Quadrature rules and distribution of points on manifolds

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Abstract. We study the error in quadrature rules on a compact manifold. Our estimates are in the same spirit of the Koksma-Hlawka inequality and they depend on a sort of discrepancy of the sampling points and a generalized variation of the function. In particular, we give sharp quantitative estimates for quadrature rules of functions in Sobolev classes.

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1. Introduction

In what follows, $\mathcal{M}$ is a smooth compact $d$-dimensional Riemannian manifold without boundary, with Riemannian measure $dx$, normalized so that the total volume of the manifold is 1, and $\Delta$ is the Laplace-Beltrami operator. This operator is self-adjoint in $L^2(\mathcal{M})$, it has a sequence of eigenvalues $\{\lambda_k^2\}$ and an orthonormal complete system of eigenfunctions $\{\varphi_k(x)\}$, $\Delta \varphi_k(x) = \lambda_k^2 \varphi_k(x)$. The eigenvalues, possibly repeated, are ordered with increasing modulus. In particular, the first eigenvalue is 0 and the associated eigenfunction is 1. An example is the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with the Laplace operator $-\sum \partial^2/\partial x_j^2$, eigenvalues $\{4\pi^2|k|^2\}_{k \in \mathbb{Z}^d}$ and eigenfunctions $\{\exp(2\pi i k x)\}_{k \in \mathbb{Z}^d}$. Another example is the sphere $S^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ with $dx$ the normalized surface measure and with $\Delta$ the angular component of the Laplacian in the space $\mathbb{R}^{d+1}$, eigenvalues $\{n(n+d-1)^{+\infty}_{n=0}\}$ and eigenfunctions the restriction to the sphere of homogeneous harmonic polynomials in space. With a small abuse of notation and in analogy with the Euclidean space, the Riemannian distance between $x$ and $y$ will be denoted $|x - y|$.

A classical problem is to approximate an integral $\int_{\mathcal{M}} f(x)dx$ with Riemann sums $N^{-1} \sum_{j=1}^N f(z_j)$, or weighted analogues $\sum_{j=1}^N \omega_j f(z_j)$, and what follows will be concerned with the discrepancy between integrals and sums for functions in

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