SPECIAL LAGRANGIAN 4-FOLDS WITH $SO(2) \rtimes S_3$-SYMMETRY IN COMPLEX SPACE FORMS

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ABSTRACT. In this article we obtain a classification of special Lagrangian submanifolds in complex space forms subject to an $SO(2) \rtimes S_3$-symmetry on the second fundamental form. The algebraic structure of this form has been obtained by Marianty Ionel in [7]. However, the classification of special Lagrangian submanifolds in $\mathbb{C}^4$ having this $SO(2) \rtimes S_3$ symmetry in [7] is incomplete. In this paper we give a complete classification of such submanifolds, and extend the classification to special Lagrangian submanifolds of arbitrary complex space forms with $SO(2) \rtimes S_3$-symmetry.

1. Introduction

A space $(N, J, g)$ is called a Hermitian manifold with complex structure $J$ and Riemannian metric $g$, if $g(JX, JY) = g(X, Y)$ for all $X$ and $Y$. The $(0, 2)$-tensor $\omega(X, Y) = g(X, JY)$ is its symplectic form. If $\omega$ is closed, then $(N, J, g)$ is said to be a Kähler manifold. In this case the Levi-Civita connection $D$ of $g$ satisfies $D\omega = 0$ as well, see [1]. A complex space form is a Kähler manifold for which the curvature tensor is given by

$$R(X, Y)Z = \epsilon (X \wedge Y + JX \wedge JY + 2g(X, JY)J) Z,$$

where $\epsilon$ is a real constant and $X \wedge Y$ is defined as

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Every complete, simply connected complex space form of dimension $n$ with constant holomorphic sectional curvature $4\epsilon$ is isometric to one of the following manifolds:

1. the standard complex space $\mathbb{C}^n$ when $\epsilon = 0$,
2. the complex projective space $\mathbb{C}P^n(4\epsilon)$ when $\epsilon > 0$,
3. the complex hyperbolic space $\mathbb{C}H^n(4\epsilon)$ when $\epsilon < 0$.

Because we consider submanifolds of a complex space form locally, we can restrict ourselves to those ambient spaces. By rescaling, we can even assume that $\epsilon = 0, 1, -1$.

A Lagrangian submanifold $M$ of a Kähler manifold $(N, J, g)$ is a submanifold such that $\omega$ vanishes identically on $M$ and the (real) dimension of $M$ is half the (complex) dimension of $N$, see [1]. This implies that $J$ induces an orthogonal isomorphism between the tangent and the normal bundle on the submanifold. The Gauss formula is given by

$$D_X Y = \nabla_X Y + h(X, Y) = \nabla_X Y + JA(X, Y),$$

where $A = -Jh$ defines a symmetric $(1, 2)$-tensor on the submanifold, and the Weingarten formula is given by

$$D_X (JY) = J(\nabla_X Y) - A(X, Y).$$

It is easy to see that the cubic form $C$, defined by

$$C(X, Y, Z) = g(A(X, Y), Z)$$
is totally symmetric. For Lagrangian submanifolds of complex space forms, the equations of Gauss and Codazzi simplify to

\begin{align}
R(X, Y)Z &= \epsilon (X \wedge Y) Z + [A_X, A_Y] Z, \\
\nabla A &= \text{symmetric.}
\end{align}

The following theorem holds, see [3] and [5].

**Theorem 1.1.** Suppose \((M^n, g)\) is a Riemannian manifold equipped with a symmetric and \(g\)-symmetric \((1, 2)\)-tensor \(A\) such that (2) and (3) are satisfied for some constant \(\epsilon\). Then for every point \(p \in M\) there exists a neighborhood \(U\) and a Lagrangian isometric immersion \(\phi : U \to N^{2n}(4\epsilon)\) into the complex space form \(N^{2n}(4\epsilon)\) such that \(g\) and \(JA\) are induced as first and second fundamental form. Such an immersion is unique up to isometries of the ambient space.

We focus on a particular form of \(A\) assuming that there is a pointwise \(G\)-symmetry of \(A\) (or equivalently of the cubic form \(C\)), where \(G\) is a subgroup of the special orthognal group \(SO(n)\). We say that \(A\) has pointwise \(G\)-symmetry at \(p\) if for all tangent vectors \(X, Y\) in \(p\), and all \(g \in G\) the relation \(A(gX, gY) = gA(X, Y)\) holds (or equivalently \(C(gX, gY, gZ) = C(X, Y, Z)\) for all \(X, Y, Z\)). Furthermore, we impose a minimality condition on \(A\) at \(p\), so for every \(X\) at \(p\), we assume that \(\text{Tr}(A_X) = 0\). These manifolds are interesting, since in \(\mathbb{C}^n\) the minimal Lagrangian submanifolds are precisely the special Lagrangian submanifolds of \(\mathbb{C}^n\) as introduced by Harvey and Lawson [6]. If a special Lagrangian submanifold of \(\mathbb{C}^n\) has \(G\)-symmetry at every point, for the same group \(G\), then a classification result for the dimension equal to 3 has been obtained by Bryant [2]. An explicit classification for special (we also use the word “special” for “minimal” in case \(c \neq 0\)) Lagrangian submanifolds of complex space forms with pointwise symmetric cubic form is not yet done, but can be easily obtained from a similar classification for affine spheres in [13].

In the present paper we consider the 4-dimensional case. In particular we consider special Lagrangian 4-folds in complex space forms with pointwise symmetry. The shape of the \((1, 2)\)-tensor \(A\), invariant under subgroups of \(SO(4)\), has been described by M. Ionel in [7]. In the same article, the author classifies special Lagrangian 4-folds of \(\mathbb{C}^4\) according to their symmetry groups. However, the classification in case the symmetry group is given by \(SO(2) \rtimes S_3\) in that article is incomplete; several possible subcases including the most general one is omitted. In the present article, we give a complete classification of all special Lagrangian 4-folds in any complex space form having this particular symmetry. This settles the problem for \(SO(2) \rtimes S_3\)-symmetry for all \(\epsilon\). The classification for other symmetry groups remains open if \(\epsilon \neq 0\).

The \(SO(2) \rtimes S_3\)-symmetry implies that \(A\) can be expressed as

\begin{align}
A(X_1, X_1) &= rX_1, & A(X_1, X_2) &= -rX_2, & A(X_1, X_3) &= 0, & A(X_1, X_4) &= 0, \\
A(X_2, X_1) &= -rX_2, & A(X_2, X_2) &= -rX_1, & A(X_2, X_3) &= 0, & A(X_2, X_4) &= 0, \\
A(X_3, X_1) &= 0, & A(X_3, X_2) &= 0, & A(X_3, X_3) &= 0, & A(X_3, X_4) &= 0, \\
A(X_4, X_1) &= 0, & A(X_4, X_2) &= 0, & A(X_4, X_3) &= 0, & A(X_4, X_4) &= 0,
\end{align}

in a well-chosen local orthonormal frame \(\{X_1, X_2, X_3, X_4\}\). In this expression \(r\) is a strictly positive function. The \(SO(2)\)-symmetry is given by the free rotation in the \(\{X_3, X_4\}\) plane and the \(S_3\)-symmetry is essentially obtained by rotations over an angle \(2\pi/3\) in the \(\{X_1, X_2\}\) plane and reflections in the \(\{X_1, X_4\}\) plane. We can remark that the form of \(A\)
is exactly that of Lagrangian submanifolds attaining equality in Chen’s inequality, see [4] and [5].

In order to list the different possible subcases, we introduce distributions

\[ N_1 = \text{span}\{X_1, X_2\}, \quad N_+ = \text{span}\{X_1, X_2, [X_1, X_2]\}, \quad N_2 = \text{span}\{X_3, X_4\}. \]

We will see that \( N_2 \) is always integrable. We obtain:

1. If \( N_1 = N_+ \), then the submanifold is a double warped product \( \mathbb{R} \times f \mathbb{R} \times g N^2 \) where \( N^2 \) is a minimal Lagrangian submanifold in an appropriate space form.

2. If \( N_1 \subset N_+ \) and \( N_+ \) is integrable, then the submanifold is a single warped product \( \mathbb{R} \times f N^3 \) where \( N^3 \) is a special Lagrangian 3-fold with \( S_3 \)-symmetry in an appropriate space form.

3. If the smallest integrable distribution containing \( N_1 \) is \( TM \), then for this final case, we do not obtain an explicit expression for the immersion, but we will rewrite the equations (7) to a system of partial differential equations in 2 coordinates out of 4 coordinates defined on the submanifold. Here, techniques will be used similar to those in [8].

When we consider the different cases, we will assume the defining conditions hold on an open neighborhood of the considered point.

2. Preliminaries

2.1. Complex space forms. We briefly recall the basic properties of \( \mathbb{C}^n \) and show how Lagrangian submanifolds of \( \mathbb{C}P^n \) and \( \mathbb{C}H^n \) can be lifted to subsets of \( \mathbb{C}^{n+1} \).

Consider the complex vector space \( \mathbb{C}^n \). Its elements can be written as n-tuples of complex numbers, so they are given as

\[ \vec{z} = (z_1, \cdots, z_n), \quad z_j = x_j + iy_j, \quad x_j, y_j \in \mathbb{R}. \]

Through the map

\[ \phi : \mathbb{C}^n \to \mathbb{R}^{2n} : (z_1, \cdots, z_n) \to (x_1, y_1, \cdots, x_n, y_n) \]

the space \( \mathbb{C}^n \) is a real 2n-dimensional manifold. The multiplication with the imaginary unit \( i \) translates to a linear map on \( \mathbb{R}^{2n} \) given as

\[ i (x_1, y_1, \cdots, x_n, y_n) = (-y_1, x_1, \cdots, -y_n, x_n). \]

and its derivative \( J \) is given as

\[ J \partial_{x_k} = \partial_{y_k}, \quad J \partial_{y_k} = -\partial_{x_k}. \]

This squares to \(-I\) and thus defines a complex structure on \( \mathbb{C}^n \). On \( \mathbb{C}^n \) there is also a Hermitian form given by

\[ s(\vec{z}, \vec{w}) = \sum_{j=1}^n z_j \bar{w}_j = \sum_{j=1}^n (x_j u_j + y_j v_j) - i \sum_{j=1}^n (x_j v_j - y_j u_j). \]

The real part, which can be denoted as \( \langle \vec{z}, \vec{w} \rangle \) defines the Euclidean scalar product on \( \mathbb{R}^{2n} \) and induces a natural Riemannian metric on \( \mathbb{C}^n \). We can see that \( J \) is an isometry and the induced Kähler form, which also coincides with the imaginary part of the Hermitian form, is closed. These structures make \( \mathbb{C}^n \) into a flat Kähler manifold.
The manifold $\mathbb{C}P^n$ can be modeled as the quotient $S^{2n+1}/S^1$, where

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | \sum_{i=0}^{n} |z_i|^2 = 1\}.$$

The equivalence is given by

$$\vec{z} \sim \vec{w} \iff \exists \phi \in \mathbb{R} \forall j \in \{0, \cdots, n\} : z_j = e^{i\phi}w_j.$$

So the unit sphere $S^{2n+1}$ is the preimage of the Hopf fibration

$$\pi : S^{2n+1} \to \mathbb{C}P^n : \vec{z} \to [\vec{z}].$$

On $S^{2n+1} \subset \mathbb{C}^{n+1}$ the complex structure $J$ induces a contact structure and the standard metric on $\mathbb{C}^{n+1}$ induces a Riemannian metric. The metric on $\mathbb{C}P^n$ that makes $\pi$ a Riemannian submersion has constant holomorphic sectional curvature $4$. An immersion $\phi : M \to S^{2n+1}$ is then said to be C-totally real or horizontal if $i\phi$ is orthogonal to the submanifold. It can be shown that every minimal C-totally real submanifold of $S^{2n+1}$ can be projected onto a special Lagrangian submanifold of $\mathbb{C}P^n$ through $\pi$ and conversely that a special Lagrangian submanifold in $\mathbb{C}P^n$ has a 1-parameter family of mutually isometric horizontal lifts as a minimal C-totally real submanifold in $S^{2n+1}$. So in order to classify special Lagrangian submanifolds in $\mathbb{C}P^n$, we can consider minimal C-totally real submanifolds in $S^{2n+1} \subset \mathbb{C}^{n+1}$, see [12]. For those submanifolds, the Gauss identity is given as

$$D_XY = \nabla_XY + JA(X, Y) - \langle X, Y \rangle \phi,$$

where $D$ is the Levi Civita connection of $\mathbb{C}^{n+1}$.

Similarly, the space $\mathbb{C}H^n$ can be modeled as $H^{2n+1}/S^1$, where

$$H^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}_1^{n+1} ||z_0|^2 - \sum_{i=1}^{n} |z_i|^2 = 1\}.$$

The equivalence relationship determined by $S^1$ is the same as the one used in the projective space. The ambient space $\mathbb{C}_1^{n+1}$ is essentially the space $\mathbb{C}^{n+1}$, but equipped with the scalar product

$$\langle \vec{z}, \vec{w} \rangle_1 = \Re \left( \sum_{j=1}^{n} z_j \bar{w}_j - z_0 \bar{w}_0 \right).$$

The complex structure is still obtained through multiplication with the imaginary unit $i$ and induces a Kähler structure on $\mathbb{C}^{n+1}$. This metric induces a Lorentzian metric on $H^{2n+1}$ and a metric of constant holomorphic sectional curvature $-4$ on $\mathbb{C}H^n$. Similar to the projective case C-totally real submanifolds $\phi : M \to H^{2n+1}$ can be defined having $i\phi$ as a normal. Each minimal C-totally real submanifold corresponds to the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}H^n$. The Gauss identity is given as

$$D_XY = \nabla_XY + JA(X, Y) + \langle X, Y \rangle \phi,$$

where $D$ is the Levi Civita connection of $\mathbb{C}^{n+1}$.
2.2. Structure equations. We can return briefly to the equations (2) and (3). We can choose an orthogonal frame \( \{X_1, X_2, X_3, X_4\} \) corresponding to (4) and define the components \( \Gamma^k_{ij} \) and \( A^k_{ij} \) as

\[
\nabla_{X_i}X_j = \sum_{k=1}^{4} \Gamma^k_{ij}X_k,
\]

\[
A(X_i, X_j) = \sum_{k=1}^{4} A^k_{ij}X_k.
\]

Then the equations (2) and (3) can be rewritten as

\[
X_i (\Gamma^l_{jk}) - X_j (\Gamma^l_{ik}) = \epsilon (\delta_{jk} \delta^l_i - \delta_{ik} \delta^l_j) + A^r_{jk} A^l_{ir} - A^r_{ik} A^l_{jr} + \Gamma^r_{lk} \Gamma^l_{jr} - \Gamma^r_{jl} \Gamma^l_{ir} + \Gamma^l_{ik} \Gamma^l_{jr} - \Gamma^l_{ij} \Gamma^l_{kr},
\]

(7)

\[
X_i (A^l_{jk}) - X_j (A^l_{ik}) = (\Gamma^r_{ij} - \Gamma^r_{ji}) A^l_{rk} + \Gamma^r_{ik} A^l_{jr} - \Gamma^r_{jr} A^l_{ir} + \Gamma^l_{ir} A^l_{jk} + \Gamma^l_{jr} A^l_{ik},
\]

(8)

where we have used the Einstein convention. We split the connection \( \nabla \) into its components and write

\[
\nabla_{X_1}X_1 = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4,
\]

\[
\nabla_{X_2}X_1 = b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4,
\]

\[
\nabla_{X_3}X_1 = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4,
\]

\[
\nabla_{X_4}X_1 = d_1X_1 + d_2X_2 + d_3X_3 + d_4X_4,
\]

\[
\nabla_{X_1}X_2 = -a_1X_1 + a_4X_3 + a_5X_4,
\]

\[
\nabla_{X_2}X_2 = -b_1X_1 + b_4X_3 + b_5X_4,
\]

\[
\nabla_{X_3}X_2 = -c_1X_1 + c_4X_3 + c_5X_4,
\]

\[
\nabla_{X_4}X_2 = -d_1X_1 + d_4X_3 + d_5X_4,
\]

\[
\nabla_{X_1}X_3 = -a_2X_1 + a_4X_2 + a_6X_4,
\]

\[
\nabla_{X_2}X_3 = -b_2X_1 + b_4X_2 + b_6X_4,
\]

\[
\nabla_{X_3}X_3 = -c_2X_1 + c_4X_2 + c_6X_4,
\]

\[
\nabla_{X_4}X_3 = -d_2X_1 + d_4X_2 + d_6X_4,
\]

\[
\nabla_{X_1}X_4 = -a_3X_1 + a_5X_2 + a_6X_3,
\]

\[
\nabla_{X_2}X_4 = -b_3X_1 + b_5X_2 + b_6X_3,
\]

\[
\nabla_{X_3}X_4 = -c_3X_1 + c_5X_2 + c_6X_3,
\]

\[
\nabla_{X_4}X_4 = -d_3X_1 + d_5X_2 + d_6X_3.
\]

Equation (8) induces linear relations between the components, independent of the ambient space. The Gauss equations give further information about \( \nabla \) but use differential equations and depend on the ambient space form.

Lemma 2.1. On a special Lagrangian submanifold \( M \) having a local \( SO(2) \times S_3 \)-symmetry there exists a frame corresponding to (4) such that:

\[
\nabla_{X_1}X_1 = a_1X_2 + a_2X_3 + a_3X_4,
\]

\[
\nabla_{X_2}X_1 = b_1X_2 + b_2X_3 + b_3X_4,
\]

\[
\nabla_{X_3}X_1 = b_2X_2 + b_3X_3 + b_4X_4,
\]

\[
\nabla_{X_4}X_1 = 0,
\]

\[
\nabla_{X_1}X_2 = -a_1X_1 - b_2X_3,
\]

\[
\nabla_{X_2}X_2 = -b_1X_1 + a_2X_3 + a_3X_4,
\]

\[
\nabla_{X_3}X_2 = -b_3X_1 + b_5X_2 - b_6X_3,
\]

\[
\nabla_{X_4}X_2 = -d_3X_1 + d_5X_2 - d_6X_3,
\]

\[
\nabla_{X_1}X_3 = -a_2X_1 - a_4X_2 + a_6X_4,
\]

\[
\nabla_{X_2}X_3 = -b_2X_1 - a_2X_2 + b_6X_4,
\]

\[
\nabla_{X_3}X_3 = -c_2X_1 - c_4X_2 + c_6X_4,
\]

\[
\nabla_{X_4}X_3 = -d_2X_1 - d_4X_2 + d_6X_4,
\]

\[
\nabla_{X_1}X_4 = -a_3X_1 - a_5X_2 - a_6X_3,
\]

\[
\nabla_{X_2}X_4 = -b_3X_1 - b_5X_2 - b_6X_3,
\]

\[
\nabla_{X_3}X_4 = -c_3X_1 - c_5X_2 - c_6X_3,
\]

\[
\nabla_{X_4}X_4 = -d_3X_1 - d_5X_2 - d_6X_3.
\]

Furthermore, the derivatives of \( r \) are given by

\[
(X_1 + iX_2)(r) = 3ir(a_1 + ib_1),
\]

(10)

\[
X_3(r) = ra_2,
\]

(11)

\[
X_4(r) = ra_3.
\]

(12)
Proof. This is just a straightforward application of equation (8). For instance
\[(\nabla_{X_2} A)(X_1, X_1) = X_2(r)X_1 + 3rb_1X_2 + rb_2X_3 + rb_3X_4,
(\nabla_{X_1} A)(X_2, X_1) = 3ra_1X_1 - X_1(r)X_2 - ra_4X_3 - ra_5X_4.\]

Then the corresponding coordinates of both derivatives are the same. Finally we can set \(b_3 = 0\), by rotating the distribution \(N_2\) such that \(X_3\) lies in the direction of \(\nabla_{X_1} X_2\), projected on \(N_2\).

\[\square\]

It is interesting to note that \(N_2\) is an integrable distribution. The distribution \(N_1\) however is integrable if and only if \(b_2 = 0\). Applying (7), we obtain the following result.

**Lemma 2.2.** The equations (7) on our frame of choice induce a system of differential equations given by:

\[\begin{align*}
(13) & \quad (X_1 + iX_2)(a_2 - ib_2) = a_3(a_6 + ib_6), \\
(14) & \quad X_3(a_2 + ib_2) = \epsilon + a_3^2 + (a_2 + ib_2)^2, \\
(15) & \quad X_4(a_2 + ib_2) = a_3(a_2 + ib_2), \\
(16) & \quad (X_1 + iX_2)(a_3) = -(a_2 - ib_2)(a_6 + ib_6), \\
(17) & \quad X_3(a_3) = 0, \\
(18) & \quad X_4(a_3) = a_3^2 + \epsilon, \\
(19) & \quad X_1(b_6) - X_2(a_6) = -(a_1a_6 + b_1b_6), \\
(20) & \quad X_3(a_6 + ib_6) = \frac{5}{3}ib_2(a_6 + ib_6), \\
(21) & \quad X_4(a_6 + ib_6) = 2a_3(a_6 + ib_6), \\
(22) & \quad X_1(b_1) - X_2(a_1) = 2r^2 - (\epsilon + a_3^2) - \frac{5}{3}b_2^2 - a_2^2 - a_1^2 - b_1^2, \\
(23) & \quad X_2(b_1) + X_1(a_1) = -\frac{2}{3}a_2b_2, \\
(24) & \quad 3X_3(a_1) - X_1(b_2) = 3a_1a_2 - 2b_1b_2, \\
(25) & \quad 3X_3(b_1) - X_2(b_2) = 2b_2a_1 + 3b_1a_2, \\
(26) & \quad X_4(a_1 + ib_1) = a_3(a_1 + ib_1) + \frac{b_2}{3}(a_6 + ib_6).
\end{align*}\]

**Proof.** This is also a straightforward application of (7). For example:

\[\begin{align*}
X_1(\Gamma^1_{23}) \times X_2(\Gamma^1_{13}) &= \Gamma^r_{13}\Gamma^1_{2r} - \Gamma^r_{23}\Gamma^1_{1r} + \Gamma^1_{r3}\Gamma^r_{12} - \Gamma^1_{r3}\Gamma^r_{21} \\
&= a_3b_6, \\
X_1(\Gamma^2_{23}) \times X_2(\Gamma^2_{13}) &= \Gamma^r_{13}\Gamma^2_{2r} - \Gamma^r_{23}\Gamma^2_{1r} + \Gamma^2_{r3}\Gamma^r_{12} - \Gamma^2_{r3}\Gamma^r_{21} \\
&= -a_3a_6.
\end{align*}\]

Combining both equations using the usual complex notations leads to (13). The other equations are obtained in a similar way. \[\square\]

### 2.3. Warped Products

In the analysis that follows, we will often encounter warped products of manifolds. When we consider a warped product of Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) with warping function \(e^f\), where \(f : M_1 \to \mathbb{R}\), we get a Riemannian
manifold \( (M_1 \times M_2, g_f) \) where \( M_1 \times M_2 \) as a differentiable manifolds is the product of \( M_1 \) and \( M_2 \) and the metric \( g_f \) is given as
\[
g_f(X, Y) = g_1(X_1, Y_1) + e^{2f} g_2(X_2, Y_2),
\]
where a vector field \( X \) is uniquely decomposed into a part \( X_1 \) tangent to \( M_1 \) and \( X_2 \) tangent to \( M_2 \). We denote this warped product as \( M_1 \times_{e^f} M_2 \). The following result can be obtained, see [9].

**Theorem 2.1.** Consider a Riemannian manifold \( (M, g) \) with Levi-Civita connection \( \nabla \) and suppose that on a neighborhood of \( p \in M \) there are orthogonal distributions \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) such that
\[
\forall X, Y \in \mathcal{N}_1 (i.e. \ X \ and \ Y \ are \ sections \ of \ \mathcal{N}_1) : \nabla_X Y \in \mathcal{N}_1,
\]
\[
\forall X, Y \in \mathcal{N}_2 : \nabla_X Y = \tilde{\nabla}_X Y + g(X, Y)H,
\]
where \( \tilde{\nabla} \) is the projection of \( \nabla \) on \( \mathcal{N}_2 \) and \( H \in \mathcal{N}_1 \). Then there exists a function \( f : M \rightarrow \mathbb{R} \) such that on a neighborhood of \( p \), \( M \) can be written as \( M_1 \times_{e^f} M_2 \), where \( M_i \) is an integral manifold of \( \mathcal{N}_i \).

If furthermore \( H = \lambda H_0 \), where \( \|H_0\| = 1 \), and \( X(\lambda) = 0 \) for every \( X \in \mathcal{N}_2 \), then \( f : M_1 \rightarrow \mathbb{R} \) and \( H = -\text{grad} e^f \).

The first part of the theorem constructs a twisted product, the second part reduces this to a warped product. This will be useful in choosing coordinates, since the product structures allows for coordinates to be chosen on each factor separately. In particular, if \( \dim(\mathcal{N}_1) = 1 \), then any non vanishing vector field in \( \mathcal{N}_1 \) can be fixed as a useful coordinate vector field on \( M \).

3. **Submanifolds in \( \mathbb{C}^4 \).**

3.1. **The case where** \( b_2 = 0 \). The assumption that \( X_3 \) lies along \( \nabla_X_1 X_2 \) becomes redundant since the latter has no \( \mathcal{N}_2 \) component. Instead, we can choose \( X_3 \) in the direction of \( \nabla_X_1 X_1 \), projected on \( \mathcal{N}_2 \). Hence without loss of generality we can assume that \( a_3 = 0 \). The equations (7) show that in this case either \( a_2 = 0 \) or \( a_6 = b_6 = c_6 = d_6 = 0 \). First we will assume that \( a_2 \neq 0 \).

**Theorem 3.1.** Consider \( M \) a special Lagrangian submanifold in \( \mathbb{C}^4 \) having \( SO(2) \times S_3 \)-symmetry and an orthogonal frame corresponding to (4). Suppose that \( \mathcal{N}_1 \) is an integrable distribution and \( \nabla_{X_1} X_1 \) is nowhere contained within this distribution. Then \( M \) is locally congruent to
\[
F(t, s, u, v) = (t, s\phi(u, v))
\]
where \( \phi \) is the horizontal lift of a special Lagrangian submanifold of \( \mathbb{C}P^2 \) to the unit sphere in \( \mathbb{C}^3 \).
Proof. Taking into account every component that vanishes in (9), we find

\[
\begin{array}{c|c}
\nabla X_i X_1 = a_1 X_1 + a_2 X_3 & \nabla X_i X_2 = -a_1 X_1, \\
\nabla X_2 X_1 = b_1 X_2 & \nabla X_2 X_2 = -b_1 X_1 + a_2 X_3, \\
\nabla X_3 X_1 = 0 & \nabla X_3 X_2 = 0, \\
\n\nabla X_4 X_1 = 0 & \nabla X_4 X_2 = 0, \\
\n\nabla X_i X_3 = -a_2 X_1 & \nabla X_i X_4 = 0, \\
\n\nabla X_2 X_3 = -a_2 X_2 & \nabla X_4 X_4 = 0, \\
\n\nabla X_3 X_3 = 0 & \nabla X_4 X_3 = 0.
\end{array}
\]

We find that the distributions span\{X_1\} and span\{X_1, X_2, X_3\} satisfy the conditions for a warped product $\mathbb{R} \times f N^3$. But $X_4(f) = 0$, hence $f$ is a constant. $M$ is a standard Riemannian product $\mathbb{R} \times N^3$ and its immersion can be written, up to an isometry as

\[
F(t, x) = (t, \psi(x)), \, \psi : N^3 \to \mathbb{C}^4.
\]

The immersion $\psi$ is contained in the subspace orthogonal to both $X_4$ and $JX_4$, since they both are constant unit normals along $N^3$. Now it is also obvious that span\{X_3\} and $N_1$ satisfy the conditions for a warped product. So $N^3$ can be decomposed as $\mathbb{R} \times e^f N^2$ and $X_3(g) = -a_2$. Then $X_3$ can be associated with a coordinate $s$ on the manifold and it follows that

\[
D_X X_3 = \frac{\partial^2 F}{\partial s^2} = 0 \Rightarrow F = A s + B.
\]

Both $A$ and $B$ are independent of $(s, t)$. Calculating (7), one has

\[
X_3(a_2) = \frac{\partial a_2}{\partial s} = a_2.
\]

The solution of this equation, after a translation of the $s$-coordinate is given as $a_2 = -\frac{1}{s}$. The derivatives of $X_3$ to $X_1$ and $X_2$ are

\[
\begin{align*}
D_X X_3 &= \frac{\partial F X_i}{\partial s} = A s X_i \\
&= \frac{1}{s} X_i = A s X_i + \frac{B s X_i}{s} \\
&\Rightarrow B s = 0.
\end{align*}
\]

So $B$ is a constant vector along the submanifold and vanishes when applying a translation. It is easy to see that $X_3 = A$ and is orthogonal to $X_i = s A s(X_i)$, for $i \in \{1, 2\}$. Hence everywhere along $A$, the position vector is orthogonal to the tangent space. Thus $A$ has constant length. Calculating the other covariant derivatives yields for example for $i, j \in \{1, 2\}$ that

\[
A s X_i = \frac{F s X_i}{s},
\]

(28)

\[
D_X (A s X_j) = \frac{D_X (F s X_j)}{s} = A s \left( \tilde{\nabla} X_i X_j \right) + J A s \left( K(X_i, X_j) \right) - \frac{1}{s} \delta_{ij} \phi.
\]

Here $\tilde{\nabla}$ is the connection restricted to $N^2$. Combining this with the other equations in (7), it follows that $A$ is a C-totally real immersion in $S^5 \subset \mathbb{C}^3$. Furthermore, the components $a_1$ and $b_1$ have no other restrictions on them except satisfying the Gauss equations for a minimal C-totally real submanifold of $S^5$. This proves the theorem.
Using the fact that 

\[ \text{N} \]

containing the simplest case one can hope for is that there is a 3-dimensional integrable distribution and \( \nabla X_1 X_1 \) is contained within this distribution. Then \( M \) is locally congruent to

\[ (29) \quad F(t, s, u, v) = (t, s, \phi(u, v)) \]

where \( \phi : \mathbb{C} \to \mathbb{C}^2 \) is a special Lagrangian surface.

**Remark 3.1.** As proved in [7], a special Lagrangian surface in \( \mathbb{C}^2 \), with complex coordinates \( x_1 + iy_1 \) and \( x_2 + iy_2 \) is a holomorphic curve in \( \mathbb{C}^2 \) with complex coordinates \( x_1 - ix_2 \) and \( y_1 + iy_2 \), and conversely.

### 3.2. The case where \( b_2 \neq 0 \)

Now the distribution \( \mathcal{N}_1 \) is no longer integrable. The simplest case one can hope for is that there is a 3-dimensional integrable distribution containing \( \mathcal{N}_1 \). Such a distribution should contain at least \( X_3 \) since

\[ [X_1, X_2] \mod \mathcal{N}_1 \parallel X_3. \]

Using the fact that \( b_2 \neq 0 \), the equations (7) reduce (9) to

\[ (30) \]

\[
\begin{align*}
\nabla X_1 X_1 &= a_1 X_2 + a_2 X_3 + a_3 X_4 \\
\nabla X_2 X_1 &= b_1 X_2 + b_2 X_3 \\
\nabla X_3 X_1 &= b_3 X_2 \\
\nabla X_4 X_1 &= 0 \\
\nabla X_1 X_3 &= -a_2 X_1 + b_2 X_2 + a_6 X_4 \\
\nabla X_2 X_3 &= -b_2 X_1 - a_2 X_2 + b_6 X_4 \\
\nabla X_3 X_3 &= a_3 X_4 \\
\nabla X_4 X_3 &= 0
\end{align*}
\]

It is apparent that the condition that \( \mathcal{N}_+ \) is integrable is given by \( a_6 + ib_6 = 0 \). We consider this case first.

**Theorem 3.3.** Suppose \( M \) is a special Lagrangian submanifold in \( \mathbb{C}^4 \) with \( \text{SO}(2) \times S_3 \)-symmetry, such that \( \mathcal{N}_1 \) is not an integrable distribution, but \( \mathcal{N}_+ \) is. Then the submanifold, up to isometry, can be given locally by either

\[ (31) \quad F(t, s, u, v) = (t, \phi(s, u, v)), \]

where \( \phi \) is a special Lagrangian submanifold with \( S_3 \)-symmetry in \( \mathbb{C}^3 \), or

\[ (32) \quad F(t, s, u, v) = t\phi(s, u, v) \]

where \( \phi \) is the horizontal lift of a special Lagrangian submanifold with local \( S_3 \)-symmetry in \( \mathbb{C}P^3 \) to the unit sphere in \( \mathbb{C}^4 \).

**Proof.** We find according to (30) and (7) that span\( \{X_1\} \) and \( \mathcal{N}_+ \) satisfy the conditions for a warped product. So \( M \) can be decomposed as \( \mathbb{R} \times e^f N^3 \), where \( X_4(f) = -a_3 \). We can solve

\[ X_4(a_3) = \frac{\partial a_3}{\partial t} = a_3^2. \]
This equation has 2 possible solutions. First, we assume \( a_3 = 0 \). In this case \( M \) is simply the manifold \( \mathbb{R} \times N^3 \). Hence the immersion, up to isometry, can be given as

\[
F(t, s, u, v) = (t, \phi(s, u, v)),
\]

where \( \phi \) is a 3-fold immersed in the subspace \( \mathbb{C}^3 \) orthogonal to \( X_4 \) and \( JX_4 \). Similar calculations as in (28) show that this can be any special Lagrangian submanifold in \( \mathbb{C}^3 \), given the presence of an \( S_3 \)-symmetry in the second fundamental form.

The second solution, after a translation of \( t \), is given by \( a_3 = -\frac{1}{t} \). The calculations are similar to the case where \( b_2 = 0 \) and \( a_2 \neq 0 \). This gives the required result. \( \square \)

The last case in \( \mathbb{C}^4 \) is the one where there is no integrable distribution containing \( \mathcal{N}_1 \) other than the whole tangent bundle. In this case, we can no longer rely on an obvious warped product structure. We can attempt to introduce a set of independent coordinates and reduce (7) to a system of PDE’s on \( \mathbb{C}^4 \) using as little functions as possible. We now use (13) to (26) to construct a coordinate frame from \( \{X_1, X_2, X_3, X_4\} \). Since \( \mathcal{N}_2 \) is integrable, it is a good idea to choose \( X_4 = T \) and \( \mu X_3 = S \). Requiring that \([S, T] = 0\) implies that

\[
[\mu X_3, X_4] = \mu [X_3, X_4] - X_4(\mu)X_3 = -(\mu a_3 + X_4(\mu))X_3 = 0.
\]

We can find such a \( \mu \) by taking \( \mu = \frac{1}{\sqrt{|\epsilon + a_3^2|}} \). The equation \( a_3^2 + \epsilon = 0 \) implies that \( a_3 \) is a constant and hence \((a_2 - i b_2)(a_6 + i b_6) = 0\). This will correspond to the integrability of either \( \mathcal{N}_1 \) or \( \mathcal{N}_4 \). Therefore \( \mu \) is well defined.

Vector fields \( U \) and \( V \) can be sought such that every couple out of \( \{S, T, U, V\} \) commutes. Such an attempt can be made, writing

\[
(33) \quad U + iV = (\rho_1 - i \rho_2) \left( (X_1 + i X_2) + (\alpha_1 + i \beta_1)S + (\alpha_2 + i \beta_2)T \right)
\]

We rename the following expressions:

\[
\rho = \rho_1 - i \rho_2,
\]

\[
\gamma_j = \alpha_j + i \beta_j \quad j \in \{1, 2\}.
\]
After calculating the Lie brackets of these four vector fields, the following conditions on the introduced functions make the vector fields commute:

\[ X_4(\rho) = -a_3 \rho, \]  
\[ X_3(\rho) = -\left( a_2 + \frac{2}{3} i b_2 \right) \rho, \]  
\[ (X_1 - i X_2)(\rho) = (b_1 + i a_1) \rho, \]  
\[ X_4(\gamma_1) = -\frac{1}{\mu} (a_6 + i b_6) + a_3 \gamma_1, \]  
\[ X_3(\gamma_1) = \frac{1}{\mu^2} (X_1 + i X_2)(\mu) + \frac{2}{3} i b_2 \rho, \]  
\[ X_2(\alpha_1) - X_1(\beta_1) = a_1 \alpha_1 + b_1 \beta_1 - \frac{2}{\mu} b_2, \]  
\[ X_4(\gamma_2) = a_3 \gamma_2, \]  
\[ X_3(\gamma_2) = (a_6 + i b_6) + \left( a_2 + \frac{2}{3} i b_2 \right) \gamma_2, \]  
\[ X_2(\alpha_2) - X_1(\beta_2) = a_1 \alpha_2 + b_1 \beta_2. \]

The following result can be obtained.

**Lemma 3.1.** Suppose \( f \) and \( g \) are real valued functions on the manifold satisfying

\[ S(f) = 0, \quad T(f) = -1, \quad S(g) = -1, \quad T(g) = 0, \]

and defining

\[ X_1(f) = \alpha_2, \quad X_2(f) = \beta_2, \quad X_1(g) = \alpha_1, \quad X_2(g) = \beta_1, \]

then the functions \( \alpha_i \) and \( \beta_i \) obtained this way satisfy the conditions (37) to (42).

It is interesting to see that this way the vector fields

\[ \tilde{U} = X_1 + \alpha_1 S + \alpha_2 T, \]
\[ \tilde{V} = X_2 + \beta_1 S + \beta_2 T, \]

satisfy \( \tilde{U}(f) = \tilde{U}(g) = \tilde{V}(f) = \tilde{V}(g) = 0 \). Furthermore \( \tilde{U} \) and \( \tilde{V} \) are independent of one-another and they span the distribution which is the intersection of the kernel of \( df \) and \( dg \). Note that this distribution is indeed 2-dimensional since both forms have a hyperplane as a kernel and these kernels do not coincide, since the 1-forms are linearly independent. Using the dimension theorem, they have a 2-dimensional intersection. Construction (33) is just a complex rotation of these two vector fields in that distribution. This way, it is clear that \( f \) and \( g \) serve as coordinates \( s \) and \( t \) conjugate to \( S \) and \( T \).

**Proof.** Apply the relation

\[ [X_i, X_j](f) = X_i X_j(f) - X_j X_i(f) = \nabla_{X_i} X_j(f) - \nabla_{X_j} X_i(f) \]
on both functions, using (30). \( \square \)
A suitable function for $f$ is easily found, since $S(a_3) = 0$. Let $f$ be a function of $a_3$, then

$$X_4(f) = f'(\epsilon + a_3^2) = -1 \iff f' = -\frac{1}{\epsilon + a_3^2}.$$ 

Hence $f$ can be given by

$$f = -\int \frac{1}{\epsilon + a_3^2} \, da_3.$$ 

This also determines $\gamma_2$ completely, since using (13) yields

$$\gamma_2 = (X_1 + iX_2)(f) = f'(X_1 + iX_2)(a_3) = \frac{a_2a_6 + b_2b_6}{\epsilon + a_3^2 + i\frac{a_2b_6 - b_2a_6}{\epsilon + a_3^2}}.$$ 

As for the function $g$, the complex valued function $z = \mu(a_2 + ib_2)$ can be considered and calculations show

$$X_4(z) = -\mu a_3(a_2 + ib_2) + \mu a_3(a_2 + ib_2) = 0,$$

$$S(z) = \mu^2 \left( \epsilon + a_3^2 + (a_2 + ib_2)^2 \right) = \text{sg} \left( \epsilon + a_3^2 \right) + z^2.$$ 

Rewriting $\bar{\epsilon} = \text{sg}(\epsilon + a_3^2)$, we find that $z$ is useful as long as $z^2 + \bar{\epsilon} \neq 0$. When $\bar{\epsilon} = +1$, this occurs when $a_2 = 0$ and $|b_2| = \sqrt{\epsilon + a_3^2}$. When $\bar{\epsilon} = -1$, this occurs when $|a_2| = \sqrt{\epsilon + a_3^2}$ and $b_2 = 0$, resulting in $\mathcal{N}_1$ being integrable.

First we assume that $z^2 \neq -\bar{\epsilon}$. Then the function $g$ can be calculated as the real part of a function $G$ of $z$ given by

$$S(G) = (\bar{\epsilon} + z^2)G' = -1 \iff G' = -\frac{1}{\bar{\epsilon} + z^2}.$$ 

A function $\rho$ still has to be constructed satisfying (34) to (36). Define a function $H$ as

$$H = \rho^3 r(z^2 + \bar{\epsilon}) |\epsilon + a_3^2|.$$ 

This function is a constant on the submanifold and can be taken to be equal to 1. This defines a function $\rho$ satisfying the necessary conditions.

Using the Frobenius theorem in [10], a coordinate frame on the submanifold is given by

$$X_4 = T,$$

$$X_3 = \frac{1}{\mu} S,$$

$$X_1 + iX_2 = \frac{U + iV}{\rho} - \gamma_1 S - \gamma_2 T.$$ 

We can describe the dependence of $a_6 + ib_6$ on $(s, t)$ by writing

$$a_6 + ib_6 = \frac{k_3 + ik_4}{\rho} \sqrt{|a_3^2 + \epsilon| (z^2 + \bar{\epsilon})^{\frac{1}{2}}}.$$ 

The functions $k_3$ and $k_4$ depend solely on $(u, v)$. This expression is obtained from (20) and (21). The rest of the equations in (7) can be rewritten and solved. Applying our
method for $\epsilon = 0$, we find after a translation of the coordinates that

$$a_3 = -\frac{1}{t},$$

$$x = \frac{\sin(2s)}{\cos(2s) + \cosh(2k_1)} \Rightarrow a_2 = -\frac{\sin(2s)}{t(\cos(2s) + \cosh(2k_1))},$$

$$y = \frac{\sinh(2k_1)}{\cos(2s) + \cosh(2k_1)} \Rightarrow b_2 = -\frac{\sinh(2k_1)}{t(\cos(2s) + \cosh(2k_1))},$$

$$r = \frac{e^{k_2}}{t\sqrt{\cos(2s) + \cosh(2k_1)}}$$

Here the functions $k_1$ and $k_2$ depend solely on $(u, v)$. Then we can use (13) to find an expression for $\gamma_1$ in terms of the coordinates. Equation (10) can be used to find an expression for $a_1$ and $b_1$ in terms of the coordinates. We obtain

$$\gamma_1 = \frac{(k_3 + ik_4) \cos(s - ik_1) + t \left( \frac{\partial k_3}{\partial v} - i \frac{\partial k_4}{\partial u} \right)}{t\rho},$$

$$a_1 = \frac{2^{\frac{4}{3}} e^{\frac{2}{3}k_2}}{3t^3 (\cos(2s) + \cosh(2k_1))^2} \left( t((\cos(2s) + \cosh(2k_1))(\rho_1 \frac{\partial k_2}{\partial v} + \rho_2 \frac{\partial k_2}{\partial u}) 
\quad + \sin(2s)(\rho_1 \frac{\partial k_1}{\partial u} - \rho_2 \frac{\partial k_1}{\partial v}) - \sinh(2k_1)(\rho_1 \frac{\partial k_1}{\partial v} + \rho_2 \frac{\partial k_1}{\partial u}) 
\quad + \sinh(2k_1)(\cos(s) \cosh(k_1)(k_4 \rho_2 - k_3 \rho_1) + \sin(s) \sinh(k_1)(k_4 \rho_1 + k_3 \rho_2)) \right),$$

$$b_1 = \frac{2^{\frac{4}{3}} e^{\frac{2}{3}k_2}}{3t^3 (\cos(2s) + \cosh(2k_1))^2} \left( t((\cos(2s) + \cosh(2k_1))(\rho_2 \frac{\partial k_2}{\partial v} - \rho_1 \frac{\partial k_2}{\partial u}) 
\quad + \sin(2s)(\rho_2 \frac{\partial k_1}{\partial u} + \rho_1 \frac{\partial k_1}{\partial v}) + \sinh(2k_1)(\rho_1 \frac{\partial k_1}{\partial u} + \rho_2 \frac{\partial k_1}{\partial v}) 
\quad + \sinh(2k_1)(\sin(s) \sinh(k_1)(k_4 \rho_2 - k_3 \rho_1) - \cos(s) \cosh(k_1)(k_4 \rho_1 + k_3 \rho_2)) \right).$$

Now every function on the submanifold is expressed in terms of $(s, t, u, v)$, possibly indirectly through $\{k_1, k_2, k_3, k_4\}$. Demanding that the other Gauss equations are satisfied gives partial differential equations for $k_i$, given by

$$\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} = 2 \tanh(k_1) \left( k_3 \frac{\partial k_1}{\partial v} - k_4 \frac{\partial k_1}{\partial u} \right),$$

$$\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} = -2 \coth(k_1) \left( k_3 \frac{\partial k_1}{\partial u} + k_4 \frac{\partial k_1}{\partial v} \right),$$

$$\Delta k_2 = 3 * 2^{\frac{1}{3}} e^{-\frac{2k_2}{3}} (e^{2k_2} + \cosh(2k_1)), $$

$$\Delta k_1 = -2^{\frac{1}{3}} e^{-\frac{2k_2}{3}} \sinh(2k_1).$$

Now we return to the case where $-1 = z^2$ and $\tilde{e} = 1$. We assume first that $\epsilon$ isn’t specified. In this case $a_2 = 0$, $b_2 = \pm \sqrt{\epsilon + a_3^2}$ and $S(z) = 0$, so $z$ is insufficient to construct the
function \( g \). Equations (7) are reduced to

\[
(X_1 - iX_2)(a_6 + ib_6) = -i(b_1 + ia_1)(a_6 + ib_6),
\]

(44)

\[
X_3(a_6 + ib_6) = \pm i \frac{5}{3} \sqrt{\epsilon + a_3^2} (a_6 + ib_6),
\]

(45)

\[
X_4(a_6 + ib_6) = 2a_3(a_6 + ib_6),
\]

(46)

\[
X_1(b_1) - X_2(a_1) = 2r^2 - \frac{8}{3} (\epsilon + a_3^2) - a_1^2 - b_1^2,
\]

(47)

\[
X_1(a_1) + X_2(b_1) = 0,
\]

(48)

\[
X_3(a_1 + ib_1) = \frac{i}{3} \left( a_3(a_6 + ib_6) \pm 2 \sqrt{\epsilon + a_3^2} (a_1 + ib_1) \right).
\]

(49)

The first equation is obtained from applying integrability on \( a_3 \). Now we define

\[
w = \frac{a_6 + ib_6}{\epsilon + a_3^2},
\]

which after derivation gives

\[
X_4(w) = -2a_3 \frac{(a_6 + ib_6)}{\epsilon + a_3^2} + 2a_3 \frac{a_6 + ib_6}{\epsilon + a_3^2} = 0,
\]

\[
S(w) = \pm i \frac{5}{3} w.
\]

The resulting differential equation for a function \( G \) of \( w \) will be

\[
S(G) = \pm G' \frac{5}{3} w = -1 \Leftrightarrow G' = \pm i \frac{3}{5w}.
\]

The solution is that \( G \) is a logarithm of \( w \). We find that \( H \) defined by

\[
H = w^2 (\epsilon + a_3^2)^2 \rho^5 r
\]

is a constant and hence can be used to express \( \rho \). We can thus solve \( w \) as

\[
w = e^{k_1 \pm \frac{i 5}{3} s} = e^{k_1} \left( \cos \left( \frac{5}{3} s \right) \pm i \sin \left( \frac{5}{3} s \right) \right).
\]

Applying (18), (46), (45), (12) and (11) when \( \epsilon = 0 \) yields

\[
a_3 = -\frac{1}{t},
\]

\[
a_6 = e^{k_1} \frac{\cos \left( \frac{5}{3} s \right)}{t^2},
\]

\[
b_6 = \pm e^{k_1} \frac{\sin \left( \frac{5}{3} s \right)}{t^2},
\]

\[
r = \frac{e^{k_2}}{t}.
\]

The equation (10) now gives \( a_1 + ib_1 \) immediately without going through \( \gamma_1 \) because of (11). The final unknown, \( \gamma_1 \) can then be determined using (44). When we pick \( b_2 = a_3 \),
we obtain
\[
\gamma_1^+ = -5e^{k_1+\frac{2s}{3}} + e^{2k_1+k_2+\frac{2s}{3}} t \left( \frac{\partial k_2}{\partial v} - 3 \frac{\partial k_1}{\partial u} \right) - i \left( \frac{\partial k_2}{\partial u} - 3 \frac{\partial k_1}{\partial v} \right),
\]
\[
a_{1+} = \frac{e^{k_1} \cos(\frac{5a}{3}) + e^{2k_1+k_2} t \left( \cos(\frac{2a}{3}) \frac{\partial k_2}{\partial v} + \sin(\frac{2a}{3}) \frac{\partial k_2}{\partial u} \right)}{3t^2},
\]
\[
b_{1+} = \frac{e^{k_1} \sin(\frac{5a}{3}) - e^{2k_1+k_2} t \left( \cos(\frac{2a}{3}) \frac{\partial k_2}{\partial u} - \sin(\frac{2a}{3}) \frac{\partial k_2}{\partial v} \right)}{3t^2},
\]
and for \(b_2 = -a_3\) we obtain
\[
\gamma_1^- = -5e^{k_1-\frac{5a}{3}} + e^{2k_1+k_2-\frac{2s}{3}} t \left( 3 \frac{\partial k_2}{\partial v} - \frac{\partial k_2}{\partial u} \right) - i \left( 3 \frac{\partial k_2}{\partial u} - \frac{\partial k_2}{\partial v} \right),
\]
\[
a_{1-} = \frac{-e^{k_1} \cos(\frac{5a}{3}) + e^{2k_1+k_2} t \left( \cos(\frac{2a}{3}) \frac{\partial k_2}{\partial v} - \sin(\frac{2a}{3}) \frac{\partial k_2}{\partial u} \right)}{3t^2},
\]
\[
b_{1-} = \frac{-e^{k_1} \sin(\frac{5a}{3}) - e^{2k_1+k_2} t \left( \cos(\frac{2a}{3}) \frac{\partial k_2}{\partial u} + \sin(\frac{2a}{3}) \frac{\partial k_2}{\partial v} \right)}{3t^2}.
\]

Equations (47) and (39) result in restrictions on the functions \(k_1\) and \(k_2\) of \((u, v)\) given by
\[
\Delta k_2 = e^{-\frac{2}{t}(2k_1+k_2)}(8 - 6e^{2k_2}),
\]
\[
\Delta k_1 = e^{-\frac{2}{t}(2k_1+k_2)}(6 - 2e^{2k_2}).
\]
These equations are valid for both \(b_2 = \pm a_3\). Using the constructed functions, the rest of the Gauss equations don’t impose further conditions. We can summarize this result in the following theorem.

**Theorem 3.4.** Each special Lagrangian submanifold of \(\mathbb{C}^4\) with \(SO(2) \times S_3\)-symmetry where the only integral distribution containing \(\mathcal{N}_1\) is the tangent bundle, can be constructed in the way above using either functions \(\{k_1, k_2, k_3, k_4\}\) subject to (43) or functions \(\{k_1, k_2\}\) subject to (51). Conversely, each such a construction results in such a submanifold, unique up to local isometry.

In the upcoming sections we will consider the construction for \(\epsilon = \pm 1\).

4. **Submanifolds in \(\mathbb{C}P^4\).**

4.1. **The case where** \(b_2 = 0\). This means that both \(\mathcal{N}_1\) and \(\mathcal{N}_2\) are integrable distributions. We can assume \(a_3 = 0\). However, the Gauss equation
\[
X_3(a_2) = 1 + a_2^2
\]
no longer allows for \(a_2\) being a constant. The following result is obtained:

**Theorem 4.1.** Suppose \(M\) is a special Lagrangian submanifold in \(\mathbb{C}P^4\) having \(SO(2) \times S_3\)-symmetry. Suppose \(\mathcal{N}_1\) is integrable. Then \(M\) can be lifted horizontally to a submanifold in the unit sphere of \(\mathbb{C}^5\) through \(F\) and this lift is congruent to
\[
F(t, s, u, v) = (\phi(u, v) \cos(s), \sin(s) \cos(t), \sin(s) \sin(t)),
\]
To find a suitable \( \mu \) assume \( X \)

Hence we can write \( M \)

where \( N \)

The distributions \( N_1 \) and \( N_2 \) satisfy the conditions for a warped product \( N_2 \times_{ef} N_1 \).

Furthermore, the distributions span \{ \( X_3 \) \} and span \{ \( X_4 \) \} satisfy those of a warped product and we can write \( M = \mathbb{R} \times_{ef} \mathbb{R} \times_{ef} N_1 \).

The functions \( f \) and \( g \) depend solely on the parameter corresponding to \( X_3 \) and are given by \( X_3(f) = -a_2 \) and \( X_3(g) = \frac{1}{a_2} \). We can assume \( X_3 = \frac{\partial}{\partial s} \) on the submanifold. We can also find a function \( \mu(s) \) such that \( \mu X_4 = \frac{\partial}{\partial t} \).

To find a suitable \( \mu \), we solve

\[
[X_3, \mu X_4] = \left( X_3(\mu) - \frac{\mu}{a_2^2} \right) X_4 = \left( \mu' \left(1 + a_2^2\right) - \frac{\mu}{a_2^2} \right) X_4 = 0.
\]

The function \( \mu = \frac{a_2}{\sqrt{1 + a_2^2}} \) satisfies this equation. We can find \( a_2(s) \) by solving

\[
\frac{\partial a_2}{\partial s} = 1 + a_2^2 \Rightarrow a_2 = \tan(s).
\]

Hence \( \mu(s) = \sin(s) \) and we calculate for \( i \in \{1, 2\} \) that

\[
D_{\frac{\partial}{\partial s}} = \frac{\partial^2 F}{\partial s^2} = -F
\]

\[
\Rightarrow F = A \cos(s) + B \sin(s),
\]

\[
D_{\frac{\partial}{\partial t}} = \frac{\partial^2 F}{\partial t \partial s} = -\frac{\partial A}{\partial t} \sin(s) + \frac{\partial B}{\partial t} \cos(s)
\]

\[
= \cot(s) \frac{\partial F}{\partial t} = \frac{\cos(s)^2}{\sin(s)} \frac{\partial A}{\partial t} + \cos(s) \frac{\partial B}{\partial t}
\]

\[
\Rightarrow \frac{\partial A}{\partial t} = 0,
\]

\[
D_{X_i} = \frac{\partial F}{\partial s} = -A_s X_i \sin(s) + B_s X_i \cos(s)
\]

\[
= -\tan(s) X_i = -A_s X_i \sin(s) - \frac{\sin(s)^2}{\cos s} B_s X_i
\]

\[
\Rightarrow B_s X_i = 0.
\]

So \( A \) is the immersion of \( N_1 \) and \( B \) is a curve tangent to \( X_4 \). Because \( F \) lies in the unit sphere, one has

\[
\langle F, F \rangle = \cos(s)^2 \langle A, A \rangle + \sin(s)^2 \langle B, B \rangle + \sin(2s) \langle A, B \rangle
\]
which implies that $A$ and $B$ have both unit length and are orthogonal. We can also calculate

\[ D_x \frac{\partial}{\partial t} = -\cos(s) \sin(s) \frac{\partial F}{\partial s} - \sin(s)^2 F = -\sin(s) B \]
\[ = \frac{\partial^2 F}{\partial t^2} = \sin(s) \frac{\partial^2 B}{\partial t^2} \]
\[ \Rightarrow B = B_1 \cos(t) + B_2 \sin(t). \]

Vector fields $B_1$ and $B_2$ are constant, normalized and orthogonal. This follows from the fact that $\langle B, B \rangle = 1$. Finally similar to (28), $A$ can be shown to be any special Lagrangian submanifold in $\mathbb{C}P^2$ lifted to the unit sphere in $\mathbb{C}^3$ orthogonal to $B_1$ and $B_2$ directions. Fixing $B_1$ and $B_2$ by an isometry leads to (53). \hfill \Box

4.2. The case where $b_2 \neq 0$. When $N_+^+$ is integrable, so when $a_6 = b_6 = 0$, the equations for $\nabla$ are given by (30). We have:

**Theorem 4.2.** Suppose $M$ is a special Lagrangian submanifold in $\mathbb{C}P^4$ having a local $SO(2) \rtimes S_3$ symmetry. Suppose $N_+^+$ is integrable. Then $M$ can be lifted horizontally to a submanifold in the unit sphere of $\mathbb{C}^5$ through $F$ and is locally isometric to

\[ F(t, s, u, v) = (\phi(s, u, v) \cos(t), \sin(t)), \]

where $\phi$ is the horizontal lift of a special Lagrangian submanifold in $\mathbb{C}P^3$ with $S_3$-symmetry to the unit sphere in $\mathbb{C}^4$.\n
**Proof.** The manifold is a warped product $R \times e^F N^3$. Solving the Gauss equation

\[ X_4(a_3) = \frac{\partial a_3}{\partial t} = 1 + a_3^2 \]

yields $a_3 = \tan(t)$. For $i \in \{1, 2, 3\}$ this implies

\[ D_{X_i} X_4 = \frac{\partial^2 F}{\partial t^2} = -F \]
\[ \Rightarrow F = A \cos(t) + B \sin(t), \]
\[ D_{X_i} X_4 = -A_i X_i \sin(t) + B_i X_i \cos(t) \]
\[ = -\tan(t) X_i = -A_i X_i \sin(t) - B_i X_i \frac{\sin(t)^2}{\cos(t)} \]
\[ \Rightarrow B_i = 0. \]

Thus $B$ is a constant vector field along the submanifold and $A$ is an immersion of a 3-fold $N^3$. Using the fact that $F$ is of unit length, $A$ and $B$ are orthogonal and of unit length. Using calculations similar to (28) $A$ is a C-totally real submanifold in $S^7$ having local $S_3$-symmetry, where $S^7$ lies in the subspace orthogonal to $B$ and $JB$. Applying a suitable isometry results in (54). \hfill \Box

The method to solve the case where the only integrable distribution containing $N_1$ is the tangent bundle, has been analyzed earlier for a non-specific complex space form. We
can now fill in $\epsilon = 1$ and we find for $z^2 \neq -1$ that

\[
a_3 = \tan(t),
\]
\[
a_2 = \frac{\sin(2s)}{\cos(t)(\cos(2s) + \cosh(2k_1))},
\]
\[
b_2 = \frac{\sinh(2k_1)}{\cos(t)(\cos(2s) + \cosh(2k_1))},
\]
\[
r = \frac{e^{k_2}}{\cos(t) \sqrt{\cos(2s) + \cosh(2k_1)}},
\]
\[
a_6 + ib_6 = \frac{k_3 + ik_4}{\rho} \sqrt{1 + a_2^2 (1 + z^2)^{-\frac{3}{2}}},
\]

where the functions $k_i$ depend only on $(u, v)$. Solving for (7), we obtain furthermore that

\[
\gamma_1 = -\tan(t)(k_3 + ik_4) \cos(s - ik_1) + \left(\frac{\partial k_4}{\partial u} - i \frac{\partial k_3}{\partial u}\right),
\]
\[
a_1 = \frac{2^2 e^{\frac{2}{3}k_2}}{3 \cos(t)^2 (\cos(2s) + \cosh(2k_1))^2} \left((\cos(2s) + \cosh(2k_1)) \left(\rho_1 \frac{\partial k_2}{\partial v} + \rho_2 \frac{\partial k_3}{\partial u}\right) + \sin(2s) \left(\rho_1 \frac{\partial k_1}{\partial u} - \rho_2 \frac{\partial k_1}{\partial v}\right) - \sinh(2k_1) \left(\rho_1 \frac{\partial k_1}{\partial v} - \rho_2 \frac{\partial k_1}{\partial u}\right) - \tan(t) \sinh(2k_1) \left(\cos(s) \cosh(k_1)(k_4\rho_2 - k_3\rho_1) + \sin(s) \sinh(k_1)(k_4\rho_1 + k_3\rho_2)\right)\right),
\]
\[
b_1 = \frac{2^2 e^{\frac{2}{3}k_2}}{3 \cos(t)^2 (\cos(2s) + \cosh(2k_1))^2} \left((\cos(2s) + \cosh(2k_1)) \left(\rho_2 \frac{\partial k_2}{\partial v} - \rho_1 \frac{\partial k_3}{\partial u}\right) + \sin(2s) \left(\rho_2 \frac{\partial k_1}{\partial u} + \rho_1 \frac{\partial k_1}{\partial v}\right) + \sinh(2k_1) \left(\rho_2 \frac{\partial k_1}{\partial v} - \rho_1 \frac{\partial k_1}{\partial u}\right) - \sinh(2k_1) \tan(t) \left(\sin(s) \sinh(k_1)(k_4\rho_2 - k_3\rho_1) - \cos(s) \cosh(k_1)(k_4\rho_1 + k_3\rho_2)\right)\right).
\]

The other equations in (7) impose restrictions on \(\{k_1, k_2, k_3, k_4\}\) given by

\[
\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} = 2 \tanh(k_1) \left(\frac{k_3}{\partial v} - k_4 \frac{\partial k_1}{\partial u}\right),
\]
\[
\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} = -2 \coth(k_1) \left(\frac{k_3}{\partial u} + k_4 \frac{\partial k_1}{\partial v}\right),
\]
\[
\Delta k_1 = e^{-\frac{2k_3}{3}} \sinh(2k_1) \left(-2^\frac{4}{3} + e^{\frac{2k_2}{3}} (k_3^2 + k_4^2)\right),
\]
\[
\Delta k_2 = 3 \ast 2^\frac{1}{3} e^{-\frac{2k_3}{3}} \left(\cosh(2k_1) - e^{2k_2}\right).
\]

\[\text{(55)}\]
When \( a_2 = 0 \) and \( b_2 = \pm \sqrt{1 + a_3^2} \), we find
\[
\begin{align*}
a_6 &= \frac{e^{k_1} \cos\left(\frac{5}{3} s\right)}{\cos(t)^2}, \\
b_6 &= \pm \frac{e^{k_1} \sin\left(\frac{5}{3} s\right)}{\cos(t)^2}, \\
r &= \frac{e^{k_2}}{\cos(t)}.
\end{align*}
\]
Furthermore, we obtain for \( b_2 = \sqrt{1 + a_3^2} \) that
\[
\begin{align*}
\gamma_1 &= -5e^{k_1 + \frac{2k_4}{3}} \tan(t) + e^{2k_1 + k_2 + \frac{2k_4}{3}} \left((\frac{\partial k_1}{\partial v} - 3\frac{\partial k_2}{\partial w}) - i(\frac{\partial k_2}{\partial w} - 3\frac{\partial k_1}{\partial v})\right), \\
a_1 &= e^{k_1} \cos\left(\frac{5}{3} s\right) \tan(t) + e^{\frac{2k_1 + k_2}{3}} \left(\cos\left(\frac{2s}{3}\right) \frac{\partial k_2}{\partial w} + \sin\left(\frac{2s}{3}\right) \frac{\partial k_2}{\partial v}\right), \\
b_1 &= e^{k_1} \sin\left(\frac{5}{3} s\right) \tan(t) - e^{\frac{2k_1 + k_2}{3}} \left(\cos\left(\frac{2s}{3}\right) \frac{\partial k_2}{\partial v} - \sin\left(\frac{2s}{3}\right) \frac{\partial k_2}{\partial w}\right),
\end{align*}
\]
and for \( b_2 = -\sqrt{1 + a_3^2} \) we obtain
\[
\begin{align*}
\gamma_1 &= -5e^{k_1 - \frac{2k_4}{3}} \tan(t) + e^{2k_1 + k_2 - \frac{2k_4}{3}} \left((\frac{\partial k_1}{\partial v} - 3\frac{\partial k_2}{\partial w}) - i(\frac{\partial k_2}{\partial w} - 3\frac{\partial k_1}{\partial v})\right), \\
a_1 &= -e^{k_1} \cos\left(\frac{5}{3} s\right) \tan(t) + e^{\frac{2k_1 + k_2}{3}} \left(\cos\left(\frac{2s}{3}\right) \frac{\partial k_2}{\partial w} - \sin\left(\frac{2s}{3}\right) \frac{\partial k_2}{\partial v}\right), \\
b_1 &= e^{k_1} \sin\left(\frac{5}{3} s\right) \tan(t) - e^{\frac{2k_1 + k_2}{3}} \left(\cos\left(\frac{2s}{3}\right) \frac{\partial k_2}{\partial v} + \sin\left(\frac{2s}{3}\right) \frac{\partial k_2}{\partial w}\right).
\end{align*}
\]
Solving the last equations in (7) implies restrictions on the functions \( k_1(u, v) \) and \( k_2(u, v) \) given by
\[
\begin{align*}
\Delta k_1 &= 2e^{-\frac{2(2k_1 + k_2)}{3}} \left(3 - e^{2k_1} - e^{2k_2}\right), \\
\Delta k_2 &= e^{-\frac{2(2k_1 + k_2)}{3}} \left(8 - e^{2k_1} - 6e^{2k_2}\right).
\end{align*}
\]
These equations are valid for both \( b_2 = \pm \sqrt{1 + a_3^2} \). We summarize this in the following theorem.

**Theorem 4.3.** Each special Lagrangian submanifold of \( \mathbb{C}P^4 \) with \( SO(2) \rtimes S_3 \)-symmetry where the only integral distribution containing \( \mathcal{N}_1 \) is the tangent bundle, can be constructed in the way above using either functions \( \{k_1, k_2, k_3, k_4\} \) subject to (54) or functions \( \{k_1, k_2\} \) subject to (56). Conversely, each such a construction results in such a submanifold, unique up to local isometry.
5. Submanifolds in $\mathbb{C}H^4$.

5.1. **The case where** $b_2 = 0$. This is the case where $N_1$ is an integrable distribution. We assume that $a_3 = 0$. Similar to the case in $\mathbb{C}P^4$, we have that $M$ is a double warped product $\mathbb{R} \times e^t \mathbb{R} \times e^t N^2$. The function $a_2$ depends only on the coordinate $s$ and is given by

$$\frac{\partial a_2}{\partial s} = a_2^2 - 1.$$ 

This equation has 3 possible solutions, depending on the initial conditions. For $a_2(0) = 1$, it is a constant. For $a_2(0) > 1$ it is given as $a_2 = -\coth(s)$. Finally for $a_2(0) < 1$, it is given as $a_2(s) = -\tanh(s)$. The connection $\nabla$ is given by

\[
\begin{align*}
\nabla_{X_1}X_1 &= a_1X_2 + a_2X_3, \\
\nabla_{X_2}X_1 &= b_1X_2, \\
\nabla_{X_3}X_1 &= 0, \\
\nabla_{X_4}X_1 &= 0, \\
\nabla_{X_1}X_3 &= -a_2X_1, \\
\nabla_{X_2}X_3 &= -a_2X_2, \\
\nabla_{X_3}X_3 &= 0, \\
\nabla_{X_4}X_3 &= -\frac{X_4}{a_2}, \\
\n\nabla_{X_1}X_2 &= -a_1X_1, \\
\nabla_{X_2}X_2 &= -b_1X_1 + a_2X_3, \\
\nabla_{X_3}X_2 &= 0, \\
\nabla_{X_4}X_2 &= 0, \\
\n\nabla_{X_1}X_4 &= 0, \\
\nabla_{X_2}X_4 &= 0, \\
\n\nabla_{X_3}X_4 &= 0, \\
\n\nabla_{X_4}X_4 &= \frac{X_4}{a_2}.
\end{align*}
\]

We have the following result.

**Theorem 5.1.** Suppose $M$ is a special Lagrangian submanifold in $\mathbb{C}H^4$ having a local $SO(2) \times S_3$-symmetry. Suppose $N_1$ is integrable. Then $M$ can be lifted horizontally to a submanifold in $H^9$ through $F$ and is locally isometric to either

\[
(57) \quad F(t, s, u, v) = (\sin(t) \sinh(s), \cos(t) \sinh(s), \phi(u, v) \cosh(s)),
\]

where $\phi$ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}H^2$ to $H^5$ in case $a_2^2 < 1$, or

\[
(58) \quad F(t, s, u, v) = (\phi(u, v) \sinh(s), \cos(t) \cosh(s), \sin(t) \cosh(s)),
\]

where $\phi$ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}P^2$ to $S^5$ in case $a_2^2 > 1$, or

\[
(59) \quad F(t, s, u, v) = (\phi(u, v), t) e^{-s}, -\frac{1}{2} e^{-s}, (||\phi(u, v), t||^2 + if(u, v)) e^{-s} + e^s,
\]

where $\phi$ is a special Lagrangian surface in $\mathbb{C}^2$ and $f$ is the integral of the differential form

\[
2 \sum_{i=1}^{2} (x^i \, dy^i - y^i \, dx^i)
\]

on $\mathbb{C}^2$ in case $a_2^2 = 1$.

**Proof.** We can check similar to the case in $\mathbb{C}P^4$ that $M = \mathbb{R} \times e^t \mathbb{R} \times e^t N^2$, where $f$ and $g$ are functions on the first factor, determined by $X_3(g) = -\frac{1}{a_2}$ and $X_3(f) = -a_2$. We can treat the cases separately for each solution to $a_2(s)$.
Assume $a_2 = -\tanh(s)$, then it is easy to see that $\frac{\partial}{\partial t} = -\sinh(s)X_4$ commutes with $\frac{\partial}{\partial s}$. The Gauss identity now implies for $i \in \{1, 2\}$ that

\[
\begin{align*}
D_s \frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial s^2} = F \\
&\Rightarrow F = A \sinh(s) + B \cosh(s),
\end{align*}
\]

\[
\begin{align*}
D_s \frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial t \partial s} = \frac{\partial A}{\partial t} \cosh(s) + \frac{\partial B}{\partial t} \sinh(s) \\
&= \coth(s) \frac{\partial F}{\partial t} = \frac{\partial A}{\partial t} \cosh(s) + \frac{\partial B}{\partial t} \sinh(s) \\
&\Rightarrow \frac{\partial B}{\partial t} = 0,
\end{align*}
\]

\[
\begin{align*}
D_{X_i} \frac{\partial}{\partial s} &= \frac{\partial F}{\partial s} X_i = A_s X_i \cosh(s) + B_s X_i \sinh(s) \\
&= \tanh(s) X_i = A_s X_i \frac{\sinh(s)^2}{\cosh(s)} + B_s X_i \sinh(s) \\
&\Rightarrow A_s X_i = 0.
\end{align*}
\]

Using the fact that $\langle F, F \rangle_1 = -1$, we get that $\langle B, B \rangle_1 = -\langle A, A \rangle_1 = -1$ and $\langle A, B \rangle_1 = 0$. Furthermore, we find

\[
\begin{align*}
D_t \frac{\partial}{\partial t} &= \frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 A}{\partial t^2} \sinh(s) \\
&= - \cosh(s) \sinh(s) \frac{\partial F}{\partial s} + \sinh(s)^2 F = -A \sinh(s) \\
&\Rightarrow A = A_1 \cos(t) + A_2 \sin(t).
\end{align*}
\]

Because $A$ has unit length, so do $A_1$ and $A_2$ and they are both orthogonal. Calculations similar to (28) show that $B$ can be taken as the horizontal lift of any special Lagrangian submanifold in $\mathbb{C}H^2$ and applying a suitable isometry gives (37).

For $a_2 = -\coth(s)$ calculations similar to the previous case result in (58).

Finally we assume $a_2 = 1$. Then the vector field given by $\frac{\partial}{\partial t} = e^{-s}X_4$ commutes with $\frac{\partial}{\partial s}$. We can calculate for $i \in \{1, 2\}$ that

\[
\begin{align*}
D_s \frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial s^2} = F \Rightarrow F = Ae^s + Be^{-s},
\end{align*}
\]

\[
\begin{align*}
D_s \frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial t \partial s} = \frac{\partial A}{\partial t} e^s - \frac{\partial B}{\partial t} e^{-s} \\
&= - \frac{\partial F}{\partial t} = - \frac{\partial A}{\partial t} e^s - \frac{\partial B}{\partial t} e^{-s} \\
&\Rightarrow \frac{\partial A}{\partial t} = 0,
\end{align*}
\]

\[
\begin{align*}
D_{X_i} \frac{\partial}{\partial s} &= \frac{\partial F}{\partial s} X_i = A_s X_i e^s - B_s X_i e^{-s} \\
&= -F_s X_i = - (A_s X_i e^s + B_s X_i e^{-s}) \\
&\Rightarrow A_s = 0.
\end{align*}
\]
Using the fact that $\langle F, F \rangle_1 = -1$, we obtain that $A$ and $B$ are vector fields with 0 length and they satisfy $\langle A, B \rangle_1 = -\frac{1}{2}$. Further calculations show

$$D \frac{\partial}{\partial t} = \frac{\partial^2 F}{\partial t^2} = e^{-s} \frac{\partial^2 B}{\partial t^2} = e^{-2s} \left( \frac{\partial F}{\partial s} + F \right) = 2A e^{-s}$$

$$\Rightarrow B = At^2 + B_1 t + B_2,$$

$$D X_1 \frac{\partial}{\partial t} = \frac{\partial F}{\partial t} X_i = B_1 X_i e^{-s} = 0$$

$$\Rightarrow B_1 = 0.$$

We can conclude that $F$ has the form

$$F = (At^2 + B_1 t + \phi) e^{-s} + Ae^s$$

Here, $\phi$ is an immersion of a 2-fold in $C^5_1$ tangent to $N_1$. Calculating the scalar products of $B$ and $A$, we get

$$\langle A, B \rangle_1 = t \langle A, B_1 \rangle_1 + \langle A, \phi \rangle_1 = -\frac{1}{2}$$

$$\Rightarrow \langle A, B_1 \rangle_1 = 0 \land \langle A, \phi \rangle_1 = -\frac{1}{2}$$

$$\langle B, B \rangle_1 = t^2 (\langle B_1, B_1 \rangle_1 - 1) + 2t \langle B_1, \phi \rangle_1 + \langle \phi, \phi \rangle_1 = 0$$

$$\Rightarrow \langle B_1, B_1 \rangle_1 = 1 \land \langle B_1, \phi \rangle_1 = 0 \land \langle \phi, \phi \rangle_1 = 0.$$  

(60)

We can shift to a different standard basis of $C^5_1$ such that

$$\langle \bar{z}, \bar{w} \rangle = \Re \left( \sum_{j=1}^{3} z_j \bar{w}_j + z_4 \bar{w}_5 + z_5 \bar{w}_4 \right).$$

In this case the constant light-like vector $A$ and time-like $B_1$, after applying a suitable isometry, can be chosen to be

$$A = (0, 0, 0, 0, 1),$$

$$B_1 = (0, 0, 1, 0, 0).$$

We can write $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$ where $\phi_j = x_j + iy_j$. Then (60) implies

$$x_4 = -\frac{1}{2},$$

$$x_3 = 0,$$

$$x_5 - 2y_4 y_5 = \sum_{j=1}^{3} |\phi_j|^2.$$
We can use the fact that both $F$ and $iF$ are orthogonal to the tangent space in $\mathbb{C}^5$ and this results in
\[
\phi_4 = -\frac{1}{2}, \\
\phi_3 = 0, \\
dy_5 = 2 \sum_{i=1}^{2} (x_i \, dy_i - y_i \, dx_i).
\]
This last equation is integrable if and only its derivative equals 0 along the submanifold. But this derivative is nothing more than a multiple of the Kähler form on $\mathbb{C}^2$ spanned by the first 2 complex coordinates. In other words, for such a submanifold to exist, the projection of $\phi$ onto the first 2 coordinates should be a Lagrangian submanifold in $\mathbb{C}^2$. Calculating the Gauss identity on $D_x X_j$ we find that the metric on this immersion is given by
\[
\langle \phi^* X_i, \phi^* X_j \rangle = e^{2s} \delta_{ij},
\]
where $\langle a, b \rangle$ is the standard scalar product on $\mathbb{C}^2$ and $\phi$ here is the restriction to the first 2 complex coordinates. Because $\langle F,X_i,F,X_j \rangle = \delta_{ij}$ and because $\phi_{3*} = 0$ and $\phi_{4*} = 0$ this condition is included in the warped product structure. Using calculations like (28) we conclude that $(\phi_1,\phi_2)$ can be any special Lagrangian 2-fold in $\mathbb{C}^2$. The result is summarized in (59).

5.2. The case where $b_2 \neq 0$. First we assume that $\mathcal{N}_+$ is an integrable distribution. This is equivalent to $a_6 + ib_6 = 0$. The connection is given by (30), resulting in a warped product structure $\mathbb{R} \times e^t \mathcal{N}_3$. The equation
\[
X_4(a_3) = \frac{\partial a_3}{\partial t} = a_3^2 - 1
\]
has a solution given as $|a_3| = 1$, $a_3 = -\tanh(t)$ or $a_3 = -\coth(t)$, depending on the initial value of $a_3$. Using an analysis similar to the case of $\mathbb{C}P^4$ and the case above gives the following result.

**Theorem 5.2.** Suppose $M$ is a special Lagrangian submanifold in $\mathbb{C}H^4$ having a local $SO(2) \rtimes S_3$-symmetry. Suppose $\mathcal{N}_1$ is non-integrable, but is contained in the integrable $\mathcal{N}_+$ distribution. Then $M$ can be lifted horizontally to a submanifold in $H^9$ through $F$ and is locally isometric to either
\begin{align*}
(61) & \quad F(t,s,u,v) = (\sinh(t), \phi(s,u,v) \cosh(t)) , \\
(62) & \quad F(t,s,u,v) = (\phi(s,u,v) \sinh(t), \cosh(t)) , \quad \text{where } \phi \text{ is the horizontal lift of a special Lagrangian submanifold of } \mathbb{C}H^3 \text{ with a local } S_3 \text{-symmetry to } H^7 \text{ in case } a_3^2 < 1, \text{ or} \\
(63) & \quad F(t,s,u,v) = (\phi(s,u,v) e^{-t} - e^{-t}/2, (\|\phi(s,u,v)\|^2 + if(s,u,v)) e^{-t} + e^t) , \quad \text{where } \phi \text{ is the horizontal lift of a special Lagrangian submanifold of } \mathbb{C}P^3 \text{ with a local } S_3 \text{-symmetry to } S^7 \text{ in case } a_3^2 > 1, \text{ or}
\end{align*}
where $\phi$ is a special Lagrangian submanifold with a local $S_3$-symmetry in $\mathbb{C}^3$ and $f$ is the integral of the differential form

$$2 \sum_{i=1}^{3} (x^i \, d\, y^i - y^i \, d\, x^i).$$

Finally we assume that there is no integrable distribution that contains $\mathcal{N}_1$ except for the tangent bundle. We return to the analysis as done for $\mathbb{C}^4$, but set $\epsilon = -1$. The result will depend on the initial value of $a_3$. First assume that $a_3^2 < 1$, then $\tilde{\epsilon} = -1$. We find functions $\{k_1, k_2, k_3, k_4\}$ of $(u, v)$ such that

$$a_3 = -\tanh(t),$$
$$a_2 = - \frac{\sinh(2s)}{\cosh(t)(\cosh(2s) + \cos(2k_1))},$$
$$b_2 = - \frac{\sin(2k_1)}{\cosh(t)(\cosh(2s) + \cos(2k_1))},$$
$$r = \frac{e^{k_2}}{\cosh(t) \sqrt{\cosh(2s) + \cos(2k_1)}},$$
$$a_6 + ib_6 = \frac{k_3 + ik_4}{\rho} \sqrt{1 - a_3^2 \left(1 - \bar{z}^2\right)}^{-\frac{1}{2}}.$$

Using (7) as earlier, we obtain $a_1, b_1, \gamma_1$ as

$$\gamma_1 = \frac{-\tanh(t)(k_3 + ik_4) \cosh(s - ik_1) + \left(\frac{\partial k_1}{\partial v} - i \frac{\partial k_1}{\partial u}\right)}{\rho},$$
$$a_1 = \frac{2\frac{\tilde{\epsilon}}{e^{k_2}}}{3 \cosh(t)^2 (\cosh(2s) + \cos(2k_1))^2} \left( (\cosh(2s) + \cos(2k_1)) \left( \rho_1 \frac{\partial k_2}{\partial v} + \rho_2 \frac{\partial k_2}{\partial u} \right) \\
+ \sinh(2s) (\rho_2 \frac{\partial k_1}{\partial v} - \rho_1 \frac{\partial k_1}{\partial u}) + \sin(2k_1) (\rho_1 \frac{\partial k_1}{\partial v} + \rho_2 \frac{\partial k_1}{\partial u}) \\
+ \sin(2k_1) \tanh(t) \cosh(s) \cos(k_1)(k_4\rho_2 - k_3\rho_1) - \sinh(s) \sin(k_1)(k_4\rho_1 + k_3\rho_2) \right),$$
$$b_1 = \frac{2\frac{\tilde{\epsilon}}{e^{k_2}}}{3 \cosh(t)^2 (\cosh(2s) + \cos(2k_1))^2} \left( (\cosh(2s) + \cos(2k_1)) \left( \frac{\partial k_2}{\partial v} - \rho_1 \frac{\partial k_2}{\partial u} \right) \\
- \sinh(2s) (\rho_2 \frac{\partial k_1}{\partial u} + \rho_1 \frac{\partial k_1}{\partial v}) - \sin(2k_1) (\rho_1 \frac{\partial k_1}{\partial u} - \rho_2 \frac{\partial k_1}{\partial v}) \\
- \sin(2k_1) \tanh(t) \sin(k_1)(k_4\rho_2 - k_3\rho_1) + \cosh(s) \cos(k_1)(k_4\rho_1 + k_3\rho_2) \right).$$
The other equations in (7) put restrictions on \( \{k_1, k_2, k_3, k_4\} \) given by

\[
\begin{align*}
\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} &= 2 \tan(k_1) \left( k_4 \frac{\partial k_1}{\partial u} - k_3 \frac{\partial k_1}{\partial v} \right), \\
\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} &= -2 \cot(k_1) \left( k_3 \frac{\partial k_1}{\partial u} + k_4 \frac{\partial k_1}{\partial v} \right), \\
\Delta k_1 &= \frac{\sin(2k_1)}{2} \left( 2^{3/4} e^{\frac{3}{4} k_2} + k_3^2 + k_4^2 \right), \\
\Delta k_2 &= -3 \cdot 2^{1/2} e^{-\frac{3}{2} k_2} \left( e^{2k_2} + \cos(2k_1) \right).
\end{align*}
\]

Then we set \( a_3^2 > 1 \) and assume \( z^2 \neq -1 \). We then find

\[
\begin{align*}
a_3 &= -\coth(t), \\
a_2 &= \frac{\sin(2s)}{\sinh(t) \left( \cos(2s) + \cosh(2k_1) \right)} , \\
b_2 &= \frac{\sinh(2k_1)}{\sinh(t) \left( \cos(2s) + \cosh(2k_1) \right)}, \\
r &= \frac{e^{k_2}}{\sinh(t) \sqrt{\cos(2s) + \cosh(2k_1)}}, \\
a_6 + i b_6 &= \frac{k_3 + i k_4}{\rho} \sqrt{a_3^2 - 1} \left( 1 + \bar{z}^2 \right)^{-\frac{1}{2}}.
\end{align*}
\]

We obtain

\[
\begin{align*}
\gamma_1 &= \frac{\coth(t) \left( k_3 + i k_4 \right) \cos(s - ik_1) + \left( \frac{\partial k_4}{\partial v} - i \frac{\partial k_3}{\partial u} \right)}{\rho}, \\
a_1 &= \frac{2^{\frac{3}{2}} e^{\frac{3}{2} k_2}}{3 \sinh(t)^2 \left( \cos(2s) + \cosh(2k_1) \right)} \left( \left( \cos(2s) + \cosh(2k_1) \right) \left( \rho_1 \frac{\partial k_2}{\partial v} + \rho_2 \frac{\partial k_2}{\partial u} \right) \\
&\quad + \sin(2s) \left( \rho_1 \frac{\partial k_3}{\partial v} - \rho_2 \frac{\partial k_3}{\partial u} \right) - \sinh(2k_1) \left( \rho_1 \frac{\partial k_1}{\partial v} + \rho_2 \frac{\partial k_1}{\partial u} \right) \\
&\quad + \sin(2k_1) \coth(t) \left( \cos(s) \cosh(k_1) \left( k_4 \rho_2 - k_3 \rho_1 \right) + \sin(s) \sinh(k_1) \left( k_4 \rho_1 + k_3 \rho_2 \right) \right) \right), \\
b_1 &= \frac{2^{\frac{3}{2}} e^{\frac{3}{2} k_2}}{3 \sinh(t)^2 \left( \cosh(2s) + \cos(2k_1) \right)} \left( \left( \cosh(2s) + \cos(2k_1) \right) \left( \rho_2 \frac{\partial k_2}{\partial v} - \rho_1 \frac{\partial k_2}{\partial u} \right) \\
&\quad + \sin(2s) \left( \rho_2 \frac{\partial k_1}{\partial v} + \rho_1 \frac{\partial k_1}{\partial u} \right) + \sinh(2k_1) \left( \rho_1 \frac{\partial k_1}{\partial v} - \rho_2 \frac{\partial k_1}{\partial u} \right) \\
&\quad - \sinh(2k_1) \coth(t) \left( \sin(s) \sinh(k_1) \left( k_3 \rho_1 - k_4 \rho_2 \right) + \cos(s) \cosh(k_1) \left( k_4 \rho_1 + k_3 \rho_2 \right) \right) \right).
\end{align*}
\]
The functions \( \{k_1, k_2, k_3, k_4\} \) have to satisfy
\[
\begin{align*}
\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} &= 2 \tanh(k_1) \left( k_3 \frac{\partial k_1}{\partial v} - k_4 \frac{\partial k_1}{\partial u} \right), \\
\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} &= -2 \coth(k_1) \left( k_3 \frac{\partial k_1}{\partial u} + k_4 \frac{\partial k_1}{\partial v} \right), \\
\Delta k_1 &= -\sinh(2k_1) \left( 2^{\frac{1}{3}} e^{-\frac{2}{3}k_2} + \frac{k_2^2 + k_4^2}{2} \right), \\
\Delta k_2 &= 3 * 2^{\frac{1}{3}} e^{-\frac{2}{3}k_2} \left( \cosh(2k_1) - e^{2k_2} \right).
\end{align*}
\]

Finally, assume \( a_3^2 > 1 \) and \( z^2 = -1 \). Then we find
\[
\begin{align*}
a_3 &= -\coth(t), \\
\alpha_6 &= \frac{e^{k_1} \cos(\frac{5}{3}s)}{\sinh(t)^2}, \\
b_6 &= \pm \frac{e^{k_1} \sin(\frac{5}{3}s)}{\sinh(t)^2}, \\
r &= \frac{e^{k_2}}{\sinh(t)}.
\end{align*}
\]

We obtain for \( b_2 = \sqrt{a_3^2 - 1} \) that
\[
\begin{align*}
\gamma_{1+} &= \frac{5 e^{k_1 + \frac{5s}{3}} \coth(t) + e^{2k_1 + k_2}}{5 \sinh(t)} \left( \frac{\partial k_2}{\partial v} - \frac{3 \partial k_1}{\partial u} \right) - i \left( \frac{\partial k_2}{\partial u} - \frac{3 \partial k_1}{\partial v} \right), \\
a_{1+} &= -\frac{e^{k_1} \cos(\frac{5}{3}s) \coth(t) + e^{2k_1 + k_2}}{3 \sinh(t)} \left( \cos(\frac{2}{3}s) \frac{\partial k_2}{\partial v} + \sin(\frac{2}{3}s) \frac{\partial k_2}{\partial u} \right), \\
b_{1+} &= -\frac{e^{k_1} \sin(\frac{5}{3}s) \coth(t) - e^{2k_1 + k_2}}{3 \sinh(t)} \left( \cos(\frac{2}{3}s) \frac{\partial k_2}{\partial u} - \sin(\frac{2}{3}s) \frac{\partial k_2}{\partial v} \right),
\end{align*}
\]

and for \( b_2 = -\sqrt{a_3^2 - 1} \) we obtain
\[
\begin{align*}
\gamma_{1-} &= \frac{5 e^{k_1 - \frac{5s}{3}} \coth(t) + e^{2k_1 + k_2}}{5 \sinh(t)} \left( \frac{3 \partial k_2}{\partial v} - \frac{\partial k_1}{\partial u} \right) - i \left( \frac{3 \partial k_2}{\partial u} - \frac{\partial k_1}{\partial v} \right), \\
a_{1-} &= \frac{e^{k_1} \cos(\frac{5}{3}s) \coth(t) + e^{2k_1 + k_2}}{3 \sinh(t)} \left( \cos(\frac{2}{3}s) \frac{\partial k_2}{\partial v} - \sin(\frac{2}{3}s) \frac{\partial k_2}{\partial u} \right), \\
b_{1-} &= -\frac{e^{k_1} \sin(\frac{5}{3}s) \coth(t) - e^{2k_1 + k_2}}{3 \sinh(t)} \left( \cos(\frac{2}{3}s) \frac{\partial k_2}{\partial u} + \sin(\frac{2}{3}s) \frac{\partial k_2}{\partial v} \right).
\end{align*}
\]

Solving the other Gauss equations results in the relations
\[
\begin{align*}
\Delta k_1 &= e^{-\frac{2}{5}(2k_1 + k_2)} (6 + 2e^{2k_1} - 2e^{2k_2}), \\
\Delta k_2 &= e^{-\frac{2}{5}(2k_1 + k_2)} (8 + e^{2k_1} - 6e^{2k_2}).
\end{align*}
\]
These equations are valid for both \( b_2 = \pm \sqrt{a_3^2 - 1} \). We can conclude with the following proposition.

**Theorem 5.3.** Each special Lagrangian submanifold of \( \mathbb{C}H^4 \) with \( \text{SO}(2) \ltimes S_3 \)-symmetry where the only integral distribution containing \( N_1 \) is the whole tangent bundle, can be constructed in the way above using functions \( \{k_1, k_2, k_3, k_4\} \) subject to (64) in case \( a_3^2(0) < 1 \), subject to (65) in case \( a_3^2(0) > 1 \), or functions \( \{k_1, k_2\} \) subject to (66) when \( a_3^2(0) > 1 \) and \( z^2 = -1 \). Conversely, each such a construction results in such a submanifold, unique up to local isometry.

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