Stochastic wave equation with thermal noise in an expanding universe

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We discuss Einstein-Klein-Gordon system in an environment of an infinite number of scalar fields leading to an external thermal noise. In the lowest order of metric and field perturbations the quadratic fluctuations consist of a sum of quantum and thermal fluctuations. We calculate these fluctuations.

I. INTRODUCTION

We consider a model of Einstein gravity with an infinite number of scalar fields (in addition to the inflaton which generates the inflation). Such a model has been discussed in [1][2]. In a Markovian approximation the dynamics of the infinite number of fields with some unknown initial (random) values can be approximated by a random force according to the well-known scheme of Brownian motion [3]. In comparison to the standard model of inflation (cold inflation) an extra thermal noise (satisfying the fluctuation-dissipation theorem) on a general metric appears (accompanied with a friction term) on the rhs of the wave equation as an expression of the influence of an infinite number of unobserved scalar fields. The unobserved scalar fields on the rhs of Einstein equations are also replaced by a noise term in such a way that the conservation law of the total energy-momentum (required for the consistency of Einstein equations) is satisfied. As in the standard perturbation expansion [4][5][6][7][8][9] we can express the perturbations of the metric appearing in the wave equation of the inflaton by perturbations of the inflaton itself. In this way we obtain a linearized stochastic wave equation for inflaton fluctuations dependent on the thermal noise and on the solutions of the time-dependent Einstein-Klein-Gordon equations. The solution of the wave equation is a superposition of a solution of the homogeneous equation (without source) and a solution of the inhomogeneous equation with the thermal noise as a source. We quantize the solution of the homogeneous equation. Then, the quadratic fluctuations consist of quantum fluctuations and thermal fluctuations.

II. EINSTEIN EQUATIONS

The energy-momentum tensor of the inflaton $T_{\mu\nu}$ in the presence of other scalar fields is not conserved. We have to compensate the energy-momentum by means of a compensating energy-momentum $T_{de}$ which we associate with the dark sector so that

$$T_{\mu\nu} = T_{\mu\nu}^\text{tot} + T_{de}^{\mu\nu}$$

where $T_{de}^{\mu\nu}$ is the energy-momentum of the dark sector. The energy-momentum tensor of the inflaton $T_{de}$ of an ideal fluid has the form

$$T_{de}^{\mu\nu} = (\rho_{de} + p_{de})u^\mu u^\nu - g^{\mu\nu}p_{de},$$

where $\rho$ is the energy density and $p$ is the pressure. The velocity $u^\mu$ satisfies the normalization condition

$$g_{\mu\nu}u^\mu u^\nu = 1.$$ 

In the case of the inflaton we have the representation

$$u^\mu = \partial^\mu \phi (\partial^\nu \phi \partial_{\nu} \phi)^{-\frac{1}{2}},$$

$$\rho + p = \partial^\mu \phi \partial_{\mu} \phi,$$

$$p = \frac{1}{2} \partial^\mu \phi \partial_{\mu} \phi - V.$$ 

We consider flat FLWR metric

$$ds^2 = dt^2 - a^2 dx^2$$

Einstein equations are written in the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T_{\mu\nu}^\text{tot},$$

where $G$ is the Newton constant.

The Friedman equation in the FRLW flat metric has the form

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_{de}).$$

III. EXPANSION AROUND THE HOMOGENEOUS SOLUTION

The inflaton equation with a friction $\gamma$ on a flat FLWR metric (6) derived in [1][2] reads

$$\partial^2 \phi_c - a^{-2}\Delta \phi_c + (3H + 2\gamma)\partial_t \phi_c + V'(\phi_c) + \frac{3}{2}\gamma^2 H \phi_c = 0.$$ 

(9)

The inflaton equation with a noise (satisfying the fluctuation-dissipation theorem) on a general metric $g_{\mu\nu}$ according to the derivation in [1][2] takes the form

$$g^{-\frac{1}{2}}\partial_\mu g^\frac{1}{2}\partial^\nu \phi + \gamma^2 \partial_t \phi + V'(\phi) + \frac{3}{2}\gamma^2 H \phi = \gamma a^{-\frac{3}{2}}W,$$

(10)
where $g = |\det(g_{\mu\nu})|$. The white noise $W$ is the Gaussian process (related to the Brownian motion $B$) with the covariance

$$\langle W_t(x)W_s(y) \rangle dt = \langle dB_t(x)dB_s(y) \rangle = \delta(t-s)\delta(x-y)dt$$

(11)

From the definitions (2)-(5) of the inflaton energy-momentum and eq.(10) we obtain the (non)conservation law

$$(T^{\mu\nu};_{\mu}) = \partial^\nu \phi (a^{-\frac{3}{2}}W - \gamma^2 \partial_t \phi - \frac{3}{2} \gamma^2 H \phi)$$

(12)

The zero component part $T^{0\nu}$ of eq.(12) is interpreted as a stochastic differential equation in the sense of Stratonovich [10](the circle denotes the Stratonovich multiplication of the Brownian differentials $dB$)

$$d\rho + 3(1+w_1)H \rho dt = \gamma \partial_t \phi \circ a^{-\frac{3}{2}} dB$$

$$-\frac{3}{2} \gamma^2 H \phi \partial_t \phi dt - \gamma^2 (\partial_t \phi)^2 dt,$$

where

$$w_1 = \left( \frac{1}{2} (\partial_t \phi)^2 - V \right) \left( \frac{1}{2} (\partial_t \phi)^2 + V \right)^{-1}.$$  

(13)

From the conservation law

$$(T^{\mu\nu};_{\mu}) = -(T^{\mu\nu};_{\nu})$$

(15)

the compensating energy density must have the (non)conservation law with an opposite sign

$$d\rho_{de} + 3H(1+w)\rho_{de} dt = \frac{3}{2} \gamma^2 H \phi \partial_t \phi dt$$

$$+ \gamma^2 (\partial_t \phi)^2 dt - \gamma \partial_t \phi a^{-\frac{3}{2}} \circ dB,$$

where

$$w = \frac{p_{de}}{\rho_{de}}.$$  

(16)

Eqs.(16)-(17) determine $\rho_{de}$ and the energy-momentum tensor of the ideal fluid (2). We write

$$\phi = \phi_c + \delta \phi$$  

(18)

We perturb the metric $g_{\mu\nu}$ around the flat FLRW metric (6) in the uniform curvature gauge [8][9] and eliminate the perturbed metric from the Einstein-Klein-Gordon system (10). Then, in the linear approximation for the inflaton perturbation $\delta \phi$ we get the equation

$$\partial_t^2 \delta \phi - a^{-2} \Delta \delta \phi + (3H + \gamma^2) \partial_t \delta \phi$$

$$+ V''(\phi_c) \delta \phi - 6\epsilon H^2 \delta \phi + \frac{3}{2} \gamma^2 H \delta \phi = \gamma a^{-\frac{3}{2}} W_t,$$

where

$$\epsilon = -H^{-2} \partial_t H.$$  

(19)

\section{Power Spectrum of the Linearized Wave Equation}

We introduce the conformal time

$$\tau = \int dt a^{-1}.$$  

(21)

With a slowly varying $H$ we have approximately

$$aH = -(1 - \epsilon)^{-1} \frac{1}{\tau}.$$  

(22)

In terms of $\tau$ eq.(19) for the Fourier transform $\delta \phi(k)$ reads (\textit{k} = |\textbf{k}|)

$$\left( \partial^2_{\tau} - \frac{2 - 3\epsilon}{1 - \epsilon} \frac{1}{\tau} \partial_{\tau} + k^2 + \frac{3\eta - 6\epsilon}{(1 - \epsilon)^2} \frac{1}{\tau^2} \right) \delta \phi = \gamma W_{\tau},$$

(23)

where

$$3\eta = V'' H^{-2}.$$  

(24)

and

$$\Gamma = \frac{\gamma^2}{3H}.$$  

(25)

Let

$$\delta \phi = \tau^\alpha \Psi$$

(26)

with

$$\alpha = \frac{1 - \frac{3}{2} \Gamma}{1 - \epsilon}.$$  

(27)

Then

$$\left( \partial^2_{\tau} + k^2 + \frac{-2 + 3\eta - 5\epsilon - 2\Gamma(1 - 2) - \frac{3}{2} \Gamma}{(1 - \epsilon)^2} \frac{1}{\tau^2} \right) \Psi = \gamma \tau^{-\alpha} W_{\tau}.$$  

(28)

The lhs of this equation agrees with Bassett et al [7] for $\gamma = 0$. Let us still use another form of the stochastic equation. Let

$$\delta \phi = \tau^\beta \Phi$$

(29)

with

$$\beta = (1 - \epsilon)^{-1} \frac{3}{2} - \frac{\epsilon}{2} - \frac{3}{2} \Gamma.$$  

(30)

Then

$$\left( \partial^2_{\tau} + \tau^{-1} \partial_{\tau} + (k^2 - \nu^2 \tau^{-2}) \right) \Phi = \gamma \tau^{-\beta} W_{\tau}.$$  

(31)

where

$$\nu^2 = (1 - \epsilon)^{-2} \left( \frac{3}{2} - \frac{\epsilon}{2} - \frac{3}{2} \Gamma \right)^2 - 3\eta + 6\epsilon.$$  

(32)

Without noise the solution of eq.(31) is the Hankel function $H_{\nu}(k\tau)$. The solution of eq.(28) without noise is $\psi_{\nu} = \tau^{-\alpha + \beta} H_{\nu}$. Then, the solution of the stochastic wave equation for $\Psi$ is
\[
\Psi(\zeta) = \gamma k^{-2} \int_\zeta^\infty G(\zeta, \zeta') (\zeta')^{\nu - \alpha} k^\alpha \sqrt{k} W_{\zeta'} d\zeta'
\]

where \( G \) is the Green function and

\[
\zeta = k\tau
\]

The Green function can be constructed from the two independent solutions of the homogeneous equation (28) (without noise)

\[
\psi_1 = \zeta^{-\alpha + \beta} J_\nu(\zeta)
\]

\[
\psi_2 = \zeta^{-\alpha + \beta} Y_\nu(\zeta)
\]

where the Bessel functions \( J \) and \( Y \) can be defined by the Hankel function \( H_\nu^{(1)} = J_\nu + iY_\nu \). The Green function for \( \zeta < \zeta' \) is

\[
G(\zeta, \zeta') = w^{-1}(\psi_1(\zeta)\psi_2(\zeta') - \psi_2(\zeta)\psi_1(\zeta'))
\]

where the constant \( w \) is the wronskian. The solution (33) satisfies the boundary condition at \( \zeta = \infty \). Namely, \( \Psi(\infty) = 0 \) and \( \partial_\zeta \Psi(\zeta)(\infty) = 0 \) because \( G(\zeta, \zeta) = 0 \) (these boundary conditions are imposed at \( ka = 0 \) according to the definition of \( \tau \) in eq.(22)). The classical system (9) is non-Hamiltonian. Hence, strictly speaking cannot be quantized by means of the standard methods. Its proper way of quantization is by means of the Lindblad theory of quantum dissipative systems. However, for a small dissipation \( \gamma \) and small \( H \) the wave equation (9) can be transformed into a wave equation of a harmonic oscillator with a time dependent frequency. The quantization of this oscillator determines the quadratic fluctuations

\[
\langle \delta \phi_\eta^2 \rangle \simeq \tau^{2\beta} |H^{(1)}_\nu(\zeta)|^2 \simeq k^{-2\nu}
\]

(38)

(for a small \( k \)). If \( \Gamma \) is small then from eq. (32)

\[
\nu = \frac{3}{2} + 3\epsilon - \eta - \frac{3}{2} \Gamma
\]

(39)

At \( \Gamma = 0 \) the formula (39) coincides with the well-known result. The thermal fluctuations are calculated from eq.(33)

\[
\langle \delta \phi_{th}^2 \rangle = k^{-3} \int_\zeta^\infty (\zeta')^{2\alpha} \left( G(\zeta, \zeta') \right)^2 d\zeta'
\]

(40)

For small \( k \) we can set the lower limit in the integral (40) to zero. Then

\[
\langle \delta \phi_{th}^2 \rangle \simeq k^{-3} \int_\zeta^\infty (\zeta')^{2\alpha} \left( G(\zeta, \zeta') \right)^2 d\zeta'
\]

(41)

For a small \( \Gamma \) we have from eqs.(30) and (32)

\[
\langle \delta \phi_{th}^2 \rangle \simeq k^{-3 - 4\epsilon + 2\eta}
\]

(42)

(for a small \( \Gamma \) the dependence of \( 2\beta - 2\nu \) on \( \Gamma \) surprisingly cancels).

V. SUMMARY

We have calculated fluctuations in an Einstein-Klein-Gordon system interacting with an environment (described by a Gaussian noise). The resulting stochastic equations took a form depending on the special environment of scalar fields interacting linearly with the inflaton (in particular this leads to the \( \frac{1}{2}\gamma^2 H \) term in eq.(10)). The fluctuations arise from quantization as well as from the interaction with the environment (thermal fluctuations). In an inflationary expansion both fluctuations are nearly scale invariant. The spectral indices in eqs.(38) and (41) are close to each other. The amplitude of thermal fluctuations depends on the friction \( \gamma \) (there are some estimates on \( \gamma \) in the warm inflation models [11]). If this amplitude is of the same order as the quantum one then it may be difficult to distinguish on the basis of CMB measurements quantum and thermal fluctuations. The procedure which we have applied to calculate the inflaton fluctuations follows the standard one [7][8][9] using an expression of metric fluctuations by inflaton fluctuations. Our results for the spectral index are different from earlier results of ref.[12]. The reason may be that the authors [12] do not base their methods solely on the Einstein-Klein-Gordon equations but on some thermodynamic arguments which may involve another form of the energy-momentum tensor (the noise on the rhs of the wave equation enforces a modification of the energy-momentum tensor in order to satisfy the conservation law).

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