EQUIVARIANT COHOMOLOGY OF THE MODULI SPACE OF
GENUS THREE CURVES WITH SYMPLECTIC LEVEL TWO
STRUCTURE VIA POINT COUNTS

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Abstract. We make cohomological computations related to the moduli space
of genus three curves with symplectic level two structure by means of counting
points over finite fields. In particular, we determine the cohomology groups of
the quartic locus as representations of the symmetric group on seven elements.

1. Introduction

Let \( n \) be a positive integer and let \( C \) be a curve. A level \( n \) structure on \( C \) is a
choice of basis for the \( n \)-torsion of the Jacobian of \( C \). The purpose of this paper is to
study the cohomology of the moduli space \( \mathcal{M}_3[2] \) of genus 3 curves with symplectic
level 2 structure.

A genus 3 curve which is not hyperelliptic is embedded as a plane quartic via
its canonical linear system. The corresponding locus in \( \mathcal{M}_3[2] \) is called the quartic
locus and it is denoted \( \mathcal{Q}[2] \). A plane quartic with level 2 structure is specified, up
to isomorphism, by an ordered septuple of points in general position in \( \mathbb{P}^2 \), up to
the action of \( \text{PGL}(3) \). This identification will be the basis for our investigation of
\( \mathcal{Q}[2] \).

Our main focus will be on \( \mathcal{Q}[2] \) but we will also consider its complement in \( \mathcal{M}_3[2] \),
i.e. the hyperelliptic locus \( \mathcal{H}_3[2] \). In both cases, the computations will be carried
out via point counts over finite fields. By virtue of the Lefschetz trace formula, such
point counts give cohomological information in the form of Euler characteristics.
However, both \( \mathcal{Q}[2] \) and \( \mathcal{H}_3[2] \) satisfy certain strong purity conditions which allow
us to deduce Poincaré polynomials from these Euler characteristics.

The group \( \text{Sp}(6, \mathbb{Z}/2\mathbb{Z}) \) acts on \( \mathcal{M}_3[2] \) as well as on \( \mathcal{Q}[2] \) and \( \mathcal{H}_3[2] \) by changing
level structures. The cohomology groups thus become \( \text{Sp}(6, \mathbb{Z}/2\mathbb{Z}) \)-representations
and our computations will therefore be equivariant. However, the action of
\( \text{Sp}(6, \mathbb{Z}/2\mathbb{Z}) \) is rather subtle on \( \mathcal{Q}[2] \) when \( \mathcal{Q}[2] \) is identified with the space of septuples of points in general position in \( \mathbb{P}^2 \). On the other hand, the action of the symmetric group \( S_7 \) on seven elements is very clear and we will therefore restrict
our attention to this subgroup. The full action of \( \text{Sp}(6, \mathbb{Z}/2\mathbb{Z}) \) is the topic of ongoing
research.

The main results are presented in Table 2 and Table 5 where we give the coho-
mology groups of \( \mathcal{Q}[2] \) and \( \mathcal{H}_3[2] \) as representations of \( S_7 \).

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2. SYMPLECTIC LEVEL STRUCTURES

Let \( K \) be an algebraically closed field of characteristic zero and let \( C \) be a smooth and irreducible curve of genus \( g \) over \( K \). The \( n \)-torsion part \( \text{Jac}(C)[n] \) of the Jacobian of \( C \) is isomorphic to \( (\mathbb{Z}/n\mathbb{Z})^{2g} \) as an abelian group and the Weil pairing is a nondegenerate and alternating bilinear form on \( \text{Jac}(C)[n] \).

**Definition 2.1.** A symplectic level \( n \) structure on a curve \( C \) is an ordered basis \((D_1, \ldots, D_{2g})\) of \( \text{Jac}(C)[n] \) such that the Weil pairing has basis \((0 I_g - I_g 0)\), with respect to this basis. Here, \( I_g \) denotes the \( g \times g \) identity matrix.

For more information about the Weil pairing and level structures, see for example [2] or [12]. Since we shall only consider symplectic level structures we shall refer to symplectic level structures simply as level structures.

A tuple \((C, D_1, \ldots, D_{2g})\) where \( C \) is a smooth and irreducible curve and \((D_1, \ldots, D_{2g})\) is a level \( n \) structure on \( C \) is called a curve with level \( n \) structure. Let \((C', D'_1, \ldots, D'_{2g})\) be another curve with level \( n \) structure. An isomorphism of curves with level \( n \) structures is an isomorphism of curves \( \phi : C \to C' \) such that \( \phi^*(D'_i) = D_i \) for \( i = 1, \ldots, n \). We denote the moduli space of genus \( g \) curves with level \( n \) structure by \( M_g[n] \). We remark that we shall consider these moduli spaces as coarse spaces and not as stacks. For \( n \geq 3 \), this remark is somewhat vacuous, see [14], but for \( n = 2 \) this is not the case. The group \( \text{Sp}(2g, \mathbb{Z}/n\mathbb{Z}) \) acts on \( M_g[n] \) by changing level structures.

In the following we shall only be interested in level \( 2 \) structures. A concept closely related to level \( 2 \) structures is that of theta characteristics.

**Definition 2.2.** Let \( C \) be a smooth and irreducible curve and let \( K_C \) be its canonical class. An element \( \theta \in \text{Pic} (C) \) such that \( 2\theta = K_C \) is called a theta characteristic. We denote the set of theta characteristics of \( C \) by \( \Theta(C) \).

Let \( C \) be a curve of genus \( g \). Given two theta characteristics \( \theta_1 \) and \( \theta_2 \) on \( C \) we obtain an element \( D \in \text{Jac}(C)[2] \) by taking the difference \( \theta_1 - \theta_2 \). Conversely, given a theta characteristic \( \theta \) and a 2-torsion element \( D \) we obtain a new theta characteristic as \( \theta' = \theta + D \). More precisely we have that \( \Theta(C) \) is a \( \text{Jac}(C)[2] \)-torsor and the set \( \tilde{\Theta}(C) = \Theta(C) \cup \text{Jac}(C)[2] \) is a vector space of dimension \( 2g + 1 \) over the field \( \mathbb{Z}/2\mathbb{Z} \) of two elements.

**Definition 2.3.** An ordered basis \( A = (\theta_1, \ldots, \theta_{2g+1}) \) of theta characteristics of the vector space \( \tilde{\Theta}(C) \) is called an ordered Aronhold basis if the expression

\[
 h^0(\theta) \mod 2,
\]

only depends on the number of elements in \( A \) that is required to express \( \theta \) for any theta characteristic \( \theta \).

**Proposition 2.4.** Let \( C \) be a smooth an irreducible curve. There is a bijection between the set of ordered Aronhold bases on \( C \) and the set of level 2 structures on \( C \).
For a proof of Proposition 2.4 as well as a more thorough treatment of theta characteristics and Aronhold bases we refer to [13].

Proposition 2.4 provides a more geometric way to think about level 2 structures. In the case of a plane quartic curve, which shall be the case of most importance to us, we point out that each theta characteristic occurring in an Aronhold basis is cut out by a bitangent line. Thus, in the case of plane quartics one can think of ordered Aronhold bases as ordered sets of bitangents (although not every ordered set of bitangents constitute an ordered Aronhold basis).

3. Plane quartics

Let $K$ be an algebraically closed field of characteristic zero and let $C$ be a smooth and irreducible curve of genus $g$ over $K$. If $C$ is not hyperelliptic it is embedded into $\mathbb{P}^{g-1}$ via its canonical linear system. Thus, a non-hyperelliptic curve of genus 3 is embedded into $\mathbb{P}^2$ and by the genus-degree formula we see see that the degree of the image is 4. We shall therefore refer to the complement of the hyperelliptic locus in $\mathcal{M}_3$ as the quartic locus and denote it by $Q = \mathcal{M}_3 \setminus \mathcal{H}_3$. Similarly, we denote the complement of the hyperelliptic locus in $\mathcal{M}_3[2]$ by $Q[2]$. Clearly, the action of $\text{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ on $\mathcal{M}_3[2]$ restricts to an action on $Q[2]$.

The purpose of this section is to give an explicit, combinatorial description of $Q[2]$. This description will be in terms of points in general position. Intuitively, a set of points in the projective plane is in general position if there is no “unexpected” curve passing through all of them. In our case, this is made precise in by the following definition.

**Definition 3.1.** Let $(P_1, \ldots, P_7)$ be a septuple of points in $\mathbb{P}^2$. We say that the septuple is in general position if there is no line passing through any three of the points and no conic passing through any six of them. We denote the moduli space of septuples of points in general position up to projective equivalence by $\mathcal{P}_7^2$.

Let $T = (P_1, \ldots, P_7)$ be a septuple of points in general position in the projective plane and let $\mathcal{N}_T$ be the net of cubics passing through $T$. If we let $F_0, F_1$ and $F_2$ be generators for $\mathcal{N}_T$, then the equation

$$\det \left( \frac{\partial F_i}{\partial x_j} \right) = 0, \quad i, j = 0, 1, 2,$$

describes a plane sextic curve $S_T$ with double points precisely at $P_1, \ldots, P_7$. By the genus-degree formula we see that $S_T$ has geometric genus 3 and it turns out that its smooth model is not hyperelliptic. Moreover, if we let $\rho : C_T \to S_T$ be a resolution of the singularities, then $D_i = \rho^{-1}(P_i)$ is a theta characteristic and $(D_1, \ldots, D_7)$ is an ordered Aronhold basis.

**Theorem 3.2** (van Geemen [9]). Sending a septuple $T = (P_1, \ldots, P_7)$ of points in general position in the projective plane to $(C_T, D_1, \ldots, D_7)$ gives a $\text{Sp}(6, \mathbb{Z}/2\mathbb{Z})$-equivariant isomorphism

$$\mathcal{P}_7^2 \to Q[2].$$

It should be pointed out that while the action of $\text{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ is clear on $Q[2]$ its action on $\mathcal{P}_7^2$ is much more subtle. However, we can at least plainly see the symmetric group $S_7 \subset \text{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ act on $\mathcal{P}_7^2$ by permuting points.
4. The Lefschetz Trace Formula

We are interested in the spaces $M_3[2]$, $Q[2]$ and $H_3[2]$ and in particular we want to know their cohomology. The Lefschetz trace formula provides a way to obtain cohomological information about a space via point counts over finite fields.

Let $p$ be a prime number, let $n \geq 1$ be an integer and let $q = p^n$. Also, let $F_q$ denote a finite field with $q$ elements, let $F_{q^m}$ denote a degree $m$ extension of $F_q$ and let $F_q$ denote a finite field with $q$ elements, let $F_{q^m}$ denote a degree $m$ extension of $F_q$ and let $F_q$ denote an algebraic closure of $F_q$. Let $X$ be a scheme defined over $F_q$ and let $F$ denote its geometric Frobenius endomorphism induced from $F_q$. Finally, let $l$ be another prime number, different from $p$, and let $H^k_{\text{ét},c}(X, \mathbb{Q}_l)$ denote the $k$'th compactly supported étale cohomology group of $X$ with coefficients in $\mathbb{Q}_l$.

Let $\Gamma$ be a finite group of rational automorphisms of $X$. Then each cohomology group $H^k_{\text{ét},c}(X, \mathbb{Q}_l)$ is a $\Gamma$-representation. The Lefschetz trace formula allows us to obtain information about these representations by counting the number of fixed points of $F_\sigma$ for different $\sigma \in \Gamma$.

**Theorem 4.1** (Lefschetz trace formula). Let $X$ be a separated scheme of finite type over $\mathbb{F}_q$ with Frobenius endomorphism $F$ and let $\sigma$ be a rational automorphism of $X$ of finite order. Then

$$|X^{F_{\sigma}}| = \sum_{k \geq 0} (-1)^k \cdot \text{Tr}(F_{\sigma}, H^k_{\text{ét},c}(X, \mathbb{Q}_l)),$$

where $X^{F_{\sigma}}$ denotes the fixed point set of $F_{\sigma}$.

For a proof, see [7], Rapport - Théorème 3.2.

**Remark 4.2.** This theorem is usually only stated in terms of $F$. To get the above version one simply applies the “usual” theorem to the twist of $X$ by $\sigma$.

**Remark 4.3.** If $\Gamma$ is a finite group of rational automorphisms of $X$ and $\sigma \in \Gamma$, then $|X^{F_{\sigma}}|$ will only depend on the conjugacy class of $\sigma$.

Let $R(\Gamma)$ denote the representation ring of $\Gamma$ and let the compactly supported $\Gamma$-equivariant Euler characteristic of $X$ be defined as the virtual representation

$$\text{Eul}_{X,c}^F = \sum_{k \geq 0} (-1)^k \cdot H^k_{\text{ét},c}(X, \mathbb{Q}_l) \in R(\Gamma).$$

By Theorem 4.1 we may determine $\text{Eul}_{X,c}^F$ by computing $|X^{F_{\sigma}}|$ for each $\sigma \in \Gamma$ and by Remark 4.3 it is enough to do so for one representative of each conjugacy class. This motivates the following definition.

**Definition 4.4.** Let $X$ be a separated scheme of finite type over $\mathbb{F}_q$ with Frobenius endomorphism $F$ and let $\Gamma$ be a finite group of rational automorphisms of $X$. The determination of $|X^{F_{\sigma}}|$ for all $\sigma \in \Gamma$ is then called a $\Gamma$-equivariant point count of $X$ over $\mathbb{F}_q$.

5. Minimal Purity

Let $X$ be a scheme over the finite field $\mathbb{F}_q$ and let $\Gamma$ be a group of rational automorphisms of $X$. We define the compactly supported $\Gamma$-equivariant Poincaré polynomial of $X$ as

$$P^F_{X,c}(t) = \sum_{k \geq 0} H^k_{\text{ét},c}(X, \mathbb{Q}_l) \cdot t^k \in R(\Gamma)[t].$$
In the previous section we saw that equivariant point counts give equivariant Euler characteristics. Poincaré polynomials contain more information and are therefore more desirable to obtain but are typically more complicated to compute. However, if \( X \) satisfies a certain purity condition one can recover the Poincaré polynomial from the Euler characteristic. See also [4], [3], [18] and [5] where similar phenomena for compact spaces have been exploited.

**Definition 5.1** (Dimca and Lehrer [8]). Let \( X \) be an irreducible and separated scheme of finite type over \( \mathbb{F}_q \) with Frobenius endomorphism \( F \) and let \( l \) be a prime not dividing \( q \). The scheme \( X \) is called minimally pure if \( F \) acts on \( H^k_{\text{ét}}(X, \mathbb{Q}_l) \) with all eigenvalues equal to \( q^{k-\dim(X)} \).

A pure dimensional and separated scheme \( X \) of finite type over \( \mathbb{F}_q \) is minimally pure if for any collection \( \{X_1, \ldots, X_r\} \) of irreducible components of \( X \), the irreducible scheme \( X_1 \setminus (X_2 \cup \cdots \cup X_r) \) is minimally pure.

Thus, if \( X \) is minimally pure, then a term \( q^{k-\dim(X)} \) in \( |X^F| \) can only come from \( H^k_{\text{ét}}(X, \mathbb{Q}_l) \) and we can determine the \( \Gamma \)-equivariant Poincaré polynomial of \( X \) via the relation

\[
\text{Eul}_X^\Gamma(\sigma) = q^{-2\dim(X)} \cdot P_X^\Gamma(\sigma)(-q^2).
\]

We will see that the moduli space \( \mathbb{Q}[2] \) is minimally pure.

Let \( C \subset \mathbb{P}^2 \) be a plane quartic, let \( P \in C \) be a point and let \( T_P C \) denote the tangent line of \( C \) at \( P \). We say that \( P \) is a bitangent point if

\[
C \cdot T_P C = 2P + 2Q
\]

for some point \( Q \) that might coincide with \( P \). If \( P \neq Q \) we say that \( P \) is a genuine bitangent point. We denote the moduli space of plane quartics with level 2 structure marked with a bitangent point by \( \mathbb{Q}_{\text{btg}}[2] \) and we denote the moduli space of plane quartics with level 2 structure marked with a genuine bitangent point by \( \mathbb{Q}_{\text{btg}}[2] \).

The space \( \mathbb{Q}_{\text{btg}}[2] \) is an open subvariety of \( \mathbb{Q}_{\text{btg}}[2] \).

**Lemma 5.2.** \( \mathbb{Q}_{\text{btg}}[2] \) is minimally pure.

**Proof.** Looijenga [15] has shown that \( \mathbb{Q}_{\text{btg}}[2] \) is isomorphic to a finite disjoint union of varieties, each isomorphic to the complement of an arrangement of tori in an ambient torus of dimension 6. Dimca and Lehrer [8] has shown that such complements of arrangements are minimally pure and it thus follows that \( \mathbb{Q}_{\text{btg}}[2] \) is minimally pure. On the other hand, Looijenga [15] has shown that there is an injection

\[
H^k_{\text{ét},c}(\mathbb{Q}_{\text{btg}}[2], \mathbb{Q}_l) \hookrightarrow H^k_{\text{ét},c}(\mathbb{Q}_{\text{btg}}[2], \mathbb{Q}_l).
\]

We thus see that \( H^k_{\text{ét},c}(\mathbb{Q}_{\text{btg}}[2], \mathbb{Q}_l) \) is an \( F \)-invariant subspace of \( H^k_{\text{ét},c}(\mathbb{Q}_{\text{btg}}[2], \mathbb{Q}_l) \).

Since the eigenvalues of \( F \) on \( H^k_{\text{ét},c}(\mathbb{Q}_{\text{btg}}[2], \mathbb{Q}_l) \) are all equal to \( q^{k-6} \) we conclude that the same is true for \( H^k_{\text{ét},c}(\mathbb{Q}_{\text{btg}}[2], \mathbb{Q}_l) \). Hence, \( \mathbb{Q}_{\text{btg}}[2] \) is minimally pure. \( \square \)

**Proposition 5.3.** \( \mathbb{Q}[2] \) is minimally pure.

**Proof.** A plane quartic has 28 bitangents so the morphism

\[
\pi : \mathbb{Q}_{\text{btg}}[2] \to \mathbb{Q}[2],
\]

forgetting the marked bitangent point, is finite of degree \( 2 \cdot 28 = 56 \). Thus, the map

\[
\pi_* \circ \pi^* : H^k_{\text{ét},c}(\mathbb{Q}[2], \mathbb{Q}_l) \to H^k_{\text{ét},c}(\mathbb{Q}[2], \mathbb{Q}_l)
\]
is multiplication with \( \deg(\pi) = 56 \). In particular, the map
\[
\pi : H^k_{\text{ét},c}(\mathbb{Q}[2], \mathbb{Q}_\ell) \to H^k_{\text{ét},c}(\mathbb{Q}_{\text{ét,alg}}[2], \mathbb{Q}_\ell)
\]
is injective and we may thus conclude that \( \mathbb{Q}[2] \) is minimally pure as in the proof of Lemma 5.2.

Since \( \mathbb{Q}[2] \) is isomorphic to \( \mathbb{P}^2 \), we may compute the cohomology of \( \mathbb{Q}[2] \) as a representation of \( S_7 \) by making \( S_7 \)-equivariant point counts of \( \mathbb{P}^2 \).

### 6. Equivariant point counts

In this section we shall perform a \( S_7 \)-equivariant point count of \( \mathbb{P}^2 \). This amounts to the computation of
\[
\left| \left( \mathbb{P}^2 \right)^{F_\sigma} \right|,
\]
for one representative \( \sigma \) of each of the fifteen conjugacy classes of \( S_7 \). The computations will be rather different in the various cases but at least the underlying idea will be the same. Throughout this section we shall work over a finite field \( \mathbb{F}_q \) where \( q \) is odd.

Let \( U \) be a subset of \( \left( \mathbb{P}^2 \right)^7 \) and interpret each point of \( U \) as an ordered septuple of points in \( \mathbb{P}^2 \). Define the discriminant locus \( \Delta \subset U \) as the subset consisting of septuples which are not in general position. If \( U \) contains the subset of \( \left( \mathbb{P}^2 \right)^7 \) consisting of all septuples which are in general position, then
\[
\mathbb{P}^2 = (U \setminus \Delta) / \text{PGL}(3).
\]

An element of PGL(3) is completely specified by where it takes four points in general position. Therefore, the points of \( \mathbb{P}^2 \) do not have any automorphisms and we have the simple relation
\[
\left| \left( \mathbb{P}^2 \right)^{F_\sigma} \right| = \frac{|U^{F_\sigma}| - |\Delta^{F_\sigma}|}{|\text{PGL}(3)|}.
\]

We will choose the set \( U \) in such a way that counting fixed points of \( F_\sigma \) in \( U \) is easy. We shall therefore focus on the discriminant locus.

The discriminant locus can be decomposed as
\[
\Delta = \Delta_l \cup \Delta_c,
\]
where \( \Delta_l \) consists of septuples where at least three points lie on a line and \( \Delta_c \) consists of septuples where at least six points lie on a conic. The computation of \( |\Delta^{F_\sigma}| \) will consist of the following three steps:

- the computation of \( |\Delta_l^{F_\sigma}| \),
- the computation of \( |\Delta_c^{F_\sigma}| \),
- the computation of \( |(\Delta_l \cap \Delta_c)^{F_\sigma}| \).

We can then easily determine \( |\Delta^{F_\sigma}| \) via the principle of inclusion and exclusion.

In the analysis of \( \Delta_l \cap \Delta_c \) the following definition will sometimes be useful, see Figure 1.

**Definition 6.1.** Let \( C \) be a smooth conic over \( \mathbb{F}_q \) and let \( P \in \mathbb{P}^2 \) be a \( \mathbb{F}_q \)-point. We then say

- that \( P \) is on the \( \mathbb{F}_q \)-inside of \( C \) if there is no \( \mathbb{F}_q \)-tangent to \( C \) passing through \( P \),
- that \( P \) is on \( C \) if there is precisely one \( \mathbb{F}_q \)-tangent to \( C \) passing through \( P \),
- that \( P \) is on the \( \mathbb{F}_q \)-outside of \( C \) if there are two \( \mathbb{F}_q \)-tangents to \( C \) passing through \( P \).
For a motivation of the terminology, see Figure 1.

![Figure 1. A conic C with a point P on the outside of C, a point Q on the inside of C and a point R on C.](image)

It is not entirely clear that the above definition makes sense. To see that it in fact does, we need the following lemma.

**Lemma 6.2.** Let $C \subset \mathbb{P}^2$ be a smooth conic over a field $k$. If there is a point $P$ such that three tangents of $C$ pass through $P$, then the characteristic of $k$ is 2.

**Proof.** Since $C$ is smooth, the three points of tangency will be in general position, so by a projective change of coordinates they can be transformed to $[1:0:0], [0:1:0]$ and $[0:0:1]$ and $C$ will then be given by a polynomial $F = XY + \alpha XZ + \beta YZ$, where $\alpha, \beta \in k^*$. The tangent lines thus become $Y + \alpha Z$, $X + \beta Z$ and $\alpha X + \beta Y$.

Let the coordinates of $P$ be $[a:b:c]$. Since these lines all pass through $P$, the first tangent equation gives that $b = -\alpha c$ and the second gives $a = -\beta c$. Inserting these expressions into the third tangent equation gives $-2\alpha\beta c = 0$. If $c = 0$, then also $a = b = 0$ which is impossible. Since also $\alpha$ and $\beta$ are nonzero we see that the only possibility is that the characteristic of $k$ is 2. □

Let $\sigma^{-1} = (i_1 \ldots i_r)$ be a cycle in $S_r$. An ordered septuple $(P_1, \ldots, P_7)$ of points in $\mathbb{P}^2$ will be fixed by $F\sigma$ if and only if $FP_{i_j} = P_{i_{j+1}}$ for $i = 1, \ldots, r-1$ and $FP_{i_r} = P_{i_1}$. This is the motivation for the following definition.

**Definition 6.3.** Let $X$ be a $\mathbb{F}_q$-scheme with Frobenius endomorphism $F$ and let $Z \subset X_{\mathbb{F}_q}$ be a subscheme. If

$$\left| \{ F^i Z \}_{i \geq 0} \right| = m,$$

we say that $Z$ is a *strict* $\mathbb{F}_q^m$-subscheme.
If \( Z \) is a strict \( \mathbb{F}_q^m \)-subscheme, the \( m \)-tuple \((Z, \ldots, F^{m-1}Z)\) is called a conjugate \( m \)-tuple. Let \( r \) be a positive integer and let \( \lambda = [1^{\lambda_1}, \ldots, r^{\lambda_r}] \) be a partition of \( r \). An \( r \)-tuple \((Z_1, \ldots, Z_r)\) of closed subschemes of \( X \) is called a conjugate \( \lambda \)-tuple if it consists of \( \lambda_1 \) conjugate 1-tuples, \( \lambda_2 \)-conjugate 2-tuples and so on. We denote the set of conjugate \( \lambda \)-tuples of closed points of \( X \) by \( X(\lambda) \).

Since the conjugacy class of an element in \( S_7 \) is given by its cycle type, we want to count the number of conjugate \( \lambda \)-tuples in both \( U \) and \( \Delta \) for each partition of seven. In this pursuit, the following formula is helpful. Its proof is a simple application of the principle of inclusion and exclusion.

**Lemma 6.4.** Let \( X \) be a \( \mathbb{F}_q \)-scheme and let \( \lambda = [1^{\lambda_1}, \ldots, n^{\lambda_n}] \) be a partition. Then

\[
|X(\lambda)| = \prod_{i=1}^{\nu} \prod_{j=0}^{\lambda_i - 1} \left[ \sum_{d|i} \mu \left( \frac{i}{d} \right) \cdot |X(d)| \right] - i \cdot j,
\]

where \( \mu \) is the Möbius function.

We now recall a number of basic results regarding point counts. First, note that if we apply Lemma 6.4 to \( X = (\mathbb{P}^n)^{\vee} \), the dual projective space, we see that the number of conjugate \( \lambda \)-tuples of hyperplanes is equal to the number of conjugate \( \lambda \)-tuples of points in \( \mathbb{P}^n \). We also recall that

\[
|\mathbb{P}^n_{\mathbb{F}_q}| = \sum_{i=0}^{n} q^i,
\]

and that

\[
|\text{PGL}(3)| = q^3 \cdot (q^3 - 1) \cdot (q^2 - 1).
\]

A slightly less elementary result is that the number of smooth conics defined over \( \mathbb{F}_q \) is

\[
q^5 - q^2.
\]

To see this, note that there is a \( \mathbb{P}^5 \) of conics. Of these there are \( q^2 + q + 1 \) double \( \mathbb{F}_q \)-lines, \( \frac{1}{2} \cdot (q^2 + q + 1) \cdot (q^3 + q) \) intersecting pairs of \( \mathbb{F}_q \)-lines and \( \frac{1}{2} \cdot (q^4 - q) \) conjugate pairs of \( \mathbb{F}_q \)-lines.

We are now ready for the task of counting the number of conjugate \( \lambda \)-tuples for each element of \( S_7 \).

**Remark 6.5.** Since \( \mathcal{P}_2^2 \) is minimally pure, Equation 6.1 gives that \( |(\mathcal{P}_2^2)^{F^\sigma}| \) is a monic polynomial in \( q \) of degree six so it is in fact enough to make counts for six different finite fields and interpolate. This is however hard to carry out in practice, even with a computer, as soon as \( \lambda \) contains parts of large enough size (where “large enough” means 3 or 4). However, one can always obtain partial information which provides important checks for our computations.

### 6.1. The case \( \lambda = [7] \)

Let \( \lambda = [7] \). Since we only need to make the computation for one permutation \( \sigma \) of cycle type \( \lambda \), we may as well assume that \( \sigma^{-1} = (1234567) \) so that \( F \) acts as \( FP_i = P_{i+1} \) for \( i = 1, \ldots, 6 \) and \( FP_7 = P_1 \). In this case, we simply take \( (\mathbb{P}^2)^7 \) as our set \( U \).

The main observation is the following.

**Lemma 6.6.** If \( (P_1, \ldots, P_7) \) is a \( \lambda \)-tuple with three of its points on a line, then all seven points lie on a line defined over \( \mathbb{F}_q \).
Proof. Suppose that the set $S = \{P_1, P_2, P_7\}$ is contained in the line $L$. Then $L$ is either defined over $\mathbb{F}_q$ or $\mathbb{F}_q'$. One easily checks that for each of the $\binom{7}{3} = 35$ possible choices of $S$ there is an integer $1 \leq i \leq 6$ such that $|F^i S \cap S| = 2$. Since a line is defined by any two points on it we have that $L = F^i L$. Hence, we have that $L$ is defined over $\mathbb{F}_q$ and that $\{P_1, FP_1, \ldots, FP_7\} = \{P_1, \ldots, P_7\} \subset L$. \hfill $\Box$

Lemma 6.7. If $(P_1, \ldots, P_7)$ is a $\lambda$-tuple with six of its points on a smooth conic, then all seven points lie on a smooth conic defined over $\mathbb{F}_q$.

Proof. Suppose that the set $S = \{P_1, \ldots, P_6\}$ lies on a smooth conic $C$. We have $|FS \cap S| = 5$ and since a conic is defined by any five points on it we have $FC = C$. Hence, we have that $C$ is defined over $\mathbb{F}_q$ and that all seven points lie on $C$. \hfill $\Box$

We conclude that $\Delta_l$ and $\Delta_e$ are disjoint. We obtain $|\Delta_l|$ by first choosing a $\mathbb{F}_q$-line $L$ and then picking a $\lambda$-tuple on $L$. We thus have

$$|\Delta_l| = (q^2 + q + 1) \cdot (q^7 - q).$$

To obtain $|\Delta_e|$ we first choose a smooth conic $C$ and then a conjugate $\lambda$-tuple on $C$. We thus have

$$|\Delta_e| = (q^5 - q^2) \cdot (q^7 - q).$$

Equation 6.1 now gives

$$\left|\left(\mathbb{P}^2\right)^F\right| = q^6 + q^3.$$

6.2. The case $\lambda = [1, 6]$. Let $\lambda = [1, 6]$. Since we only need to make the computation for one permutation $\sigma$ of cycle type $\lambda$, we may as well assume that $\sigma^{-1} = (123456)(7)$ so that $F$ acts as $FP_1 = P_{i+1}$ for $i = 1, \ldots, 5$, $FP_6 = P_1$ and $FP_7 = P_2$. Also in this case we take $(\mathbb{P}^2)^7$ as our set $U$.

The main observation is the following.

Lemma 6.8. If a $\lambda$-tuple has three points on a line, then either

1. the first six points of the $\lambda$-tuple lie on a $\mathbb{F}_q$-line or,
2. the first six points lie on two conjugate $\mathbb{F}_q^2$-lines, the $\mathbb{F}_q^2$-lines contain three $\mathbb{F}_q^3$-points each and these triples are interchanged by $F$, or,
3. the first six points lie pairwise on three conjugate $\mathbb{F}_q^3$-lines which intersect in $P_7$.

Proof. Suppose that $S = \{P_1, P_2, P_7\}$ lie on a line $L$. Then $L$ is either defined over $\mathbb{F}_q$, $\mathbb{F}_q^2$, $\mathbb{F}_q^3$ or $\mathbb{F}_q^6$. One easily checks that for each of the $\binom{7}{3} = 35$ possible choices of $S$ there is an integer $1 \leq i \leq 3$ such that $|F^i S \cap S| = 2$ so $L$ is defined over $\mathbb{F}_q$, $\mathbb{F}_q^2$ or $\mathbb{F}_q^3$, i.e. we are in one of the three cases above. \hfill $\Box$

Let $\Delta_{l,i}$ be the subset of $\Delta_l$ corresponding to case (i) in Lemma 6.8. The set $\Delta_{l,1}$ is clearly disjoint from the other two.

Lemma 6.9. If six of the points of a $\lambda$-tuple lie on a smooth conic, then all of the first six points of the tuple lie on the conic and the conic is defined over $\mathbb{F}_q$. 

Proof. Suppose that \( S = \{P_1, \ldots, P_6\} \) lie on a smooth conic \( C \). Then \(|FS \cap S| \geq 5\) so \( FC = C \). Let \( P \in S \) be a \( \mathbb{F}_{q^6} \)-point. Then \( \{P, FP, \ldots, F^5\} = \{P_1, \ldots, P_6\} \subset C \).

Since a smooth conic does not contain a line, we have that \( \Delta_c \) only intersects \( \Delta_{l,3} \).

We compute \(|\Delta_{l,1}|\) by first choosing a \( \mathbb{F}_q \)-line \( L \) and then a \( \mathbb{F}_{q^6} \) point on \( L \). Finally we choose a \( \mathbb{F}_q \)-point \( P_7 \) anywhere. We thus have

\[
|\Delta_{l,1}| = (q^2 + q + 1) \cdot (q^6 - q^3 - q^2 + q) \cdot (q^2 + q + 1).
\]

To obtain \(|\Delta_{l,2}|\) we first choose a \( \mathbb{F}_{q^2} \)-line, \( L \). By Lemma 6.4 there are \( q^4 - q \) such lines. The other \( \mathbb{F}_{q^2} \)-line must then be \( FL \). We then choose a \( \mathbb{F}_{q^6} \)-point \( P_1 \) on \( L \). The points \( P_2 = FP_1, \ldots, P_6 = F^5P_1 \) will then be the rest of our conjugate sextuple. By Lemma 6.4 (with \( \mathbb{F}_q^2 \) as the ground field) there are \( q^6 - q^2 \) choices. We now have two \( \mathbb{F}_{q^2} \)-lines with three of our six \( \mathbb{F}_{q^6} \)-points on each so all that remains is to choose a \( \mathbb{F}_q \)-point anywhere we want in \( q^2 + q + 1 \) ways. Hence,

\[
|\Delta_{l,2}| = (q^4 - q)(q^6 - q^2)(q^2 + q + 1).
\]

To count \(|\Delta_{l,3}|\) we first choose a \( \mathbb{F}_{q^2} \)-point \( P_7 \) in \( q^2 + q + 1 \) ways. There is a \( \mathbb{P}^1 \) of lines through \( P_7 \) and we want to choose a \( \mathbb{F}_{q^2} \)-line \( L \) through \( P \). By Lemma 6.4 there are \( q^3 - q \) choices. Finally, we choose a \( \mathbb{F}_{q^6} \)-point \( P_1 \) on \( L \). By Lemma 6.4 there are \( q^6 - q^3 \) possible choices. We thus have

\[
|\Delta_{l,3}| = (q^2 + q + 1)(q^3 - q)(q^6 - q^3).
\]

In order to finish the computation of \( \Delta_l \), we need to compute \(|\Delta_{l,2} \cap \Delta_{l,3}|\). We first choose a pair of conjugate \( \mathbb{F}_{q^2} \)-lines in \( \frac{1}{2}(q^4 - q) \) ways. These intersect in a \( \mathbb{F}_q \)-point and we choose \( P_7 \) away from this point in \( q^2 + q \) ways. We then choose a \( \mathbb{F}_{q^2} \)-line through \( P_7 \) in \( q^3 - q \) ways. This line intersects the two \( \mathbb{F}_{q^2} \)-lines in 2 distinct points which clearly must be defined over \( \mathbb{F}_{q^6} \). We choose one of them to become \( P_1 \) in 2 ways. Thus, in total we have

\[
|\Delta_{l,2} \cap \Delta_{l,3}| = (q^4 - q) \cdot (q^2 + q) \cdot (q^3 - q).
\]

To compute \(|\Delta_c|\) we first choose a smooth conic \( C \) in \( q^5 - q^2 \) ways and then use Lemma 6.4 to see that we have \( q^6 - q^3 - q^2 + q \) ways of choosing a conjugate sextuple on \( C \). Finally, we choose \( P_7 \) anywhere we want in \( q^2 + q + 1 \) ways. We thus see that

\[
|\Delta_c| = (q^5 - q^2)(q^6 - q^3 - q^2 + q)(q^2 + q + 1).
\]

It remains to compute the size of the intersection between \( \Delta_l \) and \( \Delta_c \). To compute do this, we begin by choosing a smooth conic \( C \) in \( q^5 - q^2 \) ways and then a \( \mathbb{F}_q \)-point \( P_7 \) not on \( C \) in \( q^2 + q + 1 - (q + 1) = q^2 \) ways. By Lemma 6.4 there are \( q^3 - q \) strict \( \mathbb{F}_{q^3} \)-lines passing through \( P \). All of these intersect \( C \) in two \( \mathbb{F}_{q^3} \)-points since, by Lemma 6.2, these lines cannot be tangent to \( C \) since the characteristic of \( \mathbb{F}_q \) is odd. More precisely, choosing any of the \( q^3 - q \) strict \( \mathbb{F}_{q^3} \)-points of \( C \) gives a strict \( \mathbb{F}_q \)-line, and since every such line cuts \( C \) in exactly two points we conclude that there are precisely \( \frac{1}{2}(q^3 - q) \) strict \( \mathbb{F}_{q^3} \)-lines through \( P \) intersecting \( C \) in two \( \mathbb{F}_{q^3} \)-points. Thus, the remaining

\[
q^3 - q - \frac{1}{2}(q^3 - q) = \frac{1}{2}(q^3 - q)
\]
\( \mathbb{F}_q^3 \)-lines through \( P_7 \) will intersect \( C \) in two \( \mathbb{F}_q^3 \)-points. If we pick one of them and label it \( P_1 \) we obtain an element in \( \Delta_i \cap \Delta_c \). Hence,

\[
|\Delta_i \cap \Delta_c| = (q^5 - q^2)q^2(q^3 - q).
\]

We now conclude that

\[
\left| (P_7^2)^{F^\sigma} \right| = q^6 - 2q^3 + 1.
\]

6.3. The case \( \lambda = [2, 5] \). Throughout this section, \( \lambda \) will denote the partition [2, 5]. We take \( U = (\mathbb{P}^2)^7 \).

**Lemma 6.10.** If \( (P_1, \ldots, P_7) \) is a \( \lambda \)-tuple with three of its points on a line, then all five \( \mathbb{F}_q^3 \)-points lie on a line defined over \( \mathbb{F}_q \). If six of the points lie on a smooth conic \( C \), then all seven points lie on \( C \) and \( C \) is defined over \( \mathbb{F}_q \).

**Proof.** The proof is very similar to the proofs of Lemmas 6.6 and 6.7 and is therefore omitted. \( \square \)

By Lemma 6.4, there are \( q^{10} + q^5 - q^2 - q \) conjugate quintuples whereof \((q^5 + q + 1)(q^5 - q) \) lie on a line. We may thus choose a conjugate quintuple whose points do not lie on a line in \( q^{10} - q^7 - q^6 + q^3 \) ways. This quintuple defines a smooth conic \( C \). By Lemma 6.10, it is enough to choose a conjugate pair outside \( C \) in order to obtain an element of \((\mathbb{P}^2)^7 \setminus \Delta\) of the desired type. Since there are \( q^4 - q \) conjugate pairs of which \( q^2 - q \) lie on \( C \) there are \( q^4 - q^2 \) remaining choices. We thus obtain

\[
\left| (P_7^2)^{F^\sigma} \right| = q^6 - q^2.
\]

6.4. The case \( \lambda = [1^2, 5] \). The computation in this case is very similar to that of the case \( \lambda = [2, 5] \) and we therefore simply state the result:

\[
\left| (P_7^2)^{F^\sigma} \right| = q^6 - q^2.
\]

6.5. The case \( \lambda = [3^1, 4^1] \). Throughout this section, \( \lambda \) shall mean the partition \([3^1, 4^1]\). Since we only need to make the computation for one permutation, we shall assume that the Frobenius permutes points \( P_1, P_2, P_3, \ldots, P_7 \) according to \((1234)\) and the three points \( P_5, P_6, P_7 \) according to \((567)\). We take \( U = (\mathbb{P}^2)^7 \).

**Lemma 6.11.** If a conjugate \( \lambda \)-tuple has three points on a line, then either

1. the four \( \mathbb{F}_q^3 \)-points lie on a \( \mathbb{F}_q \)-line, or
2. the three \( \mathbb{F}_q^3 \)-points lie on a \( \mathbb{F}_q \)-line.

**Proof.** It is easy to see that if three \( \mathbb{F}_q^3 \)-points lie on a line, then all four \( \mathbb{F}_q^3 \)-points lie on that line and even easier to see the corresponding result for three \( \mathbb{F}_q^3 \)-points.

Suppose that two \( \mathbb{F}_q^3 \)-points \( P_1 \) and \( P_2 \) and a \( \mathbb{F}_q^3 \)-point \( P \) lie on a line \( L \). Since \( F^4P_1 = P_1 \) and \( F^4P_2 = P_2 \) we see that \( FF^4L = L \). However, \( F^4P = FP \neq P \). Repeating this argument again, with \( FP \) in the place of \( P \), shows that also \( FP \) lies on \( L \). We are thus in case (1).

If we assume that two \( \mathbb{F}_q^3 \)-points and a \( \mathbb{F}_q^3 \)-point lie on a line, then a completely analogous argument shows that all four \( \mathbb{F}_q^3 \)-points lie on that line. \( \square \)
We decompose $\Delta_l$ as

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of tuples with the four $F_{q^4}$-points on a line and $\Delta_{l,2}$ consists of tuples with the three $F_{q^3}$-points on a line. The computations of $|\Delta_{l,1}|$, $|\Delta_{l,2}|$ and $|\Delta_{l,1} \cap \Delta_{l,2}|$ are completely straightforward and we get

$$|\Delta_l| = q^{13} + 2q^{12} - 3q^{10} - 2q^9 + q^8 + q^7 - q^6 - q^5 + q^4 + q^3.$$

To compute $|\Delta_c|$ we start by noting that if six of the points of a $\lambda$-tuple lie on a smooth conic $C$, then all seven points lie on $C$ and $C$ is defined over $F_q$. Thus, the problem consists of choosing a smooth conic $C$ over $F_q$ and then picking a $\lambda$-tuple on $C$. We thus have

$$|\Delta_c| = (q^5 - q^2)(q^4 - q^2)(q^3 - q).$$

Since no three points on a smooth conic lie on a line we conclude that the intersection $\Delta_l \cap \Delta_c$ is empty. We now obtain

$$\left(\left(\mathbb{P}^2\right)^{F_q}\right) = q^6 - q^5 - 2q^4 + q^3 + q^2.$$

6.6. The case $\lambda = [1, 2, 4]$. Throughout this section, $\lambda$ shall mean the partition $[1^1, 2^1, 4^1]$. Since we only need to make the computation for one permutation, we shall assume that the Frobenius permutes points $P_1, P_2, P_3, P_4$ according to $(1234)$, switches the two points $P_5, P_6$ and fixes the point $P_7$. The computation will turn out to be quite a bit more complicated in this case than in the previous cases, mainly because both 1 and 2 divide 4. We take $U = \left(\mathbb{P}^2\right)^7$.

We have the following trivial decomposition of $\Delta_l$

$$\Delta_l = \bigcup_{i=1}^{6} \Delta_{l,i},$$

where

- $\Delta_{l,1}$ consists of $\lambda$-tuples with three $F_{q^4}$-points lying on a line,
- $\Delta_{l,2}$ consists of $\lambda$-tuples with two $F_{q^4}$-points and a $F_{q^2}$-point lying on a line,
- $\Delta_{l,3}$ consists of $\lambda$-tuples with two $F_{q^4}$-points and the $F_{q^3}$-point lying on a line,
- $\Delta_{l,4}$ consists of $\lambda$-tuples with a $F_{q^4}$-point and two $F_{q^2}$-points lying on a line,
- $\Delta_{l,5}$ consists of $\lambda$-tuples with a $F_{q^4}$-point, a $F_{q^2}$-point and a $F_{q^3}$-point lying on a line, and,
- $\Delta_{l,6}$ consists of $\lambda$-tuples with two $F_{q^2}$-points and the $F_{q^3}$-point lying on a line.

This decomposition is of course naive and is not very nice to work with since none of the possible intersections are empty. The reader can surely think of many other decompositions which a priori look more promising. However, the more “clever” approaches we have tried have turned out to be quite hard to work with in practice.

The positive thing about the above decomposition is that most intersections are rather easily handled and that quadruple intersections (and higher) all consist of tuples where all seven points lie on a $F_q$-line.

The two slightly more complicated sets in the above list are $\Delta_{l,2}$ and $\Delta_{l,3}$. We shall therefore comment a bit about the computations involving them.
The set $\Delta_{l,2}$ splits into three disjoint subsets

$$\Delta_{l,2} = \Delta_{l,2}^1 \cup \Delta_{l,2}^2 \cup \Delta_{l,2}^3,$$

where

- $\Delta_{l,2}^1$ consists of $\lambda$-tuples such that the four $\mathbb{F}_q$-points and the two $\mathbb{F}_{q^2}$-points lie on a $\mathbb{F}_q$-line, or,
- $\Delta_{l,2}^2$ consists of $\lambda$-tuples such that the two $\mathbb{F}_q$-points and the $\mathbb{F}_{q^2}$-point lie on a $\mathbb{F}_q$-line $L$ (and the other two $\mathbb{F}_q$-points and the second $\mathbb{F}_{q^2}$-point lie on $FL$), or,
- $\Delta_{l,2}^3$ consists of $\lambda$-tuples such that the four $\mathbb{F}_q$-points and the two $\mathbb{F}_{q^2}$-points are intersection points of four conjugate $\mathbb{F}_{q^2}$-lines.

The sets $\Delta_{l,2}^2$ and $\Delta_{l,2}^3$ are illustrated in Figure 2 below. The cardinality of $\Delta_{l,2}^1$ is easily computed to be $(q^2 + q + 1)^2(q^4 - q^2)(q^2 - q)$. To get the cardinality of $\Delta_{l,2}^2$, we first choose a $\mathbb{F}_{q^2}$-line in $q^4 - q$ ways and then a $\mathbb{F}_q$-point $P_1$ on $L$ in $q^4 - q^2$ ways. This determines all the four $\mathbb{F}_{q^2}$ points since they must be $P_2 = FP_1$, $P_3 = F^2P_1$ and $P_4 = F^3P_1$. We must now decide if $P_5$ should lie on $L$ or $FL$. We then choose a $\mathbb{F}_q$-point on the chosen line. The lines $L$ and $FL$ both contain $q^2 + 1$ points defined over $\mathbb{F}_{q^2}$ of which precisely one is defined over $\mathbb{F}_q$ (namely the point $L \cap FL$). Hence, there are $q^2$ choices for $P_5$. It now only remains to choose $P_7$ in one of $q^2 + q + 1$ ways. We thus have

$$|\Delta_{l,2}^2| = 2(q^4 - q)(q^4 - q^2)(q^2 - q^2 + q + 1).$$

It remains to compute $|\Delta_{l,2}^3|$. We first choose a $\mathbb{F}_{q^2}$-point $P_5$ not defined over $\mathbb{F}_q$ in one of $q^4 - q$ ways. There are $q^4 - q^2$ lines $L$ strictly defined over $\mathbb{F}_{q^2}$ through $P_5$ and we choose one. We thus get four $\mathbb{F}_{q^2}$-lines which intersect in the two $\mathbb{F}_{q^2}$-points $P_5$ and $P_6$ as well as in four $\mathbb{F}_q$-points. We choose one of these to become $P_1$ and the labels of the other three points are then given. However, we could as well have chosen the line $F^2L$ and ended up with the same four $\mathbb{F}_{q^2}$-points. We therefore must divide by 2. Finally, we choose any of the $q^2 + q + 1$ $\mathbb{F}_q$-points to become $P_7$. We thus have

$$|\Delta_{l,2}^3| = 2(q^4 - q)(q^4 - q^2)(q^2 + q + 1).$$

**Figure 2.** Illustration of elements of the sets $\Delta_{l,2}^2$ and $\Delta_{l,2}^3$.

The set $\Delta_{l,3}$ splits into two disjoint subsets

$$\Delta_{l,3} = \Delta_{l,3}^1 \cup \Delta_{l,3}^2,$$

where
• \( \Delta_{1,3} \) consists of \( \lambda \)-tuples such that the four \( \mathbb{F}_q \)-points and the \( \mathbb{F}_q \)-point lie on a \( \mathbb{F}_q \)-line, or,

• \( \Delta_{2,3}^1 \) consists of \( \lambda \)-tuples such that there are two conjugate \( \mathbb{F}_q \)-lines intersecting in the \( \mathbb{F}_q \)-point, each \( \mathbb{F}_q \)-line containing two of the \( \mathbb{F}_q \)-points.

To compute \( |\Delta_{1,3}^1| \) we first choose a \( \mathbb{F}_q \)-line \( L \), then a conjugate quadruple and a \( \mathbb{F}_q \)-point on \( L \) and finally a conjugate pair of \( \mathbb{F}_q \)-points anywhere. Hence

\[
|\Delta_{1,3}^1| = (q^2 + q + 1)(q^4 - q^2)(q + 1)(q^4 - q).
\]

To compute \( |\Delta_{2,3}^1| \) we first choose a \( \mathbb{F}_q \)-line \( L \) not defined over \( \mathbb{F}_q \), then a \( \mathbb{F}_q \)-point \( P_1 \) not defined over \( \mathbb{F}_q \) on \( L \) and finally a pair of conjugate \( \mathbb{F}_q \)-points anywhere. We thus have

\[
|\Delta_{2,3}^1| = (q^4 - q)^2(q^4 - q^2).
\]

We now consider the intersection \( \Delta_{1,2} \cap \Delta_{1,3} \). The decompositions above yield a decomposition

\[
\Delta_{1,2} \cap \Delta_{1,3} = \bigcup_{i,j} \Delta_{i,j}^{1,j} \cap \Delta_{i,j}^{1,3}.
\]

The intersection \( \Delta_{1,2} \cap \Delta_{1,3} \) consists of configurations where all seven points lie on a \( \mathbb{F}_q \)-line. There are

\[
(q^2 + q + 1)(q^4 - q^2)(q^2 - q)(q + 1)
\]

such \( \lambda \)-tuples. Both the intersections \( \Delta_{1,2} \cap \Delta_{2,3}^1 \) and \( \Delta_{2,2} \cap \Delta_{1,3}^1 \) are empty.

To compute the cardinality of \( \Delta_{1,2} \cap \Delta_{1,3} \) we first choose a \( \mathbb{F}_q \)-line \( L \) in \( q^4 - q \) ways and then a strict \( \mathbb{F}_q \)-point \( P_1 \) on \( L \) in \( q^4 - q^2 \) ways. We must now decide if \( P_2 \) should lie on \( L \) or \( FL \). We then choose a strict \( \mathbb{F}_q \)-point on the chosen line in \( q^2 \) ways. We are now sure to have a tuple in \( \Delta_{2,2} \). To make sure that the tuple also lies in \( \Delta_{1,3} \) we have no choice but to put \( P_7 \) at the intersection of \( L \) and \( FL \). There are thus

\[
2(q^4 - q)(q^4 - q^2)q^2
\]

elements in the intersection \( \Delta_{1,2} \cap \Delta_{1,3}^1 \).

The intersection \( \Delta_{1,2} \cap \Delta_{1,3} \) is empty so there is only one intersection remaining. As explained in the computation of \( |\Delta_{1,2}^1| \), there are \( 2(q^4 - q)(q^4 - q^2) \) ways to obtain four strict \( \mathbb{F}_q \)-points and two strict \( \mathbb{F}_q \)-points which are the intersection points of four conjugate \( \mathbb{F}_q \)-lines. We now note that there are precisely six lines through pairs of points among the four \( \mathbb{F}_q \)-points. Four of these lines are of course the four \( \mathbb{F}_q \)-lines. The remaining two lines are defined over \( \mathbb{F}_q \) and therefore intersect in a \( \mathbb{F}_q \)-point. To obtain a tuple in \( \Delta_{1,3}^1 \) we have no choice but to choose \( P_7 \) as this intersection point. Therefore, there are

\[
2(q^4 - q)(q^4 - q^2),
\]

elements in the intersection \( \Delta_{1,2} \cap \Delta_{1,3}^1 \).

The remaining computations are rather straightforward. One obtains

\[
|\Delta_l| = q^{13} + 5q^{12} - 4q^{10} - 5q^9 - 3q^8 + 2q^7 + q^6 + 3q^5 + q^4 - q^3.
\]

We now turn to \( \Delta_c \). We have that if six points of a conjugate \( \lambda \)-tuple lie on a smooth conic \( C \), then the four \( \mathbb{F}_q \)-points and the two \( \mathbb{F}_q \)-points lie on \( C \) and \( C \) is defined over \( \mathbb{F}_q \). Thus, the computation of \( |\Delta_c| \) consists of choosing a smooth conic
C defined over $\mathbb{F}_q$, choose a $\mathbb{F}_{q^1}$-point and a $\mathbb{F}_{q^2}$-point on $C$ and finally choose a $\mathbb{F}_q$-point anywhere. We thus have

$$|\Delta_c| = (q^5 - q^2)(q^4 - q^2)(q^2 - q)(q^2 + q + 1).$$

It remains to investigate the intersection $\Delta_l \cap \Delta_c$. Since a smooth conic cannot contain three collinear points, we only have nonempty intersection between $\Delta$ and the sets $\delta_{l,3}^F$ and $\delta_{l,6}^F$.

To compute $|\Delta_{l,3}^F \cap \Delta_c|$ we first choose a smooth conic $C$, then a conjugate quadruple on $C$ and finally a pair of conjugate $\mathbb{F}_{q^2}$-points on $C$. The $\mathbb{F}_q$-point is then uniquely defined as the intersection point of the two $\mathbb{F}_{q^2}$-lines through pairs of the four $\mathbb{F}_{q^1}$-points. We thus have

$$|\Delta_{l,3}^F \cap \Delta_c| = (q^5 - q^2)(q^4 - q^2)(q^2 - q).$$

The same construction as above works for the intersection $\Delta_{l,6} \cap \Delta_c$, if we remember that we now have some choice for the $\mathbb{F}_q$-point since it can lie anywhere on the line through the two $\mathbb{F}_{q^2}$-points. We thus have

$$|\Delta_{l,6} \cap \Delta_c| = (q^5 - q^2)(q^4 - q^2)(q^2 - q)(q + 1).$$

The only thing that remains to compute is the cardinality of the triple intersection $\Delta_{l,3}^F \cap \Delta_{l,6} \cap \Delta_c$. We first assume that the $\mathbb{F}_q$-point is on the outside of a smooth conic $C$ containing the other six points. We first choose $C$ in $q^5 - q^2$ ways. There are $\frac{1}{2}(q + 1)q$ ways to choose two $\mathbb{F}_q$-points $P$ and $Q$ on $C$. Intersecting the tangents $T_P C$ and $T_Q C$ gives a $\mathbb{F}_q$-point $P_l$ which will clearly lie on the outside of $C$. Hence, there are precisely $\frac{1}{2}(q + 1)q$ ways to choose a $\mathbb{F}_q$-point on the outside of $C$.

We now want to choose a $\mathbb{F}_q$-line through $P_l$ intersecting $C$ in two $\mathbb{F}_{q^2}$-points. There are $q + 1$ $\mathbb{F}_q$-lines through $P_l$ of which two are tangents to $C$. These tangent lines contain a $\mathbb{F}_q$-point of $C$ each so there are $q - 1$ remaining $\mathbb{F}_q$-points on $C$. Picking such a point gives a line through this point, $P_l$ and one further point on $C$. We thus see that exactly $\frac{1}{2}(q - 1)$ of the $\mathbb{F}_q$-lines through $P_l$ intersect $C$ in two $\mathbb{F}_q$-points. Hence, there are precisely

$$q + 1 - 2 - \frac{1}{2}(q - 1) = \frac{1}{2}(q - 1)$$

$\mathbb{F}_q$-lines through $P_l$ which intersect $C$ in two $\mathbb{F}_{q^2}$-points. These points are clearly conjugate under $F$. We label one of them as $P_3$.

We shall now choose a conjugate pair of $\mathbb{F}_{q^2}$-lines through $P_l$ intersecting $C$ in four $\mathbb{F}_{q^3}$-points. There are $q^2 - q$ conjugate pairs of $\mathbb{F}_{q^2}$-lines through $P_l$. No $\mathbb{F}_{q^2}$-line through $P_l$ is tangent to $C$ so each $\mathbb{F}_{q^2}$-line through $P_l$ will intersect $C$ in two points. The conic $C$ contains $q^2 - q$ points which are defined over $\mathbb{F}_{q^2}$ but not $\mathbb{F}_q$. Picking such a point gives a line through this point and $P_l$ as well as one further $\mathbb{F}_{q^2}$-point not defined over $\mathbb{F}_q$. Thus, there are $\frac{1}{2}(q^2 - q)$ lines obtained in this way. Typically, such a line will be defined over $\mathbb{F}_{q^2}$ but not $\mathbb{F}_q$. We saw above that the number of such lines which are defined over $\mathbb{F}_q$ is precisely $\frac{1}{2}(q - 1)$. Thus, there are precisely

$$\frac{1}{2}(q^2 - q) - \frac{1}{2}(q - 1) = \frac{1}{2}(q^2 - 2q + 1)$$
\( F_{q^2} \)-lines, not defined over \( F_q \), which intersect \( C \) in two \( F_{q^4} \)-points. Thus, the remaining

\[
(6.2) \quad q^2 - q = \frac{1}{2}(q^2 - 2q + 1) = \frac{1}{2}(q^2 - 1)
\]

\( F_{q^2} \)-lines must intersect \( C \) in two \( F_{q^4} \)-points. Picking such a line and labelling one of the points \( P_1 \) gives a configuration belonging to \( \Delta_{l,3} \cap \Delta_{l,6} \cap \Delta_c \) and we thus see that there are

\[
\frac{1}{2}(q^5 - q^2)q(q + 1)(q - 1)(q^2 - 1)
\]
such configurations with \( P_7 \) on the outside of \( C \).

We now assume that \( P_7 \) is on the inside of \( C \). We first choose \( C \) in \( q^5 - q^2 \) ways. Since the number of \( F_q \)-points is \( q^2 + q + 1 \) and \( q + 1 \) of these lie on \( C \) the number of \( F_q \)-points not on \( C \) is precisely \( q^2 \). We just saw that \( \frac{1}{2}(q + 1)q \) of these lie on the outside of \( C \) so there must be

\[
\frac{1}{2}(q^2 - q)(q^2 - q^2 + 1) = \frac{1}{2}(q^2 - 1)
\]

\( F_q \)-points which lie on the inside of \( C \).

Since \( P_7 \) now lies on the inside of \( C \), every \( F_q \)-line through \( P_7 \) will intersect \( C \) in two points. Exactly \( \frac{1}{2}(q + 1) \) will intersect \( C \) in two \( F_q \)-points so the remaining \( \frac{1}{2}(q + 1) \) will intersect \( C \) in two conjugate \( F_{q^2} \)-points. We pick such a pair of points and label one of them \( P_5 \).

We now choose a conjugate pair of \( F_{q^2} \)-lines through \( P_7 \) intersecting \( C \) in a conjugate quadruple of \( F_{q^4} \)-points. The number of \( F_{q^2} \)-lines, not defined over \( F_q \), through \( P_7 \) is \( q^2 - q \). Two of these are tangent to \( C \) so, using ideas similar to those above, we see that

\[
\frac{1}{2}(q^2 - q - 2) - \frac{1}{2}(q + 1) + 2 = \frac{1}{2}(q^2 - 2q + 1),
\]
of these lines intersect \( C \) in points defined over \( F_{q^2} \). Hence, the remaining

\[
(6.3) \quad q^2 - q - \frac{1}{2}(q^2 - 2q + 1) = \frac{1}{2}(q^2 - 1)
\]

lines intersect \( C \) in two \( F_{q^4} \)-points which are not defined over \( F_{q^2} \). If we pick one of these points to become \( P_1 \) we end up with a configuration in \( \Delta_{l,3} \cap \Delta_{l,6} \cap \Delta_c \). We thus have

\[
\frac{1}{2}(q^5 - q^2)(q^2 - q)(q + 1)(q^2 - 1)
\]
such configurations with \( P_7 \) on the inside of \( C \). One may note that the expression above actually is the same as the expression when \( P_7 \) was on the outside, but this will not always be the case.

We thus have

\[
|\Delta_l \cap \Delta_c| = q^{12} + q^{11} - 4q^{10} - 2q^9 + 3q^8 + 4q^7 - 4q^5 + q^3,
\]

and finally also

\[
\left| (P_7^2)^{F_q} \right| = q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3.
\]
6.7. The case $\lambda = [1^3, 4]$. Throughout this section, $\lambda$ shall mean the partition $[1^3, 4]$. We shall assume that the Frobenius automorphism permutes the points $P_1, P_2, P_3, P_4$ according to $(1234)$ and fixes the $F_q$-points $P_5$, $P_6$ and $P_7$. Let $U$ be the open subset of $(\mathbb{P}^2)^7$ consisting of septuples such that the last three points of the tuple are not collinear. In other words, we choose any conjugate quadruple and three $F_q$-points which do not lie on a line.

We can decompose $\Delta_l$ into a disjoint as

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of septuples such that all four $F_q$-points lie on a $F_q$-line and $\Delta_{l,2}$ consists of septuples such that the $F_q^2$-line through $P_1$ and $P_3$ intersects the $F_q^2$-line through $P_2$ and $P_4$ in $P_5$, $P_6$ or $P_7$.

To compute $|\Delta_{l,1}|$ we simply choose a $F_q$-line $L$, a conjugate quadruple on $L$ and finally place the three $F_q$-points in such a way that they do not lie on a line. This can be done in

$$(q^2 + q + 1)^2(q^4 - q^2)(q^2 + q)q^2$$

ways. To compute $|\Delta_{l,2}|$ we first choose $P_5$, $P_6$ or $P_7$ then a $F_q^2$-line $L$ not defined over $F_q$ through this point. Finally, we choose a $F_q^3$-point $P_1$ which is not defined over $F_q^2$ on $L$ and make sure that the final two $F_q$-points are not collinear with the first. This can be done in

$$3(q^2 + q + 1)(q^2 - q)(q^4 - q^2)(q^2 + q)q^2$$

ways. This gives that

$$|\Delta_l| = 4q^{12} + 6q^{11} + q^{10} - 4q^9 - 5q^8 - q^7 - q^5.$$

We now turn to investigate $\Delta_c$. It is not hard to see that if six of the points lie on a smooth conic $C$, then the four $F_q$-points must lie on that conic and $C$ must be defined over $F_q$. We thus choose a smooth conic $C$ over $F_q$ and a conjugate quadruple on $C$. Then we choose $P_5$, $P_6$ or $P_7$ to possibly not lie on $C$ and place the other two on $C$. Finally, we place the final $F_q$-point anywhere except on the line through the other two $F_q$-points. This gives the number

$$3(q^5 - q^2)(q^4 - q^2)(q + 1)q \cdot q^2.$$  

However, we have now counted the configurations where all seven points lie on a smooth conic three times. There are

$$(q^5 - q^2)(q^4 - q^2)(q + 1)q(q - 1)$$

such configurations and it thus follows that

$$|\Delta_c| = 3q^{13} + q^{12} - 3q^{11} - 2q^{10} - q^9 + q^8 - q^7 + 2q^5.$$

We now turn to the intersection $\Delta_l \cap \Delta_c = \Delta_{l,1} \cap \Delta_c$. We begin by choosing one of the $F_q^3$-points $P_5$, $P_6$ and $P_7$ to not lie on the conic $C$. We call the chosen point $P$ and the remaining two points $P_i$ and $P_j$ where $i < j$. We now have three disjoint possibilities:

(i) the point $P$ may lie on the outside of $C$ with one of the tangents through $P$ also passing through $P_i$,

(ii) the point $P$ may lie on the outside of $C$ with none of the tangents through $P$ passing through $P_i$,

(iii) the point $P$ may lie on the inside of $C$. 

We consider the three cases (i)-(iii) separately.

(i). We begin by choosing a smooth conic $C$ in $q^5 - q^2$ ways and an $\mathbb{F}_q$-point $P$ on the outside of $C$ in $\frac{1}{2}(q + 1)q$ ways. By Equation 6.2, there are now $q^2 - 1$ ways to choose $P_1$. Since we require $P_i$ to lie on a tangent to $C$ which passes through $P$, we only have 2 choices for $P_i$. Finally, we may choose $P_j$ as any of the $q$ remaining points on $C$. We thus have

$$3(q^5 - q^2)(q + 1)q(q^2 - 1)q$$

possibilities in this case.

(ii). Again, we begin by choosing a smooth conic $C$ in $q^5 - q^2$ ways and an $\mathbb{F}_q$-point $P$ on the outside of $C$ in $\frac{1}{2}(q + 1)q$ ways and choose $P_1$ in one of $q^2 - 1$ ways. The point $P_i$ should now not lie on a tangent to $C$ which passes through $P$ so we have $q - 1$ choices. Since the line between $P$ and $P_i$ is not a tangent to $C$, there is one further intersection point between this line and $C$. We must choose $P_j$ away from this point and $P_i$ and thus have $q - 1$ possible choices. Hence, we have

$$\frac{3}{2}(q^5 - q^2)(q + 1)q(q^2 - 1)(q - 1)^2$$

possibilities in this case.

(iii). We start by choosing a smooth conic $C$ in $q^5 - q^2$ ways and then a point $P$ on the inside of $C$ in $\frac{1}{2}(q^2 - q)$ ways. We continue by using Equation 6.3 to see that we can choose $P_1$ in $q^2 - 1$ ways. We now choose $P_i$ as any of the $\mathbb{F}_q$-points on $C$ and thus have $q + 1$ choices. Finally, we may choose $P_j$ as any $\mathbb{F}_q$-point on $C$, except in the intersection of $C$ with the line through $P_i$ and $P$. Hence, we have $q - 1$ choices. We thus have

$$\frac{3}{2}(q^5 - q^2)(q^2 - q)(q^2 - 1)(q + 1)(q - 1)$$

possibilities in this case.

We now conclude that

$$|\Delta_i \cap \Delta_e| = 3q^{11} - 3q^9 - 3q^5 + 3q^3,$$

and, finally,

$$|\langle \mathcal{P}_i^2 \rangle^{\mathcal{F}^q}| = q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3.$$

6.8. The case $\lambda = [1, 3^2]$. Throughout this section, $\lambda$ shall mean the partition $[1^1, 3^2]$. We shall use the notation $P_1, P_2, P_3$ for the first conjugate triple and $Q_1$, $Q_2, Q_3$ for the second. The $\mathbb{F}_q$-point will be denoted by $R$. We take $U = (\mathbb{F}^2)^3$.

We decompose $\Delta_i$ as

$$\Delta_i = \bigcup_{i=1}^{5} \Delta_{i,i},$$

where

- $\Delta_{i1}$ consists of $\lambda$-tuples such that the points $P_1, P_2$ and $P_3$ lie on a $\mathbb{F}_q$-line,
- $\Delta_{i2}$ consists of $\lambda$-tuples such that the points $P_1, P_2$ and $P_3$ are the intersection points of a conjugate triple of $\mathbb{F}_q$-lines with each of the lines containing one of the points $Q_1, Q_2$ and $Q_3$,
• \(\Delta_{i,3}\) consists of \(\lambda\)-tuples such that the points \(Q_1, Q_2\) and \(Q_3\) are the intersection points of a conjugate triple of \(\mathbb{F}_q^3\)-lines with each of the lines containing one of the points \(P_1, P_2\) and \(P_3\).

• \(\Delta_{i,4}\) consists of \(\lambda\)-tuples such that the points \(Q_1, Q_2\) and \(Q_3\) lie on a \(\mathbb{F}_q\)-line, and

• \(\Delta_{i,5}\) consists of \(\lambda\)-tuples such that the point \(R\) is the intersection of three conjugate \(\mathbb{F}_q^3\)-lines with each of the \(\mathbb{F}_q^3\)-lines containing one of the points \(P_1, P_2\) and \(P_3\) and one of the points \(Q_1, Q_2\) and \(Q_3\).

\[
\begin{align*}
\Delta_{i,1} & \quad \Delta_{i,2} \\
\Delta_{i,3} & \quad \Delta_{i,4} \\
\Delta_{i,5} & 
\end{align*}
\]

**Figure 3.** Illustration of the decomposition of \(\Delta_i\).

6.8.1. \(\Delta_{i,1}\) and \(\Delta_{i,4}\). The cardinalities of \(\Delta_{i,1}\) and \(\Delta_{i,4}\) are obviously the same so we only make the computation for \(\Delta_{i,1}\). We thus choose a \(\mathbb{F}_q\)-line \(L\), a conjugate \(\mathbb{F}_q^3\)-tuple \(P_1, P_2, P_3\) on \(L\), a conjugate \(\mathbb{F}_q^3\)-tuple \(Q_1, Q_2, Q_3\) anywhere except equal to the other \(\mathbb{F}_q^3\)-tuple and, finally, a \(\mathbb{F}_q\)-point anywhere. We thus get

\[
|\Delta_{i,1}| = |\Delta_{i,4}| = (q^2 + q + 1)^2(q^3 - q)(q^6 + q^3 - q^2 - q - 3).
\]

6.8.2. \(\Delta_{i,2}\) and \(\Delta_{i,3}\). The cardinalities of \(\Delta_{i,2}\) and \(\Delta_{i,3}\) are of course also the same. To compute \(|\Delta_{i,2}|\) we first choose a conjugate triple of lines, \(L, FL, F^2L\), which do not intersect in a point. There are \(q^6 + q^3 - q^2 - q\) conjugate triples of lines of which \((q^2 + q + 1)(q^3 - q)\) intersect in a point. There are thus \(q^6 - q^5 - q^4 + q^3\) possible triples. We give the label \(P_1\) to the point \(L \cap FL\) which determines the labels of the other two intersection points. We must now choose if \(Q_1\) should lie on \(L, FL\) or \(F^2L\) and then place \(Q_1\) on the chosen line. There are \(3(q^3 - 1)\) ways to do this. Finally, we choose any \(\mathbb{F}_q\)-point. We thus have

\[
|\Delta_{i,2}| = |\Delta_{i,3}| = 3(q^6 - q^5 - q^4 + q^3)(q^3 - 1)(q^2 + q + 1).
\]

6.8.3. \(\Delta_{i,5}\). We first choose a \(\mathbb{F}_q\)-point \(R\) anywhere and then a conjugate triple of lines, \(L, FL, F^2L\) through \(R\). We then choose a point \(P_1\) somewhere on \(L\) in \(q^3\) ways. We must now decide if \(Q_1\) should lie on \(L, FL\) or \(F^2L\) and then pick a point on the chosen line in one of \(q^3 - 1\) ways. We thus see that

\[
|\Delta_{i,5}| = 3(q^2 + q + 1)(q^3 - q)q^3(q^3 - 1).
\]

We must now compute the cardinalities of the different intersections. Firstly, note that the intersection between \(\Delta_{i,1}\) and \(\Delta_{i,2}\) is empty. Secondly, the size of the intersection of \(\Delta_{i,1}\) and \(\Delta_{i,3}\) is equal to that of the intersection of \(\Delta_{i,2}\) and \(\Delta_{i,4}\). To obtain \(|\Delta_{i,1} \cap \Delta_{i,3}|\) we first choose a conjugate triple of lines, \(L, FL, F^2L\), which do not intersect in a point and label the intersection \(L \cap FL\) by \(Q_1\). We then choose a \(\mathbb{F}_q\)-line \(L'\) and thus get three \(\mathbb{F}_q^3\)-points \(L' \cap L, L' \cap FL\) and \(L' \cap F^2L\). We label
one of these by $P_1$. We may now choose any $\mathbb{F}_q$-point to become $R$. We thus see that that

$$|\Delta_{l,1} \cap \Delta_{l,3}| = |\Delta_{l,2} \cap \Delta_{l,4}| = 3(q^6 - q^5 - q^4 + q^3)(q^2 + q + 1)^2.$$ 

When we consider the intersection between $\Delta_{l,1}$ and $\Delta_{l,4}$ we must distinguish between the cases where the two triples lie on the same line and when they do not. A simple computation then gives

$$|\Delta_{l,1} \cap \Delta_{l,4}| = (q^2 + q + 1)^2(q^3 - q)(q^3 - q - 3) + (q^2 + q + 1)^2(q^3 + q)(q^3 - q)^2.$$ 

We continue by observing that $|\Delta_{l,1} \cap \Delta_{l,5}| = |\Delta_{l,4} \cap \Delta_{l,5}|$. To compute $|\Delta_{l,1} \cap \Delta_{l,5}|$ we first choose a $\mathbb{F}_q$-point $R$ and then a conjugate $\mathbb{F}_q$-tuple of lines $L$, $FL$, $F^2L$ through $R$. We continue by choosing a $\mathbb{F}_q$-line $L'$ not through $R$ in one of $q^2$ ways and then label $L' \cap L$, $L' \cap FL$ or $L' \cap F^2L$ by $Q_1$. Finally, we choose one of the remaining $q^3 - 1$ points of $L$ to become $P_1$. Hence,

$$|\Delta_{l,1} \cap \Delta_{l,1}| = |\Delta_{l,4} \cap \Delta_{l,5}| = 3(q^2 + q + 1)(q^3 - q)q^2(q^3 - 1).$$

The sets $\Delta_{l,2}$ and $\Delta_{l,3}$ do not intersect and neither do the sets $\Delta_{l,3}$ and $\Delta_{l,4}$. Hence, there are only two intersections left, namely the one between $\Delta_{l,2}$ and $\Delta_{l,5}$ and the one between $\Delta_{l,3}$ and $\Delta_{l,5}$. These have equal cardinalities. To compute $|\Delta_{l,2} \cap \Delta_{l,5}|$ we first choose a conjugate triple of lines, $L$, $FL$, $F^2L$, which do not intersect in a point and label the intersection $L \cap FL$ by $Q_1$. We then choose a $\mathbb{F}_q$-point $R$. The lines between $R$ and the points $Q_1$, $Q_2$ and $Q_3$ intersect the lines $L$, $FL$ and $F^2L$ in three points and we label one of them by $P_1$. We thus have

$$|\Delta_{l,2} \cap \Delta_{l,5}| = |\Delta_{l,3} \cap \Delta_{l,5}| = 3(q^6 - q^5 - q^4 + q^3)(q^2 + q + 1).$$

There is only one triple intersection, namely between $\Delta_{l,1}$, $\Delta_{l,4}$ and $\Delta_{l,5}$. To compute the size of this intersection we first choose a $\mathbb{F}_q$-point $R$ and then a conjugate triple of lines, $L$, $FL$, $F^2L$ through $R$. We then choose a $\mathbb{F}_q$-line $L'$ not through $R$ and label the intersection $L \cap L'$ by $P_1$. Finally, we choose another $\mathbb{F}_q$-line $L''$ and label one of the intersections $L'' \cap L$, $L'' \cap FL$ and $L'' \cap F^2L$ by $Q_1$. This shows that

$$|\Delta_{l,1} \cap \Delta_{l,4} \cap \Delta_{l,5}| = 3(q^2 + q + 1)(q^3 - q)q^2(q^2 - 1).$$

We now turn to the computation of $|\Delta_c|$. If six points of a $\lambda$-tuple lie on a smooth conic $C$, then both of the conjugate triples must lie on $C$ and $C$ must be defined over $\mathbb{F}_q$. Hence, to obtain $|\Delta_c|$ we only have to choose a smooth $\mathbb{F}_q$-conic $C$, two conjugate triples on $C$ and a $\mathbb{F}_q$-point anywhere. We thus have that,

$$|\Delta_c| = (q^5 - q^2)(q^3 - q)(q^3 - q - 3)(q^2 + q + 1).$$

Since the sets $\Delta_{l,1}$, $\Delta_{l,2}$, $\Delta_{l,3}$ and $\Delta_{l,4}$ all require three of the $\mathbb{F}_q$-points to lie on a line, they will have empty intersection with $\Delta_c$. This is however not true for the set $\Delta_{l,5}$. To obtain such a configuration we first choose a smooth conic $C$ and a $\mathbb{F}_q$-point $R$. Now choose a $\mathbb{F}_q$-point $P_1$ on $C$ which is not defined over $\mathbb{F}_q$ in $q^3 - q$ ways. Since both $C$ and $R$ are defined over $\mathbb{F}_q$ we know that any tangent to $C$ which passes through $R$ must either be defined over $\mathbb{F}_q$ (or $\mathbb{F}_q$). Hence, the line through $R$ and $P_1$ will intersect $C$ in $P_1$ and one point more. We label this point with $Q_1$, $Q_2$ or $Q_3$. We thus have

$$|\Delta_c \cap \Delta_{l,5}| = 3(q^5 - q^2)q^2(q^3 - q).$$
We now obtain
\[
\left| \left( \mathbb{P}_7^2 \right)^{F \sigma} \right| = q^6 - 2q^5 - 2q^4 - 8q^3 + 16q^2 + 10q + 21.
\]

### 6.9. The case $\lambda = [2^2, 3]$.
Throughout this section, $\lambda$ shall mean the partition $[2^2, 3^1]$. We shall use the notation $P_1, P_2, P_3$ for the conjugate triple and $Q_1, Q_2$ and $R_1, R_2$ for the two conjugate pairs. We let $U = (\mathbb{P}^2)^7$.

We can decompose $\Delta_l$ as
\[
\Delta_l = \Delta_{l,1} \cup \Delta_{l,2},
\]
where $\Delta_{l,1}$ consists of septuples such that the three $\mathbb{F}_{q^2}$-points lie on a line and $\Delta_{l,2}$ consists of septuples such that the four $\mathbb{F}_q$-points lie on a line.

We have
\[
\left| \Delta_{l,1} \right| = (q^2 + q + 1)(q^3 - q)(q^4 - q)(q^4 - q - 2),
\]
and
\[
\left| \Delta_{l,2} \right| = (q^2 + q + 1)(q^2 - q)(q^2 - q - 2)(q^6 + q^3 - q^2 - q).
\]
The cardinality of the intersection is easily computed to be
\[
\left| \Delta_{l,1} \cap \Delta_{l,2} \right| = (q^2 + q + 1)^2(q^2 - q)(q^2 - q - 2).
\]
This allows us to compute $|\Delta_l|$.

We have that if six of the points of a $\lambda$-tuple lie on a smooth conic $C$, then all seven points lie on $C$ and $C$ is defined over $\mathbb{F}_q$. We thus have that $\Delta_c$ is disjoint from $\Delta_l$. We also see that
\[
\left| \Delta_c \right| = (q^5 - q^2)(q^3 - q)(q^2 - q)(q^2 - q - 2),
\]
so,
\[
\left| \left( \mathbb{P}_7^2 \right)^{F \sigma} \right| = q^6 - 2q^5 - 2q^4 + 3q^3 + q^2 - 2q.
\]

### 6.10. The case $\lambda = [1^2, 2, 3]$.
Throughout this section, $\lambda$ shall mean the partition $[1^2, 2^1, 3^1]$. We shall use the notation $P_1, P_2, P_3$ for the conjugate triple, $Q_1, Q_2$ for the conjugate pair and use $R_6$ and $R_7$ to denote the two $\mathbb{F}_{q^2}$-points. Let $U = (\mathbb{P}^2)^7$.

We decompose $\Delta_l$ as
\[
\Delta_l = \Delta_{l,1} \cup \Delta_{l,2},
\]
where $\Delta_{l,1}$ consists of $\lambda$-tuples such that the three $\mathbb{F}_{q^2}$-points lie on a $\mathbb{F}_q$-line and $\Delta_{l,2}$ consists of $\lambda$-tuples such that the two $\mathbb{F}_{q^2}$-points and one of the $\mathbb{F}_q$-points lie on a $\mathbb{F}_{q^2}$-line.

We have
\[
\left| \Delta_{l,1} \right| = (q^2 + q + 1)^2(q^3 - q)(q^4 - q)(q^2 + q).
\]
We decompose $\Delta_{l,2}$ as
\[
\Delta_{l,2} = \Delta_{l,2}^6 \cup \Delta_{l,2}^7,
\]
where $\Delta_{l,2}^6$ consists of tuples such that the line through the two $\mathbb{F}_{q^2}$-points passes through $R_i$. We have
\[
\left| \Delta_{l,2}^6 \right| = \left| \Delta_{l,2}^7 \right| = (q^2 + q + 1)(q^2 - q)(q + 1)(q^6 + q^3 - q^2 - q)(q^2 + q).
\]
We now turn to the double intersections. We have
\[
\left| \Delta_{l,1} \cap \Delta_{l,2}^6 \right| = \left| \Delta_{l,1} \cap \Delta_{l,2}^7 \right| = (q^2 + q + 1)^2(q^3 - q)(q^2 - q)(q + 1)(q^2 + q),
\]
and
\[
\left| \Delta_{l,2}^6 \cap \Delta_{l,2}^7 \right| = (q^2 + q + 1)(q^2 - q)(q + 1)q(q^6 + q^3 - q^2 - q).
\]
Finally, we compute the cardinality of the intersection of all three sets
\[ |\Delta_{l,1} \cap \Delta_{l,2}^6 \cap \Delta_{l,2}^7| = (q^2 + q + 1)^2(q^3 - q)(q^2 - q)(q + 1)q. \]
This now allows us to compute \( |\Delta_4| \).

If six points of a \( \lambda \)-tuple lie on a smooth conic \( C \), then the three \( \mathbb{F}_{q^3} \)-points and the two \( \mathbb{F}_{q^2} \)-points lie on \( C \) and \( C \) is defined over \( \mathbb{F}_q \). Thus, to compute \( |\Delta_4| \) we begin by choosing a smooth conic \( C \) over \( \mathbb{F}_q \). We then choose a conjugate triple and a conjugate pair of \( \mathbb{F}_{q^2} \)-points on \( C \). Then, we choose either \( R_6 \) or \( R_7 \) and place the chosen point on \( C \). Finally, we place the remaining point anywhere we want. We thus obtain the number
\[ 2(q^5 - q^2)(q^3 - q)(q^2 - q)(q + 1)(q + 2). \]
However, in the above we have counted the configurations where all seven points lie on the conic twice. We thus have to take away
\[ (q^5 - q^2)(q^3 - q)(q^2 - q)(q + 1), \]
in order to obtain \( |\Delta_4| \).

It only remains to compute the cardinality of the intersection \( \Delta_4 \cap \Delta_c \). We only have nonempty intersection between the set \( \Delta_c \) and the set \( \Delta_{l,2} \). To compute the cardinality of this intersection, we only have to make sure to choose the point \( R_6 \) (resp. \( R_7 \)) on the line through the two \( \mathbb{F}_{q^2} \)-points. Hence, we have
\[ |\Delta_{l,2}^6 \cap \Delta_c| = |\Delta_{l,2}^7 \cap \Delta_c| = (q^5 - q^2)(q^3 - q)(q^2 - q)(q + 1)^2, \]
and, therefore,
\[ |\Delta_4 \cap \Delta_c| = 2(q^5 - q^2)(q^3 - q)(q^2 - q)(q + 1)^2. \]
This gives us
\[ |(\mathbb{P}^2_{q^2})^{F_0}| = q^6 - 3q^5 - q^4 + 5q^3 - 2q. \]

6.11. The case \( \lambda = [1^4, 3^1] \). Throughout this section, \( \lambda \) shall mean the partition \( [1^4, 3^1] \). We shall denote the four \( \mathbb{F}_q \)-points by \( P_1, P_2, P_3 \) and \( P_4 \) and denote the conjugate triple by \( Q_1, Q_2, Q_3 \). Let \( U \subset (\mathbb{P}^2)^7 \) be the subset consisting of septuples of points with the first four in general position.

A septuple in \( \Delta_4 \) will have the three \( \mathbb{F}_{q^3} \)-points on a \( \mathbb{F}_q \)-line. Thus, to compute the size of \( \Delta_4 \), we only need to place the four \( \mathbb{F}_{q^2} \)-points in general position, choose \( \mathbb{F}_q \)-line \( L \) and place the conjugate \( \mathbb{F}_{q^3} \)-tuple on \( L \). We thus have,
\[ |\Delta_4| = (q^2 + q + 1)(q^2 + q)(q^2 - 2q + 1)(q^2 + q + 1)(q^3 - q). \]

A septuple in \( \Delta_c \) will have the three \( \mathbb{F}_{q^3} \)-points on a smooth conic \( C \) defined over \( \mathbb{F}_q \). Thus, to compute \( |\Delta_c| \), we first choose a smooth conic \( C \) defined over \( \mathbb{F}_q \) and then a conjugate triple on \( C \). We then choose one of the points \( P_1, P_2, P_3 \) and \( P_4 \) to possibly not lie on \( C \). Call this point \( P \). We then place the other three points on \( C \). These three points define three lines which, in total, contain \((q + 1) + q + (q - 1) = 3q \) points. As long as we choose \( P \) away from these points, the four \( \mathbb{F}_q \)-points will be in general position. We thus obtain
\[ 4(q^5 - q^2)(q^3 - q)(q + 1)q(q - 1)(q^2 - 2q + 1). \]
However, we have counted the septuples with all seven points on a smooth conic four times. We thus need to take away
\[ 3(q^5 - q^2)(q^3 - q)(q + 1)q(q - 1)(q - 2). \]
To compute \(\Delta\) element of.

The cardinalities of these sets are easily computed to be that no three lie on a line, then the line through a point lie on a

6.12. The case \(\lambda = [1, 2^3]\). Throughout this section, \(\lambda\) shall mean the partition [1, 1, 2]. We shall denote the three conjugate pairs of \(F_q\)-points by \(P_1\), \(P_2\), \(Q_1\), \(Q_2\) and \(R_1\), \(R_2\) and the \(\mathbb{F}_q\)-point by \(O\). Let \(U \subset (\mathbb{P}^2)^7\) be the subset consisting of septuples of points such that the first six points have no three on a line.

We decompose \(\Delta_1\) as

\[
\Delta_1 = \Delta_{1,1} \cup \Delta_{1,2},
\]

where \(\Delta_{1,1}\) consists of those septuples where two conjugate \(F_q\)-points and the \(\mathbb{F}_q\)-point lie on a \(\mathbb{F}_q\)-line and \(\Delta_{1,2}\) consists of those septuples where two conjugate \(F_q\)-lines, containing two \(F_q\)-points each, intersect in the point defined over \(\mathbb{F}_q\).

The set \(\Delta_{1,1}\) naturally decomposes into three equally large, but not disjoint, subsets:

- the set \(\Delta_{1,1}^a\) where \(P_1\), \(P_2\) and \(O\) lie on a \(\mathbb{F}_q\)-line,
- the set \(\Delta_{1,1}^b\) where \(Q_1\), \(Q_2\) and \(O\) lie on a \(\mathbb{F}_q\)-line, and
- the set \(\Delta_{1,1}^c\) where \(R_1\), \(R_2\) and \(O\) lie on a \(\mathbb{F}_q\)-line.

Similarly, the set \(\Delta_{1,2}\) decomposes into six disjoint and equally large subsets:

- the two sets \(\Delta_{1,2}^{P_1,Q}\) where the line through the points \(P_1\) and \(Q_1\) also passes through the point \(O\),
- the two sets \(\Delta_{1,2}^{P_1,R}\) where the line through the points \(P_1\) and \(R_1\) also passes through the point \(O\), and
- the two sets \(\Delta_{1,2}^{Q_1,R}\) where the line through the points \(Q_1\) and \(R_1\) also passes through the point \(O\).

The cardinalities of these sets are easily computed to be

\[
|\Delta_{1,1}^a| = |\Delta_{1,1}^b| = |\Delta_{1,1}^c| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8)(q + 1),
\]

and

\[
|\Delta_{1,2}^{P_1,Q}| = |\Delta_{1,2}^{P_1,R}| = |\Delta_{1,2}^{Q_1,R}| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8).
\]

To compute \(|\Delta_{1,1}^a \cap \Delta_{1,1}^b|\) we note that if we place the three pairs of \(F_q\)-points such that no three lie on a line, then the line through \(P_1\) and \(P_2\) and the line through \(Q_1\) and \(Q_2\) will intersect in a \(\mathbb{F}_q\)-point. By choosing this point as \(O\) we obtain an element of \(\Delta_{1,1}^a \cap \Delta_{1,1}^b\). We now see that

\[
|\Delta_{1,1}^a \cap \Delta_{1,1}^b| = |\Delta_{1,1}^a \cap \Delta_{1,1}^c| = |\Delta_{1,1}^b \cap \Delta_{1,1}^c| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8).
\]

To compute \(|\Delta_{1,1}^a \cap \Delta_{1,2}^{P_1,R}|\) we first choose a conjugate pair \(Q_1\), \(Q_2\) and then a conjugate pair of \(\mathbb{F}_q\)-points \(R_1\), \(R_2\) which do not lie on the line through \(Q_1\) and \(Q_2\). We now only have one choice for \(O\). We choose a \(\mathbb{F}_q\)-line \(L\) through \(O\). There are two possibilities: either \(L\) will pass through the intersection point \(P\) of the line through \(Q_1\) and \(R_2\) and the line through \(Q_2\) and \(R_1\) or it will not. If \(L\) passes through \(P\), then we have \(q^2 - q\) possible choices for \(P_1\) and \(P_2\) on \(L\). Otherwise, we only have \(q^2 - q - 2\) choices. Hence

\[
|\Delta_{1,1}^a \cap \Delta_{1,2}^{P_1,R}| = |\Delta_{1,1}^b \cap \Delta_{1,2}^{P_1,R}| = |\Delta_{1,1}^c \cap \Delta_{1,2}^{P_1,R}| = |\Delta_{1,1}^a \cap \Delta_{1,2}^{Q_1,R}| = (q^4 - q)(q^4 - q^2)(q^2 - q) + (q^4 - q)(q^4 - q^2)q(q^2 - q - 2).\]
The only nonempty triple intersection is $\Delta_{1,1}^a \cap \Delta_{1,1}^b \cap \Delta_{i,1}^c$. A computation very similar to the one for $|\Delta_{1,1}^a \cap \Delta_{1,2}^{P_1,R_1}|$ gives

$$|\Delta_{1,1}^a \cap \Delta_{1,1}^b \cap \Delta_{i,1}^c| = 2(q^4 - q)(q^2 - q^2(q^2 - q - 2) + (q^4 - q)(q^4 - q^2)(q - 3)(q^2 - q - 4)).$$

This finishes the investigation of $\Delta_1$.

We now turn to $\Delta_c$. If six points of a $\lambda$-tuple lie on a smooth conic $C$, then the six $F_{q^2}$-points lie on $C$ and $C$ is defined over $\mathbb{F}_q$. Since no three points of a smooth conic can lie on a line, we shall obtain an element of $\Delta_c$ simply by choosing a smooth conic $C$, three conjugate pairs on $C$ and, finally, a $\mathbb{F}_q$-point anywhere. We thus have

$$|\Delta_c| = (q^5 - q^3)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4)(q^2 + q + 1).$$

We shall now compute the cardinality of the intersection between $\Delta_1$ and $\Delta_c$. The intersections with the cases $\Delta_{1,1}^a$, $\Delta_{1,1}^b$, and $\Delta_{i,1}^c$ are easily handled: we simply choose a smooth conic with three conjugate pairs on it and then place $O$ on the line through the right conjugate pair. We thus get

$$|\Delta_{1,1}^a \cap \Delta_c| = |\Delta_{1,1}^b \cap \Delta_c| = |\Delta_{1,1}^c \cap \Delta_c| = (q^5 - q^3)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4)(q + 1).$$

The intersections with the sets $\Delta_{1,2}^{P_1,Q_1}$, $\Delta_{1,2}^{P_1,R_1}$, and $\Delta_{1,2}^{Q_1,R_1}$ are perhaps even simpler: once we have chosen our conic $C$ and our conjugate pairs we have only one choice for $O$. Hence,

$$|\Delta_{1,1}^a \cap \Delta_c| = |\Delta_{1,1}^b \cap \Delta_c| = |\Delta_{1,1}^c \cap \Delta_c| = (q^5 - q^3)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4).$$

An analogous argument shows that

$$|\Delta_{1,1}^a \cap \Delta_{i,1}^b \cap \Delta_c| = |\Delta_{1,1}^b \cap \Delta_{i,1}^c \cap \Delta_c| = |\Delta_{1,1}^c \cap \Delta_{i,1}^b \cap \Delta_c| = (q^5 - q^3)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4).$$

The remaining intersections are quite a bit harder than the previous ones. We consider $\Delta_{1,1}^a \cap \Delta_{1,1}^b \cap \Delta_{1,1}^c \cap \Delta_c$, but the other intersections of this type are completely analogous and have the same size.

We first consider the case when $O$ is on the outside of $C$. There are $q + 1$ lines through $O$. Of these, precisely two are tangents and $\frac{1}{3}(q - 1)$ intersect $C$ in $F_{q^2}$-points. Thus, the remaining $\frac{1}{6}(q - 1)$ lines will intersect $C$ in two conjugate $F_{q^2}$-points. We thus pick one of these lines and label one of the intersection points by $P_1$.

Picking a $F_{q^2}$-point not defined over $F_q$ on $C$ will typically define a $F_{q^2}$-line through $O$ which is not defined over $F_q$. However, some of these choices will give $F_q$-lines and we saw above that the number of such $F_q$-lines is $\frac{1}{2}(q - 1)$. Thus, the number of $F_{q^2}$-lines, not defined over $F_q$, intersecting $C$ in two $F_{q^2}$-points is

$$\frac{1}{2}(q^2 - q) - \frac{1}{2}(q - 1) = \frac{1}{2}(q^2 - 2q + 1).$$

We pick one such line, label one of the intersection points $Q_1$ and the other intersection point $R_1$. This gives us a configuration of the desired type.
Hence, the number of tuples in $\Delta^q_{1,1} \cap \Delta^{Q,R_1}_{1,2} \cap \Delta_c$ with $O$ on the outside of $C$ is

$$\frac{1}{2}(q^5 - q^2)(q + 1)q(q - 1)(q^2 - 2q + 1).$$

We now turn to the case when $O$ is on the inside of $C$. Of the $q + 1$ lines defined over $\mathbb{F}_q$ which pass through $O$, $\frac{1}{2}(q + 1)$ will now intersect $C$ in $\mathbb{F}_q$-points and equally many in conjugate $\mathbb{F}_q$-points. We thus pick a line that intersects $C$ in two conjugate $\mathbb{F}_q$-points and label one of them by $P_1$.

We now want to pick a $\mathbb{F}_q$-line through $O$ which is not defined over $\mathbb{F}_q$ and which intersects $C$ in two $\mathbb{F}_q$-points that are not defined over $\mathbb{F}_q$. To obtain such a line we pick a $\mathbb{F}_q$-point which is not defined over $\mathbb{F}_q$ on $C$. However, such two points define tangents to $C$ which pass through $O$ and $\frac{1}{2}(q + 1)$ of the lines obtained in this way are actually defined over $\mathbb{F}_q$. We thus have

$$\frac{1}{2}(q^2 - q - 2) - \frac{1}{2}(q + 1) = \frac{1}{2}(q^2 - 2q - 3)$$

choices. We pick such a line and label the intersection points by $Q_1$ and $R_1$. Hence, the number of tuples in $\Delta^q_{1,1} \cap \Delta^{Q,R_1}_{1,2} \cap \Delta_c$ with $O$ on the inside of $C$ is

$$\frac{1}{2}(q^5 - q^2)(q^2 - q)(q + 1)(q^2 - 3).$$

This finishes the computation of $|\Delta^q_{1,1} \cap \Delta^{Q,R_1}_{1,2} \cap \Delta_c|$, $|\Delta^b_{1,1} \cap \Delta^{P,R_1}_{1,2} \cap \Delta_c|$ and $|\Delta^c_{1,1} \cap \Delta^{P,Q,R_1}_{1,2} \cap \Delta_c|$.

The only remaining intersection is $\Delta^q_{1,1} \cap \Delta^b_{1,1} \cap \Delta^c_{1,1} \cap \Delta_c$ which we shall handle in a way similar to that above. Fortunately, much of the work has already been done. To start, if $O$ is on the outside of $C$, then there are $\frac{1}{2}(q - 1)$ lines though $O$ which are defined over $\mathbb{F}_q$ and intersect $C$ in conjugate pairs of $\mathbb{F}_q$-points. Thus, there are

$$(q - 1)(q - 3)(q - 5)$$

ways to pick three lines and label the intersection points with $P_1$ and $P_2$, $Q_1$ and $Q_2$ and $R_1$ and $R_2$. Hence, the number of $\lambda$-tuples in $\Delta^q_{1,1} \cap \Delta^b_{1,1} \cap \Delta^c_{1,1} \cap \Delta_c$ with $O$ on the outside of $C$ is

$$\frac{1}{2}(q^5 - q^2)(q + 1)q(q - 1)(q - 3)(q - 5).$$

Similarly, if $O$ lies on the inside of $C$ we have seen that there are $\frac{1}{2}(q + 1)$ lines through $O$ which are defined over $\mathbb{F}_q$ and which intersect $C$ in a pair of conjugate $\mathbb{F}_q$-points. Thus, there are

$$(q + 1)(q - 1)(q - 3)$$

ways to pick three lines and label the intersection points with $P_1$ and $P_2$, $Q_1$ and $Q_2$ and $R_1$ and $R_2$. Hence, the number of $\lambda$-tuples in $\Delta^q_{1,1} \cap \Delta^b_{1,1} \cap \Delta^c_{1,1} \cap \Delta_c$ with $O$ on the inside of $C$ is

$$\frac{1}{2}(q^5 - q^2)(q + 1)q(q + 1)(q - 1)(q - 3).$$

We finally obtain

$$\left|\left(P^2_7\right)^{F^\sigma}\right| = q^6 - 3q^5 - 6q^4 + 19q^3 + 6q^2 - 24q + 7.$$
6.13. **The case** \( \lambda = [1^3, 2^2] \). Throughout this section, \( \lambda \) shall mean the partition \( [1^3, 2^2] \). We shall denote the \( \mathbb{F}_q \)-points by \( P_1, P_2 \) and \( P_3 \) and the two conjugate pairs of \( \mathbb{F}_q \)-points by \( Q_1, Q_2 \) and \( R_1, R_2 \). Let \( U \subset (\mathbb{P}^2)^7 \) be the subset consisting of septuples of points such that the first five points lie in general position.

The set \( \Delta_i \) can be decomposed as

\[
\Delta_i = \Delta_{i,1} \cup \Delta_{i,2} \cup \Delta_{i,3},
\]

where

- \( \Delta_{i,1} \) consists of tuples such that the line through \( R_1 \) and \( R_2 \) also passes through \( P_1, P_2 \) or \( P_3 \),
- \( \Delta_{i,2} \) consists of tuples such that the points \( R_1 \) and \( R_2 \) lie on the line through \( Q_1 \) and \( Q_2 \), and
- \( \Delta_{i,3} \) consists of tuples such that a line through \( Q_1 \) and one of the points \( P_1, P_2 \) and \( P_3 \) also contains \( R_1 \) or \( R_2 \).

The set \( \Delta_{i,1} \) decomposes as a union of the sets \( \Delta_{i,1}^{1}, \Delta_{i,1}^{2}, \) and \( \Delta_{i,1}^{3} \), consisting of tuples with the line through \( R_1 \) and \( R_2 \) passing through \( P_1, P_2 \) and \( P_3 \), respectively. Similarly, the set \( \Delta_{i,3} \) is the union of the six sets \( \Delta_{i,3}^{i,j}, \) \( i = 1, 2, \) \( j = 1, 2, 3 \), where \( \Delta_{i,3}^{i,j} \) contains all tuples such that \( Q_1, R_i \) and \( P_j \) lie on a line.

The cardinalities of the above sets are easily computed to be

\[
|\Delta_{i,1}^{1}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q + 1)(q^2 - q),
\]

\[
|\Delta_{i,2}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q^2 - q - 2),
\]

and

\[
|\Delta_{i,3}^{i,j}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q^2 - 1).
\]

The cardinality of \( \Delta_{i,1}^{i} \cap \Delta_{i,1}^{j} \), \( i \neq j \), is also easily computed:

\[
|\Delta_{i,1}^{i} \cap \Delta_{i,1}^{j}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q^2 - q).
\]

There is only nonempty intersection between the set \( \Delta_{i,1}^{i} \) and the set \( \Delta_{i,3}^{j,k} \) if \( k \neq i \). We then place the first five points in general position and choose a \( \mathbb{F}_q \)-line through \( P_k \) which does not pass through \( P_i \) in \( q \) ways. This gives a tuple of the desired form. We thus see that

\[
|\Delta_{i,1}^{i} \cap \Delta_{i,3}^{j,k}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)q.
\]

We also have nonempty intersection between the sets \( \Delta_{i,3}^{i} \) and the set \( \Delta_{i,3}^{j} \) where \( i \neq j \). Such a configuration is actually given by specifying the first five points in general position since we must then take \( R_1 \) as the intersection point of the line between \( Q_1 \) and \( P_i \) and the line between \( Q_2 \) and \( P_j \) and similarly for \( R_2 \). Hence,

\[
|\Delta_{i,3}^{i} \cap \Delta_{i,3}^{j}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q).
\]

Since the set \( \Delta_{i,2} \) cannot intersect any of the other sets, because this would require \( Q_1 \) and \( Q_2 \) to lie on a line through one of the \( \mathbb{F}_q \)-points, it is now time to consider the triple intersections.

Since \( P_1, P_2 \) and \( P_3 \) do not lie on a line we have that the intersection of \( \Delta_{i,1}^{1} \), \( \Delta_{i,1}^{2} \), and \( \Delta_{i,1}^{3} \) is empty. We thus only have two types of triple intersections, namely \( \Delta_{i,1}^{1} \cap \Delta_{i,1}^{j} \cap \Delta_{i,3}^{i,k} \) and \( \Delta_{i,1}^{1} \cap \Delta_{i,3}^{i} \cap \Delta_{i,3}^{j,k} \) where, of course, \( i, j \) and \( k \) are assumed to be distinct.
An element of $\Delta_{i,1} \cap \Delta_{i,j} \cap \Delta_3^{2,k}$ is specified by choosing the first five points in general position. The point $R_i$ must then be chosen as the intersection point of the line between $P_i$ and $P_j$ and the line between $Q_1$ and $P_i$ and similarly for $FR_j$. We thus have

$$|\Delta_{i,1} \cap \Delta_{i,j} \cap \Delta_3^{2,k}| = (q^2 + q + 1)(q^2 + q)(q^2 - q^3 + 3q^2 - q).$$

To compute the cardinality of the intersection $\Delta_{i,1} \cap \Delta_{i,j} \cap \Delta_3^{2,k}$ we first choose two $\mathbb{F}_q$-points $P_j$ and $P_k$. We then choose a conjugate pair of $\mathbb{F}_q$-lines through each of these points. The intersections of these lines give four $\mathbb{F}_q$-points which we only have one way to label with $Q_1$, $Q_2$, $R_1$ and $R_2$. We must now place the point $P_i$ somewhere on the line $L$ through $R_1$ and $R_2$. The line through $P_i$ and $P_k$ intersects $L$ in one $\mathbb{F}_q$-point and the line through $Q_1$ and $Q_2$ intersects $L$ in another. Thus, we have $q - 1$ choices for $P_i$. We thus see that

$$|\Delta_{i,1} \cap \Delta_{i,j} \cap \Delta_3^{2,k}| = (q^2 + q + 1)(q^2 + q)(q^2 - q^3 + 3q^2 - q).$$

This completes the investigation of $\Delta_i$.

If a smooth conic $C$ contains six of the points, then $C$ contains both the conjugate $\mathbb{F}_q$-pairs and $C$ is defined over $\mathbb{F}_q$. Thus, to compute $|\Delta_i|$ we first choose a smooth conic $C$ over $\mathbb{F}_q$ and then pick one of the points $P_1$, $P_2$ and $P_3$ to possibly lie outside $C$. We call the chosen point $P$. We then place the other two points and the two $\mathbb{F}_q$-pairs on $C$. Finally, we must place $P$ somewhere to make $P_1$, $P_2$, $P_3$, $Q_1$ and $Q_2$ lie in general position. Hence, we must choose $P$ away from the line through the two other $\mathbb{F}_q$-points and away from the line through $Q_1$ and $Q_2$. This gives us

$$(q^5 - q^3)(q + 1)q(q^2 - q)(q^2 - q - 2)(q^2 - q).$$

However, in the above we have counted the configurations where all seven points lie on $C$ three times. We must therefore take away

$$2 \cdot (q^5 - q^3)(q + 1)q(q - 1)(q^2 - q)(q^2 - q - 2)$$

in order to obtain $|\Delta_i|$.

The intersection $\Delta_{i,2} \cap \Delta_3$ is empty but the intersections of the other sets in the decomposition of $\Delta_i$ are not. To compute $|\Delta_{i,2} \cap \Delta_3|$ we shall assume that $P_i$ lies on the outside of $C$. Of the $q + 1$ lines through $P_i$ which are defined over $\mathbb{F}_q$ we have that 2 are tangent to $C$ and $\frac{1}{2}(q - 1)$ intersect $C$ in two $\mathbb{F}_q$-points. Thus, there are $\frac{1}{2}(q - 1)$ lines left which must intersect $C$ in a pair of conjugate $\mathbb{F}_q$-points. We pick such a line and label the intersection points by $R_1$ and $R_2$ in one of two ways. We shall now place the other two $\mathbb{F}_q$-points on $C$. There are $\frac{1}{2}(q + 1)q$ ways to choose two $\mathbb{F}_q$-points on $C$ of which $\frac{1}{2}(q - 1)$ pairs lie on a $\mathbb{F}_q$-line through $P_i$. There are thus $\frac{1}{2}(q^2 + 1)$ pairs which do not lie on a line through $P_i$ and, since there are two ways to label each pair, we thus have $q^2 + 1$ choices for the two $\mathbb{F}_q$-points. Finally, we shall place $Q_1$ and $Q_2$ somewhere on $C$ but we have to make sure that the points $P_1$, $P_2$, $P_3$, $Q_1$ and $Q_2$ are in general position. Since the lines between $P_i$ and the other two $\mathbb{F}_q$-points intersect $C$ only in $\mathbb{F}_q$-points, the only thing that might go wrong when choosing $Q_1$ and $Q_2$ is that the line through $Q_1$ and $Q_2$ might also go through $P_i$. As seen above, there are exactly $q - 1$ choices for $Q_1$ and $Q_2$ for which this happens, so the remaining $q^2 - q - (q - 1) = q^2 - 2q + 1$ choices will give a configuration of the desired type. We thus have that the number
of elements in $\Delta_{1,1}^i \cap \Delta_c$ such that $P_i$ lies on the outside of $C$ is

$$\frac{1}{2}(q^5 - q^2)(q + 1)q(q - 1)(q^2 + 1)(q^2 - 2q + 1).$$

We now assume that $P_i$ lies on the inside of $C$. We proceed similarly to the above. First we observe that the number of $\mathbb{F}_q$-lines through $P_i$ is $q + 1$ of which half intersect $C$ in two $\mathbb{F}_q$-points and half intersect $C$ in conjugate pairs of $\mathbb{F}_q$-points. We choose a line which intersects $C$ in two conjugate $\mathbb{F}_q$-points and label the intersection points by $R_1$ and $R_2$. We now choose a $\mathbb{F}_q$-point $P_j$ on $C$ in one of $q + 1$ ways. The line through $P_i$ and $P_j$ intersects $C$ in another $\mathbb{F}_q$-point and we choose the final $\mathbb{F}_q$-point away from this intersection point and $P_j$. Finally, we shall place the points $Q_1$ and $Q_2$ on $C$ in a way so that the points $P_1$, $P_2$, $P_3$, $Q_1$ and $Q_2$ are in general position. As above, the only thing that might go wrong is that the line through $Q_1$ and $Q_2$ might go through $P_i$ and there are precisely $q + 1$ choices for $Q_1$ and $Q_2$ for which this happens. Thus, there are $q^2 - q - (q + 1) = q^2 - 2q - 1$ valid choices for $Q_1$ and $Q_2$. Hence, there are

$$\frac{1}{2}(q^5 - q^2)(q^2 - q)(q + 1)(q + 1)(q - 1)(q^2 - 2q - 1)$$

elements in $\Delta_{1,1}^i \cap \Delta_c$ such that $P_i$ lies on the inside of $C$.

To compute the intersection $\Delta_{1,3}^i \cap \Delta_c$, we note that if we place $P_i$ outside of $C$ and then choose two $\mathbb{F}_q$-points on $C$ and two conjugate $\mathbb{F}_q$-points $Q_1$ and $Q_2$ on $C$ such that $P_1$, $P_2$, $P_3$, $Q_1$ and $Q_2$ are in general position, then we must choose $R_i$ as the other intersection point of $C$ with the line through $Q_1$ and $P_j$. We may thus use constructions analogous to those above to see that there are

$$\frac{1}{2}(q^5 - q^2)(q + 1)q(q^2 + 1)(q^2 - 2q + 1)$$

elements in $\Delta_{1,1}^i \cap \Delta_c$ with $P_j$ on the outside of $C$ and

$$\frac{1}{2}(q^5 - q^2)(q^2 - q)(q + 1)(q - 1)(q^2 - 2q - 3)$$

elements with $P_j$ on the inside of $C$.

We may now put all the pieces together to obtain

$$\left|\left(P_2^2\right)^{\mathbb{F}_q}\right| = q^6 - 7q^5 + 10q^4 + 15q^3 - 26q^2 - 8q + 15.$$

6.14. The case $\lambda = [1^5, 2]$. Throughout this section, $\lambda$ shall mean the partition $[1^5, 2]$. We shall denote the $\mathbb{F}_q$-points by $P_1$, $P_2$, $P_3$, $P_4$ and $P_5$ and the points of the conjugate pair of $\mathbb{F}_q$-points by $Q_1$ and $Q_2$. Let $U \subset (\mathbb{P}^2)^7$ be the subset consisting of septuples of points such that the first five points lie in general position.

If three points of a conjugate $\lambda$-tuple in $U(\lambda)$ lie on a line, then $Q_1$ and $Q_2$ lie on a line passing through one of the $\mathbb{F}_q$-points. There are

$$(q + 1) + q + (q - 1) + (q - 2) + (q - 3) = 5q - 5,$$

$\mathbb{F}_q$-lines passing through $P_1$, $P_2$, $P_3$, $P_4$ or $P_5$ (or possibly two of them). Each of these lines contains $q^2 - q$ conjugate pairs and no conjugate pair lies on two such lines. We thus have

$$|\Delta| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(q^2 - 5q + 6)(5q - 5)(q^2 - q).$$

If six of the points of a conjugate $\lambda$-tuple lie on a smooth conic $C$, then $C$ is defined over $\mathbb{F}_q$ and contains $Q_1$ and $Q_2$. Therefore, to compute the cardinality
of $\Delta$, we first choose a smooth conic $C$ defined over $\mathbb{F}_q$ and one of the points $P_1$, $P_2$, $P_3$, $P_4$ or $P_5$ to possibly lie outside $C$. We call the chosen point $P$. Then, we choose four $\mathbb{F}_q$-points and a conjugate pair on $C$. Finally, we choose $P$ away from the six lines through pairs of the other four $\mathbb{F}_q$-points. We thus get

\[
5(q^3 - q^2)(q + 1)q(q - 1)(q - 2)(q^2 - q)(q^2 - 5q + 6).
\]

In the above we have counted the $\lambda$-tuples with all seven points on a conic five times. We therefore must take away

\[
4(q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q^2 - q),
\]

in order to obtain $|\Delta_c|$.

To compute the size of the intersection $\Delta_1 \cap \Delta_c$ we shall decompose this set into a disjoint disjoint union of five subsets $A_i$, $i = 1, \ldots, 5$, where $A_i$ consists of those tuples where $P_i$ does not lie on the conic $C$ through the other six points. Each of the sets $A_i$ is then decomposed further into a union of the sets $A_i^{\text{out}}$ and $A_i^{\text{in}}$ where $A_i^{\text{out}}$ consists of those tuples with $P_i$ on the outside of $C$ and $A_i^{\text{in}}$ consists of those with $P_i$ on the inside of $C$. Finally, we shall decompose $A_i^{\text{out}}$ into a union of the three disjoint subsets:

- the set $A_{i,0}^{\text{out}}$ consisting of $\lambda$-tuples such that the tangent lines to $C$ passing through $P_i$ do not pass through any of the other points of the $\lambda$-tuple,
- the set $A_{i,1}^{\text{out}}$ consisting of $\lambda$-tuples such that exactly one of the tangent lines to $C$ passing through $P_i$ pass through one of the other points of the $\lambda$-tuple,
- the set $A_{i,2}^{\text{out}}$ consisting of $\lambda$-tuples such that both the tangent lines to $C$ passing through $P_i$ pass through another point of the $\lambda$-tuple.

To compute $|A_i^{\text{out}}|$, we first choose a smooth conic $C$ defined over $\mathbb{F}_q$ in $q^5 - q^2$ ways and then a point $P_i$ outside $C$ in $\frac{1}{2}(q + 1)q$ ways. As seen many times before, there are exactly $\frac{1}{2}(q - 1)$ lines through $P_i$ which are defined over $\mathbb{F}_q$ and which intersect $C$ in a conjugate pair of points. We pick such a line and label the points $Q_1$ and $Q_2$ in one of two ways. From this point on, the computations are a little bit different for the three subsets of $A_i^{\text{out}}$.

**The subset $A_{i,0}^{\text{out}}$.** We shall now pick the other four $\mathbb{F}_q$-points of the $\lambda$-tuple. Since we should not pick points whose tangents pass through $P_i$, we have $q - 1$ choices for the first point. For the second point, we should stay away from the tangent points, the first point and the other intersection point of $C$ and the line through $P_i$ and the first point. Hence, we have $q - 3$ choices. In a similar way, we see that we have $q - 5$ choices for the third point and $q - 7$ for the fourth. Hence,

\[
|A_{i,0}^{\text{out}}| = \frac{1}{2}(q^5 - q^2)(q + 1)q(q - 1)(q - 1)(q - 3)(q - 5)(q - 7).
\]

**The subset $A_{i,1}^{\text{out}}$.** We begin by choosing one of the four $\mathbb{F}_q$-points to lie on a tangent to $C$ passing through $P_i$ and then we pick the tangent it should lie on. For the first of the remaining three points we now have $q - 1$ choices and, similarly to the above case, we have $q - 3$ choices for the second and $q - 5$ for the third. Thus,

\[
|A_{i,1}^{\text{out}}| = 4 \cdot 2 \cdot \frac{1}{2}(q^5 - q^2)(q + 1)q(q - 1)(q - 1)(q - 3)(q - 5).
\]

**The subset $A_{i,2}^{\text{out}}$.** We begin by choosing two of the four $\mathbb{F}_q$-points to lie on tangents to $C$ passing through $P_i$ and then we pick which point should lie on which tangent.
For the first of the remaining two points we now have $q - 1$ choices and we then have $q - 3$ choices for the second. Thus,

$$|A_{t,2}^{out}| = \binom{4}{2} \cdot 2 \cdot \frac{1}{2}(q^5 - q^2)(q + 1)q(q - 1)(q - 3)(q - 5).$$

It remains to compute $|A_{t}^{in}|$. We first choose a smooth conic $C$ defined over $\mathbb{F}_q$ in $q^5 - q^2$ ways and then a point $P_i$ on the inside of $C$ in $\frac{1}{2}(q^2 - q)$ ways. We have already seen that there now are $\frac{1}{2}(q + 1)$ lines passing through $P_i$ which are defined over $\mathbb{F}_q$ and which intersect $C$ in a conjugate pair of points. We thus pick such a line and label the intersection points by $Q_1$ and $Q_2$. Since any $\mathbb{F}_q$-line through $P_i$ will intersect $C$ in precisely two points, we have $(q + 1)(q - 1)(q - 3)(q - 5)$ choices for the remaining four $\mathbb{F}_q$-points of the $\lambda$-tuple. We thus see that

$$|A_{t}^{in}| = \frac{1}{2}(q^5 - q^2)(q^2 - q)(q + 1)(q - 1)(q - 3)(q - 5).$$

We now conclude that

$$|\left(\mathcal{P}_{\mathbb{F}_q}^2\right)^F| = q^6 - 15q^5 + 90q^4 - 265q^3 + 374q^2 - 200q + 15.$$

6.15. **The case** $\lambda = [1^7]$. Throughout this section, $\lambda$ shall mean the partition $[1^7]$. Since we shall almost exclusively be interested in objects defined over $\mathbb{F}_q$, we shall often omit the decoration “$\mathbb{F}_q$". For instance, we shall simply write “point" to mean “$\mathbb{F}_q$-point". Let $U \subset (\mathbb{F}^2)^7$ be the subset consisting of septuples of points such that the first four points lie in general position. We thus have

$$|U(\lambda)| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(q^2 + q - 3)(q^2 + q - 4)(q^2 + q - 5).$$

The following notation will be quite convenient.

**Definition 6.12.** If $P$ and $Q$ are two points in $\mathbb{F}^2$, then the line through $P$ and $Q$ shall be denoted $PQ$.

Since we shall often want to stay away from lines through two of the first four points we define

$$\mathcal{S} = \bigcup_{1 \leq i < j \leq 4} P_iP_j.$$  

We note that $\mathcal{S}$ contains

$$6(q - 2) + 4 + 3 = 6q - 5$$

points.

6.15.1. **The set** $\Delta_t$. The set $\Delta_t$ decomposes into a disjoint union of three sets

$$\Delta_t = \Delta_{t,1} \cup \Delta_{t,2} \cup \Delta_{t,3},$$

where

- the points of $\Delta_{t,1}$ are such that at least one of the points $P_5$, $P_6$ or $P_7$ lies in $\mathcal{S}$,
- the points of $\Delta_{t,2}$ are such that one of the lines $P_iP_j$, $5 \leq i < j \leq 7$, contains one of the points $P_1$, $P_2$, $P_3$ and $P_4$, but $\{P_5, P_6, P_7\} \cap \mathcal{S} = \emptyset$, and
- the points of $\Delta_{t,3}$ are such that the three points $P_5$, $P_6$ and $P_7$ lie on a line which does not pass through $P_1$, $P_2$, $P_3$ or $P_4$. 

We shall consider the three subsets separately.

The set $\Delta_{t,1}$. For each subset $I \subset \{5, 6, 7\}$, let $\Delta_{t,1}(I)$ denote the set of points in $\Delta_{t,1}$ such that $P_i \in \mathcal{S}$ for all $i \in I$. We can then decompose $\Delta_{t,1}$ further as

$$\Delta_{t,1} = \Delta_{t,1}(\{5\}) \cup \Delta_{t,1}(\{6\}) \cup \Delta_{t,1}(\{7\}).$$

Clearly, $\Delta_{t,1}(\{i\}) \cap \Delta_{t,1}(\{j\}) = \Delta_{t,1}(\{i, j\})$.

![Figure 4. A typical element of $\Delta_{t,1}(\{i\})$.](image)

A typical element of $\Delta_{t,1}(\{i\})$ is illustrated in Figure 4 above. To compute $|\Delta_{t,1}(\{i\})|$ we first place the first four points in general position, then choose $P_i$ as any point in $\mathcal{S}$ and finally place the remaining two points anywhere. Hence

$$|\Delta_{t,1}(\{i\})| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(6q - 9)(q^2 + q - 4)(q^2 + q - 5).$$

Similarly, we have

$$|\Delta_{t,1}(\{i, j\})| = |\text{PGL}(3)| \cdot (6q - 9)(6q - 10)(q^2 + q - 5),$$

and

$$|\Delta_{t,1}(\{5, 6, 7\})| = |\text{PGL}(3)| \cdot (6q - 9)(6q - 10)(6q - 11).$$

This allows us to compute $|\Delta_{t,1}|$ as

$$|\Delta_{t,1}| = |\text{PGL}(3)| \cdot (18q^5 - 99q^4 + 252q^3 - 414q^2 + 417q - 180).$$

The set $\Delta_{t,2}$. Let $\{i, j\} \in \{5, 6, 7\}$, $r \in \{1, 2, 3, 4\}$ and let $\Delta_{t,2}(\{i, j\})$ be the subset of points in $\Delta_{t,2}$ such that $P_i P_j \cap \{P_1, P_2, P_3, P_4\} = \{P_r\}$. We also define

$$\Delta_{t,2}(\{i, j\}) = \bigcup_{r=1}^{4} \Delta_{t,2}(\{i, j\}).$$

A typical element of $\Delta_{t,2}(\{i, j\})$ is illustrated in Figure 5. To obtain an element of $\Delta_{t,2}(\{i, j\})$ we first place $P_1$, $P_2$, $P_3$ or $P_4$ in general position. There are $q + 1$ lines through $P_r$ of which 3 are contained in $\mathcal{S}$. We choose $P_i P_j$ as one of the remaining $q - 2$ lines. Note that $P_i P_j$ will not pass through any of the points

$$Q_1 = P_1 P_4 \cap P_2 P_3, \quad Q_2 = P_2 P_4 \cap P_1 P_3, \quad Q_3 = P_3 P_4 \cap P_1 P_2.$$
Hence, \( P_iP_j \) will intersect \( \mathcal{S} \) in \( P_r \) and three further points. There are thus \( q - 3 \) ways to choose \( P_i \) and then \( q - 4 \) ways to choose \( P_j \). Finally, there are
\[
|\mathbb{P}^2 \setminus \mathcal{S}| - 2 = q^2 + q + 1 - (6q - 5) - 2 = q^2 - 5q + 4,
\]
choices for the seventh point. We thus have
\[
|\Delta_{r,2}^i\{i,j\}| = |\text{PGL}(3)| \cdot (q - 2)(q - 3)(q - 4)(q^2 - 5q + 4).
\]

**Figure 5.** A typical element of \( \Delta_{r,2}^i\{i,j\} \).

We have counted some tuples several times. To begin with, the points of
\[
\Delta_{r,2}^i\{5,6\} \cap \Delta_{r,2}^i\{5,7\} \cap \Delta_{r,2}^i\{6,7\},
\]
have been counted three times. There are
\[
|\text{PGL}(3)| \cdot (q - 2)(q - 3)(q - 4)(q - 5),
\]
of these.

Further, the sets \( \Delta_{r,2}^i\{i,j\} \) and \( \Delta_{r,2}^i\{i,k\} \) will intersect if \( r \neq s \) and \( j \neq k \). A typical element is illustrated in Figure 6.

**Figure 6.** A typical element of \( \Delta_{r,2}^i\{i,j\} \cap \Delta_{r,2}^i\{i,k\} \).

To compute \( |\Delta_{r,2}^i\{i,j\} \cap \Delta_{r,2}^i\{i,k\}| \) we begin by choosing \( P_1, P_2, P_3 \) and \( P_4 \) in general position and continue by choosing \( P_1 \) outside \( \mathcal{S} \) in \( q^2 - 5q + 6 \) ways. This
gives us two lines $P_iP_r$ and $P_iP_s$ which intersect $\mathcal{I}$ in four points each. We choose $P_j$ on $P_iP_r$ away from $P_i$ and $\mathcal{I}$ in $q - 4$ ways and similarly for $P_k$. This gives
\[
|\Delta_{i,2}^r\{i,j\} \cap \Delta_{i,2}^s\{i,k\}| = |\text{PGL}(3)| \cdot (q^2 - 5q + 6)(q - 4)^2.
\]

Finally, we must compute the cardinality of the triple intersection
\[
\Delta_{i,2}^r\{5,6\} \cap \Delta_{i,2}^s\{5,7\} \cap \Delta_{i,2}^t\{6,7\},
\]
where $r$, $s$ and $t$ are distinct. A typical element of the intersection is illustrated in Figure 7.

![Figure 7](image-url)

**Figure 7.** A typical element of $\Delta_{i,2}^r\{5,6\} \cap \Delta_{i,2}^s\{5,7\} \cap \Delta_{i,2}^t\{6,7\}$.

This is where we have to pay for the awkward requirement that $P_5$, $P_6$ and $P_7$ should not be in $\mathcal{I}$. We shall view $\Delta_{i,2}^r\{5,6\} \cap \Delta_{i,2}^s\{5,7\} \cap \Delta_{i,2}^t\{6,7\}$ as an open subset of the set $T_{r,s,t}^r$ consisting of tuples such that
\begin{itemize}
  \item the line $P_5P_6$ passes through $P_r$, $P_5P_7$ passes through $P_s$ and $P_6P_7$ passes through $P_t$ but,
  \item we allow $P_5$, $P_6$ and $P_7$ to lie in $\mathcal{I}$, but,
  \item we do not allow the lines $P_iP_j$, $5 \leq i < j \leq 7$ to be contained in $\mathcal{I}$.
\end{itemize}

The complement of $\Delta_{i,2}^r\{5,6\} \cap \Delta_{i,2}^s\{5,7\} \cap \Delta_{i,2}^t\{6,7\}$ in $T_{r,s,t}^r$ can be decomposed into a union of three subsets $T_{i}^{r,s,t}$, $i = 5,6,7$, consisting of those tuples with $P_i$ in $\mathcal{I}$.

We begin with the computation of $|T_{r,s,t}^r|$. To obtain such a tuple, we begin by choosing a line $L_r$ through $P_r$ in $q - 2$ ways. We shall then choose a line $L_s$ through $P_s$. There are however two cases that may occur. Typically, the intersection point $P_5 = L_r \cap L_s$ will lie outside $\mathcal{I}$ but for one choice of $L_s$ it will lie in $\mathcal{I}$. The situation is illustrated in Figure 8.

There are $q - 3$ ways to choose $L_s$ so that $L_r \cap L_s$ lies outside $\mathcal{I}$. When we choose the line $L_t$ through $P_t$ we must make sure that $L_t$ is not contained in $\mathcal{I}$ and that $L_t$ does not pass through $L_r \cap L_s$, since we want to end up with three distinct intersection points. We thus have $q - 3$ choices. On the other hand, if we choose $L_s$ as the one line making the intersection point $L_r \cap L_s$ lie in $\mathcal{I}$ we only need to make sure that $L_t$ is not contained in $\mathcal{I}$ and we thus have $q - 2$ choices. Hence, we see that
\[
|T_{r,s,t}^r| = |\text{PGL}(3)| \cdot \left((q - 2)(q - 3)^2 + (q - 2)^2\right).
\]
We now turn to the computation of $|T_{r,s,t}^i|$, $i = 5, 6, 7$. We then begin by choosing a line $L_r$ through $P_r$ in $q - 2$ ways. The line $L_s$ through $P_s$ is then completely determined since we must have $P_i \in \mathcal{S}$. This gives us $q - 2$ choices for the final line $L_t$ through $P_t$. Hence,

$$|T_{r,s,t}^i| = |PGL(3)| \cdot (q - 2)^2.$$

We now turn to the computation of $|T_{r,s,t}^i \cap T_{r,s,t}^j|$, $5 \leq i < j \leq 7$. As above, we begin by choosing a line $L_r$ through $P_r$ in $q - 2$ ways. Since $P_i$ must lie in $\mathcal{S}$ we have only one choice for $L_s$. Since $P_j = L_s \cap L_t$ we see that we now have precisely one choice for $L_t$ also. Hence,

$$|T_{r,s,t}^i \cap T_{r,s,t}^j| = |PGL(3)| \cdot (q - 2).$$

We now consider $T_{5,r,s,t}^r \cap T_{6,r,s,t}^r \cap T_{7,r,s,t}^r$. It turns out that once the four points $P_1, P_2, P_3$ and $P_4$ have been placed in general position, there is precisely one such tuple. The situation is illustrated in Figure 9.

This finally allows us to compute

$$\Delta_{1,2} = |PGL(3)| \cdot (12q^5 - 212q^4 + 1504q^3 - 5320q^2 + 9296q - 6360).$$
The set $\Delta_{l,3}$. Recall the definition of the three points $Q_1, Q_2$ and $Q_3$ from Equation 6.4. Using these three points we may decompose $\Delta_{l,3}$ into a disjoint union of the following subsets:

- $\Delta_{l,3}({Q_r, Q_s})$ consisting of those tuples of $\Delta_{l,3}$ where $P_3, P_6$ and $P_7$ lie on the line $Q_rQ_s$, $1 \leq r < s \leq 3$, and,
- $\Delta_{l,3}({Q_r})$ consisting of those tuples of $\Delta_{l,3}$ with $P_3, P_6$ and $P_7$ on a line through $Q_r$, $1 \leq r \leq 3$, which does not pass through any of the other $Q_i$, and
- $\Delta_{l,3}({})$ consisting of those tuples of $\Delta_{l,3}$ with $P_3, P_6$ and $P_7$ on a line which does not pass through $Q_1, Q_2$ or $Q_3$.

We begin by considering $\Delta_{l,3}({Q_r, Q_s})$. The line $Q_rQ_s$ contains $q + 1$ points of which four lie in $\mathcal{S}$. There are thus $q - 3$ choices for $P_5$, $q - 4$ choices for $P_6$ and $q - 5$ choices for $P_7$. Hence,

$$|\Delta_{l,3}({Q_r, Q_s})| = |\text{PGL}(3)| \cdot (q - 3)(q - 4)(q - 5).$$

We continue with $|\Delta_{l,3}({Q_r})|$. There are $q + 1$ lines through $Q_r$ of which two are contained in $\mathcal{S}$ and two are the lines through the other two $Q_i$. Hence, there are $q - 3$ choices for a line $L$ through $Q_r$. The line $L$ intersects $\mathcal{S}$ in five points so we have $q - 4$ choices for $P_5$, $q - 5$ choices for $P_6$ and $q - 6$ choices for $P_7$. We conclude that

$$|\Delta_{l,3}({Q_r})| = |\text{PGL}(3)| \cdot (q - 3)(q - 4)(q - 5)(q - 6).$$

To compute $|\Delta_{l,3}({})|$ we begin by choosing a line $L$ which does not pass through any of the points $P_1, P_2, P_3, P_4, Q_1, Q_2$ and $Q_3$. There are $q^2 + q + 1$ lines in $\mathbb{P}^2$, of which $q + 1$ passes through $P_i$, $i = 1, 2, 3, 4$. There is exactly one line through each of these points so there are

$$q^2 + q + 1 - 4(q + 1) + 6 = q^2 - 3q + 3$$

lines which do not pass through $P_1, P_2, P_3$ and $P_4$. Of the $q + 1$ lines through $Q_i$, $i = 1, 2, 3$, precisely two have been removed above and the line $Q_iQ_j$ passes through both $Q_i$ and $Q_j$. Hence, we have

$$q^2 - 3q + 3 - 3(q - 1) + 3 = q^2 - 6q + 9,$$

choices for $L$.

The line $L$ intersects $\mathcal{S}$ in six points. We therefore have $q - 5$ choices for $P_5$, $q - 6$ choices for $P_6$ and $q - 7$ choices for $P_7$. Hence,

$$|\Delta_{l,3}({})| = |\text{PGL}(3)| \cdot q^2 - 6q + 9)q - 5)(q - 6)(q - 7).$$

We now add everything together to obtain

$$|\Delta_{l,3}| = |\text{PGL}(3)| \cdot (q^5 - 21q^4 + 173q^3 - 693q^2 + 1338q - 990)$$

and, finally,

$$|\Delta| = |\text{PGL}(3)| \cdot (31q^5 - 332q^4 + 1929q^3 - 6427q^2 + 11051q - 7530).$$

6.15.2. The set $\Delta_c$. We decompose $\Delta_c$ as

$$\Delta_c = \Delta_{c,1} \cup \Delta_{c,2},$$

where $\Delta_{c,1}$ consists of tuples where six points lie on a smooth conic $C$ with one of the points $P_1, P_2, P_3$ or $P_4$ possibly outside $C$ and $\Delta_{c,2}$ consists of tuples where six points lie on a smooth conic $C$ with one of the points $P_5, P_6$ or $P_7$ possibly outside $C$. 
To obtain an element of $\Delta_{c,1}$ we first choose one of the points $P_1, P_2, P_3$ and $P_4$ and call it $P$. Then we choose a smooth conic $C$ in $q^5 - q^2$ ways and place all of the seven points except $P$ on $C$ in 

$$(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4),$$

ways. There are three lines through pairs of points in $\{P_1, P_2, P_3, P_4\} \setminus \{P\}$ which together contain $3q$ points. These lines do not contain $P_5, P_6$ and $P_7$ so we have

$$q^2 + q + 1 - 3q - 3 = q^2 - 2q - 2,$$

choices for $P$. Multiplying everything together we obtain

$$N_1 := 4(q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4)(q^2 - 2q - 2),$$

which is almost $|\Delta_{c,1}|$ except that we have counted the tuples where all seven points lie on $C$ four times.

To obtain an element of $\Delta_{c,2}$ we first choose $P_5, P_6$ and $P_7$ and call the chosen point $P$. We then choose a smooth conic $C$ and place all but the chosen points on $C$. Finally, we place $P$ anywhere in $\mathbb{P}^2$ except at the six chosen points. In this way we obtain the number

$$N_2 := 3(q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4)(q^2 + q - 5),$$

which is almost equal to $|\Delta_{c,2}|$ except that we have counted the tuples with all seven points on $C$ three times.

We now want to compute the number of tuples with all seven points on a smooth conic $C$. We thus choose a smooth conic $C$ and place all seven points on it in

$$N_7 := (q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4)(q - 5),$$

ways. We thus have

$$|\Delta_c| = |\text{PGL}(3)| \cdot (7q^5 - 74q^4 + 288q^3 - 517q^2 + 446q - 168)$$

6.15.3. The set $\Delta_f \cap \Delta_c$. We introduce the filtration $\mathcal{F}_3 \subset \mathcal{F}_2 \subset \mathcal{F}_1 = \Delta_f \cap \Delta_c$ where

- the set $\mathcal{F}_1$ consists of tuples such that at least one line contains three points of the tuple,
- the set $\mathcal{F}_2$ consists of tuples such that at least two lines contain three points of the tuple,
- the set $\mathcal{F}_3$ consists of tuples such that at least three lines contain three points of the tuple.

The strategy will be to compute the numbers:

$$N_1 = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3|,$$

$$N_2 = |\mathcal{F}_2| + 2|\mathcal{F}_3|,$$

$$N_3 = |\mathcal{F}_3|,$$

and thereby obtain the desired cardinality.

Since the points $P_1, P_2, P_3$ and $P_4$ are assumed to constitute a frame, we must do things a little bit differently depending on whether the point not on the conic is one of these four or not. We therefore make further subdivisions.
The subsets with \( P_3, P_4 \) or \( P_5 \) not on the conic. We shall denote the subsets in question by \( \mathcal{F}_{i}^{5,6,7} \) and, similarly

\[
\begin{align*}
N_{1}^{5,6,7} &= |\mathcal{F}_{1}^{5,6,7}| + |\mathcal{F}_{2}^{5,6,7}| + |\mathcal{F}_{3}^{5,6,7}|, \\
N_{2}^{5,6,7} &= |\mathcal{F}_{2}^{5,6,7}| + 2|\mathcal{F}_{3}^{5,6,7}|, \\
N_{3}^{5,6,7} &= |\mathcal{F}_{3}^{5,6,7}|. 
\end{align*}
\]

To compute \( N_{1}^{5,6,7} \), we first choose one of the points \( P_3, P_4 \) or \( P_5 \) to be the point \( P \) not on the smooth conic \( C \) and call the remaining two points \( P_i \) and \( P_j \). We then choose \( C \) in \( q^5 - q^2 \) ways and choose two points among \( \{P_1, P_2, P_3, P_4, P_5, P_6\} \) and call them \( R_1 \) and \( R_2 \). There are \( (q+1)q \) ways to place \( R_1 \) and \( R_2 \) on \( C \) and there are then \( q - 1 \) ways to place \( P \) on the line \( R_1R_2 \). Finally, we place the remaining four points on \( C \) in \( (q-1)(q-2)(q-3)(q-4) \) ways. Multiplying everything together we obtain

\[
N_{1}^{5,6,7} := 3 \cdot \binom{6}{2} \cdot (q^5 - q^2)(q+1)q(q-1)^2(q-2)(q-3)(q-4).
\]

In order to compute \( N_{2}^{5,6,7} \), we first choose one of the points \( P_3, P_4 \) or \( P_5 \) to be the point \( P \) not on the smooth conic \( C \) and call the remaining two points \( P_i \) and \( P_j \). We then choose \( C \) in \( q^5 - q^2 \) ways and choose two unordered pairs of unordered points among \( \{P_1, P_2, P_3, P_4, P_5, P_6\} \). This can be done in \( \frac{1}{2} \cdot \binom{6}{4} \cdot \binom{4}{2} \) ways. We call the points of the first pair \( R_1 \) and \( R_2 \) and those of the second \( O_1 \) and \( O_2 \). There are \( (q+1)q(q-1)(q-2)(q-3)(q-4) \) ways to place \( \{P_1, P_2, P_3, P_4, P_5, P_6\} \) on \( C \) and the point \( P \) is then completely determined as \( P = R_1R_2 \cap O_1O_2 \). Thus

\[
N_{2}^{5,6,7} = 3 \cdot \frac{1}{2} \cdot \binom{6}{4} \cdot \binom{4}{2} \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4).
\]

The computation of \( N_{3}^{5,6,7} \) is slightly more complicated since we need to subdivide into two subcases depending on if \( P \) is on the outside or on the inside of \( C \). We call the two corresponding numbers \( N_{3,\text{out}}^{5,6,7} \) and \( N_{3,\text{in}}^{5,6,7} \).

To compute \( N_{3,\text{out}}^{5,6,7} \) we first choose one of the points \( P_3, P_4 \) or \( P_5 \) to be the point \( P \) not on the smooth conic \( C \). We proceed by choosing the smooth conic \( C \) in \( q^5 - q^2 \) ways and then the point \( P \) on the outside of \( C \) in \( \frac{1}{2}(q+1)q \) ways. We now place \( P_1 \) at one of the \( q-1 \) points of \( C \) whose tangent does not pass through \( P \) and choose one of the remaining 5 points as the other intersection point in \( C \cap P_1P \). There are now four remaining points \( P_i \), \( P_j \), \( P_k \) and \( P_l \) to place on \( C \). We place \( P_i \) at one of the \( q-3 \) remaining points of \( C \) whose tangent does not pass through \( P \) and choose one of the remaining three points as the other intersection point in \( C \cap P_iP \). There are now two points \( P_j \) or \( P_k \) to place on \( C \). We place \( P_j \) at one of the \( q-5 \) possible points and the point \( P_k \) is then determined. We thus have

\[
N_{3,\text{out}}^{5,6,7} = 3 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q+1)q \cdot (q-1) \cdot 5 \cdot (q-3) \cdot 3 \cdot (q-5).
\]

We proceed by computing \( N_{3,\text{in}}^{5,6,7} \). We first choose one of the points \( P_3, P_4 \) or \( P_5 \) to be the point \( P \) not on the smooth conic \( C \). We proceed by choosing the smooth conic \( C \) in \( q^5 - q^2 \) ways and then the point \( P \) on the outside of \( C \) in \( \frac{1}{2}(q+1)q \) ways.

We now place \( P_1 \) at one of the \( q+1 \) points of \( C \) whose tangent does not pass through \( P \) and choose one of the remaining 5 points as the other intersection point in \( C \cap P_1P \). There are now four remaining points \( P_i \), \( P_j \), \( P_k \) and \( P_l \) to place on
C. We place $P_r$ at one of the $q - 1$ remaining points of $C$ and choose one of the remaining three points as the other intersection point in $C \cap P_rP$. There are now two points $P_r$ and $P_s$ to place on $C$. We place $P_r$ at one of the $q - 3$ possible points and the point $P_s$ is then determined. We now see that

$$N^{5,6,7}_{3,\text{in}} = 3 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q - 1)q \cdot (q + 1) \cdot 5 \cdot (q - 1) \cdot 3 \cdot (q - 3).$$

The subsets with $P_1, P_2, P_3$ or $P_4$ not on the conic. We shall denote the subsets in question by $\mathcal{F}_{1,2,3,4}$ and, similarly

$$N_{1}^{1,2,3,4} = |\mathcal{F}_{1}^{1,2,3,4}| + |\mathcal{F}_{2}^{1,2,3,4}| + |\mathcal{F}_{3}^{1,2,3,4}|,$$

$$N_{2}^{1,2,3,4} = |\mathcal{F}_{1}^{1,2,3,4}| + 2|\mathcal{F}_{3}^{1,2,3,4}|,$$

$$N_{3}^{1,2,3,4} = |\mathcal{F}_{3}^{1,2,3,4}|.$$

In order to compute $N_{1}^{1,2,3,4}$, we first choose one of the points $P_1, P_2, P_3$ or $P_4$ to be the point $P$ not on the smooth conic $C$ and call the remaining three points $P_r$, $P_s$ and $P_t$. We continue by choosing a smooth conic $C$ in $q^5 - q^2$ ways.

We first assume that $P$ lies on a line $R_1R_2$ where $\{R_1, R_2\} \subset \{P_5, P_6, P_7\}$. We therefore choose the two points in 3 ways and call the remaining point $P_t$. We then place $R_1$ and $R_2$ on $C$ in $(q+1)q$ ways. We continue by choosing the three points $P_r$, $P_s$ and $P_t$ on $C$ in $(q-1)(q-2)(q-3)$ ways. The lines $P_rP_s$, $P_rP_t$ and $P_sP_t$ intersect the line $R_1R_2$ in three distinct points so there are $q - 4$ ways to choose the point $P$ on $R_1R_2$ but away from these three points and $R_1$ and $R_2$. Finally, we place $P_1$ at one of the $q - 4$ remaining points of $C$. Multiplying everything together we get

$$4 \cdot 3 \cdot (q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4).$$

We now assume that $P$ lies on a line $ab$ with $a \in \{P_1, P_2, P_3, P_4\}$ and $b \in \{P_5, P_6, P_7\}$. We thus first choose $a$ as one of the points in $\{P_r, P_s, P_t\}$ and the point $b$ as one of the points $\{P_5, P_6, P_7\}$ and place $a$ and $b$ on $C$ in one of $(q+1)q$ ways. We then place the remaining two points, $c$ and $d$, of $\{P_1, P_2, P_3, P_4\}$ on $C$ in $(q-1)(q-2)$ ways. The line $cd$ intersects $ab$ in a point outside of $C$ so there are $q-2$ ways to choose $P$ on $ab$ but away from this intersection point and $a$ and $b$. Finally, we place the remaining two points of $\{P_5, P_6, P_7\}$ on $C$ in one of $(q-3)(q-4)$ ways. Multiplying everything together we obtain

$$4 \cdot 3 \cdot 3 \cdot (q^5 - q^2)(q + 1)q(q - 1)(q - 2)^2(q - 3)(q - 4).$$

We now add the two answers above together to get

$$N_{1}^{1,2,3,4} = 24q^3(q-2)(q-3)(q-4)(2q-5)(q+1)(q^2 + q + 1)(q-1)^2.$$
point $P$ is now given as $P = R_1 R_2 \cap O_1 O_2$ and no matter how we place $P_u$ and $P_v$, the three lines $P_u P_s$, $P_v P_t$ and $P_s P_t$ will not go through $P$. We can now multiply everything together to obtain

$$4 \cdot 3 \cdot 3 \cdot 2 \cdot (q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4).$$

The other possibility is that $P$ lies on two lines $R_1 R_2$ and $R_3 b$ where $\{R_1, R_2, R_3\}$ is the set $\{P_5, P_6, P_7\}$ and $b \in \{P_r, P_s, P_t\}$. We thus choose $b$ in three ways and rename the remaining two points in $\{P_r, P_s, P_t\}$ to $P_u$ and $P_v$. From now on, we must differentiate between when $P$ is on the outside and on the inside of $C$.

First, we choose $P$ on the outside of $C$ in $\frac{1}{2}(q + 1)q$ ways. We then choose $b$ as a point on $C$ whose tangent does not pass through $P$ in $q - 1$ ways. We then choose one of the points $P_5$, $P_6$ and $P_7$ to become the second intersection point in $C \cap bP$. Then, we place the remaining two points among $\{P_5, P_6, P_7\}$ on $C$ such that the line through them passes through $P$ in $q - 3$ ways. There are now $q - 5$ ways to choose $P_u$ and $P_v$ such that the line $P_u P_v$ will pass through $P$. Thus, the remaining $(q - 3)(q - 4) - (q - 5) = q^2 - 8q + 17$ choices must give $P_u$ and $P_v$ such that none of the lines $P_r P_s$, $P_v P_t$ and $P_s P_t$ will contain $P$. We may now multiply everything together to obtain

$$4 \cdot (q^5 - q^2) \cdot 3 \cdot \frac{1}{2} (q + 1)q \cdot (q - 1) \cdot 3 \cdot (q - 3) \cdot (q^2 - 8q + 17).$$

Now we choose $P$ on the inside of $C$ in one of $\frac{1}{2}(q - 1)q$ ways. We then choose $b$ as a point on $C$ whose tangent does not pass through $P$ in $q + 1$ ways. We then choose one of the points $P_5$, $P_6$ and $P_7$ to become the second intersection point in $C \cap bP$. Then, we place the remaining two points among $\{P_5, P_6, P_7\}$ on $C$ such that the line through them passes through $P$ in $q - 1$ ways. There are now $q - 3$ ways to choose $P_u$ and $P_v$ such that the line $P_u P_v$ will pass through $P$. Thus, the remaining $(q - 3)(q - 4) - (q - 3) = (q - 3)(q - 5)$ choices must give $P_u$ and $P_v$ such that none of the lines $P_r P_s$, $P_v P_t$ and $P_s P_t$ will contain $P$. We may now multiply everything together to obtain

$$4 \cdot (q^5 - q^2) \cdot 3 \cdot \frac{1}{2} (q - 1)q \cdot (q + 1) \cdot 3 \cdot (q - 1) \cdot (q - 3)(q - 5).$$

We may now add everything together to get

$$N_{2,3,4}^2 = 36q^3(q + 1)(q^2 + q + 1)(5q^3 - 37q^2 + 82q - 60)(q - 1)^2.$$
way. We thus have

\[ N_{3,\text{out}}^{1,2,3,4} = 4 \cdot (q^6 - q^2) \cdot \frac{1}{2}(q + 1)q \cdot (q - 1) \cdot 3 \cdot (q - 3) \cdot 2 \cdot (q - 5). \]

We now turn to computing \( N_{3,\text{in}}^{1,2,3,4} \). We thus choose the point \( P \) as a point on the inside of \( C \) in \( \frac{1}{2}(q - 1)q \) ways. We begin by placing \( P_3 \) at one of the points of \( C \) whose tangent does not pass through \( P \) in \( q + 1 \) ways. We label the second intersection point of \( C \cap P_3P \) with \( P_r, P_s \) or \( P_t \) and call the remaining two points \( P_u \) and \( P_v \). We then place \( P_6 \) at one of the \( q - 1 \) remaining points of \( C \) and then choose one of the points \( P_u \) and \( P_v \) to become the other intersection point of \( C \cap P_6P \). Finally, we place \( P_7 \) at one of the remaining \( q - 3 \) points and label the other point of \( C \cap P_7P \) in the only possible way. We now see that

\[ N_{3,\text{in}}^{1,2,3,4} = 4 \cdot (q^6 - q^2) \cdot \frac{1}{2}(q - 1)q \cdot (q + 1) \cdot 3 \cdot (q - 1) \cdot 2 \cdot (q - 3), \]

and we get

\[ N_{3}^{1,2,3,4} = 192q^3(q + 1)(q^2 + q + 1)(q^2 - 3q + 3)(q - 1)^2. \]

We now obtain

\[ |\Delta_1 \cap \Delta_2| = |\text{PGL}(3)| \cdot (93q^4 - 1245q^3 + 6195q^2 - 13470q + 10737), \]

and, finally,

\[ \left| (\mathcal{P}_7^2)^{\mathcal{F}_\sigma} \right| = q^6 + 35q^5 + 490q^4 - 3485q^3 + 13174q^2 - 24920q + 18375. \]

This concludes the equivariant point count of \( \mathcal{O}[2] \). In Section 9 we provide a summary of the results of the computations.

7. The Hyperelliptic Locus

Up to this point we have almost exclusively discussed plane quartics. We shall now briefly turn our attention to the other type of genus 3 curves - the hyperelliptic curves. There are many possible ways to approach the computation of the cohomology of \( \mathcal{H}_3[2] \). Our choice is by means of equivariant point counts as in the previous section.

Recall that a hyperelliptic curve \( C \) of genus \( g \) is determined, up to isomorphism, by \( 2g + 2 \) distinct points on \( \mathbb{P}^1 \), up to projective equivalence and that any such collection \( S \) of \( 2g + 2 \) points determines a double cover \( \pi: C \to \mathbb{P}^1 \) branched precisely over \( S \) (and \( C \) is thus a hyperelliptic curve). Moreover, if we pick \( 2g + 2 \) ordered points \( P_1, \ldots, P_{2g+2} \) on \( \mathbb{P}^1 \), the curve \( C \) also attains a level 2-structure. In the genus 3 case, we get 8 points \( Q_i = \pi^{-1}(P_i) \) which determine \( \binom{8}{2} = 28 \) odd theta characteristics \( Q_i + Q_j, \ i < j \) and \( \{ Q_1 + Q_8, \ldots, Q_7 + Q_8 \} \) is an ordered Aronhold basis, see [13] and [2], Appendix B.32-33, and an ordered Aronhold basis determines a level 2-structure.

However, not all level 2-structures on the hyperelliptic curve \( C \) arise from different orderings of the points. Nevertheless, there is an intimate relationship between the moduli space \( \mathcal{H}_3[2] \) of hyperelliptic curves with level 2-structure and the moduli space \( \mathcal{M}_{0,2g+2} \) of \( 2g + 2 \) ordered points on \( \mathbb{P}^1 \) given by the following theorem which can be found in [9], Theorem VIII.1.
Theorem 7.1. Each irreducible component of $\mathcal{H}_g[2]$ is isomorphic to the moduli space $\mathcal{M}_{0,2g+2}$ of $2g+2$ ordered points on the projective line.

Dolgachev and Ortland [9] pose the question whether the irreducible components of $\mathcal{H}_g[2]$ also are the connected components or, in other words, if $\mathcal{H}_g[2]$ is smooth. In the complex case, the question was answered positively by Tsuyumine in [17] and later, by a shorter argument, by Runge in [16]. Using the results of [1], the argument of Runge carries over word for word to an algebraically closed field of positive characteristic different from 2.

Theorem 7.2. If $g \geq 2$, then each irreducible component of $\mathcal{H}_g[2]$ is also a connected component.

We have a natural action of $S_{2g+2}$ on the space $\mathcal{M}_{0,2g+2}$. Since different orderings of the points correspond to different symplectic level 2 structures, $S_{2g+2}$ sits naturally inside $\text{Sp}(2g,\mathbb{Z}/2\mathbb{Z})$ and, in fact, for $g = 3$ and for even $g$ it is a maximal subgroup, see [10]. With Theorems 7.1 and 7.2 at hand, the following slight generalization of a corollary in [9] (p.145) is clear.

Corollary 7.3. Let $g \geq 2$ and let $X_{[\tau]} = \mathcal{M}_{0,2g+2}$ for each left coset $[\tau] \in \mathcal{T} := \text{Sp}(2g,\mathbb{Z}/2\mathbb{Z})/S_{2g+2}$. Then

$$\mathcal{H}_g[2] \cong \bigsqcup_{[\tau] \in \mathcal{T}} X_{[\tau]},$$

and the group $\text{Sp}(2g,\mathbb{Z}/2\mathbb{Z})$ acts transitively on the set of connected components $X_{[\tau]}$ of $\mathcal{H}_g[2]$. In particular, there are

$$\frac{|\text{Sp}(2g,\mathbb{Z}/2\mathbb{Z})|}{|S_{2g+2}|} = \frac{2^{2g^2 - 1} \cdot 2^{2g^2 - 2} \cdots (2^2 - 1)}{(2g + 2)!},$$

connected components of $\mathcal{H}_g[2]$.

Remark 7.4. As pointed out in [16], the argument to prove the corollary stated in [9] is not quite correct in full generality as it is given there. However, it is enough to prove the result for $g = 3$ and for even $g$, and in [16] it is explained how to obtain the full result.

Let us now, once and for all, choose a set $T$ of representatives of $\text{Sp}(2g,\mathbb{Z}/2\mathbb{Z})/S_{2g+2}$. If we denote the elements of $X_{[\text{id}]}$ by $x$, then any element in $X_{[\tau]}$ can be written as $\tau x$ for some $x \in X_{[\text{id}]}$. Let $\alpha$ be any element of $\text{Sp}(2g,\mathbb{Z}/2\mathbb{Z})$. Then

$$\alpha \tau = \tau' \sigma,$$

for some $\sigma \in S_{2g+2}$ and some $\tau' \in T$. Since the Frobenius commutes with the action of $\text{Sp}(2g,\mathbb{Z}/2\mathbb{Z})$ we have that

$$F(\alpha \tau x) = \tau x,$$

if and only if

$$F(\tau' \sigma x) = \tau' (F \sigma x) = \tau x.$$

But the Frobenius acts on each of the components of $\mathcal{H}_g[2]$ so we see that $F(\alpha \tau x) = \tau x$ if and only if $\tau' = \tau$ and $F \sigma x = x$.

We now translate the above observation into more standard representation theoretic vocabulary. Define a class function $\psi$ on $S_{2g+2}$ by

$$\psi(\sigma) = |X_{[\text{id}]}^{\text{F} \sigma}|,$$
and define a class function $\hat{\psi}$ on $\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ by setting

$$\hat{\psi}(\alpha) = |H_g[2]^{F\alpha}|,$$

for any $\alpha \in \text{Sp}(6, \mathbb{Z}/2\mathbb{Z})$. By the above observation we have that

$$\hat{\psi}(\alpha) = \sum_{\tau \in \mathcal{P}} \tilde{\psi}(\tau^{-1} \alpha \tau),$$

where

$$\tilde{\psi}(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \in S_{2g+2}, \\ 0 & \text{otherwise}. \end{cases}$$

In other words, $\hat{\psi}$ is the class function $\psi$ induced from $S_{2g+2}$ up to $\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$. Thus, to make an $S_{2g+2}$-equivariant point count of $H_g[2]$ we can make an $S_{2g+2}$-equivariant point count of $M_{0,2g+2}$ and then use the representation theory of $S_{2g+2}$ and $\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ in order to first induce the class function up to $\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ and then restrict it down again to $S_{2g+2}$. Once this is done, we can obtain the $S_{2g+1}$-equivariant point count by restricting from $S_{2g+2}$ to $S_{2g+1}$.

Using Lemma 6.4, the $S_8$-equivariant point count of $H_3[2]$ is very easy. We first compute the number of $\lambda$-tuples of $\mathbb{F}^1$ for each partition of $\lambda$ of 8 and then divide by $[\text{PGL}(2)]$ in order to obtain $|M_{0,8}^{F_2}|$, where $\sigma$ is a permutation in $S_8$ of cycle type $\lambda$. The result is given in Table 3. Once this is done, we induce up to $\text{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ in order to obtain the $\text{Sp}(6, \mathbb{Z}/2\mathbb{Z})$-equivariant cohomology of $H_3[2]$. The results are given in Table 4. Finally, we restrict to $S_7$ to get the results of Table 5 and 6. The computations present no difficulties whatsoever. We also mention that the equivariant Poincaré polynomials of $M_{0,n}$ and $\overline{M}_{0,n}$ have been computed for all $n \geq 3$ in [11].

It is not very hard to see that $M_{0,2g+2}$ is isomorphic to the complement of a hyperplane arrangement. One way to see this is to start by placing the first three points at 0, 1 and $\infty$. Then $M_{0,2g+2}$ is isomorphic to $(\mathbb{A}^1 \setminus \{0,1\})^{2g-1} \setminus \Delta$, where $\Delta \subset (\mathbb{A}^1 \setminus \{0,1\})^{2g-1}$ is the subset of points where at least two coordinates are equal. Thus, by the results of Section 5 we can deduce the cohomology of $H_3[2]$ from the equivariant point counts. In Section 9 we provide a summary of the results of the computations.

8. THE TOTAL MODULI SPACE

We now know the cohomology groups of both $Q[2]$ and $H_3[2]$ as representations of $S_7$. Unfortunately, we have not been able to obtain the cohomology of $M_3[2]$. In order to say something, we shall use the comparison theorem in étale cohomology and switch to work over the complex numbers and with re Rham cohomology. There, we have the following result.

**Lemma 8.1** (Looijenga, [15]). Let $X$ be a variety of pure dimension and let $Y \subset X$ be a hypersurface. Then there is a Gysin exact sequence of mixed Hodge structures

$$\cdots \to H^{k-2}(Y) (-1) \to H^k(X) \to H^k(X \setminus Y) \to H^{k-1}(Y) (-1) \to \cdots$$

Looijenga, [15], also showed that $H^k(Q_{\text{bigr}})$ is of pure Tate type $(k,k)$. The space $H_3[2]$ is isomorphic to a disjoint union of complements of hyperplane arrangements and the $k$'th cohomology group of such spaces are known to have Tate type $(k,k)$, also by a result of Looijenga [15]. Thus, if we apply Lemma 8.1 to $X = M_3[2]$,
\( Y = \mathcal{H}_3[2] \) and \( X \setminus Y = \mathbb{Q}[2] \) we have that the long exact sequence splits into four term sequences

\[
0 \to W_k H^k(X) \to H^k(X \setminus Y) \to H^{k-1}(Y)(-1) \to W_k H^{k+1}(X) \to 0,
\]

where \( W_k H^k(X) \) denotes the weight \((k, k)\) part of \( H^k(X)\). Moreover, let \( m^k_X(\lambda) \) denote the multiplicity of \( s_\lambda \) in \( H^k(X) \) and let \( n^k(\lambda) = m^k_{\mathbb{Q}[2]}(\lambda) - m^k_{\mathcal{H}_3[2]}(\lambda) \). If \( n^k(\lambda) \geq 0 \), then \( s_\lambda \) occurs with multiplicity at least \( n^k(\lambda) \) in \( W_k H^k(\mathcal{H}_3[2]) \) and if \( n^k(\lambda) \leq 0 \), then \( s_\lambda \) occurs with multiplicity at least \( -n^k(\lambda) \) in \( W_k H^{k+1}(\mathcal{H}_3[2]) \).

Thus, Tables 2 and 6 provide explicit bounds for the cohomology groups of \( \mathcal{H}_3[2] \).

9. Summary of computations

We summarize the computations related to \( \mathbb{Q}[2] \) in Table 1 and in Proposition 9.1 we give the Poincaré polynomial of \( \mathbb{Q}[2] \). In Table 2 we give the cohomology of \( \mathbb{Q}[2] \) as a representation of \( S_7 \). The rows correspond to the cohomology groups and the columns correspond to the irreducible representations of \( S_7 \). The symbol \( s_\lambda \) denotes the irreducible representation of \( S_7 \) corresponding to the partition \( \lambda \) and a number \( n \) in row \( H^k \) and column \( s_\lambda \) means that \( s_\lambda \) occurs in \( H^k \) with multiplicity \( n \).

**Proposition 9.1.** The Poincaré polynomial of \( \mathbb{Q}[2] \) is

\[
PS_{\mathbb{Q}[2]}(t) = 1 + 35t + 490t^2 + 3485t^3 + 13174t^4 + 24920t^5 + 18375t^6.
\]

Tables 3, 4 and 5 give equivariant point counts for various spaces and groups related to the equivariant point count of \( \mathcal{H}_3[2] \) and in Table 6 we give the cohomology groups of \( \mathcal{H}_3[2] \) as representations of \( S_7 \). For convenience of we also give the Poincaré polynomial of \( \mathcal{H}_3[2] \).

**Proposition 9.2.** The Poincaré polynomial of \( \mathcal{H}_3[2] \) is

\[
PS_{\mathcal{H}_3[2]}(t) = 36 + 720t + 5580t^2 + 20880t^3 + 37584t^4 + 25920t^5.
\]
\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\(\lambda\) & \(|Q[2]^{S_7,\sigma_{\lambda}}|\) \\
\hline
[7] & \(q^6 + q^3\) \\
[6, 1] & \(q^6 - 2q^3 + 1\) \\
[5, 2] & \(q^6 - q^2\) \\
[5, 1^2] & \(q^6 - q^2\) \\
[4, 3] & \(q^6 - q^5 - 2q^4 + q^3 + q^2\) \\
[4, 2, 1] & \(q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3\) \\
[4, 1^3] & \(q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3\) \\
[3^2, 1] & \(q^6 - 2q^5 - 2q^4 - 8q^3 + 16q^2 + 10q + 21\) \\
[3, 2^2] & \(q^6 - q^5 - 2q^4 + 3q^3 + q^2 - 2q\) \\
[3, 2, 1^2] & \(q^6 - 3q^5 + 5q^3 - q^2 - 2q\) \\
[3, 1^4] & \(q^6 - 5q^5 + 10q^4 - 5q^3 - 11q^2 + 10q\) \\
[2^3, 1] & \(q^6 - 3q^5 - 6q^4 + 19q^3 + 6q^2 - 24q + 7\) \\
[2^2, 1^3] & \(q^6 - 7q^5 + 10q^4 + 15q^3 - 26q^2 - 8q + 15\) \\
[2, 1^5] & \(q^6 - 15q^5 + 90q^4 - 265q^3 + 374q^2 - 200q + 15\) \\
[1^7] & \(q^6 - 35q^5 + 490q^4 - 3485q^3 + 13174q^2 - 24920q + 18375\) \\
\hline
\end{tabular}
\caption{The \(S_7\)-equivariant point count of \(Q[2]\). We use \(\sigma_{\lambda}\) to denote any permutation in \(S_7\) of cycle type \(\lambda\).}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
& \(H^0\) & \(H^1\) & \(H^2\) & \(H^3\) & \(H^4\) & \(H^5\) & \(H^6\) \\
\hline
\(s_7\) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(s_6, 1\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(s_5, 2\) & 0 & 3 & 4 & 4 & 3 & 5 & 1 & 3 & 1 & 1 \\
\(s_5, 1^2\) & 1 & 8 & 14 & 18 & 14 & 30 & 16 & 16 & 12 & 18 \\
\(s_4, 3\) & 4 & 20 & 44 & 47 & 44 & 99 & 56 & 56 & 54 & 83 \\
\(s_4, 2, 1\) & 6 & 33 & 76 & 76 & 72 & 178 & 97 & 104 & 105 & 169 \\
\(s_4, 1^3\) & 6 & 23 & 51 & 54 & 54 & 127 & 74 & 76 & 77 & 126 \\
\hline
\(s_{3, 1^4}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(s_{2^3, 1}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(s_{2^2, 1^3}\) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(s_{2, 1^5}\) & 4 & 6 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\(s_1^7\) & 32 & 31 & 25 & 26 & 6 & 1 & 2 & 3 & 4 & 1 \\
\hline
\end{tabular}
\caption{The cohomology of \(Q[2]\) as a representation of \(S_7\).}
Table 3. The $S_8$-equivariant point count of $\mathcal{M}_{0,8}$. We use $\sigma_\lambda$ to denote any permutation in $S_8$ of cycle type $\lambda$. 

| $\lambda$ | $|\mathcal{M}_{0,8}^{F,\sigma_\lambda}|$ |
|-----------|-------------------------------------|
| [8]       | $(q^2 + 1)q^3$                      |
| [7, 1]    | $(q + 1)(q^2 + q + 1)(q^2 - q + 1)$ |
| [6, 2]    | $q(q - 1)(q^3 + q - 1)$             |
| [6, 1$^2$]| $q(q + 1)(q^3 + q - 1)$             |
| [5, 3]    | $q(q - 1)(q + 1)(q^2 + 1)$          |
| [5, 2, 1] | $q(q - 1)(q + 1)(q^2 + 1)$          |
| [5, 1$^3$]| $q(q - 1)(q + 1)(q^2 + 1)$          |
| [4$^2$]   | $q(q^4 - q^2 - 4)$                  |
| [4, 3, 1] | $(q - 1)q^2(q + 1)^2$               |
| [4, 2$^2$]| $(q - 1)(q - 2)(q + 1)q^2$          |
| [4, 2, 1$^2$] | $(q - 1)(q + 1)q^3$         |
| [4, 1$^4$]| $(q - 1)(q - 2)(q + 1)q^2$          |
| [3$^2$, 2]| $q(q - 1)(q^3 - q - 3)$             |
| [3$^2$, 1$^2$]| $q(q + 1)(q^3 - q - 3)$         |
| [3, 2$^2$, 1]| $q(q - 1)(q - 2)(q + 1)^2$ |
| [3, 2, 1$^3$]| $(q + 1)q^2(q - 1)^2$               |
| [3, 1$^5$]| $q(q - 1)(q - 2)(q - 3)(q + 1)$     |
| [2$^4$]   | $(q - 2)(q - 3)(q + 2)(q^2 - q - 4)$ |
| [2$^3$, 1$^2$]| $q(q - 2)(q + 1)(q^2 - q - 4)$ |
| [2$^2$, 1$^4$]| $q(q - 1)(q + 1)(q - 2)^2$ |
| [2, 1$^6$]| $q(q - 1)(q - 2)(q - 3)(q - 4)$    |
| [1$^8$]   | $(q - 2)(q - 3)(q - 4)(q - 5)(q - 6)$ |
| $\lambda$ | $|\mathcal{H}_3[2]^{F_{\sigma_\lambda}}|$ |
|---|---|
| [8] | $2q^5 + 2q^3$ |
| [7, 1] | $q^5 + q^4 + q^3 + q^2 + q + 1$ |
| [6, 2] | $3q^5 + 3q^3 - 6q^2 - 3q^4 + 3q$ |
| [6, 1^2] | $q^5 + q^4 + q^3 - q$ |
| [5, 3] | $q^5 - q$ |
| [5, 2, 1] | $q^5 - q$ |
| [5, 1^3] | $q^5 - q$ |
| [4^2] | $4q^5 - 16q - 4q^3$ |
| [4, 3, 1] | $2q^5 + 2q^4 - 2q^3 - 2q^2$ |
| [4, 2^2] | $6q^5 + 12q^2 - 12q^4 + 6q^3$ |
| [4, 2, 1^2] | $2q^5 - 2q^3$ |
| [4, 1^4] | $2q^5 - 4q^4 - 2q^3 + 4q^2$ |
| [3^2, 2] | $q^5 - q^4 - q^3 - 2q^2 + 3q$ |
| [3^2, 1^2] | $3q^5 + 3q^4 - 3q^3 - 12q^2 - 9q$ |
| [3, 2^2, 1] | $2q^5 - 2q^4 - 6q^3 + 2q^2 + 4q$ |
| [3, 2, 1^3] | $4q^5 - 4q^4 - 4q^3 + 4q^2$ |
| [3, 1^5] | $6q^5 - 30q^4 + 30q^3 + 30q^2 - 36q$ |
| [2^4] | $12q^5 + 48q - 60q^3 + 336q^2 - 48q^4 - 576$ |
| [2^3, 1^2] | $4q^5 - 8q^4 - 20q^3 + 24q^2 + 32q$ |
| [2^2, 1^4] | $8q^5 - 32q^4 + 24q^3 + 32q^2 - 32q$ |
| [2, 1^6] | $16q^5 - 160q^4 + 560q^3 - 800q^2 + 384q$ |
| [1^8] | $36q^5 - 720q^4 + 5580q^3 - 20880q^2 + 37584q - 25920$ |

Table 4. The $S_8$-equivariant point count of $\mathcal{H}_3[2]$. We use $\sigma_\lambda$ to denote any permutation in $S_8$ of cycle type $\lambda$. 
\[ \lambda \quad |\mathcal{H}_3[2]^{F_{\sigma_\lambda}}| \\
[7] \quad q^5 + q^4 + q^3 + q^2 + q + 1 \\
[6, 1] \quad q^5 + q^4 + q^3 - q \\
[5, 2] \quad q^5 - q \\
[5, 1^2] \quad q^5 - q \\
[4, 3] \quad 2q^5 + 2q^4 - 2q^3 - 2q^2 \\
[4, 2, 1] \quad 2q^5 - 2q^3 \\
[4, 1^3] \quad 2q^5 - 4q^4 - 2q^3 + 4q^2 \\
[3^2, 1] \quad 3q^5 + 3q^4 - 3q^3 - 12q^2 - 9q \\
[3, 2^2] \quad 2q^5 - 2q^4 - 6q^3 + 2q^2 + 4q \\
[3, 2, 1^2] \quad 4q^5 - 4q^4 - 4q^3 + 4q^2 \\
[3, 1^4] \quad 6q^5 - 30q^4 + 30q^3 + 30q^2 - 36q \\
[2^3, 1] \quad 4q^5 - 8q^4 - 20q^3 + 24q^2 + 32q \\
[2^2, 1^3] \quad 8q^5 - 32q^4 + 24q^3 + 32q^2 - 32q \\
[2, 1^5] \quad 16q^5 - 160q^4 + 560q^3 - 800q^2 + 384q \\
[1^7] \quad 36q^5 - 720q^4 + 5580q^3 - 20880q^2 + 37584q - 25920 \\

Table 5. The $S_7$-equivariant point count of $\mathcal{H}_3[2]$. We use $\sigma_\lambda$ to denote any permutation in $S_7$ of cycle type $\lambda$. 

| $H^0$ | $s_7$ | $s_{6,1}$ | $s_{5,2}$ | $s_{5,1^2}$ | $s_{4,3}$ | $s_{4,2,1}$ | $s_{4,1^3}$ | $s_{3,2^1}$ | $s_{3,2^2}$ | $s_{3,2,1^2}$ |
|-------|--------|------------|------------|------------|----------|----------|----------|----------|----------|----------|
| $H^1$ | 2      | 1          | 1          | 0          | 1        | 0        | 0        | 0        | 0        | 0        |
| $H^2$ | 2      | 7          | 9          | 5          | 5        | 7        | 1        | 3        | 2        | 1        |
| $H^3$ | 3      | 18         | 30         | 31         | 25       | 50       | 20       | 26       | 19       | 26       |
| $H^4$ | 6      | 35         | 74         | 80         | 72       | 162      | 86       | 92       | 83       | 129      |
| $H^5$ | 8      | 48         | 114        | 117        | 109      | 271      | 150      | 157      | 158      | 254      |

| $s_{3,1^4}$ | $s_{2^3,1}$ | $s_{2^2,1^3}$ | $s_{2,1^5}$ | $s_{1^7}$ |
|-------------|-------------|---------------|------------|-----------|
| $H^0$       | 0           | 0             | 0          | 0         |
| $H^1$       | 0           | 0             | 0          | 0         |
| $H^2$       | 5           | 7             | 4          | 0         |
| $H^3$       | 43          | 45            | 36         | 10        | 1         |
| $H^4$       | 105         | 96            | 92         | 35        | 4         |
| $H^5$       | 77          | 72            | 72         | 31        | 5         |

Table 6. The cohomology of $\mathcal{H}_3[2]$ as a representation of $S_7$. 

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