Homology of coloured posets: a generalisation of Khovanov’s cube construction

Brent Everitt and Paul Turner *

Abstract. We define a homology theory for a certain class of posets equipped with a representation. We show that when restricted to Boolean lattices this homology is isomorphic to the homology of the “cube” complex defined by Khovanov.

Introduction

Given a representation of a Boolean lattice one can construct a chain complex by using Khovanov’s “cube” construction in his celebrated paper on the categorification of the Jones polynomial [6]. Recall that a representation of a Boolean lattice assigns to each vertex $x$ a finite dimensional vector space $V_x$ and to each edge $x \leq y$ a linear map $V_x \to V_y$. The homology of complexes arising in this way plays a central role in link homology theories such as Khovanov homology and Khovanov-Rozansky homology. Recently, Heegaard-Floer knot homology has also been interpreted in terms of the homology of a the complex coming from a Boolean lattice equipped with a particular representation. Khovanov’s construction relies on specific properties of Boolean lattices and the question that motivates the current paper is: can one define a homology theory for a more general class of posets equipped with a representation, which for Boolean lattices gives the homology arising from Khovanov’s cube complex?

Indeed one can: we define a chain complex for an arbitrary poset with 1 equipped with a representation. We refer to such posets as coloured posets which form the objects of a category and by passing to homology we get a functor to graded modules. This functor satisfactorily answers the above question: for coloured Boolean lattices the result is isomorphic to the homology of Khovanov’s cube complex, an outcome not a priori obvious.

We begin in Section 1 by studying the category of coloured posets, $\mathcal{CP}_R$ over a ring $R$. We provide a number of examples and several basic constructions. In Section 2 we define a functor $S_*$ from $\mathcal{CP}_R$ to chain complexes over $R$ which generalises the well-known order homology of a poset to the situation where one has a local system of coefficients. The resulting homology $H_*(P,\mathcal{F})$ is what we refer to as the homology of the coloured poset $(P,\mathcal{F})$. We show that the chain complex $S_*(P,\mathcal{F})$ is homotopy equivalent to a much smaller complex $C_*(P,\mathcal{F})$, paralleling the situation in topology where the full simplicial chain complex on a space is cut down by throwing away degeneracies.

The main technical result is presented in Section 3 where we show that a coloured poset obtained by gluing two coloured posets together by a morphism gives rise to a long exact sequence in homology (see Theorem 1). We give a brief tutorial in Section 4 on Khovanov’s cube complex, which in the context of this paper is a chain complex $\mathcal{K}_p(B,\mathcal{F})$ associated to a coloured Boolean lattice $(B,\mathcal{F})$. We denote its homology by $H_p^*(B,\mathcal{F})$. In Section 5 we present the main result, namely the agreement of the coloured poset homology with the Khovanov’s cube homology for coloured

BRENT EVERITT: Department of Mathematics, University of York, York YO10 5DD, United Kingdom. e-mail: bje1@york.ac.uk. PAUL TURNER: School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh, EH1 4AS, United Kingdom and Département de mathématiques, Université de Fribourg, CH-1700 Fribourg, Switzerland. e-mail: paul@ma.hw.ac.uk.

* The first author was partially supported by the London and Edinburgh Mathematical Societies, and is grateful to the Institute for Geometry and its Applications, University of Adelaide, Australia, for their hospitality during an extended visit. The second author was partially supported by the Royal Society and is grateful to the Glenelg Maths Institute for their hospitality.
Boolean lattices. We construct a chain map $\phi$ from the cube complex $\mathcal{K}_n(B, \mathcal{F})$ to $\mathcal{C}_n(B, \mathcal{F})$, and our main result, given as Theorem 2 in [5] is

**Main Theorem.** Let $(B, \mathcal{F})$ be a coloured Boolean lattice. Then $\phi : \mathcal{K}_n(B, \mathcal{F}) \rightarrow \mathcal{C}_n(B, \mathcal{F})$ is a quasi-isomorphism, yielding isomorphisms,

$$H^2_n(B, \mathcal{F}) \xrightarrow{\sim} H_n(B, \mathcal{F}).$$

### 1. Coloured posets

The principal characters in our story, coloured posets, are partially ordered sets (posets) whose elements are labeled by $R$-modules so that there is a homomorphism between the labels of comparable elements. More concisely, a coloured poset is a representation of a poset with maximal element.

We begin by recalling basic poset terminology, for which we will generally follow [11] Chapter 3. A poset $(P, \leq)$ is a set $P$ together with a reflexive, anti-symmetric, transitive binary relation $\leq$, and a map of posets $f : (P, \leq) \rightarrow (Q, \leq')$ is a set map preserving the respective relations, i.e. $f(x) \leq' f(y)$ in $Q$ if $x \leq y$ in $P$. One writes $x < y$ when $x \leq y$ and $x \neq y$. If $x < y$ and there is no $z$ with $x < z < y$ then we say that $y$ covers $x$, and write $x <_c y$. The covering relation is illustrated via the Hasse diagram: the graph with vertices the elements of $P$, and an edge joining $x$ to $y$ iff $x <_c y$. We will follow the convention that Hasse diagrams will be presented vertically on the page with $y$ drawn above $x$ whenever $x <_c y$.

An ordered *multi-sequence* is a sequence $x_1 \leq \cdots \leq x_k$, of comparable elements. An ordered *sequence* is a multi-sequence with $x_1 < \cdots < x_k$. A sequence is saturated when it has the form $x_1 <_c \cdots <_c x_k$. This differs from the standard poset terminology (where a multi-sequence is called a multi-chain and a sequence a chain) justified by our giving preference to homological notions, where the term chain is already taken. There is the obvious notion of a 0: an element with $x \geq 0$ for all $x \in P$; similarly for a 1. A poset $P$ is graded of rank $r$ if every saturated sequence, maximal under inclusion of sequences, has the same length $r$. There is then a unique grading or rank function $\text{rk} : P \rightarrow \{0, 1, \ldots, r\}$ with $\text{rk}(x) = 0$ if and only if $x$ is minimal, and $\text{rk}(y) = \text{rk}(x) + 1$ whenever $x <_c y$. The rank 1 elements are called the *atoms*.

Sometimes our posets will turn out to be lattices: posets for which any $x$ and $y$ have a supremum or least upper bound $x \lor y$ (the join of $x$ and $y$) and an infimum, or greatest lower bound $x \land y$ (the meet of $x$ and $y$). A lattice is atomic if every element can be expressed (not necessarily uniquely) as a join of atoms.

Without explicitly mentioning it, we will often consider a poset as a category whose objects are the elements of the poset, and with a unique morphism $x \rightarrow y$ between any two comparable elements $x \leq y$. A *representation* of a poset is a covariant functor to some category of modules.

Here is a primordial example: the Boolean lattice $B = B(X)$ on the set $X$ is a lattice isomorphic, by a bijective poset mapping, to the lattice of subsets of $X$ under inclusion. We will often suppress the isomorphism and identify the elements of $B$ with subsets of $X$. If $X$ is finite, then $B$ is graded with $\text{rk}(S) = |S|$ for $S \subseteq X$, and atomic, with atoms the singletons. If the atoms are given some fixed ordering $a_1, \ldots, a_r$, then every $x \in B$ can be expressed uniquely as a join,

$$x = \bigvee a_{i_j} = a_{i_1} \lor a_{i_2} \lor \cdots \lor a_{i_k},$$

where $i_1 < \cdots < i_k$. For $x = a_{i_1} \lor \cdots \lor a_{i_k}$, one has the covering relation $x <_c y$ if and only if the unique expression for $y$ is $y = (a_{i_1} \lor \cdots \lor a_{i_j}) \lor a_{i_{j+1}} \lor \cdots \lor a_{i_k}$.

With these preliminaries out of the way we now make the principal definition. Fix a unital commutative ring $R$ and let $\text{Mod}_R$ be the category of $R$-modules.

**Definition 1.** A coloured poset $(P, \mathcal{F})$ consists of

- a poset $P$ having a unique maximal element $1_P$, and
- a covariant functor $\mathcal{F} : P \rightarrow \text{Mod}_R$. 
The functor $\mathcal{F}$ will be referred to as the colouring.

A morphism of coloured posets $(P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ is a pair $(f, \tau)$ where

- $f : P_1 \to P_2$ is a map of posets, and
- $\tau$ is a collection $\{\tau_x\}_{x \in P_1}$ where $\tau_x : \mathcal{F}_1(x) \to \mathcal{F}_2(f(x))$ is an $R$-module homomorphism.

This data satisfies the following two conditions

1. $f(x) = 1_{P_2}$ if and only if $x = 1_{P_1}$, and
2. (naturality) for all $x \leq y$ in $P_1$, the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{F}_1(x) & \xrightarrow{\mathcal{F}_1(x \leq y)} & \mathcal{F}_1(y) \\
\downarrow{\tau_x} & & \downarrow{\tau_y} \\
\mathcal{F}_2(f(x)) & \xrightarrow{\mathcal{F}_2(f(x) \leq f(y))} & \mathcal{F}_2(f(y)).
\end{array}
$$

Coloured posets and morphisms between them form a category denoted $\mathcal{C}P_R$.

Thus the colouring associates to each element of the poset an $R$-module, $\mathcal{F}(x)$ and if $x \leq y$ then there is an associated map $\mathcal{F}(x \leq y) : \mathcal{F}(x) \to \mathcal{F}(y)$. We will often find it convenient to write $\mathcal{F}^y_x$ instead of $\mathcal{F}(x \leq y)$. Also, we usually define the morphisms $\mathcal{F}(x \leq y)$ just in the cases where $x <_c y$, as all the others can be recovered from these by repeated composition.

Given a map of (uncoloured) posets $f : P_1 \to P_2$, the composite $\mathcal{F}_2 \circ f : P_1 \to \text{Mod}_R$ defines another colouring on $P_1$. Condition (2) in the above definition is merely stating that $\tau$ is a natural transformation (of functors $P_1 \to \text{Mod}_R$) from $\mathcal{F}_1$ to $\mathcal{F}_2 \circ f$.

The notion of a coloured poset is a rather general one encompassing many interesting examples as the following illustrate.

**Example 1.** Let $P$ be a poset with unique maximal element and $A$ an $R$-module. The constant colouring on $P$ by $A$ is defined by the colouring functor $\mathcal{F} : P \to \text{Mod}_R$ given by $\mathcal{F}(x) = A$ and $\mathcal{F}_P^y = \text{id}_A$ for all $x \leq y$.

**Example 2 (Pre-sheaves).** Let $X$ be a topological space and $P$ the poset of open subsets partially ordered by reverse inclusion. A colouring is equivalent to a pre-sheaf of $R$-modules on $X$.

**Example 3 (abelian subgroups).** Let $G$ be a group. Then the poset of abelian subgroups is a naturally a coloured poset, by colouring an element of the poset with the subgroup it corresponds to. Homomorphisms are just the inclusions.

**Example 4 (The Khovanov colouring).** This is a colouring of a Boolean lattice associated to a link diagram. Let $D$ be a projection of a link i.e. a link diagram, and let $\mathbb{B}$ be the Boolean lattice on the crossings of the diagram. Each crossing can be resolved into a 0 or a 1-resolution as shown on the left in Figure 1. If $S$ is some subset of crossings, then the complete resolution $D(S)$ is what results from 1-resolving the crossings in $S$ and 0-resolving the crossings not in $S$: it is a collection of planar circles. These complete resolutions are central in Kauffman’s formulation of the Jones polynomial and also in Khovanov’s definition of his homology for links [1],[6].

Now let $V$ be a graded commutative Frobenius algebra over $R$ with multiplication $m$ and comultiplication $\mu$ both of degree $-1$. We define a (graded) colouring $\mathcal{F} : \mathbb{B} \to \text{GrMod}_R$ as follows: for $S \in \mathbb{B}$, let $\mathcal{F}(S) = V^\otimes \mu[\text{rk}(S)]$, with a tensor factor corresponding to each connected component of $D(S)$ shifted by the rank of $S$ in $\mathbb{B}$. The notation is that for $W$ a graded module $(W[a])_i = W_{i-a}$. If $S <_c T$ in $\mathbb{B}$ then $D(T)$ results from 1-resolving a crossing that was 0-resolved in $D(S)$, with the qualitative effect being that two of the circles in $D(S)$ fuse into one in $D(T)$, or one of the circles in $D(S)$ bifurcates into two in $D(T)$. In the first case $\mathcal{F}(S <_c T) : V^\otimes \mu[\text{rk}(S)] \to V^\otimes \mu-1[\text{rk}(T)]$ is the map using $m$ on the tensor factors corresponding to the fused circles, and the identity on the others. In the second case, $\mathcal{F}(S <_c T) : V^\otimes \mu[\text{rk}(S)] \to V^\otimes \mu+1[\text{rk}(T)]$ is the map using $\mu$ on the tensor factor corresponding to the bifurcating circles, and the identity on the others. In both cases $\mathcal{F}(S <_c T)$ is a grading preserving map. The properties of a Frobenius algebra guarantee that $\mathcal{F}$ is a well-defined functor.
It is worth noting that (un-normalised) Khovanov homology is then defined as the homology of a certain complex obtained from this “cube”. In §3 we explain Khovanov’s construction of this complex, which we will call the cube complex of a coloured Boolean lattice. For a very restrictive class of graded Frobenius algebras this results in a bi-graded homology theory which after normalisation (depending on an orientation) gives an invariant of oriented links. More recent link homology theories, such as Khovanov-Rozansky homology, are also defined as the homology of the cube complex of a certain coloured Boolean lattice associated to a link diagram.

Example 5 (The Ozsváth-Szabó colouring). Although it has more geometric origins, knot Floer homology now has a completely combinatorial description involving a coloured Boolean lattice, along the lines of Khovanov homology. Let $\mathbb{B}$ be the Boolean lattice on the $n$ crossings of a link diagram $D$, so that as before each crossing can be resolved into a 0 or a 1-resolution, this time as shown on the right of Figure 1. Call the resolutions along the top row smoothings and singularizations respectively. If $S$ is a set of crossings, then the complete resolution $D(S)$ is the graph resulting from 1-resolving the crossings in $S$ and 0-resolving the crossings not in $S$. Let $R = \mathbb{Z}[t, s_0, \ldots, s_{2n}]$, with the $s_i$ corresponding to the edges of a completely resolved diagram. For $S \in \mathbb{B}$, let $\mathcal{F}(S)$ be the quotient $A_S$ of $R$ obtained by introducing certain relations determined by $D(S)$. If $S <_c T$ then there is a single crossing which is smoothed (resp. singularized) in $S$ that is singularized (resp. smoothed) in $T$. The map $\mathcal{F}(S <_c T) : A_S \to A_T$ is then a certain zip (resp. unzip) homomorphism. The precise details, which are a little more elaborate than in the previous example, can be found in §8.

It turns out that $\mathcal{F}$ is a colouring of the Boolean lattice associated to an oriented knot diagram, and the homology of the associated cube complex is isomorphic to the Heegaard-Floer knot homology of the knot.

Example 6 (The colouring of a Boolean lattice associated to a graph). Let $\Gamma$ be a graph, $\mathbb{B}$ the Boolean lattice on the edge set, and $M$ an $R$-algebra with multiplication $m$. If $S$ is some set of edges then let the graph $\Gamma(S)$ have the same vertex set as $\Gamma$ and edge set $S$. Define $\mathcal{F} : \mathbb{B} \to \text{Mod}_R$ as follows: if $S \in \mathbb{B}$, let $\mathcal{F}(S) = M^{\otimes k}$, with a tensor factor corresponding to each connected component of the graph $\Gamma(S)$. If $S <_c T$ in $\mathbb{B}$ then $T = S \cup \{e\}$ for some edge $e$. In particular, the graph $\Gamma(T)$ either has the same number of components as $\Gamma(S)$, or the edge $e$ connects two components, reducing the overall number by one. Define $\mathcal{F}_S^T = \text{id}$ in the first case, and in the second, $\mathcal{F}_S^T : M^{\otimes k} \to M^{\otimes k-1}$ is the map using $m$ on the tensor factors corresponding to the components connected by $e$, and the identity on the others. This procedure gives a colouring $\mathcal{F}$ of the Boolean lattice of a graph, first defined by Helme-Guizon and Rong (see §4), and the homology of the associated cube complex is related to the chromatic polynomial of $\Gamma$.

There are a number of interesting constructions with coloured posets which we now discuss.

Unions. Let $(P_1, \mathcal{F}_1)$ and $(P_2, \mathcal{F}_2)$ be coloured posets. Their union, $(P_1, \mathcal{F}_1) \cup (P_2, \mathcal{F}_2)$ is defined by taking the disjoint union of $P_1$ and $P_2$ and then identifying $1_{P_1}$ with $1_{P_2}$ (so the underlying poset of the union is almost, but not quite, the union of the underlying posets). The colouring is defined by $\mathcal{F}_1$ and $\mathcal{F}_2$ with the modification that 1 is coloured by $\mathcal{F}_1(1_{P_1}) \oplus \mathcal{F}_2(1_{P_2})$, and for $x \in P_1$ we have $\mathcal{F}_x^1 = \mathcal{F}_1 \oplus 0$ (and similarly $\mathcal{F}_y^2 = 0 \oplus \mathcal{F}_2$ for $y \in P_2$).
**Products.** The product \((P_1, \mathcal{F}_1) \times (P_2, \mathcal{F}_2) = (P, \mathcal{F})\), has underlying poset \(P\) the direct product of the \(P_i\), i.e: the poset with elements \((a, b) \in P_1 \times P_2\) with \((a, b) \leq (a', b')\) iff \(a \leq a'\) and \(b \leq b'\). The colouring is \(\mathcal{F}(a, b) = \mathcal{F}_1(a) \otimes_R \mathcal{F}_2(b)\) and \(\mathcal{F}(a', b') = (\mathcal{F}_1)^{a'} \otimes (\mathcal{F}_2)^{b'}\).

For example, if \(\mathbb{B}_i, (i = 1, 2)\) are Boolean lattices of rank \(r_i\) (isomorphic to the lattice of subsets of \(X_i\)), then \(\mathbb{B}_1 \times \mathbb{B}_2\) is Boolean of rank \(r_1 + r_2\) (isomorphic to the lattice of subsets of \(X_1 \sqcup X_2\)). If the \(\mathbb{B}_i\) are coloured by \(\mathcal{F}_i\), we have a picture like Figure 2 in the case \(r_1 = 1, r_2 = 2\), and where we have abbreviated \(U_x := \mathcal{F}_1(x)\), \(V_x := \mathcal{F}_2(x)\).

**Gluing along a morphism.** Let \((P_1, \mathcal{F}_1)\) and \((P_2, \mathcal{F}_2)\) be coloured posets and let \((f, \tau): (P_1, \mathcal{F}_1) \rightarrow (P_2, \mathcal{F}_2)\) be a morphism of coloured posets. We can construct a new coloured poset \((P_1, \mathcal{F}_1) \cup_f (P_2, \mathcal{F}_2)\) by “gluing” \(P_1\) to \(P_2\) using the map \(f\).

The underlying set of \((P_1, \mathcal{F}_1) \cup_f (P_2, \mathcal{F}_2)\) is \(P_1 \cup P_2\), the union of elements on \(P_1 \) and \(P_2\). The partial order on this set is defined as follows.

- If \(a, a' \in P_i\) then \(a \leq a'\) iff \(a \leq a'\) in \(P_i\);
- If \(a \in P_1\) and \(a' \in P_2\) then \(a \leq a'\) iff \(f(a) \leq a'\) in \(P_2\).

We will denote this poset by \(P_1 \cup_f P_2\).

The colouring functor \(\mathcal{F}: P_1 \cup_f P_2 \rightarrow \text{Mod}_R\) is defined as follows. For an object \(a \in P_i\) set \(\mathcal{F}(a) = \mathcal{F}_1(a)\). For a morphism \(a \leq a'\) we define \(\mathcal{F}_a': \mathcal{F}(a) \rightarrow \mathcal{F}(a')\) as follows.

- If \(a, a' \in P_i\) then \(\mathcal{F}_a' = (\mathcal{F}_i)^{a'}\), and
- If \(a \in P_1\) and \(a' \in P_2\) then as part of the morphism \((f, \tau)\) there is a map \(\tau_a: \mathcal{F}_1(a) \rightarrow \mathcal{F}_2(f(a))\).

Since \(f(a) \leq a'\) in \(P_2\) there is a map \((\mathcal{F}_2)^{f(a)}: \mathcal{F}_2(f(a)) \rightarrow \mathcal{F}_2(a')\). In this case set

\[
\mathcal{F}_a' = (\mathcal{F}_2)^{f(a)} \circ \tau_a.
\]

**Lemma 1.** \((P_1 \cup_f P_2, \mathcal{F})\) is a coloured poset.

**Proof.** It is routine to check that \(P_1 \cup_f P_2\) is a poset and moreover that \(1_{P_2}\) provides a unique maximal element. To verify that \(\mathcal{F}\) is a functor the only real issue is composition. Suppose \(a \leq a' \leq a''\) then we must check \(\mathcal{F}_a' \circ \mathcal{F}_a'' = \mathcal{F}_a''\). There are a number of cases to consider. If \(a, a' \in P_1\) and \(a'' \in P_2\), then the identity to check is given by the outside routes around the following diagram.

![Diagram](image)

The righthand triangle commutes courtesy of the functoriality of \(\mathcal{F}_2\) and the lefthand triangle commutes because of the naturality of \(\tau\), thus the square commutes. The other cases are simpler and omitted. \(\square\)
the following notation: an ordered multi-sequence \( x = x_1 x_2 \cdots x_k \), and we will write \( B = (B_0, F_0) \) if \( B_0 \) is a poset then the order complex on the poset with \( \mathcal{V} \in B_0 \). The last will play a key role in §4.6). We use \( \mathcal{F} \) for \( \mathcal{F}(x) \) for \( x \in B_0 \), define \( f(x) = x \cup a_\ell \), and \( \tau_x := F_x(f(x)) \). Figure 3 illustrates the rank three case, with the three decompositions (clockwise from the main picture) for \( \ell = 2, 3 \) and 1. The last will play a key role in §4.6).

### 2. The homology of a coloured poset

Poset homology was pioneered by Folkman and Rota, amongst others. We will make no attempt to summarize this vast area beyond our immediate needs, but to whet the readers appetite we mention a couple of fruitful applications: it provides an organizing principle in group representation theory, where group actions on posets lead to representations on the poset homology [3, §66]; in the theory of hyperplane arrangements it plays a key role, where \( P \) is the intersection lattice of the arrangement, see eg: [7] §4.5]. The basic principle is to pass from posets to abstract simplicial complexes. Recall that an abstract simplicial complex with vertex set \( X \) is a subset \( \Delta \subset 2^X \) such that \( \{x\} \in \Delta \) iff \( x \in X \), and \( \sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta \). The \( k \)-simplicies \( \Delta_k \) are the \( k+1 \)-element subsets and the empty set \( \emptyset \in \Delta \) is the unique \((-1)\)-simplex. If \( P \) is a poset then the order complex \( \Delta(P) \) has \( X = P \) and \( k \)-simplicies the ordered sequences \( \sigma = (x_0 < x_1 < \cdots < x_k) \) of length \( k+1 \). If \( P \) has a 1, then \( \Delta(P) \) is a cone on \( \Delta(P \setminus 1) \), hence contractible (this is standard, but see for example [7] Lemma 4.96). A similar thing is true if \( P \) has a 0, so this procedure is normally applied to the order complex on the poset with \( 0 \)'s removed: the so-called Folkman complex. More details on poset topology and homology can be found in [2]. One can rephrase most statements in traditional poset topology in terms of classifying spaces of categories if one wishes.

Our purpose in this section is to incorporate a colouring of \( P \) into this scheme. Essentially this amounts to considering a local coefficient system (given by the colouring) on the order complex. This has already made a brief appearance in the poset literature (see [7] §4.6). We use the following notation: an ordered multi-sequence \( x_1 \leq x_2 \leq \cdots \leq x_n \) will be abbreviated to \( x = x_1 x_2 \cdots x_n \), and we will write \( 1 := 1_p \).

If \( (P, \mathcal{F}) \) is a coloured poset we define the chain complex \( S_\bullet(P, \mathcal{F}) \) by setting,

\[
S_k(P, \mathcal{F}) = \bigoplus_{x_1 x_2 \cdots x_k \in P \setminus 1} \mathcal{F}(x_1),
\]

for \( k > 0 \). Thus we have one direct summand for each length \( k \) multi-sequence \( x_1 \leq x_2 \leq \cdots \leq x_k \) in \( P \setminus 1 \). A typical element can thus be written as \( \sum_x \lambda \cdot x \), where the sum is over all length \( k \)
sequences $x = x_1 x_2 \cdots x_k$ and $\lambda \in \mathcal{F}(x_1)$, and when it is important to remember that the sequence $x_1 x_2 \cdots x_k$ may contain the same element repeated a number of times. For $k = 0$ set

$$S_0(P, \mathcal{F}) = \mathcal{F}(1),$$

and for $k < 0$, we have $S_k(P, \mathcal{F}) = 0$. The differential $d_k : S_k(P, \mathcal{F}) \to S_{k-1}(P, \mathcal{F})$ is defined for $k > 1$ by

$$d_k(\lambda x_1 x_2 \cdots x_k) = \mathcal{F}^{x_2}_x(\lambda) x_2 \cdots x_k - \sum_{i=2}^{k} (-1)^{i} \lambda x_1 \cdots \hat{x}_i \cdots x_k,$$

and $d_1$ is defined by

$$d_1(\lambda x) = \mathcal{F}^1_x(\lambda).$$

**Lemma 2.** $S_*(P, \mathcal{F})$ is a chain complex.

**Proof.** We need to show $d^2 = 0$. It is not hard to see that $d_{k-1}(d_k(\lambda x_1 x_2 \cdots x_k))$ is a sum of terms of the form $\mu x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_k$, where each indexing multi-sequence appears exactly twice. We need to check that such pairs have opposite signs. One such term arises by the deletion of $x_i$ and then $x_j$, with its sign being $(-(-1)^i) \times (-(-1)^{j-1}) = (-1)^{i+j-1}$. On the other hand if $x_j$ is deleted first then the sign is $(-(-1)^j) \times (-(-1)^i) = (-1)^{i+j}$, hence the pairs cancel.

Given a morphism of coloured posets $(f, \tau) : (P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)$ there is an induced map

$$f_* : S_*(P_1, \mathcal{F}_1) \to S_*(P_2, \mathcal{F}_2)$$

defined by

$$\lambda x_1 x_2 \cdots x_k \mapsto \tau_{x_1}(\lambda) f(x_1) f(x_2) \cdots f(x_k).$$

**Lemma 3.** $f_*$ is a well-defined chain map.

**Proof.** Clearly $\tau_{x_1}(\lambda) f(x_1) f(x_2) \cdots f(x_k)$ is an element of $S_*(P_2, \mathcal{F}_2)$ and by the first condition for a morphism of coloured posets we also have $f(x_k) \neq 1_{P_2}$. To see that $f_*$ is a chain map we calculate

$$d(f_*(\lambda x_1 x_2 \cdots x_k)) = \mathcal{F}^{f(x_2)}_{f(x_1)}(\tau_{x_1}(\lambda)) f(x_2) \cdots f(x_k) + \Phi,$$

and

$$f_*(d(\lambda x_1 x_2 \cdots x_k)) = \tau_{x_2}(\mathcal{F}^{x_2}_x(\lambda)) f(x_2) \cdots f(x_k) + \Phi,$$

for $\Phi = - \sum_{i=2}^{k} (-1)^{i} \tau_{x_1}(\lambda) f(x_1) \cdots \hat{f}(x_i) \cdots f(x_k)$. These are equal by the naturality of $\tau$.

We have thus defined a covariant functor

$$S_* : \mathcal{CP}_R \to \text{Ch}_R,$$

from coloured posets to chain complexes over $R$. Finally, we define the homology of the coloured poset $(P, \mathcal{F})$ to be

$$H_n(P, \mathcal{F}) = H_n(S_*(P, \mathcal{F})).$$

Since homology is a functor from chain complexes to graded $R$-modules we therefore have a covariant functor

$$H_* : \mathcal{CP}_R \to \text{GrMod}_R.$$

Just as for homology of spaces we can cut down the size of the chain complex by factoring out redundancies. We define $\mathcal{C}_*(P, \mathcal{F})$ identically to $S_*(P, \mathcal{F})$, but with the additional requirement that the $x = x_1 x_2 \cdots x_k$ appearing in (2) are now sequences $x = x_1 < x_2 \cdots < x_k$. More precisely, for $k > 0$ let

$$\mathcal{C}_k(P, \mathcal{F}) = \bigoplus_{x_1 < x_2 \cdots < x_k} \mathcal{F}(x_1),$$

and for $k < 0$, we have $\mathcal{C}_k(P, \mathcal{F}) = 0$. The differential $\mathcal{C}_k : \mathcal{C}_k(P, \mathcal{F}) \to \mathcal{C}_{k-1}(P, \mathcal{F})$ is defined for $k > 1$ by

$$\mathcal{C}_k(\lambda x_1 x_2 \cdots x_k) = \mathcal{F}^{x_2}_x(\lambda) x_2 \cdots x_k - \sum_{i=2}^{k} (-1)^{i} \lambda x_1 \cdots \hat{x}_i \cdots x_k,$$

and $\mathcal{C}_1$ is defined by

$$\mathcal{C}_1(\lambda x) = \mathcal{F}^1_x(\lambda).$$

**Lemma 4.** $\mathcal{C}_*(P, \mathcal{F})$ is a chain complex.
with the $x_i \in P \setminus 1$ as before. Thus we have a direct summand for each length $k$ ordered sequence $x_1 < x_2 < \cdots < x_k$ in $P \setminus 1$. For $k = 0$ we set

$$C_0(P, \mathcal{F}) = \mathcal{F}(1),$$

and $C_k(P, \mathcal{F}) = 0$ for $k < 0$. Clearly $C_k \subseteq S_k$, and we define the differential to be the restriction to $C_k$ of the differential on $S_k$ (and so $C_\ast$ is a subcomplex of $S_\ast$). Note that if there is a maximal length $r_0$ of an ordered sequence in $P$, then we have $C_k(P, \mathcal{F}) = 0$ for $k > r_0$, which is not the case for $S_\ast$. Nevertheless, it turns out that $C_\ast(P, \mathcal{F})$ is homotopy equivalent to $S_\ast(P, \mathcal{F})$ as we will now see.

Let $D_k \subseteq S_k$ be the sub-module containing those summands indexed by sequences with at least one repeat in $i$: generated by elements of the form $\lambda x_1 x_2 \cdots x_k$ where $x_i = x_{i+1}$ for at least one $i$. If $\lambda x_1 \cdots x x \cdots x_k$ is one such, then there are only two terms in $d(\lambda x_1 \cdots x x \cdots x_k)$ without the repeated $x$’s, and these have opposite signs, hence cancel (if $\lambda xxx \cdots x_k$ is the term, then recall that $\mathcal{F}_x(z) = \text{id}$. Thus $d$ is closed on $D_\ast$ so $D_\ast \subseteq S_\ast$ is a subcomplex, and we have proved,

**Lemma 4.** There is a decomposition of complexes

$$S_\ast(P, \mathcal{F}) = C_\ast(P, \mathcal{F}) \oplus D_\ast(P, \mathcal{F}).$$

But the complex $D_\ast$ proves not to be interesting, as the following proposition shows. The result is similar to standard results in algebraic topology, but it is simple enough to write down an explicit proof, so we include it for completeness.

**Proposition 1.** There is a homotopy equivalence of chain complexes

$$D_\ast(P, \mathcal{F}) \simeq 0 := (\cdots 0 0 0 0 \cdots)$$

**Proof.** It suffices to show that the identity map on $D_\ast = D_\ast(P, \mathcal{F})$ is null homotopic. For this we need a family of maps $h_i : D_i \to D_{i+1}$ such that

$$\text{id} = h_{i-1}d_i + d_{i+1}h_i.$$  \hfill (3)

Given a multi-sequence $x = x_1 x_2 \cdots x_k$ in $D_\ast$ we define $p = p(x) = \min \{i \mid x_i = x_{i+1}\}$, so $p(x)$ is the position of the first repeating element, and let $n = n(x)$ be the number of times $x$ repeats. Thus we can write a multi-sequence $x = x_1 x_2 \cdots x_k$ as $x_1 \cdots x_{p-1} x_{p}^n x_{p+n} \cdots x_k$, where $x_i \neq x_{i+1}$ for $1 \leq i < p$ and $n \geq 2$. Define $h_i : D_i \to D_{i+1}$ by

$$h_i(\lambda x_1 \cdots x_{p-1} x_{p}^n x_{p+n} \cdots x_k) = \begin{cases} (-1)^{p+1} \lambda x_1 \cdots x_{p-1} x_{p+1} x_{p+n} \cdots x_k & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

To show (3) we consider separately the two cases $n$ odd and even.

**The case: $n$ odd.** We consider separately the two cases $n$ odd and even.

To show (3) we consider separately the two cases $n$ odd and even.

**The case: $n$ odd.** We have,

$$d_i(\lambda x) = \mathcal{F}_x^2(\lambda x) x_1(\cdots x_{p-1} x_p^n x_{p+n} \cdots x_k - \sum_{j=2}^{p-1} (-1)^{j} \lambda x_1 \cdots \hat{x}_j \cdots x_{p-1} x_p^n x_{p+n} \cdots x_k$$

$$- (-1)^{p} \lambda x_1 \cdots x_{p-1} x_p^{n-1} x_{p+n} \cdots x_k - \sum_{j=p+n}^{k} (-1)^{j} \lambda x_1 \cdots x_{p-1} x_p^n x_{p+n} \cdots \hat{x}_j \cdots x_k.$$  

Note how the $n$ terms indexed by $x_1 \cdots x_{p-1} x_p^{n-1} x_{p+n} \cdots x_k$ cancel to give a single term when $n$ is odd. Applying $h_{i-1}$ then gives zero on all terms except $-(-1)^{p} \lambda x_1 \cdots x_{p-1} x_p^{n-1} x_{p+n} \cdots x_k$, so that,

$$h_{i-1}d_i(\lambda x) = h_{i-1}(-(-1)^{p} \lambda x_1 \cdots x_{p-1} x_p^{n-1} x_{p+n} \cdots x_k)$$

$$= -(-1)^{p}(-1)^{p+1} \lambda x_1 \cdots x_{p-1} x_p^n x_{p+n} \cdots x_k = \lambda x,$$

resulting in $h_{i-1}d_i(\lambda x) + d_{i+1}h_i(\lambda x) = \lambda x + d_{i+1}(0) = \lambda x.$
The case: $n$ even. We compute,
\[
h_{i-1}d_i(\lambda x) = (-1)^p \sum_{j=2}^{p-1} (-1)^j \lambda x_1 \cdots \hat{x}_j \cdots x_{p-1} x_p^{n+1} x_{p+n} \cdots x_k
\]
\[
- (-1)^p \sum_{j=2}^{p-1} (-1)^j \lambda x_1 \cdots \hat{x}_j \cdots x_{p-1} x_p^{n+1} x_{p+n} \cdots x_k
\]
\[
- 0 - (-1)^{p+1} \sum_{j=p+n}^{k} (-1)^j \lambda x_1 \cdots \hat{x}_j \cdots x_{p-1} x_p^{n+1} x_{p+n} \cdots x_k.
\]

We also have
\[
d_{i+1}h_i(\lambda x) = d_{i+1}((-1)^{p+1} \lambda x_1 \cdots x_{p-1} x_p^{n+1} x_{p+n} \cdots x_k)
\]
\[
= (-1)^{p+1} \sum_{j=2}^{p-1} (-1)^j \lambda x_1 \cdots \hat{x}_j \cdots x_{p-1} x_p^{n+1} x_{p+n} \cdots x_k
\]
\[
- (-1)^{p+1} (-1)^p \lambda x_1 \cdots x_{p-1} x_p^{n} x_{p+n} \cdots x_k
\]
\[
- (-1)^{p+1} \sum_{j=p+n}^{k} (-1)^j \lambda x_1 \cdots \hat{x}_j \cdots x_{p-1} x_p^{n+1} x_{p+n} \cdots x_k.
\]

Thus
\[
h_{i-1}d_i(\lambda x) + d_{i+1}h_i(\lambda x) = -(-1)^{p+1}(-1)^p \lambda x_1 \cdots x_{p-1} x_p^{n} x_{p+n} \cdots x_k = \lambda x,
\]
as required. \hfill \Box

Corollary 1. There is a homotopy equivalence of chain complexes $C_*(P,\mathcal{F}) \simeq S_*(P,\mathcal{F})$.

In particular, $H_n(P,\mathcal{F}) \cong H_n(C_*(P,\mathcal{F}))$, a form more amenable to calculation.

We now briefly elaborate on the connection with traditional (uncoloured) poset homology and in particular the assertion that we have a local coefficient system on the order complex. An abstract simplicial complex may be viewed as a category with objects $\Delta$ and a unique morphism $\sigma \to \tau$ whenever $\tau \subset \sigma$. A system of local coefficients on $\Delta$ is a (covariant) functor $\Delta \to \text{Mod}_R$ (cf \[\S\text{2.4}]). One can form the chain complex $B_*(\Delta,\mathcal{F})$ with
\[
B_k = \bigoplus_{\sigma \in \Delta_k} \mathcal{F}(\sigma),
\]
the direct sum over the $k$-simplicies. If $\sigma = (x_0 < \cdots < x_k)$ is one such and $\sigma_j = (x_0 < \cdots < \hat{x}_j < \cdots < x_k)$, then the differential is
\[
d(\lambda \sigma) = \sum_{j=0}^{k} (-1)^j \mathcal{F}(\sigma \to \sigma_j)(\lambda)\sigma_j.
\]

If $\mathcal{F}$ is a constant system of local coefficients, $\mathcal{F}(\sigma) = A$ for all $\sigma \in \Delta$ and some $A \in \text{Mod}_R$, and $\mathcal{F}(\sigma \to \sigma_j) = \text{id}_A$, then this complex is the one appearing in traditional poset topology: if $\Delta$ is the Folkman complex of $P$ (i.e. the order complex of $P \setminus 1$) then its homology is the order homology of $P$ with coefficients in $A$.

There is an augmented version $\widetilde{B}_*(\Delta,\mathcal{F})$ with $\widetilde{B}_k = B_k$ for $k \geq 0$, and $\widetilde{B}_{-1} = \mathcal{F}(\emptyset)$. The extended differential $d : \widetilde{B}_0 \to \widetilde{B}_{-1}$ is given by the augmentation $d(\lambda \sigma) = \mathcal{F}(\sigma \to \emptyset)(\lambda)$.

Now if $P$ is a poset with $1$ and $\mathcal{F} : P \to \text{Mod}_R$ a colouring, then we get a system of local coefficients on the order complex $\mathcal{F}_P : \Delta(P) \to \text{Mod}_R$ given by $\mathcal{F}_P(\sigma) = \mathcal{F}(x_0)$ when $\sigma = (x_0 < \cdots < x_k)$, and $\mathcal{F}_P(\emptyset) = \mathcal{F}(1)$. If $\sigma \to \sigma_j$ is a morphism in $\Delta(P)$ where $\sigma_j = (x_0 < \cdots < \hat{x}_j < \cdots < x_k)$, then $\mathcal{F}_P(\sigma \to \sigma_j) = \mathcal{F}(x_{j+1})$ when $j = 0$, and is the identity otherwise. We may restrict $\mathcal{F}_P$ to a system of local coefficients on the subcomplex $\Delta(P \setminus 1)$, and in doing so we get the explicit connection we are looking for.
**Proposition 2.** \( \mathcal{C}_s(P, \mathcal{F}) = \tilde{\mathcal{B}}_{s-1}(\Delta(P \setminus 1), \mathcal{F}_P) \).

The proof is just of matter of unraveling the various definitions. The advantage of this formulation is that to a certain extent it allows us to appeal to the existing theory of lattice homology. For example, if \( P \) is a poset with 0, then the order complex \( \Delta(P) \) is a cone on \( \Delta(P \setminus 0) \), and so the (reduced) order homology of an uncoloured poset with 0 is trivial. Indeed, the same happens in the coloured case when the colouring is constant:

**Example 8.** Let \( P \) be a poset with minimal element \( 0_P \) and \( \mathcal{F} \) the constant colouring by the \( R \)-module \( A \). Then \( \mathcal{C}_s(P, \mathcal{F}) \) is acyclic i.e. \( H_n(P, \mathcal{F}) = 0 \) for all \( n \). This follows immediately from the above and Proposition 2 together with a little care in degrees zero and one.

**Example 9.** If \( (P_1, \mathcal{F}_1) \) and \( (P_2, \mathcal{F}_2) \) are coloured posets then there is a decomposition of complexes

\[
S_s(P_1 \cup P_2, \mathcal{F}_1 \cup \mathcal{F}_2) \cong S_s(P_1, \mathcal{F}_1) \oplus S_s(P_2, \mathcal{F}_2),
\]

inducing an isomorphism,

\[
H_s((P_1, \mathcal{F}_1) \cup (P_2, \mathcal{F}_2)) \xrightarrow{\cong} H_s(P_1, \mathcal{F}_1) \oplus H_s(P_2, \mathcal{F}_2).
\]

The essential point here is that elements of \( P_1 \) and elements of \( P_2 \) are incomparable in \( P_1 \cup P_2 \) so an ordered sequence \( x \) in \( (P_1 \cup P_2) \setminus 1 \) is either completely in \( P_1 \setminus 1 \) or completely in \( P_2 \setminus 1 \). Moreover the differential respects this splitting.

**Cohomology.** By defining

\[
S^s(P, \mathcal{F}) = \text{Hom}_R(S_s(P, \mathcal{F}), R)
\]

one can define cohomology \( H^s(P, \mathcal{F}) \) as the homology of the resulting cochain complex. If \( R \) is a field then the universal coefficient theorem gives an isomorphism \( H^s(P, \mathcal{F}) \cong H_s(P, \mathcal{F}) \).

**Example 10.** Let \( P \) be graded of rank \( n \) with both a 0 and a 1. Let \( P^{\text{op}} \) be the opposite poset defined by \( x \leq y \) in \( P^{\text{op}} \) if and only if \( y \leq x \) in \( P \). If we consider \( P \) as a category, \( P^{\text{op}} \) is simply the opposite category. Since \( P \) has a 0, \( P^{\text{op}} \) has a 1. If we have a colouring functor \( \mathcal{F} : P \to \text{Mod}_R \) then by composing this with the functor \( (\cdot)^\vee : \text{Mod}_R \to \text{Mod}_R \) taking a module \( A \) to its dual \( \text{Hom}_R(A, R) \), we get a contravariant functor \( P \to \text{Mod}_R \). Equivalently, we have a covariant functor \( P^{\text{op}} \to \text{Mod}_R \), or in other words a colouring \( \mathcal{F}^\vee \) of \( P^{\text{op}} \). Explicitly, \( \mathcal{F}^\vee(x) = \mathcal{F}(x)^\vee = \text{Hom}_R(\mathcal{F}(x), R) \) and for \( g \in \text{Hom}_R(\mathcal{F}(x), R) \) we have \( \mathcal{F}^\vee(x < y)(g) = g \circ \mathcal{F}(y < x) \).

In this situation we have the following duality result

\[
H_k(P^{\text{op}}, \mathcal{F}^\vee) \cong H^{n-k}(P, \mathcal{F}),
\]

seen by observing that

\[
S^{n-k}(P, \mathcal{F}) = \text{Hom}_R(S_{n-k}(P, \mathcal{F}), R) = \text{Hom}_R\left( \bigoplus_{x} \mathcal{F}(x_1), R \right)
\]

\[
= \bigoplus_{x} \text{Hom}_R(\mathcal{F}(x_1), R) = \bigoplus_{x} \mathcal{F}(x_1)^\vee = S_k(P^{\text{op}}, \mathcal{F}^\vee).
\]

3. A long exact sequence for the poset obtained by gluing along a morphism

In this section we show that a map \((f, \tau) : (P_1, \mathcal{F}_1) \to (P_2, \mathcal{F}_2)\) of coloured posets yields a long exact sequence in homology for the coloured poset \( P_1 \cup f \) \( P_2 \) obtained by gluing along the morphism. The main spin-off occurs when we focus on Boolean lattices, where the decomposition of Example 7 yields a long exact sequence in the homology of the three ingredients. This is the main technical tool needed to show that for a Boolean lattice, the coloured poset homology defined in the last section agrees with Khovanov’s cube homology which we discuss in Section 4.
Given coloured posets \((P_1, \mathcal{F}_1)\) and \((P_2, \mathcal{F}_2)\) and a morphism \((f, \tau) : (P_1, \mathcal{F}_1) \rightarrow (P_2, \mathcal{F}_2)\) we can form the three complexes \(\mathcal{C}_*(P_i, \mathcal{F}_i)\) for \(i = 0, 1\) and \(\mathcal{C}_*(P_1 \cup_f P_2, \mathcal{F})\). It is clear that \(\mathcal{C}_*(P_2, \mathcal{F}_2)\) is a sub-module of \(\mathcal{C}_*(P_1 \cup_f P_2, \mathcal{F})\), but also, one easily checks that \(d(\mathcal{C}_*(P_2, \mathcal{F}_2)) \subset \mathcal{C}_*(P_2, \mathcal{F}_2)\), and so there is a short exact sequence of complexes

\[
0 \rightarrow \mathcal{C}_*(P_2, \mathcal{F}_2) \xrightarrow{i} \mathcal{C}_*(P_1 \cup_f P_2, \mathcal{F}) \xrightarrow{q} Q_* \rightarrow 0,
\]

where by definition, \(Q_*\) is the quotient. This yields a long exact sequence in homology,

\[
\cdots \rightarrow H_n(P_2, \mathcal{F}_2) \xrightarrow{i_*} H_n(P_1 \cup_f P_2, \mathcal{F}) \xrightarrow{q_*} H_n(Q_*) \xrightarrow{\delta} H_{n-1}(P_2, \mathcal{F}_2) \xrightarrow{i_*} \cdots
\]

The \(n\)-chain module \(Q_n\) of the quotient complex is isomorphic to

\[
\bigoplus_{\mathfrak{x}} \mathcal{F}(x_1),
\]

the direct sum over those sequences \(\mathfrak{x}\) in \(P\) not entirely contained in \(P_2\), i.e. \(\mathfrak{x} = x_1 \ldots x_n\) with \(x_i \in P_1\) or \(\mathfrak{x} = x_1 \ldots x_j y_1 \ldots y_{n-j}\) where \(0 < j < n\) and the \(x_i \in P_1\), \(y_i \in P_2 \setminus 1\). We will write \(\mathfrak{x} = x_1 \ldots x_j y_1 \ldots y_{n-j}\) for the generic sequence., with the understanding that \(\mathfrak{x} = x_1 \ldots x_n\) when \(j = n\). The differential is given by \(d(\lambda \mathfrak{x}) = \alpha_j + \beta_j\), where \(\alpha_1 = 0\) and

\[
\alpha_j = \mathcal{F}^{x_2}_{x_1}(\lambda)x_2 \ldots x_j y_1 \ldots y_{n-j} + \sum_{k=2}^{j} (-1)^{k-1} \lambda x_1 \ldots \hat{x}_k \ldots x_j y_1 \ldots y_{n-j},
\]

for \(j > 1\), and \(\beta_0 = 0\), and

\[
\beta_j = \sum_{k=1}^{n-j} (-1)^{j+k-1} \lambda x_1 \ldots x_j y_1 \ldots \hat{y}_k \ldots y_{n-j},
\]

for \(j < n\).

We now define \(\pi_n : Q_n \rightarrow \mathcal{C}_{n-1}(P_1, \mathcal{F}_1)\) by

\[
\pi(\lambda x_1 \ldots x_j y_1 \ldots y_{n-j}) = \begin{cases} 
\lambda x_1 \ldots x_{n-1}, & \text{if } j = n \text{ and } x_n = 1_{P_1}, \\
0, & \text{otherwise.}
\end{cases}
\]

It is routine to check that,

**Lemma 5.** \(\pi : Q_* \rightarrow \mathcal{C}_{*}(P_1, \mathcal{F}_1)\) is a chain map.

In fact while \(Q_*\) is \textit{a priori} a great deal bigger than \(\mathcal{C}_*(P_1, \mathcal{F}_1)\), it turns out to contain a number of acyclic subcomplexes, allowing us to establish an isomorphism \(H_n(Q_*) \cong H_{n-1}(P_1, \mathcal{F}_1)\).

**Proposition 3.** The induced map \(\pi_* : H_n(Q_*) \rightarrow H_{n-1}(P_1, \mathcal{F}_1)\) is an isomorphism.

Delaying the proof of this momentarily, we now combine this proposition with the long exact sequence (4) to obtain the main result of this section.

**Theorem 1.** Let \((P_i, \mathcal{F}_i), i = 1, 2\), be coloured posets and \((f, \tau) : (P_1, \mathcal{F}_1) \rightarrow (P_2, \mathcal{F}_2)\) a morphism of coloured posets. Then there is a long exact sequence,

\[
\cdots \rightarrow H_n(P_2, \mathcal{F}_2) \xrightarrow{i_*} H_n(P_1 \cup_f P_2, \mathcal{F}) \xrightarrow{(\pi q)_*} H_{n-1}(P_1, \mathcal{F}_1) \xrightarrow{\delta \pi^{-1}_*} H_{n-1}(P_2, \mathcal{F}_2) \xrightarrow{i_*} \cdots
\]

where \((P_1 \cup_f P_2, \mathcal{F})\) is the coloured poset obtained by gluing along the morphism.
Before proving Proposition 3, we introduce a number of auxiliary complexes that play a role in the analysis of the homology of $Q_s$. For $p > 0$, fix $x = x_1 \ldots x_p$ in $P_1$, and define a complex $A^x_s$ by setting

$$A^x_q = \bigoplus_{x_{y_1} \ldots y_q} \mathcal{F}(x_1),$$

ie: the direct sum over those $x_1 \ldots x_p y_1 \ldots y_q$ where the $x$'s are fixed and the $y$'s in $P_2$ are allowed to vary. Let $A^x_0 = \mathcal{F}(x_1)$. The differential is given by

$$d(x_1 \ldots x_p y_1 \ldots y_q) = \sum_{k=1}^q (-1)^{p+k-1} \lambda x_1 \ldots \hat{y_k} \ldots y_q,$$

and $d(\lambda x y_1 \ldots y_q) \mapsto \lambda \in A^x_0$.

Let $x \in P_1$ and $P^x$ those $y \in P_2$ with $x < y$ in $P$. Then is is easy to see that $P^x$ is a subposet of $P$ with unique minimal element $f(x)$. Let $\varepsilon_p = 1$ when $p$ is even and $\varepsilon_p = (-1)^{p+q}$ when $p$ is odd.

**Lemma 6.** The map $\lambda x y_1 \ldots y_q \mapsto \varepsilon_p \lambda y_1 \ldots y_q$ is an isomorphism of complexes

$$A^x_s \cong \mathcal{C}_s(P^x, \mathcal{F}_x),$$

where $x = x_1 \ldots x_p$ and $\mathcal{F}_x$ is the constant colouring $\mathcal{F}_x(x) = \mathcal{F}(x_1)$. In particular, $A^x_s$ is acyclic.

The proof is elementary, and no doubt the reader can provide the details by scrutinizing the picture,

while keeping a close eye on the signage. Acyclic-ness follows from Example 8.

**Proof (of Proposition 3).** Let $A_n \subset Q_n$ be the direct sum $\oplus_x \mathcal{F}(x_1)$ over those $x = x_1 \ldots x_{n-1} x_n$ where $x_n = 1_n := 1 P_1$, the unique maximal element in $P_1$, and decompose (as modules),

$$Q_n = A_n \oplus D_n.$$  

Thus, $D_n$ is the direct sum over those $x$ in $Q_n$ not finishing at $1_1$. It is readily checked that $d(D_n) \subset D_{n-1}$, giving that $(D_s, d)$ is a subcomplex of $(Q_s, d)$.

We now show that $D_s$ is acyclic i.e. $H_s(D_s) = 0$ for all $n$. We show this by filtering $D_s$ and analyzing the associated spectral sequence. Let $p > 0$, and set

$$F_p D_n = \bigoplus_x \mathcal{F}(x_1),$$

the direct sum over those $x = x_1 \ldots x_j y_1 \ldots y_{n-j}$ where now $1 \leq j \leq p$ (and as usual, the $x_i \in P_1$ and the $y_i \in P_2 \setminus 1$). Set $F_p D_n = 0$ for $p < 1$. Thus, $F_p D_n$ consists of those modules indexed by sequences of length $n$ which have exited $P_1 \subset P$ after the $p$-th element.

It is easy to check that $F_p D_s$ is a sub-complex of $D_s$, yielding for each $n$ a bounded filtration,

$$0 = F_0 D_n \subset F_1 D_n \subset \cdots \subset F_{p-1} D_n \subset F_p D_n \subset F_{p+1} D_n \subset \cdots \subset F_n D_n = D_n.$$  

This gives rise to a first quadrant spectral sequence converging to $H_s D$. The $E^0$-page is

$$E^0_{p, q} = \frac{F_p D_{p+q}}{F_{p-1} D_{p+q}} \bigoplus_{x_1 \ldots x_p y_1 \ldots y_q} \mathcal{F}(x_1),$$


with differential $d^0 : E^0_{p,q} \to E^0_{p,q-1}$ defined by
\[
d^0(\lambda x_1 \ldots x_p y_1 \ldots y_q) = \sum_{k=1}^q (-1)^{p+k-1} \lambda x_1 \ldots x_p y_1 \ldots \hat{y}_k \ldots y_q.
\]
Note that $E^0_{0,q} = 0$ for all $q$, and thus $E^n_{0,q} = 0$. To compute the $E^1$-page we fix $p$ and consider separately the cases $q > 0$ and $q = 0$.

**The case** $q > 0$. We show that any cycle in $E^0_{p,q}$ is also a boundary, and thus $E^1_{p,q} = 0$. Let
\[
\sigma = \sum_i \lambda_i x_{i,1} \ldots x_{i,p} y_{i,1} \ldots y_{i,q}
\]
be a general element of $E^0_{p,q}$. Let $\mathbf{x} = x_1 \ldots x_p$ be a fixed sequence in $P_1$
\[
\sigma^\mathbf{x} = \sum_j \lambda \mathbf{x} y_{j,1} \ldots y_{j,q}
\]
the sum of those terms in $\sigma$ with $x_{i,1} \ldots x_{i,p} = \mathbf{x}$. Then $\sigma = \sum_{\mathbf{x}} \sigma^\mathbf{x}$, the sum over those $\mathbf{x}$ appearing as initial segments in $\sigma$, and
\[
d^0 \sigma = \sum_{\mathbf{x}} d^0 \sigma^\mathbf{x}.
\]
Thus, if $A^x_s$ is the complex defined immediately prior to Lemma 6, now a subcomplex of $E^0_{p,s}$, then $\sigma^\mathbf{x} \in A^x_s \subseteq E^0_{p,s}$, and $d^0 \sigma^\mathbf{x} \in A^x_{s-1} \subseteq E^0_{p,s-1}$. Also, if $\mathbf{x} \neq \mathbf{w}$ then $A^x_s \cap A^w_s = \{ 0 \}$ as subcomplexes of $E^0_{p,s}$, and so $\sigma$ is a cycle if and only if each $\sigma^\mathbf{x}$ is a cycle. But the $A^x_s$ are acyclic, so $\sigma^\mathbf{x} = d^0 \tau^\mathbf{x}$ for some $\tau^\mathbf{x} \in A^x_{s+1} \subseteq E^0_{p,s+1}$, giving $\sigma = d^0(\sum \tau^\mathbf{x})$, and thus $E^1_{p,q} = 0$ as claimed.

**The case** $q = 0$. Here $d^0 = 0$ and so the cycles are all of $E^0_{p,0} = \bigoplus_{\mathbf{x}} \mathcal{F}(x_1)$, where $\mathbf{x} = x_1 \ldots x_p$ with $x_p \neq 1$. We show that $d^0 : E^0_{p,1} \to E^0_{p,0}$ is onto and conclude that $E^1_{p,0} = 0$. If $\lambda x_1 \ldots x_p$ is an element with $x_p \neq 1$, then $f(x_p) \in P_2$ is $\neq 1$ and $x_p < f(x_p)$. We then have $d^0(\lambda x_1 \ldots x_p f(x_p)) = \lambda x_1 \ldots x_p$ as required.

Thus the $E^1$-page of the spectral sequence is entirely trivial, so that in the induced filtration of $H_*(D)$,
\[
\cdots \subset F_{p-1}H_n(D_s) \subset F_pH_n(D_s) \subset F_{p+1}H_n(D_s) \subset \cdots \subset H_n(D_s),
\]
we have trivial quotients. Thus $F_{p-1}H_n = F_pH_n$ for all $p$ and $n$. As $F_0H_nD = 0$, we conclude that $H_n(D_s) = 0$ as claimed.

To finish the proof observe that there is a short exact sequence
\[
\begin{array}{ccccccc}
0 & \rightarrow & D_* & \rightarrow & Q_* & \pi & \rightarrow & A_* & \rightarrow & 0
\end{array}
\]
whose associated homology long exact sequence, together with the acyclic-ness of $D_*$, gives that the quotient map $\pi : Q_* \to A_*$ induces isomorphisms $\pi_* : H_n(Q_*) \to H_n(A_*)$. Now, $A_n = \bigoplus_{x_1, \ldots, x_n} \mathcal{F}(x_1)$ with $x_n = 1$, and thus the complex $A_*$ can be identified with $C_{s-1}(P_1, \mathcal{F}_1)$. Under this identification the map $\pi$ above is the map $\pi : Q_* \to C_{s-1}(P_1, \mathcal{F}_1)$ of complexes defined in (5), finishing the proof. \qed

Let $(\mathbb{B}, \mathcal{F})$ be a coloured Boolean lattice of rank $r$ and $\mathbb{B} = \mathbb{B}_0 \cup \mathbb{B}_1$ a decomposition of the form given in Example 7.

**Corollary 2.** There is a long exact sequence
\[
\begin{array}{ccccccc}
\cdots & \rightarrow & H_n(\mathbb{B}_1, \mathcal{F}_1) & \overset{i_*}{\rightarrow} & H_n(\mathbb{B}, \mathcal{F}) & \overset{(\pi \delta)}{\rightarrow} & H_{n-1}(\mathbb{B}_0, \mathcal{F}_0) & \overset{\delta \pi_*^{-1}}{\rightarrow} & H_{n-1}(\mathbb{B}_1, \mathcal{F}_1) & \rightarrow & \cdots
\end{array}
\]
4. The cube complex of a Boolean lattice and its homology

We now recall a construction, first due to Khovanov [6], of a complex from a coloured Boolean lattice. It is central to the definition of the Khovanov homology of a link and is used in one of the recent combinatorial formulations of Heegaard-Floer knot homology. The reader should be aware that we are grading everything homologically, whereas in the applications cited above it is traditional to use cohomological conventions.

Let \( \mathbb{B} \) be a Boolean lattice of rank \( r \) with ordered atoms \( a_1, \ldots, a_r \), and colouring \( \mathcal{F} : \mathbb{B} \to \text{Mod}_{\mathbb{R}} \), and recall the unique expression (1) for an element of \( \mathbb{B} \) as a join of the \( a_i \) (this replaces the conventions in earlier, non-lattice oriented, literature on Khovanov homology, where the elements of \( \mathbb{B} \) were \( r \)-strings of 0’s and 1’s, and the atoms those \( r \)-strings containing a single 1). Write \( 1 := 1_{\mathbb{B}} \), the join of all the atoms.

If \( x <_c y \), then let \( \varepsilon(x <_c y) = (-1)^j \) where \( j \) is the number of atoms appearing before \( a_i \) in the unique expression for \( y \) (see (1) and the comments following it). If \( x = x_1 <_c x_2 <_c \cdots <_c x_k \) is a saturated sequence in \( \mathbb{B} \), let

\[
\varepsilon_x = \varepsilon(x_1 \ldots x_k) = \varepsilon(x_1 <_c x_2 <_c \cdots <_c x_k) = \prod \varepsilon(x_i <_c x_{i+1}).
\]

If \( 1_0 = a_2 \lor \cdots \lor a_r \), then observe that \( \varepsilon(1_0 <_c 1) = 1 \).

Khovanov’s cube complex \( \mathcal{K}_*(\mathbb{B}, \mathcal{F}) \) is then defined to have chain modules,

\[
\mathcal{K}_k = \bigoplus_{r \varepsilon x = r-k} \mathcal{F}(x),
\]

and differential \( d_k : \mathcal{K}_k(\mathbb{B}, \mathcal{F}) \to \mathcal{K}_{k-1}(\mathbb{B}, \mathcal{F}) \),

\[
d_k(\lambda) = \sum \varepsilon(x <_c y)\mathcal{F}_x^y(\lambda),
\]

where \( \lambda \in \mathcal{F}(x) \) with \( r \varepsilon x = r - k \), and the sum is over all \( y \) covering \( x \). Thus, \( d(\mathcal{K}_k) \subset \mathcal{K}_{k-1} \) with \( d = \sum r \varepsilon x = r-k \varepsilon(x <_c y)\mathcal{F}_x^y \). Observe that in degree zero the chains are just \( \mathcal{F}(1) \), in degree \( r \) they are \( \mathcal{F}(0) \), and \( \mathcal{K}_k \) is 0 outside of the range \( 0 \leq k \leq r \). To see that \( d \) is a differential, observe that if \( x <_c z <_c y \) in \( \mathbb{B} \), then there is a unique \( z' \) with \( x <_c z' <_c y \), and that \( \varepsilon(x <_c z <_c y) = \varepsilon(x <_c z' <_c y) \), i.e. consecutive edges of the Hasse diagram for a Boolean lattice can always be completed to form a square in a unique way, and all squares anticommute. As \( d \) is a sum over such squares we get \( d^2 = 0 \).

Write \( H_*^\circ(\mathbb{B}, \mathcal{F}) = H_*(\mathcal{K}_*(\mathbb{B}, \mathcal{F})) \) for the homology of the cube complex. It should be noted that \( H^\circ(-) \) is not natural with respect to morphisms of coloured Boolean lattices in general. It is, however, natural with respect to morphisms \( (f, \tau) \) for which \( f \) is a co-rank preserving injection.

The Khovanov homology of an oriented link diagram is defined as (a normalised version of) the homology of the cube complex associated to the coloured Boolean lattice defined in Example 4. A small class of (graded) Frobenius algebras result in a homology theory that is invariant under Reidemeister moves of diagrams, thus giving a genuine invariant of links. The reader wishing to make this precise should be warned that here our homological grading conventions conflict with Khovanov’s cohomological ones, and so care is needed (see 6).

Similarly, the combinatorial interpretation of Heegaard-Floer knot homology is defined as the homology of the cube complex associated to the coloured Boolean lattice of Example 5.

The decomposition \( \mathbb{B} = \mathbb{B}_0 \cup_f \mathbb{B}_1 \) of Example 7 yields a long exact sequence similar to that obtained for coloured poset homology described in the last section. As the \( \mathbb{B}_i \) \( (i = 0, 1) \) are Boolean of rank \( r - 1 \), we may form the associated cube complexes \( \mathcal{K}_*(\mathbb{B}_i, \mathcal{F}_i) \) where \( \mathcal{F}_i \) is the restriction of \( \mathcal{F} \) to \( \mathbb{B}_i \). As with the complex \( \mathcal{C}_* \) in 8 \( \mathcal{K}_*(\mathbb{B}_1, \mathcal{F}_1) \) is a subcomplex of \( \mathcal{K}_*(\mathbb{B}, \mathcal{F}) \), but now the quotient is considerably simpler, for the map,

\[
\sum_{r \varepsilon x = r-k} \lambda_x = \sum_{x \varepsilon \mathbb{B}_0} \mu_x + \sum_{x \varepsilon \mathbb{B}_0} \nu_x \rightarrow \sum_{x \varepsilon \mathbb{B}_0} \nu_x,
\]
have the induced long exact sequence in homology, below is a quasi-isomorphism, yielding isomorphisms, than the subcomplex, as we are grading

diagrams

Theorem 2. the following, whose proof appears at the end of the section.

Fig. 4. The cube complex $\mathcal{K}_s(\mathbb{B}; \mathcal{F})$ for the Boolean lattice of rank 3 (after Bar-Natan [1]). The join $a_i \vee a_j$ has been abbreviated $a_{ij}$. The edges $x <_c y$ of the Hasse diagram for $\mathbb{B}$ have been labelled with the Khovanov sign $\varepsilon(x <_c y)$.

gives an isomorphism of complexes, $\mathcal{K}_s(\mathbb{B}; \mathcal{F})/\mathcal{K}_s(\mathbb{B}_1; \mathcal{F}_1) \to \mathcal{K}_{s-1}(\mathbb{B}_0; \mathcal{F}_0)$, and thus a short exact sequence,

$$0 \to \mathcal{K}_s(\mathbb{B}_1; \mathcal{F}_1) \to \mathcal{K}_s(\mathbb{B}; \mathcal{F}) \to \mathcal{K}_{s-1}(\mathbb{B}_0; \mathcal{F}_0) \to 0.$$ 

This sequence is well known, although the degree drop in our version happens in the quotient, rather than the subcomplex, as we are grading $\mathcal{K}_s$ homologically, rather than cohomologically. Finally, we have the induced long exact sequence in homology,

$$\cdots \to H^\circ_n(\mathbb{B}_1; \mathcal{F}_1) \to H^\circ_n(\mathbb{B}; \mathcal{F}) \to H^\circ_{n-1}(\mathbb{B}_0; \mathcal{F}_0) \to H^\circ_{n-1}(\mathbb{B}_1; \mathcal{F}_1) \to \cdots$$

In Khovanov homology for links, if $(\mathbb{B}; \mathcal{F})$ is the coloured Boolean lattice of a diagram $D$ (see Example 4) then $(\mathbb{B}_0; \mathcal{F}_0)$ and $(\mathbb{B}_1; \mathcal{F}_1)$ can be interpreted as the coloured lattices associated to diagrams $D_0$ and $D_1$ obtained from $D$ by resolving a chosen crossing in $D$ to a 0- and 1-smoothing respectively. In this case the above long exact sequence is a homological incarnation of the kind of skein relation found in the definition of certain knot polynomials.

5. A quasi-isomorphism

We now have two chain complexes, and their homologies, associated to a coloured Boolean lattice: the coloured poset homology $H_s(\mathbb{B}; \mathcal{F})$ of the complex $\mathcal{C}_s(\mathbb{B}; \mathcal{F})$ from [2] and the homology $H^\circ(\mathbb{B}; \mathcal{F})$ of the cube complex defined in §4. In this section we describe a chain map $\phi$ from the cube complex to $\mathcal{C}_s(\mathbb{B}; \mathcal{F})$, and show that it turns out to be a quasi-isomorphism. The main result is the following, whose proof appears at the end of the section.

**Theorem 2.** Let $(\mathbb{B}; \mathcal{F})$ be a coloured Boolean lattice. Then $\phi : \mathcal{K}_s(\mathbb{B}; \mathcal{F}) \to \mathcal{C}_s(\mathbb{B}; \mathcal{F})$ defined below is a quasi-isomorphism, yielding isomorphisms,

$$H^\circ_n(\mathbb{B}; \mathcal{F}) \xrightarrow{\cong} H_n(\mathbb{B}; \mathcal{F}).$$

We now define the map $\phi$. Let $\lambda \in \mathcal{F}(x)$ for $x \in \mathbb{B}$, and $x = x_1 <_c \cdots <_c x_k$ a saturated sequence in $\mathbb{B}$ from $x$ to $1$, i.e: with $x_1 = x$ and $x_k = 1$, and let $x^\circ = x_1 <_c \cdots <_c x_{k-1}$. Recalling the definition of $\varepsilon_x \in \{\pm 1\}$ from [4] set $\phi : \mathcal{K}_n(\mathbb{B}; \mathcal{F}) \to \mathcal{C}_n(\mathbb{B}; \mathcal{F})$ to be

$$\phi(\lambda) = \sum_x \varepsilon_x \lambda x^\circ,$$

the sum over all saturated sequences $x \in \mathbb{B}$ from $x$ to $1$. 

rank 3

[Diagram]

rank 2

rank 1

rank 0

deg 0

deg 1

deg 2

deg 3

$\mathcal{F}(1)$

$\mathcal{F}(a_{ij})$

$\mathcal{F}(a_i)$

$\mathcal{F}(0)$
Fig. 5. The inclusion chain map \( \phi : \mathcal{K}_s(\mathbb{B}, \mathcal{F}) \to \mathcal{C}_s(\mathbb{B}, \mathcal{F}) \): the Boolean lattice of rank 3 (left) is marked with the Khovanov signage \( \varepsilon(x < c \ y) \); the saturated chains \( x \) starting at \( x = 0 \) in this example and finishing at 1 are marked (middle) with the resulting \( \varepsilon_x \), and the image (right) of \( \lambda \in \mathcal{F}(x) \).

**Proposition 4.** \( \phi : \mathcal{K}_n(\mathbb{B}, \mathcal{F}) \to \mathcal{C}_n(\mathbb{B}, \mathcal{F}) \) is a chain map.

*Proof.* Is accomplished by a brute force comparison of the maps \( \phi d \) and \( d \phi \) (where the \( d \)'s are the differentials in \( \mathcal{K}_s \) and \( \mathcal{C}_s \) respectively). Let \( \lambda \in \mathcal{F}(x) \subset \mathcal{K}_n \), and \( x_1, \ldots, x_{r-n} \) be the \( r-n \) elements of \( \mathbb{B} \) covering \( x \). Then,

\[
\lambda \xrightarrow{d} \sum_{j=1}^{r-n} \varepsilon(x x_j) \mathcal{F}_x^j(\lambda) \xrightarrow{\phi} \sum_{j=1}^{r-n} \varepsilon(x x_j) \sum_{x} \varepsilon(x_j \ldots x_{j_1} < c 1) \mathcal{F}_x^j(\lambda) x_j \ldots x_{j_1},
\]

with the second summation over the saturated sequences \( x \) from \( j_1 \) to 1. On the other hand,

\[
\lambda \xrightarrow{\phi} \sum_{j=1}^{r-n} \varepsilon(x x_j) \sum_{x} \varepsilon(x_j \ldots x_{j_1} < c 1) \lambda x_j \ldots x_{j_1},
\]

with again the second sum over the saturated chains \( x \) from \( j_1 \) to 1. In the image of this under the differential \( d \) of the complex \( \mathcal{C}_s \), each of the \( r-n \) terms contributes a term of the form \( \varepsilon(x x_j) \sum_{x} \varepsilon(x_j \ldots x_{j_1} < c 1) \mathcal{F}_x^j(\lambda) x_j \ldots x_{j_1} \), obtained by dropping the \( x \) from the chain \( x x_j \ldots x_{j_1} \). All other terms have the form

\[
\varepsilon(x x_j) \varepsilon(x_j \ldots x_{j_1} < c 1)(-1)^k \lambda x_j \ldots x_{j_1},
\]

for \( j \leq k \leq j_1 \), and where \( \varepsilon(x x_j) \varepsilon(x_j \ldots x_{j_1} < c 1) = \varepsilon(x x_j) \ldots x_{j_1} < c 1 \). The proof is thus completed by showing that all these terms cancel. As already observed, for any chain \( x_{k-1} < c x_k < c x_{k+1} \) in \( \mathbb{B} \) there is a unique \( y_k \neq x_k \) with \( x_{k-1} < c y_k < c x_{k+1} \), and \( \varepsilon(x_{k-1} x_k x_{k+1}) = -\varepsilon(x_k y_k x_{k+1}) \). Thus, there is a matching term to (6), indexed by \( x x_j \ldots y_k \ldots x_{j_1} \), and otherwise identical in all respects except for having opposite sign. This completes the proof. \( \square \)

We now bring in the decomposition \( \mathbb{B} = \mathbb{B}_0 \cup \mathbb{B}_1 \) of the Boolean lattice of Example 7 for \( \ell = 1 \). Notice that if \( x \in \mathbb{B}_1 \) and \( \varepsilon \) is a sequence (saturated or not) starting at \( x \), then \( \varepsilon \) is completely contained in the sublattice \( \mathbb{B}_1 \). Thus in particular, when \( \lambda \in \mathcal{F}(x) \), we have that \( \phi(\lambda) \) is in the subcomplex \( \mathcal{C}_s(\mathbb{B}_1, \mathcal{F}_1) \subset \mathcal{C}_s(\mathbb{B}, \mathcal{F}) \), and so \( \phi \mathcal{K}_s(\mathbb{B}_1, \mathcal{F}_1) \subset \mathcal{C}_s(\mathbb{B}_1, \mathcal{F}_1) \). We therefore have an induced map of complexes,

\[
\phi' : \mathcal{K}_s(\mathbb{B}_0, \mathcal{F}_0) = \frac{\mathcal{K}_s(\mathbb{B}, \mathcal{F})}{\mathcal{K}_s(\mathbb{B}_1, \mathcal{F}_1)} \to \mathcal{C}_s(\mathbb{B}, \mathcal{F}) = \mathcal{C}_s(\mathbb{B}_1, \mathcal{F}_1) = Q_s.
\]

**Lemma 7.** Let \( \pi : Q_s \to \mathcal{C}_s(\mathbb{B}_0, \mathcal{F}_0) \) be the map defined in Section 3 by equation (5) and \( \phi, \phi' \) as above. Then the following diagram of chain maps commutes.

\[
\begin{array}{ccc}
\mathcal{K}_s(\mathbb{B}_0, \mathcal{F}_0) & \xrightarrow{\phi} & Q_s \\
\downarrow{\phi'} & & \\
\mathcal{C}_s(\mathbb{B}_0, \mathcal{F}_0) & \xrightarrow{\pi} & \mathcal{C}_s(\mathbb{B}_0, \mathcal{F}_0)
\end{array}
\]
Note that the $\phi$ that appears in the diagram is the $\phi$ associated to the sublattice $\mathbb{B}_0$ (not $\mathbb{B}$).

**Proof.** Let $x \in \mathbb{B}_0$ and $S$ be the set of all saturated sequences $x = x_1 \cdots x_j y_1 \cdots y_{n-j} = x_1 <_c \cdots <_c x_j <_c y_1 <_c \cdots <_c y_{n-j} <_c 1$ in $\mathbb{B}$ with the $x_i \in \mathbb{B}_0$, $y_i \in \mathbb{B}_1$ and $x_1 = x$. Let $S' \subset S$ consist of those saturated sequences of the form $x_1 \cdots x_n$, where the $x_i \in \mathbb{B}_0$, $x_1 = x$ and $x_n = 1$, the unique maximal element of $\mathbb{B}_0$. Then, for $\lambda \in \mathcal{F}(x)$ we have

$$
\lambda \mapsto \phi' \sum_{x \in S} \varepsilon_x \lambda^x \xrightarrow{\pi} \sum_{x \in S'} \varepsilon_x \lambda x_1 \cdots x_{n-1} .
$$

Now, the $\varepsilon_x$ that appears on the righthand side above satisfies $\varepsilon_x = \varepsilon(x_1 <_c \cdots <_c x_{n-1} <_c 10 <_c 1) = \varepsilon(x_1 <_c \cdots <_c 10)\varepsilon(10 <_c 1)$, which in turn is just $\varepsilon(x_1 <_c \cdots <_c 10)$, as $\varepsilon(10 <_c 1) = 1$.

In particular we have a commuting diagram in homology: $\phi_* = \pi_* \phi'_*$. We now have everything we need for the,

**Proof (of the Main Theorem).** The short exact sequences in [3] and [4] can be assembled into a diagram,

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{K}_s(\mathbb{B}_1, \mathcal{F}_1) & \longrightarrow & \mathcal{K}_s(\mathbb{B}, \mathcal{F}) & \longrightarrow & \mathcal{K}_{s-1}(\mathbb{B}_0, \mathcal{F}_0) & \longrightarrow & 0 \\
\phi & & \phi & & \phi & & \phi & & \\
0 & \longrightarrow & \mathcal{C}_s(\mathbb{B}_1, \mathcal{F}_1) & \longrightarrow & \mathcal{C}_s(\mathbb{B}, \mathcal{F}) & \longrightarrow & Q_* & \longrightarrow & 0
\end{array}
$$

where by definition, $\phi'$ is the map making the righthand square commute, while it is easy to check that the left-hand square commutes. By the functorality of the long exact sequence in homology, we have the following commutative diagram

$$
\begin{array}{cccccc}
\cdots & \longrightarrow & H_n^0(\mathbb{B}, \mathcal{F}_0) & \delta & H_n^0(\mathbb{B}_1, \mathcal{F}_1) & \longrightarrow & H_n^0(\mathbb{B}, \mathcal{F}) & \delta & H_n^0(\mathbb{B}_0, \mathcal{F}_0) & \delta & H_n^{s-1}(\mathbb{B}_1, \mathcal{F}_1) & \longrightarrow & \cdots \\
\phi_* & & \phi_* & & \phi_* & & \phi_* & & \phi_* & & \phi_* & & \\
\cdots & \longrightarrow & H_{n+1}(\mathcal{F}_0) & \delta & H_{n+1}(\mathbb{B}_1, \mathcal{F}_1) & \longrightarrow & H_n(\mathbb{B}, \mathcal{F}) & \delta & H_n(\mathcal{F}_0) & \delta & H_{n+1}(\mathbb{B}_1, \mathcal{F}_1) & \longrightarrow & \cdots
\end{array}
$$

with exact rows. The proof then proceeds by induction on the rank, noting that the result is obviously true for Boolean lattices of rank 1. If $\mathbb{B}$ is rank $r + 1$ then both $\mathbb{B}_0$ and $\mathbb{B}_1$ are rank $r$, so assuming the result for rank $r$ gives that the second and fifth vertical maps in the above diagram are isomorphisms. Furthermore, the first and fourth maps are also isomorphisms: Lemma[7] gives that the $\phi'_* = \pi^{-1}_* \phi_*$, where $\phi$ is again an isomorphism because $\mathbb{B}_0$ has rank $r$, and $\pi$ is an isomorphism by Proposition[3].

By the 5-lemma, the middle map is thus an isomorphism too.

The main theorem can be strengthened somewhat: if $P$ is a poset, then call $\mathcal{F} : P \rightarrow \text{Mod}_R$ a **colouring by projectives** if $\mathcal{F}(x)$ is a projective module for all $x \in P$. As the direct sum of projectives is projective, and a quasi-isomorphism between bounded below chain complexes of projectives is a homotopy equivalence (see, eg: [12], §10.4)), we get that,

**Corollary 3.** If $\mathcal{F}$ is a colouring by projectives then $\phi : \mathcal{K}_n(\mathbb{B}, \mathcal{F}) \rightarrow \mathcal{C}_n(\mathbb{B}, \mathcal{F})$ is a homotopy equivalence.

In particular, as vector spaces are projective we have,

**Corollary 4.** If the ground ring $R$ of the colouring $\mathcal{F} : \mathbb{B} \rightarrow \text{Mod}_R$ is a field, then $\phi : \mathcal{K}_n(\mathbb{B}, \mathcal{F}) \rightarrow \mathcal{C}_n(\mathbb{B}, \mathcal{F})$ is a homotopy equivalence.
6. Normalisation for link homology

For the motivating example, namely the Khovanov colouring of a Boolean lattice associated to a link diagram, the modules are in fact graded and in order to obtain an invariant result some shifts are required. We record these shifts here in order to minimize the potential confusion arising from our grading conventions.

Let $V$ be the graded Frobenius algebra used in the construction of Khovanov homology and let $(\mathbb{B}, \mathcal{F})$ be the Boolean lattice associated to a given link diagram coloured with the Khovanov colouring of Example [4]. The grading on $V$ induces an internal grading on the associated complex. Using the convention that $(W_{a,b})_{i,j} = W_{-a,-b}$, the shifted complex we wish to consider is $\tilde{S}_{*,*}(\mathbb{B}, \mathcal{F}) = S_{*,*}(\mathbb{B}, \mathcal{F})[-n_+, n_+ - 2n_-]$, i.e.

$$\tilde{s}_{i,j}(\mathbb{B}, \mathcal{F}) = s_{i+n_+ - n_- + 2n_-}(\mathbb{B}, \mathcal{F})$$

where $n_+$ and $n_-$ are the number of positive and negative crossings of the (oriented) diagram. The homology $\tilde{H}_{*,*}(\mathbb{B}, \mathcal{F})$ is then a bigraded link invariant. To compare with the more usual grading in Khovanov homology we have $KH^{i,j} \cong \tilde{H}_{-i,-j}$.

References

[1] Dror Bar-Natan, On Khovanov’s categorification of the Jones polynomial, Algebr. Geom. Topol. 2 (2002), 337–370 (electronic).MR1917056 (2003h:57014)

[2] A. Björner, Topological methods, Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 1819–1872.MR1373690 (96m:52012)

[3] Charles W. Curtis and Irving Reiner, Methods of representation theory. Vol. II, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1987. With applications to finite groups and orders; A Wiley-Interscience Publication.MR892316 (88f:20002)

[4] Laure Helme-Guizon and Yongwu Rong, A categorification for the chromatic polynomial, Algebr. Geom. Topol. 5 (2005), 1365–1388 (electronic).MR2171813 (2006g:57020)

[5] S. I. Gelfand and Yu. I. Manin, Homological algebra, Springer-Verlag, Berlin, 1999. Translated from the 1989 Russian original by the authors; Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [Algebra, V. Encyclopaedia Math. Sci., 38, Springer, Berlin, 1994; MR1309679 (95g:18007)].MR1698374 (2000b:18016)

[6] Mikhail Khovanov, A categorification of the Jones polynomial, Duke Math. J. 101 (2000), no. 3, 359–426.MR1740682 (2002j:57025)

[7] Peter Orlik and Hiroaki Terao, Arrangements of hyperplanes, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992.MR1217488 (94e:52014)

[8] Peter Ozsváth and Zoltán Szabó, A cube of resolutions for knot Floer homology, available at arXiv:math.SG/07053852

[9] , Holomorphic disks and topological invariants for closed three-manifolds, Ann. of Math. (2) 159 (2004), no. 3, 1027–1158.MR2113019 (2006b:57016)

[10] , Holomorphic disks and knot invariants, Adv. Math. 186 (2004), no. 1, 58–116.MR2065507 (2005e:57044)

[11] Richard P. Stanley, Enumerative combinatorics. Vol. 1, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota; Corrected reprint of the 1986 original.MR1442260 (98a:05001)

[12] Charles A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.MR1269324 (95f:18001)