Arc consistency for soft constraints

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Abstract

The notion of arc consistency plays a central role in constraint satisfaction. It is known since [19, 4, 5] that the notion of local consistency can be extended to constraint optimisation problems defined by soft constraint frameworks based on an idempotent cost combination operator. This excludes non idempotent operators such as + which define problems which are very important in practical applications such as Max-CSP, where the aim is to minimize the number of violated constraints.

In this paper, we show that using a weak additional axiom satisfied by most existing soft constraints proposals, it is possible to define a notion of soft arc consistency that extends the classical notion of arc consistency and this even in the case of non idempotent cost combination operators.

A polynomial time algorithm for enforcing this soft arc consistency exists and its space and time complexities are identical to that of enforcing arc consistency in CSPs when the cost combination operator is strictly monotonic (for example Max-CSP).

A directional version of arc consistency, first introduced in [6] is potentially even stronger than the non-directional version, since it allows non local propagation of penalties. We demonstrate the utility of directional arc consistency by showing that it not only solves soft constraint problems on trees, but that it also implies a form of local optimality, which we call arc irreducibility.

Introduction

Compared to other combinatorial optimisation frameworks, the CSP framework is essentially characterised by the ubiquitous use of so-called local consistency properties and enforcing algorithms among which arc consistency is certainly preeminent.

The notion of local consistency can be characterised by a set of desirable properties:

- local consistency is a relaxation of consistency, which means that for any consistent CSP there is an equivalent non empty locally consistent CSP.

- this equivalent locally consistent CSP, which is unique, can be found in polynomial time by so-called enforcing or filtering algorithms.

Several papers have tried to extend the classical notion of arc consistency to weighted constraint frameworks. In such frameworks, the aim is to find an
assignment that minimises combined violations. The first work in this direction is probably [15] which defined arc consistency filtering for conjunctive (max-min) fuzzy CSP.

This extension was rather straightforward and one might be tempted to think that this would be the case for other frameworks such as MAX-CSP, introduced in [20, 9], where the aim is to find an assignment which minimises the (weighted) number of violated constraints. This turned out not to be the case. Later works tried to extend arc consistency in a systematic way using axiomatic frameworks to characterise the properties of the operator used to combine violations:

- the Semi-Ring CSP framework was introduced in [4, 5]. In this work, the extension of arc consistency enforcing is induced by a generalisation of the fundamental relational operators such as projection, intersection and join. The essential conclusion of this work is that extended arc consistency works as long as the operator used to combine violations is idempotent. This includes the case of conjunctive fuzzy CSP (in which we try to minimise the violation of the most violated constraint) and also some other cases with partial orders. For MAX-CSP and other related cases, the algorithm may not terminate and may also provide non equivalent CSPs.

- the Valued CSP framework was introduced in [19]. Here, the extension of the arc consistency property is essentially based on the notion of relaxation. The same conclusion as in the Semi-Ring CSP framework was reached for idempotent operators. For other frameworks such as MAX-CSP, it was shown that the problem of checking the extended arc consistency property defines an NP-complete problem.

Parallel to these tentative extensions of arc consistency, other research such as [21, 1, 13, 12] tried to provide improved lower bounds for MAX-CSP. The idea of extending arc consistency was abandoned in order to simply provide the most important service, i.e. the ability to detect that a CSP has no solution whose cost is below a given threshold.

Globally, each of these proposals violates some of the desirable properties of local consistency. In this paper we show that it is possible, by the addition to the Valued CSP framework of a single axiom, to define an extended arc consistency notion that has all the desirable properties of classical arc consistency except for the uniqueness of the arc consistency closure. It has also the pleasant property that in the idempotent operator cases, it reduces to existing working definitions and uniqueness is recovered.

It has been shown [18] that a lower bound can easily be built from any of the arc consistency closures and that this lower bound generalises and improves upon existing lower bounds [21, 1, 13, 12]. In this paper, we also consider a directional version of arc consistency that improves lower bounds by propagating partial inconsistencies and not only value deletions as [13, 12]. In fact, we show that directional arc consistency, first defined in [6], defines a locally optimal lower bound.

1 Notations and definitions

A constraint satisfaction problem (CSP) is a triple $\langle X, D, C \rangle$. $X$ is a set of $n$ variables $X = \{1, \ldots, n\}$. Each variable $i \in X$ has a domain of values $d_i \in D$.
and can be assigned any value \( a \in d_i \), also noted \((i, a)\). \( d \) will denote the cardinality of the largest domain of a CSP. \( C \) is a set of constraints. Each constraint \( c_P \in C \) is defined over a set of variables \( P \subseteq X \) (called the scope of the constraint) by a subset of the Cartesian product \( \prod_{i \in P} d_i \) which defines all consistent tuples of values. The cardinality \(|P|\) is the arity of the constraint \( c_P \). \( r \) will denote the largest arity of a CSP. We assume, without loss of generality, that at most one constraint is defined over a given set of variables. The set \( C \) is partitioned into two sets \( C = C^1 \cup C^+ \) where \( C^1 \) contains all unary constraints. For simplification, the unary constraint on variable \( i \) will be denoted \( c_i \), binary constraints being denoted \( c_{ij} \). \( e = |C^+| \) will denote the number of non unary constraints in a CSP. If \( J \subseteq X \) is a set of variables, then \( \ell(J) \) denotes the set of all possible labellings for \( J \) i.e., the Cartesian product \( \prod_{i \in J} d_i \) of the domains of the variables in \( J \). The projection of a tuple of values \( t \) onto a set of variables \( V \subseteq X \) is denoted by \( t \downarrow V \). A tuple of values \( t \) satisfies a constraint \( c_P \) if \( t \downarrow P \in c_P \).

Finally, a tuple of values over \( X \) is a solution iff it satisfies all the constraints in \( C \).

2 Valued CSP

Valued CSP (or VCSP) were initially introduced in [19]. A valued CSP is obtained by associating a valuation with each constraint. The set \( E \) of all possible valuations is assumed to be totally ordered and its maximum element is used to represent total inconsistency. When a tuple violates a set of constraints, its valuation is computed by combining the valuations of all violated constraints using an aggregation operator, denoted by \( \oplus \). This operator must satisfy a set of properties that are captured by a set of axioms defining a so-called valuation structure.

**Definition 2.1** A valuation structure is defined as a tuple \( \langle E, \oplus, \succeq \rangle \) such that:

- \( E \) is a set, whose elements are called valuations, which is totally ordered by \( \succeq \), with a maximum element denoted by \( \top \) and a minimum element denoted by \( \bot \);
- \( E \) is closed under a commutative, associative binary operation \( \oplus \) that satisfies:
  - Identity: \( \forall \alpha \in E, \alpha \oplus \bot = \alpha \);
  - Monotonicity: \( \forall \alpha, \beta, \gamma \in E, (\alpha \succeq \beta) \Rightarrow ((\alpha \oplus \gamma) \succeq (\beta \oplus \gamma)) \);
  - Absorbing element: \( \forall \alpha \in E, (\alpha \oplus \top) = \top \).

When \( E \) is restricted to \([0, 1]\), this structure of a totally ordered commutative monoid with a monotonic operator is also known in uncertain reasoning, as a triangular co-norm [7].

It is now possible to define valued CSPs. Note that, for the sake of generality, rather than considering that a valuation is associated with each constraint, as in [19], we consider that a valuation is associated with each tuple of each constraint. As observed in [3], the two approaches are essentially equivalent.
Definition 2.2 A valued CSP is a tuple \( \langle X, D, C, S \rangle \) where \( X \) is a set of \( n \) variables \( X = \{1, \ldots, n\} \), each variable \( i \in X \) has a domain of possible values \( d_i \in D \). \( C = C^1 \cup C^+ \) is a set of constraints and \( S = \langle E, \oplus, \succeq \rangle \) is a valuation structure. Each constraint \( c_P \in C \) is defined over a set of variables \( P \subseteq X \) as a function \( c_P : \prod_{i \in P} d_i \rightarrow E \).

An assignment \( t \) of values to some variables \( J \subseteq X \) can be simply evaluated by combining, for all assigned constraints \( c_P \) (i.e., such that \( P \subseteq J \)), the valuations of the projection of the tuple \( t \) on \( P \):

\[
\mathcal{V}_V(t) = \bigoplus_{c_P \in C, P \subseteq J} [c(t_P)]
\]

The problem usually considered is to find a complete assignment with a minimum valuation. Globally, the semantics of a VCSP is defined by the valuations \( \mathcal{V}(t) \) of assignments \( t \) to \( X \).

The choice of axioms is quite natural and is usual in the field of uncertain reasoning. The ordered set \( E \) simply allows us to express different degrees of constraint violation. The commutativity and associativity guarantee that the valuation of an assignment is independent of the order in which valuations are combined. The monotonicity of \( \oplus \) guarantees that assignment valuations cannot decrease when constraint violations increase. For a more detailed analysis and justification of the VCSP axioms, we invite the reader to consult [19, 12] which also emphasise the difference between idempotent and strictly monotonic aggregation operators \( \oplus \).

Definition 2.4 An operator \( \oplus \) is idempotent if \( \forall \alpha \in E, (\alpha \oplus \alpha) = \alpha \). It is strictly monotonic if \( \forall \alpha, \beta, \gamma \in E, (\alpha \triangleright \beta) \land (\gamma \neq \top) \Rightarrow (\alpha \oplus \gamma) \triangleright (\beta \oplus \gamma) \)

As shown in [19], these two properties are incompatible as soon as \( |E| > 2 \). The only valuation structures with an idempotent operator correspond to classical and possibilistic CSP [16] (min-max dual to the conjunctive fuzzy CSP framework) which use \( \oplus = \max \) as the aggregation operator. Other soft CSP frameworks such as MAX-CSP, lexicographic CSP or probabilistic CSP use a strictly monotonic operator.

Arc consistency enforcing must yield an equivalent problem, the so-called arc-consistency closure. Several notions of equivalence were introduced in [19, 12] that enabled us to compare pairs of VCSP with different valuations structure. In this paper, the notion of equivalence will only be used to compare pairs of VCSP with the same valuation structure and can therefore be simplified and strengthened.

Definition 2.5 Two VCSP \( V = \langle X, D, C, S \rangle \) and \( V' = \langle X, D', C', S \rangle \) are equivalent iff for all complete assignment \( t \) to \( X \), we have:

\[
\mathcal{V}_V(t) = \mathcal{V}_{V'}(t)
\]
3 Fair valuation structures

We start with an introductory example. In the remainder of the paper, in order to illustrate the notions introduced on concrete examples, we will consider binary weighted \textsc{Max-CSP}s which correspond to valued \textsc{CSP}s using the strictly monotonic valuation structure $\langle \mathbb{N} \cup \{\infty\}, +, \geq \rangle$. To describe such problems, we use an undirected graph representation where vertices represent values. For all pairs of variables $i, j \in X$ such that $c_{ij} \in C$, for all values $a \in d_i$, $b \in d_j$ such that $c_{ij}(a, b) \neq \bot = 0$, an edge connect the values $(i, a)$ and $(j, b)$. The weight of this edge is set to $c_{ij}(a, b)$. Unary constraints are represented by weights associated with vertices, weights equal to 0 being omitted.

Let us consider the weighted \textsc{Max-CSP} in figure 1(a). It has two variables numbered 1 and 2, each with two values $a$ and $b$ together with a single constraint. The constraint forbids pair $((1, b), (2, a))$ with cost 1 and forbids pairs $((1, a), (2, a))$ and $((1, b), (2, a))$ completely (with cost $\infty$). The pair $((1, a), (2, b))$ is completely authorised and the corresponding edge is therefore omitted.

![Figure 1: Four equivalent instances of Max-CSP](image)

If we assign the value $b$ to variable 1, it is known for sure that a cost of 1 must be paid since all extensions of $(1, b)$ to variable 2 incur a cost of at least 1. Projecting this minimum cost down from $c_{12}$ would make this explicit and induce a unary constraint on 1 that forbids $(1, b)$ with cost 1. However if we simply add this constraint to the \textsc{Max-CSP}, as was proposed in [4] for problems with an idempotent operator, the resulting \textsc{CSP} is not equivalent. The complete assignment $((1, b), (2, b))$ which initially had a cost of 1 would now have a cost of 2. In order to preserve equivalence, we must “compensate” for the induced unary constraint. This can be done by simply subtracting 1 from all the tuples that contain the value $(1, b)$. The corresponding equivalent \textsc{CSP} is shown in figure 1(b): the edge $((1, b), (2, b))$ of cost 1 has disappeared (the associated weight is now 0) while the edge $((1, b), (2, a))$ is unaffected since
it has infinite weight. We can repeat this process for variable 2: all extensions of value \((2, a)\) have infinite cost. Thus we can add a unary constraint that completely forbids value \((2, a)\). In this specific case, and because the valuation \(\infty\) satisfies \(\infty \oplus \infty = \infty\), we can either compensate for this (Figure 1(c)) or not (Figure 1(d)). In both cases, an equivalent Max-CSP is obtained. Between the problems in Figure 1(c) and 1(d), we prefer the problem in Figure 1(d) because it makes information explicit both at the domain and constraint level.

This type of projection mechanism underlies most of the lower bounds defined for Max-CSP [21, 1, 13, 12]. To our knowledge, the introduction of a “compensation” mechanism for preserving equivalence was first introduced by [11] on Max-CSP, independently of any notion of arc consistency. The use of such mechanism for the definition and establishment of arc consistency appeared in [18] and in a related form in [10] (for enforcing so-called probabilistic arc consistency).

![Figure 2: Two equivalent instances of Max-CSP](image-url)

Suppose now that the problem in Figure 1 is part of an instance of Max-CSP on four variables, as shown in Figure 2(a). As in crisp CSP, inconsistencies can propagate from domains up to constraints. The cost of \(\infty\) for \((2, a)\) can be duplicated in the costs of the pairs \(((2, a), (3, a))\) and \(((2, a), (3, b))\). Since \(c_{23}(b, b) = \infty\), this in turn implies that the assignment \((b, 3)\) inevitably has a cost of \(\infty\). No further propagation of infinite costs can be performed.

A similar process can be applied to finite costs but one must take care to compensate any cost change. The cost 1 of \((1, b)\) can be first shifted to the constraint \(c_{01}\): the costs of the pairs \(((0, a), (1, b))\) and \(((0, b), (1, b))\) become equal to 1 and the cost of the value \((1, b)\) is set to 0. Since now \(c_{01}(a, a) = c_{01}(a, b) = 1\), a cost of 1 can be projected onto value \((0, a)\). Figure 2(b) shows the result of such propagations. In the case of finite costs, the process is obviously not terminated since one could forever shift this cost back and forth between values \((0, a)\) and \((1, b)\).
3.1 A new axiom for VCSPs

To formalise and generalise the ideas presented in the previous section to other valuation structures, we have to be able to compensate for the information added by projecting weights down onto domains. This is made possible by the following additional axiom:

**Definition 3.1** In a valuation structure \( S = \langle E, \oplus, \succeq \rangle \), if \( \alpha, \beta \in E \), \( \alpha \preceq \beta \) and there exists a valuation \( \gamma \in E \) such that \( \alpha \oplus \gamma = \beta \), then \( \gamma \) is known as a difference of \( \beta \) and \( \alpha \).

The valuation structure \( S \) is fair if for any pair of valuations \( \alpha, \beta \in E \), with \( \alpha \preceq \beta \), there exists a maximal difference of \( \beta \) and \( \alpha \). This unique maximal difference of \( \beta \) and \( \alpha \) is denoted by \( \beta \ominus \alpha \).

**Lemma 3.2** Let \( S = \langle E, \oplus, \succeq \rangle \) be a fair valuation structure. Then \( \forall u, v, w, \in E, w \preceq v \), we have \( (v \ominus w) \preceq v \) and \( (u \oplus w) \oplus (v \ominus w) = (u \oplus v) \).

**Proof:** By definition, \( (v \ominus w) \oplus w = v \). From the monotonicity of \( \oplus \), this proves that \( (v \ominus w) \preceq v \) (this inequality becomes strict if \( \oplus \) is strictly monotonic and \( v \neq \top \)). The second property follows from the commutativity and associativity of \( \oplus \): we have \( (u \oplus w) \oplus (v \ominus w) = u \oplus ((v \ominus w) \oplus w) = (u \oplus v) \). \( \square \)

Most existing concrete soft constraint frameworks, including all those with either an idempotent or strictly monotonic operator \( \oplus \) are fair.

**Example 1** If \( \oplus \) is idempotent, then it can easily be shown that \( \oplus = \max \) [19]. Classical CSPs can be defined as VCSPs over the valuation structure \( S = \langle \{\bot, \top\}, \max, \succeq \rangle \), where \( \bot \) represents true and \( \top \) false. The operator \( \oplus \) is also idempotent in possibilistic CSPs [16] which define a min-max problem which is dual to the max-min problem of conjunctive fuzzy CSPs [15, 6]. When \( \oplus = \max \), we have \( \ominus = \max \), since \( \max(\max(\alpha, \beta), \alpha) = \beta \) whenever \( \alpha \preceq \beta \). When \( \alpha = \beta \), then any valuation \( \gamma \prec \alpha \) is also a valid difference of \( \beta \) and \( \alpha \) but it is clearly not maximal.

**Example 2** In the strictly monotonic valuation structure \( \langle \mathbb{N} \cup \{\infty\}, +, \geq \rangle \), \( \ominus \) is defined by \( \beta \ominus \alpha = \beta - \alpha \) for finite valuations \( \alpha, \beta \in \mathbb{N}, \alpha \leq \beta \) and \( (\infty \ominus \alpha) = \infty \) for all \( \alpha \in \mathbb{N} \cup \{\infty\} \). In the general case of any strictly monotonic operator \( \oplus \), the difference operator may not exist in \( E \), but it has been proved in [6] that the difference operator can always be constructed by embedding the valuation structure in a larger valuation structure derived from the set \( E \times E \), where \( (\beta, \alpha) \) represents the imaginary \( \beta \ominus \alpha \). This can be compared with embedding \( \mathbb{R} \) in \( \mathbb{C} \) so as to allow us to take square roots of negative numbers. This construction is interesting for lexicographic CSPs [8] for which differences are not always defined in the original valuation structure. Another possible approach is to transform the lexicographic CSP into a VCSP on the valuation structure \( \langle \mathbb{N} \cup \{\infty\}, +, \geq \rangle \) using the simple transformation described in [19].

3.2 Equivalence preserving transformations

As it has been demonstrated in the examples of Figures 1 and 2, it is possible to transform a MAX-CSP into an equivalent but different MAX-CSP using local
transformations (involving only one non unary constraint). Such operations will be called equivalence-preserving transformations:

**Definition 3.3** The subproblem of a VCSP $V = \langle X, D, C, S \rangle$ on $J \subseteq X$ is the VCSP $V(J) = \langle J, D_J, C_J, S \rangle$, where $D_J = \{ d_j : j \in J \}$ and $C_J = \{ c_P \in C : P \subseteq J \}$.

**Definition 3.4** For a VCSP $V$, an equivalence-preserving transformation of $V$ on $J \subseteq X$ is an operation which transforms the subproblem of $V$ on $J$ into an equivalent VCSP. If $C_J = \{ c_P \in C : P \subseteq J \}$ contains only one non unary constraint, such an operation is called an equivalence-preserving arc transformation.

**Example 3** The procedures **Project** and **Extend** described in Algorithm 1 are examples of equivalence-preserving transformations.

**Project** transform a VCSP by shifting valuations from the tuples of a given non unary constraint $c_P$ to the value $(i, a)$, where $i \in P, a \in d_i$. In order to preserve equivalence, any increase at the unary level is compensated at the tuple level (line 1 of Algorithm 1).

Conversely, **Extend** shifts the valuation from value $(i, a)$ to the tuples of the constraint $c_P$ where $i \in P$. Again, the fairness of the valuation structure allows us to compensate for the possible increase of the tuple valuations by an operation at the unary level (line 2 of Algorithm 1).

**Algorithm 1:** Two basic equivalence-preserving arc-transformations

```plaintext
Procedure Project(c_P, i, a)
    \beta \leftarrow \min_{t \in \ell(P - \{i\})} (c_P(t, a));
    c_i(a) \leftarrow c_i(a) \oplus \beta;
    \text{foreach } (t \in \ell(P - \{i\}) \text{ do})
    \quad c_P(t, a) \leftarrow c_P(t, a) \oplus \beta;

Procedure Extend(i, a, c_P)
    \text{foreach } (t \in \ell(P - \{i\}) \text{ do})
    \quad c_P(t, a) \leftarrow c_P(t, a) \oplus c_i(a);
    c_i(a) \leftarrow c_i(a) \oplus c_i(a);
```

**Theorem 3.5** Given any fair VCSP $V = \langle X, D, C, S \rangle$, for any $c_P \in C^+, i \in P, a \in d_i$, the application of **Project** or **Extend** on $V$ yields an equivalent VCSP.

**Proof:** To demonstrate equivalence, it is sufficient to prove that the value of $c_P(t, a) \oplus c_i(a)$ is an invariant of **Project(c_P, i, a)** and **Extend(i, a, c_P)**. For any $t \in \ell(P - \{i\})$, let $\gamma$ be the initial value of $c_P(t, a)$ and $\delta$ the initial value of $c_i(a)$. After the execution of **Project**, we have $(c_P(t, a) \oplus c_i(a)) = (\gamma \oplus \beta) \oplus (\delta \oplus \beta) = \gamma \oplus \delta$. After the execution of **Extend**, we have $(c_P(t, a) \oplus c_i(a)) = (\gamma \oplus \delta) \oplus (\delta \oplus \delta) = \gamma \oplus \delta$. This proves the invariances.

As the example of Figure 2 showed in the case of MAX-CSP, the iterated application of equivalence-preserving transformations such as **Project** and **Extend**
does not necessarily lead to a quiescent state. The two following sections show how a limited application of carefully designed equivalence-preserving transformations can guarantee that a quiescent state will always be reached.

4 Soft Arc consistency

In classical CSPs, arc consistency enforcing always increases the information available on each variable. In the case of soft arc consistency, application of arc transformations will be limited to operations that either increase the information available at the variable level or that increase information available at the constraint level as long as they do not lower the information available at the variable level. In the next section, we try to better characterise when this is possible.

4.1 On the structure of valuation structures

**Definition 4.1** In a valuation structure \( \langle E, \oplus, \succ \rangle \), an element \( \alpha \in E \) is an absorbing element iff \( \alpha \oplus \alpha = \alpha \).

Absorbing elements can be duplicated without affecting valuations. They can be propagated, in the same way as inconsistencies are in crisp CSPs. Non-absorbing elements \( \alpha \) can be shifted from one constraint to another, but each addition of \( \alpha \) must be compensated by a subtraction elsewhere.

In a valuation structure \( \langle E, \oplus, \succ \rangle \), if \( \oplus \) is idempotent then all elements of \( E \) are absorbing. If \( \oplus \) is a strictly monotonic operator then the only absorbing elements are \( \perp \) and \( \top \). Intermediate cases occur in the following examples:

**Example 4** Imagine the possible sentences for driving offences. Suppose that penalty points (up to a maximum of \( 12 \)) are awarded for minor offences, whereas serious offences are penalised by suspension of the offender’s driving license for a period of \( y \) years, for some positive integer \( y \). A driver who accumulates 12 penalty points receives an automatic one-year suspension of his/her license. The set of sentences can be modelled by a valuation structure \( S = \langle E, \oplus, \succ \rangle \) of the form:

\[
E = \{ (p, 0) : p \in \{0, \ldots, 12\} \} \cup \{ (0, y) : y \in \mathbb{N}^* \cup \{\infty\} \}
\]

\[
(p, y) \prec (p', y') \iff (y < y') \lor ((y = y') \land (p < p'))
\]

\[
(p, 0) \oplus (p', 0) = (\min(p + p', 12), 0)
\]

\[
(p, y) \oplus (p', y') = (0, y + y') \quad \text{if } (y + y' \neq 0)
\]

Note that \((12, 0) \prec (0, 1)\) even though they both give rise to a one-year license suspension. The penalty \((0, 1)\) is deemed to be worse because it can be cumulated. For example \((0, 1) \oplus (0, 1) = (0, 2)\), whereas \((12, 0) \oplus (12, 0) = (12, 0)\). Apart from \( \perp = (0, 0) \) and \( \top = (0, \infty) \), this valuation structure contains another absorbing
valuation, namely $(12,0)$. This is a fair valuation structure since $\oplus$ has the following inverse operation $\ominus$:

$$(p, 0) \ominus (p', 0) = (p - p', 0) \quad \text{if } p < 12$$

$$(12, 0) \ominus (p', 0) = (12, 0)$$

$$(0, y) \ominus (p', y') = (0, y - y')$$

**Example 5** Another interesting case occurs if, for example, a company wants to minimise both financial loss $F$ and loss of human life $H$ if a fire should break out in its factory. Supposing that the company considers that no price can be put on human life, we must have

$$(F, H) < (F', H') \iff (H < H') \lor (H = H' \land F < F')$$

If a financial loss of $F_{\text{max}}$ represents bankruptcy, then

$$(F, H) \oplus (F', H') = (\min\{F + F', F_{\text{max}}\}, H + H')$$

and $(F_{\text{max}}, 0)$ is an absorbing element which is strictly less than $\top$. Note that this valuation structure is not fair, since it is impossible to define $\alpha = (0, 1) \oplus (F_{\text{max}}, 0)$ such that $\alpha \oplus (F_{\text{max}}, 0) = (0, 1)$.

**Example 6** Consider a valuation structure $S = \langle \mathbb{N} \cup \{\infty, \top\}, \oplus, \geq \rangle$ composed of prison sentences. Sentences may be of $n$ years, life imprisonment (represented by $\infty$) or the death penalty (represented by $\top$). There is a rule that states that two life sentences lead automatically to a death sentence: in other words $(\infty + \infty) = \top$. Otherwise, sentences are cumulated in the obvious way: $\forall m, n \in \mathbb{N}, (m \oplus n = m + n); \forall n \in \mathbb{N}, (\infty + n = \infty); \forall \alpha \in E, (\top \oplus \alpha = \top)$. Although every pair $\beta, \alpha \in E, \alpha \leq \beta$ possesses a difference, this valuation structure is not fair since the set of differences of $\infty$ and $\infty$ is $\mathbb{N}$ and hence no maximal difference of $\infty$ and $\infty$ exists. However, $S$ can easily be rendered fair by replacing $\mathbb{N}$ by $\{0, 1, 2, \ldots, 150\}$, for example.

The following results show that all fair valuation structures are composed of slices separated by absorbing values, each slice being independent of the others.

**Lemma 4.2** Let $S = \langle E, \oplus, \geq \rangle$ be a valuation structure. If $\alpha, \beta \in E$, $\alpha$ is an absorbing element and $\beta \not\leq \alpha$ then $\alpha \oplus \beta = \alpha$. If $S$ is fair, then $\alpha \ominus \beta = \alpha$.

**Proof:** Since $\beta \not\leq \alpha$, it follows that $\alpha < \alpha \oplus \beta < \alpha \oplus \alpha = \alpha$, by monotonicity. Thus, $\alpha \oplus \beta = \alpha$. Furthermore, $\alpha \oplus \beta = \alpha$ shows that $\alpha$ is a difference of $\alpha$ and $\beta$. It is the maximal difference since $\alpha \ominus \beta = (\alpha \ominus \beta) \ominus \perp \not\leq (\alpha \ominus \beta) \ominus \beta = \alpha$, by monotonicity. Thus $\alpha \ominus \beta = \alpha$. \qed

**Lemma 4.3** Let $S = \langle E, \oplus, \geq \rangle$ be a valuation structure. If $\alpha, \beta \in E$, $\alpha$ is an absorbing element and $\beta \not\geq \alpha$ then $\alpha \ominus \beta = \beta$ and $\beta \ominus \alpha = \beta$.

**Proof:** Since $\alpha$ is absorbing, $\beta \ominus \alpha = (\beta \ominus \alpha) \ominus \alpha = (\beta \ominus \alpha) \ominus \alpha = \beta$. Furthermore, this shows that $\beta$ is a difference of $\beta$ and $\alpha$. It is the maximum difference since $\beta \ominus \alpha \not\leq \beta$, by Lemma 3.2. Thus, $\beta \ominus \alpha = \beta$. \qed
Theorem 4.4 (Slice Independence Theorem) Let \( S = \langle E, \oplus, \succ \rangle \) be a fair valuation structure. Let \( \beta, \gamma \in E, \beta \preceq \gamma \), and let \( \alpha_0, \alpha_1 \in E \) be absorbing valuations such that \( \alpha_0 \preceq \gamma \preceq \alpha_1 \). Then \( \alpha_0 \preceq (\gamma \oplus \beta) \preceq \alpha_1 \) and \( \alpha_0 \preceq (\gamma \ominus \beta) \preceq \alpha_1 \).

Proof: By monotonicity, \( \beta \ominus \gamma \preceq \alpha_1 \ominus \gamma = \alpha_1 \) by Lemma 4.2. By Lemma 4.3, \( \gamma = \gamma \ominus \alpha_0 = (\gamma \ominus \beta) \ominus \beta \ominus \alpha_0 = (\gamma \ominus \beta) \ominus \alpha_0 \ominus \beta \). Therefore, \((\gamma \ominus \beta) \ominus \alpha_0 \) is a difference of \( \gamma \) and \( \beta \). Since \( \gamma \ominus \beta \) is a maximal difference, \( \gamma \ominus \beta \succ (\gamma \ominus \beta) \ominus \alpha_0 \succ \alpha_0 \) by monotonicity. The remaining equalities follow from monotonicity.

The following results will be useful for the proof of correctness of the arc consistency enforcing algorithm given in the next section.

Theorem 4.5 Let \( S = \langle E, \oplus, \succ \rangle \) be a fair valuation structure. For all \( \alpha \in E \), \( \alpha \ominus \alpha \) is the maximal absorbing valuation less than or equal to \( \alpha \).

Proof: Let \( \beta = \alpha \ominus \alpha \). Now \( \alpha \oplus (\beta \ominus \beta) = (\alpha \oplus \beta) \ominus \beta = \alpha \ominus \beta = \alpha \), which shows that \( \beta \ominus \beta \) is a difference of \( \alpha \) and \( \alpha \). By definition 3.1, \( \beta \) is the maximal difference. Therefore, \( \beta \succ \beta \ominus \beta \). Since \( \beta \preceq \beta \ominus \beta \) by monotonicity, we have \( \beta = \beta \ominus \beta \) and hence \( \beta \) is absorbing. Maximality follows from Theorem 4.4, since for all absorbing valuations \( \alpha_0 \preceq \alpha \), we have \( \alpha_0 \preceq (\alpha \ominus \alpha) \).

Lemma 4.6 Let \( S = \langle E, \oplus, \succ \rangle \) be a fair valuation structure. For all \( \alpha, \beta \in E \), \((\alpha \ominus \beta) \ominus \beta = (\alpha \ominus \beta) \ominus (\alpha \ominus \beta) \).

Proof: Let \( \gamma = (\alpha \ominus \beta) \ominus (\alpha \ominus \beta) \). By Theorem 4.5, \( \gamma \) is absorbing. Now \( \gamma \preceq (\alpha \ominus \beta) \), by Lemma 3.2. Let \( \delta = ((\alpha \ominus \beta) \ominus (\alpha \ominus \beta) \ominus (\alpha \ominus \beta) \ominus (\alpha \ominus \beta) \). Then the fact that \( \delta \succ \gamma \) follows from two applications of the Slide Independence Theorem. But \( \delta \ominus (\alpha \ominus \beta) = (\alpha \ominus \beta) \). Therefore \( \delta \) is a difference of \( (\alpha \ominus \beta) \) and \( (\alpha \ominus \beta) \). \( \gamma \) being the maximal difference, this shows that \( \delta \preceq \gamma \) and \( \delta = \gamma \).

Theorem 4.7 Let \( S = \langle E, \oplus, \succ \rangle \) be a fair valuation structure. For all \( \alpha, \beta \in E \), either \( (\alpha \ominus \beta) \ominus \beta = \alpha \) or \( (\alpha \ominus \beta) \ominus \beta = (\alpha \ominus \beta) \ominus (\alpha \ominus \beta) \) which is absorbing and strictly greater than \( \alpha \).

Proof: Let \( \gamma = (\alpha \ominus \beta) \ominus (\alpha \ominus \beta) \). Now \( (\alpha \ominus \beta) \ominus \beta = ((\alpha \ominus \beta) \ominus (\alpha \ominus \beta)) \ominus (\alpha \ominus \beta) = \gamma \ominus \alpha \), by Lemma 4.6. Since \( \gamma \) is absorbing, \( \gamma \ominus \alpha \) equals either \( \alpha \) (if \( \alpha \succ \gamma \), Lemma 4.3) or \( \gamma \) (if \( \gamma \succ \alpha \), Lemma 4.2). In this case, the fact that \( \gamma \) is absorbing follows directly from Theorem 4.5.

Theorem 4.8 Any VCSP \( V = \langle X, D, C, S \rangle \) on a fair valuation structure \( S = \langle E, \oplus, \succ \rangle \) is equivalent to a VCSP on a valuation structure \( S' \) with no more than \( 2ed^2 + 2 \) absorbing valuations.

Proof: For any \( \beta \in E \) define

\[
\text{Slice}(\beta) = \{ \alpha \in E : \not\exists \gamma \text{ absorbing in } E \text{ s.t. } (\alpha \prec \gamma \preceq \beta) \lor (\alpha \succ \gamma \succ \beta) \}.
\]

If \( \beta \) is absorbing then \( \text{Slice}(\beta) = \{ \beta \} \), otherwise \( \text{Slice}(\beta) \) is the set of valuations \( \alpha \) for which there is no intermediate absorbing valuation \( \gamma \) lying between \( \alpha \) and \( \beta \). Each \( \text{Slice}(\beta) \) contains at most two absorbing valuations, namely the
maximum absorbing valuation less than or equal to $\beta$ (which is in fact $\beta \odot \beta$, by Theorem 4.5) and the minimum absorbing valuation greater than or equal to $\beta$ (which may or may not exist).

Let $E_0$ be the set of valuations taken on by the cost functions $c_P \in C$ in the VCSP and let

$$E' = \{ \perp, \top \} \bigcup_{\beta \in E_0} \text{Slice}(\beta)$$

Clearly $E'$ contains at most $2ed' + 2$ absorbing valuations. It is sufficient to show that $E'$ is closed under $\oplus$ and $\ominus$.

Consider $\alpha, \beta \in E'$ such that $\alpha \preceq \beta$ and $\beta \in \text{Slice}(\beta_0)$ for some $\beta_0 \in E_0$.

Suppose that $(\alpha \oplus \beta) \not\in E'$. This implies that $\exists \gamma$ absorbing in $E$ such that $\alpha \oplus \beta \succ \gamma \succeq \beta_0$. But, since $\beta \preceq \gamma$ by the definition of $\text{Slice}(\beta_0)$, this contradicts Theorem 4.4. Similarly, $\beta \ominus \alpha \not\in E'$ implies that $\exists \gamma$ absorbing in $E$ such that $\beta \ominus \alpha \prec \gamma \preceq \beta_0$. But, since $\beta \succeq \gamma$ by the definition of $\text{Slice}(\beta_0)$, this again contradicts Theorem 4.4.

By theorem 4.8, we can now assume, without loss of generality, that the number of absorbing valuations in the valuation structure is finite.

### 4.2 A Definition of Soft Arc Consistency

Before giving an arc consistency enforcing algorithm which is valid over any fair valuation structure, we require a formal definition of arc consistency for fair VCSPs. We first consider the usual restriction to binary VCSPs.

**Definition 4.9** A fair binary VCSP is arc consistent if for all $i, j \in X$ such that $c_{ij} \in C^+$, for all $a \in d_i$ we have:

1. $\forall b \in d_j, c_{ij}(a, b) = (c_i(a) \oplus c_{ij}(a, b) \oplus c_j(b)) \ominus (c_i(a) \oplus c_j(b))$.

2. $c_i(a) = \min_{b \in d_j} (c_i(a) \oplus c_{ij}(a, b))$

Condition 1 states that $c_{ij}(a, b)$ has been increased to the maximal element in $E$ which does not increase the valuation $(c_i(a) \oplus c_{ij}(a, b) \oplus c_j(b))$ of $(a, b)$ on $\{i, j\}$. If $\oplus$ is strictly monotonic or idempotent, then this is equivalent to saying that absorbing valuations have been propagated from $c_i(a)$ to $c_{ij}(a, b)$. Condition 2 says that we have propagated as much weight as possible from the constraint $c_{ij}$ onto $c_i$.

To gain a better understanding of condition 1 of Definition 4.9 in the most general case, consider a simple valuation structure in which penalties lies in the range $\{0, 1, 2, 3, 4, 5\}$ and $\forall \alpha, \beta \in E, (\alpha \oplus \beta = \min(5, \alpha + \beta))$. $5$ is absorbing and verifies $5 \ominus \alpha = 5$ for all $\alpha \preceq 5$. Figure 3(a) shows a 2-variable VCSP over this valuation structure. Figure 3(b) shows the result of enforcing condition 1 of Definition 4.9: $c_{12}(a, a)$ and $c_{12}(b, a)$ can both be increased to 5 without changing the valuations of the solutions $(a, a)$ and $(b, a)$. Figure 3(c) shows the result of then enforcing condition 2: penalties are projected down from constraints to domains, as we have seen in the example of Figure 1.

Definition 4.9 can be generalised to non binary VCSP. We call this generalised arc consistency, to be consistent with the terminology employed in the CSP literature [14].
Definition 4.10 A fair VCSP is generalised arc consistent if for all $c_P \in C^+$, we have:

1. $\forall t \in \ell(P), c_P(t) = (c_P(t) \oplus \beta) \ominus \beta$, where $\beta = \bigoplus_{j \in P} c_j(t_{i(j)})$.

2. $\forall i \in P, \forall a \in d_i, c_i(a) = \min_{t \in \ell(P - \{i\})}(c_i(a) \oplus c_P(t, a))$

Having given the necessary definitions, we can now define a generalised arc consistency enforcing algorithm.

4.3 Enforcing generalised arc consistency in fair VCSPs

Arc consistency is established by repeated calls to two subroutines denoted by AC-Project and AC-Extend (see Algorithms 2 and 3 respectively), called arc consistency operations. The data structure $Q$ is a queue containing elements $(t, P, \alpha)$, where $\alpha = c_P(t)$. The subroutine AC-Project($c_P, i, a$) is a simple modification of the basic equivalence-preserving transformation Project that memorises VCSP modifications in the queue for further propagation. This simple modification obviously does not alter the fact that it is an equivalence-preserving transformation.

Algorithm 2: Projection for Soft AC enforcing

Procedure AC-Project($c_P, i, a$)

\[ \beta \leftarrow \min_{t \in \ell(P - \{i\})}(c_P(t, a)); \]

if $(c_i(a) \oplus \beta \succ c_i(a))$ then

\[ c_i(a) \leftarrow c_i(a) \oplus \beta; \]

Add $(a, \{i\}, c_i(a))$ to $Q$;

foreach $(t \in \ell(P - \{i\}))$ do

\[ c_P(t, a) \leftarrow c_P(t, a) \ominus \beta; \]
Lemma 4.11 For any fair VCSP \( V = \langle X, D, C, S \rangle \), if \( c_p \in C, i \in P, a \in d_i \) then the result of applying \( \text{AC-Project}(c_p, i, a) \) to a fair VCSP \( V \) is an equivalent VCSP in which
\[
c_i(a) = \min_{t \in \ell(P - \{i\})} (c_i(a) \oplus c_p(t, a))
\]

Proof: This property follows from the fact that either \( c_i(a) \oplus \beta \) is not strictly greater than \( c_i(a) \) and in this case \( c_i(a) = c_i(a) \oplus \min_{t \in \ell(P - \{i\})}(c_p(t, a)) = \min_{t \in \ell(P - \{i\})}(c_i(a) \oplus c_p(t, a)) \) or else let \( t_{\text{min}} \) be the tuple such that \( c_p(t_{\text{min}}, a) = \min_{t \in \ell(P - \{i\})}(c_p(t, a)) \). If \( t \) is an equivalent valuation then the result of applying \( \text{AC-Extend}(i, a, c_p) \) to all the valuations \( t \in \ell(P - \{i\}) \) when this can be done without any compensation at the unary level. It also memorises the new valuation \( \gamma \) of each tuple of \( c_p \) in \( Q \) for further propagation. In this case, \( \gamma \) is always an absorbing valuation (by Theorem 4.7).

Algorithm 3: Extension for Soft AC enforcing

Procedure \( \text{AC-Extend}(i, a, c_p) \)

foreach \( (t \in \ell(P - \{i\})) \) do
    \( \beta \leftarrow \bigoplus_{j \in P} c_j((t, a)_{\{j\}}); \)
    \( \gamma \leftarrow (c_p(t, a) \oplus \beta) \oplus \beta; \)
    if \( \gamma \geq c_p(t, a) \) then
        \( c_p(t, a) \leftarrow \gamma; \)
        Add \( ((t, a), P, c_p(t, a)) \) to \( Q; \)

Lemma 4.12 For any fair VCSP \( V = \langle X, D, C, S \rangle \), if \( c_p \in C, i \in P, a \in d_i \) then the result of applying \( \text{AC-Extend}(i, a, c_p) \) to a fair VCSP \( V \) is an equivalent VCSP in which
\[
\forall t \in \ell(P - \{i\}), (c_p(t, a) = (c_p(t, a) \oplus \beta) \oplus \beta), \text{ where } \beta = \bigoplus_{j \in P} c_j((t, a)_{\{j\}})\]

Proof: To demonstrate equivalence, it is sufficient to prove that \( \forall t \in \ell(P - \{i\}), \beta \oplus c_p(t, a) \) is an invariant of \( \text{AC-Extend}(i, a, c_p) \), where \( \beta = \bigoplus_{j \in P} c_j((t, a)_{\{j\}}) \). But this is certainly the case because, if \( c_p(t, a) = \alpha \) before execution of \( \text{AC-Extend}(i, a, c_p) \), then \( \beta \oplus c_p(t, a) = \beta \oplus ((\alpha \oplus \beta) \oplus \beta) = \beta \oplus \alpha \) after \( c_p(t, a) \) is updated by \( \text{AC-Extend}(i, a, c_p) \).

We know that \( \alpha \ll (\alpha \oplus \beta) \oplus \beta \), since \( \alpha \) is clearly a difference of \( \alpha \oplus \beta \) and \( \beta \). If \( \alpha \ll (\alpha \oplus \beta) \oplus \beta \), then \( c_p(t, a) \) is assigned \( (\alpha \oplus \beta) \oplus \beta \). Hence, after \( c_p(t, a) \) is updated by \( \text{AC-Extend}(i, a, c_p) \), \( \beta \oplus c_p(t, a) = \beta \oplus \alpha \) and \( (c_p(t, a) \oplus \beta) \oplus \beta = (\alpha \oplus \beta) \oplus \beta = c_p(t, a). \)

We are now in a position to give an algorithm (Algorithm 4) for generalised arc consistency in fair VCSPs.
Algorithm 4: Generalised Arc Consistency enforcing for fair VCSPs

Procedure GAC()

\{ Initialisation phase \};

foreach \( i \in X \) do

  foreach \( a \in d_i \) do

    foreach \( c \in P \) s.t. \( i \in P \) do

      AC-Extend\((i, a, c_i)\);

      AC-Project\((c_i, i, a)\);

\{ Propagation phase \};

while \( Q \neq \emptyset \) do

  Extract the first element \((t, P, \alpha)\) from \( Q \);

  if \( c_i(t) = \alpha \) then

    if \( P \) is a singleton \( \{i\} \) then

      foreach \( c \in P \) s.t. \( i \in P \) do

        AC-Extend\((i, t_{\downarrow i}, c)\);

    else

      foreach \( i \in P \) do

        AC-Project\((c, i, t_{\downarrow i})\);

\end{algorithm}

Theorem 4.13 When GAC terminates, the resulting VCSP is generalised arc consistent.

Proof: Consider \((c_i, i, a)\) such that \( c_i \in C, i \in P, a \in d_i \). We know that AC-Project\((c_i, i, a)\) is called at least once during GAC, since it is called in the initialisation phase. After the last call to AC-Project\((c_i, i, a)\)

\[
  c_i(a) = \min_{t \in \ell(P_{\setminus i})} (c_i(a) \oplus c_P(t, a))
\]

by Lemma 4.11. This can only later become invalid by an increase in some \( c_P(t) \) by AC-Extend, which would necessarily be accompanied by the addition of \((t, P, c_P(t))\) to \( Q \) and would hence entail another call of AC-Project\((c_p, i, a)\). This contradiction demonstrates that Condition 2 in Definition 4.10 of generalised arc consistency holds when GAC terminates.

Consider \( t \in \ell(P) \) where \( c_P \in C^+ \). We know that AC-Extend\((i, t_{\downarrow i}, c_P)\) is called during the initialisation phase for each \( i \in P \). After the last such call of AC-Extend\((i, t_{\downarrow i}, c_P)\) for any \( i \in P \),

\[
  \forall t \in \ell(P), c_P(t) = (c_P(t) \oplus \beta) \ominus \beta, \text{ where } \beta = \bigoplus_{j \in P} c_j(t_{\downarrow j})
\]

by Lemma 4.12. This could only later become invalid by an update of \( c_P(t) \) or some \( c_j(t_{\downarrow j}) \) by AC-Project\((c_N, j, t_{\downarrow j})\) for some \( c_N \in C \) such that \( j \in P \cap N \). But then a call of AC-Extend\((j, t_{\downarrow j}, c_P)\) would ensue. This contradiction shows that Condition 2 of Definition 4.10 of generalised arc consistency also holds when GAC terminates.

Theorem 4.14 GAC has polynomial time complexity.
Proof: Let \( n_1 \) be the number of elements \((t, P, \alpha)\) extracted from \( Q \) during GAC such that \( P \) is a singleton, and let \( n_2 \) be the number of elements \((t, P, \alpha)\) extracted from \( Q \) during GAC such that \(|P| \geq 2\). By Theorem 4.8, we can assume that the valuation structure contains at most \( 2ed^r + 2 \) absorbing valuations.

Let \( c_P \in C^+ \) and \( t \in \ell(P) \). By Theorem 4.7, AC-Extend can only increase \( c_P(t) \) to an absorbing valuation strictly greater than its previous valuation. AC-Project can decrease \( c_P(t) \) but, by the Slice Independence Theorem, only from a non-absorbing valuation \( \gamma \) to a valuation larger than or equal to \( \delta = \gamma \ominus \gamma \), the maximal absorbing valuation less than or equal to \( \gamma \). Thus the sequence of absorbing valuations taken on by \( c_P(t) \) during GAC are strictly increasing. Thus for each of the \( 2ed^r + 2 \) attainable absorbing valuations \( \alpha \), \((t, P, \alpha)\) is added to \( Q \) at most once. Thus \( n_2 \leq ed^r(2ed^r + 2) \).

Now \( n_1 \) cannot exceed the number of calls of AC-Project during GAC since tuples \((t, P, \alpha)\) such that \( P \) is a singleton are only added to \( Q \) by AC-Project. The number of calls of AC-Project is clearly bounded above by \( erd + n_2r \). Thus \( n_1 + n_2 \leq erd + (r+1)n_2 = O(e^2d^{2r}) \). Thus the total number of iterations of the while loop in GAC is a polynomial function of \( e \) and \( d \). The result follows immediately.

**Definition 4.15** An arc consistent closure of a VCSP \( V \) is a VCSP which is arc consistent and which can be obtained from \( V \) by a finite sequence of applications of arc consistency operations AC-Extend and AC-Project.

Note that confluence of arc consistency enforcing is lost and therefore the arc consistent closure of a problem is not necessarily unique as it is in classical CSPs. Figure 4(a) shows a 2-variable VCSP on the valuation structure \( \langle \mathbb{N} \cup \{\infty\}, +, \geq \rangle \). Each edge has a weight of 1. Figures 4(b) and 4(c) show two different arc consistency closures of this VCSP.

![Figure 4](image_url)

Figure 4: A Max-CSP and two different equivalent arc consistent closures

### 4.4 Maximum arc consistency

One of the practical use of arc consistency in VCSP is the computation of lower bounds on the valuation of an optimal solution. Obviously, given a VCSP \( V \), the following valuation \( f_{\min}(V) \) is always a lower bound on the cost of an optimal solution:

\[
f_{\min}(V) = \bigoplus_{c_P \in C} \left[ \min_{t \in \ell(P)} c_P(t) \right]
\]
From this point of view, the closure in 4(b) is preferable to the closure in Figure 4(c) since it makes explicit the fact that 1 is a lower bound on the valuation of all solutions.

**Definition 4.16** A VCSP $V$ is said to be maximally arc consistent if it is arc consistent and if the associated lower bound $f_{\min}(V)$ is maximum over all arc consistent closures of $V$.

The problem of enforcing maximal arc consistency is certainly practically important and some closely related problems are known to be NP-hard [17, 19]. Consider the following problem:

**Problem 1 (Max-AC)** Given a VCSP $V = (X, D, C, S)$ and a valuation $\kappa \in S$, does there exist an arc consistency closure $V'$ of $V$ such that $f_{\min}(V') \geq \kappa$?

**Theorem 4.17** The decision problem Max-AC is NP-complete.

**Proof:** Max-AC is clearly in NP. It is therefore sufficient to give a polynomial reduction from 3-Sat to Max-AC.

Let $I_{3\text{-}Sat}$ be an instance of 3-Sat, consisting of $n$ variables and $d$ clauses, each clause being the disjunction of exactly 3 literals.

We assume that each boolean variable $u$ and its negation $\neg u$ occur exactly the same number of times in $I_{3\text{-}Sat}$. Note that if this is not initially the case, it can easily be achieved by adding the required number of tautological clauses of the form $(u \lor u \lor \neg u)$ or $(u \lor \neg u \lor \neg u)$. We will now construct an instance $V$ of MAX-CSP on $4d$ variables such that $V$ has an arc consistency closure $V'$ with $f_{\min}(V') \geq \frac{5d}{2}$ iff $I_{3\text{-}Sat}$ is satisfiable. Note that $d$ is necessarily even by our assumption that each variable $u$ and its negation occur the same number of times.

Suppose that the boolean variable $u$ (and its negation $\neg u$) occur in exactly $m(u)$ clauses in $I_{3\text{-}Sat}$. Then we add the gadget $G(u)$ shown in Figure 5 to $V$, containing the $2m(u)$ variables $i(u, r), i(\neg u, r) (r = 1, \ldots, m(u))$, each with domain-size 3. Each edge joining value $a$ at variable $i$ and value $b$ at variable $j$ represents a penalty of 1, i.e. $c_{ij}(a, b) = 1$. For each clause $c$ in $I_{3\text{-}Sat}$, we add the gadget $H(c)$ shown in Figure 6, involving a new variable $i(c)$ connected to three existing variables. In the example shown, $c \equiv (u \lor v \lor \neg w)$ where $c$ contains the $p^{th}$ occurrence of the boolean variable $u$ in $I_{3\text{-}Sat}$, the $q^{th}$ occurrence of $v$ and the $r^{th}$ occurrence of $\neg w$. In the gadget $H(c)$ shown in Figure 6, $i(c)$ is connected to $i(\neg u, p), i(\neg v, q)$ and $i(w, r)$.
Let $M_i = \max_{a \in d_i} c'_i(a)$, where $c'$ represents the constraint functions in $V'$. Each variable $i(c)$ can contribute at most 1 to $f_{\text{min}}(V')$, i.e. $M_i(c) \leq 1$. By construction of $G(u)$, if any variable (e.g. $i(u, r)$) contributes 1 to $f_{\text{min}}(V')$, then its adjacent variables ($i(\neg u, r - 1)$ and $i(\neg u, r)$) cannot contribute to $f_{\text{min}}(V')$. This means that the maximum value of $f_{\text{min}}(V')$ is $d + \frac{3d}{2} = \frac{5d}{2}$, since the total number of variables in the gadgets $G(u)$ is $3d$. Indeed, $f_{\text{min}}(V') = \frac{5d}{2}$ iff for all clauses $c$, $M_i(c) = 1$ and for all boolean variables $u$,

$$\left( \forall r \in \{1, \ldots, m(u)\}, M_i(u, r) = 1 \land M_i(\neg u, r) = 0 \right) \lor$$

$$\left( \forall r \in \{1, \ldots, m(u)\}, M_i(\neg u, r) = 1 \land M_i(u, r) = 0 \right) \quad (1)$$

Consider the clause $c \equiv (u \lor v \lor \neg w)$, whose gadget $H(c)$ is shown in Figure 6. By construction of $H(c)$, $M_i(c) = 1$ implies that

$$(M_i(\neg u, p) = 0) \lor (M_i(\neg w, q) = 0) \lor (M_i(w, r) = 0)$$

which, in turn, implies from (1), that

$$(M_i(u, 1) = 1) \lor (M_i(v, 1) = 1) \lor (M_i(w, 1) = 0) \quad (2)$$

Suppose that $V$ has an arc consistency closure $V'$ with $f_{\text{min}}(V') \geq \frac{5d}{2}$. For each variable $u$ in $I_{3.\text{SAT}}$, set $u = \text{true}$ iff $M_i(u, 1) = 1$. For each clause $c$, for example $c \equiv (u \lor v \lor \neg w)$, we know from (2) that either $u = \text{true}, v = \text{true}$ or $w = \text{false}$. Hence $I_{3.\text{SAT}}$ is satisfied.

Suppose that $\omega$ is a model of $I_{3.\text{SAT}}$. In each gadget $G(u)$ in $V$ and for each $r \in \{1, \ldots, m(u)\}$, if $u = \text{true}$ in $\omega$ then project penalties onto $i(u, r)$ from its constraints with the adjacent variables $i(\neg u, r - 1)$ and $i(\neg u, r)$; if $u = \text{false}$ in $\omega$, then project penalties onto $i(\neg u, r)$ from its constraints with the adjacent variables $i(u, r)$ and $i(u, r + 1)$. Consider a gadget $H(c)$ in $V$, such as the gadget illustrated in Figure 6 for $c \equiv (u \lor v \lor \neg w)$. If $u = \text{true}$ in $\omega$, then project penalties onto $i(c)$ from its constraint with $i(\neg u, p)$; if $u = \text{false}$ in $\omega$, then project penalties onto $i(\neg u, p)$ from its constraint with $i(c)$. Similarly,
if \( \neg w = \text{true} \) in \( \omega \), then project penalties onto \( i(c) \) from its constraint with \( i(w, r) \); if \( \neg w = \text{false} \) in \( \omega \), then project penalties onto \( i(w, r) \) from its constraint with \( i(c) \). Let \( V' \) be the resulting arc consistent VCSP. Since each clause is satisfied by \( \omega \), \( M_{i(c)} = 1 \). Furthermore, if \( u = \text{true} \) in \( \omega \), then \( M_{i(u, r)} = 1 \) for \( r \in \{1, \ldots, m(u)\} \) and if \( u = \text{false} \) in \( \omega \), then \( M_{i(\neg u, r)} = 1 \) for \( r \in \{1, \ldots, m(u)\} \). Thus \( f_{\min}(V') = d + \frac{3d}{2} = \frac{5d}{2} \).

### 4.5 The case of strictly monotonic VCSPs

For strictly monotonic VCSPs, the previous algorithm can be improved using an alternative equivalent definition of arc consistency based on the notion of the underlying CSP.

**Definition 4.18** The underlying CSP of a VCSP \( V \) has the same variables and domains as \( V \) together with, for each constraint \( c_p \in C \), a crisp constraint \( c'_p \), satisfying \( \forall t \in \ell(P)(t \in c'_p \Leftrightarrow c_p(t) < \top) \) (i.e., \( t \) is not a totally forbidden labelling).

In strictly monotonic VCSPs, the only absorbing elements are \( \top \) and \( \bot \). This allows us to give an equivalent but simpler definition of generalised arc consistency:

**Theorem 4.19** If \( \oplus \) is a strictly monotonic operator, then a VCSP is generalised arc consistent if:

1. its underlying CSP is generalised arc consistent
2. \( \forall c_p \in C^+, \forall i \in P, \forall a \in d_i \text{ such that } c_i(a) < \top \text{ then } \exists t \in \ell(P - \{i\})(c_p(t, a) = \bot) \).

**Proof:** (\( \Rightarrow \)) Suppose that a strictly monotonic VCSP is generalised arc consistent but its underlying CSP is not. Then \( \exists c_p \in C, i \in P, a \in d_i \) such that \( (c_i(a) < \top) \land (\forall t \in \ell(P - \{i\})(c_p(t, a) = \top \land \exists j \in P - \{i\} c_j(t_{i(j)}) = \top)) \). But, by Condition 1 of Definition 4.10, \( (c_j(t_{i(j)}) = \top) \) implies \( c_p(t, a) = \top \). Hence, \( c_i(a) = \top \), by Condition 2 of Definition 4.10, which is a contradiction. Suppose on the other hand that the VCSP is generalised arc consistent but Condition 2 of Theorem 4.19 is not satisfied. Then \( \exists c_p \in C^+, i \in P, a \in d_i \) such that \( c_i(a) < \top \) and \( \forall t \in \ell(P - \{i\}), (c_p(t, a) > \bot) \). But by Condition 2 of Definition 4.10, for some \( t \in \ell(P - \{i\}), c_i(a) = c_i(a) \oplus c_p(t, a) > c_i(a) \) by strict monotonicity, which is impossible.

(\( \Leftarrow \)) Suppose that a strictly monotonic VCSP satisfies Condition 1 and 2 of Theorem 4.19. Condition 2 clearly implies Condition 2 of Definition 4.10. Suppose that Condition 1 of Definition 4.10 is not satisfied. Then \( \exists c_p \in C, t \in \ell(P) \) such that \( c_p(t) \neq \delta = (c_p(t) \oplus \beta) \ominus \beta \) where \( \beta = \bigoplus_{j \in P} c_j(t_{i(j)}) \). By Theorem 4.7, \( \delta \) is absorbing, which is only possible if \( c_j(t_{i(j)}) = \top \) for some \( j \in P \). But the generalised arc consistency of the underlying CSP implies \( c_p(t) = \top \), which provides the necessary contradiction.

A possible way to enforce soft arc consistency on a strictly monotonic VCSP is therefore to first enforce classical arc consistency on the underlying CSP, assign a valuation of \( \top \) to all deleted values in the original VCSP and then
Algorithm 5: Enforcing soft arc consistency on strictly monotonic VCSPs

| Enforce arc consistency in the underlying CSP; |
| foreach $c_P \in C^+$ do |
| foreach $i \in P$ do |
| | foreach $a \in d_i$ do Project($c_P, i, a$); |

enforce the second property by applying Project($c_P, i, a$) once for all $c_P \in C^+$, all $i \in P$ and all $a \in d_i$ (see Algorithm 5).

The only additional result needed to prove that this algorithm works is the following one:

**Theorem 4.20** Let $V$ be a strictly monotonic VCSP whose underlying CSP is arc consistent. Then, $\forall c_P \in C^+, \forall i \in P, \forall a \in d_i$, the application of the equivalence-preserving transformation $\text{Project}(c_P, i, a)$ yields a VCSP whose underlying CSP is unchanged (and therefore arc consistent).

**Proof:** Project($c_P, i, a$) cannot increase a valuation $c_i(a) \neq \top$ to $\top$ since arc consistency on the underlying CSP would have deleted $(i, a)$ in this case and therefore we would have set $c_i(a) = \top$. It cannot decrease the valuation of any tuple such that $c_P(t, a) = \top$ since $\forall \beta \in E, \top \ominus \beta = \top$.

### 4.5.1 Improving space complexity

If an optimal $O(ed^r)$ arc consistency enforcing algorithm such as generalised AC7 [2] is used to enforce arc consistency on the underlying CSP of a fair VCSP, Algorithm 5 establishes arc consistency in $O(ed^r)$ too. However, the space complexity of this algorithm is dominated by the space complexity of the modified constraints which requires $O(ed^r)$ valuations. This is extremely expensive, especially for constraints defined using a cost function. This space requirement can be reduced using a simple data structure for representing modifications of costs induced by basic equivalence-preserving transformations such as Project and Extend.

Let us denote by $c_P^r \in C$ the original definition of a constraint in a fair VCSP by any possible way: explicitly by a table of valuations or implicitly by a cost function from $\ell(P) \rightarrow E$. For each constraint $c_P$, for each variable $i \in P$, we use 2 tables of $d$ valuations noted $\Delta^+_P[i]$ and $\Delta^-_P[i]$, initialised to $\bot$. Let $a \in d_i$:

- $\Delta^-_P[i][a]$ contains the combination of all the valuations that are projected from $c_P$ onto $(i, a)$;
- $\Delta^+_P[i][a]$ contains the aggregation of all the valuations that are extended from $(i, a)$ to $c_P$.

At any time, the valuation of a tuple $t \in \ell(P)$ in the modified constraint $c_P$ can simply be obtained by:

$$c_P(t) = c_P^r(t) \oplus (\bigoplus_{i \in P} \Delta^+_P[i][t|\{i\}]) \ominus (\bigoplus_{i \in P} \Delta^-_P[i][t|\{i\}])$$
By definition of projection, \((\bigoplus_{i \in P} \Delta_{P_i}[(t_{i|i(1)}]) \ll (c_{P(t)}(t) \oplus (\bigoplus_{i \in P} \Delta_{P_i}[(t_{i|i(1)})])\)
and the difference always exists in a fair VCSP. The space complexity is now reduced to \(O(edr)\) instead of \(O(ed)\) and our two basic equivalence-preserving transformations \text{Project} and \text{Extend} become space tractable even for large arity constraints defined using cost functions.

Algorithm 6: Projection and Extension for Soft AC enforcing

**Procedure** \text{Project}(c_{ij}, i, a)

\[
\begin{align*}
\alpha & \leftarrow \min_{t \in \ell(P-\{i\})} (c_{P(t)}(t, a) \oplus (\bigoplus_{j \in P} \Delta_{P_j}[(t, a)_{i(j)}]) \oplus (\bigoplus_{j \in P} \Delta_{P_j}[(t, a)_{i(j)}])); \\
\tilde{c}_i(a) & \leftarrow \tilde{c}_i(a) \oplus \alpha; \\
\Delta_{P_i}^{+}[a] & \leftarrow \Delta_{P_i}^{+}[a] \oplus \alpha;
\end{align*}
\]

**Procedure** \text{Extend}(i, a, c_P)

\[
\begin{align*}
\alpha & \leftarrow \tilde{c}_i(a); \\
\Delta_{P_i}^{+}[a] & \leftarrow \Delta_{P_i}^{+}[a] \oplus \alpha; \\
\tilde{c}_i(a) & \leftarrow \tilde{c}_i(a) \ominus \alpha;
\end{align*}
\]

Algorithm 6 describes the procedures that implement these transformations with these data structures. The time complexity of \text{Extend} is reduced to \(O(1)\) but since computing \(c_P(t)\) requires \(O(r) \oplus\) operations, \text{Project} is \(O(ed^{r-1})\) instead of \(O(d^{r-1})\). This makes generalised arc consistency enforcing on strictly monotonic VCSPs \(O(red^r)\) in time. For binary constraints, we recover the usual \(O(ed^2)\) time and \(O(ed)\) space complexities for arc consistency enforcing.

### 5 Directional arc consistency

In CSPs, directional arc consistency is a weak version of arc consistency. In VCSPs, the order imposed on variables by directional arc consistency (first defined in [6] for strictly monotonic operators) makes it possible to use the unlimited version of \text{Extend} (instead of \text{AC-Extend}) together with a terminating algorithm. For this reason, soft directional arc consistency may be stronger than arc consistency.

Consider the Max-CSP in figure 7(a). It includes one binary constraint that forbids pair ((1, b)(2, b)) and two unary constraints that forbid values (1, a) and (2, a). This VCSP is already arc consistent and the corresponding lower bound \(f_{\min}(V)\) is equal to 0. However, we can apply \text{Extend} on value (1, a),
getting the equivalent VCSP 7(b) which is not arc consistent. We can then apply Project on value (2, b) and obtain the CSP 7(c) with a corresponding lower bound \( f_{\min}(V) = 1 \).

This improved lower bound has been obtained because we have decided to pool all the unary valuations on one of the variables. This can be done successively on all variables using any given variable order. In the context of branch and bound (or other tree-based search), weights can for example be propagated towards those variables which occur earlier in the instantiation order.

**Definition 5.1** A binary VCSP is directional arc consistent according to an order \(<\) on variables if for all \( c_{ij} \in C^+ \) such that \( i < j \), \( \forall a \in d_i \)

\[
C_i(a) = \min_{b \in d_j} (C_i(a) \oplus c_{ij}(a, b) \oplus c_j(b))
\]

Provided that the VCSP is fair, directional arc consistency can be established in polynomial time by the procedure DAC in Algorithm 7, where Project and Extend are as given in Algorithm 6.

**Algorithm 7:** Enforcing directional arc consistency on fair binary VCSPs

```
Procedure DAC()
  for (i ← (n - 1) downto 1) do
    foreach (j ∈ X s.t. j > i and c_{ij} ∈ C) do
      {Start of propagation of c_{ij}};
      foreach (b ∈ d_j) do Extend(j, b, c_{ij});
      foreach (a ∈ d_i) do Project(c_{ij}, i, a);
      {End of propagation of c_{ij}};
    {Assert A(i, j) : ∀a ∈ d_i, C_i(a) = \min_{b ∈ d_j} (C_i(a) \oplus c_{ij}(a, b) \oplus c_j(b))};
```

Since DAC only applies equivalence-preserving transformations, it yields an equivalent VCSP. The following lemma is needed to prove that the VCSP obtained is directional arc consistent.

**Lemma 5.2** If \( \oplus \) is fair, then

\[
(\alpha = \alpha \oplus \beta \oplus \gamma) \land (\alpha' \succsim \alpha) \land (\gamma' \prec \gamma) \Rightarrow (\alpha' = \alpha' \oplus \beta \oplus \gamma')
\]

**Proof:** Suppose that \( (\alpha = \alpha \oplus \beta \oplus \gamma) \land (\alpha' \succsim \alpha) \land (\gamma' \prec \gamma) \). Then \( \alpha' \prec \alpha' \oplus \beta \oplus \gamma \prec \alpha' \oplus \beta \oplus \gamma \prec (\alpha' \oplus \alpha) \oplus (\alpha \oplus \beta \oplus \gamma) = (\alpha' \oplus \alpha) \oplus \alpha = \alpha' \). It follows that \( \alpha' = \alpha' \oplus \beta \oplus \gamma' \).

**Theorem 5.3** If the binary VCSP is fair, then directional arc consistency can be established in \( O(ed^2) \) time and \( O(ed) \) space complexity.

**Proof:** The assertion \( A(i, j) \) is clearly true during execution of DAC at line 1 of Algorithm 7 when constraint \( c_{ij} \) has just been propagated.

It suffices to show that \( A(i, j) \) cannot be invalidated by later propagations of constraints \( c_{ij'} \) where \( i' < j' \) and \( (i' < i) \lor (i' = i \land j' \neq j') \). Such operations
may increase \( c_i(a) \) and may decrease \( c_j(b) \) but cannot modify \( c_{ij}(a,b) \). From Lemma 5.2, \( c_j(a) = c_i(a) \odot c_{ij}(a,b) \odot c_j(b) \) remains true and hence assertion \( A(i,j) \) cannot be invalidated by later propagations and DAC yields a directional arc consistent VCSP.

As for time complexity, procedures \texttt{Extend} and \texttt{Project} are called \( O(ed) \) times and are both \( O(d) \) for binary VCSP. DAC is therefore \( O(ed^2) \) in time. The \( O(ed) \) space complexity can be attained using the implementations of \texttt{Project} and \texttt{Extend} presented in Algorithm 6.

When restricted to strictly monotonic VCSP, Theorem 5.3 can be related to the result, proved in [6], that full directional arc consistency, a stronger version of directional arc consistency, can be established in \( O(ed^2) \) time and space complexity. A VCSP is full directional arc consistent if and only if it is simultaneously arc consistent and directional arc consistent.

**Theorem 5.4** Suppose that the constraint graph of a binary fair VCSP \( V \) is a tree \( T \) and that \( V \) is directional arc consistent according to some topological ordering of the tree \((i \text{ is the father of } j \text{ in } T \Rightarrow (i < j))\). Then, for all \( a \in d_1 \), \( c_1(a) \) is the optimal valuation over all solutions to \( V \) in which variable 1 is assigned value \( a \in d_1 \).

**Proof:** Let \( S \) be the set of all the sons of variable 1 in \( T \) and for each \( j \in S \), let \( T_j \) be the set of all variables in the subtree rooted in \( j \). By induction, we assume that \( \forall j \in S, \forall b \in d_j, \) the valuation \( c_j(b) \) is the optimal valuation over all solutions to the subproblem on \( T_j \). Let \( t_{b,j} \) be one corresponding optimal tuple over \( T_j \).

Let \( a \in d_1 \) and for each \( j \in S \), let \( b_{ja} \) be a value in \( d_j \) that minimises \( c_1(a) \odot c_{1j}(a,b_{ja}) \odot c_j(b_{ja}) \). Since \( \forall i,j \in S, i \neq j, T_i \cap T_j = \emptyset \), we can build a tuple \( t \) over \( X \) by concatenation of each \( t_{b_{ja}} \) for all \( j \in S \) and by assigning value \( a \) to variable 1. The valuation of the tuple \( t \), is \( \mathcal{V}_V(t) = c_1(a) \odot \bigoplus_{j \in S} (c_j(b_{ja}) \odot c_{1j}(a,b_{ja})) = c_1(a) \) since the VCSP is directional arc consistent.

Suppose there exists \( t' \) such that \( t_{1\{1\}} = a \) and \( \mathcal{V}_V(t') \prec \mathcal{V}_V(t) \). Since the problem is tree-structured, the valuation of \( t' \) can be written as

\[
\mathcal{V}_V(t') = c_1(a) \odot \bigoplus_{j \in S} (c_{1j}(a,t'_j) \odot \mathcal{V}_{V(T_j)}(t'_{T_j})) \quad \text{(tree structure)}
\]

\[
\geq c_1(a) \odot \bigoplus_{j \in S} (c_{1j}(a,t'_j) \odot c_j(t'_j)) \quad \text{(induction, monotonicity)}
\]

\[
\geq c_1(a) \odot \bigoplus_{j \in S} (c_j(b_{ja}) \odot c_{1j}(a,b_{ja})) \quad \text{(definition of } b_{ja})
\]

\[
= c_1(a) = \mathcal{V}_V(t)
\]

which shows that no such \( t' \) exists.

**6 Arc irreducibility**

We have seen that the cost of generalising arc consistency from crisp to valued constraint satisfaction problems is the loss of uniqueness of the arc consistent closure. As Figure 4 showed, two different arc consistent closures may also induce different lower bounds via \( f_{\min} \).

If a VCSP \( V \) is equivalent to another VCSP \( V' \) which is better than \( V \) (according to some formally defined criterion \( C \)) then we say that \( V \) is reducible for this criterion. Reducing a VCSP to an equivalent irreducible problem is, in
general, an NP-hard problem. To see this, consider a CSP. If it is inconsistent (i.e., has no solution), then its best equivalent problem, for any reasonable criterion C, is a CSP in which this fact is made explicit, for example, by having \( \forall a \in d_i, c_i(a) = \top \) for some variable \( i \in X \). Since testing consistency of a CSP is an NP-complete problem, we can deduce that testing global irreducibility is NP-hard.

Fortunately, irreducibility has local versions which are analogous to local consistency. For example, a binary VCSP \( V \) is arc-irreducible if, for all pairs of variables \( i, j \in X \), \( V \) cannot be improved by replacing \( c_i, c_j, c_{ij} \) by an equivalent set of constraints \( c'_i, c'_j, c'_{ij} \). However, before giving a formal definition of arc-irreducibility, we have to consider which criteria we could use to compare equivalent VCSPs.

**Definition 6.1** A problem evaluation function \( f \) is a function which, for each VCSP \( V \), assigns a value to \( V \) in a totally ordered range. When comparing two equivalent VCSPs, \( V_1 \) and \( V_2 \), \( V_1 \) is considered as a better expression of the problem if \( f(V_1) > f(V_2) \).

**Example 7** The function \( f_{\text{min}} \) previously defined as

\[
 f_{\text{min}}(V) = \bigoplus_{c_P \in C} \left( \min_{t \in \ell(P)_t} (c_P(t)) \right)
\]

is a problem evaluation function.

The main result presented in this section concerns \( f_{\text{min}} \), but the definitions are valid for any problem evaluation function \( f \).

**Definition 6.2** A binary VCSP \( V \) is arc-irreducible with respect to the problem evaluation function \( f \) (or \((2, f_{\text{min}})\)-irreducible), if \( \forall J \subseteq X, |J| = 2 \), for all VCSP \( V' \) derived from \( V \) by an equivalence-preserving transformation on \( J \), \( f(V) \geq f(V') \).

Note that for certain choices of the problem evaluation function \( f \), an arc-irreducible VCSP is not necessarily arc-consistent. For example, if \( \forall i, j \in X, \exists a \in d_i, b \in d_j \) such that \( c_i(a) = c_j(b) = c_{ij}(a, b) = \bot \), then the VCSP is \((2, f_{\text{min}})\)-irreducible but it is not necessarily arc-consistent. Conversely, the VCSP in Figure 4(c) is arc consistent, but not \((2, f_{\text{min}})\)-irreducible. The following theorem shows that there is an important relationship between directional arc consistency and \((2, f_{\text{min}})\)-irreducibility.

**Theorem 6.3** A fair binary VCSP \( V = (X, D, C, S) \) which is directional arc consistent is \((2, f_{\text{min}})\)-irreducible.

**Proof:** Suppose that \( V \) is directional arc consistent and let \( i, j \in X \) be such that \( i < j \). Then, by definition, \( \forall a \in d_i, \exists b \in d_j \) such that

\[
c_i(a) = \min_{b \in d_j} (c_i(a) \oplus c_{ij}(a, b) \oplus c_j(b))
\]

Thus the minimum valuation of a solution to the subproblem of \( V \) on \( \{i, j\} \) is

\[
\min_{a \in d_i} (c_i(a))
\]

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Suppose that this minimum is attained for $a = u \in d_i$ and that $v \in d_j$ is such that

$$c_i(u) = (c_i(u) \oplus c_{ij}(u, v) \oplus c_j(v))$$  \hspace{1cm} (3)

Now consider a VCSP $V'$ obtained by an equivalence-preserving transformation of $V$ on $\{i, j\}$ which replaces $c_i, c_j, c_{ij}$ by $c'_i, c'_j, c'_{ij}$. Then

$$f_{\text{min}}(V') = (f_{\text{min}}(V) \oplus \left( \bigoplus_{P \subseteq \{i, j\}} \left( \min_{t \in \ell(P)} c_P(t) \right) \right) \oplus \left( \bigoplus_{P \subseteq \{i, j\}} \left( \min_{t \in \ell(P)} c'_P(t) \right) \right)$$

since $V$ and $V'$ only differ on $\{i, j\}$. Thus

$$f_{\text{min}}(V') \preceq f_{\text{min}}(V)$$

by equation 3. Thus $f_{\text{min}}(V') \preceq f_{\text{min}}(V)$ due to the equivalence of the subproblems of $V$ and $V'$ on $\{i, j\}$. Thus, $V$ is $(2, f_{\text{min}})$-irreducible.

This result must be considered with care. Given any VCSP $V$, Theorem 6.3. states that any directional arc consistent closure $V_{\text{dac}}$ of $V$ is always $(2, f_{\text{min}})$-irreducible. However, an arc consistent closure $V_{\text{ac}}$ may exist such that $f_{\text{min}}(V_{\text{ac}}) \succ f_{\text{min}}(V_{\text{dac}})$.

Corollary 6.4 Arc-irreducibility with respect to $f_{\text{min}}$ can be established in $O(ed^2)$ time complexity and $O(ed)$ space complexity on fair binary VCSPs.

Proof: This follows directly from Theorems 5.3 and 6.3.

Conclusion

The concept of arc consistency plays an essential role in constraint satisfaction as a problem simplification operation and as a tree-pruning technique during search through the detection of local inconsistencies among the uninstantiated variables. We have shown that it is possible to generalise arc consistency to any instance of the valued CSP framework provided the operator for aggregating penalties has an inverse.

A polynomial-time algorithm for establishing soft arc consistency exists. Its space and time complexity is identical to that of establishing arc consistency in CSPs whenever the aggregation operator of the VCSP is strictly monotonic, which is the case in MAX-CSP, for example. Contrarily to classical CSP arc consistency, it does not define a unique arc consistency closure. This algorithm nevertheless provides an efficient technique for generating lower bounds on the value of a solution which can be used during branch-and-bound search as in [13, 12]. The problem of finding the maximal lower bound is however NP-hard.

We have also defined a directional version of soft arc consistency which is potentially stronger since it allows non-local propagation of penalties. Directional soft arc consistency implies a form of local optimality in the expression of the VCSP, called arc irreducibility. Furthermore, the complexity of establishing directional arc consistency is identical to that of establishing arc consistency in CSPs.
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