Analytical solution of the Bohr-Mottelson equation in the presence of minimal length for yukawa potential using hypergeometric method.

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Abstract. The relativistic energy and wave function of the Bohr-Mottelson equation in the presence of minimal length are investigated by the hypergeometric method. The Bohr-Mottelson equation in the presence of minimal length is influenced by Yukawa potential. The energy relativistic is calculated by using Matlab software, and the wave function is expressed in the Hypergeometric term. The existence of minimal length caused increased of the energy relativistic of Bohr-Mottelson equation.

1. Introduction

The behavior of nucleus is described using the Bohr-Mottelson equation [1-6]. The Bohr-Mottelson equation is studied using potentials such as Infinite Square well potential [1,2], Eckart potential [3], Kratzer potential [4] and Davidson potential [5]. The method is used to solve the Bohr-Mottelson equation such as Supersymmetric Quantum Mechanics (SUSYQM) [5], Nikiforov-Uvarov [3] and Asymptotic Iteration Method (AIM) [6].

In mechanics quantum commutation relations between position and momentum operators is explained by Heisenberg Uncertainty Principle. At Heisenberg Uncertainty Principle is influenced by gravity quantum, Heisenberg Uncertainty Principle with additional a small constant as follow [1,2]

\[ [X, P] \geq i\hbar \left(1 + \alpha (\Delta P)^2\right) \]

(1)

Here \( \alpha \) is parameter minimal length that has a very small positive value, the \( X \) and \( P \) are position and momentum, respectively. The equation (1) is General Uncertainty Principle or minimal length [1,2].

This paper is studied the Bohr-Mottelson equation for rigid deformed nucleus in the presence of minimal length for Yukawa potential. To solve the Bohr-Mottelson equation in the presence of minimal length is used Hypergeometric method. By using Hypergeometric method, the relativistic energy and wave function is obtained. The organized this paper as follows, the Bohr-Mottelson Equation in the presence of minimal length and Hypergeometric method is described in section 2. The result and discussion in section 3 and the last section is conclusion.
2. Experimental

2.1. The Bohr-Mottelson equation in presence of minimal length

The General Uncertainty Principle (GUP) modify of Heisenberg Uncertainty Principle at influenced by gravity quantum. At energy much lower than the Planck mass, the equation (1) is reduced to Heisenberg Uncertainty Principle. By using the deformed canonical commutation relation in equation (1) becomes [2],

\[ \hat{X}_i = \hat{x}_i \]

\[ \hat{P}_i = \left( 1 + \alpha \hat{P}^2 \right) \hat{p}_i \]

By applying \( p^2 = -\hbar^2 \Delta \) in equation (3), so we get \( p^2 = -\frac{\hbar^2}{2B_m} \left( 1 - 2\alpha \Delta \right) \Delta \)

where \( \Delta \) is Laplacian operator. The Laplacian operator for nucleus is given

\[ \Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial q_i} \sqrt{g} g_i^j \frac{\partial}{\partial q_j} \]

with \( g \) is determinant and \( g_i^j \) is inverse of the matrix \( g_{ij} \) that has three degrees of freedom \( q_1 = \phi, q_2 = \theta, q_3 = \beta \), so we obtain Laplacian operator for nucleus

\[ \Delta = \left[ \frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^2 \frac{\partial}{\partial \beta} + \frac{1}{3\beta^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi^2} \right] \right] \]

The Hamiltonian equation which is expressed

\[ H = T + V(\beta) = \frac{p^2}{2B_m} + V(\beta) \]

where \( V(\beta) \) is potential energy in \( \beta \) function and \( B_m \) is a mass parameter. Then, equation (4) is inserted in equation (7), we get

\[ \left[ -\frac{\hbar^2}{2B_m} \Delta + \frac{\alpha \hbar^4}{B_m} \Delta^2 + V(\beta, \theta, \phi) - E \right] \chi(\beta, \theta, \phi) = 0 \]

The equation (8) is Bohr-Mottelson equation in the presence of minimal length. In the case of Bohr-Mottelson equation without the minimal length is setting \( \alpha_{ML} = 0 \) [12] for equation (8), so obtain square term is given as [2],

\[ \Delta^2 = \frac{4B_m^2}{\hbar^2} \left( V(\beta) - E_n \right)^2 \]

Equations (6) and (9) are inserted in equation (8) and multiplied by \( -\frac{2B_m}{\hbar^2} \) and \( \hbar = 1 \) (natural unit), we obtain
By applying the suitable variable change in equation and reduced to standard hypergeometric equation,

\[
\begin{bmatrix}
\frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^2 \frac{\partial}{\partial \beta} + \\
1 \\
\frac{1}{3\beta^2} \\
+ \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \theta^2}
\end{bmatrix}
\] 

\[+ 2B_m \left( 4\alpha_{ml} B_m \left( V(\beta) - E^0 \right)^2 \right) \left( V(\beta) - E \right) \right] \chi(\beta, \theta, \varphi) = 0 \quad (10)

By using the separation variable method \( \chi(\beta, \theta, \varphi) = R(\beta) \Theta(\theta) \Phi(\varphi) \) in equation (10), we have,

\[
\left\{- \frac{1}{\Phi(\varphi)} \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi(\varphi)}{\partial \varphi^2} + \frac{1}{\Theta(\theta)} \frac{1}{\sin \theta} \frac{\partial \Theta(\theta)}{\partial \theta} \right\} = \lambda
\]

Equation (11) is Euler angles part of Bohr-Mottelson equation in presence of minimal length. The \( \beta \) – part of Bohr-Mottelson equation in presence of minimal length is given,

\[
\left\{ \frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^2 \frac{\partial R(\beta)}{\partial \beta} + 2B_m \left( E - V(\beta) \right) R(\beta) \right\} = \frac{\lambda}{3\beta^2} R(\beta)
\]

By applying \( R(\beta) = Z(\beta)/\beta \) and \( \lambda = L(L+1) \) in equation (12) so we have,

\[
\left\{ \frac{d^2 F(\beta)}{d\beta^2} - \frac{L(L+1)}{3\beta^2} Z(\beta) + 2B_m \left( E - V(\beta) \right) Z(\beta) - 8B_m^2 \alpha_{ml} \left( E^2 - 2EV(\beta) + V^2(\beta) \right) \right\} = 0
\]

The equation (13) is Bohr-Mottelson equation for a \( \beta \) – part in presence of minimal length effect for rigid deformed nucleus case.

2.2. Hypergeometric Method

The second-order differential equation of hypergeometric function as follow [7,8],

\[
m(1-m) \frac{d^2 \Psi}{dm^2} + \left( c - (a + b + 1)m \right) \frac{d \Psi}{dm} - ab \Psi = 0
\]

The energy eigenvalue is obtained from the condition in equation (14) [7,8],

\[
a = -n \quad \text{or} \quad b = -n
\]

where \( n = 0,1,2,3, \ldots \) Equation (15) can be finite series of polynomials of rank \( n \) by equation (14). The solution of a wave function is given as

\[
\Psi(m) = \sum_{k=0} (a)_k (b)_k \frac{(c)_k}{n!} m^k = 1 + \frac{ab}{c} m + \frac{a(a+1)b(b+1)}{c(c+1)} m^2 + \ldots
\]

By applying the suitable variable change in equation and reduced to standard hypergeometric equation, we get energy eigenvalue and wave function [7,8].
3. Results and Discussion

The Yukawa potential is used to describe nucleon-nucleon interactions in meson theory, the general Yukawa potential as follow [9,10],

\[ V(\beta) = \frac{-V_0 e^{-\eta \beta}}{\beta} \] (17)

where \( V_0 \) is potential depth and \( \eta \) is range of nucleon force. The function of \( 1/\beta \) is approximated by

\[ \frac{1}{\beta} = \frac{2\eta e^{-\eta \beta}}{1-e^{-2\eta \beta}} \] (18)

\[ \frac{1}{\beta^2} = \frac{2\eta^2}{\sinh^2 \eta \beta} \] (19)

Then the Bohr-Mottelson in equation (13) is inserted by equations (18) and (19), we get

\[
\frac{d^2 Z(\beta)}{d\beta^2} - \left[ \frac{2\eta^2 L(L+1)}{3} + 2\alpha B_m^2 V_o^2 \right] \frac{Z(\beta)}{\sinh^2 \eta \beta} \left[ - \left( B_m V_o - 8\alpha B_m^2 E^0 V_o + 4\alpha B_m^2 V_o^2 \right) \coth \eta \beta \right. \\
- \left( 2B_m E - 8\alpha B_m^2 E^{02} + 8\alpha B_m^2 V_o E^0 - 4\alpha B_m^2 V_o^2 - B_m V_o \right) \right] = Z(\beta) = 0 \] (20)

Equation (20) is the Bohr-Mottelson equation in presence of minimal length for Yukawa potential. The equation (20) is reduced becomes

\[
\frac{d^2 Z(\beta)}{d\beta^2} - \left[ \frac{v(v-1)}{\sinh^2 \eta \beta} - 2q \coth \eta \beta + k^2 \right] Z(\beta) = 0 \] (21)

Equation (21) is differential equation with,

\[ v(v-1) = \left( \frac{2\eta^2 L(L+1)}{3} + 2\alpha B_m^2 V_o^2 \right) \] (22)

\[ 2q = - \left( B_m V_o - 8\alpha B_m^2 E^0 V_o + 4\alpha B_m^2 V_o^2 \right) \] (23)

\[ -k^2 = \left( 2B_m E - 8\alpha B_m^2 E^{02} + 8\alpha B_m^2 V_o E^0 - 4\alpha B_m^2 V_o^2 - B_m V_o \right) \] (24)

By reducing equation (21) to hypergeometric differential equation type by using \( \coth \eta \beta = 1 - 2y \), we get

\[ y(1-y) \frac{d^2 Z(\beta)}{dy^2} + (1-2y) \frac{dZ(\beta)}{dy} + \left[ v(v-1) - \frac{4\alpha^2}{4y} - \frac{4\beta^2}{4(1-y)} \right] Z(\beta) = 0 \] (25)

with,

\[ \frac{-2q + k^2}{\eta^2} = 4\alpha^2 \] (26)

\[ \frac{-2q + k^2}{\eta^2} = 4\beta^2 \] (27)
\[ v' (v-1) = \frac{v(v-1)}{\eta^2} \]  

by applying the new wave function is given,

\[ Z (\beta) = y^{\alpha_\mu} (1-y)^{\beta_\mu} f(y) \]  

in equation (25), we obtain

\[ y(1-y) \frac{f^2(y)}{dy^2} + \left[ (2\alpha_H + 1) - (2\alpha_H + 2\beta_H + 2) y \right] \frac{f(y)}{dy} \]

\[ + \left[ v' (v-1) - (\alpha_H + \beta_H) (\alpha_H + \beta_H + 1) \right] f(y) = 0 \]  

The equation (30) is hypergeometric differential equation that obtain parameter hypergeometric, yield

\[ a = \alpha_H + \beta_H - (v-1), \quad b = \alpha_H + \beta_H + v, \quad c = 2\alpha_H + 1 \]  

By inserting equations (22)-(24),(26)-(28), and (31), we obtain

\[ E = \frac{\eta^2}{2B_m} \left[ \left( \frac{2L(L+1)}{3} + \frac{2\alpha B_m^2 V_o^2}{\eta^2} + \frac{1}{4} \right) - \frac{1}{2} - n \right] \]

\[ - \left( \frac{(B_m V_o^2 - 8\alpha B_m^2 E V_o^2 + 4\alpha B_m^2 V_o^2)}{\eta^2} \right)^2 \]

\[ + 4\alpha B_m E^{0.5} - 4\alpha B_m V_o E^0 + 2\alpha B_m V_o^2 + V_o \]  

The equation (32) is relativistic energy equation of Bohr-Mottelson in the presence of minimal length for Yukawa potential. To get relativistic energy, we calculated numerically equation by using Matlab software. The result is shown in Table 1.

|\( l \)| | \( \alpha = 0 \) | \( \alpha = 0.002 \) | \( \alpha = 0.004 \) | \( \alpha = 0.006 \) | \( \alpha = 0.008 \) | \( \alpha = 0.01 \) |
|---|---|---|---|---|---|---|
| 0 | 0 | -139.216 | -35.6483 | -16.2225 | -9.34032 | -6.11683 |
| 2 | 0.000852 | 0.000854 | 0.000855 | 0.000857 | 0.000858 | 0.00086 |
| 4 | 0.010209 | 0.010306 | 0.010402 | 0.010497 | 0.010593 | 0.010688 |
| 6 | 0.022921 | 0.023398 | 0.023874 | 0.024349 | 0.024822 | 0.025295 |
| 8 | 0.040441 | 0.041999 | 0.043554 | 0.045107 | 0.046658 | 0.048208 |
| 10 | 0.062944 | 0.066839 | 0.07073 | 0.074619 | 0.078504 | 0.082386 |

Table 1 show that energy relativistic without minimal length a lower that energy relativistic with minimal length. The relativistic energy is increased when the minimal length parameter and angular momentum quantum number (l) are increased.

Then, to obtain the wave function use equations (16), (29), and \( \coth \eta \beta = (1 - 2z) \), we get

\[ Z (\beta) = \frac{(1-\coth \eta \beta)^{\beta_\mu}}{2} \frac{(1+\coth \eta \beta)^{\alpha_\mu}}{2} \sum_{l=1}^{\alpha} \sum_{e_{1},e_{2}} F_{1} (a, b, c, y) \]  

And by applying equation (31) in equation (33), we obtain wave function in Table 2.
Table 2. The wave function of the Bohr-Mottelson equation in the presence of minimal length

\[
Z(\beta) = \frac{(1 - \coth \eta \beta)^{\alpha_n}}{2} \left(1 + \coth \eta \beta\right)^{\beta_n}
\]

\[
Z(\beta) = \frac{(1 - \coth \eta \beta)^{\alpha_n}}{2} \left(1 + \coth \eta \beta\right)^{\beta_n} \left[1 + \frac{(-n)(\alpha_H + \beta_H + \nu)}{2} \left(\frac{1 - \coth \eta \beta}{2}\right)\right]
\]

Table 2 is wave function of Bohr-Mottelson equation in presence of minimal length for \( n=0 \) and \( n=1 \), respectively. The value of wave function depends on the value of parameter hypergeometric.

4. Conclusion
The Bohr-Mottelson equation is investigated in the presence of minimal length effect for Yukawa potential by using Hypergeometric method. From the investigation of the Bohr-Mottelson equation is obtained relativistic energy and wave function in minimal length effect. From the result can be conclude that relativistic energy of the Bohr-Mottelson equation in the presence of minimal length for Yukawa potential increase duo to increase minimal length parameter.

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