Infinitely many Equilibria and Some Codimension One Bifurcations in a Subsystem of a Two-Preys One-Predator Dynamical System

M Marwan¹² and J M Tuwankotta¹

¹Analysis and Geometry Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10, Bandung, 40132, Indonesia
²Mathematics Study Program, Faculty of Mathematics and Natural Sciences, Universitas Mataram, Indonesia

marwanmath@yahoo.co.id

Abstract. In this paper we study a two dimensional dynamical system depending on four parameters. This is a subsystem of a three dimensional system of two-preys one-predator system. Our focus is in the dynamics of the system and bifurcations with respect to the variation on the mortality rate of the predator. Interesting co-dimension one bifurcation such as transcritical and Hopf bifurcation have been observed. Furthermore, the system exhibit the existence of infinitely many equilibria as the bifurcation parameter become zero.

1. Introduction

In [6], [7] and [8] a type of dynamical systems known as the Predator-Prey system is considered. The author there consider the situation where the response function is of non-monotonic nature (following the study of [9]). Another type of Predator-Prey which used a different response function is considered in [3].

In this paper, we consider an extension of the Predator-Prey system which is two dimensional in the previously described studies, to three-dimensional by considering Two-Preys and One-Predator systems. This system is introduced in [4] and [5]. It is assumed that the two-preys population comes from the same species but different in age class: the young prey and the adult prey. We also assume different response function for each of these classes, i.e. the one considered in [3] and the one which is considered in [9].

The focus of this paper is to describe the dynamics and bifurcations in a subsystem of the three dimensional system. This subsystem is achieved by restricting the original system on a special plane which is invariant under a certain condition.

2. Setting of the problem

Let us consider \( \mathbb{R}^2 \) with coordinate: \((x, z)\) and a system of two ordinary differential equations:
where \( k_i, \mu, c \) and \( \delta \) are real valued parameters. This system is derived from the so-called Two Preys-One Predator dynamical system:

\[
\begin{align*}
\dot{x} &= x - \frac{1}{k_i} x^2 - \frac{\mu x z}{cx + 1} \\
\dot{y} &= \alpha x - \beta y - \frac{1}{k_z} y^2 - \frac{\eta y z}{ay^2 + by + 1} \\
\dot{z} &= -\delta z - \frac{x z}{cx + 1} - \frac{y z}{ay^2 + by + 1}
\end{align*}
\]  

Here \( x \) and \( y \) denote the densities of the prey populations, and \( z \) denotes the density of predator population. We assume that both prey populations come from the same species. However they are classified into two age classes: the young prey \( x \) and the adult prey \( y \). The parameter \( \alpha \) measures the rate of migration from young prey \( x \) into adult prey \( y \) while \( \beta \) and \( \delta \) measure the mortality rate of the adult prey and the predator. The predation rate of young prey and adult prey are measured by the parameters \( \mu \) and \( \eta \) respectively. \( c \) is a parameter that controls the saturation factor of predation (see [1]) while \( a \) and \( b \) control the group defense mechanism (see [2]). We also have the parameters of carrying capacity: \( k_1 \) and \( k_2 \).

Let us consider the situation where the migration rate is zero. Then, it is natural to look at the manifold \( y = 0 \) in the \((x, y, z)\)-space. In that manifold, the system (2) reduces to system (1). Our main concern in this paper is to study the dynamics and bifurcations in (1).

Remark 2.1. In [3] a system of two ordinary differential equations:

\[
\begin{align*}
\dot{u} &= u - \frac{uv}{1 + p_4 u} - p_3 u^2 \\
\dot{v} &= -p_4 v + \frac{uv}{1 + p_4 u} - p_3 v^2
\end{align*}
\]

is studied. Here \( u \) and \( v \) represent the population densities of the prey and the predator, respectively, while \( p_k, k = 1,2,3,4 \) are parameters. This system is known in the literature as the Bazykin Predator-Prey Dynamical System. The System (1) for \( \mu = 1 \) is comparable with the Bazykin's system for \( p_4 = 0 \).

3. Some properties of System (1)

3.1. Equilibria and their local stability

The equilibria of system (1) can be found by solving the system of equations:
There are three equilibria of System (1). The three equilibria with their related eigenvalues are summarized in the following Proposition:

**Proposition 3.1. (Existence of Equilibria)**

For all positive real values of parameters of System (1), there are three equilibria of the system, i.e.:

1) \( E_1 = (0,0) \) with eigenvalues: \( \{1, -\delta\} \) if \( \delta > 0 \). This equilibrium is a saddle point with stable manifold \( x=0 \) and unstable manifold \( z=0 \).

2) \( E_2 = (k_1,0) \) with eigenvalues: \( \{-1, k_1 - \delta\} \)

3) \( E_3 = \left( \frac{\delta}{1-k_1}, \frac{1}{k_1 \mu (1-k_1)} \right) \), \( \delta \neq 1 \) with eigenvalues:

\[
\begin{bmatrix}
\Gamma + \sqrt{\Delta} & \Gamma - \sqrt{\Delta} \\
-2k_1 \mu (1-\delta c) & -2k_1 \mu (1-\delta c)
\end{bmatrix}
\]

where
\[
\Gamma = \mu \delta (1+\delta c - ck_1 (1-\delta c)) \\
\Delta = \mu^2 \delta \left( \delta (1+\delta c)^2 + 2 \delta k_1(1-\delta c)(2(1-\delta c) - c(1+\delta c)) \right)
\]

4) For \( \delta = 0 \) System (1) has infinitely many equilibria \( E = (0,z) \), \( z \in \mathbb{R} \) with related eigenvalues \( \{1, -\mu z, 0\} \)

5) If \( \delta \to \frac{1}{k_1} \), then \( E_3 \) disappears to infinity.

The proposition can be proved by straightforward computation of the solution of the (3), thus omitted. Linearizing the vector field in the vicinity of each of the equilibria gives the results for the eigenvalues.

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**Figure 1.** Phase portraits of the System (1) when \( k_1 = 6, \ c = 1, \ \mu = 0.75 \) and \( \delta = 0.70 \) (left), \( \delta = 0.75 \) (right). The dashed curves are nullclines of the system.
As an illustration for the equilibria and their phase portrait of System (1), see figure 1. One can observe from the figure that for the setting parameter values, there are two different phase portraits i.e. at $\delta = 0.70$ and $\delta = 0.75$. At $\delta = 0.70$ we have (unstable) repelled equilibrium $E_3$ surrounding by a limit cycle. On the other hand, at $\delta = 0.75$ the equilibrium $E_3$ turns to a stable focus equilibrium. This phenomenon indicates that the system undergoes Hopf bifurcation.

In the following proposition we provide two bifurcation points of the system, i.e. transcritical bifurcation point and Hopf bifurcation point.

**Proposition 3.2. (Bifurcations of Equilibria)**

1) At $\delta = \frac{k_z}{1+k_c}$, the equilibria $E_2$ and $E_3$ undergoes transcritical bifurcation, where one eigenvalue of each equilibria passes through zero there. The equilibria then switch stability types as the parameter passes through:

2) At $\delta = \frac{k_z}{1+k_c} - \frac{1}{c(1+k_c)}$, $E_3$ undergoes Hopf bifurcation. The periodic solution created through this bifurcation exists for $k_z > 1$

One can see from the proposition 3.1, that for $\delta = \frac{k_z}{1+k_c}$, the equilibria $E_2$ and $E_3$ collide, i.e. $E_2 = (k_1,0) = E_3$ with the related eigenvalues $\{-1,0\}$. This situation indicates a bifurcation of transcritical type occur. Furthermore, when $\delta = \frac{k_z}{1+k_c} - \frac{1}{c(1+k_c)}$, from the proposition 3.1, we have $\Gamma = 0$ and $\Delta = \frac{16k_z\mu^2(k_c-1)}{c^2(k_c+1)^3}$. For $k_z > 1$ we have $\Delta < 0$. Hence the related eigenvalues are $\pm \frac{\mu(k_z-1)}{c^2(k_c+1)}i$ which indicates Hopf bifurcation. The less part on construction of the proof of the proposition 3.2 is to transform the System (1) into the normal form of transcritical bifurcation and Hopf bifurcation respectively (one can refer to [10]). The three ordered points where transcritical bifurcation, Hopf bifurcation and the infinitely many equilibria occur as the parameter $\delta$ varied are illustrated in figure 2.

![Figure 2](image_url)

**Figure 2.** An illustration of the three points in an interval including the parameter $\delta$ where the bifurcations and the infinitely many equilibria occur.

In the next section we discuss the numerical bifurcation analysis as a numerical confirmation for the previous propositions. To do this we have used AUTO2000 to do the continuation of equilibria.

4. Numerical bifurcation analysis and diagrams

Let us start by fixing a value for each of the parameters, i.e.: $k_1 = 6, \ c = 1$ and $\mu = 0.75$. The parameter $\delta$ is taken to be in the interval $[0,1]$. As a starting point we choose $\delta = 0.90$ to derive the equilibria as follows.
\[ E_1 = (0,0), \ E_2 = (6,0) \text{ and } E_3 = (9,-6.66667) \]

Following the equilibrium \( E_3 \), as we vary the parameter \( \delta \), we have found a transcritical bifurcation and a Hopf bifurcation (see figure 3). The transcritical point \( \delta = 0.857142 \) and the Hopf point \( \delta = 0.714286 \). These results are in agreement with Proposition 3.2.

![Figure 3](image)

**Figure 3.** A branch of continuation curves of equilibrium \( E_2 \) and \( E_3 \) plotted in \((\delta, x)\)-plane. The equilibrium \( E_3 \) undergoes Hopf bifurcation (HB) at \( \delta = 0.714286 \) and undergoes transcritical bifurcation (TC) with equilibrium \( E_2 \) at \( \delta = 0.857142 \).

After Hopf bifurcation, a limit cycle is created. Following this periodic solution we have the variation of the period of the periodic solution. In table 1 we have listed ten examples of the period of the limit cycle. Those periodic solutions are plotted in figure 4.

**Table 1.** The period of limit cycles as the parameter \( \delta \) is varied

| Limit Cycle | \( \delta \) | Period |
|-------------|-------------|--------|
| \( s_1 \)   | 0.7051028   | 18.68508 |
| \( s_2 \)   | 0.6773892   | 20.07643 |
| \( s_3 \)   | 0.6161866   | 22.16587 |
| \( s_4 \)   | 0.5147461   | 24.41203 |
| \( s_5 \)   | 0.4260551   | 27.38236 |
| \( s_6 \)   | 0.3479062   | 31.77746 |
| \( s_7 \)   | 0.2776356   | 38.25942 |
| \( s_8 \)   | 0.2147976   | 47.96934 |
| \( s_9 \)   | 0.1595298   | 63.09256 |
| \( s_{10} \)| 0.1348680   | 73.91731 |
Figure 4. Some periodic solutions as the parameter $\delta$ varied.

Behavior of the system in the neighborhood $\delta = 0$. In this case, as the value $\delta$ tends to zero, the period of periodic solution will tend to infinity. Numerically it shows in figure 5 as a plot of $x(t)$ and $z(t)$ in two different values of the parameter $\delta$. We also have the graph of $\lambda_{1,2}(\delta)$ the eigenvalues as a function of $\delta$ in figure 6. From the figure, one can see the variation values of the eigenvalues starting at the left side to the right side of imaginary axis. Note that crossing this axis related to the occurrence of Hopf bifurcation.

Figure 5. Graph of $x(t)$ (solid curve) and $z(t)$ (dashed curve) at two different value of $\delta$: $\delta = 0.1$ (left) and $\delta = 0.001$ (right).
When the parameter $\delta = 0$ the system has no periodic solution. At this situation it is created the equilibria $E = (0, z), \ z \in \mathbb{R}$ called the infinitely many equilibria. Note that both $E_1$ and $E_3$ are in the collection of the equilibria. We show the situation in figure 7.

5. Concluding remarks
The behavior of System (1) in the neighborhood of $\delta = 0$, in terms of dynamics and bifurcations, is very interesting. It is indicated by the birth of infinitely many equilibria. Clearly, each of this equilibria has at least one zero eigenvalue except for the point $(0, \frac{1}{\mu})$ which has two zero eigenvalues. Furthermore, each of the equilibria $(0, z_u)$ with $z_u > \frac{1}{\mu}$ is connected with an orbit to another equilibria $(0, z_l)$ with $z_l < \frac{1}{\mu}$ (see figure 7). This behavior is quite singular since as $\delta \neq 1$ but nonzero, we have a periodic solution (a limit cycle created by Hopf bifurcation, see $\gamma_{10}$ in figure 4) which can be exponentially closed to the $z$-axis. We suspect that in order to understand this singular behavior, one should find a blow up transformation which unfold this singular behavior. This is a subject of future investigation.
Acknowledgement

We acknowledge the financial support of Riset KK-ITB 2019.

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