A New Asymptotic Series and Estimates Related to Euler Mascheroni Constant

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ABSTRACT. In this article, we give a new asymptotic series for a sequence \((q_n)\) that converges to Euler-Mascheroni’s constant with the convergence speed as \(n^{-4}\). We present and prove a theorem about how to get the sequence \((q_n)\). Using this asymptotic series, we establish the lower and upper bounds for the sequence \((q_n)\).

Keywords: Euler-Mascheroni’s constant, asymptotic series, inequalities.

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1. INTRODUCTION

One of the famous constants in mathematics is the Euler-Mascheroni’s constant \(\gamma = 0,57721566490153286\ldots\). It is defined as the limit of the sequence:

\[
\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n
\]

in honor of the Swiss mathematician Leonhard Euler (1707-1783) and the Italian mathematician Lorenzo Mascheroni (1750-1800), who studied the Euler-Mascheroni’s constant \(\gamma\). The sequence \((\gamma_n)_{n \geq 1}\) and the constant \(\gamma\) have many applications in several branches of mathematics as probability, analysis, special functions and number theory. The sequence \((\gamma_n)_{n \geq 1}\) converges very slowly to the constant \(\gamma\), with the convergence speed as \(n^{-1}\). In the beginning, Tims and Tyrell [18], and then Young [19] got the lower and upper bounds for the sequence \((\gamma_n)_{n \geq 1}\) as the following:

\[
\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}
\]

with the convergence speed as \(n^{-1}\). Many authors [2, 3, 6, 7, 10, 12–17] interested in obtaining sequences that converge very fast to the limit \(\gamma\). One of them is DeTemple [6], who introduced the sequence

\[
R_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln \left( n + \frac{1}{2} \right)
\]

that converges to the limit \(\gamma\) as \(n^{-2}\). Then Mortici [12] has introduced the sequence

\[(1.1) \quad t_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \frac{1}{2} \ln \left( n^2 - \frac{1}{6} \right)\]

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in order to obtain a faster convergence to the limit \( \gamma \) with the convergence speed as \( n^{-4} \) and the following limit:

\[
\lim_{n \to \infty} n^4(t_n - \gamma) = \frac{11}{720}.
\]

Then, Cristea [4] has showed in 2014, the following double inequality

\[
\frac{11}{720n^4} - \frac{29}{9072n^6} < t_n - \gamma < \frac{11}{720n^4}
\]

for all integers \( n \geq 1 \) and has got the following asymptotic series for the sequence \( (t_n) \) given in (1.1)

\[
t_n = \gamma + \sum_{k=2}^{\infty} \frac{1}{2k} \left\{ \frac{1}{6^k} - B_{2k} \right\} \frac{1}{n^{2k}}
\]

or

\[
t_n = \gamma + \frac{11}{720n^4} - \frac{29}{9072n^6} + \frac{221}{51840n^8} - \frac{6469}{855360n^{10}} + \cdots
\]

Cristea and Mortici [5] have introduced the sequence

\[
s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n - 2} + \frac{13}{12(n - 1)} + \frac{5}{12n} - \ln n
\]

that converges to the limit \( \gamma \) with the convergence speed as \( n^{-3} \) and have demonstrated the following double inequality

\[
\frac{1}{12n^3} + \frac{11}{120n^4} < s_n - \gamma < \frac{1}{12n^3} + \frac{13}{120n^4}.
\]

Then, X. Hu, D. Lu, X. Wang [9] have presented the following sequence:

\[
r^3_{n,2} = 1 + \frac{1}{2} + \cdots + \frac{1}{n - 2} + \ln n - \frac{1}{2} \ln \left( 1 + \frac{1}{n - \frac{3}{n+1}} \right)
\]

that converges to the limit \( \gamma \) with the convergence speed as \( n^{-4} \), with the following approximation:

\[
\frac{1}{180(n + 1)^4} < \gamma - r^3_{n,2} < \frac{1}{180n^4}.
\]

The aim of the paper is to introduce a new sequence \( (q_n) \) that converges very fast to the limit \( \gamma \) and to establish the lower and upper bounds for this sequence. Motivated by Mortici [12] and Hu [9], we introduce new sequence

\[
q_n(a, b, c) = 1 + \frac{1}{2} + \cdots + \frac{1}{n - 2} + \frac{an + b}{n(n - 1)} - \frac{1}{3} \ln \left( n^3 + c \right),
\]

where \( a, b, c \) are real parameters and for \( a = \frac{3}{2}, b = -\frac{5}{12}, c = \frac{1}{4} \) the new sequence given by

\[
q_n = q_n \left( \frac{3}{2}, -\frac{5}{12}, \frac{1}{4} \right) = 1 + \frac{1}{2} + \cdots + \frac{1}{n - 2} + \frac{13}{12(n - 1)} + \frac{5}{12n} - \frac{1}{3} \ln \left( n^3 + \frac{1}{4} \right)
\]

converges to the limit \( \gamma \) with the convergence speed as \( n^{-4} \). We will show the following double inequality

\[
\frac{11}{120n^4} + \frac{1}{12n^5} + \frac{181}{2016n^6} < q_n - \gamma < \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{182}{2016n^6}
\]

for all integers \( n \geq 2 \) in the left side inequality and for all integers \( n \geq 225 \) in the right side inequality. We will also construct the asymptotic series

\[
q_n = \gamma + \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{181}{2016n^6} + \frac{1}{12n^7} + \cdots
\]
for the sequence \((q_n)\) (1.4).

2. THE RESULTS

We consider the sequence \((q_n(a, b, c))\) given by (1.3). To obtain the best real parameters \(a, b, c\), for which the sequence \((q_n(a, b, c))\) converges to \(\gamma\) with the highest convergence speed, we prove the following theorem:

**Theorem 2.1.** (i) If \(a \neq \frac{3}{2}, b \neq -\frac{5}{12}\) and \(c \neq \frac{1}{4}\) then the sequence \((q_n(a, b, c))_{n \geq 1}\) has the convergence speed as \(n^{-1}\).

(ii) If \(a = \frac{3}{2}, b \neq -\frac{5}{12}\) and \(c \neq \frac{1}{4}\) then the sequence \((q_n(a, b, c))_{n \geq 1}\) has the convergence speed as \(n^{-2}\).

(iii) If \(a = \frac{3}{2}, b = -\frac{5}{12}\) and \(c \neq \frac{1}{4}\) then the sequence \((q_n(a, b, c))_{n \geq 1}\) has the convergence speed as \(n^{-3}\).

(iv) If \(a = \frac{3}{2}, b = -\frac{5}{12}\) and \(c = \frac{1}{4}\) then the sequence \((q_n(a, b, c))_{n \geq 1}\) has the convergence speed as \(n^{-4}\).

We will use the following:

**Lemma 2.1.** If the sequence \((x_n)_{n \geq 1}\) converges to \(x\) and if there exists the limit \(\lim_{n \to \infty} n^k (x_n - x_{n+1}) = l \in \mathbb{R}\)

with \(k > 1\), then there exists the limit

\[
\lim_{n \to \infty} n^{k-1} (x_n - x) = \frac{l}{k-1}.
\]

For the proof see [11]. This lemma is a form of Cesaro-Stolz’s lemma. We utilize it in the construction of the asymptotics series and in order to estimate the convergence speed.

**Proof.** We compute the difference

\[
q_n(a, b, c) - q_{n+1}(a, b, c) = \frac{an + b}{n(n - 1)} - \frac{1}{n - 1} - \frac{an + a + b}{n(n + 1)} - \frac{1}{3} \ln (n^3 + c) + \frac{1}{3} \ln ((n + 1)^3 + c).
\]

Using a computer program as Maple, we get

\[
q_n(a, b, c) - q_{n+1}(a, b, c) = \left(a - \frac{3}{2}\right) \frac{1}{n^2} + \left(a + 2b - \frac{2}{3}\right) \frac{1}{n^3} + \left(a - c - \frac{5}{4}\right) \frac{1}{n^4} + \left(a + 2b + 2c - \frac{4}{5}\right) \frac{1}{n^5} + O\left(\frac{1}{n^6}\right).
\]

(2.5)

(i) If \(a - \frac{3}{2} \neq 0\), then

\[
\lim_{n \to \infty} n^2 (q_n(a, b, c) - q_{n+1}(a, b, c)) = \left(a - \frac{3}{2}\right) \neq 0
\]

and Lemma 2.1 says that

\[
\lim_{n \to \infty} n (q_n(a, b, c) - \gamma) = \left(a - \frac{3}{2}\right) \neq 0.
\]
We get that the sequence \( (q_n(a, b, c))_{n \geq 1} \) has the convergence speed as \( n^{-1} \).

(ii) If \( a = \frac{3}{2}, b \neq -\frac{5}{12} \) and \( c \neq \frac{1}{4} \) then the relation (2.5) is written as

\[
q_n(a, b, c) - q_{n+1}(a, b, c) = \left( \frac{1}{4} - c \right) \frac{1}{n^4} + \left( -\frac{2}{15} + 2c \right) \frac{1}{n^5} + O \left( \frac{1}{n^6} \right).
\] (2.6)

If \( b \neq -\frac{5}{12} \), then from the relation (2.6), we get

\[
\lim_{n \to \infty} n^3 (q_n(a, b, c) - q_{n+1}(a, b, c)) = \left( \frac{1}{4} + \frac{5}{6} \right) \neq 0
\]

and Lemma 2.1 says that

\[
\lim_{n \to \infty} n^2 (q_n(a, b, c) - \gamma) = \frac{1}{2} \left( \frac{2}{3} + \frac{5}{6} \right) \neq 0.
\]

We obtain that the sequence \( (q_n(\frac{3}{2}, b, c))_{n \geq 1} \) has the convergence speed as \( n^{-2} \).

(iii) If \( a = \frac{3}{2}, b = -\frac{5}{12} \) and \( c \neq \frac{1}{4} \) then the relation (2.5) is written as

\[
q_n(a, b, c) - q_{n+1}(a, b, c) = \left( \frac{1}{4} - c \right) \frac{1}{n^4} + \left( -\frac{2}{15} + 2c \right) \frac{1}{n^5} + O \left( \frac{1}{n^6} \right).
\] (2.7)

Then from the relation (2.7), we get

\[
\lim_{n \to \infty} n^4 (q_n(a, b, c) - q_{n+1}(a, b, c)) = \left( \frac{1}{4} - c \right) \neq 0
\]

and Lemma 2.1 says that

\[
\lim_{n \to \infty} n^3 (q_n(a, b, c) - \gamma) = \frac{1}{3} \left( \frac{1}{4} - c \right) \neq 0.
\]

We get that the sequence \( (q_n(\frac{3}{2}, -\frac{5}{12}, c))_{n \geq 1} \) has the convergence speed as \( n^{-3} \).

(iv) If \( a = \frac{3}{2}, b = -\frac{5}{12} \) and \( c = \frac{1}{4} \) then the relation (2.5) is written as

\[
q_n(a, b, c) - q_{n+1}(a, b, c) = \frac{11}{30n^5} + O \left( \frac{1}{n^6} \right)
\] (2.8)

and Lemma 2.1 says that

\[
\lim_{n \to \infty} n^4 (q_n(a, b, c) - \gamma) = \frac{11}{120}.
\]

We get that the sequence \( (q_n(\frac{3}{2}, -\frac{5}{12}, \frac{1}{4}))_{n \geq 1} \) has the convergence speed as \( n^{-4} \). \hfill \square

We notice that (2.8) gives us the approximation

\[ q_n - \gamma \approx \frac{11}{120n^4} \text{ as } n \to \infty. \]

We give the following theorem related to the estimates of \( (q_n) \) given in (1.4):

**Theorem 2.2.** We have the following double inequality for all integers \( n \geq 2 \) in the left side inequality and for all integers \( n \geq 225 \) in the right side inequality:

\[
\frac{11}{120n^4} + \frac{1}{12n^4} + \frac{181}{2016n^6} < q_n - \gamma < \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{182}{2016n^6}.
\]
Proof. We consider the following sequences

\[ a_n = (q_n - \gamma) - \left( \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{181}{2016n^6} \right) \]

and

\[ b_n = (q_n - \gamma) - \left( \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{182}{2016n^6} \right) \]

that converges to zero. To prove that \( a_n > 0 \) and \( b_n < 0 \), it suffices to show that \( (a_n)_{n \geq 1} \) is strictly decreasing and \( (b_n)_{n \geq 1} \) is strictly increasing. Let \( f_1(n) = a_{n+1} - a_n \) and \( f_2(n) = b_{n+1} - b_n \), where

\[
 f_1(x) = \frac{8}{12x} + \frac{5}{12(x+1)} - \frac{1}{12(x-1)} + \frac{1}{3} \ln \left( x^3 + \frac{1}{4} \right) - \frac{1}{3} \ln \left( (x+1)^3 + \frac{1}{4} \right) \\
- \left( \frac{11}{120(x+1)^4} - \frac{11}{120x^4} \right) - \left( \frac{1}{12(x+1)^5} - \frac{1}{12x^5} \right) - \left( \frac{181}{2016(x+1)^6} - \frac{181}{2016x^6} \right) 
\]

and

\[
 f_2(x) = \frac{8}{12x} + \frac{5}{12(x+1)} - \frac{1}{12(x-1)} + \frac{1}{3} \ln \left( x^3 + \frac{1}{4} \right) - \frac{1}{3} \ln \left( (x+1)^3 + \frac{1}{4} \right) \\
- \left( \frac{11}{120(x+1)^4} - \frac{11}{120x^4} \right) - \left( \frac{1}{12(x+1)^5} - \frac{1}{12x^5} \right) - \left( \frac{182}{2016(x+1)^6} - \frac{182}{2016x^6} \right) .
\]

We get

\[
 (2.9) \quad f'_1(x) = \frac{P(x-2)}{1680(x+1)^7(x-1)^2(4x^3+1)} \left( 12x + 12x^2 + 4x^3 + 5 \right)^1 x^5 > 0 
\]

for all real numbers \( x \geq 2 \) and

\[
 (2.10) \quad f'_2(x) = -\frac{Q(x-225)}{120(x+1)^7(x-1)^2(4x^3+1)} \left( 4x^3 + 1 \right)^1 x^7 < 0 
\]

for all real numbers \( x \geq 225 \), where

\[
 P(x) = 8615781393 + 48322358 \cdot 535x + 124 \cdot 451770884x^2 + 195088765300x^3 \\
+ 207843366162x^4 + 159018283386x^5 + 89932803430x^6 + 38082594545x^7 \\
+ 12078804629x^8 + 2834912752x^9 + 478671564x^{10} + 55071128x^{11} \\
+ 3869824x^{12} + 125440x^{13} 
\]

\[
 Q(x) = 177781393 + 98322358 \cdot 535x + 364 \cdot 901770884x^2 + 195088765300x^3 \\
+ 207843366162x^4 + 159018283386x^5 + 89932803430x^6 + 38082594545x^7 \\
+ 12078804629x^8 + 2834912752x^9 + 478671564x^{10} + 55071128x^{11} \\
+ 3869824x^{12} + 125440x^{13} 
\]
and
\[ Q(x) = 22876348962124636919596278035200 + 156125891358302962485888825003x^3 + 3874001939295299660913x^4 + 42953509800254866165809975x^5 + 3429542980886586835373x^6 + 2028513740325127816093x^7 + 8999214295901801973x^8 + 29943893833882652x^9 + 7380584698144x^{10} + 130981721712x^{11} + 158491784x^{12} + 117200x^{13} + 40x^{14} \]
are two polynomials with positive integers coefficients for all real numbers \( x \geq 2 \) and respectively for all real numbers \( x \geq 225 \). Then, from (2.9), we have \( f_1 \) is strictly increasing on \([2, \infty)\) and from (2.10), we have \( f_2 \) is strictly decreasing on \([225, \infty)\). It follows that from \( f_1(\infty) = f_2(\infty) = 0 \), we have \( f_1 < 0 \) on \([2, \infty)\) and \( f_2 > 0 \) on \([225, \infty)\). Thus, \((a_n)_{n \geq 2}\) is strictly decreasing and \((b_n)_{n \geq 225}\) is strictly increasing. This concludes the proof. \(\square\)

We can get the asymptotic series of the sequence \((q_n)\), using the sequence \((h_n)\)
\[ h_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} \]
harmonic sum in terms of digamma function \(\psi\)
\[ h_n = \gamma + \frac{1}{n} + \psi(n), \]
with the digamma function defined by
\[ \psi(x) = \frac{d}{dx} \left( \ln \Gamma(x) \right) = \frac{\Gamma'(x)}{\Gamma(x)}. \]
See, e.g., [1, p. 258, Rel. 6.3.2]. We have the following asymptotic expansion for the digamma function \(\psi\) that
\[ \psi(x) = \ln x - \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}}, \]
where \(B_j\) is the \(j\)th Bernoulli numbers given by
\[ \frac{1}{e^t - 1} + \frac{1}{2} - \frac{1}{t} = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{t^{2j}}{(2j)!} B_j. \]
We will demonstrate the following theorem related to the asymptotic expansion of \(q_n\):
Theorem 2.3. We get the following asymptotic expansion of \((q_n)\) as \(n \to \infty\):
\[
q_n = \gamma + \frac{1}{12n (n-1)} - \sum_{k=1}^{\infty} 1 \left\{ \frac{(-1)^{k-1}}{3 \cdot 4^k n^{3k}} + \frac{B_{2k}}{2n^{2k}} \right\}.
\]

Proof. We get
\[
q_n = h_n - \frac{1}{n} + \frac{1}{12(n-1)} + \frac{5}{12n} - \frac{1}{3} \ln \left( n^\frac{1}{4} + 1 \right)
\]
\[
= \gamma + \psi(n) + \frac{1}{12(n-1)} + \frac{5}{12n} - \frac{1}{3} \ln \left( n^\frac{1}{4} + 1 \right)
\]
\[
= \gamma + \psi(n) - \ln n + \frac{1}{12(n-1)} + \frac{5}{12n} - \frac{1}{3} \ln \left( 1 + \frac{1}{4n^3} \right)
\]
\[
= \gamma + \frac{1}{12(n-1)} - \frac{1}{2n} + \frac{5}{12n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}} - \frac{1}{3} \ln \left( 1 + \frac{1}{4n^3} \right)
\]
\[
= \gamma + \frac{1}{12n(n-1)} - \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \frac{(-1)^{k-1}}{3 \cdot 4^k n^{3k}} + \frac{B_{2k}}{2n^{2k}} \right\}.
\]

Using the binomial theorem given in [8], we get
\[
\frac{1}{12n(n-1)} = \frac{1}{12n^2 (1 - \frac{1}{n})} = \frac{1}{12n^2} + \frac{1}{12n^3} + \frac{1}{12n^4} + \frac{1}{12n^5} + \cdots
\]
We get an explicite form as
\[
(2.11)
q_n = \gamma + \frac{11}{120n^4} + \frac{1}{12n^5} + \frac{181}{2016n^6} + \frac{1}{12n^7} + \cdots
\]
We notice that the three terms of the asymptotic series (2.11) were used for the estimate of \(q_n\). We give the table with the above sequences:

| \(n\)  | \(|t_n - \gamma|\) | \(|s_n - \gamma|\) | \(|r_{n,2}^3 - \gamma|\) | \(|q_n - \gamma|\) |
|-------|--------------------|-------------------|-------------------|-------------------|
| 250   | \(1.30935 \times 10^{-17}\) | \(4.26667 \times 10^{-12}\) | \(2.25298 \times 10^{-14}\) | \(2.03175 \times 10^{-18}\) |
| 500   | \(2.04586 \times 10^{-19}\) | \(7.66667 \times 10^{-13}\) | \(7.07570 \times 10^{-16}\) | \(3.1746 \times 10^{-20}\) |
| 1000  | \(3.19665 \times 10^{-21}\) | \(1.66667 \times 10^{-14}\) | \(2.21668 \times 10^{-17}\) | \(4.96032 \times 10^{-22}\) |
| 10000 | \(3.19665 \times 10^{-27}\) | \(1.66667 \times 10^{-18}\) | \(2.22167 \times 10^{-22}\) | \(4.96032 \times 10^{-28}\) |
| 50000 | \(2.04586 \times 10^{-31}\) | \(2.66667 \times 10^{-21}\) | \(7.11076 \times 10^{-26}\) | \(3.1746 \times 10^{-32}\) |

Using the values from the above table, we conclude the superiority of the sequence \((q_n)_{n \geq 225}\) over Mortici’s sequence \((t_n)_{n \geq 225}\), Lu’s sequence \((r_{n,2}^3)_{n \geq 225}\), Cristea and Mortici’s sequence \((s_n)_{n \geq 225}\). \(\square\)

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