Longitudinal and transverse static spin fluctuations in layered ferro- and antiferromagnets

A. Katanin

Institute of Metal Physics, 620990, Ekaterinburg, Russia
Ural Federal University, 620002, Ekaterinburg, Russia

We analyse the momentum dependence of static non-uniform susceptibilities of layered local-moment systems below Curie (Neel) temperature within the $1/S$ expansion, the renormalization-group approach, and first order of $1/N$ expansion. We argue that the previously known results of the spin-wave theory and renormalization-group approach for the transverse spin susceptibility acquire strong corrections already at sufficiently low temperatures, which appear due to the interaction of the incomping magnon having momentum $q$ with magnons with momenta $k < q$. Such corrections can not be treated in the standard renormalization-group approach, but can be described by both, $1/S$ and $1/N$ expansions. The results of these expansions can be successfully extrapolated to $T = T_M$, yielding the correct weight of static spin fluctuations, determined by the $O(3)$ symmetry. For the longitudinal susceptibility, the summation of leading terms of $1/S$ expansion within the parquet approach allows to fulfill the sumrule for the weights of transverse and longitudinal fluctuations in a broad temperature region below $T_M$ outside the critical regime. We also discuss the effect of longitudinal spin fluctuations on the (sublattice) magnetization of layered systems.

Layered local-moment systems (e.g. layered perovskites,$^1$ R$_2$MO$_4$ (R is some element, M is the transition metal) and undoped parent compounds for high-Tc superconductors) are distinctly different from the cubic magnets because of the reduced value of magnetic transition (Curie or Neel) temperature, since it is determined in these systems mainly by weak interlayer exchange or anisotropy. Relatively small values of magnetic transition temperatures allow for both theoretical and experimental study of the evolution of magnetic properties in layered systems in the whole temperature range $T < T_N$ and some temperature region above $T_N$.

At low temperatures magnetic excitations in layered (as well, as cubic systems) are described by the spin wave theory, considering periodic twists of spins with respect to the ordered state. This theory however does not describe correctly the thermodynamic properties of layered systems in a broad temperature range, since it is applicable only at the temperatures, which are much lower than the magnetic transition temperature. In particular, the transition temperatures, predicted by this theory appear too large and the critical exponents are not described correctly (see, e.g. discussion in Ref. $^2$).

This situation is reminiscent of weak itinerant magnets$^3$, where the Stoner mean-field theory does not reproduce correctly thermodynamic properties. For itinerant systems, the shortcomings of Stoner theory had led to the formulation of spin-fluctuation theory by Murata and Doniach$^4$, Dzyaloshinskii and Kondratenko$^5$, and Moriya$^6$, who considered the effect of collective excitations (paramagnons). Recently, an improvement of the spin-wave treatment of layered Heisenberg magnets by considering the effect of spin fluctuations within the RPA (ladder)-type analysis was proposed in Ref. $^6$. This analysis points to a similarity between the two classes of the systems, since it shows, that (static) longitudinal spin fluctuations in layered Heisenberg magnets, described by RPA diagrams, play the role, analogous to the paramagnons in weak itinerant magnets. The corrections to the sublattice magnetization due to critical (non-spin-wave) fluctuations appear however also to be important at not too low temperatures, as can be explicitly shown within the $1/N$ expansion$^7$.

These results motivate to study the effect of different type of excitations on the momentum $q$-dependence of the static transverse $\chi^-(q,0)$ and longitudinal $\chi^+(q,0)$ spin susceptibility in layered systems (second argument corresponds to vanishing frequency). In particular, while the momentum dependence of $\chi^+(q,\omega_n)$ is expected to be dominated by one, three, and higher-magnon processes, $\chi^\perp(q,\omega_n)$ is determined by two-magnon processes already in the lowest order of $1/S$ expansion. These two susceptibilities are not however fully independent, since they must fulfill the sumrule $\sum_{\mathbf{q},\omega_n} [\chi^\perp(q,\omega_n) + \chi^-(q,\omega_n)] = S(S+1)$, which constrains relative magnitude of longitudinal and transverse spin fluctuations. In other words, the longitudinal spin fluctuations should be also important for the transverse susceptibility.

The spin susceptibility of two-dimensional systems was analyzed previously within the Schwinger boson and modified spin-wave approaches$^8$,$^11$, the renormalization-group approach$^12$,$^13$ and first-order $1/N$ expansion$^14$. As we discuss in the present paper, in the symmetric phase the three former approaches overestimate the spectral weight by a factor 3/2 (see also Refs.$^8$,$^10$,$^12$), which is important for the fulfillment of the abovediscussed sumrule. The sumrule appears to be fulfilled in $1/N$ expansion, but violated in the renormalization-group and spin-wave approaches.

Extending the results for the non-uniform magnetic susceptibilities to the magnetically-ordered phase allows to study both, the regime of low temperatures, where the spin-wave approach is applicable, and the critical regime, where the results of $1/N$ expansion can be ap-
plied. Such an analysis gives also a possibility to investigate the crossover between the two regimes, which corresponds physically to changing relevant magnetic fluctuations from spin-wave to the critical ones. The sum rule for spin susceptibilities can serve as a criterion of the validity of the results of theoretical approaches.

In the following we analyze the results of 1/S expansion for transverse and longitudinal spin susceptibilities, as well 1/N expansion for the transverse spin susceptibility, which allows us to understand the momentum dependence of these quantities in different temperature regimes.

I. THE MODEL AND THE SPIN-WAVE APPROACH

We consider two-dimensional ferro- and antiferromagnets with the easy-axis anisotropy, described by the Hamiltonian

\[ H = -\frac{1}{2} \sum_{ij} J_{ij} S_i S_j - \frac{1}{2} \eta J \sum_i S_i^z S_j^z - D \sum_i (S_i^z)^2, \]  

(1)

where \( J_{i,i+\delta_z} = J, J_{i,i+\delta_x} = \alpha J/2 \), and \( J_{ij} = 0 \) otherwise, \( \delta_x \) and \( \delta_z \) being the vectors, connecting nearest neighbor sites in the same plane and in different planes; \( \eta > 0 \) and \( D > 0 \) are the two-site and single-site easy-axis anisotropy parameters.

The 1/S expansion can be performed using the Dyson-Maleev representation

\[ S_i^+ = \sqrt{2S}a_i^+, \quad S_i^- = S - a_i^\dagger a_i, \]  

(2)

\[ S_i^z = \sqrt{2S}(a_i^\dagger - \frac{1}{2S}a_i^\dagger a_i), \]

where \( a_i^\dagger, a_i \) are the Bose ideal magnon operators. To quadratic order we obtain from Eq. (1) in the ferromagnetic case \( J > 0 \) the Hamiltonian of interacting spin waves

\[ H = H_{sw} + \frac{1}{4} \sum_{q_1,\ldots,q_4} \varphi(q_1, q_2; q_3, q_4) (a_{q_1}^\dagger a_{q_2}^\dagger a_{q_3} a_{q_4}) \times \delta_{q_1+q_2, q_3+q_4}, \]  

(3)

where

\[ H_{sw} = \sum_q E_q a_q^\dagger a_q, \]  

\[ E_q = S(J_0 - J_q) + |J| S f, \]

\[ J_q = 2J(\cos q_x + \cos q_y) + \alpha J \cos q_z \]

is the Fourier transform of the exchange integrals, \( f = (2S - 1)/|JS| + (J_0/J)S \). The interaction in Eq. (3) can be treat perturbatively.

In the case of a two-sublattice antiferromagnet we separate the lattice into \( A \) and \( B \) sublattices. On the sublattice \( A \) we use the representation \( 2 \), while on the sublattice \( B \) the “conjugate” representation:

\[ S_i^+ = \sqrt{2S}b_i^+, \quad S_i^- = -S + b_i^\dagger b_i, \]

\[ S_i^z = \sqrt{2S}(b_i - \frac{1}{2S}b_i^\dagger b_i), \]

(5)

where \( b_i^\dagger, b_i \) are the Bose operators. Introducing operators \( B_q \)

\[ a_q = (B_q + B_{q+Q})/2, \]

\[ b_q^\dagger = (B_q - B_{q+Q})/2, \]  

(6)

where \( Q = (\pi, \pi, \ldots) \) is the wavevector of the antiferromagnetic structure, up to a constant term, we have the Hamiltonian of the same form \( 4 \), but for the operators \( B_q \).

Note that in this case \( E_q \) in Eq. (3) does not have a meaning of an excitation spectrum because of non-Bose commutation relations \( [B_q, B_p^\dagger] = \delta_{q,p} + \delta_{q,p+Q} \); the true excitation spectrum is determined by the diagonalization of the Hamiltonian with respect to the states with the momenta \( q \) and \( q + Q \); in the present paper, however, we consider only static (classical) contributions to susceptibilities and (sublattice) magnetization, therefore commutation relations between operators \( B_q \) appear to be not important.

At not too low temperatures \( T \gg T_q, T_q = rJ \) for ferromagnets and \( T_q = r/2J \) for antiferromagnets, where \( r = \max(f, \alpha/2) \) the result for the sublattice magnetization in the spin-wave theory reads

\[ \langle S_q \rangle = \langle S_0 \rangle - \frac{T}{2\pi JS} \ln \frac{q_0}{r \gamma^2}, \]  

(7)

where \( \langle S_0 \rangle \) is the ground-state sublattice magnetization, \( q_0 \) corresponds to the ultraviolet cutoff parameter in momentum space for classical spin fluctuations, which is determined by temperature for quantum magnets \( S \sim 1 \): \( q_0 = [T/(JS)]^{1/2} \) in the ferromagnetic quantum case, \( q_0 = T/c \) in the antiferromagnetic quantum case \( c = \sqrt{\langle S \rangle} \) is the spin-wave velocity), and \( q_0 = \sqrt{32} \) in the classical case \( S \gg 1 \), see Ref. \( 2 \).

The spin-wave interaction can be treat in the lowest order of 1/S perturbation theory the so-called self-consistent spin-wave theory by performing decoupling of quartic terms in Eq. (3), see, e.g. Ref. \( 2 \). This theory yields renormalization of the magnon spectrum: for the renormalized anisotropy and interlayer coupling parameters one finds \( \Gamma_T = f \langle S \rangle^2 \) and \( \alpha_T = \alpha \langle S \rangle \); these renormalizations are expected to be qualitatively correct outside the critical region (see discussion in Ref. \( 2 \)), in the latter region the three-dimensional critical fluctuations due to interlayer coupling or domain wall topological contributions due to easy-axis anisotropy are expected to be important. For antiferromagnets, one has to perform additional renormalization \( J \to J \gamma/S \) in the first term of Eq. (4), where \( \gamma = S + 0.079 \) is the ground state (quantum) renormalization of the exchange interaction (its temperature renormalization can be neglected.
in the considering cases, in the following we also assume γ = S for ferromagnetic case). In case of considering the renormalization of the spectrum within the self-consistent spin-wave theory, the remaining interaction in Eq. (3) should be considered as one-particle irreducible, to avoid double counting.

II. TRANSVERSE SPIN SUSCEPTIBILITY

A. Spin-wave theory

We consider first the momentum dependence of the static transverse spin susceptibility \( \chi^{+-}(q, 0) = J_0^{1/T}d\tau\langle S^+_q(\tau)S^-_q(0)\rangle \). Using the representations (2) and (5) and decoupling quartic terms within the spin-wave approach, we obtain for both, ferro- and antiferromagnets in the small q limit

\[
\chi^{+-}_{sw}(q, 0) = \frac{2S}{|J\gamma q^2|} \tag{8}
\]

(for antiferromagnetic case the shift \( q \to q + Q \) is to be performed). Note that the result (8) can be also obtained within the modified spin-wave theory [11]. The result (8) implies that the static transverse spin susceptibility in the spin-wave approach would vanish at the magnetic phase transition temperature, which contradicts the requirement \( \chi^{+-}(q, 0) = 2\chi^{xx}(q, 0) \) at \( T = T_M \) and finiteness of the sum of averaged squares of spin components, see discussion below, in Sect. IIIB.

To overcome the latter drawback of the spin-wave approach, we consider the lowest-order correction to the result (8) due to the magnon interaction [9]

\[
\chi^{+-}_{c-sw}(q, 0) = \chi^{+-}_{sw}(q, 0) + \frac{T^2}{|J\gamma q^2|} \sum_{q_1, q_2} \frac{\varphi(q, q_1; q_2, q + q_1 - q_2)}{(q_1 + r)(q_2 + r)(q_1 + q_2)^2 + r}, \tag{9}
\]

where ‘c-sw’ stands for the spin-wave result, corrected with account of the spin-wave interaction. For the case of vanishing interlayer coupling we have

\[
\varphi(q_1, q_2; q_3, q_4) = J_{q_3} + J_{q_4} - J_{q_1 - q_3} - J_{q_1 - q_4} \approx -2|J||q_1q_2 + f|. \tag{10}
\]

The calculation of the integral yields

\[
\chi^{+-}_{c-sw}(q, 0) = \frac{2}{|J\gamma q^2|} \left[ \frac{1}{S_0} + \frac{T}{2\pi|J\gamma|}\ln\frac{q}{f^{1/2}} \right]^2. \tag{11}
\]

This result keeps its form also in the presence of the interlayer coupling with the replacement \( f \to r \). We would like to stress that the term proportional to \( \ln^2(q/f^{1/2}) \) comes from the integration over momenta \( f^{1/2} < k < q \), and therefore inaccessible for the standard renormalization-group techniques [12, 13], which consider only scales, which are larger than all the infrared cutoffs, i.e. \( k > \max(q, f^{1/2}) \).

At the same time, as we argue below, the second term in Eq. (11) appears to be crucially important for fulfillment of the sumrule already by the lowest-order perturbative correction. Indeed, considering the sum of squares of the transverse spin components for the result (11) we obtain

\[
\langle (S^x_t)^2 \rangle_{stat} + \langle (S^y_t)^2 \rangle_{stat} = T \sum_{r^{1/2} < q < q_0} \chi^{+-}_{c-sw}(q, 0) = 2S(S_0 - S) + \frac{2}{3S_0}(S_0 - S)^2. \tag{12}
\]

The result (12) describes correctly not only the low-temperature behavior of the weight of transverse fluctuations, but also the limit \( T \to T_M = T_C(T_N) \). Indeed, in this limit we have \( S = 0 \) and Eq. (12) yields \( (2/3)S_0^2 \) which, as we see in the following, is the weight, required by the \( O(3) \) symmetry.

B. 1/N expansion and comparison of different approaches

To treat the momentum-dependent transverse susceptibility beyond lowest order of \( 1/S \)-perturbation theory, we perform \( 1/N \) expansion in a way, which is similar to that described in Ref. [14] for 2D antiferromagnets (see Appendix A). The result of the corresponding expansion outside the critical regime is

\[
\chi^{+-}_{1/N, non-crit}(q, 0) = \frac{2}{|J\gamma q^2|} \left[ \frac{S}{S_0} + \frac{T}{2\pi|J\gamma|}\ln\frac{q}{r^{1/2}} \right]^2. \tag{13}
\]

It has the same structure, as the result of the \( 1/S \)-perturbation theory (11), except for the denominator in this limit we have \( S = 0 \) and Eq. (12) yields \( (2/3)S_0^2 \) which, as we see in the following, is the weight, required by the \( O(3) \) symmetry.

\[
\chi^{+-}_{1/N, T=T_M}(q, 0) = \frac{2}{|J\gamma q^2|} \left( \frac{aT}{2\pi|J\gamma|}\ln\frac{q}{r^{1/2}} \right), \tag{14}
\]

with \( a = 2/3 \), which is similar to the two-dimensional case [14]. The results (13) at \( T \to T_M \) and (14) differ only by a coefficient in front of logarithm: Eq. (13) corresponds to \( a = 1/2 \). This difference appears due to the fact, that apart from the spin-wave and longitudinal excitations, the result of the \( 1/N \) expansion in the critical regime [14] accounts for non-spin-wave (critical) fluctuations, which change the coefficient in front of the logarithm to \( a = 2/3 \).

From the result (13) we find the weight of the transverse spin fluctuations in \( 1/N \) expansion outside the crit-
FIG. 1: (Color online) The weight of the transverse spin fluctuations vs. (sublattice) magnetisation in different approaches (from top to bottom): renormalization-group approach, Eq. (17), corrected spin-wave theory, Eq. (11), 1/N-expansion outside the critical regime, Eq. (13), and standard spin-wave theory, Eq. (8). The semi-circle at the left axis shows the value of the weight $a = 2/3$, required by O(3) symmetry.

The weight of the transverse spin fluctuations in the discussed approaches is shown on Fig. 1. One can see that the weights of the corrected spin-wave theory and 1/N expansion outside the critical regime are numerically close to each other. Both these theories yield also results of the renormalization-group analysis\[12, 13, 15], as well as 1/N expansion, Eq. (13) are distinctly different from the results of RG or Schwinger boson approaches, as they contain second power of logarithm instead of the first. This leads to a difference of the weight of the transverse spin fluctuations of the abovementioned approaches already at sufficiently low temperatures ($\sigma$ close to 1), see Fig. 1. On the other hand, in comparison with the result of 1/N expansion at $T = T_N$, Eq. (14), the result (17) yields different coefficient $a_{RG} = 1$ in Eq. (14), which yields incorrect weight $\langle (S_i^x)^2 \rangle_{stat} + \langle (S_i^y)^2 \rangle_{stat}$ in the RG approach at $T \to T_M$ [17].

III. LONGITUDINAL SPIN FLUCTUATIONS

A. Spin-wave and RG analysis of longitudinal spin susceptibility

Let now analyze the longitudinal spin susceptibility within the Schwinger boson spin-wave approach [9,10]. (Note that the modified spin-wave theory [11] also yields the result (17), but for the longitudinal spin susceptibility; as discussed above the transverse spin susceptibility vanishes in the latter approach in the symmetric phase).

One can see, that the low-temperature results of systematic first-order 1/S expansion [11], as well as 1/N expansion, Eq. (13) are distinctly different from the results of RG or Schwinger boson approaches, as they contain second power of logarithm instead of the first. This leads to a difference of the weight of the transverse spin fluctuations of the abovementioned approaches already at sufficiently low temperatures ($\sigma$ close to 1), see Fig. 1. On the other hand, in comparison with the result of 1/N expansion at $T = T_N$, Eq. (14), the result (17) yields different coefficient $a_{RG} = 1$ in Eq. (14), which yields incorrect weight $\langle (S_i^x)^2 \rangle_{stat} + \langle (S_i^y)^2 \rangle_{stat}$ in the RG approach at $T \to T_M$ [17].
by magnon scattering in three different channels. In the lowest order of the perturbation theory we obtain

\[
\Phi(\mathbf{k}, \mathbf{p} - \mathbf{q}; \mathbf{k} - \mathbf{q}, \mathbf{p}) = -2Jk(p - q) - 4TJ^2 \sum_s \frac{|s(p - q)|s(s - q)|}{E_s E_{s - q}} - 4TJ^2 \sum_s \frac{|s + k - p||p - q|}{E_s E_{s + k - p}} + 4TJ^2 \sum_s \frac{|s - k - p + q|}{E_s E_{s - k + p + q}}.
\]

Using the relation \( \sum_s (s_a - q_s)s_b/(E_s E_{s - q}) = \delta_{ab}/(4\pi) \ln(q_0/q) \), we find to second order in \( 1/S \)

\[
\Phi(\mathbf{k}, \mathbf{p} - \mathbf{q}; \mathbf{k} - \mathbf{q}, \mathbf{p}) = -2Jk(p - q) \times \left[ 1 + \frac{T}{2\pi|J|\gamma^2} \left( \ln \frac{q_0}{q} + \ln \frac{q_0}{|k - p|} - \ln \frac{q_0}{|k + p - q|} \right) \right].
\]

To go beyond second order of the \( 1/S \)-perturbation theory, we apply the one-loop renormalization-group approach, which is also equivalent to summation of logarithmic divergencies of the parquet diagrams. To this end we replace infrared cutoffs in the particle-hole and particle-particle contributions in Eq. (20) by the cutoff parameter \( \mu \), differentiate r.h.s. of Eq. (20) over \( \mu \), and replace the bare vertices in the right-hand side of equation (20) by renormalized vertices [we assume that the momentum dependence

\[
\Phi(\mathbf{k}, \mathbf{p} - \mathbf{q}; \mathbf{k} - \mathbf{q}, \mathbf{p}) = -2Jk(p - q)
\]

does not change, as follows from the lowest-order correction (20)]. In this way we obtain

\[
\frac{d\phi}{d\mu} = \frac{T}{2\pi|J|\gamma^2} S^2.
\]

The equation (21) was earlier obtained within the Holstein-Primakoff representation of spin operators in Ref. [18]. Using of the Dyson-Maleev representation makes however transparent that due to the change of sign in the particle-particle channel, which occur due to bilinear structure of the interaction vertex, the contribution of the two of the three channels cancel each other.

For \( q \sim |k - p| \sim |k + p - q| \) we stop the flow at \( \mu = \max(q, |k - p|, |k + p - q|) \) to obtain the result

\[
\phi = \frac{1}{1 - \frac{T}{2\pi|J|\gamma^2} \ln \frac{q_0}{\max(q, |k - p|, |k + p - q|)}}.
\]

When the momentum transfer in one (some) channel(s) is much smaller than in the others, one has to continue scaling with the contribution of the channels with smaller momentum transfer retained. For \( q \ll |k - p|, |k + p - q| \) we obtain again the equation (22) with the replacement \( \max(q, |k - p|, |k + p - q|) \rightarrow q \). On the other hand, at \( q \gg |k - p|, |k + p - q| \) the result (23) remains valid, since the contributions of the two remaining channels cancel each other. Therefore, for arbitrary \( q \) and \( |k - p| \sim |k + p - q| \) we obtain with the logarithmic accuracy the result (cf. Ref. [2])

\[
\Phi(\mathbf{k}, \mathbf{p} - \mathbf{q}; \mathbf{k} - \mathbf{q}, \mathbf{p}) = \frac{|J|k(q - p)}{S/(2\gamma) + q^2\chi_0(q)} + O(|J|f),
\]

where \( \chi \) is given by Eq. (2). Thus, as well as in RPA for itinerant magnets [3], the effective interaction is enhanced by fluctuations.

For the static (staggered) non-uniform longitudinal susceptibility (for antiferromagnetic case the shift \( q \rightarrow q + Q \) has to be performed) we obtain

\[
\chi^{zz}(q, 0) = \frac{(S^2/\chi_0(q))}{1 - (T/2\pi|J|\gamma^2 S^0) \ln[q_0/\max(\Delta^1/2, q)]} = \frac{S^2\chi_0(q)}{S + (|J|\gamma/2)q^2\chi_0(q)}.
\]

The vanishing of the weight of longitudinal fluctuations at \( T \rightarrow T_M \) is the unphysical result of the parquet approach, which do not consider critical fluctuations in the vicinity of the magnetic transition temperature. For the actual layered systems the range \( \sigma < 0.5 \) corresponds to the critical regime, which occur in the rather narrow temperature range near the transition temperature (the magnetization drops rather sharply near \( T_M \), see Sect. IIIc). The description of this regime requires considering critical fluctuations of either \( d = 3 \) \( O(3) \) or \( d = 2 \) \( Z_2 \) symmetry, depending whether interlayer coupling or anisotropy dominates, see Refs. [2, 4, 8]. At the same time, the results (23) and (27) are expected to be valid outside the critical regime, i.e. for \( \sigma > 0.5 \).

**B. Sum rule and the weight of longitudinal fluctuations**

In the following we analyze the fulfillment of the sum rule

\[
\langle (S^x)^2 \rangle_{\text{stat}} + \langle (S^y)^2 \rangle_{\text{stat}} + \langle (S^z)^2 \rangle_{\text{stat}} = \sum_q [\chi^+(q, 0) + \chi^{zz}(q, 0)] = S^2
\]

by the considered approaches; ‘stat’ stands for contribution of classical spin fluctuations (with the ultraviolet cutoff, given by \( q_0 \)). This sumrule follows from the standard relation \( S^2 = S(S+1) \); the difference \( S(S+1) - S^2 \) in the right-hand side appears due to retaining only classical term in equation (25). The possibility of considering separately classical and quantum contributions is realized for layered systems at temperatures \( T_q \ll T \ll |J|S^2 \) and reflects the ‘frozeness’ of magnetic moments (i.e. separation of the contributions of quantum and classical spin fluctuations) in the state with long-range or strong short-range order. The sum rule (25) can be rigorously proven...
within the nonlinear-sigma model treatment of the model by excluding the dynamic spin fluctuations which implies quantum renormalization of the model parameters similarly to the $\phi^4$ model analysis, see also Refs. [8, 15].

It can be easily verified that the results of the spin-wave theory [8] and [15] fulfill the sum rule (25). As discussed in previous sections, this theory is however expected to be insufficient at not very low temperatures. Fulfillment in previous sections, this theory is however expected to be insufficient at not very low temperatures. Fulfillment of the sum rule (25) by the other approaches, considered in Sect. II, requires the weight of the longitudinal fluctuations to be

\[
\langle (S_z)^2 \rangle_{\text{stat}} = \frac{\langle S^2 \rangle_{\text{stat}} - \langle S \rangle^2}{S_0^2}
\]

\[
= \begin{cases} 
\frac{1}{2} [1 - \sigma^2 \arctan h(1 - \sigma)] & 1/N, \text{non-crit} \ \\
1/3 & 1/N, T \to T_M \ \\
0 & \text{RG}
\end{cases}
\]

The substitution of $\langle S^2 \rangle$ in the left hand side of Eq. (26) corresponds extracting the uniform (Bragg) contribution, the result therefore reflects the contribution of the longitudinal fluctuations. As discussed above, only the corrected spin-wave theory and the $1/N$ expansion fulfill the $O(3)$ symmetry, which require that at $T \to T_M$ the weight of the transverse components should be twice bigger than the longitudinal ones. For the scaling analysis, the whole spectral weight (excluding the Bragg contribution) is in fact entirely contained in the transverse part (see also Ref. [15]), which explicitly violates $O(3)$ symmetry at $T = T_N$.

To verify the conformance of the parquet (RG) approach to longitudinal fluctuations to the approximations, listed in Eq. (26), we perform summation over momenta of Eq. (21) to obtain:

\[
T \sum_q \chi^z(q, 0) = 2S \left[ S_0 - S + S \ln(S/S_0) \right]
\]

\[
= \begin{cases} 
\langle S_0 - S \rangle^2 (1 - (2/3)(S_0 - S)), & S \to S_0 \ \\
2S_0 S & S \to 0
\end{cases}
\]

Comparing the first row of the second line of Eq. (27) with Eq. (26) we find that at low temperatures it agrees with the corrected spin-wave analysis of the transverse part implying that the considered parquet approach yields correct asymptotic behavior of the spectral weight at low temperatures. In fact, the corresponding range of sublattice magnetizations $\sigma > 0.4$, where the results for the weight of longitudinal fluctuations in the parquet analysis, corrected spin-wave theory and $1/N$ expansion outside the critical regime agree with each other, appears to be rather broad (see Fig. 2); as discussed above, it corresponds to the temperatures below the (narrow) critical regime near $T_M$. As discussed in Sect. IIa, the result of the low-temperature $1/N$ expansion [13] also allows to fulfill approximately the sumrule at $\sigma > 0.4$; while the result [13] fulfills the sum rule at $T = T_M$.

\[
\sigma^{1/\beta_2} = 1 - \frac{T}{4\pi \rho_s} \left[ (N - 2) \ln \frac{q_0^2}{\Delta(T)} + \frac{1}{\max(\sigma^{1/\beta_2}, T/(4\pi \rho_s))} \right] + 2 \frac{1}{2(1 - \sigma^{1/\beta_2})} + F \left( T/(4\pi \rho_s \sigma^{1/\beta_2}) \right)
\]

where $\rho_s$ is the ground-state spin stiffness (e.g. $\rho_s = J_\gamma S_0$ in the self-consistent spin-wave theory), $\Delta(T) = \max(\alpha_T/2, f_T)$, and $F(x)$ is some non-singular function, which accounts for the contribution of higher-loop terms. The “critical exponent” $\beta_2$, which is the limit of the exponent $\beta_{2+\varepsilon}$ in $d = 2 + \varepsilon$ dimensions at $\varepsilon \to 0$, is given by $\beta_2 = (N - 1)/(2(N - 2))$. At $N = 3$ we have $\beta_2 = 1$. The leading logarithmic term in Eq. (28) corresponds in this case to the self-consistent spin-wave theory, while the subleading logarithmic term coincides with the result of the parquet approximation, discussed in Sect. IIIB (cf. Ref. [2]), and therefore describes the contribution of the static longitudinal fluctuations to (sublattice) magnetization. The nonsingular term $-2(1 - \sigma)$ does not follow directly from the parquet approach and at low temperatures compensates the second term in the square brackets. The function $F$ stands for the contribution of non-
parquet diagrams, not accounted by renormalization-group analysis.

The result (28) can be compared to that of the first-order $1/N$ expansion of the sublattice magnetization, which reads\[7\]

\[\sigma = \left\{ 1 - \frac{T}{4\pi \rho_s} \left[ (N-2) \ln \frac{2T^2}{\Delta(\alpha_r, f_r)} + B_2 \ln \frac{1}{\sigma^2} \right] - 2(1 - \sigma^2) - I_1(x_\sigma) \right\}^{\beta_2}, \]

where $\alpha_r = \alpha_{T=0}$ and $f_r = f_{T=0}$ are the ground-state quantum-renormalized parameters of the anisotropy and interlayer coupling, $B_2 = 3 + f_r/\sqrt{f_r^2 + 2\alpha_r f_r}$,

\[x_\sigma = \frac{4\pi \rho_s}{(N-2)T} \sigma^2. \]

For the cases, when only anisotropy or interlayer coupling is present ($\alpha_r = 0$ or $f_r = 0$), the results (28) and (29) differ by the replacement $\sigma^{1/\beta_2} \rightarrow \sigma^2$ in the right-hand side, which is related to the property of the first-order $1/N$ expansion, that the $N = \infty$ value of the exponent $\beta_2 = 1/2$ is corrected only in the leading (proportional to $N - 2$), but not in the subleading (proportional to $N^0$) terms in Eq. (29). Although for the abovementioned cases the result (28) can be therefore considered as a self-consistent modification of the first-order $1/N$ result, it appears that non-self-consistent equation (29) better agree with experimental data (see below), which can be related to the importance of non-spin-wave fluctuations, not captured by the renormalization-group approach.

Let us also discuss the relation of the results of $1/N$ expansion for sublattice magnetization in the presence of interlayer coupling only ($f_r = 0$) to the approach, proposed recently in Ref. 19. The result of Ref. 19 up to the logarithmic accuracy reads

\[\sigma = \left( 1 - \frac{T}{2\pi \rho_s} \ln \frac{2T^2}{c^2 \alpha \sigma} \right)^{1/2}. \]

This result can be compared to the results of first-order $1/N$ expansion (29). One can see, that although Eq. (31) contains some of the first-order $1/N$ corrections, it borrows the exponent of magnetization $1/2$ from the result of the zeroth-order $1/N$ expansion in the intermediate temperature regime and neglects subleading terms in Eq. (29), which account for the longitudinal spin fluctuations.

Comparison of different approaches is presented in Fig. 3. We choose the parameters, which correspond the compound La$_2$CuO$_4$: $\rho_s = 290K$, $c = 2618K$, and $\alpha = 10^{-3}$. One can see consequent reduction of the Neel temperature from spin-wave and self-consistent spin-wave analysis, and further to the renormalization-group approach. The results (29) and (31) appear to be numerically close to each other, and therefore Eq. (31) provides successful extrapolation of the sublattice magnetization from its low-temperature limit.

IV. CONCLUSION

In the present paper we have considered momentum dependence of static transverse- and longitudinal spin susceptibilities, analyzed the fulfillment of the sumrule (7) and discussed the effect of longitudinal spin fluctuations on the (sublattice) magnetization.

For the transverse susceptibility, we have shown that the result of the corrected spin-wave theory, accounting for the spin-wave interaction in the lowest order in $1/S$, yields the correct weight of transverse spin fluctuations in both, the low-temperature limit and $T \rightarrow T_M$. The correction to the pure spin-wave result is proportional to $\ln^2(q/r^{1/2})$ and comes from the integration over the range $r^{1/2} < k < q$ of the momenta $k$ of virtual magnons, interacting with the physical state at momentum $q$. This momentum range is inaccessible for the standard renormalization-group approach, which treats only contribution of momenta $k > q$. Therefore, standard renormalization-group approach by Nelson and Pelkovitz [12] and Chakraverty, Halperin, and Nelson [13], applied to the non-uniform spin susceptibility, has to be supplemented by considering the RG flow at the scales $k < q$. This can be done in particular within the functional renormalization-group analysis [22], which is lefted as a subject for future studies.

The first-order $1/N$ expansion allows to determine the static transverse spin susceptibility outside the critical regime and in the limit of Curie (Neel) temperature. The corresponding correction to the spin-wave result for the transverse nonuniform spin susceptibility changes from $\ln^2(q/r^{1/2})$ at low temperatures to $\ln(q/r^{1/2})$ near $T = T_M$. The result of $1/N$ expansion at $T = T_M$ differs by a factor 2/3 from the results of Schwinger-boson and the renormalization-group analysis, which happens because the latter approaches assume spin-wave picture of the excitation spectrum instead of a critical one. Fur-
ther analysis of the crossover between the spin-wave and critical regime seems to be of the certain interest.

The applicability of the ladder (parquet) approximation for the longitudinal spin susceptibility was analyzed. It was shown that the results of these approximations allow to fulfill the sumrule at the temperatures outside the critical regime, where the relative (sublattice) magnetization $\sigma > 0.4$. Extending the results of $1/N$ analysis to the order $1/N^2$, which would make possible to describe both, longitudinal and transverse spin susceptibilities within this approach with higher accuracy, are of the certain interest. The extension of the presented analysis to calculate the dynamic magnetic susceptibilities on the basis of the discussed approaches would allow to study the difference of the spin-wave and critical dynamics.

Acknowledgements. The author is grateful to V. Yu. Irkhin, A. F. Barabanov, A. N. Ignatenko, and O. Sushkov for stimulating discussions. The work is supported by the Partnership program of the Max-Planck Society.

Appendix A. 1/N expansion for the transverse spin susceptibility

Following to [6, 14, 20], to construct $1/N$ expansion we pass to the continuum classical $O(N)$-model with the partition function

$$Z = \int D\sigma D\lambda \exp \left\{ \frac{\rho_s}{2} \int_0^{1/T} d\tau \int d^2r \left[ (\nabla \sigma)^2 + f(\sigma_1^2 + \ldots + \sigma_{N-1}^2 + i\lambda(\sigma^2 - 1)) \right] \right\},$$

where $\sigma = \{\sigma_1, \ldots, \sigma_N\}$ is the vector field, $\rho_s$ is the quantum-renormalized spin stiffness, $f$ is the dimensionless anisotropy parameter (see main text, we consider here for simplicity the two-dimensional case $\alpha = 0$). In the ordered phase (only this will be considered) the excitation spectrum in the zeroth order in $1/N$, which is given by the poles of the unperturbed longitudinal and transverse Green’s functions, contains a gap $f^{1/2}$ for all the components $\sigma_m$ except for $m = N$:

$$G_0^m(q) = (q^2 + f)^{-1}, \quad G_0^N(q) = q^{-2}. \quad (33)$$

The (dimensionless) self-energy of the transverse fluctuations is given by

$$\Sigma_\sigma(k) = \frac{2T}{N} \int \frac{d^2q}{(2\pi)^2} \frac{G_0^N(k - q)}{\Pi(q)}, \quad (34)$$

where

$$\Pi(q) = T \int \frac{d^2p}{(2\pi)^2} G_0^N(p) G_0^N(q + p) + \frac{2\sigma^2}{g} G_0^N(q),$$

and $\bar{g} = (N-1)/\rho_s$ is the quantum-renormalized coupling constant. Note that Eq. (34) represents a contribution of longitudinal spin fluctuations to the transverse spin susceptibility, as $\Pi^{-1}(q)$ has a very similar structure to the vertex (23), renormalized by the longitudinal fluctuations.

Evaluation of the integral in (34) yields

$$\Sigma_\sigma(k) = -\frac{k^2}{N} \ln \left( 1 - \frac{Tg}{2\pi} \ln \frac{q_0}{q} \right) \quad (37)$$

$$+ \frac{k^2}{N} \left( \frac{\ln(k/f^{1/2})}{\ln(k/f^{1/2}) + 2\pi\sigma^2/(Tg)} + \frac{k^2}{N} O\left( \frac{1}{\ln(k/f^{1/2})} \right) \right). \quad (35)$$

Using the result of the zeroth-order $1/N$ expansion

$$\sigma^2 = 1 - (\bar{g}T/(2\pi)) \ln(q_0/f^{1/2}) \quad (36)$$

(which is consistent with the calculating the first-order $1/N$ correction to the self-energy), we represent Eq. (35) in the form

$$\Sigma_\sigma(k) = -\frac{k^2}{N} \ln \left( 1 - \frac{Tg}{2\pi} \ln \frac{q_0}{q} \right) \quad (37)$$

$$+ \frac{k^2}{N} \left( \frac{\ln(k/f^{1/2})}{\ln(k/f^{1/2}) + 2\pi\sigma^2/(Tg)} + \frac{k^2}{N} O\left( \frac{1}{\ln(k/f^{1/2})} \right) \right). \quad (35)$$

The first term in Eq. (37) can be transformed to the form, given by the renormalization-group approach (generalization of the Eq. (17) to arbitrary $N$); to this end we perform in Eq. (37) the standard replacement $N \to N - 2 \quad [14]$, which is required because of considering first-order $1/N$ terms only, and collect logarithm into the power. Collecting other terms to comply the lowest orders of perturbation result [11] yields

$$\chi^{+\sigma}(q, 0) = \frac{N - 1 - \frac{Tg}{2\pi} \ln \frac{q_0}{q}} {\rho_s q^2} \left[ \frac{1}{\ln(\bar{g}T/(2\pi))} \right]^{-1/(N-2)} \quad (38)$$

$$\times \left[ 1 + \frac{Tg}{2\pi} \ln(q/f^{1/2}) + \frac{Tg}{2\pi} \ln \frac{q_0}{q} \right]^{-1/(N-2)}.$$

Replacement $N \to N - 2$ and collecting terms into power in the multiplier in the second line of Eq. (38) is however unjustified near the magnetic transition temperature, since in that case this multiplier describes contribution of non-spin-wave degrees of freedom, which reduces the transverse susceptibility $(N - 1)/N$ times [14]. Instead, we find in that case, similarly to the two-dimensional case [14]

$$\chi^{+\sigma}(q, 0) = \frac{N - 1 - \frac{Tg}{2\pi} \ln \frac{q_0}{q}} {\rho_s q^2} \left[ \frac{1}{\ln(\bar{g}T/(2\pi))} \right]^{-1/(N-2)} \quad (38)$$

$$\times \left[ 1 + \frac{Tg}{2\pi} \ln(q/f^{1/2}) + \frac{Tg}{2\pi} \ln \frac{q_0}{q} \right]^{-1/(N-2)}.$$
Since the above-considered calculation accounted for logarithmic terms only, the same results with the replacement \( f \rightarrow r \) hold in the presence of the interlayer coupling. Comparing this to the spin-wave results (8) and (11) one should have in mind that \( N = 3 \) in the physical case and the used zeroth-order \( 1/N \) result for \( \sigma^2 \) (30) is analogous to the spin-wave result for \( \overline{\sigma}^2/\overline{\sigma}_0^2 \).

[1] *Magnetic Properties of Layered Transition Metal Compounds*, ed. L.J. de Jongh, Cluver, Dordrecht, 1989.
[2] V. Yu. Irkhin, A. A. Katanin, and M. I. Katsnelson, Phys. Rev. B 60, 1082 (1999); A. A. Katanin and V. Yu. Irkhin, Physica-Uspekhi 50, 613 (2007).
[3] T. Moriya, *Spin Fluctuations in Itinerant Electron Magnetism*, Springer, Berlin, 1985.
[4] K. K. Murata and S. Doniach, Phys. Rev. Lett. 29, 285, (1972).
[5] I. E. Dzyaloshinskii and P.S. Kondratenko, ZhETF 70, 1987 (1976) [Sov. Phys. JETP 43, 1036 (1976)]
[6] V.Yu. Irkhin and A.A. Katanin, Phys.Rev. B 55, 12318 (1997).
[7] V.Yu. Irkhin and A.A. Katanin, Phys. Lett. A 232, 143 (1997).
[8] V. Yu. Irkhin and A. A. Katanin, Phys.Rev. B 57, 379 (1998).
[9] D. P. Arovas and A. Auerbach, Phys. Rev. B 38, 316 (1988).
[10] D. Yoshioka, J. Phys. Soc. Jpn 58, 3733 (1989).
[11] M. Takahashi, Phys. Rev. B 40, 2494 (1989).
[12] D. R. Nelson and R. A. Pelkovitz Phys. Rev. B 16, 2191 (1977).
[13] S. Chakravarty, B.I.Halperin, and D.R. Nelson, Phys. Rev. B 39, 2344 (1989).
[14] A. V. Chubukov, S. Sachdev, and J. Ye, Phys. Rev. B 49 11919 (1994).
[15] A. A. Katanin and O. P. Sushkov, Phys. Rev. B 83, 094426 (2011).
[16] S. Sachdev, Phys. Rev. B 55, 142 (1997).
[17] Note that it was incorrectly concluded in Ref. [14], that the result of the RG approach agrees with the result of the \( 1/N \) expansion, Eq. (14). In fact, the result of Ref. [13] (Eqs. (D11) and (D12)) differ by a factor \( (N-1)/N \) from that of Eqs. (5.21) and (5.22) in Ref. [14] (\( N = 3 \) for the Heisenberg model), while both papers [13, 14] considered the correlation function of a single spin component.
[18] Yu. A. Kosevich, A.V. Chubukov, Zh. Eksp. Teor. Fiz. 91, 1105 (1986) [Sov. Phys. JETP 64, 654 (1986)].
[19] O. P. Sushkov, Phys. Rev. B 84, 094532 (2011).
[20] A. Auerbach, *Interacting Electrons and Quantum Magnetism*, Springer-Verlag, New York, 1994
[21] Similar corrections for itinerant antiferromagnets were considered recently in the paper by T. A. Sedrakyan and A. V. Chubukov, Phys. Rev. B 81, 174536 (2010).
[22] C. Wetterich, Phys. Lett. B 301, 90 (1993); J. Berges, N. Tetradis, and C. Wetterich, Phys. Rept. 363, 223 (2002).