Sensitivity analysis of error-contaminated time series data under autoregressive models with the application of COVID-19 data

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ABSTRACT

Autoregressive (AR) models are useful in time series analysis. Inferences under such models are distorted in the presence of measurement error, a common feature in applications. In this article, we establish analytical results for quantifying the biases of the parameter estimation in AR models if the measurement error effects are neglected. We consider two measurement error models to describe different data contamination scenarios. We propose an estimating equation approach to estimate the AR model parameters with measurement error effects accounted for. We further discuss forecasting using the proposed method. Our work is inspired by COVID-19 data, which are error-contaminated due to multiple reasons including those related to asymptomatic cases and varying incubation periods. We implement the proposed method by conducting sensitivity analyses and forecasting the fatality rate of COVID-19 over time for the four most populated provinces in Canada. The results suggest that incorporating or not incorporating measurement error effects may yield rather different results for parameter estimation and forecasting.

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1. Introduction

Time series data are common in the fields of epidemiology, economics, and engineering. Various models and methods have been developed for analyzing such data. The validity of these methods hinges on the condition that time series data are precisely collected. This condition is, however, restrictive in applications, and measurement error is often inevitable. For example, in the study of air pollution, it is difficult or even impossible to precisely obtain the true measurements of daily air pollution levels.

Some early work on time series subject to measurement error dated back to Pagano [16] and Granger and Morris [9] who discussed the equivalence of error-prone autoregressive (AR) models to the autoregressive moving average (ARMA) models without measurement
error. Chanda [4] discussed the asymptotic property of autocovariance estimators with a large lag order under the error-prone autoregressive model. Tanaka [18] proposed a Lagrange multiplier test to assess the presence of measurement error in time series data. Staudenmayer and Buonaccorsi [17] explored the classical measurement error model for the autoregressive process. Tripodis and Buonaccorsi [19] studied measurement error in forecasting using the Kalman filter. Nagata [15] presented an Edgeworth approximation for the finite sample distribution of the ordinary least square estimators of error-prone AR(1) models. Dedecker et al. [7] considered a non-linear AR(1) model with measurement error.

Despite available discussions of measurement error in time series, some limitations restrict the application scope of the existing work. Most available methods consider only autoregressive models without the drift and assume the simplest additive measurement error model. Furthermore, most work involves a complex formulation to adjust for the measurement error effects, which is not straightforward to implement for practitioners. In addition, relatively less attention has been directed to address measurement error effects on prediction under autoregressive models.

In this article, we systematically explore the analysis of error-prone time series data under autoregressive models. We propose two types of models to delineate measurement error processes: additive regression models and multiplicative models. These modeling schemes offer us additional flexibility in facilitating different applications. We investigate the impact of the naive analysis which ignores the feature of measurement error in inferential procedures, and we obtain analytical results for characterizing the biases incurred in the naive analysis. We develop an estimating equation approach to adjust for the measurement error effects on time series analysis. We describe a block bootstrap algorithm for computing standard errors of the proposed estimators. Finally, we establish asymptotic results for the proposed estimators and discuss forecasting of times series in the presence of measurement error.

Our work is partially motivated by data on COVID-19, a widespread disease that has become a global health challenge causing over two hundred million infections and four million deaths as of 1 August 2021 [8]. Because of the special features of the disease, the COVID-19 data introduce a number of new challenges: (1) due to asymptomatic infected cases and patients with light symptoms who are not being tested, the number of reported cases with COVID-19 is typically smaller than the true number of infected cases; (2) due to the limited test resources at the beginning of the pandemic, many infected cases are not able to be identified instantly; and (3) varying incubation periods for different patients lead to the delay of the identification of the infections. Consequently, the discrepancy between the reported case number and the true case number can be substantial, and ignoring these features and applying traditional time series analysis methods would no longer produce valid results.

In this paper, we apply the developed method to analyze the COVID-19 data. We are interested in studying how the fatality rate in a region may change over time and describing the trajectory of the death rate. While the fatality rate of a disease is defined as the death number divided by the case number, the determination of the fatality rate of COVID-19 is challenging. In contrast to the standard definition, Baud et al. [1] estimated fatality rates by dividing the number of deaths before a given day by the number of patients with confirmed COVID-19 infections 14 days earlier, driven by the consideration that the maximum incubation time is estimated to be 14 days. Due to the unique features of COVID-19, there
does not seem to be a precise way to define the fatality rate of COVID-19. In this paper, we conduct sensitivity analyses to assess the severity of the pandemic by using different definitions of the fatality rate and considering different ways of modeling measurement error in the data.

The remainder of the article is organized as follows. In Section 2, we introduce the notation and the setup for autoregressive time series models. In Section 3, we propose two measurement error models and explore the impact of measurement error on the analysis of time series data. In Section 4, we develop an estimating equation approach to adjust for the biases due to measurement error and discuss a forecasting procedure. In Section 5, we implement the proposed method to analyze the COVID-19 data about four Canadian provinces. The article is concluded with a discussion presented in Section 6.

2. Model setup and framework

2.1. Time series model

Consider a $T \times 1$ vector of time series, $X^{(T)} = (X_1, X_2, \ldots, X_T)^T$. We are interested in modeling the dependence of $X_t$ on its previous observations $X^{(t-1)}$ and consider it to be postulated by the autoregressive model with lag $p$, denoted AR($p$),

$$X_t = \phi_0 + \sum_{j=1}^{p} \phi_j X_{t-j} + \epsilon_t, \quad (1)$$

where $p$ is a positive integer smaller than $T$, $\epsilon^{(t)} = (\epsilon_1, \ldots, \epsilon_t)^T$ is independent of $X^{(t)} = (X_1, \ldots, X_t)^T$ with each $\epsilon_t$ independently having zero mean and variance $\sigma^2_\epsilon$, $\phi_0$ is a constant drift, $\phi = (\phi_1, \ldots, \phi_p)^T$ is the regression coefficient, and $t = 1, \ldots, T$. For ease of exposition, $X_{t-j}$ in (1) is taken as null if the index $t-j$ is negative or zero.

The additive form in (1) and the zero-mean assumption of $\epsilon_t$ show that $\phi_0$ and $\phi$ are constrained by

$$\phi_0 = E(X_t) - \{E(\tilde{X}_{t-1})\}^T \phi, \quad (2)$$

where $\tilde{X}_{t-1} = (X_{t-1}, \ldots, X_{t-p})^T$. To make the process of $X_t$ stationary, $\phi_1, \ldots, \phi_p$ are further constrained such that

$$z^p - \phi_1 z^{p-1} - \cdots - \phi_p \neq 0$$

for all $|z| = 1$ [3, Sec.3.1]. For example, a stationary AR(1) process requires that $|\phi_1| \neq 1$, and a stationary AR(2) process needs that $\phi_1 - \phi_2 \neq 1$ and $\phi_1 + \phi_2 \neq -1$. Here we are interested in the estimation of parameters, $\phi$ and $\phi_0$. Let $\mu$ denote the mean $E(X_t)$ of the time series, which equals $\frac{\phi_0}{1-\phi_1-\cdots-\phi_p}$ if $X_t$ is stationary. When $p = 1$, the stationarity of a time series implies that $\text{Var}(X_t) = \frac{\sigma^2_\epsilon}{1-\phi^2_1}$ for $t = 1, \ldots, T$.

2.2. Estimation of model parameters

The estimation of the parameters in AR($p$) model (1) can be carried out by the least-squares method. To see this, we first focus on estimation of $\phi = (\phi_1, \ldots, \phi_p)^T$. Let $S(\phi) =
\[\sum_{t=p+1}^{T} [X_t - (\phi_0 + \sum_{j=1}^{p} \phi_j X_{t-j})]^2\] be the sum of the squared difference between \(X_t\) and its linearly combined history with lag \(p\). Then applying constraint (2) gives \(S(\phi) = \sum_{t=p+1}^{T} [(X_t - E(X_t)) - (\tilde{X}_{t-1} - E(\tilde{X}_{t-1}))^T \phi]^2\).

To minimize \(S(\phi)\) with respect to \(\phi\), we solve \(\frac{\partial S(\phi)}{\partial \phi} = 0\) for \(\phi\) and obtain the solution

\[
\hat{\phi}^{(LS)} = \left( \sum_{t=p+1}^{T} \{\tilde{X}_{t-1} - E(\tilde{X}_{t-1})\} \{\tilde{X}_{t-1} - E(\tilde{X}_{t-1})\}^T \right)^{-1} \times \sum_{t=p+1}^{T} \{\tilde{X}_{t-1} - E(\tilde{X}_{t-1})\} \{X_t - E(X_t)\},
\]

where for \(t = 1, \ldots, T\), \(E(X_t)\) can be estimated by \(\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} X_t\).

Next, by constraint (2), replacing \(E(X_t)\) by \(\hat{\mu}\) gives an estimator of \(\phi_0\):

\[
\hat{\phi}_0^{(LS)} = \hat{\mu} - \hat{\mu} \sum_{j=1}^{p} \phi_j.
\]

Re-expressing (1) as \(\epsilon_t = X_t - (\phi_0 + \sum_{j=1}^{p} \phi_j X_{t-j})\) and by the definition of \(S(\phi)\), we may estimate \(\text{Var}(\epsilon_t) = \sigma^2\) by

\[
\hat{\sigma}^{2(\text{LS})} = \frac{1}{T-p} S(\hat{\phi})
\]

\[
= \frac{1}{T-p} \sum_{t=p+1}^{T} [X_t - E(X_t)]^2 - \frac{2}{T-p} \sum_{t=p+1}^{T} [X_t - E(X_t)] \{\tilde{X}_{t-1} - E(\tilde{X}_{t-1})\}^T \hat{\phi}
\]

\[
+ \frac{1}{T-p} \sum_{t=p+1}^{T} \phi^T \{\tilde{X}_{t-1} - E(\tilde{X}_{t-1})\} [\tilde{X}_{t-1} - E(\tilde{X}_{t-1})]^T \phi
\]

with \(E(X_t)\) estimated by \(\hat{\mu}\).

Estimators (3)–(5) can be derived in an alternative way. First, by the stationarity of the \(X_t\), for \(k = 0, \ldots, p\) and \(t \geq p\), \(\text{Cov}(X_t, X_{t-k})\) is time-invariant, denoted by \(\gamma_k\); it is clear that \(\gamma_0\) represents \(\text{Var}(X_t)\) for any \(t\). Let \(\Gamma\) be the autocovariance matrix

\[
\Gamma = \begin{pmatrix}
\gamma_0 & \cdots & \gamma_{p-1} \\
\vdots & \ddots & \vdots \\
\gamma_{p-1} & \cdots & \gamma_0
\end{pmatrix}.
\]

Let \(\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p)^T\) with \(\tilde{\gamma}_k = \frac{1}{T-k} \sum_{t=k+1}^{T} (X_t - \hat{\mu})(X_{t-k} - \hat{\mu})\) being an estimator of \(\gamma_k\) for \(k = 0, \ldots, p\), and let \(\tilde{\Gamma}\) be the estimator of \(\Gamma\) with \(\gamma_k\) replaced by \(\tilde{\gamma}_k\) for \(k = 0, \ldots, p-1\).

Next, we examine the summation terms in (3) and (5) by using the fact that as \(T \to \infty\), 
\[
\frac{1}{T-p} \sum_{t=p+1}^{T} [X_t - E(X_t)]^2 \to \gamma_0, \quad \frac{1}{T-p} \sum_{t=p+1}^{T} [X_t - E(X_t)] \{\tilde{X}_{t-1} - E(\tilde{X}_{t-1})\}^T \to \gamma_0
\]
\( \gamma \), and \( \frac{1}{TP} \sum_{t=p+1}^{T} (\tilde{X}_{t-1} - E(\tilde{X}_{t-1}))(\tilde{X}_{t-1} - E(\tilde{X}_{t-1}))^{T} \overset{P}{\rightarrow} \Gamma \). Then, (3)–(5) motivate an alternative method of finding estimators for \( \phi, \phi_0, \) and \( \sigma^2_\varepsilon \) by solving the estimating equations:

\begin{align*}
\phi &= \hat{\Gamma}^{-1}\hat{\gamma}, \\
\phi_0 &= \left( 1 - \sum_{i=1}^{p} \phi_i \right) \hat{\mu}, \\
\sigma^2_\varepsilon &= \hat{\gamma}_0 - 2\phi^{T}\hat{\gamma} + \phi^{T}\hat{\Gamma}\phi,
\end{align*}

for \( \phi, \phi_0, \) and \( \sigma^2_\varepsilon \). Let \( \hat{\phi}, \hat{\phi}_0 \) and \( \hat{\sigma}^2_\varepsilon \) denote the resultant estimators of \( \phi, \phi_0, \) and \( \sigma^2_\varepsilon \), respectively. These estimators are asymptotically equivalent to the least-squares estimators \( \hat{\phi}^{(LS)}, \hat{\phi}_0^{(LS)}, \) and \( \hat{\sigma}^2_\varepsilon^{(LS)} \) in a sense that \( \hat{\phi} \overset{P}{\rightarrow} 0, \hat{\phi}_0 \overset{P}{\rightarrow} 0, \) and \( \hat{\sigma}^2_\varepsilon \overset{P}{\rightarrow} 0 \), as \( T \rightarrow \infty \), and hence, they are consistent [2, Ch.7, A.7.4].

Estimating Equation (6) offers a unified estimation framework in its connections with not only the least-squares estimation but also the maximum-likelihood method under the assumption of Gaussian errors for the \( \varepsilon_t \) in (1) as well as the Yule–Walker method. Similar to the least-squares method, finding estimators using one of those approaches is asymptotically equivalent to solving (6) for \( \phi, \phi_0, \) and \( \sigma^2_\varepsilon \) [2, Ch.7, A.7.4].

3. Measurement error and impact

3.1. Measurement error models

Suppose that for \( t = 1, \ldots, T \), the observation of \( X_t \) is subject to measurement error and the precise measurement of \( X_t \) may not be observed, but its surrogate measurement \( X_t^* \) is available. We consider two measurement error models.

The first measurement error model takes an additive form

\( X_t^* = \alpha_0 + \alpha_1 X_t + e_t \)  

(7)

for \( t = 1, \ldots, T \), where the error term \( e_t \) is assumed to be independent for \( t = 1, \ldots, T \) as well as independent of \( X_t \). \( e_t \) is assumed to have mean 0 and time-invariant variance \( \sigma^2_\varepsilon \), and \( \alpha = (\alpha_0, \alpha_1)^T \) is the parameter vector. Here, \( \alpha_0 \) represents the systematic error and \( \alpha_1 \) represents the constant inflation (or shrinkage) rate due to measurement error. For instance, if \( \alpha_0 = 0 \), then setting \( \alpha_1 < 1 \) (or \( \alpha_1 > 1 \)) features the scenario where \( X_t^* \) tends to be smaller (or larger) than \( X_t \) if the noise term is ignored. This model generalizes the classical additive model considered by Staudenmayer and Buonaccorsi [17] who considered the case with \( \alpha_0 = 0 \) and \( \alpha_1 = 1 \).

By the stationarity of \( X_t \), we note that model (7) yields \( E(X_t^*) = \alpha_0 + \alpha_1 \mu \) and

\[ \text{Var}(X_t^*) = \alpha_1^2 \gamma_0 + \sigma^2_\varepsilon; \]

(8)

the variability of the \( X_t^* \) can be greater or smaller than that of \( X_t \), depending on the value of \( \alpha_1 \) and \( \sigma^2_\varepsilon \).
The second measurement error model assumes a multiplicative form:

\[ X_t^* = \beta_0 u_t X_t \]  

for \( t = 1, \ldots, T \), where \( \beta_0 \) is a positive scaling parameter, and the \( u_t \) are the error terms which are independent of each other as well as of \( X_t \). The \( u_t \) are assumed to have mean 1 and time-invariant variance \( \sigma_u^2 \). Imposing different distributions of the error term \( u_t \) allows (9) to facilitate different types of discrepancies between \( X_t \) and \( X_t^* \).

The stationarity of \( X_t \) together with model (9) implies \( E(X_t^*) = \beta_0 \mu \), and

\[ \text{Var}(X_t^*) = \beta_0^2 \left\{ (\sigma_u^2 + 1)\gamma_0 + \sigma_u^2 \mu^2 \right\} , \]  

under the assumption of independence between \( X_t \) and \( u_t \).

Since \( E(X_t^*) \) is time-invariant for both (7) and (9), in the following discussion, we let \( \mu^* \) denote \( E(X_t^*) \) for \( t = 1, \ldots, T \). The modeling of the measurement error process by (7) or (9) introduces extra parameters \( \{\alpha_0, \alpha_1, \sigma_e^2\} \) or \( \{\beta_0, \sigma_u^2\} \), where the variance of the error term is bounded by the variability of \( X_t^* \) together with others. Clearly, (8) shows that \( \sigma_e^2 < \text{Var}(X_t^*) \) and (10) implies that \( \sigma_u^2 < \beta_0^{-2} (\gamma_0 + \mu^2)^{-2} \text{Var}(X_t^*) \).

### 3.2. Naive estimation and bias analysis

Estimating Equation (6) is useful when measurements of \( X_t \) are available. However, due to the measurement error, \( X_t \) is not observed but its surrogate \( X_t^* \) is available. If we directly apply (6) to estimate the parameters of model (1) with \( X_t \) replaced by \( X_t^* \), then biased results would be produced.

To be specific, if we naively replace \( X_t \) in (1) by \( X_t^* \), then time series model (1) becomes

\[ X_t^* = \phi_0^* + \sum_{j=1}^{p} \phi_j^* X_{t-j}^* + \epsilon_t^* , \]  

where we use \( \phi^* = (\phi_1^*, \ldots, \phi_p^*)^T \) and \( \epsilon_t^* \) to indicate possible differences from the corresponding symbol in (1).

If naively using (6) with \( X_t \) replaced by \( X_t^* \) to estimate the parameters, then we let \( \hat{\phi}^* = (\hat{\phi}_1^*, \ldots, \hat{\phi}_p^*)^T \), \( \hat{\mu}^* \) and \( \hat{\sigma}_e^2 \) denote the resultant estimators. Similar to \( \hat{\mu} \) and \( \hat{\gamma}_k \), which are defined after (3) and (5), respectively, we define \( \hat{\mu}^* = \frac{1}{T} \sum_{t=1}^{T} X_t^* \) and \( \hat{\gamma}_k^* = \frac{1}{T-k} \sum_{t=1}^{T-k} (X_t^* - \hat{\mu}^*)(X_{t+k}^* - \hat{\mu}^*) \) for \( k = 0, 1, \ldots, p \), and write \( \hat{\gamma}^* = (\hat{\gamma}_1^*, \ldots, \hat{\gamma}_p^*)^T \). We call such estimators the naive estimators and such an analysis the naive analysis in the following development.

The naive estimators are expected to incur biases in estimating the corresponding parameters. Such biases depend on the form of the measurement error model, as demonstrated by the following theorems whose proofs are included in Appendix A.2 of the Supplementary Material.

**Theorem 1:** Assume regularity conditions in Appendix A.1 in the Supplementary Material. Consider model (1) with \( p = 1 \) and measurement error model (7). Let \( \omega_1 = \frac{\alpha_1 \sigma_e^2}{\alpha_1^2 \sigma_e^2 + \sigma_e^2 (1-\phi_1^2)} \), \( \phi_1^* = \phi_1 \omega_1 \), and \( \phi_0^* = (\alpha_0 + \frac{\alpha_1 \phi_0}{1-\phi_1})(1 - \phi_1 \omega_1) \). Then
(1) \( \hat{\phi}_1^* \xrightarrow{P} \phi_1^* \) and \( \hat{\phi}_0^* \xrightarrow{P} \phi_0^* \) as \( T \to \infty \);

(2) \( \epsilon_t^* = \alpha_0(1 - \phi_1^*) + \alpha_1\phi_0 - \phi_0^* + \alpha_1(\phi_1 - \phi_1^*)X_{t-1} + (1 - \phi_1^*)e_t + \alpha_1\epsilon_t \) for \( t = 1, \ldots, T; \) and hence

\[
\text{Var}(\epsilon_t^*) = \phi_1^2\alpha_1^2(1 - \omega_1)^2\left( \frac{\sigma_e^2}{1 - \phi_1^2} \right) + (1 - \omega_1\phi_1^2)^2\sigma_e^2 + \alpha_1^2\sigma_e^2.
\]

This theorem essentially implies that the naive estimator under the additive measurement error model in (7) is inconsistent because \( \phi_1^* \neq \phi_1 \) and \( \phi_0^* \neq \phi_0 \). The naive estimator \( \hat{\phi}_1^* \) attenuates and the attenuation factor \( \omega_1 \) depends on the parameters \( \alpha_1 \) and \( \sigma_e^2 \) of the measurement error model (7) as well as \( \phi_1 \) and \( \sigma_e^2 \) in time series model (1). The coefficient \( \alpha_1 \) in measurement error model (7) affects the estimation of both naive estimators \( \hat{\phi}_1^* \) and \( \hat{\phi}_0^* \), while the intercept \( \alpha_0 \) influences the estimation of \( \phi_0^* \) only but not \( \phi_1^* \) or \( \text{Var}(\epsilon^*) \).

**Theorem 2:** Assume regularity conditions in Appendix A.1 in the Supplementary Material. Consider model (1) with \( p = 1 \) and measurement error model (9). Let \( \omega_2 = 1 + \sigma_u^2 + \frac{(1 + \phi_1^2)(\sigma_u^2\sigma_e^2)}{(1 - \phi_1^2)} \) and \( \phi_1^* = \phi_1 \omega_2 \), and \( \phi_0^* = \frac{\beta_0\phi_0}{1 - \phi_1}(1 - \omega_2\phi_1) \). Then

(1) \( \hat{\phi}_1^* \xrightarrow{P} \phi_1^* \) and \( \hat{\phi}_0^* \xrightarrow{P} \phi_0^* \) as \( T \to \infty \);

(2) \( \epsilon_t^* = \beta_0\phi_0u_t - \phi_0^* + \beta_0X_{t-1}(\phi_1u_t - \omega_2\phi_1u_{t-1}) + \beta_0u_t\epsilon_t \) for \( t = 1, \ldots, T; \)

and hence \( \text{Var}(\epsilon_t^*) = \beta_0^2(\sigma_u^2\phi_0^2 + (1 + \sigma_u^2)\sigma_e^2) + \beta_0^2\phi_1^2\frac{(1 + \omega_2^2)}{\omega_2}(1 - \phi_1^2) \).

This theorem says the attenuation effect resulting from the measurement error on the estimation of \( \phi_1 \). The constant scaling parameter \( \beta_0 \) in measurement error model (9) does not influence the estimation of \( \phi_1 \) but affects the estimation of \( \phi_0 \) and \( \sigma_e^2 \). The attenuation factor \( \omega_2 \) is determined by the magnitude \( \sigma_u^2 \) of measurement error as well as the values of \( \phi_0, \phi_1 \), and \( \sigma_e^2 \) of the time series model (1).

The results of Theorems 1 and 2 can be generalized to AR(\( p \)) model (1) with \( p > 1 \); the details are deferred to Appendix A.3 in the Supplementary Material. While Theorems 1 and 2 basically quantify the asymptotic bias of the naive estimators for model (1) with \( p = 1 \), they also demonstrate nonidentifiability issues in the presence of measurement error. Merely with the availability of surrogate measurements \( \{X_t^* : t = 1, \ldots, T\} \), the parameters \( \phi_0 \) and \( \phi_1 \) of primary interest are not identifiable, and thus, inestimable. To overcome this, one strategy is to assume the parameters of the measurement error model, (7) or (9), to be known. Though this assumption appears restrictive, it is viable to allow us to examine how the estimation of \( \phi_0 \) and \( \phi_1 \) may be impacted by measurement error in the data. In applications, one may specify values for the parameters of the measurement error model using prior information. When such information is not available, one may conduct sensitivity analyses by setting different parameter values for (7) or (9), and then apply a valid estimation method (to be discussed in the next section) to estimate \( \phi_0 \) and \( \phi_1 \) accordingly. It is thereby useful to evaluate how sensitive the estimation of \( \phi_0 \) and \( \phi_1 \) would be to different scenarios of measurement error. As a result, in the following development, we treat the parameters in (7) or (9) to be known and focus the discussion on the estimation of the parameters in model (1).
4. Methodology of correcting measurement error effects

4.1. Estimation of model parameters

In this section, we develop new estimators accounting for the effects of measurement error under either additive model (7) or multiplicative model (9).

Our idea is still to employ (6) to find consistent estimators of \( \phi, \phi_0, \) and \( \sigma^2_\epsilon \), but instead of replacing \( \mu \) and \( \gamma_k \) with \( \mu^* \) and \( \gamma_k^* \) as in the naive analysis, we replace \( \mu \) and \( \gamma_k \) in (6) with new functions of the \( X_t^* \), denoted as \( \tilde{\mu} \) and \( \tilde{\gamma}_k \), which adjust for the measurement error effects. Specifically, if we can find \( \tilde{\mu} \) and \( \tilde{\gamma}_k \) such that they resemble \( \mu \) and \( \gamma_k \) in the sense that as \( T \to \infty \),

\[
\tilde{\mu} \text{ and } \mu \text{ have the same limit in probability,}
\]

and \( \tilde{\gamma}_k \text{ and } \gamma_k \text{ have the same limit in probability for } k = 0, \ldots, p, \)

then substituting \( \tilde{\mu} \) and \( \tilde{\gamma}_k \) with \( \mu \) and \( \gamma_k \) in (6) yields consistent estimators of \( \phi, \phi_0 \) and \( \sigma^2_\epsilon \).

With the availability of \( \gamma_k \) satisfying (12), let \( \tilde{\Gamma} \) denote \( \Gamma \) with the \( \gamma_k \) replaced by \( \gamma_k \). Then provided regularity conditions, consistent estimators of \( \phi, \phi_0 \) and \( \sigma^2_\epsilon \) can be obtained by solving the estimating equations for \( \phi, \phi_0, \) and \( \sigma^2_\epsilon \):

\[
\phi = \tilde{\Gamma}^{-1}\tilde{\gamma},
\]

\[
\phi_0 = \left(1 - \sum_{i=1}^{p} \phi_i\right)\tilde{\mu},
\]

\[
\sigma^2_\epsilon = \gamma_0 - 2\phi^T\tilde{\gamma} + \phi^T\tilde{\Gamma}\phi.
\]

It is immediate to obtain the following result.

Theorem 3: Assume regularity conditions in Appendix A.1 in the Supplementary Material. Suppose that \( \tilde{\mu} \) and \( \tilde{\gamma}_k \) are functions of the \( X_t^* \) with \( t = 1, \ldots, T \) which satisfy (12), and let \( \tilde{\phi}, \tilde{\phi}_0, \) and \( \tilde{\sigma}^2_\epsilon \) denote the estimators for \( \phi, \phi_0 \) and \( \sigma^2_\epsilon \), respectively, obtained by solving (13). Then, as \( T \to \infty \),

1. \( \tilde{\phi} \xrightarrow{p} \phi, \tilde{\phi}_0 \xrightarrow{p} \phi_0, \) and \( \tilde{\sigma}^2_\epsilon \xrightarrow{p} \sigma^2_\epsilon \),
2. \( \sqrt{n} (\tilde{\phi} - \phi) \xrightarrow{d} N(0, GQG^T), \)

where \( G \) is the matrix of derivatives of \( \tilde{\phi} \) with respect to \( (\tilde{\gamma}_0^*, \tilde{\gamma}^*)^T \). Here \( Q = Q_1 \), the matrix defined in Theorem 4 in Appendix A.3 of the Supplementary Material, if measurement error follows model (7); and \( Q = Q_2 \), the matrix defined in Theorem 5 in Appendix A.3 of the Supplementary Material, if measurement error follows model (9).

Now we discuss explicitly how to determine \( \tilde{\mu} \) and \( \tilde{\gamma}_k \) under measurement error model (7) or (9). With model (7), Theorem 4 in the Supplementary Material, characterizes asymptotic biases of the naive estimators, where we define \( \mu^* = \alpha_0 + \alpha_1\mu, \gamma^* = \alpha_1^2\gamma, \) and \( \gamma_0^* = \alpha_1^2\gamma_0 + \sigma^2_\epsilon \). By Theorem 4(1), it is immediate that \( \frac{\hat{\mu}^*}{\alpha_1} - \alpha_0 \xrightarrow{p} \mu, \frac{1}{\alpha_1}(\hat{\gamma}_0^* - \sigma^2_\epsilon) \xrightarrow{p} \gamma_0, \) and \( \frac{\hat{\gamma}_k^*}{\alpha_1^2} \xrightarrow{p} \gamma \) as \( T \to \infty \). Thus, taking \( \tilde{\mu} = \frac{\hat{\mu}^*}{\alpha_1} - \alpha_0, \tilde{\gamma}_0 = \frac{1}{\alpha_1}(\hat{\gamma}_0^* - \sigma^2_\epsilon), \) and \( \tilde{\gamma}_k = \frac{\hat{\gamma}_k^*}{\alpha_1^2} \) for
\( k = 1, \ldots, p \) makes (12) hold under measurement error model (7). On the other hand, with model (9), we apply Theorem 5 in the Supplementary Material, where we define \( \mu^* = \beta_0 \mu, \gamma^* = \beta_2^2 \gamma, \) and \( \gamma_0^* = \beta_0^2 ((\sigma_\epsilon^2 + 1) \gamma_0 + \sigma_\mu^2 \mu^2) \). By Theorem 5(1), it is seen that 
\[
\frac{\tilde{\mu}}{\tilde{\sigma}_0} = \frac{\tilde{\gamma}_0}{\tilde{\sigma}_0^2} - \frac{\sigma_\epsilon^2}{\sigma_\mu^2} \rightarrow \mu, \quad \frac{\tilde{\gamma}_0}{\tilde{\sigma}_0^2} + \frac{\sigma_\epsilon^2}{\sigma_\mu^2} \rightarrow \gamma
\]
for \( \tilde{\mu}, \tilde{\gamma}_0, \) and \( \tilde{\gamma}_k \) for \( k = 1, \ldots, p \).

We conclude this section with a procedure of estimating the asymptotic covariance matrix for the estimator \( \tilde{\phi} \). While Theorem 3 presents the sandwich form of the asymptotic covariance matrix of \( \tilde{\phi} \), its evaluation involves lengthy calculations. We may alternatively employ the block bootstrap algorithm [14] to obtain variance estimates for \( \tilde{\phi} \) using the following steps. First, we set a positive integer, say \( N \), as the number for the bootstrap sampling; \( N \) can be set as a large number such as 1000. Next, we repeat through the following five steps:

**Step 1:** At iteration \( n \in \{1, \ldots, N\} \), we initialize a null time series \( X^{(n,0)} \) of dimension 0 and specify a block length, say \( b \), which is an integer between 0 and \( T \). Initialize \( m = 1 \).

**Step 2:** Sample an index, say \( i \), from \( \{0, \ldots, T - b\} \), and then define \( X^{(m-1)} = \{X_{i+1}, \ldots, X_{i+b}\} \).

**Step 3:** Update the previous time series \( X^{(n,m-1)} \) by appending \( X^{(m-1)} \) to it, and let \( X^{(n,m)} \) denote the new time series.

**Step 4:** If the dimension \( X^{(n,m)} \) is smaller than \( T \) then return to Steps 2 and 3; otherwise drop the elements in the time series with the index greater than \( T \) to ensure the dimension of \( X^{(n,m)} \) to be identical to \( T \) and then go to Step 5.

**Step 5:** Obtain an estimate \( \tilde{\phi}^{(n)} \) of parameter \( \phi \) by applying the time series \( X^{(n,m)} \) to (13).

As a result, the bootstrap variance of \( \tilde{\phi} \) is given by 
\[
\text{Var}_{\text{boot}}(\tilde{\phi}) = \frac{1}{N} \sum_{n=1}^{N} (\tilde{\phi}^{(n)} - \tilde{\phi})^2
\]
where \( \tilde{\phi} = \frac{1}{N} \sum_{n=1}^{N} \tilde{\phi}^{(n)} \).

### 4.2. Forecasting and prediction error

In the absence of measurement error, with the observed time series \( X_T = \{x_1, \ldots, x_T\} \), we may be interested in predicting \( \{X_{T+1}, \ldots, X_{T+H}\} \) for a positive integer \( H \), which is done one by one starting from the nearest time point \( T + 1 \) to the farthest time point \( T + H \). To be specific, let \( h = 1, \ldots, H \), the \( h \)-step forecasting of \( X_{T+h} \) is based on its history of lag \( p \), \( \{X_{T+h-1}, \ldots, X_{T+h-p}\} \), by using the conditional expectation \( \hat{X}_{T+h} \triangleq E(X_{T+h} | x_{T+h-1}, \ldots, x_{T+h-p}) \), where for \( j = T + h - 1, \ldots, T + h - p \), \( x_j \) is the observed value of \( X_j \) if \( j \leq T \); and \( x_j \) is the predicted value of \( X_j \), \( \hat{X}_j \), if \( j > T \). This prediction minimizes the squared prediction error \( E(\hat{X}_{T+h} - X_{T+h})^2 \) [2, e.g. p.131].

By the zero mean of the random error term \( \epsilon_t \) in AR\((p)\) model (1), the conditional expectation can be calculated by

\[
\hat{X}_{T+h} = \phi_0 + \phi_1 x_{T+h-1} + \ldots + \phi_p x_{T+h-p}
\]
for \( h = 1, \ldots, H \).

When measurement error appears, the observe values \( x_j \) for \( j = T, \ldots, T - p + 1 \) in (14) are no longer available but their surrogates \( X^*_j \) are available. We now provide a sensible estimate of \( X_j \) by using the relationship of \( X_j \) and \( X^*_j \). If measurement error follows (7), we ‘predict’ \( X_j \) by

\[
\hat{X}_j = \frac{1}{\alpha_1} (X^*_j - \alpha_0) \quad \text{for } j = t, \ldots, t - p + 1; \quad (15)
\]

if the measurement error follows (9), then \( \hat{X}_j \) is ‘predicted’ by

\[
\hat{X}_j = \frac{X^*_j}{\beta_0} \quad \text{for } j = t, \ldots, t - p + 1. \quad (16)
\]

These ‘predictions’ are unbiased in the sense that \( E(\hat{X}_j) = X_j \) for \( j = t, \ldots, t - p + 1 \). Consequently, for \( h = 1, \ldots, H \), \( X_{T+h} \) is predicted as

\[
\hat{X}_{T+h} = \phi_0 + \phi_1 \hat{X}_{T+h-1} + \cdots + \phi_p \hat{X}_{T+h-p}. \quad (17)
\]

In contrast to the prediction based on (14), the prediction based on (17) incurs additional prediction error due to the use of the estimates determined by (15) or (16). We consider \( p = 1 \) to illustrate the recursive calculation of the prediction error; the prediction error with higher orders of the autoregressive process can be derived recursively in a similar way but with more complex expressions.

If the measurement error follows (7), the mean-squared prediction error of the 1-step prediction is given by

\[
P^{(1)}_e = E \{ (\hat{X}_{T+1} - X_{T+1})^2 \}
= E \left[ \phi_1 \left( X_t + \frac{\epsilon_t}{\alpha_1} \right) - \phi_1 X_T - \epsilon_{T+1} \right]^2
= \frac{\phi_1^2 \sigma^2_e}{\alpha_1^2} + \sigma^2_e,
\]

where the last step is due to the independence between \( \epsilon_t \) and \( \epsilon_{t+1} \), as well as \( E(\epsilon_t^2) = \sigma^2_e \) and \( E(\epsilon_{t+1}^2) = \sigma^2_e \). Then, the \( h \)-step prediction error is given by

\[
P^{(h)}_e = E \{ (\hat{X}_{T+h} - X_{T+h})^2 \}
= E \left[ \phi_1 (\hat{X}_{T+h-1} - X_{T+h-1}) - \epsilon_{T+1} \right]^2
= \phi_1^2 P^{(h-1)}_e + \sigma^2_e
= \frac{\phi_1^{2h} \sigma^2_e}{\alpha_1^2} + \sum_{i=0}^{h-1} \phi_1^{2i} \sigma^2_e, \quad (18)
\]

where the last step comes from the recursive evaluation of \( P^{(h-1)}_e \).
Similarly, if the measurement error follows (9), the mean-squared prediction error is given by

\[ P_e^{(1)} = E \{ (\hat{X}_{T+1} - X_{T+1})^2 \} \]
\[ = \phi_1^2 \left( \frac{\sigma_u^2}{1 - \phi_1^2} + \mu^2 \right) \sigma_u^2 + \sigma_e^2, \]

where we use the independence of \( \epsilon_{t+1}, u_t \) and \( X_t \), \( E(u_t) = 1 \), and \( \text{Var}(X_t) = \frac{\sigma_u^2}{1 - \phi_1^2} \) due to the stationary AR(1) process. Hence,

\[ P_e^{(h)} = E \{ (\hat{X}_{T+h} - X_{T+h})^2 \} \]
\[ = \phi_1^2 P_e^{(h-1)} + \sigma_e^2 \]
\[ = \phi_1^{2h} \left( \frac{\sigma_u^2}{1 - \phi_1^2} + \mu^2 \right) \sigma_u^2 + \sum_{i=0}^{h-1} \phi_1^{2i} \sigma_u^2, \quad \text{(19)} \]

The evaluation of the mean-squared prediction error \( P_e^{(h)} \) is carried out by replacing the parameters with their estimates. We comment that the common second term in (18) and (19), \( \sum_{i=0}^{h-1} \phi_1^{2i} \sigma_u^2 \), is the mean-squared prediction error for the AR(1) model for error-free settings [2, e.g. p.152], which equals \( \frac{1 - \phi_1^{2h}}{1 - \phi_1^2} \sigma_u^2 \).

For \( \alpha \) with \( 0 < \alpha < 1 \), an \( h \)-step \( (1 - \alpha) \)-prediction interval is constructed as

\[ \left[ \hat{X}_{T+h} - q_{\alpha} P_e^{(h)}, \hat{X}_{T+h} + q_{\alpha} P_e^{(h)} \right], \]

where \( q_{\alpha} \) the \( \alpha \)-level quantile of the distribution of \( \hat{X}_{T+h} - X_{T+h} \). In practice, under the normality assumption of \( \epsilon_t \) and \( e_t \), one can take \( q_{\alpha} \) to be the \( \alpha \)-level quantile of the standard normal distribution [3, p.108].

5. Analysis of COVID-19 death rates

5.1. Study objectives

Using Canadian provincial COVID-19 data containing the daily confirmed cases and deaths from 10 April 2020 to 9 January 2021, we compare the time series of death rates for British Columbia, Ontario, Quebec, and Alberta, the four provinces in Canada that experience multiple waves of outbreaks prior to the implementation of the vaccination program to the public. The daily confirmed cases and fatalities are taken from JHU CSSE COVID-19 Data (https://github.com/CSSEGISandData/COVID-19).

In epidemiology, the fatality rate, defined as the proportion of cumulative deaths of the disease in the total number of people diagnosed with the disease [12], is often used to measure the severeness of an infectious disease. For COVID-19, determining the fatality rate is not trivial. Due to the limited test capacity, individuals with light symptoms are not being tested, and asymptomatic infections and varying incubation periods make it difficult to acquire an accurate number of infections. To circumvent this, we explore different definitions of death rates. Definition 1 is from Baud et al. [1] who estimated fatality rates by
dividing the number of deaths before a given day by the number of patients who had been confirmed COVID-19 infection 14 days before, with the consideration that the maximum incubation time is regarded to be 14 days. On the other hand, the median time from symptom onset to intensive care unit admission is about 10 days [1], so we consider Definition 2 which is the number of deaths of COVID-19 by day $t$ divided by the number of confirmed cases by day $(t - 10)$. In comparison, we also consider Definition 3 by calculating the death rate on day $t$ as the ratio of the number of deaths by day $t$ to the number of confirmed cases by day $t$.

Figure 1 displays the trajectory of the fatality rates of COVID-19 in the four provinces calculated from applying the three definitions to the reported data.

To reflect the discrepancy between the reported and the true fatality rates for each province, for each definition of the fatality rate, we let $X_{it}$ with $i = 1, 2, 3, 4$ represent its true value and let $X_{it}^*$ denote the reported value on day $t$ for British Columbia, Ontario, Quebec, and Alberta, respectively. The objective is to use the reported fatality rates $\{X_{it}^*: t = 1, \ldots, T\}$ with $T = 274$ to infer the true fatality rates $X_{it}$ which are modeled by (1) separately for $i = 1, \ldots, 4$. As there is no exact information to guide us on how to characterize the relationship between $X_{it}^*$ and $X_{it}$, here we conduct sensitivity studies by considering measurement error model (7) or (9). We use the observed data $X_{it}^*$ from 10 April 2020 to 9 January 2021, i.e., $\{X_{it}^*: t = 1, \ldots, T\}$, to estimate the model parameters in (1) with measurement error effects accounted for, and then forecast the fatality rate of COVID-19, from 10 January 2021 to 14 January 2021, in British Columbia, Ontario, Quebec and Alberta, Canada.

### 5.2. Models building

To assess the stationarity of $X_{it}^*$, we conduct the augmented Dickey–Fuller (ADF) tests [5] to time series $\{X_{it}^*: t = 1, \ldots, T\}$ obtained using each definition and its differencing transformation $\{X_{i,t+1}^* - X_{it}^*: t = 1, \ldots, T\}$ for $i = 1, \ldots, 4$. The results of the test statistics (TSV) and $p$-values are reported in Web Table 4. Further, we produce the autocorrelation and partial autocorrelation plots for those time series data and show them in Web Figures 4 and 5. When both the ADF test and the diagnostic plots indicate the evidence of stationarity, we then consider the time series to be stationary. After taking 1-order differencing, the data
of all provinces derived from Definitions 1 and 2 appear to be fairly stationary, yet only the
data for Alberta derived from Definition 3 seem to be stationary.

To determine the lag value \( p \) for autoregression model (1) used for the time series \( \{X_{it} : t = 1, \ldots, T\} \) for \( i = 1, \ldots, 4 \), we fit model (11) for the observed data with \( \epsilon_{it}^* \) assumed to follow a normal distribution with mean zero and use the AIC criterion by minimizing

\[
-2 \sum_{t=p}^{T} \log f(x_{it}^*|x_{i(t-1)}^*, \ldots, x_{i(t-p)}^*) + 2p,
\]

where \( f(x_{it}^*|x_{i(t-1)}^*, \ldots, x_{i(t-p)}^*) \) is the conditional probability of \( X_{it}^* \) given \( \{X_{i(t-1)}^*, \ldots, X_{i(t-p)}^*\} \). An approximation of AIC, suggested by Hurvich and Tsai [11], is given by

\[
\log \left\{ \frac{1}{T-P+1} \sum_{t=p}^{T} (X_{it}^* - \hat{X}_{it}^*)^2 \right\} + \frac{2}{T} (p + 1),
\]

with \( \hat{X}_{it}^* \) determined by using (17) recursively if \( t \geq p \), and \( \hat{X}_{it}^* \) set as \( X_{it}^* \) if \( t \leq p \). The results are summarized in Web Table 5, where 1-differencing is applied. Further, we display quantile–quantile plots in Web Figure 6 for the standardized residuals of the autoregressive models, which fairly show the feasibility of assuming the normality of \( \epsilon_{it}^* \) in (20).

As the true process \( \{X_{it} : t = 1, \ldots, T\} \) are unavailable, we now use the preceding results for the observed data \( \{X_{it}^* : t = 1, \ldots, T\} \) to guide us to choose a suitable lag value \( p \) for using AR(\( p \)) model (1), in the hope that this treatment gives us a good approximation to the underlying truth. To be specific, we analyze only those data derived from Definitions 1 and 2 for all the four provinces as well as the data derived from Definition 3 for Alberta, whose true values \( \{X_{it} : t = 1, \ldots, T\} \) are treated as stationary after being transformed with 1-order differencing, where an AR(\( p \)) model is employed with \( p \) determined by the values shown in Web Table 5.

That is, with the 1-order differencing, the British Columbia data derived from Definitions 1 and 2 are, respectively, analyzed with AR(4) and AR(2) models; the Ontario data derived from Definitions 1 and 2 are, respectively, analyzed with AR(2) and AR(1) models; the Quebec data derived from Definitions 1 and 2 are, respectively, analyzed with AR(4) and AR(8) models; and the Alberta data derived from Definitions 1–3 are, respectively, analyzed with AR(2), AR(2) and AR(4) models.

5.3. Sensitivity analyses

As there are no additional data available for estimating the parameters for model (7) or (9), we conduct sensitivity analyses using the findings in the literature. Different studies showed different estimates of the asymptomatic infection rates, ranging from 17.9% to 78.3% [6,13]. To accommodate the heterogeneity of different studies, He et al. [10] carried out a meta-analysis and obtained an estimate of the asymptomatic infection rate to be 46%. If under-reported confirmed cases are only caused by undetected asymptomatic cases, then \( X_{it} = (1 - \tau_A)X_{it}^* \), or equivalently,

\[
X_{it}^* = \frac{1}{1 - \tau_A}X_{it},
\]
where $\tau_A$ represents the rate of asymptomatic infections.

Now we use (21) as a starting point to conduct sensitivity analyses. In multiplicative model (9), we take $\beta_0u_{it} = \frac{1}{1-\tau_A}$. With $E(u_{it}) = 1$, we set $\beta_0 = \frac{1}{1-\tau_A}$ by setting $\tau_A = 46\%$, the value from the meta-analysis of He et al. [10]. To describe different degrees of error, we consider $\sigma^2$ to take a small value, say $\sigma^2_{u1}$, and a large value, say, $\sigma^2_{u2}$, which is alternatively reflected by the change of the coefficient of variation, $CV = \frac{\sigma_u}{E(u_{it})}$, of the error term $u_{it}$ from $\sigma_{u1} \times 100\%$ to $\sigma_{u2} \times 100\%$.

When using additive model (7) to characterize the measurement error process, motivated by (21), we set $\alpha_0 = 0$ and $\alpha_1 = \frac{1}{1-46\%}$, and let $\sigma^2_e$ take a small value, say $\sigma^2_{e1}$, and a large value, say $\sigma^2_{e2}$, to feature an increasing degree of measurement error. Due to the constraints for the parameters discussed for (8) and (10), we set the values for $\sigma_{u1}, \sigma_{u2}, \sigma_{e1}$, and $\sigma_{e2}$ case by case for each definition and for each province, which are recorded in Web Table 6.

The model fitting results are reported in Tables 1–2 and Web Table 7 for the three definitions of fatality rates, where the point estimates (EST), the associated standard errors (SE), and the $p$-values for the model parameters are included. Table 1 shows that with Definition 1, the estimates of $\phi_0$ from the proposed method are smaller than those of the naive method which ignores the discrepancy of the observed and true data, whereas the absolute values of the estimates of $\phi_j$ ($j = 1, \ldots, p$) produced from the proposed method are no smaller than those obtained from the naive method. As expected, the standard errors for estimation of $\phi_j$ ($j = 1, \ldots, p$) derived from the proposed method is generally larger than those of the naive method, yet it is somewhat surprising that the standard errors associated with the estimation of $\phi_0$ produced from the proposed method are smaller than those of the naive method, though the differences are small. Both methods find no evidence to support that $\phi_0$ is different from zero for the data of British Colombia, Ontario, and Alberta, suggesting that the fatality rates of these three provinces remain statistically unchanged. However, for the Quebec data, the naive and the proposed methods show different evidence. The naive method suggests a likely upward trend with a $p$-value .032 for testing of $\phi_0$, whereas the proposed method finds no such evidence. Regarding the data of Ontario, Quebec, and Alberta, all the methods found no evidence that $\phi_j$ differs from zero for $j = 1, \ldots, p$. But for the data of British Colombia, the significance level of $\phi_j$ can be revealed differently under different measurement error settings.

Table 2 displays the results for all the four provinces data derived from Definition 2, which exhibit similar patterns we observe for Definition 1 regarding the point estimates as well as the standard errors. However, no evidence is revealed for the significance of $\phi_0$ from the naive and proposed methods. The $p$-values of testing $\phi_j$ for $j = 1, \ldots, p$ suggest fairly the same type of evidence under different measurement error settings, and those values appear to be fairly close, which is especially the case for the Quebec data.

Web Table 7 shows the results for the Alberta data derived from Definition 3, which show the patterns similar to those for Definitions 1 and 2 regarding point estimates and standard errors. Overall, $p$-values for testing $\phi_j$ with $j = 0, \ldots, p$ do not seem to change considerably for different measurement error settings. All the methods suggest that $\phi_1$ is significantly different from zero whereas all other $\phi_j$’s are not.
Table 1. Definition 1: The parameter estimation under different measurement error models: the AR(1) model with ‘order-1 differencing’ is used to fit the data of British Columbia, Ontario, Quebec, and Alberta.

| Method                              | Error Degree | Parameter | British Columbia |               | Ontario       |               | Quebec        |               | Alberta       |               |
|-------------------------------------|--------------|-----------|------------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
|                                     |              |           | EST   | SE   | p-Value | EST   | SE   | p-Value | EST   | SE   | p-Value | EST   | SE   | p-Value |
| Naive                               |              | $\phi_0$  | 0.002 | 0.009 | .824    | 0.007 | 0.012 | .556    | 0.009 | 0.004 | .322    | 0.004 | 0.002 | .052    |
|                                     |              | $\phi_1$  | 0.236 | 0.119 | .052    | 0.060 | 0.125 | .632    | 0.094 | 0.250 | .707    | 0.016 | 0.092 | .864    |
|                                     |              | $\phi_2$  | 0.280 | 0.087 | .002    | 0.099 | 0.133 | .460    | 0.068 | 0.102 | .509    | -0.086 | 0.101 | .398    |
|                                     |              | $\phi_3$  | -0.003 | 0.055 | .964    | -    | -    | -    | 0.094 | 0.103 | .365    | -    | -    | -    |
|                                     |              | $\phi_4$  | 0.134 | 0.079 | .097    | -    | -    | -    | 0.005 | 0.003 | .067    | 0.002 | 0.001 | .094    |
|                                     | Small        | $\phi_2$  | 0.284 | 0.089 | .002    | 0.099 | 0.139 | .478    | 0.068 | 0.102 | .509    | -0.086 | 0.106 | .420    |
|                                     | Small        | $\phi_3$  | -0.006 | 0.068 | .936    | -    | -    | -    | 0.094 | 0.110 | .397    | -    | -    | -    |
| The Proposed Method with Additive Error | Large       | $\phi_2$  | 0.296 | 0.105 | .007    | 0.099 | 0.151 | .514    | 0.068 | 0.511 | .895    | -0.087 | 0.120 | .474    |
|                                     | Large        | $\phi_3$  | -0.016 | 0.110 | .887    | -    | -    | -    | 0.094 | 0.550 | .865    | -    | -    | -    |
|                                     | Large        | $\phi_4$  | 0.137 | 0.108 | .208    | -    | -    | -    | 0.005 | 0.003 | .068    | 0.002 | 0.001 | .095    |
|                                     | Large        | $\phi_0$  | 0.001 | 0.005 | .836    | 0.004 | 0.008 | .616    | 0.005 | 0.003 | .068    | 0.002 | 0.001 | .094    |
|                                     | Large        | $\phi_1$  | 0.249 | 0.179 | .171    | 0.060 | 0.151 | .690    | 0.094 | 0.450 | .835    | 0.016 | 0.132 | .904    |
|                                     | Large        | $\phi_2$  | 0.296 | 0.105 | .007    | 0.099 | 0.151 | .514    | 0.068 | 0.511 | .895    | -0.087 | 0.120 | .474    |
|                                     | Large        | $\phi_3$  | -0.016 | 0.110 | .887    | -    | -    | -    | 0.094 | 0.550 | .865    | -    | -    | -    |
|                                     | Large        | $\phi_4$  | 0.137 | 0.108 | .208    | -    | -    | -    | 0.005 | 0.003 | .068    | 0.002 | 0.001 | .095    |
|                                     | Small        | $\phi_2$  | 0.283 | 0.087 | .002    | 0.100 | 0.140 | .478    | 0.068 | 0.101 | .503    | -0.087 | 0.104 | .407    |
|                                     | Small        | $\phi_3$  | -0.005 | 0.063 | .941    | -    | -    | -    | 0.094 | 0.107 | .384    | -    | -    | -    |
| The Proposed Method with Multiplicative Error | Large       | $\phi_2$  | 0.291 | 0.091 | .002    | 0.103 | 0.156 | .512    | 0.068 | 0.102 | .508    | -0.089 | 0.107 | .406    |
|                                     | Large        | $\phi_3$  | -0.011 | 0.067 | .866    | -    | -    | -    | 0.094 | 0.112 | .406    | -    | -    | -    |
|                                     | Large        | $\phi_4$  | 0.136 | 0.083 | .107    | -    | -    | -    | 0.088 | 0.106 | .409    | -    | -    | -    |
Table 2. Definition 2: The parameter estimation under different measurement error models: the AR(1) model with ‘order-1 differencing’ is used to fit the data of British Columbia and the AR(4) model with ‘order-1 differencing’ is used to fit the data of Ontario.

| Method | Error degree | Parameter | British Columbia | Ontario | Quebec | Alberta |
|--------|--------------|-----------|------------------|---------|--------|---------|
|        |              |           | EST   | SE    | p-Value | EST   | SE    | p-Value | EST   | SE    | p-Value | EST   | SE    | p-Value |
| Naive  | –            | φ₀       | 0.002 | 0.008 | .752    | 0.006 | 0.009 | .495    | 0.003 | 0.003 | .191    | 0.003 | 0.002 | .114    |
|        |              | φ₁       | 0.340 | 0.080 | < .001  | 0.223 | 0.076 | .005    | 0.139 | 0.215 | .520    | 0.451 | 0.109 | < .001  |
|        |              | φ₂       | 0.216 | 0.077 | .007    | –     | –     | –       | 0.130 | 0.101 | .207    | -0.214 | 0.073 | .005    |
|        |              | φ₃       | –     | –     | –       | –     | –     | –       | 0.086 | 0.113 | .451    | –     | –     | –       |
|        |              | φ₄       | –     | –     | –       | –     | –     | –       | 0.027 | 0.116 | .816    | –     | –     | –       |
|        |              | φ₅       | –     | –     | –       | –     | –     | –       | 0.040 | 0.092 | .666    | –     | –     | –       |
|        |              | φ₆       | –     | –     | –       | –     | –     | –       | 0.052 | 0.130 | .690    | –     | –     | –       |
|        |              | φ₇       | –     | –     | –       | –     | –     | –       | 0.135 | 0.102 | .190    | –     | –     | –       |
|        |              | φ₈       | –     | –     | –       | –     | –     | –       | 0.124 | 0.101 | .225    | –     | –     | –       |
| Small  | (σₑ₁²)      | φ₀       | 0.001 | 0.004 | .754    | 0.003 | 0.005 | .527    | 0.002 | 0.002 | .255    | 0.001 | 0.001 | .154    |
|        |              | φ₁       | 0.345 | 0.083 | < .001  | 0.224 | 0.084 | .010    | 0.139 | 0.220 | .529    | 0.454 | 0.116 | < .001  |
|        |              | φ₂       | 0.217 | 0.081 | .010    | –     | –     | –       | 0.130 | 0.108 | .234    | -0.216 | 0.077 | .007    |
|        |              | φ₃       | –     | –     | –       | –     | –     | –       | 0.086 | 0.118 | .470    | –     | –     | –       |
|        |              | φ₄       | –     | –     | –       | –     | –     | –       | 0.027 | 0.120 | .822    | –     | –     | –       |
|        |              | φ₅       | –     | –     | –       | –     | –     | –       | 0.040 | 0.094 | .673    | –     | –     | –       |
|        |              | φ₆       | –     | –     | –       | –     | –     | –       | 0.052 | 0.128 | .685    | –     | –     | –       |
|        |              | φ₇       | –     | –     | –       | –     | –     | –       | 0.135 | 0.104 | .199    | –     | –     | –       |
|        |              | φ₈       | –     | –     | –       | –     | –     | –       | 0.124 | 0.099 | .219    | –     | –     | –       |
| Large  | (σₑ₂²)      | φ₀       | 0.001 | 0.004 | .760    | 0.003 | 0.009 | .708    | 0.002 | 0.002 | .254    | 0.001 | 0.001 | .153    |
|        |              | φ₁       | 0.359 | 0.094 | < .001  | 0.224 | 0.674 | .731    | 0.139 | 0.235 | .556    | 0.462 | 0.110 | < .001  |
|        |              | φ₂       | 0.220 | 0.089 | .016    | –     | –     | –       | 0.130 | 0.121 | .290    | -0.223 | 0.083 | .010    |
|        |              | φ₃       | –     | –     | –       | –     | –     | –       | 0.086 | 0.124 | .488    | –     | –     | –       |
|        |              | φ₄       | –     | –     | –       | –     | –     | –       | 0.027 | 0.123 | .827    | –     | –     | –       |
|        |              | φ₅       | –     | –     | –       | –     | –     | –       | 0.040 | 0.100 | .690    | –     | –     | –       |
|        |              | φ₆       | –     | –     | –       | –     | –     | –       | 0.052 | 0.137 | .706    | –     | –     | –       |
|        |              | φ₇       | –     | –     | –       | –     | –     | –       | 0.135 | 0.112 | .235    | –     | –     | –       |
|        |              | φ₈       | –     | –     | –       | –     | –     | –       | 0.124 | 0.100 | .222    | –     | –     | –       |
|        |              | φ₀       | 0.001 | 0.004 | .753    | 0.003 | 0.005 | .530    | 0.002 | 0.002 | .255    | 0.001 | 0.001 | .155    |
|        |              | φ₁       | 0.343 | 0.081 | < .001  | 0.226 | 0.079 | .006    | 0.139 | 0.221 | .532    | 0.457 | 0.119 | < .001  |
|       | Small $\phi_2$ | 0.217 | 0.079 | .009 | – | – | – | 0.130 | 0.109 | .239 | -0.219 | 0.077 | .006 |
|-------|---------------|-------|-------|------|---|---|---|-------|-------|------|--------|-------|------|
|       | $\phi_3$     | –     | –     | –    | – | – | – | 0.086 | 0.119 | .471 | –       | –     | –    |
|       | $\phi_4$     | –     | –     | –    | – | – | – | 0.027 | 0.120 | .822 | –       | –     | –    |
|       | $\phi_5$     | –     | –     | –    | – | – | – | 0.040 | 0.095 | .674 | –       | –     | –    |
|       | $\phi_6$     | –     | –     | –    | – | – | – | 0.052 | 0.128 | .686 | –       | –     | –    |
|       | $\phi_7$     | –     | –     | –    | – | – | – | 0.135 | 0.105 | .203 | –       | –     | –    |
|       | $\phi_8$     | –     | –     | –    | – | – | – | 0.124 | 0.099 | .219 | –       | –     | –    |
| Large | $\phi_0$     | 0.001 | 0.004 | .758 | 0.003 | 0.005 | .526 | 0.002 | 0.002 | .253 | 0.001 | 0.001 | .157 |
|       | $\phi_1$     | 0.353 | 0.083 | <.001 | 0.232 | 0.088 | .011 | 0.139 | 0.243 | .569 | 0.477 | 0.124 | <.001 |
|       | $\phi_2$     | 0.219 | 0.082 | .010 | – | – | – | 0.130 | 0.132 | .329 | -0.233 | 0.080 | .006 |
|       | $\phi_3$     | –     | –     | –    | – | – | – | 0.086 | 0.127 | .501 | –       | –     | –    |
|       | $\phi_4$     | –     | –     | –    | – | – | – | 0.027 | 0.125 | .830 | –       | –     | –    |
|       | $\phi_5$     | –     | –     | –    | – | – | – | 0.040 | 0.101 | .693 | –       | –     | –    |
|       | $\phi_6$     | –     | –     | –    | – | – | – | 0.052 | 0.142 | .715 | –       | –     | –    |
|       | $\phi_7$     | –     | –     | –    | – | – | – | 0.135 | 0.120 | .263 | –       | –     | –    |
|       | $\phi_8$     | –     | –     | –    | – | – | – | 0.124 | 0.099 | .219 | –       | –     | –    |
5.4. Forecasting

With the fitted model for each time series in Section 5.3, we forecast the true fatality rate for the subsequent five days (10–14 January 2021) using the method described in Section 4.2. Specifically, since the true fatality rates are not observable, we ‘estimate’ them using (15) and (16), respectively, for measurement error models (7) and (9), and then we forecast the values of \( \{X_{it} : t = 275, 276, 277, 278, 279\} \) using (17).

To quantify the forecasting performance, we calculate \( P_e^{(h)} \) for \( h = 1, \ldots, H \) for each specified model of the fatality rates \( X_{it} \), and we report the results, together with the total \( \sum_{h=1}^{H} P_e^{(h)} \), in Web Tables 8–10, where \( H \) is set as 5. For \( h = 1, \ldots, H \), we report the observed prediction error \( (X_{T+h} - \hat{X}_{T+h})^2 \), and the expected prediction error defined in (18) and (19).

We apply the proposed method to forecast fatality rates based on the three definitions and report the results in Figures 2–3 and Web Figures 7–13 for the four provinces, where the prediction results after 9 January 2021 are marked in orange and red with solid circles for measurement error models (7) and (9), respectively, together with the prediction areas marked in shaded parts; and the prediction results obtained from the naive method by using (17) are marked in dark blue with hollow triangles. In comparison, as suggested by a referee, we also apply the neural network autoregressive (NNAR) model to do prediction and report the results in light blue with solid triangles; the technical details of the NNAR method are deferred to Appendix A.4 in the Supplementary Material.

![Figure 2](image_url). British Columbia by Definition 1 (AR(4), order-1 differencing): A 5-day forecasting of the true fatality rate (10 January to 15 January 2021) based on the additive (in orange) or multiplicative (in red) versus the naive model (in dark blue marked with hollow triangles) and the NNAR model (in light blue marked with solid triangles); the reported fatality rates (in black), the adjusted true fatality rate accounting for the asymptomatic cases (in green) from 7 December 2020 to 15 January 2020, and the fitted fatality rate (in blue dash line) from 7 December 2020 to 9 January 2020. (See the color display in the electronic version)
Further, we display the reported fatality rate (in black) from 10 April 2020 to 15 January 2021, together with the adjusted fatality rates (in green) and the fitted values from 10 April 2020 to 9 January 2021 (in blue dash line), where fitted values are determined based on (14) with $\phi$ estimated from the proposed methods, and adjusted fatality rates aim to adjust for those unreported asymptomatic cases by multiplying the reported fatality rates with the factor $\frac{1}{1-\tau_A}$ where $\tau_A = 46\%$, as explained for (21). The results corresponding to a mild and a large degree of measurement error are placed as top and bottom subfigures, respectively.

The results for British Columbia are presented in Figure 2 and Web Figure 7. With Definition 1, the naive and the proposed methods produce closer fatality rates over time than the NNAR method does, with the predicted values varying around $[0.010, 0.015]$; the predicted values from the former methods appear flat over time, whereas the values from the NNAR method show a rapidly increasing trend over time. With Definition 2, the naive and the proposed methods indicate that the fatality rates vary around $[0.009, 0.010]$ over time with a slightly downward trend, whereas the predicted values from the NNAR surge up.

The results for Ontario are presented in Figure 3 and Web Figure 8. With Definition 1, the naive and the proposed methods suggest that the fatality rate changes around $[0.014, 0.015]$ over time, a lot smaller than the reported fatality rate which varies around $[0.025, 0.027]$ over time. With Definition 2, the naive and the proposed methods indicate that the fatality rate varies around $[0.015, 0.017]$ over time. In contrast, the fatality rates for both definitions predicted by the NNAR method are larger than 0.020 over time.
The results for Quebec are presented in Web Figures 9–10. With Definition 1, the naive and the proposed methods show that the fatality rate changes around $[0.022, 0.025]$ over time with a slightly upward trend, yet the reported fatality rate varies $[0.040, 0.045]$. With Definition 2, the naive and the proposed methods show that the fatality rates over time are around $[0.022, 0.025]$ with a slight downward trend. In both definitions, the NNAR model suggests the results to be rapidly growing whereas the proposed method suggests a slightly downward trend.

The results for Alberta are presented in Web Figures 11–13. For the data derived from all the three definitions, the naive and the proposed methods suggest that the fatality rates are slightly increasing over time around $[0.006, 0.008]$, whereas the NNAR model suggests increasing fatality rates which can be larger than the reported fatality rates.

5.5. Model assessment

The specification of lag $p$ for model (1) of the true fatality rates $\{X_{it} : t = 1, \ldots, T\}$ is based on (20), which is derived from the reported fatality rates $\{X_{it}^* : t = 1, \ldots, T\}$ but not from $\{X_{it} : t = 1, \ldots, T\}$ itself. This discrepancy introduces the possibility of model misspecification when featuring the series $X_{it}$ using (1). To investigate this, we conduct a sensitivity analysis by considering the AR($p$) for the $X_{it}$ from Definition 1 for all the four provinces to forecast for the period from 9 January 2021 to 14 January 2021, where we set $p$ to be values different from those displayed in Web Table 5.

In Table 3, we report the observed and expected prediction errors of the forecasting using AR(3) models for the British Columbia and Quebec data and AR(1) for the Ontario and Alberta data. The results suggest that the AR($p$) models with $p$ determined by those in Web Table 5 perform better, or at least not worse, than those with $p$ set differently. Both the observed prediction errors and the expected prediction errors associated with the proposed method tend to be less sensitive to the degrees of measurement error but more sensitive to the type of the measurement error model.

6. Discussion

In this article, we investigate the impact of measurement error on time series analysis under autoregressive models and establish analytic results under the additive and multiplicative measurement error models. We propose an estimating equation method to correct for the biases induced from the naive analysis which disregards the differences between the true measurements and their surrogate measurements. We rigorously establish the theoretical results for the proposed method. As a genuine application, we apply the proposed method to analyze the fatality rates of COVID-19 data in four Canadian provinces, British Columbia, Ontario, Quebec, and Alberta, which have the most severe virus outbreaks in Canada.

In the development here, we consider two types of measurement error models to reflect the faulty nature of COVID-19 data. We stress that our intention is not to claim the feasibility of those measurement error models for characterizing the discrepancies between the reported COVID-19 data and the truth. Instead, we intend to raise the awareness that ignoring the error effects in data analysis can yield considerably different results than accommodating the error effects (e.g., [20]). As more information related to the pandemic
Table 3. The observed prediction error and expected prediction error for different lag order of autoregressive models.

| Model       | Method     | $\sigma^2$ (or $\sigma^2_u$) | Day 1 | Day 2 | Day 3 | Day 4 | Day 5 | $\sum_{h=1}^{H} OPE(h)$ | Day 1 | Day 2 | Day 3 | Day 4 | Day 5 | $\sum_{h=1}^{H} EPE(h)$ |
|-------------|------------|-------------------------------|-------|-------|-------|-------|-------|---------------------------|-------|-------|-------|-------|-------|---------------------------|
| British Columbia | AR(3) | Naive             | –     | 0.007 | 0.435 | 0.440 | 0.555 | 0.650 | 2.087                     | 0.006 | 0.006 | 0.006 | 0.006 | 0.006 | 0.029                     |
|             | Additive  | Mild               | 0.014 | 0.533 | 0.298 | 0.326 | 0.327 | 1.498 | 0.005                     | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.027                     |
|             |           | Moderate           | 0.015 | 0.531 | 0.299 | 0.329 | 0.332 | 1.505 | 0.017                     | 0.013 | 0.004 | 0.003 | 0.002 | 0.003 | 0.039                     |
|             | Multiplicative | Mild         | 0.014 | 0.533 | 0.297 | 0.326 | 0.327 | 1.497 | 0.067                     | 0.049 | 0.012 | 0.008 | 0.003 | 0.013 | 0.139                     |
|             |           | Moderate           | 0.015 | 0.531 | 0.298 | 0.328 | 0.330 | 1.502 | 0.067                     | 0.049 | 0.012 | 0.008 | 0.003 | 0.013 | 0.139                     |
| AR(3)       | Naive     | –                 | 0.026 | 0.684 | 0.261 | 0.327 | 0.331 | 1.630 | 0.005                     | 0.006 | 0.006 | 0.006 | 0.006 | 0.006 | 0.028                     |
|             | Additive  | Mild               | 0.037 | 0.799 | 0.168 | 0.183 | 0.142 | 1.330 | 0.005                     | 0.006 | 0.006 | 0.006 | 0.006 | 0.006 | 0.028                     |
|             |           | Moderate           | 0.039 | 0.813 | 0.164 | 0.178 | 0.136 | 1.330 | 0.005                     | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.026                     |
|             | Multiplicative | Mild      | 0.037 | 0.798 | 0.169 | 0.183 | 0.143 | 1.330 | 0.016                     | 0.010 | 0.003 | 0.002 | 0.002 | 0.002 | 0.034                     |
|             |           | Moderate           | 0.038 | 0.808 | 0.166 | 0.180 | 0.138 | 1.330 | 0.063                     | 0.037 | 0.009 | 0.005 | 0.002 | 0.011 | 0.116                     |
| Ontario     | AR(1)    | Naive             | –     | 0.139 | 1.424 | 2.465 | 4.582 | 6.963 | 15.572                    | 0.147 | 0.148 | 0.148 | 0.148 | 0.148 | 0.738                     |
|             | Additive  | Mild               | 0.067 | 0.918 | 1.471 | 2.761 | 4.152 | 9.369 | 0.147                     | 0.148 | 0.148 | 0.148 | 0.148 | 0.148 | 0.737                     |
|             |           | Moderate           | 0.067 | 0.918 | 1.471 | 2.761 | 4.152 | 9.369 | 0.044                     | 0.043 | 0.043 | 0.043 | 0.043 | 0.043 | 0.215                     |
|             | Multiplicative | Mild      | 0.066 | 0.917 | 1.469 | 2.758 | 4.149 | 9.359 | 0.048                     | 0.041 | 0.041 | 0.041 | 0.041 | 0.041 | 0.214                     |
|             |           | Moderate           | 0.065 | 0.879 | 1.414 | 2.676 | 4.049 | 9.084 | 0.146                     | 0.146 | 0.146 | 0.148 | 0.148 | 0.148 | 0.734                     |
| AR(2)a      | Naive     | –                 | 0.128 | 1.329 | 2.323 | 4.378 | 6.709 | 14.866 | 0.146                    | 0.146 | 0.146 | 0.148 | 0.148 | 0.148 | 0.734                     |
|             | Additive  | Mild               | 0.065 | 0.879 | 1.414 | 2.676 | 4.049 | 9.084 | 0.047                     | 0.046 | 0.043 | 0.043 | 0.043 | 0.043 | 0.221                     |
|             |           | Moderate           | 0.065 | 0.878 | 1.413 | 2.676 | 4.047 | 9.079 | 0.061                     | 0.056 | 0.042 | 0.042 | 0.042 | 0.041 | 0.242                     |
|             | Multiplicative | Mild      | 0.065 | 0.876 | 1.410 | 2.671 | 4.041 | 9.062 | 0.061                     | 0.056 | 0.042 | 0.042 | 0.042 | 0.041 | 0.242                     |
| Quebec      | AR(3)    | Naive             | –     | 0.424 | 1.954 | 5.459 | 9.963 | 15.727 | 33.527                    | 0.140 | 0.141 | 0.141 | 0.143 | 0.143 | 0.709                     |
|             | Additive  | Mild               | 0.259 | 1.207 | 3.489 | 6.258 | 9.738 | 20.950 | 0.140                    | 0.141 | 0.141 | 0.143 | 0.143 | 0.143 | 0.709                     |
|             |           | Moderate           | 0.259 | 1.207 | 3.489 | 6.258 | 9.738 | 20.950 | 0.041                    | 0.041 | 0.041 | 0.041 | 0.041 | 0.041 | 0.208                     |
|             | Multiplicative | Mild      | 0.259 | 1.206 | 3.487 | 6.254 | 9.733 | 20.938 | 0.043                    | 0.043 | 0.043 | 0.043 | 0.043 | 0.043 | 0.212                     |
|             |           | Moderate           | 0.217 | 0.994 | 2.928 | 5.192 | 8.201 | 17.533 | 0.139                    | 0.140 | 0.140 | 0.141 | 0.142 | 0.142 | 0.702                     |
| AR(4)a      | Naive     | –                 | 0.355 | 1.609 | 4.566 | 8.235 | 13.218 | 27.982 | 0.139                    | 0.140 | 0.140 | 0.141 | 0.142 | 0.142 | 0.702                     |
|             | Additive  | Mild               | 0.217 | 0.994 | 2.928 | 5.191 | 8.201 | 17.532 | 0.139                    | 0.140 | 0.140 | 0.141 | 0.142 | 0.142 | 0.702                     |
|             |           | Moderate           | 0.217 | 0.994 | 2.928 | 5.191 | 8.200 | 17.529 | 0.041                    | 0.041 | 0.041 | 0.041 | 0.041 | 0.041 | 0.206                     |

(continued).
Table 3. Continued.

| Model | Method | \( \sigma^2_e \) (or \( \sigma^2_u \)) | Observed Prediction Error (x 1000) | Expected Prediction Error |
|-------|--------|---------------------------------|----------------------------------|---------------------------|
|       |        |                                | Day 1  | Day 2  | Day 3  | Day 4  | Day 5  | \( \sum_{h=1}^H \) OPE(\( h \)) | Day 1  | Day 2  | Day 3  | Day 4  | Day 5  | \( \sum_{h=1}^H \) EPE(\( h \)) |
| Alberta | AR(1)  | Naive –                        | 0.000  | 0.005  | 0.174  | 0.389  | 0.630  | 1.198  | 0.013  | 0.013  | 0.013  | 0.013  | 0.013  | 0.064  |
|        |        | Additive Mild                  | 0.002  | 0.031  | 0.332  | 0.699  | 1.123  | 2.186  | 0.013  | 0.013  | 0.013  | 0.013  | 0.013  | 0.063  |
|        |        | Additive Moderate              | 0.002  | 0.031  | 0.332  | 0.699  | 1.123  | 2.186  | 0.004  | 0.004  | 0.004  | 0.004  | 0.004  | 0.018  |
|        |        | Multiplicative Mild           | 0.002  | 0.031  | 0.332  | 0.699  | 1.122  | 2.185  | 0.004  | 0.004  | 0.004  | 0.004  | 0.004  | 0.018  |
|        |        | Multiplicative Moderate        | 0.005  | 0.039  | 0.357  | 0.735  | 1.168  | 2.303  | 0.013  | 0.013  | 0.013  | 0.013  | 0.013  | 0.063  |
|        | AR(2)a | Naive –                        | 0.000  | 0.008  | 0.189  | 0.411  | 0.659  | 1.267  | 0.013  | 0.013  | 0.013  | 0.013  | 0.013  | 0.063  |
|        |        | Additive Mild                  | 0.005  | 0.039  | 0.357  | 0.735  | 1.168  | 2.304  | 0.013  | 0.013  | 0.013  | 0.013  | 0.013  | 0.063  |
|        |        | Additive Moderate              | 0.005  | 0.039  | 0.357  | 0.735  | 1.168  | 2.304  | 0.004  | 0.004  | 0.004  | 0.004  | 0.004  | 0.019  |
|        |        | Multiplicative Mild           | 0.005  | 0.039  | 0.358  | 0.736  | 1.169  | 2.307  | 0.004  | 0.004  | 0.004  | 0.004  | 0.004  | 0.019  |
|        |        | Multiplicative Moderate        | 0.005  | 0.039  | 0.358  | 0.736  | 1.169  | 2.307  | 0.004  | 0.004  | 0.004  | 0.004  | 0.004  | 0.019  |

\( \text{a The selected model.} \)
becomes available, one may consider more flexible measurement error models by including associated covariates such as testing capacity and hospital facilities. As it is difficult or even impossible to obtain the true number of infected cases with COVID-19, conducting sensitivity analyses seems to be the only way for us to gain a good understanding of the trajectory of the pandemic.

Our method has the flexibility in that full distributional assumptions are not required to describe the measurement error process and the time series process. While our development here is directed to using autoregressive models to delineate error-contaminated time series data, the same principles can be applied to other model forms such as moving average models or autoregressive moving average models which may be used to handle error-prone time series data, where technical details can be more notationally involved.

When checking the stationarity of time series, we apply the ADF test to the observed time series \( X_t^* \), which is mainly driven by the unavailability of the true values of \( X_t \), as well as the fact that the weakly stationarity of observed time series implies the weakly stationarity of the true time series if measurement error is featured with (7) or (9). It is interesting to rigorously develop a formal test similar to the ADF test to handle time series subject to measurement error.

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No potential conflict of interest was reported by the author(s).

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**Data availability**

All codes and data involved in this paper are available on Github (https://github.com/QihuangZhang/COVID-AR-Error). Preprint available at Zhang and Yi [21].

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