In this paper, we present a detailed analysis of the diagonalization of the higher spin Heisenberg model using its quantum affine symmetry $U_q(\widehat{sl}(2))$. In particular, we describe the bosonizations of the latter algebra, its highest weight representations, vertex operators and screening operators. Finally, we use this bosonization method to compute the vacuum-to-vacuum expectation values and the form factors of any local operator.
1 Introduction

Ever since the remarkable work of Bethe \cite{1} on the Heisenberg model \cite{2}, a tremendous amount of work has been accomplished towards its exact solution \cite{3}-\cite{29}. The reason is that this model exhibits strong quantum effects due to many-body interactions that can be both experimentally measured and exactly calculable. In particular, its behavior in the antiferromagnetic regime is highly nontrivial because each eigenstate of the Heisenberg Hamiltonian is built from a superposition of many spin configurations. It is even more interesting in the thermodynamic limit where its Hilbert space acquires a particle-like structure. As an example of this quantum behavior, the spin of the elementary excitations, called spinons or quasi particles, is always equal to $s = 1/2$ regardless of the value of the local spin \cite{13,16}. Moreover, spinons are always created in pairs by local operators. While the latter statement can be intuitively expected from the former, only rigorous analysis based on the Bethe ansatz and later confirmed through the crystal base theory led to the former. That spinons are always created in pairs is due to the fact that the total spin can only change by discrete integer values, which are then shared among an even number of spinons.

The Bethe ansatz method has played a major role in deriving the bulk of the existing results related to this model, namely the structure of its Hilbert space, its spectrum, S matrix, long distance correlation functions etc... To confirm and complete these results another method was found in Refs. \cite{27,26}. Its main new feature is that it makes use of the quantum group symmetry of this model from the start. This symmetry appears spontaneously in the thermodynamic limit and in the antiferromagnetic regime. The latter two cases are obviously of great interest because the other ones, namely ferromagnetic and massless regimes, are almost fully solved through the Bethe ansatz or conformal field theory approaches.

The above quantities have been extensively studied in the framework of the quantum group symmetry method for the spin-1/2 case. Its main conceptual outcome is that it allows for the identification of the dynamical fields responsible for the creation of spinons from a vacuum state. They have the form of vertex operators similar to those frequently used\footnote{This list is by no means exhaustive and many other interesting references are available in the literature.}.
in the context of string theory and conformal field theory. In fact, most of the detailed
calculation of correlation functions relies on technics already familiar in the context of the
latter theories. More precisely, they are analogous to those used for the calculation of
correlation functions on a torus. While, it has benefited considerably from string theory and
conformal field theory, this method has also paid back in the sense that a simpler expression
for the trace of vertex operators (i.e., correlation functions on a torus) has been found in
the context of the quantum group symmetry [30, 31]. Once the above identification has
been made the whole problem of evaluating form factors and correlation functions reduces
to a technical problem of calculating traces of vertex operators over irreducible highest
weight representations (IHWR’s) of the quantum affine algebra $U_q(\widehat{sl}(2))$. Unfortunately,
these traces are nontrivial because the latter IHWR’s are infinite-dimensional and the vertex
operators have complicated (generalized) commutation relations. This problem is solved
through the bosonization method where the latter elements, i.e., $U_q(\widehat{sl}(2))$, its IHWR’s,
and the vertex operators, are all realized in terms of harmonic oscillators satisfying simple
Heisenberg algebras.

Let us note here that in the spin-1/2 case there is an appealing identity that made
the use of the bosonization method straightforward. It is the fact that each Fock space
of the harmonic oscillators coincides precisely with one of the IHWR’s. Since evaluation
of traces over Fock spaces is simple (especially due the trace formula mentioned before)
the short distance form factors and correlation functions have been easily evaluated in Ref.
[27]. However, in the higher spin case, i.e., $s \geq 1$ such a coincidence does no longer hold.
While the bosonization technic is still applicable the Fock spaces are much larger than the
IHWR’s. Since both of them are infinite-dimensional spaces the projection from the former
to the latter is subtle. It requires a detailed and relatively complicated cohomology analysis
of these Fock spaces [32, 33]. This accounts for the main technical difficulty in the evaluation
of the form factors and correlation functions in this case. Let us note that the difficulty in the
spin-1 case is somewhat intermediary. Here, the IHWR’s coincide with the Fock spaces built
from a set of harmonic oscillators and another set of modes satisfying the Clifford algebra.
This has also allowed for the calculation of correlation functions [34].

Since all of the spin-1/2 results obtained through this symmetry method were completed
and compiled in Ref. [27], we will not touch upon this case here. We will instead focus on
the remaining goal of evaluating the correlation functions and form factors for \( s > 1 \). For
completeness we will review and compile most of the developments leading up to this goal.
Obviously, we expect the final results to be complicated. However, they are not necessarily
more complicated than their counterparts in string theory and conformal field theory on
genus 1 and higher surfaces. Also, they are still accessible through symbolic or numeric
manipulations. Although, a finite chain model can also be solved directly through these
manipulations, it will not yield the subtle features that arise just in the thermodynamic
limit. Furthermore, the solution through the symmetry method allows for the identification
of the contributions to form factors from each sector specified by the number of spinon pairs.
As far as we know this is not yet possible through a direct numerical calculation which only
leads to the total contribution. Also, one can always derive simple perturbative results from
the exact ones. The perturbation parameter is the anisotropy parameter \( q \) which coincides
with the deformation parameter in \( U_q(sl(2)) \). Finally, despite the fact that the bulk of the
final results is complicated one can still derive simple selection rules and identities among
correlation functions and form factors. For example, we will see that the fact that spinons
arise just in pairs is a consequence of a selection rule. Also from a fundamental point of
view this method allows us to check that Smirnov’s axioms [35], which were first intended
for integrable models of quantum field theory, are still satisfied (with a little modification)
by the form factors of the present lattice model. This fact, might lay the foundation of
yet another method to the evaluation of form factors using the modified Smirnov’s axioms.
Form factors are also known to satisfy the q-KZ equation which plays a crucial role in many
fields of physics and mathematics. Therefore, the expressions that we derive here are explicit
solutions to this equation.

This paper is organized as follows. In section one we review the quantum affine algebra
\( U_q(sl(2)) \). In section two we review its explicit bosonization, that is, its realization in terms
of bosonic fields. We also introduce its associated screening currents and their bosonization.
In section three we recall the diagonalization of the higher spin Heisenberg model through
\( U_q(sl(2)) \) and its vertex operators. The main consequence of this diagonalization is that
the correlation functions and form factors are expressed as traces of these vertex operators.
For the purpose of evaluating these traces we review in section four the bosonizations of the vertex operators and the $U_q(\widehat{sl(2)})$-IHWM’s. In section five we use this bosonization to evaluate explicitly the general N-point correlation functions and N-point form factors of local operators of this model. This is the main new result of this paper. In doing so we derive certain selection rules and an identity for form factors. We also make a consistency check using the known results of the $s = 1/2$ case. For completeness, we compile some other results concerning these quantities. In particular, we complete the proof that the form factors satisfy all (modified) Smirnov’s axioms. We also review the fact that they satisfy the q-KZ equation. Finally, we present in the appendix some useful relations, the operator product expansions and the trace formula of the operators used in this paper.
2 The quantum affine algebra $U_q(\hat{sl}(2))$ and its representation theory

Since the key idea here is to solve the higher spin Heisenberg model through its quantum affine symmetry $U_q(\hat{sl}(2))$, let us briefly review this algebra and its representations.

2.1 $U_q(\hat{sl}(2))$

This algebra is generated by the Chevalley elements $\{e_i, f_i, k_i^{\pm 1}, q^\pm d, i = 0, 1\}$, with defining relations \[[36, 37]\]

\[
\begin{align*}
k_i k_j &= k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \\
k_i e_i k_i^{-1} &= q^2 e_i, \quad k_i e_j k_i^{-1} = q^{-2} e_j, \quad i \neq j, \\
k_i f_i k_i^{-1} &= q^{-2} f_i, \quad k_i f_j k_i^{-1} = q^2 f_j, \quad i \neq j, \\
[e_i, f_j] &= \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\
(\delta e_j) - [3](\delta e_i)^2 e_je_i + [3]e_ie_j(\delta e_i)^2 - e_i(\delta e_j)^3 &= 0, \\
(\delta f_j) - [3](\delta f_i)^2 f_jf_i + [3]f_i f_j(\delta f_i)^2 - f_i(\delta f_j)^3 &= 0, \\
q^d e_i q^{-d} &= q^{\delta_{i,0}} e_i, \quad q^d f_i q^{-d} = q^{-\delta_{i,0}} f_i, \quad q^d k_i q^{-d} = k_i, 
\end{align*}
\]

(2.1)

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ is a $q$-integer and $q$ is a deformation parameter. The main property of this algebra is that it is an associative Hopf algebra with comultiplication $\Delta$, antipode $a$, and counit $\epsilon$ given by

\[
\begin{align*}
\Delta(e_i) &= e_i \otimes 1 + k_i \otimes e_i, \\
\Delta(f_i) &= f_i \otimes k_i^{-1} + 1 \otimes f_i, \\
\Delta(k_i) &= k_i \otimes k_i, \quad \Delta(q^d) = q^d \otimes q^d, \\
a(e_i) &= -k_i^{-1} e_i, \quad a(f_i) = -f_i k_i, \\
a(k_i) &= k_i^{-1}, \quad a(q^d) = q^{-d}, \\
\epsilon(e_i) &= \epsilon(f_i) = 0, \quad \epsilon(k_i) = 1, \quad i = 0, 1.
\end{align*}
\]

(2.2)

The special element $\gamma = k_0 k_1$ commutes with the whole $U_q(\hat{sl}(2))$ algebra and acts as $q^k$ on its highest weight representations. Here, $k$ is referred to as the level of the representation.
The Chevalley generators are only associated with the simple roots \( \alpha_0 \) and \( \alpha_1 \) of \( U_q(\hat{sl}(2)) \). As in the classical case, it is important to find all \( U_q(\hat{sl}(2)) \) generators associated with the infinite-dimensional set of roots \( \{ \pm \alpha + n\delta; n \in \mathbb{Z} \} \cup \{ n\delta; n \in \mathbb{Z}\setminus\{0\} \} \), with \( \alpha = \alpha_1 \) and \( \delta = \alpha_0 + \alpha_1 \). The Serre relations will become redundant in the commutation relations of such generators. These generators, found by Drinfeld, \[38\] are \( \{ E_{n}^{\pm}, \alpha_m, K^{\pm 1}, q^{\pm d}, \gamma^{\pm 1/2}; n \in \mathbb{Z}, m \in \mathbb{Z}^* = \mathbb{Z}\setminus\{0\} \} \) with defining relations

\[
\gamma^{1/2} \gamma^{-1/2} = \gamma^{-1/2} \gamma^{1/2} = 1, \quad [\gamma^{\pm 1/2}, y] = 0, \quad \forall y \in U_q(\hat{sl}(2)),
\]

\[
KK^{-1} = K^{-1} K = 1, \quad K \alpha_n K^{-1} = \alpha_n, \quad KE_n^{\pm} K^{-1} = q^{\pm 2} E_n^{\pm},
\]

\[
q^n q^d = q^{-d} q^d = 1, \quad K q^{\pm d} K^{-1} = q^{\pm d}, \quad q^d E_n^{\pm} q^{-d} = q^n E_n^{\pm}, \quad q^d \alpha_n q^{-d} = q^n \alpha_n,
\]

\[
[a_n, \alpha_m] = \frac{[2n][\gamma^n - \gamma^{-n}]}{n(q - q^{-1})} \delta_{n+m,0},
\]

\[
[a_n, E_m^{\pm}] = \frac{\gamma^{|n|} [2n]}{n} E_{n+m}^{\pm},
\]

\[
[E_n^{+}, E_m^{-}] = \frac{\gamma^{(n-m)/2} \Psi_{n+m} - \gamma^{(m-n)/2} \Phi_{n+m}}{q - q^{-1}},
\]

\[
E_{n+1}^{\pm} E_m^{\pm} - q^{\pm 2} E_m^{\pm} E_{n+1}^{\pm} = q^{\pm 2} E_n^{\pm} E_{m+1}^{\pm} - E_{m+1}^{\pm} E_n^{\pm}, \quad (2.3)
\]

where \( \Psi_n \) and \( \Phi_n \) are given by the mode expansions of the currents \( \Psi(z) \) and \( \Phi(z) \), which are themselves defined by

\[
\Psi(z) = \sum_{n \geq 0} \Psi_n z^{-n} = K \exp\{(q - q^{-1}) \sum_{n > 0} \alpha_n z^{-n}\},
\]

\[
\Phi(z) = \sum_{n \leq 0} \Phi_n z^{-n} = K^{-1} \exp\{-(q - q^{-1}) \sum_{n < 0} \alpha_n z^{-n}\}. \quad (2.4)
\]

Here \( z \) is a formal variable and

\[
K = \Psi_0 = \Phi_0^{-1} = q^{\alpha_0}. \quad (2.5)
\]

The algebra isomorphism \( \rho \) between the Chevalley generators and the Drinfeld generators is given explicitly by

\[
\rho : \quad k_0 \to \gamma K^{-1}, \quad k_1 \to K, \quad e_1 \to E_0^{+}, \quad f_1 \to E_0^{-}, \quad e_0 \to E_1^{-} K^{-1}, \quad f_0 \to K E_{-1}^{+}. \quad (2.6)
\]
Using this isomorphism one can re-express the comultiplication (2.2) in terms of the Drinfeld generators as

\[
\Delta(E^+_n) = E^+_n \otimes \gamma^{kn} + \gamma^{2kn} K \otimes E^+_n + \sum_{i=0}^{n-1} \gamma^{k(n+3i)/2} \Psi_{n-i} \otimes \gamma^{k(n-i)} E^+_i \mod N_- \otimes N^2_+,
\]

\[
\Delta(E^-_m) = E^-_m \otimes \gamma^{-km} + K^{-1} \otimes E^-_m + \sum_{i=0}^{m-1} \gamma^{k(i-m)/2} \Phi_{m+i} \otimes \gamma^{k(i-m)} E^-_{i} \mod N_- \otimes N^2_+,
\]

\[
\Delta(E^-_n) = E^-_n \otimes \gamma^{-2kn} K^{-1} + \gamma^{-kn} \otimes E^-_m + \sum_{i=0}^{n-1} \gamma^{-k(n-i)} E^+_i \otimes \gamma^{-k(n+3i)/2} \Phi_{i-n} \mod N^2_- \otimes N_+,
\]

\[
\Delta(E^-_m) = \gamma^{km} \otimes E^-_m + E^-_m \otimes K + \sum_{i=1}^{m-1} \gamma^{k(m-i)} E^-_m \otimes \gamma^{-k(m-i)/2} \Psi_{m-i} \mod N^2_- \otimes N_+,
\]

\[
\Delta(\alpha_m) = \alpha_m \otimes \gamma^{km/2} + \gamma^{3km/2} \otimes \alpha_m \mod N_- \otimes N_+,
\]

\[
\Delta(\alpha_m) = \alpha_m \otimes \gamma^{-3km/2} + \gamma^{-km/2} \otimes \alpha_m \mod N_- \otimes N_+,
\]

\[
\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},
\]

\[
\Delta(\gamma^{\pm 1}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2},
\]

\[
\Delta(q^{\pm d}) = q^{\pm d} \otimes q^{\pm d},
\]

(2.7)

where \(m > 0, n \geq 0\), and \(N_- \) and \(N^2_+\) are left \(Q(q)[\gamma^\pm, \Psi_m, \Phi_{-n}; m, n \in \mathbb{Z}_{\geq 0}]\) representations generated by \(\{E^\pm_m; m \in \mathbb{Z}\}\) and \(\{E^\pm_mE^\pm_n; m, n \in \mathbb{Z}\}\) respectively [33, 40].

Let us now use the formal variable approach to re-express \(U_q(\widehat{sl}(2))\) as a quantum current algebra with elements \(\{\Psi(z), \Phi(z), E^\pm(z), \gamma^{\pm 1/2}, q^{\pm d}\}\), where

\[
E^\pm(z) = \sum_{n \in \mathbb{Z}} E^\pm_n z^{-n},
\]

(2.8)

and with defining relations

\[
\gamma^{1/2} \gamma^{-1/2} = \gamma^{-1/2} \gamma^{1/2} = 1, \quad [\gamma^{1/2}, y] = 0, \quad \forall y \in U_q(\widehat{sl}(2)),
\]

(2.9)

\[
[\Psi(z), \Psi(w)] = 0,
\]

(2.10)

\[
[\Phi(z), \Phi(w)] = 0,
\]

(2.11)

\[
\Psi(z) \Phi(w) = g(wz^{-1} \gamma) g(wz^{-1} \gamma^{-1})^{-1} \Phi(w) \Psi(z),
\]

(2.12)

\[
\Psi(z) E^\epsilon(w) = g(wz^{-1} \gamma^{-\epsilon/2})^{-\epsilon} E^\epsilon(w) \Psi(z),
\]

(2.13)

\[
\Phi(z) E^\epsilon(w) = g(zw^{-1} \gamma^{-\epsilon/2})^{\epsilon} E^\epsilon(w) \Phi(z),
\]

(2.14)

\[
[E^\epsilon(z), E^-\epsilon(w)] = \frac{\delta(z w^{-1} \gamma^{-\epsilon}) \Psi(w \gamma^{\epsilon/2}) - \delta(z w^{-1} \gamma^{\epsilon}) \Phi(z \gamma^{-\epsilon/2})}{q - q^{-1}},
\]

(2.15)
Here $\epsilon = \pm 1$ and $g(z)$ is meant to be the following formal power series in $z$:

$$g(z) = \sum_{n \in \mathbb{N}} c_n z^n,$$

where the coefficients $c_n, n \in \mathbb{N}$ are determined from the Taylor expansion of the function

$$f(\xi) = \frac{q^2 \xi - 1}{\xi - q^2} = \sum_{n \in \mathbb{Z}_+} c_n \xi^n$$

at $\xi = 0$. In the above relations we have also introduced the $\delta$-function $\delta(z)$ which is defined as the formal Laurent series

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n,$$

and which plays a key role in the formal calculus approach. Its main properties are summarized by the following relations:

$$\delta(z) = \delta(z^{-1}),$$
$$\delta(z) = \frac{1}{z} + \frac{z^{-1}}{1 - z^{-1}},$$
$$G(z, w) \delta(azw^{-1}) = G(z, az) \delta(azw^{-1}) = G(a^{-1} w, w) \delta(azw^{-1}), \quad a \in \mathbb{C}^*, \quad (2.23)$$

where $G(z, w)$ is any operator whose formal Laurent expansion in $z$ and $w$ is given by

$$G(z, w) = \sum_{n, m \in \mathbb{Z}} G_{n, m} z^n w^m. \quad (2.24)$$

Note that it is crucial that both $\delta(z)$ and $G(z, w)$ have expansions in integral powers of $z$ and $w$, otherwise the above properties of the $\delta$-function will not hold.

The three relations (2.17), (2.18) and (2.19) translate the fact that $E_n^\pm, \Psi_n$ and $\Phi_n$ are homogeneous of the same degree $n$.

Unfortunately, the Hopf algebra structure of the current algebra (2.9)-(2.19), which would follow from (2.7), is not available yet in a closed form. However, Drinfeld has succeeded in
deriving a closed formula for the comultiplication, although it is not clear yet how it is related to the one defined in \((2.2)\). It is given by \([\text{2.2}]\):

\[
\Delta(\Psi(z)) = \Psi(zq^{b_1}) \otimes \Psi(zq^{b_2}),
\]

\[
\Delta(\Phi(z)) = \Phi(zq^{a_1}) \otimes \Phi(zq^{a_2}),
\]

\[
\Delta(E^+(z)) = E^+(zq^{d_0}) \otimes 1 + \Psi(zq^{d_1}) \otimes E^+(zq^{d_2}),
\]

\[
\Delta(E^-(z)) = 1 \otimes E^-(zq^{e_0}) + E^-(zq^{e_1}) \otimes \Phi(zq^{e_1}),
\]

(2.25)

where \(a_i, b_i, c_i, d_i\) satisfy the following constraints:

\[
b_1 = a_1 + 1 \otimes c, \quad e_0 = a_2 - c/2 \otimes 1,
\]

\[
d_0 = a_1 + 1 \otimes c/2, \quad d_2 = a_2 - 3c/2 \otimes 1,
\]

\[
d_1 = a_1 + 1 \otimes c/2 - c/2 \otimes 1, \quad e_2 = a_2 - c/2 \otimes 1 - 1 \otimes c/2,
\]

\[
e_1 = a_1 - 1 \otimes c/2, \quad b_2 = a_2 - c \otimes 1.
\]

(2.26)

\[2.2 \quad \text{Representation theory of } U_q(\widehat{sl}(2))\]

Let us briefly recall the definitions of some \(U_q(\widehat{sl}(2))\) representations [31, 27]. For this, we still need some notions from \(\widehat{sl}(2)\) affine algebra, which is generated by \(\{e_i, f_i, h_i, d; i = 0, 1\}\).

We define on its Cartan subalgebra \(\mathcal{h} = \mathbb{C}h_0 + \mathbb{C}h_1 + \mathbb{C}d\) an invariant symmetric bilinear form \((\quad, \quad)\) by

\[
(h_i, h_i) = 2, \quad (h_i, h_{1-i}) = -2, \quad (h_i, d) = \delta_{i,0}, \quad (d, d) = 0, \quad i = 0, 1.
\]

(2.27)

Let \(\hat{\mathcal{h}}^* = \mathbb{C}\Lambda_0 + \mathbb{C}\Lambda_1 + \mathbb{C}\delta = \mathbb{C}\alpha_0 + \mathbb{C}\alpha_1 + \mathbb{C}\Lambda_0\) be the dual space to \(h\) with

\[
<\Lambda_i, h_j> = \delta_{i,j}, \quad <\delta, d >= 1, \quad <\Lambda_i, d >= 0, \quad <\delta, h_i> = 0,
\]

where

\[
<\quad, \quad>: \hat{\mathcal{h}}^* \otimes \mathcal{h} \rightarrow \mathbb{C}.
\]

(2.29)

is the natural pairing, \(\Lambda_i\) are the fundamental weights, \(\alpha_i\) are the positive roots and \(\delta = \alpha_0 + \alpha_1\) is the null root. One can induce a symmetric bilinear form \((\quad, \quad)\) on \(\hat{\mathcal{h}}^*\) by

\[
(\Lambda_i, \Lambda_j) = \frac{1}{2}\delta_{i,1}\delta_{j,1}, \quad (\Lambda_i, \delta) = 1, \quad (\delta, \delta) = 0, \quad (\alpha_i, \alpha_i) = 2,
\]

\[
(\alpha_i, \alpha_{1-i}) = 2, \quad (\alpha_i, \Lambda_0) = \delta_{i,0}, \quad (\Lambda_0, \Lambda_0) = 0, \quad i, j = 0, 1.
\]

(2.30)
The weights $\lambda \in \hat{h}^*$ such that
\[
\lambda = n_0 \Lambda_0 + n_1 \Lambda_1, \quad n_0, n_1 \in \mathbb{N} \setminus \{0\},
\] (2.31)
are called regular dominant integral weights, and $n_0 + n_1 = k$ is the level that we have introduced previously.

Let us now turn to the representation theory of $U_q(\widehat{sl}(2))$ which is generated by \{\(e_i, f_i, K^\pm, \gamma^\pm, q^\pm, d\); \(i = 0, 1\)\}. Let $V$ be a $U_q(\widehat{sl}(2))$ representation and $\mu \in \hat{h}$, the subspace $V_\mu \subset V$ defined by
\[
V_\mu = \{v \in V/K^\pm v = q^{\pm<\mu, h_i>}v, \quad \gamma^\pm v = q^{\pm k}v, \quad q^\pm d v = q^{\pm<\mu, d>}v\},
\] (2.32)
is called a $\mu$-weight space, and any $v \in V_\mu$ is referred to as a $\mu$-weight vector. The representation $V$ becomes a weight representation if it is the direct sum of its weight spaces. A $U_q(\widehat{sl}(2))$-highest weight vector $v_\lambda$ in $V$ is a $\lambda$-weight vector which satisfies the additional condition
\[
e_i v_\lambda = 0, \quad i = 0, 1.
\] (2.33)
The space $V$ is called a $U_q(\widehat{sl}(2))$-highest weight representation if it is generated from a $\lambda$-highest weight vector $v_\lambda$. In this case, $v_\lambda$ is unique (up to a multiplication by a scalar), and hence $V$ is labelled by the weight $\lambda$ as $V(\lambda)$. This $V(\lambda)$ is called standard if it is generated from a highest weight vector $v_\lambda$ with a dominant integral weight $\lambda$ and such that
\[
f_i^{<\lambda, h_i>} + 1 v_\lambda = 0, \quad i = 0, 1.
\] (2.34)
In this case it is also irreducible. Let us now review the explicit constructions of the irreducible highest weight representations (IHWR) in the case $k = 1$, where they are also called basic representations.

### 2.3 Explicit construction of the $U_q(\widehat{sl}(2))$ basic representations

Let $U_q(\hat{h})$ be the enveloping Heisenberg algebra embedded in $U_q(\widehat{sl}(2))$ and generated by \{\(\alpha_n, \gamma^{\pm 1/2}, q^{\pm d}; \ n \in \mathbb{Z}^*\)\} with relations \[ [\alpha_n, \gamma^{\pm 1/2}] = 0, \quad \gamma^{1/2} \gamma^{-1/2} = \gamma^{-1/2} \gamma^{1/2} = 1, \]
(2.35)
\[
[\alpha_n, \alpha_m] = \delta_{n+m,0} \frac{[2n]}{n}.
\] (2.36)
Let $U_q(\hat{h}^-)$ denote the Abelian subalgebra of $U_q(\hat{h})$ generated by \{\(\alpha_n; n < 0\}\). Let $F$ be a Fock space constructed as $F = U_q(\hat{h}^-)|0>$, where $|0>$ is a vacuum state with respect to the Heisenberg algebra, i.e.,

\[
\alpha_n|0> = 0, \quad n \geq 0.
\] (2.37)

Now we define the action of $U_q(\hat{h}^-)$ on $F$ as

\[
\gamma^{\pm 1} : \quad x|0> \mapsto q^{\pm 1}x|0>,
\]

\[
\alpha_n : \quad x|0> \mapsto \alpha_n x|0>, \quad n < 0,
\]

\[
\alpha_n : \quad x|0> \mapsto [\alpha_n, x]|0>, \quad n > 0,
\]

\[
q^{\pm d} : \quad x|0> \mapsto q^{\pm d}xq^{\mp d}|0> = q^{\pm d}x|0>, \quad \forall x \in U_q(\hat{h}^-).
\] (2.38)

Then it is clear that $F$ is an irreducible representation of $U_q(\hat{h})$. However, since $F$ is annihilated by $\alpha_0$ it cannot be a $U_q(\hat{sl}(2))$ IHWR, otherwise it would also be annihilated by $E^{\pm}(z)$, according to $q^{\alpha_0}E^{\pm}(z)q^{-\alpha_0} = q^{\pm 2}E^{\pm}(z)$. This is obviously not the case. Therefore, we must extend $F$ by some orthogonal space $W$ in order to make the resulting space $G = F \otimes W$ an IHWR. In order to find $W$ and the action of $U_q(\hat{sl}(2))$ on $G$ let us introduce two vertex operators:

\[
S_{\epsilon}(z) = \exp\{\pm \epsilon \sum_{n>0} \frac{\alpha_{\epsilon n}}{n} q^{-\epsilon n/2} z^{\mp n}\}, \quad \epsilon = \pm,
\] (2.39)

which are viewed as formal Laurent series in $z$ with coefficients acting on $F$. Using (2.36) we find the following commutation relations:

\[
S_{\epsilon}^{\pm}(z)S_{\epsilon'}^{\mp}(w) = (1 - q^{-1-(\epsilon+\epsilon')/2}wz^{-1})^{\epsilon'}(1 - q^{1-(\epsilon+\epsilon')/2}wz^{-1})^{\epsilon}\ S_{\epsilon}^{\pm}(w)S_{\epsilon'}^{\mp}(z),
\]

\[
S_{\epsilon}^{\pm}(z)S_{\epsilon'}^{\pm}(w) = S_{\epsilon'}^{\pm}(w)S_{\epsilon}^{\pm}(z),
\]

\[
[\alpha_n, S_{\epsilon}^{\pm}(z)] = 0,
\]

\[
[\alpha_n, S_{\epsilon}^{\pm}(z)] = -\frac{\epsilon q^{-\epsilon n/2}z^{-n}[2n]}{n} S_{\epsilon}^{\pm}(z),
\]

\[
[\alpha_n, S_{\epsilon}^{\pm}(z)] = 0,
\]

\[
[\alpha_n, S_{\epsilon}^{\pm}(z)] = -\frac{\epsilon q^{-\epsilon n/2}z^{n}[2n]}{n} S_{\epsilon}^{\pm}(z),
\] (2.40)

where $n > 0$ and $\epsilon = \pm$. To appreciate the usefulness of $S_{\epsilon}^{\pm}(z)$ in the construction of $W$, let us introduce the currents

\[
Z^{\epsilon}(z) = S_{\epsilon}^{\pm}(z)E^{\epsilon}(z)S_{\epsilon}^{\pm}(z).
\] (2.41)
Then, using
\[
[\alpha_n, E^\epsilon(z)] = \frac{\epsilon q^{-|\epsilon|/2} z^n [2n]}{n} E^\epsilon(z), \quad \epsilon = \pm; \quad n \in \mathbb{Z}\setminus\{0\},
\] (2.42)
which is equivalent to (2.13) and (2.14), we find
\[
[\alpha_n, Z^\epsilon(z)] = 0, \quad n \in \mathbb{Z}\setminus\{0\},
\]
\[
[\gamma^{\pm 1/2}, Z^\epsilon(z)] = 0, \quad \epsilon = \pm.
\] (2.43)

This means that the actions of $Z^\pm(z)$ on $G$ are orthogonal to that of $U_q(\hat{h}^-)$ and therefore $Z^\pm(z)$ act nontrivially only on $W$. To be more specific the actions of $U(\hat{h})$ and $Z^\pm(z)$ on $G = F \otimes W$ are given by
\[
x : \ u \otimes v \rightarrow xu \otimes v, \quad x \in U_q(\hat{h}),
\]
\[
Z^\pm(z) : \ u \otimes v \rightarrow u \otimes Z^\pm(z)v, \quad u \in F, \quad v \in W.
\] (2.44)

Equations (2.13) and (2.14) imply
\[
\Psi_0 E^\pm(w) = q^{\pm 2} E^\pm(w) \Psi_0,
\]
\[
\Psi_0 Z^\pm(w) = q^{\pm 2} Z^\pm(w) \Psi_0.
\] (2.45)

This means that $\Psi_0$ and $\Phi_0$ also act nontrivially only on $W$, i.e.,
\[
\Psi_0 : \ u \otimes v \rightarrow u \otimes \Psi_0 v,
\]
\[
\Phi_0 : \ u \otimes v \rightarrow u \otimes \Phi_0 v, \quad u \in F, \quad v \in W,
\] (2.46)

The currents $\Psi(z)$ and $\Phi(z)$ can also be written as
\[
\Psi(z) = S^+_\epsilon(zq^{-3\epsilon/2}) S^-_\epsilon(zq^{3\epsilon/2}) \otimes \Psi_0,
\]
\[
\Phi(z) = S^+_\epsilon(zq^{3\epsilon/2}) S^-_\epsilon(zq^{-3\epsilon/2}) \otimes \Phi_0, \quad \epsilon = \pm.
\] (2.47)

For completeness let us also define the actions of $q^{\pm d}$ and $\gamma^{\pm 1}$ on $F \otimes W$ as
\[
q^{\pm d} : \ u \otimes v \rightarrow q^{\pm d} u \otimes q^{\pm d} v,
\]
\[
\gamma^{\pm 1} : \ u \otimes v \rightarrow q^{\pm 1} u \otimes v.
\] (2.48)

The homogeneous gradation of the $U_q(\widehat{\mathfrak{s}l(2)})$ currents is also inherited by $Z^\epsilon(z)$, that is,
\[
q^d Z^\epsilon(z) q^{-d} = Z^\epsilon(zq^{-1}).
\] (2.49)
From the above analysis it is clear that the action of the currents $E^\varepsilon(z)$ on $F \otimes W$ decomposes as

$$E^\varepsilon(z) = S^-_\varepsilon(zq^{-\varepsilon})S^+_\varepsilon(zq^\varepsilon) \otimes Z^\varepsilon(z), \quad \varepsilon = \pm,$$

(2.50)

where we have used

$$(S^\pm_\varepsilon(z))^{-1} = S^\pm_{-\varepsilon}(zq^{\pm\varepsilon}), \quad \varepsilon = \pm,$$

(2.51)

and (2.40). Substituting $E^\varepsilon(z)$ by the latter expression in (2.9)-(2.19), we find the following algebra for $Z^\varepsilon(z)$:

$$Z^\varepsilon(z)Z^{-\varepsilon}(w) - Z^{-\varepsilon}(w)Z^\varepsilon(z) = \frac{\varepsilon}{q - q^{-1}} \left( \Psi_0 \delta(zw^{-1}q^{-\varepsilon}) - \Phi_0 \delta(zw^{-1}q^\varepsilon) \right)$$

$$= \frac{1}{q - q^{-1}} \left( q^{\varepsilon\alpha(0)} \delta(zw^{-1}q^{-1}) - q^{-\varepsilon\alpha(0)} \delta(zw^{-1}q) \right),$$

$$w^2Z^\varepsilon(z)Z^{-\varepsilon}(w) = z^2Z^{-\varepsilon}(w)Z^\varepsilon(z).$$

(2.52)

These relations and (2.49) can in fact be solved explicitly for $Z^\varepsilon(z)$. One finds

$$Z^\varepsilon(z) = e^{\varepsilon\alpha}z^{\varepsilon\alpha + \frac{1}{2}(\alpha,\alpha)} = e^{\varepsilon\alpha}z^{\varepsilon\alpha + 1}, \quad \varepsilon = \pm,$$

(2.53)

where $\alpha$ is the $sl(2)$ positive simple root and $e^{\alpha} \in \mathbb{C}[P], P = Q \cup (Q + \alpha/2)$ and $Q$ are the $sl(2)$ weight and root lattices, whereas $\mathbb{C}[P]$ and $\mathbb{C}[Q]$ are the corresponding Abelian group algebras respectively. The elements $z^{\varepsilon\alpha}$ and $q^d$ act on $\mathbb{C}[P]$ as

$$z^{\varepsilon\alpha}e^\beta = z^{(\alpha,\beta)}e^\beta z^{\varepsilon\alpha},$$

$$q^de^\beta = e^\beta q^{-\varepsilon\alpha - (\beta,\beta)/2}, \quad \beta \in P.$$

(2.54)

Finally, let us now summarize this construction. The currents $E^\varepsilon(z)$ act on $F \otimes \mathbb{C}[P]$ as

$$E^\varepsilon(z) = S^-_\varepsilon(zq^{-\varepsilon})S^+_\varepsilon(zq^\varepsilon) \otimes e^{\varepsilon\alpha}z^{\varepsilon\alpha + 1}.$$

(2.55)

The subspaces $F \otimes \mathbb{C}[Q]$ and $F \otimes e^{\pm\frac{\alpha}{2}}\mathbb{C}[Q]$ whose direct sum is $F \otimes \mathbb{C}[P]$ are $U_q(sl(2))$ invariant and irreducible. They are in fact isomorphic to the basic representations $V(\Lambda_0)$ and $V(\Lambda_1)$ with highest weight states $1 \otimes 1$ and $1 \otimes e^{\pm\frac{\alpha}{2}}$, respectively. This construction was first derived in Ref. [11]. Unfortunately, in both cases the currents $Z^\pm(z)$ satisfy complicated generalized commutation relations which, contrary to the case $k = 1$, cannot be solved in terms of just
a group algebra \[ [\mathbb{H}] \]. It turns out that one needs two more sets of Heisenberg generators in addition to the set \( \{\alpha_n\} \). This solution seems to be necessary for the purpose of evaluating physical quantities which are based on explicit knowledge of the scalar product on \( U_q(\widehat{sl}(2)) \) representations.

Although explicit bases have been constructed for these representations in terms of the modes of \( Z^*(z) \), the scalar product on them has so far remained inaccessible. In the next section, we review the realization of \( U_q(\widehat{sl}(2)) \) in terms of three bosonic fields, whose modes satisfy simple Heisenberg algebras.
3 Bosonization of higher level \( U_q(sl(2)) \) algebra

Here, we extend the bosonic realization of \( U_q(sl(2)) \) given in \( (2.55) \) for \( k = 1 \) to arbitrary level \( k \). For this purpose, it is most convenient to re-express \( (2.3) \) as a quantum current algebra generated by the set of currents \( \{ E^\pm(z), \Psi(z), \Phi(z) \} \). This current algebra is given in terms of the following operator product expansions (OPE’s):

\[
\Psi(z) \Phi(w) = \frac{(z - wq^{2+k})(z - wq^{-2-k})}{(z - wq^{2-k})(z - wq^{-2+k})} \Phi(w) \Psi(z),
\]

\[
\Psi(z) E^\pm(w) = q^{\pm 2} \frac{(z - wq^{\mp(2+k/2)})}{z - wq^{\pm(2-k/2)}} E^\pm(w) \Psi(z),
\]

\[
\Phi(z) E^\pm(w) = q^{\pm 2} \frac{(z - wq^{\mp(2-k/2)})}{z - wq^{\pm(2+k/2)}} E^\pm(w) \Phi(z),
\]

\[
E^+(z) E^-(w) \sim \frac{1}{w(q - q^{-1})} \left\{ \Psi(wq^{k/2}) - \Phi(wq^{-k/2}) \right\}, \quad |z| > |wq^{\pm k}|,
\]

\[
E^-(z) E^+(w) \sim \frac{1}{w(q - q^{-1})} \left\{ \Psi(wq^{-k/2}) - \Phi(wq^{k/2}) \right\}, \quad |z| > |wq^{\pm k}|,
\]

\[
E^\pm(z) E^\pm(w) = \frac{(q^2z^2 - w^2)}{z - wq^{\pm 2}} E^\pm(w) E^\pm(z).
\]

Here, the symbol \( \sim \) means that the regular terms as \( z \) approaches \( wq^{\pm k} \) are being omitted. In the above OPE’s the identification \( \sum_{n \geq 0} z^n = (1 - z)^{-1} \) with \( |z| < 1 \) is used.

The construction for general \( k \) will be based on that for \( k = 1 \) \[14\]. The natural generalization of \( (2.55) \) that is compatible with the OPE’s \( (3.56) \) is given by

\[
\Psi(z) = : V^+(zq^{k/2}) V^-(zq^{-k/2}) : = q^{\sqrt{2k} \varphi_0} \exp \left( \sqrt{2k}(q - q^{-1}) \sum_{n>0} \varphi_n z^{-n} \right),
\]

\[
\Phi(z) = : V^+(zq^{-k/2}) V^-(zq^{k/2}) : = q^{-\sqrt{2k} \varphi_0} \exp \left( -\sqrt{2k}(q - q^{-1}) \sum_{n<0} \varphi_n z^{-n} \right),
\]

\[
E^\pm(z) = \sqrt{|k|} \psi^\pm(z) V^\pm(z),
\]

where

\[
V^\pm(z) = : e^{\pm i \varphi^\pm(z)} :,
\]

\[
\varphi^\pm(z) = \varphi - i \varphi_0 \ln z + ik \sum_{n \neq 0} \frac{q^{\mp n|k/2}}{|nk|} \varphi_n z^{-n}.
\]
Here $\psi^\pm(z)$ are deformed parafermionic fields, $\varphi^\pm(z)$ are deformed bosonic fields. Their modes $\{\varphi, \varphi_n; n \in \mathbb{Z}\}$ obey the following Heisenberg algebra:

$$
[\varphi_n, \varphi_m] = \frac{[2n][nk]}{2kn} \delta_{n+m,0},
$$

$$
[\varphi, \varphi_0] = i. \tag{3.59}
$$

Moreover, the bosonic normal ordering symbol :: indicates that in the product of fields between the two colons, the creation modes $\{\varphi_n, \varphi; n < 0\}$ should be moved to the left of the annihilation modes $\{\varphi_n; n \geq 0\}$. Henceforth, we will use the standard convention that operators defined at the same point are understood to be normal ordered. Substituting $\Psi(z), \Phi(z)$ and $E^\pm(z)$ in terms of $V^\pm(z)$ and $\psi^\pm(z)$ in (3.56) we find the following boson-parafermion algebra [14]:

$$
V^\pm(z)V^\mp(w) = z^{-\frac{2}{3}} \left( \frac{q^{k+2}wz^{-1}; q^{2k}}{(q^{k-2}wz^{-1}; q^{2k})_\infty} \right) : V^\pm(z)V^\mp(w) :,
$$

$$
V^\pm(z)V^\pm(w) = z^\frac{2}{3} \left( \frac{q^{k+k-2}wz^{-1}; q^{2k}}{(q^{k+k+2}wz^{-1}; q^{2k})_\infty} \right) : V^\pm(z)V^\pm(w) :,
$$

$$
\psi^\pm(z)\psi^\mp(w) = z^\frac{2}{3} \left( \frac{q^{k-2}wz^{-1}; q^{2k}}{(q^{k+2}wz^{-1}; q^{2k})_\infty} \right) \frac{1}{(z-wq^k)(z-wq^{-k})} \text{ regular as } z \rightarrow wq^\pm k,
$$

$$
\psi^\pm(z)\psi^\pm(w) = z^\frac{2}{3} \left( \frac{q^{k+k-2}wz^{-1}; q^{2k}}{(q^{k+k+2}wz^{-1}; q^{2k})_\infty} \right) \psi^\pm(w)\psi^\pm(z),
$$

$$
V^\pm(z)\psi^\pm(w) = \psi^\pm(w)V^\pm(z) = \text{ regular terms as } z \rightarrow w,
$$

$$
V^\pm(z)\psi^\pm(w) = \psi^\pm(w)V^\pm(z) = \text{ regular terms as } z \rightarrow w, \tag{3.60}
$$

where

$$
(x; y) = \prod_{i=0}^{\infty} (1 - xy^i). \tag{3.61}
$$

When $k = 1$ the latter algebra simplifies to

$$
V^\pm(z)V^\mp(w) = \frac{1}{(z-wq)(z-wq^{-1})} : V^\pm(z)V^\mp(w) :,
$$

$$
V^\pm(z)V^\pm(w) = (z-wq^{1+1})(z-wq^{-1+1}) : V^\pm(z)V^\pm(w) :,
$$

$$
\psi^\pm(z)\psi^\mp(w) = 1 + \text{ regular terms as } z \rightarrow w,
$$

$$
\psi^\pm(z)\psi^\pm(w) = \psi^\pm(w)\psi^\pm(z),
$$

$$
V^\pm(z)\psi^\pm(w) = \psi^\pm(w)V^\pm(z) = \text{ regular terms as } z \rightarrow w,
$$

$$
V^\pm(z)\psi^\pm(w) = \psi^\pm(w)V^\pm(z) = \text{ regular terms as } z \rightarrow w. \tag{3.62}
$$
From these relations we see that the parafermionic fields $\psi^\pm(z)$ can be identified with the identity operator. Therefore, we reproduce the construction of Ref. [41] in the present notations as

$$
\Psi(z) = V^+(z q^{1/2}) V^- (z q^{-1/2}) : ,
$$
$$
\Phi(z) = V^+(z q^{-1/2}) V^- (z q^{1/2}) : ,
$$
$$
E^\pm(z) = V^\pm(z). \tag{3.63}
$$

To be more precise, we have $\alpha_n = \sqrt{2}\varphi_n$.

However, for $k > 1$, $\psi^\pm(z)$ cannot be identified with the identity operator since they satisfy nontrivial OPE’s. Unfortunately, these OPE’s are generalized commutation relations. While one can still use these relations to construct bases of the $U_q(\hat{sl}(2))$ IHWR’s, one cannot easily use them for the purpose of evaluating scalar products of these IHWR’s. As we will see later however, it is precisely the scalar products defined on the IHWR’s that would lead to the physical quantities, such as correlation functions and form factors. For this reason we need to re-express the parafermionic fields $\psi^\pm(z)$ in terms of two bosonic fields since we can always easily define scalar products on Heisenberg representations.

### 3.1 Bosonization of higher level $U_q(\hat{sl}(2))$

We refer to the bosonic realization of $U_q(\hat{sl}(2))$ for arbitrary $k$, as the Wakimoto construction. For this we need two extra Heisenberg algebras generated by $\{\varphi^2, \varphi_n^2\}$ and $\{\varphi^{(3)}, \varphi_n^{(3)}\}$ that bosonize $\psi^\pm(z)$. In the classical case, the Wakimoto bosonization of $\psi^\pm(z)$ has the form

$$
\psi^\pm(z) = i(a\partial \varphi^{(2)}(z) + b\partial \varphi^{(3)}(z))e^{ic\varphi^{(2)}(z) + id\varphi^{(3)}(z)}, \tag{3.64}
$$

for some real parameters $a, b, c, d$ and free bosonic fields $\varphi^{(2)}(z)$ and $\varphi^{(3)}(z)$. Therefore, quantum derivatives are naturally expected to replace the classical ones. Moreover, this bosonization must be consistent with the classical limit, i.e., $q \to 1$. Finally, the OPE $\psi^+(z)\psi^-(w)$ is fixed to be that given in (3.60). Keeping track of the latter requirements, we
define the general form of the bosonization of $\psi^\pm(z)$ as:

$$[k]\psi^\pm(z) = \frac{\exp\{\pm i \sqrt{2/k} \varphi^{1,\pm}(z)\}}{z(q-q^{-1})} \left( \exp\{\pm i \sqrt{2/k} X_A^{(\pm)}(z)\} - \exp\{\pm i \sqrt{2/k} X_B^{(\pm)}(z)\} \right), \quad (3.65)$$

where $\varphi^{1,\pm}(z) \equiv \varphi^\pm(z)$ and

$$X_A^{(\pm)}(z) = \varphi^{(2)} - i\varphi_0^{(2)} \ln q A_2^{(\pm)} - i\varphi_0^{(3)} \ln q A_3^{(\pm)} + i \sum_{n \neq 0} \{ A_2^{(\pm)}(n)\varphi_n^{(2)} + A_3^{(\pm)}(n)\varphi_n^{(3)} \} \frac{z^n}{n}. \quad (3.66)$$

The operators $X_B^{(\pm)}(z)$ is given by a similar expression to (3.66) with $A$ being replaced by $B$. The bosonic modes $\{\varphi^{(2)}, \varphi_n^{(2)}\}$ and $\{\varphi^{(3)}, \varphi_n^{(3)}\}$ satisfy the following Heisenberg algebras:

$$[\varphi_n^{(j)}, \varphi_m^{(\ell)}] = (-1)^{j-1} n I_j(n) \delta^{j,\ell} \delta_{n+m,0}, \quad \text{and} \quad [\varphi_n^{(j)}, \varphi_0^{(\ell)}] = (-1)^{j-1} i \delta^{j,\ell}, \quad j, \ell = 2, 3. \quad (3.67)$$

Here no sum with respect to $j$ is meant.

First, consistency with (3.56) requires the following relations to be satisfied:

$$\exp\{i \sqrt{2/k} X_B^{(+)}(z)\} \exp\{-i \sqrt{2/k} X_A^{(-)}(w)\} = \frac{z-wq^{k+2}}{q(z-wq^{k})} \exp\left\{ -\frac{2}{k} \langle \varphi^{1,+}(z)\varphi^{1,-}(w) \rangle \right\} \times \exp\{i \sqrt{2/k} (X_B^+(z) - X_B^-(w))\} :; \quad (3.68)$$

$$\exp\{i \sqrt{2/k} X_A^{(+)}(z)\} \exp\{-i \sqrt{2/k} X_B^{(-)}(w)\} = \frac{q(z-wq^{-k-2})}{z-wq^{-k}} \exp\left\{ -\frac{2}{k} \langle \varphi^{1,+}(z)\varphi^{1,-}(w) \rangle \right\} \times \exp\{i \sqrt{2/k} (X_A^+(z) - X_A^-(w))\} :; \quad (3.70)$$

$$\exp\{i \sqrt{2/k} X_A^{(+)}(z)\} \exp\{-i \sqrt{2/k} X_A^{(-)}(w)\} = q \exp\left\{ -\frac{2}{k} \langle \varphi^{1,+}(z)\varphi^{1,-}(w) \rangle \right\} \times \exp\{i \sqrt{2/k} (X_A^+(z) - X_A^-(w))\} :; \quad (3.72)$$

$$\exp\{i \sqrt{2/k} X_B^{(+)}(z)\} \exp\{-i \sqrt{2/k} X_B^{(-)}(w)\} = q^{-1} \exp\left\{ -\frac{2}{k} \langle \varphi^{1,+}(z)\varphi^{1,-}(w) \rangle \right\} \times \exp\{i \sqrt{2/k} (X_B^+(z) - X_B^-(w))\} :; \quad (3.73)$$

Any solution to the above constraints for the free parameters $A_2^{(\pm)}(n), A_2^{(\pm)}, A_3^{(\pm)}(n), A_3^{(\pm)}, B_2^{(\pm)}(n), B_2^{(\pm)}, B_3^{(\pm)}(n), B_3^{(\pm)}, I_2(n)$ and $I_3(n)$ leads to a deformation of the Wakimoto
construction. From these relations we find

\[ A^\pm_2(n) = q^{-nk}B^\mp_2(n), \]
\[ A^\pm_3(n) = q^{-nk}B^\mp_3(n), \]
\[ A^\pm_2 = -B^\pm_2 = k/2, \]
\[ A^\pm_3 = B^\mp_3 = \pm \sqrt{\frac{k(k + 2)}{2}}, \]

(3.74)

and

\[ A^\pm_2(n)A^\pm_2(-n)I_2(n) - A^\pm_3(n)A^\pm_3(-n)I_3(n) = \frac{k}{2} \left( \frac{[n(k + 2)]}{[nk]} - 1 \right), \quad n > 0, \]
\[ A^\pm_2(n)A^\pm_2(-n)I_2(n) - A^\pm_3(n)A^\pm_3(-n)I_3(n) = \frac{k}{2} \left( \frac{[2n]}{[nk]} \right), \quad n > 0. \]

(3.75)

Therefore, the only free parameters to be determined are \( A^\pm_2(n), A^\pm_3(n), I_2(n), \) and \( I_3(n). \) These are restricted to satisfy the set of general “master” equations (3.75). Each solution to these equations yields a particular Wakimoto construction. In this manner, all these constructions were re-derived in Ref. [45]. For example, the first one is obtained by fixing the above parameters as [46]:

\[ A^\pm_2(n) = q^{-nk/2}, \]
\[ A^\pm_3(n) = \pm \frac{1}{2} \sqrt{\frac{k + 2}{k}} (q^{-nk} - 1), \]
\[ I_2(n) = \frac{k}{4} \left( \frac{[n(k + 2)]}{[nk]} + [2n] - [nk] \right), \]
\[ I_3(n) = \frac{k^2}{k + 2} \left( \frac{[n][n(k + 2)]}{[nk][nk/2]} \right). \]

(3.76)

The second Wakimoto bosonization corresponds to the following values for the parameters [47]:

\[ A^{(e)}_2(n) = \frac{nk}{[nk]}q^{-nk/2}, \]
\[ A^{(-e)}_2(n) = \frac{nkq^{-nk/2}}{[2n]} \left( \frac{[n(k + 2)]}{[nk]} - 1 \right), \]
\[ A^{(e)}_3(n) = 0, \]
\[ A^{(-e)}_3(n) = \sqrt{k(k + 2)nq^{-nk/2}(q^{-1}) \left( \frac{[n]}{[2n]} \right)}. \]
\[ I_2(n) = \frac{[2n][nk]}{2nk^2}, \]
\[ I_3(n) = \frac{[2n][n(k + 2)]}{2n^2(k + 2)}, \]  
(3.77)

where \( \epsilon \) is equal to + or − if \( n > 0 \) or \( n < 0 \), respectively. The third bosonization corresponds to the following choices of parameters [45]:

\[ A_2^{(\pm)}(n) = \sqrt{\frac{k + 2}{2}} \frac{nk}{[nk]} q^{n(1-2)k/2}, \quad n > 0, \]
\[ A_2^{(\pm)}(n) = \sqrt{\frac{k + 2}{2}} \frac{nk}{[nk]} q^{\pm nk/2}, \quad n < 0, \]
\[ A_3^{(+)}(n) = -\sqrt{2k} \frac{n}{[2n]} q^{n(f-k/2)}, \]
\[ A_3^{(-)}(n) = -\sqrt{2k} \frac{n}{[2n]} q^{n(f-2-3k/2)}, \]
\[ I_2(n) = \frac{[nk][n(k + 2)]}{n^2k(k + 2)} q^{kn}, \]
\[ I_3(n) = \frac{[2n]^2}{4n^2}. \]  
(3.78)

Here \( f \) is a free parameter.

In Ref. [45], it is shown that all these bosonizations can be obtained from one another through linear combinations of their modes. Let us now show that another linear combination of these modes leads to the bosonization of Ref. [48]. We will be using the latter in the rest of this paper for simplification reasons. Indeed, from the linear combinations

\[ a_{X_1,n} = \sqrt{2kq^{-2n-|n|}} \frac{[n(k + 2)]}{[nk]} \varphi_n^{(1)} + \sqrt{k(2 + k)q^{-2n-(k+1)|n|}} \frac{[2n]}{[nk]} \varphi_n^{(2)}, \]
\[ a_{X_2,n} = -\sqrt{2kq^{-n(2+k)-(k+2)|n|}} \frac{[2n]}{[nk]} \varphi_n^{(1)} - \sqrt{k(2 + k)q^{-n(2+k)-(k+2)|n|}} \frac{[2n]}{[nk]} \varphi_n^{(2)}, \]
\[ a_{X_3,n} = 2\varphi_n^{(3)}, \]
\[ \sqrt{\frac{k}{2}} a_{X_1,0} = (2 + k)\varphi_0^{(1)} + \sqrt{2(2 + k)} \varphi_0^{(2)}, \]
\[ \sqrt{\frac{k}{2}} a_{X_2,0} = -2\varphi_0^{(1)} - \sqrt{2(2 + k)} \varphi_0^{(2)}, \]
\[ \sqrt{\frac{k}{2}} a_{X_3,0} = \sqrt{2k} \varphi_0^{(3)}, \]
\[ \sqrt{\frac{k}{2}} Q_{X_1} = i(2 + k)\varphi_0^{(1)} + i\sqrt{2(2 + k)} \varphi_0^{(2)}, \]
we reproduce the bosonization of Ref. [48] in terms of three fields \( X_1, X_2, X_3 \) as:

\[
E^+ (z) = \frac{1}{z(q - q^{-1})} \sum_{\epsilon = \pm 1} \epsilon E_{\epsilon}^+ (z),
\]

\[
E^- (z) = \frac{1}{z(q - q^{-1})} \sum_{\epsilon = \pm 1} \epsilon E_{\epsilon}^- (z),
\]

where

\[
E_{\epsilon}^+(z) = \exp \{ \partial X_1^{(\epsilon)} (q^{-2} z; -k + \frac{2}{2}) + X_2 (2 q^{(\epsilon-1) (k+2)} z; -1) + X_3 (2 q^{(\epsilon-1) (k+1)-1} z; 0) \},
\]

\[
E_{\epsilon}^-(z) = \exp \{ -X_2 (2 q^{-k-2} z; 1) - X_3 (2 q^{-k-2+\epsilon} z; 0) \},
\]

and

\[
\partial X_1^{(\epsilon)} (q^{-2} z; -k + \frac{2}{2}) = \epsilon \{ (q - q^{-1}) \sum_{n=1}^{\infty} a_{X_1,\epsilon,n} z^{-n} q^{(2\epsilon-k+2)\frac{n}{2}} + a_{X_1,0} \log(q) \}. \tag{3.82}
\]

Here a deformed bosonic field is denoted by

\[
X(L; M, N|z, \alpha) = -\sum_{n>0} [Ln][Mn][Nn] a_{X,n} z^{-n} q^{n|\alpha} + \frac{L}{MN} a_{X,0} \ln(z) + \frac{L}{MN} Q_x,
\]

\[
X(N|z, \alpha) = X(L; L, N|z, \alpha) = -\sum_{n>0} \frac{a_{X,n} z^{-n} q^{n|\alpha}}{[Nn]} + \frac{a_{X,0} \ln(z)}{N} + \frac{Q_x}{N}. \tag{3.83}
\]

It has the following two-point matrix elements:

\[
< X(L; M, N|z, \alpha) X(L'; M', N'|w, \alpha') > = \sum_{n>0} \frac{[Ln][L'n][a_{X,n}, a_{X,-n}]}{[Mn][M'n][Nn][N'n]} z^{-n} w^n q^{n(\alpha+\alpha')} + \frac{LL'[a_{X,0}, Q_x]}{MM'NN'} \ln(z),
\]

\[
< X(N|z, \alpha) X(N'|w, \alpha') > = \sum_{n>0} \frac{[a_{X,n}, a_{X,-n}]}{[Nn][N'n]} z^{-n} w^n q^{n(\alpha+\alpha')} + \frac{[a_{X,0}, Q_x]}{NN'} \ln(z), \tag{3.84}
\]

with

\[
[a_{X_1,n}, a_{X_1,m}] = \frac{[2n][(k+2)n]}{n} \delta_{n+m,0},
\]

\[
\sqrt{\frac{k}{2}} Q_{X_2} = -2i \varphi^{(1)} - i \sqrt{2(2+k)} \varphi^{(2)},
\]

\[
\sqrt{\frac{k}{2}} Q_{X_3} = i \sqrt{2k} \varphi^{(3)},
\]

(3.79)
\[ [a_{X_2,n}, a_{X_2,m}] = -\frac{[2n]^2}{n} \delta_{n+m,0}, \]
\[ [a_{X_3,n}, a_{X_3,m}] = \frac{[2n]^2}{n} \delta_{n+m,0}, \]
\[ [a_{X_1,0}, Q_{X_1}] = 2(k + 2), \]
\[ [a_{X_2,0}, Q_{X_2}] = -4, \]
\[ [a_{X_3,0}, Q_{X_3}] = 4. \]

(3.85)

### 3.2 Screening currents and their bosonization

A screening current \( S(z) \) has OPE’s with the \( U_q(\widehat{sl}(2)) \) currents that are either regular or \( q \)-total derivatives, i.e.,

\[
x(z)S(w) = \pm S(w)x(z) \sim \text{regular terms as } z \to w,
\]
\[
x(z)S(w) = \pm S(w)x(z) \sim \pm \alpha D_w \left( \frac{Z(w)}{z - w} \right), \quad x(z) = \Psi(z), \Phi(z), E^\pm(z),
\]

(3.86)

where \( Z(w) \) is some normal ordered operator and the \( q \)-derivative is defined by

\[
\alpha D_z(f(z)) = \frac{f(zq^\alpha) - f(zq^{-\alpha})}{z(q - q^{-1})}.
\]

(3.87)

The solution to the above relations (3.86) that is widely used in the literature is given by

\[
S(z) = -\frac{1}{z(q - q^{-1})} \sum_{\delta = \pm 1} \delta S_\delta(z),
\]

(3.88)

where

\[
S_\delta(z) = \exp\{-X_1(k + 2|q^{-2}z; -\frac{k + 2}{2}) - X_2(2|q^{-k-2}z; -1) - X_3(2|q^{-k-2+\delta}z; 0)\}.
\]

(3.89)

Let us note that in Ref. [19] a general theory for constructing screening currents of "pure exponential type" was presented and it allows for the derivation of two more screening currents. This construction is quite complicated for \( U_q(\widehat{sl}(n)) , n \geq 3 \) but in the case of \( U_q(\widehat{sl}(2)) \) it can simply be summarised as follows: A screening current can be automatically derived from each step current, i.e., \( E^+(z) \) and \( E^-(z) \). All it takes is to write \( E^\pm(z) \) in the form

\[
E^\pm(z) = e^{X^\pm(z)} D_z e^{Y^\pm(z)},
\]

(3.90)
then up to coefficients $\epsilon_+, \epsilon_- = \pm 1$ one can read off automatically two screening currents $S^\pm(z)$ as
\begin{equation}
S^\pm(z) = e^{\epsilon_+ Y^\pm(z)}. \tag{3.91}
\end{equation}

The coefficients $\epsilon_\pm$ are determined from the OPE’s $E^\pm(z)S^\pm(w)$ which are required to be $q$-total derivatives. However, there is some redundancy, in that, we need only one of them, say $S^+(z)$ which we denote by $\eta(z)$ (to follow the usual notations in the literature). From the simple bosonization of $E^+(z)$, that of $\eta(z)$ is then found as
\begin{equation}
\eta(z) = e^{X_4(2\vert q^{-k-2z};0)}. \tag{3.92}
\end{equation}
4 Diagonalization of the higher spin Heisenberg model

4.1 Vertex operators

The objects of main interest that bridge the mathematical aspects of the quantum affine algebra $U_q(sl(2))$ and the physical aspects of the Heisenberg model are called vertex operators. More specifically, the matrix elements of local operators of this model (i.e., form factors) are expressed as traces of products of vertex operators. Since the latter operators act as intertwiners of $U_q(sl(2))$ representations let us briefly recall certain features of these representations that are relevant for our purposes.

As described in section 2.2 the $U_q(sl(2))$ highest weight representations are denoted by $V(\lambda)$ where $\{\lambda = \lambda_m = (k-m)\Lambda_0 + m\Lambda_1, m = 0, \ldots, k\}$ and $\{\Lambda_0, \Lambda_1\}$ are the sets of $U_q(sl(2))$ dominant highest weights and fundamental weights, respectively. A $U_q(sl(2))$ evaluation representation $V^{(\ell)}(z)$ ($0 \leq \ell \leq k$) is the affinitization of the spin $\ell/2$ representation. It is isomorphic to $V^{(\ell)} \otimes \mathbb{C}[z, z^{-1}]$, where $V^{(\ell)}$ is the $U_q(sl(2))$ $(\ell + 1)$-dimensional representation, with a basis $\{v^{(\ell)}_m, 0 \leq m \leq \ell\}$ such that:

$$
e_1 v^{(\ell)}_m = [m] v^{(\ell)}_{m-1}, \quad f_1 v^{(\ell)}_m = [\ell - m] v^{(\ell)}_{m+1}, \quad t_1 v^{(\ell)}_m = q^{\ell-2m} v^{(\ell)}_m, \quad e_0 v^{(\ell)}_m = [\ell - m] v^{(\ell)}_{m+1}, \quad f_0 v^{(\ell)}_m = [m] v^{(\ell)}_{m-1}, \quad t_0 v^{(\ell)}_m = q^{2m-\ell} v^{(\ell)}_m, \quad (4.93)$$

where it is understood that $v^{(\ell)}_m = 0$ if $m > \ell$ or $m < 0$. $V^{(\ell)}(z)$ is equipped with the following $U_q(sl(2))$ module structure $[20]$

$$
e_1 v^{(\ell)}_m \otimes z^n = [m] v^{(\ell)}_{m-1} \otimes z^n, \quad e_0 v^{(\ell)}_m \otimes z^n = [\ell - m] v^{(\ell)}_{m+1} \otimes z^{n+1}, \quad f_1 v^{(\ell)}_m \otimes z^n = [\ell - m] v^{(\ell)}_{m+1} \otimes z^n, \quad f_0 v^{(\ell)}_m \otimes z^n = [m] v^{(\ell)}_{m-1} \otimes z^{n-1}, \quad t_1 v^{(\ell)}_m \otimes z^n = q^{\ell-2m} v^{(\ell)}_m \otimes z^n, \quad t_0 v^{(\ell)}_m \otimes z^n = q^{2m-\ell} v^{(\ell)}_m \otimes z^n. \quad (4.94)$$

In terms of the Drinfeld realization this becomes

$$\gamma^{\pm 1/2} v^{(\ell)}_m \otimes z^s = v^{(\ell)}_m \otimes z^s,$$

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Let \( V^{(\ell)}(z) \) be two evaluation representations dual to \( V^{(\ell)}(z) \) through the action of the antipode \( a \) and its inverse \( a^{-1} \), respectively. Then we have the following isomorphisms \([26]\):

\[
C_{\pm}^{(\ell)} : V^{(\ell)}(zq^{\pm 2}) \xrightarrow{\sim} V^{(\ell)a^\pm 1}(z),
\]

where

\[
C_{\pm}^{(\ell)}(v_m^{(\ell)} \otimes (zq^{-2})^n) = C_{\pm m}^{(\ell)}v_{m-\ell}^{(\ell)*} \otimes z^n,
\]

\[
C_{\pm m}^{(\ell)} = (-1)^m q^{m^2-m(\ell+1)} \begin{pmatrix} \ell \\ m \end{pmatrix}^{-1}, \quad 0 \leq m \leq \ell.
\]

Here \( \{v_m^{(\ell)*}, 0 \leq m \leq \ell\} \) is the basis of \( V^{(\ell)*} \) dual to \( V^{(\ell)} \), and the notation \( \begin{pmatrix} x \\ y \end{pmatrix} \) defines the \( q \)-analogue of the binomial coefficient as

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x! \\ y! (x-y)! \end{pmatrix},
\]

\[
[x]! = [x][x-1] \ldots [1].
\]

Following the terminology of Refs. \([25, 26]\) there are two types of vertex operators referred to as type I vertex operators \( \Phi_{\lambda}^{\mu,v^{(\ell)}}(z) \) and type II vertex operators \( \Psi_{\lambda}^{V^{(\ell)},\mu}(z) \) respectively. They are defined as maps between \( U_q(sl(2)) \) representations in the following way:

\[
\Phi_{\lambda}^{\mu,v^{(\ell)}}(z) : V(\lambda) \to V(\mu) \otimes V^{(\ell)}(z),
\]

\[
\Psi_{\lambda}^{V^{(\ell)},\mu}(z) : V(\lambda) \to V^{(\ell)}(z) \otimes V(\mu).
\]

Their main feature is that they satisfy the following intertwining properties:

\[
\Phi_{\lambda}^{\mu,v^{(\ell)}}(z) \circ x = \Delta(x) \circ \Phi_{\lambda}^{\mu,v^{(\ell)}}(z),
\]

\[
\Psi_{\lambda}^{V^{(\ell)},\mu}(z) \circ x = \Delta(x) \circ \Psi_{\lambda}^{V^{(\ell)},\mu}(z), \quad \forall x \in U_q(sl(2)).
\]

This simply means that they commute with \( U_q(sl(2)) \) when they act as above. Let us also note that all vertex operators conserve the homogeneous gradation of \( U_q(sl(2)) \) highest.
weight modules, i.e.,
\[(d \otimes \text{id})\tilde{\Phi}^\mu_{\lambda}V(\ell)(z) - \tilde{\Phi}^\mu_{\lambda}V(\ell)(z)d = -z\frac{d}{dz}\tilde{\Phi}^\mu_{\lambda}V(\ell)(z),\]
\[(d \otimes \text{id})\tilde{\Psi}^{\nu}_{\lambda}V(\ell)(z) - \tilde{\Psi}^{\nu}_{\lambda}V(\ell)(z)d = -z\frac{d}{dz}\tilde{\Psi}^{\nu}_{\lambda}V(\ell)(z).\] (4.101)

It is also convenient to define components of the above vertex operators
\[\tilde{\Phi}^\mu_{\lambda}V(\ell)(z) = \sum_{j=0}^{\ell} \tilde{\Phi}^\mu_{\lambda,j}(z) \otimes \psi^{(\ell)}_j,\]
\[\tilde{\Psi}^{\nu}_{\lambda}V(\ell)(z) = \sum_{j=0}^{\ell} \psi^{(\ell)}_j \otimes \tilde{\Psi}^{\nu}_{\lambda,j}(z).\] (4.102)

From a physical point of view only type I vertex operators with \(\ell = k\) are relevant since they realize the action of local operators on the eigenspace of the spin \(s = k/2\) Heisenberg model. Moreover, only type II vertex operators with \(\ell = 1\) are important for physical purposes because the elementary excitations which are created by these operators are always spin-1/2 spinons, regardless of the value \(s = k/2\) of the local spin variables. In these physical situations, the vertex operators are normalized as follows:
\[\tilde{\Phi}^\sigma(\lambda),V(k)(z)|\lambda\rangle = |\sigma(\lambda) > \otimes \psi_m^{(k)} + \cdots, \quad \lambda = m\Lambda_0 + n\Lambda_1,\]
\[\tilde{\Psi}^{(1)}V(1),\lambda_\pm(z)|\lambda\rangle = v_\mp \otimes |\lambda_\pm > + \cdots, \quad \lambda_\pm = (m \mp 1)\Lambda_0 + (n \pm 1)\Lambda_1.\] (4.103)

Moreover, since the latter operators will be interpreted as creation operators of local spin states and annihilation operators of eigenstates, respectively, one needs their conjugate vertex operators which play opposite roles. They are obtained through the isomorphisms \(\Phi^\dagger\) and the substitution of \(V(z)\) by of \(V^{\ast a_{\pm 1}}(z)\) in \((4.99)\), that is,
\[\tilde{\Phi}^\sigma(\lambda),V(k)\ast a_{\pm 1}(z) = \alpha_{\lambda,\pm}^{(k)}(id \otimes C^{(k)}_{\pm})\tilde{\Phi}^\sigma(\lambda),V(k)(zq^{\mp 2}),\]
\[\tilde{\Psi}^{(1)\ast a_{\pm 1},\lambda}(z) = \alpha_{\lambda,\pm}^{(1)}(C^{(1)}_{\pm} \otimes \text{id})\tilde{\Psi}^{(1),\lambda}(zq^{\mp 2}), \quad \epsilon = \pm,\] (4.104)
where the normalization constants are fixed by \([26]\)
\[\tilde{\Phi}^\sigma(\lambda),V(k)\ast a_{\pm}(z)|\lambda\rangle = |\sigma(\lambda) > \otimes \psi_m^{(k)\ast} + \cdots,\]
\[\tilde{\Psi}^{(1)\ast a_{\pm},\lambda}(z)|\lambda\rangle = v_\epsilon^\ast \otimes |\lambda_\epsilon > + \cdots.\] (4.105)
Here for notational convenience \( v_0^* = v_+^* \) and \( v_1^* = v_-^* \). One then finds
\[
\alpha_{\lambda, \pm}^{(\lambda)} = \frac{1}{C_{\pm, k-n}^{(k)}},
\alpha_{\lambda, \pm}^{\pm} = \frac{1}{C_{\pm, \pm}^{(1)}}.
\] (4.106)

For the purpose of translating the action of a local operator on the Hilbert space \( \mathcal{F}^{(m)} = \text{End}(V(\lambda_m)) \), inverse vertex operators have also been introduced in Ref. [26]. They intertwine \( U_q(\widehat{sl}(2)) \) representations in the following order:
\[
\tilde{\Phi}_{\sigma(\lambda_m), V(k)}^\lambda(z) : V(\sigma(\lambda_m)) \otimes V^{(k)}(z) \rightarrow V(\lambda_m),
\] (4.107)
where \( \lambda_m = (k-m)\Lambda_0 + m\Lambda_1 \) and \( \sigma(\lambda_m) = m\Lambda_0 + (k-m)\Lambda_1 \). They have \( k + 1 \) components \( \tilde{\Phi}_{\sigma(\lambda_m), V(k), j}^\lambda(z) \) defined by
\[
\tilde{\Phi}_{\sigma(\lambda_m), V(k), j}^\lambda(z)|u> = \tilde{\Phi}_{\sigma(\lambda_m), V(k)}^\lambda(z)(|u> \otimes v_j) = (\text{id}_{V(\lambda_m)} \otimes <v_j, .>)\tilde{\Phi}_{\sigma(\lambda_m), V(k)}^\lambda(z)|u>,
\] (4.108)
for any \(|u> \in V(\sigma(\lambda_m))\). Using the above normalizations one then finds
\[
\tilde{\Phi}_{\sigma(\lambda_m), V(k), j}^\lambda(z) = \frac{C_{\pm, k-j}^{(k)} \tilde{\Phi}_{\sigma(\lambda_m), k-j}(zq^{-2})}{C_{\pm, m}^{(k)}}, \quad 0 \leq m \leq k.
\] (4.109)
Furthermore, from the above definitions it has been shown in Ref. [26] that
\[
\tilde{\Phi}_{\sigma(\lambda_m), V(k)}^\lambda(z) \circ \tilde{\Phi}_{\sigma(\lambda_m), V(k)}^{\lambda_m}(z) = g_{\lambda_m} \text{id}_{V(\lambda_m)};
\tilde{\Phi}_{\sigma(\lambda_m), V(k)}^{\lambda}(z) \circ \tilde{\Phi}_{\sigma(\lambda_m), V(k)}^{\lambda_m}(z) = g_{\lambda_m} \text{id}_{V(\lambda_{k-m})},
\] (4.110)
where \( g_{\lambda_m} \) are scalar functions given by
\[
g_{\lambda_m} = q^{(k-m)m} \left[ k \right] \frac{(q^{2(k+1)}; q^4)_\infty}{(q^2; q^4)_\infty}.\] (4.111)

### 4.2 Commutation relations

In this section, we briefly summarize the commutation relations of the vertex operators following Ref. [26]. There, they have been derived from the q-KZ equation. For this purpose we need to introduce an \( R \) matrix whose entries are the Boltzmann weights of the classical
two dimensional lattice model, and which is equivalent to the higher spin Heisenberg model. The mathematical interpretation of this matrix is that it intertwines the action of \( U_q(sl(2)) \) evaluation representations, that is,

\[
R(z_1, z_2) \Delta(x) = \Delta'(x) R(z_1, z_2), \quad R(z_1, z_2) \in \text{End}(V^{(m)}_{z_1} \otimes V^{(n)}_{z_2}), \quad \forall x \in U_q(sl(2)),
\]

where \( \Delta \) is the comultiplication of \( U_q(sl(2)) \), \( \Delta' = \Delta P \), with \( P \) being the transposition operator, i.e., \( P(u \otimes v) = v \otimes u \). This \( R \) matrix, denoted by \( \tilde{R}_{mn}(z_1/z_2) \), is uniquely defined through the above intertwining relations and the following normalization:

\[
\tilde{R}_{mn}(z_1/z_2)(v^{(m)}_0 \otimes v^{(n)}_0) = v^{(m)}_0 \otimes v^{(n)}_0.
\]

In the case of most physical interest, i.e., \( m = n = k \), one finds explicitly \[31\]

\[
\tilde{R}_{kk}(z)P = \sum_{l=1}^{k} \prod_{r=1}^{l} \left( \frac{1 - zq^{2k-2r+2}}{z - q^{2k-2r+2}} \right) P_l,
\]

where \( P_l \) is the projector defined as:

\[
V^{(k)} \otimes V^{(k)} = V^{(2k)} \oplus V^{(2k-2)} \oplus \cdots \oplus V^{(2)} \oplus V^{(0)},
\]

\[
P_l : \quad V^{(k)} \otimes V^{(k)} \rightarrow V^{(2k-2l)}.
\]

In Ref. [20] other \( R \) matrices, which are different from \( \tilde{R} \) only by an overall scale factor, are also introduced as

\[
R_{kk}(z) = r_{kk} \tilde{R}_{kk}(z), \quad R_{VV}(z) = -R_{11}(z), \quad R_{VV^*}(z) = (\text{id} \otimes C_-)R_{VV}(zq^{-2})(\text{id} \otimes C_-)^{-1}, \quad R_{V^*V^*}(z) = (C_- \otimes C_-)R_{VV}(z)(C_- \otimes C_-)^{-1}, \quad r_{kk}(z) = z^{-k/2}(q^{2}; q^4)^{\infty}(q^{2k+2}; q^4)^{\infty}(q^{2k+2}; q^4)^{\infty}.
\]

From the solution to the q-KZ equation for the two-point vacuum to vacuum matrix elements of vertex operators, the following key commutation relations have been found [20]:

\[
\Phi^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_2)\Phi^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_1) = R_{kk}(z_1/z_2)\Phi^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_1)\Phi^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_2), \quad \tilde{\Phi}^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_2)\tilde{\Phi}^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_1) = r(z_1/z_2)\tilde{\Phi}^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_1)\tilde{\Phi}^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_2),
\]

\[
\Psi^{V^{(k)}_{\sigma(\lambda)}}, z_1 \Psi^{V^{(k)}_{\sigma(\lambda)}}, z_2 = \tau(z_1/z_2)\tilde{\Psi}^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_2)\tilde{\Psi}^{\lambda, V^{(k)}_{\sigma(\lambda)}}(z_1),
\]

28
where $\mu = \lambda_\pm$. The last commutation relation still holds if $\tilde{\Psi}_{\lambda}(z)$ and $R_{V\lambda}(z)$ are replaced by $\tilde{\Psi}_{\lambda}(z)$ and $R_{\lambda}(z)$, respectively. The $W$ factors are the face type Boltzmann weights and are given in the appendix, and

$$\tau(z) = z^{-1/2} \frac{\theta_{q^4}(qz)}{\theta_{q^4}(qz^{-1})},$$

$$\theta_{p}(z) = (z;p)_{\infty}(pz^{-1};p)_{\infty}(p;p)_{\infty}. \quad (4.118)$$

Moreover, it is worth noting that in the limit $|z_1/z_2| \to 1$, the products $\tilde{\Psi}_{\lambda}(z)\tilde{\Psi}_{\lambda}(z)$ and $\tilde{\Psi}_{\lambda}(z)\tilde{\Psi}_{\lambda}(z)$ are holomorphic, whereas $\tilde{\Psi}_{\lambda}(z)\tilde{\Psi}_{\lambda}(z)$ have simple poles when $\lambda_\pm = \nu$, and whose residues are given by

$$\text{Res}_{z_1 = z_2} \tilde{\Psi}_{\lambda}(z)\tilde{\Psi}_{\lambda}(z) = (v_+ \otimes v_+ + v_- \otimes v_-) \otimes g_{\lambda} \text{id}_{V(\lambda)}, \quad (4.119)$$

where

$$g_{\lambda} = q^{-1} \xi(q^2;1,q^4)(q^2;p)_{\infty}(q^{2n+2};p)_{\infty},$$

$$g_{\lambda} = \xi(q^2;1,q^4)(q^2;p)_{\infty}(pq^{2n-2};p)_{\infty},$$

$$\xi(z; x, y) = \frac{(xz;p,q^4)_{\infty}(x^{-1}yz;p,q^4)_{\infty}}{(q^2; xz;p,q^4)_{\infty}(q^{-2}x^{-1}yz;p,q^4)_{\infty}},$$

$$\left(\frac{z}{x,y}\right)_{\infty} = \prod_{m,n=0}^\infty \left(1 - zx^m y^n\right). \quad (4.120)$$

### 4.3 Diagonalization of the higher spin Heisenberg model

In this section, we briefly summarize the diagonalization of the higher spin Heisenberg model as performed in Ref. [26]. We do not go into the details of the proofs which are already well presented in this reference.
In this framework, it is more convenient to diagonalize the transfer matrix instead. This is of course equivalent to the diagonalization of the Hamiltonian. For this crucial purpose, let us describe the main steps and results of Ref. [26]:

1. Given the $R$ matrix $R_{kk}(z)$ (4.116) the higher spin Hamiltonian is obtained from it as

$$PR_{kk}(z_1/z_2) = \text{const}(1 + uh_{l+1,l} + \cdots), \quad u \to 0, \quad z_1/z_2 = e^u,$$

$$\mathcal{H} = \sum_{l \in \mathbb{Z}} h_{l+1,l} = \text{const} \frac{d}{dz} \log(T(z))|_{z=1}, \quad (4.121)$$

where $T(z)$ is the transfer matrix.

2. Due to the intertwining relations (4.112) and the thermodynamic limit (i.e. $\sum_{l \in \mathbb{Z}}$), $\mathcal{H}$ is invariant under the quantum affine symmetry $U_q(\widehat{sl}(2))$

$$[\Delta^{(\infty)}(x), \mathcal{H}] = 0, \quad \forall x \in U_q(\widehat{sl}(2)), \quad (4.122)$$

where $\Delta^{(\infty)}$ is the infinite comultiplication. This is defined from $\Delta$ in (2.2) as

$$\Delta^{(\infty)} = \lim_{n \to \infty} \Delta^{(n)}, \quad \Delta^{(n)} = (\Delta \otimes \text{id})\Delta^{(n-1)}, \quad \Delta^{(1)} = \Delta. \quad (4.123)$$

3. The Hilbert space $F$ of $\mathcal{H}$ is given by

$$F = \sum_{\lambda,\mu} F_{\lambda\mu},$$

$$F_{\lambda\mu} = \sum_{\lambda,\mu} V(\lambda) \otimes V(\mu)^{a} = \sum_{\lambda,\mu} \text{Hom}(V(\mu), V(\lambda)), \quad (4.124)$$

where $V(\mu)^{a}$ is the dual representation obtained from $V(\mu)$ through the antipode $a$. The second definition of $F_{\lambda\mu}$ is more convenient in defining the action of $U_q(\widehat{sl}(2))$ on it. To see this, let $f \in \text{Hom}(V(\mu), V(\lambda))$ then the left action of $U_q(\widehat{sl}(2))$ is defined by

$$xf = \sum x_{(1)} \circ f \circ a(x_{(2)}), \quad \Delta(x) = \sum x_{(1)} \otimes x_{(2)}, \quad \forall x \in U_q(\widehat{sl}(2)). \quad (4.125)$$

Moreover, let $F_{\lambda\mu}^r$ denote $F_{\lambda\mu}$ as a $U_q(\widehat{sl}(2))$ right representation, then we have

$$fx = \sum a^{-1}(x_{(2)}) \circ f \circ x_{(1)}, \quad f \in F_{\lambda\mu}^r. \quad (4.126)$$

There is a unique vacuum state $|vac>_{\lambda} \in F_{\lambda\lambda}$, and which is identified with the identity element $\text{id}_{V(\lambda)}$. This also holds for $\lambda < vac| \in F_{\mu\lambda}^r$. There is a natural scalar product

$$<f|g> = \frac{tr_{V(\lambda)}(q^{-2\rho}fg)}{tr_{V(\lambda)}(q^{-2\rho})}, \quad f \in F_{\lambda\mu}^r, \quad g \in F_{\mu\lambda} \quad (4.127)$$
with the property
\[ < f | x g > = < f | x g >, \quad \forall x \in U_q(sil(2)). \] (4.128)

Let us mention that as defined above, \( F \) is free from the potential divergences, which plug the formal space \( \cdots \otimes V^{(k)} \otimes V^{(k)} \otimes \cdots \) on which \( H \) acts naturally.

4. The eigenspace \( F \) has a Fock space structure (i.e., a particle or spinon picture). This means that all its states can be obtained through successive actions of a set of creation operators on the vacuum state. They can also be annihilated through a set of annihilation operators. These operators are defined through type II vertex operators with \( \ell = 1 \). More specifically, creation operators \( \varphi^{*\lambda}_{\lambda, \epsilon}(z) \) acting on \( F_{\lambda \mu} = \text{Hom}(V(\mu), V(\lambda)) \) are defined by

\[ \varphi^{*\lambda}_{\lambda, \epsilon}(z) : F_{\lambda \mu} \to F_{\lambda' \mu'}, \quad f \to \tilde{\Psi}^{*\lambda}_{\lambda, \epsilon}(z) \circ f, \quad f \in F_{\lambda \mu}. \] (4.129)

Similarly, the annihilation operators \( \varphi^{\prime\mu}_{\mu, \epsilon}(z) \) acting on \( F_{\lambda \mu}^{\prime} \) are defined by

\[ \varphi^{\prime\mu}_{\mu, \epsilon}(z) : F_{\lambda \mu}^{\prime} \to F_{\lambda' \mu'}, \quad f \to f \circ \tilde{\Psi}^{\prime\mu}_{\mu, \epsilon}(z), \quad f \in F_{\lambda \mu}^{\prime}. \] (4.130)

With these definitions, it has been shown in Ref. [26] that the following commutation relations hold:

\[
\begin{align*}
\varphi^{\prime\mu}_{\mu, \epsilon_1}(z_1)\varphi^{\mu}_{\lambda, \epsilon_2}(z_2) & = \sum_{\mu', \epsilon_1', \epsilon_2'} R_{V,V'}(z_1/z_2)_{\epsilon_1, \epsilon_2}^{\epsilon_1', \epsilon_2'} \varphi^{\prime\mu'}_{\mu, \epsilon_1'}(z_2)\varphi^{\mu}_{\lambda, \epsilon_2}(z_1)W_{\lambda, \mu, \nu}(z_1/z_2), \\
\varphi^{\mu\prime}_{\mu, \epsilon_1}(z_1)\varphi^{*\lambda}_{\lambda, \epsilon_2}(z_2) & = \sum_{\mu', \epsilon_1', \epsilon_2'} R_{V',V}(z_1/z_2)_{\epsilon_1, \epsilon_2}^{\epsilon_1', \epsilon_2'} \varphi^{\mu\prime}_{\mu', \epsilon_1'}(z_2)\varphi^{*\lambda}_{\lambda, \epsilon_2}(z_1)W_{\lambda, \mu, \nu}(z_1/z_2), \\
\varphi^{\prime\mu}_{\mu, \epsilon_1}(z_1)\varphi^{*\lambda}_{\lambda, \epsilon_2}(z_2) & = \sum_{\mu', \epsilon_1', \epsilon_2'} R_{V',V}(z_1/z_2)_{\epsilon_1, \epsilon_2}^{\epsilon_1', \epsilon_2'} \varphi^{\prime\mu'}_{\mu, \epsilon_1'}(z_2)\varphi^{*\lambda}_{\lambda, \epsilon_2}(z_1)W_{\lambda, \mu, \nu}(z_1/z_2), \\
& \times (-q)^{\pm \delta_{\lambda, \nu} \delta_{\mu, \mu'}} W_{\lambda, \mu, \nu}(z_1q^{-2}/z_2) \\
& + g^2_{\lambda} \delta_{\lambda, \nu} \delta_{\epsilon_1, \epsilon_2} \delta(z_1/z_2).
\end{align*}
\] (4.131)

Here \( \mu = \lambda_{\pm} \). These commutations relations are typical for creation and annihilation operators in integrable models of quantum field theory.

Therefore, a typical \( n \)-particle state in \( F_{\lambda_{n+1}, \epsilon_{n+1}} \) is given by

\[ |z_1, \ldots, z_1 >_{\lambda_{n+1}, \epsilon_{n+1}} = \varphi^{*\lambda_{n+1}}_{\lambda_{n+1}, \epsilon_{n+1}}(z_1) \cdots \varphi^{*\lambda_{1}}_{\lambda_{1}, \epsilon_{1}}(z_1) |vac >_{\lambda} = \tilde{\Psi}^{*\lambda_{n+1}}_{\lambda_{n+1}, \epsilon_{n+1}}(z_1) \cdots \tilde{\Psi}^{*\lambda_{1}}_{\lambda_{1}, \epsilon_{1}}(z_1). \] (4.132)
The scalar product of the latter two states then reads
\begin{equation}
\langle \lambda, \lambda', \ldots, \lambda_m < z_1', \ldots, z_m' | \lambda, \lambda', \ldots, \lambda_m, \lambda_1, \lambda_2, \ldots, \lambda_1 | z_1, \ldots, z_m \rangle = \langle \lambda \rangle \langle \lambda' \rangle \cdots \langle \lambda_m \rangle \langle \lambda'_1 \rangle \cdots \langle \lambda'_m \rangle = \bar{\Psi}^\lambda_{\lambda_1, \epsilon_1} (z'_1) \cdots \bar{\Psi}^\lambda_{\lambda_m, \epsilon_m} (z'_m). \tag{4.133}
\end{equation}

Similarly, a typical state in \( \mathcal{F}_{\lambda_{\epsilon_1}} \) is represented as
\begin{equation}
\langle \lambda, \lambda', \ldots, \lambda_m < z_1', \ldots, z_m' | \lambda, \lambda', \ldots, \lambda_m, \lambda_1, \lambda_2, \ldots, \lambda_1 | z_1, \ldots, z_m \rangle = \langle \lambda \rangle \langle \lambda' \rangle \cdots \langle \lambda_m \rangle \langle \lambda'_1 \rangle \cdots \langle \lambda'_m \rangle = \bar{\Psi}^\lambda_{\lambda_1, \epsilon_1} (z'_1) \cdots \bar{\Psi}^\lambda_{\lambda_m, \epsilon_m} (z'_m). \tag{4.133}
\end{equation}

The scalar product of the latter two states then reads
\begin{equation}
\langle \lambda, \lambda', \ldots, \lambda_m < z_1', \ldots, z_m' | z_1, \ldots, z_m \rangle = \frac{\text{Tr}_{V(\lambda)} \left( q^{-2\mu} \bar{\Psi}^\lambda_{\lambda_1, \epsilon_1} (z'_1) \cdots \bar{\Psi}^\lambda_{\lambda_m, \epsilon_m} (z'_m) \bar{\Psi}^\lambda_{\lambda_1, \epsilon_1} (z_1) \cdots \bar{\Psi}^\lambda_{\lambda_m, \epsilon_m} (z_1) \right)}{\text{Tr}_{V(\lambda)} (q^{-2\mu})}. \tag{4.134}
\end{equation}

5. To see that the Fock space of the above creation and annihilation operators is the eigenspace of the Hamiltonian let us recall the definition of the transfer matrix \( T(z) \) in this framework. To simplify the notations let us write
\begin{align*}
\Phi^{\sigma(\lambda)V} (z) &= \sum_j \Phi_j (z) \otimes v_j^{(k)}, \\
\Phi^{\mu V^{a-1}} (z) &= \sum_j \Phi_j^* (z) \otimes v_j^{(k)*}, \tag{4.135}
\end{align*}
and let \( \Phi^* (z) \) denote the transpose of \( \Phi^{\mu V^{a-1}} (z) \). Then, \( T(z) \) is defined by
\begin{align*}
T(z) &= T^{\sigma(\lambda)\sigma(\mu)} (z) : \mathcal{F}_{\lambda \mu} = V(\lambda) \otimes V(\mu) \otimes V(\lambda^*) \otimes V(\mu^*) \otimes V(\lambda) \otimes V(\mu)^* \\
&= \Phi_{\sigma(\lambda)} (z) \otimes \Phi_{\sigma(\mu)} (z) \otimes \Phi_{\lambda^*} (z) \otimes \Phi_{\mu^*} (z) \otimes \Phi_{\lambda} (z) \otimes \Phi_{\mu} (z). \tag{4.136}
\end{align*}
This means
\begin{equation}
T(z)(f) = \sum_j \Phi_j (z) \circ f \circ \Phi_j^* (z), \quad \forall f \in \mathcal{F}_{\lambda \mu}. \tag{4.137}
\end{equation}
In particular, \( g_{\lambda}^{-1} T(1) \) is identified with the translation operator. Using the last equation, the commutation relations of vertex operators \([4.117]\), and the identity
\begin{equation}
\sum_j \Phi_j (z) \circ \Phi_j^* (z) = g_{\lambda} \text{id}, \tag{4.138}
\end{equation}
one finds the following important relations \([4.16]\):
\begin{align*}
T^{\sigma(\lambda)\sigma(\lambda)}_{\lambda \lambda} (z)vac \rangle \langle vac &= g_{\lambda} \langle vac | vac \rangle, \\
T^{\sigma(\lambda')\sigma(\lambda)}_{\lambda' \lambda} (z_2) \varphi^{\lambda'}_{\lambda', \epsilon} (z_1) &= \tau(z_1/z_2) \varphi^{\sigma(\lambda')}_{\sigma(\lambda), \epsilon} (z_1) T^{\sigma(\lambda)\sigma(\lambda)}_{\lambda' \lambda} (z_2), \\
T^{\sigma(\lambda')\sigma(\lambda)}_{\lambda' \lambda} (z_2) \varphi^{\lambda'}_{\lambda', \epsilon} (z_1) &= \tau(z_1/z_2)^{-1} \varphi^{\sigma(\lambda')}_{\sigma(\lambda), \epsilon} (z_1) T^{\sigma(\lambda)\sigma(\lambda)}_{\lambda' \lambda} (z_2), \\
T (z_2) \varphi^{\mu}_{\lambda', \epsilon} (z_1) &= \tau(z_1/z_2)^2 \varphi^{\mu}_{\lambda', \epsilon} (z_1), \\
T (z_2) \varphi^{\mu}_{\lambda', \epsilon} (z_1) T (z_2)^{-1} &= \tau(z_1/z_2)^{-2} \varphi^{\mu}_{\lambda', \epsilon} (z_1), \tag{4.139}
\end{align*}
where \( \tilde{T}(z) = T_{\lambda \mu}^{(\lambda \sigma(\mu))}(z) T_{\lambda \mu}^{(\sigma(\lambda \mu))}(z) \). Consequently, (4.121) implies that

\[
[H, \varphi_{\lambda \mu}^\mu(z)] = -e(z)\varphi_{\lambda \mu}^\mu(z),
\]

\[
[H, \varphi_{\lambda \mu}^\sigma(z)] = e(z)\varphi_{\lambda \mu}^\sigma(z),
\]

with

\[
e(z) = \text{const } z \frac{d}{dz} \log(\tau(z)) = \text{const } \sqrt{1 - k^2 \cos^2(p(z))},
\]

where the momentum \( p(z) \) of an elementary excitation (spinon) is as usual defined by

\[
\tau(z) = e^{-ip(z)},
\]

and \( k \) should not be confused with the level of \( U_q(\widehat{sl}(2)) \) or the spin \( k/2 \) of the Heisenberg model. It is the elliptic module associated with a nome \(-q\). Obviously, \( e(z) \) is nothing but the energy of a spinon, which is a spin 1/2 excitation. Equation (4.141) is then the dispersion relation of a spinon and which is independent of the value \( k/2 \) of local spin variables.

6. Let \( L \) be a local operator acting on the subspace \( V^{(k)\otimes n} \), i.e., \( L \in \text{End}(V^{(k)\otimes n}) \), then its action on \( \mathcal{F}_{\lambda \mu} \) reads

\[
L(f) = \mathcal{L}_{\lambda} \circ f, \quad f \in \mathcal{F}_{\lambda \mu},
\]

and where \( \mathcal{L}_{\lambda} \), in turn, is defined in terms of type I vertex operators as

\[
\mathcal{L}_{\lambda} = (g_\lambda g_{\lambda(1)} \cdots g_{\lambda(n-1)})^{-\frac{1}{2}} \tilde{\Phi}_{\lambda(1)}^{\lambda_{(1)}}(z_1) \circ (\tilde{\Phi}_{\lambda(2)}^{\lambda_{(2)}}(z_2) \otimes \text{id}_V) \circ \cdots \circ (\tilde{\Phi}_{\lambda(n)}^{\lambda_{(n)}}(z_n) \otimes \text{id}_V) \circ (\text{id}_V \otimes L)
\]

\[
\circ (\tilde{\Phi}_{\lambda(n-1)}^{\lambda_{(n-1)}}(z_n) \otimes \text{id}_V) \circ \cdots \circ (\tilde{\Phi}_{\lambda(1)}^{\lambda_{(1)}}(z_2) \otimes \text{id}_V) \circ \tilde{\Phi}_\lambda^{\lambda_{(1)}}(z_1).
\]

where \( \lambda^{(l)} = \sigma(\lambda^{(l-1)}) \) and \( V = V^{(k)} \). In this sense we can think of type I vertex operators as the creation of local spin states belonging to the subspace \( V^{(k)\otimes n} \). Furthermore, from the commutation relations (4.117) it can be shown that local operators always commute with the creation operators, that is,

\[
\mathcal{L}_{\mu} \tilde{\Psi}_{\lambda,\epsilon}^\mu(z) = \tilde{\Psi}_{\lambda,\epsilon}^\mu(z) \mathcal{L}_{\lambda}.
\]

A matrix element of \( L \) (form factor) is then given by

\[
\lambda < \text{vac}|L|z_n, \ldots, z_1>_{\epsilon_n, \ldots, \epsilon_1}^{\lambda_n, \ldots, \lambda_1} = \frac{T_{\text{r}_V(\lambda)}(q^{-2\rho} \mathcal{L}_{\lambda} \tilde{\Psi}_{\lambda_n, \epsilon_n}^{\lambda_n}(z_n) \cdots \tilde{\Psi}_{\lambda_1, \epsilon_1}^{\lambda_1}(z_1))}{T_{\text{r}_V(\lambda)(q^{-2\rho})}}.
\]
In particular, the correlation functions of $L$ (its vacuum to vacuum expectation values) simplify to

$$\lambda < \text{vac}|L|\text{vac}>\lambda = \frac{\text{Tr}_{V(\lambda)}(q^{-2\rho}L_{\lambda})}{\text{Tr}_{V(\lambda)}(q^{-2\rho})}. \quad (4.147)$$

Let us however note that the evaluation of the above traces is a nontrivial task because the representations are infinite-dimensional and so the vertex operators. For this purpose we use the powerful bosonization technique, that is, the representation of $U_q(sl(2))$ currents, screening currents, irreducible highest weight modules and vertex operators in terms of modes satisfying a simple Heisenberg algebra. We have already presented the bosonization of the currents and screening currents and therefore in the next section we focus just on the bosonization of the vertex operators and $U_q(sl(2))$ irreducible highest weight representations.
5 Bosonization of vertex operators and IHWM’s

5.1 Bosonization of vertex operators

Here we will be using the bosonization of \( U_q(\widehat{sl}(2)) \) as given in (3.80). The results of this section is spread throughout many papers in the literature but we will be following closely Ref. [33], especially as far as the cohomology analysis is concerned. Due to the reducibility of the Fock spaces as \( U_q(\widehat{sl}(2)) \) modules the bosonization of the vertex operators is a little trickier. To see this, let us construct Fock spaces \( F_{\ell,s,t} \) from the successive actions of the creation modes \( \{ a_{X_1,-n_1}, a_{X_2,-n_2}, a_{X_3,-n_3}, n_1, n_2, n_3 > 0 \} \) on the vector \( |\ell; s,t> = e^{\frac{i}{\sqrt{2}}(Q_{X_1} + \frac{i}{2}Q_{X_2} + \frac{i}{2}Q_{X_3})}|0; 0> \). Because the currents do not depend explicitly on \( Q_{X_1} \) they act naturally on the larger Fock spaces \( F_{\ell} = \oplus_{s,t \in \mathbb{Z}} F_{\ell,s,t} \). However, as can easily be verified from their characters, these Fock spaces are larger than the \( U_q(\widehat{sl}(2)) \) Verma modules \( L(\lambda_{\ell}) \) with highest weights \( \lambda_{\ell} = (k - \ell)\Lambda_0 + \ell\Lambda_1 \). Note that the particular state

\[
|\ell> = |\ell; 0, 0> = |vac>_{\lambda_{\ell}} = |\lambda_{\ell}>
\]

is a common highest weight state to \( F_{\ell}, L(\lambda_{\ell}) \), and to the irreducible highest weight module \( V(\lambda_{\ell}) \). Therefore a projector is needed to project the larger space \( F_{\ell} \) on \( L(\lambda_{\ell}) \). Such a projector is easily constructed from the screening current \( \eta(z) \) (3.92) and also the “screening operator” \( a_{X_2} + a_{X_3} \) acting on \( F_{\ell} \). Both these operators commute up to total derivatives with all \( U_q(\widehat{sl}(2)) \) currents. The bosonizations of \( \eta(z) \) (3.92) and its “conjugate” \( \xi(z) \) are given by

\[
\eta(z) = \exp(X_3(2|q^{-k-2}z; 0)),
\]

\[
\xi(z) = \exp(-X_3(2|q^{-k-2}z; 0)).
\]

One can easily check that these two fields are conjugate to each other in the following sense:

\[
\xi(z)\eta(w) = -\eta(w)\xi(z) = \frac{q^3}{z - w} + :\xi(z)\eta(w):.
\]

This means that their zero modes in \( \eta(z) = \sum_{n \in \mathbb{Z}} z^{-n-1}\eta_n \) and \( \xi(z) = \sum_{n \in \mathbb{Z}} z^{-n}\xi_n \) satisfy the anticommutation relation

\[
\{\xi_0, \eta_0\} = 1.
\]
Therefore, \( L(\lambda_\ell) \) is isomorphic to the following restricted space \( \tilde{F}_\ell \):

\[
V(\lambda_\ell) \sim \tilde{F}_\ell = \oplus_{s,t \in \mathbb{Z}} \text{Ker}_{\eta_0} \left( \text{Ker}_{a_{x_2} + a_{x_3}} (F_{\ell,s,t}) \right) = \oplus_{s \in \mathbb{Z}} \text{Ker}_{\eta_0} (F_{\ell,s,s}).
\] (5.152)

Let us now define the actions of the vertex operators so that they intertwine the Fock spaces \( \tilde{F}_\ell \). Let the bosonization of type I vertex operator be

\[
\tilde{\Phi}_{\lambda_{t_1}}^{\lambda_{t_2},V(\ell)}(z) = g_{\lambda_{t_1}}^{\lambda_{t_2},V(\ell)}(z) \sum_{m=0}^{\ell} \tilde{\Phi}_{\ell,m}^{(r)}(z) \otimes v_m^{(\ell)}, \quad 2r = \ell + \ell_1 - \ell_2,
\] (5.153)

where the screened vertex operator components \( \tilde{\Phi}_{\ell,m}^{(r)}(z) \) are given in terms of the bare ones \( \tilde{\Phi}_{\ell,m}(z) \) by

\[
\tilde{\Phi}_{\ell,m}^{(r)}(z) = \int_{q^p z}^{q^p \infty} d_p t_1 \ldots \int_{q^p z}^{q^p \infty} d_p t_r S(t_1) \ldots S(t_r) \tilde{\Phi}_{\ell,m}(z).
\] (5.154)

The bosonization of the latter bare components \( \tilde{\Phi}_{\ell,m}^{(r)}(z) \) is in turn derived from that of \( \tilde{\Phi}_{\ell}^{(r)}(z) \) through the successive applications of the intertwining relations (4.100). More specifically, using these relations (4.100), the comultiplication (2.7), and the fact that \( N^{+v}_0(F_{\ell,m}) = N^{-v}_0(F_{\ell,m}) = 0 \), \( N^{\pm v}_0(z) \in F[z, z^{-1}]v_m^{(\ell)} \), one arrives at:

\[
[E^+(w), \tilde{\Phi}_{\ell}^{(r)}(z)] = 0,
\]

\[
[H_n, \tilde{\Phi}_{\ell}^{(r)}(z)] = q^{2n} z^n [n/2] q^{k(n + |n|)/2} \tilde{\Phi}_{\ell}^{(r)}(z), \quad n \neq 0,
\]

\[
K \tilde{\Phi}_{\ell}^{(r)}(z) K^{-1} = q^r \tilde{\Phi}_{\ell}^{(r)}(z),
\]

\[
\tilde{\Phi}_{m}^{(r)}(z) = \frac{1}{[\ell - m]!} \left[ \ldots [\tilde{\Phi}_{\ell}^{(r)}(z), E_0^-]_q, E_0^-]_q^2, \ldots, E_0^-]_q^{\ell-m} \right].
\] (5.155)

Here the quantum commutator \([A, B]_{q^x}\) is defined by

\[
[A, B]_{q^x} = AB - q^x BA.
\] (5.156)

These relations can easily be solved for the bosonization of \( \tilde{\Phi}_{\ell}^{(r)}(z) \), and one finds

\[
\tilde{\Phi}_{\ell}^{(r)}(z) = \exp \left( X_1(\ell; 2, k + 2|q^k z; k + 2/2) \right).
\] (5.157)

Finally, due to the commutation relations

\[
[\tilde{\Phi}_{\ell}^{(r)}(z), \eta_0] = 0,
\]

\[
[\tilde{\Phi}_{\ell}^{(r)}(z), a_{x_2,0} + a_{x_3,0}] = 0,
\] (5.158)
one can easily see that the screened vertex operators intertwine the restricted Fock spaces as:

\[ \tilde{\Phi}^{(r)}_{\ell,m}(z) : \tilde{F}_{\ell_1} \to \tilde{F}_{\ell_2} \otimes V^{(\ell)}(z) \]  

(5.159)

The normalization functions \( g_{\lambda_{\ell_2}, V^{(\ell)}}(z) \) are fixed by.

\[ \tilde{\Phi}_{\lambda_{\ell_1}}^{\lambda_{\ell_2}, V^{(\ell)}}(z)|\ell_1; 0, 0 >= |\ell_2; 0, 0 > \otimes \psi + \cdots. \]  

(5.160)

In fact, unlike the case \( k = 1 \) with the simple bosonization (2.53), there is a second bosonization here of the vertex operator satisfying the intertwining relations. To distinguish it from the first one given in (5.157), let us denote it by

\[ \hat{\Phi}_{\lambda_{\ell_1}}^{\lambda_{\ell_2}, V^{(\ell)}}(z) = g_{\lambda_{\ell_2}, V^{(\ell)}}(z) \sum_{m=0}^{\ell} \tilde{\Phi}^{(r)}_{\ell,m}(z) \otimes v_m^{(\ell)}, \]  

(5.161)

with the bare components \( \tilde{\Phi}^{(r)}_{\ell,m}(z) \) being related to \( \hat{\Phi}^{(\ell)}_{\ell}(z) \) in the same manner as described by (5.155). Moreover, the bosonization of \( \hat{\Phi}^{(\ell)}_{\ell}(z) \) is given by

\[ \hat{\Phi}^{(\ell)}_{\ell}(z) = \exp(X_1(k + 1 - \ell; 2, \frac{k + 2}{2}|q^r; 0) + X_2(k + 1 - \ell; 1, 2|z; -1) + X_3(k - \ell; 1, 2|z; 0)). \]  

(5.162)

It is typical in the context of conformal field theory to refer to this second bosonized vertex operator as a conjugate vertex operator.

With obvious modifications, a type II vertex operator intertwines also the restricted Fock spaces when they are screened as

\[ V^{(\ell)}(z) = \sum_{m=0}^{\ell} v_m^{(\ell)} \otimes \tilde{\Psi}^{(r)}_{\ell,m}(z), \]  

(5.163)

with

\[ \tilde{\Psi}^{(r)}_{\ell,m}(z) = \int_{0}^{q_{k-\ell}z} d_p t_1 \cdots \int_{0}^{q_{k-\ell}z} d_p t_r \tilde{\Psi}^{(\ell)}_{m}(z) S(t_1) \cdots S(t_r). \]  

(5.164)

Similarly, the bosonizations of the bare components \( \tilde{\Psi}^{(\ell)}_{m}(z) \) are derived recursively from that of \( \tilde{\Phi}^{(\ell)}_{0}(z) \) as shown by the following explicit form of the intertwining relations (4.100):

\[ [E^{-}(w), \tilde{\Phi}^{(\ell)}_{0}(z)] = 0, \]

\[ [H_n, \tilde{\Phi}^{(\ell)}_{0}(z)] = -z^n \frac{nf}{n} q^{k(n-|n|/2)} \tilde{\Phi}^{(\ell)}_{0}(z), \quad n \neq 0, \]

\[ K \tilde{\Phi}^{(\ell)}_{0}(z) K^{-1} = q^{-\ell} \tilde{\Phi}^{(\ell)}_{0}(z), \]

\[ \tilde{\Phi}^{(\ell)}_{0}(z) = \frac{1}{[m]!} \cdots \frac{1}{[\tilde{\Phi}^{(\ell)}_{0}(z), E_0^+] q^\ell, E_0^+] q^{\ell-2}, \cdots, E_0^+] q^{\ell-2(m-1)}. \]  

(5.165)
From the first three commutation relations one finds the bosonization of $\tilde{\Psi}_0^{(r)}(z)$

$$\tilde{\Psi}_0^{(r)}(z) = \exp(X_1(\ell; 2, k + 2|q^{-2}z; -\frac{k + 2}{2}) + X_2(\ell; 2, 1|q^{-2}z; 0) + X_3(\ell; 2, 1|q^{-2}z; 0)).$$

(5.166)

Moreover, one can easily check

$$[\tilde{\Psi}_0^{(r)}(z), \eta_0] = 0,$$

$$[\tilde{\Psi}_0^{(r)}(z), a_{X_2,0} + a_{X_3,0}] = 0,$$

(5.167)

and hence the screened components intertwine the restricted Fock spaces as

$$\tilde{\Psi}_{\ell,m}^{(r)}(z) : \tilde{F}_{\ell_1} \rightarrow V^{(r)}(z) \otimes \tilde{F}_{\ell_2}.$$  

(5.168)

Here also there exists a second bosonization given by

$$\hat{\Psi}_{\lambda_1,\lambda_2}^{(r)}(z) = \hat{h}_{\lambda_1}^{(r)}(z) \sum_{m=0}^{\ell} v_m^{(l)} \otimes \hat{\Psi}_{\ell,m}^{(r)}(z),$$

(5.169)

with

$$\hat{\Psi}_0^{(r)}(z) = \exp(-X_1(\ell + 2; 2, k + 2|q^{-2}z; -\frac{k + 2}{2}) - X_2(2|q^{-2}z; 0)).$$

(5.170)

For the same reason as in the case of type I vertex operators, we refer to $\hat{\Psi}_0^{(r)}(z)$ as a type II conjugate vertex operator.

Let us however recall that in conformal field theory conjugate vertex operators play an important role in simplifying the calculation of correlation functions of vertex operators only on a genus zero manifold, i.e., the sphere. They do not enter in the calculation of higher genus correlation functions, and in particular in the case of the correlation functions on a torus. As we have seen in (4.146) the physical correlation functions and form factors of the Heisenberg model are expressed as traces of vertex operators in a similar manner as genus one correlation functions in conformal field theory, conjugate vertex operators are then not particularly useful for our purposes. Therefore, we will not consider them any further.

Let us now briefly describe how the above contours of the Jackson integrals were defined in Ref. [33]. The basic requirement is the following commutation relation:

$$[\tilde{\Psi}_{\ell,0}^{(r)}(z), E^-(w)] = 0,$$

(5.171)
which is one of the intertwining relations. Here, a q-difference appears in the OPE

\[ \tilde{\Psi}_{\ell,0}^{(t)}(z) \::\! S(t) :: E^-(w) \sim_{k+2} t \frac{(q^\ell t z^{-1}; p)_{\infty}}{(w-t)(q^{-\ell+2} t z^{-1}; p)_{\infty}} \tilde{\Psi}_{\ell,0}^{(t)}(z) e^{-X_1(k+2|q^{-2}; k+2)} : \]

(5.172)

With this OPE one can easily check using the definition of a Jackson integral (8.306) that the following integral of a total q-difference

\[ \int_0^a dp t^{k+2} \partial_t \frac{(q^\ell t z^{-1}; p)_{\infty}}{(w-t)(q^{-\ell+2} t z^{-1}; p)_{\infty}} \tilde{\Psi}_{\ell,0}^{(t)}(z) e^{-X_1(k+2|q^{-2}; k+2)} : \]

(5.173)

vanishes if

\[ a = q^{k-\ell} p^{-j} z, \quad j = 0, 1, \ldots \] (5.174)

In the above contours the first choice \( j = 0 \) has been picked. The normalization functions are obtained through the conventions made in (4.103) and the various OPE’s. One finds explicitly

\[ h_{\lambda_\ell}^{V(1), \lambda_{\ell+1}}(z) = q^{-2(k-2)s} z^{-2s}, \]
\[ h_{\lambda_\ell}^{V(1), \lambda_{\ell-1}}(z) = -q^{2+\frac{\ell}{2}-2(3\ell+4)s} z^{(\ell+2)s} B_q(1-2\ell s, -2s)^{-1}, \] (5.175)

with \( s = 1/2(k+2) \).

### 5.2 Irreducible highest weight representations

Since correlation functions and form factors are given as traces over irreducible highest weight representations \( V(\lambda_\ell) \), which are embedded in the Verma modules \( L(\lambda_\ell) \), one needs then another projection from the latter to the former. This is provided through a BRST cohomology analysis. As in the classical case a BRST operator \( Q_n \) can be defined as

\[ Q_n = \int_0^\infty dp t J_n(t), \] (5.176)

with the BRST current \( J_n(t) \) being given by

\[ J_n(t) = \frac{1 - A_n}{1 - A_s} \int_0^{q^t} dp t_2 \cdots \int_0^{q^t} dp t_n S(t_1) S(t_2) \cdots S(t_n), \] (5.177)

and being single valued on the restricted Fock spaces \( \tilde{F}_{\ell,n,\tilde{n}} \equiv \tilde{F}_{n,\tilde{n}} \). Here, \( A_s \) is defined by

\[ S(t_1) S(t_2) = A_s(t_1/t_2) S(t_2) S(t_1), \]
\[ A_s(t) = \frac{\theta_p(pq^{-2} t)}{\theta_p(pq^2 t)}, \]
\[ A_s = A_s(q^{-2-\epsilon}), \] (5.178)
where, $\epsilon$ is a regularization parameter and $A_s(t)$ is a pseudo constant, that is, $A_s(tp^n) = A_s(t)$, $n \in \mathbb{Z}$.

Following the analysis of the general case we consider $\ell_{n,\bar{n}} = n - \bar{n}\frac{P}{P'} - 1$ with $n$ and $\bar{n}$ being integers, and $P$ and $P'$ coprime integers satisfying $\frac{P}{P'} = k + 2$. However, the highest weights $\lambda_{\ell_{n,\bar{n}}} = (k - \ell_{n,\bar{n}})A_0 + \ell_{n,\bar{n}}A_1$ of $V(\lambda_t)$ are such that $1 \leq n \leq P$ and $0 \leq \bar{n} \leq P'$ (our present case corresponds to $P = k + 2$ and $P' = 1$). Given this, it has been partially shown in Ref. [33] that

$$ Q_n Q_{P-n} = Q_{P-n} Q_n = 0, \quad (5.179) $$

and that the following sequence is a complex:

$$ \cdots \to \tilde{F}_{n+2P,\bar{n}} \to \tilde{F}_{n,\bar{n}} \to \tilde{F}_{n,\bar{n}} \to \tilde{F}_{n-2P,\bar{n}} \to \cdots $$

As a consequence, it was claimed that the $U_q(sl(2))$ irreducible highest weight representation $V(\lambda_{\ell_{n,\bar{n}}})$ with $1 \leq n \leq P$, $0 \leq \bar{n} \leq P'$ is isomorphic to the single nonvanishing BRST cohomology group as follows:

$$ \text{Ker}Q_n^{[s]} / \text{Im}Q_n^{[s-1]} \sim V(\lambda_{\ell_{n,\bar{n}}}) \quad \text{if} \quad s \in \mathbb{Z} \setminus \{0\}, $$

$$ \text{Ker}Q_n^{[s]} / \text{Im}Q_n^{[s-1]} \sim 0 \quad \text{if} \quad s \neq 0, \quad (5.180) $$

where

$$ Q_n^{[2s]} = Q_n : \tilde{F}_{n-2sP,\bar{n}} \to \tilde{F}_{n-2sP,\bar{n}}, \quad s \in \mathbb{Z}, $$

$$ Q_n^{[2s+1]} = Q_{P-n} : \tilde{F}_{n-2sP,\bar{n}} \to \tilde{F}_{n-2(s+1)P,\bar{n}}, \quad s \in \mathbb{Z}. \quad (5.181) $$

This is a natural generalization of the classical (i.e., $q = 1$) theorem. From the Lefschetz formula the trace of a BRST invariant operator $O$ over $V(\lambda_{\ell_{n,\bar{n}}})$ follows as:

$$ Tr_{V(\lambda_{\ell_{n,\bar{n}}})} O = \sum_{s \in \mathbb{Z}} (-1)^s Tr_{\tilde{F}_{n,\bar{n}}^{[s]}} O^{[s]} \quad (5.182) $$

where

$$ \tilde{F}_{n,\bar{n}}^{[s]} = \tilde{F}_{n-Ps,\bar{n}}, \quad s \in 2\mathbb{Z}, $$

$$ \tilde{F}_{n,\bar{n}}^{[s]} = \tilde{F}_{n-P(s-1),\bar{n}}, \quad s \in 2\mathbb{Z} + 1. \quad (5.183) $$

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and the graded BRST invariant operators $O^{[s]}$ are obtained recursively from $O$ through the relations

$$Q_n^{[s]} O^{[s]} = O^{[s+1]} Q_n^{[s]}, \quad O^{[0]} = O.$$  \hfill (5.184)

In particular, using (5.154) and the fact

$$\mu^{-d} S_{\delta,0} = S_{\delta,0} \mu^{-d},$$  \hfill (5.185)

one can show that the screened vertex operators are BRST invariant and satisfy

$$\mu^{-d} \nu^{-\alpha_0} Q_n^{(r)} \Phi^{(r)}_{\ell,m}(z) = A_{\Phi,\ell}^{n_2} \mu^{-d} \nu^{-\alpha_0} \Phi^{(r+n_3-n_1)}_{\ell,m}(z) Q_n^{(r)},$$

$$Q_{P-n_2} \mu^{-d} \nu^{-\alpha_0} \Phi^{(r+n_3-n_1)}_{\ell,m}(z) = A_{\Phi,\ell}^{P-n_2} \mu^{-d} \nu^{-\alpha_0} \Phi^{(r)}_{\ell,m}(z) Q_{P-n_1},$$  \hfill (5.186)

where $A_{\Phi,\ell}$ is defined by

$$S(t) \Phi^{(r)}_{\ell}(z) = A_{\Phi,\ell}(z/t) \Phi^{(r)}_{\ell}(z) S(t),$$

$$A_{\Phi,\ell}(z) = (q^{k+2} z)^{\frac{\epsilon-1}{\epsilon-2}} \frac{\theta_p(q^{-\ell} z)}{\theta_p(q^\ell z)},$$

$$A_{\Phi,\ell} = A_{\Phi,\ell}(q^{\ell} z).$$  \hfill (5.187)

Here, $\epsilon$ is a regularization parameter and $A_{\Phi,\ell}(z)$ a pseudo constant, that is, $A_{\Phi,\ell}(z p^n) = A_{\Phi,\ell}(z), n \in \mathbb{Z}$. Similarly, physical type II vertex operators are BRST invariant and satisfy

$$\mu^{-d} \nu^{-\alpha_0} Q_n^{(r)} \Psi^{(r)}_{1,m}(z) = A_{\Phi,1}^{n_2} \mu^{-d} \nu^{-\alpha_0} \Psi^{(1-r)}_{1,m}(z) Q_n^{(r)}, \quad n_2 - n_1 = 1 - 2r, \quad r = 0, 1;$$

$$Q_{P-n_2} \mu^{-d} \nu^{-\alpha_0} \Psi^{(1-r)}_{1,m}(z) = A_{\Phi,1}^{P-n_2} \mu^{-d} \nu^{-\alpha_0} \Psi^{(r)}_{1,m}(z) Q_{P-n_1}, \quad n_2 - n_1 = 1 - 2r, \quad r = 0, 1;$$  \hfill (5.188)

where $A_{\Phi,1}$ is given by the following relations:

$$S(t) \Psi^{(1)}_{1}(z) = A_{\Phi,1}(z/t) \Psi^{(1)}_{1}(z) S(t),$$

$$A_{\Phi,1}(z) = (q^{-2} z)^{\frac{\epsilon-1}{\epsilon-2}} \frac{\theta_p(q^{k+1} z)}{\theta_p(q^{k-1} z)},$$

$$A_{\Phi,1} = A_{\Phi,1}(q^{k+1} z).$$  \hfill (5.189)

Here also, $\epsilon$ is a regularization parameter and $A_{\Phi,1}(z)$ is a pseudo constant, i.e., $A_{\Phi,1}(z p^n) = A_{\Phi,1}(z), n \in \mathbb{Z}$. 

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Now let us make some remarks about the explicit evaluation of the traces over $\tilde{F}^{[s]}_{n,\bar{n}}$. The bosonization of the currents is independent of the zero mode $\xi_0$ since

$$E^+(z) = \partial_\bar{z}\xi(z) \exp -X_2(2|q^{-k-2}z; 1). \quad (5.190)$$

Therefore, states with $\xi_0$ must not be created by vertex operators and propagate. This means that the traces over the restricted Fock spaces of the BRST invariant operators can be evaluated in terms of the simpler traces over the unrestricted Fock spaces as follows:

$$Tr_{\tilde{F}^{[s]}_{n,\bar{n}}} O^{[s]} = Tr_{F^{[s]}_{n,\bar{n}}} \left( O^{[s]} \oint \frac{dy}{2\pi i} \eta(y)\xi(y') \right). \quad (5.191)$$

Let us remind that the restricted Fock spaces is in the kernel of $a_{X_1,0} + a_{X_2,0}$. The above expression (5.191) is slightly different from the proposed one in Ref. [33] in two aspects: First, we have displaced $O^{[s]}$ to the left of $\eta(y)$ and $\xi(y')$. Second, We have displaced $\xi(y')$ to the right of $\eta(y)$. The reason for the first choice is that by doing so we arrange for the smallest number of poles from the OPE’s to contribute to the $y$ contour integral. This is particularly useful in the limit $q \to 0$, where the leading term of the trace reduces to that of the matrix element. The latter will be easily computed since only the pole at $y = y'$ contributes to the $y$ integral. The reason for the second modification is merely due to simplification purposes of the trace formula over the $X_2$ and $X_3$ parts of the vertex operators.

Later we will illustrate explicitly these statements through the simplest example of the 1-point correlation function of the spin-1/2 Heisenberg model. Obviously, this is already known through Baxter’s methods, but we will rederive its leading term through this general bosonization method, which is applicable to any spin.
6 Correlation functions and form factors

6.1 N-point correlation functions

Here, we build on the previous chapters to evaluate the N-point correlation functions of local operators. By this, we mean their vacuum to vacuum expectation values. We attempt to make this evaluation as explicit as possible but there are some expressions which can be written in complete explicit forms only once the value of the local spin \((= k/2)\) is fixed to some number.

Since a general local operator \(L\) acting on \(V^{(k)\otimes n}\), which is embedded in \(V^{(k)\otimes \infty}\) can be expressed as a linear combination of local operators of the form \(E_{i_n} \otimes \cdots \otimes E_{i_1}\), it is sufficient to consider just the latter basis operators. Here \(E_{ij}\) are the \(k \times k\) unit matrices.

As (4.147) shows, a single basis local operators has many correlation functions depending on the sectors it is acting in. Each sector is characterized by the boundary conditions imposed on both ends of the chain of spins. These boundary conditions, in turn, are in one to one correspondence with the highest weights \(\lambda\), that is, with the vacuum states \(|\text{vac}\rangle_{\lambda}\). Therefore, the correlation function of a basis operator in the sector \(\lambda_m = (k - m)\Lambda_0 + m\Lambda_1\), is given by

\[
< E_{i_n} \otimes \cdots \otimes E_{i_1} >^{(\lambda_m)}_{z_n, \ldots, z_1} = A(q) \frac{\Tr_{V^{(\lambda_m)}}(\mu^{-d} \nu^{-\alpha_0} \Phi^{(\lambda^{(1)})}_{\lambda_m, V, z_1} (z_1) \cdots \Phi^{(n-1)}_{\lambda^{(n-1)}, V, z_n} (z_n) \Phi^{(n)}_{\lambda(n), V, z_1} (z_1) \cdots \Phi^{(1)}_{\lambda_1, V, z_1} (z_1))}{\Tr_{V^{(\lambda_m)}}(\mu^{-d} \nu^{-\alpha_0})}
\]

(6.192)

where we have used (4.144), (4.147), \(\rho = 2d + \alpha_0/2\), and

\[
\begin{align*}
\mu &= q^4, \\
\nu &= q, \\
A(q) &= \frac{1}{g_{\lambda_m} g_{\lambda^{(1)}} \cdots g_{\lambda^{(n-1)}}}, \\
\lambda^{(1)} &= \sigma(\lambda_m), \\
\sigma(\lambda^{(l)}) &= \lambda^{(l-1)}.
\end{align*}
\]

(6.193)
Using (4.109), we rewrite the above correlation function as

\[ <E_{i_1j_1} \otimes \cdots \otimes E_{i_nj_n}>_{\lambda_{n}}^{(\lambda_m)} = B(q) \frac{T_{j_{2n}, \ldots, j_{n}}^{(\lambda_m)}(z_{2n}, \ldots, z_{1} | \mu, \nu)}{\text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-\alpha_0})}, \tag{6.194} \]

where

\[ T_{j_{2n}, \ldots, j_{1}}^{(\lambda_m)}(z_{2n}, \ldots, z_{1} | \mu, \nu) = \text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-\alpha_0} \tilde{ \Phi }_{\lambda_m, V}^{(1)}(\lambda_{2n-1}, j_{2n}) \cdots \tilde{ \Phi }_{\lambda_m, j_{1}}^{(1)}(z_{1})), \]

\[ \lambda^{(1)}(n-l) = \lambda^{(n-l)}, \quad 1 \leq l \leq n, \]

\[ z_{n+l} = q^{-2} z_{n-l+1}, \quad 1 \leq l \leq n, \]

\[ j_{n+l} = k - i_{n-l+1}, \quad 1 \leq l \leq n, \]

\[ B(q) = A(q) \frac{\prod_{l=1}^{n} C_{+, k-i_{l}}^{(k)}}{\prod_{l=1}^{n} C_{+, l}^{(k)}}, \quad C_{+, s}^{(k)} = C_{+, k-m}^{(k)} \text{ if } s \in 2N; \quad C_{+, s}^{(k)} = C_{+, m}^{(k)} \text{ if } s \in 2N + 1. \tag{6.195} \]

The evaluation of this correlation functions reduces then to the evaluations of the traces \( T_{j_{2n}, \ldots, j_{1}}^{(\lambda_m)}(z_{2n}, \ldots, z_{1} | \mu, \nu) \) and \( \text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-\alpha_0}) \). Let us first focus on the calculation of the latter one, which is just the character of \( V(\lambda_m) \). From (5.182)-(5.184), this can be expressed in general as

\[ \text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-\alpha_0}) = \sum_{s \in \mathbb{Z}} (-1)^s \text{Tr}_{\tilde{F}^{[s]}_{\tilde{h}, \tilde{n}}} O^{[s]}, \tag{6.196} \]

where

\[ O^{[s]} = \mu^{-d} \nu^{-\alpha_0}, \]

\[ \text{Tr}_{\tilde{F}^{[s]}_{\tilde{h}, \tilde{n}}} O^{[s]} = \text{Tr}_{F^{[s]}_{\tilde{h}, \tilde{n}}} (\mu^{-d} \nu^{-\alpha_0} \int \frac{dy}{2\pi i} \eta(y) \xi(y')) \]

\[ = \text{Tr}_{F_{\tilde{h}, \tilde{n}}}^{X_{1}, \ldots, X_{s}} (\mu^{-d} X_{1} \nu^{-a_{X_{1}}}) \text{Tr}_{F_{\tilde{h}, \tilde{n}}}^{X_{2}, X_{s}, \ldots, X_{s}} (\mu^{-d} X_{2} - d^{X_{3}} \nu^{-a_{X_{2}}}) \int \frac{dy}{2\pi i} \eta(y) \xi(y'). \tag{6.197} \]

Here

\[ d^{X_{1}} = - \sum_{n>0} \frac{n^{2} a_{X_{1}, -n} a_{X_{1}, n}}{[2n]^{2} ([k+2]n)} - \frac{a_{X_{1}, 0}(a_{X_{1}, 0} + 2)}{4(k+2)}, \]

\[ d^{X_{2}} = \sum_{n>0} \frac{n^{2} a_{X_{2}, -n} a_{X_{2}, n}}{[2n]^{2}} + \frac{a_{X_{2}, 0}(a_{X_{2}, 0} - 2)}{8}, \]

\[ d^{X_{3}} = - \sum_{n>0} \frac{n^{2} a_{X_{3}, -n} a_{X_{3}, n}}{[2n]^{2}} - \frac{a_{X_{3}, 0}(a_{X_{3}, 0} + 2)}{8}. \tag{6.198} \]
Let us note that our physical situation corresponds to $m = \tilde{n} - 1$ and $\tilde{n} = 0$.

Using (5.183), the general expression of a trace of a product of vertex operators (8.302) and the Fourier double transformation formula (8.313) given in the appendix we find

$$\begin{align*}
\text{Tr}_{F_{\tilde{a},\tilde{n}}} (\mu^{-dX_1} \nu^{-aX_1,0}) &= \frac{\nu^{\frac{1}{2}}}{\eta(\mu)} \mu^{PP'(s-\frac{\tilde{\alpha}P+\tilde{\beta}P}{2PP'})^{2}} \nu^{P(s-\frac{\tilde{\alpha}P+\tilde{\beta}P}{2PP'})}, \quad s \in 2\mathbb{Z}, \\
\text{Tr}_{F_{\tilde{a},\tilde{n}}} (\mu^{-dX_1} \nu^{-aX_1,0}) &= \frac{\nu^{\frac{1}{2}}}{\eta(\mu)} \mu^{PP'(s+1+\frac{\tilde{\alpha}P+\tilde{\beta}P}{2PP'})^{2}} \nu^{P(s+1+\frac{\tilde{\alpha}P+\tilde{\beta}P}{2PP'})}, \quad s \in 2\mathbb{Z} + 1, \\
\text{Tr}_{F_{\tilde{a},\tilde{n}}} (\mu^{-dX_2} \nu^{-aX_2,0}) &= \int \frac{dy}{2\pi i} \eta(y)(y') = \frac{q^3 \nu^{-2} \mu^{\frac{1}{2}} \eta(\mu)}{\theta_{\mu}(\nu^{-2})}.
\end{align*}$$

(6.199)

Here the eta function $\eta$ and the Jacobi theta function $\theta_{\mu}(z)$ are given in the appendix (8.304) and (8.305). Putting all contributions together one arrives at

$$\begin{align*}
\text{Tr}_{V_{\lambda_m}} (\mu^{-d\nu^{-a_0}}) &= \frac{q^3 \nu^{-1}}{\theta_{\mu}(\nu^{-2})} \sum_{s \in \mathbb{Z}} (PP'(s-\frac{\tilde{\alpha}P+\tilde{\beta}P}{2PP'})^{2}) \nu^{2P(s-\frac{\tilde{\alpha}P+\tilde{\beta}P}{2PP'})} - \mu^{PP'(s+\frac{\tilde{\alpha}P+\tilde{\beta}P}{2PP'})^{2}} \nu^{2P(s+\frac{\tilde{\alpha}P+\tilde{\beta}P}{2PP'})}.
\end{align*}$$

(6.200)

Up to an overall factor this expression coincides with the Weyl-Kac character formula for the $U_q(sl(2))$ irreducible highest weight module.

Let us now turn to the evaluation of the more complicated $T_{j_2n, \cdots, j_1}(z_2n, \cdots, z_1 | \mu, \nu)$ trace (6.193). For this purpose, we introduce a useful c-function $f$ which will be a building block for most of the expressions that we will encounter henceforth. Let $A =: \prod_{i=1}^{n} A_i :$ and $B =: \prod_{j=1}^{m} B_j :$ be two normal ordered composite operators expressed as products of normal ordered elementary operators $A_i, B_j$, respectively. By elementary operator we mean any operator of type $E_{\epsilon}^{\pm}(z), \ S_{\delta}(z), \ \tilde{\Phi}_{k}^{(b)}(z), \ \tilde{\Phi}_{1}^{(1)}(z), \ \eta(z), \ or \ \xi(z)$. The function $f(A, B)$ is then simply defined by

$$A : B := AB : f(A, B) := AB : \prod_{i=1}^{n} \prod_{j=1}^{m} f(A_i, B_j).$$

(6.201)

Due to (5.153), the above trace can be re-expressed as

$$T_{j_2n, \cdots, j_1}(z_2n, \cdots, z_1 | \mu, \nu) = C_{j_2n, \cdots, j_1}(z_2n, \cdots, z_1) F_{j_2n, \cdots, j_1}(z_2n, \cdots, z_1 | \mu, \nu),$$

(6.202)

with

$$\begin{align*}
F_{j_2n, \cdots, j_1}(z_2n, \cdots, z_1 | \mu, \nu) &= \text{Tr}_{V_{\lambda_m}} (\mu^{-d\nu^{-a_0}} \tilde{\Phi}_{k,j_2n}^{(r_2n)}(z_2n) \cdots \tilde{\Phi}_{k,j_1}^{(r_1)}(z_1)), \\
C_{j_2n, \cdots, j_1}(z_2n, \cdots, z_1) &= g_{\lambda_m, V}^{\lambda_m, V}(z_2n) \cdots g_{\lambda_m, V}^{\lambda_m, V}(z_1), \\
r_i &= k - m \quad \text{if} \quad l \in 2\mathbb{N}; \quad r_i = m \quad \text{if} \quad l \in 2\mathbb{N} + 1.
\end{align*}$$

(6.203)
Now the problem reduces further to the calculation of the normalization functions \( g^{(i)}_{\lambda_{(i-1)}}(z_i) \) and the trace \( F_{\delta_{(i)}}^{(\lambda_{(i-1)})} (z_{2n}, \cdots, z_1 | \mu, \nu) \). To simplify the solution to this problem we use the table of OPE’s (8.284)-(8.290) given in the appendix to re-write each component \( \tilde{\Phi}_{(r_i)}^{(k)}(z_i) \) as

\[
\tilde{\Phi}_{k_j, t_j}(z_l) = \sum_{i(t)} \cdots \sum_{i(t)} \int_{q^{k_1}p_2}^{q^{k_\infty}} \cdots \int_{q^{k_1}p_2}^{q^{k_\infty}} d_{p_1}^{(l)} \cdots d_{p_j}^{(l)} \frac{d_{x_j}^{(l)}}{2\pi i} \frac{d_{x_j}^{(l)}}{2\pi i} \times H_{\delta_1, \cdots, \delta_j, \epsilon_1, \cdots, \epsilon_j}(z_1, t_1, \cdots, t_j, \xi_1, \cdots, \xi_j, \cdots, \xi_{k-1}, C_i) D_l, \tag{6.204}
\]

where

\[
D_l =: \tilde{\Phi}_{k_j}^{(k)}(z_l) S_{\delta_1}^{(l)}(t_1) \cdots S_{\delta_j}^{(l)}(t_j) E_{\epsilon_1}^{(l)}(\xi_1) \cdots E_{\epsilon_j}^{(l)}(\xi_j) : , \tag{6.205}
\]

and \( H_{\delta_1, \cdots, \delta_j, \epsilon_1, \cdots, \epsilon_j}(z_1, t_1, \cdots, t_j, \xi_1, \cdots, \xi_j, \cdots, \xi_{k-1}, C_i) \) is a c-function which can be obtained from (5.154) and the OPE’s given in the Appendix. Moreover, \( C_i \) defines the contours of the above integrals depending on the domains of convergence of the various OPE’s involved in the relation (5.154). Unfortunately, it does not seem possible to write down a simple closed formula for this latter function. However, once the level \( k \) and the vacuum parameter \( m \) are fixed numerically then it becomes relatively simpler to write down such functions. Therefore, we assume that they are known functions with the help of the OPE table. Taking this into account and relation (4.103) the functions \( g^{(i)}_{\lambda_{(i-1)}}(z_i) \) are then determined from the normalizations

\[
g^{(i)}_{\lambda_{(i-1)}}(z_i) \tilde{\Phi}_{k, k-m}^{(m)}(z_1) | m; 0, 0 > | k - m; 0, 0 > \otimes v_{k-m}^{(k)} + \cdots, \quad i = 1, 3, \cdots 2n - 1,
\]

\[
g^{(i)}_{\lambda_{(i-1)}}(z_i) \tilde{\Phi}_{k, k-m}^{(m)}(z_1) | k - m; 0, 0 > | m; 0, 0 > \otimes v_{k-m}^{(k)} + \cdots, \quad i = 2, 4, \cdots 2n . \tag{6.206}
\]

Using the OPE’s table given in the Appendix we find

\[
g^{(i)}_{\lambda_{(i-1)}}(z_i) = h(m)^{-1}, \quad i = 1, 3, \cdots 2n - 1,
\]

\[
g^{(i)}_{\lambda_{(i-1)}}(z_i) = h(k - m)^{-1}, \quad i = 2, 4, \cdots 2n, \tag{6.207}
\]

with

\[
h(m) = \left( \sum_{\delta_1, \cdots, \delta_m, \epsilon_1, \cdots, \epsilon_m} \int_{q^{k_1}p_2}^{q^{k_\infty}} \cdots \int_{q^{k_1}p_2}^{q^{k_\infty}} d_{p_1} \cdots d_{p_1} \frac{d_{x_j}^{(l)}}{2\pi i} \frac{d_{x_j}^{(l)}}{2\pi i} \times H_{\delta_1, \cdots, \delta_m, \epsilon_1, \cdots, \epsilon_m}(z_1, t_1, \cdots, t_m, \xi_1, \cdots, \xi_m, C_i)(z q^k) \frac{m k}{2(k+2)} q^m \sum_{j=1}^{m} \epsilon_j \prod_{j=1}^{m} (t_j q^{-2})^{-\frac{m}{k+2}} \right) . \tag{6.208}
\]
Now the trace $F_{j_2, \ldots, j_1}^{(\lambda_m)}(z_{2n}, \ldots, z_1|\mu, \nu)$ reduces to

$$F_{j_2, \ldots, j_1}^{(\lambda_m)}(z_{2n}, \ldots, z_1|\mu, \nu) = \prod_{l=1}^{2n} \left( \sum_{\delta_1^{(l)}, \ldots, \delta_{i_l}^{(l)}, \epsilon_1^{(l)}, \ldots, \epsilon_{k-j_l}^{(l)}} \int_{q^2_{pz}} q^{2\infty} \frac{d\delta_1^{(l)}(t_1^{(l)}) \cdots d\delta_{i_l}^{(l)}(t_{i_l}^{(l)}) \cdots d\epsilon_1^{(l)}(\xi_1^{(l)}) \cdots d\epsilon_{k-j_l}^{(l)}(\xi_{k-j_l}^{(l)})}{2\pi i} \prod_{2n \geq l_1 > l_2 \geq 1} f(D_{l_1}, D_{l_2}) \right) \times \text{Tr}_{V(\lambda_m)}(\mu^{-d}\nu^{-\alpha}O_{r_1, \ldots, r_{2n}}),$$

(6.209)

with $O_{r_1, \ldots, r_{2n}}$ being defined by

$$O_{r_1, \ldots, r_{2n}} = \prod_{l=1}^{2n} \Phi_k^{(l)}(z_l)S_{\delta_1^{(l)}(t_1^{(l)})} \cdots S_{\delta_{i_l}^{(l)}(t_{i_l}^{(l)})}E_{\epsilon_1^{(l)}(\xi_1^{(l)})} \cdots E_{\epsilon_{k-j_l}^{(l)}(\xi_{k-j_l}^{(l)})}.$$

(6.210)

The function $f(D_{l_1}, D_{l_2})$ can be read off from the OPE’s (8.284)-(8.290) of elementary operators given in the appendix.

The problem then reduces further to the calculation of the trace $\text{Tr}_{V(\lambda_m)}(\mu^{-d}\nu^{-\alpha}O_{r_{2n}, \ldots, r_1})$. To this end, we apply the method of the previous section, that is,

$$\text{Tr}_{V(\lambda_m)}(\mu^{-d}\nu^{-\alpha}O_{r_1, \ldots, r_{2n}}) = \sum_{s \in \mathbb{Z}} (-1)^s \text{Tr}_{F_{\tilde{\Phi}, n}}^{[s]}(\mu^{-d}\nu^{-\alpha}O_{r_1, \ldots, r_{2n}} \oint \frac{dy}{2\pi i} \eta(y)\xi(y)).$$

(6.211)

Using (5.180), we define the $O_{s}^{[s]}$ as

$$O_{r_1, \ldots, r_{2n}}^{[s]} = O_{r_{2n}, \ldots, r_1},$$

$$O_{r_1, \ldots, r_{2n}}^{[s]} = A_{\Phi, k}^{s/p}O_{r_{2n}, \ldots, r_1}, \quad \text{if } s \in 2\mathbb{Z},$$

$$O_{r_1, \ldots, r_{2n}}^{[s]} = A_{\Phi, k}^{(k+sp-p)}O_{r_{2n}, \ldots, r_1}, \quad \text{if } s \in 2\mathbb{Z} + 1.$$

(6.212)

This means that it is sufficient to calculate the following trace for a fixed even number $s$:

$$R(s, r_{2n}, \ldots, r_1) = Tr_{F_{\tilde{\Phi}, n'}}^{[s]}(\mu^{-d}\nu^{-\alpha}O_{r_{2n}, \ldots, r_1} \oint \frac{dy}{2\pi i} \eta(y)\xi(y)).$$

(6.213)

The modes of $X_2$ and $X_3$ satisfy the same Heisenberg algebra but with opposite signatures. Therefore one expects many simplifications if they are treated simultaneously. For this reason, it is convenient to split the trace into the product of two parts. One involves only the modes of $X_1$ and the other involves only those of $X_2$ and $X_3$, i.e.,

$$R(s, r_{2n}, \ldots, r_1) = R^{X_1}(s, r_{2n}, \ldots, r_1)R^{X_2, X_3}(r_{2n}, \ldots, r_1)$$

(6.214)
where

\[ R^{X_1}(s;r_{2n}, \ldots, r_1) = \text{Tr}_{\mathcal{E}_{\delta_1}^{X_1}} \left( \mu^{-d_{X_1}} \nu^{-a_{X_1}a} \mathcal{O}_{X_1} \right), \]

\[ R^{X_2,X_3}(r_{2n}, \ldots, r_1) = \text{Tr}_{\mathcal{E}_{\delta_1}^{X_2,X_3}} \left( \mu^{-d_{X_2} - d_{X_3}} \nu^{-a_{X_2}a} \mathcal{O}_{X_2,X_3} \right) \int \frac{dy}{2\pi i} \eta(y) \xi(y'), \]

(6.215)

with

\[ \mathcal{O}_{X_1}^{r_1, \ldots, r_{2n}} = \prod_{l=1}^{2n} \Phi_k^{(k)}(z_l) S_{X_1}^{X_1}(t_1^{(l)}) \cdots S_{X_1}^{X_1}(t_{r_l}^{(l)}) E_{X_1}^{X_1}(\xi_1^{(l)}) \cdots E_{X_1}^{X_1}(\xi_{l_{k-j_1}}^{(l)}); \]

\[ \mathcal{O}_{X_2,X_3}^{r_1, \ldots, r_{2n}} = \prod_{l=1}^{2n} S_{X_1}^{X_2,X_3}(t_1^{(l)}) \cdots S_{X_1}^{X_2,X_3}(t_{r_l}^{(l)}) E_{X_1}^{X_2,X_3}(\xi_1^{(l)}) \cdots E_{X_1}^{X_2,X_3}(\xi_{l_{k-j_1}}^{(l)}); \]

(6.216)

Here we have introduced the notations

\[ E^{-X_1}_c(\xi) = \exp(\partial X_1^{(c)}(q^{-2}\xi; -\frac{k+2}{2})) \]

\[ E^{-X_2,X_3}_c(\xi) = \exp(X_2(2|q^{(c-1)(k+2)}\xi; -1) + X_3(2|q^{(c-1)(k+1)}\xi; 0)) \]

\[ S_{X_1}^{X_1}(t) = \exp(-X_1(k + 2|q^{-2}t; -\frac{k+2}{2})) \]

\[ S_{X_2,X_3}^{X_1}(t) = \exp(-X_2(2|q^{-k-2}t; -1) - X_3(2|q^{-k-2+\delta}t; 0)). \]

(6.217)

Using the general traces given in relations \((8.297)\) and \((8.302)\) of the appendix, we find then

\[ R^{X_1}(s;r_{2n}, \ldots, r_1) = T^{X_1}_{\ell_s}(2n, 0, N, M, n_4|X, Y, Z, W), \]

\[ R^{X_2,X_3}(r_{2n}, \ldots, r_1) = T^{X_2,X_3}(A, B, C, D, G, H|X, Y, Z, W), \]

(6.218)

where

\[ \ell_s = \ell_{n-P_s,n'}, \]

\[ N = \sum_{l=1}^{2n} r_l = nk, \]

\[ M = 2kn - \sum_{l=1}^{t+1} j_l = nk + \sum_{l=1}^{t_n} i_l - \sum_{l=1}^{t_m} j_l; \]

\[ t_{a+\sum_{l=1}^{r_l} r_l} = t_a^{(l)}, \quad 1 \leq a \leq r_l, \]

\[ \xi_{a+\sum_{l=1}^{r_l}(k-j_l)} = \xi_a^{(l)}, \quad 1 \leq a \leq k - j_l, \]

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Though these expressions are cumbersome they still allow us to deduce a simple selection rule. Indeed, from the trace formulas in the appendix, we have

\[ R^{X_1}(s; r_{2n}, \cdots, r_1) = T^X_1(2n, 0, N, M, n_4|X, Y, Z, W) = \delta_{2nk, 2N}(\cdots), \]

\[ R^{X_2, X_3}(r_{2n}, \cdots, r_1) = T^{X_2, X_3}(A, B, C, D, G, H|\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = \delta_{N, M}(\cdots). \] (6.220)

The first constraint \( N = nk \) is trivially satisfied, and the second one, which less trivial reads

\[ \sum_{l=1}^{n} i_l = \sum_{l=1}^{n} j_l. \] (6.221)

Thus, only local operators which satisfy the latter sum rule, i.e., conserve the spin, lead to nonvanishing correlation functions. This selection rule is a generalization of the spin \( 1/2 \) case [27].

Let us now recapitulate the explicit result of the latter calculation. The correlation function \( < E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} |_{(\lambda_m)_{2n, \cdots, z_1}} > \) of the basis local operator \( E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} \) is given
by

\[ \langle E_{n_1 j_1} \otimes \cdots \otimes E_{i_1 j_1} \rangle_{Z, \ldots, Z_1} = \frac{B(q)}{\text{Tr}_{V(\Lambda_0)}(\mu^{-d\nu-\alpha_0})} C_{j_2 n, \ldots, j_1} (Z_{2n}, \ldots, z_1) \times \prod_{l=1}^{2n} \left\{ \sum_{\delta_l \cdots \delta_l} \sum_{t_l, \ldots, t_l} \int_{q^k \infty} d_{l_{\delta_l}} \cdot \cdots \cdot \int_{q^k \infty} d_{l_{\delta_l}} \cdot \int_{q^k \infty} d_{l_{\delta_l}} \cdot \frac{d_{l_{\delta_l}}}{2\pi i} \cdot \cdots \cdot \frac{d_{l_{\delta_l}}}{2\pi i} \right\} \]

\[ \times H_{\delta_l \cdots \delta_l} (z_1, t_1, \ldots, t_1, \xi_1, \ldots, \xi_1, \xi_{k-j_1}, C_l) \prod_{2n \geq l_1 > l_2 \geq 1} f(D_{l_1}, D_{l_2}) \times \left( \sum_{s \in 2Z} A_{\Phi,k}^{(s, p)} R(s, r_{2n}, \ldots, r_1) - \sum_{s \in 2Z+1} A_{\Phi,k}^{(s, p)} R(s, k - r_{2n}, \ldots, k - r_1) \right), \]

(6.222)

where all the various symbols are defined throughout the preceding analysis and all the \( z_i \) are set to 1. Moreover, for this correlation function to be non-vanishing the selection rule \( \sum_{i=1}^{n} i_l = \sum_{i=1}^{n} j_i \) must be fulfilled.

Let us now briefly check, in the simplest case \( s = 1/2 \) that the leading order term of the 1-point correlation function coincides with the known result. More specifically, we consider the 1-point correlation function of the local operator \( E_{00} \) in the sector \( \Lambda_0 \). This means that this case corresponds to the specializations

\[ k = 1, \]
\[ P = k + 2 = 3, \]
\[ P' = 1, \]
\[ \tilde{n} = 1, \]
\[ \tilde{n} = 0. \quad (6.223) \]

Following the general prescription this correlation function is defined by

\[ \langle E_{00} \rangle_{\Lambda_0} = P_0(q^{-2z}, z), \]

(6.224)

where

\[ P_0(z, w) = \frac{1}{g_{\Lambda_0}} \frac{\text{Tr}_{V(\Lambda_0)}(\mu^{-d\nu-\alpha_0}) \check{\Phi}_{\Lambda_0,V,0}(zq^2) \check{\Phi}_{\Lambda_1,V,0}(w)}{\text{Tr}_{V(\Lambda_0)}(\mu^{-d\nu-\alpha_0})} \]
\[ = -q \frac{\text{Tr}_{V(\Lambda_0)}(\mu^{-d\nu-\alpha_0}) \check{\Phi}_{\Lambda_0,V,0}(z) \check{\Phi}_{\Lambda_1,V,0}(w)}{g_{\Lambda_0} \text{Tr}_{V(\Lambda_0)}(\mu^{-d\nu-\alpha_0})} \]
\[ = -q \frac{g_{\Lambda_0} \check{\Phi}_{\Lambda_0,V,0}(z) \check{\Phi}_{\Lambda_0,V,0}(w)}{g_{\Lambda_0} \text{Tr}_{V(\Lambda_0)}(\mu^{-d\nu-\alpha_0})} \frac{\text{Tr}_{V(\Lambda_0)}(\mu^{-d\nu-\alpha_0}) \check{\Phi}_{\Lambda_1,V,0}(w)}{\text{Tr}_{V(\Lambda_0)}(\mu^{-d\nu-\alpha_0})}, \quad (6.225) \]
with
\[
\begin{align*}
\mu &= q^4, \\
\nu &= q, \\
g_{\Lambda_0} &= \frac{(q^4; q^4)_\infty}{(q^2; q^4)_\infty}.
\end{align*}
\] (6.226)

Here the component vertex operators are explicitly given by
\[
\begin{align*}
\tilde{\Phi}^{(1)}_{1,1}(z) &= \int_{qz}^{qz\infty} dp t S(t) \Phi_k(z), \\
\tilde{\Phi}^{(0)}_{1,0}(z) &= \oint d\xi \quad \left[ \Phi_k(z), E^-(\xi) \right]_q, \\
g^{\Lambda_0, V}_{\Lambda_1}(z) &= \frac{q^{-\frac{5}{6}z^2}}{B_p\left(\frac{2}{3}, \frac{2}{3}\right)}, \\
g^{\Lambda_1, V}_{\Lambda_0}(z) &= 1,
\end{align*}
\] (6.227)

where \( p = q^6 \) and the quantum beta function \( B_p(x, y) \) is defined in the appendix (8.308).

The normalization constants are determined from
\[
\begin{align*}
\tilde{\Phi}^{\Lambda_1; V}_{\Lambda_0}(z)|\Lambda_0 &= |\Lambda_1 > + \cdots, \\
\tilde{\Phi}^{\Lambda_0, V}_{\Lambda_1}(z)|\Lambda_1 &= |\Lambda_0 > + \cdots.
\end{align*}
\] (6.228)

The latter relations translate into
\[
\begin{align*}
g^{\Lambda_1, V}_{\Lambda_0}(z)\tilde{\Phi}^{(0)}_{1,1}(z)|\Lambda_0 &= |\Lambda_1 > + \cdots, \\
g^{\Lambda_0, V}_{\Lambda_1}(z)\tilde{\Phi}^{(1)}_{1,0}(z)|\Lambda_1 &= |\Lambda_0 > + \cdots.
\end{align*}
\] (6.229)

From the Lefschetz formula and (5.212) we have
\[
Tr_{V(\Lambda_0)}(\mu^{-d}\nu^{-\alpha_0}\tilde{\Phi}^{(1)}_{1,1}(z)\tilde{\Phi}^{(0)}_{1,0}(w)) = \sum_{s \in \mathbb{Z}} (-1)^s Tr_{F^{[s]}_{\Lambda_0, \Lambda_1}} \left( \mathcal{O}^{[s]} \oint d\eta(y) \eta(y') \right),
\] (6.230)

with
\[
\begin{align*}
\mathcal{O} &= \mathcal{O}^{[0]} = \mu^{-d}\nu^{-\alpha_0}\tilde{\Phi}^{(1)}_{1,1}(z)\tilde{\Phi}^{(0)}_{1,0}(w), \\
\mathcal{O}^{[s]} &= A_{\ell}^{1+2n_1+(s-1)p}\mu^{-d}\nu^{-\alpha_0}\tilde{\Phi}^{(1)}_{1,1}(z)\tilde{\Phi}^{(0)}_{1,0}(w), \quad s \in 2\mathbb{Z} + 1, \\
\mathcal{O}^{[s]} &= A_{\ell}^{sp}\mu^{-d}\nu^{-\alpha_0}\tilde{\Phi}^{(1)}_{1,1}(z)\tilde{\Phi}^{(0)}_{1,0}(w), \quad s \in 2\mathbb{Z}.
\end{align*}
\] (6.231)
Since the vacuum state $|\Lambda_0>$ has the lowest degree with respect to $-d$, which is zero, clearly the leading term in $q$ (in the limit $q \to 0$) of the 1-point function must coincide with the leading term of the $s = 0$ term. The latter, in turn, is obtained from the leading term of the following matrix element of vertex operators:

$$
\langle \Lambda_0 | \tilde{\Phi}^{\Lambda_0, V}_{\Lambda_1} (z) \tilde{\Phi}^{\Lambda_1, V}_{\Lambda_0, 0} (w) | \Lambda_0 >_F = g^{\Lambda_0, V}_{\Lambda_1} (z) g^{\Lambda_1, V}_{\Lambda_0, 0} (w) \int_{qz}^\infty d_p t S(t) \Phi (z) \int \frac{d \xi}{2 \pi i} [\Phi (\xi), \Phi (z)] \int \frac{dy}{2 \pi i} \eta(y) \xi (y') | 0 >
$$

This means that

$$
P_0 (zq^{-2}, z) \sim \frac{q^2}{g_{\Lambda_0}}.
$$

and therefore the leading term of the physical 1-point correlation function is given by

$$
P_0 (q^{-2}, z) \sim O(q^2),
$$

and coincides with the known result [15, 40, 31]. Note that up to a factor $q^{-3}$ originating from the OPE of $\eta(y)$ and $\xi(y')$ this matrix element is equal to the matrix element of the vertex operators acting on the unrestricted Fock space, i.e.,

$$
\langle \Lambda_0 | \tilde{\Phi}^{\Lambda_0, V}_{\Lambda_1, 1} (z) \tilde{\Phi}^{\Lambda_1, V}_{\Lambda_0, 0} (w) | \Lambda_0 >_F = g^{\Lambda_0, V}_{\Lambda_1, 1} (z) g^{\Lambda_1, V}_{\Lambda_0, 0} (w) \times < 0 | \int_{qz}^\infty d_p t S(t) \Phi_1 (z) \int \frac{d \xi}{2 \pi i} [\Phi_1 (z), \Phi^{-} (\xi)] | 0 >
$$

$$
= -q^4 \frac{(q^6 w / z; q^4)_{\infty}}{(q^4 w / z; q^4)_{\infty}}
$$

$$
= q^{-3} < \Lambda_0 | \tilde{\Phi}^{\Lambda_0, V}_{\Lambda_1, 1} (z) \tilde{\Phi}^{\Lambda_1, V}_{\Lambda_0, 0} (w) | \Lambda_0 >_F.
$$

The reason is that the vertex operators are BRST invariant and the highest weight states $|\Lambda_i>$ are BRST states and therefore the above relation is expected.

### 6.2 N-point form factors of local operators

In this section we derive the N-point form factors of local operators. Although the physically interesting ones are those where local operators act on a single site, we consider the most
general case where local operators act on a set of neighboring sites. Let us recall that the physical type II vertex operators are the spin-1/2 ones. They are bosonized as follows:

$$\tilde{\Psi}_{\lambda}^{V^{(1)}(1),\lambda_{\pm 1}}(z) = h_{\lambda}^{V^{(1)}(1),\lambda_{\pm 1}}(z) \sum_{m=0}^{1} v_{m}^{(1)} \otimes \tilde{\Psi}_{1,m}^{(r_{\pm})}(z), \quad (6.236)$$

with $r_{+} = 0$ and $r_{-} = 1$, and

$$\tilde{\Psi}_{1,m}^{(0)}(z) = \tilde{\Psi}_{1,m}^{(1)}(z),$$
$$\tilde{\Psi}_{1,m}^{(1)}(z) = \int_{0}^{q^{k-1}z} d_{p} t \tilde{\Psi}_{m}^{(1)}(z) S(t). \quad (6.237)$$

The bosonizations of the bare components are derived from (5.165), that is,

$$\tilde{\Psi}_{1,0}^{(0)}(z) = \tilde{\Psi}_{0}^{(1)}(z) = \exp(X_{1}(1; 2, k + 2 |q^{-2}z; - \frac{k + 2}{2}) + X_{2}(2 |q^{-2}z; 0) + X_{3}(2 |q^{-2}z; 0)),$$
$$\tilde{\Psi}_{1,0}^{(1)}(z) = \oint \frac{dq}{2\pi i} [\tilde{\Psi}_{0}^{(1)}(z), E^{+}(u)] q,$$
$$\tilde{\Psi}_{1,1}^{(0)}(z) = \int_{0}^{q^{k-1}z} d_{p} t \tilde{\Psi}_{0}^{(1)}(z) S(t),$$
$$\tilde{\Psi}_{1,1}^{(1)}(z) = \int_{0}^{q^{k-1}z} d_{p} t \oint \frac{dq}{2\pi i} [\tilde{\Psi}_{0}^{(1)}(z), E^{+}(u)] q S(t). \quad (6.238)$$

Moreover, one can easily check that

$$[\tilde{\Psi}_{0}^{(l)}(z), \eta_{0}] = 0,$$
$$[\tilde{\Psi}_{0}^{(l)}(z), a_{X_{2},0} + a_{X_{3},0}] = 0, \quad (6.239)$$

and hence the screened components intertwine the restricted Fock spaces as

$$\tilde{\Psi}_{1,m}^{(r)}(z) : \tilde{F}_{1} \rightarrow V^{(1)}(z) \otimes \tilde{F}_{2}. \quad (6.240)$$

Just as in the case of Type I vertex operators, it is convenient to write the bosonized bare components of the vertex operators in the following form:

$$\tilde{\Psi}_{1,0}^{(0)}(z) = \sum_{\epsilon' = \pm 1} \oint \frac{du}{2\pi i} I_{\epsilon'}^{(0),(1)}(z, u|C) : \tilde{\Psi}_{0}^{(1)}(z) E^{+}_{\epsilon'}(u) :,$$
$$\tilde{\Psi}_{1,0}^{(1)}(z) = \sum_{\delta' = \pm 1} \int_{0}^{q^{k-1}z} d_{p} t I_{\delta'}^{(1),(0)}(z, t|C) : \tilde{\Psi}_{0}^{(1)}(z) S_{\delta'}(t) :,$$
$$\tilde{\Psi}_{1,1}^{(1)}(z) = \sum_{\delta' = \pm 1} \sum_{\epsilon' = \pm 1} \int_{0}^{q^{k-1}z} d_{p} t \oint \frac{du}{2\pi i} I_{\delta'}^{(1),(1)}(z, t, u|C) : \tilde{\Psi}_{0}^{(1)}(z) S_{\delta'}(t) E^{+}_{\epsilon'}(u) :. \quad (6.241)$$

53
The various c-functions \( I \) and corresponding contours \( C \) are given in the appendix (5.291).

As mentioned previously any local operator can be written as a linear combination of the basis local operators \( E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} \), and therefore we consider only the following general form factor:

\[
< \lambda_m | E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} | \rho_{n'}, \cdots, \rho_1 >^{\beta_{n'}, \cdots, \beta_1}_{1-\alpha_{n'}, \cdots, 1-\alpha_1} = \frac{\text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-a_0} E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} \Psi_{X_{n', \cdots, 1-\alpha_{n'}}^V}^{V', \cdots, \alpha_1}(\rho_{n'}) \cdots \Psi_{X_{1, \cdots, 1-\alpha_1}^V}(\rho_1))}{\text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-a_0})}
\]

\[
= B(q) \prod_{i=1}^{n'} d_{\alpha_i, \beta_i} \frac{\text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-a_0} \Phi_{\lambda(2n-1), \cdots, 1}^{(i)}(z_{2n}) \cdots \Phi_{\lambda(0), \cdots, 1}^{(i)}(z_1) \Psi_{X_{n', \cdots, 1-\alpha_{n'}}^V}^{V', \cdots, \alpha_1}(\rho_{n'}) \cdots \Psi_{X_{0, \cdots, 1}^V}(\rho_1))}{\text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-a_0})}
\]

(6.242)

where

\[
\Psi_{X_{i-1, \cdots, 1-\alpha_i}^V}(\rho_i) = d_{\alpha_i, \beta_i} \Psi_{X_{i-1, \cdots, 1-\alpha_i}^V}(\bar{\rho}_i), \quad 1 \leq i \leq n',
\]

\[
d_{1,1} = 1, \quad d_{1,-1} = -q^{-1}, \quad d_{0,1} = -q, \quad d_{0,-1} = 1,
\]

\[
\bar{\rho}_i = q^2 \rho_i,
\]

\[
\alpha_i = 0, 1;
\]

\[
\lambda_i' = \lambda_{1-i} + \beta_i (\Lambda_1 - \Lambda_0), \quad \beta_i = \pm 1, \quad 1 \leq i \leq n',
\]

\[
\lambda_0' = \lambda_{n'} = \lambda_m,
\]

(6.243)

and all other notations are defined in (5.193). We have already evaluated the trace in the denominator of the above form factor. Let us then focus on the trace in the numerator, i.e.,

\[
T_1 = \text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-a_0} \Phi_{\lambda(2n-1), \cdots, 1}^{(i)}(z_{2n}) \cdots \Phi_{\lambda(0), \cdots, 1}^{(i)}(z_1) \Psi_{X_{n', \cdots, 1-\alpha_{n'}}^V}^{V', \cdots, \alpha_1}(\bar{\rho}_{n'}) \cdots \Psi_{X_{0, \cdots, 1}^V}(\bar{\rho}_1))
\]

\[
= C_{j_2, \cdots, j_1}(z_{2n}, \cdots, z_1) \left( \prod_{i=1}^{n'} h^{\lambda_i}_{\lambda_{i-1}}(\bar{\rho}_i) \right) \times T_2
\]

(6.244)

where we have used (5.236) and (5.202) to reduce the trace down to

\[
T_2 = \text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-a_0} \Phi_{k, j_2, \cdots, 1}^{(r_2)}(z_{2n}) \cdots \Phi_{k, j_1}^{(r_1)}(z_1) \Psi_{1, \cdots, 1}^{(r_{n'})}(\bar{\rho}_{n'}) \cdots \Psi_{1, \cdots, 1}^{(r_1)}(\bar{\rho}_1)),
\]

(6.245)

where \( r_i = (1 - \beta_i)/2 \). Because any order of products of vertex operators can be obtained from an initial one through the commutation relations (4.117), let us assume for simplicity
that in the above trace, type II vertex operators are ordered in the following manner:

\[
\Psi_{1,a''}(\bar{\rho}_{a''}) \cdots \Psi_{1,a_1}(\bar{\rho}_{a_1}) = \Psi_{1,0}(\bar{\rho}_{a''}) \cdots \Psi_{1,0}(\bar{\rho}_{a_1}) \\
\times \Psi_{1,1}(\bar{\rho}_{a+b+c}) \cdots \Psi_{1,1}(\bar{\rho}_{a+b+1}) \Psi_{1,0}(\bar{\rho}_{a+b}) \cdots \Psi_{1,0}(\bar{\rho}_{a_1}) \Psi_{1,1}(\bar{\rho}_a) \cdots \Psi_{1,1}(\bar{\rho}_1). \tag{6.246}
\]

We can then simply normal order the latter operator as

\[
\Psi_{1,a''}(\bar{\rho}_{a''}) \cdots \Psi_{1,a_1}(\bar{\rho}_{a_1}) = \prod_{i=1}^{a'} \left( \sum_{\delta_{i'}, \epsilon_{i'}}^q \int_0^{q^{-1}z} d_p t_i(a) \int \frac{d u_i(a)}{2 \pi i} f(\bar{\rho}_i, t_i(a), u_i(a) | C) \right) \\
\times \prod_{i=a+1}^{b} \left( \sum_{j_{i'}, \epsilon_{i'}}^q \int_0^{q^{-1}z} d_p t_i(b) I_{\delta_{i'}, \epsilon_{i'}}(\bar{\rho}_i, t_i(b) | C) \right) \prod_{i=b+1}^{c} \left( \sum_{\epsilon_{i'}} f(\bar{\rho}_i, u_i(c) | C) \right) \\
\times \prod_{1 \leq i' < i \leq a'} f(K_{i, K_{i'}} \mathcal{O}_{r_{i'}} \cdots r_i) \tag{6.247}
\]

where we have introduced the following notations for the normal ordered composite operators:

\[
\mathcal{O}_{r_{a''}, \cdots r_{a_1}} = \prod_{i=1}^{a'} K_i, \quad K_i = \begin{cases} \Psi_0(\bar{\rho}_i) S_{\delta_i}(t_i(a)) E_{\epsilon_i(i)}(u_i(a)) : & 1 \leq i \leq a, \\
\Psi_0(\bar{\rho}_i) S_{\delta_i}(t_i(b)) : & a+1 \leq i \leq a+b, \\
\Psi_0(\bar{\rho}_i) E_{\epsilon_i(i)}(u_i(c)) : & a+b+1 \leq i \leq a+b+c, \\
\Psi_0(\bar{\rho}_i) & a+b+c+1 \leq i \leq a'. \tag{6.248}
\end{cases}
\]

Using (6.204) and the latter normal ordering, we express \(T_2\) as

\[
T_2 = \prod_{i=1}^{2n} \left( \sum_{\xi_{1}^{(0)}, \cdots, \xi_{1}^{(\ell_{1})}} \int \frac{d \xi_{1}^{(l_1)}}{2 \pi i} \cdots \int \frac{d \xi_{1}^{(l_1)}}{2 \pi i} \right) \prod_{2n \geq l_1 \geq 1} f(D_{l_1}, D_{l_2}) \\
\times H_{\xi_{1}^{(0)}, \cdots, \xi_{1}^{(\ell_{1})}}(\xi_{1}^{(l_1)}, \cdots, \xi_{1}^{(l_1)}, \xi_{2}^{(l_1)}, \cdots, \xi_{2}^{(l_1)}, \xi_{3}^{(l_1)}, \cdots, \xi_{3}^{(l_1)}, C_1) \prod_{2n \geq l_1 > l_2 \geq 1} f(D_{l_1}, D_{l_2}) \\
\times \left( \prod_{i=1}^{a} \left( \sum_{\delta_{i'}, \epsilon_{i'}} \int_0^{q^{-1}z} d_p t_i(a) \int \frac{d u_i(a)}{2 \pi i} f(\bar{\rho}_i, t_i(a), u_i(a) | C) \right) \right) \\
\times \left( \prod_{i=a+1}^{b} \left( \sum_{j_{i'}, \epsilon_{i'}} \int_0^{q^{-1}z} d_p t_i(b) I_{\delta_{i'}, \epsilon_{i'}}(\bar{\rho}_i, t_i(b) | C) \right) \right) \\
\times \left( \prod_{i=b+1}^{c} \left( \sum_{\epsilon_{i'}} f(\bar{\rho}_i, u_i(c) | C) \right) \right) \\
\times \prod_{1 \leq i' < i \leq a'} f(K_{i, K_{i'}} \mathcal{O}_{r_{i'}} \cdots r_i) \times T_3. \tag{6.249}
\]
In this expression which term is the integrand of which integral should be clear from the various indices. The trace \( T_3 \) is given by

\[
T_3 = \text{Tr}_{V(\lambda_m)}(\mu^{-d} \nu^{-\alpha a} O_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'})
\]

\[
= \sum_{s \in \mathbb{Z}} (-1)^s \text{Tr}_{F_E^{[s]}}(\mu^{-d} \nu^{-\alpha a} O^{[s]}_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'}) \oint \frac{dy}{2\pi i} \eta(y) \xi(y').
\]  

(6.250)

Here the operators \( O^{[s]}_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'} \) are defined through (5.186) and (5.188 as follows:

\[
O^{[0]}_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'} = O_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'},
\]

\[
O^{[s]}_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'} = \mathcal{A}_{\Phi_1}^{n(k+sp-p)} \mathcal{A}_{\Phi, 1}^{s p m/2} O_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'}, \quad s \in 2\mathbb{Z}
\]

\[
O^{[s]}_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'} = \mathcal{A}_{\Phi, k}^{n(k+sp-p)} \mathcal{A}_{\Phi_1}^{(2n-n'+(s-1)p+1)n'/2+2} \sum_{l=1}^{n'-1} l r_{l+1}^{n'} O_{1-r_{2n}, \ldots, 1-r_1; 1-r_{n'}, \ldots, 1-r_1'}.
\]

(6.251)

Let us note that the last relation is satisfied if this condition on the various parameters \( r'_i \) holds:

\[
2 \sum_{l=1}^{n'} r'_i = n'.
\]

(6.252)

Here also, in order to evaluate the above trace it is sufficient to calculate the following one for a fixed even number \( s \):

\[
S(s, r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1') = \text{Tr}_{F_E^{[s]}}(O_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'}) \oint \frac{dy}{2\pi i} \eta(y) \xi(y').
\]

(6.253)

Again, it is convenient to split the trace into the product of two parts. One involves only the modes of \( X_1 \) and the other involves only those of \( X_2 \) and \( X_3 \), i.e.,

\[
S(s, r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1') = S^{X_1}(s, r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1') S^{X_2, X_3}(r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'),
\]

(6.254)

where

\[
S^{X_1}(s, r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1') = \text{Tr}_{F_{X_1}^{[s]}}(\mu^{-d X_1} \nu^{-a X_1 \alpha} O^{X_1}_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'}),
\]

\[
S^{X_2, X_3}(r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1') = \text{Tr}_{F_{X_2, X_3}^{[s]}}(\mu^{-d X_2 - d X_3} \nu^{-a X_2 \alpha} O^{X_2, X_3}_{r_{2n}, \ldots, r_1; r_{n'}, \ldots, r_1'}) \oint \frac{dy}{2\pi i} \eta(y) \xi(y').
\]

(6.255)
and

\[
\mathcal{O}^{X_1}_{r_{2n}, \ldots, r_1; r'_{2n}, \ldots, r'_1} = \mathcal{O}^{X_1}_{r_{2n}, \ldots, r_1} \prod_{i=1}^{n'} \tilde{\Psi}_0^{(1), X_1}(\bar{\rho}_i) \times \prod_{i=1}^{a} S^{X_1}_{\delta^{(a)_i}}(t^{(a)}_i) \times \prod_{i=a+1}^{b} S^{X_1}_{\delta^{(b)_i}}(t^{(b)}_i);
\]

\[
\mathcal{O}^{X_2, X_3}_{r_{2n}, \ldots, r_1; r'_{2n}, \ldots, r'_1} = \mathcal{O}^{X_2, X_3}_{r_{2n}, \ldots, r_1} \prod_{i=1}^{n'} \tilde{\Psi}_0^{(1), X_2, X_3}(\bar{\rho}_i) \times \prod_{i=1}^{a} S^{X_2, X_3}_{\delta^{(a)_i}}(t^{(a)}_i) \times \prod_{i=a+1}^{b} S^{X_2, X_3}_{\delta^{(b)_i}}(t^{(b)}_i)
\times \prod_{i=1}^{a} E^{+}_{\epsilon_i}(u^{(a)}_i) \times \prod_{i=a+1}^{a+b+c} E^{+}_{\epsilon_i}(u^{(c)}_i),
\]

(6.256)

with

\[
\tilde{\Psi}_0^{(1), X_1}(\bar{\rho}) = \exp(X_1(1; 2, k + 2|q^{k-2}\bar{\rho}; -\frac{k+2}{2})),
\]

\[
\tilde{\Psi}_0^{(1), X_2, X_3}(\bar{\rho}) = \exp(X_2(2|q^{-2}\bar{\rho}; 0) + X_3(2|q^{-2}\bar{\rho}; 0)).
\]

(6.257)

Using the latter expressions and the general trace formulas in the appendix we find then

\[
S^{X_1}(s, r_{2n}, \ldots, r_1; r'_{2n}, \ldots, r'_1) = T^{X_1}_{\ell_s}(2n, n', N + n_s, M, n_4|X, Y, Z, W),
\]

\[
S^{X_2, X_3}(r_{2n}, \ldots, r_1; r'_{2n}, \ldots, r'_1) = T^{X_2, X_3}(A, B, C, D, G, H|\bar{X}, \bar{Y}, \bar{Z}, \bar{W}),
\]

(6.258)

with

\[
\ell_s = \ell_{\tilde{n}-P, \tilde{n}} = \tilde{n} - Ps - 1 - \tilde{n}\frac{P}{P_t},
\]

\[
N = \sum_{l=1}^{2n} r_l = nk,
\]

\[
M = 2kn - \sum_{l=1}^{2n} j_l = nk + \sum_{l=1}^{n} i_l - \sum_{l=1}^{n} j_l,
\]

\[
t_{\sum_{l=1}^{l-1} r_l} = t_{\sum_{l=1}^{l-1} r_l}^{(l)}, \quad 1 \leq g \leq r_l,
\]

\[
t_{N+i} = t_i^{(a)}, \quad 1 \leq i \leq a,
\]

\[
t_{N+a+i} = t_i^{(b)}, \quad 1 \leq i \leq b,
\]

\[
\xi_{g+\sum_{l=1}^{l-1} (k-j_l)} = \xi_{g+\sum_{l=1}^{l-1} (k-j_l)}^{(l)}, \quad 1 \leq g \leq k - j_l,
\]

\[
u_i = u_i^{(a)}, \quad 1 \leq i \leq a,
\]

\[
u_{a+i} = u_i^{(c)}, \quad 1 \leq i \leq c,
\]

\[
\delta_{g+\sum_{l=1}^{l-1} r_l} = \delta_{g+\sum_{l=1}^{l-1} r_l}^{(l)}, \quad 1 \leq g \leq r_l,
\]

\[
\delta_{N+i} = \delta_i^{(a)}, \quad 1 \leq i \leq a,
\]

\[
\delta_{N+a+i} = \delta_i^{(b)}, \quad a + 1 \leq i \leq b.
\]
\[ \epsilon_i' = \epsilon_i^{(a)}, \quad 1 \leq i \leq a, \]
\[ \epsilon_{a+i}' = \epsilon_i^{(c)}, \quad 1 \leq i \leq c, \]
\[ X = \{ z_1, \ldots, z_{2n} \}, \]
\[ Y = \{ \bar{\rho}_1, \ldots, \bar{\rho}_n \}, \]
\[ Z = \{ t_1, \ldots, t_{N+a+b} \}, \]
\[ W = \{ \xi_1, \ldots, \xi_M \}, \]
\[ A = \{ \delta_1 - k - 2, \ldots, \delta_{N+a+b} - k - 2, \epsilon_1^{(c)} - 1 - 2, \ldots, \epsilon_{a+c}^{(c)} - k - 2 \}, \]
\[ B = \{ (\epsilon_1 - 1)(k+1) - 1, \ldots, (\epsilon_M - 1)(k+1) - 1, -2, \ldots, -2 \}, \]
\[ C = \{ -k - 2, \ldots, -k - 2 \}, \]
\[ D = \{ -1, \ldots, -1, 1, \ldots, 1 \}, \]
\[ G = \{ (\epsilon_1 - 1)(k+2), \ldots, (\epsilon_M - 1)(k+2), -2, \ldots, -2 \}, \]
\[ H = \{ -1, \ldots, -1, 0, \ldots, 0 \}, \]
\[ \tilde{X} = \{ t_1, \ldots, t_{N+a+b}, u_1, \ldots, u_{a+c} \}, \]
\[ \tilde{Y} = \{ \xi_1, \ldots, \xi_M, \bar{\rho}_1, \ldots, \bar{\rho}_n \}, \]
\[ \tilde{Z} = \{ t_1, \ldots, t_{N+a+b}, u_1, \ldots, u_{a+c} \}, \]
\[ \tilde{W} = \{ \xi_1, \ldots, \xi_M, \bar{\rho}_1, \ldots, \bar{\rho}_n \}. \]

(6.259)

The dimensions of the latter sets and any other details should be clear from the contexts.

Just as in the previous case of the correlation functions, though the above expression for the
N-point form factor is very complicated, one can still deduce from it some simple selection
rules. Indeed, the general trace formulas (8.297) and (8.302) in the appendix imply

\[ S^{X_1} (s, r_{2n}, \ldots, r_1; r_{n}', \ldots, r_1') = \delta_{n'-2(a+b), 0}(\cdots), \]
\[ S^{X_2 \cdot X_3} (r_{2n}, \ldots, r_1; r_{n}', \ldots, r_1') = \delta_{\sum_{l=1}^n i_l + n', \sum_{l=1}^n j_l + a + c}(\cdots). \]

(6.260)

Therefore the following selection rules must be satisfied in order for the form factors to be
non-vanishing:

\[ a + b = n'/2, \]
\[ \sum_{l=1}^n i_l + \frac{n'}{2} = \sum_{l=1}^n j_l + a + c. \]

(6.261)
Since \( a + b \) is an integer number, \( n' \) is an even integer number. These selection rules are also consistent with those of the spin 1/2 case.

Let us now recapitulate the main result of this calculation. The form factor \( <\lambda_0|E_{i_{n_j}n_j}|\rho_{n',\cdots,n} >_1 \) of the basis local operator \( E_{i_{n_j}n_j} \cdots E_{i_{n_j}n_j} \) is given by

\[
<\lambda_0|E_{i_{n_j}n_j} \cdots E_{i_{n_j}n_j}|\rho_{n',\cdots,n} >_1 = B(q) \prod_{i=1}^{n'} \left( d_{\mathcal{A}_i} h_{\lambda_i} C_{j_{2n},\cdots,j_1} (z_{2n},\cdots,z_1) \right)
\]

\[
= \frac{2n}{\prod_{i=1}^{n'} \sum_{\delta_1^{(i)},\cdots,\delta_{k-j_i}^{(i)}}, \epsilon_1^{(i)},\cdots,\epsilon_{k-j_i}^{(i)} \int_{q_{p_1}}^{q_{p_2}} \int_{q_{p_1}}^{q_{p_2}} \int_{q_{p_1}}^{q_{p_2}} \int_{q_{p_1}}^{q_{p_2}} \frac{d\xi_1^{(i)}}{2\pi i} \cdots \frac{d\xi_{k-j_i}^{(i)}}{2\pi i} \prod_{i=1}^{n'} f(D_{i_1},D_{i_2}) \}
\]

\[
\prod_{i=1}^a \left( \sum_{\delta_1^{(i)},\cdots,\delta_{k-j_i}^{(i)},\epsilon_1^{(i)},\cdots,\epsilon_{k-j_i}^{(i)} \int_{q_{p_1}}^{q_{p_2}} \int_{q_{p_1}}^{q_{p_2}} \int_{q_{p_1}}^{q_{p_2}} \int_{q_{p_1}}^{q_{p_2}} \frac{d\xi_1^{(i)}}{2\pi i} \cdots \frac{d\xi_{k-j_i}^{(i)}}{2\pi i} \prod_{i=1}^{n'} f(K_{i_1},K_{i_2}) \}
\]

\[
\prod_{1 \leq j_1 < i \leq n'} \frac{\prod_{i=1}^{n'} f(\mathcal{O}_{r_2n},\cdots,r_1,\mathcal{O}_{r_2n'},\cdots,r_1')}{\prod_{i=1}^{n'} f(\mathcal{O}_{r_2n},\cdots,r_1,\mathcal{O}_{r_2n'},\cdots,r_1')}
\]

\[
\sum_{s \in 2Z} A_{\Phi_s}^{n(k+sp-p)} A_{\Phi_{s'}}^{n'(k+sp-p)} S(s,r_2n',\cdots,r_1') \]

\[
= \sum_{s \in 2Z} A_{\Phi_s}^{n(k+sp-p)} A_{\Phi_{s'}}^{n'(k+sp-p)} S(s,r_2n',\cdots,r_1') \]

\[
(6.262)
\]

Here also all the various symbols are c-functions and are defined throughout the above analysis. Moreover, the selection rules leading to non-vanishing form factor can be rewritten in the form

\[
\sum_{i=1}^n i + \frac{n'}{2} = \sum_{i=1}^{n'} \beta_i = \sum_{i=1}^{n'} \alpha_i = 0.
\]

(6.263)

It is clear from the latter selection rules that since \( n' \) must be even, local operators create spinons only in pairs, which is identical to the spin 1/2 case.
Now that we understand how the vertex operators are the building blocks of the form factors we can derive more relations satisfied by them using the various basic relations among the vertex operators. For example, if we define a general matrix element of a local operator $\mathcal{O}$ as

$$
\frac{\beta_{m+1} \cdots \beta_{m+1}}{\alpha_{m}, \cdots, \alpha_{m+1}} < \rho_{m+1}, \cdots, \rho_{m} | \mathcal{O} | \rho_{m}, \cdots, \rho_{1} > \frac{\beta_{m+1} \cdots \beta_{m+1}}{\alpha_{m}, \cdots, \alpha_{1}}
$$

$$
= \frac{\text{Tr}_{V(\lambda_{n})}(\mu^{-d_{\nu}} \Psi_{\lambda_{m}}^{\lambda_{n}}(\rho_{n}) \cdots \Psi_{\lambda_{m}, \alpha_{m+1}}^{\lambda_{m}}(\rho_{m+1}))(\mathcal{O}) \Psi_{\lambda_{m-1}, \alpha_{m}}^{\lambda_{m}}(\rho_{m}) \cdots \Psi_{\lambda_{1}, \alpha_{1}}^{\lambda_{1}}(\rho_{1}))}{\text{Tr}_{V(\lambda_{n})}(\mu^{-d_{\nu}})}.
$$

(6.264)

then using the first relation of (6.243) we arrive at the following identity:

$$
\frac{\beta_{m+1} \cdots \beta_{m+1}}{\alpha_{m}, \cdots, \alpha_{m+1}} < \rho_{n}, \cdots, \rho_{m+1} | \mathcal{O} | \rho_{m}, \cdots, \rho_{1} > \frac{\beta_{m+1} \cdots \beta_{m+1}}{\alpha_{m}, \cdots, \alpha_{1}}
$$

$$
= \left( \prod_{i=m+1}^{n} d_{\alpha_{i}, \beta_{i}}^{-1} \right) < O | q^{-2} \rho_{n}, \cdots, q^{-2} \rho_{m+1}, \rho_{m}, \cdots, \rho_{1} > \frac{\beta_{m+1} \cdots \beta_{m+1}}{\alpha_{m}, \cdots, \alpha_{1}}
$$

(6.265)

This identity is particularly useful in situations where matrix elements of local operators mapping excited states to excited states are needed. It means that one needs just to evaluate the form factors of local operators mapping excited states to the vacuum. An example of such a situation is the evaluation of the dynamic correlation functions at arbitrary nonzero temperature [50].

For completeness, let us now summarise some of the interesting results previously found in the literature with respect to the form factors. It has been shown in Ref. [29] that the form factors satisfy the higher level q-KZ equation. This is given in a matrix form. Let us keep most of the same notations as in this reference where the form factor is defined in a matrix form (up to normalization factor) as

$$
F_{\lambda_{n}, \cdots, \lambda_{0}}(\rho_{n}, \cdots, \rho_{1}) = \text{Tr}_{V(\lambda_{0})}(q^{-2p} \mathcal{O}_{\lambda_{0}} \Psi_{\lambda_{m-1}, \alpha_{m}}^{\lambda_{m}}(\rho_{m}) \cdots \Psi_{\lambda_{1}, \alpha_{1}}^{\lambda_{1}}(\rho_{1})).
$$

(6.266)

Then using the commutation relations, the cyclic property of a trace it has been shown that the following q-KZ equation is satisfied:

$$
F_{\lambda_{n}, \cdots, \lambda_{0}}(\rho_{n}, \cdots, q^{4} \rho_{1}, \cdots, \rho_{1})
$$

$$
= R^{*i-1i}(\rho_{i-1}/q^{4} \rho_{i})^{-1} \cdots R^{*i+1i}(\rho_{1}/q^{4} \rho_{1})^{-1}
$$
\[
\times \pi V^* (q^{-2\bar{\rho}}) R^{\text{strin}}(\rho_i/\rho_n) \cdot \cdot \cdot R^{\text{strin}+1}(\rho_i/\rho_{i+1}) \\
\times \sum_{\lambda_i', \ldots, \lambda_{i+1}', \lambda_{i-1}', \ldots, \lambda_i'} q^{4(\Delta \lambda_i' - \Delta(\lambda_0))} F_{\lambda_i', \lambda_{i+1}', \lambda_{i-1}', \ldots, \lambda_i'} (\rho_n, \ldots, \rho_1) \\
\times W \left( \begin{array}{c|c}
\lambda_0 & \lambda_1 \\
\lambda_1' & \lambda_2' \\
\end{array} \right) \cdot \cdot \cdot \times W \left( \begin{array}{c|c}
\lambda_{i-2} & \lambda_{i-1} \\
\lambda_{i-1}' & \lambda_i' \\
\end{array} \right) q^{4(\rho_i/\rho_1)} \\
\times W \left( \begin{array}{c|c}
\lambda_i & \lambda_{i+1} \\
\lambda_{i+1}' & \lambda_{i+2}' \\
\end{array} \right) \cdot \cdot \cdot \times W \left( \begin{array}{c|c}
\lambda_{n-1} & \lambda_n \\
\lambda_n' & \lambda_1' \\
\end{array} \right) q^{4(\rho_i/\rho_{i+1})},
\]  
(6.267)

where \( \bar{\rho} \) is the restriction of \( \rho = 2d + \alpha_0/2 \) to a finite dimensional representation, \( R^{ij} (z) \) is \( R(z) \) acting on \( V_i \otimes V_j \), and similarly for \( R^* (z) \). Moreover,

\[
V^* = V^{(1)*a^{-1}}, \\
\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + 2)}.
\]
(6.268)

This means that the explicit expression for the form factors we found in (6.262) provides the unique physical solution for the q-KZ equation. This is nontrivial because typically difference equations such as q-KZ admit an infinite number of solutions which differ from each other by pseudo constants. Further analyticity requirements based on physical considerations are needed in order to single out a unique solution. It would be interesting to try to solve directly the q-KZ equation consistently with the analyticity requirements which can be read off from the explicit expressions above. This might lead to other more compact forms for the form factors.

The other interesting result that has been obtained in the literature is the fact that the form factors satisfy a lattice version of Smirnov’s axioms. This is an important step towards making Smirnov’s axioms universal by including not only integrable models of quantum field theory, as originally thought, but also lattice models with different dispersion relations. Let us briefly remind Smirnov’s axioms as applied to integrable quantum field theories [35], and as summarised in Ref. [33]:

Axiom 1: Form factors \( F(\beta_1, \ldots, \beta_n)_{\epsilon_1, \ldots, \epsilon_n} \) have the S-matrix symmetry

\[
F(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n)_{\epsilon_1, \ldots, \epsilon_i, \epsilon_{i+1}, \ldots, \epsilon_n} S^{\epsilon_i \epsilon_{i+1}}_{\epsilon_i^1 \epsilon_{i+1}^1} = F(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n)_{\epsilon_1, \ldots, \epsilon_i^1, \epsilon_{i+1}^1, \ldots, \epsilon_n}.
\]
(6.269)
Axiom 2: Form factors satisfy the difference equation

\[ F(\beta_1, \cdots, \beta_n + 2\pi i)_{\epsilon_1, \cdots, \epsilon_n} = F(\beta_n, \beta_1, \cdots, \beta_{n-1})_{\epsilon_{n-1}, \epsilon_1, \cdots, \epsilon_{n-1}}. \]  

(6.270)

Axiom 3: Form factors have the annihilation property. This means that the residue of the \( n \)-particle form factor \( F(\beta_1, \cdots, \beta_n)_{\epsilon_1, \cdots, \epsilon_n} \) can be expressed in terms of \((n-2)\)-particle ones. Moreover, they have simple poles at \( \beta_i = \beta_j + i\pi, i > j \), with residues at \( \beta_n = \beta_{n-1} + i\pi \) given by

\[ 2\pi i \ \text{Res} F(\beta_1, \cdots, \beta_n)_{\epsilon_1, \cdots, \epsilon_n} \]

\[ = F(\beta_1, \cdots, \beta_{n-2})_{\epsilon_1', \cdots, \epsilon_{n-1}'} \delta_{\epsilon_{n-2}, -\epsilon_{n-1}} \]

\[ \times \left( S_{\epsilon_1 \epsilon_1'}^{\epsilon_{n-2} \epsilon_{n-1}'}(\beta_{n-1} - \beta_1) \cdots S_{\epsilon_{n-1} \epsilon_{n-2}}^{\epsilon_{n-3} \epsilon_{n-2}}(\beta_{n-1} - \beta_{n-2}) \right). \]  

(6.271)

These axioms provide a powerful alternative method for the exact derivation of form factors of local operators in integrable quantum field theories. They also lead to the \( q \)-KZ equation for form factors.

A crucial result was obtained in Ref. [51] for the spin 1/2 case where it was shown that Smirnov’s axioms are still satisfied by the form factors of the Heisenberg model. Then, this result has been generalized for the third axiom to the higher spin Heisenberg model. To describe this result let us use the following ”light” definition for a component form factor, which is sufficient for this purpose:

\[ F_{\lambda_{\lambda-1}^{\cdots \cdots} \lambda}^{\epsilon_{\epsilon_{\cdots} \cdots} \epsilon}(\rho_n, \cdots, \rho_1) = \text{Tr}_{V(\lambda)} \left( q^{-2\rho} \alpha \Psi^A_{\lambda_{n-1}, \epsilon_n}(\rho_n) \cdots \Psi^A_{\lambda_1, \epsilon_1}(\rho_1) \right). \]  

(6.272)

Recall that \( \epsilon = + \) and \( \epsilon = - \) correspond to the components \( m = 0 \) and \( m = 1 \) respectively. That only simple poles appear in the factorization property of the latter form factors is traced to the following OPE’s of type II vertex operators in the limit \( q^{-2}\rho_2/\rho_1 \to 1 \):

\[ \Psi^\mu_{\lambda_+ \epsilon_2}(\rho_2) \Psi^\lambda_{\lambda_1, \epsilon_1}(\rho_1) \sim N_{\lambda_+}^{\lambda_1}(\epsilon_2, \epsilon_1) \delta_{\epsilon_2, -\epsilon_1} \frac{1}{1 - q^{-2}\rho_2/\rho_1} \text{id} + O(1), \]  

(6.273)

where the constants \( N_{\lambda_+}^{\lambda_1}(\epsilon_2, \epsilon_1) \) are given by

\[ N_{\lambda_+}^{\lambda_1}(+, -) = -q N_{\lambda_+}^{\lambda_1}(-, +) = -q^4(1 - p) \frac{(q^2 p; p)_\infty}{(q^{-2} p; p)_\infty} \frac{\xi(q^2; 1, q^4)}{B_p(1 - 2s, -2s)}; \]

\[ N_{\lambda_+}^{\lambda_1}(+, -) = -q N_{\lambda_+}^{\lambda_1}(-, +) = q(1 - p) \frac{(q^2 p; p)_\infty}{(q^{-2} p; p)_\infty} \frac{\xi(q^2; 1, q^4)}{B_p(6s, -2s)}. \]  

(6.274)
Using these OPE’s, the cyclic property of a trace and the commutation relations of type II vertex operators (4.110), the following residue formula at $q^{-2}\rho_2/\rho_1 = 1$ was derived in Ref. [33]:

$$
\text{Res} F_{\lambda,\cdots,\lambda_1,\lambda}^{\varepsilon_n,\cdots,\varepsilon_1}(\rho_n, \cdots, \rho_1)
$$

\begin{align*}
&= \delta_{\lambda}^{\lambda_1} \delta_{\varepsilon_2, -\varepsilon_1} N_{\lambda_1}^{\lambda} (-\varepsilon_1, \varepsilon_1) F_{\lambda,\cdots,\lambda_1,\lambda}^{\varepsilon_n,\cdots,\varepsilon_3}(\rho_n, \cdots, \rho_3) \\
& \quad - N_{\lambda_1}^{\lambda}(\varepsilon_1, -\varepsilon_1) \delta_{\varepsilon_{n-2}, -\varepsilon_1} R_{\varepsilon_{n-3}^{\varepsilon_{n-2}^{\vdots} \varepsilon_1}}^{\varepsilon_{n-3}^{\vdots} \varepsilon_1}(\rho_n/\rho_2) R_{\varepsilon_{n-4}^{\varepsilon_{n-3}^{\vdots} \varepsilon_1}}^{\varepsilon_{n-4}^{\vdots} \varepsilon_1}(\rho_{n-1}/\rho_2) \cdots R_{\varepsilon_{n-1}^{\varepsilon_1}}^{\varepsilon_{n-1}^{\varepsilon_1}}(\rho_{n-1}/\rho_2) R_{\varepsilon_{n}^{\varepsilon_1}}^{\varepsilon_{n}^{\varepsilon_1}}(\rho_3/\rho_2) \\
& \quad \times \sum_{\mu_1, \cdots, \mu_{n-3}} W \left( \begin{array}{c} \mu_{n-3} \\ \lambda_{n-1} \\ \lambda_{1} \\ \lambda \\ \lambda_1 \\ \lambda \end{array} \right)_{\rho_n/\rho_2} W \left( \begin{array}{c} \mu_{n-4} \\ \lambda_{n-2} \\ \lambda_{n-3} \\ \lambda_{1} \\ \lambda \end{array} \right)_{\rho_{n-1}/\rho_2} \cdots W \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \rho_3/\rho_2 \\ \rho_1 \\ \rho_2 \end{array} \right) \\
& \quad \times F_{\lambda_1,\mu_3,\mu_{n-3},\cdots,\mu_1,\lambda}^{\varepsilon_n,\cdots,\varepsilon_1}(\rho_n, \cdots, \rho_3).
\end{align*}

(6.275)

As noted in Ref. [33], the main new ingredient of the lattice version of the third axiom is that now the face type Boltzmann weights $W \left( \begin{array}{c} \lambda \\ \mu \\
\mu' \\ \nu \end{array} \right)_{\rho}$ enter the residue formula.

On can also derive the higher spin lattice version of the first and second axioms. Indeed, using the commutation relations of type II vertex operators (4.117) we find the lattice analogue of the first axiom

$$
F_{\lambda,\cdots,\lambda_1,\lambda}^{\varepsilon_n,\cdots,\varepsilon_1}(\rho_n, \cdots, \rho_{i+1}, \rho_i, \cdots, \rho_1) = \\
\sum_{\mu=\lambda_1} F_{\lambda,\cdots,\lambda_{i+1},\lambda}^{\varepsilon_n,\cdots,\varepsilon_1}(\rho_n, \cdots, \rho_i, \rho_{i+1}, \cdots, \rho_1) R_{\varepsilon_{i+1}^{\varepsilon_1}}^{\varepsilon_{i+1}^{\varepsilon_1}}(\rho_{i+1}/\rho_i) W \left( \begin{array}{c} \lambda_{i+1} \\ \mu \\ \lambda_i \\ \rho_{i+1}/\rho_i \end{array} \right). 
$$

(6.276)

Moreover, (5.165) and (4.101) imply

$$
q^{-2\rho} \Psi^\mu_{\lambda,\varepsilon}(z) q^{2\nu} = q^\varepsilon \Psi^\mu_{\lambda,\varepsilon}(z q^4).
$$

(6.277)

This relation and the cyclic property of a trace lead to the lattice version of the second axiom, i.e.,

$$
F_{\lambda,\lambda_{n-1},\cdots,\lambda_1,\lambda}^{\varepsilon_n,\varepsilon_{n-1},\cdots,\varepsilon_1}(\rho_n, \cdots, \rho_1) = q^\varepsilon F_{\lambda,\cdots,\lambda_1,\lambda,\lambda}^{\varepsilon_n,\cdots,\varepsilon_1}(\rho_{n-1}, \cdots, \rho_1, \rho_n q^4).
$$

(6.278)

Therefore, the crucial point is that Smirnov’s axioms extend to integrable lattice models, which is beyond their original scope of integrable quantum field theories. However, when applied to lattice models they also involve in general the face type Boltzmann weights $W$ in addition to the usual $R$ matrix.
We see that despite the fact that the final expression for the form factors is quite complicated one can still obtain simple information from it, such as the selection rules and the way spinons are created by local operators. One of the main lessons one learns from this type of analysis is that the identification of the creation and annihilation operators of eigenstates of a Hamiltonian is an important step towards the complete solution of a model. The next crucial step is handling the commutation relations of the operators which are typically generalized commutation relations. Since the only commutation relations we know how to handle completely are those of Heisenberg and Clifford types we try then to express the creation and annihilation of the eigenstates in terms of the generators of the latter algebras. If we succeed in overcoming this step then the main goal of evaluating the relevant physical quantities like correlation functions becomes a matter of technical details, as was shown here. Needless, to say that one would like to extend this method to as many models as possible, and in particular the Heisenberg model with nonzero temperature and/or nonzero magnetic field, the 8-vertex model, the Potts model etc... The more we learn about the generalized commutation relations the more we learn about the exact solution of integrable models at the level of correlation functions and form factors and not just the spectrum.

7 Acknowlegment

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8 Appendix

8.1 Operator product expansions (OPE’s) of the elementary operators

The elementary operators that appear in the evaluation of the correlation functions and form factors are the vertex operators \( \tilde{\Phi}_\ell^{(t)}(z) \) (type I) and \( \tilde{\Psi}_0^{(t)}(z) \) (type II), the currents \( E_c^\pm(z) \), the screening currents \( S_\delta(z) \), \( \eta(z) \), and \( \xi(z) \). They are bosonized as follows:

\[
\begin{align*}
\tilde{\Phi}_\ell^{(t)}(z) &= : \exp(X_1(\ell; 2, k + 2|q^k z; \frac{k + 2}{2}) ) : , \\
\tilde{\Psi}_0^{(t)}(z) &= : \exp(X_1(\ell; 2, k + 2|q^{k-2} z; -\frac{k + 2}{2}) + X_2(\ell; 2, 1|q^{-2} z; 0) + X_3(\ell; 2, 1|q^{-2} z; 0) ) : , \\
E_c^-(z) &= : \exp(\partial X_1^{(c)}(q^{-2} z; -\frac{k + 2}{2}) + X_2(2|q^{(c-1)(k+2)} z; -1) + X_3(2|q^{(c-1)(k+1)-1} z; 0) ) : , \\
E_c^+(z) &= : \exp(-X_2(2|q^{-k-2} z; 1) - X_3(2|q^{-k+2} z; 0) ) : , \\
S_\delta(z) &= : \exp(-X_1(k + 2|q^{-2} z; -\frac{k + 2}{2}) - X_2(2|q^{-k-2} z; -1) - X_3(2|q^{-k+2} z; 0) ) : , \\
\eta(z) &= : \exp(X_3(2|q^{-k-2} z; 0) ) : , \\
\xi(z) &= : \exp(-X_3(2|q^{-k-2} z; 0) ) : .
\end{align*}
\]

where

\[
\partial X_1^{(c)}(q^{-2} z; -\frac{k + 2}{2}) = \epsilon\{(q - q^{-1}) \sum_{n=1}^{\infty} a_{X_1, c} z^{-cn} q^{(2c-k+2)n} + a_{X_1, 0} \log(q) \}. \tag{8.280}
\]

The correlation functions and form factors are given in terms of the c-function \( f \) defined by:

\[
: A :: B := AB : f(A, B) = : AB : \prod_{i=1}^{n} \prod_{j=1}^{m} f(A_i, B_j), \tag{8.281}
\]

for the composite normal ordered operators \( A =: \prod_{i=1}^{n} A_i : \) and \( B =: \prod_{j=1}^{m} B_j : \), with \( A_i \) and \( B_j \) being elementary operators. Therefore to evaluate \( f(A, B) \) for any composite operators \( A \) and \( B \), it is sufficient to know just the OPE’s among all elementary operators. The latter OPE’s are presented below.

OPE’s:

\[
\begin{align*}
\tilde{\Phi}_\ell^{(t)}(z) \tilde{\Phi}_\ell^{(t)}(w) &= (zq^k)^{\ell^2/(2k+2)} \frac{(pq^{2(1-\ell)} w z^{-1}; p, q^4)_{\infty}(pq^{2(1+\ell)} w z^{-1}; p, q^4)_{\infty}}{(pq^2 w z^{-1}; p, q^4)_{\infty}^2} : \tilde{\Phi}_\ell^{(t)}(z) \tilde{\Phi}_\ell^{(t)}(w) : , \\
\tilde{\Psi}_0^{(t)}(z) \tilde{\Psi}_0^{(t)}(w) &= (zq^{k-2})^{\ell^2/(2k+2)} \frac{(q^{2(1-\ell)} w z^{-1}; p, q^4)_{\infty}(q^{2(1+\ell)} w z^{-1}; p, q^4)_{\infty}}{(q^2 w z^{-1}; p, q^4)_{\infty}^2} : \tilde{\Psi}_0^{(t)}(z) \tilde{\Psi}_0^{(t)}(w) : ,
\end{align*}
\]

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\[ \tilde{\Phi}^{(\ell)}(z) \tilde{\Psi}^{(\ell)}_0(w) = \left( z q^k \right)^{\ell/2(k+2)} \frac{(q^{k+2(1-\ell)} w z^{-1}; p q^4 \infty)}{(q^{k+2 w z^{-1}}; p q^4 \infty)} (q^{k+2(1+\ell)} w z^{-1} ; p ; q^4) \]

\[ \times : \tilde{\Phi}^{(\ell)}(z) \tilde{\Psi}^{(\ell)}_0(w) : , \]  

\[ \tilde{\Psi}^{(\ell)}_0(z) \tilde{\Phi}^{(\ell)}(w) = \left( z q^{k-2} \right)^{\ell/2(k+2)} \frac{(q^{k+2(3-\ell)} w z^{-1}; p q^4 \infty)}{(q^{k+6 w z^{-1}}; p q^4 \infty)} (q^{k+2(3+\ell)} w z^{-1} ; p ; q^4) \]

\[ \times : \tilde{\Psi}^{(\ell)}_0(z) \tilde{\Phi}^{(\ell)}(w) : . \]  

\[ \tilde{\Phi}^{(\ell)}(z) E^{-1}(w) = : \tilde{\Phi}^{(\ell)}(z) E^{-1}(w) : , \]

\[ \tilde{\Phi}^{(\ell)}(z) E^{-1}(w) = \frac{z - q^{-\ell-k-2} w}{z - q^{-\ell-k-2} w} : \tilde{\Phi}^{(\ell)}(z) E^{-1}(w) : , \]

\[ E^{-1}(z) \tilde{\Phi}^{(\ell)}(w) = \frac{q^{\ell} z - q^{k+2} w}{z - q^{k+2} w} : E^{-1}(z) \tilde{\Phi}^{(\ell)}(w) : , \]

\[ E^{-1}(z) \tilde{\Phi}^{(\ell)}(w) = q^{-\ell} : E^{-1}(z) \tilde{\Phi}^{(\ell)}(w) : , \]

\[ \tilde{\Psi}^{(\ell)}_0(z) E^{+1}(w) = : \tilde{\Psi}^{(\ell)}_0(z) E^{+1}(w) : , \]

\[ \tilde{\Psi}^{(\ell)}_0(z) E^{+1}(w) = \frac{z - q^{-\ell-k} w}{z - q^{-\ell-k} w} : \tilde{\Psi}^{(\ell)}_0(z) E^{+1}(w) : , \]

\[ E^{+1}(z) \tilde{\Psi}^{(\ell)}_0(w) = \frac{q^{-\ell} z - q^{k+2} w}{z - q^{k+2} w} : E^{+1}(z) \tilde{\Psi}^{(\ell)}_0(w) : , \]

\[ E^{+1}(z) \tilde{\Psi}^{(\ell)}_0(w) = q^{\ell} : E^{+1}(z) \tilde{\Psi}^{(\ell)}_0(w) : , \]
\[ S_{-1}(z) \tilde{\Psi}_0^{(\ell)}(w) = (q^{-2}z)^{-1/k+2} \frac{(q^{1+k}pwz^{-1}; p)_\infty}{(q^{k-1}pwz^{-1}; p)_\infty} : S_{-1}(z) \tilde{\Psi}_0^{(\ell)}(w) :, \quad |z| > |q^{k-1}pw|, \] (8.286)

\[
\begin{align*}
E_1^+(z)E_1^+(w) &= q \frac{z-w}{z-wq^2} : E_1^+(z)E_1^+(w) :, \quad |z| > |q^2w|, \\
E_1^+(z)E_{-1}^+(w) &= q \frac{z-w}{z-wq^2} : E_1^+(z)E_{-1}^+(w) :, \quad |z| > |q^2w|, \\
E_1^+(z)E_1^-(w) &= q^{-1} : E_1^+(z)E_1^-(w) :, \\
E_1^+(z)E_{-1}^-(w) &= q^{-1} \frac{z-w}{z-wq^2} : E_{-1}^+(z)E_{-1}^-(w) :, \quad |z| > |q^2w|, \\
E_1^-(z)E_1^-(w) &= q \frac{z-w}{z-wq^2} : E_1^-(z)E_1^-(w) :, \quad |z| > |w|, \\
E_1^-(z)E_{-1}^+(w) &= q^{-1} \frac{z-wq^2}{z-wq^{-2}} : E_{-1}^-(z)E_1^+(w) :, \\
E_1^-(z)E_{-1}^-(w) &= q : E_{-1}^+(z)E_{-1}^-(w) :, \quad |z| > |w|. \tag{8.287}
\end{align*}
\]

\[
\begin{align*}
S_1(z)S_1(w) &= z^{-k} q^{k+2} \frac{(z-wq^{-2}; p)_\infty}{(z^{-1}wq^2; p)_\infty} (z-w)(z-w^2p; p)_\infty, \quad |z| > |w|, \\
S_1(z)S_{-1}(w) &= z^{k+2} q^{-k-2} \frac{(z^{-1}wq^{-2}; p)_\infty}{(z^{-1}wq^2; p)_\infty}, \quad |z| > |q^{-2}w|, \\
S_{-1}(z)S_1(w) &= z^{k+2} q^{-k-6} \frac{(z^{-1}wq^{-2}; p)_\infty}{(z^{-1}wq^2; p)_\infty}, \quad |z| > |q^{-2}pw|, \\
S_{-1}(z)S_{-1}(w) &= z^{-k} q^{k+4} \frac{(z-wq^{-2}; p)_\infty}{(z^{-1}wq^2; p)_\infty}, \quad |z| > |w|. \tag{8.288}
\end{align*}
\]

\[
\begin{align*}
E_1^+(z)S_1(w) &= q : E_1^+(z)S_1(w) :, \\
E_1^+(z)S_{-1}(w) &= q \frac{z-q^{-2}w}{z-w} : E_1^+(z)S_{-1}(w) :, \quad |z| > |q^{-2}w|, \\
E_{-1}^+(z)S_1(w) &= q^{-1} \frac{z-q^2w}{z-w} : E_{-1}^+(z)S_1(w) :, \quad |z| > |w|, \\
E_{-1}^+(z)S_{-1}(w) &= q^{-1} : E_{-1}^+(z)S_{-1}(w) :, \\
S_1(z)E_1^+(w) &= q : S_1(z)E_1^+(w) :, \\
S_1(z)E_{-1}^+(w) &= q \frac{z-q^{-2}w}{z-w} : S_1(z)E_{-1}^+(w) :, \quad |z| > |q^{-2}w|.
\end{align*}
\]
\[ S_1(z)E_1^+(w) = q^{-1} \frac{z - q^2 w}{z - w} : S_1(z)E_1^+(w) :, \quad |z| > |w|, \]
\[ S_1(z)E_1^+(w) = q^{-1} : S_1(z)E_1^+(w) :, \]
\[ E_1^+(z)S_1(w) = q^{-1} : E_1^+(z)S_1^+(w) :, \]
\[ E_1^+(z)S_1(w) = q^{-1} z - q^{-k-1} w : E_1^+(z)S_1^+(w) :, \quad |z| > |q^{-k-2} w|, \]
\[ E_1^+(z)S_1(w) = q z - q^{k+1} w : E_1^+(z)S_1^+(w) :, \quad |z| > |q^k w|, \]
\[ E_1^+(z)S_1(w) = q : E_1^+(z)S_1(w) :, \]
\[ S_1(z)E_1^+(w) = q^{-1} : S_1(z)E_1^+(w) :, \]
\[ S_1(z)E_1^+(w) = q^{-1} z - q^{-k-1} w : S_1(z)E_1^+(w) :, \quad |z| > |q^{-k-2} w|, \]
\[ S_1(z)E_1^+(w) = q z - q^{k+1} w : S_1(z)E_1^+(w) :, \quad |z| > |q^k w|, \]
\[ S_1(z)E_1^+(w) = q : S_1(z)E_1^+(w) :, \quad (8.289) \]
\[ \eta(z)\xi(w) = \frac{q^{k+2}}{z - w} : \eta(z)\xi(w) :, \quad |z| > |w|, \]
\[ \eta(z)\tilde{\phi}_{\epsilon}^{(l)}(w) = \eta(z)\tilde{\phi}_{\epsilon}^{(l)}(w) :, \]
\[ \tilde{\phi}_{\epsilon}^{(l)}(z)\eta(w) = : \tilde{\phi}_{\epsilon}^{(l)}(z)\eta(w) :, \]
\[ \eta(z)\tilde{\psi}_0^{(l)}(w) = q^{-k-2}(z - w q^k) : \eta(z)\tilde{\psi}_0^{(l)}(w) :, \quad |z| > |q^k w|, \]
\[ \tilde{\psi}_0^{(l)}(z)\eta(w) = q^{-2}(z - w q^{-k}) : \tilde{\psi}_0^{(l)}(z)\eta(w) :, \quad |z| > |q^{-k} w|, \]
\[ \eta(z)E^+_\rho(w) = \frac{q^{k+2}}{z - w q^\rho} : \eta(z)E^+_\rho(w) :, \quad |z| > |q^\rho w|, \]
\[ E^+_\rho(z)\eta(w) = \frac{q^{k+2}}{z q^\rho - w} : E^+_\rho(z)\eta(w) :, \quad |z| > |q^{-\rho} w|, \]
\[ \eta(z)E^-_\epsilon(w) = (z q^{-k-2} - w q^{(-1)(k+1)-1}) : \eta(z)E^-_\epsilon(w) :, \quad |z| > |q^{(k+1)} w|, \]
\[ E^-_\epsilon(z)\eta(w) = (z q^{(-1)(k+1)-1} - w q^{-k-2}) : E^-_\epsilon(z)\eta(w) :, \quad |z| > |q^{(-k+1)} w|, \]
\[ \eta(z)S_\delta(w) = \frac{q^{k+2}}{z - w q^{\delta}} : \eta(z)S_\delta(w) :, \quad |z| > |q^{\delta} w|, \]
\[ S_\delta(z)\eta(w) = \frac{q^{k+2}}{z q^{\delta} - w} : S_\delta(z)\eta(w) :, \quad |z| > |q^{-\delta} w|, \]
\[ \xi(z)E^+\rho(w) = q^{-k-2}(z - w q^\rho) : \xi(z)E^+\rho(w) :, \quad |z| > |q^\rho w|, \]
\[ E^+\rho(z)\xi(w) = q^{-k-2}(z q^\rho - w) : E^+\rho(z)\xi(w) :, \quad |z| > |q^{-\rho} w|, \]
\[ \xi(z)E^-\epsilon(w) = \frac{1}{z q^{-k-2} - w q^{(-1)(k+1)-1}} : \xi(z)E^-\epsilon(w) :, \quad |z| > |q^{(k+1)} w|, \]

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\[ E_\varepsilon^-(z)\xi(w) = \frac{1}{zq^{(\varepsilon+1)(k+1)-1} - wzq^{-k-2}} : E_\varepsilon^-(z)\xi(w) :, \quad |z| > |q^{-k+1}w|, \]
\[ \xi(z)S_\delta(w) = q^{-k-2}(z - wz) : \xi(z)S_\delta(w) :, \quad |z| > |q^\delta w|, \]
\[ S_\delta(z)\eta(w) = q^{-k-2}(zq^\delta - w) : S_\delta(z)\eta(w) :, \quad |z| > |q^{-\delta}w|. \] (8.290)

### 8.2 Type II vertex operators

The various \( c \)-functions \( H \) and contours \( C \) defining the bosonization of the physical type II vertex operators in relation (8.241) are given by

\[
\begin{align*}
I_1^{(1),(0)}(z,t|C) &= -\frac{(q^{k-2}z)^{-\frac{1}{k+2}}(q^{-k+1}tpz^{-1};p)_{\infty}}{t(q - q^{-1})(q^{-k+1}tpz^{-1};p)_{\infty}}, \quad \text{with } C : |z| > |q^{-k-1}t|, \\
I_{-1}^{(1),(0)}(z,t|C) &= \frac{(q^{k-2}z)^{-\frac{1}{k+2}}(q^{-k+1}tz^{-1};p)_{\infty}}{t(q - q^{-1})(q^{-k+1}tz^{-1};p)_{\infty}}, \quad \text{with } C : |z| > |q^{-k-1}t|, \\
I_1^{(0),(1)}(z,u|C) &= -\frac{zq^k}{u(u - q^{-k+1}z)}, \quad \text{with } C : |u| > |q^{-k+1}z|, \\
I_{-1}^{(0),(1)}(z,u|C) &= -\frac{zq^{k+2}}{u(u - q^{-k+1}z)}, \quad \text{with } C : |z| > |q^{-k-1}u|, \\
I_{1,1}^{(1),(1)}(z,t,u|C) &= -\frac{zq^{k+1}(q^{k-2}z)^{-\frac{1}{k+2}}(q^{-k+1}tpz^{-1};p)_{\infty}}{ut(u - q^{-k+1}z)(q - q^{-1})(q^{-k+1}tpz^{-1};p)_{\infty}}, \quad \text{with } C : |u| > |q^{-k-1}z|, |z| > |q^{-k-1}pt|, \\
I_{1,-1}^{(1),(1)}(z,t,u|C) &= -\frac{zq^{k+1}(q^{k-2}z)^{-\frac{1}{k+2}}u - q^2t(q^{-k+1}tpz^{-1};p)_{\infty}}{ut(u - q^{-k+1}z)(q - q^{-1})u - t(q^{-k+1}tpz^{-1};p)_{\infty}}, \quad \text{with } C : |z| > |q^{-k-1}z|, |z| > |q^{-k-1}pt|, |u| > |t|, \\
I_{-1,1}^{(1),(1)}(z,t,u|C) &= -\frac{zq^{k+1}(q^{k-2}z)^{-\frac{1}{k+2}}u - q^{-2}t(q^{-k+1}tz^{-1};p)_{\infty}}{ut(u - q^{-k+1}z)(q - q^{-1})u - t(q^{-k+1}tz^{-1};p)_{\infty}}, \quad \text{with } C : |u| > |q^{-k-1}z|, |z| > |q^{-k+1}t|, |u| > |q^{-2}t|, \\
I_{-1,-1}^{(1),(1)}(z,t,u|C) &= \frac{zq^{k+1}(q^{k-2}z)^{-\frac{1}{k+2}}(q^{-k+1}tz^{-1};p)_{\infty}}{ut(u - q^{k+1}z)(q - q^{-1})(q^{-k+1}tz^{-1};p)_{\infty}}, \quad \text{with } C : |z| > |q^{-k+1}u|, |z| > |q^{-k-1}t|. \quad \text{(8.291)}
\end{align*}
\]

### 8.3 Trace of vertex operators over Fock spaces

Let \( \{a_n, n \neq 0\} \) generate a Heisenberg algebra with

\[ [a_n, a_m] = h(n)\delta_{n+m,0}. \] (8.292)
Define as usual a vacuum vector \( |0> \) by

\[
a_n|0> = 0, \quad n > 0. \tag{8.293}
\]

Let \( F \) be a Fock space generated from the successive actions of the creation operators on \( |0> \) then we have the following formula for the evaluation of a trace of vertex operators

\[
\text{Tr}_F(\mu \sum_{n>0} n a_n a_n / h(n) \exp(- \sum_{n=0} \mu \cdot a_n A_n) \exp(\sum_{n>0} a_n A_n)) = \frac{1}{\prod_{n=1}^\infty (1 - \mu^n)} \exp(- \sum_{m=1}^\infty h(m) A_m A_m \frac{\mu^m}{1 - \mu^m}). \tag{8.294}
\]

Here \( A_n \) and \( A_{-n}, n > 0 \), are given c-functions.

### 8.4 Applications of the trace formula

Let us consider the general trace in the sector \( X_1 \)

\[
T^{X_1}_\ell (N_1, N_2, N_3, N_4, n_4 | X, Y, Z, W) = \text{Tr}_{F_{\ell}^{X_1}}(\mu^{-d_{X_1}} \nu^{-s_{X_1,0}} \prod_{i=1}^{N_1} e^{X_1(k; 2, k+2|q^k x_i; \frac{k+2}{2})} \prod_{j=1}^{N_2} e^{X_1(1; 2, k+2|q^k y_j; \frac{k+2}{2})} \prod_{r=1}^{N_3} e^{-X_1(k+2|q^k z_r; \frac{k+2}{2})} \prod_{s=1}^{n_4} e^{\partial X_1(q^k w_s; \frac{k+2}{2})} :)} \tag{8.295}
\]

where the sets \( X, Y, Z, W \) are defined by

\[
X = \{x_1, \cdots, x_{N_1}\}, \\
Y = \{y_1, \cdots, y_{N_2}\}, \\
Z = \{z_1, \cdots, z_{N_3}\}, \\
W = \{w_1, \cdots, w_{n_4}\}. \tag{8.296}
\]

Applying the general formula (8.294) for the trace we find

\[
T^{X_1}_\ell (N_1, N_2, N_3, N_4, n_4 | X, Y, Z, W) = \frac{\delta_{N_1+N_2-2N_3} G_1(X)G_2(Y)G_3(Z)G_4(X, Y)G_5(X, Z) \times G_6^+(X, W)G_6^-(X, W)G_7(Y, Z)G_8^+(Y, W)G_8^-(Y, W) \times G_9^+(Z, W)G_9^-(Z, W)G_{10}}{(\mu; \mu)_\infty} \tag{8.297}
\]
where

\[
G_1(X) = \prod_{i \leq j} \frac{(q^{2k-2}p\mu_{x_j}x_{ij}^{-1}; q^4; p; \mu)_\infty (q^{2k+2}p\mu_{x_i}x_{ij}^{-1}; q^4; p; \mu)_\infty}{(q^{2k}p\mu_{x_i}x_{ij}^{-1}; q^4; p; \mu)_\infty (q^{2k+2}p\mu_{x_j}x_{ij}^{-1}; q^4; p; \mu)_\infty} \times (q^{-2k+2}p\mu_{x_j}x_{ij}^{-1}; q^4; p; \mu)_\infty (q^{-2k+2}p\mu_{x_i}x_{ij}^{-1}; q^4; p; \mu)_\infty,
\]

\[
G_2(Y) = \prod_{i \leq j} \frac{(q^{4}p\mu_{y_j}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_i}y_{ij}^{-1}; q^4; p; \mu)_\infty}{(q^{4}p\mu_{y_j}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_i}y_{ij}^{-1}; q^4; p; \mu)_\infty},
\]

\[
G_3(Z) = \prod_{i \leq j} \frac{(q^{-2}p\mu_{z_j}z_{ij}^{-1}; p; \mu)_\infty (q^{-2}p\mu_{z_j}z_{ij}^{-1}; p; \mu)_\infty (q^{-2}p\mu_{z_i}z_{ij}^{-1}; p; \mu)_\infty}{(q^{2}p\mu_{z_i}z_{ij}^{-1}; p; \mu)_\infty (q^{2}p\mu_{z_j}z_{ij}^{-1}; p; \mu)_\infty (q^{2}p\mu_{z_i}z_{ij}^{-1}; p; \mu)_\infty},
\]

\[
G_4(X, Y) = \prod_{i, j} \frac{(q^{4}p\mu_{x_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}x_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{x_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}x_{ij}^{-1}; q^4; p; \mu)_\infty}{(q^{4}p\mu_{x_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}x_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{x_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}x_{ij}^{-1}; q^4; p; \mu)_\infty},
\]

\[
G_5(X, Z) = \prod_{i, j} \frac{(q^{4}p\mu_{x_i}z_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{x_i}z_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{z_j}x_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{z_j}x_{ij}^{-1}; q^4; p; \mu)_\infty}{(q^{4}p\mu_{z_i}z_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{z_i}z_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{z_j}x_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{z_j}x_{ij}^{-1}; q^4; p; \mu)_\infty},
\]

\[
G_6^+ (X, W) = \prod_{i=1}^{n_4} \frac{(q^4p\mu_{x_i}w_i^{-1}; \mu)_\infty}{(q^{2k+2}p\mu_{x_i}w_i^{-1}; \mu)_\infty},
\]

\[
G_6^- (X, W) = \prod_{i=n_4+1}^{n_3} \frac{(q^{-2k-2}p\mu_{x_i}^{-1}w_i; \mu)_\infty}{(q^{-2k-2}p\mu_{x_i}^{-1}w_i; \mu)_\infty},
\]

\[
G_7 (Y, Z) = \prod_{i, j} \frac{(q^{4}p\mu_{y_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}y_{ij}^{-1}; q^4; p; \mu)_\infty}{(q^{4}p\mu_{y_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_i}y_{ij}^{-1}; q^4; p; \mu)_\infty (q^{4}p\mu_{y_j}y_{ij}^{-1}; q^4; p; \mu)_\infty},
\]

\[
G_8^+ (Y, W) = \prod_{i=1}^{n_4} \frac{(q^{-3}p\mu_{y_i}w_i^{-1}; \mu)_\infty}{(q^{-3}p\mu_{y_i}w_i^{-1}; \mu)_\infty},
\]

\[
G_8^- (Y, W) = \prod_{i=n_4+1}^{n_3} \frac{(q^{-3}p\mu_{y_i}^{-1}w_i; \mu)_\infty}{(q^{-3}p\mu_{y_i}^{-1}w_i; \mu)_\infty},
\]

\[
G_9^+ (Z, W) = \prod_{i=1}^{n_4} \frac{(q^{-k-3}p\mu_{z_i}w_i^{-1}; \mu)_\infty}{(q^{-k-3}p\mu_{z_i}w_i^{-1}; \mu)_\infty},
\]

\[
G_9^- (Z, W) = \prod_{i=n_4+1}^{n_3} \frac{(q^{-k-3}p\mu_{z_i}^{-1}w_i; \mu)_\infty}{(q^{-k-3}p\mu_{z_i}^{-1}w_i; \mu)_\infty},
\]

\[
G_{10} = \mu^{i+\frac{(k+2)}{12}} n^{-\ell} q^{2n_4-N_4} \prod_{i=1}^{N_1} (q^k x_i) \prod_{i=1}^{N_2} (q^{k-2} y_i) \prod_{i=1}^{N_3} (q^{2} z_i^{-1}) = \text{Tr}_{\mathcal{F}_{\mathcal{X}_2, \mathcal{X}_3}} \mathcal{O} = \text{Tr}_{\mathcal{F}_{\mathcal{X}_2, \mathcal{X}_3}} \left( \mathcal{O} \oint_{2\pi i} dy \eta(y) \xi(w_0) \right) |_{a_{X_2,0}+a_{X_3,0}=0} \quad (8.299)
\]
with
\[
\mathcal{O} = \mu^{-a_{X_2} - a_{X_3}} \nu^{-a_{X_2} \delta} \prod_{i=1}^{\bar{N}} e^{-X_3(2|q^a_i x_i;0)} \prod_{j=1}^{\bar{N}} e^{X_3(2|q^b_j y_j;0)} \prod_{r=1}^{\bar{M}} e^{-X_2(2|q^c_r z_r;\nu)} \prod_{s=1}^{\bar{M}} e^{X_2(2|q^d_s w_s;\mu)},
\]

(8.300)

and the various sets of parameters are defined by

\[
A = \{a_1, \ldots, a_{\bar{N}}\},
B = \{b_1, \ldots, b_{\bar{N}}\},
C = \{c_1, \ldots, c_{\bar{M}}\},
D = \{d_1, \ldots, d_{\bar{M}}\},
G = \{g_1, \ldots, a_{\bar{M}}\},
H = \{h_1, \ldots, a_{\bar{M}}\},
\tilde{X} = \{x_1, \ldots, x_{\bar{N}}\},
\tilde{Y} = \{y_1, \ldots, y_{\bar{N}}\},
\tilde{Z} = \{z_1, \ldots, z_{\bar{M}}\},
\tilde{W} = \{w_1, \ldots, w_{\bar{M}}\},
\]

(8.301)

Applying (8.294) we find

\[
T^{X_2,X_3}(A, B, C, D, G, H|\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = \frac{\delta_{\bar{N},\bar{M}^\gamma}}{\prod_{n=1}^{\infty}(1 - \mu^n)^2} G_1(\tilde{X}, w_0)G_2(\tilde{Y}, \gamma)G_3(\tilde{Z})G_4(\tilde{W})G_5(\tilde{X}, \tilde{Y}, w_0, \nu)G_6(\tilde{Z}, \tilde{W})G_7(\tilde{X}, \tilde{Y}, w_0, \gamma)
\]

(8.302)

where the delta function \(\delta_{N,M}\) ensures that the above trace is restricted to the space \(\tilde{F}^{X_2,X_3}\), and

\[
G_1(\tilde{X}, w_0) = \prod_{i,i'=0}^{\bar{N}} (\mu q^{a_i-a_{i'}} x_i x_{i'}^{-1}; \mu)_\infty, \quad \text{with} \quad x_0 = w_0, \quad a_0 = -k - 2,
\]

\[
G_2(\tilde{Y}, \gamma) = \prod_{j,j'=0}^{\bar{N}} (\mu q^{b_j-b_{j'}} y_j y_{j'}^{-1}; \mu)_\infty, \quad \text{with} \quad y_0 = \gamma, \quad b_0 = -k - 2,
\]

\[
G_3(\tilde{Z}) = \prod_{r,r'=1}^{\bar{M}} (\mu q^{c_r+d_r-c_{r'}+d_{r'}-z_r z_{r'}^{-1}}; \mu)_\infty^{-1},
\]

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\[ G_4(\tilde{W}) = \prod_{s, s' = 1}^{M} (\mu q^{g_s + h_s - g_{s'} w_s w_{s'}^{-1}}; \mu)^{-1}, \]
\[ G_5(\tilde{X}, \tilde{Y}, w_0, \gamma) = \prod_{i, j = 0}^{N} (\mu q^{a_i - b_j x_i y_j^{-1}}; \mu)^{-1} (\mu q^{b_j - a_i y_j x_i^{-1}}; \mu)^{-1}, \]
\[ G_6(\tilde{Z}, \tilde{W}) = \prod_{r, s = 1}^{M} (\mu q^{x_i - b_j y_j^{-1}} z_r w_s^{-1}; \mu) (\mu q^{y_j - a_i x_i^{-1}} z_r w_s^{-1}; \mu)^{-1}, \]
\[ G_7(\tilde{X}, \tilde{Y}, w_0, \gamma) = \frac{q^{k+2}}{\gamma - w_0} \prod_{i = 1}^{N} q^{a_i x_i} - q^{-k-2} w_0 \prod_{j = 1}^{M} q^{b_j y_j} - q^{-k-2} \gamma, \]
\[ \gamma = w_0 \nu^{-2} \prod_{i = 1}^{N} q^{a_i - b_j x_i y_i^{-1}} \prod_{r = 1}^{M} q^{b_j - a_i x_i^{-1}} w_r z_r^{-1}. \quad (8.303) \]

8.5 Useful functions and integrals

Eta function:
\[ \eta(z) = z^{\frac{1}{24}} \prod_{n = 1}^{\infty} (1 - z^n). \quad (8.304) \]

Jacobi theta function:
\[ \theta_p(z) = (z; p) (p z^{-1}; p) (p; p), \quad \text{with} \quad (z; p) = \prod_{n = 0}^{\infty} (1 - z p^n). \quad (8.305) \]

Jackson integrals:
\[ \int_{0}^{c} d_p t f(t) = c(1 - p) \sum_{n = 0}^{\infty} f(c p^n) p^n, \]
\[ \int_{0}^{c \infty} d_p t f(t) = c(1 - p) \sum_{n = -\infty}^{\infty} f(c p^n) p^n. \quad (8.306) \]

q-gamma function:
\[ \Gamma_p(z) = \frac{(p; p) (p z^{-1}; p) (1 - p)^{1-z}}{(p^z; p)}. \quad (8.307) \]

q-beta function:
\[ B_p(x, y) = \int_{0}^{1} d_p t (t) (p t; p) (p^x; p) (p^{y-t}; p)^{\infty}. \quad (8.308) \]

Basic hypergeometric series:
\[ F_p(a, b, c; z) = \frac{\Gamma_p(c)}{\Gamma_p(a) \Gamma_p(c - a)} \int_{0}^{1} d_p t (t) (p t; p) (p^{a-t}; p) (p^{b-t}; p) (p^{c-a-t}; p) (z t; p)^{\infty}. \quad (8.309) \]
8.6 Useful relations

\[ B_p(x, y) = \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x + y)}. \quad (8.310) \]
\[ \Gamma_p(z + 1) = \frac{1 - p^z}{1 - p} \Gamma_p(z). \quad (8.311) \]

\[ \int_0^c d_p t f(t) = c \int_0^1 d_p t f(ct), \]
\[ \int_{c\infty} d_p t f(t) = c \int_p^\infty d_p t f(ct), \]
\[ \int_p^\infty d_p t f(t) = \int_1^1 d_p t t^{-2} f(t^{-1}). \quad (8.312) \]

Fourier double transformation formula:

\[ \sum_{n \in \mathbb{Z}} \oint_{C_0} \frac{dz}{2\pi i} z^n f(z) = f(1). \quad (8.313) \]

Triple Jacobi identity

\[ \sum_{n \in \mathbb{Z}} x^{n^2/2} y^{2n} = (x; x)_\infty (-x^{1/2} y^2; x)_\infty (-x^{1/2} y^2; x)_\infty. \quad (8.314) \]

8.7 Face type Boltzmann weights

The face type Boltzmann weights \( W \left( \begin{array}{c|c} \lambda & \mu \\ \mu' & \nu \end{array} \right| z \) that appear in the commutation relations of the vertex operators (4.117) are given by

\[ W \left( \begin{array}{c|c} \lambda & \mu \\ \mu' & \nu \end{array} \right| z \) = -z^{2\lambda+\lambda_0-\lambda_0-1/2} \xi(z; 1, pq^4)^{-1/2} \times \hat{W} \left( \begin{array}{c|c} \lambda & \mu \\ \mu' & \nu \end{array} \right| z \),

\[ \hat{W} \left( \begin{array}{c} \lambda \lambda_+ \\ \lambda_+ \lambda \end{array} \right| z \) = \frac{\theta_p(pq^2)}{\theta_p(pq^{-2n-2}z)} \frac{\theta_p(pq^2)}{\theta_p(pq^2z)} ,

\[ \hat{W} \left( \begin{array}{c} \lambda \lambda_+ \\ \lambda_+ \lambda \end{array} \right| z \) = q^{-1} \times \frac{\Gamma_p((2n + 2)s)\Gamma_p((2n + 2)s)}{\Gamma_p((2n + 4)s)\Gamma_p(2ns)} \frac{\theta_p(pz)}{\theta_p(pq^2z)} ,

\[ \hat{W} \left( \begin{array}{c} \lambda \lambda_- \end{array} \right| z \) = q^{-1} \times \frac{\Gamma_p(1 - (2n + 2)s)\Gamma_p(1 - (2n + 2)s)}{\Gamma_p(1 - (2n + 4)s)\Gamma_p(1 - 2ns)} \frac{\theta_p(pz)}{\theta_p(pq^2z)} ,

\[ \hat{W} \left( \begin{array}{c} \lambda \lambda_- \end{array} \right| z \) = z^{-1} \times \frac{\theta_p(pq^2)}{\theta_p(pq^2z)} \frac{\theta_p(q^{2n+2}z)}{\theta_p(pq^2z)} ,

\[ \hat{W} \left( \begin{array}{c|c} \lambda \lambda_0 \\ \lambda_0 \lambda \end{array} \right| z \) = 1, \]
\[ \hat{W} \left( \begin{array}{cc} \lambda & \lambda \pm \\ \lambda \pm & \lambda \pm \end{array} \bigg| z \right) = 0 \quad \text{otherwise.} \] (8.315)
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