INFINESIMAL SPECTRAL FLOW
AND SCATTERING MATRIX

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Abstract. In this note we introduce the absolutely continuous and the singular parts of the spectral shift function as integrals of the absolutely continuous and, respectively, of the singular parts of the infinitesimal spectral flow. Under certain assumption, we show that this definition is independent of the piecewise linear path of integration. The proof is based on a representation of the scattering operator of a pair of trace compatible operators as chronological exponential of the infinitesimal scattering matrix, and on the fact that the trace of the infinitesimal scattering matrix is equal to the absolutely continuous part of the infinitesimal spectral flow. As a corollary, a variant of the Birman-Krein formula is derived. An interpretation of Pushnitski’s μ-invariant is given.

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INTRODUCTION

Let $H_0$ be a self-adjoint operator, and let $V$ be a trace class operator on a Hilbert space $\mathcal{H}$. Then M. G. Krein’s famous result [20] says that there is a unique $L^1$-function $\xi_{H_0+V,H_0}(\lambda)$, known as the Lifshits-Krein spectral shift function, such that for any $C_0^\infty(\mathbb{R})$-function $f$

$$\text{Tr}(f(H_0 + V) - f(H_0)) = \int_{-\infty}^{\infty} f'(\lambda)\xi_{H_0+V,H_0}(\lambda)\,d\lambda.$$  

(1)

The notion of the spectral shift function was discovered by the physicist I. M. Lifshits [22]. An excellent survey on the theory of the spectral shift function can be found in [9].

In 1975, Birman and Solomyak [10] proved the following remarkable formula for the spectral shift function

$$\xi(\lambda) = \frac{d}{d\lambda} \int_0^1 \text{Tr}(VE_{(-\infty,\lambda)}^{H_r})\,dr,$$

where $H_r = H_0 + rV$, $r \in \mathbb{R}$, and $E_{(-\infty,\lambda)}^{H_r}$ is the spectral projection (see also [29]). On the basis of the Birman-Solomyak formula, the notion of infinitesimal spectral flow was introduced in [4]. The infinitesimal spectral flow of a self-adjoint operator $H$ under a trace compatible perturbation $V$ (see the definition in the

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text) is defined by the formula
\[ \Phi_H(V)(\varphi) = \text{Tr}(V \varphi(H)), \quad \varphi \in C_c^\infty(\mathbb{R}). \]

It was shown in [1] that for any two operators \( H_0, H_1 \in \mathcal{A} \) from a trace compatible affine space \( \mathcal{A} \) (see the definition in the text) one can define the spectral shift function \( \xi_{H_1,H_0} \) of the pair \( H_0, H_1 \) as the integral of the infinitesimal spectral flow
\[ \xi_{H_1,H_0}(\varphi) = \int_\Gamma \Phi_{H_\lambda}(\dot{H}_\lambda)(\varphi) \, dr, \quad \varphi \in C_c^\infty(\mathbb{R}), \]
where \( \Gamma = \{ H_\lambda \} \) is any piecewise smooth path in \( \mathcal{A} \), connecting \( H_0 \) and \( H_1 \). It was shown in [4] that the integral does not depend on the choice of the path \( \Gamma \), and the spectral shift function, as defined above, is an absolutely continuous measure.

Examples of trace compatible affine spaces include the classical case of \( H_0 + L^1_{sa}(\mathcal{H}) \) with an arbitrary self-adjoint operator \( H_0 \) on \( \mathcal{H} \), \( D_0 + B_{sa}(\mathcal{H}) \) with a self-adjoint operator \( D_0 \) with compact resolvent [3] and an affine space of Schrödinger operators of the form \( -\Delta + V + \ell^1(L^\infty) \), where \( V \in L^\infty(\mathbb{R}^d) \) [28, Section B9] (for definition of \( \ell^1(L^\infty) \) see e.g. [27, Chapter 4]).

The well-known Birman-Kreĭn formula for the spectral shift function ([8], see also [30, §8.4]) asserts that for a.e. \( \lambda \in \mathbb{R} \)
\[ \det S(\lambda) = e^{-2\pi i \xi(\lambda)}, \]
where \( S(\lambda) \) is the scattering matrix of the pair \( H_0, H_0 + V, \ V \in \mathcal{S}_1(\mathcal{H}) \). This formula was discovered for the first time by V. S. Buslaev and L. D. Faddeev in the case of Sturm-Liouville operators on a half-line [14].

In this note we introduce the absolutely continuous, \( \Phi^{(a)} \), and the singular, \( \Phi^{(s)} \), parts of the infinitesimal spectral flow \( \Phi \) by formulas (19) and (17), and a decomposition of the spectral shift function
\[ \xi_{H_1,H_0} = \xi_{H_1,H_0}^{(a)} + \xi_{H_1,H_0}^{(s)}, \]
where \( \xi^{(a)} = \int_\Gamma \Phi^{(a)} \) and \( \xi^{(s)} = \int_\Gamma \Phi^{(s)} \), \( \Gamma \) being a piecewise linear path connecting \( H_0 \) and \( H_1 \). Similar notions in the context of Herglotz functions were considered also in [17], where one can also find historical comments on the subject.

Under certain assumption, which includes a class of Schrödinger operators, it is proved that the definition of \( \xi^{(a)} \) and \( \xi^{(s)} \) does not depend on the choice of the piecewise linear path \( \Gamma \). The proof is based on the following formula for the scattering matrix
\[ S(H_1, H_0) = \text{Exp} \left( -2\pi i \int_0^1 W_+(H_0, H_r) \Pi_{H_r}(\dot{H}_r) W_+(H_r, H_0) \, dr \right), \]
where \( \{ H_r \}_{r \in [0,1]} \) is a piecewise linear path in \( \mathcal{A} \), connecting \( H_0 \) and \( H_1 \). Though this formula is an almost straightforward consequence of the stationary formula for the scattering matrix, it seems to be new (to the best of the author’s knowledge).

As is well-known, the Birman-Kreĭn formula (3) determines the spectral shift function up to an integer-valued function. We show that \( \xi \) function in (3) can be replaced by \( \xi^{(s)} \). This suggests that the above mentioned integer-valued function may be \( \xi^{(s)} \). At the end of the note an interpretation of Pushnitski’s \( \mu \)-invariant is given.
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1. Notation and preliminaries

1.1. Trace compatibility. We recall the notion of a trace compatible affine space of operators, which was introduced in [4]. Let \( \mathcal{A} = H_0 + \mathcal{A}_0 \) be an affine space of self-adjoint operators on a Hilbert space \( \mathcal{H} \), where \( H_0 \) is a self-adjoint operator on \( \mathcal{H} \) and \( \mathcal{A}_0 \) is a vector subspace of the real Banach space of all bounded self-adjoint operators on \( \mathcal{H} \). We say that \( \mathcal{A} \) is trace compatible, if for all \( \varphi \in C^\infty_c(\mathbb{R}) \), \( V \in \mathcal{A}_0 \) and \( H \in \mathcal{A} \)

\[
V \varphi(H) \in \mathcal{S}_1(\mathcal{H}),
\]

where \( \mathcal{S}_1(\mathcal{H}) \) is the ideal of trace class operators, and if \( \mathcal{A}_0 \) is endowed with a locally convex topology which coincides with or is stronger than the uniform topology, such that the map \( (V_1, V_2) \in \mathcal{A}_0^2 \mapsto V_1 \varphi(H_0 + V_2) \) is \( \mathcal{S}_1(\mathcal{H}) \)-continuous for all \( H_0 \in \mathcal{A} \) and \( \varphi \in C^\infty_c(\mathbb{R}) \). In particular, \( \mathcal{A} \) is a locally convex affine space. We also assume, that for any \( V \in \mathcal{A}_0 \) there exist non-negative \( V_1, V_2 \in \mathcal{A}_0 \), such that \( V = V_1 - V_2 \). This technical condition is satisfied in all interesting examples. We say that a pair of operators \( H_0, H_1 \) is trace compatible, if they belong to some trace compatible affine space, we say that an operator \( V \) is a trace compatible perturbation of \( H \), if the pair \( H, H + V \) is trace compatible.

The infinitesimal spectral flow is a distribution valued 1-form on a trace compatible affine space of operators \( \mathcal{A} \), defined by formula

\[
\Phi_H(V)(\varphi) = \text{Tr}(V \varphi(H)), \quad H \in \mathcal{A}, \ V \in \mathcal{A}_0, \ \varphi \in C^\infty_c(\mathbb{R}).
\]

Actually, \( \Phi_H(V) \) is a measure on the spectrum of \( H \) [4]. For example, for the trace compatible affine space \( D + C^\infty_c(\mathbb{R}) \), \( D = \frac{i}{\pi} d \), if \( a \in C^\infty_c(\mathbb{R}) \), then for any \( v \in C^\infty_c(\mathbb{R}) \)

\[
\Phi_{D+a}(v) = \frac{1}{2\pi} \int_{\mathbb{R}} v(x) dx \cdot \text{Lebesgue measure}.
\]

We note, that this formula is a sort of trace analogue for \( \mathbb{R} \) of Connes’ formula for Dixmier trace [15, 26].

Lemma 1.1. If operators \( H_0 \) and \( H_1 \) are trace compatible, then their essential spectra coincide.

Proof. Let \( U \) be an interval, which does not intersect the essential spectrum of \( H_0 \), and let \( \Delta \subseteq U \) be a segment. The projection \( E_{\Delta}^{H_0} \) is finite-dimensional, and, hence, it is trace class. If \( \varphi \) is a smoothed indicator of \( \Delta \) whose support is a subset of \( U \), then it follows from [4, Lemma 2.1] that \( \varphi(H_1) - \varphi(H_0) \) is also trace class. Hence, \( \varphi(H_1) \) is trace class as well, which is possible only if \( \Delta \) does not intersect the essential spectrum of \( H_1 \). \( \square \)

Lemma 1.2. If one (and hence all) operator from \( \mathcal{A} \) is semibounded, then for any \( H_0, H_1 \in \mathcal{A} \) the spectral shift function \( \xi_{H_1, H_0} \) is integer-valued outside of their common essential spectrum.

Proof. We can assume that elements of \( \mathcal{A} \) are semibounded from below. Let \( [a, b] \subseteq \mathbb{R} \setminus \sigma_{\text{ess}}(H_0) \) be a segment, and let \( f \in C^\infty \) be a decreasing function such
that $f(x) = 1$ for $x < a$, $f(x) = 0$ for $x > b$. It follows that $f(H_0), f(H_1) \in F_{0,1}$ [3 Subsection 1.5]. Since $H_0, H_1$ are semibounded, $f(H_1) - f(H_0)$ is compact (even trace class) by [4 Lemma 2.1]. Hence, by [3 Theorem 3.18], the spectral shift function $\xi_f$ of the pair $f(H_0), f(H_1)$ is integer-valued. The invariance principle (see e.g. [23 (1.9)] or [4 (13)]) now implies that $\xi_{H_1, H_0}$ is integer valued on $[a, b]$. □

If it is known that $E^H_{\lambda} - E^H_0$ is compact for $\lambda \notin \sigma_{ess}(H_0)$, then the proof of this lemma shows that the spectral shift function is integer valued outside the essential spectrum. That $E^H_{\lambda} - E^H_0$ is compact for $\lambda \notin \sigma_{ess}(H_0)$ presumably can be shown by methods of [24].

1.2. Scattering theory. We recall some notions of the mathematical scattering theory from [30] (see also [5, 11, 25]). The wave operators of a pair of self-adjoint operators $H_0, H$ on $\mathcal{H}$ are the operators $\mathcal{W}_{\pm}(H, H_0)$ (when the respective limits exist)

$W_{\pm}(H, H_0) := \lim_{t \to \pm \infty} e^{-itH} P_a(H_0) e^{-itH_0} P_a(H_0)$,

where $P_a(H_0)$ denotes the projection onto the absolutely continuous subspace $\mathcal{H}_0^{ac}$ of $H_0$, and the limit is the strong operator limit. Further, $W_{\pm}(H, H_0) = W_{\pm}(H_0, H)$ [30 (2.2.2)]. The wave operator $W_{\pm}(H, H_0)$ is an isometry of $\mathcal{H}_0^{ac}$ into $\mathcal{H}_0^{ac}$, where $\mathcal{H}_0^{ac}$ is the absolutely continuous subspace of $H$, i.e.

$W_{\pm}(H, H_0) W_{\pm}(H, H_0) = P_a(H_0)$ and $W_{\pm}(H, H_0) W_{\pm}(H, H_0) \leq P_a(H)$.

If $W_{\pm}(H, H_0) W_{\pm}(H, H_0) = P_a(H)$ then the wave operator $W_{\pm}(H, H_0)$ is called complete. The completeness of $W_{\pm}(H, H_0)$ is equivalent to the existence of $W_{\pm}(H_0, H)$ [30 Theorem 2.3.6]. If the wave operators $W_{\pm}(H, H_0)$ exist and are complete then the scattering operator of a pair of operators $H_0, H$ is defined as $\mathcal{S}(H, H_0) = W_{+}^{*}(H, H_0) W_{-}(H, H_0)$ and it is a unitary operator on the absolutely continuous subspace of $H_0$. If the wave operators $W_{\pm}(H_2, H_1)$ and $W_{\pm}(H_1, H_0)$ exist, then $W_{\pm}(H_2, H_0)$ also exists and $W_{\pm}(H_2, H_0) = W_{\pm}(H_2, H_1) W_{\pm}(H_1, H_0)$ [30 Theorem 2.1.7].

Let $\mathcal{A}$ be a trace compatible affine space of operators on a Hilbert space $\mathcal{H}$. If $H_0, H_1 \in \mathcal{A}$ then the wave operators $W_{\pm}(H_1, H_0)$ exist and are complete. Existence of $W_{\pm}(H_1, H_0)$ follows from Birman’s local criteria for existence of wave operators [30 Theorem 6.1.4]. Completeness of $W_{\pm}(H_1, H_0)$ follows from existence of $W_{\pm}(H_0, H_1)$. Hence, for any $H_0, H_1 \in \mathcal{A}$, the scattering operator $\mathcal{S}(H_1, H_0)$ exists and is a unitary operator on $\mathcal{H}_0^{ac}$. This implies that the absolutely continuous spectrum of all $H \in \mathcal{A}$ coincide.

If $V$ is a trace compatible perturbation of $H$, then $G := \sqrt{V}$ is $H$-weakly smooth [30 Definition 5.1.1]. This follows from $H$-weak smoothness of the Hilbert-Schmidt operator $GE^\Lambda_a$ and [30 Lemmas 5.1.3, 5.1.4].

1.3. Direct integrals of Hilbert spaces. Let $\mathcal{A}$ be a trace compatible affine space of operators on $\mathcal{H}$. Let $H_0 \in \mathcal{A}$ and let $V \in \mathcal{A}_0$. Let $\sigma^{(a)}_{H_0}$ be the spectrum
of the absolutely continuous part of $H_0$. Let
\[ T_0: H^a_0 \to \int_{\sigma^a_0} \mathcal{H}_\lambda d\lambda, \quad T_0: \eta \mapsto \tilde{\eta}(\cdot), \]
be an isomorphism of the Hilbert space $H^a_0$ onto a direct integral of Hilbert spaces $\mathcal{H}_\lambda$, such that $(T_0 H_0 \eta)(\lambda) = \lambda T_0 \eta(\lambda) = \lambda \tilde{\eta}(\lambda)$, $\eta \in H^a_0$. The scattering operator $S(H_1, H_0)$ is diagonal in this representation \cite[§2.4]{30} and it acts as multiplication by a unitary operator $S(\lambda; H_1, H_0)$ on $\mathcal{H}_\lambda$. The function $\lambda \mapsto S(\lambda; H_1, H_0)$ is called the scattering matrix.

Let $\mathcal{K}$ be an auxiliary Hilbert space. Let $V = G^* J G$, where $J = J^*$ is an invertible operator on $\mathcal{K}$ and $G: \mathcal{H} \to \mathcal{K}$, and let
\[ T_r(z) = GR_z(H_r)G^*, \quad z \in \mathbb{C} \setminus \mathbb{R}, \]
where $H_0 \in \mathcal{A}$, $V \in \mathcal{A}_0$, $H_r = H_0 + rV$. One can take $\mathcal{K} = \mathcal{H}$, $G = \sqrt{|V|}$ and $J = \text{sgn}(V)$, but other decompositions are also useful \cite{23}.

We denote by $\Lambda = \Lambda_\mathcal{A}$ the inner part of the common absolutely continuous spectrum of operators $H \in \mathcal{A}$.

**Assumption 1.3.** (i) There exists $p \in [1, \infty]$ such that for any $H \in \mathcal{A}$ and $V \in \mathcal{A}_0$, the function $T(z) = GR_z(H)G^*$ takes values in $\mathfrak{S}_p(\mathcal{K})$ for $\text{Im} z \neq 0$, and for all $\lambda \in \Lambda_\mathcal{A}$ it has non-tangential limit values $T(\lambda \pm i0)$ in $\mathfrak{S}_p(\mathcal{K})$.

(ii) The imaginary part $B(\lambda \pm i0) := \text{Im} T(\lambda \pm i0)$ of $T(\lambda \pm i0)$ belongs to $\mathfrak{S}_1(\mathcal{K})$ for all $\lambda \in \Lambda_\mathcal{A}$.

This assumption holds, for example, for the space
\[ -\Delta + \{ V \in L^\infty(\mathbb{R}^d); \exists \varepsilon > 0 \ \forall C > 0 \ \forall x \in \mathbb{R}^d |V(x)| \leq C(1 + |x|)^{-d-\varepsilon} \} \]
with $\Lambda_\mathcal{A} = (0, \infty)$, where $\Delta$ is the Laplace operator on $\mathbb{R}^d$ with $\text{dom}(\Delta) = \mathcal{H}_2(\mathbb{R}^d)$ Sobolev space (see e.g. \cite{11}, see also \cite{1, 6, 18, 21}).

If Assumption 1.3 holds, then the formula
\[ Z(\lambda; G) \eta = (T_0 G^* \eta)(\lambda), \quad \lambda \in \Lambda_\mathcal{A}, \]
defines operators $Z(\lambda; G) \in \mathcal{B}(\mathcal{K}, \mathcal{H}_\mathcal{A})$ unambiguously \cite{1, 11, 30}. Letting $Z_r(\lambda) = Z(\lambda; G)$ with respect to the direct integral decomposition of $H_r$, by \cite[§5.4.4]{30}
\[ \pi Z_r^*(\lambda) Z_r(\lambda) = B_r(\lambda + i0). \]

It follows that $Z_r(\lambda)$ is a Hilbert-Schmidt operator. Under Assumption 1.3 if $H_0 \in \mathcal{A}$ and $V \in \mathcal{A}_0$, then $V$ is an integral operator with respect to the representation \cite{27}, and has kernel
\[ v(\lambda, \lambda') = Z_0(\lambda) J Z_0^*(\lambda'), \quad \lambda, \lambda' \in \Lambda_\mathcal{A}, \]
which takes trace class values. This follows from \cite[Lemma 5.4.3]{30} and (7).

We give the proof of the following lemma for completeness (see e.g. \cite{1, 11, 30}).

**Lemma 1.4.** If $T_0(\lambda + i0)$ exists then $T_r(\lambda + i0)$ exists if and only if the operator $1 + rT_0(\lambda + i0)J$ is invertible.
Proof. The second resolvent identity \( R(z) = R_0(z) - R(z)V R_0(z) \) implies that
\[
T_r(z)(1 + rJT_0(z)) = T_0(z).
\]
Taking limits, one gets
\[
T_r(\lambda + i0)(1 + rJT_0(\lambda + i0)) = T_0(\lambda + i0).
\]
Since \( T_0 \) is compact, \( 1 + rJT_0(\lambda + i0) \) is not invertible if and only if there exists a non zero \( \psi \in \mathcal{H} \), such that \( (1 + rJT_0(\lambda + i0))\psi = 0 \). This and (9) imply that \( T_0(\lambda + i0)\psi = 0 \). Hence \( \psi = 0 \). \( \square \)

2. Results

Definition 2.1. The infinitesimal scattering matrix \( \Pi_{H_0}(V) \) of the operator \( H_0 \in \mathcal{A} \) under a trace compatible perturbation by \( V \in \mathcal{A}_0 \) is a self-adjoint operator, which in the representation (8) acts by multiplication by
\[
\Pi_{H_0}(V)(\lambda) := v(\lambda, \lambda) = Z_0(\lambda)J Z_0^*(\lambda), \quad \lambda \in \Lambda_\mathcal{A}.
\]

The map \( V \in \mathcal{A}_0 \mapsto \Pi_{H}(V) \) is not linear, but if perturbations \( V_1, V_2 \in \mathcal{A}_0 \) are disjoint (i.e. \( V_1 V_2 = V_2 V_1 = 0 \)) then
\[
\Pi_{H}(V_1 + V_2) = \Pi_{H}(V_1) + \Pi_{H}(V_2).
\]
If Assumption 1.3 holds, then for each \( \lambda \in \Lambda_\mathcal{A} \) the operator \( \Pi_{H}(V)(\lambda) \) is trace class.

The following theorem is a classical stationary representation for the scattering matrix \( \Pi_{H_0}(V) \) [7, 11, Theorem 5.7.1].

Theorem 2.2. If \( H_0 \in \mathcal{A}, \ V \in \mathcal{A}_0, \ H_r = H_0 + rV, \ r \in [0, 1] \) and if Assumption 1.3 holds, then for all \( \lambda \in \Lambda_\mathcal{A} \) and for all \( r \in [0, 1] \) the scattering matrix
\[
S(\lambda; H_r, H_0) \exists,
\]
and the stationary representation for the scattering matrix
\[
S(\lambda; H_r, H_0) = 1_\lambda - 2\pi i rZ_0(\lambda)J(1 + rT_0(\lambda + i0)J)^{-1}Z_0^*(\lambda)
\]
holds. Moreover, \( S(\lambda) - 1_\lambda \in \mathcal{S}_1(\mathcal{H}_\lambda) \) for all \( \lambda \in \Lambda_\mathcal{A} \).

Lemma 2.3. If \( H_r = H_0 + rV, \) where \( H_0 \in \mathcal{A}, \ V \in \mathcal{A}_0, \) and Assumption 1.3 holds, then
\[
\frac{d}{dr}S(\lambda; H_r, H_0)\big|_{r=r_0} = -2\pi i \Pi_{H_0}(V)(\lambda)
\]
for all \( \lambda \in \Lambda_\mathcal{A} \), where the derivative is taken in \( \mathcal{S}_1(\mathcal{H}_\lambda) \)-topology.

Proof. The formula (12) implies that for \( \lambda \in \Lambda_\mathcal{A} \)
\[
S(\lambda; H_r, H_r) = 1_\lambda - 2\pi i (r - r_0)Z_\lambda(\lambda)J(1 + (r - r_0)T_\lambda(\lambda + i0)J)^{-1}Z_\lambda^*(\lambda),
\]
where \( 1 + (r - r_0)T_\lambda(\lambda + i0)J \) is invertible by Lemma 1.4. It follows that in \( \mathcal{S}_1(\mathcal{H}_\lambda) \)
\[
\frac{d}{dr}S(\lambda; H_r, H_r) = -2\pi i [Z_\lambda(\lambda)J(1 + (r - r_0)T_\lambda(\lambda + i0)J)^{-1}Z_\lambda^*(\lambda)
- (r - r_0)Z_\lambda(\lambda)J(1 + (r - r_0)T_\lambda(\lambda + i0)J)^{-2}T_\lambda(\lambda + i0)JZ_\lambda^*(\lambda)]
= -2\pi i [Z_\lambda(\lambda)J(1 + (r - r_0)T_\lambda(\lambda + i0)J)^{-2}Z_\lambda^*(\lambda)].
\]
This and (10) complete the proof. \(\square\)

**Proposition 2.4.** If \(H_0 \in \mathcal{A}, V \in \mathcal{A}_0,\) if \(H_r = H_0 + rV,\) \(r \in [0, 1]\) and if Assumption 1.3 holds, then for all \(\lambda \in \Lambda_A\)

\[
S(\lambda; H_1, H_0) = \text{Tr} \left( -2\pi i \int_0^1 w_+(\lambda; H_0, H_r) \Pi_{H_r}(H_r) w_+(\lambda; H_r, H_0) \, dr \right).
\]

**Proof.** It follows from Lemma 2.3 and [30, Corollary 7.1.2] that for all \(\lambda \in \Lambda_A\)

\[
\frac{d}{dr} S(\lambda; H_r, H_0) = -2\pi i w_+(\lambda; H_0, H_r) \Pi_{H_r}(V)(\lambda) w_+(\lambda; H_r, H_0) S(\lambda; H_r, H_0),
\]

where the derivative is taken in \(\mathcal{S}_1(\mathcal{H}_\lambda)\)-topology. The derivative \(dS(\lambda; H_r, H_0)\) is \(\mathcal{S}_1(\mathcal{H}_\lambda)\)-continuous by (14). Since \(S(\lambda; H_r, H_0)\) is also \(\mathcal{S}_1(\mathcal{H}_\lambda)\)-continuous, by the last formula the function \(r \mapsto w_+(\lambda; H_0, H_r) \Pi_{H_r}(V)(\lambda) w_+(\lambda; H_r, H_0)\) is \(\mathcal{S}_1(\mathcal{H}_\lambda)\)-continuous. Hence, integration of the last equation by Lemma A.1 gives (15). \(\square\)

**Theorem 2.5.** Let Assumption 1.3 holds for \(\mathcal{A}.\) If \(H_0, H_1 \in \mathcal{A}\) and if \(\{H_r\}_{r \in [0, 1]}\) is a piecewise linear path in \(\mathcal{A}\) connecting \(H_0\) and \(H_1,\) then for all \(\lambda \in \Lambda_A\) (19) holds.

**Proof.** This follows from [30, Corollary 7.1.2], Proposition 2.4, the multiplicative property of wave operators and Lemma A.2 \(\square\)

The question of whether the piecewise linear path in this theorem can be replaced by piecewise smooth path is open.

It is known that the infinitesimal spectral flow \(\Phi_H(V)\) is a measure on \(\sigma_H.\) Let \(\Phi_H^{(a)}(V)\) and \(\Phi_H^{(s)}(V)\) denote the absolutely continuous and singular parts of this measure. In other words

\[
\Phi_H^{(a)}(V)(\varphi) = \text{Tr} V \varphi(H^{(a)}), \quad \varphi \in C_c(\mathbb{R}),
\]

and

\[
\Phi_H^{(s)}(V)(\varphi) = \text{Tr} V \varphi(H^{(s)}), \quad \varphi \in C_c(\mathbb{R}),
\]

where \(H^{(a)}\) and \(H^{(s)}\) are absolutely continuous and singular parts of \(H.\) We define the "absolutely continuous" and "singular" parts of the spectral shift function \(\xi\) by formulas

\[
\xi^{(a)}(\varphi) = \int_{\Gamma} \Phi^{(a)}(\varphi), \quad \xi^{(s)}(\varphi) = \int_{\Gamma} \Phi^{(s)}(\varphi),
\]

where \(\Gamma\) is any piecewise linear path in \(\mathcal{A},\) connecting \(H_0\) and \(H_1.\) Independence of this definition from the choice of the path \(\Gamma\) will be shown in Corollary 2.8. The absolute continuity of \(\Phi_H^{(a)}(V)\) follows from the following proposition.

**Proposition 2.6.** If \(V\) is a trace compatible perturbation of \(H\) satisfying Assumption 1.3 then for any \(\varphi \in C_c^\infty(\mathbb{R})\) the operator \(V \varphi(H)\) is an integral operator with trace class valued kernel \(v(\lambda, \lambda'), \lambda, \lambda' \in \Lambda_A,\) and

\[
\Phi_H^{(a)}(V)(\varphi) = \int_{\Lambda_A} \text{Tr}_{H_\lambda}(\Pi_H(V)(\lambda)) \varphi(\lambda) \, d\lambda.
\]
The proof of this proposition is standard (see e.g. [7, 30]).

As a corollary we get a variant of the Birman-Kreĭn formula.

**Theorem 2.7.** If $H_0, H_1 \in \mathcal{A}$ and Assumption 1.3 holds, then

\[ -2\pi i \xi^{(a)}_{H_1, H_0}(\lambda) = \log \det S(\lambda; H_1, H_0), \quad \text{a.e. } \lambda \in \Lambda_A, \]

where the branch of the logarithm is chosen in such a way, that the function $r \in [0, 1] \mapsto \log \det S(\lambda; H_r, H_0)$ is continuous, and $\xi^{(a)}$ is defined by (18) with $\Gamma$ a straight line.

**Proof.** Let, as usual, $H_r = H_0 + rV$, $H_0 \in \mathcal{A}$, $V \in \mathcal{A}_0$. By definition, for any $\varphi \in C_c^\infty(\mathbb{R})$

\[ \xi^{(a)}(\varphi) = \int_0^1 \Phi^{(a)}_{H_r}(V)(\varphi) \, dr = \int_0^1 \text{Tr} \left( V \varphi(H_r^{(a)}) \right) \, dr. \]

Hence, by Proposition 2.6, it follows that

\[ \xi^{(a)}(\varphi) = \int_0^1 \int_{\Lambda_A} \text{Tr}_{\mathcal{H}_\lambda^{(r)}} [w(\lambda; H_r, H_0)] \varphi(\lambda) \, d\lambda \, dr, \]

where the Hilbert spaces $\mathcal{H}_\lambda^{(r)}$ are from the direct integral decomposition for $H_r$. Since the operators $w(\lambda; H_r, H_0): \mathcal{H}_\lambda \to \mathcal{H}_\lambda^{(r)}$ are unitary [30 Proposition 5.7.3], from this and Fubini’s theorem it follows that

\[ \xi^{(a)}(\varphi) = \int_{\Lambda_A} \int_0^1 \text{Tr}_{\mathcal{H}_\lambda} [w(\lambda; H_0, H_r) w_r(\lambda, \lambda) w(\lambda; H_r, H_0)] \varphi(\lambda) \, dr \, d\lambda. \]

Proposition 2.4 and Lemma A.3 now imply

\[ -2\pi i \xi^{(a)}(\varphi) = \int_{\Lambda_A} \log \det S(\lambda; H_1, H_0) \varphi(\lambda) \, d\lambda, \]

where the branch of the logarithm is chosen as in the statement of the theorem. Since $\xi^{(a)}$ is absolutely continuous by [4, Theorem 2.9], so is $\xi^{(a)}$. Hence, for a.e. $\lambda \in \Lambda_A$

\[ -2\pi i \xi^{(a)}(\lambda) = \log \det S(\lambda; H_1, H_0). \]

\[ \square \]

**Corollary 2.8.** Definition (18) of $\xi^{(a)}$ and $\xi^{(s)}$ is independent of the choice of the piecewise linear path $\Gamma$ in $\mathcal{A}$.

**Proof.** For $\xi^{(a)}$ this follows from Theorems 2.5 and 2.7 (this also follows from the proof of Theorem 2.7 with reference to Theorem 2.5 instead of Proposition 2.4). For $\xi^{(s)}$ this follows from $\xi = \xi^{(a)} + \xi^{(s)}$ and [4, Theorem 2.9].

**Corollary 2.9.** For the affine space (6) the Birman-Kreĭn formula (3) holds.

**Proof.** Kato’s theorem implies that Schrödinger operators of the class (6) don’t have singular spectrum on the absolutely continuous spectrum. This and Theorem 2.7 imply that for $\lambda \in \Lambda_A$ (5) holds. For $\lambda \notin \Lambda_A$ this follows from the equality $\sigma_{ess} = \sigma_{ac}$, which is known to be true for Schrödinger operators of the class (6), and from Lemma 1.2. \[ \square \]
Let \( e^{i\theta_1(r)} , e^{i\theta_2(r)}, \ldots \) be the set of eigenvalues of \( S(\lambda; H_r, H_0) \). The functions \( \theta_j(r), j = 1, 2, \ldots, \) are continuous functions of \( r \) (see e.g. \cite{30} or \cite{19} IV.3.5). For any \( \theta \in [0, 2\pi) \), one can define Pushnitski’s invariant \( \mu(\theta; \lambda) = \mu(\theta; \lambda, H_1, H_0) \) as the spectral flow through the point \( e^{i\theta} \) by eigenvalues \( e^{i\theta_1(r)}, e^{i\theta_2(r)}, \ldots \) of the path \( \{S(\lambda; H_r, H_0)\}_{r \in [0,1]} \) \cite{32}, i.e.

\[
(20) \quad \mu(\theta; \lambda) = \sum_{j=1}^{\infty} \left( 1 + \frac{\theta_j(1) - \theta}{2\pi} \right),
\]

where \([x]\) is the integer part of \( x \). The formulas \((19)\) and \((20)\) imply that for a.e. \( \lambda \in \Lambda_A \)

\[
(21) \quad \xi(a)(\lambda) = -\frac{1}{2\pi} \sum_{j=1}^{\infty} \theta_j(1) = -\frac{1}{2\pi} \int_{0}^{2\pi} \mu(\theta; \lambda) \, d\theta,
\]

which is Pushnitski’s formula \cite{23} (1.12)]. Theorem 2.5 and \cite{30} §7.8 imply that the definition of \( \mu(\theta; \lambda) \), given by \(20\), does not depend on the choice of the piecewise linear path \( H_r \), connecting \( H_0 \) and \( H_1 \), and it is well-defined in the sense that the eigenvalues \( \theta_j(r) \) do not make excessive windings around the unit circle. We note that in \cite{32} the scattering matrix is connected with 1 by sending the imaginary part \( y \) of the spectral parameter \( \lambda + iy \) to \(+\infty\).

In \cite{2} it will be shown that in the case of a class of Schrödinger operators which admit embedded eigenvalues, the "singular" part \( \xi(s) \) of the spectral shift function is an integer-valued function, which is non-zero on the absolutely continuous spectrum. It will be shown that in this case Pushnitski’s \( \mu \)-invariant admits a natural decomposition \( \mu(\theta, \lambda) = \mu(a)(\theta, \lambda) + \mu(s)(\theta, \lambda) \), where \( \mu(a) \) and \( \mu(s) \) are integer-valued functions, \( \mu(a) \) is non-zero only on absolutely continuous spectrum, while \( \mu(s) \) does not depend on \( \theta \), and actually is equal to \(-\xi(s)\). For \( \mu(a) \) the formulas \(20\) and \(21\) hold. In the case, considered in this note, \( \mu(s) = \xi(s) = 0 \) on the absolutely continuous spectrum. Proofs of these results will be based on the analysis of the behaviour of the eigenvalues the unitary-valued functions \( M(z; H_r, H_0), S(z; H_0, G, rJ), \) introduced by A.B. Pushnitski in \cite{32}, and that of \( S(\lambda; H_r, H_0) \), as \( \text{Im} \, z \to 0^+ \) and \( \text{Im} \, z \to +\infty \). One of the elements of the proof is that the scattering matrix \( S(\lambda; H_r, H_0) \), considered as a function of the coupling constant \( r \), is a meromorphic function, which admits analytical continuation to the real poles \( r_0 \) of \( (J^{-1} + rT_0(\lambda + i0))^{-1} \). As is well known, these poles correspond to embedded eigenvalues (see e.g. \cite{30} Lemma 4.7.8 or the proof of \cite{11} Theorem 4.2]). This result can be interpreted as a jump by an integer multiple of \( 2\pi \) of one of the scattering phases \( \theta_j(r, \lambda) \), when \( r \) crosses the "resonance" point \( r_0 \), i.e. the point, for which the equation \( H_r \psi = \lambda \psi, \lambda \in \Lambda \), has an \( L^2 \) solution, in accordance with Pushnitski’s formula \cite{23} (1.12]). This also agrees with physical interpretation, given in \cite{13} XVIII.6].

It seems to be likely that \( \xi(s) \) is always an integer-valued function.

**Appendix A. Chronological exponential**

In this appendix an exposition of the chronological exponential is given. See e.g. \cite{16} and \cite{12} Chapter 4].
Let \( p \in [1, \infty] \) and let \( a < b \). Let \( A(\cdot) : [a, b] \to \mathfrak{S}_p(\mathcal{H}) \) be a piecewise continuous path of self-adjoint operators from \( \mathfrak{S}_p(\mathcal{H}) \). Consider the equation

\[
\frac{dX(t)}{dt} = \frac{1}{i} A(t)X(t), \quad X(a) = 1,
\]

where the derivative is taken in \( \mathfrak{S}_p(\mathcal{H}) \). By definition, the left chronological exponent is

\[
\text{Exp} \left( \frac{1}{i} \int_a^t A(s) \, ds \right) = 1 + \sum_{k=1}^{\infty} \frac{1}{i^k} \int_a^t dt_1 \int_{t_1}^{t_2} dt_2 \ldots \int_{t_{k-1}}^{t_k} dt_k A(t_1) \ldots A(t_k),
\]

where the series converges in \( \mathfrak{S}_p(\mathcal{H}) \)-norm.

**Lemma A.1.** The equation (22) has a unique continuous solution \( X(t) \), given by formula

\[
X(t) = \text{Exp} \left( \frac{1}{i} \int_a^t A(s) \, ds \right).
\]

**Proof.** Substitution shows that (23) is a continuous solution of (22). Let \( Y(t) \) be another continuous solution of (22). Taking the integral of (22) one gets

\[
Y(t) = 1 + \frac{1}{i} \int_a^t A(s)Y(s) \, ds.
\]

Iteration of this integral and the bound \( \sup_{t \in [a, b]} \| A(t) \|_1 \leq \text{const} \) show that \( Y(t) \) coincides with (23). \( \square \)

**Lemma A.2.** The following equality holds

\[
\text{Exp} \left( \int_s^u A(s) \, ds \right) = \text{Exp} \left( \int_s^t A(s) \, ds \right) \text{Exp} \left( \int_t^u A(s) \, ds \right).
\]

**Proof.** Both sides of this equality are solutions of the equation \( \frac{dX(u)}{du} = \frac{1}{i} A(u)X(u) \) with the initial condition \( X(t) = \text{Exp} \left( \int_s^t A(s) \, ds \right) \). \( \square \)

By \( \det \) we denote the classical Fredholm determinant (see e.g. [27]).

**Lemma A.3.** If \( p = 1 \) then the following equality holds

\[
\det \text{Exp} \left( \frac{1}{i} \int_a^t A(s) \, ds \right) = \exp \left( \frac{1}{i} \int_a^t \text{Tr}(A(s)) \, ds \right).
\]

**Proof.** Let \( F(t) \) and \( G(t) \) be the left and the right hand sides of this equality respectively. Then \( \frac{dG(t)}{dt} = \frac{1}{i} \text{Tr}(A(t))G(t), \quad G(a) = 1 \). Further, by Lemma A.2

\[
\frac{d}{dt} F(t) = \lim_{h \to 0} \frac{1}{h} \left( \det \text{Exp} \left( \frac{1}{i} \int_t^{t+h} A(s) \, ds \right) - 1 \right) F(t) = \frac{1}{i} \text{Tr}(A(t))F(t),
\]

where the last equality follows from definitions of determinant [27 (3.5)], \( \text{Exp} \)
and piecewise continuity of \( A(s) \). \( \square \)
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