Gorenstein objects in the $n$-Trivial extensions of abelian categories.

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Abstract

Given an abelian category, we introduce a categorical concept of (strongly) Gorenstein projective (resp., injective) objects, by defining a new special class of objects. Then we study the transfer of these properties when passing to an abelian category and its $n$–trivial extension category and also give a characterization of Gorenstein object over it. We give, at the end, applications of this study on the category of modules over an associative ring and triangular matrix rings.

1 Introduction

Enochs and all., in [7], introduced the so-called Gorenstein projective and injective modules over a ring $R$ and developed the first properties of Gorenstein homological algebra. Since then, many other works studied several properties of these Gorenstein objects. In particular, in [5], Mahdou and Bennis introduced strongly Gorenstein projective (resp., injective) modules as an analogue structure to free modules. Moreover, they gave a characterization of Gorenstein modules via strongly Gorenstein modules, see [5]. Also, in [14], Mahdou and Ouarghi studied the transfer of some Gorenstein homological properties between a ring and its trivial extension. Furthermore, in [1] Anderson, Bennis and co-authors gave a generalisation of $R \ltimes M$, raising a new ring called $n$–trivial ring extension by a family of modules $M = (M_i)_{i \in I}$ where, $I = \{1, 2, \ldots, n\}$ and denoted by $R \ltimes_n M$. In [1] they extensively studied its algebraic properties.

Motivated by the work done in [11], recently in [3], the author, Bennis and J.R. García gave a categorical generalisation of $R \ltimes_n M$ by constructing a new category, called the $n$–trivial extension of abelian category by a family of functors $F = (F_i)_{i \in I}$. Using a set of functors $T, C, U, Z$ and other tools, they were able to give a wide range of properties such as: generators, projective and injective objects, also ensuring the existence of limits and colimits in such categories.

In the present paper, we give a categorical approach for some Gorenstein properties. We define at first the meaning of a (strongly) Gorenstein projective (resp., injective) object in an abelian category,
then we give results concerning the transfer of these concepts from an abelian category to an \( n \)-trivial extension of this category. We, also, study the Gorenstein dimension in this categorical setting.

Our paper is structured in the following manner: In section 2, we collect definitions and results that we need throughout the paper. In section 3, we first define the class of so-called \( SP(\text{resp.}, SI) \)–objects which will be a cornerstone in establishing a definition of (Strongly) Gorenstein projective, injective objects. We conclude the section with a study of the homological algebra of Gorenstein objects. In section 4, under the light of the new categorical tools, we explore the Gorenstein homological modules over the \( n \)-trivial ring extension, for \( n \in \mathbb{N} \), also we characterize under some conditions Gorenstein flat modules. Finally, we apply our theorems to characterize Gorenstein projective modules over triangular matrix rings of the form

\[
\begin{pmatrix}
R_3 & 0 & 0 \\
M_1 & R_2 & 0 \\
M_2 & M_1 & R_3
\end{pmatrix}
\]

2 Preliminary

We start this section by collecting background materials that will be necessary in the sequel. For more details, we refer the reader to [15, 14, 10, 12, 3].

Throughout this paper \( R \) will be an associative ring with identity, and all modules will be, unless otherwise specified, unital left \( R \)-modules. The category of all left (right if needed) \( R \)-modules will be denoted by \( R\text{-Mod} \). In [4], the authors introduced three classes of modules called strongly Gorenstein projective, injective and flat modules. These modules allowed more subtle characterizations of Gorenstein projective and injective modules, similar to the characterization of projective modules via free modules. Also, in [3], the Gorenstein homological dimension of a ring \( R \), namely the Gorenstein global dimension of \( R \), denoted \( G - \text{gldim}(R) \) was studied.

In this paper, we propose a new categorical approach to give a more elaborated and global study of these notions but first we recall the structure that is the subject of our approach. In [3], the construction of the \( n \)-trivial extension of a ring \( R \) by a family \( M = (M_i)_{i=1}^n \) of \( R \)-bimodules and a family \( \varphi = (\varphi_{i,j})_{1 \leq i,j \leq n-1} \) of bilinear maps such that each \( \varphi_{i,j} \) is written multiplicatively

\[
\varphi_{i,j} : M_i \otimes M_j \longrightarrow M_{i+j}
\]

whenever \( i + j \leq n \) with

\[
\varphi_{i+j,k}(\varphi_{i,j}(m_i, m_j), m_k) = \varphi_{i,j+k}(m_i, \varphi_{j,k}(m_j, m_k)).
\]

The \( n \)-trivial extension of \( R \) by \( M = (M_i)_{i=1}^n \) is the additive group \( R \oplus M_1 \oplus \cdots \oplus M_n \) with the multiplication defined by:

\[
(m_0, ..., m_n)(m'_0, ..., m'_n) = \sum_{i+j=n} m_i m'_j.
\]

Notice that when \( n = 1 \), \( R \times R \) is the classical trivial extension of \( R \) by \( M \) which is denoted by \( R \times M \). Along this paper we maintain the notation \( R \times R \) for the \( n \)-trivial extension of the \( R \) by \( M = (M_i)_{i=1}^n \). In [3], the concept of the \( n \)-trivial extension of an abelian category by a family of functors was introduced as follows:

Let \( \mathcal{A} \) be an abelian category, \( F = (F_i)_{i \in I} \) be a family of additive covariant endofunctors of \( \mathcal{A} \) with \( I = \{1, \ldots, n\} \) and \( \Phi = (\Phi_{i,j} : F_i F_j \longrightarrow F_{i+j}) \) be a family of natural transformations such that, for
every $i, j, k \in I$ with $i + j + k \in I$, the diagram:

$$F_iF_jF_kX \xrightarrow{(\Phi_{i,j})X} F_iF_{j+k}X$$

commutes.

The category of right $n$-trivial extension of $\mathbb{A}$ by $F$ is given by:

**Objects**: An object of this category is of the form $(X, f)$ with $X$ is an object of $\mathbb{A}$ and $f = \langle f_i \rangle_{i \in I}$ is a family of morphisms $f_i : F_i X \to X$ such that for every $i, j \in I$, the diagram

$$F_iF_jX \xrightarrow{(\Phi_{i,j})X} F_iF_{j+k}X$$

commutes when $i + j \in I$, if not, the composition

$$F_iF_jX \xrightarrow{F_iF_j} F_iX \xrightarrow{f_i} X$$

is zero.

**Morphisms**: For every two objects $(X, \alpha)$ and $(Y, \beta)$, a morphism $\gamma : (X, \alpha) \to (Y, \beta)$ in $\mathbb{A} \ltimes F$ is a morphism $X \to Y$ in $\mathbb{A}$ such that, for every $i \in I$, the diagram

$$F_iX \xrightarrow{F_i\gamma} F_iY$$

commutes.

By a dual statement, the left $n$–trivial extension of $\mathbb{A}$ by a family $G = \langle G_i \rangle_{i \in I}$ of additive covariant endofunctors was also defined and denoted by $G \rtimes_n \mathbb{A}$ see [3].

This notion was elaborated in [3] by using couples of functors $(T, C)$, $(H, K)$ relating the category $\mathbb{A}$ with $\mathbb{A} \ltimes_n F$ and $G \rtimes_n \mathbb{A}$ defining as follows:

The functor $T : \mathbb{A} \to \mathbb{A} \ltimes_n F$ given, for every object $X$, by $T(X) = (X \oplus (\bigoplus_{i \in I} F_i X), \kappa)$, where

$$\kappa = (\kappa_i)$$

such that, for every $i \in I$,

$$\kappa_i = (a_{i,0}^X) : F_i X \oplus (\bigoplus_{j \in I} F_j X) \to X \oplus (\bigoplus_{j \in I} F_j X)$$

defined by

$$\begin{cases}
a_{i+1,1}^X = id_{F_i X}.
a_{i+j+1,j+1}^X = \Phi_{i,j} X & \text{for } j \text{ with } i + j \in I. 
0 & \text{otherwise.}
\end{cases}$$
and for morphisms by:

\[
T(\alpha) = \begin{pmatrix}
\alpha & 0 & \cdots & 0 \\
\vdots & F_1 \alpha & \vdots \\
0 & \cdots & 0 \\
0 & \cdots & 0 & F_n \alpha
\end{pmatrix}
\]

The cokernel functor \( C : \mathcal{A} \times_n F \rightarrow \mathcal{A} \) is given by \( C((X, f)) = \text{Coker}(f : \oplus_{i \in I} F_i X \rightarrow X) \) for every object \((X, f)\) and for a morphism \( \alpha : (X, f) \rightarrow (Y, g) \) in \( \mathcal{A} \times_n F \), \( C(\alpha) \) is the induced map.

The functor \( H : \mathcal{A} \rightarrow G \times_n \mathcal{A} \) given, for every \( X \in \mathcal{A} \), by \( H(X) = (G_n X \oplus G_{n-1} X \oplus \cdots \oplus G_1 X) \oplus X, \lambda = (\lambda_i) \), where, for every \( i \in I \),

\[
\lambda_i = (a_{i,\beta}^n) : G_i G_n X \oplus G_i G_{n-1} X \oplus \cdots \oplus G_i G_1 X \oplus G_i X \rightarrow G_n X \oplus G_{n-1} X \oplus \cdots \oplus G_1 X \oplus X
\]

defined by

\[
\begin{align*}
a_{n+1, n+1-i}^i &= id_{G_i X} \\
a_{n-j+1, n+1-(i+j)}^i &= \Psi_{ij} X & \text{for } j \text{ where } i + j \in I. \\
0 & & \text{otherwise.}
\end{align*}
\]

and on morphisms by

\[
H(\alpha) = \begin{pmatrix}
G_n \alpha & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & G_1 \alpha \\
0 & \cdots & 0 & \alpha
\end{pmatrix}
\]

The kernel functor \( K : G \times_n \mathcal{A} \rightarrow \mathcal{A} \) is given, for every object in \((G_n \times \cdots \times G_1 \times \mathcal{A})\) by \( K(X, g : X \rightarrow G_i X) = \text{Ker}(g : X \rightarrow \oplus_{i \in I} G_i X) \), and for a morphism \( \beta \), \( K\beta \) is the induced morphism.

It is shown in [3] that these functors verify the following equalities \( CT \cong id_{\mathcal{A}} \) and \( KH \cong id_{\mathcal{A}} \).

## 3 Gorenstein object

In this section, we are going to study the transfer of (Strongly) Gorenstein homological dimensions between \( \mathcal{A} \) and \( \mathcal{A} \times_n F \). To this purpose we proceed as follows, first we introduce a special type of objects verifying certain properties then we define (Strongly) Gorenstein objects and give the transfer theorems.

**Definition 3.1** An object of a category \( \mathcal{A} \) is called \( SP \) (resp., \( SI \))-object if it admits an exact sequence of the form \( 0 \rightarrow X \rightarrow P \rightarrow X \rightarrow 0 \) (resp., \( 0 \rightarrow X \rightarrow I \rightarrow X \rightarrow 0 \)), with \( P \) (resp., \( I \)) a projective (resp. injective) object of \( \mathcal{A} \).

The class of \( SP \) (resp., \( SI \)) objects is denoted \( \mathcal{SP} \) (resp., \( \mathcal{SI} \)). We say that an endofunctor \( F \) of \( \mathcal{A} \) is \( SP \)-left (resp., right) exact if \( 0 \rightarrow FX \rightarrow FP \rightarrow FX \) (resp., \( FX \rightarrow FI \rightarrow FX \rightarrow 0 \)) is still exact.

We propose the following definition of Strongly gorenstein objects on an abelian category.

**Definition 3.2** We say that an object \( X \) in \( \mathcal{A} \) is strongly Gorenstein projective noted \( \mathcal{SGP}(\mathcal{A}) \) if \( X \) is \( SP \), such that the functor hom\((- , Q)\) is \( SP \)-left exact for any projective object \( Q \) in \( \mathcal{A} \).
One can remark that this definition coincide with the one, in [4], in the category of modules.

In [17], the notion of perfect functor was introduced to study the Gorenstein case, but for the strongly Gorenstein a lesser restriction is needed which we define as follows:

**Definition 3.3** An endofunctor $F : \mathcal{A} \rightarrow \mathcal{A}$ is said to be quasi-endofunctor if

(GP1) $F$ is left $SP$-exact.

(GP2) $\text{Ext}(X, FQ) = 0$, for any $X \in SGP(\mathcal{A})$ and $Q$ projective object of $\mathcal{A}$.

The Strongly injective gorenstein case is dual to this statement and called quasi-coperfect.

**Lemma 3.4** The following statements are equivalent

(i) $F_i$, for every $i \in I$, satisfies (GP2)

(ii) $\text{Ext}^1(\mathcal{A}, F_iQ) = 0$, for any $X \in SGP(\mathcal{A})$ and $Q$ projective object of $\mathcal{A}$.

(iii) $\text{Ext}^{n+1}(\mathcal{A}, F_iQ) = 0$, for any $X \in SGP(\mathcal{A})$ and $Q$ a projective object of $\mathcal{A}$.

First, we start by objects in $\mathcal{A} \otimes_n F$ of the form $T(X)$, for an object $X$ in $\mathcal{A}$.

**Proposition 3.5** We assume that $F_i$’s are quasi-perfect, if $X \in SGP(\mathcal{A})$ then $T(X) \in SGP(\mathcal{A} \otimes_n F)$.

**Proof.** If $X$ is $SGP(\mathcal{A})$ then there is an exact sequence

$$0 \rightarrow X \xrightarrow{\delta_1} P \xrightarrow{\delta_2} X \rightarrow 0$$

with $P$ is a projective object in $\mathcal{A}$. Then, we get

$$T(X) \xrightarrow{T(\delta_1)} T(P) \xrightarrow{T(\delta_2)} T(X) \rightarrow 0$$

where $T(\delta_1) = \begin{pmatrix} \delta_1 & 0 & \ldots & 0 \\ \vdots & F_1 \delta_1 & \vdots & \\ 0 & \ldots & \ddots & 0 \\ 0 & \ldots & 0 & F_n \delta_1 \end{pmatrix}$. Since we assume GP1, we have indeed the following exact sequence.

$$0 \rightarrow T(X) \xrightarrow{T(\delta_1)} T(P) \xrightarrow{T(\delta_2)} T(X) \rightarrow 0$$

Now, for every projective object $T(Q)$ in $\mathcal{A} \otimes_n F$, we have

$$0 \rightarrow \text{Hom}(T(X), T(Q)) \xrightarrow{T^*(\delta_2)} \text{Hom}(T(P), T(Q)) \xrightarrow{T^*(\delta_1)} \text{Hom}(T(X), T(Q)).$$

by GP2 we have

$$\text{Ext}^1(T(X), T(Q)) \cong \text{Ext}^1(\mathcal{A}, UT(Q)) = \text{Ext}^n(\mathcal{A}, UT(Q)) = 0.$$

Hence

$$0 \rightarrow \text{Hom}(T(X), T(Q)) \xrightarrow{T^*(\delta_2)} \text{Hom}(T(P), T(Q)) \xrightarrow{T^*(\delta_1)} \text{Hom}(T(X), T(Q)) \rightarrow 0$$
Remark 1  Suppose that the functor $T$ is fully faithful. For every object $X$ in $\underline{A}$, if $T(X)$ is strongly Gorenstein projective then $X$ is also strongly Gorenstein projective.

Proposition 3.6  We assume that $(F_i)_{i \in I}$ are quasi-perfect functors. Let $(X, f) \in \text{SGP}(\underline{A} \ltimes_n F)$ object of $\underline{A} \ltimes_n F$ then $C(X, f) \in \text{SGP}(\underline{A})$.

Proof. For $(X, f) \in \text{SGP}(\underline{A} \ltimes_n F)$, we have the following exact sequence for $P$ is projective object of $\underline{A}$

$$0 \rightarrow (X, f) \xrightarrow{\delta_1} T(P) \xrightarrow{\delta_2} (X, f) \rightarrow 0 \quad (1)$$

and we have the following diagram

$$\begin{array}{cccccccc}
0 & \rightarrow & \text{Im} f & \rightarrow & X & \xrightarrow{\pi} & C(X, f) & \rightarrow & 0 \\
& & \downarrow{\alpha_1} & & \downarrow{C(\delta_1)} & & & \\
0 & \rightarrow & \tilde{F} P & \rightarrow & P \oplus \tilde{F} P & \rightarrow & P & \rightarrow & 0
\end{array}$$

By the snake lemma, we get the exact sequence

$$0 \rightarrow \text{Ker}(\delta_1) \rightarrow \text{Im} f \xrightarrow{Ker(\pi)} C(X, f) \rightarrow 0$$

Since $\text{Ker}(\pi)$ is a monomorphism, we get that $\text{Ker}(\delta_1) = 0$. Hence we have the short exact sequence.

$$0 \rightarrow C(X, f) \rightarrow X \rightarrow C(X, f) \rightarrow 0$$

Since for every projective object $Q$ of $\underline{A}$ we have $\text{Ext}^1(\underline{C}(X, f), Q) \cong \text{Ext}^1(\underline{(X, f), Q}, 0)$ and $(X, f)$ is SGP, then the second term is zero.

Therefore the sequence

$$0 \rightarrow \text{hom}(C(X, f), Q) \xrightarrow{\gamma_1} \text{hom}(P, Q) \xrightarrow{\gamma_2} \text{hom}(C(X, f), Q) \rightarrow 0$$

is exact, where $\gamma_1 = C^*(\delta_2)$ and $\gamma_2 = C^*(\delta_1)$ is exact. ■

By the duality, pointed in [3], between $\underline{A} \ltimes_n F$ and $G \ltimes_n \underline{A}$, without stating a proof we give the strongly gorenstein injective context. Similarly to [22]

We say that $(G_i)_{i \in I}$ are quasi-coperfect:

$(GI1)$ $(G_i)_{i \in I}$ are right SI–exact;

$(GI2)$ $\text{Ext}(G_i Q, X) = 0$, for any $X$ strongly Gorenstein injective object in $\underline{A}$ and $Q$ injective object of $\underline{A}$.

Definition 3.7  We say that an object $X$ in $\underline{A}$ is strongly Gorenstein injective or simply $\text{SGI}(\underline{A})$, if $X$ is $SI$ and for any injective object $Q$, the functor $\text{hom}(Q, -)$ is $SI$–exact for any injective object $Q$ in $\underline{A}$.

Proposition 3.8  If $X \in \text{SGI}(\underline{A})$, then $H(X)\text{SGI}(G \ltimes_n \underline{A})$.

The converse holds if $H$ is fully faithful.

Similarly to theorem 3.6 we give the following theorem.
Theorem 3.9 We assume that $G_i$ is quasi-coperfect functors. Let $(X, f) \in SGI(G \times_n \Delta)$ then $K(X, f)$ is also in $SGI(\Delta)$.

Inspired by [3] we define Gorenstein projective (resp., injective) objects as follows

Definition 3.10 An object is Gorenstein projective (resp., injective) if, and only if, it is a direct summand of a Strongly Gorenstein projective (resp., injective) object.

In [3], they characterized the class $Add(T(P))$ of objects that are isomorphic to direct sums of copies of $T(P)$, where $P$ is an object of $\Delta$. Dually we gave a similar result concerning the class $Prod(H(I))$, of product of copies of $H(I)$.

Lemma 3.11 a) Let $P$ be an object of $\Delta$. An object $(X, f)$ of $\Delta \times_n F$ is in $Add(T(P))$ if and only if $C(X, f) \in Add(P)$ and $(X, f) \cong T(C(X, f))$.

b) Let $I$ be an object of $\Delta$. An object $(X, f)$ of $G \times_n \Delta$ is in $Prod(H(I))$ if and only if $K(X, f) \in Prod(I)$ and $(X, f) \cong H(K(X, f))$.

Remark 2 One could remark that if we take $P$ a projective (resp., $I$ an injective) object in $\Delta$, then the above theorem gives a characterization of projective (resp., injective) objects in $\Delta \times_n F$ (resp., $G \times_n \Delta$).

Using 3.10 and 3.11 we get the following

Theorem 3.12 Assuming that $(F_i)_{i \in I}$ are all quasi-perfect, then

- an object $(X, f)$ in $\Delta \times_n F$ is Gorenstein projective if and only if $C(X, f)$ is also Gorenstein projective and $(X, f) \cong T(C(X, f))$.

- $P$ is finitely generated (strongly) Gorenstein projective in $\Delta$ if and only if also is $T(P)$ in $\Delta \times_n F$.

Dually, we assume that (GI1), (GI2) are satisfied in $G \times_n \Delta$.

Theorem 3.13 An object $(X, f)$ in $G \times_n \Delta$ is Gorenstein injective if and only if $K(X, f)$ is also Gorenstein injective and $(X, f) \cong H(K(X, f))$.

4 Gorenstein dimension

In the following, we focus on the Gorenstein homological dimensions. First, we recall the definition of Gorenstein projective (resp., injective) dimension.

Definition 4.1 [13 Definition 2.8] The Gorenstein projective dimension, $Gpd_R M$, of an $R$–module $M$ is defined by declaring that $Gpd_R M \leq n$ ($n \in \mathbb{N}_0$) if, and only if, $M$ has a Gorenstein projective resolution of length $n$. Similarly, one defines the Gorenstein injective dimension.

Proposition 4.2 We assume that all $(F_i)$’s, $i \in I$, are quasi-perfect.

a) Suppose that $C$ is $SP$–left exact. Let $(X, f)$ be an object in $\Delta \times_n F$ such that $X = X_1 \oplus X_2$ with $\text{Im}(f_i) \subset X_2$, for all $i \in I$, then

$$Gpd_\Delta X_1 \leq Gpd_{\Delta \times_n F}(X, f).$$
We assume $G_i$’s are quasi-coperfect.

b) Let $(X,f)$ be an object in $G \times_n \Lambda$ such that $X = X_1 \oplus X_2$ with $X_2 \in \text{Ker}(f_i)$, for all $i \in I$, then

$$\text{Gid}_X(X_1) \leq \text{Gid}_{G \times_n \Lambda}(X,f).$$

**Proof.** Since a) and b) are proved dually, we assume that $\text{Gpd}_{G \times_n \Lambda}(X,f) = m < \infty$. For $m = 0$, we have $(X,f)$ is a Gorenstein projective object of $\Lambda \times_n F$. Hence $C(X,f)$ is also Gorenstein projective in $\Lambda$ by \[\text{1.7}\]. Since $\text{Im}(f_i) \in X_2$ and $X = X_1 \oplus X_2$, we get that $X_1$ is a direct summand of $C(X,f)$. Therefore $X_1$, is Gorenstein projective in $\Lambda$.

Now assume that $m \geq 1$. Consider two exact sequences in $\Lambda$

$$0 \rightarrow K \rightarrow P_1 \xrightarrow{\epsilon_1} X_1 \rightarrow 0 \quad \text{and} \quad P_2 \xrightarrow{\epsilon_2} X_2 \rightarrow 0$$

with $P_1$ is Gorenstein projective and $P_2$ is projective in $\Lambda$. Since $\text{Im}(f_i) \in X_2$, we can set $f_i = (f_{i,1}, f_{i,2})$ with $f_{i,1} : F_1 X_1 \rightarrow X_2$ and $f_{i,2} = F_1 X_2 \rightarrow X_2$. Now, let $L$ to be the kernel of the morphism:

$$\lambda = ((f_{i,1} \circ F_1 \epsilon_1)_{i \in I}, \epsilon_2, (f_{i,2} \circ F_1 \epsilon_2)_{i \in I}) : \bigoplus_{i \in I} F_1 P_1 \oplus P_2 \bigoplus_{i \in I} F_1 P_2 \rightarrow X_2$$

and consider $\mu = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \lambda \end{pmatrix} : P_1 \bigoplus_{i \in I} F_1 P_1 \oplus P_2 \bigoplus_{i \in I} F_1 P_2 \rightarrow X_1 \oplus X_2$.

Then we obtain the following exact sequence:

$$(K \oplus L, \alpha) \xrightarrow{\text{Ker}(P_1)} (T(P_1) \oplus T(P_2)) \xrightarrow{\mu} (X,f_i) \rightarrow 0$$

with $\alpha = (\alpha_{i,1} : F_1 K \rightarrow L, \alpha_{i,2} : F_1 L \rightarrow L)$. The middle term represents a Gorenstein projective object in $\Lambda \times_n F$, hence $\text{Gpd}_{\Lambda \times_n F}(X,f_i) = 1 + \text{Gpd}_{\Lambda \times_n F}(K \oplus L, \alpha_i)$. By induction, we get that $\text{Gpd}_{\Lambda \times_n F}(R \oplus L, \alpha) \geq \text{Gpd}_{\Lambda \times_n F}(R \oplus L, \alpha_i) \geq 1 + \text{Gpd}_{\Lambda \times_n F}(K, \alpha_i)$. Therefore $\text{Gpd}_{\Lambda \times_n F}(X,f_i) \geq 1 + \text{Gpd}_{\Lambda \times_n F}(X,f)$. \[\blacksquare\]

**Proposition 4.3** Suppose that $I^J F_i = 0$ (resp., $R^J G_i$), for $i \in I$, $j > 1$, then:

$\text{Gpd}_{\Lambda}(X) = \text{Gpd}_{\Lambda \times_n F}(T(X))$ (resp., $\text{Gid}_{\Lambda}(X) = \text{Gid}_{G \times_n \Lambda}(H(X))$).

**Corollary 4.4** For every object $X \in \Lambda$ we have:

- $\text{Gpd}_{\Lambda}(X) \leq \min(\text{Gpd}_{\Lambda \times_n F}(Z(X), \text{Gpd}_{\Lambda \times_n F}(T(X)))$.

- $\text{Gid}_{\Lambda}(X) \leq \min(\text{Gid}_{G \times_n \Lambda}(Z(X), \text{Gid}_{G \times_n \Lambda}(H(X)))$.

- $G - \text{gldim}_{\Lambda}(\Lambda) \leq G - \text{gldim}_{\Lambda \times_n F}(F)$.

Considering the ring $R \times_n M$ and observing that $R \times_n M = R \otimes_R R \times_n M$ i.e $(T(R))$. These results have the following concretisation.

**Corollary 4.5** $G - \text{gldim}_{\Lambda}(R) \leq G - \text{gldim}_{\Lambda \times_n F}(R \times_n M)$. Suppose that all the objects $M_i$ are flat, for $i \in I$, then $G - \text{gldim}_{\Lambda}(R) = G - \text{gldim}_{\Lambda \times_n F}(R \times_n M)$.
5 Application on the category of modules over n-trivial extension

Now, we look upon the realisation of the Gorenstein categorical properties on the category of modules over the n–trivial extension. Throughout this part, we adopt the following notations: Let $R$ be an associative ring and $(M_i)_{i \in I}$ such that $S := R \times_n M$ is the n–trivial extension defined in $[1]$. In $[3]$, they identified the category of modules over $S$ to the category $A \times_n F$, by choosing $A := R \text{ Mod}$ and $F_i = - \otimes M_i$, for $i \in I$, with $\Phi_{(i,j)} = id_A \otimes \Phi_{(i,j)}$. In fact we have the following theorem.

**Theorem 5.1** $[3]$ Theorem 2.2] The category $R \text{ Mod} \times_n (M_i \otimes -)_i$ is isomorphic to the category of left modules over $R \times_n M_1 \times \cdots \times M_n$.

**Remark 3** Denoting $G_i = \text{hom}(M_i, -)$, for $i \in I$, to be the adjoint of $F_i$. In $[3]$, it was proved that $A \times_n F \cong G \times_n A$. Also, the functors $T$, $U$, $C$, $H$ and $K$ between $R \text{ Mod}$ and $S \text{ Mod}$ have a concrete realisation as follows $C = - \otimes_R R$, $T = - \otimes_R S$, $K = \text{hom}_S(R, -)$, $H = \text{hom}_R(S, -)$, see $[2]$.

Under these identification, we investigate some aspect of strongly Gorenstein homological properties of the 2–trivial extension and extend some of the result established in $[15]$. We recall the results in $[15]$ that are the subject of our applications.

**Theorem 5.2** $[15]$ theorem 2.1] Let $N$ be an $R$–module. Then:

1) a) Suppose that $pd_R(M) < \infty$. If $N$ is strongly Gorenstein projective $R$–modules, then $M \otimes_R S$ is a strongly Gorenstein projective $S$–module.

b) Conversely, suppose that $M$ is flat $R$–module. If $N \otimes_R S$ is a strongly Gorenstein projective $S$–module, then $N$ is a strongly Gorenstein projective $R$–module.

2) Suppose that $\text{Ext}_R^p(S,M) = 0$ for all $p \geq 1$ and $fd_R(S) < \infty$. If $N$ is a strongly Gorenstein injective $R$–module, then $\text{Hom}_R(S,N)$ is a strongly Gorenstein projective $R$–module.

For sake of simplicity, we will restrain to the case of $R \times M_1 \times M_2$, since the results are the same for $R \times_n M$ with $n \in \mathbb{N}$.

**Remark 4** One should remark that if, for every $i \in I$, $M_i$ has a finite projective dimension then $M_i \otimes_R -$ is quasi-perfect. Also if, for every $i \in I$, $M_i$ is a flat $R$–modules then every $\text{Hom}(M_i, -)$ is quasi-coperfect, for every $i \in I$.

The following theorem comes directly from $[15]$ and $[20]$ where we establish the transfer of strongly Gorenstein between $R$ and $R \times M_1 \times M_2$, giving results as the ones in $[15]$.

**Corollary 5.3** Let $N$ be an $R$–module and Then:

a) Assume that $- \otimes_R M_i$ is quasi-perfect, for $i \in \{1, 2\}$, if $N$ is strongly gorenstein projective as an $R$–module then $N \otimes_R S$ is also strongly gorenstein projective as an $S$–module.

b) Conversely, suppose that $(M_i)_{i \in I}$ are flat $R$–modules, if $N \otimes_R S$ is strongly gorenstein projective then so that $N$ as an $R$–module.

**Corollary 5.4** Let $N$ be an $R$–module. Then:
a) For \( i \in I \), we assume that \( \text{fd}_R(M_i) < \infty \), if \( N \) is strongly gorenstein injective as an \( R \)-module then \( \text{hom}_R(S, N) \) is also strongly gorenstein injective \( S \)-module.

Under the above assumptions we give the converse

b) Conversely, if an \( N \) strongly gorenstein injective \( S \)-module then \( \text{hom}_S(R, N) \) is strongly Gorenstein injective.

**Corollary 5.5** We suppose that \( M_i \otimes_R - \) is quasi-perfect, for \( i \in I \). An \( S \)-module \( N \) is Gorenstein projective if and only if \( N / M_i N \) are Gorenstein projective and

\[
\begin{array}{ccccccccc}
M_i & \otimes_R M_i & \otimes_R N & \xrightarrow{M_i \otimes f_i} & M_i & \otimes_R N & \xrightarrow{f_i} & N & \rightarrow & \text{Coker}(f_i) & \rightarrow & 0
\end{array}
\]

is exact, for \( i \in I \).

The injective case is dual.

### 5.1 Gorenstein flat

Also under the realisation established in remark 3 and for simplicity sake, we will prove the following result in the case of \( n = 2 \), since iteratively we can have the general case.

**Proposition 5.6** Let \((X, f)\) represent an object of \( G \times_n \Delta \) of finite injective dimension, if \( R_i K(X, f) = 0 \) for \( i \geq 1 \), then

\[
\text{id}_{G \times_n \Delta}(X, f) = \text{id}_{\Delta}(\text{Ker}(X, f))
\]

**Proof.** If \( \text{id}_{G \times_n \Delta}(X, f) = 0 \), the equality is straightforward from the construction of injective objects. We shall continue by way of induction on \( \text{id}_{G \times_n \Delta}(X, f) = n \geq 1 \). We take a one step injective resolution

\[
\begin{array}{cccccccccc}
0 & \rightarrow & X & \rightarrow & I \oplus_{i \in I} G_i I & \rightarrow & B & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{cccccccccc}
0 & \rightarrow & G_i X & \rightarrow & \oplus_{i \in I} G_i(I \oplus_{i \in I} G_i I) & \rightarrow & \oplus_{i \in I} G_i B & \rightarrow & 0
\end{array}
\]

Since \( R_i K(X, f) = 0 \) for \( i \geq 1 \), it follows that

\[
\begin{array}{cccccccccc}
0 & \rightarrow & K(X, f) & \rightarrow & I & \rightarrow & K(B, \beta) & \rightarrow & 0
\end{array}
\]

is exact and \( R_i K(B, \beta) = 0 \). By induction, we have

\[
\text{id}_{G \times_n \Delta}(X, f) = 1 + \text{id}_{G \times_n \Delta}(B, \beta) = 1 + \text{id}_{\Delta}(K(B, \beta)) = \text{id}_{\Delta}(K(X, f)).
\]
Proposition 5.7 If $\text{Ext}^i_S(R, S) = 0$ for $i \geq 0$, then

$$\text{id}_S(S) = \text{id}_R(B_1 \oplus B_2 \oplus M_2).$$

where

- $B_1 = \text{Ann}_R(M_1) \cap \text{Ann}_R(M_2),$
- $B_2 = \{(0, m_1, 0) \in S/(0, m_1, 0), (0, m'_1, 0) = 0$ for $m'_1 \in M_1\}$

**Proof.** We shall present the proof within the framework of the category $G \cong \Lambda$. One also can remark that $S$ as an $S$-module is represented by the object $T(R)$. And under the category isomorphism of theorem [7, Theorem 5.8] it is represented in $G \cong \Lambda$ by $T^{op}(R) = (R \oplus M_1 \oplus M_2, \kappa = (\kappa^1, \kappa^2))$, with

$$\kappa^1 : R \oplus M_1 \oplus M_2 \longrightarrow G_1 R \oplus G_1 M_1 \oplus G_1 M_2$$

and

$$\kappa^2 : R \oplus M_1 \oplus M_2 \longrightarrow G_2 R \oplus G_2 M_1 \oplus G_2 M_2$$

where $\kappa^1_{(1, 2)} : R \longrightarrow G_1 M_1$, $\kappa^1_{(2, 3)} : M_1 \longrightarrow G_1 M_2$, $\kappa^2_{(1, 3)} : R \longrightarrow G_2 M_2$ and zero morphism otherwise. An observation on the following maps

$$R \oplus M_1 \oplus M_2 \xrightarrow{\kappa^1} G_1 R \oplus G_1 M_1 \oplus G_1 M_2 \xrightarrow{G_1 \kappa^2} G_1 G_2 R \oplus G_1 G_2 M_1 \oplus G_1 G_2 M_2$$

and

$$R \oplus M_1 \oplus M_2 \xrightarrow{\kappa^2} G_2 R \oplus G_2 M_1 \oplus G_2 M_2 \xrightarrow{G_2 \kappa^2} G_2 G_2 R \oplus G_2 G_2 M_1 \oplus G_2 G_2 M_2$$

yields the following formula, $\text{Ker}(\kappa^1 \oplus \kappa^2) = B_1 \oplus B_2 \oplus M_2$.

**Corollary 5.8** If $B_1 \oplus B_2 \oplus M_2$ have a finite injective dimension then $S$ is $n$-Gorenstein.

**Proposition 5.9** We assume that $S$ is $n$-Gorenstein $S$-module. Let $(X, f)$ represent a left $S$-module. $(X, f)$ is Gorenstein flat if and only if $\text{Coker}(f)$ is Gorenstein flat and

$$F_1 F_1 X \xrightarrow{F_1 f_1} F_1 X \xrightarrow{f_1} X \longrightarrow \text{Coker}(f_1) \longrightarrow 0$$

is exact.

**Proof.** From [8, Theorem 10.3.8], we have that $(X, f) \cong \varinjlim T(P_i)$ were $P_i$ are finitely generated Gorenstein projective. Following the same method done in the proof of [5, Theorem 3.4], we get the wanted result.

### 5.2 Triangular matrix ring

Let $T = \begin{pmatrix} R_1 & 0 & 0 \\ M_1 & R_2 & 0 \\ M_2 & M_1 & R_3 \end{pmatrix}$ be a formal triangular matrix ring.

We recall from [7, Theorem 5.2], the category of left $T$-module is isomorphic to $S = R_1 \times R_2 \times R_3 \text{ Mod} \times - \otimes R M_1 \times R M_2$. Applying the description of objects over $S$, we get that a left $T$-module $V$ is represented by

$$V = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where $X \in R_1 - \text{mod}$, $Y \in R_2 - \text{mod}$ and $Z \in R_3 - \text{mod}$ together with the following maps
Corollary 5.10 We suppose that $M_i \otimes -$ are quasi-perfect for $i \in \{1, 2\}$. Let

$V = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ be left $T$-modules. The following assertion are equivalent:

i) $V$ is Gorenstein projective left $T$-module;

ii) $X$, $\text{Coker}(f_1)$ and $\text{Coker}(f_3)$ are Gorenstein projective left module over there respective rings and $f_i$ for $i \in \{1, 2\}$ are injective and $f_3$ is surjective.

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