ON THE VARIETY OF ALMOST COMMUTING NILPOTENT MATRICES

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Abstract. We study the variety of \( n \times n \) matrices with commutator of rank at most one. We describe its irreducible components; two of them correspond to the pairs of commuting matrices, and \( n - 2 \) components of smaller dimension corresponding to the pairs of rank one commutator. In our proof we define a map to the zero fiber of the Hilbert scheme of points and study the image and the fibers.

1. Introduction

Let \( V \) be a vector space of dimension \( n \) over a field \( K \) of characteristic equal to 0 or \( \geq n/2 \). Let \( g = gl_n(V) \) and \( n \) be the nilcone of \( g \), i.e., the cone of nilpotent matrices of \( g \). We write elements of \( V \) and \( V^* \) as column and row vectors, respectively. In this paper we study the variety

\[ N := \{(X, Y, i, j) \in n \times n \times V \times V^* \mid [X, Y] + ij = 0\} \]

and prove that it has \( n \) irreducible components: 2 of dimension \( n^2 + n - 1 \) corresponding to the case where the matrices commute, and \( n - 2 \) of dimension \( n^2 + n - 2 \), corresponding to the noncommutative pairs.

The pairs of almost commuting matrices have been studied recently in [7], where Gan and Ginzburg study the structure of the scheme

\[ M := \{(X, Y, i, j) \in g \times g \times V \times V^* \mid [X, Y] + ij = 0\}. \]

They prove that the irreducible components of \( M \) are the closures of the sets \( \mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_n \), defined as

\[ \mathcal{M}_t = \{ (X, Y, i, j) \in \mathcal{M} \mid \text{Y has pairwise distinct eigenvalues and} \dim K\langle X, Y \rangle i = t, \dim jK\langle X, Y \rangle = n - t \} \]

where \( K\langle X, Y \rangle i \) (resp. \( jK\langle X, Y \rangle \)) is the smallest subspace of \( V \) (resp. \( V^* \)) containing \( i \) (resp. \( j \)) and invariant under \( X \) and \( Y \).

Let \( \mathcal{K}_n = \{ (X, Y) \in n \times n \mid [X, Y] = 0 \} \), the variety of commuting nilpotent matrices. Baranovsky proved in [1] that \( \mathcal{K}_n \) is irreducible and has dimension \( n^2 - 1 \).

In his proof, he shows that \( U = \{ (X, Y, i) \in \mathcal{K}_n \times V \mid K[X, Y] i = V \} \) is irreducible, dense in \( \mathcal{K}_n \times V \) and has dimension \( n^2 + n - 1 \). \( GL(V) \) acts faithfully on \( U \), and the quotient \( U/GL(V) \) is a fiber of the Hilbert scheme of points under the Hilbert-Chow morphism.

Let

\[ \mathcal{N}_{r,s} = \{ (X, Y, i, j) \in \mathcal{N} \mid \dim K\langle X, Y \rangle i = r, \dim jK\langle X, Y \rangle = s \}, \]

\[ \mathcal{N}'_{r,s} = \{ (X, Y, i, j) \in \mathcal{N}' \mid \dim K\langle X, Y \rangle i \leq r, \dim jK\langle X, Y \rangle \leq s \} \]

and \( \overline{\mathcal{N}}_{r,s} \) the Zariski closure of \( \mathcal{N}_{r,s} \). Clearly \( \mathcal{N}_{r,s} \subseteq \overline{\mathcal{N}}_{r,s} \subseteq \mathcal{N}'_{r,s} \).
are located two positions above the diagonal are zero too. Therefore,
\[ \mathcal{N} = \mathcal{N}_{0,0} \cup \mathcal{N}_{n,0} \cup \bigcup_{0 < r + s < n} \mathcal{N}_{r,s}. \]

Following Baranovsky, we have that \( \mathcal{N}_{0,n} = \mathcal{N}_{0,n}^\prime \) and \( \mathcal{N}_{n,0} = \mathcal{N}_{n,0}^\prime \) which can be identified with \( \mathcal{K}_n \times V \) and \( \mathcal{K}_n \times V^* \), respectively.

Our main theorem is the following.

**Theorem 1.** (a) The irreducible components of \( \mathcal{N} \) are precisely \( \mathcal{N}_{t,n-1-t} \), 1 \( \leq \) \( t \leq n-2 \), \( \mathcal{N}_{0,n}^\prime \) and \( \mathcal{N}_{n,0}^\prime \).
(b) \( \dim \mathcal{N}_{t,n-1-t} = n^2 + n - 2 \) for 1 \( \leq t \leq n-2 \) and \( \dim \mathcal{N}_{0,n}^\prime = \dim \mathcal{N}_{n,0}^\prime = n^2 + n - 1 \).

In the last section we study the variety
\[ \mathcal{S} = \{(A, B, i, j) \in n \times n \times V \times V^* \mid A + B = ij\}. \]
and describe its irreducible components. The key fact is that if \( (A, B, i, j) \in \mathcal{S} \) then \( A \) and \( B \) are simultaneously triangularizable. The considerations that we make for this variety are simpler that the ones for \( \mathcal{N} \) since in this case it is easy to deform an element of \( \mathcal{S} \) and stay inside \( \mathcal{S} \), one just has to consider matrices that are strictly upper triangular in the given basis.

2. The Hilbert scheme

In this section we establish a connection between the Hilbert scheme and \( \mathcal{N}_{t,n-1-t} \) to prove that the later is irreducible if \( \text{char} \mathcal{K} = 0 \) or \( \geq n/2 \). This will also allow us to prove part (b) of Theorem 1.

Clearly every element of \( \mathcal{K}(X, Y) \) (resp. \( j \mathcal{K}(X, Y) \)) can be written as \( p(x, Y) \) (resp. \( j p(x, Y) \)) where \( p(x, y) \in \mathcal{K}(x, y) \) is a polynomial in the noncommutative variables \( x \) and \( y \).

**Lemma 2.** \( j \mathcal{K}(X, Y) \) (resp. \( j \mathcal{K}(X, Y) \)) is a right (resp. left) \( \mathcal{K}[x, y] \)-module and its perpendicular complement (\( (j \mathcal{K}(X, Y))^\perp \)) = \( \{v \in V \mid uv = 0 \ \forall u \in j \mathcal{K}(X, Y)\} \) (resp. \( \langle X, Y \rangle^\perp \)) = \( \{z \in V^* \mid zw = 0 \ \forall w \in \mathcal{K}(X, Y)\} \) is a left (resp. right) \( \mathcal{K}[x, y] \)-module, where \( x \) and \( y \) act as \( X \) and \( Y \) respectively.

**Proof.** \( j \mathcal{K}(X, Y) \) and \( (j \mathcal{K}(X, Y))^\perp \) are right and left \( \mathcal{K}(x, y) \)-modules, respectively. We have to prove that the two-sided ideal generated by \( [x, y] \) acts as zero on both of them.

Let \( q_1(x, y), q_2(x, y) \in \mathcal{K}(x, y) \). Then for every \( v \in (j \mathcal{K}(X, Y))^\perp \),
\[ q_1(X, Y)(X\ Y - Y\ X)q_2(X, Y)v = q_1(X, Y)i(jq_2(X, Y)v) = 0 \]
since \( jq_2(X, Y) \in \mathcal{K}(X, Y) \). This proves that \( (j \mathcal{K}(X, Y))^\perp \) is a \( \mathcal{K}[x, y] \)-module.

To prove that \( j \mathcal{K}(X, Y) \) itself is a \( \mathcal{K}[x, y] \)-module, note that \( j \mathcal{K}(X, Y) \subseteq (j \mathcal{K}(X, Y))^\perp \)
since the elements of \( j \mathcal{K}(X, Y) \) and \( \mathcal{K}(X, Y)i \) have the form \( j p(x, Y), q(x, Y)i \) for \( p(x, y), q(x, y) \in \mathcal{K}(x, y) \), and \( j p(X, Y)q(X, Y)i = 0 \) (\[ \mathcal{L} \], Lemma 2.1.3.). This proves the claim.

From now on, we write \( j \mathcal{K}[X, Y] = j \mathcal{K}(X, Y), \mathcal{K}[X, Y]i = \mathcal{K}(X, Y)i \).
We now define a map \( \mathcal{N}_{r,s} \to \mathcal{H}^m_0 \times \mathcal{H}^n_0 \) where \( \mathcal{H}^m_0 \) is the fiber of the Hilbert-Chow morphism \( \mathcal{H}^m(A^2) \to S^m(A^2) \) over the point \( m \cdot [0] \in S^m(A^2) \) and \( \mathcal{H}^m(A^2) \) denotes the Hilbert scheme of points in the affine plane. The image of \( \text{Chow morphism} \ H \) to \( X \) is the ideal \( \{ p(x,y) \in \mathbb{K}[x,y] \mid p(X,Y)i = 0 \} \). This ideal is also equal to \( \{ p(x,y) \in \mathbb{K}[x,y] \mid p(X,Y)v = 0 \ \forall v \in \mathbb{K}[X,Y]i \} \) since \( i \) is a cyclic vector for \( X \) and \( Y \) on \( \mathbb{K}[X,Y]i \). Similarly, we define \( \mathcal{N}_{r,s} \to \mathcal{H}^n_0 \) as the ideal \( \{ p(x,y) \in \mathbb{K}[x,y] \mid jp(X,Y) = 0 \} \).

Since \( (\mathbb{K}[X,Y]i)^\perp \) and \( (j\mathbb{K}[X,Y])^\perp \) are \( \mathbb{K}[x,y] \)-modules, one could try to induce maps \( \mathcal{N}_{r,s} \to \mathcal{H}^m_0 \), \( \mathcal{H}^n_0 \), but there may not be a cyclic vector for the actions of \( X \) and \( Y \) on those spaces. However, if \( r + s = n - 1 \) the following theorem implies that such map does exist.

**Theorem 3.** There is a well-defined regular map \( \mathcal{N}_{r,n-1-t} \to \mathcal{H}^{r+t}_0(A^2) \) induced by the actions of \( X \) and \( Y \) on \( (j\mathbb{K}[X,Y])^\perp \). This map is dominant and the image of any element of \( \mathcal{N}_{r,n-1-t} \) has the form \( \langle y_1^{t+1}, x-a_1y-\cdots-a_ty^t \rangle \) or \( \langle x^t, y-a_1x-\cdots-a_tx^t \rangle \) for some \( a_1, \ldots, a_t \in \mathbb{K} \).

Similar considerations hold for the map \( \mathcal{N}_{t,n-1-t} \to \mathcal{H}^{n-t}_0(A^2) \) induced by the actions of \( X \) and \( Y \) on \( (\mathbb{K}[X,Y]i)^\perp \). In order to prove Theorem 1 we study the image and the fibers of the map \( \Psi : \mathcal{N}_{t,n-1-t} \to \mathcal{H}^{t}_0 \times \mathcal{H}^{n-t}_0 \). Recall that \( GL_n(\mathbb{K}) \) acts on \( \mathcal{N} \) by \( G \cdot (X,Y,i,j) = (GXG^{-1},GYG^{-1},Gi,jG^{-1}) \).

**Theorem 4.** The fibers of the map \( \Psi : \mathcal{N}_{t,n-1-t} \to \mathcal{H}^{t}_0 \times \mathcal{H}^{n-t}_0 \) are the \( GL_n(\mathbb{K}) \)-orbits, and the isotropy of each element of \( \mathcal{N}_{t,n-1-t} \) is one-dimensional.

In order to prove Theorems 3 and 4 we need a technical result.

**Lemma 5.** Let \( (X,Y,i,j) \in \mathcal{N}_{r,s} \). There is a basis \( \{ e_1, \ldots, e_n \} \) of \( V \) with dual basis \( \{ e_1^*, \ldots, e_n^* \} \) of \( V^* \) so that \( e_r = i \), \( e_{r+1-s} = j \), and \( X \), \( Y \) are upper triangular in this basis.

**Proof.** Since \( j\mathbb{K}[X,Y] \) annihilates \( \mathbb{K}[X,Y]i \) we can decompose \( V = V_1 \oplus V_2 \oplus V_3 \) so that \( V_1 = \mathbb{K}[X,Y]i \) and \( V_3^* = (j\mathbb{K}[X,Y])^* \). We are to find elements \( e_1, \ldots, e_r \in V_1, e_{r+1}, \ldots, e_{n-s} \in V_2, e_{n+1-s}, \ldots, e_n \in V_3 \) satisfying the conditions.

Consider the lex deg order of the monomials in \( \mathbb{K}[x,y] \):

\[
1 < x < y < x^2 < xy < y^2 < x^3 < x^2y < xy^2 < y^3 < \ldots
\]

Choose the largest (according to \( < \)) monomial \( m_1 \) so that \( m_1(X,Y)i \neq 0 \) (it exists since \( X^aY^bi = 0 \) if \( a+b \geq n \)), and inductively choose \( m_k \) as the largest monomial so that \( m_k(X,Y)i \) is not a linear combination of \( m_1(X,Y)i, \ldots, m_{k-1}(X,Y)i \). This gives us \( r \) monomials \( m_1, \ldots, m_r \) so that \( \mathbb{K}[X,Y]i = (m_1(X,Y)i, \ldots, m_r(X,Y)i) \).

We set \( e_1 = m_1(X,Y)i, \ldots, e_r = m_r(X,Y)i = i \). The action of \( X \) and \( Y \) in this basis is triangular, since multiplying by \( x \) or \( y \) “increases” monomials, and for every monomial \( m \), either \( m(X,Y)i \) is in the basis or is a linear combination of larger monomials. That \( e_r = i \) is a consequence of the following lemma which will also be useful later.

**Lemma 6.** Let \( m(x,y) = x^a y^b \) and \( \lambda = \{(a_1, b_1), \ldots, (a_r, b_r)\} \). If \( (a, b) \in \lambda \) and \( 0 \leq a' \leq a, 0 \leq b' \leq b \) then \( (a', b') \in \lambda \).
Proof. If \((a', b') \notin \lambda\) then \(X^a Y^{b'} i\) can be written as a linear combination of larger monomials \(\sum_{x^a y^{b'} < x^c y^d} \alpha_{c, d} X^c Y^d i\). Then

\[
m_i(X, Y) i = X^{a-a'} Y^{b-b'} \left( \sum_{x^a y^{b'} < x^c y^d} \alpha_{c, d} X^c Y^d i \right) = \sum_{x^a y^{b'} < x^c y^d} \alpha_{c, d} X^{c+a-a'} Y^{d+b-b'} i,
\]

but this is a contradiction since \(x^a y^{b'} < x^c y^d\) implies \(x^a y^b < x^{c+a-a'} y^{d+b-b'}\). \(\square\)

In particular this implies that \(m_r(x, y) = 1\) and therefore \(e_r = i\).

Now we can follow the same procedure in \(V_2 = jK[X, Y]\) to find elements \(e_{n+1-s}, \ldots, e_n \in jK[X, Y]\) which in turn give rise to \(e_{n+1-s}, \ldots, e_n \in V_2 \subseteq V\).

To find the remaining elements of the basis, note that since \(V_2 \subseteq (jK[X, Y])^1\) we have that \(jK[X, Y]X V_2, jK[X, Y]Y V_2 \subseteq jK[X, Y]V_2 = 0\). This means that \(X|_{V_2}, Y|_{V_2} : V_2 \rightarrow V_1 \oplus V_2\). Since \(X, Y : V_1 \rightarrow V_1\), the actions of \(X\) and \(Y\) as endomorphisms of \(V_2\) are nilpotent, and they are commutative since \([X, Y] = ij\) which acts as 0 on this space. Therefore we can find \(e_{r+1}, \ldots, e_{n-s} \in V_2\) that make both \(X\) and \(Y\) upper triangular when restricted to \(V_2\). The basis \(\{e_1, \ldots, e_n\}\) of \(V\) satisfies the conditions.

Therefore we can assume that \(X\) and \(Y\) are upper triangular,

\[
i = \begin{pmatrix} 0 & & & & \\ 0 & \ddots & & & \\ \vdots & \ddots & \ddots & & \\ 0 & \ddots & \ddots & 0 & \\ 0 & \ddots & \ddots & \ddots & 0 \end{pmatrix}, \quad j = \begin{pmatrix} * & & & & \\ & \ddots & & & \\ & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & * \\ & & & \ddots & \ddots \end{pmatrix}, \quad jK[X, Y] = \begin{pmatrix} 0 & 0 & \ldots & 0 & * & \ldots & * \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},
\]

where the 1's in \(i\) and \(j\) are located in the \(r\)-th and \((n+1-s)\)-th position, respectively.

**Proof of Theorem 3.** We have to prove that \((jK[X, Y])^1\) admits a cyclic vector and that one of \(X, Y\) is regular when restricted to that space.

Let \(w = (jK[X, Y])^1 \setminus K[X, Y]i\). Since \(X w, Y w = (jK[X, Y])^1 = kW \oplus K[X, Y]\) and \(X, Y\) are nilpotent, we have that \(X w, Y w \in K[X, Y]i\); let \(X w = P(X, Y) i, Y w = Q(X, Y) i\), where \(P\) and \(Q\) are polynomials. Let \(Q(x, y) = Q_1(x, y) + c\) where \(Q_1(0, 0) = 0\). If \(c \neq 0\) then we have

\[
\frac{1}{c} (YP(X, Y) - XQ_1(X, Y)i) = \frac{1}{c} (YP(X, Y) - XQ(X, Y) + cX)i
\]

and every monomial in the left-hand side is \(> x\) in the lex deg order. Multiplying by \(X^{a-1}Y^b\), \(a > 0, b \geq 0\); we conclude that \(X^a Y^b i\) is a linear combination of larger monomials. According to the construction of the basis in Lemma 5, we have that
the basis for \( \mathbb{K}[X,Y] \) is \{\( Y^t i \), \( Y^{t-1} i \), \( Y^{t-2} i \), \( Y^{t-3} i \), \( Y^{t-4} i \), \( Y^{t-5} i \)\} and this implies that \( Y \) acts regularly in \( \mathbb{K}[X,Y] \).

If \( P \) has a constant term we can reverse the roles of \( x, y \) in the lex deg order. Now we prove that at least one of the polynomials \( P, Q \) has a nonzero constant term.

We can choose \( u \in V \setminus (j\mathbb{K}[X,Y])^\perp \) so that \( Xu,Yu \in (j\mathbb{K}[X,Y])^\perp \): to do this take any \( u_0 \in V \setminus (j\mathbb{K}[X,Y])^\perp \) if \( Xu_0,Yu_0 \in (j\mathbb{K}[X,Y])^\perp \), take \( u = u_0 \). If not, say \( Xu_0 \notin (j\mathbb{K}[X,Y])^\perp \), take \( u_1 = Xu_0 \) and repeat the process.

Then \( ju \neq 0 \) and \( Xu = \alpha w + R(X,Y)i, Yu = \beta w + S(X,Y)i \) for some \( R, S \in \mathbb{K}[x, y] \), \( \alpha, \beta \in \mathbb{K} \). If \( ju = 0 \) then \( j\mathbb{K}[X,Y]u = 0 \) since \( Xu, Yu \in (j\mathbb{K}[X,Y])^\perp \). We can assume, normalizing \( u \) if necessary, that \( ju = 1 \). Therefore

\[
i = iju = XYu - YXu = X(\beta w + S(X,Y)i) - Y(\alpha w + R(X,Y)i) = (\beta P(X,Y) - \alpha Q(X,Y) + XS(X,Y) - YR(X,Y))i.
\]

If \( P \) and \( Q \) have no constant term then all the monomials in the right-hand side have positive degree, a contradiction.

Therefore one of \( X \) or \( Y \) acts regularly on \( \mathbb{K}[X,Y] \). Assume without loss of generality that it is \( Y \). It is easy to see that then \( X|_{\mathbb{K}[X,Y]} = A(Y)|_{\mathbb{K}[X,Y]} \) where \( A(y) = a_1 + \cdots + a_{t-1}y^{t-1} \in \mathbb{K}[y] \).

Now we prove that there exists \( i' \in (j\mathbb{K}[X,Y])^\perp \setminus \mathbb{K}[X,Y] \) so that \( Yi' = i' \). Since \( Xw, Yw \in \mathbb{K}[X,Y] \), let

\[
Xw = D(Y)i = (d_0 + d_1 Y + \cdots + d_{t-1} Y^{t-1}), \quad Yw = C(Y)i = (c_0 + c_1 Y + \cdots + c_{t-1} Y^{t-1})i.
\]

If \( c_0 \neq 0 \) let \( i' = \frac{1}{c_0}(w - c_1 i - \cdots - c_{t-1} Y^{t-2} i) \in \mathbb{K}[w] + \mathbb{K}[Y]i = (j\mathbb{K}[X,Y])^\perp \). So assume by contradiction that \( c_0 = 0 \). Then \( jw = 0 \) implies

\[
0 = ijw = (XY - YX)w = (A(Y)C(Y) - YD(Y))i
\]

and therefore, comparing the coefficient of \( Y, d_0 = 0 \).

Let \( u \in V \) as before. Then

\[
i = iju = XYu - YXu = X(\beta w + S(X,Y)i) - Y(\alpha w + R(X,Y)i) = (\beta D(Y) - \alpha C(Y) + XS(X,Y) - YR(X,Y))i;
\]

but all the monomials on the right-hand side have positive degree, which is impossible. Therefore we can find such \( i' \).

But \( X i' = \frac{1}{c_0}(Xw - c_1 X i - \cdots - c_{t-1} X Y^{t-2} i) \in \mathbb{K}[X,Y]i \subseteq (j\mathbb{K}[X,Y])^\perp \). Therefore \( \mathbb{K}[X,Y]i \subseteq \mathbb{K}[X,Y]i' \subseteq (j\mathbb{K}[X,Y])^\perp \). This and \( \dim(j\mathbb{K}[X,Y])^\perp = 1 + \dim \mathbb{K}[X,Y]i \) imply that \( j\mathbb{K}[X,Y]i' = \mathbb{K}[X,Y]i' \) which means that \( i' \) is a cyclic vector in \( j\mathbb{K}[X,Y]i' \) and therefore we have a map \( \mathcal{N}_{t,n-1-t} \to \mathcal{H}_{10}^{t+1}(A^2) \).

If \( X i' = (a_1 Y + a_2 Y^2 + \cdots + a_t Y^t) i' \) then the image of \( (X, Y, i, j) \) in \( \mathcal{H}_{10}^{t+1}(A) \) is \( \langle y^{t+1}, x - a_1 y - a_2 y^2 - \cdots - a_t y^t \rangle \).

**Proof of Theorem 4.** Let \( x_{r,s} \) and \( y_{r,s} \) denote the entries in the \( r \)-th row and \( s \)-th column of the matrices that represent \( X \) and \( Y \) respectively in the basis described in Lemma 5.

Since

\[
\begin{vmatrix}
  x_{t+1,t} & y_{t+1,t} \\
  x_{t+2,t+1} & y_{t+2,t+1}
\end{vmatrix} \neq 0
\]

we can assume without loss of generality that \( x_{t+1,t}, y_{t+2,t+1} \neq 0 \). Therefore

\[
\mathbb{K}[X,Y]i = \langle i, Yi, \ldots, Y^{t-1}i \rangle, \quad j\mathbb{K}[X,Y] = \langle j, jX, \ldots, jX^{n-1-t}i \rangle.
\]
Consider the filtration $W_0 = (jK[X,Y])^\perp = \langle j, jX, \ldots, jX^{n-1} \rangle^\perp$, $W_1 = \\
\langle jX, \ldots, jX^{n-1} \rangle^\perp$, $W_{n-1-t} = (jX^{n-1-t})^\perp$.

We choose a basis $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ of $V$ so that $v_1 = Y^{t-1}i, v_i = i' \in W_0 \setminus \mathbb{K}[X,Y]i$, $v_{i+2} \in W_1 \setminus W_0, \ldots, v_n \in V \setminus W_{n-1-t}$, and we can do this in such a way that

$$x_{p,q} = \begin{cases} 0 & q - p \geq 0 \\ a_1 & q - p = 1, 1 \leq p \leq t \\ 1 & q - p = 1, t + 1 \leq p \leq n - 1 \\ 0 & q - p \neq 1, t + 1 \leq p \leq n, t + 1 \leq q \leq n \\ \end{cases}$$

$$y_{p,q} = \begin{cases} 0 & q - p \geq 0 \\ b_1 & q - p = 1, 1 \leq p \leq t \\ 1 & q - p = 1, t + 1 \leq p \leq n - 1 \\ 0 & q - p \neq 1, t + 1 \leq p \leq t + 1, 1 \leq q \leq t + 1 \\ \end{cases}$$

where $\Psi(X, Y, i, j) = (y^{q+1}, \langle x - a_1 y - \cdots - a_4 y', \langle x^{n-t}, y - b_1 x - \cdots - b_{n-1-t,x^{n-1-t}} \rangle)$ and $a_4b_1 \neq 1$.

For example, for $n = 7, t = 4$:

$$X = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & x_{16} & x_{17} \\ 0 & 0 & a_1 & a_2 & a_3 & x_{26} & x_{27} \\ 0 & 0 & 0 & a_1 & a_2 & x_{36} & x_{37} \\ 0 & 0 & 0 & 0 & a_1 & x_{46} & x_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & y_{16} & y_{17} \\ 0 & 0 & 1 & 0 & 0 & y_{26} & y_{27} \\ 0 & 0 & 0 & 1 & 0 & y_{36} & y_{37} \\ 0 & 0 & 0 & 0 & 1 & y_{46} & y_{47} \\ 0 & 0 & 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, i' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, j = \begin{pmatrix} 0 & 0 & 0 & 0 & a_1b_1 - 1 & 0 \end{pmatrix}.$$

(Note that we are changing $j$ from our previous notation, this is to simplify our expressions for $X$ and $Y$).

First we are to prove that the isotropy is one-dimensional. Let $Z = G - I$ where $G \cdot (X, Y, i, j) = (X, Y, i, j)$. $ZK[Y]i = 0$ and $jK[X]Z = 0$ imply $z_{p,q} = 0$ if $p \geq t + 2$ or $q \leq t$. Therefore $Z$ is upper triangular. We want to prove that $z_{p,q} = 0$ unless $p = 1, q = n$. We proceed by induction on $q - p$. $[Z, X] = [Z, Y] = 0$ imply

$$\sum_{r \leq m \leq s} \begin{vmatrix} x_{r,m} & z_{r,m} \\ x_{m,s} & z_{m,s} \end{vmatrix} = \sum_{r \leq m \leq s} \begin{vmatrix} y_{r,m} & z_{r,m} \\ y_{m,s} & z_{m,s} \end{vmatrix} = 0$$

for every $r \leq s$ (see [2]).

For $r = t, s = t + 1$ we have

$$0 = \begin{vmatrix} y_{t,t} & z_{t,t} \\ y_{t,t+1} & z_{t,t+1} \end{vmatrix} + \begin{vmatrix} y_{t+1,t} & z_{t+1,t} \\ y_{t+1,t+1} & z_{t+1,t+1} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = z_{t+1,t+1}.$$
This proves the case \( q - p = 0 \).

Assume that \( z_{p,q} = 0 \) if \( q - p < u \) and let \( r \) and \( s \) be so that \( s - r = u + 1 \). We are to prove that \( z_{r,s-1} = z_{r+1,s} = 0 \).

\[
0 = \sum_{r \leq m \leq s} \begin{vmatrix} x_{r,m} & z_{r,m} \\ x_{m,s} & z_{m,s} \end{vmatrix} = \begin{vmatrix} x_{r,r} & z_{r,r} \\ x_{r,s} & z_{r,s} \end{vmatrix} + \begin{vmatrix} x_{r,r+1} & z_{r,r+1} \\ x_{r+1,s} & z_{r+1,s} \end{vmatrix} + \begin{vmatrix} x_{r,s-1} & z_{r,s-1} \\ x_{s-1,s} & z_{s-1,s} \end{vmatrix} + \begin{vmatrix} x_{r,s} & z_{r,s} \\ x_{s,s} & z_{s,s} \end{vmatrix}
\]

Similarly, \( 0 = \begin{vmatrix} y_{r,r+1} & z_{r,r+1} \\ y_{s-1,s} & z_{s+1,s} \end{vmatrix} \).

\( r \leq t \) and \( s - 1 > t \) we have

\[
0 = \begin{vmatrix} a_1 & z_{r,s-1} \\ 1 & z_{r+1,s} \end{vmatrix} = \begin{vmatrix} 1 & z_{r,s-1} \\ b_1 & z_{r+1,s} \end{vmatrix} \Rightarrow z_{r,s-1} = z_{r+1,s} = 0.
\]

If \( r > t \) then \( r + 1 \geq t + 2 \), so \( z_{r+1,s} = 0 \) and

\[
0 = \begin{vmatrix} x_{r,r+1} & z_{r,s-1} \\ x_{s-1,s} & z_{r+1,s} \end{vmatrix} = \begin{vmatrix} 1 & z_{r,s-1} \\ 1 & 0 \end{vmatrix} = -z_{r,s-1}.
\]

Similar considerations apply if \( s - 1 \leq t \).

This proves that \( z_{p,q} = 0 \) if \( p - q \leq n - 2 \) (since in the computations above we require \( r \geq 1, s \leq n \)). Therefore \( z_{p,q} = 0 \) unless \( p = 1, q = n \).

It is easy to check that if \( z_{1,n} \) is arbitrary and \( z_{p,q} = 0 \) for \( (p,q) \neq (1,n) \) then \( G = I + Z \) fixes \( (X,Y,i,j) \). Therefore the isotropy is one-dimensional.

Now we want to prove that the conjugacy class is uniquely determined by the image under \( \Psi \). In order to do this we are to find a suitable basis in which \( X \) and \( Y \) are easy to describe.

**Lemma 7.** There exists a vector \( v \in V \) so that

\( (a) \ v \notin \text{Im} X + \text{Im} Y \);

\( (b) \ Z_m v \in \mathbb{K}[X,Y]i \) where \( Z_m = (Y - b_1X - \cdots - b_{n-1-t-m}X^{n-1-t-m})X^m; 0 \leq m \leq n - 2 - t \);

\( (c) \ Z_0 v = (Y - b_1X - \cdots - b_{n-1-t}X^{n-1-t}) v = 0 \);

**Proof.** We will construct \( v \) one entry at a time (using the basis from before).

\[
v = \begin{pmatrix} v_1 \\ \vdots \\ v_{t+1} \\ v_{t+2} \\ \vdots \\ v_n \end{pmatrix}.
\]

Let \( v_n = 1 \), this guarantees (a).

Now we prove that \( \text{Im} Z_m \subseteq \mathbb{K}[X,Y]i \). This means that the last \( n - t \) rows of \( Z_m \) vanish. Let \( \tilde{Y} \) and \( \tilde{X} \) be the \( (n-t) \times (n-t) \) lower right submatrices of \( Y \) and \( X \) respectively; so in fact \( \tilde{X} \) is a regular nilpotent matrix, and \( \tilde{Y} = b_1 \tilde{X} + \cdots + b_{n-1-t} \tilde{X}^{n-1-t} \). Therefore

\[
(\tilde{Y} - b_1 \tilde{X} - \cdots - b_{n-1-t-m} \tilde{X}^{n-1-t-m}) \tilde{X}^m =
\]
(b_{n-t-m}X^{n-t-m} + \cdots + b_{n-1-t}X^{n-1-t})\hat{X}^m = 0.

It is easy to see that the \((t + 1) \times (t + 1)\) upper left submatrix of \(Y - b_1X - \cdots - b_{n-1-t-m}X^{n-1-t-m}\) is a regular nilpotent matrix, in fact, the entries just above the main diagonal are all equal to \(1 - a_1b_1 \neq 0\).

Now consider the \(t\)-th row of \(Z_m\). The \((t + 1)\)-th entry of that row of \((Y - b_1X - \cdots - b_{n-1-t-m}X^{n-1-t-m})\) is \(1 - a_1b_1\), and all the preceding entries are equal to 0. After multiplying by \(X^{n-1-t-m}\), the \((t + 1 + m)\)-th entry of the \(t\)-th row of \(Z_m\) is equal to \(1 - a_1b_1\), and all the preceding entries are equal to 0. So we use \(Z_{n-2-t}, Z_{n-3-t}, \ldots, Z_1\) to choose \(v_{n-1}, v_{n-2}, \ldots, v_{t+2}\) so that the \(t\)-th entry of \(Z_mv\) is zero, i.e., \(Z_mv \in \text{Im}[X,Y]i = \text{Im}[Y]i = \langle Y, Y^2, \ldots, Y^t \rangle\).

Since the \((t + 1) \times (t + 1)\) upper left submatrix of \(Z_0\) is regular then, given \(v_{t+2}, \ldots, v_n \in \mathbb{K}\), there are unique numbers \(v_2, \ldots, v_{t+1} \in \mathbb{K}\) so that the vector \(w\) formed in this way is in \(\text{Im}\ Z_0\) (\(w_1\) is arbitrary).

\[
\begin{align*}
v &= \begin{pmatrix} * \\ \vdots \\ * \\ 1 \end{pmatrix}, & Xv &= \begin{pmatrix} * \\ \vdots \\ * \\ 0 \end{pmatrix}, & X^2v &= \begin{pmatrix} * \\ \vdots \\ 1 \\ 0 \end{pmatrix}, & \ldots, & X^{n-1-t}v &= \begin{pmatrix} * \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \\
YX^{n-1-t}v &= \begin{pmatrix} * \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix}, & YtX^{n-1-t}v &= \begin{pmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
\]

Therefore \(V = \langle v, Xv, \ldots, X^{n-1-t}v, YX^{n-1-t}v, \ldots, Y^tX^{n-1-t}v \rangle, \mathbb{K}[X,Y]i = \langle YX^{n-1-t}v, \ldots, Y^tX^{n-1-t}v \rangle, (j\mathbb{K}[X,Y])^\perp = \langle X^{n-1-t}v, YX^{n-1-t}v, \ldots, Y^tX^{n-1-t}v \rangle\).

Now we write \(X\) and \(Y\) as matrices in this basis. Since \(Y\) acts regularly in \((j\mathbb{K}[X,Y])^\perp\) and \(X\) acts as \(a_1Y + \cdots + a_{t+1}Y^{t+1}\) in this space we have that

\[
x_{p,q} = \begin{cases} a_{q-p} & 1 \leq p < q \leq t \\ 1 & q = p, t + 1 \leq p \leq n - 1 \\ 0 & \text{otherwise} \end{cases}
\]

\(v \in \text{ker}\ Z_0\) means

\(Yv = b_1Xv - \cdots - b_{n-1-t}X^{n-1-t}v\)

and for \(m = 0, \ldots, n - 2 - t\); the coefficient of \(YX^{n-1-t}v\) in \(Y(X^mv)\) is 0. This implies that

\[
y_{p,q} = \begin{cases} 1 & q - p = 1, 1 \leq p \leq t \\ b_{q-p} & t + 1 \leq p < q \leq n \\ 0 & q - p \geq 0, p = t \text{ or } q = n \end{cases}
\]
For example, for $n = 8$, $t = 4$:

$$X = \begin{pmatrix}
0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 \\
0 & 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & a_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad Y = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & y_{16} & y_{17} & 0 \\
0 & 0 & 1 & 0 & 0 & y_{26} & y_{27} & 0 \\
0 & 0 & 0 & 1 & 0 & y_{36} & y_{37} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 \\
0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

To conclude the proof of the Theorem \[4\] we use following

**Lemma 8.** The entries $y_{p,q}$ for $1 \leq p < t$, $t + 2 \leq q < n$, the vector $i$ and the covector $j$ are uniquely determined by $a_1, \ldots, a_t, b_1, \ldots, b_{n-1-t}$ and by the condition $\text{rk}([X,Y]) = 1$.

**Proof.** We split $X$ and $Y$ in blocks of sizes $t$, $1$, $n-1-t$:

$$X = \begin{pmatrix} X_{11} & X_{12} & 0 \\
0 & 0 & X_{23} \\
0 & 0 & X_{33} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\
0 & 0 & Y_{23} \\
0 & 0 & Y_{33} \end{pmatrix},$$

so

$$X_{11} = \begin{pmatrix} 0 & a_1 & \ldots & a_{t-2} & a_{t-1} \\
0 & 0 & \ldots & a_{t-3} & a_{t-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & a_1 \\
0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad X_{12} = \begin{pmatrix} a_t \\
0 \\
\vdots \\
a_2 \\
a_1 \end{pmatrix}, \quad \text{etc.}$$

Then $X_{11}Y_{13} + X_{12}Y_{23} - Y_{12}X_{23} - Y_{13}X_{33} = \hat{ij}$, a rank one matrix, where $\hat{i}$ and $\hat{j}$ are the truncated vector and covector respectively. Note that the entry in the lower left position of $ij$ is equal to $a_1b_1 - 1 \neq 0$, and that $X_{12}Y_{23}$ and $Y_{12}X_{23}$ are determined by $a_1, \ldots, a_t, b_1, \ldots, b_{n-1-t}$.

Let

$$h_1 = (1 \ 0 \ \ldots \ 0 \ 0), \quad h_2 = (0 \ 1 \ \ldots \ 0 \ 0), \ldots, \quad h_t = (0 \ 0 \ \ldots \ 0 \ 1).$$

Since $h_tX_{11} = 0$ and $h_tY_{13} = 0$ we conclude that $h_t\hat{ij} = h_t(X_{12}Y_{23} - Y_{12}X_{23}) \neq 0$. From here we get that $\hat{j}$ is uniquely determined by the parameters $a_1, \ldots, a_t, b_1, \ldots, b_{n-1-t}$ (up to a constant multiple, that we ignore).

In general, after we know $h_mY_{13}$ for $m > k$, we calculate

$$h_k\hat{ij} = h_k(X_{11}Y_{13} + X_{12}Y_{23} - Y_{12}X_{23} - Y_{13}X_{33}).$$

Since $h_kX_{11} = a_1h_{k-1} + a_2h_{k-2} + \ldots$ and $h_kY_{13}X_{33} = (0 \ y_{k,t+2} \ \ldots \ y_{k,n-1})$ we conclude that the first entry of $h_k(X_{11}Y_{13} + X_{12}Y_{23} - Y_{12}X_{23} - Y_{13}X_{33})$ does not depend on $Y_{13}$, so neither does $h_k\hat{ij}$. This means that $h_k\hat{i}$ depends only on $a_1, \ldots, a_t, b_1, \ldots, b_{n-1-t}$ (the key fact here is that $a_1b_1 - 1 \neq 0$). And with this, we can find $h_kY_{13}X_{33} = (0 \ y_{k,t+2} \ \ldots \ y_{k,n-1})$. Since the last entry of $h_kY_{13}$ is $h_kY_{13}X_{33} = (y_{k,t+2} \ \ldots \ y_{k,n-1} \ 0)$ is zero, we are not loosing any information.

□

This concludes the proof of Theorem \[4\]
From Theorem 4 and the fact the $\Psi$ is dominant we conclude that $N_{t,n-1-t}$ (and therefore $N_{t,n-1-t}$) is irreducible and its dimension is $\dim H^{n-1} + \dim H^{n-t} + \dim GL(V) - 1 = n^2 + n - 2$. $\dim N_{0,n} = \dim N'_{n,0} = n^2 + n - 1$ as was proved in [1].

3. Proof of Theorem 11

First we prove the non-redundancy of $N_{t,n-1-t}; 1 \leq t \leq n - 2$.

**Lemma 9.** $N_{t,n-1-t} \cap N_{r,s} = \emptyset$ unless $r = t$ and $s = n - 1 - t$.

**Proof.** Since $N_{r,s} \subseteq N_{t,n-1-t}$, then $N_{t,n-1-t} \cap N_{r,s} \neq \emptyset$ implies $t \leq r$, $n - 1 - t \leq s$, so $n - 1 \leq r + s$, but $N_{r,s} = \emptyset$ if $r + s \geq n$ and $r, s > 0$, therefore $n - 1 = r + s$ and we conclude that $t = r$, $n - 1 - t = s$. $\square$

To complete the proof of Theorem 11 we have to prove that

$$N = N'_{0,n} \cup N'_{n,0} \cup \bigcup_{t=1}^{n-2} N_{t,n-1-t}.$$ 

**Theorem 10.** If $0 < r + s < n - 1$ then $N_{r,s} \subseteq \bigcup_{0 < t < n - 1} N_{t,n-1-t}$.

Let $(X, Y, i, j) \in N_r$, $r = \dim \mathbb{K}[X, Y]$i. Let $\lambda$ be as in Lemma 8. The conclusion of the Lemma means that $\lambda$ represents a Young diagram of size $r$. Let $\lambda_x = \{(a, b) \in \lambda|(a + 1, b) \in \lambda\}$, $\lambda_y = \{(a, b) \in \lambda|(a, b + 1) \in \lambda\}$.

Fix a decomposition $V = \mathbb{K}[X, Y]i \oplus V'$ and write $X$ and $Y$ in blocks accordingly:

$$X = \begin{pmatrix} X_1 & X_3 \\ 0 & X_2 \end{pmatrix}, Y = \begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix}$$

Here $X_1$ and $Y_1$ (resp. $X_3$ and $Y_3$) are nilpotent commuting endomorphisms of $\mathbb{K}[X, Y]i$ (resp. $V'$) while $X_3, Y_3: V' \rightarrow \mathbb{K}[X, Y]i$. We want to describe the subvariety $N_{X_1, Y_1, i} = \{(X', Y', j') \in n \times n \times V^* | X' = \begin{pmatrix} X_1' & X_3' \\ 0 & X_2' \end{pmatrix}, Y' = \begin{pmatrix} Y_1' & Y_3' \\ 0 & Y_2' \end{pmatrix}, [X', Y'] = ij'\}$

**Lemma 11.** $N_{X_1, Y_1, i}$ is birationally equivalent to $\mathcal{K}_{n-r} \times (V'^{r+1})$ and therefore irreducible.

In concrete terms, we have the following

**Lemma 12.** Let

$$X_3' = \sum_{(a,b) \in \lambda} X^a Y^b i\alpha_{(a,b)}, Y_3' = \sum_{(a,b) \in \lambda} X^a Y^b i\beta_{(a,b)}$$

where $\alpha_{(a,b)}, \beta_{(a,b)} \in V'^{r+1}$. Then $j', \{(a, b) | (a, b) \in \lambda_x\}$ and $\{(0, b) | (0, b) \in \lambda_y\}$ are uniquely determined by $X_3', Y_3'$ and the remaining $\alpha_{(a,b)}, \beta_{(a,b)}$.

**Proof.** The condition $ij' = X_1 Y_3' + Y_3' X_1' - Y_1' X_3' - X_1' Y_3'$ means

1. $ij' = \sum_{(a,b) \in \lambda} (X^{a+1} Y^b i)(\beta_{(a,b)}) + (X^a Y^b i)(\beta_{(a,b)} X_2') - (X^a Y^b i)(\alpha_{(a,b)}) - (X^a Y^b i)(\alpha_{(a,b)} Y_2')$
Lemma 13. □

monomials in the lex deg order.

Proof. Since cyclic vector for $X$ is regular and $2$, let $W = \sum_{(a,b)\in\Lambda} (X^aY^b)(\alpha(a,b) - w(a,b))$, where $\lambda\in\Lambda$. Consider the appearances of $X^aY^b$ in equation (1). For $(a,b) = (0,0)$, we have $j' = \beta(0,0)X'_2 - \alpha(0,0)Y'_2$, and for every other $(a,b) \in \lambda$,

\begin{equation}
0 = \beta(a,b)X'_2 - \alpha(a,b)Y'_2 + \alpha(1,b) - \alpha(a,b-1) + w(a,b)
\end{equation}

where $\beta(1,b) = \alpha(a,-1) = 0$.

Therefore we can define

\[
\beta(a,b) = -\beta(a+1,b)X'_2 + \alpha(a+1,b)Y'_2 + \alpha(a+1,b-1) - w(a+1,b), \quad (a,b) \in \lambda; \\
\alpha(0,b) = \beta(0,b+1)X'_2 - \alpha(0,b+1)Y'_2 + w(0,b+1), \quad (0,b) \in \lambda.
\]

This guarantees (2) for every $(a,b) \in \lambda \setminus (0,0)$. The definition is non-recursive since every $\alpha(a,b)$, $\beta(a,b)$ is defined as a regular function of covectors associated to higher monomials in the lex deg order. □

A particular case of Lemma (2) is important in its own right.

Lemma 13. Let $X_1 \in \mathfrak{gl}_r$, $Y_2 \in \mathfrak{gl}_{n-r}$ be nilpotent and regular. Then for generic $\alpha(a,b)$, $\beta(a,b)$ we have that $\dim j'\mathbb{K}[X,Y] = n - 1 - r$.

Proof. Since $X_1$ is regular then $Y_1 = \sum c_{r-1} X_1^r$ with $c_1, \ldots, c_{r-1} \in \mathbb{K}$, $i$ is a cyclic vector for $X_1$ and $\lambda = \{(0,0), \ldots, (r-1,0)\}$. For simplicity we denote $\alpha_t = \alpha(t,0)$, $\beta_t = \beta(t,0)$ so

\[
X'_1 = \sum_{t=0}^{r-1} X'^t \alpha_t, \quad Y'_2 = \sum_{t=0}^{r-1} X'^t \beta_t.
\]

It follows that

\[
ij' = \beta_t X'_2 + \alpha_t Y'_2
\]

\[
= X_1 \left( \sum_{t=0}^{r-1} X'^t \beta_t \right) + \left( \sum_{t=0}^{r-1} X'^t \beta_t \right) X'_2 - \sum_{n=1}^{r-1} c_n X'^n - \left( \sum_{t=0}^{r-1} X'^t \alpha_t \right) - \sum_{u=1}^{r-1} c_u \alpha_t u.
\]

where we define $\beta_{-1} = 0$. Then $j'' = \beta_0 X'_2 - \alpha_0 Y'_2$ and $\beta_{-1} + \beta_t X'_2 - \alpha_t Y'_2 - \sum_{u=1}^{r-1} c_u \alpha_{t-u} = 0$ for every $t \geq 1$. This implies that $\alpha_0, \ldots, \alpha_{r-1}, \beta_{r-1} \in V'$ are arbitrary and $\beta_{r-2}, \ldots, \beta_0$ are defined recursively by $\beta_t = -\beta_{t-1} + \alpha_{t-1} + \beta_{t-1}$ in $V'$.

Now we want to calculate $\dim j'\mathbb{K}[X',Y'] = \dim j'\mathbb{K}[X'_2,Y'_2]$ in the case where $Y'_2$ is regular and $X'_2 = \sum_{n=0}^{n-r-1} d_n Y'_n$.

It turns out that $j'' = (c_1 d_1 - 1) Y'_n + \text{terms with higher powers of } Y'_n$. It follows that $\dim j'\mathbb{K}[X',Y'] = n - r - 1$ when $c_1 d_1 - 1 \neq 0$ and $\alpha_0$ is a cyclic vector for $Y'_2$. □
Proof of Theorem 10. Let \((X, Y, i, j) \in \mathcal{N}_{r,s}\). We want to prove that every open set \(U\) in \(\mathcal{N}\) containing \((X, Y, i, j)\) intersects \(\mathcal{N}_{t,n-1-t}\) for some \(t\). Since \(\mathcal{N}_X, Y, i\) is birationally equivalent to \(\mathbb{K}_{n-r} \times (V')^{r+1}\) and the pairs of nilpotent matrices where one is regular is dense in \(\mathbb{K}_{n-r}\) (see [4]), we can find \((X', Y', i', j') \in U \cap \mathcal{N}_X, Y, i\) so that \(X_2\) is regular. Now we can reverse the roles of \(i\) and \(j\) to apply Lemma 13 and find \((X'', Y'', i', j') \in U \subseteq \mathcal{N}_{t,n-1-t}\) where \(t = \dim j' \mathbb{K}[X', Y']\).

\[\square\]

Remark. The hypothesis about the characteristic of \(\mathbb{K}\) has only been used for the irreducibility of the zero fiber of the Hilbert scheme. In other characteristics it is still true that \(\mathcal{N} = \mathcal{N}_{0,n} \cup \mathcal{N}_{n,0} \cup \bigcup_{t=1}^{n-2} \mathcal{N}_{t,n-1-t}\) but the closed sets on the right-hand side may not be irreducible.

4. \(\text{rk}(A + B) \leq 1\)

Consider the variety \(S = \{(A, B, i, j) \in n \times n \times V \times V^* | A + B = ij\}\).

Lemma 14. If \((A, B, i, j) \in S\) then \(A\) and \(B\) can be simultaneously triangularized.

Proof. We proceed by induction on \(\dim V\), the first case being obvious. Let \(v \in \ker B \setminus \{0\}\). If \(jv = 0\) then \(Av = 0\) and we can apply induction on \(V/\mathbb{K}v\). Otherwise we can assume that \(jv = 1\). Then \(i = iju = Av + Bv = Av\) and therefore \(B = ij - A = A(vj - I)\). Choose \(w \in V^*\) so that \(wA = 0\). Then \(wB = 0\) and \(wi = 0\), so we can apply the induction on \(V^*/w\).

\[\square\]

Recall that \(\mathbb{K}(A, B)\) be the smallest subspace of \(V\) containing \(i\) and invariant under \(A\) and \(B\).

Lemma 15. \((A + B)|_{\mathbb{K}(A, B)} = 0\).

The proof of this Lemma is essentially the proof of Lemma 2.1.3 in [7] using our Lemma [4]

Proof. We have to prove that \((A + B)p(A, B)i = ijp(A, B)i = 0\) for any polynomial \(p\) in two noncommutative variables. But

\[jp(A, B)i = \text{tr}(p(A, B)ij) = \text{tr}(p(A, B)(A + B)) = 0\]

since \(A\) and \(B\) are upper triangular in the same basis.

\[\square\]

As a consequence we can write \(\mathbb{K}[A]i = \mathbb{K}[A, B]i = \mathbb{K}(A, B)i\) and similarly \(j\mathbb{K}[A] = j\mathbb{K}[A, B] = j\mathbb{K}(A, B)\).

Now we want to find the irreducible components of \(S\). Let

\[S_{r,s} = \{(A, B, i, j) \in S | \dim \mathbb{K}[A]i \leq r, \dim j\mathbb{K}[A] \leq s\}\]

This is clearly a closed set in the Zariski topology.

Theorem 16. The irreducible components of \(S\) are \(S_{r,n-r}, r = 0, 1, \ldots, n - 1\).
We are to prove that $S$ is used to triangularize $S$. Then $(\tau, L, i, i^\tau)$ where the 1's in $ij$ the Lemma, the matrix $S$ sets $\dim K[L]$ implies $\dim K[S] = n$. Therefore

$$S = \bigcup_{r=0}^{n} S_{r,n-r}.$$ 

Let

$$L = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}, \ 0 \leq r \leq n-1, \ j_r = (0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0)$$

where the 1's in $i_r$ and $j_r$ are located in the $r$-th and $(r+1)$-th positions, respectively. Then $(L, i_r, j_r, j_r) \in S_{r,n-r}$ if and only if $r = r'$. This proves that the closed sets $S_{r,n-r}, r = 0, 1, \ldots, n-1$ are non-redundant.

Now we have to prove that $S_{r,n-r}$ is irreducible. Let

$$S'_{r,n-r} = \{(A, B, i, j) \in S \mid \dim K[A] = r, \dim jK[A] = s\}.$$ 

We are to prove that $S'_{r,n-r}$ is irreducible and that its closure in the Zariski topology is $S_{r,n-r}$.

Let $(A, B, i, j) \in S_{r,n-1-r}$ and define $L, i_r, j_r$ as before in the same basis that is used to triangularize $A$ and $B$. For $\tau \in K$ consider $(A(\tau), B(\tau), i(\tau), j(\tau))$ where $A(\tau) = \tau A + (1 - \tau)L, \ i(\tau) = \tau i + (1 - \tau)i_r, \ j(\tau) = \tau j + (1 - \tau)j_r, B(\tau) = i(\tau)j(\tau) - A(\tau)$. Clearly the curve $\{(A(\tau), B(\tau), i(\tau), j(\tau)) \mid \tau \in K\}$ is contained in $S_{r,n-r}$ and $\dim K[A(\tau)] = n - r, \dim j(\tau)K[A(\tau)] = n - r$ except for finitely many values of $\tau$. Therefore $(A, B, i, j)$ is in the closure of $S'_{r,n-r}$.

Every $(A, B, i, j) \in S'_{r,n-r}$ can be conjugated into

$$A = \begin{pmatrix} J & A' \\ 0 & J \end{pmatrix}, \ i = i_r, \ j = j_r, \ B = ij - A$$

where the blocks in $A$ have sizes $r$ and $n - r$, respectively, and $J$ represents the Jordan block of the appropriate size. The elements of the isotropy have the form

$$G = \begin{pmatrix} I & G' \\ 0 & I \end{pmatrix}, \ G'J = JG'.$$

The space of $r \times (n - r)$ blocks $G'$ with $G'J = JG'$ is isomorphic to $K^{\min(r,n-r)}$ and therefore irreducible. We conclude that $S'_{r,n-r}$ is irreducible and its dimension is $r(n - r) + n^2 - \min\{r, n - r\}$.

\[\square\]

References

[1] Baranovsky, V. The variety of pairs of commuting nilpotent matrices is irreducible. Transform. Groups 6 (2001), no. 1, 3–8.
[2] Basili, Roberta; Iarrobino, Anthony. Pairs of commuting nilpotent matrices, and Hilbert function. arXiv:0709.2301
[3] Basili, Roberta. On the irreducibility of varieties of commuting matrices. J. Pure Appl. Algebra 149 (2000), no. 2, 107–120.

[4] Basili, Roberta. On the irreducibility of commuting varieties of nilpotent matrices. J. Algebra 268 (2003), no. 1, 58–80.

[5] Basili, Roberta. Some remarks on varieties of pairs of commuting upper triangular matrices and an interpretation of commuting varieties. [arXiv:0803.0722]

[6] Etingof, Pavel; Ginzburg, Victor. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. Invent. Math. 147 (2002), no. 2, 243–348.

[7] Gan, Wee Liang; Ginzburg, Victor. Almost-commuting variety, $D$-modules, and Cherednik algebras. With an appendix by Ginzburg. IMRP Int. Math. Res. Pap. 2006, 26439, 1–54.

[8] Nakajima, Hiraku. Lectures on Hilbert schemes of points on surfaces. University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999.

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