Functional Meyer-Tanaka Formula

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Abstract

The functional Itô formula, firstly introduced in Dupire (2009) for continuous semimartingales, might be extended in two directions: different dynamics for the underlying process and/or weaker assumptions on the regularity of the functional. In this paper, we pursue the former type by proving the functional version of the Meyer-Tanaka Formula for the class of convex functionals. Following the idea of the proof of the classical Meyer-Tanaka formula, we study the mollification of functionals and its convergence properties. As an example, we apply the theory to the running maximum functional.

1 Introduction

Our goal in this article is to prove the functional extension of the well-known Meyer-Tanaka formula. The theory of functional Itô calculus was presented in the seminal paper Dupire (2009) and it was further developed and applied to diverse topics in, for instance, Ekren et al. (2012a,b), Peng.
and Wang (2011); Ma et al. (2012); Siu (2012); Ji et al. (2013); Xu (2013); Ji and Yang (2013); Jazaerli and Saporito (2013); Cont and Fournié (2013, 2010b, a) and Oberhauser (2012). Before proceeding, a remark regarding nomenclature. In this text, the adjective classical will always refer to the finite-dimensional Itô stochastic calculus.

The Meyer-Tanaka formula is the extension of Itô formula to convex functions. More precisely, in the classical case, if \( f : \mathbb{R} \to \mathbb{R} \) is convex and \((x_t)_{t \geq 0}\) is a continuous semimartingale, then

\[
f(x_t) = f(x_0) + \int_0^t f'(x_s) \, ds + \int_{\mathbb{R}} L^x(t, y) \, df'(y),
\]

where \( f' \) is the left-derivative of \( f \) and \( L^x(t, y) \) is the local time of the process \( x \) at \( y \); see Karatzas and Shreve (1988), for example. This formula is easily generalized to functions \( f \) that are absolutely continuous with derivative of bounded variation, which is equivalent to say that \( f \) is the difference of two convex functions. We would like to remind the reader that the local time here is given by the limit in probability:

\[
L^x(t, y) = \lim_{\varepsilon \to 0^+} \frac{1}{4\varepsilon} \int_0^t 1_{[y-\varepsilon, y+\varepsilon]}(x_s) \, d\langle x \rangle_s,
\]

where \( \langle x \rangle \) is the quadratic variation of the process \( x \). We are adhering the convention \( 4\varepsilon \) instead of \( 2\varepsilon \). For \( y \in \mathbb{R} \) fixed, the process \((L^x(t, y, \omega))_{t \geq 0}\) is a.s continuous and increasing in \( t \) and càdlàg in \( y \). The following extension to time-dependent functions was studied in Elworthy et al. (2007):

\[
f(t, x_t) = f(0, x_0) + \int_0^t f_t(s, x_s) \, ds + \int_0^t f_x(s, x_s) \, dx_s + \int_{\mathbb{R}} L^x(t, y) \, df_x(t, y) - \int_{\mathbb{R}} \int_0^t L^x(s, y) \, ds \, df_x(s, y),
\]

where \( f_t \) and \( f_x \) are the time and space left-derivatives, respectively. It is assumed that \( f \) is left-continuous and locally bounded, \( f_t \) is left-continuous and \( f_x \) is left-continuous and of locally finite variation in \( \mathbb{R}_+ \times \mathbb{R} \). We forward the reader to the reference cited above for some other different generalizations of Meyer-Tanaka formula (1.1) and for the precise definition of the Lebesgue-Stieltjes integral \( \int_{\mathbb{R}} \int_0^t L_y^x(s, t, \omega) \, ds \, df_x(s, t, \omega) \).

Since a functional extension of the Meyer-Tanaka would be inherently time-dependent, Equation (1.2) is of utmost importance for our goal. However, we will not pursue a functional extension of (1.2) in its full generality of assumptions. It is clear that many assumptions of the results presented in our work could be weakened along the lines of Elworthy et al. (2007), but in order to provide a clear exposition of the subject we will consider assumptions that are general enough to introduce the important techniques without adding a cumbersome notation.

There are several other generalizations of the Itô formula that could be extended to the functional framework. For instance, Al-Hussaini and Elliott (1987); Peskir (2005); Lowther (2010); Ghomrasni and Peskir (2003); Elworthy et al. (2007); Russo and Vallois (1996); Föllmer et al. (1995); Feng and Zhao (2007) and Carlen and Protter (1992). We will not pursue
them here, of course, but we hope that the foundations laid in this work might help in this task.

Meyer-Tanaka formula and its generalizations have many interesting applications in Finance, as, for instance, Mijatović (2010); Duffie and Harrison (1993) and J. Detemple and Tian (2003). Other applications can be found in the theory of Local Volatility of Dupire (1994), see for example Klebaner (2002) and Musiela and Rutkowski (2008).

The paper is organized as follows: we finish this introduction with a presentation of functional Itô calculus and we define the mollification of functionals in Section 2. This is a very important tool that will be used in Section 3 in order to prove the functional extension of the Meyer-Tanaka formula. We will apply the theory to the running maximum example throughout our exposition.

1.1 A Brief Primer on Functional Itô Calculus

In this section we will present a short primer of the functional Itô calculus introduced in Dupire (2009). The goal is to familiarize the reader with the notation, main definitions and theorems needed for the results that follow.

The space of càdlàg paths in $[0,t]$ will be denoted by $\Lambda_t$. For a fixed time horizon $T > 0$, we define the space of paths as

$$\Lambda = \bigcup_{t \in [0,T]} \Lambda_t.$$  

We will denote elements of $\Lambda$ by upper case letters and often the final time of its domain will be subscripted, e.g. $Y \in \Lambda_t \subset \Lambda$ will be denoted by $Y_t$. The value of $Y_t$ at a specific time will be denoted by lower case letters:

$$y_s = Y_t(s),$$  

for any $s \leq t$. Moreover, if a path $Y_t$ is fixed, the path $Y_s$, for $s \leq t$, will denote the restriction of the path $Y_t$ to the interval $[0,s]$.

The following important path deformations are always defined in $\Lambda$. For $Y_t \in \Lambda$ and $t \leq s \leq T$, the flat extension of $Y_t$ up to time $s \geq t$ is defined as

$$Y_{t,s-t}(u) = \begin{cases} 
  y_u, & \text{if } 0 \leq u \leq t, \\
  y_t, & \text{if } t \leq u \leq s,
\end{cases}$$  

see Figure 1. For $h \in \mathbb{R}$, the bumped path, see Figure 2, is defined by

$$Y_{t}^h(u) = \begin{cases} 
  y_u, & \text{if } 0 \leq u < t, \\
  y_t + h, & \text{if } u = t.
\end{cases}$$  

For any $Y_t, Z_s \in \Lambda$, where it is assumed without loss of generality that $s \geq t$, we define the following metric in $\Lambda$,

$$d_\Lambda(Y_t, Z_s) = \|Y_{t,s-t} - Z_s\|_\infty + |s-t|,$$

where

$$\|Y_t\|_\infty = \sup_{u \in [0,t]} |y_u|.$$  

One could easily show that $(\Lambda, d_\Lambda)$ is a complete metric space.
Additionally, a functional is any function $f : \Lambda \rightarrow \mathbb{R}$. Continuity with respect to $d\Lambda$ is defined as the usual definition of continuity in a metric space and is denominated $\Lambda$-continuity.

For a functional $f$ and a path $Y_t$ with $t < T$, the time functional derivative of $f$ at $Y_t$ is defined as

$$\Delta_t f(Y_t) = \lim_{\delta t \rightarrow 0^+} \frac{f(Y_{t+\delta t}) - f(Y_t)}{\delta t},$$

whenever this limit exists. The space functional derivative of $f$ at $Y_t$ is defined as, if the following limit exists,

$$\Delta_x f(Y_t) = \lim_{h \rightarrow 0} \frac{f(Y_{t+h}) - f(Y_t)}{h}.$$ (1.4)

In this case, $t = T$ is allowed.

Finally, for any $i, j \in \{0\} \cup \mathbb{N} \cup \{+\infty\}$, a functional $f : \Lambda \rightarrow \mathbb{R}$ is said to belong to $C^{i,j}$ if it is $\Lambda$-continuous and has $\Lambda$-continuous derivatives $\Delta_t^k f$ and $\Delta_x^m f$, for $k = 1, \ldots, i$ and $m = 1, \ldots, j$. Here, clearly, $\Delta_t^k = \Delta_t (\Delta_t^{k-1})$ and $\Delta_x^m = \Delta_x (\Delta_x^{m-1})$. Moreover, we use the notation $\Delta_{xx} = \Delta_x^2$.

We state now the functional Itô formula. The proof can be found in Dupire (2009).

**Theorem 1.1** (Functional Itô Formula; Dupire (2009)). Let $x$ be a continuous semimartingale and $f \in C^{1,2}$. Then, for any $t \in [0, T]$,

$$f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) ds + \int_0^t \Delta_x f(X_s) dx_s + \frac{1}{2} \int_0^t \Delta_{xx} f(X_s) d\langle x \rangle_s.$$ (1.5)

One should notice that the Itô formula above is of the same form as the classical Itô formula for continuous semimartingale, the only change being the definition of the time and space functional derivatives given by Equations (1.3) and (1.4). This theorem was extended in terms of weakening the regularity of $f$ and generalizing the dynamics of $x$, see Cont and Fournié (2013, 2010b,a) and Oberhauser (2012). Here, we will examine a different class of functionals than it was consider in these previous works. Namely, we will consider the class of convex functionals as defined in Definition 3.2 The main result of this article is the next theorem:
Theorem 1.2 (Functional Meyer-Tanaka Formula). Consider a functional \( f : \Lambda \rightarrow \mathbb{R} \) satisfying Hypotheses [3.3]. Then

\[
\begin{align*}
(1.5) \quad f(X_t) &= f(X_0) + \int_0^t \Delta_t f(X_s) \, ds + \int_0^t \Delta_s f(X_s) \, dx_s \\
&\quad + \int_{\mathbb{R}} L^x(t, y) d_y \partial_y^- \mathcal{F}(X_t, y) - \int_0^t \int_{\mathbb{R}} L^x(s, y) d_y \partial_y^- \mathcal{F}(X_s, y)
\end{align*}
\]

See Equation (3.4) for the precise definition of \( \mathcal{F} \).

2 Functional Mollification

In this section, we investigate the mollification of functionals. The goal is to create a sequence of very smooth functionals converging to the original one in various senses. This technique will be used to prove the functional Meyer-Tanaka formula as it is similarly used in the proof of its classical version.

The main example of non-smooth functional to think of is the running maximum:

\[
(2.1) \quad m(Y_t) = \sup_{0 \leq s \leq t} y_s.
\]

Let us first verify that this is a \( \Lambda \)-continuous functional. Notice \( m(Y_t) = m(Y_t, r) \), for any \( Y_t \in \Lambda \) and \( r \geq 0 \). Hence, if we fix \( Y_t, Z_s \in \Lambda \) with \( s \geq t \), we find

\[
|m(Y_t) - m(Z_s)| = |m(Y_{t,s-t}) - m(Z_s)| = \sup_{0 \leq u \leq s} |Y_{t,s-t}(u) - Z_s(u)| \leq d_\Lambda(Y_t, Z_s).
\]

Therefore, the running maximum is (Lipschitz) \( \Lambda \)-continuous. Moreover, one could also verify that \( \Delta_t m(Y_t) = 0 \). Define now the subset of \( \Lambda \) where the supremum is attained at the last value:

\[
\mathcal{S} = \{ Y_t \in \Lambda : m(Y_t) = y_t \}.
\]

For paths in \( \mathcal{S} \), the space functional derivative is not defined: the right space functional derivative is 1 and the left space functional derivative is 0, where these one-sided derivatives are obviously defined as

\[
\Delta^\pm_x f(Y_t) = \lim_{h \to 0^\pm} \frac{f(Y_t^h) - f(Y_t)}{h}.
\]

For paths outside \( \mathcal{S} \), the space functional derivative is well-defined and it is 0: \( \Delta_x f(Y_t) = 0 \), for \( Y_t \notin \mathcal{S} \). Therefore, with this example in mind we proceed to study the mollification of functionals.

Consider a functional \( f : \Lambda \rightarrow \mathbb{R} \) and define \( F : \Lambda \times \mathbb{R} \rightarrow \mathbb{R} \) as

\[
(2.2) \quad F(Y_t, h) = f(Y_t^h).
\]
When denoting functionals, capital letters will be used as above, i.e. it will denote a function with domain $\Lambda \times \mathbb{R}$ where the first variable is the path and the second variable is the bump applied to this path. This notation will be carried out in the remainder of the paper. We choose to use this notation to help the analysis of the space functional derivative of the mollification.

A mollifier in $\mathbb{R}$ is a function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ such that $\rho \in C^\infty_c(\mathbb{R})$, the space of compactly supported smooth functions; $\int_{\mathbb{R}} \rho(z)dz = 1$; and $\rho_n(x) := n\rho(nx)$ converges to Dirac delta, $\delta(x)$, in the sense of distributions. We also refer to the sequence $(\rho_n)_{n \in \mathbb{N}}$ as the mollifiers. Notice that $\rho_n \in C^\infty_c(\mathbb{R})$.

Remark 2.1. The mollifier will be taken as follows:

$$\rho(z) = c \exp \left\{ \frac{1}{(z-1)^2 - 1} \right\} 1_{[0,2]}(z),$$

where $c$ is chosen in order to have $\int_{\mathbb{R}} \rho(z)dz = 1$.

We thus define the sequence of mollified functionals as

$$F_n(Y_t, h) = \int_{\mathbb{R}} \rho_n(h-\xi)F(Y_t, \xi)d\xi = \int_{\mathbb{R}} \rho_n(\xi)F(Y_t, h-\xi)d\xi. \quad (2.3)$$

This mollification is well-defined as long as the real function $F(Y_t, \cdot)$ is locally integrable for any path $Y_t \in \Lambda$. See [Evans (2010)] for instance, for details on the mollification in the case of real functions.

Notice that if the functional $f$ is $\Lambda$-continuous, $F(Y_t, \cdot)$ is then continuous for fixed $Y_t \in \Lambda$, because $d_\Lambda(Y_t^h, Y_t^{h_2}) = |h_1 - h_2|$. This implies $F(Y_t, \cdot)$ is locally integrable, and therefore the mollification $F_n$ is well-defined when $f$ is $\Lambda$-continuous.

Notice now that $F(Y_t^z, h) = F(Y_t, h + z)$ and then

$$F_n(Y_t^z, h) = \int_{\mathbb{R}} \rho_n(h-\xi)F(Y_t, \xi + z)d\xi = \int_{\mathbb{R}} \rho_n(h-(\xi - z))F(Y_t, \xi)d\xi = F_n(Y_t, h + z). \quad (2.4)$$

Thus, for any $k \in \mathbb{N}$,

$$\Delta_x^{(k)} F_n(Y_t, h) = \partial_h^{(k)} F_n(Y_t, h),$$

where $\partial_h^{(k)}$ denotes the $k$-th derivative with respect to the $h$ variable and $\Delta_x^{(k)}$ is the $k$-th composition of $\Delta_x$. This is the main property of the mollified functionals. Therefore, $F_n$, as a functional, is infinitely differentiable with respect to the space variable.

We would like also to point it out that a particular mollification of the running maximum was considered in [Dupire (2009)] to derive a pathwise version of the famous formula due to Lévy:

$$\max_{0 \leq s \leq t} x_s = x_0 + L^{x-m}_t (t, 0),$$

where $m$ is the running maximum process. The reader is forwarded to [Karatzas and Shreve (1988) Section 6.3.C] for more details on these results in the Brownian motion case.

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2.1 Continuity of the Mollified Functionals and its Derivatives

In this section we will study the relation of continuity of $f$ and continuity of the mollification $F_n$.

We already saw that, if $F(Y_t, \cdot)$ is locally integrable for any given $Y_t \in \Lambda$, then $F_n(Y_t, \cdot)$ is infinitely differentiable in $\mathbb{R}$, and therefore it is also continuous. However, differentiability in the functional sense does not imply $\Lambda$-continuity. Hence, it is necessary to consider a slightly stronger assumption on the continuity of the functional $f$ in order to be able to conclude the $\Lambda$-continuity of $F_n$. We will thus assume throughout that:

**Assumption 2.2** (Continuity on $f$). There exists $\phi: \mathbb{R} \rightarrow \mathbb{R}$ positive and locally bounded depending only on $f$ such that

$$\forall \varepsilon > 0, \forall Y_t \in \Lambda, \exists \delta > 0, d_\Lambda(Y_t, Z_s) < \delta \Rightarrow |F(Y_t, \xi) - F(Z_s, \xi)| < \varepsilon \phi(\xi), \forall \xi \in \mathbb{R}. \tag{2.5}$$

Notice that Assumption 2.2 implies that $f$ is $\Lambda$-continuous. Moreover, if $\phi \equiv 1$, then the family of functionals $\{F(\cdot, \xi)\}_{\xi \in \mathbb{R}}$ is $\Lambda$-equicontinuous.

The weakening of this assumption could be pursued, but it is not in the scope of this work.

By Equation (2.3), we see

$$|F_n(Y_t, h) - F_n(Z_s, h)| \leq \int_{\mathbb{R}} \rho_n(h - \xi)|F(Y_t, \xi) - F(Z_s, \xi)|d\xi.$$

Hence, fixing $\varepsilon > 0$, $n \in \mathbb{N}$ and $h \in \mathbb{R}$, and choosing $\delta > 0$ from the continuity assumption on $f$ with $\varepsilon$ equals

$$\int_{\mathbb{R}} \rho_n(h - \xi)\phi(\xi)d\xi,$$

we have, for $Y_t, Z_s \in \Lambda$ satisfying $d_\Lambda(Y_t, Z_s) < \delta$,

$$|F_n(Y_t, h) - F_n(Z_s, h)| \leq \int_{\mathbb{R}} \rho_n(h - \xi)|F(Y_t, \xi) - F(Z_s, \xi)|d\xi < \varepsilon.$$

Therefore, assuming that $f$ satisfies Assumption 2.2 we conclude that $F_n(\cdot, h)$ is $\Lambda$-continuous for any $n \in \mathbb{N}$ and $h \in \mathbb{R}$. Considering now the derivatives of $F_n$, we see

$$\Delta^{(k)}_h F_n(Y_t, h) = \partial^{(k)}_h F_n(Y_t, h) = \int_{\mathbb{R}} \partial^{(k)}_h (\rho_n(h - \xi))F(Y_t, \xi)d\xi,$$

and since $\partial^{(k)}_h (\rho_n(h - \cdot))$ are in $C^\infty(\mathbb{R})$, the same argument employed above for the $\Lambda$-continuity of $F_n$ can be used to conclude the $\Lambda$-continuity of $\partial^{(k)}_h F_n(\cdot, h)$.

We have thus proved the following result:

**Proposition 2.3.** Suppose $f$ satisfies Assumption 2.2. Then, for any $n \in \mathbb{N}$ and $h \in \mathbb{R}$, $F_n(\cdot, h)$ and $\Delta^{(k)}_h F_n(\cdot, h)$ are $\Lambda$-continuous, for any $k \in \mathbb{N}$.

We now show below that the running maximum verifies Assumption 2.2 with $\phi \equiv 1$. 

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Example 2.4 (Running Maximum). The running maximum is defined by Equation (2.1). One can easily show

\[ M(Y_t, \xi) = m(Y_t^\xi) = \max \{m(Y_t), y_t + \xi^+\} \]

Moreover, for any \(Y_t, Z_s \in \Lambda\),

\[ |M(Y_t, \xi) - M(Z_s, \xi)| \leq |m(Y_t) - m(Z_s)| \cdot 1_{(m(Y_t) \geq y_t + \xi^+, m(Z_s) \geq z_s + \xi^+)} + |m(Y_t) - (z_s + \xi^+)| \cdot 1_{(m(Y_t) \geq y_t + \xi^+, m(Z_s) < z_s + \xi^+)} + |(y_t + \xi^+) - m(Z_s)| \cdot 1_{(m(Y_t) < y_t + \xi^+, m(Z_s) \geq z_s + \xi^+)} + |y_t - z_s| \cdot 1_{(m(Y_t) < y_t + \xi^+, m(Z_s) < z_s + \xi^+)} \]

Analyzing the terms above, one can deduce

\[ |M(Y_t, \xi) - M(Z_s, \xi)| \leq \max\{|m(Y_t) - m(Z_s)|, d_\Lambda(Y_t, Z_s)\}. \]

Since the right-hand side bound is independent of \(\xi\), the running maximum satisfies Assumption 2.2 with \(\varphi \equiv 1\).

2.2 The Issue with the Time Derivative

As we have seen, the functional \(F_n\) is smooth with respect to the space variable. In this section, we will study the question of the existence of the time functional derivative. Notice that

\[ F_n(Y_t, h) = \int_{\mathbb{R}} \rho_n(h - \xi)F(Y_t, \xi) d\xi \]

When is \(F_n\) time differentiable? We should clearly assume that \(\Delta_t f\) exists everywhere in \(\Lambda\). Moreover, we observe

\[ F(Y_t, \xi) = f((Y_t)_{t, \cdot}) - f((Y_t^\xi)_{t, \cdot}) + f((Y_t^\xi)_{t, \cdot}) - f(Y_t^\xi). \]

So, if we additionally assume

\[ \lim_{\delta t \to 0^+} \frac{f((Y_t, \delta t)_{t, \cdot}) - f((Y_t^\xi)_{t, \cdot})}{\delta t} = 0, \quad (2.6) \]

for all \(\xi \in \mathbb{R}\), we conclude, under mild integrability assumptions, that

\[ \Delta_t F_n(Y_t, h) = \int_{\mathbb{R}} \rho_n(h - \xi)\Delta_t F(Y_t, \xi) d\xi = (\Delta_t F)_n(Y_t, h). \]

The equation above should be understood as: \(\Delta_t F_n(Y_t, h)\) is the mollification of \(\Delta_t F(Y_t, \cdot)\) at \(h\).

In general, assumption (2.6) is not true. Indeed, the running integral functional, \(f(Y_t) = \int_0^t y_u du\), does not satisfy (2.6) for \(\xi \neq 0\). However, if we define

\[ g(Y_t, \xi) = \lim_{\delta t \to 0^+} \frac{f((Y_t, \delta t)_{t, \cdot}) - f((Y_t^\xi)_{t, \cdot})}{\delta t}, \quad (2.7) \]
where we are assuming that the limit above exists for all \( Y_t \in \Lambda \) and \( \xi \in \mathbb{R} \), we conclude

\[
\Delta_tF_n(Y_t, h) = (\Delta_tF)_n(Y_t, h) + \int_{\mathbb{R}} \rho_n(h - \xi)g(Y_t, \xi)d\xi.
\]

If \( g(Y_t, \cdot) \) is a continuous real function for any fixed \( Y_t \in \Lambda \), the second term of the right-hand side of the equation above converges to \( g(Y_t, h) \) when \( n \to +\infty \). Therefore, since \( g(Y_t, 0) = 0 \), we will find that this term has no contribution on convergence computations presented in Section 3.4. Hence, assumption (2.6) serves more to simplify the computations than as an additional restriction to the existence and continuity assumptions on \( g \).

Example 2.5 (Running Maximum). Notice that

\[
m((Y_t, \delta t)_{\xi}) = \max \{ m(Y_t), y_t + \xi \} = m((Y_t^\xi)_t, \delta t),
\]

and therefore, \( m \) satisfies Equation (2.6). Furthermore, this shows that the running maximum is weakly path-dependent, i.e., the Lie bracket is zero (in the limit characterization), see (Jazaerli and Saporito, 2013, Lemma 3.2).

2.2.1 Time and Joint Mollification

We will not pursue this here, but it is important to mention two different mollification techniques:

(i) Time Mollification:

\[
F_n(Y_t, \delta t) = \int_{\mathbb{R}} \rho_n(\delta t - \eta)f(Y_t, \eta)d\eta.
\]

(ii) Joint Mollification:

\[
F_n(Y_t, \delta t, h) = \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_n(h - \xi)\rho_n(\delta t - \eta)f((Y_t^\xi)_t, \eta)d\xi d\eta.
\]

An obvious issue with the joint mollification is the choice between \( f((Y_t^\xi)_t, \eta) \) and \( f((Y_t^\xi)_t, \eta) \); both would be initially valid choices. This is not a problem when we restrict ourselves to the path-independent case: \( f(Y_t) = h(t, y_t) \). However, as it was noted in (Jazaerli and Saporito, 2013), the different ordering of bump and flat extension is a very important characteristic of the functional calculus. Therefore, in order to focus on the main ideas we will not pursue these mollification techniques in the present work and hence we assume the existence and \( \Lambda \)-continuity of the functional time derivative of the original functional \( f \).

Additionally, as it happened in the aforesaid paper in a different circumstance, the Lie bracket of the operators \( \Delta_t \) and \( \Delta_x \) would probably play an important role when the joint mollification is used. We would like also to point out the similarity of the condition (2.6) and the limit characterization of the Lie bracket given in (Jazaerli and Saporito, 2013, Lemma 3.2).
3 Functional Meyer-Tanaka Formula

3.1 Local Time

We start by fixing a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a continuous semi-martingale \(x\) in this space. If we denote the local time of the process \(x\) at level \(y\) by \(L^x(s, y)\), it is known that, for any bounded measurable function \(\psi: \mathbb{R}_+ \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}\),

\[
\int_0^t \psi(s, \omega, x_s) d\langle x \rangle_s = 2 \int_{\mathbb{R}} \left( \int_0^t \psi(s, \omega, y) L^x(ds, y) \right) dy, \quad t \geq 0 \text{ a.s. (3.1)}
\]

This is called the occupation times formula, see [Revuz and Yor 2004, Exercise 1.15, Chapter VI]. Let us prove this formula. It is known that if \(\varphi: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}\) is bounded and measurable, then

\[
\int_0^t \varphi(s, x_s) d\langle x \rangle_s = 2 \int_{\mathbb{R}} \left( \int_0^t \varphi(s, y) L^x(\omega)(ds, y) \right) dy, \quad t \geq 0 \text{ a.s. ,}
\]

which means there exists \(\Omega_0 \in \mathcal{F}\) with \(\mathbb{P}(\Omega_0) = 1\) such that, for each \(\omega \in \Omega_0\),

\[
\int_0^t \varphi(s, x_s(\omega)) d\langle x(\omega) \rangle_s = 2 \int_{\mathbb{R}} \left( \int_0^t \varphi(s, y) L^x(\omega)(ds, y) \right) dy, \quad t \geq 0,
\]

where \(\langle x(\omega) \rangle\) and \(L^x(\omega)\) are the realizations of the quadratic variation and the local time, respectively. Hence, since \(\psi(\cdot, \omega, \cdot)\) is bounded and measurable

\[
\int_0^t \psi(s, \omega, x_s(\omega)) d\langle x(\omega) \rangle_s = 2 \int_{\mathbb{R}} \left( \int_0^t \psi(s, \omega, y) L^x(\omega)(ds, y) \right) dy, \quad t \geq 0,
\]

yielding the desired result.

We will denote the path realization of the semimartingale \(x\) over the time interval \([0, t]\) by \(X(\omega)_t \in \Lambda_t\). For a given functional \(f\), we define \(\psi_f(s, \omega, y) = f(X(\omega)_{y_u})\), where, for any \(Y_t \in \Lambda\) and \(y \in \mathbb{R}\),

\[
Y^y_{t_u}(u) = \begin{cases} 
  y_u, & \text{if } 0 \leq u < t, \\
  y, & \text{if } u = t.
\end{cases}
\]

(3.2)

Notice that \(Y^y_{t_u} = Y^y_{t_u-y_t} \in \Lambda_t\). Therefore, for every functional \(f\) such that \(\psi_f\) above is bounded and measurable, we have

\[
\int_0^t f(X_s) d\langle x \rangle_s = 2 \int_{\mathbb{R}} \left( \int_0^t F(X_s, y) L^x(ds, y) \right) dy,
\]

with

\[
F(Y_s, y) = f(Y^y_{t_u}).
\]

The definition of the function \(F\) above serves two purposes. Firstly, alleviates notation. Secondly, it helps us take derivatives with respect to the \(Y_s\) and the last value \(y\) separately. Capital calligraphic letters will always be used as above meaning that it will denote a function with domain \(\Lambda \times \mathbb{R}\) where the first variable is the path and the second variable is the value to will replace the last value of the path. We will keep this notation through out the paper.
Example 3.1 (Running Integral). Consider the running integral functional $f(Y_t) = \int_0^t y_u\,du$. We clearly have $F(Y_s, y) = f(Y_s)$ and moreover, we find

- $\int_0^t f(X_s)\,d\langle x \rangle_s = \int_0^t \left(\int_0^s y_u\,du\right)\,d\langle x \rangle_s = \int_0^t (\langle x \rangle_t - \langle x \rangle_u) x_u\,du$.

- $\int_0^t \left(\int_0^t F(X_s, y)L^s(ds, y)\right)\,dy = \int_0^t \left(\int_0^s y_u\,du\right) L^s(ds, y)\,dy = \int_0^t \left(\int_0^s L^s(t, y)\,dy - \int_0^s L^s(u, y)\,dy\right) x_u\,du$.

Therefore, since $\langle x \rangle_t = 2\int_0^t L^s(t, y)\,dy$, we verify Equation (3.3) for this particular example.

3.2 Convex Functionals

In this section we define the notion of convexity for functionals and then discuss some of its basic properties.

Definition 3.2 (Convex Functionals). We say $f$ is a convex functional if $F(Y_t, \cdot)$ is a convex real function for any $Y_t \in \Lambda$.

Notice that, for $f \in C^{1,2}$, convexity implies that $\Delta_{xx} f(Y_t) \geq 0$, for any $Y_t \in \Lambda$.

Remark 3.3. Another possible definition for convexity of a functional would be

$$f(\lambda Y_t + (1 - \lambda) Z_t) \leq \lambda f(Y_t) + (1 - \lambda) f(Z_t),$$

for all $\lambda \in [0,1]$ and $Y_t, Z_t \in \Lambda$. Observe $Y_t$ and $Z_t$ must be in the same $\Lambda_t$ space. This clearly implies the previous definition of convexity because

$$F(Y_t, \lambda h_1 + (1 - \lambda) h_2) = f(\lambda Y_t h_1 + (1 - \lambda) Y_t h_2).$$

However, condition (3.5) is stronger than necessary for what follows.

Example 3.4 (Running Maximum). An example of a (non-smooth) convex functional is the running maximum $m(Y_t)$. It is actually convex in the stronger sense of (3.5).

For a convex functional $f$, for any $Y_t \in \Lambda$, $F(Y_t, \cdot)$ is continuous, $\partial_{h}^\pm F(Y_t, h)$ exist for any $h \in \mathbb{R}$ and is non-decreasing in $h$. Moreover, $\partial_{h}^\pm F(Y_t, h) = \Delta_{x}^\pm F(Y_t, h)$.

3.3 Integration by Parts

We will now derive some integration by parts computations that will be useful later.
We start by noticing that the following identity is obviously true:

\[
\int_{\mathbb{R}} \left( \int_{0}^{t} F(X_{s}, y) L^{x}(ds, y) \right) dy = \int_{\mathbb{R}} F(X_{t}, y) L^{x}(t, y) dy
- \int_{\mathbb{R}} \left( \int_{0}^{t} \partial_{t} F(X_{s}, y) L^{x}(s, y) ds \right) dy,
\]

where \( F \) is given by Equation (3.4) and

\[
\partial_{t} F(Y_{s}, y) = \lim_{u \to s} \frac{F(Y_{s}, y) - F(Y_{u}, y)}{s - u},
\]

the usual time derivative of a function. Let us now verify that this derivative exists under certain regularity assumptions. Notice that \( F(Y_{s}, y) \) does not depend on the last value of the path \( Y_{s} \), and hence \( \Delta_{x} F(Y_{s}, y) = 0 \).

So, assuming that (2.6) is true and that \( \Delta_{t} f \) exists, \( \Delta_{t} F(X_{s}, y) \) also exists. Thus, \( F(\cdot, y) \in C^{1,2} \) and one can show by Theorem 1.1

\[
\partial_{t} F(X_{s}, y) = \Delta_{t} F(X_{s}, y).
\]

Notice that, similarly to what was argued in Section 2.2, \( \partial_{t} F(X_{s}, y) \) exists even if we do not assume (2.6). Moreover,

\[
(3.6) \quad F(Y_{t}, y + h) = F(Y_{t-h}, h) = f(Y_{t-h})
\]

and then

\[
\partial_{y}^{(k)} F(Y_{t}, y) = \partial_{y}^{(k)} F(Y_{t-i}, 0) = \Delta_{x}^{(k)} f(Y_{t-i}).
\]

Before proceeding, we would like to comment on the commutation issue of \( \partial_{t} \) and \( \partial_{y} \). It is well-known now that \( \Delta_{t} \) and \( \Delta_{x} \) do not commute and that the class of functionals where they do actually commute plays an important role in the functional Itô calculus theory, see Jazaerli and Saporito (2013). However, we do not experience a similar problem here. \( \partial_{t} \) and \( \partial_{y} \) do commute: \( \partial_{y} \partial_{t} F(Y_{t}, y) = \partial_{t} \partial_{y} F(Y_{t}, y) \), as one can easily verify by direct computation. The reason is that in the definition of \( F(Y_{t}, y) \) it is implied that the “bump” \( y \) happens always at the end of the path \( Y_{t} \). Therefore, there is no ambiguity in the order of the time perturbation and the bump that we experience in the case of \( \Delta_{x} \) and \( \Delta_{t} \).

For \( g \in C^{1}(\mathbb{R}) \) and \( f \in C^{1,2} \),

\[
(3.7) \quad \int_{\mathbb{R}} g(y) \Delta_{x} f(X_{t-y}) dy = \int_{\mathbb{R}} g(y) \partial_{yy} F(X_{t}, y) dy
= - \int_{\mathbb{R}} g'(y) \partial_{y} F(X_{t}, y) dy
\]
Furthermore, if we consider $q(s, y)$ smooth with compact support, we find

$$
\int_0^t \int_{\mathbb{R}} \partial_t \Delta_{xx} f(X^y_s) q(s, y) dy ds = \int_0^t \int_{\mathbb{R}} \partial_y \partial_y f(X^y_s) q(s, y) dy ds
$$

$$
= \int_0^t \int_{\mathbb{R}} \partial_t \partial_y \partial_y F(X_s, y) q(s, y) dy ds
$$

$$
= - \int_0^t \int_{\mathbb{R}} \partial_t \partial_y F(X_s, y) \partial_y q(s, y) dy ds
$$

$$
= - \int_{\mathbb{R}} \partial_y F(X_s, y) \partial_y q(s, y) \bigg|_0^t dy + \int_0^t \int_{\mathbb{R}} \partial_y F(X_s, y) \partial_y q(s, y) dy ds,
$$

where

$$
q(s, y) \bigg|_0^t = q(t, y) - q(0, y).
$$

### 3.4 Convergence Properties

The idea behind the proof of the classical Meyer-Tanaka formula (see [Karatzas and Shreve 1988](#), for example) is to apply Itô formula to the smooth mollification of the convex function in consideration, let $n$ go to infinity to approximate the original function and then analyze the limit of all the terms of the Itô formula. Having this strategy in mind, in this section we will investigate the convergence of certain quantities that will be important when proving the functional Meyer-Tanaka formula.

We firstly define the functional $f_n : \Lambda \rightarrow \mathbb{R}$ as

$$
(3.8) \quad f_n(Y_t) = F_n(Y_t, 0) = \int_{\mathbb{R}} \rho_n(\xi) F(Y_t, -\xi),
$$

where $F_n$ mollification of $F$ given in Equation (2.3). It is important to notice that, by Equation (2.4), $f_n(Y_t^h) = F_n(Y_t, h)$. Notice then the following facts:

1. If $f$ is convex, then $f_n$ is still convex. Indeed,

$$
F_n(Y_t, (1 - \lambda)h_1) = \int_{\mathbb{R}} F(Y_t, (1 - \lambda)h_1, (1 - \lambda)h_2, y) \rho_n(y) dy
$$

$$
= \int_{\mathbb{R}} F(Y_t, (1 - \lambda)h_1, (1 - \lambda)h_2, y, (1 - \lambda)h_2, y) \rho_n(y) dy
$$

$$
\leq (1 - \lambda) F_n(Y_t, h_1) + (1 - \lambda) F_n(Y_t, h_2).
$$

Hence, since $f_n$ is smooth, $\Delta_{xx} f_n(Y_t) \geq 0$.

2. For all $Y_t \in \Lambda$,

$$
\lim_{n \rightarrow +\infty} f_n(Y_t) = f(Y_t),
$$

$$
\lim_{n \rightarrow +\infty} \Delta_x f_n(Y_t) = \lim_{n \rightarrow +\infty} \partial_x F_n(Y_t, 0) = \partial_x F(Y_t, 0) = \Delta_x f(Y_t),
$$

$$
\lim_{n \rightarrow +\infty} \Delta_t f_n(Y_t) = \Delta_t f(Y_t).
$$
The last convergence follows because $\Delta_t f_n$ is the mollification of $\Delta_t f$ when (2.6) is true.

3. $\Delta_x f_n(Y_t)$ increasingly converges to $\Delta_x f(Y_t)$. Indeed, one can write

$$\Delta_x f_n(Y_t) = \int_0^2 \rho(\xi) \partial^- h F \left( Y_t, -\frac{\xi}{n} \right) d\xi,$$

and it is easy to see the desired result using the fact that $\partial^- h F(Y_t, h)$ is non-decreasing in $h$, because of the convexity of $f$.

4. We would like to apply the Dominated Convergence Theorem for stochastic integrals (see (Protter, 2005, Theorem 32, Chapter IV)) and conclude

$$\lim_{n \to +\infty} \int_0^t \Delta_x f_n(X_s) ds = \int_0^t \Delta_x f(X_s) ds \quad \text{u.c.p.,}$$

where u.c.p. means uniformly on compacts in probability. So, it is necessary to bound the sequence of stochastic processes $(\Delta_x f_n(X_s))_{n \in \mathbb{N}}$ by a $x$-integrable process. Since $\partial^- h F(Y_t, h)$ is non-decreasing in $h$, by (3.9),

$$\Delta^- F(Y_s, -2) \leq \Delta^- F \left( Y_s, -\frac{2}{n} \right) \leq \Delta_x f_n(Y_s) \leq \Delta^- F(Y_s).$$

Hence

$$|\Delta_x f_n(Y_s)| \leq |\Delta^- f(Y_s)| + |\Delta^- F(Y_s, -2)|$$

and therefore, if we assume that $(\partial^- F(X_s, h))_{s \in [0, T]}$ is $x$-integrable for $h = 0$ and $h = -2$, we would be able to conclude the convergence (3.10).

5. An easier convergence issue is

$$\lim_{n \to +\infty} \int_0^t \Delta_t f_n(Y_s) ds = \int_0^t \Delta_t f(Y_s) ds.$$

Indeed, one can easily notice

$$\int_0^t \Delta_t f_n(Y_s) ds = \int_0^t \int_{-\infty}^{\infty} \rho_n(-\xi) \Delta_t F(Y_s, \xi) d\xi ds = \int_{-\infty}^{\infty} \rho_n(-\xi) \left( \int_0^t \Delta_t F(Y_s, \xi) ds \right) d\xi.$$

Moreover, for any given $Y_t \in \Lambda$, the real function $\xi \mapsto \int_0^t \Delta_t F(Y_s, \xi) ds$ is continuous. In fact, for a given path $Y_t \in \Lambda$, the function $(s, \xi) \mapsto \Delta_t F(Y_s, \xi)$ is uniformly continuous on compacts. Therefore, these facts imply the desired convergence.

### 3.5 Preliminary Result

We start this section by stating the assumptions on the functional $f$.

**Hypotheses 3.5.** We henceforth assume that

1. $f \in C^{1,0}$, meaning that $f$ is $\Lambda$-continuous, and $\Delta_t f$ exists everywhere in $\Lambda$ and it is also $\Lambda$-continuous;
2. Assumption 2.2 is verified;
3. Equation (2.6) is satisfied:
   \[
   \lim_{\delta t \to 0^+} \frac{f((Y_t, \delta t)_{\xi}) - f((Y_\xi)_{t, \delta t})}{\delta t} = 0,
   \]
   for any \( Y_t \in \Lambda \) and \( \xi \in \mathbb{R} \);
4. \( f \) is convex;
5. \( (\partial h^\nu F(X_s, h))_{s \in [0, T]} \) is \( x \)-integrable for \( h = 0 \) and \( h = -2 \).

The main non-technical assumption above is the convexity of the functional \( f \).
Therefore, as we studied in Section 3.4, \( f_n \) belongs to \( C^{1, \infty} \), and the following convergences hold

(3.11) \[ \lim_{n \to +\infty} f_n(Y_t) = f(Y_t), \]

(3.12) \[ \lim_{n \to +\infty} \int_0^t \Delta_x f_n(X_s) dx_s = \int_0^t \Delta_x f(X_s) dx_s \text{ u.c.p.,} \]

(3.13) \[ \lim_{n \to +\infty} \int_0^t \Delta_t f_n(Y_s) ds = \int_0^t \Delta_t f(Y_s) ds, \]

for any \( Y_t \in \Lambda \).

Similarly as in Protter (2005), we first analyze the limit of \( f_n(X_t) = F_n(X_t, 0) \) without identifying the limit of the Itô term. By the Functional Itô’s Formula, Theorem 1.1, we find

\[
    f_n(X_t) = f_n(X_0) + \int_0^t \Delta_t f_n(X_s) ds + \int_0^t \Delta_x f_n(X_s) dx_s + \frac{1}{2} \int_0^t \Delta_{xx} f_n(X_s) d\langle x \rangle_s.
\]

Consider now the continuous process

\[
    A_n^f = \int_0^t \Delta_{xx} f_n(X_s) d\langle x \rangle_s.
\]

This process is increasing because \( f_n \) is convex, and then \( \Delta_{xx} f_n \geq 0 \). Hence, by Equations (3.11) – (3.13), \( A_n^f \) converges u.c.p. to a continuous increasing process \( A_f^f \) that satisfies

(3.14) \[ f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) ds + \int_0^t \Delta_x f(X_s) dx_s + \frac{1}{2} A_f^f. \]

As in the classical case, Equation (3.14) shows that the convex functional of a continuous semimartingale is also a continuous semimartingale.

### 3.6 The Functional Meyer-Tanaka Formula

The purpose of this section is to identify the term \( A_f^f \) of Equation (3.14) and, in the process of doing so, to prove the functional Meyer-Tanaka formula. Remember \( F \) is defined by Equation (3.4). If we denote the
mollification of $\mathcal{F}$ with respect to the $y$ variable by $F_n$, we can easily conclude, by Equation (3.6),

$$ F_n(Y_t^{y}, h) = F_n(Y_t, y + h) $$

and then

$$ \partial_h^{(k)} F_n(Y_t^{y}, h) = \partial_y^{(k)} F_n(Y_t, y + h). $$

In particular, $\Delta_x^{(k)} f_n(Y_t^{y}) = \partial_y^{(k)} F_n(Y_t, y)$. So, by Equation (3.3),

$$ \frac{1}{2} \int_0^t \Delta_{xx} f_n(X_s) d(x) = \int_0^t \left( \int_0^t \partial_{yy} F_n(X_s, y)L^x(ds, y) \right) dy $$

$$ = \int_0^t \partial_{yy} F_n(X_t, y)L^x(t, y) dy - \int_0^t \int_0^t \Delta_{yy} F_n(X_s, y)L^x(s, y) dy ds, $$

$$ \Delta_{xx} f_n(Y_s^{y}) q(s, y) dy ds $$

Hence, for $g : \mathbb{R} \rightarrow \mathbb{R}$ and $q : \mathbb{R}^2 \rightarrow \mathbb{R}$ smooth and compactly supported, we have, by the computations performed in Section 3.3,

$$ \int_0^t \partial_{yy} F_n(Y_t, y) g(y) dy = - \int_0^t g'(y) \partial_y F_n(Y_t, y) dy $$

$$ \xrightarrow{n \rightarrow +\infty} - \int_0^t g'(y) \partial_y F(Y_t, y) dy = \int_0^t g(y) d_y \partial_y F(Y_t, y) $$

and

$$ \int_0^t \int_0^t \Delta_{xx} f_n(Y_s^{y}) q(s, y) dy ds = - \int_0^t \partial_y F_n(Y_s, y) \partial_y q(s, y) dy ds $$

$$ \xrightarrow{n \rightarrow +\infty} - \int_0^t \partial_y F(Y_s, y) \partial_y q(s, y) dy ds $$

$$ + \int_0^t \int_0^t \partial_{yy} F(Y_s, y) \partial_y q(s, y) dy ds $$

$$ = \int_0^t \int_0^t q(s, y) d_{s, y} \partial_y F(Y_s, y) $$

$$ \xrightarrow{n \rightarrow +\infty} \int_0^t \int_0^t q(s, y) d_{s, y} \partial_y F(Y_s, y). $$

Therefore, using well-known arguments along the lines of Elworthy et al. 2007 Proof of Theorem 2.1, we can extend the formulas above for non-smooth $g$ and $q$, and conclude

$$ \lim_{n \rightarrow +\infty} \frac{1}{2} \int_0^t \Delta_{xx} f_n(X_s) d(x) = \int_0^t L^x(t, y) d_y \partial_y F(X_t, y) $$

$$ - \int_0^t L^x(s, y) d_{s, y} \partial_y F(X_s, y). $$
We have finally proved the Functional Meyer-Tanaka Formula,

\begin{equation}
(3.17) \quad f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) \, ds + \int_0^t \Delta^-_t f(X_s) \, dx_s \\
+ \int_\mathbb{R} L^x(t, y) d_y \partial_y^- F(X_t, y) \\
- \int_0^t \int_\mathbb{R} L^x(s, y) d_{s, y} \partial_y^- F(X_s, y)
\end{equation}

**Remark 3.6.** Let us consider a continuous semimartingale \( x \). Following the idea of \cite{Elworthy2007} (Theorem 2.3), we define the process \( x^* \)

\[ x^*_t = x_t - a_t, \text{ where } (a_t)_{t \geq 0} \text{ is a continuous process of finite variation}. \]

It is obvious that \( x^* \) is also a semimartingale. Denote the local time of \( x^* \) by \( L_{x^*} \). Therefore, the same argument of \cite{Elworthy2007} (Theorem 2.3) applied to the computation we have just performed in (3.15) and (3.16) gives us the following version of the functional Meyer-Tanaka formula

\begin{equation}
(3.18) \quad f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) \, ds + \int_0^t \Delta^-_t f(X_s) \, dx_s \\
+ \int_\mathbb{R} L^{x^*}(t, y) d_y \partial_y^- F(X_t, y + a_t) \\
- \int_0^t \int_\mathbb{R} L^{x^*}(s, y) d_{s, y} \partial_y^- F(X_s, y + a_s)
\end{equation}

This version of the formula will be used in the following example.

**Example 3.7 (Running Maximum).** Let us apply the functional Meyer-Tanaka formula, Equation (3.18), to the running maximum \( m(Y_t) \). Firstly, we have already concluded that

\[ \Delta_t m(Y_t) = 0 \text{ and } \Delta^-_t m(Y_t) = 0, \forall Y_t \in \Lambda. \]

Moreover, we have shown in Examples 2.4, 2.5, and 3.4 that \( m \) satisfies Hypotheses 3.5. Define now the functional

\[ z_t := z(Y_t) := \sup_{0 \leq s < t} y_s \]

and notice that

\[ M(Y_t, y + h) = m(Y_{t^+}) = z_t 1_{\{y + h \leq z_t\}} + (y + h) 1_{\{y + h > z_t\}}. \]

Hence, we can compute

\[ \partial_y^- M(Y_t, y) = 1_{\{y > z_t\}} \Rightarrow d_y \partial_y^- M(Y_t, y) = \delta_{z_t}(dy). \]

We then face a problem because \( d_{s, y} \partial_y^- M(Y_s, y) \) is not easily computed. However, we notice that

\[ \partial_y^- M(Y_t, y + z_t) = 1_{\{y > 0\}} \Rightarrow d_y \partial_y^- M(Y_t, y) = \delta_0(dy) \]

and \( d_t \partial_y^- M(Y_t, y + z_t) = 0. \)
Hence, we will apply formula (3.18) with $a_t = m(X_t) = z(X_t)$, which is clearly a continuous process of finite variation. These equalities hold because the process $x$ is continuous. Therefore, we finally find the important formula of Lévy

$$\max_{0 \leq s \leq t} x_s = x_0 + L^{x-m}(t, 0),$$

where $L^{x-m}$ is the local time of the process $(x_t - m(X_t))_{t \in [0, T]}$.

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