Bethe ansatz solution for one-dimensional directed polymers in random media

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Abstract. We study the statistical properties of one-dimensional directed polymers in a short-range random potential by mapping the replicated problem to a many-body quantum boson system with attractive interactions. We find the full set of eigenvalues and eigenfunctions of the many-body system and perform the summation over the entire spectrum of excited states. The analytic continuation of the obtained exact expression for the replica partition function from integer to non-integer replica parameter $N$ turns out to be ambiguous. Performing the analytic continuation simply by assuming that the parameter $N$ can take arbitrary complex values, and going to the thermodynamic limit of the original directed polymer problem, we obtain the explicit universal expression for the probability distribution function of free energy fluctuations.

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1. Introduction

Directed polymers in a quenched random potential have been the subject of intense investigations during the past two decades [1]. Diverse physical systems such as domain walls in magnetic films [2], vortices in superconductors [3], wetting fronts on planar systems [4] or Burgers turbulence [5] can be mapped to this model, which exhibits numerous non-trivial features derived from the interplay between elasticity and disorder. The best understanding has been achieved for a string confined to a plane. In this case we deal with an elastic string directed along the $\tau$ axis within an interval $[0, L]$. Randomness enters the problem through a disorder potential $V[\phi(\tau), \tau]$, which competes against the elastic energy. The problem is defined by the Hamiltonian

$$H[\phi(\tau), V] = \int_0^L d\tau \left\{ \frac{1}{2} \partial_\tau \phi(\tau)^2 + V[\phi(\tau), \tau] \right\};$$

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where in the simplest case the disorder potential $V[\phi, \tau]$ is Gaussian-distributed with a zero mean $\overline{V(\phi, \tau)} = 0$ and the $\delta$ correlations:

$$\overline{V(\phi, \tau)V(\phi', \tau')} = u\delta(\tau - \tau')\delta(\phi - \phi').$$

(2)

Here the parameter $u$ describes the strength of the disorder. Historically, the problem of central interest was the scaling behavior of the polymer mean squared displacement which, in the thermodynamic limit ($L \to \infty$), is believed to have a universal scaling form:

$$\overline{\langle \phi^2 \rangle}(L) \propto L^{2\zeta}$$

(3)

(where $\langle \cdots \rangle$ and $\langle \cdots \rangle$ denote the thermal and the disorder averages), with $\zeta$ the so-called wandering exponent. A more general problem for all directed polymer systems of the type, equation (1), is the statistical properties of their free energy fluctuations. Besides the usual extensive (linear in $L$) self-averaging part $f_0L$ (where $f_0$ is the linear free energy density), the total free energy $F$ of such systems contains a disorder-dependent fluctuating contribution $\tilde{F}$, which is characterized by non-trivial scaling in $L$. It is generally believed (at least in the systems with short-ranged correlations) that in the limit of large $L$ the typical value of the free energy fluctuations scales with $L$ as

$$\tilde{F} \propto L^\omega,$$

(4)

i.e. they are characterized by a single universal exponent $\omega < 1$. In other words, in the limit of large $L$ the total (random) free energy of the system can be represented as

$$F = f_0L + \tilde{f}L^\omega$$

(5)

where $\tilde{f}$ is the random quantity which in the thermodynamic limit $L \to \infty$ is described by a non-trivial universal distribution function $P_r(\tilde{f})$. The derivation of this function for the system with $\delta$-correlated random potential, equations (1) and (2), is the central issue of the present work.

One can easily note that the above two exponents $\zeta$ and $\omega$ are not independent. Indeed, since the free energy fluctuation $\tilde{F} \sim L^\omega$ can be estimated by the typical value of the elastic energy, $\tilde{F} \propto \phi^2/L$, where, according to equation (3), the typical deviation $\phi \sim L^\zeta$, one finds $\omega = 2\zeta - 1$.

It is generally believed that, for all short-range correlated disorder potentials, the free energy fluctuations exponent assumes a universal value $\omega = 1/3$. Numerical studies [6] as well as the solution via mapping to the Burgers equation [7] confirm this conjecture. One arrives at the same conclusion studying scaling properties of the free energy by mapping the replicated problem to an $N$-particle quantum boson system [8] and using the Bethe ansatz solution. However, in this latter case, the resulting distribution function $P_r(\tilde{f})$ exhibits severe pathologies such as the vanishing of its second moment, which assumes that the distribution function is not positively defined.

Let us consider this point in more detail. For the string with the zero boundary conditions at $\tau = 0$ and at $L$ the partition function of a given sample is

$$Z[V] = \int_{\phi(0)=0}^{\phi(L)=0} D[\phi(\tau)] e^{-\beta H[\phi,V]}$$

(6)

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where $\beta = 1/T$ denotes the inverse temperature. On the other hand, the partition function is related to the total free energy $F[V]$ via

$$Z[V] = \exp(-\beta F[V]).$$

The free energy $F[V]$ is defined for a specific realization of the random potential $V$ and thus represents a random variable. Let us take the $N$th power of both sides of equation (7) and perform the averaging over the random potential $V$:

$$Z[N, L] = \langle \exp(-\beta NF[V]) \rangle.$$

(8)

The quantity in the lhs of the above equation:

$$Z[N, L] \equiv Z^N[V]$$

(9)

is called the replica partition function, and it is defined originally for an arbitrary integer parameter $N$. Substituting $F = f_0L + \tilde{f}L^\omega$ into equation (8) and redefining

$$Z[N, L] = \tilde{Z}[N, L]e^{-\beta N f_0 L}$$

we get

$$\tilde{Z}[N, L] = \langle \exp(-\beta NL^\omega \tilde{f}) \rangle.$$

(10)

The averaging in the rhs of the above equation can be represented in terms of the distribution function $P_L(\tilde{f})$ (which depends on the system size $L$). In this way we arrive at the following general relation between the replica partition function $\tilde{Z}[N, L]$ and the distribution function of the free energy fluctuations $P_L(\tilde{f})$:

$$\tilde{Z}[N, L] = \int_{-\infty}^{+\infty} d\tilde{f} P_L(\tilde{f})e^{-\beta NL^\omega \tilde{f}}.$$

(12)

The above equation is the bilateral Laplace transform of the function $P_L(\tilde{f})$, and at least formally it allows us to restore this function in terms of the replica partition function $\tilde{Z}[N, L]$. In order to do so we have to compute $\tilde{Z}[N, L]$ for an arbitrary integer $N$ and then perform an analytical continuation of this function from integer to arbitrary complex values of $N$. Introducing a new complex variable

$$\tilde{s} = \beta NL^\omega$$

(13)

and denoting

$$\tilde{Z} \left[ \frac{\tilde{s}}{\beta L^\omega} , L \right] \equiv \tilde{Z}_L(\tilde{s})$$

(14)

we could reconstruct the distribution function $P_L(\tilde{f})$ via the inverse Laplace transform:

$$P_L(\tilde{f}) = \int_{-\infty}^{+\infty} \frac{d\tilde{s}}{2\pi i} \tilde{Z}_L(\tilde{s})e^{\tilde{s}\tilde{f}}.$$

(15)

Finally, provided there exists a finite thermodynamic limit function

$$\lim_{L \to \infty} \tilde{Z}_L(\tilde{s}) \equiv \tilde{Z}_*(\tilde{s})$$

(16)
we can find the distribution function

\[ P_\ast(\tilde{f}) = \int_{-\infty}^{+\infty} \frac{d\tilde{s}}{2\pi i} \tilde{Z}_\ast(\tilde{s}) e^{\tilde{s}\tilde{f}}, \]  

(17)

which would describe the statistics of the rescaled free energy fluctuations \( \tilde{f} \) in the infinite system. The above equation defining \( P_\ast(\tilde{f}) \) contains no parameters and hence is expected to be universal. Therefore according to the relation \( \tilde{s} = \beta NL^\omega \) we see that in the thermodynamic limit the relevant values of the original replica parameter are

\[ N \sim L^{-\omega} \rightarrow 0. \]  

(18)

This explains why the two limits \( L \rightarrow \infty \) and \( N \rightarrow 0 \) do not commute [9], and the approximation of the replica partition function through the ground state wavefunction fails (see also [10]). In Kardar’s original solution [8], after mapping the replicated problem to interacting quantum bosons, one arrives at the replica partition function for positive integer parameters \( N > 1 \). Assuming a large \( L \rightarrow \infty \) limit, one is tempted to approximate the result by the ground state contribution only, as for any \( N > 1 \) the contributions of excited states are exponentially small for \( L \rightarrow \infty \). However, in the analytic continuation for arbitrary complex \( N \) the contributions which are exponentially small at positive integer \( N > 1 \) can become essential in the region \( N \rightarrow 0 \), which according to equation (17) defines the function \( P_\ast(\tilde{f}) \). Thus, it is the neglect of the excited states which is the origin of the non-physical nature of the obtained solution. In other words, for the proper analytic continuation of the replica solution to the region \( N \rightarrow 0 \), first one has to calculate the replica partition function \( Z[N,L] \) exactly for arbitrary integer \( N \), and only after that one can take the thermodynamic limit \( L \rightarrow \infty \) while keeping the value of the parameter \( \tilde{s} = \beta NL^\omega \) finite.

In the present paper, we report the results of the calculation of the replica partition function \( Z(N,L) \) for arbitrary integer \( N \) which, in terms of the Bethe ansatz solution for quantum bosons with attractive \( \delta \) interactions (sections 2 and 3), involves the summation over the entire spectrum of excited states (section 4). Unfortunately the analytic continuation of the obtained exact expression \( Z(N,L) \) from integer to non-integer \( N \) turns out to be ambiguous, since our replica partition function grows as \( \exp(N^3) \) at large \( N \) (a similar problem to the analytic continuation to the region \( N \rightarrow 0 \) one faces in the replica theory of the mean-field spin glasses where the replica partition function grows as \( \exp(N^2) \)). Performing a kind of a ‘replica symmetric’ analytic continuation, i.e. just assuming that the originally integer-value parameter \( N \) can take arbitrary complex values and taking the thermodynamic limit \( L \rightarrow \infty \) in \( Z(N,L) \) (section 5), allows us to compute an inverse Laplace transformation, cf equation (17), which provides us with the explicit expression for the distribution function of the free energy fluctuations (section 6, equation (102)). Although up to the present moment, we have not uncovered any unphysical properties in the obtained probability function \( P_\ast(f) \), this solution could be considered as a distant analog of the ‘replica symmetric approximation’ in the mean-field spin glasses. In particular, it should be noted that our result is different from the Tracy–Widom distribution [11], which describes the statistics of fluctuations in various statistical systems [12]–[17] which are widely believed to belong to the same universality class as the present model [18]–[20] (for further discussion of this issue see section 7).

5 We thank B Derrida for turning our attention to the issue of analytic continuation.
Various technical aspects of the calculations are moved to the appendices. In particular, in appendices A–C, we analyze the structure and properties of \( N \)-particle wavefunctions of one-dimensional quantum bosons, both with repulsive and with attractive interactions.

2. Mapping to quantum bosons

Explicitly, the replica partition function, equation (9), of the system described by the Hamiltonian, equation (1), is

\[
Z(N,L) = \prod_{a=1}^{N} \int_{\phi_a(L)=0}^{\phi_a(0)=0} D\phi_a(\tau) \exp \left[ -\beta \int_{0}^{L} d\tau \sum_{a=1}^{N} \{ \frac{1}{2} [\partial_\tau \phi_a(\tau)]^2 + V[\phi_a(\tau),\tau] \} \right].
\] (19)

Since it is assumed that the random potential \( V[\phi,\tau] \) has the Gaussian distribution the disorder average \( \langle \cdots \rangle \) in the above equation is very simple:

\[
\exp \left[ -\beta \int_{0}^{L} d\tau \sum_{a=1}^{N} V[\phi_a(\tau),\tau] \right] = \exp \left[ \frac{\beta^2}{2} \int_{0}^{L} d\tau \int_{0}^{L} d\tau' \sum_{a,b=1}^{N} V[\phi_a(\tau),\tau]V[\phi_b(\tau'),\tau'] \right].
\] (20)

Using equation (2) we have

\[
Z(N,L) = \prod_{a=1}^{N} \int_{\phi_a(L)=0}^{\phi_a(0)=0} D\phi_a(\tau)
\times \exp \left[ -\frac{1}{2} \beta \int_{0}^{L} d\tau \left\{ \sum_{a=1}^{N} [\partial_\tau \phi_a(\tau)]^2 \right. \right.
\left. - \beta u \sum_{a \neq b=1}^{N} \delta [\phi_a(\tau) - \phi_b(\tau)] \right\} \right].
\] (21)

It should be noted that the second term in the exponential of the above equation contains formally divergent contributions proportional to \( \delta(0) \) (due to the terms with \( a = b \)). In fact, this is just an indication that the continuous model, equations (1) and (2), is ill defined at short distances and requires proper lattice regularization. Of course, the corresponding lattice model would contain no divergences, and the terms with \( a = b \) in the exponential of the corresponding replica partition function would produce irrelevant constant \( \frac{1}{2} L \beta^2 u N \delta(0) \) (where the lattice version of \( \delta(0) \) has a finite value). Since the lattice regularization has no impact on the continuous long distance properties of the considered system this term will just be omitted in our further study.

Introducing the \( N \)-component scalar field replica Hamiltonian

\[
H_N[\phi] = \frac{1}{2} \int_{0}^{L} d\tau \left( \sum_{a=1}^{N} [\partial_\tau \phi_a(\tau)]^2 \right. \right.
\left. - \beta u \sum_{a \neq b}^{N} \delta [\phi_a(\tau) - \phi_b(\tau)] \right)
\] (22)

for the replica partition function, equation (21), we obtain the standard expression:

\[
Z(N,L) = \prod_{a=1}^{N} \int_{\phi_a(L)=0}^{\phi_a(0)=0} D\phi_a(\tau) e^{-\beta H_N[\phi]}
\] (23)

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where \( \phi \equiv \{ \phi_1, \ldots, \phi_N \} \). According to the above definition this partition function describe the statistics of \( N \delta \)-interacting (attracting) trajectories \( \phi_a(\tau) \) all starting (at \( \tau = 0 \)) and ending (at \( \tau = L \)) at zero: \( \phi_a(0) = \phi_a(L) = 0 \).

In order to map the problem to one-dimensional quantum bosons, let us introduce a more general object:

\[
\Psi(x; t) = \prod_{a=1}^{N} \int_{\phi_a(0) = 0}^{\phi_a(t) = x_a} D\phi_a(\tau) e^{-\beta H_N[\phi]} \tag{24}
\]

which describes \( N \) trajectories \( \phi_a(\tau) \) all starting at zero (\( \phi_a(0) = 0 \)), but ending at \( \tau = t \) in arbitrary given points \( \{ x_1, \ldots, x_N \} \). One can easily show that, instead of using the path integral, \( \Psi(x; t) \) may be obtained as the solution of the linear differential equation:

\[
\partial_t \Psi(x; t) = \frac{1}{2\beta} \sum_{a=1}^{N} \delta^2_{x_a} \Psi(x; t) + \frac{1}{2} \beta^2 u \sum_{a \neq b}^{N} \delta(x_a - x_b) \Psi(x; t) \tag{25}
\]

with the initial condition

\[
\Psi(x; 0) = \prod_{a=1}^{N} \delta(x_a). \tag{26}
\]

One can easily see that equation (25) is the imaginary-time Schrödinger equation:

\[
-\partial_t \Psi(x; t) = H \Psi(x; t) \tag{27}
\]

with the Hamiltonian

\[
\hat{H} = -\frac{1}{2\beta} \sum_{a=1}^{N} \delta^2_{x_a} - \frac{1}{2} \beta^2 u \sum_{a \neq b}^{N} \delta(x_a - x_b) \tag{28}
\]

which describes \( N \) Bose particles of mass \( \beta \) interacting via the attractive two-body potential \( -\beta^2 u \delta(x) \). The original replica partition function, equation (23), is then obtained via a particular choice of the final-point coordinates:

\[
Z(N, L) = \Psi(0; L). \tag{29}
\]

The standard general strategy of the further calculations is in the following. Let us denote the eigenfunctions of the Hamiltonian equation (28) by \( \Psi_\xi(x) \), where the index \( \xi \) (which can be both integer and continuous) labels the eigenstates. Provided the wavefunctions \( \Psi_\xi(x) \) constitute the orthonormal and complete set, the time-dependent solution of equation (25) with the initial conditions, equation (26), is given by

\[
\Psi(x; t) = \sum_\xi \Psi_\xi(x) \Psi_\xi^*(0) e^{-E(\xi)t} \tag{30}
\]

where \( E(\xi) \) denotes the energy of the \( \xi \)th eigenstate:

\[
\hat{H} \Psi_\xi(x) = E(\xi) \Psi_\xi(x). \tag{31}
\]

Then, according to equation (29), the replica partition function \( Z(N, L) \) of the original polymer system is obtained just by the summation over all eigenstates of the quantum Hamiltonian (28):

\[
Z(N, L) = \sum_\xi |\Psi_\xi(0)|^2 e^{-E(\xi)L} \tag{32}
\]

Thus, the crucial point of the present approach is finding the eigenfunctions and the energy spectrum of the Hamiltonian (28), which is the topic of section 3.
3. Eigenstates of the one-dimensional quantum boson system

3.1. Repulsive bosons

The eigenfunctions of one-dimensional δ-interacting repulsive \((u < 0)\) quantum bosons, equation (28), have been derived by Lieb and Liniger in 1963 [21]. An eigenstate of this system is characterized by \(N\) continuous momenta \(\{q_1, \ldots, q_N\} \equiv q\) with the wavefunction (see appendix A)

\[
\Psi_q^{(N)}(x) = C^{(N)}(q) \sum_P (-1)^{|P|} \left( \prod_{a<b}^{N} ((q_a - q_b) + i\kappa \text{sgn}(x_a - x_b)) \right) \exp \left[ i \sum_{a=1}^{N} q_a x_a \right] \tag{33}
\]

where we have introduced the notation

\[
\kappa = \beta^3 u. \tag{34}
\]

The summation in equation (33) goes over all permutations \(P\) of the \(N\) momenta \(\{q_1, \ldots, q_N\}\) over \(N\) particles located at \(\{x_1, \ldots, x_N\}\) and \([P]\) denotes the parity of the permutation. The normalization constant \(C^{(N)}(q)\) is

\[
C^{(N)}(q) = \frac{1}{\sqrt{N! \prod_{a<b}^{N} (q_a - q_b)^2 + \kappa^2}} \tag{35}
\]

and the associated energy \(E_N(q)\) is

\[
E_N(q) = \frac{1}{2\beta} \sum_{a=1}^{N} q_a^2. \tag{36}
\]

A useful alternative representation of these wavefunctions is

\[
\Psi_q^{(N)}(x) = C^{(N)}(q) \sum_P (-1)^{|P|} \left( \prod_{a<b}^{N} [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right) \exp \left[ i \sum_{a=1}^{N} q_a x_a \right] \tag{37}
\]

\[
= C^{(N)}(q) \left( \prod_{a<b}^{N} [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right) \det[\exp(i\kappa x)] \tag{38}
\]

where the symbol \(\exp(i\kappa x)\) denotes the \(N \times N\) matrix with the elements \(\exp(iq_a x_b)\) \((a, b = 1, \ldots, N)\). Here, by definition, the differential operators \(\partial_{x_a}\) act only on the exponential terms and \(not\) on the signum functions \(\text{sgn}(x_a - x_b)\). One can easily see that the above wavefunctions \(\Psi_q^{(N)}(x)\) represent a set of plane waves in any sector of the type \(x_{a_1} < x_{a_2} < \cdots < x_{a_N}\) with a finite jump (equal to \(\kappa\)) of the derivatives at all ‘boundary’ points \(x_{a_i} = x_{a_j}\). These functions are symmetric with respect to any permutation of the particle coordinate \(\{x_1, \ldots, x_N\}\) and \(antisymmetric\) with respect to permutations of the momenta \(\{q_1, \ldots, q_N\}\). It can be proven that the wavefunctions, equation (33), are orthonormal and constitute the complete set (see appendix A; one can find a detailed discussion, for example, in [22, 23]). Specifically, for any two functions \(\Psi_q^{(N)}(x)\) and \(\Psi_{q'}^{(N)}(x)\) considered in the sectors \(q_1 < q_2 < \cdots < q_N\) and \(q'_1 < q'_2 < \cdots < q'_N\),

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orthonormality implies that
\[
\int_{-\infty}^{+\infty} dx_1 \cdots dx_N \Psi_{q'}^{(N)^*}(x)\Psi_{q}^{(N)}(x) = (2\pi)^N \delta(q_1 - q_1)\delta(q_2 - q_2) \cdots \delta(q_N - q_N). \tag{39}
\]
Similarly, for any two functions \(\Psi_{q}^{(N)}(x)\) and \(\Psi_{q'}^{(N)}(x')\) considered in the sectors \(x_1 < x_2 < \cdots < x_N\) and \(x_1' < x_2' < \cdots < x_N'\), completeness implies that
\[
\int_{-\infty}^{+\infty} dq_1 \cdots dq_N \Psi_{q}^{(N)*}(x)\Psi_{q}^{(N)}(x') = (2\pi)^N \delta(x_1 - x_1')\delta(x_2 - x_2') \cdots \delta(x_N - x_N'). \tag{40}
\]
Thus, for a repulsive interaction \((u < 0)\) the time-dependent solution of the differential equation (25) (with the starting condition, equation (26)) would be sufficiently simple:
\[
\Psi^{(N)}(x; t) = \int_{q_1 < \cdots < q_N} dq_1 \cdots dq_N \Psi_{q}^{(N)*}(x)\Psi_{q}^{(N)}(0) \exp \left[-\frac{t}{2\beta} \sum_{a=1}^{N} q_a^2\right]. \tag{41}
\]

Unfortunately, from the point of view of the replica theory of disordered polymers the repulsive bosons make no physical meaning since the parameter \(u\) (according to its definition, equation (2)) is positively defined.

### 3.2. Attractive bosons

The situation with attractive \((u > 0)\) bosons is more complicated. One can easily prove that (irrespective of the sign of the parameter \(u\)) the functions \(\Psi_{q}^{(N)}(x)\), equation (33), are orthonormal eigenfunctions of the Hamiltonian, equation (28). However, unlike the repulsive case, this set of functions is not complete. In other words, for \(u > 0\) the completeness conditions, equation (40), are not satisfied. Physically this indicates that, besides the continuous spectrum (or free particles) states, our system must have another type of eigenstate, in which the particles are bound into localized clusters. The spectrum and some properties of the eigenfunctions for attractive one-dimensional quantum bosons have been derived by McGuire [24] and Yang [25] (see also [26,27]). However, since attractive bosons do not have a proper thermodynamic limit (in the number of particles \(N \rightarrow \infty\)) due to the scaling \(E_N \propto -N^3\), the interest in this system has been rather limited.

We first consider the ground state wavefunction \(\Psi_{q}^{(1)}(x)\) in which all \(N\) particles are bound into one cluster with the free center-of-mass motion controlled by the momentum \(q\) (see appendix B):
\[
\Psi_{q}^{(1)}(x) = C^{(1)}(q) \exp \left[iq \sum_{a=1}^{N} x_a - \frac{1}{\kappa} \sum_{a \neq b}^{N} [x_a - x_b]\right] \tag{42}
\]
where
\[
C^{(1)}(q) = \sqrt{\frac{\kappa^N N!}{\kappa N}} \tag{43}
\]
is the normalization constant defined by the orthonormality condition:
\[
\int_{-\infty}^{+\infty} dx_1 \cdots dx_N \Psi_{q}^{(1)}(x)\Psi_{q'}^{(1)*}(x) = (2\pi)\delta(q - q'). \tag{44}
\]

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The energy of this state is
\[ E_1(q; N) = \frac{N}{2\beta}q^2 - \frac{\kappa^2}{24\beta}(N^3 - N). \] (45)

On the other hand, one can also easily prove that the above ground state wavefunction, equation (42), can be represented in a form similar to the free particle structure, equation (33), by introducing (discrete) imaginary parts for the momenta \( q_a \). Indeed, due to the symmetry of this function with respect to permutations of \( \{x_1, x_2, \ldots, x_N\} \) it is sufficient to consider it in the sector \( x_1 < x_2 < \cdots < x_N \). Defining
\[ q_a = q - \frac{i}{2}\kappa(N + 1 - 2a) \] (46)
and substituting these momenta into the general expression for the wavefunction, equation (33) one easily recovers equation (42) (see appendix B for details). Also, substituting equation (46) into the general expression for the energy spectrum, equation (36), one can also recover equation (45).

Now, using the above scheme, one can construct the eigenfunctions of a generic excited state. It consists of \( M (1 \leq M \leq N) \) ‘clusters’ \( \{\Omega_\alpha\} \) of bound particles, where \( \alpha = 1, \ldots, M \) labels a given cluster. Each cluster is characterized by the momentum \( q_\alpha \) of its center-of-mass motion, and by the number \( n_\alpha \) of particles contained in it (such that \( \sum_{\alpha=1}^{M} n_\alpha = N \)). Instead of \( N \) independent real momenta \( q_a \ (a = 1, \ldots, N) \) one introduces \( M \) complex ‘vector’ momenta:
\[ q_\alpha \rightarrow q_\alpha^\alpha = q_\alpha - \frac{i}{2}\kappa(n_\alpha + 1 - 2r) \] (47)
where \( r = 1, 2, \ldots, n_\alpha \) and
\[ \sum_{\alpha=1}^{M} n_\alpha = N. \] (48)

The corresponding wavefunction \( \Psi_{q,n}^{(M)}(x_1, \ldots, x_N) \) is characterized by \( M \) continuous parameters \( q = (q_1, \ldots, q_M) \) (which are the momenta of the center-of-mass motion of the clusters) and \( M \) integer parameters \( n = (n_1, \ldots, n_M) \) (which are the numbers of particles of each cluster). Explicitly this wavefunction is given by equation (38) where the \( N \times N \) matrix \( \exp(iq\mathbf{x}) \) is now composed of \( N \) columns:
\[ \{q_1^1, q_1^2, \ldots, q_1^n_1; q_2^1, q_2^2, \ldots, q_2^n_2; \ldots; q_M^1, q_M^2, \ldots, q_M^n_M\} \] (49)
and \( N \) rows \( x_a \). One can easily see that, as in the case of repulsive bosons, this wavefunction is symmetric with respect to the permutation of particle coordinates. However, for practical applications the general representation, equation (38), is not very convenient. Writing the determinant of the matrix \( \exp(iq\mathbf{x}) \) explicitly, after a few efforts in simple algebra one can derive a more transparent structure of the wavefunction (see appendix C for details). Assuming first that the position of particles are ordered, \( x_1 < x_2 < \cdots < x_N \), let us consider a permutation \( P \) of \( N \) momenta \( q_\alpha^\alpha \), equation (49), over \( N \) particles \( x_a \).
so that a particle number \( a \) is attributed a momentum component \( q_r(a) \). The particles getting the momenta with the same \( \alpha \) (having the same real part \( q_\alpha \)) will be termed as belonging to a cluster \( \Omega_\alpha \). For a given permutation \( P \) the particles belonging to the same cluster are numbered by the ‘internal’ index \( r = 1, \ldots, n_\alpha \). From now on we have to take into account not all the permutations, but only those for which the internal particle number \( r(a) \) is the growing function of the particle number \( a \) in every cluster. Namely, let a cluster \( \Omega_\alpha \) consist of the particles \( x_{a_1}, x_{a_2}, \ldots, x_{a_{n_\alpha}} \) (where \( a_1 < a_2 < \cdots < a_{n_\alpha} \) and the positions of particles are ordered: \( x_{a_1} < x_{a_2} < \cdots < x_{a_{n_\alpha}} \)). Then, among all \( n_\alpha \) ‘internal’ permutations of the momenta components \( q_1^\alpha, q_2^\alpha, \ldots, q_{n_\alpha}^\alpha \), the non-zero contribution is given only by the one in which \( r(a_{i+1}) = r(a_i) + 1 \). In this case the explicit form of the wavefunction (for \( M \geq 2 \)) is (see appendix C)

\[
\Psi_{q,n}^{(M)}(x) = C_{q,n}^{(M)} \sum_P (-1)^{|P|} \prod_{a \in \Omega_\alpha} \left[ q_\alpha(a) - \frac{i}{2} \kappa (n_\alpha(a) + 1 - 2r(a)) \right] - \left( q_\alpha(b) - \frac{i}{2} \kappa (n_\alpha(b) + 1 - 2r(b)) \right) - i \kappa \right] 
\times \exp \left[ \sum_{a=1}^{n_\alpha} q_\alpha(a) x_a + \frac{\kappa}{2} \sum_{a=1}^{n_\alpha} (n_\alpha(a) + 1 - 2r(a)) x_a \right]
\]

(50)

where the product goes only over pairs of particles belonging to different clusters and the symbol \( \sum_P \) means that the summation goes only over the permutations \( P \) in which the ‘internal’ indices \( r(a) \) are ordered inside each cluster.

For generic positions of the particles (beyond the sector \( x_1 < x_2 < \cdots < x_N \)) the expression for the wavefunction reduces to

\[
\Psi_{q,n}^{(M)}(x) = C_{q,n}^{(M)} \sum_P (-1)^{|P|} \prod_{a \in \Omega_\alpha} \left[ q_\alpha(a) - q_\alpha(b) + \frac{i \kappa}{2} \sum_{c \in \Omega_\alpha} \text{sgn}(x_a - x_c) \right] 
- \frac{i \kappa}{2} \sum_{c \in \Omega_\alpha} \text{sgn}(x_b - x_c) + i \kappa \text{sgn}(x_a - x_b) \right] 
\times \exp \left[ \sum_{a=1}^{n_\alpha} q_\alpha(a) x_a - \frac{\kappa}{4} \sum_{a=1}^{n_\alpha} \sum_{a' \in \Omega_\alpha} |x_a - x_{a'}| \right].
\]

(51)

Here the summation only goes over the permutations in which ‘internal’ indices \( r(a) \) in the clusters are ordered according to the spatial ordering of the particles belonging to these clusters. For example, let a cluster \( \Omega_\alpha \) be composed of the particles \( \{x_{a_1}, x_{a_2}, \ldots, x_{a_{n_\alpha}}\} \) and the spatial positions of these particles are such that \( x_{a_1} < x_{a_2} < \cdots < x_{a_{n_\alpha}} \), while the particles numbering \( a_1, a_2, \ldots, a_{n_\alpha} \) are now arbitrary. Then the ordering of the internal index \( r(a) \) of the permutations involved in equation (51) is such that \( r(a_{i+1}) = r(a_i) + 1 \) (i.e. it goes from the smallest \( x_a \) in the cluster to the largest one).

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According to equation (38), the permutation of any two momenta $M$ wavefunction of unbound particles (the case $M = N$ and $n_1 = n_2 = \cdots = n_N = 1$) is fully antisymmetric with respect to any momenta permutation. For the generic case, $2 \leq M < N$ the situation is slightly more tricky as the symmetry with respect to the momenta permutations depends on the values of the corresponding integer parameters $n_\alpha$. According to equation (38), the permutation of any two momenta $q_{a_1}$ and $q_{a_2}$ belonging to the clusters which have the same numbers of particles, $n_{a_1} = n_{a_2} = n$, produces the factor $(-1)^n$ (this operation corresponds to the permutation of $n$ columns of the matrix $\exp(i q_{a_1} x_b)$). Hence two eigenstates which differ one from another only by such momenta permutations could be called equivalent. On the other hand, the permutation of the momenta between two clusters with different numbers of particles reveals no specific symmetry at all. In other words, the wavefunctions with permuted momenta belonging to two clusters with different numbers of particles are just two different wavefunctions describing two different eigenstates. Thus for comparing the wavefunctions described by the parameters $(q_\alpha, n_\alpha)$ ($\alpha = 1, \ldots, M$) (in particular for the study of their orthogonality) it is crucial to specify ‘subsets’ of equal $n$s. Namely, a generic eigenstate $(q, n)$ with $M$ clusters could be specified in terms of the following set of parameters:

$$(q, n) = \{ (q_{s_1}, m_1), \ldots, (q_{s_k}, m_k); (q_{s_1+1}, m_2), \ldots, (q_{s_k}, m_2); \cdots; (q_{s_1+s_{k-1}+1}, m_k), \ldots, (q_{s_1+s_{k}}+1, m_k) \}$$

where $s_i$ ($i = 1, \ldots, k; 1 \leq k \leq M$) are the numbers of clusters which have the same numbers of particles and $k$ denotes the number of different cluster types. For a given $k$

$$s_1 + s_2 + \cdots + s_k = M$$

and

$$\sum_{\alpha=1}^{M} n_\alpha = \sum_{i=1}^{k} s_im_i = N.$$  

In this representations all the integers $\{m_i\}$ are assumed to be different:

$$1 \leq m_1 < m_2 < \cdots < m_k.$$
Due to the symmetry with respect to the momenta permutations inside the subsets of equal \( ns \) it is sufficient to consider the wavefunctions in the sectors

\[
q_1 < q_2 < \cdots < q_{s_1};
\]

\[
q_{s_1+1} < q_{s_1+2} < \cdots < q_{s_1+s_2};
\]

\[\cdots\cdots\cdot\]

\[
q_{s_1+s_2+\cdots+s_{k-1}+1} < q_{s_1+s_2+\cdots+s_{k-1}+2} < \cdots < q_{s_1+s_2+\cdots+s_k}.
\]

(57)

It can be shown (see appendix C) that any two wavefunctions \( \Psi^{(M)}_{q,n}(x) \) and \( \Psi^{(M')}_{q',n'}(x) \) in which the parameters \((q,n)\) and \((q',n')\) are assumed to have the structure described above (equations (53)–(57)) are orthogonal:

\[
\int_{-\infty}^{+\infty} dx_1 \cdots dx_N \Psi^{(M)}_{q,n}(x) \Psi^{(M')}_{q',n'}(x) = \delta(M, M') \left( \prod_{\alpha=1}^{M} \delta(n_{\alpha}, n'_{\alpha}) \right) \left( \prod_{\alpha=1}^{M} (2\pi) \delta(q_{\alpha} - q'_{\alpha}) \right)
\]

(58)

where \( \delta(n,m) \) is the Kronecker symbol and \( \delta(q) \) is the \( \delta \) function. The above orthonormality condition defines the normalization constant:

\[
C^{(M)}(q,n) = \left[ \frac{1}{N! \prod_{\alpha<\beta}^{M} \prod_{r=1}^{n_{\alpha}} \prod_{r'=1}^{n'_{\beta}} [(q_{\alpha} - q_{\beta} - i\kappa((n_{\alpha} - n_{\beta})/2 - r + r'))^2 + \kappa^2]} \times \prod_{\alpha=1}^{M} \frac{(n_{\alpha}!)^2 \kappa^{n_{\alpha}}}{n_{\alpha} \kappa} \right]^{1/2}.
\]

(59)

In other words, the wavefunctions, equations (51) or (52), form the orthonormal set. Although, at present we are not able to prove that this set is complete, the suggestion of completeness (which assumes that there exist no other eigenstates besides those described above) looks quite natural.

Finally, substituting equations (47) and (48) into (36), for the energy spectrum one easily obtains

\[
E_M(q,n) = \frac{1}{2\beta} \sum_{\alpha=1}^{M} \sum_{r=1}^{n_{\alpha}} (q_{\alpha}^2) = \frac{1}{2\beta} \sum_{\alpha=1}^{M} n_{\alpha} q_{\alpha}^2 - \frac{\kappa^2}{24\beta} \sum_{\alpha=1}^{M} (n_{\alpha}^3 - n_{\alpha}).
\]

(60)

4. Replica partition function

The time-dependent solution \( \Psi(x,t) \) of equation (25), satisfying the starting condition equation (26), can be represented in terms of the linear combination of the eigenfunctions \( \Psi^{(M)}_{q,n}(x) \), equations (50)–(52):

\[
\Psi(x,t) = \sum_{M=1}^{N} \sum_{n} \int Dq \Psi^{(M)}_{q,n}(x) \Psi^{(M)*}_{q,n}(0) \exp[-E_M(q,n)t].
\]

(61)
Here the summations over \( n_\alpha \) are performed in terms of the parameters \( \{ s_i, m_i \} \), equations (54)–(56):

\[
\sum'_{\mathbf{n}} \equiv \sum_{k=1}^{M} \sum_{s_1 \cdots s_k = 1}^{\infty} \sum_{1 \leq m_1 \cdots m_k}^{\infty} \delta \left( \sum_{i=1}^{k} s_i, M \right) \delta \left( \sum_{i=1}^{k} s_i m_i, N \right)
\]

(62)

where for simplicity, due to the presence of the Kronecker symbols, the summations over \( m_i \) and \( s_i \) are extended to infinity. The symbol \( \int' \mathcal{D} \mathbf{q} \) in equation (61) denotes the integration over \( M \) momenta \( q_a \) in the sectors, equation (57); the energy spectrum \( E_M(q, n) \) is given by equation (60).

Now, according equation (29) for the replica partition function of the original directed polymer problem we get

\[
Z(N, L) = \Psi(0; L) = \sum_{M=1}^{\infty} \sum'_{\mathbf{n}} \int' \mathcal{D} \mathbf{q} |\Psi^{(M)}(0)|^2 \exp[-E_M(q, n)L]
\]

(63)

where, due to the presence of the Kronecker symbols in equation (62), the summation over \( M \) can also be extended to infinity. Using equation (50), and taking into account antisymmetry with respect to the momenta permutations, one can easily prove that for \( M \geq 2 \)

\[
\Psi^{(M)}(0) = C^{(M)}_{q, n} \sum'_{P} (-1)^{|P|} \times \prod_{a<b, \alpha(a)\neq\alpha(b)}^{N} \left( q_{\alpha(a)} - q_{\alpha(b)} - i\kappa \left[ \frac{n_{\alpha(a)} - n_{\alpha(b)}}{2} - r(a) + r(b) \right] - i\kappa \right)
\]

\[
= C^{(M)}_{q, n} \sum'_{P} (-1)^{|P|} \prod_{a<b, \alpha(a)\neq\alpha(b)}^{N} \left( q_{\alpha(a)} - q_{\alpha(b)} - i\kappa \left[ \frac{n_{\alpha(a)} - n_{\alpha(b)}}{2} - r(a) + r(b) \right] \right).
\]

(64)

Given the antisymmetry of the product with respect to permutations of the momenta, it is sufficient to consider only one (trivial) permutation and multiply the result by the total number of permutations:

\[
\Psi^{(M)}(0) = C^{(M)}_{q, n} \frac{N!}{n_1!n_2! \cdots n_M!} \prod_{\alpha<\beta}^{M} n_{\alpha} n_{\beta} \prod_{\alpha<\beta, r=1}^{M} r' \left( q_{\alpha} - q_{\beta} - i\kappa \left[ \frac{n_{\alpha} - n_{\beta}}{2} - r + r' \right] \right).
\]

(65)

For \( M = 1 \), according to equations (42)–(43):

\[
\Psi^{(1)}_{q}(0) = \sqrt{\frac{\kappa^N N!}{\kappa N}}.
\]

(66)

We see that the expression \(|\Psi^{(M)}(0)|^2 \exp[-E_M(q, n)L]\) in equation (63) is symmetric with respect to the permutations of the momenta \( q_a \) belonging to the clusters with the same number of particles. In this case the expression for the partition function can be written.
in terms of the *unconstrained* integration over the momenta \(q_\alpha\):

\[
Z(N, L) = \sum_{M=1}^{\infty} \sum_{k=1}^{M} \sum_{s_1 \cdots s_k = 1}^{\infty} \sum_{1 \leq m_1 \cdots < m_k}^{\infty} \frac{\delta(\sum_i s_i, M) \delta(\sum_i s_im_i, N)}{s_1!s_2! \cdots s_k!} \left( \prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi} \right)
\]

\[\times |\Psi_{q,n}^{(M)}(0)|^2 e^{-E_M(q,n)L} \tag{67}\]

where

\[n \equiv \{m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots, m_k, \ldots, m_k\}. \tag{68}\]

Equation (67) contains the summations of the quantity

\[f(n_1, n_2, \ldots, n_M) = \left( \prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi} \right) |\Psi_{q,n}^{(M)}(0)|^2 e^{-E_M(q,n)L} \tag{69}\]

which is the function of \(M\) integer parameters \(n_\alpha\). Using explicit expressions, equation (60), (59) and (65), one can easily prove that this function is fully symmetric with respect to permutations of all its \(M\) arguments. In this case

\[
\sum_{k=1}^{M} \sum_{s_1 \cdots s_k = 1}^{\infty} \sum_{1 \leq m_1 \cdots < m_k}^{\infty} \frac{\delta(\sum_i s_i, M) \delta(\sum_i s_im_i, N)}{s_1!s_2! \cdots s_k!} f(n) = \frac{1}{M!} \sum_{n_1=1}^{\infty} \cdots \sum_{n_M=1}^{\infty} \delta \left( \sum_{\alpha=1}^{M} n_\alpha, N \right) f(n) \tag{70}\]

so that the summations in the expression for the replica partition function, equation (67), can be essentially simplified:

\[
Z(N, L) = \sum_{M=1}^{\infty} \frac{1}{M!} \left( \prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi} \right) \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_M=1}^{\infty} \delta \left( \sum_{\alpha=1}^{M} n_\alpha, N \right)
\]

\[\times |\Psi_{q,n}^{(M)}(0)|^2 e^{-E_M(q,n)L}. \tag{71}\]

Substituting here the explicit expressions for \(\Psi_{q,n}^{(M)}(0)\), equations (65) and (66), for \(E_M(q,n)\), equation (60), as well as for the normalization constant \(C^{(M)}(q,n)\), equation (59), one gets

\[
Z(N,L) = N!\kappa^N \int_{-\infty}^{+\infty} \frac{dq}{2\pi\kappa N} e^{-(NL/2\gamma^2 + (\kappa^2L/24\beta)(N^3-N)}
\]

\[+ N!\kappa^N \sum_{M=2}^{\infty} \frac{1}{M!} \left[ \prod_{\alpha=1}^{M} \sum_{n_\alpha=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi\kappa n_\alpha} \right] \delta \left( \sum_{\alpha=1}^{M} n_\alpha, N \right)
\]

\[\times e^{-\left(\frac{L}{2\beta}\right)\sum_{\alpha=1}^{M} n_\alpha^2 + \left(\kappa^2L/24\beta\right)\sum_{\alpha=1}^{M} (n^3_\alpha - n_\alpha)}
\]

\[\times \prod_{\alpha<\beta} \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left| q_\alpha - q_\beta - (\kappa/2)(n_\alpha - n_\beta - 2r + 2r') \right|^2 + \kappa^2. \tag{72}\]

The first term in the above expression is the contribution of the ground state \((M = 1)\) and the next terms \((M \geq 2)\) are the contributions of the rest of the energy spectrum.

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Next, after a few lines of slightly cumbersome transformations (see appendix D) equation (72) can be reduced to the form (cf equation (10))

\[ Z(N, L) = N! \kappa^N e^{-\beta NL f_0} \hat{Z}(N, L) \]  

(73)

where \( f_0 \) is the linear (self-averaging) free energy density:

\[ f_0 = \frac{1}{4} \beta^4 u^2, \]  

(74)

and

\[
\hat{Z}(N, \lambda) = \int_{-\infty}^{+\infty} \frac{dp}{4\pi} \int_{0}^{+\infty} dt \ e^{-\lambda N t - \lambda N p^2 + \frac{1}{3} \lambda^3 N^3} 
+ \sum_{M=2}^{\infty} \frac{1}{M!} \sum_{n_1, \ldots, n_M=1}^{\infty} \left[ \prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{dp_{\alpha}}{4\pi} \int_{0}^{+\infty} dt_{\alpha} \ e^{-\lambda n_{\alpha} t_{\alpha} - \lambda n_{\alpha} p_{\alpha}^2 + (1/3) \lambda^3 n_{\alpha}^3} \right] 
\times \delta \left( \sum_{\alpha=1}^{M} n_{\alpha}, N \right) \prod_{\alpha<\beta}^{M} \frac{|p_{\alpha} - p_{\beta} - i\lambda(n_{\alpha} - n_{\beta})|^2}{|p_{\alpha} - p_{\beta} - i\lambda(n_{\alpha} + n_{\beta})|^2}. \]  

(75)

Here instead of the system length \( L \) we have introduced a new parameter:

\[ \lambda(L) = \frac{1}{2} \left( \frac{L}{\beta^2} \right)^{1/3} = \frac{1}{2} \left( \beta^3 u^2 L \right)^{1/3}. \]  

(76)

Next, we linearize the terms cubic in \( n_{\alpha} \) in the exponentials of equation (75) with the help of Airy functions, using the standard relation

\[ \exp \left( \frac{1}{3} \lambda^3 n^3 \right) = \int_{-\infty}^{+\infty} dy \ Ai(y) \exp(\lambda n). \]  

(77)

After shifting the Airy function parameters of integration \( y_{\alpha} \to y_{\alpha} + t_{\alpha} + p_{\alpha}^2 \) the expression for \( \hat{Z}(N, \lambda) \) becomes sufficiently compact:

\[
\hat{Z}(N, \lambda) = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \frac{dp}{4\pi} \int_{0}^{+\infty} dt \ Ai(y + t + p^2) \exp(\lambda N y) 
+ \sum_{M=2}^{\infty} \frac{1}{M!} \left[ \prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} dy_{\alpha} \int_{-\infty}^{+\infty} \frac{dp_{\alpha}}{4\pi} \int_{0}^{+\infty} dt_{\alpha} \ Ai(y_{\alpha} + t_{\alpha} + p_{\alpha}^2) \right] 
\times \sum_{n_1, \ldots, n_M=1}^{\infty} \delta \left( \sum_{\alpha=1}^{M} n_{\alpha}, N \right) \prod_{\alpha<\beta}^{M} \frac{|p_{\alpha} - p_{\beta} - i\lambda(n_{\alpha} - n_{\beta})|^2}{|p_{\alpha} - p_{\beta} - i\lambda(n_{\alpha} + n_{\beta})|^2} \prod_{\alpha=1}^{M} \exp(\lambda n_{\alpha} y_{\alpha}). \]  

(78)

Finally, after performing summations over \( \{n_1, \ldots, n_M\} \) (see appendix E) the above expression can be represented as an analytic function of two parameters \( \lambda N \) and \( \lambda \):

\[ \hat{Z}(N, \lambda) = Z_1(\lambda N) + \sum_{M=2}^{\infty} Z_M(\lambda N; \lambda) \]  

(79)

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where

$$Z_1(\lambda N) = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dp \int_{0}^{+\infty} dt \, \text{Ai}(y + t + p^2) \exp(\lambda Ny)$$  \hspace{1cm} (80)

and

$$Z_M(\lambda N; \lambda) = \frac{1}{(M-1)!} \int \mathcal{D}_M(y, p) \int \hat{G}_M(p; \psi, \phi, \chi) e^{\lambda N(y_1 + i\eta_1)} \prod_{a=2}^{M} \frac{e^{\lambda(y_a + i\eta_a)}}{e^{\lambda(y_1 + i\eta_1)} - e^{\lambda(y_a + i\eta_a)}}$$  \hspace{1cm} (81)

with the definition

$$\eta_a \equiv \eta_a(\psi, \phi, \chi) = \frac{1}{2} \sum_{\beta \neq \alpha}^{M} (\psi_{\alpha\beta}^2 + \psi_{\beta\alpha}^2 - \phi_{\alpha\beta}^2 - \phi_{\beta\alpha}^2 + 2\chi_{\alpha\beta} - 2\chi_{\beta\alpha}).$$  \hspace{1cm} (82)

Above we have introduced the integration operator

$$\int \mathcal{D}_M(y, p) \equiv \prod_{a=1}^{M} \int_{-\infty}^{+\infty} dy_a \int_{-\infty}^{+\infty} dp_a \int_{0}^{+\infty} dt_a \, \text{Ai}(y_a + t_a + p_a^2)$$  \hspace{1cm} (83)

as well as the integro-differential operator

$$\int \hat{G}_M(p; \psi, \phi, \chi) \equiv \left[ M \prod_{\alpha \neq \beta}^{M} \int \int_{-\infty}^{+\infty} \frac{d\psi_{\alpha\beta}}{2\pi} \frac{d\phi_{\alpha\beta}}{2\pi} \frac{\partial}{\partial \chi_{\alpha\beta}} \right] e^{-(1/2) \sum_{\alpha \neq \beta}^{M} \left( |p_{\alpha} - p_{\beta}|(\psi_{\alpha\beta}^2 + \phi_{\alpha\beta}^2 - 2\chi_{\alpha\beta}) \right)}$$  \hspace{1cm} (84)

where it is assumed that the derivatives over \{\chi_{\alpha\beta}\} are taken at \chi_{\alpha\beta} = 0.

The crucial point is that all these factors: \int \mathcal{D}_M(y, p), \int \hat{G}_M(p; \psi, \phi, \chi), \eta_a(\psi, \phi, \chi) as well as the last product in equation (81) do not contain the replica parameter N; the latter enters the expression only in the combination \lambda N within the exponentials of equations (80) and (81). Thus, we have obtained the exact expression for the partition function, equations (79)–(81), in the form of an analytic function of the replica parameter N, which until now was assumed to be an arbitrary integer. In the following, we will consider the analytic continuation of this function to arbitrary complex values of N and, in particular, in the limit N \to 0. Unfortunately this crucial step of the analytic continuation is ambiguous, as our partition function grows as \exp(N^3) at large N (it is well known that in this case there can exist many distribution functions which have the same values of N). We will return to this problem with a further discussion in section 7.

5. Thermodynamic limit

Assuming now that the parameter N is an arbitrary complex quantity we are going to take the thermodynamic limit \( L \to \infty \) in the replica partition function, equations (79)–(81), where, according to equation (76), the length \( L \) enters in terms of the parameter \( \lambda(L) = \frac{1}{2}(\beta^3 u^2 L)^{1/3} \). It is crucial that the replica parameter \( N \) appears in equations (79)–(81) only in the combination \( \lambda N \). Keeping in mind further calculations of the free energy distribution function, equations (12)–(18), in the limit \( \lambda \to \infty \) we have to keep the value

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of the parameter

\[ s \equiv \lambda N = \frac{1}{2} (\beta^5 u^2 L)^{1/3} N \quad (85) \]

finite. In other words, in the thermodynamic limit \( L \to \infty \), the replica parameter \( N \sim 1/\lambda \sim L^{-1/3} \to 0 \).

It turns out that in the thermodynamic limit the expression for the replica partition function, equations (79)–(81), simplifies dramatically. Indeed, since the functions \( \eta_\alpha \), equation (82), take only real values, we have

\[
\lim_{\lambda \to \infty} \exp[\lambda(y_\alpha + i \eta_\alpha)] - \exp[\lambda(y_1 + i \eta_1)] = \begin{cases} 
0, & \text{for } y_\alpha < y_1 \\
-1, & \text{for } y_\alpha > y_1.
\end{cases} \quad (86)
\]

Substituting this into equation (81) we get

\[
\lim_{\lambda \to \infty} Z_M(s; \lambda) \equiv \tilde{Z}_M(s) = \left( -1 \right)^{M-1} \frac{1}{(M-1)!} \int \mathcal{D}_M(\mathbf{y}, \mathbf{p}) e^{s \eta_1} \left[ \prod_{\alpha=2}^M \theta(y_\alpha - y_1) \right] 
\times \int \hat{G}_M(\mathbf{p}; \mathbf{\psi}, \mathbf{\phi}, \mathbf{\chi}) e^{i s \eta_1 (\mathbf{\psi}, \mathbf{\phi}, \mathbf{\chi})}. \quad (87)
\]

It can be shown (see appendix F) that the last term in the above equation is unity:

\[
\int \hat{G}_M(\mathbf{p}; \mathbf{\psi}, \mathbf{\phi}, \mathbf{\chi}) e^{i s \eta_1 (\mathbf{\psi}, \mathbf{\phi}, \mathbf{\chi})} \equiv 1. \quad (88)
\]

Thus, introducing the function

\[
\Phi(x) \equiv \int_{-\infty}^{+\infty} \frac{dp}{4\pi} \int_0^{+\infty} dt \ Ai(x+t+p^2) \quad (89)
\]

and substituting the definition of the operator \( \int \mathcal{D}_M(\mathbf{y}, \mathbf{p}) \), equation (83), into (87) one gets

\[
\tilde{Z}_M(s) = \left( -1 \right)^{M-1} \frac{1}{(M-1)!} \int_{-\infty}^{+\infty} dx \Phi(x) e^{sx} \left[ \int_{-\infty}^{+\infty} dy \Phi(y) \right]^{M-1}. \quad (90)
\]

Now, substituting equations (90) and (80) into equation (79) one finds

\[
\lim_{\lambda \to \infty} \tilde{Z}(N, \lambda) \equiv \tilde{Z}(s) = \int_{-\infty}^{+\infty} dx \Phi(x) e^{sx} \sum_{M=1}^{\infty} \frac{(-1)^{M-1}}{(M-1)!} \left[ \int_{-\infty}^{+\infty} dy \Phi(y) \right]^{M-1}. \quad (91)
\]

The summation of this series leads to the result

\[
\tilde{Z}(s) = \int_{-\infty}^{+\infty} dx \Phi(x) \exp \left( sx - \int_{-\infty}^{+\infty} dy \Phi(y) \right) \quad (92)
\]

with the function \( \Phi(x) \) defined in equation (89). Note that the prefactor \( N! \kappa^N \) in the expression for the full replica partition function, equation (73), is irrelevant in the thermodynamic limit \( \lambda \to \infty \), since for fixed parameter \( s = N \lambda \):

\[
N! \kappa^N = \Gamma(1+N) \kappa^N = \Gamma \left( 1 + \frac{s}{\lambda} \right) \kappa^{s/\lambda} \bigg|_{\lambda \to \infty} \to 1 \quad (93)
\]

and hence

\[
\lim_{L \to \infty} \left[ Z(N, L) e^{\beta NL_f 0} \right] = \tilde{Z}(s). \quad (94)
\]

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6. Free energy distribution function

The distribution function of the free energy fluctuations can now be derived following the lines of the general approach discussed in section 1, equations (6)–(17). According to the definition of the replica partition function, equation (8):

\[ Z(N, L) = \int_{-\infty}^{+\infty} dF \mathcal{P}_L(F)e^{-\beta N F} \]  

(95)

where \( \mathcal{P}_L(F) \) is the distribution function of the total free energy of the system. Redefining the replica partition function according to equation (73) and introducing rescaled free energy fluctuations \( f \) according to the definition

\[ F = f_0 L + \frac{1}{\beta} \lambda f \]  

(96)

(where \( f_0 \) and \( \lambda \) are defined in equations (74) and (76)) instead of equation (95) we get

\[ \tilde{Z}(N, L) = \int_{-\infty}^{+\infty} df \mathcal{P}_L(f)e^{-\lambda N f} \]  

(97)

where \( \mathcal{P}_L(f) \) is the distribution function of the free energy fluctuations, which is related to \( \mathcal{P}_L(F) \) via

\[ \mathcal{P}_L(f) = \frac{\lambda}{\beta N! \kappa^N} \mathcal{P}_L\left( f_0 L + \frac{\lambda}{\beta} f \right). \]  

(98)

Taking now the limit \( L \to \infty \) in both sides of equation (97), at fixed \( \lambda N \equiv s \), we obtain

\[ \tilde{Z}(s) = \int_{-\infty}^{+\infty} df \mathcal{P}_*(f)e^{-sf} \]  

(99)

where \( \tilde{Z}(s) \) is given in equation (92) and

\[ \mathcal{P}_*(f) = \lim_{L \to \infty} \mathcal{P}_L(f) \]  

(100)

is the universal thermodynamic limit distribution function of the free energy fluctuations. This function is obtained from the relation, equation (99), via the inverse Laplace transform

\[ \mathcal{P}_*(f) = \int_{-\infty}^{+\infty} \frac{ds}{2\pi i} \tilde{Z}(s)e^{sf}. \]  

(101)

Substituting here the explicit expression for \( \tilde{Z}(s) \), equation (92), we find after a trivial integration,

\[ \mathcal{P}_*(f) = \Phi(-f) \exp \left[ -\int_{-f}^{+\infty} dy \Phi(y) \right] \]  

(102)

where the function \( \Phi(x) \) is defined in equation (89).

This is the central result of this paper. The shape of the distribution function \( \mathcal{P}_*(f) \) is represented in figure 1. One can easily check that the function \( \mathcal{P}_*(f) \), equation (102), is positively defined and properly normalized. Indeed, the function \( \Phi(x) \), equation (89),

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is positive for all values of \( x \) \((-\infty < x < +\infty)\) and its asymptotics are
\[
\Phi(x \to +\infty) \sim \exp\left(-\frac{2}{3}x^{3/2}\right) \to 0 \quad \Phi(x \to -\infty) \sim |x|^{1/2} \to +\infty.
\]
(103)

Thus, according to equation (102),
\[
\int_{-\infty}^{+\infty} df \, P_*(f) = -\int_{-\infty}^{+\infty} df \frac{d}{df} \exp\left[-\int_{-f}^{+\infty} dy \Phi(y)\right] = 1.
\]
(104)

The asymptotic forms of the left and the right tails are
\[
P_*(f \to -\infty) \sim \exp\left(-\frac{2}{3}|f|^{3/2}\right)
\]
(105)
\[
P_*(f \to +\infty) \sim \exp(-c_0 f^{3/2})
\]
(106)

where \(c_0 \approx 0.11\).

7. Discussion

The obtained result for the distribution function of the free energy fluctuations, equation (102), is rather surprising. At present there exists an appreciable list of statistical systems for which similar distribution functions have been computed exactly in the thermodynamic limit. These systems are: the polynuclear growth (PNG) model [12], the longest increasing subsequences (LIS) model [13], the longest common subsequences (LCS) model [14], the oriented digital boiling model [15], the ballistic decomposition model [16] and finally the zero-temperature lattice version of the directed polymers with a specific (non-Gaussian) site-disorder distribution [17]. It is remarkable that in all these systems (which do not always look similar) the fluctuations of the quantities which play the role of ‘energy’ are described by the same distribution function, the so-called the Tracy–Widom (TW) distribution [11].

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The result obtained in the present work is not the Tracy–Widom distribution, although it is widely believed that the system considered here belongs to the same universality class as the models listed above. It is not clear, however, which way the term ‘universality class’ should be understood. As far as only the scaling exponents are concerned, all these systems, including the present one, indeed belong to the same universality class. On the other hand, the shapes of the distribution functions of the systems listed above and the result obtained for the present system are different. What is really surprising is that they are different not just at the level of ‘details’ but in the qualitative properties, such as the asymptotic behavior of the left and right tails. The TW distribution $P_{\text{TW}}(f)$ is strongly asymmetric in its asymptotics:

$$P_{\text{TW}}(f \to -\infty) \sim \exp \left( -\frac{2}{3} f^{3/2} \right)$$  \hspace{1cm} (107)

$$P_{\text{TW}}(f \to +\infty) \sim \exp \left( -\frac{1}{12} f^3 \right)$$  \hspace{1cm} (108)

hence, the right tail decays much faster than the left one. In contrast, our distribution function is ‘almost symmetric’ in its right and left tails, cf equations (105) and (106), and moreover, due to numerical factors, its left tail decays slightly faster than the right one.

One may propose three possible explanations of the discrepancy discussed above:

1. All the systems described by the TW distributions are essentially the zero-temperature models, while our system by its definition is the ‘high-temperature’ one. Moreover, formally, our system has no zero-temperature limit at all. To study the limit $T \to 0$, one would need to introduce a lattice or a ‘finite width’ $\delta$-function regularization in the model, cf equations (1)–(2). In both cases, the Bethe ansatz solution leading to the result, equation (102), would no longer be valid. On the other hand, from a general physical point of view the conclusion that the thermodynamic limit of directed polymers at $T = 0$ and in the limit $T \to 0$ look different would be quite surprising. In particular, the solution in the thermodynamic limit of the present system confined to a cylinder geometry reveals no such difference [18].

2. Unlike the present system, the disorder in all the systems described by the TW distribution is essentially non-Gaussian. Formally, any deviation from the Gaussian statistics of the disorder would again ruin our Bethe ansatz solution, and at present it is not clear to what extent this solution is ‘stable’ with respect to non-Gaussian deviations.

3. It is much more likely that the technical problem given by this third option appears. According to equation (99) the distribution function $P_s(f)$ is defined by the replica partition function $\tilde{Z}(s)$ at finite (both positive and negative) values of the parameter $s \propto L^{1/3} N$, which means that in the thermodynamic limit $L \to \infty$ the function $P_s(f)$ is defined by the replica partition function with the ‘number of replicas’ $N \to 0$ (both positive and negative). In the present work we have computed the replica partition function for arbitrary positive integer $N$ with the aim to perform an analytic continuation for the values $N \to 0$. Unfortunately, the analytic continuation from integers to arbitrary (real or complex) $N$ is unambiguous only if the corresponding function growth at infinity is not faster than $\exp(N)$. In our case the partition function grows as $\exp(N^3)$, and hence the knowledge of this function at arbitrary integer $N$ does not guarantee an unambiguous reconstruction in the region $|N| \ll 1$.

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(see footnote 5). The classical example of such a type of situation is well known in the theory of the mean-field spin glasses, such as the Sherrington and Kirkpatrick (SK) model [28] and the random energy model (REM) [29]. For both models one can relatively easily compute the replica partition functions for arbitrary positive integer number of replicas $N$, and in both cases the replica partition function grows as $\exp(N^2)$ at large $N$. Further ‘direct’ analytic continuation of these solutions to the region $0 < N < 1$ yields nothing else but the replica symmetric solutions, which at first sight look sufficiently reasonable (at least in the SK model), but more detailed investigation reveals that they are unphysical. As we know, the solution which is believed to be valid in the region $0 < N < 1$ reveals the Parisi replica symmetry breaking (RSB) structure (one-step RSB in the case of REM), and it is derived in terms of a heuristic procedure (directly in the interval $0 < N < 1$) and not as a proper analytic continuation from integer to non-integer values of $N$. Moreover, in the case of REM there is a kind of phase transition at $N = 0$, which means that at negative $N$ the replica partition function should be computed separately [30].

Although up to the present moment we have not uncovered any unphysical properties in the probability function $P_*(f)$ obtained, the above arguments indicate that the present solution, equation (102), could as well be a kind of distant analog of the ‘replica symmetric approximation’, while the derivation of the ‘true’ solution would require more sophisticated ideas.

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Appendix A. Wavefunctions of quantum bosons with repulsive interactions

Explicitly the eigenstate equation (31) is

$$\frac{1}{2\beta} \sum_{a=1}^{N} \partial^2_{x_a} \Psi(x) + \frac{1}{2} \beta^2 u \sum_{a \neq b}^{N} \delta(x_a - x_b) \Psi(x) = -E \Psi(x). \quad (A.1)$$

Due to the symmetry of the wavefunction with respect to permutations of its arguments it is sufficient to consider it in the sector

$$x_1 < x_2 < \cdots < x_N \quad (A.2)$$

as well as at its boundary. Inside this sector the wavefunction $\Psi(x)$ satisfies the equation

$$\frac{1}{2\beta} \sum_{a=1}^{N} \partial^2_{x_a} \Psi(x) = -E \Psi(x) \quad (A.3)$$

which describes $N$ free particles, and its generic solution is the linear combination of $N$ plane waves characterized by $N$ momenta $\{q_1, q_2, \ldots, q_N\} \equiv q$. Integrating equation (A.1)
over the variable \((x_{i+1} - x_i)\) in a small interval around zero, \(|x_{i+1} - x_i| < \epsilon \to 0\), and assuming that the other variables \(\{x_j\}\) (with \(j \neq i, i + 1\)) belong to the sector, equation (A.2), one easily finds that the wavefunction \(\Psi(x)\) must satisfy the following boundary conditions:

\[
(\partial_{x_{i+1}} - \partial_x + \kappa)\Psi(x)|_{x_{i+1}=x_i+0} = 0
\]  

(A.4)

where \(\kappa = \beta^3 u\). Functions satisfying both equation (A.3) and the boundary conditions equation (A.4) can be written in the form

\[
\Psi_{q_1\ldots q_N}(x_1, \ldots, x_N) \equiv \Psi_q^{(N)}(x) = C \left( \prod_{a<b}^N [\partial_{x_a} - \partial_{x_b} + \kappa] \right) \det[\exp(iqx)]
\]  

(A.5)

where \(C\) is the normalization constant to be defined later, and the symbol \(\exp(iqx)\) denotes the \(N \times N\) matrix with the elements \(\exp(iq_a x_b)(a, b = 1, \ldots, N)\). First of all, it is evident that, being a linear combination of the plane waves, the above wavefunction satisfies equation (A.3). To demonstrate which way this function satisfies the boundary conditions, equation (A.4), let us check it, as an example, for the case \(i = 1\). According to equation (A.5), the wavefunction \(\Psi_q^{(N)}(x)\) can be represented in the form

\[
\Psi_q^{(N)}(x) = -(\partial_{x_2} - \partial_{x_1} - \kappa)\tilde{\Psi}_q^{(N)}(x)
\]  

where

\[
\tilde{\Psi}_q^{(N)}(x) = C \left( \prod_{a=3}^N [\partial_{x_a} - \partial_{x_1} + \kappa][\partial_{x_a} - \partial_{x_2} + \kappa] \right) \left( \prod_{3\leq a<b}^N [\partial_{x_a} - \partial_{x_b} + \kappa] \right)
\]

(A.7)

\[
\times \det[\exp(iqx)].
\]

One can easily see that this function is antisymmetric with respect to the permutation of \(x_1\) and \(x_2\). Substituting equation (A.6) into (A.4) (with \(i = 1\)) we get

\[
-[(\partial_{x_2} - \partial_{x_1})^2 - \kappa^2] \tilde{\Psi}_q^{(N)}(x)|_{x_1 = x_2} = 0.
\]  

(A.8)

Given the antisymmetry of the lhs expression with respect to the permutation of \(x_1\) and \(x_2\) the above condition is indeed satisfied at boundary \(x_1 = x_2\).

Since the eigenfunction \(\Psi_q^{(N)}(x)\) satisfying equation (A.1) must be symmetric with respect to permutations of its arguments, the function, equation (A.5), can be easily continued beyond the sector, equation (A.2), to the entire space of variables \(\{x_1, x_2, \ldots, x_N\} \in \tilde{R}_N\):

\[
\Psi_q^{(N)}(x) = C \left( \prod_{a<b}^N [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right) \det[\exp(iqx)]
\]  

(A.9)

where, by definition, the differential operators \(\partial_{x_a}\) act only on the exponential terms and not on the \(\text{sgn}(x)\) functions, and for further convenience we have redefined \(i^{N(N-1)/2}C \to C\). Explicitly the determinant in the above equation is

\[
\det[\exp(iqx)] = \sum_P (-1)^{|P|} \exp \left[ i \sum_{a=1}^N q_{pa} x_a \right]
\]  

(A.10)
where the summation goes over the permutations \( P \) of \( N \) momenta \( \{q_1, q_2, \ldots, q_N\} \) over \( N \) particles \( \{x_1, x_2, \ldots, x_N\} \), and \([P]\) denotes the parity of the permutation. In this way the eigenfunction, equation (A.9), can be represented as follows:

\[
\Psi^{(N)}_{q}(x) = C \sum_{P} (-1)^{|P|} \left( \prod_{a<b}^{N} [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right) \exp \left[ i \sum_{a=1}^{N} q_p x_a \right]. \quad (A.11)
\]

Taking the derivatives, we obtain

\[
\Psi^{(N)}_{q}(x) = C \sum_{P} (-1)^{|P|} \left( \prod_{a<b}^{N} [q_{pa} - q_{pb} + i\kappa \text{sgn}(x_a - x_b)] \right) \exp \left[ i \sum_{a=1}^{N} q_p x_a \right]. \quad (A.12)
\]

It is evident from these representations that the eigenfunctions \( \Psi^{(N)}_{q}(x) \) are \textit{antisymmetric} with respect to permutations of the momenta \( q_1, \ldots, q_N \).

Substituting the expression for the eigenfunctions, equation (A.5) (which is valid in the sector, equation (A.2)), into equation (A.3) for the energy spectrum we find

\[
E = \frac{1}{2\beta} \sum_{a=1}^{N} q_a^2. \quad (A.13)
\]

Now one can easily prove that the above eigenfunctions with different momenta are orthogonal to each other. Let us consider two wavefunctions \( \Psi^{(N)}_{q}(x) \) and \( \Psi^{(N)}_{q'}(x) \), where it is assumed that

\[
q_1 < q_2 < \cdots < q_N \quad q'_1 < q'_2 < \cdots < q'_N. \quad (A.14)
\]

Using the representation, equation (A.11), for the overlap of these two function we get

\[
\overline{\Psi^{(N)}_{q}(x)} \Psi^{(N)}_{q'}(x) \equiv \int_{-\infty}^{+\infty} d^N x \overline{\Psi^{(N)}_{q}(x)} \Psi^{(N)}_{q'}(x) = |C|^2 \sum_{P, P'} (-1)^{|P|+|P'|}
\times \int_{-\infty}^{+\infty} d^N x \left\{ \left( \prod_{a<b}^{N} [i(\partial_{x_a} - \partial_{x_b}) - i\kappa \text{sgn}(x_a - x_b)] \right) \exp \left[ -i \sum_{a=1}^{N} q'_{pa} x_a \right] \right\}
\times \left\{ \left( \prod_{a<b}^{N} [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right) \exp \left[ i \sum_{a=1}^{N} q_p x_a \right] \right\}. \quad (A.15)
\]

Integrating by parts we obtain

\[
\overline{\Psi^{(N)}_{q}(x)} \Psi^{(N)}_{q'}(x) = |C|^2 \sum_{P, P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N x \exp \left[ -i \sum_{a=1}^{N} q'_{pa} x_a \right]
\times \left( \prod_{a<b}^{N} [-i(\partial_{x_a} - \partial_{x_b}) - i\kappa \text{sgn}(x_a - x_b)] [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b)] \right)
\times \exp \left[ i \sum_{a=1}^{N} q_p x_a \right]. \quad (A.16)
\]
or

\[
\overline{\Psi_{q'}^{(N)^*}(x)} \Psi_q^{(N)}(x) = |C|^2 \sum_{P,P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N x \exp \left[ -i \sum_{a=1}^{N} q'_p x_a \right] \\
\times \left( \prod_{a<b} \left[-(\partial_{x_a} - \partial_{x_b})^2 + \kappa^2 \right] \right) \exp \left[ i \sum_{a=1}^{N} q_p x_a \right].
\]

(A.17)

Taking the derivatives and performing the integrations we find

\[
\overline{\Psi_{q'}^{(N)^*}(x)} \Psi_q^{(N)}(x) = |C|^2 \sum_{P,P'} (-1)^{|P|+|P'|} \left( \prod_{a<b} \left[(q_p - q'_p)^2 + \kappa^2 \right] \right) \left[ \prod_{a=1}^{N} (2\pi) \delta(q_p - q'_p) \right].
\]

(A.18)

Taking into account the constraint, equation (A.14), one can easily note that the only terms which survive in the above summation over the permutations are \( P = P' \), all contributing equal values. Thus, we finally get

\[
\overline{\Psi_{q'}^{(N)^*}(x)} \Psi_q^{(N)}(x) = |C|^2 N! \left( \prod_{a<b} \left[(q_a - q_b)^2 + \kappa^2 \right] \right) \left[ \prod_{a=1}^{N} (2\pi) \delta(q_a - q'_a) \right].
\]

(A.19)

With the normalization constant

\[
|C| = \frac{1}{\sqrt{N! \prod_{a<b} [(q_a - q_b)^2 + \kappa^2]}}
\]

(A.20)

we conclude that the set of eigenfunctions, equation (A.11) or (A.12), are orthonormal. The proof of completeness of this set is given in [23]. It should be noted that the above wavefunctions present the orthonormal set of eigenfunctions of the problem, equation (A.1), for any sign of the interactions \( \kappa \), i.e. both for the repulsive, \( \kappa < 0 \), and for the attractive, \( \kappa > 0 \), cases. However, only in the case of repulsion is this set complete, while in the case of attractive interactions, \( \kappa > 0 \), in addition to the solutions, equation (A.11), which describe the continuous free particles’ spectra, one finds the whole family of discrete bound eigenstates (which do not exist in the case of repulsion) (see appendices B and C).

Appendix B. Ground state of quantum bosons with attractive interactions

The simplest example of the bound state is the one in which all \( N \) particles are bound into one finite size ‘cluster’:

\[
\Psi_q^{(1)}(x) = C \exp \left[ iq \sum_{a=1}^{N} x_a - \frac{i}{2} \kappa \sum_{a,b=1}^{N} |x_a - x_b| \right]
\]

(B.1)
where \( C \) is the normalization constant (see below) and \( q \) is the (continuous) momentum of the free center-of-mass motion. Substituting this function in equation (A.1), one can easily check that this is indeed the eigenfunction with the energy spectrum given by the relation

\[
E = -\frac{1}{2\beta} \sum_{a=1}^{N} \left[ i q - \frac{1}{2} \kappa \sum_{b=1}^{N} \text{sgn}(x_a - x_b) \right]^2
\]

(B.2)

where it is assumed (by definition) that \( \text{sgn}(0) = 0 \). Since the result of the above summations does not depend on the mutual particle positions, for simplicity we can order them according to equation (A.2). Then, using the well-known relations

\[
\sum_{a=1}^{N} \sum_{b=1}^{N} |x_a - x_b| = -(N + 1 - 2a)
\]

(B.3)

\[
\sum_{a=1}^{N} a = \frac{1}{2} N(N + 1)
\]

(B.4)

\[
\sum_{a=1}^{N} a^2 = \frac{1}{6} (N + 1)(2N + 1)
\]

(B.5)

for the energy spectrum, equation (B.2), we get

\[
E = \frac{N}{2\beta} q^2 - \frac{\kappa^2}{24\beta} (N^3 - N) \equiv E_1(q; N).
\]

(B.6)

The normalization constant \( C \) is defined by the orthonormality condition

\[
\Psi^{(1)*}_{q'}(x) \Psi^{(1)}_q(x) \equiv \int_{-\infty}^{+\infty} dx_1 \cdots dx_N \Psi^{(1)*}_{q'}(x) \Psi^{(1)}_q(x) = (2\pi)\delta(q - q').
\]

(B.7)

Substituting here equation (B.1) we get

\[
\Psi^{(1)*}_{q'}(x) \Psi^{(1)}_q(x) = |C|^2 \int_{-\infty}^{+\infty} dx_1 \cdots dx_N \exp \left[ i(q - q') \sum_{a=1}^{N} x_a - \frac{1}{2} \kappa \sum_{a,b=1}^{N} |x_a - x_b| \right]
\]

\[
= |C|^2 N! \int_{-\infty}^{+\infty} dx_1 \int_{x_1}^{+\infty} dx_2 \cdots \int_{x_{N-1}}^{+\infty} dx_N \times \exp \left[ i(q - q') \sum_{a=1}^{N} x_a + \kappa \sum_{a=1}^{N} (N + 1 - 2a)x_a \right]
\]

(B.8)

where for the ordering, equation (A.2), we have used the relation

\[
\frac{1}{2} \sum_{a,b=1}^{N} |x_a - x_b| = -\sum_{a=1}^{N} (N + 1 - 2a)x_a.
\]

(B.9)
Integrating first over \( x_N \), then over \( x_{N-1} \), and proceeding until \( x_1 \), we find
\[
\Psi_q^{(1)*}(x)\Psi_q^{(1)}(x) = |C|^2 N! \left( \prod_{r=1}^{N-1} \frac{1}{r[(N-r)\kappa - i(q-q')]} \right) \int_{-\infty}^{+\infty} dx_1 \exp \left[ iN(q-q')x_1 \right]
\]
\[
= |C|^2 N! \left( \prod_{r=1}^{N-1} \frac{1}{r(N-r)\kappa} \right) (2\pi)\delta(N(q-q'))
\]
\[
= |C|^2 \frac{N\kappa}{N!\kappa^N} (2\pi)\delta(q-q').
\]
(B.10)

According to equation (B.7) this defines the normalization constant
\[
|C| = \sqrt{\frac{\kappa^N N!}{\kappa^N}} = C^{(1)}(q).
\]
(B.11)

Note that the eigenstate described by the considered wavefunction, equation (B.1), exists only in the case of attraction, \( \kappa > 0 \), otherwise this function is divergent at infinity and consequently it is not normalizable.

It should be noted that the wavefunction, equation (B.1), can also be derived from the general eigenfunction structure, equation (A.12), by introducing (discrete) imaginary parts for the momenta \( q_a \). We assume again that the position of particles are ordered according to equation (A.2) and define the particles’ momenta according to the rule
\[
q_a = q - \frac{i}{2}\kappa(N + 1 - 2a).
\]
(B.12)

Substituting this into equation (A.12) we get
\[
\Psi_q^{(1)}(x_1 < x_2 < \cdots < x_N) \propto \sum_P (-1)^{|P|} \left( \prod_{a<b} \left[ (q - \frac{i}{2}\kappa(N + 1 - 2P_a)) \right. \right.
\]
\[
- \left. \left( q - \frac{i}{2}\kappa(N + 1 - 2P_b) \right) - i\kappa \right]\]
\[
\times \exp \left[ \frac{iq}{2} \sum_{a=1}^{N} x_a + \frac{\kappa}{2} \sum_{a=1}^{N} (N + 1 - 2P_a)x_a \right]
\]
\[
\propto \sum_P (-1)^{|P|} \left( \prod_{a<b} \left[ P_b - P_a + 1 \right] \right.
\]
\[
\times \exp \left[ \frac{iq}{2} \sum_{a=1}^{N} x_a + \frac{\kappa}{2} \sum_{a=1}^{N} (N + 1 - 2P_a)x_a \right].
\]
(B.13)

Here one can easily note that, due to the presence of the product \( \prod_{a<b}[P_b - P_a + 1] \) in the summation over permutations, only the trivial one, \( P_a = a \), gives a non-zero contribution (if we permute any two numbers in the sequence 1, 2, \ldots, \( N \) then we can always find two numbers \( a < b \), such that \( P_b = P_a - 1 \)). Thus
\[
\Psi_q^{(1)}(x_1 < x_2 < \cdots < x_N) \propto \exp \left[ \frac{iq}{2} \sum_{a=1}^{N} x_a + \frac{\kappa}{2} \sum_{a=1}^{N} (N + 1 - 2a)x_a \right].
\]
(B.14)

Here one can easily note that, due to the presence of the product \( \prod_{a<b}[P_b - P_a + 1] \) in the summation over permutations, only the trivial one, \( P_a = a \), gives a non-zero contribution (if we permute any two numbers in the sequence 1, 2, \ldots, \( N \) then we can always find two numbers \( a < b \), such that \( P_b = P_a - 1 \)). Thus
\[
\Psi_q^{(1)}(x_1 < x_2 < \cdots < x_N) \propto \exp \left[ \frac{iq}{2} \sum_{a=1}^{N} x_a + \frac{\kappa}{2} \sum_{a=1}^{N} (N + 1 - 2a)x_a \right].
\]
(B.15)
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Taking into account the relation, equation (B.9), we recover the function, equation (B.1), which is symmetric with respect to its $N$ arguments and therefore can be extended beyond the sector, equation (A.2), for arbitrary particle positions. Finally, substituting the momenta, equation (B.12), into the general expression for the energy spectrum, equation (A.13), we get

$$E = \frac{1}{2\beta} \sum_{a=1}^{N} \left[ q - \frac{i}{2}\kappa(N + 1 - 2a) \right]^2.$$  (B.16)

Performing here simple summations (using equations (B.4) and (B.5)) one recovers equation (B.6).

Appendix C. Wavefunctions of quantum bosons with attractive interactions

C.1. Eigenfunctions

The general expression for the eigenfunctions both for the case of repulsion and for the case of attraction is given in equation (33) or (37) and (38). A generic eigenfunction is characterized by $N$ momenta parameters $\{q_a\} (a = 1, 2, \ldots, N)$ which in the case of attractive interactions may have imaginary parts. It is convenient to group these parameters into $M (1 \leq M \leq N)$ ‘vector’ momenta:

$$q^\alpha_r = q^\alpha - \frac{i}{2}\kappa(n^\alpha + 1 - 2r)$$  (C.1)

where $q^\alpha (\alpha = 1, 2, \ldots, M)$ are the continuous (real) parameters and the (discrete) imaginary components of each ‘vector’ are labeled by an index $r = 1, 2, \ldots, n^\alpha$. With the given total number of particles equal to $N$, the integers $n^\alpha$ have to satisfy the constraint

$$\sum_{\alpha=1}^{M} n^\alpha = N.$$  (C.2)

In other words, a generic eigenstate is characterized by the discrete number $M$ of complex ‘vector’ momenta, by the set of $M$ integer parameters $\{n_1, n_2, \ldots, n_M\} \equiv n$ (which are the numbers of imaginary components of each ‘vector’) and by the set of $M$ real continuous momenta $\{q_1, q_2, \ldots, q_M\} \equiv q$.

To understand the structure of the determinant of the $N \times N$ matrix $\exp(i q_a x_b)$, which defines the wavefunctions, equations (37) and (38), the $N$ momenta $q_a = q^\alpha_r$ can be ordered as follows:

$$\{q_a\} = \{q_1^1, q_2^1, \ldots, q_{n_1}^1; q_1^2, q_2^2, \ldots, q_{n_2}^2; \ldots; q_1^M, q_2^M, \ldots, q_{n_M}^M\}.$$  (C.3)

By definition

$$\det[\exp(i q x)] = \sum_P (-1)^{|P|} \exp \left[ \sum_{a=1}^{N} q_a x_a \right]$$  (C.4)

where the summation goes over the permutations of $N$ momenta $\{q_a\}$, equation (C.3), over $N$ particles $\{x_1, x_2, \ldots, x_N\}$ and $|P|$ denotes the parity of the permutation. For a given permutation $P$ a particle number $a$ is attributed a momentum component $q^\alpha_r(a)$.

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The particles getting the momenta with the same $\alpha$ (having the same real part $q_a$) will be termed as belonging to a cluster $\Omega_\alpha$. For a given permutation $P$ the particles belonging to the same cluster are numbered by the ‘internal’ index $r = 1, \ldots, n_\alpha$. Thus, according to equation (38):

$$
\Psi^{(M)}_{q,n}(x) = C^{(M)}_{q,n} \sum_P (-1)^P \left( \prod_{a<b} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \right) \exp \left[ i \sum_{a=1}^N q_{\alpha(a)} x_a \right]
$$

(C.5)

where $C^{(M)}_{q,n}$ is the normalization constant to be defined later. Substituting here equation (C.1) and taking derivatives we get

$$
\Psi^{(M)}_{q,n}(x) = C^{(M)}_{q,n} \sum_P (-1)^P \times \prod_{a<b} \left[ q_{a(a)} - q_{a(b)} - i\kappa \left( \frac{n_{\alpha(a)} - n_{\alpha(b)}}{2} - r(a) + r(b) - \text{sgn}(x_a - x_b) \right) \right]

\times \exp \left[ i \sum_{a=1}^N q_{a(a)} x_a + \frac{\kappa}{2} \sum_{a=1}^N \left( n_{\alpha(a)} + 1 - 2r(a) \right) x_a \right].
$$

(C.6)

The pre-exponential product in the above equation contains two types of terms: the pairs of points $(a, b)$ which belong to different clusters ($\alpha(a) \neq \alpha(b)$) and pairs of points which belong to the same cluster ($\alpha(a) = \alpha(b)$). In the last case, the product $\Pi_\alpha$ over the pairs of points which belong to a cluster $\Omega_\alpha$ reduces to

$$
\Pi_\alpha \propto \prod_{a<b \in \Omega_\alpha} [r(b) - r(a) - \text{sgn}(x_a - x_b)].
$$

(C.7)

As for the ground state wavefunction equations (B.14) and (B.15), one can easily note that, due to the presence of this product in the summations over $n_\alpha!$ ‘internal’ (inside the cluster $\Omega_\alpha$) permutations $r(a)$, only one permutation gives a non-zero contribution. To prove this statement, we note that the wavefunction $\Psi^{(M)}_{q,n}(x)$ is symmetric with respect to permutations of its $N$ arguments $\{x_a\}$; it is then sufficient to consider the case where the positions of the particles are ordered, $x_1 < x_2 < \cdots < x_N$. In particular, the particles $\{x_{a_k}\} (k = 1, 2, \ldots, n_\alpha)$ belonging to the same cluster $\Omega_\alpha$ are also ordered $x_{a_1} < x_{a_2} < \cdots < x_{a_{n_\alpha}}$. In this case

$$
\Pi_\alpha \propto \prod_{k<l} [r(l) - r(k) + 1].
$$

(C.8)

Now it is evident that the above product is non-zero only for the trivial permutation, $r(k) = k$ (since if we permute any two numbers in the sequence 1, 2, \ldots, $n_\alpha$, we can always find two numbers $k < l$, such that $r(l) = r(k) - 1$). In this case

$$
\Pi_\alpha \propto \prod_{k<l} [l - k + 1].
$$

(C.9)
Including the values of all these ‘internal’ products, equation (C.9), in the redefined normalization constant \( C_{q,n}^{(M)} \), the wavefunction, equation (C.6) (with \( x_1 < x_2 < \cdots < x_N \)), is

\[
\Psi_{q,n}^{(M)}(x) = C_{q,n}^{(M)} \sum_P (-1)^P \prod_{a < b} \left[ q_{a(a)} - q_{a(b)} - i\kappa \left( \frac{n_{a(a)} - n_{a(b)}}{2} - r(a) + r(b) + 1 \right) \right]
\]

\[
\times \exp \left[ \sum_{a=1}^{N} q_{a(a)} x_a + \frac{\kappa}{2} \sum_{a=1}^{N} (n_{a(a)} + 1 - 2r(a)) x_a \right]
\]

where the product now goes only over the pairs of particles belonging to different clusters and the symbol \( \sum_P \) means that the summation goes only over the permutations \( P \) in which the ‘internal’ indices \( r(a) \) are ordered inside each cluster.

Note that, although the positions of particles belonging to the same cluster are ordered, the mutual positions of particles belonging to different clusters could be arbitrary, so that geometrically the clusters are free to ‘penetrate’ each other. In other words, the name ‘cluster’ does assume geometrically compact particle positions.

Now, using the symmetry of the wavefunction \( \Psi_{q,n}^{(M)}(x) \) with respect to the permutations of its arguments the expression in equation (C.10) can be easily continued beyond the sector \( x_1 < x_2 < \cdots < x_N \) for the entire coordinate space \( R_N \). Using the relations

\[
\sum_{a \in \Omega_a} (n_a + 1 - 2r(a)) x_a = \sum_{k=1}^{n_a} (n_a + 1 - 2k) x_{a_k} = -\frac{1}{2} \sum_{k,l=1}^{n_a} |x_{a_k} - x_{a_l}|
\]

and

\[
(n_a + 1 - 2k) = -\sum_{l=1}^{n_a} \text{sgn} (x_{a_k} - x_{a_l})
\]

(where \( x_{a_1} < x_{a_2} < \cdots < x_{a_{n_a}} \)), the wavefunction \( \Psi_{q,n}^{(M)}(x) \), equation (C.10), with arbitrary particle positions is

\[
\Psi_{q,n}^{(M)}(x) = C_{q,n}^{(M)} \sum_P (-1)^P \prod_{a < b} \left[ q_{a(a)} - q_{a(b)} + i\kappa \sum_{c \in \Omega_a(a)} \text{sgn}(x_a - x_c) \right.
\]

\[
- \frac{i\kappa}{2} \sum_{c \in \Omega_b(b)} \text{sgn}(x_b - x_c) + i\kappa \text{sgn}(x_a - x_b) \bigg]
\]

\[
\times \exp \left[ \sum_{a=1}^{M} q_{a} \sum_{a \in \Omega_a} x_a - \frac{\kappa}{2} \sum_{a=1}^{M} \sum_{a' \in \Omega_a} |x_a - x_{a'}| \right].
\]

Here the summation goes only over the permutations \( P \) of the momenta, equation (C.3), in which ‘internal’ indices \( r(a) \) in the clusters are ordered according to the actual spatial ordering of the particles belonging to these clusters (i.e. \( r(a) \) increases from the smallest \( x_a \) in the cluster to the largest one). This wavefunction can also be rewritten in the more
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compact form:

\[ \Psi^{(M)}_{q,n}(x) = C^{(M)}_{q,n} \sum_{P} (-1)^{|P|} \prod_{a<b}^{N} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \]

\times \exp \left[ \sum_{a=1}^{M} q_{a} \sum_{a \in \Omega_a} x_a - \frac{\kappa}{4} \sum_{a=1}^{M} \sum_{a' \in \Omega_a} |x_a - x_a'| \right]. \quad (C.14)

C.2. Orthogonality

We define the overlap of two wavefunctions characterized by the two sets of parameters, \((M, n, q)\) and \((M', n', q')\) as

\[ Q^{(M,M',n,n')}_{q,q'} = \int_{-\infty}^{+\infty} d^N x \Psi^{(M',n')}_{q,q'}(x) \Psi^{(M,n)}_{q,n}(x). \quad (C.15) \]

Substituting here equation (C.14) we get

\[ Q^{(M,M',n,n')}_{q,q'} = C^{(M)}_{q,n} C^{(M')}_{q',n'} \sum_{P} \sum_{P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N x \]

\times \left( \prod_{a<b}^{N} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \right) \]

\times \exp \left[ -i \sum_{a=1}^{M'} q'_{a} \sum_{a \in \Omega_a} x_a - \frac{\kappa}{4} \sum_{a=1}^{M'} \sum_{a \in \Omega_a} |x_a - x_{a'}| \right] \]

\times \left( \prod_{a<b}^{N} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \right) \]

\times \exp \left[ \sum_{a=1}^{M} q_{a} \sum_{a \in \Omega_a} x_a - \frac{\kappa}{4} \sum_{a=1}^{M} \sum_{a' \in \Omega_a} |x_a - x_{a'}| \right]. \quad (C.16) \]

where \(\{\Omega_a\}\) and \(\{\Omega'_a\}\) denote the clusters of the permutations \(P\) and \(P'\), respectively. Integrating by parts we obtain

\[ Q^{(M,M',n,n')}_{q,q'} = C^{(M)}_{q,n} C^{(M')}_{q',n'} \sum_{P} \sum_{P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N x \]

\times \exp \left[ -i \sum_{a=1}^{M'} q'_{a} \sum_{a \in \Omega_a} x_a - \frac{\kappa}{4} \sum_{a=1}^{M'} \sum_{a \in \Omega_a} |x_a - x_{a'}| \right] \]

\times \left( \prod_{a<b}^{N} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \right) \]

\times \exp \left[ \sum_{a=1}^{M} q_{a} \sum_{a \in \Omega_a} x_a - \frac{\kappa}{4} \sum_{a=1}^{M} \sum_{a' \in \Omega_a} |x_a - x_{a'}| \right].
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\[ \times \left( \prod_{\alpha < b, \alpha(\alpha) \neq \alpha(b)}^{N} \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \text{sgn}(x_a - x_b) \right] \right) \]

\[ \times \exp \left[ i \sum_{\alpha=1}^{M} q_{\alpha} \sum_{a \in \Omega_\alpha} x_a - \frac{\kappa}{4} \sum_{a,b \in \Omega_\alpha} \frac{M}{M} \sum_{\alpha} |x_a - x_b| \right]. \tag{C.17} \]

First, let us consider the case when the integer parameters of the two functions coincide, \( M = M' \) and \( n = n' \), and for the moment let us suppose that all these integer parameters \( \{n_{\alpha}\} \) are different, \( 1 \leq n_1 < n_2 < \cdots < n_M \). Then, in the summations over the permutations in equation (C.17), we find two types of terms:

(A) the ‘diagonal’ ones in which the two permutations coincide, \( P = P' \);

(B) the ‘off-diagonal’ ones in which the two permutations are different, \( P \neq P' \).

The contribution of the ‘diagonal’ ones is

\[ Q_{n,n}^{(M,M)(A)}(q, q') = C_{q,n}^{(M)} C_{q',n}^{(M)} \sum_{p}^{'} + \infty \left\{ -i \frac{M}{M} \sum_{\alpha} q_{\alpha} \sum_{a \in \Omega_\alpha} x_a - \frac{\kappa}{4} \sum_{a,b \in \Omega_\alpha} \sum_{\alpha} |x_a - x_b| \right\}. \]

It is evident that all permutations \( \alpha(\alpha) \) in the above equation give the same contribution and therefore it is sufficient to consider only the contribution of the ‘trivial’ permutation which is represented by the line in equation (C.3). The cluster ordering given by this permutation we denote by \( \alpha_0(\alpha) \). For this particular configuration of clusters we can redefine the particle numbering, so that instead of a ‘plane’ index \( a = 1, 2, \ldots, N \) the particles would be counted by two indices \( (\alpha, r) \). Due to the symmetry of the integrated expression in equation (C.18) with respect to the permutations of the particles inside the clusters, we can introduce the ‘internal’ particle ordering for every cluster: \( x_1^{\alpha} < x_2^{\alpha} < \cdots < x_{n_{\alpha}}^{\alpha} \). In this way, using the relation, equation (C.11), we get

\[ Q_{n,n}^{(M,M)(A)}(q, q') = C_{q,n}^{(M)} C_{q',n}^{(M)} \frac{N!}{n_1! n_2! \cdots n_M!} \]

\[ \times \left[ \prod_{\alpha=1}^{M} (n_{\alpha}! \int_{-\infty}^{+\infty} dx_1^{\alpha} \int_{x_1^{\alpha}}^{+\infty} dx_2^{\alpha} \cdots \int_{x_{n_{\alpha}-1}}^{+\infty} dx_{n_{\alpha}}^{\alpha}) \right] \]

\[ \times \exp \left[ -i \sum_{\alpha=1}^{M} q_{\alpha}^{\prime} \sum_{r=1}^{n_{\alpha}} x_r^{\alpha} + \frac{\kappa}{4} \sum_{\alpha=1}^{M} \sum_{r=1}^{n_{\alpha}} (n_{\alpha} + 1 - 2r)x_r^{\alpha} \right] \]

\[ \text{doi:10.1088/1742-5468/2010/03/P03022} \]
\[
\times \left( \prod_{\alpha<\beta} \prod_{r=1}^{M} \prod_{r'=1}^{M} \left[ -\left( \partial_{x_{\alpha}} - \partial_{x_{\beta}} \right)^2 + \kappa^2 \right] \right)
\]

\[
\times \exp \left[ i \sum_{\alpha=1}^{M} q_{\alpha} \sum_{r=1}^{N} x_{\alpha}^{r} + \frac{\kappa}{2} \sum_{\alpha=1}^{M} \sum_{r=1}^{M} (n_{\alpha} + 1 - 2r) x_{\alpha}^{r} \right]
\]

where the factor \( N!/n_1! \cdots n_M! \) is the total number of permutations of \( M \) clusters over \( N \) particles. Taking the derivatives and reorganizing the terms we obtain

\[
Q^{(M,M)}_{n,n}(q, q') = C^{(M)}_{q,n} C^{(M)}_{q',n} N! \left( \prod_{\alpha<\beta} \prod_{r=1}^{M} \prod_{r'=1}^{M} \left[ q_{\alpha} - q_{\beta} - \frac{i\kappa}{2} (n_{\alpha} - n_{\beta} - 2r + 2r') \right]^2 + \kappa^2 \right)
\]

\[
\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{\alpha=1}^{M} \left[ \frac{n_{\alpha} \kappa}{(n_{\alpha}!)^2 \kappa^{n_{\alpha}}} (2\pi) \delta(q_{\alpha} - q_{\alpha}') \right] .
\]

Now let us prove that the ‘off-diagonal’ terms of equation (C.17), in which the permutations \( P \) and \( P' \) are different, give no contribution. Here we can also choose one of the permutations, say the permutation \( P \), to be the ‘trivial’ one represented by the line in equation (C.3) with the cluster ordering denoted by \( \alpha_0(a) \). Given the symmetry of the wavefunctions it will be sufficient to consider the contribution of the sector \( x_1 < x_2 < \cdots < x_N \). According to equation (C.17), we get

\[
Q^{(M,M)}_{n,n}(q, q') \propto \sum_{P' < P} (-1)^{|P'|}
\]

\[
\times \int_{x_1 < \cdots < x_N} d^N x \exp \left[ -i \sum_{\alpha=1}^{M} q'_{\alpha} \sum_{a \in \Omega'_{\alpha}} x_{a} - \frac{\kappa}{4} \sum_{\alpha=1}^{M} \sum_{a,b \in \Omega'_{\alpha}} |x_{a} - x_{b}| \right]
\]

\[
\times \left( \prod_{\alpha_0(a) \neq \alpha_0(b)} [-i(\partial_{x_{a}} - \partial_{x_{b}}) + i\kappa \operatorname{sgn}(x_{a} - x_{b})] \right)
\]

doi:10.1088/1742-5468/2010/03/P03022
we can always find a cluster $\Omega$ which contains no ‘internal’ products among particles belonging to the cluster $\Omega$ carefully at the structure of the products in equation (C.22). Unlike the first product, not to the cluster (denoted by the symbol ‘$o$’), some of the clusters $\Omega$ must be different from $\Omega$. As an illustration, let us consider a particular case of $N = 10$, with three clusters $n_1 = 5$ (denoted by the symbol ‘$\bigcirc$’), $n_2 = 2$ (denoted by the symbol ‘$\times$’), and $n_3 = 3$ (denoted by the symbol ‘$\triangle$’):

$$\begin{array}{cccccccccccc}
\text{Particle number} & a & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text{Permutation $\alpha o(a)$} & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \times & \times & \triangle & \triangle & \triangle & \triangle \\
\text{Permutation $\alpha'(a)$} & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \times & \times & \triangle & \triangle & \triangle & \triangle & \triangle.
\end{array}$$

Here in the permutation $\alpha'(a)$ the particle $a = 4$ belongs to the cluster $\alpha = 3$ (and not to the cluster $\alpha = 1$ as in the permutation $\alpha o(a)$) and the particle $a = 8$ belongs to the cluster $\alpha = 1$ (and not to the cluster $\alpha = 3$ as in the permutation $\alpha o(a)$). Now let us look carefully at the structure of the products in equation (C.22). Unlike the first product, which contains no ‘internal’ products among particles belonging to the cluster $\Omega$, the second product does. Besides, the signs of the differential operators ($\partial x_a - \partial x_b$) in the second product are opposite to the ‘normal’ ones in the first product (cf equations (C.7)–(C.9)). It is these two factors (the presence of the ‘internal’ products and the ‘wrong’ signs of the differential operators) which make the ‘off-diagonal’ contributions, equation (C.22), to be zero. Indeed, in the above example, the second product contains the term

$$\begin{align*}
\Pi_{4,5}' &\equiv [-i(\partial x_4 - \partial x_5) + i\kappa] \exp \left[ i \sum_{\alpha=1}^{3} q_{\alpha} \sum_{a \in \Omega_{\alpha}} x_a + \frac{\kappa}{2} \sum_{\alpha=1}^{3} \sum_{a \in \Omega_{\alpha}} (n_{\alpha} + 1 - 2r(a))x_a \right] \\
&\text{(C.23)}
\end{align*}$$

(we remind ourselves that the particles in the clusters $\Omega_{\alpha}$ are ordered, and in particular $x_4 < x_5$). Taking the derivatives, we get

$$\begin{align*}
\Pi_{4,5}' &\propto \left[ -i q_1 + \frac{\kappa}{2} (n_1 + 1 - 2r(4)) - i q_1 - \frac{\kappa}{2} (n_1 + 1 - 2r(5)) \right] + i \kappa \\
&\quad \propto [r(4) - r(5) + 1] = 0 \\
&\text{(C.24)}
\end{align*}$$

since in the first cluster $r(a) = a$.

One can easily understand that the above example reflects the general situation. Since all the cluster sizes $n_{\alpha}$ are supposed to be different, whatever the permutation $\alpha'(a)$ is, we can always find a cluster $\Omega$ such that some of its particles belong to the same cluster number $\alpha$ in the permutation $\alpha'(a)$ while the others do not. Then one has to consider the contribution of the product of two neighboring number points:

$$\begin{align*}
\Pi_{k,k+1}' &\equiv [-i(\partial x_k - \partial x_{k+1}) + i\kappa] \exp \left[ i \sum_{\alpha=1}^{3} q_{\alpha} \sum_{a \in \Omega_{\alpha}} x_a + \frac{\kappa}{2} \sum_{\alpha=1}^{3} \sum_{a \in \Omega_{\alpha}} (n_{\alpha} + 1 - 2r(a))x_a \right] \\
&\text{(C.25)}
\end{align*}$$

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where in the permutation $\alpha'(a)$ the particle $k$ belongs to the cluster number $\alpha$ and the particle $(k + 1)$ belongs to some other cluster. Taking the derivatives one gets
\[
\Pi_{k,k+1} \propto [r(k) - r(k + 1) + 1] = 0 \tag{C.26}
\]
as $r(a)$ is the ‘internal’ particle number in the cluster $\Omega^a$, where $r(k + 1) = r(k) + 1$ (cf equations (C.7)–(C.9)).

Thus, the only non-zero contribution to the overlap, equation (C.15), of two wavefunctions $\Psi^{(M)}_{\mathbf{q}'\mathbf{n}}(\mathbf{x})$ and $\Psi^{(M)}_{\mathbf{q}\mathbf{n}}(\mathbf{x})$ (having the same number of clusters $M$ and characterized by the same set of integer parameters $1 \leq n_1 < n_2 < \cdots < n_M$) comes from the ‘diagonal’ terms, equation (C.21):
\[
Q^{(M,M)}_{\mathbf{q}\mathbf{n}}(\mathbf{q}, \mathbf{q}') = (C^{(M)}_{\mathbf{q}\mathbf{n}})^2 N! \prod_{\alpha=1}^{M} \left[ \frac{n_{\alpha} \kappa}{(n_{\alpha})^2 \kappa^\alpha} \right] \\
\times \left( \prod_{\alpha,\beta=r=1}^{M} \prod_{\alpha<\beta,r'=1}^{M} \left[ r_{\alpha} - r_{\beta} - \frac{1}{2}(n_{\alpha} - n_{\beta} - 2r + 2r') \right]^2 + \kappa^2 \right) \\
\times \prod_{\alpha=1}^{M} [(2\pi) \delta(q_{\alpha} - q'_{\alpha})]. \tag{C.27}
\]

The situation when there are clusters which have the same numbers of particles $n_{\alpha}$ is somewhat more complicated. Let us consider the overlap between two wavefunction $\Psi^{(M)}_{\mathbf{q}'\mathbf{n}}(\mathbf{x})$ and $\Psi^{(M)}_{\mathbf{q}\mathbf{n}}(\mathbf{x})$ (which, as before, have the same $M$ and $\mathbf{n}$) such that in the set of $M$ integers $n_1, n_2, \ldots, n_M$ there are two $n_{\alpha}$s which are equal, say $n_{\alpha_1} = n_{\alpha_2}$ (where $\alpha_1 \neq \alpha_2$). In the eigenstate $(\mathbf{q}', \mathbf{n})$ these two clusters have the center-of-mass momenta $q_{\alpha_1}'$ and $q_{\alpha_2}'$ and in the eigenstate $(\mathbf{q}, \mathbf{n})$ they have the momenta $q_{\alpha_1}$ and $q_{\alpha_2}$, respectively. According to the above discussion, the non-zero contributions in the summation over the cluster permutations $\alpha(a)$ and $\alpha'(a)$ in equation (C.17) appear only if the clusters $\{\Omega^a\}$ of the permutation $\alpha(a)$ totally coincide with the clusters $\{\Omega^\prime_a\}$ of the permutation $\alpha'(a)$. In the case when all $n_{\alpha}$ are different this is possible only if the permutation $\alpha(a)$ coincides with the permutation $\alpha'(a)$. In contrast to that, in the case when we have $n_{\alpha_1} = n_{\alpha_2}$, there are two non-zero options. The first one, as before, is given by the ‘diagonal’ terms with $\alpha(a) = \alpha'(a)$ (so that the clusters $\{\Omega^a\}$ and $\{\Omega^\prime_a\}$ are just the same) and this contribution is proportional to $\delta(q_{\alpha_1} - q_{\alpha_1}') \delta(q_{\alpha_2} - q_{\alpha_2}')$. The second (‘off-diagonal’) contribution is given by such a permutation $\alpha'(a)$ in which the cluster $\Omega_{\alpha_1}'$ (of the permutation $\alpha'(a)$) coincides with the cluster $\Omega_{\alpha_2}$ (of the permutation $\alpha(a)$) and the cluster $\Omega_{\alpha_2}'$ (of the permutation $\alpha'(a)$) coincide with the cluster $\Omega_{\alpha_1}$ (of the permutation $\alpha(a)$) while the rest of the clusters of these two permutations are the same, $\Omega_{\alpha}' = \Omega_{\alpha}(\alpha \neq \alpha_1, \alpha_2)$. Correspondingly, this last contribution is proportional to $\delta(q_{\alpha_1} - q_{\alpha_1}') \delta(q_{\alpha_2} - q_{\alpha_1}) (-1)^{n_{\alpha_1}}$. In fact, this situation with two equivalent contributions is the consequence of the symmetry of the wavefunction $\Psi^{(M)}_{\mathbf{q}'\mathbf{n}}(\mathbf{x})$: the permutation of two momenta $q_{\alpha_1}$ and $q_{\alpha_2}$ belonging to the clusters which have the same numbers of particles, $n_{\alpha_1} = n_{\alpha_2}$, produces just the factor $(-1)^{n_{\alpha_1}}$ (see the discussion below equation (52)). Therefore considering the clusters with equal numbers of particles as equivalent and restricting analysis to the sectors $q_{\alpha_1} < q_{\alpha_2}$: $q_{\alpha_1}' < q_{\alpha_2}'$ we find that the second contribution, $\delta(q_{\alpha_1} - q_{\alpha_1}') \delta(q_{\alpha_2} - q_{\alpha_1})$, is identically equal to zero, thus returning to the above result equation (C.27).
In a generic case the $M$-component vector $\mathbf{n}$ can be represented in the form

$$\mathbf{n} = \{m_1, \ldots, m_s, m_2, \ldots, m_s, \ldots, m_k, \ldots, m_k\}$$  \hspace{1cm} (C.28)

where $s_1 + s_2 + \cdots + s_k = M$ and $k$ integers $\{m_k\}$ ($1 \leq k \leq M$) are all supposed to be different:

$$1 \leq m_1 < m_2 < \cdots < m_k.$$  \hspace{1cm} (C.29)

Due to the symmetry with respect to the momenta permutations inside the subsets of equal $n$s it is sufficient to consider the wavefunctions in the sectors

$$q_1 < q_2 < \cdots < q_{s_1};$$

$$q_{s_1+1} < q_{s_1+2} < \cdots < q_{s_1+s_2};$$

$$\cdots \cdots \cdots$$

$$q_{s_1+\cdots+s_k-1+1} < q_{s_1+\cdots+s_k-1+2} < \cdots < q_{s_1+\cdots+s_k+1}.$$  \hspace{1cm} (C.30)

In this representation we again recover the above result equation (C.27)

Finally, let us consider the overlap of two eigenstates described by two different sets of integer parameters, $\mathbf{n}' \neq \mathbf{n}$. In fact, this situation is quite simple because if the clusters of the two states are different from each other, it means that in the summation over the pairs of permutations $P$ and $P'$ in equation (C.17) there exist no two permutations for which these two sets of clusters $\{\Omega_\alpha\}$ and $\{\Omega'_\alpha\}$ would coincide. Which, according to the above analysis, means that this expression is equal to zero. Note that the condition $M' \neq M$ automatically implies that $\mathbf{n}' \neq \mathbf{n}$.

Thus we have proved that

$$Q_{\mathbf{n},\mathbf{n}'}^{(M,M')} (q, q') = \left(C^{(M)}_{\mathbf{n},\mathbf{n}'}\right)^2 \delta(M, M') \left(\prod_{\alpha=1}^{M} \delta(n_\alpha, n'_\alpha)\right) \left(\prod_{\alpha=1}^{M} (2\pi)\delta(q_\alpha - q'_\alpha)\right)$$

$$\times N! \prod_{\alpha=1}^{M} \left[\frac{n_\alpha \kappa}{(n_\alpha')^2 \kappa'^2}\right]$$

$$\times \left(\prod_{\alpha<\beta} \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n'_\beta} \left[q_\alpha - q_\beta - \frac{i\kappa}{2}(n_\alpha - n_\beta - 2r + 2r') + \kappa^2\right]^2 + \kappa^2\right)$$  \hspace{1cm} (C.31)

where the integer parameters $\{n_\alpha\}$ and $\{n'_\alpha\}$ are assumed to have the generic structure represented in equations (C.28) and (C.29), and the momenta $\{q_\alpha\}$ and $\{q'_\alpha\}$ of the clusters with equal numbers of particles are restricted in the sectors, equation (C.30). Finally, according to equation (C.31), the orthonormality condition defines the normalization constant $C^{(M)}_{\mathbf{n},\mathbf{n}}$ as it is given in equation (59).

**Appendix D**

In this appendix we simplify the last term in the rhs of equation (72) and prove that

$$\Pi \equiv \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \frac{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta - 2r + 2r')|^2}{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2 + \kappa^2} = \frac{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)|^2}.$$  \hspace{1cm} (D.1)

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We rewrite $\Pi$ as

$$\Pi = \left\{ \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left| \left( q_\alpha - \frac{i\kappa}{2} n_\alpha \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) + i\kappa(r-r') \right|^2 \right\}$$

$$\times \left\{ \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left[ \left( q_\alpha - \frac{i\kappa}{2} n_\alpha + i\kappa r \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta + i\kappa r' \right) - i\kappa \right] \right. \right.$$  

$$\times \left. \left[ \left( q_\alpha - \frac{i\kappa}{2} n_\alpha \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) - i\kappa (r-r'+1) \right] \right\}^{-1}. \quad (D.2)$$

Redefining the indices $r$ and $r'$ of the product in the left brackets $[\cdots]$ of the denominator

$$r \to n_\alpha + 1 - r \quad r' \to n_\beta + 1 - r' \quad (D.3)$$

we find

$$\Pi = \left\{ \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left| \left( q_\alpha - \frac{i\kappa}{2} n_\alpha \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) + i\kappa(r-r') \right|^2 \right\}$$

$$\times \left\{ \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left[ \left( q_\alpha + \frac{i\kappa}{2} n_\alpha \right) - \left( q_\beta + \frac{i\kappa}{2} n_\beta \right) - i\kappa (r-r'+1) \right] \right. \right.$$  

$$\times \left. \left[ \left( q_\alpha - \frac{i\kappa}{2} n_\alpha \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) + i\kappa (r-r'+1) \right] \right\}^{-1}. \quad (D.4)$$

or

$$\Pi = \frac{\prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r-r')|^2}{\prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r-r'+1)|^2}. \quad (D.5)$$

Now, shifting the product over $r$ in the denominator by 1 we obtain

$$\Pi = \frac{\prod_{r'=1}^{n_\beta} \prod_{r=1}^{n_\alpha} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r-r')|^2}{\prod_{r'=2}^{n_\beta} \prod_{r=1}^{n_\alpha} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r-r'+1)|^2}$$

$$= \frac{\prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(1-r')|^2}{\prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(n_\alpha + 1 - r')|^2}. \quad (D.6)$$

Redefining $r'$ in the product in the denominator

$$r' \to n_\beta + 1 - r' \quad (D.7)$$

we obtain

$$\Pi = \frac{\prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r'-1)|^2}{\prod_{r'=1}^{n_\beta} |(q_\alpha + (i\kappa/2)n_\alpha) - (q_\beta + (i\kappa/2)n_\beta) + i\kappa r'|^2}$$

$$= \frac{\prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) - i\kappa(r'-1)|^2}{\prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) - i\kappa r'|^2}. \quad (D.8)$$

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Finally, shifting the product in the numerator by \((-1)\), we get
\[
\Pi = \prod_{n=0}^{n_3-1} \frac{(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) - i\kappa r'}{2} \frac{(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) - i\kappa r'}{2} = \frac{|q_\alpha - (i\kappa/2)n_\alpha| - (q_\beta - (i\kappa/2)n_\beta)|}{2} \frac{|q_\alpha - (i\kappa/2)n_\alpha| - (q_\beta - (i\kappa/2)n_\beta)|}{2} = \frac{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)|^2} \tag{D.9}
\]
which proves equation (D.1).

Next, we substitute equation (D.1) into the expression equation (72) for the partition function \(Z(N,L)\) and represent it in the form of equation (73), where
\[
\tilde{Z}(N,L) = \int_{-\infty}^{+\infty} \frac{dq}{2\pi \kappa N} e^{-\frac{(NL/2\beta)q^2 + (\kappa^2/24\beta)N}{2}} + \sum_{M=2}^{\infty} \frac{1}{M!} \sum_{n_1 \cdots n_M} \left[ \prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi \kappa n_\alpha} \right] \delta \left( \sum_{\alpha=1}^{M} n_\alpha, N \right) \\
\times e^{-(L/2\beta)\sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 + (\kappa^2/24\beta)\sum_{\alpha=1}^{M} n_\alpha^2} \prod_{\alpha < \beta} \frac{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)|^2} \tag{D.10}
\]
Redefining all the momenta
\[
q_\alpha = \left( \frac{\beta \kappa}{L} \right)^{1/3} p_\alpha \tag{D.11}
\]
and introducing a new parameter
\[
\lambda = \frac{1}{2} \left( \frac{L \kappa^2}{\beta} \right)^{1/3} = \frac{1}{2} (\beta^3 u^2 L)^{1/3} \tag{D.12}
\]
we rewrite
\[
\frac{L}{2\beta} q_\alpha^2 = \lambda p_\alpha^2 \tag{D.13}
\]
\[
\frac{q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)}{q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)} = \frac{p_\alpha - p_\beta - i\lambda (n_\alpha - n_\beta)}{p_\alpha - p_\beta - i\lambda (n_\alpha + n_\beta)} \tag{D.14}
\]
and obtain the integrals
\[
\int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi \kappa n_\alpha} \cdots = \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi \kappa} \int_{-\infty}^{+\infty} \frac{\lambda}{2\pi \kappa} \int_{0}^{+\infty} dt e^{-\lambda n_\alpha t} \cdots = \int_{-\infty}^{+\infty} \frac{dp_\alpha}{4\pi} \int_{0}^{+\infty} dt e^{-\lambda n_\alpha t} \cdots \tag{D.15}
\]
Substituting the transformations, equations (D.13)–(D.15), into equation (D.10), we get equation (75).

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Appendix E

To perform the summation over \( n_1, \ldots, n_M \) in equation (78), let us modify the pre-exponential factor:

\[
F(p; n) = \prod_{\alpha < \beta}^M \frac{|p_\alpha - p_\beta - i\lambda(n_\alpha - n_\beta)|^2}{|p_\alpha - p_\beta - i\lambda(n_\alpha + n_\beta)|^2} = \prod_{\alpha \neq \beta}^M \frac{(|p_\alpha - p_\beta| + i\lambda(n_\alpha - n_\beta))}{\sqrt{(|p_\alpha - p_\beta| + i\lambda(n_\alpha + n_\beta))(|p_\alpha - p_\beta| - i\lambda(n_\alpha + n_\beta))}}. \tag{E.1}
\]

We introduce the auxiliary fields \( \chi_{\alpha\beta}, \phi_{\alpha\beta}, \) and \( \psi_{\alpha\beta} \) to generate the numerator

\[
(|p_\alpha - p_\beta| + i\lambda(n_\alpha - n_\beta)) = \frac{\partial}{\partial \chi_{\alpha\beta}} \exp[(|p_\alpha - p_\beta| + i\lambda(n_\alpha - n_\beta)) \chi_{\alpha\beta}]
\]

and the denominators

\[
\frac{1}{\sqrt{(|p_\alpha - p_\beta| + i\lambda(n_\alpha + n_\beta))}} = \int_{-\infty}^{+\infty} \frac{d\phi_{\alpha\beta}}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( |p_\alpha - p_\beta| + i\lambda(n_\alpha + n_\beta) \right) \phi_{\alpha\beta}^2 \right] \tag{E.2}
\]

\[
\frac{1}{\sqrt{(|p_\alpha - p_\beta| - i\lambda(n_\alpha + n_\beta))}} = \int_{-\infty}^{+\infty} \frac{d\psi_{\alpha\beta}}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( |p_\alpha - p_\beta| - i\lambda(n_\alpha + n_\beta) \right) \psi_{\alpha\beta}^2 \right]. \tag{E.3}
\]

Combining together equations (E.2)–(E.4) we rewrite the prefactor as

\[
F(p; n) = \left( \prod_{\alpha \neq \beta}^M \int_{-\infty}^{+\infty} \frac{d\psi_{\alpha\beta} d\phi_{\alpha\beta}}{2\pi} \frac{\partial}{\partial \chi_{\alpha\beta}} \right) \times \exp \left[ -\frac{1}{2} \sum_{\alpha \neq \beta}^M |p_\alpha - p_\beta| \left( \phi_{\alpha\beta}^2 + \psi_{\alpha\beta}^2 - 2\chi_{\alpha\beta} \right) + i\lambda \sum_{\alpha = 1}^M n_\alpha \right] \bigg|_{\chi_{\alpha\beta} = 0}. \tag{E.5}
\]

where

\[
\eta_\alpha \equiv \eta_\alpha(\psi, \phi, \chi) = \frac{1}{2} \sum_{\beta \neq \alpha}^M (\psi_{\alpha\beta}^2 + \psi_{\beta\alpha}^2 - \phi_{\alpha\beta}^2 - \phi_{\beta\alpha}^2 + 2\chi_{\alpha\beta} - 2\chi_{\beta\alpha}). \tag{E.6}
\]

Introducing now the integro-differential operator

\[
\int \tilde{G}_M(p; \psi, \phi, \chi) \equiv \left[ \prod_{\alpha \neq \beta}^M \int_{-\infty}^{+\infty} \frac{d\psi_{\alpha\beta} d\phi_{\alpha\beta}}{2\pi} \frac{\partial}{\partial \chi_{\alpha\beta}} \right] e^{-\frac{1}{2} \sum_{\alpha \neq \beta}^M |p_\alpha - p_\beta| \left( \psi_{\alpha\beta}^2 + \phi_{\alpha\beta}^2 - 2\chi_{\alpha\beta} \right)} \tag{E.7}
\]

as well as the integration operator

\[
\int D_M(y, p) \equiv \prod_{\alpha = 1}^M \int_{-\infty}^{+\infty} dy_\alpha \int_{-\infty}^{+\infty} dp_\alpha \int_0^{+\infty} dt_\alpha \exp \left[ -\frac{1}{4\pi} \int_0^{+\infty} dt_\alpha \mathrm{Ai}(y_\alpha + t_\alpha + p_\alpha^2) \right] \tag{E.8}
\]

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the partition function, equation (78), can be represented as follows:

\[
\hat{Z}(N, \lambda) = \int \mathcal{D}_1(y, p) \exp(\lambda Ny) + \sum_{M=2}^{\infty} \frac{1}{M!} \int \mathcal{D}_M(y, p) \int \mathcal{G}_M(p; \psi, \phi, \chi) \times \sum_{n_1, \ldots, n_M=1}^{\infty} \delta \left( \sum_{\alpha=1}^{M} n_{\alpha}, N \right) \prod_{\alpha=1}^{M} e^{(y_{\alpha} + i\eta_{\alpha})n_{\alpha}}.
\]

Next we prove that

\[
S(N, M) \equiv \sum_{n_1 \ldots n_M=1}^{\infty} a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots a_M^{n_M} \delta \left( \sum_{\alpha=1}^{M} n_{\alpha}, N \right)
= a_1^{N} \frac{a_2}{(a_1 - a_2)} \frac{a_3}{(a_1 - a_3)} \cdots \frac{a_M}{(a_1 - a_M)} + a_2^{N} \frac{a_1}{(a_2 - a_1)} \frac{a_3}{(a_2 - a_3)} \cdots \frac{a_M}{(a_2 - a_M)}
+ \ldots + a_M^{N} \frac{a_1}{(a_M - a_1)} \frac{a_2}{(a_M - a_2)} \cdots \frac{a_{M-1}}{(a_M - a_{M-1})}.
\]

To unbound the summations over \(n_1, n_2, \ldots, n_M\) in the above series, let us introduce the integral representation of the Kronecker symbol:

\[
\delta(k, m) = \int_C \frac{dz}{2\pi i} z^{m-k-1}
\]

where both \(k\) and \(m\) are assumed to be positive integers and the integration in the complex plane goes over a closed contour \(C\) around zero. Using this representation we obtain

\[
S(N, M) = \int_C \frac{dz}{2\pi i} z^{N-1} \prod_{\alpha=1}^{M} \sum_{n_{\alpha}=1}^{\infty} \left( \frac{a_{\alpha}}{z} \right)^{n_{\alpha}}
\]

where it is assumed that the radius \(R_C\) of the contour \(C\) is big enough, namely \(R_C > \max_{\alpha} |a_{\alpha}|\), so that all the complex numbers \(a_1, a_2, \ldots, a_M\) are contained inside \(C\). In this case all the summations in equation (E.12) are convergent, and we find

\[
S(N, M) = \int_C \frac{dz}{2\pi i} z^{N-1} \frac{a_1}{(z - a_1)} \frac{a_2}{(z - a_2)} \cdots \frac{a_M}{(z - a_M)}.
\]

Since the above integral is equal to the sum of \(M\) pole contributions at \(z = a_{\alpha}(\alpha = 1, 2, \ldots, M)\), one gets equation (E.10).

Now, substituting equation (E.10) (with \(a_{\alpha} = \exp[\lambda(y_{\alpha} + i\eta_{\alpha})]\)) into equation (E.9) we obtain

\[
\hat{Z}(N, \lambda) = \int \mathcal{D}_1(y, p) \exp(\lambda Ny) + \sum_{M=2}^{\infty} \frac{1}{M!} M \times \int \mathcal{D}_M(y, p) \int \mathcal{G}_M(p; \psi, \phi, \chi) e^{\lambda N(y_{\alpha} + i\eta_{\alpha})} \prod_{\alpha=2}^{M} e^{\lambda(y_{\alpha} + i\eta_{\alpha}) - e^{\lambda(y_{\alpha} + i\eta_{\alpha})}}
\]

where we have used the symmetry of the expressions in equations (E.7) and (E.8) with respect to permutations of \(\{(y_{\alpha} + i\eta_{\alpha})\}\).

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Appendix F

In this appendix, we show that, independently of the values of the momenta \(p_\alpha\) and the parameter \(s\), the integro-differential factor in equation (87) provides unity:

\[
\mathcal{G} \equiv \int \hat{G}_M(p; \psi, \phi, \chi)e^{i\eta(p, \phi, \chi)} = 1. \tag{F.1}
\]

Substituting here the definitions, equations (E.6) and (E.7), the factor \(\mathcal{G}\) can be split into two factors:

\[
\mathcal{G} = \mathcal{G}_A \mathcal{G}_B
\]

where

\[
\mathcal{G}_A = \left[ \prod_{2 \leq \alpha \neq \beta} \int_{-\infty}^{+\infty} \frac{d\psi_{\alpha\beta} d\phi_{\alpha\beta}}{2\pi} \frac{\partial}{\partial \chi_{\alpha\beta}} \right] e^{-\left(1/2\right) \sum_{\alpha, \beta} |p_\alpha - p_\beta| \left(\psi_{\alpha\beta}^2 + \phi_{\alpha\beta}^2 - 2\chi_{\alpha\beta}\right)} \bigg|_{\chi_{\alpha\beta} = 0}
\]

and

\[
\mathcal{G}_B = \left[ \prod_{\alpha = 2}^{M} \int_{-\infty}^{+\infty} \frac{d\psi_{\alpha 1} d\phi_{\alpha 1}}{2\pi} \frac{\partial}{\partial \chi_{\alpha 1}} \int_{-\infty}^{+\infty} \frac{d\psi_{1\alpha} d\phi_{1\alpha}}{2\pi} \frac{\partial}{\partial \chi_{1\alpha}} \right] \times \exp \left[ -\frac{1}{2} \sum_{\alpha = 2}^{M} \left| p_\alpha - p_1 \right| \left(\psi_{\alpha 1}^2 + \phi_{\alpha 1}^2 + \psi_{1\alpha}^2 + \phi_{1\alpha}^2\right) + \sum_{\alpha = 2}^{M} \left| p_\alpha - p_1 \right| \left(\chi_{\alpha 1} + \chi_{1\alpha}\right) + \frac{i}{2} s \sum_{\alpha = 2}^{M} \left(\psi_{\alpha 1}^2 - \phi_{\alpha 1}^2 + \psi_{1\alpha}^2 - \phi_{1\alpha}^2\right) + is \sum_{\alpha = 2}^{M} \left(\chi_{\alpha 1} - \chi_{1\alpha}\right) \right] \bigg|_{\chi_{\alpha 1}, \chi_{1\alpha} = 0}.
\]

For the factor \(\mathcal{G}_A\), equation (F.3), we easily find

\[
\mathcal{G}_A = \prod_{2 \leq \alpha \neq \beta} \frac{1}{\sqrt{|p_\alpha - p_\beta|}} \cdot \frac{1}{\sqrt{|p_\alpha - p_\beta|}} \cdot |p_\alpha - p_\beta| \equiv 1. \tag{F.5}
\]

In a similar way we obtain for the factor \(\mathcal{G}_B\)

\[
\mathcal{G}_B = \prod_{\alpha = 2}^{M} \int_{-\infty}^{+\infty} \frac{d\psi_{\alpha 1}}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left| |p_\alpha - p_1| - is \right| \psi_{\alpha 1}^2 \right)
\]

\[
\times \prod_{\alpha = 2}^{M} \int_{-\infty}^{+\infty} \frac{d\phi_{\alpha 1}}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left| |p_\alpha - p_1| + is \right| \phi_{\alpha 1}^2 \right)
\]

\[
\times \prod_{\alpha = 2}^{M} \int_{-\infty}^{+\infty} \frac{d\psi_{1\alpha}}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left| |p_\alpha - p_1| - is \right| \psi_{1\alpha}^2 \right)
\]

\[
\times \prod_{\alpha = 2}^{M} \int_{-\infty}^{+\infty} \frac{d\phi_{1\alpha}}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left| |p_\alpha - p_1| + is \right| \phi_{1\alpha}^2 \right)
\]

\[
\times \prod_{\alpha = 2}^{M} \frac{\partial}{\partial \chi_{\alpha 1}} \exp(\left| |p_\alpha - p_1| - is \chi_{\alpha 1}\right)|_{\chi_{\alpha 1} = 0}
\]
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\[
\times \prod_{\alpha=2}^{M} \frac{\partial}{\partial \chi_{1\alpha}} \exp([|p_{\alpha} - p_{1}| + is] \chi_{1\alpha}) \right|_{\chi_{1\alpha}=0} \tag{F.6}
\]

\[
= \prod_{\alpha=2}^{M} \frac{1}{(p_{\alpha} - p_{1})^2 + s^2} \times [(p_{\alpha} - p_{1})^2 + s^2] \equiv 1 \tag{F.7}
\]

which proves equation (F.1).

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