DIVISIBILITY AND DISTRIBUTION OF 5-REGULAR PARTITIONS

QI-YANG ZHENG

Abstract. In this paper we study \( b_5(n) \), the 5-regular partitions of \( n \). Using the
theory of modular forms, we prove several theorems on the divisibility and distribution
properties of \( b_5(n) \) modulo prime \( m \geq 5 \). In particular, we prove that there are
infinitely many Ramanujan-type congruences modulo prime \( m \geq 5 \).

1. Introduction and statement of results

1.1. Introduction. In number theory, we usually denote \( p(n) \) as the number of the
partitions of \( n \). Ramanujan found the three remarkable congruences as follows:
\[
p(5n + 4) \equiv 0 \pmod{5}, \\
p(7n + 5) \equiv 0 \pmod{7}, \\
p(11n + 6) \equiv 0 \pmod{11}.
\]
Such congruences are called Ramanujan-type congruences.

For \( k \in \mathbb{Z}_{>1} \), we define the \( k \)-regular partitions \( b_k(n) \) by
\[
\sum_{n=0}^{\infty} b_k(n)q^n = \prod_{n=1}^{\infty} \frac{1 - q^{kn}}{1 - q^n}.
\]
We study the arithmetical properties of \( b_5(n) \) in this paper. Hirschhorn and Sellers\,[3] prove that there are infinitely many Ramanujan-type congruences of \( b_5(n) \) modulo 2. Gordon and Ono\,[2] prove that
\[
b_5(5n + 4) \equiv 0 \pmod{5}.
\]

Up to now, Ramanujan-type congruence of \( b_5(n) \) modulo prime \( m \geq 7 \) has not
been found. Our main result is that for each prime \( m \geq 5 \), there exist infinitely many
Ramanujan-type congruences modulo \( m \). For example, we obtain
\[
b_5(2023n + 99) \equiv 0 \pmod{7}
\]
satisfied for each nonnegative integer \( n \). We will give more examples in Section 5.

1.2. Statement of results.

Theorem 1.1. Let \( m \geq 5 \) be a prime. Then a positive density of primes \( l \) have the
property that
\[
b_5 \left( \frac{mln - 1}{6} \right) \equiv 0 \pmod{m}
\]
satisfied for each integer \( n \) with \( (n, l) = 1 \).

The theorem immediately implies that there are infinitely many Ramanujan-type congruences of \( b_5(n) \) modulo \( m \). Moreover, together with the Chinese Remainder Theorem, we obtain that if \( m \) is a squarefree integer coprime to 3, then there are infinitely many Ramanujan-type congruences of \( b_5(n) \) modulo \( m \).

Surprisingly, we do not know whether there is Ramanujan-type congruence of \( b_5(n) \) modulo 3.

Theorem 1.1 also implies that
\[
\#\{0 \leq n \leq X \mid b_5(n) \equiv 0 \pmod{m}\} \gg X,
\]
where \( m \geq 5 \) is a prime. For other residue classes \( i \not\equiv 0 \pmod{m} \), we also provide a useful criterion to obtain similar result.

**Theorem 1.2.** Let \( m \geq 5 \) be a prime. If there exists one \( k \in \mathbb{Z} \) such that
\[
b_5\left(mk + \frac{m^2 - 1}{6}\right) \equiv e \not\equiv 0 \pmod{m},
\]
then for each \( i = 1, 2, \cdots, m - 1 \), we have
\[
\#\{0 \leq n \leq X \mid b_5(n) \equiv i \pmod{m}\} \gg \frac{X}{\log X}.
\]
Moreover, if such \( k \) exists, then \( k < 10(m - 1) \).

The congruence of Gordon and Ono show that our criterion is inapplicable for the case \( m = 5 \).

2. **Notation and definitions**

Our proof is depending on the theory of modular forms. First recall that the Dedekind’s eta function is defined by:
\[
\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),
\]
where \( q = e^{2\pi iz} \). It is well-known that \( \eta(z) \) is holomorphic and never vanishes in the upper half plane.

Now we introduce the \( U \) operator. If \( j \) is a positive integer, we define \( U(j) \) as follows:
\[
(2.1) \quad \left( \sum_{n=0}^{\infty} a(n)q^n \right) | U(j) = \sum_{n=0}^{\infty} a(jn)q^n.
\]
Sometimes the following expression of $U(j)$ operator is more convenient for computation.

$$\left( \sum_{n=0}^{\infty} a(n)q^n \right) \mid U(j) = \sum_{n=0}^{\infty} a(n)q^{\frac{n^2}{j}}. \quad (2.2)$$

We define $M_k(\Gamma_0(N), \chi)_m$ as the reduction mod $m$ of the $q$-expansions of modular forms in $M_k(\Gamma_0(N), \chi)$ with integral coefficients. Moreover, we define $S_k(\Gamma_0(N), \chi)_m$ in a similar way.

3. Proof of Theorem 1.1

Before proving Theorem 1.1, we list some useful results. The following theorem is due to Gordon and Hughes\[1\].

**Theorem 3.1** (B. Gordon, K. Hughes). Let $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ be an $\eta$-quotient for which

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24},$$

$$\sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

$$k := \frac{1}{2} \sum_{\delta \mid N} r_\delta \in \mathbb{Z},$$

then $f(z)$ satisfies

$$f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^k f(z)$$

for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Here $\chi$ is a Dirichlet character modulo $N$ and

$$\chi(n) = \left( \frac{-1}{n} \right) \prod_{\delta \mid N} \delta r_\delta,$$

for each positive odd number $n$.

Though Theorem 3.1 ensures that $f(z)$ is weakly modular, we need to show that the order at the cusps of $\Gamma_0(N)$ are nonnegative(resp. positive) to obtain that $f(z)$ is a modular(resp. cusp) form. The following theorem of Martin\[4, 6\] provides the explicit expression of the order at cusps.

**Theorem 3.2** (Y. Martin). Let $c, d, N$ be positive integers with $d \mid N$ and $(c, d) = 1$ and $f(z)$ is an $\eta$-quotient satisfying the conditions of Theorem 3.1, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\frac{N}{24} \sum_{\delta \mid N} r_\delta(d^2, \delta^2) \frac{1}{\delta(d^2, N)}.$$
The following theorem due to Serre[7] show that cusp forms have very nice arithmetic properties.

**Theorem 3.3** (J.-P. Serre). The set of primes \( l \equiv -1 \pmod{Nm} \) such that
\[
f \mid T(l) \equiv 0 \pmod{m}
\]
for each \( f(z) \in S_k(\Gamma_0(N), \psi)_m \) has positive density, where \( T(l) \) denotes the usual Hecke operator acting on \( S_k(\Gamma_0(N), \psi) \).

**Proof of Theorem 1.1.** For a fixed prime \( m \), let
\[
f(m; z) := \frac{\eta(5z)}{\eta(z)} \eta^a(5mz) \eta^b(mz),
\]
where \( m' := (m \mod 6) \) and \( a := 5 - m', \ b := m' - 1 \). It is easy to show that \( f(m; z) \equiv \eta^{am+1}(5z)\eta^{bm-1}(z) \pmod{m} \) and
\[
\eta^{am+1}(5z)\eta^{bm-1}(z) \in S_{2m}(\Gamma_0(5), \chi_5),
\]
where \( \chi_5(n) = \left( \frac{1}{5} \right) \). On the other hand,
\[
f(m; z) = \sum_{n=0}^{\infty} b_5(n) q^{\frac{24n + m(5n + b) + 4}{24}} \prod_{n=1}^{\infty} (1 - q^{5mn})^a (1 - q^{mn})^b.
\]
Acting the \( U(m) \) operator on \( f(z) \) and since \( U(m) \equiv T(m) \pmod{m} \), obtaining
\[
\sum_{n=0}^{\infty} b_5(n) q^{\frac{24n + m(5n + b) + 4}{24}} | U(m) \equiv \frac{\eta^{am+1}(5z)\eta^{bm-1}(z)}{\prod_{n=1}^{\infty} (1 - q^{5mn})^a (1 - q^{bn})^b} \pmod{m},
\]
where \( T(m) \) denotes usual Hecke operator acting on \( S_{2m}(\Gamma_0(5), \chi_5) \). As for the LHS of (3.3), we have
\[
\sum_{n=0}^{\infty} b_5(n) q^{\frac{24n + m(5n + b) + 4}{24}} | U(m) = \sum_{\substack{n=0 \\ m \mid 6n + 1}}^{\infty} b_5(n) q^{\frac{24n + m(5n + b) + 4}{24}}.
\]
Using Theorem 3.1 and 3.2, one can verify that \( \eta^4(5z)\eta^4(z) \in S_4(\Gamma_0(5)) \) and have the order of 1 at all cusps. Thus we can write \( \eta^{am+1}(5z)\eta^{bm-1}(z) | T(m) = \eta^4(5z)\eta^4(z)g(m; z) \), where \( g(m; z) \in M_{2m-4}(\Gamma_0(5), \chi_5)_m \). Hence
\[
\sum_{\substack{n=0 \\ m \mid 6n + 1}}^{\infty} b_5(n) q^{\frac{6n + 1}{m}} \equiv \eta^{4-a}(5z)\eta^{4-b}(z)g(m; z) \pmod{m}.
\]
Replacing \( q \) by \( q^6 \) shows that
\[
\sum_{\substack{n=0 \\ m \mid 6n + 1}}^{\infty} b_5(n) q^{\frac{6n + 1}{m}} \equiv \eta^{4-a}(30z)\eta^{4-b}(6z)g(m; 6z) \pmod{m}.
\]
Since $b_5(n)$ vanishes for non-integer $n$, so
\begin{equation}
\sum_{n=0}^{\infty} b_5 \left( \frac{mn-1}{6} \right) q^n \equiv \eta^{4-a}(30z) \eta^{4-b}(6z) g(m; 6z) \pmod{m}.
\end{equation}

Moreover, one can verify that $\eta^{4-a}(30z) \eta^{4-b}(6z) \in S_2(\Gamma_0(180))$. Let
\begin{equation}
\sum_{n=0}^{\infty} a(n) q^n = \eta^{4-a}(30z) \eta^{4-b}(6z) g(m; 6z) \in S_{2m-2}(\Gamma_0(180), \chi_5).
\end{equation}

By Theorem 3.3, the set of primes $l$ such that
\begin{equation}
\sum_{n=0}^{\infty} a(n) q^n \mid T(l) \equiv 0 \pmod{m}
\end{equation}
has positive density, where $T(l)$ denotes Hecke operator acting on $S_2(\Gamma_0(180), \chi_5)$.

Moreover, by the theory of Hecke operator, we have
\begin{equation}
\sum_{n=0}^{\infty} a(n) q^n \mid T(l) = \sum_{n=0}^{\infty} \left( a(ln) + \left( \frac{l}{5} \right) l^{2m-3} a \left( \frac{n}{l} \right) \right) q^n.
\end{equation}

Since $a(n)$ vanishes for non-integer $n$, $a(n/l) = 0$ when $(n, l) = 1$. Thus $a(ln) \equiv 0 \pmod{m}$ when $(n, l) = 1$. Recalling that $a(n) \equiv b_5 \left( \frac{mn-1}{6} \right) \pmod{m}$, we obtain
\begin{equation}
b_5 \left( \frac{mln-1}{6} \right) \equiv 0 \pmod{m}
\end{equation}
satisfied for each integer $n$ with $(n, l) = 1$.

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4. Proof of Theorem 1.2

First we recall another important theorem of Serre\cite{7}.

\begin{theorem}[J.-P. Serre] The set of primes $l \equiv 1 \pmod{Nm}$ such that
\begin{equation}
a(nl^r) \equiv (r+1)a(n) \pmod{m}
\end{equation}
for each $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in S_k(\Gamma_0(N), \psi)_m$ has positive density, where $r$ is a positive integer and $n$ is coprime to $l$.
\end{theorem}

Here we introduce a theorem of Sturm, which provide a useful criterion to get some congruences via finite computation. Variants of Sturm’s Theorem are stated in \cite{5, 6, 8}.

\begin{theorem}[J. Sturm] Suppose $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi)_m$ such that
\begin{equation}
a(n) \equiv 0 \pmod{m}
\end{equation}
for all $n \leq \frac{kN}{12} \prod_{p | N} \left( 1 + \frac{k}{p} \right)$. Then $a(n) \equiv 0 \pmod{m}$ for all $n \in \mathbb{Z}$.
\end{theorem}
Proof of Theorem 1.2. Let $m \geq 5$ be a prime. Suppose $k \in \mathbb{Z}$ such that

$$b_5 \left( mk + \frac{m^2 - 1}{6} \right) \equiv e \not\equiv 0 \pmod{m},$$

let $s = 6k + m$. Since $b_5(n)$ vanishes for negative $n$, we have $mk + (m^2 - 1)/6 \geq 0$. Hence $s = 6k + m > 0$ and

$$b_5 \left( \frac{ms - 1}{6} \right) \equiv b_5 \left( mk + \frac{m^2 - 1}{6} \right) \equiv e \pmod{m}.$$

For a fix prime $m \geq 5$, let $S(m)$ denote the set of primes $l$ such that

$$a(nl^r) \equiv (r + 1)a(n) \pmod{m} \quad \text{for each} \quad f(z) = \sum_{n=0}^{\infty} a(n)q^n \in S_{2m-2}(\Gamma_0(180),\chi_5)_m,$$

where $r$ is a positive integer and $(n,l) = 1$. Recalling that

$$\sum_{n=0}^{\infty} b_5 \left( \frac{mn - 1}{6} \right) q^n \in S_{2m-2}(\Gamma_0(180),\chi_5)_m,$$

since $S(m)$ is infinite by Theorem 4.1, choose $l \in S(m)$ such that $l > s$, then

$$b_5 \left( \frac{ml^rs - 1}{6} \right) \equiv (r + 1) b_5 \left( \frac{ms - 1}{6} \right) \equiv (r + 1)e \pmod{m}.$$

Now we fix $l$, choose $\rho \in S(m)$ such that $\rho > l$, then

$$(4.1) \quad b_5 \left( \frac{m\rho n - 1}{6} \right) \equiv 2b_5 \left( \frac{mn - 1}{6} \right) \pmod{m}$$

satisfied for each $n$ coprime to $\rho$. For each $i = 1, 2, \ldots, m - 1$, let $r_i \equiv i(2e)^{-1} - 1 \pmod{m}$ and $r_i > 0$. Let $n = l^r_s$ in (4.1), we obtain

$$b_5 \left( \frac{ml^rs - 1}{6} \right) \equiv 2b_5 \left( \frac{ml^r_i s - 1}{6} \right) \equiv 2(r_i + 1)e \equiv i \pmod{m}.$$

Since the variables except $\rho$ are fixed, it suffices to prove that the estimate of the choices of $\rho \gg X/\log X$ and which is derived from Theorem 4.1 and the Prime Number Theorem.

The upper bound $10(m - 1)$ of $k$ is obtained by Sturmb’s Theorem.

□

5. Examples of Ramanujan-type congruences

Using Sturm’s Theorem, we compute that

$$\sum_{n=0}^{\infty} b_5 \left( \frac{mn - 1}{6} \right) q^n \mid T(l) \equiv 0 \pmod{m}$$

satisfied for $(m,l) = (7,17), (11,41), (13,16519)$. Some elementary computation yields that

Examples.

$$b_5(2023n + 99) \equiv 0 \pmod{7},$$

$$b_5(18491n + 75) \equiv 0 \pmod{11},$$

$$b_5(3547405693n + 35791) \equiv 0 \pmod{13}.$$
Moreover, the congruence $b_5(5n + 4) \equiv 0 \pmod{5}$ implies that

$$\sum_{n=0}^{\infty} b_5 \left( \frac{5n - 1}{6} \right) q^n \mid T(l) \equiv 0 \pmod{5}$$

satisfied for each prime $l$.

6. OPEN PROBLEMS

We have the following conjecture of the existence of Ramanujan-type congruence.

**Conjecture 6.1.** Let $m$ be a positive integers, then there are infinitely many Ramanujan-type congruences modulo $m$.

We also have the following conjecture analogous to Newman’s Conjecture for the usual partition function $p(n)$.

**Conjecture 6.2.** Let $m$ be a positive integers, then for each integer $i$, there are infinitely many $n$ for which

$$b_5(n) \equiv i \pmod{m}.$$ 

**Remark.** By Theorem 1.2, we verify that Conjecture 6.2 is true for prime $7 \leq m \leq 40$.

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DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY(ZHUHAI CAMPUS), ZHUHAI

*Email address: zhengqy29@mail2.sysu.edu.cn*