Research article

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Liouville property of fractional Lane-Emden equation in general unbounded domain

Abstract: Our purpose of this paper is to consider Liouville property for the fractional Lane-Emden equation

\[ (-\Delta)^{\alpha} u = u^p \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \]

where \( \alpha \in (0, 1) \), \( N \geq 1 \), \( p > 0 \) and \( \Omega \subset \mathbb{R}^{N-1} \times [0, +\infty) \) is an unbounded domain satisfying that \( \Omega_t := \{ x' \in \mathbb{R}^{N-1} : (x', t) \in \Omega \} \) with \( t \geq 0 \) has increasing monotonicity, that is, \( \Omega_t \subset \Omega_{t'} \) for \( t' \geq t \). The shape of \( \Omega_\infty := \lim_{t \to \infty} \Omega_t \) in \( \mathbb{R}^{N-1} \) plays an important role to obtain the nonexistence of positive solutions for the fractional Lane-Emden equation.

Keywords: Fractional Laplacian; Lane-Emden equation; Nonexistence

MSC: 35J60; 35B53

1 Introduction

In this paper, we consider Liouville property for the fractional Lane-Emden equation

\[ (-\Delta)^{\alpha} u = u^p \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \tag{1.1} \]

where \( \alpha \in (0, 1) \), \( p > 0 \), \( \Omega \) is an unbounded domain in \( \mathbb{R}^N \) with \( N \geq 1 \), and \( (-\Delta)^{\alpha} \) with \( \alpha \in (0, 1) \) is the fractional Laplacian defined in the principle value sense,

\[ (-\Delta)^{\alpha} u(x) = c_{N,\alpha} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x) - u(x+z)}{|z|^{N+2\alpha}} \, dz, \]

here \( B_\varepsilon(0) \) is the ball with radius \( \varepsilon \) centered at the origin and \( c_{N,\alpha} > 0 \) is the normalized constant. We say that \( u \) is a bounded solution of (1.1) if \( u \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and \( u \) satisfies (1.1) pointwisely.

As an important property, the Liouville theorem for Lane-Emden equation has attracted a lot of attention by many mathematician by the application in the derivation of uniform bound via blowing-up analysis. Note that the nonexistence of stable solution is studied in [1] by finite morse index with restrictions on the boundary and at infinity. Without the zero Dirichlet boundary condition, Liouville results could be obtained by Hadamard property in [2, 3], by iterating the decaying rate at infinity in [4] and by Hardy estimates in [5].

It is known that the Leray-Schauder degree theory is a very useful method for deriving solutions of elliptic equations on bounded domains. The essential step is to obtain a uniform bound by considering a sequence of solutions \( \{u_n\}_n \) such that \( u_n(x_n) = u_n L^\infty = M_n \to +\infty \) as \( n \to +\infty \) for some \( \{x_n\}_n \subset \Omega \). Then let \( \Omega_n = \{ x \in \mathbb{R}^N : \frac{1}{M_n} x + x_n \in \Omega \} \) and \( v_n(x) = \frac{1}{M_n} u_n(\frac{1}{M_n} x + x_n) \) for some \( k > 0 \), then \( \{v_n\}_n \) is uniformly bounded,
and the limit $v_\infty$ of $(v_n)_n$ is a solution of related limit semilinear equation in the limit domain $\Omega^\infty$. Note that for $C^2$ domain $\Omega$, the limit domain of $\Omega_n$ is either $\Omega^\infty = \mathbb{R}^N$ or $\Omega^\infty = \mathbb{R}^{N-1} \times (0, \infty)$. While if the original domain contains a cone point on the boundary, then the limit domain has the third possibility that $\Omega^\infty$ is a cone. As a consequence, the nonexistence of positive solutions to the limit equation in a cone has to be involved additionally. As far as we know, the nonexistence of elliptic equation depends on the shape of limit domain at infinity in some direction. Our concern in this article is to consider the non-existence of elliptic equations in one type of unbounded domain.

Recently, qualitative properties of solutions for nonlocal elliptic equations have been studied extensively, such as the existence of weak solutions or very weak solutions in [6, 7] by variational methods, a survey in [8] on variational methods, large solution [9] by Perron’s method, the regularities in [10], Pohozaev’s identity [11] and Liouville results [12–14]. In [15] it develops the method of moving plane in fractional setting to obtain the classification of critical elliptic equations in an integral form and then it is used to obtain nonexistence of bounded solutions for semilinear elliptic equations in half space [12, 13, 16] subject to various boundary type conditions. In particular, Fall and Weth in [13] obtained the nonexistence of positive bounded solution of (1.1) when $\Omega = \mathbb{R}^N$ and $p < \frac{N+2\alpha}{N-1-2\alpha}$, by applying the method of moving plane, via some interesting estimates of Green’s kernel in $\mathbb{R}^N$. It is worth noting that star-shaped domain with respect to infinity is involved in obtaining the nonexistence of fractional Lane-Emden equation in [14] and it is an important notation in our derivation of nonexistence to (1.1).

Before stating our main result, we introduce the following notations.

(i) Given $\Omega \subset \mathbb{R}^M \times [0, +\infty)$ and $t \in [0, +\infty)$, with $M \geq 1$ being an integer, denote

$$O_t := \{x' \in \mathbb{R}^M : (x', t) \in \Omega\}.$$  

For $t \geq 0$, $O_t$ is nonempty, bounded and the mapping $t \mapsto O_t$ is increasing in the following sense

$$O_t \subset O_{t'} \quad \text{if} \quad 0 \leq t \leq t' < +\infty.$$

Denote

$$O_{\infty} = \bigcup_{t \in [0, \infty)} O_t. \quad (1.2)$$

Now we introduce two types domains:

(\text{S}_0) \text{star-shaped domain } \Omega \subset \mathbb{R}^M, \text{ if there exists } e \in \partial \Omega \text{ such that for every point } x \in \partial \Omega, \text{ the segment } \{te + (1 - t)(x - e) : t \in [0, 1]\} \text{ is contained in } \Omega;

(\text{S}_\infty) \text{star-shaped domain } \Omega \subset \mathbb{R}^M \text{ with respect to infinity, if there exists a point } e \in \mathbb{R}^M \setminus \Omega \text{ such that for every point } x \in \Omega, \text{ the half-line } \{e + t(x - e) : t \geq 1\} \text{ is contained in } \Omega.

The main results state as follows.

\textbf{Theorem 1.1.} Assume that $\alpha \in (0, 1)$, $\Omega \subset \mathbb{R}^{N-1} \times [0, +\infty)$ is a $C^{\alpha,1}$ domain verifying (i) and $\Omega_{\infty}$ is given as (1.2) with $\Omega = \Omega$. Then problem (1.1) has no nonnegative, nontrivial and bounded solutions if one of the following holds:

\begin{itemize}
  \item[(i)] $\Omega_{\infty} = \mathbb{R}^{N-1}$ and $0 < p < \frac{N-1+2\alpha}{N-1-2\alpha}$ if $N > 1 + 2\alpha$, otherwise for $p > 0$, $1 \leq N \leq 1 + 2\alpha$;
  \item[(ii)] $\Omega_{\infty} \subset \mathbb{R}^{N-1}$ is a star-shaped domain with respect to infinity and $1 \leq p < \frac{N-1+2\alpha}{N-1-2\alpha}$ with $N > 1 + 2\alpha$;
  \item[(iii)] $\Omega_{\infty} = \mathbb{R}^{N-2} \times [0, +\infty)$ and $1 < p < \frac{N-2+2\alpha}{N-2-2\alpha}$ for $N > 2 + 2\alpha$, otherwise for $p > 1$, $1 \leq N \leq 2 + 2\alpha$;
  \item[(vi)] $\Omega_{\infty}$ is bounded, $C^2$ star-shape in $\mathbb{R}^{N-1}$ and $N > 1 + 2\alpha$, $p \geq \frac{N-1+2\alpha}{N-1-2\alpha}$.
\end{itemize}

Our basic tool is the traditional method of moving plane, involved by [17] in the fractional setting, we develop this traditional method of moving planes to obtain the increasing monotonicity in the direction $x_N$ and reduce problem (1.1) into

$$(-\Delta)^{\alpha}_{\mathbb{R}^{N-1}} u = u^p \quad \text{in} \quad \Omega_{\infty}, \quad (1.3)$$

subject to zero Dirichlet boundary condition when $\Omega_{\infty} \not\subset \mathbb{R}^{N-1}$, where $(-\Delta)^{\alpha}_{\mathbb{R}^{N-1}}$ is the fractional Laplacian in $\mathbb{R}^{N-1}$. Then the nonexistence results could be obtained for (1.3) as in [13, 14].
Remark 1.1. In the particular case that $\Omega$ is a cone such as $\{x = (x', x_N) \in \mathbb{R}^N : x_N > \theta |x'|\}$ for some $\theta > 0$, problem (1.1) has no nonnegative, nontrivial and bounded solutions.

If $\Omega_{\infty}$ also verifies the similar assumptions of Theorem 1.1, we can repeat our above procedure and derive the following corollary directly:

Corollary 1.1. Under the assumptions of Theorem 1.1, we assume that the domain $\Omega_{\infty} \subset \mathbb{R}^{N-2} \times [0, +\infty)$ is a $C^2$ domain verifying (D) and $\Omega_{\infty, \infty}$ is given by (1.2) with $\Omega_{\infty} = \Omega_{\infty, \infty}$. Then problem (1.1) does not admit nonnegative, nontrivial and bounded solutions if one of the following holds:

(i) $\Omega_{\infty, \infty} = \mathbb{R}^{N-2}$ and $0 < p < \frac{N-2+2\alpha}{N-2-2\alpha}$ if $N > 2 + 2\alpha$, otherwise for $p > 0$, $1 \leq N \leq 1 + 2\alpha$;

(ii) $\Omega_{\infty, \infty} \subset \mathbb{R}^{N-2}$ is a star-shaped domain with respect to infinity and $1 \leq p \leq \frac{N-2+2\alpha}{N-2-2\alpha}$ with $N > 2 + 2\alpha$;

(iii) $\Omega_{\infty, \infty} = \mathbb{R}^{N-3} \times [0, +\infty)$ and $1 < p < \frac{N-2+2\alpha}{N-2-2\alpha}$ for $N > 3 + 2\alpha$, otherwise for $p > 1$, $1 \leq N \leq 3 + 2\alpha$;

(vi) $\Omega_{\infty, \infty}$ is bounded, star-shape $C^2$ domain in $\mathbb{R}^{N-2}$ and $N > 2 + 2\alpha$, $p > \frac{N-2+2\alpha}{N-2-2\alpha}$.

2 The proof of nonexistence results

For the domain $\Omega$ verifying (D), we shall prove that the solution of (1.1) has the $x_N$-increasing property by using the method of moving planes. To this end, we introduce the following notations. For $\lambda > 0$, denote

$$\Sigma_\lambda = \{x = (x', x_N) \in \mathbb{R}^N \cap \Omega : x_N < \lambda\},$$

$$T_\lambda = \{x = (x', x_N) \in \mathbb{R}^N : x_N = \lambda\},$$

$$u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x),$$

where $x_\lambda = (x', 2\lambda - x_N)$ for $x = (x', x_N) \in \mathbb{R}^N$. For any subset $A$ of $\mathbb{R}^N$, we write $A_\lambda = \{x_\lambda : x \in A\}$ the reflection of $A$ with regard to $T_\lambda$. Since $\Omega_t$ is bounded for $t \geq 0$, then the domain $\Sigma_\lambda$ is always bounded for any $\lambda > 0$.

Proposition 2.1. Under the hypotheses of Theorem 1.1, let $u$ be a nonnegative nontrivial solution of (1.1), then

$$u(x', t) \leq u(x', s) \quad \text{for} \quad s \geq t, \quad \forall x' \in \mathbb{R}^{N-1}.$$  \hspace{1cm} (2.4)

Proof. We divide the proof into two steps.

Step 1: By the assumption of $\Omega \subset \mathbb{R}^N_+$, we may assume

$$\lambda_0 = \inf\{\lambda \geq 0 : w_\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda\}.$$  \hspace{1cm} (2.5)

The purpose of this step is to show that if $\lambda > \lambda_0$ is close to $\lambda_0$, then $w_\lambda > 0$ in $\Sigma_\lambda$. To this end, let $\Sigma^-_\lambda = \{x \in \Sigma_\lambda : w_\lambda(x) < 0\}$, we first claim that if $\lambda > \lambda_0$ is close to $\lambda_0$, then

$$\Sigma^-_\lambda = \emptyset.$$  \hspace{1cm} (2.5)

By contradiction, we assume (2.5) is not true, that is $\Sigma^-_\lambda \neq \emptyset$. We denote

$$w^-_\lambda(x) = w_\lambda(x), \quad w^-_\lambda(x) = 0, \quad x \in \mathbb{R}^N \setminus \Sigma^-_\lambda,$$  \hspace{1cm} (2.6)

$$w^-_\lambda(x) = 0, \quad x \in \Sigma^-_\lambda, \quad w^-_\lambda(x) = w_\lambda(x), \quad x \in \mathbb{R}^N \setminus \Sigma^-_\lambda$$  \hspace{1cm} (2.7)

It is obvious that $w^-_\lambda(x) = w_\lambda(x) - w^-_\lambda(x)$ for all $x \in \mathbb{R}^N$. By direct computation, for $x \in \Sigma^-_\lambda$, we have

$$(-\Delta)^{\alpha} w^-_\lambda(x) = - \int_{(\Lambda \setminus \Omega_\lambda) \cup (\Omega_\lambda \setminus \Omega)} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz.$$
We look at each of these integrals separately. Since \( u = 0 \) in \( \Omega \setminus \Omega \) and \( u_A = 0 \) in \( \Omega \setminus \Omega_A \), then

\[
I_1 = \int_{\Omega \setminus \Omega_A} \frac{u_A(z)}{|x - z|^{N + 2a}} dz - \int_{\Omega(A)} \frac{u(z)}{|x - z|^{N + 2a}} dz \\
= \int_{\Omega \setminus \Omega_A} u_A(z) \frac{1}{|x - z|^{N + 2a}} dz - \frac{1}{|x - z|^{N + 2a}} dz \\
\geq 0,
\]

by the fact that \( u_A \geq 0 \) and \( |x - z| > |x - z| \) for all \( x \in \Sigma_A \) and \( z \in \Omega \setminus \Omega_A \).

In order to fix the sign of \( I_2 \), note that \( w_A(x) = -w_A(z) \) for any \( z \in \mathbb{R}^N \) and then

\[
I_2 = \int_{\Sigma_A \setminus \Sigma_A} \frac{w_A(z)}{|x - z|^{N + 2a}} dz + \int_{\Sigma_A \setminus \Sigma_A} \frac{w_A(z)}{|x - z|^{N + 2a}} dz \\
= \int_{\Sigma_A \setminus \Sigma_A} w_A(z) \frac{1}{|x - z|^{N + 2a}} dz - \frac{1}{|x - z|^{N + 2a}} dz \\
\geq 0,
\]

since \( w_A \geq 0 \) in \( \Sigma_A \setminus \Sigma_A \) and \( |x - z| > |x - z| \) for all \( x \in \Sigma_A \) and \( z \in \Sigma_A \setminus \Sigma_A \). Finally, since \( w_A(z) < 0 \) for \( z \in \Sigma_A \),

we have

\[
I_3 = \int_{\Sigma_A} \frac{w_A(z)}{|x - z|^{N + 2a}} dz = - \int_{\Sigma_A} \frac{w_A(z)}{|x - z|^{N + 2a}} dz \geq 0.
\]

Hence, we obtain that for all \( \lambda > \lambda_0 \),

\[
(-\Delta)^a w_A(x) \leq 0, \quad \forall x \in \Sigma_A
\]

(2.8)

and then for \( x \in \Sigma_A \),

\[
(-\Delta)^a w_A^+(x) \geq (-\Delta)^a w_A^-(x) = (-\Delta)^a u_A(x) - (-\Delta)^a u(x).
\]

(2.9)

Combining (1.1) with (2.9) and (2.6), we have that

\[
(-\Delta)^a w_A^+(x) \geq (-\Delta)^a u_A(x) - (-\Delta)^a u(x) \\
= u_A^p(x) - u^p(x) = -\varphi(x)w_A^+(x), \quad x \in \Sigma_A
\]

where

\[
\varphi(x) = \frac{u_A^p(x) - u^p(x)}{u_A(x) - u(x)}, \quad |\varphi| \leq 2^p |u_A^{p-1} + u^{p-1}| \leq 2^p |u|^{p-1}.
\]

Hence, we have

\[
\Delta^a w_A^+(x) \leq \varphi(x)w_A^+(x), \quad x \in \Sigma_A
\]

(2.10)

and observe that \( w_A = 0 \) in \( (\Sigma_A)^c \), then we have that

\[
\sup_{x \in \Sigma_A} w_A^+(x) \leq cR(\Sigma_A)^{2a} \|\varphi w_A^+\|_{L^\infty(\Sigma_A)} \leq cR(\Sigma_A)^{2a} \|\varphi\|_{L^\infty(\Sigma_A)} \sup_{x \in \Sigma_A} w_A^+(x),
\]

that is,

\[
1 \leq c2^p R(\Sigma_A)^{2a} |u|^{p-1},
\]

(2.11)

where \( c \) is a positive constant independent of \( \Sigma_A \) and

\[
R(\Sigma_A) = \inf \left\{ r > 0 : |B_r(x) \setminus \Sigma_A| \geq \frac{1}{2} |B_r(x)|, \quad \forall x \in \Sigma_A \right\}.
\]
Choosing \( \lambda > \lambda_0 \) close enough to \( \lambda_0 \), \( R(\Sigma_A) \) is small, a contraction is derived from (2.11) and then
\[
w_A = w_A^* \geq 0 \quad \text{in} \quad \Sigma_A.
\]
But this is a contradiction with the definition of \( \Sigma_A \), so we have that \( w_A \equiv 0 \) in \( \Sigma_A \).

We next claim that if \( w_A \geq 0 \) and \( w_A \not\equiv 0 \) in \( \Sigma_A \), then \( w_A > 0 \) in \( \Sigma_A \). Assuming this claim is true, we complete the proof, since the function \( u \) is positive in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), so that \( w_A \) is positive on \( \partial \Omega \cap \partial \Sigma_A \) and then, by continuity \( w_A \not\equiv 0 \) in \( \Sigma_A \).

Now we prove above claim. In fact, assume there exists \( x_0 \in \Sigma_A \) such that \( w_A(x_0) = 0 \), that is, \( u_\lambda(x_0) = u(x_0) \). One and we have that
\[
(-\Delta)^a w_A(x_0) = (-\Delta)^a u_A(x_0) - (-\Delta)^a u(x_0) = u_\lambda^p(x_0) - u^p(x_0) = 0.
\] (2.12)

On the other hand, let \( A_\lambda = \{(x_1, x') \in \mathbb{R}^N | x_1 > \lambda \} \), since \( w_A(z) = -w_A(z) \) for any \( z \in \mathbb{R}^N \) and \( w_A(x_0) = 0 \), we find
\[
(-\Delta)^a w_A(x_0) = -\int_{A_\lambda} \frac{w_A(z)}{|x_0 - z|^{N+2a}} \, dz - \int_{A_\lambda} \frac{w_A(z)}{|x_0 - z|^{N+2a}} \, dz
\]
\[
= -\int_{A_\lambda} w_A(z) \left(\frac{1}{|x_0 - z|^{N+2a}} - \frac{1}{|x_0 - z|^{N+2a}}\right) \, dz.
\]

Since \( |x_0 - z| > |x_0 - z| \) for \( z \in A_\lambda \), \( w_A(z) \geq 0 \) and \( w_A(z) \not\equiv 0 \) in \( A_\lambda \), then
\[
(-\Delta)^a w_A(x_0) < 0,
\] (2.13)
which contradicts (2.12).

**Step 2:** Our purpose of this step is to move the planes forward for any \( \lambda > \lambda_0 \) up to
\[
\lambda_1 := \sup\{ \lambda > \lambda_0 | w_\mu > 0 \quad \text{in} \quad \Sigma_\mu \quad \text{for any} \quad \mu \in (\lambda_0, \lambda) \} = +\infty.
\]

By contradiction, we assume that \( \lambda_1 < +\infty \). Since for any \( \lambda \in (0, \lambda_1) \), we have that \( w_\lambda > 0 \) in \( \Sigma_\lambda \), by the continuity, we derive that \( w_{\lambda_1} \equiv 0 \) in \( \Sigma_{\lambda_1} \) and \( w_A \equiv 0 \) in \( \Sigma_A \). Thus, by the claim just proved above, we have \( w_{\lambda_1} > 0 \) in \( \Sigma_{\lambda_1} \).

Now we claim that if \( w_{\lambda} > 0 \) in \( \Sigma_{\lambda} \), then there exists \( \varepsilon > 0 \) such that \( w_{\lambda} > 0 \) in \( \Sigma_{\lambda} \) for all \( \lambda \in [\lambda_1, \lambda_1 + \varepsilon] \). This claim directly implies that \( \lambda_1 = +\infty \), completing Step 2.

In fact, if \( \lambda_1 < +\infty \), then under our hypotheses, we have that \( \Sigma_A \) is compact. Denote
\[
D_\mu = \{ x \in \Sigma_A | \text{dist}(x, \partial \Sigma_A) \geq \mu \}
\]
for \( \mu > 0 \) small. Since \( w_{\lambda_1} > 0 \) in \( \Sigma_{\lambda_1} \) and \( D_\mu \) is compact, then there exists \( \mu_0 > 0 \) such that \( w_{\lambda_1} \geq \mu_0 \) in \( D_\mu \). By continuity of \( w_\lambda(x) \), for \( \varepsilon > 0 \) small enough and any \( \lambda \in (\lambda_1, \lambda_1 + \varepsilon) \), we have that
\[
w_\lambda \geq 0 \quad \text{in} \quad D_\mu.
\]
As a consequence,
\[
\Sigma_{\lambda_1} \subset \Sigma_{\lambda_1} \setminus D_\mu
\]
and \( R(\Sigma_{\lambda_1}) \) is small if \( \varepsilon \) and \( \mu \) are small. Using (2.8) and proceeding as in Step 1, we have that for all \( x \in \Sigma_{\lambda_1} \),
\[
(-\Delta)^a w_{\lambda_1}^*(x) = (-\Delta)^a u_{\lambda_1}(x) - (-\Delta)^a u(x) - (-\Delta)^a w_{\lambda_1}(x)
\]
\[
\geq (-\Delta)^a u_{\lambda_1}(x) - (-\Delta)^a u(x) = u_\lambda^p(x) - u^p(x)
\]
\[
= -\varphi(x) w_{\lambda_1}^*(x),
\]
where \( \varphi(x) = -\frac{u_\lambda^p(x) - u^p(x)}{u_\lambda(x) - u(x)} \) is bounded. Since \( w_{\lambda_1}^* = 0 \) in \( \Sigma_{\lambda_1}^c \) and \( |\Sigma_{\lambda_1}^c| \) is small, for \( \varepsilon \) and \( \mu \) small, from the analysis Step 1, we obtain that \( w_{\lambda_1} \geq 0 \) in \( \Sigma_{\lambda_1} \). Since \( \lambda_1 > 0 \) and \( w_{\lambda_1} \not\equiv 0 \) in \( \Sigma_{\lambda_1} \), as before we have \( w_{\lambda_1} > 0 \) in \( \Sigma_{\lambda_1} \), completing the proof of the claim. The proof ends. \( \square \)
Proof of Theorem 1.1. We prove this argument by contradiction. Assume that \( u \geq 0 \) is nontrivial solution of \((1.1)\). When \( \Omega \) verifies \((\mathcal{D})\), then by Proposition 2.1, we have that \( u \) satisfies \((2.4)\). Let
\[
u_m(x) = u(x', x_N + m), \quad \forall x \in \mathbb{R}^N,
\]
then \( \{u_m\}_m \) is an increasing and bounded sequence of functions and satisfies
\[
(-\Delta)^a u = u^p \quad \text{in} \quad \Omega - m.
\]

Note that \( \Omega_t \subset \Omega_{t'} \) for \( 0 \leq t \leq t' < +\infty \),
\[
\Omega - m \subset \Omega - k \quad \text{if} \quad k > m, \quad \lim_{m \to +\infty} \Omega - m = \mathbb{R}^N,
\]
from the regularity results in \([9, \text{Theorem 2.1}]\) and the stability property \([9, \text{Theorem 2.4}]\), we obtain that
\[
u_\infty := \lim_{m \to +\infty} u_m
\]
is a bounded classical solution to
\[
(-\Delta)^a u = u^p \quad \text{in} \quad \Omega_\infty \times \mathbb{R}, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus (\Omega_\infty \times \mathbb{R}) \quad \text{if} \quad \Omega_\infty \neq \mathbb{R}^{N-1}.
\]
By the increasing property of \( \{u_m\}_m \), we have that
\[
u_\infty(x', x_N) = u_\infty(x', x_N)
\]
for any \( x_N, y_N \in \mathbb{R} \),
which implies that \( u_\infty \) is \( x_N \)-independent. Letting \( v_\infty(x') = u_\infty(x', x_N) \), by the standard argument, we have that
\[
(-\Delta)^a_{\mathbb{R}^N} v_\infty(x') = c_N (-\Delta)^a u_\infty(x), \quad \forall x \in \Omega_\infty \times \mathbb{R},
\]
then \( v_\infty \) is a positive, bounded and classical solution for
\[
(-\Delta)^a_{\mathbb{R}^N} v_\infty = c_N v_\infty^p \quad \text{in} \quad \Omega_\infty, \tag{2.14}
\]
subject to \( v_\infty = 0 \) in \( \mathbb{R}^{N-1} \setminus \Omega_\infty \) if \( \Omega_\infty \neq \mathbb{R}^{N-1} \).

A contradiction is derived by the fact that problem \((2.14)\) has no positive bounded classical solution when
(i) \( \Omega_\infty = \mathbb{R}^{N-1} \) and \( 0 < p < \frac{N-1+2\alpha}{N-2-2\alpha} \) if \( N > 1 + 2\alpha \), otherwise for \( p > 0, 1 \leq N \leq 1 + 2\alpha \) by \([13, \text{Theorem 1.2}]\);
(ii) \( \Omega_\infty \subset \mathbb{R}^{N-1} \) is a star-shaped domain with respect to infinity and \( 1 < p \leq \frac{N-1+2\alpha}{N-2-2\alpha} \) with \( N > 1 + 2\alpha \) by \([14, \text{Theorem 1.5}]\);
(iii) \( \Omega_\infty = \mathbb{R}^{N-2} \times \{0, +\infty\} \) and \( 1 < p < \frac{N-2+2\alpha}{N-2-2\alpha} \) for \( N > 2 + 2\alpha \), otherwise for \( p > 1, 1 \leq N \leq 2 + 2\alpha \) by \([13, \text{Theorem 1.2}]\);
(vi) \( \Omega_\infty \) is bounded, \( C^2 \) and star-shape in \( \mathbb{R}^{N-1}, N > 1 + 2\alpha, \) and \( p \geq \frac{N-1+2\alpha}{N-2-2\alpha} \) by \([14, \text{Corollary 1.2}]\).

As a consequence, we obtain the nonexistence results of \((1.1)\) in Theorem 1.1.

\[ \square \]

Proof of Corollary 1.1. When problem \((1.1)\) has a positive solution \( u \), then Proposition 2.1 and regularity results guarantee that \( v_\infty(x') = \lim_{m \to +\infty} u(x', x_N + m) \) is a positive, bounded and classical solution for problem
\[
(-\Delta)^a_{\mathbb{R}^N} v_\infty = c_N v_\infty^p \quad \text{in} \quad \Omega_\infty \quad \text{and} \quad v_\infty = 0 \quad \text{in} \quad \mathbb{R}^{N-1} \setminus \Omega_\infty, \tag{2.15}
\]
Furthermore, we note that \( v_{\infty, \infty}(x'') = \lim_{m \to +\infty} v_{\infty}(x''', x_{N-1} + m) \) is a positive, bounded and classical solution of
\[
(-\Delta)^a_{\mathbb{R}^N} v_{\infty, \infty} = c_N v_{\infty, \infty}^p \quad \text{in} \quad \Omega_{\infty, \infty} \quad \text{and} \quad v_{\infty, \infty} = 0 \quad \text{in} \quad \mathbb{R}^{N-2} \setminus \Omega_{\infty, \infty}, \tag{2.16}
\]
subject to \( v_{\infty, \infty} = 0 \) in \( \mathbb{R}^{N-2} \setminus \Omega_{\infty, \infty} \) if \( \Omega_{\infty, \infty} \neq \mathbb{R}^{N-2} \). Then the same conclusion is obtained as the proof of Theorem 1.1.

\[ \square \]

Acknowledgements: Y. Wang is supported by NNSF of China, No: 11661045, by the Jiangxi Provincial Natural Science Foundation, No: 20202ACBL201001, 20202BAB201005, and by Key R&D plan of Jiangxi Province, No:20181ACE50029. Y. Wei is supported by NNSF of China, No: 11871242.
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