Structure of Amplitude Correlations in open chaotic Systems

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The Verbaarschot-Weidenmüller-Zirnbauer (VWZ) model is believed to correctly represent the correlations of two S-matrix elements for an open quantum chaotic system, but the solution has considerable complexity and is presently only accessed numerically. Here a procedure is developed to deduce its features over the full range of the parameter space in a transparent and simple analytical form preserving accuracy to a considerable degree. The bulk of the VWZ correlations are described by the Gorin-Seligman expression for the 2-amplitude correlations of the Ericson-Gorin-Seligman model. The structure of the remaining correction factors for correlation functions is discussed with special emphasis on the rôle of the level correlation hole both for inelastic and elastic correlations.

I. INTRODUCTION

The Verbaarschot-Weidenmüller-Zirnbauer (VWZ) model for chaotic scattering for an open system is generally believed to be a realistic example of such processes. It is based on a Random Matrix Model for which the random Hamiltonian is dynamically coupled to the external channels. For the correlations of two S-matrix elements it gives an explicit general solution, which unfortunately is very complex. In spite of the importance of this result, the dominant physical features are mostly not transparent. Hence they cannot be reliably exploited in simplified form for application to higher order correlation functions such as cross-section correlations for which the corresponding VWZ solutions are not known presently. The VWZ amplitude correlations can presently be evaluated only using numerical methods with specified values for the numerous parameters, but for special cases. Approximate expressions exist for small correlation times, but the convergence radius of the expansion is small.

The properties of higher order correlation functions in the VWZ model are largely unaccessible with present approaches and their properties are unexplored with few exceptions. The cross-section correlations (4-point amplitude correlations) have been obtained for a vanishing correlation time. Their behavior is however only accessed numerically. A realistic study of these correlation functions requires then a simplified approach in which the properties of the amplitude correlations closely mirror those of the VWZ model. In the following we will demonstrate that the model due to Ericson-Gorin-Seligman (EGS) gives a good general representation of the VWZ solution for amplitude correlations, but with some characteristic deviations. Its explicit solution is in some respects more general than the one of the VWZ model and has been used in a study to be published to explore the structure of cross-section correlations both for near-chaotic and for chaotic scattering from open systems.

The EGS model is inspired by reaction theory such as in Ref. It approximates the S-matrix by a sum of complex poles, the positions of which are assumed not to be influenced by the coupling to the external channels. In its main version these poles have partial width amplitudes with a normal (Gaussian) distribution; the pole positions are assumed to have the same spacing distribution as for the Random Matrix Model, i. e., generated by matrix elements with a normal distribution. Although the EGS model implements a number of features of the VWZ one, it differs in particular by not accounting fully for the dynamical feedback of the open channels on the spacing distribution. In addition, the price for the solvability of the EGS model is that it violates unitarity sum-rules, which potentially may produce unrealistic properties. It is shown however in Ref. that these restrictions are of unimportant in most situations; this conclusion is concretely vindicated by the results of the present article. As a check on the effectiveness of this approach the Gorin-Seligman (GS) results for the 2-point correlations of the EGS model have been numerically compared to the ones in the VWZ model. The finding is that this model with normally distributed partial width amplitudes describes the gross features of the VWZ model surprisingly well for a variety of conditions. This suggests that the structure of the GS result might be used as a guide to develop an accurate and transparent analytical approximation to the VWZ results. This is the goal of the present article. It gives a valuable insight in addition to the accurate numerical results for the 2-point function. This permits one to identify explicitly the origin and nature of the differences of the VWZ result and the GS one. The understanding of this point gives confidence that the EGS model gives a reliable approximate description of the chaotic cross-section correlations.

The strategy, therefore, will be to extract the main variation of the key VWZ result with the GS solution as guide to identify appropriate expansion parameters for the deviations. Using this information a natural procedure is developed for obtaining explicit closed expressions for the additional factors over the entire parameter space.

In Sec. the standard VWZ 2-point function (8.10) in
Ref. [1] is converted to the time representation and expressed more conveniently in terms of the time variable $\tau$. This representation is technically easier to handle than the energy one, since it gives products and not folding integrals. In addition, it is immediately apparent that there are no singularities in the integrals. The basic correlation functions and their notation are defined here.

In Sec. III some basic identities associated with VWZ integral are given as well as their relations to the level spacing correlations. This section also defines notations for the weighted functions generated by the integrations.

Sec. IV describes the construction of the small expansion parameter, the basic expansion and the expansion of the factors coupling to the observed channels referred to as the 'leads'. It gives in addition the structure of the general expansion term.

Sec. V gives the explicit expressions for small and large $\tau$ as well as a convenient interpolation connecting the two regions.

Sec. VI is devoted to a discussion of the characteristic features of the amplitude correlation functions in the light of the results. It also discusses possible generalizations and some practical consequences.

II. THE VWZ CORRELATION FUNCTION

The key result in Ref. [1] is Eq. (8.10), which gives the correlation function for two conjugate S-matrix amplitudes $S(E(1))$ and $S^*(E(2))$ at energies $E(1, 2)$ differing by $\epsilon = E(1) - E(2)$. The system is assumed to have no secular energy dependence with $|E(1) + E(2)|^2/2$ and threshold effects in the different channels are neglected. The observed channels (the 'leads') are denoted by $a, b, c, d$ while open channels without restrictions are denoted by $e$ with corresponding transmission coefficients $T_e$. Ensemble averages are denoted by $\langle >$. The energy scale has units such that the average level spacing $d = 1$.

The general VWZ correlation function in the energy representation is given by the triple integral

$$C_{abcd}(\epsilon) = \langle S_{ab}(E(1)) S^*_{cd}(E(2)) \rangle = \langle < S_{ab} > < S^*_{cd} > \rangle$$

$$= \frac{1}{8} \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \int_0^\infty d\lambda_3 \frac{(1 - \lambda)(\lambda_1 - \lambda_2)}{[(1 + \lambda_1)(1 + \lambda_2)(1 + \lambda_3)^{1/2}]^2} \times$$

$$\exp \left[-i\pi \epsilon (\lambda_1 + \lambda_2 + 2\lambda)\right] \prod_e \frac{1}{(1 + T_e\lambda)^{1/2}(1 + T_e\lambda_2)^{1/2}} \times$$

$$\left\{ \delta_{ab}\delta_{cd} < S_{aa} > < S^*_{cc} > T_a T_c \left( \frac{\lambda_1}{1 + T_a\lambda_1} + \frac{\lambda_2}{1 + T_a\lambda_2} + \frac{2\lambda}{1 - T_a\lambda} \right) \left( \frac{\lambda_1}{1 + T_c\lambda_1} + \frac{\lambda_2}{1 + T_c\lambda_2} + \frac{2\lambda}{1 - T_c\lambda} \right) + \right.$$

$$\left( \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \right) T_a T_b \left[ \frac{\lambda_1(1 + \lambda_1)}{(1 + T_a\lambda_1)(1 + T_b\lambda_1)} + \frac{\lambda_2(1 + \lambda_2)}{(1 + T_a\lambda_2)(1 + T_b\lambda_2)} + \frac{2\lambda(1 - \lambda)}{(1 - T_a\lambda)(1 - T_b\lambda)} \right] \right\}$$

The normalization is determined by the triple integral of a weight factor and gives no singular contribution from the region near the origin

$$\frac{1}{8} \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \int_0^\infty d\lambda_3 \frac{(1 - \lambda)(\lambda_1 - \lambda_2)}{[(1 + \lambda_1)(1 + \lambda_2)(1 + \lambda_3)^{1/2}]^2} = 1$$

While these expressions present nicely in these variables, it is not the most convenient representation for applications. The equivalent correlation function in the time representation is obtained by taking the Fourier transform using that $\int_{-\infty}^\infty d\epsilon \exp[2\pi i\epsilon x] = \delta(x)$, where the energy is measured in units of the average level spacing $d$. The time $\tau \geq 0$ is measured in standard units $2\pi/d$ as, e. g., in Ref. [12], Eq. (8). This means in particular that the condition for overlapping levels $\Gamma > d$ corresponds to $\sum_e T_e > 2\pi$. It is convenient to make the variable substitutions $\lambda_{1,2} = 2\tau x_{1,2}$ and $\lambda = \tau x$. These transformations give immediately the Fourier transform as

$$\tilde{C}_{abcd}(\tau) = \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^{\tau_{min}(1,1/\tau)} \frac{1}{(1 + 2\tau x_1)^{1/2}(1 + 2\tau x_2)^{1/2}} \times$$

$$\frac{x|x_1 - x_2|}{[(1 + 2\tau x_1)(1 + 2\tau x_2)(1 + 2\tau x_2)^{1/2}(x + 2x_1)^{1/2}(x + 2x_2)^{1/2}(x + 2x_2)^{1/2}]^2} \times$$

$$\delta(1 - (x_1 + x_2 + x)) \prod_e \frac{1}{(1 + 2\tau x_1 T_e)^{1/2}(1 + 2\tau x_2 T_e)^{1/2}} \times$$

$$\left\{ \delta_{ab}\delta_{cd} < S_{aa} > < S^*_{cc} > 2\tau T_a T_c \left( \frac{x_1}{1 + 2\tau T_a x_1} + \frac{x_2}{1 + 2\tau T_a x_2} + \frac{x}{1 - \tau T_a x} \right) \left( \frac{x_1}{1 + 2\tau T_c x_1} + \frac{x_2}{1 + 2\tau T_c x_2} + \frac{x}{1 - \tau T_c x} \right) + \right.$$

$$\left. \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \right\} T_a T_b \left[ \frac{x_1(1 + x_1)}{(1 + 2\tau T_a x_1)(1 + 2\tau T_b x_1)} + \frac{x_2(1 + x_2)}{(1 + 2\tau T_a x_2)(1 + 2\tau T_b x_2)} + \frac{x}{(1 - 2\tau T_a x)(1 - 2\tau T_b x)} \right] \right\}$$
\((\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) T_a T_b \left\{ \frac{x_1(1 + 2\tau x_1)}{(1 + 2\tau T_a x_1)(1 + 2\tau T_b x_1)} + \frac{x_2(1 + 2\tau x_2)}{(1 + 2\tau T_a x_2)(1 + 2\tau T_b x_2)} + \frac{x(1 - \tau x)}{(1 - \tau T_a x)(1 - \tau T_b x)} \right\} \) 

The expression \([9]\) is the basis for the following. It is convenient to work with the correlation functions \(\tilde{C}_{aa,cc}^1(\tau)\) and \(\tilde{C}_{ab,ab}^2(\tau)\) defined from this expression:

\[
\tilde{C}_{ab,cd}(\tau) = \delta_{ab}\delta_{cd}\tilde{C}_{aa,cc}^1(\tau) + (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \tilde{C}_{ab,ab}^2(\tau)
\]  

There are 3 basic situations: elastic autocorrelations, correlations of two different elastic amplitudes and inelastic autocorrelations. Other correlations between two S-matrix amplitudes vanish for the VWZ 2-point function. The relevant cases are explicitly

\[
\begin{align*}
\tilde{C}_{aa,aa}(\tau) &= \tilde{C}_{aa,aa}^1(\tau) + 2\tilde{C}_{aa,aa}^2(\tau) & \text{elastic autocorrelations} \\
\tilde{C}_{aa,cc}(\tau) &= \tilde{C}_{aa,cc}^1(\tau) & \text{elastic correlations with } a \neq c \\
\tilde{C}_{ab,ab}(\tau) &= \tilde{C}_{ab,ab}^2(\tau) & \text{inelastic autocorrelations with } a \neq b
\end{align*}
\]  

### III. BASIC IDENTITIES

The following identities are crucial in the expansions for small and moderate \(\tau\). They result in the limit of vanishing transmission coefficients and are central in deriving non-trivial approximations in the following. Exact integrals are obtained from Eq. \([9]\), where the index \(\tau\) on a quantity means that it is an averaged integral weighted by the parts of the integrands that are independent of the transmission coefficients as in Eq. \([8]\) below. In the limit of vanishing transmission coefficients the correlation function \(\tilde{C}_1(\tau)\) defines the following function

\[
I(\tau) \equiv \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^{\min(1,1/\tau)} dx \frac{(1 - \tau x)}{(1 + 2\tau x_1)^{1/2}(1 + 2\tau x_2)^{1/2}(x_1 x_2)^{1/2}(x + 2x_1)^2(x + 2x_2)^2} \times \delta[1 - (x_1 + x_2 + x)]
\]

Here \(I(\tau = 0) = 1\). One notes that the time-dependent factor formally appears as an additional channel \(e\) with \(T_e = 1\). We will in the following denote by \(\{f(x_1, x_2, x)\}, I(\tau)\) the integral with the same time-dependent integrand weighted by \(f(x_1, x_2, x)\). In Eq. \([8]\) this definition of \(I(\tau)\) corresponds to the integration of the integrand multiplied by 1 with \([1]_\tau \equiv 1\).

\[
2\tau I(\tau) = 1 - b_1(\tau)
\]

The last term in this expression for \(I(\tau)\) is exactly the Dyson level correlation function \(b_1(\tau)\) \([17]\), while the first term 1 in Eq. \([9]\) comes from the self-correlation of a narrow level with itself. Here \(b_1(\tau) = \int ds Y_{2,1}(s) \exp(2\pi i \tau s)\) of the corresponding level correlation in the energy representation. It has the explicit form \([15]\)

\[
\begin{align*}
\text{For } 0 < \tau < 1 & \quad b_1(\tau) = 1 - 2\tau + \tau \ln[1 + 2\tau] \simeq 1 - 2\tau + 2\tau^2 - 2\tau^3 + 8\tau^4/3 - 4\tau^5 + \ldots \\
\text{For } \tau > 1 & \quad b_1(\tau) = -1 + \tau \ln\left[\frac{2\tau + 1}{2\tau - 1}\right] \simeq 0 + (1/12)\tau^{-2} + \ldots
\end{align*}
\]

Here \(b_1(1) \simeq 0.0986\) and it vanishes rapidly for larger \(\tau\). This level correlation function is an explicit result in the VWZ model found by Efetov and Verbaarschot \([3, 18]\).

A similar, but simpler, relation is obtained similarly from the correlation function \(\tilde{C}_2(\tau)\) in the limit of all \(T_e = 0\). In this case the origin is also a level correlation, but here only the self-correlation of the levels contributes and gives only the first term in Eq. \([9]\).

\[
[1 + 2\tau(x_1^2 + x_2^2 - (1/2)x^2)], I(\tau) = 1
\]

It is convenient to regroup the terms in Eq. \([11]\) using \(x_1 + x_2 + x = 1\) so as to display the parameters \(x_1, x_2\) and \(x\) which will serve as small expansion parameters in the following:

\[
x_1^2 + x_2^2 - (1/2)x^2 \equiv 1 - 2(x_1 x_2 - (1/4)x^2)
\]
With the expression (9) for $I(\tau)$ in terms of the Dyson function

$$4\tau \left[ x_1 x_2 + x - (1/4)x^2 \right] e \tau I(\tau) = (1 + 2\tau)I(\tau) - 1 = \frac{1 - (1 + 2\tau)b_1(\tau)}{2\tau} \simeq$$

$$\tau(1 - \tau + 2\tau^2/3 + O(\tau^3)) \quad (\text{for } \tau < 1) : \quad \frac{1}{2\tau} - \frac{1}{12\tau^2} \quad (\text{for } \tau > 1)$$

This combination is used later in the perturbative expansion terms.

### IV. STRATEGY FOR EXPANSIONS

Our method is illustrated by the inelastic case assuming at first for simplicity that the transmission coefficients in the leads $T_{a,b,c}$ are small. In the case of the GS result, the dependence on the transmission coefficients in Eq. (11) is then governed by the factor $\prod_e (1 + 2\tau T_e)^{-1/2}$. A related factor $\prod_e (1 - \tau x T_e)(1 + 2\tau x T_e)^{-1/2}(1 + 2\tau x_2 T_e)^{-1/2}$ appears in the VWZ model as the product contribution to the integrand of Eq. (3), which approximately gives such a factor for small and moderate $T_e$. This suggests that one should avoid power expansions based on $2\tau T_e < 1$ as a small parameter, since such expansions mostly have a narrow radius of convergence for the GS case. A far better expansion is obtained by the following procedure.

Since $x_1 + x_2 = 1 - x$ and with the notation $T'_e = T_e/(1 + 2\tau T_e)$ one can write (10) after breaking out the factor $(1 + 2\tau T_e)^{-1/2}:

$$\frac{(1 - \tau T'_{e,x})}{(1 + 2\tau x_1 T_e)^{1/2}(1 + 2\tau x_2 T_e)^{1/2}} \equiv \frac{(1 - \tau T'_{e,x})}{(1 - 2\tau T'_{e,x} + 4x_1 x_2 \tau^2 T_e T'_e)^{1/2}} \times (1 + 2\tau T_e)^{-1/2}$$

The first term of this product is then exponentiated. The individual contributions in the exponent are expanded assuming sufficiently small expansion parameters

$$\prod_e \frac{(1 - \tau T'_{e,x})}{(1 + 2\tau x_1 T_e)^{1/2}(1 + 2\tau x_2 T_e)^{1/2}} =$$

$$\exp \left\{ \sum_e \left[ \ln(1 - \tau x T_e) - (1/2) \ln \left( 1 - 2\tau T'_e x + 4x_1 x_2 \tau^2 T_e T'_e \right) \right] \right\} \times \prod_e (1 + 2\tau T_e)^{-1/2} \simeq$$

$$\exp \left\{ -2\tau \sum_e T_e T'_e (x_1 x_2 + x - (1/4)x^2) \right\} \times \prod_e (1 + 2\tau T_e)^{-1/2}$$

In this expansion the linear terms in $x_1 x_2$ and $x$ regroup naturally with the term in $x^2$ for small $\tau$. For $\tau > 1$ this is no longer the case, but terms of second order and higher in these expansion quantities are negligible in the relevant regions of integration as compared to the linear terms.

Note that the expansion is unrelated to the issue of whether the resonances are overlapping. Corresponding approximations can be made for the lead correction factors $g_{\tilde{C}_1,2}(\tau)$ which according to Eq. (4) are

$$g_{\tilde{C}_1}(x_1, x_2, x; \tau) = \left( \frac{x_1}{1 + 2\tau T_a x_1} + \frac{x_2}{1 + 2\tau T_a x_2} + \frac{x}{1 - \tau T_a x} \right) \times \left( \frac{x_1}{1 + 2\tau T_c x_1} + \frac{x_2}{1 + 2\tau T_c x_2} + \frac{x}{1 - \tau T_c x} \right)$$

$$g_{\tilde{C}_2}(x_1, x_2, x; \tau) = \left[ \frac{x_1 (1 + 2\tau x_1)}{(1 + 2\tau T_a x_1)(1 + 2\tau T_b x_1)} + \frac{x_2 (1 + 2\tau x_2)}{(1 + 2\tau T_a x_2)(1 + 2\tau T_b x_2)} + \frac{x (1 - \tau x)}{(1 - \tau T_a x)(1 - \tau T_b x)} \right]$$

Assuming as previously that no single decay channel dominates contribution from the lead channels and extracting an overall factor $(1 + 2\tau T_{a,b,c})^{-1}$ one finds a qualitatively different contribution, which now contains terms which are

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1 The following expansion assumes that $2x_1 x_2 \tau T_e < 1$, typically. Equation (13) suggests the characteristic value of $x_1 x_2$ is of order $O(4\tau^{-1})$ or less in typical integrals for $\tau > 1$. One can verify that deviations lead to small and negligible contributions to the correction terms even in the limit of very strong transmission $T_e \simeq 1$. 
linear in $\tau T_{a,b,c}$ which is of significance for the behavior of the correlation functions for small $\tau$. This is contrary to the quadratic dependence on $\tau^2 T_e^2$ in the sum $\tau^2 \sum_e T_e^2$ of Eq. (15). The next order terms in the transmission coefficients of the leads are readily included, but do not qualitatively change the results. In accordance with the previous assumption that no channel dominates and for simplicity only the linear contribution from the leads is retained here. The correction term to this order in the integrand becomes for the lead factor $g_{C_i}(\tau)$,

$$2\tau g_{C_1}(x_1, x_2, x; \tau) \simeq 2\tau(1 + 2\tau T_a)^{-1}(1 + 2\tau T_c)^{-1}[1 + 4\tau(T_a + T_c)(x_1x_2 + x - x^2/4) + ...]$$

(18)

For $g_{C_2}(\tau)$ the leading terms in $\tau T_{a,b}$ are with $\tau' = \tau/(1 + 2\tau)$

$$g_{C_2}(x_1, x_2, x; \tau) \simeq (1 + 2\tau)(1 + 2\tau T_a)^{-1}(1 + 2\tau T_b)^{-1}[1 - 4\tau'(1 - (1 + \tau)(T_a + T_b))(x_1x_2 + x - x^2/4) + ...]$$

(19)

The factors given in Eqs. (16-17) with Eqs. (18) and (19), respectively, will be used in the following integrands.

The relevant terms weighted in the VWZ integral therefore occur as exponents proportional to $x_1x_2 + x - x^2/4$ characteristically scaled by a multiplying term dependent on the transmission coefficients. When the linear approximation to the lead correction is valid over the important regions of the integrals we can exponentiate them and combine the results with those of Eq. (15). Introduced into the integrand of Eq. (8) gives

$$\tilde{C}_{aa,cc}(\tau) \simeq \langle \nu_{aa} \rangle \langle \nu_{cc}^* \rangle 2\tau T_a T_c(1 + 2\tau T_a)^{-1}(1 + 2\tau T_c)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \left[ \exp\{-\kappa_1(x_1x_2 + x - (1/4)x^2)\} \right]_\tau I(\tau)$$

(20)

with $\kappa_1 = -4\tau(T_a + T_c) + 2\tau^2 \sum_e T_e T'_e$

(21)

and

$$\tilde{C}_{ab,ab}(\tau) \simeq (1 + 2\tau)T_a T_b(1 + 2\tau T_a)^{-1}(1 + 2\tau T_b)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \left[ \exp\{-\kappa_2(x_1x_2 + x - (1/4)x^2)\} \right]_\tau I(\tau)$$

(22)

with $\kappa_2 = 4\tau' - 4\tau'(1 + \tau)(T_a + T_b) + 2\tau^2 \sum_e T_e T'_e$

(23)

This approximation is expressed here in a global form valid for all values of the time variable $\tau$. It will now be evaluated in detail in different regions.

V. EXPLICIT RESULTS FOR THE APPROXIMATION

A. Small and moderate $\tau$ with $O(\kappa_{1,2}) < \max[1, \tau]$

From Eqs. (4) and (13) this gives expanding Eq. (20) for the exponent smaller than unity

$$\tilde{C}_{aa,cc}(\tau) \simeq \langle \nu_{aa} \rangle \langle \nu_{cc}^* \rangle 2\tau T_a T_c(1 + 2\tau T_a)^{-1}(1 + 2\tau T_c)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \left[ 1 - \frac{b_1(\tau)}{2\tau^2} + \frac{\tau(T_a + T_c) - \frac{1}{2}\tau^2 \sum_e T_e T'_e}{2\tau^2} \right] \left( 1 - \frac{(1 + 2\tau)b_1(\tau)}{2\tau^2} \right)$$

(24)

The restrictions from the integration region are automatically taken into account. The corresponding result for $\tilde{C}_{ab,ab}(\tau)$ from Eq. (22) is

$$\tilde{C}_{ab,ab}(\tau) \simeq T_a T_b(1 + 2\tau T_a)^{-1}(1 + 2\tau T_b)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \left[ 1 + \frac{(1 + \tau)(T_a + T_b) - \frac{1}{2}\tau^2(1 + 2\tau) \sum_e T_e T'_e}{2\tau^2} \right] \left( 1 - \frac{(1 + 2\tau)b_1(\tau)}{2\tau^2} \right)$$

(25)
For $\tau > 1$ the Dyson function in Eq. (10) is very small with $b_1(\tau) \approx 1/2\tau^{-2}$ and gives a negligible contribution. Within the expansion region $\hat{C}_{aa;cc}(\tau > 1)$ becomes

$$\hat{C}^1_{aa;cc}(\tau > 1) \simeq$$

$$\langle S_{aa} \rangle \langle S_{cc}^* \rangle T_a T_e (1 + 2\tau T_a)^{-1} (1 + 2\tau T_e)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left( 1 + \left( T_a + T_e - \frac{1}{4} \sum_e T_e T'_e \right) (1 + O(\tau^{-1})) \right)$$

$$\rightarrow \langle S_{aa} \rangle \langle S_{cc}^* \rangle T_a T_e (1 + 2\tau T_a)^{-1} (1 + 2\tau T_e)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left( 1 + \frac{1}{2} \left( T_a + T_e - \tau \sum_e T_e T'_e \right) (1 + O(\tau^{-1})) \right)$$

for $\tau \rightarrow \infty$ and $\sum_e T_e < 2$

The corresponding result for $\hat{C}^2_{ab;ab}(\tau)$ is

$$\hat{C}^2_{ab;ab}(\tau > 1) \simeq T_a T_b (1 + 2\tau T_a)^{-1} (1 + 2\tau T_b)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left( 1 + \frac{1}{2} \left( T_a + T_b - \tau \sum_e T_e T'_e \right) (1 + O(\tau^{-1})) \right)$$

for $\tau \rightarrow \infty$ and $\sum_e T_e < 2$

Note that the limit $\tau \rightarrow \infty$ applies only to the case of non-overlapping levels.

**B. Moderate and large $\tau$ with $O(\kappa_{1,2}) > max(1, \tau)$**

In this limit the previous expansion breaks down and the problem is dominated by the exponent appearing in Eqs. (20) and (22). In its simplest form the lead terms are weak. Then the unobserved channels determine the correlations. The typical integrand has the factor $\prod_e (1 + 2\tau T_e)^{-1/2} \exp \left[ \sum_{x=1}^{x=2} \left( x_1 x_2 + x - (1/4)x^2 \right) \right]$. It contains in addition the contribution to the integrand from the time-factor which formally corresponds to an additional channel with $T_e = 1$ which can be expanded as for any $T_e$ with $\tau' = \tau/(1 + 2\tau)$. These two decreasing factors set the scale for contributions such that for the weakly coupled inelastic case these come effectively from the region for which

$$(x_1 x_2 + x - (1/4)x^2) \sim O \left( \frac{1}{\tau^{2'} + \tau^2 \sum_e T_e T'_e} \right)$$

Since $x_1 + x_2 + x = 1$ and $x_1$ enter symmetrically we can take $x_1 > x_2$ and multiply by 2. When the integration variable in Eq. (28) is limited to small values, we have $x_1 \rightarrow 1$, while $x_2$ and $x$ are both effectively small with $x_2 << 1; x < 1$. In this limit the integrand can be carried out explicitly. Note however that for $\tau > 1$ we have $x < 1/\tau$, while the integration region for $x_2$ is limited only by $x_2 < 1/2$. The full integration region corresponds approximately to a long narrow rectangle for large $\tau$. Equation (28) can be used to choose $\tau$ such that the contributions come effectively only from a region for which both variables are much smaller than $1/\tau$. These conditions should be introduced into the basic integration as given in Eq. (8) but in addition the explicit time-dependence of the integrand should be expressed correspondingly and exponentiated.

Under these conditions the integral over $x_1$ is immediate. The integral over $x_2$, $x$ concentrates well within the integration region which can be extended to $\infty$ for both $x_2, x$ without restrictions. From Eqs. (20) and (22) this corresponds to the explicit evaluation of the terms $\exp [-\kappa_{1,2} \left( x_1 x_2 + x - (1/4)x^2 \right) \pi \tau \tau']$ using that the time-factor is $\left( 1 - \tau x \right) (1 + 2\tau x_1)^{-1/2} (1 + 2\tau x_2)^{-1/2} \simeq (1 + 2\tau)^{-1/2} \exp [-2\tau' \left( x_1 x_2 + x - (1/4)x^2 \right)]$ to a good approximation in this region. Consequently, as described in Appendix B, we have in terms of the integral $I(a, \tau)$ of Eqs. (111) and (112)

$$\hat{C}^1_{aa;cc}(\tau) \simeq \langle S_{aa} \rangle \langle S_{cc}^* \rangle T_a T_e (1 + 2\tau T_a)^{-1} (1 + 2\tau T_e)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \frac{1}{2} \left( \frac{\pi}{2\tau^2 + (1 + 2\tau)(\kappa_1 - 1/2)} \right)^{1/2}$$

where $\kappa_1 = -4\tau(T_a + T_e) + 2\tau^2 \sum_e T_e T'_e$

The corresponding expression for $\hat{C}^2$ is closely similar:

$$\hat{C}^2_{ab;ab}(\tau) \simeq T_a T_b (1 + 2\tau T_a) T_b (1 + 2\tau T_b)^{-1} (1 + 2\tau T_b)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \frac{1}{2} \left( \frac{\pi}{2\tau^2 + (1 + 2\tau)(\kappa_2 - 1/2)} \right)^{1/2}$$

where $\kappa_2 = 4\tau' - 4\tau' (1 + \tau)(T_a + T_b) + 2\tau^2 \sum_e T_e T'_e$
In the region $\tau > 1$ these expressions simplify neglecting contributions of order $\tau^{-1}$ in the correction terms.

\begin{equation}
\tilde{C}_{aa,cc}^1(\tau) \simeq \langle S_{aa} \rangle \langle S_{cc}^* \rangle T_a T_c (1 + 2\tau T_a)^{-1} (1 + 2\tau T_c)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left( \frac{\pi}{2} \right)^{1/2} \frac{1}{1 - 4(T_a + T_c) + 2\tau \sum_e T_e T'_e} \right)^{1/2}
\end{equation}

\begin{equation}
\tilde{C}_{ab,ab}^2(\tau \gg 1) \rightarrow T_a T_b (1 + 2\tau T_a)^{-1} (1 + 2\tau T_b)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left( \frac{\pi}{2} \right)^{1/2} \frac{1}{1 - 2(T_a + T_b) + 2\tau \sum_e T_e T'_e} \right)^{1/2}
\end{equation}

For asymptotically large $\tau$ these expressions become

\begin{equation}
\tilde{C}_{aa,cc}^1(\tau \rightarrow \infty) \rightarrow \langle S_{aa} \rangle \langle S_{cc}^* \rangle T_a T_c (1 + 2\tau T_a)^{-1} (1 + 2\tau T_c)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left( \frac{\pi}{2} \right)^{1/2} \frac{1}{1 - 4(T_a + T_c) + \sum_e T_e} \right)^{1/2}
\end{equation}

\begin{equation}
\tilde{C}_{ab,ab}^2(\tau \rightarrow \infty) \rightarrow T_a T_b (1 + 2\tau T_a)^{-1} (1 + 2\tau T_b)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left( \frac{\pi}{2} \right)^{1/2} \frac{1}{1 - 2(T_a + T_b) + \sum_e T_e} \right)^{1/2}
\end{equation}

C. An Interpolation Formula

The behavior of the correlation functions is most easily visualized in terms of a closed analytical expression valid for the entire parameter space, even at the cost of a slightly reduced accuracy. The previous expressions cover most regions of the time variable, but the small and large $\tau$ expansions do not quite overlap. The interpolation between these two regions is easily obtained by interpolating by hand. These expressions depend on the combination $-2(1 - \tau')(T_a + T_b) + \tau \sum_e T_e T'_e$ for $\tilde{C}_{ab,ab}^2(\tau)$ and on a similar one, $-2(T_a + T_c) + \tau \sum_e T_e T'_e$, in the case of $\tilde{C}_{aa,bb}^1(\tau)$. Except for a minor effect at small $\tau$ produced by the linear dependence on the lead parameters $T_{a,b,c}$, the correction term as compared to the GS expression is a smooth, monotonically decreasing positive function of $\tau \sum_e T_e T'_e$. For $\tilde{C}_{ab,ab}^2(\tau)$, a simple interpolated expression is obtained by replacing the small $\tau$ expansion in Eq. (25) which contains a term of the type $1 - y$ with a small parameter $y$ by an expression of the type $(1 + 2y)^{-1/2} \simeq 1 - y$.

This gives, over the entire region, the interpolated value

\begin{equation}
\tilde{C}_{ab,ab}^2(\tau) \simeq T_a T_b (1 + 2\tau T_a)^{-1} (1 + 2\tau T_b)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left\{ 1 + 2 \left( -\tau (1 + \tau) (T_a + T_b) + \frac{1}{2} \tau^2 (1 + 2\tau) \sum_e T_e T'_e \right) \frac{1 - (1 + 2\tau) b_1(\tau)}{2\tau^2} \right\}^{-1/2}
\end{equation}

In this form the same expression can be used both for isolated and for overlapping levels. This expression describes the global shape of the VWZ result very well over the entire range, including its absolute value even at large $\tau$ for which the correlation functions become extremely small. It differs there from the more exact expressions by an overall factor $2/\sqrt{\pi} = 1.128$. for $\tau \sum_e T_e T'_e >> 1$. For very large $\tau$ the VWZ factor in Eq. (35) is dominated by the sum over those open channels for which $\tau T_e >> 1$ since $2\tau \sum_e T_e T'_e \rightarrow \sum_e T_e$.

In the region $\tau > 1$ nearly the same expression is valid also for the elastic correlation function $\tilde{C}_{aa,cc}^1(\tau)$ both for isolated and for overlapping levels. The dependence of the correlation functions $\tilde{C}_{ab,ab}^2$ on the level correlation function $b_1(\tau)$ is nearly negligible in this region and is indicated in Eqs. (35) and (36) only to emphasize the continuity of the expressions. For the region $\tau < 1$, however, the expression (21) should be used, however, so as to correctly incorporate the effects of the level correlations. One has

\begin{equation}
\tilde{C}_{aa,cc}^1(\tau) \simeq \langle S_{aa} \rangle \langle S_{cc}^* \rangle T_a T_c (1 + 2\tau T_a)^{-1} (1 + 2\tau T_c)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \times \left\{ 1 + 2 \left( -2\tau (1 + \tau) (T_a + T_c) + \frac{1}{2} \tau^2 (1 + 2\tau) \sum_e T_e T'_e \right) \frac{1 - (1 + 2\tau) b_1(\tau)}{2\tau^2} \right\}^{-1/2}
\end{equation}

for $\tau > 1$

Here the level correlation function $b_1(\tau)$ is nearly negligible for $\tau > 1$, but it is included above to emphasize the generality of the expressions (35) and (36) for all values of $\tau$. The implications of these results will be discussed in the next section.

VI. DISCUSSION

The previous results demonstrate that the chaotic amplitude correlations of the VWZ model can be accurately described by a closed analytical expression which is at a
To avoid misunderstandings, we remind the reader that the results have been obtained assuming for simplicity that no single channel dominates the sum of the partial widths nor the sum of their squares. This assumption is not basic and can be generalized. In particular, the case of a small number of channels can be obtained using the same technique even without this assumption. The price is a considerably more complicated discussion.

The analytical representation reveals that the gross shape of the correlation function is dominated by a characteristic product factor which is identical to the one in Eq. (11) for the amplitude correlations in the GS solution of the EGS model. This property stands out particularly well in the case of the inelastic correlation function $C^2_{abab}(\tau)$. For the EGS model it is a direct consequence of the assumption that the partial width amplitudes are random, i.e., have individually Gaussian distributions uncorrelated between channels. The result for the VWZ model demonstrates that in this case these amplitudes have become dynamically random to a high degree. This strong dependence on the width amplitudes and their distributions means that the correlation functions are particularly sensitive to any deviations from chaotic conditions in this sector and this will be so for the VWZ model as well. This is of considerable practical interest, since it opens the possibility of direct investigations of the sensitivity of systems approaching full chaoticity to amplitude distributions which are not normal. The consequences of possible modifications have been investigated in the case of the GS solution \[13\]. Such results are immediately relevant the VWZ case as well.

Corresponding observations are also valid for the elastic correlation function $C^3_{aaac}(\tau)$ which for $\tau > 1$ rapidly converges to the same expression as for $C^2_{abab}(\tau)$ but for an overall factor. The elastic and inelastic correlation functions differ for $\tau < 1$ mainly due to the dominance of the level correlation function $b_1(\tau)$ in Eq. (11) in $C^3_{aaac}(\tau)$ in this region; a natural form for this function is the Dyson function which explicitly is a consequence of the VWZ model \[8\], but introduced phenomenologically "ad hoc" in the EGS model and its solutions. From the present approximate VWZ results one observes directly that its global properties are the important ones: the level repulsion on the scale of the level spacing and the detailed normalization of the "correlation hole" in the spacing distribution which it produces. These level spacing correlations rapidly become unimportant for larger $\tau$. In this region the correlation function explores predominantly the average structure of individual broadened levels.

A general feature of the VWZ elastic correlation function is the linear dependence on $\tau$ near the origin. It is presently clear from Eq. (20) that this is a consequence of the level repulsion by which a level creates an adjacent "hole" corresponding to exactly one missing level. Since the Fourier transform at $\tau = 0$ of the correlation function $C^3_{aaac}(\tau)$ is the energy integral over the entire range, the joint contribution of the level and its "hole" vanishes in this case. This effect has previously been well investigated in the context of the EGS model\[4\]; the present result reveals a more detailed, yet still simple, picture of the interplay of the level distribution and the transmission coefficients.

It is frequently taken for granted in the literature that the correlation functions decrease exponentially for large $\tau$ for strongly overlapping levels. The present analytical VWZ description shows clearly that this is only approximately correct as is also the case for the GS expression. The correct asymptotic dependence of the correlations in the VWZ model is an inverse power law $\tau^{-1(M/2+2)}$ with $M$ is the number of open channels $e$ in agreement with the behavior previously suggested by the EGS model\[4\]. With increasing $\tau$ a power $\tau^{-1/2}$ is added successively every time $\tau T_e$ becomes larger than 1 for any channel. For experimental tests of the properties of such systems, it is then useful to consider not only the the main global exponential decay parameter $\sum e T_e$, but also $\sum e T_e^2$ which governs the onset of the deviation from the exponential law.

The differences of the 2-point function for the exact VWZ solution to that of the EGS model appear mainly as 3 characteristic effects.

1. The 2-point function of the VWZ model differs from the one of the simplified EGS model by an overall modulation function given in Eqs. (55,56). It is a considerable experimental challenge to display this function explicitly owing to the high statistical precision required. Its effects are very interesting from a theoretical viewpoint, however. They can be investigated in detail both using exact numerical VWZ results, using our approximations as well as using numerical computer simulations of chaotic systems. This modification stands out particularly clearly in case of the inelastic correlation function $C^2_{abab}(\tau)$. It has a typical behavior apparent from the expressions in Eqs. (21,25) and (29,30).

2. For $\tau > 1$ the dominant term is modified by a factor $(1 + 2\tau \sum e T_e T'_e)^{-1/2}$, which decreases monotonically with $\tau$. In the asymptotic power law limit of very large $\tau$ this factor becomes approximately $(1 + \sum e T_e)^{-1/2}$ both for isolated and for overlapping levels\[4\]. This asymptotic limit is of little practical importance since

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Footnote 2: A frequent practical situation is that of open channels which divide effectively into two groups, one with transmission coefficients contributing both to $\sum e T_e$ and $\sum e T'_e$ at the relevant $\tau$ and another one with small transmission coefficients contributing only to $\sum e T_e$, but with a negligible effective contribution to $\sum e T'_e$. These latter weak channels give a contribution indistinguishable from the one for true absorption with an exponential contribution to the correlation function with a decay parameter
2. The VWZ model has a typical suppression factor \( < S_{aa} >= (1 - T_a)^{1/2}(1 - T_c)^{1/2} \) in the case of the correlation function \( C^1 \) contributing to elastic correlations as apparent already in the general expression (11) as well as in the detailed expressions (24,29) and (35,36). This factor has the consequence that the correlation function for 2 elastic amplitudes vanishes in the extreme limit when one or the other of the transmission coefficients reaches the limit of very strong transmission, i.e., for \( T_{a,c} = 1 \). Similarly, in this same limit the elastic autocorrelation function is partly suppressed, but only by about 30%. The GS solution used here for comparison has the factor 1 instead of the factor \( < S_{aa} >= 1 \) is outside its range of its applicability as discussed in Ref. [13]. The difference originates in the unitarity constraint, which is not imposed on the EGS model. For smaller \( T_{a,c} \) the effect of this difference becomes minor. The GS expression can be improved by introducing this VWZ factor phenomenologically. One should however realize that the case of very strong transmission is of limited practical interest. This multiplicative factor in the elastic channels is the single most important difference between the VWZ and EGS models for chaotic amplitude correlations.

3. For \( \tau < 1 \) the modulation function depends explicitly on the level spacing distribution also in the inelastic case. In the region of very small \( \tau \) this inelastic factor is \[ 1 + \tau(T_a + T_b) - 1/2 \tau^2 \sum_e T_e^2 + \ldots \]. At small \( \tau \) this leads to a minor initial increase of the modulation factor with a maximum returning to 1 near \( \tau = 2(T_a + T_b)/\sum_e T_e^2 \) and thereafter joining the previously described monotonic decrease for \( \tau > 1 \). A closely related factor modifies the correlation function \( \tilde{C}_{aa;cc}(\tau) \), but it is then difficult to disentangle from the level spacing correlation function.

The present results provide an encouraging structure for the approximate description of higher order correlation functions for chaotic and nearly chaotic systems for which exact results are known only in special cases. The close correspondence the present results from the VWZ model to those of the EGS model suggests that the latter should provide a reliable, albeit approximate, guide to this situation. This feature also suggests more generally that the simplified EGS model is a valid laboratory for exploring the sensitivity of various features which govern the approach to the fully chaotic situation of the VWZ model. In particular, the physics of the sensitivity of the correlation functions to symmetry violations can be reliably clarified using the simpler EGS model.

VII. ACKNOWLEDGEMENTS

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Appendix A: The 2-point Correlations in the EGS Model: Time Representation

The EGS model gives the following results for the present case of normally distributed partial width amplitudes in each of the uncorrelated channels [4, 13], i.e., partial widths with Porter-Thomas distributions [19]:

\[
\tilde{C}[S_{ab}S_{ab}^{*}](\tau) = T_a T_b \Pi_{e;ab}(\tau) \quad \text{for } a \neq b \quad \text{(inelastic autocorrelations)} \tag{A1}
\]

\[
\tilde{C}[S_{aa}S_{aa}^{*}](\tau) = T_a T_e \left( \Pi_{e;aa}(\tau) - \rho_0 b(\tau) \Pi_{e;aa}(\tau/2) \Pi_{e;cc}(\tau/2) \right) \quad \text{(elastic correlations)}
\]

for \( \tau > 0 \) and 0 for \( \tau < 0 \).

Here

\[
\Pi_{e;\alpha,\beta}(\tau) = (1 + 2\tau T_\alpha)^{-1}(1 + 2\tau T_\beta)^{-1} \prod_e (1 + 2\tau T_e)^{-1/2} \tag{A2}
\]

\[ \Lambda_{ab} = \sum_e T_e^{\text{small}}. \] In this case these weak channels do not contribute to terms \( \sum_e T_e T'_e \) in Eqs. (31,32) nor to the asymptotic expression containing \( \sum_e T_e \), nor do they effectively occur in the asymptotic power expansion. This observation in no way invalidates the asymptotic power law. It only means that the region of exploration should be sufficiently large with \( \tau T_e > 1 \) for all of the channels.
for $\alpha \neq \beta$. It is multiplied by 3 for $\alpha = \beta$ in the case a normal distribution.

The exponential approximation to the time-variation holds if the variance of the total width obeys the inequality:

$$(2\pi)^2 \tau^2 < (\Gamma - \langle \Gamma \rangle)^2 \Rightarrow \tau^2 \sum \mathcal{E}_n^2 < 1.$$  

\textbf{Appendix B: The integral $I(a, \tau)$ in the large $a$ limit}

Consider an integral of the type $I(a, \tau)$, symmetrical in $x_1$ and $x_2$, for large values of the parameter $a >> \max(1, \tau)$ to leading order neglecting corrections of order $1/a$.

$$I(a, \tau) \equiv \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^{\min(1, 1/\tau)} dx \delta(x_1 + x_2 + x - 1) \frac{x|x_1 - x_2|}{(x_1 x_2)^{1/2}(x + 2x_1)^2(x + 2x_2)^2} \exp \left[-a[x_1 x_2 + x - (1/4)x^2] \right]$$

For the case $x_1 = 1 - x_2 - x > x_2 \simeq 1$ it follows that the relevant region of integration over $x_2 + x << 1$ can then be freely extended to $\infty$, since this gives negligible extra contributions. In this case the term $(x + 2x_1)^2 \simeq 4[1 - 2x_2 - x] = 4(x_1 - x_2)$ which is the numerator term such that the ratio to a good approximations is 1 which simplifies the discussion (an omitted term $(x_2 + x_2/2)^2$ is assumed small compared to 1). Denoting $x_2 = y = u^2$ and $x = z = v^2$ with $\rho^2 = u^2 + v^2$ and $u = \rho \cos \phi; v = \rho \sin \phi$ gives on exponentiating the contributions associated with $x_1$,

$$I(a, \tau) \simeq \frac{2}{4} \int_0^\infty dy \int_0^\infty dz \frac{z}{y^{1/2}(z + 2y)^2} \exp \left[-(a - 1/2)[y + z] \right] =$$

$$2 \int_0^\infty d\rho \int_0^{\pi/2} d\phi \frac{\sin^3 \phi}{(1 + \cos^2 \phi) \rho^2} \exp \left[-(a - 1/2)\rho^2 \right] \to \left(\frac{\pi}{2(2a - 1)} \right)^{1/2} \text{ for } a >> \max(1, \tau)$$

(B2)

where the the last step is obtained using $\int_0^\infty dt \exp(-kt^2) = (\pi/4k)^{1/2}$ ; $\int_0^{\pi/2} d\phi \sin^3 \phi (1 + \cos^2 \phi)^{-2} = 1/2$.

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