STABILITY OF SYMMETRIC SPACES OF NONCOMPACT TYPE UNDER RICCI FLOW

RICHARD H BAMLER

Abstract. In this paper we establish stability results for symmetric spaces of noncompact type under Ricci flow, i.e. we will show that any small perturbation of the symmetric metric is flown back to the original metric under an appropriately rescaled Ricci flow.

It will be important for us which smallness assumptions we have to impose on the initial perturbation. We will find that as long as the symmetric space does not contain any hyperbolic or complex hyperbolic factor, we don’t have to assume any decay on the perturbation. Furthermore, in the hyperbolic and complex hyperbolic case, we show stability under a very weak assumption on the initial perturbation. This will generalize a result obtained by Schulze, Schr"{u}rer and Simon ([SSS2]) in the hyperbolic case.

The proofs of those results make use of an improved $L^1$-decay estimate for the heat kernel in vector bundles as well as elementary geometry of negatively curved spaces.

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1. Introduction

Consider a locally symmetric space $(M, g)$, i.e. a Riemannian manifold which locally has a reflection symmetry at every point (for more details see section 3). By the de Rham Decomposition Theorem, its universal cover $\tilde{M}$ can be expressed as a product $M_1 \times \ldots \times M_m$ of irreducible symmetric spaces. All $M_i$ are Einstein metrics with Einstein constants $\lambda_i$. If all $\lambda_i$ are negative, then $M$ is said to be of noncompact type. Furthermore, if all $\lambda_i$ are equal to some $\lambda < 0$, then $g$ is an Einstein metric with Einstein constant $\lambda$. Hence it is a fixed point of the rescaled Ricci flow equation
\begin{equation}
\partial_t g_t = -2 \text{Ric}_{g_t} - 2\lambda g_t.
\end{equation}

In this paper, we will prove stability results for the metric $g$, i.e. we will show that every sufficiently small perturbation $g_0 = g + h$ flows back to $g$ under (1.1) as $t \to \infty$. Surprisingly, for most of the symmetric spaces we don’t have to impose any spatial decay assumption on the perturbation $h$.

**Theorem 1.1.** Let $(M, \overline{g})$ be a locally symmetric space of noncompact type which is Einstein of Einstein constant $\lambda < 0$ and assume that the de Rham decomposition of $M$ contains no factors which are homothetic to $\mathbb{H}^n$, $(n \geq 2)$ or $\mathbb{C} \mathbb{H}^{2n}$, $(n \geq 1)$. Then there is an $\varepsilon > 0$ depending only on $M$ such that if
\begin{equation}
(1 - \varepsilon)\overline{g} \leq g_0 \leq (1 + \varepsilon)\overline{g},
\end{equation}
and if $(g_t)$ evolves by (1.1), then $g_t$ exists for all time $t$ and as $t \to \infty$ we have convergence $g_t \to \overline{g}$ in the pointed Cheeger-Gromov sense, i.e. there is a family of diffeomorphisms $\Psi_t$ of $M$ such that $\Psi_t g_t \to \overline{g}$ in the smooth sense on every compact subset of $M$. 
In the hyperbolic or complex hyperbolic case, we have to impose stronger assumptions on the perturbation.

**Theorem 1.2.** Let \((M, \overline{g})\) be either \(\mathbb{H}^n\) for \(n \geq 3\) or \(\mathbb{C}H^{2n}\) for \(n \geq 2\), choose a basepoint \(x_0 \in M\) and let \(r = d(\cdot, x_0)\) denote the radial distance function. There is an \(\varepsilon_1 > 0\) and for every \(q < \infty\) an \(\varepsilon_2 = \varepsilon_2(q) > 0\) such that the following holds: If \(g_0 = \overline{g} + h\) and \(h = h_1 + h_2\) satisfies

\[
|h_1|(p) < \frac{\varepsilon_1}{r + 1} \quad \text{and} \quad \sup_M |h_2| + \left( \int_M |h_2|^q dx \right)^{1/q} < \varepsilon_2,
\]

then Ricci flow (1.1) exists for all time and we have convergence \(g_t \rightarrow \overline{g}\) in the pointed Cheeger-Gromov sense.

In the case \(M = \mathbb{H}^n\), \(n \geq 4\) Schulze, Schnürer and Simon (SSS2) have shown stability for every perturbation \(h\) for which \(\|h\|_{L^\infty(M)}\) is bounded by a small constant depending on \(\|h\|_{L^2(M)}\). This result is implied by Theorem 1.2 by the interpolation inequality. Li and Yin (LY) have shown a stability result for \(M = \mathbb{H}^n\), \(n \geq 3\) when the Riemannian curvature approaches the hyperbolic curvature like \(\varepsilon \delta e^{-\delta r}\).

One drawback of the decay assumption of Theorem 1.2 is that we cannot generalize the stability to quotients of \(M\) under a group action which does not fix the distance function \(r\). Results of this kind have to be proven separately. For example, for compact quotients of hyperbolic space, this stability was established by Ye in Ye and for finite volume quotients (i.e. if there are cusps) by the author in Bam2.

Theorem 1.1 and 1.2 have the following immediate consequences:

**Corollary 1.3.** Let \((M, \overline{g})\) be a locally symmetric space of noncompact type which is Einstein of Einstein constant \(\lambda < 0\) and assume that the de Rham decomposition of \(\tilde{M}\) contains no factors which are homothetic to \(\mathbb{H}^n\), \((n \geq 2)\) or \(\mathbb{C}H^{2n}\), \((n \geq 1)\). Then there is an \(\varepsilon > 0\) depending only on \(\tilde{M}\) such the following holds: If \(g\) is an Einstein metric on \(M\) with Einstein constant \(\lambda\) and

\[
(1 - \varepsilon)\overline{g} \leq g \leq (1 + \varepsilon)\overline{g},
\]

then \(g\) is isometric to \(\overline{g}\).

**Corollary 1.4.** Let \((M, \overline{g})\) be either \(\mathbb{H}^n\) for \(n \geq 3\) or \(\mathbb{C}H^{2n}\) for \(n \geq 2\), choose a basepoint \(x_0 \in M\) and let \(r = d(\cdot, x_0)\) denote the radial distance function. There is an \(\varepsilon_1 > 0\) and for every \(q < \infty\) an \(\varepsilon_2 = \varepsilon_2(q) > 0\) such that the following holds: If \(g\) is an Einstein metric on \(M\) of the same Einstein constant as \(\overline{g}\) and \(g = \overline{g} + h_1 + h_2\) with

\[
|h_1|(p) < \frac{\varepsilon_1}{r + 1} \quad \text{and} \quad \sup_M |h_2| + \left( \int_M |h_2|^q dx \right)^{1/q} < \varepsilon_2,
\]

then \(g\) is isometric to \(\overline{g}\).
By results of Graham-Lee ([GL]) and Biquard ([Biq]), the spaces $\mathbb{H}^n, (n \geq 4)$ and $\mathbb{C}\mathbb{H}^{2n}, (n \geq 2)$ admit deformations $g$ which are Einstein, are not isometric to $\overline{g}$ and satisfy

$$(1 - \varepsilon)\overline{g} \leq g_0 \leq (1 + \varepsilon)\overline{g}.$$  

Hence for those spaces we cannot assume a result which is as strong as that of Theorem 1.1 resp. Corollary 1.3. However, we can ask whether in this case we always have longtime existence of the Ricci flow and convergence to an Einstein metric under certain assumptions. And for $\mathbb{H}^3$ we can still ask whether we have a result as in Theorem 1.1. Both questions seem to be very difficult.

Our results will rely on a geometric analysis of the heat kernel associated to the linearized Ricci deTurck flow equation. It will turn out that the geometry of the vector bundle $\text{Sym}_2 T^* M$ in which the perturbation lives, improves the $L^1$-decay rate of the heat kernel. We will see that the obstruction against a good decay, comes from cusp deformations (see subsection 5.9). Those correspond to the “trivial Einstein deformations” in [Bam1] and can be seen as algebraic deformations of cusp cross-sections. Cusp deformations turn out to exist only for the spaces $\mathbb{H}^n, (n \geq 3)$ and $\mathbb{C}\mathbb{H}^{2n}, (n \geq 2)$. Theorem 1.1 and 1.2 for type $h_2$ perturbations will then follow immediately from this heat kernel estimate. In order to allow type $h_1$ perturbations, we will use a trick from the geometry of negatively curved spaces.

The paper is organized as follows: In section 2, we discuss the Ricci flow and Ricci deTurck flow equation and give a short overview over all analytical tools needed in this paper. Section 3 contains a quick introduction into the geometry of symmetric spaces. In section 4, we prove more abstract bounds on heat kernels in twisted vector bundles over symmetric spaces. The results obtained in this section are very general and are of independent interest. They involve certain constants which we will then estimate for our particular purpose in section 5. Finally, section 6 contains the proofs of Theorems 1.1 and 1.2.

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2. Analytical preliminaries

2.1. Ricci deTurck flow. In order to establish the desired stability results, we will analyze Ricci deTurck flow. This flow is a modification of Ricci flow via a continuous family of diffeomorphisms.

Recall that the rescaled Ricci flow equation reads

$$\dot{g}_{RF} = -2 \text{Ric}_{g_{RF}} - 2\lambda g_{RF}. \quad (2.1)$$

In order to define the Ricci deTurck flow, we need to make use of a distinguished background metric $\overline{g}$ which we will always choose to be the given symmetric metric on $M$. Define the divergence operator

$$\text{div}_{\overline{g}} : C^\infty(M; \text{Sym}_2 T^* M) \rightarrow C^\infty(M; TM), \quad h \mapsto -\sum_i (\nabla^g_i h(\overline{e}_i, \cdot))_{\overline{g}}$$
where we sum over a local $\mathcal{G}$-orthonormal frame field $(\mathcal{T}_i)$ and the musical operator $\mathcal{G}$ is also taken with respect to $\mathcal{G}$. Set

$$X_{\mathcal{G}}(h) = \text{div}_{\mathcal{G}} h + \frac{1}{2} \nabla \text{tr}_{\mathcal{G}} h.$$ 

Then the Ricci deTurck flow equation reads

$$\dot{g}_t^{DT} = -2 \text{Ric}_{g_t^{DT}} - 2\lambda g_t^{DT} - \mathcal{L}_{X_{g_t^{DT}}} g_t^{DT}. \quad (2.2)$$

The advantage of Ricci deTurck flow over Ricci flow is that its linearization at $g_t = \overline{g}$ is strongly elliptic. This fact has been used by deTurck to give a simplified proof for the short-time existence of Ricci flow ([DeT]). In fact, if we express equation (2.2) in terms of the perturbation $h_t = g_t^{DT} - \overline{g}$, we obtain

$$\partial_t h_t + L h_t = Q_t \quad (2.3)$$

where $L$ is called Einstein operator with

$$(Lh)_{ab} = -\triangle h_{ab} - 2\mathcal{G}^{uv} \mathcal{G}^{pq} R_{aupb} h_{vq}$$

and $Q_t$ only contains terms of higher order:

$$Q_{ab} = -g^{uv} g^{pq} (\nabla_u h_{pa} \nabla_v h_{qb} - \nabla_p h_{ua} \nabla_v h_{qb} + \frac{1}{2} \nabla_a h_{up} \nabla_b h_{vq})$$

$$- g^{uv} g^{pq} (\nabla_u h_{vp} + \frac{1}{2} \nabla_p h_{uv}) (\nabla_v h_{qb} + \nabla_b h_{qa} - \nabla_q h_{ab})$$

$$- \mathcal{G}^{uv} \mathcal{G}^{pq} (\nabla_u h_{vp} + \frac{1}{2} \nabla_p h_{uv}) \nabla_a h_{ab}$$

$$- \mathcal{G}^{uv} \mathcal{G}^{pq} (\nabla_v h_{qv} + \frac{1}{2} \nabla_q h_{vq}) h_{au} - \mathcal{G}^{uv} \mathcal{G}^{pq} (\nabla_a h_{uv} + \frac{1}{2} \nabla_v h_{pq} + \frac{1}{2} \nabla_a h_{uv} + \frac{1}{2} \nabla_v h_{pq}) h_{bu}$$

$$- (g^{uv} - \mathcal{G}^{uv}) (\nabla_a h_{uc} + \nabla_c h_{av} - \nabla_a h_{uv} - \nabla_a h_{uv}).$$

Hence if $|h| < 0.1$, we can estimate $|Q| \leq C(|\nabla h|^2 + |h| |\nabla^2 h|)$. We will also sometimes make use of the identity

$$Q_t = R_t + \nabla^a S_t,$$

where

$$R_{ab} = -g^{uv} g^{pq} (\nabla_u h_{pa} \nabla_v h_{qb} - \nabla_p h_{ua} \nabla_v h_{qb} + \frac{1}{2} \nabla_a h_{up} \nabla_b h_{vq})$$

$$- g^{uv} g^{pq} (\nabla_u h_{vp} + \frac{1}{2} \nabla_p h_{uv}) (\nabla_v h_{qb} + \nabla_b h_{qa} - \nabla_q h_{ab})$$

$$+ \mathcal{G}^{uv} \mathcal{G}^{pq} (\nabla_u h_{vp} + \frac{1}{2} \nabla_p h_{uv}) (\nabla_v h_{qb} + \nabla_b h_{qa} - \nabla_q h_{ab})$$

$$+ \mathcal{G}^{uv} \mathcal{G}^{pq} (\nabla_a h_{uc} + \nabla_c h_{av} - \nabla_a h_{uv} - \nabla_a h_{uv})$$

and $(\nabla^a S)_{ab} = -\mathcal{G}^{kl} \nabla_k S_{lab}$ with

$$S_{lab} = \mathcal{G}^{uv} \mathcal{G}^{pq} (\nabla_p h_{qv} + \frac{1}{2} \nabla q h_{pv}) (\mathcal{G}_{lb} h_{ua} + \mathcal{G}_{la} h_{bu})$$

$$+ \mathcal{G}_{lp} (g^{uv} - \mathcal{G}^{uv}) (\nabla_a h_{bv} + \nabla_b h_{av} - \nabla_v h_{ab}) - \mathcal{G}_{la} (g^{uv} - \mathcal{G}^{uv}) \nabla_b h_{uv}.$$ 

Observe that as long as $|h| < 0.1$ we have

$$|R| \leq C |\nabla h|^2 \quad \text{and} \quad |S| \leq C |h| |\nabla h|.$$ 

The following Proposition expresses the equivalence of Ricci deTurck flow and Ricci flow.
Proposition 2.1. Let \((g_t^{DT})_{t \in [0, T]}\) be a smooth solution to the Ricci deTurck flow equation (2.2) and assume that \(|g_t^{DT} - \overline{g}| < 0.1\) everywhere. Define the time dependent vector field \(X_t = X_{\Psi_t}(g_t^{DT})\). Then \(X_t\) has a flow \((\Psi_t)_{t \in [0, T]}\), i.e. there is a family of diffeomorphisms \(\Psi_t : M \to M\) such that
\[
\Psi_t = X_t \circ \Psi_0 \quad \text{and} \quad \Psi_0 = \text{id}_M,
\]
and \(g_t = \Psi_t^* g_t^{DT}\) solves the normalized Ricci flow equation (2.1).

Proof. For the existence of the flow \((\Psi_t)\) observe that we have \(|X_t| \leq C t^{-1/2}\) by Corollary 2.3 below. The fact that \(g_t\) satisfies the normalized Ricci flow equation can be checked easily. \(\square\)

Hence, in order to establish Theorems 1.1 and 1.2, it suffices to prove the stability for Ricci deTurck flow instead of Ricci flow. As we will see later, the main work will go into establishing the stability of the linearized Ricci deTurck flow equation
\[
\partial_t h_t + Lh_t = 0 \tag{2.4}
\]

2.2. A priori derivative estimates. We will recall an a priori derivative estimate for linear or a certain type of nonlinear parabolic equations. If \(\Omega \subset \mathbb{R}^n \times \mathbb{R}\) denotes some parabolic neighborhood in space-time (e.g. \(\Omega = B_r(0) \times [0, T]\)), then we will denote by \(C^{2m;m}(\Omega)\) the space of scalar functions on \(\Omega\) which are \(i\) times differentiable in spatial direction and \(j\) times differentiable in time direction if \(i + 2j \leq 2m\). For \(\alpha \in (0, \frac{1}{2})\), the corresponding Hölder space will be denoted by \(C^{2m,2\alpha;m,\alpha}(\Omega)\).

In order to present our results in a scale invariant way, we will use the following weights to define the Hölder norm on \(C^{2m,2\alpha;m,\alpha}(\Omega)\): Assume
\[
r = \min\{r' : \Omega \subset B_{r'}(p) \times [t - (r')^2, t]\} \text{ for some } p, t \text{ such that } t < \infty.
\]
Then set
\[
\|u\|_{C^{2m,2\alpha;m,\alpha}(\Omega)} = \sum_{|\iota| + 2k \leq 2m} r^{1|\iota| + 2k}(\|D^\iota \partial_t^k u\|_{C^0} + r^{2\alpha}[D^\iota \partial_t^k u]_{2\alpha,\alpha}),
\]
where \(\iota\) runs over products of spatial derivatives.

Set \(B_r = B_r(0) \subset \mathbb{R}^n\).

Proposition 2.2. Let \(r > 0\) and consider the parabolic neighborhoods \(\Omega = B_r \times [-r^2, 0]\) and \(\Omega' = B_{2r} \times [-4r^2, 0]\).

Assume that \(u \in C^{2,1}(\Omega')\) satisfies the equation
\[
(\partial_t - L)u = R[u] = r^{-2} f_1(r^{-1} x, u) + r^{-1} f_2(r^{-1} x, u) \cdot u \otimes \nabla u + f_3(r^{-1} x, u) \cdot \nabla u \otimes \nabla u + f_4(r^{-1} x, u) \cdot u \otimes \nabla^2 u,
\]
where \(f_1, \ldots, f_4\) are smooth functions in \(x\) and \(u\) such that \(f_2, f_3, f_4\) can be paired with the tensors \(u \otimes \nabla u, \nabla u \otimes \nabla u\) resp. \(u \otimes \nabla^2 u\). Assume that the linear operator \(L\) has the form
\[
Lu = a_{ij}(x) \partial^2_{ij} u + b_i(x) \partial_i u + c(x) u.
\]
Now assume that we have the following bounds for $m \geq 1$, $\alpha \in (0, \frac{1}{2})$:
\[
\frac{1}{\Lambda} < a_{ij} < \Lambda, \quad \|a_{ij}\|_{C^{2m-2,2\alpha;m-1,\alpha}(\Omega')} < \Lambda, \quad \|h_t\|_{C^{2m-2,2\alpha;m-1,\alpha}(\Omega')} < r^{-1}\Lambda, \quad \|c\|_{C^{2m-2,2\alpha;m-1,\alpha}(\Omega')} < r^{-2}\Lambda.
\]
Then there are constants $\varepsilon_m > 0$ and $C_m < \infty$ depending only on $\Lambda$, $\alpha$, $n$, $m$ and the $f$, such that if
\[
H = \|u\|_{L^\infty(\Omega')} < \varepsilon_m,
\]
then
\[
\|u\|_{C^{2m,2\alpha;m,\alpha}(\Omega)} < C_m H.
\]

For a proof see e.g. [Bam2].

We will frequently make use of the following consequence of Proposition 2.2.

**Corollary 2.3.** Let $T > 0$ and assume that $(h_t)_{t \in [0,T)}$ satisfies either the Ricci deTurck flow equation (2.3) or the linearized Ricci deTurck flow equation (2.4) on a domain $D \subset M$, where $M^n$ denotes any complete Riemannian manifold.

Then for any $m$, there exist constants $\varepsilon_m > 0$, $C_m < \infty$ depending only on $m$, $n$ and bounds on the curvature tensor of $M$ as well as its derivatives, such that if
\[
H = \|h\|_{L^\infty(D' \times [0,T))} < \varepsilon_m,
\]
then
\[
\|\nabla^m h_t\|_{L^\infty(D)} < C_m t^{-m/2} H \quad \text{for all } t \in [0,T).
\]

Observe that $\varepsilon_m, C_m$ are in particular independent of the injectivity radius of $M$.

**Proof.** At each point $p \in D$ pass over to a local cover and consider the domains $\Omega = B_r(p) \times [3r^2, 4r^2] \subset B_{2r}(p) \times [0, 4r^2] = \Omega'$ for $0 < r < \frac{1}{2}T^{1/2}$. Proposition 2.2 then yields the desired result. \qed

### 2.3. Short-time existence.

In this subsection let $(M, \overline{g})$ be an arbitrary Riemannian manifold.

From (2.3), we see that the Ricci deTurck flow equation is strongly parabolic if $h_t$ is small enough. We will quote a general short-time existence result which follows by a standard inverse function theorem argument. For more details see [Shi], [LS] and [SSSI, sec 4].

**Proposition 2.4 (Short-time existence).** Let $(M, \overline{g})$ be a complete Riemannian manifold. Assume that its curvature tensor is globally bounded in the $C^{0,\alpha}$-sense. Then there are $\varepsilon_{s.e.}, \sigma_{s.e.} > 0$, $C_{s.e.,m} < \infty$ which only depend on $M$ and $\overline{g}$ such that the following holds:

Let $g_0$ be a smooth metric on $M$. If
\[
\|g_0 - \overline{g}\|_{L^\infty(M)} < \varepsilon_{s.e.},
\]
then there is a unique $L^\infty$-bounded smooth solution $(g_t) \in C^\infty(M \times [0, \sigma_{s.e.}^2])$ to the Ricci deTurck flow equation (2.7) with initial metric $g_0$. Moreover, we have the bound
\[
\|g_t - \overline{g}\|_{L^\infty(M \times [0, \sigma_{s.e.}^2])} \leq C_{s.e.,m}\|g_0 - \overline{g}\|_{L^\infty(M)}.
\]
2.4. **Short-time estimates for the heat kernel.** For small times, we can estimate the heat kernel using a result by Cheng, Li and Yau (CLY):

**Proposition 2.5.** Let $M^n$ be a complete Riemannian manifold of uniformly bounded curvature, $E$ a vector bundle over $M$ and $p_0 \in M$. Then for every $T < \infty$ and $\delta > 0$ there are constants $C_m = C_m(M, E, p_0, T, \delta)$ such that the following holds:

Let $(k_t)_{0 < t < T} \in C^\infty(M; E) \otimes E^*_p$ be the heat kernel in $p_0$, i.e. $\partial_t k_t = \triangle k_t$ and $k_t \to^{t \to 0} \delta_{p_0} \text{id}_{E_{p_0}}$.

Then we have the estimates

$$|\nabla^m k_t|(p) \leq C_m t^{-(n+m)/2} \exp \left( -\frac{r^2}{(4 + \delta)t} \right),$$

where $r = d(p_0, p)$ and $0 < t < T$.

**Proof.** Observe that by Kato’s inequality we have $\partial_t |k_t| \leq \triangle |k_t|$ and hence the scalar heat kernel on $M$ bounds $|k_t|$. The bounds on the derivatives follow with Proposition 2.2. \hfill \square

3. **The geometry of symmetric spaces**

3.1. **Introduction.** We give a short introduction to the geometry of symmetric spaces. More detailed expositions can be found e.g. in [Hel], [Ebe], [Bal], ...

Let $(M, \mathfrak{g})$ be a Riemannian manifold and $p \in M$. We call an isometry $\Phi : M \to M$ with $\Phi(p) = p$, a reflection at $p$, if $d\Phi = -\text{id}_{T_p M}$. $M$ is called a (globally) symmetric space, if it admits a reflection at every point. A Riemannian manifold which locally admits a reflection at every point is called a locally symmetric space. Every locally symmetric space is the quotient of a simply connected symmetric space by a properly discontinuous group action and vice versa.

Assume now that $M$ is simply connected and choose a basepoint $p_0 \in M$. Denote by $G$ the connected component of its isometry group and by $K < G$ the isotropy group at $p_0$, i.e. the stabilizer subgroup of $p_0$. Then $M = G/K$ where we identify $p_0$ with $1 \cdot K$.

We call $M$ irreducible, if it does not split as a product $M = M' \times M''$. In this case, $M$ is automatically Einstein. If the Einstein constant is zero, then $M$ is isometric to Euclidean space. If it is positive, then $M$ is compact and $M$ is said to be of compact type and if it is negative, then $M$ is diffeomorphic to $\mathbb{R}^n$ and $M$ is said to be of noncompact type. By the de Rham Decomposition Theorem, every simply connected symmetric space splits uniquely as the product

$$M = M_1 \times \ldots \times M_m$$

of irreducible factors $M_i$. Generally, we say that a locally symmetric space is of compact (resp. noncompact) type, if all factors in the de Rham decomposition of its universal cover are of compact (resp. noncompact) type.

For a list of all irreducible symmetric spaces, see [Bes, p. 200].
3.2. The infinitesimal structure. Let $M = G/K$ be a simply connected symmetric space of noncompact type. We will now discuss its infinitesimal structure. Let $\mathfrak{g}$ be the Lie algebra of $G$. The elements of $\mathfrak{g}$ correspond to Killing fields on $M$. There is an involutory isomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ which corresponds to the reflection at the basepoint $p_0$. This isomorphism induces the splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ into $-1$ and $1$ eigenspaces where $\mathfrak{k}$ is the Lie algebra of $K$. Moreover, we see that

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}, \quad [\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t}.$$ 

For $v, w \in \mathfrak{g}$ we define the Killing form by

$$\langle v, w \rangle = \text{tr}[v, [w, \cdot]].$$

It can be seen easily that the splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is orthogonal with respect to the Killing form which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra, i.e. an abelian subalgebra which is not contained in a bigger abelian subalgebra in $\mathfrak{p}$. The dimension $r = \dim \mathfrak{a}$ is called the rank of $M$. Obviously, to every $v \in \mathfrak{p}$ there is a maximal abelian subalgebra containing $v$ and contained in $\mathfrak{p}$. It can be proven (cf. [Bal]) that all such algebras are conjugate under the adjoint action of $K$. Hence, the rank of $M$ is well defined.

Now consider the infinitesimal adjoint action $[v, \cdot] : \mathfrak{g} \to \mathfrak{g}$ of any $v \in \mathfrak{a}$ on $\mathfrak{g}$. Since it is antisymmetric with respect to the Killing form and interchanges $\mathfrak{p}$ and $\mathfrak{k}$, we can diagonalize $[v, \cdot]$ with real eigenvalues. Moreover, since $\mathfrak{a}$ is abelian, we can find a simultaneous eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta \subset \mathfrak{a}^*$ is called the root system and

$$[v, x_\alpha] = \alpha(v)x_\alpha$$

for any $v \in \mathfrak{a}$ and $x_\alpha \in \mathfrak{g}_\alpha$. The subspaces $\mathfrak{g}_\alpha$ are pairwise orthogonal with respect to the Killing form and for all $\alpha \in \Delta \setminus \{0\}$ the subspace $\mathfrak{g}_\alpha$ is isotropic.

It is easy to see that $-\Delta = \Delta$ and that the involution $\sigma$ maps $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$. So if we set $\mathfrak{p}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$ and $\mathfrak{k}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$, we have $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{p}_\alpha \oplus \mathfrak{k}_\alpha$. Let $v_0 \in \mathfrak{a}$ be an arbitrary vector such that $\alpha(v_0) \neq 0$ for all nonzero $\alpha \in \Delta$ and define the set of positive roots by $\Delta^+ = \{\alpha \in \Delta : \alpha(v_0) > 0\}$. Then we have the following root space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \oplus \mathfrak{k}_0$$

$$= \mathfrak{p} \oplus \mathfrak{k} = \left( \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{p}_\alpha \right) \oplus \left( \bigoplus_{\alpha \in \Delta^+} \mathfrak{k}_\alpha \oplus \mathfrak{k}_0 \right).$$

These splittings are orthogonal with respect to the Killing form. The subspace $\mathfrak{k}_0$ is a Lie algebra. Its geometric meaning will be described below.

Using the Jacobi identity, we can conclude that for any two $\alpha, \beta \in \Delta$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. Hence $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$ are nilpotent.
Lie algebras with $\sigma(n^+) = n^-$. The spaces $n^+$ and $n^-$ are isotropic with respect to the Killing form, but on $n \oplus n^-$

$$\langle \cdot, \cdot \rangle = -\langle \cdot, \sigma \cdot \rangle$$

is a positive definite scalar product. Let $\alpha_1, \ldots, \alpha_{n-r}$ be the roots of $\Delta^+$ occurring with the appropriate multiplicities and let $x_1, \ldots, x_{n-r}$ be an orthonormal basis of $n^+$ with respect to $\langle \cdot, \cdot \rangle$ such that $x_i \in g_{\alpha_i}$. Then $[x_i, x_j] \in g_{\alpha_i + \alpha_j}$. Set $y_i = \sigma x_i \in g_{-\alpha_i} \subset n^-$. So $\langle x_i, y_j \rangle = -\delta_{ij}$ and it is easy to see that $[x_i, y_j] \in g_{\alpha_i - \alpha_j}$ and $[y_i, y_j] \in g_{-\alpha_i - \alpha_j}$. We set

$$p_i = \frac{1}{\sqrt{2}}(x_i - y_i), \quad k_i = \frac{1}{\sqrt{2}}(x_i + y_i).$$

Hence, $p_1, \ldots, p_{n-r}$ form an orthonormal basis of the orthogonal complement of $a$ in $p$ and $k_1, \ldots, k_{n-r}$ are a negative orthonormal basis of the orthogonal complement of $t_0$ in $t$. We also choose an orthonormal basis $v_1, \ldots, v_r$ of $a$ with respect to $\langle \cdot, \cdot \rangle$.

Observe that $\sigma [x_i, y_i] = [y_i, x_i] = -[x_i, y_i]$, hence $[x_i, y_i] \in p$. Moreover, $[x_i, y_i] \in g_0$, so $[x_i, y_i] \in a$. Since for any $v \in a$, we have

$$\langle [x_i, y_i], v \rangle = -\langle [x_i, v], y_i \rangle = \alpha_i(v)\langle x_i, y_i \rangle = -\alpha_i(v),$$

we obtain

$$[x_i, y_i] = -\alpha_i^\#.$$  \hfill (3.1)

Finally, we apply our knowledge on the infinitesimal structure to find out more about the global geometry. The subgroup $A = \exp(a) < G$ corresponding to $a$ is abelian and isomorphic to $\mathbb{R}^r$. The orbit $F = A.p_0$ is a geodesic submanifold of $M$ isometric to $\mathbb{R}^r$ and is called a maximal flat of $M$. The subgroup $K_0 = \exp(t_0) < K$ corresponding to $t_0$ is the point stabilizer of the flat $F$. Observe that there are symmetric spaces with trivial $K_0$, such as $SL(n)/SO(n)$, however many symmetric spaces, e.g. hyperbolic space, have nontrivial $K_0$. The stabilizer (not the point stabilizer) $\text{Stab}_K(F)$ of the flat $F$ however consists of several components of $K_0$. Forming the quotient $W = \text{Stab}_K(F)/K_0$ yields a discrete group, called the Weyl group. It follows that the orbit $K.p$ of every point $p \in M$ under the isotropy group $K$ intersects $F$ in a nonempty set which is invariant under $W$. Moreover, one can see that $F$ can be decomposed into fundamental domains for the action of $W$ which are called Weyl chambers and that $W$ is generated by reflections along the walls of an arbitrary Weyl chamber. Finally, consider the subgroups $N$ resp. $N^-$ corresponding to $n$ resp. $n^-$. The product subgroups $P = AN$ and $P^- = AN^-$ are called Borel subgroups. They act simply transitively on $M$ and stabilize a Weyl chamber at infinity.

3.3. Homogeneous vector bundles over symmetric spaces. Let $M = G/K$ as before. We can regard $M$ as the base of a right $K$-principal bundle $\pi : G \to M$. Given any representation $\rho : K \to GL(E)$ (where $E$ is a real vector space) we can form the associated vector bundle $G \times_\rho E = (G \times E)/\sim$ where

$$(gg', e) \sim (g, \rho(g')e).$$
We will denote this associated vector bundle, the vector space as well as the representation simply by $E$ and we will also say that $E$ is a **homogeneous vector bundle**. We remark that the pullback $\pi^*E$ is the trivial bundle $G \times E$.

A **principal connection** on $G$ is a $\mathfrak{t}$-valued 1-form $\theta \in \Omega^1(G; \mathfrak{t})$ satisfying the following two properties (compare e.g. [Roe])

(i) **Equivariance**: For any $v \in TG$, $k \in K$ and right translate $v.k$, we have $\theta(v.k) = Ad(k^{-1})\theta(v)$. Here $Ad : K \to GL(\mathfrak{t})$ is the adjoint representation with $Ad_* : \mathfrak{t} \to gl(\mathfrak{t}), u \mapsto (w \mapsto [u, w])$.

(ii) **Being a projection**: For $u \in \mathfrak{t}$ denote by $R_u$ the vector field $R_u : G \to TG$ generated by the infinitesimal right action $g \mapsto g.u$. Then, we impose $\theta(R_u) = u$.

A principal connection gives us a connection on every homogeneous vector bundle $E$. Let $f \in C^\infty(M; E)$ be a section of $E$ and consider its pullback $\tilde{f} = \pi^*f$ as a function $G \to E$. Then we set for any $v \in T_pM$

$$\nabla^E_v f = (d\tilde{f}(v') + \rho_* \theta(v') \tilde{f})/\sim$$

where $v' \in T_pG$ is any vector projecting to $v$, i.e. $\pi(p') = p$ and $d\pi(v') = v$.

There is a canonical principal connection $\theta$ on $G$ with which we will always work from now on: Identify all tangent spaces of $G$ with $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ by the left $G$-action and define $\theta$ everywhere to be the projection $\mathfrak{g} \to \mathfrak{k}$. We can easily see that this is the only connection which is invariant by the left $G$-action and the reflection at the basepoint. Now, consider the adjoint representation $Ad : K \to GL(\mathfrak{p})$ and its associated vector bundle, the tangent bundle $E = TM$. The principal connection $\theta$ induces a connection $\nabla^E$ on $E$. It is not difficult to see that this connection is exactly the Levi-Civita connection on $TM$.

### 3.4 Killing fields and Lie derivatives.

Consider a homogeneous vector bundle $E$ over a symmetric space $M = G/K$ corresponding to a representation $\rho : K \to GL(E)$. Moreover, let $\theta$ be the principal connection on $\pi : G \to M$ from the last subsection.

For each $x \in \mathfrak{g}$ there is a Killing field $X = \frac{d}{dt}\big|_{t=0} \exp(tx) \in C^\infty(M; TM)$ and a right-invariant vector field $\tilde{X} \in C^\infty(G; TG)$ with $\tilde{X}(1) = x$. Then $d\pi(\tilde{X}) = X$.

Consider now a section $f \in C^\infty(M; E)$ and the corresponding function $\tilde{f} = \pi^*f : G \to E$. We define $\tilde{f}' : G \to E$ as the derivative on $G$ in the direction $\tilde{X}$

$$\tilde{f}' = d\tilde{f}(\tilde{X}).$$

Since $\tilde{f}'(gg') = \rho((g')^{-1})\tilde{f}'(g)$, we find that $\tilde{f}' = \pi^*f'$ for some section $f' \in C^\infty(M; E)$. We call $f'$ the **Lie derivative** of $f$ with respect to $X$ or $x$ and write

$$f' = \mathcal{L}_X f = \mathcal{L}_x f.$$

It is then easy to see that for $x, y \in \mathfrak{g}$, we have (observe that since the vector fields $\tilde{X}, \tilde{Y}$ are right-invariant, $[\tilde{X}, \tilde{Y}]$ corresponds to $-[x, y]$)

$$\mathcal{L}_x \mathcal{L}_y f - \mathcal{L}_y \mathcal{L}_x f = -\mathcal{L}_{[x, y]} f. \tag{3.2}$$
We now relate the Lie derivative $\mathcal{L}_X$ to the covariant derivative $\nabla_X$. At any point $p \in M$, we can decompose $X = X_0 + X_1$ where $X_0$ and $X_1$ are Killing fields such that for the corresponding right-invariant vector fields $\tilde{X}_0, \tilde{X}_1 \in C^\infty(G; TG)$, we have $\theta(\tilde{X}_0) = 0$ and $\theta(\tilde{X}) = \tilde{X}_1$ on $\pi^{-1}(p)$. Then $X_1(p) = 0$ and at $p$

$$\mathcal{L}_{X_0} f = \nabla_{X_0} f \quad \text{and} \quad \mathcal{L}_{X_1} f = \rho_* \theta(\tilde{X}) f.$$ 

This implies

$$\mathcal{L}_X f = \nabla_X f + \rho_* \theta(\tilde{X}) f. \quad (3.3)$$

Finally, we compute the Riemannian curvature of $M$ at $p_0$. Let $x, y, z \in \mathfrak{p}$ and denote by $X, Y, Z$ the corresponding Killing fields. Then $\theta(\tilde{X}) = \theta(\tilde{Y}) = \theta(\tilde{Z}) = 0$ and hence by (3.3) applied to $E = TM$, we must have $\nabla X = \nabla Y = \nabla Z$ at $p_0$.

Hence

$$-[[X, Y], Z] = -\nabla_{[X,Y]} Z + \nabla^2_{Z,X} Y - \nabla^2_{X,Z} Y = R(Y, X) Z - R(Y, Z) X = R(X, Y) Z.$$

So expressed on $\mathfrak{p}$

$$R(x, y) z = -[[x, y], z]. \quad (3.4)$$

3.5. Cross-sections of symmetric spaces. Consider the splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{t}$, fix a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and consider the set $\Delta^+ \subset \mathfrak{a}^*$ of positive roots of $\mathfrak{g}$. We call a root $\alpha \in \Delta^+$ simple if there is no decomposition $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \in \Delta^+$. We know (cf [Ebe]) that the set $\mathcal{B}^+ = \{ \beta_1, \ldots, \beta_r \}$ of simple roots forms a basis of the vector space $\mathfrak{a}^*$ and that every $\alpha \in \Delta^+$ can be expressed as a linear combination $\sum_{i=1}^r k_i \beta_i$ of the simple roots with nonnegative integer coefficients $k_i$.

We define the positive Weyl chamber

$$\mathcal{C} = \{ v \in \mathfrak{a} : \alpha(v) \geq 0 \text{ for all } \alpha \in \Delta^+ \} = \{ v \in \mathfrak{a} : \beta(v) \geq 0 \text{ for all } \beta \in \mathcal{B}^+ \}.$$

$\mathcal{C}$ has the structure of a polytope and for every splitting $\mathcal{B}^+ = \overline{\mathcal{B}}^+ \cup \mathcal{B}^+$ we can consider the corresponding wall

$$\mathcal{W} = \mathcal{C} \cap \{ v \in \mathfrak{a} : \beta(v) = 0 \text{ for all } \beta \in \mathcal{B}^+ \}.$$ 

So for $\overline{\mathcal{B}}^+ = \emptyset$, we have $\mathcal{W} = \mathcal{C}$ and for $\mathcal{B}^+ = \mathcal{B}^+$, we have $\mathcal{W} = \{0\}$. From now on, given a wall $\mathcal{W} \subset \mathcal{C}$, we will denote the corresponding splitting sets by $\overline{\mathcal{B}}_W$ and $\mathcal{B}_W^+$. Every wall $\mathcal{W} \subset \mathcal{C}$ has a boundary $\partial \mathcal{W}$ which consists of walls $\mathcal{W}' \subset \partial \mathcal{W}$ which are smaller than $\mathcal{W}$ by one dimension. Those walls $\mathcal{W}'$ correspond to the splitting sets $\overline{\mathcal{B}}_{W'} = \overline{\mathcal{B}}_W \cup \{ \beta \}$ for $\beta \in \mathcal{B}_W$.

For every wall $\mathcal{W} \subset \mathcal{C}$, let $\overline{\Delta}_W \subset \Delta^+$ be the set of roots which can be represented by linear combinations of the simple roots $\overline{\mathfrak{a}}_W$ and let $\overline{\Delta}_W^+ = \Delta^+ \setminus \overline{\Delta}_W$. Setting, $\overline{\mathfrak{a}}_W = \text{span}(\overline{\mathfrak{a}}_W)^\#$, we moreover obtain an orthogonal splitting $\mathfrak{a} = \overline{\mathfrak{a}}_W \oplus \mathfrak{a}_W$. Now set $\overline{\mathfrak{p}}_W = \overline{\mathfrak{a}}_W \oplus \bigoplus_{\alpha \in \overline{\Delta}_W} \mathfrak{m}_\alpha$, $\overline{\mathfrak{t}}_W = [\overline{\mathfrak{p}}_W \oplus \overline{\mathfrak{a}}_W, \overline{\mathfrak{p}}_W \oplus \overline{\mathfrak{a}}_W]$ and $\overline{\mathfrak{g}}_W = \overline{\mathfrak{p}}_W \oplus \overline{\mathfrak{a}}_W \oplus \overline{\mathfrak{t}}_W$.

It is not hard to see that $\overline{\mathfrak{p}}_W$ and $\overline{\mathfrak{t}}_W$ are Lie algebras. Denote by $\overline{G}_W$ and $\overline{K}_W$ the corresponding Lie groups. Then $\overline{M}_W = \overline{G}_W / \overline{K}_W$ is a symmetric space which we will call a cross-section of $M$. Hence $\overline{M}_{(0)} = M$ and $\overline{M}_{\mathcal{C}} = \{ \text{pt} \}$. We remark
that not every symmetric space \( M = G'/K' \) with \( G' < G \) and \( K' < K \) arises by this construction, e.g. \( \mathbb{H}^2 \) is not a cross-section of \( \mathbb{H}^3 \).

We need to discuss a few more properties of \( \overline{M}_W \): Its Weyl group \( \overline{W}_W \) acting on \( \overline{a}_W \) is generated by reflections along the walls corresponding to the roots \( \overline{\Delta}_W \). Hence \( \overline{W}_W \) is a subgroup of the Weyl group \( W \) of \( M \) and its fixed point set in \( a \) is exactly \( \overline{a}_W \). Next, consider the nilpotent Lie algebras

\[
\overline{\pi}_W = \bigoplus_{\pi \in \overline{\pi}_W} g_{\pi} \quad \text{and} \quad \overline{\mu}_W = \bigoplus_{\alpha \in \overline{\Delta}_W^+} g_{\alpha}.
\]

Let \( \overline{N}_W, \overline{A}_W < G \) be the corresponding Lie groups, let \( \overline{A}_W, \overline{A}_W < G \) be the Lie groups corresponding to \( \overline{a}_W \) resp. \( \overline{a}_W \) and set \( \overline{P}_W = \overline{A}_W \overline{N}_W \) and \( \overline{P}_W = \overline{A}_W \overline{N}_W \). Observe that \( N = \overline{N}_W \overline{N}_W \) and \( P = \overline{P}_W \overline{P}_W \). We will now show that \( \overline{G}_W \) normalizes \( \overline{P}_W \): To do this, it suffices to establish \( [\overline{P}_W, \overline{a}_W] \subset \overline{a}_W \). Obviously, \( [\overline{P}_W, \overline{a}_W] = 0 \). For the second factor observe that for \( \overline{\pi} \in \overline{\Delta}_W^+ \) and \( \beta \in \overline{\Delta}_W^+ \), we have \( [\overline{\pi}, g_{\beta}] \subset \overline{g}_{\pi + \beta} \). Now if \( -\overline{\pi} + \overline{\beta} \) were not positive, \( \overline{\alpha} - \overline{\beta} \) would be, but expressing this root as a linear combination of the roots in \( \mathcal{B}^+ \) would lead to a negative coefficient in front of one of the roots of \( \overline{\Delta}_W^+ \). This shows that \( [\overline{\pi}, g_{\beta}] \subset \overline{a}_W \) and hence the claim.

Consider now a homogeneous vector bundle \( E \) over \( M \). It corresponds to a representation \( \rho : K \to GL(E) \) on a vector space which we also denote by \( E \). Restriction to \( \overline{K}_W \) yields a representation \( \rho_{\overline{W}} : \overline{K}_W \to GL(E) \). We will denote the associated homogeneous vector bundle over \( \overline{M}_W \) by \( E_{\overline{W}} \). Let now \( f \in C^\infty(\overline{M}_W; E_{\overline{W}}) \) be a section. It corresponds to a smooth map \( \hat{f} : \overline{G}_W \to E \) such that \( \hat{f}(gk) = \rho_{\overline{W}}(k^{-1})\hat{f}(g) \) for all \( g \in \overline{G}_W, k \in \overline{K}_W \). Using the fact that \( P \) and \( \overline{P}_W \) operate simply transitively on \( M \) resp. \( \overline{M}_W \), and the identity \( P = \overline{P}_W \overline{P}_W \), it is easy to see that there is a unique smooth extension \( \hat{f} : G \to E \) of \( \hat{f} \) such that the following is holds: \( \hat{f}(gk) = \rho(k^{-1})\hat{f}(g) \) for all \( g \in G, k \in K \) and \( \hat{f}(hg) = \hat{f}(g) \) for all \( h \in \overline{P}_W, g \in G \). Hence \( \hat{f} \) corresponds to a smooth \( P_{\overline{W}} \)-invariant section \( \hat{f} \in C^\infty(M; E) \) which we call the lift of \( f \). Since the isometry group \( \overline{G}_W \) of \( \overline{M}_W \) normalizes \( \overline{P}_W \), the construction of the lift is equivariant under \( \overline{G}_W \).

4. The heat kernel in homogeneous vector bundles

4.1. Statement of the results. In this section, we will prove a general decay result about the heat kernel in homogeneous vector bundles over symmetric spaces. Let \( M = G/K \) be a simply-connected symmetric space of noncompact type and consider a homogeneous vector bundle \( E \) over \( M \). Choose a basepoint \( p_0 \in M \), set \( E_0 = E_{p_0} \) and consider the heat kernel \( (k_t)_{t > 0} \in C^\infty(M; E) \otimes E_0 \) centered in \( p_0 \), i.e. for all \( e \in E_0 \)

\[
\partial_t k_t e = \Delta k_t e \quad \text{and} \quad k_t e \xrightarrow{t \to 0} \delta_{p_0} e.
\]

In the following, we will explain how to compute a constant \( \lambda_0 = \lambda_{M,E} \), depending on the space \( M \) and the bundle \( E \), which bounds the exponential \( L^1 \)-decay rate
of $k_t$, i.e. for which $||k_t||_{L^1(M)} \leq Ce^{-\lambda_0 t}$ for all $t > 0$ and some $C < \infty$. In many cases, this bound already turns out to be the exact decay rate, by which we mean that we even have $ce^{-\lambda_0 t} \leq ||K_t||_{L^1(M)} \leq Ce^{-\lambda_0 t}$ for all $t > 0$ and some $c > 0, C < \infty$. After stating the exact Theorem, we will discuss its implications on the longtime behavior of general solutions of the heat equation in $E$, $L^1$-bounds on the corresponding Green’s kernel and $L^\infty$-estimates for the Poisson equation. In the subsequent section we will apply our result to the case in which $M$ is Einstein and $E = \text{Sym}_2 T^*$ is the vector bundle of symmetric bilinear forms (i.e. variations of the metric). This will lead to a decay result for the linearized Ricci deTurck equation.

The constant $\lambda_0 = \lambda_{M,E}$ is defined to be the minimum over certain constants each corresponding to a cross-section of the symmetric space $M$. In order to give an idea about the concept behind this, we will first discuss the case in which $M$ has rank 1. Then $\lambda_0 = \min\{\lambda_L, \lambda_B\}$ where $\lambda_L$ and $\lambda_B$ are defined as follows:

**The constant $\lambda_L$:** Consider a Borel subgroup $P = AN < G$ (see subsection 3.2), i.e. $P$ acts simply transitively on $M$ and fixes a point at infinity. Let $V_{par} \subset C^\infty(M; E)$ be the vector space of $P$-invariant sections (we will later call those sections *parabolically invariant*). Evaluation at $p_0$ induces an isomorphism $V_{par} \cong E_0$. Observe that for every $f \in V_{par}$, its Laplacian $\triangle f$ is also contained in $V_{par}$ and hence we can define the operator $S_{par} = -\triangle : V_{par} \to V_{par}$. As we will see in the next subsection, $S_{par}$ is self-adjoint and using the isomorphism $V_{par} \cong E_0$, we will compute that $S_{par}(e) = -\sum_{i=} k_i k_i$. Now define $\lambda_L$ to be the smallest eigenvalue of $S_{par}$. We will see that always $\lambda_L \geq 0$.

**The constant $\lambda_B$:** Here we consider all Bochner formulas for sections in $E$, i.e. expressions

$$-\triangle = D^*D + \lambda$$  \hspace{1cm} (4.1)

for some linear first order operator $D : C^\infty(E; M) \to C^\infty(E'; M)$ and its formal adjoint $D^* : C^\infty(E'; M) \to C^\infty(E; M)$. Let $\lambda_B$ be the maximum of all such $\lambda$. Obviously, $\lambda_B \geq 0$, since we always have the trivial Bochner formula $-\triangle = \nabla^*\nabla$. The constant $\lambda_B$ bounds the $L^2$-decay of $K_t$, i.e. $||k_t||_{L^2(M)} \leq Ce^{-\lambda_B t}$ for all $t > 1$ and some $C < \infty$.

We remark that in the rank 1 case, we could replace this definition by setting $\lambda_B$ to be the supremum over all $\lambda$ for which we have $||k_t||_{L^2(M)} \leq Ce^{-\lambda t}$ for all $t > 1$ and some $C < \infty$. This might improve the constant $\lambda_0$ and lead to a stronger result. However, it would make the computation of $\lambda_B$ unnecessarily complicated for our purposes and it is also not clear to us how to carry this concept over to the higher rank case.

The main theorem of this section in the rank 1 case now reads

**Theorem 4.1** (Rank 1 case). Let $M$ be of rank 1 and let $\lambda_L$ and $\lambda_B$ be defined as above.

If $\lambda_B > \lambda_L$, then the exponential decay rate is exactly $\lambda_L$, i.e. there are constants
\[ c > 0, C < \infty \text{ such that} \]
\[ c e^{-\lambda t} \leq \|k_t\|_{L^1(M)} \leq C e^{-\lambda t} \quad \text{for all} \quad t > 0. \]

If \( \lambda_B < \lambda_L \), then we have at least
\[ c e^{-\lambda t} \leq \|k_t\|_{L^1(M)} \leq C e^{-\lambda_B t} \quad \text{for all} \quad t > 0. \]
Finally, if \( \lambda_B = \lambda_L \), the upper bound still holds with \( \lambda_B \) replaced by any \( \lambda < \lambda_B \) (where \( C \) depends on \( \lambda \)). More precisely, we have
\[ c e^{-\lambda t} \leq \|k_t\|_{L^1(M)} \leq C(\log(t+2))^{1/2} (t+2)^{a/2} e^{-\lambda t} \]
where \( a = \max\{\sum_{i=1}^{n+r} |\alpha_i|\}/\min_i |\alpha_i|, 2\} \).

We give two examples which illustrate the possible constellations of \( \lambda_L \) and \( \lambda_B \):

**Example A.** Consider \( M = \mathbb{H}^n \) and \( E = T^* \), the bundle of 1-forms. In this case, we have \( \lambda_L = 1 \) and \( \lambda_B = n-1 \). So for \( n > 2 \), we have an exact exponential decay with rate \( \lambda_0 = 1 \). For \( n = 2 \), we can show that \( e^{t\lambda}k_t\|_{L^1(M)} \) does not stay bounded for large \( t \):

Assume the opposite and consider the disc model of \( \mathbb{H}^2 \) imbedded in \( \mathbb{R}^2 \), i.e. \( g_{\mathbb{H}^2} = (1-r^2)^{-2}(dx^2 + dy^2) \), and choose \( p_0 = 0 \). Then for \( f = dx \in C^\infty(M; E) \), we have \( \Delta f = f = 0 \) and hence \( \int_M \langle f, e^{t\lambda}k_t \rangle \) is constant in time and nonzero. But since \( |f| = 1 - r^2 \) and since the \( L^1 \)-norm of \( e^{t\lambda}k_t \) is assumed to stay bounded, we conclude that the supremum of \( e^{t}\|k_t\| \) over a sufficiently large ball around \( p_0 \) has to stay bounded from below as \( t \to \infty \).

However, \( e^{t}\lambda k_t \) stays bounded in \( L^\infty \) along with all its derivatives for the following reason: By Cauchy-Schwarz and the convolution property of \( k_t \), we conclude that for \( t > 2 \) we have \( |\nabla^m k_t| = |\nabla^m k_1 \ast k_{t-1}| \leq \|k_{t-1}\|_{L^2(M)} \leq Ce^{-t} \).

So by Arzela-Ascoli, we find a subsequence \( e^{t}\lambda k_{t_i} \) which converges to some nonzero \( k_{\infty} \in C^\infty(M; E) \). This \( k_{\infty} \) must be bounded in \( L^2 \) and \( L^1 \) and by the right choice of the \( t_i \), we can guarantee that \( dk_{\infty} = 0 \) and \( d^2 k_{\infty} = 0 \). Since \( k_{\infty} \) is also spherical (see the next subsection), we conclude that \( k_{\infty} \) must be a nonzero multiple of \( f \), but \( f \) is unbounded in \( L^1 \).

**Example B.** Consider the case \( M = \mathbb{H}^2 \) and \( E = \text{Sym}^0 \mathbb{H}^2 T^* \), the space of quadratic differentials. Then \( \lambda_L = 4 \) and \( \lambda_B = 2 \), so the exponential decay rate lies between \( -4 \) and \( -2 \). Since \( f = dx dy \in C^2(M; E) \) is a bounded section satisfying \( \Delta f + 2f = 0 \), we conclude that \( \int_M \langle f, e^{2\lambda}k_t \rangle \) is constant in time and nonzero. Hence, \( \|e^{2\lambda}k_t\|_{L^1(M)} \) has to stay bounded from below and the exponential decay rate is exactly \( -2 \).

We will now discuss the case in which \( M \) has general rank. As explained in subsection 3.3 for every wall \( \mathcal{W} \subset \mathcal{C} \) of the positive Weyl chamber \( \mathcal{C} \), there is a cross-sectional symmetric space \( \overline{M}_\mathcal{W} \) of \( M \). For example, \( \overline{M}_{\{0\}} = M \) and \( \overline{M}_{\mathcal{C}} \) is just a point. The vector bundle \( E \) restricts to a homogeneous vector bundle \( E_\mathcal{W} \) over \( \overline{M}_\mathcal{W} \) and to every section \( f \in C^\infty(\overline{M}_\mathcal{W}; E_\mathcal{W}) \), we find a lift \( \hat{f} \in C^\infty(M; E) \) which is invariant under the parabolic subgroup \( P_\mathcal{W} \). Obviously, then also \( \Delta \hat{f} \) is invariant under \( P_\mathcal{W} \) and hence \( \Delta \hat{f} = \hat{f}' \) for some \( f' \in C^\infty(\overline{M}_\mathcal{W}; E_\mathcal{W}) \). There is a
linear (zero-order) bundle endomorphism $S_W : C^\infty(\overline{M}_W; E_W) \to C^\infty(\overline{M}_W; E_W)$ which satisfies
\[ f' = \triangle_W f = \overline{\triangle} f - S_W f, \tag{4.2} \]
where $\overline{\triangle}$ denotes the Laplacian on $\overline{M}_W$. In the next subsection, we will see that $S_W$ is self-adjoint at that at $p_0$, we have $S_W(e) = -\sum_{\alpha \in \Delta(W)} k_\alpha k_\alpha$. We remark that in the case $W = \{0\}$, we have $\triangle_W = \overline{\triangle} = \triangle$ and $S_W = 0$. In the case $W = C$, we have $\triangle_W = S_W = S_{par}$. Now consider all possible Bochner formulas
\[ -\triangle_W = D^* D + \lambda \tag{4.3} \]
on $\overline{M}_W$ and let $\lambda_W$ be the maximum of all such $\lambda$. We have $\lambda_W \geq 0$, since there is always the trivial Bochner formula $-\triangle_W = \nabla^* \nabla + S_W$. Observe that in the two extreme cases we get $\lambda(0) = \lambda_B$ and $\lambda_C = \lambda_L$. We finally set $\lambda_0 = \lambda_{M,E} = \min_{W \subseteq C} \lambda_W$ and $\lambda_1 = \min_{W \subseteq C} \lambda_W$

We can now state the main theorem of this section in its full generality:

**Theorem 4.2** (General rank case). Let $M$ be a simply-connected symmetric space of noncompact type and let the constants $(\lambda_W)_{W \subseteq C}$, $\lambda_0$ and $\lambda_1$ be defined as above. If $\lambda_1 > \lambda_C$, then the exponential decay rate is exactly $\lambda_0 = \lambda_C$, i.e. there are constants $c > 0, C < \infty$ such that
\[ ce^{-\lambda_0 t} \leq \|k_t\|_{L^1(M)} \leq Ce^{-\lambda_0 t} \quad \text{for all} \quad t > 0. \]

If $\lambda_1 \leq \lambda_C$, then the upper bound still holds with $\lambda_0$ replaced by any $\lambda < \lambda_0$ (where $C$ depends on $\lambda$). More precisely, we have for $c > 0, A, C < \infty$
\[ ce^{-\lambda_0 t} \leq \|k_t\|_{L^1(M)} \leq C(t + 2)^A e^{-\lambda_0 t}. \]

As an immediate corollary we obtain

**Corollary 4.3.** Assume that we are in the setting of Theorem 4.2. Consider a solution $(s_t)_{t \geq 0} \in C^\infty(M; E)$ to the heat equation $\partial_t s_t = \triangle s_t$ which is bounded on compact time intervals. Then if $\lambda < \lambda_0$ or $\lambda \geq \lambda_0$ and $\lambda_1 \neq \lambda_C$, we have
\[ \|s_t\|_{L^\infty(M)} \leq Ce^{-\lambda t}\|s_0\|_{L^\infty(M)} \]
and in the case $\lambda = \lambda_1 = \lambda_C$ we get
\[ \|s_t\|_{L^\infty(M)} \leq C(t + 2)^A e^{-\lambda t}\|s_0\|_{L^\infty(M)}. \]

Observe that we did not impose any spatial decay or compact support assumptions on $s_0$.

Furthermore, we can use Theorem 4.2 to find an $L^1$-estimate on the associated Green’s kernel which leads to an $L^\infty$-estimate of the Poisson equation:

**Corollary 4.4.** Assume, we are in the setting of Theorem 4.2. Let $\lambda < \lambda_0$ and consider the Green’s kernel $g \in C^\infty(M \setminus \{p_0\}; E) \otimes E^*_0$ of the operator $-\triangle - \lambda$ centered in $p_0$. Then $\Lambda = \|g\|_{L^1} < \infty$.

As a consequence, we obtain the estimate
\[ \Lambda\|\triangle s + \lambda s\|_{L^\infty} \geq \|s\|_{L^\infty} \tag{4.4} \]
for all bounded sections $s \in C^\infty(M; E)$. In particular, $-\triangle - \lambda$ does not have $L^\infty$-bounded kernel elements.
We remark that for these results it is essential that the homogeneous vector bundle $E$ has curvature. Otherwise, we would only get the estimate $\|K_t\|_{L^1} < C$ in Theorem 4.2. As for Corollary 4.3 observe that a simple application of the maximum principle would give us $\|s_t\|_{L^\infty} \leq \|s_0\|_{L^\infty}$. So the curvature of the vector bundle $E$ leads to an extra decay rate, but only for large $t$. Corollary 4.4 maybe shows the effect of the curvature of $E$ in the most demonstrative way: If $E$ were flat, then any constant section $s \in C^\infty(M; E)$ would contradict inequality (4.4) already for $\lambda = 0$. However, in the non-flat case, the curvature forces the section to have nontrivial Laplacian.

The next result is in the same spirit as Theorem 4.1, but it gives a pointwise bound on the heat kernel.

**Theorem 4.5.** Assume we are in the setting of Theorem 4.1 and that $\text{rank } M = 1$. There is a constant $C < \infty$ such that for $\lambda_0 = \min\{\lambda_L, \lambda_B\}$

$$|k_t(p)| \leq \frac{C}{\text{vol } B_r(p_0)} e^{-\lambda_0 t} \quad \text{where} \quad r = d(p_0, p).$$

A few remarks on the proofs of Theorems 4.1, 4.2 and 4.5: Obviously, Theorem 4.2 implies Theorem 4.1 in the main case $\lambda_L > \lambda_B$ (which is the one needed here). Despite of this fact, we first carry out the proof of Theorem 4.1 in subsection 4.4, since it is much simpler than the proof of Theorem 4.2 which is described in subsection 4.5. Subsection 4.2 contains a preparatory discussion on spherical models which will be used to describe the heat kernel $k_t$. In subsection 4.3, we discuss some basic bounds which will be needed in both the rank 1 as well as the general rank case.

### 4.2. Spherical sections in symmetric spaces.

Consider a homogeneous vector bundle $E$ over $M$ coming from a representation $\rho : K \to GL(E)$. We will analyze two classes of sections of $E$, namely spherical and parabolically invariant ones. Later, we will generalize the discussion of the parabolically invariant sections to $P_W$-invariant ones what will then allow us to compute the endomorphisms $S_W$. The last part will only be needed for the proof of Theorem 4.2 and can be skipped for the rank 1 case.

We first introduce spherical sections. Let $p_0 \in M$ be a basepoint and $K$ its stabilizer. Then $K$ naturally acts on the space of sections $C^\infty(M; E)$ of $E$ and on the fiber $E_0 = E_{p_0}$ over $p_0$. Hence it also acts on $C^\infty(M; E) \otimes E_0^\vee$. We will now consider elements of this space rather than sections of $E$.

**Definition 4.6.** A section $f \in C^\infty(M; E) \otimes E_0^\vee$ is called spherical if it is invariant under the action of $K$.

Obviously, the Laplacian $\triangle f$ of a spherical section is also spherical and $(\triangle f)(e) = \triangle(f(e))$ for all $e \in E_0$.

Let now $f$ be a spherical section, consider a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and the flat $\mathcal{F} = \exp(\mathfrak{a})p_0$ (cf subsection 3.2). Recall that the orbit of any point $p \in M$ under the action of $K$ intersects $\mathcal{F}$ in a nonempty set which is invariant under the Weyl group $W$. So $f$ is already determined by its restriction
to $F$. Since the curvature along $F$ vanishes, the vector bundle $E$ restricted to $F$ becomes trivial and we can view the restriction of $f$ to $F$ as a function

$$F : a \rightarrow \text{End}(E_0), \quad v \mapsto f(\exp(v).p_0).$$

It is easy to see that $F$ actually only takes values in the subspace of $K_0$-equivariant endomorphisms of $\text{End}_{K_0}(E_0)$ of $E_0$ and is equivariant under the Weyl group, i.e. for every $w \in W$ and $v \in a$, we have $F(w.v) = w.F(v) = w \circ F(v) \circ w^{-1}$.

**Definition 4.7.** The smooth function $F : a \rightarrow \text{End}_{K_0}(E_0)$ is called a spherical model of $f$.

Reversely, we can state the following result:

**Lemma 4.8.** If $F : a \rightarrow \text{End}_{K_0}(E_0)$ is a smooth function which is equivariant under the action of the Weyl group $W$, then there is a unique smooth spherical $f \in C^\infty(M; E) \otimes E_0^*$ such that $F$ arises in the way described above.

**Proof.** It is clear that there is such a unique continuous spherical $f \in C^0(M; E) \otimes E_0^*$. Using the calculus described below, we can successively compute its covariant derivatives $\nabla^m f \in C^0(M; (T^*)^\otimes m \otimes E) \otimes ((T^*)^\otimes m \otimes E)_0^*$ in terms of their spherical models and show that these are bounded. \hfill \Box

We will now compute the Laplacian $f' = \Delta f$ of a spherical section $f$ in terms of its radial model $F$, i.e. for each $v \in a$ we will compute $F'(v)$. Recall the vectors $v_i, k_i, p_i, x_i, y_i \in g$ as defined in subsection 3.2. The conjugates $k'_i = \text{Ad}(\exp(v)) \cdot k_i$, $p'_i = \text{Ad}(\exp(v)) \cdot p_i$ correspond to rotations resp. translations at $p = \exp(v).p_0$. Hence

$$f'(p) = \sum_{i=1}^{r} \mathcal{L}_{v_i} \mathcal{L}_{v_i} f(p) + \sum_{i=1}^{n-r} \mathcal{L}_{p'_i} \mathcal{L}_{p'_i} f(p).$$

Now observe that since

$$\text{Ad}(\exp(v)) x_i = \exp(\alpha_i(v)) x_i \quad \text{and} \quad \text{Ad}(\exp(v)) y_i = \exp(-\alpha_i(v)) y_i,$$

we find

$$p'_i = -\frac{1}{\text{sh} \alpha_i(v)} k_i + \frac{1}{\text{th} \alpha_i(v)} k'_i.$$

So since by (3.1) $[k_i, k'_i] = \text{sh}(\alpha_i(v)) \alpha_i^\#$, we conclude by (3.2)

$$\mathcal{L}_{p'_i} \mathcal{L}_{p'_i} f = \frac{1}{\text{sh}^2 \alpha_i(v)} \mathcal{L}_{k_i} \mathcal{L}_{k_i} f + \frac{\text{ch}^2 \alpha_i(v)}{\text{sh}^2 \alpha_i(v)} \mathcal{L}_{k'_i} \mathcal{L}_{k'_i} f - \frac{\text{ch} \alpha_i(v)}{\text{sh}^2 \alpha_i(v)} (\mathcal{L}_{k'_i} \mathcal{L}_{k_i} f + \mathcal{L}_{k_i} \mathcal{L}_{k'_i} f)
$$

$$= \frac{1}{\text{sh}^2 \alpha_i(v)} \mathcal{L}_{k_i} \mathcal{L}_{k_i} f + \frac{\text{ch}^2 \alpha_i(v)}{\text{sh}^2 \alpha_i(v)} \mathcal{L}_{k'_i} \mathcal{L}_{k'_i} f - 2 \frac{\text{ch} \alpha_i(v)}{\text{sh}^2 \alpha_i(v)} \mathcal{L}_{k'_i} \mathcal{L}_{k_i} f + \text{cth}(\alpha_i(v)) \mathcal{L}_{\alpha_i^\#} f.$$
Now observe that at $p$ we have $\mathcal{L}_{k_i} f(e) = f(k_i e)$ and $\mathcal{L}_{k_i} f(e) = k_i f(e)$. Hence, we obtain

$$F'(v)(e) = \Delta F(v)(e) + \sum_{i=1}^{n-r} \left[ \frac{1}{\sh^2 \alpha_i(v)} F(v)(k_i k_i e) + \frac{\ch^2 \alpha_i(v)}{\sh^2 \alpha_i(v)} k_i k_i F(v)(e) - 2 \frac{\ch \alpha_i(v)}{\sh^2 \alpha_i(v)} k_i (F(v)(k_i e)) + \cth(\alpha_i(v))(\partial_{\alpha_i^*} F)(v)(e) \right].$$

So for $v \to \infty$ in the sense that $\alpha_i(v) \to \infty$ for all $i$, the expression becomes in the limit

$$F'(v)(e) = \Delta F(v)(e) + \sum_{i=1}^{n-r} \left[ k_i k_i F(v)(e) + (\partial_{\alpha_i^*} F)(v)(e) \right]. \quad (4.5)$$

Next, let $P = AN < G$ be a Borel subgroup and call a $P$-invariant section $f \in C^\infty(M; E)$ parabolically invariant. Since $P$ acts transitively on $M$, the section $\hat{f}$ is determined by its value $f(p_0) \in E_0$ at $p_0$. Observe that with $f$ its Laplacian $\Delta f$ is also parabolically invariant. We have

$$(\Delta f)(p_0) = \sum_{i=1}^{\gamma} \mathcal{L}_{v_i} \mathcal{L}_{v_i} f(p_0) + \sum_{i=1}^{n-r} \mathcal{L}_{p_i} \mathcal{L}_{p_i} f(p_0). \quad (4.6)$$

Using the fact that $\mathcal{L}_{v_i} f = \mathcal{L}_{v_i} f = 0$ and $p_i = -k_i + \sqrt{2}x_i$, we obtain

$$(\Delta f)(p_0) = \sum_{i=1}^{n-r} \left[ \mathcal{L}_{k_i} \mathcal{L}_{k_i} f(p_0) - \sqrt{2} \mathcal{L}_{x_i} \mathcal{L}_{k_i} f(p_0) \right].$$

Since $\mathcal{L}_{x_i} \mathcal{L}_{k_i} f = \mathcal{L}_{k_i} \mathcal{L}_{x_i} f + \frac{1}{\sqrt{2}} \mathcal{L}_{\alpha_i^*} f = 0$, we obtain

$$(\Delta f)(p_0) = \sum_{i=1}^{n-r} k_i k_i f(p_0). \quad (4.7)$$

Observe that the right hand side is exactly the zero order term in $(4.6)$. It is easy to see that the map $E_0 \to E_0$, $e \mapsto \sum_{i=1}^{n-r} k_i k_i e$ is self-adjoint.

Now consider a cross-section $M_W$ corresponding to a wall $W \subset C$ and let $P_W = T_W N_W \subset G_\mathfrak{g}_W$ be the corresponding parabolic subgroup. Let $f \in C^\infty(M_W; E_W)$ and consider its ($P_W$-invariant) lift $\hat{f} \in C^\infty(M; E)$. Then $\Delta \hat{f}$ is also $P_W$-invariant and hence there is an $f' \in C^\infty(M_W; E_W)$ such that $\hat{f} = \Delta \hat{f}$. We will calculate $f'$. First, recall that the isometry group $O_W$ of $M_W$ normalizes $P_W$, so if $f$ is $P_W$-invariant, then so are its translates by the action of $O_W$. Hence, it suffices to compute $f'(p_0)$. Using $(4.6)$ and the fact that $\mathcal{L}_v \hat{f} = 0$ for $v \in \mathfrak{a}_W$ and $\mathcal{L}_{x_0} \hat{f} = 0$, we obtain as above

$$\hat{f}'(p_0) = (\Delta \hat{f})(p_0) = \sum_{i} \mathcal{L}_{v_i} \mathcal{L}_{v_i} \hat{f} + \sum_{p \in \pi^v W} \mathcal{L}_{p^v} \mathcal{L}_{p^v} \hat{f} + \sum_{a \in \Delta^v} k_a k_a \hat{f}(p_0).$$
So
\[ f'(p_0) = (\overline{\Delta} f)(p_0) + \sum_{\alpha \in \Delta_n^+} k_{\alpha}.k_{\alpha}.f(p_0). \]
and comparing this with (4.2) yields
\[ (S_W f)(p_0) = - \sum_{\alpha \in \Delta_n^+} k_{\alpha}.k_{\alpha}.f(p_0). \quad (4.8) \]

4.3. First bounds on the heat kernel. Now consider the heat kernel \((k_t)_{t > 0} \in C^\infty(M; E) \otimes E_0^*\). By an elementary uniqueness argument one can show:

**Lemma 4.9.** \((k_t)_{t > 0}\) is a spherical section and its spherical model \((K_t)_{t > 0} : a \to \text{End}_{K_0} E_0\) satisfies
\[ \partial_t K_t(v) = \triangle K_t(v)(e) + \sum_{i=1}^{n-r} \left[ \frac{1}{\text{sh}^2 \alpha_i(v)} K_i(v)(k_i.k_i.e) + \frac{1}{\text{sh}^2 \alpha_i(v)} k_i.(K_i(v)(k_i.e)) + \text{cth}(\alpha_i(v))(\partial_{\alpha_i}# K_i)(v)(e) \right]. \]
Moreover, \(K_t(v) \in \text{End}_{K_0} E_0\) is always self-adjoint.

Observe that by (4.7) \(\lambda_L\) (in the rank 1 case) resp. \(\lambda_C\) (in the general rank case) is the smallest eigenvalue of the endomorphism
\[ E \to E, \quad e \mapsto - \sum_{i=1}^{n-r} k_i.k_i.e. \]
Denote moreover by \(\mu_i\) the largest eigenvalue of the endomorphism
\[ E \to E, \quad e \mapsto - k_i.k_i.e \]
and consider the differential operator
\[ -L^0 = \triangle - \lambda_C + \sum_{i=1}^{n-r} 2 \mu_i \frac{\text{ch}\alpha_i(v)}{\text{sh}^2 \alpha_i(v)} \left[ \mu_i - 1 \right] + \sum_{i=1}^{n-r} \text{cth}(\alpha_i(v))\partial_{\alpha_i}# \]
acting on scalar functions on \(a\). By Lemma 4.8 the third term is a spherical model coming from a spherical function \(\mu \in C^\infty(M; \mathbb{R})\). Using the discussion from subsection 4.2 (in the case \(E = \mathbb{R}\)), we find that \(L^0\) corresponds to the differential operator \(\triangle - \lambda_C + \mu\) acting on spherical functions on \(M\). Let \((k_t^0)_{t > 0}\) be the fundamental solution of \(\triangle - \lambda_C + \mu\) centered at \(p_0\), i.e.
\[ \partial_t k_t^0 = \triangle k_t^0 + (-\lambda_C + \mu)k_t^0 \quad \text{and} \quad k_t^0 \xrightarrow{t \to 0} \delta_{p_0}. \]
By uniqueness, \(k_t^0\) is spherical and its spherical model \(K_t^0\) satisfies
\[ \partial_t K_t^0 = -L^0 K_t^0 \]
We can show that \(K_t^0\) bounds \(K_t\):
Lemma 4.10. For every $t > 0$ and $v \in a$ denote by $K_t(v)(\min)$ resp. $K_t(v)(\max)$ the minimal resp. maximal eigenvalue of the endomorphism $K_t(v)$. Then

$$0 < K_t(v)(\min) \leq K_t(v)(\max) \leq K^0_t(v).$$

Moreover, $K_t(v)(\max)$ is a subsolution to the heat operator $\partial_t + L^0$ in the following sense: If $(G_t)_{t \geq t_0} \in C^\infty(a)$ is a solution to the equation $\partial_t G_t = -L^0 G_t$ and a spherical model, then $K_{t_0}(v)(\max) \leq G_{t_0}$ implies $K_t(v)(\max) \leq G_t$ for all $t \geq t_0$.

Proof. The proof makes use of the maximum principle. We will first establish the bound $K_t(v)(\min) > 0$. If the inequality was not true then, by the local behavior of the heat kernel for small times, we would find some $\varepsilon > 0$ and some first time $t' > 0$ such that there are $v' \in a$ and $e' \in E_0$ with $|e'| = 1$ such that

$$\langle K_t(v)e, e \rangle \geq -\varepsilon$$

holds for all $t \leq t'$, $v \in a$ and $e \in E_0$ with $|e| = 1$ with equality for $t = t'$, $v = v'$ and $e = e'$. This implies $K_{t'}(v')e' = K_{t'}(\min)e' = -\varepsilon e'$ and

$$\langle \partial_t K_{t'}(v')e', e' \rangle \leq 0.$$  \hfill (4.9)

as well as $\langle \partial_t K_{t'}(v')e', e' \rangle = 0$ for any direction $u \in a$ and $\langle \Delta K_{t'}(v')e', e' \rangle \geq 0$. As for the zero order terms we compute

$$\langle K_{t'}(v')(k_i, k_i, e') \rangle = \langle K_{t'}(v')(e'), k_i, k_i, e' \rangle = K_{t'}(v')(min) \langle k_i, k_i, e', e' \rangle = \varepsilon \langle k_i, e', k_i, e' \rangle$$

and

$$\langle k_i, (K_{t'}(v')(v_i, e')) \rangle = \varepsilon \langle k_i, k_i, e', e' \rangle \leq \varepsilon \langle k_i, e', k_i, e' \rangle.$$  

So

$$\langle \partial_t K_{t'}(v')e', e' \rangle \geq \sum_{i=1}^{n-r} \left( \frac{1}{sh^2 \alpha_i(v')} K_{t'}(v')(k_i, k_i, e') + \frac{ch \alpha_i(v')}{sh^2 \alpha_i(v')} k_i, k_i, K_{t'}(v')(e') \right) - 2 \frac{ch \alpha_i(v')}{sh^2 \alpha_i(v')} k_i, (K_{t'}(v')(k_i, e')) \right) \varepsilon \sum_{i=1}^{n-r} \left( \frac{1 - ch \alpha_i(v')}{sh \alpha_i(v')} \right)^2 |k_i, e'|^2 > 0,$$

contradicting (4.9).

We will now prove the claim that $K_t(v)(\max)$ is a subsolution to the heat operator $\partial_t + L^0$. The bound $K_t(v)(\max) \leq K^0_t(v)$ follows with a little more effort and will not be proven here since we will not directly need it.

Denote by $A$ a fixed constant which will be determined later. Let $\varepsilon > 0$ and assume again that there is some first time $t' > t_0$ such that there are some $v' \in a$, $e' \in E_0$ with $|e'| = 1$ such that

$$\langle K_{t}(v)e, e \rangle \leq G_{t}(v) + \varepsilon e^{At}$$

holds for all $t_0 \leq t \leq t'$, $v \in a$ and $e \in E_0$ with $|e| = 1$ and equality is true for $t = t'$, $v = v'$ and $e = e'$. This implies $K_{t'}(v')e' = K_{t'}(v')(\max)e'$ and $K_{t'}(v')(\max) = G_{t'}(v') + \varepsilon e^{At'}$.

Obviously,  \hfill (4.10)

$$\langle \partial_t K_{t'}(v')e', e' \rangle \geq \varepsilon \partial_t G_{t'}(v') + \varepsilon A e^{At'}.$$
Moreover, for any direction \( u \in \mathfrak{a} \), we have \( \langle \partial_u K_{v'}(v'), e' \rangle = \partial_u G_{v'}(v') \) and \( \langle \Delta K_{v'}(v')e', e' \rangle \leq \Delta G_{v'}(v') \). So

\[
\left\langle \Delta K_{v'}(v')e' + \sum_{i=1}^{n-r} \text{cth}(\alpha_i(v'))(\partial_{\alpha_i^*} K_{v'})(v')(e'), e' \right\rangle \\
\leq \Delta G_{v'}(v') + \sum_{i=1}^{n-r} \text{cth}(\alpha_i(v))\partial_{\alpha_i^*} G_{v'}(v').
\]  

(4.11)

Since \( K_{v'}(v')(e') = K_{v'}(v'(\max))e' \) and \( K_{v'}(v'(\max)) > 0 \), we can estimate

\[
\sum_{i=1}^{n-r} \langle k_i k_i, K_{v'}(v')(e'), e' \rangle \leq -\lambda_C K_{v'}(v'(\max)).
\]

Moreover,

\[
-\langle k_i k_i, K_{v'}(v'(k_i, e')) \rangle = \langle K_{v'}(v') k_i, e', k_i, e' \rangle \\
\leq K_{v'}(v'(\max))(k_i, e', k_i, e') = -K_{v'}(v'(\max))(e', k_i, e').
\]

This implies (we use \( \chi = \frac{\alpha_i(v')}{\text{sh}^2 \alpha_i(v')} \) here)

\[
\sum_{i=1}^{n-r} \left\langle \frac{1}{\text{sh}^2 \alpha_i(v')} K_{v'}(v')(k_i, k_i, e') + \frac{\chi}{\text{sh}^2 \alpha_i(v')} k_i k_i, K_{v'}(v')(e') \right\rangle \\
-2 \frac{\chi}{\text{sh}^2 \alpha_i(v')} k_i (K_{v'}(v')(k_i, e')), e' \rangle \leq \left( -\lambda_C + \sum_{i=1}^{n-r} 2\mu_i \frac{\chi \alpha_i(v') - 1}{\text{sh}^2 \alpha_i(v')} \right) K_{v'}(v'(\max)).
\]

So with (4.11) we obtain

\[
\langle \partial_i K_{v'}(v')e', e' \rangle \leq -L^2 G_{v'}(v') - \left( -\lambda_C + \sum_{i=1}^{n-r} 2\mu_i \frac{\chi \alpha_i(v') - 1}{\text{sh}^2 \alpha_i(v')} \right) G_{v'}(v') \\
+ \left( -\lambda_C + \sum_{i=1}^{n-r} 2\mu_i \frac{\chi \alpha_i(v') - 1}{\text{sh}^2 \alpha_i(v')} \right) K_{v'}(v'(\max)).
\]

Combining this with (4.10) and using the fact that \( K_{v'}(v'(\max)) = G_{v'}(v') + \varepsilon e^{At}, \) we conclude

\[
\varepsilon Ae^{At'} \leq \varepsilon \left( -\lambda_C + \sum_{i=1}^{n-r} 2\mu_i \frac{\chi \alpha_i(v') - 1}{\text{sh}^2 \alpha_i(v')} \right) e^{At'}.
\]

For sufficiently large \( A \) this yields a contradiction. \( \square \)

4.4. **The rank 1 case.** Assume in this subsection that \( M \) has rank 1. Then \( \mathfrak{a} \cong \mathbb{R} \) and we simply write \( \alpha_i = \alpha_i(1) \), i.e. \( \alpha_i(r) = \alpha_i r \). We will now prove Theorems 4.4 and 4.5.
Proof of Theorem 4.1. For the lower bounds observe that by definition of \( \lambda_L \), there is a \((\text{parabolically invariant})\) section \( f \in C^\infty(M; E) \) with \( \Delta f = -\lambda_L f \). Hence its convolution with the heat kernel satisfies \( f * k_t = e^{-\lambda_L t} f \), and thus the \( L^1 \)-norm of \( e^{\lambda_L t} k_t \) must be bounded from below.

In order to establish the upper bounds, set

\[
H_t^{(p)} = \left( \int_0^\infty \left| \prod_{i=1}^{n-1} \text{sh}(\alpha_i r) \right| \cdot (K_t(r) \text{(max)})^p \, dr \right)^{1/p}
\]

for \( p \geq 1 \). It is easy to see that there are constants \( c_p \) and \( C_p \) depending on \( p \) such that

\[
c_p H_t^{(p)} \leq \| k_t \|_{L^p(M)} \leq C_p H_t^{(p)}.
\]

By definition of \( \lambda_B \), there is a Bochner formula of the form \( -\Delta = D^* D + \lambda_B \). A basic application of Stoke’s Theorem yields

\[
\frac{d}{dt} \| k_t \|_{L^2(M)}^2 = -2\lambda_B \| k_t \|_{L^2(M)}^2 - 2 \| Dk_t \|_{L^2(M)}^2 \leq -2\lambda_B \| k_t \|_{L^2(M)}^2.
\]

Hence for \( t \geq 1 \)

\[
H_t^{(2)} \leq \frac{1}{c_2} \| k_t \|_{L^2} \leq C e^{-\lambda_B t}.
\]

(4.12)

Now from Lemma 4.10 we know that \( K_t(v) \text{(max)} \) is a subsolution to the heat operator \( \partial_t + L^0 \) and hence

\[
\frac{d}{dt} H_t^{(1)} + \lambda_L H_t^{(1)} \\
\leq \int_0^\infty \left| \prod_{i=1}^{n-1} \text{sh}(\alpha_i r) \right| \left( \partial_t^2 K_t(r) \text{(max)} + \sum_{i=1}^{n-1} \text{cth}(\alpha_i r) \partial_{\alpha_i} K_t(r) \text{(max)} \right) dr
\]

\[
+ \int_0^\infty \left| \prod_{i=1}^{n-1} \text{sh}(\alpha_i r) \right| \cdot \sum_{i=1}^{n-1} 2\mu_i \frac{\text{ch}(\alpha_i r) - 1}{\text{sh}^2(\alpha_i r)} K_t(r) \text{(max)} dr.
\]

By partial integration or Green’s formula applied to the corresponding spherical function, the first integral vanishes. It remains to bound the second integral.

Let \( \delta > 0 \). Choose the \( i_0 \) for which \( \alpha_{i_0} \) is minimal and recall that \( a = \max \{ (\sum_{i=1}^{n-1} \alpha_i) / \alpha_{i_0}, 2 \} \). Using Hölders inequality, we can bound the second integral by

\[
C \left( \int_0^\infty \left| \prod_{i=1}^{n-1} \text{sh}(\alpha_i r) \right| \left( \frac{\text{ch}(\alpha_{i_0} r) - 1}{\text{sh}^2(\alpha_{i_0} r)} \right)^{a+\delta} \, dr \right)^{\frac{1}{a+\delta}}
\]

\[
\times \left( \int_0^\infty \left| \prod_{i=1}^{n-1} \text{sh}(\alpha_i r) \right| (K_t(r) \text{(max)})^{1+\frac{1}{a+\delta-1}} dr \right)^{\frac{a+\delta-1}{a+\delta}}
\]

\[
\leq C \delta^{-\frac{1}{a+\delta}} H_t^{(1)} \left( H_t^{(1)} \right)^{\frac{a+\delta}{a+\delta-1}} \left( H_t^{(2)} \right)^{\frac{2}{a+\delta}}.
\]
We can rewrite this inequality as
\[
\frac{d}{dt} \left( e^{\lambda L t} H_t^{(1)} \right)^{\frac{2}{n+2}} \leq \frac{2}{a + \delta} C \delta^{\frac{n}{n+2}} \left( e^{\lambda L t} H_t^{(2)} \right)^{\frac{2}{n+2}}.
\]
In the cases \( \lambda < \lambda_B \) and \( \lambda > \lambda_B \), we can integrate this inequality using (4.12) to conclude that \( e^{\lambda L t} H_t^{(1)} \) stays bounded. If \( \lambda = \lambda_B \), we find
\[
\frac{d}{dt} \left( e^{\lambda L t} H_t^{(1)} \right)^{\frac{2}{n+2}} \leq C \delta^{\frac{n}{n+2}}.
\]
Hence for \( \delta < 1 \)
\[
e^{\lambda L t} H_t^{(1)} \leq C \left( \delta^{\frac{1}{n+2}} t + 2 \right)^{\frac{n+4}{n+2}} \leq C \delta^{-1/2} (t + 2)^{\frac{n+4}{n+2}}.
\]
The theorem follows for \( \delta = \frac{1}{\log(t+2)} \). □

Corollary 4.3 follows by convolution.

**Proof of Corollary 4.4.** For the first statement use the identity
\[
g = \int_0^\infty e^{\lambda t} k_t dt.
\]
The second statement follows from the first by convolution. □

**Proof of Theorem 4.5.** Note first that for an appropriate constant \( V_0 \)
\[
\text{vol } B_r(p_0) = V_0 \int_0^r \prod_{i=1}^{n-1} \text{sh}(\alpha_i r') dr'
\]
and hence if we set \( a = \sum_{i=1}^{n-1} \alpha_i \), we find that \( \text{vol } B_r(p_0) \) is asymptotic to \( e^{ar} \). So for \( r \geq 1 \) we have \( ce^{ar} \leq \text{vol } B_r(p_0) \leq Ce^{ar} \) for large \( r \). For \( r \leq 1 \), can simply compare with the Euclidean volume growth: \( cr^n < \text{vol } B_r(p_0) < C r^n \).

Now consider the heat kernel \( k_t \). For \( t < \frac{1}{4} \), the inequality follows from Proposition 2.5. For \( t \geq \frac{1}{4} \), we can argue as in the proof of Theorem 4.1 and conclude
\[
\|k_t\|_{L^2(M)} \leq Ce^{-\lambda_B t}.
\]
By convolution and Cauchy-Schwarz, this gives us an \( L^\infty \)-bound for \( t \geq \frac{1}{2} \)
\[
|k_t|(p) = |k_{t/2} \ast k_{t/2}|(p) \leq C_0 e^{-\lambda_B t} \leq C_0 e^{-\lambda t}.
\]
(4.13) Observe that the quantities \( k_t(\text{max}) \) and \( |k_t| \) are comparable: \( C^{-1} |k_t| \leq k_t(\text{max}) \leq |k_t| \). Now recall that by Lemma 4.10, the spherical model \( K_t(\text{max}) \) is a subsolution to the heat operator \( \partial_t + L^o \) where
\[
-L^o = \partial_r^2 + \sum_{i=1}^{n-1} \alpha_i \text{cth}(\alpha_i r) \partial_r - \lambda_L + \sum_{i=1}^{n-1} 2 \mu_i \frac{\text{ch}(\alpha_i r)}{\text{sh}^2(\alpha_i r)}.
\]
Let \( \delta = \frac{1}{2} \min_i \alpha_i \) and set \( F(r) = e^{-ar} - e^{-(a+\delta)r} \). Then there is some \( r_0 \) such that \( F'(r) < 0 \) for \( r \geq r_0 \) and we have

\[
-L^\circ F + \lambda_L F < F''(r) + \sum_{i=1}^{n-1} \alpha_i F'(r) + \sum_{i=1}^{n-1} 2\mu_i \frac{\text{ch}(\alpha_i r) - 1}{\text{sh}^2(\alpha_i r)} F(r) < -\delta (a + \delta) e^{-(a+\delta)r} + C e^{-2\delta r} (e^{-ar} - e^{-(a+\delta)r}).
\]

So, possibly after increasing \( r_0 \), we can assume that \((-L^\circ F + \lambda_L F)(r) < 0\) for all \( r \geq r_0 \). This shows that \( e^{-\lambda_L t} F(r) \) and hence also \( e^{-\lambda t} F(r) \) is a supersolution for the heat operator \( \partial_t + L^\circ \) on the domain \( \{ r \geq r_0 \} \).

Now choose \( C_1 \) so large that \( C_1 f(r_0) \geq C_0 \). By (4.13), we have the boundary estimate \( K_i(r)(\max)(r_0) \leq C_1 e^{-\lambda t} F(r_0) \) for \( t \geq \frac{1}{2} \). Moreover, using Proposition 2.25 we find that after possibly increasing \( C_0 \) that \( K_i(r)(\max) \leq C_1 e^{-\lambda t} F(r) \) for \( r \geq r_0 \). So, by the maximum principle this implies \( K_i(r)(\max)(r) \leq C_1 e^{-\lambda t} F(r) \) for all \( t > \frac{1}{2} \) and \( r \geq r_0 \). Since \( F(r) < e^{-ar} \), this proves the claim.

4.5. The \( L^1 \)-decay in the general rank case. We will now carry out the proof of Theorem 4.2 for the case in which \( M \) can have rank higher than 1. The difficulty here comes from the fact that the functions \( \frac{\text{ch}(\alpha_i r) - 1}{\text{sh}^2(\alpha_i r)} \) are only decaying towards one coordinate direction. We will resolve this issue by controlling certain \( L^2 \)-norms of \( K_i \) which allow us to reduce dimensions step by step. Those norms will correspond to the possible splittings \( a = \mathfrak{a}_W \oplus \mathfrak{g}_W \) for walls \( W \subset \mathcal{C} \) (see subsection 3.5) and will be controlled using Bochner formulae for the corresponding symmetric spaces \( \overline{M}_W \). In the case in which \( M \) is a product of rank 1 symmetric spaces, this program can be carried out without any problems: The spaces \( \overline{M}_W \) are factors of \( M \) and the spherical model \( K_i \) on \( a \) can be approximated by a spherical model on \( \mathfrak{a}_W \) and an \( N_W \)-invariant section on \( \mathfrak{g}_W \) as long as we are far enough away from the origin on \( \mathfrak{g}_W \). However, in the general case, the domain on which the spherical model resembles this mixed spherical-parabolic model, has a more complicated geometry due to the lack of orthogonality of the roots. Therefore, a more careful localization has to be carried out.

For the moment let \( \lambda_0 \) be an arbitrary constant. We will need the following results:

**Lemma 4.11.** There are constants \( C_m < \infty \) such that the following holds: Assume that \( \|k_t\|_{L^1(M)} \leq H e^{-\lambda t} \) for \( t \in [0, T] \). Then

\[
\|\nabla^m k_t\|_{L^1(M)} \leq C_m H e^{-\lambda t} \quad \text{for} \quad t \in [1, T].
\]

**Proof.** This follows from the fact that \( \nabla^m k_t = \nabla^m k_1 \ast k_{t-1} \) and Young’s inequality. \( \square \)

**Lemma 4.12.** There are constants \( d > 0 \) and \( C < \infty \) such that for every wall \( W \subset \mathcal{C} \) we have the following inequality on the cross-section \( \overline{M}_W \) of dimension \( \overline{\pi} \): Let \( g \in C^\infty(\overline{M}_W; E_W) \otimes E_0 \) be a spherical section. Then for any \( p \in \overline{M}_W \) we have

\[
|g|(p) \leq C e^{-dr} \|g\|_{L^1(\overline{M}_W)}
\]
where \( r = d(p_0, p) \) and \( W^{1,p} \) denotes the Sobolev norm.

**Proof.** Since there are only finitely many cross-sections \( \overline{M}_\mathcal{W} \) of \( M \), it suffices to show the inequality on \( M \). By Sobolev imbedding, the inequality is true for \( r \leq 10 \). Assume now \( r > 10 \) and consider the orbit \( O = K.p \). It can be seen easily that there are constants \( c, d > 0 \) which are independent of \( p \) such that we can find \( N = \lceil ce^{dr} \rceil \) points \( p_1, \ldots, p_N \in O \) whose pairwise distance is greater than 2. Then on each \( B_k = B_1(p_k) \) we have by Sobolev imbedding

\[
|g|(p) = |g|(p_k) \leq C\|g\|_{W^{1,\eta}(B_k)}.
\]

Hence

\[
N|g|(x) \leq \sum_{k=1}^N C\|g\|_{W^{1,\eta}(B_k)} \leq C\|g\|_{W^{1,\eta}(M)}.
\]

This yields the desired bound. \( \Box \)

We will need appropriate cutoff functions which specify the regions in which we compare the spherical model \( K_t \) with some mixed spherical-parabolic models. We will use a parameter \( \sigma > 10 \) here to specify the accuracy with which this comparison holds. For every wall \( \mathcal{W} \subset C \) and consider the splitting \( a = \overline{a}_W \oplus a_W \). Corresponding to \( \mathcal{W} \) and the parameter \( \sigma \), we will define cutoff functions \( \eta^\sigma_W \in C^\infty(\overline{a}_W) \) and \( \eta^\sigma_W \in C^\infty(a_W) \) such that the support of \( \eta^\sigma_W = \tau_W \eta^\sigma_W \subset C^\infty(a_W) \) and \( \mathcal{W} \) equals 1 resemble the wall \( \mathcal{W} \) in a coarse sense.

In order to do this, we first define regions which will help us to characterize the behaviour of the \( \eta^\sigma_W \). Let \((a_W)_{\mathcal{W} \subset C}, (b_W)_{\mathcal{W} \subset C}\) be numbers greater than 1 which we will determine in the next Lemma and define the regions \( X^\sigma_W, S^\sigma_W, R^\sigma_W \subset a \)

\[
X^\sigma_W = \{ \overline{\alpha} + \underline{v} \in \overline{a}_W \oplus a_W : |\overline{\alpha}| \leq a_W(\sigma - 1), \underline{\alpha}(v) \geq 0 \text{ for all } \underline{v} \in B^+_W \}
\]

\[
S^\sigma_W = \{ \overline{\alpha} + \underline{v} \in \overline{a}_W \oplus a_W : |\overline{\alpha}| \leq a_W \sigma, \underline{\alpha}(v) \geq b_W \sigma \text{ for all } \underline{v} \in B^+_W \}
\]

\[
R^\sigma_W = \{ \overline{\alpha} + \underline{v} \in \overline{a}_W \oplus a_W : |\overline{\alpha}| \leq a_W(\sigma - 1), \underline{\alpha}(v) \geq b_W(\sigma + 1) \text{ for all } \underline{v} \in B^+_W \}
\]

We will later identify \( S^\sigma_W \) as containing the support of \( \eta^\sigma_W \), \( R^\sigma_W \) as a region in which \( \eta^\sigma_W \) is constantly equal to 1 and the regions \( X^\sigma_W \) for \( \mathcal{W} \in \partial \mathcal{W} \) will serve to cover a certain part of the support of \( \partial \eta^\sigma_W \) (namely \( \text{supp} \eta^\sigma_W \partial \eta^\sigma_W \)). We need the following geometric identity:

**Lemma 4.13.** There are choices for \( a_W, b_W > 1 \) (which we will henceforth fix) such that for any wall \( \mathcal{W} \subset C \) and all \( \sigma > 10 \):

1. \( \underline{\alpha}(v) \geq \sigma \) whenever \( v \in S^\sigma_W \) for all \( \underline{v} \in B^+_W \).
2. We can cover a certain boundary part of \( S^\sigma_W \) by \( X^\sigma_W \):

\[
\{ \overline{\alpha} + \underline{v} \in S^\sigma_W : \underline{\beta}(v) \leq b_W(\sigma + 1) \text{ for some } \underline{\beta} \in B^+_W \} \subset \bigcup_{W \in \partial \mathcal{W}} X^\sigma_W.
\]

Recall, that \( \partial \mathcal{W} \) denotes the set of all codimension 1 walls of \( \mathcal{W} \).

3. For any \( f \geq 1 \) we have

\[
X^\sigma_W \subset R^\sigma_f \cup \bigcup_{W \in \partial \mathcal{W}} X^\sigma_f.
\]
Figure 1. The regions $X^W_\sigma$, $S^W_\sigma$ and $R^W_\sigma$.

**Proof.** Recall that the walls $W$ of $C$ stand in one-to-one correspondence with splittings $B^+ = B^+_W \cup B^-_W$ of the basis $B^+$ and that $\overline{a_W} = \text{span}(B^+_W)^\#$.

For property (1) observe that for $v = \overline{v} + \overline{v} \in S^W_\sigma$ and $\alpha \in \Delta^+$ we have

$$\alpha(v) = \alpha(\overline{v}) + \alpha(\overline{v}) \geq -C_0 a_W \sigma + b_W \sigma$$

for some large constant $C_0$. So property (1) can be ensured if

$$b_W - C_0 a_W \geq 1$$

for all $W \subset C$. (4.14)

As for property (2) consider $v = \overline{v} + \overline{v} \in S^W_\sigma$ and assume that $\beta(\overline{v}) \leq b_W(\sigma + 1)$ for some $\beta \in B^+_W$. Let $W' \in \partial W$ bet the wall for which $B^+_{W'} = B^+_W \setminus \{\beta\}$. Then $\overline{a_{W'}} = \text{span}(\overline{a_W} \cup \{\beta^\#\})$. Hence, if we consider the splitting $v = \overline{v} + \overline{v}' \in \overline{a_W} \oplus \overline{a_{W'}}$, we find $|\overline{v}'| \leq |\overline{v}| + C_1 \beta(\overline{v}) \leq a_W \sigma + C_1 b_W(\sigma + 1)$. So, if we choose

$$a_{W'} \geq 2 a_W + 4 C_1 b_W$$

for all $W' \in \partial W$, (4.15)

we can ensure that $|\overline{v}'| \leq a_W(\sigma - 1)$. In order to conclude that $v \in X^W_\sigma$, we still have to show that $\alpha(\overline{v}') \geq 0$ for all $\alpha \in B^+_W$. Since $\overline{v}' = \overline{v} - |\beta^\#|^{-2} \beta(\overline{v}) \beta^\#$, it suffices to show that $\langle \alpha^\#, \beta^\# \rangle \leq 0$ if $\alpha \neq \beta$. For this choose $x_\alpha \in g_\alpha$, $x_\beta \in g_\beta$ and $y_\alpha = \sigma x_\alpha$, $y_\beta = \sigma x_\beta$ such that $[x_\alpha, y_\alpha] = -\alpha^\#$ and $[x_\beta, y_\beta] = -\beta^\#$ (compare with (3.1)). Now since $\alpha$ and $\beta$ are simple, $\alpha - \beta = 0$ cannot be a root and hence $[x_\alpha, y_\beta] = 0$. Hence we conclude by the Jacobi identity

$$\langle \alpha^\#, \beta^\# \rangle = \langle [x_\alpha, y_\alpha], [x_\beta, y_\beta] \rangle = \langle [x_\beta, y_\alpha], [x_\alpha, y_\beta] \rangle + \langle [x_\alpha, x_\beta], [y_\alpha, y_\beta] \rangle = \langle [x_\alpha, x_\beta], \sigma [x_\alpha, x_\beta] \rangle \leq 0.$$ (4.16)

Finally, we analyze property (3): Let $v \in X^W_\sigma \setminus R^W_\sigma$. Then there is a $\beta \in B^+_W$ such that $\beta(\overline{v}) \leq b_W(\sigma + 1)$. As in the previous paragraph, we conclude that property (3) holds whenever (4.15) is satisfied.
It is now easy to see that we can choose the constants \((a_W)_{W \subset C}\) and \((b_W)_{W \subset C}\) to satisfy (4.14) and (4.15).

In the following Lemma we finally introduce the cutoff functions.

**Lemma 4.14.** We can define cutoff functions \(\eta_{\sigma}^{W} \in C^\infty(\overline{a_{W}})\), \(\eta_{\sigma}^{W} \in C^\infty(\overline{a_{W}})\) and \(\eta_{\sigma}^{W} = \eta_{\sigma}^{W} \eta_{\sigma}^{W} \in C^\infty(a)\) with the following properties (for \(\sigma > 1\)):

1. \(0 \leq \eta_{\sigma}^{W} \leq 1\) and \(|\partial \eta_{\sigma}^{W}|, |\partial^2 \eta_{\sigma}^{W}| \leq C\) everywhere and independently of \(\sigma\) and \(W\). Moreover, \(\eta_{\sigma}^{W}\) is invariant under the Weyl group \(W_W\).
2. \(\text{supp } \eta_{\sigma}^{W} \subset S_{\sigma}^{W}\) and \(\{\eta_{\sigma}^{W} = 1\} \supset R_{\sigma}^{W}\).
3. On \(\text{supp } \eta_{\sigma}^{W}\) we have \(a \geq \sigma\) for all \(a \in \Delta_{\sigma}^{W}\).
4. \(\text{supp } \eta_{\sigma}^{W} \partial \eta_{\sigma}^{W}, \text{supp } \eta_{\sigma}^{W} \partial^2 \eta_{\sigma}^{W} \subset \bigcup_{W \in \partial W} X_{\sigma}^{W}\).

**Proof.** Let \(\eta_{\sigma}^{W} \in C^\infty(\overline{a_{W}})\) be a radially symmetric cutoff function which is 1 on \(B_{avv(\sigma-1)}(0) \subset \overline{a}\) and vanishes outside \(B_{avv}(0)\). For \(W = C\), we just set \(\eta_{\sigma}^{W} = 1\). In order to define \(\eta_{\sigma}^{W} \in C^\infty(\overline{a_{W}})\), we choose a cutoff function \(\varphi_{\sigma}^{W} \in C^\infty(\mathbb{R})\) which is 1 on \([b_{W}(\sigma + 1), \infty)\) and vanishes on \((-\infty, b_{W}\sigma]\) and we set

\[
\eta_{\sigma}^{W} = \prod_{\alpha \in \Delta_{\sigma}^{W}} \varphi_{\sigma}^{W}(\alpha \circ \text{proj}_{a_{W}}).
\]

For \(W = \{0\}\), we set \(\eta_{\sigma}^{W} = 1\). Properties (1) and (2) trivially hold. Property (3) is just a restatement of Lemma 4.13 (1) and property (4) follows from Lemma 4.13 (2).

Now consider the heat kernel \((k_{t})_{t \geq 0}\) and its spherical model \((K_{t})_{t \geq 0}\). Let \(\lambda_0\) still be an arbitrary constant and assume that \(\|k_{t}\|_{L^1(M)} \leq H e^{-\lambda_0 t}\) for \(t \in [0, T]\). In the following analysis will always assume that \(t \in [1, T]\).

Fix a wall \(W \subset C\) and some \(\sigma > 10\). Unless denoted otherwise, we will most often leave out \(W\) in the index, e.g. \(\eta_{\sigma} = \eta_{\sigma}^{W}\) and \(\varphi_{\sigma} = \varphi_{\sigma}^{W}\). Consider the splitting \(a = \overline{a} \oplus \underline{a}\) associated to \(W\) and define the time-dependent function \(G : \overline{a} \rightarrow \text{End}_{K_{0}} E_{0}\) by

\[
G_{t} = \int_{\overline{a}} \eta_{\sigma} K_{t} \prod_{\alpha \in \Delta^{+}} e^{\underline{a}}.
\]

In the case \(W = \{0\}\) we just have \(G_{t} = K_{t}\). Observe that since \(K_{t}(\text{min}) > 0\) by Lemma 4.10 we can use \(G_{t}\) to bound the weighted \(L^1\)-norm of \(K_{t}\) along \(\overline{a}\):

\[
\int_{\overline{a}} |K_{t}| \prod_{\alpha \in \Delta^{+}} e^{\underline{a}} \leq C|G_{t}|.
\]

Since \(G_{t}\) is still equivariant under \(W\), it is a spherical model on \(\overline{M}\). Let \(g_{t} \in C^\infty(\overline{M}; E) \otimes E_{0}\) be the associated spherical section. We can estimate that on
supp \(\overline{\pi}_\sigma \subset \overline{a}\) we have (using Lemma 4.14 (3))

\[
|\partial^m G_t| \leq \int_{\overline{a}} \eta |\partial^m K_t| \prod_{\alpha} e^\alpha \leq C \int_{\overline{a}} \eta |\partial^m K_t| \prod_{\alpha} \text{sh} \alpha.
\]

So using Lemma 4.11 we can conclude that for \(t \in [1, T]\)

\[
\|\nabla^m g_t\|_{L^1(\text{supp} \overline{\pi}_\sigma)} \leq C_m \|\nabla^m k_t\|_{L^1(M)} \leq C_m H e^{-\lambda_0 t}.
\] (4.17)

So by Lemma 4.12 we obtain for \(t \in [1, T]\) and \(p \in \text{supp} \overline{\pi}_\sigma \subset M\)

\[
|g_t|(p) \leq C e^{-dr} H e^{-\lambda_0 t}.
\] (4.18)

We can compute the evolution of \(G_t\) using the evolution equation for \(K_t\) from Lemma 4.9. To simplify notation we will denote all indices \(i\) corresponding to roots \(\alpha_i \in \Delta^+\) by \(\overline{i}\) and the same for \(\overline{\alpha}\).

\[
\partial_t G_t = \int_{\overline{a}} \eta \left[ \Delta K_t + \Delta K_t + \sum_{\overline{\alpha}} \text{cth} \overline{\alpha} \partial_{\overline{\alpha}} K_t + \sum_{\overline{\alpha}} (\text{cth} \overline{\alpha} - 1) \partial_{\overline{\alpha}} K_t 
+ \sum_{\overline{i}} \partial_{\overline{\alpha}} K_t \right] \prod_{\overline{\alpha}} e^{\alpha}
\]

\[
= \overline{\Delta} G_t + \sum_{\overline{\alpha}} \text{cth} \overline{\alpha} \partial_{\overline{\alpha}} G_t + \sum_{\overline{i}} \left( \frac{1}{\text{sh}^2 \alpha_i} K_{i,k_i,k_i} + \frac{\text{ch}^2 \alpha_i}{\text{sh}^2 \alpha_i} \right) G_t
- 2 \frac{\text{ch} \overline{\alpha}}{\text{sh}^2 \alpha_\overline{\alpha}} G_t \overline{G_t} + %_1 + %_2 + %_3
\] (4.19)

where

\[
%_1 = \int_{\overline{a}} \eta \left[ \Delta K_t + \sum_{\overline{\alpha}} \partial_{\overline{\alpha}} K_t \right] \prod_{\overline{\alpha}} e^{\alpha}
%_2 = \int_{\overline{a}} \eta \sum_{\overline{\alpha}} (\text{cth} \overline{\alpha} - 1) \partial_{\overline{\alpha}} K_t \prod_{\overline{\alpha}} e^{\alpha}
%_3 = \int_{\overline{a}} \eta \sum_{\overline{i}} \left( \frac{1}{\text{sh}^2 \alpha_i} K_{i,k_i,k_i} + \left( \frac{\text{ch}^2 \alpha_i}{\text{sh}^2 \alpha_i} - 1 \right) \right) k_{i,k_i,k_i} K_i
- 2 \frac{\text{ch} \alpha_i}{\text{sh}^2 \alpha_i} k_{i,k_i,k_i} \right] \prod_{\overline{\alpha}} e^{\alpha}
\]

Recall that all but the \(\%\)-terms in (4.19) are just the operator \(\Delta_W g\) from (4.13) in terms of spherical models on \(\overline{M}\). Hence, by the definition of \(\lambda_W\), there is a first order differential operator \(D\) such that

\[
\partial_t g_t = -D^* D g_t - \lambda_W g_t + %_1 + %_2 + %_3.
\] (4.20)
Now, we define the time-dependent quantity
\[ B^W_\sigma = \int_M \eta_\sigma^2 |G_t|^2 \prod_\alpha \text{sh} \eta_\sigma^\alpha = \| \eta_\sigma g_t \|^2_{L^2(M)}. \]

If \( W = C \), we just set \( B^W_\sigma = |G_t|^2 \). We can compute its time derivative using (4.20):
\[ \frac{1}{2} \partial_t B^W_\sigma = -\| \eta_\sigma D g_t \|^2_{L^2(M)} - \lambda_W B^W_\sigma \]
\[ + \int_M \eta_\sigma^2 \nabla \eta_\sigma \nabla g_t \nabla g_t + \int_M \eta_\sigma^2 (\%_1 + \%_2 + \%_3) g_t. \quad (4.21) \]

The last two error terms can be estimated by the following Lemma.

**Lemma 4.15.** There are constants \( C, A < \infty \) and \( c > 0 \) such that we have the following estimates: Assume that \( \| K_t \|_{L^1(M)} \leq H e^{-\lambda_0 t} \) for \( t \in [0, T] \). Then we have for times \( [1, T] \):
\[ \left| \int_M \eta_\sigma^2 (\%_2 + \%_3) g_t \right| \leq C e^{-\sigma} H^2 e^{-2\lambda_0 t}, \]
\[ \int_M \eta_\sigma \nabla \eta_\sigma \| g_t \| \nabla g_t \leq C e^{-c\sigma} H^2 e^{-2\lambda_0 t}, \]

Moreover, for every \( \varepsilon > 0 \) we have the following estimate: For \( W' \subset W \) set \( f_{W'} = e^{\dim W' - \dim W} \). Then
\[ \left| \int_M \eta_\sigma^2 \%_1 g_t \right| \leq \left( B^W_\sigma \right)^{1/2} \| \eta_\sigma \%_1 \|_{L^2(M)} \leq C \left( B^W_\sigma \right)^{1/2} \sum_{W' \subset W} e^{\varepsilon A f_{W'}} \left( B^W_{\sigma \nu} \right)^{1/2}. \]

**Proof.** We start with the first inequality. Observe that by property (3) of Lemma 4.14 we know that on \( \text{supp} \eta_\sigma \subset \Omega \)
\[ |\%_2| + |\%_3| \leq C e^{-\sigma} \int_M \eta_\sigma (K_t + |\partial K_t|) \prod_\alpha e^{\alpha}. \]

Hence, by (4.18), and Lemma 4.11 we conclude (using \( e^{-dr} \leq 1 \))
\[ \left| \int_M \eta_\sigma^2 (\%_2 + \%_3) g_t \right| \leq C e^{-\sigma} H e^{-\lambda_0 t} \int_M \eta_\sigma^2 \left( \int_M \eta_\sigma (K_t + |\partial K_t|) \prod_\alpha e^{\alpha} \right) \prod_\alpha \text{sh} \eta_\sigma^\alpha \]
\[ \leq C e^{-\sigma} H e^{-\lambda_0 t} \left( \| k_t \|_{L^1(M)} + \| \nabla k_t \|_{L^1(M)} \right) \leq C e^{-\sigma} H^2 e^{-2\lambda_0 t}. \]

Analogously, we establish the second inequality. This time, we make use of the \( e^{-dr} \)-factor in (4.18) and of (4.17) for \( m = 1 \). We find that for some \( c > 0 \) depending on \( d \)
\[ \int_M \eta_\sigma |\nabla \eta_\sigma| \| g_t \| \nabla g_t \leq C e^{-c\sigma} H e^{-\lambda_0 t} \int_M \eta_\sigma \| g_t \| \nabla g_t \leq C e^{-c\sigma} H^2 e^{-2\lambda_0 t}. \]
We now establish the third inequality. To avoid confusion, we will write out the $W$-index again. Observe first that $\sum_{a} a^\#$ is invariant under $W_W$ and hence it is contained in $a_{W_W}$. Let now $\mathfrak{v} \in \mathfrak{a}_W$. Then, by partial integration

\[
(\eta^-_W\%_1)(\mathfrak{v}) \leq \int_{\{\mathfrak{v}\} \times \mathfrak{a}_{W_W}} \eta^-_W(|\partial \eta^-_W| + |\partial^2 \eta^-_W|)|K_i| \prod_{a \in \Delta^+_W} e^a \\
\leq C \int_{\{\mathfrak{v}\} \times \mathfrak{a}_{W_W}} \eta^-_W(|\partial \eta^-_W| + |\partial^2 \eta^-_W|)|K_i| \prod_{a \in \Delta^+_W} \sh a.
\]  

(4.22)

Since by Lemma 4.14 (4) we know that the support of its integrand is covered by the $X^-_W$ for $W' \subset \partial W$, we can bound the last quantity by $C \sum_{W' \in \partial W} Y^-_W$ where

\[ Y^-_W = \int_{\{\mathfrak{v}\} \times \mathfrak{a}_{W_W} \cap X^-_W} |K_i| \prod_{a \in \Delta^+_W} \sh a. \]

Now consider any $W' \subseteq W$ (not necessarily of codimension 1 in $W$) and let $f \geq 1$. Then we can bound $Y^-_W$ using Lemma 4.13 (3) for $f = \varepsilon^{-1}$

\[ Y^-_W \leq \int_{\{\mathfrak{v}\} \times \mathfrak{a}_{W_W} \cap X^-_W} \eta^-_{-1/2} |K_i| \prod_{a \in \Delta^+_W} \sh a + \sum_{W' \in \partial W} Y^-_{W'} \]

In order to bound the integral, let us first analyze its domain $\{\mathfrak{v}\} \times \mathfrak{a}_{W_W} \cap X^-_W$. Observe that the set $X^-_W$ can be written as a direct product $X^-_W = Y^-_W \times X^-_W$ with respect to the splitting $a_W \oplus a_W$. Moreover, since $a_W \subset a_W$, there is an orthogonal splitting $a_W = a_{\perp} \oplus a_W$ and we have $a_W = a_W \oplus a_{\perp}$. So we can represent the domain of the integral as a product with respect to the splitting $\mathfrak{v} = \mathfrak{a}_W \oplus \mathfrak{a}_{W^'}$:

\[
(\{\mathfrak{v}\} \times \mathfrak{a}_{W_W}) \cap X^-_W = (\{\mathfrak{v}\} \times a_{\perp} \times \mathfrak{a}_{W'}) \cap (Y^-_W \times X^-_W) = (\{\mathfrak{v}\} \times a_{\perp} \cap X^-_W) \times X^-_W
\]

So by Cauchy-Schwarz

\[
Y^-_W \leq \left( \int_{\{\mathfrak{v}\} \times a_{\perp} \cap X^-_W} (\eta^-_W)^2 \left( \int_{Y^-_W} \eta^-_{-1/2} |K_i| \prod_{a \in \Delta^+_W} \sh a \right)^2 \prod_{a \in \Delta^+_W} \sh a \right)^{1/2} \\
\times \left( \int_{\{\mathfrak{v}\} \times a_{\perp} \cap X^-_W} \prod_{a \in \Delta^+_W} \sh a \right)^{1/2} + \sum_{W' \in \partial W} Y^-_{W'}.
\]

The last integral can be bounded by $Ce^{2A\varepsilon}$ for an appropriate $A < \infty$. Now recall that $f_W = \varepsilon^{\dim W - \dim W}$ for $W' \subset W$. Substituting in the identity above $\varepsilon f_W$ for $\sigma$, applying it recursively and plugging it back into (4.22) yields

\[
\int_{\mathfrak{a}_W} (\eta^-_W\%_1)^2 \prod_{\mathfrak{v} \in \mathfrak{a}_W} \sh a \leq C \sum_{W' \subseteq W} B_{f_W}^W e^{2\varepsilon f_W} \]

and hence the desired result. \qed
So combining (4.21) with Lemma 4.15 we conclude

**Lemma 4.16.** There are constants $C, A < \infty$ and $c > 0$ such that the following holds: Let $\varepsilon > 0$, consider a wall $\mathcal{W} \subset \mathcal{C}$ and set $f_{\mathcal{W}} = \varepsilon^{\dim \mathcal{W} - \dim \mathcal{C}}$ for each $\mathcal{W}' \subset \mathcal{W}$.

Then, under the assumption that $\|k_i\|_{L^1(M)} \leq H\varepsilon^{-\lambda t}$ for $t \in [0, T]$, we have for $\sigma > 10$ and times $t \in [1, T]$

\[
\frac{1}{2} \partial_t B^W_\sigma \leq -\lambda W B^W_\sigma + C e^{-\sigma} H^2 e^{-2\lambda t} + C (B^W_\sigma)^{1/2} \sum_{W' \subset W} e^{Af_{W'}} (B^{W'}_{W'})^{1/2}.
\]

We will come back to this evolution inequality later. First, we estimate the evolution of $\|k_t\|_{L^1(M)}$ in terms of the $B^W_\sigma$. For this, we define the quantity

\[ S_t = \int_{\mathcal{C}} K_t(\max) \prod_{\alpha} \sh \alpha \]

and observe that $S_t$ is comparable with $\|k_t\|_{L^1(M)}$, i.e.

\[ cQ_t \leq \|k_t\|_{L^1(M)} \leq CQ_t \quad \text{for all } t > 0. \]

**Lemma 4.17.** There are constants $C, A < \infty$ such that: Let $\varepsilon > 0$ and set $f_{\mathcal{W}} = \varepsilon^{\dim \mathcal{W} - \dim \mathcal{C}}$. Then at any time $t > 0$ and for $\sigma > 10$ we have the estimate

\[
\partial_t S_t \leq - (\lambda C - Ce^{-\sigma}) S_t + C \sum_{W \subset \mathcal{C}} e^{Af_{W}} (B^W_{W})^{1/2}.
\]

**Proof.** Recall that by Lemma 4.10 (see also the proof of Theorem 4.1)

\[
(\partial_t + \lambda C) \int_{\mathcal{C}} K_t(\max) \prod_{\alpha} \sh \alpha \leq \int_{\mathcal{C}} \left( \sum_i 2 \mu_i \frac{\ch \alpha_i - 1}{\sh^2 \alpha_i} \right) K_t(\max) \prod_{\alpha} \sh \alpha.
\]

By Lemma 4.13 (3) we have $\mathcal{C} = X^\mathcal{C}_C \subset R^C \cup \bigcup_{W \in \partial \mathcal{C}} X^\mathcal{W}_W$. So by Lemma 4.14 (3), we conclude that outside the regions $X^\mathcal{W}_W$, ($\mathcal{W} \in \partial \mathcal{C}$), the term inside the parentheses can be bounded by $Ce^{-\sigma}$. Hence, in order to establish the Lemma, it suffices to show that for every $\mathcal{W} \in \partial \mathcal{C}$, we have

\[
\int_{X^\mathcal{W}_W} |K_t| \prod_{\alpha} \sh \alpha \leq C \sum_{W \subset \mathcal{W}} e^{Af_{W}} (B^W_{W})^{1/2}.
\]

Analogously to the proof of Lemma 4.15, we set for any wall $\mathcal{W} \subset \mathcal{C}$ (not only for codimension 1 walls)

\[ Y^\mathcal{W}_\sigma = \int_{X^\mathcal{W}_W} |K_t| \prod_{\alpha} \sh \alpha \]

Now, using Lemma 4.13 (3) with $f = \varepsilon^{-1}$ and the splitting $X^\mathcal{W}_\sigma = X^\mathcal{W}_W \times X^\mathcal{W}_W$, we get

\[
Y^\mathcal{W}_{f_{W}} \leq \int_{X^\mathcal{W}_W} K_t(\max) \prod_{\alpha} \sh \alpha + \sum_{W \subset \mathcal{W}} Y^\mathcal{W}_{f_{W}}.
\]
If \( \sigma > c > 0 \) such that the inequality is true for any \( W \).

**Proof.** We proceed by induction over the dimension of \( W \). The solutions of a system of evolution inequalities. In the first step, we use Lemma 4.16 to estimate the \( B_\sigma^W \) assuming a bound on \( \|k_t\|_{L^1(M)} \).

**Lemma 4.18.** There are constants \( C, A < \infty \) and \( 1 > c > 0 \) such that: Assume that \( \|K_t\|_{L^1(M)} \leq He^{-\lambda_0 t} \) for \( t \in [0, T] \). Consider a wall \( W \subset C \), set \( \lambda_W \) and assume \( \lambda_W^\min > \lambda_W \).

If \( \sigma > 10 \) is so large that \( e^{-\sigma} < \lambda_W^\min - \lambda_0 \), then we have for times \( t \in [1, T] \)

\[
B_\sigma^W \leq Ce^{-\sigma}H^2e^{-2\lambda_0 t} + Ce^{A\sigma} \exp(-2(\lambda_W^\min - e^{-\sigma})t).
\]

**Proof.** We proceed by induction over the dimension of \( W \). Fix \( W \subset C \) and assume that the inequality is true for any \( W' \subsetneq W \). Let \( \varepsilon > 0 \) be a constant whose value will be determined later and choose \( (f_{W'})_{W' \subset W} \) according to Lemma 4.16, i.e. \( f_{W'} = e^{\dim W' - \dim W} \). Then

\[
\frac{1}{2} \partial_t B_\sigma^W \leq -\lambda_W B_\sigma^W + Ce^{-\sigma}H^2e^{-2\lambda_0 t} + C(B^W_\sigma)^{1/2} \sum_{W' \subsetneq W} e^{2A\lambda'_{W'}} (B^W_{f_{W'}})^{1/2}
\]

\[
\leq - (\lambda_W - e^{-\sigma}) B_\sigma^W + Ce^{-\sigma}H^2e^{-2\lambda_0 t} + Ce^{\sigma} \sum_{W' \subsetneq W} e^{2A\lambda'_{W'}} B^W_{f_{W'}}
\]

\[
\leq - (\lambda_W - e^{-\sigma}) B_\sigma^W + Ce^{-\sigma}H^2e^{-2\lambda_0 t} + C \sum_{W' \subsetneq W} e^{(2A\lambda'_{W'} + 1)\sigma} e^{A\lambda'_{W'}} B^W_{f_{W'}} \exp(-2(\lambda_W^\min - e^{-\sigma})t)
\]

Now choose \( \varepsilon \) small enough such that \( 2\varepsilon A\lambda'_{W'} + 1 - cf_{W'} \leq -2c \) for all \( W' \subsetneq W \) and set \( A' = 2\varepsilon A\lambda(0) + 1 + A\lambda(0) \). Since \( \sigma > 10 \), we can find a constant \( c' > 0 \) such that \( e^{-\sigma} - e^{-\varepsilon^{-1}} > c'e^{-\sigma} \). Applying those assumptions, we obtain the evolution inequality

\[
\frac{1}{2} \partial_t B_\sigma^W \leq -(\lambda_W^\min - e^{-\sigma}) B_\sigma^W + Ce^{-2\sigma}H^2e^{-2\lambda_0 t}
\]

\[
+ Ce^{A'\sigma} \exp(-2c'e^{-\sigma}t - 2(\lambda_W^\min - e^{-\sigma})t).
\]

Iterating this inequality yields the desired result \( \Box \).
So
\[ \frac{1}{2} \partial_t \left( \exp(2(\lambda_W^{\text{min}} - e^{-\sigma})t) B^W_\sigma \right) \]
\[ \leq C e^{-2\sigma} H^2 \exp(2(\lambda_W^{\text{min}} - \lambda_0 - e^{-\sigma})t) + C e^{A'\sigma} \exp(-2c'e^{-\sigma}t). \]

Integrating this inequality and using the fact that 2(\lambda_W^{\text{min}} - \lambda_0 - e^{-\sigma}) > 2e^{-c\sigma} yields
\[ \exp(2(\lambda_W^{\text{min}} - e^{-\sigma})t) B^W_\sigma \leq C B^W_\sigma(t = 1) \]
\[ + C e^{-c\sigma} H^2 \exp(2(\lambda_W^{\text{min}} - \lambda_0 - e^{-\sigma})t) + C e^{A'\sigma + \sigma} \]
and hence the desired result. □

We can finally combine Lemmas 4.17 and 4.18 to prove Theorem 4.2.

**Proof of Theorem 4.2.** For small times, the theorem follows from Proposition 2.5. Assume for the moment that \( \lambda_0 \) is an arbitrary constant satisfying \( \lambda_0 < \lambda_1 = \min_{\mathcal{W} \subseteq \mathcal{C}} \lambda_{\mathcal{W}} \) and set \( H_t = \sup_{t \in [0, t]} S_t e^{\lambda_0 t} \). Then by Lemma 4.17 and Lemma 4.18 as long as \( 2e^{-c\sigma} < \lambda_1 - \lambda_0 \) and \( \sigma > 10 \), we have for \( t \geq 1 \)
\[ \partial_t S_t \leq -(\lambda_1 - C - e^{-\sigma})S_t \]
\[ + C \sum_{\mathcal{W} \subseteq \mathcal{C}} e^{Af_W \sigma} \left( e^{-cf_W \sigma/2} H_t e^{-\lambda_0 t} + e^{Af_W \sigma/2} \exp(-(\lambda_W^{\text{min}} - e^{-f_W \sigma})t) \right) \]
So, if \( \varepsilon \) is chosen small enough as to ensure \( \varepsilon Af_W - cf_W/2 < -1 \) for all \( \mathcal{W} \subseteq \mathcal{C} \), we obtain for \( A' = Af_W/2 \)
\[ \partial_t (S_t e^{\lambda_0 t}) \leq C e^{-\sigma} H_t e^{(\lambda_1 - \lambda_0) t} + C e^{A' \sigma} \exp(-(\lambda_1 - \lambda_0 - e^{-\sigma})t). \] (4.23)

Now consider first the case \( \lambda_1 > \lambda_C \) and set \( \lambda_0 = \lambda_C = \min_{\mathcal{W} \subseteq \mathcal{C}} \lambda_{\mathcal{W}} \). Choose \( \delta > 0 \) small enough such that \( A' \delta - \lambda_1 + \lambda_C < -2\delta \) and set \( \sigma = \delta t \). Then for large \( t \) we can assume \( e^{-\sigma} < \delta \) and \( 2e^{-c\sigma} < \lambda_1 - \lambda_0 \) and we get
\[ \partial_t (S_t e^{\lambda_0 t}) \leq C e^{-\delta t} H_t + C e^{-\delta t} = C e^{-\delta t}(H_t + 1). \]
So whenever \( H_t \leq 2S_t e^{\lambda_0 t} \), we find
\[ \partial_t \log(S_t e^{\lambda_0 t} + 1) \leq 2C e^{-\delta t}. \]
Since the right hand side is integrable for \( t \to \infty \), we conclude that \( S_t e^{\lambda_0 t} \) stays bounded.

Consider now the case \( \lambda_1 \leq \lambda_C \). Plugging \( \lambda_0 = \lambda_1 - 2e^{-c\sigma} \) into (4.23) yields
\[ \partial_t (S_t e^{\lambda_0 t}) \leq C e^{-\sigma} H_t + C e^{A' \sigma} \exp((e^{-\sigma} - 2e^{-c\sigma})t) \leq C e^{-\sigma} H_t + C e^{A' \sigma}. \]
So whenever \( H_t \leq 2S_t e^{\lambda_0 t} \), we find
\[ \partial_t (S_t e^{\lambda_0 t}) \leq C e^{-\sigma} (S_t e^{\lambda_0 t}) + C e^{A' \sigma}. \]
By Gronwall’s Lemma, we conclude
\[ S_t e^{\lambda_0 t} \leq C \exp(C e^{-\sigma} t + (A' + 1)\sigma). \]
Choosing \( \sigma = \log(t + 2) \) yields the desired result. □
5. Analysis of the Einstein operator

5.1. Introduction. In this section, we will apply the results from section 4 to the linearized Ricci deTurck equation $\partial_t h_t = -Lh_t$ where $L = -\Delta - 2R$ is the Einstein operator (see (2.1)). Let $E = \text{Sym}_2 T^*$ be the bundle of symmetric bilinear forms. Observe that the zero order term $2R$ is a fiberwise self-adjoint endomorphism on $E$ and hence it can be diagonalized with eigenvalues $\varphi_1, \ldots, \varphi_m$ with respect to a splitting $E = E_1 \oplus \ldots \oplus E_m$ of vector bundles.

We can integrate the extra term $2R$ into Theorem 4.1 in the following way: Redefine $\lambda_L$ to be the smallest eigenvalue of the operator $S_{\text{par}} = -\Delta - 2R : V_{\text{par}} \to V_{\text{par}}$ acting on parabolically invariant sections and $\lambda_B$ as the optimal constant $\lambda$ for Bochner formulas $-\Delta - 2R = D^*D + \lambda$. Those new constants $\lambda_L$ and $\lambda_B$ are then just the old constants minus one of the $\varphi_i$ each. Now, redefine $(k_t)_{t \geq 0} \in C^\infty(M; E) \otimes E_0^*$ to be the heat kernel for the operator $\partial_t + L$, i.e. $\partial_t k_t = -Lk_t$. Obviously, $k_t$ is just the old heat kernel with some extra exponential $\varphi_i$-decay on the $E_i$-component. It is now easy to conclude that with these redefinitions, Theorem 4.1 stays valid in its original reading: For $\lambda_0 = \min \{\lambda_L, \lambda_B\}$, we have $\|k_t\|_{L^1(M)} \leq Ce^{-\lambda_0 t}$. The same is true for Theorem 4.2.

Analogously, we can integrate the $2R$-term into Theorem 4.2. This time, we have to redefine the constants $(\lambda_W)_{W \subset C}$ introduced in subsection 4.1 to include the zero order term. In order to do this, we replace the Laplace operator $\Delta$ by $-L$ in the paragraph preceding equation (4.2), i.e. we set $-L\hat f = \hat f$. This changes the definition of $\Delta_W$ and $-S_W$ by an extra $2R$ summand and hence gives us a new $\lambda_W$. Now Theorem 4.2 continues to hold for the redefined heat kernel.

So in order to estimate the $L^1$-decay rate of $k_t$, we need to get a good bound on the redefined constants $\lambda_W$. In this section we will solely be concerned with the analysis of these constants. Our result will be:

**Proposition 5.1.** Assume that $M$ is a symmetric space of noncompact type which is Einstein. Consider the Einstein operator $L = -\Delta - 2R$ acting on the vector bundle $E = \text{Sym}_2 T^*$ of symmetric bilinear forms over $M$. Let $(\lambda_W)_{W \subset C}$ be the constants associated to $M$, $E$ and $L$ as redefined above. Then

(i) $\lambda_W \geq 0$ for all walls $W \subset C$ and hence $\lambda_0 = \min_{W \subset C} \lambda_W \geq 0$.

(ii) We have $\lambda_0 > 0$ if and only if $M$ does not contain any hyperbolic or complex hyperbolic factor in its de Rham decomposition.

(iii) If $M = \mathbb{H}^n$ for $n \geq 3$ or $M = \mathbb{C}\mathbb{H}^{2n}$ for $n \geq 2$, then $\lambda_L = \lambda_C = 0$ and $\lambda_B = \lambda_{\{0\}} > 0$.

(iv) If $M = \mathbb{H}^2$, then $\lambda_L = \lambda_C > 0$ and $\lambda_B = \lambda_{\{0\}} = 0$.

Hence in the last case in which does not contain a hyperbolic or complex hyperbolic factor, the $L^1$-norm of the heat kernel $k_t$ associated to $\partial_t + L$ is exponentially decaying as $t \to \infty$. If $M = \mathbb{H}^n$ or $\mathbb{C}\mathbb{H}^{2n}$, then $\|k_t\|_{L^1(M)}$ stays bounded and by Theorem 4.3, we have the bound $|k_t| < C(\text{vol } B_r(p_0))^{-1}$ for all $t$.

This section is organized as follows: In subsection 5.2, we recall the important identities and carry out some of the basic calculations. The reader who is only interested in the rank 1 case, then finds an estimate on $\lambda_B$ in subsection 5.3.
For an estimate on $\lambda_L$ he or she can then immediately jump to subsections 5.7 through 5.10 (where he or she can always replace the index $m$ by $m$ and leave out any term with index $T$). In order to understand the higher rank case, subsection 5.3 will still be important since it discusses a Bochner formula which will later be applied to cross-section $\Omega_W$ of $M$. In subsection 5.4 we will find that the problem of estimating $\lambda_W$ splits into an estimate on three vector bundles $E_2$, $E_{\text{Sym}^2 \overline{p}}$ and $E_{\text{Sym}^2 \overline{p}}$. The estimates on $E_2$ and $E_{\text{Sym}^2 \overline{p}}$ will be carried out in subsections 5.5 and 5.6. The estimate on $E_{\text{Sym}^2 \overline{p}}$ is the most difficult one and will be carried out in subsections 5.7 and 5.8. During this discussion, a possible nullspace arises which we will then analyze in subsection 5.9. Finally, subsection 5.10 contains the proof of Proposition 5.1.

5.2. Preliminary calculations. Fix a wall $W \subset C$ and consider the splitting $\Delta^+ = \Delta^+_W \cup \Delta^+_W$. As explained in subsection 5.5 we obtain orthogonal splittings $a = p_W + \overline{a}_W$, $\mathfrak{p} = \overline{p}_W + p_W$ and $\mathfrak{t} = \overline{t}_W + t_W$. Here

$$\overline{p}_W = \overline{a}_W + \bigoplus_{\pi \in \Xi_W} p_{\pi} \quad \text{and} \quad p_W = a_W + \bigoplus_{\alpha \in \Delta^+_W} p_{\alpha}.$$ 

Moreover, we set

$$\overline{t}_W = [\overline{p}_W, \overline{a}_W] \quad \text{and} \quad t_W = \bigoplus_{\alpha \in \Delta^+_W} t_{\alpha}.$$ 

We remark that $\overline{k}_W$ is not a Lie algebra and in general $\mathfrak{t} \neq \overline{t}_W + t_W$. In the following, we will often make use of the fact that $\overline{a}_W = \overline{p}_W + \overline{t}_W$ is a Lie algebra and that moreover

$$[\overline{p}_W, \overline{a}_W], [\overline{t}_W, t_W] \subset \overline{t}_W, \quad \text{and} \quad [\overline{p}_W, \overline{t}_W], [\overline{p}_W, t_W] \subset \overline{p}_W. \quad (5.1)$$

From now on, we will leave out the index $W$. Recall the orthonormal systems $k_1, \ldots, k_{n-r} \in \mathfrak{t}$ and $p_1, \ldots, p_{n-r} \in \mathfrak{p}$ from subsection 3.2. They split into systems $\{ k_i \}, \{ p_i \}$ resp. $\{ k_i \}, \{ p_i \}$ corresponding to roots $\alpha_i \in \Delta^+$ resp. $\alpha_i \in \Delta^+$. In the following, we will denote by $e_1, \ldots, e_n \in \mathfrak{p}$ an arbitrary orthonormal basis of $\mathfrak{p}$ which obeys the splitting $\mathfrak{p} = \overline{\mathfrak{p}} \oplus \mathfrak{p}$, i.e. the index set $\{ i \}$ splits into $\{ \overline{i} \}$ and $\{ i \}$ such that $\{ e_i \}$ is an orthonormal basis for $\overline{\mathfrak{p}}$ and $\{ e_{\overline{i}} \}$ one for $\mathfrak{p}$.

Let $E = \text{Sym}_2 T^*$ be the vector bundle of symmetric bilinear forms. We will identify $T^* \cong T$. At the basepoint $p_0 \in M$, we can identify $T \cong \mathfrak{p}$ and hence $E_0 = \text{Sym}_2 \mathfrak{p}$. The splitting $\mathfrak{p} = \overline{\mathfrak{p}} \oplus \mathfrak{p}$ induces a splitting $\text{Sym}_2 \mathfrak{p} = \text{Sym}_2 \overline{\mathfrak{p}} \oplus \text{Sym}_2 \mathfrak{p} \oplus \overline{\mathfrak{p}}$. Observe that this splitting comes from a splitting

$$E_W = E_{\text{Sym}^2 \overline{\mathfrak{p}}} \oplus E_{\text{Sym}^2 \mathfrak{p}} \oplus E_{\overline{\mathfrak{p}}} \quad (5.2)$$

over the whole space $\Omega_W$. We will denote the elements of $\text{Sym}_2 \mathfrak{p}$ by $v \cdot w = w \cdot v$ for $v, w \in \mathfrak{p}$ and set $\langle v \cdot w, v' \cdot w' \rangle = \frac{1}{2} \langle v, v' \rangle \langle w, w' \rangle + \frac{1}{2} \langle v, w' \rangle \langle w, v' \rangle$. Hence $\{ \sqrt{2} e_i \cdot e_j, e_k \cdot e_k : i < j \}$ is an orthonormal basis for $\text{Sym}_2 \mathfrak{p}$. 


From (3.4), we obtain that $R(v \cdot w) = \sum_i e_i \cdot R(e_i, v) w = - \sum_i e_i \cdot [e_i, v, w]$. Hence using (4.8), we can compute that for $v \cdot w \in \text{Sym}_2 \mathfrak{p}$

$$S_W(v \cdot w) = - \sum_m ([k_m, [k_m, v]] \cdot w + v \cdot [k_m, [k_m, w]]) + 2[k_m, v] \cdot [k_m, w])$$

$$+ 2 \sum_i e_i \cdot [e_i, v, w].$$

Pairing this with $v' \cdot w' \in \text{Sym}_2 \mathfrak{p}$ yields

$$2\langle S_W(v \cdot w), v' \cdot w' \rangle = \sum_m \left( \langle [k_m, v], [k_m, v'] \rangle \langle w, w' \rangle + \langle [k_m, v], [k_m, w] \rangle \langle v, v' \rangle 
+ \langle [k_m, w], [k_m, v'] \rangle \langle v, w' \rangle + \langle [k_m, w], [k_m, w] \rangle \langle v, v' \rangle 
- 2\langle [k_m, v], v' \rangle \langle [k_m, w], w' \rangle - 2\langle [k_m, v], w' \rangle \langle [k_m, w], v' \rangle 
+ 2\langle [v', v], w \rangle, w' \rangle + 2\langle [w', v], w \rangle, v' \rangle \right).$$

(5.3)

We remark, that in order to determine the constants $\lambda_L$ resp. $\lambda_B$ in the rank 1 case we have to analyze the operators $S_C$ resp. $S_{(0)}$.

5.3. A Bochner formula. We will now derive a Bochner formula for $L$ on $M$ and hence get a lower bound for $\lambda_{(0)} = \lambda_B$. In the higher rank case, this Bochner formula will be applied to cross-sections $M_W$ of $M$ in subsection 5.6. So in order to allow for this further application, we will not require $M$ to be Einstein in this subsection.

Recall the definition of the divergence operator

$$\text{div} : C^\infty(M; \text{Sym}_2 T^*) \to C^\infty(M; T^*), \quad h_{ij} \mapsto -\sum_i \nabla_i h_{ij}$$

and define the exterior derivative with coefficients in $T^*$

$$d : C^\infty(M; \text{Sym}_2 T^*) \to C^\infty(M; \Lambda_2 T^* \otimes T^*), \quad h_{ij} \mapsto \nabla_i h_{jk} - \nabla_j h_{ik}.$$ 

Their formal adjoints are

$$\text{div}^* : C^\infty(M; T^*) \to C^\infty(M; \text{Sym}_2 T^*), \quad \gamma_i \mapsto \frac{1}{2} (\nabla_i \gamma_j + \nabla_j \gamma_i)$$

and

$$d^* : C^\infty(M; \Lambda_2 T^* \otimes T^*) \to C^\infty(M; \text{Sym}_2 T^*), \quad \gamma_{ijk} \mapsto -\frac{1}{2} (\nabla_k \gamma_{kij} + \nabla_k \gamma_{kji}).$$

We can then calculate that

$$(Lh)_{ij} = (\text{div}^* \text{ div} + d^* d)h_{ij} - R(h)_{ij} - \frac{1}{2} \sum_k \left( R_{ikj} h_{kj} + h_{ik} R_{kj} \right).$$

(5.4)

In the following Lemma we will show that the zero order term is nonnegative and most often even positive definite. Hence, by choosing $D = \text{div} + d : C^\infty(M; E) \to C^\infty(M; T^* \oplus \Lambda_2 T^* \otimes T^*)$, we conclude that $\lambda_{(0)} = \lambda_B$ is nonnegative resp. positive (compare with (4.1) and (4.3)).
Lemma 5.2. Let $R$ resp. $\text{Ric}$ be the Riemannian curvature resp. Ricci curvature at a point of a symmetric space $M$ of noncompact type and let $T$ be the tangent space at that point. Then the operator

$$A : \text{Sym}_2 T^* \longrightarrow \text{Sym}_2 T^*, \quad h_{ij} \mapsto -R(h)_{ij} - \frac{1}{2} \sum_k \left( \text{Ric}_{ik} h_{kj} + h_{jk} \text{Ric}_{ki} \right)$$

is self-adjoint and nonnegative definite.

Moreover, if we consider the splitting $T = T_1 \oplus \ldots \oplus T_m$ associated to the de Rham decomposition $M = M_1 \times \ldots \times M_m$ and assume that the $M_1, \ldots, M_m$ are the only $\mathbb{H}^2$-factors, then the nullspace of $A$ is

$$\{h = h_1 + \ldots + h_m : h_i \in \text{Sym}_2 T_i^*, \text{tr} h_i = 0\}.$$  

Hence, if $M$ does not contain any $\mathbb{H}^2$-factor, then $A$ is positive definite.

Proof. Assume first that $M = M' \times M''$ is reducible and let $T = T' \oplus T''$ be the corresponding splitting. Choose an orthonormal basis $e_1, \ldots, e_n$ of $T$ which obeys this splitting. We first show that $A$ preserves the induced splitting $\text{Sym}_2 T^* = \text{Sym}_2(T')^* \oplus \text{Sym}_2(T'')^*$: If $h \in \text{Sym}_2(T')^*$, then $R(h)_{ij} = \sum_{s,t} R_{ijst} h_{st}$ is only nonzero if $e_i, e_j \in T'$, since $R_{ijst}$ is only nonzero if either all indices $i, s, t, j$ belong to $T'$ or to $T''$. Furthermore since $\text{Ric}_T = \text{Ric}_{T'} + \text{Ric}_{T''}$, we see that $\sum_k \text{Ric}_{ik} h_{kj}$ is only nonzero if $e_i, e_j \in T'$. So $A(h) \in \text{Sym}_2(T')^*$. Analogously, we see that $A$ maps $\text{Sym}_2(T'')^*$ into itself and by self-adjointness it also has to preserve $(T')^*(T'')^*$.

Next, we show that $A$ is positive definite on $(T')^*(T'')^*$. Let $h \in (T')^*(T'')^*$ and observe that $R(h) = 0$. Then $\langle A(h), h \rangle = -\sum_{i,j,k} \text{Ric}_{ik} h_{kj} h_{ij} > 0$ if $h \neq 0$. So, we can restrict ourselves to the case in which $M$ is irreducible.

Let $h \in \text{Sym}_2 T^*$ and choose an orthonormal basis $e_1, \ldots, e_n$ for which $h$ is diagonal, i.e. $h = \sum_{i=1}^n \lambda_i e_i^* \otimes e_i^*$. Observe, that since $M$ is of noncompact type, the sectional curvatures $K_{ij} = \langle R(e_i, e_j) e_j, e_i \rangle \leq 0$. We can compute that

$$\langle R(h), h \rangle = \sum_{i,j,j'} R_{ijj'} h_{ii'} h_{jj'} = \sum_{i,j} K_{ij} \lambda_i \lambda_j.$$  

Hence, since $\text{Ric}_{ii} = \sum_j K_{ij}$

$$\langle A(h), h \rangle = \left(-\sum_{i,j} K_{ij} \lambda_i \lambda_j + \frac{1}{2} K_{ij} \lambda_i^2 + \frac{1}{2} K_{ij} \lambda_j^2 \right) = -\sum_{i,j} \frac{1}{2} K_{ij} (\lambda_i + \lambda_j)^2 \geq 0.$$  

Assume now that $h$ lies in the nullspace. Then we must have $K_{ij} = 0$ whenever $\lambda_i \neq \lambda_j$. We can split $T = T'_1 \oplus \ldots \oplus T'_m$ such that $T'_k$ is spanned by all $e_k$ for which $|\lambda_k|$ is a given constant. Then for $e_i$ and $e_j$ belonging to different $T'_k$, we have $K_{ij} = 0$ and since the Riemannian curvature even has to property of having nonnegative curvature operator, we conclude that all sectional curvature between different $T'_k$ vanish and hence $T'_1 \oplus \ldots \oplus T'_m$ corresponds to a geometric splitting $M = M_1 \times \ldots \times M_m$. Since we assumed $M$ to be irreducible, we find that $m = 1$ and hence $h$ only has two eigenvalues $-\lambda$ and $\lambda$.

Let $T = T_- \oplus T_+$ be the orthogonal splitting of eigenspaces of $h$. As before, we conclude that the sectional curvatures on $T_-$ and $T_+$ all vanish and hence those
subspaces correspond to flats in $M$ or abelian subspace $a_-, a_+ \subset p$. Now since the rank $r$ of $M$ is equal to the the number of simple roots and hence is not larger than $n - r$, we conclude $2r \leq n$. But since $\dim a_- + \dim a_+ = n$, we must have $2r = n$ and $\dim a_- = \dim a_+ = r$. In this case, all positive roots of $M$ have to be simple and hence orthogonal to each other (the argument for this is analogous to the one involving equation (4.16)). It follows that $M = \mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ and by irreducibility $M = \mathbb{H}^2$.

5.4. Block form of the Einstein operator. The splitting $p = \bar{p} \oplus p$ induces a splitting $\text{Sym}_2 p = \text{Sym}_2 \bar{p} \oplus \text{Sym}_2 p \oplus \bar{p}p$. We will find that $S_{\mathcal{W}} : \text{Sym}_2 p \to \text{Sym}_2 p$ preserves this splitting, i.e. the differential operator $-\Delta + S_{\mathcal{W}}$ acting on $C^\infty(M_{\mathcal{W}}, \mathcal{E}_{\mathcal{W}})$ preserves the splitting $(5.2)$ if we view it as a bundle endomorphism. Hence we can analyze the operator $-\Delta + S_{\mathcal{W}}$ on each component separately in the following subsections.

**Lemma 5.3.** The map $p \to p$, $v \mapsto -\sum_m \langle [k_m, [k_m, v]] \rangle$ is self-adjoint and preserves the splitting $p = \bar{p} \oplus p$.

In particular, $\sum_m \langle [k_m, v], [k_m, w] \rangle = 0$ for any $v \in \bar{p}$ and $w \in p$.

*Proof.* Let $e_1, \ldots, e_n \in p$ be an orthonormal basis which respects the splitting $p = \bar{p} \oplus p$. We find for $v \in \bar{p}$ and $w \in p$,

$$-\sum_m \langle [k_m, [k_m, v]], w \rangle = \sum_m \langle [k_m, v], [k_m, w] \rangle = \sum_{m,l} \langle [k_m, v], e_l \rangle \langle [k_m, w], e_l \rangle = \sum_{m,l} \langle [v, e_l], k_m \rangle \langle [w, e_l], k_m \rangle.$$  

(5.5)

Now recall from (5.1) that if the index $l$ is of type $\bar{f}$, we have $[w, e_l] \in \bar{f}$ and if it is of type $f$, we have $[v, e_l] \in f$. So one of the two expressions $[v, e_l]$ and $[w, e_l]$ is always contained in $\bar{f}$ and we conclude that (5.5) is equal to

$$-\sum_l \langle [v, e_l], [w, e_l] \rangle = -\text{Ric}(v, w) = (n - 1)\langle v, w \rangle = 0.$$  

□

**Lemma 5.4.** For every wall $\mathcal{W} \subset \mathcal{C}$, the splitting $p = \bar{p} \oplus p$ induces a splitting $\text{Sym}_2 p = \text{Sym}_2 \bar{p} \oplus \text{Sym}_2 p \oplus \bar{p}p$. The operator $S_{\mathcal{W}} : \text{Sym}_2 p \to \text{Sym}_2 p$ is self-adjoint and acts on each part of the splitting independently.

*Proof.* Let $v, w, v', w' \in p$. We conclude from (5.3) that

$$2\langle S_{\mathcal{W}}(v \cdot w), v' \cdot w' \rangle = \sum_m \left( \langle [k_m, v], [k_m, v'] \rangle \langle w, w' \rangle + \langle [k_m, v], [k_m, w'] \rangle \langle w, v' \rangle + \langle [k_m, w], [k_m, v'] \rangle \langle v, v' \rangle + \langle [k_m, w], [k_m, w'] \rangle \langle v, w' \rangle \right)$$

$$-2 \sum_m \langle [v, v'], k_m \rangle \langle [w, w'], k_m \rangle - 2 \langle [v, v'], [w, w'] \rangle$$

$$-2 \sum_m \langle [v, w'], k_m \rangle \langle [w, v'], k_m \rangle - 2 \langle [v, w'], [w, v'] \rangle.$$
The fact that $S_{W}$ is self-adjoint can be seen easily from this expression. Observe that the last two lines in this formula are equal to

$$-2\langle \text{proj}_{\mathfrak{k}}[v, v'], [w, w'] \rangle - 2\langle \text{proj}_{\mathfrak{k}}[w, v'], [v, w'] \rangle,$$

(5.6)

where $\mathfrak{k}^\perp$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{k}$.

Now assume that $v, w \in \mathfrak{p}$ and $v', w' \in \mathfrak{p}$, i.e. $v \cdot w \in \text{Sym}_2 \mathfrak{p}$ and $v' \cdot w' \in \text{Sym}_2 \mathfrak{p}$. Then $[v, v'], [w, w'] \in \mathfrak{k}$, so expression (5.6) vanishes. Since $\langle w, w' \rangle = \langle w, v' \rangle = \langle v, w' \rangle = 0$, we conclude $\langle S_{W}(v \cdot w), v' \cdot w' \rangle = 0$.

Secondly, assume that $v, w, v', w' \in \mathfrak{p}$, i.e. $v \cdot w \in \text{Sym}_2 \mathfrak{p}$ and $v' \cdot w' \in \mathfrak{p}$. Then $\mathfrak{k}$ and $\mathfrak{p}$ vanish again. Moreover, $\langle w, w' \rangle = 0$ and by Lemma 5.3 we conclude $\sum \langle [k_m, v], [k_m, w'] \rangle = \sum \langle [k_m, w], [k_m, w'] \rangle = 0$. So $\langle S_{W}(v \cdot w), v' \cdot w' \rangle = 0$.

Finally, assume that $v \in \mathfrak{p}$ and $v, v', w' \in \mathfrak{p}$, i.e. $v \cdot w \in \mathfrak{p}$ and $v' \cdot w' \in \text{Sym}_2 \mathfrak{p}$. Then $\mathfrak{k}$ vanishes again since $[v, v'], [v, w'] \in \mathfrak{k}$ and by Lemma 5.3 as well as $\langle v, v' \rangle = \langle v, w' \rangle = 0$, we conclude $\langle S_{W}(v \cdot w), v' \cdot w' \rangle = 0$. \hfill $\square$

5.5. The Einstein operator on $\mathfrak{p}$. We now analyze the operator $-\overline{\Delta} + S_{W}$ on $\mathfrak{p}$. We will use the trivial Bochner formula $-\overline{\Delta} = \nabla^* \nabla$ and hence it suffices to analyze $S_{W}$ acting on $\mathfrak{p}$.

**Lemma 5.5.** For every wall $W \subset C$, the restricted operator $S_{W} : \mathfrak{p} \rightarrow \mathfrak{p}$ is positive definite.

**Proof.** Let $v, v' \in \mathfrak{p}$ and $w, w' \in \mathfrak{p}$. Using the calculations from the proof of Lemma 5.3, the fact that $[v, v'] \in \mathfrak{k}$ and $[w, w'] \in \mathfrak{k}$ and hence $\langle \text{proj}_{\mathfrak{k}}[w, v'], [v, w'] \rangle = 0$, we conclude

$$2\langle S_{W}(v \cdot w), v' \cdot w' \rangle = \sum \left( \langle [k_m, v], [k_m, v'] \rangle \langle w, w' \rangle + \langle v, v' \rangle \langle [k_m, w], [k_m, w'] \rangle \right)$$

$$- 2\langle [v, v'], [w, w'] \rangle.$$

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathfrak{p}$ which respects the splitting $\mathfrak{p} = \mathfrak{p} \oplus \tilde{\mathfrak{p}}$ and express $h = \sum_{j} h_{\gamma_{j}} e_{\gamma_{j}} \cdot e_{\gamma_{j}} \in \mathfrak{p}$ Then summing over all free indices

$$2\langle S_{W}h, h \rangle = h_{\gamma_{1}} h_{\gamma_{2}} \langle [k_m, e_{\gamma_{1}}], [k_m, e_{\gamma_{2}}] \rangle + h_{\gamma_{1}} h_{\gamma_{2}} \langle [k_m, e_{\gamma_{2}}], [k_m, e_{\gamma_{1}}] \rangle - 2h_{\gamma_{1}} h_{\gamma_{2}} \langle [e_{\gamma_{1}}, e_{\gamma_{2}}] \rangle$$

We can rewrite the last coefficient as

$$\langle [e_{\gamma_{1}}, e_{\gamma_{2}}], [e_{\gamma_{2}}, e_{\gamma_{1}}] \rangle = -\langle [e_{\gamma_{1}}, e_{\gamma_{2}}], e_{\gamma_{1}} \rangle = -\langle [e_{\gamma_{2}}, e_{\gamma_{2}}], e_{\gamma_{1}} \rangle = \langle [e_{\gamma_{1}}, e_{\gamma_{2}}], [e_{\gamma_{1}}, e_{\gamma_{2}}] \rangle =$$

$$\langle [e_{\gamma_{1}}, e_{\gamma_{2}}], [e_{\gamma_{1}}, e_{\gamma_{2}}] \rangle + \langle [e_{\gamma_{2}}, e_{\gamma_{2}}], [e_{\gamma_{2}}, e_{\gamma_{2}}] \rangle =$$

$$-\sum_{j} \langle [e_{\gamma_{1}}, e_{\gamma_{2}}], k_{m_{j}} \rangle \langle [e_{\gamma_{2}}, e_{\gamma_{1}}], k_{m_{j}} \rangle + \langle [e_{\gamma_{1}}, e_{\gamma_{2}}], [e_{\gamma_{1}}, e_{\gamma_{2}}] \rangle$$
\[
\begin{aligned}
&= \frac{1}{2} \sum_{m} \langle [k_m, e_T^r], e_{T'}^r \rangle \langle [k_m, e_T^s], e_{T'}^s \rangle + \frac{1}{2} \sum_{m} \langle [k_m, e_{T'}^r], [k_m, e_T^s], e_{T'}^s \rangle \\
&\quad + \langle [e_T^r, e_{T'}^r], [e_T^s, e_{T'}^s] \rangle
\end{aligned}
\]

Hence

\[
2 \langle S_W h, h \rangle = h_{T} h_{T'} \langle [k_m, e_T^r], e_{T'}^r \rangle \langle [k_m, e_T^s], e_{T'}^s \rangle + h_{T} h_{T'} \langle [k_m, e_{T'}^r], [k_m, e_T^s], e_{T'}^s \rangle \\
+ h_{T} h_{T'} \langle [k_m, e_{T'}^r], [k_m, e_T^s], e_{T'}^s \rangle + h_{T} h_{T'} \langle [k_m, e_T^r], e_{T'}^r \rangle \langle [k_m, e_{T'}^r], e_T^s \rangle \\
- h_{T} h_{T'} \langle [k_m, e_T^r], e_{T'}^r \rangle \langle [k_m, e_{T'}^r], e_T^s \rangle - h_{T} h_{T'} \langle [k_m, e_{T'}^r], [k_m, e_T^s], e_{T'}^s \rangle \\
- 2 h_{T} h_{T'} \langle [e_T^r, e_{T'}^r], [e_T^s, e_{T'}^s] \rangle
\]

Since \([k_m, e_T^r] \in V\), the first term vanishes and we can regroup the expression as follows:

\[
= \frac{1}{2} \sum_{m,i,j} \left( h_{T} h_{T'} \langle [k_m, e_T^i], e_{T'}^j \rangle - h_{T} h_{T'} \langle [k_m, e_T^i], e_{T'}^j \rangle \right)^2 \\
+ \frac{1}{2} \sum_{m,i,j} \left( h_{T} h_{T'} \langle [k_m, e_T^i], e_{T'}^j \rangle - h_{T} h_{T'} \langle [k_m, e_T^i], e_{T'}^j \rangle \right)^2 \\
+ \sum_{m,i,j} \left( h_{T} h_{T'} \langle [k_m, e_T^i], e_{T'}^j \rangle \right)^2 - 2 \left( h_{T} h_{T'} \langle [e_T^i, e_{T'}^j] \rangle \right)^2 \geq 0
\]

Observe that this expression is nonnegative since the Killing form is negative definite on \(\mathfrak{g}\). If the expression is zero, then all the squared terms have to vanish, in particular

\[
0 = \langle [k_m, H_T], e_{T'}^r \rangle = \langle [k_m, e_{T'}^r], H_T \rangle
\]

for all \(T, T'\) and \(m\) where \(H_T = \sum_{T'} h_{T} e_{T'} \in V\). We will show that \(V \subset V\). This implies then that \(H_T = 0\) for all \(T\) and hence \(h = 0\), establishing the Lemma.

It remains to prove \(V \subset V\). First observe that by (5.11) we have \([k_m, p_m] = \alpha_m^p\) and hence \(a \subset V\). Moreover, for any index \(m\) there is a \(v \in a\) with \(\alpha_m(v) \neq 0\) and we have \([k_m, v] = -\alpha_m(v)p_m\), so \(p_m \in V\) what establishes the claim. \(\Box\)

5.6. The Einstein operator on \(\text{Sym}_2 \overline{\mathbf{F}}\). We will now analyze the operator on \(-\Delta + S_W\) on \(E_{\text{Sym}_{2}} \overline{\mathbf{F}}\). Let \(\overline{\nabla}\) be the Riemannian curvature operating on \(\overline{M}_{W}\) acting on symmetric bilinear forms \(h \in \text{Sym}_2 \overline{\mathbf{F}} \cong E_{\text{Sym}_{2}} \overline{\mathbf{F}}\). We now make use of the same Bochner formula as in (5.4), but this time on \(\overline{M}_{W}\), to conclude that

\[
(-\Delta + S_W(h))_{ij} = (\overline{\text{div}} \overline{\text{div}} + \overline{d} \overline{d}) h_{ij} \\
+ \overline{R}(h)_{ij} - \frac{1}{2} \sum_{k} (\overline{\text{Ric}}_{ik} h_{kj} + h_{ik} \overline{\text{Ric}}_{kj}) + S_W(h)_{ij}
\]

Here \(\overline{\text{div}}, \overline{d}\) and \(\overline{\text{Ric}}\) denote the corresponding operators and tensors on \(\overline{M}_{W}\). It remains to analyze the last line. This can be carried out on \(\text{Sym}_2 \overline{\mathbf{F}}\).
Lemma 5.6. The operator $B : \text{Sym}_2 \overline{\mathbf{p}} \rightarrow \text{Sym}_2 \overline{\mathbf{p}}$, 

$$h \mapsto \overline{R}(h)_{ij} - \frac{1}{2} \sum_k (\text{Ric}_{ik} h_{kj} + h_{ik} \text{Ric}_{kj}) + S_W(h)_{ij}$$

is nonnegative definite. Moreover, if $M$ does not contain any $\mathbb{H}^2$-factor in its deRham decomposition, then $B$ is even positive definite.

Proof. For every $h \in \text{Sym}_2 \overline{\mathbf{p}}$, we have 

$$\langle B(h), h \rangle = \langle \overline{A}(h), h \rangle + \langle S_W(h) + 2\overline{R}(h), h \rangle,$$

where $A : \text{Sym}_2 \overline{\mathbf{p}} \rightarrow \text{Sym}_2 \overline{\mathbf{p}}$ is the expression from Lemma 5.2 on $M_W$. By the same Lemma we know that $A$ is nonnegative definite and we will now show that $S_W + 2\overline{R}$ is nonnegative definite as well.

Let $v, w, v', w' \in \overline{\mathbf{p}}$. Using the calculation from the proof of Lemma 5.3 we conclude

$$2\langle S_W(v \cdot w), v' \cdot w' \rangle = \sum_{m} \left( \langle [k_m, v], [k_m, v'] \rangle \langle w, w' \rangle + \langle [k_m, v], [k_m, w'] \rangle \langle w, v' \rangle + \langle v, v' \rangle \langle [k_m, w], [k_m, w'] \rangle + \langle v, w' \rangle \langle [k_m, w], [k_m, v'] \rangle \right) - 4\langle \overline{R}(v \cdot w), v' \cdot w' \rangle$$

Hence, for any $h = \sum_{j, \overline{m}} h_{ij, \overline{m}} e_{\overline{m}}$ with $h_{ij} = h_{\overline{m}}$, we find

$$\langle S_W(h) + 2\overline{R}(h), h \rangle = 2 \sum_{j, \overline{m}} h_{ij, \overline{m}} \left( \langle [k_m, e_{\overline{m}}], [k_m, e_{\overline{m}}] \rangle \right) = 2 \sum_{j, \overline{m}} \left| \sum_{\overline{m}} h_{ij, \overline{m}} e_{\overline{m}} \right|^2$$

This proves nonnegativity of $B$.

Assume now that $h$ lies in the nullspace of $B$. Hence, it lies in the nullspace of $A$ and for all $j$ and $\overline{m}$ we have $[k_m, \sum_{\overline{m}} h_{ij, \overline{m}} e_{\overline{m}}] = 0$. By Lemma 5.2, we have $h = h_1 + \ldots + h_m'$ corresponding to a splitting $M_W = \mathbb{H}^2 \times \ldots \times \mathbb{H}^2 \times M_{m'+1} \times \ldots \times M_m$ and each $h_k$ is traceless. Without loss of generality, we can assume that all the $h_k$ are nonzero and hence the vectors $\sum_{\overline{m}} h_{ij, \overline{m}} e_{\overline{m}}$ span a subspace $\mathbf{p}' \subset \overline{\mathbf{p}}$ which corresponds to the tangent space of the $\mathbb{H}^2 \times \ldots \times \mathbb{H}^2$ factor. We have $[k, \mathbf{p}'] = 0$ and for every $e \in \overline{\mathbf{p}}'$ pointing in the direction of one of the $\mathbb{H}^2$-factors, we have $[k_m, e] = 0$ for all but one $\overline{m}$ which corresponds to this $\mathbb{H}^2$-factor. This implies that this $\mathbb{H}^2$-factor is already an $\mathbb{H}^2$-factor of $M$. \hfill \square

5.7. The Einstein operator on $\text{Sym}_2 \mathbf{p}$—Parts involving $a$. In the following three sections, we will analyze the operator $-\Delta + S_W$ on $E_{\mathbf{p}}$. We will use the trivial Bochner formula $-\Delta = \nabla^* \nabla$ on $M_W$ and we will show that $S_W$ is nonnegative definite on $\text{Sym}_2 \mathbf{p}$ and characterize the nullspace.

First observe that we have the splitting $\mathbf{p} = a \oplus a^\perp$ where $a^\perp = \sum_{\mathbf{a}} p_{\mathbf{a}}$, which induces a splitting $\text{Sym}_2 \mathbf{p} = \text{Sym}_2 a \oplus a \cdot a^\perp \oplus \text{Sym}_2 a^\perp$.

Lemma 5.7. For every wall $W \subset C$, the restricted operator $S_W : \text{Sym}_2 \mathbf{p} \rightarrow \text{Sym}_2 \mathbf{p}$ preserves the splitting $\text{Sym}_2 \mathbf{p} = \text{Sym}_2 a \oplus a \cdot a^\perp \oplus \text{Sym}_2 a^\perp$ and it is positive definite on $\text{Sym}_2 a \oplus a \cdot a^\perp$. 

Proof. Let \( v, w \in \mathfrak{a} \). We will need the following identity:

\[
\langle v, w \rangle = \text{tr}[v, [w, \cdot]] = \sum_{l=1}^{n-r} \langle [v, [w, p_l]], p_l \rangle - \sum_{l=1}^{n-r} \langle [v, [w, k_l]], k_l \rangle = 2 \sum_{l=1}^{n-r} \alpha_l(v)\alpha_l(w).
\]

Hence \( \sum_{l=1}^{n-r} \alpha_l^# \alpha_l(v) = \frac{1}{2} v \). Recall also that by (3.11) \( [k_l, p_l] = -[x_l, y_l] = \alpha_l^# \).

Let now \( v, w \in \mathfrak{a}, \) i.e. \( v \cdot w \in \text{Sym}_2 \mathfrak{a} \). Choose an orthonormal basis \( e_1, \ldots, e_r \) of \( \mathfrak{a} \) and consider the orthonormal basis \( e_1, \ldots, e_r, p_1, \ldots, p_{n-r} \) of \( \mathfrak{p} \). Then, since \( \alpha_l(v) = \alpha_l(w) = 0 \) and \([p_l, v] = 0\)

\[
S_W(v \cdot w) = \sum_m (\alpha_m(v)[k_m, p_m] \cdot w + \alpha_m(w)[k_m, p_m] - 2\alpha_m(v)\alpha_m(w)p_m \cdot p_m) \]

\[- 2 \sum_l \alpha_l(v)p_l \cdot [k_l, w]) = \sum_m (\alpha_m(v)\alpha_m^# \cdot w + \alpha_m(w)v \cdot \alpha_m^#) = v \cdot w.
\]

So \( S_W \) is positive definite on \( \text{Sym}_2 \mathfrak{a} \).

Now assume that \( v \in \mathfrak{a} \) and \( w \in \mathfrak{a}^\perp \), i.e. \( v \cdot w \in \mathfrak{a} \cdot \mathfrak{a}^\perp \). Then

\[
S_W(v \cdot w) = \sum_m (\alpha_m(v)[k_m, p_m] \cdot w - v \cdot [k_m, k_m, w]) + 2\alpha_m(v)p_m \cdot [k_m, w]) \]

\[- 2 \sum_l \alpha_l(v)p_l \cdot [k_l, w] = \frac{1}{2} v \cdot w - \sum_k v \cdot [k_m, [k_m, w]]
\]

Hence \( S_W \) maps the space \( \mathfrak{a} \cdot \mathfrak{a}^\perp \) to itself and it can be expressed as a tensor product of the identity on \( \mathfrak{a} \) and the map

\[
\mathfrak{a}^\perp \rightarrow \mathfrak{a}^\perp, \quad w \mapsto \frac{1}{2} w - \sum_k [k_m, [k_m, w]]
\]

This map is positive definite, hence the tensor product, too. \( \square \)

5.8. **The Einstein operator on** \( \text{Sym}_2 \mathfrak{p} \)—**The part** \( \text{Sym}_2 \mathfrak{a}^\perp \). It remains to analyze the operator \( S_W \) on \( \text{Sym}_2 \mathfrak{a}^\perp \). This case is the most complicated one since we have to deal with the richer nilpotent structure on \( \mathfrak{n} \). We will find out that the \( S_W \) nonnegative definite on this space and that the nullspace corresponds exactly to certain deformations of \( \mathfrak{n} \). In the next subsection, we will then show that in many cases such deformations do not exist and hence \( S_W \) is positive definite.

As a first step it will be essential to express the operator \( S_W \) in terms of the nilpotent structure on \( \mathfrak{n} \). In order to do this, we first need to discuss how we can recover the complete structure of the Lie-algebra \( \mathfrak{g} \) from the nilpotent structure on \( \mathfrak{n} \) and the roots \( \alpha_i \). Recall that \( \mathfrak{n} \) is spanned by the basis vectors \( x_1, \ldots, x_{n-r} \) which are orthonormal with respect to the scalar product \( \langle \cdot, \cdot \rangle = -\langle \cdot, \sigma \cdot \rangle \). We define the symbol \( \left( \begin{smallmatrix} i & j \\ k \end{smallmatrix} \right) \) by the following identity:

\[
[x_i, x_j] = \sum_k \left( \begin{smallmatrix} i & j \\ k \end{smallmatrix} \right)x_k.
\]
Then \( (i^j_k) \) is a \((1, 2)\)-tensor on \( n \) in the indices \( i, j, k \) and it is antisymmetric in \( i \) and \( j \). Since \( n \) is nilpotent, we know that \( (i^i_l) = 0 \) for all \( i \) and \( l \), so in particular
\[
\sum_l (i^i_l) = \text{tr}(i^i) = 0. \tag{5.7}
\]
Moreover, by nilpotency we know that \( (i^j_k) \) or \( (i^k_j) \) cannot both be nonzero. Hence
\[
\sum_{j,k} (i^j_k) (i^k_j) = 0. \tag{5.8}
\]
Observe that equations (5.7) and (5.8) are tensorial, i.e. they also stay true if we change the orthonormal basis \( x_1, \ldots, x_{n-r} \).

The symbol \( (i^j_l) \) contains all the information on the nilpotent Lie group \( n \). Using this information and the roots \( \alpha_1, \ldots, \alpha_{n-r} \in a^* \), we will now reconstruct the structure of the Lie algebra \( g \). First, we analyze terms of the form \([x_i, y_j]\).

We have for any \( l \):
\[
\langle [x_i, y_j], y_l \rangle = \langle [y_j, y_l], x_i \rangle = \langle [x_j, x_l], y_i \rangle = -\langle j^l_i \rangle.
\]
So if \( \text{pr}_n \) denotes the projection on \( n \), we have
\[
\text{pr}_n([x_i, y_j]) = \sum_l (j^l_i) x_l.
\]
Furthermore, we can compute that
\[
\langle [x_i, y_j], x_l \rangle = -\langle [x_i, x_l], y_j \rangle = \langle i^l_j \rangle.
\]
Hence
\[
\text{pr}_{n^-}([x_i, y_j]) = -\sum_k (i^k_j) y_k.
\]
Finally, for any \( v \in a \),
\[
\langle [x_i, y_j], v \rangle = \langle [v, x_i], y_j \rangle = \alpha_i(v) \langle x_i, y_j \rangle = -\delta_{ij} \alpha_i(v).
\]
So
\[
\text{pr}_a([x_i, y_j]) = -\delta_{ij} \alpha_i^\#.
\]
We can thus write down the projection of \([x_i, y_j]\) onto the space \( t_0^a = a \oplus n \oplus n^- \):
\[
\text{pr}_{t_0^a}([x_i, y_j]) = \sum_l (j^l_i) x_l - \sum_l (i^l_j) y_l - \delta_{ij} \alpha_i^#
\]
\[
= \frac{1}{\sqrt{2}} \sum_l \left[ (j^l_i) - (i^l_j) \right] k_l + \frac{1}{\sqrt{2}} \sum_l \left[ (j^l_i) + (i^l_j) \right] p_l - \delta_{ij} \alpha_i^\# \tag{5.9}
\]
Understanding the part of \([x_i, y_j]\) in \(\xi_0\) is more difficult because we did not introduce an orthonormal basis on this space. Hence, the best we can do here is to determine the scalar product of two such terms.

\[
\langle [x_i, y_j], [x_{i'}, y_{j'}] \rangle = -\langle [[x_i, y_j], y_{j'}], x_{i'} \rangle - \langle [x_i, [y_j, y_{j'}]], x_{i'} \rangle \\
= \langle [x_i, y_{j'}], [x_{i'}, y_j] \rangle + \langle [x_i, x_{i'}], [y_j, y_{j'}] \rangle \\
= \langle [x_i, y_{j'}], [x_{i'}, y_j] \rangle - \sum_l \left( i' j' \right) \left( j' i' \right) \\
= \langle [x_i, y_{j'}], \sigma[x_{i'}, y_j] \rangle + 2\langle [x_i, y_{j'}], \text{pr}_p([x_{i'}, y_j]) \rangle - \sum_l \left( i' j' \right) \left( j' i' \right) \\
= -\langle [x_i, y_{j'}], [x_j, y_{i'}] \rangle + \sum_l \left\{ \left( i' j' \right) + \left( j' i' \right) \right\} \left\{ \left( j' \right) + \left( i' \right) \right\} \\
- \sum_l \left( i' j' \right) \left( j' i' \right) + 2\delta_{i'j'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle.
\]

We will now repeat this process twice while permuting \(j \to j' \to i' \to j\).

\[
\langle [x_i, y_{j'}], [x_j, y_{i'}] \rangle = -\langle [x_i, y_{j'}], [x_{i'}, y_{j'}] \rangle + \sum_l \left\{ \left( i' j' \right) + \left( j' i' \right) \right\} \left\{ \left( j' \right) + \left( i' \right) \right\} \\
- \sum_l \left( i' j' \right) \left( j' i' \right) + 2\delta_{i'j'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle,
\]

\[
\langle [x_i, y_{i'}], [x_{j'}, y_j] \rangle = -\langle [x_i, y_{i'}], [x_{j'}, y_{j'}] \rangle + \sum_l \left\{ \left( j' \right) + \left( i' \right) \right\} \left\{ \left( j' \right) + \left( i' \right) \right\} \\
- \sum_l \left( i' j' \right) \left( j' i' \right) + 2\delta_{i'j'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle.
\]

So if we add the first and third equation and subtract the second one, we obtain

\[
\langle [x_i, y_j], [x_{i'}, y_{j'}] \rangle = \frac{1}{2} \sum_l \left\{ \left( j' \right) + \left( j' \right) \right\} \left\{ \left( j' \right) + \left( j' \right) \right\} \\
\left\{ \left( j' \right) + \left( j' \right) \right\} + \left\{ \left( j' \right) + \left( j' \right) \right\} \left\{ \left( j' \right) + \left( j' \right) \right\} \right\} \\
- \frac{1}{2} \sum_l \left\{ \left( i' j' \right) \left( j' i' \right) \left( j' i' \right) \right\} \right\} + \left\{ \left( i' j' \right) \left( j' i' \right) \right\} \left\{ \left( i' j' \right) \left( j' i' \right) \right\} \right\} \\
+ \delta_{i'j'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle - \delta_{i'j'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle + \delta_{i'j'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle.
\]

For any \(i, i', j, j'\) set

\[
A_{ij'j'} = \delta_{ij'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle - \delta_{i'j'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle + \delta_{i'j'}\delta_{j'i'}\langle \alpha_i^\#, \alpha_j^\# \rangle
\]

and interpret \(A_{ij'j'}\) as a \((0, 4)\)-tensor on \(n\).
We will now calculate the curvature. In order to simplify calculations later,
we will for the moment assume that $x_1, \ldots, x_{n-r}$ is any orthonormal basis of $\mathfrak{n}$
with respect to the scalar product $(\cdot, \cdot)$, i.e. the $x_i$ do not satisfy the grading of $\mathfrak{n}$
anymore and there is no root associated to them. We furthermore set 
$k_i = \frac{1}{\sqrt{2}}(x_i + y_i)$ and $p_i = \frac{1}{\sqrt{2}}(x_i - y_i)$. Observe that we can still use the identities
above as long as they were tensorial.

We now calculate the sectional curvature on the plane span$\{p_a, p_b\}$ using (3.4):

$$R_{abba} = 4 \langle [p_a, p_b], [p_a, p_b] \rangle = \langle [x_a - y_a, x_b - y_b], [x_a - y_a, x_b - y_b] \rangle$$

$$= 2 \langle [x_a, x_b], [y_a, y_b] \rangle - 4 \langle [x_a, x_b], [x_a, y_b] \rangle - 4 \langle [x_a, x_b], [y_a, x_b] \rangle$$

$$+ 2 \langle [x_a, y_b], [x_a, y_b] \rangle + 2 \langle [x_a, y_b], [y_a, x_b] \rangle$$

$$= -2 \sum I \left( \frac{a b}{l} \right)^2 - 4 \sum I \left( \frac{a b}{l} \right) \langle x_l, [x_a, y_b] \rangle - 4 \left( \frac{a b}{l} \right) \langle x_l, [y_a, x_b] \rangle$$

$$+ \sum I \left\{ \left( \frac{b l}{a} + \frac{a l}{b} \right) \right\} \left\{ \left( \frac{b l}{a} + \frac{a l}{b} \right) \right\}$$

$$- \left\{ \left( \frac{a l}{b} + \frac{a l}{b} \right) \right\} \left\{ \left( \frac{b l}{a} + \frac{b l}{a} \right) \right\} - \sum I \left( \frac{a l}{b} \right) \left( \frac{b l}{a} \right) + \frac{2 A_{abba}}{A_{abba}}$$

$$= -6 \sum I \left( \frac{a b}{l} \right)^2 + 2 \sum I \left( \frac{a l}{b} \right) + \frac{2 A_{abab}}{A_{abab}}$$

$$- 8 \sum I \left( \frac{a b}{l} \right) \left( \frac{b l}{a} \right) - 4 \sum I \left( \frac{a l}{b} \right) \left( \frac{b l}{a} \right) + 2 A_{abab} - 2 A_{abba}$$

We will also need

$$4 \langle [k_m, p_a], [k_m, p_a] \rangle = \langle [x_m + y_m, x_a - y_a], [x_m + y_m, x_a - y_a] \rangle$$

$$= -2 \langle [x_m, x_a], [y_m, y_a] \rangle - 4 \langle [x_m, x_a], [x_m, y_a] \rangle + 4 \langle [x_m, x_a], [y_m, x_a] \rangle$$

$$+ 2 \langle [x_m, y_a], [x_a, y_m] \rangle + 2 \langle [x_m, y_a], [x_m, y_a] \rangle$$

$$= \sum I \left( \frac{2 m a}{l} \right)^2 - 4 \left( \frac{m a}{l} \right) \langle x_l, [x_m, y_a] \rangle + 4 \left( \frac{m a}{l} \right) \langle x_l, [y_m, x_a] \rangle$$
new orthonormal basis

We can use these calculations to express the operator $S_W$ on $\text{Sym}_2 \mathfrak{a}^\perp$. Consider a symmetric bilinear form $h \in \text{Sym}_2 \mathfrak{a}^\perp$. Let $\{p_\alpha\}$ be an orthonormal basis of $\mathfrak{a}^\perp$ which diagonalizes $h$ and which obeys the splitting $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. On $\bigoplus \mathfrak{p}_\sigma \mathfrak{p}_\tau$ we can just choose the standard orthonormal basis $\{x_\sigma\}$. Corresponding to this new orthonormal basis $p_1, \ldots, p_{n-r}$ of $\bigoplus \mathfrak{p}_\alpha \mathfrak{p}_a$ there is then an orthonormal basis $x_1, \ldots, x_{n-r}$ of $\mathfrak{n}$ such that for $y_\alpha = \sigma x_\alpha$, we have $p_\alpha = \frac{1}{\sqrt{2}}(x_\alpha + y_\alpha)$. Moreover, we set $k_\alpha = \frac{1}{\sqrt{2}}(x_\alpha + y_\alpha)$. Observe that $x_1, \ldots, x_{n-r}$ respects the splitting $\mathfrak{n} = \mathfrak{\bar{n}} \oplus \mathfrak{n}$, i.e. $\{x_\sigma\}$ is a basis for $\mathfrak{\bar{n}}$ and $\{x_\alpha\}$ one for $\mathfrak{n}$.

Let $\{\lambda_\alpha\}$ be the eigenvalues of $h$, hence $h = \sum_{\alpha} \lambda_\alpha p_\alpha \cdot p_\alpha$. We can compute

$$4 \langle R(h), h \rangle = \sum_{a,b,a',b'} 4R_{aba'b'} h_{aa'} h_{bb'} = \sum_{a,b} 4R_{aba'b} \lambda_a \lambda_{\bar{b}}$$

$$= \sum_{a,b} \left(-6 \left(\frac{a}{b}\right)^2 + 2 \left(\frac{a}{b}\right)^2 \right) - 8 \left(\frac{a}{b}\right)^2 \left(\frac{b}{a}\right)^2$$

$$- 4 \left(\frac{a}{b}\right)^2 \left(\frac{b}{a}\right)^2 + 4 \left(\frac{b}{a}\right)^2 \left(\frac{a}{b}\right)^2 \lambda_a \lambda_{\bar{b}} + 2 \sum_{a,b} (A_{aba'b} - A_{a'b'a}) \lambda_a \lambda_{\bar{b}}$$

and

$$4 \sum_{a,b,m} \langle [k_m, [k_m, p_\alpha]], p_\alpha \rangle \lambda_a^2 + 4 \sum_{a,b,m} \langle [k_m, p_\alpha], p_\beta \rangle^2 \lambda_a \lambda_{\bar{b}}$$

$$= - \sum_{a,b,m} \left(2 \left(\frac{a}{b}\right)^2 + 2 \left(\frac{a}{m}\right)^2 \left(\frac{m}{a}\right)^2 \right) - 4 \left(\frac{a}{b}\right)^2 \left(\frac{b}{a}\right)^2$$

$$- 4 \left(\frac{a}{m}\right)^2 \left(\frac{m}{a}\right)^2 \lambda_a^2$$
\[-2 \sum_{a,b} (A_{a b a} + A_{b b a}) \lambda_a^2 + 2 \sum_{a,b,m} \left\{ \left( \frac{m}{a} \right) - \left( \frac{m}{b} \right) \right\}^2 \lambda_a \lambda_b \]

\[= -2 \sum_{a,b} \left( \lambda_a^2 + \lambda_b^2 + \lambda_z^2 \right) \left( \frac{x}{z} \right)^2 - 2 \sum_{a,b,m} \left\{ \left( \frac{a}{m} \right) + \left( \frac{b}{m} \right) \right\}^2 \lambda_a^2 \]

\[+ 2 \sum_{a,b,m} \left\{ \left( \frac{a}{b} \right) - \left( \frac{b}{m} \right) \right\}^2 \lambda_a \lambda_b - 2 \sum_{a,m} (A_{a a m} + A_{m a m}) \lambda_a^2. \]

Here, we have used the following two identities: First, by (5.8) and the fact that \((\frac{a}{m}) = 0\)

\[\sum_{a,b,l} \left( \frac{a}{l} \right) \left( \frac{b}{l} \right) = \sum_{m,l} \left( \frac{m}{l} \right) \left( \frac{a}{l} \right) = 0\]

and secondly by exchange of \(l\) and \(m\)

\[\sum_{a,b,l} \left\{ \left( \frac{a}{l} \right) - \left( \frac{b}{l} \right) \right\}^2 \lambda_a^2 = 0.\]

Using these two identities, we can finally express the operator \(S_W\):

\[2\langle S_W h, h \rangle = 2 \sum_{a,b} (\lambda_a^2 + \lambda_b^2) \left( \frac{x}{z} \right)^2 + 8 \sum_{a,b,l} \left( \frac{a}{l} \right) \left( \frac{b}{l} \right) \lambda_a \lambda_b \]

\[+ 2 \sum_{a,b,m} \left\{ \left( \frac{a}{m} \right) + \left( \frac{b}{m} \right) \right\}^2 \lambda_a^2 - 2 \sum_{a,b,l} \left\{ \left( \frac{a}{l} \right) + \left( \frac{b}{l} \right) \right\}^2 \lambda_a \lambda_b \]

\[- 2 \sum_{a,b} (A_{a b b} - A_{b a b}) \lambda_a \lambda_b - 2 \sum_{a,m} (A_{a a m} + A_{m a m}) \lambda_a^2 \]

\[= 2 \sum_{a,b} (\lambda_a^2 + \lambda_b^2) \left( \frac{x}{z} \right)^2 + 8 \sum_{a} \left\{ \sum_{a} \lambda_a \left( \frac{a}{a} \right) \right\}^2 \]

\[+ \sum_{a,b,l} \left\{ \left( \frac{a}{l} \right) + \left( \frac{b}{l} \right) \right\}^2 (\lambda_a - \lambda_b)^2 \]

\[- 2 \sum_{a,b} (A_{a b b} - A_{b a b}) \lambda_a \lambda_b + 2 \sum_{a,m} (A_{a a m} + A_{m a m}) \lambda_a^2.\]

We will now show that the last line is always nonnegative. This will then establish the nonnegativity of \(S_W\). To carry out the calculation, we rewrite the last line in tensorial form:

\[-2 \sum_{a,a', b,b'} (A_{a b a'} - A_{b b a'}) h_{a a'} h_{b b'} + 2 \sum_{a,a', m} (A_{a a' m} + A_{m a m}) h_{a a'} h_{a a'} \]
and we return to the original orthonormal basis \( x_1, \ldots, x_{n-r} \) which obeyed the nilpotent grading of \( n \). Then the expression above becomes

\[
-2 \sum_{\alpha \beta} (h_{\alpha \beta} - h_{\alpha \beta} h_{\beta \alpha} + h_{\alpha \beta} h_{\beta \alpha} - h_{\alpha \beta} h_{\beta \alpha} - h_{\alpha \beta} h_{\beta \alpha}) (\alpha^\#_\alpha, \alpha^\#_\beta) + 4 \sum_{\alpha \beta} h_{\alpha \beta}^2 |\alpha^\#_\alpha|^2
\]

\[= 4 \left| \sum_{\alpha} h_{\alpha \alpha} \alpha^\#_\alpha \right|^2 + 2 \left| \sum_{\alpha \beta} h_{\alpha \beta} (\alpha^\#_\alpha - \alpha^\#_\beta) \right|^2.
\]

So \( S_W \) is indeed nonnegative definite on \( \text{Sym}_2 a^\perp \) and the nullspace consists exactly of those \( h = \sum_{\alpha \beta} h_{\alpha \beta} p_\alpha \cdot p_\beta \) (for \( h_{\alpha \beta} = h_{\beta \alpha} \) and the original orthonormal basis \( x_1, \ldots, x_{n-r} \) which respects the grading of \( n \)) which satisfy the following five identities (5.10)-(5.14) below

\[
\sum_i h_{\alpha i} \left( \begin{array}{c} i \\ \beta \\ \gamma \end{array} \right) + \sum_i h_{\beta i} \left( \begin{array}{c} \alpha \\ i \\ \gamma \end{array} \right) = \sum_i h_{\gamma i} \left( \begin{array}{c} \alpha \\ \beta \\ i \end{array} \right)
\]

(5.10)

\[
\sum_i h_{\alpha i} \alpha_i = 0
\]

(5.11)

\[
h_{\alpha \beta} = 0 \quad \text{if} \quad \alpha_\alpha \neq \alpha_\beta
\]

(5.12)

\[
\sum_{\alpha \beta} h_{\alpha \beta} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = 0
\]

(5.13)

Observe that condition (5.12) implies that if \( h \) lies in the nullspace of \( S_W \), then it has block form with respect to the splitting \( \bigoplus \alpha p_\alpha \) and hence we can find an orthonormal basis \( x_1, \ldots, x_{n-r} \) which both respects the nilpotent grading of \( n \) and for which the associated orthonormal basis \( p_1, \ldots, p_{n-r} \) diagonalizes \( h \) with eigenvalues. In this basis, we see that identity (5.13) is redundant. In the said basis, the fifth identity characterizing the nullspace is

\[
h_{\alpha \alpha} = h_{\beta \beta} \quad \text{if} \quad \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) + \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) \neq 0 \quad \text{for some} \ \alpha.
\]

(5.14)

5.9. Analysis of the nullspace. Let \( h \in \text{Sym}_2 a^\perp_W \) be a symmetric bilinear form on \( a^\perp_W \). In the last subsection we found that \( h \) lies in the nullspace \( N_W \subset \text{Sym}_2 a^\perp \) of \( S_W \) if it satisfies the identities (5.10)-(5.14). Set \( a^\perp = a^\perp_W = \bigoplus \alpha p_\alpha \) and let \( N = N_C \subset \text{Sym}_2 a^\perp \) be the nullspace corresponding to \( C \). In other words, \( N \) is the space of bilinear forms \( h \in \text{Sym}_2 a^\perp \) satisfying

\[
\sum_i h_{\alpha i} \left( \begin{array}{c} i \\ \beta \\ \gamma \end{array} \right) + \sum_i h_{\beta i} \left( \begin{array}{c} \alpha \\ i \\ \gamma \end{array} \right) = \sum_i h_{\gamma i} \left( \begin{array}{c} \alpha \\ \beta \\ i \end{array} \right)
\]

(5.15)

\[
\sum_i h_{\alpha i} \alpha_i = 0
\]

(5.16)

\[
h_{\alpha \beta} = 0 \quad \text{if} \quad \alpha_\beta \neq \alpha_\beta
\]

(5.17)

(Recall that identity (5.13) is redundant.) In other words, we can say that \( N \) is the space of all \( h \in \text{Sym}_2 a^\perp \) which are in block form with respect to the splitting
$a^\perp = \bigoplus \alpha p_\alpha$ and which satisfy the following two identities if $h = \sum_{a=1}^{n-r} \lambda_a p_a \cdot p_a$ for an orthonormal basis $p_1, \ldots, p_{n-r}$ for which the associated $x_1, \ldots, x_{n-r}$ respect the nilpotent grading of $n$:

\[ \lambda_a + \lambda_b = \lambda_c \quad \text{if} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0 \]  
(5.18)

\[ \sum_i \lambda_i \alpha_i = 0 \]  
(5.19)

**Lemma 5.8.** For every wall $W \subset C$ consider the imbedding $\text{Sym}_2 a^\perp_W \subset \text{Sym}_2 a^\perp$. Then $N_W = N \cap \text{Sym}_2 a^\perp_W$.

**Proof.** Let $h \in N_W$ and interpret $h$ as a symmetric bilinear form on $a^\perp$. Then identity (5.19) is obviously satisfied as well as (5.18) in the case in which $a, b, c$ are of type $a, b, c$. If $c$ is of type $\overline{a}$, then $a$ and $b$ must be of $\overline{a}$ and $\overline{b}$ to guarantee $(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0$, but in this case both sides vanish. If $a$ is of type $\overline{a}$ and $c$ of type $\overline{c}$, then $b$ must be of type $\overline{b}$. So since not both expressions $(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0$, and $(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0$, can vanish, we can use (5.14) to conclude (5.19). The same is true reversing the role of $a$ and $b$.

Let now on the other hand $h \in N \cap \text{Sym}_2 a^\perp_W$ and consider the diagonalizing basis $p_1, \ldots, p_{n-r}$. Since (5.10)-(5.12) are trivially satisfied and (5.13) is redundant, we only need to establish (5.14). This follows from identity (5.18) for $b = l$. □

We will now analyze the nullspace $N$. By the following Lemma, we can restrict our analysis to irreducible symmetric spaces.

**Lemma 5.9.** Assume that $\mathfrak{g}$ has a de Rham decomposition $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$. Then $N = N_1 \oplus \ldots \oplus N_m$ where $N_i$ is the nullspace corresponding to $\mathfrak{g}_i$.

**Proof.** By (5.17) every $h \in N$ takes block form with respect to the (coarse) splitting $a^\perp = a_1^\perp \oplus \ldots a_m^\perp$ coming from the de Rham decomposition. The other direction is clear. □

**Lemma 5.10.** Consider an $h \in N$ and choose a diagonalizing orthonormal basis $x_1, \ldots, x_{n-r}$ as above.

Assume that for two indices $a, b$ we have $\alpha_a = \alpha_b = \alpha_0$ and that there is a representation

\[ x_a = \sum_{u,v} A_{uv}[x_u, x_v] + \sum_{u,v} B_{uv}[x_u, y_v] \]  
(5.20)

such that $B_{uv} = 0$ whenever $\alpha_v = \alpha_0$. Then $\lambda_a = \lambda_b$.

**Proof.** Using the subspaces

\[ \mathfrak{g}_{\alpha,\lambda} = \text{span}\{x_i : \alpha_i = \alpha, \lambda_i = \lambda\} \]

\[ \mathfrak{g}_{-\alpha,-\lambda} = \text{span}\{y_i : \alpha_i = \alpha, \lambda_i = \lambda\} = \sigma \mathfrak{g}_{\alpha,\lambda}, \]

we obtain refined splittings

\[ n = \bigoplus_{\alpha \in \Delta^+,\lambda} \mathfrak{g}_{\alpha,\lambda} \quad \text{and} \quad n^- = \bigoplus_{\alpha \in \Delta^+,\lambda} \mathfrak{g}_{-\alpha,-\lambda}. \]
By (5.18) and (5.9), we conclude

\[
\begin{align*}
\mathfrak{g}_{\alpha, \lambda} \cdot \mathfrak{g}_{\alpha', \lambda'} & \subset \mathfrak{g}_{\alpha + \alpha', \lambda + \lambda'} \\
\mathfrak{g}_{\alpha, \lambda} \cdot \mathfrak{g}_{-\alpha', -\lambda} & \subset \mathfrak{g}_{-\alpha, -\lambda, \lambda} \quad \text{if } \alpha \neq \alpha'.
\end{align*}
\]

Consider the representation (5.20) of \(x_a\) and observe that \([x_u, x_v] \in \mathfrak{g}_{\alpha_u + \alpha_v, \lambda_u + \lambda_v}\) and \([x_u, y_v] \in \mathfrak{g}_{\alpha_u - \alpha_v, \lambda_u - \lambda_v}\). So if we set \(A_{uv} = 0\) whenever \(\alpha_u + \alpha_v \neq \alpha_0\) or \(\lambda_u + \lambda_v \neq \lambda_a\) as well as \(B_{uv} = 0\) whenever \(\alpha_u - \alpha_v \neq \alpha_0\) or \(\lambda_u - \lambda_v \neq \lambda_a\), representation (5.20) continues to hold. We will assume this property from now on.

Now observe that by (5.9), we have \(a = u + v\). Using the representation (5.20), we find

\[
\begin{align*}
x_a & = \sum_{u,v} A_{uv} [[x_u, x_v], x_a] + \sum_{u,v} B_{uv} [[x_u, y_v], y_a, x_a] \\
& = \sum_{u,v} A_{uv} \left( [[y_v, x_v], x_u], x_a \right) + \left( [[x_u, y_v], x_v], x_a \right) \\
& \quad + \sum_{u,v} B_{uv} \left( [[y_v, y_v], x_u], x_a \right) + \left( [[x_u, y_v], y_v], x_a \right) \\
& = \sum_{u,v} A_{uv} \left( [[x_a, x_u], y_v, x_v], x_a \right) + \left( [[y_b, y_v], x_u], x_a \right) + \left( [[x_u, y_v], y_v], x_a \right) \\
& \quad + \left( [[x_a, x_u], y_v], x_a \right) + \left( [[y_b, y_v], x_v], x_a \right) + \left( [[x_u, y_v], x_a], x_v \right) \\
& \quad + \sum_{u,v} B_{uv} \left( [[x_a, x_u], y_v, x_v], x_a \right) + \left( [[y_b, y_v], x_u], x_a \right) + \left( [[x_u, y_v], y_v], x_a \right) \\
& \quad + \left( [[x_a, x_u], y_v], x_a \right) + \left( [[y_b, y_v], x_v], x_a \right) + \left( [[x_u, y_v], x_a], x_v \right)
\end{align*}
\]

This implies \([x_a, y_v], x_a \in \mathfrak{g}_{\alpha_0, 2\lambda_a - \lambda_b}\) since none of the successive Lie brackets lie in \(\mathfrak{g}_0\). Note here that for the seventh term, we have used the property that \(B_{uv} = 0\) if \(\alpha_v = \alpha_0\).

**Lemma 5.11.** Assume that \(\mathfrak{g}\) is the Lie algebra of an irreducible symmetric space. If its rank is greater than 1, then every \(x_a\) has a representation (5.20).

If its rank is equal to 1, then \(\Delta = \{-2\alpha', -\alpha', 0, \alpha', 2\alpha'\}\) and every \(x_a \in \mathfrak{g}_{2\alpha'}\) has a representation (5.20).
Proof. Set $\alpha_0 = \alpha_1$ and consider the following subspace of $g_{\alpha_0}$:

$$V = \left\{ \sum_{u,v} A_{uv}[x_u,x_v] + \sum_{u,v} B_{uv}[x_u,y_v] : B_{uv} = 0 \text{ if } \alpha_v = \alpha_0 \right\} \cap g_{\alpha_0}.$$ 

Assume that $V \neq g_{\alpha_0}$. Then there is an $x \in g_{\alpha_0}$ such that for $y = \sigma x$ we have

$$\langle [x_u,x_v], y \rangle = 0 \text{ for all } u,v \text{ and } \langle [x_u,y_v], y \rangle = 0 \text{ if } \alpha_v \neq \alpha_0.$$ 

This implies that

$$[x_u,y] \text{ has no component in } g_{-\alpha_v} \text{ for all } u,v.$$ 

$$[x_u,y] \text{ has no component in } g_{\alpha_v} \text{ if } \alpha_v \neq \alpha_0.$$ 

$$[y_v,y] \text{ has no component in } g_{-\alpha_u} \text{ if } \alpha_v \neq \alpha_0.$$ 

Hence, we conclude that

$$[g_{\beta},y] = 0 \quad \text{if } \beta \in \Delta \setminus \{-2\alpha_0,-\alpha_0,0,\alpha_0,2\alpha_0\}.$$ 

Applying $\sigma$ yields $[g_{\beta},x] = 0$ for the same $\beta$'s. So we also have

$$0 = [g_{\beta},[y,x]] = [g_{\beta},\alpha^\#].$$ 

This implies that $\langle \alpha^\#, \beta^\# \rangle = 0$ for all $\beta \in \Delta \setminus \{-2\alpha_0,-\alpha_0,0,\alpha_0,2\alpha_0\}$. In the higher rank case this contradicts the irreducibility of $g$.

In the rank 1 case, the Lemma follows from the fact that $g_{2\alpha} = [g_{\alpha'},g_{\alpha'}]$. \qed

We will now completely analyze the case in which $g$ is the Lie algebra of a rank 1 symmetric space $M$. The only possibilities here are real, complex, quaternionic and octonionic hyperbolic space:

$$R^h, \quad C^{h2}, \quad H^{h4}, \quad O^{16}$$

where $n \geq 2$. The symbols $H$ and $O$ denote the division algebras of the quaternions and the octonions. We left out the spaces $C^2$ and $H^4$ since they are isometric to $R^2$ resp. $R^4$. Observe that octonionic hyperbolic space only exists in dimension 16.

Obviously, $g$ has dimension 1. The set of positive roots $\Delta^+$ consists of a single root $\alpha$ in the real case and two roots $\alpha, 2\alpha$ in the other cases. We can model the algebraic structure of the root spaces in the following way (see e.g. [Mos]): Let $K = R, C, H$ or $O$ depending on which space we look at. Denote by $\text{Im } K = \{ v \in K : \tau = -v \}$ the imaginary subspace. Observe that $\dim \text{Im } K = \dim K - 1$. In the case $K = O$ let $n = 2$. We have the identifications

$$g_{\alpha} = K^{n-1}, \quad g_{2\alpha} = \text{Im } K.$$  \hfill (5.21)

For $v, w \in g_{\alpha} = K^{n-1}$ set

$$(v,w) = \tau_1 w_1 + \ldots + \tau_{n-1} w_{n-1}.$$ 

Then we can describe the Lie algebra structure on $n = g_{\alpha} \oplus g_{2\alpha}$ by

$$[v,w] = 2 \text{Im}(v,w).$$ 

**Lemma 5.12.** If $g$ is the Lie algebra of a rank 1 symmetric space $M$, then we can describe the nullspace $N$ as follows ($n \geq 2$)
If $M = \mathbb{R}H^n$, then $a^\perp \cong \mathbb{R}^{n-1}$ and $\mathcal{N} = \{ h \in \text{Sym}_2 \mathbb{R}^{n-1} : \text{tr} \ h = 0 \}$.

If $M = \mathbb{C}H^{2n}$, then $a^\perp \cong \mathbb{C}^{n-1} \oplus \mathbb{R}$. View $\mathbb{C}^{n-1}$ as $\mathbb{R}$-vector space and let the endomorphism $J$ denote multiplication by $i$.

Hence the case $K$ yields $\mathcal{N} = \{ h \in \text{Sym}_2 \mathbb{C}^{n-1} : Jh + hJ = 0 \}$.

If $M = \mathbb{H}H^n$ or $M = \mathcal{O}H^{16}$, then $\mathcal{N} = \{0\}$.

Proof. Let $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathcal{O}$ and recall the identifications (5.21). We will furthermore identify $n$ with $a^\perp$ via the map $x \mapsto \frac{1}{\sqrt{2}}(x - \sigma x)$ and sometimes view symmetric bilinear forms on $a^\perp$ as endomorphism on $n$. Then a symmetric bilinear form $h$ lies in $\mathcal{N}$ if and only if

(i) $h = h^0 + h^1$ for symmetric bilinear forms $h^0, h^1$ on $g_a = \mathbb{K}^{n-1}$ resp. $g_{2a} = \text{Im} \ K$ (see (5.17)).

(ii) For all $v, w \in \mathbb{K}^{n-1}$, we have $[h^0(v), w] + [v, h^0(w)] = h^1([v, w])$ (see (5.15)).

(iii) $\text{tr} \ h^0 + 2 \text{tr} \ h^1 = 0$ (see 5.14).

Hence, the case $K = \mathbb{R}$ is settled.

Now assume that $\text{Im} \ K$ is nontrivial. By Lemmas 5.10 and 5.11, we conclude that $h^1 = \lambda \text{id}$ for some $\lambda$. Let $x_1, \ldots, x_{d-1}$ be an orthonormal basis of $g_{2a} = \text{Im} \ K$ and set $y_i = x_i$. For each $i = 1, \ldots, d-1$ define an endomorphism $J_i \in \text{End} g_a$ by

\[ \langle J_i v, w \rangle = \langle \text{Im}(v, w), y_i \rangle. \]

The $J_i$ are antisymmetric and satisfy $J_i^2 = -1$ (in the case $K = \mathbb{C}$, we have $J_1 = J$). So condition (ii) reads

\[ J_i h^0 + h^1 J_i = \lambda J_i \quad \text{for all} \quad i = 1, \ldots, d-1. \]

We conclude that $2 \text{tr} \ h^0 = (d-1)\lambda$ and hence by condition (iii) it follows that $0 = \text{tr} \ h^0 + 2 \text{tr} \ h^1 = 4(d-1) \text{tr} \ h^0$ and thus $\lambda = 0$. This establishes the case $K = \mathbb{C}$.

In the case $K = \mathbb{H}$ we can choose $x_1, x_2, x_3 \in \text{Im} \ H$ such that $x_1 x_2 = x_3$ and hence $J_1 J_2 = J_3$. Then

\[ h^0 J_3 = h^0 J_1 J_2 = -J_1 h^0 J_2 = J_1 J_2 h^0 = J_3 h^0. \]

Together with $J_5 h^0 + h^0 J_5 = 0$ this yields $h^0 = 0$.

Finally using a multiplication table, we see that in the case $K = \mathcal{O}$ we can choose $x_1, \ldots, x_7 \in \text{Im} \ O$ such that $J_1 J_2 J_3 J_4 J_5 J_6 = J_7$. Hence with

\[ h^0 J_7 = h^0 (J_1 \cdots J_6) = -J_1 h^0 (J_2 \cdots J_6) = \cdots = (J_1 \cdots J_6) h^0 = J_7 h^0 \]

and $h^0 J_7 + J_7 h^0 = 0$, we conclude $h^0 = 0$. \qed

5.10. Final conclusion. We can finally give a proof of Proposition 5.1

Proof of Proposition 5.1. The nonnegativity of $\lambda_W$ follows from Lemmas 5.4, 5.5, 5.6, 5.7 and the calculations of subsection 5.8. In the case $M = \mathbb{H}^n$ or $M = \mathbb{C}H^{2n}$, the proposition follows from Lemmas 5.2 and 5.12.

In order to show $\lambda_0 > 0$, in the case in which $M$ does not contain any hyperbolic or complex hyperbolic factor, we only need to show that $\mathcal{N}_W = \{0\}$. By Lemma
we can assume that $M$ is irreducible. For the rank 1 case we use Lemma 5.12.

Assume now that $M$ is irreducible and of higher rank. Lemmas 5.10 and 5.11 yield that $\lambda_a = \lambda_b$ whenever $\alpha_a = \alpha_b$. Hence the eigenvalue $\lambda_a$ depends only on the root $\alpha_a$. Consider the simple roots $\beta_1, \ldots, \beta_r$ of $\Delta^+$ and let $\lambda'_1, \ldots, \lambda'_r$ be the corresponding eigenvalues. We now use the fact that $g$ is generated by the linear subspace

$$\bigoplus_{i=1}^r (g_{\beta_i} + g_{-\beta_i}).$$

This follows by the fact that any $\alpha \in \Delta^+$ has $\langle \alpha^\# , \beta_i^\# \rangle > 0$ for some $\beta_i$ and the computation (1.16). Hence, if $\alpha_a = \sum_{i=1}^r k_i \beta_i$ we conclude that $\lambda_a = \sum_{i=1}^r k_i \lambda'_i$. So there is an element $v \in a$ such that $\lambda_a = \alpha_a(v)$ for all $a$. By (5.19) and the first identity in the proof of Lemma 5.7 we conclude

$$0 = \sum_{a=1}^{n-r} \alpha_a(v) \alpha_a^\# = \frac{1}{2} v$$

and hence $h = 0$. \qed

6. Proofs of the main theorems

6.1. Introduction. In this section, we will prove the stability results Theorem 1.1 and 1.2. Consider a solution $(g_t)_{t \in [0, T)}$ to Ricci deTurck flow (2.2). Recall from subsection 2.1 that we can write the evolution equation for $h_t = g_t - \bar{g}$ as

$$\partial_t h_t + L h_t = Q_t = R_t + \nabla^* S_t$$

(6.1)

where

$$|Q_t| \leq C(|\nabla h_t|^2 + |h_t| |\nabla^2 h_t|), \quad |R_t| \leq C |\nabla h_t|^2, \quad |S_t| \leq C |h_t| |\nabla h_t|.$$ 

Let $k_t \in C^\infty (\text{Sym}_2^* T^* M \otimes (\text{Sym}_2^* T^* M)^*; M \times M)$ be the kernel of the Einstein operator $L$, i.e.

$$\partial_t k_t(\cdot, x_1) = -L k_t(\cdot, x_1) \quad \text{and} \quad k_t(\cdot, x_1) \xrightarrow{t \to 0} \delta_{x_1} \text{id}_{\text{Sym}_2^* T^*_x M}.$$ 

For $(x_1, t_1) \in M \times [0, T)$ and $0 \leq t_0 < t_1$, we obtain by convolution

$$h(x_1, t_1) = \int_M k_{t_1-t_0} (x_1, x) h_{t_0} (x) dx + \int_0^{t_1} \int_M k_{t_1-t} (x_1, x) Q_t (x) dx dt$$

$$= \int_M k_{t_1-t_0} (x_1, x) h_{t_0} (x) dx$$

$$+ \int_0^{t_1} \int_M (k_{t_1-t} (x_1, x) R_t (x) + \nabla k_{t_1-t} (x_1, x) S_t (x)) dx dt.$$  (6.2)

We will frequently make use of this identity.

In the next subsection, we prove Theorem 1.1. In order to establish Theorem 1.2 we first derive some more precise short-time estimates in subsection 6.3. Then, we present a trick involving the geometry of negative sectional curvature
to obtain a good estimate on the linearized equation in subsection 6.4. Finally, we prove Theorem 1.1 in subsection 6.5.

6.2. Proof of Theorem 1.1.

Proof of Theorem 1.1. Observe that by passing over to its universal cover, we can always assume $M$ to be simply connected.

By Theorem 4.2 and Proposition 5.1, we know that there are constants $\lambda > 0$ and $C < \infty$ such that we have the following bound on the heat kernel

$$\|k_t(x_1, \cdot)\|_{L^1(M)} \leq Ce^{-\lambda t}$$

for all $x_1 \in M$ and $t > 0$.

Let $\varepsilon_0 > 0$ be a small constant which we will determine in the course of the proof and define $T_{\text{max}}$ to be the maximum over all $T$ such that Ricci deTurck flow $h_t$ starting from $h_0 = g_0 - \mathcal{F}$ exists on $[0, T)$ and satisfies $\|h_t\|_{L^\infty(M)} < \varepsilon_0$ everywhere. In the following, we will show that for sufficiently small $\varepsilon_0$, we have $\|h_t\|_{L^\infty(M)} \leq C_1 \varepsilon e^{-\lambda t}$ where $\varepsilon$ is the constant which controls $h_0$. Hence, if $\varepsilon$ is small enough, we conclude that $\|h_t\|_{L^\infty(M)} < \varepsilon_0/2$ on $[0, T_{\text{max}})$ and Proposition 2.4 yields $T_{\text{max}} = \infty$.

Now set for every $t_1 \in [0, T_{\text{max}})$

$$Z_{t_1} = \max_{t \in [0, t_1)} e^{\lambda t} \|h_t\|_{L^\infty(M)}$$

Again, by Proposition 2.4, we find that if we choose $\varepsilon$ small enough, we have $T_{\text{max}} > \sigma^2$ and $Z_{\sigma^2} \leq C \varepsilon$. By Corollary 2.3 we conclude that for sufficiently small $\varepsilon_0$, we have for all $t < [\sigma^2, T_{\text{max}})$

$$\|\nabla^m h_t\|_{L^\infty(M)} \leq C_m Z_t e^{-\lambda t} \implies \|Q_t\|_{L^\infty(M)} \leq C Z_t^2 e^{-2\lambda t}.$$ 

We now use (6.2) for $t_1 > t_0 = \sigma^2$:

$$h_{t_1}(x_1) = \int_M k_{t_1 - \sigma^2}(x_1, x) h_{\sigma^2}(x) dx + \int_{\sigma^2}^{t_1} \int_M k_{t_1 - t}(x_1, x) Q_t(x) dx dt$$

to obtain the estimate

$$|h_{t_1}(x_1)| \leq C \varepsilon e^{-\lambda t_1} + C Z_{t_1}^2 \int_{\sigma^2}^{t_1} e^{-\lambda(t_1 - t)} e^{-2\lambda t} dt \leq C \varepsilon e^{-\lambda t_1} + C Z_{t_1}^2 e^{-\lambda t_1}.$$ 

We conclude that there is a constant $C_0 < \infty$ such that

$$Z_{\sigma^2} \leq C_0 \varepsilon$$

and

$$Z_t \leq C_0 (\varepsilon + Z_t^2)$$

for all $t \in [\sigma^2, T_{\text{max}})$. Now assume $\varepsilon < (2C_0)^{-2}$. Observe that since $Z_t$ is continuous in $t$, either $Z_t \leq 2C_0 \varepsilon$ holds for all times $t \in [\sigma^2, T_{\text{max}})$ or there is a time $t \in [\sigma^2, T_{\text{max}})$ with $Z_t = 2C_0 \varepsilon$. However, the latter case immediately gives a contradiction:

$$2C_0 \varepsilon = Z_t \leq C_0 (\varepsilon + Z_t^2) = C_0 (\varepsilon + 4C_0^2 \varepsilon^2) < 2C_0 \varepsilon.$$ 

This implies the claim for $C_1 = 2C_0$. \qed
6.3. Short-time estimates. In this subsection, we establish some analytical facts which are needed later in the proof of Theorem 1.2. Our main result will be Lemma 6.2 which states that a bound on perturbation in terms of two components as in the assumption of Theorem 1.2 will persist for a small time and there are suitable a priori derivative estimates. We will partially make use of methods developed in [KL].

**Lemma 6.1.** There is an \( \varepsilon_0 > 0 \) and constants \( \varepsilon_m > 0 \) such that: Let \( r_1 < 1 \), \( t_1 = \frac{r_1^2}{r_1^2} \), \( x_1 \in M \) and assume that \( (h_t)_{t \in [0, t_1]} \) is a solution to \( (6.1) \) on \( B_{2r_1}(x_1) \) which satisfies \( |h_t| < \varepsilon_0 \) everywhere. Then if \( |h_t| < \varepsilon_0 \) everywhere

\[
r_1^{-\frac{1}{2}(n+2)} \|
abla h \|_{L^2(B_{2r_1}(x_1) \times [0, r_1^2])} \leq C r_1^{-1-\frac{1}{2}(n+2)} \|
abla h \|_{L^2(B_{2r_1}(x_1) \times [0, r_1^2])} + C r_1^{-\frac{1}{2}n} \| h_0 \|_{L^2(B_{2r_1}(x_1))}.
\]

Moreover, if \( |h_t| < \varepsilon_m \) everywhere, we have for all \( (x, t) \in B_{r_1}(x_1) \times [\frac{1}{2}t_1, t_1] \)

\[
|\nabla^m h|(x, t) \leq C_m r_1^{-m-\frac{1}{2}(n+2)} \| h \|_{L^2(B_{2r_1}(x_1) \times [0, r_1^2])}.
\]

**Proof.** As for the first estimate consider a cutoff function \( \eta \in C^\infty(M) \) which is equal to 1 on \( B_{r_1}(x_1) \), vanishes outside \( B_{2r_1}(x_1) \) and satisfies \( |\nabla \eta| \leq C r_1^{-1} \). We use \( \partial_t h_t + \nabla \cdot \nabla h_t = R_t + \nabla^* S_t \) and \( |R_t| \leq C |\nabla h_t|^2 \), \(|S_t| \leq C |h_t| |\nabla h_t| \) to carry out the following computation (integration will always be over \( B_{2r_1}(x_1) \) and \( |h_t| < \varepsilon_0 \) is assumed to be sufficiently small)

\[
\frac{1}{2} \partial_t \int \eta^2 |h_t|^2 + \int \eta^2 |\nabla h_t|^2
\]
\[
\leq \int \eta^2 |R_t||h_t| + \int \eta^2 |S_t||\nabla h_t| + \int \eta |\nabla \eta||h_t||\nabla h_t| + \int \eta |\nabla \eta||S_t||h_t|
\]
\[
\leq \int \eta^2 |R_t||h_t| + \int \eta^2 |S_t|^2 + \frac{1}{2} \int \eta^2 |\nabla h_t|^2 + \int |\nabla \eta|^2 |h_t|^2
\]
\[
+ \frac{1}{4} \int \eta^2 |\nabla h_t|^2 + \frac{1}{2} \int \eta^2 |S_t|^2 + \frac{1}{2} \int |\nabla \eta|^2 |h_t|^2
\]
\[
\leq \frac{3}{4} \int \eta^2 |\nabla h_t|^2 + 2 \int |\nabla \eta|^2 |h_t|^2
\]

Hence

\[
\int_0^{t_1} \int_{B_{2r_1}(x_1)} \eta^2 |\nabla h_t|^2 \leq C r_1^{-2} \int_0^{t_1} \int_{B_{2r_1}(x_1)} |h_t|^2 + 2 \int_{B_{2r_1}(x_1)} \eta^2 |h_0|^2.
\]

This establishes the first inequality.

In order to prove the second inequality, we first choose constants \( \rho_m = 1 + 2^{-m} \) and \( \tau_m = \frac{1}{2} - 2^{-m-1} \) for \( m \geq 0 \). Observe that \( 1 \leq \rho_m \leq 2 \) decreases and \( 0 \leq \tau_m < \frac{1}{2} \) increases with \( m \). For the following fix \( m \geq 0 \) and consider a cutoff function \( \eta \in C^\infty(M) \) which is equal to 1 on \( B_{\rho_m+1}(x_1) \) vanishes outside \( B_{\rho_m r_1}(x_1) \) and satisfies \( |\nabla \eta| < C_m r_1^{-1} \). If we differentiate \( (6.1) \) \( m \) times, we obtain

\[
\partial_t (\nabla^m h_t) + \nabla^* \nabla (\nabla^m h_t) = [\nabla^* \nabla, \nabla^m] h_t + Q_t^{(m)}
\]
where

\[ Q_t^{(m)} = \sum_{i_1 + \ldots + i_k = m+2, \; t_i \geq 0, \; t_2, \ldots, t_{k-1} \geq 1} \nabla^{i_1} h_t \ast \ldots \ast \nabla^{i_k} h_t. \]

Now observe that by Corollary 2.3 and the bound \(|h_t| < \varepsilon_m\), we have \(|\nabla^i h| \leq C_i \varepsilon_m r_1^{-i}\) for all \(i \leq m + 2\) on \(M \times [\tau_m, t_1]\) if \(\varepsilon_m\) is sufficiently small. Hence, we can bound the two extra terms as follows

\[ |Q_t^{(m)}| \leq C_m \sum_{i=0}^{m+1} r_1^{-m-2+i} |\nabla^i h_t|, \]

\[ |[\nabla^s \nabla, \nabla^m] h_t| \leq C_m \sum_{i=0}^{m} |\nabla^i h_t| \leq C_m \sum_{i=0}^{m} r_1^{-m-2+i} |\nabla^i h_t|. \]

Hence similarly as before

\( \frac{1}{2} \partial_t \int \eta^2 |\nabla^m h_t|^2 + \int \eta^2 |\nabla^{m+1} h_t|^2 \)

\[ \leq \int \eta^2 (Q_t^{(m)} \|\nabla^m h_t\| + \int \eta^2 \|[\nabla^s \nabla, \nabla^m] h_t\| |\nabla^m h_t| + \int \eta \|\nabla \eta\| |\nabla^{m+1} h_t| |\nabla^m h_t| \]

\[ \leq C_m \sum_{i=0}^{m} \int \eta^2 |\nabla^i h_t| |\nabla^m h_t| + C m r_1^{-1} \int \eta \|\nabla^{m+1} h_t| |\nabla^m h_t| \]

\[ \leq C_m \sum_{i=0}^{m} \int_{B_{p_m r_1}(x_1)} |\nabla^i h_t|^2 + \frac{1}{2} \int \eta^2 |\nabla^{m+1} h_t|^2. \]

So

\[ \partial_t \int \eta^2 |\nabla^m h_t|^2 + \int \eta^2 |\nabla^{m+1} h_t|^2 \leq C_m \sum_{i=0}^{m} \int_{B_{p_m r_1}(x_1)} |\nabla^i h_t|^2. \]

We now multiply this inequality by \(t/t - \tau_{m+\frac{1}{2}}\) and integrate it first from \(\tau_{m+\frac{1}{2}}\) to some \(t' \in [\tau_{m+\frac{1}{2}}, t_1]\) and then from \(\tau_{m+\frac{1}{2}}\) to \(t_1\) to find

\[ \|\nabla^m h_{t'}\|_{L^2(B_{p_m+1 r_1}(x_1))} \leq C_m \sum_{i=0}^{m} r_1^{-m-1+i} \|\nabla^i h\|_{L^2(B_{p_m+1 r_1}(x_1) \times [\tau_1, t_1])}, \]

\[ \|\nabla^{m+1} h\|_{L^2(B_{p_m+1 r_1}(x_1) \times [\tau_{m+1} t_1])} \leq C_m \sum_{i=0}^{m} r_1^{-m-1+i} \|\nabla^i h\|_{L^2(B_{p_m+1 r_1}(x_1) \times [\tau_1, t_1])}. \]

Hence, by induction

\[ \|\nabla^m h_{t'}\|_{L^2(B_{p_m+1 r_1}(x_1))} \leq C m r_1^{-m-1} \|h\|_{L^2(B_{2 r_1}(x_1) \times [0, t_1])} \]

and Sobolev embedding for large \(m\) yields the desired result. □

In the following let \(\sigma_0 = \sigma_{s.e.}\), where \(\sigma_{s.e.}\) is the constant from Proposition 2.4.
Lemma 6.2. There are constants $A_m < \infty$ and $\varepsilon_m > 0$ such that for all $a \geq 0$, $b_1, b_2 \geq 0$, $\sigma \leq \sigma_0$, $m \geq 0$ and $q \geq 2$ we have: Let $(h_t)_{\in [0, \sigma^2]}$ be a solution to (6.7) and assume that $|h_0| < \varepsilon_m$ and $h_0 = h_0^1 + h_0^2$ with $|h_0^1|, |h_0^2| < \varepsilon_m$ and
\[
|h_0^1| \leq \frac{b_1}{r + 1 + a}, \quad \left( \int_M |h_0^2|^{1/|q|} \right)^{1/q} \leq b_2.
\]
Then, there are continuous families $(h_t^1)_{\in [0, \sigma^2]}, (h_t^2)_{t \in [0, \sigma^2]}$ with $h_t = h_t^1 + h_t^2$ such that for $m = 0$ and all $t \in [0, \sigma^2]$ or $m \geq 1$ and all $t \in \left[ \frac{1}{2} \sigma_0^2, \sigma^2 \right]$ we have $|\nabla^m h_t^1|, |\nabla^m h_t^2| < A_m \varepsilon_m$ and
\[
|\nabla^m h_t^1| \leq \frac{A_m b_1}{r + 1 + a}, \quad \left( \int_M |\nabla^m h_t^2|^{1/|q|} \right)^{1/q} \leq A_m b_2.
\]
Moreover, for all $t \in \left[ \frac{1}{2} \sigma_0^2, \sigma^2 \right]$ we have $|\nabla^m h_t^1| \leq A_m b_2$.

Proof. Let $B_1, B_2$ be positive numbers which will be determined later and assume first that all $\varepsilon_m$ are bounded by some constant $\varepsilon_0 > 0$. By Proposition 2.4, we have $|h_t| < C \varepsilon_0$ on $M \times [0, \sigma^2]$ for some $C$. Hence, there is some number $w \leq C \varepsilon_0$ such that
\[
|h_t| \leq u + v + w \quad \text{ (6.3)}
\]
where $u, v \in C^\infty(M)$ are nonnegative scalar functions with
\[
u = \frac{B_1 b_1}{r + 1 + a}, \quad ||v||_{L^q(M)} \leq B_2 b_2 \quad \text{ (6.4)}
\]
Imagine $w$ to be close to the infimum with this property. In the following we will show that we can rechoose $v$ such that (6.3) even holds for $\frac{1}{2} w$. Hence, by induction it holds for $w = 0$.

Consider some $0 < r_1 \leq \sigma$, set $t_1 = r_1^2$ and for any $x \in M$ \[
H_{r_1}(x) = r_1^{-\frac{1}{2}(n+2)} ||h||_{L^2(B_{r_1}(x) \times [0,t_1])} + r_1^{-\frac{1}{2}n} ||h_0||_{L^2(B_{r_1}(x))} \leq C u(x) + C w + 2 r_1^{-\frac{n}{2}} ||v||_{L^2(B_{r_1}(x))}.
\]
Observe, that also $H_{r_1}(x) \leq C \varepsilon_0$. By Lemma 6.1 we have the following estimates (for $(x', t') \in B_{r_1}(x) \times [\frac{1}{2} t_1, t_1]$ and $m$ not too large)
\[
r_1^{-\frac{1}{2}(n+2)} ||\nabla h||_{L^2(B_{r_1}(x) \times [0,t_1])} \leq C r_1^{-1 - \frac{1}{2}(n+2)} ||h||_{L^2(B_{r_1}(x) \times [0,t_1])} \leq C r_1^{-1} H_{r_1}(x),
\]
\[
|\nabla^m h|(x', t') \leq C m r_1^{-m - \frac{1}{2}(n+2)} ||h||_{L^2(B_{r_1}(x) \times [0,t_1])} \leq C m r_1^{-m} H_{r_1}(x).
\]
From this, we obtain estimates on $R = \nabla h \ast \nabla h$ (again for $(x', t') \in B_{r_1}(x) \times [\frac{1}{2} t_1, t_1]$)
\[
r_1^{-1}(n+2) ||R||_{L^1(B_{r_1}(x) \times [0,t_1])} \leq C r_1^{-2} H_{r_1}^2(x) \leq C r_1^{-2} \varepsilon_0 H_{r_1}(x),
\]
\[
|R|(x', t') \leq C r_1^{-2} H_{r_1}^2(x) \leq C r_1^{-2} \varepsilon_0 H_{r_1}(x).
\]
as well as on $S = h \ast \nabla h$

$$r_1^{-(n+2)} \|S\|_{L^1(B_{r_1}(x) \times [0,t_1])} \leq r_1^{-\frac{1}{2}(n+2)} \|S\|_{L^2(B_{r_1}(x) \times [0,t_1])} \leq Cr_1^{-1} \epsilon_0 H_{r_1}(x).$$

$|S|(x', t') \leq Cr_1^{-1} \epsilon_0 H_{r_1}(x)$.

We now use (6.2) for $x_1 \in M$ and $t_0 = 0$

$$h(x_1, t_1) = \int_M k_{t_1}(x_1, x) h(x, 0) dx$$

$$+ \int_0^{t_1} \int_M (k_{t_1}(x, t) R(x, t) + \nabla k_{t_1}(x, t) S(x, t)) dx dt.$$

We can bound the first integral $\int_M$ using the fact that $|k_{t_1}(x, x)| \leq C \Phi_{r_1}(x, x)$ where $\Phi_{r_1}(x, x) = r_1^{-n} \exp(-\frac{1}{8} r_1^{-2} d^2(x, x))$ (see Proposition 2.5)

$$|\int_M| \leq \int_M \frac{C b_1 \Phi_{r_1}(x_1, x) dx}{r(x) + 1 + a} + C \int_M \Phi_{r_1}(x_1, x) h_0^2(x) dx$$

$$\leq \frac{C b_1}{r(x_1) + 1 + a} + C \int_M \Phi_{r_1}(x_1, x) h_0^2(x) dx.$$

As for the second integral, we split the domain of integration $M \times [0, t_1]$ into two parts: $\Omega = B_{r_1}(x_1) \times [\frac{1}{2} t_1, t_1]$ and its complement. For the integral over $\Omega$, we use the pointwise bounds on $R$ and $S$ as well as the fact that by Proposition 2.5

$$\int_{B_{r_1}(x_1) \times [0, \frac{1}{2} t_1]} |k_{t_1}(x_1, x) dx dt \leq Cr_1^2,$$

$$\int_{B_{r_1}(x_1) \times [0, \frac{1}{2} t_1]} |\nabla k_{t_1}(x_1, x) dx dt \leq Cr_1$$

to conclude

$$|\int_{\Omega}| \leq C \epsilon_0 H_{r_1}(x_1) \leq C \epsilon_0 u(x_1) + C \epsilon_0 w + C \epsilon_0 r_1^{-\frac{n}{2}} \|v\|_{L^q(B_{2r_1}(x_1))}.$$

On $M \times [0, t_1] \setminus \Omega$, we use the fact that by Proposition 2.5 we have the bounds $|k_{t_1-t}(x_1, x)| < C \Phi_{r_1}(x_1, x)$ and $|\nabla k_{t_1-t}(x_1, x)| < C r_1^{-1} \Phi_{r_1}(x_1, x)$ to conclude

$$|\int_{M \times [0, t_1] \setminus \Omega}| \leq C \epsilon_0 \int_M \Phi_{r_1}(x_1, x) H_{r_1}(x) dx$$

$$\leq C \epsilon_0 u(x_1) + C \epsilon_0 w + C \epsilon_0 r_1^{-\frac{n}{2}} \int_M \Phi_{r_1}(x_1, x) \|v\|_{L^q(B_{2r_1}(x))} dx.$$

Hence for some $C_1$

$$|h|(x_1, t_1) \leq C_1 (1 + \epsilon_0 B_1) b_1 \frac{r_1}{r(x_1) + 1 + a} + C_1 \epsilon_0 w + \tilde{v}_{r_1}(x_1).$$

where using $\tilde{v}_{r_1}(x) = r_1^{-\frac{n}{2}} \|v\|_{L^q(B_{2r_1}(x))}$

$$\tilde{v}_{r_1}(x_1) = C \int_M \Phi_{r_1}(x_1, x) \|h_0^2(x) + \epsilon_0 \tilde{v}_{r_1}(x)\| dx + C \epsilon_0 \tilde{v}_{r_1}(x_1).$$
Set \( \tilde{v} = \sup_{0<r_1<\sigma} \tilde{v}_{r_1} \). Denote by \( \mathcal{M}_\sigma \) the Hardy-Littlewood maximal operator up to scale \( \sigma \), i.e. for any nonnegative function \( f \in C^\infty(M) \), we set
\[
(\mathcal{M}_\sigma f)(x_1) = \sup_{0<r_1<\sigma} \frac{1}{\text{vol } B_{r_1}(x_1)} \int_{B_{r_1}(x_1)} f(x) \, dx.
\]
Observe that for every \( 0 < r_1 \leq \sigma \)
\[
\int_M \Phi_{r_1}(x_1, x) f(x) \, dx = \int_{M \setminus B_{r_1}(x_1)} \Phi_{r_1}(x_1, x) f(x) \, dx + \int_{B_{r_1}(x_1)} \Phi_{r_1}(x_1, x) f(x) \, dx
- \int_0^\sigma \frac{d}{dr'} \left( r_1^{-n} e^{-\frac{(r')^2}{4t'}} \right) \left( \int_{B_{r'}(x_1)} f(x) \, dx \right) \, dr'
\leq C \int_M \Phi_\sigma(x_1, x) f(x) \, dx - (\mathcal{M}_\sigma f)(x_1) \int_0^\sigma \frac{d}{dr'} \left( r_1^{-n} e^{-\frac{(r')^2}{4t'}} \right) \left( \int_{B_{r'}(x_1)} f(x) \, dx \right) \, dr'
\leq C \int_M \Phi_\sigma(x_1, x) f(x) \, dx + C(\mathcal{M}_\sigma f)(x_1).
\]
Hence
\[
\tilde{v}(x_1) \leq (\mathcal{M}_\sigma(|h_0^2| + \varepsilon_0 \hat{v}_{r_1}))(x_1)
+ C \int_M \Phi_\sigma(x_1, x) (|h_0^2|(x) + \varepsilon_0 v(x)) \, dx + C\varepsilon_0 \hat{v}_{r_1}(x_1).
\]
By the Hardy-Littlewood maximal inequality (cf \( [SW] \) and Young’s inequality there is some \( C_2 \) (which is independent of \( q \)) such that
\[
\| \tilde{v} \|_{L^q(M)} \leq C \| h_0^2 \|_{L^q(M)} + C\varepsilon_0 \| \hat{v}_{r_1} \|_{L^q(M)}
\leq C_2 \| h_0^2 \|_{L^q(M)} + C_2 \varepsilon_0 \| v \|_{L^q(M)} \leq C_2 (1 + \varepsilon_0 B_2) b_2.
\]
Now choose \( B_1 = 2C_1 \) and \( B_2 = 2C_2 \). This allows us to choose \( \varepsilon_0 \) small enough such that \( C_1 (1 + \varepsilon_0 B_1) \leq B_1, C_2 (1 + \varepsilon_0 B_2) \leq B_2 \) and \( \varepsilon_0 < \frac{1}{2} \). We find that \( |h_t| \leq u + \tilde{v} + \hat{w} \) with \( \tilde{w} = \frac{1}{2} w \) and \( \| \tilde{v} \|_{L^q(M)} \leq B_2 b_2 \). Iterating this argument shows that we can find \( u, v \in C^\infty(M) \) satisfying (6.3) and (6.3) for \( u = 0 \).

Now consider such \( u \) and \( v \) and recall that for every \( x \in M \) and \( (x', t') \in B_{\sigma_0}(x) \times [\frac{1}{2} \sigma_0^2, \sigma^2] \) we have
\[
|\nabla^m h|(x', t') \leq C_m \sigma_0^{-m} H_{\sigma_0}(x) \leq C_m \sigma_0^{-m} (u(x) + \| v \|_{L^q(B_{2\sigma_0}(x))})
\]
and recall that by Corollary 2.3 we have \( |\nabla^m h_t| < C_m \varepsilon_m \) for \( t \in [\frac{1}{2} \sigma_0^2, \sigma^2] \). So on \( B_{\sigma_0}(x) \times [\frac{1}{2} \sigma_0^2, \sigma^2] \), we can find a splitting \( h_t = h_t^1 + h_t^2 \) such that \( |\nabla^m h_t^1|, |\nabla^m h_t^2| < C_m \varepsilon_m \) and
\[
|\nabla^m h_t^1| \leq \frac{C_m b_1}{r + a + 1}, \quad |\nabla^m h_t^2| \leq C_m \| v \|_{L^q(B_{2\sigma_0}(x))} \leq C_m B_2 b_2.
\]
Using a suitable partition of unity, we can glue those splittings together. It is clear that we can extend this splitting to the time interval \([0, \sigma^2]\) such that the zero order bounds hold on \([0, \frac{1}{2} \sigma_0^2] \). \( \square \)
6.4. Hyperbolic geometry and bounds on the linear equation. We need the following elementary observation.

**Lemma 6.3.** Let $M = \mathbb{H}^n$ or $\mathbb{CH}^{2n}$. There are constants $C < \infty$ and $\mu > 0$ such that:

Consider two distinct points $x_0, x_1 \in M$ and let $r_0 > 0$, $0 < \alpha < \frac{\pi}{2}$. Let $v \in T_{x_1}M$ be the vector pointing towards $x_0$ and define the sector

$$S_{v,\alpha} = \{ \exp_{x_1}(u) : u \in T_{x_1}M, \angle_{x_1}(u, v) \leq \alpha \}.$$

Then for $d = d(x_0, x_1) - r_0$ we have

$$\text{vol} \left( B_{r_0}(x_0) \setminus S_{v,\alpha} \right) \leq C e^{-\mu d} \alpha^{-2(n-1)}.$$

**Proof.** By rescaling we can assume that the sectional curvatures are $\leq -1$. We will then show the volume estimate for $\mu = n - 1$.

Choose $a$ such that $\text{sh} a = e^{-d}(1 - \cos \alpha)^{-1} \leq C e^{-d} \alpha^{-2}$. In the following, we will show that

$$B_{r_0}(x_0) \setminus S_{v,\alpha} \subset B_a(x_1).$$

Since $\text{vol} B_a(x_1) \leq C(\text{sh} a)^{n-1}$, this will give us the desired estimate.

Consider a point $x' \in B_{r_0}(x_0) \setminus S_{v,\alpha}$. Let $a' = d(x_1, x')$, $r_0' = d(x_0, x')$, $u \in T_{x_1}M$ such that $\exp_{x_1}(u) = x'$ and $\alpha' = \angle_{x_1}(u, v) > \alpha$. By the triangle inequality we have $a' \geq d$. Consider a comparison triangle $\triangle x_0 x_1 x'$ in $\mathbb{H}^2$ and let $\alpha'$ be the angle at $x_1$. By triangle comparison, we have $\alpha' \geq \alpha' \geq \alpha$ and hence by the law of cosines in $\mathbb{H}^2$

$$\text{ch} r_0' \geq \text{ch}(r_0 + d) \text{ch} a' - \text{sh}(r_0 + d) \text{sh} a' \cos \alpha.$$

Moreover, since

$$\text{ch} r_0' \leq \text{ch} r_0 = \text{ch}(r_0 + d) \text{ch} d - \text{sh}(r_0 + d) \text{sh} d,$$

we conclude

$$\text{th}(r_0 + d) \left( \text{sh} a' \cos \alpha - \text{sh} d \right) \geq \text{ch} a' - \text{ch} d.$$
Observe that since \( a' \geq d \), either the right hand side is positive or \( d < 0 \) and hence the left hand side is positive. So
\[
\text{sh} a' \cos \alpha - \text{sh} d \geq \text{ch} a' - \text{ch} d.
\]
This implies
\[
\text{sh} a'(1 - \cos \alpha) \leq e^{-d}
\]
and hence \( \text{sh} a' \leq \text{sh} a \) which establishes the claim. \( \square \)

**Lemma 6.4.** Let \( M = \mathbb{H}^n \), \( n \geq 3 \) or \( \mathbb{C}\mathbb{H}^{2n} \), \( n \geq 2 \) choose a basepoint \( x_0 \in M \) and consider the radial distance function \( r = d(\cdot, x_0) \). For every \( w > 0 \) there is a constant \( C = C(w) < \infty \) such that:
Assume that \( h \in C^\infty(M; \text{Sym}_2 T^*M) \) and that
\[
|h|(x) < \frac{1}{(r(x) + 1 + a)^w}
\]
for some \( a \geq 0 \). Then for all \( x_1 \in M \) and \( r_1 = r(x_1) \) and \( t \geq 0 \)
\[
\int_M |k_1|(x_1, x)|h|(x)dx < \frac{C}{(r_1 + 1 + a + t)^w}.
\]

**Proof.** For small times \( t \leq 1 \), the estimate follows with the help of Proposition 2.5. So assume that \( t > 1 \).
Recall \( \lambda_B > 0 \) from subsection 4.1. If \( r_1 + \frac{\lambda_B}{\mu}t \leq 1 + a \), then we find by the \( L^1 \)-boundedness of \( k_1 \) (cf. Theorem 4.1 and Proposition 5.1)
\[
\int_M |k_1|(x_1, x)|h|(x)dx \leq \frac{C}{(1 + a)^w} \leq \frac{C'}{(r_1 + 1 + a + t)^w}.
\]
Assume from now on \( r_1 + \frac{\lambda_B}{\mu}t > 1 + a \) and hence \( r_2 := \frac{1}{2}r_1 - \frac{1}{2}(1 + a) + \frac{\lambda_B}{2\mu}t > 0 \).
We can then bound
\[
\int_{M \setminus B_{r_2}(x_0)} |k_1|(x_1, x)|h|(x)dx \leq \frac{C}{(r_2 + 1 + a)^w} \leq \frac{C'}{(r_1 + 1 + a + t)^w}
\]
and hence, it remains to bound the integral on \( B_{r_2}(x_0) \). Set \( \alpha = \exp(-\frac{\mu}{8(n-1)}r_1 - \frac{\lambda_B}{4(n-1)}t) \).
Let \( v \in T_{x_0}M \) be the vector which points in the direction of \( x_0 \) and consider the sector \( S_{v, \alpha} \).
By Lemma 6.3 we have
\[
\text{vol}(B_{r_2}(x_0) \setminus S_{v, \alpha}) \leq Ce^{\mu(r_2-r_1)}\alpha^{-(n-1)}.
\]
So by Cauchy-Schwarz and the bound \( \|k_1\|_{L^2(M)} \leq Ce^{-(n-2)t} \) (cf. (4.12) in the proof of Theorem 4.1)
\[
\int_{B_{r_2}(x_0) \setminus S_{v, \alpha}} |k_1|(x_1, x)|h|(x)dx \leq Ce^{\frac{\mu}{2}(r_2-r_1)}\alpha^{-(n-1)}e^{-\lambda_Bt}
\]
\[
= C \exp \left( -\frac{\mu}{8}r_1 - \frac{\mu}{4}(1 + a) - \frac{\lambda_B}{4}t \right) \leq \frac{C}{(r_1 + 1 + a + t)^w}.
\]
In order to bound the integral on the remaining part $B_2(x_0) \cap S_{v,a}$, we use the fact that $\|k_t\|_{L^1(S_{v,a})} \leq C\alpha^{n-1}$ (observe that $k_t$ is spherical and that the set of angles pointing into the sector $S_{v,a}$ at $x_1$ has measure \( \sim \alpha^{n-1} \)):

$$
\int_{B_2(x_0) \cap S_{v,a}} |k_t|(x_1, x) |h|(x) \, dx \leq C\alpha^{n-1} \frac{1}{(1+a)^w} \leq \frac{C}{(r_1 + 1 + a + t)^w}. \quad \Box
$$

### 6.5. Proof of Theorem 1.2

We will need the following linear estimates:

**Lemma 6.5.** Assume that $2 \leq q < \infty$ and let $h_0 \in C^\infty(M; \text{Sym}_2 T^* M)$ such that $\|h_0\|_{L^q(M)} < \infty$. Consider $h_t(x) = \int_M k_t(x, x') h_0(x') \, dx'$, the solution of $\partial_t h_t = -Lh_t$. Then, for $\lambda = \frac{2}{q} \lambda_B > 0$ we have

$$
\|h_t\|_{L^q(M)} \leq C e^{-\lambda t}\|h_0\|_{L^q(M)}.
$$

**Proof.** By the $L^1$-boundedness of the heat kernel, the inequality is true for $q = \infty$ with $\lambda = 0$ and by the Bochner formula (cf. (4.12)), it holds for $q = 2$ and $\lambda = \lambda_B$. Hence, by the Marcinkiewicz interpolation theorem, it holds for any $2 \leq q < \infty$ with $\lambda = \frac{2}{q} \lambda_B$. \quad \Box

**Proof of Theorem 1.2.** Observe first that the Theorem is more general for larger $q$. Hence, we can assume $q \geq 2$.

Let $\varepsilon_0 > 0$ be a small constant which we will determine in the course of the proof. As in the proof of Theorem 1.1 let $T_{\text{max}}$ be the maximum over all $T$ such that Ricci deTurck flow $h_t$ exists on $[0, T)$ and satisfies $|h_t| < \varepsilon_0$ everywhere. We will show that for sufficiently small $\varepsilon$ (independent of $T_{\text{max}}$) we even have $|h_t| < \varepsilon_0/2$ and hence $T_{\text{max}} = \infty$. Recall that by Proposition 2.1 we have $T_{\text{max}} > \sigma_0^2$.

By Lemma 6.2 (applied successively to the time intervals $[0, \sigma_0^2], [\sigma_0^2, \frac{1}{2} \sigma_0^2], ...$), we conclude that for every $t \in [0, T_{\text{max}})$ there is a splitting $h_t = h_t^1 + h_t^2$ with

$$
\|h_t\|_{L^q(M)} \leq C e^{-\frac{\lambda}{2} t}, \quad \left( \int_M |h_t^2|^q \right)^{1/q} \leq \varepsilon_2 Y t \leq e^{-\frac{\lambda}{2} t}
$$

for some $Y_t < \infty$ (here $\lambda$ is the constant from Lemma 5.5). For every $t \in [0, T_{\text{max}})$ let $Y_t$ be the infimum over all possible $Y'_t$ for all splittings $h_t = h_t^1 + h_t^2$ and set $Z_t = \max_{t' \in [0, t]} Y_{t'}$.

By Lemma 6.2 we have $Z_{\sigma_0} \leq C_0$. Applying Lemma 6.2 at positive times, we find that for any $t_1, t_2 \in [0, T_{\text{max}})$, we have $Z_{t_2} \leq C_1 Z_{t_1}$ whenever $t_2 < t_1 + \sigma_0^2$. Moreover, we conclude that for times $[\sigma_0^2, T_{\text{max}})$ we can rechoose $h_t^1$ and $h_t^2$ piecewise continuously in time such that $h_t = h_t^1 + h_t^2$ and for $m = 0, 1, 2$ we have $|\nabla^m h_t^1|, |\nabla^m h_t^2| < A \varepsilon_0$ and

$$
|\nabla^m h_t^1| \leq \frac{A \varepsilon_1 Z_t}{r + 1 + t}, \quad \sup_M |\nabla^m h_t^2| + \left( \int_M |\nabla^m h_t^2|^q \right)^{1/q} \leq A \varepsilon_2 Z_t e^{-\frac{\lambda}{2} t}.
$$

Hence, since $|Q_t| \leq C(|\nabla h_t|^2 + |h_t| |\nabla^2 h_t|)$, we find that $Q_t = Q_t^1 + Q_t^2$ with

$$
|Q_t^1| \leq \frac{C \varepsilon_1 Z_t^2}{(r + 1 + t)^2}, \quad \left( \int_M |Q_t^1|^q \right)^{1/q} \leq C \varepsilon_2 Z_t e^{-\frac{\lambda}{2} t}.
$$
Let now $t_1 \in [\sigma_0^2, T_{\text{max}})$, $x_1 \in M$ and $r_1 = r(x_1)$ and recall from (6.1) that

$$h_{t_1}(x_1) = \int_M k_{t_1 - \sigma_0^2}(x, x) \sigma_0^2(x) dx + \int_{\sigma_0^2}^{t_1} \int_M k_{t_1 - t}(x_1, x) Q_t(x) dx dt$$

Hence, $h_{t_1} = \hat{h}_{t_1}^1 + \hat{h}_{t_1}^2$, where ($i = 1, 2$)

$$\hat{h}_{t_1}^i(x_1) = \int_M k_{t_1 - \sigma_0^2}(x_1, x) h_{\sigma_0^2}^i(x) dx + \int_{\sigma_0^2}^{t_1} \int_M k_{t_1 - t}(x_1, x) Q_t^i(x) dx dt.$$

We will estimate $\hat{h}_{t_1}^1(x_1)$ and $\hat{h}_{t_1}^2(x_1)$.

Observe first that by Lemma 6.4 for $w = 1$

$$\left| \int_M k_{t_1 - \sigma_0^2}(x_1, x) h_{\sigma_0^2}^1(x) dx \right| \leq \frac{C\varepsilon_1}{r_1 + 1 + t_1}$$

and for $w = 2$ and $t \in [\sigma_0^2, t_1]$

$$\left| \int_M k_{t_1 - t}(x_1, x) Q_t^1(x) dx \right| \leq \frac{C\varepsilon_2 Z_t^2}{(r_1 + 1 + t_1)^2} \leq \frac{C\varepsilon_2 Z_t^2}{t_1(r_1 + 1 + t_1)}.$$

Hence

$$|\hat{h}_{t_1}^1(x_1)| \leq \frac{C_2(\varepsilon_1 + \varepsilon_2 Z_t^2)}{r_1 + 1 + t_1}.$$

Secondly, by Lemma 6.5 we find

$$\left( \int_M |\hat{h}_{t_1}^2|^q \right)^{1/q} \leq C\varepsilon_2 e^{-\lambda t_1} + C_3\varepsilon_2 Z_t \int_{\sigma_0^2}^{t_1} e^{-\lambda(t_1 - t)} e^{-\frac{\lambda}{z_t} dt}$$

$$\leq (C_3\varepsilon_2 + C_4(q)\varepsilon_2 Z_t) e^{-\frac{\lambda}{z_t} t_1}.$$ 

Here $C_4(q)$ depends on $\lambda$ and hence on $q$. By the minimality of $Y_{t_1}$ we conclude

$$Y_{t_1} \leq \max\{C_2(1 + \varepsilon_1 Z_{t_1}^2), C_3 + C_4(q)\varepsilon_2 Z_{t_1} \}.$$ 

Let $C_5 = \max\{C_0, C_2, C_3\}$ and observe that

$$Z_{\sigma_0^2} \leq C_5 \quad \text{and} \quad Z_t \leq \max\{C_5(1 + \varepsilon_1 Z_{t_1}^2), C_5 + C_4(q)\varepsilon_2 Z_t \}.$$ 

Now set $\varepsilon_1 = (2C_1C_5)^{-2}$ and $\varepsilon_2(q) = (2C_1C_4(q))^{-1}$. If $Z_t \leq 2C_5$ did not hold for all $t \in [\sigma_0^2, T_{\text{max}})$, then there must be a jump, i.e. two times $t_1 < t_2$ with $t_2 - t_1 < \sigma_0^2$ such that $Z_{t_1} \leq 2C_5$, but $Z_{t_2} > 2C_5$. By the fact that $Z_{t_2} \leq C_1 Z_{t_1} \leq 2C_1C_5$, we find

$$2C_5 < Z_{t_2} \leq \max\{C_5(1 + \varepsilon_1(2C_1C_5)^2), C_5 + C_4(q)\varepsilon_2(2C_1C_5) \} = 2C_5,$$

a contradiction. Hence, we have $Z_t \leq 2C_5$ for all $t \in [\sigma_0^2, T_{\text{max}})$ and the claim follows. \qed
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