrow{0} \ldots$$

$$\text{ker}_n(M) = \ldots \to M_{n+1} \xrightarrow{\delta'_n} \text{ker}(\delta_n) \xrightarrow{0} \ldots$$

For 2) consider:

$$\frac{M}{\text{cok}_n(M)} = \ldots \to 0 \xrightarrow{0} \ldots \xrightarrow{\delta_n(M_n)\delta_{n+1}} M_{n-2} \xrightarrow{\ldots}$$

$$\frac{M}{\text{ker}_n(M)} = \ldots \to 0 \xrightarrow{\delta_n(M_n)\delta_{n+1}} M_{n-1} \xrightarrow{m_n} M_{n-2} \xrightarrow{\ldots}$$

For 3) consider:

$$\frac{\text{cok}_n(M)}{\text{ker}_n(M)} = \ldots \to 0 \xrightarrow{\delta_n(M_n)\delta_{n+1}} \xrightarrow{\cong} M_{n} \xrightarrow{0} \ldots$$

For 4) and 5) the following chain complexes have the same homology:

$$\frac{\text{ker}_n(M)}{\text{ker}_m(M)} = \ldots \delta_m(M_m) \xrightarrow{0} M_{m-1} \xrightarrow{0} \ldots \xrightarrow{\delta_n(M_n)\delta_{n+1}} M_{n-2} \xrightarrow{\ldots}$$

$$\frac{\text{ker}_n(M)}{\text{cok}_m(M)} = \ldots \xrightarrow{0} \delta_m(M_m) \xrightarrow{\delta_n(M_n)\delta_{n+1}} M_{n-1} \xrightarrow{0} \ldots$$

$$\frac{\text{cok}_n(M)}{\text{ker}_m(M)} = \ldots \delta_m(M_m) \xrightarrow{0} \ldots \xrightarrow{\delta_n(M_n)\delta_{n+1}} M_{n-2} \xrightarrow{\ldots}$$

$$\frac{\text{cok}_n(M)}{\text{cok}_m(M)} = \ldots \xrightarrow{0} \delta_m(M_m) \xrightarrow{\delta_n(M_n)\delta_{n+1}} \ldots \xrightarrow{0} \ldots$$

5 **Torsion theories induced by $tr_n \dashv cosk_n$**

We define the torsion theories $\mu_{\geq n}$, simplicial analogues for $\mathcal{MN}_n$ which similarly to the case of chain complexes, are defined by a localization $tr_n \dashv cosk_n$ in simplicial objects. The torsion category of $\mu_{\geq n}$ is the category of simplicial groups such that they are trivial below degree $n$. First, we recall some basic properties of simplicial objects.

Following [13], the simplicial category $\Delta$ has, as objects, finite ordinals $[n] = \{0, 1, \ldots, n\}$ and as morphisms monotone functions. In particular, we have the morphisms $\delta^n_i : [n-1] \to [n]$ the injection which does not take the value $i \in [n]$ and $\sigma^n_i : [n+1] \to [n]$ the surjection where $\sigma(i) = \sigma(i+1)$.

Any morphism $\mu$ in $\Delta$ can written uniquely as:

$$\mu = \delta^n_{i_s} \delta^{n-1}_{i_{s-1}} \cdots \delta^{n-t+1}_{i_1} \sigma^{m-t}_{j_t} \cdots \sigma^{m-2}_{j_2} \sigma^{m-1}_{j_1}.$$ 

such that $n \geq i_s > \cdots > i_1 \geq 0$, $0 \leq j_t < \cdots < j_1 < m$ and $n = m - t + s$.
The category of simplicial objects in a category $X$ is the functor category $Simp(X) = [\Delta^{op}, X]$. Thus, a simplicial object $X: \Delta^{op} \to X$ corresponds to a family of objects $\{X_n\}_{n \in \mathbb{N}}$ in $X$, the face morphisms $d_i: X_n \to X_{n-1}$ and the degeneracies morphisms $s_i: X_n \to X_{n+1}$ satisfying the simplicial identities:

\[
d_i d_j = d_{j-1} d_i \quad \text{if} \quad i < j \\
s_i s_j = s_{j+1} s_i \quad \text{if} \quad i \leq j \\
d_i s_j = \begin{cases} 
    s_{j-1} d_i & \text{if} \quad i < j \\
    1 & \text{if} \quad i = j \text{ or } i = j + 1 \\
    s_j d_{i-1} & \text{if} \quad i > j + 1.
\end{cases}
\]

5.1. Let $X$ be a simplicial object in a pointed category with pullbacks. The Moore normalization functor $N: \text{Simp}(X) \to \text{ch}(X)$ is given by $N(X)_0 = X_0$,

\[
N(X)_n = \bigcap_{i=0}^{n-1} \ker(d_i : X_n \to X_{n-1})
\]
and differentials $\delta_n = d_n \circ \cap_i \ker(d_i) : N(X)_n \to N(X)_{n-1}$ for $n \geq 1$.

The functor $N$ preserves finite limits. In [12], it was proved that if $X$ is a semi-abelian category $N$ also preserves regular epimorphisms, and hence, it preserves short exact sequences. Moreover, for a simplicial object $X$ the Moore complex $N(X)$ is a proper chain complex and we can define the $n$-homology object of a simplicial object $X$ as:

\[
H_n(X) = H_n(N(X)).
\]

The objects $H_n(N(X))$ are internal abelian objects for $n \geq 1$ (see [12]).

This generalizes the results proven by Moore for the case of simplicial groups, where the homotopy groups of a simplicial group $X$ can be calculated as $\pi_n(X) = H_n(N(X))$ (see, for example [22]).

5.2. If $\Delta_n$ is the full subcategory of $\Delta$ with objects $[m]$ for $m \leq n$, an $n$-truncated simplicial object $X$ in $X$ is a functor $X: \Delta^{op}_n \to X$. Let $\text{Simp}_n(X)$ be the category of $n$-truncated simplicial objects, then there is truncation functor:

\[
\text{tr}_n: \text{Simp}(X) \longrightarrow \text{Simp}_n(X)
\]

which simply forgets the objects $X_i$ and the morphisms $s_i, d_i$ for degrees $i > n$.

It is a standard application of Kan extensions that if $X$ is finitely complete/cocomplete (as is the case if $X$ is semi-abelian) each functor $\text{tr}_n$ admits a left/ right adjoint named the $n$-skelton and $n$-coskeleton, respectively: $\text{sk}_n \dashv \text{tr}_n \dashv \text{cosk}_n$. We will write $S_k = \text{sk}_n \text{tr}_n$ and $\text{Cosk}_n = \text{cosk}_n \text{tr}_n$.

If $X$ has finite limits, the $n$-coskeleton of an $n$-truncated simplicial object $X$ is described as follows. For an $n$-truncated simplicial object $X$ the simplicial kernel of the face morphisms $d_0, \ldots, d_n: X_n \to X_{n-1}$ is an object $\Delta_{n+1}$ with morphisms $\pi_0, \ldots, \pi_{n+1}: \Delta_{n+1} \to X_n$ such that $d_i \pi_j = d_{j-1} \pi_i$ for all $i < j$ and it is universal with this property: given a family of morphisms $p_0, \ldots, p_{n+1} : Y \to X_n$ such that $d_i p_j = d_{j-1} p_i$ for all $i < j$ then there is a unique morphism $\alpha : Y \to \Delta_{n+1}$ such that $\pi_i \alpha = p_i$. Moreover, the universal property of the simplicial kernel $\Delta_{n+1}$ allows to define degeneracies morphisms $s_i: X_n \to \Delta_{n+1}$. So, the simplicial kernel of $X$ defines an $(n+1)$-truncated simplicial object. Finally, the coskeleton is given by the iteration of successive simplicial kernels.

For $n = 0$, the 0-coskeleton is known as the indiscrete functor $\text{Ind}: X \to \text{Simp}(X)$ given by:

\[
\text{Ind}(X) = \begin{array}{cccc}
    \pi_4 & \longrightarrow & X^4 & \longrightarrow & \pi_4 \\
    \pi_0 & \longrightarrow & \pi_0 & \longrightarrow & \pi_0 \\
    \pi_3 & \longrightarrow & X^3 & \longrightarrow & \pi_3 \\
    \pi_0 & \longrightarrow & \pi_0 & \longrightarrow & \pi_0 \\
\end{array}
\]

where $X^n$ is $n$-fold product of $X$ and the degeneracies are defined by the product projections.
For $\mathbb{X} = \text{Grps}$, the simplicial kernel $\Delta_{n+1}$ of a $n$-truncated simplicial group $X$ can be described as the subgroup of $X^{n+2}_{n+2}$ of $(n + 2)$-tuples $(x_0, \ldots, x_{n+1})$ such that $\pi_i(x_j) = \pi_{j-1}(x_i)$ for $i < j$ and where $\pi_i$ are the product projections.

The following result was first proved for simplicial groups in [9]. The generalization is straightforward.

**Theorem 5.3.** Let $\mathbb{X}$ be a pointed category with finite limits. For a simplicial object $X$ with corresponding Moore complex $M$, then the Moore complex of $n$-coskeleton $\text{Cosk}_n(X)$ satisfies:

- $N(\text{Cosk}_n(X)) = M_i$ for $n \leq i$;
- $N(\text{Cosk}_n(X))_{n+1} = \ker(\delta_n : M_n \to M_{n+1})$;
- $N(\text{Cosk}_n(X)) = 0$ for $i > n + 1$.

In other words, the Moore normalization $N$ commutes up to isomorphism with the coskeleton functors for simplicial objects and chain complexes:

\[
\begin{align*}
\text{Simp}(\mathbb{X}) & \xrightarrow{N} \text{ch}(\mathbb{X}) \\
\text{tr}_n \left\{ \begin{array}{c} \text{cosk} \\
\text{cosk} \end{array} \right\} & \xrightarrow{\mu} \text{ch}(\mathbb{X})_{n \geq}
\end{align*}
\]

**Definition 5.4.** Let $\mathbb{X}$ be a semi-abelian category. For each $n > 1$ we have that $\text{tr}_{n-1} \dashv \text{cosk}_{n-1}$ is a localization since $\text{tr}_{n-1}$ admits a left adjoint, namely $\text{sk}_{n-1}$. Then, by Theorem [2.9] it defines a hereditary torsion theory in $\text{Simp}(\mathbb{X})$ which will be written as:

\[
\mu_{\geq n} := (\mathcal{T}_{\text{tr}_{n-1}}, \mathcal{F}_{\text{tr}_{n-1}}).
\]

We will also write $\mu_{\geq n} : \text{Simp}(\mathbb{X}) \to \text{Simp}(\mathbb{X})$ for the associated idempotent radical.

By definition, the category $\mathcal{T}_{\text{tr}_{n-1}}$ is the full subcategory of simplicial objects $X$ with $X_i = 0$ for $n - 1 \geq i$.

The torsion theory $\mu_{\geq 1} = (\mathcal{T}_{\text{tr}_0}, \mathcal{F}_{\text{tr}_0})$ naturally extends the torsion theory $(\text{Ab}(\mathbb{X}), \text{Eq}(\mathbb{X}))$ in $\text{Grpd}(\mathbb{X})$, to this end we need to recall a result about Mal’tsev categories.

5.5. A category $\mathbb{X}$ is called a Mal’tsev category if any internal reflexive relation is an equivalence relation. Semi-abelian categories are Mal’tsev categories.

**Proposition 5.6.** Let $\mathbb{X}$ be a regular Mal’tsev category. The category $\text{Grpd}(\mathbb{X})$ is closed under subobjects in $\text{Simp}(\mathbb{X})$.

**Proposition 5.7.** Let $\mathbb{X}$ be a semi-abelian category and, for $n = 0$, consider the torsion theory $\mu_{\geq 1} = (\mathcal{T}_{\text{tr}_0}, \mathcal{F}_{\text{tr}_0})$ in $\text{Simp}(\mathbb{X})$. The torsion-free subcategory $\mathcal{F}_{\text{tr}_0}$ is equivalent to $\text{Eq}(\mathbb{X})$. On the other hand, $\text{Ab}(\mathbb{X}) \subset \mathcal{T}_{\text{tr}_0}$.

**Proof.** For $n = 0$ we have that $\text{tr}_0 \dashv \text{cosk}_0 \cong (\_0 \dashv \text{Ind})$. The unit of $(\_0 \dashv \text{Ind})$ for a simplicial object $X$ is given by

\[
\begin{align*}
X = & \quad \ldots \quad \xrightarrow{d_4} \quad X_3 \quad \xrightarrow{d_3} \quad X_2 \quad \xrightarrow{d_2} \quad X_1 \quad \xleftarrow{d_1} \quad X_0 \\
\text{Ind}(X) = & \quad \ldots \quad \xrightarrow{\pi_4} \quad X_3 \quad \xrightarrow{\pi_3} \quad X_2 \quad \xrightarrow{\pi_2} \quad X_1 \quad \xleftarrow{\pi_1} \quad X_0
\end{align*}
\]

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Since $X$ is a Mal’tsev category and by Proposition 5.6, if $X$ has a monic unit $\eta_X$ then $X$ is a groupoid since $\text{Ind}(X)$ is an equivalence relation. Finally, since $(d_0, d_1) : X_1 \to X_0$ is monic then $X$ is an equivalence relation and $\mathcal{F}_{tr_0} \subseteq \text{Eq}(X)$. Conversely, an equivalence relation always has a monic unit $\eta_X$.

On the other hand, an internal abelian group $X$ has $X_0 = 0$, so $\text{Ab}(X) \subseteq \mathcal{T}_{tr_0}$.

**Theorem 5.8.** Let $X$ be a semi-abelian category and $X$ a simplicial object with Moore chain complex $M$. Then, the normalization functor $N$ maps the short exact sequence of $X$ given the torsion theory $\mu \geq n + 1$ in $\text{Simp}(X)$ into the short exact sequence of $M$ given by the torsion theory $\mathcal{KE}\mathcal{R}_{n+1}$ in $\text{pch}(X)$.

Moreover, $N$ maps the torsion theory $\mu \geq n + 1$ into $\mathcal{KE}\mathcal{R}_{n+1}$:

$$
\begin{array}{ccc}
\mathcal{T}_{tr_n} & \rightarrow & \text{Simp}(X) \\
N & \downarrow & N \\
\text{pch}(X)_{\geq n + 1} & \rightarrow & \mathcal{F}_{tr_n}
\end{array}
$$

i.e. the subcategory $\mathcal{T}_{tr_n}$ is mapped into $\text{pch}(X)_{\geq n + 1}$ and $\mathcal{F}_{tr_n}$ into $\mathcal{M}\mathcal{N}_n$.

**Proof.** Since $X$ is semi-abelian the normalization functor $N$ preserves short exact sequences and also preserves the normal epi/mono factorization of morphisms in $\text{Simp}(X)$. Since $N$ commutes (up to isomorphism) with the truncation and coskeleton functors we have that for a simplicial object $X$ and its Moore complex $M$, the functor $N$ maps the short exact sequence in $\text{Simp}(X)$:

$$
\begin{array}{ccccccccccc}
0 & \rightarrow & \ker(\eta_X) & \rightarrow & X & \rightarrow & \eta_X(X) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ker(\eta_X) & \rightarrow & X & \rightarrow & \eta_X(X) & \rightarrow & 0
\end{array}
$$

into

$$
\begin{array}{cccccccccc}
0 & \rightarrow & M_{n+2} & \rightarrow & \ker(\delta_{n+1}) & \rightarrow & 0 & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_{n+2} & \rightarrow & \ker(\delta_{n+1}) & \rightarrow & 0 & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_{n+1} & \rightarrow & \ker(\delta_{n+1}) & \rightarrow & 0 & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_n & \rightarrow & \ker(\delta_{n+1}) & \rightarrow & 0 & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_{n-1} & \rightarrow & \ker(\delta_{n+1}) & \rightarrow & 0 & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \rightarrow & \ldots & \rightarrow & \ldots & \rightarrow & \ldots
\end{array}
$$

Since the short exact sequence of the torsion theories is preserved from $\mu \geq n + 1$ to $\mathcal{KE}\mathcal{R}_{n+1}$, it follows that $N(\mathcal{T}_{tr_n}) \subseteq \text{pch}(X)_{\geq n + 1}$ and $N(\mathcal{F}_{tr_n}) \subseteq \mathcal{M}\mathcal{N}_n$.

6 Torsion theories of truncated Moore complexes in simplicial groups

In order to define the torsion theories $\mu \geq n$, analogues for the torsion theories $\mathcal{CO}\mathcal{K}_n$ in simplicial objects, we will restrict ourselves to the case of the category of $X = \text{Grps}$. With this stronger
assumption the subcategories $\mathcal{M}_{\geq n}$ and $\mathcal{M}_{n\geq}$ of simplicial groups with trivial Moore complex below/above at order $n$ will appear as torsion/torsion-free subcategories of the torsion theories $\mu_{n\geq}$ and $\mu_{\geq n}$, respectively.

We need to recall some results of D. Conduché [9]. In particular, in a simplicial group $X$ each $X_n$ can be decomposed as successive semi-direct products of the objects of its Moore complex $M_i$ with $i \leq n$.

6.1. [9] In order to avoid multiple subscripts we will write $\sigma_i = \overline{i}$ for the degeneracy maps of $\Delta$.

For any object $[n] = \{0 < 1 < \cdots < n\}$ of the simplicial category $\Delta$ we will introduce an order in $S(n)$ the set of surjective maps of $\Delta$ with domain $[n]$. Any surjective map $\sigma : [n] \rightarrow [m]$ is written uniquely as $\sigma = \overline{i_1} \overline{i_2} \cdots \overline{i_{n-m}}$ with $i_1 < i_2 < \cdots < i_{n-m}$. We introduce the inverse lexicographic order in $S(n, m)$ the set of surjective maps form $[n]$ to $[m]$:

$$\overline{i_1} \overline{i_2} \cdots \overline{i_{n-m}} < \overline{j_1} \overline{j_2} \cdots \overline{j_{n-m}} \text{ if } i_{n-1} = j_{n-m}, \ldots, i_{s+1} = j_{s+1}, \text{ and } i_s > j_s.$$ 

This order extends to $S(n)$ by setting $S(n, m) < S(n, l)$ if $m > l$.

As an example, for $S(4)$ we have:

$$\text{id}_{[4]} < 3 < 2 < 23 < 1 < 13 < 12 < 123 < 0 < 03 < 02 < 023 < 01 < 013 < 012 < 0123$$

For a simplicial group $X$ with Moore complex $M$ and a surjective map $i = \overline{i_1} \overline{i_2} \cdots \overline{i_r}$ we have $s_1 = s_{i_1}, \ldots, s_{i_1}$, and $d_1 = d_{i_1}, \ldots, d_{i_r}$. Using the order of $S(n)$ we have a filtration of $X_n$ by the subgroups

$$G_{n,1} = \bigcap_{j \geq 1} \ker(d_j).$$

Notice that $G_{n,\text{id}} = 0$ and $G_{n,n-1} = M(X)_n$. The order $S(n)$ satisfies for a surjective map $i : [n] \rightarrow [r]$ and its successor $j$ we have the semidirect product

$$G_{n,j} \cong G_{n,i} \times_{s_{i_1}} M_r.$$ 

Finally, this implies that $X_n$ decomposes as a sequence of semi-direct products:

$$X_n = (\cdots (M_n \times_{s_{n-1}} M_{n-1}) \times_{s_{n-2}} \cdots) \times_{s_{n-1}} \cdots s_0 M_0$$

**Corollary 6.2.** For each $n \in \mathbb{N}$, the category $\mathcal{M}_{\geq n+1}$ and $\mathcal{T}_{\text{tr}_n}$ are equivalent.

Moreover, the category $\mathcal{M}_{\geq n+1}$ is a torsion subcategory in $\text{Simp}(\mathcal{X})$;

$$(\mathcal{T}_{\text{tr}_n}, \mathcal{F}_{\text{tr}_n}) \cong (\mathcal{M}_{\geq n+1}, \mathcal{F}_{\text{tr}_n}).$$

**Proof.** It follows immediately from the semidirect decomposition that a simplicial group $X$ has $X_i = 0$ for $n > i$ if and only if $M_i = 0$ for $n > i$. 

The analogue of the cotruncation functor for simplicial groups was introduced by T. Porter as follows.

6.3. [23] There is a cotruncation functor

$$\text{Cot}_n : \text{Simp}(	ext{Grp}) \rightarrow \text{Simp}(	ext{Grp})$$

such that

$$\begin{align*}
\text{Simp}(	ext{Grp}) & \xrightarrow{\text{N}} \text{chn}(	ext{Grp}) \\
\text{Simp}(	ext{Grp}) & \xrightarrow{\text{Cot}_n} \text{chn}(	ext{Grp})
\end{align*}$$

such that

$$\begin{align*}
\text{Simp}(	ext{Grp}) & \xrightarrow{\text{N}} \text{chn}(	ext{Grp}) \\
\text{Simp}(	ext{Grp}) & \xrightarrow{\text{Cot}_n} \text{chn}(	ext{Grp})
\end{align*}$$

(8)
commutes up to natural isomorphism, where \( N \) is the Moore normalization functor. The functor \( \text{Cot}_n(X) \) is defined as follows:

\[
\text{Cot}_n(X)_i = X_i \quad \text{for} \quad n > i ,
\]

\[
\text{Cot}_n(X)_n = X_n ,
\]

and for \( i > n \) the object \( \text{Cot}_n(X)_i \) is obtained by deleting all \( M_k \) for \( k > n \) and replacing \( M_n \) by \( M_n/\delta_{n+1}(M_{n+1}) \) in the semi-direct decomposition.

We recall some useful properties of this functor.

**Proposition 6.4.** \([23] \) Let \( \mathcal{M}_{n\geq} \) be the full subcategory of \( \text{Simp}(\text{Grp}) \) defined by those simplicial groups whose Moore complex is trivial for dimensions greater than \( n \). Let \( i_n : \mathcal{M}_{n\geq} \to \text{Simp}(\text{Grp}) \) the inclusion functor then

1. \( \text{Cot}_n \) is left adjoint of \( i_n \);
2. the unit \( \eta_X : X \to \text{Cot}_n(X) \) of the adjunction is a regular epimorphism which induces an isomorphism in \( \pi_i(X) \) for \( i \leq n \);
3. for any simplicial group \( X \), \( \pi(Cot_n(X)) = 0 \) for \( i > n \);
4. the inclusion \( \mathcal{M}_{n\geq} \to \mathcal{M}_{n+1\geq} \) correspond to a natural epimorphism

\[
\eta_n : \text{Cot}_{n+1} \to \text{Cot}_n
\]

and, for a simplicial group \( X \), then \( \text{Ker}(\eta_n(X)) \) is a \( K(\pi_{n+1}(X), n + 1) \)-simplicial group (an Eilenberg-Mac Lane simplicial group).

This cotruncation functor for simplicial groups is normal and thus defines a torsion theory in \( \text{Simp}(\text{Grp}) \) as in Theorem 2.7.

**Corollary 6.5.** The subcategory \( \mathcal{M}_{n\geq} \) of \( \text{Simp}(\text{Grp}) \) given by the simplicial groups with trivial Moore complex for dimension greater than \( n \) is a torsion-free subcategory of \( \text{Simp}(\text{Grp}) \). The torsion theory is given by the pair \( \mu_{n\geq} = (\text{Ker}(\text{Cot}_n), \mathcal{M}_{n\geq}) \).

**Proof.** By Theorem 2.7 it suffices to prove that the functor \( \text{Cot}_n \) is normal. Let \( \eta \) be the unit as in 6.3 for a simplicial group \( X \) with a Moore complex \( M \). Since taking normalization preserves short exact sequences we have that the Moore complex of \( ker(\eta_X) \) is:

\[
\ldots \to M_{n+2} \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{e_{n+1}} d_{n+1}(M_{n+1}) \to 0 \to \ldots
\]

which is trivial under the chain cotruncation \( \text{cotr}_n \). Since the functor \( \text{Cot}_n \) and \( \text{Cot}_n \) commute with the Moore normalization as in [5,3] we have that \( \text{Cot}_n(ker(\eta_X)) = 0 \) for any simplicial group \( X \). \( \square \)

**Definition 6.6.** For each \( n \) we will denote \( \mu_{n\geq} \) the torsion theory in \( \text{Simp}(\text{Grp}) \) given by the functor \( \text{Cot}_n \), i.e.:

\[
\mu_{n\geq} = (\text{Ker}(\text{Cot}_n), \mathcal{M}_{n\geq})
\]

and also the associated idempotent radical will be denoted by \( \mu_{n\geq} : \text{Simp}(\mathbb{X}) \to \text{Simp}(\mathbb{X}) \).

The category \( \mathcal{M}_{0\geq} \) is equivalent to the category \( \text{Dis}(\text{Grp}) \) of discrete simplicial groups, simplicial groups where all degeneracies and face morphisms are the identity. And it is well-know from Loday’s article [20] that the category \( \mathcal{M}_{1\geq} \) is equivalent to the category of internal grupoids \( \text{Grpd}(\text{Grp}) \).
Corollary 6.7. The categories $\text{Dis}(\text{Grp})$ of discrete simplicial groups and the category $\text{Grpd}(\text{Grp})$ of internal groupoids are torsion-free subcategories of $\text{Simp}(\mathbb{X})$.

Theorem 6.8. Let be $X$ a simplicial group with Moore complex $M$. The normalization functor $N$ maps the short exact sequence of $X$ given by the torsion theory $\mu_{n \geq}$ into the short exact sequence of $M$ given by $\text{COK}_{n+1}$.

Moreover, $N$ maps the torsion category $\text{Ker}(\text{Cot}_n)$ into the torsion category $\text{EP}_{n+1}$ and, respectively, the torsion-free category $\mathcal{M}_{n \geq}$ into $\text{pch}(\text{Grp})_{n \geq}$:

$$
\begin{array}{cccc}
\text{Ker}(\text{Cot}_n) & \perp & \text{Simp}(\mathbb{X}) & \perp & \mathcal{M}_{n \geq} \\
N & & N & & N \\
\text{EP}_{n+1} & \perp & \text{pch}(\mathbb{X}) & \perp & \text{pch}(\text{Grp})_{n \geq}
\end{array}
$$

Proof. Since the cotruncation functors commute up to isomorphism with normalization as in diagram 8 and $N$ preserves short exact sequences, the short exact sequence in $\text{Simp}(\text{Grp})$:

$$0 \longrightarrow \text{ker}(\eta_X) \longrightarrow X \xrightarrow{\eta_X} \text{Cot}_n(X) \longrightarrow 0$$

is mapped under $N$ into the short exact sequence (written vertically) in $\text{pch}(\text{Grp})$:

$$N(\text{ker}(\eta_x)) = \ldots \longrightarrow M_{n+1} \xrightarrow{\epsilon_{n+1}} \delta_{n+1}(M_{n+1}) \xrightarrow{m_{n+1}} 0 \longrightarrow \ldots$$

$$M = \ldots \longrightarrow M_{n+1} \xrightarrow{\delta_{n+1}} M_{n} \xrightarrow{\delta_{n}} M_{n-1} \longrightarrow \ldots$$

$$N(\text{Cot}_n(X)) = \ldots \longrightarrow 0 \longrightarrow M_n/\delta_{n+1}(M_{n+1}) \xrightarrow{\delta_{n+1}} M_{n-1} \longrightarrow \ldots$$

Since the associated short exact sequence of the torsion theory is preserved it follows that $N(\text{Ker}(\text{Cot}_n)) \subseteq \text{EP}_{n+1}$ and $N(\mathcal{M}_{n \geq}) \subseteq \text{pch}(\text{Grp})_{n \geq}$.  

Theorem 6.9. The torsion subcategories of the torsion theories $\mu_{n \geq}$ and $\mu_{n \geq+1}$ in $\text{Simp}(\text{Grp})$ are linearly ordered as:

$$0 \subseteq \ldots \subseteq \text{Ker}(\text{Cot}_{n+1}) \subseteq \mathcal{M}_{n \geq+1} \subseteq \text{Ker}(\text{Cot}_n) \subseteq \mathcal{M}_{n \geq} \subseteq \ldots \subseteq \text{Simp}(\text{Grp}).$$

Moreover, the torsion theories $\mu_{n \geq}$ and $\mu_{n \geq+1}$ form a linearly ordered lattice $\mu(\text{Grp})$:

$$0 \leq \ldots \leq \mu_{n+1 \geq} \leq \mu_{n+1} \leq \mu_{n \geq} \leq \mu_{n \geq} \leq \ldots$$

$$\ldots \leq \mu_{2 \geq} \leq \mu_{1 \geq} \leq \mu_{2 \geq} \leq \mu_{0 \geq} \leq \text{Simp}(\text{Grp}).$$

Proof. First we will prove $\mathcal{M}_{n \geq+1} \subseteq \text{Ker}(\text{Cot}_n)$. For a simplicial group $X$ and $M$ its Moore normalization and $\eta$ the unit as in Proposition 6.4. Then, if $M_i = 0$ for $n \geq i$ then $X_i = 0$ for $n \geq i$ and since $\eta_X$ is a normal epimorphism then $\text{Cot}_n(X)_i = 0$ for $n \geq i$. It follows from the semi-direct decomposition that $\text{Cot}_n(X) = 0$.

Now we prove $\text{Ker}(\text{Cot}_n) \subseteq \mathcal{M}_{n \geq}$. From it is clear that if $\text{Cot}_n(X) = 0$ we have $X_i = 0$ for $n-1 \geq i$ then $M_i = 0$ for $n-1 \geq i$ and $X$ is in $\mathcal{M}_{n \geq}$.  

Definition 6.10. We will write $\mu(\text{Grp})$ for the linearly order lattice of torsion theories in $\text{Simp}(\text{Grp})$ given by $\mu_{n \geq}$ and $\mu_{n \geq}$.  

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Theorem 6.11. The torsion theory $\mu_{n\geq}$ is hereditary and $\mu_{\geq n}$ is cohereditary. Moreover, the subcategories $M_{n\geq}$ and $M_{\geq n}$ are semi-abelian.

Proof. It follows from the fact that $N$ is an exact functor. Then, $M_{n\geq}$ and $M_{\geq n}$ are semi-abelian by theorem 2.14 and corollary 2.15 respectively. \qed

Theorem 6.12. 1. For $n \geq 1$, a simplicial group $X$ is torsion for $\mu_{n\geq}$ (i.e. it belongs to $\operatorname{Ker}(\operatorname{Cot})$) if and only if and $tr_{n-1}(X) = 0$ and $\eta_X$ is a normal epimorphism, where $\eta$ is the unit of $tr_{n-1} \circ \cosk_n$.

Proof. 1) Recall that $\mu_{n\geq} \leq \mu_{\geq n}$ we have the inclusion of torsion subcategories $\operatorname{Ker}(\operatorname{Cot}) \subseteq \operatorname{Ker}(\operatorname{tr}_{n-1})$. Then, a simplicial group $X$ belongs to $\operatorname{Ker}(\operatorname{tr}_{n-1})$ if and only if its Moore complex $M$ is trivial for degrees $n - 1 \geq i$, and moreover $X$ belongs to $\operatorname{Ker}(\operatorname{Cot})$ if and only if, in addition, $\delta_{n+1} : M_{n+1} \rightarrow M_n$ is a normal epimorphism.

Let $X$ belong to $\operatorname{Ker}(\operatorname{tr}_{n-1})$ and $\eta_X : X \rightarrow \cosk_n$ where $\eta$ is the unit of the adjunction $tr_{n-1} \circ \cosk_n$. The normalization of $\eta_X$ is

$$
\begin{align*}
M(X) = \ldots & \rightarrow M_{n+2} \rightarrow M_{n+1} \xrightarrow{\delta_{n+1}} M_n \rightarrow 0 \rightarrow \ldots \\
\downarrow M(\eta_X) & \downarrow \downarrow \downarrow \\
M(\cosk_n) = \ldots & \rightarrow 0 \rightarrow M_n \xrightarrow{1} M_n \rightarrow 0 \rightarrow \ldots
\end{align*}
$$

Since the normalization functor is conservative we have that $\delta_{n+1}$ is a normal epimorphism if and only if $\eta_X$ is a normal epimorphism. 2) It is similar to 1). \qed

Notice, as expected from the torsion theory $\langle \operatorname{Conn}(\operatorname{Grpd}), \operatorname{Dis}(\operatorname{Grpd}) \rangle$ in $\operatorname{Grpd}(\operatorname{Grpd})$, that the torsion category $\operatorname{Ker}(\operatorname{Cot})$ in $\operatorname{Simp}(\operatorname{Grp})$ contains the subcategory of connected internal groupoids $\operatorname{Conn}(\operatorname{Grpd})$, i.e. internal groupoids $X$ with the condition that $(d_0, d_1) : X_1 \rightarrow X_0^2$ is a normal epimorphism.

7 Homotopy groups and torsion subobjects

Definition 7.1. For $m \geq n$ and the idempotent radicals of the torsion theories of $\mu(\operatorname{Grp})$:

$$
\cdots \leq \mu_{m\geq} \leq \mu_{\geq m} \leq \cdots \leq \mu_{n\geq} \leq \mu_{\geq n} \leq \cdots,
$$

we consider the quotients of preradicals of $\operatorname{Simp}(\operatorname{Grp})$:

$$
\Pi_{m\geq}^{\geq n} := \mu_{m\geq}^{\geq n} / \mu_{m\geq}, \quad \Pi_{\geq m}^{\geq n} := \mu_{\geq m}^{\geq n} / \mu_{\geq m}, \quad \Pi_{\geq m}^{\geq n} := \mu_{\geq m}^{\geq n} / \mu_{\geq m}, \quad \Pi_{n\geq}^{\geq n} := \mu_{n\geq} / \mu_{n\geq};
$$
as well as, for all $n$ the trivial quotients:

$$
\Pi_{\geq n}^{\geq n} := \mu_{\geq n} / 0 \cong \mu_{\geq n}, \quad \Pi_{\geq n} := \mu_{\geq n} / 0 \cong \mu_{\geq n}, \quad \Pi_{n\geq} := \mu_{n\geq} / \mu_{n\geq}.
$$

For a simplicial group $X$ the objects

$$
\Pi_{m\geq}^{\geq n}(X), \quad \Pi_{\geq m}^{\geq n}(X), \quad \Pi_{\geq m}^{\geq n}(X), \quad \Pi_{\geq m}^{\geq n}(X), \quad \Pi_{\geq n}(X), \quad \Pi_{\geq n}(X), \quad \Pi_{\geq n}(X), \quad \Pi_{\geq n}(X)
$$

will be called the fundamental simplicial groups of $X$. Accordingly, the family of functors:

$$
\Pi_{m\geq}^{\geq n}, \Pi_{\geq m}^{\geq n}, \Pi_{\geq m}^{\geq n}, \Pi_{\geq n}^{\geq n}, \Pi_{\geq n}^{\geq n}, \Pi_{\geq n}^{\geq n}, \Pi_{\geq n}^{\geq n}, \Pi_{\geq n}^{\geq n} : \operatorname{Simp}(\operatorname{Grp}) \longrightarrow \operatorname{Simp}(\operatorname{Grp})
$$

will be called fundamental simplicial functors.
Following Proposition 6.4 the homotopy groups of $\Pi_{n \geq i}^n(X) = Id/\mu_{n \geq i}(X) = Cot_{n}^i(X)$ are the same as $X$ for $n \geq i$ and trivial elsewhere. The homotopy groups of the fundamental simplicial groups are the same as $X$ or trivial at some degrees. The following result generalizes 3) and 4) of 6.4.

**Theorem 7.2.** Let be $X$ a simplicial group with Moore complex $M$. The homotopy groups of the fundamental simplicial group of $X$ are calculated as follows:

1. For all $n \geq 0$
   \[
   \pi_i(\Pi^n_{n \geq i}(X)) = \pi_i(\Pi^{n+1}_{n \geq i}(X)) = \begin{cases} 
   \pi_i(M) & i \geq n + 1 \\
   0 & n + 1 > i. 
   \end{cases}
   \]

2. For all $n \geq 0$
   \[
   \pi_i(\Pi^n_{n \geq i}(X)) = \pi_i(\Pi^{n+1}_{n \geq i}(X)) = \begin{cases} 
   0 & i \geq n + 1 \\
   \pi_i(X) & n + 1 > i. 
   \end{cases}
   \]

3. For all $n \geq 0$
   \[
   \pi_i(\Pi^n_{n \geq i}(X)) = 0 \text{ for all } i.
   \]

4. For $m > n \geq 0$
   \[
   \pi_i(\Pi^{n \geq i}_{m \geq i}(X)) = \begin{cases} 
   \pi_i(X) & m + 1 > i \geq n + 1 \\
   0 & \text{otherwise}. 
   \end{cases}
   \]

5. Moreover, for $m > n \geq 0$ and for all $i$
   \[
   \pi_i(\Pi^n_{m \geq i}(X)) = \pi_i(\Pi^{n+1}_{m \geq i}(X)) = \pi_i(\Pi^{n+1}_{m \geq i}(X)) = \pi_i(\Pi^{n+1}_{m \geq i}(X)).
   \]

6. In particular, for $m = n + 1$
   \[
   \pi_i(\Pi^{n \geq i}_{n + 2}(X)) = \begin{cases} 
   \pi_i(X) & i = n + 1 \\
   0 & i \neq n. 
   \end{cases}
   \]

**Proof.** Since the Moore Normalization preserves short exact sequences and the preradicals $\mu_{n \geq i}$ and $\mu_{m \geq i}$ are mapped into the preradicals $cok_n$ and $ker_n$, this follows from the calculations of 4.3. \qed

7.3. For an abelian group $A$, a simplicial group $X$ is an Eilenberg-Mac Lane simplicial group of type $K(A, n)$ or a $K(A, n)$-simplicial group, if it has $\pi_n(X) = A$ and all other homotopy groups trivial.

In particular, the $n$-th Eilenberg-Mac Lane simplicial group $K(A, n)$ for an abelian group $A$ (in symmetric form) is defined as follows. Consider the $(n + 1)$-truncated simplicial group $k(A, n)$:

\[
k(A, n) = \begin{array}{c}
A^{n+1} \\
d_n \\
d_{n-1} \\
\vdots \\
d_0 \\
\end{array}
\begin{array}{c}
\rightarrow 0 \\
\rightarrow 0 \\
\rightarrow 0 \\
\rightarrow 0 \\
\rightarrow 0 \\
\end{array}
\begin{array}{c}
d_n+1 \\
d_n \\
d_{n-1} \\
\vdots \\
d_0 \\
\end{array}
\]

where the non-trivial face morphisms are

\[
(d_0, d_1, \ldots, d_{n+1}) = (p_0, p_0 + p_1 + p_1 + p_2, \ldots, p_n - 1 + p_n, p_n),
\]

where $p_i$ are the product projections and the degeneracies are given by $s_i = (0, \ldots, 1_A, \ldots, 0)$ with $1_A$ in the $i$th-place for $0 \leq i \leq n$. Then, we define

\[
K(A, n) = cosk_{n+1}(k(A, n)).
\]

It can be observed that for $m \geq n + 1$, $K(A, n)_m = A^{(\frac{m}{n})}$ where $\binom{m}{n}$ is the binomial coefficient.
Indeed, it is easy to see that the the Moore complex of $K(A, n)$ is:

$$M(K(A, n)) = \cdots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots \rightarrow 0.$$

(9)

This construction yields an embedding of the category $Ab$ of abelian groups into the category of simplicial groups $Simp(\text{Grp})$ at any degree $n \geq 1$:

$$K(\_ , n) : Ab \rightarrow Simp(\text{Grp}) .$$

At $n = 1$, it correspond with the usual definition of an abelian group as a simplicial group. For $n = 0$ we will also consider the embedding of discrete simplicial groups $\text{Dis}(\text{Grp}) \rightarrow Simp(\text{Grp})$.

7.4. The Dold-Kan Theorem gives an equivalence between the categories of simplicial abelian groups $Simp(\text{Ab})$ and chain complexes in abelian groups $\text{chn}(\text{Ab})_{\geq 0}$, where the equivalence is given by the Moore normalization. In [6] this equivalence was further extended to an equivalence between $Simp(\text{Grp})$ and the category of hypercrossed modules in $\text{Grp}$. A hypercrossed module is a group chain complex $M$ with group actions for all $\alpha$:

$$\Phi^n_\alpha : M_{r(\alpha)} \rightarrow \text{Aut}(M_n) \text{ for } \alpha \in S(n)$$

and binary operations

$$\Gamma^n_{\alpha, \sigma} : M_{r(\alpha)} \times M_{r(\sigma)} \rightarrow M_n \text{ for } \alpha, \sigma \in S(n), 1 < \sigma < \alpha, \alpha \cap \sigma = \emptyset$$

satisfying some equations. $S(n)$ has the order introduced in [6,1] and $\alpha \cap \sigma = \emptyset$ means that the maps $\alpha, \sigma$ do not share a common index in their factorization by the degeneracies $i$.

**Lemma 7.5.** Let $X$ be a simplicial group with Moore complex

$$M = \cdots \rightarrow 0 \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots$$

then $X$ isomorphic to the Eilenbeg-Mac Lane simplicial group $K(A, n)$.

**Proof.** Consider the chain complex $M$ as above. Since all degrees of $M$ are trivial except one, any morphism $M_i \rightarrow \text{Aut}(M_j)$ with $j > i$ is trivial as well as any binary mappings $M_i \times M_j \rightarrow M_k$ with $k > i, j$. This means that the structure of hypercrossed module is necessarily unique. Thus, it follows from the equivalence of hypercrossed modules and simplicial groups (see [6] that $X \cong K(A, n)$.

The next corollary generalizes part 4) of Proposition 6.4.

**Corollary 7.6.** For $n \geq 0$ and a simplicial group $X$ the simplicial groups:

$$\Pi_{n+2}^{\geq n+2}(X), \Pi_{n+1}^{\geq n+1}(X), \Pi_{n+2}^{\geq n+1}(X), \Pi_{n+1}^{\geq n+1}(X)$$

are $K(\pi_{n+1}(X), n + 1)$-simplicial groups.

Moreover, $\Pi_{n+1}^{\geq n+1}(X)$ is isomorphic to $K(\pi_{n+1}(X), n + 1)$ the $(n + 1)$-th Eilenberg-Mac Lane simplicial group of $\pi_{n+1}(X)$.

**Proof.** It follows by definition from 6) of theorem 7.2.

Moreover, the Moore complex of $\Pi_{n+1}^{\geq n+1}(X)$ isomorphic to the chain complex $\frac{\ker_{n+1}(M)}{\text{cok}_{n+2}(M)}$, i.e.:

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow \pi_{n+1}(X) \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

It follows from lemma 7.5 that $\Pi_{n+1}^{\geq n+1}(X) \cong K(\pi_{n+1}(X), n + 1)$.
Corollary 7.7. 1. The fundamental functor $\Pi_{0 \geq}$ is naturally isomorphic to the connected component functor $\pi_0$ followed by the discrete functor:

$$\xymatrix{ \text{Simp}(\text{Grp}) \ar[r]^{\Pi_{0 \geq}} \ar[d]_{\pi_0} & \text{Simp}(\text{Grp}) \ar[d]_{\text{Dis}} \\
\text{Grp} \ar[ur]_{\text{Cot}} & }$$

2. For $n \geq 1$, the fundamental functor $\Pi_{n+1 \geq}$ is naturally isomorphic to the homotopy group functor $\pi_n$ followed by the embedding $K(\omega, n)$:

$$\xymatrix{ \text{Simp}(\text{Grp}) \ar[r]^{\Pi_{n+1 \geq}} \ar[d]_{\pi_n} & \text{Simp}(\text{Grp}) \ar[d]_{\text{Ab}} \\
\text{Grp} \ar[ur]_{K(\omega, n)} & }$$

Proof. 1) From Corollary 6.7, the torsion theory $\mu_0 \geq$ has as torsion-free reflector the functor $\Pi_{0 \geq} = \text{Cot}_0 = \text{Dis} \pi_0$ since $\pi_0(X) = \text{coeq}(d_0, d_1) = X_0/\delta_1(M_1)$. 2) It follows from Corollary 7.6.

Following [13], the fundamental groupoid or Poincaré groupoid $\Pi_1(X)$ of a simplicial set $X$ has as objects the set $X_0$, the vertices of $X$, and morphisms are generated by the elements of $X_1$ and their formal inverses and the relations $s_0(x) = 1_x$ if $x \in X_0$ and $(d_0\sigma)(d_2\sigma) = d_1\sigma$ if $\sigma \in X_2$. Recently, in [10] the fundamental groupoid $\Pi_1 : \text{Simp}(X) \to \text{Grpd}(X)$ has been studied for simplicial objects in an exact Mal’tsev category $X$ as the left adjoint of the nerve functor $N : \text{Grpd}(X) \to \text{Simp}(X)$. Indeed, if $X$ is semi-abelian (in particular the category of groups as in our case) for a simplicial object $X$, $\Pi_1(X)$ is the unique groupoid structure that has

$$X_1/(d_2(Ker(d_0) \cap Ker(d_1))) \cong X_0$$

as the underlying reflexive graph, which for simplicial groups corresponds to the cotruncation functor $\text{Cot}_1$. Thus we have:

Corollary 7.8. The fundamental functor $\Pi_{1 \geq}$ is naturally isomorphic to the fundamental groupoid functor as indicated in the diagram:

$$\xymatrix{ \text{Simp}(\text{Grp}) \ar[r]^{\Pi_{1 \geq}} \ar[d]_{\Pi_1} & \mathcal{M}_{1 \geq} \ar[d]_{\mathcal{N}} \\
\text{Grpd}(\text{Grp}) \ar[ur]_{\text{M}_1} & }$$

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