Metric properties of the lamplighter group as an automata group

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Abstract. We examine the geometry of the Cayley graph of the lamplighter group with respect to the generating set rising from its interpretation as an automata group due to Grigorchuk and Zuk. We find some metric behavior with respect to this generating set analogous to the metric behavior in the standard group theoretic generating set. The similar metric behavior includes expressions for geodesic paths and families of ‘dead-end’ elements, which are endpoints of terminating geodesic rays. We also exhibit some different metric behavior between these two generating sets related to the existence of ‘seesaw’ elements.

1. Introduction

There are several generating sets of interest for the lamplighter group $L$, which is the wreath product of the group of order two with the integers. The standard presentation of $L$ arises from this wreath product structure, namely

$$L = \langle a, t \mid a^2, [t^i at^{-i}, t^j at^{-j}], i, j \in \mathbb{Z} \rangle.$$

We will refer to the set of elements $\{a, t\}$ as the wreath product generating set of $L$.

Automata groups form a rich class of groups with a number of remarkable properties. For example, Grigorchuk’s groups of intermediate growth are realized by automata groups, as described by Grigorchuk, Nekrashevich and Sushchanski˘ı in [4]. Grigorchuck and Žuk show in [5] that the lamplighter group $L$ can also be regarded as an automata group. The automaton they use to realize $L$ in this way is shown in Figure 1. The generating set arising from this interpretation is $\{t, ta\}$, where $a$ and $t$ are the wreath product generators of $L$. We will refer to the generating set $\{t, ta\}$ as the automata generating set of $L$. The generator $t$ corresponds to the initial automaton with start state labelled ‘$t$’ and the generator $ta$ corresponds to the initial automaton with start state labelled ‘$ta$’. Grigorchuk and Žuk compute the spectral radius of $L$ with respect to this generating set; they find remarkably that it is a discrete measure.

We explored the geometry of the Cayley graph of $L$ with respect to the wreath product generating set $\{a, t\}$ in [2]. Although the perspective which gives rise to the automata generating set is quite different from the wreath product perspective, we show below that the metric properties of the

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group are similar in several ways with respect to either generating set. In particular, we find that dead-end elements of arbitrary depth occur with respect to both generating sets. With respect to the wreath product generating set there are ‘seesaw’ elements \([2]\), which do not occur with respect to the automaton generating set. However, we find and describe seesaw-like behavior with respect to the automaton generating set. It is not known the extent to which the existence of dead-end elements of arbitrary depth depends upon the choice of generating set. We show here that the existence of seesaw words does depend upon generating set.

2. Normal forms and geodesics in \(L\)

As described in \([2]\), we consider an element in \(L = \mathbb{Z}_2 \wr \mathbb{Z}\) geometrically as a particular configuration of a bi-infinite string of light bulbs together with a “lamplighter” or cursor. Each bulb has two states, on and off, and the cursor which indicates the current bulb under consideration. An element can be viewed as a set of light bulbs which are illuminated, together with a position of the cursor. Figure 2 gives an example of an element in \(L\) represented in this way.

We consider a word in the wreath product generators \(\{a, t\}\) representing an element \(w \in L\). We can view this word as a sequence of instructions to create the configuration of illuminated bulbs represented by the element \(w\). We begin with the bi-infinite string of bulbs all in the off state with the cursor at the origin, then read the generators in the word one at a time from left to right. The generator \(a\) changes the state of the bulb at the current cursor position, and the generators \(t^\pm 1\) move the cursor one position right or left, respectively. In this way, successive prefixes of the word representing \(w\) create a sequence of configurations of light bulbs and cursor positions, ending with the one representing \(w\). We often view the cursor position as “moving” as dictated by these prefixes, ending in the position corresponding to \(w\). We view the question of finding minimal length representatives for \(w\) as finding methods of constructing \(w\) in this way as efficiently as possible.

The wreath product generators \(a\) and \(t\) of \(L\) encapsulate the two instructions necessary to create an element \(w \in L\). The automata generators of \(L\) combine these two basic operations of moving the cursor and illuminating a bulb. The generator \(ta\) moves the cursor one step to the right and changes the state of the bulb at the new location of the cursor, and its inverse \((ta)^{-1} = at^{-1}\) changes the state of the bulb at the current location and then moves the cursor one step to the left.

The location of the cursor in a word \(w \in L\) is easily computed as the exponent sum of \(t\) if the word is described in terms of the wreath product generating set. For the automata generating
set, the location of the cursor is the total exponent sum of $t$ and $ta$. Since the relators have all have total $t$ and $ta$ exponent sum zero, the total exponent sums for different words representing the same element will be identical.

2.1. Normal forms. We would like to be able to calculate the word length of $w \in L$ with respect to the automata generating set, in a manner analogous to the computation of word length with respect to the wreath product generating set presented in [2]. This was accomplished in [2] through the use of normal forms for elements of $L$. Below, we use the same normal forms to aid us in computing the word length of $w$ with respect to the automata generating set.

Normal forms for elements of $L$ with respect to the wreath product generators $a$ and $t$ are given in terms of the conjugates $a_k = t^k a t^{-k}$, which move the cursor to the $k$-th bulb, turn it on, and return the cursor to the origin.

We present two normal forms for an element $w \in L$ with respect to the wreath product generators, the right-first normal form given by

$$rf(w) = a_{i_1} a_{i_2} \ldots a_{i_m} a_{-j_1} a_{-j_2} \ldots a_{-j_l} t^r$$

and the left-first normal form given by

$$lf(w) = a_{-j_1} a_{-j_2} \ldots a_{-j_l} a_{i_1} a_{i_2} \ldots a_{i_m} t^r$$

with $i_m > \ldots i_2 > i_1 > 0$ and $j_l > \ldots j_2 > j_1 \geq 0$. In the first $m$ prefixes of the right-first normal form, only bulbs at positive indices are illuminated, in increasing order. In longer prefixes, the illuminated bulbs at nonpositive indices are added as well. Once our prefix includes $a_{i_1} a_{i_2} \ldots a_{i_m} a_{-j_1} a_{-j_2} \ldots a_{-j_l}$, all subsequent prefixes move the cursor closer its position in $w$.

The left-first normal form has prefixes which first illuminate bulbs at nonpositive integers. Note that these normal forms are slightly different from the normal forms in [2], as the bulb in position zero is grouped together with the bulbs at negative integers rather than the bulbs at positive integers. The reason for this grouping will become clear below.

One or possibly both of these normal forms will lead to a minimal length representative for $w \in L$ with respect to the wreath product generating set, depending upon the location of the cursor relative to the origin. That position is easy to detect from the sign of the exponent sum of $t$, given as $r$ above. If $r \geq 0$, then the left-first normal form will naturally lead to a minimal length representative, and if $r \leq 0$, then the right-first one will lead to a minimal length representative. If the exponent sum of $t$ is zero, then both of these normal forms yield minimal length representatives. Words of this type, with the cursor position at the origin, play an important role in the geometry of the Cayley graph, as described in Section 3.
While these normal forms are defined in terms of conjugates of $a$ in wreath product generating set, we use them to compute the word length of an element with respect to the automata generating set as well.

We first note one difference between the two generating sets discussed above, with consequences for computing the word length of elements. In the wreath product generating set, travel and bulb illumination are accomplished separately by the two generators. That is, there must be at least one instance of the generator $a$ for each light bulb illuminated in the configuration of bulbs representing a group element, and enough instances of the generator $t$ to move the cursor to the appropriate bulbs. In the automata generating set, these two operations are combined in the generator $ta$. A single generator can either move the cursor or move the cursor and illuminate a bulb in one step.

Second, we note the following about exponent sum in the prefixes of a word representing $w \in L$. As described above, each prefix in a word representing $w$ determines a configuration of illuminated bulbs, with the final configuration represented by $w$. Suppose that $w \in L$ has an illuminated bulb in position $n$. Then there must be a prefix of this word in which the cursor is at position $n-1$. We prove this in the following lemma.

**Lemma 2.1.** Let $w \in L$ be represented by a word $\gamma$ in the generators $\{t, ta\}$. If $w$ has a bulb illuminated in position $n$, there is some prefix of $\gamma$ for which the total exponent sum of $ta$ and $t$ is $n-1$.

**Proof.** The only two generators which can illuminate a bulb are $ta$ and $(ta)^{-1} = at^{-1}$. Let $\eta$ be the shortest prefix of $\gamma$ in which the bulb in position $n$ is illuminated. Recall that the generator $t$ moves the cursor one unit to the right. If $\eta$ ends in the generator $ta$, then the prefix $\eta(ta)^{-1}$ of $\gamma$ must have the cursor in position $n-1$; that is, the exponent sum of all the generators in that prefix is $n-1$.

If $\eta$ ends in the generator $(ta)^{-1} = at^{-1}$, then the cursor is at position $n-1$ in $\eta$, since after turning on the bulb in position $n$ via the generator $a$, the generator $t^{-1}$ moves the cursor one position to the left. Thus, there is a prefix of $\gamma$ with total exponent sum $n-1$. □

We again contrast the two generating sets. With respect to the wreath product generators, the cursor must be in position $-n$ to illuminate a bulb in position $-n$, for $n \geq 0$. With respect to the automata generating set, the cursor must move to position $-n-1$ as part of illuminating a bulb in position $-n$. If this illumination is done via $ta$, then the cursor begins in position $-n-1$ and moves to position $n$ to illuminate the bulb there. If this illumination is done via $(ta)^{-1} = at^{-1}$, then the cursor begins in position $-n$, illuminates the bulb, and then moves to position $-n-1$. Thus the automata generating set requires the cursor to move farther from the origin in order to illuminate the same bulb. With respect to illuminating bulbs at positive indices, the need to visit bulb $n-1$ to illuminate bulb $n$ does not result in any extra travel for the cursor, since in that case the cursor will pass through position $n-1$ on its way from the origin to position $n$.

We note that the length of $a$ with respect to the automata generating set is two, as the cursor must be at position $-1$ at some prefix of any word representing this element to ensure that the bulb in position zero is illuminated. The minimal length representatives of $a$ in this generating set are $t^{-1}(ta)$ and $(ta)^{-1}t$. 
To compute the length of an element with respect to the automata generating set, the only data we require are the position of the leftmost and rightmost illuminated bulbs and the cursor position. The configuration of the intermediate bulbs does not affect the word length of an element. In the prefixes of minimal length representatives, the cursor moves toward the extreme illuminated positions, then to its position in the element. At each position, we choose \( t \) or \( ta \) appropriately to ensure that each bulb is left in the desired state. The information about left-most and right-most illuminated bulbs and cursor position can easily be obtained from either normal form given above with respect to the wreath product generating set, and thus we do not define new normal forms for the automata generating set.

With these three pieces of information, it is easy to see how to define minimal length representatives for elements of \( L \) with respect to the automata generators, in the spirit of the left- and right-first normal forms. If the cursor position in \( w \in L \) is to the right of the origin, we follow the example of the left-first normal form and create a word whose initial prefixes contain only the generators \( t^{-1} \) and \( (ta)^{-1} \), and successively illuminate the correct bulbs to the left of the origin. Then we add enough instances of the generator \( t \) to create a prefix with the cursor at the origin. We can then add a suffix to this word containing only the generators \( t \) and \( ta \) to illuminate the appropriate bulbs in positive positions and then finally move the cursor to its position in \( w \) with \( t \) and \( t^{-1} \) as appropriate. We will see below that this path gives a minimal length representative for \( w \). If the cursor position in \( w \) is at or to the left of the origin, we follow the example of the right-first normal form and create a minimal length representative in that way. These representatives are not unique, as discussed below.

2.2. Computing word length in \( L \). In [2], we used the normal forms given above to compute the word length of \( w \in L \) with respect to the wreath product generating set as follows.

**Proposition 2.2** ([2], Proposition 3.6). Let \( w = a_{i_1}a_{i_2} \ldots a_{i_m}a_{-j_1}a_{-j_2} \ldots a_{-j_l}t^r \in L \), with \( 0 < i_1 < i_2 \cdots < i_m \) and \( 0 \leq j_1 < j_2 \cdots < j_l \). We define

\[
D(w) = m + l + \min\{2j_l + i_m + |r - i_m|, 2i_m + j_l + |r + j_l|\}.
\]

The word length of \( w \) with respect to the generating set \( \{a, t\} \) is exactly \( D(w) \).

Thus, word length with respect to the wreath product generating set is the number of illuminated bulbs plus the total necessary travel of the cursor.

We now define a similar quantity, \( D'(w) \), also computed from either normal form of \( w \), which will give the word length of \( w \) with respect to the automata generating set.

**Definition 2.3.** Let \( w = a_{i_1}a_{i_2} \ldots a_{i_m}a_{-j_1}a_{-j_2} \ldots a_{-j_l}t^r \in L \), with \( 0 < i_1 < i_2 \cdots < i_m \) and \( 0 \leq j_1 < j_2 \cdots < j_l \). If \( l = 0 \), there are no bulbs illuminated at or to the left of the origin and we set \( D'(w) = i_m + |r - i_m| \). Otherwise, we set

\[
D'(w) = \min\{2(j_l + 1) + i_m + |r - i_m|, 2i_m + j_l + 1 + |r + j_l + 1|\}.
\]

We now prove that the quantity \( D'(w) \), which is computed from the normal form of \( w \) in the wreath product generating set, yields the word length of \( w \) with respect to the automata generating set.

**Proposition 2.4.** The word length of \( w \in L \) with respect to the automata generating set \( \{t, ta\} \) is given by \( D'(w) \).
Thus, word length with respect to the automata generating set is merely the total travel required for the cursor, including the necessary travel to the bulb one to the left of the leftmost illuminated bulb. Proposition 2.4 is proved via the following two lemmas.

**Lemma 2.5.** The length of an element \( w \in L \) with respect to the generating set \( \{t, ta\} \) is at most \( D'(w) \).

**Proof.** We first describe a word in \( t \) and \( ta \) representing the element \( w \) with the generators arranged in the order suggested by the left-first normal form. That is, we first have a sequence of \( j_l + 1 \) occurrences of the \( t^{-1} \) or \( at^{-1} \) generators whose prefixes move the cursor to the position \( -j_l - 1 \), illuminating the appropriate bulbs. We add to this initial string of generators \( j_l + 1 \), so that this word now represents a configuration of illuminated bulbs at non-positive positions with the cursor at the origin. We then add a suffix in the generators \( t \) and \( ta \) of length \( i_m \), so that the prefixes of this suffix move the cursor to position \( i_m \), illuminating the appropriate bulbs at positive indices in increasing order. Finally, there are \( |r - i_m| \) occurrences of \( t \) or \( t^{-1} \) so that in the combined word, the cursor is at the required position for the element \( w \). The length of this word is the first of the two terms over which the minimum for \( D'(w) \) is taken. Expressing \( w \) as a word in the manner suggested by the right-first normal form gives the second of those two terms and thus we have the desired upper bound.

It is easy to see that if \( l = 0 \), there are no bulbs illuminated in positions at or to the left of the origin. In this case, we can avoid moving the cursor to the left of the origin in the prefixes and we see that a similar argument to above produces a word representing \( w \) of length \( i_m + |r - i_m| = D'(w) \).

**Lemma 2.6.** The length of an element \( w \in L \) with respect to the generating set \( \{t, ta\} \) is at least \( D'(w) \).

**Proof.** To compute a lower bound on the word length of \( w \in L \), we consider the minimum number of generators necessary to move the cursor to the positions of the leftmost and rightmost illuminated bulbs, and leave it in the proscribed position. As pointed out in Lemma 2.4, to illuminate the bulb in position \( n \), the cursor must visit position \( n - 1 \), which will result in additional cursor travel in the prefixes of a representative in the case where there is an illuminated bulb at or to the left of the origin.

We now compute this lower bound explicitly. First, we suppose that \( m > 0 \) and \( l > 0 \), so that there is at least one illuminated bulb to the right of the origin and at least one illuminated bulb at the origin or to the left of the origin. We consider a word \( \gamma \) in \( \{t^{\pm 1}, ta^{\pm 1}\} \) representing \( w \) with specific prefixes. We know that the bulbs in positions \( i_m \) and \( -j_l \) are illuminated in \( w \), and that the cursor position is \( r \in \mathbb{Z} \). Thus \( \gamma \) must have prefixes with exponent sum, in order, \( 0, i_m, -j_l - 1, r \) or \( 0, -j_l - 1, i_m, r \). To accomplish this first possible order, there must be at least \( i_m + i_m + j_l + 1 + |r + j_l + 1| \) occurrences of the generators, and to accomplish this in the second order, there must be at least \( j_l + 1 + j_l + 1 + i_m + |r - i_m| \) occurrences of the generators, giving the desired bounds.

If \( m = 0 \), there are no bulbs illuminated to the right of the origin, and the word \( \gamma \) must have prefixes with exponent sums \( 0, -j_l - 1, \) and \( r \), giving a lower bound of \( j_l + 1 + |r + j_l + 1| \) on the word length of \( w \).
If \( t = 0 \), there are no bulbs illuminated at the origin or to the left of the origin, and the word \( \gamma \) must have prefixes with exponent sums 0, \( i_m \), and \( r \), giving a lower bound of \( i_m + |i_m - r| \) on the word length of \( w \).

Combining these lemma proves Proposition 2.4 that the length of \( w \) with respect to the automata generating set \( \{t, ta\} \) is exactly \( D'(w) \).

As with the wreath product generating set, minimal length representatives for group elements in \( L \) are not generally unique. If the total \( t \) and \( ta \) exponent sum in a word \( w \) is zero, the cursor is left at the origin in \( w \) and there will be minimal length representatives for \( w \) arising from both the left-first and right-first manners of construction described above. If there is any bulb which is “visited twice” by the cursor during the construction of the element, there also will be more than one minimal length representative for that element. That is, if \( \gamma \) is a minimal length representative for \( w \in L \), and \( \gamma \) has two different prefixes \( \gamma_1 \) and \( \gamma_2 \) for which the cursor position is \( n \), then we can create another minimal length representative for \( w \). If the bulb in position \( n \) is off in \( w \), then we can either have it remain off during the construction of \( w \) or switch it on in \( \gamma_1 \) then later off in \( \gamma_2 \). If the bulb in position \( n \) is on in \( w \), we again have a choice as to which prefix illuminates the bulb- it can be switched on in \( \gamma_1 \) and remain on in \( \gamma_2 \) or be left off in \( \gamma_1 \) and then switched on in \( \gamma_2 \).

For example, we consider the word \( w = a_4a_5a_6\ldots a_{-6}1^{-2} \) pictured in Figure 2. Since this word has the cursor in position \( -2 \), to the left of the origin, only words arising from the right-first normal form will be minimal and the length of any minimal length representative is 24. The bulb in position 1 will be visited twice during the construction of this element, as will in fact all bulbs except those in positions 0 and \( -1 \). Focusing on the bulb in position 1, we see that there is a choice as to how many times the state of this bulb changes- either twice or never, since it is left off in \( w \). So there is a minimal length representative \( t^3(ta)^3t^{-7}(ta)^{-1}t^{-4}(ta)^{-1}t^5 \) of \( w \) which never turns on the bulb in position 1, and there is also the minimal length representative \( (ta)2(ta)^3t^{-5}(ta)^{-1}t^{-1}(ta)^{-1}t^{-4}(ta)^{-1}t^5 \) of \( w \) which switches the bulb in position 1 on during the first visit and switches it off during the second visit. We will have similar choices for bulbs in positions 2, 3, 4, 5 and 6, as well as all bulbs in positions from \(-2\) to \(-5\), resulting in many possible geodesic representatives for \( w \).

This analysis shows there are at least \( 2^u \) possible geodesic representatives for group elements which have \( u \) pairs of different prefixes yielding the same cursor position and thus \( u \) bulbs which are visited twice by the cursor during the construction of the word. Furthermore, bulbs with the cursor position at the origin will have \( 2^u \) geodesics in each of the left-first and right-first directions, giving \( 2^{u+1} \) total.

3. Properties of the Cayley graph of \( L \) as an automata group

3.1. Dead-end elements. We found in 2 that \( L \) contained elements which we called dead-end elements with respect to the wreath product generating set, meaning that a geodesic ray from the identity to such an element could not be extended further. We show that \( L \) also contains dead-end elements with respect to the automata generating set \( \{t, ta\} \).

Definition 3.1. An element \( w \) in a finitely generated group \( G \) is a dead-end element with respect to a finite generating set \( X \) for \( G \) if \( |w| = n \) and \( |wx| \leq n \) for all generators \( x \) in \( X \cup X^{-1} \), where \( |\cdot| \) represents word length with respect to the generating set \( X \).
These elements are called dead-end elements because a geodesic ray in the Cayley graph $\Gamma(G, X)$ from the origin to a dead-end element $w$ cannot be extended beyond $w$. Note that in groups such as $L$ (with either generating set under consideration) where all relators are of even length, if $w$ is a dead-end element and $x \in X$, the word length of $wx$ will be necessarily $n - 1$.

There are different “strengths” of dead-end behavior, measured by the notion of depth.

**Definition 3.2.** An element $w$ in a finitely generated group $G$ is a dead-end element of depth $k$ with respect to a finite generating set $X$ if $k$ is the largest integer with the following property: if the word length of $w$ in $n$, then $|wx_1x_2 \ldots x_l| \leq n$ for $1 \leq l \leq k$ and all choices of generators $x_i \in X \cup X^{-1}$.

A dead-end element in the Cayley graph $\Gamma(G, X)$ is a point from which it is impossible to make immediate progress away from the identity. The depth of a dead-end element $w$ reflects how long a path from $w$ must be to reach an element further away from the identity than $w$.

We show in [1] that all dead-end elements in Thompson’s group $F$ with respect to the standard finite generating set $\{x_0, x_1\}$ have depth two, and in [2] that there are dead-end elements of arbitrary depth in $L$ with respect to the wreath product generating set. The results of [2] extend to a larger class of wreath products as well. We now show that these results can be extended to the automata generating set of $L$.

**Theorem 3.3.** The lamplighter group $L$ contains dead-end words of arbitrary depth with respect to the generating set $\{t, ta\}$.

**Proof.** We define a family of elements which we show to be dead-end elements with respect to the automata generating set. Let $d_m$ denote a group element which has the bulbs at positions $m$ and $-m + 1$ illuminated, and the cursor at the origin. No bulbs may be illuminated beyond positions $m$ and $-m + 1$, and the state of the intermediate bulbs between $m$ and $-m + 1$ is irrelevant. In the example given in Figure 3, we take all these intermediate bulbs to be illuminated yielding the element $d_m = a_0a_1a_2 \ldots a_na_{-1}a_{-2} \ldots a_{-m+1}$. These words all have length $4m$. Since the cursor is at the origin, we can find minimal length representatives for these words in the manner of either the left-first or right-first normal forms, as described in Section 2.1.

We now check that these are dead-end elements.

- $w(t)$ and $w(ta)$ will have the cursor at position 1, and their word length will now be $4m - 1$.
- $w(t^{-1})$ and $w(at^{-1})$ will have the cursor at position -1, and the total length will again be $4m - 1$.

To see that the depth of $d_m$ is at least $m$, we note the following. An element with any possible configuration of bulbs illuminated only between positions $-m + 1$ and $m$, and with the cursor remaining anywhere between $-m + 1$ and $m$, will lie in the $4m$ ball around the identity. Thus, a path from $d_m$ to any point in the $4m + 1$ ball will have length at least $m + 1$, since the cursor must leave the range $[-m, m]$ in order to illuminate a bulb with an index outside this range. □

We note that all elements of $L$ with the cursor at the origin and illuminated bulbs to the right and at or to the left of the origin are dead-end elements. The argument above shows that the depth
is at least the distance to the closer of the rightmost illuminated bulb or one to the left of the leftmost illuminated bulb. In fact, since the cursor motions to move to the rightmost or leftmost illuminated bulbs all reduce word length, we see that the depth is in fact twice the smaller of these two distances. The specific words $d_n$ used in the proof above to exhibit dead-end elements of arbitrary depth have depth $2n$.

3.2. Seesaw elements. An element $w$ in a finitely generated group $G$ is a seesaw element with respect to a generating set $X$ if there are only two possible suffixes that a geodesic representative for $w$ can have, of the form $g^k$ for a group generator $g$. Phrased in terms of word length, we say that $w$ is a seesaw element if there is a unique generator and its inverse which reduce the word length of $w$, and for which $k$ subsequent reductions in word length only occur through successive applications of that generator. These elements may present difficulty for the construction of canonical minimal length representatives for group elements. In [3] we show that Thompson’s group $F$ in the standard finite generating set $\{x_0, x_1\}$ contains seesaw elements, and use them to show that $F$ is not combable by geodesics. In [2] we show that the lamplighter group $L$ in the wreath product generating set as well as a wide class of wreath products contain seesaw elements with respect to at least one generating set.

We now precisely define seesaw elements.

**Definition 3.4.** A element $w$ in a finitely generated group $G$ with finite generating set $X$ is a seesaw element of swing $k > 0$ with respect to a generator $g$ if the following conditions hold. Let $|w|$ represent the word length of $w$ with respect to the generating set $X$.

1. Right multiplication by both $g$ and $g^{-1}$ reduces the word length of $w$; that is, $|wg^{\pm 1}| = |w| - 1$, and for all $h \in X \setminus \{g^{\pm 1}\}$, we have $|wh^{\pm 1}| \geq |w|$.
2. Additionally, $|wg^l| = |wg^{l-1}| - 1$ for integral $l \in [1, k]$, and $|wg^mh^{\pm 1}| \geq |wg^m|$ for all $h \in X \setminus \{g\}$ and integral $m \in [1, k-1]$.
3. Similarly, $|wg^{-l}| = |wg^{-l+1}| - 1$ for integral $l \in [1, k]$, and $|wg^{-m}h^{\pm 1}| \geq |wg^{-m}|$ for all $h \in X \setminus \{g^{-1}\}$ for integral $m \in [1, k-1]$.

These are called seesaw elements because they behave like a seesaw resting in balance. When in balance, there is a choice about which of two opposite directions to go down, but once that initial choice is made, there we can continue further downward only in that chosen direction, for the number of steps described as the swing.

**Theorem 3.5.** The lamplighter group $L$ with respect to the automata generating set does not contain any seesaw elements.

**Proof.** Suppose $w \in L$ is a seesaw element with respect to a generator $g$. Then both $g$ and $g^{-1}$ must reduce the word length of $w$. If $g$ is $t$ or $t^{-1}$, then in order for both $t$ and $t^{-1}$ to reduce word length, the cursor position in $w$ must be at the origin with at least one bulb illuminated to the right of the origin and at least one bulb illuminated at the origin or to the left of the origin.
Since the cursor is at the origin in a dead-end element. Let $g = \alpha_{-1}a_5$, which has the cursor at the origin and length 20 with respect to the automata generating set $\{t, ta\}$.

In this case, both $ta$ and $at^{-1}$ will also reduce word length, and $w$ cannot be a seesaw word as the first condition of Definition 3.4 is not satisfied.

Similarly, if $w$ is a seesaw element with respect to $ta$ or $(ta)^{-1}$, in order for both $ta$ and $(ta)^{-1}$ to reduce word length, the position of the cursor in $w$ must again be the origin, with at least one bulb illuminated to the right of the origin and at least one bulb illuminated at the origin or to the left of the origin. In this case, both $t$ and $t^{-1}$ will also reduce word length, and $w$ cannot be a seesaw word as the first condition of Definition 3.4 is not satisfied.

Thus, for any $w \in L$ and $g \in \{t^{\pm 1}, (ta)^{\pm 1}\}$, if $w$ is a seesaw word then the position of the cursor in $w$ must be the origin, and there must be bulbs illuminated at both positive and non-positive indices. However, in such a word multiplication by all four generators will decrease the word length. Thus $w$ cannot be a seesaw word as the first condition of Definition 3.4 is not satisfied.

\[ \square \]

### 3.3. Seesaw-like elements.

While $L$ with respect to the automata generating set does not contain seesaw elements as it does with respect to the generating set $\{a, t\}$, there are elements which exhibit similar behavior. Seesaw words can be multiplied by a unique pair $g^{\pm 1}$ in order to reduce their word length. For the ‘seesaw-like’ elements we describe below, there are two distinct families of generators analogous to the role of the generators $g$ and $g^{-1}$ for seesaw words.

**Theorem 3.6.** The lamplighter group $L$ with respect to the automata generating set contains seesaw-like elements $w_k$, which satisfy the following conditions with respect to the two sets of generators $\{t, ta\}$ and $\{t^{-1}, at^{-1}\}$.

1. All generators reduce the length of $w_k$, that is $|w_kg^{\pm 1}| = |w_k| - 1$ for all $g \in \{t, ta\}$.
2. Additionally, successive multiplication by the generators $\{t, ta\}$ reduces word length for up to $k$ iterations. That is, $|w_kg_1g_2 \ldots g_l| = |w_k| - l$ for integral $l \in [1, k]$ and $g_l \in \{t, ta\}$. Furthermore, any multiplication by the other generators to these shortened words increases word length, so we have $|w_kg_1g_2 \ldots g_lh| = |w_k| - l + 1$ for integral $l \in [1, k-1]$, $g_l \in \{t, ta\}$, and $h \in \{t^{-1}, at^{-1}\}$.
3. Similarly, successive multiplication by the generators $\{t^{-1}, at^{-1}\}$ reduces word length for up to $k$ iterations and we have $|w_kh_1h_2 \ldots h_l| = |w_k| - l$ for integral $l \in [1, k]$ and $h_l \in \{t^{-1}, at^{-1}\}$. Again, any multiplication by the other generators to these shortened words increases word length, so we have $|w_kh_1h_2 \ldots h_lg| = |w_k| - l + 1$ for integral $l \in [1, k-1]$, $g \in \{t, ta\}$, and $h_l \in \{t^{-1}, at^{-1}\}$.

**Proof.** We consider the group elements $w_k = a_ka_{-k+1}$, which have length $4k$. The element $w_5$ is pictured in Figure 4. For a given $k$, this word has two illuminated bulbs in positions $k$ and $-k+1$, and the cursor at the origin.

Since the cursor is at the origin in $w_k$, all generators reduce word length and the element is a dead-end element. Let $g \in \{t^{\pm 1}, (ta)^{\pm 1}\}$. Then in $w_kg$, the cursor is either to the right or left of
the origin in \( \mathbb{Z} \). It is clear from the definition of \( D' \) and Proposition 2.4 that once the cursor is to the right of the origin, an application of \( t \) or \( ta \) will decrease \( D' \) and thus will continue to reduce word length until the position of the rightmost illuminated bulb is reached. So for \( l \in [1, k] \), the group element \( w_k g_1 g_2 \ldots g_l \) with \( g_i \in \{ t, ta \} \) has word length \( |w_k| - l \). Applications of \( t^{-1} \) and \( at^{-1} \) increase the word length of these elements by moving the cursor to the left, except possibly for the last one with \( l = k \). Similarly, a group element \( w_k g_1 g_2 \ldots g_l \) with \( g_i \in \{ t^{-1}, (ta)^{-1} \} \) has word length \( |w_k| - l \), and computing \( D' \) again shows that applications of \( t \) and \( ta \) to those shortened words increase the word length of these elements.

Although these elements \( w_k \) are not seesaw elements in the sense of 3, they do share an important property with seesaw elements. In a true seesaw element, there is a choice between a single generator and its inverse which both reduce word length. After that initial choice, there are no further options for reducing word length for the next \( k \) steps; only repeated applications of that same generator will reduce the word length to \( |w| - k \). Thus, this initial choice has long term consequences for continued length reduction.

In the words described above, the choice for decreasing word length is not between two generators, but two sets of generators: \( \{ t, ta \} \) and \( \{ t^{-1}, (ta)^{-1} \} \). Each of these sets moves the cursor position in a different direction, right and left, respectively. Once one of these sets of generators is chosen to decrease the word length of a seesaw-like word \( w_k \), only generators from that set of generators can further decrease the word length for \( k \) iterations. As in true seesaw words, these two different directions diverge as quickly as possible, though both directions reduce word length. For example, for \( s \leq k \), the distance between the shortened elements \( w_k t^s \) and \( w_k t^{-s} \) is \( 2s \).

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