Research Article

Prescribed Thermal Activity in the Radiative Bidirectional Flow of Magnetized Hybrid Nanofluid: Keller-Box Approach

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In this exploration, we decided to investigate the significance of prescribed thermal conditions on unsteady 3D dynamics of water-based radiative hybrid nanofluid with the impact of cylindrical-shaped nanosized particles (alumina (Al₂O₃) and titania (TiO₂)). For physical relevancy, the impact of the Lorentz force is also included. The combination of suitable variables has been used to transform the transport equations into the system of ordinary differential equations and then numerically solved via the Keller-Box approach. Graphical illustrations have been used to predict the impact of the involved parameters on the thermal setup. Convergence analysis is presented via the grid independence approach. Skin frictions and local Nusselt numbers against various choices of involved parameters are plotted and arranged in tabular forms. It is observed through the present investigation that temperature distribution is increased with the higher choices of radiation parameter (i.e., 0 ≤ Rd ≤ 2) and decreased with the improvement in the choices of temperature maintaining indices (i.e., −2.0 ≤ r, s ≤ 2.0). Moreover, the thermophysical properties except specific heat for hybrid nanofluid are improved with the involvement of cylindrical-shaped nanoparticles. The temperature of the hybrid nanofluid is observed to be higher for variable thermal conditions as compared to uniform thermal conditions. Outfalls for a limited version of the report have been compared with a previous published paper.

1. Introduction

Nanofluids have been widely used in many technological and industrial processes like nuclear reactors, automobile radiators, and solar aircrafts because their superior thermal conductivity as compared to conventional fluids as the rate of heating/cooling is extremely reliant on the performance of thermal conductivity of the nanosized particles. Various mathematical relations [1, 2] have been adopted to investigate the thermophysical properties of nanosized particles but the most appropriate mathematical relations regarding these properties are developed/discussed by Masoumi et al. [3]. The boundary regime flow of nanofluid due to an expanding device is firstly numerically deliberated by Khan and Pop [4] with random motion and thermodiffusion effects of tiny-sized particles. Sheikholeslami and Rokni et al. [5] elaborated the heat transfer mechanism by using single-phase and double-phase estimations of magneto-nanofluid and proved that higher estimation of tiny particle concentration augments the temperature gradient. Ganvir et al. [6] summarized the combined performance of convective heat transference, particle size, thermophysical properties, inlet velocity, volume concentration, and liquid temperature, both analytically and numerically.

An improved approach to overcome the thermal needs of industrial and engineering processes is famous with the name of hybrid nanofluid. The investigation about the
importance of hybrid nanofluid has gained a tremendous height due to their wide applications in domestic refrigerators, engine cooling, microelectronics, heat exchanger devices, pharmaceutical processes, fuel cells, grinding processes, ultrasonic radiations, thermal diffusion processes, and many more. Hybrid nanofluid can be formed by submerging two or more nanoparticles into the host liquid. These nanoparticles may include metals, dielectrics, liquid materials, polymeric, lipids, and semiconductors. Some common examples of nanoparticles are copper, zinc-oxide, carbon nanotubes, phosphates, zinc sulfide, cadmium telluride, etc. Alumina and titania nanoparticles are widely used in engineering applications, and these are prepared from metal precursors. These tiny-sized particles can be synthesized by electrochemical, chemical, or photochemical methods. These nanoparticles have commercial applications due their structure, high strength, electron affinity, and electrical conductivity. The most widely used oxide ceramic material is alumina, and it has applications in cutting tools, tap washers, spark plugs, etc. The most abundantly used nanomaterial for human life is titania, and it is used in sunscreen, biomedical applications, photovoltaic devices, pharmaceutical drugs, and waste water treatment and as a food additive. Sarkar et al. [7] reviewed the advantages and disadvantages of nano as well as hybrid nanofluids and recommended that hybrid nanofluids have various advantages as compared to conventional nanofluids because of their auspicious heat transfer enhancement, pressure drop ability, improved thermal network, and favorable aspect ratio. Sidik et al. [8] disclosed the recent progress related to the field of hybrid nanofluids by discussing the factors affecting their thermal properties and stability. Unsteady dynamics of hybrid nanofluid with heat transfer characteristic due to the expanding/contracting device is numerically explored by Waini et al. [9] and presented the stability analysis regarding the dual solutions. Numerical evaluation regarding pure water-based hybrid nanofluids with the combination of nanoparticles (alumina, titania, and silica) is explored by Minea [10]. Investigation concerning water-based hybrid nanofluids with the optical properties of titania and alumina nanoparticles is made by Leong et al. [11] along the stability of the obtained solution. Moldoveanu et al. [12] glimpsed the experimental evaluation regarding the hybrid mixture of alumina and titania nanoparticles with viscosity as the main focus of the investigation. Moldoveanu et al. [13] also evaluated the hybrid mixture of titania and alumina nanoparticles with thermal conductivity as the foremost point of the exploration. Shirazi et al. [14] experimentally discussed the mixture of titania nanoparticles and water in order to observe the level of the oil recovery enhancement process. Khan et al. [15] elaborated the mixed convective stagnation dynamics of the radioactive mixture of titania, copper, and water in the magnetic environment towards an expandable device. An experimental evaluation regarding the stability, characterization, and dynamic viscosity of the hybrid mixture of titania and cupric oxide with water as working fluid is completed by Asadi et al. [16]. Ahmad et al. [17] explained the heat/mass transference attributes in the hybrid interpretation of alumina and copper nanoparticles through porous media. A comparative depiction regarding unsteady transport of magnetically influenced water-based fluid with the hybrid mixture of nanoparticle combinations (copper-alumina and alumina-titania) towards an expanding device using finite element approach is elaborated by Ali et al. [18]. Some recent scientific contributions about hybrid nanofluids have been addressed by some scholars/researchers (refs. [19–21]) and their applications (refs. [22–26]).

In the last few years, researchers are interested to discuss the novel impact of the shape of nanoparticles in the improvement of heating/cooling processes, solar aircrafts, effective thermal conductivity, and other thermophysical characteristics. Zhang et al. [27] measured the values of effective thermal conductivity and thermal diffusivity of nanofluids by considering the spherical as well as cylindrical-shaped nanoparticles using transient technique and predicted that Hamilton and Crosser models provide the best approximation for these thermophysical characteristics. Later on, Timofeeva et al. [28] theoretically explained the effect of the nanoparticle shape on thermophysical behaviour of alumina with the help of experimental data. Yang and Ma [29] provided the computer simulation for the understanding of translocation processes of nanoparticles with the usages of different shapes (ellipsoids, discs, rods, and spheres) of nanoparticles across a lipid bilayer. Maheshwary et al. [30] experimentally discussed the significance of the particle shape, particle size, and concentration on thermal conductivity of water-conveying titania nanofluid and predicted that thermal conductivity of the mixture is improved by intensifying the choices of the particle shape, size, and concentration. Sheikholeslami [31] discussed the effect of various shapes (platelet, brick, cylinder, and sphere) of nanoparticles on the forced convective flow of water-conveying cupric oxide nanofluid within a permeable lid-driven enclosure in the magnetic environment by using CVFEM and indicated that the Nusselt number declines with the augment of the Lorentz force. Rashid and Liang [32] numerically investigated the implication of the nanoparticle shape (sphere and lamina) on the dynamics of magnetized nanofluid with heat transfer enhancement phenomenon, thermal radiation effect, and joule heating process towards an expanding disk through a porous space in a rotating frame. Dinarvand and Rostami [33] analyzed the shape factor influence of nanoparticles (graphene oxide and magnetite) for bidirectional unsteady dynamics of water-conveying hybrid nanofluid squeezed between two parallel surfaces using the Tiwari-Das model and predicted that the shape factor effect of nanoparticles has a crucial role in food processing, polymer processing, injection modeling, lubrication, etc. Bhattad and Sarkar [34] theoretically examined the significances of the nanoparticle size and shapes (brick, platelet, sphere, and cylinder) on the thermohydraulic enactment of a sheet evaporator by using hybrid nanofluids having various combinations of mixtures.

The combined significances of thermal radiation and Lorentz force space contribute a vital role in the development of combustion processes, nuclear weapons, electron ramifications, polarization process, stellar evolution, heat conduction process, petroleum reservoirs, etc. Turkyilmazoglu and Pop [35] numerically addressed the natural convection dynamics of radiative water-conveying nanofluid containing copper, alumina, titania, cupric oxide, and silver nanoparticles across a flat device. Devi and Devi [36] numerically examined the
effects of Newtonian heating and Lorentz force on bidirectional
dynamics of water-conveying hybrid nanofluid with the
mixture of copper and alumina. From this exploration, it
is conveyed that hybrid nanofluid provides the better rate of
heat transference than conventional nanofluid. Sheikholeslami
and Sadoughi [37] numerically disclosed the MHD effect on
the flow of nanofluid inside a porous enclosure with four
square heat sources by considering the importance of nano-
particle shapes. Hayat et al. [38] discussed the radiative and
heat transfer characteristics for hybrid mixture of silver and
cupric oxide nanoparticles with water as base liquid in a rotating
frame. It is deduced that rotation and radiation phenomena
boost the thermal environment of the hybrid mixture. Some
more recent exploration regarding the implications of the
Lorentz force and porous media is found in the refs. [39–43].

The variation in the temperature fluctuation at the geo-
metric surfaces is beneficial for several industrial and engi-
neering applications. Liu and Andersson [44] implemented the
variable thermal conditions to investigate the heat transference
characteristics for 3D dynamics of a liquid towards a bidirec-
tional expanding device. Both the heating processes, namely,
PST (prescribed surface temperature) and PHF (prescribed
heat flux), have been discussed by Liu and Andersson, and it
was predicted that variable thermal conditions provide an
improved rate of heat transference than arbitrary thermal con-
ditions at the geometric surface. Oliveira et al. [45] discussed
the practical applications of variable thermal conditions in
the engineering processes like power converters, motor con-
trollers, passive thermosyphons, and air conditioning process.
Waini et al. [46] discussed the heat transference characteristic
for the steady dynamics of hybrid nanofluid past a vertical thin
needle by considering the variable heat flux at the geometric
surface. Waini et al. [47] also investigated the heat transference
process along with variable heat flux for the stagnated dynamics
of hybrid nanofluid (alumina and copper) with water as working
liquid on a contracting cylindrical geometry. Some more recent contributions related to the variable temperature of the geometrical surfaces are found in the refs. [48–52].

In the view of abovementioned comprehensive literature
survey, it is noticed that much attention has not been given to
the dynamics of hybrid nanofluid towards bidirectional elon-
gating geometry. The main theme of the present contribution
is to predict the effects of cylindrical-shaped nanoparticles (alu-
mina and titania) for radiative water-conveying hybrid nano-
fluid flow towards an unsteady bidirectional elongating device
with prescribed thermal conditions, and this type of contribu-
tion is not found in literature to the best of the author’s knowl-
egde. Additionally, influence of the Lorentz force is also
incorporated in the mathematical model. Suitable mathematical
relations have been used to transform the transport equations
into dimensionless forms, and then, computer simulation is
made via the Keller-Box method [53–59]. Finally, the foremost
outcomes obtained through present numerical investigation are
presented through various plots and tables.

2. Mathematical Formulation

In order to frame the unsteady mathematical model for bidi-
rectional dynamics of water-conveying hybrid nanofluid
with cylindrical-shaped nanoparticles (alumina (Al₂O₃)
and titania (TiO₂)), the Cartesian configuration is adopted. The
mathematical relation of the Lorentz force is used to inspect the
MHD (magnetohydrodynamics) effects with strength \( B_0 = b_0/\sqrt{1 - ct} \) (\( b_0 \) represents the initial strength of
the magnetic field). The mathematical relation of Rosse-
land approximation is then followed to examine the effects
of thermal radiation. The tiny particles are considered in
thermal equilibrium. The no-slip phenomenon is considered
at the surface to keep the flow incompressible as well as lam-
inar. Expansion velocity \( u_\alpha = ax/(1 - ct) \); \( a > 0, c > 0 \) is
opted along the \( x \)-axis, and the expansion velocity \( v_\alpha = by/
(1 - ct) \; b \geq 0 \) is decided along the \( y \)-axis, whereas \( 0 < z < \infty \) is the region covered by the hybrid nanofluid (as sketched
in Figure 1). In order to provide the variable temperature
mechanism at the surface, two types of thermal conditions,
namely, PST and PHF, are applied. Table 1 is constructed
to summarize the thermophysical characteristics of water
\( H_2O \), alumina, and titania.

In the continuation of the abovementioned assumptions
with a boundary layer theory, the transport equations are
manifested as follows (refs. [38, 52]):

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \mu_{\text{hnd}} \frac{\partial^2 u}{\partial z^2} - \frac{\sigma_{\text{hnd}}}{\rho_{\text{hnd}}} B_0^2 u,
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \mu_{\text{hnd}} \frac{\partial^2 v}{\partial z^2} - \frac{\sigma_{\text{hnd}}}{\rho_{\text{hnd}}} B_0^2 v,
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha_{\text{hnd}} \frac{\partial^2 T}{\partial z^2} - \frac{1}{(\rho C_p)_{\text{hnd}}} \frac{\partial q_{\text{rad}}}{\partial z}.
\]

The velocity and thermal conditions for equation (1) are
conveyed as follows (ref. [41]):

\[
\begin{align*}
z &= 0 : u = u_w(x, t) = \frac{ax}{1 - ct}, \\
v &= v_w(y, t) = \frac{by}{1 - ct}, \\
w &= 0, \\
z &\rightarrow \infty : u \rightarrow 0, \\
v &\rightarrow 0, \\
\text{PST case} : z = 0 : T = T_w(x, y, t) = T_{\infty} + T_0 \left( \frac{x^2y}{1 - ct} \right), \\
z &\rightarrow \infty : T \rightarrow T_{\infty}, \\
\text{PHF case} : z = 0 : -k_{\text{hnd}} \left( \frac{\partial T}{\partial z} \right)_w = q_w(x, y, t) = T_1 \left( \frac{x^2y}{1 - ct} \right), \\
z &\rightarrow \infty : T \rightarrow T_{\infty}.
\end{align*}
\]
Here, \((u, v, w)\) designates the velocity components along the \(x-, y-,\) and \(z\)-directions, respectively, \(T\) shows the temperature at the surface, time factor is symbolized by \(t\), \((r, s)\) are the indices that are used to control the temperature at the surface, \(T_0\) and \(T_1\) are dimensional constants, \(\mu_{hmf}\) is opted to describe the effective viscosity of the hybrid mixture, \(\rho_{hmf}\) is chosen to label the density of the hybrid mixture, \(k_{hmf}\) is taken to mark the thermal conductivity of the hybrid mixture, \(\alpha_{hmf} = k_{hmf}/(\rho C_p)_{hmf}\) is picked to state the thermal diffusivity of the hybrid mixture, \(C_{pbf}\) is selected to represent the specific heat capacity, and \(\sigma_{hmf}\) is typified to express the influence of electrical conductivity of the hybrid nanofluid.

The mathematical relations to introduce the cylindrical shaped nanoparticles for present evolution of hybrid nanomaterial are composed as follows (refs. [2, 27, 31]):

\[
\begin{align*}
\rho_{hmf} &= \psi_1 \rho_{p1} + \psi_2 \rho_{p2} + (1 - \psi_1 - \psi_2) \rho_f, \\
(\rho C_p)_{hmf} &= \psi_1 (\rho C_p)_{p1} + \psi_2 (\rho C_p)_{p2} + (1 - \psi_1 - \psi_2) (\rho C_p)_{p}, \\
k_{hmf} &= \frac{k_{p2} + 3.82 k_{bf} - 3.82 \psi_2 (k_{p2} - k_{bf})}{k_{p2} + 3.82 k_{bf} + \psi_2 (k_{p2} - k_{bf})}, \\
k_{bf} &= \frac{(k_{p1} + 3.82 k_f) - 3.82 \psi_1 (k_{p1} - k_f)}{(k_{p1} + 3.82 k_f) + \psi_1 (k_{p1} - k_f)}, \\
\sigma_{hmf} &= 1 + \frac{3 \left( (\sigma_{p2}/\sigma_{bf}) - 1 \right) \psi_2}{(\sigma_{p2}/\sigma_{bf}) + 2 - \left( (\sigma_{p2}/\sigma_{bf}) - 1 \right) \psi_2}, \\
\sigma_{bf} &= 1 + \frac{3 \left( (\sigma_{p1}/\sigma_f) - 1 \right) \psi_1}{(\sigma_{p1}/\sigma_f) + 2 - \left( (\sigma_{p1}/\sigma_f) - 1 \right) \psi_1}, \\
\mu_{hmf} &= \frac{k_{hmf}}{(\rho C_p)_{hmf}}, \\
\mu_{bf} &= 1 + 13.5 \psi_2 + 904.4 \psi_2^2, \\
\mu_f &= 1 + 13.5 \psi_1 + 904.4 \psi_1^2. 
\end{align*}
\]

The mathematical relations to introduce the cylindrical shaped nanoparticles for present evolution of hybrid nanomaterial are composed as follows (refs. [2, 27, 31]):

\[
\begin{align*}
\rho_{hmf} &= \psi_1 \rho_{p1} + \psi_2 \rho_{p2} + (1 - \psi_1 - \psi_2) \rho_f, \\
(\rho C_p)_{hmf} &= \psi_1 (\rho C_p)_{p1} + \psi_2 (\rho C_p)_{p2} + (1 - \psi_1 - \psi_2) (\rho C_p)_{p}, \\
k_{hmf} &= \frac{k_{p2} + 3.82 k_{bf} - 3.82 \psi_2 (k_{p2} - k_{bf})}{k_{p2} + 3.82 k_{bf} + \psi_2 (k_{p2} - k_{bf})}, \\
k_{bf} &= \frac{(k_{p1} + 3.82 k_f) - 3.82 \psi_1 (k_{p1} - k_f)}{(k_{p1} + 3.82 k_f) + \psi_1 (k_{p1} - k_f)}, \\
\sigma_{hmf} &= 1 + \frac{3 \left( (\sigma_{p2}/\sigma_{bf}) - 1 \right) \psi_2}{(\sigma_{p2}/\sigma_{bf}) + 2 - \left( (\sigma_{p2}/\sigma_{bf}) - 1 \right) \psi_2}, \\
\sigma_{bf} &= 1 + \frac{3 \left( (\sigma_{p1}/\sigma_f) - 1 \right) \psi_1}{(\sigma_{p1}/\sigma_f) + 2 - \left( (\sigma_{p1}/\sigma_f) - 1 \right) \psi_1}, \\
\mu_{hmf} &= \frac{k_{hmf}}{(\rho C_p)_{hmf}}, \\
\mu_{bf} &= 1 + 13.5 \psi_2 + 904.4 \psi_2^2, \\
\mu_f &= 1 + 13.5 \psi_1 + 904.4 \psi_1^2. 
\end{align*}
\]

Here, \((u, v, w)\) designates the velocity components along the \(x-, y-,\) and \(z\)-directions, respectively, \(T\) shows the temperature at the surface, time factor is symbolized by \(t\), \((r, s)\) are the indices that are used to control the temperature at the surface, \(T_0\) and \(T_1\) are dimensional constants, \(\mu_{hmf}\) is opted to describe the effective viscosity of the hybrid mixture, \(\rho_{hmf}\) is chosen to label the density of the hybrid mixture, \(k_{hmf}\) is taken to mark the thermal conductivity of the hybrid mixture, \(\alpha_{hmf} = k_{hmf}/(\rho C_p)_{hmf}\) is picked to state the thermal diffusivity of the hybrid mixture, \(C_{pbf}\) is selected to represent the specific heat capacity, and \(\sigma_{hmf}\) is typified to express the influence of electrical conductivity of the hybrid nanofluid.

The mathematical relations to introduce the cylindrical shaped nanoparticles for present evolution of hybrid nanomaterial are composed as follows (refs. [2, 27, 31]):

\[
\begin{align*}
\rho_{hmf} &= \psi_1 \rho_{p1} + \psi_2 \rho_{p2} + (1 - \psi_1 - \psi_2) \rho_f, \\
(\rho C_p)_{hmf} &= \psi_1 (\rho C_p)_{p1} + \psi_2 (\rho C_p)_{p2} + (1 - \psi_1 - \psi_2) (\rho C_p)_{p}, \\
k_{hmf} &= \frac{k_{p2} + 3.82 k_{bf} - 3.82 \psi_2 (k_{p2} - k_{bf})}{k_{p2} + 3.82 k_{bf} + \psi_2 (k_{p2} - k_{bf})}, \\
k_{bf} &= \frac{(k_{p1} + 3.82 k_f) - 3.82 \psi_1 (k_{p1} - k_f)}{(k_{p1} + 3.82 k_f) + \psi_1 (k_{p1} - k_f)}, \\
\sigma_{hmf} &= 1 + \frac{3 \left( (\sigma_{p2}/\sigma_{bf}) - 1 \right) \psi_2}{(\sigma_{p2}/\sigma_{bf}) + 2 - \left( (\sigma_{p2}/\sigma_{bf}) - 1 \right) \psi_2}, \\
\sigma_{bf} &= 1 + \frac{3 \left( (\sigma_{p1}/\sigma_f) - 1 \right) \psi_1}{(\sigma_{p1}/\sigma_f) + 2 - \left( (\sigma_{p1}/\sigma_f) - 1 \right) \psi_1}, \\
\mu_{hmf} &= \frac{k_{hmf}}{(\rho C_p)_{hmf}}, \\
\mu_{bf} &= 1 + 13.5 \psi_2 + 904.4 \psi_2^2, \\
\mu_f &= 1 + 13.5 \psi_1 + 904.4 \psi_1^2. 
\end{align*}
\]
Here, quantities of volume fractions for alumina and titania nanoparticles are expressed through $\psi_1$ and $\psi_2$, respectively. The case of conventional fluid can be recovered by considering $\psi_1 = \psi_2 = 0$.

The equation for radiative heat transfer is defined as follows (refs. [35, 38]):

$$q_{\text{rad}} = -\frac{16\sigma^*}{3k^*}T_0^3 \frac{\partial T}{\partial z}. \quad (4)$$

Here, $q_{\text{rad}}$ describes the radiative heat transference, $\sigma^*$ illustrates the Stefan Boltzmann factor, and $k^*$ explains the effect of the mean absorption factor.

The set of relations used to nondimensionalize the present mathematical model is conveyed as follows (refs. [41, 52]):

$$u = \frac{ax}{1 - ct} f' (\eta),$$
$$v = \frac{ay}{1 - ct} g' (\eta),$$
$$w = - \left( \frac{a\partial f}{1 - ct} \right)^{1/2} [f (\eta) + g(\eta)],$$
$$\eta = \left( \frac{a}{\theta_j(1 - ct)} \right)^{1/2} z, \quad \theta_j = \frac{T(x, y, z, t) - T_\infty}{T_{\text{w}}(x, y, t) - T_\infty}, \quad u_{\text{PST}} = \frac{T_1}{k_f} \left( \frac{\theta_j}{a(1 - ct)} \right)^{1/2} x^5 y^3 \phi(\eta). \quad (5)$$

PST case: $\theta_j = \frac{T(x, y, z, t) - T_\infty}{T_{\text{w}}(x, y, t) - T_\infty},$ PHF case: $T - T_\infty = \frac{T_1}{k_f} \left( \frac{\theta_j}{a(1 - ct)} \right)^{1/2} x^5 y^3 \phi(\eta). \quad (6)$

With the involvement of equations (6) and (7), the transport equations become

$$\varepsilon_1 f'''' - g'' + (f + g) f'' - S \left( f^2 + \frac{\eta}{2} f'' \right) - \varepsilon_2 M^2 f' = 0, \quad (7)$$

$$\varepsilon_1 g'''' - f'' + (f + g) g'' - S \left( g^2 + \frac{\eta}{2} g'' \right) - \varepsilon_2 M^2 g' = 0, \quad (8)$$

PST case : $\varepsilon_1 (1 + R_s) \theta'' - \text{Pr} (f' + g') \theta' - \left( \frac{f' + 3 g'}{3} \right) \theta - \frac{\theta_j}{\theta_j + \frac{\eta}{2} \theta_j} = 0, \quad (9)$

PHF case : $\varepsilon_1 (1 + R_s) \phi'' - \text{Pr} (f' + g') \phi' - \left( \frac{f' + 3 g'}{3} \right) \phi - \frac{\phi_j}{\phi_j + \frac{\eta}{2} \phi_j} = 0, \quad (10)$

with boundary restrictions

$$f(0) + g(0) = 0,$$
$$f'(0) = 1,$$
$$g'(0) = \alpha,$$
$$f'(\infty) \rightarrow 0,$$
$$g'(\infty) \rightarrow 0, \quad \theta(\infty) \rightarrow 0, \quad \phi'(\infty) \rightarrow - \frac{k_f}{k_{\text{htf}}}, \phi(\infty) \rightarrow 0. \quad (11)$$

Here, the Hartmann number is recognized by $M = (\sigma_j/\rho_j)^{1/2} b_0$, the unstable factor is stated by $S = c/a$, the elongation ratio is expressed by $\alpha = b/a$, the Prandtl factor is indicated by $\text{Pr} = v/\alpha$, $R_s = (16\sigma^*/3k^*k_t)T_\infty^3$ is the radiation factor, and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are the relations for the present hybrid mixture and these relations are elaborated as follows:

$$\varepsilon_1 = \frac{1 + 13.5 \rho_j + 904.4 \psi_2^5}{(\psi_1(1/\rho_j) + \psi_2(1/\rho_j) + (1 - \psi_1 - \psi_2))},$$
$$\varepsilon_2 = \frac{1 + 3 \psi_2}{(\rho_j(1/\rho_j) + \psi_2(1/\rho_j) + (1 - \psi_1 - \psi_2))},$$
$$\varepsilon_3 = \frac{(k_{p_1} + 3.82k_{p_2}) - 3.82k_{p_2}(k_{p_2} - k_{p_1})}{(k_{p_2} + 3.82k_{p_2}) + 3.82k_{p_2}(k_{p_2} - k_{p_1})},$$

The most fascinating quantities for thermal processes and most used in the improvement of heat exchanger devices are termed as skin-friction coefficients (i.e., $C_{f_{\text{FX}}}$ and $C_{f_{\text{FY}}}$) and the local Nusselt number (i.e., $Nu_x$). The
3. Keller-Box Simulation

The final form of the system of equations obtained through the aforementioned modeling along with boundary conditions is simulated via the Keller-Box approach. This numerical approach has accuracy of up to second order and has rapid convergence ability than other routine work numerical approaches (shooting method, RK-method, BVP4c, etc.). This approach also provides the flexibility about adoption of the step size for the computational domain and is more appropriate for the solution of boundary layer flow problems. The major advantage of this method over other numerical methods is its unique conversion procedure of differential equations into algebraic equations using central difference approximations. Foremost steps to implement this numerical approach are stated below and summarized via the flow chart (Figure 2):

(i) The first step is to transform the higher-order differential equations into first-order differential equations

(ii) The next step is to transmute the obtained differential system into the difference equation system via central difference numerical approximations

(iii) Linearization of the system of difference equations is completed with the courtesy of Newton’s linearization standard method

This numerical approach is its unique conversion procedure of differential equations into algebraic equations using central difference approximations. Foremost steps to implement

\[
C_{fx} = \frac{\tau_{wx}}{\rho_f \mu_w}, \\
C_{fy} = \frac{\tau_{wy}}{\rho_f \mu_w}, \\
\tau_{wx} = \mu_{hnf} \left( \frac{\partial u}{\partial z} \right)_{z=0}, \\
\tau_{wy} = \mu_{hnf} \left( \frac{\partial v}{\partial z} \right)_{z=0}, \\
PST \ case: \ Nu_x = \frac{xq_h}{k_f (T_w - T_\infty)}, \\
PHF \ case: \ Nu_x = \frac{xq_h}{k_f (T_w - T_\infty)}, \\
q_h = -k_{hnf} \left( \frac{\partial T}{\partial z} \right)_{z=0} + (q_{rad})_w, \\
PHF \ case: \ Nu_x = \frac{xq_h}{k_f (T_w - T_\infty)}, \\
q_h = -k_{hnf} \left( \frac{\partial T}{\partial z} \right)_{z=0} + (q_{rad})_w.
\]

\[
\text{Re}_x^{1/2} C_{fx} = (1 + 13.5 \psi_1 + 904.4 \psi_1^2) \cdot (1 + 13.5 \psi_2 + 904.4 \psi_2^2) f''(0), \\
\text{Re}_y^{1/2} C_{fy} = \alpha^{3/2} (1 + 13.5 \psi_1 + 904.4 \psi_1^2) \cdot (1 + 13.5 \psi_2 + 904.4 \psi_2^2) g''(0), \\
\text{Re}_x^{-1/2} Nu_x = \left\{ \begin{array}{ll}
-\frac{k_{hnf}}{k_f} (1 + R_d) \theta'(0) & \text{PPF case}, \\
(1 + R_d) \frac{1}{\phi(0)} & \text{PHF case}.
\end{array} \right.
\]

\[\begin{array}{cccc}
\text{Table 2: Convergence of the Keller-Box simulation for } & -f''(0) & -g''(0) & -\theta'(0) & 1/\phi(0) \\
\text{PST case} & 500 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 1000 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 1500 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 2000 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 2500 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 3000 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 3500 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 4000 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 4500 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 5000 & 0.6718217 & 0.317969 & 2.631693 & 3.556174 \\
& 10000 & 0.6718217 & 0.317969 & 2.631693 & 3.556174
\end{array}\]
Table 3: Outcomes in the nonappearance of nanoparticles, magnetic and unsteadiness aspects.

| \( \alpha = 1.0 \) | \(-f''(0)\) | \( f(\infty) \) | \(-g''(0)\) | \( g(\infty) \) |
|-------------------|------------|------------|------------|------------|
| Present           | 1.173722   | 0.751498   | 1.173722   | 0.751498   |
| Liu and Andersson [44] | 1.173721   | 0.751494   | 1.173721   | 0.751494   |
| \( \alpha = 0.5 \) | \(-f''(0)\) | \( f(\infty) \) | \(-g''(0)\) | \( g(\infty) \) |
| Present           | 1.093095   | 0.842387   | 0.465205   | 0.451678   |
| Liu and Andersson [44] | 1.093096   | 0.842360   | 0.465206   | 0.451663   |
| \( \alpha = 0.0 \) | \(-f''(0)\) | \( f(\infty) \) | \(-g''(0)\) | \( g(\infty) \) |
| Present           | -1.0       | 1.0        | 0.0        | 0.0        |
| Liu and Andersson [44] | -1.0       | 1.0        | 0.0        | 0.0        |

Figure 3: (a, b) Temperature fluctuation against the variation of radiation factor \( R_d \) for the PST case and for the PHF case.
The linearized equations are then arranged into matrix-vector forms.

The LU decomposition technique is opted to solve the obtained matrix-vector problem.

Finally, the value of the unknown vector provides the numerical solution of the aforementioned mathematical problem.

During the implementation of the abovementioned steps, the computational domain \( [0, \infty) \) is truncated into the finite domain \([\eta_0, \eta_\infty]\). In order to obtain the first approximation of the numerical solution, we selected \( \eta_0 = 0, \eta_\infty = 20, n_p = 500, h = (\eta_\infty - \eta_0)/n_p \), and then, the desired accuracy, i.e., \( \varepsilon = 10^{-6} \), is achieved by varying the value of \( n_p \) (the numbers of grid points) with the reduction in the value of \( h \) (step size).

Table 2 is designed to estimate the rate of convergence of the Keller-Box simulation as well as to find the best choice of \( n_p \) for the simulation of the local Nusselt number and skin-friction coefficients. It is deduced through Table 2 that one thousand grid points are enough for the convergent solution of \( f''(0) \) and five hundred grid points are sufficient to achieve the convergence criteria for \( g''(0) \), whereas five thousand grid points are necessary to attain the convergent approximation for both \( \theta'(0) \) as well as \( 1/\phi(0) \). In order to check the stability of the Keller-Box solution, the value of \( n_p \) is increased up to ten thousand and the solution is found consistent. The convergent solution obtained through Table 2 is used for further manipulations in order to find the impact of involved parameters on the thermal setup, local Nusselt number, and skin-friction coefficients.

4. Code Validation

In order to validate the numeric code for the solution of the considered problem, the outfalls reported in the present contribution have been compared with the outfalls discussed in...
the article (Liu and Andersson [44]) for \( f''(0), g''(0), f(\infty) \) and \( g(\infty) \) in the absence of nanoparticles. A convincing scientific connection has been found between the present scrutiny and the published activity. In this regard, Table 3 is arranged in the analysis.

5. Results and Discussion

The present heading describes the importance of the involved important parameters like radiation factor \( R_d \) and temperature maintaining indices \((r, s)\) on thermal setups \([\theta(\eta), \phi(\eta)]\) via various plots, whereas the physical quantities like the local Nusselt number and skin friction coefficients are discussed for various estimations of involved constraints via various graphs and tables. Moreover, the thermophysical properties for the present hybrid mixture are also computed at the end of this section and discussed deeply. Figure 3 describes the impact of radiation factor \( R_d \) on \( \theta(\eta) \) and on \( \phi(\eta) \). It is detected through Figures 3(a) and 3(b) that the escalating choice of \( R_d \) enhances the worth of thermal setups. The thickness of the thermal layer is observed larger for smaller choices of \( R_d \) as compared to larger estimations of \( R_d \). The maximum temperature for the PST mechanism is observed as one, whereas the maximum temperature for the PHF case is noticed as 0.5 when \( R_d \) is increased from 0 to 2.0. Overall, the strength of the thermal setup for the PHF case is observed to be more prominent than that for the PST case. Physically, the abovementioned changes are produced because \( R_d \) is the mathematical ratio of Stefan Boltzman number \( \sigma^* \) to mean absorption factor \( k^* \). The value of \( \sigma^* \) is increased whereas the value of \( k^* \) is diminished with the positive tendency of \( R_d \). Figure 4 explains the influence of temperature maintaining index \( r \) corresponding to the \( x \)-direction on \( \theta(\eta) \) and on \( \phi(\eta) \) while other parameters are being fixed. The temperature of hybrid nanofluid is diminished for both the PST and PHF cases with the escalating amount of \( r \). The value of temperature is detected higher for the PHF case than the PST case for \( r \leq -2 \) and is deduced.

![Figure 5: (a, b) Temperature fluctuation against the variation of the index \( s \) for the PST case and for the PHF case.](image-url)
higher for the PST case than the PHF case for other choices of \( r \). Physically, the temperature distribution is well dominant for variable thermal conditions than arbitrary thermal conditions. The wideness of the thermal layer is larger in the PHF case than the PST case. Figure 5 is plotted to explain the worth of temperature maintaining index \( s \) corresponding to the \( y \)-direction on \( \theta(\eta) \) and on \( \phi(\eta) \) by keeping the other involved factors fixed. The temperature with the higher estimations of \( s \) is reduced for both the PHF and PST mechanisms. The value of temperature fluctuation is attained higher for PST conditions than PHF conditions for all the choices of \( s \), but the thermal thickness is achieved quite dominant for the PHF case as associated with the PST case.

Figure 6(a) manifests the combined influence of \( \psi_1 \) and \( M \) on \( C_{fx} \) for fixed values of other involved parameters. Skin-friction coefficient \( C_{fx} \) is reduced with increasing amounts of both \( M \) and \( \psi_1 \). Physically, more electric conduction is produced in the flow of the hybrid mixture with the positive growth in \( M \) from 0.5 to 3.5. Skin-friction coefficient \( C_{fx} \) is reduced with the phenomenon of electric conduction. Moreover, the electrical conductivity of alumina is much higher than the electrical conductivity of host liquid and this phenomenon produces the reduction in the value of \( C_{fx} \). The smaller selection of \( \psi_1 \) produces the stream of moderate thickness, whereas the thickness of the stream is observed to be double when \( \psi_1 \) reached 0.10. Figure 6(b) explains the combined influence of \( \psi_1 \) and \( S \) on \( C_{fx} \) for fixed selections of other involved constraints. Skin-friction coefficient \( C_{fx} \) is reduced with the higher estimation of \( S \). Physically, expansion rate \( \alpha \) is reduced with the improvement in the value of \( S \), and therefore, reduction in the computation of \( C_{fx} \) is attained. Figure 7(a) reveals the collective effect of \( \psi_2 \) and \( \lambda \) on \( C_{fy} \) with other parameters being kept fixed. Skin-friction coefficient \( C_{fy} \) is abridged with growing amounts of both \( \lambda \) and \( \psi_2 \). Actually, a less elongation rate is produced with the development in \( \lambda \) from 0.2 to 1.4.

\[
\begin{align*}
\alpha &= s = 0.5, \lambda = 0.2, \psi_2 = 0.01 \\
M &= 0.5, 1.3, 2.1, 2.9, 3.5 \\
\psi_1 &= 0.02, 0.04, 0.06, 0.08, 0.10 \\
\end{align*}
\]
Skin-friction coefficient $C_{fy}$ is diminished with the productions of a less elongation rate. Moreover, the electrical conductivity of titania is much higher than the electrical conductivity of host liquid and this phenomenon produces the reduction in the computation of $C_{fy}$. As electrical conductivity of titania is higher than the electrical conductivity of alumina, therefore, faster reduction in $C_{fy}$ is achieved than $C_{fx}$. The smaller assortment of $\psi_2$ yields the stream of minute thickness, whereas the thickness of the stream is observed triple when $\psi_2$ reached 0.10. Figure 7(b) elucidates the joined impact of $\psi_2$ and $\lambda$ on $C_{fy}$ for stationary values of other mathematical constraints. Skin-friction coefficient $C_{fy}$ is enhanced with the advancement in $\alpha$. Precisely, expansion rate $b$ is enhanced and expansion rate $a$ is diminished with the development in $\alpha$, and therefore, enhancement in the computation of $C_{fy}$ is attained.

Figure 8(a) explores the graphical assessment of Nusselt number $Nu_x$ against the variations in $R_d$ and $r$ with other parameters being retained as fixed. Nusselt number $Nu_x$ is improved with the enhancement in $r$ from −2.0 to 2.0, and also, it is increased with the escalation in $R_d$ from 0.0 to 2.0. Energy is transformed in the form of electromagnetic waves with the involvement of thermal radiation, and it is more efficient in porous space or vacuum because it is the ideal situation for full transmission of the radiation energy. The value of $Nu_x$ is augmented with the escalation in $R_d$ from 0.0 to 2.0. The rate of heat transference is observed slower for negative values of $r$ than positive values of $r$ because the surface temperature is mentioned higher for the negative value of $r$ than the positive value of $r$, and therefore, the thermal flux across the surface will be higher for the positive value of $r$ than its negative value. As a result, the Nusselt number $Nu_x$ is tremendously improved.

Figure 8(b) describes the graphical evaluation of Nusselt number $Nu_y$ against the variations in $S$ and $s$ with other parameters being taken as fixed. Nusselt number $Nu_y$ is improved with the enhancement in $s$ from −2.0 to 2.0, and
Figure 8: Nusselt number $N_u_x$ against the variations in (a) $R_d$ & $r$ and $N_u_y$ against the variations in (b) $S$s.

Table 4: Contribution of solid volume fractions $\psi_1$ and $\psi_2$ on thermal interest quantities for $\alpha = S = M = 0.5$, $r = s = 1.0$, $R_d = 0.5$.

| Nanoparticle volume fractions | $-Re_x^{1/2}C_{fx}$ | $-Re_y^{1/2}C_{fy}$ | $(Re_x^{-1/2}(1 + R_d))N_u_d$ |
|------------------------------|---------------------|---------------------|-------------------------------|
| $\psi_1$ $\psi_2$           | VST case            | VHF case            |                               |
| 0.01 0.01                    | 1.818652            | 2.354086            | 3.165999 3.165999             |
| 0.04 0.01                    | 3.199621            | 4.200442            | 3.392241 3.392241             |
| 0.07 0.01                    | 5.443228            | 7.263307            | 3.584352 3.584352             |
| 0.10 0.01                    | 8.510418            | 11.50723            | 3.751383 3.751383             |
| 0.13 0.01                    | 12.42493            | 16.96354            | 3.903712 3.903712             |
| 0.16 0.01                    | 17.20563            | 23.65449            | 4.046176 4.046176             |
| 0.01 0.04                    | 3.207035            | 4.209267            | 3.351655 3.351655             |
| 0.01 0.07                    | 5.46183             | 7.285107            | 3.50311  3.50311              |
| 0.01 0.10                    | 8.542308            | 11.54414            | 3.630213 3.630213             |
| 0.01 0.13                    | 12.47104            | 17.01642            | 3.743252 3.743252             |
| 0.01 0.16                    | 17.26637            | 23.72367            | 3.846784 3.846784             |
Table 5: Fluctuations in the thermophysical behaviours for the existing hybrid mixture with $\psi_1 = 0.01$.

| $\psi_2$ | $\rho_{huf}$ | $(\rho C_p)_{huf} \times 10^3$ | $k_{huf}$ | $\sigma_{huf} \times 10^{-6}$ |
|----------|-------------|-----------------|-----------|------------------|
| 0.00     | 1026.829    | 4.14486         | 0.640742  | 86666.77         |
| 0.01     | 1059.358    | 4.109932        | 0.663426  | 86666.84         |
| 0.03     | 1124.416    | 4.040076        | 0.709807  | 9624.44          |
| 0.05     | 1189.474    | 3.97022         | 0.757953  | 66140.04         |
| 0.07     | 1254.532    | 3.900364        | 0.806847  | 69423.7          |
| 0.09     | 1319.59     | 3.830508        | 0.857368  | 734798.5         |

Table 6: Fluctuations in the thermophysical behaviours for the existing hybrid mixture with $\psi_2 = 0.01$.

| $\psi_1$ | $\rho_{huf}$ | $(\rho C_p)_{huf} \times 10^3$ | $k_{huf}$ | $\sigma_{huf} \times 10^{-6}$ |
|----------|-------------|-----------------|-----------|------------------|
| 0.00     | 1029.629    | 4.144072        | 0.634980  | 66666.77         |
| 0.01     | 1059.358    | 4.109932        | 0.663426  | 66666.84         |
| 0.03     | 1118.816    | 4.041652        | 0.721891  | 6124.44          |
| 0.05     | 1178.274    | 3.973372        | 0.782554  | 65140.04         |
| 0.07     | 1237.732    | 3.905092        | 0.845547  | 69423.7          |
| 0.09     | 1297.19     | 3.836812        | 0.911013  | 734798.5         |

also, it is increased with the escalation in $S$ from 0.0 to 2.0. The transmission of heat is perceived slower for negative selections of $s$ than positive selections of $s$ because the stretching device is maintained at higher temperature for the negative value of $s$ than the positive value of $s$, and therefore, thermal flux across the stretching device will be higher for the positive value of $s$ than its negative value. As a result, the Nusselt number $Nu_s$ is extremely enriched. Physically, unsteady expansion parameter $S$ is involved in equation (9) with the sum of dimensionless temperature and its derivative with respect to $t$. As a result, the rate of heat transfer across the $yz$ plane is increased.

Roles of solid volume fractions $\psi_1$ for alumina and $\psi_2$ for titania on the local Nusselt number as well as on skin friction coefficients are discussed in Table 4. It is attained in Table 4 that escalating selections of $\psi_1$ and $\psi_2$ improve the skin-friction coefficients for the present model. It is also observed that the impact of $\psi_1$ and $\psi_2$ is more dominant for the flow along the $y$-direction as compared to the $x$-direction. Mathematically, the involvement of $\alpha$ is absent in equation (14), whereas it is involved in the reciprocal form as described in equation (15). In our study, the value of $\alpha$ is selected to be 0.5, so the numerical value of $g'(0)$ is observed to be higher than the numerical value of $f'(0)$. Moreover, the roles of $\psi_1$ and $\psi_2$ on the local Nusselt number are also discussed in Table 3 and it is deduced that higher estimations of $\psi_1$ and $\psi_2$ in the range $[0.01,0.16]$ improve the rate of heat transferance which is beneficial for many industrial and engineering applications. Furthermore, the rate of heat transference is observed to be equal quantitatively for both the PST and PHF mechanisms. The thermophysical features for the present hybrid mixture are discussed in Table 5 for various choices of $\psi_2$ (the weightage of titania) by adjusting $\psi_1 = 0.01$. The escalating estimation of $\psi_2$ enhances the density, thermal conductivity, and electrical conductivity of the hybrid mixture whereas the heat capacity is reduced with the present development in $\psi_2$. Physically, the mixture of alumina and titania with the working fluid, i.e., water, provides the improvement in the thermophysical features as compared to that in Table 1. In Table 6, thermophysical characteristics for present hybrid combination are computed for different selections of $\psi_1$ (the solid volume fraction of alumina) by taking $\psi_2 = 0.01$. With the increase of $\psi_1$, the density, the electrical conductivity, and the thermal conductivity are improved but the heat capacity is condensed with this variation. The base fluid, i.e., water, delivers enlargement in thermophysical properties of hybrid nanofluid associated to Table 1 with the chemically mixture of alumina and titania.

6. Conclusions

This study provides the mathematical analysis for unsteady bidirectional dynamics of radiative water-conveying hybrid nanofluid (i.e., the combination of titania and alumina) with the significance of variable thermal conditions (PST and PHF) and impact of the cylindrical shape of nanoparticles. The influence of the Lorentz force is also incorporated to make the investigation more impactful. Numerical simulation is made via the Keller-Box approach, and key observations are listed as follows:

(i) The thermophysical features of the hybrid nanofluid except specific heat are improved with the involvement of the cylindrical shape of nanoparticles, i.e., titania and alumina

(ii) The rate of heat transference is observed identical for both the PST and PHF cases

(iii) The magnitude of the stress applied on the $y$-direction is observed to be higher than the magnitude of the stress applied along the $x$-direction with the positive estimations of volume fractions of nanoparticles

(iv) The temperature of the hybrid nanofluid is increased with the development in the choice of $R_d$ and decreased with the improvement in the choices of temperature maintaining indices $(r, s)$

(v) Skin-friction coefficients are reduced with progressions in the values of the Hartmann number, unsteady parameter, and volume percentages of nanoparticles

(vi) The local Nusselt number is improved with developments in the amounts of radiation and unsteady parameters

This scientific contribution has many mechanical, biomedical, and commercial applications. These are coating a sheet with hybrid nanomaterials, manufacturing of printing ink, degrading organic contaminants, manufacturing of sodium vapour lamps, etc. This study is also really helpful
for the researchers working in the field of nanomaterials and can be protracted in the future by considering different geometries.

**Nomenclature**

- \( B_0 \): Strength of magnetic field
- \( a, b \): Stretching rates
- \( t \): Time
- \( c \): Time coefficient
- \( x, y, z \): Space coordinates
- \( u_0, v_0 \): Stretching velocities
- \( \text{PHF} \): Prescribed heat flux
- \( T_w \): Surface temperature
- \( r, s \): Thermal indices
- \( T'_r, T'_1 \): Dimensional constants
- \( T_{co} \): Ambient temperature
- \( u, v, w \): Velocity field components
- \( T \): Temperature
- \( h_{nf} \): Hybrid nanofluid
- \( \mu_{h_{nf}} \): Dynamic viscosity of hnf
- \( \rho_{h_{nf}} \): Density of hnf
- \( \sigma_{h_{nf}} \): Electrical conductivity of hnf
- \( k_{h_{nf}} \): Thermal conductivity of hnf
- \( \alpha_{h_{nf}} \): Thermal diffusivity of hnf
- \( C_{ph_{nf}} \): Specific heat capacity of hnf
- \( \varphi_1 \): Volume fraction of alumina
- \( \varphi_2 \): Volume fraction of titania
- \( f \): Fluid
- \( b_f \): Base fluid
- \( \rho_1 \): Alumina nanoparticles
- \( \rho_2 \): Titania nanoparticles
- \( q_{rad} \): Radiative heat transfer
- \( \sigma^* \): Stefan Boltzmann constant
- \( k^* \): Mean absorption coefficient
- \( \eta \): Similarity variable
- \( f', g' \): Dimensionless velocities
- \( \theta, \phi \): Dimensionless temperatures
- \( Pr \): Prandtl number
- \( R_d \): Radiation parameter
- \( S \): Unsteady parameter
- \( M \): Hartmann number
- \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \): Dimensionless quantities
- \( \text{Re}_x, \text{Re}_y \): Reynolds numbers
- \( N_u \): Nusselt number
- \( C_{f_{x}}, C_{f_{y}} \): Skin friction coefficients
- \( \alpha \): Stretching ratio parameter
- \( \rho, k, C_p, \sigma \): Thermophysical properties
- \( h \): Step size
- \( \varepsilon \): Convergence criterion
- \( n_p \): Numbers of grid points.

**Data Availability**

The raw data supporting the conclusion of this report will be made available by the corresponding author without undue reservation.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All authors contributed equally to this work, and all the authors have read and approved the final version of the report.

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