PARTIAL AVERAGING NEAR A RESONANCE IN PLANETARY DYNAMICS

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Abstract. Following the general numerical analysis of Melita and Woolfson (1996), I showed in a recent paper that a restricted, planar, circular planetary system consisting of Sun, Jupiter and Saturn would be captured in a near (2:1) resonance when one would allow for frictional dissipation due to interplanetary medium (Haghhighipour, 1998). In order to analytically explain this resonance phenomenon, the method of partial averaging near a resonance was utilized and the dynamics of the first-order partially averaged system at resonance was studied. Although in this manner, the finding that resonance lock occurs for all initial relative positions of Jupiter and Saturn was confirmed, the first-order partially averaged system at resonance did not provide a complete picture of the evolutionary dynamics of the system and the similarity between the dynamical behavior of the averaged system and the main planetary system held only for short time intervals. To overcome these limitations, the method of partial averaging near a resonance is extended to the second order of perturbation in this paper and a complete picture of dynamical behavior of the system at resonance is presented. I show in this study that the dynamics of the second-order partially averaged system at resonance resembles the dynamical evolution of the main system during the resonance lock in general, and I present analytical explanations for the evolution of the orbital elements of the main system while captured in resonance.

Key words: planetary dynamics, resonance capture, averaging.

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1. Introduction

Inspired by the results of extensive numerical investigations by Melita and Woolfson (1996) and in the framework of a restricted planar circular three-body system, I have recently shown that a planetary system consisting of a star and two planets, subject to dynamical friction due to a freely rotating uniform interplanetary medium, will be captured in resonance when the inner planet is more massive (Haghighipour, 1998, hereafter paper I). As an example of such a planetary system, the three-body system of Sun-Jupiter-Saturn was numerically integrated and it was shown that the orbital period of Saturn would be captured in a near (2:1) commensurability with that of Jupiter (Figure 1). In this system, the motion of Sun was neglected and Jupiter was assumed to be moving on a circular orbit with a known constant frequency. As a result of this resonance trapping, the eccentricity and semimajor axis of Saturn’s osculating ellipse and also its orbital angular momentum and total energy became essentially constant (paper I).

![Figure 1. Graph of the ratio of orbital period of Saturn to that of Jupiter versus time. The system is captured in a near (2:1) resonance. Integrations were performed on a timescale of $10^4(T_J/2\pi)$ years where $T_J$ is the orbital period of Jupiter. The system was integrated with Sun at the origin of an inertial plane-polar coordinate system, Jupiter initially on the x-axis and Saturn initially at $(a, e, \theta, \hat{v}) = (1.8380462, 0.0556, 0, 0)$ where $a$ and $e$ are the semimajor and the eccentricity of Saturn’s osculating ellipse, respectively, and $\hat{v}$ is its true anomaly. The density of the interplanetary medium was taken to be equal to 16 times the mass of Jupiter uniformly spread in a spherical volume with 50 au radius and the masses of Sun, Jupiter and Saturn were taken to be constant and equal to their present values.](image-url)
This paper is devoted to the analysis of the dynamical behavior of the planetary system described above while captured in resonance. A preliminary analysis of the dynamics of this system was presented in paper I using the principle of averaging; in fact in paper I, the first-order partially averaged system at resonance was obtained and its dynamics was studied. It turned out that in the first order of approximation, the partially averaged system was Hamiltonian with a periodic potential function that guaranteed the occurrence of resonance lock for all initial relative positions of the two planets. The dynamics of the first-order averaged system agreed with that of the main three-body system only for short time intervals and it was mentioned that in order to obtain an agreement which would hold for longer times, one had to extend the analysis to the study of the dynamics of the second-order partially averaged system at resonance.

In this paper, a complete picture of the dynamical behavior of the three-body system of Sun-Jupiter-Saturn during resonance capture is presented. This is accomplished by extending the calculations to the second-order partially averaged system at resonance, where one can also explain the behavior of Saturn’s orbital elements and angular momentum during the resonance lock.

The model under investigation is presented in section 2. Section 3 contains a brief discussion of the method of partial averaging near a resonance and in section 4, the application of this method to the Sun-Jupiter-Saturn system is discussed. Section 5 concludes this study by reviewing and summarizing the results.

2. The Model

The model under investigation is a restricted planar circular three-body system consisting of Sun, Jupiter and Saturn. In this model, the motion of Sun due to gravitational attraction of Jupiter and Saturn is neglected and Jupiter is assumed to have a uniform circular motion with a known constant period $T_J$. The only source of dissipation in this system is the dynamical friction due to interplanetary medium. It has been shown by
Dodd and McCrea (1952) and also by Binney and Tremaine (1987) that the effect of the dynamical friction on the orbital motion of a general spherical body \( b \) with mass \( m_b \) around a general star with mass \( M \) appears as a deceleration with a magnitude given by

\[
R = \frac{2\pi G^2 \rho_o}{V_{rel}^2} m_b \ln\left[1 + \frac{(S V_{rel}^2)}{G m_b}\right],
\]

and a direction that is opposite to the velocity of body \( b \) with respect to the medium. In this equation, \( \rho_o \) is the uniform density of the interplanetary medium, \( G \) is Newton’s constant and \( S = r_b (m_b/2M)^{1/3} \), where \( r_b \) is the radial distance of \( b \) to the central star. The relative velocity \( \vec{V}_{rel} \) in equation (1) is given by \( \vec{V}_{rel} = \vec{V} - \vec{V}_m \), where \( \vec{V} \) is the velocity of the body and \( \vec{V}_m \) is that of the interplanetary medium at the position of the body. For the planetary system presented in this paper, it is assumed that the interplanetary medium is freely rotating around the central star in the same sense as body \( b \). Therefore, \( \vec{V}_m \) is perpendicular to the position vector of \( b \) and has a magnitude equal to \( (GM/r_b)^{1/2} \).

In an inertial coordinate system with its origin on Sun, the equation of motion of Saturn in dimensionless form is given by (paper I)

\[
\frac{d^2 \vec{r}}{dt^2} = -\frac{\vec{r}}{r^3} - \varepsilon \frac{\vec{r} - \hat{r}_J}{|\vec{r} - \hat{r}_J|^3} - \vec{R}.
\]

In this equation, \( \varepsilon = m_j/M \), where \( m_j \) is the mass of Jupiter, \( \hat{r}_J \) is the unit vector along Jupiter’s orbital radius and \( \vec{R} = \mathcal{R} \vec{V}_{rel}/V_{rel} \) where

\[
\mathcal{R} = \frac{A}{V_{rel}^2} \ln(1 + Br_{rel}^4),
\]

is the dimensionless magnitude of Saturn’s deceleration due to dynamical friction and \( V_{rel} \) is dimensionless magnitude of its velocity with respect to interplanetary medium and is given by

\[
V_{rel}^2 = \dot{r}^2 + r^2(\dot{\theta} - \omega_m)^2,
\]

where \( \omega_m = r^{-3/2} \), is the dimensionless angular frequency of the medium at distance \( r \) from Sun. Denoting the mass of Saturn by \( m_s \), one finds that \( A = 2\pi(\rho_o r_j^3/M)(m_s/M) \)
and \( B = 2^{-2/3}(m_s/M)^{-4/3} \). For any physical system \( A << 1 \) and \( \delta \equiv AB << 1 \). It is important to mention that in deriving equation (2) a set of units has been chosen such that \((T_J/2\pi)^2 = r_J^3/GM\) and all positions and time variables have been scaled by the orbital radius of Jupiter \((r_J)\) and its orbital period divided by \(2\pi\), respectively (see paper I for details).

3. Partially Averaged System at Resonance

In order to apply the method of partial averaging near a resonance to the problem presented in this paper, it is useful to write equation (2) in terms of the Delaunay action-angle variables (Appendix A). Denoting the semimajor axis and the eccentricity of Saturn’s osculating ellipse by \( a \) and \( e \) respectively, the appropriate Delaunay variables for the relative motion (2) are given by \( L = a^{1/2}, G = L(1 - e^2)^{1/2}, l = \dot{u} - e \sin \dot{u} \) and \( g = \theta - \dot{\upsilon} \), where \( \dot{u}, \dot{\upsilon} \) and \( l \) are the eccentric, the true and the mean anomalies of Saturn’s orbit, respectively (Kovalevsky, 1967; Sternberg, 1969; Hagihara, 1972). In these relations, \( L \) and \( G \) are action variables and \( l \) and \( g \) are their corresponding angular variables. From Appendix A, it is evident that For the pure Kepler problem, i.e. in the absence of perturbations in equation (2), the solutions of the Delaunay equations are given by constant \( L, G \) and \( g \) and

\[
l = \omega_K t + l_0 ,
\]

where \( \omega_K = L^{-3} \) is the Keplerian frequency and \( l_0 \) is a constant of integration.

The problem under investigation here is a Kepler system with period \( T = 2\pi/\omega_K \) that is perturbed by forces that involve a period \( T_p \). At resonance, it is necessary, but not sufficient that \( T \) and \( T_p \) become commensurate, i.e. relatively prime integers \( n \) and \( n' \) exist such that \( nT_p = n'T \). To explain the dynamical behavior of the system near the resonance manifold in phase space, it is more convenient to imagine a system with a single degree of freedom and action-angle variables \((I, \Theta)\) such that
\[ \dot{I} = \epsilon \mathcal{P}(I, \Theta, t, \epsilon), \] (6)

and

\[ \dot{\Theta} = \Omega(I) + \epsilon \mathcal{N}(I, \Theta, t, \epsilon), \] (7)

where \( \mathcal{P} \) and \( \mathcal{N} \) are explicitly time-dependent with period \( T_p \).

In the absence of perturbation (\( \epsilon = 0 \)), equations (6) and (7) represent a Hamiltonian system whose action variable \( I \) has a constant value \( I_0 \). At resonance, this action variable fluctuates around its unperturbed value. Let us denote these fluctuations by

\[ I - I_0 = \epsilon' \mathcal{D}, \] (8)

where \( \epsilon' \) is a small parameter which has yet to be determined. In a similar fashion, let us show deviations of \( \Theta \) from its unperturbed value \( \Omega(I_0) t \) by

\[ \Theta - \Omega(I_0) t = \Phi. \] (9)

Substituting for \( I \) and \( \Theta \) from equations (8) and (9) in equations (6) and (7), the dynamical equations of the system will be given by

\[ \dot{\mathcal{D}} = \frac{\epsilon}{\epsilon'} \mathcal{P}(I_0, \Theta, t) + \epsilon \mathcal{D} \frac{\partial \mathcal{P}}{\partial I}(I_0, \Theta, t) + O(\epsilon \epsilon'), \] (10)

and

\[ \dot{\Phi} = \epsilon' \mathcal{D} \frac{\partial \Omega}{\partial I}(I_0) + \epsilon \mathcal{N}(I_0, \Theta, t) + \frac{1}{2} \epsilon^2 \mathcal{D}^2 \frac{\partial^2 \Omega}{\partial I^2}(I_0) + O(\epsilon \epsilon') + O(\epsilon^3). \] (11)

Using the averaging theorem (Sanders, 1985; Wiggins, 1996; Chicone et al., 1996a&amp;b and 1997a&amp;b), we can now study the averaged dynamics of the system (10) and (11) at resonance by introducing the averaging integration

\[ \bar{Q} = \frac{1}{nT_p} \int_{0}^{nT_p} Q(I_0, \Omega(I_0) t + \Phi, t) \, dt. \] (12)
for any function $Q(I, \Theta, t)$. Also, the averaging theorem requires $\dot{D}$ and $\dot{\Phi}$ to have the same power of perturbation parameter in the lowest order, that is $\epsilon'^2 = \epsilon$. The partially averaged dynamics of the system (6) and (7) at resonance is now given by

$$\dot{D} = \epsilon^{1/2} \bar{P}(I_0, \bar{\Phi}, t) + \epsilon D \left( \frac{\partial P}{\partial I}(I_0, \bar{\Phi}, t) \right),$$

and

$$\dot{\Phi} = \epsilon^{1/2} \bar{N}(I_0, \bar{\Phi}, t) + \epsilon \left[ \bar{N}(I_0, \bar{\Phi}, t) + \frac{1}{2} \bar{D}^2 \left( \frac{\partial^2 \Omega}{\partial I^2}(I_0) \right) \right],$$

where $\langle \cdot \rangle$ represent an averaged quantity. Solutions to equations (13) and (14) are expected to represent the average dynamical behavior of the perturbation system (6) and (7) over time intervals of order $\epsilon^{-1/2}$ (Sanders et al., 1985; Wiggins, 1996; Chicone et al., 1996a&b and 1997a&b).

4. Second Order Partially Averaged System at Resonance

Mathematically speaking, equation (2) represents a dynamical system with perturbation parameters $\epsilon$ and $\delta$ (see Appendix A). In a parameter space, $\epsilon$ and $\delta$ represent a plane where the full two-parameter perturbation problem (2) must be studied. This, however, requires consideration of all curves in this parameter space. However, in order to simplify the calculations and to avoid complexities, I consider a linear relation between $\epsilon$ and $\delta$. That is, $\delta = \epsilon \Delta$, with $\Delta$ fixed. This will simplify the analysis by reducing the two-parameter perturbation problem (2) to a perturbation problem with only one parameter, $\epsilon$.

At resonance, the orbital periods of Saturn and Jupiter become commensurate and therefore, the semimajor axis of Saturn’s osculating ellipse becomes essentially constant. Following the formalism presented in section 3, $L$ and $l$ can be written as

$$L = L_0 + \epsilon^{1/2} D,$$

and

$$l = \frac{1}{L_0^3} t + \varphi,$$
where $D$ represents the deviation of $L$ from its resonant value $L_0$ and $\varphi$ represents the deviation of $l$ from its resonant value $L_{0}^{-3}t$. Dynamics of the system near resonance is obtained by writing equation (2) in terms of the new action-angle variables ($D, G, \varphi, g,$).

The result is (paper I)

\[
\dot{D} = -\varepsilon^{1/2} F_{11} - \varepsilon D F_{12} + O(\varepsilon^{3/2}) ,
\]

\[
\dot{G} = -\varepsilon F_{22} + O(\varepsilon^{3/2}) ,
\]

\[
\dot{\varphi} = -\varepsilon^{1/2} \left( \frac{3D}{L_0^4} \right) + \varepsilon \left( \frac{6D^2}{L_0^5} + F_{32} \right) + O(\varepsilon^{3/2}) ,
\]

\[
\dot{g} = \varepsilon F_{42} + O(\varepsilon^{3/2}) ,
\]

where

\[
F_{11}(L_0, G, L_{0}^{-3}t + \varphi, g) = \frac{\partial H_{ext}}{\partial L} (L_0, G, L_{0}^{-3}t + \varphi, g) - \Delta R_L(L_0, G, L_{0}^{-3}t + \varphi, g) ,
\]

\[
F_{22}(L_0, G, L_{0}^{-3}t + \varphi, g) = \frac{\partial H_{ext}}{\partial g} (L_0, G, L_{0}^{-3}t + \varphi, g) - \Delta R_G(L_0, G, L_{0}^{-3}t + \varphi, g) ,
\]

\[
F_{32}(L_0, G, L_{0}^{-3}t + \varphi, g) = \frac{\partial H_{ext}}{\partial L} (L_0, G, L_{0}^{-3}t + \varphi, g) + \Delta R_l(L_0, G, L_{0}^{-3}t + \varphi, g) ,
\]

\[
F_{42}(L_0, G, L_{0}^{-3}t + \varphi, g) = \frac{\partial H_{ext}}{\partial G} (L_0, G, L_{0}^{-3}t + \varphi, g) + \Delta R_g(L_0, G, L_{0}^{-3}t + \varphi, g) ,
\]

and

\[
F_{12}(L_0, G, L_{0}^{-3}t + \varphi, g) = \frac{\partial F_{11}}{\partial L} (L_0, G, L_{0}^{-3}t + \varphi, g) .
\]

In equations (21)-(25), the effect of gravitational attraction of Jupiter appears in the external Hamiltonian $H_{ext}$, where

\[
H_{ext} = -\frac{1}{|\vec{r} - \hat{r}_J|} .
\]

Also the contribution of the frictional force of the interplanetary medium is given by (Appendix C)

\[
\mathcal{R}_L = \frac{a}{\delta} (1 - e^2)^{-1/2} (\mathcal{R}_x A - \mathcal{R}_y B) ,
\]

\[
\mathcal{R}_G = \frac{1}{\delta} \left[ \mathcal{R}_x \sin(g + \dot{\varphi}) - \mathcal{R}_y \cos(g + \dot{\varphi}) \right] ,
\]

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\[ R_l = -\frac{r}{e\delta} a^{-1/2} (R_x C + R_y W) , \]
\[ R_g = \frac{G}{e\delta} (R_x E + R_y F) , \]

where

\[ A = \sin (g + \hat{v}) + e \sin g , \]
\[ B = \cos (g + \hat{v}) + e \cos g , \]
\[ C = (-2e + \cos \hat{v} + e \cos^2 \hat{v}) \cos (g + \hat{v}) + (2 + e \cos \hat{v}) \sin (g + \hat{v}) \sin \hat{v} , \]
\[ W = (-2e + \cos \hat{v} + e \cos^2 \hat{v}) \sin (g + \hat{v}) - (2 + e \cos \hat{v}) \cos (g + \hat{v}) \sin \hat{v} , \]
\[ E = \cos g + \left( \frac{\sin \hat{v}}{1 + e \cos \hat{v}} \right) \sin (g + \hat{v}) , \]
\[ F = \sin g - \left( \frac{\sin \hat{v}}{1 + e \cos \hat{v}} \right) \cos (g + \hat{v}) , \]

and \( R_x \) and \( R_y \) are the Cartesian components of \( \vec{R} \). For the three-body system of Sun-Jupiter-Saturn in a \((2:1)\) resonance, the dynamics of the second-order partially averaged system at resonance is given by

\[ \dot{\bar{D}} = -\varepsilon^{1/2} F_{11} - \varepsilon \bar{D} F_{12} , \]
\[ \dot{\bar{G}} = -\varepsilon \bar{F}_{22} , \]
\[ \dot{\bar{\varphi}} = -\varepsilon^{1/2} \left( \frac{3\bar{D}}{L_0^4} \right) + \varepsilon \left( \frac{6\bar{D}^2}{L_0^5} + \bar{F}_{32} \right) , \]
\[ \dot{\bar{g}} = \varepsilon \bar{F}_{42} . \]

where

\[ \bar{F}_{ij}(\bar{G}, \bar{g}, \bar{\varphi}) = \frac{1}{4\pi} \int_0^{4\pi} F_{ij}(\bar{G}, \frac{1}{L_0^3} t + \bar{\varphi}, \bar{g}, t) dt . \]

In order to study these equations, one needs to calculate \( \bar{F}_{ij} \). From equations (21)-(25), calculations of these terms involve the effect of gravitational attraction of Jupiter and also the frictional effect of the interplanetary medium. Appendices B and C contain these calculations, respectively. From Appendix C and to the lowest order in eccentricity, the quantities \( R_L, R_G, R_l \) and \( R_g \) are given by
\[ R_L = -\frac{1}{2} a^2 e^2 \cos l \left( 1 - \frac{3}{4} \cos^2 l \right)^{1/2}, \quad (42) \]
\[ R_G = -\frac{1}{2} a e^2 \cos l \left( 1 - \frac{3}{4} \cos^2 l \right)^{1/2}, \quad (43) \]
\[ R_l = \frac{1}{a} e^2 (137 \sin l - 23 \sin 3l) \left( 1 - \frac{3}{4} \cos^2 l \right)^{1/2}, \quad (44) \]
\[ R_g = \frac{1}{8} G a e^2 (61 \sin l + 23 \sin 3l) \left( 1 - \frac{3}{4} \cos^2 l \right)^{1/2}. \quad (45) \]

Substituting for \( l \) from equation (16) and using the averaging integration (41), the averaged values of these quantities will be equal to zero. The non-vanishing contribution of friction appears as terms proportional to \( e^3 \). Since at resonance the numerical value of \( e \) is only a few percent (paper I), the dynamical friction will not play a vital role in the dynamical evolution of the averaged system. Therefore, in the rest of these calculations, I neglect the contribution of dynamical friction at resonance and will consider only the gravitational attraction of Jupiter as the main source of perturbation. Also, in order to simplify the calculations, all quantities will be expanded only to the first order in eccentricity.

Neglecting the frictional perturbation of the medium, the quantities \( \bar{F}_{ij} \) will then be obtained only from the external Hamiltonian \( \mathcal{H}_{ext} \). From Appendix B these quantities are given by

\[ \bar{F}_{11} = \frac{e}{a_0^2} \sigma_{11} \sin (2\varphi + g), \quad (46) \]
\[ \bar{F}_{12} = \frac{1}{e a_0^{5/2}} \sigma_{11} \sin (2\varphi + g), \quad (47) \]
\[ \bar{F}_{22} = \frac{e}{2a_0^3} \left[ \sigma_{22}^{(3/2)} + 3 \sigma_{22}^{(5/2)} \right] \sin (2\varphi + g), \quad (48) \]
\[ \bar{F}_{32} = -\frac{1}{2 e a_0^{5/2}} \sigma_{11} \cos (2\varphi + g), \quad (49) \]
\[ \bar{F}_{42} = \frac{G}{2 e a_0^3} \sigma_{11} \cos (2\varphi + g), \quad (50) \]
where
\[ \sigma_{11} = \sum_{h=0}^{\infty} \left[ \frac{\Gamma\left(\frac{3}{2} + h\right)}{a_0^h h! \Gamma\left(\frac{3}{2}\right)} \right]^2 \left\{ 1 + \left(\frac{2h + 3}{h + 1}\right) \left[ 1 - \frac{3}{4a_0^2} \left(\frac{2h + 5}{h + 2}\right) \right] \right\}, \] (51)
\[ \sigma_{22}^{(3/2)} = \sum_{h=0}^{\infty} \left[ \frac{\Gamma\left(\frac{3}{2} + h\right)}{a_0^h h! \Gamma\left(\frac{3}{2}\right)} \right]^2 \left\{ 1 + \frac{3}{4a_0^2} \left(\frac{2h + 5}{h + 2}\right) \left(\frac{2h + 3}{h + 1}\right) \right\}, \] (52)
\[ \sigma_{22}^{(5/2)} = \sum_{h=0}^{\infty} \left[ \frac{\Gamma\left(\frac{5}{2} + h\right)}{a_0^h h! \Gamma\left(\frac{5}{2}\right)} \right]^2 \left\{ 1 - \frac{1}{4a_0^2} \left(\frac{2h + 5}{h + 1}\right) \left[ 3 + \left(\frac{2h + 7}{h + 2}\right) \left(1 - \frac{3}{4a_0^2} \frac{2h + 9}{h + 3}\right) \right] \right\}. \] (53)

Dynamical equations of the second-order averaged system at resonance can now be written as
\[
\dot{D} = -\frac{e \varepsilon^{1/2}}{a_0^2} \sigma_{11} \left(1 + \frac{\varepsilon^{1/2}}{e^2 a_0^{1/2}} D\right) \sin(2\varphi + g), \] (54)
\[
\dot{G} = -\frac{e \varepsilon}{2a_0^2} \left[ \sigma_{22}^{(3/2)} + 3 \sigma_{22}^{(5/2)} \right] \sin(2\varphi + g), \] (55)
\[
\dot{\varphi} = -\frac{3 \varepsilon^{1/2}}{a_0^2} D + \frac{6 \varepsilon}{a_0^{5/2}} \left[D^2 - \frac{1}{12e} \sigma_{11} \cos(2\varphi + g) \right], \] (56)
\[
\dot{g} = \frac{G \varepsilon}{2e a_0^3} \sigma_{11} \cos(2\varphi + g). \] (57)

where the overbars have been dropped for the sake of simplicity.

Equations (54)-(57) allow us to analyze the averaged dynamics of the perturbation system (2) at resonance. Numerical integrations of these equations indicate that the rate of change of orbital angular momentum of Saturn (\(G\)) is almost zero (Figure 2) and its resonant value agrees with the value obtained from the direct integrations of the main system (paper I).

![Figure 2](image-url)

Figure 2. Graph of orbital angular momentum of Saturn versus time at resonance. The initial conditions are given by \((D, G, \varphi, g) = (0, 1.275, -5.4, 0.882)\) and the timescale is \(10^4(T_J/2\pi)\) years.
One can also use $G = L(1 - e^2)^{1/2}$ along with equation (15) to show that in the first order in eccentricity and second order in perturbation parameter $\varepsilon^{1/2}$,

$$\frac{de}{dt} = \frac{G \varepsilon}{2a^3} \left[ \sigma^{(3/2)}_{22} + 3\sigma^{(5/2)}_{22} - 2G a^{-1/2}_0 \sigma_{11} \right] \sin(2\varphi + g). \quad (58)$$

Figure 3 shows that at resonance, the right hand side of equation (58) is essentially zero which implies that eccentricity becomes almost constant at resonance.

Let us now turn our attention to evolution of action variable $D$ with time. As mentioned in section 2, $D$ represents the deviation of $L$ from its Keplerian value. One, therefore, expects $L$ and $D$ to have the same frequency. In a comparison with the result of the first order partially averaged system at resonance presented in paper I where the similarity between $D$ and $L$ was only pronounced for time intervals of order $\varepsilon^{-1/2}$, the second-order partially averaged equations reveal this similarity in general (Figure 4).
Figure 4. Graph of $D$ (left) and the action variable $L$ (right) versus time while the system is at resonance. The original system shows a periodic behavior (right) with a frequency which is in a very good agreement with the second-order averaged system at resonance (left). The initial conditions for the graph of $D$ (left) are the same as Figure 2 and for the graph of $L$ are the same as Figure 1. The timescale for both graphs is $10^4(T_J/2\pi)$ years.

It is necessary to mention that the graph of $L$ versus time shows two different frequencies (Figure 4). One is the frequency of its envelope and the other is the Keplerian frequency of Saturn. From the principle of averaging presented in section 3, it is the frequency of the envelope that has to be similar to that of $D$. Figure 5 shows the frequency of the envelope and also the Keplerian frequency of action variable $L$. The (2:1) commensurability is simply obtained by calculating Saturn’s Keplerian period from this figure and multiplying it by the timescale $10^4(T_J/2\pi)$.

Figure 5. Graph of the action variable $L$ versus time while the system is at resonance. There are two frequencies associated with this quantity: A slow frequency related to the time variation of its envelope and a fast frequency which is equal to the Keplerian frequency of Saturn at resonance. The near (2:1) commensurability is simply obtained by dividing the time interval of this graph by the number of revolutions of Saturn during this interval and multiply the result by $10^4(T_J/2\pi)$. 
5. Summary

It has been shown by Haghhighipour (1998) and in a more extensive way by Melita and Woolfson (1996) that the three-body system of Sun-Jupiter-Saturn will be captured in a near (2:1) resonance when one allows for the frictional force of the interplanetary medium. In an attempt to study these results analytically, the method of partial averaging near a resonance was employed and the system was considered to be restricted, planar and circular. The first-order partially averaged system confirmed the occurrence of resonance capture for all initial relative positions of the two planets (Haghhighipour, 1998); however, it was unable to fully illustrate the dynamical behavior of the system. The agreement between evolutionary dynamics of the first-order partially averaged system at resonance and the main system was observed only for time intervals of order $(m_J/M)^{-1/2}$. In order to obtain a better agreement, dynamics of the second-order partially averaged system at resonance has been studied. Numerical integrations of the averaged equations at this order have confirmed that the orbital angular momentum of Saturn and the eccentricity of its osculating ellipse become essentially constant at resonance and demonstrated the close similarity between the dynamics of the averaged system at resonance and the main system over long time intervals.

The effect of dynamical friction has not appeared in this study. It has turned out that the first non-vanishing terms of the averaged frictional force will be proportional to $e^3$, which has been neglected in these calculations. It is important to mention that it is only through dynamical friction that one can study the effect of the density of the interplanetary medium on the dynamics of the system. However, calculations presented in this paper indicate that considering dynamical friction as the only source of perturbation, the density of the medium will not play a vital role in time variation of the orbital elements of Saturn at resonance. It is expected that accretion will provide the appropriate ground for the study of the effects of the density of the medium on resonance phenomena.
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Appendix A

Let us assume that the motions of Jupiter and Saturn are co-planar. In a plane-polar coordinate system with its origin on Sun, the equation of motion of Saturn, i.e. equation (2) can be written as (paper I)

\[ P_r = \dot{r} \quad , \]  
\[ P_\theta = r^2 \dot{\theta} \quad , \]  
\[ \dot{P}_r = \frac{P_\theta^2}{r^3} - \frac{1}{r^2} - \frac{\epsilon}{|\vec{r} - \vec{r}_J|^3} \left[ r - \cos (\theta - \theta_J) \right] - \left( R_x \cos \theta + R_y \sin \theta \right) \quad , \]  
\[ \dot{P}_\theta = -\epsilon \frac{r}{|\vec{r} - \vec{r}_J|^3} \sin (\theta - \theta_J) - r \left( -R_x \sin \theta + R_y \cos \theta \right) \quad , \]

and

where

\[ R_x = \frac{A}{V_{rel}^3} \ln (1 + Br^2 V_{rel}^4) \left[ \dot{r} \cos \theta - r (\dot{\theta} - \omega_m) \sin \theta \right] \quad , \]  
\[ R_y = \frac{A}{V_{rel}^3} \ln (1 + Br^2 V_{rel}^4) \left[ \dot{r} \sin \theta + r (\dot{\theta} - \omega_m) \cos \theta \right] \quad . \]

The Delaunay variables presented in section 4 are related to the plane-polar coordinates by \( P^2_\theta = G^2 = r(1 + e \cos \hat{v}) \), \( P_r = e \sin \hat{v} / G \) and \( g = \theta - \hat{v} \). From these equations, the dynamical equations (A1)-(A4) can be written as

\[ \frac{dL}{dt} = a \left( 1 - e^2 \right)^{-1/2} \left[ F_r e \sin \hat{v} + F_{\theta} (1 + e \cos \hat{v}) \right] \quad , \]  
\[ \frac{dG}{dt} = r F_{\theta} \quad , \]  
\[ \frac{dl}{dt} = \omega_K + \frac{r}{e} a^{-1/2} \left[ F_r (-2e + \cos \hat{v} + e \cos^2 \hat{v}) - F_{\theta} \left( 2 + e \cos \hat{v} \right) \sin \hat{v} \right] \quad , \]  
\[ \frac{dg}{dt} = \frac{1}{e} \left[ a(1 - e^2) \right]^{1/2} \left[ -F_r \cos \hat{v} + F_{\theta} \left( 1 + \frac{1}{1 + e \cos \hat{v}} \right) \sin \hat{v} \right] \quad , \]
where

\[ F_r = \varepsilon \frac{\cos(\theta - \theta_J) - r}{|\vec{r} - \vec{r}_J|^3} - (\mathcal{R}_x \cos \theta + \mathcal{R}_y \sin \theta) , \quad (A11) \]

\[ F_\theta = -\varepsilon \frac{\sin(\theta - \theta_J)}{|\vec{r} - \vec{r}_J|^3} + (\mathcal{R}_x \sin \theta - \mathcal{R}_y \cos \theta) , \quad (A12) \]

and the Keplerian frequency of the osculating ellipse is given by \( \omega_K = L^{-3} \). The quantities \( F_r \) and \( F_\theta \) are associated with perturbations due to gravitational attraction of Jupiter and also dynamical friction of the interplanetary medium.

### Appendix B

The external Hamiltonian \( \mathcal{H}_{ext} \) can be written as

\[ \mathcal{H}_{ext} = - \left[ r^2 - 2r \cos(\theta - \theta_J) + 1 \right]^{-1/2} . \quad (B1) \]

Following the simplifications presented in paper I, quantities \( r \) and \( \cos(\theta - \theta_J) \) are approximately given by

\[ r \simeq a \left( 1 - e \cos l \right) , \quad (B2) \]

and

\[ \cos(\theta - \theta_J) \simeq \cos(l + g - \theta_J) + e \left[ \cos(2l + g - \theta_J) - \cos(g - \theta_J) \right] , \quad (B3) \]

where the \( O(e^2) \) terms have been neglected. In this approximation,

\[ \mathcal{H}_{ext} \simeq - \left[ 1 + a^2 - 2a \cos(l + g - \theta_J) \right]^{-1/2} \left[ 1 + \frac{1}{2} e a \frac{2a \cos l + \cos(2l + g - \theta_J) - 3 \cos(g - \theta_J)}{1 + a^2 - 2a \cos(l + g - \theta_J)} \right] . \quad (B4) \]

From equations (21)-(25), contribution of gravitational attraction of Jupiter in dynamics of the second-order partially averaged system at resonance appears as partial derivatives of \( \mathcal{H}_{ext} \) with respect to \( L, l, G \) and \( g \). Substituting \( a^2 \) by \( L \) and \( e \) by \( (1 - G^2/L^2)^{1/2} \), one
will obtain

\[
\frac{\partial H_{\text{ext}}}{\partial L} \simeq a^{1/2} \left[ 1 + a^2 - 2a \cos(l + g - \theta_j) \right]^{-3/2} \left[ 2a - \frac{a}{e} \cos l + \frac{3}{2e} \cos(g - \theta_j) - 2 \cos(l + g - \theta_j) - \frac{1}{2e} \cos(2l + g - \theta_j) \right] \\
+ \frac{3}{2} e a^{3/2} \left[ 1 + a^2 - 2a \cos(l + g - \theta_j) \right]^{-5/2} \left[ 2(1 + 2a^2) \cos l - 8a \cos(g - \theta_j) - \cos(3l + 2g - 2\theta_j) + 3 \cos(l + 2g - 2\theta_j) \right],
\]

(B5)

\[
\frac{\partial H_{\text{ext}}}{\partial l} \simeq a \left[ 1 + a^2 - 2a \cos(l + g - \theta_j) \right]^{-3/2} \left[ ea \sin l + \sin(l + g - \theta_j) + e \sin(2l + g - \theta_j) \right] \\
+ \frac{3}{2} e a^2 \left[ 1 + a^2 - 2a \cos(l + g - \theta_j) \right]^{-5/2} \sin(l + g - \theta_j) \\
\left[ 2a \cos l + \cos(2l + g - \theta_j) - 3 \cos(g - \theta_j) \right],
\]

(B6)

\[
\frac{\partial H_{\text{ext}}}{\partial G} \simeq \frac{G}{2e} \left[ 1 + a^2 - 2a \cos(l + g - \theta_j) \right]^{-3/2} \\
\left[ 2a \cos l + \cos(2l + g - \theta_j) - 3 \cos(g - \theta_j) \right],
\]

(B7)

and

\[
\frac{\partial H_{\text{ext}}}{\partial g} \simeq a \left[ 1 + a^2 - 2a \cos(l + g - \theta_j) \right]^{-3/2} \\
\left\{ \sin(l + g - \theta_j) + \frac{1}{2} e \left[ \sin(2l + g - \theta_j) - 3 \sin(g - \theta_j) \right] \right\} \\
- \frac{3}{4} e a^2 \left[ 1 + a^2 - 2a \cos(l + g - \theta_j) \right]^{-5/2} \\
\left[ 4 \sin l - 2a \sin(g - \theta_j) + 3 \sin(l + 2g - 2\theta_j) \\
- 2a \sin(2l + g - \theta_j) - \sin(3l + 2g - 2\theta_j) \right].
\]

(B8)

Quantities above can be simplified noting that at resonance \(a_0 \simeq 1.625\) (paper I).

Therefore one can use the relation

\[
(1 - 2\xi \cos \alpha + \xi^2)^{-\lambda} = \sum_{q=0}^{\infty} C_q^\lambda (\cos \alpha) \xi^q, \quad |\xi| < 1,
\]

(B9)
with $\xi = a_0^{-1}$ and $\alpha = (l + g - \theta_j)$, to write equations (B5)-(B8) in a simple form. In this relation, $C^\lambda_q(\cos \alpha)$ are Gegenbauer polynomials which are given by

$$C^\lambda_q(\cos \alpha) = \sum_{h=0}^{q} \frac{\Gamma(\lambda + h) \Gamma(\lambda + q - h)}{h!(q-h)! [\Gamma(\lambda)]^2} \cos\left[(q-2h)\alpha\right].$$  \hspace{1cm} (B10)

It is important to note that integration (41) applies certain restrictions to the admissible values of $q$ and $h$. Due to the harmonic nature of the terms produced by equation (B10), the only terms that survive the integration (41) are the ones proportional to $\cos(2l+g-\theta_j)$ and $\sin(2l+g-\theta_j)$. On the other hand, expansion of quantities (B5)-(B8) using Gegenbauer polynomials, will result in producing terms proportional to $\cos[(q-2h)(l+g-\theta_j)]$ and $\sin[(q-2h)(l+g-\theta_j)]$. After simplifying equations (B5) to (B8), one realizes that the only terms that can have non-zero contributions are the ones with angular arguments equal to $l$, $(g-\theta_j)$, $(l+2g-2\theta_j)$, $(2l+g-\theta_j)$ and $(3l+2g-2\theta_j)$ which require $q$ to be equal to $(2h \pm 1)$, $(2h \pm 2)$, $(2h \pm 3)$, $(2h)$ and $(2h \pm 1)$, respectively. Denoting the contributing part of the external Hamiltonian by $\mathcal{H}^*_\text{ext}$, the non-vanishing terms in equations (B5)-(B8) can be written as

$$\frac{\partial \mathcal{H}^*_\text{ext}}{\partial L} \simeq -\frac{1}{2 e a_0^{5/2}} \sigma_{11} \cos(2l+g-\theta_j),$$ \hspace{1cm} (B11)

$$\frac{\partial \mathcal{H}^*_\text{ext}}{\partial l} \simeq \frac{e}{a_0^2} \sigma_{11} \sin(2l+g-\theta_j),$$ \hspace{1cm} (B12)

$$\frac{\partial \mathcal{H}^*_\text{ext}}{\partial g} \simeq \frac{e}{2a_0^2} \left[ \sigma_{22}^{(3/2)} + 3 \sigma_{22}^{(5/2)} \right] \sin(2l+g-\theta_j),$$ \hspace{1cm} (B13)

and

$$\frac{\partial \mathcal{H}^*_\text{ext}}{\partial G} \simeq \frac{G}{2 e a_0^3} \sigma_{11} \cos(2l+g-\theta_j).$$ \hspace{1cm} (B14)

where $\sigma_{11}$, $\sigma_{22}^{(3/2)}$ and $\sigma_{22}^{(5/2)}$ are given by equations (51)-(53). Substituting equations (B11)-(B14) in equations (21)-(25) and integrating the results using integration (41), $\bar{F}_{11}, \bar{F}_{12}, \bar{F}_{22}, \bar{F}_{32}$ and $\bar{F}_{42}$ can be written as in equation (46)-(50).
Appendix C

I have shown in paper I that equations of motion in terms of the Delaunay variables can also be written as

\[
\begin{align*}
\frac{dL}{dt} &= -\varepsilon \frac{\partial H_{\text{ext}}}{\partial l} + \varepsilon \Delta R_L, \\
\frac{dG}{dt} &= -\varepsilon \frac{\partial H_{\text{ext}}}{\partial g} + \varepsilon \Delta R_G, \\
\frac{dl}{dt} &= \frac{1}{L^3} + \varepsilon \frac{\partial H_{\text{ext}}}{\partial L} + \varepsilon \Delta R_l,
\end{align*}
\]  
\quad \text{(C1)}
\]

\[
\begin{align*}
\frac{dg}{dt} &= \varepsilon \frac{\partial H_{\text{ext}}}{\partial G} + \varepsilon \Delta R_g.
\end{align*}
\]  
\quad \text{(C4)}

Comparing equations (C1)-(C4) with equations (A7)-(A10), \( R_L, R_G, R_l \) and \( R_g \) can be written as in equations (27)-(30). After substituting for \( R_x \) and \( R_y \) from equations (A5) and (A6) in these equations, the effect of the frictional force of the interplanetary medium on dynamics of the averaged system appears as

\[
R_L = -\frac{a}{BV_{\text{rel}}^3}(1 - e^2)^{-1/2} \left[ a(1 - e^2)(\dot{\theta} - \omega_m) + e\dot{r}\sin \hat{v} \right] \ln(1 + Br^2V_{\text{rel}}^4),
\]  
\quad \text{(C5)}
\]

\[
R_G = -\frac{r}{BV_{\text{rel}}^3}(\dot{\theta} - \omega_m) \ln(1 + Br^2V_{\text{rel}}^4),
\]  
\quad \text{(C6)}
\]

\[
R_l = -\frac{ra^{-1/2}}{eBV_{\text{rel}}^3} \left[ \dot{r}(-2e \cos \hat{v} + e \cos^2 \hat{v}) - r(\dot{\theta} - \omega_m)(2 + e \cos \hat{v}) \sin \hat{v} \right] \ln(1 + Br^2V_{\text{rel}}^4),
\]  
\quad \text{(C7)}
\]

\[
R_g = \frac{G}{eBV_{\text{rel}}^3} \left[ \dot{r} \cos \hat{v} - r(\dot{\theta} - \omega_m) \left( 1 + \frac{1}{1 + e \cos \hat{v}} \right) \sin \hat{v} \right] \ln(1 + Br^2V_{\text{rel}}^4),
\]  
\quad \text{(C8)}
\]

I have shown in paper I that at resonance, the numerical value of the logarithmic term \( \ln(1 + Br^2V_{\text{rel}}^4) \) becomes so small that one can use the approximation \( \ln(1 + \zeta) \simeq \zeta, \zeta << 1 \) to simplify equations (C5)-(C8). Using this approximation and also substituting for \( \omega_m, \dot{\theta}, \dot{r} \) and \( V_{\text{rel}} \) by (paper I)
\[ \omega_m \simeq a^{-3/2} \left( 1 + \frac{3}{2} e \cos l \right), \quad (C9) \]

\[ \dot{\theta} \simeq a^{-3/2} \left( 1 + 2 e \cos l \right), \quad (C10) \]

\[ \dot{r} \simeq e a^{-1/2} \sin l. \quad (C11) \]

and

\[ \mathcal{V}_{rel} \simeq a^{-1/2} e \left( 1 - \frac{3}{4} \cos^2 l \right)^{1/2}, \quad (C12) \]

respectively, \( \mathcal{R}_L, \mathcal{R}_G, \mathcal{R}_I \) and \( \mathcal{R}_g \) can be written as in equations (42)-(45).