A class of dynamic frictional contact problems governed by a system of hemivariational inequalities in thermoviscoelasticity *

Stanislaw Migórski † and Paweł Szafraniec
Jagiellonian University
Institute of Computer Science
Faculty of Mathematics and Computer Science
ul. Łojasiewicza 6, 30348 Krakow, Poland

Dedicated to the memory of Professor Zdzisław Naniewicz

Abstract. In this paper we prove the existence and uniqueness of the weak solution for a dynamic thermoviscoelastic problem which describes frictional contact between a body and a foundation. We employ the nonlinear constitutive viscoelastic law with a long-term memory, which include the thermal effects and consider the general non-monotone and multivalued subdifferential boundary conditions for the contact, friction and heat flux. The model consists of the system of the hemivariational inequality of hyperbolic type for the displacement and the parabolic hemivariational inequality for the temperature. The existence of solutions is proved by using recent results from the theory of hemivariational inequalities and a fixed point argument.

Keywords: Dynamic contact; thermoviscoelastic; evolution hemivariational inequality; Clarke subdifferential; nonconvex; hyperbolic; parabolic; viscoelastic material; frictional contact; weak solution.

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† The corresponding author. Email: stanislaw.migorski@ii.uj.edu.pl
1 Introduction

Problems involving thermoviscoelastic contact arise naturally in many situations, particularly those involving industrial processes when two or more deformable bodies may come in contact or may lose contact as a result of thermoviscoelastic expansion or contraction. For this reason there is a considerable literature devoted to this topic. The first existence and uniqueness results for contact problems with friction in elastodynamics were obtained by Duvaut and Lions [11]. Later, Martins and Oden [21] studied the normal compliance model of contact with friction and showed existence and uniqueness results for a viscoelastic material. These results were extended by Figueiredo and Trabucho [12] to thermoelastic and thermoviscoelastic models. In these papers the authors used the classical Galerkin method combined with a regularization technique and compactness arguments. Recently dynamic viscoelastic frictional contact problems with or without thermal effects have been investigated in a large number of papers, see e.g. Adly et al. [1], Amassad et al. [2], Andrews et al. [3, 4], Chau et al. [5], Han and Sofonea [14], Jarusek [16], Kuttler and Shillor [20], Migorski [25], Migorski and Ochal [27], Migorski et al. [28, 29], Rochdi and Shillor [33] and the references therein.

In this paper we consider the frictional contact problem between a nonlinear thermoviscoelastic body and an obstacle. We suppose that the process is dynamic and the material is viscoelastic with long memory and thermal effect. Our main interest lies in general nonmonotone and possibly multivalued subdifferential boundary conditions. More precisely, it is supposed that on the contact part of the boundary of the body under consideration, the subdifferential relations hold, the first one between the normal component of the velocity and the normal component of the stress, the second one between the tangential components of these quantities and the third one between temperature and the heat flux vector. These three subdifferential boundary conditions are the natural generalizations of the normal damped response condition, the associated friction law and the well known Fourier law of heat conduction, respectively. For examples, applications and detailed explanations concerning the boundary conditions we refer to Panagiotopoulos [31, 32], Naniewicz and Panagiotopoulos [30], and Migorski et al. [29].

The thermoviscoelastic phenomena can be divided into three classes: static, quasistatic, and full dynamic. The quasistatic problems can be viewed as being of mixed elliptic–parabolic type, while the dynamic case is of mixed hyperbolic–parabolic type. The latter is more complicated, and we have in the literature only a few results concerning existence and uniqueness. We investigate a fully dynamic contact problem which consists of the energy-elasticity equations of hyperbolic type together with a nonlinear parabolic equation for the temperature. Because of the multivalued multidimensional boundary conditions, the problem is formulated as a system of two coupled evolution hemivariational inequalities. All subdifferentials are understood in this paper in the sense of Clarke and are considered for locally Lipschitz, and in general nonconvex and nonsmooth superpotentials. This allows to incorporate in our model several types of boundary conditions considered earlier e.g. in [30, 31, 32, 29]. We note that when the superpotentials involved in the problem are convex functions, the
hemivariational inequalities reduces to variational inequalities.

The goal of the paper is to provide the result on existence and uniqueness of a global weak solution to the system. The existence of solutions is obtained by combining recent results on the hyperbolic hemivariational inequalities \([23, 24, 29, 17, 18]\) and the results on the parabolic hemivariational inequalities \([22, 26]\), and by applying a fixed point argument. In spite of importance of the subject in applications, to the best of the authors' knowledge, the existence of solutions to the system of hemivariational inequalities in dynamic thermoviscoelasticity has studied in very few papers \([7, 8, 9]\). However, in all aforementioned papers, there is a coupling between the displacement (and velocity) and the temperature in the constitutive law which is assumed to be linear. In this paper we deal with the fully nonlinear constitutive relation and assume the coupling also in the heat flux boundary condition on the contact surface. Finally, we note that for linear thermoelastic materials a system of hemivariational inequalities was formulated by Panagiotopoulos in Chapter 7.3 of \([32]\). However, the regularity hypotheses on the multivalued terms were quite unnatural and the data were assumed to be very regular (cf. Proposition 7.3.2 in \([32]\)).

The content of the paper is as follows. After the preliminary material of Section 2, in Section 3 we present the physical setting and the classical formulation of the problem. In Section 4 we deliver the variational formulation of the mechanical problem and state our main existence and uniqueness result. The proof of the main result is provided in Section 5. Some examples of nonmonotone and multivalued subdifferential boundary conditions are given in Section 6.

## 2 Preliminaries

In this section we introduce notation and recall some definitions and results needed in the sequel, cf. \([14, 10, 29, 7, 31]\).

We denote by \(S^d\) the linear space of second order symmetric tensors on \(\mathbb{R}^d\), \(d = 2, 3\), or equivalently, the space \(\mathbb{R}^{d \times d}\) of symmetric matrices of order \(d\). We recall that the canonical inner products and the corresponding norms on \(\mathbb{R}^d\) and \(S^d\) are given by

\[
    u \cdot v = u_i v_i, \quad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{1/2} \quad \text{for all } u = (u_i), v = (v_i) \in \mathbb{R}^d,
\]

\[
    \sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\|_{S^d} = (\tau : \tau)^{1/2} \quad \text{for all } \sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in S^d,
\]

respectively. Here and below, the indices \(i\) and \(j\) run from 1 to \(d\), and the summation convention over repeated indices is adopted.

Let \(\Omega\) be an open bounded subset of \(\mathbb{R}^d\) with a Lipschitz continuous boundary \(\Gamma\) and let \(\nu\) denote the outward unit normal vector to \(\Gamma\). We introduce the spaces

\[
    H = L^2(\Omega; \mathbb{R}^d), \quad \mathcal{H} = \{ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \}, \quad \mathcal{H}_1 = \{ \tau \in \mathcal{H} \mid \text{Div } \tau \in H \}.
\]
It is well known that the spaces $H$, $\mathcal{H}$ and $\mathcal{H}_1$ are Hilbert spaces equipped with the inner products

$$\langle u, v \rangle_H = \int_{\Omega} u \cdot v \, dx, \quad \langle \sigma, \tau \rangle_{\mathcal{H}} = \int_{\Omega} \sigma : \tau \, dx, \quad \langle \sigma, \tau \rangle_{\mathcal{H}_1} = \langle \sigma, \tau \rangle_H + \langle \text{Div} \sigma, \text{Div} \tau \rangle_H,$$

where $\varepsilon : H^1(\Omega; \mathbb{R}^d) \to \mathcal{H}$ and $\text{Div} : \mathcal{H}_1 \to H$ denote the deformation and the divergence operator, respectively, given by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad \text{Div} \, \sigma = (\sigma_{ij,j}).$$

An index that follows a comma indicates a derivative with respect to the corresponding component of the spatial variable $x \in \Omega$. Given $v \in H^1(\Omega; \mathbb{R}^d)$ we denote by $\gamma_0 v$ its trace on $\Gamma$, where $\gamma_0 : H^1(\Omega; \mathbb{R}^d) \to H^{1/2}(\Gamma; \mathbb{R}^d) \subset L^2(\Gamma; \mathbb{R}^d)$ is the trace map. If $d = 1$, then the trace operator from $H^1(\Omega)$ into $L^2(\Gamma)$ is denoted by $\gamma_0$. For $v \in L^2(\Gamma; \mathbb{R}^d)$ we denote by $v_\nu$ and $v_\tau$ the usual normal and tangential components of $v$ on the boundary $\Gamma$, i.e., $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu$. Similarly, for a regular tensor field $\sigma : \Omega \to \mathbb{S}^d$, we define its normal and tangential components by $\sigma_\nu = (\sigma v) \cdot \nu$ and $\sigma_\tau = \sigma v - \sigma_\nu \nu$, respectively. The following two Green–type formulas can be found in Chapter 2 of [29]:

$$\int_{\Omega} (u \text{ div } v + \nabla u \cdot v) \, dx = \int_{\Gamma} u (v \cdot \nu) \, d\Gamma, \quad (1)$$

for $u \in H^1(\Omega)$ and $v \in H^1(\Omega; \mathbb{R}^d)$, and

$$\int_{\Omega} \sigma : \varepsilon(v) \, dx + \int_{\Omega} \text{Div} \sigma \cdot v \, dx = \int_{\Gamma} \sigma v \cdot v \, d\Gamma \quad (2)$$

for $v \in H^1(\Omega; \mathbb{R}^d)$ and $\sigma \in C^1(\overline{\Omega}, \mathbb{S}^d)$.

We recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $\varphi : X \to \mathbb{R}$, where $X$ is a Banach space (see [6]). The generalized directional derivative of $\varphi$ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \to x, \ t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$  

The generalized gradient of $\varphi$ at $x$, denoted by $\partial \varphi(x)$, is a subset of a dual space $X^*$ given by $\partial \varphi(x) = \{ \zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X \}$.

We denote by $L(X, Y)$ the space of linear continuous mappings from $X$ to $Y$. Given a reflexive Banach space $Y$, we denote by $\langle \cdot, \cdot \rangle_{Y^* \times Y}$ the duality pairing between the dual space $Y^*$ and $Y$. In what follows different positive constants, which may change from line to line, will be denoted by the same letter $c$.

Finally, we recall the following result (cf. Lemma 7 in [18]) which is a consequence of the Banach contraction principle and which will be used in the proof of the main theorem of this paper.

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Let be a Banach space with a norm \( \| \cdot \|_X \) and \( T > 0 \). Let \( \Lambda : L^2(0; T; X) \to L^2(0; T; X) \) be an operator satisfying

\[
\| (\Lambda \eta_1)(t) - (\Lambda \eta_2)(t) \|^2_X \leq c \int_0^t \| \eta_1(s) - \eta_2(s) \|^2_X \, ds
\]

for every \( \eta_1, \eta_2 \in L^2(0; T; X) \), a.e. \( t \in (0; T) \) with a constant \( c > 0 \). Then \( \Lambda \) has a unique fixed point in \( L^2(0; T; X) \), i.e. there exists a unique \( \eta^* \in L^2(0; T; X) \) such that \( \Lambda \eta^* = \eta^* \).

3 Physical setting and classical formulation

In this section we introduce the physical setting of the problem, describe the classical model and list the hypotheses on the data.

Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^d \), \( d = 2, 3 \), with a Lipschitz continuous boundary \( \Gamma = \partial \Omega \). The boundary \( \Gamma \) is composed of three sets \( \Gamma_D, \Gamma_N, \Gamma_C \), with mutually disjoint relatively open sets \( \Gamma_D, \Gamma_N \) and \( \Gamma_C \), such that \( \text{meas} (\Gamma_D) > 0 \). We consider a viscoelastic body, which in the reference configuration, occupies volume \( \Omega \) and which is supposed to be stress free and at a constant temperature, conveniently set as zero. We assume that the temperature changes accompanying the deformations are small and they do not produce any changes in the material parameters which are regarded temperature independent. We are interested in a mathematical model that describes the evolution of the mechanical state of the body and its temperature during the time interval \( [0; T] \) where \( 0 < T < \infty \). To this end, we denote by \( \sigma = \sigma(x, t) = (\sigma_{ij}(x, t)) \) the stress field, by \( u = u(x, t) = (u_i(x, t)) \) the displacement field, and by \( \theta = \theta(x, t) \) the temperature, where \( x \in \Omega \) and \( t \in [0; T] \) denote the spatial and the time variables, respectively. The functions \( u : \Omega \times [0; T] \to \mathbb{R}^d \), \( \sigma : \Omega \times [0; T] \to \mathbb{S}^d \) and \( \theta : \Omega \times [0; T] \to \mathbb{R} \) will play the role of the unknowns of the frictional contact problem. From time to time, we suppress the explicit dependence of the quantities on the spatial variable \( x \), or both \( x \) and \( t \).

We suppose that the body is clamped on \( \Gamma_D \), the volume forces of density \( f_0 = f_0(x, t) \) act in \( \Omega \) and the surface tractions of density \( f_1 = f_1(x, t) \) are applied on \( \Gamma_N \). Moreover, the body is subjected to a heat source term per unit volume \( g = g(x, t) \) and it comes in contact with an obstacle, the so-called foundation, over the contact surface \( \Gamma_C \). We also use the notation \( Q = \Omega \times (0; T) \), \( \Sigma_D = \Gamma_D \times (0; T) \), \( \Sigma_N = \Gamma_N \times (0; T) \) and \( \Sigma_C = \Gamma_C \times (0; T) \). Without loss of generality we can assume that the material density and the specific heat at constant deformation are constants, both set equal to one. Assuming small displacements, the system of the equation of motion and the law of conservation of energy take the form

\[
\begin{align*}
    u''(t) - \text{Div} \sigma(t) &= f_0(t) \quad \text{in} \quad Q \\
    \theta'(t) + \text{div} q(t) &= R(t, u'(t)) + g(t) \quad \text{in} \quad Q.
\end{align*}
\]

For the thermal diffusion, we adopt the following law with the heat flux vector \( q \) of the form

\[
q(t) = -K(x, t, \nabla \theta(t)) \quad \text{in} \quad Q.
\]
In the case $K(x, t, \cdot)$ is a linear function, this law reduces to the Fourier law of heat conduction of the form $q(t) = -k(x, t)\nabla\theta(t)$ in $Q$ where $k = k(x, t)$ represents the thermal conductivity tensor. In the heat equation, we suppose that $R$ is a nonlinear function of the velocity. A model with a linear function $R$ of the form $R(x, t, v) = -\sum_{i,j=1}^{d} c_{ij}(x, t) \frac{\partial v}{\partial x_i}$ for $v \in H^1(\Omega; \mathbb{R}^d)$, a.e. $(x, t) \in Q$, where $c_{ij} \in L^\infty(Q)$ are the components of the tensor of thermal expansion was considered in [1, 5]. The behavior of the material is described by the nonlinear thermoviscoelastic constitutive law of Kelvin-Voigt type with a long-term memory of the form

$$\sigma(t) = A(t, \varepsilon(u'(t))) + B(t, \varepsilon(u(t))) + \int_0^t C(t - s)\varepsilon(u(s)) ds + C_e(t, \theta(t)) \quad \text{in} \quad Q.$$ 

We allow the viscosity operator $A$, the elasticity operator $B$, the relaxation operator $C$ and the thermal expansion operator $C_e$ to depend on the time. This law generalizes the following classical equation of the linear thermoviscoelasticity theory of the form

$$\sigma_{ij} = a_{ijkl} \varepsilon_{kl}(u') + b_{ijkl} \varepsilon_{kl}(u) - c_{ij} \theta \quad \text{in} \quad Q,$$

where $a = (a_{ijkl})$ and $b = (b_{ijkl})$, $i, j, k, l = 1, \ldots, d$ are the viscosity and elasticity fourth order tensors, respectively, and $(c_{ij})$ are the so-called coefficients of thermal expansion.

Our main interest lies in the contact and friction boundary conditions on the surface $\Gamma_C$. As concerns the contact condition we assume that the normal stress $\sigma_{\nu}$ and the normal velocity $u'_\nu$ satisfy the nonmonotone normal damped response condition of the form

$$-\sigma_{\nu} \in \partial j_{\nu}(x, t, u'_\nu) \quad \text{on} \quad \Sigma_C.$$

The friction relation is given by

$$-\sigma_{\tau} \in \partial j_{\tau}(x, t, u'_\tau) \quad \text{on} \quad \Sigma_C$$

and describes the multivalued law between the tangential force $\sigma_{\tau}$ on $\Gamma_C$ and the tangential velocity $u'_\tau$. Moreover, we suppose that there is heat exchange between the surface $\Gamma_C$ and the foundation and that the dependence between the heat flux vector and the boundary temperature is described by the possibly multivalued relation of the subdifferential type with a nonconvex potential $j$. Since the power that is generated by the frictional contact forces is proportional to the tangential velocity, we introduce the function $h_{\tau}$ in the following relation $q(t) \cdot \nu = h_{\tau}(t, \|u'_\tau(t)\|_{\mathbb{R}^d}) - \partial j(x, t, \theta(t))$ on $\Sigma_C$. We rewrite it in the following form

$$- \frac{\partial \theta}{\partial u'} \in \partial j(x, t, \theta(t)) - h_{\tau}(x, t, \|u'_\tau(x, t)\|_{\mathbb{R}^d}) \quad \text{on} \quad \Sigma_C,$$

(3)

where $\frac{\partial \theta}{\partial u'} = K(x, t, \nabla\theta(t)) \cdot \nu$. In a simple case, when $h_{\tau} \equiv 0$ (there is no coupling between the temperature and the tangential velocity on $\Sigma_C$) and $j(x, t, r) = \frac{1}{2} k_e (r - \theta_R)^2$ for $r \in \mathbb{R}$, a.e. $(x, t) \in \Sigma_C$, $k_e$ being the heat exchange coefficient between
the body and the foundation and $\theta_R$ being the temperature of the foundation, the condition (3) reduces to the equation

$$-\frac{\partial \theta}{\partial v_K} = k_c (\theta - \theta_R) \quad \text{on} \quad \Sigma_C$$

which was studied in [1, 5]. As a simple tangential function $h_\tau$ in (3), we may take

$$h_\tau(x, t, r) = \lambda(x, t) r \quad \text{for all} \quad r \in \mathbb{R}_+, \quad \text{a.e.} \quad (x, t) \in \Sigma_C,$$

where $\lambda \in L^\infty(\Sigma_C)$ represents a time-dependent rate coefficient for the gradient of the temperature. Here $j_\nu: \Sigma_C \times \mathbb{R} \rightarrow \mathbb{R}$, $j_\tau: \Sigma_C \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $j: \Sigma_C \times \mathbb{R} \rightarrow \mathbb{R}$ are locally Lipschitz functions in their last variables and $\partial j_\nu$, $\partial j_\tau$, $\partial j$ represent their Clarke subdifferentials. Many various possibilities of nonconvex potentials $j_\nu$, $j_\tau$, $j$ can be considered to model boundary conditions, see e.g. [29] for examples and applications.

For the sake of simplicity, we assume that the temperature vanishes on $\Gamma_D \cup \Gamma_N$, i.e. $\theta = 0$ on $(\Gamma_D \cup \Gamma_N) \times (0, T)$. Finally, we denote by $u_0$, $v_0$ and $\theta_0$ the initial displacement, the initial velocity and the initial temperature, respectively. Under these assumptions, the classical formulation of the mechanical problem of frictional contact for the thermoviscoelastic body is the following.

**Problem $P$:** find a displacement field $u: Q \rightarrow \mathbb{R}^d$ and a temperature $\theta: Q \rightarrow \mathbb{R}$ such that

$$u''(t) - \text{Div} \, \sigma(t) = f_0(t) \quad \text{in} \quad Q$$

$$\sigma(t) = A(t, \epsilon(u'(t))) + B(t, \epsilon(u(t))) + \int_0^t C(t - s) \epsilon(u(s)) \, ds + C_e(t, \theta(t)) \quad \text{in} \quad Q$$

$$\theta'(t) - \text{div} \, K(x, t, \nabla \theta(t)) = R(t, u'(t)) + g(t) \quad \text{in} \quad Q$$

$$u(t) = 0 \quad \text{on} \quad \Sigma_D$$

$$\sigma(t) \nu = f_1(t) \quad \text{on} \quad \Sigma_N$$

$$-\sigma_\nu \in \partial j_\nu(x, t, u'(t)),$$  

$$-\sigma_\tau \in \partial j_\tau(x, t, u'(t)) \quad \text{on} \quad \Sigma_C$$

$$-\frac{\partial \theta}{\partial v_K} \in \partial j(x, t, \theta(t)) - h_\tau(x, t, \|u'(x, t)\|_{\mathbb{R}^d}) \quad \text{on} \quad \Sigma_C$$

$$\theta(t) = 0 \quad \text{on} \quad (\Gamma_D \cup \Gamma_N) \times (0, T)$$

$$u(0) = u_0, \quad u'(0) = v_0, \quad \theta(0) = \theta_0 \quad \text{in} \quad \Omega.$$

In order to provide the variational formulation of Problem $P$, we need some additional notation. We introduce the following spaces

$$E = \{ v \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \quad \text{on} \quad \Gamma_D \} \quad \text{and} \quad V = \{ \eta \in H^1(\Omega) \mid \eta = 0 \quad \text{on} \quad \Gamma_D \cup \Gamma_N \}.$$  

On $E$ we consider the inner product and the corresponding norm given by

$$(u, v)_E = (\epsilon(u), \epsilon(v))_{L^2(\Omega; \mathbb{R}^d)}, \quad \|v\|_E = \|\epsilon(v)\|_{L^2(\Omega; \mathbb{R}^d)} \quad \text{for} \quad u, v \in E.$$
From the Korn inequality $\|v\|_{H^1(Ω;\mathbb{R}^d)} \leq c\|\varepsilon(v)\|_{L^2(Ω;\mathbb{S}^d)}$ for $v \in E$ with $c > 0$, it follows that $\|\cdot\|_{H^1(Ω;\mathbb{R}^d)}$ and $\|\cdot\|_E$ are the equivalent norms on $E$. Let $H = L^2(Ω;\mathbb{R}^d)$ and $Z = H^δ(Ω;\mathbb{R}^d)$ with a fixed $δ \in (1/2, 1)$. Denoting by $i: E \to Z$ the embedding injection and by $γ: Z \to L^2(Γ;\mathbb{R}^d)$ the trace operator, for all $v \in E$, we have $γ_v = γ(iv)$. For simplicity we omit the notation of the embedding and write $γ_0v = γv$ for $v \in E$. Identifying $H$ with its dual, we have the following evolution fivefold of spaces with dense, continuous and compact embeddings

$$E \subset Z \subset H \subset Z^* \subset E^*.$$ 

We also introduce the following spaces of vector valued functions $E = L^2(0,T;E)$, $Z = L^2(0,T;Z)$, $\hat{H} = L^2(0,T;\hat{H})$ and $E = \{v \in E \mid v' \in E^*\}$, where the time derivative is understood in the sense of vector valued distributions. Endowed with the norm $\|v\|_E = \|v\|_E + \|v'\|_{E^*}$, the space $E$ becomes a separable reflexive Banach space. We have

$$E \subset E \subset Z \subset \hat{H} \subset Z^* \subset E^*,$$

with dense and continuous embeddings. The duality for the pair $(E,E^*)$ is denoted by $\langle w, z \rangle_{E^*×E} = \int_0^T \langle w(s), z(s) \rangle_{E^*×E} ds$. It is well known (see e.g. [10, 34]) that the embeddings $E \subset C(0,T;H)$ and $\{v \in E \mid v' \in E\} \subset C(0,T;E)$ are continuous and $E \subset Z$ is compact.

Similarly, we introduce the space $Y = H^4(Ω)$ with the same $δ \in (1/2, 1)$ and we obtain the evolution fivefold of spaces

$$V \subset Y \subset L^2(Ω) \subset Y^* \subset V^*$$

with dense, continuous and compact embeddings. Let $V = L^2(0,T;V)$, $\mathcal{Y} = L^2(0,T;Y)$ and $W = \{η \in V | η' \in V^*\}$. We have

$$W \subset V \subset Y \subset L^2(0,T;L^2(Ω)) \subset Y^* \subset V^*,$$

where all the embeddings are dense and continuous. We also know that the embeddings $W \subset C(0,T;L^2(Ω))$ and $\{η \in V | η' \in W\} \subset C(0,T;V)$ are continuous and $W \subset Y$ is compact. Furthermore, we denote by $γ_s: Y \to L^2(Γ)$ the trace operator for scalar valued functions and we write $γ_0v = γ_sv$ for $v \in V$.

The following assumptions on the data of Problem $P$ will be needed throughout the paper. We assume that the viscosity operator $A$, the elasticity operator $B$, the relaxation operator $C$ and the thermal expansion operator $C_e$ satisfy the following hypotheses.

**$H(A)$:** $A: Q × \mathbb{S}^d \to \mathbb{S}^d$ is such that

(a) $A(·, ·, ε)$ is measurable on $Q$ for all $ε \in \mathbb{S}^d$.

(b) $A(x, t, ·)$ is continuous on $\mathbb{S}^d$ for a.e. $(x, t) \in Q$.

(c) $\|A(x, t, ε)\|_{\mathbb{S}^d} \leq a_0(x, t) + a_1\|ε\|_{\mathbb{S}^d}$ for all $ε \in \mathbb{S}^d$, a.e. $(x, t) \in Q$ with $a_0 \in L^2(Q)$, $a_0 ≥ 0$, $a_1 > 0$. 

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(d) $(\mathcal{A}(x, t, \varepsilon_1) - \mathcal{A}(x, t, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq m_\mathcal{A} \|\varepsilon_1 - \varepsilon_2\|^2_{S^d}$ for all $\varepsilon_1, \varepsilon_2 \in S^d$, a.e. $(x, t) \in Q$ with $m_\mathcal{A} > 0$.

(e) $\mathcal{A}(x, t, \varepsilon) : \varepsilon \geq \alpha_\mathcal{A} \|\varepsilon\|^2_{S^d}$ for all $\varepsilon \in S^d$, a.e. $(x, t) \in Q$ with $\alpha_\mathcal{A} > 0$.

**H(\mathcal{B})**: $\mathcal{B} : Q \times S^d \rightarrow S^d$ is such that

(a) $\mathcal{B}(\cdot, \cdot, \varepsilon)$ is measurable on $Q$ for all $\varepsilon \in S^d$.

(b) $\|\mathcal{B}(x, t, \varepsilon)\|_{S^d} \leq b_0(x, t) + b_1 \|\varepsilon\|_{S^d}$ for all $\varepsilon \in S^d$, a.e. $(x, t) \in Q$ with $b_0 \in L^2(Q)$, $b_0, b_1 \geq 0$.

(c) $\|\mathcal{B}(x, t, \varepsilon_1) - \mathcal{B}(x, t, \varepsilon_2)\|_{S^d} \leq L_B \|\varepsilon_1 - \varepsilon_2\|_{S^d}$ for all $\varepsilon_1, \varepsilon_2 \in S^d$, a.e. $(x, t) \in Q$ with $L_B > 0$.

**H(\mathcal{C})**: $\mathcal{C} : Q \times S^d \rightarrow S^d$ is such that

(a) $\mathcal{C}(x, t, \varepsilon) = c(x, t) \varepsilon$ for all $\varepsilon \in S^d$, a.e. $(x, t) \in Q$.

(b) $c(x, t) = (c_{ijkl}(x, t))$ with $c_{ijkl} = c_{ikjl} \in L^2(0, T; L^\infty(\Omega))$.

**H(\mathcal{C}_e)**: $\mathcal{C}_e : Q \times \mathbb{R} \rightarrow S^d$ is such that

(a) $\mathcal{C}_e(\cdot, \cdot, r)$ is measurable on $Q$ for all $r \in \mathbb{R}$.

(b) $\|\mathcal{C}_e(x, t, r)\|_{S^d} \leq c_{e0}(x, t) + c_{e1} |r|$ for all $r \in \mathbb{R}$, a.e. $(x, t) \in Q$ with $c_{e0} \in L^2(Q)$, $c_{e0}, c_{e1} \geq 0$.

(c) $\|\mathcal{C}_e(x, t, r_1) - \mathcal{C}_e(x, t, r_2)\|_{S^d} \leq L_e |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $(x, t) \in Q$ with $L_e > 0$.

The contact and frictional potentials $j_\nu$ and $j_\tau$ and the potential $j$ satisfy the following hypotheses.

**H(j_\nu)**: $j_\nu : \Sigma_C \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

(a) $j_\nu(\cdot, \cdot, r)$ is measurable on $\Sigma_C$ for all $r \in \mathbb{R}$ and there exists $e_0 \in L^2(\Gamma_C)$ such that $j_\nu(\cdot, e_0(\cdot)) \in L^1(\Sigma_C)$.

(b) $j_\nu(x, t, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $(x, t) \in \Sigma_C$.

(c) $|\partial j_\nu(x, t, r)| \leq c_{0\nu}(x, t) + c_{1\nu} |r|$ for all $r \in \mathbb{R}$, a.e. $(x, t) \in \Sigma_C$ with $c_{0\nu} \in L^\infty(\Sigma_C)$, $c_{0\nu}, c_{1\nu} \geq 0$.

(d) $(\zeta_1 - \zeta_2)(r_1 - r_2) \geq -m_\nu |r_1 - r_2|^2$ for all $\zeta_i \in \partial j_\nu(x, t, r_i)$, $r_i \in \mathbb{R}$, $i = 1, 2$, a.e. $(x, t) \in \Sigma_C$ with $m_\nu \geq 0$.

**H(j_\tau)**: $j_\tau : \Sigma_C \times \mathbb{R}^d \rightarrow \mathbb{R}$ is such that

(a) $j_\tau(\cdot, \cdot, \xi)$ is measurable on $\Sigma_C$ for all $\xi \in \mathbb{R}^d$ and there exists $e_1 \in L^2(\Gamma_C; \mathbb{R}^d)$ such that $j_\tau(\cdot, e_1(\cdot)) \in L^1(\Sigma_C)$.

(b) $j_\tau(x, t, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $(x, t) \in \Sigma_C$.

(c) $|\partial j_\tau(x, t, \xi)|_{\mathbb{R}^d} \leq c_{0\tau}(x, t) + c_{1\tau} |\xi|$ for all $\xi \in \mathbb{R}^d$, a.e. $(x, t) \in \Sigma_C$ with $c_{0\tau} \in L^\infty(\Sigma_C)$, $c_{0\tau}, c_{1\tau} \geq 0$.

(d) $(\zeta_1 - \zeta_2) \cdot (\xi_1 - \xi_2) \geq -m_\tau |\xi_1 - \xi_2|^2\|_\mathbb{R}^d$ for all $\zeta_i \in \partial j_\tau(x, t, \xi_i)$, $\xi_i \in \mathbb{R}^d$, $i = 1, 2$, a.e. $(x, t) \in \Sigma_C$ with $m_\tau \geq 0$.

**H(j)**: $j : \Sigma_C \times \mathbb{R} \rightarrow \mathbb{R}$ is such that
(a) $j(\cdot,\cdot, r)$ is measurable on $\Sigma_C$ for all $r \in \mathbb{R}$ and there exists $c_2 \in L^2(\Gamma_C)$ such that $j(\cdot, \cdot, c_2(\cdot)) \in L^1(\Sigma_C)$.

(b) $j(\mathbf{x}, t, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $(\mathbf{x}, t) \in \Sigma_C$.

(c) $|\partial_j(\mathbf{x}, t, r)| \leq c_0(x, t) + c_1|r|$ for all $r \in \mathbb{R}$, a.e. $(\mathbf{x}, t) \in \Sigma_C$ with $c_0 \in L^\infty(\Sigma_C)$, $c_0, c_1 \geq 0$.

(d) $(\zeta_1 - \zeta_2)(r_1 - r_2) \geq -m_0|r_1 - r_2|^2$ for all $\zeta_i \in \partial_j(\mathbf{x}, t, r_i)$, $r_i \in \mathbb{R}$, $i = 1, 2$, a.e. $(\mathbf{x}, t) \in \Sigma_C$ with $m_0 \geq 0$.

The thermal conductivity operator $K$, the operator $R$ in the heat equation, and the tangential function $h_\tau$ satisfy the following assumptions.

**$H(K):$**

(a) $K(\cdot, \cdot, \xi)$ is measurable on $Q$ for all $\xi \in \mathbb{R}^d$.

(b) $K(x, t, \cdot)$ is continuous on $\mathbb{R}^d$ for a.e. $(x, t) \in Q$.

(c) $\|K(x, t, \xi)\|_{\mathbb{R}^d} \leq k_0(x, t) + k_1 \|\xi\|_{\mathbb{R}^d}$ for all $\xi \in \mathbb{R}^d$, a.e. $(x, t) \in Q$ with $k_0 \in L^2(Q)$, $k_0 \geq 0$.

(d) $(K(x, t, \xi_1) - K(x, t, \xi_2)) \cdot (\xi_1 - \xi_2) \geq m_K \|\xi_1 - \xi_2\|_{\mathbb{R}^d}^2$ for all $\xi_1, \xi_2 \in \mathbb{R}^d$, a.e. $(x, t) \in Q$ with $m_K > 0$.

(e) $K(x, t, \xi) \cdot \xi \geq \alpha_K \|\xi\|_{\mathbb{R}^d}^2$ for all $\xi \in \mathbb{R}^d$, a.e. $(x, t) \in Q$ with $\alpha_K > 0$.

**$H(R):$**

(a) $R(\cdot, \cdot, v) \in L^2(Q)$ for all $v \in E$.

(b) $\|R(x, t, v_1) - R(x, t, v_2)\|_{L^2(\Omega)} \leq L_R \|v_1 - v_2\|_E$ for all $v_1, v_2 \in E$, a.e. $(x, t) \in Q$ with $L_R > 0$.

**$H(h_\tau):$**

(a) $h_\tau(\cdot, r) \in L^2(\Gamma_C)$ for all $r \in \mathbb{R}_+$.

(b) $|h_\tau(x, r_1) - h_\tau(x, r_2)| \leq L_\tau |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}_+$, a.e. $x \in \Gamma_C$ with $L_\tau > 0$.

We assume that the body forces, surface tractions, the density of heat sources and the initial conditions have the following regularity.

**$H(f):$**

$f_0 \in L^2(0, T; E^*)$, $f_1 \in L^2(0, T; L^2(\Gamma_N; \mathbb{R}^d))$, $g \in L^2(0, T; V^*)$, $u_0 \in E$, $v_0 \in H$ and $\theta_0 \in L^2(\Omega)$.

### 4 Variational formulation of the problem

In this section, we obtain the variational formulation of Problem $P$, establish the properties of the operators involved in the problem and formulate the main result on the unique solvability of Problem $P$.

First, we define the function $f: (0, T) \rightarrow E^*$ by

$$
(f(t), v)_{E^*} = \langle f_0(t), v \rangle_{E^*} + \langle f_1(t), v \rangle_{L^2(\Gamma_N; \mathbb{R}^d)} \quad \text{for} \quad v \in E \quad \text{and a.e.} \ t \in (0, T). \quad (13)
$$
We now combine (13)–(16) to see that which implies

Next, we use (17) and the constitutive law (5) to obtain the following inequality

On the other hand, from the definition of the Clarke subdifferential combined with (9), we have

for all $v \in E$ and a.e. $t \in (0, T)$. We multiply the equation of motion (4) by $P t r i p l e$ of sufficiently smooth functions which solve Problem $P$, $v \in E$ and $t \in (0, T)$. We take into account the boundary conditions (8) and the fact that $v = 0$ on $\Gamma_D$ to obtain

On the other hand, from the definition of the Clarke subdifferential combined with (9), we have

which implies

We now combine (13)–(16) to see that

for all $v \in E$ and a.e. $t \in (0, T)$, where the operators $A, B, C: (0, T) \times E \to E^*$ and $C_1: (0, T) \times L^2(\Omega) \to E^*$ are defined by

for all $u, v \in E$. Assume that $(u, \sigma, \theta)$ is a triple of sufficiently smooth functions which solve Problem $P$, $v \in E$ and $t \in (0, T)$. We multiply the equation of motion (4) by $v$ and use the Green formula (2) to find that

We take into account the boundary conditions (8) and the fact that $v = 0$ on $\Gamma_D$ to obtain

On the other hand, from the definition of the Clarke subdifferential combined with (9), we have

for all $v \in E$ and a.e. $t \in (0, T)$. We multiply the equation of motion (4) by $P t r i p l e$ of sufficiently smooth functions which solve Problem $P$, $v \in E$ and $t \in (0, T)$. We take into account the boundary conditions (8) and the fact that $v = 0$ on $\Gamma_D$ to obtain

On the other hand, from the definition of the Clarke subdifferential combined with (9), we have

which implies

We now combine (13)–(16) to see that

for all $v \in E$ and a.e. $t \in (0, T)$, where the operators $A, B, C: (0, T) \times E \to E^*$ and $C_1: (0, T) \times L^2(\Omega) \to E^*$ are defined by

for all $u, v \in E$. Assume that $(u, \sigma, \theta)$ is a triple of sufficiently smooth functions which solve Problem $P$, $v \in E$ and $t \in (0, T).
for a.e. \( t \in (0,T) \). Next, let \( \zeta \in V \) and \( t \in (0,T) \). Multiplying the equation (10) by \( \zeta \), using (11) and the Green formula (1), we have

\[
\langle \theta'(t), \zeta \rangle_{V^* \times V} + \int_{\Omega} K(x,t,\nabla \theta(t)) \cdot \nabla \zeta \, dx - \int_{\Gamma_C} \frac{\partial \theta}{\partial \nu_K} \zeta \, d\Gamma = \langle R(t, u'(t)) + g(t), \zeta \rangle_{V^* \times V}.
\]

From the definition of the Clarke subdifferential and the condition (10), it follows that

\[
- \int_{\Gamma_C} \frac{\partial \theta}{\partial \nu_K} \zeta \, d\Gamma \leq \int_{\Gamma_C} j^0(t, \theta(t); \zeta) \, d\Gamma - \int_{\Gamma_C} h_r(||u'(t)||_{\mathbb{R}^d}) \zeta \, d\Gamma.
\]

By (23) and (24), we deduce the following inequality

\[
\langle \theta'(t), C_2(t, \theta(t)), \zeta \rangle_{V^* \times V} + \int_{\Gamma_C} j^0(t, \theta(t); \zeta) \, d\Gamma \geq \langle C_3(t, u'(t)) + g(t), \zeta \rangle_{V^* \times V} \tag{25}
\]

for all \( \zeta \in V \) and a.e. \( t \in (0,T) \), where the operators \( C_2 : (0,T) \times V \rightarrow V^* \) and \( C_3 : (0,T) \times E \rightarrow V^* \) are given by

\[
\langle C_2(t, \theta), \zeta \rangle_{V^* \times V} = \langle K(x, t, \nabla \theta), \nabla \zeta \rangle_{L^2(\Omega)} \quad \text{for all} \quad \theta, \zeta \in V, \tag{26}
\]

\[
\langle C_3(t, v), \zeta \rangle_{V^* \times V} = \langle R(t, v), \zeta \rangle_{V^* \times V} + \int_{\Gamma_C} h_r(||v||_{\mathbb{R}^d}) \zeta \, d\Gamma \tag{27}
\]

for all \( v \in E \), \( \zeta \in V \) and a.e. \( t \in (0,T) \). Finally, we use (18), (25) and the initial conditions (12) to obtain the following system of hemivariational inequalities which is the variational formulation of Problem \( P \).

**Problem \( P_V \):** find \( u \in \mathcal{E} \) with \( u' \in \mathbb{E} \) and \( \theta \in \mathcal{W} \) such that

\[
\langle u''(t) + A(t, u'(t)) + B(t, u(t)) + \int_0^t C(t - s)u(s) \, ds + C_1(t, \theta(t)), v \rangle_{E^* \times E} + \\
\quad + \int_{\Gamma_C} \left( j^0_\nu(x, t, u'(t); v_\nu) + j^0_\tau(t, u'(t); v_\tau) \right) \, d\Gamma \geq \langle f(t), v \rangle_{E^* \times E}
\]

for all \( v \in E \) and a.e. \( t \in (0,T) \)

\[
\langle \theta'(t) + C_2(t, \theta(t)), \zeta \rangle_{V^* \times V} + \int_{\Gamma_C} j^0(t, \theta(t); \zeta) \, d\Gamma \geq \langle C_3(t, u'(t)) + g(t), \zeta \rangle_{V^* \times V}
\]

for all \( \zeta \in V \) and a.e. \( t \in (0,T) \)

\[
u(0) = u_0, \quad u'(0) = v_0, \quad \theta(0) = \theta_0.
\]

In what follows we establish the properties of the operators involved in Problem \( P_V \). For the proofs of Lemmata 2, 3 and 4 we refer to Lemmata 8, 9 and 10, respectively, in [17].
Lemma 2 Under the hypothesis \( H(A) \), the operator \( A: (0,T) \times E \to E^* \) defined by (17) satisfies the properties

(a) \( A(\cdot, v) \) is measurable on \((0,T)\) for all \( v \in E \).
(b) \( A(t, \cdot) \) is strongly monotone for a.e. \( t \in (0,T) \), i.e. \( \langle A(t,v) - A(t,u), v - u \rangle_{E^* \times E} \geq m_A \|v - u\|_E^2 \) for all \( u, v \in E \), a.e. \( t \in (0,T) \).
(c) \( \|A(t,v)\|_{E^*} \leq \tilde{a}_0(t) + \tilde{a}_1 \|v\|_E \) for all \( v \in V \), a.e. \( t \in (0,T) \) with \( \tilde{a}_0 \in L^2(0,T) \), \( \tilde{a}_0 \geq 0 \) and \( \tilde{a}_1 > 0 \).
(d) \( \langle A(t,v), v \rangle_{E^* \times E} \geq \alpha_A \|v\|_E^2 \) for all \( v \in E \), a.e. \( t \in (0,T) \).
(e) \( A(t, \cdot) \) is pseudomonotone for a.e. \( t \in (0,T) \),

where \( \tilde{a}_0(t) = \sqrt{2} \|a_0(t)\|_{L^2(\Omega)} \) and \( \tilde{a}_1 = \sqrt{2} a_1 \).

Lemma 3 Under the hypothesis \( H(B) \), the operator \( B: (0,T) \times E \to E^* \) defined by (20) satisfies the properties

(a) \( B(\cdot, v) \) is measurable on \((0,T)\) for all \( v \in E \).
(b) \( B(t, \cdot) \) is Lipschitz continuous for a.e. \( t \in (0,T) \), i.e. \( \|B(t,u) - B(t,v)\|_{E^*} \leq L_B \|u - v\|_E \) for all \( u, v \in E \), a.e. \( t \in (0,T) \).
(c) \( \|B(t,v)\|_{E^*} \leq \bar{b}_0(t) + \bar{b}_1 \|v\|_E \) for all \( v \in E \), a.e. \( t \in (0,T) \) with \( \bar{b}_0 \in L^2(0,T) \) and \( \bar{b}_0, \bar{b}_1 \geq 0 \).

where \( \bar{b}_0(t) = \sqrt{2} \|b_0(t)\|_{L^2(\Omega)} \) and \( \bar{b}_1 = \sqrt{2} b_1 \).

Lemma 4 Under the hypothesis \( H(C) \), the operator \( C \) defined by (21) satisfies \( C \in L^2(0,T; \mathcal{L}(E,E^*)) \).

The proofs of Lemmata 5 and 7 are elementary and therefore they are omitted.

Lemma 5 Under the hypothesis \( H(C_e) \), the operator \( C_1: (0,T) \times L^2(\Omega) \to E^* \) defined by (22) satisfies the properties

(a) \( C_1(\cdot, \theta) \) is measurable on \((0,T)\) for all \( \theta \in L^2(\Omega) \).
(b) \( C_1(t, \cdot) \) is Lipschitz continuous for a.e. \( t \in (0,T) \), i.e. \( \|C_1(t,\theta_1) - C_1(t,\theta_2)\|_{E^*} \leq L_e \|\theta_1 - \theta_2\|_{L^2(\Omega)} \) for all \( \theta_1, \theta_2 \in L^2(\Omega) \), a.e. \( t \in (0,T) \).
(c) \( \|C_1(t, \theta)\|_{E^*} \leq \tilde{c}_{0e}(t) + \tilde{c}_{1e} \|\theta\|_{L^2(\Omega)} \) for all \( \theta \in L^2(\Omega) \), a.e. \( t \in (0,T) \) with \( \tilde{c}_{0e} \in L^\infty(0,T) \) and \( \tilde{c}_{0e}, \tilde{c}_{1e} \geq 0 \).

where \( \tilde{c}_{0e}(t) = \sqrt{2} \|c_{0e}(t)\|_{L^\infty(\Omega)} \) and \( \tilde{c}_{1e} = \sqrt{2} c_{1e} \).

Lemma 6 Under the hypothesis \( H(K) \), the operator \( C_2: (0,T) \times V \to V^* \) defined by (20) satisfies the properties

(a) \( C_2(\cdot, \theta) \) is measurable on \((0,T)\) for all \( \theta \in V \).
(b) \( C_2(t, \cdot) \) is strongly monotone for a.e. \( t \in (0,T) \), i.e. there exists \( m_1 > 0 \) such that \( \langle C_2(t,\theta_1) - C_2(t,\theta_2), \theta_1 - \theta_2 \rangle_{V^* \times V} \geq m_1 \|\theta_1 - \theta_2\|_V^2 \) for all \( \theta_1, \theta_2 \in V \).
(c) \( \|C_2(t, \theta)\|_{V^*} \leq \tilde{k}_0(t) + \tilde{k}_1 \|\theta\|_V \) for all \( \theta \in V \), a.e. \( t \in (0,T) \) with \( \tilde{k}_0 \in L^2(0,T) \), \( \tilde{k}_0 \geq 0 \) and \( \tilde{k}_1 > 0 \).
(d) \( \langle C_2(t, \theta), \theta \rangle_{V^*} \geq \alpha_K \| \theta \|^2 \) for all \( \theta \in V \), a.e. \( t \in (0, T) \).

(e) \( C_2(t, \cdot) \) is pseudomonotone for a.e. \( t \in (0, T) \),

where \( \bar{k}_0(t) = \sqrt{2} \| k_0(t) \|_{L^\infty(\Omega)} \) and \( \bar{k}_1 = \sqrt{2} k_1 \).

**Proof.** The properties (a)–(d) are direct consequences of the hypothesis \( H(K) \). For the proof of (e), we apply Proposition 26.12 of [34, p.572] to deduce that the operator \( C_2(t, \cdot) \) is monotone, coercive, bounded and continuous. In particular, it is monotone and hemicontinuous, so by Proposition 27.7(a) of [34, p.586], we infer that \( C_2(t, \cdot) \) is pseudomonotone for a.e. \( t \in (0, T) \). \( \square \)

**Lemma 7** Under the hypotheses \( H(R) \) and \( H(h_r) \), the operator \( C_3: (0, T) \times E \to V^* \) defined by \( \langle 27 \rangle \) satisfies the properties:

(a) \( C_3(\cdot, v) \) is measurable on \( (0, T) \) for all \( v \in E \).

(b) \( C_3(\cdot, v) \) is Lipschitz continuous for a.e. \( t \in (0, T) \), i.e. \( \| C_3(t, v_1) - C_3(t, v_2) \|_{V^*} \leq L_R \| v_1 - v_2 \|_E \) for all \( v_1, v_2 \in E \), a.e. \( t \in (0, T) \).

(c) \( \| C_3(t, v) \|_{V^*} \leq c_{31}(t) + c_{32} \| v \|_E \) for all \( v \in E \), a.e. \( t \in (0, T) \) with \( c_{31} \in L^2(0, T) \) and \( c_{31}, c_{32} \geq 0 \).

We state the properties of the potential \( J: (0, T) \times L^2(\Gamma_C) \to \mathbb{R} \) defined by

\[
J(t, \theta) = \int_{\Gamma_C} j(x, t, \theta(x)) \, d\Gamma \quad \text{for all } \theta \in L^2(\Gamma_C), \text{ a.e. } t \in (0, T).
\]

(28)

The proof of the Lemma 8 below follows the lines of the proof of Lemma 3.1 of [28] and Lemma 5 of [25].

**Lemma 8** Under the hypothesis \( H(j) \) the functional \( J(\cdot) \) given by (28) has the following properties:

(a) \( J(\cdot, \theta) \) is measurable on \( (0, T) \) for all \( \theta \in L^2(\Gamma_C) \) and \( J(\cdot, 0) \in L^1(0, T) \).

(b) \( J(t, \cdot) \) is locally Lipschitz on \( L^2(\Gamma_C) \) (in fact, Lipschitz on bounded subsets of \( L^2(\Gamma_C) \) for a.e. \( t \in (0, T) \).

(c) \( \| \partial J(t, \theta) \|_{L^2(\Gamma_C)} \leq \| c_0(t) \|_{L^2(\Gamma_C)} + c_1 \| \theta \|_{L^2(\Gamma_C)} \) for \( \theta \in L^2(\Gamma_C) \), a.e. \( t \in (0, T) \).

(d) \( \langle z_1 - z_2, \theta_1 - \theta_2 \rangle_{L^2(\Gamma_C)} \geq -m_0 \| \theta_1 - \theta_2 \|^2_{L^2(\Gamma_C)} \) for all \( z_i \in \partial J(t, \theta_i), \theta_i \in L^2(\Gamma_C), i = 1, 2 \), a.e. \( t \in (0, T) \).

(e) for all \( \theta, \zeta \in L^2(\Gamma_C) \) and a.e. \( t \in (0, T) \), we have

\[
J^0(t, \theta; \zeta) \leq \int_{\Gamma_C} j^0(x, t, \theta(x); \zeta(x)) \, d\Gamma.
\]

Our main existence and uniqueness result for Problem \( P_V \) is formulated below. We denote by \( \tau_e \) the embedding constant of \( E \) into \( Z \) and by \( c_e \) the embedding constant of \( V \) into \( Y \).
Theorem 9: Under the hypotheses $H(A)$, $H(B)$, $H(C)$, $H(C_e)$, $H(j_\nu)$, $H(j_\tau)$, $H(j)$, $H(f)$, $H(K)$, $H(R)$, $H(h\tau)$, and the following conditions

\[
\begin{aligned}
 \text{either } & j_\nu(x, t, \cdot) \text{ and } j_\tau(x, t, \cdot) \text{ are regular} \\
 \text{or } & -j_\nu(x, t, \cdot) \text{ and } -j_\tau(x, t, \cdot) \text{ are regular}
\end{aligned}
\]

\[
\begin{aligned}
\text{either } & j(x, t, \cdot) \text{ or } -j(x, t, \cdot) \text{ is regular}
\end{aligned}
\]

\[
\begin{aligned}
m_A \geq \max \{m_\nu, m_\tau\} \pi_e^2 \|\gamma\|^2 \\
\alpha_A > 6 \max \{c_{1\nu}, c_{1\tau}\} \pi_e^2 \|\gamma\|^2 \\
m_K \geq m_0 c_e^2 \|\gamma_s\|^2 \\
\alpha_K > c_1 c_e^2 \|\gamma_s\|^2
\end{aligned}
\]

Problem $P_V$ has a unique solution $\{u, \theta\}$ such that $u \in \mathcal{E}$, $u' \in \mathcal{E}$ and $\theta \in \mathcal{W}$.

5 Proof of Theorem 9

The proof of Theorem 9 will be carried out in several steps. It is based on recent arguments of first and second order hemivariational inequalities and a fixed point argument. In the proof we consider two auxiliary intermediate problems.

**Step 1.** Let $\eta \in \mathcal{E}^*$ be given. We consider the following second order hemivariational inequality.

**Problem $P_{\eta}^1$:** find $u_\eta \in \mathcal{E}$ such that $u'_{\eta} \in \mathcal{E}$ and such that

\[
\begin{aligned}
\langle u''_{\eta}(t) + A(t, u'_{\eta}(t)), v \rangle_{\mathcal{E}^* \times \mathcal{E}} + \\
\int_{\Gamma_C} \left( j_\nu^0(x, t, u'_{\eta\nu}(t); v_\nu) + j_\tau^0(t, u'_{\eta\tau}(t); v_\tau) \right) d\Gamma \geq \langle f(t) - \eta(t), v \rangle_{\mathcal{E}^* \times \mathcal{E}}
\end{aligned}
\]

for all $v \in \mathcal{E}$ and a.e. $t \in (0, T)$

\[
u_\eta(0) = u_0, \quad u'_{\eta}(0) = v_0.
\]

The unique solvability of Problem $P_{\eta}^1$ is established by our next lemma.

Lemma 10: For $\eta \in \mathcal{E}^*$, Problem $P_{\eta}^1$ has a unique solution $u_\eta \in \mathcal{E}$ such that $u'_{\eta} \in \mathcal{E}$. Moreover, if $u_i$ denotes the solution to Problem $P_{\eta}^1$ corresponding to $\eta = \eta_i \in \mathcal{E}^*$, $i = 1, 2$, then there exists $c > 0$ such that

\[
\|u_1(t) - u_2(t)\|_{\mathcal{E}}^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{E}^*}^2 ds \text{ for all } t \in [0, T].
\]
Problem. It follows from the hypotheses \( H(A), H(j_\nu), H(j_\nu) \), (29), (31) and (32) that we are able to apply Theorem 8.6 in [29] from which we infer that Problem \( P_1^\eta \) has a unique solution \( u_\eta \in \mathcal{E} \) such that \( u_\eta' \in \mathcal{E} \). Exploiting the method used for evolution hemivariational inequalities in Theorem 5.17 of [29] (cf. (5.86) and (5.88) in [29]), we are able to show (35) and the following estimate for the first-order derivatives
\[
\|u_1'(t) - u_2'(t)\|_E^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_E^2 \, ds \quad \text{for all } t \in [0, T].
\] (36)

For details we refer to Chapter 5 of [29]. This completes the proof of the lemma.

\[ \square \]

**Step 2.** We use the displacement field \( u_\eta \) obtained in Lemma 10 and consider the following first order hemivariational inequality.

**Problem** \( P_2^\eta \): find \( \theta_\eta \in \mathcal{W} \) such that such that
\[
\langle \theta_\eta'(t) + C_2(t, \theta_\eta(t)), \zeta \rangle_{V^*_\times V} + \int_{\Gamma_C} j^0(t, \theta_\eta(t); \zeta) \, d\Gamma \geq \langle C_3(t, u_\eta'(t)) + g(t), \zeta \rangle_{V^*_\times V}
\]

for all \( \zeta \in V \) and a.e. \( t \in (0, T) \)
\[
\theta_\eta(0) = \theta_0.
\]

The following result ensures the existence and uniqueness of a solution to Problem \( P_2^\eta \).

**Lemma 11** For \( \eta \in \mathcal{E}^* \), Problem \( P_2^\eta \) has a unique solution \( \theta_\eta \in \mathcal{W} \). Moreover, if \( \theta_i \) denotes the solution to Problem \( P_2^\eta \) corresponding to \( \eta = \eta_i \in \mathcal{E}^* \), \( i = 1, 2 \), then there exists \( c > 0 \) such that
\[
\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_E^2 \, ds \quad \text{for all } t \in [0, T].
\] (37)

**Proof.** The proof of the lemma will be done in four steps. Consider the following evolution inclusion associated with Problem \( P_2^\eta \).
\[
\begin{cases}
\text{find } \theta \in \mathcal{W} \text{ such that } \\
\theta'(t) + C_2(t, \theta(t)) + \gamma_s^* \partial J(t, \gamma_s \theta(t)) \ni C_3(t, u'(t)) + g(t) \quad \text{for a.e. } t \in (0, T) \\
\theta(0) = \theta_0.
\end{cases}
\] (38)

**Step 1°.** Under the hypotheses \( H(j) \) and (30), we prove that \( \theta \in \mathcal{W} \) is a solution to Problem \( P_2^\eta \) if and only if \( \theta \) solves (38).

Let \( \theta \in \mathcal{W} \) be a solution to (38), i.e. there exists \( \zeta \in \mathcal{Y}^* \) such that \( \xi(t) = \gamma_s^* \z(t) \), \( z(t) \in \partial J(t, \gamma_s \theta(t)) \) for a.e. \( t \in (0, T) \) and
\[
\theta'(t) + C_2(t, \theta(t)) + \z(t) = C_3(t, u'(t)) + g(t) \quad \text{for a.e. } t \in (0, T).
\] (39)

By the definition of the subdifferential, we have
\[
\langle \z(t), w \rangle_{L^2(\Gamma_C)} \leq J^0(t, \gamma_s \theta(t); w) \quad \text{for all } w \in L^2(\Gamma_C), \text{ a.e. } t \in (0, T).
\] (40)
Combining Lemma 8(e), (39) and (40), we obtain
\[
\langle C_3(t, u'(t)) + g(t) - \theta'(t) - C_2(t, \theta(t)), \zeta \rangle =
\]
\[
= \langle \xi(t), \zeta \rangle_{Y^* \times Y} = \langle z(t), \gamma_s \zeta \rangle_{L^2(\Gamma_C)} \leq J^0(t, \gamma_s \theta(t); \gamma_s \zeta) \leq \int_{\Gamma_C} j^0(x, t, \theta(t); \zeta) d\Gamma
\]
for all $\zeta \in V$, a.e. $t \in (0, T)$. Hence, $\theta$ is a solution to Problem $P^0_2$.

Vice versa, let $\theta$ be a solution to Problem $P^0_2$. We note that the regularity hypothesis (30) implies that either $J(t, \cdot)$ or $-J(t, \cdot)$ is regular for a.e. $t \in (0, T)$, and the inequality in Lemma 8(e) holds with equality, cf. Clarke [6]. Using this equality, we obtain
\[
\langle \theta'(t) + C_2(t, \theta(t)) - C_3(t, u'(t)) - g(t), \zeta \rangle_{V^* \times V} + J^0(t, \gamma_s \theta(t); \gamma_s \zeta) \geq 0
\]
for all $\zeta \in V$ and a.e. $t \in (0, T)$. By Proposition 2.1(ii) of [28], the previous inequality implies that
\[
\langle C_3(t, u'(t)) + g(t) - \theta'(t) - C_2(t, \theta(t)), \zeta \rangle_{V^* \times V} \leq (J \circ \gamma_s)^0(t, \theta(t); \zeta)
\]
for all $\zeta \in V$ and a.e. $t \in (0, T)$. Using the definition of the subdifferential and Proposition 2.1(ii) of [28], the previous inequality implies that
\[
C_3(t, u'(t)) + g(t) - \theta'(t) - C_2(t, \theta(t)) \in \partial(J \circ \gamma_s)(t, \theta(t)) = \gamma_s^* \partial J(t, \gamma_s \theta(t))
\]
for a.e. $t \in (0, T)$. Thus $\theta$ is a solution to (38). This completes the proof of Step 1°.

Step 2°. Under the hypotheses $H(C_3), H(j), H(K), H(R), H(h_r)$ and (30), we prove that the evolution inclusion (38) has a unique solution $\theta \in W$.

The proof of this step follows from the argument of Theorem 7 of [26]. First, we suppose temporarily that the initial condition $\theta_0 \in V$. Let $\tilde{C}_2: V \to V^*$ be the Nemitsky operator corresponding to $C_2$ and defined by $(\tilde{C}_2 \theta)(t) = C_2(t, \theta(t) + \theta_0)$ for $\theta \in V$ and a.e. $t \in (0, T)$. Let $N: V \to 2^{V^*}$ be the multivalued Nemitsky operator corresponding to $\gamma_s^* \circ \partial J(t, \gamma_s \cdot)$, i.e.
\[
N \theta = \{ w \in V^* \mid w(t) \in \gamma_s^* \partial J(t, \gamma_s(\theta(t) + \theta_0)) \text{ a.e. } t \in (0, T) \} \text{ for } \theta \in V.
\]
Under these notation, the problem (38) can be written as the operator inclusion:
\[
\begin{cases}
\theta' + \tilde{C}_2 \theta + N \theta \ni \tilde{C}_3(u') + g \\
\theta(0) = 0,
\end{cases}
\]
where $\tilde{C}_3: \mathcal{E} \to V^*$ is given by $(\tilde{C}_3 z)(t) = C_3(t, z(t))$ for $z \in \mathcal{E}$. Note that $\theta \in W$ is a solution to problem (38) if and only if $\theta - \theta_0 \in W$ solves (41).

Let $L: D(L) \subset V \to V^*$ be the operator defined by $L \theta = \theta'$ with $D(L) = \{ \theta \in W \mid \theta(0) = 0 \}$. It is known (see e.g. [34]) that $L$ is densely defined maximal monotone operator. Let $\mathcal{F}: V \to 2^{V^*}$ be the operator given by $\mathcal{F} \theta = \tilde{C}_2 \theta + N \theta$ for $\theta \in V$. Now, the problem (41) is equivalent to
\[
\text{find } \theta \in D(L) \text{ such that } L \theta + \mathcal{F} \theta \ni \tilde{C}_3(u') + g.
\]
In order to prove the existence of a solution to the problem (41), we show that the operator $F$ is bounded, coercive and $L$-pseudomonotone. The proof of boundedness and $L$-pseudomonotonicity is quite similar to that given in Theorem 7 of [26]. We show the coercivity of $F$. To this end, from the equality

$$\langle \tilde{C}_2 \theta, \theta \rangle_{V^* \times V} = \int_0^T \langle C_2(t, \theta(t) + \theta_0), \theta(t) + \theta_0 \rangle_{V^* \times V} dt - \int_0^T \langle C_2(t, \theta(t) + \theta_0), \theta_0 \rangle_{V^* \times V} dt$$

for $\theta \in V$, using (c) and (d) of Lemma 6, and the Hölder inequality, we obtain

$$\langle \tilde{C}_2 \theta, \theta \rangle_{V^* \times V} \geq \alpha_K \| \theta + \theta_0 \|^2_V - c \| \theta \|_V - c \geq \alpha_K \| \theta \|^2_V - c \| \theta \|_V - c$$ (42)

with a positive constant $c > 0$. Next, let $\theta \in V$, $w \in N$. So $w \in \mathcal{Y}^*$, $w(t) = \gamma_s^* \xi(t)$ and $\xi(t) \in \partial J(t, \gamma_s(\theta(t) + \theta_0))$ for a.e. $t \in (0, T)$. Exploiting Lemma 8(c), the continuity of the embedding $V \subset Y$ and of the trace operator $\gamma_s$, it follows that

$$\langle w, z \rangle_{V^* \times V} = \int_0^T \langle w(t), z(t) \rangle_{V^* \times V} dt = \int_0^T \langle \xi(t), \gamma_s z(t) \rangle_{L^2(\Gamma_C)} dt \leq$$

$$\leq c_e \| \gamma_s \| \int_0^T \| \xi(t) \|_{L^2(\Gamma_C)} \| z(t) \|_V dt \leq$$

$$\leq c_e \| \gamma_s \| \int_0^T \left( \| c_0(t) \|_{L^2(\Gamma_C)} + c_1 c_e \| \gamma_s \| \| \theta(t) + \theta_0 \|_V \right) \| z(t) \|_V dt \leq$$

$$\leq c_e \| \gamma_s \| \| c_0 \|_{L^2(\Sigma_C)} \| z \|_V + c_1 c_e^2 \| \gamma_s \|^2 \| \theta + \theta_0 \|_V \| z \|_V$$

for all $z \in V$. Hence, we infer

$$\| w \|_{V^*} \leq c_e \| \gamma_s \| \| c_0 \|_{L^2(\Sigma_C)} + c_1 c_e^2 \| \gamma_s \|^2 (\| \theta \|_V + T \| \theta_0 \|_V)$$

and

$$| \langle N \theta, \theta \rangle_{V^* \times V} | = | \langle w, \theta \rangle_{V^* \times V} | \leq \| w \|_{V^*} \| \theta \|_V \leq c_1 c_e^2 \| \gamma_s \|^2 \| \theta \|^2_V + c \| \theta \|_V$$

with a positive constant $c$. The latter and (42) implies

$$\langle F \theta, \theta \rangle_{V^* \times V} = \langle \tilde{C}_2 \theta, \theta \rangle_{V^* \times V} + \langle N \theta, \theta \rangle_{V^* \times V} \geq (\alpha_K - c_1 c_e^2 \| \gamma_s \|^2) \| \theta(t) \|_V - c \| \theta \|_V - c$$

Finally, by the hypothesis (34), we deduce that the operator $F$ is coercive.

Since the multivalued operator $F$ is bounded, coercive and $L$-pseudomonotone, from Theorem 6.3.73 in [10], it follows that the problem (41) has a solution $\theta \in D(L)$, so $\theta + \theta_0$ solves (38) in the case $\theta_0 \in V$. Subsequently, exploiting the method used in Theorem 7 of [26], we are able to prove that the problem (38) has a solution $\theta \in \mathcal{W}$ in the case $\theta_0 \in L^2(\Omega)$.

Step 3. We claim that the solution to Problem $P_2^0$ is unique. From Step 1, it is enough to prove that the problem (38) has a unique solution. Let $\theta_1, \theta_2 \in \mathcal{W}$ be solutions to (38), i.e.

$$\theta_1'(t) + C_2(t, \theta_1(t)) + \xi_1(t) = C_3(t, u'(t)) + g(t) \text{ a.e. } t \in (0, T),$$ (43)
From (45), we have \( \xi_i = 1, 2 \). By Lemma 8(d), we deduce 
\[ \theta_1(0) = \theta_2(0) = \theta_0. \]

Subtracting (44) from (43), multiplying the result by the unique solutions to Problem 2, we obtain 
\[ \theta_1(t) - \theta_2(t) = 0 \quad \text{for all } t \in [0, T]. \]

From (43), we have \( \xi_i(t) = \gamma_s^* z_i(t) \) with \( z_i(t) \in \partial J(t, \gamma_s \theta_i(t)) \) for a.e. \( t \in (0, T) \) and \( i = 1, 2 \). By Lemma 8(d), we deduce 
\[ \int_0^t \langle \xi_1(s) - \xi_2(s), \theta_1(s) - \theta_2(s) \rangle_{V^* \times V} \, ds = 0 \]
for all \( t \in [0, T] \). Inserting the inequality (48) into (47), using Lemma 6(b) and (33), we obtain 
\[ \frac{1}{2} \| \theta_1(t) - \theta_2(t) \|^2_{L^2(\Omega)} + \int_0^t \langle C_2(s, \theta_1(s)) - C_2(s, \theta_2(s)), \theta_1(s) - \theta_2(s) \rangle_{V^* \times V} \, ds = 0 \]
for all \( t \in [0, T] \) with \( c = m_K - m_0 c_2^2 \| \gamma_s \|^2 \geq 0 \). Hence we deduce that \( \theta_1 = \theta_2 \) which completes the proof of the uniqueness of solution.

Step 4'. We will establish the estimate (37). Let \( \eta_i \in \mathcal{E}^* \) and let \( \theta_i = \theta_{n_i} \) be the unique solutions to Problem \( P_2^n \) corresponding to \( \eta_i, i = 1, 2 \). We use the same technique as in Step 3'. Subtracting the equations satisfied by \( \theta_i \), multiplying the result by \( \theta_1(t) - \theta_2(t) \) and integrating on \([0, t]\), we deduce 
\[ \frac{1}{2} \| \theta_1(t) - \theta_2(t) \|^2_{L^2(\Omega)} + \int_0^t \langle C_2(s, \theta_1(s)) - C_2(s, \theta_2(s)), \theta_1(s) - \theta_2(s) \rangle_{V^* \times V} \, ds + \]
\[ + \int_0^t \langle \xi_1(s) - \xi_2(s), \theta_1(s) - \theta_2(s) \rangle_{V^* \times V} \, ds = \]
\[ = \int_0^t \langle C_3(s, u'_1(s)) - C_3(s, u'_2(s)), \theta_1(s) - \theta_2(s) \rangle_{V^* \times V} \, ds, \]
where $\xi_i(t) = \gamma_i^* z_i(t)$, $z_i(t) \in \partial J(t, \gamma_i \theta_i(t))$ for a.e. $t \in (0, T)$, $i = 1, 2$. Exploiting Lemma 6(b), Lemma 7(b), (48) and the Young inequality with $\varepsilon > 0$, we have

$$\frac{1}{2} \| \theta_1(t) - \theta_2(t) \|_{L^2(\Omega)}^2 + m_K \int_0^t \| \theta_1(s) - \theta_2(s) \|_{C(\Omega)}^2 \, ds \leq$$

$$\leq \int_0^t \| C_3(s, u_1'(s)) - C_3(s, u_2'(s)) \|_{L^\infty} \| \theta_1(s) - \theta_2(s) \|_{L^\infty} \, ds \leq$$

$$\leq \frac{L_R}{2\varepsilon^2} \int_0^t \| u_1'(s) - u_2'(s) \|_{L^2(E)}^2 \, ds + \frac{\varepsilon^2}{2} \int_0^t \| \theta_1(s) - \theta_2(s) \|_{C(\Omega)}^2 \, ds$$

for all $t \in [0, T]$. Choosing $\varepsilon = \sqrt{2m_K}$, we conclude

$$\frac{1}{2} \| \theta_1(t) - \theta_2(t) \|_{L^2(\Omega)}^2 \leq \frac{L_R}{4m_K} \int_0^t \| u_1'(s) - u_2'(s) \|_{L^2(E)}^2 \, ds$$

for all $t \in [0, T]$. Finally, we use the estimate (36) and the previous inequality to obtain (37). This completes the proof of the lemma. \qed

**Step 3.** In this step, we apply a fixed point argument. Let $u_\eta \in E$ with $u_\eta' \in \mathbb{E}$ be the solution to Problem $P_1^\eta$ and let $\theta_\eta \in \mathcal{W}$ be the solution to Problem $P_2^\eta$ obtained in Lemma 10 and Lemma 11, respectively. We define the operator $\Lambda : \mathcal{E}^* \to \mathcal{E}^*$ by

$$\langle \Lambda \eta(t), v \rangle_{E^* \times E} = \langle B(t, u_\eta(t)) + \int_0^t C(t - s) u_\eta(s) \, ds + C_1(t, \theta_\eta(t)), v \rangle_{E^* \times E} \quad (49)$$

for all $v \in E$ and a.e. $t \in (0, T)$.

**Lemma 12** The operator $\Lambda$ defined by (49) has a unique fixed point $\eta^* \in \mathcal{E}^*$.

**Proof.** It is easy to check that the operator $\Lambda$ is well defined. Indeed, from Lemmata 3(c) and 5(c) and the inequality

$$\| \int_0^t C(t - s) u_\eta(s) \, ds \|_{E^*} \leq \int_0^t \| C(t - s) \|_{L(E, E^*)} \| u_\eta(s) \|_{E} \, ds \leq$$

$$\leq \left( \int_0^t \| C(\tau) \|_{L(E, E^*)}^2 \, d\tau \right)^{1/2} \left( \int_0^t \| u_\eta(\tau) \|_{E}^2 \, d\tau \right)^{1/2} \leq \| C \|_{L^2(0, t; L(E, E^*))} \| u_\eta \|_{L^2(0, t; E)}$$

for all $t \in [0, T]$, we have

$$\| \Lambda \eta \|_{E^*}^2 = \int_0^T \| (\Lambda \eta)(s) \|_{E^*}^2 \, ds \leq c \int_0^T \left( \| B(s, u_\eta(s)) \|_{E^*}^2 + \| \int_0^s C(t - s') u_\eta(s') \, ds' \|_{E^*}^2 + \| C_1(s, \theta(s)) \|_{E^*}^2 \right) \, ds \leq$$

$$\leq c \left( 1 + \| u_\eta \|_{E}^2 + \| \theta_\eta \|_{E^*}^2 \right)$$
where \( c > 0 \). Hence \( \| \Lambda \eta \|_{E^*} \leq c \left( 1 + \| u_0 \|_E + \| \theta_0 \|_W \right) \) which implies that the operator \( \Lambda \) is well defined and takes values in \( E^* \).

Subsequently, we will show that the operator \( \Lambda \) has a unique fixed point. Let \( \eta_1, \eta_2 \in E^* \). By (49), we have

\[
\| \Lambda \eta_1(t) - \Lambda \eta_2(t) \|_{E^*}^2 \leq c \left( \| B(t, u_1(t)) - B(t, u_2(t)) \|_{E^*}^2 + \right. \\
+ \left. \| \int_0^t C(t-s)(u_1(s) - u_2(s)) \, ds \|_{E^*}^2 + \| C_1(t, \theta_1(t)) - C_1(t, \theta_2(t)) \|_{E^*}^2 \right).
\]

Using Lemmata 3(b) and 5(b), and the inequality

\[
\| \int_0^t C(t-s)(u_1(s) - u_2(s)) \, ds \|_{E^*}^2 \leq \| C \|_{L^2(0,T;\mathcal{L}(E,E^*))}^2 \int_0^t \| u_1(s) - u_2(s) \|_E^2 \, ds
\]

for all \( t \in [0,T] \), we deduce

\[
\| \Lambda \eta_1(t) - \Lambda \eta_2(t) \|_{E^*}^2 \leq c \left( \| u_1(t) - u_2(t) \|_E^2 + \int_0^t \| u_1(s) - u_2(s) \|_E^2 \, ds + \\
+ \| \theta_1(t) - \theta_2(t) \|_{L^2(\Omega)}^2 \right).
\]

Hence, by (35) and (37), we obtain

\[
\| \Lambda \eta_1(t) - \Lambda \eta_2(t) \|_{E^*}^2 \leq c \int_0^t \| \eta_1(s) - \eta_2(s) \|_{E^*}^2 \, ds
\]

for all \( t \in [0,T] \) with \( c > 0 \). Applying Lemma 1, we infer that there exists a unique \( \eta^* \in E^* \) such that \( \Lambda \eta^* = \eta^* \). This completes the proof of the lemma.

**Step 4.** We have now all ingredients to prove the theorem. Let \( \eta^* \in E^* \) be the unique fixed point of the operator \( \Lambda \) established in Lemma 12 i.e.

\[
\eta^*(t) = B(t, u_{\eta^*}(t)) + \int_0^t C(t-s)u_{\eta^*}(s) \, ds + C_1(t, \theta_{\eta^*}(t))
\]

for a.e. \( t \in (0,T) \). Let \( u^* = u_{\eta^*} \) be the unique solution of Problem \( P^{\eta^*}_1 \) corresponding to \( \eta^* \) established in Lemma 10. Moreover, let \( \theta^* = \theta_{\eta^*} \) be the unique solution of Problem \( P^{\eta^*_2} \) proved in Lemma 11. Hence, \( \{u^*, \theta^*\} \) is the unique solution to Problem \( P_V \) with the regularity \( u^* \in \mathcal{E}, u^* \in \mathcal{E} \) and \( \theta^* \in \mathcal{W} \). The uniqueness part of the theorem is a consequence of the uniqueness of the fixed point of \( \Lambda \) and Lemmata 10 and 11. This completes the proof of the theorem.

**6 Examples**

We now give a simple example of the functional which satisfies hypothesis \( H(J)_1 \).
Example 13 Let us consider the functional $J_1 : L^2(\Gamma_C; \mathbb{R}^d) \to \mathbb{R}$ defined by

$$J_1(v) = \int_{\Gamma_C} \left( \int_0^{v_N(x)} \beta(s) \, ds \right) \, d\Gamma(x) \quad \text{for all } v \in L^2(\Gamma_C; \mathbb{R}^d)$$

(for simplicity we drop the $(x,t)$-dependence in the integrand of $J$), where the function $\beta$ satisfies the following hypothesis (cf. $H(p_N)$ in Section ??):

$$H(\beta) : \quad \beta \in L^\infty_{\text{loc}}(\mathbb{R}) \text{ is a function such that } |\beta(s)| \leq \beta_0(1 + |s|) \text{ for } s \in \mathbb{R} \quad \text{with } \beta_0 > 0,$$

$$\lim_{\tau \to \xi^\pm} \beta(\tau) \text{ exist for every } \xi \in \mathbb{R} \quad \text{and}$$

$$\frac{\text{ess inf}_{\xi_1 \neq \xi_2} \beta(\xi_1) - \beta(\xi_2)}{\xi_1 - \xi_2} \geq -m_2 \quad \text{with some } m_2 > 0. \quad (50)$$

We define the multivalued map $\hat{\beta} : \mathbb{R} \to 2^\mathbb{R}$ which is obtained from $\beta$ by "filling in the gaps" at its discontinuity points, i.e. $\hat{\beta}(\xi) = [\beta(\xi), \overline{\beta}(\xi)]$, where

$$\beta(\xi) = \lim_{\delta \to 0^+} \text{ess inf}_{|t-\xi| \leq \delta} \beta(t), \quad \overline{\beta}(\xi) = \lim_{\delta \to 0^+} \text{ess sup}_{|t-\xi| \leq \delta} \beta(t)$$

and $[\cdot, \cdot]$ denotes the interval. It is well known (see e.g. [?]) that a locally Lipschitz function $j_N : \mathbb{R} \to \mathbb{R}$ can be determined, up to an additive constant, by the relation $j_N(s) = \int_0^s \beta(\tau) \, d\tau$ and $\partial j_N(s) = \hat{\beta}(s)$ for $s \in \mathbb{R}$. It can be shown (see [25] for the details) that $j_N$ satisfies $H(j_N)$ and the functional $J_1$ satisfies $H(J_1)$.

References

[1] S. Adly, O. Chau and M. Rochdi, Solvability of a class of thermal dynamical contact problems with subdifferential conditions, Numerical Algebra, Control and Optimization, 2 (2012), 91–104.

[2] A. Amassad, K. L. Kuttler, M. Rochdi and M. Shillor, Quasi-static thermoviscoelastic contact problem with slip dependent friction coefficient, Math. Comp. Modeling, 36 (2002), 839–854.

[3] K. T. Andrews, K. L. Kuttler, M. Rochdi and M. Shillor, One-dimensional dynamic thermoviscoelastic contact with damage, J. Math. Anal. Appl., 272 (2002), 249–275.

[4] K. T. Andrews, M. Shillor, S. Wright and A. Klarbring, A dynamic thermoviscoelastic contact problem with friction and wear, Int. J. Engng Sci., 35 (1997), 1291–1309.

[5] O. Chau, R. Oujja and M. Rochdi, A mathematical analysis of a dynamical frictional contact model in thermoviscoelasticity, Discrete and Cont. Dyn. Systems, Ser. S, 1 (2008), 61–70.

[6] F. H. Clarke, Optimization and Nonsmooth Analysis, Wiley - Interscience, New York (1983).
[7] Z. Denkowski and S. Migórski, A system of evolution hemivariational inequalities modeling thermoviscoelastic frictional contact, *Nonlinear Analysis*, 60 (2005), 1415–1441.

[8] Z. Denkowski and S. Migorski, Hemivariational inequalities in thermoviscoelasticity, *Nonlinear Analysis*, 63 (2005), 87–97.

[9] Z. Denkowski, S. Migorski and A. Ochal, Optimal control for a class of mechanical thermoviscoelastic frictional contact problems, *Control and Cybernetics*, 36 (2007), 611–632.

[10] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer/Plenum, New York (2003).

[11] G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin (1976).

[12] I. Figueiredo and L. Trabucho, A class of contact and friction dynamic problems in thermoelasticity and in thermoviscoelasticity, *Int. J. Engng Sci.*, 33 (1995), 45–66.

[13] D. Goeleven, M. Miettinen and P.D. Panagiotopoulos, Dynamic hemivariational inequalities and their applications, *J. Optimiz. Theory and Appl.*, 103 (3) (1999), 567–601.

[14] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, American Mathematical Society, International Press (2002).

[15] J. Haslinger, M. Miettinen and P.D. Panagiotopoulos, *Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications*, Kluwer Academic Publishers, Boston, Dordrecht, London (1999).

[16] J. Jarusek, Dynamic contact problems with given friction for viscoelastic bodies, *Czech. Math. J.*, 46 (1996), 475–487.

[17] A. Kulig, Hyperbolic hemivariational inequalities for dynamic viscoelastic contact problems, *Journal of Elasticity*, 110 (2013), 1–31.

[18] A. Kulig and S. Migórski, Solvability and continuous dependence results for second order nonlinear evolution inclusions with a Volterra-type operator, *Nonlinear Analysis*, 75 (2012), 4729–4746.

[19] K. L. Kuttler, Dynamic friction contact problem with general normal and friction laws, *Nonlinear Analysis*, 28 (1997), 559–575.

[20] K. L. Kuttler and M. Shillor, Dynamic bilateral contact with discontinuous friction coefficient, *Nonlinear Analysis*, 45 (2001), 309–327.

[21] J. A. C. Martins and J. T. Oden, Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, *Nonlinear Analysis*, 11 (1987), 407–428.

[22] S. Migórski, On the existence of solutions for parabolic hemivariational inequalities, *Journal of Computational and Applied Mathematics*, 129 (2001), 77–87.

[23] S. Migórski, Evolution hemivariational inequalities in infinite dimension and their control, *Nonlinear Analysis*, 47 (2001), 101–112.
[24] S. Migórski, Boundary hemivariational inequalities of hyperbolic type and applications, *J. Global Optimiz.*, 31 (2005), 505–533.

[25] S. Migórski, Dynamic hemivariational inequality modeling viscoelastic contact problem with normal damped response and friction, *Applicable Analysis*, 84 (2005), 669–699.

[26] S. Migórski and A. Ochal, Boundary hemivariational inequality of parabolic type, *Nonlinear Analysis*, 57 (2004), 579–596.

[27] S. Migórski and A. Ochal, Existence of solutions for second order evolution inclusions with application to mechanical contact problems, *Optimization*, 55 (2006), 101–120.

[28] S. Migórski, A. Ochal and M. Sofonea, Integrodifferential hemivariational inequalities with applications to viscoelastic frictional contact, *Mathematical Models and Methods in Applied Sciences*, 18 (2008), 271–290.

[29] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics, vol. 26, Springer, New York (2013).

[30] Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, Inc., New York - Basel - Hong Kong (1995).

[31] P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions*, Birkhäuser, Basel (1985).

[32] P. D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin (1993).

[33] M. Rochdi and M. Shillor, Existence and uniqueness for a quasistatic frictional bilateral contact problem in thermoviscoelasticity, *Quart. Appl. Math.*, 58 (2000), 543–560.

[34] E. Zeidler, *Nonlinear Functional Analysis and Applications II A/B*, Springer, New York (1990).