The index of self-adjoint Shapiro-Lopatinskii boundary problems of order 1

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1. Introduction

**Elliptic local boundary problems.** Let $X$ be a compact manifold with boundary $Y = \partial X$ and let $E$ be a Hermitian vector bundle over $X$. In order to state a *boundary problem* for sections of $E$ one needs a differential or pseudo-differential operator $P$ acting on sections of $E$ and some *boundary conditions* on sections of $E$. In this paper we are interested in *local* boundary conditions described by a *boundary operator* $B$ taking sections of $E$ over $X$ to sections of a vector bundle $G$ over $Y$. The boundary problem defined by $P$, $B$ is called an *elliptic boundary problem* if $P$ is elliptic and $B$ satisfies the Shapiro-Lopatinskii condition.

In this paper we consider only operators $P$ of order 1. In this case one can assume that $B$ has the form $B = B_Y \circ \gamma$, where $\gamma$ takes sections of $E$ over $X$ to their restrictions to $Y$, and $B_Y$ is a differential or pseudo-differential operator from sections of $E|_Y$ to sections of $G$. A crucial role is played by the kernel $N$ of the symbol of $B_Y$ over the unit cotangent sphere bundle $S_Y$ of $Y$. It is a subbundle of the lift to $S_Y$ of the restriction $E|_Y$. We will call the subbundle $N$ the *kernel-symbol* of $B_Y$ and $B$. If $N$ is equal to the lift to $S_Y$ of a subbundle of $E|_Y$, we will say that $N$ is *bundle-like*, borrowing a term of Bandara, Goffeng, and Saratchandran [BGS].

**The Atiyah–Bott–Singer index theorem.** If $P$, $B$ is elliptic, then $P \oplus B$ defines Fredholm operators in appropriate Sobolev spaces and hence the index of $P \oplus B$ is defined. By the Atiyah–Bott–Singer index theorem [AB] this index is equal to the *topological index* of $P$, $B$ defined in terms of the principal symbols of $P$ and $B_Y$. The Atiyah–Bott–Singer proof of this index theorem was supposed to be included in the series of papers [AS1] – [AS5] by Atiyah and Singer. See the introduction to [AS1]. But this proof was never published. An improved version of the analytic part was written down by Hörmander [H]. The crux of the matter is the definition of the topological index. See Atiyah [A1], [A2]. The definition is not quite direct and is based on using the boundary condition to canonically deform the symbol of $P$ to a special form, which we will call *bundle-like*. This deformation, in turn, is based on a deformation of the kernel-symbol $N$ to a bundle-like one.

**Self-adjoint elliptic operators on closed manifolds.** For individual self-adjoint Fredholm operators there is no analogue of index. But if one considers families of operators instead of individual operators, the index theory for self-adjoint Fredholm operators turns out to be as rich as the one for Fredholm operators. Such a theory was developed by Atiyah–Singer [AS] and applied by Atiyah–Patodi–Singer [APS] to prove an analogue of the Atiyah–Singer theorem for families of self-adjoint elliptic operators on closed manifolds.

**Self-adjoint local elliptic boundary problems.** It is only natural to ask for an analogue of the Atiyah–Bott–Singer index theorem for families of self-adjoint local boundary problems, but, apparently, this question hardly attracted any serious attention until the last decade.
About ten years ago the physics of graphene and, in particular, questions posed by M. Katsnelson, served as a stimulus to address this question at least for operators of order 1 and families parameterized by circle, i.e. for the spectral flow. The relevant physics is discussed in [KN]. The case when \( \dim X = 2 \) was addressed by Prokhorova [P1], Katsnelson and Nazaikinskii [KN], and then by Prokhorova [P2], [P3] in greater generality. The spectral flow of linear families of Dirac operators was considered by Gorokhovsky and Lesch [GL].

In the present paper this question is addressed for operators of order 1 in arbitrary dimension. Many of the features specific to the self-adjoint case are already present in this case. Following Hörmander [H], we do not strive for the greatest possible generality and consider only pseudo-differential operators belonging to a class introduced in [H], Chapter 20.

We will consider families of boundary problems \( P, B \), parameterized by a topological space \( Z \), such that the operators \( P \) are self-adjoint and induce unbounded self-adjoint operators in appropriate Sobolev spaces when restricted to the kernel \( \text{Ker} B \). The latter property requires the kernel-symbol \( N \) of \( B \) to be a lagrangian subbundle with respect to an indefinite metric defined in terms of the symbol of \( P \) at \( Y \). It turns out that one also need to assume that the boundary operators \( B \) are bundle-like in the sense that they are induced by a morphisms of bundles \( E|Y \to G \). Then the kernel-symbols \( N \) are also bundle-like.

For such families one can define both the analytical index reflecting behavior of the induced unbounded self-adjoint operators and the topological index depending only on the families of the symbols of \( P \) and the kernel-symbols \( N \). Both indices are elements of \( K^1(Z) \), as they should be. We will prove that the analytical index is equal to the topological one under a moderate topological assumption about the symbol of \( P \), and that, in general, these indices may differ only by an element of order 2 in \( K^1(Z) \). See Theorems 8.1 and 8.2.

**The topological index.** Similarly to Atiyah–Bott–Singer theorem [AB], defining the topological index is a key part. As in [AB], this is done by using the boundary conditions to deform the symbol \( \sigma \) of \( P \) to a standard form on the boundary. Similarly to [AB], we first consider what happens over a point \( u \in S \). This involves some nice geometry of self-adjoint operators and lagrangian subspaces in finite dimensions and is the subject of Section 2.

In Section 3 we begin by performing the deformations of Section 2 simultaneously for every \( u \in S \). This brings \( \sigma \) into a standard form depending almost only on \( N \), but in order to get an analogue of Atiyah–Bott definition one needs to further deform the kernel-symbol \( N \) and then the symbol \( \sigma \) to bundle-like ones. In [AB] this is done by the Fifth Homotopy. Somewhat disappointingly, this homotopy does not work in the self-adjoint case. Moreover, there is a non-trivial obstruction to deforming the symbol \( \sigma \) and the kernel-symbol \( N \) to bundle-like ones. See Section 3. By these reasons one needs to require at least that \( N \) can be deformed to a bundle-like kernel-symbol after some stabilization. Moreover, the deformation should continuously depend on the parameter \( z \in Z \). When such a deformation exists, the remaining part of the definition is similar to the Atiyah–Singer definition [AS4].
The analytical index. Defining the analytical index also encounters new difficulties. The analytical index of families of self-adjoint elliptic operators on closed manifolds was defined by Atiyah–Patodi–Singer [APS]. This definition is based on Atiyah–Singer [AS] theory of self-adjoint (actually, skew-adjoint) bounded operators in a Hilbert space, and requires, as the first step, replacing given operators by operators of order 0. With the theory of pseudo-differential operators at hand, this is very simple on closed manifolds, but not so on manifolds with boundary. Author's approach to the analytical index [I_2] works equally well for families of bounded and unbounded operators with minimal continuity properties.

Section 4 is devoted to a little theory of abstract boundary problems serving as a bridge between pseudo-differential boundary problems and the theory developed in [I_2]. This little theory may be considered as an axiomatic parameterized version of some aspects of the classical theory of boundary problems as presented, for example, in [W], Section 15. The framework of this little theory resembles that of the theory of boundary triples (see, for example, Schmüdgen [S], Chapter 14 for the latter), but the goals are different.

Section 5 begins by recalling the definition of Hörmander's class of operators and a description of pairs P, B leading to abstract boundary problems in the sense of Section 4. After this we turn to the problem of realization of symbols and kernel-symbols by such pairs.

Realization of kernel-symbols. This is the topic of the rest of Section 5. For the Atiyah–Singer [AS1], [AS4] approach to index theorems one needs to be able to define not only the analytical index of operators and families of operators, but also the analytical index of symbols and families of symbols. This is done by using pseudo-differential operators to construct realizations of symbols and families of symbols. It turns out that for self-adjoint boundary problems there is a non-trivial obstruction to the existence of self-adjoint realizations. It is similar to the obstruction to defining the topological index (actually, it is weaker than the latter). Moreover, when this obstruction vanish, the realization is not unique. This is the main reason for eventually requiring boundary conditions to be bundle-like: if a kernel-symbol N is bundle-like, it has a canonical bundle-like realization.

A similar phenomenon of the non-uniqueness of realizations in the context of Atiyah–Patodi–Singer boundary conditions was encountered and thoroughly investigated by Melrose and Piazza [MP1], [MP2]. These boundary conditions are canonical realizations of some kernel-symbols, but, in contrast with bundle-like realizations of bundle-like kernel-symbols, these realizations do not continuously depend on parameters.

The proof of the index theorems. With all preliminary work done, we turn to the proof of our index theorems. We adapt to the self-adjoint boundary problems Hörmander's approach to Atiyah–Bott-Singer index theorem. See [H], Chapter 20. The first step is to construct sufficiently many standard boundary problems guaranteed to have vanishing analytical index. This is done in Section 6, which is largely based on the proof of Proposition 20.3.1 in [H]. The next step is to take (families of) boundary problems standard near the boundary
and glue them to standard ones to get self-adjoint operators on the double of $X$, which is a
closed manifold. This is the topic of Section 7. This constructions does not affect neither
the analytical, nor the topological index. The proof for the analytical index is based on the
proof of Proposition 20.3.2 in [H]. The proof for the topological index is fairly routine. Fi-
nally, in Section 8 we apply the results of Sections 6 and 7 to reduce the index problem to
the case of closed manifolds, and then apply the Atiyah–Patodi–Singer index theorem [APS].
See [APS], Theorem (3.4). This proves our first index theorem, Theorems 8.1.

The additional assumption in Theorem 8.1 is caused by the limitations of the construction
of boundary problems with index zero used in Section 6. A simple doubling trick shows
that this assumption always holds for the direct sum of a boundary problem with a copy of
it. This proves Theorem 8.2, the index theorem modulo elements of order 2 in $K^1(Z)$.

**Special boundary conditions.** We will say that a boundary condition is *special* if it satis-
fies not only the Shapiro-Lopatinskii condition, but also a natural “dual” form of it. For
example, if $P$ is a *differential operator* of order 1, then every elliptic boundary condition
is automatically special. Section 9 is devoted to such boundary conditions for self-adjoint
symbols minimally resembling at the boundary $Y$ symbols of Dirac operators. We call such
symbols *anti-commuting*. See Section 9 for the precise definition. The main result of this
section is an Agranovich–Dynin type theorem, Theorem 9.3.

Given an anti-commuting self-adjoint symbol $\sigma$ together with a special boundary condition,
one can define a natural boundary condition “dual” to the given one. One can also do the
same for families. Theorem 9.3 expresses the difference of the topological indices of $\sigma$ with
these two boundary conditions in terms of the restriction of $\sigma$ to the boundary. An inter-
esting aspect of this result is the natural appearance of the original Bott periodicity map [B]
in the proof. See Theorem 9.1 and its proof.

**Dirac-like boundary problems.** In Section 10 we consider an even more narrow class of
self-adjoint operators $P$ and boundary conditions. We call the corresponding boundary prob-
lems *Dirac-like*, although we require only a small part of the rich algebraic structure of Dirac
operators to be present. Basically, the operator $P$ is required to be odd with respect to a
$\mathbb{Z}/2$-grading of the bundle $E$, its symbol is required to induce a skew-adjoint symbol on a
“half” of the bundle $E|Y$, and the boundary condition is required to commute with this
induced symbol. See Section 10 for the precise definitions.

The main result of Section 10 is a theorem expressing the topological and the analytical
index of families of such boundary problems in terms of the topological index of some fam-
ilies of self-adjoint operators induced on the boundary. See Theorem 10.3. This theorem is
motivated by the results of Gorokhovsky and Lesch [GL], who considered only Dirac opera-
tors (but indicated that many of their arguments are valid in greater generality) and only the
spectral flow of linear families. The proof of Theorem 10.3 is almost entirely topological, in
contrast with the heat equation methods used in [GL].
2. Geometric algebra at the boundary

**Elliptic pairs.** Let $E$ be a finitely dimensional vector space over $C$ equipped with a Hermitian positive definite scalar product $\langle \cdot, \cdot \rangle$, and let $\sigma, \tau: E \to E$ be two linear operators. We will say that $\sigma, \tau$ is an elliptic pair if the operator $a \sigma + b \tau$ is invertible for every $a, b \in \mathbb{R}$ such that $(a, b) \neq 0$. Then, in particular, $\sigma$ and $\tau$ are invertible. Moreover, $\lambda + \sigma^{-1}\tau$ is invertible for every $\lambda \in \mathbb{R}$. In other words, $\sigma^{-1}\tau$ has no real eigenvalues. For the rest of this section we will assume that $\sigma, \tau$ is an elliptic pair. Let $\rho = \sigma^{-1}\tau$.

**The ordinary differential equation associated with $\sigma, \tau$.** Let $t$ be a variable running over $\mathbb{R}$ and let $\partial = dt/dt$ be the usual differentiation operator. Let $D = -i \partial$. We are interested in the differential equation $(\sigma D + \tau)(f) = 0$ for functions $f: \mathbb{R} \to E$. Obviously, it is equivalent to the equation $D(f) + \rho(f) = 0$. It is well known that every solution of such an equation is a linear combination of solutions of the form $f(t) = p(t) e^{-i\lambda t}$, where $\lambda \in \mathbb{C}$ is an eigenvalue of $\rho$ and $p(t)$ is a polynomial with coefficients in the generalized eigenspace $\mathcal{E}_\lambda(\rho) \subset E$ of $\rho$ corresponding to $\lambda$.

A solution of the form $p(t) e^{-i\lambda t}$ is bounded on $\mathbb{R}_{\geq 0}$ if and only if $\text{Re} - i\lambda \leq 0$, or, equivalently, $\text{Im} \lambda \leq 0$. Since $\rho$ has no real eigenvalues, the last condition is equivalent to $\text{Im} \lambda < 0$ and such solutions are actually exponentially decreasing when $t \to \infty$. The solutions of the form $p(t) e^{-i\lambda t}$ with $\text{Im} \lambda > 0$ are exponentially increasing when $t \to \infty$.

Let us denote by $\mathcal{M}_+(\rho)$ and $\mathcal{M}_-(\rho)$ the spaces of solutions exponentially decreasing and, respectively, increasing when $t \to \infty$.

At the same time a solution $f(t)$ is determined by its initial value $f(0)$. It follows that $\mathcal{M}_+(\rho)$ is canonically isomorphic to the space $\mathcal{L}_-(\rho) \subset E$, the sum of the generalized eigenspaces of $\rho$ corresponding to the eigenvalues $\lambda$ with $\text{Im} \lambda < 0$. Similarly, $\mathcal{M}_-(\rho)$ is canonically isomorphic to the space $\mathcal{L}_+(\rho) \subset E$, the sum of the generalized eigenspaces of $\rho$ corresponding to the eigenvalues $\lambda$ with $\text{Im} \lambda > 0$.

**Self-adjoint elliptic pairs and the associated Pontrjagin spaces.** From now on we will assume that the elliptic pair $\sigma, \tau$ is self-adjoint in the sense that the operators $\sigma, \tau$ are self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle$. As usual, for a linear operator $\varphi: E \to E$ we will denote by $\varphi^*$ its adjoint with respect to $\langle \cdot, \cdot \rangle$. Let us define a new scalar product $[\cdot, \cdot]$ on $E$, which in general will be indefinite. Namely, for $u, v \in E$ let

$$[u, v] = \langle \sigma(u), v \rangle.$$  

Since the operator $\sigma$ is self-adjoint and $\langle \cdot, \cdot \rangle$ is Hermitian, $\langle \sigma(u), v \rangle = \overline{\langle \sigma(v), u \rangle}$ and hence $[u, v] = [v, u]$ for every $u, v \in E$, i.e. the scalar product $[\cdot, \cdot]$ is Hermitian. Since $\sigma$ is invertible, the scalar product $[\cdot, \cdot]$ is non-degenerate. Therefore $E$ together with the product $[\cdot, \cdot]$ is a Pontrjagin space, or an indefinite inner product space.
The operator \( \rho \) turns out to be \textit{self-adjoint with respect to} \( \langle \cdot , \cdot \rangle \), i.e. such that
\[
[\rho(u), v] = [u, \rho(v)]
\]
for every \( u, v \in E \). Indeed, since \( \tau \) is self-adjoint, \( \langle \tau(u), v \rangle = \langle u, \tau(v) \rangle \), or, what is the same, \( \langle \sigma \circ \rho(u), v \rangle = \langle u, \sigma \circ \rho(v) \rangle \) for every \( u, v \in E \). This implies that
\[
[\rho(u), v] = \langle \sigma \circ \rho(u), v \rangle = \langle u, \sigma \circ \rho(v) \rangle = \langle \sigma(u), \rho(v) \rangle = [u, \rho(v)],
\]
as claimed. The operator \( \rho \) cannot be self-adjoint with respect to \( \langle \cdot , \cdot \rangle \). Indeed,
\[
\rho^* = (\sigma^{-1} \tau)^* = \tau^* (\sigma^{-1})^* = \tau \sigma^{-1} = \sigma \sigma^{-1} \tau \sigma^{-1} = \sigma \rho \sigma^{-1}.
\]
Hence \( \rho^* = \rho \) if and only if \( \rho \) commutes with \( \sigma \), or, equivalently, \( \tau \) commutes with \( \sigma \). But, if \( \tau \) commutes with \( \sigma \), then there exists a common eigenvector of \( \sigma \) and \( \tau \), and this contradicts the ellipticity assumption. At the same time \( \rho \) is often skew-adjoint with respect to \( \langle \cdot , \cdot \rangle \), i.e. such that \( \rho^* = -\rho \). Clearly, this happens if and only if \( \rho \) anti-commutes with \( \sigma \), or, equivalently, \( \tau \) anti-commutes with \( \sigma \). Such elliptic pairs \( \sigma, \tau \) usually arise from Clifford modules, but, as we will see, they are quite ubiquitous.

By introducing the factor \( i = \sqrt{-1} \) at various places one can rephrase our discussion in the language of \textit{complex symplectic spaces}, but the language of Pontrjagin spaces seems to be more natural for our purposes.

\textbf{2.1. Lemma.} If \( \lambda \neq \overline{\mu} \), then the generalized eigenspaces \( E_{\lambda}(\rho) \) and \( E_{\mu}(\rho) \) are orthogonal with respect to the scalar product \( [\cdot , \cdot] \).

\textbf{Proof.} The proof is similar to the one in the positive definite case. See Pontrjagin \cite{Po}, Section 2(A), or Bongár \cite{Bog}, Theorem 3.3. \( \square \)

\textbf{2.2. Corollary.} The restrictions of \( [\cdot , \cdot] \) to \( L_+^{\rho} \) and to \( L_-^{\rho} \) are equal to zero. \( \square \)

\textbf{2.3. Corollary.} The dimensions of the spaces \( L_+^{\rho} \) and \( L_-^{\rho} \) are equal.

\textbf{Proof.} Since \( [\cdot , \cdot] \) is non-degenerate, for every linear map \( l: L_+^{\rho} \rightarrow \mathbb{C} \) there exists \( v \in E \) such that \( l(u) = [u, v] \) for \( u \in L_+^{\rho} \). Since \( [u, w] = 0 \) for \( u, w \in L_+^{\rho} \), there exists \( v \in L_-^{\rho} \) with this property. It follows that the natural map from \( L_-^{\rho} \) to the dual space of \( L_+^{\rho} \) defined by \( [\cdot , \cdot] \) is surjective. Hence \( \dim L_+^{\rho} \leq \dim L_-^{\rho} \). By the same reasons \( \dim L_-^{\rho} \leq \dim L_+^{\rho} \). \( \square \)

\textbf{Lagrangian subspaces.} Here we largely follow F. Latour \cite{L}, Section I.1. A subspace \( L \subset E \) is said to be \textit{lagrangian} if \( L \) is equal to its orthogonal complement with respect to \( [\cdot , \cdot] \).
Since $\langle \cdot, \cdot \rangle$ is non-degenerate, the dimension of a lagrangian subspace is equal to the half of the dimension of $E$. In particular, if a lagrangian subspace exists, then $\dim E$ is even. Two lagrangian subspaces $L, L'$ are said to be transverse if $L \cap L' = 0$, or, equivalently, $L + L' = E$. Our main example of transverse lagrangian subspaces is $\mathcal{L}_+(\rho), \mathcal{L}_-(\rho)$.

For a lagrangian subspace $L \subset E$ let us denote by $\Lambda_L$ the set of all lagrangian subspaces transverse to $L$. Let $M \in \Lambda_L$ and let $q : E \rightarrow L$ be the projection along $M$. Then

$$q(x) = x \text{ for every } x \in L \text{ and}$$

$$[q(u), v] + [u, q(v)] = [u, v] \text{ for every } u, v \in E.$$ 

Indeed, the first property is trivial, and it is sufficient to check the second one in the case when each of $u, v$ belongs to either $M$ or $L$. But in this case it is trivial. Conversely, if $q : E \rightarrow L$ has these properties, then an immediate verification shows that $M = \ker q$ is a lagrangian subspace transverse to $L$. If $q'$ is another map with these properties, then $\delta = q - q'$ is a linear map equal to zero on $L$ and such that

$$(1) \quad [\delta(u), v] + [u, \delta(v)] = 0$$

for every $u, v \in E$. Therefore $\Lambda_L$ is an affine space over the vector space of such maps $\delta$. In particular, $\Lambda_L$ is contractible. The structure of an affine space on $\Lambda_L$ does not depend on $M$. But if a base point $M \in \Lambda_L$ is fixed, then we can identify $\Lambda_L$ with the space of such maps $\delta$. The lagrangian subspace $M'$ corresponding to $\delta$ is equal to the kernel $\ker (q - \delta)$ and hence is equal to the graph $\{u + \delta(u) \mid u \in M\}$ of the restriction $\delta|_M$ of $\delta$ to $M$.

**Boundary conditions.** A boundary condition for the elliptic pair $\sigma, \tau$ is defined as a subspace $N$ complementary to $\mathcal{L}_-(\rho)$, i.e. such that $N \cap \mathcal{L}_-(\rho) = 0$ and $N + \mathcal{L}_-(\rho) = E$. A boundary condition $N$ is said to be self-adjoint if $N$ is lagrangian. In this case it is sufficient to assume that $N$ is transverse to $\mathcal{L}_-(\rho)$, i.e. $N \cap \mathcal{L}_-(\rho) = 0$. For the rest of this section we will assume that $N$ is a self-adjoint boundary condition for $\sigma, \tau$.

**Parameters and deformations.** In our applications the elliptic pair and the boundary condition will continuously depend on a parameter $x \in X$, where $X$ is a topological space. Moreover, the vector space $E$ will depend on $x \in X$. More precisely, we will be given a family $E_x, x \in X$ of vector spaces forming a vector bundle over $X$, scalar products $\langle \cdot, \cdot \rangle_x, x \in X$ on the vector spaces $E_x$ defining a Hermitian structure on this vector bundle, self-adjoint operators $\sigma_x, \tau_x : E_x \rightarrow E_x$, and subspaces $N_x \subset E_x$ such that

$$E = E_x, \sigma = \sigma_x, \tau = \tau_x \quad \text{and} \quad N = N_x$$

satisfy our assumptions for every $x \in X$ and continuously depend on $x$. Usually $X$ is a compact manifold, but in this section it is sufficient to assume that $X$ is paracompact.
The rest of this section is devoted to deforming the elliptic pair \( \sigma, \tau \) together with the boundary condition \( N \) into a more or less canonical form. The deformation itself will be sufficiently canonical to continuously depend on \( x \) when a parameter \( x \in X \) as above is present. The deformation will be carried out in several stages, and this continuity property will be more or less obvious at each stage.

**The first deformation.** Our first goal is to make the operator \( \sigma \) is not only self-adjoint, but is also unitary, i.e. such that \( \sigma = \sigma^* = \sigma^{-1} \). Let \( |\sigma| = \sqrt{\sigma^* \sigma} \), and for \( \alpha \in [0, 1/2] \) let

\[
\sigma_\alpha = |\sigma|^{-\alpha} \sigma |\sigma|^{-\alpha}, \quad \tau_\alpha = |\sigma|^{-\alpha} \tau |\sigma|^{-\alpha}, \quad \text{and}
\]

\[
\rho_\alpha = \sigma_\alpha^{-1} \tau_\alpha = |\sigma|^\alpha \rho |\sigma|^{-\alpha}.
\]

Then the ellipticity assumption holds for \( \sigma_\alpha, \tau_\alpha \) in the role of \( \sigma, \tau \). Indeed,

\[
a \sigma_\alpha + b \tau_\alpha = |\sigma|^{-\alpha} (a \sigma + b \tau) |\sigma|^{-\alpha}
\]

is invertible together with \( a \sigma + b \tau \) for every \( a, b \in \mathbb{R} \) such that \((a, b) \neq 0\). Let

\[
[ u, v ]_\alpha = \langle \sigma_\alpha(u), v \rangle.
\]

Then \([\cdot, \cdot]_\alpha\) is a non-degenerate Hermitian form on \( E \), and \( \rho_\alpha \) is self-adjoint with respect to \([\cdot, \cdot]_\alpha\). Clearly, the operators \( \sigma_\alpha, \tau_\alpha \), and \( \rho_\alpha \) continuously depend on \( \alpha \), as also the form \([\cdot, \cdot]_\alpha\). The operators \( \rho_\alpha \) are conjugate to \( \rho \) and hence have the same eigenvalues, and the spaces \( \mathcal{L}_+(\rho_\alpha) \) and \( \mathcal{L}_-(\rho_\alpha) \) also continuously depend on \( \alpha \). For \( \alpha = 1/2 \) the operator \( \sigma_\alpha = \sigma_{1/2} = |\sigma|^{-1} \sigma \) is self-adjoint and has only 1 and \(-1\) as eigenvalues. It follows that \( \sigma_{1/2} \) is an involution, i.e. \( \sigma_{1/2}^2 = 1 \). In turn, this implies that \( \sigma_{1/2}^{-1} = \sigma_{1/2} \) and hence \( \sigma_{1/2} \) is a unitary operator.

We need also deform the boundary condition \( N \). Let \( N_\alpha = |\sigma|^\alpha(N), \alpha \in [0, 1/2] \). Then

\[
\langle \sigma_\alpha(N_\alpha), N_\alpha \rangle = \langle |\sigma|^{-\alpha} \sigma |\sigma|^{-\alpha} |\sigma|^\alpha(N), |\sigma|^\alpha(N) \rangle
\]

\[
= \langle |\sigma|^{-\alpha} \sigma(N), |\sigma|^\alpha(N) \rangle = \langle |\sigma|^\alpha \sigma |\sigma|^{-\alpha} \sigma(N), N \rangle
\]

\[
= \langle \sigma(N), N \rangle = 0,
\]

i.e. \( \sigma_\alpha(N_\alpha) \) is orthogonal to \( N_\alpha \) and hence \( N_\alpha \) is lagrangian subspace with respect to \([\cdot, \cdot]_\alpha\). A standard verification shows that the subspace \( N_\alpha \) is transverse to \( \mathcal{L}_-(\rho_\alpha) \) and hence is a self-adjoint boundary condition for \( \sigma_\alpha, \tau_\alpha \).

**The second deformation.** Now we would like to deform the operator \( \rho \), while keeping its properties, to an operator having only \( i \) and \(-i\) as the eigenvalues. This deformation is
independent of the first one and can be done either after or before it. Let \( p_+ \) be the projection of \( E \) onto \( \mathcal{L}_+(\rho) \) along \( \mathcal{L}_-(\rho) \), and let \( p_- \) be the projection of \( E \) onto \( \mathcal{L}_-(\rho) \) along \( \mathcal{L}_+(\rho) \). Let \( \rho_0 = i p_+ - i p_- \). We claim that \( \rho_0 : E \rightarrow E \) is self-adjoint with respect to \( [\cdot, \cdot] \). Let \( u = u_+ + u_- \) and \( v = v_+ + v_- \) with

\[
\begin{align*}
u_+, v_+ &\in \mathcal{L}_+(\rho) \quad \text{and} \quad u_-, v_- \in \mathcal{L}_-(\rho).
\end{align*}
\]

Then

\[
[p_0(u), v] = [i u_+ - i u_-, v] = [i u_+, v] - [i u_-, v]
\]

\[
= [i u_+, v_+ + v_-] - [i u_-, v_+ + v_-] = [i u_+, v_-] + [i u_-, v_+]
\]

\[
= [u_+, -i v_-] + [u_-, i v_+] = [u_+ + u_-, i v_+ - i v_-] = [u, \rho_0(v)].
\]

This proves our claim. Clearly, the only eigenvalues of \( \rho_0 \) are \( i \) and \( -i \). Let \( \tau_0 = \sigma \rho_0 \). Then \( \langle \tau_0(u), v \rangle = \langle \sigma \rho_0(u), v \rangle = [p_0(u), v] \) and \( \langle u, \tau_0(v) \rangle = [u, \rho_0(v)] \). It follows that \( \tau_0 \) is self-adjoint with respect to \( \langle \cdot, \cdot \rangle \). Let us connect \( \rho, \tau \) with \( \rho_0, \tau_0 \) respectively by linear homotopies

\[
\rho_\beta = (1 - \beta) \rho + \beta \rho_0 \quad \text{and} \quad \tau_\beta = (1 - \beta) \tau + \beta \tau_0,
\]

where \( \beta \in [0, 1] \). Then \( \rho_\beta = \sigma^{-1} \tau_\beta \) for every \( \alpha \). Clearly,

\[
\mathcal{L}_+(\rho_\beta) = \mathcal{L}_+(\rho) \quad \text{and} \quad \mathcal{L}_-(\rho_\beta) = \mathcal{L}_-(\rho)
\]

for every \( \alpha \). The ellipticity assumption holds for \( \sigma, \tau_\beta \) and every \( \beta \). Indeed,

\[
a \sigma + b \tau_\beta = \sigma(a + b \rho_\beta)
\]

is invertible if \( a, b \in \mathbb{R}^2 \setminus \{0\} \) because \( \sigma \) is invertible and \( \rho_\beta \) has no real eigenvalues.

This deformation does not affect the scalar product \( [\cdot, \cdot] \) and the space \( \mathcal{L}_+(\rho) \). Therefore \( N \) remains a boundary condition during the second deformation \( \sigma_\beta, \tau_\beta, \rho_\beta, \beta \in [0, 1] \).

**Lagrangian subspaces when the operator \( \sigma \) is unitary.** Suppose that the operator \( \sigma \) is self-adjoint and unitary. Let \( L \) be a lagrangian subspace. Since \( \sigma \) is unitary, \( \sigma(L) \) is also lagrangian. Indeed, the lagrangian property of \( L \) means that \( \sigma(L) \) is orthogonal to \( L \) with respect to \( \langle \cdot, \cdot \rangle \). Since the operator \( \sigma \) is unitary, this implies that \( \sigma(\sigma(L)) = L \) is orthogonal to \( \sigma(L) \) with respect to \( \langle \cdot, \cdot \rangle \), and hence the space \( \sigma(L) \) is lagrangian.

Since \( \sigma \) is self-adjoint and unitary, \( E \) can be decomposed into a direct sum \( E = E^+ \oplus E^- \), where \( E^+ \) and \( E^- \) are the eigenspaces of \( \sigma \) corresponding to the eigenvalues \( 1 \) and \( -1 \) respectively. Since \( \sigma \) is self-adjoint, the spaces \( E^+ \), and \( E^- \) are orthogonal with respect
to $\langle \bullet, \bullet \rangle$. Clearly, the scalar product $[\bullet, \bullet]$ is equal to $\langle \bullet, \bullet \rangle$ on $E^+$ and to $-\langle \bullet, \bullet \rangle$ on $E^-$. In the orthogonal decomposition $E = E^+ \oplus E^-$ the operator $\sigma$ takes the matrix form

$$(2) \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ \hspace{1cm}

The lagrangian subspace $L$ is transverse to both $E^+$ and $E^-$, i.e. $L \cap E^+ = L \cap E^- = 0$. This implies that the projection of $L$ onto $E^+$ along $E^-$ is injective, as also the projection onto $E^-$ along $E^+$. Since the dimension of $L$ is equal to the half of the dimension of $E$, it follows that the dimensions of $E^+$ and $E^-$ are also equal to the half of the dimension of $E$. Also, $L$ is equal to the graph $\{ u + \varphi(u) \mid u \in E^+ \}$ of a unique linear map $\varphi : E^+ \longrightarrow E^-$. We claim that $\varphi$ is an isometry with respect to $\langle \bullet, \bullet \rangle$. Indeed, if $u, v \in E^+$, then

$$0 = [u + \varphi(u), v + \varphi(v)] = \langle \sigma(u + \varphi(u)), v + \varphi(v) \rangle$$

$$= \langle u - \varphi(u), v + \varphi(v) \rangle = \langle u, v \rangle - \langle \varphi(u), v \rangle + \langle u, \varphi(v) \rangle - \langle \varphi(u), \varphi(v) \rangle.$$ \hspace{1cm}

The spaces $E^+$ and $E^-$ are orthogonal with respect to the product $\langle \bullet, \bullet \rangle$ and therefore $\langle \varphi(u), v \rangle = \langle u, \varphi(v) \rangle = 0$. Hence $\langle \varphi(u), \varphi(v) \rangle = \langle u, v \rangle$ for every $u, v \in E^+$, i.e. $\varphi$ is an isometry. If $L'$ is another lagrangian subspace and $\varphi'$ is the corresponding isometry, then $L'$ is transverse to $L$ if and only if $\varphi(u) \neq \varphi'(u)$ for every nonzero $u \in E^+$.

Let us use the isometry $\varphi$ to identify $E^-$ with $E^+$. This identification turns $L$ into the diagonal $\Delta = \{(u, u) \mid u \in E^+ \}$. Let us denote by $F^\perp$ the orthogonal complement of $F \subset E$ with respect to $\langle \bullet, \bullet \rangle$. Then $\Delta^\perp = \{(u, -u) \mid u \in E^+ \}$ is also a lagrangian subspace.

**The third deformation.** After two deformations the operator $\sigma$ is self-adjoint and unitary, and $\rho$ has only $i$ and $-i$ as eigenvalues. In particular, $\rho$ is uniquely determined by the spaces $\mathcal{L}_+(\rho)$ and $\mathcal{L}_-(\rho)$. In the third deformation we will use the boundary condition $N$ to bring $\mathcal{L}_-(\rho)$ into a standard form. To begin with, let $\varphi : E^+ \longrightarrow E^-$ be the isometry corresponding to the lagrangian subspace $N$. Let us identify $E^+$ with $E^-$ by the isometry $\varphi$, and let $F = E_+ = E_-$. With these identifications $N$ turns into the diagonal $\Delta = \{(u, u) \mid u \in F \}$ and $\varphi$ into the identity map $\text{id}_F : F \longrightarrow F$.

The lagrangian subspace $\mathcal{L}_-(\rho)$ corresponds to some isometry $\psi_+ : F \longrightarrow F$. As we saw, the transversality of $\mathcal{L}_-(\rho)$ and $N$ implies that $\psi_+(u) \neq u$ for every $u \in F$. In other terms, 1 is not an eigenvalue of $\psi_-$. Therefore there is a canonical spectral deformation $\psi_-(\alpha)$, $\alpha \in [0, 1]$ in the class of isometries not having 1 as an eigenvalue, such that $\psi_- = \psi_-(0)$ and $\psi_-(1)$ has only $-1$ as an eigenvalue, i.e. $\psi_-(1) = -\text{id}_F$. The corresponding lagrangian subspace is the anti-diagonal $\Delta^\perp = \{(u, -u) \mid u \in F \}$.

Let $L_-(\alpha)$ be the lagrangian subspace corresponding to $\psi_\alpha$. Then $L_-(\alpha)$ is transverse to $N$ for every $\alpha$. Also, $L_+(0) = \mathcal{L}_+(\rho)$ is transverse to $L_-(0) = \mathcal{L}_-(\rho)$. By the discussion
of the spaces $\Lambda_L$ above, for each $\alpha \in [0, 1]$ the space of lagrangian subspaces transverse to $L_-(\alpha)$ is an affine space over the vector space of linear maps $\delta: E \rightarrow L_-(\alpha)$ such that (1) holds. These affine spaces form a locally trivial bundle over $[0, 1]$ (since $[0, 1]$ is contractible, this bundle is actually trivial). By the homotopy lifting property the deformation $L_-(\alpha)$, $\alpha \in [0, 1]$ can be accompanied by a deformation $L_+(\alpha)$, $\alpha \in [0, 1]$ starting with $L_+(0)$ and such that $L_+(\alpha)$ is transverse to $L_-(\alpha)$ for every $\alpha$. Moreover, the deformation $L_+(\alpha)$, $\alpha \in [0, 1]$ is unique up to homotopy (between homotopies) when parameters are present. Cf. the discussion of loops and bundles in [I1], Section 16. It follows that the $L_+(1)$ is unique up to a homotopy when parameters are present. It is worth to point out that the deformation $L_+(\alpha)$, $\alpha \in [0, 1]$ is not a spectral deformation of the isometry corresponding $\mathcal{L}_+(\rho)$. Let $\rho_\alpha: E \rightarrow E$ be the linear map such that

$$\rho_\alpha(u) = i u \text{ for } u \in L_+(\alpha) \text{ and } \rho_\alpha(u) = -i u \text{ for } u \in L_-(\alpha).$$

Let $\tau_\alpha = \sigma \rho_\alpha$. As in the discussion of the second deformation, we see that $\rho_\alpha$ is self-adjoint with respect to $[\cdot, \cdot]$ and hence $\tau_\alpha$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$ for every $\alpha$. Clearly, $\mathcal{L}_+(\rho_\alpha) = L_+(\alpha)$ and $\mathcal{L}_-(\rho_\alpha) = L_-(\alpha)$ and that $\rho_\alpha$ has only $i$ and $-i$ as eigenvalues for every $\alpha$. Since $N$ is transverse to $\mathcal{L}_-(\rho_\alpha)$ by the construction, $N$ is a boundary condition for $\sigma, \tau_\alpha$ for every $\alpha$ and stays intact during this deformation. To sum up, at the end of the third deformation $\sigma$ has the standard form (2), the boundary condition $N$ is equal to the diagonal $\Delta$, and the eigenspace $\mathcal{L}_-(\rho)$ is equal to $\Delta^\perp$.

**The fourth deformation.** It is similar to the third one. Now we will bring the space $\mathcal{L}_+(\rho)$ to a standard form. After the third deformation it is a lagrangian subspace transverse to $\mathcal{L}_-(\rho) = \Delta^\perp$. The lagrangian subspace $N = \Delta$ is also transverse to $\mathcal{L}_-(\rho) = \Delta^\perp$. Since the space of lagrangian subspaces transverse to a given one, say to $\Delta^\perp$, is an affine space, we can deform $\mathcal{L}_+(\rho)$ to $N$ by a linear deformation in this affine space. Moreover, this is a canonical deformation. As in the third deformation, this deformation of $\mathcal{L}_+(\rho)$ defines a deformation of $\rho$ and $\tau$, and $N$ remains a boundary condition during this deformation. The fourth deformation brings $\sigma, \rho, \tau$ and $N$ into a normal form. As we already saw, $\sigma$ has the form (2) and treating $\varphi$ as the identification map and $N = \Delta$ if we treat $\varphi$ as an identification of $E^+$ with $E^-$. Since $\Delta$ and $\Delta^\perp$ are the eigenspaces of $\rho$ with eigenvalues $i$ and $-i$ respectively, one can easily compute $\rho$ and $\tau$. We see that

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

**The normal form.** When everything depends on parameters, it may happen that the vector spaces $E^+, E^-$ depend on a parameter, or that $E^+, E^-$ do not depend on a parameter, but the isometry $\varphi$ does. In this case it is not possible to identify $E^+$ with $E^-$ by $\varphi$ and one
has to use a normal form explicitly involving $\varphi$. For every isometry $\varphi : E^+ \to E^-$ let

$$\Delta(\varphi) = \{(u, \varphi(u)) \mid u \in E^+\} \quad \text{and} \quad \Delta^\perp(\varphi) = \{(u, -\varphi(u)) \mid u \in E^+\}.$$ (4)

If $\sigma$ has the form (2) with respect to the decomposition $E = E^+ \oplus E^-$ and $\Delta(\varphi), \Delta^\perp(\varphi)$ are the eigenspaces of $\rho$ with the eigenvalues $i, -i$ respectively, then

$$\Delta(\varphi) = \{(u, \varphi(u)) \mid \rho = \begin{pmatrix} 0 & i \varphi^* \\ i \varphi & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & i \varphi^* \\ -i \varphi & 0 \end{pmatrix},$$ (5)

as a simple computation, taking into account that $\varphi^* = \varphi^{-1}$, shows. If also

$$N = \Delta(\varphi) \quad \text{and} \quad N^\perp = \Delta^\perp(\varphi),$$ (6)

we will say that $\sigma, \tau$ and $N$ are normalized. Equivalently, $\sigma, \tau$ and $N$ are normalized if $\sigma$ is a unitary map, $L_+(\rho) = N$, $L_-(\rho) = N^\perp$, and the spaces $L_+(\rho), L_-(\rho)$ are the eigenspaces of $\rho$ with the eigenvalues $i, -i$ respectively. Clearly, in this case $\sigma$ anti-commutes with $\tau$ and $\rho$.

**Normalized** $\sigma, \rho, \tau$ and $N$ in terms of the decomposition $E = \Delta(\varphi) \oplus \Delta^\perp(\varphi)$. Clearly,

$$\sigma(u, \varphi(u)) = (u, -\varphi(u)), \quad \sigma(u, -\varphi(u)) = (u, \varphi(u)), \quad \rho(u, \varphi(u)) = \begin{pmatrix} i \varphi^{-1} \circ \varphi(u) \\ i \varphi(u) \end{pmatrix} = i(u, \varphi(u)), \quad \rho(u, -\varphi(u)) = \begin{pmatrix} -i \varphi^{-1} \circ \varphi(u) \\ i \varphi(u) \end{pmatrix} = -i(u, -\varphi(u)).$$

One can always identify $\Delta(\varphi)$ with $\Delta^\perp(\varphi)$ by the map $(u, v) \mapsto (u, -v)$. In terms of this identification and the decomposition $E = \Delta(\varphi) \oplus \Delta^\perp(\varphi)$ we have

$$\sigma(u, \varphi(u)) = (u, 0), \quad \rho = \begin{pmatrix} 0 & i \varphi^* \\ i \varphi & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$ (7)

which is often more convenient because $\varphi$ is hidden in the formulas (6) for $\Delta(\varphi), \Delta^\perp(\varphi)$.

**Graded normal form.** Suppose that for some orthogonal decomposition $E = E_+ \oplus E_-$ and an isometry $\varphi : E_+ \to E_-$ the operators $\sigma, \tau, \rho$ have the form

$$\sigma = \begin{pmatrix} 0 & \varphi^* \\ \varphi & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & -i \varphi^* \\ i \varphi & 0 \end{pmatrix}$$ (8)

and $N = E_+ \oplus 0$. In this case we will say that $\sigma, \tau, \rho$ and $N$ are graded normalized.
Clearly, in this case \( \sigma \) is unitary, \( L_+(\rho) = N \), \( L_-(\rho) = N^\perp \), and \( \rho \) acts on \( L_+(\rho) \) and \( L_-(\rho) \) as the multiplication by \( i \) and \( -i \) respectively. Hence \( \sigma, \tau, \rho \) and \( N \) are also normalized, but the corresponding decomposition is different from \( E = E_+ \oplus E_- \). More explicitly, let us define subspaces \( \Delta(\phi), \Delta^\perp(\phi) \) by (4) and identify them with \( E_+ \) by the orthogonal projection \( E = E_+ \oplus E_- \rightarrow E_+ \). In terms of the resulting decomposition \( E = \Delta(\phi) \oplus \Delta^\perp(\phi) = E_+ \oplus E_+ \) the operators \( \sigma, \rho, \tau \) take the same form

\[
\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

as in (3), but with \( \Delta(\phi), \Delta^\perp(\phi) \) playing the same role as \( E^+, E^- \) in (3), and with \( E_+, E_- \) playing the same role as \( \Delta(\phi), \Delta^\perp(\phi) \) in (3).

The positive eigenspaces of \( \sigma \cos \theta + \tau \sin \theta \). Let \( \theta \) run over the interval \([0, \pi]\). Then the points \( (\cos \theta, \sin \theta) \) run over a standard half-circle. If \( \sigma, \tau \) is a self-adjoint elliptic pair, then \( \sigma \cos \theta + \tau \sin \theta, \theta \in [0, \pi] \) is a continuous family of self-adjoint invertible operators. Let \( E^+(\theta), E^-(\theta) \subset E \) be the sum of eigenspaces of \( \sigma \cos \theta + \tau \sin \theta \) corresponding to the positive and negative eigenvalues respectively. Clearly, \( E^+(0) = E^+, E^+(\pi) = E^- \), and \( E^+(\theta), \theta \in [0, \pi] \) is a path connecting \( E^+ \) with \( E^- \). The subspaces \( E^+ \) and \( E^- \) do not change during our deformations, and \( E^+(\theta), \theta \in [0, \pi] \) remains to be a path connecting \( E^+ \) with \( E^- \). Similarly, \( E^-(\theta), \theta \in [0, \pi] \) is a continuous path connecting \( E^- \) with \( E^+ \). These paths play a crucial role in the definition of the topological index.

It is instructive to compute the path \( E^+(\theta), \theta \in [0, \pi] \) for \( \sigma, \rho, \tau \) as in (5). After passing to the decomposition \( E = \Delta(\phi) \oplus \Delta^\perp(\phi) \) and the standard form (7) we get

\[
\sigma \cos \theta + \tau \sin \theta = \begin{pmatrix} 0 & \cos \theta - i \sin \theta \\ \cos \theta + i \sin \theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix},
\]

where \( z = \cos \theta + i \sin \theta \). It follows that

\[
E^+(\theta) = \left\{ (a, za) \mid a \in \Delta(\phi) \right\} \subset \Delta(\phi) \oplus \Delta^\perp(\phi).
\]

By passing to the standard form (5) and taking into account (4), we see that

\[
E^+(\theta) = \left\{ (u, \varphi(u)) + z (u, -\varphi(u)) \mid u \in E^+ \right\}
\]

\[
= \left\{ ((1 + z)u, (1 - z)\varphi(u)) \mid u \in E^+ \right\} \quad \text{and hence}
\]

\[
E^+(\theta) = \left\{ u, \frac{1 - z}{1 + z} \varphi(u) \mid u \in E^+ \right\},
\]

where for \( z = -1 \) the last expression is interpreted as \( E^- \).
3. Self-adjoint symbols and boundary conditions

**Self-adjoint symbols.** Let \( X \) be a compact manifold. For convenience, we will assume that \( X \) is equipped with a riemannian metric. Let \( TX \) be the tangent bundle of \( X \) and let \( BX \) and \( SX \) be the bundles of unit balls and unit spheres in \( TX \) respectively. Let \( \pi : TX \to X \) be the bundle projection. When there is no danger of confusion, we will denote by \( \pi \) also various related projections, such as \( SX \to X \) and \( BX \to X \). We will denote the fiber of \( TX \) at \( x \in X \) by \( T_x X \) and use similar notations for other bundles.

Let \( E \) be a complex vector bundle on \( X \) equipped with a Hermitian metric. In contrast with the riemannian metric on \( X \) the Hermitian metric on \( E \) will play a key role. An (elliptic) self-adjoint symbol is a self-adjoint automorphism \( \sigma \) of the bundle \( \pi^* E \) over \( SX \). Such a symbol \( \sigma \) defines a decomposition \( \pi^* E = E^+ (\sigma) \oplus E^- (\sigma) \) over \( SX \) into the sums \( E^+ (\sigma) \) and \( E^- (\sigma) \) of eigenspaces of \( \sigma \) corresponding to the positive and negative eigenvalues of \( \sigma \) respectively. Up to homotopy \( \sigma \) is determined by the subbundle \( E^+ (\sigma) \subset \pi^* E \).

**Self-adjoint symbols of order 1.** Suppose that \( X \) has non-empty boundary \( Y = \partial X \). Let \( SY \) be the bundle of unit spheres in the tangent bundle \( TY \), and let \( BX_Y \) and \( SX_Y \) be the restrictions to \( Y \) of the bundles \( BX \) and \( SX \) respectively. For every \( y \in Y \) let \( \nu_y \) be unit normal to the tangent space \( T_y Y \) in the tangent space \( T_y X \) pointing into \( X \).

We will say that a self-adjoint symbols \( \sigma \) is a self-adjoint symbol of order 1 if for every \( y \in Y \) and \( u \in S_y Y \) the restriction of \( \sigma \) to the half-circle

\[
\{ \nu_y \cos \theta + u \sin \theta \mid 0 \leq \theta \leq \pi \} \subset S_y X
\]

has the form

\[
\sigma (\nu_y \cos \theta + u \sin \theta) = \sigma_y \cos \theta + \tau_u \sin \theta
\]

for some self-adjoint endomorphisms \( \sigma_y \) and \( \tau_u \) of the fiber of \( E_y \) of \( E \) over \( y \). Of course, such a symbol \( \sigma \) should be thought as the symbol of a pseudo-differential operator of order 1. Clearly, \( \sigma_y = \sigma (\nu_y) \). Since \( \sigma (u) \) is invertible for every \( u \in S_Y X \), the pair \( \sigma_y, \tau_u \) is an elliptic pair in the sense of Section 2. For \( u \in S_Y X \) let

\[
\rho_u = \sigma_y^{-1} \tau_u.
\]

Clearly, the point \( y = \pi (u) \) is determined by the tangent vector \( u \). We will always assume that \( y, u \) are related in this way. For \( y \in Y \) let \( E^+_y \) and \( E^-_y \) be the fibers at \( \nu_y \in SY \subset SX \) of the bundles \( E^+ (\sigma) \) and \( E^- (\sigma) \) respectively. Then

\[
E_y = E^+_y \oplus E^-_y.
\]
**Bundle-like symbols and extensions of the bundles** \( E^+(\sigma) \) and \( E^-(\sigma) \). We will say that a self-adjoint symbol \( \sigma \) of order 1 is **bundle-like** if \( \tau_u \) depends only on \( y = \pi(u) \). This property depends only on the restriction of \( \sigma \) to \( SX_Y \). If \( \sigma \) is bundle-like, then

\[
\sigma(v_y \cos \theta + u \sin \theta) = \sigma_y \cos \theta + \tau_u \sin \theta
\]

depends only on \( y \) and \( \theta \), and hence the fibers \( E^+(u, \theta) \) and \( E^-(u, \theta) \) of the bundles \( E^+(\sigma) \) and \( E^-(\sigma) \) respectively at the point \( v_y \cos \theta + u \sin \theta \) also depend only on \( y \) and \( \theta \). Therefore, for every \( r \in [0, 1] \) and

\[
w = v_y \cos \theta + ru \sin \theta \in BY_X
\]

we can define vector spaces \( E^+(w) \) and \( E^-(w) \) by

\[
E^+(w) = E^+(u, \theta) \quad \text{and} \quad E^-(w) = E^-(u, \theta).
\]

Of course, the bundle-like property of \( \sigma \) matters only for \( r = 0 \). It ensures that these vector spaces define extensions of the bundles \( E^+(\sigma) \) and \( E^-(\sigma) \) to \( BX_Y \), which we will denote by \( E^+(\sigma) \) and \( E^-(\sigma) \) respectively. The extended bundles lead to their classes

\[
e^+(\sigma), \ e^-(\sigma) \in K^0(SX \cup BX_Y)
\]

in K-theory. Let \( e^+(\sigma), \ e^-(\sigma) \) be their images under the coboundary map

\[
K^0(SX \cup BX_Y) \longrightarrow K^1(BX, SX \cup BX_Y).
\]

The kernel of the coboundary map is equal to the image of the natural map

\[
K^0(BX) \longrightarrow K^0(SX \cup BX_Y).
\]

It follows that \( e^+(\sigma) = 0 \) if \( E^+(\sigma) \) can be extended to \( BX \). Also, \( E^+(\sigma) \oplus E^-(\sigma) \) is lifted from \( X \) together with \( E^+(\sigma) \oplus E^-(\sigma) \), and hence \( e^+(\sigma) + e^-(\sigma) = 0 \).

**Boundary conditions.** A **boundary condition** for a self-adjoint symbol \( \sigma \) of degree 1 is a subbundle \( N \) of the restriction of the bundle \( \pi^*E \) to \( SX_Y \). Let \( \sigma \) be a self-adjoint symbol of order 1 and \( N \) be a boundary condition for \( \sigma \).

\( N \) is said to be **self-adjoint** if for every \( u \in SY \) the fiber \( N_u \) is a lagrangian subspace of \( E_y \) with respect to the form \( [a, b]_Y = \langle \sigma_y a, b \rangle \), where \( \langle \cdot, \cdot \rangle \) is the Hermitian metric in \( E \).

\( N \) is said to be an **elliptic** or satisfying the **Shapiro–Lopatinskii condition** if for every \( u \in SY \) the fiber \( N_u \) is transverse to \( L_+(\rho_u) \), i.e.

\[
N_u \cap L_+(\rho_u) = 0 \quad \text{and} \quad N_u + L_+(\rho_u) = E_y,
\]

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where \( y = \pi(u) \). If \( N \) is elliptic, then \( N \) is self-adjoint if and only if for every \( u \in S_X \) the fiber \( N_u \) is a self-adjoint boundary condition for \( \sigma_y, \tau_u \) in the sense of Section 2.

\( N \) is said to be \textit{bundle-like} if the fibers \( N_u \) depend only on \( y = \pi(u) \), i.e. if \( N \) is equal to the lift to \( S_X \) of a subbundle of the bundle \( E_Y \), the restriction of the bundle \( E \) to \( Y \).

\textbf{Self-adjoint boundary conditions.} The following facts immediately follow from Section 2. If \( N \) is a self-adjoint boundary condition, then the dimension of \( E \) is even, the dimension of \( N \) is equal to the half of the dimension of \( E \), and

\[
N_u \cap E^+_y = N_u \cap E^-_y = 0
\]

for every \( u \in S_X \). It follows that the dimensions of \( E^+_y \) and \( E^-_y \) are also equal to the half of \( \dim E \), and \( N_u \) is equal to the graph of an isomorphism \( \varphi_u : E^+_y \rightarrow E^-_y \), i.e.

\[
N_u = \left\{ a + \varphi_u(a) \mid a \in E^+_y \right\}.
\]

If \( \sigma \) is self-adjoint and unitary, i.e. \( \sigma = \sigma^* = \sigma^{-1} \), then the isomorphisms \( \varphi_u \) are isometries, and every family of such isometries defines a self-adjoint boundary condition.

\textbf{Normalized symbols and boundary conditions.} Suppose that \( \sigma \) is a self-adjoint symbol of order 1 and \( N \) is an elliptic self-adjoint boundary condition for \( \sigma \). We will say that the pair \( \sigma, N \) is \textit{normalized} if \( \sigma_y, \rho_u, \tau_u \) and \( N_u \) are normalized in the sense of Section 2 for every \( u \in S_X \). If \( \sigma, N \) is normalized, then the operators \( \rho_u \) and \( \tau_u \) are determined by \( \sigma_y \) and the boundary condition \( N_u \). In particular, if \( \sigma, N \) is normalized and \( N \) is bundle-like, then \( \sigma \) is also bundle-like.

\textbf{Deformations of symbols and boundary conditions.} In Section 2 we constructed a deformation of an arbitrary elliptic pair together with a self-adjoint boundary condition to a normalized elliptic pair and boundary condition. Moreover, we saw that this construction is nearly canonical and can be applied in continuous families of elliptic pairs and self-adjoint boundary conditions. The resulting deformations of families are unique up to homotopy. In particular, such deformation can be applied to the family of elliptic pairs \( \sigma_y, \tau_u \) and boundary conditions \( N_u \) parameterized by \( u \in S_Y \). By the usual homotopy extension property this deformation can be extended to a deformation of the symbol \( \sigma \). This deformation ends with new pair \( \sigma, N \), and this new pair \( \sigma, N \) is normalized.

If \( N \) was a bundle-like boundary condition before the deformation, it remains bundle-like during the deformation. Indeed, the boundary condition may be changed only during the first stage of the deformation. This stage is canonical and depends only on \( \sigma_y, y \in Y \), and hence keeps \( N \) being bundle-like. Also, if both \( \sigma \) and \( N \) were bundle-like before the deformation, then, obviously, the deformation can be arranged to keep this property.
**Elementary symbols and boundary conditions.** Let us construct some standard symbols and boundary conditions which we will call *elementary* ones.

Suppose that a collar neighborhood of $Y$ in $X$ is identified with $Y \times [0, 1)$, and points in this collar are denoted by $(y, x_n)$, where $y \in Y$ and $x_n \in [0, 1)$. We will treat $x_n$ as the function $(y, x_n) \mapsto x_n$ on the collar. Let $\varphi: (-1, 1) \to [0, 1]$ be a smooth function with compact support such that $\varphi$ is equal to 1 in a neighborhood of 0 (in this section we will use $\varphi$ only on $[0, 1)$). Suppose that $F$ is a Hermitian vector bundle on $X$ and that the restriction $F|_{Y \times [0, 1)}$ is presented as an orthogonal direct sum $F^+ \oplus F^−$. Let $\lambda, \lambda^+, \lambda^−$ be some positive real numbers, and let us use the same notations for the automorphisms

$$\lambda: F \to F, \quad \lambda^+: F^+ \to F^+, \quad \lambda^−: F^− \to F^−,$$

defined as the multiplications by the numbers $\lambda, \lambda^+, \lambda^−$ respectively. Let us identify the tangent bundle over the collar with the pull-back of its restriction $TX|_Y$ to the boundary by the map $(y, x_n) \mapsto y$. Then we can define endomorphisms

$$\sigma^i = i(1 - \varphi(x_n)) \lambda (1 - \varphi(x_n)) = i(1 - \varphi(x_n))^2 \lambda,$$

$$\sigma^b_+ (v_y \cos \theta + u \sin \theta) = \varphi(x_n) (\cos \theta + i\lambda^+ \sin \theta),$$

$$\sigma^b_- (v_y \cos \theta + u \sin \theta) = \varphi(x_n) (-\cos \theta + i\lambda^- \sin \theta),$$

of the bundles $E, F^+, F^-$. Let

$$\sigma^b = \sigma^b_+ \oplus \sigma^b_- : F^+ \oplus F^- \to F^+ \oplus F^-.$$ 

Since $\varphi(x_n) = 0$ for $x_n$ close to 1, the endomorphisms $\sigma^b$ canonically extends to an endomorphism $\sigma^b: F \to F$. Let

$$\sigma = \sigma^b + \sigma^i: F \to F,$$

and let us define an endomorphism $\sigma^{sa}$ of the bundle $E = F \oplus F$ by the matrix

$$\sigma^{sa} = \begin{pmatrix} 0 & \sigma \\ \sigma^* & 0 \end{pmatrix}.$$

A trivial verification shows that $\sigma^{sa}$ is a self-adjoint symbol of order 1.

By the very definition the symbol $\sigma^{sa}$ is bundle-like. Moreover, the operator $\sigma^{sa}$ over $u \in SX$ depends only on $y = \pi(u)$ over the whole $X$ and hence $\sigma^{sa}$ can be extended to an isomorphism $\pi^* E \to \pi^* E$ over $BX$. It follows that the bundles $E^+(\sigma^{sa})$ and $E^−(\sigma^{sa})$ can be extended to $BX$. In turn, this implies that $\varepsilon^+(\sigma^{sa}) = \varepsilon^−(\sigma^{sa}) = 0$. 

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There are natural boundary conditions for $\sigma^{sa}$. For every $y \in Y$ the endomorphism $\sigma_y^{sa}$ of

$$E_y = F_y \oplus F_y = \left( F_y^+ \oplus F_y^- \right) \oplus \left( F_y^+ \oplus F_y^- \right)$$

is given by the formula

$$(u^+, u^-, v^+, v^-) \mapsto (v^+, -v^-, u^+, -u^-),$$

and the corresponding indefinite Hermitian product $[\cdot, \cdot]_y$ has the form

$$[ (u^+, u^-, v^+, v^-), (a^+, a^-, b^+, b^-) ] = \langle u^+, b^+ \rangle - \langle u^-, b^- \rangle + \langle v^+, a^+ \rangle - \langle v^-, a^- \rangle.$$

Let

$$N_u^{sa} = \mathcal{L}_+(\rho_u) = \left( F_y^+ \oplus 0 \right) \oplus \left( 0 \oplus F_y^- \right),$$

where $y = \pi(u)$. Then $\mathcal{L}_-(\rho_u) = (N_u^{sa})^\perp$ and since

$$[ (u^+, 0, 0, v^-), (a^+, 0, 0, b^-) ] = \langle u^+, 0 \rangle - \langle 0, b^- \rangle + \langle 0, a^+ \rangle = 0,$$

the vector subspaces $N_u^{sa} \subset E_y$ are lagrangian and hence define a self-adjoint boundary condition $N$ for $\sigma^{sa}$. Clearly, $\sigma^{sa}, N^{sa}$ are bundle-like and graded normalized. We will call symbols and boundary conditions of the form $\sigma^{sa}, N^{sa}$ elementary.

An obstruction to deforming boundary conditions to bundle-like ones. Let $\sigma$ be a self-adjoint symbol of the order 1, and let $E_Y^+$ and $E_Y^-$ be the bundles over $Y$ having as fibers over $y \in Y$ the spaces $E_y^+$ and $E_y^-$ respectively.

Let $N$ be a self-adjoint elliptic boundary condition for $\sigma$, and suppose that $\sigma, N$ are normalized. Then $N$ defines for every $u \in SY$ an isometry $\varphi_u : E_y^+ \to E_y^-$, and these isometries define an isometric isomorphism of bundles

$$\varphi : \pi^* E_Y^+ \to \pi^* E_Y^-.$$

These bundles are well defined also over $BY$ and the isometry $\varphi$ leads to a class

$$\mathcal{I}(N) \in K^0(BY, SY) = K^0(TY),$$

where the last group is the K-theory with compact supports. Clearly, if $N$ is bundle-like, then $\varphi$ extends to $BY$ and hence $\mathcal{I}(N) = 0$. In fact, a converse is also true.
3.1. Proposition. $\mathcal{I}(N) = 0$ if and only if there exist elementary $\sigma^{sa}, N^{sa}$ such that $N \oplus N^{sa}$ can be deformed to a bundle-like boundary condition for $\sigma \oplus \sigma^{sa}$.

Proof. If $\sigma^{sa}, N^{sa}$ are elementary, then $N^{sa}$ is bundle-like and hence $\mathcal{I}(N^{sa}) = 0$. It follows that $\mathcal{I}(N) = \mathcal{I}(N \oplus N^{sa})$. This implies the “if” part of the proposition.

Suppose now that $\mathcal{I}(N) = 0$. Then there exists a bundle $F$ over $Y$ such that the direct sum of $\varphi$ and the identity isomorphism $\text{id}_{\pi^* F} : \pi^* F \to \pi^* F$ can be extended to $\text{BY}$. Moreover, by adding to $F$ another bundle $F'$ over $Y$, if necessary, we can ensure that $F$ is a trivial bundle, and, in particular, can be extended to $X$. Let us apply the construction of elementary symbols and boundary conditions to the bundles $F, F^+ = F, F^- = 0$, and $\lambda = \lambda^+ = \lambda^- = 1$ (the value of $\lambda^-$ is actually irrelevant because the corresponding bundle is zero). Let $E' = F \oplus F$, and let $\sigma' = \sigma^{sa}, N' = N^{sa}$ be the resulting elementary symbol and the boundary condition $E'$. The above construction provides $\sigma', N'$ in the graded normalized form. Passing to the “non-graded” normalized form shows that taking the direct sum of $\sigma, N$ with $\sigma', N'$ results in adding $F$ to both $E^+_y$ and $E^-_y$ and adding the identity morphism $\text{id}_{\pi^* F}$ of $\pi^* F$ to $\varphi$. Therefore it is sufficient to show that $N$ can be deformed to a bundle-like boundary condition if $\varphi$ can be extended to an isomorphism $\tilde{\varphi}$ over $\text{BY}$. But, clearly, the restriction of $\tilde{\varphi}$ to the zero section of $\text{BY}$ defines a bundle-like boundary condition homotopic to $N$. ■

Boundary conditions in the classical form. Classically, the boundary conditions for $\sigma$ are described by a vector bundle $G$ over $Y$ and a surjective bundle map $\beta : \pi^* E|_{\text{SY}} \to G$ over the projection $\text{SY} \to Y$. The corresponding subbundle $N$ is simply the kernel $\ker \beta$. By the definition, $\beta$ satisfies the Shapiro–Lopatinskii condition if $N$ has this property, and is self-adjoint if $N$ is. In the non-self-adjoint case the existence of such $G, \beta$ ensures that after adding a symbol of the form $\sigma^b + \sigma^i$ one can deform the symbol to a bundle-like one. See the Fifth homotopy of Atiyah and Bott [AB], or the Step II in Hörmander’s proof of Proposition 20.3.3 in [H]. In its turn, deforming the symbol to a bundle-like one is the key step in defining the topological index. See Atiyah [A1], [A2].

In the self-adjoint case the situation is different. Namely, every self-adjoint boundary condition $N$ can be realized as the kernel of a bundle map $\beta$ as above. Indeed, for every $u \in \text{SY}$ the subspace $N_u \subset E_y$ is lagrangian and hence transverse to $E^+_y$. Therefore there is a unique projection $\beta_u : E_y \to E^+_y$ with the kernel $N_u$. These projections define a surjective bundle map $\beta : \pi^* E|_{\text{SY}} \to E^+$ with the kernel $N$ covering $\text{SY} \to Y$. At the same time every element of $K^0(\text{BY}, \text{SY})$ can be realized as $\mathcal{I}(N)$ for some self-adjoint $N, \sigma$ satisfying the Shapiro–Lopatinskii condition. Namely, one can define the operators $\rho_u, \tau_u$ by (5). By these reasons in the self-adjoint case the classical form of boundary conditions does not help to define the topological index.
The class $\varepsilon^+(\sigma, N)$. We would like to define an analogue of the class $\varepsilon^+(\sigma)$ when $\sigma$ is not necessarily a bundle-like symbol. Such an analogue exist if a self-adjoint elliptic boundary condition $N$ for $\sigma$ is given, and, moreover, $N$ is bundle-like or at least $\mathcal{F}(N) = 0$.

Proposition 3.1 implies that $N \oplus N^{sa}$ can be deformed to a bundle-like boundary condition $N'$ for $\sigma \oplus \sigma^{sa}$ for some elementary $\sigma^{sa}, N^{sa}$. After this we can use $N'$ to deform $\sigma \oplus \sigma^{sa}, N'$ to bundle-like pair $\sigma', N'$ (while keeping $N'$ intact). As we saw, $\varepsilon^+(\sigma^{sa}) = 0$. Since the invariant $\varepsilon^+(\bullet)$ is obviously additive with respect to the direct sums, this suggests to introduce the class $\varepsilon^+(\sigma, N) = \varepsilon^+(\sigma')$. Let us check that $\varepsilon^+(\sigma, N)$ does not depend on the choices made. Since the normalizing deformations can be applied to families, in particular, to homotopies, $\varepsilon^+(\sigma, N)$ does not depend on the choice of deformation of $N \oplus N^{sa}$ to a bundle-like boundary condition for fixed $\sigma^{sa}, N^{sa}$. Different $\sigma^{sa}, N^{sa}$ lead to the same class because one can simultaneously add two of them and then use either one of them. If $\sigma, N$ are already normalized and bundle-like, then there is no need to deform $N \oplus N^{sa}$ and $\sigma \oplus \sigma^{sa}$ (and even to add $\sigma^{sa}, N^{sa}$) and hence

$$
\varepsilon^+(\sigma, N) = \varepsilon^+(\sigma \oplus \sigma^{sa}) = \varepsilon^+(\sigma) + \varepsilon^+(\sigma^{sa}) = \varepsilon^+(\sigma).
$$

If both $\sigma$ and $N$ are bundle-like, but the pair $\sigma, N$ is not normalized, we can deform $\sigma, N$ to a normalized pair in the class of bundle-like pairs. This deformation will induce a deformation of the subbundle $\mathcal{E}^+(\sigma)$. It follows that in this case $\varepsilon^+(\sigma, N) = \varepsilon^+(\sigma)$.

The topological index. Let us embed $X$ into $\mathbb{R}^{n \geq 0} = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ for some $n$ in such a way that $Y = X \cap \mathbb{R}^{n-1} \times 0$ and $X$ is transverse to $\mathbb{R}^{n-1} \times 0$ in $\mathbb{R}^n$. Let $N$ be the normal bundle to $X$ in $\mathbb{R}^n$. The normal bundle of $TX$ in $T\mathbb{R}^{n}_{\geq 0} = \mathbb{R}^{n}_{\geq 0} \times \mathbb{R}^n$ can be identified with the lift of the bundle $N \oplus N$ to $TX$, where $N \oplus 0$ is simply the normal bundle $N$, and $0 \oplus N$ consists of vectors tangent to the fibers of $N$. The bundle $N \oplus N$ has a natural complex structure resulting from identifying it with $N \oplus iN = N \otimes \mathbb{C}$.

The previous paragraph implies that a tubular neighborhood of $TX$ in $T\mathbb{R}_{\geq 0}^n$ can be identified with the bundle $\cup$ of unit balls in $N \oplus N$. Let $S$ be the bundle of unit spheres in $N \oplus N$. Let us consider the restrictions $\cup|BX, S|BX, \cup|SX$, and $\cup|BX_Y$ to $BX, BX, SX$, and $BX_Y$ respectively. The complex structure on $N \oplus N$ leads to the Thom isomorphism

$$
\text{Th}: K^1(BX, SX \cup BX_Y) \longrightarrow K^1(\cup|BX, (S|BX) \cup (\cup|SX) \cup (\cup|BX_Y)).
$$

Clearly, $S \cup (\cup|SX) \cup (\cup|BX_Y)$ is equal to the boundary of $\cup|BX \subset T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, and $\cup|BX_Y$ is equal to the intersection of this boundary with $(\mathbb{R}^{n-1} \times 0) \times \mathbb{R}^n$. Let $S^{2n}$ be the one point compactification of $\mathbb{R}^n \times \mathbb{R}^n$. There is a canonical map

$$
S^{2n} \longrightarrow \cup|BX/(S \cup (\cup|BX) \cup (\cup|BX_Y))
$$

from $S^{2n}$ to the quotient of the subspace $\cup|BX \subset \mathbb{R}^n \times \mathbb{R}^n$ by its boundary, and this map
induces a canonical homomorphism

\[ \iota : K^1(\bigcup |BX, S \cup (\bigcup |SX) \cup (\bigcup |BX_Y)) \to K^1(S^{2n}). \]

Of course, \( K^1(S^{2n}) = 0 \), but let us pretend that we don’t know this and define

\[ t(\sigma, N) \in K^1(S^{2n}) \]

as the image of the class \( \epsilon^+(\sigma, N) \in K^1(BX, SX \cup BX_Y) \) under the composition \( \iota \circ \text{Th} \).

This is the topological index of \( \sigma, N \), but in order to get something non-zero, one needs to allow symbols and boundary conditions depending on a parameter.

**Families of self-adjoint symbols and boundary conditions.** Suppose that our symbols and boundary conditions continuously depend on a parameter \( z \in Z \), where \( Z \) is a topological space. Then everything in this section can be arranged to be continuously depending on \( z \). In fact, our constructions are either canonical and hence continuously depend on \( z \), or are based on the extension of homotopies, which can be arranged to be continuously depending on \( z \). Moreover, one can allow manifolds \( X = X(z) \) continuously depending on \( z \), in the sense that \( X(z) \) is the fiber over \( z \) of a locally trivial bundle \( p : W \to Z \), as in [AS4].

Then the boundaries \( \partial X(z) \) form another bundle \( V \to Z \), and one can define bundles \( TW, SW, SW_V \), etc., involving, of course, only vectors tangent to the fibers. Moreover, for a family \( \sigma(z), N(z), z \in Z \) of self-adjoint symbols and boundary conditions, one can define the class \( \mathcal{I}(N) \in K^0(BV, SV) \) and if \( \mathcal{I}(N) = 0 \), define the class

\[ \epsilon^+(\sigma, N) \in K^1(BW, SW \cup BW_V). \]

Following Atiyah and Singer [AS4], we will now construct the forward image of \( \epsilon^+(\sigma, N) \) in \( K^1(Z) \), assuming that \( Z \) is compact. There is a natural number \( n \) and a continuous map \( f : W \to R^n \) such that the restriction of \( f \) to every fiber \( X(z) = p^{-1}(z) \) is a smooth embedding. Moreover, we can assume that the image of \( f \) is contained in the positive half-space \( R^n_+ = R^{n-1} \times R_{\geq 0} \), and that for every \( z \in Z \)

\[ \partial X(z) = f^{-1}(R^{n-1} \times 0). \]

and \( f(X(z)) \) is transverse to \( R^{n-1} \times 0 \) in \( R^n \). Let us identify \( W \) with its image in \( Z \times R^n \) under the map \( x \mapsto (p(x), f(x)) \). Now we can carry the constructions of the previous subsection with the parameter \( z \in Z \) and get an element \( t(\sigma, N) \) of the group \( K^1(S^{2n} \times Z) \) as the image of \( \epsilon^+(\sigma, N) \) under the Thom isomorphism and the parametrized analogue of the homomorphism \( \iota \). Finally, we define the topological index

\[ t-\text{ind}(\sigma, N) \in K^1(Z) \]

of the family \( \sigma(z), N(z), z \in Z \) as the image of \( t(\sigma, N) \) under the Bott periodicity map.
4. Abstract boundary problems

The framework. Let \( H_0 \) be a separable Hilbert space and let \( H_1 \) be a dense vector subspace of \( H_0 \), which is a Hilbert space in its own right (the Hilbert space structure of \( H_1 \) is not induced from \( H_0 \)). Suppose that the inclusion \( \iota : H_1 \rightarrow H_0 \) is a bounded operator with respect to these Hilbert space structures. Let \( H^\delta \) also be a Hilbert space and let \( H^{1/2}_1 \) be a dense subspace of \( H^\delta \), which is a Hilbert space in its own right. Let

\[ \gamma : H_1 \rightarrow H^{1/2}_1 \]

be a surjective bounded operator such that the kernel \( \ker \gamma \) is dense in \( H_1 \) and hence in \( H_0 \). Let \( \langle \cdot, \cdot \rangle_0 \) and \( \langle \cdot, \cdot \rangle_\delta \) be the scalar products in \( H_0 \) and \( H^\delta \) respectively.

Projections. A projection (not necessarily orthogonal) in a Hilbert space \( H \) is a bounded operator \( p : H \rightarrow H \) such that \( p \circ p = p \) and hence \( p \circ (1 - p) = (1 - p) \circ p = 0 \). If \( p \) is a projection, then \( 1 - p \) is also a projection, and, clearly, \( \im (1 - p) \subset \ker p \) and \( \im p \subset \ker (1 - p) \). The identity \( p + (1 - p) = 1 \) implies that \( \im p + \im (1 - p) = H \) and \( \ker p \cap \ker (1 - p) = 0 \). By taking into account the above inclusions we see that \( \im p \cap \im (1 - p) = 0 \) and \( \ker p + \ker (1 - p) = H \). It follows that \( \im (1 - p) = \ker p \) and \( \im p = \ker (1 - p) \). Of course, all this is well known.

Boundary conditions. Let \( A : H_1 \rightarrow H_0 \) be a bounded operator. A boundary condition for \( A \) is a projection \( \Pi : H^\delta \rightarrow H^\delta \) leaving the subspace \( H^{1/2}_1 \) invariant. We will denote by \( \Pi_{1/2} \) the projection \( H^\delta \rightarrow H^{1/2}_1 \) induced by \( \Pi \).

The boundary problem associated with \( A \) and \( \Pi \) is the problem of finding solutions \( u \) of the equation \( Au = 0 \) subject to the condition \( (1 - \Pi_{1/2}) \circ \gamma (u) = 0 \). Let

\[ G_{1/2} = \ker \Pi_{1/2} = \im (1 - \Pi_{1/2}) \quad \text{and} \]
\[ G = \ker \Pi = \im (1 - \Pi) . \]

Clearly, \( G_{1/2} = \ker \Pi_{1/2} = \ker \Pi \cap H_{1/2} = G \cap H_{1/2} . \) Let

\[ \Gamma : H_1 \rightarrow G_{1/2} \]

be the map induced by \( (1 - \Pi_{1/2}) \circ \gamma , \) and

\[ A \oplus \Gamma : H_1 \rightarrow H_0 \oplus G_{1/2} \]

be the operator defined by \( u \rightarrow (Au, \Gamma u) . \) In terms of the operator \( A \oplus \Gamma \) the boundary problem associated with \( A \) and \( \Pi \) is the problem of finding the kernel \( \ker A \oplus \Gamma \).
The Lagrange identity. Let $\Sigma : \mathcal{H}^\partial \rightarrow \mathcal{H}^\partial$ be a skew-adjoint invertible bounded operator leaving the subspace $\mathcal{H}^\partial_{1/2}$ invariant and such that the abstract Lagrange identity

$$\langle Au, v \rangle_0 - \langle u, Av \rangle_0 = \langle \Sigma \gamma u, \gamma v \rangle_\partial$$

holds for every $u, v \in \mathcal{H}_1$. Such identities are also known as abstract Green formulas. Note that $\Sigma \gamma u$ and $\gamma v$ belong to $\mathcal{H}_{1/2}^\partial$, but the scalar product on the right is taken in $\mathcal{H}^\partial$.

Self-adjoint boundary conditions. A boundary condition $\Pi$ for $A$ is said to be self-adjoint if $\Pi$ is a self-adjoint projection in the Hilbert space $\mathcal{H}^\partial$ and

$$\Sigma(\text{Im } \Pi) = \ker \Pi$$

and symmetric if the projection $\Pi$ is self-adjoint and $\Sigma(\text{Im } \Pi) \subset \ker \Pi$.

Let us equip $\mathcal{H}_0 \oplus G$ with the structure of Hilbert space induced from $\mathcal{H}_0 \oplus \mathcal{H}^\partial$.

4.1. Theorem. If $\Pi$ is a self-adjoint boundary condition, then the orthogonal complement of $A \oplus \Gamma(\mathcal{H}_1)$ in $\mathcal{H}_0 \oplus G$ is equal to the image of $\ker A \oplus \Gamma$ under the map $u \mapsto (u, \Sigma \circ \gamma u)$.

Proof. Let $h \in \mathcal{H}_1$, $u \in \mathcal{H}_0$, and $w \in G = \ker \Pi$. Then $A \oplus \Gamma(h) = (Ah, \Gamma h)$ is orthogonal to $(u, w)$ if and only if

$$\langle Ah, u \rangle_0 + \langle \Gamma h, w \rangle_\partial = 0.$$  

Since $w \in \ker \Pi$ and $\Pi_{1/2}$ is equal to $\Pi$ on the image of $\gamma$,

$$\langle \Gamma h, w \rangle_\partial = \langle \gamma h, w \rangle_\partial - \langle \Pi \circ \gamma h, w \rangle_\partial$$

$$= \langle \gamma h, w \rangle_\partial - \langle \gamma h, \Pi w \rangle_\partial = \langle \gamma h, w \rangle_\partial.$$  

It follows that the left hand side of (12) it equal to

$$\langle Ah, u \rangle_0 + \langle \gamma h, w \rangle_\partial.$$  

Now (11) together with the skew-adjointness of $\Sigma$ imply that it is equal to

$$\langle h, Au \rangle_0 + \langle \Sigma \circ \gamma h, \gamma u \rangle_\partial + \langle \gamma h, w \rangle_\partial$$

$$= \langle h, Au \rangle_0 - \langle \gamma h, \Sigma \circ \gamma u \rangle_\partial + \langle \gamma h, w \rangle_\partial$$

$$= \langle h, Au \rangle_0 + \langle \gamma h, w - \Sigma \circ \gamma u \rangle_\partial.$$
Suppose that (12) holds for every $h \in H_1$. Then $\langle h, Au \rangle_0 = 0$ if $\gamma h = 0$. By our assumptions such $h$ are dense in $H_0$. It follows that $Au = 0$. In turn, this implies that

$$\langle \gamma h, w - \Sigma \circ \gamma u \rangle_\delta = 0$$

for every $h \in H_1$. Since $\gamma$ is a map onto $H^\delta_{1/2}$ and $H^\delta_{1/2}$ is dense in $H^\delta$, it follows that

$$w - \Sigma \circ \gamma u = 0.$$

Since $w \in G = \text{Ker } \Pi$ and $\Pi$ is a self-adjoint boundary condition, it follows that

$$\gamma u \in \Sigma^{-1}(\text{Ker } \Pi) = \text{Im } \Pi = \text{Ker } (1 - \Pi)$$

and hence $u \in \text{Ker } \Gamma$. Since $Au = 0$, this implies that $u \in \text{Ker } A \oplus \Gamma$. It follows that if $(u, w)$ is orthogonal to $A \oplus \Gamma(H_1)$, then $u \in \text{Ker } A \oplus \Gamma$ and $w = \Sigma \circ \gamma u$.

Conversely, let $u \in \text{Ker } A \oplus \Gamma$ and $w = \Sigma \circ \gamma u$. Then $\gamma u \in \text{Ker } (1 - \Pi) = \text{Im } \Pi$ and

$$w = \Sigma \circ \gamma u \in \Sigma(\text{Im } \Pi) = \text{Ker } \Pi = G.$$

As we saw above, this implies that $\langle Ah, u \rangle_0 + \langle \Gamma h, w \rangle_\delta$ is equal to

$$\langle h, Au \rangle_0 + \langle \gamma h, \Sigma \circ w - \gamma u \rangle_\delta = \langle h, 0 \rangle_0 + \langle \gamma h, 0 \rangle_\delta = 0,$$

i.e. $(Ah, \Gamma h)$ is orthogonal to $(u, w)$ for every $h \in H_1$. The lemma follows.

\textbf{4.2. Corollary.} \textit{If the boundary condition }$\Pi$\textit{ is self-adjoint, then the orthogonal complement of }$A \oplus \Gamma(H_1)$\textit{ in }$H_0 \oplus G$\textit{ is contained in }$H_1 \oplus G_{1/2}$\textit{.}

\textbf{Proof.} Theorem 4.1 implies that if $(u, w) \in H_0 \oplus G$ is orthogonal to $A \oplus \Gamma(H_1)$, then $u \in H_1$ and $w = \Sigma \circ \gamma u$. Since $\Sigma$ leaves $H_{1/2}$ invariant, it follows that $w \in H_{1/2}$ and hence $w \in G \cap H_{1/2} = G_{1/2}$.

\textbf{The induced unbounded operator.} Suppose that $\Pi$ is a symmetric boundary condition. Let $A_\Gamma : \text{Ker } \Gamma \rightarrow H^0$ be the operator induced by $A \oplus \Gamma$. It is an unbounded operator in $H_0$. Since $\text{Ker } \gamma$, and hence $\text{Ker } \Gamma$, are dense in $H_0$, the operator $A_\Gamma$ is densely defined.

Since $\Pi$ is a self-adjoint projection, the kernels $\text{Ker } (1 - \Pi)$ and $\text{Ker } \Pi$ are orthogonal in $H^\delta$. Since $\Pi$ is a symmetric boundary condition, $\Sigma$ maps $\text{Ker } (1 - \Pi) = \text{Im } \Pi$ into $\text{Ker } \Pi$. It follows that $\langle \Sigma \gamma u, \gamma v \rangle_\delta = 0$ for every $u, v \in \text{Ker } \Gamma$, and (11) implies that $A_\Gamma$ is a symmetric operator. But our current assumptions are not sufficient to prove $A_\Gamma$ is a closed operator, and that $A_\Gamma$ is a self-adjoint if the boundary condition $\Pi$ is self-adjoint.
4.3. **Theorem.** If the boundary condition $\Pi$ is self-adjoint, then the kernel $\text{Ker } A_\Gamma$ is equal to the orthogonal complement of the image $A_\Gamma(H_1)$ in $H_0$.

**Proof.** Suppose that $v \in H_0$ is orthogonal to $A_\Gamma(H_1)$. Then $\langle Au, v \rangle_0 = 0$ for every $u \in \text{Ker } \Gamma$, and, in particular, for every $u \in \text{Ker } \gamma$. For such $u$ the identity (11) implies that $\langle u, A v \rangle_0 = 0$. Since $\text{Ker } \gamma$ is dense in $H_0$, it follows that $A v = 0$. Since $\Sigma$ is skew-adjoint, $A v = 0$ together with the identity (11) implies that $\langle u, v \rangle_0 + \langle \gamma u, \Sigma \gamma v \rangle_0 = \langle u, v \rangle_0 - \langle \Sigma \gamma u, \gamma v \rangle_0 = 0$ for every $u \in H_1$. Therefore $(v, \Sigma \gamma v)$ belongs to the orthogonal complement of $A \oplus \Gamma(H_1)$ in $H_0 \oplus G$. By Theorem 4.1 this implies that $v \in \text{Ker } A \oplus \Gamma$. But the latter condition is equivalent to $v \in \text{Ker } A_\Gamma$.

Conversely, if $u \in \text{Ker } \Gamma$ and $v \in \text{Ker } A_\Gamma$, then (11) implies that $\langle Au, v \rangle_0 = \langle \Sigma \gamma u, \gamma v \rangle_0 = -\langle \gamma u, \Sigma \gamma v \rangle_0$. At the same time $u, v \in \text{Ker } \Gamma$ and hence $\gamma u, \gamma v \in \text{Ker } (1 - \Pi) = \text{Im } \Pi$. Since $\Pi$ is a self-adjoint boundary condition, this implies that $\Sigma \gamma v \in \text{Ker } \Pi$. Since $\Pi$ is a self-adjoint projection, this, in turn, implies that $\langle \gamma u, \Sigma \gamma v \rangle_0 = 0$ and hence $\langle Au, v \rangle_0 = 0$. It follows that every $v \in \text{Ker } A_\Gamma$ is orthogonal to $A_\Gamma(H_1)$. This completes the proof. ■

**Further assumptions.** Let us assume that the inclusion $\iota: H_1 \rightarrow H_0$ is a compact operator. Let us assume also that $\Pi$ is a self-adjoint boundary condition and that the bounded operator $A \oplus \Gamma: H_1 \rightarrow H_0 \oplus G_{1/2}$ is Fredholm.

4.4. **Theorem.** $A_\Gamma$ defines an unbounded self-adjoint operator in $H_0$. Moreover, this operator is Fredholm, has discrete spectrum, and is an operator with compact resolvent.

**Proof.** As usual, for a number $\lambda$ we denote by $A - \lambda$ the operator $u \mapsto Au - \lambda u$. More precisely, $A - \lambda$ is an abbreviated notation for the operator $A - \lambda \iota: H_1 \rightarrow H_0$.

In particular, $A - \lambda$ is a compact perturbation of $A: H_1 \rightarrow H_0$. It follows that

$$(A - \lambda) \oplus \Gamma: H_1 \rightarrow H_0 \oplus G_{1/2}$$

is a compact perturbation of $A \oplus \Gamma$. Since $A \oplus \Gamma$ is assumed to be Fredholm, $(A - \lambda) \oplus \Gamma$ is a Fredholm operator for every $\lambda$. Clearly, the kernel of $A_\Gamma - \lambda$ is equal to the kernel of $(A - \lambda) \oplus \Gamma$. Since the operator $(A - \lambda) \oplus \Gamma$ is Fredholm, the kernel of $(A - \lambda) \oplus \Gamma$, and
hence also the kernel of $A - \lambda$, is finitely dimensional for every $\lambda$. Similarly, the image of $A - \lambda$ is equal to the intersection of the image of $(A - \lambda) \oplus \Gamma$ with $H_0 \oplus 0 \subset H_0 \oplus G_{1/2}$. Since $(A - \lambda) \oplus \Gamma$ is Fredholm, the image of $(A - \lambda) \oplus \Gamma$ is closed in $H_0 \oplus G_{1/2}$. It follows that the image of $A - \lambda$ is closed in $H_0$.

Clearly, for every $\lambda \in \mathbb{R}$ the identity (11) holds also for $A - \lambda$ in the role of $A$. Therefore for real $\lambda$ Theorems 4.1 and 4.3 hold for $A - \lambda$ in the role of $A$. Since $\text{Ker} \ A - \lambda$ is finitely dimensional, Lemma 4.3 implies that the codimension of this image is finite. It follows that the image of $A - \lambda$ is closed in $H_0$.

The continuity of $A$ as a function of $\gamma$, $A$, $\Sigma$, $\Pi$. Let us fix Hilbert spaces $H_0$, $H_1$, $H^0$, $H_{1/2}^0$, as also the inclusions $\iota: H_1 \longrightarrow H_0$ and $H_{1/2}^0 \longrightarrow H^0$, and consider the space of quadruples $\gamma$, $A$, $\Sigma$, $\Pi$ satisfying all our assumptions. Let us equip this space with the topology induced by the norm topologies on the spaces of bounded operators and the space of orthogonal projections $\Pi$.

Let us equip the space of unbounded self-adjoint operators in $H_0$ by the topology of the convergence in the norm resolvent sense. See [ReS], Section VIII.7, for the latter.

4.5. Lemma. $\Lambda$ continuously depends on $\gamma$, $A$, $\Sigma$, $\Pi$. If the operator $A \oplus \Gamma$ is invertible, then $\Lambda$ is invertible as an operator $\text{Ker} \ \Gamma \longrightarrow H_0$ and the inverse $\Lambda^{-1}: H_0 \longrightarrow H_1$ continuously depends on the quadruple.

Proof. Let us fix some quadruple $\gamma$, $A$, $\Sigma$, $\Pi$ and use the boldface font for objects related to this quadruple. Let consider quadruples $\gamma$, $A$, $\Sigma$, $\Pi$ close to this quadruple. Let

$$\Lambda \oplus \Gamma : H_1 \oplus \text{Im} \ \Pi_{1/2} \longrightarrow H_0 \oplus H_{1/2}^0$$

be the bounded operator defined by the rule

$$\Lambda \oplus \Gamma : (h, w) \longmapsto (Ah, \Gamma h + w).$$

The operator $\Lambda \oplus \Gamma$ is Fredholm, being as the direct sum of the Fredholm operator $A \oplus \Gamma$ and the identity $\text{Im} \ \Pi_{1/2} \longrightarrow \text{Im} \ \Pi_{1/2}$. Hence the operator $\Lambda \oplus \Gamma$ is Fredholm for quadruples close to the fixed one. Suppose first that $A \oplus \Gamma$ is invertible. Then $\Lambda \oplus \Gamma$ is also invertible, and hence $\Lambda \oplus \Gamma$ is invertible for quadruples close to the fixed one. The inverse of the
latter operator continuously depends on the quadruple. If $\Gamma h = 0$, then

$$\Lambda \oplus \Gamma^{-1}(Ah, 0) = \Lambda \oplus \Gamma^{-1}(Ah, \Gamma h + 0) = (h, 0)$$

and hence the operator $A_\Gamma : \text{Ker} \Gamma \to H_0$ is invertible and

$$A_\Gamma^{-1}(u) = \Lambda \oplus \Gamma^{-1}(u, 0)$$

for every $u \in H_0$. Since $\Lambda \oplus \Gamma^{-1}$ continuously depends on the quadruple, this implies that $A_\Gamma^{-1} : H_0 \to \text{Ker} \Gamma \subset H_1$ continuously depends on the quadruple. This proves the second claim of the lemma. By taking the composition of these inverses with the inclusion $\iota : H_1 \to H_0$ we conclude that $A_\Gamma^{-1} : H_0 \to H_0$ continuously depends on the quadruple. But this means exactly that $A_\Gamma$ continuously depends on the quadruple in the topology of the convergence in the norm resolvent sense. This proves the lemma in the case when $A \oplus \Gamma$ is invertible. In general, $(A - \lambda) \oplus \Gamma$ is invertible for some $\lambda$. By the already proved special case $(A - \lambda) \Gamma$ continuously depends on the quadruple for quadruples close to the fixed one. But $(A - \lambda) \Gamma = A_\Gamma - \lambda$, and hence $A_\Gamma$ also continuously depends on the quadruple. ■

**Families of abstract self-adjoint boundary problems.** Let $X$ be a topological space. Suppose that for every $x \in X$ we are given a quadruple $\gamma(x), A(x), \Sigma(x), \Pi(x)$ satisfying all our assumptions and continuously depending on $x$. Then for every $x \in X$ a bounded operator $\Gamma(x) : H^1 \to G_{1/2}(x) = \text{Ker} \Pi_{1/2}(x)$ and an unbounded operator

$$A_\Gamma(x) = A(x) \Gamma(x)$$

in $H_0$ are defined. By Lemma 4.5 the map $x \mapsto A_\Gamma(x)$ is continuous with respect the topology of $X$ and the topology of the convergence in the norm resolvent sense on unbounded self-adjoint operators in $H_0$. It follows that if two real numbers $a < b$ do not belong to the spectrum of $A_\Gamma(z)$ for some $z \in X$, then $a, b$ do not belong to the spectrum of $A_\Gamma(x)$ for $x \in X$ sufficiently close to $z$, and the spectral projections

$$P_{[a, b]}(A_\Gamma(x))$$

continuously depend on $x$ for such $x$. See [ReS], Theorem VIII.23. This immediately implies that $A_\Gamma(x), x \in X$ is a Fredholm family of self-adjoint operators in the sense of [I$_2$] and hence the analytical index of this family is defined.
5. Pseudo-differential operators and boundary conditions

Operators. As in Section 3, let $X$ be a compact manifold with the non-empty boundary $Y$, and let $E$ be a Hermitian vector bundle on $X$. We will keep the notations and assumptions of Section 3 related to $X$, $Y$. In particular, we assume that a collar neighborhood of $Y$ in $X$ is identified with $Y \times [0, 1)$, denote by $x_n$ the $[0, 1)$-coordinate in the collar, and fix a smooth function $\phi: (-1, 1) \to [0, 1]$ with compact support equal to 1 near 0. We will identify the restriction of $E$ to the collar with the pull-back of $E|Y$. Recall that by $E_y$ we denote the fiber of $E$ over $y \in Y$.

We will consider operators of order 1 acting on sections of $E$ and belonging to the class introduced by Hörmander [H], Chapter 20. They have the form $P = P^b + P^i$, where

$$P^i \in \Psi^1_{phg} (X \sim Y; E, E)$$

and the kernel of $P^i$ has compact support in $(X \sim Y) \times (X \sim Y)$, and

$$P^b = \Sigma \circ D_n + T(x_n),$$

where $\Sigma$ is a vector bundle map $E \to E$ over the collar and

$$T: (-1, 1) \to \Psi^1_{phg} (Y; E|Y, E|Y)$$

is a smooth function vanishing outside $(-1/2, 1/2)$, say. The principal symbol of $P$ is defined as the sum of the principal symbols of $P^b$ and $P^i$. We will need only the restriction of the principal symbol to $S_X$, and will call this restriction simply the symbol of $P$. It is a bundle map $\sigma: \pi^* E \to \pi^* E$. For $u \in SY$ and $y = \pi(u)$ we will denote by $\sigma_y$ and $\tau_y$ the maps $E_y \to E_y$ induced by $\Sigma$ and the symbol of $T(0)$ respectively. Let $\rho_y = \sigma_y^{-1} \tau_y$. As usual, $P$ and $\sigma$ are said to be elliptic if $\sigma$ is an automorphism of $\pi^* E$, and $\sigma$ is said to be self-adjoint if $\sigma$ is self-adjoint in every fiber.

Boundary operators for general elliptic operators of order 1. Recall that $\gamma$ is the operator of taking sections of $E$ over $X$ to their restriction to $Y$. Let $G$ be a vector bundle over $Y$. A boundary operator maps sections of $E$ over $X$ to sections of $G$ over $Y$. Hörmander [H], Chapter 20, considers boundary operators of the form $B = B_Y \circ \gamma$, where $B_Y$ is a pseudo-differential operator of order 0 from $E|Y$ to $G$.

Let $P$, $\sigma$ and $B$ be as above, and suppose that $P$, $\sigma$ are elliptic. For every $u \in SY$ and $y = \pi(u)$ the principal symbol of $B_Y$ defines a linear map $\beta_u: E_y \to G_y$. As is well

* Strictly speaking, these operators are not standard pseudo-differential operators, but can be approximated by such. This standard issue does not affect neither Hörmander’s, nor our arguments.
known, since $\sigma$ is elliptic, the operators $\rho_u$ have no real eigenvalues. Recall that we denote by $\mathcal{M}_+(\rho_u) \subset E_Y$ the space of bounded solutions $f : \mathbb{R} \to E_Y$ of the ordinary differential equation $D(f) + \rho_u(f) = 0$, where $D = -i\partial$ and, as usual, $y = \pi(u)$. The boundary operator $B$ is said to satisfy Shapiro-Lopatinskii condition for $P$ if for every $u \in SY$ the map $\beta_u$ induces an isomorphism of the space of the initial values $f(0)$ of solutions in $\mathcal{M}_+(\rho_u)$ with $G_y$. In this case one also says that the pair $(P, B)$ is elliptic.

As explained in Section 2, the space of the initial values of solutions in $\mathcal{M}_+(\rho_u)$ is equal to $L_-(\rho_u) \subset E_Y$, the sum of the generalized eigenspaces of $\rho_u$ corresponding to eigenvalues $\lambda$ with $\text{Im} \lambda < 0$. Therefore the Shapiro-Lopatinskii condition holds if and only if for every $u \in SY$ the map $\beta_u$ induces an isomorphism $L_-(\rho_u) \to G_y$, or, equivalently, the kernel $N_u = \text{Ker} \beta_u$ is transverse to $L_-(\rho_u)$. The kernels $N_u$ are fibers of a bundle $N$ over $SY$, which we will call the kernel-symbol of $B_Y$ and $B$.

**Fredholm operators.** Suppose that $P, B$ are as above and $B$ satisfies Shapiro-Lopatinskii condition for $P$. Then the rule $u \mapsto (Pu, Bu)$ defines a map

$$P \oplus B : \overline{H}(1)(X \to Y, E) \to \overline{H}(0)(X \to Y, E) \oplus \overline{H}(1/2)(Y, G)$$

of Sobolev spaces, which turns out to be Fredholm. Moreover, its kernel consists of $C^\infty$-smooth sections, and its image is the orthogonal complement of a subspace consisting of pairs of $C^\infty$-smooth sections. See [H], Theorem 20.1.8’.

**Bundle-like boundary operators in the self-adjoint case.** Suppose now that the operator $P$ and its symbol $\sigma$ are elliptic and self-adjoint. Then $\sigma$ is an elliptic self-adjoint symbol of order 1 in the sense of Section 3. In this case we would like to impose additional conditions on $B$ ensuring that the restriction of $P$ to the kernel of $B$ in an appropriate Sobolev space is a self-adjoint operator in the sense of the theory of unbounded operators in Hilbert spaces. The action of $P$ on smooth sections satisfies a Green formula of the form (11). Namely,

$$\langle Pu, v \rangle_0 - \langle u, P v \rangle_0 = \langle i \Sigma \gamma u, \gamma v \rangle_\partial,$$

where $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \gamma \cdot \rangle_\partial$ are the $L_2$-products of sections of $E$ and $E|Y$ respectively defined by the Hermitian structure, and $\gamma$ being the trace operator taking a section over $X$ to its restriction to $Y$. The proof essentially amounts to the integration by parts. See [Gr], Proposition 1.3.2, for the general case of not necessarily self-adjoint operators of arbitrary order. By continuity the above Green formula extends to the operator

$$P : \overline{H}(1)(X \to Y, E) \to \overline{H}(0)(X \to Y, E)$$

with $\gamma$ being the trace operator

$$\gamma : \overline{H}(1)(X \to Y, E) \to H(1/2)(Y, E|Y).$$
Let us consider the case when $B_Y$ is induced by a bundle morphism $E|Y \to G$. In this case the kernel-symbol $N$ of $B_Y$ is a bundle-like boundary conditions for the symbol $\sigma$ and may be considered as a subbundle of $E|Y$, and the kernel of $B$ on $\overline{H}_{(1)}(X \sim Y, E)$ consists of sections $u$ such that $\gamma u \in \overline{H}_{(1/2)}(Y, N)$. Since only the kernel of $B$ matters, we may assume that $G$ is equal to the orthogonal complement of $N$ in $E|Y$ and $B_Y$ is induced by the orthogonal projection of $E|Y$ onto $G$. Let

$$
\Pi : H_{(0)}(Y, E|Y) \to H_{(0)}(Y, E|Y) \quad \text{and} \quad 
\Pi_{1/2} : H_{(1/2)}(Y, E|Y) \to H_{(1/2)}(Y, E|Y)
$$

be the operators induced by the orthogonal projection of $E|Y$ onto $N$. Then we can replace $B_Y$ by $1 - \Pi_{1/2}$ and $B$ by $\Gamma = (1 - \Pi_{1/2}) \circ \gamma$ without affecting the kernels. Clearly, such a replacement can be done also when parameters are present.

If $N$ is a self-adjoint boundary condition for $\sigma$, then $\Pi$ is a self-adjoint boundary condition for the operator $P$ acting on the Sobolev spaces as above. In more details, let

$$
H_0 = \overline{H}_{(0)}(X \sim Y, E), \quad H_1 = \overline{H}_{(1)}(X \sim Y, E),
$$

$$
H^\partial = H_{(0)}(Y, E|Y), \quad H^\partial_{1/2} = H_{(1/2)}(Y, E|Y)
$$

and $\gamma : H_1 \to H^\partial_{1/2}$ be the above trace operator. Then we are in the framework of Section 4 and can apply its results to $\Pi, \Pi_{1/2}$ and to $P, i\Sigma$ in the roles of $A, \Sigma$ respectively. If $N$ is a self-adjoint boundary condition for $\sigma$, then $\Sigma(N) = i\Sigma(N)$ is orthogonal to $N$ and hence $i\Sigma(\text{Im} \Pi) = \text{Ker} \Pi$, i.e. $\Pi$ is a self-adjoint boundary condition for $P$ in the sense of Section 4. Since the inclusion

$$
H_{(1/2)}(Y, E|Y) \to H_{(0)}(Y, E|Y)
$$

is compact and the Shapiro-Lopatinskii condition ensures that $P \oplus \Gamma$ is Fredholm, Theorem 4.4 applies. Hence the restriction $P_\Gamma = P_B$ of $P$ to the kernel $\text{Ker} \Gamma = \text{Ker} B$ is an unbounded self-adjoint operator. Moreover, it is a Fredholm operator with discrete spectrum and compact resolvent. As explained at the end of Section 4, Lemma 4.5 implies that this construction leads to a Fredholm family of self-adjoint operators when parameters are present. Of course, one should assume that $P$ and $B$ continuously depend on parameters.

**General boundary operators in the self-adjoint case.** Let $B_Y$ be a pseudo-differential operator of order 0, not necessarily induced by a bundle morphism. Let $N$ be the kernel-symbol of $B_Y$. Suppose that $B = B_Y \circ \gamma$ satisfies the Shapiro-Lopatinskii condition for $P$, and hence $N$ satisfies the Shapiro-Lopatinskii condition for the symbol $\sigma$ of $P$. Let

$$
\Pi : H_{(0)}(Y, E|Y) \to H_{(0)}(Y, E|Y)
$$
be the orthogonal projection onto the kernel of $B_Y$. By the results of Seeley $\Pi$ is a pseudo-differential operator of order 0. See [S$_2$], Theorem IV.6 and the proof of Theorem IV.7, pp. 252–257. In particular, $\Pi$ induces a projection

$$\Pi_{1/2} : H_{(1/2)}(Y, E|Y) \longrightarrow H_{(1/2)}(Y, E|Y)$$

As before, we can replace $B_Y$ by $1 - \Pi_{1/2}$ and $B$ by $\Gamma = (1 - \Pi_{1/2}) \circ \gamma$ without affecting the kernels and the induced unbounded operators. But assuming that $N$ is self-adjoint is not sufficient to ensure that the unbounded operator $P_\Gamma = P_B$ is self-adjoint. Moreover, the property of being self-adjoint is not determined by the principal symbol of $B_Y$.

In order to discuss this in more details, it is convenient to change the form of boundary conditions. Seeley [S$_2$] suggested to consider arbitrary pseudo-differential operators $\Pi$ of order 0 acting in $E|Y$ such that $\text{Im} \, \Pi$ is closed in $H_{(0)}(Y, E|Y)$ and an analogue of the Shapiro-Lopatinskii condition holds. This analogue means that the kernel-symbol of $\Pi$ satisfies the Shapiro-Lopatinskii condition for $\sigma$. Seeley called such operators $\Pi$ well-posed for $P$. See [S$_2$], Definition VI.3 (p. 289). For such $\Pi$ one can define the boundary operator $\Gamma$ and the unbounded operator $P_\Gamma$ exactly as before. While for general elliptic operators well-posed operators define a strictly larger class of boundary problems than the classical boundary operators, in the self-adjoint case these classes are the same. Cf. the discussion of boundary conditions in the classical form in Section 3.

The results of Section 4 imply that $P_\Gamma$ is self-adjoint if $i \Sigma(\text{Im} \, \Pi) = \Sigma(\text{Im} \, \Pi)$ is equal to the orthogonal complement to $\text{Im} \, \Pi$ in $H_{(0)}(Y, E|Y)$. When $\Pi$ is a projection, this is equivalent to $\Sigma(\text{Im} \, \Pi) = \text{Ker} \, \Pi$. It is only natural to ask when such an operator $\Pi$ exists.

**Realization of self-adjoint kernel-symbols by self-adjoint operators.** If $P$ is elliptic, then $\Sigma$ is an automorphism of the bundle $E$. For the rest of this section we will assume that, moreover, $\Sigma$ is an isometry. The general case can be reduced to this one by replacing the $\Sigma$ by $|\Sigma|^{-1} \Sigma$, the subbundle $N$ by $|\Sigma|^{-1/2}(N)$, etc. Cf. the first homotopy in Section 2.

Let $N$ be an elliptic self-adjoint boundary condition for the symbol $\sigma$ in the sense of Section 3. For every $u \in SY$ and $y = \pi(u)$ let $\pi_u : E_y \longrightarrow E_y$ be the orthogonal projection with the image $N_u$. These projections define an endomorphism

$$\pi : \pi^* E|SY \longrightarrow \pi^* E|SY.$$

A realization of the boundary condition $N$ is a pseudo-differential operator of order 0 from $E|Y$ to $E|Y$ with the principal symbol $\pi$ such that induced operator

$$\Pi : H_{(0)}(Y, E|Y) \longrightarrow H_{(0)}(Y, E|Y)$$

is a self-adjoint projection and $\Sigma(\text{Im} \, \Pi) = \text{Ker} \, \Pi$. The last condition is the crucial one.
The results of Seeley [S2] imply that there always exists a pseudo-differential operator \( \Pi \) of order 0 with the principal symbol \( \pi \) such that \( \Pi \) is a projection, i.e. \( \Pi \circ \Pi = \Pi \). After replacing \( \Pi \) by \( (\Pi + \Pi^*)/2 \) we will get a self-adjoint projection with the same symbol. The boundary condition \( N \) also defines for every \( u \in SY \) an isometry \( \varphi_u : E^+_Y \rightarrow E^-_Y \) having \( N_u \) as its graph. In turn, these isometries define an isometric isomorphism of bundles

\[
\varphi : \pi^*E^+_Y \rightarrow \pi^*E^-_Y
\]

already used in Section 3. An isometric realization of \( \varphi \) is a pseudo-differential operator of order 0 from \( E^+_Y \) to \( E^-_Y \) with the principal symbol \( \varphi \) such that induced operator

\[
\Phi : H_{(0)}(Y, E^+_Y) \rightarrow H_{(0)}(Y, E^-_Y)
\]

is an isometry of Hilbert spaces.

The isometric realizations of \( \varphi \) are closely related to the realizations of \( N \). Let

\[
H^\delta_+ = H_{(0)}(Y, E^+_Y), \quad H^\delta_- = H_{(0)}(Y, E^-_Y), \quad \text{and let}
\]

\[
\text{pr}_+ : H^\delta_+ \oplus H^\delta_- \rightarrow H^\delta_+, \quad \text{pr}_- : H^\delta_+ \oplus H^\delta_- \rightarrow H^\delta_-
\]

be the canonical projections. Then \( H^\delta = H^\delta_+ \oplus H^\delta_- \) and \( \Sigma \) acts as the identity on \( H^\delta_+ \) and the minus identity on \( H^\delta_- \). Suppose that \( \Pi : H^\delta_+ \rightarrow H^\delta_- \) is a self-adjoint projection such that \( \Sigma (\text{Im} \Pi) = \text{Ker} \Pi \). We claim that \( \text{Im} \Pi \) is the graph of an isometry \( \Phi : H^\delta_+ \rightarrow H^\delta_- \).

Since \( \Sigma (\text{Im} \Pi) = \text{Ker} \Pi \), the image \( \text{Im} \Pi \) intersects each of the spaces \( H^\delta_+ \) and \( H^\delta_- \) by zero. Suppose that \( v \in H^\delta_+ \) is orthogonal to \( \text{pr}_+(\text{Im} \Pi) \). Clearly, then \( v \) is orthogonal to \( \text{Im} \Pi \). Since \( \Sigma(v) = v \), this implies that \( v \) is orthogonal to \( \text{Ker} \Pi \) also, and hence \( v = 0 \). It follows that \( \text{pr}_+ \) induces a surjective map \( \text{Im} \Pi \rightarrow H^\delta_+ \). Similarly, \( \text{pr}_- \) induces a surjective map \( \text{Im} \Pi \rightarrow H^\delta_- \). It follows that \( \text{Im} \Pi \) is equal to the graph of a linear map \( \Phi : H^\delta_+ \rightarrow H^\delta_- \). Arguing as in Section 2 we see that \( \Phi \) is an isometry (see the discussion of lagrangian subspaces when \( \sigma \) is unitary). This proves our claim.

Let us express \( \Pi \) in terms of \( \Phi \) and vice versa. The orthogonal complement to the graph of \( \Phi \) is equal to the graph of \( -\Phi \). Since \( \Pi \) is a self-adjoint projection, it is also equal to \( \text{Im}(1 - \Pi) \). It follows that every \( (x, y) \in H^\delta_+ \oplus H^\delta_- \) admits a unique presentation

\[
(x, y) = (a, \Phi(a)) + (b, -\Phi(b)),
\]

where \( a, b \in H^\delta_+ \). An easy calculation shows that

\[
a = \frac{x + \Phi^{-1}(y)}{2}, \quad b = \frac{x - \Phi^{-1}(y)}{2}.
\]
This leads to the formulas

\[ \Pi(x, y) = \left( \frac{x + \Phi^{-1}(y)}{2}, \frac{\Phi(x) + y}{2} \right) \quad \text{and} \]

(13)

\[ \Phi(x) = \text{pr}_- \left( 2 \Pi(x, 0) \right) \]

relating \( \Pi \) and \( \Phi \).

Conversely, suppose that \( \Phi : H^0_+ \rightarrow H^0_- \) is an isometry. Let us define \( \Pi \) by the above formula. Direct calculations show that \( \Pi \circ \Pi = \Pi \), i.e. \( \Pi \) is a projection, and

\[
(1 - \Pi)(x, y) = \left( \frac{x - \Phi^{-1}(y)}{2}, \frac{y - \Phi(x)}{2} \right)
\]

\[= \Sigma \left( \frac{x - \Phi^{-1}(y)}{2}, \frac{y - \Phi(x)}{2} \right) = \Sigma \left( \Pi(x, -y) \right). \]

Therefore \( \Sigma(\text{Im } \Pi) = \text{Im } (1 - \Pi) = \text{Ker } \Pi \). Another direct calculation, using the assumption that \( \Phi \) is an isometry, shows that \( \text{Im } \Pi \) is orthogonal to \( \text{Im } (1 - \Pi) \). Since \( \Pi \) is a projection, this implies that \( \Pi \) is a self-adjoint projection.

5.1. Theorem. Suppose that \( \Pi \) and \( \Phi \) are related as in (13). Then \( \Pi \) is a realization of the boundary condition \( N \) if and only if \( \Phi \) is an isometric realization of \( \varphi \).

Proof. The formulas (13) imply that \( \Pi \) is a pseudo-differential operator of order 0 if and only if \( \Phi \) is. Moreover, if \( \Pi \) and \( \Phi \) are pseudo-differential operators of order 0, then their principal symbols are related in the same way as the operators themselves. It follows that the principal symbol of \( \Pi \) is equal to \( \pi \) if and only if the principal symbol of \( \Phi \) is equal to \( \varphi \). This proves the theorem. ■

5.2. Corollary. There exists a realization of the boundary condition \( N \) if and only if there exists an isometric realization of \( \varphi \). ■

5.3. Theorem. There exists an isometric realization of \( \varphi \) if and only if the analytical index of some pseudo-differential operator of order 0 with the principal symbol \( \varphi \) is zero.

Proof. The “only if” part is trivial. Let \( \Psi_0 \) be some pseudo-differential operator of order 0 with the principal symbol \( \varphi \). If the analytical index of \( \Psi_0 \) is zero, then \( \Psi_1 = \Psi_0 + k \) is an isomorphism for some operator \( k \) of finite rank. The principal symbol of \( \Psi + k \) is equal to that of \( \Psi \), i.e. to \( \varphi \). Let \( \Phi = \Psi_1 |\Psi_1|^{-1} \), where \( |\Psi_1| = (\Psi_1^* \Psi_1)^{1/2} \). Then \( \Phi \) is an isometry. The results of Seeley \([S_2]\) imply that \( |\Psi_1| \) is a pseudo-differential operator of
order 0 with the principal symbol $|\varphi|$. Since $\varphi$ is an isometry, $|\varphi|$ is the identity bundle map. It follows that the principal symbol of $\Phi$ is equal to that of $\Psi$, i.e. to $\varphi$. Since $\Phi$ is an isometry, this proves the “if” part. ■

Realization of self-adjoint kernel-symbols in families. Suppose now, as at the end of Sections 3, that our symbols and operators depend on a parameter $z \in \mathbb{Z}$, where $\mathbb{Z}$ is a topological space. As in Sections 3, we allow also the manifold $X = X(z)$ to vary among the fibers of a bundle. Let $Y(z) = \partial X(z)$. We will assume that $Z$ is a compactly generated and paracompact space. For example, every metric space has these properties. Like above, we will consider only the case when the bundle maps $\Sigma(z)$ are isometries.

Let $N(z), z \in \mathbb{Z}$ be a continuous family of self-adjoint elliptic boundary conditions. Then the families of the corresponding bundle maps $\pi(z), z \in \mathbb{Z}$ and $\varphi(z), z \in \mathbb{Z}$ are defined. A realization of the family $N(z), z \in \mathbb{Z}$ is defined as a continuous family $\Pi(z), z \in \mathbb{Z}$ of pseudo-differential operators of order 0 such that $\Pi(z)$ is a realization of $N(z)$ for every $z \in \mathbb{Z}$. The isometric realizations of $\varphi(z), z \in \mathbb{Z}$ are defined in the same manner.

5.4. Theorem. Suppose that $\Pi = \Pi(z)$ and $\Phi = \Phi(z)$ are related as in (13) for every $z \in \mathbb{Z}$. Then $\Pi(z), z \in \mathbb{Z}$ is a realization of $N(z), z \in \mathbb{Z}$ if and only if $\Phi(z), z \in \mathbb{Z}$ is an isometric realization of $\varphi(z), z \in \mathbb{Z}$.

Proof. It is sufficient to repeat the proof of Theorem 5.1 with parameters added. ■

5.5. Corollary. There exists a realization of the family of boundary conditions $N(z), z \in \mathbb{Z}$ if and only if there exists an isometric realization of $\varphi(z), z \in \mathbb{Z}$. ■

Fredholm families. In order to prove an analogue of Theorem 5.3 for families, we need a general result about families of Fredholm operators with analytical index zero, namely, Theorem 5.6 below. It depends on [I$_1$], [I$_2$]. Let $\mathbb{H}$ be a Hilbert bundle with the base $Z$, considered as a family $H_z, z \in \mathbb{Z}$ of Hilbert spaces. Let $\mathbb{A}$ be a family of Fredholm operators $A_z: H_z \rightarrow H_z, z \in \mathbb{Z}$, continuous in an appropriate sense. It is sufficient to assume that $\mathbb{A}$ is fully Fredholm in the sense of [I$_3$]. This is the case if $\mathbb{A}$ is a continuous family of elliptic pseudo-differential operators by the results of Atiyah and Singer [AS$_4$]. The continuity in the following theorem can be understood in the same sense.

5.6. Theorem. For every $z \in \mathbb{Z}$ let $A_z = U_z|A_z|$ be the polar decomposition of $A_z$. If the analytical index of the family $\mathbb{A}$ is zero, then there exists a continuous family of isometries $U'_z: H_z \rightarrow H_z, z \in \mathbb{Z}$ such that $U'_z - U_z$ is a finite rank operator for every $z \in \mathbb{Z}$.

Proof. Note that, in general, the family $U_z, z \in \mathbb{Z}$ is not continuous in any reasonable sense because the dimension of the kernels $\text{Ker } U_z$ may jump. Suppose that the analytical
index of $A$ is zero. Then for every $z \in Z$ the index of $A_z$ is zero and $U_z$ is a partial isometry with the same kernel and cokernel as $A_z$. Since the index of $A_z$ is zero, the dimensions of the kernel and cokernel are equal and hence there exist isometries $H_z \to H_z$ equal to $U_z$ on the orthogonal complement $H_z \ominus \ker U_z$ of the kernel $\ker U_z = \ker A_z$.

Let $U^{\text{fin}}(z)$ be the set of isometries $H_z \to H_z$ equal to $U_z$ on a closed subspace of finite codimension in $H_z \ominus \ker U_z$, and let us equip $U^{\text{fin}}(z)$ with the norm topology. Since $A$ is a Fredholm family, the family of spaces $U^{\text{fin}}(z)$, $z \in Z$ forms a locally trivial bundle over $Z$ having $U^{\text{fin}}(z)$ as the fiber over $z \in Z$. Let us denote this bundle by

$$\pi^{\text{fin}}(A) : U^{\text{fin}}(A) \to Z.$$  

This is an analogue of the Grassmannian bundle $\text{Gr}(A)$ defined in [I2] for strictly Fredholm families of self-adjoint Fredholm operators. There is also a “universal” analogue

$$\pi^{\text{fin}} : U^{\text{fin}} \to |\mathcal{PS}_0|$$  

of $\pi^{\text{fin}}(A)$, where $|\mathcal{PS}_0|$ is a classifying space. See [I1], Section 15, where $U^{\text{fin}}$ is denoted by $U$. The bundle $\pi^{\text{fin}}(A)$ is induced from the bundle $\pi^{\text{fin}}$ by a polarized index map $Z \to |\mathcal{PS}_0|$. This is an analogue of Theorem 5.1 in [I2] with a similar proof.

Let us use the definition of the analytical index of families from [I2], Section 3. It is equivalent to the classical one when the latter applies. See [I2], Section 7. With this definition, the analytical index is zero if and only if the polarized index map is homotopic to a constant map (compare [I2], the proof of Theorem 5.2). Since the analytical index of $A$ is zero, this implies that $\pi^{\text{fin}}(A)$ is a trivial bundle and hence admits a continuous section. If $z \mapsto U'_z$ is a section, then $U'_z$ is an isometry $H_z \to H_z$ equal to $U_z$ on a subspace of finite codimension. It follows that the rank of $U'_z - U_z$ is finite for every $z \in Z$. ■

**Remark.** With somewhat more work one can prove that if the analytical index of $A$ is zero, then there exists a continuous family of isomorphisms $A'_z : H_z \to H_z$, $z \in Z$ such that $A'_z - A_z$ has finite rank for every $z \in Z$.

**5.7. Theorem.** There exists an isometric realization of $\varphi(z)$, $z \in Z$ if and only if the analytical index of some family of pseudo-differential operators of order 0 with the principal symbols $\varphi(z)$, $z \in Z$ is zero.

**Proof.** The “only if” part is trivial. Let $\Psi(z)$, $z \in Z$ be some family of pseudo-differential operators of order 0 with the principal symbols $\varphi(z)$, $z \in Z$, and suppose that the analytical index of this family is 0. Then the index of the operator $\Psi(z)$ is zero for every $z \in Z$.

Let $\Psi(z) = U(z) |\Psi(z)|$ be the polar decomposition of $\Psi(z)$. We claim that $U(z)$ is a pseudo-differential operator of order 0 with the (principal) symbol $\varphi(z)$. Since the index
of the operator $\Psi(z)$ is zero, there exists an operator $k$ of finite rank such that $\Psi(z) + k$ is invertible. By the results of Seeley [S$_2$], the operator

$$|\Psi(z) + k| = \left( (\Psi(z)^* + k^*) (\Psi(z) + k) \right)^{1/2}$$

is a pseudo-differential operator of order 0 with the symbol $|\varphi(z)|$. Since $\varphi$ is unitary, this symbol is equal to the identity. Clearly, $|\Psi(z) + k|$ is equal to $|\Psi(z)|$ on the intersection of the kernels $\text{Ker} k$ and $\text{Ker} k^* \circ \Psi(z)$, a subspace of finite codimension. It follows that $U(z)$ is equal to $\Psi(z) |\Psi(z) + k|^{-1}$ on a subspace of finite codimension. In turn, this implies that $U(z)$ is a pseudo-differential operator of order 0 with the same symbol as $\Psi(z) |\Psi(z) + k|^{-1}$. Since the symbol of $|\Psi(z) + k|^{-1}$ is equal to the identity, it follows that the symbol of $U(z)$ is equal to $\varphi(z)$. This proves our claim.

By Theorem 5.6 there exists a continuous family of isometries $U'(z), z \in Z$ such that the differences $U'(z) - U(z)$ have finite rank. This implies that $U'(z)$ is a pseudo-differential operator of order 0 with the same symbol as $U(z)$, i.e. with the symbol $\varphi(z)$. Therefore $U'(z), z \in Z$ is an isometric realization of $\varphi(z), z \in Z$. This proves the “if” part.

**The two obstructions.** The analytical index of a family $\Phi(z), z \in Z$ of pseudo-differential operators of order 0 with with the symbols $\varphi(z), z \in Z$ depends only on $\varphi(z), z \in Z$ and is called the analytical index of the latter. Let $P(z), z \in Z$ be a continuous family of self-adjoint pseudo-differential operators of order 1 belonging to the Hörmander’s class, and let $N(z), z \in Z$ be a family of self-adjoint boundary conditions for the symbols $\sigma(z)$ of these operators. The above results show that the analytical index of the corresponding family $\varphi(z), z \in Z$ is an obstruction, and the only obstruction to the existence of realizations of the family $N(z), z \in Z$. As in Section 3, let $V \rightarrow Z$ be the bundle having $Y = \partial X(z)$ as the fiber over $z \in Z$. The families of bundles

$$BY(z), \ SY(z), \ E(z)^+_{Y(z)}, \ \text{and} \ E(z)^-_{Y(z)}$$

lead to the corresponding bundles over $V$, which we will denote by

$$BV, \ SV, \ E^+_{V}, \ \text{and} \ E^-_{V}$$

respectively. Clearly, the family $\varphi(z), z \in Z$ defines an isometry

$$\varphi: \pi^* E^+_{V} \rightarrow \pi^* E^-_{V},$$

where now $\pi$ is the projection $SV \rightarrow V$, and hence defines a class

$$\mathcal{I}(N) \in K^0(BV, SV) = K^0(TV),$$

the parametrized version of the class $\mathcal{I}(N)$ from Section 3. By the Atiyah–Singer index
Theorem for families [AS4] the analytical index of the family $\varphi(z), z \in \mathbb{Z}$ is equal to the forward image of the class $\mathcal{I}(N)$ in $K^0(\mathbb{Z})$ (cf. the definition of the topological index in Section 3). In particular, if $\mathcal{I}(N)$ is equal to zero, then the analytical index of $\varphi(z), z \in \mathbb{Z}$ is also equal to zero. At the same time the class $\mathcal{I}(N)$ is the obstruction to defining the topological index of $(\sigma, N)$, as the parametrized version of Proposition 3.1 shows. Therefore, if the topological index of $(\sigma, N)$ is defined, then there exists a realization of the family of boundary conditions $N(z), z \in \mathbb{Z}$.

**The non-uniqueness of realizations.** It turns out that even when a realization of a family of boundary conditions $N(z), z \in \mathbb{Z}$ exists, it is usually not unique, even up to homotopy. Since the realizations of $N(z), z \in \mathbb{Z}$ are in one-to-one correspondence with the isometric realizations of $\varphi(z), z \in \mathbb{Z}$, in order to prove this, it is sufficient to prove that the latter are not unique. Let $\Phi(z), z \in \mathbb{Z}$ be an isometric realization of $\varphi(z), z \in \mathbb{Z}$.

Let us return to the proof of Theorem 5.6 in the case of the family $A_z = \Phi(z), z \in \mathbb{Z}$. Sections of the bundle $\pi^{\text{fin}}(\Phi) : U^{\text{fin}}(\Phi) \rightarrow \mathbb{Z}$ define isometric realizations $\Phi'(z), z \in \mathbb{Z}$ such that $\Phi'(z)$ is equal to $\Phi'(z)$ on a closed subspace of finite codimension. Since an isometric realization exists, this bundle is trivial, as was explained in the proof of Theorem 5.6. In general, an isometric realization $\Phi'(z), z \in \mathbb{Z}$ will differ from $\Phi(z), z \in \mathbb{Z}$ by compact operators. This suggests to consider the bundle

$$
\pi^{\text{comp}}(\Phi) : U^{\text{comp}}(\Phi) \rightarrow \mathbb{Z}
$$

having as a fiber over $z \in \mathbb{Z}$ the space of isometries $U$ such that $U - \Phi(z)$ is a compact operator. There is also a “universal” bundle

$$
\pi^{\text{comp}} : U^{\text{comp}} \rightarrow | \mathcal{P}\mathcal{J}_0 |
$$

over the same base $| \mathcal{P}\mathcal{J}_0 |$ as $\pi^{\text{fin}}$, and $\pi^{\text{comp}}(\Phi)$ is induced from $\pi^{\text{comp}}$ by the same polarized index map. Therefore the bundle $\pi^{\text{comp}}(\Phi)$ is trivial by the same reason as $\pi^{\text{fin}}(\Phi)$. The fibers of the bundles $\pi^{\text{fin}}(\Phi)$ and $\pi^{\text{comp}}(\Phi)$ are homeomorphic, respectively, to the space $U^{\text{fin}}$ of isometries of a fixed Hilbert space equal to the identity on a closed subspace of finite codimension, and to the space $U^{\text{comp}}$ of isometries differing from the identity by a compact operator. Both of them are homotopy equivalent to the infinite unitary group $U(\infty)$ and hence are classifying spaces for $K^1$-theory. For $U^{\text{fin}}$ see [I1], Corollary 12.10. For $U^{\text{comp}}$ this is implicit in [AS]. See also [I1], Section 16.

Therefore every two isometric realizations define an element of the group $K^1(\mathbb{Z})$, invariant under continuous deformations. Moreover, given an isometric realization, every element of $K^1(\mathbb{Z})$ results from some other isometric realization. This means that, in general, the natural data on the level of principal symbols do not lead to a well defined, even only up to homotopy, family of self-adjoint boundary problems. But if we limit ourselves by bundle-like boundary conditions, we will get a preferred class of canonical realizations.
6. Some operators with the index zero

The framework. We will work in the framework of Hörmander [H], Section 20.3. In particular, $X$ is a compact $n$-dimensional manifold with non-empty boundary $\partial X$, a collar neighborhood of $\partial X$ is identified with $\partial X \times [0, 1)$, and points in this collar are denoted by $(x', x_n)$, where $x' \in \partial X$ and $x_n \in [0, 1)$. This identification allows us to define the partial derivative $\partial / \partial x_n$ in the collar. Let $D_n = -i \partial / \partial x_n$. Also, $E$ is a complex vector bundle over $X$, identified over $\partial X \times [0, 1)$ with the bundle induced from the restriction $E|_{\partial X}$ by the projection $\partial X \times [0, 1) \to \partial X$. The bundle $E$ is equipped with a Hermitian metric and $X$ is equipped with a riemannian metric or just a density. Deviating from [H], we denote by $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_{\partial}$ the Hermitian scalar product of sections of $E$ followed by the integration over $X$ and $\partial X$ respectively. We refer to [H] for any unexplained notions and results.

The standard operators. Let us review the construction of operators from Proposition 20.3.1 in [H]. Suppose that $E|_{\partial X}$ is presented as an orthogonal direct sum $E|_{\partial X} = E^+ \oplus E^-$. This leads to a similar decomposition of $E$ over the collar $\partial X \times [0, 1)$. By an abuse of notations we denote the summands of the latter also by $E^+$ and $E^-$. Let $\varphi: (-1, 1) \to [0, 1]$ be a smooth function with compact support such that $\varphi$ is equal to 1 in a neighborhood of 0. We consider $\varphi(x_n)$ as a smooth function on $X$ with the support in the collar $\partial X \times [0, 1)$. Let $\lambda^+, \lambda^-, \lambda > 0$ be positive real numbers. Let us choose pseudo-differential operators

$$
\Lambda^+ \in \Psi^1_{phg}(\partial X, E^+, E^+), \quad \Lambda^- \in \Psi^1_{phg}(\partial X, E^-, E^-), \quad \text{and}
$$

$$
\Lambda \in \Psi^1_{phg}(X \sim \partial X, E, E)
$$

with principal symbols equal to $\lambda^+, \lambda^-$, and $\lambda$ times the identity respectively. We may assume that these operators are formally self-adjoint and positive. Given a section $u$ of $E$, we can represent its restriction to the collar as $(u^+, u^-)$, where $u^+, u^-$ are sections of $E^+, E^-$ respectively. Let us define the operator $P^b$ acting on sections of $E$ by

$$
P^b u = \varphi(x_n) \left( (D_n + i \Lambda^+) u^+ + (-D_n + i \Lambda^-) u^- \right).
$$

Let $Q^i$ be the operator acting on sections of $E$ by

$$
Q^i u = (1 - \varphi(x_n)) \Lambda \left( (1 - \varphi(x_n)) u \right),
$$

and let $P^i = i Q^i$. Finally, let $P = P^b + P^i$ and $B u = u^-|_{\partial X}$. It is easy to check that the boundary operator $B$ satisfies Shapiro-Lopatinskii condition with respect to $P$, i.e. $(P, B)$ is an elliptic boundary problem. By [H], Proposition 20.3.1, the operator $(P, B)$ induces Fredholm operators of index zero in appropriate Sobolev spaces. Since we need not so much this result, but rather its proof, we present some details.
Self-adjointness and Green formulas. The operator $\Lambda$ is formally self-adjoint, and hence the operator $Q^i$ is also formally self-adjoint. Since $1 - \phi(x_n)$ vanishes near the boundary,

$$\langle Q^i u, v \rangle_X = \langle u, Q^i v \rangle_X$$

for every two sections $u, v$ of $E$. Since $P^i = i Q^i$, it follows that

$$\langle P^i u, v \rangle_X = - \langle u, P^i v \rangle_X$$

for every two sections $u, v$ of $E$. The operators $\Lambda^+$ and $\Lambda^-$ are also formally self-adjoint. Since $\partial X$ is a compact manifold without boundary,

$$\langle \Lambda^+ u, v \rangle_\partial = \langle u, \Lambda^+ v \rangle_\partial \quad \text{and} \quad \langle \Lambda^- u, v \rangle_\partial = \langle u, \Lambda^- v \rangle_\partial ,$$

for every two sections $u, v$ of $E|\partial X$. The action of operators $\phi(x_n)\Lambda^+$ and $\phi(x_n)\Lambda^-$ on a section $u$ of $E$ depends only on the restriction of $u$ to the collar, and over the collar these operators act only in the direction of $\partial X$. It follows that for every two sections $u, v$ of $E$

$$\langle \phi(x_n)\Lambda^+ u, v \rangle_X = \langle u, \phi(x_n)\Lambda^+ v \rangle_X \quad \text{and}$$

$$\langle \phi(x_n)\Lambda^- u, v \rangle_X = \langle u, \phi(x_n)\Lambda^- v \rangle_X ,$$

and hence

$$\langle \phi(x_n)i\Lambda^+ u, v \rangle_X = - \langle u, \phi(x_n)i\Lambda^+ v \rangle_X \quad \text{and}$$

$$\langle \phi(x_n)i\Lambda^- u, v \rangle_X = - \langle u, \phi(x_n)i\Lambda^- v \rangle_X .$$

Let us denote by $\gamma$ the operator of taking the restriction to $\partial X$. The operator $D_n$ is formally self-adjoint, and the Green formula for the operator $D_n$ is

$$(14) \quad \langle D_n u, v \rangle_X - \langle u, D_n v \rangle_X = \langle \gamma u, -i\gamma v \rangle_\partial = \langle \gamma u, \gamma v \rangle_\partial .$$

Now we can deduce the Green formula for $\phi(x_n)D_n$. The Green formula (14) implies that

$$\langle \phi(x_n)D_n u, v \rangle_X - \langle u, \phi(x_n)D_n v \rangle_X$$

$$= \langle D_n u, \phi(x_n) v \rangle_X - \langle u, \phi(x_n)D_n v \rangle_X$$

$$= \langle u, D_n(\phi(x_n) v) \rangle_X + i \langle \gamma u, \gamma(\phi(x_n) v) \rangle_\partial - \langle u, \phi(x_n)D_n v \rangle_X .$$

$$= \langle u, D_n(\phi(x_n) v) \rangle_X + i \langle \gamma u, \gamma v \rangle_\partial - \langle u, \phi(x_n)D_n v \rangle_X .$$
where at the last step we used the fact that \( \varphi(0) = 1 \). By the Leibniz formula

\[
D_n(\varphi(x_n) v) = (D_n \varphi(x_n)) v + \varphi(x_n) D_n v.
\]

Since \( D_n(\varphi(x_n)) = -i \varphi'(x_n) \), it follows that

\[
\langle \varphi(x_n) D_n u, v \rangle_X - \langle u, \varphi(x_n) D_n v \rangle_X = i \langle u, \varphi'(x_n) v \rangle_X + i \langle \gamma u, \gamma v \rangle_{\partial}.
\]

This is the Green formula for the operator \( \varphi(x_n) D_n \).

**Gårding inequalities.** Since \( \Lambda^+ \) is a positive pseudo-differential operator,

\[
\text{Re} \langle \Lambda^+ u, u \rangle_X \geq -C'' \langle u, u \rangle_X
\]

for some constant \( C'' > 0 \). See [H], Theorem 18.1.14. It follows that

\[
\text{Re} \langle \varphi(x_n) \Lambda^+ u, u \rangle_X = \text{Re} \langle \varphi(x_n)^{1/2} \Lambda^+ u, \varphi(x_n)^{1/2} u \rangle_X
\]

\[
\geq -C' \langle \varphi(x_n)^{1/2} u, \varphi(x_n)^{1/2} u \rangle_X \geq -C \langle u, u \rangle_X
\]

for some constants \( C', C > 0 \), and hence

\[
\text{Im} \langle \varphi(x_n) i \Lambda^+ u, u \rangle_X \geq -\langle u, u \rangle_X
\]

for some constant \( C > 0 \). Similarly, for some constant \( C > 0 \)

\[
\text{Im} \langle \varphi(x_n) i \Lambda^- u, u \rangle_X \geq -\langle u, u \rangle_X \quad \text{and} \quad \text{Im} \langle P^i u, u \rangle \geq -C \langle u, u \rangle_X.
\]

**The main estimates.** They are concerned with the operators \( P^b \) and \( P^i \). Let us consider the action of \( P^b \) separately on \( E^+ \) and \( E^- \). If \( u \) is section of \( E^+ \), then

\[
2 \text{Im} \langle P^b u, u \rangle_X = \left( \langle P^b u, u \rangle_X - \overline{\langle P^b u, u \rangle_X} \right) / i
\]

\[
= \left( \langle P^b u, u \rangle_X - \langle u, P^b u \rangle_X \right) / i
\]

\[
= 2 \langle \varphi(x_n) \Lambda^+ u, u \rangle_X + \langle u, \varphi'(x_n) u \rangle_X + \langle \gamma u, \gamma u \rangle_{\partial}
\]

\[
\geq -C \langle u, u \rangle_X + \langle \gamma u, \gamma u \rangle_{\partial}
\]

for some constant \( C > 0 \). Here we used the fact that \( \Lambda^+ \) commutes with the multiplication by \( \varphi(x_n) \) and both these operators are symmetric, and also the elementary inequality

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\[ \langle u, \varphi'(x_n)u \rangle_X \geq -C \langle u, u \rangle_X. \] Similarly, if \( u \) is section of \( E^- \), then
\[
2 \text{Im} \langle P^b u, u \rangle_X \geq -C \langle u, u \rangle_X - \langle \gamma u, \gamma u \rangle_{\partial}.
\]
The second minus sign on the right is caused by the minus sign in front of \( D_n \) in the definition of the action of \( P^b \) on sections of \( E^- \). By combining the estimates for sections of \( E^+ \) and of \( E^- \) we see that if \( u \) is a section of \( E \), then
\[
2 \text{Im} \langle P^b u, u \rangle_X \geq \langle \gamma u^+, \gamma u^+ \rangle_{\partial} - \langle \gamma u^-, \gamma u^- \rangle_{\partial} - C \langle u, u \rangle_X
\]
for some constant \( C > 0 \). Similarly,
\[
2 \text{Im} \langle P^1 u, u \rangle_X = \left( \langle i Q^1 u, u \rangle_X - \langle u, i Q^1 u \rangle_X \right)/i
\]
\[
= \langle Q^1 u, u \rangle_X + \langle u, Q^1 u \rangle_X
\]
\[
= \langle \Lambda(1 - \varphi(x_n))u, (1 - \varphi(x_n))u \rangle_X + \langle (1 - \varphi(x_n))u, \Lambda(1 - \varphi(x_n))u \rangle_X
\]
\[
\geq -C \langle u, u \rangle_X
\]
for some constant \( C > 0 \). Finally, for \( P = P^b + P^1 \) and some \( C > 0 \) we get the estimate
\[
\text{Im} \langle Pu, u \rangle_X \geq \langle \gamma u^+, \gamma u^+ \rangle_{\partial} - \langle \gamma u^-, \gamma u^- \rangle_{\partial} - C \langle u, u \rangle_X.
\]

6.1. Lemma. If \( t \in \mathbb{R} \) is sufficiently large, then
\[
\text{Im} \langle (P + it \text{id})u, u \rangle_X \geq \langle u, u \rangle_X,
\]
for every \( u \) such that \( Bu = \gamma u^- = 0 \), and the operator \( (P + it \text{id}) \oplus B \) is injective.

Proof. The inequality preceding the lemma implies that if \( \gamma u^- = 0 \), then
\[
\text{Im} \langle Pu, u \rangle_X \geq -C \langle u, u \rangle_X
\]
where \( C > 0 \) is a constant not depending on \( u \). Hence the inequality of the lemma holds if \( t \geq C + 1 \) (and \( \gamma u^- = 0 \)). This inequality implies the injectivity of \( (P + it \text{id}) \oplus B \). ■

Formally adjoint operators. The operators \( -P^i, -\varphi(x_n)i\Lambda^+ \), and \( -\varphi(x_n)i\Lambda^- \) are formally adjoint to \( P^i \), \( \varphi(x_n)i\Lambda^+ \), and \( \varphi(x_n)i\Lambda^- \) respectively. The operator formally adjoint to \( \varphi(x_n)D_n \) is the operator \( u \mapsto D_n(\varphi(x_n)u) \), by the Leibniz formula equal to
\[
u \mapsto -i\varphi'(x_n)u + \varphi(x_n)D_n u.
\]

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Let us define the operators $\tilde{P}^b$, $\tilde{P}$, and $\tilde{P}'$ by
\[
\tilde{P}^b u = \varphi(x_n) \left( (-D_n + i\Lambda^+)u^+ + (D_n + i\Lambda^-)u^- \right),
\]
\[
\tilde{P} = \tilde{P}^b + P^i, \quad \text{and} \quad \tilde{P}' = \tilde{P} + i\varphi'(x_n).
\]

Then $-\tilde{P}'$ is formally adjoint to $P$, and we the corresponding Green formula is
\[
\langle Pu, v \rangle_X = \langle u, -\tilde{P}'v \rangle_X + \langle u, -i\varphi'(x_n)v \rangle_X + \gamma \langle u^+, \gamma v^+ \rangle_\partial - i\langle \gamma u^-, \gamma v^- \rangle_\partial.
\]

The operators $\tilde{P}^b$ and $\tilde{P}$ are operators of the same nature as $P^b$ and $P$, but with the roles of $E^+$ and $E^-$ interchanged. As above, this implies that
\[
\text{Im} \langle \tilde{P}'u, u \rangle_X \geq \langle \gamma u^-, \gamma u^- \rangle_\partial - \langle \gamma u^+, \gamma u^+ \rangle_\partial - C' \langle u, u \rangle_X
\]
for every section $u$ and some constant $C' > 0$. It follows that for some $C > 0$
\begin{equation}
\text{Im} \langle (P + it \text{id})u, u \rangle_X \geq C \langle u, u \rangle_X
\end{equation}
for every $u$ such that $\gamma u^+ = 0$ if $t$ is sufficiently large (independently of $u$).

### 6.2. Lemma
If $t \in \mathbb{R}$ is sufficiently large, then the operator $(P + it \text{id}) \oplus B$ is surjective.

**Proof.** Let us choose a constant $C > \sup |\varphi'(x_n)|$ and a sufficiently large $t$ so that (15) holds with this $C$. Since $(P + it \text{id}) \oplus B$ is Fredholm, it is sufficient to prove that every $(v, h)$ orthogonal to the image of $(P + it \text{id}) \oplus B$ is equal to $(0, 0)$. For such $(v, h)$
\begin{equation}
\langle (P + it \text{id})u, v \rangle_X + \langle \gamma u^-, h \rangle_\partial = 0
\end{equation}
for every $u$. The Green formula for $P$ implies that
\[
0 = \langle \gamma u^-, h \rangle_\partial + \langle (P + it \text{id})u, v \rangle_X
\]
\[
= \langle \gamma u^-, h \rangle_\partial + \langle u, -(\tilde{P}' + it \text{id})v \rangle_X
\]
\[
+ \langle u, -i\varphi'(x_n)v \rangle_X + i\langle \gamma u^+, \gamma v^+ \rangle_\partial - i\langle \gamma u^-, \gamma v^- \rangle_\partial
\]
for every $u$. It follows that if $\gamma u = 0$, then
\[
\langle u, -(\tilde{P}' + it \text{id})v \rangle_X + \langle u, -i\varphi'(x_n)v \rangle_X = 0.
\]
Since sections \( u \) with \( \gamma u = 0 \) are dense in \( L_2(X) \), this implies that

\[
(17) \quad (\bar{P}' + it \text{id}) \nu + i \phi'(x_n) \nu = 0
\]

and hence

\[
\langle \gamma u^-, h \rangle_\partial + i \langle \gamma u^+, \gamma v^+ \rangle_\partial - i \langle \gamma u^-, \gamma v^- \rangle_\partial = 0
\]

for every \( u \). If \( \gamma u^- = 0 \), this identity implies that \( \langle \gamma u^+, \gamma v^+ \rangle_\partial = 0 \). By choosing \( u \) such that \( \gamma u^- = 0 \) and \( \gamma u^+ = \gamma v^+ \) we conclude that \( \gamma v^+ = 0 \) and hence

\[
\text{Im} \langle (\bar{P}' + it \text{id}) \nu, \nu \rangle_X \geq C \langle \nu, \nu \rangle_X
\]

by the inequality (15). By combining this with (17) we see that

\[
\sup |\phi'(x_n)| \langle \nu, \nu \rangle_X \geq \langle \phi'(x_n) \nu, \nu \rangle_X \geq C \langle \nu, \nu \rangle_X.
\]

By the choice of \( C \) this implies that \( \langle \nu, \nu \rangle_X = 0 \) and \( \nu = 0 \). Now (16) simplifies to \( \langle \gamma u^-, h \rangle_\partial = 0 \) and implies \( h = 0 \). Therefore \( \langle \nu, h \rangle = 0 \). ■

**Standard self-adjoint isomorphisms.** Lemmas 6.1 and 6.2 imply that for sufficiently large \( t \) the operator \((P + it \text{id}) \oplus B\) is an isomorphism between appropriate Sobolev spaces. Recall that \( B u = \gamma u^- \) and let \( \bar{B} u = \gamma u^+ \). The inequality (15) implies that \((\bar{P}' + it \text{id}) \oplus \bar{B}\) is injective for sufficiently large \( t \), and an argument similar to the proof of Lemma 6.2 shows that this operator is surjective. Therefore it is an isomorphism between appropriate Sobolev spaces. This implies that for sufficiently large \( t \) the operator \((-\bar{P}' - it \text{id}) \oplus \bar{B}\) is also an isomorphism. The point is that \(-\bar{P}' - it \text{id}\) is formally adjoint to \(P + it \text{id}\). This allows to combine these two operators into a formally self-adjoint operator acting on sections of the bundle \( E \oplus E \). Namely, let us define the operator \( P^{sa}\) by the matrix

\[
P^{sa} = \begin{pmatrix} 0 & P + it \text{id} \\ -\bar{P}' - it \text{id} & 0 \end{pmatrix}.
\]

Then the operator \( P^{sa}\) is formally self-adjoint. Together with the boundary operator

\[
B^{sa} : (u, v) \mapsto (\gamma v^+, \gamma u^-),
\]

where \((u, v) \in E \oplus E\), it satisfies Shapiro–Lopatinskii condition. The kernel \( N^{sa}\) of \( B^{sa}\) is equal to \((E^+ \oplus 0) \oplus (0 \oplus E^-)\). By the construction, the symbol of \( P^{sa}\) is an elementary symbol, and \( N^{sa}\) is the corresponding boundary condition. Hence the topological index of the boundary problem defined by \( P^{sa}, N^{sa}\) is zero. By the above \( P^{sa} \oplus B^{sa}\) is an isomorphism between appropriate Sobolev spaces for sufficiently large \( t \), and hence the analytical index is also zero. Moreover, the indices vanish even when parameters are present.
7. Glueing and cutting

The framework. We will keep the assumptions and notations of Section 6. Following Hörmander [H], Section 20.3, we introduce the double $\hat{X}$ of $X$ consisting of two copies $X_1$ and $X_2$ of $X$. Let $Y$ be the common boundary of $X_1$ and $X_2$. We will identify the copy of the collar $\partial X \times [0, 1)$ in $X_1$ with $Y \times [0, 1)$ and the copy in $X_2$ with $Y \times (-1, 0]$, with $x_n$ changing the sign in the second copy. The smooth structure on $\hat{X}$ is defined by taking the smooth structures on $X_1$ and $X_2$ and the product smooth structure on $Y \times (-1, 1)$. This smooth structure depends on the identification of the collar with $\partial X \times [0, 1)$. Recall that $E$ is a bundle over $X$. Let $E_1$ and $E_2$ be the copies of the bundle $E$ over $X_1$ and $X_2$ respectively, and let $\hat{E}$ be the bundle over $\hat{X}$ obtained by glueing $E_1$ and $E_2$ over $Y$. Let $F_2 = E_2 \oplus E_2$ and let $F_1$ be a Hermitian bundle over $X_1$ which is equal to the bundle $E_1 \oplus E_1$ over the collar $Y \times [0, 1) \subset X_1$. Let $\hat{F}$ be the bundle over $\hat{X}$ obtained by glueing $F_1$ and $F_2$ over $Y$.

The operators. As in Section 6, let $E|Y$ be presented as a direct sum $E|Y = E^+ \oplus E^-$. Let us define operators $P^b$ and $\tilde{P}^b$ acting on sections of $\hat{E}$ by the same formulas as in Section 6, and then define the operator $\hat{P}^{b,sa}$ acting on sections of $\hat{E} \oplus \hat{E}$ by the matrix

$$
\hat{P}^{b,sa} = \begin{pmatrix} 0 & P^b \\ -\tilde{P}^b - i \varphi'(x_n) & 0 \end{pmatrix}.
$$

Since the cut-off function $\varphi$ used to define $P^b$ and $\tilde{P}^b$ has support in $(-1, 1)$ and $\hat{F}$ is equal to $\hat{E} \oplus \hat{E}$ over $Y \times (-1, 1)$, we may, and from now on will, consider $\hat{P}^{b,sa}$ as an operator acting on sections of $\hat{F}$. Let $P^{1,sa}_1$ and $P^{2,sa}_2$ be the operators defined in the same way, but acting on the sections of $F_1$ and $F_2 = E_2 \oplus E_2$ over $X_1$ and $X_2$ respectively.

Let $P^{i,sa}_1$ be a formally self-adjoint operator belonging to

$$
\Psi_{phg}^1 (X_1 \sim Y; F_1, F_1)
$$

and such that its kernel has compact support in $(X_1 \sim Y) \times (X_1 \sim Y)$. Let

$$
P^{sa}_1 = P^{i,sa}_1 + P^{b,sa}_1
$$

and let $B^{sa}_1$ be the operator $B^{sa}_1$ from Section 6. Then $P^{sa}_1$ together with the boundary operator $B^{sa}_1$ satisfies Shapiro-Lopatinskii condition.

The main difference between $P^{sa}_1$ and the standard operator $P^{sa}$ from Section 6 is in allowing an arbitrary formally self-adjoint “interior” operator $P^{i,sa}_1$. We also omitted the “correction terms” $\pm i t \text{id}$, which will reappear soon.
Let $P^i_2$ be the copy over $X_2$ of the operator $P^i$ from Section 6, and let

$$P^{i\text{sa}}_2 = \begin{pmatrix} 0 & P^i_2 \\ -P^i_2 & 0 \end{pmatrix}$$

and $P^{\text{sa}}_2 = P^{b\text{sa}}_2 + P^{i\text{sa}}_2$. Finally, let

$$\hat{P}^{\text{sa}} = P^{i\text{sa}}_1 + \hat{P}^{b\text{sa}} + P^{i\text{sa}}_2.$$ 

The operator $\hat{P}^{\text{sa}}$ should be thought as the result of “glueing” to a fairly general operator $P^{\text{sa}}_1$ the standard operator $P^{\text{sa}}_2$.

The operator $P^{\text{sa}}_2$ has almost the same form as the operator $P^{\text{sa}}$ from Section 6, but restricting $\hat{P}^{b\text{sa}}$ to $X_2$ changes the sign of $x_n$. Changing the sign of $x_n$ back replaces $D_n$ by $-D_n$. Therefore $P^{\text{sa}}_2$ has the same form as $P^{i\text{sa}}_1$ with the roles of $E^+$ and $E^-$ interchanged, and the relevant boundary operator is

$$B^{\text{sa}}_2: (u, v) \mapsto (\gamma u^+, \gamma v^-)$$

instead of the boundary operator $B^{\text{sa}}_1 = B^{\text{sa}}$.

**Cutting $\hat{P}^{\text{sa}}$.** Hörmander’s Proposition 20.3.2 from [H] suggests that the analytical index of $\hat{P}^{\text{sa}}$ should be equal to that of $P^{i\text{sa}}_1$ with the boundary conditions defined by $B^{\text{sa}}_1$. Of course, both of them are equal to zero because we are dealing with self-adjoint operators, but we need to forget this fact and prove the equality in a way allowing to add parameters.

Let us begin by adding to the operators $P^{\text{sa}}_1$, $P^{\text{sa}}_2$, and $\hat{P}^{\text{sa}}$ the “correction term”

$$\begin{pmatrix} 0 & it \text{id} \\ -it \text{id} & 0 \end{pmatrix}$$

If $t$ is sufficiently large, then $P^{\text{sa}}_2 \oplus B^{\text{sa}}_2$ is an isomorphism between appropriate Sobolev spaces. Adding this term does not change the analytical index of $\hat{P}^{\text{sa}}$ even in families, and does not change the topological index because this does not affect the principal symbol.

Let $X_1 \sqcup X_2$ be the disjoint union of $X_1$ and $X_2$, and let $F_1 \sqcup F_2$ be the disjoint union of the bundles $F_1$ and $F_2$, a bundle over $X_1 \sqcup X_2$. Recall that $F_1$ is equal to $E_1 \oplus E_1$ over $Y \times (-1, 1)$, and let us represent sections of $F_1 \sqcup F_2$ over $Y \times (-1, 0] \sqcup Y \times [0, 1)$ by quadruples $(u_1, v_1, u_2, v_2)$, where $(u_1, v_1)$ and $(u_2, v_2)$ are sections of $E_1 \oplus E_1$ and $E_2 \oplus E_2$ respectively.
For every $\tau \in R$ let $B_{\tau}$ be the boundary operator

$$B_\tau (u_1, v_1, u_2, v_2) = (\gamma u_1^- - \tau \gamma u_2^-, \gamma v_1^+ - \tau \gamma v_2^+, \gamma u_2^+ - \tau \gamma u_1^+, \gamma v_2^- - \tau \gamma v_1^-)$$

Then

$$B_0 (u_1, v_1, u_2, v_2) = (\gamma u_1^-, \gamma v_1^+, \gamma u_2^+, \gamma v_2^-)$$

and

$$B_1 (u_1, v_1, u_2, v_2) = (\gamma u_1^- - \gamma u_2^-, \gamma v_1^+ - \gamma v_2^+, \gamma u_2^+ - \gamma u_1^+, \gamma v_2^- - \gamma v_1^-).$$

The boundary operators $B_{\tau}$ are non-local for $\tau \neq 0$. This issue is ignored by Hörmander in the proof of Proposition 20.3.2 in [H]. But one can interpret $P_{1a} \sqcup P_{2a}$ as an operator over $X$ acting on sections of the bundle $F_1 \oplus (E_1 \oplus E_1)$, by $P_{1a}$ on the summand $F_1 \oplus 0$, and by $P_{2a}$ on the summand $0 \oplus (E_1 \oplus E_1)$. With this interpretations the boundary operators $B_{\tau}$ are local. Moreover, this interpretation of $P_{1a} \sqcup P_{2a}$ together with the boundary operators $B_{\tau}$ satisfies Shapiro-Lopatinskii condition, and the corresponding boundary conditions are self-adjoint. In order to prove this, it is sufficient to examine the kernel of $B_{\tau}$ over an arbitrary $u \in SY$. In the obvious notations, this kernel is described by the equations

$$u_1^+ = v_1^- = 0, \quad u_2^- = v_2^+ = 0.$$

Together they imply that also $u_2^+ = v_2^- = 0$ and $u_1^- = v_1^+ = 0$. It follows that the kernel of $B_{\tau}$ is transverse to $L_-(\rho_u)$ and hence is a boundary condition for the elliptic pair $\Sigma_y$, $\tau_u$ at $u$. Equivalently, $B_{\tau}$ satisfies Shapiro-Lopatinskii condition. In order to prove self-adjointness we need to check that the kernel is lagrangian. The relevant Hermitian product is the direct sum of the product $[\bullet, \bullet]_1$ and $[\bullet, \bullet]_2$ on the two copies of $E_y \oplus E_y$, and $[\bullet, \bullet]_2$ has the same form as $[\bullet, \bullet]_1$, but with the roles of $E^+$ and $E^-$ interchanged. Therefore the description of the product $[\bullet, \bullet]_1$ at the end of Section 6 implies that

$$\langle u_1^+, b_1^+ \rangle - \langle u_1^-, b_1^- \rangle + \langle v_1^+, a_1^+ \rangle - \langle v_1^-, a_1^- \rangle$$

$$+ \langle u_2^-, b_2^+ \rangle - \langle u_2^+, b_2^- \rangle + \langle v_2^-, a_2^- \rangle - \langle v_2^+, a_2^+ \rangle.$$
If \((u_1, v_1, u_2, v_2), (a_1, b_1, a_2, b_2) \in \text{Ker } B_\tau\), then this expression is equal to
\[
\langle u_1^+, \tau b_2^- \rangle - \langle \tau u_2^-, b_1^- \rangle + \langle \tau v_2^+, a_1^+ \rangle - \langle v_1^-, \tau a_2^- \rangle \\
+ \langle u_2^-, \tau b_1^- \rangle - \langle \tau u_1^+, b_2^- \rangle + \langle \tau v_1^-, a_2^- \rangle - \langle v_2^+, \tau a_1^+ \rangle.
\]

Since \(\tau\) is real, the terms in the last expression cancel pair-wise, and hence it is equal to zero. It follows that the kernel \(\text{Ker } B_\tau\) is lagrangian, and therefore the operators \(B_\tau\) define self-adjoint elliptic boundary conditions for \(P_1^{sa} \sqcup P_2^{sa}\). Clearly, the boundary operators \(B_\tau\) continuously depend on \(\tau\).

The operator \(B_0\) is equal to the direct sum \(B_1^{sa} \oplus B_2^{sa}\) and hence \(P_1^{sa} \sqcup P_2^{sa}\) together with the boundary condition \(B_0\) is the direct sum of \(P_1^{sa}\) and \(P_2^{sa}\) together with the boundary conditions \(B_1^{sa}\) and \(B_2^{sa}\) respectively. Since \(P_0^{sa} \oplus B_2^{sa}\) is an isomorphism, the analytical index of \(P_1^{sa} \sqcup P_2^{sa}\) with the boundary condition \(B_0\) is equal to the analytical index of \(P_1^{sa}\) with the boundary conditions \(B_1^{sa}\). Moreover, this is true and the proof remains the same when \(P_i^{sa}\) depends on a parameter.

Let us consider now the boundary operator \(B_1\). Up to a permutation of summands,
\[
B_1(u_1, v_1, u_2, v_2) = \gamma(u_1, v_1) - \gamma(u_2, v_2).
\]

Let us return to the original interpretation of \(P_1^{sa} \sqcup P_2^{sa}\) as an operator over \(X_1 \sqcup X_2\). Then vanishing of \(B_1\) on a of \(F_1 \sqcup F_2\) means simply that the restrictions of this section to \(X_1\) and \(X_2\) agree on the boundaries and hence define a section of \(\tilde{F}\) over \(\tilde{X}\). In particular, restricting \(P_1^{sa} \sqcup P_2^{sa}\) to the kernel \(\text{Ker } B_1\) almost recovers the operator \(\hat{P}^{sa}\). The only problem is that the sections over \(\tilde{X}\) defined by elements of \(\text{Ker } B_1\) are, in general, only continuous on \(Y\) in the directions transverse to \(Y\), even if we consider our operators as acting in Sobolev spaces ensuring existence of several derivatives.

But we are interested only in the analytical index of \(P_1^{sa} \sqcup P_2^{sa}\) with the boundary conditions \(B_1\). Of course, we implicitly assume that the operators \(P_i^{sa}\) and \(\hat{P}^{b,sa}\) and hence \(P_1^{sa} \sqcup P_2^{sa}\) continuously depend on a parameter. By the results of \([I_2]\) the analytical index is determined by a somewhat enhanced family of kernels
\[
\text{Ker } \left( (P_1^{sa} \sqcup P_2^{sa}) \oplus B_1 \right).
\]

The enhancement is concerned with the eigenvectors of the operator \(P_1^{sa} \sqcup P_2^{sa}\) restricted to the kernel \(\text{Ker } B_1\), i.e. with the kernels
\[
\text{Ker } \left( (P_1^{sa} \sqcup P_2^{sa} - \lambda) \oplus B_1 \right)
\]
with \(\lambda \in \mathbb{R}\). By the elliptic regularity the sections of \(\hat{F}\) over \(\tilde{X}\) defined by elements of these
kernels are $C^\infty$-smooth. It follows that the eigenvalues and eigenspaces defining the analytical index of the family of operators $P_1^{sa} \sqcup P_2^{sa}$ with the boundary conditions $B_1$ are exactly the same as the eigenvalues and eigenspaces defining the analytical index of the family of operators $\hat{P}^{sa}$. Therefore these two families have the same analytical index.

Since $B_\tau$ continuously depends on $\tau$, the analytical index of $P_1^{sa} \sqcup P_2^{sa}$ with the boundary conditions $B_\tau$ is independent of $\tau$. In particular, the analytical index for $\tau = 1$ is the same as for $\tau = 0$. It follows that the analytical index of $P_1^{sa}$ with the boundary conditions $B_1^{sa}$ is equal to the analytic index of $\hat{P}^{sa}$.

The topological index. Our next task is to prove that the topological index of $P_1^{sa}$ with the boundary conditions $B_1^{sa}$ is equal to the topological index of $\hat{P}^{sa}$.

The symbols $\Sigma_1$ and $\Sigma_2$ of the operators $P_1^{sa}$ and $P_2^{sa}$ respectively are bundle-like in the sense of Section 3, and hence the extensions $E^+(\Sigma_1)$ and $E^+(\Sigma_2)$ of the bundles $E^+(\Sigma_1)$ and $E^+(\Sigma_2)$. These extensions depend only on symbols $\Sigma_1$, $\Sigma_2$, but not on the boundary conditions. Since the restrictions of these symbols to

$$BX_1Y = BX_2Y = BX_Y$$

are the same, the restrictions of the bundles $E^+(\sigma_1)$ and $E^+(\sigma_2)$ are equal. By identifying these bundles over $BX_Y$ we get a bundle $E^+$ over

$$SX_1 \cup SX_2 \cup BX_Y = S\hat{X} \cup BX_Y.$$

The restriction $E^+|_{S\hat{X}}$ is the bundle $E^+(\hat{\sigma})$ associated with the symbol $\hat{\sigma}$ of $\hat{P}^{sa}$. Let

$$e^+ \in K^0(S\hat{X} \cup BX_Y)$$

be the class of $E^+$, and let $e^+$ be its image under the coboundary map

$$K^0(S\hat{X} \cup BX_Y) \longrightarrow K^1(B\hat{X}, S\hat{X} \cup BX_Y).$$

Let us consider the commutative diagram

$$
\begin{array}{c}
K^0(S\hat{X} \cup BX_Y) \\
\downarrow \\
K^1(B\hat{X}, S\hat{X} \cup BX_Y)
\end{array} 
\longrightarrow 
\begin{array}{c}
K^0(SX_1 \cup BX_1Y) \oplus K^0(SX_2 \cup BX_2Y) \\
\downarrow \\
K^1(BX_1, SX_1 \cup BX_1Y) \oplus K^1(BX_2, SX_2 \cup BX_2Y)
\end{array},
$$

where the horizontal arrows are the direct sums of the restriction maps, and the vertical
arrows are the coboundary maps. The lower horizontal arrow is an isomorphism because it is placed between the groups $K^0(BX, BX) = 0$ and $K^1(BX, BX) = 0$ in a Mayer-Vietoris sequence. The classes $e^+, e^+(\sigma_1), e^+(\sigma_2), \text{etc.}$ are mapped by the arrows of this diagram as in the following diagram.

$$
\begin{array}{ccc}
\vdots & \downarrow & \vdots \\
\varepsilon & \longrightarrow & \varepsilon^+(\sigma_1) \oplus \varepsilon^+(\sigma_2) \\
\end{array}
$$

Since the lower horizontal arrow is an isomorphism, the class $\varepsilon^+$ is uniquely determined by $\varepsilon^+(\sigma_1)$ and $\varepsilon^+(\sigma_2)$. Clearly, the image of $\varepsilon^+$ under the map

$$K^1(B\hat{X}, S\hat{X} \cup BX) \longrightarrow K^1(B\hat{X}, S\hat{X})$$

is nothing else but $\varepsilon^+(\hat{\sigma})$. It follows that the class $\varepsilon^+(\hat{\sigma})$ is uniquely determined by the classes $\varepsilon^+(\sigma_1)$ and $\varepsilon^+(\sigma_2)$.

In order to pass from these classes to topological indices, let us embed the manifold $\hat{X}$ into $\mathbb{R}^n$ for some $n$ in such a way that

$$X_1 = \hat{X} \cap \mathbb{R}^n_{\geq 0} \quad \text{and} \quad X_2 = \hat{X} \cap \mathbb{R}^n_{\leq 0}$$

and $X_2$ is the reflection of $X_1$ in the hyperplane $\mathbb{R}^{n-1} \times 0$. Let $\mathbb{N}$ be the normal bundle to $\hat{X}$ in $\mathbb{R}^n$. The normal bundle of $T\hat{X}$ in $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ can be identified with the lift of the bundle $\mathbb{N} \oplus \mathbb{N}$ to $T\hat{X}$. As in Section 3, the bundle $\mathbb{N} \oplus \mathbb{N}$ has a natural complex structure, and a tubular neighborhood of $T\hat{X}$ in $T\mathbb{R}^n$ can be identified with the bundle $\mathbb{U}$ of unit balls in $\mathbb{N} \oplus \mathbb{N}$. Let $\mathbb{S}$ be the bundle of unit spheres in $\mathbb{N} \oplus \mathbb{N}$.

The complex structure on $\mathbb{N} \oplus \mathbb{N}$ leads to Thom isomorphisms

$$K^1(BX_1, SX_1 \cup BX) \longrightarrow K^1(\cup BX_1, (\mathbb{S} \cup BX_1) \cup (\cup SX_1) \cup (\cup BX)),$$

$$K^1(BX_2, SX_2 \cup BX) \longrightarrow K^1(\cup BX_2, (\mathbb{S} \cup BX_2) \cup (\cup SX_2) \cup (\cup BX)),$$

$$K^1(B\hat{X}, S\hat{X} \cup BX) \longrightarrow K^1(\cup B\hat{X}, (\mathbb{S} \cup B\hat{X}) \cup (\cup S\hat{X}) \cup (\cup BX)),$$

$$K^1(B\hat{X}, S\hat{X}) \longrightarrow K^1(\cup B\hat{X}, (\mathbb{S} \cup B\hat{X}) \cup (\cup S\hat{X})).$$

These Thom isomorphisms commute with various restriction maps, and the analogue of the
lower horizontal arrow in the first diagram is an isomorphism by the same reason as that arrow. It follows that the image \( \text{Th}(\epsilon^+) \) of the class \( \epsilon^+ \) under the third Thom isomorphism above is uniquely determined by the images of the classes \( \epsilon^+(\sigma_1) \) and \( \epsilon^+(\sigma_2) \) under the first two Thom isomorphisms. Similarly to the class \( \epsilon^+ (\hat{\sigma}) \) itself, the image of \( \epsilon^+ (\hat{\sigma}) \) under the last Thom isomorphism is equal to the image of \( \text{Th}(\epsilon^+) \) under the map
\[
K^1\left( \bigcup B\hat{X}, (\bigcup B\hat{X}) \cup (\bigcup S\hat{X}) \cup (\bigcup B\hat{X}_Y) \right) \to K^1\left( \bigcup B\hat{X}, (\bigcup B\hat{X}) \cup (\bigcup S\hat{X}) \right)
\]
induced by the quotient map
\[
q : \bigcup B\hat{X} / \left( (\bigcup B\hat{X}) \cup (\bigcup S\hat{X}) \right) \to \bigcup B\hat{X} / \left( (\bigcup B\hat{X}) \cup (\bigcup S\hat{X}) \cup (\bigcup B\hat{X}_Y) \right).
\]
The composition of the canonical map
\[
S^{2n} \to \bigcup B\hat{X} / \left( \bigcup (\bigcup S\hat{X}) \right)
\]
with \( q \) is equal to the sum (as in the definition of homotopy groups) of the maps
\[
S^{2n} \to \bigcup B\hat{X}_1 / \left( \bigcup (\bigcup S\hat{X}_1) \cup (\bigcup B\hat{X}_Y) \right) \quad \text{and}
\]
\[
S^{2n} \to \bigcup B\hat{X}_2 / \left( \bigcup (\bigcup S\hat{X}_2) \cup (\bigcup B\hat{X}_Y) \right).
\]
Since \( X_2 \) is the reflection of \( X_1 \) in the hyperplane \( R^{n-1} \times 0 \), the second map is minus the map defining the index of \( \sigma_2 \). It follows that
\[
(18) \quad \text{t-ind} (\hat{\sigma}) = \text{t-ind} (\sigma_1, N_1) - \text{t-ind} (\sigma_2, N_2),
\]
where \( N_1, N_2 \) are the boundary conditions corresponding to \( B^{sa}_1, B^{sa}_2 \) respectively. In fact, since the symbols \( \sigma_1, \sigma_2 \) are already bundle-like, the corresponding topological indices are well defined without boundary conditions. As usual, the additivity property \( (18) \) is valid also when parameters are present (and is non-vacuous only with parameters).

We are still not used the assumption that \( \sigma_2 \) is the symbol of a standard operator. In particular, the additivity property \( (18) \) holds when \( \sigma_2 \) is a general symbol of the same type as \( \sigma_1 \). When \( \sigma_2 \) is the symbol of a standard operator, \( \epsilon^+(\sigma_2) = 0 \), as we saw in Section 3. It follows that in this case \( \text{t-ind} (\sigma_2, N_2) = 0 \) and hence
\[
\text{t-ind} (\hat{\sigma}) = \text{t-ind} (\sigma_1, N_1).
\]
In other words, the topological index of \( \hat{P}^{sa}_1 \) with the boundary conditions \( B^{sa}_1 \) is equal to the topological index of \( \hat{P}^{sa}_1 \). To sum up, for families of operators and boundary conditions having the standard form near the boundary the determination of both the analytical and the topological index can be reduced to corresponding problem on closed manifolds.
8. Index theorems

The framework. We will speak about various objects parameterized by a topological space $Z$, which will be assumed to be compactly generated and paracompact. For example, every metric space has these properties. In particular, we will consider manifolds $X(z)$, $z \in Z$ continuously depending on $z$ in the sense that $X(z)$ is the fiber over $z$ of a locally trivial bundle, as at the end of Section 3. In order to avoid cumbersome notations and phrases, we will omit the parameter $z$ from the notations and speak about $X$, vector bundles and pseudo-differential operators on $X$, etc. having in mind families parameterized by $z \in Z$.

Let $P$ be a pseudo-differential operator in a bundle $E$ over $X$ belonging to the Hörmander class, and let $\sigma$ be the symbol of $P$. Suppose that $\sigma$ is elliptic and self-adjoint. Let $N$ be an elliptic self-adjoint boundary condition for $\sigma$ in the sense of Section 3. By the reasons discussed in Section 5 we assume also that $N$ is bundle-like. Then we can consider $N$ as a subbundle of the restriction $E|Y$, where $Y = \partial X$. The symbol $\sigma$ of $P$ induces a self-adjoint endomorphism $\Sigma: E|Y \to E|Y$. Recall that $E^+_Y$ and $E^-_Y$ are the subbundles of $E|Y$ generated by the eigenvectors of $\Sigma$ with positive and negative eigenvalues respectively.

Let $\Pi: E|Y \to E|Y$ be the orthogonal projection onto $N$, and let $\Gamma = (1 - \Pi) \circ \gamma$, where $\gamma$ is the operator taking the sections of $E$ to their restrictions to the boundary $Y$. We may consider $\Gamma$ as an operator taking values in section of the orthogonal complement $N^\perp$ of $N$ in $E|Y$. Then the operators $P \oplus \Gamma$, considered as operators in appropriate Sobolev spaces, are Fredholm (and continuously depend on $z$). The results of Section 4 show that by restricting $P$ to $\text{Ker} \Gamma$ we get an unbounded self-adjoint operator $P_\Gamma$, and that the analytical index of $P_\Gamma$ is well-defined. We will denote this index by $\text{a-ind}(P,N)$. Since, in fact, everything depends on $z \in Z$, the analytical index $\text{a-ind}(P,N) \in K^1(Z)$.

Since $N$ is bundle-like, the topological index $\text{t-ind}(\sigma,N) \in K^1(Z)$ is also defined.

8.1. Theorem. If the bundle $E^+_Y$ extends to a bundle over $X$ in a manner continuously depending on the parameter $z$, then $\text{a-ind}(P,N) = \text{t-ind}(\sigma,N)$.

Proof. We may assume that the operators $T(x_n)$ from Section 5 do not depend on $x_n$ for sufficiently small $x_n$. See [H], the proof of Proposition 20.3.3. Let $\tau$ be the symbol of $T(0)$. The boundary condition $N$ defines an isometric isomorphism $\varphi: E^+_Y \to E^-_Y$, and we will identify the bundles $E^+_Y$ and $E^-_Y$ by $\varphi$. By the results of Sections 2 and 3 we can deform $\sigma, N$ by a deformation preserving all our assumptions to a normalized new pair $\sigma, N$. This deformation can be lifted to a deformation of $P$. After this deformation $\sigma$ is a bundle-like symbol and, moreover, $\sigma, N$ have a standard form over $Y$. Namely,

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$
and \( N = \Delta \) with respect to the decomposition \( E|Y = E^+_Y \oplus E^-_Y \). See (3). Let us pass to
the decomposition \( E|Y = \Delta \oplus \Delta^\perp \), where the subbundles \( \Delta \) and \( \Delta^\perp \) are defined in the
obvious way, and identify these subbundles with \( E^+_Y \) by the projection. Then

\[
\Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\quad \text{and} \quad
\tau = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

and \( N \) is equal to the first summand \( \Delta \oplus 0 \) of this decomposition. Now the symbol \( \sigma \) is
equal near \( Y \) to an elementary symbol in the sense of Section 3. More precisely, \( \sigma \) is equal
to the elementary symbol \( \sigma^{sa} \) corresponding to \( F^+ = \Delta, F^- = 0 \), and \( \lambda^+ = 1 \) (near
\( Y \) we don’t need the bundle \( F \)). The corresponding boundary condition \( N^{sa} \) is equal to
\( F^+ \oplus 0 = \Delta \oplus 0 \), i.e. to \( N \). Let us further deform \( P \) without changing its symbol to a new
operator \( P \) equal near \( Y \) to a standard self-adjoint operator \( P^{sa} \) from Section 6.

The bundle \( F^+ \) is canonically isomorphic to \( E^+_Y \) and hence can be extended to a bundle \( F \)
over \( X \). Therefore the constructions of Section 7 apply to \( P \) in the role of \( P^{sa}_1 \) and \( N \) in
the role of the boundary condition \( B^{sa}_1 \) from that section (more precisely, one takes as \( B^{sa}_1 \)
the boundary operator defined by \( N \)). In more details, a copy of \( E \) over \( X_1 \) plays the role
of \( F_1 \), and a copy of \( F \oplus F \) over \( X_2 \) plays the role of \( F_2 \). The constructions of Section 7
lead to an operator \( \hat{P}^{sa} \) on the double \( \hat{X} = X_1 \cup X_2 \) of \( X \) such that

\[
a\text{-ind} (P, N) = a\text{-ind} (\hat{P}^{sa}) \quad \text{and} \quad t\text{-ind} (P, N) = t\text{-ind} (\hat{P}^{sa}).
\]

But \( \hat{X} \) is a closed manifold, and for closed manifolds our definitions of the analytical and
the topological index agree with the classical definitions of Atiyah-Patodi-Singer [APS]. For
the topological index this is obvious, and for the analytical index this follows from [I_2],
Theorem 8.5, and the fact that on closed manifolds families of self-adjoint pseudo-differential
operators are strictly Fredholm in the sense of [I_2]. This fact follows from the results of
Seeley [S_1]. By the results of Atiyah, Patodi, and Singer

\[
a\text{-ind} (\hat{P}^{sa}) = t\text{-ind} (\hat{P}^{sa}).
\]

See [APS], Theorem (3.4), and the discussion below. It follows that

\[
a\text{-ind} (P, N) = t\text{-ind} (P, N).
\]

Since the topological and analytical indices are defined for families, they are invariant under
homotopies. Hence our deformations of \( P, N \) do not change neither the topological nor the
analytical index. Since we were implicitly taking about families, the theorem follows.

\[\blacksquare\]

The index theorem for self-adjoint operators on closed manifolds. The index theorem for
the families of self-adjoint operators on closed manifolds is stated by Atiyah, Patodi, and
Singer [APS] only for families of operators parameterized by a compact space on a fixed
manifold $X$. The assumption that the manifold $X$ is fixed is not essential, and the tools
developed by Atiyah and Singer [AS$_4$] allow to extend this index theorem to the families of
closed manifolds parameterized by a compact space.

It seems that the simplest way to deal with more general spaces of parameters is to rewrite
the proof of Theorem (3.4) in [APS] using author’s definition of the analytical index $[I_2]$ from
the very beginning. The proof in [APS] is based on the Atiyah-Singer [AS] correspondence
between self-adjoint Fredholm operators and loops of Fredholm operators, which is, in
fact, build into the definition of the analytical index in [APS]. If the definition of $[I_2]$ is used,
this tool should be replaced by Theorem 8.5 from $[I_2]$. The rest is routine.

**8.2. Theorem.** In general, $a$-$\text{ind}(P, N) - t$-$\text{ind}(\sigma, N)$ is an element of order 2 in $K^1(Z)$.

**Proof.** Let us consider the bundle $E \oplus E$ over $X$ and the operator $P \oplus P$ in this bundle
together with the boundary condition $N \oplus N$. Clearly,

$$a$-$\text{ind}(P \oplus P, N \oplus N) = 2 \ a$-$\text{ind}(P, N) \quad \text{and} \quad t$-$\text{ind}(P \oplus P, N \oplus N) = 2 \ t$-$\text{ind}(P, N).$$

Also, the bundle $(E \oplus E)^+_Y$ is equal to $E^+_Y \oplus E^+_Y$. The boundary condition $N$ induces an
isomorphism $E^+_Y \longrightarrow E^-_Y$, and therefore $(E \oplus E)^+_Y$ is isomorphic to $E^+_Y \oplus E^-_Y = E|Y$. Since
the bundle $E|Y$ obviously extends to $X$, Theorem 8.1 applies to $P \oplus P$, $N \oplus N$, and hence

$$a$-$\text{ind}(P \oplus P, N \oplus N) = t$-$\text{ind}(P \oplus P, N \oplus N).$$

The theorem follows. ■

**Extending $E^+_Y$ to $X$.** Sometimes simple topological considerations ensure that $E^+_Y$ extends
to $X$. Let us assume for simplicity that $X$ and $E$ do not depend on $z \in Z$, but the boundary
problems $P, N$ are still allowed to depend on $z$. Suppose that the vector field $\nu_y, y \in Y$
of vectors normal to the boundary extends to a nowhere zero vector field $x \mapsto \nu_x$ on $X$.
Then $\sigma(\nu_x), x \in X$ is a family of self-adjoint endomorphisms of fibers $E_x$ of $E$, and the
sums of eigenspaces of $\sigma(\nu_x)$ with positive eigenvalues define a bundle extending $E^+_Y$.

More interestingly, suppose that the dimension of $X$ is even and let us choose an extension
of the vector field $\nu_y, y \in Y$ to a vector field $\nu_x, x \in X$ which is non-zero except, perhaps,
of a single point $b \in X$. Let $B$ be a ball in $X$ with the center $b$ and $S$ be its boundary. As
in the previous paragraph, one can extend $E^+_Y$ to $X$ with the interior of $B$ removed. Such
an extensions, in particular, defines a vector bundle on $S$, and it is sufficient to extend this
bundle to $B$. Since $S$ is an odd-dimensional sphere, the Bott periodicity implies that such
extension exists if the dimension of the bundle is large enough. But one can always increase
the dimension of the bundle $E^+_Y$ without affecting neither the topological nor the analytical
index by taking the direct sum with a standard boundary problem from Section 6.
9. Special boundary conditions

Anti-commuting elliptic pairs and special boundary conditions. For a major part of this section we will work in the framework of Section 2. We will say that an elliptic pair $\sigma, \tau$ is anti-commuting if $\sigma$ anti-commutes with $\tau$ and hence with $\rho = \sigma^{-1} \circ \tau$. In this case $\sigma \rho \sigma^{-1} = -\rho$ and hence $\sigma (L_{+}(\rho)) = L_{-}(\rho)$ and $\sigma (L_{-}(\rho)) = L_{+}(\rho)$. We will say that a boundary condition $N$ for an elliptic pair $\sigma, \tau$ is special if $N$ is transverse not only to $L_{-}(\rho)$, but also to $L_{+}(\rho)$.

Suppose that $\sigma, \tau$ is a self-adjoint elliptic pair and $N$ is a special boundary condition for $\sigma, \tau$. We would like to deform $\sigma, \rho, \tau, N$ to a normal form while keeping $N$ special.

To begin with, let us apply to $\sigma, \tau$ the first two deformations from Section 2. Clearly, during the first deformation $\sigma_{\alpha}$ remains anti-commuting with $\tau_{\alpha}$ and $N_{\alpha}$ stays transverse to $L_{-}(\rho_{\alpha})$ and by the same reason stays transverse to $L_{+}(\rho_{\alpha})$. The second deformation does not affect $\sigma$, the product $\{\bullet, \bullet\}$, and the spaces $L_{+}(\rho), L_{-}(\rho)$. Hence during the second deformation $N$ remains transverse to $L_{+}(\rho)$ and $L_{-}(\rho)$. Since $\sigma (L_{+}(\rho)) = L_{-}(\rho)$ and $\sigma (L_{-}(\rho)) = L_{+}(\rho)$, the map $\rho_{0}$ anti-commutes with $\sigma$ and hence during the second deformation $\rho, \tau$ remain anti-commuting with $\sigma$. After the second deformation the spaces $L_{+}(\rho), L_{-}(\rho)$ are the eigenspaces of $\rho$ with the eigenvalues $i, -i$ respectively.

Let us represent $L_{+}(\rho)$ as the graph of an isometry $\varphi: E^{+} \rightarrow E^{-}$. Then

$$L_{+}(\rho) = \Delta(\varphi) \quad \text{and} \quad L_{-}(\rho) = \Delta^{\perp}(\varphi),$$

and hence $\sigma, \rho, \tau$ have the standard form (5). But now $N$ does not have the form (6) and $\sigma, \tau, N$ are not normalized. Still, already after the first deformation $\sigma$ is unitary and hence $N^{\perp} = \sigma(N)$. It follows that $N^{\perp}$ is a lagrangian subspace transverse to both $L_{+}(\rho)$ and $L_{-}(\rho) = \sigma(L_{+}(\rho))$. Equivalently, $N^{\perp}$ is a special boundary condition for $\sigma, \tau$.

Now we are ready for the third and the last step of our deformation. Let us present $N$ as the graph of an isometry $\psi: E^{+} \rightarrow E^{-}$. Since $L_{+}(\rho)$ is transverse to $N$ and $N^{\perp}$, the isometry $\psi^{-1} \circ \varphi$ has no eigenvalues equal to 1 or $-1$. Therefore we can deform $\varphi$ in the class of isometries with this property to a new isometry $\varphi$ such that $\psi^{-1} \circ \varphi$ has only $i$ and $-i$ as eigenvalues. This deformation of $\varphi$ induces a deformation of the graph of $\varphi$, i.e. of $L_{+}(\rho)$. Simultaneously we deform $L_{-}(\rho)$ as the image of $L_{+}(\rho)$ under $\sigma$. This deformation of $L_{+}(\rho)$ and $L_{-}(\rho)$ defines a deformation of $\rho$ and hence a deformation of $\tau = \sigma \circ \rho$. We do not move $\sigma$ and $N$ during this deformation and hence keep the anti-commutation property and $N$ being a special boundary condition. This third deformation partially replaces both the third and the fourth deformations from Section 2.

Bott periodicity maps. Let us recall some constructions of Bott [B]. Let $F$ be a complex vector space with a Hermitian metric. As usual, for a subspace $A \subset F$ we will denote by $A^{\perp}$
its orthogonal complement. For a linear isomorphism \( a : F \rightarrow F \) let

\[
\lambda(a, \theta) = \{ (u \cos \theta/2, a(u) \sin \theta/2) \mid u \in F \} \subset F \oplus F, \quad \theta \in [0, \pi].
\]

The path \( \lambda(a, \theta) \), \( \theta \in [0, \pi] \) connects \( F \oplus 0 \) with \( 0 \oplus F \). Cf. Bott [B], (4.4) and (4.5). Now, let \( A \) be a vector subspace of \( F \). For \( \eta \in [0, \pi] \) let us define a map \( f(A, \eta) : F \rightarrow F \) by

\[
f(A, \eta)(u) = ue^{i\eta} \quad \text{for } u \in A \quad \text{and}
\]

\[
f(A, \eta)(u) = ue^{-i\eta} \quad \text{for } u \in A^\perp.
\]

The path \( f(A, \eta) \), \( \theta \in [0, \pi] \) connects \( \text{id} \) with \( -\text{id} \). Cf. Bott [B], (5.1). One can extend this path by a path \( f(A, \eta) \), \( \theta \in [\pi, 2\pi] \) returning from \( -\text{id} \) to \( \text{id} \) and actually independent from \( A \). One can take \( f(A, \eta) = \exp(i\eta) \) for \( \eta \in [\pi, 2\pi] \), as we will do, although Bott uses the path \( f(A_0, \eta) \), \( \eta \in [\pi, 2\pi] \) for a fixed subspace \( A_0 \). Bott [B] assumes that \( \dim A = \dim A^\perp \), but the definitions make sense without this assumption. These two constructions can be combined as follows. For a vector subspace \( A \subset F \) let

\[
\gamma(A, \eta, \theta) = \lambda(f(A, \eta), \theta) \subset F \oplus F, \quad \eta \in [0, 2\pi], \quad \theta \in [0, \pi].
\]

More explicitly, \( \gamma(A, \eta, \theta) \) is the sum of subspaces

\[
\left\{ (u \cos \theta/2, ue^{i\eta} \sin \theta/2) \mid u \in A \right\} \quad \text{and}
\]

\[
\left\{ (u \cos \theta/2, ue^{-i\eta} \sin \theta/2) \mid u \in A^\perp \right\}.
\]

The map assigning to a subspace \( A \) of \( F \) the family \( \gamma(A, \eta, \theta) \), \( \eta \in [0, 2\pi], \theta \in [0, \pi] \) of subspaces of \( F \oplus F \) is essentially the Bott periodicity map. In order to get the Bott periodicity map itself, one needs to turn the rectangle \([0, 2\pi] \times [0, \pi]\) into a sphere by identifying the sides \( 0 \times [0, \pi] \) and \( 2\pi \times [0, \pi] \) and collapsing each of the sides \([0, 2\pi] \times 0 \) and \([\pi, 2\pi] \times 0 \) to a pole of the sphere.

In Bott's construction a subspace \( A \subset F \) is encoded by the operator \( f(A, \pi/2) \) equal to the multiplication by \( i \) on \( A \) and to the multiplication by \( -i \) on \( A^\perp \). This is a skew-adjoint operator. For our purposes it is more convenient to encode \( A \) by the self-adjoint operator equal to the identity on \( A \) and to the minus identity on \( A^\perp \), i.e. by the operator \(-if(A, \pi/2)\). This leads to the following modification of the Bott periodicity map. Let

\[
f'(A, \eta) = -if(A, \eta) \quad \text{and} \quad \gamma'(A, \eta, \theta) = \lambda(f'(A, \eta), \theta).
\]

The family \( f_t(A, \eta) = e^{-it}f(A, \eta), \quad t \in [0, \pi/2] \) substituted into \( \lambda(\bullet, \theta) \) leads to a homotopy between the map assigning \( A \) families \( \gamma(A, \eta, \theta) \) and \( \gamma'(A, \eta, \theta) \) respectively.
We will use the latter. Clearly, $\gamma'(A, \eta, \theta)$ is the sum of subspaces
\[
\left\{ \left( u \cos \theta/2, \; u e^i(\eta - \pi/2) \sin \theta/2 \right) \mid u \in A \right\} \quad \text{and} \quad \left\{ \left( u \cos \theta/2, \; u e^{-i}(\eta + \pi/2) \sin \theta/2 \right) \mid u \in A^\perp \right\}.
\]

The positive eigenspaces of $\sigma \cos \theta + \tau \sin \theta$. For $\sigma, \rho, \tau$ having the standard form (5), the family $E^+(\theta)$, $\theta \in [0, \pi]$ of these eigenspaces was determined at the end of Section 2. A reparameterized version of this family better matches the Bott periodicity map. When $\theta$ runs over $[0, \pi]$, the fraction $(1 - z)/(1 + z)$, where $z = \cos \theta + i \sin \theta$, runs over the negative imaginary half-line $iR_{\leq 0}$. The formula (9) implies that up to a canonical homotopy the path $E^+(\theta)$, $\theta \in [0, \pi]$ is the same as
\[
\left\{ \left( u, -i \varphi(u) \frac{\sin \theta/2}{\cos \theta/2} \right) \mid u \in E^+ \right\}
\]

\[
= \left\{ \left( u \cos \theta/2, -i \varphi(u) \sin \theta/2 \right) \mid u \in E^+ \right\}, \quad \theta \in [0, \pi].
\]

If the spaces $E^+$ and $E^-$ are both identified with a vector space $F$, then this path is equal to the path $\lambda(-i \varphi, \theta)$, $\theta \in [0, \pi]$. This fact depends on $\sigma, \rho, \tau$ having the standard form (5), but not on the boundary condition $N$.

Comparing $N$ and $N^\perp$. Let us identify $E^+$ with $E^-$ by $\psi$. Then $\psi = 1$, $N = \Delta$, and the eigenvalues of $\varphi$ are $i$ or $-i$. We can bring $\sigma, \rho, \tau, N$ into the normal form in the sense of Section 2 by moving the eigenvalue $i$ of $\varphi$ to 1 clockwise along the unit circle in $C$ and moving the eigenvalue $-i$ counterclockwise to 1. In more details, let
\[
\varphi_\eta(u) = \varphi(u) e^{-i\eta} \quad \text{for} \quad u \in \mathcal{L}_+(\varphi),
\]
\[
\varphi_\eta(u) = \varphi(u) e^{i\eta} \quad \text{for} \quad u \in \mathcal{L}_-(\varphi).
\]

The family $\varphi_\eta$, $\eta \in [0, \pi/2]$ is our deformation of $\varphi$. It deforms $\varphi$ into the identity map and leads to a deformation of $\rho$ and $\tau$, while $\sigma$ and $N$ are not moved. At the end $\eta = \pi/2$ of this deformation $\mathcal{L}_+(\varphi) = \Delta(\varphi) = \Delta(\text{id}) = N$ and $\sigma, \rho, \tau$ have the standard form (3). Therefore $\sigma, \rho, \tau, N$ are normalized. The deformation $\varphi_{-\eta}$, $\eta \in [-\pi/2, 0]$ moves in the opposite direction and connects $\text{id}$ with $-\text{id}$.

Similarly, we can bring $\sigma, \rho, \tau, N^\perp$ into the normal form by moving the eigenvalue $i$ of $\varphi$ to $-1$ clockwise and moving the eigenvalue $-i$ counterclockwise to $-1$. More precisely, this deformation is $\varphi_{-\eta}$, $\eta \in [0, \pi/2]$. Together the deformations $\varphi_{-\eta}$, $\eta \in [0, \pi/2]$ and $\varphi_{-\eta}$, $\eta \in [-\pi/2, 0]$ define a deformation $\varphi_{-\eta}$, $\eta \in [-\pi/2, \pi/2]$ connecting $\text{id}$ with $-\text{id}$.
It is convenient to reparameterize this deformation as $\omega_\eta = \varphi_{-(\eta - \pi/2)}$, $\eta \in [0, \pi]$. Then

$$\omega_\eta(u) = \varphi(u) e^{i(\eta - \pi/2)} = -i \varphi(u) e^{i\eta} = u e^{i\eta} \quad \text{for} \quad u \in \mathcal{L}_+(\varphi),$$

$$\omega_\eta(u) = \varphi(u) e^{-i(\eta - \pi/2)} = i \varphi(u) e^{-i\eta} = u e^{-i\eta} \quad \text{for} \quad u \in \mathcal{L}_-(\varphi).$$

The space $\lambda(-i\omega_\eta, \theta)$ is equal to the sum of subspaces

$$\left\{ \left. \begin{pmatrix} u \cos \theta/2, & u e^{i(\eta - \pi/2)} \sin \theta/2 \end{pmatrix} \right| u \in \mathcal{L}_+(\varphi) \right\} \quad \text{and}$$

$$\left\{ \left. \begin{pmatrix} u \cos \theta/2, & u e^{-i(\eta + \pi/2)} \sin \theta/2 \end{pmatrix} \right| u \in \mathcal{L}_-(\varphi) \right\}.$$

By comparing this with the similar presentation of $\gamma'(A, \eta, \theta)$, we see that

$$\lambda(-i\omega_\eta, \theta) = \gamma'\left(\mathcal{L}_+(\varphi), \eta, \theta\right). \quad (19)$$

This also follows directly from the definitions.

**Canonical homomorphisms** $\beta: K^i(SY) \longrightarrow K^i(SX \cup BX_Y)$, $i = 0, 1$. Now we move to the framework of Section 3. Recall that $X$ is a compact manifold and $Y = \partial X$. Let $I^2$ be the rectangle $[0, \pi] \times [0, 2\pi]$ and let $\partial I^2$ be its boundary. The key element in the construction of the homomorphism $\beta$ are the Bott periodicity isomorphisms

$$K^i(SY) \longrightarrow K^i\left(SY \times I^2, SY \times \partial I^2\right),$$

where $i = 0, 1$. Let us take the product $SY \times [0, \pi]$ and collapse $SY \times 0$ and $SY \times \pi$ into two different copies $Y_0$ and $Y_\pi$ of $Y$. The result $SY$ can be identified with $SX_Y$ using the parameterization $v_\eta \cos \theta + u \sin \theta$ of half-circles from Section 3. Naturally, we agree that $Y_0$ corresponds to points with $\theta = 0$ and $Y_{\pi}$ to points with $\theta = \pi$. The quotient map $SY \times [0, \pi] \longrightarrow SY$ together with this identification lead to a continuous map

$$SY \times I^2 = \left(SY \times [0, \pi]\right) \times [0, 2\pi] \longrightarrow SX_Y \times [0, 2\pi].$$

There is also a natural homeomorphism

$$SX \cup (SX_Y \times [0, 2\pi]) \cup (BX_Y \times 2\pi) \longrightarrow SX \cup BX_Y.$$

The composition of the last two maps leads to canonical homomorphisms

$$K^i\left(SY \times I^2, SY \times \partial I^2\right) \longrightarrow K^i(SX \cup BX_Y), \quad (20)$$
where $i = 0, 1$. The homomorphisms $\beta$ are defined as the compositions of these homomorphisms with the Bott periodicity isomorphisms. When everything depends on a parameter $z \in \mathbb{Z}$, as at the end of Section 3, this construction leads to canonical homomorphisms

$$\beta : K^i(SV) \to K^i(SW \cup BW_V),$$

where the spaces involved are as in Section 3.

**Anti-commuting symbols and special boundary conditions.** In the framework of Section 3, let $\sigma$ be a self-adjoint symbol such that $\sigma_y$ is unitary for every $y \in Y$ and $\sigma$ is **anti-commuting** in the sense that $\sigma_y$ anti-commutes with $\tau_u$ for every $u \in SY$ and $y = \pi(u)$. Let $N$ be a self-adjoint elliptic bundle-like boundary condition for $\sigma$. Then the subspace $N_u$, where $u \in SY$, depends only on $y = \pi(u)$ and we may denote it by $N_y$. We will say that the boundary condition $N$ for $\sigma$ is **special** if $N_y$ is transverse not only to $L^-(\rho_u)$, but also to $L^+(\rho_u)$ for every $u \in SY$ and $y = \pi(u)$. If $N$ is special, the anti-commuting property of $\sigma$ implies that $N^\perp$ is also a special boundary condition for $\sigma$. Since $N$ and $N^\perp$ are bundle-like boundary conditions for $\sigma$, the invariants $e^+(\sigma, N)$ and $e^+(\sigma, N)$, as also $e^+(\sigma, N^\perp)$ and $e^+(\sigma, N^\perp)$ are well defined. We would like to see how they differ.

For $u \in SY$ and $y = \pi(u)$ let $\psi_y$ and $\varphi_u$ be the isometries $E^+_y \to E^-_y$ having as their graphs $N_y$ and $L^+(\rho_u)$ respectively. Let us identify the bundles $E^+(\sigma)$ and $E^-(\sigma)$ by the isometries $\psi_y$, $y \in Y$. Then we may treat the isometries $\varphi_u$ as isometries of $E^+_y$. Since $N$ is a special boundary condition, these isometries have no eigenvalues equal to 1 or $-1$.

It turns out that, as one may expect, the difference is determined by the behavior of $\sigma$ and $N$ at the boundary $Y$. For $u \in SY$ and $y = \pi(u)$ let $\psi_y$ and $\varphi_u$ be the isometries $E^+_y \to E^-_y$ having as their graphs $N_y$ and $L^+(\rho_u)$ respectively. Then $\psi_y^{-1} \circ \varphi_u$ is an isometry having no eigenvalues equal to 1 or $-1$. For every $u \in SY$ consider subspaces

$$L^+\left(\psi_y^{-1} \circ \varphi_u\right) \quad \text{and} \quad L^-\left(\psi_y^{-1} \circ \varphi_u\right)$$

and let $\nu_u : E^+_y \to E^+_y$ be the operator acting on the first subspace as the identity $\text{id}$ and on the second one as $-\text{id}$. Clearly, the operators $\nu_u$ are self-adjoint and invertible, and hence define a self-adjoint elliptic symbol $\nu$, the **boundary symbol** $\nu$. Since $Y$ is a closed manifold, $\nu$ requires no boundary conditions and hence there are well defined classes

$$e^+(\nu) \in K^0(SY) \quad \text{and} \quad e^+(\nu) \in K^1(BY, SY).$$

When a parameter $z \in \mathbb{Z}$ is present, as at the end of Section 3, we get classes

$$e^+(\nu) \in K^0(SV) \quad \text{and} \quad e^+(\nu) \in K^1(BV, SV),$$

where we used the notations from Section 3.
9.1. Theorem. Under the above assumptions \( e^+(\sigma, N) - e^+(\sigma, N^\perp) = \beta\left(e^+(\nu)\right) \).

Proof. Let us consider first the case of a single manifold \( X \) with the boundary \( Y \). For every \( u \in SY \) and \( y = \pi(u) \) let \( \psi_y \) and \( \varphi_u \) be the isometries \( E^+_y \longrightarrow E^-_y \) having as their graphs \( N_y \) and \( L_+(\rho_u) \) respectively. Let us deform \( \sigma_y, \rho_u, \tau_u \) as explained at the beginning of this section (actually, by the assumptions, the first deformation is not needed). These deformations can be arranged to continuously depend on \( u \). Moreover, they can be extended to a continuous deformation of the symbol \( \sigma \). At the end of these deformations the isometries \( \psi_y^{-1} \circ \varphi_u \) have only \( i \) and \( -i \) as eigenvalues. In general, the corresponding eigenspaces depend on \( u \), and the deformed symbol \( \sigma \) is not bundle-like. Let us identify \( E^+_y \) with \( E^-_y \) by the isometry \( \psi_y \) for every \( y \in Y \).

Now we can further deform \( \sigma_y, \rho_u, \tau_u \) and bring \( \sigma_y, \rho_u, \tau_u, N_y \) into a normal form in the sense of Section 2 by moving the eigenvalues \( i \) and \( -i \) to 1 as explained above. For the deformed symbol \( \varphi_u = \text{id} \) for every \( u \) and hence the deformed symbol is bundle-like. Of course, we need to extend these deformations to a deformation of the whole symbol \( \sigma \) over \( X \). A convenient way to do this is to identify \( SX \) with the result of glueing \( SX \times 0 = SY \) and placing \( \sigma \) before this deformation on \( SX \) and placing the deformation on \( SX \times [0, \pi/2] \). At the end we will get a bundle-like symbol, which we denote by \( \sigma \). Since \( \sigma \) is bundle-like, the bundle \( E^+(\sigma) \) over

\[
SX \cup BX_Y = SX \cup (SX \times [0, \pi/2]) \cup (BX_Y \times \pi/2)
\]

is well defined, and the invariant \( e^+(\sigma, N) \) is the class of this bundle.

Similarly, we can bring \( \sigma_y, \rho_u, \tau_u, N^\perp \) to a normal form by moving the eigenvalues \( i \) and \( -i \) to \( -1 \). For the deformed symbol \( \varphi_u = -\text{id} \) for every \( u \) and hence it is bundle-like. As above, we will place the deformation on \( SX \times [0, \pi/2] \) and get another bundle-like symbol, which we denote by \( \sigma^\perp \). The invariant \( e^+(\sigma, N^\perp) \) is the class of the bundle \( E^+(\sigma^\perp) \). The bundles \( E^+(\sigma) \) and \( E^+(\sigma^\perp) \) differ only over

\[
(21) \quad (SX \times [0, \pi/2]) \cup (BX_Y \times \pi/2).
\]

In order to compare them, let us construct an intermediate, a bundle \( E^+(\sigma) \) over

\[
(22) \quad (SX \times [0, \pi/2 + 2\pi]) \cup (BX_Y \times [\pi/2 + \pi, \pi/2 + 2\pi]).
\]

Over the subset (21) we set the bundle \( E^+(\sigma) \) to be equal to \( E^+(\sigma) \). In order to define \( E^+(\sigma) \) over \( SX \times [\pi/2, \pi/2 + 2\pi] \), recall that we identified \( SX \) with the quotient \( SY \) of \( SY \times [0, \pi] \). For every \( u \in SY \) and \( \theta \in [0, \pi] \), \( \eta \in [\pi/2, \pi/2 + 2\pi] \), let

\[
y'\left(L_+(\varphi_u), \eta - \pi/2, \theta\right)
\]
be the fiber of $E^+(\sigma)$ over the point represented by $(\nu, \theta, \eta)$. Here $\varphi_{\nu}$ refers to the symbol before the last deformations. By the equality (19) and the description of the positive subspaces, for $\eta = \pi/2$ these fibers are the same as the fibers of $E^+(\sigma)$. In particular, the bundle $E^+(\sigma)$ is correctly defined over $S_X \times [0, \pi/2 + 2\pi]$. For $\eta \in [\pi/2 + \pi, \pi/2 + 2\pi]$ these fibers do not depend on $\nu$ and hence the family of these fibers can be extended to $B_X \times [\pi/2 + \pi, \pi/2 + 2\pi]$ in the same way as in Section 3 the bundles $E^+(\sigma)$ were extended to $B_X \times \pi$. The resulting bundle over the space (22) is the promised bundle $E^+(\sigma)$.

By the construction, the bundle $E^+(\sigma)$ over $S_X \times [0, \pi/2]$ is equal to the bundle $E^+(\sigma)$, and over $S_X \times [\pi/2, \pi]$ is equal to a “reflected” copy of the restriction of the bundle $E^+(\sigma)$ to $S_X \times [0, \pi/2]$. Hence over $S_X \times [0, \pi]$ we can deform the bundle $E^+(\sigma)$ to

$$\left( E^+(\sigma) \big| S_X \times [0, \pi] \right)$$

without affecting it over $S_X \times 0$ and $S_X \times \pi$. Instead of this, we can simply cut out the subspace $S_X \times [0, \pi]$ and the bundle over it. This will replace the interval $[0, \pi/2 + 2\pi]$ by the shorter interval $[0, \pi/2 + \pi]$ and result in a bundle over $S_X \times [0, \pi/2 + \pi]$. This bundle is equal to $E^+(\sigma^\perp)$ over the subspace (21). After a reparameterization replacing the interval $[0, \pi/2 + \pi]$ by the interval $[0, \pi/2]$, the restriction of this new bundle to

$$\left( S_X \times [0, \pi/2 + \pi] \right) \cup \left( B_X \times (\pi/2 + \pi) \right)$$

may be considered as another extension of $E^+(\sigma^\perp)$ from $S_X \times [0, \pi/2]$ to $B_X \times \pi/2$. Since our new bundle is also defined over $B_X \times [\pi/2, \pi/2 + \pi]$, the two extensions are isomorphic. By placing back the cut out piece, we see that the restriction

$$\tilde{E}^+(\sigma^\perp) = E^+(\sigma) \big| \left( S_X \times [0, \pi/2 + 2\pi] \right) \cup \left( B_X \times (\pi/2 + 2\pi) \right)$$

is isomorphic to $E^+(\sigma^\perp)$ after a reparameterization replacing $[0, \pi/2 + 2\pi]$ by $[0, \pi/2]$.

Let us now replace in $\tilde{E}^+(\sigma)$ the part over $S_X \times [\pi/2, \pi/2 + 2\pi]$ by the product

$$\left( \tilde{E}^+(\sigma) \big| S_X \times \pi/2 \right) \times [\pi/2, \pi/2 + 2\pi],$$

and let $\tilde{E}^+(\sigma)$ be the resulting bundle. If we cut out this product from $\tilde{E}^+(\sigma)$, we will get the bundle $E^+(\sigma)$. By placing back the cut out piece, we see that $\tilde{E}^+(\sigma)$ is isomorphic, after a reparameterization replacing $[0, \pi/2 + 2\pi]$ by $[0, \pi/2]$, to $E^+(\sigma)$. Denoting by $[F]$ the K-theory class of a vector bundle $F$, we see that

$$(23) \quad [E^+(\sigma)] - [E^+(\sigma^\perp)] = [\tilde{E}^+(\sigma)] - [\tilde{E}^+(\sigma^\perp)].$$

The bundles $\tilde{E}^+(\sigma)$ and $\tilde{E}^+(\sigma^\perp)$ differ only over $S_X \times [\pi/2, \pi/2 + 2\pi]$, and their fibers
over the point represented by \((u, \theta, \eta)\) are
\[
\gamma' \left( \mathcal{L}_+ (\varphi_u), \eta - \pi/2, \theta \right) \quad \text{and} \quad \lambda(-i \text{id}, \theta)
\]
respectively, and these fibers are equal for \(\eta = \pi/2\) and \(\eta = \pi/2 + 2\pi\). Let us rename \(\eta - \pi/2\) as \(\eta\). Then applying the difference construction to two bundles over
\[
SX \times [0, 2\pi] = SY \times I^2
\]
with these fibers and their identity isomorphism over \(SX \times \partial I^2\) leads to an element
\[
D \in K^0 \left( SY \times I^2, SY \times \partial I^2 \right).
\]
By the construction the image of \(D\) under the homomorphism (20) is equal to the difference (23) and hence to \(e^+(\sigma, N) - e^+(\sigma, N^\perp)\).

At the same time \(D\) is equal to image under the Bott periodicity map of the K-theory class of the vector bundle \(\mathcal{L}_+ (\bullet)\) over \(SY\) having the vector space \(\mathcal{L}_+ (\varphi_u)\) as the fiber over \(u \in SY\). By the definition, \(\mathcal{L}_+ (\varphi_u)\) is the only eigenspace of \(\varphi_u\) with positive eigenvalue. It follows that \(\mathcal{L}_+ (\bullet) = E^+(\varphi)\), and hence \(e^+(\varphi)\) is equal to the K-theory class of \(\mathcal{L}_+ (\bullet)\). This implies that \(e^+(\sigma, N) - e^+(\sigma, N^\perp)\) is equal to the result of applying to \(e^+(\varphi)\) first the Bott periodicity map and then the homomorphism (20), i.e. applying \(\beta\).

This proves the theorem in the case of a single manifold \(X\). Including parameters into this proof is a routine matter. \(\blacksquare\)

9.2. Corollary. \textit{Under the same assumptions} \(e^+(\sigma, N) - e^+(\sigma, N^\perp) = \beta \left( e^+(\varphi) \right)\).

\textbf{Proof.} It sufficient to note that the Bott periodicity commutes with the coboundary maps in K-theory. \(\blacksquare\)

9.3. Theorem. \textit{Under the same assumptions} \(\text{t-ind} (\sigma, N) - \text{t-ind} (\sigma, N^\perp) = \text{t-ind} (\varphi)\).

\textbf{Proof.} Of course, this is non-trivial only for families \(\sigma(z), N(z), z \in \mathbb{Z}\). In the construction of the topological indices of \((\sigma, N), (\sigma, N^\perp)\) and \(\varphi\) passing to elements
\[
t(\sigma, N), t(\sigma, N^\perp), t(\varphi) \in K^1 (S^{2n} \times \mathbb{Z})
\]
absorbs homomorphisms (20) and they disappear. The next and last step in the construction of the topological index is an application of the Bott periodicity map, or, rather, in the context of this section, of its inverse. This step cancels the application of the Bott periodicity map in the definition of \(\beta\). Therefore the theorem follows from Corollary 9.2. \(\blacksquare\)

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10. Dirac-like boundary problems

**Odd operators.** In the framework of Section 5, suppose that the bundle $E$ has the form $E = F \oplus F$ for some Hermitian bundle $F$ over $X$. Let $P$ be a self-adjoint elliptic operator on sections of $E$ belonging to the Hörmander class and let $\Sigma$ be the corresponding endomorphism of $E$ over the collar. Suppose that the operator $P$ is *odd*, i.e. has the form

$$P = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix},$$

and that $\Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

with respect to the decomposition $E = F \oplus F$. Let $\sigma$ be the symbol of $P$.

**A class of boundary conditions.** Let $f : F|Y \to F|Y$ be a bundle automorphism, and let $B_Y$ be the bundle map $(F \oplus F)|Y \to F|Y$ defined by $B_Y(a, b) = b - f(a)$. Then $B_Y \circ \gamma$ is a boundary operator and the corresponding kernel-symbol $N$ is such that $N_u$ is equal to the graph $\{(a, f_y(a)) \mid a \in F_y\}$ of the action of $f$ on the fiber $F_y$ for $y = \pi(u)$. Since

$$\langle \Sigma(a, f_y(a)), (b, f_y(b)) \rangle = \langle (f_y(a), a), (b, f_y(b)) \rangle$$

and

$$= \langle f_y(a), b \rangle + \langle a, f_y(b) \rangle,$$

the kernel-symbol $N$ is self-adjoint if and only if $f$ is skew-adjoint, or, equivalently, if $f$ is self-adjoint. In general, such a kernel-symbol $N$ is not an elliptic boundary condition for $\sigma$. Let $\tau_u, \rho_u$ be as in Section 5. Since $P$ is odd, the operators $\tau_u$ have the form

$$\tau_u = \begin{pmatrix} 0 & \tau_u^* \\ \tau_u & 0 \end{pmatrix}$$

for some operators $\tau_u : F_y \to F_y$ (which are closely related to $\tau_u$ from Section 9). We will say that $f$ is *equivariant* if $f_y$ commutes with $\tau_u$ when $y = \pi(u)$. The basic properties of the boundary conditions defined by self-adjoint automorphisms $f$ are contained in the following two lemmas, which are essentially due to A. Gorokhovsky and M. Lesch [GL].

**10.1. Lemma.** Suppose that $f$ is skew-adjoint and either if $f$ is positive or negative definite, or $f$ is equivariant. Then $N$ is a special boundary condition for $\sigma$.

**Proof.** Cf. [GL], Proposition 3.1(a). If $\tau_u$ has the above form, then

$$\rho_u = \begin{pmatrix} \tau_u & 0 \\ 0 & \tau_u^* \end{pmatrix}.$$
Let \((a, b) \in F \oplus F\) be an eigenvector of \(\rho_u\) with the eigenvalue \(\lambda\) belonging to \(N_u\). Then \(b = f_y(a)\) and \(\tau_u(a) = \lambda a\), \(\tau_u^*(b) = \lambda b\). It follows that \(\tau_u^* \circ f_y(a) = \lambda f_y(a)\) and hence \(\tau_u^* \circ f_y \circ \tau_u(a) = \lambda^2 f_y(a)\). In turn, this implies that

\[
\langle \tau_u^* \circ f_y \circ \tau_u(a), a \rangle = \langle \lambda^2 f_y(a), a \rangle \quad \text{and hence}
\]

\[
\langle f_y \circ \tau_u(a), \tau_u(a) \rangle = \lambda^2 \langle f_y(a), a \rangle.
\]

Since \(f_y\) is skew-adjoint, both scalar products here are purely imaginary and hence \(\lambda^2 \in \mathbb{R}\). If \(i f_y\) is positive, then both these scalar products belong to \(i \mathbb{R}\) and hence \(\lambda^2 \geq 0\). It follows that \(\lambda \in \mathbb{R}\). Since \(\rho_u\) has no non-zero real eigenvalues, it follows \(\lambda = 0\) and \((a, b) = 0\). Therefore \(N_u\) intersects both \(\mathcal{L}_+(\rho_u)\) and \(\mathcal{L}_-(\rho_u)\) by \(0\). This proves the lemma when the operators \(i f_y\) are positive definite. The case when the operators \(i f_y\) are negative definite is completely similar. Since \(f\) is skew-adjoint, the bundle \(F\) is equal to the direct sum of two subbundles with fibers \(\mathcal{L}_+(f_y)\) and \(\mathcal{L}_-(f_y)\) respectively. If \(f_y\) commutes with \(\tau_u\) when \(y = \pi(u)\), then both these subbundles are invariant under operators \(\tau_u\). It follows that \(\sigma_y, \tau_u\) and \(N_u\) are also direct sums, and that the already proved part of the lemma applies to both summands. The lemma follows. 

**10.2. Lemma.** Suppose that the above framework depends on a parameter \(z \in Z\). If the automorphism \(i f(z)\) is positive or negative definite for every \(z \in Z\), then the analytical index of the family \(P(z), z \in Z\) with the boundary conditions \(N(z), z \in Z\) is equal to zero.

**Proof.** Cf. [GL], the proof of Corollary 3.4. We will deal with the case of a single operator first, ignoring the fact that in this case the index is zero. Recall the Green formula

\[
\langle Pu, v \rangle_0 - \langle u, Pv \rangle_0 = \langle i \Sigma y u, y v \rangle_0.
\]

Let \(u = (x, y)\) and \(v = (a, b)\), where \(x, y, a, b\) are sections of \(F\). Then

\[
\langle i \Sigma y u, y v \rangle_0 = \langle i(y y, y x), (y a, y b) \rangle_0 = i \langle y y, y a \rangle_0 + i \langle y x, y b \rangle_0,
\]

\[
\langle Pu, v \rangle_0 - \langle u, Pv \rangle_0 = \langle Ay, a \rangle_0 + \langle A^* x, b \rangle_0 - \langle x, A b \rangle_0 - \langle y, A^* a \rangle_0.
\]

Suppose that \(i f\) is positive definite. Let \(\varepsilon\) be the automorphism of \(E\) defined by

\[
\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and let us consider the operator \(P + \varepsilon\) with the same boundary conditions \(N\). Since \(\varepsilon\) is a self-adjoint bundle automorphism, the left hand side of the Green formula for \(P + \varepsilon\) is the same as for \(P\). The right hand side is also the same, depending only on \(\Sigma\). Suppose that
\((P + \varepsilon)(s_+, s_-) = 0\). Then \(s_+ + As_- = 0\) and \(A^*s_+ - s_- = 0\). Suppose also that the section \((s_+, s_-)\) satisfies the boundary condition \(N\), i.e. \(\gamma s_- = f(\gamma s_+)\), and apply the Green formula to \(u = (x, y) = (s_+, 0)\) and \(v = (a, b) = (0, s_-)\). Then the left hand side of the Green formula is equal to
\[
\langle A y, a \rangle_0 + \langle A^* x, b \rangle_0 - \langle x, A b \rangle_0 - \langle y, A^* a \rangle_0
\]
\[
= \langle A 0, 0 \rangle_0 + \langle A^* s_+, s_- \rangle_0 - \langle s_+, As_- \rangle_0 - \langle 0, A^* 0 \rangle_0
\]
\[
= \langle A^* s_+, s_- \rangle_0 - \langle s_+, As_- \rangle_0 = \langle s_-, s_- \rangle_0 + \langle s_+, s_+ \rangle_0 = \langle s, s \rangle_0 ,
\]
and the right hand side is equal to
\[
i \langle \gamma y, \gamma a \rangle_\partial + i \langle \gamma x, \gamma b \rangle_\partial = i \langle \gamma 0, \gamma 0 \rangle_\partial + i \langle \gamma s_+, \gamma s_- \rangle_\partial
\]
\[
= -\langle \gamma s_+, i f(\gamma s_+) \rangle_\partial .
\]
Therefore the Green formula takes the form
\[
\langle s, s \rangle_0 = -\langle \gamma s_+, i f(\gamma s_+) \rangle_\partial = -\langle i f(\gamma s_+), \gamma s_+ \rangle_\partial .
\]
The left hand side is \(\geq 0\) and, since \(if\) is positive definite, the right hand side is \(\leq 0\). Hence \(\langle s, s \rangle_0 = 0\) and \(s = 0\). It follows that the kernel of \((P + \varepsilon) \oplus \Gamma\), where \(\Gamma\) is the boundary operator associated with \(N\), is zero. Theorem 4.1 implies that \((P + \varepsilon) \oplus \Gamma\) is an isomorphism and hence the index of the boundary problem defined by \(P + \varepsilon\) and \(N\) is zero. Since \(\varepsilon\) is an operator of order 0, the index of the boundary problem defined by \(P\) and \(N\) is also zero. When \(if\) is negative definite, the same proof with \(P + \varepsilon\) replaced by \(P - \varepsilon\) works. Clearly, this proof works also when parameters are present. ■

**Another form of** \(P, N\). In order to apply the results of Section 9, we need to bring \(P\) and \(N\) to the standard “ungraded” form. Consider the subbundles
\[
\Delta_F = \{(a, a) \mid a \in F\} \quad \text{and} \quad \Delta_F^\perp = \{(b, -b) \mid b \in F\}
\]
of \(E = F \oplus F\). Clearly, \(\Delta_F = E^+\), \(\Delta_F^\perp = E^-\), and we can identify both \(\Delta_F\) and \(\Delta_F^\perp\) with \(F\) by the projection \((x, y) \mapsto x\). Suppose that \(f\) is skew-adjoint and write down the corresponding boundary condition \(N\) in terms of the decomposition \(E = \Delta_F \oplus \Delta_F^\perp = F \oplus F\). An easy calculation shows that for every \(u \in SY\) and \(y = \pi(u)\)
\[
N_u = N_y = \left\{ \left( a, \frac{1 - f_y}{1 + f_y}(a) \right) \mid a \in F_y \right\} .
\]
Note that $1 + f_y$ is invertible because $f$ is skew-adjoint. By the same reason

$$\psi_y = \frac{1 - f_y}{1 + f_y}$$

is unitary and invertible. If we consider $\psi_y$ as an operator $E_y^+ \to E_y^-$, then $N_y$ is the graph of $\psi_y$. If $f$ is equivariant, then $N$ is a special boundary condition by Lemma 10.1. But in order to apply the results of Section 9 we need to impose further conditions on $P$.

**Dirac-like boundary problems.** Suppose that $f$ is self-adjoint and equivariant and that the operators $\tau_u : F \to F$ are skew-adjoint for every $u \in SY$. In this case the symbol $\sigma$ is anti-commuting, $N$ is a special boundary condition, and hence we are in the situation of Section 9. One may call the boundary problem defined by such $P, f$ *Dirac-like*.

Since $f$ is equivariant, the operators $\tau_u$ leave invariant $L_+(f_y)$ and $L_-(f_y)$. Let

$$\tau_u^+ : L_+(f_y) \to L_+(f_y) \quad \text{and} \quad \tau_u^- : L_-(f_y) \to L_-(f_y)$$

be the operators induced by $\tau_u$. Let $L_+(f)$ and $L_-(f)$ be the bundles having $L_+(f_y)$ and $L_-(f_y)$, respectively, as the fibers over $y \in Y$. Then

$$F = L_+(f) \oplus L_-(f)$$

and the operators $\tau_u^+$ and $\tau_u^-$ define a skew-adjoint symbols $\tau^+$ and $\tau^-$ over $Y$ in the bundles $L_+(f)$ and $L_-(f)$ respectively. Multiplying $\tau^+$ and $\tau^-$ by $i$ results in self-adjoint symbols $i\tau^+$ and $i\tau^-$.

**10.3. Theorem.** $t\text{-ind}(\sigma, N) = t\text{-ind}(i\tau^-) = t\text{-ind}(-i\tau^+)$.

**Proof.** After a spectral deformation of $f$ we may assume that $i, -i$ are the only eigenvalues of $f_y$ for every $y$. Then $i, -i$ are also the only eigenvalues of $\psi_y$ for every $y$, and

$$L_+(\psi_y) = L_-(f_y), \quad L_-(\psi_y) = L_+(f_y).$$

Let $f^+ : F|Y \to F|Y$ be the multiplication by $i$, and $N^+$ be the corresponding boundary condition. Since $if^+$ is negative definite, $a\text{-ind}(\sigma, N^+) = 0$ by Lemma 10.2. Clearly, the bundle $E_Y^+$ is isomorphic to $E|Y$ and hence extends to $X$. Hence $t\text{-ind}(\sigma, N^+) = 0$ by Theorem 8.1, and in order to prove the first equality it is sufficient to prove that

$$\varepsilon^+(\sigma, N) - \varepsilon^+(\sigma, N^+) = \beta\left(e^+(i\tau^-)\right).$$

The boundary conditions $N$ and $N^+$ differ only in the summand $L_-(f)$, and in this sum-
mand they differ in the same way as the boundary conditions $N$ and $N^\perp$ in Theorem 9.1 and Corollary 9.2. In this summand the proof of Theorem 9.1 applies and will imply (24) once we prove that the sum of the positive eigenspaces of the operators $i\tau_u^-$ are equal to the positive eigenspaces of the operators $\psi_u$ from Section 9 for the summand $\mathcal{L}_-(f)$.

Since in Section 9 we worked with “ungraded” operators, we need to pass to the decomposition $E = \Delta F \oplus \Delta^\perp F = F \oplus F$. We already computed the isometries $\psi_y$ in this decomposition, and on $\mathcal{L}_-(f)$ they are equal to the multiplication by $i$. At the same time

$$\rho_u = \begin{pmatrix} \tau_u & 0 \\ 0 & -\tau_u \end{pmatrix}$$

with respect to the original decomposition $E = F \oplus F$, and hence

$$\mathcal{L}_+(\rho_u) = \mathcal{L}_+(\tau_u) \oplus \mathcal{L}_-(\tau_u) \subset F \oplus F$$

with respect to the original decomposition. It follows that with respect to the decomposition $E = \Delta F \oplus \Delta^\perp F = F \oplus F$ the subspace $\mathcal{L}_+(\rho_u)$ is the graph of the isometry $\varphi_u$ such that

$$\varphi_u(a) = a \quad \text{if} \quad a \in \mathcal{L}_+(\tau_u) \quad \text{and}$$

$$\varphi_u(a) = -a \quad \text{if} \quad a \in \mathcal{L}_-(\tau_u).$$

It follows that in the summand $\mathcal{L}_-(f)$

$$\mathcal{L}_+\left(\psi_y^{-1} \circ \varphi_u\right) = \mathcal{L}_-(\tau_u^-).$$

But $\mathcal{L}_-(\tau_u^-)$ is exactly the sum of the positive eigenspaces of $i\tau_u^-$. By the previous paragraph, this implies the equality (24).

Next, let $f^- : F|Y \rightarrow F|Y$ be the multiplication by $-i$, and $N^-$ be the corresponding boundary condition. In order to complete the proof, it is sufficient to prove that

$$\epsilon^+(\sigma, N) - \epsilon^+(\sigma, N^-) = \beta\left(e^+(-i\tau^+\right)).$$

(25)

The boundary conditions $N$ and $N^+$ differ only in the summand $\mathcal{L}_+(f)$, and in this summand they differ in the same way as the boundary conditions $N$ and $N^\perp$ in Section 9. After passing to the decomposition $E = \Delta F \oplus \Delta^\perp F = F \oplus F$ we see that in the summand $\mathcal{L}_-(f)$

$$\mathcal{L}_+\left(\psi_y^{-1} \circ \varphi_u\right) = \mathcal{L}_+(\tau_u^+).$$

Since $\mathcal{L}_+(\tau_u^+)$ is the sum of the positive eigenspaces of $-i\tau_u^-$, this implies (25). □
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