SHORT INTERVAL RESULTS FOR A CLASS OF ARITHMETIC FUNCTIONS

OLIVIER BORDELLÈS

ABSTRACT. Using estimates on Hooley’s Δ-function and a short interval version of the celebrated Dirichlet hyperbola principle, we derive an asymptotic formula for a class of arithmetic functions over short segments. Numerous examples are also given.

1. Introduction and result

Studying the behaviour of arithmetic functions in short intervals is a quite long-standing problem in number theory. By 'short intervals' we mean the study of sums of the shape

\[ \sum_{x<n\leq x+y} F(n) \]

where \( y = o(x) \) as \( x \to \infty \).

One of the first results to be published dealing with this problem was Ramachandra’s benchmarking paper [11] revisited by Kátai & Subbarao [8]. More recently, Cui & Wu [3] derived a short interval version of the Selberg-Delange method, developed by Selberg and Delange between 1954 and 1971 in order to provide a quite general theorem giving the right order of magnitude of the usual arithmetic functions.

For multiplicative functions \( F \) such that \( F(p) \) is close to 1 for every prime \( p \), another method was developed in [2] using profound theorems from Filaseta-Trifonov and Huxley-Sargos results on integer points near certain smooth curves. This leads to very precise estimates of a large class of multiplicative functions.

In this work, we derive asymptotic results for short sums of arithmetic functions \( F \) such that there exist \( s \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{\geq 1} \) and real numbers \( \kappa \in [0,1), \beta \geq 0, \delta \geq 0 \) and \( A > 0 \) such that

\[ (1) \quad \sum_{n\leq z} (F \ast \mu)(n) = z \sum_{j=0}^{s} a_j (\log z)^j + O\left(z^{\kappa}(\log ez)^\beta\right) \quad (z \geq 1, \ a_j \in \mathbb{C}, \ a_s \neq 0) \]

\[ (2) \quad |(F \ast \mu)(n)| \leq A \tau_m(n) (\log en)^\delta \quad (n \in \mathbb{Z}_{\geq 1}) \]

where, as usual, \( \mu \) is the Möbius function, \( \tau_m \) is the \( m \)th Dirichlet-Piltz divisor function and \( f \ast g \) is the Dirichlet convolution product of the arithmetic functions.

2010 Mathematics Subject Classification. 11A25, 11N37, 11L07.

Key words and phrases. Short sums, Dirichlet hyperbola principle.
Let \( e^e < y \leq x \) be real numbers and \( F \) be an arithmetic function satisfying (1) and (2). Then, for any \( \varepsilon \in (0, \frac{1}{2}] \) and \( x^{1/2} e^{-\frac{1}{2} (\log x)^{1/4}} \leq y \leq x e^{-(\log x)^{1/4}} \)

\[
\sum_{x < n \leq x+y} F(n) = \frac{ya_s}{s+1} (\log x)^{s+1} + O\left(y^\kappa e^{(\kappa-1)(\log x)^{1/4}} (\log x)^\beta \right) \\
+ y(\log x)^{\max\left(s,\delta+m-\frac{1}{2}+\varepsilon_{m+1}(x)\right)} + x^\varepsilon.
\]

The implied constant depends only on \( A, \varepsilon, m, s, a_0, \ldots, a_s \) and on the implied constant arising in (1). For any integer \( r \geq 2 \), the function \( \epsilon_r(x) = o(1) \) as \( x \to \infty \) is given in (3) below. It should be mentioned that if the error term in (1) is of the form \( O(z^{\kappa+\varepsilon}) \) instead of \( O\left(z^\kappa (\log ez)^\beta\right) \), then the proof shows that the term \( xy^{\kappa-1} e^{(\kappa-1)(\log x)^{1/4}} (\log x)^\beta \)

has to be replaced by

\[
xy^{\kappa-1+\varepsilon} e^{(\kappa-1+\varepsilon)(\log x)^{1/4}}
\]

the other terms remaining unchanged.

2. NOTATION

In what follows, \( e^e < y \leq x \) are large real numbers, \( N \geq 1 \) is a large integer and \( \varepsilon > 0 \) is an arbitrary small real number which does not need to be the same at each occurrence.

For any \( t \in \mathbb{R} \), \( \lfloor t \rfloor \) is the integer part of \( t \), \( \psi(t) = t - \lfloor t \rfloor - \frac{1}{2} \) is the first Bernoulli function, \( e(t) = e^{2\pi it} \) and \( \|t\| \) is the distance from \( t \in \mathbb{R} \) to its nearest integer.

For any integer \( r \geq 2 \) and any real number \( x > e^e \), we set

\[
\epsilon_r(x) = \sqrt{r \log \log \log x \left( r - 1 + \frac{30}{\log \log \log x} \right)}.
\]

Besides the arithmetic functions already listed above, let \( \Delta_r \) be the \( r \)th Hooley’s divisor function defined by

\[
\Delta_r(n) = \max_{u_1, \ldots, u_{r-1} \in \mathbb{R}} \sum_{d_1 \cdots d_{r-1} | n, e^{u_1} < d_i \leq e^{u_i+1}} 1
\]

where \( r \geq 2 \) is a fixed integer.

Finally, let \( F \) be any arithmetic function and let \( f = F \ast \mu \) be the Eratosthenes transform of \( F \). We set

\[
\Sigma_F(N, x, y) := \sum_{N < n < 2N} f(n) \left( \psi \left( \frac{x+y}{n} \right) - \psi \left( \frac{x}{n} \right) \right) \quad (y < N \leq x).
\]
3. First technical tools

Lemma 2. Let $F$ be any arithmetic function satisfying (1), and $f = F \ast \mu$ be the Eratosthenes transform of $F$. Then, for any real numbers $1 \leq t \leq z$

$$
\sum_{z < n \leq z+t} f(n) = ta_s (\log z)^s + O \left\{ t (\log ez)^{s-1} + z^s (\log ez) \beta \right\}.
$$

Proof. Follows from (1) and usual arguments. We leave the details to the reader. \hfill \Box

Lemma 3. Let $F$ be any arithmetic function and $f = F \ast \mu$ be the Eratosthenes transform of $F$. Then, for any integers $N \in (y, x]$ and $H \geq 1$

$$
\Sigma_F(N, x, y) = - \int_x^{x+y} \sum_{0 < |h| \leq H} \sum_{N < n \leq 2N} \frac{f(n)}{n} e \left( \frac{ht}{n} \right) \frac{1}{2\pi i h} + O \left( \max_{x \leq z \leq x+y} \sum_{N < n \leq 2N} |f(n)| \min \left( 1, \frac{1}{H \|z/n\|} \right) \right)
$$

where $\Sigma_F(N, x, y)$ is given in (4).

Proof. We use the representation

$$
\psi(t) = - \sum_{0 < |h| \leq H} \frac{e(ht)}{2\pi i h} + O \left\{ \min \left( 1, \frac{1}{H \|t\|} \right) \right\} \quad (H \in \mathbb{Z}_{\geq 1})
$$

to get

$$
\sum_{N < n \leq 2N} f(n) \psi \left( \frac{x}{n} \right) = - \sum_{0 < |h| \leq H} \sum_{N < n \leq 2N} \frac{1}{2\pi i h} f(n) e \left( \frac{hx}{n} \right)
$$

$$
+ O \left( \sum_{N < n \leq 2N} |f(n)| \min \left( 1, \frac{1}{H \|x/n\|} \right) \right)
$$

and the result follows using the identity

$$
e(a(x + y)) - e(ax) = 2\pi i a \int_x^{x+y} e(at) dt.
$$

The proof is complete. \hfill \Box

4. Dirichlet’s hyperbola principle in short intervals

The Dirichlet hyperbola principle was first developed by Dirichlet circa 1840 which enabled him to prove the first non-trivial asymptotic result for the so-called Dirichlet divisor problem. It was later generalized in order to deal with long sums of convolution products of the shape $f \ast g$ and usually provides good estimates when $f(n)$ or $g(n)$ is $\ll n^\epsilon$.

The next result is a short sum version of this very useful principle.
Let $x, y, T \in \mathbb{R}$ such that $1 \leq \max \left( y, \frac{y}{x} \right) \leq T \leq x$. Then, for any arithmetic functions $f$ and $g$

$$\sum_{x < n \leq x + y} (f \ast g)(n) = \sum_{d \leq T} f(d) \sum_{\frac{x}{d} < k \leq \frac{x+y}{d}} g(k) \sum_{z \leq x/T} g(z) \sum_{\frac{z}{d} < \frac{x+y}{d}} f(d) + O \left( \max_{k \leq 2x/T} \left| g(k) \right| \sum_{T < d \leq T(1+y/x)} |f(d)| \right).$$

**Proof.** Let $1 \leq \max \left( y, \frac{y}{x} \right) \leq T \leq x$. Then

$$\sum_{x < n \leq x + y} (f \ast g)(n) = \sum_{d \leq x+y} f(d) \sum_{\frac{x}{d} < k \leq \frac{x+y}{d}} g(k) \sum_{T < d \leq x+y} f(d) \sum_{\frac{y}{d} < \frac{x+y}{d}} g(k) = \sum_{d \leq T} f(d) \sum_{\frac{x}{d} < k \leq \frac{x+y}{d}} g(k) + S_1(T)$$

where

$$S_1(T) = \sum_{T < d \leq x+y} f(d) \sum_{\frac{x}{d} < k \leq \frac{x+y}{d}} g(k) - \sum_{T < d \leq x+y} f(d) \sum_{k \leq \frac{x}{d}} g(k) = \sum_{k \leq \frac{x+y}{d}} g(k) \sum_{T < d \leq \frac{x+y}{d}} f(d) - \sum_{k \leq \frac{x}{d}} g(k) \sum_{T < d \leq \frac{x}{d}} f(d) = \sum_{k \leq \frac{x+y}{d}} g(k) \sum_{T < d \leq \frac{x+y}{d}} f(d) = \sum_{k \leq \frac{x+y}{d}} g(k) \sum_{T < d \leq \frac{x+y}{d}} f(d) = \sum_{k \leq \frac{x+y}{d}} g(k) \sum_{T < d \leq \frac{x+y}{d}} f(d) = \sum_{k \leq \frac{x+y}{d}} g(k) \sum_{T < d \leq \frac{x+y}{d}} f(d) = \sum_{k \leq \frac{x+y}{d}} g(k) \sum_{T < d \leq \frac{x+y}{d}} f(d) + S_2(T)$$

say. Since $y \leq T \leq x$, the interval $(\frac{x}{T}, \frac{x+y}{T}]$ contains at most an integer, namely $\lfloor \frac{x+y}{T} \rfloor$, so that

$$S_2(T) = g \left( \left\lfloor \frac{x+y}{T} \right\rfloor \right) \left( \left\lfloor \frac{x+y}{T} \right\rfloor - \left\lfloor \frac{x}{T} \right\rfloor \right) \sum_{T < d \leq \frac{x+y}{1+y/x}} f(d).$$

Note that $S_2(T) = 0$ in the following cases:

- either the interval $(\frac{x}{T}, \frac{x+y}{T}]$ does not contain any integer;
- or $\frac{x+y}{T} \in \mathbb{Z}$, for then $\frac{x+y}{1+y/x} = T$ and the inner sum of $S_2(T)$ is empty.
Now if \( \frac{x+y}{T} \notin \mathbb{Z} \) and if the interval \( \left( \frac{x}{T}, \frac{x+y}{T} \right) \) contains an integer, then \( \left\lfloor \frac{x+y}{T} \right\rfloor > \frac{x}{T} \) so that
\[
0 \leq \frac{x+y}{[(x+y)/T]} - T < \frac{x+y}{x/T} - T = \frac{Ty}{x}
\]
achieving the proof.

Remark 5. Applying Lemmas 2 and 4 to our problem yields the estimate
\[
\sum_{x<n \leq x+y} F(n) = \frac{y a_s}{s+1} (\log x)^{s+1} + O_{\varepsilon} \left( xy^{\alpha-1} (\log x)^\beta + y (\log x)^{\max(s,\delta+m-1)} + x^\varepsilon \right)
\]
as soon as \( x^{1/2} \leq y \leq x \), uniformly for any function \( F \) satisfying (11) and (2). Thus, Theorem 1 substantially improves on the error term, and the rest of the text is devoted to show how one can get such an improvement.

5. Estimates for Hooley-type \( \Delta \)-functions

We easily derive from [7, Theorem 70] and partial summation the following result.

Lemma 6. For any \( x \geq 1 \) sufficiently large and any fixed integer \( r \geq 2 \)
\[
\sum_{n\leq x} \frac{\Delta_r(n)}{n} \ll r, \varepsilon (\log x)^{1+\varepsilon_r(x)} + \varepsilon
\]
where \( \varepsilon_r(x) \) is defined in (3).

Lemma 6 implies the following short interval result for Hooley’s \( \Delta_r \)-functions.

Corollary 7. For any large real numbers \( 1 \leq y \leq x \), small real number \( \varepsilon \in (0, 1/2] \) and any fixed integer \( r \geq 2 \)
\[
\sum_{x<n \leq x+y} \Delta_r(n) \ll r, \varepsilon y (\log x)^{\varepsilon_r(x)} + x^\varepsilon
\]
where \( \varepsilon_r(x) \) is defined in (3).

Proof. If \( 1 \leq y \leq x^{1/2} \), then
\[
\sum_{x<n \leq x+y} \Delta_r(n) \ll y \max_{x<n \leq x+y} \Delta_r(n) \ll r, \varepsilon y \max_{x<n \leq x+y} n^{\varepsilon/2} \ll r, \varepsilon x^\varepsilon
\]
whereas the case \( x^{1/2} \leq y \leq x \) follows from [10, Corollary 4] and Lemma 6.

Borrowing an idea from [4], we get the following upper bound.

Lemma 8. For all \( r, n, N \in \mathbb{Z}_{\geq 1} \)
\[
\sum_{\substack{d\mid n \\ N<d\leq 2N}} \tau_r(d) \leq (\log 2eN)^{r-1} \Delta_{r+1}(n).
\]

Proof. The result is obvious for \( r = 1 \), so that we may suppose \( r \geq 2 \). For all \( n \in \mathbb{Z}_{\geq 1} \) and \( k_1, \ldots, k_{r-1} \in \mathbb{R} \), we set
\[
\Delta(n; k_1, \ldots, k_{r-1}) := \sum_{d_1, \ldots, d_{r-1} | n \atop e_{k_1} \leq d_1 \leq e_{k_r+1}} 1
\]
so that from [7, page 122]

\[ \tau_r(d) = \sum_{k_1=0}^{[\log d]} \cdots \sum_{k_{r-1}=0}^{[\log d]} \Delta(d; k_1, \ldots, k_{r-1}) \]

and hence

\[
\sum_{\substack{d \mid n \\ N < d \leq 2N}} \tau_r(d) &= \sum_{\substack{d \mid n \\ N < d \leq 2N}} \sum_{k_1=0}^{[\log d]} \cdots \sum_{k_{r-1}=0}^{[\log d]} \Delta(d; k_1, \ldots, k_{r-1}) \\
&\leq \sum_{k_1=0}^{[\log 2N]} \cdots \sum_{k_{r-1}=0}^{[\log 2N]} \sum_{\substack{d \mid n \\ N < d \leq 2N}} \sum_{e^{k_1} < d_1 \leq e^{k_1+1}} \sum_{e^{k_1} < d_2 \leq e^{k_1+1}} \cdots \sum_{e^{k_1} < d_{r-1} \leq e^{k_1+1}} \sum_{N/(d_1 \cdots d_{r-1}) < d_r \leq 2N/(d_1 \cdots d_{r-1})} 1 \\
&\leq \sum_{k_1=0}^{[\log 2N]} \cdots \sum_{k_{r-1}=0}^{[\log 2N]} \Delta_{r+1}(n) \leq (\log 2eN)^{r-1} \Delta_{r+1}(n)
\]

as asserted. \(\square\)

Now from Corollary 7 and Lemma 8, we are in a position to establish the main result of this section.

**Proposition 9.** Let \(z \geq 1\) be any real number, \(4 \leq H \leq N \leq z\) and \(m \geq 1\) be integers. Then

\[
\sum_{N < n \leq 2N} \tau_m(n) \min\left(1, \frac{1}{H \|z/n\|}\right) \ll NH^{-1} \log H (\log N)^{m-1} (\log z)^{r_{m+1}(z)} + z^\varepsilon
\]

where \(\varepsilon_{m+1}(z)\) is defined in (3).

**Proof.** Define \(K := [\log H / \log 2] \geq 2\). Generalizing [11, Lemma 2.2], we get

\[
\sum_{N < n \leq 2N} \tau_m(n) \min\left(1, \frac{1}{H \|z/n\|}\right) \ll H^{-1} \sum_{N < n \leq 2N} \tau_m(n) + \sum_{k=0}^{K-2} 2^{-k} \sum_{N < n \leq 2N} \tau_m(n)_{\|z/n\| < 2^k H^{-1}}
\]
and using Lemma 8 and Corollary 7 we obtain for any \( k \in \{0, \ldots, K-2\} \)
\[
\sum_{N < n \leq 2N} \tau_m(n) \leq \sum_{N < n \leq 2N} \tau_m(n) \left( \left\lfloor \frac{z}{n} + 2^k H^{-1} \right\rfloor - \left\lfloor \frac{z}{n} - 2^k H^{-1} \right\rfloor \right)
\]
\[
\leq \sum_{z-2^{k+1} NH^{-1} < \ell \leq z + 2^{k+1} NH^{-1}} \sum_{N < n \leq 2N} \tau_m(n) \Delta_m(\ell)
\]
\[
\leq (\log 2eN)^{m-1} \sum_{z-2^{k+1} NH^{-1} < \ell \leq z + 2^{k+1} NH^{-1}} \Delta_m(\ell)
\]
\[
\leq (\log N)^{m-1} \left\{ 2^k NH^{-1} (\log z)^{\epsilon_{m+1}(z)} + z^\varepsilon \right\}
\]
and thus
\[
\sum_{N < n \leq 2N} \tau_m(n) \min\left(1, \frac{1}{H \|z/n\|}\right)
\]
\[
\ll NH^{-1} (\log N)^{m-1} + \sum_{k=0}^{K-2} 2^{-k} \left\{ 2^k NH^{-1} (\log z)^{\epsilon_{m+1}(z)} + z^\varepsilon \right\} (\log N)^{m-1}
\]
\[
\ll NH^{-1} \log H (\log N)^{m-1} (\log z)^{\epsilon_{m+1}(z)} + z^{2\varepsilon}
\]
achieving the proof.

6. PROOF OF THEOREM 1

Let \( e^\varepsilon < y \leq x \) and \( \max\left(y, \frac{z}{y}\right) \leq T \leq x \). From Lemma 4 with \( g = 1 \), (2) and Shiu’s theorem [12], we get
\[
\sum_{x < n \leq x + y} F(n) = \sum_{d \leq T} f(d) \left( \left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right) + \sum_{k \leq T} \sum_{\frac{k}{x} < d \leq \frac{x+k}{y}} f(d)
\]
\[
+ O_A \left( (\log T)^\delta \sum_{T < n \leq T+y} \tau_m(n) \right)
\]
\[
= y \sum_{d \leq T} \frac{f(d)}{d} - \sum_{y < d \leq T} f(d) \left( \psi\left( \frac{x+y}{d} \right) - \psi\left( \frac{x}{d} \right) \right) + \sum_{k \leq T} \sum_{\frac{k}{x} < d \leq \frac{x+k}{y}} f(d)
\]
\[
+ O\left( \sum_{d \leq y} |f(d)| \right) + O_{A,\varepsilon} \left( y (\log T)^{\delta+m-1} + x^\varepsilon \right)
\]
\[
:= y S_1 - S_2 + S_3 + O_{A,\varepsilon} \left( y (\log T)^{\delta+m-1} + x^\varepsilon \right).
\]
For \( S_1 \), using partial summation and (1), we get
\[
\sum_{d \leq T} \frac{f(d)}{d} = \sum_{j=0}^s a_j (\log T)^j + \sum_{j=0}^s a_j \int_1^T \frac{(\log t)^j}{t} dt + O(1)
\]
\[
= \frac{a_s}{s+1} (\log T)^{s+1} + O(\log^s T) .
\]
For $S_3$, using Lemma \[2\] we derive
\[
\begin{align*}
S_3 &= \sum_{k \leq \frac{x}{s}} \left\{ \frac{y a_s}{k} \log^s \frac{x}{k} + O \left( \frac{y}{k} \log^{s-1} x + \left( \frac{x}{k} \right)^\kappa \log^\beta x \right) \right\} \\
&= \frac{y a_s}{s+1} \sum_{k \leq \frac{x}{s}} \frac{1}{k} \log^s \frac{x}{k} + O \left( y \log^s x + x T^{\kappa-1} \log^\beta x \right)
\end{align*}
\]
and using
\[
\sum_{k \leq z} \frac{1}{k} \left( \log \frac{x}{k} \right)^j = \frac{1}{j+1} \left\{ (\log x)^{j+1} - (\log \frac{x}{z})^{j+1} \right\} + O_j (\log^j x) \quad (j \in \mathbb{Z}_{>0}, \ 1 \leq z \leq x)
\]
we get
\[
S_3 = \frac{y a_s}{s+1} \left\{ (\log x)^{s+1} - (\log T)^{s+1} \right\} + O \left( y \log^s x + x T^{\kappa-1} \log^\beta x \right).
\]

It remains to estimate $S_2$. To this end, we first split the interval $(y, T]$ into $O \left( \log \frac{T}{y} \right)$ dyadic sub-intervals of the shape $(N, 2N)$ giving
\[
|S_2| \ll \max_{y < N < T} |\Sigma_F(N, x, y)| \log(T/y)
\]
where $\Sigma_F(N, x, y)$ is given in \[4\]. From Lemma \[3\] and Abel summation, we get
\[
\begin{align*}
|\Sigma_F(N, x, y)| &\ll \frac{y}{N} \max_{N < N_1 < 2N} \max_{x \leq z \leq x+y} \sum_{N < n < N_1} |f(n)| \left| \sum_{h < H} e \left( \frac{hz}{n} \right) \right| \\
&\quad + \max_{x \leq z \leq x+y} \sum_{N < n < 2N} |f(n)| \min \left( 1, \frac{1}{H \| z/n \|} \right) \\
&\ll_A \left\{ \frac{y H}{N} \max_{N < N_1 < 2N} \sum_{N < n < N_1} \tau_m(n) \min \left( 1, \frac{1}{H \| z/n \|} \right) \\
&\quad + \max_{x \leq z \leq x+y} \sum_{N < n < 2N} \tau_m(n) \min \left( 1, \frac{1}{H \| z/n \|} \right) \right\} (\log N)^6
\end{align*}
\]
and Proposition \[9\] implies that, for $H \geq 4$ and $\max(H, y, \frac{x}{y}) \ll N < T < x$, we have
\[
|\Sigma_F(N, x, y)| \ll_{A, \varepsilon} \log H (\log N)^{\delta+m-1} (\log x)^{\epsilon_{m+1}(x)} (y + NH^{-1}) + x^{2\varepsilon} (y H N^{-1} + 1)
\ll_{A, \varepsilon} y \log(N/y)(\log N)^{\delta+m-1} (\log x)^{\epsilon_{m+1}(x)} + x^{2\varepsilon}
\]
with the choice of $H = 4 \left[ Ny^{-1} \right]$. If $\max \left( y, \frac{x}{y} \right) \leq T < x$, adding the contributions from \[5\] and \[6\] we deduce that
\[
\sum_{x < n \leq x+y} F(n) = \frac{y a_s}{s+1} (\log x)^{s+1}
\]
and
\[
\begin{align*}
&+ O \left\{ x T^{\kappa-1} (\log x)^{\beta} + y \left( \log \frac{x}{y} \right)^2 (\log T)^{\delta+m-1} (\log x)^{\epsilon_{m+1}(x)} \\
&\quad + y \log^s x + x^{2\varepsilon} \right\}
\end{align*}
\]
the term $y(\log T)^{\delta+m-1}$ being absorbed by $y \left( \log(T/y) \right)^2 (\log T)^{\delta+m-1}(\log x)^{\epsilon_{m+1}(x)}$. The asserted result then follows from choosing $T = y \exp \left( (\log x)^{1/4} \right)$.

\[\square\]

## 7. Applications and examples

In this section, the following additional arithmetic functions will appear.

- For any $k \in \mathbb{Z}_{\geq 2}$, let $\mu_k$ be the characteristic function of the set of $k$-free numbers. Note that $\mu_2 = \mu^2$.

- For any $n, k \in \mathbb{Z}_{\geq 2}$, $\tau(k)(n)$ counts the number of $k$-free divisors of $n$, with the convention $\tau(k)(1) = 1$. Note that $\tau(2) = 2^\omega$ where, as usual, $\omega(n)$ is the number of distinct prime factors of $n$.

- For any $k \in \mathbb{Z}_{\geq 1}$, $\Lambda_k$ is the $k$th von Mangoldt’s function defined by $\Lambda_k = \mu \ast \log_k$. Similarly, if $g$ is any arithmetic function satisfying $g(1) \neq 0$, the von Mangoldt function $\Lambda_g$ attached to $g$ is implicitly defined by the equation $\Lambda_g \ast g = g \ast \log$.

### 7.1. Example 1.

In [4], it is shown that, uniformly for $x^{1/2} \log x < y < x^{1/2}(\log x)^{5/2}$

$$
\sum_{x < n \leq x+y} \tau_4(n) = \frac{1}{6} y (\log x)^3 + O \left( (xy)^{1/3}(\log x)^{2/3} \right)
$$

Let $k \in \mathbb{Z}_{\geq 2}$ and take $f = \tau_{k-1}$. Applying Theorem [1] with $m = k - 1$, $s = k - 2$, $\delta = 0$, $\kappa = 1 - \frac{2}{k}$, we get

**Corollary 10.** For any $x^{1/2} e^{-\frac{1}{4}(\log x)^{1/4}} \leq y \leq x e^{-\left(\log x\right)^{1/4}}$ and any $\varepsilon \in \left[0, \frac{1}{2}\right]$,

$$
\sum_{x < n \leq x+y} \tau_k(n) = \frac{y (\log x)^{k-1}}{(k-1)!} + O \left( xy^{-2/k+\varepsilon}(\log x)^{1/4} + y (\log x)^{k-3/2+\varepsilon_4(x)} + x^\varepsilon \right)
$$

**the term** $y^\varepsilon$ **being omitted when** $k = 2$. In particular,

$$
\sum_{x < n \leq x+y} \tau_4(n) = \frac{1}{6} y (\log x)^3 + O \left( y (\log x)^{3/2+\varepsilon_4(x)} \right)
$$

**for all** $x^{2/3+\varepsilon} e^{-\frac{1}{4}(\log x)^{1/4}} \leq y \leq x e^{-\left(\log x\right)^{1/4}}$.

### 7.2. Example 2.

In this example, $F(n)$ is either $\tau(n)^2$, or $\tau(n^2)$. Improving on a result from [3], Zhai [13, Corollary 4] showed that

$$
\sum_{x < n \leq x+y} F(n) \sim C_F y (\log x)^3
$$

for $y = o(x)$ with

$$
\frac{y}{x^{1/2} \log x} \to \infty \quad \text{lorsque} \quad x \to \infty
$$
and where \( C_F = \frac{1}{6\zeta(2)} \) if \( F = \tau^2 \), and
\[
C_F = \frac{1}{6} \prod_p \left( 1 - \frac{1}{p} \right)^2 \left( 1 + \frac{2}{p} \right) \quad \text{if} \quad F = \tau \circ \text{Id}^3.
\]

Let \( f \) be the Eratosthenes transform of \( F \). Note that, if \( F = \tau^2 \), then \( f = \tau \circ \text{Id}^2 \), and if \( F = \tau \circ \text{Id}^3 \), then \( f = 3^\omega \), so that in both cases
\[
|f(n)| \leq \tau_3(n) \quad (n \in \mathbb{Z}_{>1})
\]
and by [13] Main Theorem], we have in both cases
\[
\sum_{n \leq x} f(n) = 3C_F x (\log x)^2 + Ax \log x + Bx + O \left( x^{1/2} (\log x)^4 \right)
\]
where \( A, B \in \mathbb{R} \). We may apply Theorem [2] with \( \delta = 0 \), \( A = 1 \), \( s = 2 \), \( m = 3 \), \( \beta = 4 \)
and \( \kappa = \frac{1}{4} \), giving the following more precise version of Zhai’s result.

**Corollary 11.** The function \( F \) being indifferently either \( \tau^2 \) or \( \tau \circ \text{Id}^3 \), for any \( x^{1/2} e^{-\frac{3}{4} (\log x)^{1/4}} \leq y \leq x e^{-\log x^{1/4}} \) and any \( \varepsilon \in \left[ 0, \frac{1}{2} \right] \)
\[
\sum_{x < n \leq x + y} F(n) = C_F y (\log x)^3
\]
\[
+ O_{\varepsilon} \left( y^{-1/2} y^{-\frac{3}{4} (\log x)^{1/4}} (\log x)^4 + y (\log x)^{5/2 + \varepsilon_4(x) + \varepsilon} \right)
\]
where \( C_F = \frac{1}{6\zeta(2)} \) if \( F = \tau^2 \), and
\[
C_F = \frac{1}{6} \prod_p \left( 1 - \frac{1}{p} \right)^2 \left( 1 + \frac{2}{p} \right)
\]
if \( F = \tau \circ \text{Id}^3 \). In particular, if \( x^{2/3} e^{-\frac{3}{4} (\log x)^{1/4}} \log x \leq y \leq x e^{-\log x^{1/4}} \)
\[
\sum_{x < n \leq x + y} F(n) = C_F y (\log x)^3 + O_{\varepsilon} \left( y (\log x)^{5/2 + \varepsilon_4(x)} \right).
\]

**7.3. Example 3.** Let \( k \in \mathbb{Z}_{>2} \) and take \( f = \mu_k \). Theorem [2] applied with the values \( \delta = s = \beta = 0 \), \( m = 1 \) and \( \kappa = \frac{1}{4} \) gives the following corollary.

**Corollary 12.** For \( x^{1/2} e^{-\frac{3}{4} (\log x)^{1/4}} \leq y \leq x e^{-\log x^{1/4}} \) and any \( \varepsilon \in \left( 0, \frac{1}{2} \right] \)
\[
\sum_{x < n \leq x + y} \tau_{(k)}(n) = \frac{y \log x}{\zeta(k)} + O_{\varepsilon} \left( y^{1-1/k} e^{-1+1/k} (\log x)^{1/4} + y (\log x)^{1/2 + \varepsilon_2(x) + \varepsilon} \right).
\]
In particular, if \( x^{1/2} e^{-\frac{3}{4} (\log x)^{1/4}} \leq y \leq x e^{-\log x^{1/4}} \), then
\[
\sum_{x < n \leq x + y} \tau_{(k)}(n) = \frac{y \log x}{\zeta(k)} + O \left( y (\log x)^{1/2 + \varepsilon_2(x)} \right).
\]

---

\( ^{1(1)} \)There is a misprint on \( C_F \) in [5] Corollary 1] for which the coefficient \( \frac{1}{6} \) has been forgotten in both cases.
For instance, when $k = 2$, we get
\[
\sum_{x < n \leq x + y} 2^{\omega(n)} = \frac{y \log x}{\zeta(2)} + O(y(\log x)^{1/2 + \epsilon_3(x)})
\]
for all $x^{2/3} e^{-\frac{1}{8} (\log x)^{1/4}} \leq y \leq xe^{-\log(x)^{1/4}}$.

7.4. **Example 4.** Let $k \in \mathbb{Z}_{\geq 2}$ and take $f = \tau(k)$. Theorem 1 with $s = 1$, $m = 2$, $\delta = 0$ and $\kappa$ given below gives

\[\sum_{x < n \leq x + y} (\tau * \mu_k)(n) = \frac{y(\log x)^2}{2\zeta(k)} + O_{\epsilon}(xy^{\kappa-1}e(\kappa-1)(\log x)^{1/4} + y(\log x)^{3/2 + \epsilon_3(x)} + x^\epsilon)\]

where
\[
\kappa := \begin{cases} 
1/k, & \text{if } k \in \{2, 3\} \\
\frac{131}{416} + \epsilon, & \text{if } k \geq 4.
\end{cases}
\]

7.5. **Example 5.** We use Theorem 1 with $f = \mu^2 \times 2^\omega$. In [6], it is shown that, for any real number $z \geq 1$ sufficiently large\(\text{[2]}\)
\[
\sum_{n \leq z} \mu^2(n)2^{\omega(n)} = Az \log z + Bz + O(z^{1/2}(\log z)^6)
\]
where $A = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right) \frac{1}{1 - \frac{1}{p}}$ and $B = A \left(2\gamma - 1 + 6 \sum_{p} \frac{(p-1) \log p}{p^{(p+2)}}\right)$. Thus, we can apply Theorem 1 with $s = 1$, $m = 2$, $\delta = 0$, $\beta = 6$ and $\kappa = \frac{1}{2}$, giving

\[\sum_{x < n \leq x + y} 3^{\omega(n)} = \frac{1}{2} \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right) y(\log x)^2 + O(xy^{-1/2} e^{-\frac{1}{4} (\log x)^{1/4}} (\log x)^6 + y(\log x)^{3/2 + \epsilon_3(x)} + x^\epsilon).\]

7.6. **Example 6.** The function $F$ does not need to be multiplicative. Suppose that we have at our disposal an arithmetic function $g$ such that $|g(n)| \leq A$, $g(1) \neq 0$ and
\[
\sum_{n \leq z} g(n) = az + O(z^\kappa(\log z)^\beta) \quad (z \geq 1)
\]
with $a \in \mathbb{C} \setminus \{0\}$, $\kappa \in [0, 1)$, $\beta \geq 0$. By Abel summation, we get
\[
\sum_{n \leq z} g(n) \log n = az(\log z - 1) + O(z^\kappa(\log z)^{\beta+1})
\]
so that Theorem 1 may be used with $f = g \times \log$ and $m = s = \delta = 1$, giving the following estimate.

\(\text{[2]}\)In fact, the authors in [6] give an error term of the shape $O_{\epsilon}(z^{1/2+\epsilon})$, but a close inspection of their proof reveals that it can be sharpened to $\ll z^{1/2}(\log z)^6$. 

\[\text{SHORT INTERVAL RESULTS FOR A CLASS OF ARITHMETIC FUNCTIONS 11}\]
Corollary 15. For any $\varepsilon \in \left(0, \frac{1}{2}\right]$ and all $x^{1/2}e^{-\frac{3}{4}(\log x)^{1/4}} \leq y \leq xe^{-(\log x)^{1/4}}$

$$\sum_{x<n \leq x+y} \Lambda_y(n) G(n) = \frac{1}{2}ay(\log x)^2 + O\left(xy^{k-1}e^{(\kappa-1)(\log x)^{1/4}}(\log x)^{\beta+1}\right)$$

$$+ y(\log x)^{3/2+\varepsilon_2(x)} + x^{\varepsilon}$$

where $G := g \ast 1$.

For instance, when $g = \mu^2$, so that $a = \zeta(2)^{-1}, \kappa = \frac{1}{2}$ and $\beta = 0$, we get for all $x^{1/2}e^{-\frac{3}{4}(\log x)^{1/4}} \leq y \leq xe^{-(\log x)^{1/4}}$ and any $\varepsilon \in \left(0, \frac{1}{4}\right]$

$$\sum_{x<n \leq x+y} \Lambda_{\mu^2}(n) 2^\omega(n) = \frac{y(\log x)^2}{2\zeta(2)} + O\left(xy^{-1/2}e^{-\frac{3}{4}(\log x)^{1/4}} \log x \right)$$

$$+ y(\log x)^{3/2+\varepsilon_2(x)} + x^{\varepsilon}.$$

In particular, if $x^{2/3}e^{-\frac{1}{3}(\log x)^{1/4}} \leq y \leq xe^{-(\log x)^{1/4}}$, then

$$\sum_{x<n \leq x+y} \Lambda_{\mu^2}(n) 2^\omega(n) = \frac{y(\log x)^2}{2\zeta(2)} + O\left(y(\log x)^{\frac{3}{2}+\varepsilon_2(x)}\right).$$

7.7. Example 7. Let $k \in \mathbb{Z}_{\geq 1}$ and take $f = \log^k \ast \log^k$. For any $n \in \mathbb{Z}_{\geq 1}$, we have $0 \leq f(n) \leq 4^{-k} \tau(n)(\log n)^{2k}$. Furthermore, it is known \([9]\) that for any $\varepsilon \in \left(0, \frac{1}{3}\right]$ and any real number $x \geq 1$

$$\sum_{n \leq z} f(n) = zP_{2k+1}(\log z) + O_\varepsilon \left(z^{1/3+\varepsilon}\right)$$

where $P_{2k+1}$ is a polynomial of degree $2k + 1$ and leading coefficient $\frac{(k!)^2}{(2k+1)!}$, so that Theorem \([\square]\) can be used with $s = 2k + 1, m = 2, \delta = 2k$ and $\kappa = \frac{1}{3}$, giving

Corollary 16. For all $x^{1/2}e^{-\frac{3}{4}(\log x)^{1/4}} \leq y \leq xe^{-(\log x)^{1/4}}$, any $k \in \mathbb{Z}_{\geq 1}$ and any $\varepsilon \in \left(0, \frac{1}{2}\right]$,

$$\sum_{x<n \leq x+y} \left(\Lambda_k \ast \tau \ast \log^k\right)(n) = \frac{(k!)^2 y(\log x)^{2k+2}}{(2k+2)!}$$

$$+ O_{k,\varepsilon}\left(xy^{-2/3+\varepsilon}e^{-\frac{3}{4}(\log x)^{1/4}} + y(\log x)^{2k+\frac{3}{2}+\varepsilon_3} + x^{\varepsilon}\right).$$

References

[1] O. Bordellès, Short sums of restricted Möbius functions, *Acta Arith.* 142 (2008), 367–375.

[2] O. Bordellès, Multiplicative functions over short segments, *Acta Arith.* 157 (2013), 1–10.

[3] Z. Cui & J. Wu, The Selberg-Delange method in short intervals with an application, *Acta Arith.* 163 (2014), 247–260.

[4] M. Z. Gareev, F. Luca & W. G. Nowak, The divisor problem for $d_4(n)$ in short intervals, *Arch. Math.* 86 (2006), 60–66.
Short interval asymptotics for a class of arithmetic functions, *Acta Math. Hungar.* **113** (2006), 85–99.

B. Gordon & K. Rogers, Sums of the divisor function, *Canad. J. Math.* **16** (1964), 151–158.

R. R. Hall & G. Tenenbaum, *Divisors*, Cambridge University Press, 1987.

I. Kátai & M. V. Subbarao, Some remarks on a paper of Ramachandra, *Lithuanian Math. J.* **43** (2003), 410–418.

I. Kuichi & M. Minamide, On the Dirichlet convolution of completely additive functions, *J. Integer Sequences* **17** (2014), Art. 14.8.7.

M. Nair & G. Tenenbaum, Short sums of certain arithmetic functions, *Acta Math.* **180** (1998), 119–144.

K. Ramachandra, Some problems of analytic number theory, *Acta Arith.* **31** (1976), 313–324.

P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math. **313** (1980), 161–170.

W. Zhai, Asymptotics for a class of arithmetic functions, *Acta Arith.* **170** (2015), 135–160.

2 allée de la combe, 43000 Aiguilhe, France

E-mail address: borde43@wanadoo.fr