Some results on ideal impacts of billiard balls

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Abstract
We analyze the impact of two equal billiard balls in three ideal situations: when
the balls freely slide on the plane of the billiard, when they roll without sliding
and when one of them freely slides and the other rolls. In all the cases we
suppose that the contact between the balls is smooth. We base our analysis
on some recent general theoretical results on ideal impacts obtained by means
of Differential Geometric Impulsive Mechanics. We use symbolic computation
software to solve the computational difficulties arising by the high number of
degrees of freedom of the system. Some particular but significative impacts,
with opportunely assigned left velocities and positions of the balls, are analyzed
in details. The results admit easy interpretations that turn out to be in good
agreement with the reasonable forecasts and the behaviours of real systems.

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Introduction
The impact problems arising in the study of billiard games has always represented,
for their possibility of being modelled with a lower or higher degree of simplification
and the facility of comparison of the theoretical results with the real behaviour of
the balls in the game, interesting challenges and severe tests for all who study Classi-
cal Impulsive Mechanics. The geometric approach to these mechanical problems has
recently given important theoretical results, when techniques of modern Differential
Geometry were fruitfully applied to investigate the concept of ideality of unilateral
constraints ([1]), even in presence of kinetic constraints ([2]). Unfortunately, because
of computational difficulties due to the number of degrees of freedom of the systems,
often theoretical results cannot be straitly applied to analyze practical problems:
the impact between two equal billiard balls can be modelled with an impulsive me-
chanical system with 10 degrees of freedom (two for the position of the center of mass
and three for the Euler angles for each ball) simultaneously subject to an instanta-
aneous unilateral constraint (the contact condition), possibly together with permanent
and/or instantaneous kinetic constraints (the rolling conditions), all of them involving
several parameters. The calculations arising in the study of this impact are almost
prohibitive if approached with hand techniques. Luckily, nowadays the use of symbolic
computation softwares can give a very great help to solve (part of) these problems.

In this paper we present the application of the theoretical results of [1, 2] to the
classical example of impact of identical billiard balls in three ideal but nevertheless
significative cases: the smooth case, when both the balls slide on the horizontal plane; the rolling case, when both the balls roll without sliding on the horizontal plane; a mixed case, when a rolling ball impacts with a ball that slides. In everyone of these cases, the theory is powerful enough to allow a very general approach, with both the balls moving on the plane, and with general orientations of the balls at the impact. Even in this general situation, some results (for example concerning the angle formed by the directions of the balls after the impact) will appear in a very clear form. However, in order to have a clearer interpretation of some results, it will be almost unavoidable the analysis of particular impacts. This choice will appear even more necessary in the mixed case, if we want to avoid almost meaningless situations. In these cases, the results assume very simple aspects, that can be easily compared with those of well known experiences.

In the first and simplest case, we had to solve a $11 \times 11$ linear system (fortunately with a very simple structure), while in the second and more complicated case we had to solve a $15 \times 15$ linear system, both involving several parameters, such as the trigonometric functions of the Euler angles of the balls: to this aim, we used CoCoA 4.7, a freeware software for exact symbolic computation [3].

We choose to focus our attention to the most meaningful results from the point of view of the physical behaviour of the balls. Then the computational aspect of the analysis is not treated in details in the paper. Moreover, we point out that, once the results of [1, 2] are assumed, the algorithm to determine the behaviour of the balls after the impact starting by the knowledge of positions and motions of the balls before the impact (is based on but) do not involve refined geometric techniques. Then, for the same reason written above, we describe in details the algorithm (that is, we describe in details the results of [1, 2]), and then we analyze in details the results obtained, but we expose only a very concise description of the geometry underlying the problem.

1 Preliminaries

The theoretical foundation of ideality for unilateral constraints acting on a mechanical system with a finite number of degrees of freedom can be fruitfully framed in the context of Modern Differential Geometry, in particular using the concepts of fibre bundles, their jet–extensions, subbundles and tangent spaces, additionally endowed with suitable metrics. For the reader interested in the general theoretical approach, we refer to [1, 2] and the references therein.

1.1 The geometric environment of the problem

To study the particular case of two equal billiard balls of radius $r$ and mass $m$ moving on the billiard plane, we introduce the configuration space–time $\mathcal{V}$, a differentiable manifold fibered over the time line and parameterized by 11 coordinates. We can choose these coordinates as $(t, x_1, y_1, \psi_1, \vartheta_1, \varphi_1, x_2, y_2, \psi_2, \vartheta_2, \varphi_2)$, where $t$ is the time coordinate, $x_i, y_i$ are the coordinates of the center of mass of the $i$-th ball and $\psi_i, \vartheta_i, \varphi_i$ are the Euler angles giving the orientation of the $i$-th ball. The simultaneous presence of the balls on the plane implies that not all the configurations of $\mathcal{V}$ are admissible, since at any instant the coordinates must satisfy the relation $(x_2 - x_1)^2 + (y_2 - y_1)^2 \geq 4r^2$. There is then a natural positional constraint $\mathcal{S} \subset \mathcal{V}$ expressed by the condition

$$\mathcal{S} := \{(x_2 - x_1)^2 + (y_2 - y_1)^2 - 4r^2 = 0\}$$

that must be satisfied in the moment of the impact (and then called the impact constraint).

The absolute–velocity space–time $J_1(\mathcal{V})$ is the first jet–extension of $\mathcal{V}$, parameterized by 21 coordinates: the first 11 are the coordinates running on $\mathcal{V}$, the remaining
can be grouped in a vector $\mathbf{p} = (1, \dot{x}_1, \dot{y}_1, \dot{\psi}_1, \dot{\varphi}_1, \dot{x}_2, \dot{y}_2, \dot{\psi}_2, \dot{\varphi}_2)$: the presence of the first constant coordinate stresses the necessity of respecting the affine nature of the velocity space–time.

The space $V(V)$ describing the possible impulses acting on the system is the vertical fiber bundle of $V$. It is a differentiable manifold parameterized by 21 coordinates, whose first 11 are the coordinates running on $V$ and the remaining 10 can grouped in a vector $\mathbf{I} = (0, I_{x1}, I_{y1}, I_{\psi1}, I_{\varphi1}, I_{x2}, I_{y2}, I_{\psi2}, I_{\varphi2}, I_{p})$: this time the presence of the first constant coordinate is due to the necessity of respecting the vector nature of the impulse space, and points out that the fibers of $V(V)$ are the vector spaces modelling the affine spaces given by the fibers of $J_1(V)$.

The relation between the space of velocities and the space of impulses is cleared by the very nature of the impact problem: given an input velocity $\mathbf{p}_L$ (the so called “left–velocity”) of the system and an impulse $\mathbf{I}$, the sum

$$\mathbf{p}_R = \mathbf{p}_L + \mathbf{I}$$

is required to be an output velocity (“right–velocity”) of the system. A similar situation is perfectly framed in the relations between affine spaces and modelling vector spaces. However, once the situation is cleared up of possible confusions, from the computational point of view the significative parts of velocities $\mathbf{p}$ and impulses $\mathbf{I}$ are given by the 10 components $((\dot{x}_1, \dot{y}_1, \dot{\psi}_1, \dot{\varphi}_1, \dot{x}_2, \dot{y}_2, \dot{\psi}_2, \dot{\varphi}_2))$ and $(I_{x1}, I_{y1}, I_{\psi1}, I_{\varphi1}, I_{x2}, I_{y2}, I_{\psi2}, I_{\varphi2}, I_{p})$ respectively. These 10–uples will be used without ulterior specifications in the calculations.

A scalar product $\Phi$ acting on the space of impulses $V(V)$ is defined by using the positive definite matrix $g$ that gives the kinetic energy of the system as a quadratic form in the velocity coordinates $(\dot{x}_1, \dot{y}_1, \dot{\psi}_1, \dot{\varphi}_1, \dot{x}_2, \dot{y}_2, \dot{\psi}_2, \dot{\varphi}_2)$. The elements of $g$ are in general functions of the positional coordinates $(x_1, y_1, \vartheta_1, \varphi_1, x_2, y_2, \vartheta_2, \varphi_2)$ and take automatically into account the massive properties of the system. Considering the significative part of the impulse vectors and denoting with $A$ the inertia momentum of the balls with respect to the center of mass, we have

$$g = \begin{pmatrix}
    m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & A & 0 & A \cos \vartheta_1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & A \cos \vartheta_1 & 0 & A & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & A & 0 & A \cos \vartheta_2 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & A \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & A \cos \vartheta_2 & A \\
\end{pmatrix}.$$

A simple calculation shows that this scalar product is positive definite if and only if $\sin \vartheta_1 \neq 0$ and $\sin \vartheta_2 \neq 0$.

Some subspaces of the velocity space–time $J_1(V)$ can be naturally introduced. The presence of the constraint (1) determines the space $T$ of the velocities that are tangent to the positional constraint. Straightforward arguments (of Differential or even Elementary Geometry, since the relative velocities of the centers of mass must be orthogonal to the radii of the balls at the instant of the impact) show that in our case $T$ is given by those velocities obeying the condition

$$(\dot{x}_2 - \dot{x}_1)(x_2 - x_1) + (\dot{y}_2 - \dot{y}_1)(y_2 - y_1) = 0.$$  

Moreover, the possible presence of rolling conditions

$$\begin{cases}
    \dot{x}_i - r \dot{\vartheta}_i \sin \psi_i + r \dot{\varphi}_i \sin \vartheta_i \cos \psi_i = 0 \\
    \dot{y}_i + r \dot{\vartheta}_i \cos \psi_i + r \dot{\varphi}_i \sin \vartheta_i \sin \psi_i = 0
\end{cases}$$

...
for one or both the balls restricts the space of admissible velocities to a subspace $A \subset J_1(V)$. Note that, when the system is subject to (3), the space $A$ is restricted before, during and after the impact. However, due to the linearity of (4), $A$ still owns the geometric structure of affine space.

2 Ideality of the unilateral constraint

The geometric setup synthetically described above allows the introduction of a subspace $A \cap T \subset A \subseteq J_1(V)$ of the space of admissible velocities $A$ of the system before and after the impact given by the admissible velocities of the system when the balls are in contact. We assume that the contact between the balls is smooth (otherwise an ulterior restriction of the space of admissible velocities of the system when the balls are in contact must be introduced for the presence of an ulterior kinetic condition).

Due to the affine structure of these spaces and the existence of the metric structure given by (2), for every element $p$ in $A$, there exists a closest element $P(p)$ in $A \cap T$. The element $P(p)$ can be thought of as the component of $p$ tangent to the impact constraint, while the difference

$$I^\perp(p) = p - P(p)$$

can be thought of as the component of $p$ orthogonal to the impact constraint.

The most natural behaviour of the impact constraint is then given by a “reflection”. The impact with the constraint leaves invariant the tangent component $P(p_L)$ of the left velocity $p_L$ and reflects the orthogonal component $I^\perp(p_L)$. Then, for every admissible left velocity $p_L$, the corresponding right velocity $p_R$ is given by

$$p_R = p_L - 2I^\perp(p_L).$$

It can be shown (see [1, 2]) that the right velocity $p_R$ given by (6) belongs to $A$ if $p_L \in A$. Moreover, it preserves the kinetic energy in the impact (for every observer for which the constraint is at rest, that are the only ones for which the conservation of kinetic energy has a clear meaning [4]). This property motivates the ideality of the chosen reaction.

3 The algorithm

The algorithm to determine the right velocity $p_R$ starting from a left velocity $p_L$ can be easily described by the following procedure:

1) we start from an a priori knowledge of the space of admissible velocities $A$ (coinciding with the whole $J_1(V)$ when no rolling constraints act on the balls), its subspace $A \cap T$ (determined by the impulse kinetic constraint [3] and the scalar product [2])

2) we have as input an assigned left–velocity $p_L \in A$;

3) we determine the tangent component $P(p_L) \in A \cap T$ and the orthogonal component $I^\perp(p_L)$;

4) we determine the right–velocity $p_R \in A$ using the relation $p_R = p_L - 2I^\perp(p_L)$.

Clearly, from the operative point of view, the core of the algorithm is point 3): it consists in the solution of the Least Square Problem determining the projection $P(p_L)$ of the assigned left velocity $p_L \in A$ onto the space $A \cap T$, together with the condition [3] in the smooth case, together with the conditions [3] and [1] for both
\[ i = 1, 2 \] in the rolling case, and together with the conditions (3) and (4) for only one \( i \in \{1, 2\} \) for the mixed case.

Starting from the 10–uple \( \mathbf{p}_L \approx (x_1, \dot{x}_1, \ddot{x}_1, \psi_1, \dot{\psi}_1, \phi_1, \dot{\phi}_1, \dot{x}_2, \ddot{x}_2, \psi_2, \dot{\psi}_2, \phi_2) \), we introduce the function of the 10 unknown \((x_1, \dot{x}_1, \psi_1, \dot{\psi}_1, \phi_1, \dot{\phi}_1, \dot{x}_2, \ddot{x}_2, \psi_2, \dot{\psi}_2, \phi_2)\)

\[
\| \mathbf{p} - \mathbf{p}_L \|^2 = m (\dot{x}_1 - \dot{x}_1) + m (\dot{y}_1 - \dot{y}_1)^2 \\
+ A (\dot{\psi}_1 - \dot{\psi}_1)^2 + A (\dot{\phi}_1 - \dot{\phi}_1)^2 \\
+ 2 A \cos \psi_1 (\dot{\psi}_1 - \dot{\psi}_1) (\dot{\phi}_1 - \dot{\phi}_1) \\
+ m (\dot{x}_2 - \dot{x}_2)^2 + m (\dot{y}_2 - \dot{y}_2)^2 \\
+ A (\dot{\psi}_2 - \dot{\psi}_2)^2 + A (\dot{\phi}_2 - \dot{\phi}_2)^2 \\
+ 2 A \cos \psi_2 (\dot{\psi}_2 - \dot{\psi}_2) (\dot{\phi}_2 - \dot{\phi}_2). 
\]

Using the Lagrange Multipliers Method, in the smooth case we look (neglecting the parameters introduced by the LMM) for a 10–uple minimizing the function

\[
\mathcal{L}_{\text{smooth}} = \| \mathbf{p} - \mathbf{p}_L \|^2 + \lambda (|x_2 - x_1(x_2 - x_1) + (y_2 - y_1(y_2 - y_1)|. 
\]

In the rolling case we look (once again neglecting the parameters introduced by the LMM) for a 10–uple minimizing the function

\[
\mathcal{L}_{\text{roll}} = \| \mathbf{p} - \mathbf{p}_L \|^2 + \lambda (|x_2 - x_1(x_2 - x_1) + (y_2 - y_1(y_2 - y_1)| + \mu_{11} \dot{x}_1 \sin \psi_1 + \mu_{12} \dot{\psi}_1 \cos \psi_1 \\
+ \mu_{21} \dot{x}_2 \sin \psi_2 + \mu_{22} \dot{\psi}_2 \cos \psi_2). 
\]

In the mixed case (supposing the ball nr. 2 rolling without sliding) we look for a 10–uple minimizing the function

\[
\mathcal{L}_{\text{mixed}} = \| \mathbf{p} - \mathbf{p}_L \|^2 + \lambda (|x_2 - x_1(x_2 - x_1) + (y_2 - y_1(y_2 - y_1)| + \mu_{21} \dot{x}_2 \sin \psi_2 + \mu_{22} \dot{\psi}_2 \cos \psi_2). 
\]

In all cases the minimizing 10–uple gives the required projection \( \mathcal{P}(\mathbf{p}_L) \), and then the main object for the analysis of the impact.

4 The possible ideal impacts

In this section we present the results of the computation in the three cases described above. To perform this computation, we used CoCoA 4.7, a freeware software for polynomial symbolic computation (3). For each situation, we present some general (in the sense of “depending by the whole set of parameters”) but significative results, and some results obtained for particular impacts (in the sense of “having chosen some parameters in a simple but significative way”).

4.1 Case I. The smooth situation

It can be easily proved that, when the contact between the balls and the plane of the billiard is smooth, the impact does not affect the behaviour of the “angle” variables \((\dot{\psi}_1, \dot{\psi}_1, \dot{\phi}_1, \dot{\phi}_2, \dot{\phi}_2)\) and the unique variations regard the “linear” variables
The "usual" scalar product. Then the results (11) have complicated expressions, without easy interpretation. For example, we smooth and the rolling cases, so that (11) still hold, together with (12, 13).

In particular, the "linear" velocity variables have the same behaviour in the roll without sliding on the billiard plane, in general all the 10 component representing it can be easily foreseen that, differently from the smooth case, when both the balls at the impact.

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Unfortunately, although explicitly computable, the changes of the "angle" velocity variables have complicated expressions, without easy interpretation. For example, we

\( (\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2) \) of the velocities of the centers of mass of the balls. In particular, we have the following results:

\[
\begin{align*}
\dot{x}_1^R &= \frac{\dot{x}_1^f(y_2 - y_1)^2 + \dot{x}_1^f(x_2 - x_1)^2 + (\dot{y}_2^f - \dot{y}_1^f)(x_2 - x_1)(y_2 - y_1)}{4r^2} \\
\dot{y}_1^R &= \frac{\dot{y}_1^f(x_2 - x_1)^2 + \dot{y}_1^f(y_2 - y_1)^2 + (\dot{x}_2^f - \dot{x}_1^f)(x_2 - x_1)(y_2 - y_1)}{4r^2} \\
\dot{x}_2^R &= \frac{\dot{x}_2^f(x_2 - x_1)^2 + \dot{x}_2^f(y_2 - y_1)^2 - (\dot{y}_2^f - \dot{y}_1^f)(x_2 - x_1)(y_2 - y_1)}{4r^2} \\
\dot{y}_2^R &= \frac{\dot{y}_2^f(y_2 - y_1)^2 + \dot{y}_2^f(x_2 - x_1)^2 - (\dot{x}_2^f - \dot{x}_1^f)(x_2 - x_1)(y_2 - y_1)}{4r^2}.
\end{align*}
\]  

Note that, while the exit velocity depends on the relative position of the centers of mass of the balls at the impact, it does not depend on the orientations of the balls at the impact.

Let us denote with \( \mathbf{v}_i^L, \mathbf{v}_i^R, i = 1, 2 \) the "usual" left and right velocities of the centers of mass of the balls (so that, for example, the component of \( \mathbf{v}_1^L \) along the \( x \) direction is \( \dot{x}_1^f \)) and with \( \mathbf{u} \cdot \mathbf{w} \) the "usual" scalar product. Then the results (11) satisfy the relation

\[
\mathbf{v}_1^L \cdot \mathbf{v}_2^L = \mathbf{v}_1^R \cdot \mathbf{v}_2^R.
\]

In particular (12) implies that, when the impact happens with one of the two balls at rest, independently of the impact angle, the exit velocities of the two centers of mass are orthogonal.

The computation shows also that the reactive impulse \(-2I^\perp(p_L)\) has (null components relative to the "angle" variables and) components relative to the "linear" variables

\[
\begin{align*}
I_{x_1}^\perp &= -I_{x_2}^\perp = -\frac{(\dot{x}_2^f - \dot{x}_1^f)(x_2 - x_1)^2 + (\dot{y}_2^f - \dot{y}_1^f)(x_2 - x_1)(y_2 - y_1)}{2r^2} \propto (x_2 - x_1) \\
I_{y_1}^\perp &= -I_{y_2}^\perp = -\frac{(\dot{x}_1^f - \dot{x}_2^f)(x_2 - x_1)(y_2 - y_1) + (\dot{y}_2^f - \dot{y}_1^f)(y_2 - y_1)^2}{2r^2} \propto (y_2 - y_1).
\end{align*}
\]  

Therefore, each reactive impulse relative to a single ball turns out to be parallel to the direction determined by the centers of mass of the balls. This property, together with the smoothness of the contact between balls and plane, motivate also why the impact does not affect the "spin" of the balls (see [5]).

Moreover, simple tests show the reasonableness of (11). When \( \mathbf{v}_1^L = 0 \), and the impact is direct central, then \( \mathbf{v}_2^R = 0, \mathbf{v}_1^R = \mathbf{v}_2^L \). More generally, when \( \mathbf{v}_1^L = 0 \), then \( \mathbf{v}_1^R \) turns out to be parallel to the direction determined by the centers of mass at the impact.

### 4.2 Case II. The pure rolling situation

It can be easily foreseen that, differently from the smooth case, when both the balls roll without sliding on the billiard plane, in general all the 10 component representing the velocities of the balls change with the impact. The computation support this forecast. In particular, the "linear" velocity variables have the same behaviour in the smooth and the rolling cases, so that (11) still hold, together with (12, 13).

Unfortunately, although explicitly computable, the changes of the "angle" velocity variables have complicated expressions, without easy interpretation. For example, we
have,
\[
\dot{\theta}^R_1 = \frac{1}{4r^3} \left\{ \left( \dot{x}^L_2 \sin \psi_1 - \dot{y}^L_1 \cos \psi_1 \right) (x_2 - x_1)^2 \\
+ \left[ r \dot{\dot{\theta}}_2^L - (\dot{y}^L_2 - \dot{y}^L_1) \right] \cos \psi_1 (y_2 - y_1)^2 \\
+ \left[ (\dot{y}^L_2 - \dot{y}^L_1) \sin \psi_1 - (\dot{x}^L_2 - \dot{x}^L_1) \cos \psi_1 \right] (x_2 - x_1)(y_2 - y_1) \right\}.
\]

It is then convenient to restrict our analysis to particular situations, such as when the left velocity of one ball is fixed, and possibly null, and fixing the angle between \(\vec{V}^L_2\) and the direction determined by the centers of mass.

For example, for an impact with the ball nr. 1 at rest, \(\vec{V}^L_2\) parallel to the \(x\)-direction and angle between \(\vec{V}^L_2\) and the direction determined by the centers of mass of \(\vec{\Omega}_1\), we have

\[
\begin{align*}
\dot{x}^R_1 &= \frac{1}{2} \dot{x}^L_2; \\
\dot{y}^R_1 &= \frac{1}{2} \dot{y}^L_2; \\
\dot{\psi}^R_1 &= \frac{\left( \sin \psi_1 + \cos \psi_1 \right) \cos \vartheta_1}{2 \sin \vartheta_1} \frac{\dot{x}^L_2}{r}; \\
\dot{\vartheta}^R_1 &= \frac{\left( \sin \psi_1 - \cos \psi_1 \right)}{2} \frac{\dot{x}^L_2}{r}; \\
\dot{\phi}^R_1 &= -\frac{\left( \sin \psi_1 + \cos \psi_1 \right)}{2 \sin \vartheta_1} \frac{\dot{x}^L_2}{r}; \\
\dot{x}^R_2 &= \frac{1}{2} \dot{x}^L_2; \\
\dot{y}^R_2 &= -\frac{1}{2} \dot{y}^L_2; \\
\dot{\psi}^R_2 &= \dot{\psi}^L_2 - \frac{\cos \vartheta_2}{\sin \vartheta_2} \frac{r \dot{\dot{\vartheta}}_2^L + \dot{x}^L_2 \cos \psi_2}{2r}; \\
\dot{\vartheta}^R_2 &= \frac{r \dot{\vartheta}_2^L + \dot{x}^L_2 \cos \psi_2}{2r}; \\
\dot{\phi}^R_2 &= \frac{1}{\sin \vartheta_2} \frac{r \dot{\vartheta}_2^L - \dot{x}^L_2 \cos \psi_2}{2r}.
\end{align*}
\]

Another simple example is given by the direct central impact with the ball nr. 1 at rest and \(\vec{V}^L_2\) parallel to the \(x\)-direction. We have

\[
\begin{align*}
\dot{x}^R_1 &= \dot{x}^L_2; \\
\dot{y}^R_1 &= 0 \\
\dot{\psi}^R_1 &= \frac{\cos \psi_1 \cos \vartheta_1}{\sin \vartheta_1} \frac{\dot{x}^L_2}{r},
\end{align*}
\]
\[ \dot{\theta}^R_1 = \sin \psi_1 \frac{\dot{x}_L^2}{r}; \]
\[ \dot{\phi}^R_1 = -\frac{\cos \psi_1}{\sin \theta_1} \frac{\dot{x}_L^2}{r}; \]
\[ \dot{x}_2^R = 0; \]
\[ \dot{y}_2^R = 0 \]
\[ \dot{\psi}_2^R = \frac{\cos \psi_2 \cos \vartheta_2}{\sin \theta_2} \frac{\dot{x}_L^2}{r}; \]
\[ \dot{\theta}_2^R = 0; \]
\[ \dot{\phi}_2^R = 0. \]

4.3 Case III. The mixed situation

The mixed situation, when one of the balls rolls without sliding and the other slides on the billiard plane seems a very artful situation. However, taking into account the results presented in [6] about the behaviour of a billiard ball moving with friction on the billiard plane, the case of a rolling ball impacting with another ball at rest so that the second ball starts to move sliding on the billiard plane is the ideal case that better models a “real world” situation. Once again, some simple particular impacts give clearer outlooks than general results. For example, let us consider once again an impact with the ball nr. 1 at rest (on a smooth part of the billiard plane), the ball nr. 2 rolling on the billiard plane with \( \mathbf{V}_2^L \) parallel to the \( x \)-direction and angle between \( \mathbf{V}_2^L \) and the direction determined by the centers of mass of \( \frac{\pi}{4} \). If we set \( A = amr^2 \), we obtain

\[ \dot{x}_1^R = \frac{a + 1}{a + 2} \dot{x}_2^L; \]
\[ \dot{y}_1^R = \frac{a + 1}{a + 2} \dot{x}_2^L; \]
\[ \dot{\psi}_1^R = 0; \]
\[ \dot{\theta}_1^R = 0; \]
\[ \dot{\phi}_1^R = 0 \]
\[ \dot{x}_2^R = \frac{a + 1}{a + 2} \dot{x}_2^L; \]
\[ \dot{y}_2^R = -\frac{1}{a + 2} \dot{x}_2^L \]
\[ \dot{\psi}_2^R = \frac{\cos \vartheta_2}{a + 2 \sin \theta_2} \left( \dot{\theta}_2^L + \frac{\dot{x}_L^2}{r} \cos \psi_2 \right); \]
\[ \dot{\theta}_2^R = \frac{1}{a + 2} \left( (a + 1) \dot{\theta}_2^L + \frac{\dot{x}_L^2}{r} \cos \psi_2 \right); \]
\[ \dot{\varphi}^{R}_2 = \frac{1}{a + 2} \frac{1}{\sin \vartheta_2} \left( \dot{\vartheta}^{L}_2 - (a + 1) \frac{\dot{x}^{L}_2}{r} \cos \psi_2 \right). \]

However, in this case the most interesting result is given by the scalar product \( \mathbf{V}^R_1 \cdot \mathbf{V}^R_2 \) of the right “linear” velocities: a straightforward calculation shows that

\[ \mathbf{V}^R_1 \cdot \mathbf{V}^R_2 = \frac{a(a + 1)}{(a + 2)^2} (\dot{x}^{L}_2)^2 \neq 0. \]

Therefore, in this situation, the exit velocities of the balls are not orthogonal. Since this is, for every billiard player, a very well known “real” fact, the result gives an ulterior proof of the good agreement of the result of the mixed model to the real situation.

5 Conclusions

We used a symbolic computation software to obtain detailed results in the study of the behaviour of two equal billiard balls colliding in three ideal situations. The use of computation software was almost essential in order to obtain effective results by the known theoretical procedure, since the high number of degrees of freedom of the system involves calculations that could hardly be approached by hand techniques.

Although the computation were performed in very general conditions, often the results obtained can be interpreted only for particular impacts, such as when one of the ball is at rest. Therefore, only a few of the numerous interesting impacts where analyzed in details.

The results in the smooth and the pure rolling situations respect the forecasts (easier in the smooth case, more complicated in the other), especially about the angle formed by the directions of the balls after the impact, and about the spins of the balls after the impact. The analysis of the mixed situation, suggested by similar analyses of analogous problems that can be found in literature, although the apparent artfulness of the case, give results that are in good agreement to well known real behaviours of the billiard balls.

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