Automorphic forms: a physicist’s survey

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Summary. Motivated by issues in string theory and M-theory, we provide a pedestrian introduction to automorphic forms and theta series, emphasizing examples rather than generality.

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References
Automorphic forms play an important rôle in physics, especially in the context of string and M-theory dualities. Notably, U-dualities, first discovered as symmetries of classical toroidal compactifications of 11-dimensional supergravity by Cremmer and Julia [1] and later on elevated to quantum postulates by Hull and Townsend [2], motivate the study of automorphic forms for exceptional arithmetic groups $E_n(\mathbb{Z})$ ($n = 6, 7, 8$, or their $A_n$ and $D_n$ analogues for $1 \leq n \leq 5$) – see e.g. [3] for a review of U-duality. These notes are a pedestrian introduction to these (seemingly abstract) mathematical objects, designed to offer a concrete footing for physicists. The basic concepts are introduced via the simple $SL(2)$ Eisenstein and theta series. The general construction of continuous representations and of their accompanying Eisenstein series is detailed for $SL(3)$. Thereafter we present unipotent representations and their theta series for arbitrary simply-laced groups, based on our recent work with D. Kazhdan [5]. We include a (possibly new) geometrical interpretation of minimal representations, as actions on pure spinors or generalizations thereof. We close with some comments about the physical applications of automorphic forms which motivated our research.

1 Eisenstein and Jacobi Theta series disembodied

The general mechanism underlying automorphic forms is best illustrated by taking a representation-theoretic tour of two familiar $SL(2,\mathbb{Z})$ examples:

1.1 $SL(2,\mathbb{Z})$ Eisenstein series

Our first example is the non-holomorphic Eisenstein series

$$E_s^{SL(2)}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \left( \frac{\tau_2}{m + n\tau_1^2} \right)^s,$$

which, for $s = 3/2$, appears in string theory as the description of the complete, non-perturbative, four-graviton scattering amplitude at low energies [6]. It is a function of the complex modulus $\tau$, taking values on the Poincaré upper half plane, or equivalently points in the symmetric space $\mathcal{M} = K\backslash G = SO(2)\backslash SL(2,\mathbb{R})$ with coset representative

$$e = \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \in SL(2,\mathbb{R}).$$

The Eisenstein series (1) is invariant under the modular transformation

$$\tau \to (a\tau + b)/(c\tau + d),$$

which is the right action of $g \in SL(2,\mathbb{Z})$ on $\mathcal{M}$. Invariance follows simply from that of the lattice $\mathbb{Z} \times \mathbb{Z}$. This set-up may be formalized by introducing:

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1 The more mathematically minded reader may consult the excellent review [4].
(i) The linear representation $\rho$ of $\text{SL}(2, \mathbb{R})$ in the space $\mathcal{H}$ of functions of two variables $f(x, y)$,

$$[\rho(g) \cdot f](x, y) = f(ax + by, cx + dy), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$  (4)

(ii) An $\text{SL}(2, \mathbb{Z})$-invariant distribution

$$\delta_Z(x, y) = \sum_{(m, n) \in \mathbb{Z}^2 \setminus (0, 0)} \delta(x - m)\delta(y - n)$$  (5)

in the dual space $\mathcal{H}^*$. 

(iii) A vector

$$f_K(x, y) = (x^2 + y^2)^{-s}$$  (6)

invariant under the maximal compact subgroup $K = \text{SO}(2) \subset G = \text{SL}(2, \mathbb{R})$.

The Eisenstein series (1) may now be recast in a general notation for automorphic forms

$$E_{s}^{\text{SL}(2)}(e) = \langle \delta_Z, \rho(e) \cdot f_K \rangle, \quad e \in G.$$  (7)

The modular invariance of $E_{s}^{\text{SL}(2)}$ is now manifest: under the right action $e \mapsto eg$ of $g \in \text{SL}(2, \mathbb{Z})$, the vector $\rho(e) \cdot f_K$ transforms by $\rho(g)$, which in turn hits the $\text{SL}(2, \mathbb{Z})$ invariant distribution $\delta_Z$. Furthermore (7) is ensured to be a function of the coset $K \backslash G$ by invariance of the vector $f_K$ under the maximal compact $K$. Such a distinguished vector is known as spherical. All the automorphic forms we shall encounter can be written in terms of a triplet $(\rho, \delta_Z, f_K)$.

Clearly any other function of the $\text{SO}(2)$ invariant norm $|x, y|_{\infty} \equiv \sqrt{x^2 + y^2}$ would be as good a candidate for $f_K$. This reflects the reducibility of the representation $\rho$ in (4). However, its restriction to homogeneous, even functions of degree $2s$,

$$f(x, y) = \lambda^{2s} f(\lambda x, \lambda y) = y^{-2s} f\left(\frac{x}{y}, 1\right),$$  (8)

is irreducible. The restriction of the representation $\rho$ acts on the space of functions of a single variable $z = x/y$ by weight $2s$ conformal transformations $z \mapsto (az + b)/(cz + d)$ and admits $f_K(z) = (1 + z^2)^{-s}$ as its unique spherical vector. In these variables, the distribution $\delta_Z$ is rather singular as its support is on all rational values $z \in \mathbb{Q}$. A related problem is that the behavior of $E_{s}^{\text{SL}(2)}(\tau)$ at the cusp $\tau \to i\infty$ is difficult to assess – yet of considerable interest to physicists being the limit relevant to non-perturbative instantons [6].

These two problems may be evaded by performing a Poisson resummation on the integer $m \to \tilde{m}$ in the sum (5), after first separating out terms with $n = 0$. The result may be rewritten as a sum over the single variable $N = \tilde{m}n$, except for two degenerate – or “perturbative” – contributions:
\[ E_s^{\text{Sl}(2)} = 2 \zeta(2s) \tau_2^s + \frac{2\sqrt{\pi} \tau_2^{1-s} \Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) + \frac{2\pi^{3/2} \tau_2}{\Gamma(s)} \sum_{N \in \mathbb{Z} \setminus \{0\}} \mu_s(N) N^{s-1/2} \Gamma(s-1/2)(2\pi\tau_2 N) e^{2\pi i \tau_1 N}. \]  

In this expression, the summation measure

\[ \mu_s(N) = \sum_{n \mid N} n^{-2s+1}, \]  

is of prime physical interest, as it is connected to quantum fluctuations in an instanton background [7, 8, 9].

First focus on the non-degenerate terms in the second line. Analyzing the transformation properties under the Borel and Cartan \( \text{Sl}(2) \) generators \( \rho \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \tau_1 \to \tau_1 + t \) and \( \rho \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} : \tau_2 \to t^2 \tau_2 \), we readily see that they fit into the framework (7), upon identifying \( f_K(z) = z^{s-1/2} K_{s-1/2}(z) \), \( \delta_Z(z) = \sum_{N \in \mathbb{Z} \setminus \{0\}} \mu_s(N) \delta(z - N) \), and the representation \( \rho \) as

\[ E_+ = iz, \quad E_- = i(z \partial_z + 2 - 2s) \partial_z, \quad H = 2z \partial_z + 2 - 2s. \]  

This is of course equivalent to the representation on homogeneous functions (8), upon Fourier transform in the variable \( z \). The power-like degenerate terms in (9) may be viewed as regulating the singular value of the distribution \( \delta \) at \( z = 0 \). They may, in principle, be recovered by performing a Weyl reflection on the regular part. It is also easy to check that the spherical vector condition, \( K \cdot f_K(z) \equiv (E_+ - E_-) \cdot f_K(z) = 0 \), is the modified Bessel equation whose unique decaying solution at \( z \to \infty \) is the spherical vector in (11).

While the representation \( \rho \) and its spherical vector \( f_K \) are easily understood, the distribution \( \delta_Z \) requires additional technology. Remarkably, the summation measure (10) can be written as an infinite product

\[ \mu_s(z) = \prod_{p \text{ prime}} f_p(z), \quad f_p(z) = \frac{1 - p^{2s+1}|z|_p^{2s-1}}{1 - p^{-2s+1}} \gamma_p(z). \]  

(A simple trial computation of \( \mu_s(2 \cdot 3^2) \) will easily convince the reader of this equality.) Here \( |z|_p \) is the \( p \)-adic\(^4 \) norm of \( z \), i.e. \( |z|_p = p^{-k} \) with \( k \) the largest integer such that \( p^k \) divides \( z \). The function \( \gamma_p(z) \) is unity if \( z \) is a \( p \)-adic

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\(^4 \) A useful physics introduction to \( p \)-adic and adelic fields is [10]. It is worth noting that a special function theory analogous to that over the complex numbers exists for the \( p \)-adics.
integer \(|z|_p \leq 1\) and vanishes otherwise. Therefore \(\mu(z)\) vanishes unless \(z\) is an integer \(N\). Equation (7) can therefore be expressed as

\[
\mathcal{E}_s^{SL(2)}(e) = \sum_{z \in \mathbb{Q}} \prod_{p = \text{prime}, \infty} f_p(z) \rho(e) \cdot f_K(z),
\]

(14)

The key observation now is that \(f_p\) is in fact the spherical vector for the representation of \(SL(2, \mathbb{Q}_p)\), just as \(f_\infty := f_K\) is the spherical vector of \(SL(2, \mathbb{R})\)!

In order to convince herself of this important fact, the reader may evaluate the \(p\)-adic Fourier transform of \(f_p(y)\) on \(y\), thereby reverting to the \(SL(2)\) representation on homogeneous functions (8): the result

\[
\tilde{f}_p(x) = \int_{\mathbb{Q}_p} dz f_p(z) e^{ixz} = |1, x|_p^{-2s} \equiv \max(1, |x|_p)^{-2s},
\]

(15)

is precisely the \(p\)-adic counterpart of the real spherical vector \(f_K(x) = (1 + x^2)^{-s} \equiv |1, x|_{\infty}^{-2s}\). The analogue of the decay condition is that \(f_p\) should have support over the \(p\)-adic integers only, which holds by virtue of the factor \(\gamma_p(y)\) in (13). It is easy to check that the formula (14) in this representation reproduces the Eisenstein series (1).

The \(SL(2, \mathbb{Z})\)-invariant distribution \(\delta_\mathbb{Z}\) can be straightforwardly obtained by computing the spherical vector over all \(p\)-adic fields \(\mathbb{Q}_p\). More conceptually, the Eisenstein series (1) may be written \textit{ad elically} (or \textit{globally}) as

\[
\mathcal{E}_s^{SL(2)}(e) = \sum_{z \in \mathbb{Q}} \rho(e) \cdot f_A(z), \quad f_A(z) = \prod_{p = \text{prime}, \infty} f_p(z),
\]

(16)

where the sum \(z \in \mathbb{Q}\) is over principle adeles\(^5\), and \(f_A\) is the spherical vector of \(SL(2, A)\), invariant under the maximal compact subgroup \(K(A) = \prod_p SL(2, \mathbb{Z}_p) \times U(1)\) of \(SL(2, \mathbb{A})\). This relation between functions on \(G(\mathbb{Z}) \backslash G(\mathbb{R}) / K(\mathbb{R})\) and functions on \(G(\mathbb{Q}) \backslash G(\mathbb{A}) / K(\mathbb{A})\) is known as the Strong Approximation Theorem, and is a powerful tool in the study of automorphic forms (see e.g. [4] for a more detailed introduction to the adelic approach).

### 1.2 Jacobi theta series

Our next example, the Jacobi theta series, demonstrates the key rôle played by Fourier invariant Gaussian characters – “the Fourier transform of the Gaussian is the Gaussian”. Our later generalizations will involve cubic type characters invariant under Fourier transform.

In contrast to the Eisenstein series, the Jacobi theta series

\[
\theta(\tau) = \sum_{m \in \mathbb{Z}} e^{i\pi \tau m^2},
\]

(17)

\(^5\) Adeles are infinite sequences \((z_p)_{p=\text{prime}, \infty}\) where all but a finite set of \(z_p\) are \(p\)-adic integers. Principle adeles are constant sequences \(z_p = z \in \mathbb{Q}\), isomorphic to \(\mathbb{Q}\) itself.
is a modular form for a congruence subgroup $\Gamma_0(2)$ of $SL(2,\mathbb{Z})$ with modular weight $1/2$ and a non-trivial multiplier system. It may, nevertheless, be cast in the framework (7), with a minor caveat. The representation $\rho$ now acts on functions of a single variable $x$ as

$$E_+ = i\pi x^2, \quad H = \frac{1}{2}(x\partial_x + \partial_x x), \quad E_- = \frac{i}{4\pi} \partial_x^2,$$

(18)

Here, the action of $E_+$ and $H$ may be read off from the usual Borel and Cartan actions of $SL(2)$ on $\tau$ while the generator $E_-$ follows by noting that the Weyl reflection $S : \tau \to -1/\tau$ can be compensated by Fourier transform on the integer $m$. The invariance of the “comb” distribution $\delta_Z(x) \equiv \sum_{m \in \mathbb{Z}} \delta(x - m)$ under Fourier transform is just the Poisson resummation formula.

Finally (the caveat), the compact generator $K = E_+ - E_-$ is exactly the Hamiltonian of the harmonic oscillator, which notoriously does not admit a normalizable zero energy eigenstate, but rather the Fourier-invariant ground state $f_\infty(x) = e^{-\pi x^2}$ of eigenvalue $i/2$. This relaxation of the spherical vector condition is responsible for the non-trivial modular weight and multiplier system. Correspondingly, $\rho$ does not represent the group $SL(2,\mathbb{R})$, but rather its double cover, the metaplectic group.

Just as for the Eisenstein series, an adelic formula for the summation measure exists: note that the $p$-adic spherical vector must be invariant under the compact generator $S$ which acts by Fourier transform. Remarkably, the function $f_p(x) = \gamma_p(x)$, imposing support on the integers only is Fourier invariant – it is the $p$-adic Gaussian! One therefore recovers the “comb” distribution with uniform measure. Note that the $SL(2) = Sp(1)$ theta series generalizes to higher symplectic groups under the title of Siegel theta series, relying in the same way on Gaussian Poisson resummation.

## 2 Continuous representations and Eisenstein series

The two $SL(2)$ examples demonstrate that the essential ingredients for automorphic forms with respect to an arithmetic group $G(\mathbb{Z})$ are (i) an irreducible representation $\rho$ of $G$ and (ii) corresponding spherical vectors over $\mathbb{R}$ and $\mathbb{Q}_p$. We now explain how to construct these representations by quantizing coadjoint orbits.

### 2.1 Coadjoint orbits, classical and quantum: $SL(2)$

As emphasized by Kirillov, unitary representations are quite generally in correspondence with coadjoint orbits [11]. For simplicity, we restrict ourselves to finite, simple, Lie algebras $\mathfrak{g}$, where the Killing form $\langle \cdot, \cdot \rangle$ identifies $\mathfrak{g}$ with its dual. Let $\mathcal{O}_j$ be the orbit of an element $j \in \mathfrak{g}$ under the action of $G$ by the adjoint representation $j \to gg^{-1} = j$. Equivalently, $\mathcal{O}_j$ may be viewed as an homogeneous space $S\backslash G$, where $S$ is the stabilizer (or commutant) of $j$. 

The (co)adjoint orbit $O_j$ admits a (canonical, up to a multiplicative constant) $G$-invariant Kirillov–Kostant symplectic form, defined on the tangent space at a point $\tilde{j}$ on the orbit by $\omega(x, y) = (\tilde{j}, [x, y])$. Non-degeneracy of $\omega$ is manifest, since its kernel, the commutant $S$ of $\tilde{j}$, is gauged away in the quotient $S\backslash G$. Parameterizing $O_j$ by an element $e$ of $S\backslash G$, one may rewrite $\omega = d\theta$ where the “contact” one-form $\theta = (j, de e^{-1})$, making the closedness and $G$-invariance of $\omega$ manifest. The coadjoint orbit $O_j = S\backslash G$ therefore yields a classical phase space with a $G$-invariant Poisson bracket and hence a set of canonical generators representing the action of $G$ on functions of $O_j$.

The representation $\rho$ associated to $j$ follows by quantizing this classical action, i.e. by choosing a Lagrangian subspace $L$ (a maximal commuting set of observables) and representing the generators of $G$ as suitable differential operators on functions on $L$.

This apparently abstract construction is simply illustrated for $Sl(2)$: consider the coadjoint orbit of the element

$$j = \left( \begin{array}{cc} \frac{l}{2} & \frac{-i}{2} \\ \frac{i}{2} & \frac{l}{2} \end{array} \right),$$

with stabilizer $S = \mathbb{R} j$. The quotient $S\backslash G$ may be parameterized as

$$e = \left( \begin{array}{c} 1 \\ \gamma \end{array} \right), \quad \left( \begin{array}{c} 1 \\ \beta \end{array} \right).$$

(20)

The contact one-form is

$$\theta = tr j de e^{-1} = -l\gamma d\beta.$$  

(21)

The group $G$ acts by right multiplication on $e$, followed by a compensating left multiplication by $S$ maintaining the choice of gauge slice (20). The resulting infinitesimal group action is expressed in terms of Hamiltonian vector fields

$$E_+ = il\gamma, \quad H = 2i\beta\partial_\beta - 2\gamma\partial_\gamma, \quad E_- = -i\beta^2\partial_\beta + i(1 + 2\beta\gamma)\partial_\gamma.$$  

(22)

We wish to express these transformations in terms of the Poisson bracket determined by the Kirillov–Kostant symplectic form

$$\omega = d\theta = l \, d\gamma \wedge d\beta,$$

(23)

namely

$$\{\gamma, \beta\}_{PB} = \frac{1}{l}.$$  

(24)

Indeed, it is easily verified that the generators (22) can be represented canonically

$$E_+ = il\gamma, \quad H = 2i\beta\gamma, \quad E_- = -il(1 + \beta\gamma),$$

(25)

with respect to the Poisson bracket (24). The next step is to quantize this classical mechanical system:
The quantized coadjoint orbit representation follows directly by substituting (26) in (25) and the result is precisely the Eisenstein series representation (12). The physicist reader will observe that the parameter $s$ appearing there arises from quantum orderings of the operators $\beta$ and $\gamma$.

The construction just outlined, based on the quantization of an element $j$ in the hyperbolic conjugacy class of $SL(2, \mathbb{R})$, leads to the continuous series representation of $SL(2, \mathbb{R})$. Recall that conjugacy classes of $SL(2)$ are classified by the value $6$ of $C \equiv 2 \text{tr}j^2 = l^2 > 0$. The elliptic case $C < 0$ with $j$ conjugate to an antisymmetric matrix leads to discrete series representations and will not interest us in these Notes. However, the non-generic parabolic (or nilpotent) conjugacy class $C = 0$ is of considerable interest, being key to theta series for higher groups. There is only a single nilpotent conjugacy class with representative

$$j = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad j^2 = 0. \quad (27)$$

The stabilizer $S \subset SL(2, \mathbb{R})$ is the parabolic group of lower triangular matrices so the nilpotent orbit $S \setminus G$ may be parameterized as

$$e = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \gamma \\ 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 1 & 1 \end{pmatrix}. \quad (28)$$

The contact and symplectic forms are now

$$\theta = \gamma d\beta, \quad \omega = d\gamma \wedge d\beta, \quad (29)$$

and the action of $SL(2)$ may be represented by the canonical generators

$$E_+ = i\gamma, \quad H = 2i\beta\gamma, \quad E_- = -i\beta^2\gamma \quad (30)$$

accompanied by Poisson bracket \{\gamma, \beta\}_{PB} = 1. This representation also follows by the contraction $l \to 0$ holding $l\gamma$ fixed in (25). The relation to theta series is exhibited by performing a canonical transformation $\gamma = y^2$ and $\beta = \frac{1}{\sqrt{l}} p/y$ which yields

$$E_+ = iy^2, \quad H = ipy, \quad E_- = -\frac{i}{4} p^2. \quad (31)$$

Upon quantization, this is precisely the metaplectic representation in (18). In contrast to the continuous series, there is no quantum ordering parameter (although a peculiarity of $SL(2)$ is that it appears as the $s = 1$ instance of the continuous series representation (12)).

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6 The geometry of the three coadjoint orbits is exhibited by parameterizing the $sl(2)$ Lie algebra as $\mathfrak{g} = \begin{pmatrix} k_1 & k_2 + k_0 \\ k_2 - k_0 & -k_1 \end{pmatrix}$. The orbits are then seen to correspond to massive, lightlike and tachyonic $2 + 1$ dimensional mass-shells $k_\mu k^\mu = -k_0^2 + k_1^2 + k_2^2 = -\frac{1}{l^2}$. 

2.2 Coadjoint orbits: general case

For general groups $G$, the orbit method predicts the Gelfand-Kirillov dimension\(^7\) of the generic continuous irreducible representation to be $(\dim G - \text{rank } G)/2$: a generic non-compact element may be conjugated into the Cartan algebra, whose stabilizer is the Cartan (split) torus. There are, therefore, rank $G$ parameters corresponding to the eigenvalues in the Cartan subalgebra. Non-generic elements arise when eigenvalues collide, and lead to representations of smaller functional dimension. When all eigenvalues degenerate to zero, there are a finite set of conjugacy class of nilpotent elements with non-trivial Jordan patterns, hence a finite set of parameter-less representations usually called “unipotent”. The nilpotent orbit of smallest dimension, namely the orbit of any root, leads to the minimal unipotent representation, which plays a distinguished rôle as the analog of the $Sl(2)$ (Jacobi theta series) metaplectic representation [12].

2.3 Quantization by induction: $Sl(3)$

Given a symplectic manifold with $G$-action, there is no general method to resolve the quantum ordering ambiguities while maintaining the $g$-algebra. However, (unitary) induction provides a standard procedure to extend a representation $\rho_H$ of a subgroup $H \subset G$ to the whole of $G$. Let us illustrate the first non-trivial case: the generic orbit of $Sl(3)$.

Just as for $Sl(2)$ in (20), the coadjoint orbit of a generic $sl(3)$ Lie algebra element

$$j = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, \quad (32)$$

can be parameterized by the gauge-fixed $Sl(3)$ group element

$$e = \begin{pmatrix} 1 & y & 1 \\ y & 1 & 1 \\ w + yu & u & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x + vz \\ v & 1 & z \\ z & 1 \end{pmatrix}, \quad (33)$$

whose six-dimensional phase space is equipped with the contact one-form

$$\theta = (l_2 - l_1)ydx + (l_3 - l_2)wdz + [(l_3 - l_1)w + (l_3 - l_2)y](dv + xdz). \quad (34)$$

(The canonical generators are easily calculated.) To quantize this orbit, a natural choice of Lagrangian submanifold is $w = y = u = 0$ so that $Sl(3)$ is realized on functions of three variables $(x, z, v)$. These variables parameterize the coset $P\setminus G$, where $P = P_{1,1,1}$ is the (minimal) parabolic subgroup of $G$.

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\(^7\) The Gel’fand–Kirillov, or functional dimension counts the number of variables – being unitary, all these representations of non-compact groups are of course infinite dimensional in the usual sense.
lower triangular matrices (look at equation (33)). A set of one-dimensional representations on $P$ are realized by the character

$$\chi(p) = \prod_{i=1}^{3} |a_{ii}|^{\rho_i} \text{sgn}^\epsilon(a_{ii}), \quad p = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in P, \quad (35)$$

where $\rho_i$ are three constants (defined up to a common shift $\rho_i \rightarrow \rho_i + \sigma$) and $\epsilon_i \in \{0, 1\}$ are three discrete parameters. The representation of $G$ on functions of $P \setminus G$ induced from $P$ and its character representation (35) acts by

$$g : f(e) \mapsto \chi(p)f(eg^{-1}), \quad (36)$$

where $eg^{-1} = pe'$ and $e' \in P \setminus G$ (coordinatized by $\{x, z, v\}$). It is straightforward to obtain the corresponding generators explicitly,

$$E_\beta = \partial_x - z\partial_v \quad E_{-\beta} = x^2 \partial_x + vz\partial_v + (\rho_2 - \rho_1)x$$
$$E_\gamma = \partial_x \quad E_{-\gamma} = z^2 \partial_x + vz\partial_v - (v + xz)\partial_x + (\rho_3 - \rho_2)z$$
$$E_\omega = \partial_v \quad E_{-\omega} = vz\partial_v + x(v + xz)\partial_x + (\rho_3 - \rho_1)v + (\rho_2 - \rho_1)xz$$

$$H_\beta = 2x\partial_x + vz\partial_v - z\partial_z + (\rho_2 - \rho_1) \quad H_\gamma = -x\partial_x + vz\partial_v + 2z\partial_z + (\rho_3 - \rho_2), \quad (37)$$

where $sl(3)$ generators are defined by,

$$sl(3) \ni X = \begin{pmatrix} -\frac{2}{3}H_\beta - \frac{1}{3}H_\gamma & E_\beta & E_\omega \\ -E_{-\beta} & -\frac{1}{3}H_\gamma + \frac{4}{3}H_\beta & E_\gamma \\ -E_{-\omega} & -E_{-\gamma} & \frac{2}{3}H_\gamma + \frac{4}{3}H_\beta \end{pmatrix}. \quad (38)$$

For later use, we evaluate the action of the Weyl reflection $A$ with respect to the root $\beta$ which exchanges the first and second rows of $e$ up to a compensating $P$ transformation,

$$[A \cdot f](x, v, z) = x^{\rho_2 - \rho_1} f(-z, v, -1/x). \quad (39)$$

The quadratic and cubic Casimir invariants $C_2 = \frac{1}{12} \Tr X^2$ and $C_3 = \frac{27}{2} \det X$,

$$C_2 = \frac{1}{6} \left[ (\rho_1 - \rho_2)^2 + (\rho_2 - \rho_3)^2 + (\rho_3 - \rho_1)^2 \right] + (\rho_1 - \rho_3), \quad (40)$$
$$C_3 = -\frac{1}{2} \left[ (\rho_1 + \rho_2 - 2\rho_3 + 3) [\rho_2 + \rho_3 - 2\rho_1 - 3] [\rho_3 + \rho_1 - 2\rho_2] \right], \quad (41)$$

agree with those of the classical representation on the 6-dimensional phase space $\{x, y, z, u, v, w\}$, upon identifying $\ell_i = \rho_i$ and removing the subleading “quantum ordering terms”.

The same procedure works in the case of a nilpotent coadjoint orbit. As an Exercise, the reader may show that the maximal nilpotent orbit of a single $3 \times 3$ Jordan block has dimension 6 and can be quantized by induction from the same minimal parabolic $P_{1,1,1}$. The nilpotent orbit corresponding to an $2 + 1$
block decomposition on the other hand has dimension 4, leading to a unitary, functional dimension 2, representation of $\text{SL}(3)$ induced from the (maximal) parabolic $P_{2,1}$. This is the minimal representation of $\text{SL}(3)$, or simpler, the $\text{SL}(3)$ action on functions of projective $\mathbb{RP}^3$.

In fact, all irreducible unitary representations of $\text{SL}(3,\mathbb{R})$ are classified as representations induced from (i) the maximal parabolic subgroup $P_{1,1,1}$ by the character $\chi(p)$ (with $\rho_i \in i\mathbb{C}$), or (ii) the parabolic subgroup $P_{1,2}$ by an irreducible unitary representation of $\text{SL}(2)$ of the discrete, supplementary or degenerate series [13].

2.4 Spherical vector and Eisenstein series

The other main automorphic form ingredient, the spherical vector, turns out to be straightforwardly computable in the $\text{SL}(n)$ representation unitarily induced from the parabolic subgroup $P$. We simply need a $P$-covariant, $K$-invariant function on $G$. For simplicity, consider again $\text{SL}(3)$ and denote the three rows of the second matrix in (33) as $e_1, e_2, e_3$. Under left multiplication by a lower triangular matrix $p = (a_{i,j} \leq j) \in P$, $e_1 \mapsto a_{11}e_1$ and $e_2 \mapsto a_{21}e_1 + a_{22}e_2$. Therefore the norms of $|e_1|_{\infty}$ and $|e_1 \wedge e_2|_{\infty}$ are $P$-covariant and maximal compact $K = \text{SO}(3)$-invariant. The spherical vector over $\mathbb{R}$ is the product of these two norms raised to powers corresponding to the character $\chi$ in (35),

$$f_{\infty} = |1, x, v + xz|^{\rho_1 - \rho_2}_{\infty} |1, v, z|^{\rho_2 - \rho_3}_{\infty}. \quad (42)$$

(Recall that $|\cdot|_{\infty}$ is just the usual orthogonal Euclidean norm.) Similarly, the spherical vector over $\mathbb{Q}_p$ is the product of the $p$-adic norms,

$$f_p = |1, x, v + xz|^{\rho_1 - \rho_2}_p |1, v, z|^{\rho_2 - \rho_3}_p. \quad (43)$$

The $\text{SL}(3,\mathbb{Z})$, continuous series representation, Eisenstein series follows by summing over principle adeles,

$$\mathcal{E}_{\rho_i}^{\text{SL}(3)}(e) = \sum_{(x,z,v) \in \mathbb{Q}^3} \left[ \prod_{p \text{ prime}} f_p \right] \rho(e) \cdot f_{\infty}. \quad (44)$$

Writing out the adelic product in more mundane terms,

$$\mathcal{E}_{\rho_i}^{\text{SL}(3)}(e) = \sum_{(m^i,n^j) \in \mathbb{Z}^6, \atop m^i j \neq 0} \left[ \frac{(m^i)^2}{2} \right]^{\rho_i - \rho_j} \left[ \frac{(m^j)^2}{2} \right]^{\rho_j - \rho_3}, \quad (45)$$

where $m^i j = m^i n^j - m^j n^i$. As usual, the sum is convergent for $\text{Re}(\rho_i - \rho_j)$ sufficiently large and can be analytically continued to complex $\rho_i$ using functional relations representing the Weyl reflections on the weights $(\rho_i)$. The above procedure suffices to describe Eisenstein series for all finite Lie groups.
2.5 Close encounters of the cube kind

Cubic characters are central to the construction of minimal representations and their theta functions for higher simply laced groups $D_n$ and $E_{6,7,8}$. They can also be found in a particular realization of the $SL(3)$ continuous series representation at $\rho_i = 0$ (which also turns out to arise by restriction of the minimal representation of $G_2$ [12]) : let us perform the following (mysterious) sequence of transformations: (i) Fourier transform over $v, z$, and call the conjugate variables $\partial_z = ix_0, \partial_v = iy$. (ii) Redefine $x = 1/(py^2) + x_0/y$. (iii) Fourier transform over $p$ and redefine the conjugate variable $p_p = x_1^3$. These operations yield generators,

$$E_\beta = y\partial_0$$
$$E_{-\beta} = -x_0\partial + i\frac{x_0^3}{y^2}$$
$$E_\gamma = ix_0$$
$$E_{-\gamma} = -i(y\partial + x_0\partial_0 + x_1\partial_1)\partial_0 + \frac{1}{27}y\partial_1^3 + \frac{4x_0\partial_1}{y^2} + \frac{28y\partial_0}{27x_1^2} - 6i\partial_0$$
$$E_\omega = iy$$
$$E_{-\omega} = -i(y\partial + x_0\partial_0 + x_1\partial_1)\partial - \frac{1}{27}x_0\partial_1^3 - \frac{4x_0\partial_1}{y^2} - \frac{28y\partial_0}{27x_1^2} - 6i\partial$$
$$H_\beta = -y\partial + x_0\partial_0$$
$$H_{-\beta} = -y\partial - 2x_0\partial_0 - x_1\partial_1 - 2 - 4s.$$  (46)

where $\partial \equiv \partial_y$ and $\partial_0 \equiv \partial_{x_0}$. The virtue of this presentation is that the positive root Heisenberg algebra $[E_\beta, E_\gamma] = E_\omega$ is canonically represented. In addition, the Weyl reflection with respect to the root $\beta$ is now very simple,

$$[A \cdot f](y, x_0, x_1) = e^{i\frac{x_0^3}{y^2}}f(-x_0, y, x_1)$$  (47)

and the phase is cubic! Notice that the same cubic term appears in the expression for $E_{-\beta}$. Indeed, the spherical vector condition for the compact generator $K_\beta = E_\beta + E_{-\beta}$ has solution

$$f_K(y, x_0, x_1) = \exp\left[-\frac{ix_0^3}{y(y^2 + x_0^2)}\right] g(y^2 + x_0^2),$$  (48)

which implies an automorphic theta series formula summing over cubic rather than Gaussian characters [14].

3 Unipotent representations and theta series

The above construction of $SL(3)$ Eisenstein series based on continuous series representations extends easily to $SL(n)$ and (modulo some extra work) any

This sequence of transformations also makes sense at $\rho_2 \neq \rho_3$ as long as $\rho_1 = \rho_2$. 

\[8\]
simple Lie group: they generalize the non-holomorphic $Sl(2)$ Eisenstein series (1). However, the Jacobi theta series (17) and its generalizations, without any dependence on free parameters, is often more suited to physical applications. Theta series can be obtained as residues of Eisenstein series at special points in their parameter space. Instead, here we wish to take a representation theoretic approach to theta series, based on automorphic forms coming from nilpotent orbits.

### 3.1 The minimal representation of (A)DE groups

The first step in gathering the various components of formula (7) is to construct the minimal representation $\rho$ associated to a nilpotent orbit of simple Lie groups $G$ other than $A_n$ (there are many different constructions of the minimal representation in the literature, e.g. [15, 16, 17, 18, 19], see also [20] for a physicist’s approach based on Jordan algebras; we shall follow [15]). We will always consider the maximally split real form of $G$. Minimality is ensured by selecting the nilpotent orbit of smallest dimension: the orbit of the longest root $E_{-\omega} = j$ is a canonical choice. This orbit can be described by grading the Lie algebra $\mathfrak{g}$ with the Cartan generator $H_\omega = [E_\omega, E_{-\omega}]$ (or equivalently studying the branching rule for the adjoint representation under the $Sl(2)$ subgroup generated by $\{E_\omega, H_\omega, E_{-\omega}\}$). The resulting 5-grading of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

where the one-dimensional spaces $\mathfrak{g}_{\pm 2}$ are spanned by the highest and lowest roots $E_{\pm \omega}$. Therefore the space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Heisenberg algebra of dimension $\dim \mathfrak{g}_1 + 1$ with central element $E_\omega$. Furthermore, since $[\mathfrak{g}_0, \mathfrak{g}_{\pm 2}] = \mathfrak{g}_{\pm 2}$, we have $\mathfrak{g}_0 = m \oplus H_\omega$ where $[m, E_{\pm \omega}] = 0$. The Lie algebra $m$ generates the Levi subgroup $M$ of a parabolic group $P = MU$ with unipotent radical $U = \exp \mathfrak{g}_1$. Hence the coadjoint orbit of $E_{-\omega}$ is parameterized by $H_\omega \oplus \mathfrak{g}_1 \oplus E_\omega$, the orthogonal complement of its stabilizer. Its dimension is twice the dimension $d$ of the minimal representation obtained through its quantization and is listed in Table 1.

To quantize the minimal nilpotent orbit, note that the symplectic vector space $\mathfrak{g}_1$ admits a canonical polarization chosen by taking as momentum variables the positive root $\beta_0$ attached to the highest root $\omega$ on the extended Dynkin diagram, along with those positive roots $\beta_i = \ldots, d - 2$ with Killing inner products $\beta_0, \beta_i) = 1$. The conjugate position variables are then $\gamma_i = \omega - \beta_i$. These generators are given by the Heisenberg representation $\rho_M$ acting on functions of $d$ variables,

$$E_\omega = iy, \quad E_{\beta_i} = y \partial x_0, \quad E_{\gamma_i} = ix_0, \quad i = 0, \ldots, d - 2$$

---

9 Recall that a parabolic group $P$ of upper block-triangular matrices (with a fixed given shape) decomposes as $P = MU$ where the unipotent radical $U$ is the subgroup with unit matrices along the diagonal blocks while the Levi $M$ is the block diagonal subgroup.
So far the generator $y$ is central. By the Shale–Weil theorem [21], $\rho_H$ extends to a representation of the double cover of the symplectic group $Sp(d-1)$. The latter contains the Levi $M$ with trivial central extension of $Sp(2d)$ over $M$. In physics terms, the Levi $M$ acts linearly on the positions and momenta by canonical transformations. In particular, the longest element $S$ in the Weyl group of $M$ is represented by Fourier transform,

$$[S \cdot f](y, x_0, \ldots, x_{d-2}) = \int \prod_{i=0}^{d-2} \frac{dp_i}{\sqrt{2\pi \lambda}} f(y, p_0, \ldots, p_{d-2}) e^{\frac{i}{\lambda} \sum_{i=0}^{d-2} p_i x_i}.$$  

(51)

The subgroup $L \subset M$ commuting with $E_{\beta_0}$, does not mix positions and momenta and therefore acts linearly on the variables $x_{i=1\ldots d-2}$ while leaving $(y, x_0)$ invariant. The representation of the parabolic subgroup $P$ can be extended to $P_0 = P \times \exp tH_{\beta_0}$ (where $\exp tH_{\beta_0}$ is the one-parameter subgroup generated by $H_{\beta_0} = [E_{\beta_0}, E_{-\beta_0}]$) by defining

$$H_{\beta_0} = -y \partial_y + x_0 \partial_{x_0},$$  

(52)

(here $\partial \equiv \partial_y$ and $\partial_0 \equiv \partial_{x_0}$). Notice that the element $y$, which played the rôle of $h$ before, is no longer central. To extend this representation to the whole of $G$, note that Weyl reflection with respect to the root $\beta_0$ acts just as in the $SL(3)$ case (47),

$$[A \cdot f](y, x_0, x_1, \ldots, x_{d-2}) = e^{-iI_3/y} f(-x_0, y, x_1, \ldots, x_{d-2}).$$  

(53)

In this formula, $I_3(x_i)$ is the unique $L$-invariant (normalized) homogeneous, cubic, polynomial in the $x_{i=1\ldots d-2}$ (see Table 1). Remarkably, the Weyl group relation

$$(AS)^3 = (SA)^3$$  

holds, thanks to the invariance of the cubic character $e^{-iI_3/y}$ under Fourier transform over $x_{i=0\ldots d-2}$ [22] (see also [23]). This is the analog of the Fourier invariance of the Gaussian character for the symplectic theta series. It underlies the minimal nilpotent representation and its theta series.
The remaining generators are obtained by applying the Weyl reflections $A$ and $S$ to the Heisenberg subalgebra (50). In particular, the negative root $E_{-\beta_0}$ takes the universal form,

$$E_{-\beta_0} = -x_0 \partial + \frac{iI_3}{y^2}$$

which we first encountered in the $SL(3)$ example (46).

It is useful to note that this construction can be cast in the language of Jordan algebras: $L$ is in fact the reduced structure group of a cubic Jordan algebra $J$ with norm $I_3$; $M$ and $G$ can then be understood as the “conformal” and “quasi-conformal” groups associated to $J$. The minimal representation arises from quantizing the quasi-conformal action – see [28] for more details on this approach, which generalizes to all semi-simple algebras including the non simply-laced cases.

3.2 $D_4$ minimal representation and strings on $T^4$

As illustration, we display the minimal representation of $SO(4,4)$ [24] (see [25] for an alternative construction). The extended Dynkin diagram is

```
    3
   / \  \
  \   \  /
   \ / \ 
    1 -- 2 -- 4
     \   /  \
      \ / \
      \ / \
       \ / \
        \ / \
        \ / \
        \ / 
```

and the affine root $-\omega$ attaches to the root $\beta_0$. The grade-1 symplectic vector space $g_1$ is spanned by $4 + 4$ roots

$$\begin{align*}
\beta_0 &= \beta_0 + \alpha_1 + \alpha_2 + \alpha_3 \\
\beta_i &= \beta_0 + \alpha_i \\
\gamma_i &= \beta_0 + \alpha_j + \alpha_k \quad \{i,j,k\} = \{1,2,3\}.
\end{align*}$$

(56)

The positive roots are represented as in (50), while the negative roots read

$$\begin{align*}
E_{-\beta_0} &= -x_0 \partial + \frac{i x_1 x_2 x_3}{y^2} \\
E_{-\beta_i} &= x_1 \partial + \frac{x_1}{y} (1 + x_2 \partial_2 + x_3 \partial_3) - i x_0 \partial_2 \partial_3 \\
E_{-\gamma_0} &= 3i \partial_0 + iy \partial_0 - y \partial_1 \partial_2 \partial_3 + i(x_0 \partial_0 + x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3) \partial_0 \\
E_{-\gamma_1} &= iy \partial_1 \partial + i(2 + x_0 \partial_0 + x_1 \partial_1) \partial_1 - \frac{x_2 x_3}{y} \partial_0 \\
E_{-\omega} &= 3i \partial + iy \partial^2 + \frac{i}{y} + ix_0 \partial_0 \partial + \frac{x_1 x_2 x_3}{y^2} \partial_0 + x_0 \partial_1 \partial_2 \partial_3 \\
&\quad + \frac{i}{y} (x_1 x_2 \partial_1 \partial_2 + \text{cyclic}) + i(x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3) (\partial + \frac{1}{y}),
\end{align*}$$

(57)
as well as cyclic permutations of \(\{1, 2, 3\}\). The Levi \(M = [\text{Sl}(2)]^3\), obtained
by removing \(\beta_0\) from the extended Dynkin diagram, acts linearly on positions
and momenta and has generators

\[
E_{\alpha_i} = -x_0 \partial_i - \frac{ix_j x_k}{y}, \quad E_{-\alpha_i} = x_i \partial_0 + iy \partial_j \partial_k.
\]

Finally, the Cartan generators are obtained from commutators \([E_{\alpha}, E_{-\alpha}]\),

\[
H_{\beta_0} = -y \partial + x_0 \partial_0 \\
H_{\alpha_i} = -x_0 \partial_0 + x_i \partial_i - x_j \partial_j - x_k \partial_k - 1.
\]

This representation also arises in a totally different context: the one-loop
amplitude for closed strings compactified on a 4-torus! T-duality requires this
amplitude to be an automorphic form of \(SO(4, 4, \mathbb{Z}) = D_4(\mathbb{Z})\). In fact, it may
be written as an integral of a symplectic theta series over the fundamental
domain of the genus-1 world-sheet moduli space,

\[
\mathcal{A}(g_{ij}, B_{ij}) = \int_{\text{SO}(2) \setminus \text{Sl}(2, \mathbb{R})/\text{Sl}(2, \mathbb{Z})} d^2 \tau \frac{\tau^2}{\tau^2} \theta_{\text{Sp}(8, \mathbb{Z})}(\tau, \bar{\tau}; g_{ij}, B_{ij}).
\]

Here \((g_{ij}, B_{ij})\) are the metric and Neveu-Schwarz two-form on \(T^4\) parameterizing the moduli space \([\text{SO}(4) \times \text{SO}(4)]/\text{SO}(4, \mathbb{R})\). The symplectic theta series \(\theta_{\text{Sp}(8, \mathbb{Z})}\) is the partition function of the 4 + 4 string world-sheet winding modes \(m_a^i\), \(i = 1, \ldots, 4, a = 1, 2\) around \(T^4\). Like any Gaussian theta series, it is invariant under the (double cover of the) symplectic group over integers, \(\text{Sp}(8, \mathbb{Z})\) in this case. The modular group and T-duality group arise
as a dual pair \(\text{Sl}(2) \times \text{SO}(4, 4)\) in \(\text{Sp}(8)\) – in other words, each factor is the
commutant of the other within \(\text{Sp}(8)\). Therefore, after integrating over the
\(\text{Sl}(2)\) moduli space, an \(\text{SO}(4, 4, \mathbb{Z})\) automorphic form, based on the minimal
representation remains. Dual pairs are a powerful technique for constructing
new automorphic forms from old ones.

To see the minimal representation of \(D_4\) emerge explicitly, note that \(\text{Sl}(2)\)-
invariant functions of \(m_a^i\) must depend on the cross products,

\[
m^{ij} = \epsilon^{ab} m^i_a m^j_b,
\]

which obey the quadratic constraint

\[
m^{[ij} m^{kl]} = 0,
\]

and therefore span a cone in \(\mathbb{R}^6\). The 5 variables \((y, x_0, x_i)\) are mapped to
this 5-dimensional cone by diagonalizing the action of the maximal commuting
set of six observables \(\mathcal{C} = (E_{\alpha_3}, E_{\beta_3}, E_{\gamma_1}, E_{\gamma_2}, E_{\alpha_0}, E_{\omega})\) whose eigenvalues
may be identified with the constrained set of six coordinates on the cone
\(i(m^{43}, m^{24}, m^{14}, m^{23}, m^{13}, m^{12})\). The intertwiner between the two represen-
tations is a convolution with the common eigenvector of the generators \(\mathcal{C}\).
amounting to a Fourier transform over $x_3$. This intertwiner makes the hidden triality symmetry, which is crucial for heterotic/type II duality [26], of the \( \text{Sl}(4) \)-covariant string representation manifest.

An advantage of the covariant realization is that the spherical vector follows by directly computing the integral (61). The real spherical vector is read-off from worldsheet instanton contributions

\[
f_\infty(m_{ij}) = \frac{e^{-2\pi \sqrt{(m_{ij})^2}}}{\sqrt{(m_{ij})^2}},
\]

while its \( p \)-adic counterpart follows from the instanton summation measure

\[
f_p(m_{ij}) = \gamma_p(m_{ij}) \frac{1 - p |m_{ij}|_p}{1 - p}.
\]

Intertwining back to the triality invariant realization gives

\[
f_\infty(y, x_0, x_i) = \frac{e^{\frac{1}{2} \sum_{i=1}^3 (y^2 + x_0^2 + x_i^2)}}{\sqrt{y^2 + x_0^2}} K_0 \left( \frac{\prod_{i=1}^3 (y^2 + x_0^2 + x_i^2)}{y^2 + x_0^2} \right).
\]

This is the prototype for spherical vectors of all higher simple Lie groups.

### 3.3 Spherical vector, real and \( p \)-adic

To find the spherical vector for higher groups, one may either search for generalizations of the covariant string representation in which the result is a simple extension of the world-sheet instanton formula (64) – see Section 3.5, or try and solve by brute force the complicated set of partial differential equations \( (E_\alpha + E_{-\alpha}) f = 0 \) demanded by \( K \)-invariance. Fortunately, knowing the exact solution (66) for \( D_4 \) gives enough inspiration to solve the general case [5].

To see this, note that the phase in (66) has precisely the right anomalous transformation under \( (y, x_0) \to (-x_0, y) \) to cancel the cubic character of the Weyl generator (53), or equivalently the cubic term appearing in \( E_\beta_0 + E_{-\beta_0} \).

The real part of the spherical vector therefore depends on \( (y, x_0) \) through their norm \( R = \sqrt{y^2 + x_0^2} = |y, x_0|_\infty \). Moreover, invariance under the linearly acting maximal compact subgroup of \( L \) restricts the dependence on \( x_i \) to its quadratic \( I_2 \), cubic \( I_3 \) and quartic \( I_4 \) invariants. Choosing a frame where all but three of the \( x_i \) vanish, the remaining equations are then essentially the same as for the known \( D_4 \) case. The universal result is

\[
f_\infty(X) = \frac{1}{R^{n+1}} K_{n/2} (|X, \nabla X|_3/R) \exp \left( -\frac{x_0 I_3}{y R^2} \right),
\]

where \( X \equiv (y, x_0, \ldots, x_{d-2}) \) and \( I_3 \) is given in Table 1. Notice that the result depends on the pullback of the Euclidean norm to the Lagrangian subspace.
\((X, \nabla_X[I_3/R])\) of the coadjoint orbit. The function \(K_t(x)\) is related to the usual modified Bessel function by \(K_t(x) \equiv x^{-t}K_t(x)\), and the parameter \(s = 0, 1, 2, 4\) for \(G = D_4, E_6, E_7, E_8\), respectively.\(^{10}\)

The \(p\)-adic spherical vector computation is much harder since the generators cannot be expressed as differential operators. It was nevertheless completed in [27] by very different techniques, again inspired by the \(D_4\) result (65), intertwined to the triality invariant representation. The result mirrors the real case, namely for \(|y|_p < |x_0|_p\),

\[
f_p(X) = \frac{1}{R^{s+1}} K_{p,s/2}(X, \nabla_X[I_3/R]) \exp \left( -i \frac{I_3}{x_0 y} \right),
\]

where \(R = |y, x_0|_p = |x_0|_p\) is now the \(p\)-adic norm, and \(K_{p,t}\) is a \(p\)-adic analogue of the modified Bessel function,

\[
K_{p,t}(x) = \frac{1 - p^s |x|_p^{-s}}{1 - p^s} \gamma_p(x),
\]

\((\gamma_p(x))\) generalizes to a function of several arguments by \(\gamma_p(X) = 0\) unless \(|X|_p \leq 1\). The result for \(|y|_p > |x_0|_p\) follows by the Weyl reflection \(A\).

### 3.4 Global theta series

Having obtained the real and \(p\)-adic spherical vectors for any \(p\), one may now insert them in the adelic formula (14) to construct exceptional theta series. Equivalently, we may use the representation (7),

\[
\theta_G(e) = \langle \delta_{G(Z)}, \rho_G(e)f_\infty \rangle, \quad \delta(X) = \prod_p f_p(X),
\]

where \(X = (x_0, x_1, \ldots, x_{d-2})\), with a similar expression in the \(p\)-adic case.

However, there still remains a further divergence when \(y = x_0 = 0\). It can be shown that these terms may be regularized to give a sum of two terms,\(^{10}\) For \(D_n>4\) the result is slightly more complicated, see [5]. It is noteworthy that the ratio \(I_3/R\) is invariant under Legendre transform with respect to all entries in \(X\), although the precise meaning of this observation is unclear.
namely a constant plus a theta series based on the minimal representation of the Levi subgroup $M$. Altogether, the global formula for the theta series in the minimal representation of $G$ reads [27]

$$\theta_G(e) = \sum_{X \in [\mathbb{Z} \setminus \{0\}] \times \mathbb{Z}^{d-1}} \mu(X) \rho(e) \cdot f_\infty(X)$$

$$+ \sum_{X \in [\mathbb{Z} \setminus \{0\}] \times \mathbb{Z}^{d-2}} \mu(X) \rho(e) \cdot f_\infty(X) + \alpha_1 + \alpha_2 \theta_M(e). \quad (72)$$

Notice that the degenerate contributions in the second line will mix with the non-degenerate ones under a general right action of $G(\mathbb{Z})$.

### 3.5 Pure spinors, tensors, 27-sors, . . .

We end the mathematical discussion by returning to the $Sl(4)$-covariant presentation of the minimal representation of $SO(4, 4)$ on functions of 6 variables $m^{ij}$ with a quadratic constraint (62). The existence of this presentation may be traced to the 3-grading $28 = 6 \oplus (15 + 1) \oplus 6$ of the Lie algebra of $SO(4, 4)$ under the Abelian factor in $Gl(4) \subset SO(4, 4)$; the top space in this decomposition is an Abelian group, whose generators in the minimal representation of $SO(4, 4)$ can be simultaneously diagonalized. The eigenvalues transform linearly under $Sl(4)$ as a two-form, but satisfy one constraint in accord with the functional dimension 5 of the minimal representation of $SO(4, 4)$.

This phenomenon also occurs for higher groups: for $D_n$, the branching of $SO(n, n)$ into $Gl(n)$ leads to a dimension $n(n-1)/2$ Abelian subgroup, whose generators transform linearly as antisymmetric $n \times n$ matrices $m^{ij}$. Their simultaneous diagonalization in the minimal representation of $D_n$ leads to the same constraints as in (62), solved by rank 2 matrices $m^{ij}$. The number of independent variables is thus $2n - 3$, in accord with the functional dimension of the minimal representation. This is in fact the presentation obtained from the dual pair $SO(n, n) \times Sl(2) \subset Sp(2n)$, and just as in (64), the spherical vector is a Bessel function of the norm $\sqrt{m^{ij}}$.

For $E_6$, the 3-grading $78 = 16 \oplus (45 + 1) \oplus 16$ from the branching into $SO(5, 5) \times \mathbb{R}$ leads to a realization of the minimal representation of $E_6$ on a spinor $Y$ of $SO(5, 5)$, with 5 quadratic constraints $\sqrt{Y} Y = 0$. The solutions to these constraints are in fact the pure spinors of Cartan and Chevalley. The spherical vector was computed in [5] by Fourier transforming over one column of the $3 \times 3$ matrix $X$ appearing in the canonical polarization, and takes the remarkably simple form

$$f_\infty(Y) = K_1 \left( \sqrt{YY} \right). \quad (73)$$

Its $p$-adic counterpart, obtained by replacing orthogonal with $p$-adic norms, also simplifies accordingly. We thus conclude that functions of pure spinors...
of $SO(5,5)$ (as well as other real forms of $D_5$) carry an action of $E_{6(6)}(\mathbb{R})^{11}$. Given that pure spinors of $SO(9,1)$ provide a convenient covariant reformulation of ten-dimensional super-Yang-Mills theory and string theory [29], it is interesting to ponder the physical consequences of this hidden $E_6$ symmetry.

For $E_7$, the 3-grading corresponding to the branching $133 = 27 \oplus (78 + 1) \oplus 27$ into $E_6 \times \mathbb{R}$, leads to a realization of the minimal representation of $E_7$ on a 27 representation of $E_6$, denoted $Y$, subject to the condition that the 27 part in the symmetric tensor product $27 \otimes_s 27$ vanishes – in other words, $\partial_Y I_3(Y) = 0$. This corresponds to 10 independent quadratic conditions, whose solutions may aptly be dubbed pure 27-sors. The spherical vector was computed in [5] by Fourier transforming over one column of the antisymmetric $6 \times 6$ matrix $X$ in the canonical polarization, and is again extremely simple

$$f_\infty(Y) = K_{3/2} \left( \sqrt{YY} \right). \quad (74)$$

Unfortunately, $E_8$ does not admit any 3-grading. However, the 5-grading $248 = 1 \otimes 56 \otimes (133 + 1) \otimes 56 \otimes 1$ from the branching into $E_7 \times SL(2)$ leads to an action of $E_8$ on functions of “pure” 56-sors $Y$ of $E_7$ together with an extra variable $y$. For the minimal representation of $E_8$, the appropriate notion of purity requires the quadratic equations $\partial_Y \otimes \partial_Y I_4(Y) = 0$, where $I_4$ is the quartic invariant of $E_7$. As explained in [18], less stringent purity conditions lead to unipotent representations with larger dimension. This kind of construction based on a 5-grading is in fact available for all semi-simple groups in the quaternionic real form, and is equivalent to the “canonical” construction of the minimal representation in the simply-laced case [18].

4 Physical applications

Having completed our brief journey into the dense forest of unipotent representations and automorphic forms, we now return to a more familiar ground, and describe some physical applications of these mathematical constructions.

4.1 The automorphic membrane

The primary motivation behind our study of exceptional theta series was the conjecture of [31]: the exact four-graviton $R^4$ scattering amplitude, predicted by $U$-duality and supersymmetry, ought be derivable from the eleven-dimensional quantum supermembrane – an obvious candidate to describe fundamental $M$-theory excitations. For example, in eight dimensions, in analogy with the one-loop string amplitude, the partition function of supermembrane zero-modes should be a theta series of $E_6(\mathbb{Z})$, which subsumes both

\footnote{In contrast to the conformal realization of $E_6$ on 21 variables discussed in [28], this representation is irreducible.}
the $U$-duality group $SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$, and the toroidal membrane modular group $SL(3, \mathbb{Z})$. Integrating the partition function over world-volume moduli $\mathbb{R}^+ \times SL(3)$, yields by construction a $U$-duality invariant result which should reproduce the exact four-graviton $R^4$ scattering amplitude for M-theory on a $T^3$, including membrane instantons, namely a sum of $SL(3)$ and $SL(2)$ Eisenstein series $[30, 32]$.

Having constructed explicitly the $E_6$ theta series, we may now test this conjecture $[34]$. Recall that in the canonical realization, the $E_6$ minimal representation contains an $SL(3) \times SL(3)$ group acting linearly from the left and right on a $3 \times 3$ matrix of integers $m_{AM}^A$, together with two singlets $y, x_0$. In addition, there is an extra $SL(3)$ built from the non-linearly acting generators $E_{\beta_0, \gamma_0, \omega_0}$, which further decomposes into the $\mathbb{R}^+ \times SL(2)$ factors mentioned above. The integers $m_{AM}^A$ are interpreted as winding numbers of a toroidal membrane wrapping the target-space $T^3$, $X^M = m_{AM}^A \sigma^A$. The two extra integers $y, x_0$ do not appear in the standard membrane action but may be interpreted as a pair of world-volume 3-form fluxes – an interesting prediction of the hidden $E_6$ symmetry, recently confirmed from very different arguments $[33]$.

The integration over the membrane world-volume $SL(3)$ moduli amounts to decomposing the minimal representation with respect to the left acting $SL(3)$ and keeping only invariant singlets. For a generic matrix $m_{AM}^A$, the unique such invariant is its determinant, which we preemptively denote $x_3^1 = \text{det}(M)$. This leaves a representation of the non-linear $SL(3)$ acting on functions of three variables $(y, x_0, x_1)$ (the right $SL(3)$ acts trivially): this is precisely the representation studied in Section 2.5. In addition, non-generic matrices contribute further representations charged under both left and right $SL(3)$s.

It remains to carry out the integration over the membrane world-volume factor $\mathbb{R}^+$ inside the non-linear $SL(3)$. This integral is potentially divergent. Instead, a correct mathematical prescription is to look at the constant term with respect to a parabolic $P_{1,2} \subset SL(3)_{NL}$: indeed we find that this produces the result predicted by the conjecture $[34]$.

This is strong evidence that membranes are indeed the correct degrees of freedom of M-theory, although the construction only treats membrane zero-modes. It would be very interesting to see if the $E_6$ symmetry can be extended to fluctuations and in turn to lead to a quantization of the complete toroidal supermembrane.

### 4.2 Conformal quantum cosmology

The dynamics of spatially separated points decouple as a space-like singularity is approached. Only effective 0+1-dimensional quantum mechanical degrees of freedom remain at each point. Classically, these correspond to a particle on a hyperbolic billiard, whose chaotic motion translates into a sequence of Kasner flights and bounces of the spatial geometry $[35]$. Originally observed for 3+1-dimensional Einstein gravity, this chaotic behavior persists for 11-dimensional
supergravity, whose the billiard is the Weyl chamber of a generalized $E_{10}$ Kac-Moody group [36]. Upon accounting for off-diagonal metric and gauge degrees of freedom, the hyperbolic billiard can be unfolded onto the fundamental domain of the arithmetic group $E_{10}(\mathbb{Z})$. Automorphic forms for $E_{10}(\mathbb{Z})$ should therefore be relevant in to the wave function of the universe!

Automorphic forms for generalized Kac-Moody groups are out of our present reach. However, automorphic forms for finite Lie groups may still be useful in a cosmological context because their corresponding minimal representations can be viewed as conformal quantum mechanical systems of the type that arising near cosmological singularities [38]. Indeed, changing variables $y = \rho^2 / 2$, $x_i = \rho q_i / 2$ in the canonical minimal representation, the generators of the grading $SL(2)$ subalgebra become

$$E_\omega = \frac{1}{2} \rho^2 \quad H_\omega = \rho p \quad E_{-\omega} = \frac{1}{2} \left( \rho^2 + \frac{4A}{\rho^2} \right). \quad (75)$$

Here $A$ is a quartic invariant of the coordinates and momenta $\{q_i, \pi_i\}$ corresponding (up to an additive constant) to the quadratic Casimir of the Levi $M$. Choosing $E_{-\omega}$ as the Hamiltonian, the resulting mechanical system has a dynamical, $d = 0 + 1$ conformal, $SL(2) = SO(2, 1)$ symmetry. In contrast to the one-dimensional conformal quantum mechanics of [37], the conformal symmetry $SL(2)$ is enlarged to a much larger group $G$ mixing the radial coordinate $\rho$ with internal ones $x_i$. It can be shown that these conformal systems appear upon dimensional reduction of Einstein’s equations near a space-like singularity [38].

4.3 Black hole micro-states

Finally, minimal representations and automorphic forms play an important rôle in understanding the microscopic origin of the Bekenstein-Hawking entropy of black holes. From thermodynamic arguments, these stationary, spherically symmetric classical solutions of Einstein-Maxwell gravity are expected to describe an exponentially large number of quantum micro-states (on the order of the exponential of the area of their horizon in Planck units). It is an important question to determine the exact degeneracy of micro-states for a given value of their charges – as always, U-duality is a powerful constraint on the result. An early conjecture in the framework of $N = 4$ supergravity relates the degeneracies to Fourier coefficient of a certain modular form of $Sp(4, \mathbb{Z})$ constructed by Igusa [39]. A more recent study suggests that the 3-dimensional U-duality group (manifest after timelike dimensional reduction of the 4-dimensional stationary solution) should play the rôle of a “spectrum generating symmetry” for the black hole degeneracies [40]. For M-theory compactified on $T^7$ or $K3 \times T^3$, the respective $E_8(\mathbb{Z})$ or $SO(8, 24, \mathbb{Z})$ symmetry may be sufficiently powerful to determine these degeneracies, and there are strong indications that the minimal representation and theta series are the appropriate objects [40, 41, 42].
5 Conclusion

In this Lecture, we hope to have given a self-contained introduction to automorphic forms, based on string theory experience – rigor was jettisoned in favor of simplicity and utility. Our attempt will be rewarded if the reader is preempted to study further aspects of this rich field: non-minimal unipotent representations, non-simply laced groups, non-split real forms, reductive dual pairs, arithmetic subgroups, Fourier coefficients, L-functions... Alternatively, he or she may solve any of the homework problems outlined in Section 4.

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