Homotopy Perturbation and Adomian Decomposition Methods for a Quadratic Integral Equations with Erdelyi-Kober Fractional Operator

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Abstract

This paper is devoted with two analytical methods; Homotopy perturbation method (HPM) and Adomian decomposition method (ADM). We display an efficient application of the ADM and HPM methods to the nonlinear fractional quadratic integral equations of Erdelyi–Kober type. The existence and uniqueness of the solution and convergence will be discussed. In particular, the well-known Chandrasekhar integral equation also belong to this class, recent will be discussed. Finally, two numerical examples demonstrate the efficiency of the method.

Keywords: Adomian decomposition; Integral equations; Homotopy; Functional equations

Introduction

It is well-known that the theory of integral equations has many applications in describing numerous events and problems of the real world. Nonlinear quadratic integral equations (NQIE) are also often encountered in the theories of radiative transfer and neutron transport [1,2].

Many research about (QIE) appear in the literature, numerous research papers have appeared devoted to nonlinear fractional quadratic integral equation (NFQIE) [3-8]. However, there few work on NFQIE with Erdelyi–Kober fractional operator.

Hashem [9], studied the existence of maximal and minimal at least one continuous solution for NFQIE of Erdelyi–Kober type

\[ x(t) = a(t) + g(t, x(t)) \left[ t^\alpha - s^\alpha \right] f(s, x(s)), \quad t \in I, \alpha > 0, m > 0 \quad (1) \]

In this paper, we investigated the existence, uniqueness of the solution and convergence for NFQIE (1), using two methods; HPM and ADM. The homotopy perturbation method (HPM) was suggested by Ji-Huan [10-15] in 1999. In this method, the solution can be expressed by an infinite series, which commonly converges fast to the exact solution. It is a coupling of the traditional perturbation method and homotopy in topology, which is solved differential and integral equations, linear and nonlinear. The HPM does not require a small parameter in equations. Also, it has an important advantage which enlarges the application of nonlinear problem in applied science.

The Adomian decomposition method (ADM) solves many of functional equations, for example, differential, integro-differential, differential-delay, and partial differential equations. The solution usually appears in a series form, this method has many significant advantages, it does not require linearization, perturbation and other restrictive methods. Also, it might change the problem to a solved one [16-19]. It is worth mentioning that our results are motivated by the generalization of the work.

**Theorem 1:** Assume that

\[ f: [0,1] \to \mathbb{R} = [0, +\infty) \] is a continuous function on [0,1],

\[ g, u: [0,1] \times \mathbb{R} \to \mathbb{R} \] are continuous and bounded with

\[ M_1 = \sup_{x \in [0,1]} |g(t, x)| \] and

\[ M_2 = \sup_{x \in [0,1]} |u(t, x)|, \]

there exist constant \( L_x \) and \( L_u \) such that

\[ |a(t, x) - a(t, y)| \leq L_x |x - y|, \]

\[ (L_x + L_u) < 1. \]

Then the nonlinear fractional quadratic integral (Theorem 1) has a unique positive solution \( x \in C \).

**Proof:** For \( x, y \in S \) and for each \( t \in [0,1] \), we obtain

\[ (Tx)(t) - (Ty)(t) = g(t, x) \left[ t^\alpha - s^\alpha \right] f(s, x(s))ds \]

\[ -g(t, y) \left[ t^\alpha - s^\alpha \right] f(s, y(s))ds \]

\[ +g(t, x) \left[ t^\alpha - s^\alpha \right] f(s, y(s))ds \]

\[ -g(t, y) \left[ t^\alpha - s^\alpha \right] f(s, x(s))ds, \]

\[ = \left[ g(t, x) - g(t, y) \right] \left[ t^\alpha - s^\alpha \right] f(s, x(s))ds \]

\[ +g(t, x) \left[ t^\alpha - s^\alpha \right] f(s, y(s))ds \]

\[ +M_1 L_x \left[ t^\alpha - s^\alpha \right] f(s, y(s))ds \]

\[ \leq M_2 L_u \left[ t^\alpha - s^\alpha \right] f(s, y(s))ds \]

\[ +M_1 L_u \left[ t^\alpha - s^\alpha \right] f(s, x(s))ds \]

\[ \leq \max_{t \in [0,1]} \left[ (Tx)(t) - (Ty)(t) \right]. \]

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\[S \leq (L_1M_1 + L_2M_2) \|x - y\| t^\beta \Gamma(\beta + 1),\]
\[S \leq (L_1M_2 + L_2M_1) \|x - y\| t^\beta \Gamma(\beta + 1).
\]

By (H4), the operator \(T\) is a contraction map from \(S\) into \(S\), hence the conclusion of the theorem follows.

**Main Results**

In this section, we prove the existence and uniqueness of continuous solutions and the convergence for Equation

\[x(t) = a(t) + g(t, x(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds + f(s, x(s)) \tag{2}\]

we denote by \(C = C(I)\) the space of all real-valued functions which are continuous on \(I = [0,1]\). We can transform (2) into an equivalent fixed point problem \(Tx = x\), where the operator \(T : C \to C\) is defined by

\[(Tx)(t) = f(t) + g(t, x(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds \tag{3}\]

Observe that the existence of a fixed point for the operator \(T\) implies the existence of a solution for (2).

Now define a subset \(S\) of \(C\) as

\[S = \{x \in C : x - f(t) \leq k, k = M_1 M_2 \Gamma(\beta + 1)\},\]

Then operator \(T\) maps \(S\) into \(S\), since for \(x \in S\)

\[|x(t) - f(t)| \leq M_1 M_2 \Gamma(\beta + 1) \int_0^t \alpha(s) ds \leq M_1 M_2 \Gamma(\beta + 1).\]

It is clear that \(S\) is a closed subset of \(C\).

**Homotopy Perturbation Method**

The He's homotopy perturbation technique [10,11] defines the homotopy \(u(t, \lambda) : \Omega \times [0,1] \to \mathbb{R}\) which satisfies

\[H(u, \lambda) = (1 - \lambda) F(u) + \lambda L(u) = 0 \tag{4}\]

Where \(\lambda \in \Omega\) and \(p \in [0,1]\) is an impeding parameter, \(u_0\) is an initial approximation which satisfies the boundary conditions, we can define \(H(u, p)\) by

\[H(u, 0) = F(u), H(u, 1) = L(u),\]

where \(F(u)\) is an integral operator such that \(F(u) = u(t) - a(t)\), and \(L(u)\) has the form,

\[L(u) = u(t) - a(t) - g(t, x(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds \tag{5}\]

and continuously trace an implicitly defined curve from a starting point \(H(u_0, 0)\) to a solution function \(H(x, t)\). The embedding parameter \(p\) monotonically increases from zero to one as the trivial problem \(F(u) = 0\) is continuously deformed to the original problem \(L(u) = 0\).

The embedding parameter \(p \in [0,1]\) can be considered as an expanding parameter [20].

\[u = \sum_{n=0}^{\infty} p^n u_n, \tag{6}\]

when \(p \to 1\), (6) corresponds to (4) and give an approximation to the solution of (2) as follows,

\[x(t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n, \tag{7}\]

The series (7) converges in most cases, and the rate of convergence depends on \(L(u)\) [21].

We substitute (6) into (4) and equate the terms with identical powers of \(p\), obtaining

\[p^0 : u_0(t) = a(t),\]

\[p^1 : u_1(t) = g(t, u_0(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds + f(s, x(s)) \tag{8}\]

\[p^2 : u_2(t) = g(t, u_1(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds + f(s, x(s)) \tag{9}\]

\[p^3 : u_3(t) = g(t, u_2(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds + f(s, x(s)) \tag{10}\]

\[\vdots \]

\[p^n : u_n(t) = g(t, u_{n-1}(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds + f(s, x(s)) \tag{11}\]

\[p^0 : u_0(t) = a(t),\]

\[p^1 : u_1(t) = g(t, u_0(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds + f(s, x(s)) \tag{8}\]

\[p^2 : u_2(t) = g(t, u_1(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds + f(s, x(s)) \tag{9}\]

\[p^3 : u_3(t) = g(t, u_2(t))\int_0^t (\tau - s)^{\beta-1} \Gamma(\beta) \alpha(s) ds + f(s, x(s)) \tag{10}\]
We can be constructed a distinct convex homotopy as
\[
+\sum_{i=1}^{\infty} A_i(t) \left( t^i \sum_{j=1}^{i} B_{ij}(t) \right),
\]
and has the exact solution \( x(t)=t^2 \). First applying homotopy perturbation method.

**Case 1:** We can be constructed a homotopy as follows
\[
H(u, p) = \left( 1 - p \right) \left( u(t) - g(t) \right) + p \left( u(t) - g(t) \right),
\]
substituting (6) into (13), and equating the same powers of \( p \)
\[
p^0: u_0(t) = \frac{1}{2} \left[ 12 s^{-12} \Gamma(12)(t^{12} - s^{12})^2 x(s)ds \right],
\]
and so on. Then the approximate solution is
\[
x(t) = \frac{1}{2} \left[ 12 s^{-12} \Gamma(12)(t^{12} - s^{12})^2 x(s)ds \right],
\]

Second applying (ADM) to equation (12), we get
\[
x_i(t) = \frac{1}{2} \left[ 12 s^{-12} \Gamma(12)(t^{12} - s^{12})^2 x(s)ds \right], \quad i \geq 1,
\]

Where \( A_i \) are Adomian polynomials of the nonlinear term \( x^i \), and the solution will be
\[
x(t) = \sum_{i=1}^{\infty} x_i(t).
\]

Table 1 shows a comparison between the absolute error of (HPM) (when \( n=1 \)) and (ADM) solutions (when \( q=1 \)), (Figure 1).

**Case 2:** We can be constructed a distinct convex homotopy as follows
\[
H(u, p) = \left( 1 - p \right) u(t) + p \left( u(t) - g(t) \right),
\]
\[
-\left[ t^{-1} \frac{1}{2} \left[ 12 s^{-12} \Gamma(12)(t^{12} - s^{12})^2 x(s)ds \right] \right] = 0,
\]

It can continuously trace an implicitly defined curve from a starting point \( H(u_0) \) to a solution function \( H(u_1) \), and equating the coefficients of the same powers of \( p \), we obtain
\[
p^0: u_0(t) = 0,
\]
\[
p^1: u_1(t) = t^{-1} - t^{-12} \frac{1}{2} \left[ 12 s^{-12} \Gamma(12)(t^{12} - s^{12})^2 x(s)ds \right],
\]
and so on.
Table 2 shows a comparison between the absolute error of (HPM) (when \( n = 1 \)) and (ADM) solutions (when \( q = 1 \)), (Figure 2).

**Example 2**

\[
x(t) - t^2 + 14(t^2 + 1) + 14(t^2 + 1)x(t)\int_0^t \Gamma(0.3)(t-s)^0 \cos(x^2(s)+x^2(s))ds
\]

**Case 1:** Applying homotopy perturbation method, we can be constructed a homotopy as follows

\[
H(u_p, p) = (1 - p)u(t) - g(t) + p(u(t) - g(t))
\]

\[
-14(t^2 + 1)u(t)\int_0^t \Gamma(0.3)(t-s)^0 \cos(u^2(s) + u^2(s))ds = 0
\]

Substituting (6) into (16), and equating the same powers of \( p \), we obtain

\[
p^0 : u_0 = t^2 + 14(t^2 + 1),
\]

\[
p^1 : u_1 = 14(t^2 + 1)u_0 \int_0^t \Gamma(0.3)(t-s)^0 H_1(s)ds,
\]

\[
p^2 : u_2 = 14(t^2 + 1)u_1 \int_0^t \Gamma(0.3)(t-s)^0 H_1(s)ds
\]

\[
+14(t^2 + 1)u_1 \int_0^t \Gamma(0.3)(t-s)^0 H_1(s)ds,
\]

and so on. Then the approximate solution is

\[
x(t) = \lim_{n \to \infty} u_0 + u_1 + u_2 + \ldots + u_n,
\]

Second applying (ADM) to equation (15), we get

\[
\begin{align*}
x_n(t) &= t^2 + 14(t^2 + 1), \\
x_q(t) &= 14(t^2 + 1)x_n(t)\int_0^t \Gamma(0.3)(t-s)^0 A_n(s)ds.
\end{align*}
\]

Where \( A_n \) are Adomian polynomials of the nonlinear term \( \cos(x^2 + x^2) \) and the solution will be

\[
x(t) = \sum_{n=0}^{\infty} x_n(t),
\]

**Case 2:** We can be constructed a distinct convex homotopy as follows

\[
H(u_p, p) = (1 - p)u(t) + p(u(t) - g(t))
\]

\[
-14(t^2 + 1)u(t)\int_0^t \Gamma(0.3)(t-s)^0 \cos(x^2(s) + x^2(s))ds.
\]

It can continuously trace an implicitly defined curve from a starting point \( H(u_0) \) to a solution function \( H(u_1) \), and equating the coefficients of the same powers of \( p \), we obtain

\[
p^0 : u_0 = 0,
\]

\[
p^1 : u_1 = 14(t^2 + 1)u_0 \int_0^t \Gamma(0.3)(t-s)^0 H_1(s)ds,
\]

\[
= 14(t^2 + 1)u_0 \int_0^t \Gamma(0.3)(t-s)^0 H_1(s)ds.
\]

Table 3 shows a comparison between (HPM) and (ADM) solutions (when \( n = 2, q = 2 \), (Figure 3)).

**Figure 1:** The difference between exact and approximate solutions by (HPM) and (ADM).
Comparison between the absolute error between (HPM) and (ADM)

Table 4 shows a comparison between the absolute error between (HPM) and (ADM).

Table 4: Comparison between the absolute error between (HPM) and (ADM) (when $n=2, q=2$).

| t  | $u_{HPM}$ | $u_{ADM}$ |
|----|-----------|-----------|
| 0.1| 0.28924531| 0.28924531|
| 0.2| 0.31590677| 0.31590677|
| 0.3| 0.36286938| 0.36286938|
| 0.4| 0.44089584| 0.44089584|
| 0.5| 0.56123678| 0.56123678|
| 0.6| 0.73670859| 0.73670859|
| 0.7| 0.98233639| 0.98233639|
| 0.8| 1.31593221| 1.31593221|
| 0.9| 1.75867138| 1.75867138|
| 1  | 2.33568188| 2.33568188|

Table 3: Comparisons between (HPM) and (ADM) solutions (when $n=2, q=2$).

References

1. Baruch C, Eskin M (1981) Existence theorems for an integral equation of the Chandrasekhar H-equation with perturbation. Journal of Mathematical Analysis and Applications 83: 159-171.
2. Hashem HHG (2015) On successive approximation method for coupled systems of Chandrasekhar quadratic integral equations. Journal of the Egyptian Mathematical Society 23: 108-112.
3. Banas J, Caballero J, Rocha J, Sadarangani K (2005) Monotonic solutions of a class of quadratic integral equations of Volterra type. Computers Mathematics with Applications 49: 943-952.
4. Banas J, Martinon A (2004) Monotonic solutions of a quadratic integral equation of Volterra type. Computers Mathematics with Applications 47: 271-279.
5. Banas J, Martin JR, Sadarangani K (2006) On solutions of a quadratic integral equation of Hammerstein type. Mathematical and Computer Modelling 43: 97-104.
6. Caballero J, Darwish MA, Sadarangani K (2014) Solvability of a quadratic integral equation of Fredholm type in Holder spaces. Electronic Journal of Differential Equations 31: 1-10.
7. Shou-Zhong F, Zhong W, Jun-Sheng D (2013) Solution of quadratic integral equations by the Adomian decomposition method. CMES-Comput Model Eng Sci 92: 369-385.
8. Ziada AA (2013) Adomian solution of a nonlinear quadratic integral equation. Journal of the Egyptian Mathematical Society 21: 52-56.
9. Hashem HH, Zaki MS (2013) Caratheodory theorem for quadratic integral equations of Erdelyi-Kober type. Journal of Fractional Calculus and Applications 4: 56-72.
10. He JH (1999) Homotopy perturbation technique. Computer methods in applied mechanics and engineering 178: 257-262.
11. He JH (2000) A coupling method of a homotopy technique and a perturbation technique for non-linear problems. International Journal of Non-Linear Mechanics 35: 37-43.
12. He JH (2003) Determination of limit cycles for strongly non-linear oscillators. Phys Rev Lett 90: 174301.
13. He JH (2003) Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and computation 135: 73-79.
14. He JH (2004) Asymptotology by homotopy perturbation method. Applied Mathematics and Computation 156: 591-596.
15. He JH (2006) Homotopy perturbation method for solving boundary value problems. Physics letters A 350: 87-88.
16. Abbaoui K, Cherruault Y (1994) Convergence of Adomian’s method applied to differential equations. Comput Math 28: 103-109.
17. Adomian G (1983) Stochastic system. Academic press, New York.
18. Adomian G (1995) Solving frontier problems of physics: the decomposition method. Kluwer, Dordercht.
19. Adomian G, Rach R, Mayer R (1992) Modified decomposition. J Math Comput 23: 17-23.
20. He JH (2006) New interpretation of homotopy perturbation method. Internat J Modern Phys B 20: 2561-2568.
21. Liao SJ (1997) Boundary element method for general nonlinear differential operators. Engineering Analysis with Boundary Elements 20: 91-99.
22. Ghorbani A (2009) Beyond Adomian polynomials: he polynomials. Chaos, Solitons Fractals 39: 1486-1492.
23. Baranis J, O’Regan D (2008) On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order. Journal of Mathematical Analysis and Applications 345: 573-582.
24. Darwish MA, Beata R (2014) Asymptotically stable solutions of a generalized fractional quadratic functional-integral equation of Erdélyi-Kober type. J Funct Spaces.
25. Darwish MA, Sadarangani K (2015) On a quadratic integral equation with supremum involving Erdélyi-Kober fractional order. Mathematische Nachrichten 288: 566-576.
26. El-Sayed AM, Hashem HHG, Ziada EAA (2010) Picard and Adomian methods for quadratic integral equation. Comp Appl Math 29: 447-463.
27. El-Sayed AM, Hashem HHG, Ziada EAA (2014) Picard and Adomian decomposition methods for a quadratic integral equation of fractional order. Computational and Applied Mathematics 33: 95-109.
28. Wang J, Dong X, Yong Z (2012) Analysis of nonlinear integral equations with Erdélyi–Kober fractional operator. Communications in Nonlinear Science and Numerical Simulation 17: 3129-3139.