Characterizing entanglement using quantum discord over state extensions

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We propose a framework to characterize entanglement with quantum discord, both asymmetric and symmetric, over state extensions. In particular, we show that the minimal Bures distance of discord over state extensions is equivalent to Bures distance of entanglement. This equivalence places quantum discord at a more primitive position than entanglement conceptually in the sense that entanglement can be interpreted as an irreducible part of discord over all state extensions. Based on this equivalence, we also offer an operational meaning of Bures distance of entanglement by connecting it to quantum state discriminations. Moreover, for the relative entropy part, we prove that the entanglement measure introduced by Devi and and Rajagopal [Phys. Rev. Lett. 100, 140502 (2008)] is actually equivalent to the relative entropy of entanglement. We also provide several quantifications of entanglement based on discord measures.

I. INTRODUCTION

Quantum correlations [1, 2] are defined from different viewpoints and they, in turn, are expected to offer different advantages. Hence, the characterization and quantification of quantum correlations is instrumental in exploring and exploiting the quantum phenomena. The remarkable advantages that quantum correlations offer make quantum information theory more powerful than classical theory. Entanglement [1] is an important quantum resource which plays a crucial role in quantum information processing, quantum algorithms, quantum computation and cryptography [3]. The notion of quantum correlations and resources beyond entanglement [2, 4–6] such as quantum discord [2, 7–10], quantum coherence [6, 11–13] etc. is also very prominent and useful in quantum information theory. For instance, quantum discord is the genuine resource in the DQC1 algorithm [14, 15].

Among several quantum correlations, entanglement and quantum discord are significant, and are usually regarded as very distinct in nature. While entanglement belongs to the entanglement-separability paradigm, quantum discord belongs to the information-theoretic paradigm. Nevertheless, entanglement and quantum discord have both essential similarities and significant differences. Several remarkable investigations relating to entanglement and discord have been studied [16–26] in recent years. Especially, the correspondence between classical states versus separable states [27], the characterization and quantification of entanglement with the generalized information-theoretic measure [28], the minimal quantum discord of bipartite state over state extensions [29] and the minimal correlated coherence over symmetric state extensions [30, 31] are of special interest.

It should be noted that the framework that quantifies entanglement with quantum discord or coherence over state extensions is quite different from the existing entanglement measures such as entanglement of formation and entanglement cost [32], distillable entanglement [33], relative entropy of entanglement and Bures distance of entanglement [34, 35], robustness of entanglement [36], and squashed entanglement [37, 38], which are mostly based on operational meaning, information principles, and mathematics. Thus far, the relationship between these two different categories of entanglement measures is not clear.

In this paper, we provide a link and clarify the relationship of two kinds of entanglement measures. Firstly, we propose a framework to characterize entanglement with quantum discord over state extensions. Actually the general quantum discord including geometric discord [39], measurement-induced geometric discord [40] and geometric correlated partial coherence [41], over all possible extended states is shown to be a kind of quantification of entanglement, which also provides an alternative perspective to understand entanglement from the viewpoint of quantum discord. We show further, for the Bures distance [42, 43], that the corresponding discord measure over all state extensions is equivalent to the Bures distance of entanglement [35, 44, 45]. Moreover, Bures distance of entanglement was proved to be equal to its corresponding convex roof [45], which plays a key role in our study. In fact, we prove the equivalence by showing that the minimal Bures distance of discord over state extensions is bounded by the Bures distance of entanglement and its convex roof. For relative entropy, we prove that the corresponding discord measure...
is equal to the measurement-induced discord, which implies that the quantifier of entanglement proposed in Ref. [28] is equivalent to the relative entropy of entanglement [35].

Bures distance of entanglement is defined from the geometric viewpoint whose operational meaning is not very clear. A correspondence, however, between Bures distance of discord and quantum state discriminations has been established in Ref. [46]. Based on this correspondence and the equivalence between Bures distance of entanglement and discord over state extensions, we offer an operational meaning of Bures distance of entanglement by linking it to quantum state discriminations.

This paper is structured as follows. In Sec. II, we recall various concepts which are prerequisite for our study. We discuss characterization of entanglement using asymmetric and symmetric quantum discord in Sec. III and Sec. IV respectively. We give an operational meaning of the Bures distance of entanglement in Sec. III. C. We conclude our findings in Sec. V. Proofs of some theorems and symbols with their meanings are presented in Sec. VI (Appendix) at the end of the bibliography.

II. PRELIMINARIES

In this section, we briefly recall various concepts which are prerequisite for our study.

A. Separable states

Let \( \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \) be the composite Hilbert space of a bipartite system, and \( \mathcal{D}(\mathcal{H}) \) be the set of density matrices on \( \mathcal{H} \). A quantum state \( \rho_{ab} \in \mathcal{D}(\mathcal{H}) \) shared between two parties \( a \) and \( b \) is called separable if it can be represented as

\[
\rho_{ab} = \sum_i p_i \rho_a^i \otimes \rho_b^i,
\]

where \( p_i \geq 0 \), \( \sum_i p_i = 1 \) and \( \rho_a^i \), \( \rho_b^i \) are local states for parties \( a \) and \( b \). Otherwise, it is called entangled. We denote the set of separable states by \( \mathcal{S} \). The separable states are prepared by local (quantum) operations and classical communications (LOCC). That is, to prepare a separable state, Alice first prepares a state \( \rho_a^i \) with probability \( p_i \) and communicate this to Bob. Based on this information, Bob prepares the corresponding state \( \rho_b^i \). However, it is not possible to create entangled states in this way. With respect to a quantum measurement, a bipartite separable state is classified into following types: (i) classical-classical (CC) if \( \rho_{ab} = \sum_{i,j} \Pi_a^i \otimes \Pi_b^j \rho_{ab} (\Pi_a^i \otimes \Pi_b^j) = \sum_{i,j} p_i \langle i|a \rangle \langle i|a \rangle \otimes \langle j|b \rangle \langle j|b \rangle \), (ii) classical-quantum (CQ) if \( \rho_{ab} = \sum_i \Pi_a^i \otimes \rho_{ab} (\Pi_a^i \otimes I_b) = \sum_i p_i \langle i|a \rangle \langle i|a \rangle \otimes \rho_b^i \), and (iii) quantum-classical (QC) if \( \rho_{ab} = \sum_{i,j} \Pi_a^i \otimes \Pi_b^j \rho_{ab} (I_a \otimes I_b) = \sum_{i,j} p_i \rho_a^i \otimes \rho_b^j \), where the probabilities are nonnegative and sum to unity. \( \{i\}^a \) and \( \{j\}^b \) are orthonormal basis of parties \( a \) and \( b \), respectively.

B. Relative entropy

Relative entropy is defined as

\[
S(\rho||\sigma) := \text{tr}[\rho (\log \rho - \log \sigma)].
\]

Throughout, unless stated otherwise, the base of the log should be taken 2. For any classical-quantum state \( \sigma_{ab}^{(cq)} = \sum_j p_j \pi_a^j \otimes \pi_b^j \), we observe that \( \text{tr}[\rho_{ab} \log \sigma_{cq}] = \text{tr}[\Pi_a(\rho_{ab}) \log \sigma_{cq}] \), where \( \Pi_a(\rho_{ab}) = \sum_i (\pi_a^i \otimes I_b) \rho_{ab} (\pi_a^i \otimes I_b) \). Actually, if we assume \( \sigma_{ab}^{(cq)} = \sum_{ijk} q_{ijk} \pi_a^i \otimes \pi_b^j \), then

\[
\begin{align*}
\text{tr}[\Pi_a(\rho_{ab}) \log \sigma_{ab}^{(cq)}] &= \sum_{ijkl} \log q_{ijkl} \text{tr}[(\pi_a^i \otimes I_b)(\pi_a^j \otimes I_b)\pi_b^k] \\
&= \sum_{ijkl} \log q_{ijkl} \text{tr}[(\pi_a^i \otimes I_b)\pi_b^k] = \text{tr}[\rho_{ab} \log \sigma_{ab}^{(cq)}].
\end{align*}
\]

Hence,

\[
S(\rho_{ab}||\sigma_{ab}^{(cq)}) = \text{tr}[\rho_{ab} \log \rho_{ab}] - \text{tr}[\rho_{ab} \log \sigma_{ab}^{(cq)}] = \text{tr}[\Pi_a(\rho_{ab}) \log \Pi_a(\rho_{ab})] + \text{tr}[\Pi_a(\rho_{ab}) \log \Pi_a(\rho_{ab})] - \text{tr}[\Pi_a(\rho_{ab}) \log \sigma_{ab}^{(cq)}] = S(\Pi_a(\rho_{ab})) - S(\rho_{ab}) + S(\Pi_a(\rho_{ab}))|\sigma_{ab}^{(cq)}|.
\]

We remark that a similar equation is established in block coherence theory [47].

C. Bures distance

The Bures distance is defined as [42, 43]

\[
d_B(\rho, \sigma) := \sqrt{2 - 2F(\rho, \sigma)},
\]

where \( F(\rho, \sigma) \) is the fidelity \( F(\rho, \sigma) := \text{tr}\sqrt{\rho \sigma \sqrt{\rho \sigma}} \) between \( \rho \) and \( \sigma \). Since \( F(\rho, \sigma) \in [0, 1] \) and is unity iff \( \rho = \sigma \), \( d_B(\rho, \sigma) \) is nonnegative and vanishes iff \( \rho = \sigma \). Moreover, the monotonicity and joint concavity of fidelity [3] implies that \( d_B^2 \) is contractive and jointly convex.

D. Entanglement

An entanglement measure \( E \) is a functional on \( \mathcal{D}(\mathcal{H}) \) satisfying (1) Faithfulness: \( E(\rho) \geq 0 \), where the equality holds iff \( \rho \in \mathcal{S} \), and (2) Monotonicity: \( E(\rho) \geq E(\Phi(\rho)) \) for any LOCC operation \( \Phi \).

\textbf{Bures distance of entanglement.}–The Bures distance of entanglement is defined as the minimal square of Bures distance to separable states [34],

\[
E_B(\rho_{ab}) := \min_{\sigma_{ab} \in \mathcal{S}} d_B^2(\rho_{ab}, \sigma_{ab}).
\]

Obviously, \( E_B(\rho_{ab}) \) is nonnegative, and vanishes iff \( \rho_{ab} \) is separable. Furthermore, \( E_B \) is convex and non-increasing.
under LOCC operations \([34, 35]\). Note that for the bipartite pure state \(|\psi\rangle = \sum \sqrt{\lambda_i} |x_i\rangle_a |y_i\rangle_b\) with \(\lambda_1 \geq \ldots \geq \lambda_n\), \(E_B(|\psi\rangle) = 2(1 - \sqrt{\lambda_1})\) \([45]\). In fact, if we assume that \(\sigma_{ab} \in S\) has a separable pure state decomposition \(\sigma_{ab} = \sum_j q_j |\phi_j\rangle_{ab} \langle \phi_j|\), then

\[
F(|\psi\rangle, \sigma_{ab}) = \sum_j q_j |\langle \phi_j|\phi_{\text{max}}\rangle|^2 \\
\leq \sum_j q_j |\langle \phi_{\text{max}}|\phi_j\rangle|^2 = |\langle \psi|\phi_{\text{max}}\rangle|,
\]

where \(|\langle \psi|\phi_{\text{max}}\rangle| := \max_j |\langle \psi|\phi_j\rangle|\) is the maximal over all \(j\). Therefore, \(E_B(|\psi\rangle) = 2(1 - \sqrt{\lambda_1})\) and the corresponding closest separable state is \(|x_1, y_1\rangle_{ab}\).

**Convex roof of Bures distance of entanglement.**–The convex roof of Bures distance of entanglement is defined as

\[
E_{B, \text{cr}}(\rho_{ab}) := \min_{p_i, |\psi_i\rangle} \sum_i p_i E_B(|\psi_i\rangle),
\]

where the minimal is taken over all pure state decompositions \(\rho_{ab} = \sum_i p_i |\psi_i\rangle \langle \psi_i|\).

**E. Quantum discord**

**Asymmetric quantum discord.**–A functional \(D^a\) on \(D(\mathcal{H})\) is called a discord measure (asymmetric or local) if it satisfies the following properties:

(D1) \(D^a\) is faithful, i.e., \(D^a(\rho_{ab}) \geq 0\) and the equality holds if and only if \(\rho \in \mathcal{CQ}\).

(D2) \(D^a\) is non-increasing for any quantum operation on subsystem \(b\), i.e., \(D^a(\rho_{ab}) \geq D^a(I_a \otimes \Phi_b(\rho_{ab}))\) for any local operation \(\Phi_b\).

(D3) \(D^a\) is invariant under local unitary transformations, i.e., \(D^a(U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger) = D^a(\rho_{ab})\) for any unitary \(U_a \otimes U_b\) acting on \(H_a \otimes H_b\).

(D4) \(D^a\) reduces to an entanglement monotone for pure states.

Quantum discord has been studied extensively from different viewpoints in the forms of geometric discord, measurement-induced geometric discord \([48]\) and geometric correlated partial coherence \([41]\). Quantum discord, obviously, captures quantum correlation beyond entanglement in the sense that a separable state may have nonzero quantum discord.

**Symmetric quantum discord.**–A functional \(D\) in state space (or in state space) is called (symmetric or global) discord \([9, 10]\) measure if it satisfies the following properties:

(D1') \(D\) is faithful, i.e., \(D(\rho_{ab}) \geq 0\) and the equality holds if and only if \(\rho_{ab} \in \mathcal{C}\).

(D2') \(D\) is invariant under local unitary transformations, i.e., \(D(U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger) = D(\rho_{ab})\) for any unitary acting on \(H_a \otimes H_b\).

**III. CHARACTERIZING ENTANGLEMENT VIA ASYMMETRIC QUANTUM DISCORD**

In this section, we propose the framework to quantify entanglement via quantum discord (asymmetric or local), and introduce several quantifiers with geometric discord and measurement-induced geometric discord.

**A. General results**

In this section, we explore and interpret the relationship between entanglement and quantum discord (asymmetric) over state extensions.

**Definition 1.** For a bipartite state \(\rho_{ab} \in D(\mathcal{H})\), the minimal discord over state extensions is defined as

\[
\mathcal{E}^a(\rho_{ab}) := \min_{\sigma_{aa'b} = \rho_{ab}} D^a(\rho_{aa'b}),
\]

where the minimization is taken along the bipartition \(aa' : b\) with \(tr_{aa'} \rho_{aa'b} = \rho_{ab}\) (see Fig. 1).

**Theorem 1.** If \(D^a\) is a discord measure satisfying (D1-D4), the corresponding minimal discord, \(\mathcal{E}^a\), over state extensions has the following remarkable properties:

(E1) \(\mathcal{E}^a(\rho_{ab}) \geq 0\) with the equality if and only if \(\rho \in \mathcal{S}\).

(E2) \(\mathcal{E}^a\) is invariant under local unitary transformations.

(E3) \(\mathcal{E}^a\) is nonincreasing under local partial trace,

\[
\mathcal{E}^a(\rho_{ab}) \leq \mathcal{E}^a(\rho_{aa'b})
\]
for any state extension $\rho_{aa'b}$ of $\rho_{ab}$.

(E4) $\mathcal{E}^a$ is nonincreasing under local operations in party $a$.

(E5) $\mathcal{E}^a$ reduces to an entanglement monotone for pure states.

Proof. (E1) The nonnegativity of quantum discord implies that $\mathcal{E}^a$ is always nonnegative. Moreover, any separable state $\rho_{ab} = \sum_i p_i \rho_i^a \otimes \rho_i^b$ can be embedded into a larger classical-quantum state $\rho_{aa'b} = \sum_i p_i |\alpha_i\rangle_{aa'} \langle \alpha_i| \otimes \rho_i^b$ such that $\rho_{ab} = tr_{a'}[\rho_{aa'b}]$, where $a'$ is the ancillary system pertinent to party $a$ and $|\alpha_i\rangle_{aa'}$ is the purification of $\rho_i^a$ for each $i$ [28]. However, an entangled state does not admit such an extension. Therefore, $\mathcal{E}^a$ is faithful in separable states.

(E2) Assuming $\rho_{aa'b}$ is the state extension of $\rho_{ab}$, $tr_{a'}[U_a \otimes U_b \rho_{aa'b} U_a^\dagger \otimes U_b^\dagger] = U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger$ and the local unitary invariance of $D^a$ implies that $\mathcal{E}^a(\rho_{ab}) \geq \mathcal{E}^a(\rho_{aa'b})$. On the contrary, we can also show $\mathcal{E}^a(\rho_{ab}) \leq \mathcal{E}^a(U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger)$ implying that $\mathcal{E}^a$ is invariant under local unitary transformation.

(E3) This follows trivially because any state extension $\rho_{aa'a'b}$ of state $\rho_{aa'b}$ is also the extension of $\rho_{aa'a'b}$.

(E4) Using Stinespring representation [50], the local operation in party $a$ can be realized by adding a pure state ancilla, followed by a global unitary operation and tracing out the ancilla system, i.e., $\sum_i K_i^aa \rho_i^a K_i^{a\dagger} = tr_{a'} U_{aa'}(\rho_{aa'b} \otimes |0\rangle_{a'} \langle 0|) U_{aa'}^\dagger$. Therefore, one has

$$\mathcal{E}^a(\rho_{ab}) \geq \mathcal{E}^a(\rho_{ab} \otimes |0\rangle_{a'} \langle 0|) = \mathcal{E}^a(U_{aa'} \rho_{ab} \otimes |0\rangle_{a'} \langle 0| U_{aa'}^\dagger) \geq \mathcal{E}^a(\sum_i K_i^a \rho_i^a K_i^{a\dagger}),$$

where the inequality in the first and the third lines follows from property (E3).

(E5) Firstly, combining the definition of $\mathcal{E}^a$ and property (E3),

$$\mathcal{E}^a(|\psi\rangle \langle \psi|) \geq \min_{|\phi\rangle} D^a(|\psi\rangle \otimes |\phi\rangle) \geq D^a(|\psi\rangle) = E(|\psi\rangle).$$

Further, considering the special case, that is, the extension space is one-dimensional, $\mathcal{E}^a(|\psi\rangle) = D^a(|\psi\rangle) = E(|\psi\rangle).$}

Remark 1. The above results provide an alternative avenue to understand quantum entanglement from the viewpoint of discord over state extensions. We can see that $\mathcal{E}^a$ is a good candidate of an entanglement measure. Moreover, $D^a$ reduces to $\mathcal{E}^a$ for pure states.

A pseudo-distance $d$ in state space $D(\mathcal{H})$ is called contractive, if it satisfies $d(\rho, \sigma) \geq d(\Phi(\rho), \Phi(\sigma))$ for any quantum operation $\Phi$ and $\rho, \sigma \in D(\mathcal{H})$. We remark that by pseudo-distance, we refer to a nonnegative bivariate function $d$ in state space with $d(\rho, \sigma) = 0$ iff $\rho = \sigma$. Relative entropy and Bures distance are examples of pseudo-distances. We will simply write “distance” for simplicity.

Definition 2. For a bipartite state $\rho_{ab} \in D(\mathcal{H})$, the minimal geometric discord over state extensions (GDSE) is defined as

$$\mathcal{E}_d^a(\rho_{ab}) := \min_{\Pi_{aa'} \in \mathcal{C}_Q} \min_{tr_{a'}[\rho_{aa'b}] = \rho_{ab}} d(\rho_{aa'b}, \Pi_{aa'} \otimes I_b(\rho_{aa'b})), \tag{2}$$

where the minimization is taken over all extended states $\rho_{aa'b}$ and classical-quantum states in $D(\mathcal{H}_{aa'b})$. Here $d$ is a contractive distance in $D(\mathcal{H}_{aa'b})$. Furthermore, the minimal measurement-induced geometric discord over state extensions (MIDSE) is defined as

$$\mathcal{E}_d^a(\rho_{ab}) := \min_{\rho_{aa'a'b}} \min_{tr_{a'}[\rho_{aa'a'b}] = \rho_{ab}} d(\rho_{aa'a'b}, \Pi_{aa'a'} \otimes I_b(\rho_{aa'b})), \tag{3}$$

where the minimization is taken over extended states and local projection in subsystem $aa'$.

Remark 2. For both relative entropy and Bures distance, the corresponding quantifiers of discord satisfy (D1-D4). Theorem 1 then implies that the corresponding GDSE and MIDSE are good candidates of entanglement measures.

Next, we consider the quantification of entanglement by performing partial trace; a slight modification of Eq. (2). Define quantum correlation by

$$\hat{\mathcal{E}}_d^a(\rho_{ab}) := \min_{\sigma_{aa'b} \in \mathcal{C}_Q} d(\rho_{ab}, tr_{a'}[\sigma_{aa'b}]),$$

where the minimum is taken over all classical-quantum state in $D(\mathcal{H}_{aa'b})$. Obviously, $\hat{\mathcal{E}}_d^a$ is equivalent to $E_d := \min_{\sigma_{aa'b} \in \mathcal{C}_Q} d(\rho_{ab}, \sigma_{aa'b})$, which is the corresponding entanglement quantification of distance $d$. In other words, geometric entanglement can be linked to discord over state extensions. For measurement-induced geometric discord, however, it is not the trivial case, as shown below.

Definition 3. For $\rho_{ab} \in D(\mathcal{H}_{ab})$, we define a quantity

$$\hat{\mathcal{E}}_d^a(\rho_{ab}) := \min_{\Pi_{aa'} \in \mathcal{C}_Q} \min_{tr_{a'}[\rho_{aa'a'b}] = \rho_{ab}} d(\rho_{aa'a'b}, \Pi_{aa'} \otimes I_b(\rho_{aa'b}))),$$

where the minimal is taken over all extended states $\rho_{aa'a'b}$ and local projection in subsystem $aa'$.

This quantity is related to the previous ones, via the following inequalities.

Theorem 2. For $\rho_{ab} \in D(\mathcal{H}_{ab})$,

$$\mathcal{E}_d^a(\rho_{ab}) \geq \hat{\mathcal{E}}_d^a(\rho_{ab}) \geq E_d(\rho_{ab}),$$

$$\hat{\mathcal{E}}_d^a(\rho_{ab}) \geq \hat{\mathcal{E}}_d^a(\rho_{ab}) = E_d(\rho_{ab}),$$

and

$$\mathcal{E}_d^a(\rho_{ab}) \geq \mathcal{E}_d^a(\rho_{ab}),$$

$$\hat{\mathcal{E}}_d^a(\rho_{ab}) \geq \hat{\mathcal{E}}_d^a(\rho_{ab}) = E_d(\rho_{ab}).$$

Proof. These inequalities can be derived from the definition directly.

B. Characterization with Bures distance discord

Here, we prove two important theorems related to the minimal Bures distance discord, $\mathcal{E}_B^a$, over state extensions. In the following theorem we show that $\mathcal{E}_B^a$ is convex.
Theorem 3. $E_B^n$ is convex,

$$E_B^n(p_{i|j}) \leq \sum_i p_i E_B^n(\rho_{ab}) ,$$

where $p_i$ are probabilities and $\rho_{ab}$ are bipartite states shared between parties $a$ and $b$.

Proof. See Appendix VI A.

Using Theorem 3, we arrive at another theorem below.

Theorem 4. For $\rho_{ab} \in \mathcal{D}(\mathcal{H})$, the minimal Bures distance of discord over state extensions is equivalent to the Bures distance of entanglement.

$$E_B^n(\rho_{ab}) = E_B(\rho_{ab}).$$

Proof. Since $E_B(\rho_{ab}) = E_B^n(\rho_{ab})$ [45], we just need to show that for each $\rho_{ab} \in \mathcal{D}(\mathcal{H})$, we have

$$E_B(\rho_{ab}) \leq E_B^n(\rho_{ab}) \leq E_B(\rho_{ab}).$$

See Appendix VI B for the complete proof.

Having established these results, Theorem 2 for the Bures distance together with Theorem 4 imply

$$E_B^n(\rho_{ab}) \geq \hat{E}_B^n(\rho_{ab}) \geq E_B(\rho_{ab}) = E_B^n(\rho_{ab}) = \hat{E}_B^n(\rho_{ab}).$$

C. Operational meaning of Bures distance of entanglement

The quest for an operational meaning or interpretation of an entanglement measure lies at the very heart of the entanglement theory. While entanglement of formation, distillable entanglement or entanglement cost are measures of entanglement having an operational meaning [1, 3], the Bures distance of entanglement is an entanglement measure defined from the geometric viewpoint and its physical meaning is not clear. By means of the equivalence in Theorem 4, and the operational meaning of Bures distance of discord [46], we provide an operational meaning of the Bures distance of entanglement.

Let us briefly review the ambiguous quantum state discrimination (QSD) protocol [51]. Suppose Alice selects a state $\rho_i$ from a set of states $\{\rho_i\}_{i=1}^N$ with probability $\eta_i$ and sends it to Bob. Bob’s job is to determine which state he has received by performing a positive-operator-valued measure (POVM) on each $\rho_i$, and declares that the state is $\rho_j$ when the measurement outcome reads $j$. The probability to get the result $j$ given state $\rho_i$ is $p(j|i) = tr(M_j \rho_i)$, and the corresponding optimal success probability is

$$P_{\text{opt}}^{(\nu\nu)}(\{\rho_i, \eta_i\}_{i=1}^N) := \max_{\Lambda_i} \sum_i \eta_i tr(M_i \rho_i) ,$$

where the maximization is done over all POVMs. Similarly, $P_{\text{opt}}^{(\nu\nu)}(\{\rho_i, \eta_i\}_{i=1}^N) := \max_{\Lambda_i} \sum_i \eta_i tr(\Pi_i \rho_i)$ is the optimal success probability to discriminate $\{\rho_i, \eta_i\}_{i=1}^N$ with von Neumann measurement.

Based on the operational meaning of the Bures distance of asymmetric discord [46], the connection between geometric entanglement and QSD is as follows.

Corollary. For a bipartite quantum state $\rho_{ab} \in \mathcal{D}(\mathcal{H})$, the square of the maximal fidelity to the set of separable states, $F(\rho_{ab}, S) := \max_{\sigma_{ab} \in S} F(\rho_{ab}, \sigma_{ab})$, is equal to the optimal success probability to discriminate a set of quantum states with von Neumann measurement, i.e.,

$$F^2(\rho_{ab}, S) = \max_{\rho_{ab}, \eta_i} P_{\text{opt}}^{(\nu\nu)}(\{\rho_i, \eta_i\}) ,$$

where $\eta_i = tr(\rho_i |\rho_{ab} \rangle \langle \rho_{ab}|)$, $\rho_i = \eta_i^{-1} \sqrt{\rho_{ab} |\alpha_i \rangle \langle \alpha_i| \rho_{ab} \rangle}$ and the maximum is taken over all possible extended states $\rho_{ab} \rho_{ab}$. von Neumann measurement $\Pi_i$ and orthogonal basis $\{|\alpha_i\rangle\}$ on $\mathcal{H}_{aa'}$.

This offers an operational meaning to the Bures distance of entanglement.

D. Characterization with relative entropy discord

We define relative entropy of discord by

$$D_r(\rho) := \min_{\mu \in \mathcal{Q}} S(\rho || \mu)$$

where $\mathcal{Q}$ is the set of all states. Using Eq. (1) we have

$$D_r(\rho_{ab}) = \min_{\rho_{ab}, \eta_i} S(\rho_{ab} || \sigma_{ab})$$

$$= \min_{\rho_{ab}}[S(\Pi_a(\rho_{ab})) - S(\rho_{ab}) + \min_{\sigma_{ab}} S(\Pi_a(\rho_{ab}) || \sigma_{ab})]$$

$$= \min_{\rho_{ab}} S(\Pi_a(\rho_{ab})) - S(\rho_{ab})$$

$$= \min_{\rho_{ab}} S(\rho_{ab}) || \Pi_a(\rho_{ab}) = D_r^s(\rho_{ab}).$$

Next, for the quantities $E_r, \hat{E}_r, E_r^n, \hat{E}_r^n, \tilde{E}_r$ and $\tilde{E}_r^n$, where symbols have their usual meanings but now for the relative entropy, Theorem 2 and Theorem 5 lead to the following result.

Theorem 6. For a bipartite state $\rho_{ab} \in \mathcal{D}(\mathcal{H})$,

$$\tilde{E}_r^n(\rho_{ab}) = \hat{E}_r^n(\rho_{ab}) \geq E_r(\rho_{ab}) = \tilde{E}_r(\rho_{ab}) = \hat{E}_r(\rho_{ab}).$$

Proof. From Theorem 5, we have

$$E_r^n(\rho_{ab}) = E_r^s(\rho_{ab}),$$

and

$$\tilde{E}_r(n)(\rho_{ab}) = \tilde{E}_r^s(n)(\rho_{ab}) = E_r(\rho_{ab}).$$

The inequality follows from Theorem 2.

Remark 3. In Ref. [28], authors proved that $\tilde{E}_r(\rho_{ab}) = \min_{tr[a(\sigma_{ab} \rangle \langle \sigma_{ab}|)} = \rho_{ab} S(\rho_{ab} || \sigma_{ab}) \geq E_r(\rho_{ab})$, where the minimum is taken over all separable states $\sigma_{ab}$ with $tr[a(\sigma_{ab} \rangle \langle \sigma_{ab}|) = \rho_{ab}$. Our results affirm that they are equal.
Theorem 7. If $D$ is a symmetric discord measure satisfying (D’1-D’3), the minimal discord, $E$, over state extensions has the following desirable and remarkable properties:

- (E’1) $E(\rho_{ab}) \geq 0$ with the equality if and only if $\rho \in CC$.
- (E’2) $E$ is invariant under local unitary transformations.
- (E’3) $E$ is nonincreasing under local partial trace,

$$E(\rho_{ab}) \leq E(\rho_{aa\prime bb\prime}),$$

for any state extension $\rho_{aa\prime bb\prime}$ of $\rho_{ab}$.

(E’4) $E$ is nonincreasing under local operations.

(E’5) $E$ reduces to an entanglement monotone for pure states.

Proof. The proof is similar to that of Theorem 1.

### B. Minimal Bures distance of symmetric discord

In this part, we define the minimal Bures distance of symmetric discord.

Definition 5. For a bipartite state $\rho_{ab} \in \mathcal{H}$, the minimal Bures distance of discord over state extensions is defined as

$$E_B(\rho_{ab}) := \min_{\sigma \in CC} \min_{\rho_{aa\prime bb\prime} = \rho_{ab}} d^2_B(\rho_{aa\prime bb\prime}, \sigma),$$

where the minimum is taken with respect to the bipartition $aa\prime : bb\prime$.

Following, we state that the minimal Bures distance of symmetric discord is convex, and is equivalent to the Bures distance of entanglement.

Theorem 8. $E_B$ is convex,

$$E_B\left(\sum_i p_i \rho_{ab}^i\right) \leq \sum_i p_i E_B(\rho_{ab}^i),$$

where $p_i$ are probabilities and $\rho_{ab}^i$ are bipartite states shared between parties $a$ and $b$.

The proof is similar to that of Theorem 3. With above Theorem, one has the following result.

Theorem 9. For $\rho_{ab} \in D(\mathcal{H})$, $E_B(\rho_{ab}) = E_B(\rho_{ab})$.

The proof is similar to that of Theorem 4.

Remark 4. It is interesting to note that $E_B(\rho_{ab}) = E_B^3(\rho_{ab}) = E_B(\rho_{ab})$, which means that Bures distance of entanglement is equivalent to the minimal Bures distance of discord over state extensions, both on one subsystem and two subsystems.

Remark 5. It was Luo who first proposed to quantify entanglement as the minimal quantum discord over state extensions [29]. Above theorem offers an affirmative evidence that this kind of entanglement quantification is consistent with the previous entanglement measures. However, it is not clear whether this equivalence still holds for other entanglement measures such as entanglement measures based on distances [1] and relative entropy of entanglement [34].

### C. More definitions

Definition 6. For a bipartite state $\rho_{ab} \in D(\mathcal{H})$, the minimal geometric discord over state extensions (GDSE) is defined as

$$E_d(\rho_{ab}) := \min_{\sigma_{aa\prime bb\prime} \in CC} \min_{\rho_{aa\prime bb\prime} = \rho_{ab}} d(\rho_{aa\prime bb\prime}, \sigma_{aa\prime bb\prime}),$$

where the minimum is taken with respect to the bipartition $aa\prime : bb\prime$. Moreover, the minimal measurement-induced geometric discord over state extensions (MIDSE) is defined as

$$E_d(\rho_{ab}) := \min_{\Pi_a, \Pi_b} \min_{\rho_{ab}} d(\rho_{pp(a)\Pi_a \otimes \Pi_b(\rho_{pp(a)})},$$

where the minimum is taken over all local projections in both subsystems $aa\prime$ and $bb\prime$ and along the bipartition $aa\prime : bb\prime$.
Remark 6. For relative entropy or Bures distance, the corresponding geometric discord satisfies \((D')-D'3)\). From Theorem 7, the corresponding quantification is a good candidate of entanglement measure.

Next, let us consider the quantification by performing partial trace,

\[
\hat{E}_d(\rho_{ab}) := \min_{\sigma_{aa'} \in \mathcal{C}_C} d(\rho_{ab}, \text{tr}_{a'b'}(\sigma_{aa'})),
\]

where the minimum is taken over all classical-classical states. It is easy to see that \(\hat{E}_d\) is equivalent to the general geometric entanglement measure \(E_d\), which is defined as the minimal distance between the state and the set of separable states. Nevertheless, it is not the case for the measurement-induced geometric discord, as shown below.

**Definition 7.** For \(\rho_{ab} \in D(H_{ab})\), we define a quantity \(\hat{E}_d'(\rho_{ab}) := \min_{\Pi_{aa'}} \min_{\rho_{a'a'b'}} d(\rho_{ab}, \text{tr}_{a'b'}(\Pi_{aa'} \otimes \Pi_{bb'})),\) where the minimum is taken over all extended states \(\rho_{a'a'b'}\) and local projection in both subsystems \(aa'\) and \(bb'\).

Certainly, we obtain \(\hat{E}_d'(\rho_{ab}) \geq E_d(\rho_{ab})\) for any \(\rho_{ab} \in D(H_{ab})\). Moreover, based on the discussion in Remark 3, the equality holds for relative entropy. And whether the equality holds for other distances as well would be an interesting investigation.

In conclusion, we have the following result.

**Theorem 10.** For \(\rho_{ab} \in D(H_{ab})\), the following inequalities are true.

\[
E'_d(\rho_{ab}) \geq E_d(\rho_{ab}) \geq E_d(\rho_{ab}),
\]

\[
\hat{E}_d(\rho_{ab}) \geq \hat{E}_d(\rho_{ab}) = E_d(\rho_{ab}),
\]

and

\[
E'_d(\rho_{ab}) \geq \hat{E}_d'(\rho_{ab}),
\]

\[
E_d(\rho_{ab}) \geq \hat{E}_d(\rho_{ab}) = E_d(\rho_{ab}).
\]

**Proof.** The inequalities can be derived directly from the definitions. \(\square\)

D. Characterization with Bures distance discord and relative entropy discord

For the Bures distance, Theorem 9 implies

\[
E'_B(\rho_{ab}) \geq E'_B(\rho_{ab}) \geq E_B(\rho_{ab}) = E_B(\rho_{ab}) = \hat{E}_B(\rho_{ab}).
\]

and for relative entropy,

\[
E'_r(\rho_{ab}) = E_r(\rho_{ab}) \geq E_r(\rho_{ab}) = \hat{E}_r'(\rho_{ab}) = \hat{E}_r(\rho_{ab}).
\]

V. CONCLUSION

In this paper, we have proposed a framework to quantify entanglement from the viewpoint of discord, which unifies previous works including [27–29]. Several quantifications of entanglement are introduced based on quantum discord over state extensions.

Especially, for the Bures distance, we prove that the minimal discord (both asymmetric and symmetric) over state extensions is equivalent to the Bures distance of entanglement, which not only establishes an equivalence between this kind of entanglement quantification with existing entanglement measure, but also provides an operational meaning for the Bures distance of entanglement.

Moreover, for relative entropy, by proving that the corresponding discord measure is equivalent to the measurement induced relative entropy of discord, we show that the MIDSE (both asymmetric and symmetric) is equivalent to the relative entropy of entanglement, which reinforces the result in Ref. [28].

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VI. APPENDIX

A. Proof of Theorem 3

Proof. Note that

\[ \rho_{aa'ba'} := \sum_i p_i \rho_{aa'b}^i \otimes |i\rangle_{a'} \langle i| \]

is a state extension of \( \rho_{ab} = \sum_i p_i \rho_{a'b}^i \) whenever \( \rho_{a'a'b}^i \) is a state extension of \( \rho_{aa'b}^i \) for all \( i \). Without loss of generality, suppose \( \rho_{a'a'b}^i \) is the optimal state extension of \( \rho_{aa'b}^i \) for each \( i \), and \( \sigma^*_a \) is the corresponding closest classical-quantum state.
Then, we have
\[
\sum_i p_i \mathcal{E}_B^a(\rho_{ab}^i) = \sum_i p_id_B^a(\rho_{ab}^{i*}, \sigma_i^* ) \\
= \sum_i p_i d_B^a(\rho_{ab}^{i*} \otimes |i\rangle_a \langle i|, \sigma_i^* \otimes |i\rangle_a \langle i|) \\
\geq d_B^a(\sum_i p_i \rho_{ab}^{i*} \otimes |i\rangle_a \langle i|, \sum_i p_i \sigma_i^* \otimes |i\rangle_a \langle i|) \\
\geq d_B^a(\sum_i p_i \rho_{ab}^{i*} \otimes |i\rangle_a \langle i|) \\
\geq \mathcal{E}_B^a(\sum_i p_i \rho_{ab}^{i})
\]
where the first inequality follows from the joint convexity of \(d_B^a\) and the second inequality is based on the definition of \(D_B^a\) and the fact that \(\sum_i p_i \sigma_i^* \otimes |i\rangle_a \langle i|\) is a classical-quantum state along the bipartition \(a\alpha'' : b\). The last inequality follows because \(\sum_i p_i \rho_{ab}^{i} \otimes |i\rangle_a \langle i|\) is a state extension of \(\sum_i p_i \rho_{ab}^{i}\).

\[\Box\]

B. Proof of Theorem 4

Here we restate the claim to be proved. That is, for each \(\rho_{ab} \in D(\mathcal{H})\),
\[
E_B(\rho_{ab}) \leq \mathcal{E}_B^a(\rho_{ab}) \leq E_B^c(\rho_{ab}).
\]

**Proof.** Suppose \(\rho_{ab}^{i*}\) is the optimal state extensions of \(\rho_{ab}\) and \(\sigma^*\) is the corresponding closest classical-quantum state. Then
\[
\mathcal{E}_B^a(\rho_{ab}) = d_B^a(\rho_{ab}^{i*}, \sigma^*) \geq d_B^a(\rho_{ab}, tr_{a'}\sigma^*) \geq E_B(\rho_{ab}).
\]

The first \(\geq\) is the result of the contractibility of Bures distance and the second \(\geq\) is because \(tr_{a'}\sigma^*\) is a separable state. In fact, if \(\sigma^* = \sum_i p_i |\alpha_i\rangle_{a'} \langle \alpha_i| \otimes \rho_i\), then tracing out the subsystem \(a'\) will lead to a decomposition of the form \(tr_{a'}\sigma^* = \sum_{i,k} p_{ik} \lambda_k |x_{ik}^i\rangle_a \langle x_{ik}^i| \otimes \rho_i\), that must be a separable state.

In particular, let us consider the pure state case. Suppose \(|\psi\rangle = \sum_i \sqrt{\lambda_i} |x_i\rangle_a |y_i\rangle_b\) with \(\lambda_1 \geq \ldots \geq \lambda_n\). Then
\[
\mathcal{E}_B^a(|\psi\rangle) \leq \min_{\sigma_{a''aa'\rho} \in CQ} d_B^a(|\psi\rangle \otimes |u\rangle_{a'} \langle u|, \sigma_{a''aa'\rho} ) \\
\leq d_B^a(|\psi\rangle \otimes |u\rangle_{a'} \langle u| ) \\
\leq E_B(|\psi\rangle).
\]

Combining the above two results, one has \(\mathcal{E}_B^a(|\psi\rangle) = E_B(|\psi\rangle)\) for any pure state \(|\psi\rangle\).

On the other hand, for any mixed state with pure state decomposition \(\rho_{ab} = \sum_i p_i |\psi_i\rangle \langle \psi_i|\), Theorem 1 tells us that
\[
E_B(\rho_{ab}) \leq \sum_i p_i E_B(|\psi_i\rangle) = \sum_i p_i E_B(\rho_{ab}).
\]

Taking the minimum over all pure state decompositions,
\[
\mathcal{E}_B^a(\rho_{ab}) \leq E_B^c(\rho_{ab}).
\]

\[\Box\]

C. Symbols and meanings

| Symbol | Explanation |
|--------|-------------|
| \(S\)  | set of separable states |
| \(CQ\) | set of classical-quantum states |
| \(CC\) | set of classical-classical states |
| \((D^a)\ D\) | (a)symmetric discord |
| \((D_B^a)\ D_B\) | (a)symmetric Bures distance of discord |
| \(d_B\) | Bures distance |
| \(E_d\) | geometric entanglement |
| \(E_B\) | Bures distance of entanglement |
| \(E_B\) | convex roof of Bures distance of entanglement |
| \((\mathcal{E}^a)\ \mathcal{E}\) | minimal (a)symmetric DSE |
| \((\mathcal{E}_B^a)\ \mathcal{E}_B\) | minimal (a)symmetric Bures distance of DSE |
| \((\mathcal{E}_B^a)\ \mathcal{E}_d\) | minimal (a)symmetric DSE |
| \((\mathcal{E}_B^c)\ \mathcal{E}_d\) | minimal (a)symmetric MIDSE |
| \((\mathcal{E}_d^a)\ \mathcal{E}_d\) | minimal partial trace of (a)symmetric DSE |
| \((\mathcal{E}_d^c)\ \mathcal{E}_d\) | minimal partial trace of (a)symmetric MIDSE |

**Table I.** Symbols and their meanings. Here DSE stands for discord over state extensions and MIDSE is for measurement-induced geometric discord over state extensions.