STRONG APPROXIMATION AND DESCENT

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ABSTRACT. We introduce descent methods to the study of strong approximation on algebraic varieties. We apply them to two classes of varieties defined by $P(t) = N_{K/k}(z)$: firstly for quartic extensions of number fields $K/k$ and quadratic polynomials $P(t)$ in one variable, and secondly for $k = \mathbb{Q}$, an arbitrary number field $K$ and $P(t)$ a product of linear polynomials over $\mathbb{Q}$ in at least two variables. Finally, we illustrate that a certain unboundedness condition at archimedean places is necessary for strong approximation.

CONTENTS

1. Introduction 1
2. Strong approximation on singular varieties 5
3. A descent lemma 8
4. Quadratic polynomials represented by quartic norms 9
5. Totally split polynomials represented by norms 17
6. A counterexample 26
References 28

1. INTRODUCTION

Let $k$ be a number field. We study strong approximation with Brauer–Manin obstruction for two families of algebraic varieties $X \subset \mathbb{A}^{n+s}_k$ defined by equations of the form

$$P(t) = N_{K/k}(z),$$

(1.1)

where $P(t) \in k[t_1, \ldots, t_s]$ is a polynomial in $s$ variables over $k$ and $N_{K/k}$ is a norm form for an extension $K/k$ of degree $n$.

In our proofs, we use descent to reduce the problem to strong approximation on their universal torsors. While descent has been applied frequently to weak approximation, the precise formulation of our descent lemma seems to be crucial for its first applications to strong approximation.

To prove strong approximation on universal torsors, we reduce it to quadrics by the fibration method in one case, and we generalize a result of Browning and Matthiesen, based on methods of Green and Tao from additive combinatorics, in the other case.

In our results, we encounter an unboundedness condition at the archimedean places. We give a counterexample to strong approximation that shows that such a condition is generally necessary.

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Background. Let $X$ be an algebraic variety defined over a number field $k$. We say that strong approximation holds for $X$ off a finite set $S$ of places of $k$ if the image of the set $X(k)$ of rational points on $X$ is dense in the space $X(A^S_k)$ of adelic points on $X$ outside $S$. Strong approximation for $X$ off $S$ implies the Hasse principle for $S$-integral points on any $S$-integral model of $X$.

For a proper variety $X$, strong approximation off $S = \emptyset$ is the same as weak approximation. For an affine variety $X$, however, studying strong approximation and the Hasse principle for integral points on integral models seems to be generally much harder than studying weak approximation and the Hasse principle for rational points on proper models of $X$.

Failures of weak approximation and the Hasse principle for rational points on proper varieties are often explained by Brauer–Manin obstructions, introduced by Manin [Man71]. Only recently this was generalized to strong approximation by Colliot-Thélène and Xu [CTX09].

For algebraic groups and their homogeneous spaces, weak and strong approximation and the Hasse principle have been widely studied. For certain simply connected semisimple groups and their principal homogeneous spaces, we have the classical work of Kneser, Harder, Platonov and others. For certain homogeneous spaces of connected algebraic groups with connected or abelian stabilizers, see for example [Bor96] [CTX09] [Har08] [Dem11] [WX12] [BD13] [WX13] and the references therein for weak and strong approximation with Brauer–Manin obstruction. This includes varieties defined by (1.1) for constant $P$ and arbitrary $K/k$, and for $s = 1$ with $\deg(P(t)) = 2$ and $[K:k] = 2$.

Much less is known for more general varieties that are not homogeneous spaces of algebraic groups. For weak approximation with Brauer–Manin obstruction, let us mention the now classical example of Châtelet surfaces [CTSSD87a] [CTSSD87b], which are actually smooth proper models of certain varieties defined by (1.1). See [DSW12] [Wei12] and the references therein for weak approximation results for several further classes of varieties defined by (1.1).

A recent breakthrough is the introduction of deep results from additive combinatorics due to Green–Tao and Matthiesen to deduce weak approximation for varieties of the form (1.1) when $P(t)$ is a product of arbitrarily many linear polynomials over $\mathbb{Q}$ [BMS12] [HSW13] [BM13].

For strong approximation with Brauer–Manin obstruction for affine varieties defined by equations of the form

$$P(t) = q(z_1, z_2, z_3),$$

see [CTX13]. For more general fibrations over $\mathbb{A}^1_k$ with split (e.g., geometrically integral) fibers, see [CTH12].

Colliot-Thélène and Harari [CTH12] p. 4] ask for the integral Hasse principle and strong approximation for the equation (1.1) with $s = 1$, a separable polynomial $P(t)$ of degree at least 3, and $[K:k] = 2$. They say that this is out of reach of the current techniques because of the following two essential difficulties: on the one hand, $\text{Br}(X_t)/\text{Br}(k)$ is infinite for each smooth $k$-fiber $X_t$ for the natural fibration $X \to \mathbb{A}^1_k$ via projection to the $t$-coordinate,
and on the other hand, the fibers over the roots of $P(t)$ are not split. Note that the same difficulties occur for $P(t)$ of degree at least 2, and $[K:k] \geq 3$.

Counterexamples to strong approximation explained by Brauer–Manin obstructions can be found in [KT08, Gun13]. See also [CTW12] for computations of Brauer–Manin obstructions for integral points on certain cubic surfaces.

Our results. We obtain the first strong approximation results for varieties defined by (1.1). Here, both of the difficulties occur that were mentioned in [CTH12, p. 4]. In our main theorems, we consider two families of such varieties. Our notation is mostly standard; see the end of the introduction for a reminder.

**Theorem 1.1.** Let $K/k$ be an extension of number fields. Assume that $P(t) = c(t^2 - a) \in k[t]$ is an irreducible quadratic polynomial, and $[K:k] = 4$ with $\sqrt{a} \in K$. Let $X \subset \mathbb{A}_k^n$ be defined by (1.1) with $s = 1$.

Assume that there is an archimedean place $v_0$ such that $p(X(k_{v_0}))$ is not bounded. Then strong approximation with Brauer–Manin obstruction holds for $X$ off $v_0$.

We also compute $\text{Br}(X)$ for $X$ as in Theorem 1.1, and in certain cases we deduce the Hasse principle for integral points and strong approximation (without Brauer–Manin obstructions), see Corollaries 4.6 and 4.7.

**Theorem 1.2.** Let $k = \mathbb{Q}$. For $s \geq 2$, let $P(t) \in \mathbb{Q}[t_1, \ldots, t_s]$ be a product of pairwise proportional or affinely independent linear polynomials over $\mathbb{Q}$. Let $K/\mathbb{Q}$ be an arbitrary extension of number fields of degree $n$. Let $X \subset \mathbb{A}_\mathbb{Q}^{n+s}$ be the affine variety defined by (1.1).

Let $C$ be the union of the connected components of $p(X_{\text{sm}}(\mathbb{R}))$ that contain balls of arbitrarily large radius, where $p : X \to \mathbb{A}_k^n$ is the projection to the $t$-coordinates. Then $X_{\text{sm}}(k)$ is dense in the image of $(p^{-1}(C) \times X_{\text{sm}}(\mathbb{A}_k^n))^{\text{Br}}$ in $X(\mathbb{A}_k^n)$.

If $K$ is not totally imaginary or if the factors of $P(t)$ are linear forms, then $C = p(X_{\text{sm}}(\mathbb{R}))$, and $X$ satisfies smooth strong approximation with algebraic Brauer–Manin obstruction off $\infty$ (see Definition 2.1).

Furthermore, we can deduce a smooth Hasse principle with Brauer–Manin obstruction for integral points on $X$ as in Theorem 1.2. See Corollary 5.6 for details.

**Techniques.** There are two fundamental techniques to reduce the study of weak approximation and the Hasse principle for rational points (possibly with Brauer–Manin obstruction) for one class of varieties to the same questions for other varieties, namely the fibration method and the descent method. One may ask whether these techniques can also be applied in the context of strong approximation.

The fibration method typically applies to fibrations $f : X \to Y$ where weak approximation or the Hasse principle is known both for the fibers of $f$ and the base $Y$. An example is the deduction of the Hasse principle for quadratic forms in five variables from the four-variable-case. See [CTSSD87a, CTSSD87b] for more involved applications, and [Har94] for its combination with Brauer–Manin obstructions.
Generalizing the fibration method to strong approximation is achieved in some generality in [CTX13, CTH12].

The descent method reduces the study of weak approximation and the Hasse principle with (algebraic) Brauer–Manin obstruction to their study on torsors under tori over the original variety (for example universal torsors). This is expected to simplify the task since typically no (algebraic) Brauer–Manin obstruction occurs on universal torsors. See [CTCS80, CTSSD87a, CTSSD87b] for applications of descent to weak approximation.

To our knowledge, Theorems 1.1 and 1.2 are the first applications of the descent method to strong approximation. For this, an important auxiliary result is the descent lemma presented in Section 3. A subtle point was to find the right formulation that makes it applicable in practice. Then the proof of this lemma is an easy application of descent theory as introduced by Colliot-Thélène and Sansuc, see [CTS87].

Our descent lemma reduces strong approximation with algebraic Brauer–Manin obstruction on the original variety to strong approximation on auxiliary varieties containing open subsets of universal torsors.

For $X$ as in Theorem 1.1 we have determined a local description of universal torsors in [DSW12]. Here, we observe that these are essentially fibrations over $\mathbb{A}^k$ whose fibers are smooth 3-dimensional quadrics, so that an application of the fibration method yields the result.

For $X$ as in Theorem 1.2 we show that the universal torsors are essentially varieties for which weak approximation was proved in recent work of Browning and Matthiesen [BM13] (on $X$ defined over $k = \mathbb{Q}$ by (1.1) in cases where $P(t)$ is totally split over $\mathbb{Q}$, with $s = 1$), based on results from additive combinatorics by Green–Tao and Matthiesen. We generalize the key technical result [BM13, Theorem 5.2] of Browning and Matthiesen from linear forms to linear polynomials, and we deduce that the varieties containing our universal torsors also satisfy strong approximation.

**An unboundedness condition.** We observe that Theorems 1.1 and 1.2 include unboundedness conditions at an archimedean place. This has some resemblance with conditions at archimedean places appearing in strong approximation results on homogeneous spaces of algebraic groups, e.g., [PR94, Theorem 7.12].

Example 6.2 shows that strong approximation with Brauer–Manin obstruction off $\infty$ does not hold for the variety $X \subset \mathbb{A}^3$ defined by

$$t(t - 2)(t - 10) = x^2 + y^2,$$

which is an example of (1.1). Here, $X(\mathbb{R})$ has a bounded and an unbounded connected component.

This counterexample and Theorems 1.1 and 1.2 lead us to the expectation that only the following version of smooth strong approximation can be true for varieties defined by (1.1). See also [CTH12].

**Question 1.3.** Let $K/k$ be an extension of number fields of degree $n$, let $P(t) \in k[t_1, \ldots, t_s]$ be a non-constant polynomial. Let $X \subset \mathbb{A}^{n+s}_k$ be the affine variety over $k$ defined by (1.1). Let $v_0$ be an archimedean place.
Let $p : X \to \mathbb{A}_k^s$ be the projection to the $t$-coordinates. Let $C$ be the union of the connected components of $p(X^{\text{sm}}(k_{\text{eq}}))$ that contain balls of arbitrarily large radius.

Is $X^{\text{sm}}(k)$ dense in the image of $(p^{-1}(C) \times X^{\text{sm}}(\mathbb{A}_k^{\{v\}}))^{\text{Br}(\mathbb{X}^{\text{sm}})}$ in $X(\mathbb{A}_k^{\{v\}})$ with respect to the adelic topology?

Theorems 1.1 and 1.2 give an affirmative answer to this question for two families of varieties.

**Terminology.** For a field $k$ of characteristic 0, fix an algebraic closure $\overline{k}$, and let $\Gamma_k$ be the absolute Galois group. Let $\text{Br}(k)$ be the Brauer group of $k$. For a scheme $X$ over $k$, let $X^{\text{sm}}$ be its smooth locus, and let $\mathbb{X} := X \times_k \overline{k}$. Let $\text{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ be the cohomological Brauer group, $\text{Br}_0(X)$ its subgroup of constant elements, namely the image of the natural map $\text{Br}(k) \to \text{Br}(X)$, and $\text{Br}_1(X)$ its algebraic Brauer group, namely the kernel of the natural map $\text{Br}(X) \to \text{Br}(\mathbb{X})$.

Now let $k$ be a number field. Then $\Omega_k$ denotes the set of places of $k$, and $\infty_k$ denotes its subset of archimedean places. We write $v < \infty_k$ for $v \in \Omega_k \setminus \infty_k$. The ring of integers in $k$ is denoted by $\mathcal{O}_k$. For $v \in \Omega_k$, let $k_v$ be the completion of $k$ at the place $v$, and let $\mathcal{O}_v$ be the ring of integers in $k_v$. The adele ring with its usual adelic topology is denoted by $\mathbb{A}_k$. For a finite subset $S \subset \Omega_k$, let $\mathcal{O}_S$ be the ring of $S$-integers of $k$, and let $\mathbb{A}_k^S = \prod_{v \in \Omega_k \setminus S} k_v$ be the adeles without $v$-component for all $v \in S$, which also comes with a natural adelic topology. In particular, we write $\mathbb{A}_k^\infty := \mathbb{A}_k^\infty$ for the adeles without archimedean components.

Let $X$ be a geometrically integral variety over a number field $k$, and let $S \subset \Omega_k$ be a finite set of places. We say that strong approximation holds for $X$ off $S$ if $X(k)$ is dense in the image of $X(\mathbb{A}_k)$ in $X(\mathbb{A}_k^S)$.

One says that strong approximation with (algebraic) Brauer–Manin obstruction holds for $X$ off $S$ if $X(k)$ is dense in the image of $X(\mathbb{A}_k)^{\text{Br}(X)}$ (resp. $X(\mathbb{A}_k)^{\text{Br}_1(X)}$) in $X(\mathbb{A}_k^S)$ under the natural projection (see [CTX13, Definition 2.4]). Here, $X(\mathbb{A}_k)^B$ is the set of all adelic points $(x_v) \in X(\mathbb{A}_k)$ satisfying $\sum_{v \in \Omega_k} \text{inv}_v(\beta(x_v)) = 0$ for all $\beta$ in a subset $B \subset \text{Br}(X)$, where $\text{inv}_v : \text{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ is the invariant map from local class field theory.

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## 2. Strong approximation on singular varieties

Let $k$ be a number field. For smooth varieties $X$ over $k$, it is interesting to study strong approximation because we can derive the existence of integral points on any integral model of $X$. 
For singular varieties $X$ over $k$, the implicit function theorem may fail, and hence we cannot hope to prove strong approximation on $X$. On the other hand, integral models of $X^{\text{sm}}$ often have far less integral points than integral models of $X$, hence strong approximation on $X^{\text{sm}}$ is generally too much to ask for.

Instead, we introduce the following notion of smooth strong approximation on $X$ which is suitable to determine the existence of integral points on any integral model of the singular variety $X$ (see Remark 2.2 and Corollary 5.6). Below, we will study the relation of this notion to central strong approximation, as introduced by Colliot-Thélène and Xu for the study of strong approximation on singular varieties in [CTX13, §8].

**Definition 2.1.** Let $X$ be a geometrically integral variety over a number field $k$. Let $S$ be a finite set of places of $k$. We say that $X$ satisfies smooth strong approximation off $S$ if $X^{\text{sm}}(k)$ is dense in the image of $X^{\text{sm}}(\mathbb{A}_k)$ in $X(\mathbb{A}_S)$ under the natural projection.

Analogously, we say that $X$ satisfies smooth strong approximation with (algebraic) Brauer–Manin obstruction off $S$ if $X^{\text{sm}}(k)$ is dense in the image of $X^{\text{sm}}(\mathbb{A}_k)^{\text{Br}}(X^{\text{sm}})$ (resp. $X^{\text{sm}}(\mathbb{A}_k)^{\text{Br}_1}(X^{\text{sm}})$) in $X(\mathbb{A}_S)$.

We say that $X$ satisfies the smooth integral Hasse principle (with (algebraic) Brauer–Manin obstruction) if the following holds for any integral model $\mathcal{X}$ of $X$: If

$$X^{\text{sm}}(\mathbb{A}_k)^B \cap \left( \prod_{v \in \infty} X(k_v) \times \prod_{v < \infty} \mathcal{X}(\mathcal{O}_v) \right)$$

is non-empty (where $B$ is $\emptyset$ resp. $\text{Br}(X^{\text{sm}})$ resp. $\text{Br}_1(X^{\text{sm}})$), then there are integral points on $\mathcal{X}$.

If $X$ is smooth, then smooth strong approximation off $S$ is the same as strong approximation off $S$, and the smooth integral Hasse principle is the same as the integral Hasse principle; similarly with (algebraic) Brauer–Manin obstructions.

For singular $X$, however, note that smooth strong approximation on $X$ off $S$ is not the same as strong approximation on $X^{\text{sm}}$, the latter means that $X^{\text{sm}}(k)$ is dense in the image of $X^{\text{sm}}(\mathbb{A}_k)$ in $X^{\text{sm}}(\mathbb{A}_S^\infty)$, whose adelic topology is stronger than the topology induced by $X(\mathbb{A}_S^\infty)$.

**Remark 2.2.** Assume that $X$ is a variety over a number field $k$ satisfying smooth strong approximation (with (algebraic) Brauer–Manin obstruction) off $\infty_k$. Then the smooth integral Hasse principle (with (algebraic) Brauer–Manin obstruction) holds on any integral model of $X$.

Indeed, for any integral model $\mathcal{X}$, we have $(q_v)$ in the set (2.1). An open neighborhood of $(q_v)_{v < \infty} \in X(\mathbb{A}_k)$ is given by $\prod_{v < \infty} \mathcal{X}(\mathcal{O}_v)$. Then smooth strong approximation off $\infty_k$ gives a $q \in X^{\text{sm}}(k)$ which lies in this open neighborhood. This ensures $q \in \mathcal{X}(\mathcal{O}_k)$.

Next, we compare smooth strong approximation to central strong approximation [CTX13, §8]. Recall that it can be defined as follows:

**Definition 2.3** (Colliot-Thélène, Xu). Let $X$ be a geometrically integral variety over a number field $k$. Let $S$ be a finite set of places of $k$. Let $f :
\( \tilde{X} \to X \) be a resolution of singularities for \( X \). The following two definitions do not depend on the choice of \( \tilde{X} \).

One says that **central strong approximation holds for \( X \) off \( S \)** if the diagonal image of \( X^\text{sm}(k) \) is dense in the natural image of \( \tilde{X}(\mathbb{A}_k) \) in \( X(\mathbb{A}_k^0) \).

Assume that \( \text{Br}(\tilde{X})/\text{Br}_0(\tilde{X}) \) (resp. \( \text{Br}_1(\tilde{X})/\text{Br}_0(\tilde{X}) \)) is finite. One says that **central strong approximation with (algebraic) Brauer–Manin obstruction holds for \( X \) off \( S \)** if the diagonal image of \( X^\text{sm}(k) \) is dense in the natural image of \( \tilde{X}(\mathbb{A}_k)^{\text{Br}(\tilde{X})} \) (resp. \( \tilde{X}(\mathbb{A}_k)^{\text{Br}_1(\tilde{X})} \)) in \( X(\mathbb{A}_k^0) \).

Note that the finiteness assumption on the Brauer group is generally necessary to ensure that the definition of central strong approximation with Brauer–Manin obstruction is independent of the choice of \( \tilde{X} \), but that no such finiteness condition is needed in our definition of smooth strong approximation.

To compare central and smooth strong approximation, we start with the following lemma, which is similar to [CTS00, Proposition 1.1] and [CTX13, Proposition 2.6]

**Lemma 2.4.** Let \( X \) be a smooth geometrically integral variety over a number field \( k \), and let \( U \subset X \) be an open subset. Let \( B \) be a subgroup of \( \text{Br}(U) \) such that \( B/(B \cap \text{Br}_0(U)) \) is finite. Then \( U(\mathbb{A}_k)^B \) is dense in \( (X(\mathbb{A}_k))^B/\text{Br}(X) \).

**Proof.** Let \( A \) be a finite subset of \( \text{Br}(U) \) that generates \( B/(B \cap \text{Br}_0(U)) \). There are a finite set \( T \) of places of \( k \) containing \( \infty \) and smooth \( \mathcal{O}_T \)-schemes \( \mathfrak{U} \subset \mathfrak{X} \) with geometrically integral fibers over the points of \( \text{Spec}(\mathcal{O}_T) \) such that the restriction of \( \mathfrak{U} \subset \mathfrak{X} \) over \( \text{Spec}(k) \subset \text{Spec}(\mathcal{O}_T) \) is \( U \subset X \), with \( A \subset \text{Br}(\mathfrak{U}) \) and \( \mathfrak{U}(\mathcal{O}_v) \neq \emptyset \) for each \( v \notin T \).

We must show: Given \( T_1 \supset T \) and an open set \( W_v \subset X(k_v) \) for any \( v \in T_1 \) with

\[
\left( \prod_{v \in T_1} W_v \times \prod_{v \notin T_1} \mathfrak{X}(\mathcal{O}_v) \right)^{B \cap \text{Br}(X)} \neq \emptyset, \tag{2.2}
\]

the intersection of the set in (2.2) with \( U(\mathbb{A}_k)^B \) is not empty.

For any place \( v \), the evaluation of elements of \( \text{Br}(X) \) in \( X(k_v) \) is locally constant, and \( U(k_v) \) is dense in \( X(k_v) \), hence the set in \( \mathfrak{X}(\mathcal{O}_v) \) contains a point \( p_v \in U(k_v) \) for any place \( v \).

As in the proof of the version [CTS00, Théorème 1.4] of Harari’s formal lemma [Har94, Corollaire 2.6.1], there is a finite set \( T_2 \) of places with \( T_2 \cap T_1 = \emptyset \) and \( q_v \in \mathfrak{X}(\mathcal{O}_v) \cap U(k_v) \) for any \( v \in T_2 \) such that, for any \( \beta \in A \),

\[
\sum_{v \in T_1} \beta(p_v) + \sum_{v \in T_2} \beta(q_v) = 0.
\]

For any \( v \in T_1 \), let \( q_v := p_v \). For any \( v \notin T_1 \cup T_2 \), choose an arbitrary \( q_v \in \mathfrak{U}(\mathcal{O}_v) \neq \emptyset \), hence \( \beta(q_v) \in \text{Br}(\mathcal{O}_v) = 0 \) for any \( \beta \in A \). Therefore, \( (q_v) \in X(\mathbb{A}_k) \) lies in \( U(\mathbb{A}_k)^B \) and also in (2.2), hence our claim holds. 

**□**

**Lemma 2.5.** Let \( X \) be a geometrically integral variety over a number field \( k \), and let \( S \) be a finite set of places of \( k \). Then smooth strong approximation on \( X \) off \( S \) is equivalent to central strong approximation on \( X \) off \( S \).
Assume that $\text{Br}(X^{sm})/\text{Br}_0(X^{sm})$ (resp. $\text{Br}_1(X^{sm})/\text{Br}_0(X^{sm})$) is finite. Then smooth strong approximation with (algebraic) Brauer–Manin obstruction on $X$ off $S$ is equivalent to central strong approximation with (algebraic) Brauer–Manin obstruction on $X$ off $S$.

Proof. Assume that smooth strong approximation holds on $X$ off $S$, i.e., $X^{sm}(k)$ is dense in the image of $X^{sm}(\mathbb{A}_k)$ in $X(\mathbb{A}_k^S)$. For a resolution of singularities $f: \tilde{X} \to X$ with $\tilde{X} \supset X^{sm}$, we must show that $X^{sm}(k)$ is dense in the image of $\tilde{X}(\mathbb{A}_k)$ in $X(\mathbb{A}_k^S)$.

By assumption, $X^{sm}$ is geometrically integral, and we can regard it as a dense open subset of the smooth variety $\tilde{X}$. Hence $X^{sm}(\mathbb{A}_k)$ is dense in $\tilde{X}(\mathbb{A}_k)$. A given point $(p_v) \in \tilde{X}(\mathbb{A}_k)$ can therefore be approximated arbitrarily well by a point $(q_v) \in X^{sm}(\mathbb{A}_k)$. Applying $f: \tilde{X} \to X$ to these adelic points, we see that $(f(p_v)) \in X(\mathbb{A}_k)$ is also very close to the image of $(f(q_v)) = (q_v) \in X^{sm}(\mathbb{A}_k)$ in $X(\mathbb{A}_k)$. By smooth strong approximation on $X$ off $S$, we can find a $p \in X^{sm}(k)$ arbitrarily close to the image of $(q_v)$ in $X(\mathbb{A}_k^S)$, which is very close to the image of $(f(p_v))$ in $X(\mathbb{A}_k^S)$. Hence central strong approximation holds on $X$ off $S$.

The converse holds since $X^{sm}(\mathbb{A}_k)$ can be considered as a subset of $\tilde{X}(\mathbb{A}_k)$.

Since $X^{sm} \subset \tilde{X}$ is open and $\text{Br}(X^{sm})/\text{Br}_0(X^{sm})$ is finite, $X^{sm}(\mathbb{A}_k)^{\text{Br}(X^{sm})}$ is dense in $\tilde{X}(\mathbb{A}_k)^{\text{Br}(\tilde{X})}$ by Lemma 2.3 and similarly for $\text{Br}_1$. A similar argument as above deduces from this that smooth strong approximation with (algebraic) Brauer–Manin obstruction on $X$ off $S$ is equivalent to central strong approximation with (algebraic) Brauer–Manin obstruction on $X$ off $S$. \hfill \square

3. A descent lemma

The following lemma, based on descent theory of Colliot-Thélène and Sansuc [CTSS], is central in our proofs of strong approximation in the following sections.

**Lemma 3.1.** Let $k$ be a number field. Let $X$ be an integral variety over $k$ with $\overline{k}[X^{sm}] = \overline{k}^\times$ and $\text{Pic}(X^{sm})$ of finite type, and let $U$ be a dense open subset of $X^{sm}$.

Let $S$ be a finite subset of $\Omega_k$. For any universal torsor $f: T \to X^{sm}$, assume that its restriction $T_U := T \times_{X^{sm}} U$ is geometrically integral, and that there is a commutative diagram

$$
\begin{array}{ccc}
\overline{T}_U & \to & \overline{Y} \\
\downarrow{i|T_U} & & \downarrow{g} \\
U & \to & X
\end{array}
$$

where $Y$ is a variety over $k$ satisfying smooth strong approximation off $S$, and $i: \overline{T}_U \to \overline{Y}$ is an open immersion.

Then $X$ satisfies smooth strong approximation with algebraic Brauer–Manin obstruction off $S$.

Proof. We must find a rational point $p \in X^{sm}(k)$ which approximates the projection of a given $(p_v) \in X^{sm}(\mathbb{A}_k)^{\text{Br}(X^{sm})}$ in $X(\mathbb{A}_k^S)$. 

By descent theory (see [CTSS77, Sko99 Theorem 3]), there is a universal torsor \( f: \mathcal{T} \to X^{\text{sm}} \) and \((r_v) \in \mathcal{T}(\mathbb{A}_k)\) such that \((f(r_v)) = (p_v)\).

Since \( \mathcal{T}_U \) is geometrically integral, any integral model \( \Xi_U \) of \( \mathcal{T}_U \) satisfies \( \Xi_U(\mathcal{O}_v) \neq \emptyset \) for almost all \( v < \infty \). This implies that \( \mathcal{T}_U(\mathbb{A}_k) \) is dense in \( \mathcal{T}(\mathbb{A}_k) \) since \( \mathcal{T} \) is smooth and \( \mathcal{T}_U \) is open and dense in it. Therefore, we can find \((r'_v) \in \mathcal{T}_U(\mathbb{A}_k)\) very close to \((r_v) \in \mathcal{T}(\mathbb{A}_k)\).

By assumption, we have \( \mathcal{T}_U \subset Y^{\text{sm}} \subset Y \) for some \( Y \) satisfying smooth strong approximation off \( S \). Hence we obtain a point \( r \in Y^{\text{sm}}(k) \) arbitrarily close to the projection of \((r'_v) \in \mathcal{T}_U(\mathbb{A}_k) \subset Y^{\text{sm}}(\mathbb{A}_k) \in Y(\mathbb{A}_k^S)\); for \( r \) close enough, we have \( r \in \mathcal{T}_U(k) \).

Let \( p := f(r) \in U(k) \). Since \((r'_v) \) is very close to \((r_v) \) in \( \mathcal{T}(\mathbb{A}_k) \), we know that \((p'_v) := (f(r'_v))\) is very close to \((p_v) \) in \( X^{\text{sm}}(\mathbb{A}_k) \). Since \( X^{\text{sm}} \) is open in \( X \), also \((p'_v) \) is very close to \((p_v) \) with respect to the adelic topology of \( X(\mathbb{A}_k) \). Furthermore, since \( r \) is very close to the projection of \((r'_v) \) in \( Y(\mathbb{A}_k^S) \), we have \( p \) very close to the projection of \((p'_v) = (g(r'_v))\) in \( X(\mathbb{A}_k^S) \). Hence \( p \) is very close to the projection of \((p_v) \) in \( X(\mathbb{A}_k^S) \). \( \square \)

4. Quadratic polynomials represented by quartic norms

In this section, we apply our Descent Lemma 3.1 to prove Theorem 1.1. The main work lies in Proposition 4.1 proving strong approximation for the varieties \( Y \subset \mathbb{A}_k^4 \) defined by (4.2), which essentially are the universal torsors, using their fibration over \( \mathbb{A}_k^4 \) whose fibers are three-dimensional quadrics.

Additionally, we compute the Brauer group of \( X \) in Proposition 4.5 and obtain results on the integral Hasse principle.

Proof of Theorem 1.1. Let \( X \) be as in Theorem 1.1. Since \( P(t) \) is separable, \( X \) is smooth over \( k \). Let \( U := X \cap \{ P(t) \neq 0 \} \). By the local description of its universal torsors [DSW12 Proposition 2] (see also [DSW12 proof of Proposition 3]), we have

\[ \mathcal{T}_U \subset \mathbb{A}_k^4 \times R_{K/k}(\mathbb{G}_m,K)^2 \]

defined by

\[ t - \sqrt{a} = \rho \cdot N_{K/L}(x) \cdot \sigma(N_{K/L}(y)) \neq 0 \]  

(4.1)

for \( k \subset L := k(\sqrt{a}) \subset K \), some \((\rho, \xi) \in L^\times \times K^\times \) satisfying \( cN_{L/k}(\rho) = N_{K/k}(\xi) \), and \( \sigma \in \Gamma_k \) with \( \sigma(\sqrt{a}) = -\sqrt{a} \). The restriction of \( f: \mathcal{T} \to X \) to \( \mathcal{T}_U \) is given by \((t, x, y) \mapsto (t, x\xi y)\).

We write \( K = L(\sqrt{u + v\sqrt{a}}) \) with \( u, v \in k \). For \( x = (x_1 + x_2\sqrt{a}) + (x_3 + x_4\sqrt{a})u + v\sqrt{a} \in K \) and \( y \in K \), we have

\[ N_{K/L}(x) = (x_1 + x_2\sqrt{a})^2 - (x_3 + x_4\sqrt{a})(u + v\sqrt{a}) \]

\[ = g_0(x) + g_1(x)\sqrt{a}, \]

\[ \rho \cdot \sigma(N_{K/L}(y)) = \lambda_0(y) + \lambda_1(y)\sqrt{a}, \]

for quadratic forms

\[ g_0(x) = x_1^2 + ax_2^2 - x_3^2 - ax_4^2 - 2ux_3x_4, \]

\[ g_1(x) = 2x_1x_2 - 2ux_3x_4 - v^2 - avx_4^2. \]
in \( x = (x_1, \ldots, x_4) \) and some quadratic forms \( \lambda_0(y), \lambda_1(y) \) in \( y = (y_1, \ldots, y_4) \). Then

\[
\rho \cdot N_{K/L}(x) \cdot \sigma(N_{K/L}(y)) = g_0(x)\lambda_0(y) + ag_1(x)\lambda_1(y) + (g_0(x)\lambda_1(y) + g_1(x)\lambda_0(y))\sqrt{a}.
\]

This gives an open embedding of \( U \) into the affine variety \( \tilde{Y} \subset \mathbb{A}_k^8 \) with coordinates \((t, x, y)\) defined by

\[
t = g_0(x)\lambda_0(y) + ag_1(x)\lambda_1(y), \quad -1 = g_0(x)\lambda_1(y) + g_1(x)\lambda_0(y).
\]

Clearly \( \tilde{Y} \cong Y \) for \( Y \subset \mathbb{A}_k^8 \) with coordinates \((x, y)\) defined by

\[
-1 = g_0(x)\lambda_1(y) + g_1(x)\lambda_0(y). \tag{4.2}
\]

We have a morphism \( g : Y \to X \) defined by

\[(x, y) \mapsto (g_0(x)\lambda_0(y) + ag_1(x)\lambda_1(y), \xi x y).\]

We observe that \( g : Y \to X \) and \( f : T \to X \) have the same restriction to \( U \).

Note that \( X \) is smooth over \( k \). Also \( Y \) is smooth over \( k \) since

\[
\overline{Y} \cong \{2\sqrt{a} = \sigma(\rho)w_1 w_2 w_3 w_4 - \rho w'_1 w'_2 w'_3 w'_4 \} \subset \mathbb{A}_k^8,
\]

using \([4,1] \). By \([DSW12, \text{Proposition 2}] \), \( \text{Pic}(X) \) is torsion-free of finite rank, hence \( T_U \) is a torsor under a torus over the geometrically integral variety \( U \), and hence \( T_U \) is geometrically integral.

Therefore, we can apply Lemma \([L,1] \) to get strong approximation with Brauer–Manin obstruction for \( X \) off \( v_0 \) once we have shown that \( Y \) satisfies strong approximation off \( v_0 \), which is done in Proposition \([L,1] \) below. \(\square\)

The following result completes the proof of Theorem \([1,1] \).

**Proposition 4.1.** The affine variety \( Y \subset \mathbb{A}_k^8 \) defined by \((4.2) \) satisfies strong approximation off \( v_0 \).

**Proof.** Consider the projection \( \pi : Y \to \mathbb{A}_k^4 \) to the \( y \)-coordinate. Over \( V := \{N_{K/k}(y) \neq 0\} \subset \mathbb{A}_k^4 \), its fibers are smooth 3-dimensional quadrics. Indeed, note that we can write

\[
g_0(x)\lambda_1(y) + g_1(x)\lambda_0(y) = g_0(x_1, x_2) + q_1(x_3, x_4)
\]

where \( g_0, q_1 \) are the following binary quadratic forms with coefficients in \( k[y_1, \ldots, y_4] \):

\[
g_0(x_1, x_2) = \lambda_1(y)x_1^2 + 2\lambda_0(y)x_1 x_2 + a\lambda_1(y)x_2^2,
\]

\[
q_1(x_3, x_4) = -(u\lambda_1(y) + v\lambda_0(y))x_3^2 - 2(u v \lambda_1(y) + u \lambda_0(y))x_3 x_4
\]

\[- a(u \lambda_1(y) + v \lambda_0(y))x_4^2.
\]

Its discriminants are

\[
\text{disc}(g_0) = a\lambda_1(y)^2 - \lambda_0(y)^2 = -N_{L/k}(\rho)N_{K/k}(y),
\]

\[
\text{disc}(q_1) = (u^2 - a v^2)(a\lambda_1(y)^2 - \lambda_0(y)^2)
\]

\[
= -N_{L/k}(u + v \sqrt{a})N_{L/k}(\rho)N_{K/k}(y).
\]

For \( N_{K/k}(y) \neq 0 \), the binary forms \( g_0, q_1 \) have full rank, hence each fiber \( Y_y \) is a three-dimensional quadric.
For \( v_0 \) as above, we claim that \( Y_\mathcal{Y}(k_{v_0}) \) is not compact for any \( y \in V(k_{v_0}) \).
Indeed, if \( v_0 \) is complex, this claim is obvious, so we assume that \( v_0 \) is real and consider everything in the following with respect to the corresponding real embedding of \( k \). If one of \( \text{disc}(q_0) \) and \( \text{disc}(q_1) \) is negative, then one of \( q_0 \) and \( q_1 \) is indefinite, and the claim is true. So we can assume \( \text{disc}(q_0), \text{disc}(q_1) > 0 \), and we claim that one of \( q_0, q_1 \) is positive definite and the other is negative definite, i.e., \( \lambda_1(y)(-u\lambda_1(y) + v\lambda_0(y)) < 0 \).

For this, we show first that \( u > 0 \). By assumption, \( a\lambda_1(y)^2 - \lambda_0(y)^2 > 0 \) (so \( a > 0 \)) and \( u^2 - av^2 > 0 \) (so \( u - v\sqrt{a} \) and \( u + v\sqrt{a} \) have the same sign).
If \( u < 0 \), then both \( u - v\sqrt{a} \) and \( u + v\sqrt{a} \) are negative, so all places of \( K \) above \( v_0 \) are complex, and \( N_{K/k} \) is positive definite.

By unboundedness of \( p(X(k_{v_0})) \), we have \( c > 0 \). Therefore, \( N_{K/k}(\rho) = c^{-1} N_{K/k}(\xi) > 0 \), hence \( \text{disc}(q_0) < 0 \), contradicting our assumption of definiteness of \( q_0 \). Hence \( u > 0 \).

Therefore, \( u^2 - av^2 > 0 \) implies \( u > |v|\sqrt{a} \). Furthermore, \( a\lambda_1(y)^2 - \lambda_0(y)^2 > 0 \) implies \( |\lambda_1(y)| > |\lambda_0(y)|/\sqrt{a} \). Therefore,

\[
\lambda_1(y)(u\lambda_1(y) + v\lambda_0(y)) > (\sqrt{a}|v|) \frac{|\lambda_0(y)|}{\sqrt{a}} |\lambda_1(y)| + v\lambda_0(y)\lambda_1(y) = |v\lambda_0(y)\lambda_1(y)| + v\lambda_0(y)\lambda_1(y) \geq 0,
\]

so one of \( q_0, q_1 \) is positive definite and the other is negative definite, and \( Y_\mathcal{Y}(k_{v_0}) \) is not compact also in this case.

For strong approximation, by smoothness of \( Y \), it is enough to show the following: Let \( \mathcal{Y} \) be the integral model of \( Y \) defined by clearing denominators from \( Y(k) \). Let \( W := \pi^{-1}(V) \). For any finite set \( S \subset \Omega_k \setminus \{v_0\} \) containing \( \infty_k \setminus \{v_0\} \) and any

\[
(p_v)_{v \in \Omega_k} \in \prod_{v \in S \cup \{v_0\}} W(k_v) \times \left( \prod_{v \notin S \cup \{v_0\}} \mathcal{Y}(\mathcal{O}_v) \cap W(k_v) \right),
\]

we must find \( p \in W(k) \) arbitrarily close to \( p_v \) for all \( v \in S \) with \( p \in \mathcal{Y}(\mathcal{O}_v) \) for all \( v \notin S \cup \{v_0\} \).

We prove this by the fibration method. First we restrict the base \( \mathbb{A}_k^1 \) to a line in it. It is enough to find a line \( \ell \subset \mathbb{A}_k^1 \) such that

1. \( \pi^{-1}(\ell)(k_v) \neq \emptyset \) for all \( v \in \Omega_k \),
2. \( \pi^{-1}(\ell)(k_v) \) contains a point \( p_v \) very close to \( p_v \) for all \( v \in S \), and
3. \( \pi^{-1}(\ell)(k_v) \cap \mathcal{Y}(\mathcal{O}_v) \neq \emptyset \) for all \( v \notin S \cup \{v_0\} \),

and to prove that \( \pi^{-1}(\ell) \) satisfies strong approximation off \( v_0 \).

For this, we construct a suitable line \( \ell \). Let \( r_v = \pi(p_v) \in V(k_v) \) for all \( v \in \Omega_k \).
We can choose a point \( M \in V(k) \) arbitrarily close to \( r_v \) for \( v \in S \cup \{v_0\} \) because \( \mathbb{A}_k^1 \) satisfies weak approximation. Then the fiber \( Y_M(k_v) \) contains a point very close to \( p_v \) for all \( v \in S \cup \{v_0\} \) by the implicit function theorem, since all fibers above \( V \) are smooth. Since \( Y_M \) is a quadric of dimension 3, there is a finite set \( S' \subset \Omega_k \setminus (S \cup \{v_0\}) \) such that, for all \( v \notin S' \cup S \cup \{v_0\} \), we have \( M \in \mathbb{A}_k^1(\mathcal{O}_v) \) and \( \mathcal{Y}_M(\mathcal{O}_v) \neq \emptyset \). This gives (1) outside \( S' \), (2), and (3) outside \( S' \) for any line through \( M \).
For $\pi^{-1}(\ell)$ to satisfy strong approximation off $v_0$, we need the following technical condition: We want $\ell \cap Z = \emptyset$ for

$$Z = \{ \lambda_1(y) = \text{disc}(q_0) = 0 \} \cup \{ u\lambda_1(y) + v\lambda_0(y) = \text{disc}(q_1) = 0 \}.$$ 

Recall that $\text{disc}(q_0)$ and $\text{disc}(q_1)$ differ only by a constant in $k^\times$. Since $\text{disc}(q_0)$ is irreducible of degree 4 and $\lambda_1(y)$ has degree 2, and similarly for the second part, we know that $Z$ has dimension 2. The closure of the union of all lines through $M$ meeting $Z$ is a 3-dimensional subvariety of $A_k^1$. Therefore, $V' := (A_k^1 \setminus Z') \cap V$ is a dense open subset of $A_k^1$. For all $v \in S'$, we choose arbitrary $p_v \in \mathfrak{y}(\mathcal{O}_v)$. Then we have $N \in V'(k)$ very close to $\pi(p_v)$ such that $Y_N(k_v) \cap \mathfrak{y}(\mathcal{O}_v) \neq \emptyset$ for all $v \in S'$ by the implicit function theorem. This gives (1) and (3) for $S'$ for any line through $N$.

Let $\ell$ be the line through $M$ and $N$. This satisfies (1), (2), (3), and it does not meet $Z$, which we now use to prove strong approximation off $v_0$ for $Y' := \pi^{-1}(\ell)$.

We choose an isomorphism $\ell \cong A_k^1$. This implies that $Y'$ is defined by

$$-1 = q_0'(x_1, x_2) + q_1'(x_3, x_4)$$

(4.3) with

$$q_0'(x_1, x_2) = a_0(t)x_1^2 + b_0(t)x_1x_2 + c_0(t)x_2^2,$n$$q_1'(x_3, x_4) = a_1(t)x_3^2 + b_1(t)x_3x_4 + c_1(t)x_4^2,$n

where $a_0(t), \ldots, c_1(t) \in k[t]$. Their discriminants are the same polynomials in $t$ up to a constant in $k^\times$. We have

$$(a_0(t), \text{disc}(q_0')) = (a_1(t), \text{disc}(q_1')) = 1$$

since $q'_0$ is the restriction of $q_0$ to $\ell$ and $a_0(t)$ is the restriction of $\lambda_1(y)$ to $\ell$, so by construction, they do not both vanish in a point on $\ell$, hence are coprime; and similarly for $q_1$.

Let $T \subset \Omega_k$ be a finite set of places containing $\infty_k$ such that 2 and $(a_i(t), \text{disc}(q_i'))$ for $i = 0, 1$ and all non-zero coefficients of the polynomials $a_0(t), \ldots, c_1(t), \text{disc}(q_0'), \text{disc}(q_1')$ are units in $\mathcal{O}_T$. Then the equation (4.3) considered over $\mathcal{O}_T$ defines an $\mathcal{O}_T$-model $\mathfrak{y}'$ of $Y'$.

For some finite set $T' \subset \Omega_k$ containing $T$, let

$$(q_v)_{v \in \Omega_k} \in \prod_{v \in T'} Y'(k_v) \times \prod_{v \notin T'} \mathfrak{y}'(\mathcal{O}_v).$$

We must find $q \in Y'(k)$ arbitrarily close to $q_v$ for $v \in T' \setminus \{ v_0 \}$ with $q \in \mathfrak{y}'(\mathcal{O}_v)$ for $v \notin T'$.

Let $r_v := \pi(q_v) \in A_k^1$. By strong approximation on $A_k^1$ off $v_0$, we can choose $r \in \{ \text{disc}(q_0') \neq 0 \} \subset A_k^1(k)$ very close to $r_v$ for all $v \in T' \setminus \{ v_0 \}$ (which gives us $q_v \in Y'(k_v)$ very close to $q_v$ for all such $v$ by the implicit function theorem), with $r \in A_k^1(\mathcal{O}_v)$ for all $v \notin T'$.

We have seen that $Y'(k_{v_0}) = Y'(k_v)$ is not compact, hence non-empty. Next, we show that $\mathfrak{y}'(\mathcal{O}_v) \neq \emptyset$ for all $v \notin T'$. Indeed, if $\text{disc}(q_0'(r)) = a_0(r)c_0(r) - b_0(r)^2/4$ is non-zero modulo $\pi_v$, where $\pi_v$ is a uniformizer of $\mathcal{O}_v$, then $q_0(r)$ is a binary quadratic form over $\mathcal{O}_v/(\pi_v)$, so $\mathfrak{y}'(\mathcal{O}_v) \neq \emptyset$ by
Hensel’s lemma. If it is zero modulo \( \pi_v \), then \( a_0(r) \) and \( a_1(r) \) are non-zero modulo \( \pi_v \). Then \( \mathcal{Y}_r' \) modulo \( \pi_v \) is isomorphic to
\[
\{a_0(r)x^2 + a_1(r)y^2 = -1\} \times \mathbb{A}_k^2,
\]
so \( \mathcal{Y}_r'(\mathcal{O}_v) \neq \emptyset \) by Hensel’s lemma.

Therefore, we have a point
\[
(q'_v) \in \prod_{v \in T'} Y'_r(k_v) \times \prod_{v \notin T'} \mathcal{Y}_r'(\mathcal{O}_v)
\]
with \( q'_v \) very close to \( q_v \) for all \( v \in T' \setminus \{v_0\} \). Note that \( Y'_r \) is a quadric of dimension 3. Therefore, \( Y'_r \) satisfies strong approximation off \( v_0 \) since \( Y'_r(k_{v_0}) \) is not compact (see for example [CTX09, Theorem 3.7, §5.3]). Therefore, there exists a \( q \in Y'_r(k) \) such that \( q \) is very close to \( q'_v \) and also to \( q_v \) for all \( v \in T' \setminus \{v_0\} \), with \( q \in \mathcal{Y}_r'(\mathcal{O}_v) \) for all \( v \notin T' \). Therefore, \( Y'_r \) satisfies strong approximation off \( v_0 \).

**Remark 4.2.** Our proof of Proposition 4.1 uses the fibration method for \( \pi : Y \to \mathbb{A}_k^1 \). The key parts of our proof could alternatively be used to check the conditions for an application of the version of the fibration method stated in [CTX13, Proposition 3.1].

Our next goal is to compute the Brauer group of \( X \) as in Theorem 1.1. We start with some lemmas that will also be used in our counterexample in Section 6.

Let \( X \subset \mathbb{A}_k^{n+1} \) be defined by \( \{1\} \) for \( s = 1 \), a finite extension \( K/k \) of fields of characteristic 0 and a non-constant polynomial \( P(t) \in k[t] \). Recall that \( X^{\mathrm{sm}} \subset X \) is the smooth locus of \( X \). If \( P(t) \) is separable, then \( X^{\mathrm{sm}} = X \).

The following lemma shows that \( \operatorname{Br}(X^{\mathrm{sm}}) = 0 \).

**Lemma 4.3.** Let \( k \) be an algebraically closed field of characteristic 0. Let \( P(t) = c(t - a_1)^{r_1} \cdots (t - a_r)^{r_r} \in k[t] \) with \( r \geq 1 \), \( \gcd(e_1, \ldots, e_r) = 1 \) and \( a_i \neq a_j \) for \( i \neq j \). Let \( X \subset \mathbb{A}_k^{n+1} \) be the affine variety defined by \( P(t) = z_1 \cdots z_n \). Then \( \operatorname{Br}(X^{\mathrm{sm}}) = 0 \).

**Proof.** We prove the claim by induction on \( n \). If \( n = 1 \), then \( X^{\mathrm{sm}} = X \cong \mathbb{A}_k^1 \).

By Tsen’s theorem, \( \operatorname{Br}(X^{\mathrm{sm}}) = 0 \).

For \( n > 1 \), consider the projection
\[
X \to \mathbb{A}_k^1, \quad (t, z_1, \ldots, z_n) \mapsto z_n.
\]
Let \( \eta \) be the generic point of \( \mathbb{A}_k^1 \). Then the generic fiber \( X^{\mathrm{sm}}_\eta \) is just the smooth locus of the affine variety over \( k(z_n) \) defined by
\[
\frac{1}{z_n} P(t) = z_1 \cdots z_{n-1}.
\]
We have \( \operatorname{Br}(X^{\mathrm{sm}}) \subset \operatorname{Br}(X^{\mathrm{sm}}_\eta) \) for any smooth variety, and we will show that the latter is trivial.

Let \( X^{\mathrm{sm}}_\eta = X^{\mathrm{sm}}_\eta \times_{k(z_n)} k(z_n) \). Since \( k[X^{\mathrm{sm}}_\eta]^\times = k(z_n)^\times \) by the same residue computation as in the proof of [DSW12, Proposition 2], we have \( H^2(k(z_n), k[X^{\mathrm{sm}}_\eta]^\times) = \operatorname{Br}(k(z_n)) \). Therefore, the Hochschild–Serre spectral sequence gives the exact sequence
\[
\operatorname{Br}(k(z_n)) \to \ker \left( \operatorname{Br}(X^{\mathrm{sm}}) \to \operatorname{Br}(X^{\mathrm{sm}}_\eta) \right) \to H^1(k(z_n), \operatorname{Pic}(X^{\mathrm{sm}}_\eta)).
\]
By Tsen’s theorem, \( Br(k(z_n)) = 0 \) since \( k = \overline{k} \). By the induction hypothesis, \( Br(X^\text{sm}) = 0 \).

Since \( \gcd(e_1, \ldots, e_r) = 1 \), the Picard group \( \text{Pic}(X^\text{sm}) \) is a constant torsion-free module (i.e., the natural action of \( \Gamma(k(z_n)) \) on it is trivial), hence we have \( H^1(k(z_n), \text{Pic}(X^\text{sm})) = 0 \). Therefore, the exact sequence above gives \( Br(X^\text{sm}) = 0 \). \( \square \)

For \( X \) as above, let \( U \subset X^\text{sm} \) be the open subset defined by \( P(t) \neq 0 \). Denote \( \hat{T} = \mathbb{F}[U]^\times / \overline{\mathbb{F}}^\times \) and \( \hat{M} = \text{Div}_{X, U}(\overline{X}) \). We have the natural exact sequence

\[
0 \to \hat{T} \to \hat{M} \to \text{Pic}(\overline{X}^\text{sm}) \to 0.
\]  

\( \text{(4.4)} \)

**Lemma 4.4.** Let \( k \) be a field of characteristic 0 with \( H^3(k, \overline{k}^\times) = 0 \). Assume that \( P(t) = c_{g_1}(t)^{e_1} \cdots g_r(t)^{e_r} \) for \( c \in k^\times \), pairwise distinct irreducible monic polynomials \( g_1(t), \ldots, g_r(t) \in k[t] \) and \( e_1, \ldots, e_r \) positive integers with \( \gcd(e_1, \ldots, e_r) = 1 \). Let \( X \) be defined by \( (1.1) \) with \( s = 1 \). Then

\[
Br(X^\text{sm})/Br_0(X^\text{sm}) \cong H^1(k, \text{Pic}(\overline{X}^\text{sm})) \cong \ker \left( H^2(k, \hat{T}) \to H^2(k, \hat{M}) \right).
\]

**Proof.** By Lemma 4.3, we have \( Br(\overline{X}^\text{sm}) = 0 \), hence \( Br(X^\text{sm}) = Br_1(X^\text{sm}) \). Note that \( \mathbb{F}[X^\text{sm}]^\times = \overline{k}^\times \).

By the Hochschild–Serre spectral sequence, we have the exact sequence

\[
Br(k) \to Br_1(X^\text{sm}) \to H^1(k, \text{Pic}(\overline{X}^\text{sm})) \to H^3(k, \overline{k}^\times) = 0,
\]

hence \( Br(X^\text{sm})/Br_0(X^\text{sm}) \cong H^1(k, \text{Pic}(\overline{X}^\text{sm})) \).

Since \( \hat{M} \) is a permutation \( \Gamma_1 \)-module, \( H^1(k, \hat{M}) = 0 \). Hence \( \text{(4.4)} \) implies that

\[
0 \to H^1(k, \text{Pic}(\overline{X}^\text{sm})) \to H^2(k, \hat{T}) \to H^2(k, \hat{M})
\]

is exact, and the result follows. \( \square \)

To compute the Brauer group in the situation of Theorem 1.1, we argue similarly as in Lemma 4.3. The condition \( H^3(k, \overline{k}^\times) = 0 \) holds when \( k \) is a number field, for example.

**Proposition 4.5.** Let \( k \) be a field of characteristic 0 with \( H^3(k, \overline{k}^\times) = 0 \). Let \( K/k \) be of degree 4 and \( P(t) = c(t^2 - a) \) irreducible over \( k \) and split over \( K \). Let \( X \) be defined by \( (1.1) \) with \( s = 1 \).

We have

\[
Br(X)/Br_0(X) = \begin{cases} 
0, & \text{if } K/k \text{ cyclic or non-Galois}, \\
\mathbb{Z}/2\mathbb{Z}, & \text{otherwise}.
\end{cases}
\]

**Proof.** First, suppose that \( K/k \) is Galois. Since \( \text{Pic}(\overline{X}) \) is split by \( K \) and torsion-free, we have

\[
H^1(k, \text{Pic}(\overline{X})) \cong H^1(K/k, \text{Pic}(X_K))
\]

by the inflation-restriction sequence.

Let \( L := k(\sqrt{a}) \). Note that \( \text{Div}_{X_K}U_K(X_K) \cong \mathbb{Z}[L/k] \otimes \mathbb{Z}[K/k] \), hence \( H^i(K/k, \text{Div}_{X_K}U_K(X_K)) = 0 \) for \( i > 0 \). With the exact sequence

\[
0 \to K[U]^\times / K^\times \to \text{Div}_{X_K}U_K(X_K) \to \text{Pic}(X_K) \to 0,
\]  

\( \text{(4.5)} \)
this gives $H^1(K/k, \text{Pic}(X_K)) \cong H^2(K/k, K[U]^\times / K^\times)$.

To compute the latter, we consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[L/k] \oplus \mathbb{Z}[K/k] \to K[U]^\times / K^\times \to 0$$  \hspace{1cm} (4.6)

from [DSW12, Proposition 2]. We consider

$$H^2(K/k, \mathbb{Z}) \to H^2(K/k, \mathbb{Z}[L/k] \oplus \mathbb{Z}[K/k]) \to H^2(K/k, K[U]^\times / K^\times)$$

$$\to H^3(K/k, \mathbb{Z}) \to H^3(K/k, \mathbb{Z}[L/k] \oplus \mathbb{Z}[K/k]).$$

We have $H^2(K/k, \mathbb{Z}[K/k]) = 0$ since this is an induced module, and $H^2(K/k, \mathbb{Z}[L/k]) \cong H^2(K/L, \mathbb{Z})$ by Shapiro’s lemma. Hence the first map is the natural map $H^2(K/k, \mathbb{Z}) \to H^2(K/L, \mathbb{Z})$, which is surjective since any quartic Galois extension $K/k$ is abelian.

Similarly, $H^3(K/k, \mathbb{Z}[K/k]) = 0$ and $H^3(K/k, \mathbb{Z}[L/k]) \cong H^3(K/L, \mathbb{Z})$. Since $\text{Gal}(K/L)$ is cyclic,

$$H^3(K/L, \mathbb{Z}) \cong H^1(K/L, \mathbb{Z}) \cong \text{Hom}(\text{Gal}(K/L), \mathbb{Z}) = 0.$$ Therefore, $H^3(K/k, \mathbb{Z}[L/k] \oplus \mathbb{Z}[K/k]) = 0$.

Therefore, the third map in our part of the long exact sequence above is an isomorphism. Now

$$H^3(K/k, \mathbb{Z}) = \begin{cases} 0, & \text{if } K/k \text{ is cyclic}, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \text{Gal}(K/k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \end{cases}$$

Indeed, in the first case, we argue as in the previous paragraph. In the second case, we can refer to a classical computation of Schur (see [Kar87, Corollary 2.2.12]).

Next, suppose that $K/k$ is non-Galois. By the inflation-restriction sequence, we have

$$0 \to H^1(L/k, \text{Pic}(\overline{X})^{\Gamma_L}) \to H^1(k, \text{Pic}(\overline{X})) \to H^1(L, \text{Pic}(\overline{X})).$$  \hspace{1cm} (4.7)

We will show that the first and third groups are trivial.

For the triviality of $H^1(L/k, \text{Pic}(\overline{X})^{\Gamma_L})$, we note that (4.4) gives

$$0 \to \overline{T}^{\Gamma_L} \to \overline{M}^{\Gamma_L} \to \text{Pic}(\overline{X})^{\Gamma_L} \to H^1(\Gamma_L, \overline{T}).$$  \hspace{1cm} (4.8)

Analogously to (4.10), we have over $\overline{k}$ the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[L/k] \oplus \mathbb{Z}[K/k] \to \overline{T} \to 0.$$  \hspace{1cm} (4.9)

Since the action of $\Gamma_L$ on $\mathbb{Z}[L/k]$ is trivial, this exact sequence has a $\Gamma_L$-equivariant splitting, hence $\overline{T}$ is a permutation $\Gamma_L$-module, so $H^1(L, \overline{T}) = 0$.

Therefore, (4.8) is a short exact sequence, giving

$$H^1(L/k, \overline{M}^{\Gamma_L}) \to H^1(L/k, \text{Pic}(\overline{X})^{\Gamma_L})$$

$$\to H^2(L/k, \overline{T}^{\Gamma_L}) \to H^2(L/k, \overline{M}^{\Gamma_L}).$$

We note that $\overline{M} \cong \mathbb{Z}[L/k] \otimes \mathbb{Z}[K/k]$, hence $\overline{M}^{\Gamma_L} \cong \mathbb{Z}[L/k] \otimes \mathbb{Z}[K/k]^{\Gamma_L}$ is an induced $\text{Gal}(L/k)$-module. Therefore, $H^i(L/k, \overline{M}^{\Gamma_L}) = 0$ for $i > 0$, hence the second map in our part of the long exact sequence is an isomorphism.
We have $\mathbb{Z}[K/k]^{\Gamma_k} \cong \mathbb{Z}[L/k]$. We choose $\sigma \in \Gamma_k$ and $\beta \in \Gamma_L$ such that $\Gamma_k/\Gamma_L = \{\Gamma_L, \sigma \Gamma_L\}$ and $\Gamma_L/\Gamma_K = \{\Gamma_K, \beta \Gamma_K\}$. Then

$$
\Gamma_k/\Gamma_K = \{\Gamma, \sigma \Gamma_K, \beta \Gamma_K, \sigma \beta \Gamma_K\}.
$$

Now the orbits of the natural action of $\Gamma_L = \Gamma_K \cup \beta \Gamma_K$ on this are $\{\Gamma_K, \beta \Gamma_K\}$ (since $\Gamma_K$ is normal in $\Gamma_L$ because of $[K : L] = 2$) and $\{\sigma \Gamma_K, \sigma \beta \Gamma_K\}$ (since the action of $\Gamma_L$ on this set is non-trivial because $\beta \sigma \sigma^{-1}$ maps $\sigma \Gamma_K$ to $\beta \sigma \Gamma_K$ and lies in $\Gamma_L$ because $\Gamma_L$ is normal in $\Gamma_K$ since $[L : k] = 2$). Therefore, $\mathbb{Z}[K/k]^{\Gamma_k} = (\Gamma_K + \beta \Gamma_K, \sigma \Gamma_K + \sigma \beta \Gamma_K)$ of rank 2, with a non-trivial action of $\text{Gal}(L/k)$.

Taking $\Gamma_L$-invariants of $\langle \mathbb{Z} \rangle$ gives

$$
0 \to \mathbb{Z} \to \mathbb{Z}[L/k] \oplus \mathbb{Z}[L/k] \to \hat{T}^{\Gamma_L} \to H^1(L, \mathbb{Z}) = 0.
$$

The long exact sequence gives

$$
0 = H^2(L/k, \mathbb{Z}[L/k] \oplus \mathbb{Z}[L/k]) \to H^2(L/k, \hat{T}^{\Gamma_L}) \to H^3(L/k, \mathbb{Z}) = 0,
$$

where the latter is trivial because $L/k$ is cyclic. Hence $H^2(L/k, \hat{T}^{\Gamma_L}) = 0$, which implies $H^1(L/k, \text{Pic}(\mathbb{X})) = 0$ in $\langle 13 \rangle$.

For the triviality of $H^1(L, \text{Pic}(\mathbb{X}))$ in $\langle 14 \rangle$, we write $K = k(\sqrt{u + v\sqrt{a}})$ for some $u \in k$ and $v \in k^\times$. Then $K \otimes_k L \cong K \oplus K'$ with $K' = k(\sqrt{u - v\sqrt{a}})$. We note that $X \times_k L$ is defined by the equation

$$
c(t - \sqrt{a})(t + \sqrt{a}) = N_{K/L}(z_1)N_{K'/L}(z_2).
$$

Over $\overline{K}$, we have $N_{K/L}(z_1) = u_1u_2$ and $N_{K'/L}(z_2) = u_3u_4$.

As a sequence of $\Gamma_L$-modules, $\langle 13 \rangle$ becomes

$$
0 \to \mathbb{Z} \to \mathbb{Z}^2 \oplus \mathbb{Z}[K/L] \oplus \mathbb{Z}[K'/L] \to \hat{T} \to 0,
$$

hence $\hat{T} \cong \mathbb{Z} \oplus \mathbb{Z}[K/L] \oplus \mathbb{Z}[K'/L]$, with basis $(t - \sqrt{a}, u_1, u_2, u_3, u_4)$. Therefore, $H^2(L, \hat{T}) \cong H^2(L, \mathbb{Z}) \oplus H^2(K, \mathbb{Z}) \oplus H^2(K', \mathbb{Z})$ by Shapiro’s lemma.

Then

$$
D^+_i = \{t = \sqrt{a}, u_i = 0\}, \quad D^-_i = \{t = -\sqrt{a}, u_i = 0\}
$$

for $i = 1, \ldots, 4$ are a basis of $\hat{M}$. Considering the action of $\Gamma_L$ on this basis shows that

$$
\mathbb{Z}D_i^+ \oplus \mathbb{Z}D_i^+ \cong \mathbb{Z}D_i^- \oplus \mathbb{Z}D_i^- \cong \mathbb{Z}[K/L],
$$

$$
\mathbb{Z}D_i^+ \oplus \mathbb{Z}D_i^+ \cong \mathbb{Z}D_i^- \oplus \mathbb{Z}D_i^- \cong \mathbb{Z}[K'/L],
$$

hence $\hat{M} \cong (\mathbb{Z}[K/L] \oplus \mathbb{Z}[K'/L])^2$. Therefore, $H^2(L, \hat{M}) \cong (H^2(K, \mathbb{Z}) \oplus H^2(K', \mathbb{Z}))^2$, again by Shapiro’s lemma.

The map

$$
\hat{T} \to \hat{M}, \quad t - \sqrt{a} \mapsto D^+_1 + D^+_2 + D^+_3 + D^+_4, \quad u_i \mapsto D^+_i + D^-_i
$$

from $\langle 13 \rangle$ induces a map $H^2(L, \hat{T}) \to H^2(L, \hat{M})$ that can be explicitly described as

$$
(\chi_1, \chi_2, \chi_3) \mapsto (\chi_1|K + \chi_2, \chi_1|K' + \chi_3, \chi_2, \chi_3).
$$

The sequence $\langle 14 \rangle$ gives

$$
0 = H^1(L, \hat{M}) \to H^1(L, \text{Pic}(\mathbb{X})) \to H^2(L, \hat{T}) \overset{\delta}{\to} H^2(L, \hat{M})
$$

ULRICH DERENTHAL AND DASHENG WEI
Corollary 4.7. Let \( \omega \) be a basis of \( K/k \) over \( k \). Let \( \chi_1, \chi_2, \chi_3 \) be a basis of \( \text{Pic}(X) \) over \( k \) and let \( \xi \) be a primitive \( n \)-th root of unity for \( n \geq 0 \), then the Hasse principle holds for integral points on \( X \). In particular, if \( K/k \) is cyclic or non-Galois, then the Hasse principle holds for integral points on \( X \).

Proof. This follows directly from Theorem 1.1 and Proposition 4.5. □

Corollary 4.6. In the situation of Theorem 1.1, assume additionally that the extension \( K/k \) is cyclic or non-Galois. Then strong approximation holds for \( X \). □

Using Lemma 4.4 and (4.7), we deduce

\[
\text{Br}(X)/\text{Br}_0(X) \cong H^1(k, \text{Pic}(X)) = 0.
\]

5. Totally split polynomials represented by norms

For the proof of strong approximation for \( X \) as in Theorem 1.2, we determine its universal torsors in Proposition 5.1 below. If the factors of \( P(t) \) are linear forms, these turn out to be essentially the varieties \( \mathcal{Y} \) that satisfy weak approximation by \cite{BM13} Theorem 1.3 based on \cite{BM13} Theorem 5.2; for Theorem 1.2 we must generalize the latter as in Theorem 5.2 below. By our Descent Lemma 3.1, it then remains to prove strong approximation for \( \mathcal{Y} \), which we do in Proposition 5.3.

Proposition 5.1. Let \( k \) be a field of characteristic 0. Let \( K/k \) be a field extension of degree \( n \). Let \( P(t) = c g_1(t)^{e_1} \cdots g_r(t)^{e_r} \) with \( c \in k^\times \) and pairwise non-proportional linear polynomials \( g_i(t) \in k[t_1, \ldots, t_s] \) in \( s \geq 2 \) variables, and \( e_1, \ldots, e_r \in \mathbb{Z}_{>0} \). Let \( X \) be defined by (1.1).

Universal torsors over \( X^{\text{sm}} \) exist if and only if there are \( \lambda_1, \ldots, \lambda_r \in k^\times \) and \( \xi \in K^\times \) satisfying

\[
c^{e_1} \cdots c^{e_r} = N_{K/k}(\xi).
\]
Then restrictions $T_U \subset \mathbb{A}_k^{rn+s}$ of universal torsors $f : T \to X^{sm}$ to $U := X \cap \{ P(t) \neq 0 \}$ are geometrically rational and defined by

$$N_{K/k}(z_i) = \lambda_i^{-1} g_i(t) \neq 0,$$

for $i = 1, \ldots, r$, for some $\lambda_1, \ldots, \lambda_r \in k^\times$ and $\xi \in K^\times$ satisfying (5.1). The map $f : T \to X^{sm}$ is defined on $T_U$ by $(t, z_1, \ldots, z_r) \mapsto (t, \xi z_1^1 \cdots z_r^r)$.

Proof. We observe that $\overline{X}$ over $\overline{k}$ is defined by

$$e g_1(t)^{e_1} \cdots g_r(t)^{e_r} = u_1 \cdots u_n. \quad (5.2)$$

With $U' := \mathbb{A}_k^n \setminus \{ g_1(t) \cdots g_r(t) = 0 \}$, we have

$$\overline{U} = U \times_k \overline{k} \cong U' \times \mathbb{G}_m^{n-1},$$

hence $\text{Pic}(\overline{U}) = 0$ using [CTS77, Lemme 11].

The singular locus of $\overline{X}$ is the union of all $\{ g_i(t) = g_j(t) = u_i = u_m = 0 \}$ for $k \neq l \in \{1, \ldots, n\}$ and $i, j \in \{1, \ldots, r\}$ with either $i = j$ and $e_i > 1$ or $i \neq j$.

The abelian group $\text{Div}(\overline{X})$ is free of rank $rn$, generated by $D_{i,j} = \{ \lambda_i = \lambda_j \}$.

We have $\overline{k}[X^{sm}]^\times = \overline{k}^\times$. Indeed, since every $f \in \overline{k}[X^{sm}]^\times$ has the form

$$f = c' g_1(t)^{s_1} \cdots g_r(t)^{s_r} u_1^{m_1} \cdots u_n^{m_n}$$

with $c' \in \overline{k}^\times$ and $s_1, \ldots, s_r, m_1, \ldots, m_n \in \mathbb{Z}$, and

$$0 = \text{div}(f) = \sum_{i=1}^r \sum_{j=1}^n (s_i + m_j e_i) D_{i,j},$$

we have $s_i = -m_j e_i$ for all $i$ and $j$, hence $m_1 = \cdots = m_n$ and $s_i = -m_1 e_i$ for all $i$, so $f$ is constant by (5.2).

The abelian group $\overline{k}[U]^\times / \overline{k}^\times$ is free of rank $r + n - 1$, generated by the classes of the functions $g_1(t), \ldots, g_r(t), u_1, \ldots, u_n$ with an obvious $\Gamma_k$-action and the relation

$$\sum_{i=1}^r e_i [g_i(t)] - \sum_{j=1}^n [u_j] = 0.$$ 

Indeed, by a result of Rosenlicht ([Ros61, Theorem 2], applied as in [CTS77, Lemme 10]),

$$\overline{k}[U]^\times / \overline{k}^\times \cong (\overline{k}[U]^\times / \overline{k}^\times \times (\overline{k}[\mathbb{G}_m,k]^\times / \overline{k}^\times)^{n-1} \cong \mathbb{Z}^r \times \mathbb{Z}^{n-1},$$

with basis $g_1(t), \ldots, g_r(t), u_1, \ldots, u_{n-1}$. Adding the additional generator $u_n$, the relation above is obtained from the equation defining $X$.

Hence $\overline{k}[U]^\times / \overline{k}^\times \cong (\mathbb{Z}^r \oplus \mathbb{Z} [\Gamma_k/\Gamma_K]) / \mathbb{Z}$, and, as in the proof of [DSW12, Proposition 2], the short exact sequence

$$1 \to \overline{k}^\times \to \overline{k}[U]^\times \to \overline{k}[U]^\times / \overline{k}^\times \to 1$$

has a $\Gamma_k$-equivariant splitting under the condition that $\lambda_1, \ldots, \lambda_r \in k^\times$ and $\xi \in K^\times$ satisfying (5.1) exist. This is necessary and sufficient for the existence of universal torsors on $X$ [CTS87, Corollary 2.3.4].
By these computations, we have made the short exact sequence
\[ 1 \to \bar{E}[U]^\times / \bar{E}^\times \to \text{div} \bar{X}^m \setminus \{U\} \to \text{Pic}(\bar{X}^m) \to 0, \]
explicit, which can be used for the local description of universal torsors via \cite[Theorem 2.3.1, Corollary 2.3.4]{CTS87}, \cite[Theorem 4.3.1]{Sk01}. The computation is very similar to \cite[Proposition 2]{DSW12}, with the result as in \cite[Remark 2]{DSW12}. Note that in our case, \( \text{Pic}(\bar{X}^m) \) has torsion if and only if \( \gcd(e_1, \ldots, e_r) > 1 \).

Furthermore, \( T_U \) is geometrically rational. Indeed, over \( \bar{K} \), it is defined by
\[ u_{i,1} \cdots u_{i,n} = \lambda^1_{i}^{-1} g_i(t) \neq 0 \]
for \( i = 1, \ldots, r \), hence it can be parameterized by the \( r(n-1) + s \) variables \( t_1, \ldots, t_s \) and \( u_{i,1}, \ldots, u_{i,n-1} \) for \( i = 1, \ldots, r \).

To prove strong approximation on the universal torsors as in Proposition \cite[Definition 5.1]{BM13} in the situation of Theorem \cite[Proposition 2]{BM13}, we must extend the main analytic result \cite[Theorem 5.2]{BM13} of Browning and Matthesen from linear forms to (not necessarily homogeneous) linear polynomials; see also \cite[Remark 5.3]{BM13}. To state the result, we introduce some notation.

Let \( K \) be a number field of degree \( n \), and let \( N_{K/\mathbb{Q}}(z) \in \mathbb{Z}[z_1, \ldots, z_n] \) be an associated norm form with integral coefficients. Let \( f_1(t), \ldots, f_r(t) \in \mathbb{Z}[t_1, \ldots, t_s] \) be linear polynomials that are pairwise affinely independent over \( \mathbb{Q} \), i.e., for \( i \neq j \in \{1, \ldots, r\} \), the homogeneous parts \( f_i(t) - f_j(t) \) are not proportional over \( \mathbb{Q} \). Consider the system of equations
\[ N_{K/\mathbb{Q}}(z_i) = f_i(t), \quad (i = 1, \ldots, r). \tag{5.3} \]

Let \( \mathcal{D}_+ \subset \mathbb{R}^n \) be the fundamental domain for the action of the free part of the norm-1-subgroup of the unit group of \( K \) as in \cite[equation (2.3)]{BM13}. Let \( \bar{R} \subset [-1, 1]^s \subset \mathbb{R}^s \) be a convex bounded set with \( |f_i(T\bar{R})| \leq T \), for \( 1 \leq i \leq r \) and sufficiently large \( T \in \mathbb{R} \).

From \cite[Definition 5.1]{BM13}, recall the definition of the representation function
\[ R(m; \mathcal{D}_+, z', M) := 1_{m \neq 0} \cdot \# \left\{ z'' \in \mathbb{Z}^n \cap \mathcal{D}_+: N_{K/\mathbb{Q}}(z'') = m, z'' \equiv z' \pmod{M} \right\}, \]
for \( m \in \mathbb{Z}, M \in \mathbb{Z}_{>0} \) and \( z' \in \mathbb{Z}^n \).

Let \( q' = (t', z'_1, \ldots, z'_s) \in \mathbb{Z}^{r+n+s} \) and \( M \in \mathbb{Z}_{>0} \). Then
\[ N(T) := \sum_{t'' \in T\bar{R}} \prod_{i=1}^{n} R(f_i(t''); \mathcal{D}_+, z'_i, M) \tag{5.4} \]
is the number of solutions \( q'' = (t'', z''_1, \ldots, z''_s) \in \mathbb{Z}^{r+n+s} \) of \cite{BM13} lying in \( T\mathbb{R} \times (\mathcal{D}_+)^r \) with
\[ q'' \equiv q' \pmod{M} \]
and \( f_1(t'') \cdots f_r(t'') \neq 0 \).

Let \( r_1 \) resp. \( r_2 \) be the number of real resp. complex places of \( K \), let \( D_K \) be its discriminant, and let \( R_K^{(+)} \) be its modified regulator as in \cite{BM13}. 


Remark 5.4. Let 
\[ c_K := \frac{2^{r_1-1}(2\pi)^r \sqrt{\Delta_K}}{|D_K|}. \]

**Theorem 5.2.** Let \( f_1, \ldots, f_r \in \mathbb{Z}[t_1, \ldots, t_s] \) be linear polynomials that are pairwise affinely independent, let \( K \) be a number field, let \( M \in \mathbb{Z} \), and let \( \mathbf{q}' = (t', \mathbf{z}'_1, \ldots, \mathbf{z}'_r) \in \mathbb{Z}^{rn+s} \). Let \( \epsilon \) be an arbitrary element of \( \{\pm\}^r \) if \( K \) is not totally imaginary, and let \( \epsilon = (+, \ldots, +) \) if \( K \) is totally imaginary. Let \( \mathfrak{R} \subset [-1, 1]^s \) be a convex bounded set whose closure is contained in \( \mathfrak{R} := \{ t \in \mathbb{R}^s : 0 < \epsilon_i(f_i(t) - f_i(0)) < 1 \text{ for } i = 1, \ldots, r \} \).

For \( N(T) \) as in (5.4), we have
\[ N(T) = \beta_{\infty} \prod_p \beta_p \cdot T^s + o(T^s), \quad (T \to \infty), \]
where
\[ \beta_{\infty} := c_K \operatorname{vol}(\mathfrak{R}), \]
and \( \prod_p \beta_p \) is absolutely convergent, with
\[ \beta_p := \lim_{m \to \infty} \frac{1}{p^{\alpha_p}} \sum_{t'' \equiv t' \pmod{p^{\alpha_p}(M)}} \prod_{i=1}^r \rho(p^m, f_i(t''), \mathbf{z}'_i; p^{\alpha_p}(M)) \]
and
\[ \rho(p^m, A, \mathbf{z}'; p^{\alpha_p}(M)) := \# \left\{ \mathbf{z}'' \in (\mathbb{Z}/p^m\mathbb{Z})^s : \mathbf{N}_{K/\mathbb{Q}}(\mathbf{z}'') \equiv A \pmod{p^m}, \mathbf{z}'' \equiv \mathbf{z}' \pmod{p^{\alpha_p}(M)} \right\} \]
for \( m \geq \alpha_p(M) \) and \( A \in \mathbb{Z}/p^m\mathbb{Z} \).

**Proof.** First, we note that the condition on \( \mathfrak{R} \) implies that
\[ 0 < \epsilon_i(f_i(T\mathfrak{R})) < T \quad (5.5) \]
for \( i = 1, \ldots, r \) and sufficiently large \( T \). Indeed, since the closure of \( \mathfrak{R} \) is compact, there is a \( \delta > 0 \) such that \( \delta < \epsilon_i(f_i(t) - f_i(0)) < 1 - \delta \) for all \( t \in \mathfrak{R} \).

Multiplying these inequalities by \( T \) and using the identity \( f_i(Tt) - f_i(0) = T(f_i(t) - f_i(0)) \), we have
\[ \epsilon_i(f_i(0)) + T\delta < \epsilon_i(f_i(Tt)) < T + \epsilon_i(f_i(0)) - T\delta \]
for all \( t \in \mathfrak{R} \), which gives the claim for all \( T \geq \delta^{-1}|f_i(0)| \).

The proof follows [BM13, §6–§10]. For \( \epsilon \in \{\pm\} \) and \( T > 0 \), let
\[ \mathfrak{D}_+(T) := \operatorname{vol}\left\{ \mathbf{z} \in \mathfrak{D}_+ : 0 < \epsilon N_{K/\mathbb{Q}}(\mathbf{z}) \leq T \right\}, \]
which has an \((n - 1)\)-Lipschitz parameterizable boundary by [BM13 (2.4)].

We start with the \( W \)-trick, see [GT08] and [BM13, §6]. Let \( W := \prod_{p \leq w(T)} p^{\alpha(p)} \) as in [BM13 (6.1)] with \( w(T) := \log \log T \) and \( \alpha(p) := [C_1 \log p \log T + 1] \) for a constant \( C_1 \geq 1 \) that will be chosen below, and assume that \( T \) is large enough so that \( W \) is divisible by \( M \).
We proceed as in the proof of [BM13, Proposition 8.2], but without the need of splitting into $2^s$ subsets.

By choosing our parameter $C_1$ large enough, we can apply [BM13, Proposition 7.10], which is formulated for pairwise affinely independent linear polynomials, to the functions $R_i(m) := R(\epsilon_i m; D_+, z'_i, M)$ and $R'_i(m) := R'(\epsilon_i m; D_+, z'_i, M)$ on \{1, \ldots, T\} because $\epsilon_i f_i(t)$ takes these values for $t \in T\mathcal{R}$ by (7.5). Hence

$$N(T) = \sum_{t_0 \in W} \sum_{t_1 \in \mathbb{Z}} \prod_{i=1}^r R'_i(W t_1 + t_0) + o(T^s).$$

Comparing the definitions of $W$ and $R'$ shows

$$N(T) = \sum_{t_0 \in W} \sum_{W t_1 + t_0 \in T\mathcal{R}} \prod_{i=1}^r R'_i(W t_1 + t_0) + o(T^s).$$

For $t_0 \in W$, we have $f_i(t_0) \not\equiv 0 \pmod W$. Hence we can write $f_i(W t_1 + t_0) = W h_i(t_1) + A_i(t_0)$ with a linear polynomial $h_i(t) \in \mathbb{Z}[t]$ and $0 < A_i(t_0) < W$ with $A_i(t_0) \equiv f_i(t_0) \pmod W$. Note that $h_1(t), \ldots, h_r(t)$ are also pairwise affinely independent.

For $t_0 \in W$, define the convex set

$$\mathcal{R}_{t_0,T} := \{ t_1 \in \mathbb{R}^s : W t_1 + t_0 \in T\mathcal{R} \}. $$

We have $\text{vol}(\mathcal{R}_{t_0,T}) = (T/W)^{-s} \text{vol}(\mathcal{R})$.

For $t_0 \in W$, we have $f_i(t_0) \in \mathcal{A}_i$, hence we can apply the results of [BM13, §8–9]. In particular, the functions

$$\tilde{R}_i(m) := \left( \frac{\rho(W, f_i(t_0), z'_i; M)}{W^{n-1}} \right)^{-1} R'(W \epsilon_i m + A_i(t_0))$$

Recall the definition [BM13, (6.2)] of the set of unexceptional residue classes

$$A_i := \left\{ A \in \mathbb{Z}/W \mathbb{Z} : 0 \leq v_p(A) < v_p(W)/3 \text{ for all } p < w(T) \right\}.$$
on \( \{ m : 0 < Wm + \epsilon_i A_i(t_0) \leq T \} \) have a simultaneous pseudorandom majorant as in [BM13 (8.27)] by [BM13 Proposition 9.1, 9.2].

Using \( A_i(t_0) \in A_i \), [BM13 Proposition 6.3] and [BM13 (8.2)] hold, so that the inverse theorem for the Gowers uniformity norms [GTZ12 Theorem 1.3] gives

\[
\max_{1 \leq i \leq r} \| \hat{R}_i - \text{vol}(\mathcal{D}_+^i(1)) \|_{U^{r-1}} = o(1).
\]

Therefore, the generalized von Neumann theorem [GT10 Proposition 7.1], which applies to possibly inhomogeneous linear polynomials, gives

\[
N(T) = \sum\limits_{t_0 \in W} \text{vol}(\mathfrak{A}_{t_0,T}) \prod_{i=1}^r \frac{\rho(W, f_i(t_0), z_i'; M)}{W^{n-1}} \text{vol}(\mathcal{D}_+^i(1)) + o(T^n).
\]

Defining

\[
V_\infty(T) := T^n \text{vol}(\mathfrak{A}) \prod_{i=1}^r \text{vol}(\mathcal{D}_+^i(1))
\]

and

\[
\Sigma(T) := \frac{1}{W^n} \sum_{t_0 \in W} \prod_{i=1}^r \prod_{p \leq w(T)} \frac{\rho(p^{\alpha(p)}, f_i(t_0), z_i'; p^{\gamma_p}(M))}{p^{\alpha(p)(n-1)}},
\]

we have \( N(T) = \Sigma(T) \cdot V_\infty(T) + o(T^n) \).

For the evaluation of \( V_\infty(T) \), recall that \( \text{vol}(\mathcal{D}_+^i(1)) = c_K > 0 \) unless \( K \) is totally imaginary and \( \epsilon_i = -1 \) [BM13 Remark 5.4]. By our assumption on \( \epsilon \) in the totally imaginary case, we have \( V_\infty(T) = c_K' \text{vol}(\mathfrak{A}) T^n = \beta_\infty T^n \).

The proof that \( \Sigma(T) \) may be replaced by \( \prod_p \beta_p \) with an error of order \( o(T^n) \) remains the same as in [BM13 §10] as the lifting results for solutions modulo prime powers from [BM13 §3.3, §4] are unchanged, and [BM13 Proposition 5.5] also holds for our possibly inhomogeneous linear polynomials \( f_i(t) \).

**Proposition 5.3.** Let \( s \geq 2 \). Let \( Y \subset \mathbb{A}^{rn+s}_Q \) be the variety defined by

\[
N_{K/Q}(z_i) = \lambda_i^{-1} g_i(t), \quad (i = 1, \ldots, r)
\]

with \( g_i(t_1, \ldots, t_s) \in \mathbb{Q}[t_1, \ldots, t_s] \) pairwise affinely independent linear polynomials and \( \lambda_i \in \mathbb{Q}^* \).

If \( K \) is not totally imaginary or if the polytope

\[
Q := \{ t \in \mathbb{R}^s : \lambda_i^{-1} g_i(t) > 0 \text{ for } i = 1, \ldots, r \}
\]

contains balls of arbitrarily large radius, then smooth strong approximation holds for \( Y \) off \( \infty \).

**Proof.** We may assume that the norm form \( N_{K/Q} \in \mathbb{Z}[z_1, \ldots, z_n] \) has integral coefficients. By rescaling \( t_1, \ldots, t_s \) and \( z_1, \ldots, z_r \), we may assume that

\[
\lambda_i^{-1} g_i(t) \in \mathbb{Z}[t_1, \ldots, t_s]
\]

for \( i = 1, \ldots, r \) are linear forms with integral coefficients. We define the integral model \( \mathfrak{M} \subset \mathbb{A}^{rn+s}_\mathbb{Z} \) of \( Y \) by the same equations, but considered over \( \mathbb{Z} \).

For smooth strong approximation on \( Y \), we must show: for any finite set of places \( S \subset \Omega_Q \setminus \{ \infty \} \), any \( (q_v) = (q_v, z_{1,v}, \ldots, z_{r,v}) \in Y^{sm}(\mathbb{A}_Q) \) with \( q_v \in \mathfrak{M}(\mathbb{Z}_v) \) for all \( v \notin S \cup \{ \infty \} \), there is a \( q \in Y^{sm}(\mathbb{Q}) \) with \( q \in \mathfrak{M}(\mathbb{Z}) \) for
all \( v \notin S \cup \{ \infty \} \) and \( q \) arbitrarily close to \( q_v \) for all \( v \in S \). This is the same as finding \( q \in Y^{\text{sm}}(\mathbb{Q}) \cap \mathfrak{Q}(\mathbb{Z}_S) \) arbitrarily close to \( q_v \) for all \( v \in S \).

By the implicit function theorem, we may assume that the given \((q_v)\) satisfies
\[
g_1(t_v) \cdots g_r(t_v) \neq 0
\]
for all \( v \in \Omega_S \).

Let \( C \in \mathbb{Z} \) with \( C^{-1} \in \mathbb{Z}_S \) (i.e., all prime factors of \( C \) lie in \( S \)) be such that
\[
q'_v = (t'_v, z'_1, \ldots, z'_r) := (Ct_v, Cz'_1, \ldots, Cz'_r) \in \mathbb{Z}^{rn+s}_v
\]
for all \( v \in S \).

Let \( Y' \) be the variety obtained from \( Y \) by the change of coordinates replacing \((t, z_1, \ldots, z_r)\) by \((Ct, Cz_1, \ldots, Cz_r)\). An integral model \( \mathfrak{Y}' \) of \( Y' \) is given by
\[
N_{K/\mathbb{Q}}(z_i) = f_i(t), \quad (i = 1, \ldots, r)
\]
with \( f_i(t) := C^n \lambda^{-1}_i q_i(C^{-1}t) \in \mathbb{Z}[t] \). This maps \((q_v) \in Y^{\text{sm}}(\mathbb{A}_Q)\) to \((q'_v) \in Y^{\text{sm}}(\mathbb{A}_Q)\) as above with \( q'_v \in \mathfrak{Y}'(\mathbb{Z}_v) \) for all \( v \in S \) (and of course still \( q'_v \in \mathfrak{Y}'(\mathbb{Z}_v) \) for all \( v \notin S \cup \{ \infty \} \)).

By strong approximation on \( \mathbb{A}_\mathbb{Z}^{rn+s} \), we find \( q' = (t', z'_1, \ldots, z'_r) \in \mathbb{Z}^{rn+s} \)

arbitrarily close to \( q_v \) for all \( v \in S \).

Now we are looking for a point \( q'' = (t'', z''_1, \ldots, z''_r) \in \mathbb{Z}^{rn}(\mathbb{Q}) \cap \mathfrak{Y}'(\mathbb{Z}) \)

very close to \( q' \) in the \( v \)-adic topology for all \( v \in S \). This translates into the condition
\[
q'' \equiv q' \pmod{M} \quad (5.9)
\]
for a positive integer \( M = \prod_{p \in S} p^{m'} \) for some sufficiently large integer \( m' \).

Once we have found such a \( q'' \in \mathbb{Z}^{rn}(\mathbb{Q}) \cap \mathfrak{Y}'(\mathbb{Z}) \), then this is also very close to \( q'_v \in \mathfrak{Y}'(\mathbb{Z}_v) \) for all \( v \in S \), and then \( q := (C^{-1}t'', C^{-1}z''_1, \ldots, C^{-1}z''_r) \in \mathbb{Z}^{rn}(\mathbb{Q}) \cap \mathfrak{Y}'(\mathbb{Z}_S) \)

is very close to \( q_v \) for all \( v \in S \), completing the proof.

By definition of \( N(T) \) in \((5.4)\), where the condition \( f_1(t''_1) \cdots f_r(t''_r) \neq 0 \)

ensures \( q'' \in \mathbb{Z}^{rn}(\mathbb{Q}) \), it is enough to show \( N(T) > 0 \) for some \( T \).

There is a set \( \mathfrak{K} \) with positive volume satisfying the conditions of Theorem \( 5.2 \).

Indeed, if \( K \) is not totally imaginary, there is an \( \epsilon \in \{ \pm \}^r \) such that
\[
Q_{\epsilon} := \{ t \in \mathbb{R}^n : \epsilon_i(f_i(t) - f_i(0)) > 0 \text{ for } i = 1, \ldots, r \},
\]
(which is a cone whose vertex is the origin) is non-empty. If \( K \) is totally imaginary, note that our condition on \( Q \) is equivalent to the condition that
\[
Q' := \{ t \in \mathbb{R}^n : f_i(t) > 0 \text{ for } i = 1, \ldots, r \}
\]
contains balls of arbitrarily large radius because of the way we defined \( f_i(t) \) from \( \lambda_i^{-1} g_i(t) \). For \( R \in \mathbb{R}_{>0} \), let \( t_R \) be the center of a ball of radius \( R \) contained in \( Q' \). For sufficiently large \( R \), we have \( f_i(t_R) > f_i(0) \), hence \( t_R \) lies in the cone \( Q_{\epsilon} \) for \( \epsilon = (+, \ldots, +) \), which is therefore non-empty. For any \( K \), a point \( t \) in \( Q_{\epsilon} \), sufficiently close to the origin satisfies \( f_i(t) < 1 \) for \( i = 1, \ldots, r \), hence lies in \( \Omega_{\epsilon} \) as in Theorem \( 5.2 \) and the same is true for a small enough closed ball \( \mathfrak{K} \) around \( t \).

Hence an application of Theorem \( 5.2 \) with this \( \mathfrak{K} \) gives \( N(T) = \beta_{\infty} \prod_p \beta_{p} T^\epsilon + o(T^\epsilon) \) for large enough \( T \), where \( \beta_{\infty} = \epsilon_K \text{ vol}(\mathfrak{K}) > 0. \)
Finally, we show that $\prod_p \beta_p > 0$. By [BM13] Proposition 5.5 (which also holds for our inhomogeneous linear polynomials) there is an $L' > 0$ such that $\prod_{p > L'} \beta_p > 0$. Now we show that $\beta_p > 0$ for any prime $v = p$. Let

$$m' := 2(v_p(M) + \max_{i=1,...,r} v_p(f_i(t'_p)) + v_p(n)) + 1.$$

Let $q'' = (t'', z'', ..., z'_r) \in \mathbb{Z}^{rn+s}$ be such that $q'' \equiv q'_p (mod p^{m'})$. Then $\rho(p^{m'}, f_i(t''), z''_i, p^{v_p(M)}) > 0$. Since $f_i(t'') \neq 0 \in \mathbb{Z}/p^{m'}\mathbb{Z}$ because $f_i(t'_p) \neq 0 \in \mathbb{Z}_p$ by (5.7), we can apply [BM13] Lemma 3.4 to obtain

$$\frac{\rho(p^m, f_i(t''), z''_i, p^{v_p(M)})}{p^{m(n-1)}} = \frac{1}{p^{m'(n-1)}}$$

for all $m \geq m'$ and all $t'' \in (\mathbb{Z}/p^{m'}\mathbb{Z})^s$ satisfying $t'' \equiv t'' (mod p^{m'})$. Since $t'$ was chosen very close to $t'_p$ for $p \in S$, and since $v_p(M) = 0$ for $p \notin S \cup \{\infty\}$, we note that $t'' \equiv t'_p (mod p^{m'})$ implies $t'' \equiv t' (mod p^{v_p(M)})$. Therefore, the $m$-th approximation $\beta_{p,m}$ of $\beta_p$ has at least $p^{(m-m')s}$ summands from these $t''$ that are at least $p^{-ms} \cdot (p^{-m'(n-1)} - 1)$, hence $\beta_{p,m} \geq p^{-m'(s+r(n-1))}$. Therefore, $\beta_p > 0$ for all primes $p$.

The complement of the hyperplanes \{ $g_1(t) = 0$, ..., $g_r(t) = 0$ \} consists of open polytopes. The following lemma shows that essentially every other such polytope occurs as the interior of a connected component of $p(X_{sm}^r(\mathbb{R}))$ if $K$ is totally imaginary.

**Lemma 5.4.** Let $X$ be as in Theorem [1.3] with $P(t) = cg_1(t)^{e_1} \cdots g_r(t)^{e_r}$ as in Proposition [5.1] for $k = \mathbb{Q}$.

If $K$ is not totally imaginary, then $p(X_{sm}^r(\mathbb{R})) = \mathbb{R}^s$. If $K$ is totally imaginary, then the connected components of $p(X_{sm}^r(\mathbb{R}))$ are precisely the sets

$$Q_{c} := \left\{ t \in \mathbb{R}^s : \begin{array}{l} g_i(t) = 0 \text{ for at most one } i \in \{1, \ldots, r\} \text{ with } e_i = 1 \\ e_ig_i(t) > 0 \text{ for all other } i \in \{1, \ldots, r\} \end{array} \right\}$$

for all $c = (c_1, \ldots, c_r) \in \{\pm\}^r$ such that $(c_1)^{e_1} \cdots (c_r)^{e_r} \cdot c > 0$.

**Proof.** Recall from the proof of Proposition [5.1] that the singular locus of $X$ is the union of all \{ $g_i(t) = g_j(t) = u_l = u_m = 0$ \} for $l \neq m \in \{1, \ldots, n\}$ and $i, j \in \{1, \ldots, r\}$ with either $i = j$ and $e_i > 1$ or $i \neq j$.

If $K$ is not totally imaginary, the norm form $N_{K/\mathbb{Q}}$ is indefinite over $\mathbb{R}$, and for any $t \in \mathbb{R}^s$, the number $P(t)$ can be represented by $N_{K/\mathbb{Q}}(z)$ for some $z \in \mathbb{R}^n$ with $(t, z) \in X_{sm}^r(\mathbb{R})$.

If $K$ is totally imaginary, the norm form $N_{K/\mathbb{Q}}$ is positive definite over $\mathbb{R}$, hence for any $(t, z) \in X(\mathbb{R})$ with $P(t) = 0$, we have $z = 0$, hence $t \in p(X_{sm}^r(\mathbb{R}))$ if and only if $g_i(t) = 0$ for at most one $i \in \{1, \ldots, r\}$ that satisfies additionally $e_i = 1$. This clearly splits into the connected components described in the statement of the lemma.

**Proof of Theorem 1.3.** We want to apply Lemma [5.1] Let $U := X \cap \{ P(t) \neq 0 \}$. Let $f : T \to X_{sm}^r$ be a universal torsor, with local description $\mathcal{T}_U \to U$ as in Proposition [5.1] with associated $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}^\times$ and $\xi \in \mathbb{K}^\times$. Also by Proposition [5.1] $\mathcal{T}_U$ is geometrically rational, hence geometrically integral.
Let $Y \subset \mathbb{A}_Q^{\infty n+s}$ be defined by
\[ N_{K/Q}(z_i) = \lambda_i^{-1} g_i(t), \quad (i = 1, \ldots, r). \]
By Proposition 5.1, we have an open immersion $\mathcal{T}_U \subset Y$.

The morphism $g : Y \to X$ defined by $(t, z_1, \ldots, z_r) \mapsto (t, \xi z_1^e \cdots z_r^e)$ clearly agrees on $\mathcal{T}_U$ with $f : T \to X^{sm}$.

If $K$ is not totally imaginary, every such $Y$ satisfies strong approximation off $\infty$ by Proposition 5.3, hence Lemma 3.1 implies Theorem 1.2.

Furthermore, $C = p(X^{sm}(\mathbb{R})) = \mathbb{R}^s$ by Lemma 5.4.

If $K$ is totally imaginary, Proposition 5.3 gives smooth strong approximation off $\infty$ not necessarily for all $Y$. Following the proof of Lemma 3.1, we see that $X^{sm}(k)$ is dense in the image of $V^{Br(X^{sm})}$ in $X(K)$, where $V$ is the set of all $(p_v) \in X^{sm}(K)$ that can be lifted to $(r_v) \in T(K)$ for a universal torsor $f : T \to X^{sm}$ such that the associated $Y$ satisfies strong approximation off $\infty$.

To complete the proof of Theorem 1.2, it remains to show that $p^{-1}(C) \times X^{sm}(K)$ is contained in $V$. Indeed, consider $(p_v) = (t_v, z_v) \in X^{sm}(K)$ with $t_{\infty}$ in an unbounded connected component $Q_\epsilon$ of $C$ for some $\epsilon \in \{\pm\}^r$ as in Lemma 5.4. Then we have its lift $(r_v) \in T(K)$ for some universal torsor $f : T \to X^{sm}$, and very close to it $(r'_v) = (t'_v, z'_{1,v} \cdots z'_{r,v}) \in \mathcal{T}_U(K)$. By choosing $(r'_v)$ close enough to $(r_v)$, we know that $(f(r'_v)) = (t'_v, \xi z_1^{e_1} \cdots z_r^{e_r}) \in U(K)$ has $t'_{\infty}$ on the same $Q_\epsilon$ as $t_{\infty}$. Since $P(t'_{\infty}) \neq 0$, we have $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ with $\epsilon_i = \text{sign}(g_i(t'_{\infty})) \in \{\pm\}$. Since $(r'_v)$ satisfies the equations defining $\mathcal{T}_U$ as in Proposition 5.1, we have $\lambda_i^{-1} g_i(t'_{\infty}) = N_{K/Q}(z'_{i,\infty}) > 0$, hence $\text{sign}(\lambda_i^{-1}) = \text{sign}(g_i(t'_{\infty})) = \epsilon$. Therefore, the associated $Q$ in Proposition 5.3 is precisely the interior of $Q_\epsilon$. Since $Q_\epsilon$ contains balls of arbitrarily large radius by assumption, the same holds for $Q$, and Proposition 5.3 tells us that $Y$ satisfies smooth strong approximation, hence $(p_v) \in V$ as required.

If $g_1(t), \ldots, g_r(t)$ are linear forms, then the conditions $\epsilon_i g_i(t) > 0$ define open halfspaces in $\mathbb{R}^s$ bounded by the hyperplanes $\{g_i(t) = 0\}$ through the origin. Therefore, the interior of every connected component of $p(X^{sm}(\mathbb{R}))$ is a cone whose vertex is the origin, hence every connected component contains balls of arbitrarily large radius, and $C = p(X^{sm}(\mathbb{R}))$.

**Remark 5.5.** For $X$ as in Theorem 1.2, write $P(t) = c g_1(t)^{e_1} \cdots g_r(t)^{e_r}$ as in Proposition 5.1. Under the assumption that $\text{gcd}(e_1, \ldots, e_r) = 1$, the Picard group Pic$(X^{sm})$ is free of finite rank, hence $\text{Br}_1(X^{sm})/\text{Br}_0(X^{sm}) \cong H^1(k, \text{Pic}(X^{sm}))$ is finite.

Hence by Lemma 2.3, $X$ also satisfies central strong approximation with algebraic Brauer–Manin obstruction off $\infty$ under the assumption that $K$ is not totally imaginary or that $g_1(t), \ldots, g_r(t)$ are linear forms.

By Remark 2.2, Theorem 1.2 implies that $X$ also satisfies the smooth integral Hasse principle (in the sense of Definition 2.1). This can be stated explicitly as follows.

**Corollary 5.6.** Let $P(t) \in \mathbb{Z}[t_1, \ldots, t_s]$ be a product of linear polynomials over $\mathbb{Z}$ that are over $\mathbb{Q}$ pairwise proportional or affinely independent. Let $K/Q$ be an extension of number fields of degree $n$, and let $N_{K/Q}(z) \in \mathbb{Z}$.
\[ \mathbb{Z}[z_1, \ldots, z_n] \] be an associated norm form with integral coefficients. Suppose that \( K \) is not totally imaginary or that the factors of \( P(t) \) are linear forms.

Let \( X \subset \mathbb{A}^{n+1}_\mathbb{Z} \) be the affine scheme defined by \[ P(t) = N_{K/Q}(z). \]

Let \( X = X_Q \) be the generic fiber.

If there are \( q_v = (t_v, z_v) \in X(\mathbb{Z}) \) for all finite places \( v \in \Omega_Q \) and \( q_\infty = (t_\infty, z_\infty) \in X(\mathbb{R}) \) such that \( (q_v)_{v \in \Omega_Q} \) is orthogonal to \( Br_1(X^{\text{sm}}) \), with \( P(t_v) \in \mathbb{Z}_v^{\times} \) for almost all \( v < \infty \) and \( P(t_v) \in \mathbb{Z}_v^{\times} \) for all other \( v \in \Omega_Q \), then there are integral points on \( X \).

**Proof.** The conditions on \( P(t_v) \) ensure that \( (q_v) \in U(\mathbb{A}_k) \subset X^{\text{sm}}(\mathbb{A}_k) \), where \( U := X \cap \{ P(t) \neq 0 \} \subset X^{\text{sm}} \). Hence \( (q_v) \) lies in the set \( \{2, 1\} \). As in Remark 2.2 we obtain an integral point on \( X \). \( \square \)

6. A counterexample

Now we consider the case that \( [K : k] = 2 \) and \( P(t) \) is the product of pairwise distinct linear factors.

**Proposition 6.1.** Let \( k \) be a field of characteristic 0 with \( H^3(k, \mathbb{Q}^\times) = 0 \). Let \( K := k(\sqrt{d}) \) be a quadratic extension of \( k \). Let \( P(t) = c(t - a_1) \cdots (t - a_r) \in k[t] \) with pairwise distinct \( a_1, \ldots, a_r \in k \). Let \( X \) be defined by \( \{ t \} \) with \( s = 1 \).

Then \( Br(X)/Br_0(X) \cong (\mathbb{Z}/2\mathbb{Z})^{r-1} \) is generated by the classes of the quaternion algebras \( (t - a_i, d) \in Br(X) \), for \( i = 1, \ldots, r-1 \).

**Proof.** Let \( U \) be defined again by \( P(t) \neq 0 \). Analogously to (4.9), we have the exact sequence

\[ 0 \to \mathbb{Z} \to \mathbb{Z}^r \oplus \mathbb{Z}[K/k] \to \widehat{T} \to 0, \]

hence \( \widehat{T} \cong \mathbb{Z}^{m-1} \oplus \mathbb{Z}[K/k] \) as a \( \Gamma_k \)-module. Furthermore, we have \( \widehat{M} \cong \mathbb{Z}[K/k]^m \).

We have \( H^1(k, \text{Pic}(X)) \cong H^1(K/k, \text{Pic}(X_K)) \cong H^2(K/k, K[U]^{\times}/K^{\times}) \cong (\mathbb{Z}/2\mathbb{Z})^{m-1} \).

Indeed, for the first isomorphism, we note that \( \text{Pic}(X) \) is torsion-free and split by \( K \), for the second isomorphism, we use \( \mathbb{H} \) and \( H^1(K/k, \widehat{M}) = 0 \) for \( i > 0 \) since \( \widehat{M} \cong \mathbb{Z}[K/k]^m \) is an induced Gal(\( K/k \))-module, and for the third isomorphism, we note that \( K[U]^{\times}/K^{\times} \) is \( \mathbb{Z}^{m-1} \oplus \mathbb{Z}[K/k] \) as a Gal(\( K/k \))-module, as in our computation of \( \widehat{T} \) above.

Since \( X \) is smooth, Lemma 4.4 implies \( Br(X)/Br_0(X) \cong (\mathbb{Z}/2\mathbb{Z})^{m-1} \).

Now we describe \( Br(X)/Br_0(X) \) explicitly. Let \( \beta_i \) be the quaternion algebra \( (t - a_i, d) \) over \( k(X) \).

First we show that \( \beta_i \in Br(X) \). Indeed, \( (t - a_i, d) \) is clearly well-defined on \( U_i = \{ t \neq a_i \} \subset X \), and \( (P(t)/(t - a_i), d) \) is well-defined on \( U_i' = \bigcap_{j \neq i} \{ t \neq a_j \} \). Since \( U_i \cup U_i' = X \) and \( (t - a_i, d) = (P(t)/(t - a_i), d) \) in \( Br(k(X)) \) (since \( P(t) = N_{K/k}(z) \) implies \( (P(t), d) = (N_{K/k}(z), d) = 0 \)), we can extend \( (t - a_i, d) \) to a well-defined element of \( Br(X) \).
Next we show that $\beta_1, \ldots, \beta_{r-1}$ are $\mathbb{Z}/2\mathbb{Z}$-linearly independent elements in $\text{Br}(k(X))/\text{Br}_0(X)$. Again let $p : X \to \mathbb{A}^1_k$ be the projection to the $t$-coordinate. This induces a map $\text{Br}(k(t)) \to \text{Br}(k(X))$. Let $\eta$ be the generic point of $\mathbb{A}^1_k$ and $X_\eta$ the generic fiber of $p$. By [Wit35 Satz, p. 465], the kernel of $\text{Br}(k(t)) \to \text{Br}(k(X))$ is generated by $(P(t), d)$, hence the induced map $\phi : \text{Br}(k(t)) \to \text{Br}(k(X))/\text{Br}_0(X)$ has kernel generated by $(P(t), d)$ and Br$(k)$. For $n_1, \ldots, n_{r-1} \in \mathbb{Z}/2\mathbb{Z}$, let $\beta := \sum_{i=1}^{r-1} n_i \beta_i \in \text{Br}(k(t))$. We claim that $\beta \in \ker \phi$ if and only if $\beta = 0$, which would imply the linear independence of $\beta_1, \ldots, \beta_{r-1}$ and complete the proof of this theorem.

Indeed, let $\chi$ be the non-trivial character of $\text{Gal}(K/k)$, and let $D_i$ be the divisor $\{t = a_i\}$ of $\mathbb{A}^1_k$, for $i = 1, \ldots, r$. By [CTSD94 Proposition 1.1.3], the residue of $\beta_i$ on $D_i$ is $\chi$, and the residue of $\beta_i$ on any other divisor of $\mathbb{A}^1_k$ is 0. Hence the residue of $(P(t), d)$ is $\chi$ on $D_1, \ldots, D_r$ and 0 on any other divisor, and any element of $\ker \phi$ has the same residue (with 0) on all of $D_1, \ldots, D_r$. Since $\beta$ has residue 0 on $D_r$, it must have residue 0 on $D_1, \ldots, D_{r-1}$ as well, which is only possible for $n_1 = \cdots = n_{r-1} = 0$, i.e., $\beta = 0$. \hfill $\square$

The following example illustrates that in Question 1.3, an unboundedness condition at an archimedean place is necessary. We construct a variety over $\mathbb{Q}$ with one bounded and one unbounded connected component over $\mathbb{R}$ and a point orthogonal to the Brauer group lying on the bounded component that cannot be approximated arbitrarily close. In particular, it is not enough to require that $X(k_v)$ is unbounded for one archimedean $v$.

**Example 6.2.** Let $X \subset \mathbb{A}^3_\mathbb{Q}$ be defined by

$$t(t - 2)(t - 10) = x^2 + y^2.$$  \hfill (6.1)

We note that the projection $p : X \to \mathbb{A}^1_\mathbb{Q}$ to the $t$-coordinate has the bounded connected component $[0, 2]$ and the unbounded connected component $[10, \infty)$. In fact, $X(\mathbb{R})$ has precisely two connected components, namely $X_1 := p^{-1}([0, 2])$ bounded, $X_2 := p^{-1}([10, \infty))$ unbounded.

Our variety $X$ has an adelic point $(q_v) \in X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p)$ given by

$$q_v = (t_v, x_v, y_v) := \begin{cases} (5, x_5, y_5), & v = 5, \\ (1, 3, 0), & v \neq 5 \end{cases}$$

where $(x_5, y_5) \in \mathbb{Z}_5^2$ is a solution of $x_5^2 + y_5^2 = -75$, which exists by Hensel’s lemma.

This point is orthogonal to $\text{Br}(X)$. Indeed, $\text{Br}(X)/\text{Br}_0(X)$ is generated by $\beta_1 := (t, -1)$ and $\beta_2 := (t - 2, -1)$ by Proposition 6.1. We have

$$\sum_{v \in \Omega_\mathbb{Q}} \beta_i(q_v) = \sum_{v \neq 5} \text{inv}_v(\beta_i((1, 3, 0))) + \text{inv}_5(\beta_i((5, x_5, y_5)))$$

$$= \text{inv}_5(\beta_i((1, 3, 0))) + \text{inv}_5(\beta_i((5, x_5, y_5))) = 0$$

since $(1, -1)_5 = (5, -1)_5 = (-1, -1)_5 = (3, -1)_5 = 0$.

Our goal is to show by contradiction that there is no integral point $q = (t, x, y) \in X(\mathbb{Z})$ that is very close to our adelic point $(q_v)$ at the places 2 and 5. More precisely, we show that $|t - 2|_2 \leq \frac{1}{8}$ and $|t - 5|_5 \leq \frac{1}{2}$ are impossible. This is done via the following claim.
Claim: Given \((q'_t) = (t'_v, x'_v, y'_v) \in (\mathbb{X}(\mathbb{R}) \times \prod_p X(\mathbb{Z}_p))^\text{Br}(X)\) satisfying \(|t'_v - 1|_v \leq \frac{1}{5}\), then \(q'_\infty\) lies in the bounded real component \(X_1\).

This claim leads to our goal as follows. A point \(q \in X(\mathbb{Z})\) very close to \((q_t)\) at 2 and 5 lies on \(X_1\) by the claim. By the product formula, we get the contradiction

\[
1 = \prod_{v \in \Omega_\mathbb{k}} |t|_v \leq |t|_\infty \cdot |t|_5 \leq 2 \cdot \frac{1}{5} < 1,
\]

using \(t \in \mathbb{Z} \setminus \{0\}\) and \(|t - 5|_5 \leq \frac{1}{25}\). Hence such a \(q \in X(\mathbb{Z})\) does not exist, and our adelic point \((q_t)\) cannot be approximated closely.

It remains to prove the claim. For finite \(p \neq 2, 5\) and any \(q'_p = (t'_p, x'_p, y'_p) \in X(\mathbb{Z}_p)\), we claim

\[
\beta_i(q_p) = 0 \text{ in } \text{Br}(\mathbb{Q}_p) \text{ for } i = 1, 2.
\]

(6.2)

If the Legendre symbol \((\frac{\cdot}{p}) = 1\), then \((t'_p, -1) = 0 \in \text{Br}(\mathbb{Q}_p)\) for any \(t'_p \in \mathbb{Q}_p^\times\). Therefore, we may assume \((\frac{\cdot}{p}) = -1\), hence \(\mathbb{Q}_p(\sqrt{\cdot})/\mathbb{Q}_p\) is unramified of degree 2, so \(x'^2 + y'^2\) has even valuation at \(p\). If \(v_p(t'_p) = 0\), then \((t'_p, -1) = 0 \in \text{Br}(\mathbb{Q}_p)\). If \(v_p(t'_p) \geq 1\), then \(v_p(t'_p - 2) = v_p(t'_p - 10) = 0\); because of \((6.1)\), \(v_p(t'_p)\) is also even, hence \((t'_p, -1) = 0 \in \text{Br}(\mathbb{Q}_p)\). Similarly, we prove \((t'_p - 2, -1) = 0\).

For any \((q'_t) = (t'_v, x'_v, y'_v) \in (\mathbb{X}(\mathbb{R}) \times \prod_p X(\mathbb{Z}_p))^\text{Br}(X)\) with \(|t - 1|_2 \leq \frac{1}{5}\), obviously \(t \neq 0\). By \((6.2)\),

\[
\sum_{v \in \Omega_\mathbb{k}} \text{inv}_v(\beta_i(q)) = \text{inv}_2(\beta_i(q)) + \text{inv}_5(\beta_i(q)) + \text{inv}_\infty(\beta_i(q))
\]

\[
\begin{cases}
(t, -1)_\infty, & i = 1, \\
1 + (t - 2, -1)_\infty, & i = 2,
\end{cases}
\]

since \((t, -1)_2 = (t - 2, -1)_2 = 0\) (because \((\frac{\cdot}{2}) = 1\)) and \((t, -1)_2 = (1, -1)_2 = 0\) and \((t - 2, -1) = (1 - 2, -1)_2 = 1\) (because \(|t - 1|_2 \leq \frac{1}{5}\).

By global class field theory, \(\sum_{v \in \Omega_\mathbb{k}} \text{inv}_v(\beta_i(q)) = 0\), hence \((t, -1)_\infty = 0\) and \((t - 2, -1)_\infty = 1\). Therefore, \(0 \leq t \leq 2\), hence \(q'_t \in X_1\), which was our claim.

References

[BD13] M. Borovoi and C. Demarche. Manin obstruction to strong approximation for homogeneous spaces. Comment. Math. Helv., 88(1):1–54, 2013.

[BM13] T. D. Browning and L. Matthiesen. Norm forms for arbitrary number fields as products of linear polynomials, arXiv:1307.7641, 2013.

[BMS12] T. D. Browning, L. Matthiesen, and A. N. Skorobogatov. Rational points on pencils of conics and quadrics with many degenerate fibres, arXiv:1209.0207, 2012.

[Bor96] M. Borovoi. The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer. J. reine angew. Math., 473:181–194, 1996.

[CT03] J.-L. Colliot-Thélène. Points rationnels sur les fibrations. In Higher dimensional varieties and rational points (Budapest, 2001), volume 12 of Bolyai Soc. Math. Stud., pages 171–221. Springer, Berlin, 2003.

[CTCS80] J.-L. Colliot-Thélène, D. Coray, and J.-J. Sansuc. Descente et principe de Hasse pour certaines variétés rationnelles. J. reine angew. Math., 320:150–191, 1980.
[CTH12] J.-L. Colliot-Thélène and D. Harari. Approximation forte en famille. *J. reine angew. Math.*, to appear, arXiv:1209.0717, 2012.

[CTS77] J.-L. Colliot-Thélène and J.-J. Sansuc. La R-équivalence sur les tores. *Ann. Sci. École Norm. Sup.* (4), 10(2):175–229, 1977.

[CTS87] J.-L. Colliot-Thélène and J.-J. Sansuc. La descente sur les variétés rationnelles. II. *Duke Math. J.*, 54(2):375–492, 1987.

[CTS00] J.-L. Colliot-Thélène and A. N. Skorobogatov. Descent on fibrations over $\mathbf{P}^1_k$ revisited. *Math. Proc. Cambridge Philos. Soc.*, 128(3):383–393, 2000.

[CTSD94] J.-L. Colliot-Thélène and P. Swinnerton-Dyer. Hasse principle and weak approximation for pencils of Severi-Brauer and similar varieties. *J. reine angew. Math.*, 453:49–112, 1994.

[CTSSD87a] J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer. Intersections of two quadrics and Châtelet surfaces. I. *J. reine angew. Math.*, 373:37–107, 1987.

[CTSSD87b] J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer. Intersections of two quadrics and Châtelet surfaces. II. *J. reine angew. Math.*, 374:72–168, 1987.

[CTW12] J.-L. Colliot-Thélène and O. Wittenberg. Groupe de Brauer et points entiers de deux familles de surfaces cubiques affines. *Amer. J. Math.*, 134(5):1303–1327, 2012.

[CTX09] J.-L. Colliot-Thélène and F. Xu. Brauer-Manin obstruction for integral points of homogeneous spaces and representation by integral quadratic forms. *Compos. Math.*, 145(2):309–363, 2009. With an appendix by D. Wei and Xu.

[CTX13] J.-L. Colliot-Thélène and F. Xu. Strong approximation for the total space of certain quadratic fibrations. *Acta Arith.*, 157(2):169–199, 2013.

[Dem11] C. Demarche. Le défaut d’approximation forte dans les groupes linéaires connexes. *Proc. Lond. Math. Soc.* (3), 102(3):563–597, 2011.

[DSW12] U. Derenthal, A. Smeets, and D. Wei. Universal torsors and values of quadratic polynomials represented by norms, arXiv:1202.3567, 2012.

[GT08] B. Green and T. Tao. The primes contain arbitrarily long arithmetic progressions. *Ann. of Math.* (2), 167(2):481–547, 2008.

[GT10] B. Green and T. Tao. Linear equations in primes. *Ann. of Math.* (2), 171(3):1753–1850, 2010.

[GTZ12] B. Green, T. Tao, and T. Ziegler. An inverse theorem for the Gowers $U^{s+1}[N]$-norm. *Ann. of Math.* (2), 176(2):1231–1372, 2012.

[Gun13] F. Gundlach. Integral Brauer-Manin obstructions for sums of two squares and a power. *J. Lond. Math. Soc.* (2), 88(2):599–618, 2013.

[Har94] D. Harari. Méthode des fibrations et obstruction de Manin. *Duke Math. J.*, 75(1):221–260, 1994.

[Har08] D. Harari. Le défaut d’approximation forte pour les groupes algébriques commutatifs. *Algebra Number Theory*, 2(5):595–611, 2008.

[HSW13] Y. Harpaz, A. N. Skorobogatov, and O. Wittenberg. The Hardy–Littlewood conjecture and rational points, arXiv:1304.3333, 2013.

[Kar87] G. Karol’evsky. *The Schur multiplier*, volume 2 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1987.

[KT08] A. Kresch and Yu. Tschinkel. Two examples of Brauer-Manin obstruction to integral points. *Bull. Lond. Math. Soc.*, 40(6):995–1001, 2008.

[Man71] Yu. I. Manin. Le groupe de Brauer-Grothendieck en géométrie diophantienne. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 401–411. Gauthier-Villars, Paris, 1971.

[PR94] V. Platonov and A. Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[Ros61] M. Rosenlicht. Toroidal algebraic groups. *Proc. Amer. Math. Soc.*, 12:984–988, 1961.
[Sko99] A. N. Skorobogatov. Beyond the Manin obstruction. *Invent. Math.*, 135(2):399–424, 1999.

[Sko01] A. N. Skorobogatov. *Torsors and rational points*, volume 144 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[Wei12] D. Wei. On the equation $N_{K/k}(Ξ) = P(t)$. arXiv:1202.4115, 2012.

[Wit35] E. Witt. Über ein Gegenbeispiel zum Normensatz. *Math. Z.*, 39(1):462–467, 1935.

[WX12] D. Wei and F. Xu. Integral points for multi-norm tori. *Proc. Lond. Math. Soc. (3)*, 104(5):1019–1044, 2012.

[WX13] D. Wei and F. Xu. Integral points for groups of multiplicative type. *Adv. Math.*, 232:36–56, 2013.

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