THE UNIVERSALITY CLASS OF MONOPOLE CONDENSATION
IN NON-COMPACT, QUENCHED LATTICE QED

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Abstract
Finite size scaling studies of monopole condensation in noncompact quenched lattice \textit{QED} indicate an authentic second order phase transition lying in the universality class of four dimensional percolation. Since the upper critical dimension of percolation is six, the measured critical indices are far from mean-field values. We propose a simple set of ratios as the exact critical indices for this transition. The implication of these results for critical points in Abelian gauge theories are discussed.
Monopole condensation was identified long ago as the physical mechanism driving the confinement transition in $U(1)$ lattice gauge theory [1]. This model illustrates the dual Meissner effect which is presumed to occur in non-Abelian theories as discussed by t’Hooft [2] and Mandelstam [3]. The complexity of these models, however, has led to slow progress in sharpening our understanding of the condensation mechanism and its physical implications. It is, therefore, interesting to consider a particularly simple model of monopole condensation in four dimensions, where exacting work can be done. In particular, we shall consider monopole condensation in non-compact quenched lattice $QED$ and argue that it is in the same universality class as four dimensional percolation. The simplicity of the model will allow us to do accurate numerical studies, which will determine the monopole susceptibility critical index $\gamma$, the monopole percolation (“magnetic” critical index $\beta$, and the correlation length critical index $\nu$ to a few percent. We shall see that the critical indices coincide with those of four dimensional percolation. These results are interesting because they indicate that monopole condensation in non-compact quenched $QED$ is an authentic second order phase transition even though the local field theory in which it is embedded is just a free field. In addition, since the upper critical dimensionality of percolation is six, the critical indices associated with the phase transition are far from mean-field values. We begin by discussing our numerical determinations of the critical indices $\gamma$, $\beta$ and $\nu$. The lattice action we simulated is

$$S_{gauge} = \frac{1}{2} \sum_{n\mu\nu} (\theta_\mu(n) + \theta_\nu(n + \mu) - \theta_\mu(n + \nu) - \theta_\nu(n))^2 \equiv \frac{1}{2} \sum_{n\mu\nu} \Theta_{\mu\nu}^2(n),$$

where the gauge fields $\theta_\mu(n)$ are oriented, real variables in the range $(-\infty, +\infty)$ defined on lattice links. Although Eq.(1) is just a free field, we can define a magnetic charge on the lattice as was already done in compact lattice $QED_4$. Introduce an electric charge $e$ and define an integer-valued Dirac string by,

$$e\Theta_{\mu\nu}(n) = e\bar{\Theta}_{\mu\nu}(n) + 2\pi S_{\mu\nu},$$

where the integer $S_{\mu\nu}$ determines the strength of the string threading the plaquette and $e\bar{\Theta}_{\mu\nu}$ is defined to lie in the interval $(-\pi, +\pi]$. The integer-valued monopole current $m_\mu(\tilde{n})$, defined on links of the dual lattice, is then

$$m_\mu(\tilde{n}) = \frac{1}{2} \varepsilon_{\mu\nu\kappa\lambda} \Delta^+_{\nu}(n + \hat{\mu}) S_{\kappa\lambda}(n + \hat{\mu}).$$
where $\Delta_+^\nu$ is the forward lattice difference operator, and $m_\mu$ is the oriented sum of the $S_{\mu\nu}$ around the faces of an elementary cube. This definition, which is gauge-invariant, implies the conservation law $\Delta_\mu m_\mu(\vec{n}) = 0$ which means that monopole world lines form closed loops. A fuller discussion of these variables can be found in ref.[5] where useful contrasts are made with the same constructions in compact lattice $QED$.

As emphasized originally in ref.[5], the constructions and concepts of percolation [6-8] are useful in quantifying the meaning of monopole condensation. Introduce the idea of a connected cluster of monopoles on the dual lattice: one counts the number of dual sites joined into clusters by monopole line elements. The oriented (vector-like) nature of the monopole elements is ignored. The problem of identifying and counting clusters is now exactly the same as occurs in bond percolation problems [6-8]. In the simplest models of bond percolation the entire problem is one of counting. One assumes that bonds are occupied randomly with probability $\rho$. At some critical concentration $\rho_c$ (called the percolation threshold), the largest connected cluster becomes infinite in extent, occupying a macroscopic fraction of the dual lattice, and signalling a phase transition. A natural order parameter for the phase transition is $M = n_{\text{max}}/n_{\text{tot}}$ where $n_{\text{max}}$ is the number of sites in the largest cluster and $n_{\text{tot}}$ is the total number of connected sites [8]. Its associated susceptibility reads,

$$
\chi = \frac{\left< \sum_{n_{\text{min}}}^{n_{\text{max}}} g_n n^2 - n_{\text{max}}^2 \right>}{n_{\text{tot}}}
$$

(4)

where $n$ labels the size of a cluster occurring $g_n$ times on the dual lattice. In general $n_{\text{min}} = 2$, but for monopoles $n_{\text{min}} = 4$ because of the conservation law.

The critical indices for monopole condensation are then defined as in bond percolation[8]. For a lattice of infinite extent, $M$ should be nonzero for a strong coupling $e$ and vanish identically for weak coupling. At some critical point, $M$ should turn on non-analytically with a "magnetic" exponent $\beta$,

$$
M \sim \left( \frac{1}{e_c^2} - \frac{1}{e^2} \right)^\beta, \quad e \geq e_c
$$

(5)

where we have written $1/e^2$ (rather than $e$ itself), following the standard conventions of strong coupling lattice gauge theory [1]. The susceptibility should diverge at $e_c$ with a susceptibility index $\gamma$. 

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\[ \chi \sim \left( \frac{1}{e_c^2} - \frac{1}{e^2} \right)^{-\gamma} \]

and the linear size \( \xi \) of the largest cluster should also diverge:

\[ \xi \sim \left( \frac{1}{e_c^2} - \frac{1}{e^2} \right)^{-\nu}, \quad e \geq e_c \]

where \( \nu \) is the correlation length exponent.

In order to measure \( \gamma, \beta \) and \( \nu \), we did three types of computer experiment. First we measured \( \chi \) and \( M \) as a function of coupling \( 1/e^2 \) on lattices of volume \( L^4 \), with \( L \) ranging from 10 to 20. Since \( S_{gauge} \) is a quadratic form, independent gauge field configurations could be generated by FFT methods avoiding critical slowing down entirely [9]. According to standard finite-size scaling arguments [10], the peak of the susceptibility should grow with lattice size \( L \) as

\[ \chi_{\text{max}}(L) \sim L^{\gamma/\nu}. \] (8)

In addition, the value of the order parameter \( M \) should vanish

\[ M(L) \sim L^{-\beta/\nu}, \quad e = e_c \] (9)

at the critical point. And finally, Eq.(5) yields an estimate of the magnetic exponent \( \beta \) as long as we can study a range of coupling \( 1/e^2 \) where the functional dependence of Eq.(5) is not distorted by finite size effects. Consider the susceptibility first. Data of \( \chi \) versus \( 1/e^2 \) for \( L \) values 10, 12, 16, 18 and 20 are given in Table 1. Note that the peak occurs at \( 1/e_c^2 = 0.244 \) independent of \( L \). In Fig. 1 we plot the logarithms of the peaks, \( \ln \chi_{\text{max}} \) versus \( \ln L \), and find an excellent straight-line fit with the slope,

\[ \gamma/\nu = 2.24(2) \] (10)

The order parameter \( M \) was measured in parallel with \( \chi \), and the results are recorded in Table 2. A plot of \( \ln M \) as a function of \( \ln L \) at \( 1/e_c^2 = 0.244 \) is shown in Fig. 2. Again a straight line fit compatible with finite-size scaling emerges and the slope is determined to be

\[ \beta/\nu = 0.88(2) \] (11)
Finally we attempted a direct measurement of Eq.(5) using the \(16^4, 18^4\), and \(20^4\) lattices. The measurements of \(M\) indicate that its value at \(1/e^2 = 0.242\) decreases significantly as \(L\) ranges from 16 to 20, and are not useful here. However, the couplings \(1/e^2 = 0.240, 0.238, 0.236\) yield stable \(M\) values. We plot \(\ln M\) against \(\ln(1/e_c^2 - 1/e^2)\) in Fig.3, and find that a linear fit is acceptable for the three points \(1/e^2 = 0.236, 0.238, 0.240\). The fit gives an estimate for the slope, the magnetic exponent \(\beta\),

\[
\beta = 0.58(2) \tag{12}
\]

Clearly the measurements of \(\gamma/\nu\) and \(\beta/\nu\) in Eqs.(10) and (11) are our most precise. It is interesting to test whether the critical indices of the monopole condensation transition satisfy hyperscaling (which is expected of any model with a single divergent correlation length, which controls the system’s non-analyticities at the critical point), and deduce other critical exponents of the transition. According to hyperscaling,

\[
\gamma/\nu = 2 - \eta \tag{13a}
\]

\[
\beta/\nu = (d - 2 + \eta)/2, \tag{13b}
\]

where \(d = 4\) here. From Eq.(13.a) the critical index \(\eta = -0.24\) and from Eq.(13.b) \(\eta = -0.24\), as well. The agreement with hyperscaling is perfect! The third hyperscaling relation \(2\beta/\nu + \gamma/\nu = d\), becomes \(1.76 + 2.24 = 4.00\) and works perfectly also, while the fourth hyperscaling relation, \(2\beta\delta - \gamma = dv\), gives the critical index \(\delta = 3.55(2)\). Finally, the fifth and last hyperscaling relation, \(dv = 2 - \alpha\), requires additional input for the specific heat index \(\alpha\). Now use Eq.(12), our determination of the magnetic exponent. For example, combining Eqs.(10) and (12) with the hyperscaling relations gives all the critical indices of the transition,

\[
\gamma = 1.48(3), \quad \nu = 0.66(3), \quad \eta = -0.24(2),
\]

\[
\alpha = -0.64(3), \quad \beta = 0.58(2), \quad \delta = 3.55(3) \tag{14}
\]

We complete the discussion of our numerical results with two observations. First, we should classify this critical point, if at all possible. Naturally, we suspect that it is simply related to a four-dimensional percolation problem, although we have not proved this analytically, because of the vector character of the monopole problem and the conservation law \((\Delta^- \mu m_\mu(\vec{n}))\). In fact, the results of Eq.(14) are in excellent agreement with the critical
exponents of four-dimensional percolation, as estimated both by numerical simulation [6] and from series expansions [7], so we suggest that these two transitions lie in the same universality class. This is particularly intriguing since the indices of Eq.(14) are far from mean field indices. In fact, the upper critical dimensionality of percolation is 6, where mean field considerations become exact and $\gamma = \beta = 1, \nu = 1/2, \eta = 0, \alpha = -1$ and $\delta = 2$ [11].

Second, since the results of Eq.(14) are so close to ratios of small integers, we conjecture that the exact critical indices of this universality class are,

\[
\begin{align*}
\gamma &= 3/2, \quad \nu = 2/3, \quad \eta = -1/4, \\
\alpha &= -2/3, \quad \beta = 7/12, \quad \delta = 25/7
\end{align*}
\]

A comment about the negative value of $\eta$ is in order. The hyperscaling relations (Eq.13) are derived on the assumption that the physics of the critical region can be described by a local scalar field theory. In general for a field of dimension $d\phi$ they read:

\[
\frac{\gamma}{\nu} = d - 2d_{\phi}, \quad \frac{\beta}{\nu} = d_{\phi},
\]

which for $d_{\phi} = (d - 2 + \eta)/2$ reproduces Eq.(13). If $\eta$ is negative, we see that $d_{\phi}$ is smaller than the canonical value $d/2 - 1$, which leads to an infrared behaviour that is inconsistent with unitarity. The numerical success of the hyperscaling relations, the conjectured exact fractional form of the critical indices Eq.(15), and the field theoretic description of the critical point remain to be integrated into a single comprehensive theory of percolation.

The fact that quenched non-compact $QED$ monopoles condense with the same exponents as four-dimensional percolation is perhaps not too surprising. In ref.[5] it was shown that the concentration of monopole world lines varies smoothly as $1/e^2$ is made to vary across the critical region, i.e. there exists a smooth function $\rho(1/e^2)$ for the probability of bond occupation, and so the only source of non-analytic behavior can be the percolation threshold itself. Dynamical considerations are entirely subsumed by geometrical ones. This is in marked contrast to the compact $U(1)$ model, in which the monopole density falls sharply across the deconfining transition [4]. It would be interesting to repeat these measurements in non-compact lattice $QED$ including the effects of dynamical fermions.

We have argued elsewhere [12] that in this case, because of the compact nature of the $U(1)$ connection required to couple fermions to the model, the monopoles might have a direct dynamical influence on the physics of chiral symmetry breaking - there is no reason $a \text{ priori}$ to expect that monopole condensation in this case lies in the same universality class.
Preliminary results with $N_f = 2$ suggest that the two cases are difficult to distinguish [13]. Work continues on this interesting problem.

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Table 1
Monopole susceptibility $\chi$ as a function of coupling $1/e^2$ for lattices $10^4, 12^4, 16^4, 18^4$ and $20^4$.

| $1/e^2$ | $10^4$   | $12^4$   | $16^4$   | $18^4$   | $20^4$   |
|---------|----------|----------|----------|----------|----------|
| .254    | 38.66(38)| 50.27(45)| 67.35(60)| 72.80(91)| 75.2(1.5)|
| .252    | 41.89(45)| 56.64(57)| 75.58(78)| 88.3(1.2)| 94.8(2.2)|
| .250    | 45.93(57)| 63.44(72)| 95.6(1.0)| 109.1(1.7)| 117.5(2.9)|
| .248    | 49.16(72)| 70.85(93)| 116.4(1.6)| 140.7(2.9)| 153.1(4.8)|
| .246    | 51.37(83)| 75.9(1.1)| 136.9(2.5)| 172.9(2.5)| 211.4(4.3)|
| .244    | 52.02(97)| 77.2(1.4)| 146.2(3.5)| 192.9(3.4)| 248.8(7.4)|
| .242    | 51.9(1.1)| 73.8(1.6)| 134.9(4.4)| 167.4(5.0)| 215.8(10.3)|
| .240    | 48.2(1.1)| 66.2(1.8)| 106.4(4.5)| 114.5(9.6)| 105.7(12.8)|
| .238    | 43.1(1.3)| 56.9(1.9)| 63.8(3.2)| 64.6(5.0)| 62.8(6.1)|
| .236    | 36.3(1.2)| 42.1(1.6)| 38.7(2.1)| 35.7(1.4)| 37.1(3.1)|
Table 2
Same as Table 1, but for the order parameter $M = n_{\text{max}}/n_{\text{tot}}$.

| $1/e^2$ | $10^4$ | $12^4$ | $16^4$ | $18^4$ | $20^4$ |
|---------|--------|--------|--------|--------|--------|
| .254 | .0952(15) | .0652(11) | .0311(60) | .0225(6) | .0165(7) |
| .252 | .1096(19) | .0776(13) | .0392(8) | .0282(9) | .0196(9) |
| .250 | .1282(22) | .0942(16) | .0507(11) | .0368(13) | .0282(17) |
| .248 | .1519(26) | .1162(19) | .0684(16) | .0518(19) | .0390(27) |
| .246 | .1809(29) | .1468(23) | .0968(21) | .0816(16) | .0688(20) |
| .244 | .2172(33) | .1882(28) | .1410(29) | .1283(23) | .1165(29) |
| .242 | .2559(37) | .2374(31) | .2052(33) | .1993(27) | .1950(34) |
| .240 | .3033(38) | .2949(33) | .2775(34) | .2722(55) | .2792(67) |
| .238 | .3557(39) | .3530(32) | .3518(28) | .3505(38) | .3497(52) |
| .236 | .4099(37) | .4147(28) | .4154(22) | .4162(29) | .4135(42) |
Figure captions

1. Finite-size scaling for the peaks of the monopole susceptibility $\ln \chi_{\text{max}}$ as a function of $\ln L$.

2. Same as Fig. 1, but for the order parameter $M$ at $e_c$.

3. $\ln M$ against $\ln(1/e_c^2 - 1/e^2)$ for $1/e^2 = 0.242, 0.240, 0.238$ and $0.236$. The size of the symbols for $16^4, 18^4$ and $20^4$ lattices include the statistical error bars except in the case $1/e^2 = 0.242$, which is not used in the fit because of the finite-size effects.