Increasing Potentials in Non-Abelian and Abelian Gauge Theories

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Abstract

An exact solution for an SU(2) Yang-Mills field coupled to a scalar field is given, which has potentials with a linear and a Coulomb part. This may have some physical importance since many phenomenological QCD studies assume a linear plus Coulomb potential. Usually the linear potential is motivated with lattice gauge theory arguments. Here the linear potential is an exact result of the field equations. We also show that in the Nielsen-Olesen Abelian model there is an exact solution in the BPS limit, which has a Coulomb-like electromagnetic field and a logarithmically rising scalar field. Both of these solutions must be cut-off from above to avoid infinite field energy.

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I. INTRODUCTION

In this paper we will examine some simple, exact solutions to the field equations of both non-Abelian and Abelian gauge theories. In both cases the gauge fields will be coupled to a scalar field in the BPS limit \[1\] \[2\] (i.e. the scalar field has zero mass and zero self coupling). We will not apply the usual boundary conditions that the fields vanish at spatial infinity. This means that these solutions will have infinite field energy. Prasad and Sommerfield \[1\] have given an exact, classical solution to the SU(2) Yang-Mills-Higgs field equations in the BPS limit, that had non-singular fields and finite energy. Recently a new exact solution to these field equations was discovered \[3\]. This new solution was found using the analogy between general relativity and Yang-Mills theory. It was similar to the Schwarzschild solution, but the character of the spherical singularity of the “event horizon” was different. Although the fields of this Yang-Mills solution vanished at spatial infinity, the field energy was infinite due to the singularities in the fields.

In this paper we will examine another exact solution to this SU(2) Yang-Mills-Higgs system, which has neither fields that vanish at infinity, nor finite energy. The time component of the gauge fields and the scalar fields are both found to increase linearly with distance from the origin, while the space components of the gauge fields have a Coulomb-like behaviour. This is of interest since some phenomenological studies of QCD use a linear plus Coulomb potential \[4\]. Usually the linear, confining part of the potential is motivated using lattice gauge theory arguments \[5\]. In this paper the linear potential is an exact, analytical result of the field equations. We will also take a look at a model with an Abelian gauge field coupled to a scalar field. The solution for this model gives a Coulomb-like potential for the gauge field and a rising logarithmic scalar field. This solution suffers from both a singularity at the origin, due to the Coulomb-like potential, and infinite field energy, due to the logarithmic scalar field.

We will briefly setup the field equations and simplify them using a generalized Wu-Yang ansatz \[6\]. A short review of the Schwarzschild-like classical solutions will be given so that...
comparisons between these solutions can be made.

II. LINEAR POTENTIAL FOR SU(2) YANG-MILLS THEORY

The system which we consider is an SU(2) gauge field coupled to a scalar field in the triplet representation. The Lagrangian for this system is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} (D_\mu \phi^a)(D^\mu \phi^a)$$

(1)

where the field tensor is defined in terms of the gauge fields, $W_\mu^a$, by

$$F_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon^{abc} W_\mu^b W_\nu^c$$

(2)

and the covariant derivative of the scalar field is

$$D_\mu \phi^a = \partial_\mu \phi^a + g\epsilon^{abc} W_\mu^b \phi^c$$

(3)

The Lagrangian of Eq. (1) is in the BPS limit where the scalar fields’ mass and self interaction are taken as zero. In order to simplify the Euler-Lagrange equations which result from this system one uses a generalized Wu-Yang ansatz

$$W_i^a = \epsilon_{aij} \frac{r_j}{gr^2} [1 - K(r)] + \left( \frac{r_i r_a}{r^2} - \delta_{ia} \right) \frac{G(r)}{gr^2}$$

$$W_0^a = \frac{r^a}{gr^2} J(r)$$

$$\phi^a = \frac{r^a}{gr^2} H(r)$$

(4)

The second term of $W_i^a$ is usually not written down in the Wu-Yang ansatz, and it is symmetric in its free indices as compared to the first term which is antisymmetric. In terms of this ansatz the Euler-Lagrange equations for this system are simplified into the following set of four coupled, non-linear differential equations

$$r^2 K'' = K(K^2 + G^2 + H^2 - J^2 - 1)$$

$$r^2 G'' = G(K^2 + G^2 + H^2 - J^2 - 1)$$

$$r^2 J'' = 2J(K^2 + G^2)$$

$$r^2 H'' = 2H(K^2 + G^2)$$

(5)
where the primes denote differentiation with respect to $r$. In Refs. [3] and [7] it was found that these equations have the following solution

\[
K(r) = \frac{\mp \cos \theta \ C r}{1 \pm C r}
\]

\[
G(r) = \frac{\mp \sin \theta \ C r}{1 \pm C r}
\]

\[
J(r) = \frac{\sinh \gamma}{1 \pm C r}
\]

\[
H(r) = \frac{\cosh \gamma}{1 \pm C r}
\]

(6)

where $C$, $\gamma$, and $\theta$ are arbitrary constants. In [3] and [7] only the special case $\theta = 0$ was given. Inserting these functions back into the expressions for the gauge and scalar fields of Eq. (4) it is found that both the plus and minus solutions of Eq. (3) have singularities at $r = 0$ (plus and minus refer to the signs in the denominators of Eq. (6)). This singularity is of the same kind as singularities which are found in other classical field theory solutions (e.g. the Coulomb solution and the Schwarzschild solution). In addition the minus solution has a spherical singularity at $r = r_0 = 1/C$. This feature motivated a loose comparison between this solution and the Schwarzschild solution of general relativity. It was speculated that the spherical barrier at $r_0$ might give a confinement mechanism similar to the confinement mechanism of the true Schwarzschild solution. By treating this solution as a background field it was shown that it did tend to confine a scalar, color charged particle to the region $r < r_0$ [8]. In addition this confined scalar particle behaved as a fermion due to the spin from isospin mechanism [9]. However even though the character of the singularity at $r = 0$ is the same for both the Schwarzschild solution and its Yang-Mills counterpart, the nature of the spherical singularity is different. In the Schwarzschild case the event horizon singularity is not a true singularity, but is rather a coordinate singularity, which can be removed by choosing a different coordinate system in which to express the solution. Transforming the Schwarzschild solution to Kruskal coordinates leaves only a singularity at $r = 0$. In the case of the minus solution given in Eq. (3) the singularity at $r = 1/C$ is a true singularity of the fields.
The new solutions to Eqs. (5) are

\[
K(r) = \cos \theta \quad G(r) = \sin \theta \\
J(r) = H(r) = Ar^2 + \frac{B}{r}
\]

where \(A\), \(B\) and \(\theta\) (and therefore \(K\) and \(G\)) are arbitrary constants. It is interesting to note that this solution cannot be obtained from the first order Bogomolny equations \[2\]. This emphasizes the fact that although all solutions of the Bogomolny equations will also satisfy the Euler-Lagrange equations, the reverse is not necessarily true. Inserting these functions into the expressions for the gauge and scalar fields of Eq. (4) we find that the magnitude of time component of the gauge field and the scalar field increase linearly with \(r\) as well as having a \(1/r^2\) part. The space components of the gauge fields have a Coulomb-like behaviour. Usually, when one talks about a Coulomb plus linear potential in QCD, this is in reference to the time component of the gauge fields only \[4\] (i.e. the time component of the gauge field is the sum of a linear plus Coulomb term). The linear term is thought to be the result of the non-perturbative character of the interaction, and it is conjectured to give the confinement property of the theory. Usually this linear term is motivated using lattice gauge theory, but here it falls out as an analytical result. It is obvious that this field configuration will yield an infinite field energy since some of the fields do not fall off at large \(r\) but rather increase as long as \(A \neq 0\). (If \(A = 0\) and \(\theta = \pi/2\) we find that the fields given by the new solution have the same asymptotic behaviour as \(r \to \infty\) as our previous Schwarzschild-like solution). Also for \(\theta \neq 0\) and/or \(B \neq 0\) one will have singularities at \(r = 0\) for both the time and space components of the gauge fields. Both of these features are undesirable. However if one thinks of the solution of Eq. (7) as an isolated spin 0 “quark” (the scalar field in this model) then the fact that one finds an infinite energy is what is expected from other heuristic arguments of confinement, which contend that it should cost an infinite amount of energy to create an isolated quark. For the special cases when \(\theta = 0\) and \(B = 0\) the singularities vanish in \(W_\lambda^a\), and \(\phi^a\) and \(W_0^a\) respectively. (For the case \(\theta = 0\) the space component of the gauge fields vanish altogether). The calculation of the field energy is straightforward.
Using the functions of Eq. (7) the energy in the fields is

\[ E = \int T^{00} d^3x \]

\[ = \frac{4\pi}{g^2} \int_{r_b}^{r_c} \left( \frac{J^2}{r^2} + \frac{(rJ' - J)^2}{2r^2} + \frac{H^2}{r^2} + \frac{(rH' - H)^2}{2r^2} \right) dr \]

\[ = \frac{4\pi A^2}{g^2} (r_c^3 - r_b^3) - \frac{8\pi B^2}{g^2} \left( \frac{1}{r_c^3} - \frac{1}{r_b^3} \right) \quad (8) \]

We have used the fact that \( K^2 + G^2 = 1 \) and \( K' = G' = 0 \). The integral is cut off from above because of the linearly increasing gauge and scalar fields, and it is cut off from below because of the singularity in \( \phi^a \) and \( W_0^a \) at \( r = 0 \). If one lets \( r_c \to \infty \) the \( A^2 \) term becomes infinite due to the linearly increasing fields. The linearly increasing fields (especially \( W_0^a \)) are similar to some phenomenological QCD potentials which are thought to give the confinement property. However, these potentials are usually motivated using lattice gauge theory arguments, rather than being analytical results. If one lets \( r_b \to 0 \) then the \( B^2 \) term gives an infinite field energy due to the singular fields at \( r = 0 \). If one takes the special case \( B = 0 \) then only the linearly increasing fields make the field energy infinite. It is interesting to note that the Coulomb-like singularities in \( W_i^a \) apparently do not lead to a divergence in the field energy if one integrates down to \( r = 0 \).

Both this new solution and the previous general relativity inspired solutions suffer from infinite field energy unless the integrals of the energy density are cut off. In the case of the Schwarzschild-like solutions the infinite energy came from integrating through the two singularities of the solution (i.e. at \( r = 0 \) and \( r = 1/C \)). The gauge and scalar fields of the Schwarzschild-like solution, however, vanished rapidly as \( r \to \infty \). The present linearly increasing solution also has singularities at \( r = 0 \) in \( W_i^a \) if \( \theta \neq 0 \), and in \( \phi^a \) and \( W_0^a \) if \( B \neq 0 \). Unlike the singularities of the Schwarzschild-like solution, the singularity in \( W_i^a \) is fairly benign in that it does not make the integral of the energy density diverge. The singularities in \( \phi^a \) and \( W_0^a \) on the other hand still lead to a divergent field energy. Thus, unless one takes the trivial case where \( A = B = \theta = 0 \), the total field energy still diverges (from the linearly increasing fields and/or from the singularities at \( r = 0 \)), when integrated over all space. In
the special case $B = 0$ the infinite field energy comes entirely from the linearly increasing fields, and the solution can be considered well behaved in the sense the the Coulomb-like singularity in $W_i^a$ at $r = 0$ does not make the energy diverge. As commented earlier, if one views this solution as an isolated spin 0 “quark”, then the infinite energy fits in with the idea that it should take an infinite amount of energy to separate two quarks.

Finally if the present solution is used as a background field in which to study the motion of a test particle one finds that the particle exhibits the spin from isospin phenomenon due to the antisymmetric part of $W_i^a$. If $\theta \neq 0$ a test scalar particle moving in the background field of the solution will behave as a spin 1/2 particle, while a test fermion will have integer spin. The same thing occured when we examined a test scalar particle moving in the background field of the Schwarzschild-like solution. We will present the details of a similar study for the present solution in an upcoming paper.

III. LOGARITHMIC SOLUTION OF THE NIELSEN-OLESEN MODEL

Having looked at a non-Abelian system we now turn to the somewhat easier Abelian electrodynamics model in (2 + 1) dimensions considered by Nielsen and Olesen. The Lagrangian for this model looks the same as that in Eq. (1) with $F_a^{\mu\nu} \rightarrow F_{\mu\nu}$ and $(D_\mu \phi^a)(D_\mu \phi^a) \rightarrow (D_\mu \phi)^*(D_\mu \phi)$. The Lorentz indices only run over $(0,1,2)$ here. The field strength tensor in this case is

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu $$

and the covariant derivative for the scalar field is now

$$ D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi $$

The scalar field $\phi$ is complex and carries no group index. Again we will take the scalar field to have no mass and no self interaction. In order to simplify the field equations which result from the Lagrangian we make the following ansatz
where we are using polar coordinates \((r, \theta)\). In terms of this ansatz the Euler-Lagrange equations of this (2+1) Abelian gauge theory become

\[rF'' + F' = rF \left( \frac{n}{r} - eA \right)^2 \]
\[A'' + \frac{1}{r} A' - \frac{1}{r^2} A = \left( e^2 A - \frac{ne}{r} \right) F^2\] (12)

This system of coupled equations is simpler than the model considered by Nielsen and Olesen who included a mass and self interaction term for the scalar field. If one does not require that the fields vanish as \(r \to \infty\) then a solution to Eq (12) is

\[A(r) = \frac{n}{er} \quad F(r) = F_0 \ln(Cr) + B\] (13)

where \(F_0, B\) and \(C\) are arbitrary constants. As with the solution of the previous section, this solution has a field which increases without bound as \(r \to \infty\). Here it is only the scalar field which increases, and it increases logarithmically rather than linearly. The solution has a Coulomb singularity at \(r = 0\) due to \(A_\theta\), and it has an infinite field energy when integrated over all space.

**IV. CONCLUSIONS**

We have given exact infinite energy solutions to two gauge theories in the BPS limit. The first system was an SU(2) gauge theory coupled to a scalar field in the triplet representation. This system has been well studied and several other infinite energy and finite energy solutions to this model are known. In the solution given here the scalar fields and the time component of the gauge fields increased linearly with \(r\), while the space part of the gauge fields had a Coulomb-like behaviour. The field energy of this solution was infinite due to the increasing fields (and also the singularities in \(\phi^a\) and \(W_0^a\) if \(B \neq 0\)), but surprisingly the Coulomb-like singularity at \(r = 0\) in \(W_i^a\) did not cause the integral of the energy density to
diverge. This should be contrasted with the Coulomb solution of electrodynamics, where the singularity at $r = 0$ does cause the integral of the energy density to diverge. This solution has some features which may be of interest to the study of the confinement problem in QCD. Many phenomenological models of QCD employ a linear plus Coulomb potential to study the spectra of various strong interaction bound states [4]. The Coulomb term is thought to arise for the same reasons as in QED, while the linear, confining term is said to be a result of the non-perturbative nature of the interaction. Usually lattice gauge theory arguments are used to motivate the linearly confining potential. In the solution given here the linearly increasing potential comes out as an exact result of the classical field equations. The break up of the linear and Coulomb parts of the solution given here is not the same as that used in the phenomenological studies. One could do similar studies with the current solution as a background field to determine if it can reproduce some of the successes of these phenomenological studies. In order to do this it would be necessary to give the SU(3) version of this solution. It would be useful to extend the solution to SU(N), which would then incorporate the SU(3) case. As with the SU(N) generalization of the BPS solution [11], and the SU(N) generalization of the Schwarzschild-like solutions [12], it is possible to extend the present solution to SU(N) by using a maximal embedding of the SU(2) solution in SU(N).

The second solution which we considered was for the Nielsen-Olesen model in the BPS limit. This system also gave a solution with an increasing field. In this case it was only the scalar field which increased, and the increase was logarithmic rather than linear. The gauge field of this solution was a simple Coulomb-like potential which had a singularity at $r = 0$. While the physical uses of this infinite energy solution to the Nielsen-Olesen model are unclear, it does share some of the characteristics of the Yang-Mills-Higgs solution despite arising from an Abelian gauge theory.

The view taken here and in our previous paper [3] is that some of the characteristics of the strong interaction (the confinement property in particular) may be explained, at least partially, by considering classical solutions of the Yang-Mills system. Both the Schwarzschild-like solution and the present solution do lead, classically, to a type of confinement [3].
Whether any of these solutions are in fact connected to the actual confinement mechanism remains to be seen. The present solution has the advantage over our previous Schwarzschild-like solution in that it agrees in a loose way with the heuristic expectations of how the confining potential for QCD should behave. It may be that the actual confinement mechanism of QCD is an entirely quantum effect which cannot be studied using these classical solutions. However, given the rich structure displayed by the classical non-Abelian gauge system, and the suggestive nature of these field potentials, it is worthwhile to examine the possibility that the confinement mechanism may be connected, at least partially, to these, or possibly other undiscovered, classical solutions.

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