Heat kernel asymptotics: more special case calculations

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Special case calculations are presented, which can be used to put restrictions on the general form of heat kernel coefficients for transmittal boundary conditions and for generalized bag boundary conditions.

I. INTRODUCTION

The heat equation asymptotics plays a prominent role in mathematics [27] and theoretical physics [16,15,23,2,25,12,24]. For manifolds without boundaries efficient schemes have been developed and the calculation of the volume part is nowadays nearly automatic [26,3,42]. When the manifold has a boundary, suitable boundary conditions have to be imposed and the heat equation asymptotics receives additional contributions depending on the boundary conditions considered [41,40]. In recent years a conglomerate of methods has been proven to be very effective in the determination of this asymptotics for Laplace type operators. The conglomerate of methods consists of functorial techniques, index theorems and special case calculations. All ingredients are very agil in that they can be adopted to the specific boundary condition considered. A characteristic feature is that on its own none of the methods is able to determine the coefficients fully, but instead each determines only part of the complete answer. As a rule, the information obtained by each ingredient overlaps with the information obtained from the others and this provides crucial cross checks for the calculation. Roughly one could say, that special case calculations are “responsible” for the group of extrinsic curvature terms, the index theorem allows to extract information on the gauge connection terms and functorial techniques, most prominently conformal transformation techniques [10], determine whatever is left over. Employing this procedure, the heat equation asymptotics for the classic as well as for “exotic” boundary conditions has been determined. A summary is provided in the conference contribution [30]. As further approaches let us mention [36,35,37,14].

The special case calculations that have been employed so far, consist of the following examples. (1) Dirichlet and Robin boundary conditions: scalar fields on the generalized cone $I \times N$, $I = [0, 1]$, $N$ a Riemannian manifold [7,8,34,22]. (2) Mixed boundary conditions: spinor fields and p-forms on the generalized cone [21,11]. (3) Oblique boundary conditions: again the generalized cone, in addition $B^2 \times T^{D-2}$ with the two-ball $B^2$ and the $(D-2)$-dimensional torus $T^{D-2}$ [20]. (4) Spectral boundary conditions: generalized cone plus $B^2 \times N$ [21,19,29].

In the present contribution we will present two further special case calculations, one on a different manifold and one on the ball with different boundary conditions. First we provide a calculation on a manifold obtained when glueing together a hemisphere and a ball. This example determines crucial information for transmittal boundary conditions [31]. Second we consider the Dirac operator on the ball with boundary conditions as they occur in gauge theories in Euclidean bags. These boundary conditions involve an angle $\theta$, which is a substitute for introducing small quark masses to drive the breaking of chiral symmetry [33,43]. An analysis of the associated heat equation asymptotics has barely started [43,28].

II. HEMISPHERE BALL GLUED TOGETHER

The first example we present is of relevance in the context of transmittal boundary conditions. To define transmittal boundary conditions, see e.g. [31], consider the $d$-dimensional manifold $\mathcal{M} = \mathcal{M}^+ \cup_\Sigma \mathcal{M}^-$, which is the union of two compact manifolds $\mathcal{M}^\pm$ along their common boundary $\Sigma$. Let $D^\pm$ be Laplace type operators on $\mathcal{M}^\pm$ written invariantly as

$$D^\pm = -g^{ij} \nabla_i^\pm \nabla_j^\pm - E^\pm.$$
The operator $D = (D^+, D^-)$ acts on a pair $\phi = (\phi^+, \phi^-)$. The transmittal boundary condition is defined by the operator

$$B_U \phi = \{ \phi^+|_{\Sigma} - \phi^-|_{\Sigma} \} \oplus \{(\nabla^+_m \phi^+)|_{\Sigma} + (\nabla^-_m \phi^-)|_{\Sigma} + U \phi^+|_{\Sigma} \},$$

where $\nabla^\pm_{m,\pm}$ is the respective exterior normal derivative on $M^\pm$ to the boundary $\Sigma$.

Let us consider a specific example of the above situation and we start with the two-dimensional case.

Let $M^+ = H^2$ be the unit hemisphere and $M^- = B^2$ the two-ball. On the hemisphere we consider the “massive” Laplacian

$$D^+ = -\Delta_{H^2} + \frac{1}{4}.$$ 

Adding the curvature term $(1/8)R = (1/4)$ has the advantage that the eigenvalues $\lambda^2$ of $D^+$ are complete squares. In detail, the eigenfunctions are

$$\phi_{\text{hemis}}(\theta, \varphi) = N_h e^{im\varphi} P_{\lambda-1/2}^-(\cos \theta), \quad m \in \mathbb{Z},$$

with the associated Legendre functions $P_
u^m(x)$ and $N_h$ a normalisation constant. The values of $\lambda$ have to be fixed by boundary conditions.

On the ball we simply take

$$D^- = -\Delta_{B^2},$$

with the well known eigenfunctions

$$\phi_{\text{disc}}(r, \varphi) = N_d e^{im\varphi} J_{|m|}(\lambda r), \quad m \in \mathbb{Z}.$$ 

Here, $J_\nu(x)$ are the Bessel functions and $N_d$ is a normalisation constant.

Next we glue the hemisphere and the disc together along their boundary, which is a circle. As a specific example of the transmittal boundary conditions we impose that the eigenfunctions of $D^+$ and $D^-$ as well as their normal derivatives agree along the circle, this is we choose $U = 0$.

Matching the eigenfunctions eliminates one of the normalization constants,

$$N_d J_{|m|}(\lambda) = N_h P_{\lambda-1/2}^-(0),$$

so

$$N_d = N P_{\lambda-1/2}^- (0), \quad N_h = N J_{|m|}(\lambda).$$

Next we match the normal derivatives. The natural normal derivatives in the example are the exterior normal derivatives $(\partial / \partial r)$ and $(\partial / \partial \theta)$. In these the condition reads

$$0 = \frac{\partial}{\partial \theta} \phi_{\text{hemis}} + \frac{\partial}{\partial r} \phi_{\text{disc}},$$

and this gives the implicit eigenvalue equation

$$0 = \lambda J_{|m|}'(\lambda) P_{\lambda-1/2}^-(0) - J_{|m|}(\lambda) \frac{d}{dx} P_{\lambda-1/2}^- (x) \big|_{x=0}.$$ 

In the following we will simplify the notation to

$$\frac{d}{dx} P_{\lambda-1/2}^- (x) \big|_{x=0} = \frac{d}{dx} P_{\lambda-1/2}^- (0).$$

Once this implicit eigenvalue equation is known, the procedure developed in [7] can be used in order to determine the heat equation asymptotics for the specific problem at hand. Starting point of the procedure is the contour integral representation

$$\zeta(s) = \sum_{m=-\infty}^{\infty} \int \frac{dk}{2\pi i} \frac{(k^2 + M^2)^{-s}}{k} \ln \left[ k J_{|m|}'(k) P_{\lambda-1/2}^-(0) - J_{|m|}(k) \frac{d}{dx} P_{\lambda-1/2}^- (0) \right],$$

where $M$ is the mass and $\zeta(s)$ is the zeta function.
where the contour $\gamma$ encloses counterclockwise all real positive zeroes of equation (3) and where a small mass $M^2$ (as a infrared regulator put to zero later) has been introduced.

The next step is to deform the contour to the imaginary axis, obtaining

$$
\zeta(s) = \frac{\sin(\pi s)}{\pi} \sum_{m=-\infty}^{\infty} \int_{M}^\infty dk \, (k^2 - M^2)^{-s} \frac{\partial}{\partial k} \ln \left[ kI_{m}(k)P_{\nu-1/2}^{-m}(0) - I_{m}(k) \frac{d}{dx}P_{\nu-1/2}^{-m}(0) \right],
$$

where $P_{\nu}^{-m}(x) = P_{\nu-1}^{-m}(x)$, this implies $P_{\nu-1/2}^{-m}(x) = P_{\nu-1/2}^{-m}(x)$, has been used \[32\].

As explained in great detail in [7], the heat equation asymptotics is encoded in the asymptotic behavior of the implicit eigenvalue equation (3). For the Bessel functions everything needed is provided by Olver’s uniform asymptotic expansions. For $m \to \infty$ and $z = km$ fixed, they read \[39\],

$$
I_m(z) \sim \frac{1}{\sqrt{2\pi m}} \left(1 + \frac{z^2}{2} \right)^{1/4} \left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{m^k}\right],
$$

$$
I'_m(z) \sim \frac{1}{2\sqrt{\pi m}} \left(1 + \frac{z^2}{2} \right)^{1/4} \left[1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{m^k}\right],
$$

with $t = 1/\sqrt{1 + z^2}$ and $\eta = \sqrt{1 + z^2} + \ln[z/(1 + \sqrt{1 + z^2})]$. The polynomials are defined recursively through

$$
u_{k+1}(t) = \frac{1}{2} t^2(1 - t^2)u_k'(t) + \frac{1}{8} \int_0^t d\tau \, (1 - 5\tau^2)u_k(\tau),$$

$$v_k(t) = u_k(t) + t(t^2 - 1) \left[\frac{1}{2} u_{k-1}(t) + t u'_{k-1}(t)\right],$$

starting with $u_0(t) = 1$. The first few coefficients are listed in \[32\], higher coefficients are immediate to obtain by the above recursions using a simple computer program.

The relevant information for the Legendre functions is \[2\]

$$
P_{\lambda-1/2}^{-m}(0) = \frac{2^{-m} \sqrt{\pi}}{\Gamma(3/4 + (m + \lambda)/2)1(3/4 + (m - \lambda)/2)},
$$

$$
\frac{d}{dx}P_{\lambda-1/2}^{-m}(0) = \frac{2^{-m+1} \sqrt{\pi}}{\Gamma(1/4 + (m + \lambda)/2) \Gamma(1/4 + (m - \lambda)/2)},
$$

Further expansion of the $\Gamma$-functions is achieved using \[32\]

$$
\ln \Gamma(z) \sim z \ln z - z - \frac{1}{2} \ln 2\pi + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}},
$$

with the Bernoulli numbers $B_{2k}$. We use this in the slightly different form

$$
\Gamma(u + a) \sim \exp \left(u \ln u - u - (a - 1/2) \ln u + \ln \sqrt{2\pi} + \sum_{l=1}^{\infty} \frac{\gamma_l(a)}{u^l}\right),
$$

respectively

$$
\frac{1}{\Gamma(u + a)} \sim \exp \left(-u \ln u - u - (a - 1/2) \ln u - \ln \sqrt{2\pi} + 1 + \sum_{l=1}^{\infty} \frac{\gamma_l(a)}{u^l}\right),
$$

with the $\gamma_l(a)$ easily determined by an algebraic computer program.

These expansions are used for $m \neq 0$ after substituting $k \to km$ in the integral \[3\]: a suitable choice turns out to be $u = m(1 + ik)/2$. As known from the ball, $m = 0$ needs special treatment and a simple large argument consideration is sufficient.
Without going into the details of this calculation, as to be expected the sum over \( m \) produces Riemann zeta functions and a typical expression for the asymptotic contribution is

\[
A_l(s) = 2\frac{\sin(\pi s)}{\pi} \zeta_R(2s + l) \int_0^\infty dk \, k^{-2s} \frac{\partial}{\partial k} D_l(k),
\]

where, e.g.,

\[
D_1(k) = \frac{-2 - 3k^2 + \sqrt{1 + k^2}}{24(1 + k^2)^{3/2}}, \quad D_2(k) = -\frac{k^4}{32(1 + k^2)^3}.
\]

The \( k \)-integrals are standard and residues and values of \( \zeta(s) \) are easily determined. As a pleasant feature let us mention that the whole calculation can be and has been fully automated with a simple algebraic computer program.

The transition from two dimensions to arbitrary dimension \( d \) is surprisingly easy once the suitable organising functions are known. On ball and sphere settings the best organisation is obtained via the use of the Barnes zeta function \([4,5,17,8]\),

\[
\zeta_B(s,b) = \sum_{\vec{m} = 0}^{\infty} \frac{1}{(b + m_1 + \ldots + m_d)^s} = \sum_{l=0}^{\infty} \frac{(l + d - 1)}{d} (l + b)^{-s}.
\]

In the present context this is seen as follows. On the hemisphere, eigenfunctions can be represented as Gegenbauer polynomials \([38]\). The relevant dependence on the normal coordinate \( \theta \) is

\[P_{\lambda-l/2}(\cos \theta), \quad l \in \mathbb{N}_0,\]

where again a coupling

\[E^+ = \frac{d - 1}{4d} R^+ = \frac{1}{4}(d-1)^2\]

has been included. The radial dependence on the ball is

\[J_{l+(d-2)/2}(\lambda r)\]

such that

\[\frac{\partial}{\partial r} J_{l+(d-2)/2}(\lambda r) = \lambda J_{l+(d-2)/2}(\lambda r) - \frac{d - 2}{2} J_{l+(d-2)/2}(\lambda r).\]

The angular tangential dependences of both solutions agree. Eq. (2) suggests, that the most convenient choice to proceed is the implicit eigenvalue equation

\[0 = \lambda J'_{l+(d-2)/2}(\lambda) P_{\lambda-1/2}^{-l-(d-2)/2}(0) - J_{l+(d-2)/2}(\lambda) \frac{d}{dx} P_{\lambda-1/2}^{-l-(d-2)/2}(0).\]

The index \( m \) is replaced by \( l + (d - 2)/2 \) and the degeneracy

\[d(l) = (2l + d - 2)\frac{(l + d - 3)!}{l!(d-2)!}\]

of the spherical harmonics has to be taken into account. Thus final answers can be found from the two dimensional result once the Riemann zeta function is replaced by a sum of the Barnes zeta function \([17,8]\),

\[\sum_{l=0}^{\infty} d(l) \left( l + \frac{d-2}{2} \right)^{-2s} = \zeta_B(2s, (d-2)/2) + \zeta_B(2s, d/2).\]

Note, however, that eq. (10) does not correspond to matching the normal derivatives at the boundary, but instead they are assumed to have a jump described by \( U = (d - 2)/2 \).
\[
\left( \frac{\partial}{\partial r} \phi_{\text{disc}} \right) \bigg|_{r=1} + \left( \frac{\partial}{\partial \theta} \phi_{\text{hemis}} \right) \bigg|_{\theta=\pi/2} = -\frac{d-2}{2} \phi_{\text{disc}} \bigg|_{r=1}.
\]

In summary, the meromorphic structure of the zeta function for the \(d\)-dimensional problem is completely clear and can be used to calculate the leading heat kernel coefficients. Subtracting the volume contributions of \(\mathcal{M}^+\) and \(\mathcal{M}^-\), the “boundary part” of some leading coefficients is

\[

c_1 = 0,
\]

\[

c_2 = \frac{(4-d)2^{-d}}{3 \Gamma(d/2)},
\]

\[

c_3 = \frac{(d-5)(d-3)\sqrt{\pi}2^{-d}}{64 \Gamma(d/2)},
\]

\[

c_4 = \frac{2^{-d}}{180 \Gamma(d/2)} \left( \frac{250}{7} + \frac{2839}{42} - \frac{191}{7}d^2 + \frac{61}{21}d^3 \right).
\]

These results can and have been used to put restrictions on the general form of the coefficients for transmittal boundary conditions \([31]\). To exemplify the procedure consider the leading coefficients. Let \(K^\pm\) be the extrinsic curvature on \(\Sigma\) as induced from \(\mathcal{M}^\pm\). Invariance theory shows that for a localizing test function

\[

c_1^\Sigma = \int_\Sigma dy \, c_1 f \text{Tr} (1),
\]

\[

c_2^\Sigma = (4\pi)^{-d/2} \frac{1}{6} \int_\Sigma dy \, \text{Tr} \left( c_1 f (K^+ + K^-) 1 + e_2 (f^+_{\gamma^m} + f^-_{\gamma^n}) 1 + e_3 f U \right).
\]

For the hemisphere ball example, we have \(K^+ = 0\), \(K^- = d-1\), \(U = (d-2)/2\), and derive the equations

\[

c_1 = 0,
\]

\[

(d-1)c_1 + \frac{d-2}{2}c_3 = 4-d,
\]

and so \(c_1 = 2\), \(c_3 = -6\). This information, as well as the remaining one \(c_2 = 0\), is easily obtained by different means, but for the higher coefficients the special case input is crucial.

**III. GENERALIZED EUCLIDEAN BAG BOUNDARY CONDITIONS**

In theories of Euclidean bags, chiral symmetry breaking is triggered by imposing the boundary condition

\[
0 = \Pi^- \psi \big|_{\partial \mathcal{M}} := \frac{1}{2} \left( 1 + ie^{\cdot \Gamma^5 \Gamma^m} \right) \big|_{\partial \mathcal{M}} \tag{11}
\]

on the spinor field \(\psi\). For \(\theta = 0\) this is a well studied problem which can be understood as a mixed boundary problem of an associated Laplace type operator \(D\). Let \(e_j\) be a \(d\)-bein system, \(\Gamma_{jkl}\) the Christoffel symbols relative to the orthonormal frame and \(\Gamma^j\) the \(\Gamma\)-matrices projected along this frame. Let \(P\) be the Dirac operator on \(\mathcal{M}\),

\[
P = -i \Gamma^j \nabla_j,
\]

where \(\nabla_j = e_j + \omega_j\) is the covariant derivative with the spin connection

\[
\omega_j = -\frac{1}{4} \Gamma^k_{jkl} \Gamma^l.
\]

Consider the associated second order problem for \(D = P^2\) with domain

\[
\text{domain}(D) = \{ \psi \in C^\infty(V) : \Pi^- \psi \big|_{\partial \mathcal{M}} \oplus \Pi^- \left( -i \gamma^j \nabla_j \right) \psi \big|_{\partial \mathcal{M}} = 0 \}.
\]

For \(\theta = 0\), this leads to the mixed problem.

5
\[
\Pi_- \psi |_{\partial M} + (\nabla_m - S) \Pi_+ \psi |_{\partial M} = 0, \tag{12}
\]

where
\[
\Pi_+ = \frac{1}{2} \left( 1 - i \Gamma^5 \Gamma_m \right)
\]

and
\[
S = -\frac{1}{2} K \Pi_+.
\]

In order to derive eq. (12), it is crucial that \([\Pi_-, \Gamma^a] = 0\), see [9], a relation that does not hold for \(\theta \neq 0\). For the case \(\theta = 0\) the special case calculation on the ball has been performed in [8,21] and subsequently used to restrict the general form of the coefficients for mixed boundary conditions [11]. It is the aim of this section, to generalize the special case calculation to \(\theta \neq 0\) and to use the results in order to determine the \(a_1\)-coefficient for a general manifold.

We start the analysis by solving the eigenvalue problem on the \(d\)-dimensional ball. The eigenspinors of the Dirac operator on the ball have the form [18]
\[
\psi^{(+)}_\pm &= \frac{C}{r^{(d-2)/2}} \left( \pm J_{n+(d-2)/2}(kr) Z_{\nu}^{(m)}(\Omega) \right), \tag{13}
\]
\[
\psi^{(-)}_\pm &= \frac{C}{r^{(d-2)/2}} \left( \pm i J_{n+(d-2)/2}(kr) Z_{\nu}^{(m)}(\Omega) \right), \tag{14}
\]

with \(n \in \mathbb{N}_0\). Here, \(Z_{\nu}^{(m)}(\Omega)\) are the spinor modes on the sphere, see [13], and \(C\) is a normalisation constant. The condition (11),
\[
\Pi_- \psi |_{\partial M} = \frac{1}{2} \begin{pmatrix} 1 & -ie^\theta \\ ie^{-\theta} & 1 \end{pmatrix},
\]

is easily applied to (13) and (14), and the implicit eigenvalue equations read
\[
J_{n+d/2}(k) \mp e^\theta J_{n+d/2-1}(k) = 0
\]
for \(\psi^{(+)}_\pm\), and
\[
J_{n+d/2}(k) \pm e^{-\theta} J_{n+d/2-1}(k) = 0
\]
for \(\psi^{(-)}_\pm\). Suitably combined, we write these as
\[
J_{n+d/2-1}(k) - e^{-2\theta} J_{n+d/2}(k) = 0, \tag{15}
\]
\[
J_{n+d/2-1}(k) - e^{2\theta} J_{n+d/2}(k) = 0. \tag{16}
\]

Clearly it is sufficient to deal with eq. (13), the contributions from (14) follow by replacing \(\theta \rightarrow -\theta\).

For convenience we introduce \(\nu = n + (d-2)/2\). To analyse the zeta function we start again with the contour integral representation
\[
\zeta(s) = \sum d(\nu) \int_\gamma \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln \left( J_{\nu}^2(k) - e^{-2\theta} J_{\nu+1}^2(k) \right),
\]

where \(d(\nu)\) is the degeneracy associated with the implicit eigenvalue equation (13) and \(\gamma\) again encloses counterclockwise the positive real zeroes of (13). (For convenience, we omit writing the infrared cutoff \(M^2\). The procedure in principle is as in the previous section.) Denoting by \(d_s\) the dimension of spinor space, \(d_s = 2^{d/2}\), and taking into account the sphere eigenspinor degeneracies, one finds
\[ d(\nu) = \frac{1}{2} d_s \left( \frac{d + n - 2}{n} \right). \]

Proceeding in the manner described, we shift the contour to the imaginary axis to find

\[ \zeta(s) = \frac{\sin \pi s}{\pi} \sum d(\nu) \nu^{-2s} \int_0^\infty dk \, k^{-2s} \frac{\partial}{\partial k} \ln \left( I_0^2(k) + e^{-2\theta} I_{\nu+1}^2(k) \right). \] (17)

To simplify the analysis of the asymptotic behavior of the integrand it is convenient to rewrite (17) in terms of Bessel functions involving only one index. For this purpose we use (12)

\[ I_{\nu+1}(z) = I_\nu(z) - \frac{\nu}{z} I_\nu(z), \]

which allows to write

\[ \zeta(s) = \frac{\sin \pi s}{\pi} \sum d(\nu) \nu^{-2s} \int_0^\infty dz \, z^{-2s} \times \]

\[ \frac{\partial}{\partial z} \ln \left( e^{-\theta} I_\nu^2(z\nu) + \left[ e^\theta + \frac{e^{-\theta}}{z^2} \right] I_\nu^2(z\nu) - \frac{2}{z} e^{-\theta} I_\nu(z\nu) I'_\nu(z\nu) \right), \]

when irrelevant factors in the logarithm are neglected.

The uniform asymptotics is completely determined by eqs. (3) and (4). Here we will concentrate only on the terms contributing to the heat equation coefficients \( a_0 \) and \( a_1 \). It is easy to see, that the relevant pieces from the argument of the logarithm are

\[ (...) = \frac{e^{2\nu t} e^{-\theta (1 + z^2)^{1/2}}}{2\pi \nu} \frac{\partial}{\partial z} \ln \left( 1 + \frac{1 + t}{2} [e^{2\theta} - 1] \right) + \text{irr..} \] (18)

The contribution to \( \zeta(s) \) resulting from the first line has already been dealt with in the calculation for the case \( \theta = 0 \) and it reads (11)

\[ \zeta_1(s) = \frac{d_s}{4\sqrt{\pi}} \frac{\Gamma(s - 1/2)}{\Gamma(s + 1)} \zeta_B(2s - 1, d/2 - 1) - \frac{d_s}{4\sqrt{\pi}} \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} \zeta_B(2s, d/2 - 1). \]

The same contribution comes from (14). The second line in (13) contributes the \( \theta \)-dependent piece

\[ \zeta_\theta(s) = \frac{\sin \pi s}{\pi} \sum d(\nu) \nu^{-2s} \int_0^\infty dz \, z^{-2s} \frac{\partial}{\partial z} \ln \left( 1 + \frac{1 + t}{2} [e^{2\theta} - 1] \right). \] (19)

Using standard Taylor series expansion, one first finds

\[ \frac{\partial}{\partial z} \ln \left( 1 + \frac{1 + t}{2} [e^{2\theta} - 1] \right) = - \sum_{l=0}^{\infty} (-1)^l (\tanh \theta)^{l+1} \frac{z^l}{(1 + z^2)^{l+3/2}}. \]

This allows the \( z \)-integrals to be done and an intermediate answer is

\[ \zeta_\theta(s) = - \frac{d_s}{4 \Gamma(s)} \zeta_B(2s, d/2 - 1) \sum_{l=0}^{\infty} (-1)^l (\tanh \theta)^{l+1} \frac{\Gamma(s + (1 + l)/2)}{\Gamma((l + 3)/2)}. \]

Simplifications occur if \( \zeta_{-\theta} \) from (16) is added,

\[ \zeta_\theta(s) + \zeta_{-\theta}(s) = \frac{d_s}{2 \Gamma(s)} \zeta_B(2s, d/2 - 1) \sum_{l=0}^{\infty} (\tanh \theta)^{2l+3} \frac{\Gamma(s + 1 + l)}{\Gamma(2 + l)} \]

\[ = \frac{d_s}{2} \zeta_B(2s, d/2 - 1) \sum_{l=0}^{\infty} \Gamma(s + 1 + l) \frac{\Gamma(2 + l)}{\Gamma(2 + l)} \]

\[ = \frac{d_s}{2} \zeta_B(2s, d/2 - 1) (\cosh 2s - \theta). \]
As expected, for $\theta = 0$ the new term $\zeta_\theta(s) + \zeta_{-\theta}(s)$ vanishes. From here, the residues $\text{Res} \, \zeta(d/2)$ and $\text{Res} \, \zeta((d - 1)/2)$ are easily determined. The normalization coefficient $a_0$ is confirmed, for $a_1$ we find

$$a_1 = (4\pi)^{-(d-1)/2} \frac{1}{4} ds |S^{d-1}| \left( \cosh^{d-1} \theta - 1 \right),$$

with $|S^{d-1}|$ the volume of the $(d - 1)$-sphere. On an arbitrary manifold $\mathcal{M}$ the coefficient is written in the form

$$a_1^{\mathcal{M}} = (4\pi)^{-(d-1)/2} \int_{\partial \mathcal{M}} dy \delta \text{Tr}(1),$$

and we read off

$$\delta = \frac{1}{4} \left( \cosh^{d-1} \theta - 1 \right).$$

As expected, the coefficient $\delta$ depends on $\theta$ as well as on the dimension $d$. The dimension dependence is a result of the fact that the number of $\Gamma$-matrices depends on the dimension $d$.

IV. CONCLUSIONS

This contribution provides further evidence that special case calculations play an important role in the determination of the boundary contributions to the heat equation asymptotics.

Whereas previous applications involved only Bessel functions as special case solutions, in our first calculation we have provided here an example where the associated Legendre functions are of relevance. The example involved was the union of a hemisphere and a ball of radius one, as it turned out to be useful in the analysis of transmittal boundary conditions [31]. The choice of a hemisphere simplifies the calculations considerably, because the asymptotic behavior of the Legendre functions needed is relatively simple, see eqs. (7) and (8). If instead of the hemisphere a spherical cap is considered, further contributions will arise because the boundary is not geodesically complete any more and so $K_{ab} \neq 0$. Presumably, also for this more general situation an analysis should be possible, for example along the lines of [6], where the needed asymptotic behavior of the Legendre functions has already been determined.

The nice feature of using the zeta function for the analysis of the heat kernel asymptotics is that it also allows for the consideration of other spectral functions like the determinant and the Casimir energy. So as a side effect of the present calculation an investigation of the influence of edges on these quantities can be envisaged.

In the second application presented, we have provided the first steps into the thorough analysis of the boundary condition (11). The explicit determination of the heat kernel coefficients is in its infancy. A special case calculation is sufficient to obtain the full $a_1$-coefficient, a result which is already new. For the higher coefficients, techniques particularly adapted to the Dirac operators are needed [29], and further results will be presented in [28].

Acknowledgement: I would like to thank my coauthors M. Bordag, T. Branson, G. Cognola, J.S. Dowker, E. Elizalde, G. Esposito, P.B. Gilkey, A.Yu. Kamenshchik, J.H. Park and D.V. Vassilevich for the very pleasant collaborations on the subject of heat equation asymptotics during the course of the last years. Special thanks also to the organisers for the invitation and for making this wonderful conference possible.

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