Holographic Approach to Nonequilibrium Dynamics of Moving Mirrors Coupled to Quantum Critical Theories

Chen-Pin Yeh, Jen-Tsung Hsiang and Da-Shin Lee

Department of Physics, National Dong-Hwa University,
Hualien 97401, Taiwan, Republic of China

Abstract

We employ the holographic method to study fluctuations and dissipation of an n-dimensional moving mirror coupled to quantum critical theories in d spacetime dimensions. The bulk counterpart of the mirror with perfect reflection is a n + 1-dimensional membrane in the Lifshitz geometry of d + 1 dimensions. The motion of the mirror can be realized from the dynamics of the brane at the boundary of the bulk. The excited modes of the brane in the bulk render the mirror undergoing Brownian motion. For small displacement of the mirror, we derive the analytical results of the correlation functions and response functions. The dynamics of the mirror due to small fluctuations around the brane vacuum state in the bulk is found supraohmic so that after initial growth, the velocity fluctuations approach a saturated value at late time with a power-law behavior. On the contrary, in the Lifshitz black hole background, the mirror in thermal fluctuations shows that its relaxation dynamics becomes ohmic, and the saturation of velocity fluctuations is reached exponentially in time. Finally a comparison is made with the result of a moving mirror driven by free fields.

PACS numbers: 11.25.Tq 11.25.Uv 05.30.Rt 05.40.-a
I. INTRODUCTION

Understanding of the microscopic origin of dissipation and fluctuations in a nonequilibrium system is the main concern in statistical mechanics. One of the ubiquitous nonequilibrium phenomena in nature is Brownian motion in which an object moves under a fluctuating environment. In this case, the Langevin equation is known to provide a satisfactory description of the Brownian particle, and its generic form is given by

$$m \ddot{X}(t) + \int dt' \theta(t - t') \mu(t - t') X(t') = R(t), \tag{1}$$

where $X(t)$ is the position of the particle. This Langevin equation is a classical equation of motion modified phenomenologically by two terms that incorporate both dissipation and fluctuation effects upon the particle in a random medium. The memory kernel $\mu(t)$ accounts for the dissipation effect, and in general depends on the past histories of the particle. The noise force $R(t)$ that mimics the random environment and is correlated over time scales determined by the typical scales of the medium. The statistic properties of the noise are specified by

$$\langle R(t) \rangle = 0, \quad \langle R(t) R(t') \rangle = \eta(t - t'). \tag{2}$$

These two effects are ultimately crucial for the system to evolve into thermodynamic equilibrium of the Brownian particle with the environment, and are thus related by the fluctuation-dissipation theorem.

A microscopic description that leads to the Langevin equation (1) has been developed by Caldeira and Leggett [1] within the context of one-particle quantum mechanics. The idea is to consider a specific system-environment model where the particle interacts bilinearly with an environment. The effects of environmental degrees of freedom on the particle can be summarized with the method of Feynman-Vernon influence functional [2] by integrating out environment variables. In the classical approximation where the intrinsic quantum uncertainties of the particle is ignored, the Langevin equation can be obtained by minimizing the corresponding effective stochastic action. This effective action can be exactly derived if the environment variables are Gaussian and their coupling with the system is linear [3, 4]. However many interesting strongly coupled environments, such as condensed matter systems, are out of reach by the method of influence functional.

The study of the Brownian particle has been extended to the nonequilibrium dynamics of
a charged oscillator in nontrivial quantized electromagnetic-field backgrounds [3, 6] and to
a perfectly reflecting mirror moving in a quantum field [7, 8]. The moving mirror problem
is of interest in its own right. For example, this problem can be related to the dynamical
Casimir effect. When the mirror undergoes nonuniform acceleration, it is expected to create
quantum radiation that in turn damps out the motion of the mirror as a result of the motion-
induced radiation reaction force. In 3+1-dimensional spacetime, this problem is solved only
for small mirror displacement. The force acting on the mirror is the radiation pressure of
the environmental field that arises from the area integral of the stress tensor. Then the
course-grained effective action is obtained by integrating out the quantum field, and thus
the corresponding semiclassical Langevin equation of the form [11] is derived. In the small
displacement and slow motion limit of the mirror, the emission of quantum radiations that
accounts for damping is found hardly detectable in a quantum vacuum environment. It is
still unclear whether or not this effect can be observed in a strong coupling environment.

To understand these issues, in this paper we plan to employ holographic duality to study
the nonequilibrium dynamics of a mirror moving in a quantum/thermal bath. The prototype
of the holographic duality is the AdS/CFT correspondence, which is a weak-strong coupling
duality between the type IIB string theory in AdS background and the \( \mathcal{N} = 4 \) super Yang-
Mills theory \([9]\). It is soon generalized to other backgrounds and field theories, and has been
proven fruitful in applying to the strong coupling problems in condensed matter systems
and the hydrodynamics of the quark-gluon plasma etc. The first investigations in applying
AdS/CFT to study the dissipation behavior in the strongly coupled field theory were done
independently in \([10–12]\), where the probed particle is represented by a string hanging from
the boundary of the AdS black hole. Later progresses have been made to understand the
fluctuations of this end point of the string as Brownian motion \([13–17]\). For a general review
on application in the nonequilibrium dynamics, see \([18]\). In recent studies \([19]\), Tong and
Wong have discussed Brownian motion in the environments at their quantum critical points
using the holographic duality. The quantum critical point is a fixed point theory with the
following scaling symmetry:

\[
  t \rightarrow \Lambda^z t, \quad x \rightarrow \Lambda x.
\]

For \( z = 1 \), it is the usual scale invariance from conformal symmetry. Other values of \( z \)
can arise from finite temperature multicritical points (\( z = 2 \) for the Lifshitz point) of the
condensed matter systems. Quantum critical points can also be realized in the strongly correlated electron system, for example, the dimer model \[20\]. The holographic dual for such quantum critical theories has been proposed in \[21\], where the gravity theory is in the Lifshitz background:

\[
ds^2 = L^2 \left( -r^{2z}dt^2 + r^2 dx^2 + \frac{dr^2}{r^2} \right).
\]

In the following, the curvature radius \( L \) is set to be unity. In \[19\], the authors derived the analytic results of dissipation and random force correlation functions in quantum critical theories via holographic duality in the Lifshitz background.

The idea of this paper is to model the problem of a mirror moving under quantum critical theories in terms of the gravity theory with scaling symmetry characterized by the exponent \( z \) via the holographic duality. We can then compare the results with the findings in \[7\] for \( z = 1 \) in the case of the relativistic quantum field, and further generalize it to other quantum field theories for \( z \neq 1 \). In the next section, our holographic setup will be explained in details, and the response function to an external force on the two-dimensional mirror will be derived. In Sec. III, we generalize to a mirror of general dimensions and also compute the response function and the correlation function that exhibit essential properties of Brownian motion. The fluctuation-dissipation theorem is then verified. In Sec. IV, the finite temperature Brownian motion of a moving mirror in the holographic setup will be studied. We then conclude and point out some future works in Sec. V.

**II. HOLOGRAPHIC SETUP FOR THE MOVING MIRROR PROBLEM**

In this section we propose a holographic setup for coupling a mirror to quantum critical theories. Here we employ the “bottom-up” method for the holography duality, where we leave the derivation of the duality in general backgrounds as an open question but are content in assuming that there is a field theory dual to the gravity setup we consider here. The gravity counterpart for the two-dimensional mirror is a three-dimensional membrane (3-brane). We consider it as a probed brane moving in the \( d + 1 \)-dimensional Lifshitz background with the metric given in \(4\). Here the nature of this 3-brane is left unspecified, while the field theory dual to Lifshitz geometry is still unclear (see \[21\] and the follow-up papers). But we expect the behaviors we found in this paper can be general for large classes of strongly coupled field theories. Let us consider another \( d - 1 \)-brane extended in all spatial
directions other than $r$ and located at $r = r_b$. We would like to interpret this as a boundary
brane where the boundary theory lives. And $1/r_b$ can be regarded as the UV cutoff in the
boundary theory [19], and its physical interpretation will be given later. Let the probed
3-brane end on the boundary $d - 1$-brane and extend in the $x_1$ and $x_2$ directions. We then
treat these two directions as spatial directions of a two-dimensional mirror and other spatial
directions $x^I$, with $I = 3, 4, \ldots, d$ as perpendicular directions to the mirror’s surface. In the
bulk, the position of the 3-brane can be parametrized (in the static gauge) by $x^I(t, r, x_1, x_2)$. We assume a rigid mirror, so $x^I$ is independent of $x_1$ and $x_2$. The 3-brane is governed by
the DBI action

$$S_{DBI} = -T_3 \int dr \, dt \, dx_1 \, dx_2 \sqrt{\det h_{ab}},$$

where $h_{ab}$ is the induced metric on the
brane and $T_3$ is the brane tension. Here, for simplicity, we have assumed the trivial dilaton
background and turn off the gauge fields on the 3-brane. In the Lifshitz geometry it reduces
to the following action:

$$S_{DBI} = -T_3 \int dr \, dt \, dx_1 \, dx_2 \, r^{z+1} \left( 1 + r^4 x'^I x'^I - \frac{\dot{x}'^I \ddot{x}'^I}{r^{2z-2}} + \frac{(x'^I x'^I) \ddot{x}'^I}{r^{2z-6}} - \frac{(x'^I x'^I) (\ddot{x}'^J \ddot{x}'^J)}{r^{2z-6}} \right), \quad (5)$$

where $x'^I = \partial_r x^I$ and $\dot{x}'^I = \partial_t x^I$ and the last expression in (5) is obtained by assuming small
variations of $x^I$ around the minimal energy configuration. Since all modes in the $I$ directions
are independent, we assume that the motion of the mirror along one of them denoted by $x$.

The equation of motion for the expectation value of $x$ in frequency space becomes

$$\frac{\partial}{\partial r} \left( r^{z+5} \frac{\partial}{\partial r} \langle x \rangle \right) + \frac{\omega^2}{r^{z-3}} \langle x \rangle = 0 . \quad (6)$$

In [7, 8], we investigate the dynamics of a perfectly reflecting mirror when it couples with the
quantum field. Their mutual coupling can be derived from the Dirichlet boundary conditions
of the field we imposed on the mirror. In the field-theoretic approach, the effective coupling
in the limit of small displacement is shown to take the form

$$\int dt \, F(t) \, X(t) , \quad (7)$$

where $X$ is the position of the mirror. The radiation pressure of the field $F(t)$ on the mirror
is given by the expectation value of the energy momentum tensor in either the vacuum or
the thermal state of the field

$$F(t) = \int dx_1 \, dx_2 \langle T_{xx} \rangle . \quad (8)$$
Here $T_{x,x}$ is the component of the energy momentum tensor of in the direction of mirror’s motion. It is found that the force arising from the quantum field cannot be evaluated infinitesimally close to the surface of the mirror due to short-distance divergences [7], which later can be resolved by introducing a fluctuating boundary [22]. Thus, the introduced $1/r_b$, a short-distance scale, naturally gives uncertainties of the location of the mirror’s surface at $r = r_b$.

In the holographic setup, the variable

$$X(t) = x(t, r = r_b) \quad (9)$$

is the boundary value of the 3-brane position. Here we assume the boundary of the probed 3-brane can have an effective coupling like the one in [7]. Additionally, we would like to emphasize that although this type of the coupling is obtained for the problem of a moving mirror influenced from radiation fields, our following holographic approach is also applied for an extended object as long as its coupling to quantum critical theory can be described by [7]. Varying the action gives

$$T_3 S r_b^{z+5} \frac{\partial \langle x \rangle}{\partial r} \bigg|_{r=r_b} = F \quad (10)$$

where $\langle x \rangle$ satisfies the equation of motion (6) and $S$ is the area of the mirror. Similar to the study of the Brownian particle in [19], we calculate the response function in this holographic setup by first solving (6) with the incoming-wave boundary condition, which is a usual holographic prescription for the retarded Green function [19]. Then the solution to the equation of motion is the Hankel function of the first kind

$$\langle x(t, r) \rangle = \frac{1}{r^{z+\frac{5}{2}}} H_{\frac{1}{2} + \frac{1}{2}}^{(1)} \left( \frac{\omega}{z r^2} \right) e^{-i\omega t}. \quad (11)$$

The force acting on the mirror can be calculated by (10):

$$F(\omega) = T_3 S \omega r_b^{-\frac{3}{2}+2} H_{\frac{1}{2} - \frac{1}{2}}^{(1)} \left( \frac{\omega}{z r_b^2} \right) e^{-i\omega t}. \quad (12)$$

The linear response to this force is described by

$$\langle X(\omega) \rangle = \chi(\omega, z) F(\omega). \quad (13)$$

Thus according to the identification (9), we find the response function

$$\chi(\omega, z) = \frac{1}{\omega r_b^5 T_3 S} \frac{H_{\frac{1}{2} + \frac{1}{2}}^{(1)} \left( \frac{\omega}{z r_b} \right)}{H_{\frac{1}{2} - \frac{1}{2}}^{(1)} \left( \frac{\omega}{z r_b} \right)}. \quad (14)$$
It is then instructive to examine the low-frequency behavior of the response function, expressed in the form

$$\chi(\omega, z) = \frac{1}{m(z)(i\omega)^2 + \mu(\omega, z)}, \quad (15)$$

where $m$ is an inertial mass and the $\mu$ term is the self-energy. The low-frequency expansion, i.e. $\omega \ll r_b^2$, gives

$$m(z) = \frac{T_3 S}{(4 - z)r_b^{\frac{8}{3}-4}}, \quad \mu(\omega, z) = \gamma(z)(-i\omega)^{1+\frac{4}{z}} + \delta(z)(-i\omega)^4 + \ldots \quad (16)$$

with

$$\gamma(z) = \frac{T_3 S}{(2z)^{4/z}} \frac{\Gamma(\frac{1}{2} - \frac{2}{z})}{\Gamma(\frac{1}{2} + \frac{2}{z})}, \quad \delta(z) = -\frac{T_3 S}{(4 - 3z)(4 - z)^{2r_b^{\frac{3}{3}-4}}} \quad (17)$$

To avoid the breakdown of a low-frequency expansion near $z = 4/3$ and $z = 4$, we have to restrict the value of $\omega$ such that in the expansion, the next order correction can not be larger than the order of interest. This restriction imposes a condition, $\omega < |(z - 4)(z - 4/3)| r_b$. Apart from $z = 4/3$ and $z = 4$, the $\gamma$ term with a frequency dependence $\omega^{1+\frac{4}{z}}$ will give the damping effect on the mirror. Additionally, the self-energy $\mu$ has a term proportional to $\omega^4$, which is the next order result in a small $\omega$ expansion. As in the case of the Brownian particle [19], the similar nonanalytic term in the self-energy, which has the power-law dependence on frequency, is also found. Since the object of interest is a two-dimensional mirror, as compared with a point particle in [19], the power of the $r$ dependence in action $S_{DBI}$ is increased by 2 to account for the additional degrees of freedom. As a result, the critical value shifts to $z = 4$. It has also been discussed in [19] that in spite that both $m$ and $\gamma$ are changed from positive to negative values when $z$ goes from $1 < z < 4$ to $z > 4$, its ratio $\gamma/m$ remains positive in a way that they still give sensible results for describing the dynamics of the mirror.

Before closing this section, let us compare the $z = 1$ case with the field theoretic calculations in [7], where the mirror is coupled to a relativistic free scalar field. Here when $z = 1$, the self-energy term obtained from a holographic approach is given by

$$\mu(\omega, z = 1) = T_3 S \left( \frac{\omega^4}{9r_b^2} + \frac{i\omega^5}{6} + \ldots \right) \quad (18)$$

with an ultraviolet energy cutoff $r_b$. The dominant terms in a small $\omega$ expansion have the same $\omega$ dependence in both cases, but different coefficients. The difference in the coefficients may lie in the fact that the environment assumed in the holographic approach is a strongly
coupled field rather than a free field in the field-theoretic approach. In particular, we observe that the coefficient of the $\gamma$ term is proportional to $T_3$, so it will increase in accordance with the coupling constant $\lambda$ of the corresponding strongly coupled boundary field. The connection that $T_3 \propto \lambda$ merely reflects the fact that the 3-brane tension $T_3$ is proportional to $\alpha'^{-2}$ that in turn can be related to the coupling strength $\lambda$ by $\lambda = L^4/\alpha'^2$ via AdS/CFT correspondence where $L$ is the curvature radius in the Lifshitz background.

III. GENERAL DIMENSION MIRRORS AND THE FLUCTUATION-DISSIPATION THEOREM

We now generalize our previous results to an $n$-dimensional mirror with its bulk counterpart as a $n + 1$-brane with the coordinates, $x^I_n(t, r, x_1, x_2, ..., x_n)$, where $I_n = n + 1, ..., d − 1$ are the directions normal to the brane. Under the same assumptions used in the 3-brane case, when the mirror moves along one of the $I_n$ directions, the corresponding equation of motion for $\langle x \rangle$ is given by

$$\frac{\partial}{\partial r} \left( r^{z+n+3} \frac{\partial}{\partial r} \langle x \rangle \right) + \frac{\omega^2}{r^{z-n-1}} \langle x \rangle = 0 . \quad (19)$$

Assuming the coupling described by a similar surface integral of the stress tensor of the fields as in (7), we then have the response function of an $n$-dimensional mirror given by

$$\chi_n(\omega, z) = \frac{1}{\omega T_{n+1} S_n} \frac{H^{(1)}_{n+2} \left( \frac{\omega}{z r_b} \right)}{H^{(1)}_{n+2} \left( \frac{\omega}{z r_b} \right) + \frac{1}{2}}, \quad (20)$$

where $S_n$ is the mirror’s surface area and $T_{n+1}$ is the $(n + 1)$-brane tension. In the low frequency limit, the response function can be cast in the form

$$\chi_n(\omega, z) = \frac{1}{m_n(z)(i\omega)^2 + \mu_n(\omega, z)} , \quad (21)$$

in which

$$m_n(z) = \frac{T_{n+1} S_n}{(n + 2 - z)r_b^{z-n-2}} , \quad \mu_n(\omega, z) = \gamma_n(z)(-i\omega)^{1+\frac{n+2}{z}} + \delta_n(z)(-i\omega)^4 + ... \quad (22)$$

with

$$\gamma_n(z) = \frac{\Gamma \left( \frac{1}{2} - \frac{n+2}{z} \right)}{(2)^{(n+2)/z} \Gamma \left( \frac{1}{2} + \frac{n+2}{z} \right)} , \quad \delta_n(\omega, z) = -\frac{T_{n+1} S_n}{(n + 2 - 3z)(n + 2 - z)^2 r_b^{z-n-2}} . \quad (23)$$
Here the critical value is changed to $z = n + 2$ as expected. The low-frequency expansion is valid as long as $\omega < \left| \frac{z - (n - 2)}{z - (n + 2)/3} \right| r_b^2$.

In the following, we will examine the long time dynamics of the mirror, in particular its saturation mechanism on velocity fluctuations.

A. Fluctuation-dissipation theorem

The stochastic behavior of the mirror is reflected by the two-point correlation function of its position. The idea of the holographic duality is to relate the two-point function of mirror’s positions to the correlation function of the position of $n + 1$-brane evaluated on the boundary of the bulk. The fluctuations associated with the mirror’s position in the holographic setup result from the fluctuations around the brane vacuum state in the bulk. In what follows, we quantize the modes normal to the $n$-dimensional mirror surface. The procedure of the canonical quantization mainly follows that in [18].

We first find the momentum conjugated to the coordinate $x$, which describes the motion of the brane, from $S_{DBI}$, straightforwardly generalized from (5), as

$$\pi(t, r) = \frac{T_{n+1}}{r^{z-1-n}} \dot{x}(t, r), \quad (24)$$

for a rigid mirror so that $x$ does not depend on $x_1, x_2, ..., x_n$. The mode expansion on the position operator $x(t, r)$ in its frequency space is given by

$$x(t, r) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} x_\omega(r) e^{-i\omega t}$$

$$= \int_0^{\infty} d\omega U_\omega(r) \left( a_\omega e^{-i\omega t} + a_\omega^\dagger e^{i\omega t} \right). \quad (25)$$

The equal-time commutation relations give the Wronskian condition of the mode functions $U_\omega$ in the Lifshitz geometry,

$$-iT_{n+1}S_n \int_0^{r_b} dr \frac{1}{r^{z-1-n}} \left\{ U_\omega(r)e^{-i\omega t}\partial_t [U_\omega(r)e^{i\omega t}] - \partial_t [U_\omega(r)e^{-i\omega t}] U_\omega(r)e^{i\omega t} \right\} = 1. \quad (26)$$

The vacuum state is annihilated by $a_\omega$ for all $\omega$ modes. The mode functions are solved from [19] with the Neumann boundary condition $x_n'(r_b, t) = 0$ and the Wronskian condition (26). Thus the two-point function associated with the rigid $D$-brane fluctuations at $r = r_b$ is obtained as

$$\langle X_\omega X_{-\omega} \rangle = \langle x_\omega(r_b) x_{-\omega}(r_b) \rangle = 2\pi U_\omega^2(r_b)$$
\begin{equation}
\frac{4zr_{b}^{2-n}}{\pi \omega^{2} T_{n+1} S_{n}} \left[ J_{\frac{n+2}{2}} - \frac{1}{2} \left( \frac{\omega}{z r_{b}^{2}} \right) + Y_{\frac{n+2}{2}} - \frac{1}{2} \left( \frac{\omega}{z r_{b}^{2}} \right) \right]^{-1}.
\end{equation}

The fluctuation-dissipation theorem relates the fluctuations of the mirror’s position to the imaginary part of its response function in (20), and can be shown to take the form
\begin{equation}
\langle X_{\omega} X_{-\omega} \rangle = 2 \text{Im} \chi_{n}(\omega, z).
\end{equation}

Evidently the bulk results can really capture the essential properties of Brownian motion in a general environment.

**B. Supraohmic behavior and velocity fluctuations**

According to the general Langevin equation (1), the effects of the environment on the system are classified according to the low-frequency behavior of the imaginary part of the self-energy \( \mu \), i.e. the friction term of the general form \( \gamma \omega^{k+1} \), as subohmic, ohmic, and supraohmic for \( k < 0 \), \( k = 0 \) and \( k > 0 \) respectively. The low-frequency expansion of the response function in (20) from the holographic approach gives \( k = (n+2)/z \), and for the positive value of \( n \) and \( z \) it corresponds to the supraohmic case. As a result of the fluctuation-dissipation relation, it is expected that the properties of the fluctuations associated with the mirror will be different from the ohmic environment, such that they result in the different mechanism for the evolution of velocity fluctuations toward their saturation.

The long-time dynamics of velocity fluctuations relies on the behavior of the correlation function in the low-frequency limit, obtained from (27), as \( \langle X_{\omega} X_{-\omega} \rangle \sim \omega^{\frac{n+2}{2}} \) for \( 1 < z < n + 2 \) and \( \langle X_{\omega} X_{-\omega} \rangle \sim \omega^{-\frac{n+2}{2}} \) for \( z > n + 2 \). Thus from the Langevin equation (1), the correlation function of the random forces is found to be
\begin{equation}
\langle R_{n;\omega} R_{n;\omega'} \rangle = \frac{\langle X_{\omega} X_{-\omega} \rangle}{\chi_{n}(\omega, z) \chi_{n}(-\omega, z)}.
\end{equation}

Then, together with the results in (21), (22), and (23), we have
\begin{equation}
\langle R_{n;\omega} R_{n;\omega'} \rangle \sim \omega^{\frac{n+2}{2}+1},
\end{equation}
in the low-frequency limit. Thus,
\begin{equation}
\int_{-\infty}^{\infty} d\tau' \langle R_{n}(\tau) R_{n}(\tau') \rangle = \int_{0}^{\infty} d\tau' \langle R_{n}(\tau) R_{n}(\tau') \rangle = \int_{-\infty}^{\infty} d\tau \eta(\tau - \tau') \propto \lim_{\omega \to 0} \langle R_{n;\omega} R_{n;\omega} \rangle = 0.
\end{equation}
We show that the integration of the force-force correlation function over the whole time regime is found vanishing. In general, the positive force-force correlation may contribute to the growth of the velocity dispersion, whereas the negative correlation may halt its growth. The cancelation between them implies that the velocity dispersion will reach a constant at asymptotical times. It certainly leads to a rather different saturation mechanism than an ohmic environment where the force-force correlation function remains positive at all times. This scenario for the supraohmic case has been discussed in [7, 8, 23]. Here we extend our precious study by considering the strongly coupled environment in quantum critical theories.

Velocity fluctuations can be computed straightforwardly as follows:

$$\langle (\delta v(t))^2 \rangle = \langle v^2(t) \rangle - \langle v(t) \rangle^2 = \int \frac{d\omega}{\pi} \omega^2 \langle X_\omega X_{-\omega} \rangle (1 - \cos \omega t).$$  \hspace{1cm} (32)

The saturated value of the velocity dispersion is

$$v_s^2 = \int \frac{d\omega}{\pi} \omega^2 \langle X_\omega X_{-\omega} \rangle = 4z^2 N \frac{r_b^{2z-2-n}}{T_{n+1} S_n},$$  \hspace{1cm} (33)

with

$$N = \int_0^{1/2z} dy \left( J_{\frac{2z+2}{2}}(y) + Y_{\frac{2z+2}{2}}(y) \right)^{-1},$$  \hspace{1cm} (34)

where we impose the frequency cutoff $r_b^z$. Using the expression of the inertial mass in (22), we have $m_n v_s^2 \sim r_b^z$ as expected on dimensional grounds. Nevertheless, the late-time saturation behavior of the velocity fluctuations follows the power law. We find that for $z > n+2$,

$$\langle (\delta v(t))^2 \rangle - v_s^2 \propto -\frac{1}{T_{n+1} S_n} (2z)^{\frac{n+2}{z}} t^{-\frac{n+2}{z}-2},$$  \hspace{1cm} (35)

and for $1 < z < n+2$,

$$\langle (\delta v(t))^2 \rangle - v_s^2 \propto -\frac{1}{T_{n+1} S_n} \left( 2z \right)^{\frac{n+2}{z}+2} r_b^{2(2+n-z)} t^{-\frac{n+2}{z}}.$$  \hspace{1cm} (36)

They are our main results in this paper. Notice that different power-law behavior in time is mainly due to the fact that the low frequency behavior of the response function is dominated, respectively, by the inertial mass term for $1 < z < n+2$ and the $\gamma$ term for $z > n+2$. The $r_b$ dependence can also be realized from the dimensional argument, and can be substituted by the inertial mass through the relation (22), which carries units with dimension $[m] = 2 - z$ in this problem.
IV. HAWKING RADIATION AND THERMAL MOTION

We now heat up the environment in this holographic model with a Lifshitz black hole background. We will study the response function and thermal fluctuations for a \( n+1 \)-brane in this background, and explicitly verify the corresponding fluctuation-dissipation theorem.

The background metric of a Lifshitz black hole in \( d+1 \) dimensions is

\[
ds^2 = -r^{2z} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2.
\]

In the low-frequency limit, the actual form of the function \( f(r) \) is irrelevant, but the function is required to satisfy the properties like \( f(r) \rightarrow 1 \) for \( r \rightarrow \infty \) and \( f(r) \simeq c(r - r_h) \) near the black hole horizon \( r_h \) with \( c = (d + z - 1)/r_h \). The temperature of the black hole and also of the boundary field theory is

\[
\frac{1}{T} = \frac{4\pi}{d + z - 1} \frac{1}{r_h^z}.
\]

The equation of motion for the expectation value of \( x_T \) moving along one of the directions normal to the \( (n+1) \)-brane in this background becomes

\[
\frac{\partial}{\partial r} \left( r^{z+n+3} f(r) \frac{\partial}{\partial r} \langle x_T \rangle \right) + \frac{\omega^2}{r^{z-n-1} f(r)} \langle x_T \rangle = 0.
\]

This equation of motion can be cast into the Schrodinger-like equation using tortoise coordinate \( r^* = \int dr f(r)^{-1}r^{z-1} \) as follows:

\[
\frac{d^2 y_T}{dr^*^2} + \left[ \omega^2 - V(r) \right] y_T = 0,
\]

where \( y_T = r^{\frac{2z}{2}+1} x_T \) and \( V(r) = r^{2z} \left( \frac{2}{2} + 1 \right) f(r) \left[ (z + 1 + \frac{n}{2}) f(r) + rf'(r) \right] \). We first find the solutions in three separate regimes specified below. The whole solution will be obtained by the matching method (see [19] and references therein). Here we just summarize the final results.

In regime (A), the near-horizon region, defined by \( r \rightarrow r_h \) and thus \( V(r) \ll \omega^2 \), the solution in the small \( \omega \) approximation is given by

\[
x_T^{(A)} \simeq D_T \left[ 1 - \frac{i \omega}{cr_h^{z+1}} \ln(r - r_h) + O(\omega^2) \right],
\]

where \( D_T \) is a constant and the in-falling boundary condition at \( r = r_h \) is chosen. In regime (B), where \( r \) takes intermediate values and \( V(r) \gg \omega^2 \), the solution is found as

\[
x_T^{(B)} \simeq D_T (1 - i \omega r_h^{n+2} k) \left[ 1 + O(\omega^2) \right] + i D_T \omega r_h^{n+2} \int_r^\infty \frac{dr'}{r'^{z+n+3} f(r')} \left[ 1 + O(\omega^2) \right],
\]
where \( \kappa \) is an \( \omega \)-independent integration constant. Finally in regime (C), as \( r \to r_b \), the equation reduces to the one in Lifshitz geometry, and its solution is the Bessel functions. In the small \( \omega \) limit, we have the expansion

\[
x^{(C)}_T \simeq i \frac{D_T}{z + n + 2} \frac{w_r n^2}{2 z - n + 2} \left[ 1 - \frac{1}{2 z - n + 2} + \frac{1}{2} (\omega/2 z r^z)^2 + \mathcal{O}(\omega^4) \right] + D_T (1 - i \omega r_h^{n+2}) \left[ 1 + \frac{1}{2 z - n + 2} + \frac{1}{2} (\omega/2 z r^z)^2 + \mathcal{O}(\omega^4) \right].
\]

Thus the response function is given by

\[
\chi_n T(\omega) = x^{(C)}_T(r_b, \omega) \frac{T_{n+1} S_n r_b^{z+n+3} x^{(C)}_T(r_b, \omega)}{m_n T(z) - \gamma_n T(z) i \omega} + O(\omega),
\]

where

\[
m_n T(z) = \frac{T_{n+1} S_n}{r_b^{z-n+2}} \left\{ \frac{1}{n + 2} + \frac{1}{z} \frac{(r_h/r_b)^{2n+4}}{n+2} \left[ (n + 2 + z) - \kappa_r^{z+n+2} \right] \right\}, \quad \gamma_n T(z) = T_{n+1} S_n r_h^{n+2}.
\]

The inertial mass \( m_n T \) and the damping coefficient \( \gamma_n T \) have the temperature dependence through the black hole temperature \((38)\). Since the damping term has linear \( \omega \) dependence, the stochastic dynamics of the mirror in the thermal environment will be expected to be ohmic.

Next we quantize the modes in this thermal background. We use the approximate solutions found above and impose the Neumann boundary condition at \( r = r_b \). Since the mode function near the horizon \( r = r_h \) exhibits logarithmic divergence, an infrared energy cutoff scale \( \epsilon \) as \( r \to r_h \) is introduced for regularization. Without proper renormalization, the result for the counterpart of \((27)\), denoted as \( \langle X T_{\omega} X T_{-\omega} \rangle \), can be pathological in the background of Lifshitz black hole. We may absorb this infrared divergence by carefully defining the density of states as \( \Delta \omega = 4 \pi^2 T / \ln(1/\epsilon) \) \((18)\). The modes expansion now becomes

\[
x_T(t, r) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2 \pi}} x_T(\omega)(r) e^{-i\omega t} = \sqrt{\frac{\ln(1/\epsilon)}{4 \pi^2 T}} \int_0^{\infty} d\omega U_T(\omega)(r) \left( a_\omega e^{-i\omega t} + a^\dagger_\omega e^{i\omega t} \right).
\]  

The corresponding Wronskian condition is

\[
- iT_{n+1} S_n \int_{r_h + \epsilon}^{r_b} dr \frac{1}{f(r)} e^{-i\omega t} \left\{ \partial_t \left[ U_T(\omega)(r) e^{i\omega t} \right] - \partial_t \left[ U_T(\omega)(r) e^{-i\omega t} \right] U_T(\omega)(r) e^{i\omega t} \right\} = 1.
\]

(47)
It is quite straightforward to find the complete solution over three separate regimes using the matching method. Imposing the Neumann boundary condition at \( r = r_b \) gives a solution with an undetermined constant \( D_T \) that can be fixed by the Wronskian condition in (47). Here we only keep the divergent parts of this integral as \( r_b \to \infty \) and \( \varepsilon \to 0 \). Thus, in the small \( \omega \) approximation, the most relevant terms in the integral of (47) come from solutions in regions (A) and (C), which take the forms,

\[
U^{(A)}_{T,\omega}(r) \simeq D_T \left[ \frac{(n + 2)^2 - z^2}{(2z)^2} (\omega/2zr_b^z)^{\frac{n+2}{2} - 1} + \frac{z + n}{d + z} - 1 \right] \left( \omega/2zr_b^z \right)^{\frac{n+2}{2}} + 1 \ln(r - r_h), \tag{48}
\]

and

\[
U^{(C)}_{T,\omega}(r) \simeq D_T \left[ \frac{(n + 2)^2 - z^2}{(2z)^2} (\omega/2zr_b^z)^{\frac{n+2}{2} - 1} \right]. \tag{49}
\]

The undetermined constant \( D_T \) is then given by

\[
|D_T|^2 = \frac{1}{2\omega T n + 1} \frac{n}{\Im(1/\varepsilon)} \left( \frac{z}{2z} \right)^{\frac{n+2}{2} - 1} \left( \frac{\omega}{\omega r_b^z} \right)^{\frac{n+2}{2}} - 1 \prod \ln(1/\varepsilon)^{n+2}, \tag{50}
\]

which apparently suffers from the \( \ln(1/\varepsilon) \) divergence. Note that in the Lifshitz black hole background, the modes of the \( D \)-brane in the bulk get excited to obey the thermal distribution \( \langle a_\omega a_\omega^\dagger \rangle = (1 - e^{-\Theta})^{-1} \). Putting all together, the leading term in the small \( \omega \) expansion of \( \langle X_{T,\omega} X_{T,\omega} \rangle \) is obtained from the mode function \( U^{(C)}_{T,\omega}(r) \) evaluated at \( r = r_b \) as

\[
\langle X_{T,\omega} X_{T,\omega} \rangle = \langle x_{T,\omega}(r_b) x_{T,\omega}(r_b) \rangle = \frac{2\pi}{1 - e^{-\Theta}} \frac{\ln(1/\varepsilon)}{4\pi^2 T} |U^{(C)}_{T,\omega}(r_b)|^2 \approx \frac{2}{\ln(1/\varepsilon)} T n + 1 S_n T r_h^{-n-2}, \tag{51}
\]

which is divergence free.

The fluctuation-dissipation theorem in the thermal environment,

\[
\langle X_{T,\omega} X_{T,\omega} \rangle = 2 \left( \frac{1}{1 - e^{-\Theta}} \right) \Im(\zeta_{nT}(\omega)), \tag{52}
\]

can be checked explicitly from the above results in their low-frequency limit. Additionally, we find that

\[
\langle R_{nT,\omega} R_{nT,\omega} \rangle = \frac{\langle X_{T,\omega} X_{T,\omega} \rangle}{\chi_{nT}(\omega, z) \chi_{nT}(\omega, z)} \approx \frac{\langle X_{T,\omega} X_{T,\omega} \rangle}{\omega^2 \gamma_{nT}^2} \approx T n + 1 S_n 2T \left( \frac{4\pi T}{d + z - 1} \right)^{n+2} \tag{53}
\]

using the above expressions of \( \gamma_{nT} \) in (46) and \( r_h \) in (38). Thus, this in turn can be translated into the white noise forces in the time domain,

\[
\langle R_{nT}(t) R_{nT}(t') \rangle = T n + 1 S_n 2T \left( \frac{4\pi T}{d + z - 1} \right)^{n+2} \delta(t - t'). \tag{54}
\]
As a result, velocity fluctuations is anticipated to evolve in the same way as in the Brownian motion in the ohmic case. They increase initially as the consequence of the energy input from the noise forces. When $t \sim \gamma^{-1}_{nT}$, the damping effect comes into the play, and then slows down their growth into saturation. The saturated value of velocity fluctuations is $v^2_{Ts} \approx T/m^2_{nT}$, where $m_{nT}$ is defined in (45). The relaxation dynamics follows an exponential behavior in time within a time scale determined by $\gamma_{nT}$ in (45). The main finding of this paper is to obtain the general result of damping in a strong coupling environment by

$$\gamma_{nT}(z) = T_{n+1} S_n \left( \frac{4\pi T^4}{d + z - 1} \right)^{n+2},$$

where from (38) we replace $r_h$ by $T$ to explicitly show the temperature dependence of the result.

In particular, for $z = 1$ (the relativistic environmental field) and for a two-dimensional mirror,

$$\gamma_T = T_3 S (2\pi T/d)^4.$$  

The ohmic dynamics and the $T^4$ dependence of $\gamma_T$ are in agreement with the findings in [7], but the proportionality constant is different between the strong coupling environment and the free field background as expected. Notice that in the strong field theory with coupling $\lambda >> 1$, it will always show enhancement, since $T_{n+1} \propto \lambda^{2+n}$ following the similar arguments in the 3-brane case.

V. SUMMARY AND OUTLOOK

In this paper, we have successfully established the holographic setup for the nonequilibrium dynamics of a moving mirror coupled to quantum critical theories. The aim of this work is to understand the quantum microphysics of nonequilibrium statistical problems via holographic duality. The mirror with perfect reflectance is realized by a $n + 1$-brane of the bulk theory in the Lifshitz geometry. The excitations of the bulk brane, due to either its vacuum state in the Lifshitz geometry or the thermal state in the Lifshitz black hole background, render the mirror undergoing Brownian motion. Nevertheless, they exhibit rather different damping behaviors, in particular, on the evolution of velocity fluctuations. The dissipation exerted on the mirror in the vacuum case is found to be supraohmic. For an initial growth of velocity fluctuations, the saturation at late times follows the power-law: when $z > n + 2$,
the saturation behavior is like $t^{\frac{n+2}{n} - 2}$, and when $1 < z < n + 2$, $t^{-\frac{n+2}{n}}$, respectively. On the contrary, in the Lifshitz black hole background, the dissipation caused by thermal excitations becomes ohmic so that the relaxation dynamics toward saturation is exponentially fast with a relaxation time scale $\propto 1/\gamma_{nT}$ where $\gamma_{nT}(z) = T_{n+1}S_n [(4\pi T)/(d + z - 1)]^{n+2}$. In the small displacement approximation, for the relativistic quantum field ($z = 1$) and a two-dimensional mirror, the dissipation/relaxation behavior of the mirror influenced from the quantum field via the holographic approach follows the same dynamics as is obtained by the field theoretic approach. However, all results based upon the holographic duality are enhanced by the brane tension $T_n$ to account for the strong coupling effects of the environment.

Finally, we would like to point out some of our future work. In view of a close relation between the holographic approach and the field-theoretical study of the nonequilibrium problems in the linear response regime, the generalized Langevin equation in [1], which can be derived from the known interactions between the system and the bath via the method of influence functional, may be obtained from holography [24]. It is then an important next step to establish this correspondence more explicitly by studying the full nonequilibrium dynamics in a strong coupling environment beyond the linear response.

Acknowledgments

This work was supported in part by the National Science Council, Taiwan.

[1] A. O. Caldeira and A. J. Leggett, “Path integral approach to quantum Brownian motion”, Physica A 121, 587 (1983).
[2] R. P. Feynman and F. L. Vernon, “The theory of a general quantum mechanical system interacting with a linear dissipative system”, Ann. Phys. (N.Y.) 24, 118 (1963); ibid. 281, 547 (2000).
[3] H. Grabert, P. Schramm, and G.-L. Ingold, “Quantum Brownian motion: The functional integral approach”, Phys. Rep. 168, 115 (1988).
[4] B. L. Hu, J. P. Paz and Y. Zang, “Quantum Brownian motion in a general environment:
Exact master equation with nonlocal and dissipation and colored noise”, Phys. Rev. D 45, 2843 (1992); B. L. Hu, J. P. Paz and Y. Zang, “Quantum Brownian motion in a general environment: II. Nonlinear coupling and perturbative approach”, Phys. Rev. D 47, 1576 (1993).

[5] J.-T. Hsiang, T.-H. Wu, and D.-S. Lee, “Stochastic Lorentz forces on a point charge moving near the conducting plate”, Phys. Rev. D 77, 105201 (2008).

[6] J.-T. Hsiang, T.-H. Wu, and D.-S. Lee, “Subvacuum effects of the quantum field on the dynamics of a test particle”, Ann. Phys. (N.Y.) 327, 522 (2012).

[7] C.-H. Wu and D.-S. Lee, “Nonequilibrium dynamics of moving mirrors in quantum fields: Influence functional and Langevin equation”, Phys. Rev. D 71, 125005 (2005).

[8] J.-T. Hsiang, T.-H. Wu, D.-S. Lee, S.-K. King and C.-H. Wu, “Quantum noise in the mirror-field system: A field theoretic approach”, Ann. Phys. (N.Y.) 329, 28 (2013).

[9] J. M. Maldacena, “The large $N$ limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)]; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory”, Phys. Lett. B 428, 105 (1998); E. Witten, “Anti-de Sitter space and holography”, Adv. Theor. Math. Phys. 2, 253 (1998).

[10] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz, and L. G. Yaffe, “Energy loss of a heavy quark moving through $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma”, J. High Energy Phys. 07 (2006), 013.

[11] S. S. Gubser, “Drag force in AdS/CFT”, Phys. Rev. D 74, 126005 (2006).

[12] J. Casalderrey-Solana and D. Teaney, “Heavy quark diffusion in strongly coupled $\mathcal{N} = 4$ Yang-Mills theory”, Phys. Rev. D 74, 085012 (2006).

[13] D. T. Son and D. Teaney, “Thermal noise and stochastic strings in AdS/CFT”, J. High Energy Phys. 07 (2009) 021.

[14] G. C. Giecold, E. Iancu, and A. H. Mueller, “Stochastic trailing string and Langevin dynamics from AdS/CFT”, J. High Energy Phys. 07 (2009) 033.

[15] R. C. Myers, A. O. Starinets, and R. M. Thomson, “Holographic spectral functions and diffusion constants for fundamental matter”, J. High Energy Phys. 11 (2007) 091.

[16] J. Casalderrey-Solana, K.-Y. Kim, and D. Teaney, “Stochastic string motion above and below the world sheet horizon”, J. High Energy Phys. 12 (2009) 066.
[17] S. Caron-Huot, P. Chesler and D. Teaney, “Fluctuation, dissipation and thermalization in non-equilibrium $AdS_5$ black hole geometries”, Phys. Rev. D 84, 026012 (2011).

[18] J. Boer, V. Hubeny, M. Rangamani and M. Shigemori, “Brownian motion in AdS/CFT”, J. High Energy Phys. 07 (2009) 094; V. Hubeny and M. Rangamani, “A holographic view on physics out of equilibrium”, Adv. High Energy Phys. 2010, 297916 (2010).

[19] D. Tong and K. Wong, “Fluctuation and dissipation at a quantum critical point”, Phys. Rev. Lett. 110, 061602 (2013).

[20] D. Rokhsar and S. Kivelson, “Superconductivity and the quantum hard-core dimer gas”, Phys. Rev. Lett. 61, 2376 (1988).

[21] S. Kachru, X. Liu and M. Mulligan, “Gravity duals of Lifshitz-like fixed points”, Phys. Rev. D 78, 106005 (2008).

[22] L. H. Ford and N. F. Svaiter, “Vacuum energy density near fluctuating boundaries”, Phys. Rev. D 58, 065007 (1998).

[23] J.-T. Hsiang, T.-H. Wu and D.-S. Lee, “Brownian motion of a charged particle under electromagnetic fluctuations at finite temperature”, Found. Phys. 41, 77 (2011).

[24] C. P. Herzog and D. T. Son, “Schwinger-Keldysh propagators from AdS/CFT correspondence”, J. High Energy Phys. 03 (2003) 046.