Orlicz-Hardy Spaces Associated with Operators Satisfying Davies-Gaffney Estimates

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Abstract. Let \( X \) be a metric space with doubling measure, \( L \) a nonnegative self-adjoint operator in \( L^2(X) \) satisfying the Davies-Gaffney estimate, \( \omega \) a concave function on \((0, \infty)\) of strictly lower type \( p_\omega \in (0, 1] \) and \( \rho(t) = t^{-1/\omega - 1}(t-1) \) for all \( t \in (0, \infty) \). The authors introduce the Orlicz-Hardy space \( H_{\omega,L}(X) \) via the Lusin area function associated to the heat semigroup, and the BMO-type space \( \text{BMO}_{\rho,L}(X) \). The authors then establish the duality between \( H_{\omega,L}(X) \) and \( \text{BMO}_{\rho,L}(X) \); as a corollary, the authors obtain the \( \rho \)-Carleson measure characterization of the space \( \text{BMO}_{\rho,L}(X) \). Characterizations of \( H_{\omega,L}(X) \), including the atomic and molecular characterizations and the Lusin area function characterization associated to the Poisson semigroup, are also presented. Let \( X = \mathbb{R}^n \) and \( L = -\Delta + V \) be a Schrödinger operator, where \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) is a non-negative potential. As applications, the authors show that the Riesz transform \( \nabla L^{-1/2} \) is bounded from \( H_{\omega,L}(\mathbb{R}^n) \) to \( L(\omega) \); moreover, if there exist \( q_1, q_2 \in (0, \infty) \) such that \( q_1 < 1 < q_2 \) and \( [\omega(t^{q_2})]^{q_1} \) is a convex function on \((0, \infty)\), then several characterizations of the Orlicz-Hardy space \( H_{\omega,L}(\mathbb{R}^n) \), in terms of the Lusin-area functions, the non-tangential maximal functions, the radial maximal functions, the atoms and the molecules, are obtained. All these results are new even when \( \omega(t) = t^p \) for all \( t \in (0, \infty) \) and \( p \in (0, 1) \).

1 Introduction

The theory of Hardy spaces \( H^p \) in various settings plays an important role in analysis and partial differential equations. However, the classical theory of Hardy spaces on \( \mathbb{R}^n \) is intimately connected with the Laplacian operator. In recent years, the study of Hardy spaces and BMO spaces associated with different operators inspired great interests; see, for example, [1, 2, 3, 11, 12, 13, 14, 18, 17, 21, 33] and their references. In [1], Auscher, Duong and McIntosh studied the Hardy space \( H^1_L(\mathbb{R}^n) \) associated to an operator \( L \) whose heat kernel satisfies a pointwise Poisson upper bound. Later, in [11, 12], Duong and Yan introduced the BMO-type space \( \text{BMO}_L(\mathbb{R}^n) \) associated to such an \( L \) and established the duality between \( H^1_L(\mathbb{R}^n) \) and \( \text{BMO}_L^*(\mathbb{R}^n) \), where \( L^* \) denotes the adjoint operator of \( L \) in \( L^2(\mathbb{R}^n) \). Yan [33] further generalized these results to the Hardy space \( H^p_L(\mathbb{R}^n) \) with \( p \in (0, 1] \) close to 1 and its dual space. Very recently, Auscher, McIntosh and Russ [2]
treated the Hardy space $H^1$ associated to the Hodge Laplacian on a Riemannian manifold with doubling measure; Hofmann and Mayboroda [18] introduced the Hardy space $H^1_L(\mathbb{R}^n)$ and its dual space adapted to a second order divergence form elliptic operator $L$ on $\mathbb{R}^n$ with complex coefficients. Notice that these operators may not have the pointwise heat kernel bounds. Furthermore, Hofmann et al [17] studied the Hardy space $H^1_L(\mathcal{X})$ on a metric measured space $\mathcal{X}$ adapted to $L$, which is nonnegative self-adjoint, and satisfies the so-called Davies-Gaffney estimate.

On the other hand, as another generalization of $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum-Orlicz in [4] and Orlicz in [23]. Since then, the theory of the Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis; see, for example, [24, 25]. Moreover, the Orlicz-Hardy spaces are also good substitutes of the Orlicz spaces in dealing with many problems of analysis. In particular, Strömberg [30], Janson [20] and Viviani [32] studied Orlicz-Hardy spaces and their dual spaces.

Recall that the Orlicz-Hardy spaces associated operators on $\mathbb{R}^n$ have been studied in [22, 21]. In [22], the heat kernel is assumed to enjoy a pointwise Poisson type upper bound; while in [21], $L$ is a second order divergence form elliptic operator on $\mathbb{R}^n$ with complex coefficients. Motivated by [18, 17, 20, 32], in this paper, we study the Orlicz-Hardy space $H_{\omega,L}(\mathcal{X})$ and its dual space associated with a nonnegative self-adjoint operator $L$ on a metric measured space $\mathcal{X}$.

Let $\mathcal{X}$ be a metric space with doubling measure and $L$ a nonnegative self-adjoint operator in $L^2(\mathcal{X})$ satisfying the Davies-Gaffney estimate. Let $\omega$ on $(0, \infty)$ be a concave function of strictly lower type $p_\omega \in (0, 1)$ and $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$ for all $t \in (0, \infty)$. A typical example of such Orlicz functions is $\omega(t) = t^p$ for all $t \in (0, \infty)$ and $p \in (0, 1]$. To develop a real-variable theory of the Orlicz-Hardy space $H_{\omega,L}(\mathcal{X})$, the key step is to establish an atomic (molecular) characterization of these spaces. To this end, we inherit a method used in [2, 21]. We first establish the atomic decomposition of the tent space $T_\omega(\mathcal{X})$, whose proof implies that if $F \in T_\omega(\mathcal{X}) \cap T^2_2(\mathcal{X})$, then the atomic decomposition of $F$ holds in both $T_\omega(\mathcal{X})$ and $T^2_2(\mathcal{X})$. Then by the fact that the operator $\pi_\Phi,L$ (see (4.6)) is bounded from $T^2_2(\mathcal{X})$ to $L^2(\mathcal{X})$, we further obtain the $L^2(\mathcal{X})$-convergence of the corresponding atomic decomposition for functions in $H_{\omega,L}(\mathcal{X}) \cap L^2(\mathcal{X})$, since for all $f \in H_{\omega,L}(\mathcal{X}) \cap L^2(\mathcal{X})$, $t^2Le^{-t^2L}f \in T^2_2(\mathcal{X}) \cap T_\omega(\mathcal{X})$. This technique plays a fundamental role in the whole paper.

With the help of the atomic decomposition, we establish the dual relation between the spaces $H_{\omega,L}(\mathcal{X})$ and $\text{BMO}_{\rho,L}(\mathcal{X})$. As a corollary, we obtain the $\rho$-Carleson measure characterization of the space $\text{BMO}_{\rho,L}(\mathcal{X})$. Having at hand the duality relation, we then obtain the atomic and molecular characterizations of the space $H_{\omega,L}(\mathcal{X})$. We also introduce the Orlicz-Hardy space $H_{\omega,S_p}(\mathcal{X})$ via the Lusin area function associated to the Poisson semigroup. With the atomic characterization of $H_{\omega,L}(\mathcal{X})$, we finally show that the spaces $H_{\omega,S_p}(\mathcal{X})$ and $H_{\omega,L}(\mathcal{X})$ coincide with equivalent norms. Let $\mathcal{X} = \mathbb{R}^n$ and $L = -\Delta + V$, where $V \in L^{1,\text{loc}}_1(\mathbb{R}^n)$ is a nonnegative potential. As applications, we show that the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$; moreover, if there exist $q_1, q_2 \in (0, \infty)$ such that $q_1 < 1 < q_2$ and $[\omega(t^{q_2})]^{q_1}$ is a convex function on $(0, \infty)$, then we
obtain several characterizations of $H_{\omega,L}(\mathbb{R}^n)$, in terms of the Lusin-area functions, the nontangential maximal functions, the radial maximal functions, the atoms and the molecules. Notice that here, the potential $V$ is not assumed to satisfy the reverse Hölder inequality.

Notice that the assumption that $L$ is nonnegative self-adjoint enables us to obtain an atomic characterization of $H_{\omega,L}(\mathcal{X})$. The method used in the proof of atomic characterization depends on the finite speed propagation property for solutions of the corresponding wave equation of $L$ and hence the self-adjointness of $L$. Without self-adjointness, as in [1, 12, 18, 22, 21, 33], where $L$ satisfies $H_{\omega}$-functional calculus and the heat kernel generated by $L$ satisfies a pointwise Poisson type upper bound or the Davies-Gaffney estimate, a corresponding (Orlicz-)Hardy space theory with the molecular (not atomic) characterization was also established in [1, 12, 18, 22, 21, 33].

Precisely, this paper is organized as follows. In Section 2, we first recall some definitions and notation concerning metric measured spaces $\mathcal{X}$, then describe some basic assumptions on the operator $L$ and the Orlicz function $\omega$ and present some properties of the operator $L$ and Orlicz functions considered in this paper.

In Section 3, we first recall some notions about tent spaces and then study the tent space $T_\omega(\mathcal{X})$ associated to the Orlicz function $\omega$. The main result of this section is that we characterize the tent space $T_\omega(\mathcal{X})$ by the atoms; see Theorem 3.1 below. As a byproduct, we see that if $f \in T_\omega(\mathcal{X}) \cap T^2_\omega(\mathcal{X})$, then the atomic decomposition holds in both $T_\omega(\mathcal{X})$ and $T^2_\omega(\mathcal{X})$, which plays an important role in the remaining part of this paper; see Corollary 3.1 below.

In Section 4, we first introduce the Orlicz-Hardy space $H_{\omega,L}(\mathcal{X})$ and prove that the operator $\pi_{\Psi,L}$ (see (4.6) below) maps the tent space $T_\omega(\mathcal{X})$ continuously into $H_{\omega,L}(\mathcal{X})$ (see Proposition 4.2 below). By this and the atomic decomposition of $T_\omega(\mathcal{X})$, we obtain that for each $f \in H_{\omega,L}(\mathcal{X})$, there is an atomic decomposition of $f$ holding in $H_{\omega,L}(\mathcal{X})$ (see Proposition 4.3 below). We should point out that to obtain the atomic decomposition of $H_{\omega,L}(\mathcal{X})$, we borrow a key idea from [17], namely, for a nonnegative self-adjoint operator $L$ in $L^2(\mathcal{X})$, then $L$ satisfies the Davies-Gaffney estimate if and only if it has the finite speed propagation property; see [17] (or Lemma 2.2 below). Via this atomic decomposition of $H_{\omega,L}(\mathcal{X})$, we further obtain the duality between $H_{\omega,L}(\mathcal{X})$ and $\text{BMO}_{\rho,L}(\mathcal{X})$ (see Theorem 4.1 below). As an application of this duality, we establish a $\rho$-Carleson measure characterization of the space $\text{BMO}_{\rho,L}(\mathcal{X})$; see Theorem 4.2 below. We point out that if $\mathcal{X} = \mathbb{R}^n$, $L = -\Delta \equiv -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $\omega$ is as above with $p_\omega \in (n/(n+1), 1]$, then the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$ in this case coincides with the Orlicz-Hardy space in [22] and it was proved there that $H_{\omega,L}(\mathbb{R}^n) = H_\omega(\mathbb{R}^n)$; see [20, 32] for the definition of $H_\omega(\mathbb{R}^n)$.

In Section 5, by Proposition 4.3 and Theorem 4.1, we establish the equivalence of $H_{\omega,L}(\mathcal{X})$ and the atomic (resp. molecular) Orlicz-Hardy $H_{\omega,\text{at}}^M(\mathcal{X})$ (resp. $H_{\omega,\text{mol}}^M(\mathcal{X})$); see Theorem 5.1 below. We notice that the series in $H_{\omega,\text{at}}^M(\mathcal{X})$ (resp. $H_{\omega,\text{mol}}^M(\mathcal{X})$) is defined to converge in the norm of $(\text{BMO}_{\rho,L}(\mathcal{X}))^*$; while in Corollary 4.1 below, the atomic decomposition holds in $H_{\omega,L}(\mathcal{X})$. Applying the atomic characterization, we further characterize the Orlicz-Hardy space $H_{\omega,L}(\mathcal{X})$ in terms of the Lusin area function associated to the Poisson semigroup; see Theorem 5.2 below.

As applications, in Section 6, we study the Hardy spaces $H_{\omega,L}(\mathbb{R}^n)$ associated to the
Schrödinger operator $L = -\Delta + V$, where $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a nonnegative potential. We characterize $H_{\omega,L}(\mathbb{R}^n)$ in terms of the Lusin-area functions, the atoms and the molecules; see Theorem 6.1 below. Moreover, we show that the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$ and from $H_{\omega,L}(\mathbb{R}^n)$ to the classical Orlicz-Hardy space $H_\omega(\mathbb{R}^n)$, if $p_\omega \in (\frac{n}{n+1},1]$; see Theorems 6.2 and 6.3 below. If there exist $q_1, q_2 \in (0,\infty)$ such that $q_1 \leq 1 < q_2$ and $[\omega(t^{q_2})]^{q_1}$ is a convex function on $(0,\infty)$, then we obtain several characterizations of $H_{\omega,L}(\mathbb{R}^n)$, in terms of the non-tangential maximal functions and the radial maximal functions; see Theorem 6.4 below. Denote $H_{\omega,L}(\mathbb{R}^n)$ by $H^p_{\omega,L}(\mathbb{R}^n)$, when $p \in (0,1]$ and $\omega(t) = t^p$ for all $t \in (0,\infty)$. We remark that the boundedness of $\nabla L^{-1/2}$ from $H^1_{\omega,L}(\mathbb{R}^n)$ to the classical Hardy space $H^1(\mathbb{R}^n)$ was established in [17]. Moreover, if $n = 1$ and $p = 1$, the Hardy space $H^1_{\omega,L}(\mathbb{R}^n)$ coincides with the Hardy space introduced by Czaja and Zienkiewicz in [9]; if $L = -\Delta + V$ and $V$ belongs to the reverse Hölder class $\mathcal{H}_q(\mathbb{R}^n)$ for some $q \geq n/2$ with $n \geq 3$, then the Hardy space $H^p_{\omega,L}(\mathbb{R}^n)$ when $p \in (n/(n+1),1]$ coincides with the Hardy space introduced by Dziubanski and Zienkiewicz [13, 14].

Finally, we make some conventions. Throughout the paper, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $X \lesssim Y$ means that there exists a positive constant $C$ such that $X \leq CY$; the symbol $\lfloor \alpha \rfloor$ for $\alpha \in \mathbb{R}$ denotes the maximal integer no greater than $\alpha$; $B(z_B, r_B)$ denotes an open ball with center $z_B$ and radius $r_B$ and $CB(z_B, r_B) \equiv B(z_B, Cr_B)$. Set $\mathbb{N} \equiv \{1,2,\cdots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. For any subset $E$ of $\mathcal{X}$, we denote by $E^c$ the set $\mathcal{X} \setminus E$. We also use $C(\gamma,\beta,\cdots)$ to denote a positive constant depending on the indicated parameters $\gamma,\beta,\cdots$.

2 Preliminaries

In this section, we first recall some notions and notation on metric measured spaces and then describe some basic assumptions on the operator $L$ studied in this paper; finally we present some basic properties on Orlicz functions and also describe some basic assumptions of them.

2.1 Metric measured spaces

Throughout the whole paper, we let $\mathcal{X}$ be a set, $d$ a metric on $\mathcal{X}$ and $\mu$ a nonnegative Borel regular measure on $\mathcal{X}$. Moreover, we assume that there exists a constant $C_1 \geq 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

\[(2.1) \quad V(x, 2r) \leq C_1 V(x, r) < \infty,\]

where $B(x,r) \equiv \{y \in \mathcal{X} : d(x,y) < r\}$ and

\[(2.2) \quad V(x,r) \equiv \mu(B(x,r)).\]

Observe that if $d$ is a quasi-metric, then $(\mathcal{X},d,\mu)$ is called a space of homogeneous type in the sense of Coifman and Weiss [8].
Notice that the doubling property (2.1) implies the following strong homogeneity property that
\[(2.3)\quad V(x, \lambda r) \leq C\lambda^n V(x, r)\]
for some positive constants $C$ and $n$ uniformly for all $\lambda \geq 1$, $x \in \mathcal{X}$ and $r > 0$. The parameter $n$ measures the dimension of the space $\mathcal{X}$ in some sense. There also exist constants $C > 0$ and $0 \leq N \leq n$ such that
\[(2.4)\quad V(x, r) \leq C \left(1 + \frac{d(x, y)}{r}\right)^N V(y, r)\]
uniformly for all $x, y \in \mathcal{X}$ and $r > 0$. Indeed, the property (2.4) with $N = n$ is a simple corollary of the strong homogeneity property (2.3). In the cases of Euclidean spaces, Lie groups of polynomial growth and more generally in Ahlfors regular spaces, $N$ can be chosen to be 0.

In what follows, for each ball $B \subset \mathcal{X}$, we set
\[(2.5)\quad U_0(B) \equiv B \quad \text{and} \quad U_j(B) \equiv 2^j B \setminus 2^{j-1} B \quad \text{for} \quad j \in \mathbb{N}.

### 2.2 Assumptions on operators $L$}

Throughout the whole paper, as in [17], we always suppose that the considered operators $L$ satisfy the following assumptions.

**Assumption (A).** The operator $L$ is a nonnegative self-adjoint operator in $L^2(\mathcal{X})$. 

**Assumption (B).** The semigroup $\{e^{-tL}\}_{t>0}$ generated by $L$ is analytic on $L^2(\mathcal{X})$ and satisfies the Davies-Gaffney estimates, namely, there exist positive constants $C_2$ and $C_3$ such that for all closed sets $E$ and $F$ in $\mathcal{X}$, $t \in (0, \infty)$ and $f \in L^2(E)$,
\[(2.6)\quad \|e^{-tL}f\|_{L^2(F)} \leq C_2 \exp \left\{ - \frac{\text{dist}(E, F)^2}{C_3 t} \right\} \|f\|_{L^2(E)},\]

where and in what follows, $\text{dist}(E, F) \equiv \inf_{x \in E, y \in F} d(x, y)$ and $L^2(E)$ is the set of all $\mu$-measurable functions on $E$ such that $\|f\|_{L^2(E)} = \left\{ \int_E |f(x)|^2 \, d\mu(x) \right\}^{1/2} < \infty$.

Examples of operators satisfying Assumptions (A) and (B) include second order elliptic self-adjoint operators in divergence form on $\mathbb{R}^n$, degenerate Schrödinger operators with nonnegative potential, Schrödinger operators with nonnegative potential and magnetic field and Laplace-Beltrami operators on all complete Riemannian manifolds; see for example, [10, 15, 28, 29].

By Assumptions (A) and (B), we have the following results which were established in [17].

**Lemma 2.1.** Let $L$ satisfy Assumptions (A) and (B). Then for any fixed $k \in \mathbb{Z}_+$ (resp. $j, k \in \mathbb{Z}_+$ with $j \leq k$), the family $\{(t^2 L)^k e^{-t^2 L}\}_{t>0}$ (resp. $\{(t^2 L)^j (I + t^2 L)^{-k}\}_{t>0}$) of operators satisfies the Davies-Gaffney estimates (2.6) with positive constants $C_2$, $C_3$ depending on $n, k$ (resp. $n, j, k$) only.
In what follows, for any operator $T$, let $K_T$ denote its integral kernel when this kernel exists. By [17, Proposition 3.4], we know that if $L$ satisfies Assumptions (A) and (B), and $T = \cos(t\sqrt{L})$, then there exists a positive constant $C_4$ such that

$$
\text{(2.7)} \quad \text{supp } K_T \subset D_t \equiv \{(x, y) \in X \times X : d(x, y) \leq C_4 t\}.
$$

This observation plays a key role in obtaining the atomic characterization of the Orlicz-Hardy space $H_{\omega,L}(X)$; see [17] and Proposition 4.3 below.

**Lemma 2.2.** Suppose that the operator $L$ satisfies Assumptions (A) and (B). Let $\varphi \in C_0^\infty(\mathbb{R})$ be even and $\text{supp } \varphi \subset (-C_4^{-1}, C_4^{-1})$, where $C_4$ is as in (2.7). Let $\Phi$ denote the Fourier transform of $\varphi$. Then for every $\kappa \in \mathbb{Z}_+$ and $t > 0$, the kernel $K_{(t^2 L)^{-1/2}(t^{2L})}$ of $(t^2 L)^{-1/2} \Phi(t^{2L})$ satisfies $\text{supp } K_{(t^2 L)^{-1/2}(t^{2L})} \subset \{(x, y) \in X \times X : d(x, y) \leq t\}$.

The following estimate is often used in this paper. Let $\mathcal{L}_{C \rightarrow C}$ denote the set of all measurable functions from $\mathbb{C}$ to $\mathbb{C}$. For $\delta > 0$, define

$$
F(\delta) \equiv \{ \psi \in \mathcal{L}_{C \rightarrow C} : \text{there exists } C > 0 \text{ such that for all } z \in \mathbb{C}, \|\psi(z)\| \leq C \frac{|z|^\delta}{1 + |z|^{2\delta}} \}.
$$

Then for any non-zero function $\psi \in F(\delta)$, we have $\int_0^\infty |\psi(t)|^2 \frac{dt}{t} < \infty$. It was proved in [17] that for all $f \in L^2(X)$,

$$
\text{(2.8)} \quad \int_0^\infty \|\psi(t\sqrt{L})f\|^2_{L^2(X)} \frac{dt}{t} = \int_0^\infty \|\psi(t)\|^2 \frac{dt}{t} \|f\|^2_{L^2(X)}.
$$

### 2.3 Orlicz functions

Let $\omega$ be a positive function defined on $\mathbb{R}_+ \equiv (0, \infty)$. The function $\omega$ is said to be of upper (resp. lower) type $p$ for some $p \in [0, \infty)$, if there exists a positive constant $C$ such that for all $t \geq 1$ (resp. $t \in (0, 1]$) and $s \in (0, \infty)$,

$$
\text{(2.9)} \quad \omega(st) \leq C t^p \omega(s).
$$

Obviously, if $\omega$ is of lower type $p$ for some $p > 0$, then $\lim_{t \to 0^+} \omega(t) = 0$. So for the sake of convenience, if it is necessary, we may assume that $\omega(0) = 0$. If $\omega$ is of both upper type $p_1$ and lower type $p_0$, then $\omega$ is said to be of type $(p_0, p_1)$. Let

$$
p_0^+ \equiv \inf \{p > 0 : \text{there exists } C > 0 \text{ such that (2.9) holds for all } t \in [1, \infty), s \in (0, \infty)\},
$$

and

$$
p_0^- \equiv \sup \{p > 0 : \text{there exists } C > 0 \text{ such that (2.9) holds for all } t \in (0, 1], s \in (0, \infty)\}.
$$

The function $\omega$ is said to be of strictly lower type $p$ if for all $t \in (0, 1)$ and $s \in (0, \infty)$, $\omega(st) \leq t^p \omega(s)$, and we define

$$
p_\omega \equiv \sup \{p > 0 : \omega(st) \leq t^p \omega(s) \text{ holds for all } s \in (0, \infty) \text{ and } t \in (0, 1)\}.
$$

It is easy to see that $p_\omega \leq p_0^- \leq p_0^+$ for all $\omega$. In what follows, $p_\omega$, $p_0^-$ and $p_0^+$ are called the strictly critical lower type index, the critical lower type index and the critical upper type index of $\omega$, respectively.
Remark 2.1. We claim that if \( p_\omega \) is defined as above, then \( \omega \) is also of strictly lower type \( p_\omega \). In other words, \( p_\omega \) is attainable. In fact, if this is not the case, then there exist some \( s \in (0, \infty) \) and \( t \in (0, 1) \) such that \( \omega(st) > t^{p_\omega} \omega(s) \). Hence there exists \( \epsilon \in (0, p_\omega) \) small enough such that \( \omega(st) > t^{p_\omega-\epsilon} \omega(s) \), which is contrary to the definition of \( p_\omega \). Thus, \( \omega \) is of strictly lower type \( p_\omega \).

Throughout the whole paper, we always assume that \( \omega \) satisfies the following assumption.

Assumption (C). Let \( \omega \) be a positive function defined on \( \mathbb{R}_+ \), which is of strictly lower type and its strictly lower type index \( p_\omega \in (0, 1] \). Also assume that \( \omega \) is continuous, strictly increasing and concave.

Notice that if \( \omega \) satisfies Assumption (C), then \( \omega(0) = 0 \) and \( \omega \) is obviously of upper type 1. Since \( \omega \) is concave, it is subadditive. In fact, let \( 0 < s < t \), then

\[
\omega(s + t) \leq \frac{s + t}{t} \omega(t) \leq \omega(t) + \frac{s}{t} \omega(s) = \omega(s) + \omega(t).
\]

For any concave function \( \omega \) of strictly lower type \( p \), if we set \( \tilde{\omega}(t) \equiv \int_0^t \frac{\omega(s)}{s} \, ds \) for \( t \in [0, \infty) \), then by [32, Proposition 3.1], \( \tilde{\omega} \) is equivalent to \( \omega \), namely, there exists a positive constant \( C \) such that \( C^{-1} \omega(t) \leq \tilde{\omega}(t) \leq C \omega(t) \) for all \( t \in [0, \infty) \); moreover, \( \tilde{\omega} \) is strictly increasing, concave, subadditive and continuous function of strictly lower type \( p \). Since all our results are invariant on equivalent functions, we always assume that \( \omega \) satisfies Assumption (C); otherwise, we may replace \( \omega \) by \( \tilde{\omega} \).

Convention. From Assumption (C), it follows that \( 0 < p_\omega \leq p_{\tilde{\omega}} \leq p_{\omega}^+ \leq 1 \). In what follows, if (2.9) holds for \( p_{\omega}^+ \) with \( t \in [1, \infty) \), then we choose \( \tilde{p}_\omega \equiv p_{\omega}^+ \); otherwise \( p_{\omega}^+ < 1 \) and we choose \( \tilde{p}_\omega \equiv (p_{\omega}^+, 1) \) to be close enough to \( p_{\omega}^+ \), the meaning will be made clear in the context.

For example, if \( \omega(t) = t^p \) with \( p \in (0, 1] \) for all \( t \in (0, \infty) \), then \( p_\omega = p_{\tilde{\omega}} = p \); if \( \omega(t) = t^{1/2} \ln(e^t + 1) \) for all \( t \in (0, \infty) \), then \( p_\omega = p_{\tilde{\omega}} = 1/2 \), but \( 1/2 < \tilde{p}_\omega < 1 \).

Let \( \omega \) satisfy Assumption (C). A measurable function \( f \) on \( \mathcal{X} \) is said to be in the space \( L(\omega) \) if \( \int_{\mathcal{X}} \omega(|f(x)|) \, d\mu(x) < \infty \). Moreover, for any \( f \in L(\omega) \), define

\[
\|f\|_{L(\omega)} \equiv \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \omega\left(\frac{|f(x)|}{\lambda}\right) \, d\mu(x) \leq 1 \right\}.
\]

Since \( \omega \) is strictly increasing, we define the function \( \rho(t) \) on \( \mathbb{R}_+ \) by

\[
(2.10) \quad \rho(t) \equiv \frac{t^{-1}}{\omega^{-1}(t^{-1})} \quad \text{for all } t \in (0, \infty),
\]

where \( \omega^{-1} \) is the inverse function of \( \omega \). Then the types of \( \omega \) and \( \rho \) have the following relation; see [32] for a proof.

Proposition 2.1. Let \( 0 < p_0 \leq p_1 \leq 1 \) and \( w \) be an increasing function. Then \( \omega \) is of type \( (p_0, p_1) \) if and only if \( \rho \) is of type \( (p_1^{-1} - 1, p_0^{-1} - 1) \).
3 Tent spaces associated to Orlicz functions

In this section, we study the tent spaces associated to Orlicz functions \( \omega \) satisfying Assumption (C). We first recall some notions.

For any \( \nu > 0 \) and \( x \in \mathcal{X} \), let \( \Gamma_\nu(x) \equiv \{(y, t) \in \mathcal{X} \times (0, \infty) : d(x, y) < \nu t\} \) denote the cone of aperture \( \nu \) with vertex \( x \in \mathcal{X} \). For any closed set \( F \) of \( \mathcal{X} \), denote by \( \mathcal{R}_\nu F \) the union of all cones with vertices in \( F \). For any open set \( O \) in \( \mathcal{X} \), denote the tent over \( O \) by \( T_\nu(O) \), which is defined as \( T_\nu(O) \equiv [\mathcal{R}_\nu(O^C)]^0 \). It is easy to see that \( T_\nu(O) = \{(x, t) \in \mathcal{X} \times (0, \infty) : d(x, O^C) \geq \nu t\} \).

In what follows, we denote by \( \mathcal{R}_\nu(F), \Gamma_\nu(x) \) and \( T_\nu(O) \) simply by \( \mathcal{R}(F), \Gamma(x) \) and \( O \), respectively.

For all measurable function \( g \) on \( \mathcal{X} \times (0, \infty) \) and \( x \in \mathcal{X} \), define

\[
A_\nu(g)(x) \equiv \left( \int_{\Gamma_\nu(x)} |g(y, t)|^2 \frac{d\mu(y)}{V(x, t) \ t} \right)^{1/2}
\]

and denote \( A_1(g) \) simply by \( A(g) \).

If \( \mathcal{X} = \mathbb{R}^n \), Coifman, Meyer and Stein [7] introduced the tent space \( T^p_2(\mathbb{R}^{n+1}) \) for \( p \in (0, \infty) \). The tent spaces \( T^p_2(\mathcal{X}) \) on spaces of homogenous type were studied by Russ [26]. Recall that a measurable function \( g \) is said to belong to the space \( T^p_2(\mathcal{X}) \) with \( p \in (0, \infty) \), if \( \|g\|_{T^p_2(\mathcal{X})} \equiv \|A(g)\|_{L^p(\mathcal{X})} < \infty \). On the other hand, Harboure, Salinas and Viviani [16] introduced the tent space \( T_\omega(\mathbb{R}^{n+1}) \) associated to the function \( \omega \).

In what follows, we denote by \( T_\omega(\mathcal{X}) \) the space of all measurable function \( g \) on \( \mathcal{X} \times (0, \infty) \) such that \( A(g) \in L(\omega) \), and for any \( g \in T_\omega(\mathcal{X}) \), define its norm by

\[
\|g\|_{T_\omega(\mathcal{X})} \equiv \|A(g)\|_{L(\omega)} = \inf \left\{ \lambda > 0 : \int_\mathcal{X} \omega \left( \frac{A(g)(x)}{\lambda} \right) d\mu(x) \leq 1 \right\}.
\]

A function \( a \) on \( \mathcal{X} \times (0, \infty) \) is called a \( T_\omega(\mathcal{X}) \)-atom if

(i) there exists a ball \( B \subset \mathcal{X} \) such that \( \text{supp} \ a \subset \overline{B} \); and

(ii) \( \int_B |a(x, t)|^2 \frac{d\mu(x)}{t} \leq [V(B)]^{-1} [\rho(V(B))]^{-2} \).

Since \( \omega^{-1} \) is concave, by the Jensen inequality and the Hölder inequality, we have

\[
\omega^{-1} \left( \frac{\int_B \omega(A(a)(x)) \ d\mu(x)}{V(B)} \right) \leq \frac{1}{V(B)} \int_B A(a)(x) \ d\mu(x) \leq \frac{\|a\|_{T^2_\omega(\mathcal{X})}}{[V(B)]^{1/2}} \leq \frac{1}{V(B) \rho(V(B))},
\]

which implies that

\[
\int_B \omega(A(a)(x)) \ d\mu(x) \leq V(B) \omega \left( \frac{1}{V(B) \rho(V(B))} \right) = 1.
\]

Thus, the claim holds.

For functions in the space \( T_\omega(\mathcal{X}) \), we have the following atomic decomposition. The proof of Theorem 3.1 is similarly to those of [7, Theorem 1], [26, Theorem 1.1] and [21, Theorem 3.1]; we omit the details.
Theorem 3.1. Let \( \omega \) satisfy Assumption (C). Then for any \( f \in T_\omega(\mathcal{X}) \), there exist \( T_\omega(\mathcal{X}) \)-atoms \( \{a_j\}_{j=1}^\infty \) and \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) such that for almost every \( (x, t) \in \mathcal{X} \times (0, \infty) \),

\[
(3.1) \quad f(x, t) = \sum_{j=1}^\infty \lambda_j a_j(x, t),
\]

and the series converges in the space \( T_\omega(\mathcal{X}) \). Moreover, there exists a positive constant \( C \) such that for all \( f \in T_\omega(\mathcal{X}) \),

\[
(3.2) \quad \Lambda(\{\lambda_j a_j\}_j) \equiv \inf \left\{ \lambda > 0 : \sum_{j=1}^\infty V(B_j) (|\lambda_j| - \lambda \Lambda(\lambda V(B_j) \rho(V(B_j)))) \leq 1 \right\} \leq C \|f\|_{T_\omega(\mathcal{X})},
\]

where \( \widehat{B}_j \) appears as the support of \( a_j \).

Remark 3.1. (i) Let \( \{\lambda^i_j\}_{i,j} \) and \( \{a^i_j\}_{i,j} \) satisfy \( \Lambda(\{\lambda^i_j a^i_j\}_j) < \infty \), where \( i = 1, 2 \). Since \( \omega \) is of strictly lower type \( p \), we have \( \Lambda(\{\lambda^i_j a^i_j\}_j)^{p_\omega} \leq \sum_{i=1}^2 \Lambda(\{\lambda^i_j a^i_j\}_j)^{p_\omega} \).

(ii) Since \( \omega \) is concave, it is of upper type 1. Thus, \( \sum_j |\lambda_j| \lesssim \Lambda(\{\lambda_j a_j\}_j) \lesssim \|f\|_{T_\omega(\mathcal{X})} \).

The following conclusions on the convergence of (3.1) play an important role in the remaining part of this paper.

Corollary 3.1. Let \( \omega \) satisfy Assumption (C). If \( f \in T^2_\omega(\mathcal{X}) \cap T_\omega(\mathcal{X}) \), then the decomposition (3.1) holds in both \( T_\omega(\mathcal{X}) \) and \( T^2_\omega(\mathcal{X}) \).

The proof of Corollary 3.1 is similar to that of [21, Proposition 3.1]; we omit the details.

In what follows, let \( T^b_\omega(\mathcal{X}) \) and \( T^{2,b}_\omega(\mathcal{X}) \) denote, respectively, the spaces of all functions in \( T_\omega(\mathcal{X}) \) and \( T^2_\omega(\mathcal{X}) \) with bounded support, where \( p \in (0, \infty) \). Here and in what follows, a function \( f \) on \( \mathcal{X} \times (0, \infty) \) having bounded support means that there exist a ball \( B \subset \mathcal{X} \) and \( 0 < c_1 < c_2 \) such that \( \text{supp} \ f \subset B \times (c_1, c_2) \).

Lemma 3.1. (i) For all \( p \in (0, \infty) \), \( T^{p,b}_\omega(\mathcal{X}) \subset T^{2,b}_\omega(\mathcal{X}) \). In particular, if \( p \in (0, 2] \), then \( T^{p,b}_\omega(\mathcal{X}) \) coincides with \( T^{2,b}_\omega(\mathcal{X}) \).

(ii) Let \( \omega \) satisfy Assumption (C). Then \( T^b_\omega(\mathcal{X}) \) coincides with \( T^{2,b}_\omega(\mathcal{X}) \).

The proof of Lemma 3.1 is similar to that of [21, Lemma 3.3] and we omit the details.

4 Orlicz-Hardy spaces and their dual spaces

In this section, we always assume that the operator \( L \) satisfies Assumptions (A) and (B), and the Orlicz function \( \omega \) satisfies Assumption (C). We introduce the Orlicz-Hardy space associated to \( L \) via the Lusin-area function and give its dual space via the atomic and molecular decompositions of the Orlicz-Hardy space. Let us begin with some notions and notation.
For all function \( f \in L^2(\mathcal{X}) \), the Lusin-area function \( S_L(f) \) is defined by setting, for all \( x \in \mathcal{X} \),

\[
S_L f(x) \equiv \left( \int \int_{\Gamma(x)} |t^2 L e^{-t^2 L} f(x)|^2 \frac{d\mu(y)}{V(x,t)} \ dt \right)^{1/2}.
\]

From (2.8), it follows that \( S_L \) is bounded on \( L^2(\mathcal{X}) \). Hofmann and Mayboroda \[18\] introduced the Hardy space \( H^1(\mathbb{R}^n) \) associated with a second order divergence form elliptic operator \( L \) as the completion of \( \{ f \in L^2(\mathbb{R}^n) : \ S_L(f) \in L^1(\mathbb{R}^n) \} \) with respect to the norm \( \| f \|_{H^1(\mathbb{R}^n)} \equiv \| S_L(f) \|_{L^1(\mathbb{R}^n)} \). Similarly, Hofmann et al \[17\] introduced the Hardy space \( H^1_\omega(\mathcal{X}) \) associated to the nonnegative self-adjoint operator \( L \) satisfying the Davies-Gaffney estimate on metric measured spaces in the same way.

Let \( \mathcal{R}(L) \) denote the range of \( L \) in \( L^2(\mathcal{X}) \) and \( \mathcal{N}(L) \) its null space. Then \( \mathcal{R}(L) \) and \( \mathcal{N}(L) \) are orthogonal and

\[
L^2(\mathcal{X}) = \mathcal{R}(L) \oplus \mathcal{N}(L).
\]

Following \[2, 17\], we introduce the Orlicz-Hardy space \( H_{\omega,L}(\mathcal{X}) \) associated to \( L \) and \( \omega \) as follows.

**Definition 4.1.** Let \( L \) satisfy Assumptions (A) and (B) and \( \omega \) satisfy Assumption (C). A function \( f \in \mathcal{R}(L) \) is said to be in \( H_{\omega,L}(\mathcal{X}) \) if \( S_L(f) \in L(\omega); \) moreover, define

\[
\| f \|_{H_{\omega,L}(\mathcal{X})} \equiv \| S_L(f) \|_{L(\omega)} \equiv \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \omega \left( \frac{S_L(f)(x)}{\lambda} \right) \ d\mu(x) \leq 1 \right\}.
\]

The Orlicz-Hardy space \( H_{\omega,L}(\mathcal{X}) \) is defined to be the completion of \( \widetilde{H}_{\omega,L}(\mathcal{X}) \) in the norm \( \| \cdot \|_{H_{\omega,L}(\mathcal{X})} \).

**Remark 4.1.** (i) Notice that for \( 0 \neq f \in L^2(\mathcal{X}) \), \( \| S_L(f) \|_{L(\omega)} = 0 \) holds if and only if \( f \in \mathcal{N}(L) \). Indeed, if \( f \in \mathcal{N}(L) \), then \( t^2 L e^{-t^2 L} f = 0 \) and hence \( \| S_L(f) \|_{L(\omega)} = 0 \). Conversely, if \( \| S_L(f) \|_{L(\omega)} = 0 \), then \( t^2 L e^{-t^2 L} f = 0 \) for all \( t \in (0, \infty) \). Hence for all \( t \in (0, \infty) \), \( (e^{-t^2 L} - I)f = \int_0^t -2sLe^{-s^2 L} f \ ds = 0 \), which further implies that \( Lf = Le^{-t^2 L} f = 0 \) and \( f \in \mathcal{N}(L) \). Thus, in Definition 4.1, it is necessary to use \( \mathcal{R}(L) \) rather than \( L^2(\mathcal{X}) \) to guarantee \( \| \cdot \|_{H_{\omega,L}(\mathcal{X})} \) to be a norm. For example, if \( \mu(\mathcal{X}) < \infty \) and \( e^{-tL} 1 = 1 \), then we have \( 1 \in L^2(\mathcal{X}) \) and \( L1 = Le^{-tL} 1 = \frac{d}{dt} e^{-tL} 1 = 0 \), which implies that \( 1 \in \mathcal{N}(L) \) and \( \| S_L(1) \|_{L(\omega)} = 0 \).

(ii) From the strictly lower type property of \( \omega \), it is easy to see that for all \( f_1, f_2 \in H_{\omega,L}(\mathcal{X}) \),

\[
\| f_1 + f_2 \|_{H_{\omega,L}(\mathcal{X})} \leq \| f_1 \|_{H_{\omega,L}(\mathcal{X})} + \| f_2 \|_{H_{\omega,L}(\mathcal{X})}.
\]

(iii) From the theorem of completion of Yosida \[34, p. 56\], it follows that \( H_{\omega,L}(\mathcal{X}) \) is dense in \( H_{\omega,L}(\mathcal{X}) \), namely, for any \( f \in H_{\omega,L}(\mathcal{X}) \), there exists a Cauchy sequence \( \{ f_k \}_{k=1}^{\infty} \) in \( H_{\omega,L}(\mathcal{X}) \) such that \( \lim_{k \to \infty} \| f_k - f \|_{H_{\omega,L}(\mathcal{X})} = 0 \).

(iv) If \( \omega(t) = t \) for all \( t \in (0, \infty) \), then the space \( H_{\omega,L}(\mathcal{X}) \) is just the space \( H^1_\omega(\mathcal{X}) \) introduced by Hofmann et al \[17\]. Moreover, if \( \omega(t) = t^p \) for all \( t \in (0, \infty) \), where \( p \in (0, 1] \), we then denote the Orlicz-Hardy space \( H_{\omega,L}(\mathcal{X}) \) by \( H^p_L(\mathcal{X}) \).
(v) If \( \mathcal{X} = \mathbb{R}^n \), \( L = -\Delta \) and \( \omega \) satisfies Assumption (C) with \( p_\omega \in (n/(n + 1), 1] \), then the Orlicz-Hardy space \( H_{\omega,L}(\mathbb{R}^n) \) coincides with the Orlicz-Hardy space in [22] and it was proved there that \( H_{\omega,L}(\mathbb{R}^n) = H_\omega(\mathbb{R}^n) \); see [20, 32] for the definition of \( H_\omega(\mathbb{R}^n) \).

We now introduce the notions of \( (\omega, M) \)-atoms and \( (\omega, M, \epsilon) \)-molecules as follows.

**Definition 4.2.** Let \( M \in \mathbb{N} \). A function \( \alpha \in L^2(\mathcal{X}) \) is called an \( (\omega, M) \)-atom associated to the operator \( L \) if there exists a function \( b \in \mathcal{D}(L^M) \) and a ball \( B \) such that

1. \( \alpha = L^M b; \)
2. \( \text{supp} L^k b \subseteq B, \ k \in \{0, 1, \cdots, M\}; \)
3. \( \|(r_B^2 L)^k b\|_{L^2(\mathcal{X})} \leq r_B^{2M} |V(B)|^{-1/2} |\rho(V(B))|^{-1}, \ k \in \{0, 1, \cdots, M\}. \)

**Definition 4.3.** Let \( M \in \mathbb{N} \) and \( \epsilon \in (0, \infty) \). A function \( \beta \in L^2(\mathcal{X}) \) is called an \( (\omega, M, \epsilon) \)-molecule associated to the operator \( L \) if there exist a function \( b \in \mathcal{D}(L^M) \) and a ball \( B \) such that

1. \( \beta = L^M b; \)
2. For every \( k \in \{0, 1, \cdots, M\} \) and \( j \in \mathbb{Z}_+ \), there holds
   \[ \|(r_B^2 L)^k b\|_{L^2(U_j(B))} \leq r_B^{2M} 2^{-j\epsilon} |V(2^j B)|^{-1/2} |\rho(V(2^j B))|^{-1}, \]
   where \( U_j(B) \) for \( j \in \mathbb{Z}_+ \) is as in (2.5).

It is easy to see that each \( (\omega, M) \)-atom is an \( (\omega, M, \epsilon) \)-molecule for any \( \epsilon \in (0, \infty) \).

**Proposition 4.1.** Let \( L \) satisfy Assumptions (A) and (B), \( \omega \) satisfy Assumption (C), \( \epsilon > n(1/p_\omega - 1/p_\omega^-) \) and \( M > n/(2(p_\omega - 1/2)) \). Then all \( (\omega, M) \)-atoms and \( (\omega, M, \epsilon) \)-molecules are in \( H_{\omega,L}(\mathcal{X}) \) with norms bounded by a positive constant.

**Proof.** Since each \( (\omega, M) \)-atom is an \( (\omega, M, \epsilon) \)-molecule, we only need to prove the proposition with an arbitrary \( (\omega, M, \epsilon) \)-molecule \( \beta \) associated to a ball \( B \equiv B(x_B, r_B) \).

Let \( \tilde{p}_\omega \) be as in Convention such that \( \epsilon > n(1/p_\omega - 1/\tilde{p}_\omega) \) and \( \lambda \in \mathbb{C} \). Then there exists \( b \in L^2(\mathcal{X}) \) such that \( \beta = L^M b \). Write

\[
\int_{\mathcal{X}} \omega(\mathcal{S}_L(\lambda \beta)(x)) \, d\mu(x)
\leq \int_{\mathcal{X}} \omega(|\lambda| \mathcal{S}_L([I - e^{-r_B^2 L}]^M \beta)(x)) \, d\mu(x) + \int_{\mathcal{X}} \omega(|\lambda| \mathcal{S}_L([I - e^{-r_B^2 L}]^M \beta)(x)) \, d\mu(x)
\leq \sum_{j=0}^{\infty} \int_{\mathcal{X}} \omega(|\lambda| \mathcal{S}_L([I - e^{-r_B^2 L}]^M (\beta \chi_{U_j(B)}))(x)) \, d\mu(x)
+ \sum_{j=0}^{\infty} \int_{\mathcal{X}} \omega(|\lambda| \mathcal{S}_L([I - e^{-r_B^2 L}]^M (L^M [b \chi_{U_j(B)}]))(x)) \, d\mu(x) \equiv \sum_{j=0}^{\infty} H_j + \sum_{j=0}^{\infty} I_j.
\]

Let us estimate the first term. For each \( j \geq 0 \), let \( B_j \equiv 2^j B \) in this proof. Since \( \omega \) is concave, by the Jensen inequality and the Hölder inequality, we obtain

\[
H_j \leq \sum_{k=0}^{\infty} \int_{U_k(B_j)} \omega(|\lambda| \mathcal{S}_L([I - e^{-r_B^2 L}]^M (\beta \chi_{U_j(B)})))(x)) \, d\mu(x)
\]
For $k = 0, 1, 2$, by the $L^2(\mathcal{X})$-boundedness of $S_L$ and $e^{-r^2_B L}$, we obtain

$$\|S_L[(I - e^{-r^2_B L})^M(\beta e^{i(x_j)})] \|_{L^2(U_k(B_j))} \lesssim \|\beta\|_{L^2(U_j(B_j))}. \tag{4.3}$$

The proof of the case $k \geq 3$ involves much more complicated calculation, which is similar to the proof of [18, Lemma 4.2]. We give the details for the completeness. Write

$$\begin{align*}
\|S_L[(I - e^{-r^2_B L})^M(\beta e^{i(x_j)})] \|_{L^2(U_k(B_j))}^2 & \leq \int_{\mathbb{R}^n} \int_{U_k(B_j)} \int_0^\infty t^2 L e^{-t^2 L} |I - e^{-r^2_B L}]^M(\beta e^{i(x_j)})|^2 \frac{d\mu(x) \, dt}{t} \\
& \lesssim \int_0^\infty \int_{\mathbb{R}^n} \int_{U_k(B_j)} t^2 L e^{-t^2 L} |I - e^{-r^2_B L}]^M(\beta e^{i(x_j)})|^2 \frac{d\mu(x) \, dt}{t} \\
& \quad + \sum_{i=0}^{k-2} \int_{2^{k-2} r_B} \int_{U_i(B_j)} \cdots \frac{d\mu(x) \, dt}{t} \equiv J + \sum_{i=0}^{k-2} J_i.
\end{align*}$$

Using the fact that $I - e^{-r^2_B L} = \int_0^{r^2_B} L e^{-sL} \, ds$, Lemma 2.1 and the Minkowski inequality, we obtain

$$\begin{align*}
J &= \int_0^\infty \int_{\mathbb{R}^n} \int_{U_k(B_j)} t^2 L^{M+1} e^{-(t^2+s_1+\cdots+s_M) L} \\
& \quad \times (\beta e^{i(x_j)}(x) \, ds_1 \cdots \, ds_M \, \frac{d\mu(x) \, dt}{t} \\
& \lesssim \left\{ \int_0^{r^2_B} \cdots \int_0^{r^2_B} \left[ \int_0^{\infty} \left( \frac{t^4 \|\beta\|^2_{L^2(U_j(B_j))}}{(t^2+s_1+\cdots+s_M)^{2(M+1)}} \right) \, dt \right]^{1/2} \, ds_1 \cdots \, ds_M \right\}^2 \\
& \lesssim r^4_B \|\beta\|^2_{L^2(U_j(B_j))} \int_0^{\infty} (2^{k+j} r_B)^{-4M} \min \left\{ \frac{2^{k+j} r_B}{t}, \frac{t}{2^{k+j} r_B} \right\} \, dt \\
& \lesssim 2^{-4M(k+j)} \|\beta\|^2_{L^2(U_j(B_j))}.
\end{align*}$$

Similarly,

$$\sum_{i=0}^{k-2} J_i = \sum_{i=0}^{k-2} \int_{U_i(B_j)} \int_{(2^{k-1-2i})^2 r_B} \cdots \int_0^{r^2_B} t^2 L^{M+1} e^{-(t^2+s_1+\cdots+s_M) L} \, ds_1 \cdots \, ds_M \, \frac{d\mu(x) \, dt}{t}.$$
Combining the estimates of J and J, we obtain that

\[(4.4) \quad ||S_L([I - e^{-r_B^2L}]^M(\beta \chi_{U_j(B)}))|L^2(U_{k_j(B)})| \lesssim \sqrt{k}2^{-2M(k+j)}||\beta||^2_{L^2(U_j(B))}.\]

By Definition 4.3, \(2Mp_\omega > n(1 - p_\omega/2)\), Assumption (C), (4.3) and (4.4), we have

\[
H \lesssim V(B_j)\omega \left( \frac{|\lambda|2^{-j\epsilon}}{V(B_j)\rho(V(B_j))} \right) + \sum_{k=3}^{\infty} V(2^kB_j)\omega \left( \frac{|\lambda|\sqrt{2^{-2M(j+k)}-j\epsilon}}{|V(2^kB_j)|^{1/2}V(B_j)^{1/2}\rho(V(B_j))} \right)
\]

\[
\lesssim 2^{-j\rho_\omega}V(B_j)\omega \left( \frac{|\lambda|}{V(B_j)\rho(V(B_j))} \right) + \sum_{k=3}^{\infty} \sqrt{k}2^{kn(1-p_\omega/2)}2^{-2Mk\rho_\omega(j+k)-j\rho_\omega}V(B_j)\omega \left( \frac{|\lambda|}{V(B_j)\rho(V(B_j))} \right)
\]

\[
\lesssim 2^{-j\rho_\omega}V(B_j)\omega \left( \frac{|\lambda|}{V(B_j)\rho(V(B_j))} \right).
\]

Since \(\rho\) is of lower type \(1/\tilde{p}_\omega - 1\) and \(\epsilon > n(1/p_\omega - 1/\tilde{p}_\omega)\), we further obtain

\[
\sum_{j=0}^{\infty} H \lesssim \sum_{j=0}^{\infty} 2^{-j\rho_\omega}V(B_j) \left\{ \frac{V(B)\rho(V(B))}{V(B_j)\rho(V(B_j))} \right\}^{p_\omega/\tilde{p}_\omega} \omega \left( \frac{|\lambda|}{V(B)\rho(V(B))} \right)
\]

\[
\lesssim \sum_{j=0}^{\infty} 2^{-j\rho_\omega}V(B_j) \left\{ \frac{V(B)}{V(B_j)} \right\}^{p_\omega/\tilde{p}_\omega} \omega \left( \frac{|\lambda|}{V(B)\rho(V(B))} \right)
\]

\[
\lesssim \sum_{j=0}^{\infty} 2^{-j\rho_\omega}2^{jn(1-p_\omega/\tilde{p}_\omega)}V(B)\omega \left( \frac{|\lambda|}{V(B)\rho(V(B))} \right) \lesssim V(B)\omega \left( \frac{|\lambda|}{V(B)\rho(V(B))} \right).
\]

Let us now estimate the remaining term \(\{I_j\}_{j \geq 0}\). Applying the Jensen inequality, we have

\[
I_j \lesssim \sum_{k=0}^{\infty} \int_{U_k(B_j)} \omega(|\lambda|S_L([I - [I - e^{-r_B^2L}]^M(\beta \chi_{U_j(B)}))|L^2(U_k(B_j)))d\mu(x)
\]

\[
\lesssim \sum_{k=0}^{\infty} V(2^kB_j)\omega \left( \frac{|\lambda|}{|V(2^kB_j)|^{1/2}} ||S_L([I - [I - e^{-r_B^2L}]^M(\beta \chi_{U_j(B)}))|L^2(U_k(B_j))) \right)
\]

Notice that

\[
||S_L([I - [I - e^{-r_B^2L}]^M(\beta \chi_{U_j(B)}))|L^2(U_k(B_j)))
\]
\[ \lesssim r_B^{-2M} \sup_{1 \leq t \leq M} \| S_L((r_B^2 L)^M e^{-t r_B^2 L} [b \chi_{U_j(B)}]) \|_{L^2(U_k(B))}. \]

For \( k = 0, 1, 2 \), by the \( L^2(\mathcal{X}) \)-boundedness of \( S_L \) and \((r_B^2 L)^M e^{-t r_B^2 L}\), we have
\[ \| S_L((r_B^2 L)^M e^{-t r_B^2 L} [b \chi_{U_j(B)}]) \|_{L^2(U_k(B))} \lesssim \| b \|_{L^2(U_j(B))}. \]

For \( k \geq 3 \), Lemma 2.1 yields that
\[ \| S_L((r_B^2 L)^M e^{-t r_B^2 L} [b \chi_{U_j(B)}]) \|_{L^2(U_k(B))}^2 \lesssim \int_B \int_{\mathbb{R}(U_k(B))} (t^2 + l r_B^2)^{2(M+1)} \| b \chi_{U_j(B)} \|_2^2 \frac{d\mu(x) dt}{t} + \int_B \int_{(2^{-k-2} t r_B)_{U_k(B)}} \cdots \| b \chi_{U_j(B)} \|_2^2 \frac{d\mu(x) dt}{t} \]
\[ \lesssim \int_B \| b \|_{L^2(U_j(B))}^2 \left[ \int_0^\infty \exp \left\{ - \frac{\text{dist}(B_j, \mathbb{R}^n \setminus 2^{-k} B_j)^2}{t^2 + l r_B^2} \right\} \frac{dt}{t} \right] + (k - 2) \int_B \frac{dt}{t^{4M+1}} \lesssim k 2^{-4M(k+j)} \| b \|_{L^2(U_j(B))}^2. \]

Combining the above estimates, similarly to the calculation of \( H_j \), we obtain
\[ \sum_{j=0}^{\infty} I_j \lesssim V(B) \omega\left( \frac{|\lambda|}{V(B) \rho(V(B))} \right), \]
which further yields that
\[ (4.5) \quad \int_{\mathcal{X}} \omega(S_L(\lambda \beta)(x)) d\mu(x) \lesssim V(B) \omega\left( \frac{|\lambda|}{V(B) \rho(V(B))} \right). \]
This implies that \( \| \beta \|_{H_{\omega,L}(\mathcal{X})} \lesssim 1 \), which completes the proof of Proposition 4.1. \( \square \)

4.1 Decompositions into atoms and molecules

In what follows, let \( M \in \mathbb{N} \) and \( M > \frac{r_B^2}{2} (\frac{1}{p_\omega} - \frac{1}{2}) \), where \( p_\omega \) is as in Assumption (C). We also let \( \Phi \) be as in Lemma 2.2 and \( \Psi(t) = t^{2(M+1)} \Phi(t) \) for all \( t \in (0, \infty) \). For all \( f \in L^2_0(\mathcal{X} \times (0, \infty)) \) and \( x \in \mathcal{X} \), define
\[ (4.6) \quad \pi_{\Psi,L} f(x) \equiv C_\Psi \int_0^\infty \Psi(t \sqrt{L})(f(\cdot, t))(x) \frac{dt}{t}, \]
where \( C_\Psi \) is the positive constant such that
\[ (4.7) \quad C_\Psi \int_0^\infty \Psi(t) t^2 e^{-t^2} \frac{dt}{t} = 1. \]
Here $L^2_b(\mathcal{X} \times (0, \infty))$ denotes the space of all function $f \in L^2(\mathcal{X} \times (0, \infty))$ with bounded support. Recall that a function $f$ on $\mathcal{X} \times (0, \infty)$ having bounded support means that there exist a ball $B \subset X$ and $0 < c_1 < c_2$ such that $\text{supp} f \subset B \times (c_1, c_2)$.

**Proposition 4.2.** Let $L$ satisfy Assumptions (A) and (B), $\omega$ satisfy Assumption (C), $M > \frac{2}{\pi} \left(\frac{1}{p_\omega} - \frac{1}{2}\right)$ and $\pi_{\Psi,L}$ be as in (4.6). Then

(i) the operator $\pi_{\Psi,L}$, initially defined on the space $T^{2,b}_2(\mathcal{X})$, extends to a bounded linear operator from $T^{2}_2(\mathcal{X})$ to $L^2(\mathcal{X})$;

(ii) the operator $\pi_{\Psi,L}$, initially defined on the space $T^{b}_b(\mathcal{X})$, extends to a bounded linear operator from $T_\omega(\mathcal{X})$ to $H_\omega, L(\mathcal{X})$.

**Proof.** (i) Suppose that $f \in T^{2,b}_2(\mathcal{X})$. For any $g \in L^2(\mathcal{X})$, by the Hölder inequality and (2.8), we have

$$|\langle \pi_{\Psi,L}(f), g \rangle| = \left| C_\Psi \int_0^\infty \langle \Psi(t\sqrt{L})f, g \rangle \frac{dt}{t} \right| \leq \int_X \int_{\Gamma(x)} \|f(y,t)\Psi(t\sqrt{L})g(y,t)\|_V \frac{d\mu(y)}{V(y,t)} \frac{dt}{t} d\mu(x) \leq \int_X \|\pi_{\Psi,L}(f)\|_{L^2(\mathcal{X})} \|g\|_{L^2(\mathcal{X})} \, d\mu(x),$$

which implies that $\|\pi_{\Psi,L}(f)\|_{L^2(\mathcal{X})} \lesssim \|f\|_{T^{2,b}_2(\mathcal{X})}$. From this and the density of $T^{2,b}_2(\mathcal{X})$ in $T^{2}_2(\mathcal{X})$, we deduce (i).

To prove (ii), let $f \in T^{b}_b(\mathcal{X})$. Then, by Lemma 3.1(ii), Corollary 3.1 and (i) of this proposition, we have

$$\pi_{\Psi,L}(f) = \sum_{j=1}^\infty \lambda_j \pi_{\Psi,L}(a_j) = \sum_{j=1}^\infty \lambda_j \alpha_j$$

in $L^2(\mathcal{X})$, where $\{\lambda_j\}_{j=1}^\infty$ and $\{a_j\}_{j=1}^\infty$ satisfy (3.2). Recall that here, $\text{supp} a_j \subset \hat{B}_j$ and $B_j$ is a ball of $\mathcal{X}$.

On the other hand, by (2.8), we have that the operator $S_L$ is bounded on $L^2(\mathcal{X})$, which implies that for all $x \in X$, $S_L(\pi_{\Psi,L}(f))(x) \leq \sum_{j=1}^\infty |\lambda_j| S_L(\alpha_j)(x)$. This, combined with the monotonicity, continuity and subadditivity of $\omega$, yields that

$$\int_X \omega(S_L(\pi_{\Psi,L}(f))(x)) \, d\mu(x) \leq \sum_{j=1}^\infty \int_X \omega(|\lambda_j| S_L(\alpha_j)(x)) \, d\mu(x).$$

We now show that $\alpha_j = \pi_{\Psi,L}(a_j)$ is a multiple of an $(\omega, M)$-atom for each $j$. Let

$$b_j \equiv C_\Psi \int_0^\infty t^{2M} L \Phi(t\sqrt{L})(a_j(t)) \frac{dt}{t}.$$ 

Then $\alpha_j = L^M b_j$. Moreover, by Lemma 2.2, for each $k \in \{0, 1, \cdots, M\}$, we have $\text{supp} L^k b_j \subset B_j$. On the other hand, for any $h \in L^2(B_j)$, by the Hölder inequality and (2.8), we have

$$\left| \int_X (r_{B_j}^2 L)^k b_j(x) h(x) \, d\mu(x) \right|$$
By (4.5), we obtain
\[
\int_X \omega(S_L(\pi_{\Psi,L}(f))(x)) \, d\mu(x) \leq \sum_{j=1}^{\infty} \int_X \omega(|\lambda_j S_L(\alpha_j)(x)|) \, d\mu(x)
\]
For each \( j \), we have
\[
\int_X \omega(S_L(\pi_{\Psi,L}(f))(x)) \, d\mu(x) \leq \sum_{j=1}^{\infty} V(B_j) \omega\left( \frac{|\lambda_j|}{V(B_j) \rho(V(B_j))} \right),
\]
which implies that \( \|\pi_{\Psi,L}(f)\|_{H_{\omega,L}(X)} \lesssim \Lambda(\{\lambda_j a_j\}_j) \lesssim \|f\|_{T_\omega(X)} \), and hence completes the proof of Proposition 4.2. \( \square \)

**Proposition 4.3.** Let \( L \) satisfy Assumptions (A) and (B), \( \omega \) satisfy Assumption (C) and \( M > \frac{1}{4} \left( \frac{1}{p_\omega} - \frac{1}{2} \right) \). Then for all \( f \in H_{\omega,L}(X) \cap L^2(X) \), there exist \( (\omega,M) \)-atoms \( \{\alpha_j\}_{j=1}^{\infty} \) and \( \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \) such that \( f = \sum_{j=1}^{\infty} \lambda_j \alpha_j \) in both \( H_{\omega,L}(X) \) and \( L^2(X) \). Moreover, there exists a positive constant \( C \) such that for all \( f \in H_{\omega,L}(X) \cap L^2(X) \),
\[
\Lambda(\{\lambda_j\alpha_j\}_j) \equiv \inf \left\{ \lambda > 0 : \sum_{j=1}^{\infty} V(B_j) \omega \left( \frac{|\lambda_j|}{\lambda V(B_j) \rho(V(B_j))} \right) \leq 1 \right\} \leq C \|f\|_{H_{\omega,L}(X)},
\]
where for each \( j \), \( \alpha_j \) is supported in the ball \( B_j \).

**Proof.** Let \( f \in H_{\omega,L}(X) \cap L^2(X) \). Then by \( H_\infty \)-functional calculus for \( L \) together with (4.7), we have
\[
f = C_{\Psi} \int_0^{\infty} \Psi(t\sqrt{L}) t^2 Le^{-t^2L} f \, dt = \pi_{\Psi,L}(t^2 Le^{-t^2L} f)
\]
in \( L^2(X) \). By Definition 4.1 and (2.8), we have \( t^2 Le^{-t^2L} f \in T_\omega(X) \cap T_2^2(X) \). Applying Theorem 3.1, Corollary 3.1 and Proposition 4.2 to \( t^2 Le^{-t^2L} f \), we obtain
\[
f = \pi_{\Psi,L}(t^2 Le^{-t^2L}) = \sum_{j=1}^{\infty} \lambda_j \pi_{\Psi,L}(a_j) = \sum_{j=1}^{\infty} \lambda_j \alpha_j
\]
in both \( L^2(X) \) and \( H_{\omega,L}(X) \), and
\[
\Lambda(\{\lambda_j\alpha_j\}_j) \lesssim \|t^2 Le^{-t^2L} f\|_{T_\omega(X)} \sim \|f\|_{H_{\omega,L}(X)}.
\]
On the other hand, by the proof of Proposition 4.2, we obtain that for each \( j \in \mathbb{N} \), \( \alpha_j \) is an \( (\omega,M) \)-atom up to a harmless constant, which completes the proof of Proposition 4.3. \( \square \)
Corollary 4.1. Let $L$ satisfy Assumptions (A) and (B), $\omega$ satisfy Assumption (C) and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Then for all $f \in H_{\omega,L}(\mathcal{X})$, there exist $(\omega, M)$-atoms $\{\alpha_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subseteq C$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $H_{\omega,L}(\mathcal{X})$. Moreover, there exists a positive constant $C$ independent of $f$ such that $\Lambda((\lambda_j \alpha_j)_{j=1}^\infty) \leq C\|f\|_{H_{\omega,L}(\mathcal{X})}$.

Let $H_{\omega,\text{fin}}^{\text{at}, M}(\mathcal{X})$ and $H_{\omega,\text{fin}}^{\text{mol}, M, \epsilon}(\mathcal{X})$ denote the spaces of finite combinations of $(\omega, M)$-atoms and $(\omega, M, \epsilon)$-molecules, respectively. From Corollary 4.1 and Proposition 4.1, we deduce the following density conclusions.

Corollary 4.2. Let $L$ satisfy Assumptions (A) and (B), $\omega$ satisfy Assumption (C), $\epsilon > n(1/p_\omega - 1/p^*_\omega)$ and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Then both the spaces $H_{\omega,\text{fin}}^{\text{at}, M}(\mathcal{X})$ and $H_{\omega,\text{fin}}^{\text{mol}, M, \epsilon}(\mathcal{X})$ are dense in the space $H_{\omega,L}(\mathcal{X})$.

4.2 Dual spaces of Orlicz-Hardy spaces

In this subsection, we study the dual space of the Orlicz-Hardy space $H_{\omega,L}(\mathcal{X})$. We begin with some notions.

Let $\phi = L^M \nu$ be a function in $L^2(\mathcal{X})$, where $\nu \in D(L^M)$. Following [18, 17], for $\epsilon > 0$, $M \in \mathbb{N}$ and fixed $x_0 \in \mathcal{X}$, we introduce the space

$$\mathcal{M}_{\omega}^{M, \epsilon}(L) \equiv \{ \phi = L^M \nu \in L^2(\mathcal{X}) : \|\phi\|_{\mathcal{M}_{\omega}^{M, \epsilon}(L)} < \infty \},$$

where

$$\|\phi\|_{\mathcal{M}_{\omega}^{M, \epsilon}(L)} \equiv \sup_{\lambda \in \mathbb{Z}_+} \left\{ 2^n [V(x_0, 2^j)]^1/2 \rho(V(x_0, 2^j)) \sum_{k=0}^M \|L^k \nu\|_{L^2(U^*_j(B(x_0, 1)))} \right\}.$$

Notice that if $\phi \in \mathcal{M}_{\omega}^{M, \epsilon}(L)$ for some $\epsilon > 0$ with norm 1, then $\phi$ is an $(\omega, M, \epsilon)$-molecule adapted to the ball $B(x_0, 1)$. Conversely, if $\beta$ is an $(\omega, M, \epsilon)$-molecule adapted to any ball, then $\beta \in \mathcal{M}_{\omega}^{M, \epsilon}(L)$.

Let $A_t$ denote either $(I + t^2 L)^{-1}$ or $e^{-t^2 L}$ and $f \in (\mathcal{M}_{\omega}^{M, \epsilon}(L))^*$, the dual space of $\mathcal{M}_{\omega}^{M, \epsilon}(L)$. We claim that $(I - A_t)^M f \in L^2_{\text{loc}}(\mathcal{X})$ in the sense of distributions. In fact, for any ball $B$, if $\psi \in L^2(B)$, then it follows from the Davies-Gaffney estimate (2.6) that $(I - A_t)^M \psi \in \mathcal{M}_{\omega}^{M, \epsilon}(L)$ for every $\epsilon > 0$. Thus,

$$|\langle (I - A_t)^M f, \psi \rangle| \equiv |\langle f, (I - A_t)^M \psi \rangle| \leq C(t, r_B, \text{dist} (B, x_0))\|f\|_{(\mathcal{M}_{\omega}^{M, \epsilon}(L))^*}\|\psi\|_{L^2(B)},$$

which implies that $(I - A_t)^M f \in L^2_{\text{loc}}(\mathcal{X})$ in the sense of distributions.

Finally, for any $M \in \mathbb{N}$, define

$$\mathcal{M}_{\omega}^M(\mathcal{X}) \equiv \bigcap_{\epsilon > n(1/p_\omega - 1/p^*_\omega)} (\mathcal{M}_{\omega}^{M, \epsilon}(L))^*.$$
Definition 4.4. Let \( L \) satisfy Assumptions (A) and (B), \( \omega \) satisfy Assumption (C), \( \rho \) be as in (2.10) and \( M > \frac{2}{3} \left( \frac{1}{p_\infty} - \frac{1}{2} \right) \). A functional \( f \in \mathcal{M}_M^\rho(\mathcal{X}) \) is said to be in \( \text{BMO}_M^\rho(\mathcal{X}) \) if

\[
\|f\|_{\text{BMO}_M^\rho(\mathcal{X})} \equiv \sup_{B \subset \mathcal{X}} \frac{1}{\rho(V(B))} \left[ \frac{1}{V(B)} \int_B |(I - e^{-r_B^2 L})^M f(x)|^2 \, d\mu(x) \right]^{1/2} < \infty,
\]

where the supremum is taken over all ball \( B \) of \( \mathcal{X} \).

The proofs of the following two propositions are similar to those of Lemmas 8.1 and 8.3 of [18], respectively; we omit the details.

Proposition 4.5. Let \( L, \omega, \rho \) and \( M \) be as in Definition 4.4. Then \( f \in \text{BMO}_M^\rho(\mathcal{X}) \) if and only if \( f \in \mathcal{M}_M^\rho(\mathcal{X}) \) and

\[
\sup_{B \subset \mathcal{X}} \frac{1}{\rho(V(B))} \left[ \frac{1}{V(B)} \int_B |(I - (I + r_B^2 L)^{-1})^M f(x)|^2 \, d\mu(x) \right]^{1/2} < \infty.
\]

Moreover, the quantity appeared in the left-hand side of the above formula is equivalent to \( \|f\|_{\text{BMO}_M^\rho(\mathcal{X})} \).

Proposition 4.6. Let \( L, \omega, \rho \) and \( M \) be as in Definition 4.4. Then there exists a positive constant \( C \) such that for all \( f \in \text{BMO}_M^\rho(\mathcal{X}) \),

\[
\sup_{B \subset \mathcal{X}} \frac{1}{\rho(V(B))} \left[ \frac{1}{V(B)} \int_B |(t^2 L)^M e^{-t^2 L} f(x)|^2 \frac{d\mu(x) \, dt}{t} \right]^{1/2} \leq C \|f\|_{\text{BMO}_M^\rho(\mathcal{X})}.
\]

The following Proposition 4.6 and Corollary 4.3 are a kind of Calderón reproducing formulae.

Proposition 4.6. Let \( L, \omega, \rho \) and \( M \) be as in Definition 4.4, \( \epsilon > 0 \) and \( \tilde{M} > M + \epsilon + \frac{N}{4} + \frac{N}{4} \left( \frac{1}{p_\infty} - 1 \right) \), where \( N \) is as in (2.4). Fix \( x_0 \in \mathcal{X} \). Assume that \( f \in \mathcal{M}_M^\rho(\mathcal{X}) \) satisfies

\[
\int_{\mathcal{X}} \frac{|(I - (I + L)^{-1})^M f(x)|^2}{1 + [d(x, x_0)]^{n+\epsilon+2N(1/p_\infty-1)}} \, d\mu(x) < \infty.
\]

Then for all \( (\omega, \tilde{M}) \)-atom \( \alpha \),

\[
\langle f, \alpha \rangle = \tilde{C}_M \int_{\mathcal{X} \times (0, \infty)} (t^2 L)^M e^{-t^2 L} f(x) \frac{t^2 L e^{-t^2 L} \alpha(x)}{t} \frac{d\mu(x) \, dt}{t},
\]

where \( \tilde{C}_M \) is the positive constant satisfying \( \tilde{C}_M \int_0^\infty t^{2(M+1)} e^{-2t^2} \, dt = 1 \).

Proof. Let \( \alpha \) be an \( (\omega, \tilde{M}) \)-atom supported in \( B \equiv B(x_B, r_B) \). Notice that (4.8) implies that

\[
\int_{\mathcal{X}} \frac{|(I - (I + L)^{-1})^M f(x)|^2}{r_B + [d(x, x_B)]^{n+\epsilon+2N(1/p_\infty-1)}} \, d\mu(x) < \infty.
\]
For $R > \delta > 0$, write

\[
\tilde{C}_M \int_\delta^R \int_X (t^2 L)^M e^{-t^2 L} f(x) \frac{d\mu(x)}{t} dt = \left\langle f, \tilde{C}_M \int_\delta^R (t^2 L)^{M+1} e^{-t^2 L} \alpha dt \right\rangle = \left\langle f, \alpha - \tilde{C}_M \int_\delta^R (t^2 L)^{M+1} e^{-2t^2 L} \alpha dt \right\rangle.
\]

Since $\alpha$ is an $(\omega, \tilde{M})$-atom, by Definition 4.2, there exists $b \in L^2(X)$ such that $\alpha = L\tilde{M}b$. Thus, by the fact that $\tilde{M} > M + \frac{n}{4} + \frac{N}{2} \omega - 2$, we obtain

\[
\alpha = L\tilde{M}b = (L - L(I + L)^{-1} + L(I + L)^{-1})^M L\tilde{M}^{-M}b
\]

\[
= \sum_{k=0}^M C_M^k (L - L(I + L)^{-1})^{M-k} (L(I + L)^{-1})^k L\tilde{M}^{-k}b
\]

where $C_M^k$ denotes the combinatorial number, which together with $H$-functional calculus further implies that

\[
\left\langle f, \alpha - \tilde{C}_M \int_\delta^R (t^2 L)^{M+1} e^{-2t^2 L} \alpha dt \right\rangle = \left\langle f, \tilde{C}_M \int_\delta^R (t^2 L)^{M+1} e^{-2t^2 L} \tilde{M}^{-k}b dt \right\rangle
\]

By (4.8), we see that up to a harmless constant, the term I is bounded by

\[
\left\langle f, \frac{\left| (I - (I + L)^{-1})^M f(x) \right|^2}{r_B + \left[ d(x, x_B) \right]^{n+\epsilon+2N(\frac{1}{\omega} - 1)}} \right\rangle_{L^2(X)} \leq \sup_{0 \leq k \leq M} \left\langle f, \int_\delta^\infty (t^2 L)^{M+1} e^{-2t^2 L} \tilde{M}^{-k}b dt \right\rangle
\]

\[
	imes \left( \int_X \frac{dt}{t} \right)^{1/2} \left( r_B + \left[ d(x, x_B) \right]^{n+\epsilon+2N(\frac{1}{\omega} - 1)} \right) \mu(x)
\]

\[
\leq \sup_{0 \leq k \leq M} \sum_{j=0}^\infty \left( 2^j r_B \right)^{n+\epsilon+2N(\frac{1}{\omega} - 1)} \int_R \left( (t^2 L)^{M+1} e^{-2t^2 L} \right)_{L^2(U_j(x_B))} dt \leq \frac{dt}{t^{2(M-k)+1}}
\]

\[
\leq \sup_{0 \leq k \leq M} \sum_{j=0}^\infty \left( 2^j r_B \right)^{n+\epsilon+2N(\frac{1}{\omega} - 1)} \left\| b \right\|_{L^2(U_j(x_B))} \int_R \frac{dt}{t^{2(M-k)+1}}
\]
as $R \to \infty$.

Similarly, the term $H$ is controlled by

\[
\left\{ \int_{\mathcal{X}} \frac{|(I - (I + L)^{-1}Mf(x)|^2}{r_B + [d(x, x_B)]^{n+2N(\frac{1}{p}-1)}} \, d\mu(x) \right\}^{1/2} \sup_{0 \leq k \leq M} \left\{ \int_0^\delta \left\{ \left( t^2 L \right)^{M+1} + e^{-2t^2L}L^{\tilde{M}-k}b(x) \right\} dt \right\} \leq \sum_{j=0}^\infty \sup_{0 \leq k \leq M} (2j r_B)^{(n+\epsilon)/2} \left\{ \int_{U_j(B)} \left\{ \int_0^\delta \left( t^2 L \right)^{M+1} + e^{-2t^2L}L^{\tilde{M}-k}b(x) \right\} dt \right\} \leq \sum_{j=0}^\infty H_j.
\]

For $j \geq 3$, we further have

\[
H_j \leq \sup_{0 \leq k \leq M} (2j r_B)^{n+\epsilon/2} \left\{ \int_0^\delta \left( t^2 L \right)^{M+1} e^{-2t^2L}L^{\tilde{M}-k}b(x) \right\} dt \leq (2j r_B)^{n+\epsilon/2} \left\{ \int_0^\delta \left( t^2 R \right)^{n/2+\epsilon/2} \right\} dt \leq 2^{-j/2} \delta^{n/2+\epsilon/2} \leq \delta^{-\epsilon} \to 0,
\]

as $R \to \infty$. The estimates (4.9) and (4.10) imply that $H \to 0$ as $\delta \to 0$, and hence complete the proof of Proposition 4.6.
To prove that Proposition 4.6 holds for all \( f \in \text{BMO}^M_{\rho,L}(X) \), we need the following “dyadic cubes” on spaces of homogeneous type constructed by Christ [6, Theorem 11].

**Lemma 4.1.** There exists a collection \( \{Q^k_\alpha \subset X : k \in \mathbb{Z}, \alpha \in I_k\} \) of open subsets, where \( I_k \) denotes some (possibly finite) index set depending on \( k \), and constants \( \delta \in (0,1) \), \( a_0 \in (0,1) \) and \( C_5 \in (0,\infty) \) such that

- (i) \( \mu(X \setminus \cup \alpha Q^k_\alpha) = 0 \) for all \( k \in \mathbb{Z} \);
- (ii) if \( i \geq k \), then either \( Q^k_\alpha \subset Q^k_\beta \) or \( Q^i_\alpha \cap Q^k_\beta = \emptyset \);
- (iii) for each \((k,\alpha)\) and each \( i < k \), there exists a unique \( \beta \) such that \( Q^k_\alpha \subset Q^i_\beta \);
- (iv) the diameter of \( Q^k_\alpha \) is no more than \( C_5 \delta^k \);
- (v) each \( Q^k_\alpha \) contains certain ball \( B(z^k_\alpha, a_0 \delta^k) \).

From Proposition 4.6 and Lemma 4.1, we deduce the following conclusion.

**Corollary 4.3.** Let \( L, \omega, \rho \) and \( M \) be as in Definition 4.4 and \( \widetilde{M} > M + \frac{n}{4} + \frac{N}{2} (\frac{1}{p_\omega} - 1) \). Then for all \((\omega, \widetilde{M})\)-atom \( \alpha \) and \( f \in \text{BMO}^M_{\rho,L}(X) \),

\[
\langle f, \alpha \rangle = C_M \int_{\mathcal{X} \times (0,\infty)} (t^2 L)^M e^{-t^2 L} f(x) \frac{(t^2 L e^{-t^2 L} \alpha(x))}{t} \, d\mu(x) \, dt, 
\]

where \( C_M \) is as in Proposition 4.6.

**Proof.** Let \( \epsilon \in (0, \widetilde{M} - M - \frac{n}{4} - \frac{N}{2} (\frac{1}{p_\omega} - 1)) \). By Proposition 4.6, we only need to show that (4.8) with such an \( \epsilon \) holds for all \( f \in \text{BMO}^M_{\rho,L}(X) \).

Let all the notation be the same as in Lemma 4.1. For each \( j \in \mathbb{Z} \), choose \( k_j \in \mathbb{Z} \) such that \( C_5 \delta^{k_j} \leq 2^j < C_5 \delta^{k_j-1} \). Let \( B = B(x_0,1) \), where \( x_0 \) is as in (4.8), and

\[ M_j \equiv \{ \beta \in I_{k_0} : Q^{k_0}_{\beta} \cap B(x_0, C_5 \delta^{k_j-1}) \neq \emptyset \}. \]

Then for each \( j \in \mathbb{Z}_+ \),

\[
U_j(B) \subset B(x_0, C_5 \delta^{k_j-1}) \subset \bigcup_{\beta \in M_j} Q^{k_0}_{\beta} \subset B(x_0, 2C_5 \delta^{k_j-1}).
\]

By Lemma 4.1, the sets \( Q^{k_0}_{\beta} \) for all \( \beta \in M_j \) are disjoint. Moreover, by (iv) and (v) of Lemma 4.1, there exists \( z^{k_0}_{\beta} \in Q^{k_0}_{\beta} \) such that

\[
B(z^{k_0}_{\beta}, a_0 \delta^{k_0}) \subset Q^{k_0}_{\beta} \subset B(z^{k_0}_{\beta}, C_5 \delta^{k_0}) \subset B(z^{k_0}_{\beta}, 1).
\]

Thus, by Proposition 4.4, (2.4) and the fact that \( \rho \) is of upper type \( 1/p_\omega - 1 \), we have

\[
H \equiv \left\{ \int_{\mathcal{X}} \frac{|(I - (1 + L)^{-1})^M f(x)|^2}{1 + [d(x,x_0)]^{n+\epsilon+2N(1/p_\omega-1)}} \, d\mu(x) \right\}^{1/2} 
\lesssim \sum_{j=0}^{\infty} 2^{-j[(n+\epsilon)/2+N(1/p_\omega-1)]} \left\{ \sum_{\beta \in M_j} \int_{Q^{k_0}_{\beta}} |(I - (1 + L)^{-1})^M f(x)|^2 \, d\mu(x) \right\}^{1/2}.
\]
By (4.11), (4.12) and (2.3), we obtain
\[ \sum_{j=0}^{\infty} 2^{-j(n+\epsilon)/2+N(1/p_{\omega}-1)} \left\{ \sum_{\beta \in M_j} \rho(V(z_{\beta}^{k_0}, 1)) \right\}^{1/2} \]
\[ \lesssim \sum_{j=0}^{\infty} 2^{-j(n+\epsilon)/2+N(1/p_{\omega}-1)} \left\{ \sum_{\beta \in M_j} 2^{2jN(1/p_{\omega}-1)} \rho(V(x_0, 1)) \right\}^{1/2} \]
\[ \lesssim \sum_{j=0}^{\infty} 2^{-j(n+\epsilon)/2} \left\{ \sum_{\beta \in M_j} V(z_{\beta}^{k_0}, 1) \right\}^{1/2} . \]

By (4.11), (4.12) and (2.3), we obtain
\[ \sum_{\beta \in M_j} V(z_{\beta}^{k_0}, 1) \lesssim \sum_{\beta \in M_j} V(z_{\beta}^{k_0}, a\delta^{k_0}) \lesssim \sum_{\beta \in M_j} V(Q_{\beta}^{k_0}) \lesssim V(x_0, 2C_5\delta^{k_j-1}) \lesssim 2^{jn}V(B), \]
which further implies that \( H < \infty \), and hence completes the proof of Corollary 4.3. \( \square \)

Using Corollary 4.3, we now establish the duality between \( H_{\omega,L}(\mathcal{X}) \) and \( \text{BMO}^M_{\rho,L}(\mathcal{X}) \).

**Theorem 4.1.** Let \( L \) satisfy Assumptions (A) and (B), \( \omega \) satisfy Assumption (C), \( \rho \) be as in (2.10), \( M > \frac{N}{2}(\frac{1}{p_{\omega}} - \frac{1}{2}) \) and \( \bar{M} > M + \frac{N}{2}(\frac{1}{p_{\omega}} - 1) \). Then \( (H_{\omega,L}(\mathcal{X}))^* \), the dual space of \( H_{\omega,L}(\mathcal{X}) \), coincides with \( \text{BMO}^M_{\rho,L}(\mathcal{X}) \) in the following sense.

(i) Let \( g \in \text{BMO}^M_{\rho,L}(\mathcal{X}) \). Then the linear functional \( \ell_g \), which is initially defined on \( H_{\omega,\text{fin}}^{\text{at},\bar{M}}(\mathcal{X}) \) by
\[ \ell_g(f) \equiv \langle g, f \rangle, \]
(4.13)
has a unique extension to \( H_{\omega,L}(\mathcal{X}) \) with \( \|\ell_g\|_{(H_{\omega,L}(\mathcal{X}))^*} \leq C\|g\|_{\text{BMO}^M_{\rho,L}(\mathcal{X})} \), where \( C \) is a positive constant independent of \( g \).

(ii) Conversely, let \( \epsilon > n(1/p_{\omega}-1/p_{\omega}^*) \). Then for any \( \ell \in (H_{\omega,L}(\mathcal{X}))^* \), \( \ell \in \text{BMO}^M_{\rho,L}(\mathcal{X}) \) with \( \|\ell\|_{\text{BMO}^M_{\rho,L}(\mathcal{X})} \leq C\|\ell\|_{(H_{\omega,L}(\mathcal{X}))^*} \) and (4.13) holds for all \( f \in H_{\omega,\text{fin}}^{\text{at},\bar{M},\epsilon}(\mathcal{X}) \), where \( C \) is a positive constant independent of \( \ell \).

**Proof.** Let \( g \in \text{BMO}^M_{\rho,L}(\mathcal{X}) \). For any \( f \in H_{\omega,\text{fin}}^{\text{at},\bar{M}}(\mathcal{X}) \), by Proposition 4.1, we have that \( t^2Le^{-t^2L}f \in T_\omega(\mathcal{X}) \). By Theorem 3.1, there exist \( \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \) and \( T_{\omega}(\mathcal{X}) \)-atoms \( \{a_j\}_{j=1}^{\infty} \) supported in \( \{\hat{B}_j\}_{j=1}^{\infty} \) such that (3.1) and (3.2) hold. This, together with Corollary 4.3, the Hölder inequality, Proposition 4.5 and Remark 3.1(ii), yields that
\[ |\langle g, f \rangle| = \left| C_{\bar{M}} \int_0^{\infty} \int_{\mathcal{X}} (t^2L)^M e^{-t^2L} g(x) t^2Le^{-t^2L} f(x) \frac{d\mu(x)dt}{t} \right| \]
\[ \lesssim \sum_{j} |\lambda_j| \int_0^{\infty} \int_{\mathcal{X}} |(t^2L)^M e^{-t^2L} g(x)a_j(x,t)| \frac{d\mu(x)dt}{t} \]
\[ \lesssim \sum_{j} |\lambda_j||a_j|_{T_2(\mathcal{X})} \left( \int_{\hat{B}_j} |(t^2L)^M e^{-t^2L} g(x)|^2 \frac{d\mu(x)dt}{t} \right)^{1/2} . \]
\[ \sum_j |\lambda_j| \|g\|_{\text{BMO}_{\rho,L}(\mathcal{X})} \lesssim \|i^2 L e^{-tL} f\|_{\mathcal{I}_\omega(\mathcal{X})} \|g\|_{\text{BMO}_{\rho,L}(\mathcal{X})} \]

\[ \sim \|f\|_{H_{\omega,L}(\mathcal{X})} \|g\|_{\text{BMO}_{\rho,L}(\mathcal{X})}. \]

Then by Proposition 4.2, we obtain (i).

Conversely, let \( \ell \in (H_{\omega,L}(\mathcal{X}))^* \). If \( g \in \mathcal{M}_{\omega,L}^M(L) \), then \( g \) is a multiple of an \((\omega, M, \epsilon)\)-molecule. Moreover, if \( \epsilon > n(1/p_\omega - 1/p_\rho^+) \), then by Proposition 4.1, we have \( g \in H_{\omega,L}(\mathcal{X}) \) and hence \( \mathcal{M}_{\omega,L}^M(L) \subset H_{\omega,L}(\mathcal{X}) \). Then \( \ell \in \mathcal{M}_{\omega,L}^M(\mathcal{X}) \).

On the other hand, for any ball \( B \), let \( \phi \in L^2(B) \) with \( \|\phi\|_{L^2(B)} \leq \frac{1}{|V(B)|^{1/2} \rho(V(B))} \) and \( \bar{\beta} = (I - [I + r_B^2 L]^{-1})M \phi \). Obviously, \( \bar{\beta} = (r_B^2 L)^M (I + r_B^2 L)^{-M} \phi = L^M \bar{b} \). Then from the fact that \( \{(r_B^2 L)^k (I + r_B^2 L)^{-M} \}_{0 \leq k \leq M} \) satisfy the Davies-Gaffney estimate (see Lemma 2.1), we deduce that for each \( j \in \mathbb{Z}_+ \) and \( k = 0, 1, \ldots, M \), we have

\[ \|(r_B^2 L)^k \bar{b}\|_{L^2(U_j(B))} = r_B^{2M} \| (I - [I + r_B^2 L]^{-1})^{k} (I + r_B^2 L)^{-M} \phi \|_{L^2(U_j(B))} \]

\[ \lesssim r_B^{2M} \exp \left\{ \frac{- \text{dist}(B, U_j(B))}{r_B} \right\} \|\phi\|_{L^2(B)} \]

\[ \lesssim r_B^{2M} 2^{-2j(M+\epsilon)} 2^j (1/p_\omega - 1/2) [V(2^j B)]^{-1/2} \rho(V(2^j B)) \]

\[ \lesssim r_B^{2M} 2^{-2j} [V(2^j B)]^{-1/2} \rho(V(2^j B))^{-1}, \]

since \( 2M > n(1/p_\omega - 1/2) \). Thus, \( \bar{\beta} \) is a multiple of an \((\omega, M, \epsilon)\)-molecule. Since \( (I - [I + t^2 L]^{-1})M \ell \) is well defined and belongs to \( L^2_{\text{loc}}(\mathcal{X}) \) for every \( t > 0 \), by the fact that \( \|\bar{\beta}\|_{H_{\omega,L}(\mathcal{X})} \lesssim 1 \), we have

\[ |\langle (I - [I + r_B^2 L]^{-1})^M \ell, \phi \rangle| = |\langle \ell, (I - [I + r_B^2 L]^{-1})^M \phi \rangle| = |\langle \ell, \bar{\beta} \rangle| \lesssim \|\ell\|_{(H_{\omega,L}(\mathcal{X}))^*}, \]

which further implies that

\[ \frac{1}{\rho(V(B))} \left( \frac{1}{V(B)} \int_B (I - [I + r_B^2 L]^{-1})^M \ell(x)^2 d\mu(x) \right)^{1/2} \]

\[ = \sup_{\|\phi\|_{L^2(B)} \leq 1} \left| \langle \ell, (I - [I + r_B^2 L]^{-1})^M \phi \rangle \right| \lesssim \|\ell\|_{(H_{\omega,L}(\mathcal{X}))^*}. \]

Thus, by Proposition 4.4, we obtain \( \ell \in \text{BMO}_{\rho,L}^M(\mathcal{X}) \), which completes the proof of Theorem 4.1.

\[ \boxed{\blacksquare} \]

**Remark 4.2.** By Theorem 4.1, we have that for all \( M > \frac{n}{2} (\frac{1}{p_\omega} - \frac{1}{2}) \), the spaces \( \text{BMO}_{\rho,L}^M(\mathcal{X}) \) coincide with equivalent norms; thus, in what follows, we denote \( \text{BMO}_{\rho,L}^M(\mathcal{X}) \) simply by \( \text{BMO}_{\rho,L}(\mathcal{X}) \).

Recall that a measure \( d\mu \) on \( \mathcal{X} \times (0, \infty) \) is called a \( \rho \)-Carleson measure if

\[ \|d\mu\|_\rho \equiv \sup_{B \subset \mathcal{X}} \left\{ \frac{1}{V(B)[\rho(V(B))]^2} \int_B |d\mu| \right\}^{1/2} < \infty, \]
where the supremum is taken over all balls $B$ of $\mathcal{X}$ and $\hat{B}$ denotes the tent over $B$; see [16].

Using Theorem 4.1 and Proposition 4.5, we obtain the following $\rho$-Carleson measure characterization of $\text{BMO}_{\rho,L}(\mathcal{X})$.

**Theorem 4.2.** Let $L$ satisfy Assumptions (A) and (B), $\omega$ satisfy Assumption (C), $\rho$ be as in (2.10) and $M > \frac{n}{2} \left( \frac{1}{p_{\omega}} - \frac{1}{2} \right)$. Then the following conditions are equivalent:

(i) $f \in \text{BMO}_{\rho,L}(\mathcal{X})$;

(ii) $f \in \mathcal{M}^M_{\omega}(\mathcal{X})$ satisfies (4.8) for some $\epsilon > 0$, and $d\mu_f$ is a $\rho$-Carleson measure, where $d\mu_f$ is defined by

$$
\|d\mu_f\|_{\rho} \equiv \| (t^2 L)^M e^{-t^2 L} f(x) \|_2 \left( \frac{dt}{t} \right).
$$

Moreover, $\|f\|_{\text{BMO}_{\rho,L}(\mathcal{X})}$ and $\|d\mu_f\|_{\rho}$ are comparable.

**Proof.** It follows from Proposition 4.5 and the proof of Corollary 4.3 that (i) implies (ii).

Conversely, let $\tilde{M} > M + \epsilon + \frac{n}{4} + \frac{N}{2} \left( \frac{1}{p_\omega} - 1 \right)$. By Proposition 4.6, we have

$$
\langle f, g \rangle = \tilde{C}_M \int_{\mathcal{X} \times (0, \infty)} (t^2 L)^M e^{-t^2 L} f(x) (t^2 L)^{-2} e^{-t^2 L} g(x) \frac{d\mu(x)}{t},
$$

where $g$ is any finite combination of $(\omega, \tilde{M})$-atoms. Then similarly to the estimate of (4.14), we obtain that

$$
\langle f, g \rangle \lesssim \|d\mu_f\|_{\rho} \|g\|_{H_{\omega,L}(\mathcal{X})}.
$$

Since, by Corollary 4.2, $H^\text{at, fin}_{\omega} (\mathcal{X})$ is dense in $H_{\omega,L}(\mathcal{X})$, this together with Theorem 4.1 and (4.15) implies that $f \in (H_{\omega,L}(\mathcal{X}))^* = \text{BMO}_{\rho,L}(\mathcal{X})$, which completes the proof of Theorem 4.2.

\[\square\]

## 5 Characterizations of Orlicz-Hardy spaces

In this section, we characterize the Orlicz-Hardy spaces by atoms, molecules and the Lusin-area function associated with the Poisson semigroup. Let us begin with some notions.

**Definition 5.1.** Let $L$ satisfy Assumptions (A) and (B), $\omega$ satisfy Assumption (C) and $M > \frac{n}{2} \left( \frac{1}{p_{\omega}} - \frac{1}{2} \right)$. A distribution $f \in (\text{BMO}_{\rho,L}(\mathcal{X}))^*$ is said to be in the space $H^M_{\omega, \text{at}}(\mathcal{X})$ if there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $(\omega, M)$-atoms $\{\alpha_j\}_{j=1}^\infty$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in the norm of $(\text{BMO}_{\rho,L}(\mathcal{X}))^*$ and $\sum_{j=1}^\infty V(B_j) \omega(V(B_j) \rho(V(B_j))) < \infty$, where for each $j$, supp $\alpha_j \subset B_j$.

Moreover, for any $f \in H^M_{\omega, \text{at}}(\mathcal{X})$, its norm is defined by $\|f\|_{H^M_{\omega, \text{at}}(\mathcal{X})} \equiv \inf \Lambda(\{\lambda_j \alpha_j\}_{j})$, where $\Lambda(\{\lambda_j \alpha_j\}_{j})$ is the same as in Proposition 4.3 and the infimum is taken over all possible decompositions of $f$.

**Definition 5.2.** Let $L$ satisfy Assumptions (A) and (B), $\omega$ satisfy Assumption (C), $M > \frac{n}{2} \left( \frac{1}{p_{\omega}} - \frac{1}{2} \right)$ and $\epsilon > n(1/p_{\omega} - 1/p_{\omega}^*)$. A distribution $f \in (\text{BMO}_{\rho,L}(\mathcal{X}))^*$ is said to be in the space $H^M_{\omega, \text{mol}}(\mathcal{X})$ if there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $(\omega, M, \epsilon)$-molecules $\{\beta_j\}_{j=1}^\infty$ such that

$$
\sum_{j=1}^\infty V(B_j) \omega(V(B_j) \rho(V(B_j))) < \infty,
$$

where $\sum_{j=1}^\infty \supp \beta_j \subset B_j$. 
The Orlicz-Hardy space $H_{M,\varepsilon}(\mathcal{X})$ is defined by $\|f\|_{H_{M,\varepsilon}(\mathcal{X})} = \inf \Lambda(\{\lambda_j\beta_j\} \in \mathbb{R})$ where $\Lambda(S)$ is the same as in Proposition 4.3 and the infimum is taken over all possible decompositions of $f$.

For all $f \in L^2(\mathcal{X})$ and $x \in \mathcal{X}$, define the Lusin area function associated to the Poisson semigroup by

$$\|f\|_{H_{\omega,S_p}(\mathcal{X})} \equiv \|S_P(f)\|_{L^2(\mathcal{X})} = \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \frac{\left( S_P(f)(x) \right)}{\lambda} \, d\mu(x) \leq 1 \right\}.$$  

The Orlicz-Hardy space $H_{\omega,S_p}(\mathcal{X})$ is defined to be the completion of $H_{\omega,S_p}(\mathcal{X})$ in the norm $\| \cdot \|_{H_{\omega,S_p}(\mathcal{X})}$.

We now show that the spaces $H_{\omega,L}(\mathcal{X})$, $H_{\omega,\mathcal{M}}(\mathcal{X})$, $H_{\omega,\mathcal{M}_{\varepsilon}}(\mathcal{X})$ and $H_{\omega,S_p}(\mathcal{X})$ coincide with equivalent norms.

### 5.1 Atomic and molecular characterizations

In this subsection, we establish the atomic and the molecular characterizations of the Orlicz-Hardy spaces. We start with the following auxiliary result.

**Proposition 5.1.** Let $L$ satisfy Assumptions (A) and (B) and $\omega$ satisfy Assumption (C) and $R(L)$ be as in (4.2). A function $f \in R(L)$ is said to be in $H_{\omega,S_p}(\mathcal{X})$ if $S_P(f) \in L(\omega)$; moreover, define

$$\|f\|_{H_{\omega,S_p}(\mathcal{X})} \equiv \|S_P(f)\|_{L^2(\mathcal{X})} = \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \frac{\left( S_P(f)(x) \right)}{\lambda} \, d\mu(x) \leq 1 \right\}.$$  

The Orlicz-Hardy space $H_{\omega,S_p}(\mathcal{X})$ is defined to be the completion of $H_{\omega,S_p}(\mathcal{X})$ in the norm $\| \cdot \|_{H_{\omega,S_p}(\mathcal{X})}$.

We now show that the spaces $H_{\omega,L}(\mathcal{X})$, $H_{\omega,\mathcal{M}}(\mathcal{X})$, $H_{\omega,\mathcal{M}_{\varepsilon}}(\mathcal{X})$ and $H_{\omega,S_p}(\mathcal{X})$ coincide with equivalent norms.
\[ (V(B))^{1/2} \rho(V(B)) \]. This together with the \( L^2(\mathcal{X}) \)-boundedness of the operator \( t^2 L e^{-t^2 L} \) (see Lemma 2.1) yields that \( H \lesssim \frac{\|\phi\|_{L^2(B)}}{[V(B)]^{1/2} \rho(V(B))} \), which is a desired estimate.

If \( B \subset 2^{k+1} B \setminus 2^{k-1} B \) for some \( k \geq 3 \), then we have \( \tilde{B} \subset 2^{k+1} B \) and \( \text{dist}(\tilde{B}, B) \geq 2^{k-2} R \). Thus, by the Davies-Gaffney estimate, we obtain

\[ H \lesssim \frac{2^{n(k+2)(1/p_\omega - 1/2)}}{[V(B)]^{1/2} \rho(V(B))} \exp \left\{ - \frac{\text{dist}(\tilde{B}, B)^2}{t^2} \right\} \|\phi\|_{L^2(\tilde{B})} = \frac{2^{n(1/p_\omega - 1/2)} \|\phi\|_{L^2(\tilde{B})}}{[V(B)]^{1/2} \rho(V(B))}, \]

which is also a desired estimate.

**Case ii) \( r_B < R \).** In this case, we further consider two subcases. If \( d(x_B, x_0) \leq 4R \), then \( \tilde{B} \subset B(x_B, 5R) \) and hence

\[ (5.2) \quad [V(\tilde{B})]^{1/2} \rho(V(\tilde{B})) \lesssim \left( \frac{R}{r_B} \right)^{n(1/p_\omega - 1/2)} [V(B)]^{1/2} \rho(V(B)). \]

On the other hand, noticing that \( I - e^{-r_B^2 L} = \int_0^{r_B^2} L e^{-r^2 L} \, dr \), by the Minkowski inequality and the \( L^2(\mathcal{X}) \)-boundedness of the operator \( t^2 L e^{-t^2 L} \), we have

\[ (5.3) \quad \left( \int_B \| (I - e^{-r_B^2 L})^M t^2 L e^{-t^2 L} \phi(x)^2 \, d\mu(x) \right)^{1/2} = \left( \int_B \left| \int_0^{r_B^2} \cdots \int_0^{r_B^2} t^2 L^{M+1} e^{-(r_1 + \cdots + r_M + t^2)} \phi(x) \, dr_1 \cdots dr_M \right|^2 \, d\mu(x) \right)^{1/2} \lesssim \|\phi\|_{L^2(\tilde{B})} \int_0^{r_B^2} \cdots \int_0^{r_B^2} \frac{t^2}{(r_1 + \cdots + r_M + t^2)^{M+1}} dr_1 \cdots dr_M \lesssim \left( \frac{r_B}{t} \right)^{2M} \|\phi\|_{L^2(\tilde{B})}. \]

By \( 2M > n(\frac{1}{p_\omega} - \frac{1}{2}) \) and the estimates (5.2) and (5.3), we obtain

\[ H \lesssim \left( \frac{R}{t} \right)^{2M} \frac{\|\phi\|_{L^2(\tilde{B})}}{[V(\tilde{B})]^{1/2} \rho(V(\tilde{B}))}, \]

which is also a desired estimate.

If \( d(x_B, x_0) > 4R \), then there exists \( k_0 \geq 3 \) such that \( d(x_B, x_0) \in (2^{k_0-1} R, 2^{k_0} R) \). Thus, \( \tilde{B} \subset B(x_B, 2^{k_0+1} R) \), dist \( (\tilde{B}, B) \geq 2^{k_0-2} R \) and

\[ (5.4) \quad [V(\tilde{B})]^{1/2} \rho(V(\tilde{B})) \lesssim \left( \frac{2^{k_0} R}{r_B} \right)^{n(1/p_\omega - 1/2)} [V(B)]^{1/2} \rho(V(B)). \]

By the Davies-Gaffney estimate, we further obtain

\[ \left( \int_B \| (I - e^{-r_B^2 L})^M t^2 L e^{-t^2 L} \phi(x)^2 \, d\mu(x) \right)^{1/2} \]
which, together with (5.4), $r_B < R$ and $2M > n(\frac{1}{p_{\omega}} - \frac{1}{2})$, yields that

$$H \lesssim \left( \frac{R + t}{t} \right)^{2M} \frac{\|\phi\|_{L^2(B)}}{[V(B)]^{1/2} \rho(V(B))},$$

This is also a desired estimate, which completes the proof of Proposition 5.1.

From Proposition 5.1, it follows that for each $f \in (\text{BMO}_{\rho, L}(\mathcal{X}))^*$, $t^2 \langle t^2 L f \rangle$ is well defined. In fact, for any ball $B \equiv B(x_B, r_B)$ and $\phi \in L^2(B)$, by Proposition 5.1, we have

$$\langle t^2 L f, \phi \rangle \equiv \langle f, t^2 L \phi \rangle \leq C(t, r_B, B)\|\phi\|_{L^2(B)}\|f\|_{(\text{BMO}_{\rho, L}(\mathcal{X}))^*},$$

which implies that $t^2 \langle t^2 L f \rangle \in L^2_{\text{loc}}(\mathcal{X})$ in the sense of distributions. Moreover, recalling that the atomic decomposition obtained in Corollary 4.1 holds in $H_{\omega, L}(\mathcal{X})$, then by Theorem 4.1, we see the atomic decomposition also holds in $(\text{BMO}_{\rho, L}(\mathcal{X}))^*$. Applying these observations, similarly to the proof of [21, Theorem 5.1], we have the following result. We omit the details here.

**Theorem 5.1.** Let $L$ satisfy Assumptions (A) and (B), $\omega$ satisfy Assumption (C), $M > \frac{\theta}{\frac{p_{\omega}}{2} - \frac{1}{2}}$ and $\epsilon > n(1/p_{\omega} - 1/p_{\omega}^+)$. Then the spaces $H_{\omega, L}(\mathcal{X})$, $H_{\omega, \text{at}}^M(\mathcal{X})$ and $H_{\omega, \text{mol}}^M(\mathcal{X})$ coincide with equivalent norms.

### 5.2 A characterization by the Lusin area function $S_P$

In this subsection, we characterize the space $H_{\omega, L}(\mathcal{X})$ by the Lusin area function $S_P$ as in (5.1). We start with the following auxiliary conclusion.

**Lemma 5.1.** Let $K \in \mathbb{Z}_+$. Then the operator $(t\sqrt{L})^K e^{-t\sqrt{L}}$ is bounded on $L^2(\mathcal{X})$ uniformly in $t$. Moreover, there exists a positive constant $C$ such that for all closed sets $E$, $F$ in $\mathcal{X}$ with $\text{dist} (E, F) > 0$, all $t > 0$ and $f \in L^2(E)$,

$$\|(t\sqrt{L})^{2K} e^{-t\sqrt{L}} f\|_{L^2(F)} + \|(t\sqrt{L})^{2K+1} e^{-t\sqrt{L}} f\|_{L^2(F)} \leq C \left( \frac{t}{\text{dist} (E, F)} \right)^{2K+1} \|f\|_{L^2(E)},$$
Proof. It was proved in [18, Lemma 5.1] and [17, Lemma 4.15] that
\[
\| t^{2K} e^{-t\sqrt{T}} f \|_{L^2(E)} \lesssim \left( \frac{t}{\text{dist} (E, F)} \right)^{2K+1} \| f \|_{L^2(E)}.
\]

To establish a similar estimate for \((t\sqrt{T})^{2K+1} e^{-t\sqrt{T}} f\), we first notice that the subordination formula
\[
e^{-t\sqrt{T}} f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{1}{4u}L} f du
\]
implies that
\[
\sqrt{t} e^{-t\sqrt{T}} f = -\frac{\partial}{\partial t} e^{-t\sqrt{T}} f = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{te^{-u} L}{u^{3/2}} e^{-\frac{1}{4u}L} f du.
\]
Then by the Minkowski inequality and Lemma 2.1, we obtain
\[
\| (t\sqrt{T})^{2K+1} e^{-t\sqrt{T}} f \|_{L^2(F)} \lesssim \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left( \frac{t^2 \sqrt{L}}{4u} \right)^{K+1} e^{-\frac{1}{4u}L} f \|_{L^2(F)} \| u^K du
\]
\[
\lesssim \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \exp \left\{ -\frac{u \text{ dist} (E, F)^2}{C_3 t^2 \sqrt{L}} \right\} u^K du \| f \|_{L^2(E)}
\]
\[
\lesssim \left( \frac{t}{\text{dist} (E, F)} \right)^{2K+1} \| f \|_{L^2(E)}.
\]

To show that \((t\sqrt{T})^{2K+1} e^{-t\sqrt{T}} f\) is bounded on \(L^2(\mathcal{X})\) uniformly in \(t\), by the boundedness of \(t^2 L e^{-t^2 L}\) on \(L^2(\mathcal{X})\) uniformly in \(t\), we have
\[
\| (t\sqrt{T})^{2K+1} e^{-t\sqrt{T}} f \|_{L^2(\mathcal{X})} \lesssim \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left( \frac{t^2 \sqrt{L}}{4u} \right)^{K+1} e^{-\frac{1}{4u}L} f \|_{L^2(\mathcal{X})} \| u^K du
\]
\[
\lesssim \int_0^\infty \frac{e^{-u}}{\sqrt{u}} u^K du \| f \|_{L^2(\mathcal{X})} \lesssim \| f \|_{L^2(\mathcal{X})}.
\]
Similarly, we have that \((t\sqrt{T})^{2K} e^{-t\sqrt{T}} f\) is bounded on \(L^2(\mathcal{X})\) uniformly in \(t\), which completes the proof of Lemma 5.1. \(\square\)

Similarly to [21, Lemma 5.1], we have the following lemma. We omit the details. Recall that a nonnegative sublinear operator \(T\) means that \(T\) is sublinear and \(Tf \geq 0\) for all \(f\) in the domain of \(T\).

Lemma 5.2. Let \(L\) satisfy Assumptions (A) and (B), \(\omega\) satisfy Assumption (C) and \(M > \frac{4}{2} \left( \frac{1}{p_0} - \frac{1}{2} \right)\). Suppose that \(T\) is a linear (resp. nonnegative sublinear) operator, which maps \(L^2(\mathcal{X})\) continuously into weak-\(L^2(\mathcal{X})\). If there exists a positive constant \(C\) such that for all \((\omega, M)\)-atom \(\alpha\),
\[
\int_{\mathcal{X}} \omega(T(\lambda\alpha)(x)) \, d\mu(x) \leq CV(B)\omega \left( \frac{|\lambda|}{V(B)\rho(V(B))} \right),
\]
\[
\int_{\mathcal{X}} \omega(T(\lambda\alpha)(x)) \, d\mu(x) \leq CV(B)\omega \left( \frac{|\lambda|}{V(B)\rho(V(B))} \right).
\]
then $T$ extends to a bounded linear (resp. sublinear) operator from $H_{\omega,L}(X)$ to $L(\omega)$; moreover, there exists a positive constant $\widetilde{C}$ such that for all $f \in H_{\omega,L}(X)$, \[ \|Tf\|_{L(\omega)} \leq \widetilde{C}\|f\|_{H_{\omega,L}(X)}. \]

**Theorem 5.2.** Let $L$ satisfy Assumptions (A) and (B) and $\omega$ satisfy Assumption (C). Then the spaces $H_{\omega,L}(X)$ and $H_{\omega,S_p}(X)$ coincide with equivalent norms.

**Proof.** Let us first prove that $H_{\omega,L}(X) \subset H_{\omega,S_p}(X)$. By (2.8), we have that the operator $S_P$ is bounded on $L^2(\mathbb{R}^n)$. Thus, by Lemma 5.2, to show that $H_{\omega,L}(X) \subset H_{\omega,S_p}(X)$, we only need to show that (5.6) holds with $T = S_P$, where $M \in \mathbb{N}$ and $M > \frac{4}{5}(\frac{1}{p_\omega} - \frac{1}{2})$.

Indeed, it is enough to show that for all $f \in H_{\omega,L}(X) \cap L^2(X)$, \[ \|S_P(f)\|_{L(\omega)} \lesssim \|f\|_{H_{\omega,L}(X)}, \]

which implies that $H_{\omega,L}(X) \cap L^2(X) \subset H_{\omega,S_p}(X)$. Then by the completeness of $H_{\omega,L}(X)$ and $H_{\omega,S_p}(X)$, we see that $H_{\omega,L}(X) \subset H_{\omega,S_p}(X)$.

Suppose that $\lambda \in \mathbb{C}$ and $\alpha$ is an $(\omega,M)$-atom supported in $B \equiv B(x_B,r_B)$. Since $\omega$ is concave, by the Jensen inequality and the Hölder inequality, we have

\[
\int_X \omega(S_P(\lambda\alpha)(x)) \, d\mu(x) = \sum_{k=0}^{\infty} \int_{U_k(B)} \omega(|\lambda|S_P(\alpha)(x)) \, d\mu(x) 
\leq \sum_{k=0}^{\infty} V(2^kB)\omega\left(\frac{|\lambda|\int_{U_k(B)} S_P(\alpha)(x) \, d\mu(x)}{V(2kB)}\right) 
\leq \sum_{k=0}^{\infty} V(2^kB)\omega\left(\frac{|\lambda|\|S_P(\alpha)\|_{L^2(U_k(B))}}{|V(2kB)|^{1/2}}\right).
\]

Since $S_P$ is bounded on $L^2(X)$, for $k = 0, 1, 2$, we have

\[ \|S_P(\alpha)\|_{L^2(U_k(B))} \lesssim \|\alpha\|_{L^2(X)} \lesssim [V(B)]^{-1/2}[\rho(V(B))]^{-1}. \]

For $k \geq 3$, write

\[
\|S_P(\alpha)\|_{L^2(U_k(B))}^2 = \int_{U_k(B)} \int_0^{d(x,x_B)/4} \int_{d(x,y)<t} \left|t\sqrt{L}e^{-t\varpi\alpha(y)}\right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \, d\mu(x) 
+ \int_{U_k(B)} \int_{d(x,x_B)/4}^{\infty} \int_{d(x,y)<t} \cdots \equiv I_k + J_k.
\]

Since $\alpha$ is an $(\omega,M)$-atom, by Definition 4.2, we have $\alpha = L^M b$ with $b$ as in Definition 4.2. To estimate $I_k$, let $F_k(B) \equiv \{y \in X : d(x,y) < d(x,x_B)/4 \text{ for some } x \in U_k(B)\}$. Then we have \[ d(y,z) \geq d(x,x_B) - d(z,x_B) - d(y,x) \geq \frac{3}{4}d(x,x_B) - r_B \geq 2^{k-2}r_B. \] By Lemma 5.1, we have

\[
I_k \lesssim \int_0^{2^{k-2}r_B} \int_{F_k(B)} \left|(t\sqrt{L})^{2M+1}e^{-t\varpi\alpha(y)}\right|^2 \, d\mu(y) \frac{dt}{t^{4M+1}} 
\lesssim \|b\|_{L^2(X)}^2 \int_0^{2^{k-2}r_B} \left(\frac{t}{\text{dist}(F_k(B),B)}\right)^{4M+2} \frac{dt}{t^{4M+1}} \lesssim 2^{-4kM}[V(B)]^{-1}[\rho(V(B))]^{-2}.
\]
For the term $J_k$, notice that if $x \in U_k(B)$, then we have $d(x, x_B) \geq 2^{k-1}r_B$, which together with Lemma 5.1 yields that
\[
J_k \lesssim \int_{2^{k-1}r_B}^{\infty} \int_X |(t\sqrt{L})^{2M+1} e^{-t\sqrt{L}} b(y)|^2 \, d\mu(y) \, \frac{dt}{t^{4M+1}}
\lesssim \int_{2^{k-1}r_B}^{\infty} \|b\|^2_{L^2(\chi)} \frac{dt}{t^{4M+1}} \lesssim 2^{-4kM} [V(B)]^{-1} [p(V(B))]^{-2}.
\]

Using the estimates of $I_k$ and $J_k$ together with the strictly lower type $p_\omega$ of $\omega$, we obtain
\[
V(2^k B) \omega \left( \frac{|\lambda||S_P(\alpha)||_{L^2(U_k(B))}}{|V(2^k B)|^{1/2}} \right) \lesssim 2^{-2kM p_\omega} V(2^k B) \left( \frac{V(B)}{V(2^k B)} \right)^{p_\omega/2} \omega \left( \frac{|\lambda|}{V(B)p(V(B))} \right)
\lesssim 2^{-k(2M p_\omega - n(1 - p_\omega)/2)} V(B) \omega \left( \frac{|\lambda|}{V(B)p(V(B))} \right),
\]
where $2M p_\omega > n(1 - p_\omega/2)$. Thus, we finally obtain that
\[
\int_X \omega(S_P(\lambda \alpha)(x)) \, d\mu(x) \lesssim V(B) \omega \left( \frac{|\lambda|}{V(B)p(V(B))} \right),
\]
that is, (5.6) holds. This finishes the proof of the inclusion of $H_{\omega,L}(\chi)$ into $H_{\omega,S_P}(\chi)$.

Conversely, for any $f \in H_{\omega,S_P}(\chi) \cap L^2(\chi)$, we have $t \sqrt{L} e^{-t \sqrt{L}} f \in T_\omega(\chi)$, which together with Proposition 4.2(ii) implies that $\pi_{\Psi,L}(t \sqrt{L} e^{-t \sqrt{L}} f) \in H_{\omega,L}(\chi)$.

On the other hand, by $H_\infty$-functional calculus, we have $f = \tilde{C}_\Psi \pi_{\Psi,L}(t \sqrt{L} e^{-t \sqrt{L}} f)$ in $L^2(\chi)$, where $\tilde{C}_\Psi$ is the positive constant such that $\tilde{C}_\Psi \int_0^\infty \Psi(t)te^{-t/4} \, dt = 1$ and $C_\Psi$ as in (4.7). This, combined with the fact $\pi_{\Psi,L}(t \sqrt{L} e^{-t \sqrt{L}} f) \in H_{\omega,L}(\chi)$, implies that $f \in H_{\omega,L}(\chi)$. Via a density argument, we further obtain $H_{\omega,S_P}(\chi) \subset H_{\omega,L}(\chi)$, which completes the proof of Theorem 5.2.

**Remark 5.1.**

(i) Since the atoms are associated with $L$, they do not have vanishing integral in general. Hofmann et al [17] introduced the so-called the conservation property of the semigroup, namely, $e^{-tL}1 = 1$ in $L^2_{\text{loc}}(\chi)$, and showed that under this assumption and Assumptions (A) and (B), then for each $(1, M)$-atom $\alpha$, $\int_X \alpha(x) \, d\mu(x) = 0$. From this and Proposition 4.3, we immediately deduce that if $L$ satisfies Assumptions (A) and (B) and has the conservation property, and $\omega$ satisfies Assumption (C) with $p_\omega \in (n/(n+1), 1]$, then $H_{\omega,L}(\chi) \subset H_\omega(\chi)$, where $\chi$ is an Ahlfors $n$-regular space (see [32]). In particular, $H^p_{L}(\chi) \subset H^p(\chi)$ for all $p \in (n/(n+1), 1]$.

(ii) Let $s \in \mathbb{Z}_+$. The semigroup $\{e^{-tL}\}_{t \geq 0}$ is said to have the $s$-generalized conservation property, if for all $\gamma \in \mathbb{Z}^+_n$ with $|\gamma| \leq s$,
\[
e^{-tL}((\cdot)^\gamma)(x) = x^\gamma \text{ in } L^2_{\text{loc}}(\mathbb{R}^n),
\]

namely, for every $\phi \in L^2(\mathbb{R}^n)$ with bounded support,
\[
\int_{\mathbb{R}^n} x^\gamma e^{-tL} \phi(x) \, dx \equiv \int_{\mathbb{R}^n} e^{-tL}((\cdot)^\gamma)(x) \phi(x) \, dx = \int_{\mathbb{R}^n} x^\gamma \phi(x) \, dx,
\]
where $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n_+$.

Notice that for any $\phi$ with bounded support and $\gamma \in \mathbb{Z}^n_+$, by the Davies-Gaffney estimate, one can easily check that $x^\gamma e^{-tL} \phi(x), x^\gamma (I + L)^{-1} \phi(x) \in L^1(\mathbb{R}^n)$. Hence, by (5.8) and the $L^2(\mathbb{R}^n)$-functional calculus, we obtain

\begin{equation}
\int_{\mathbb{R}^n} x^\gamma (I + L)^{-1} \phi(x) \, dx = \int_0^\infty e^{-t} \left[ \int_{\mathbb{R}^n} x^\gamma e^{-tL} \phi(x) \, dx \right] \, dt = \int_{\mathbb{R}^n} x^\gamma \phi(x) \, dx.
\end{equation}

Let $\alpha$ be an $(\omega, M)$-atom and $s \equiv \lfloor n(\frac{1}{p_\omega} - 1) \rfloor$. By Definition 4.2, there exists $b \in D(L^M)$ such that $\alpha = L^M b$. Thus, if $L$ satisfies (5.7), then for all $\gamma \in \mathbb{Z}^n_+$ and $|\gamma| \leq s$, by (5.8) and (5.9), we obtain

\[
\int_{\mathbb{R}^n} x^\gamma \alpha(x) \, d\mu(x) \\
= \int_{\mathbb{R}^n} x^\gamma (I + L)^{-1} \alpha(x) \, d\mu(x) \\
= \int_{\mathbb{R}^n} x^\gamma (I + L)^{-1} (I + L) L^{M^{-1}} b(x) \, d\mu(x) - \int_{\mathbb{R}^n} x^\gamma (I + L)^{-1} L^{M^{-1}} b(x) \, d\mu(x) \\
= \int_{\mathbb{R}^n} x^\gamma L^{M^{-1}} b(x) \, d\mu(x) - \int_{\mathbb{R}^n} x^\gamma (I + L)^{-1} L^{M^{-1}} b(x) \, d\mu(x) = 0,
\]

which implies that $\alpha$ is a classical $H_{\omega}(\mathbb{R}^n)$-atom; for the definition of $H_{\omega}(\mathbb{R}^n)$-atoms, see [32].

Thus, if $L$ satisfies (5.7) and Assumptions (A) and (B), and $\omega$ satisfies Assumption (C), then by Proposition 4.3, we know that $H_{\omega, L}(\mathbb{R}^n) \subset H_{\omega}(\mathbb{R}^n)$. In particular, $H^p_{L}(\mathbb{R}^n) \subset H^p(\mathbb{R}^n)$ for all $p \in (0, 1]$.

### 6 Applications to Schrödinger operators

In this section, let $\mathcal{X} \equiv \mathbb{R}^n$ and $L \equiv -\Delta + V$ be a Schrödinger operator, where $V \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ is a nonnegative potential. We establish several characterizations of the corresponding Orlicz-Hardy spaces $H_{\omega, L}(\mathbb{R}^n)$ by beginning with some notions.

Since $V$ is a nonnegative function, by the Feynman-Kac formula, we obtain that $h_t$, the kernel of the semigroup $e^{-tL}$, satisfies that for all $x, y \in \mathbb{R}^n$ and $t \in (0, \infty)$,

\begin{equation}
0 \leq h_t(x, y) \leq (4\pi t)^{-n/2} \exp \left( -\frac{|x - y|^2}{4t} \right).
\end{equation}

It is easy to see that $L$ satisfies Assumptions (A) and (B).

From Theorems 5.1 and 5.2, we deduce the following conclusions on Hardy spaces associated with $L$.

**Theorem 6.1.** Let $\omega$ be as in Assumption (C), $M > \frac{n}{2} (\frac{1}{p_\omega} - \frac{1}{2})$ and $\epsilon > n(1/p_\omega - 1/p_\omega^+)$.

Then the spaces $H_{\omega, L}(\mathbb{R}^n), H^M_{\omega, \text{at}}(\mathbb{R}^n), H^{M, \epsilon}_{\omega, \text{mol}}(\mathbb{R}^n)$ and $H_{\omega, \text{Sp}}(\mathbb{R}^n)$ coincide with equivalent norms.
Let us now establish the boundedness of the Riesz transform $\nabla L^{-1/2}$ on $H_{\omega,L}(\mathbb{R}^n)$. We first recall a lemma established in [17].

**Lemma 6.1.** There exist two positive constants $C$ and $c$ such that for all closed sets $E$ and $F$ in $\mathbb{R}^n$ and $f \in L^2(E)$,

$$\|t \nabla e^{-t^2 L} f\|_{L^2(F)} \leq C \exp \left\{ -\frac{\text{dist}(E,F)^2}{ct^2} \right\} \|f\|_{L^2(E)}.$$  

**Theorem 6.2.** Let $\omega$ be as in Assumption (C). Then the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$.

**Proof.** It was proved in [17] that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$; thus, to prove Theorem 6.2, by Lemma 5.2, it suffices to show that (5.6) holds.

Suppose that $\lambda \in \mathbb{C}$ and $\alpha = L^M b$ is an $(\omega, M)$-atom supported in $B \equiv B(x_B, r_B)$, where $b$ is as in Definition 4.2 and we choose $M \in \mathbb{N}$ such that $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$.

For $j = 0, 1, 2$, by the Jensen inequality, the Hölder inequality and the $L^2(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$, we obtain

$$\int_{U_j(B)} \omega(|\nabla L^{-1/2} \alpha(x)|) \, dx \lesssim |B| \omega \left( \frac{\|\lambda \nabla L^{-1/2} \alpha\|_{L^2(U_j(B))}}{|B|^{1/2}} \right) \lesssim |B| \omega \left( \frac{|\lambda|}{\rho(|B|)|B|} \right).$$

Let us estimate the case $j \geq 3$. By [18, Lemma 2.3], we see that the operator $t \nabla (t^2 L^M e^{-t L})$ also satisfies the Davies-Gaffney estimate. By this, the fact that $\omega^{-1}$ is convex, the Jensen inequality, the Hölder inequality and Lemma 6.1, we obtain

$$\omega^{-1} \left( \frac{1}{|U_j(B)|} \int_{U_j(B)} \omega(|\nabla L^{-1/2} \alpha(x)|) \, dx \right) \lesssim \frac{1}{|U_j(B)|} \int_{U_j(B)} \omega^{-1} \circ \omega \left( \left| \int_0^\infty \lambda \nabla e^{-t^2 L} \alpha(x) \, dt \right| \right) \, dx \lesssim \frac{1}{|U_j(B)|} \int_0^\infty \int_{U_j(B)} |\lambda| \nabla (t^2 L^M e^{-t L} b(x)) \, dx \, \frac{dt}{t^{2M+1}} \lesssim \frac{|\lambda||b|_{L^2(\mathbb{R}^n)}}{|U_j(B)|^{1/2}} \int_0^\infty \exp \left\{ -\frac{\text{dist}(B, U_j(B))^2}{ct^2} \right\} \, \frac{dt}{t^{2M+1}} \lesssim \frac{|\lambda||b|_{L^2(\mathbb{R}^n)}}{|U_j(B)|^{1/2}} \int_0^\infty (2jr_B)^{-2M} \min \left\{ \frac{t}{2jr_B}, \frac{2jr_B}{t} \right\} \, \frac{dt}{t} \lesssim 2^{-j(2M+n/2)} \frac{|\lambda|}{\rho(|B|)|B|},$$

where $c$ is a positive constant. Since $\omega$ is of lower type $p_\omega$, we obtain

$$\int_{U_j(B)} \omega(|\nabla L^{-1/2} \alpha(x)|) \, dx \lesssim |U_j(B)| \omega \left( 2^{-j(2M+n/2)} \frac{|\lambda|}{\rho(|B|)|B|} \right) \lesssim 2^{-j(2Mp_\omega + n(p_\omega/2 - 1))} |B| \omega \left( \frac{|\lambda|}{\rho(|B|)|B|} \right).$$

Combining the above estimates and using $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$, we obtain that (5.6) holds for $\nabla L^{-1/2}$, which completes the proof of Theorem 6.2. \qed
It was proved in [17] that the Riesz transform $\nabla L^{-1/2}$ is bounded from $H^1_1(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. Similarly to [21, Theorem 7.4], we have the following result. We omit the details here; see [20, 32, 27] for more details about the Hardy-Orlicz space $H_\omega(\mathbb{R}^n)$.

**Theorem 6.3.** Let $\omega$ be as in Assumption (C) and $p_\omega \in (\frac{n}{n+1}, 1]$. Then the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $H_\omega(\mathbb{R}^n)$.

We next characterize the Orlicz-Hardy space $H_{\omega,L}(\mathbb{R}^n)$ via maximal functions. To this end, we first introduce some notions.

Let $\nu > 0$. Recall that for all measurable function $g$ on $\mathbb{R}^{n+1}_+$ and $x \in \mathbb{R}^n$, the Lusin area function $A_\nu(g)(x)$ is defined by $A_\nu(g)(x) \equiv (\int_{\Gamma_\nu(x)} |g(y,t)|^2 \frac{dy dt}{t^{n+1}})^{1/2}$. Also the non-tangential maximal function is defined by $N_\nu(g)(x) \equiv \sup_{(y,t) \in \Gamma_\nu(x)} |g(y,t)|$.

**Lemma 6.2.** Let $\eta, \nu \in (0, \infty)$. Then there exists a positive constant $C$, depending on $\eta$ and $\nu$, such that for all measurable function $g$ on $\mathbb{R}^{n+1}_+$,

\[
(6.2) \quad C^{-1} \int_{\mathbb{R}^n} \omega(A_\eta(g)(x)) \, dx \leq \int_{\mathbb{R}^n} \omega(A_\nu(g)(x)) \, dx \leq C \int_{\mathbb{R}^n} \omega(A_\eta(g)(x)) \, dx
\]

and

\[
(6.3) \quad C^{-1} \int_{\mathbb{R}^n} \omega(N_\eta(g)(x)) \, dx \leq \int_{\mathbb{R}^n} \omega(N_\nu(g)(x)) \, dx \leq C \int_{\mathbb{R}^n} \omega(N_\eta(g)(x)) \, dx.
\]

**Proof.** (6.2) was established in [21, Lemma 3.2], while (6.3) can be proved by an argument similar to those used in the proofs of [5, Theorem 2.3] and [21, Lemma 5.3]. We omit the details, which completes the proof of Lemma 6.2.

For any $\beta \in (0, \infty)$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $N^\beta_h(f)(x) \equiv N^\beta(e^{-\nu^2 L} f)(x)$, $N^\beta_p(f)(x) \equiv N^\beta(e^{-\nu^2 L} f)(x)$, $R_h(f)(x) \equiv \sup_{t > 0} |e^{-\nu^2 L} f(x)|$ and $R_p(f)(x) \equiv \sup_{t > 0} |e^{-\nu^2 L} f(x)|$.

We denote $N^\beta_h(f)$ and $N^\beta_p(f)$ simply by $N_h(f)$ and $N_p(f)$, respectively.

Similarly to Definition 4.1, we introduce the space $H_{\omega,N_h}(\mathbb{R}^n)$ as follows.

**Definition 6.1.** Let $\omega$ be as in Assumption (C) and $R(L)$ as in (4.2). A function $f \in R(L)$ is said to be in $H_{\omega,N_h}(\mathbb{R}^n)$ if $N_h(f) \in L(\omega)$; moreover, define

\[
\|f\|_{H_{\omega,N_h}(\mathbb{R}^n)} \equiv \|N_h(f)\|_{L(\omega)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \omega \left( \frac{N_h(f)(x)}{\lambda} \right) \, dx \leq 1 \right\}.
\]

The Hardy space $H_{\omega,N_h}(\mathbb{R}^n)$ is defined to be the completion of $\tilde{H}_{\omega,N_h}(\mathbb{R}^n)$ in the norm $\| \cdot \|_{H_{\omega,N_h}(\mathbb{R}^n)}$.

The spaces $H_{\omega,N_p}(\mathbb{R}^n)$, $H_{\omega,R_h}(\mathbb{R}^n)$ and $H_{\omega,R_p}(\mathbb{R}^n)$ are defined in a similar way.

The following Moser type local boundedness estimate was established in [17].
Lemma 6.3. Let $u$ be a weak solution of $\bar{L}u \equiv Lu - \partial_t^2 u = 0$ in the ball $B(Y_0, 2r) \subset \mathbb{R}^{n+1}_+$. Then for all $p \in (0, \infty)$, there exists a positive constant $C(n, p)$ such that

$$\sup_{Y \in B(Y_0, r)} |u(Y)| \leq C(n, p) \left( \frac{1}{r^{n+1}} \int_{B(Y_0, 2r)} |u(Y)|^p dY \right)^{1/p}.$$

To establish the maximal function characterizations of $H_{\omega, L}(\mathbb{R}^n)$, an extra assumption on $\omega$ is needed.

Assumption (D). Let $\omega$ satisfy Assumption (C). Suppose that there exist $q_1, q_2 \in (0, \infty)$ such that $q_1 \leq 2 < q_2$ and $[\omega(t^{q_2})]^{q_1}$ is a convex function on $(0, \infty)$. \hfill $\Box$

Notice that if $\omega(t) = t^p$ with $p \in (0, 1]$ for all $t \in (0, \infty)$, then for all $q_1 \in (0, 1)$ and $q_2 \in [1/(pq_1), \infty)$, $[\omega(t^{q_2})]^{q_1}$ is a convex function on $(0, \infty)$; if $\omega(t) = t^{1/2} \ln(e^4 + t)$ for all $t \in (0, \infty)$, then it is easy to check that $[\omega(t^{1/2})]^{1/2}$ is a convex function on $(0, \infty)$.

Theorem 6.4. Let $\omega$ be as in Assumption (D). Then the spaces $H_{\omega, L}(\mathbb{R}^n)$, $H_{\omega, N_h}(\mathbb{R}^n)$, $H_{\omega, N_p}(\mathbb{R}^n)$, $H_{\omega, \mathcal{R}_h}(\mathbb{R}^n)$ and $H_{\omega, \mathcal{R}_p}(\mathbb{R}^n)$ coincide with equivalent norms.

Proof. We first show that $H_{\omega, L}(\mathbb{R}^n) \subset H_{\omega, N_h}(\mathbb{R}^n)$. By (6.1), for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$N_h(f)(x) \leq \sup_{y \in B(x,t)} t^{-n} \int_{\mathbb{R}^n} \exp \left( -\frac{|y-z|^2}{4t^2} \right) |f(z)| \, dz$$

$$\leq \sum_{j=0}^{\infty} \sup_{y \in B(x,t)} t^{-n} \int_{U_j(B(y,2t))} \exp \left( -\frac{|y-z|^2}{4t^2} \right) |f(z)| \, dz$$

$$\lesssim M(f)(x) + \sum_{j=2}^{\infty} \sup_{y \in B(x,t)} t^{-n} 2^{-j(n+1)} \int_{U_j(B(y,2t))} |f(z)| \, dz \lesssim M(f)(x),$$

where $M$ is the Hardy-Littlewood maximal function on $\mathbb{R}^n$. Thus, $N_h$ is bounded on $L^2(\mathbb{R}^n)$.

Thus, by Lemma 5.2 and the completeness of $H_{\omega, L}(\mathbb{R}^n)$ and $H_{\omega, N_h}(\mathbb{R}^n)$, similarly to the proof of Theorem 5.2, we only need to show that for each $(\omega, M)$-atom $\alpha$, (5.6) holds with $T = N_h$, where $M \in \mathbb{N}$ and $M > \frac{n}{2} \left( \frac{1}{p_\omega} - \frac{1}{2} \right)$.

To this end, suppose that $\alpha$ is an $(\omega, M)$-atom and supp $\alpha \subset B \equiv B(x_B, r_B)$. For $j = 0, \cdots, 10$, since $N_h$ is bounded on $L^2(\mathbb{R}^n)$, by the Jensen inequality and the Hölder inequality, we have that for any $\lambda \in \mathbb{C},$

$$\int_{U_j(B)} \omega(\xi) \, dx \lesssim |U_j(B)| \omega \left( \frac{\|\xi\|_{L^2(U_j(B))}}{|B|^{1/2}} \right) \lesssim |B| \omega \left( \frac{\|\xi\|_{L^2(\mathbb{R}^n)}}{\rho(|B|)|B|} \right).$$

For $j \geq 11$ and $x \in U_j(B)$, let $a \in (0, 1)$ such that $ap_\omega(2M + n) > n$. Write

$$N_h(\alpha)(x) \leq \sup_{y \in B(x,t), t \leq 2^{j-2}r_B} |e^{-t^2L}(\alpha)(y)| + \sup_{y \in B(x,t), t > 2^{j-2}r_B} |e^{-t^2L}(\alpha)(y)| \equiv H_j + I_j.$$
To estimate $H_j$, observe that if $x \in U_j(B)$, then we have $|x - x_B| > 2^{j-1}r_B$, and if $z \in B$ and $y \in F_j(B) \equiv \{y \in X : |x - y| < 2^{j-2}r_B$ for some $x \in U_j(B)\}$, then we have

$$|y - z| \geq |x - x_B| - |z - x_B| - |y - x| \geq 2^{j-1}r_B - r_B - 2^{j-2}r_B \geq 2^{j-3}r_B.$$ 

By (6.1), we obtain

$$H_j \lesssim \sup_{y \in B(x,t), \ t \leq 2^{n+2}r_B} \frac{1}{t^n} \int_B e^{-\frac{|x-y|^2}{4t^2}}|\alpha(z)| \, dz$$

$$\lesssim \sup_{t \leq 2^{n+2}r_B} \frac{1}{t^n} \left(\frac{t}{2^n r_B}\right)^{N+n} \|\alpha\|_{L^1(B)} \lesssim 2^{-j[n+(1-a)N]}|B|^{-1}[\rho(|B|)]^{-1},$$

where $N \in \mathbb{N}$ satisfies that $p_\omega[n + (1-a)N] > n$.

For the term $I_j$, notice that since the kernel $h_t$ of $\{e^{-t^2L}\}_{t>0}$ satisfies (6.1), we have that for each $k \in \mathbb{N}$, there exist two positive constants $c_k$ and $\tilde{c}_k$ such that for almost every $x, y \in \mathbb{R}^n$,

$$(6.4) \quad \left| \frac{\partial^k}{\partial t^k} h_t(x,y) \right| \leq \frac{\tilde{c}_k}{t^{k+n/2}} \exp\left\{ -\frac{|x - y|^2}{c_k t}\right\};$$

see [10, 17]. On the other hand, since $\alpha$ is an $(\omega,M)$-atom, by Definition 4.2, we have $\alpha = L^Mb$ with $b$ as in Definition 4.2, which together with (6.4) implies that

$$I_j = \sup_{y \in B(x,t), \ t > 2^{n+2}r_B} t^{-2M} \|(t^2L)^ Me^{-t^2L}(b)(y)\|$$

$$\lesssim \sup_{y \in B(x,t), \ t > 2^{n+2}r_B} t^{-2M-n} \int_B e^{-\frac{|x-y|^2}{2M|B|}} |b(z)| \, dz \lesssim 2^{-n(2M+n)}|B|^{-1}[\rho(|B|)]^{-1}.$$

Combining the above two estimates, we obtain

$$\sum_{j=11}^{\infty} \int U_j(B) \omega(\mathcal{N}_h(\lambda\alpha)(x)) \, dx$$

$$\lesssim \sum_{j=11}^{\infty} |U_j(B)| \left[ 2^{-j p_\omega[n+(1-a)N]} + 2^{-j p_\omega[n+2M]} \right] \omega\left( \frac{|\lambda|}{\rho(|B|)|B|} \right) \lesssim |B| \omega\left( \frac{|\lambda|}{\rho(|B|)|B|} \right).$$

Thus, (5.6) holds with $T = \mathcal{N}_h$, and hence $H_{\omega,L}(\mathbb{R}^n) \subset H_{\omega,\mathcal{N}_h}(\mathbb{R}^n)$.

From the fact that for all $f \in L^2(\mathbb{R}^n)$, $\mathcal{R}_h(f) \leq \mathcal{N}_h(f)$, it follows that for all $f \in H_{\omega,\mathcal{N}_h}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $\|f\|_{H_{\omega,\mathcal{N}_h}(\mathbb{R}^n)} \leq \|f\|_{H_{\omega,\mathcal{N}_h}(\mathbb{R}^n)}$, which together with a density argument implies that $H_{\omega,\mathcal{N}_h}(\mathbb{R}^n) \subset H_{\omega,\mathcal{R}_h}(\mathbb{R}^n)$.

To show that $H_{\omega,\mathcal{R}_h}(\mathbb{R}^n) \subset H_{\omega,\mathcal{R}_P}(\mathbb{R}^n)$, by (5.5), we have that for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{R}_P(f)(x) = \sup_{t>0} |e^{-t^2L} f(x)| \lesssim \sup_{t>0} \int_0^\infty e^{-\frac{u}{u^2}} e^{-\frac{u^2}{4t^2}} f(x) \, du.$$
\[ \lesssim \mathcal{R}_h(f)(x) \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \, du \lesssim \mathcal{R}_h(f)(x), \]

which implies that for all \( f \in H_\omega, \mathcal{R}_h(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), \( \|f\|_{H_\omega, \mathcal{R}_h(\mathbb{R}^n)} \lesssim \|f\|_{H_\omega, \mathcal{R}_h(\mathbb{R}^n)} \). Then by a density argument, we obtain \( H_\omega, \mathcal{R}_h(\mathbb{R}^n) \subset H_\omega, \mathcal{R}_p(\mathbb{R}^n) \).

Let us now show that \( H_\omega, \mathcal{R}_p(\mathbb{R}^n) \subset H_\omega, \mathcal{N}_p(\mathbb{R}^n) \). Since \( \omega \) satisfies Assumption (D), there exist \( q_1, q_2 \in (0, \infty) \) such that \( q_1 < 1 < q_2 \) and \( [\omega(t^{q_2})]^{q_1} \) is a convex function on \((0, \infty)\).

For all \( x \in \mathbb{R}^n \), \( t \in (0, \infty) \) and \( f \in L^2(\mathbb{R}^n) \), let \( u(x, t) \equiv e^{-t^\alpha f}(x) \). Then \( \tilde{L}u = Lu - \partial_t^2 u = 0 \). Applying Lemma 6.3 to such a \( u \) with \( 1/q_2 \), we obtain that for all \( y \in B(x, t/4) \),

\[ |e^{-t^\alpha f(y)}|^{1/q_2} \lesssim \frac{1}{t^{n+1}} \int_{t/2}^{3t/2} \int_{B(x,t/2)} |e^{-t^\alpha f}(z)|^{1/q_2} \, dz \, ds \lesssim \frac{1}{t^n} \int_{B(x,t)} \|\mathcal{N}_p(f)(z)\|^{1/q_2} \, dz. \]

Since \([\omega(t^{q_2})]^{q_1} \) is convex on \((0, \infty)\), by the Jensen inequality, we obtain

\[ \left[ \omega\left(|e^{-t^\alpha f(y)}|\right)\right]^{q_1} \leq \left[ \omega\left(\left( \frac{1}{t^n} \int_{B(x,t)} \|\mathcal{N}_p(f)(z)\|^{1/q_2} \, dz \right)^{q_2} \right) \right]^{q_1} \]

\[ \lesssim \frac{1}{t^n} \int_{B(x,t)} \omega(\mathcal{N}_p(f)(z)) \, dz \lesssim \mathcal{M}\left(\omega(\mathcal{N}_p(f))^{q_1}\right)(x), \]

which together with the fact that \( \omega \) is continuous implies that for all \( x \in \mathbb{R}^n \),

\[ \omega\left(\mathcal{N}_p^{1/4}(f)(x)\right) \lesssim \mathcal{M}\left(\omega(\mathcal{N}_p(f))^{q_1}\right)(x)^{1/q_1}. \]

Now by (6.3) and the fact that \( \mathcal{M} \) is bounded on \( L^{1/q_1}(\mathbb{R}^n) \), we obtain

\[ \|\omega(\mathcal{N}_p(f))\|_{L^{1/q_1}(\mathbb{R}^n)} \lesssim \|\omega(\mathcal{N}_p^{1/4}(f))\|_{L^{1/q_1}(\mathbb{R}^n)} \lesssim \|\mathcal{M}\left(\omega(\mathcal{N}_p(f))^{q_1}\right)\|_{L^{1/q_1}(\mathbb{R}^n)} \lesssim \|\omega(\mathcal{N}_p(f))\|_{L^{1}(\mathbb{R}^n)}, \]

and hence \( \|f\|_{H_\omega, \mathcal{N}_p(\mathbb{R}^n)} \lesssim \|f\|_{H_\omega, \mathcal{R}_p(\mathbb{R}^n)} \). Then by a density argument, we obtain that \( H_\omega, \mathcal{R}_p(\mathbb{R}^n) \subset H_\omega, \mathcal{N}_p(\mathbb{R}^n) \).

Finally, let us show that \( H_\omega, \mathcal{N}_p(\mathbb{R}^n) \subset H_\omega, \mathcal{L}(\mathbb{R}^n) \). For all \( x \in \mathbb{R}^n \), \( \beta \in (0, \infty) \) and \( f \in L^2(\mathbb{R}^n) \), define \( \tilde{S}_\beta^3 f(x) \equiv (\int_{\Gamma_\beta(x)} |\tilde{\nabla} e^{-t^\alpha f}(y)|^2 \, dy \, dt)^{1/2} \), where \( \tilde{\nabla} \equiv (\nabla, \partial_t) \) and \( |\tilde{\nabla}|^2 = |\nabla|^2 + (\partial_t)^2 \). It is easy to see that \( S_\beta f \lesssim \tilde{S}_\beta^3 f \).

It was proved in the proof of [17, Theorem 8.2] that for all \( f \in L^2(\mathbb{R}^n) \) and \( u > 0 \),

\[ (6.5) \quad \sigma_{\tilde{S}_\beta^3}^{1/2}(u) \lesssim \frac{1}{u^{2}} \int_0^u \frac{\sigma_{\mathcal{N}_p}(t)}{t} \, dt + \sigma_{\mathcal{N}_p}(u), \]

where \( \beta \in (0, \infty) \) is large enough, and \( \sigma_g \) denotes the distribution of the function \( g \).

Since \( \omega \) is of upper type 1 and lower type \( p_\omega \in (0, 1] \), we have \( \omega(t) \sim \int_0^t \frac{\omega(u)}{u} \, du \) for each \( t \in (0, \infty) \), which together with (6.2), (6.3), (6.5) and \( S_\beta f \lesssim \tilde{S}_\beta^3 f \), further implies that

\[ \int_{\mathbb{R}^n} \omega(S_\beta(f)(x)) \, dx \lesssim \int_{\mathbb{R}^n} \omega(\tilde{S}_\beta^3(f)(x)) \, dx \lesssim \int_{\mathbb{R}^n} \omega(\tilde{S}_\beta^{1/2}(f)(x)) \, dx \]
\[ \int_0^\infty \omega(t) \left[ \frac{1}{t^2} \int_0^t u \sigma_{N^\beta_p}(u) \, du + \sigma_{N_{\omega}^\beta}(t) \right] \, dt \]
\[ \lesssim \int_0^\infty \frac{\omega(t)}{t} \int_0^t u \sigma_{N^\beta_p}(u) \, du + \int_\mathbb{R}^n \omega(N_{\omega}^\beta_p(x)) \, dx \]
\[ \lesssim \int_\mathbb{R}^n \omega(N_{\omega}^\beta_p(x)) \, dx \lesssim \int_\mathbb{R}^n \omega(N_{\omega}^\beta_p(x)) \, dx. \]

Thus, we obtain that \( \|f\|_{H_{\omega,SP}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\omega,NP}(\mathbb{R}^n)} \). By Theorem 6.1, we finally obtain that \( H_{\omega,NP}(\mathbb{R}^n) \subset H_{\omega,SP}(\mathbb{R}^n) = H_{\omega,L}(\mathbb{R}^n) \), which completes the proof of Theorem 6.4.

**Remark 6.1.** (i) If \( n = 1 \) and \( p = 1 \), the Hardy space \( H^1_L(\mathbb{R}^n) \) coincides with the Hardy space introduced by Czaja and Zienkiewicz in [9].

(ii) If \( L = -\Delta + V \), and \( V \) belongs to the reverse Hölder class \( H_q(\mathbb{R}^n) \) for some \( q \geq n/2 \) with \( n \geq 3 \), then the Hardy space \( H^p_L(\mathbb{R}^n) \) when \( p \in (n/(n+1),1] \) coincides with the Hardy space introduced by Džubišni and Zienkiewicz [13, 14].

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