Ideals in Graph Algebras

Efren Ruiz · Mark Tomforde

Received: 9 February 2013 / Accepted: 18 April 2013 / Published online: 11 May 2013
© Springer Science+Business Media Dordrecht 2013

Abstract We show that the graph construction used to prove that a gauge-invariant ideal of a graph \(C^*\)-algebra is isomorphic to a graph \(C^*\)-algebra, and also used to prove that a graded ideal of a Leavitt path algebra is isomorphic to a Leavitt path algebra, is incorrect as stated in the literature. We give a new graph construction to remedy this problem, and prove that it can be used to realize a gauge-invariant ideal (respectively, a graded ideal) as a graph \(C^*\)-algebra (respectively, a Leavitt path algebra).

Keywords Graph algebras · Graph \(C^*\)-algebras · Leavitt path algebras · Gauge-invariant ideals · Graded ideals

Mathematics Subject Classifications (2010) 46L55 · 16D25

1 Introduction

An important, and often quoted result, in the theory of graph \(C^*\)-algebras is that a gauge-invariant ideal of a graph \(C^*\)-algebra is isomorphic to a graph \(C^*\)-algebra.

Presented by Paul Smith.

M. Tomforde was supported by a grant from the Simons Foundation (#210035 to Mark Tomforde).

E. Ruiz
Department of Mathematics, University of Hawaii, 200 W. Kawili St.,
Hilo, HI 96720-4091, USA
e-mail: ruize@hawaii.edu

M. Tomforde (✉)
Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA
e-mail: tomforde@math.uh.edu
Likewise, an analogous result in the theory of Leavitt path algebras states that a graded ideal of a Leavitt path algebra is isomorphic to a Leavitt path algebra. Unfortunately, it has recently been determined that the proofs of these results in the existing literature are incorrect, and moreover, the graph constructed to realize such ideals as graph $C^*$-algebras or Leavitt path algebras, does not work as intended. The purpose of this paper is to rectify this problem by giving a new graph construction that allows one to realize a gauge-invariant ideal in a graph $C^*$-algebra as a graph $C^*$-algebra, as well as realize a graded ideal in a Leavitt path algebra as a Leavitt path algebra. Thus we show that the often quoted results about gauge-invariant ideals (respectively, graded ideals) being graph $C^*$-algebras (respectively, Leavitt path algebras) are indeed true—but the graph used to realize them as such is not the graph that has been previously described in the literature.

If $E$ is a graph, then the gauge-invariant ideals of the graph $C^*$-algebra $C^*(E)$ are in one-to-one correspondence with the admissible pairs $(H, S)$, consisting of a saturated hereditary subset $H$ and a set $S$ of breaking vertices of $H$. Likewise, for any field $K$, the graded ideals of the Leavitt path algebra $L_K(E)$ are in one-to-one correspondence with the admissible pairs $(H, S)$. In [8, Definition 1.4], Deicke et al. describe a graph $HE_S$ formed from $E$ and a choice of an admissible pair $(H, S)$, and in [8, Lemma 1.6] they claim that the graph $C^*$-algebra $C^*(HE_S)$ is isomorphic to the gauge-invariant ideal $I_{(H,S)}$ corresponding to $(H, S)$. In [5, Proposition 3.7] it is claimed that for any field $K$ the Leavitt path algebra $L_K(HE_S)$ is isomorphic to the graded ideal $I_{(H,S)}$ contained in $L_K(E)$, and the proof they give is modeled after the proof of [8, Lemma 1.6]. In [6, Section 1] the authors points out that there is an error in the proofs of both [8, Lemma 1.6] and [5, Proposition 3.7], and in particular the argument that the proposed isomorphism is surjective is flawed in both cases. (According to [6], these errors were pointed out by John Clark and Iain Dangerfield.)

In this paper we seek to rectify these problems. After some preliminaries in Section 2, we continue in Section 3 to show that [8, Lemma 1.6] and [5, Proposition 3.7] are not true as stated, and we produce counterexamples to help the reader understand where the problem occurs. Specifically, we exhibit a graph $E$ with an admissible pair $(H, S)$ such that the gauge-invariant ideal $I_{(H,S)}$ in $C^*(E)$ is not isomorphic to $C^*(HE_S)$, and the graded ideal $I_{(H,S)}$ in $L_K(E)$ is not isomorphic to $L_K(HE_S)$. In Section 4 we describe a new way to construct a graph from a pair $(H, S)$, and we denote this graph by $\overline{E}_{(H,S)}$. In Section 5 we prove that if $I_{(H,S)}$ is a gauge-invariant ideal of $C^*(E)$, then $I_{(H,S)}$ is isomorphic to $C^*(\overline{E}_{(H,S)})$. In Section 6 we prove that if $I_{(H,S)}$ is a graded ideal of $L_K(E)$, then $I_{(H,S)}$ is isomorphic to $L_K(\overline{E}_{(H,S)})$. The theorems of Sections 5 and 6 show that indeed every gauge-invariant ideal of a graph $C^*$-algebra is isomorphic to a graph $C^*$-algebra, and every graded ideal of a Leavitt path algebra is isomorphic to a Leavitt path algebra.

We mention that when $S = \emptyset$ (which always occurs if $E$ is row-finite), then our graph $\overline{E}_{(H,S)}$ is the same as the graph $HE_S$ of Deicke et al. Thus [8, Lemma 1.6] and [5, Proposition 3.7] (and their proofs) are valid when $S = \emptyset$.

2 Preliminarsies

In this section we establish notation and recall some standard definitions.