Pay to change lanes: A cooperative lane-changing strategy for connected/automated driving

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Abstract

This paper proposes a cooperative lane changing strategy using a transferable utility games framework. This allows vehicles to engage in transactions where gaps in traffic are created in exchange for monetary compensation. We formulate gains in travel time, referred to as time differences, that result from achieving higher speeds. These time differences, coupled with value of time, are used to formulate a utility function where utility is transferable. We also allow for games between connected vehicles that do not involve transfer of utility. We apply Nash bargaining theory to solve the latter. A cellular automaton is developed and utilized to perform simulation experiments that explore the impact of such transactions on traffic conditions (travel-time savings, resulting speed-density relations and shock wave formation) and the benefit to vehicles. The results show that lane changing with transferable utility between drivers can help achieve win-win results, improve both individual and social benefits without resulting in any adverse effects on traffic characteristics in general and, in fact, result in slight improvement at traffic densities outside of free-flow and (bumper-to-bumper) jammed traffic.

Keywords: Cooperative game theory, lane changing, connected vehicles, transferable utility, side payment, mobile payment

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1. Introduction

Lane-changing is one of the fundamental maneuvers in vehicular traffic dynamics. Vehicles change lanes to achieve desired speeds (discretionary lane-changing), to avoid unsafe conditions or to move into turning/exit lanes (mandatory lane changing). A majority of models of both types of lane change maneuvers describe them as discrete decision processes carried out by vehicles that are considering/attempting to change lanes (Gipps, 1986; Kesting et al., 2007; Zheng, 2014; Kamal et al., 2015; Du et al., 2015; Keyvan-Ekbatani et al., 2016; Li et al., 2016a; Pan et al., 2016; Bevly et al., 2016). We refer to (Ahmed et al., 1996; Toledo et al., 2003) for a classical reference on discrete choice methods for lane changing and to (Pan et al., 2016) for a more recent background on these types of lane changing models. These tend to ignore the competition for space that may arise between vehicles and how this competition affects their decisions. This has given rise to game theoretic techniques in modeling lane-changing dynamics (Kita, 1999; Kita et al., 2002; Wang et al., 2015; Meng et al., 2016; Liu et al., 2007; Talebpour et al., 2015; Oyler et al., 2016; Li et al., 2017). A typical setting in these approaches is one in which a discrete set of maneuvers (typically two or three) are being considered by a vehicle that is attempting to change lanes (the target vehicle) and a vehicle that is in the target lane but behind the target vehicle (the lag vehicle). For example, the target vehicle may have the choice set \{change lane, do not change lane\} while the lag vehicle has the choice set \{give way, do not give way\} (Rahman et al., 2013).

The different approaches in the literature vary in how they model the payoffs associated with pure strategies, which may vary depending on whether the maneuver is mandatory or discretionary. Some papers only consider lane changing games for mandatory behavior, such as merging (Kita, 1993, 1999; Pei and Xu, 2006). Most game-theoretic approaches consider lane changing to be non-cooperative games, the outcomes of which are either Nash or Stackelberg equilibria depending on how the game is modeled (Yoo and Langari, 2013; Li et al., 2016b). Cooperative strategies have also been considered recently (Wang et al., 2015; Yao et al., 2017). A common feature of the latter is that vehicles are assumed to be selfless; one in which cooperative vehicles (typically under some form of control) will take actions that maximize the collective or group utility, not their own. This leads to winners and losers.

Automation and vehicle to vehicle (V2V) communication present an opportunity to re-think lane-changing strategies. These allow vehicles to broadcast their payoffs, which can vary from vehicle to vehicle and for
the same vehicle from trip to trip, i.e., depending on trip purpose (Hossan et al., 2016). Communication also allows vehicles to engage in bargaining (and/or repeated) games. These two features culminate in a departure from the traditional (non-cooperative) game-theoretic lane-changing approaches in which decisions are made without communication. Indeed, with vehicle to vehicle (V2V) communication capabilities, connected vehicles can easily engage in transactions based on their individual travel needs. For example, quick mobile payment without transaction costs has gained popularity in China.

In light of this, we propose a lane changing paradigm, suitable for an automated world, in which gaps in traffic are envisaged as resources (or goods) that can be traded. In simple terms, we propose a lane changing mechanism that allows vehicles to purchase right of way or compensate other vehicles for allowing them to change lanes. From a modeling standpoint, we propose modeling lane changing as transferable utility (TU) games with side payments (Thomas, 2008; Myerson, 2013). Our approach also allows for vehicles to refuse to engage in TU games as well; in this case, we consider Nash bargaining. It can be shown (when there are no transaction fees) that the outcomes of these games are at least pareto efficient (Coase, 2013). To the best of our knowledge, this is the first time TU games are applied to lane changing dynamics.

This paper is organized as follows: Sec. 2 describes the problem setting and formulates the utility functions and the game’s payoffs in Sec. 2.1 - Sec. 2.3. The remainder of Sec. 2 presents the lane change games with, transfer of utility, side payments (Sec. 2.4), and games between connected vehicles that do not wish to engage in transactions, i.e., games with non-transferable utility (Sec. 2.5). Sec. 3 presents a numerical example and a set of simulation experiments to test the proposed model, analyze the results from the aspects of cost-effectiveness and impact on traffic flow. Sec. 4 concludes the paper.

2. Methodology

2.1. Problem description

The game setting we assume in this paper is one that is played between pairs of vehicles, a lane changing (target) vehicle and a lag vehicle in the

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1 Although it is common to assume that the vehicles are perfectly knowledgeable (of the payoffs).

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target lane. We shall focus on this simple setting, but the ideas are generalizable to games involving more than two vehicles or settings involving two lane changing vehicles. For example, the proposed approach can be easily extended to situations with multiple pairwise lane change games: one simply parallelizes the proposed approach. A more general framework is one where a vehicle engaged in transactions with more than one vehicle simultaneously, e.g., a vehicle that wishes to make two immediate lane change maneuvers. The first maneuver can be analyzed in a similar way to the pairwise approach described in this paper. Pricing the second maneuver is more subtle as it depends on the outcome of the first maneuver. However, the proposed approach can still be utilized as a building block for such sophisticated scenarios.

It is assumed that vehicles involved in the game can communicate positions, speeds, accelerations, and values of time (VOTs). We assume mixed traffic: vehicles capable of (and willing to) engage in lane change transactions, referred to as transaction vehicles (TV), and those that either do not possess transaction capabilities (or do not wish to engage in transactions), which are referred to as non-transaction vehicles (NTV). This is illustrated in Fig. 1.

![Fig. 1: Transaction vehicles (TVs) and non-transaction vehicles (NTVs).](image)

The types of games considered in this paper are those in which utility can be transferred (or traded) between vehicles. While the motivations for changing lanes can include numerous factors, such as comfort, safety, and speed, not all of these factors can be traded. For example, safety cannot (and should not) be traded. Utility can only be transferred when a common currency that is valued equally by both vehicles is used. It is for this reason that we formulate the utility function below using VOT. The mathematical notation used in this paper is listed in Table 1.
| Variable | Description |
|----------|-------------|
| $v_i^1$  | the higher speed choice (of vehicle $i$) |
| $v_i^2$  | the lower speed choice (of vehicle $i$) |
| $v_i^E$  | the expected equilibrium speed (of vehicle $i$) |
| $v_i$    | the speed choice of vehicle $i$ |
| $a_{i1}$ | the acceleration from $v_1$ to $v_E$ (of vehicle $i$) |
| $a_{i2}$ | the acceleration from $v_2$ to $v_E$ (of vehicle $i$) |
| $a_{\text{pos}}$ | the value of $a_{i1}$ and $a_{i2}$ we used in simulation when they are positive |
| $a_{\text{neg}}$ | the value of $a_{i1}$ and $a_{i2}$ we used in simulation when they are negative |
| $t_a$    | the average time it takes for a vehicle to complete a lane-change |
| $t_{b1}$ | the time when the speed of vehicle changes from $v_1$ to $v_E$ |
| $t_{b2}$ | the time when the speed of vehicle changes from $v_2$ to $v_E$ |
| $t_b$    | $\max\{t_{b1}, t_{b2}\}$ |
| $S_a$    | the difference in distance achieved when choosing $v_1$ over $v_2$ from time 0 to $t_a$ |
| $S_b$    | the difference in distance achieved when choosing $v_1$ over $v_2$ from time $t_a$ to $t_b$ |
| $S$      | the difference in distance achieved when choosing $v_1$ over $v_2$ from time 0 to $t_b$, $S = S_a + S_b$ |
| $t_{iA}$ | the time difference between achieving $v_1$ and $v_2$ (for vehicle $i$) |
| $c_{i\text{vot}}$ | the coefficient representing the VOT (of vehicle $i$) |
| $w_i$    | the utility of vehicle $i$ |
| $A, B$   | the utility matrices of vehicles A and B, respectively |
| $p$      | the probability for vehicle A to change lanes in its threat strategy in a TU game |
| $q$      | the probability for vehicle B to not give way in its threat strategy in a TU game |
| $T_A, T_B$ | the payoffs of vehicles A and B, respectively, at their threat strategies |
| $Q_A, Q_B$ | the payoffs of vehicles A and B, respectively, at status quo |
| $\omega^*$ | the total maximal achievable utility by vehicles A and B in a TU game |
| $(i^*, j^*)$ | the strategy pair that achieves the maximal payoff $\omega^*$, also the final decision of TU game |
| $\sigma$ | the side payment in a TU game |
2.2. Utility function

Consider two vehicles, A and B, in the lane-changing game depicted in Fig. 2. Vehicle A has the choice set: \{change lanes, do not change lanes\} and vehicle B’s choice set is \{give way, do not give way\}. The payoffs are summarized in Table 2; \(A_{ij}\) and \(B_{ij}\) are the payoffs to vehicles A and B, respectively, associated with actions \(i\) and \(j\). If A chooses to change lanes and B chooses not to give way, we assume that both vehicles get a large negative payoff (e.g., they collide). If A changes lanes and B gives way, we denote the speeds achieved by vehicles A and B by \(v_A^1\) and \(v_B^2\), respectively. If A stays as a result of B not giving way, we denote the speeds achieved by A and B by \(v_A^2\) and \(v_B^1\), respectively. Here, we have that \(v_A^1 > v_A^2\) and \(v_B^1 > v_B^2\). When A chooses to stay despite B giving way, they both assume the lower speeds \(v_A^2\) and \(v_B^2\).

For vehicle A, the utility function, denoted \(\mu_A\) (\(\mu_B\) for B) is related to both its own speed choice \(v^A\), and that of vehicle B denoted \(v^B\). Similarly,
$u^B$ is related to both $v^B$ and $v^A$. Vehicle $A$’s utility function is given by

$$u^A(v^A, v^B) = \begin{cases} -M & v^A = v^A_1, v^B = v^B_1 \\ c^A\text{vot}t^A_d & v^A = v^A_1, v^B = v^B_2 \\ 0 & v^A = v^A_2, \end{cases}$$

where $M$ is a large positive number, $c^A\text{vot}$ is a coefficient capturing VOT of $A$, and $t^A_d$ is referred to as time difference between choosing lower speed $v^A_2$ and higher speed $v^A_1$ for $A$. Using time difference as a means of calculating the utility, the reference point (a.k.a. datum) of $u^A$ coincides with the action $v^A = v^A_2$, that is $u^A(v^A_2, \cdot) = 0$. The latter can be interpreted as: choosing/maintaining the lower speed comes with zero utility. Vehicle $B$’s utility, $u^B$, can be calculated in a similar fashion. The utility matrix in Table 2 can be rewritten as Table 3. Time difference is the travel time saved with the higher-speed choice over a short time period. How it is calculated is described next.

### 2.3. Modeling of time difference

Assume a vehicle is traveling with longitudinal speed $v = v_0$ and at time $t = 0$ an opportunity presents itself for the vehicle to achieve a higher speed (e.g., via a lane change). For simplicity, we assume that the two lanes have similar traffic conditions (on average); that is, the traffic densities are the same, and hence so are the equilibrium speeds in the two lanes. However, we do not assume that the gaps in traffic are the same and this is what creates the speed gain opportunities. The speed gains occur over a short period of time equal to the length of an equilibration process. In other words, a vehicle changing lanes accelerates to a higher speed then decelerates to the equilibrium speed. Similarly, a vehicle giving way might decelerate for a short period of time and then accelerate to the equilibrium speed.

Both the target vehicle (vehicle $A$) and the lag vehicle (vehicle $B$) can be described as a vehicle that is changing its speed to one of two possible

| Actions         | vehicle B          |         |         |
|-----------------|--------------------|---------|---------|
|                 | vehicle A          | Do not give way | Give way |
| Change lane     | $(-M, -M)$         | $c^A\text{vot}t^A_d, 0)$ |
| Do not change lane | $(0, c^B\text{vot}t^B_d)$ | $(0, 0)$ |
speeds: either \( v = v_1 \), or \( v = v_2 \), where \( v_1 > v_2 \). We denote by \( t_a > 0 \) the time instant at which the vehicle achieves its new speed. We assume, without loss of generality, that \( t_a \) is the same in both scenarios. If the first speed is adopted, illustrated in the top part of Fig. 3(a), the vehicle reaches the faster speed \( v_1 \) over a longer travel distance. If the second speed is chosen, illustrated in the bottom part of Fig. 3(a), it reaches \( v_2 \) over a shorter travel distance. Assume these two scenarios are identical in all other aspects (including traffic conditions, vehicle performance and characteristics, etc.) so that in both scenarios the vehicle achieves the same equilibrium speed eventually. We denote the equilibrium speed by \( v_E \). The equilibrium speed \( v_E \) is mainly affected by traffic density. Assume, without loss of generality, that \( v_1 \geq v_E \geq v_2 \). Let \( t_b \) denote the time instant at which the vehicle achieves the equilibrium speed \( v_E \), \( t_{b1} \) in the first scenario and \( t_{b2} \) in the second scenario. If the vehicle chooses the higher speed, it will have covered a longer distance by time \( t_b = \max\{t_{b1}, t_{b2}\} \) as shown in Fig. 3(b). We denote the difference in distance covered during the equilibration process by \( S \).

![Diagram](image.png)

Fig. 3: Influence of different speed choices at time \( t = 0 \).

We split the distance \( S \) into two parts: \( S_a \) and \( S_b \) such that \( S = S_a + S_b \). Here, \( S_a \) denotes the distance difference from time 0 to time \( t_a \) and \( S_b \) is the difference in distance covered from time \( t_a \) to time \( t_b \). \( S_a \) is given (approximately) by

\[
S_a = \frac{(v_1 - v_2)t_a}{2}.
\]  

(2)

The distance \( S_b \) consists of two parts: \( S_{b1} \) and \( S_{b2} \), pertaining to the first and second scenarios, respectively. Let \( d_{b1} \) (\( d_{b2} \)) denote the distance traveled by the vehicle from time \( t_a \) to time \( t_{b1} \) (\( t_{b2} \)) in the first (second) scenario. Then,

\[
S_{b1} = d_{b1} - v_E \Delta t_{b1}
\]

(3)
and

\[ S_{b2} = v_E \Delta t_{b2} - d_{b2}, \]  

(4)

where \( \Delta t_{b1} = t_{b1} - t_{a1} \) and \( \Delta t_{b2} = t_{b2} - t_{a2} \). To illustrate these calculations, we give two examples, one for vehicle A and one for vehicle B in Fig. 4. In both Fig. 4(a) (vehicle A) and Fig. 4(b) (vehicle B), the solid red curve is the speed profile in the first scenario, the solid blue curve is the speed profile in the second scenario, and the green line is equilibrium speed.

In Fig. 4(a), vehicle A finds a gap in the adjacent lane at time \( t = 0 \). If A changes lanes, they can temporarily achieve \( v_1 > v_E \), and then decelerate to \( v_E \); if A does not change lanes, they maintain a speed of \( v_E \). In this case, \( S_{b2} = 0 \). In Fig. 4(b), vehicle B receives a lane-change request from vehicle A. If B gives way, they decelerate to \( v_2 \) and then accelerate to \( v_E \). If B does not give way, they can temporarily speed up to \( v_1 > v_E \) to close the gap and then decelerate to \( v_E \).

It’s worth mentioning that the speed profiles in Fig. 4 need not be linear, especially from \( t_a \) to \( t_{b1} \) (or \( t_{b2} \)). However, as we do not have information for the (future) period from \( t_a \) to \( t_{b1} \) (or \( t_{b2} \)), it is safe to assume a constant acceleration \( a_1 \) (or \( a_2 \)). Hence, we can approximate \( S_{b1} \) and \( S_{b2} \) as:

\[ S_{b1} = \frac{(v_E - v_1)^2}{-2a_1} \]  

(5)

and

\[ S_{b2} = \frac{(v_E - v_2)^2}{2a_2}. \]  

(6)
Here $a_1$ and $a_2$ should fall within an appropriate range between maximum deceleration and maximum acceleration. The sign of $a_i$ should be the same as the sign of $v_E - v_i$, that is, for $i = 1$, the vehicle decelerates to $v_E$, while for $i = 2$ the vehicle accelerates to $v_E$. Combining (2), (5), and (6), i.e., $S = S_a + S_{b,1} + S_{b,2}$, we get:

$$S = \frac{1}{2} \left[ (v_1 - v_2)t_a + \frac{(v_E - v_1)^2}{-a_1} + \frac{(v_E - v_2)^2}{a_2} \right].$$ (7)

It is worth noting that, although (7) was derived under the assumption that $v_1 \geq v_E \geq v_2$, it also applies to other scenarios, $v_1 \geq v_2 \geq v_E$ and $v_E \geq v_1 \geq v_2$.

We define the time difference $t_d$ as the average amount of travel time saved during the equilibration process as a result of achieving the higher speed $v_1$ at time $t = 0$. That is:

$$t_d \equiv \frac{S}{v_E} = \frac{1}{2v_E} \left[ (v_1 - v_2)t_a + \frac{(v_E - v_1)^2}{-a_1} + \frac{(v_E - v_2)^2}{a_2} \right],$$ (8)

where the parameters $t_a$, $a_1$, and $a_2$ need to be calibrated. In the simulation experiments below, we set $a_i$ to $a_{\text{pos}}$ if $v_E - v_i \geq 0$, and $a_{\text{neg}}$ if $v_E - v_i < 0$ for simplicity. The speeds $v_1$, $v_2$ and $v_E$ are variables in different TU games, where $v_E$ is affected by density and VOT in this paper. Note that for a road with heterogeneous traffic conditions across lanes, one can generalize the proposed approach to one with different equilibrium speeds in the two lanes and the areas depicted in Fig. 4 would have more components.

2.4. Transfer of utility and side payments

In a game with transferable utility (TU), side payments from one vehicle to another are allowed. These are games that involve two (connected) vehicles that wish to engage in a transaction. This section describes utility transfer and how to calculate the side payment. Let $\sigma \in \mathbb{R}$ denote the side payment made by vehicle A to vehicle B. When $\sigma$ is negative, this is interpreted as a positive payment that is made by vehicle B to vehicle A. When $\sigma < 0$, B does not give way to A, but also makes a payment to A as a result. This is an important factor distinguishing games where vehicles agree to engage in transactions from those where a transaction does not take place (the non-transaction games described in the next section).

We define the following payoff matrices based on the values given in Table 2:

$$A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B \equiv \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$ (9)
The main idea behind transfer of utility is that through side payment the highest total payoff, denoted \( \omega^* \), can be achieved. Here

\[
\omega^* \equiv \max_{i,j \in \{1,2\} \times \{1,2\}} (A_{ij} + B_{ij}).
\] (10)

Define the strategy pair that achieves the maximal payoff by

\[
(i^*, j^*) \equiv \arg \max_{i,j \in \{1,2\} \times \{1,2\}} (A_{ij} + B_{ij}).
\] (11)

The payoffs achieved this way are \( A_{i^*, j^*} - \sigma \) for A and \( B_{i^*, j^*} + \sigma \) for B; clearly the total utility is \( \omega^* \).

For any \( (i,j) \) pair, define \( \tilde{A}_{ij} \equiv A_{ij} - \sigma \) and \( \tilde{B}_{ij} \equiv B_{ij} + \sigma \). Since, \( \tilde{A}_{ij} + \tilde{B}_{ij} = \omega_{ij} \), where \( \omega_{ij} \) is a constant\(^2\), we have that the set of payoff pairs associated with the strategy pair \( (i,j) \) in a TU game fall along a line of slope -1 that goes through the point \( (A_{ij}, B_{ij}) \). As such, the set of possible payoffs, the feasible region, for a TU game, denoted \( \Omega_{TU} \), is defined as the convex hull off all possible strategy pairs \( (\tilde{A}_{ij}, \tilde{B}_{ij}) \):

\[
\Omega_{TU} \equiv \text{Conv}\{(\tilde{A}_{ij}, \tilde{B}_{ij}) : i,j = 1,2\}.
\] (12)

The feasible region associated with a TU game (the set \( \Omega_{TU} \)) is shown in Fig. 5(a).

In order to determine what the appropriate side payment is, one needs to first investigate the strategy applied in the absence of an agreement. Such a strategy is known as the threat strategy. Define \( p \equiv [p \ 1 - p]^\top \) and \( q \equiv [q \ 1 - q]^\top \), where \( p \) is the probability that vehicle A chooses to change lanes under their threat strategy and \( q \) is the probability that vehicle B chooses to not give way under their threat strategy. Under their threat strategies, the expected payoffs to vehicles A and B are given, respectively, by \( T_A \equiv p^\top A q \) and \( T_B \equiv p^\top B q \). The pair \( (T_A, T_B) \) is known as the threat point.

The expected payoffs \( T_A \) and \( T_B \) (associated with the threat strategies) can be achieved without agreement. Then, in an agreement the payoffs should be no less than \( T_A \) for vehicle A and no less than \( T_B \) for vehicle B. Since the total payoff from the TU solution is \( \omega^* \), we have that the TU solution lies between the two points \( (T_A, \omega^* - T_A) \) and \( (\omega^* - T_B, T_B) \). For example, if \( (i^*, j^*) = (1,2) \), the range of the TU solution is depicted in

\(^2\)These constants can vary from one \( (i,j) \) pair to another and \( \omega_{i,j^*} = \omega^* \).
Fig. 5(b). Thomas (2008) suggests use of the midpoint as a “natural compromise”: the two vehicles lose equally if the agreement is broken. The
midpoint solution is also depicted in Fig. 5(b).

The payoffs associated with the midpoint solution are given by the
pair \( \left( \frac{T_A - T_B + \omega^*}{2}, \frac{T_B - T_A + \omega^*}{2} \right) \) and we immediately see that A’s threat strategy
should be chosen in a way that maximizes \( T_A - T_B \), while B’s should be
chosen in a way that minimizes it. Since

\[
T_A - T_B = \mathbf{p}^\top \mathbf{A} \mathbf{q} - \mathbf{p}^\top \mathbf{B} \mathbf{q} = \mathbf{p}^\top (\mathbf{A} - \mathbf{B}) \mathbf{q},
\]

we have that selecting the threat strategy can be described as a zero-sum
game, where A’s expected payoff is \( \mathbf{p}^\top (\mathbf{A} - \mathbf{B}) \mathbf{q} \) and B’s expected payoff is
\( \mathbf{p}^\top (\mathbf{B} - \mathbf{A}) \mathbf{q} \). From Table 3, the matrix \( \mathbf{A} - \mathbf{B} \) has the following structure

\[
\mathbf{A} - \mathbf{B} = \begin{bmatrix}
0 & c_A^A_{\text{vol}, d} \\
-c_B^B_{\text{vol}, d} & 0
\end{bmatrix}
\]

Since \( c_A^A_{\text{vol}, d} \geq 0 \) and \( c_B^B_{\text{vol}, d} \geq 0 \), we have that \( \max_i \min_j (A_{i,j} - A_{i,j}) = \min_j \max_i (A_{i,j} - A_{i,j}) = 0 \) and we always have the saddle point \{change
lanes, do no give way\}. While this strategy results in a crash, a crash will
not take place since this is a TU game: one in which a total utility of \( \omega^* \)
is guaranteed. We hence have that \( \mathbf{p} = \mathbf{q} = [1 \ 0]^\top \) and consequently
\( T_A = T_B = 0 \).

Side payment. The payoffs associated with a TU solution (representing
the natural compromise): \( \frac{1}{2}(T_A - T_B + \omega^*) \) for A and \( \frac{1}{2}(T_B - T_A + \omega^*) \) for
B. The side payment is immediately given by

\[ \sigma = A_i^* j^* - T_A - T_B + \omega^* = A_i^* j^* - \frac{\omega^*}{2} \]  

(15)

If \( \sigma > 0 \), the side payment is made from vehicle A to vehicle B and A changes lanes. If \( \sigma < 0 \), the side payment is made from vehicle B to vehicle A and B does not give way. Note that, since \( A_i^* j^* = \omega^* - B_i^* j^* \), we have from (15) that

\[ \sigma = \omega^* - B_i^* j^* - \frac{T_A - T_B + \omega^*}{2} = -\left( B_i^* j^* - \frac{\omega^*}{2} \right) . \]  

(16)

Hence, had we elected to calculate the side payment based on vehicle B’s payoffs, we obtain the same side payment with a negative sign, signifying that, in this case, the payment is made in the opposite direction.

2.5. Games with non-transferable utility

When vehicle A encounters a vehicle that does not wish to engage in a transaction, side payments are not possible. However, when the two vehicles can communicate, bargaining is possible. The motivation for considering such situations is that we wish to allow for scenarios in which vehicles do not wish to make side payments and those where it is acknowledged that payment is not always guaranteed for those that wish to receive side payments. We note that, from a methodological perspective, this does not preclude scenarios that do not involve utility transfer.

The feasible set for a non-transaction lane-change game, denoted by \( \Omega_{NTU} \), is the convex hull of the 4 points, \( (A_i, B_i)_{i,j \in \{1,2\} \times \{1,2\}} \). That is

\[ \Omega_{NTU} \equiv \text{Conv}\left( \{(A_{ij}, B_{ij}) : i,j = 1,2\} \right) . \]  

(17)

The feasible region is illustrated in Fig. 6(a). Communication between vehicle motivates a Nash bargaining solution (Nash, 1950, 1953) for the NTU game. The Nash bargaining solution, which we denote by \( (N_A, N_B) \), is the unique solution to the maximization problem

\[ \max_{(u_A, u_B) \in \Omega_{NTU}} \frac{(u_A - Q_A)(u_B - Q_B)}{u_A \geq Q_A, u_B \geq Q_B} , \]  

(18)

where \( (Q_A, Q_B) \) is known as the status quo point. The status quo point occurs before an agreement; \( Q_A \) and \( Q_B \) are the utilities achieved by vehicles A and B, respectively, if they do not play the game. It is hence natural to set
Fig. 6: Feasible set and optimal solution for a non-transaction game.

\((Q_A, Q_B) = (0, 0)\) corresponding to the strategy \{A does not change lanes, B gives way\}. Note that in the literature, \((Q_A, Q_B)\) is sometimes referred to as a threat point; we avoid the latter nomenclature to avoid confusion between \((Q_A, Q_B)\) and \((T_A, T_B)\) in the TU game above.

Returning to the bargaining problem (18), we have that

\[
(N_A, N_B) = \arg \max_{(u_A, u_B) \in \Omega_{\text{NTU}}} u_A u_B. \tag{19}
\]

The contour lines of the objective function in (19) are curves that increase in value the farther one moves away from the origin \((0, 0)\). Since the solution lies in the positive quadrant, the optimal solution lies along the line connecting the two points \((A_{21}, B_{21})\) and \((A_{12}, B_{12})\), depicted in Fig. 6(b) as a solid black line. The equation of this line is \(u_B = (-B_{21} / A_{12}) u_A + B_{21}\). Substituting this into the objective function, we have that

\[
N_A = \frac{1}{2} A_{12} \quad \text{and} \quad N_B = \frac{1}{2} B_{21}. \tag{20}
\]

Denote the final decision in NTU game as \((i', j')\). (20) is interpreted as the following strategy: the outcome of the game is either \((i', j') = (1, 2)\), meaning \{vehicle A changes lanes, vehicle B gives way\} or \((i', j') = (2, 1)\), meaning \{vehicle A does not change lanes, vehicle B does not give way\}, each outcome with probability 0.5.
2.6. Model summary and simulation

The overall process is summarized in Fig. 7. In situations where vehicles are not completely aware of the utilities of other vehicles, one can extend the present approach summarized in Fig. 7 to include games such as those in (Talebpour et al., 2015) (assuming no cooperation).

![Diagram of lane change decision process flow chart.](image)

Fig. 7: Lane change decision process flow chart.
Safety considerations can also be appended to the present framework, where mandatory lane change conditions arise: see for example the models presented in (Treiber and Kesting, 2013, Chapter 14). To simulate and test the lane changing game, we propose a cellular automaton (Nagel and Schreckenberg, 1992; Maerivoet and De Moor, 2005) given in Algorithm 1 below.

**Algorithm 1**: Simulation of Traffic Dynamics

---

- **Input**: Cell size, number of cells, discrete time step length, inflow rate, probability of slow-down ($p_{sd}$), {c volte}, percentage of different vehicle classes, max speed ($v_{max}$), total simulation time, initial speeds and positions of vehicles.

- **Iterate**: For each time step do:
  - For each vehicle, starting from downstream most do:
    - **Acceleration**: if $v < v_{max}$, $v \leftarrow v + 1$.
    - **Slow down (lead vehicles)**: for subject vehicle located in site $[l]$, $[l + m]$, and $[l + n]$ denote the positions of the immediate leader in the subject lane and other lane, respectively.
      - If strategy chosen is stay then
        $$v_{stay} \leftarrow \min(v, \left\lfloor \frac{m-1}{2} \right\rfloor).$$
      - End If
      - If strategy chosen is change lanes then
        $$v_{change} \leftarrow \min(v, v_{stay} + 1, \left\lfloor \frac{n-1}{2} \right\rfloor).$$
      - End If
    - **Randomization**: With probability $p_{sd}$, $v_{stay} \leftarrow \max(0, v_{stay} - 1)$ and $v_{change} \leftarrow \max(0, v_{change} - 1)$.
  - End For
  - For each lead/lag vehicle pair (A and B) from downstream most do:
    - **Lane change game**: Calculate A, B, solve $(i^*, j^*)$ or $(i', j')$.
    - Determine new speeds based on $(i^*, j^*)$ or $(i', j')$.
    - Update $v_{A_{stay}}, v_{A_{change}}, v_{B_{stay}},$ and $v_{B_{change}}$.
  - End For
  - For each vehicle do:
    - Update system state:
      - If $v_{change} > v_{stay}$ then change lane state
      - $v \leftarrow \max(v_{change}, v_{stay})$.
    - End If
  - End For
- End For

---

Algorithm 1 considers a two-lane road represented as a two-dimensional
uniform lattice $\mathcal{L} \times \mathcal{I}$, where $\mathcal{L}$ is the longitudinal dimension and $\mathcal{I}$ is cross-sectional dimension (i.e., the lanes). The spatial discretization is such that each site in the lattice can be occupied by at most one vehicle. The state of each (occupied) site during a discrete time step is specified by discretized speeds, which can take integer values between 0 and $v_{\text{max}}$, where $v_{\text{max}}$ is the maximum number of cells that can be traversed by a vehicle on the road during a single discrete time step. The simulation procedure is summarized in Algorithm 1. The following definitions are used in Algorithm 1: $[i]$ is cell $i$ in lane $i$, $-i$ is the complement of cell $i$ (all cells excluding $i$).

3. Experiments

3.1. Numerical example

Consider two vehicle classes on the road, one with high VOT and one with low VOT. The equilibrium mean speeds are 38 km/h for TVs with high VOT and 31 km/h for TVs with low VOT. We assume that vehicle A has low VOT and wants to make a lane change; it has two options: change lane and achieve a speed of $v_A^1 = 55$ km/h, or stay and maintain a speed of $v_A^2 = 25$ km/h. Vehicle B, with high VOT, is the competing lag vehicle and it has two options: do not give way and achieve a speed of $v_B^1 = 52$ km/h and give way to achieve a speed of $v_B^2 = 45$ km/h. Following (Hossan et al., 2016), the VOT coefficient for a low VOT vehicle is set to $c_{\text{vot}}^A = 10$ dollars/h and that for the high VOT vehicle class is set to $c_{\text{vot}}^B = 25$ dollars/h. We set $t_a = 3$ s, $a_A^1 = -4$ m/s$^2$, $a_A^2 = 1$ m/s$^2$, $a_B^1 = -3$ m/s$^2$, and $a_B^2 = -1$ m/s$^2$. Applying (8), we have that $t_A^d = 2.26$ s and $t_B^d = 0.34$ s. Then based on Table 3, we have the payoff matrix given in Table 4.

Table 4: Utility matrix for a TU game

| Actions          | vehicle B                         |
|------------------|-----------------------------------|
|                  | Do not give way | Give way |
| vehicle A        | Change lane     | $(-M, -M)$ | $(0.0062, 0)$ |
|                  | Do not change lane | $(0, 0.0023)$ | $(0, 0)$ |

Following the steps in Sec. 2.4, we get $\omega^* = 0.0062$, $T_A - T_B = 0$, and the TU solution for this game is $(0.0031, 0.0031)$. The cooperative strategy $(i^*, j^*)$ is $\{A \text{ changes lanes, B gives way}\}$, and $A_{i^*, j^*} = A_{12} = 0.0062$. Since $\sigma = A_{i^*, j^*} - t_A - T_B + \omega^* = 0.0062 - 0.0031 = 0.0031$ is positive, A would pay B 0.0031 dollars to change lanes and B receives 0.0031 dollars to give way to A. It is worth noting that, in this case, even though A has a lower VOT...
than B, it is still possible that A is willing to pay B to change lanes. But, in general, vehicles with higher VOT are more likely to be the payers.

3.2. Simulation experiments: Analysis of benefit to TVs

The defaults parameters used for the experiments in the remainder of this section are set to: cell size = 7.5 m, total length of road = 20.25 km, time step = 1 s, $v_{\text{max}} = 5$ cells/s, $a_{\text{pos}} = 1$ cell/s$^2$, $a_{\text{neg}} = -1$ cell/s$^2$, number of lanes = 2, $p_{\text{sd}} = 1/3$, High-VOT = 25 dollars/h, low-VOT = 10 dollars/h (Hossan et al., 2016), penetration rate of TV = 1, and high-to-low VOT ratio = 1:4.

Figures 8 - 11 depict the results of simulation experiments, where we change the penetration rates of TVs and traffic densities. Income and time savings per hour are illustrated in Fig. 8 and Fig. 9 under varying traffic densities for both high VOT TVs and low VOT TVs. They show that income is, in general, negative for high VOT TVs while the time saved is positive. For the low VOT TVs, we see the exact opposite. This is intuitive. To appreciate the trade-off between travel time saving and income, consider the following “benefit” index:

$$\beta = \frac{1}{T_{\text{tr}}} \sum_{i \in V} \sum_{g \in G_i} (c_i^{\text{vot}} \Delta T_{i,g} + I_{i,g}),$$

(21)

where $\beta$ denotes benefit, $T_{\text{tr}}$ is the average travel time of all vehicles $i$ in the set of TV vehicles $V$, $\Delta T_{i,g}$ is the travel time saved (can be negative) by vehicle $i$ in the lane change game $g$, $G_i$ is the set of lane change games played by vehicle $i$ and $I_{i,g}$ is the net income earned by vehicle $i$ in game $g$.
(a) high VOT TVs

(b) low VOT TVs

Fig. 9: The time saving per hour-travel for (a) high VOT TVs and (b) low VOT TVs.

(a) high VOT TVs

(b) low VOT TVs

Fig. 10: The total benefit per hour-travel for (a) high VOT TVs and (b) low VOT TVs.

(can be negative). The total benefit per hour-travel is illustrated in Fig. 10 under varying densities for high and low VOT TVs.

For both high and low VOT TVs, the benefit is positive in general. Furthermore, as TV penetration rates increase, the total benefit tends to grow. We see the same pattern as the traffic density increases from 0 to 120 veh/km. For densities approaching the jam density (133 veh/km in our simulations), lane changing becomes more difficult and benefit approaches zero. Hence, the highest benefit to both high and low VOT TVs is around heavy congestion, but where vehicles can still perform lane change maneuvers. Similarly, the simulation results (Fig. 10) show that in free flow conditions (density <10 veh/km), in total jam condition (density >131 veh/km) or when TV penetration rate is very low (<0.05), the benefit is very small (between -0.2% and 0.2%). Because the benefit of TVs is always positive,
some NTVs may be encouraged to join TU games. Moreover, higher penetration rates mean higher benefit. Hence joining TU games can result in increased benefit to all vehicles.

**Truthfulness in reporting VOT.** To investigate the impact of untruthful reporting of VOT, we consider two scenarios: one in which high-VOT vehicles declare they are low-VOT vehicles and one where low-VOT vehicles declare they are high-VOT vehicles. As with the previous experiments, we test these two scenarios under varying traffic densities and varying TV penetration rates. **Fig. 11** illustrates the total benefits per hour travel for these two scenarios; **Fig. 11(a)** is a plot of the total benefit of high VOT TVs when they are untruthful (they declare that they are low VOT TVs) and **Fig. 11(b)** is a plot of the total benefit of low VOT TVs when they are untruthful (they declare that they are high VOT TVs). In both cases, we observe negative benefits in high traffic density scenarios, which suggests that there is no incentive to lie about their VOT. This hints that the mechanism that we proposed incentivizes truthfulness. We leave this at the conjecture level and attack the problem of mechanism design and truthfulness to future research.

![Fig. 11: Benefit per hour-travel for (a) high VOT TVs and (b) low VOT TVs, when they are untruthful about their reported VOT.](image)

**“VIP” TVs.** In this experiment, we vary the $c_{vot}$ for the high VOT TV while holding all else fixed. Specifically, $c_{vot}$ for the low VOT TV is 10 dollars/h and the penetration rate of high VOT TVs is held at 1% (very small percentage). This is an example of high profile vehicle (along with their entourage). The results are illustrated in **Fig. 12.** In moderate to very high congestion, the “VIP” TVs save about 10-35% in travel time when $c_{vot}$ is increased from 10 dollars/h to 60 dollars/h. Interestingly, however, we
observe a bound on time saving of about 38%, which when reached cannot be improved with greater payment (greater $c_{\text{vot}}$ doesn’t make sense when it is larger than 40 dollars/h). This can be attributed to the simple nature of the TU game that is being tested in this paper. To be specific, the 38% bound can be attributed to the “VIP” TV being blocked by their leaders (in the subject lane). The 38% bound may be broken if the TV is allowed to engage in transactions with multiple vehicles simultaneously, namely, including the lag vehicle in the target lane and leaders in both the subject and target lanes.

### 3.3. Simulation experiments: Impact on traffic

**Speed-density relation.** We examine the impact of introducing transactions on traffic as whole. The first experiments compare the resulting speed-density relations when TVs constitute 100% of the vehicle population and when they constitute only 50% of the vehicle population. Fig. 13(a) compares the speed-density relations of NTVs in the 50% case with all vehicles (both TV and NTV) in both the 100% and 50% cases. The figure shows a very small (almost unnoticeable) decrease in mean (equilibrium) speeds of NTVs across a range of traffic densities when the percentage of TVs is 50%. We can, at the very least, say that the introduction of TVs does not adversely impact mean speeds (in fact, we see slight improvement). In Fig. 13(b), we see a more noticeable difference in mean speeds across a range of traffic densities between high and low VOT TVs. Specifically, we see that the speeds of high-VOT TVs are higher than the speeds of low-VOT TVs, outside of free flow conditions and totally jammed traffic, by as much as 60%. In the former case, lane changes are either not needed to improve speeds or can be carried out without the need to negotiate gaps; in the latter case, changing lanes will not help improve speeds.
We next investigate the impact of TV penetration on the speed-density relations pertaining to high and low VOT TVs. Fig. 14 shows that as the TV penetration rate increases, the speeds of high VOT TVs will increase, the speeds of low VOT TVs tend to drop, but very slightly. (Note that the scale of the x-axis in Fig. 14(a) and Fig. 14(b) do not include the entire range of traffic densities that were tested; this was done to illustrate the differences/similarities for the different penetration rates.) As the TV penetration rates approach zero, the speed-density relations of the high and the low VOT TVs converge. The higher TV penetration rate means a higher probability of TU games taking place, and this gives the high VOT TVs
more opportunities to increase their speeds, while low VOT TVs can also “sell” their time more frequently.

We next investigate the impact of varying low-to-high VOT TV ratios. The results are given in Fig. 15. Clearly, high VOT TVs always have higher speeds than low VOT TVs. An interesting finding is that, as the ratio of high VOT TVs increases, the speeds of both low and high VOT vehicles decrease. On one hand, a higher high VOT TV fraction results in lower frequencies of transactions with low VOT TVs, leading to a decrease in high VOT TV speeds. On the other hand, higher high VOT TV fractions results in low VOT TV giving way to high VOT TVs with frequency, leading to a decrease in low VOT TV speeds. Hence a healthy market share should have a relatively higher percentage of low VOT vehicles than high VOT vehicles.

Fig. 15: Density and speed relationship varying ratio of high VOT TVs and low VOT TVs: (a) high VOT TVs, (b) low VOT TVs.

**Shock wave formation.** Finally, a 4.5km ring road is simulated over a 1-hour time period and we compare the traffic dynamics that arise in two scenarios: (i) 100% TVs (i.e., 0% NTVs) and (ii) 0% TV (i.e., 100% NTVs). Two cases are investigated in each scenario: a case of below (but near) critical traffic density with 13.3 veh/km·lane and a case of super-critical (jammed) traffic with 33.3 veh/km·lane. Fig. 16 shows the resulting speed heat maps (average of both lanes) for all four cases. Fig. 16(a) and Fig. 16(c) are the heatmaps obtained in the sub-critical cases (average density = 13.3 veh/km·lane). We see the formation of both forward and backward waves clearly in both cases, but they appear to be less severe in the first scenario (100% TVs). Fig. 16(b) and Fig. 16(d) are the heatmaps obtained in the super-critical cases (average density = 33.3 veh/km·lane). In general,
there is no significant difference in the wave formation characteristics in high density traffic conditions. Therefore, while it is difficult to conclude that TU games can prevent stop and go waves from forming, we can comfortably conclude that TU games do not have an adverse effect on traffic conditions in general.

4. Conclusion and outlook

Vehicles available on the market today come equipped with advanced sensor technologies and in many vehicles features such as adaptive cruise control and lane departure warning come standard. This allows vehicles
(on the roads today) to respond to traffic conditions around them. It is safe to bet that communication between vehicles is right around the corner. This creates opportunities for re-thinking traffic management in radical ways. This thinking is the premise of this paper. We propose a treatment of lane changing as transferable utility (TU) games with side payments. We demonstrate that such utility transfer allows vehicles engaged in such transactions to achieve pareto efficient payoffs. The main idea is that vehicles exchange right-of-way for money. This constitutes a departure from cooperative lane changing involving winners and losers to one where all players can win.

A cellular automaton was developed to perform experiments. The simulation results indicate that the ability to play TU games had no impact on travel times in free-flow conditions, heavy (bumper-to-bumper) traffic conditions, or when the penetration of vehicles willing to engage in TU games approaches zero. Otherwise, both vehicles with low and high values of travel time (VOT) derived benefit from the proposed set-up.

The proposed model is rather simple. Our aim was to test the idea of TU games for lane changing at an approachable level. This creates numerous avenues for future research. From considering more sophisticated models of utility and traffic dynamics, to games that involve more than two players with more choices, to games that include unconnected vehicles. Along the lines of the utility function, the estimated values of the constants can have a significant impact on the time difference parameter, which can have a substantial impact on the outcomes of the games. One topic for future research involves the design of lane changing games that are robust to estimation errors. On the other hand, in the absence of estimation errors, we conjecture that the TU framework proposed in this paper disincentivizes untruthfulness when reporting value of time. We feel that an approach grounded in mechanism design will help shed light on this analytically. That the outcome of the games depends on communicated VOTs creates opportunities to game the system and it might be possible to engage in multiple transactions and exploit arbitrage opportunities. Hence, appropriate pricing schemes would be a particularly interesting avenue to investigate.

References

Ahmed, K., Ben-Akiva, M., Koutsopoulos, H., Mishalani, R., 1996. Models of freeway lane changing and gap acceptance behavior. Transportation
and traffic theory 13, 501–515.

Bevly, D., Cao, X., Gordon, M., Ozbilgin, G., Kari, D., Nelson, B., Woodruff, J., Barth, M., Murray, C., Kurt, A., et al., 2016. Lane change and merge maneuvers for connected and automated vehicles: A survey. IEEE Transactions on Intelligent Vehicles 1, 105–120.

Coase, R., 2013. The problem of social cost. The journal of Law and Economics 56, 837–877.

Du, Y., Wang, Y., Chan, C., 2015. Autonomous lane-change controller, in: Intelligent Vehicles Symposium (IV), 2015 IEEE, IEEE. pp. 386–393.

Gipps, P., 1986. A model for the structure of lane-changing decisions. Transportation Research Part B 20, 403–414.

Hossan, S., Asgari, H., Jin, X., 2016. Investigating preference heterogeneity in value of time (VOT) and value of reliability (VOR) estimation for managed lanes. Transportation Research Part A 94, 638–649.

Kamal, M., Taguchi, S., Yoshimura, T., 2015. Efficient vehicle driving on multi-lane roads using model predictive control under a connected vehicle environment, in: Intelligent Vehicles Symposium (IV), 2015 IEEE, IEEE. pp. 736–741.

Kesting, A., Treiber, M., Helbing, D., 2007. General lane-changing model mobil for car-following models. Transportation Research Record: Journal of the Transportation Research Board , 86–94.

Keyvan-Ekbatani, M., Knoop, V., Daamen, W., 2016. Categorization of the lane change decision process on freeways. Transportation Research Part C 69, 515–526.

Kita, H., 1993. Effects of merging lane length on the merging behavior at expressway on-ramps. Transportation and Traffic Theory , 37–51.

Kita, H., 1999. A merging–giveway interaction model of cars in a merging section: A game theoretic analysis. Transportation Research Part A 33, 305–312.

Kita, H., Tanimoto, K., Fukuyama, K., 2002. A game theoretic analysis of merging-giveway interaction: A joint estimation model, in: Transportation and Traffic Theory in the 21st Century: Proceedings of the 15th International Symposium on Transportation and Traffic Theory, Adelaide,
Li, K., Wang, X., Xu, Y., Wang, J., 2016a. Lane changing intention recognition based on speech recognition models. Transportation Research Part C 69, 497–514.

Li, N., Oyler, D., Zhang, M., Yildiz, Y., Girard, A., Kolmanovsky, I., 2016b. Hierarchical reasoning game theory based approach for evaluation and testing of autonomous vehicle control systems, in: Decision and Control (CDC), 2016 IEEE 55th Conference on, IEEE. pp. 727–733.

Li, N., Oyler, D., Zhang, M., Yildiz, Y., Kolmanovsky, I., Girard, A., 2017. Game theoretic modeling of driver and vehicle interactions for verification and validation of autonomous vehicle control systems. IEEE Transactions on control systems technology.

Liu, H., Xin, W., Adam, Z., Ban, J., 2007. A game theoretical approach for modeling merging and yielding behavior at freeway on-ramp section, in: Allsop, R., Bell, M., Heydecker, B. (eds.), Proceedings of the 17th International Symposium on Transportation and Traffic Theory, Elsevier, Oxford, UK. pp. 197–212.

Maerivoet, S., De Moor, B., 2005. Cellular automata models of road traffic. Physics reports 419, 1–64.

Meng, F., Su, J., Liu, C., Chen, W., 2016. Dynamic decision making in lane change: Game theory with receding horizon, in: Control (CONTROL), 2016 UKACC 11th International Conference on, IEEE. pp. 1–6.

Myerson, R., 2013. Game theory. Harvard university press.

Nagel, K., Schreckenberg, M., 1992. A cellular automaton model for freeway traffic. Journal de physique I 2, 2221–2229.

Nash, J., 1950. The bargaining problem. Econometrica: Journal of the Econometric Society, 155–162.

Nash, J., 1953. Two-person cooperative games. Econometrica: Journal of the Econometric Society, 128–140.

Oyler, D., Yildiz, Y., Girard, A., Li, N., Kolmanovsky, I., 2016. A game theoretical model of traffic with multiple interacting drivers for use in autonomous vehicle development, in: American Control Conference (ACC), 2016, IEEE. pp. 1705–1710.
Pan, T., Lam, W., Sumalee, A., Zhong, R., 2016. Modeling the impacts of mandatory and discretionary lane-changing maneuvers. Transportation Research Part C 68, 403–424.

Pei, Y., Xu, H., 2006. The control mechanism of lane changing in jam condition, in: Intelligent Control and Automation, 2006. WCICA 2006. The Sixth World Congress on, IEEE. pp. 8655–8658.

Rahman, M., Chowdhury, M., Xie, Y., He, Y., 2013. Review of microscopic lane-changing models and future research opportunities. IEEE transactions on intelligent transportation systems 14, 1942–1956.

Talebpour, A., Mahmassani, H., Hamdar, S., 2015. Modeling lane-changing behavior in a connected environment: A game theory approach. Transportation Research Part C 59, 216–232.

Thomas, F., 2008. Game theory. University of California, Los Angeles.

Toledo, T., Koutsopoulos, H., Ben-Akiva, M., 2003. Modeling integrated lane-changing behavior. Transportation Research Record 1857, 30–38.

Treiber, M., Kesting, A., 2013. Traffic flow dynamics: Data, models and simulation. Springer-Verlag Berlin Heidelberg.

Wang, M., Hoogendoorn, S., Daamen, W., van Arem, B., Happee, R., 2015. Game theoretic approach for predictive lane-changing and car-following control. Transportation Research Part C 58, 73–92.

Yao, S., Knoop, V., van Arem, B., 2017. Optimizing traffic flow efficiency by controlling lane changes: Collective, group, and user optima. Transportation Research Record: Journal of the Transportation Research Board, 96–104.

Yoo, J., Langari, R., 2013. A stackelberg game theoretic driver model for merging, in: ASME 2013 Dynamic Systems and Control Conference, American Society of Mechanical Engineers. pp. V002T30A003–V002T30A003.

Zheng, Z., 2014. Recent developments and research needs in modeling lane changing. Transportation Research Part B 60, 16–32.