Circular orbits and acceleration of particles by near-extremal dirty rotating black holes: general approach

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Abstract

We study the effect of collisions of ultrahigh energy particles near the black hole horizon (BSW effect) for two scenarios: when one of the particles either (i) moves on a circular orbit or (ii) plunges from it toward the horizon. It is shown that such circular near-horizon orbits can exist for near-extremal black holes only. This includes the innermost stable orbit (ISCO), marginally bound orbit (MBO) and photon one (PhO). We consider generic ‘dirty’ rotating black holes not specifying the metric and show that the energy in the center-of-mass frame has the universal scaling dependence on the surface gravity $\kappa$. Namely, $E_{c.m.} \sim \kappa^{-n}$ where for the ISCO, $n = \frac{1}{3}$ in case (i) or $n = \frac{1}{2}$ in case (ii). For the MBO and PhCO, $n = \frac{1}{2}$ in both scenarios that agrees with recent calculations of Harada and Kimura for the Kerr metric. We also generalize the Grib and Pavlov observations made for the Kerr metric. The magnitude of the BSW effect on the location of collision has a somewhat paradoxical character: it decreases when approaching the horizon.

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1. Introduction

Recent discovery of the unbound growth of energies in collisions of particles near the horizon of the Kerr black hole [1] provoked a series of works in which this phenomenon was investigated in detail (see, e.g., the recent works [2–4] and references therein). I call this effect, found by M Bañados, J Silk and S M West, the BSW effect according to the names of its authors. Irrespective of details, there are some general conditions for the BSW effect to occur. Particles should collide in the vicinity of the horizon in such a way that the special relation between the energy and angular momentum of one of the colliding particles should be satisfied. Both conditions (proximity to the horizon and fine-tuning of the particle’s parameters) are naturally satisfied for the innermost stable circular orbit (ISCO) in the near-extremal case. This kind of orbit was known to be important in astrophysics of accretion disks around black holes [5–7]. Therefore, the high-energetic collisions near such orbits are interesting not only from...
the theoretical viewpoint but can also have astrophysical applications. The details of such collisions are investigated in [2, 3] for the Kerr metric.

Meanwhile, astrophysical black holes are surrounded by matter and are in this sense ‘dirty’. For them, the BSW effect should also occur since it is a property inherent to generic rotating black holes [8]. Although full analysis requires knowledge of the details of the spacetime, some essential features of this effect turn out to be model independent, that can be thought of as the manifestation of universality of black hole physics near the horizon. In this paper, we mainly consider two such issues: (i) the dependence of the effect on the distance from the horizon and (ii) the properties of ISCO and some other circular orbits—such as marginally bound orbit (MBO) and the photon one (PhO) near the horizon for the near-extremal black holes. Originally, the BSW effect was discussed for maximally rotating black holes [1] that entailed some doubts with respect to attempts to relate it to the realistic astrophysical context [9, 10]. Meanwhile, later on, it was shown that the scenario of multiple scattering leads to the possibility of the BSW effect in the Kerr background even for nonextremal black holes [11] that was extended to generic dirty black holes [8].

In this paper the accent is made on the properties of nonextremal black holes, for the case when they become near-extremal, so the surface gravity $\kappa \to 0$. It turns out in this situation that the dependence of the relevant quantities on $\kappa$ is universal and holds for all models. In this way, we find the asymptotic expressions for the characteristics of different types of circular orbits. Using these results, we find the general dependence of the energy $E_{c.m.}$ of colliding particles in the center-of-mass frame on $\kappa$.

Exact values for the characteristics of the circular orbits found in [5] for the Kerr metric are rather cumbersome and even the check of corresponding formulas is not so simple. However, it turns out that if one is interested in the orbits’ characteristics near the horizon of a near-extremal black hole, the behavior of these quantities is rather simple and can be found in a general explicit form.

Quite recently, the BSW effect on the ISCO was also studied for static black holes in a magnetic field with the discussion of potential astrophysical relevance [4]. Although this is a quite different issue to which our approach does not apply directly, this shows potential relevance of the BSW effect in astrophysics and serves as an additional motivation to study the properties of this effect on the near-horizon circular orbits.

In what follows, one should distinguish between two variants of the BSW effect. The first variant implies that a particle orbiting the ISCO (MBO or PhO) collides with some other particle (we call it O-variant from ‘orbiting’). The second variant implies that one of the colliding particles plunges toward the horizon from a circular orbit having the same values of energy and angular momentum which it had there (we call it H-variant from ‘horizon’). In both variants, collisions occur in the immediate vicinity of the horizon but the scenarios of collisions are somewhat different. It is worth noting that the possibility of unbound growth of the energy in the center-of-mass frame due to collision between a particle on the circular orbit and a radially moving one was already briefly mentioned in the literature (see equation (2.55) in [12]).

2. General properties of BSW effect for nonextremal black holes

2.1. Basic formulas

Consider a generic axially symmetric rotating black hole spacetime. Its metric can be written as

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + dl^2 + g_{zz} dz^2.$$  \hspace{1cm} (1)
Here, the metric coefficients do not depend on $t$ and $\phi$. On the horizon, $N = 0$. Alternatively, one can use the coordinates $\theta$ and $r$, similar to the Boyer–Lindquist ones for the Kerr metric, instead of $l$ and $z$. In (1), we assume that the metric coefficients are even functions of $z$, so the equatorial plane $\theta = \frac{\pi}{2}$ ($z = 0$) is a symmetry one. Throughout the paper, we assume that the fundamental constants $G = c = \hbar = 1$.

In the spacetime under discussion, there are two conserved quantities $u_0 \equiv -E$ and $u_\phi \equiv L$ where $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity of a test massive particle, $\tau$ is the proper time (or the affine parameter if the particle is lightlike) and $x^\mu = (t, \phi, l, z)$ are the coordinates. The aforementioned conserved quantities have the physical meaning of the energy per unit mass (or frequency for a lightlike particle) and the azimuthal component of the angular momentum, respectively. It follows from the symmetry reasonings that there exist geodesics in such a background which lie entirely in the plane $\theta = \frac{\pi}{2}$. For them, the equations of motion read (the dot denotes the derivative with respect to the proper time $\tau$)

$$i = u_0 = \frac{E - \omega L}{N^2}.$$  \hfill (2)

We assume that $i > 0$, so that $E - \omega L > 0$ outside the horizon. On the horizon itself, $E - \omega L = 0$ is allowed:

$$\phi = \frac{L}{g_{\phi\phi}} + \frac{\omega(E - \omega L)}{N^2},$$  \hfill (3)

$$N^2 i^2 = -V_{\text{eff}} = (E - \omega L)^2 - bN^2, \quad b = \left(\gamma + \frac{L^2}{g_{\phi\phi}}\right).$$  \hfill (4)

Here, $\gamma = 0$ for lightlike geodesics and $\gamma = 1$ for timelike ones. For definiteness, we consider a pair of particles having the equal masses $m_1 = m_2 = m$. We also assume that either both particles approach the horizon or one of them stands at a fixed $l$ while the other particle moves toward the horizon, so $l \leq 0$. The particles are labeled by the subscript $i = 1, 2$.

The quantity which is relevant for us is the energy in the center-of-mass frame $E_{\text{c.m.}} = \sqrt{2m \sqrt{1 - u_{\mu(1)}u^{\mu(2)}}}$ \cite{1}. After simple manipulations, one obtains from (2)–(4) that

$$\frac{E_{\text{c.m.}}^2}{2m^2} = h + 1 - \frac{L_1 L_2}{g_{\phi\phi}},$$  \hfill (5)

where

$$h = \frac{X_1 X_2 - Z_1 Z_2}{N^2},$$  \hfill (6)

$$X_i \equiv E_i - \omega L_i,$$  \hfill (7)

$$Z_i = \sqrt{(X_i)^2 - N^2 b_i},$$  \hfill (8)

$$b_i = 1 + \frac{L_i^2}{g_{\phi\phi}}.$$  \hfill (9)

By definition, we call a particle critical if $(X_i)_+ = 0$, so

$$E_i = \omega_+ L_i$$  \hfill (10)

and usual otherwise. Here, the subscript ‘$+$’ refers to the values on the horizon.
2.2. Dependence of the BSW effect on proximity to the horizon

It was observed in [14] that, rather surprisingly, the magnitude of the BSW effect in the Kerr background decreases as a point of collision is chosen closer and closer to the horizon. Now we will show that this property has a universal character. We take particle 1 to be near-critical, so

$$E_1 = \omega_1 L_1 (1 + \delta), \; \delta \ll 1.$$  

Particle 2 is assumed to be usual. Near the horizon, $N \to 0$, so

$$Z_2 \approx X_2 \neq 0,$$

$$E_2 \approx \left( X_2 \right)^2 + Y,$$

where

$$X_1 = L(\omega - \omega_1) + \omega_1 L_1 \delta,$$  

$$\omega \sim r - r_+ \sim N^2$$

Then, the first term on the right-hand side of (12) has the order $\delta^2$ and is negligible, so

$$X_1 \approx \omega_1 L_1 \delta.$$

It is convenient to make an ansatz introducing the new function $\chi$ according to

$$N = \omega_1 L_1 + \delta \sqrt{(b_1)} \chi.$$  

Then,

$$Z_1 \approx \omega_1 L_1 \delta \sqrt{1 - \chi^2},$$

$$Y \approx \frac{b_1}{\omega_1 L_1 \delta} f,$$

where

$$f = \frac{1 - \sqrt{1 - \chi^2}}{\chi^2}.$$  

Thus, we have a universal coordinate dependence in terms of the function $f$. It is convenient to make the substitution

$$\chi = \sin \eta, \quad 0 \leq \eta \leq \frac{\pi}{2}.$$  

Then,

$$f = \frac{1}{2 \cos^2 \frac{\eta}{2}}.$$  

In the near-horizon region,

$$b = b_+ \sim O(r - r_+) = b_+ + O(N^2) = b_+ + O(\delta^2).$$  

Thus, neglecting the terms of the order $\delta^2$, we have

$$\frac{E^2}{2m^2} \approx \left( X_1 \right)_+ + \frac{b_+}{\omega_1 L_1 \delta} f(\chi).$$

We obtained a function that (for a fixed small value of $\delta$) monotonically decreases away from the horizon where $\eta = 0$, $f = \frac{1}{2}$, to the turning point of particle 1 where $Z_1 = 0$ and $\eta = \frac{\pi}{2}$, $f = 1$. In this sense, the dependence of the effect on the distance has somewhat paradoxical character: the closer the point of collision is to the horizon, the weaker its magnitude measured by $f$. This generalizes the corresponding observation made for the Kerr metric in [14].
3. Circular orbits

We restrict ourselves to orbits in the equatorial plane. It is the properties of circular orbits which we now turn to. It is convenient to introduce, instead of the proper distance $l$, the radial coordinate $\rho$ according to $dl = \frac{d\rho}{\sqrt{-g}}$. By definition, the circular orbit at $\rho = \rho_0$ is determined by the equalities

$$V_{\text{eff}}(\rho_0) = 0,$$

$$\frac{dV_{\text{eff}}}{d\rho}(\rho_0) = 0,$$  

where according to (4) and (8),

$$V_{\text{eff}} = -Z^2.$$  

These equations state that $\rho_0$ is a perpetual turning point. It follows from (4), (8) and (22) that for a particle on such an orbit,

$$X_+ = \sqrt{b}N.$$  

If this orbit lies near the horizon (see below for details), it follows from (3) that

$$\dot{\phi} \approx \frac{\omega_+ \sqrt{b}}{N} > 0$$  

assuming $\omega_+ > 0$. Thus, near the horizon, a particle rotates on the prograde orbit.

3.1. Nonexistence of near-horizon circular orbits for generic nonextremal rotating black holes

First of all, we show that for a nonextremal black hole with a fixed surface gravity $\kappa \neq 0$, there are no circular orbits in the near-horizon limit, thus generalizing the observation of [14] made there for the Kerr metric. It follows from (4) that

$$-\frac{1}{2} \frac{dV_{\text{eff}}}{d\rho} = \left[ -(E - \omega L)L \frac{d\omega}{d\rho} - \frac{dN}{dl} b - \frac{N^2}{2} \frac{db}{d\rho} \right].$$  

In the horizon limit $N \rightarrow 0$, $\frac{dN}{dl} \rightarrow \kappa$. It follows from (4) and (22) that in this limit,

$$X_+ = E - \omega_+ L \rightarrow 0.$$  

We obtain that

$$-\frac{1}{2} \frac{dV_{\text{eff}}}{d\rho} \rightarrow -b_+ \kappa \neq 0$$  

in contradiction with (23) that proves the statement.

In what follows, we will consider a near-extremal black hole, with small but nonzero $\kappa$. When $\kappa$ itself tends to zero, the aforementioned general prohibition does not work and circular orbits in the near-horizon region can exist. Now we will examine different kinds of orbits separately.

3.2. Innermost stable circular orbit

This is the circular orbit closest to the horizon on the threshold of the stability. Correspondingly, we must add to equations (22) and (23) also

$$\frac{d^2V_{\text{eff}}(\rho_0)}{d\rho^2} = 0.$$  

5
Now we will find explicit formulas for the metric and dynamic characteristics of a particle on the ISCO near the horizon of the nearly extremal rotating black hole. We can use the power expansion in terms of \( x = \rho_0 - \rho_+ \) near the horizon \( \rho = \rho_+ \):

\[
N^2 = 2\kappa x + Dx^2 + Cx^3 + \cdots ,
\]

\[
\omega = \omega_+ - B_1 x + B_2 x^2 + \cdots ,
\]

\[
L = L_0 + ax + \cdots .
\]

(31)

(32)

(33)

Let us recall that the coefficients entering expansion (31) are related to the fixed coordinate gauge in which the coefficient at \( d\rho^2 \) is equal to \( N - 1 \).

It is instructive to stress that an orbit for which calculations are being carried out lies outside the horizon. Moreover, although formally \( \rho_0 - \rho_+ \to 0 \) when \( \kappa \to 0 \), the proper distance from the horizon does not vanish and even may be large (see section 4 below for details). Therefore, \( L_0 \) is not the value of the angular momentum on the horizon (where a massive particle cannot be situated at all). Rather, it gives the value of the momentum on the near-horizon orbit under discussion in the main approximation, while the terms of the order of \( x \) and higher give corrections to it. In doing so, \( x \) is small but cannot vanish on the ISCO; our goal is just to find it (see equation (42) below and explicit comparison with the Kerr case in equation (77)).

We have two small values: \( x \) and \( \kappa \). We are interested in the near-extremal limit. By definition, its very meaning is in that in (31) the first term is small as compared to the second one:

\[
\kappa \ll Dx.
\]

(34)

Correspondingly, in the region under discussion,

\[
N = x\sqrt{D} + \frac{\kappa}{\sqrt{D}} - \frac{\kappa^2}{2D^{3/2}x} + \frac{C}{2\sqrt{D}}x^3 + \cdots .
\]

(35)

We substitute aforementioned expansions into (22), (23) where (24), (8) are used. Then, in this limit, calculating \( dV_{\text{eff}}/d\rho \) and neglecting high-order corrections, we have

\[
-\frac{1}{2} \frac{dV_{\text{eff}}}{d\rho}(\rho_0) \approx A_0 \kappa + A_1 x + A_2 x^2 + A_3 \frac{\kappa^2}{x} = 0
\]

(36)

with coefficients which are given below. Equation (30) with (22), (25) taken into account gives us

\[
-\frac{1}{2} \frac{d^2V}{d\rho^2}(\rho_0) = B_1^2 L^2 - bD - F x + O(x^2, \kappa x) = 0,
\]

(37)

where

\[
F = 4B_1 B_2 L^2 + 2L\sqrt{b_0 b_2} \sqrt{D} + 2Db' + 3bC.
\]

(38)

We must substitute into (36) the relationships that follow from equations (22), (37) and (38). After some algebraic manipulations, one finds

\[
A_0 = A_1 = 0,
\]

(39)

\[
A_2 = b_0 b_2 B_1 + b'_0 D - b_0 C,
\]

(40)

\[
A_3 = -\frac{b_0}{2D}.
\]

(41)
\[ b_0 \equiv b(\rho_0). \] Then, it follows from equation (36) that
\[ x \approx H \sqrt[3]{\kappa}, \quad H = \left( \frac{b_0}{2DA_2} \right)^{1/3}. \] (42)

To express \( L_0 \) entirely in terms of the metric coefficients on the horizon, one should take into account the definition of \( b \) in equations (9) and (37). As a result, we have
\[ L \approx L_0 + \frac{FH}{2 \sqrt{D \left( B_1^2 - \frac{D}{P} \right)}} \kappa^{1/3}, \] (43)
\[ L_0^2 = \frac{D}{B_1^2 - \frac{D}{P}} \quad b_+ = \frac{B_1^2(\tilde{g}_{\phi\phi})_+}{B_1^2(\tilde{g}_{\phi\phi})_+ - D}. \] (44)

Equations (22), (35) and (42) give us
\[ X = \sqrt{b_0}N \approx \sqrt{b_0} \sqrt{DH} \kappa^{1/3}, \] (45)
so on the ISCO,
\[ X \sim N \sim \kappa^{2/3}. \] (46)

Thus, the quantity \( X \) that according to (7) and (10) measures the deviation of particle’s parameters from the criticality has a universal dependence on the surface gravity irrespective of the concrete form of the metric.

### 3.3. Marginally bound orbit

Another type of circular orbit is a so-called marginally bound one (MBO). This means that a particle satisfying equations (22) and (23) has the zero velocity at infinity, so the energy per mass \( E = 1 \). We are interested in the prograde MBO only since for \( \rho_0 \to \rho_+ \) in the extremal limit. After some algebra, one finds from (4), (22), (23), (31) and (32) that now for small \( \kappa \),
\[ x \approx \kappa \alpha, \] (47)
where the coefficient \( \alpha \) is equal to
\[ \alpha = \frac{1}{D} \left( \frac{1}{\sqrt{1 - \frac{D}{P}}} - 1 \right), \] (48)
\[ 1 \equiv \frac{1}{P} \bigg[ \frac{1}{(\tilde{g}_{\phi\phi})_+} + \omega_+^2 \bigg] = \frac{\omega_+^2}{\omega_+^2} \] (49)
\[ E = \omega_+ + \frac{\kappa B_1}{\omega_+ \sqrt{P \sqrt{P - D}}} > 0, \] (50)
\[ N^2(\rho_0) \approx \kappa^2 \frac{1}{P - D}. \] (51)
\[ L = \frac{1}{\omega_+} - \kappa s, \quad s = \frac{B_1}{D \omega_+^2} \left( 1 - \sqrt{1 - \frac{D}{P}} \right) > 0, \] (52)
\[ b_+ = 1 + \frac{1}{(\tilde{g}_{\phi\phi})_+ \omega_+^2} = \frac{B_1^2}{\omega_+^2} P. \] (53)
In this case, equation (25) gives us
\[ X \approx \sqrt{bN} \approx \frac{B_1}{\omega_+ \sqrt{P}} \frac{\kappa}{\sqrt{P - D}}, \]  
so
\[ X \sim N \sim \kappa. \]  

### 3.4. Photon orbits (PhO)

Now, for massless particles (photons), one must put \( \gamma = 0 \) in (4). Then, we have for the reduced potential
\[ \tilde{V} \equiv \frac{\sqrt{\Phi}}{L} \]
\[ \tilde{V} = (\tilde{E} - \omega)^2 - \frac{N^2}{g_{\phi\phi}} \]
\[ \tilde{E} = \frac{E}{L}. \]  

For the photon circular orbit, \( \tilde{V}(\rho_0) = \tilde{V}'(\rho_0) = 0 \), whence at \( \rho = \rho_0 \), we have
\[ (\tilde{E} - \omega)^2 - \frac{N^2}{g_{\phi\phi}} = 0, \]
\[ 2(\tilde{E} - \omega) \frac{d\rho}{d\rho} + \left( \frac{N^2}{g_{\phi\phi}} \right)' = 0. \]  

One can verify that for the Schwarzschild case, one obtains from here the well-known result \( \rho_0 = 3M \).

For a near-extremal black hole, in the near-horizon region, it is easy to check that in expansion (31), the first and second terms have the same order, \( x = \kappa \alpha \), where in the main approximation, \( \alpha \) does not contain \( \kappa \). Collecting in equations (57) and (58) all terms of the first order in \( \kappa \) and solving the corresponding equation with respect to \( \alpha \), one obtains that equations (48) and (51) are now satisfied with
\[ P = \frac{B_1^2 (g_{\phi\phi})_+}{2}. \]

Then, we obtain for \( \tilde{E} \) the general expression
\[ \tilde{E} = \omega_+ + \kappa B_1 \left( 1 - \frac{\sqrt{P - D}}{P} \right) > 0. \]  

\[ X \approx \frac{L}{\sqrt{(g_{\phi\phi})_+}} \frac{\kappa}{\sqrt{P - D}} = \frac{B_1 L \kappa}{\sqrt{P - D}}. \]  

Now dependence (55) still holds.

### 4. Proper distance

All three kinds of orbits have \( \rho \approx \rho_+ \). However, they are separated spatially since the proper distance between them does not vanish. This generalizes the corresponding properties of the Kerr metric [5]. Namely, between the horizon and the MBO or PhO, writing \( \rho = \rho_+ = \kappa y \), we have
\[ l = \int \frac{d\rho}{N} \approx \int_0^{\alpha} \frac{dy}{\sqrt{2 y + D y^2}} = \frac{1}{\sqrt{D}} \ln \frac{\sqrt{P + D}}{\sqrt{P - D}}, \]  
where equation (48) was taken into account.

Between the horizon and the ISCO,
\[ l \approx \frac{1}{\sqrt{D}} \ln \frac{\rho_0 - \rho_+}{\kappa} \approx \frac{1}{3\sqrt{D}} \ln \frac{1}{\kappa}. \]
5. Near-extremal Kerr

It is instructive to check these formulas for the Kerr case. Then,
\[
\frac{d\rho^2}{N^2} = r^2 \frac{dr^2}{\Delta}, \quad \Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-),
\]
where \( r \) is the Boyer–Lindquist coordinate; \( r_{\pm} = M \pm \sqrt{M^2 - a^2} \), \( M \) is the black hole mass and \( a \) characterizes its angular momentum. Then,
\[
N^2 = \frac{(r - r_+)(r - r_-)}{r^2 + a^2 + \frac{2Ma^2}{r}}, \quad g_{\phi\phi} = r^2 + a^2 + \frac{2Ma^2}{r},
\]
\[
\frac{d\rho}{dr} = \frac{r}{\sqrt{r^2 + a^2 + \frac{2Ma^2}{r}}},
\]
\[
\omega = \frac{2Ma}{r^3 + a^2 r + 2Ma^2}, \quad \omega_+ = \frac{a}{2Mr_+},
\]
\[
\kappa = \frac{\sqrt{M^2 - a^2}}{2Mr_+}.
\]

In the near-extremal case
\[
a = M(1 - \epsilon),
\]
\( \epsilon \ll 1 \),
\[
\kappa \approx \frac{\sqrt{\epsilon}}{\sqrt{2M}},
\]
the horizon radius
\[
r_+ \approx M(1 + \sqrt{2\epsilon}),
\]
and the horizon angular velocity
\[
\omega_+ \approx \frac{1}{2M}(1 - \sqrt{2\epsilon}).
\]

The relevant coefficients defined above are in the main approximation equal to
\[
C = 0, \quad D = \frac{1}{M^2}, \quad B_1 = \frac{1}{M^2}, \quad B_2 = \frac{1}{2M^3}, \quad H = M^{5/3}, \quad F = \frac{4}{M^3}.
\]

Now we can obtain the radius \( r_0 \) and other main characteristics of all circular orbits near the horizon.

Innermost stable circular orbit. From (42), it follows that, neglecting high-order corrections,
\[
b_0 = \frac{4}{3}, \quad b_0' = 0, \quad A_2 = \frac{2}{3M^3}, \quad H^3 = M^5
\]
\[
\frac{r_0 - M}{M} \approx 2^{2/3} \epsilon^{1/3},
\]
\[
l \approx \frac{M}{6} \ln \frac{1}{\epsilon},
\]
\[
L \approx \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} 2^{2/3} \epsilon^{1/3},
\]
(77)
\[ X \approx \frac{2^{2/3}}{\sqrt{3}} e^{1/3}. \]  

(78)

**Marginally bound orbit.** Taking into account equations (70), (71), (66), (67) and (72), one finds that

\[ P = \frac{2}{M^2}, \quad \alpha = M^2(\sqrt{2} - 1), \quad s = \frac{\sqrt{2} - 1}{\sqrt{2}}, \]

(79)

\[ r - M \approx 2M\sqrt{\varepsilon}, \]

(80)

\[ \frac{L}{M} \approx 2 + 2\sqrt{\varepsilon}, \]

(81)

\[ E = 1, \quad \frac{L}{E} \approx 2M. \]

(82)

The proper distance from the horizon is

\[ l \approx M \ln(1 + \sqrt{2}). \]

(83)

**Photon one.** Now \( P = \frac{4}{M}, \quad \alpha = M\left(\frac{2}{\sqrt{3}} - 1\right) \) and, taking into account equation (71), we obtain

\[ r_0 - M = 2M\sqrt{\frac{2}{3}}\sqrt{\varepsilon}, \]

(84)

\[ \tilde{E} = \frac{E}{L} = \frac{1}{2M}. \]

(85)

In the limit under discussion, the proper distance between the PhO and MBO equals in this limit

\[ l = M \ln \frac{1 + \sqrt{2}}{\sqrt{3}}. \]

(86)

Equations (75)–(86) agree with equations (2.22) and (2.23) of [5]. Equation (78) agrees with equations (4.6) and (4.7) of [2] and equation (77) reproduces the first two terms of equation (4.7) of [2].

6. **O-variant of BSW effect: collisions on circular orbits**

Now we apply the results for the characteristics of circular orbits to the collisions and elucidate the properties of the BSW effect for generic near-extremal black holes. As for any circular orbits \( Z = 0, \) in the near-horizon limit, this entails that \( X \rightarrow 0 \) according to (8) and (22). The quantity \( h \) that enters the expression for energy (5) simplifies

\[ h = \frac{X_1 X_2}{N^2}. \]

(87)

It is worth noting that the dependence of the energy on the parameters of two particles has been factorized. Now, from the results of the previous sections, we have already all means to find the asymptotic behavior of \( h \) for small \( \kappa. \) When \( h \rightarrow \infty, \) for the collision of two massive particles, \( E_{c.m.} \approx m\sqrt{2h} \) according to equation (5).

Further, we will consider different kinds of circular orbits separately.
6.1. Innermost stable circular orbit

6.1.1. Both particles move on the ISCO. Then, it follows from (46) that the quantity $h$ is finite and so is the energy $E_{c.m.}$. Thus, the BSW effect is absent. This can be easily understood from the previous general results. Namely, a particle moving on the ISCO is necessarily near-critical since $X \to 0$ for it. Meanwhile, the collision between two critical particles cannot produce the BSW effect [8, 13].

6.1.2. One particle is on the ISCO, the second particle is usual. Let particle 1 be on the ISCO and particle 2 be usual. Then, it follows from (46) that $X_1 \sim \kappa^{2/3} \sim N$, so (5) gives us

$$E_{c.m.} \approx U m \sqrt{2X_2 \kappa^{-1/3}} \sim \kappa^{-1/3}. \quad (88)$$

Here, the coefficient $U$ can be found from (35), (42) and (45):

$$U = \left(\frac{b_0}{D}\right)^{1/12} (2\Lambda_2)^{1/6}. \quad (89)$$

6.2. Marginally bound orbit

6.2.1. Both particles on the MBO. It is seen from (55) that $X_1 \sim X_2 \sim \kappa^{1/2} \sim N$. Therefore, $c$ is finite and the BSW effect is absent. Explanation is similar to that for the ISCO.

6.2.2. One particle is on the MBO, the second particle is usual. Now we have from (54), (53) and (5) that

$$E_{c.m.} \approx V m \sqrt{2X_2 \kappa^{-1/2}} \sim \kappa^{-1/2}, \quad (90)$$

$$V = \sqrt{\frac{B_1}{m \omega_s}} \left(1 - \frac{D}{\bar{F}}\right)^{1/4}. \quad (91)$$

6.3. Photon one

We assume that the second particle is massive with the mass $m$, so we deal with the collision between massive and massless particles [13]. In the present situation, a photon is already critical. For collisions between such particles, the formula for the energy $E_{c.m.}$ preserves its general structure but now the expression for $b$ changes for the photon that affects also the expression for $Z$. Now, in equation (8), $b = \frac{L_2}{L_1}$ (see section 6 of [8] for details). The situation does not differ qualitatively from that for MBO.

6.3.1. Massive particle is critical. Using equation (55), we see that the energy turns out to be finite in accordance with the general conclusion of [13] that two critical particles are unable to produce the BSW effect.

6.3.2. Massive particle is usual. Now, according to [13], equation (5) is somewhat changed:

$$\frac{E_{c.m.}^2}{2m^2} = \frac{1}{2} + \frac{1}{m} \left(\frac{h - L_1 L_2}{8 \phi \phi}\right). \quad (92)$$

where $c$ is given by the same expression (6). Then, the dependence of the energy $E_{c.m.}$ on $\kappa$ in the limit under discussion is the same as in (90) although with another coefficients:

$$E_{c.m.} \approx W \sqrt{2mX^2 \kappa^{-1/2}} \sim \kappa^{-1/2}, \quad (93)$$
\[ W = \sqrt{B_1 L \left( 1 - \frac{D}{P} \right)^{1/4}}. \]  

(94)

One should bear in mind that for massive particles, \( L \) has the dimension of \( M \), while for massless ones, \( L \) does not have the dimension of \( M \) since it enters the combination \( \nu - \omega L \), where \( \nu \) is the photon frequency.

7. H-variant of BSW effect: collisions of particles plunging from circular orbits

In the previous section, we dealt with the situation when a particle moving along the circular near-horizon orbit collided with some usual particle (O-variant of the BSW effect, according to our definition). Now we consider somewhat another scenario: both particles collide on the horizon, one of them arrived there from a circular orbit (H-variant of the BSW effect, according to our definition). It means that it spent some time on that orbit and, due to the instability of the orbit or being on the threshold of stability, moved toward the horizon with the same values of the energy and momentum which it had at the circular orbit. Mathematically, it corresponds to taking the horizon limit \( N \to 0 \) first in formula (5) for \( E_{\text{c.m.}} \). Then, one can derive the general expression which can also be taken directly from equation (9) of [8]:

\[
\frac{E_{\text{c.m.}}}{m} \approx \sqrt{\frac{b_1 Y_2}{Y_1}}. 
\]  

(95)

Here,

\[
Y_i = E_i - \omega_s L_i. 
\]  

(96)

The quantities \( Y_i \) differ slightly from \( X_i \) defined in (7):

\[
Y_i = X_i + (\omega - \omega_s) L_i. 
\]  

(97)

For a usual particle 2, \( X_2 \neq 0 \), so we can safely neglect the second small term on the right-hand side of (97), whence \( Y_2 \approx X_2 \). For near-critical particle 1 with small \( X_1 \), the situation is much more subtle and one should be careful keeping both terms. As usual, we consider different kinds of circular orbits separately.

7.1. Innermost stable circular orbit

It follows from (25), (35) and (32) that

\[
Y_1 = \sqrt{bN} + (\omega - \omega_s) L_1 = \left( \sqrt{b} \sqrt{D} - L_1 \right) x_0, 
\]  

(98)

where \( L_0 \) is the value of the angular momentum on the ISCO. Meanwhile, it follows from (43) and (44) that in the main approximation, \( \sqrt{b} \sqrt{D} - L_1 = 0 \). Thus, both terms in \( Y_1 \) mutually cancel. The main nonvanishing correction comes from the second term in (35). Then, we have

\[
Y_1 \approx \sqrt{\frac{b}{\sqrt{D} \kappa}}. 
\]  

(99)

By substitution into (95), we obtain that

\[
E_{\text{c.m.}} \approx m \sqrt{\frac{b_1 Y_2}{Y_1}} \approx m \sqrt{X_2 (bD)^{1/4} \kappa^{-1/2}}. 
\]  

(100)
7.2. Marginally bound orbit

Now it follows from (49)–(52) and (97) that

$$ Y_1 = \kappa \omega_+ + \lambda, \quad (101) $$

$$ \frac{E_{c.m.}}{m} = A_{\text{mbo}} \sqrt{X_2} \kappa^{-1/2} \quad (102) $$

$$ A_{\text{mbo}} = \sqrt{\frac{DB_1}{\omega_+ B_1}} \left( 1 - \sqrt{1 - \frac{D}{P}} \right)^{-1/2}. \quad (103) $$

The ratio of $E_{c.m.}$ to that for the MBO is equal to

$$ \mu = \left( \frac{b}{D} \right)^{1/4} \sqrt{\frac{\omega_+ B_1}{B_1}} \left( 1 - \sqrt{1 - \frac{D}{P}} \right)^{1/2}. \quad (104) $$

7.3. Photon one

In a similar way, we obtain expression (102) but now with another coefficient

$$ A_{\text{pho}} = \sqrt{LDB_1} \left( 1 - \sqrt{1 - \frac{D}{P}} \right)^{-1/2}. \quad (105) $$

8. Comparison with the Kerr metric

In the particular case of the Kerr metric, using (70) and concrete values of the coefficients in (73) and (79), one can obtain from equations (5)–(89) that

$$ \frac{E_{c.m.}}{2m} \approx \beta (2\varepsilon)^{-1/6}, \quad (106) $$

$$ \beta = 2^{-1/6} 3^{-1/4} \sqrt{2X_2}. \quad \text{This coincides with equation (5.1) of [2].} $$

Equation (100) gives us

$$ \frac{E_{c.m.}}{m} \approx \left( \frac{2}{3} \right)^{1/4} \sqrt{2\varepsilon X_2} \varepsilon^{-1/4} \quad (107) $$

that agrees with equation (4.8) of [2]. Equation (104) gives us $\mu = 3^{-1/4} (2 - \sqrt{2})^{1/2}$ that also agrees with the value listed in [2] after equation (5.1). One should bear in mind that the term ‘subcritical’ of [2] corresponds to the ‘usual’ of our paper.

9. Summary and conclusion

Thus, we obtained some results valid for generic dirty rotating black holes that generalize previous observations for the Kerr metric [5, 2, 14]. These results apply to the properties of the near-horizon region and the BSW effect that can occur there. They include (i) general statements about circular orbits in the near-horizon region and the BSW effect for generic nonextremal black holes, (ii) and the properties of the near-extremal black holes. In point (i), it is shown that (a) there are no circular orbits near the horizon of a nonextremal black hole with a finite nonzero surface gravity $\kappa$ and (b) in the near-horizon region, for a fixed small deviation of the angular momentum from the critical value, the magnitude of the BSW effect grows from the horizon to the turning point.
If $\kappa \to 0$, a black hole is near-extremal, and circular orbits do exist. In point (ii), for each type of circular orbit, we showed that its parameters depend on $\kappa$ in a universal way and investigated two variants of the BSW effect—when one of the colliding particles moves on such a circular orbit (O-variant) or plunges from it toward the horizon where it collides with a usual particle (H-variant). The dependence of the collision energy $E_{\text{c.m.}}$ on $\kappa$ is found. It turned out that it has the general character of the scaling law $E_{\text{c.m.}} \sim \kappa^{-n}$ where $n$ depends both on the type of the orbit and on the variant of the BSW effect. For the ISCO, $n = \frac{1}{3}$ for the O-variant and $n = \frac{1}{2}$ for the H-variant. This difference was called ‘quite intriguing’ in [2] for the Kerr metric. Now we see that it reveals itself for a generic dirty rotating near-extremal black hole. For the MBO and PhO, $n = \frac{1}{2}$ in both variants. Thus, the universality of black hole physics revealed itself not only in the very existence of the BSW effect for generic dirty black holes but also in its properties.

These results extend the potential relevance of the BSW effect for astrophysics from the Kerr metric to dirty black hole since, say, the ISCO is an example of how a particle’s energy and angular momentum can be fine-tuned naturally in accretion disks with electromagnetic radiation or in inspiralling binaries [2].

In this paper, our generalization of [2] concerned the properties of the BSW effect for particles moving in the equatorial plane. Meanwhile, the BSW effect takes place in the Kerr background not only on the equator but also on some finite belt around it [15]. It would be of interest to generalize the properties of the BSW effect to nonequatorial motion of particles near dirty rotating black holes.

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