EMBEDDINGS OF SMOOTH FUNCTION SPACES, EXTRAPOLATIONS, AND RELATED INEQUALITIES

ÓSCAR DOMÍNGUEZ AND SERGEY TIKHONOV

ABSTRACT. We study the Sobolev embedding in

- subcritical case, that is, \((W^{k,p}_0(\Omega)) \hookrightarrow L^{p^*}_{\nu,p}(\Omega)\) for \(k < d/p\),
- critical case, that is, \((W^{k,p}_0(\Omega)) \hookrightarrow Y\) for \(k = d/p\) and appropriate \(Y\),
- and supercritical case, that is, \(W^{k,p}_0(\mathcal{X}) \hookrightarrow Y\) for \(k > d/p, \mathcal{X} \in \{\mathbb{R}^d, \mathbb{T}^d\}\) and appropriate \(Y\).

We obtain characterizations of these embeddings in terms of pointwise inequalities involving rearrangements and moduli of smoothness/derivatives of functions and via extrapolation theorems for corresponding smooth function spaces. Applications include, among others, Ulyanov–Kolyada type inequalities for rearrangements, inequalities for moduli of smoothness, sharp Jawerth–Franke embeddings for Lorentz–Sobolev spaces, various characterizations of Gagliardo–Nirenberg, Trudinger, Maz’ya–Hansson–Brézis–Wainger and Brézis–Wainger embeddings.

Contents

1. Introduction 2
1.1. An overview 2
1.2. Main results and observations 2
1.3. Methodology 10
1.4. Structure of the paper 10
2. Notation and definitions 10
2.1. Function spaces 10
2.2. Interpolation methods 13
3. Auxiliary results 14
3.1. Hardy-type inequalities 14
3.2. Properties of the modulus of smoothness 15
3.3. Some interpolation lemmas 17
4. Subcritical case 19
5. Critical case 33
6. Supercritical case 46
References 67

2010 Mathematics Subject Classification. Primary 46E35, 42B35; Secondary 26A15, 46E30, 46B70.

Key words and phrases. Lebesgue, Lorentz, Besov, Lipschitz, Sobolev spaces; Embeddings; Extrapolations; Moduli of smoothness; Interpolation; Rearrangement inequalities.
1. Introduction

1.1. An overview. Sobolev inequalities constitute an important part of functional analysis and geometry with a wide range of remarkable applications in the theory of PDE’s, calculus of variations and mathematical physics [AH, EE04, Maz, SobBook]. The classical Sobolev theorem [Sob] reads as follows:

\[ (W^k_p(\Omega))_0 \hookrightarrow L^{p^*}(\Omega), \quad p^* = \frac{dp}{d-kp} \]

provided that \( k \in \mathbb{N}, 1 \leq p < \infty \) and \( k < d/p \). Here \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^d, d \geq 2 \).

One of the beauties of Sobolev inequalities is that they link different fields, which originally were studied from distinct perspectives. Below we briefly review some of these links.

The original proof of Sobolev embedding (1.1) ([Sob], see also [SobBook] and [Maz]) uses tools from functional analysis, namely, an integral representation formula which allows us to reconstruct functions from their derivatives. This rather complicated proof was later significantly simplified by Peetre [Pc] (with [Hu] and [ON] as forerunners) applying interpolation methods. In fact, Peetre’s approach uses a more refined technique replacing the target space \( L^{p^*}(\Omega) \) in (1.1) by the smaller Lorentz space \( L^{p^*,p}(\Omega) \) (note that \( p < p^* \)), that is,

\[ (W^k_p(\Omega))_0 \hookrightarrow L^{p^*,p}(\Omega). \]

Furthermore, this embedding is optimal within the class of r.i. spaces.

The link between Sobolev inequalities and isoperimetric inequalities is found in the seminal papers of Maz’ya [Maz60, Maz61] and Federer and Fleming [FedFle]. This interplay has witnessed a tremendous progress in both research areas, yielding not only to sharp versions of Sobolev embeddings in a broader setting (see [Maz, MM10] and the references therein) but also to optimal constants in Sobolev inequalities (see, e.g., [BreLi, Mos, Tal]). However, these results strongly relied on the Pólya-Szegő principle (cf. [Mossi, AlLi]) and were mainly restricted to first-order Sobolev functions. The higher-order case causes additional obstructions due to the lack of a full analogue of the Pólya-Szegő inequality. It has been studied in a systematic way only recently by Cianchi, Pick and Slavíková [CiPS], who applied rather sophisticated iteration constructions. Earlier results in this direction had been obtained in [MM07b].

Sobolev inequalities involving r.i. norms can also be rephrased in terms of one-dimensional Hardy type operators. This approach was studied in detail in Edmunds, Kerman and Pick [EKP], Cianchi [Ci04b], Kerman and Pick [KP] and Curbera and Ricker [CuRi] under the name reduction theorems; these inequalities can also be traced in the work of Netrusov [Ne87b].

We would also like to mention the concept of envelopes, see [Tri01, Har], and the mass transportation method applied to the study of first-order Sobolev inequalities in \( \mathbb{R}^d \) given in [CorNaVi].

1.2. Main results and observations. The main objective of this paper is to provide two different characterizations of the Sobolev embedding

\[ (W^k_p(\Omega))_0 \hookrightarrow L^{p^*,p}(\Omega). \]
where \( X \) and \( Y \) are suitable function spaces. Firstly, we show that (1.3) is equivalent to certain pointwise inequalities involving rearrangements and moduli of smoothness/derivatives of functions. Secondly, the embedding (1.3) can be equivalently written via extrapolation theorems for corresponding smooth spaces. Let us give more detail to explain each approach. We shall consider three cases:

- subcritical case:
  \((W^k_p(\Omega))_0 \hookrightarrow L_{p^*,p}(\Omega)\) \(k < d/p\),

- critical case:
  \((W^k_p(\Omega))_0 \hookrightarrow Y\) \(k = d/p\) and appropriate \(Y\),

- and supercritical case:
  \(\dot{W}^k_p(X) \hookrightarrow Y\) \(k > d/p\), \(X \in \{\mathbb{R}^d, \mathbb{T}^d\}\) and appropriate \(Y\).

**Subcritical case.** In this setting \((k < d/p)\) it is well known that the embedding \((W^k_p(\Omega))_0 \hookrightarrow L_{p^*,q}(\Omega)\) holds if and only if \(q \geq p\) (cf. (1.2)). We obtain (see Theorems 4.1 and 4.3 below) the following four new equivalent conditions for this embedding to hold:

(i) for \(f \in L^p(\Omega)\), we have

\[
\left( \int_0^1 f^*(u)^p du \right)^{1/p} + t^k \left( \int_{\mathbb{R}^d} u^{q/p} f^*(u)^{q/p} du \right)^{1/q} \lesssim \omega_k(f, t)_p,
\]

(ii) for \(f \in (W^k_p(\Omega))_0\), we have

\[
\left( \int_1^t \left( u^{1/p-1} \int_0^u u^{1-k/d} f^*(u) du^{1/q} dv \right)^{1/q} \right)^{1/p} \lesssim \left( \int_t^1 (|\nabla f|^*)^p dv \right)^{1/p},
\]

(iii) there exists \(C > 0\), which is independent of \(s\), such that

\[
\|f\|_{L^d_{-sp/q}(\Omega)} \leq C(k-s)^{1/q}\|f\|_{B^s_{p,q}(\Omega)}, \quad 0 < s < k,
\]

(iv) there exists \(C > 0\), which is independent of \(r\), such that

\[
\|f\|_{L^r_{-r/q}(\Omega)} \leq C\|f\|_{(W^k_{r,p}(\Omega))_0}, \quad r < p.
\]

To understand better the strength of the previous result, let us consider the endpoint case \(p = 1\). This case deserves special attention because it corresponds to the sharp Gagliardo-Nirenberg inequality [Poo]

\[
\|f\|_{L^d_{-1/q}(\Omega)} \leq C\|\nabla f\|_{L^1(\Omega)}, \quad \frac{1}{d} + \frac{1}{d'} = 1.
\]

It is well known that the classical Gagliardo-Nirenberg inequality, which is obtained by replacing the \(L^d_{-1}(\Omega)\)-norm on the left-hand side of (1.8) by the \(L^d(\Omega)\)-norm, is equivalent to both the isoperimetric inequality (cf. [Maz]) and (1.8) (the self-improving property; cf. [Haj]). Moreover, the classical Sobolev inequality (1.1) follows from the Gagliardo-Nirenberg inequality via elementary computations (see [Sa]). Further characterizations of (1.8) were obtained in Martín, Milman and
Pustylnik [MMP]. In particular, they showed [MMP, Theorem 1] that \(^{(1.8)}\) is equivalent to the symmetrization inequality

\[
\int_0^t u^{-1/d}(f^{**}(u) - f^*(u))du \lesssim \int_0^t |\nabla f|^*(u)du.
\]

The previous characterizations of the sharp Gagliardo-Nirenberg inequality (or equivalently, the isoperimetric inequality) are now complemented by \(^{(1.4)}\) and \(^{(1.6)}\). Specifically, we obtain that \(^{(1.8)}\) holds if and only if

\[
\int_0^{t^d} f^*(u)du + t \int_{t^d}^1 u^{1/d'} f^{**}(u)\frac{du}{u} \lesssim \omega_1(f, t)_1
\]

(see \(^{(1.4)}\)). Note that the latter inequality strengthens the well-known Ulyanov–Kolyada inequality [Kol89a], which plays a central role in embedding theorems:

\[
t \left( \int_{t^d}^{\infty} u^{-p/d} \int_0^u (f^*(v) - f^*(u))^p \frac{du}{v} \right)^{1/p} \lesssim \omega_1(f, t)_p,
\]

\(1 \leq p < \infty;\)

see Remark 4.2(i) below.

We also show that the Gagliardo-Nirenberg inequality \(^{(1.8)}\), or more generally, Sobolev inequalities \(^{(1.2)}\), can be characterized by extrapolation means. Indeed, the assertion \(^{(1.6)}\) provides sharp estimates for the blow-up rates of the norm of the classical embedding \(B^s_{p,q}(\Omega) \hookrightarrow L^{d/p}_{d-2s/q}(\Omega), s < d/p,\) as the smoothness parameter \(s\) approaches the certain critical value. This extrapolation assertion fits into the research program initiated by Bourgain, Brézis, Mironescu [BBM] and Maz’ya, Shaposhnikova [MS] (see also [EEK06, EEEK07, KMX, KolLe]).

Let us examine the new connection between \(^{(1.2)}\) and \(^{(1.6)}\) more carefully. In light of the Yano extrapolation theorem (see [JM, Mil]), it follows that \(^{(1.6)}\) implies \((W^k_p(\Omega))_0 \hookrightarrow L^p^*(\Omega).\) However, it is remarkable that the converse statement also holds true, that is, we are able to obtain converse extrapolation theorems for Sobolev embeddings in the spirit of Tao’s paper [Tã]. In other words, from an endpoint embedding (i.e., \((W^k_p(\Omega))_0 \hookrightarrow L^p^*(\Omega)\)) one can derive all intermediate embeddings with sharp blows up of the norms (i.e, \(B^s_{p,q}(\Omega) \hookrightarrow L_{d-2s/q}(\Omega)\)) with the norm bound \(O((k - s)^{1/q})\) as \(s \to k^-).\) A similar comment also applies to \(^{(1.7)}\).

The problem of optimality of Sobolev embeddings can be considered from the different perspective. More specifically, so far we have dealt with the optimality of the target space in \(^{(1.2)}\). Conversely, fixing the target space, the counterpart of the embedding \((W^k_p(\Omega))_0 \hookrightarrow L^{p_*,q}(\Omega)\) is the embedding \((W^k L^p_{p,q}(\Omega))_0 \hookrightarrow L^{p_*(\Omega)}\) which holds if and only if \(q \leq p^*.\) We again give four different equivalent conditions for the latter embedding (see Theorems 4.4 and 4.7 below):

(i) for \(f \in (W^k L^p_{p,q}(\Omega))_0,\) we have

\[
\omega_k(f,t)_{p^*} \lesssim \left( \int_0^{t^d} (u^{1/p} |D^k f|^*(u))^{d/p} \frac{du}{u} \right)^{1/q} + t^k \left( \int_{t^d}^1 (|D^k f|^*(u))^{p^*} \frac{du}{u} \right)^{1/p^*},
\]
(ii) for \( f \in (W^k L_{p,q}(\Omega))_0 \), we have

\[
\left( \int_0^1 \left( u^{1/p-1} \int_0^u u^{1-k/d} f^*(u) \frac{du}{u} \right)^{p^*} \frac{dv}{v} \right)^{1/p^*} \lesssim \left( \int_0^1 (v^{1/p} |\nabla^k f|^*)^q \frac{dv}{v} \right)^{1/q},
\]

(iii) for any \( m \in \mathbb{N} \) there exists \( C > 0 \), which is independent of \( r \), such that

\[
\| f \|_{B^{(1/p-1/r)}_{p^*,q}(\Omega),m} \leq C (r-p)^{-1/q} \| f \|_{(W^k L_{r,q}(\Omega))_0}, \quad r > p,
\]

(iv) there exists \( C > 0 \), which is independent of \( r \), such that

\[
\| f \|_{L^{r^*,p^*}(\Omega)} \leq C \| f \|_{(W^k L_{r,q}(\Omega))_0}, \quad r < p.
\]

Note that the inhomogeneous counterpart of (1.9) with \( q = p \) has been recently obtained in [GNO, Theorem 3.2]; while assertions (ii) – (iv) are new. The extrapolations of \((W^k L_{p,q}(\Omega))_0 \hookrightarrow L_{p^*}(\Omega)\) given in (1.11) and (1.12) involve the Sobolev spaces \((W^k L_{r,q}(\Omega))_0\) with \( r \to p^+ \) and \( r \to p^- \), respectively. In particular, (1.11) provides sharp blow up of the norm of the recently obtained Jawerth-Franke embedding for Lorentz-Sobolev spaces [SeTr, Theorem 1.2]. It is worthwhile to mention that, unlike (1.6), the constant in (1.11) becomes arbitrary large as \( r \to p^+ \). This phenomenon shows an interesting distinction between optimal range space and optimal domain space in Sobolev inequalities.

Our approach is flexible enough to be applied to the more general Sobolev embeddings \((W^k L_{p,q}(\Omega))_0 \hookrightarrow L_{p^*,q_1}(\Omega), q_0 \leq q_1 \) (cf. [Tal, MiPu, CiPS]); see Theorem 4.9 below.

**Critical case.** In the case \( k = d/p \in \mathbb{N} \) the embedding (1.1) fails to be true, that is,

\[
(W^{d/p}(\Omega))_0 \not\hookrightarrow L_{\infty}(\Omega), \quad p > 1.
\]

To overcome this drawback we can use two different strategies. On the one hand, Sobolev embeddings can be obtained for the fixed domain space (i.e., \((W^{d/p}(\Omega))_0\)) by enlarging the target space (i.e., \(L_{\infty}(\Omega)\)). On the other hand, for the fixed target space (i.e., \(L_{\infty}(\Omega)\)) one can restrict the domain space (i.e., \((W^{d/p}(\Omega))_0\)).

Firstly, we shall concentrate on the case when \((W^{d/p}(\Omega))_0\) in (1.13) is fixed. We start with the well-known embedding

\[
(W^{d/p}(\Omega))_0 \hookrightarrow L_{\infty}(\log L)^{-1/p'}(\Omega),
\]

which is traditionally attributed to Trudinger [Tru] with Peetre [Pe], Pohozaev [Po] and Yudovich [Yu] as forerunners (cf. also [Stri]). Furthermore, it is optimal within the class of Orlicz spaces (see [Ci04a]).

In this paper, following the program suggested in a study of the subcritical case, we establish new characterizations of the Trudinger inequality via rearrangement inequalities in terms of moduli of smoothness and extrapolation means (see Theorem 5.1 below).

Despite its importance, the embedding (1.14) is not optimal among all r.i. spaces. This is illustrated by the fact that the embedding \((W^{d/p}(\Omega))_0 \hookrightarrow L_q(\Omega), q < \infty\),
can be improved involving Lorentz spaces. Indeed, we have
\begin{equation}
(W^{d/p}_p(\Omega))_0 \hookrightarrow L_{q,p}(\Omega) \quad \text{for all } q < \infty.
\end{equation}

Note that $L_{q,p}(\Omega) \subseteq L_q(\Omega)$, $q > p$. In view of (1.15), it was shown independently by Hansson [Han] and Brézis, Wainger [BreWain] that
\begin{equation}
(W^{d/p}_p(\Omega))_0 \hookrightarrow L_{\infty,p}(\log L)^{-1}(\Omega).
\end{equation}

This embedding is rooted in the work of Maz’ya on isocapacitary inequalities [Maz, p. 232] (see also [AH, (7.6.1), p. 209]). Related results may be found in the paper by Brudnyi [Bru] and Hedberg [Hed]; a more general assertion was obtained by Cwikel and Pustylnik [CwP].

Furthermore, the target space in (1.16) is the best possible among the class of r.i. spaces (see [Han] and [CwP]). In particular, we have
\begin{equation}
(W^{d/p}_p(\Omega))_0 \hookrightarrow L_{\infty,p}(\log L)^{-b}(\Omega) \iff b \geq 1.
\end{equation}

Our goal is to establish new links between the Maz’ya-Hansson-Brézis-Wainger embedding (1.16), estimates for moduli of smoothness and extrapolation constructions that complement the well-known connection between (1.16) and isocapacitary inequalities traced back to Maz’ya. To be more precise, we obtain (cf. Theorem 5.3) that (1.17) is equivalent to either of the conditions:

(i) for $f \in L_p(\Omega)$, we have
\begin{equation}
\left( \int_1^t (1 - \log u)^{-bp} f^*(u)^p \frac{du}{u} \right)^{1/p} \lesssim t^{-1/p} \omega_{d/p}(f, t^{1/d})_p,
\end{equation}

(ii) if $0 < \lambda < d/p$ then there exists $C > 0$, which is independent of $\lambda$, such that
\begin{equation}
\|f\|_{L^{d/\lambda,p}(\log L)^{-b}(\Omega)} \leq C\lambda^{1/p} \|f\|_{B^{d/p-\lambda}\text{d/p}_p(\Omega)}.
\end{equation}

The new rearrangement inequality (1.18) provides the quantitative version of the Maz’ya-Hansson-Brézis-Wainger embedding. Moreover, inequality (1.19) can be considered as the logarithmic counterpart of the Bourgain-Brézis-Mironescu-Maz’ya-Shaposhnikova theorem; see the discussion in Remark 5.4(i) below. Furthermore, our result answers a question raised by Martín and Milman in [MM07a] concerning characterizations of the Maz’ya-Hansson-Brézis-Wainger embedding in terms of limits of a family of norms (i.e., $\{\|f\|_{L^{d/\lambda,p}(\log L)^{-1}(\Omega)} : \lambda \to 0^+\}$); see Remark 5.4(ii). In fact, we go a step further and show that the converse is also true, that is, from the embedding (1.16) we achieve all intermediate Sobolev embeddings $B^{d/p-\lambda}_p(\Omega) \hookrightarrow L^{d/\lambda,p}(\log L)^{-1}(\Omega)$ with sharp behaviour $O(\lambda^{1/p})$ of the norms as $\lambda \to 0^+$. Let us now focus on Sobolev embeddings into $L^{\infty}(\Omega)$. In the one-dimensional setting, the fundamental theorem of Calculus implies $(W^1_1(a,b))_0 \hookrightarrow L^{\infty}(a,b)$. In the higher dimensional case the corresponding estimate fails to be true, that is, one can find functions $f \in (W^1_1(\Omega))_0$ such that $f \notin L^{\infty}(\Omega)$ (see (1.13) with $p = d$). This obstruction can be overcome with the help of the refined scale given by Lorentz-Sobolev spaces. Namely, Stein [Ste] showed that
\begin{equation}
(W^1 L^{d,1}_d(\Omega))_0 \hookrightarrow L^{\infty}(\Omega);
\end{equation}
an alternative proof may be found in DeVore and Sharpley [DS84a]. Note that 
$L_{d,1}(\Omega) \subset L_d(\Omega)$ if $d \geq 2$. A characterization of all r.i. spaces $X$ for which the first-order Sobolev space modelled on $X$ is formed by bounded functions was obtained by Cianchi and Pick [CiP]. In particular, they proved the optimality of $L_{d,1}(\Omega)$ within the class of r.i. spaces. Cianchi, Pick and Slavíková [CiPS] have recently extended these results to higher-order derivatives. Namely, they proved (cf. [CiPS, Theorem 6.9]) that if $k < d$ then

$$\tag{1.20} (W^k L_{d/k,q}(\Omega))_0 \hookrightarrow L_\infty(\Omega) \iff q \leq 1. \label{eq:1.20}$$

Our purpose is to provide characterizations of embedding (1.20) in terms of sharp inequalities of the $L_\infty$-moduli of smoothness, as well as extrapolation estimates of Jawerth-Franke type embeddings for Lorentz-Sobolev spaces. In more details (cf. Theorems 5.6 and 5.8) the embedding $(W^k L_{d/k,q}(\Omega))_0 \hookrightarrow L_\infty(\Omega)$ is equivalent to any of the following four conditions:

(i) for $f \in (W^k L_{d/k,q}(\Omega))_0$, we have

$$\omega_k(f,t) \lesssim \left( \int_0^t (u^{k/d} |\nabla^k f|^*)(u)^q \frac{du}{u} \right)^{1/q}, \tag{1.21}$$

(ii) for $f \in (W^k L_{d/k,q}(\Omega))_0$, we have

$$t^{k/d-1} \int_0^t u^{1-k/d} f^*(u) \frac{du}{u} \lesssim \left( \int_t^1 (u^{k/d} |\nabla^k f|^*)(u)^q \frac{du}{u} \right)^{1/q}, \tag{1.22}$$

(iii) for any $m \in \mathbb{N}$ there exists $C > 0$, which is independent of $r$, such that

$$\|f\|_{B^k_{\infty,q}(\Omega),m} \leq C \left( r - \frac{d}{k} \right)^{-1/q} \|f\|_{(W^k L_{r,q}(\Omega))_0}, \quad r > \frac{d}{k}, \tag{1.23}$$

(iv) there exists $C > 0$, which is independent of $r$, such that

$$\|f\|_{L^q_{*,q}(\Omega)} \leq C \left( \frac{d}{k} - r \right)^{-1/q} \|f\|_{(W^k L_{r,q}(\Omega))_0}, \quad r < \frac{d}{k}. \tag{1.24}$$

To the best of our knowledge, this is the first result to establish equivalence between the Stein inequality (1.20) and pointwise rearrangement inequalities. Note that inequality (1.21) with $q = 1$, i.e.,

$$\omega_k(f,t) \lesssim \int_0^t u^k |\nabla^k f|^*(u) \frac{du}{u}$$

is due to DeVore and Sharpley [DS84a, Lemma 2] if $k = 1$ and, Kolyada and Pérez Lázar [KoPe, (1.6)] for higher-order derivatives. It plays a central role in the theory of function spaces as can be seen in [Har], [GMNO], and the references within. On the other hand, the new inequality (1.22) gives a nontrivial improvement of the Kolyada inequality [Ko07, Corollary 3.2]

$$f^{**}(t) \lesssim \int_t^1 u^{k/d} |\nabla^k f|^{**}(u) \frac{du}{u}.$$

For further details, see Remark 5.9(i) below.
Another observation concerns the recent Jawerth-Franke embedding for Lorentz-Sobolev spaces obtained by Seeger and Trebels \cite{SeTr}, which asserts that if $1 < r < p < \infty$, $0 < q \leq \infty$ and $k > d(\frac{1}{r} - \frac{1}{p})$, then

\begin{equation}
(W^k L_{r,q}(\Omega))_0 \hookrightarrow B^{k-d/r+d/p}_{p,q}(\Omega). \tag{1.25}
\end{equation}

However, the limiting case $p = \infty$ was left open in \cite{SeTr}. Remarkably, this case is closely related to the Stein inequality (1.20). Indeed, as a byproduct of (1.20), we can consider the case $p = \infty$ in (1.25) and, in addition, obtain sharp estimates of the rates of blow up of the corresponding embedding constant (cf. (1.23)). Furthermore, the converse assertion is also valid, that is, the Stein inequality (1.20) follows from Jawerth-Franke embeddings (1.25) with $p = \infty$ via extrapolation. In a similar vein, we prove that the Stein inequality is equivalent to extrapolate the sharp version of Talenti’s embedding \cite{Tal}

\begin{equation}
(W^k L_{r,q}(\Omega))_0 \hookrightarrow L^{r_*,q}_{r}(\Omega), \quad r < \frac{d}{k} \tag{1.26}
\end{equation}

(cf. (1.24)). It is worth mentioning that the optimal constant in (1.26) with $k = 1$ was obtained in \cite{Al} and \cite{Tal}. To the best of our knowledge, the corresponding question for higher-order derivatives (i.e., $k > 1$) still remains open. Note that in (1.24) we derive the optimal asymptotic behaviour of the constant in (1.26) with respect to the integrability parameter $r \to \frac{d}{k} -$.

Supercritical case. It is well known that there are functions from $\dot{H}^{1+d/p}_{p}(\mathbb{T}^d)$, $1 < p < \infty$, that are not Lipschitz-continuous. However, due to the celebrated Brézis-Wainger theorem \cite{BreWain}, functions $f$ from Sobolev spaces $\dot{H}^{1+d/p}_{p}(\mathbb{T}^d)$ are almost Lipschitz-continuous in the sense that

$$|f(x) - f(y)| \leq C|x - y| \log |x - y|^{1/p'} \|f\|_{\dot{H}^{1+d/p}_{p}(\mathbb{T}^d)}$$

for all $0 < |x - y| < 1/2$. Note that this inequality can be interpreted in terms of the embedding

$$\dot{H}^{1+d/p}_{p}(\mathbb{T}^d) \hookrightarrow \operatorname{Lip}_{\infty, \infty}^{(1, -1/p')}(\mathbb{T}^d).$$

This result has found profound applications in function spaces and PDEs. Just to mention some of them, it was the starting point of the theory of continuity envelopes of function spaces \cite{Tri06, Har}. It also plays a central role in studying the eigenvalue distribution of certain pseudo-differential operators \cite{EH99, EH00}. For extensions of the Brézis-Wainger inequality to the more general class of Triebel-Lizorkin spaces and Besov spaces, we refer the reader to Edmunds and Haroske \cite{EH99, EH00}.

For convenience, we temporarily restrict our discussion to $d = 1$. We study characterizations of Brézis-Wainger embeddings for the Sobolev spaces $\dot{H}^{\alpha+1/p}_{p}(\mathbb{T})$, $\alpha > 0$. Namely, in Theorem 6.1, we show that the following statements are equivalent:

(i) \begin{equation}
\dot{H}^{\alpha+1/p}_{p}(\mathbb{T}) \hookrightarrow \operatorname{Lip}_{\infty, \infty}^{(\alpha, -b)}(\mathbb{T}), \tag{1.27}
\end{equation}

(ii) for $f \in B^{1/p}_{p,1}(\mathbb{T})$, we have

\begin{equation}
\omega_{\alpha}(f, t)_{\infty} \lesssim \int_0^t (1 - \log u)^{b/\alpha} u^{-1/p} \omega_{\alpha+1/p}(f, u) \frac{du}{u}, \tag{1.28}
\end{equation}
There exists $C > 0$, which is independent of $\alpha_0$, such that
\begin{equation}
\|f\|_{C^{\alpha_0}(\mathbb{T}),\alpha} \leq C(\alpha - \alpha_0)^{-b}\|f\|_{\dot{H}^{\alpha+1/p}(\mathbb{T})}, \quad 0 < \alpha_0 < \alpha,
\end{equation}

(iv) $b \geq 1/p'$.

This result shows that Brézis-Wainger inequalities are closely connected to Ulyanov inequalities \[Ul\]. The latter is central in approximation theory, function spaces and interpolation theory (cf. \[KoT1\] and the references therein). Taking $b = 1/p'$ in (1.28) we get a new Ulyanov-type inequality, which improves the known estimate \[Ti10, (1.5)\]

\[\omega_\alpha(f, t)_{\infty} \lesssim \int_0^t u^{-1/p}(1 - \log u)^{1/p'}\omega_{\alpha+1/p}(f, u)p\frac{du}{u}\]
in several directions; see discussion in Remark 6.2(i). The higher-dimensional version of (1.28) also holds true for functions $f$ on $\mathbb{T}^d$ or $\mathbb{R}^d$; see Remark 6.3.

The Brézis-Wainger embedding (1.27) can be complemented by the well-known Jawerth-Franke embedding (cf. \[Ja, Fra\]; see also \[Mars\] and \[Vy\])

\begin{equation}
\dot{H}^{\alpha+1/p}(\mathbb{T}^d) \hookrightarrow B^\alpha_{\infty,p}(\mathbb{T}^d).
\end{equation}

Accordingly, we study characterizations of (1.30) via pointwise inequalities for moduli of smoothness and extrapolations. More precisely, Theorem 6.13 asserts that the following statements are equivalent:

(i) $\dot{H}^{\alpha+1/p}(\mathbb{T}^d) \hookrightarrow B^\alpha_{\infty,q}(\mathbb{T}^d)$,

(ii) for $f \in B^{d/p}_{p,1}(\mathbb{T}^d)$, we have

\begin{equation}
t^\alpha \left( \int_t^\infty (u^{-\alpha}\omega_{\alpha+d/p}(f, u)_{\infty})^q\frac{du}{u} \right)^{1/q} \lesssim \int_0^t u^{-d/p}\omega_{\alpha+d/p}(f, u)p\frac{du}{u},
\end{equation}

(iii) there exists $C > 0$, which is independent of $\lambda$, such that

\begin{equation}
\|f\|_{B^{\alpha-\lambda}_{\infty,\alpha+d/p}(\mathbb{T}^d),\alpha+d/p} \leq C\lambda^{1/q}\|f\|_{B^{\alpha+1/p-\lambda}_{p,q}(\mathbb{T}^d),\alpha+d/p}, \quad \lambda > 0,
\end{equation}

(iv) $q \geq p$.

In the special case $\alpha + d/p \in \mathbb{N}$, inequality (1.31) is due to Kolyada \[Kol89b\] (see also \[Ne87a\]), while (1.32) was obtained in Kolyada and Lerner \[KolLe\]. Our result provides the non-trivial extensions of both (1.31) and (1.32) to the fractional setting $\alpha + d/p \notin \mathbb{N}$.

Our technique can also be applied to characterize embeddings involving the space of functions with bounded variation. Specifically, Theorem 6.17 establishes the connection between the embedding $BV(\mathbb{T}^d) \hookrightarrow \text{Lip}_{q,\infty}^{(d/q, -1/q)}(\mathbb{T}^d)$, $q > 1$, and the sharp Ulyanov inequality for the $L_1$-moduli of smoothness.
1.3. Methodology. Our method is partially inspired by Calderón’s program [Cal], which establishes the equivalence between the boundedness properties of (quasi-)linear operators acting on r.i. spaces and pointwise rearrangement inequalities (cf. also [JM, Mil]). In the setting of Besov spaces, similar ideas appear in Nilsson [Ni] and Trebels [Tre]. However, we need to introduce some modifications as Calderón’s method does not necessarily yield optimal results in borderline cases. This obstruction already occurs in the critical case and supercritical case of Sobolev embeddings. To circumvent this issue, we apply the machinery of limiting interpolation, more specifically, Holmstedt’s reiteration formulas for limiting interpolation spaces obtained in [EvO, EvOP].

In light of the abstract version of the theorem of Yano (cf. [JM, Mil]), one can recover endpoint estimates from intermediate estimates with sharp behaviour of the constants. Hence, we can show that extrapolation estimates imply Sobolev embeddings. On the other hand, the converse assertion requires deeper analysis. Indeed, it is well known that not all endpoint estimates can be obtained via extrapolation (for an elementary proof of this fact, we refer to [Ta]; see also the related result given in [HMPV]). However, it was shown by Tao [Ta] that, in the very special setting of translation invariant operators on compact symmetric spaces, it is still possible to establish the converse Yano extrapolation theorem. This strong result is crucial in our analysis to prove that limiting Sobolev inequalities imply extrapolation estimates.

1.4. Structure of the paper. In Section 2 we collect the main notations and definitions. Definitions of function spaces are given in Section 2.1 while the interpolation methods are discussed in Section 2.2. Section 3 contains auxiliary results, namely, Hardy-type inequalities (cf. Section 3.1), basic properties of moduli of smoothness (cf. Section 3.2) and interpolation results (cf. Section 3.3). The rest of the paper is divided into three sections: the subcritical case (Section 4), critical (Section 5), and supercritical (Section 6).

2. Notation and definitions

Given two (quasi-)Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding from $X$ into $Y$ is continuous.

As usual, $\mathbb{R}^d$ denotes the Euclidean $d$-space, $\mathbb{T}^d = [0, 2\pi]^d$ is the $d$-dimensional torus, $\mathbb{T} = \mathbb{T}^1$, $\mathbb{N}$ is the collection of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^d$ is the lattice of all points in $\mathbb{R}^d$ with integer-valued components.

For $1 \leq p \leq \infty$, $p'$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

We will assume that $A \lesssim B$ means that $A \leq CB$ with a positive constant $C$ depending only on nonessential parameters. If $A \lesssim B \lesssim A$, then $A \asymp B$.

Let $| \cdot |_d$ stand for the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$.

2.1. Function spaces. Throughout the paper, $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$ (for the precise definition, see [Tri06, Definition 4.3, p. 195]). Without loss of generality we shall assume that $|\Omega|_d = 1$. The decreasing rearrangement $f^* : [0, 1] \rightarrow [0, \infty)$ of a Lebesgue-measurable function $f$ in $\Omega$ is defined by

$$f^*(t) = \inf\{\lambda \geq 0 : |\{x \in \Omega : |f(x)| > \lambda\}|_d \leq t\}, \quad t \in [0, 1],$$

(2.1)
and the maximal function $f^{**}$ of $f^*$ is given by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(u) du.$$  

For $0 < p, q \leq \infty$, and $-\infty < b < \infty$, the Lorentz-Zygmund space $L_{p,q}(\log L)_b(\Omega)$ is formed by all Lebesgue-measurable functions $f$ on $\Omega$ having a finite quasi-norm

$$\|f\|_{L_{p,q}(\log L)_b(\Omega)} = \left( \int_0^1 (t^{1/p} (1 - \log t)^b f^*(t)) \frac{dt}{t} \right)^{1/q}$$  

(with the usual modification if $q = \infty$). Note that $L_{p,q}(\log L)_b(\Omega)$ becomes trivial when $p = \infty$, $0 < q < \infty$ and $b \geq -1/q$, or $p = q = \infty$, but $b > 0$. If $p = q$ in $L_{p,q}(\log L)_b(\Omega)$ then we obtain the Zygmund space $L_p(\log L)_b(\Omega)$. Setting $b = 0$ in $L_{p,q}(\log L)_b(\Omega)$ we recover the Lorentz spaces $L_{p,q}(\Omega)$ and if, in addition, $p = q$ then we obtain the Lebesgue spaces $L_p(\Omega)$. For more details, standard references are [BS], [BR], and [EE04].

Let $k \in \mathbb{N}, 1 \leq p \leq \infty$ and $0 < q \leq \infty$. The Lorentz-Sobolev space $W^k_{p,q}(\Omega)$ is defined as the set of all $k$-times weakly differentiable functions $f$ in $\Omega$ with $|\nabla^m f| \in L_{p,q}(\Omega)$ for $m = 0, \ldots, k$. Here, $\nabla^0 f = f$ and $\nabla^m f$, $m \in \mathbb{N}$, denotes the vector of all $m$-th order weak derivatives $D^\alpha f, |\alpha| = m$, of $f$ and $|\nabla^m f| = \sum_{|\alpha|=m} |D^\alpha f|$. The space $W^k_{p,q}(\Omega)$ is equipped with the norm

$$\|f\|_{W^k_{p,q}(\Omega)} = \sum_{m=0}^k \|\nabla^m f\|_{L_{p,q}(\Omega)}.$$  

Obviously, setting $p = q$ in $W^k_{p,q}(\Omega)$ we obtain the classical Sobolev space $W^k_p(\Omega) := W^k_{p,p}(\Omega)$.

By $C_0^\infty(\Omega)$ we denote the space of all infinitely time differentiable functions with compact support in $\Omega$. The space $(W^k_{p,q}(\Omega))_0$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^k_{p,q}(\Omega)$. Observe that, thanks to the Poincaré inequality, the space $(W^k_{p,q}(\Omega))_0$ can be equivalently normed by $\|\nabla^k f\|_{L_{p,q}(\Omega)}$, that is,

$$\|f\|_{W^k_{p,q}(\Omega)} \simeq \|\nabla^k f\|_{L_{p,q}(\Omega)}, \quad f \in (W^k_{p,q}(\Omega))_0.$$  

In order to introduce Sobolev spaces of fractional order, we first recall the concept of directional derivatives. The directional derivative of $f$ of order $s > 0$ along a vector $\zeta \in \mathbb{R}^d$ is given by

$$D^s_\zeta f(x) = (i\xi, \zeta)^s \hat{f}(\xi)^\vee(x), \quad x \in \mathbb{R}^d.$$  

For $1 \leq p \leq \infty$, the (fractional) Sobolev space $\dot{H}^s_p(\mathbb{R}^d)$ is formed by all $f$ such that

$$\|f\|_{\dot{H}^s_p(\mathbb{R}^d)} = \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D^s_\zeta f\|_{L_p(\mathbb{R}^d)} < \infty.$$  

We set $\dot{H}^0_p(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. Note that if $1 < p < \infty$ then $\dot{H}^s_p(\mathbb{R}^d)$ coincides with the Riesz potential space and

$$\|f\|_{\dot{H}^s_p(\mathbb{R}^d)} \simeq \|(|\xi|^s \hat{f}(\xi))^\vee\|_{L_p(\mathbb{R}^d)},$$
cf. [Wi79a, Wi79b], and, in particular, setting $s = k \in \mathbb{N}$ one recovers the classical Sobolev space $\dot{W}^k_p(\mathbb{R}^d)$ endowed with the semi-norm

$$\|f\|_{\dot{W}^k_p(\mathbb{R}^d)} = \|\nabla^k f\|_{L^p(\mathbb{R}^d)}.$$  

The periodic space $\dot{H}^s_p(\mathbb{T}^d)$ can be introduced similarly.

For $h \in \mathbb{R}^d$, we let $\Omega_h = \{x \in \Omega : x + th \in \Omega, 0 \leq t \leq 1\}$. As usual, we denote by $\Delta_h f = \Delta^k_h f$ the first difference of $f$ with step $h$, that is, $\Delta_h f(x) = f(x + h) - f(x)$, $x \in \Omega_h$. Given $k \in \mathbb{N}$, the higher order differences $\Delta^k_h f$ are defined inductively by $\Delta^k_h f(x) = \Delta_h(\Delta^k_h f)(x)$ for all $x \in \Omega_{(k+1)h}$. It is plain to check that

$$\Delta^k_h f(x) = \sum_{j=0}^k (-1)^j \binom{k}{j} f(x + (k-j)h), \quad x \in \Omega_{kh}.$$  

Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The $k$-th order modulus of smoothness of $f$ in $L_p(\Omega)$ is defined by

$$(2.4)\quad \omega_k(f,t)_{p,\Omega} = \sup_{|h| \leq t} \|\Delta^k_h f\|_{L_p(\Omega)}, \quad t > 0.$$  

As long as there is no danger of confusion we will sometimes write $\omega_k(f,t)_p$ instead of $\omega_k(f,t)_{p,\Omega}$.

We also recall the definition of the modulus of smoothness of fractional order for functions in $L_p(\mathbb{R}^d)$. For $s > 0$, we let

$$\Delta^s_h f(x) = \sum_{j=0}^\infty (-1)^j \binom{s}{j} f(x + (s-j)h), \quad x \in \mathbb{R}^d,$$

where $\binom{s}{j} = \frac{s(s-1)\ldots(s-j+1)}{j!}$, $\binom{s}{0} = 1$ and

$$\omega_s(f,t)_{p,\mathbb{R}^d} = \sup_{|h| \leq t} \|\Delta^s_h f\|_{L_p(\mathbb{R}^d)}, \quad t > 0.$$  

Clearly, if $v \in \mathbb{N}$ then we recover the classical modulus of smoothness (2.4) for functions $f \in L_p(\mathbb{R}^d)$. Analogously, one can define $\omega_s(f,t)_{p,\mathbb{T}^d}$ for $f \in L_p(\mathbb{T}^d)$. We shall often write $\omega_s(f,t)_p$ to denote both $\omega_s(f,t)_{p,\mathbb{R}^d}$ and $\omega_s(f,t)_{p,\mathbb{T}^d}$. This should hopefully cause no confusion, as the meaning should be clear from the context. Some properties of the fractional moduli of smoothness will be collected in Section 3.2 below.

Let $0 < s < k, k \in \mathbb{N}$ and $0 < q \leq \infty$. The (homogeneous) Besov space $B^s_{p,q}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the (quasi)-norm

$$(2.5)\quad \|f\|_{B^s_{p,q}(\Omega),k} = \left(\int_0^\infty (t^{-s} \omega_k(f,t)_{p,\Omega})^q \frac{dt}{t}\right)^{1/q}$$  

(suitably interpreted when $q = \infty$). Similarly, one can introduce the spaces $B^s_{p,q}(\mathbb{R}^d)$ and $B^s_{p,q}(\mathbb{T}^d)$. Here, we stress that homogeneous Besov spaces are commonly denoted with a dot (i.e., $\dot{B}^s_{p,q}(\Omega)$). However, since all Besov spaces treated in this paper are homogeneous spaces, we shall skip the dot in the latter notation in order not to overburden the present exposition. We hope this will not cause confusion for the reader. It is well known that $\|f\|_{B^s_{p,q}(\Omega),k_1} \approx \|f\|_{B^s_{p,q}(\Omega),k_2}$ for different values of
$k_1, k_2 > s$ (including fractional if $\Omega$ is replaced by $\mathbb{R}^d$ or $\mathbb{T}^d$) but the equivalence constants depend on $k_1, k_2$.

In particular, with $p = q = \infty$, one recovers the Hölder-Zygmund spaces $C^s(\mathbb{R}^d)$ and $C^s(\mathbb{T}^d)$.

Sometimes, it is more convenient to work with

$$
\|f\|_{B^{p,q}_{k,q}((\Omega), b)} = \left( \int_0^1 \left( t^{-s} \omega_k (f, t; \Omega) \right)^q \frac{dt}{t} \right)^{1/q}.
$$

Similarly, the functionals $\|f\|_{B^{p,q}_{k,q}((\mathbb{R}^d), \alpha)}$ and $\|f\|_{B^{p,q}_{k,q}(\mathbb{T}^d), \alpha}$ are introduced.

Let $s > 0, 1 \leq p \leq \infty, 0 < q \leq \infty$, and $-\infty < b < \infty$. The logarithmic Lipschitz space $\text{Lip}^{(s, -b)}_{p,q}((\mathbb{R}^d))$ is the collection of all $f \in L_p((\mathbb{R}^d))$ such that

$$
\|f\|_{\text{Lip}^{(s, -b)}_{p,q}((\mathbb{R}^d))} = \left( \int_0^1 \left( t^{-s} (1 - \log t)^{-b} \omega_b (f, t; p) \right)^q \frac{dt}{t} \right)^{1/q} < \infty
$$

(the usual interpretation is made when $q = \infty$). Note that, unlike Besov spaces (see (2.5)), the semi(quasi)-norms given in (2.7) are defined over the interval $(0, 1)$. In addition, we shall assume that $b > 1/q$ ($b \geq 0$ if $q = \infty$). These assumptions allow us to avoid trivial spaces. For detailed study of the spaces $\text{Lip}^{(s, -b)}_{p,q}((\mathbb{R}^d))$ with $s = k \in \mathbb{N}$ we refer the reader to [EE04], [Har], and [DHT19]. Some distinguished examples are

$$
\text{Lip}^{(k, 0)}_{p,\infty}((\mathbb{R}^d)) = \dot{W}^k_p((\mathbb{R}^d)), \quad 1 < p < \infty;
$$

$$
\text{Lip}^{(1, 0)}_{1,\infty}((\mathbb{R}^d)) = \text{BV}((\mathbb{R}^d)) \quad \text{(the space formed by bounded variation functions)};
$$

$$
\text{Lip}^{(1, 0)}_{\infty,\infty}((\mathbb{R}^d)) = \text{Lip}((\mathbb{R}^d)) \quad \text{(classical Lipschitz space)}.
$$

The periodic counterpart $\text{Lip}^{(s, -b)}_{p,q}((\mathbb{T}^d))$ can be introduced in analogy to (2.7).

### 2.2. Interpolation methods.

Let $(A_0, A_1)$ be a couple of quasi-Banach spaces. The Peetre $K$-functional is defined by

$$
K(t, f) = K(t, f; A_0, A_1) = \inf_{f_1 \in A_1} \{ \| f - f_1 \|_{A_0} + t \| f_1 \|_{A_1} \}, \quad t > 0, \quad f \in A_0 + A_1.
$$

Let $0 < \theta < 1, -\infty < b < \infty$, and $0 < q \leq \infty$. The logarithmic interpolation space $(A_0, A_1)_{\theta,q,b}$ is the set formed by all $f \in A_0 + A_1$ such that

$$
\|f\|_{(A_0, A_1)_{\theta,q,b}} = \left( \int_0^\infty \left( t^{-\theta} (1 + |\log t|)^b K(t, f) \right)^q \frac{dt}{t} \right)^{1/q} < \infty
$$

with the usual modification for $q = \infty$. See [Gu], [EvO], and [EvOP]. In particular, if $b = 0$ in $(A_0, A_1)_{\theta,q,b}$ then we obtain the classical real interpolation space $(A_0, A_1)_{\theta,q}$; see [BS, BL].

Assume now that $A_1 \hookrightarrow A_0$. Then it is plain to check that $K(t, f) \asymp \|f\|_{A_0}$ for $t > 1$. Consequently, we have

$$
\|f\|_{(A_0, A_1)_{\theta,q,b}} \asymp \left( \int_0^1 \left( t^{-\theta} (1 + |\log t|)^b K(t, f) \right)^q \frac{dt}{t} \right)^{1/q}.
$$
This fact together with the finer tuning given by logarithmic weights allows us to introduce limiting interpolation spaces with \( \theta = 1 \). Namely, the space \( (A_0, A_1)_{(1,b),q} \) is the collection of all \( f \in A_0 \) for which

\[
\|f\|_{(A_0, A_1)_{(1,b),q}} = \left( \int_0^1 (t^{-1}(1 + |\log t|)^b K(t, f))^q \frac{dt}{t} \right)^{1/q} < \infty.
\]

(2.12)

Note that this space becomes trivial if \( b \geq -1/q \ (b > 0 \text{ if } q = \infty) \). Then, we shall assume that \( b < -1/q \ (b \leq 0 \text{ if } q = \infty) \). We also remark that the integral \( \int_0^1 \) in (2.12) can be replaced (with equivalence of norms) by \( \int_a^b \) for any \( a > 0 \). For further details and properties, we refer the reader to [CFKU], [EvO], [EvOP], [FS], [GOT] and [Mil].

3. Auxiliary results

3.1. Hardy-type inequalities. Below we collect some Hardy-type inequalities for averages that we will use on several occasions in the rest of the paper.

**Lemma 3.1** ([SteW, p. 196]). Let \( \alpha > 0 \) and \( 1 \leq p < \infty \). Then for any non-negative measurable function \( f \) on \( (0, \infty) \),

\[
\left( \int_0^\infty \left( \int_0^t f(u)du \right)^p t^{-\alpha} \frac{dt}{t} \right)^{1/p} \leq \frac{p}{\alpha} \left( \int_0^\infty (tf(t))^p t^{-\alpha} \frac{dt}{t} \right)^{1/p}
\]

and

\[
\left( \int_0^\infty \left( \int_t^\infty f(u)du \right)^p t^\alpha \frac{dt}{t} \right)^{1/p} \leq \frac{p}{\alpha} \left( \int_0^\infty (tf(t))^p t^\alpha \frac{dt}{t} \right)^{1/p}.
\]

The corresponding results also hold true for non-negative measurable functions on \( (a, b) \subset (0, \infty) \).

**Lemma 3.2** ([BR, Theorem 6.4]). Let \( \alpha > 0, 1 \leq p \leq \infty \) and \( -\infty < b < \infty \). Then for any non-negative measurable function \( f \) on \( (0, 1) \),

\[
\left( \int_0^1 \left( \int_0^t f(u)du \right)^p t^{-\alpha}(1 - \log t)^b \frac{dt}{t} \right)^{1/p} \lesssim \left( \int_0^1 (tf(t))^p t^{-\alpha}(1 - \log t)^b \frac{dt}{t} \right)^{1/p}
\]

and

\[
\left( \int_0^1 \left( \int_t^1 f(u)du \right)^p t^\alpha(1 - \log t)^b \frac{dt}{t} \right)^{1/p} \lesssim \left( \int_0^1 (tf(t))^p t^\alpha(1 - \log t)^b \frac{dt}{t} \right)^{1/q}.
\]

Furthermore, if \( f(t) = t^{\lambda-1}g(t) \) with \( \lambda > 0 \) and \( g \) is a decreasing function then the inequalities (3.3) and (3.4) still hold true when \( 0 < p < 1 \).

The corresponding results also hold true for non-negative measurable functions on \( (a, b) \subset (0, \infty) \).

The limiting version of Lemma 3.2 reads as follows.

**Lemma 3.3** ([BR, Theorem 6.5]). Let \( 1 \leq p \leq \infty \) and \( b + 1/p \neq 0 \). Then for any non-negative measurable function \( f \) on \( (0, 1) \),
(i) if \( b + 1/p > 0 \),
\[ \left( \int_0^1 \left( (1 - \log t)^b \int_0^t f(u) \, du \right)^p \frac{dt}{t} \right)^{1/p} \leq \left( \int_0^1 \left( t(1 - \log t)^{b+1} f(t) \right)^p \frac{dt}{t} \right)^{1/p} \]
(ii) if \( b + 1/p < 0 \),
\[ \left( \int_0^1 \left( (1 - \log t)^b \int_0^t f(u) \, du \right)^p \frac{dt}{t} \right)^{1/p} \leq \left( \int_0^1 \left( t(1 - \log t)^{b+1} f(t) \right)^p \frac{dt}{t} \right)^{1/p} \]

**Lemma 3.4.** Let \( \alpha < 0, -\infty < \beta < \infty \), and \( 0 < q \leq \infty \). Then for any non-negative decreasing function \( f \) on \((0, 1)\),
\[ \left( \int_t^1 \left( v^\alpha \int_0^v u^\beta f(u) \, du \right)^q \frac{dv}{v} \right)^{1/q} \leq t^\alpha \int_t^1 v^\beta f(v) \, dv + \left( \int_t^1 \left( (v^\alpha + \beta + 1) f(v) \right)^q \frac{dv}{v} \right)^{1/q} \]

**Proof.** We have
\[ \left( \int_t^1 \left( v^\alpha \int_0^v u^\beta f(u) \, du \right)^q \frac{dv}{v} \right)^{1/q} \leq \left( \int_t^1 \left( v^\alpha \int_0^v u^\beta f(u) \, du \right)^q \frac{dv}{v} \right)^{1/q} \]

It is clear that (3.6) immediately follows from (3.7) and the following estimates
\[ \left( \int_{2t}^1 \left( v^\alpha + \beta + 1 f(v) \right)^q \frac{dv}{v} \right)^{1/q} \leq \left( \int_t^1 \left( v^\alpha \int_t^v u^\beta f(u) \, du \right)^q \frac{dv}{v} \right)^{1/q} \]
\[ \leq \left( \int_t^1 \left( (v^\alpha + \beta + 1) f(v) \right)^q \frac{dv}{v} \right)^{1/q} \]

To show (3.8), we first use the monotonicity of \( f \):
\[ \int_t^1 \left( v^\alpha \int_t^v u^\beta f(u) \, du \right)^q \frac{dv}{v} \geq \int_{2t}^1 \left( v^\alpha + \beta + 1 f(v) \right)^q \frac{dv}{v} \]

Second, the right-hand side estimate in (3.8) is an immediate consequence of Lemma 3.1 for \( q \geq 1 \). If \( q < 1 \), we let \( j \in \mathbb{N}_0 \) be such that \( 2^{-j-1} \leq t < 2^{-j} \). Then
\[ \int_t^1 \left( v^\alpha \int_t^v u^\beta f(u) \, du \right)^q \frac{dv}{v} \leq \sum_{k=0}^j 2^{-k\alpha q} \left( \sum_{\nu=k}^j 2^{-\nu(\beta+1) f(2^{-\nu})} \right)^q \]
\[ \leq \sum_{k=0}^j 2^{-k\alpha q} \sum_{\nu=k}^j (2^{-\nu(\beta+1) f(2^{-\nu})})^q \leq \int_t^1 \left( (v^\alpha + \beta + 1) f(v) \right)^q \frac{dv}{v}, \]
which completes the proof of (3.8). \( \square \)

**3.2. Properties of the modulus of smoothness.** For later use, we recall some well-known properties of the moduli of smoothness [BS, KoT2]. Let \( k, m \in \mathbb{N} \) and \( 1 \leq p \leq \infty \). We have
(a) \( \omega_k(f, t)_{p, \Omega} \) is a non-negative non-decreasing function of \( t \);
(b) if \( u \geq 1 \) then
\[ \omega_k(f, tu)_{p, \Omega} \leq u^k \omega_k(f, t)_{p, \Omega}; \]
(c) if \( k < m \) then
\[
\omega_m(f; t, \Omega) \lesssim \omega_k(f; t, \Omega)
\]
and
\[
\omega_k(f; t, \Omega) \lesssim t^k \int_0^\infty \omega_m(f; u, \Omega) \frac{du}{u^{1+k}}.
\]
(d) if \( |\nabla^k f| \in L_p(\Omega) \) then
\[
\omega_k(f; t, \Omega) \lesssim t^k ||\nabla^k f||_{L_p(\Omega)}.
\]

The moduli of smoothness \( \omega_k(f; t, \Omega) \) can be characterized in terms of the \( K \)-functional relative to the pair formed by \( L_p(\Omega) \) and \( (W^k_p(\Omega))_0 \). Specifically, we have (see [BS, Chapter 5, Theorem 4.12, page 339] and [JS, Theorem 1])
\[
\omega_k(f; t, \Omega) \asymp K(t^k, f; L_p(\Omega), (W^k_p(\Omega))_0).
\]
Sobolev spaces on \( \mathbb{R}^d \) can be characterized through moduli of smoothness (see (2.8)). Namely,
\[
\|f\|_{W^k_p(\mathbb{R}^d)} \asymp \sup_{t > 0} t^{-k} \omega_k(f; t, \mathbb{R}^d), \quad k \in \mathbb{N}, \quad 1 < p < \infty
\]
see, e.g., [KolLe, Proposition 2.4] and [Tri11, page 174]. The corresponding result for domains remains valid by using extension operators and (3.14). More precisely, the following statement holds.

**Lemma 3.5.** Let \( k \in \mathbb{N} \) and \( 1 < p < \infty \). Then, we have
\[
\|f\|_{W^k_p(\Omega)} \asymp \sup_{t > 0} t^{-k} \omega_k(f; t, \Omega).
\]

**Proof.** Given a function \( f \) defined on \( \Omega \), we denote by \( \tilde{f} \) the continuation of \( f \) to \( \mathbb{R}^d \) by 0 outside \( \Omega \). According to (3.14), we have
\[
\|f\|_{W^k_p(\Omega)} \asymp \|\nabla^k f\|_{L_p(\Omega)} = \|\tilde{f}\|_{W^k_p(\mathbb{R}^d)} \asymp \sup_{t > 0} t^{-k} \omega_k(\tilde{f}; t, \mathbb{R}^d).
\]
Therefore, the proof is complete by noting that \( \omega_k(\tilde{f}; t, \mathbb{R}^d) \asymp \omega_k(f; t, \Omega), f \in C^m_0(\Omega) \).

The (fractional) modulus of smoothness \( \omega_s(f; t, \Omega), s > 0 \), also satisfy the properties (a)–(c) listed above. In addition, the counterpart of (3.13) assert that
\[
\omega_s(f; t, \mathbb{R}^d) \asymp K(t^s, f; L_p(\mathbb{R}^d), \dot{H}^s_p(\mathbb{R}^d)), \quad 1 \leq p \leq \infty;
\]
see [KoT2, (1.32)]. As a consequence, if \( 1 \leq p \leq \infty, s > 0, 0 < q \leq \infty, 0 < \theta \leq 1, \) and \( b > 1/q (b \geq 0 \) if \( q = \infty \),) then
\[
\dot{B}^\theta_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), \dot{H}_p^s(\mathbb{R}^d))_{\theta,q},
\]
\[
\text{Lip}^{(s,-b)}_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), \dot{H}_p^s(\mathbb{R}^d))_{(1,-b),q},
\]
see (2.7) and (2.12). We also mention that the formulas (3.16)–(3.18) also hold true for function spaces over \( \mathbb{T}^d \).

The characterization (3.14) can also be extended to the fractional setting. More specifically, we have
Lemma 3.6. Let \( s > 0 \) and \( 1 < p < \infty \). Then, we have
\[
\|f\|_{\dot{H}^s_p(\mathbb{R}^d)} \asymp \sup_{t > 0} t^{-s} \omega_0(f, t)_{\mathbb{R}^d}.
\]
The corresponding result for periodic functions also holds true.

The previous result is a simple consequence of (3.16) and the closedness of the unit ball \( \dot{H}^s_p(\mathbb{R}^d) \) in \( L_p(\mathbb{R}^d) \) for \( p \in (1, \infty) \). Taking into account (3.9), we can rewrite (3.19) as
\[
\dot{H}^s_p(\mathbb{R}^d) = \text{Lip}_{p, \infty}^{(s, 0)}(\mathbb{R}^d), \quad 1 < p < \infty;
\]
this extends (2.8) to the fractional setting.

3.3. Some interpolation lemmas. For later use, we collect below some useful Holmstedt’s formulas.

Lemma 3.7. Assume that \( 0 < \theta < 1, 0 < q, r \leq \infty \) and \( b < -1/r \) (\( b \leq 0 \) if \( r = \infty \)). Let \( K(t, f) = K(t, f; A_0, A_1), \ 0 < t < 1 \). Then, we have

(i)
\[
K(t^\theta, f; A_0, (A_0, A_1)_{\theta, r}) \asymp t^\theta \left( \int_t^\infty (u^{-\theta}K(u, f))^r \frac{du}{u} \right)^{1/r},
\]
(ii)
\[
K(t^{1-\theta}, f; (A_0, A_1)_{\theta, q}, A_1) \asymp \left( \int_0^t (u^{-\theta}K(u, f))^q \frac{du}{u} \right)^{1/q},
\]
(iii)
\[
K(t(1 - \log t)^{-b-1/r}, f; A_0, (A_0, A_1)_{(1, b), r})
\]
\[
\asymp K(t, f) + t(1 - \log t)^{-b-1/r} \left( \int_t^1 (u^{-1}(1 - \log u)^bK(u, f))^r \frac{du}{u} \right)^{1/r},
\]
(iv)
\[
K(t^{1-\theta}(1 - \log t)^{-b-1/r}, f; (A_0, A_1)_{\theta, q}, (A_0, A_1)_{(1, b), r}) \asymp \left( \int_0^t (u^{-\theta}K(u, f))^q \frac{du}{u} \right)^{1/q} + t^{1-\theta}(1 - \log t)^{-b-1/r} \left( \int_t^1 (u^{-1}(1 - \log u)^bK(u, f))^r \frac{du}{u} \right)^{1/r}.
\]

For the proofs see [BS, Corollary 2.3, Chapter 5, page 310], [EvOP, Theorem 6.10*], and [FS, Theorem 4.1].

The following interpolation formulas were shown in [EvOP, Theorem 7.4*], [EvO, Theorem 4.7*+].

Lemma 3.8. Let \( 0 < \theta < 1, 0 < p, q \leq \infty \), and \( b < -1/q \). Then, we have
\[
(A_0, (A_0, A_1)_{(1, b), q})_{\theta, p} = (A_0, A_1)_{\theta, p; \theta(b+1/q)}
\]
and
\[
(A_0, A_1)_{\theta, q; b+1/\min\{p, q\}} \hookrightarrow (A_0, (A_0, A_1)_{\theta, p})_{(1, b), q} \hookrightarrow (A_0, A_1)_{\theta, q; b+1/\max\{p, q\}}.
\]

The characterization of the \( K \)-functional for pairs of Lorentz spaces in terms of rearrangements reads as follows.
Lemma 3.9 ([Hol, Theorem 4.1]). Assume $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1 \leq \infty$. Let $1/\alpha = 1/p_0 - 1/p_1$. Then,

\begin{equation}
K(t, f; L_{p_0,q_0}(\Omega), L_{p_1,q_1}(\Omega)) \asymp \left( \int_0^t (u^{1/p_0} f^*(u))^{q_0} \frac{du}{u} \right)^{1/q_0} + \left( \int_t^1 (u^{1/p_1} f^*(u))^{q_1} \frac{du}{u} \right)^{1/q_1} \tag{3.27}
\end{equation}

for $t \in (0, 1)$.

Assume $0 < p_0 < \infty$ and $0 < q_0 \leq \infty$. Then,

\begin{equation}
K(t, f; L_{p_0,q_0}(\Omega), L_{\infty}(\Omega)) \asymp \left( \int_0^t (u^{1/p_0} f^*(u))^{q_0} \frac{du}{u} \right)^{1/q_0} \tag{3.28}
\end{equation}

for $t \in (0, 1)$.

As far as the $K$-functional for pairs of homogeneous Sobolev spaces, the following holds

Lemma 3.10 ([DSc, Theorem 2, Section 4], [DS84b, Theorem 8.4], [MM06, Theorem 2]). Let $k \in \mathbb{N}$. If one of the conditions

\begin{equation}
\begin{cases}
- p_0 = q_0 = 1, & 1 < p_1 < \infty, \quad 0 < q_1 \leq \infty, \\
- 1 < p_0 < p_1 < \infty, & 0 < q_0, q_1 \leq \infty, \\
- 1 < p_0 < \infty, & 0 < q_0 \leq \infty, \quad p_1 = q_1 = \infty, \\
- p_0 = q_0 = 1, & p_1 = q_1 = \infty,
\end{cases}
\end{equation}

is satisfied, then

\begin{equation}
K(t, f; (W^k L_{p_0,q_0}(\Omega)), (W^k L_{p_1,q_1}(\Omega))) \asymp K(t, f; (\nabla^k f; L_{p_0,q_0}(\Omega), L_{p_1,q_1}(\Omega))). \tag{3.29}
\end{equation}

It is well known that the spaces $L_{r,q}((\log L)_b(\Omega))$ can be generated from the couple $(L_p(\Omega), L_{\infty}(\Omega)), p < r$, applying the interpolation method (2.10). Namely, if $0 < r < p < \infty, 0 < q \leq \infty$ and $-\infty < b < \infty$ then

$$(L_r(\Omega), L_{\infty}(\Omega))_{1-\frac{1}{p} q; b} = L_{p,q}((\log L)_b(\Omega));$$

see [GOT, Corollary 5.3]. Next we complement this result by showing that the spaces $L_{\infty,q}((\log L)_b(\Omega))$ can be characterized as limiting interpolation spaces (see (2.12)).

Lemma 3.11. Let $0 < p < \infty, 0 < q \leq \infty$, and $b < -1/q (b \leq 0$ if $q = \infty)$. Then, we have

\begin{equation}
(L_p(\Omega), L_{\infty}(\Omega))_{(1,b),q} = L_{\infty,q}((\log L)_b(\Omega)). \tag{3.30}
\end{equation}

Proof. Using the well-known fact

\begin{equation}
K(t, f; L_p(\Omega), L_{\infty}(\Omega)) \asymp \left( \int_0^t f^*(u)^p du \right)^{1/p} \tag{3.31}
\end{equation}

(see [BL, Theorem 5.2.1]) we arrive at

$$
\|f\|_{(L_p(\Omega), L_{\infty}(\Omega))_{(1,b),q}} \asymp \left( \int_0^t \frac{1}{t^{q/p}} (1 - \log t)^{bq} \left( \int_0^t f^*(u)^p du \right)^{q/p} \frac{dt}{t} \right)^{1/q}.
$$
Obviously, we have
\[
\|f\|_{(L_p(\Omega), L_{\infty}(\Omega))_{(1,b),q}} \gtrsim \left( \int_0^1 (1 - \log t)^{bq} f^*(t)^q \frac{dt}{t} \right)^{1/q} = \|f\|_{L_{\infty,q}(\log L)_b(\Omega)}.
\]

The converse inequality is a consequence of the Hardy’s inequality (3.3), which in fact holds for any \(0 < p < \infty\) and \(0 < q \leq \infty\) due to monotonicity of \(f^*(t)\).

Extrapolation means allow us to characterize Zygmund spaces (respectively, Lorentz-Zygmund spaces) in terms of the simpler Lebesgue spaces (respectively, Lorentz spaces). See [JM], [Mil] and [ET]. For later use, we recall the extrapolation description of \(L_\infty(\log L)_b(\Omega)\).

**Lemma 3.12 ([ET, Section 2.6.2, pages 69–75]).** Assume \(b < 0\). We have
\[
(3.32) \quad \|f\|_{L_\infty(\log L)_b(\Omega)} \asymp \sup_{j \geq 0} 2^{jb} \|f\|_{L_{2j,1}(\Omega)}.
\]

4. **Subcritical case**

We are concerned with the subcritical case of the Sobolev’s embedding theorem which claims that if \(k \in \mathbb{N}, 1 \leq p < \infty\) and \(k < d/p\), then
\[
(W_p^k(\Omega))_0 \hookrightarrow L_{p^*,q}(\Omega), \quad p^* = \frac{dp}{d - kp}.
\]
See [Hu], [ON] and [Pe].

The embedding (4.1) can be characterized as follows.

**Theorem 4.1.** Let \(1 \leq p < \infty, 0 < q \leq \infty\) and \(k \in \mathbb{N}\). Assume that \(k < d/p\). Let \(p^* = dp/(d - kp)\). The following statements are equivalent:

(i) \((W_p^k(\Omega))_0 \hookrightarrow L_{p^*,q}(\Omega)\),

(ii) for \(f \in L_p(\Omega)\), we have
\[
(4.2) \quad \left( \int_0^t f^*(u)^p du \right)^{1/p} + t^k \left( \int_0^t u^{q/p^*} f^*(u)^q \frac{du}{u} \right)^{1/q} \lesssim \omega_k(f, t)_p,
\]

(iii) there exists \(C > 0\), which is independent of \(s\), such that
\[
\|f\|_{L_{\frac{dp}{d - kp},q}(\Omega)} \leq C(k - s)^{1/q} \|f\|_{B_{p,q}(\Omega),k}, \quad 0 < s < k.
\]

In particular, \(B_{p,q}^s(\Omega) \hookrightarrow L_{\frac{dp}{d - kp},q}(\Omega)\) with norm \(O((k - s)^{1/q})\) as \(s \to k^-\), that is,
\[
\|f\|_{L_{\frac{dp}{d - kp},q}(\Omega)} \leq C(k - s)^{1/q} \|f\|_{B_{p,q}^s(\Omega),k}, \quad 0 < s < k,
\]

(iv) \(q \geq p\).

Before we proceed with the proof of this theorem, some remarks are in order.
Remark 4.2. (i) The following inequality of Kolyada [Kol89a] (see the previous results in [Ul]) plays a central role in embedding theorems. Let \( 1 \leq p < \infty \), then

\[
(4.3) \quad t \left( \int_{t^d}^{\infty} u^{-p/d} \int_0^u (f^*(v) - f^*(u))^p \frac{dv}{u} \right)^{1/p} \lesssim \omega_1(f, t)_p, \quad f \in L_p(\mathbb{R}^d).
\]

Assume \( 1 \leq p < d \). Then, inequality (4.2) with \( k = 1 \) and \( q = p \) reads as follows

\[
(4.4) \quad \left( \int_0^{t^d} f^*(u)^p \frac{du}{u} \right)^{1/p} + t \left( \int_{t^d}^{1} u^{p/p^*} f^*(u)^p \frac{du}{u} \right)^{1/p} \lesssim \omega_1(f, t)_p.
\]

Next we show that both inequalities (4.3) and (4.4) are equivalent if \( p \in (1, d) \), but (4.4) improves (4.3) if \( p = 1 \).

Assume first that \( p \in (1, d) \). Then, we claim that

\[
(4.5) \quad t \left( \int_{t^d}^{\infty} u^{-p/d} \int_0^u (f^*(v) - f^*(u))^p \frac{dv}{u} \right)^{1/p} \times \left( \int_0^{t^d} f^*(u)^p \frac{du}{u} \right)^{1/p} + t \left( \int_{t^d}^{1} u^{p/p^*} f^*(u)^p \frac{du}{u} \right)^{1/p}.
\]

To derive this, we will make use of the following result

\[
\int_0^u (f^{**}(v) - f^*(u))^p dv \lesssim \int_0^u (f^*(v) - f^*(u))^p dv \lesssim \int_0^{2u} (f^{**}(v) - f^*(v))^p dv;
\]

see [CGO, Proposition 4.5]. We have

\[
(4.6) \quad =: I + II.
\]

Obviously,

\[
(4.7) \quad I \leq \left( \int_0^{t^d} (f^{**}(v))^p dv \right)^{1/p} \quad \text{and} \quad II \leq t \left( \int_{t^d}^{\infty} v^{p/p^*} (f^{**}(v))^p \frac{dv}{v} \right)^{1/p}.
\]

On the other hand, since \((f^{**}(t))' = \frac{f(t) - f^{**}(t)}{t}\) (see (2.2)), it follows from the fundamental theorem of calculus that

\[
(4.8) \quad f^{**}(t) = - \int_t^\infty (f^{**}(u))' du = \int_t^\infty \frac{f^{**}(u) - f^*(u)}{u} du,
\]
where we have also used that \( \lim_{t \to \infty} f^{**}(t) = 0 \). By \( (4.8) \), applying Hardy’s inequality \( (3.2) \) together with Hölder’s inequality, we obtain

\[
\left( \int_{0}^{t} f^{**}(v)^p dv \right)^{1/p} \leq \left( \int_{0}^{t} \left( \int_{v}^{t} \frac{f^{**}(u) - f^*(u)}{u} du \right)^p dv \right)^{1/p} + t^{d/p} \int_{t}^{\infty} \frac{f^{**}(v) - f^*(v)}{v} dv
\]

\[
\lesssim \left( \int_{0}^{t} (f^{**}(v) - f^*(v))^p dv \right)^{1/p} + t \left( \int_{t}^{\infty} v^{p/p^*} (f^{**}(v) - f^*(v))^{p} dv \right)^{1/p}
\]

\[(4.9)\]

\[
= I + II.
\]

Similarly, we get

\[
(4.10) \quad t \left( \int_{t}^{\infty} v^{p/p^*} f^{**}(v)^p dv \right)^{1/p} \lesssim t \left( \int_{t}^{\infty} v^{p/p^*} (f^{**}(v) - f^*(v))^{p} dv \right)^{1/p} = II.
\]

So a combination of \( (4.6), (4.7), (4.9) \) and \( (4.10) \) results in

\[
t \left( \int_{t}^{\infty} u^{-p/d} \int_{0}^{u} (f^*(v) - f^*(u))^p dv du \right)^{1/p} \lesssim \left( \int_{0}^{t} f^{**}(v)^p dv \right)^{1/p} + t \left( \int_{t}^{\infty} v^{p/p^*} f^{**}(v)^p dv \right)^{1/p}
\]

\[(4.11)\]

It is clear that

\[
\left( \int_{0}^{t} f^{**}(v)^p dv \right)^{1/p} + t \left( \int_{t}^{\infty} v^{p/p^*} f^{**}(v)^p dv \right)^{1/p}
\]

\[
\geq \left( \int_{0}^{t} f^*(v)^p dv \right)^{1/p} + t \left( \int_{t}^{\infty} v^{p/p^*} f^*(v)^p dv \right)^{1/p}
\]

\[(4.12)\]

Conversely, since \( p > 1 \), we can apply Hardy’s inequality \( (3.1) \) to estimate

\[
(4.13) \quad \left( \int_{0}^{t} f^{**}(v)^p dv \right)^{1/p} \lesssim \left( \int_{0}^{t} f^*(v)^p dv \right)^{1/p}
\]

and

\[
\left( \int_{t}^{\infty} v^{p/p^*} f^{**}(v)^p dv \right)^{1/p} \lesssim t^{d/p^* - d} \int_{0}^{t} f^*(v) dv
\]

\[
\quad + \left( \int_{t}^{\infty} \left( v^{1/p^*} \frac{1}{v} \int_{v}^{t} f^*(u) du \right)^p dv \right)^{1/p}
\]

\[
\lesssim t^{-1} \left( \int_{0}^{t} f^*(v)^p dv \right)^{1/p} + t \left( \int_{t}^{\infty} v^{p/p^*} f^*(v)^p dv \right)^{1/p},
\]

\[(4.14)\]

where we have also applied Hölder’s inequality in the last step. Hence, by \( (4.11) \)-\( (4.14) \), we conclude \( (4.5) \).
Suppose now that \( p = 1 \). Then, inequalities (4.3) and (4.4) read as follows

\[
(4.15) \quad t \int_{t}^{\infty} u^{-1/d} \int_{0}^{u} (f^*(v) - f^*(u)) \frac{du}{u} \lesssim \omega_1(f, t)_1
\]

and

\[
(4.16) \quad \int_{0}^{t} f^*(u) du + t \int_{t}^{1} u^{1-1/d} f^*(u) \frac{du}{u} \lesssim \omega_1(f, t)_1.
\]

Further, (4.15) can be rewritten as

\[
(4.17) \quad \int_{0}^{t} f^*(u) du \lesssim \omega_1(f, t)_1,
\]

since the left hand sides of (4.15) and (4.17) are equivalent. Clearly, (4.16) is stronger than (4.17). Furthermore, the terms

\[
\int_{0}^{t} f^*(u) du \quad \text{and} \quad t \int_{t}^{1} u^{1-1/d} f^*(u) \frac{du}{u}
\]

given in the left-hand side of (4.16) are not comparable. To check this, consider the functions \( f_1(u) = u^{-1}(- \log u)^{-\beta} \), \( \beta > 1 \), and \( f_2(u) = (- \log u)^{-\eta}, \eta < 1 \).

(ii) Let \( 1 \leq p \leq q \leq \infty \) and \( k < d/p \). The sharp Sobolev inequality given in Theorem 4.1(iii), that is,

\[
\|f\|_{L_{d/p}^{d/p q} (\mathbb{R}^d)} \leq C(k - s)^{1/q} \|f\|_{W_{p,q}^k(\mathbb{R}^d)}, \quad s \to k^-,
\]

with \( C > 0 \) independent of \( s \), has been also obtained in [KMX, Theorem 4] based on different techniques.

**Proof of Theorem 4.1.** (i) \( \implies \) (ii) : Using (i) and (3.13), we have

\[
K(t^k; f; L_p(\Omega), L_{p^*, q}(\Omega)) \lesssim K(t^k; f; L_p(\Omega), (W_{p, q}^k(\Omega))_0) \asymp \omega_k(f, t)_p.
\]

By (3.27), we have

\[
K(t, f; L_p(\Omega), L_{p^*, q}(\Omega)) \asymp \left( \int_{0}^{t^{1/k}} f^*(u)^p du \right)^{1/p} + t \left( \int_{t^{1/k}}^{1} u^{q/p^*} f^*(u)^q \frac{du}{u} \right)^{1/q},
\]

which implies (ii).

(ii) \( \implies \) (iii): From (ii) we have

\[
t^k \left( \int_{t}^{1} u^{q/p^*} f^*(u)^q \frac{du}{u} \right)^{1/q} \lesssim \omega_k(f, t)_p.
\]

Let \( 0 < s < k \). Then,

\[
(4.18) \quad \int_{0}^{1} t^{k-s-q} \int_{t}^{1} u^{q/p^*} f^*(u)^q \frac{du}{u} dt \lesssim \int_{0}^{1} t^{-s} \omega_k(f, t)_p \frac{dt}{t}.
\]

Applying Fubini’s theorem, we get

\[
\int_{0}^{1} t^{k-s-q} \int_{t}^{1} u^{q/p^*} f^*(u)^q \frac{du}{u} \frac{dt}{t} \lesssim (k-s)^{-1} \int_{0}^{1} u^{q/p^*} f^*(u)^q \frac{du}{u} = (k-s)^{-1} \|f\|_{L_{d/p}^{d/p q} (\mathbb{R}^d)}.
\]
Combining (4.18) and (4.19) implies
\[ \|f\|_{L^q_{\frac{dp}{d-kp}}(\Omega)} \lesssim (k-s) \int_0^1 t^{-sq} \omega_k(f,t) \frac{dt}{t} = (k-s) \|f\|_{B^q_{p,q}(\Omega), k}. \]

(iii) \(\implies\) (i): Since \(s \to k-\), elementary computations yield that
\[ \|f\|_{B^q_{p,q}(\Omega), k} \lesssim \sup_{0 < t < 1} t^{-sq} \omega_k(f,t) \frac{dt}{t} \lesssim (k-s)^{-1} \|f\|_{W^k_p(\Omega)}, \]
where we have also used (3.12). Then, by (iii), we derive
\[ \|f\|_{L^q_{\frac{dp}{d-sp}}(\Omega)} \lesssim (k-s)^{1/q} \|f\|_{B^q_{p,q}(\Omega), k} \lesssim \|f\|_{W^k_p(\Omega)}. \]
Taking limits \(s \to k-\) we arrive at (i).

The equivalence between (i) and (iv) is well known (cf. [EKP, Theorem 5.11]).

We provide another characterization of the Sobolev inequality (4.1) in terms of estimates involving only rearrangements and derivatives. Namely, we obtain the following

**Theorem 4.3.** Let \(1 < p < \infty, 0 < q \leq \infty\) and \(k \in \mathbb{N}\). Assume that \(k < d/p\). Let \(p^* = dp/(d-kp)\) and \(1/\alpha = 1 - k/d\). The following statements are equivalent

(i) \((W^k_p(\Omega))_0 \hookrightarrow L^p_{r^*,q}(\Omega)\),

(ii) for \(f \in (W^k_p(\Omega))_0\), we have
\[ \left( \int \left( u^{1-p/k-1/\alpha} \int_0^u v^{1/\alpha} f^*(v) \frac{du}{u} \frac{dv}{v} \right)^{1/q} \right)^{1/p} \lesssim \left( \int \left( \|\nabla^k f^*(v)\|^p \right) dv \right)^{1/p}, \]

(iii) we have
\[ (W^k_{L,r,p}(\Omega))_0 \hookrightarrow L^r_{r^*,q}(\Omega), \quad r^* = \frac{dr}{d-kr}, \]
with norm \(O(1)\) as \(r \to p^-\). More precisely, there exists \(C > 0\), which is independent of \(r\), such that
\[ \|f\|_{L^r_{r^*,q}(\Omega)} \leq C \|f\|_{(W^k_{L,r,p}(\Omega))_0}, \quad r < p, \]

(iv) \(q \geq p\).

**Proof.** The equivalence between (i) and (iv) has already been discussed in Theorem 4.1.

(i) \(\implies\) (ii): It follows from
\[ (W^k_p(\Omega))_0 \hookrightarrow L^p_{p^*,q}(\Omega) \quad \text{and} \quad (W^k_1(\Omega))_0 \hookrightarrow L_{\alpha,1}(\Omega) \]
(see (4.1)) that
\[ K(t, f; L_{\alpha,1}(\Omega), L^p_{p^*,q}(\Omega)) \lesssim K(t, f; (W^k_1(\Omega))_0, (W^k_p(\Omega))_0). \]
By (3.27), (3.29), and Lemma 3.4, we have

\[ K(t, f; L_{\alpha,1}(\Omega), L_{p^*,q}(\Omega)) \asymp t \left( \int_0^1 \left( v^{1/p-k/d-1/\alpha} \int_0^v u^{1/\alpha-1} f^*(u) du \right)^q \frac{dv}{v} \right)^{1/q} \]

and

\[ K(t, f; (W^k_p(\Omega))_0, (W^k_p(\Omega))_0) \asymp t \left( \int_0^1 (|\nabla^k f|^*(v))^p dv \right)^{1/p} \]

Inserting these estimates into (4.22) we arrive at (4.20).

(ii) \implies (i): Applying monotonicity properties and (4.20),

\[ \|f\|_{L_{p^*,q}(\Omega)} = \left( \int_0^1 (v^{1/p-k/d} f^*(v))^q \frac{dv}{v} \right)^{1/q} \]

\[ \lesssim \left( \int_0^1 (v^{1/p-k/d-1/\alpha} \int_0^v u^{1/\alpha} f^*(u) du) \frac{dv}{v} \right)^{1/q} \]

\[ \lesssim \left( \int_0^1 (|\nabla^k f|^*(v))^p dv \right)^{1/p} \lesssim \|f\|_{(W^k_p(\Omega))_0}, \]

where the last estimate is an immediate consequence of the Hardy’s inequality (3.1) (noting that \( p > 1 \)).

(ii) \implies (iii): Let (ii) hold. Then we can assume that \( q \geq p \) (because (ii) \iff (iv)) and \( r > 1 \) (because \( r \to p^- \) and \( p > 1 \)). By Fubini’s theorem and (3.1), we have for any \( 1 < r < p \),

\[ I := \left( \int_0^1 t^{(1/r-1)p} \left( \int_0^1 (v^{1/p-k/d-1/\alpha} \int_0^v u^{1/\alpha} f^*(u) du) \frac{dv}{v} \right)^{p/q} \frac{dt}{t} \right)^{1/p} \]

\[ \lesssim \left( \int_0^1 t^{(1/r-1)p} \int_0^1 (|\nabla^k f|^*(v))^p dv dt \right)^{1/p} \]

\[ \asymp (1/r - 1/p)^{-1/p} \left( \int_0^1 (v^{1/r} |\nabla^k f|^*(v))^p dv \right)^{1/p} \]

\[ \lesssim (1/r - 1/p)^{-1/p} (1 - 1/r)^{-1} \left( \int_0^1 (v^{1/r} |\nabla^k f|^*(v))^p dv \right)^{1/p} \]

(4.23) \quad \lesssim (1/r - 1/p)^{-1/p} \|f\|_{(W^k_{r,p}(\Omega))_0}.

On the other hand, applying Minkowski’s inequality we get

\[ I \geq \left( \int_0^1 \left( v^{1/p-k/d-1/\alpha} \int_0^v u^{1/\alpha} f^*(u) du \right)^q \frac{dv}{v} \left( \int_0^1 t^{(1/r-1)p} \frac{dt}{t} \right)^{q/p} \right)^{1/q} \]

\[ \asymp (1/r - 1/p)^{-1/p} \left( \int_0^1 \left( v^{1/r-k/d-1/\alpha} \int_0^v u^{1/\alpha} f^*(u) du \right)^q \frac{dv}{v} \right)^{1/q} \]

\[ \gtrsim (1/r - 1/p)^{-1/p} \left( \int_0^1 (v^{1/r-k/d} f^*(v))^q \frac{dv}{v} \right)^{1/q} = (1/r - 1/p)^{-1/p} \|f\|_{L_{r^*,q}(\Omega)}, \]
where \( r^* = \frac{d}{d - k} \). Inserting this estimate into (4.23), we obtain
\[
\|f\|_{L_r^*,q}(\Omega) \lesssim \|f\|_{(W^kL_r,p)(\Omega)}.
\]

(iii) \implies (i): The embedding (i) follows from (4.21) by taking limits as \( r \to p^- \) and noting that \( r^* \to p^*^- \) as \( r \to p^- \).

\[\square\]

Note that the classical Sobolev inequality (1.1) can be rewritten as
\[
(W^k_{\frac{dp}{kp + d}}(\Omega))_0 \hookrightarrow L_p(\Omega), \quad d > k, \quad \frac{d}{d - k} \leq p < \infty.
\]

In Theorem 4.1 we have established characterizations of the sharp version of (4.24), which is obtained by using the finer scale of Lorentz spaces as target spaces. Alternatively, (4.24) can be strengthened by enlarging the domain space. More precisely, the following inequality holds (see [Al] and [Tal])
\[
(W^kL_{\frac{dp}{kp + d}})(\Omega) \hookrightarrow L_p(\Omega), \quad d > k, \quad \frac{d}{d - k} < p < \infty.
\]

Note that \( L_{\frac{dp}{kp + d}}(\Omega) \subsetneq L_{\frac{dp}{kp + q}}(\Omega) \).

Our next goal is to characterize (4.25) by using different means, namely, \( L_p \)-moduli of smoothness (Theorem 4.4) and rearrangements of functions in \( L_p(\Omega) \) (Theorem 4.7).

**Theorem 4.4.** Let \( k \in \mathbb{N}, d > k, \frac{d}{d - k} < p < \infty \) and \( 0 < q \leq \infty \). The following statements are equivalent:

(i) \[ (W^kL_{\frac{dp}{kp + q}})(\Omega))_0 \hookrightarrow L_p(\Omega), \]

(ii) for \( f \in (W^kL_{\frac{dp}{kp + q}})(\Omega))_0 \), we have
\[
\omega_k(f, t)_p \leq \left( \int_0^1 (u^{k/d + 1/p} |\nabla^nf|^*(u))^{qdu} \right)^{1/q} + t^k \left( \int_0^1 (|\nabla^nf|^*(u))^{pdu} \right)^{1/p},
\]

(iii) we have
\[ (W^kL_{r,q}(\Omega))_0 \hookrightarrow B_{r,q}^{k-d/r+d/p}(\Omega) \]

with norm \( \mathcal{O}(r - \frac{dp}{kp + d})^{-1/q} \) as \( r \to \frac{dp}{kp + d}^- \). More precisely, for any \( m \in \mathbb{N} \) there exists \( C > 0 \), which is independent of \( r \), such that
\[
\|f\|_{B_{r,q}^{k-d/r+d/p}(\Omega),m} \leq C \left( r - \frac{dp}{kp + d} \right)^{-1/q} \|f\|_{(W^kL_{r,q})(\Omega))_0}, \quad r > \frac{dp}{kp + d},
\]

(iv) \[ q \leq p. \]

**Remark 4.5.** (i) The inhomogeneous counterpart of (4.26) with \( q = p \) has been recently obtained by Gogatishvili, Neves and Opic [GNO, Theorem 3.2]. More precisely, they showed that if \( f \in W^kL_{\frac{dp}{kp + q}}(\Omega) \) then
\[
t^k \|f\|_{L_p(\Omega)} + \omega_k(f, t)_p \leq \sum_{l=0}^k \left( \int_0^1 (u^{k/(d+1/p} |\nabla^lf|^*(u))^{pdu} \right)^{1/p} + t^k \left( \int_0^1 (|\nabla^lf|^*(u))^{pdu} \right)^{1/p}
\]
for all $t \in (0, 1)$.

(ii) The Jawerth-Franke embedding for Lorentz-Sobolev spaces was recently obtained in [SeTr, Theorem 1.2]. Let $1 < r < p < \infty$, $0 < q \leq \infty$ and $k > d\left(\frac{1}{r} - \frac{1}{p}\right)$. Then,

$$ (W^kL_{r,q}(\Omega))_0 \hookrightarrow B^{k-d/r+d/p}_{p,q}(\Omega). $$

Hence, (4.27) consists of a strength of the embedding (4.28) with sharp estimates of the norm as the integrability parameter $r$ approaches certain critical value.

(iii) The limiting case $p = \infty$ in Theorem 4.4 will be settled in Theorem 5.6 below.

Proof of Theorem 4.4. (i) $\implies$ (ii): Using the embedding in (i), we have

$$ K(t^k, f; L^p(\Omega), (W^kL_{dp/kr+d,q}(\Omega))_0, (W^kL_{dp}(\Omega))_0) \lesssim K(t^k, f; (W^kL_{dp/kr+d,q}(\Omega))_0, (W^kL_{dp}(\Omega))_0). $$

In view of (3.29) and (3.27), we obtain

$$ K(t, f; (W^kL_{dp/kr+d,q}(\Omega))_0, (W^kL_{dp}(\Omega))_0) \asymp K(t, |\nabla^k f|; L^p_1(\Omega), L^p_1(\Omega)) $$

Therefore, combining this estimate, (4.29) and (3.13), we arrive at (4.26).

(ii) $\implies$ (iii): Since $r \to \frac{dp}{kr+d}+$, we may assume that $k - d/r + d/p < 1$. Further, it suffices to show (4.27) with $m = 1$ (see (3.10)).

By (3.11), we have

$$ \|f\|_{H^{k-d/r+d/p}_{p,q}(\Omega), 1}^{q} = \int_0^{\infty} t^{-(k-d/r+d/p)q} \omega_1(f, t) \frac{1}{t} \rho dt \leq C \int_0^{\infty} t^{(1-k+d/r-d/p)q} \left( \int_t^{\infty} \frac{\omega_k(f, u) du}{u} \right)^{q} \frac{1}{t} dt, $$

Next we show that

$$ \left( \int_0^{\infty} t^{(1-k+d/r-d/p)q} \left( \int_t^{\infty} \frac{\omega_k(f, u) du}{u} \right)^{q} \frac{1}{t} dt \right)^{1/q} \leq C \left( \int_0^{\infty} t^{-(k-d/r+d/p)q} \omega_k(f, t) \frac{1}{t} \rho dt \right)^{1/q}, $$

where $C > 0$ does not depend on $r$. Indeed, if $q \geq 1$ then (4.31) is an immediate consequence of the Hardy’s inequality (3.2) where $C \asymp 1 - k + \frac{d}{r} - \frac{d}{p}$, which is uniformly bounded as $r \to \frac{dp}{kr+d}+$. Assume now $q < 1$. By monotonicity properties
and Fubini's theorem, we have
\[
\left( \int_0^\infty t^{(1-k+d/r-d/p)q} \left( \int_t^\infty \frac{\omega_k(f,u)_p \, du}{u} \right)^q \, dt \right)^{1/q}
\times \left( \sum_{i=-\infty}^\infty 2^{i(1-k+d/r-d/p)q} \left( \sum_{j=i}^\infty \frac{\omega_k(f,2^j)_p}{2^j} \right)^q \right)^{1/q}
\leq \left( \sum_{i=-\infty}^\infty 2^{i(1-k+d/r-d/p)q} \sum_{j=i}^\infty \left( \frac{\omega_k(f,2^j)_p}{2^j} \right)^q \right)^{1/q}
\times \left( \sum_{j=-\infty}^\infty 2^{j(-k+d/r-d/p)q} \omega_k(f,2^j)_p \right)^{1/q}
\times \left( \int_0^\infty t^{-(k-d/r+d/p)q} \omega_k(f,t)_p \, dt \right)^{1/q},
\]
which completes the proof of (4.31).

Inserting the estimate (4.31) into (4.30) and invoking (ii), we establish
\[
\|f\|_q^{(k-r/r+d/p)q(\Omega)} \leq \left( \int_0^\infty t^{-(k-d/r+d/p)q} \omega_k(f,t)_p \, dt \right)^{1/q} 
\times \left( \int_0^\infty \left( |\nabla^k f|^q(u) \right)^{q/p} \, du \right)^{1/q/p} 
\times \left( \int_0^\infty \left( |\nabla^k f|^q(u) \right)^{q/p} \, du \right)^{1/q/p} 
\times \left( \int_0^\infty \left( |\nabla^k f|^q(u) \right)^{q/p} \, du \right)^{1/q/p}.
\]

We estimate I as follows:
\[
I = \int_0^\infty (u^{k/d+1/p} |\nabla^k f|^q(u))^{q/p} \, du 
\times (k/d - 1/r + 1/p)^{-1} \int_0^\infty (u^{1/r} |\nabla^k f|^q(u))^{q/p} \, du 
\times \left( \int_0^\infty \left( |\nabla^k f|^q(u) \right)^{q/p} \, du \right)^{1/q/p} 
\times \left( \int_0^\infty \left( |\nabla^k f|^q(u) \right)^{q/p} \, du \right)^{1/q/p}.
\]

To deal with II we shall distinguish two cases. Suppose first that \( q \geq p \). Note that we may assume without loss of generality that \( r < p \). In virtue of Lemma 3.1, we get
\[
II \lesssim \left( \frac{1}{r-1/p} \right)^{-q/p} \int_0^\infty t^{q/r} (|\nabla^k f|^q(t))^{q/p} \, dt = \left( \frac{1}{r-1/p} \right)^{-q/p} \|f\|_q^{(k-r/p)q(\Omega)}.
\]
Further, since
\[
r \rightarrow \frac{dp}{kp+d} \quad \Longleftrightarrow \quad \frac{1}{r} - \frac{1}{p} \rightarrow \frac{k}{d}
\]
we derive
\[
II \lesssim \|f\|_q^{(k-r/p)q(\Omega)}, \quad q \geq p.
\]
Secondly, let \( q < p \). Applying monotonicity properties of rearrangements, we obtain

\[
II \asymp \sum_{i=-\infty}^{\infty} 2^{(1/r-1/p)q} \left( \sum_{j=1}^{\infty} (|\nabla^k f|^*(2^j))^{p/2^j} \right)^{q/p} \\
\leq \sum_{i=-\infty}^{\infty} 2^{(1/r-1/p)q} \sum_{j=1}^{\infty} (|\nabla^k f|^*(2^j))^{q2^jq/p} \\
\lesssim \left( \frac{1}{r} - \frac{1}{p} \right)^{-1} \sum_{j=-\infty}^{\infty} (|\nabla^k f|^*(2^j))^{q2^jq/r} \asymp \left( \frac{1}{r} - \frac{1}{p} \right)^{-1} \int_0^{\infty} t^{q/r}(|\nabla^k f|^*(t))^{q} \frac{dt}{t}.
\]

This implies (see (4.34))

\[
(4.36) \quad II \lesssim \|f\|_{(W^k L_r,q(\Omega))_0}, \quad q < p.
\]

Combining (4.32), (4.33), (4.35), and (4.36), we obtain

\[
\|f\|_{B^k_{p,q} L^{r+d/p}(\Omega),m} \lesssim \left( 1 + \left( \frac{k}{d} - \frac{1}{r} + \frac{1}{p} \right)^{-1} \right) \|f\|_{(W^k L_r,q(\Omega))_0} \\
\asymp \left( r - \frac{dp}{kp+d} \right)^{-1} \|f\|_{(W^k L_r,q(\Omega))_0},
\]

i.e., (4.27) follows.

(iii) \( \Rightarrow \) (i): We claim that

\[
(4.37) \quad \|f\|_{L^p(\Omega)} \lesssim \left( r - \frac{dp}{kp+d} \right)^{1/q} \|f\|_{B^k_{p,q} L^{r+d/p}(\Omega),m}, \quad r \to \frac{dp}{kp+d} + .
\]

Indeed, using monotonicity properties of moduli of smoothness,

\[
\|f\|_{B^k_{p,q} L^{r+d/p}(\Omega),m} \geq \left( \int_1^{\infty} t^{-(k-d/r+d/p)q} \omega_m(f,t)^q \frac{dt}{t} \right)^{1/q} \\
\gtrsim \|f\|_{L^p(\Omega)} \left( \int_1^{\infty} t^{-(k-d/r+d/p)q} \frac{dt}{t} \right)^{1/q} \\
\asymp \left( k - \frac{d}{r} + \frac{d}{p} \right)^{-1/q} \|f\|_{L^p(\Omega)}.
\]

Now the claim (4.37) follows from (4.34).

As a combination of (4.37) and (4.27), we arrive at

\[
\|f\|_{L^p(\Omega)} \leq C \|f\|_{(W^k L_r,q(\Omega))_0}, \quad r > \frac{dp}{kp+d},
\]

where \( C \) is a positive constant which does not depend on \( r \). Then, in virtue of the monotone convergence theorem, we derive \((W^k L_{dp/(kp+d),q}(\Omega))_0 \hookrightarrow L^p(\Omega)\).

The equivalence (i) \( \iff \) (iv) is well known (see [EKP]).
Remark 4.6. The method of proof given above to show that (i) \iff (ii) \iff (iii) in Theorem 4.4 also works with the limiting case \( p = \frac{d}{d-k} \). In this case, the embedding given in (i) involves Sobolev spaces based on \( \| \cdot \|_{L^{1,q}(\Omega)} \), \( q \leq 1 \), which is not equivalent to a r.i. function norm if \( q < 1 \).

The counterpart of Theorem 4.4 in terms of estimates involving only rearrangements reads as follows.

Theorem 4.7. Let \( k \in \mathbb{N}, d > k, \frac{d}{d-k} < p < \infty, 1/\alpha = 1-k/d \) and \( 0 < q \leq \infty \). The following statements are equivalent:

(i) 
\[
(W^{k}L_{\frac{dp}{kp+d},q}(\Omega))_0 \hookrightarrow L_p(\Omega),
\]

(ii) for \( f \in (W^{k}L_{\frac{dp}{kp+d},q}(\Omega))_0 \), we have 
\[
\left( \int_{t}^{1} \left( \int_{0}^{t} v^{1/p-1/\alpha} f^*(u) \frac{du}{u} \right)^p \frac{dv}{v} \right)^{1/p} \lesssim \left( \int_{t}^{1} \left( \int_{0}^{(v^{k/d+1/p})^{*}} |\nabla f|^*(v) \right)^q \frac{dv}{v} \right)^{1/q},
\]

(iii) we have 
\[
(W^{k}L_{r,q}(\Omega))_0 \hookrightarrow L_{r^*,p}(\Omega), \quad r^* = \frac{dr}{d-kr},
\]
with norm \( \mathcal{O}(1) \) as \( r \to \frac{dp}{kp+d} \). More precisely, there exists \( C > 0 \), which is independent of \( r \), such that 
\[
\| f \|_{L_{r^*,p}(\Omega)} \leq C \| f \|_{(W^{k}L_{r,q}(\Omega))_0}, \quad r < \frac{dp}{kp+d},
\]

(iv) 
\[
q \leq p.
\]

Remark 4.8. The limiting case \( p = \infty \) in Theorem 4.7 will be investigated in Theorem 5.8 below.

Proof of Theorem 4.7. The equivalence between (i) and (iv) was already treated in Theorem 4.4.

(i) \implies (ii): According to (i) and (4.1), we have 
\[
K(t, f; L_{1,1}(\Omega), L_p(\Omega)) \lesssim K(t, f; (W^{k}L_{1}(\Omega))_0, (W^{k}L_{dp/(kp+d),q}(\Omega))_0).
\]

Since (cf. (3.27)) 
\[
K(t^{1/\alpha-1/p}, f; L_{1,1}(\Omega), L_p(\Omega)) \asymp \int_{0}^{t} v^{1/\alpha} f^*(v) \frac{dv}{v} + t^{1/\alpha-1/p} \left( \int_{t}^{1} f^*(v) p dv \right)^{1/p},
\]
and (cf. (3.29) and (3.27)) 
\[
K(t^{1/\alpha-1/p}, f; (W^{k}L_{1}(\Omega))_0, (W^{k}L_{dp/(kp+d),q}(\Omega))_0) 
\asymp \int_{0}^{t} |\nabla f|^*(v) dv + t^{1/\alpha-1/p} \left( \int_{t}^{1} (v^{k/d+1/p} |\nabla f|^*(v))^{q} \frac{dv}{v} \right)^{1/q},
\]

The method of proof given above to show that (i) \iff (ii) \iff (iii) in Theorem 4.4 also works with the limiting case \( p = \frac{d}{d-k} \). In this case, the embedding given in (i) involves Sobolev spaces based on \( \| \cdot \|_{L^{1,q}(\Omega)} \), \( q \leq 1 \), which is not equivalent to a r.i. function norm if \( q < 1 \).
it follows from (4.40) that
\[ t^{-1/\alpha + 1/p} \int_0^t v^{1/\alpha} f^*(v) \, \frac{dv}{v} + \left( \int_t^1 f^*(v)^p \, dv \right)^{1/p} \]
\[ \lesssim t^{-1/\alpha + 1/p} \int_0^t |\nabla^k f|^*(v) \, dv + \left( \int_t^1 (v^{k/d+1/p} |\nabla^k f|^*(v))^{q} \frac{dv}{v} \right)^{1/q}. \]
(4.41)

Further, in view of (3.6) (noting that \( p > d/(d-k) \)), we have
\[ t^{-1/\alpha + 1/p} \int_0^t |\nabla^k f|^*(v) \, dv + \left( \int_t^1 (v^{k/d+1/p} |\nabla^k f|^*(v))^{q} \frac{dv}{v} \right)^{1/q} \]
\[ \asymp \left( \int_t^1 (v^{k/d+1/p} |\nabla^k f|^*(v))^{q} \frac{dv}{v} \right)^{1/q} \]
and
\[ t^{-1/\alpha + 1/p} \int_0^t v^{1/\alpha} f^*(v) \, \frac{dv}{v} + \left( \int_t^1 f^*(v)^p \, dv \right)^{1/p} \]
\[ \asymp \left( \int_t^1 \left( v^{1/p-1/\alpha} \int_0^v u^{1/\alpha} f^*(u) \, \frac{du}{u} \right)^p \, dv \right)^{1/p}. \]
(4.42) (4.43)

Therefore, inserting (4.42) and (4.43) into (4.41), we arrive at (4.38).

(i) \( \implies \) (i): By (4.38), we have
\[ \left( \int_t^1 f^*(v)^p \, dv \right)^{1/p} \lesssim \left( \int_t^1 (v^{k/d+1/p} |\nabla^k f|^*(v))^{q} \frac{dv}{v} \right)^{1/q}. \]

Taking the supremum over all \( t \in (0,1) \), we obtain
\[ \|f\|_{L_p(\Omega)} = \left( \int_0^1 f^*(v)^p \, dv \right)^{1/p} \lesssim \left( \int_0^1 (v^{k/d+1/p} |\nabla^k f|^*(v))^{q} \frac{dv}{v} \right)^{1/q} \]
\[ \asymp \|f\|_{(W^k L^{dp/q^p q^{-p}}(\Omega))_0}, \]
where the last estimate is an immediate consequence of (3.3), which can also be applied even in the case \( q < 1 \) due to the monotonicity properties of rearrangements.

(ii) \( \implies \) (iii): First we note that \( q \leq p \) because we have already shown that (ii) \( \iff \) (iv). Applying monotonicity properties, Minkowski’s inequality, (4.38) and
Fubini’s theorem, we get

\[ \|f\|_{L^{r,q}(\Omega)} = \left( \int_0^1 (t^{1/r} f^*(t))^{p} \frac{dt}{t} \right)^{1/p} \]

\[ \leq \left( \int_0^1 (t^{1/r-k/d-1/p})^{p} \left( \int_0^t u^{1/\alpha} f^*(u) \frac{du}{u} \right)^{p} \frac{dt}{t} \right)^{1/p} \]

\[ \times \left( \frac{1}{r - \frac{kp + d}{dp}} \right)^{1/q} \left( \int_0^1 (t^{1/\alpha})^{p} \left( \int_0^t u^{1/\alpha} f^*(u) \frac{du}{u} \right)^{p} \frac{dt}{t} \right)^{1/p} \]

\[ \leq \left( \frac{1}{r - \frac{kp + d}{dp}} \right)^{1/q} \left( \int_0^1 (t^{1/\alpha})^{p} \left( \int_0^t u^{1/\alpha} f^*(u) \frac{du}{u} \right)^{p} \frac{dt}{t} \right)^{1/p} \]

\[ \times \left( \int_0^1 (t^{1/r} |\nabla^k f|^{*}(t))^{q} \frac{dt}{t} \right)^{1/q} \]

Therefore, to complete the proof of (4.39) it suffices to show that

\[ (4.44) \quad \left( \int_0^1 (t^{1/r} |\nabla^k f|^{*}(t))^{q} \frac{dt}{t} \right)^{1/q} \leq C \|f\|_{(W^k L_{r,q}(\Omega))_0}, \quad r < \frac{dp}{kp + d}, \]

where \( C > 0 \) is independent of \( r \).

To verify (4.44), we first assume that \( q \geq 1 \). Since \( r \to \frac{dp}{kp + d} \) and \( p > d/(d - k) \), we may assume without loss of generality that \( 1/r < 1 \). Then, by Lemma 3.1, we have

\[ \left( \int_0^1 (t^{1/r} |\nabla^k f|^{*}(t))^{q} \frac{dt}{t} \right)^{1/q} \leq \left( 1 - \frac{1}{r} \right)^{-1} \|f\|_{(W^k L_{r,q}(\Omega))_0} \]

and so, (4.44) follows because \( 1 - 1/r \to 1 - k/d + 1/p \) as \( r \to \frac{dp}{kp + d} \).

Secondly, let \( q < 1 \). Applying monotonicity properties and Fubini’s theorem, we obtain

\[ \int_0^1 (t^{1/r} |\nabla^k f|^{*}(t))^{q} \frac{dt}{t} \]

\[ \leq \sum_{j=0}^{\infty} 2^{-j(1/r - 1)} \sum_{\nu=j}^{\infty} 2^{-\nu |\nabla^k f|^{*}(2^{-\nu})} \]

\[ \leq \left( 1 - \frac{1}{r} \right)^{-1} \sum_{\nu=0}^{\infty} (2^{-\nu |\nabla^k f|^{*}(2^{-\nu})}) \]

The proof of (4.44) is complete.

To deal with the remaining implication (iii) \( \implies \) (i), it suffices to take limits as \( r \to \frac{dp}{kp + d} \) (or equivalently, \( r^* \to p^- \)) in (4.39) and apply the monotone convergence theorem. \( \square \)
We finish this section with some further generalizations of Theorems 4.3 and 4.7. Let \( k \in \mathbb{N}, 1 < p < d/k \), and \( 1/p^* = 1/p - k/d \). The embeddings
\[
(W^k_p(\Omega))_0 \hookrightarrow L_{p^*, p}(\Omega) \quad \text{and} \quad (W^k_{L_{d/p^*}^p(\Omega)})_0 \hookrightarrow L_p(\Omega),
\]
which were investigated in detail in Theorems 4.3 and 4.7, respectively, are special cases of the more general result due to Talenti [Tal, (4.6)]
\[
(W^k_{L_{p,q}(\Omega)})_0 \hookrightarrow L_{p^*, q}(\Omega), \quad 1 \leq q \leq \infty.
\]
See also [MiPu] and [CiPS]. Furthermore, the target space in (4.45) is optimal among all r.i. spaces. In particular,
\[
(W^k_{L_{p,q_0}(\Omega)})_0 \hookrightarrow L_{p^*, q_1}(\Omega) \iff q_0 \leq q_1.
\]

Our method of proof of Theorems 4.3 and 4.7 can easily be adapted to establish the corresponding characterizations of (4.45). Namely, the following result holds.

**Theorem 4.9.** Let \( 1 < p < \infty, 1 \leq q_0, q_1 \leq \infty \) and \( k \in \mathbb{N} \). Assume that \( k < d/p \). Let \( p^* = dp/(d - kp) \) and \( 1/\alpha = 1 - k/d \). The following statements are equivalent

(i) \( (W^k_{L_{p,q_0}(\Omega)})_0 \hookrightarrow L_{p^*, q_1}(\Omega) \),

(ii) for \( f \in (W^k_{L_{p,q_0}(\Omega)})_0 \), we have
\[
(\int_0^1 \left( v^{1/p - k/d - 1/\alpha} \int_0^v u^{1/\alpha} f^*(u) \frac{du}{u} \right)^{q_1} \frac{dv}{u})^{1/q_1} \leq \left( \int_0^1 (\sqrt[k]{|\nabla^k f|^{p^*}(v)})^{q_0} \frac{dv}{u} \right)^{1/q_0},
\]

(iii) we have
\[
(W^k_{L_{r,q_0}(\Omega)})_0 \hookrightarrow L_{r^*, q_1}(\Omega), \quad r^* = \frac{dr}{d - kr},
\]

with norm \( O(1) \) as \( r \to p^- \). More precisely, there exists \( C > 0 \), which is independent of \( r \), such that
\[
\|f\|_{L_{r^*, q_1}(\Omega)} \leq C\|f\|(W^k_{L_{r,q_0}(\Omega)})_0, \quad r < p,
\]

(iv) \( q_0 \leq q_1 \).

Observe that the estimates (4.46) comprise (4.20) and (4.38). More precisely, if \( q_0 = q_1 = p \) in (4.46) then we recover (4.20), i.e.,
\[
(\int_0^1 \left( v^{1/p - k/d - 1/\alpha} \int_0^v u^{1/\alpha} f^*(u) \frac{du}{u} \right)^{p} \frac{dv}{u})^{1/p} \leq \left( \int_0^1 (|\nabla^k f|^{p^*}) \frac{dv}{u} \right)^{1/p},
\]
and setting \( r > d/(d - k), 1/p = k/d + 1/r \), and \( q_0 = q_1 = r \) in (4.46) we obtain (4.38), i.e.,
\[
(\int_0^1 \left( v^{1/r - 1/\alpha} \int_0^v u^{1/\alpha} f^*(u) \frac{du}{u} \right)^{r} \frac{dv}{u})^{1/r} \leq \left( \int_0^1 (v^{k/d + 1/r} |\nabla^k f|^{p^*}) \frac{dv}{u} \right)^{1/r}.
\]
5. Critical case

Let $1 < p < \infty$ and $\frac{d}{p} \in \mathbb{N}$. The Trudinger embedding asserts that

$$\tag{5.1} (W^{d/p}_p(\Omega))_0 \hookrightarrow L_\infty(\log L)^{-1/2p'}(\Omega).$$

See [Pe, Po, Str, Tru, Yu]. Furthermore, this embedding is optimal within the class of Orlicz spaces (see [Ci04a]). In particular, we have

$$\tag{5.2} (W^{d/p}_p(\Omega))_0 \hookrightarrow L_\infty(\log L)^{-b}(\Omega) \iff b \geq 1/p'.$$

The first goal of this section is to obtain characterizations of (5.1) via growth estimates of rearrangements in terms of moduli of smoothness and extrapolation estimates. Namely, we have

**Theorem 5.1.** Let $1 < p < \infty$, $\frac{d}{p} \in \mathbb{N}$ and $b \geq 0$. The following statements are equivalent:

(i) $$(W^{d/p}_p(\Omega))_0 \hookrightarrow L_\infty(\log L)^{-b}(\Omega),$$

(ii) for $f \in L_p(\Omega)$, we have

$$\tag{5.3} f^*(t) \lesssim t^{-1/p}(1 - \log t)^b \omega_{d/p}(f,t^{1/d})_p,$$

(iii) we have

$$(W^{d/p}_p(\Omega))_0 \hookrightarrow L_q(\Omega)$$

with norm $O(q^b)$ as $q \to \infty$. More precisely, there exists $C > 0$, which is independent of $q$, such that

$$\|f\|_{L_q(\Omega)} \leq C q^b \|f\|_{(W^{d/p}_p(\Omega))_0}, \quad q < \infty,$$

(iv) $$b \geq 1/p'.$$

**Proof.** (i) $\implies$ (ii): By (i), we have

$$\tag{5.4} K(t^{d/p}, f; L_p(\Omega), L_\infty(\log L)^{-b}(\Omega)) \lesssim K(t^{d/p}, f; L_p(\Omega), (W^{d/p}_p(\Omega))_0) \asymp \omega_{d/p}(f,t)_p,$$

where the last estimate follows from (3.13).

Next we compute $K(t, f; L_p(\Omega), L_\infty(\log L)^{-b}(\Omega))$. Using that $L_\infty(\log L)^{-b}(\Omega) = (L_p(\Omega), L_\infty(\Omega))_{(1-b),\infty}$ (see (3.30)), we can apply (3.23) and (3.31) together with elementary computations to get

$$K(t(1 - \log t)^b, f; L_p(\Omega), L_\infty(\log L)^{-b}(\Omega))$$

$$\asymp K(t(1 - \log t)^b, f; L_p(\Omega), (L_p(\Omega), L_\infty(\Omega))_{(1-b),\infty})$$

$$\asymp \left( \int_0^t f^*(v)^p dv \right)^{1/p} + t(1 - \log t)^b \sup_{t^{p/q} \leq u \leq 1} u^{-1/p}(1 - \log u)^{-b} \left( \int_0^u f^*(v)^p dv \right)^{1/p}$$

$$\asymp \left( \int_0^t f^*(v)^p dv \right)^{1/p} + t(1 - \log t)^b \sup_{t^{p/q} \leq u \leq 1} u^{-1/p}(1 - \log u)^{-b} \left( \int_0^u f^*(v)^p dv \right)^{1/p}$$

$$\asymp \left( \int_0^t f^*(v)^p dv \right)^{1/p} + t(1 - \log t)^b \sup_{t^{p/q} \leq u \leq 1} (1 - \log u)^{-b} f^*(u).$$
In particular, we have
\begin{equation}
K(t(1 - \log t)^b, f; L_p(\Omega), L_\infty(\log L)^{-b} (\Omega)) \gtrsim t f^*(t^p).
\end{equation}

Now it follows from (5.5), (5.4) and (3.9) that
\[
t^{d/p} f^*(t^d) \lesssim \frac{K(t^{d/p}(1 - \log t)^b, f; L_p(\Omega), L_\infty(\log L)^{-b} (\Omega))}{\omega_{d/p}(f, t(1 - \log t)^{b/p/d})_p} \lesssim (1 - \log t)^b \omega_{d/p}(f, t)_p.
\]

(ii) \implies (iii): Using the well-known fact that rearrangements preserve Lebesgue norms (see, e.g., [BS, Proposition 1.8, Chapter 2, page 43]) together with (5.3) and (3.15), we obtain
\[
\|f\|_{L_q(\Omega)} = \left(\int_0^1 f^*(t)^q dt\right)^{1/q} \lesssim \left(\int_0^1 (t^{-1/p}(1 - \log t)^b \omega_{d/p}(f, t^{1/d})_p)^q dt\right)^{1/q} \lesssim \left(\int_0^1 (-\log t)^{bq} dt\right)^{1/q} \|f\|_{(W_p^{d/p}(\Omega))_b},
\]
where \(I_q := \left(\int_0^1 (-\log t)^{bq} dt\right)^{1/q} \). Next we show that \(I_q \approx q^b\), which leads to (iii). Indeed, employing Stirling’s formula for the Gamma function we have \((\Gamma(\xi))^{1/\xi} \approx \xi\) as \(\xi \to \infty\), then
\[
I_q = \Gamma(bq + 1) : (bq)^{1/q} \Gamma(bq)^{1/q} \times \left(\Gamma(bq)^{(1/(bq)})^b \approx q^b.
\]

(iii) \implies (i): According to (iii) we have
\[
\sup_{j \geq 0} 2^{-j^b} \|f\|_{L_{2^{d}(\Omega)}} \lesssim \|f\|_{(W_p^{d/p}(\Omega))_b},
\]
and thus, the embedding \((W_p^{d/p}(\Omega))_b \hookrightarrow L_\infty(\log L)^{-b}(\Omega)\) follows immediately from (3.32).

(ii) \implies (i): Using (ii) and simple change of variables, we get
\[
\|f\|_{L_\infty(\log L)^{-b}(\Omega)} = \sup_{0 < t < 1} (1 - \log t)^{-b} f^*(t) \lesssim \sup_{0 < t < 1} t^{-d/p} \omega_{d/p}(f, t)_p \lesssim \|f\|_{(W_p^{d/p}(\Omega))_b}
\]
where the last estimate follows from (3.15).

The equivalence between (i) and (iv) was stated in (5.2).

\[\square\]

Remark 5.2. It turns out that the statement (iii) in Theorem 5.1 can be replaced by one of the following extrapolation results:

(iii) \(a\) \((W_p^{d/p}(\Omega))_0 \hookrightarrow L_{q, \infty}(\Omega)\) with norm \(O(q^b)\) as \(q \to \infty\). More precisely, there exists \(C > 0\), which is independent of \(q\), such that
\[
\|f\|_{L_{q, \infty}(\Omega)} \leq C q^b \|f\|_{(W_p^{d/p}(\Omega))_0}, \quad q < \infty.
\]

(iii) \(b\) There exists \(C > 0\), which is independent of \(q\), such that
\[
\|f\|_{L_{q, \infty}(\log L)^{-b}(\Omega)} \leq C \|f\|_{B_{p, \infty}^{d-p/q}(\Omega), (d/p)}, \quad q > p.
\]
In particular, \( B_{p,\infty}^{d/p-d/q}(\Omega) \hookrightarrow L_{q,\infty}(\log L)_+ \) with embedding constant uniformly bounded as \( q \to \infty \), that is,

\[
\|f\|_{L_{q,\infty}(\log L)_+}(\Omega) \leq C\|f\|_{B_{p,\infty}^{d/p-d/q}(\Omega),d/p}, \quad q > p.
\]

**Proof.** We start by proving that (ii) \( \implies \) (iii)\(_a\). By (ii) and (3.15), we have

\[
f^*(t) \lesssim (1 - \log t)^b \sup_{0 < t < \infty} t^{-1/p} \omega_{d/p}(f, t^{1/d})_p \asymp (1 - \log t)^b \|f\|(W_{p}^{d/p}(\Omega))_0,
\]

which yields

\[
\|f\|_{L_{q,\infty}(\Omega)} = \sup_{0 < t < 1} t^{1/q} f^*(t) \lesssim \|f\|(W_{p}^{d/p}(\Omega))_0 \sup_{0 < t < 1} t^{1/q}(1 - \log t)^b \asymp q^b \|f\|(W_{p}^{d/p}(\Omega))_0.
\]

Next we show that (iii)\(_a\) \( \implies \) (i). According to (iii)\(_a\), we have

\[
\|f\|_{(W_{p}^{d/p}(\Omega))_0} \gtrsim q^{-b} \|f\|_{L_{q,\infty}(\Omega)} = \sup_{q > 1} f^*(t) \sup_{0 < t < 1} e^{-(-\log t)/q} q^{-b} \times \sup_{0 < t < 1} (-\log t)^{-b} f^*(t) \asymp \|f\|_{L_{q,\infty}(\log L)_+}(\Omega).
\]

The implication (ii) \( \implies \) (iii)\(_b\) is obvious. Further, (iii)\(_b\) \( \implies \) (i) holds true. Indeed, it follows from (iii)\(_b\) and (3.15) that

\[
\|f\|_{L_{q,\infty}(\log L)_+}(\Omega) \lesssim \sup_{q > 1} \sup_{0 < t < 1} t^{-d/p+d/q} \omega_{d/p}(f, t)_p = \sup_{0 < t < 1} t^{-d/p} \omega_{d/p}(f, t)_p \sup_{q > 1} t^{d/q} \lesssim \|f\|(W_{p}^{d/p}(\Omega))_0.
\]

Passing to the limits \( q \to \infty \), we obtain (i).

Some comments are in order. Assume \( 1 < p < \infty, k \in \mathbb{N} \), and \( 0 < s < d/p \leq k \). Let \( 1/r = 1/p - s/d \). According to [KMX, Theorems 5 and 7], the following sharp version of the Sobolev inequality for Besov spaces holds

\[
\|f\|_{L_{r,\infty}(\mathbb{R}^d)} \leq C(d - sp)^{-1} \|f\|_{B_{p,\infty}^s(\mathbb{R}^d),k}, \quad f \in B_{p,\infty}^s(\mathbb{R}^d),
\]

where \( C > 0 \) is independent of \( s \). Here, we remark that \( d/p \) is not necessarily positive integer and \( L_{r,\infty}(\mathbb{R}^d) \) is the Lorentz space on \( \mathbb{R}^d \) equipped with

\[
\|f\|_{L_{r,\infty}(\mathbb{R}^d)} = \sup_{0 < t < \infty} t^{1/r} f^*(t);
\]

see (2.1) and (2.3). Note that the embedding \( B_{p,\infty}^s(\mathbb{R}^d) \hookrightarrow L_{r,\infty}(\mathbb{R}^d) \) is optimal within the context of r.i. spaces (cf. [Mar] and [Ne89]) and in virtue of (5.6) its norm blows up as \( s \to d/p^- \) (according to the facts that \( W_{p}^{d/p}(\mathbb{R}^d) \not\hookrightarrow L_{\infty}(\mathbb{R}^d) \) and (3.19); cf. also (1.13)). This raises the natural question whether it is possible to obtain (non-trivial) free-smoothness Sobolev inequalities for Besov spaces as \( s \) approaches the critical value \( d/p \). To be more precise, the question about the existence of some r.i. space \( X \) such that \( X \nsubseteq L_p(\mathbb{R}^d) \) and \( \|f\|_X \leq C \|f\|_{B_{p,\infty}^s(\mathbb{R}^d),k} \) with \( C \) independent of \( f \) and \( s \). Accordingly, Theorem 5.1 with (iii)\(_b\) provides a positive answer if \( k = d/p \). Indeed, if \( s \to d/p^- \), then

\[
B_{p,\infty}^s(\Omega) \hookrightarrow L_{r,\infty}(\log L)_{-1/p'}(\Omega) \quad \text{with uniform norm with respect to} \ s.
\]
Furthermore, this result is sharp in the following sense. There exists $C > 0$ independent of $s$ such that

$$\|f\|_{L_r,\infty(\log L)_{-b}(\Omega)} \leq C\|f\|_{B^s_{p,\infty}(\Omega),d/p}$$

if and only if

$$b \geq 1/p'.$$

Let $1 < p < \infty$ and $\frac{d}{p} \in \mathbb{N}$. In Theorem 5.1 we were concerned with characterizations of the embedding

$$(W^{d/p}_p(\Omega))_0 \hookrightarrow L_{\infty}(\log L)_{-1/p'}(\Omega)$$

(see (5.1)). As already mentioned in (5.2) this embedding is sharp within the class of Orlicz spaces (and, in particular, Zygmund spaces). However, it can be improved if we are willing to allow Lorentz-Zygmund spaces. Namely, in light of the Maz’ya-Hansson-Brézis-Wainger embedding (1.16), we have

$$(5.7) \hspace{1cm} (W^{d/p}_p(\Omega))_0 \hookrightarrow L_{\infty,p}(\log L)_{-1}(\Omega)$$

and

$$(5.8) \hspace{1cm} L_{\infty,p}(\log L)_{-1}(\Omega) \subseteq L_{\infty}(\log L)_{-1/p'}(\Omega).$$

This does not contradict (5.2) because $L_{\infty,p}(\log L)_{-1}(\Omega)$ does not fit into the scale of Orlicz spaces. Furthermore, the target space in (5.7) is the best possible among the class of r.i. spaces (see [Han] and [CwP]). In particular, we have

$$(5.9) \hspace{1cm} (W^{d/p}_p(\Omega))_0 \hookrightarrow L_{\infty,p}(\log L)_{-b}(\Omega) \iff b \geq 1.$$}

We now turn to the counterpart of Theorem 5.1 for (5.7).

**Theorem 5.3.** Let $1 < p < \infty$, $\frac{d}{p} \in \mathbb{N}$ and $b > 1/p$. The following statements are equivalent:

(i) $$(W^{d/p}_p(\Omega))_0 \hookrightarrow L_{\infty,p}(\log L)_{-b}(\Omega),$$

(ii) for $f \in L_p(\Omega)$, we have

$$\left(\int_1^t (1 - \log u)^{-bp} f^*(u)^p \frac{du}{u}\right)^{1/p} \lesssim t^{-1/p} \omega_{d/p}(f, t^{1/d})_p,$$

(iii) if $0 < \lambda < d/p$ then there exists $C > 0$, which is independent of $\lambda$, such that

$$(5.10) \hspace{1cm} \|f\|_{L_{d/\lambda,p}(\log L)_{-b}(\Omega)} \leq C\lambda^{1/p}\|f\|_{B^{d/p-\lambda}_p(\Omega),d/p'}.$$$$

In particular, $B^{d/p-\lambda}_p(\Omega) \hookrightarrow L_{d/\lambda,p}(\log L)_{-b}(\Omega)$ with norm $O(\lambda^{1/p})$ as $\lambda \to 0+$, that is,

$$(5.11) \hspace{1cm} \|f\|_{L_{d/\lambda,p}(\log L)_{-b}(\Omega)} \leq C\lambda^{1/p}\|f\|_{B^{d/p-\lambda}_p(\Omega),d/p'},$$

(iv) $b \geq 1$.

Before we proceed with the proof of this theorem, some comments are in order.
Remark 5.4. (i) To the best of our knowledge, inequality (5.11) with \( b = 1 \), i.e.,
\[
\|f\|_{L_{d/\lambda,p}((\log L)^{-1}(\Omega)} \leq C\lambda^{1/p}\|f\|_{B_{p,p}^{d/p-\lambda}(\Omega)}^{1/d/p}, \quad \lambda \to 0+,
\]
is new. We now pursue a further study of this interesting estimate for its own sake. It can be considered a sharp form of Sobolev embedding theorem as the smoothness parameter approaches the critical value. This research area attracted a lot of attention starting from the papers [BBM, MS]. Among other results, Bourgain, Brézis and Mironescu [BBM] showed the following: Let \( Q \) be a cube in \( \mathbb{R}^d \) and assume \( p \in [1, \infty) \), \( s \in [1/2, 1) \) and \( s < d/p \). Then, there exists a positive constant \( c_d \), which depends only on \( d \), such that
\[
\left\| \int_Q f(x) \, dx \right\|_{L_{d/\lambda,p}^p(Q)} \leq c_d((1 - s)(d - sp)^{1-p})^{1/p} \left( \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dx \, dy \right)^{1/p}.
\]
Here, \( \int_Q f = \frac{1}{|Q|^d} \int_Q f(x) \, dx \). Afterwards, Maz’ya and Shaposhnikova [MS] extended this result to the whole range \( s \in (0, 1) \). Namely, if \( p \in [1, \infty) \), \( s \in (0, 1) \) and \( s < d/p \) then there exists a positive constant \( c_{d,p} \), which depends only on \( d \) and \( p \), such that
\[
\|f\|_{L_{d/\lambda,p}^p(\mathbb{R}^d)} \leq c_{d,p}(s(1 - s)(d - sp)^{1-p})^{1/p} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dx \, dy \right)^{1/p}
\]
for all \( f \in C_{0}^\infty(\mathbb{R}^d) \). Further, we remark that
\[
\left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dx \, dy \right)^{1/p} \leq \|f\|_{B_{p,p}^{d/p-\lambda}(\mathbb{R}^d),1} = \left( \int_0^\infty t^{-sp}\omega_1(f,t)^p \, dt \right)^{1/p}
\]
with equivalence constants independent of \( s \in (0, 1) \) (see [KolLe, Proposition 2.3] and [DoTi, Section 10.3]).

The corresponding sharp inequalities obtained by replacing the Lebesgue norms in (5.13), (5.14) by Lorentz norms were studied in [KMX], [EEK06], and [EEK07]. In particular, it was shown by Karadzhov, Milman and Xiao [KMX, Theorem 5] that
\[
\|f\|_{L_{d/\lambda,p}(\mathbb{R}^d)} \leq C\lambda^{1/p-1}\|f\|_{B_{p,p}^{d/p-\lambda}(\mathbb{R}^d),d/p}, \quad \lambda \to 0+.
\]

Comparing (5.12) and (5.15) we see that the target space in (5.15) is smaller than the corresponding one in (5.12), i.e.,
\[
\|f\|_{L_{d/\lambda,p}((\log L)^{-1}(\Omega)} \leq C\|f\|_{L_{d/\lambda,p}(\Omega)}
\]
with \( C > 0 \) independent of \( \lambda \). However, the crucial point, which makes our argument rather optimal, is the fundamental difference between the behavior of the constants in (5.12) and (5.15) with respect to \( \lambda \). Namely, the embedding constant in (5.15) behaves like \( \lambda^{1/p-1} \), which blows up as \( \lambda \to 0+ \). This makes sense because the corresponding target space \( L_{d/\lambda,p}(\Omega) \) becomes trivial if \( \lambda = 0 \) (cf. Section 2.1). On the other hand, the embedding constant in (5.12) is \( \lambda^{1/p} \), which remains bounded.
as \( \lambda \to 0^+ \). In particular, taking limits as \( \lambda \to 0^+ \) in (5.12) we find that
\[
B_{d/p}^{s/d}(\Omega) \hookrightarrow L_{\infty,p}(\log L)^{-1}(\Omega),
\]
which is optimal in the setting of r.i. spaces (see [Mar, Theorem 7]).

(ii) It is an open question whether the Brézis-Wainger embedding (5.7) can be
obtained from Sobolev embeddings via extrapolation methods. An explicit formula-
lation was stated in Martín and Milman [MM07a]. In this paper the authors observed
that limits of Lorentz norms in the following Talenti inequality (cf. [Tal, (4.6)]) for
\( k = 1 \) and iterating for \( k > 1 \); see also (4.45)
\[
\|f\|_{L^{r,q}(\Omega)} \leq C r \|f\|_{W^{k,p,q}(\Omega)}, \quad 0 < s < \frac{d}{p}, \quad \frac{1}{r} = \frac{1}{p} - \frac{k}{d}, \quad 1 \leq q \leq \infty,
\]
are not useful to achieve (5.7). Here, \( C \) is uniformly bounded as \( p \to d/k− \). How-
ever, they showed that some limiting embeddings into \( L_\infty(\Omega) \) follow from
(5.16) by extrapolation means.

Theorem 5.3 provides a positive answer to the above question and shows that
(5.7) follows from
\[
\|f\|_{L^{r,p}(\log L)^{-1}(\Omega)} \leq C r^{-1/p} \|f\|_{B_{d/p}^{s/d}(\Omega),d/p}, \quad 0 < s < \frac{d}{p}, \quad \frac{1}{r} = \frac{1}{p} - \frac{s}{d},
\]
(see (5.12)) by means of taking limits as \( s \to d/p− \) (or equivalently, \( r \to \infty \)). Hence,
we circumvent the problems on the limits of Lorentz norms in the embeddings
(5.16), as well as their counterparts for the Besov space
\[
\|f\|_{L^{r,p}(\mathbb{R}^d)} \leq C r^{1-1/p} \|f\|_{B_{d/p}^{s/d}(\mathbb{R}^d),d/p}
\]
(see (5.15)), by switching the target space from \( L_{r,p}(\Omega) \) to the bigger Lorentz-
Zygmund space \( L_{r,p}(\log L)^{-1}(\Omega) \) but substantially improving the sharp exponent
of the asymptotic decay of the norm from \( r^{-1/p} \) to \( r^{-1/p} \).

(iii) Both Theorems 5.1(ii) and 5.3(ii) establish estimates of rearrangements in
terms of moduli of smoothness. Namely, if \( 1 < p < \infty \) and \( d/p \in \mathbb{N} \) then
\[
(1 - \log t)^{-1/p'} f^*(t) \lesssim t^{1/p} \omega_{d/p}(f, t^{1/d})_p
\]
and
\[
\left( \int_t^1 (1 - \log u)^{-p} f^*(u) u^{\frac{d}{u}} \frac{du}{u} \right)^{1/p} \lesssim t^{-1/p} \omega_{d/p}(f, t^{1/d})_p.
\]
To see that left-hand sides of (5.17) and (5.18) are not comparable, consider \( f_1 \) and
\( f_2 \) such that \( f_1^+(t) = t^{-\varepsilon} \) for some \( \varepsilon > 0 \) and \( f_2^+(t) = (1 - \log t)^{1/p'} \). Thus (5.17) and
(5.18) provide independent estimates. This however does not contradict the fact that
(5.1) is weaker than (5.7). (Recall that in virtue of Theorems 5.1 and 5.3, inequality
(5.17) is equivalent to the Trudinger embedding (5.1), while (5.18) is equivalent
to the Maz’ya-Hansson-Brézis-Wainger embedding (5.7).) Here we observe that if
\( t \to 0^+ \) then the left-hand side of (5.18) dominates the corresponding one for
(5.17) (cf. (5.8)).
Proof of Theorem 5.3. (i) ⇒ (ii): Assume \((W^d/p)_p(\Omega)) \hookrightarrow L_{\infty,p}(\log L)_{-b}(\Omega)\). Then, by (3.13), we have
\begin{equation}
(5.19)
K(t^{d/p}, f; L_p(\Omega), L_{\infty,p}(\log L)_{-b}(\Omega)) \lesssim K(t^{d/p}, f; L_p(\Omega), (W^d/p)_p(\Omega)) \propto \omega_{d/p}(f, t)\nu.
\end{equation}

To estimate \(K(t, f; L_p(\Omega), L_{\infty,p}(\log L)_{-b}(\Omega))\), we first note that \(L_{\infty,p}(\log L)_{-b}(\Omega) = (L_p(\Omega), L_{\infty}(\Omega))_{(1-b),p}\) (see (3.30)), and then (3.23), (3.31) and Fubini’s theorem allow us to obtain
\begin{align}
K(t(1 - \log t)^{b-1/p}, f; L_p(\Omega), L_{\infty,p}(\log L)_{-b}(\Omega)) \\
\geq t(1 - \log t)^{b-1/p} \left( \int_0^1 (1 - \log u)^{-bp} \int_0^u f^*(u) p \frac{du}{u} \right)^{1/p}.
\end{align}

This yields
\begin{equation}
(5.20)
K(t(1 - \log t)^{b-1/p}, f; L_p(\Omega), L_{\infty,p}(\log L)_{-b}(\Omega)) \approx t(1 - \log t)^{b-1/p} \left( \int_0^1 (1 - \log u)^{-bp} f^*(u) p \frac{du}{u} \right)^{1/p}.
\end{equation}

Combining (5.20) and (5.19), we find that
\begin{align}
t^{d/p}(1 - \log t)^{b-1/p} \left( \int_{ct}^1 (1 - \log u)^{-bp} f^*(u) p \frac{du}{u} \right)^{1/p} \\
\lesssim K(t^{d/p}(1 - \log t)^{b-1/p}, f; L_p(\Omega), L_{\infty,p}(\log L)_{-b}(\Omega)) \\
\lesssim \omega_{d/p}(f, t(1 - \log t)^{(bp-1)/d}) \lesssim (1 - \log t)^{b-1/p} \omega_{d/p}(f, t)\nu,
\end{align}
where we have used (3.9) in the last step. Therefore, (ii) follows.

(ii) ⇒ (iii): Applying (ii) and a change of variables we arrive at
\begin{equation}
(5.21)
\int_0^1 t^{dp/d} \int_t^1 (1 - \log u)^{-bp} f^*(u) p \frac{du}{u} dt \lesssim \int_0^1 t^{(\lambda - d/p)p} \omega_{d/p}(f, t) p \frac{dt}{t} = \|f\|^p_{B_{d/p}^{d/p}(\Omega),d/p}.
\end{equation}
On the other hand, we have
\[
\int_0^1 t^{\lambda p/d} \int_t^1 (1 - \log u)^{-bp} f^*(u)\frac{du}{u} dt \lesssim \lambda^{-1} \int_0^1 u^{\lambda p/d} (1 - \log u)^{-bp} f^*(u)\frac{du}{u} = \lambda^{-1} \| f \|_{L_{d/\lambda,p}(\log L)^{-b}(\Omega)}^p.
\]
(5.22)

Combining (5.21) and (5.22), we obtain (5.10).

(iii) \( \Rightarrow \) (i): We have
\[
\left( \int_0^1 t^{(\lambda - d/p)p} \omega_{d/p}(f,t)\frac{dt}{t} \right)^{1/p} \lesssim \lambda^{-1/p} \sup_{0 < t < 1} t^{-d/p} \omega_{d/p}(f,t) \lesssim \lambda^{-1/p} \| f \|_{(W_p^{d/p}(\Omega))_0},
\]
(5.23)
where the last estimate follows from (3.15). Hence (iii) and (5.23) imply
\[
\| f \|_{L_{d/\lambda,p}(\log L)^{-b}(\Omega)} \leq C \| f \|_{(W_p^{d/p}(\Omega))_0}, \quad \lambda \in (0, d/p).
\]

We may now let \( \lambda \to 0^+ \) to get the desired estimate (i).

For (i) \( \iff \) (iv), see (5.9).

\[ \square \]

As application of Theorems 5.1 and 5.3 we obtain in Corollary 5.5 below the exact behaviour of the embedding constant of
\[
(W_{p}^{d/p}(\Omega))_0 \hookrightarrow L_{q,p}(\Omega),
\]
where \( p \) is fixed and \( q \to \infty \) (see (1.15)). Related questions when the target space \( L_{q,p}(\Omega) \) is replaced by the bigger Lebesgue space \( L_q(\Omega) \) were investigated in [ET, Section 2.7.2, pages 89–92]. Such questions are of great interest in connection to extrapolation theory; see [ET, page 92] and the references within.

**Corollary 5.5.** Let \( 1 < p < \infty \) and \( \frac{d}{p} \in \mathbb{N} \). Then, \( (W_{p}^{d/p}(\Omega))_0 \hookrightarrow L_{q,p}(\Omega) \) with norm \( \mathcal{O}(q) \) as \( q \to \infty \), that is, there exists \( C > 0 \), which is independent of \( q \), such that
\[
\| f \|_{L_{q,p}(\Omega)} \leq C q \| f \|_{(W_{p}^{d/p}(\Omega))_0}.
\]
Furthermore, this result is sharp in the following sense
(5.24)
\[
\| f \|_{L_{q,p}(\Omega)} \leq C q^b \| f \|_{(W_{p}^{d/p}(\Omega))_0} \iff b \geq 1.
\]

**Proof.** For \( b \in \mathbb{R} \), we have
\[
\left( \int_t^1 u^{p/q} f^*(u)\frac{du}{u} \right)^{1/p} \leq \left( \int_t^1 (1 - \log u)^{-bp} f^*(u)\frac{du}{u} \right)^{1/p} \sup_{t \leq u \leq 1} u^{1/q} (-\log u)^b
\]
\[
= q^b \left( \int_t^1 (1 - \log u)^{-bp} f^*(u)\frac{du}{u} \right)^{1/p} \sup_{t^{1/q} \leq u \leq 1} u (-\log u)^b
\]
\[
\lesssim q^b \left( \int_t^1 (1 - \log u)^{-bp} f^*(u)\frac{du}{u} \right)^{1/p}.
\]
EMBEDDINGS, EXTRAPOLATIONS, AND RELATED INEQUALITIES

Thus, applying the inequality given in Theorem 5.3(ii) with \( b = 1 \) (see also (5.18)), we obtain

\[
\left( \int_t^1 u^{p/q} f^*(u)^p \frac{du}{u} \right)^{1/p} \lesssim q t^{-1/p} \omega_{d/p}(f, t^{1/d})_p.
\]

Taking the supremum over all \( t \in (0, 1) \) on both sides and using a simple change of variables, we get

\[
\|f\|_{L^q,p(\Omega)} \lesssim q \sup_{0 < t < 1} t^{-d/p} \omega_{d/p}(f, t_1^{1/d}) \lesssim q \|f\|_{(W^{d/p}_{p}(\Omega))_0},
\]

where the last estimate follows from (3.15).

It remains to show the only-if part in (5.24). Assume that there exists \( b \) such that

\[
\|f\|_{L_{q,p}(\Omega)} \leq C q^b \|f\|_{(W^{d/p}_{p}(\Omega))_0}.
\]

Combining this inequality together with the estimate (cf. [SteW, p. 192])

\[
\|f\|_{L^q(\Omega)} \lesssim \left( \frac{p}{q} \right)^{1/p} \|f\|_{L_{q,p}(\Omega)}, \quad p < q < \infty,
\]

we arrive at

\[
\|f\|_{L^q(\Omega)} \leq C q^{b-1/p} \|f\|_{(W^{d/p}_{p}(\Omega))_0}.
\]

Then, according to Theorem 5.1, we have \( b - 1/p \geq 1/p' \). \( \square \)

In the rest of this section we shall focus on Sobolev embeddings into \( L^\infty(\Omega) \). Specifically, if \( k \leq d - 1 \) then

(5.25) \( (W^k L_{d/k,1}(\Omega))_0 \hookrightarrow L^\infty(\Omega) \),

see [Ste], [DS84a], [CiP] and [CiPS]. Furthermore, this embedding is sharp, that is, \( \epsilon \)

(5.26) \( (W^k L_{d/k,q}(\Omega))_0 \hookrightarrow L^\infty(\Omega) \iff q \leq 1; \)

cf. [CiPS, Theorem 6.9].

Next we complement Theorems 5.1 and 5.3 by characterizations of (5.25) in terms of sharp inequalities of the \( L^\infty \)-moduli of smoothness, as well as extrapolation estimates of Jawerth-Franke type embeddings for Lorentz-Sobolev spaces. More specifically, we establish the following

**Theorem 5.6.** Let \( k \in \mathbb{N}, d > k \) and \( 0 < q \leq \infty \). The following statements are equivalent:

(i) \( (W^k L_{d/k,q}(\Omega))_0 \hookrightarrow L^\infty(\Omega) \),

(ii) for \( f \in (W^k L_{d/k,q}(\Omega))_0 \), we have

(5.27) \[ \omega_k(f, t)_\infty \lesssim \left( \int_0^t u^{d/k} \left| \nabla^k f^*(u) \right|^q \frac{du}{u} \right)^{1/q}, \]

(iii) we have

(5.28) \( (W^k L_{r,q}(\Omega))_0 \hookrightarrow B_{\infty,q}^{b-d/r}(\Omega) \).
with norm $O((r - \frac{d}{k})^{-1/q})$ as $r \to \frac{d}{k^+}$. More precisely, for any $m \in \mathbb{N}$ there exists $C > 0$, which is independent of $r$, such that

\begin{equation}
\|f\|_{B_{\infty,q}^{k-d/r}(\Omega),m} \leq C \left(r - \frac{d}{k}\right)^{-1/q} \|f\|_{(W^{kL_r,q}_r(\Omega))_0}, \quad r > \frac{d}{k},
\end{equation}

Remark 5.7. (i) Inequality (5.27) with $q = 1$, i.e.,

$$\omega_k(f,t) \lesssim \int_0^t u^k |\nabla f|^*(u^d) \frac{du}{u}$$

was shown by DeVore and Sharpley [DS8a, Lemma 2] if $k = 1$ and, Kolyada and Pérez Lázaro [KolPe, (1.6)] for higher-order derivatives. It plays a central role in the theory of function spaces as can be seen in [Har], [GMNO], and the references given there.

(ii) Recall that the Jawerth-Franke embedding for Lorentz-Sobolev spaces asserts that if $1 < r < p < \infty, 0 < q \leq \infty$ and $k > d\left(\frac{1}{r} - \frac{1}{p}\right)$, then

\begin{equation}
(W^{kL_r,q}_r(\Omega))_0 \hookrightarrow B_{p,q}^{k-d/r+d/p}(\Omega);
\end{equation}

cf. (4.28). However, the interesting case $p = \infty$ in (5.30) was left open in [SeTr]. As a byproduct of (5.28), we can cover this case and, in addition, obtain sharp estimates of the rates of blow-up of the corresponding embedding constant. More specifically, we get

\begin{equation}
(W^{kL_r,q}_r(\Omega))_0 \hookrightarrow B_{\infty,q}^{k-d/r}(\Omega), \quad q \leq 1, \quad d/r < k < d,
\end{equation}

with norm $O((r - \frac{d}{k})^{-1/q})$. In particular, this shows that (5.31) does not hold in the limiting case $r = d/k$ in the case when the Besov spaces are defined in terms of moduli of smoothness. This matches the fact that $B_{\infty,q}^0(\Omega)$ endowed with $\|f\|_{B_{\infty,q}^0(\Omega),m} = (\int_0^\infty \omega_m(f,t)^q \frac{dt}{t})^{1/q}$, $m \in \mathbb{N}$, (see (2.5)) becomes trivial.

(iii) The distinction between the subcritical and critical cases given in Theorems 4.4 and 5.6, respectively, is the sharp exponent $q$. More specifically, in the subcritical case we obtain the optimal index $q = p$ (see Theorem 4.4(iv)), whereas $q = 1$ in the critical case (see Theorem 5.6(iv)).

Proof of Theorem 5.6. (i) $\implies$ (ii): By (3.13), (i), (3.29) and (3.28) we have

$$\omega_k(f,t) \propto K(t^k,f;L_\infty(\Omega),(W^{kL_\infty}_\infty(\Omega))_0) \lesssim K(t^k,f;(W^{kL_d/k,q}_d(\Omega))_0,(W^{kL_\infty}_\infty(\Omega))_0)$$

$$\times K(t^k,|\nabla f|^d;L_{d/k,q}(\Omega),L_\infty(\Omega)) \propto \left(\int_0^t u^{k/d}\omega_k(f,t)^q \frac{du}{u}\right)^{1/q}.$$

(ii) $\implies$ (iii): Since $r \to d/k^+$, we may assume, without loss of generality, that $k - d/r < 1$. In light of (3.10), it is enough to show (5.29) with $m = 1$. Following (4.30) and (4.31) (with $p = \infty$), we have

$$\|f\|_{B_{\infty,q}^{k-d/r}(\Omega),1} \leq C \left(\int_0^\infty t^{-(k-d/r)q}\omega_k(f,t)^q \frac{dt}{t}\right)^{1/q},$$
where \(C > 0\) does not depend on \(r\). Therefore, applying (ii) and a simple change of variables, we obtain
\[
\|f\|_{B_{\infty,q}^{k-d/r}(\Omega),1} \lesssim \left( \int_0^\infty t^{-(k-d/r)q} \int_0^t (u)^{k/d |\nabla^k f|^*(u))^{q} \frac{du}{u} \frac{dt}{t} \right)^{1/q} \times (kr - d)^{-1/q} \left( \int_0^\infty (t^{1/r} |\nabla^k f|^*(t))^{q} \frac{dt}{t} \right)^{1/q}.
\]
This proves (iii).

(iii) \(\implies\) (i): We claim that
\[
\|f\|_{L_\infty(\Omega)} \leq C \left( r - \frac{d}{k} \right)^{1/q} \|f\|_{B_{\infty,q}^{k-d/r}(\Omega),m}, \quad r > \frac{d}{k},
\]
where \(C > 0\) is independent of \(r\). Indeed, we have
\[
\|f\|_{B_{\infty,q}^{k-d/r}(\Omega),m} \geq \int_1^\infty t^{-(k-d/r)q} \omega_m(f,t)^{q} \frac{dt}{t} \geq \omega_m(f,1)^{q} \int_1^\infty t^{-(k-d/r)q} \frac{dt}{t}
\]
Applying (5.32) and (iii), we derive
\[
\|f\|_{L_\infty(\Omega)} \leq C \|f\|_{(W^k_{r,q}(\Omega))_0}, \quad r > \frac{d}{k},
\]
where \(C > 0\) does not depend on \(r\). Thus, from monotone convergence we deduce that \((W^k_{d/k,q}(\Omega))_0 \hookrightarrow L_\infty(\Omega)\).

The equivalence (i) \(\iff\) (iv) was already stated in (5.26).

Next we deal with the counterpart of Theorem 5.6 for estimates involving only rearrangements.

**Theorem 5.8.** Let \(k \in \mathbb{N}, d > k, 1/\alpha = 1 - k/d, \) and \(0 < q \leq \infty\). The following statements are equivalent:

(i)

\[ (W^k_{d/k,q}(\Omega))_0 \hookrightarrow L_\infty(\Omega), \]

(ii) for \(f \in (W^k_{d/k,q}(\Omega))_0\), we have
\[
(5.33) \quad t^{-1/\alpha} \int_0^t v^{1/\alpha} f^*(v) \frac{dv}{v} \lesssim \left( \int_t^1 (v^{k/d} |\nabla^k f|^*(v))^{q} \frac{dv}{v} \right)^{1/q},
\]

(iii) we have
\[
(5.34) \quad (W^k_{r,q}(\Omega))_0 \hookrightarrow L_{r^*,q}(\Omega), \quad r^* = \frac{dr}{d-kr},
\]
with norm \(O((r^*)^{1/q})\) as \(r \to \frac{d}{k}^-\). More precisely, there exists \(C > 0\), which is independent of \(r\), such that
\[
(5.35) \quad \|f\|_{L_{r^*,q}(\Omega)} \leq C (r^*)^{1/q} \|f\|_{(W^k_{r,q}(\Omega))_0}, \quad r < \frac{d}{k},
\]

(iv) \(q \leq 1\).

The following set of comments pertains to the previous result.
Remark 5.9. (i) Let $q = 1$. The inequality (5.33) reads as follows

\[
(5.36) \quad t^{-1/\alpha} \int_0^t v^{1/\alpha} f^*(v) \frac{dv}{u} \lesssim \int_t^1 v^{k/d} |\nabla^k f|^{**}(v) \frac{dv}{v}.
\]

It turns out that (5.36) provides an improvement of the following estimate given in [Kol07, Corollary 3.2]

\[
(5.37) \quad f^{**}(t) \lesssim \int_t^1 v^{k/d} |\nabla^k f|^{**}(v) \frac{dv}{v}.
\]

This follows from

\[
\int_0^t f^*(v) dv \leq t^{-1/\alpha} \int_0^t v^{1/\alpha} f^*(v) \frac{dv}{v}.
\]

In addition, it is plain to see that the terms $f^{**}(t)$ and $t^{-1/\alpha} \int_0^t v^{1/\alpha} f^*(v) \frac{dv}{v}$ are not comparable (consider, e.g., the function $f^*(t) = t^{-1/\alpha}(1 - \log t)^\varepsilon$, \( \varepsilon < -1 \) and invoke [BS, Chapter 2, Corollary 7.8, p. 86]).

(ii) Setting $q = 1$ in (5.35) we recover the sharp blow-up of the norm of the embedding (5.34) as $r \to (d/k)$—obtained by Talenti (see (5.16)).

(iii) Theorem 5.8 provides the limiting version of Theorem 4.7 with $p = \infty$. Note that there are significant distinctions between these two theorems. For instance, in virtue of (4.38) (with $q = p$) and (5.33) (with $q = 1$), we have

\[
(5.37) \quad \left( \int_t^1 \left( u^{1/p-1/\alpha} \int_0^u v^{1/\alpha} f^*(v) \frac{du}{u} \right)^p dv \frac{dv}{u} \right)^{1/p} \lesssim \left( \int_t^1 \left( v^{k/d+1/p} |\nabla^k f|^{**}(v) \right)^p dv \frac{dv}{v} \right)^{1/p}
\]

and

\[
(5.38) \quad t^{-1/\alpha} \int_0^t v^{1/\alpha} f^*(v) \frac{dv}{u} \lesssim \int_t^1 v^{k/d} |\nabla^k f|^{**}(v) \frac{dv}{v},
\]

respectively. Despite the left-hand side of (5.38) corresponds to the one given in (5.37) with $p = \infty$, this is not the case for their right-hand sides, where we switch from $L_p$ norms in (5.37) to the $L_1$ norm in (5.38).

Concerning the extrapolation estimates given in Theorems 4.7(iii) and 5.8(iii), we make the following observations. The optimal inequalities read as follows (see (4.39) and (5.35))

\[
(5.39) \quad \|f\|_{L_{r^*,p}(\Omega)} \leq C \|f\|_{(W^k L_{r,p}(\Omega))_p}, \quad \frac{d}{d-k} < p < \infty, \quad r < \frac{dp}{kp+d},
\]

and

\[
(5.40) \quad \|f\|_{L_{r^*,1}(\Omega)} \leq C r^* \|f\|_{(W^k L_{r^*,1}(\Omega))_p}, \quad r < \frac{d}{k},
\]

Here $r^* = \frac{dp}{d-kr}$ and $d > k$. On the one hand, the critical values of the integrability parameter $r$ in (5.39) and (5.40) coincide (note that, formally speaking, $\frac{dp}{kp+d} = \frac{d}{k}$ if $p = \infty$). On the other hand, we switch from the second index $p$ in Lorentz spaces in (5.39) to the index 1 in (5.40). More interestingly, the uniform behaviour of the embedding constant in (5.39) with respect to $r$ breaks down for $p = \infty$ where we obtain the blow up $r^*$ (see (5.40)). In particular, (5.40) fails to be true if $r = d/k$ (and so, $r^* = \infty$). Indeed, we observe that in this case the Lorentz space $L_{r^*,1}(\Omega) = \{0\}$ (cf. Section 2.1).
Proof of Theorem 5.8. (i) \implies (ii): It follows from (i) and the embedding (5.41)
\[(W_1^k(\Omega))_0 \hookrightarrow L_{\alpha,1}(\Omega)\]
(cf. (4.1)) that (5.42)
\[K(t, f; L_{\alpha,1}(\Omega), L_\infty(\Omega)) \leq K(t, f; (W_1^k(\Omega))_0, (W^k_{d/k,q}(\Omega))_0).\]
By the Holmstedt’s formula (3.28),
\[K(t, f; L_{\alpha,1}(\Omega), L_\infty(\Omega)) \sim \int_0^t v^{1/\alpha} f^*(v) \frac{dv}{v}.\]
On the other hand, applying (3.29) together with (3.27), we arrive at
\[K(t, f; (W_1^k(\Omega))_0, (W^k_{d/k,q}(\Omega))_0) \sim K(t, |\nabla f|; L_1(\Omega), L_{d/k,q}(\Omega))\]
\[\times \int_0^t |\nabla f|^*(v) dv + t \left(\int_0^1 (v^{k/d} |\nabla f|^*)(v) q \frac{dv}{v}\right)^{1/q}.\]
Therefore, by (5.42), (5.43) and (5.44), we have
\[t^{-1/\alpha} \int_0^t v^{1/\alpha} f^*(v) \frac{dv}{v} \leq t^{-1/\alpha} \int_0^t |\nabla f|^*(v) dv + \left(\int_0^1 (v^{k/d} |\nabla f|^*)(v) q \frac{dv}{v}\right)^{1/q}.\]
Hence, we complete the proof of (5.33) by invoking (3.6).
(ii) \implies (iii): According to (5.33), we have
\[\int_0^1 (t^{1/r^*} f^*(t))^q \frac{dt}{t} \leq \int_0^1 \left(t^{1/r^* - 1/\alpha} \int_0^t v^{1/\alpha} f^*(v) \frac{dv}{v}\right)^q \frac{dt}{t}\]
\[\leq \int_0^1 t^{q/r^*} \int_t^1 (v^{k/d} |\nabla f|^*)(v) q \frac{dv}{v} \frac{dt}{t}\]
\[\times r^{*} \int_0^1 (v^{1/r} |\nabla f|^*)(v) q \frac{dv}{v}.\]
Therefore, to complete the proof of (iii), it is enough to apply the following inequality
\[\left(\int_0^1 (v^{1/r} |\nabla f|^*)(v) q \frac{dv}{v}\right)^{1/q} \leq C \|f\|_{(W^k_{d/k,q}(\Omega))_0}, \quad r < \frac{d}{k},\]
where $C > 0$ is independent of $r$. The proof of (5.45) is completely analogous to that in the proof of (4.44), so we omit further details.
(iii) \implies (i): Firstly, we claim that (5.46)
\[\lim_{r^* \to \infty} (r^*)^{-1/q} \|f\|_{L_{r^*,q}(\Omega)} = \|f\|_{L_\infty(\Omega)}.\]
Indeed, since $\lim_{t \to 0^+} f^*(t) = \|f\|_{L_\infty(\Omega)}$, given any $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that
\[\|f\|_{L_\infty(\Omega)} - \varepsilon \leq f^*(t)^q \leq \|f\|_{L_\infty(\Omega)}^q + \varepsilon, \quad t \in (0, \delta).\]
Since $r^* \to \infty$, we may assume that $r^* > 2$. Then
\[
\frac{1}{r^*} \int_0^1 t^{q/r^*} f^*(t)^q \frac{dt}{t} = \frac{\delta^{q/r^*}}{r^*} \int_0^t \left(\frac{t}{\delta}\right)^{q/r^*} f^*(t)^q \frac{dt}{t} \leq \frac{\delta^{q(1/r^* - 1/2)}}{r^*} \int_0^1 t^{q/2} f^*(t)^q \frac{dt}{t}.\]
Now, taking limits as \( r^* \to \infty \), we derive

\[
(5.48) \quad \lim_{r^* \to \infty} \frac{1}{r^*} \int_{\delta}^{1} t^{q/r^*} f^*(t)^q \frac{dt}{t} = 0.
\]

On the other hand, it follows from (5.47) that

\[
\frac{1}{q} \delta^{q/r^*}(\|f\|^q_{L_\infty(\Omega)} - \varepsilon) = \frac{1}{r^*}(\|f\|^q_{L_\infty(\Omega)} - \varepsilon) \int_{0}^{\delta} t^{q/r^*} \frac{dt}{t}
\]

\[
\leq \frac{1}{r^*} \int_{0}^{\delta} t^{q/r^*} f^*(t)^q \frac{dt}{t}
\]

\[
\leq \frac{1}{r^*}(\|f\|^q_{L_\infty(\Omega)} + \varepsilon) \int_{0}^{\delta} t^{q/r^*} \frac{dt}{t} = \frac{1}{q} \delta^{q/r^*}(\|f\|^q_{L_\infty(\Omega)} + \varepsilon),
\]

which yields

\[
\frac{1}{q}(\|f\|^q_{L_\infty(\Omega)} - \varepsilon) \leq \lim_{r^* \to \infty} \frac{1}{r^*} \int_{0}^{\delta} t^{q/r^*} f^*(t)^q \frac{dt}{t} \leq \frac{1}{q}(\|f\|^q_{L_\infty(\Omega)} + \varepsilon).
\]

Using that \( \varepsilon > 0 \) was arbitrary, we obtain

\[
(5.49) \quad \lim_{r^* \to \infty} \frac{1}{r^*} \int_{0}^{\delta} t^{q/r^*} f^*(t)^q \frac{dt}{t} = \frac{1}{q}\|f\|^q_{L_\infty(\Omega)}.
\]

Therefore, (5.46) follows from (5.48) and (5.49).

Now we are ready to show the implication (iii) \( \Rightarrow \) (i). Taking limits in (5.35) (noting that \( r^* \to \infty \) if and only if \( r \to (d/k)^- \)) and using (5.46), we have

\[
\|f\|_{L_\infty(\Omega)} \asymp \lim_{r \to \infty} (r^*)^{-1/q\|f\|_{L_{r^*,q}(\Omega)}} \lesssim \lim_{r \to (d/k)^-} \|
abla^k f\|_{L_{r,q}(\Omega)} = \|
abla^k f\|_{L_{d/k,q}(\Omega)}.
\]

For the equivalence (i) \( \iff \) (iv), see (5.26).

\[\square\]

6. Supercritical case

It was shown by Brézis and Wainger [BreWain] that functions in the Sobolev space \( \dot{H}^{1+d/p}(\mathbb{T}^d), 1 < p < \infty \), are Lipschitz-continuous up to the logarithmic term. More precisely, they proved that

\[
\dot{H}^{1+d/p}(\mathbb{T}^d) \hookrightarrow \text{Lip}_{(1,1)}((\mathbb{T}^d).
\]

This embedding was recently extended in [DHT19] to any Sobolev space in the supercritical case, that is,

\[
(6.1) \quad \dot{H}^{\alpha+d/p}(\mathbb{T}^d) \hookrightarrow \text{Lip}_{(\alpha,1)}((\mathbb{T}^d), \quad \alpha > 0.
\]

Furthermore, if \( d = 1 \) then the previous embedding is optimal in the following sense

\[
(6.2) \quad \dot{H}^{\alpha+1/p}(\mathbb{T}) \hookrightarrow \text{Lip}_{(\alpha,\alpha-b)}((\mathbb{T}) \iff b \geq 1/p',
\]

see [DHT19]. We also remark that the counterpart of (6.1) for function spaces on \( \mathbb{R}^d \) holds. The arguments given in [DHT19], originally developed in the inhomogeneous situation, carry over to the homogeneous case as well.
We will see that (6.1) is closely connected to the so-called Ulyanov type inequalities for moduli of smoothness. Recall that the sharp Ulyanov inequality for \(1 < p < q < \infty\) states that if \(f \in L_p(\mathbb{R}^d)\), then

\[
\omega_\alpha(f, t) \lesssim \left( \int_0^t (u^{-\theta} \omega_{\alpha+\theta}(f, u)_p)^{q} \frac{du}{u} \right)^{1/q}, \quad \alpha > 0, \quad \theta = d \left( \frac{1}{p} - \frac{1}{q} \right).
\]

This inequality remains valid for functions \(f \in L_p(\mathbb{T}^d)\). In the limiting cases, if \(p = 1\) or \(q = \infty\) then for any \(t \in (0, 1)\), we have (cf. [Ti10, Theorem 1], [KoT1])

\[
\omega_\alpha(f, t) \lesssim \left( \int_0^t (u^{-\theta} (1 - \log u)^{1/\min\{p', q\}} \omega_{\alpha+\theta}(f, u)_p)^{q_\ast} \frac{du}{u} \right)^{1/q_\ast}, \quad f \in L_p(\mathbb{T}),
\]

where

\[
\theta = \frac{1}{p} - \frac{1}{q} \quad \text{and} \quad q_\ast = \left\{ \begin{array}{ll} q, & q < \infty, \\ 1, & q = \infty. \end{array} \right.
\]

Our first concern in this section is to carry out the programme developed in the previous sections for (6.1). Namely, we obtain the following

**Theorem 6.1.** Let \(1 < p < \infty, \alpha > 0\) and \(b \geq 0\). The following statements are equivalent:

(i)

\[
\dot{H}_p^{\alpha+1/p}(\mathbb{T}) \hookrightarrow \operatorname{Lip}_{\infty, \infty}^{(\alpha-b)}(\mathbb{T}),
\]

(ii) for \(f \in B_{p,1}^{1/p}(\mathbb{T})\), we have

\[
\omega_\alpha(f, t) \lesssim \int_0^t (1 - \log t)^{b/\alpha} u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u},
\]

(iii) we have

\[
\dot{H}_p^{\alpha+1/p}(\mathbb{T}) \hookrightarrow C^{\alpha_0}(\mathbb{T})
\]

with norm \(O((\alpha - \alpha_0)^-b)\) as \(\alpha_0 \to \alpha^\ast\). More precisely, there exists \(C > 0\), which is independent of \(\alpha_0\), such that

\[
\|f\|_{C^{\alpha_0}(\mathbb{T}), \alpha} \leq C(\alpha - \alpha_0)^{-b} \|f\|_{\dot{H}_p^{\alpha+1/p}(\mathbb{T})}, \quad 0 < \alpha_0 < \alpha,
\]

(iv)

\[b \geq 1/p'.\]

**Remark 6.2.** (i) Inequality (6.6) with \(b = 1/p'\), i.e.,

\[
\omega_\alpha(f, t) \lesssim \int_0^t (1 - \log t)^{1/\alpha p'} u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u}
\]

sharpens the known estimate (see (6.3))

\[
\omega_\alpha(f, t) \lesssim \int_0^t u^{-1/p} (1 - \log u)^{1/p'} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u}.
\]
Indeed, by monotonicity properties of the moduli of smoothness (cf. Section 3.2) and elementary estimates, we derive that

\[
\int_t^{(1 - \log t)^{1/\alpha p'}} u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \lesssim t^{-1/p - \alpha} \omega_{\alpha+1/p}(f, t)_p \int_t^{(1 - \log t)^{1/\alpha p'}} u^\alpha \frac{du}{u} \\
\times t^{-1/p} (1 - \log t)^{1/\alpha p'} \omega_{\alpha+1/p}(f, t)_p \\
\lesssim \int_0^t u^{-1/p} (1 - \log u)^{1/\alpha p'} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u}.
\]

(6.10)

Thus, the right-hand side of (6.8) is smaller than the corresponding one of (6.9). Note that inequality (6.8) is valid for a bigger class of functions than (6.9). For instance, take \( f \in L_p(\mathbb{T}) \) such that \( \omega_{\alpha+1/p}(f, u)_p \gg u^{1/p} (1 - \log u)^{-\varepsilon}, \ 1 < \varepsilon < 1 + 1/p' \) (see [Ti04]).

(ii) We observe that the sharp estimate (6.7) involves the semi-norm \( ||| \cdot |||_{C^a_\alpha(\mathbb{T}), \alpha} \) (cf. (2.6)) rather than the usual one \( \| \cdot \|_{C^a_\alpha(\mathbb{T}), \alpha} \) (cf. (2.5)). This technical issue is required in order to characterize the space \( \text{Lip}^{(\alpha, -b)}_{\infty, \infty}(\mathbb{T}) \) (see (2.7)) in terms of extrapolation of the scale \( C^a_\alpha(\mathbb{T}), \ a_0 < \alpha \).

(iii) The assumption \( b \geq 0 \) in Theorem 6.1 is imposed to avoid trivial spaces. Recall that \( \text{Lip}^{(\alpha, -b)}_{\infty, \infty}(\mathbb{T}) = \{0\} \) if \( b < 0 \) (see Section 2.1).

(iv) Despite the fact that (6.5) is optimal within the class \( \text{Lip}^{(\alpha, -b)}_{\infty, \infty}(\mathbb{T}) \), we will prove in Theorem 6.5 below that (6.5) can be improved with the help of the finer scale \( \text{Lip}^{(\alpha, -b)}_{\infty, q}(\mathbb{T}), q < \infty \) (see also Remark 6.6(ii) below.

(v) In the previous theorem we restrict our attention to univariate periodic functions. A comment for functions on \( \mathbb{T}^d \) and \( \mathbb{R}^d \) will be given in Remark 6.3 below.

**Proof of Theorem 6.1.** (i) \( \Rightarrow \) (ii): Using the well-known embedding

\[
B^{1/p}_{p,1}(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T})
\]

(6.11) and (i), we derive

\[
K(t, f; L_\infty(\mathbb{T}), \text{Lip}^{(\alpha, -b)}_{\infty, \infty}(\mathbb{T})) \lesssim K(t, f; B^{1/p}_{p,1}(\mathbb{T}), \dot{H}^{\alpha+1/p}_{p}(\mathbb{T})).
\]

(6.12)

Next, we proceed to estimate these \( K \)-functionals. According to (3.17), we have

\[
B^{1/p}_{p,1}(\mathbb{T}) = (L_p(\mathbb{T}), \dot{H}^{\alpha+1/p}_{p}(\mathbb{T}))^{1/(1+\alpha p)},
\]

and thus, by (3.22), we obtain

\[
K(t^{1 - \frac{1}{1+\alpha p}}, f; B^{1/p}_{p,1}(\mathbb{T}), \dot{H}^{\alpha+1/p}_{p}(\mathbb{T})) \asymp \int_0^t u^{-1/p} K(u, f; L_p(\mathbb{T}), \dot{H}^{\alpha+1/p}_{p}(\mathbb{T})) \frac{du}{u} \\
\asymp \int_0^t u^{-1/p} K(u^{\alpha+1/p}, f; L_p(\mathbb{T}), \dot{H}^{\alpha+1/p}_{p}(\mathbb{T})) \frac{du}{u}.
\]

Hence, a simple change of variables and (3.16) allow us to get

\[
K(t^{\alpha}, f; B^{1/p}_{p,1}(\mathbb{T}), \dot{H}^{\alpha+1/p}_{p}(\mathbb{T})) \asymp \int_0^t u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u}.
\]

(6.13)
Next we estimate $K(t, f; L_\infty(\mathbb{T}), \text{Lip}_{\infty, \alpha}^{(a-b)}(\mathbb{T}))$. It follows from (3.18), (3.23) and (3.16) that
\[
K(t^\alpha (1 - \log t)^b, f; L_\infty(\mathbb{T}), \text{Lip}_{\infty, \alpha}^{(a-b)}(\mathbb{T})) \\
\times K(t^\alpha (1 - \log t)^b, f; L_\infty(\mathbb{T}), \dot{H}_\infty^\alpha(\mathbb{T}))_{(1-b, \infty)} \\
\times K(t^\alpha, f; L_\infty(\mathbb{T}), \dot{H}_\infty^\alpha(\mathbb{T})) \\
+ t^\alpha (1 - \log t)^b \sup_{t^\alpha < u < 1} u^{-1}(1 - \log u)^{-b} K(u, f; L_\infty(\mathbb{T}), \dot{H}_\infty^\alpha(\mathbb{T}))
\]
(6.14) $\gtrsim K(t^\alpha, f; L_\infty(\mathbb{T}), \dot{H}_\infty^\alpha(\mathbb{T})) \propto \omega_\alpha(f, t)$. Plugging the estimates (6.14) and (6.13) into (6.12), we arrive at
\[
\omega_\alpha(f, t)_\infty \lesssim \int_0^{(1-\log t)^{b/\alpha}} u^{-1/p} \omega^{1/p + 1/p}(f, u) \frac{du}{u}.
\]
(ii) $\implies$ (iii): According to (ii), we have
\[
\omega_\alpha(f, t)_\infty \lesssim \int_0^{(1-\log t)^{b/\alpha}} u^{-1/p} \omega^{1/p + 1/p}(f, u) \frac{du}{u} \\
\lesssim t^\alpha (1 - \log t)^b \sup_{u>0} u^{-\alpha-1/p} \omega^{1/p + 1/p}(f, u) \\
\times t^\alpha (1 - \log t)^b \|f\|_{\dot{H}_p^{\alpha-1/p}(\mathbb{T})},
\]
where we have also used (3.19) in the last estimate. Therefore, applying a simple change of variables, we get
\[
\|f\|_{C_0(\mathbb{T}), \alpha} \times \sup_{0 < t < 1/2} t^{-\alpha_0} \omega_\alpha(f, t)_\infty \lesssim \|f\|_{\dot{H}_p^{\alpha-1/p}(\mathbb{T})} \sup_{0 < t < 1/2} t^{\alpha - \alpha_0} (-\log t)^b \\
\leq (\alpha - \alpha_0)^{-b} \|f\|_{\dot{H}_p^{\alpha-1/p}(\mathbb{T})} \sup_{0 < t < 1} t(-\log t)^b \\
\lesssim (\alpha - \alpha_0)^{-b} \|f\|_{\dot{H}_p^{\alpha-1/p}(\mathbb{T})}.
\]
(iii) $\implies$ (i): Let $j_0 \in \mathbb{N}_0$ be such that $2^{-j_0} < \alpha$ and set $\alpha_j = \alpha - 2^{-j}$, $j \geq j_0$. By (iii), we have
\[
2^{-j_0} \sup_{0 < t < 1/2} t^{-\alpha_j} \omega_\alpha(f, t)_\infty \leq C \|f\|_{\dot{H}_p^{\alpha-1/p}(\mathbb{T})}, \quad j \geq j_0,
\]
where $C > 0$ is independent of $j$. Hence, taking the supremum over all $j \geq j_0$, we get
\[
\sup_{j \geq j_0} 2^{-j_0} \sup_{0 < t < 1/2} t^{-\alpha_j} \omega_\alpha(f, t)_\infty \leq C \|f\|_{\dot{H}_p^{\alpha-1/p}(\mathbb{T})}
\]
and thus, we will get (i) if we show that
\[
(6.15) \quad \sup_{j \geq j_0} 2^{-j_0} \sup_{0 < t < 1/2} t^{-\alpha_j} \omega_\alpha(f, t)_\infty \propto \|f\|_{\text{Lip}_{\infty, \alpha}^{(a-b)}(\mathbb{T})}.
\]
Applying Fubini’s theorem and elementary computations, we obtain
\[
\sup_{j \geq j_0} 2^{-j_0} \sup_{0 < t < 1/2} t^{-\alpha_j} \omega_\alpha(f, t)_\infty = \sup_{0 < t < 1/2} t^{-\alpha}(1 - \log t)^{-b} \omega_\alpha(f, t)_\infty \times \|f\|_{\text{Lip}_{\infty, \alpha}^{(a-b)}(\mathbb{T})},
\]
where the penultimate estimate follows from the change of variables
\[
\sup_{j \geq j_0} 2^{-jb} t^{2^{-j}} \asymp (- \log t)^{-b} \sup_{j \geq j_0 - \log(- \log t)} 2^{-2^{-j}b} \]
together with the fact that
\[
2^{-2^{-j_0}b} \leq \sup_{j \geq j_0 - \log(- \log t)} 2^{-2^{-j}b} \leq \sup_{x > 0} 2^{-x} x^b < \infty.
\]
Hence, \((6.15)\) holds.

The equivalence between (i) and (iv) was already stated in \((6.2)\).

\[\Box\]

Remark 6.3. Imitating the proof of the implication \((i) \implies (ii)\) in Theorem 6.1, the embedding \((6.1)\) allows us to derive the following Ulyanov type inequality for functions \(f \in L_p(\mathbb{T}^d)\), \(1 < p < \infty\). Namely, we have
\[
\omega_\alpha(f, t, \infty) \lesssim \int_0^t \left( t^{-s} (1 + |\log t|)^b \omega_\alpha(f, t)^q \frac{dt}{t} \right)^{1/q}
\]
whenever the right-hand side is finite. The corresponding inequality for functions \(f \in L_p(\mathbb{R}^d)\) also holds true.

Before going further, we briefly recall the definition of Besov spaces of logarithmic smoothness. Let \(0 < s < \alpha, 1 \leq p \leq \infty, 0 < q \leq \infty\), and \(-\infty < b < \infty\). Then the (periodic) space \(B^{s,b}_{p,q}(\mathbb{T})\) is formed by all \(f \in L_p(\mathbb{T})\) such that
\[
(6.16) \quad \|f\|_{B^{s,b}_{p,q}(\mathbb{T}), \alpha} = \left( \int_0^\infty \left( t^{-s} (1 + |\log t|)^b \omega_\alpha(f, t)^q \frac{dt}{t} \right)^{1/q} \right).
\]
(with the usual change when \(q = \infty\)). Frequently, the integral in \((6.16)\) is taken over \((0, 1)\), that is,
\[
(6.17) \quad \|f\|_{B^{s,b}_{p,q}(\mathbb{T}), \alpha} = \left( \int_0^1 \left( t^{-s} (1 - \log t)^b \omega_\alpha(f, t)^q \frac{dt}{t} \right)^{1/q} \right).
\]
In particular, setting \(b = 0\) in \(\|f\|_{B^{s,b}_{p,q}(\mathbb{T}), \alpha}\) (respectively, \(\|f\|_{B^{s,b}_{p,q}(\mathbb{T})}\)) we recover \(\|f\|_{B^{s,b}_{p,q}(\mathbb{T}), \alpha}\) see \((2.5)\) (respectively, \(\|f\|_{B^{s,b}_{p,q}(\mathbb{T})}\), see \((2.6)\)). For more details on function spaces of logarithmic smoothness, we refer the reader to [DoTi].

Next we establish other sharpness assertions for \((6.8)\) which complement that given by \((ii) \iff (iv)\) in Theorem 6.1.

Remark 6.4. Inequality \((6.8)\) obtained in Theorem 6.1 is optimal in the following senses
\[
(6.18) \quad \omega_\alpha(f, t, \infty) \lesssim \left( \int_0^t (1 - \log t)^{1/p} \left( u^{-1/p} \omega_{\alpha + 1/p}(f, u)^q \frac{du}{u} \right)^{1/q} \right) \iff q \leq 1
\]
and
\[
(6.19) \quad \omega_\alpha(f, t, \infty) \lesssim \int_0^t (1 - \log t)^{1/p} \left( u^{-1/p} (1 + |\log u|)^b \omega_{\alpha + 1/p}(f, u)^q \frac{du}{u} \right) \iff b \geq 0.
\]
Proof of Remark 6.4. We show that the inequality
\[ \omega_\alpha(f, t) \lesssim \left( \int_0^{t(1 - \log t)^{1/p'}} \left( u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \right)^q \right)^{1/q} \]
yields \( q \leq 1 \). Indeed, taking \( t \) sufficiently large in the previous estimate we obtain
\[ B_{p,q}^1(T) \hookrightarrow L_\infty(T). \]
Since
\[ (6.20) \quad B_{p,q}^1(T) \hookrightarrow L_\infty(T) \iff q \leq 1, \]
see [SiTr, Theorem 3.3.1], (6.18) follows.

Next, we show that if
\[ (6.21) \quad \omega_\alpha(f, t) \lesssim \int_0^{t(1 - \log t)^{1/p'}} u^{-1/p}(1 + |\log u|)^b \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \]
holds then \( b \geq 0 \). We will proceed by contradiction, that is, assume that there exists \( b < 0 \) such that (6.21) holds. Then, taking \( t \) sufficiently large in (6.21), we derive
\[ (6.22) \quad B_{p,1}^{1/p,b}(T) \hookrightarrow L_\infty(T). \]
Let us distinguish two possible cases. Assume first that \( b \in (-1, 0) \). Let \( q \in \left( 1, \frac{1}{b+1} \right) \). Then, we have
\[ B_{p,q}^{1/p}(T) \hookrightarrow B_{p,1}^{1/p,b}(T) \] (see [DoTi, Proposition 6.1]) and, by (6.22), \( B_{p,q}^{1/p}(T) \hookrightarrow L_\infty(T) \), which is not true for \( q > 1 \) (see (6.20)). If \( b \leq -1 \) then the proof follows from the trivial embeddings \( B_{p,1}^{1/p,b_0}(T) \hookrightarrow B_{p,1}^{1/p,b}(T), b_0 > b, \) and the previous case. \( \square \)

According to (6.2) the Brézis-Wainger embedding
\[ (6.23) \quad \dot{H}_{p}^{\alpha+1/p}(T) \hookrightarrow \operatorname{Lip}_{1/p.(\alpha, -1/p')}^{(\alpha, -1/p')}(T) \]
is the best possible among the class of the logarithmic Lipschitz spaces \( \operatorname{Lip}_{1/p.\infty}^{(\alpha, -1/p)}(T) \).
However, as already mentioned in Remark 6.2(iv), (6.23) is not optimal within the broader scale of the spaces \( \operatorname{Lip}_{1/p.\infty}^{(\alpha, -1/p)}(T) \).

Theorem 6.5. Let \( \alpha > 0, 1 < p < \infty \) and \( b > 1/p \). Then, we have
\[ (6.24) \quad \dot{H}_{p}^{\alpha+1/p}(T) \hookrightarrow \operatorname{Lip}_{1/p.\infty}^{(\alpha, -1/p)}(T) \iff b \geq 1. \]

Remark 6.6. (i) For function spaces over \( \mathbb{R}^d \) and \( \alpha = 1 \), a local version of the embedding given in (6.24) was obtained by Triebel [Tri06, (1.242), page 49] and Haroske [Har, (6.18), page 96] as a part of the computation of the so-called continuity envelope of Sobolev spaces. However, their arguments do not allow to consider the fractional setting, that is, \( \alpha > 0 \). Below, we shall give a new approach which will enable us to establish (6.24) for all \( \alpha > 0 \).

(ii) The embedding (6.24) sharpens (6.23). More precisely, we will show that
\[ (6.25) \quad \operatorname{Lip}_{1/p.\infty}^{(\alpha, -1)}(T) \hookrightarrow \operatorname{Lip}_{1/p.\infty}^{(\alpha, -1/p')}(T). \]
Let \( t \in (0, 1) \). Using monotonicity properties of the moduli of smoothness (see Section 3.2), we get

\[
\| f \|_{\text{Lip}^{(\alpha,-1)}_{\infty,p}(\mathbb{T})} \geq \left( \int_0^t (u^{-\alpha}(1 - \log u)\omega_\alpha(f, u)_{\infty}^p \frac{du}{u} \right)^{1/p} \\
\geq t^{-\alpha} \omega_\alpha(f, t)_{\infty} \left( \int_0^t (1 - \log u)^{-p} \frac{du}{u} \right)^{1/p} \\
\asymp t^{-\alpha} (1 - \log t)^{-1/p'} \omega_\alpha(f, t)_{\infty}.
\]

Now the embedding (6.25) follows by taking the supremum over all \( t \in (0, 1) \).

Moreover, it is not hard to see that \( \text{Lip}^{(\alpha,-1)}_{\infty,p}(\mathbb{T}) \neq \text{Lip}^{(\alpha,-1/p')}_{\infty,\infty}(\mathbb{T}) \).

**Proof of Theorem 6.5.** We will make use of the fractional counterpart of (3.11) which states that

\[
(6.26) \quad \omega_\alpha(f, t)_{\infty} \leq t^\alpha \int_t^1 \frac{\omega_{\gamma + \alpha}(f, u)_{\infty} \, du}{u^\alpha}, \quad t \in (0, 1), \quad \alpha, \gamma > 0.
\]

See [KoT1, Theorem 4.4].

We start by showing that

\[
(6.27) \quad B_{\infty,p}^{\alpha}(\mathbb{T}) \hookrightarrow \text{Lip}^{(\alpha,-1)}_{\infty,p}(\mathbb{T}).
\]

Applying (6.26) and (3.5), we have

\[
\| f \|_{\text{Lip}^{(\alpha,-1)}_{\infty,p}(\mathbb{T})} = \left( \int_0^1 \frac{(t^{-\alpha}(1 - \log t)^{-1}\omega_\alpha(f, t)_{\infty})^p \, dt}{t} \right)^{1/p} \\
\leq \left( \int_0^1 (1 - \log t)^{-1} \int_t^1 \frac{\omega_{\alpha + \gamma}(f, u)_{\infty} \, du}{u^\alpha} \frac{p \, dt}{t} \right)^{1/p} \\
\leq \left( \int_0^1 (t^{-\alpha}\omega_{\alpha + \gamma}(f, t)_{\infty})^p \, dt \right)^{1/p} \leq \| f \|_{B_{\infty,p}^{\alpha}(\mathbb{T}), \alpha + \gamma}.
\]

Combining (6.27) with the Jawerth-Franke embedding \( \dot{H}^{\alpha+1/p}_{p}(\mathbb{T}) \hookrightarrow B_{\infty,p}^{\alpha}(\mathbb{T}) \) (see (6.42) below), we arrive at the embedding stated in (6.24).

Next we show the only-if part. We will proceed by contradiction. Assume that there exists \( b \in (1/p, 1) \) such that \( \dot{H}^{\alpha+1/p}_{p}(\mathbb{T}) \hookrightarrow \text{Lip}^{(\alpha,-b)}_{\infty,p}(\mathbb{T}) \). It is plain to see that \( \text{Lip}^{(\alpha,-b)}_{\infty,p}(\mathbb{T}) \hookrightarrow \text{Lip}^{(\alpha,-b+1/p)}_{\infty,p}(\mathbb{T}) \). Hence,

\[
\dot{H}^{\alpha+1/p}_{p}(\mathbb{T}) \hookrightarrow \text{Lip}^{(\alpha,-b+1/p)}_{\infty,p}(\mathbb{T}) \quad \text{for some} \quad b < 1.
\]

However, this contradicts (6.2). \( \square \)

**Remark 6.7.** Similarly one can show that

\[
(6.28) \quad \dot{H}^{\alpha+d/p}_{p}(\mathcal{X}) \hookrightarrow \text{Lip}^{(\alpha,-1)}_{\infty,p}(\mathcal{X}), \quad \mathcal{X} \in \{ \mathbb{T}^d, \mathbb{R}^d \}, \quad \alpha > 0, 1 < p < \infty.
\]

The characterization of the refinement of the Brézis-Wainger embedding, i.e., \( \dot{H}^{\alpha+1/p}_{p}(\mathbb{T}) \hookrightarrow \text{Lip}^{(\alpha,-1)}_{\infty,p}(\mathbb{T}) \) takes the following form.

**Theorem 6.8.** Let \( 1 < p < \infty, \alpha > 0 \) and \( b > 1/p \). The following statements are equivalent:
(i) $\hat{H}_p^{\alpha+1/p}(\mathbb{T}) \hookrightarrow \text{Lip}_{\infty,p}^{(\alpha-b)}(\mathbb{T})$,

(ii) for $f \in B_{p,1}^{1/p}(\mathbb{T})$, we have

$$\omega_\alpha(f,t)_\infty + t^\alpha (1 - \log t)^{b-1/p} \left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-b}\omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \int_0^{t(1-\log t)^{(b-1/p)/\alpha}} u^{-1/p} \omega_{\alpha+1/p}(f,u)_p \frac{du}{u},$$

(6.29)

(iii) we have

$$B_{p,p}^{\alpha+1/p-\lambda}(\mathbb{T}) \hookrightarrow B_{\infty,p}^{\alpha-\lambda-b} \big(\mathbb{T}\big)$$

with norm $O(\lambda^{1/p})$ as $\lambda \to 0^+$. More precisely, there exists $C > 0$, which is independent of $\lambda$, such that

$$\|f\|_{B_{p,p}^{\alpha-\lambda-b}(\mathbb{T})} \leq C \lambda^{1/p} \|f\|_{B_{p,p}^{\alpha+1/p-\lambda}(\mathbb{T}),\alpha+1/p}, \quad \lambda > 0,$$

(iv) $b \geq 1$.

Remark 6.9. (i) The two terms given in the left-hand side of (6.29) are not comparable. Indeed, take $f \in L_\infty(\mathbb{T})$ with $\omega_\alpha(f,t)_\infty \asymp t^\alpha (1 - \log t)^{1/p'} (1 + \log(1 - \log t))^{-\varepsilon}$ where $\varepsilon < 1/p$ (cf. [Ti04]). Elementary computations show that $\omega_\alpha(f,t)_\infty \lesssim t^\alpha (1 - \log t)^{1/p'} \left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-1}\omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p}$. On the other hand, consider $g \in L_\infty(\mathbb{T})$ such that $\omega_\alpha(g,t)_\infty \asymp t^\eta$ for some $\eta \in (0,\alpha)$ (cf. [Ti04]). It is readily seen that $t^\alpha (1 - \log t)^{1/p'} \left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-1}\omega_\alpha(g,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \omega_\alpha(g,t)$.

(ii) The Ulyanov-type inequality (6.29) with $b = 1$, that is,

$$\omega_\alpha(f,t)_\infty + t^\alpha (1 - \log t)^{1/p'} \left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-1}\omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \int_0^{t(1-\log t)^{1/\alpha p'}} u^{-1/p} \omega_{\alpha+1/p}(f,u)_p \frac{du}{u},$$

(6.31)

sharpens the estimate

$$\omega_\alpha(f,t)_\infty \lesssim \int_0^{t(1-\log t)^{1/\alpha p'}} u^{-1/p} \omega_{\alpha+1/p}(f,u)_p \frac{du}{u}$$

given in Theorem 6.1(ii) with $b = 1/p'$ (see also (6.8)). In particular, (6.31) sharpens the Ulyanov inequality (6.9). To be more precise, let

$$I(t) = \omega_\alpha(f,t)_\infty + t^\alpha (1 - \log t)^{1/p'} \left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-1}\omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p}$$

and $J(t) = \omega_\alpha(f,t)_\infty$. Obviously, $J(t) \leq I(t)$. Setting $f \in L_\infty(\mathbb{T})$ with $J(t) = \omega_\alpha(f,t)_\infty \asymp t^\alpha (1 - \log t)^{1/p'} (1 + \log(1 - \log t))^{-\varepsilon}, \varepsilon < 1/p$ (see [Ti04]), it is easy to check that $I(t) \asymp t^\alpha (1 - \log t)^{1/p'} (1 + \log(1 - \log t))^{-\varepsilon + 1/p}$. Thus, $J(t)$ and $I(t)$ are
not equivalent. Such an improvement is consistent with the fact that the embedding stated in Theorem 6.8(i), i.e,

\[ \dot{H}^{\alpha+1/p}_p(\mathbb{T}) \hookrightarrow \text{Lip}^{(\alpha,-1)}_{\infty,p}(\mathbb{T}), \]

is a refinement of that given in Theorem 6.1(i)

\[ \dot{H}^{\alpha+1/p}_p(\mathbb{T}) \hookrightarrow \text{Lip}^{(\alpha,-1/p')}_{\infty,p}(\mathbb{T}); \]

see Remark 6.6(ii).

(iii) The sharp norm estimates of the classical Sobolev embeddings for Besov spaces

\[ (6.32) \quad B^{k,-\lambda}_{p,r}(\mathbb{T}) \hookrightarrow B^{k+1/p-1/q-\lambda}_{q,r}(\mathbb{T}), \quad 1 \leq p < q \leq \infty, \quad 0 < r \leq \infty, \quad k \in \mathbb{N}, \]

as \( \lambda \to 0^+ \) or \( \lambda \to (k-1/p + 1/q)^- \) were settled in [KolLe] and [Do]. Here both Besov (semi-)norms in (6.32) are defined in terms of the corresponding modulus of smoothness with fixed order \( k \). This is in sharp contrast with (6.30) where the semi-norms \( \|f\|_{B^{\alpha,-\lambda}_{q,r}(\mathbb{T})} \) and \( \|f\|_{B^{\alpha+1/p-\lambda}_{q,r}(\mathbb{T})} \) involve the moduli of smoothness of order \( \alpha \) and \( \alpha + 1/p \), respectively. See (6.17). Furthermore, it turns out that the sharp estimates for (6.32) obtained in [KolLe] and [Do] and those given in (6.30) with \( b = 1 \) are independent of each other. Indeed, assume \( k = \alpha + 1/p \in \mathbb{N} \). According to [Do, Remark 3.3], if \( q = \infty \) and \( r = p \) in (6.32) then

\[ (6.33) \quad \|f\|_{B^{\alpha,-\lambda}_{q,r}(\mathbb{T}),k} \leq C \lambda^{1/p}\|f\|_{B^{\alpha+1/p-\lambda}_{q,r}(\mathbb{T}),k}, \quad \lambda \to 0^+. \]

At the same time, it follows from (6.30) that

\[ (6.34) \quad \|f\|_{B^{\alpha,-\lambda}_{q,r}(\mathbb{T}),\alpha} \leq C \lambda^{1/p}\|f\|_{B^{\alpha+1/p-\lambda}_{q,r}(\mathbb{T)},k}, \quad \lambda \to 0^+. \]

Notice that (6.34) is not an immediate consequence of (6.33). To be more precise, invoking Marchaud inequality (6.26) (noting that \( \alpha < k = \alpha + 1/p \)) and Hardy's inequality (3.2), it is plain to check that

\[ (6.35) \quad \|f\|_{B^{\alpha,-\lambda}_{q,r}(\mathbb{T}),\alpha} \leq \left( \int_0^\infty t^{-(\alpha-\lambda)/p} \omega_\alpha(f,t)^p \frac{dt}{t} \right)^{1/p} \lesssim \lambda^{-1}\|f\|_{B^{\alpha,-\lambda}_{q,r}(\mathbb{T}),k} \]

and so, by (6.33),

\[ \|f\|_{B^{\alpha,-\lambda}_{q,r}(\mathbb{T}),\alpha} \lesssim \lambda^{1/p-1}\|f\|_{B^{\alpha+1/p-\lambda}_{q,r}(\mathbb{T)},k}. \]

Here, the embedding constant blows up as \( \lambda \to 0^+ \), which is not the case in (6.34).

(iv) Notice that a remark parallel to Remark 6.2(ii) applies on (6.30).

(v) The assumption \( b > 1/p \) in Theorem 6.8 is imposed to avoid trivial spaces.

Recall that \( \text{Lip}^{(\alpha,-b)}_{\infty,p}(\mathbb{T}) = \{0\} \) if \( b \leq 1/p \) (see Section 2.1).

**Proof of Theorem 6.8.** (i) \( \implies \) (ii): By (i) and (6.11), we obtain

\[ (6.36) \quad K(t,f;L_\infty(\mathbb{T}),\text{Lip}^{(\alpha,-b)}_{\infty,p}(\mathbb{T})) \lesssim K(t,f;B^{1/p}_{p,1}(\mathbb{T}),\dot{H}^{\alpha+1/p}_p(\mathbb{T})). \]

We have (see (6.13))

\[ (6.37) \quad K(t^\gamma,f;B^{1/p}_{p,1}(\mathbb{T}),\dot{H}^{\alpha+1/p}_p(\mathbb{T})) \asymp \int_0^t u^{-1/p}\omega_{\alpha+1/p}(f,u)\frac{du}{u}. \]
Next we compute $K(t, f; L_\infty(\mathbb{T}), \text{Lip}_{\infty,p}^{(\alpha,-b)}(\mathbb{T}))$. Since $\text{Lip}_{\infty,p}^{(\alpha,-b)}(\mathbb{T}) = (L_\infty(\mathbb{T}), \dot{H}_\infty^{\alpha}(\mathbb{T}))_{(1,-b),p}$ (see (3.18)), we can apply (3.23) to get

$$K(t(1 - \log t)^{b-1/p}, f; L_\infty(\mathbb{T}), \text{Lip}_{\infty,p}^{(\alpha,-b)}(\mathbb{T}))$$

$$\times K(t(1 - \log t)^{b-1/p}, f; L_\infty(\mathbb{T}), (L_\infty(\mathbb{T}), \dot{H}_\infty^{\alpha}(\mathbb{T}))_{(1,-b),p})$$

$$\times K(t, f; L_\infty(\mathbb{T}), \dot{H}_\infty^{\alpha}(\mathbb{T}))$$

$$+ t(1 - \log t)^{b-1/p} \left( \int_t^1 (u^{-1}(1 - \log u)^{-b}\omega_{\alpha}(f, u)_{\infty})^p \frac{du}{u} \right)^{1/p}.$$

Therefore, by (3.16),

$$K(t^\alpha(1 - \log t)^{b-1/p}, f; L_\infty(\mathbb{T}), \text{Lip}_{\infty,p}^{(\alpha,-b)}(\mathbb{T})) \times \omega_\alpha(f, t)_\infty$$

$$\approx t^\alpha(1 - \log t)^{b-1/p} \left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-b}\omega_\alpha(f, u)_{\infty})^p \frac{du}{u} \right)^{1/p}.$$  \hfill (6.38)

Putting together (6.36), (6.37) and (6.38), it follows that

$$\omega_\alpha(f, t)_\infty + t^\alpha(1 - \log t)^{b-1/p} \left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-b}\omega_\alpha(f, u)_{\infty})^p \frac{du}{u} \right)^{1/p}$$

$$\lesssim \int_0^{t(1 - \log t)^{(b-1)/\alpha}} u^{-1/p}\omega_{\alpha+1/p}(f, u) \frac{du}{u}.$$

(ii) $\implies$ (iii): Firstly, we will show that (ii) implies

$$\left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-b}\omega_\alpha(f, u)_{\infty})^p \frac{du}{u} \right)^{1/p}$$

$$\lesssim t^{-\alpha}(1 - \log t)^{-b+1/p} \int_0^t u^{-1/p}(1 - \log u)^{b-1/p}\omega_{\alpha+1/p}(f, u) \frac{du}{u}.$$  \hfill (6.39)

Indeed, using monotonicity properties of the moduli of smoothness (see Section 3.2), we write

$$\int_0^{t(1 - \log t)^{(b-1)/\alpha}} u^{-1/p}\omega_{\alpha+1/p}(f, u) \frac{du}{u} \lesssim \int_0^t u^{-1/p}(1 - \log u)^{b-1/p}\omega_{\alpha+1/p}(f, u) \frac{du}{u}$$

$$+ \omega_{\alpha+1/p}(f, t) \frac{u^{-\alpha-1/p}}{u^{\alpha}} \int_t^{t(1 - \log t)^{(b-1)/\alpha}} \frac{du}{u}$$

$$\lesssim \int_0^t u^{-1/p}(1 - \log u)^{b-1/p}\omega_{\alpha+1/p}(f, u) \frac{du}{u}.$$
Therefore, by (ii) we derive
\[ t^\alpha (1 - \log t)^{b-1/p} \left( \int_t^1 (u^{-\alpha} (1 - \log u)^{-b}) \omega_\alpha(f, u)_\infty^p \frac{du}{u} \right)^{1/p} \leq \int_0^{t(1 - \log t)^{(b-1/p)/\alpha}} u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \leq \int_0^t u^{-1/p} (1 - \log u)^{b-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u}. \]

This yields \((6.39)\).

Let \(0 < \lambda < \alpha\). Applying Hölder’s inequality,

\[
(\int_0^t u^{-1/p} (1 - \log u)^{b-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u})^p \leq (\alpha - \lambda)^{-p/p'} \int_0^t \left( \frac{du}{u} \right)^{p} \left( (1 - \log u)^{b-1/p} \omega_{\alpha+1/p}(f, u)_p \right) \frac{du}{u}.
\]

In view of \((6.17)\), \((6.39)\) and \((6.40)\) and applying Fubini’s theorem twice, we have

\[
\|f\|_{B_{p, \alpha, \lambda, \alpha+1/p}(T)}^p = \int_0^1 u^{-\alpha p + \lambda p} (1 - \log u)^{-b p} \omega_\alpha(f, u)_\infty^p \frac{du}{u} = \int_0^1 u^{-\alpha p} (1 - \log u)^{-b p} \omega_\alpha(f, u)_\infty^p \frac{du}{t} \int_0^t t^{\lambda p} \frac{du}{u} = \int_0^1 u^{-\alpha p} (1 - \log u)^{-b p} \omega_\alpha(f, u)_\infty^p \frac{du}{u} \int_0^t t^{\lambda p} dt \leq \lambda \int_0^t t^{\lambda p - \alpha p} (1 - \log t)^{-b p + 1} \left( \int_0^1 u^{-1/p} (1 - \log u)^{b-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \right)^p \frac{dt}{t}.
\]

Thus,

\[
\lambda \int_0^t t^{\lambda p - \alpha p} (1 - \log t)^{-b p + 1} \left( \int_0^1 u^{-1/p} (1 - \log u)^{b-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \right)^p \frac{dt}{t} \leq \int_0^t t^{\lambda p - \alpha p} (1 - \log t)^{-b p + 1} \left( \int_0^1 u^{-1/p} (1 - \log u)^{b-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \right)^p \frac{dt}{t}.
\]

Therefore,

\[
\lambda \int_0^t t^{\lambda p - \alpha p} (1 - \log t)^{-b p + 1} \left( \int_0^1 u^{-1/p} (1 - \log u)^{b-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \right)^p \frac{dt}{t} \leq \int_0^t t^{\lambda p - \alpha p} (1 - \log t)^{-b p + 1} \left( \int_0^1 u^{-1/p} (1 - \log u)^{b-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u} \right)^p \frac{dt}{t}.
\]
where the last estimate follows from the fact that \((\alpha - \lambda)^{-p/p' - 1}\) is uniformly bounded with \(\lambda \to 0^+\).

(iii) \(\implies\) (i): We claim that there is a positive constant \(C\), which does not depend on \(\lambda\), satisfying
\[
\|f\|_{B_{p,p}^{\alpha+1/p-\lambda(T),\alpha+1/p}} \leq C \lambda^{-1/p} \|f\|_{H_{p}^{\alpha+1/p}(T)}, \quad \lambda \to 0^+.
\]
Indeed, we have
\[
\|f\|_{B_{p,p}^{\alpha+1/p-\lambda(T),\alpha+1/p}} = \left( \int_0^1 t^{-(\alpha+1/p-\lambda)p} \omega_{\alpha+1/p}(f, t)^{p'} \frac{dt}{t} \right)^{1/p} \leq \left( \int_0^1 t^{p} \omega_{\alpha+1/p}(f, t)^{p} \right)^{1/p} \leq \lambda^{-1/p} \|f\|_{H_{p}^{\alpha+1/p}(T)},
\]
where we have used Lemma 3.6 in the last estimate.

According to (iii) and (6.41), we obtain
\[
\|f\|_{B_{p,p}^{\alpha-\lambda,b(T),\alpha}} \lesssim \lambda^{1/p} \|f\|_{B_{p,p}^{\alpha+1/p-\lambda(T),\alpha+1/p}} \lesssim \|f\|_{H_{p}^{\alpha+1/p}(T)},
\]
where the hidden constant is independent of \(\lambda\). Then, the embedding given in (i) follows by passing to the limit \(\lambda \to 0^+\) and applying the monotone convergence theorem (see (2.7) and (6.17)).

The equivalence (i) \(\iff\) (iv) was already shown in Theorem 6.5.

\[\square\]

**Remark 6.10.** Repeating the proof of the implication (i) \(\implies\) (ii) in Theorem 6.8 line by line but now using (6.28), one can establish the multivariate counterpart of (6.31). Namely, if \(1 < p < \infty\) and \(\alpha > 0\) then
\[
\omega_{\alpha}(f, t)_{\infty} \leq t^{\alpha}(1 - \log t)^{1/p'} \left( \int_0^1 (u^{-\alpha}(1 - \log u)^{-1} \omega_{\alpha}(f, u)_{\infty})^p \frac{du}{u} \right)^{1/p} \lesssim \int_0^{t(1-\log t)^{1/\alpha'}} \left( u^{-d/p} \omega_{\alpha+d/p}(f, u)^{p} \frac{du}{u} \right), \quad f \in L_p(\mathbb{R}^d),
\]
whenever the right-hand side is finite. The corresponding inequality for functions \(f \in L_p(\mathbb{R}^d)\) also holds true.

**Remark 6.11.** The Ulyanov-type inequality (6.29) (with \(b = 1\)) obtained in Theorem 6.8 is optimal in the following senses
\[
\omega_{\alpha}(f, t)_{\infty} \leq t^{\alpha}(1 - \log t)^{1/p'} \left( \int_0^1 (u^{-\alpha}(1 - \log u)^{-1} \omega_{\alpha}(f, u)_{\infty})^p \frac{du}{u} \right)^{1/p} \lesssim \left( \int_0^{t(1-\log t)^{1/\alpha'}} (u^{-1/p} \omega_{\alpha+1/p}(f, u)^{p} \frac{du}{u}) \right)^{1/q} \iff q \leq 1,
\]
and
\[
\omega_\alpha(f, t) + t^\alpha(1 - \log t)^{1/p'} \left( \int_t^1 (u^{-\alpha}(1 - \log u)^{-1} \omega_\alpha(f, u)\,du/u)^{1/p} \right) \\
\lesssim \int_0^t (1 - \log u)^{1/\alpha p'} u^{-1/p}(1 - \log u)^b \omega_{\alpha+1/p}(f, u)\,du/u 
\]

The proofs of these assertions proceed in complete analogy with those given to show (6.18) and (6.19) and they are left to the reader.

Before we proceed further, we recall that the Jawerth-Frank embeddings establish relations between Besov spaces and Triebel-Lizorkin spaces with different metrics. See [Ja] and [Fra] (cf. also [Mars] and [Vy]). In particular, working with Sobolev spaces, the result reads as follows.

**Theorem 6.12 (Embeddings of Jawerth-Franke).** Let \(1 \leq p_0 < p < p_1 \leq \infty\) and \(\alpha \geq 0\). Then
\[
B^{\alpha+d(1/p_0-1/p)}_{p_0,p}(\mathbb{T}^d) \hookrightarrow \dot{H}^\alpha_p(\mathbb{T}^d) \hookrightarrow B^{\alpha+d(1/p_1-1/p)}_{p_1,p}(\mathbb{T}^d).
\]

The previous embeddings also hold true for function spaces over \(\mathbb{R}^d\).

Note that working with Fourier-analytically defined function spaces, these embeddings can be extended to any \(\alpha \in \mathbb{R}\).

Applying the relationships between Lipschitz spaces and Besov spaces obtained in [DHT19], we observe that the Brézis-Wainger-type embeddings (cf. Theorem 6.5 and Remark 6.7) can be strengthened by the Jawerth-Franke embeddings. To be more precise, it follows from
\[
B^\alpha_{\infty,p}(\mathbb{T}^d) \hookrightarrow \text{Lip}_{\infty,p}^{(\alpha-1)}(\mathbb{T}^d), \quad \alpha > 0, \quad 1 < p < \infty,
\]

that
\[
\dot{H}^{\alpha+d/p}_{p}(\mathbb{T}^d) \hookrightarrow \dot{H}^\alpha_{\infty,p}(\mathbb{T}^d)
\]
consists of an improvement of
\[
\dot{H}^{\alpha+d/p}_{p}(\mathbb{T}^d) \hookrightarrow \text{Lip}_{\infty,p}^{(\alpha-1)}(\mathbb{T}^d).
\]

Our next goal is to study the estimates in terms of the moduli of smoothness and extrapolation inequalities related to (6.43).

**Theorem 6.13.** Let \(1 < p < \infty\), \(\alpha > 0\) and \(0 < q \leq \infty\). The following statements are equivalent:

(i) \( \dot{H}^{\alpha+d/p}_{p}(\mathbb{T}^d) \hookrightarrow \dot{H}^\alpha_{\infty,p}(\mathbb{T}^d) \),

(ii) for \( f \in B^{d/p}_{p,1}(\mathbb{T}^d) \), we have
\[
t^\alpha \left( \int_t^\infty (u^{-\alpha} \omega_{\alpha+d/p}(f, u)\,du/u)^{1/q} \right)^{1/q} \lesssim \int_0^t u^{-d/p} \omega_{\alpha+d/p}(f, u)\,du/u,
\]
we have
\[ B_{p,q}^{\alpha+d/p-\lambda}(\mathbb{T}^d) \hookrightarrow B_{\infty,q}^{\alpha-\lambda}(\mathbb{T}^d) \]

with norm \( O(\lambda^{1/q}) \) as \( \lambda \to 0^+ \). More precisely, there exists \( C > 0 \), which is independent of \( \lambda \), such that
\[
\|f\|_{B_{\infty,q}^{\alpha-\lambda}(\mathbb{T}^d), \alpha+d/p} \leq C \lambda^{1/q} \|f\|_{B_{p,q}^{\alpha+d/p-\lambda}(\mathbb{T}^d), \alpha+d/p}, \quad \lambda > 0,
\]

(iv) \( q \geq p \).

The corresponding result also holds true for \( \mathbb{R}^d \).

Remark 6.14. (i) Inequality (6.44) with \( q = p \) is actually the fractional counterpart of the following Kolyada’s inequality [Kol89b] (see also [Ne87a]):
\[
t^{\alpha} \left( \int_t^1 (u^{d/p-k} \omega_k(f,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \int_0^t u^{-d/p} \omega_k(f,u)_p \frac{du}{u}, \quad k > d/p.
\]

(ii) Let \( d = 1 \). The inequality (6.44) with \( q = p \),
\[
t^\alpha \left( \int_t^1 (u^{-\alpha \omega_{\alpha+1/p}(f,u)_\infty})^p \frac{du}{u} \right)^{1/p} \lesssim \int_0^t u^{-1/p} \omega_{\alpha+1/p}(f,u)_p \frac{du}{u},
\]
is stronger than (6.29) with \( b = 1 \), that is,
\[
\omega_\alpha(f,t)_\infty + t^\alpha (1 - \log t)^{1/p'} \left( \int_t^1 (u^{-\alpha} (1 - \log u)^{-1} \omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \int_0^t (u^{-\alpha \omega_{\alpha+1/p}(f,u)_\infty})^p \frac{du}{u}.
\]

Indeed, assume that (6.46) holds true. Then, applying the Marchaud inequality (6.26) together with Hardy’s inequality (3.5), we get
\[
\left( \int_t^1 (u^{-\alpha} (1 - \log u)^{-1} \omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \left( \int_t^1 (1 - \log u)^{-1} \int_u^1 \omega_{\alpha+1/p}(f,v)_\infty \frac{dv}{v} \frac{du}{u} \right)^{1/p} \lesssim \left( \int_t^1 (u^{-\alpha \omega_{\alpha+1/p}(f,u)_\infty})^p \frac{du}{u} \right)^{1/p}.
\]

Hence, it follows from (6.46) that
\[
t^\alpha (1 - \log t)^{1/p'} \left( \int_t^1 (u^{-\alpha} (1 - \log u)^{-1} \omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \int_0^t (u^{-\alpha \omega_{\alpha+1/p}(f,u)_\infty})^p \frac{du}{u}.
\]

Indeed, assume that (6.46) holds true. Then, applying the Marchaud inequality (6.26) together with Hardy’s inequality (3.5), we get
\[
\left( \int_t^1 (u^{-\alpha} (1 - \log u)^{-1} \omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \left( \int_t^1 (1 - \log u)^{-1} \int_u^1 \omega_{\alpha+1/p}(f,v)_\infty \frac{dv}{v} \frac{du}{u} \right)^{1/p} \lesssim \left( \int_t^1 (u^{-\alpha \omega_{\alpha+1/p}(f,u)_\infty})^p \frac{du}{u} \right)^{1/p}.
\]

Hence, it follows from (6.46) that
\[
t^\alpha (1 - \log t)^{1/p'} \left( \int_t^1 (u^{-\alpha} (1 - \log u)^{-1} \omega_\alpha(f,u)_\infty)^p \frac{du}{u} \right)^{1/p} \lesssim \int_0^t (u^{-\alpha \omega_{\alpha+1/p}(f,u)_\infty})^p \frac{du}{u}.
\]
On the other hand, invoking again (6.26), we have

$$\omega_\alpha(f, t)_\infty \lesssim t^\alpha \int_t^1 u^{-\alpha} \omega_{\alpha+1/p}(f, u)_\infty \frac{du}{u} = I + II,$$

where

$$I = t^\alpha \int_t^{(1-\log t)^{1/\alpha p'}} u^{-\alpha} \omega_{\alpha+1/p}(f, u)_\infty \frac{du}{u},$$

and

$$II = t^\alpha \int_{(1-\log t)^{1/\alpha p'}}^1 u^{-\alpha} \omega_{\alpha+1/p}(f, u)_\infty \frac{du}{u}.$$

By Hölder’s inequality, we get

$$II \leq t^\alpha \left( \int_t^{(1-\log t)^{1/\alpha p'}} (u^{-\alpha} \omega_{\alpha+1/p}(f, u)_\infty)^p \frac{du}{u} \right)^{1/p} \left( \int_t^{(1-\log t)^{1/\alpha p'}} \frac{du}{u} \right)^{1/p'}$$

$$\propto t^\alpha (1 - \log t)^{1/p'} \left( \int_{(1-\log t)^{1/\alpha p'}}^1 (u^{-\alpha} \omega_{\alpha+1/p}(f, u)_\infty)^p \frac{du}{u} \right)^{1/p}$$

(6.50) \quad \lesssim \int_{0}^{(1-\log t)^{1/\alpha p'}} u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u}

where we have used (6.46) in the last step.

Next we estimate $I$. Using the monotonicity properties of the moduli of smoothness (see Section 3.2),

$$I \leq t^\alpha \omega_{\alpha+1/p}(f, t(1-\log t)^{1/\alpha p'})_\infty \int_t^{(1-\log t)^{1/\alpha p'}} u^{-\alpha} \frac{du}{u}$$

$$\lesssim \omega_{\alpha+1/p}(f, t(1-\log t)^{1/\alpha p'})_\infty \lesssim \frac{\omega_{\alpha+1/p}(f, t(1-\log t)^{1/\alpha p'})_\infty}{(t(1-\log t)^{1/\alpha p'})^{1/p}}$$

$$\times \frac{\omega_{\alpha+1/p}(f, t(1-\log t)^{1/\alpha p'})_\infty}{(t(1-\log t)^{1/\alpha p'})^{\alpha + 1/p}} \int_{0}^{(1-\log t)^{1/\alpha p'}} u^{-\alpha} \frac{du}{u}$$

(6.51) \quad \lesssim \int_{0}^{(1-\log t)^{1/\alpha p'}} u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u}.$$

Therefore, (6.49)–(6.51) yield that

$$\omega_\alpha(f, t)_\infty \lesssim \int_{0}^{(1-\log t)^{1/\alpha p'}} u^{-1/p} \omega_{\alpha+1/p}(f, u)_p \frac{du}{u}$$

(6.52)

Moreover, elementary computations lead to

$$t^\alpha (1 - \log t)^{1/p'} \left( \int_t^{(1-\log t)^{1/\alpha p'}} (u^{-\alpha} (1 - \log u)^{-1} \omega_\alpha(f, u)_\infty)^p \frac{du}{u} \right)^{1/p}$$

$$\lesssim (1 - \log t)^{1/p'} \omega_\alpha(f, t)_\infty \left( \int_t^{(1-\log t)^{1/\alpha p'}} (1 - \log u)^{-p} \frac{du}{u} \right)^{1/p}$$

(6.53) \quad \lesssim \omega_\alpha(f, t)_\infty.
Finally, putting together (6.48), (6.52) and (6.53), we arrive at (6.47).

(iii) Another proof of (6.45) with $\alpha + d/p \in \mathbb{N}$ was obtained in [Do, Remark 3.3]. See also Remark 6.9(iii).

Proof of Theorem 6.13. (i) $\implies$ (ii): According to (i) and the well-known embedding $B_{p,1}^{d/p}(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d)$, we have

$$K(t^\alpha, f; L_\infty(\mathbb{T}^d), B_{\infty,q}^\alpha(\mathbb{T}^d)) \lesssim K(t^\alpha, f; B_{p,1}^{d/p}(\mathbb{T}^d), \dot{H}^{\alpha+d/p}(\mathbb{T}^d)).$$

It turns out that

$$K(t^\alpha, f; B_{p,1}^{d/p}(\mathbb{T}^d), \dot{H}^{\alpha+d/p}(\mathbb{T}^d)) \asymp \int_0^t u^{-d/p} \omega_{\alpha+d/p}(f, u)_p \frac{du}{u},$$

see (6.37).

Next we estimate $K(t, f; L_\infty(\mathbb{T}^d), B_{\infty,q}^\alpha(\mathbb{T}^d))$. Since

$$B_{\infty,q}^\alpha(\mathbb{T}^d) = (L_\infty(\mathbb{T}^d), \dot{H}^{\alpha+d/p}(\mathbb{T}^d)) \frac{\alpha^q}{\alpha p + q}$$

(see (3.17)), we can invoke (3.21) together with (3.16) to establish

$$K(t^\alpha, f; L_\infty(\mathbb{T}^d), B_{\infty,q}^\alpha(\mathbb{T}^d))$$

$$\asymp t^\alpha \left( \int_t^\infty (u^{-\alpha} K(u^\alpha, f; L_\infty(\mathbb{T}^d), \dot{H}^{\alpha+d/p}(\mathbb{T}^d)))^{q/\omega_{\alpha+d/p}(f, u)_\infty} \frac{du}{u} \right)^{1/q}$$

$$\asymp t^\alpha \left( \int_t^\infty (u^{-\alpha} \omega_{\alpha+d/p}(f, u)_\infty)^{q/\omega_{\alpha+d/p}(f, u)_\infty} \frac{du}{u} \right)^{1/q}. \quad (6.56)$$

Combining (6.54), (6.55) and (6.56), we obtain

$$t^\alpha \left( \int_t^\infty (u^{-\alpha} \omega_{\alpha+d/p}(f, u)_\infty)^{q/\omega_{\alpha+d/p}(f, u)_\infty} \frac{du}{u} \right)^{1/q} \lesssim \int_0^t u^{-d/p} \omega_{\alpha+d/p}(f, u)_p \frac{du}{u}.$$

(ii) $\implies$ (iii): Let $\lambda > 0$. Applying Fubini’s theorem, we have

$$\|f\|_{B_{\infty,q}^{\alpha}(\mathbb{T}^d), \alpha+d/p} \approx \left( \int_0^1 u^{-(\alpha - \lambda)q} \omega_{\alpha+d/p}(f, u)_\infty^q \frac{du}{u} \right)^{1/q}$$

$$\asymp \lambda^{1/q} \left( \int_0^1 u^{-\alpha q} \omega_{\alpha+d/p}(f, u)_\infty^q \int_0^u t^{\lambda q} \frac{dt}{t} \frac{du}{u} \right)^{1/q}$$

$$= \lambda^{1/q} \left( \int_0^1 t^{\lambda q} \int_0^1 u^{-\alpha q} \omega_{\alpha+d/p}(f, u)_\infty^q \frac{du}{u} \frac{dt}{t} \right)^{1/q}.$$

Therefore, in virtue of (6.44) we arrive at (6.57)

$$\|f\|_{B_{\infty,q}^{\alpha}(\mathbb{T}^d), \alpha+d/p} \lesssim \lambda^{1/q} \left( \int_0^1 t^{-(\alpha - \lambda)q} \left( \int_0^t u^{-d/p} \omega_{\alpha+d/p}(f, u)_p \frac{du}{u} \right)^q \frac{dt}{t} \right)^{1/q}.$$
We distinguish two possible cases. Firstly, assume that \( q \geq 1 \). Then, noting that 
\[
\left( \int_0^1 t^{-(\alpha-\lambda)q} \left( \int_0^t u^{-d/p} \omega_{\alpha+d/p}(f, u) \frac{du}{u} \right)^q \frac{dt}{t} \right)^{1/q}
\]
(6.58) \[
\lesssim \left( \int_0^1 (t^{-\alpha+d/p-\lambda}) \omega_{\alpha+d/p}(f, t)^q \frac{dt}{t} \right)^{1/q}
\]
uniformly with respect to \( \lambda \).

Let \( q < 1 \). Notice that we may assume without loss of generality that \( \alpha > \lambda \) because \( \lambda \to 0^+ \). Then applying monotonicity properties and Fubini’s theorem, we have

\[
\int_0^1 t^{-(\alpha-\lambda)q} \left( \int_0^t u^{-d/p} \omega_{\alpha+d/p}(f, u) \frac{du}{u} \right)^q \frac{dt}{t}
\]
\[
\leq \sum_{j=0}^{\infty} 2^{j(\alpha-\lambda)q} \left( \sum_{i=j}^{\infty} 2^{i(\alpha+d/p-\lambda)} \omega_{\alpha+d/p}(f, 2^{-i}) \right)^q
\]
\[
\leq \sum_{j=0}^{\infty} 2^{j(\alpha-\lambda)q} \sum_{i=j}^{\infty} (2^{i(\alpha+d/p-\lambda)} \omega_{\alpha+d/p}(f, 2^{-i}))^q
\]
\[
\lesssim \frac{1}{2^{(\alpha-\lambda)q}} \sum_{i=0}^{\infty} (2^{i(\alpha+d/p-\lambda)} \omega_{\alpha+d/p}(f, 2^{-i}))^q
\]
\[
\lesssim \int_0^1 (t^{-(\alpha+d/p-\lambda)}) \omega_{\alpha+d/p}(f, t)^q \frac{dt}{t}
\]
since \( \lambda \to 0^+ \). Thus (6.58) holds for \( q > 0 \).

Inserting this estimate into (6.57), we get

\[
\| f \|_{\dot{H}^\alpha_{\infty,q}(\mathbb{T}^d)} \lesssim \lambda^{1/q} \left( \int_0^1 (t^{-\alpha+d/p-\lambda}) \omega_{\alpha+d/p}(f, t)^q \frac{dt}{t} \right)^{1/q}
\]
\[
= \lambda^{1/q} \| f \|_{\dot{B}^{\alpha+d/p}_{p,q}(\mathbb{T}^d)}.
\]

(iii) \( \implies \) (i): By (3.19), we have

\[
\| f \|_{\dot{B}^{\alpha+d/p-\lambda}_{p,q}(\mathbb{T}^d), \alpha+d/p} = \left( \int_0^1 (t^{-\alpha+d/p+\lambda}) \omega_{\alpha+d/p}(f, t)^q \frac{dt}{t} \right)^{1/q} \lesssim \lambda^{-1/q} \| f \|_{\dot{H}^\alpha_{p,d}(\mathbb{T}^d)}.
\]

This and (6.45) allow us to derive

\[
\| f \|_{\dot{H}^\alpha_{\infty,q}(\mathbb{T}^d), \alpha+d/p} \leq C \| f \|_{\dot{H}^\alpha_{p,d}(\mathbb{T}^d)}, \quad \lambda \to 0^+.
\]

The embedding \( \dot{H}^\alpha_{p,d}(\mathbb{T}^d) \hookrightarrow \dot{B}^\alpha_{\infty,q}(\mathbb{T}^d) \) now follows from the monotone convergence theorem.

The equivalence between (i) and (iv) is well known. See [SiTr, Theorem 3.2.1]. In this reference only inhomogeneous spaces are considered. However, the arguments carry over to homogeneous spaces.

The same method of proof also works when \( \mathbb{T}^d \) is replaced by \( \mathbb{R}^d \). \( \square \)
Remark 6.15. The method of proof of Theorem 6.13 can also be applied to show that the sharp estimate (6.45) also holds when the Besov semi-norms $\|\cdot\|_{B_{\infty,q}^{\alpha,-\lambda}(\mathbb{T}^d),\alpha+d/p}$ and $\|\cdot\|_{B_{p,q}^{\alpha+d/p-\lambda}(\mathbb{T}^d),\alpha+d/p}$ are replaced by $\|\cdot\|_{B_{\infty,q}^{\alpha,-\lambda}(\mathbb{T}^d),\alpha+d/p}$ and $\|\cdot\|_{B_{p,q}^{\alpha+d/p-\lambda}(\mathbb{T}^d),\alpha+d/p}$ respectively. Further details are left to the reader.

Our next goal is to provide additional insights on the Ulyanov inequality (6.3) with $p = 1$, i.e.,

\begin{equation}
\omega_\alpha(f, t)_q \lesssim \left( \int_0^t (u^{-1/q'}(1 - \log u)^{1/q'})^{\frac{1}{q'}} \omega_{\alpha+1/q'}(f, u_1)\frac{du}{u} \right)^{1/q}, \quad 1 < q < \infty.
\end{equation}

To get this we shall rely on the following embeddings between logarithmic Lipschitz spaces, which were recently obtained in [DHT19].

Lemma 6.16. Let $\alpha > 0$ and $1 < q < \infty$. Then, we have

\begin{equation}
\text{Lip}_{1,\infty}^{(\alpha+d/q',0)}(\mathbb{T}^d) \hookrightarrow \text{Lip}_{q,\infty}^{(\alpha,-1/q)}(\mathbb{T}^d).
\end{equation}

The corresponding embedding for Lipschitz spaces on $\mathbb{R}^d$ also holds true. In addition, if $d = 1$ and $b \geq 0$ then

\begin{equation}
\text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(\mathbb{T}) \hookrightarrow \text{Lip}_{q,\infty}^{(\alpha,-b)}(\mathbb{T}) \iff b \geq 1/q.
\end{equation}

Theorem 6.17. Let $1 < q < \infty, \alpha > 0$ and $b \geq 0$. The following statements are equivalent:

(i)

\begin{equation}
\text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(\mathbb{T}) \hookrightarrow \text{Lip}_{q,\infty}^{(\alpha,-b)}(\mathbb{T}),
\end{equation}

(ii) for $f \in B_{1,q'}^{1/q'}(\mathbb{T})$, we have

\begin{equation}
\omega_\alpha(f, t)_q \lesssim \left( \int_0^t (u^{-1/q'}(1 - \log u)^{1/q'})^{\frac{1}{q'}} \omega_{\alpha+1/q'}(f, u_1)\frac{du}{u} \right)^{1/q},
\end{equation}

(iii) we have

\begin{equation}
\text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(\mathbb{T}) \hookrightarrow B_{q,\infty}^{\alpha_0}(\mathbb{T})
\end{equation}

with norm $O((\alpha - \alpha_0)^{-b})$ as $\alpha_0 \to \alpha^-$. More precisely, there exists $C > 0$, which is independent of $\alpha_0$, such that

\begin{equation}
\|f\|_{B_{q,\infty}^{\alpha_0}(\mathbb{T}),\alpha} \leq C(\alpha - \alpha_0)^{-b}\|f\|_{\text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(\mathbb{T})}, \quad 0 < \alpha_0 < \alpha,
\end{equation}

(iv)

\begin{equation}
b \geq 1/q.
\end{equation}

Remark 6.18. (i) Inequality (6.62) with $b = 1/q$, i.e.,

\begin{equation}
\omega_\alpha(f, t)_q \lesssim \left( \int_0^t (1 - \log u)^{1/q'} \omega_{\alpha+1/q'}(f, u_1)\frac{du}{u} \right)^{1/q},
\end{equation}

improves the known estimate (6.59), i.e.,

\begin{equation}
\omega_\alpha(f, t)_q \lesssim \left( \int_0^t (u^{-1/q'}(1 - \log u)^{1/q'})^{\omega_{\alpha+1/q'}(f, u_1)}\frac{du}{u} \right)^{1/q}.
\end{equation}
Indeed, basic properties of the moduli of smoothness (cf. Section 3.2) allow us to obtain
\[
\left( \int_t^{(1-\log t)^{1/q}} (u^{-1/q'} \omega_{\alpha+1/q'}(f, u))^{\alpha q} \frac{du}{u} \right)^{1/q} \\
\lesssim t^{-\alpha - 1/q'} \omega_{\alpha+1/q'}(f, t) \left( \int_t^{(1-\log t)^{1/q}} u^{\alpha q} \frac{du}{u} \right)^{1/q} \\
\lesssim \left( \int_0^t (u^{-1/q'} (1 - \log u)^{1/q} \omega_{\alpha+1/q'}(f, u))^{\alpha q} \frac{du}{u} \right)^{1/q}.
\]

Thus the right-hand side of (6.64) is dominated by the right-hand side of (6.65). We also observe that (6.64) holds true for any \( f \in B^{1/q'}_{1,q}(\mathbb{T}) \) unlike (6.65), which can be applied for \( f \in B^{1/q',1/q}(\mathbb{T}) \). Note that \( B^{1/q',1/q}(\mathbb{T}) \subseteq B^{1/q'}_{1,q}(\mathbb{T}) \).

(ii) The higher-dimensional version of inequality (6.64) also holds. See Remark 6.20 below.

(iii) Setting \( \alpha = 1/q \) in (6.63), we deduce
\[
BV(\mathbb{T}) \hookrightarrow B^{\alpha}_{q,\infty}(\mathbb{T}), \quad 1 < q < \infty, \quad 0 < \alpha_0 < 1/q.
\]

This embedding is known because \( BV(\mathbb{T}) \hookrightarrow B^{1}_{1,\infty}(\mathbb{T}) \) and, in addition, it is plain to show that \( B^{1}_{1,\infty}(\mathbb{T}) \hookrightarrow B^{\alpha}_{q,\infty}(\mathbb{T}) \). However, the key novelty of (6.63) relies on the fact that the embedding constants of (6.66) can be estimated as follows
\[
t^{-\alpha_0} \omega_{1/q}(f, t) \leq C(1/q - \alpha_0)^{-1/q}\|f\|_{BV(\mathbb{T})},
\]
where \( C > 0 \) is independent of \( f, t \in (0,1) \) and \( \alpha_0 \to 1/q- \). Furthermore, the exponent \( 1/q \) is sharp.

(iv) In analogy to Remark 6.2(ii), the reason for working with the semi-norm \( \| \cdot \|_{B^{\alpha}_{q,\infty}(\mathbb{T}),\alpha} \) in (6.63) instead of \( \| \cdot \|_{B^{\alpha}_{q,\infty}(\mathbb{T}),\alpha} \) is to be able to characterize the space \( \text{Lip}^{(\alpha,-b)}_{q,\infty}(\mathbb{T}) \), which is endowed with (2.7), in terms of extrapolation of the scale \( B^{\alpha}_{q,\infty}(\mathbb{T}) \) as \( \alpha_0 \to \alpha^- \).

**Proof of Theorem 6.17.** (i) \( \implies \) (ii): The embedding \( B^{1/q'}_{1,q}(\mathbb{T}) \hookrightarrow L_q(\mathbb{T}) \) together with (i) allows us to derive
\[
K(t, f; L_q(\mathbb{T}), \text{Lip}^{(\alpha,-b)}_{q,\infty}(\mathbb{T})) \lesssim K(t, f; B^{1/q'}_{1,q}(\mathbb{T}), \text{Lip}^{(\alpha+1/q',0)}_{1,\infty}(\mathbb{T})).
\]

Applying (3.18) and (3.23), we have
\[
K(t(1-\log t)^b, f; L_q(\mathbb{T}), \text{Lip}^{(\alpha,-b)}_{q,\infty}(\mathbb{T})) \\
\asymp K(t(1-\log t)^b, f; L_q(\mathbb{T}), (L_q(\mathbb{T}), \dot{H}^\alpha_q(\mathbb{T}))(1-b),\infty) \\
\asymp K(t, f; L_q(\mathbb{T}), \dot{H}^\alpha_q(\mathbb{T})) \\
+ t(1-\log t)^b \sup_{t<u<1} u^{-1}(1-\log u)^{-b} K(u, f; L_q(\mathbb{T}), \dot{H}^\alpha_q(\mathbb{T})) \\
\gtrsim K(t, f; L_q(\mathbb{T}), \dot{H}^\alpha_q(\mathbb{T}))
\]
and thus, by (3.16),
\[
K(t^a(1-\log t)^b, f; L_q(\mathbb{T}), \text{Lip}^{(\alpha,-b)}_{q,\infty}(\mathbb{T})) \gtrsim \omega_\alpha(f, t)_q.
\]
Next we estimate $K(t, f; B_{1,q}^{1/q'}(T), \text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(T))$. To this end, we will make use of the following interpolation formulas (see (3.17) and (3.18))

\begin{equation}
B_{1,q}^{1/q'}(T) = (L_1(T), \dot{H}_1^{\alpha+1-1/q}(T))^{1/(1+\alpha q')},
\end{equation}

and

\begin{equation}
\text{Lip}_{1,\infty}^{(\alpha+1-1/q,0)}(T) = (L_1(T), \dot{H}_1^{\alpha+1-1/q}(T))^{(1,0),\infty}.
\end{equation}

Therefore, by (6.69), (6.70) and (3.24), applying monotonicity properties of the $K$-functional and a simple change of variables, we get

\begin{align*}
K(t^{\alpha q'/q}, f; B_{1,q}^{\frac{1}{q'}}(T), \text{Lip}_{1,\infty}^{(\alpha+1-\frac{1}{q},0)}(T)) \\
&\propto K(t^{\alpha q'/q}, f; (L_1(T), \dot{H}_1^{\alpha+1-\frac{1}{q}}(T))^{1/(1+\alpha q')}, (L_1(T), \dot{H}_1^{\alpha+1-\frac{1}{q}}(T))^{(1,0),\infty}) \\
&\propto \left( \int_0^t \left( u^{-\frac{1}{1+\alpha q'}} K(u, f; L_1(T), \dot{H}_1^{\alpha+1-\frac{1}{q}}(T)) \frac{du}{u} \right)^{\frac{1}{q}} \\
&\quad + t^{\frac{\alpha q'}{q+1}} \sup_{t<u<1} u^{-1} K(u, f; L_1(T), \dot{H}_1^{\alpha+1-\frac{1}{q}}(T)) \right) \\
&\propto \left( \int_0^t \left( u^{-\frac{1}{1+\alpha q'}} K(u, f; L_1(T), \dot{H}_1^{\alpha+1-\frac{1}{q}}(T)) \frac{du}{u} \right)^{\frac{1}{q}} \\
&\quad + t^{\frac{\alpha q'}{q+1}} K(t, f; L_1(T), \dot{H}_1^{\alpha+1-\frac{1}{q}}(T)) \right) \\
&\propto \left( \int_0^{(\alpha+1-\frac{1}{q})^{-1}} \left( u^{-\frac{1}{q'}} K(u^{\alpha+1-\frac{1}{q}}, f; L_1(T), \dot{H}_1^{\alpha+1-\frac{1}{q}}(T)) \frac{du}{u} \right)^{\frac{1}{q'}} \right)^{-1}.
\end{align*}

Applying now (3.16) in the previous estimate we obtain

\begin{equation}
K(t^{\alpha q'/q}, f; B_{1,q}^{\frac{1}{q'}}(T), \text{Lip}_{1,\infty}^{(\alpha+1-\frac{1}{q},0)}(T)) \propto \left( \int_0^{(\alpha+1-\frac{1}{q})^{-1}} \left( u^{-\frac{1}{q'}} \omega_{\alpha+1-\frac{1}{q}}(f, u) \frac{du}{u} \right)^{\frac{1}{q'}} \right)^{-1}.
\end{equation}

Combining (6.68), (6.67) and (6.71), we arrive at

\begin{align*}
\omega_\alpha(f, t)_q &\lesssim K(t^\alpha (1 - \log t)^b, f; B_{1,q}^{1/q'}(T), \text{Lip}_{1,\infty}^{(\alpha+1-1/q,0)}(T)) \\
&\lesssim \left( \int_0^{t(1-\log t)^{b/\alpha}} \left( u^{-1/q'} \omega_{\alpha+1-1/q}(f, u)_1 \frac{du}{u} \right)^{1/q} \right).
\end{align*}

(ii) $\implies$ (iii): By (6.62),

\begin{align*}
\omega_\alpha(f, t)_q &\lesssim \left( \int_0^{t(1-\log t)^{b/\alpha}} u^{\alpha q} \frac{du}{u} \right)^{1/q} \|f\|_{\text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(T)} \\
&\times t^\alpha (1 - \log t)^b \|f\|_{\text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(T)}.
\end{align*}

Let $\alpha_0 \in (0, \alpha)$. The previous estimate and (6.17) yield that

\begin{align*}
\|f\|_{B_{q,\infty}(T), \alpha} = \sup_{0<t<1} t^{-\alpha_0} \omega_\alpha(f, t)_q &\lesssim \|f\|_{\text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(T)} \sup_{0<t<1} t^{\alpha-\alpha_0} (1 - \log t)^b \\
&\times (\alpha - \alpha_0)^{-b} \|f\|_{\text{Lip}_{1,\infty}^{(\alpha+1/q',0)}(T)}.
\end{align*}
(iii) $\implies$ (i): Let $j_0 \in \mathbb{N}_0$ be such that $2^{-j_0} < \alpha$ and let $\alpha_j = \alpha - 2^{-j}$, $j \geq j_0$.

According to (6.63), we have

$$
(6.72) \quad \sup_{j \geq j_0} 2^{-j b} \|f\|_{B^{\alpha_j}_{q, \infty}(T), \alpha} \lesssim \|f\|_{\text{Lip}_{q, \infty}^{(\alpha+1/q', 0)}(T)}.
$$

On the other hand, in view of (6.17) and applying Fubini’s theorem, we derive

$$
\sup_{j \geq j_0} 2^{-j b} \|f\|_{B^{\alpha_j}_{q, \infty}(T), \alpha} = \sup_{j \geq j_0} 2^{-j b} \sup_{0 < t < 1} t^{-\alpha_j} \omega_\alpha(f, t) q
$$

$$
= \sup_{0 < t < 1} t^{-\alpha} \omega_\alpha(f, t) q \sup_{j \geq j_0} 2^{-j b} t^{2^{-j}}
$$

$$
\lesssim \sup_{0 < t < 1} t^{-\alpha} (1 - \log t)^{-b} \omega_\alpha(f, t) q = \|f\|_{\text{Lip}_{q, \infty}^{(\alpha-b)}(T)}.
$$

Inserting this estimate into (6.72), we arrive at $\text{Lip}_{q, \infty}^{(\alpha+1/q', 0)}(T) \hookrightarrow \text{Lip}_{q, \infty}^{(\alpha-b)}(T)$.

Concerning the equivalence between (i) and (iv), we refer to (6.61). □

We conclude this paper with two remarks.

**Remark 6.19.** The inequality (6.64) is optimal in the following sense

$$
(6.73) \quad \omega_\alpha(f, t) q \lesssim \left( \int_0^{t(1-\log t)^{1/q}} (u^{1/q'} \omega_{\alpha+1/q'}(f, u) t)^{1/r} \frac{du}{u} \right)^{1/r} \iff r \leq q
$$

and

$$
(6.74) \quad \omega_\alpha(f, t) q \lesssim \left( \int_0^{t(1-\log t)^{1/q}} (u^{1/q'} (1 - \log u)^b \omega_{\alpha+1/q'}(f, u) t)^{1/q} \frac{du}{u} \right)^{1/q} \iff b \geq 0.
$$

**Proof of Remark 6.19.** We start by showing (6.73). Since

$$
(6.75) \quad \left\| f - \frac{1}{2\pi} \int_0^{2\pi} f(t) dt \right\|_{L_q(T)} \asymp \sup_{t \geq 0} \omega_\alpha(f, t) q,
$$

see, e.g., [LvPi, (12)] and [DoTi, (13.12)], it follows from the inequality stated in (6.73) that $B^{1/q'}_{1,q}(T) \hookrightarrow L_q(T)$. According to [SiTr, Theorem 3.2.1], the previous embedding holds if and only if $r \leq q$.

Assume that the estimate given in (6.74) holds. Then, by (6.75), $B^{1/q', b}_{1,q}(T) \hookrightarrow L_q(T)$. Since $L_q(\log L)_{b}(T)$ is the r.i. hull of $B^{1/q', b}_{1,q}(T)$ (see [Mar]), we derive $L_q(\log L)_{b}(T) \hookrightarrow L_q(T)$. This yields $b \geq 0$.

**Remark 6.20.** The counterpart of (6.64) in higher dimensions reads as follows. Assume $\alpha > 0$ and $1 < q < \infty$. If $f \in B^{d/q'}_{1,q}(T^d)$ and $t \in (0, 1)$ then

$$
(6.76) \quad \omega_\alpha(f, t) q \lesssim \left( \int_0^{t(1-\log t)^{1/q}} (u^{-d/q'} \omega_{\alpha+d/q'}(f, u) t)^{1/q} \frac{du}{u} \right)^{1/q}.
$$

As a byproduct, we obtain the multivariate version of (6.59), namely,

$$
(6.77) \quad \omega_\alpha(f, t) q \lesssim \left( \int_0^{t} (u^{-d/q'} (1 - \log u)^{1/q} \omega_{\alpha+d/q'}(f, u) t)^{1/q} \frac{du}{u} \right)^{1/q}.
$$

Starting from the embedding (6.60), the proof of (6.76) proceeds in the same vein as was done for the implication (i) $\implies$ (ii) in Theorem 6.17. Note that this argument
can also be applied to show that (6.76) (and so, (6.77)) also holds true for functions $f$ on $\mathbb{R}^d$.

Acknowledgements The first author was partially supported by MTM 2017-84058-P. The second author was partially supported by MTM 2017-87409-P, 2017 SGR 358, and the CERCA Programme of the Generalitat de Catalunya. Part of this work was done during the visit of the authors to the Isaac Newton Institute for Mathematical Sciences, Cambridge, EPSRC Grant no EP/K032208/1.

References

[AH] Adams, D.R., Hedberg, L.I.: Function Spaces and Potential Theory. Springer, Berlin, 1999.
[AIl] Almgren, F.J., Lieb, E.H.: Symmetric decreasing rearrangement is sometimes continuous. J. Amer. Math. Soc. 2, 683–773 (1989).
[Al] Alvino, A.: Sulla diseguaglianza di Sobolev in spazi di Lorentz. Boll. Unione Mat. Ital. A 14, 148–156 (1977).
[BR] Bennett, C., Rudnick, K.: On Lorentz-Zygmund spaces. Dissertationes Math. 175, 67 pp. (1980).
[Bs] Bennett, C., Sharpley, R.: Interpolation of Operators. Academic Press, New York, 1988.
[BL] Bergh, J., Lofstrom, J.: Interpolation Spaces. An Introduction. Springer, Berlin, 1976.
[BBM] Bourgain, J., Brézis, H., Mironescu, P.: Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications. J. Anal. Math. 87, 77–101 (2002).
[BreLi] Brézis, H., Lieb, E.: Sobolev inequalities with remainder terms. J. Funct. Anal. 62, 73–86 (1985).
[BreWain] Brézis, H., Wainger, S.: A note on limiting cases of Sobolev embeddings and convolution inequalities. Comm. Partial Differential Equations 5, 773–789 (1980).
[Bru] Brudnyi, Yu.: Rational approximation and imbedding theorems. Dokl. Akad. Nauk SSSR 247, 269–272 (1979) (in Russian); English translation in Soviet Math. Dokl. 20, 681–684 (1979).
[CGO] Caetano, A.M., Gogatishvili, A., Opic, B.: Sharp embeddings of Besov spaces involving only logarithmic smoothness. J. Approx. Theory 152, 188–214 (2008).
[Cal] Calderón, A.P.: Spaces between $L^1$ and $L^\infty$ and the theorem of Marcinkiewicz. Studia Math. 26, 273–299 (1966).
[Ci04a] Cianchi, A.: Optimal Orlicz-Sobolev embeddings. Rev. Mat. Iberoam. 20, 427–474 (2004).
[Ci04b] Cianchi, A.: Symmetrization and second order Sobolev inequalities. Ann. Mat. Pura Appl. 183, 45–77 (2004).
[CiPS] Cianchi, A., Pick, L.: Sobolev embeddings into BMO, VMO, and $L^\infty$. Ark. Mat. 36, 317–340 (1998).
[Cf] Cobos, F., Fernández, D.L.: Hardy-Sobolev spaces and Besov spaces with a function parameter. In: Function Spaces and Applications, Lecture Notes Math. 1302. Springer, Berlin (1988), pp. 158–170.
[CfKU] Cobos, F., Fernández-Cabrera, L.M., Kühn, T., Ulrich, T.: On an extreme class of real interpolation spaces. J. Funct. Anal. 256, 2321–2366 (2009).
[CorNaVi] Cordero-Erausquin, D., Nazaret, B., Villani, C.: A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. Adv. Math. 182, 307–332 (2004).
[CuRi] Curbera, G.P., Ricker, W.J.: Can optimal rearrangement invariant Sobolev imbeddings be further extended? Indiana Univ. Math. J. 56, 1479–1497 (2007).
[CwP] Cwikel, M., Pustylnik, E.: Sobolev type embeddings in the limiting case. J. Fourier Anal. Appl. 4, 433–446 (1998).
[DeV] DeVore, R.A., Scherer, K.: Interpolation of linear operators on Sobolev spaces. Ann. of Math. 109, 583–599 (1979).
DeVore, R.A., Sharpley, R.C.: On the differentiability of functions in $\mathbb{R}^n$. Proc. Amer. Math. Soc. 91, 326–328 (1984).

DeVore, R.A., Sharpley, R.C.: Maximal functions measuring smoothness. Mem. Am. Math. Soc. 47, no. 293 (1984).

Domínguez, O.: Sharp estimates of the norms of embeddings between Besov spaces. Z. Anal. Anwend. 37, 127–149 (2018).

Domínguez, O., Tikhonov, S.: Function spaces of logarithmic smoothness: embeddings and characterizations. Mem. Am. Math. Soc. To appear. ArXiv:1811.06399

Domínguez, O., Haroske, D.D., Tikhonov, S.: Embeddings and characterizations of Lipschitz spaces. Preprint, (2019).

Edmunds, D.E., Evans, W.D.: Hardy Operators, Function Spaces, and Embeddings. Springer, Berlin, 2004.

Edmunds, D.E., Evans, W.D., Karadzhov, G.E.: Sharp estimates of the embedding constants for Besov spaces. Rev. Mat. Complut. 19, 161–182 (2006).

Edmunds, D.E., Evans, W.D., Haroske, D.D.: Embeddings in spaces of Lipschitz type, entropy and approximation numbers, and applications. J. Approx. Theory 104, 226–271 (2000).

Edmunds, D.E., Haroske, D.D., Neves, J.S., Opic, B.: Embeddings of Sobolev-type spaces into generalized H"older spaces involving $k$-modulus of smoothness. Ann. Mat. Pura Appl. 194, 425–450 (2015).

Federer, H., Fleming, W.: Normal and integral currents. Ann. of Math. 72, 458–520 (1960).

Fernández-Martínez, P., Signes, T.: Limit cases of reiteration theorems. Math. Nachr. 288, 25–47 (2015).

Holmstedt, T.: Interpolation of quasi-normed spaces. Math. Scand. 26, 177–199 (1970).

Hunt, R.: On $L(p, q)$ spaces. Enseignement Math. 12, 249–276 (1966).
[IvPi] Ivanov, V.I., Pichugov, S.A., Approximation of periodic functions in $L_p$ by linear positive methods, and multiple moduli of continuity. Math. Notes. 42, 925–930 (1987).

[Ja] Jawerth, B.: Some observations on Besov and Lizorkin-Triebel spaces. Math. Scand. 40, 94–104 (1977).

[JM] Jawerth, B., Milman, M.: Extrapolation theory with applications. Mem. Amer. Math. Soc. 89, 82 pp. (1991).

[JS] Johnen, H., Scherer, K.: On the equivalence of the $K$-functional and moduli of continuity and some applications. Lecture Notes in Math., vol. 571, Springer, Berlin, 1976, 119–140.

[KMX] Karadzhov, G.E., Milman, M., Xiao, J.: Limits of higher order Besov spaces and sharp reiteration theorems. J. Funct. Anal. 221, 323–339 (2005).

[KP] Kerman, R., Pick, L.: Optimal Sobolev embeddings. Forum Math 18, 535–570 (2006).

[Kol07] Kolyada, V.I.: On embedding theorems. NAFSA 8-Nonlinear Analysis, Function Spaces and Applications. Vol. 8, Czech. Acad. Sci., Prague, 2007, pp. 34–94.

[Kol89a] Kolyada, V.I.: Estimates of rearrangements and embedding theorems. Mat. Sb. 136, 3–23 (1988). English translation in Math. USSR-Sb. 64, 1–21 (1989).

[Kol89b] Kolyada, V.I.: On relations between moduli of continuity in different metric s. Trudy Mat. Inst. Steklov 181, 117–136 (1988). English translation in Proc. Steklov Inst. Math. 4, 127–148 (1989).

[KolLe] Kolyada, V.I., Lerner, A.K.: On limiting embeddings of Besov spaces. Studia Math. 171, 1–13 (2005).

[KolPe] Kolyada, V.I., Pérez Lázaro, F.J.: Inequalities for partial moduli of continuity and partial derivatives. Constr. Approx. 34, 23–59 (2011).

[Mars] Marschall, J.: Some remarks on Triebel spaces. Studia Math. 87, 79–92 (1987).

[Mar] Martín, J.: Symmetrization inequalities in the fractional case and Besov embeddings. J. Math. Anal. Appl. 344, 99–123 (2008).

[MM06] Martín, J., Milman, M.: Sharp Gagliardo-Nirenberg inequalities via symmetrization. Math. Res. Lett. 14, 49–62 (2006).

[MM07a] Martín, J., Milman, M.: A note on Sobolev inequalities and limits of Lorentz spaces. Contemp. Math. 445, 237–245 (2007).

[MM07b] Martín, J., Milman, M.: Higher-order symmetrization inequalities and applications. J. Math. Anal. Appl. 330, 91–113 (2007).

[MM10] Martín, J., Milman, M.: Pointwise symmetrization inequalities for Sobolev functions and applications. Adv. Math. 225, 121–199 (2010).

[MMP] Martín, J., Milman, M., Pustylnik, E.: Sobolev inequalities: Symmetrization and self-improvement via truncation. J. Funct. Anal. 252, 677–695 (2007).

[Maz60] Maz’ya, V.: Classes of regions and imbedding theorems for function spaces. Dokl. Akad. Nauk SSSR 133, 527–530 (1960) (in Russian); English translation in Soviet Math. Dokl. 1, 882–885 (1960).

[Maz61] Maz’ya, V.: On $p$-conductivity and theorems on embedding certain functional spaces into a $C$-space. Dokl. Akad. Nauk SSSR 140, 299–302 (1961) (in Russian).

[Maz] Maz’ya, V.: Sobolev Spaces with Applications to Elliptic Partial Differential Equations, Second, revised and augmented edition. Springer, Heidelberg, 2011.

[MS] Maz’ya, V., Shaposhnikova, T.: On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. J. Funct. Anal. 195, 230–238 (2002).

[Mil] Milman, M.: Extrapolation and Optimal Decompositions. Lecture Notes in Math. 1580, Springer, Berlin, 1994.

[MiPu] Milman, M., Pustylnik, E.: On sharp higher order Sobolev embeddings. Commun. Contemp. Math. 6, 495–511 (2004).

[Mos] Moser, J.: A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. 20, 1077–1092 (1971).
[Mossi] Mossino, J.: *Inégalités Isopérimétriques et Applications en Physique*. Travaux en cours, Hermann, Paris, 1984.

[Ne87a] Netrusov, Yu.V.: *Embedding theorems for the spaces $H_{p}^{\omega,k}$ and $H^{\omega,\omega,k}_{p}$*. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 59 (1987); Chisl. Metody i Voprosy Organiz. Vychisl. 8, 83–102, 177–178 (in Russian); translation in J. Soviet Math. 47, 2881–2895 (1989).

[Ne87b] Netrusov, Yu.V.: *Embedding theorems for Lizorkin-Triebel spaces*. Notes Sci. Sem. LOMI 159, 103–112 (1987) (in Russian); translation in J. Soviet Math. 47, 2896–2903 (1989).

[Ne89] Netrusov, Yu.V.: *Imbedding theorems of Besov spaces in Banach lattices*. J. Soviet Math. 47, 2871–2881 (1989).

[Ni] Nilsson, P.: *Reiteration theorems for real interpolation and approximation spaces*. Ann. Mat. Pura Appl. 132, 291–330 (1982).

[ON] O’Neil, R.: *Convolution operators and $L^{p,q}$ spaces*. Duke Math. J. 30, 129–142 (1963).

[Pe] Peetre, J.: *Espaces d’interpolation et théorème de Soboleff*. Ann. Inst. Fourier 16, 279–317 (1966).

[Po] Pohozhaev, S.I.: *On the imbedding Sobolev theorem for $pl = n$*. Dokl. Conf., Sect. Math. Moscow Power Inst., 158–170 (1965) (in Russian).

[Poo] Poornima, S.: *An embedding theorem for the Sobolev spaces $W^{1,1}(R^n)$*. Bull. Sci. Math. 107, 253–259 (1983).

[Sa] Saloff-Coste, L.: *Aspects of Sobolev Inequalities*. Cambridge University Press, Cambridge, 2002.

[SeTr] Seeger, A., Trebels, W.: *Embeddings for spaces of Lorentz-Sobolev type*. Math. Ann. 373, 1017–1056 (2019).

[SiTr] Sickel, W., Triebel, H.: *Hölder inequalities and sharp embeddings in function spaces of $B^{p,q}_{\nu}$ and $F^{\nu,q}_{p}$ type*. Z. Anal. Anwend. 14, 105–140 (1995).

[SiTi] Simonov, B., Tikhonov, S.: *Sharp Ul’yanov-type inequalities using fractional smoothness*. J. Approx. Theory 162, 1654–1684 (2010).

[Sob] Sobolev, S.L.: *On a theorem of functional analysis*. Mat. Sbornik 4, 471–497 (1938) (in Russian); English translation in Amer. Math. Soc. Transl. 34, 39–68 (1963).

[SobBook] Sobolev, S.L.: *Some Applications of Functional Analysis in Mathematical Physics*. Transl. of Math. Monographs, American Math. Soc., Providence, RI, 1991.

[Ste] Stein, E.M.: *The differentiability of functions in $R^n$*. Ann. of Math. 113, 383–385 (1981).

[SteW] Stein, E.M., Weiss, G.: *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, 1971.

[Stri] Strichartz, R.S.: *A note on Trudinger’s extension of Sobolev’s inequality*. Indiana Univ. Math. J. 21, 841–842 (1972).

[Tal] Talenti, G.: *Inequalities in rearrangement invariant function spaces*. Nonlinear Analysis, Function Spaces and Applications 5, 177–230 (1994).

[Ta] Tao, T.: *A converse extrapolation theorem for translation-invariant operators*. J. Funct. Anal. 180, 1–10 (2001).

[Ti10] Tikhonov, S.: *Weak type inequalities for moduli of smoothness: the case of limit value parameters*. J. Fourier Anal. Appl. 16, 590–608 (2010).

[Ti04] Tikhonov, S.: *On modulus of smoothness of fractional order*. Real. Analysis Exchange, 30, 507–518 (2004/2005).

[Tri01] Triebel, H.: *The Structure of Functions*. Birkhäuser, Basel, 2001.

[Tri06] Triebel, H.: *Theory of Function Spaces III*. Birkhäuser, Basel, 2006.

[Tri11] Triebel, H.: *Limits of Besov norms*. Arch. Math. 96, 169–175 (2011).

[Tre] Trebels, W.: *Inequalities for moduli of smoothness versus embeddings of function spaces*. Arch. Math. 94, 155–164 (2010).

[Tru] Trudinger, N.S.: *On embeddings into Orlicz spaces and some applications*. J. Math. Mech. 16, 473–483 (1967).

[U] Ulyanov, P.L.: *The imbedding of certain function classes $H_{p}^{\omega}$*. Izv. Akad. Nauk SSSR Ser. Mat. 3, 649–686 (1968) (in Russian); English translation in Math. USSR-Izv. 2, 601–637 (1968).

[Vy] Vyborn, J.: *A new proof of the Jawerth-Franke embedding*. Rev. Mat. Complut. 21, 75–82 (2008).

[Wi79a] Wilmes, G.: *On Riesz-type inequalities and K-functionals related to Riesz potentials in $R^N$*. Numer. Funct. Anal. Optim. 1, 57–77 (1979).
[Wi79b] Wilmes, G.: Some inequalities for Riesz potentials of trigonometric polynomials of several variables. Proc. Symp. Pure Math. 35 (Part 1), 175–182 (1979).

[Yu] Yudovich, V.I.: Some estimates connected with integral operators and with solutions of elliptic equations. Soviet Math. Dokl. 2, 746–749 (1961) (in Russian).

O. Domínguez, Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain.

E-mail address: oscar.dominguez@ucm.es

S. Tikhonov, Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C 08193 Bellaterra (Barcelona), Spain; ICREA, Pg. Lluïv Companys 23, 08010 Barcelona, Spain, and Universitat Autònoma de Barcelona.

E-mail address: stikhonov@crm.cat