An exact solution of the higher-order gravity in standard radiation-dominated era

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Abstract

We report that the standard evolution of radiation-dominated era (RDE) universe $a \propto t^{1/2}$ is a sufficient condition for solving a sixth order gravitational field equation derived from the Lagrangian containing $BR_{ab}^{\text{R}}R_{ab}^{\text{R}} + CR_{cc}^{R}$ as well as a polynomial $f(R)$ for a spatially flat radiation FLRW universe. By virtue of the similarity between $R_{ab}^{\text{R}}R_{ab}^{\text{R}}$ and $R^2$ models up to the background order and of the vanishing property of $R_{cc}^{R}$ for $H = 1/(2t)$, the analytical solution can be obtained from a special case to general one. This proves that the standard cosmic evolution is valid even within modified gravitational theory involving higher-order terms. An application of this background solution to the tensor-type perturbation reduces the complicated equation to the standard second order equation of gravitational wave. We discuss the possible ways to discriminate the modified gravity model on the observations such as the gravitational wave from the disturbed universe and primordial abundances.

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I. INTRODUCTION

Even though Einstein’s general relativity (GR) has successfully passed the observational tests in the solar system scale, lots of efforts to generalize GR for cosmology have been continued. One of them is to introduce additional higher-order derivative terms due to both theoretical and phenomenological reasons. For instance, sixth order $R\Box R$ and fourth order $R_{ab}R^{ab}$ terms [2,7,17] are considered here with the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi}\left(f(R) + BR_{ab}R^{ab} + CR\Box R\right) + L_m\right],$$  

where $f(R)$ is a polynomial function of the Ricci scalar $R$

$$f(R) = \sum_{n=1}^{N} A_n R^n = R + A_{(2)}R^2 + A_{(3)}R^3 + ..., \quad A_{(1)} \equiv 1, \quad A_{(2)} \equiv A, \quad (2)$$

$A_{(n)}, B, C$ are constants, $R_{ab}$ is the Ricci tensor. The d’Alembertian of $R$ is $\Box R \equiv g^{ab}R_{ab} = R_{cc}$ where the semicolon denotes covariant derivative, and the matter part Lagrangian is defined as $\delta (\sqrt{-g}L_m) \equiv \frac{1}{2}\sqrt{-g}T_{(m)}^{ab}\delta g_{ab}$. In this paper we follow the Hawking-Ellis [5] convention, but adopt (Planckian) natural units $c \equiv 1 \equiv G_N \equiv \hbar \equiv k_B$. As special cases of general $f(R)$ gravity [7–9], $N = 1$ and $N = 2$ in Eq. (2) correspond to the gravity theory of Einstein and Starobinsky [10], respectively. The $R^2$ theory is not only favored by Planck Collaboration [11] as an inflationary model that predicts successfully some observables such as the spectral index and tensor-to-scalar ratio, but also by scalaron dark matter model that estimates its mass [12]. Moreover, there have been attempts to extend the $R^2$ theory. The model with the next $N = 3$ term as a small contribution to $N = 2$ theory was also studied [13–15], recently, $N = 4$ case was also investigated [16].

Further, a fourth order $BR_{ab}R^{ab}$ theory, that is neither of $f(R)$ model nor conformally equivalent to Einstein gravity, was also introduced in the literature and textbooks [2,7,17]. For the physical meaning or motivation of $CR\Box R$ theory, e.g., a conformal equivalence to two interacting scalar fields causing inflation, we refer to Ref. [1]. According to a review for higher order gravity theory [18], $\Box R$ model was pioneered by Buchdahl [19] (1951), and quantum gravitational higher order corrections to Einstein-Hilbert action was the idea of Sakharov [20] (1967), prior to Starobinsky. The case including $N = 3$ and $CR\Box R$ without the $B$-term for an inflationary regime was studied using phase diagrams and conformal transformation [13]. Besides, the gravitational field equation for even higher-order gravity was derived by imposing the Noether symmetry [21].
In cosmology, modified gravity models are usually applied to the inflationary epoch or to the present age in order to describe the accelerated expansion of the universe. The effects of $f(R)$ gravity was considered also in the radiation-dominated era (RDE) for the study of baryogenesis \cite{22} or big bang nucleosynthesis (BBN) \cite{23,24,25,26}. In particular, besides the numerical solutions for the given gravity models, a standard solution for a scale factor describing an evolution of RDE Friedmann-Lemaître-Robertson-Walker (FLRW) universe was found for a theory with a generic Lagrangian containing almost arbitrary function of $R$ \cite{27}. In this paper, we show that the standard RDE solution is still viable for the gravity models involving the fourth or sixth order differential equations. See the $B$ and $C$ terms in Eq. (7) which were proposed in the previous literature. This proves that, even within modified gravitational theory involving higher-order terms, the standard cosmic evolution is valid in the modified gravity containing the higher-order terms.

In Section II, we introduce the generalized Einstein field equation for our model. In Section III, starting with some standard cosmological assumptions, we apply the gravity models from specific one with a basic power-law ansatz of the scale factor to general case and find a common standard RDE solution in Eq. (23), which is our main result. Section IV is dedicated to two observational aspects of a specific case of the modified theory. The final section is devoted for the mathematical conclusion and discussions.

II. GRAVITATIONAL FIELD EQUATIONS

The gravitational field equation (GFE) \cite{1,2,13} from the metric variation \cite{28,29} of the action (1) is

$$
\left[ g_{ab} \left( \frac{\partial f}{\partial R} \right)_c - \left( \frac{\partial f}{\partial R} \right)_{ab} - \frac{1}{2} f g_{ab} \right] - 8\pi (T_{ab}^{(B)} + T_{ab}^{(C)}) = 8\pi T_{ab}^{(m)},
$$

where

$$
T_{ab}^{(B)} \equiv \frac{B}{8\pi} \left( \frac{1}{2} R^{cd} R_{cd} g_{ab} + R_{ab} - 2 R^{cd} R_{acbd} - \frac{1}{2} g_{ab} R^{cd} g_{cd} - R_{ab} \right),
$$

$$
T_{ab}^{(C)} \equiv \frac{C}{8\pi} \left[ 2 R_{c,b}^{ae} - 2 R_{ab} R^{c} - R_{a,b} \right] - 8\pi (T_{ab}^{(B)} + T_{ab}^{(C)}) = 8\pi T_{ab}^{(m)},
$$

For a derivation of the GFE, we use the following relation

$$
\delta(\Box R) = -\delta g_{ab} R^{abc}_{\ c} - \delta g_{ab} R^{ab} - g^{ab} \delta g_{abc}^{\ cd}_{\ d} + \delta g_{ab}^{\ bac}_{\ c}
$$

$$
- R^{ab} \delta g_{abc}^{\ c} - 2 R_{c,b}^{ae} \delta g_{ab}^{\ c} - R^{c} \delta g_{ab}^{\ b} + \frac{1}{2} R^{c} g^{ab} \delta g_{abc},
$$

(6)
referring to the helpful work by Barth and Christensen [29]. The contraction of the GFE in Eq. (3) with an inverse metric $g^{ab}$ will be useful for RDE cosmology, which is given as

$$3 \left( \frac{\partial f}{\partial R} \right)^c_e + \frac{\partial f}{\partial R} R - 2f] + 2B R^e_c + C [6 R^e_c d + 2 R R^e_c + R^e R] = 8\pi T^{(m)c}_e.$$  

(7)

For later convenience, we also introduce an alternative equivalent form of the GFE after arranging the directly varied GFE in Eq. (3) [2, 30]

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{1}{F} \left[ 8\pi T^{(m)}_{ab} + F_{;ab} - g_{ab} \left( F^c_c - \frac{f - RF}{2} \right) + C \left( R_{;a} R_{;b} - \frac{1}{2} g_{ab} R^e R_c \right) 
+ B \left( \frac{2}{3} R R_{ab} + \frac{1}{2} R^{ed} R_{ed} g_{ab} + \frac{1}{3} R_{;ab} - 2 R^{ed} R_{;abcd} + \frac{1}{6} g_{ab} R^e R_c - R_{ab}^{;c} c \right) \right],$$

(8)

$$F \equiv f, R + \frac{2}{3} B R + 2 C R^e c, \quad f, R \equiv \frac{\partial f}{\partial R},$$

(9)

and the trace of Eq. (8)

$$-R = \frac{1}{F} \left[ 8\pi T^{(m)c}_e - 3 F^e_c + 2f - 2FR + \frac{2}{3} BR^2 - CR^e R_c \right].$$

(10)

### III. BACKGROUND UNIVERSE DURING THE RDE

We adopt a spatially flat FLRW metric representing a homogeneous and isotropic background universe

$$ds^2 = a^2(\eta) \left( -d\eta^2 + \delta_{\alpha\beta} dx^\alpha dx^\beta \right),$$

(11)

where $a(\eta)$ is the cosmic scale factor, $\eta$ is the conformal time ($dt \equiv ad\eta$), and $\dot{a}/a \equiv \frac{1}{a} \frac{da}{dt} \equiv H$ is the Hubble expansion rate. We adopt a perfect fluid in standard cosmology, whose energy-momentum tensor is composed of time-dependent energy density $\mu(t)$ and pressure $p(t)$

$$T^{(m)}_{00} = -\mu(t), \quad T^{(m)}_{\alpha\beta} = 0, \quad T^{(m)}_{\beta} = p(t) \delta^\alpha_\beta.$$  

(12)

In the RDE the cosmological constant is negligible and the equation of state (EoS) for radiation-like is simply given with the RDE density

$$p = \frac{1}{3} \mu.$$  

(13)
A. Case $N = 2, C = 0$

In this subsection, a special case with $f(R) = R + AR^2$ (Starobinsky model of $f(R)$ gravity) and $C = 0$ (no sixth order gravity) is considered. Then the temporal and spatial components of the GFE in Eq. (3), respectively, become 

$$8\pi\mu = 3H^2 + 6(3A + B)(2\ddot{H} - \dot{H}^2 + 6\dot{H}H^2),$$  \hspace{1cm} (14)

$$8\pi p = -(2\dot{H} + 3H^2) - 2(3A + B)\left(2\frac{d^3 H}{dt^3} + 12\ddot{H} + 9\dot{H}^2 + 18\dot{H}H^2\right),$$ \hspace{1cm} (15)

where it is notable that $A$- and $B$-terms play qualitatively the same role for the evolution of the background universe. The modified Friedmann equation in Eq. (14) and Eq. (15) can be checked by substituting them into the following continuity equation:

$$\dot{\mu} + 3H(\mu + p) = 0.$$ \hspace{1cm} (16)

A system of three independent ordinary differential equations (ODEs) in Eqs. (14), (15) (or (16)), and (13), contains 3 unknown functions, $a(t), \mu(t),$ and $p(t)$. Putting Eqs. (14) and (13) into Eq. (16), the set of ordinary differential equations can be reduced as an ODE for $H(t)$ that has various solutions:

$$\dot{H} + 2H^2 + 2(3A + B)\left(\frac{d^3 H}{dt^3} + 7\ddot{H} + 4\dot{H}^2 + 12\dot{H}H^2\right) = 0.$$ \hspace{1cm} (17)

This is a nonlinear third order ODE for $H(t)$ (fourth order eq. for $a(t)$); however, an exact analytical solution can be easily found by a power-law ansatz:

$$a(t) \propto t^\alpha, \quad H(t) = \frac{\alpha}{t}, \quad \dot{H} = -\frac{\alpha}{t^2}, \quad \ddot{H} = 2\frac{\alpha}{t^3}, \quad \frac{d^3 H}{dt^3} = -6\frac{\alpha}{t^4}.$$ \hspace{1cm} (18)

Among two mathematical solutions, $\alpha = 1/2$ or $\alpha = 0$, for Eq. (17), the former is selected as a standard solution.

By reducing order of the ODE with the definition of the expansion rate (from fourth to third order) and the following chain rule (from third to second order) for $X = X(t)$,

$$\dot{X} = \frac{dX}{dt}, \quad \frac{d^2 X}{dt^2} = \frac{dX}{dX} \frac{dX}{dt} = \frac{dX}{dX} \frac{d^2 X}{dt^2} + \left(\frac{dX}{dX}\right)^2,$$ \hspace{1cm} (19)

$$Y \equiv \dot{X}, \quad \frac{dY}{dX} = \frac{dX}{dX} = \frac{\dot{Y}}{Y}, \quad \frac{d^2 Y}{dX} = \frac{1}{Y^2}\left(\ddot{Y} - \frac{\dot{Y}^2}{Y}\right),$$ \hspace{1cm} (20)

we try to alleviate an instability that the fourth order ODE for $a(t)$ in Eq. (17) mathematically suffers. Thus, Eq. (17) has this mathematical form with no explicit time dependence

$$Y + 2X^2 + 2(3A + B)Y\left(\frac{d^2 Y}{dX^2}Y + \left(\frac{dY}{dX}\right)^2 + 7\frac{dY}{dX}X + 4Y + 12X^2\right) = 0.$$ \hspace{1cm} (21)
\[
\dot{H} + 2H^2 + 2(3A + B)\dot{H}^2 + \frac{d^2\dot{H}}{dH^2}\dot{H} + \left(\frac{d\dot{H}}{dH}\right)^2 + 7\frac{d\dot{H}}{dH}H + 4\dot{H} + 12H^2 = 0. \tag{22}
\]

As one simplest form of solution, \(\dot{H}(H) = -2H^2\) is an exact analytic solution of the second order ODE in Eq. (22) regardless of the value of \((3A + B)\), so it can be used to test computer code implementing the corresponding differential equations by comparing analytical plot and the numerical results \([31]\).

Consequently, the flat FLRW universe model under \(R + AR^2\) as well as \(BR^{ab}R_{ab}\) gravity has a standard RDE solution

\[
a(t) = (const)t^{1/2}, \quad H(t) = \frac{1}{2t}, \quad \dot{H}(H) = -2H^2, \tag{23}
\]

where the third one describes a flipped parabola (a 2D phase diagram) on a \(H-\dot{H}\) plane in which only quadrant IV plane is physically acceptable. We can also notice that the RDE solution (23) is still viable in more generalized models such as any polynomial \(f(R)\) or a sixth order gravity.

### B. Case \(B = 0 = C\), any natural number \(N\)

For a polynomial \(f(R)\) (Eq. (2)) model setting \(B = C = 0\), we use the same method with the previous case to get an ODE for \(H(t)\) whose counterpart is Eq. (17)

\[
\left[\frac{\dot{R}}{2} - 3(\dot{H} + 2\dot{H}H)\right]f_{,R} - 3(\dot{H} + H^2)\dot{R}\frac{d}{dR}f_{,R} + 3\dot{R}\frac{d}{dR}\left(H\dot{R}\frac{d}{dR}f_{,R}\right) + 4H\left[\frac{f}{2} - 3(\dot{H} + H^2)f_{,R} + 3H\dot{R}\frac{d}{dR}f_{,R}\right] = 0, \tag{24}
\]

where the Ricci scalar \(R\) from the metric in Eq. (11) and its time derivative are, respectively

\[
R = 6(\dot{H} + 2H^2), \quad \dot{R} = 6(\ddot{H} + 4\dot{H}H). \tag{25}
\]

Substituting Eq. (18) into Eq. (25), one can obtain

\[
R(t) = \frac{6\alpha(2\alpha - 1)}{t^2}. \tag{26}
\]

This result simply shows how the standard RDE solution in Eq. (23) from the Einstein gravity \((N = 1)\) becomes also a solution of the GFE in Eq. (24) involving any polynomial \(f(R)\) gravity because the condition of \(\alpha = 1/2\) yields

\[
R = 0 = \dot{R} = \ddot{R} = f, \quad f_{,R} = 1. \tag{27}
\]
According to Barrow and Ottewill [27], the standard RDE evolution given in Eq. (23) is a solution of not only a polynomial \( f(R) \), but also any \( f(R) \) theory in which \( f(0) = 0 \) and \( f_R(0) \neq 0 \). Those exceptional examples are \( f(R) \sim R^{-N}, \ln R \), and so on.

Meanwhile the trace in Eq. (7) in this case \((B = C = 0)\) is useful [7], especially for the perfect fluid governed by the RDE EoS in Eq. (13),

\[
3 \left( \frac{\partial f}{\partial R} \right)_c + \frac{\partial f}{\partial R} R - 2f = T^{(m)c}_c = 0.
\]

This differential equation equivalent to Eq. (24) within the flat FLRW model tells us [7] that \( f(R) \) theories have more various solutions (the exact solution (23) is just one of them) than Einstein’s theory \((f = R)\) relating \( R \) with \( T^{(m)c}_c \) not differentially but algebraically, and that the function (23) is not necessary but sufficient to be a solution of the ODE in Eq. (28).

C. General case

In analogous to the previous cases, it is now easy to see the trace in Eq. (10) (or Eq. (7)) for a RDE flat FLRW universe

\[
-R = \frac{1}{F} \left[ -3F^c_c + 2f - 2FR + \frac{2}{3} BR^2 - CR^c_c R_c \right],
\]

has an analytical solution of \( H(t) = 1/(2t) \) because the solution \((\alpha = 1/2)\) implies that

\[
R = 0 = R^c_c = -\left( \ddot{R} + 3H \dot{R} \right) = f = \dot{F}, \quad F \equiv f_R + \frac{2}{3} BR + 2CR^c_c = 1 = f_R.
\]

The traced Eq. (29) is a complicated fifth order ODE for \( H(t) \) admitting various solutions, but the standard RDE solution is a sufficient condition to satisfy Eq. (29) regardless of the constants \( A(n \geq 2), B, \) and \( C \). Among other various numerical solutions of Eq. (29) depending on those constants, it would be an intriguing problem which one could be a candidate as a physical solution describing the evolution of a RDE universe affected by the modified gravity models. Figure 1 shows one of numerical solutions of Eq. (17) for \((3A + B) = 0.1s^2\) in the Case \( N = 2, C = 0 \).

IV. OBSERVATIONAL TESTS FOR THE THEORY \( N = 2, C = 0 \)

Although we focus on a theory of the homogeneous and isotropic universe model, we would like to briefly mention two observational aspects for the Case \( N = 2, C = 0 \) (subsection III.
FIG. 1: Two solution curves for Eq. [17] (Case $N = 2, C = 0$). The analytic solution $H = 1/(2t)$ (dashed line) is plotted regardless of $(3A + B)$. An approximate numerical solution (solid line) is introduced for $(3A + B) = 0.1 s^2$.

A.), i.e. Starobinsky plus $BR^{ab}R_{ab}$ model up to the linearly perturbed order. Those are BBN in the FLRW background universe and gravitational wave from the disturbed universe.

Observations of primordial abundances such as $^2$H, $^3$He, $^4$He, and $^7$Li are currently the only data providing information of RDE. Since the change in cosmic expansion rate significantly affects the freeze-out time of the nucleosynthesis, the BBN calculation with the given gravity model could validate this model [31]. Specifically, for the BBN calculation, the cosmic expansion rate can be obtained by solving Eqs. (14) and (15) with thermodynamic variables of dominant species such as photons, neutrinos, electrons, and positrons. Then, taking the modified cosmic expansion rate, one can carry out BBN calculation with traditionally reliable [32] or user-friendly improved [33] BBN codes, whose result would depend on the coefficient $(3A + B)$. Therefore, by the observations of primordial abundances, the coefficient $(3A + B)$ whose original dimension is length to the second power could be constrained [31]. In particular, we note that there could be the potential to distinguish between Einstein’s theory of gravity and generalized gravity theories using precisely measured amounts of helium ($^4$He) and deuterium ($^2$H).

The propagation speed of tensor-type first order cosmological perturbation can distinguish $BR^{ab}R_{ab}$ from $AR^2$ gravity in principle. If those modified gravity effects are small enough to be treated perturbatively comparing to Einstein gravity, then the wave equation can be simplified to Mukhanov-Sasaki equation that implies the propagation speed slightly less than
the speed of light due to only $BR_{ab}R_{ab}$ effect \(^{34}\). If the speed of cosmological gravitational wave is determined to \(c\) in future observations with very high accuracy, then $BR_{ab}R_{ab}$ model will be inappropriate as a gravity theory and $AR^2$ model will be able to survive. While $A$- and $B$- gravity behave similarly at the background order as we see above, they are very distinguishable in the next order both mathematically \(^2\) and physically. In reality, it is very difficult to detect or measure the subtle interval of arrival between electromagnetic and gravitational wave signal. It is currently unclear which gravity model is the better explanation for the RDE universe; however, we wish to emphasize that our RDE solution for the evolution of the background universe is very available in observationally selected model(s) among the aforementioned Cases.

V. CONCLUSION AND DISCUSSIONS

We conclude that the standard RDE solution of $a(t) \propto t^{1/2}$ obtained from the Einstein gravity ($N = 1$) is also a RDE solution in a spatially flat FLRW universe filled with perfect fluid under a generalized gravity model whose Lagrangian is $$\left[ \frac{1}{16\pi} \left( \sum_{n=1}^{N} A_{(n)} R^n + BR_{ab}R_{ab} + CRR_{c}^{c} \right) + L_{m} \right].$$ Besides the polynomial $f(R)$ theory ($A_{(n)}$-terms) as a subset of more general $f(R)$ gravity \(^{27}\), the fourth order $BR_{ab}R_{ab}$ ($B$ gravity) and sixth order $CRR_{c}^{c}$ ($C$ gravity) theories in the FLRW model also allow the same solution in Eq. (23) by virtue of the qualitative sameness of the $R_{ab}R_{ab}$ with the $R^2$ theory (see Eqs. (14, 15, 17)) and of the vanishing property of $\Box R$ (Eq. (30)) with the solution. This analytical solution can be some fiducial curves (regardless of the coefficients $A_{(n \geq 2)}$, $B$, and $C$) that can test the numerical computations to find other numerical solution curves (depending on the coefficients).

At first we made efforts to obtain some numerical RDE solutions of Eq. (17) instead of the exact analytical solution in the special case ($R_{ab}R_{ab}$ model added to Starobinsky theory for $N = 2$), we inductively obtained a hypothesis that the solution in Eq. (23) is also a solution in any polynomial $f(R)$ model and deductively tried to prove it, to which the discovery for general $f(R)$ gravity by Barrow and Ottewill was prior \(^{27}\).

For the Case $N = 2, C = 0$ (subsection III. A.), we have either standard analytical solution in Eq. (23) or other numerical solutions for the third order ODE in Eq. (17). If Eq. (23) is a real physical description of the evolution of the universe during the RDE, then $A$ and $B$ terms introduced in the action serve theoretically useful purposes (such as
quantum corrections or renormalization [35]) without affecting the background evolution although the constants $A$ and $B$ are arbitrary with this solution that cannot be determined from observational constraints. However if other non-standard solutions of scale factor are allowed by the RDE universe governed by the higher-derivative gravity, then the slight effects of $A$ and $B$ terms can be constrained by the comparison of the BBN observations and calculations [31].

Even if the terms involved by $A$ and $B$ representing the four-derivative theory in the action Eq. (1) share the same solution at the background order in the homogeneous and isotropic model, the two terms behave quite differently in the perturbed universe with gravitational wave [2, 31] or density fluctuation [3]. A model with an action containing a general function of a contracted quantity $R_{ab}R_{ab}$ [36] is not likely to have the same RDE solution in Eq. (23) since the quantity $R_{ab}R_{ab} = 12(\dot{H}^2 + 3\dot{H}H^2 + 3H^4)$ is non-vanishing even if $H = 1/(2t)$. While the generic $f(R)$ and $RR^c_c$ gravity can be conformally transformed into Einstein gravity with single minimally coupled scalar field (MSF) [4, 7, 37, 38] and with two interacting MSFs [1] respectively, $R_{ab}R_{ab}$ gravity does not have such a symmetry [2, 3].

How about the standard matter-dominated era (MDE) solution $a(t) \propto t^{2/3}$? This MDE solution from the Einstein gravity is hardly an exact solution in a polynomial $f(R)$ gravity as well as in the $B$ or $C$ gravity unless the effects of the $A_{(n \geq 2)}$, $B$ and $C$ theories are small enough during the MDE because $R(t)$ cannot become zero with $\alpha = 2/3$. However, if the universe evolved like Eq. (18) during the RDE, the correction terms from the $B$ or $C$ gravity might decay enough to converge to the standard curve $a(t) \propto t^{2/3}$ (See Appendix). The MDE in $f(R)$ gravity was investigated in Ref. [39]. It is also noticed that semiclassical analysis of the Wheeler-DeWitt equation describing the universe before the inflationary epoch allows a radiation-like solution $a(t) \propto t^{1/2}$ [40] and that power-law expanding $a \propto t^\alpha$ can be an attractor solution in some fourth-order $f(R)$ models that have conformal symmetry [41].

One of our main assumptions is a spatially flat FLRW universe (three space curvature $K = 0$), beyond which our argument is unlikely established since the Ricci scalar

$$R = 6(\dot{H} + 2H^2 + \frac{K}{a^2})$$

from the FLRW metric including the closed or open universe model [42–45]

$$ds^2 = a^2\left(-d\eta^2 + \frac{dr^2}{(1-Kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right),$$

(32)
is non-vanishing with \(a(t) \propto t^{1/2}\) unless \(K/a^2\) term is negligible. For simplicity, our arguments are based on the homogeneous and isotropic assumption, so called the cosmological principle. However, there is a debate on the principle \([46–49]\) and researchers \([50, 51]\) introduce alternative metrics beyond FLRW, e.g. to explain the current accelerating expansion of the universe. Comparison between \(A\) and \(B\) models using those metrics is another issue.

Cosmological perturbations for the \(f(R)\) and \(RR^{e}\) gravity (\(B = 0 = L_m\) and general \(f(R)\) in Eq. (1)) were investigated \([30]\), where the arranged GFE in Eq. (8) is more convenient than the directly varied GFE in Eq. (3) since each component of the Einstein tensor \(G^\alpha_\beta \equiv R^\alpha_\beta - \frac{1}{2} R \delta^\alpha_\beta\) up to the linearly perturbed order is already calculated and listed, e.g. in Ref. [52]. The equation for a tensor-type (tracefree and transverse) perturbation variable \(C^{(t)}_{\alpha\beta}\) in Fourier space for the model was derived as \([30]\)

\[
\dddot{C}^{(t)}_{\alpha\beta} + \left(3H + \frac{\dot{F}}{F}\right)\dot{C}^{(t)}_{\alpha\beta} + \frac{k^2 + 2K}{a^2} C^{(t)}_{\alpha\beta} = 0,
\]

where \(F = f_{,R} + 2CR^{e}\), and \(k\) is the wavenumber of the perturbations. Interestingly, this complicated gravitational wave (GW) equation can be simplified to a standard form when the evolution of a flat background universe is described by the solution in Eq. (23) and \(f_{,R} = 1\) during the relating era so that \(F\) becomes unity. Further, the simplified GW equation can become the Bessel equation whose exact solutions are known \([30, 52–54]\). The propagation speed of cosmological gravitational wave, in principle, can play a role of a discriminator in selection of gravity models. Most of models predict that the speed is the same as \(c\) while a perturbative \(BR^{a\beta}R_{ab}\) model \([34]\) as a subset of more general theory by Weinberg \([55]\) and a model \([56]\) inspired by string theory do not. For more detail, see TABLE II in Ref. [56] for a flat universe in each generalized gravity.

Future investigations with a part of this higher-order Lagrangian would be e.g. finding other kinds of physical RDE evolution of the universe beyond the standard solution found here, and trying applications to physics of neutron star(s), to alternative cosmological metrics or to quantum cosmology where modified gravity effects may be significant.

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Appendix: Decaying higher-derivative terms during the radiation-matter coexistence and matter-dominated era

Since the universe evolves from the RDE to the MDE, we elaborate the epoch of coexistence of radiation and matter as well as the MDE with two approximate but intuitive arguments for convergence to the standard MDE solution $a \propto t^{2/3}$. According to the scalar-type density fluctuation theory, the MDE solution in generalized gravity is required to match the observed large scale structure if viscosity of cosmic fluid is negligible \[28, 54\]. As a research on the BBN in the Brans-Dicke gravity pointed out \[26\], we basically assume in this Appendix that the scale factor should evolve nearly as $t^{1/2}$ during the RDE so that the correction terms due to $A, B$ and $C$ gravity are small enough to keep up to the first order.

Firstly, if the power-law ansatz in Eq. (18) with a slight deviation from the EoS in Eq. (13), $p \simeq \mu/3$, i.e. $\alpha \simeq 1/2$, is inserted into Eq. (7) or into Eqs. (14, 15), then $C$- and $(3A+B)$-term rapidly decay proportionally to $1/t^6$ and to $1/t^4$ respectively during the RDE.

Secondly, let us narrow down to $A$ and $B$ gravity and treat the higher-derivative terms in Eqs. (14, 15) by using the standard evolution of the double-component (radiation and matter) flat universe described by the dimensionless Friedmann equation

$$\frac{H^2}{H_0^2} = \frac{\Omega_r}{y^4} + \frac{\Omega_m}{y^3} \iff \dot{y}^2 = (a_0 y)^{-2}[\alpha_r^2 + 2\alpha_m y],$$  \hspace{1cm} (A.1)

whose integration by part results in $t = t(a)$ form \[57\]:

$$H_0t = \frac{4}{3} \frac{y_{rm}^2}{\sqrt{\Omega_r}} \left[ 1 - \left( 1 - \frac{1}{2} \frac{y}{y_{rm}} \right) \sqrt{1 + \frac{y}{y_{rm}}} \right],$$  \hspace{1cm} (A.2)

where $H_0 \equiv H(t_0)$ is the Hubble constant, $t_0$ is the present time, $y \equiv a(t)/a_0$ is the normalized scale factor, $\alpha_m \equiv a_0^2 H_0^2 \Omega_m/2$ and $\alpha_r \equiv (a_0^2 H_0^2 \Omega_r)^{1/2}$ \[51\] are constants determined by observational values, $\Omega_0 \equiv \mu_0/\mu_{c0}$ represents the energy density contribution of $i$-component (e.g. subscripts $r$ and $m$ stand for radiation and matter), $\mu_{c0} \equiv \frac{3}{8\pi} H_0^2$ is the present critical density, and $y_{rm} \equiv a(t_{rm})/a_0 \equiv \Omega_{r0}/\Omega_{m0}$ is the normalized scale factor when $\mu_r = \mu_m$ at $t = t_{rm}$. Eq. (A.2) is insightful because its RDE ($a \ll a_{rm}$) and MDE limits ($a \gg a_{rm}$, but until the contribution of the cosmological constant is negligible) recover the
results from the standard cosmology [57]:

\[ a \ll a_{rm} \Rightarrow y \simeq \left(2\sqrt{\Omega_{r0}H_0 t}\right)^{1/2}, \quad (A.3) \]
\[ a \gg a_{rm} \Rightarrow y \simeq \left(\frac{3}{2}\sqrt{\Omega_{m0}H_0 t}\right)^{2/3}. \quad (A.4) \]

Eq. (A.2) is also useful because it enables us to calculate the time of radiation-matter equality when \( a = a_{rm} \) [57]:

\[ t_{rm} = \frac{4}{3} \left(1 - \frac{1}{\sqrt{2}}\right)H_0^{-1}\Omega_{r0}^{3/2}\Omega_{m0}^{2/3}, \quad (A.5) \]

whose value is about 23,000 years if we use the following observational values \( H_0 = 67.4 \text{ km/s/Mpc} = 100h \text{ km/s/Mpc}, \quad \Omega_{m0} = 0.315 \) determined from Planck observation of the cosmic microwave background (CMB) with the \( \Lambda \)CDM model [58], and \( \Omega_{r0} = 2.47 \times 10^{-5}h^{-2} \) determined by the CMB temperature, \( T = 2.7255 \text{ K} \) [59, 60].

However, since Eq. (A.2) for the radiation plus matter universe has a form of inverse function from which \( H(t) \) and its \( t \) derivatives are hardly calculated, we had better use the exact solution in terms of the dimensionless conformal time \( \eta (dt \equiv a d\eta) \) [51]

\[ a(\eta) = a_0 \left[\frac{1}{2} \alpha_m \eta^2 + \alpha_r \eta\right], \quad t = \frac{a_0}{6} \left[\alpha_m \eta^3 + 3\alpha_r \eta^2\right], \quad (A.6) \]

setting \( t = \eta = 0 \) when \( a = 0 \). Using Eq. (A.6) derived from Einstein gravity allows us to estimate approximately the decaying behavior of the higher-derivative terms in Eqs. (14) and (15), respectively, as \( \eta \) elapses,

\[ \left(2\dot{H}H - \dot{H}^2 + 6\ddot{H}H^2\right) = \frac{1}{a^4}\left[2\mathcal{H}''\mathcal{H} - (\mathcal{H}')^2 - 3\mathcal{H}^4\right] \]
\[ \simeq -\frac{\alpha_m}{2a_0^4\left(\frac{\alpha_m}{2}\eta^2 + \alpha_r \eta\right)^7}\left(8\alpha_r^2 + 18\alpha_r \alpha_m \eta + 9\alpha_m^2 \eta^2\right), \quad (A.7) \]
\[ \left(2\frac{d^3H}{dt^3} + 12\dddot{H}H + 9\dot{H}^2 + 18\ddot{H}H^2\right) \]
\[ = \frac{1}{a^4}\left[2\mathcal{H}'''' - 2\mathcal{H}''\mathcal{H} + (\mathcal{H}')^2 - 12\mathcal{H}'\mathcal{H}^2 + 3\mathcal{H}^4\right], \]
\[ \simeq -\frac{\alpha_m}{2a_0^4\left(\frac{\alpha_m}{2}\eta^2 + \alpha_r \eta\right)^7}\left(32\alpha_r^2 + 54\alpha_r \alpha_m \eta + 27\alpha_m^2 \eta^2\right), \quad (A.8) \]

where \( \mathcal{H} \equiv a'/a \equiv \frac{1}{a} \frac{da}{d\eta} = aH \). If there are only subtle deviations from the standard cosmological evolutions in the fourth order gravity, in other words, the terms from Einstein’s theory such as \( 8\pi \mu, 3H^2, 8\pi p \), and \(-2\dot{H} + 3\mathcal{H}^2\) in Eqs. (14, 15) are roughly like \( \propto \eta^{-4} \) and \( \propto \eta^{-6} \) in the RDE and the MDE respectively, then the Eqs. (A.7, A.8) show that how much
faster the higher-derivative corrections disappear than the standard terms do as time goes by. For a while let us imagine a hypothetical universe filled with only radiation literally i.e. the case of $\alpha_m = 0$, which implies that Eqs. (A.7) and (A.8) vanish. This is consistent with the fact that the exact solution in Eq. (23) makes those terms nought. Values of the dimensionless parameter $w(t) \equiv p/\mu = (p_r + p_m)/(\mu_r + \mu_m) = 1/[3(1 + y/y_{rm})]$ are nearly 1/3 for the RDE ($y/y_{rm} \ll 1$), 1/6 for the time of radiation-matter equality ($y/y_{rm} = 1$), and 0 for the MDE ($y/y_{rm} \gg 1$, but before the dark energy effects meaningfully appear), respectively.

In short, the higher-derivative terms rapidly decay to recover the standard cosmological equations that contains the MDE solution $a \propto t^{2/3}$ under the condition that the scale factor behavior in the fourth-order gravity subtly deviates from the solution from GR.

If we more narrow down to $f$ gravity (“Starobinsky model”), the theory becomes a subset of $f(R)$ cosmology. In this case, we note that there is a more general exact solution that covers not only the radiation-dust coexistence era [61] but also MDE plus dark energy epoch (almost the whole history of cosmic evolution) with a change of the equation of state for $f(R)$ gravity that corresponds to the recent comic acceleration [62]. They also pointed out that their solution successfully coincides with the standard MDE solution for redshift interval from 2 to 4 so that the requirement for the formation of large scale structure is fulfilled [62]. The problem which specific functions among generic $f(R)$ are selected by astronomical observations is still ongoing.

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