A CONTINUOUS ANALOGUE OF THE INVARIANCE PRINCIPLE AND ITS ALMOST SURE VERSION

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Abstract. We deal with random processes obtained from a homogeneous random process with independent increments by replacement of the time scale and by multiplication by a norming constant. We prove the convergence in distribution of these processes to Wiener process in the Skorohod space endowed by the topology of uniform convergence. An integral type almost sure version of this limit theorem is obtained.

2000 AMS Mathematics Subject Classification. 60F05 Central limit and other weak theorems, 60F15 Strong theorems.

Key words and phrases: functional limit theorem, almost sure limit theorem, process with independent stationary increments.

1. Introduction

The usual invariance principle asserts the convergence of the sequence of the random processes

\[ X_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nx]} \xi_i, \quad x \in \mathbb{R}^+, \quad \text{as} \quad n \to \infty, \]

to the Wiener process \( W \), where \( \xi_i \) are independent identically distributed centered random variables with variance 1. In this paper we study approximations of the Wiener process \( W \) by the random processes

\[ X_t(x) = \frac{1}{\sqrt{t}} V(tx), \quad x \in \mathbb{R}^+, \quad (1) \]

where \( t > 0 \) is a parameter and \( V \) is a centered homogeneous random process with independent increments such that \( V(0) = 0 \) and \( \mathbb{E}(V(1))^2 = \sigma^2 \). Then almost all sample paths of \( X_t \) belong to the Skorohod space \( D[0,1] \). In Section 2 we prove that \( X_t \) converges to \( \sigma W \), as \( t \to \infty \), in distribution in \( D[0,1] \).

Almost sure versions of functional limit theorems were studied in several papers. Here we mention only Lacey and Philipp [1], Chuprunov and Fazekas [2], Chuprunov and Fazekas [3].

In Section 3 we prove almost sure versions of our limit theorem. Let \( (\Omega, \mathcal{A}, \mathbb{P}) \) be the probability space on which the random process \( V \) is defined. We show that for certain sequence \( (s_n) \), the sequence of measures

\[ Q_n(\omega) = Q_n[X_{s_n}(t)](\omega) = \frac{1}{\ln n} \sum_{k=1}^{n} \frac{1}{k} \delta_{X_{sk}(t)}(\omega) \quad (1+) \]

converges weakly to the distribution of \( \sigma W \) in \( D[0,1] \) for almost all \( \omega \in \Omega \). Here and in the following \( \delta_x \) denotes the measure of unit mass, concentrated in the point \( x \).

Also we prove integral type almost sure versions of our limit theorem. In Chuprunov and Fazekas [4] a general integral type almost sure limit theorem is presented. Then the general theorem is applied to obtain almost sure versions of limit theorems for semistable and max-semistable processes, moreover for processes being in the domain of attraction of a stable law or being in the domain of geometric partial attraction of a semistable or a
max-semistable law. We mention a simple consequence of the general result of Chuprunov and Fazekas [4]. Let $V(t)$ be a process with characteristic function (3). Let $f$ be a function with property (B). Let $X_t$ be defined by (1). Then
\[
1 \log(T) \int_1^T \delta_{X_f(t)}(\omega) \frac{1}{t} dt
\]
converges weakly, as $T \to \infty$, to centered gaussian distribution with the variance $K(\infty)$.

In this paper we prove a functional version of this proposition. We show that the measures
\[
Q_T = \frac{1}{D(T)} \int_1^T \delta_{X_f(t)}(\omega) d(t) dt
\]
converge weakly, as $T \to \infty$, to the distribution of $\sigma W$ in $D[0,1]$ for almost all $\omega \in \Omega$.

The proof of this result is based on the criterion for integral type almost sure version of a limit theorem which was obtained in Chuprunov and Fazekas [4].

2. Functional limit theorems

We will denote by $\overset{d}{\to}$ the convergence in distribution, by $\overset{w}{\to}$ the weak convergence of measures, by $\mu_\zeta$ the distribution of the random element $\zeta$ and by $\mathfrak{B}(\mathcal{B})$ the $\sigma$-algebra of the Borel subsets of the metric space $\mathcal{B}$. In the paper we will denote by the same symbols the random process and the random element corresponding to this random process.

Using the Kolmogorov representation (see [5], sect. 18) we can assume that the characteristic function of the centered homogeneous random process $V(t)$ with independent increments is
\[
\phi_{V(t)}(x) = \mathbb{E} \left( e^{ixV(t)} \right) = e^{\omega(t,x,K(x))} = e^{t \int_{-\infty}^{+\infty} (e^{ixy} - 1 - ixy) \frac{1}{y^2} dK(y)}, \quad x \in \mathbb{R}.
\]
Here $K(y)$ is an bounded increasing function such that $K(-\infty) = 0$.

We will consider the sequence of the random processes
\[
Y_n = X_{s_n},
\]
where
\[
s_n \to \infty, \quad \text{as} \quad n \to \infty.
\]

We will use the following preliminary result.

**Theorem 1.** Let $Y_n$ be defined by (1) and (4) and assume that $(s_n)$ satisfies (5). Let $\sigma^2 = K(\infty)$. Then we have
\[
Y_n \overset{d}{\to} \sigma W, \quad \text{as} \quad n \to \infty,
\]
in $D[0,1]$ endowed by the topology of uniform convergence.

**Proof.** Let $0 \leq t_1 < t_2 < \infty$. By the convergence criterions in [5], sect. 19,
\[
Y_n(t_2) - Y_n(t_1) \overset{d}{\to} \sigma(W(t_2) - W(t_1)), \quad \text{as} \quad n \to \infty.
\]
Let $0 \leq t_0 < t_1 < \ldots < t_k < \infty$. Introduce the notation $\Delta Y_n = Y_n(t_i) - Y_n(t_{i-1})$ and $\Delta W_i = W(t_i) - W(t_{i-1})$. Since $\Delta Y_n$, $1 \leq i \leq k$, are independent random variables, from (6) we obtain
\[
(\Delta Y_1, \ldots, \Delta Y_k) \overset{d}{\to} (\sigma \Delta W_1, \ldots, \sigma \Delta W_k).
\]
Consequently, the finite dimensional distributions of $Y_n$ converge to the finite dimensional distributions of $\sigma W$. Also we have

$$\limsup_{n \to \infty} E|Y_n(t_2) - Y_n(t_1)|^2 = \sigma^2|t_2 - t_1|.$$ \hspace{1cm} (7)

But (7) together with the convergence of the finite dimensional distributions gives the weak convergence of $Y_n$ to $\sigma W$ (see Billingsley [6], Theorem 15.6) in $D[0,1]$ with Skorohod’s $J_1$-topology. However, in our case the limit process is a continuous one. Hence, (see Pollard [7], p. 137, and the discussion in Billingsley [6], Sect. 18), the weak convergence in Skorohod’s $J_1$-topology actually implies the weak convergence in the uniform topology of $D[0,1]$. The proof is complete.

Using Theorem 1, we can prove the following theorem.

**Theorem 2.** Let $X_t$ be defined by (1). Then it holds that

$$X_t \xrightarrow{d} \sigma W, \text{ as } t \to \infty,$$

in $D[0,1]$ endowed by the topology of uniform convergence.

**Proof.** Consider $D[0,1]$ in the topology of uniform convergence and the space $M$ of distributions on $D[0,1]$ with the topology of convergence in distribution. Then $M$ is a metric space and denote by $\rho_M$ a metric which defines this topology on $M$. Then, by Theorem 1, $\rho_M(\mu_{X_{sn}}, \mu_{\sigma W}) \to 0$, as $s \to \infty$. Therefore $\rho_M(\mu_{X_{t}}, \mu_{\sigma W}) \to 0$, as $t \to \infty$. The proof is complete.

Let $\xi_i, i \in N$, be independent identically distributed random variables with the expectation $a$ and the variance $\sigma^2$ and let $\pi(t), t \in R^+$, be a Poissonian process with the intensity 1, and let the family $\xi_i, i \in N, \text{be independent of } \pi(t), t \in R^+$. Then

$$V(x) = \sum_{i=1}^{\pi(x)} \xi_i - ax, \text{ } x \in R^+,$$

is a centered homogeneous random process with independent increments such that $V(0) = 0$ and $E(V(1))^2 = \sigma^2 + a^2$

So from Theorem 2 we obtain the following corollary.

**Corollary 1.** Let $\xi_i, i \in N$, be independent identically distributed random variables with the expectation $a$ and the variance $\sigma^2$. Let

$$X'_t(x) = \frac{\sum_{i=1}^{\pi(tx)} \xi_i - atx}{\sqrt{t}}, \text{ } x \in R^+.$$ \hspace{1cm} (8)

Then one has

$$X'_t \xrightarrow{d} \sqrt{\sigma^2 + a^2} W, \text{ as } t \to \infty,$$

in $D[0,1]$ with the topology of uniform convergence.

For $\xi_i = 1$ from Corollary 1 we obtain the following.

**Corollary 2.** Let

$$X'^*_t(x) = \frac{\pi(tx) - tx}{\sqrt{t}}, \text{ } x \in R^+.$$ \hspace{1cm} (9)
Then one has
\[ X_t^* \overset{d}{\to} W, \text{ as } t \to \infty, \]
in \( D[0,1] \) with the topology of uniform convergence.

3. Almost sure versions of functional limit theorems

We will consider the sequence of measures defined by (1+) and connected with the random processes \( X_n(t) \).

For the sequence \( s_n \) we will assume the following property
\[(A) \quad \text{for some } \beta > 0, \quad \frac{s_n}{n^\beta} \text{ is an increasing sequence.}\]

**Theorem 3.** Let (A) be valid. Then it holds that
\[ Q_n(\omega) \overset{w}{\to} \mu_{\sigma W} \quad \text{if } n \to \infty \] (10)
for almost all \( \omega \in \Omega \).

**Proof.** Let \( l < k \). Let
\[ Y_{kl}(x) = \begin{cases} 0, & 0 \leq x < \frac{s_l}{s_k}, \\ Y_k(x) - \frac{V(s_l)}{\sqrt{s_k}}, & \frac{s_l}{s_k} \leq x \leq 1. \end{cases} \]
Then \( Y_{kl}(x), 0 \leq x \leq 1 \), and \( Y_l(x), 0 \leq x \leq 1 \), are independent random processes. Let \( \rho \) be the metric of \( D[0,1] \). Using the moment inequality from [8], sect. 5, we obtain
\[
E\rho(Y_k, Y_{kl}) \leq E \sup_{0 \leq x \leq 1} |Y_k(x) - Y_{kl}(x)| \leq E \sup_{0 \leq x \leq \frac{s_l}{s_k}} |Y_k(x)| + E \left| \frac{V(s_l)}{\sqrt{s_k}} \right| \leq \\
\leq 4E \left| Y_k \left( \frac{s_l}{s_k} \right) \right| + E \left| \frac{V(s_l)}{\sqrt{s_k}} \right| \leq 5 \sqrt{E(V(s_l))^2} \leq 5\sigma \sqrt{\frac{s_l}{s_k}} \leq 5\sigma \left( \frac{l}{k} \right)^{\beta/2}. \leq 5\sigma \left( \frac{l}{k} \right)^{\beta/2}. (11)
\]
By Lemma 1 from [3], this implies (10). The proof is complete.

For the function \( f \) we will consider following the property
\[(B) \quad \text{for some } \beta > 0, \quad \frac{f(x)}{x^\beta} \text{ is an increasing function.}\]

Now we will prove the integral type almost sure version of Theorem 2. We will consider the random processes
\[ Y_t(x) = \frac{V(f(t)x)}{\sqrt{f(t)}}, \quad 0 \leq x \leq 1. \]
We will assume that
\[(C) \quad \text{the function } d(s) \text{ is a decreasing such that } \int_k^{k+1} d(s) ds \leq \log \sqrt{\frac{k+1}{k}} \text{ for all } k \in \mathbb{N} \text{ and } \int_1^\infty d(s) ds = +\infty. \text{ Let } D(S) = \int_1^S d(s) ds. \]

**Theorem 4.** Let (B) and (C) be valid. Then we have
\[ Q_S^I(\omega) = \frac{1}{D(S)} \int_1^S \delta_{Y_t(\omega)} d(s) ds \overset{w}{\to} \mu_{\sigma W}, \quad \text{as } S \to \infty, \] (12)
for almost all \( \omega \in \Omega \).

**Proof.** Let \( 0 < l < k, l, k \in \mathbb{N}, k \leq t \leq k + 1 \). Introduce the notation

\[
Y_{lkt}(s) = \begin{cases} 
0, & 0 \leq s \leq \frac{f(l+1)}{f(t)}, \\
Y_t(s) - \frac{V(f(l))}{\sqrt{f(t)}}, & \frac{f(l+1)}{f(t)} \leq s \leq 1.
\end{cases}
\]

Then \( \{Y_{lkt}(s) : k \leq t \leq k + 1\} \) and \( \{Y_t(s) : l \leq t \leq l + 1\} \) are independent families. Repeating the proof of (11), we obtain

\[
\mathbb{E} \rho(Y_t, Y_{lkt}) \leq \mathbb{E} \sup_{0 \leq s \leq 1} |X_t(s) - X_{lkt}(s)| \leq 5 \cdot 2^{3/2} \sigma \left( \frac{1}{k} \right)^{3/2}.
\]

By Corollary 2.1, from Chuprunov and Fazekas [4] this and Theorem 2 implies (12). The proof is complete.

Theorem 4 and Corollary 1 (resp. Corollary 2) of Theorem 2 imply the corollaries.

**Corollary 3.** Let the \( X' \) be defined by (8) and let \( f \) be a function with the property (B). Then

\[
\frac{1}{\ln(S)} \int_1^S \delta_{X_{f(s)}(\omega)} \frac{1}{s} ds \overset{w}{\rightarrow} \mu \sqrt{\sigma^2 + \sigma^2 W}, \quad \text{as} \quad S \to \infty,
\]

for almost all \( \omega \in \Omega \).

**Corollary 5.** Let the \( X^* \) be defined by (9) and let \( f \) be a function with the property (B). Then

\[
\frac{1}{\ln(S)} \int_1^S \delta_{X_{f(s)}(\omega)} d(s) ds \overset{w}{\rightarrow} \mu W, \quad \text{as} \quad S \to \infty,
\]

for almost all \( \omega \in \Omega \).

**Remark 1.** Corollary 1 of Theorem 2 is a functional limit theorem for random sums. So Corollary 3 of Theorem 4 is an integral type almost sure version of a functional limit theorem for random sums. (For limit theorems for random sums see Korolev and Kruglov [9].)

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