On the $|K^\lambda|$-summability of Fourier series and its conjugate series

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Abstract The $K^\lambda$-means were first introduced by Karamata. Vučković first studied the $K^\lambda$-summability of a Fourier series and later on Lal studied the $K^\lambda$-summability of a conjugate series. In the present paper, we have studied the $|K^\lambda|$-summability of Fourier series and conjugate series.

Mathematics Subject Classification 42A28

1 Definitions and notations

For $n = 0, 1, 2, \ldots$, define the numbers $[n \atop k]$ for $0 \leq k \leq n$ by

$$ \Pi_{v=0}^{n-1}(x + v) = \sum_{k=0}^{n} [n \atop k] x^k, \quad x > 0 $$

(1.1)

where $\prod_{v=0}^{n-1}(x + v) = x(x + 1) \cdots (x + n - 1) = \frac{\Gamma(x+n)}{\Gamma(x)}$. Clearly, $[n \atop k] = 0$ when $k < 1$ and $k > n$. We shall use the convention that $[0 \atop 0] = 1$. The numbers of $[n \atop k]$ are known as Stirling’s number of first kind. We know [8, p. 43] the following recursion formula

$$ [n \atop k] = [n-1 \atop k-1] + (n-1)[n-1 \atop k]. $$

(1.2)
Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with sequence of partial sums $\{s_n\}$ i.e., $s_n = \sum_{k=0}^{n} u_k$. Let $\lambda > 0$, the $K^\lambda$-mean $(t_n)$ of the sequence $\{s_n\}$ is defined by [2,5]

$$t_n = \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=0}^{n} \binom{n}{k} \lambda^k s_k. \quad (1.3)$$

If $\lim_{n \to \infty} t_n = s$, then we say that sequence $\{s_n\}$ (or the series $\sum u_n$) is summable $K^\lambda$ to $s$.

The series $\sum u_n$ (or the sequence $\{s_n\}$) is said to be absolutely $K^\lambda$-summable if $\{t_n\} \in BV$; i.e.,

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty. \quad (1.4)$$

Using (1.2) and (1.3), we obtain

$$t_n = \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=0}^{n} \binom{n}{k} \lambda^k s_k + (n - 1) \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^k s_k$$

$$= \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=1}^{n} \binom{n-1}{k-1} \lambda^k s_k + (n - 1) \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^k s_k$$

$$= \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \left[ \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^{k+1} s_{k+1} + (n - 1) \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^k s_k \right], \quad (1.5)$$

and

$$t_{n-1} = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^k s_k$$

$$= \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \left[ \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^{k+1} s_{k+1} + (n - 1) \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^k s_k \right]. \quad (1.6)$$

From (1.5) and (1.6), it follows that

$$t_n - t_{n-1} = \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^{k+1} (s_{k+1} - s_k)$$

$$= \frac{\lambda}{n - 1 + \lambda} \left[ \frac{\Gamma(\lambda)}{\Gamma(n + \lambda - 1)} \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^k u_{k+1} \right]$$

$$= \frac{\lambda \xi_{n-1}(u)}{n - 1 + \lambda}. \quad (1.7)$$

We may derive the following useful identity (Proposition 1.1) which is similar to the Kogbetliantz identity [3] for the Cesàro mean, namely,

$$n(\sigma_n^\alpha - \sigma_{n-1}^\alpha) = \tau_n^\alpha,$$

where $\sigma_n^\alpha$ is the $(C, \alpha)$ mean of $\sum a_n$ and $\tau_n$ is the $(C, \alpha)$ mean of $\{na_n\}$.

**Proposition 1.1**

$$(n - 1 + \lambda)(t_n - t_{n-1}) = \xi_{n-1}(u)$$

where $t_n$ is the $K^\lambda$ mean of $\sum u_n$ and $\xi_n$ is the $K^\lambda$ mean of $\{u_{n+1}\}$; i.e.,

$$\xi_n(u) = \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=0}^{n} \binom{n}{k} \lambda^k u_{k+1}. \quad (1.8)$$
From Proposition 1.1, it follows that \( \sum u_n \) is the \( |K^\lambda| \)-summable if and only if

\[
\sum_{n=1}^{\infty} \frac{|x_{n-1}(u)|}{n} < \infty.
\] (1.9)

**Proposition 1.2** The \( K^\lambda \)-method is absolutely conservative; that is \( |C, 0| \subset |K^\lambda| \).

**Proof** We need the following result [8, p. 43 problem 200]:

\[
\sum_{n=k}^{\infty} \left( \frac{n}{k} \right) \frac{(1-u)^n}{n!} = \frac{1}{k!} \left( \log \frac{1}{u} \right)^k.
\] (1.10)

From (1.7), it follows that

\[
t_n - t_{n-1} = \sum_{k=0}^{\infty} a_{n,k} u_k
\]

where

\[
a_{n,k} = \begin{cases} 
\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} [n-1]^{\lambda} & 0 \leq k \leq n, \\
0 & k > n.
\end{cases}
\]

The \( K^\lambda \)-method is absolutely conservative if and only if

\[
\sum_{n=k}^{\infty} |a_{n,k}| < \infty;
\]

i.e.,

\[
\sum_{n=k}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} [n-1]^{\lambda} < \infty.
\]

We have

\[
\sum_{n=k}^{\infty} \frac{1}{(n-1)!} \int_0^{1} u^{\lambda-1} (1-u)^{n-1} [\frac{n-1}{k-1}] \, du
\]

\[
= \lambda^k \int_0^{1} u^{\lambda-1} \left( \sum_{n=k-1}^{\infty} \frac{(1-u)^n}{n!} \right) \, du
\]

\[
= \lambda^k \int_0^{1} u^{\lambda-1} \frac{1}{(k-1)!} \left( \log \frac{1}{u} \right)^{k-1} \, du \quad \text{using (1.10)},
\]

\[
= \lambda^k \int_0^{\infty} \theta^{k-1} e^{-\lambda \theta} \, d\theta = 1,
\]

which shows that the \( K^\lambda \)-method is absolutely conservative.

\[\square\]
2 Application to trigonometric Fourier series

Let \( f \) be a \( 2\pi \)-periodic function and integrable in the sense of Lebesgue over \((-\pi, \pi)\). Let the trigonometric Fourier series of \( f \) at \( x \) be given by

\[
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).
\] (2.1)

The conjugate series of (2.1) is given by

\[
\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x),
\]

\[
\phi_x(t) = \frac{1}{2} \{ f(x + t) + f(x - t) - 2f(x) \},
\]

\[
\psi_x(t) = \frac{1}{2} \{ f(x + t) - f(x - t) \},
\]

\[
\tilde{f}(x; \epsilon) = \frac{2}{\pi} \int_{\epsilon}^{\pi} \psi_x(t) \left( \frac{1}{2} \cot \frac{1}{2} t \right) dt,
\]

and

\[
\tilde{f}(x) = \lim_{\epsilon \to 0^+} \tilde{f}(x; \epsilon), \quad \text{whenever the limit exists.}
\] (2.2)

The \( K^\lambda \)-means were first introduced by Karamata [2]. Lototsky [5] reintroduced the special case \( \lambda = 1 \). Vučković [10] was the first to study the \( K^\lambda \)-summability of Fourier series and his result reads as follows.

**Theorem 2.1** If

\[
(f(x + t) + f(x - t) - 2f(x)) \log \frac{1}{t} = o(1) \quad \text{as } t \to 0^+,
\]

then the trigonometric Fourier series of \( f \) at \( x = t \) is \( K^\lambda \)-summable to \( f(x) \).

Later Lal [4] obtained the following result for the conjugate series.

**Theorem 2.2** If \( \int_{0}^{t} |\psi_x(u)| \, du = o\left( \frac{1}{\log t^{-1}} \right) \) as \( t \to 0^+ \), then \( \sum_{n=1}^{\infty} B_n(x) \) is \( K^\lambda \)-summable to \( \tilde{f}(x) \), whenever it exists.

In the present paper, we study the absolute \( K^\lambda \)-summability of a Fourier series and its conjugate series. We prove

**Theorem 2.3** Let \( 0 < \delta < e^{-2} \). Then \( \phi(t) \log t^{-1} \in BV(0, \delta) \Rightarrow \sum A_n(x) \in |K^\lambda| \).

**Theorem 2.4** Let \( 0 < \delta < e^{-2} \). Then \( \psi(t) \log t^{-1} \in BV(0, \delta) \), and \( \frac{\psi(t)}{t} \in L(0, \delta) \Rightarrow \sum B_n(x) \in |K^\lambda| \).

3 Notations and lemmas

For the proofs of the theorems, we need the following additional notations:

\[
K^\lambda_n(t) = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \sum_{k=0}^{n-1} [^{n-1}\kappa_k \lambda^k \sin(k+1)t],
\]

\[
\tilde{K}^\lambda_n(t) = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \sum_{k=0}^{n-1} [^{n-1}\kappa_k \lambda^k \cos(k+1)t],
\]

\[
P_k(t) = (\lambda^2 + 2\lambda \cos t + k^2)^{\frac{1}{2}},
\]
\[ R(n, t) = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \prod_{k=0}^{n-2} P_k(t), \]
\[ \theta_k(t) = \tan^{-1} \left( \frac{\lambda \sin t}{\lambda \cos t + k} \right), \]
\[ l(n) = 2 + \lambda \sum_{k=1}^{n-2} \frac{1}{\lambda + k} \sim \log n, \]
\[ g(n, t) = \int_0^t \frac{\tilde{K}_n^\lambda(u)}{\log u} du, \]
\[ h(n, t) = \int_t^\delta \frac{\tilde{K}_n^\lambda(u)}{\log u} du, \]
\[ G(n, t) = \int_t^\delta \frac{u}{\log u} e^{-Au^2\log n} du, \]
\[ H(n, t) = \int_t^\delta \frac{u^3}{\log u} e^{-Au^2\log n} du, \]
\[ L(n, t) = \int_t^\delta \frac{e^{-Au^2\log n}}{u(\log u)^2} du. \]

We need the following lemmas to prove our theorems:

**Lemma 3.1** Let \( \theta_k(t), R(n, t), K_n^\lambda(t) \) and \( \tilde{K}_n^\lambda(t) \) be defined as above. Then

(i) \( K_n^\lambda(t) = R(n, t) \sin \left( 2t + \sum_{k=1}^{n-2} \theta_k(t) \right) \),

(ii) \( \tilde{K}_n^\lambda(t) = R(n, t) \cos \left( 2t + \sum_{k=1}^{n-2} \theta_k(t) \right) \).

**Proof** We have

\[
\tilde{K}_n^\lambda(t) + iK_n^\lambda(t) = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \sum_{k=0}^{n-1} \lambda^k e^{(k+1)t} \]
\[ = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} e^{it} \sum_{k=0}^{n-1} (\lambda e^{it})^k \]
\[ = \frac{\Gamma(\lambda) e^{it}}{\Gamma(n - 1 + \lambda)} \prod_{k=0}^{n-2} (\lambda e^{it} + k) \]
\[ = \frac{\Gamma(\lambda) e^{it}}{\Gamma(n - 1 + \lambda)} e^{it} \prod_{k=0}^{n-2} P_k(t) \exp(i\theta_k(t)) \]
\[ = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \left( \prod_{k=0}^{n-2} P_k(t) \right) \exp \left\{ i \left( 2t + \sum_{k=1}^{n-2} \theta_k(t) \right) \right\} \]
\[ = R(n, t) \exp \left\{ i \left( 2t + \sum_{k=1}^{n-2} \theta_k(t) \right) \right\}, \]

from which the lemma follows. \( \square \)
Lemma 3.2 [9, Chapter 5, Lemma 5.5] Let \( R(n, t), K_\lambda^i(n) \), and \( \tilde{K}_\lambda^i(n) \) be defined as in Sect. 3. Then for some positive constant \( A \) and all \( t \in (0, \pi) \)

\[
\begin{align*}
(i) \quad R(n, t) &= \begin{cases} 0(1) & \text{if } t = 0, \\
0(1) e^{-At^2 \log n}, & \text{if } t > 0,
\end{cases} \\
(ii) \quad K_\lambda^i(n) &= \begin{cases} 0(1) & \text{if } t = 0, \\
0(1) e^{-At^2 \log n}, & \text{if } t > 0,
\end{cases} \\
(iii) \quad \tilde{K}_\lambda^i(n) &= \begin{cases} 0(1) & \text{if } t = 0, \\
0(1) e^{-At^2 \log n}. & \text{if } t > 0,
\end{cases}
\end{align*}
\]

and 

\[
\begin{align*}
(i) \quad R(n, t) &= \Gamma(\lambda) \prod_{k=0}^{n-2} \left( \lambda^2 + 2\lambda k \cos t + k^2 \right)^{\frac{1}{2}} \\
&= \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)} \prod_{k=0}^{n-2} \left( \lambda^2 + 2\lambda k \cos t + k^2 \right)^{\frac{1}{2}} \\
&= \prod_{k=0}^{n-2} \left[ 1 - \frac{4\lambda k \sin^2 \frac{1}{2} t}{(\lambda + k)^2} \right]^{\frac{1}{2}} \\
&= \exp \left[ -\frac{1}{2} \sum_{k=0}^{n-2} \log \left( 1 - \frac{4\lambda k \sin^2 \frac{1}{2} t}{(\lambda + k)^2} \right) \right]. \quad (3.1)
\end{align*}
\]

We observe that 

\[
0 < \frac{4\lambda k \sin^2 \frac{1}{2} t}{(\lambda + k)^2} < 1
\]

for \( k = 1, 2, 3, \ldots \) and \( 0 < t < \pi \). As \( \log(1-\theta)^{-1} \geq \theta \) for \( 0 < \theta < 1 \) and \( \sin x \geq \frac{2x}{\pi} \), \( 0 \leq x \leq \frac{\pi}{2} \), we have

\[
\sum_{k=0}^{n-2} \log \left( 1 - \frac{4\lambda k \sin^2 \frac{1}{2} t}{(\lambda + k)^2} \right) \leq \sum_{k=0}^{n-2} \frac{4\lambda k \sin^2 \frac{1}{2} t}{(\lambda + k)^2} \geq \frac{4\lambda t^2 \sum_{k=0}^{n-2} k}{\pi^2 \sum_{k=0}^{n-2} (\lambda + k)^2} \geq \frac{4A t^2 \log n}{(\lambda + k)^2} \quad (3.2)
\]

where \( A \) is a positive constant. Using (3.2) in (3.1), we obtain the second estimate of Lemma 3.2 (i). The proof of Lemma 3.2(ii) and (iii) follows from Lemma 3.2(i).

Lemma 3.3 [9, Chapter 5, Lemma 5.6] Let \( 0 < t \leq \frac{\pi}{4} \). Then

\[
\begin{align*}
(i) \quad \sin \left( 2t + \sum_{k=1}^{n-2} \theta_k(t) \right) - \sin l(n)t &= O(t^3 \log n), \\
(ii) \quad \cos \left( 2t + \sum_{k=1}^{n-2} \theta_k(t) \right) - \cos l(n)t &= O(t^3 \log n).
\end{align*}
\]
Proof We have

\[
\left| \sin \left( 2t + \sum_{k=1}^{n-2} \theta_k(t) \right) - \sin l(n)t \right| \leq \left| 2t + \sum_{k=1}^{n-2} \theta_k - l(n)t \right|. \tag{3.3}
\]

Next, we note that

\[
0 < \frac{\lambda \sin t}{\lambda \cos t + k} < 1
\]

whenever \(0 < t \leq \frac{\pi}{4}\) and \(k \geq 1\). Thus for \(0 < t \leq \frac{\pi}{4}\)

\[
\theta_k = \left[ \tan^{-1} \frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda \sin t}{\lambda \cos t + k} \right] + \left[ \frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda t}{\lambda + k} \right]
\]

\[
= O \left( \left( \frac{\lambda \sin t}{\lambda \cos t + k} \right)^3 \right) + O \left( \frac{t^3}{\lambda \cos t + k} \right) + O \left[ \frac{t^3}{(\lambda \cos t + k)(\lambda + k)} \right] + \frac{\lambda t}{\lambda + k}
\]

\[
= O \left( \frac{t^3}{k^3} \right) + O \left( \frac{t^3}{k^2} \right) + O \left( \frac{t^3}{k} \right), \quad 1 \leq k \leq n - 2. \tag{3.4}
\]

Using (3.4), we have

\[
2t + \sum_{k=1}^{n-2} \theta_k(t) = t \left[ 2 + \lambda \sum_{k=1}^{n-2} \frac{1}{\lambda + k} \right] + O(t^3) \sum_{k=1}^{n-2} \frac{1}{k}
\]

\[
= tl(n) + O(t^3 \log n). \tag{3.5}
\]

Using (3.5) in (3.3), we obtain Lemma 3.3(i). The proof of Lemma 3.3(ii) is similar to that of Lemma 3.3(i).

\[\square\]

Lemma 3.4 [9, Chapter 5, Lemma 5.7] Let \(0 < t \leq \frac{\pi}{2}\). Then \(R'(n, t) = O(1) t \log n R(n, t)\).

Proof We have

\[
R(n, t) = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \prod_{k=0}^{n-2} P_k(t),
\]

and so, by logarithmic differentiation, since \(P_k(t) \geq k\)

\[
R'(n, t) = R(n, t) \sum_{k=0}^{n-2} \frac{P_k'(t)}{P_k(t)}
\]

\[
= R(n, t) \sum_{k=1}^{n-2} \frac{(-\lambda k \sin t)}{(P_k(t))^2}
\]

\[
= O(1) t R(n, t) \sum_{k=1}^{n-2} \frac{1}{k}
\]

\[
= O(1) t \log n R(n, t),
\]

from which the lemma follows. \[\square\]

Lemma 3.5 Let \(\alpha_n = \int_0^\delta \frac{\cos n u}{\log u + \pi} \, du\). Then the series \(\sum \alpha_n \in |K^\lambda|\); i.e., the series \(\sum \frac{|g(n, \delta)|}{n}\) is convergent.
\textbf{Proof} Integrating by parts, we have
\[ \alpha_n = \int_0^\delta \frac{\cos nu}{\log u^{-1}} \, du \]
\[ = \sin n\delta - \frac{1}{n} \int_0^\delta \frac{\sin nu}{u(\log u^{-1})^2} \, du \]
\[ = -\frac{1}{\log \delta - 1} \int_\delta^\pi \cos nu \, du - \frac{1}{n} \int_0^\delta \frac{\sin nu}{u(\log u^{-1})^2} \, du \]
\[ = -(\alpha_{n,1} + \alpha_{n,2}), \quad \text{say}. \]

It is known [1] that
\[ \int_0^\delta \frac{\sin nu}{u(\log 1/u)^2} \, du = O\left(\frac{1}{(\log n)^2}\right) \]
and hence \( \sum \alpha_{n,2} \) is absolutely convergent. And, since \( |K^\lambda| \)-method is absolutely conservative
\[ \sum \alpha_{n,2} \in |K^\lambda|. \]

It remains to show that \( \sum \alpha_{n,1} \in |K^\lambda| \).

By definition the series \( \sum \alpha_{n,1} \in |K^\lambda| \) if
\[ \sum \equiv \sum_{n=1}^\infty \frac{1}{n} \left| \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)} \sum_{k=0}^{n-1} \lambda^k \int_\delta^\pi \cos(k+1)u \, du \right| < \infty. \]

Using the notation of Sect. 3 and Lemma 3.2(iii), we get
\[ \sum \equiv \sum_{n=1}^\infty \frac{1}{n} \left| \int_\delta^\pi \tilde{K}_n^\lambda(t) \, dt \right| \]
\[ \leq \sum_{n=1}^\infty \frac{1}{n} \int_\delta^\pi |\tilde{K}_n^\lambda(t)| \, dt \]
\[ \leq O(1) \sum_{n=1}^\infty \frac{1}{n} \int_\delta^\pi e^{-A\delta^2 \log n} \, dt \]
\[ = O(1) \sum_{n=1}^\infty \frac{e^{-A\delta^2 \log n}}{n^{1+A\delta^2}} = O(1), \]
which implies that \( \sum \alpha_{n,1} \in |K^\lambda| \).

As \( \sum \alpha_n \in |K^\lambda| \), and collecting the above results, it follows that
\[ \sum_{n=1}^\infty \frac{1}{n} \left| \frac{\Gamma(\lambda)}{\Gamma(n+\lambda-1)} \sum_{k=0}^{n-1} \lambda^k \int_0^\delta \frac{\cos(k+1)u}{\log u^{-1}} \, du \right| < \infty; \]
that is
\[ \sum_{n=1}^\infty \frac{1}{n} \left| \int_0^\delta \frac{du}{\log u^{-1}} \frac{\Gamma(\lambda)}{\Gamma(n+\lambda-1)} \sum_{k=0}^{n-1} \lambda^k \cos(k+1)u \, du \right| < \infty; \]
that is,
\[ \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{0}^{\delta} \frac{K_n(u)}{\log u^{-1}} \, du \right| < \infty; \]
that is, \( \sum \frac{|g(n, \delta)|}{n} < \infty. \)
This completes the proof of the lemma. \( \square \)

**Lemma 3.6** For every positive \( \Delta, \) however large,

(i) \( G(n, t) = O \left( \frac{e^{-A\delta^2 \log n}}{\log n \log t^{-1}} \right) + \frac{L(n, t)}{2A \log n}, \)

(ii) \( H(n, t) = O \left( \frac{t^2 e^{-A\delta^2 \log n}}{\log t^{-1} \log n} \right) + O(1) \frac{G(n, t)}{\log n}. \)

**Proof of (i)** Integrating by parts we have,

\[
G(n, t) = -\frac{1}{2A \log n} \int_{t}^{\delta} \frac{d}{du} \left( e^{-Au^2 \log n} \right) \frac{du}{\log u^{-1}} \\
= \frac{1}{2A \log n} \left[ \frac{e^{-A\delta^2 \log n}}{\log t^{-1}} - \frac{e^{-A\delta^2 \log n}}{\log \delta^{-1}} \right] \\
+ \frac{1}{2A \log n} \int_{t}^{\delta} \frac{e^{-Au^2 \log n}}{u(\log u^{-1})^2} \frac{du}{\log u^{-1}} \\
\leq \frac{e^{-A\delta^2 \log n}}{2A(\log n) \log t^{-1}} + \frac{L(n, t)}{2A \log n},
\]
from which (i) follows. \( \square \)

**Proof of (ii)** Integrating by parts, we get

\[
H(n, t) = -\frac{1}{2A \log n} \int_{t}^{\delta} \frac{u^2}{(\log u^{-1})} \frac{d}{du} \left( e^{-Au^2 \log n} \right) \, du \\
= \frac{1}{2A \log n} \left[ \frac{t^2 e^{-A\delta^2 \log n}}{\log t^{-1}} - \frac{\delta^2 e^{-A\delta^2 \log n}}{\log \delta^{-1}} \right] \\
+ \frac{1}{2A \log n} \int_{t}^{\delta} e^{-Au^2 \log n} \frac{d}{du} \left( \frac{u^2}{\log u^{-1}} \right) \, du \\
\leq \frac{t^2 e^{-A\delta^2 \log n}}{2A \log n} + \frac{1}{A \log n} \left[ \int_{t}^{\delta} \frac{ue^{-Au^2 \log n}}{\log u^{-1}} \, du + \frac{1}{2} \int_{t}^{\delta} \frac{ue^{-Au^2 \log n}}{(\log u^{-1})^2} \, du \right] \\
\leq \frac{t^2 e^{-A\delta^2 \log n}}{2A \log n} + \frac{1}{A \log n} \left[ \int_{t}^{\delta} \frac{ue^{-Au^2 \log n}}{\log u^{-1}} \, du + \frac{1}{2} \int_{t}^{\delta} \frac{ue^{-Au^2 \log n}}{\log u^{-1}} \, du \right] \\
= \frac{t^2 e^{-A\delta^2 \log n}}{2A \log n} + O(1) \frac{1}{\log n} \int_{t}^{\delta} \frac{ue^{-Au^2 \log n}}{\log u^{-1}} \, du,
\]
from which the result follows. \( \square \)

**Lemma 3.7** Let \( 0 < t < \delta < e^{-2}. \) Then

(i) \( L(n, t) = O \left( \frac{1}{t^2 (\log t^{-1})^2 \log n} \right), \)

(ii) \( L(n, t) = O \left( \frac{1}{\log n} \right). \)
Proof of (i) Integrating by parts, we get

\[
L(n, t) = -\frac{1}{2A \log n} \int_t^1 \frac{d}{du} \left( e^{-Au^2 \log n} \right) \frac{du}{u^2 (\log u^{-1})^2}
\]

\[
= \frac{1}{2A \log n} \left[ e^{-Ar^2 \log n} - e^{-A\delta^2 \log n} \right]
\]

\[
+ \frac{1}{2A \log n} \int_t^1 \frac{d}{du} \left( \frac{1}{u^2 (\log u^{-1})^2} \right) e^{-Au \log n} du
\]

< \frac{1}{2A \log n} \left[ e^{-A \delta^2 \log n} \right],
\]

since the last integral is negative, and this completes the proof of (i).

Proof of (ii) Let \(0 < \beta < 2\). By the simple computation, we get

\[
\frac{d}{du} \left( \frac{e^{-Au^2 \log n}}{(\log 1/u)^2} \right) = \frac{2e^{-Au^2 \log n}}{(\log 1/u)^3} \left[ 1 - Au^2 \log \frac{1}{u} \log n \right].
\]

The expression \(\frac{e^{-Au^2 \log n}}{(\log 1/u)^2}\) is monotonic decreasing in \(u\) whenever \(1 - Au^2 \log \frac{1}{u} \log n < 0\). It is easy to see that

\[
\left( \frac{1}{u} \right)^{\frac{1-2\beta}{\beta}} \log \frac{1}{u} > 1,
\]

that is,

\[
u^2 \log \frac{1}{u} - u^{\frac{1}{\beta}} > 0.
\]

In view of this inequality, we get

\[
1 - Au^2 \log \frac{1}{u} \log n = 1 - Au^\frac{1}{\beta} \log n - A \left( u^2 \log \frac{1}{u} - u^{\frac{1}{\beta}} \right) \log n
\]

\[
< 1 - Au^\frac{1}{\beta} \log n < 0,
\]

which holds for \(u > (A \log n)^{-\beta}\). This ensures that \(\frac{e^{-Au^2 \log n}}{(\log 1/u)^2}\) is monotonic decreasing for \(u > (A \log n)^{-\beta}\).

We shall consider the cases \((A \log n)^{-\beta} < \delta\) and \((A \log n)^{-\beta} \geq \delta\) separately. In case \((A \log n)^{-\beta} < \delta\) writing

\[
L(n, t) = \left( \int_t^{1/(A \log n)^{\beta}} + \int_{1/(A \log n)^{\beta}}^{\delta} \right) \frac{e^{-Au^2 \log n}}{u (\log u^{-1})^2} du
\]

and using the monotonicity of \(\frac{e^{-Au^2 \log n}}{(\log u^{-1})^2}\) for the second integral, we get

\[
L(n, t) \leq e^{-Ar^2 \log n} \int_t^{1/(A \log n)^{\beta}} \frac{du}{u (\log u^{-1})^2} + e^{-A^{1-2\beta}/(\beta \log (A \log n))^{1-2\beta}} \int_{1/(A \log n)^{\beta}}^{\delta} \frac{du}{u}
\]

\[
= O \left( \frac{e^{-Ar^2 \log n}}{\beta \log (A \log n)} \right) + O \left( \frac{1}{(\log n)^{\Delta}} \right), \quad \Delta > 1
\]

\[
= O \left( \frac{1}{\log \log n} \right).
\]
In case \((A \log n)^{-\beta} \geq \delta\), we have
\[
I(n, t) = \int_{1}^{\delta} \frac{e^{-A u^2 \log n}}{u \left(\log \frac{1}{u}\right)^2} \, du,
\]
\[
\leq \int_{1}^{(A \log n)^{-\beta}} \frac{e^{-A u^2 \log n}}{u \left(\log \frac{1}{u}\right)^2} \, du,
\]
which is same as the first integral in the first case discussed above. Lastly, in case \((A \log n)^{-\beta} < t\), \(I(n, t)\) is majorized by the second integral \(\int_{(A \log n)^{-\beta}}^{\delta} \frac{e^{-A u^2 \log n}}{u \left(\log \frac{1}{u}\right)^2} \, du\), in the first case and this completes the proof of (ii).

Lemma 3.8 Let \(0 < \delta < e^{-2}\) and \(\Delta > 1\), however large. Then

(i) \(g(n, t) = O\left(\frac{t}{\log t^{-1}}\right)\),

(ii) \(h(n, t) = O\left(\frac{1}{n A^2}\right) + O(1) \frac{e^{-A t^2 \log n}}{\log t^{-1}} + O(1) \frac{t^2 e^{-A t \log n}}{\log t^{-1}} + \frac{L(n, t)}{\log n}\).

Proof of (i) As \(\tilde{K}_n^1(u) = O(1)\) by Lemma 3.2(iii), the result follows.

Proof of (ii) First using Lemma 3.1(ii) and thereafter applying Lemma 3.3(ii), we obtain
\[
h(n, t) = \int_{1}^{\delta} \frac{\tilde{K}_n^1(u)}{\log u^{-1}} \, du
\]
\[
= \int_{1}^{\delta} \frac{R(n, u)}{\log u^{-1}} \cos \left\{ 2u + \sum_{k=1}^{n-2} \theta_k(u) \right\} \, du
\]
\[
= \int_{1}^{\delta} \frac{1}{\log u^{-1}} R(n, u) \cos (l(n)u) \, du + \int_{1}^{\delta} \frac{R(n, u)}{\log u^{-1}} \left[ \cos \left\{ 2u + \sum_{k=1}^{n-2} \theta_k(u) \right\} - \cos l(n)u \right] \, du
\]
\[
= \int_{1}^{\delta} \frac{R(n, u)}{\log u^{-1}} \cos \{l(n)u\} \, du + O(1) \log n \int_{1}^{\delta} \frac{R(n, u) u^2 \, du}{\log u^{-1}}
\]
\[
= I_1 + O(1) I_2, \quad \text{say.} \tag{3.6}
\]

Integrating by parts and using Lemma 3.4 and Lemma 3.2(i), we get
\[
I_1 = \left[ \frac{R(n, u) \cos l(n)u}{\log u^{-1} l(n)} \right]_{1}^{\delta} - \int_{1}^{\delta} \left[ \frac{R'(n, u)}{\log u^{-1}} + \frac{R(n, u)}{u \left(\log u^{-1}\right)^2} \right] \frac{\sin l(n)u \, du}{l(n)}
\]
\[
= O\left(\frac{e^{-A \delta^2 \log n}}{\log n}\right) + O\left(\frac{e^{-A \delta^2 \log n}}{\log t^{-1} \log n}\right) + O(1) \int_{1}^{\delta} \frac{e^{-A u^2 \log n}}{u \left(\log u^{-1}\right)^2} \, du
\]
\[
+ O(1) \frac{1}{\log n} \int_{1}^{\delta} \frac{e^{-A u^2 \log n}}{u \left(\log u^{-1}\right)^2} \, du
\]
\[
= O\left(\frac{1}{(n A^2)}\right) + O\left(\frac{e^{-A \delta^2 \log n}}{\log n \log t^{-1}}\right) + O(1) G(n, t) + \frac{O(1) L(n, t)}{\log n}. \tag{3.7}
\]

Using Lemma 3.2 and Lemma 3.6(ii), we have
\[
I_2 = \log n H(n, t)
\]
\[
= O(1) \frac{t^2 e^{-A t^2 \log n}}{\log t^{-1}} + O(1) G(n, t). \tag{3.8}
\]

Collecting the results from (3.6) to (3.8) and using the estimate for \(G(n, t)\) from Lemma 3.6(i), we obtain the desired estimate for \(h(n, t)\). □
4 Proof of Theorem 2.3 Using [6]

For $n \geq 1$ and $0 < \delta < e^{-2}$, we write

\[ A_n(x) = \frac{2}{\pi} \left( \int_0^{\delta} + \int_{\delta}^{\pi} \right) \phi(t) \cos nt \, dt \]

\[ = \frac{2}{\pi} (P_n + Q_n), \text{ say.} \tag{4.1} \]

Let $\xi_n(Q)$ be the $n$th $K^\lambda$-mean of the sequence $\{Q_{n+1}\}$.

The series $\sum Q_n \in |K^\lambda|$, if and only if

\[ \sum \frac{|\xi_{n-1}(Q)|}{n} < \infty. \tag{4.2} \]

By simple computation and an appeal to Lemma 3.2(iii)

\[ \xi_{n-1}(Q) = \int_{\delta}^{\pi} \phi(t) \left[ \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \sum_{k=0}^{n-1} \frac{\Gamma^k}{k!} \cos(k+1)t \right] \, dt \]

\[ = \int_{\delta}^{\pi} \phi(t) \tilde{K}_n(t) \, dt \]

\[ = O(1) \int_{\delta}^{\pi} |\phi(t)| e^{-A\delta^2 \log n} \, dt \]

\[ = O(1) e^{-A\delta^2 \log n}. \]

This ensures (4.2) and consequently vindicates that the $|K^\lambda|$-summability of trigonometric Fourier series is a local property. Writing $g(t) = \phi(t) \log t^{-1}$, $\alpha_n = \int_0^{\delta} g(u) \frac{\cos nu}{\log u^{-1}} \, du$ and integrating by parts, we obtain

\[ P_n = g(\delta) \alpha_n - \int_0^{\delta} dg(t) \int_0^{t} \frac{\cos nu}{\log u^{-1}} \, du \]

\[ = g(\delta) \alpha_n - \beta_n, \text{ say.} \tag{4.3} \]

As $\sum \alpha_n \in |K^\lambda|$ by Lemma 3.5 it remains to prove that $\sum \beta_n \in |K^\lambda|$. Let $\xi_n(\beta)$ be the $n$th $K^\lambda$-mean of the sequence $\{\beta_{n+1}\}$. It is easily seen that

\[ \xi_{n-1}(\beta) = \frac{\Gamma(\lambda)}{\Gamma(n - 1 + \lambda)} \sum_{k=0}^{n-1} \frac{\Gamma^k}{k!} \left[ \int_0^{\delta} \frac{dg(t)}{\log u^{-1}} \, du \right] \]

\[ = \int_0^{\delta} g(n, t) \, dg(t). \]

By definition $\sum \beta_n \in |K^\lambda|$, if and only if

\[ \sum \frac{|\xi_{n-1}(\beta)|}{n} < \infty; \]

that is,

\[ \sum_{n=1}^{\infty} \int_0^{\delta} g(n, t) \, dg(t) < \infty. \tag{4.4} \]

As $\int_0^{\delta} |dg(t)|$ is finite, for the validity of (4.4), it is enough to show that uniformly in $0 < t \leq \delta$.

\[ \sum = \sum_{n=1}^{\infty} \frac{|g(n, t)|}{n} = O(1). \tag{4.5} \]
Putting \( T_1 = \exp(t^{-1}) \) and \( T_2 = \exp(t^{-2}) \), we write
\[
\sum = \left( \sum_{n=1}^{T_1} + \sum_{n=T_1+1}^{\infty} \right) \frac{|g(n, t)|}{n}.
\]
(4.6)

By Lemma 3.8(i)
\[
\sum_{n=1}^{T_1} = O \left( \frac{t}{\log t^{-1}} \right) \sum_{n=1}^{T_1} \frac{1}{n} = O(1).
\]
(4.7)

By Lemma 3.5 and Lemma 3.8(ii)
\[
\sum_{n=T_1+1}^{\infty} \frac{|g(n, t)|}{n} \leq \sum_{n=T_1+1}^{\infty} \frac{|g(n, \delta)|}{n} + \sum_{n=T_1+1}^{\infty} \frac{|h(n, t)|}{n}
\]
\[
= O(1) + O(1) \sum_{n=T_1+1}^{\infty} \frac{1}{n^{1+Ad^2}} + O(1) \sum_{n=T_1+1}^{\infty} \frac{1}{(\log t^{-1}) n} \sum_{n=T_1+1}^{\infty} e^{-Ar^2 \log n}
\]
\[
+ O(1) \frac{t^2}{\log t^{-1}} \sum_{n=T_1+1}^{\infty} \frac{e^{-Ar^2 \log n}}{n} + O(1) \sum_{n=T_1+1}^{\infty} \frac{L(n, t)}{n \log n}
\]
\[
= O(1) + O(1) \sum_{n=T_1+1}^{\infty} \frac{L(n, t)}{n \log n}
\]
(4.8)

since \( \int_{T_1}^{\infty} e^{-Ar^2 \log x} \frac{dx}{x \log x} = \int_{t}^{\infty} e^{-Ad^2} \frac{d\theta}{\log \theta} = O(\log t^{-1}) \) and \( \int_{T_1}^{\infty} e^{-Ar^2 \log x} \frac{dx}{x \log x} = t^{-2} \int_{T_1}^{\infty} e^{-Ad^2} d\theta = O(t^{-2}) \).

Now writing \( \sum_{n=T_1+1}^{\infty} \frac{L(n, t)}{n \log n} = \left( \sum_{n=T_1+1}^{T_2} \frac{L(n, t)}{n \log n} + \sum_{n=T_2+1}^{\infty} \frac{L(n, t)}{n \log n} \right) \) and employing Lemma 3.7(ii) and Lemma 3.7(i), respectively, for the first and second sums, we get
\[
\sum_{n=T_1+1}^{\infty} \frac{L(n, t)}{n \log n} = O(1) \sum_{n=T_1+1}^{T_2} \frac{1}{n \log n} \log \log T_2 + O(1) \frac{1}{t^2 (\log t^{-1})^2} \sum_{n=T_2+1}^{\infty} \frac{1}{n (\log n)^2}
\]
\[
= O(1) \log \log T_1 + O(1) \frac{1}{t^2 (\log t^{-1})^2 \log T_2}
\]
\[
= O(1).
\]
(4.9)

Collecting the results from (4.6)–(4.9), we get (4.5) and this completes the proof of Theorem 2.3.

5 Proof of Theorem 2.4

We need the following additional lemmas for the proof of Theorem 2.4.

Lemma 5.1 [7, p. 314, Lemma 10] \( \psi(t) \log t^{-1} \in BV(0, \delta) \), and \( \frac{\psi(t)}{t} \in L(0, \delta) \) if and only if \( \psi(+0) = 0 \) and \( \int_0^1 \log t^{-1} |d\psi(t)| < \infty \).
**Lemma 5.2** Let \( T_1 = \exp(t^{-1}) \). Then

\[
\begin{align*}
(i) \quad & \int_t^\delta K^\lambda_n(u) \, du = e^{-At^2 \log n} \left[ O(t^2) + O\left( \frac{1}{\log n} \right) \right], \\
(ii) \quad & \int_t^\delta K^\lambda_n(u) \, du = O\left( \frac{1}{\log n} \right), \quad \text{when } n \leq T_1.
\end{align*}
\]

**Proof** By Lemma 3.3(i) and Lemma 3.2(i)

\[
\int_t^\delta K^\lambda_n(u) \, du = \int_t^\delta R(n, u) \sin l(n) u \, du + O(1) \log n \int_t^\delta u^3 R(n, u) \, du
\]

\[
= \int_t^\delta R(n, u) \sin l(n) u \, du + O(1) \log n \int_t^\delta u^3 e^{-At^2 \log n} \, du
\]

\[
= J_1 + O(1)J_2, \quad \text{say (5.1)}
\]

Integrating by parts, we get

\[
J_2 = \frac{\log n}{2} \int_t^\delta v e^{-At^2 \log n} \, du
\]

\[
= \frac{1}{2A} \left( t^2 e^{-At^2 \log n} - \delta^2 e^{-A\delta^2 \log n} \right)
\]

\[
+ \frac{1}{2A^2 \log n} \left( e^{-At^2 \log n} - e^{-A\delta^2 \log n} \right)
\]

\[
\leq \frac{1}{2A} \left( t^2 + \frac{1}{2A \log n} \right) e^{-At^2 \log n}.
\]

As \( R(n, t) \) is monotonic non-increasing in \( t \), by second mean value theorem, we have for \( t < t' < \delta \).

\[
J_1 = R(n, t) \left[ \cos l(n) t - \cos l(n) t' \right]
\]

\[
= O\left( \frac{e^{-At^2 \log n}}{\log n} \right),
\]

using Lemma 3.2(i). \( \square \)

Part (i) follows from (5.1), (5.2) and (5.3). When \( n \leq T_1 \), it is easily seen that \( t^2 \) is dominated by \( (\log n)^{-1} \) and hence (ii) follows from (i).

By Lemma 5.1, Theorem 2.4 takes the following equivalent form:

**Theorem 5.3** If \( \psi(+) = 0 \) and \( \int_0^\delta |\psi(t)||\log t^{-1} < \infty \), then \( \sum_{n=1}^{\infty} B_n(x) \in |K^\lambda| \).

**Proof of Theorem 5.3** For \( n \geq 1 \), we write

\[
B_n(x) = \frac{2}{\pi} \left[ \int_0^\delta + \int_\delta^\pi \right] \psi(t) \sin nt \, dt
\]

\[
= \frac{2}{\pi} (p_n + q_n), \quad \text{say}.
\]

By adopting the argument used in proving \( \sum Q_n \in |K^\lambda| \) (see the proof of Theorem 2.3) it can be shown that \( \sum q_n \in |K^\lambda| \). Integrating by parts and using the fact that \( \psi(+) = 0 \), we get

\[
p_n = \int_0^\delta d\psi(t) \int_t^\delta \sin nu \, du.
\]
Let \( \xi_n(p) \) denote the \( n \)th \( K^\lambda \)-mean of the sequence \( \{p_{n+1}\} \). By routine simplification, we have

\[
\xi_{n-1}(p) = \int_0^\delta d\psi(t) \int_t^\delta \frac{\Gamma(\lambda)}{\Gamma(n+\lambda-1)} \left( \sum_{k=0}^{n-1} [n-1]^k \sin(k+1) u \right) du \\
= \int_0^\delta d\psi(t) \int_t^\delta K_n^\lambda(u) du. \tag{5.5}
\]

By definition \( \sum p_n \in |K^\lambda| \), if and only if

\[
\sum_{n=1}^\infty \left| \frac{\xi_{n-1}(p)}{n} \right| < \infty;
\]

that is,

\[
\sum_{n=1}^\infty \frac{1}{n} \left| \int_0^\delta d\psi(t) \int_t^\delta K_n^\lambda(u) du \right| < \infty. \tag{5.6}
\]

As \( \int_0^\delta |d\psi(t)| \log t^{-1} \) is finite, for the validity of (5.6), it suffices to show that uniformly in \( 0 < t \leq \delta \)

\[
\sum^* \equiv \sum_{n=1}^\infty \frac{1}{n} \left| \int_t^\delta K_n^\lambda(u) du \right| = O(\log t^{-1}). \tag{5.7}
\]

Writing \( \sum^* = \sum_{n=1}^{T_1} + \sum_{n=T_1+1}^\infty \) and using Lemma 5.2(ii) and Lemma 5.2(i), respectively, for the first sum and second sum, we get

\[
\sum^* = O(1) \sum_{n=1}^{T_1} \frac{1}{n \log n} + O(t^2) \sum_{n=T_1+1}^\infty \frac{e^{-At^2 \log n}}{n} \\
+ O(1) \sum_{n=T_1+1}^\infty \frac{e^{-At^2 \log n}}{n \log n} \\
= O(1) \log \log T_1 + O(t^2) \int_{T_1}^\infty \frac{e^{-At^2 \log x}}{x} dx \\
+ O(1) \int_{T_1}^\infty \frac{e^{-At^2 \log x}}{x \log x} dx \\
= O(\log t^{-1})
\]

which ensures (5.7) and this completes the proof of Theorem 5.3. \( \square \)

**Acknowledgments** The authors are thankful to the referee for his valuable suggestions and criticisms which led to the improvement of the paper.

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