The mass in terms of Einstein and Newton

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Abstract
It is shown that the mass of an asymptotically flat manifold with a noncompact boundary can be computed in terms of limiting surface integrals involving the Einstein tensor of the interior metric and the Newton tensor attached to the second fundamental form of the boundary. This extends to this setting previous results by several authors in the boundaryless case. The method outlined below, which is based on a coordinate-free approach due to Herzlich, also applies to asymptotically hyperbolic manifolds, again with a noncompact boundary, for which a similar notion of mass has been recently considered by Almaraz and the first named author, and both cases will be discussed here.

Keywords: asymptotically flat manifolds, asymptotically hyperbolic manifolds, noncompact boundary, mass, center of mass

1. Introduction
Since its inception in the context of the so-called ADM formulation of general relativity, the concept of mass, which turns out to be the fundamental numerical invariant of (time-symmetric) asymptotically flat initial data sets, has played a central role both in the original physical setting and in subsequent applications to several areas of geometric analysis, particularly in the study of questions related to the existence and multiplicity of solutions of the classical Yamabe problem; see [BM, dLPZ] and the references therein. However, since the mass is costumarily defined by a limiting process involving integration over larger and larger spheres of a certain quantity depending on the derivatives of the metric up to first order with no manifest physical or geometric meaning, it is not at all obvious from its very definition how it ties to the asymptotic geometry of the given Riemannian metric. This lack of an explicit geometric meaning for the mass naturally suggests the problem of expressing it as an asymptotic surface integral of geometric quantities directly related to the underlying Riemannian structure. This may be achieved by first establishing the expected relationship between the mass (as well as other asymptotic invariants like the center of mass, etc) and certain curvature integrals of the given...
metric for a special class of initial data sets and then appealing to a suitable density theorem making sure that such a class is dense in the space of all appropriate initial data sets satisfying the relevant dominant energy condition in a topology in which the given invariant is continuous. Besides confirming the general validity of a geometric formula for the mass already available in the physics literature [AH, Ch], this approach has been widely used in many settings and has provided considerable insight on the nature of a large class of such asymptotic invariants [CW, CS, Hu].

We remark, however, that this method of ‘improving the asymptotics’ is rather technical in nature, as it necessarily involves the previous establishment of a corresponding density theorem in suitable weighted functional spaces. Thus, it is highly desirable to circumvent the use of this piece of hard analysis by providing a direct proof of the geometric formulae for the asymptotic invariants. In the specific case of the ADM mass and the center of mass for asymptotically flat manifolds, this has been accomplished in [MT] through an ingenious (but rather lengthy) computation in local coordinates. Even more recently, by combining the penetrating analysis on the existence and well-definiteness of these asymptotic invariants due to Michel [Mi] with a Pohozaev–Schoen-type integral formula involving the Ricci and scalar curvatures, Herzlich [H] was able to present an elementary approach to the problem which in particular dispenses local computations and hence provides a manifestly conceptual definition of the invariants in geometric terms. We mention that, as already explained in [H], this approach also applies to asymptotically hyperbolic manifolds, thus furnishing a geometric formula for the mass functional attached to such manifolds in [CH].

The purpose of this note is to show that the elegant approach in [H] can be adapted to express the mass and the center of mass for the class of asymptotically flat manifolds carrying a noncompact boundary recently studied in [ABdL] in terms of the Einstein tensor of the metric in the interior, the Newton tensor attached to the second fundamental form of the boundary and suitable asymptotically conformal vector fields (theorem 2.2). The method of proof, which as in [H] combines Michel’s analysis with a generalized Pohozaev–Schoen integral formula (proposition 3.2) which we believe might have an independent interest, also allows us to express the mass functional associated to an asymptotically hyperbolic manifold with a noncompact boundary as defined in [AdL], again in terms of the Einstein and Newton tensors (theorem 4.1).

2. The asymptotically flat case

We start by recalling the main definition in [ABdL]. In the following, if \( \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n ; x_n \geq 0 \} \) is the Euclidean half-space endowed with the standard flat metric \( \delta \) and \( r = \sqrt{x_1^2 + \cdots + x_n^2} \) is the standard radial coordinate, then we set \( \mathbb{R}^n_{r_*+} = \{ x \in \mathbb{R}^n_+ ; r > r_* \} \), where \( r_* > 0 \). Moreover, \((M, g)\) will denote an oriented, smooth manifold of dimension \( n \geq 3 \) carrying a non-compact boundary \( \Sigma \). Also, \( R^g \) is the scalar curvature of \( g \) and \( H^g \) is the mean curvature of \( \Sigma \), computed with respect to the inward unit normal vector field \( \nu^g \). In general, we will denote by the same symbol the restriction to \( \Sigma \) of a metric on \( M \). The next definition isolates the class of manifolds we are interested in; they are modelled at infinity on the Euclidean half-space \( \mathbb{R}^n_+ \) above.

**Definition 2.1 ([ABdL]).** We say that \((M, g)\) as above is *asymptotically flat* (with a noncompact boundary \( \Sigma \)) if there exists a compact subset \( A \subset M \) and a diffeomorphism \( F : M \setminus A \to \mathbb{R}^n_{r_0+} \), for some \( r_0 > 0 \), so that:
(1) as $r \to +\infty$,
\[ |g_{ij} - \delta_{ij}| + r|\partial_k g_{ij}| + r^2|\partial_k \partial_l g_{ij}| = O(r^{-\tau}), \quad \tau > \frac{n-2}{2}, \tag{2.1} \]

where the Euclidean (and hence the radial) coordinates have been transplanted to $M \setminus A$ by means of the chart at infinity $F$ and used to compute the coefficients of $g$ and its derivatives;

(2) both $\int_M R^2 \mathrm{d}\nu^2_M$ and $\int_M H^2 \mathrm{d}\nu^2_M$ are finite.

As already observed, in the presence of a chart at infinity $F$ we may identify $M \setminus A = \mathbb{R}^n_{0_+}$. This allows us to define for $r_0 < r < r'$, $M_{r,r'} = \{x \in M \setminus A; r \leq |x| \leq r'\}$, $\Sigma_{r,r'} = \{x \in \partial (M \setminus A); r \leq |x| \leq r'\}$ and the coordinate $(n-1)$-sphere $S_{r,1}^{n-1} = \{x \in M \setminus A; |x| = r\}$, so that $\partial M_{r,r'} = S_{r,1}^{n-1} \cup \Sigma_{r,r'} \cup S_{r',1}^{n-1}$.

We represent by $\mu^\delta$ the outward unit normal vector field to $S_{r,1}^{n-1}$, computed with respect to the reference metric $\delta$. Also, we consider the coordinate $(n-2)$-sphere $S_{r,1}^{n-2} = \partial S_{r,1}^{n-1} \subset \partial (M \setminus A)$, endowed with its outward unit conormal vector field $\vartheta^\delta$, which is tangent to $\Sigma$. As before we set $e = g - \delta$, where we have written $\delta = F^* \delta$ for simplicity of notation, and we define the 1-form
\[ \| \mu^\delta \|_{e,1} = w(\text{div}^\delta e - \text{d} \text{tr}^\delta e) - \nabla^\delta w.e + \text{tr}^\delta e \text{d}w, \tag{2.2} \]

where $w$ is a function on $\mathbb{R}^n_+$. In the following we will make use of the universal constants
\[ c_n = \frac{1}{2(n-1)\omega_{n-1}}, \quad d_n = \frac{1}{(2-n)(n-1)\omega_{n-1}}, \]

where $\omega_{n-1}$ is the volume of the unit $(n-1)$-sphere. Finally, we represent by $\eta^\delta = -\nu^\delta$ the outward unit normal vector field along $\Sigma$. The following result, proved in [ABdL], provides the most relevant asymptotic invariant for the class of manifolds appearing in definition 2.1.

**Theorem 2.1.** If $(M,g)$ is asymptotically flat as above then the quantity
\[ m_{(M,g)} = c_n \lim_{r \to +\infty} \left( \int_{S_{r,1}^{n-1}} \| \mu^\delta \|_{e,1} \mathrm{d}\nu^\delta_{S_{r,1}^{n-2}} - \int_{S_{r,1}^{n-2}} e(\eta^\delta, \vartheta^\delta) \mathrm{d}\nu^\delta_{S_{r,1}^{n-2}} \right) \tag{2.3} \]
is finite and its value does not depend on which chart at infinity is chosen, where 1 here denotes the function identically equal to 1.

**Remark 2.1.** In order to have the mass well defined it is not required to assume a second order pointwise control on the metric as in (2.1) above; a first order control suffices (see [Mi] for a discussion of this rather subtle point in the boundaryless case). However, for our purposes, this extra assumption is crucial as it not only implies that $H^\delta = O(r^{-\tau-1})$ but also that both $\text{Ric}^\delta$ and $\text{R}^\delta$ are $O(r^{-\tau-2})$.

This expression for the mass $m_{(M,g)}$ involves integration over larger and larger (hemi-) spheres of quantities depending on the derivatives of the metric up to first order with no direct geometric meaning. Thus, as already remarked in the Introduction, it is highly desirable to obtain an expression for this invariant in terms of more fundamentally geometric objects. In the boundaryless case, it is well-known that the mass can be asymptotically written in terms of the Einstein tensor of $g$. 

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\[ E^g = \text{Ric}^g - \frac{R^g}{2} g, \]

see [H] and the references therein. Here we show how the elegant method in [H] can be adapted to express \( m_{(M,g)} \) above in terms of the Einstein tensor \( E^g \) and the Newton tensor along the boundary, namely,

\[ J^g = \Pi^g - H^g, \]

where \( \Pi^g \) is the second fundamental form of \( \Sigma \), defined with respect to \( \nu^g \), the inward unit normal, and \( H^g = \text{tr}_{\nu^g} \Pi^g \) is the mean curvature. As in the boundaryless case, these tensors should be evaluated on the radial vector field \( X_0 = r \partial_r \). More precisely, the following result holds.

**Theorem 2.2.** One has

\[ m_{(M,g)} = d_n \lim_{r \to +\infty} \left[ \int_{S_{r+1}^n} E^g(x_0, \nu^g) d\text{vol}^g_{S_{r+1}^n} + \int_{S_{r-2}^n} J^g(x_0, \nu^g) d\text{vol}^g_{S_{r-2}^n} \right]. \]

**Remark 2.2.** After the completion of this article we learned that X. Chai, working independently from us, also proved theorem 2.2 (see [C], theorem 2).

A simple variation of the procedure leading to the proof of theorem 2.1 also allows us to define the center of mass of an asymptotically flat manifold \((M, g)\) as in definition 2.1; see [Mi] for a discussion of the boundaryless case.

**Theorem 2.3.** Let \((M, g)\) be an asymptotically flat manifold with \( m_{(M,g)} \neq 0 \) and assume moreover that both \( \int_M r^2 d\text{vol}^g_M \) and \( \int_\Sigma rH^g d\text{vol}^g_\Sigma \) are finite, where the asymptotic radial coordinate \( r \) has been smoothly extended to the whole of \( M \). Then for each \( \alpha = 1, \ldots, n-1 \) the quantity

\[ c_{\alpha}^{(M,g)} = \frac{c_n}{m_{(M,g)}} \lim_{r \to +\infty} \left[ \int_{S_{r+1}^n} \Pi^g r_{\alpha,\alpha} (\nu^g) d\text{vol}^g_{S_{r+1}^n} - \int_{S_{r-2}^n} x_\alpha e(\eta^g, \nu^g) d\text{vol}^g_{S_{r-2}^n} \right] \]

is finite. Moreover, the vector \( c_{(M,g)} = (c_1^{(M,g)}, \ldots, c_{n-1}^{(M,g)}) \) does not depend on the chosen chart at infinity (up to Euclidean rigid motions) and is termed the center of mass of \((M, g)\).

**Remark 2.3.** It is known that the center of mass \( c_{(M,g)} \) is also well defined if instead of the integrability conditions on \( R^g \) and \( H^g \) in theorem 2.3 above we assume the corresponding Regge–Teitelboim (RT) evenness conditions at infinity. More precisely, for \( x = (x', x_n) \in M \setminus A \), where \( x' = (x_1, \ldots, x_{n-1}) \), define the involution \( \tau : M \setminus A \to M \setminus A \) by \( \tau(x', x_n) = (-x', x_n) \), so that for each function \( f \) on \( M \setminus A \) we may consider its odd part

\[ f^{\text{odd}}(x) = \frac{1}{2} (f(x) - f(\tau x)). \]

Then the RT requirement is that

\[ |g^{\text{odd}}_{ij} + r | \partial_r g^{\text{odd}}_{ij} | = O(r^{\alpha - 1}), \quad (R^g)^{\text{odd}} = O(r^{\alpha - 2}), \quad (H^g)^{\text{odd}} = O(r^{\alpha - 3}); \]

see [CCS, CW, Hu, H] for discussions in the boundaryless case.
In order to write the center of mass in theorem 2.3 in terms of $E^g$ and $J^g$ we make use of the conformal vector fields $X_\alpha = r^2 \partial_\alpha - 2x_\alpha \sum_i x_i \partial_i, \alpha = 1, \cdots, n-1$, which obviously are tangent to $\partial \mathbb{R}_+^n$ (similarly to $X_0$).

**Theorem 2.4.** With the notation above,

$$
\varsigma^\alpha_{(M,g)} = -\frac{d_\alpha}{2m(M,g)} \lim_{r \to +\infty} \left[ \int_{S_{r+1}^n} E^g(X_\alpha, \mu^g) d\text{vol}_r + \int_{S_{r-1}^n} J^g(X_\alpha, \partial^g) d\text{vol}_r \right].
$$

3. The proofs of theorems 2.2 and 2.4

We follow [H] closely and for $r > 4r_0$ consider a cut-off function $\chi = \chi_r(r')$ on $M \setminus A$ which vanishes for $r' \leq r/2$, equals 1 for $r' \geq 3r/4$ and satisfies the estimates

$$
|\nabla \chi| \leq C r^{-1}, \quad |\nabla^2 \chi| \leq C r^{-2}, \quad |\nabla^3 \chi| \leq C r^{-3},
$$

for some $C > 0$ independent of $r$. We then define a metric on $M_r := M_r/\mathbb{R}_+$ by

$$
h = (1 - \chi) \delta + \chi g, \quad (3.4)
$$

which is then extended to the whole of $M$ in the obvious manner. With this notation at hand, the computations in [ABdL, section 3] leading to the proof of theorem 2.1 above easily imply the following alternative expression for the mass. We remark that this also follows by adapting to this setting Michel’s analysis [Mi] in the boundaryless case.

**Proposition 3.1.** With the notation above,

$$
m(M,g) = c_n \lim_{r \to +\infty} \left[ \int_{M_r} R^g d\text{vol}_r + 2 \int_{\Sigma_r} H^g d\text{vol}_\Sigma \right], \quad (3.5)
$$

where $\Sigma_r := \Sigma_{r/4,r}$.

The next ingredient in the proof of theorem 2.2 is a Pohozaev–Schoen-type integral identity whose infinitesimal version we present now. A special case of this result appears in [H, lemma 2.1]; see also [BdLF] for another variant of this useful identity. We have chosen here to present this material in full generality as we believe it might be useful in other contexts as well. Thus, let us consider a Riemannian $p$-manifold $(N, \gamma)$ and take $K = K_{ij} \in S^2(N, \gamma)$ to be a symmetric, twice covariant tensor and $Y = Y^i \in \mathcal{X}(N)$ a vector field, where $i, j = 1, \cdots, p$.

**Proposition 3.2.** There holds

$$
\text{div}^\gamma(Y, K) = \langle \text{div}^\gamma K, Y \rangle_\gamma + \langle K, \text{div}^\gamma Y \rangle_\gamma + \frac{1}{p} \text{div}^\gamma Y \text{Tr}^\gamma K, \quad (3.6)
$$

where $\text{div}^\gamma : \mathcal{X}(N) \to S^2(N, \gamma)$ is the $L^2$ adjoint of the divergence map $\text{div}^\gamma : S^2(N, \gamma) \to \mathcal{X}(N)$ and the tilde means the trace free part.

**Proof.** Computing at the center of a normal coordinate system we have $(Y, K)_i = K_{ij} Y_j$ and hence
\[ \text{div}^\gamma (Y \lrcorner K) = (K_{ij} Y_i)_j, \]
\[ = K_{ij} Y_i + K_{ij} Y_j, \]
\[ = K_{ij} Y_i + \frac{1}{2} K_{ij} (Y_{ij} + Y_{ji}), \]

where the comma means covariant derivation. In invariant terms this means that

\[ \text{div}^\gamma (Y \lrcorner K) = \langle \text{div}^\gamma K, Y \rangle_\gamma + \frac{1}{2} \langle K, L Y \rangle_\gamma, \]

where \( L \) is the Lie derivative. Since

\[ \text{div}^\gamma Y = \frac{1}{2} L Y \gamma, \]

we get

\[ \text{div}^\gamma (Y \lrcorner K) = \langle \text{div}^\gamma K, Y \rangle_\gamma + \langle K, \text{div}^\gamma Y \rangle_\gamma. \]

Also, since

\[ (\text{div}^\gamma Y)_j = \frac{1}{2} (Y_{ij} + Y_{ji}), \]

we have

\[ \text{tr}^\gamma \text{div}^\gamma Y = \text{div}^\gamma Y, \]

so that

\[ \tilde{\text{div}}^\gamma Y = \text{div}^\gamma Y - \frac{1}{p} (\text{div}^\gamma Y)_\gamma, \]

and the result follows. \( \square \)

To proceed with the proof of theorem 2.2, we integrate (3.6) over the \( n \)-dimensional half-annulus \((M, h)\), where \( h \) is the interpolating metric in (3.4). We take \( K \) to be

\[ E^h = \text{Ric}^h - \frac{R^h}{2} h, \]

the Einstein tensor of \( h \), so that

\[ \text{div}^h E^h = 0, \quad \text{tr}^h E^h = \frac{2}{2} - n R^h. \]

Noticing that

\[ \int_{S_{r/2}^{n-1}} E^h (Y, \mu^h) d\text{vol}_{S_{r/2}^{n-1}} = 0 \]

because \( h = \delta \) there, we obtain
\[
\int_{S_{r+1}^{n-1}} E^h(Y, \mu^h) d\text{vol}_{S_{r+1}^{n-1}} + \int_{\Sigma_r} E^h(Y, \eta^h) d\text{vol}_{\Sigma_r} + \frac{2}{2n} \int_{M_r} \text{div}^h Y \cdot R^h d\text{vol}_{M_r}.
\]

We now take \( Y \) to be the radial vector field \( X_0 = r \partial_r \) so that

\[
\text{div}^h X_0 = n, \quad \text{div}^h_{X_0} = 0,
\] (3.7)

the last identity holding due to the fact that \( X_0 \) is conformal relatively to \( \delta \). As in [H] we note that, as \( r \to +\infty \),

\[
\int_{M_r} \text{div}^h X_0 \cdot R^h d\text{vol}_{M_r} = \int_{M_r} \text{div}^h \delta X_0 \cdot R^h d\text{vol}_{M_r} + o(1)
\]

\[
= n \int_{M_r} R^h d\text{vol}_{M_r} + o(1),
\]

where we used the decay on the scalar curvature described in remark 2.1. Similarly,

\[
\int_{M_r} \langle E^h, \text{div}^h_{X_0} \rangle d\text{vol}_{M_r} = \int_{M_r} \langle E^h, \text{div}^h_{\delta X_0} \rangle d\text{vol}_{M_r} + o(1)
\]

\[
= o(1),
\]

which finally gives

\[
\int_{S_{r+1}^{n-1}} E^h(X_0, \mu^h) d\text{vol}_{S_{r+1}^{n-1}} + \int_{\Sigma_r} E^h(X_0, \eta^h) d\text{vol}_{\Sigma_r} = \frac{2}{2n} \int_{M_r} R^h d\text{vol}_{M_r} + o(1).
\] (3.8)

We now integrate (3.6) over another configuration, namely, the \((n - 1)\)-dimensional annulus \( \Sigma_r \), whose boundary is \( \partial \Sigma_r = S_{r/4}^{n-2} \cup S_{r/2}^{n-2} \). This time we take \( K \) to be \( \Pi^h - H^h h \), the Newton tensor of \( \Sigma \) with respect to \( h \), so that

\[
\text{tr}^h J^h = (2 - n) H^h, \quad \text{div}^h J^h = \text{Ric}^h(\eta, \cdot),
\]

with the last identity being just the contracted Codazzi equation; see the proof of [BdLF, theorem 14]. We also take \( Y \) to be \( r \partial_{\Sigma_r} \), which we still denote by \( X_0 \). Hence,

\[
\text{div}^h X_0 = n - 1, \quad \text{div}^h_{X_0} = 0,
\] (3.9)

the last identity being true due to the fact that \( X_0|_{\Sigma} \) is conformal relatively to \( \delta = \delta|_{\Sigma} \). Using that

\[
\int_{S_{r/4}^{n-1}} \langle J^h(X, \mu^h) \rangle d\text{vol}_{S_{r/4}^{n-1}} = 0,
\]

because \( h = \delta \) there, we get
\[
\int_{S^{n-1}} J^h(X_0, \vartheta^h) d\text{vol}^h_{S^{n-1}} = \int_{\Sigma_r} \text{Ric}^h(X_0, \eta^h) d\text{vol}^h_{\Sigma_r}
+ \frac{2}{n-1} \int_{\Sigma_r} \text{div}^h X_0 \cdot H^h d\text{vol}^h_{\Sigma_r},
\]

As before, we make use of the decay on the mean curvature in remark 2.1 to get
\[
\int_{\Sigma_r} \text{div}^h X_0 \cdot H^h d\text{vol}^h_{\Sigma_r} = (n-1) \int_{\Sigma_r} H^h d\text{vol}^h_{\Sigma_r} + o(1),
\]
and similarly,
\[
\int_{\Sigma_r} \langle \text{div}^h X_0, J^h \rangle d\text{vol}^h_{\Sigma_r} = \int_{\Sigma_r} \langle \text{div}^h X_0, J^h \rangle d\text{vol}^h_{\Sigma_r} + o(1)
= o(1),
\]
so that
\[
\int_{S^{n-2}} J^h(X_0, \vartheta^h) d\text{vol}^h_{S^{n-2}} = \int_{\Sigma_r} \text{Ric}^h(X_0, \eta^h) d\text{vol}^h_{\Sigma_r}
+ \frac{2-n}{2} \int_{\Sigma_r} 2H^h d\text{vol}^h_{\Sigma_r} + o(1).
\]

Putting together (3.8) and (3.10) and using that
\[
E^h(X_0, \eta^h) - \text{Ric}^h(X_0, \eta^h) = -\frac{R^h}{2} (X_0, \eta^h)_h = 0
\]
because \(X_0\) is tangent to \(\Sigma\), we end up with
\[
d_n \left[ \int_{S^{n-1}_+} E^h(X_0, \mu^h) d\text{vol}^h_{S^{n-1}_+} + \int_{S^{n-2}} J^h(X_0, \vartheta^h) d\text{vol}^h_{S^{n-2}} \right]
= c_n \left[ \int_{M_+} R^h d\text{vol}^h_{M_+} + 2 \int_{\Sigma_r} H^h d\text{vol}^h_{\Sigma_r} \right] + o(1),
\]
so theorem 2.2 follows from proposition 3.1.

The proof of theorem 2.4 is obtained by essentially the same argument as above, where we now use that
\[
\text{div}^h X_\alpha = -2n x_\alpha, \quad \text{div}^h X_\alpha = 0,
\]
the last identity holding due to the fact that \(X_\alpha\) is conformal relatively to \(\delta\).

**Remark 3.1.** Although not explicitly mentioned so far, a crucial ingredient in the proofs of the theorems above is the fact that the reference space \(\mathbb{R}^n_+\) is *static* in the sense that
\[
\mathcal{N}_\delta := \left\{ w : \mathbb{R}^n_+ \to \mathbb{R}; (\nabla^h)^2 w = 0, \frac{\partial w}{\partial \eta^h} = 0 \right\}
\]
is non-trivial, where \( \eta^\delta = (0, 0, \cdots, -1) \) is the outward unit normal along \( \partial \mathbb{R}^n_+ \). In fact, one easily checks that \( \mathcal{N}_\delta \) is generated by the functions

\[
\begin{align*}
w_0(x) &= 1, \\
w_1(x) &= x_1, \\
&\vdots \\
w_{n-1}(x) &= x_{n-1}.
\end{align*}
\]

Using this terminology, the key point is that \( \text{div}^\delta X \in \mathcal{N}_\delta \) whenever \( X \) is conformal with respect to \( \delta \) and everywhere tangent to \( \partial \mathbb{R}^n_+ \); see [H, lemma 2.2] for the boundaryless version of this result and compare with the first identity in (3.7) and (3.11) above. In fact, the Hessian operator \((\nabla^\delta)^2\) above equals \( DR^\delta_\gamma \), the \( L^2 \) adjoint of the linearization \( DR^\delta \) of the scalar curvature operator at \( \delta \). In general, we say that a Riemannian manifold \((N, \gamma)\), possibly endowed with a noncompact totally geodesic boundary, is static if

\[
\mathcal{N}_\gamma := \left\{ w : N \to \mathbb{R}; DR^\gamma_\gamma w = 0, \frac{\partial w}{\partial \eta^\gamma} = 0 \right\}
\]  

(3.13)

is non-trivial, where as always \( \eta^\gamma \) is the outward unit normal along the boundary; see [AdL] for an application of this concept in the asymptotically hyperbolic case. Since

\[
DR^\gamma_\gamma w = (\nabla^\gamma)^2 w + (\Delta^\gamma w) - wRic^\gamma,
\]

we see that (3.13) reduces to (3.12) when \( \gamma = \delta \).

4. The asymptotically hyperbolic case

As already observed, the methods above may be used to express in geometric terms the mass of an asymptotically hyperbolic manifold with a noncompact boundary, an asymptotic invariant studied in detail in [AdL]. Here the reference space is the hyperbolic half-space in dimension \( n \geq 3 \) given by

\[
\mathbb{H}^n_+ = \{ y \in \mathcal{R}^{1,n}; y_0 > 0, \langle y, y \rangle_L = -1, y_n \geq 0 \},
\]

where \( \mathcal{R}^{1,n} \) is the Minkowski space with the standard flat metric

\[
\langle y, y \rangle_L = -y_0^2 + y_1^2 + \cdots + y_n^2.
\]

By setting \( s = \sqrt{y_1^2 + \cdots + y_n^2} \), \( \mathbb{H}^n_+ \) inherits the hyperbolic metric

\[
b = \frac{ds^2}{1 + s^2} + s^2 h_0,
\]

where \( h_0 \) stands for the canonical metric on the unit upper hemisphere \( S^{n-1}_+ \). Note that \( \mathbb{H}^n_+ \) carries a noncompact, totally geodesic boundary, namely, \( \partial \mathbb{H}^n_+ = \{ y \in \mathbb{H}^n; y_n = 0 \} \). Similarly to remark 3.1 above, we set

\[
\mathcal{N}_b := \left\{ W : \mathbb{H}^n_+ \to \mathbb{R}; (\nabla^b)^2 W = nW, \frac{\partial W}{\partial \eta^b} = 0 \right\},
\]

where, as usual, \( \eta^b \) is the outward unit normal to \( \partial \mathbb{H}^n_+ \). This space is spanned by the functions

\[
W_a(y) = y_a |_{\mathbb{H}^n_+}, \quad a = 0, 1, \cdots, n-1,
\]

so \( \mathbb{H}^n_+ \) is a static space; see again remark 3.1.

We now define the notion of an asymptotically hyperbolic manifold with a non-compact boundary having \((\mathbb{H}^n_+, b)\) as a model at infinity; see [AdL] for further details. For this it is convenient to set \( s = \sinh \rho \), so that
in the coordinates \((\rho, \theta), \theta \in S^{n-1}_+\). Notice that in this model \(\rho\) is the geodesic distance from the origin \(\rho = 0\).

**Definition 4.1 ([AdL]).** We say that \((M^n, g)\) is asymptotically hyperbolic (with a non-compact boundary \(\Sigma\)) if there exists \(\rho_0 > 0\), a compact subset \(A \subset M\) and a diffeomorphism \(F : M \setminus A \rightarrow \mathbb{H}^n_{+, \rho_0}\) such that:

1. As \(\rho \rightarrow +\infty\),
   \[|g_{ij} - \delta_{ij}| + |f_i g_{ij}| + |f_j g_{ij}| = O(e^{-\rho \tau}), \quad \tau > \frac{n}{2}, \]  
   (4.14)

   where \(\{f_i\}_{i=1}^n\) is a \(b\)-orthonormal frame with \(f_1 = \partial_\rho\) and \(g_{ij} = \langle F^* f_i, F^* f_j \rangle_b\);

2. both \(\int_M e^\theta (R^b + n(n - 1)) d\text{vol}_M^b\) and \(\int_{\mathcal{S}^n_r} e^\theta H^b d\text{vol}_{S^2_r}^b\) are finite, where the asymptotical radial coordinate \(\rho\) has been smoothly extended to the whole of \(M\).

**Remark 4.1.** The same observations as in remark 2.1 above apply here, so in particular we have that \(\text{Ric}^b + (n - 1) g, R^b + n(n - 1)\) and \(H^b\) are \(O(e^{-\rho \tau})\).

Regarding this class of manifolds, the next result has been proved in [AdL]. It provides any such manifold with an asymptotic invariant which naturally extends the corresponding notion in the boundaryless case treated in [CH, Wa].

**Theorem 4.1.** If \((M, g)\) is an asymptotically hyperbolic manifold as above then the linear map \(\mathcal{M}_{(M, g)} : N_b \rightarrow \mathbb{R}\) given by

\[\mathcal{M}_{(M, g)}(W) := c_a \lim_{\rho \rightarrow +\infty} \left[ \int_{S^{n-1}_+} U^{b}_{c,w}(\mu^b) d\text{vol}_{S^{n-1}_+} - \int_{S^{n-2}_r} W(\nu^b, \vartheta^b) d\text{vol}_{S^{n-2}_r} \right],\]

where here \(e = g - b\), is well defined and independent of the chosen chart at infinity. Here, \(S^{n-1}_r\), etc, are defined just as in the flat case.

We mention that a positive mass theorem for the mass functional \(\mathcal{M}\) above has been established in [AdL] under the assumption that the underlying manifold is spin.

With this notion at hand, it is not hard to use a simple variation of the arguments in the previous section to check that the mass functional can be asymptotically expressed in terms of the Einstein and Newton tensors. More precisely, define the modified Einstein tensor

\[\hat{\mathcal{E}}^b = \mathcal{E}^b - \frac{(n - 1)(n - 2)}{2} g,\]

and observe that the vector fields \(X_a, a = 0, 1, \cdots, n - 1\) considered above, when properly transplanted to our hyperbolic model \((\mathbb{H}^n_{+, b})\), are conformal with respect to \(b\). Thus, \(\text{div}^b X_a = 0\) and a direct computation shows that \(\text{div}^b X_a = n W_a\), in accordance with remark 3.1. Putting these facts together and proceeding exactly as above we easily obtain the following result.

**Theorem 4.2.** Under these conditions, for any \(a = 0, 1, \cdots, n - 1\) there holds

\[\mathcal{M}_{(M, g)}(W_a) = d_a \lim_{\rho \rightarrow +\infty} \left[ \int_{S^{n-1}_+} \hat{\mathcal{E}}^b(X_a, \mu^b) d\text{vol}_{S^{n-1}_+} + \int_{S^{n-2}_r} J^b(X_a, \vartheta^b) d\text{vol}_{S^{n-2}_r} \right].\]
It should be mentioned that applications of this result to rigidity questions related to asymptotically hyperbolic Einstein manifolds with a noncompact boundary are discussed in [AdL].

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