REAL ANALYTIC APPROXIMATION OF LIPSCHITZ FUNCTIONSON HILBERT SPACE AND OTHER BANACH SPACES

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Abstract. Let $X$ be a separable Banach space with a separating polynomial. We show that there exists $C \geq 1$ (depending only on $X$) such that for every Lipschitz function $f : X \to \mathbb{R}$, and every $\varepsilon > 0$, there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon$ and $\text{Lip}(g) \leq C\text{Lip}(f)$. This result is new even in the case when $X$ is a Hilbert space. Furthermore, in the Hilbertian case we also show that $C$ can be assumed to be any number greater than 1.

1. Introduction and main results

Not much is known about the natural question of approximating functions by real analytic functions on a real Banach space $X$. When $X$ is finite dimensional, a famous paper of Whitney’s [W] provides a completely satisfactory answer to this problem: a combination of integral convolutions with Gaussian kernels and real analytic approximations of partitions of unity allows to show that for every $C^k$ function $f : \mathbb{R}^n \to \mathbb{R}^m$ and every continuous $\varepsilon : \mathbb{R}^n \to (0, +\infty)$ there exists a real analytic function $g$ such that $\|D^jg(x) - D^jf(x)\| \leq \varepsilon(x)$ for all $x \in \mathbb{R}^n$ and $j = 1, \ldots, k$.

In an infinite dimensional Banach space $X$, the lack of a translation invariant measure which assigns finite, strictly positive volume to balls makes this question much more difficult to answer, as integral convolutions cannot be directly used in order to regularize a given function. By constructing a real analytic approximation of a partition of unity, Kurzweil was able to show in his classic paper [K] that for every Banach space $X$ having a separating polynomial, for every Banach space $Y$, and for every continuous function $f : X \to Y$, $\varepsilon : X \to (0, +\infty)$, there exists a real analytic function $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ for all $x \in X$.

In the special case of norms (or more generally convex functions) defined on a Banach space $X$, Deville, Fonf and Hajek [DFH1, DFH2], by introducing specific formulae suited to this problem, have proved two important results: first, in $l_p$ or $L_p$, with $p$ an even integer, any equivalent norm can

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be uniformly approximated on bounded sets by analytic norms. Second, in $X = c_0$ or $X = C(K)$, with $K$ a countable compact, any equivalent norm can be uniformly approximated by analytic norms on $X \setminus \{0\}$.

These results leave at least three important questions open. Question one, is Kurzweil’s result improvable? Is it necessary for a Banach space $X$ to have a separating polynomial in order to enjoy the property that every continuous function defined on $X$ can be uniformly approximated by real analytic functions? Fry and, independently, Cepedello and Hajek have proved that every uniformly continuous function defined on $c_0$ (which fails to have a separating polynomial) [F1] and, more generally, a Banach space with a real analytic separating function [CH], can be uniformly approximated by real analytic functions. However, the approximating functions they construct (again, refining Kurzweil’s technique, by employing a real analytic approximation of a partition of unity) are not uniformly continuous, and in any case the problem of approximating continuous functions (not uniformly continuous) defined on $c_0$ by real analytic functions remains open.

Question two. Given a $C^k$ function $f$ defined on a Banach space $X$ with a separating polynomial, and a continuous $\varepsilon : X \to (0, +\infty)$, is it possible to find a real analytic function $g$ on $X$ such that $\|D^j f - D^j g\| \leq \varepsilon$ for $j = 0, 1, \ldots, k$? Nothing is known about this problem, even in the case $k = 1$ and $X$ being a Hilbert space. It is worth mentioning that if we replace real analytic with $C^\infty$ and consider only $k = 1$, then the question has an affirmative answer, which was found by Moulis [M] in the cases $X = \ell_p, c_0$ and with range any Banach space (although her proof can be adapted to every Banach space with an unconditional basis and a $C^\infty$ smooth bump function, see [AFGJL]), and by Hajek and Johanis [HJ] very recently for certain range spaces in the far more general case of a separable Banach space $X$ with a $C^\infty$ smooth bump function. And, interestingly enough, in both cases the solution came as a corollary of a theorem on approximation of Lipschitz functions by more regular Lipschitz functions. This leads us to the following natural question, which is the subject matter of the present paper.

Question three. Is it possible to approximate a Lipschitz function $f$ defined on a Banach space $X$ having a separating polynomial by Lipschitz, real analytic functions $g$? And is it possible to have the Lipschitz constants of the functions $g$ be of the order of the Lipschitz constant of $f$ (that is $\text{Lip}(g) \leq C\text{Lip}(f)$, where $C \geq 1$ is independent of $f$)? In such case, does this hold for any number $C$ greater than one (that is, is it possible to get real analytic approximations that almost preserve the Lipschitz constant of the given function)? It is worth noting that every approximation method based on constructing real analytic approximations of partitions of unity in infinite dimensions (such as [K, F1, CH, J]) cannot work to give a solution to this problem, the reason being that in order to make the supports of the functions forming the partition locally finite one has to give up the idea of those functions having a common Lipschitz constant. That is, even if we
consider $C^p$ smooth functions instead of real analytic functions in this question, the standard approximation technique based on the use of $C^p$ smooth partitions of unity does not provide a solution to the problem.

A partial answer to this problem can be obtained by combining the Deville-Fonf-Hajek results on real analytic approximation of convex functions with the following theorem of M. Cepedello-Boiso [Ce]: a Banach space is superreflexive if and only if every Lipschitz function can be approximated by differences of convex functions which are bounded on bounded sets, uniformly on bounded sets. It follows that, for $X = \ell_p$ or $L_p$, with $p$ an even integer, and for every Lipschitz function $f : X \to \mathbb{R}$ there exists a sequence of real analytic functions $g_n : X \to \mathbb{R}$ which are Lipschitz on bounded sets, and such that $\lim_{n \to \infty} g_n = f$, uniformly on bounded sets. However, this method has two important disadvantages: 1) the approximation cannot be made to be uniform on $X$, and 2) even on a fixed bounded set $B$, we lose control on the Lipschitz constants of the approximations $g_n$: in fact, one has $\text{Lip}(g_n|_B) \to \infty$ as $n \to \infty$.

A successful new approach to the question of approximating Lipschitz functions on infinite-dimensional Banach spaces was discovered by Fry in [F2], with the introduction of what one can call sup-partitions of unity (see the following section for a definition), a tool which has been thoroughly exploited in [HJ].

Very recently, Fry and Keener [FK] have constructed a real analytic approximation of a sup partition of unity and have employed it to show that, for every Banach space $X$ having a separating polynomial, for every bounded open subset $U$ of $X$, for every bounded Lipschitz function $f : U \to \mathbb{R}$, and for every $\varepsilon > 0$ there exists a Lipschitz, real analytic function $g : U \to \mathbb{R}$ such that $|f - g| \leq \varepsilon$.

A disadvantage of the sup partition approach to approximation is that it only works for bounded functions. In this paper we will simplify the construction in [FK], and we will combine this with some new tools in order to eliminate these restrictions and obtain the following.

**Theorem 1.** Let $X$ be a separable Banach space which admits a separating polynomial. Then there exists a number $C \geq 1$ such that for every Lipschitz function $f : X \to \mathbb{R}$, and for every $\varepsilon > 0$ there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$, and $\text{Lip}(g) \leq C\text{Lip}(f)$.

As a matter of fact the proof of this result works for any Banach space $X$ having a separating function with a Lipschitz holomorphic extension to a uniformly wide neighborhood of $X$ in the complexification $\tilde{X}$.

**Definition 1.** A separating function $Q$ on a Banach space $X$ is a function $Q : X \to [0, +\infty)$ such that $Q(0) = 0$ and there exists $M, m > 0$ such that $Q(x) \geq m\|x\|$ whenever $\|x\| \geq M$. 
It is clear that $X$ has a real analytic Lipschitz separating function (with a holomorphic extension to a uniformly wide neighborhood of $X$ in $\tilde{X}$) if and only if $X$ has a real analytic Lipschitz function $Q$ which satisfies the above definition with $M = m = 1$ (and with a holomorphic extension to a uniformly wide neighborhood of $X$ in $\tilde{X}$). So we also have the following.

**Theorem 2.** Let $X$ be a separable Banach space having a separating function $Q : X \to [0, \infty)$ with a Lipschitz holomorphic extension $\tilde{Q}$ defined on a set of the form $\{x + iy \in \tilde{X} : x, y \in X, \|y\| < \delta\}$ for some $\delta > 0$. Then there exists a number $C \geq 1$ such that for every Lipschitz function $f : X \to \mathbb{R}$, and for every $\varepsilon > 0$ there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X$, and $\text{Lip}(g) \leq C \text{Lip}(f)$.

We will prove (see Lemma 2 below) that every Banach space with a separating polynomial (that is a polynomial $P : X \to \mathbb{R}$ such that $P(0) = 0 < \inf\{P(x) : \|x\| = 1\}$) also has a Lipschitz, real analytic separating function with a holomorphic extension to a uniformly wide neighborhood of $X$ in $\tilde{X}$. We do not know if the converse is true. A natural related question is: does the space $c_0$ have such a Lipschitz separating function?

By using Theorem 1 and part of its proof (namely Lemma 3 below) one can also prove the following.

**Theorem 3.** Let $X$ be a separable Banach space which admits a separating polynomial. Let $f : X \to \mathbb{R}$ be bounded, Lipschitz, and $C^1$ with uniformly continuous derivative. Then for each $\varepsilon > 0$, there exists a real analytic function $g : X \to \mathbb{R}$ with $|g - f| < \varepsilon$ and $\|g' - f'\| < \varepsilon$.

We give a detailed proof of this result in [AFK].

Finally, by combining the Lasry-Lions sup-inf convolution regularization technique [LL] with the preceding Theorem and with some of the techniques developed for the proof of Theorem 1, we will obtain the following improvement of our main result in the most important case; namely, if $X$ is a Hilbert space, then the real analytic approximations $g$ can be made to almost preserve the Lipschitz constant of the given function $f$.

**Theorem 4.** Let $X$ be a separable Hilbert space, and $f : X \to \mathbb{R}$ a Lipschitz function. Then for every $\varepsilon > 0$ there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$, and $\text{Lip}(g) \leq \text{Lip}(f) + \varepsilon$.

2. **A brief outline of the proof**

The proof of our main result is rather long, and very technical at some points so, for the reader’s convenience, we will next explain the main ideas of our construction (which we here intentionally oversimplify in order not to be burdened by important, but not very meaningful precision and notation).
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As said in the introduction, we will show in Lemma 2 that every Banach space $X$ with a separating polynomial $p$ of degree $n$ has a Lipschitz, real analytic separating function. This is done as follows: such a Banach space always has a $2n$-degree homogeneous polynomial $q$ such that $\|x\|^{2n} \leq q(x) \leq K\|x\|^{2n}$ for all $x \in X$. Then the function $Q : X \to (0, +\infty)$ defined by

$$Q(x) = (1 + q(x))^{\frac{1}{2n}} - 1$$

is real analytic, Lipschitz, and separating.

The next step will be taking a dense sequence $\{x_n\}$ in $X$ and constructing an equi-Lipschitz, real analytic analogue of a sup partition of unity of complex implicit equations depending on a parameter (see Proposition 1 below).

Next, there is a real analytic norm $\lambda : c_0 \to [0, \infty)$ which satisfies $\|y\|_\infty \leq \lambda(y) \leq 2\|y\|_\infty$ for every $y \in c_0$. Assuming $f : X \to [1, 1001]$ is 1-Lipschitz, define a function $g : X \to \mathbb{R}$ by

$$g(x) = \frac{\lambda(f(x_n)\varphi_n(x))_{n=1}^{\infty}}{\lambda(\varphi_n(x))_{n=1}^{\infty}}$$

(the sup-partition of unity approach to approximation consists in replacing the usual locally finite sum in a classical partition of unity by taking the norm $\lambda$. Since $\lambda$ is equivalent to the supremum norm of $c_0$, the function $g$ roughly behaves like $\sup_n \{f(x_n)\varphi_n(x)/\sup_n \varphi_n(x)\}$, which is a Lipschitz function approximating $f$). It can be checked that $g$ is real analytic, 8518492Lip($Q$)-Lipschitz, and $|f - g| \leq 8$. Moreover, given $\delta > 0$, there exists a neighborhood $\tilde{U}_\delta$ of $X$ in $\tilde{X}$, independent of $f$ but dependent on the interval $[1, 1001]$ and the functions $\varphi_n$, such that $g$ has a holomorphic extension $\tilde{g} : \tilde{U}_\delta \to \mathbb{C}$ satisfying

$$|\tilde{g}(z) - g(x)| \leq \delta$$

for all $z = x + iy \in \tilde{U}_\delta$.

Proving such independence of $\tilde{U}_\delta$ from $f$ is a delicate matter that will be tackled with the help of a refinement of the main result of [CHP]: an estimation of the domain of existence of the holomorphic solutions to a family of complex implicit equations depending on a parameter (see Proposition 1 below).
If we were only interested in approximating bounded functions by Lipschitz, real analytic functions and we did not care about the Lipschitz constant of the approximations, a replacement of the interval $[1, 1001]$ with a suitable translation of the range of $f$ in the above argument would finish our proof (up to scaling). However, as the size of the interval increases, so does the Lipschitz constant of $f$, and inversely, the size of $\tilde{U}_\delta$ decreases. In order to prove our main result in full generality we have to work harder.

Up to scaling, the above argument shows the following: there exists $C \geq 1$ (depending only on $X$) such that, for every $\delta > 0$ there is an open neighborhood $\tilde{U}_\delta$ of $X$ in $\tilde{X}$ such that, for every Lipschitz function $f : X \to [0, 1]$ with $\text{Lip}(f) \leq 1$, there exists a real analytic function $\tilde{g} : X \to \mathbb{R}$, with holomorphic extension $\tilde{g} : \tilde{U}_\delta \to \mathbb{C}$, such that

1. $|f(x) - g(x)| \leq 1/10$ for all $x \in X$.
2. $\tilde{g}$ is Lipschitz, with $\text{Lip}(\tilde{g}) \leq C\text{Lip}(f)$.
3. $|\tilde{g}(x + iy) - g(x)| \leq \delta$ for all $z = x + iy \in \tilde{U}_\delta$.

Now, given a 1-Lipschitz, bounded function $f : X \to [0, +\infty)$, we can cut its graph into pieces, namely we may define, for $n \in \mathbb{N}$, the functions

$$f_n(x) = \begin{cases} f(x) - n + 1 & \text{if } n - 1 \leq f(x) \leq n, \\ 0 & \text{if } f(x) \leq n - 1, \\ 1 & \text{if } n \leq f(x). \end{cases}$$

The functions $f_n$ are clearly 1-Lipschitz and take values in the interval $[0, 1]$, so, for $\delta > 0$ (to be fixed later on) there exist a neighborhood $\tilde{U}_\delta$ of $X$ in $\tilde{X}$ and C-Lipschitz, real analytic functions $g_n : X \to \mathbb{R}$, with holomorphic extensions $\tilde{g}_n : \tilde{U}_\delta \to \mathbb{C}$, such that for all $n \in \mathbb{N}$ we have that $g_n$ is C-Lipschitz, $|g_n - f_n| \leq 1/8$, and $|\tilde{g}_n(x + iy) - g_n(x)| \leq \delta$ for all $z = x + iy \in \tilde{U}_\delta$.

Observe that the function $X \ni x \mapsto \{f_n(x)\}_{n=1}^\infty$ takes values in $\ell_\infty$, and more precisely in the image of the path $\beta : [0, +\infty) \to \ell_\infty$ defined by

$$\beta(t) = \left(1, \ldots, 1, \underbrace{t - n + 1, 0, 0, \ldots}_{n^{th \text{ place}}}ight) = \sum_{j=1}^{n-1} e_j + (t-n+1)e_n \text{ if } n-1 \leq t \leq n.$$ 

The path $\beta$ is a 1-Lipschitz injection of $[0, +\infty)$ into $\ell_\infty$, with a uniformly continuous (but not Lipschitz) inverse $\beta^{-1} : \beta([0, +\infty)) \to [0, +\infty)$.

Define a uniformly continuous (not Lipschitz) function $h$ on the path $\beta$ by $h(\beta(t)) = t$ for all $t \geq 0$, that is $h(y) = \beta^{-1}(y)$ for $y \in \beta([0, +\infty))$. Then we have $f(x) = h(\{f_n(x)\}_{n=1}^\infty)$.

We will construct an open tube of radius $1/8$ (with respect to the supremum norm $\|\cdot\|_\infty$) around the path $\beta$ in $\ell_\infty$, and a real-analytic approximate extension (with bounded derivative) $H$ of the function $h$ defined on this tube. Then, since $|g_n - f_n| \leq 1/10$ and $\{f_n(x)\}_{n=1}^\infty$ takes values in the path $\beta$, the function $\{g_n(x)\}_{n=1}^\infty$ will take values in this tube, and therefore $g(x) := H(\{g_n(x)\}_{n=1}^\infty)$ will approximate $H(\{f_n(x)\}_{n=1}^\infty)$, which in turn
approximates \( h(\{f_j(x)\}_{n=1}^{\infty}) = f(x) \). Besides, since \( H \) has a bounded derivative on the tube and the functions \( g_n \) are \( C \)-Lipschitz then \( g \) will be \( C M \)-Lipschitz, where \( M \) is an upper bound of \( DH \) on the tube. Moreover, we will show that there exist \( \delta > 0 \) (this is the \( \delta \) we had to fix) and a holomorphic extension \( \tilde{H} \) of \( H \) so that if \( v \in \ell_\infty \) satisfies \( \|v\|_\infty \leq \delta \) then
\[
|H(u + iv) - H(u)| \leq 1.
\]
We will call \( H \) a real analytic gluing tube function.

Thus, resetting \( C \) to \( CM \), we will deduce that, there exists an open neighborhood \( \tilde{U} := \tilde{U}_\delta \) of \( X \) in \( \tilde{X} \) such that, for every \( 1 \)-Lipschitz, bounded function \( f : X \to \mathbb{R} \), there exists a real analytic function \( g : X \to \mathbb{R} \), with holomorphic extension \( \tilde{g} : \tilde{U} \to \mathbb{C} \), such that

1. \( |f(x) - g(x)| \leq 1 \) for all \( x \in X \).
2. \( g \) is \( C \)-Lipschitz.
3. \( \|\tilde{g}(x + iy) - g(x)\| \leq 1 \) for all \( z = x + iy \in \tilde{U} \).

The last step of the proof consists in passing from bounded to unbounded functions. This can be achieved by constructing a real analytic approximation \( \theta_n(Q(x)) \) to a partition of unity subordinated to a covering of \( X \) consisting of crowns \( C_n = \{x \in X : 2^{n-1} < Q(x) < 2^{n+1}\} \) of rapidly increasing diameter (so that \( \sum_{n=1}^{\infty} \text{Lip}(\theta_n) < 3 \)), and by approximating each restriction of \( f \) to \( C_n \) by a real analytic \( C \)-Lipschitz function \( g_n \), in order to define
\[
g(x) = \sum_{n=1}^{\infty} \theta_n(Q(x))g_n(x),
\]
which we will check is a \( 5C \)-Lipschitz real analytic approximation of \( f \). Up to scaling, our main theorem will then be proved.

3. Notation and basic definitions. Complexifications.

Our notation is standard, with \( X \) denoting a Banach space, and an open ball with centre \( x \) and radius \( r \) denoted \( B_X(x,r) \) or \( B(x,r) \) when the space is understood. If \( \{f_j\}_j \) is a sequence of Lipschitz functions defined on \( X \), then we will at times say this family is \emph{equi-Lipschitz} if there is a common Lipschitz constant for all \( j \). A \emph{homogeneous polynomial of degree} \( n \) is a map, \( P : X \to \mathbb{R} \), of the form \( P(x) = A(x,x,\ldots,x) \), where \( A : X^n \to \mathbb{R} \) is \( n \)-multilinear and continuous. For \( n = 0 \) we take \( P \) to be constant. A \emph{polynomial of degree} \( n \) is a sum \( \sum_{i=0}^{n} P_i(x) \), where the \( P_i \) are \( i \)-homogeneous polynomials.

Let \( X \) be a Banach space, and \( G \subset X \) an open subset. A function \( f : G \to \mathbb{R} \) is called \emph{analytic} if for every \( x \in G \), there are a neighbourhood \( N_x \), and homogeneous polynomials \( P_n^x : X \to \mathbb{R} \) of degree \( n \), such that
\[
f(x + h) = \sum_{n \geq 0} P_n^x(h) \text{ provided } x + h \in N_x.
\]
Further information on polynomials may be found, for example, in [SS].

For a Banach space $X$, we define its (Taylor) complexification $\tilde{X} = X \oplus iX$ with norm

$$
\| x + iy \|_{\tilde{X}} = \sup_{0 \leq \theta \leq 2\pi} \| \cos \theta x - \sin \theta y \|_X = \sup_{T \in X^*, \|T\| \leq 1} \sqrt{T(x)^2 + T(y)^2}.
$$

We will sometimes denote, for $z = x + iy \in \tilde{X}$, $x := \text{Re } z$, $y := \text{Im } z$. If $L : E \to F$ is a continuous linear mapping between two real Banach spaces then there is a unique continuous linear extension $\tilde{L} : \tilde{E} \to \tilde{F}$ of $L$ (defined by $\tilde{L}(x + iy) = L(x) + iL(y)$) such that $\| \tilde{L} \| = \| L \|$. For a continuous $k$-homogeneous polynomial $P : E \to \mathbb{R}$ there is also a unique continuous $k$-homogeneous polynomial $\tilde{P} : \tilde{E} \to \mathbb{C}$ such that $\tilde{P} = P$ on $E \subset \tilde{E}$, but the norm of $P$ is not generally preserved: one has that $\| \tilde{P} \| \leq 2^{k-1} \| P \|$. It follows that if $q(x)$ is a continuous polynomial on $X$, there is a unique continuous polynomial $\tilde{q}(z) = \tilde{q}(x + iy)$ on $\tilde{X}$ where for $y = 0$ we have $\tilde{q} = q$. It also follows that if the Taylor series $f(x + h) = \sum_{n=0}^{\infty} D^n f(x)(h)/n!$ of a real analytic function $f : X \to \mathbb{R}$ at a point $x$ has radius of convergence $r_x$, then the series $\sum_{n=0}^{\infty} D^n f(x)(h)/n!$ has radius of convergence $r_x/2e$ (see for instance [CH], Lemma 0)). Consequently $f$ has a holomorphic extension $\tilde{f}$ to the neighborhood $\{x + iy : x, y \in X, \| x + iy \|_{\tilde{X}} < r_x\}$ of $X$ in $\tilde{X}$.

We will also use the fact that for this complexification procedure the complexifications $\tilde{c_0}, \tilde{\ell}_\infty$ of the real Banach spaces $c_0$ and $\ell_\infty$ are just the usual complex versions of these spaces. That is, we have $\tilde{c_0} = \{ \{ z_j \} : z_j \in \mathbb{C}, \lim_{j \to \infty} z_j = 0 \}$, with norm $\| z \|_{\tilde{c_0}} = \| \{ z_j \} \|_{\tilde{c_0}} = \max_j \{ | z_j | \}$, and similarly for $\tilde{\ell}_\infty$. For more information on complexifications (and polynomials) we recommend [MST].

In the sequel, all extensions of functions from $X$ to $\tilde{X}$, as well as subsets of $\tilde{X}$, will be embellished with a tilde. All smoothness properties of a norm or Minkowski functional are assumed implicitly to not include the point 0.

4. Preliminary results: the Preiss norm and separating functions

4.1. The Preiss norm. As developed in [FPWZ], there is a real analytic norm on $c_0$ (hereafter referred to as the Preiss norm, $\| \cdot \|$) that is equivalent to the canonical supremum norm $\| \cdot \|_\infty$. Let us recall the construction. Let $C : c_0 \to \mathbb{R}$ be given by $C(\{ x_n \}) = \sum_{n=1}^{\infty} (x_n)^{2^n}$. Let $W = \{ x \in c_0 : C(x) \leq 1 \}$. Then $\| \cdot \|$ is the Minkowski functional of $W$; that is, $\| x \|$ is the solution for $\lambda$ to $C(\lambda^{-1}x) = 1$. The Preiss norm is analytic at all non-zero points in $c_0$. To see this, let us define the function $\tilde{C} : V \to \mathbb{C}$ by $\tilde{C}(\{ z_n \}) = \sum_{n=1}^{\infty} (z_n)^{2^n}$ where $V$ is the subset of $\tilde{c_0}$ for which the series converges. Then $\tilde{C}$ is analytic at each $z \in \tilde{c_0}$. Indeed, the partial sums are analytic as a consequence of the analyticity of the (complex linear) projection functions $p_j(\{ z_i \}) = z_j$. Since
the series in the definition of $\tilde{C}$ converges locally uniformly at each $z \in \tilde{c}_0$ the analyticity of $\tilde{C}$ on $\tilde{c}_0$ follows. Also, for $z \in \tilde{c}_0$ sufficiently close to $c_0$ and $\lambda \in \mathbb{C} \setminus \{0\}$ sufficiently close to $\mathbb{R} \setminus \{0\}$ we have $\frac{\partial \tilde{C}(\lambda^{-1} z)}{\partial \lambda} \neq 0$, hence one can apply the complex Implicit Function Theorem (see e.g. [Ca], or [D] page 265, where the real result for Banach spaces is easily extended to the analytic case) to $F(z, \lambda) = \tilde{C}(\lambda^{-1} z) - 1$ to obtain a unique holomorphic solution $\lambda(z)$ to $F(z, \lambda) = 0$, with $\lambda := \tilde{\lambda}|_{c_0} = \|\cdot\|$, defined on a neighborhood of $c_0 \setminus \{0\}$ in $\tilde{c}_0$. Now if $x = \{x_n\}$ satisfies $\|x\|_\infty = 1$, then $\sum_{n=1}^\infty \left(\|x\|^{-1} x_n\right)^{2n} = 1$ implies $\|x\| \geq 1$. On the other hand, if $\|x\|_\infty = 1/2$, then $C(x) \leq \sum_{n=1}^\infty (1/2)^{2n} < 1$, implying $\|x\| < 1$. Hence, $(1/2) \|x\| \leq \|x\|_\infty \leq \|x\|$ for all $x$ in $c_0$. We shall use the above notation throughout this article.

4.2. Separating polynomials and separating Lipschitz analytic functions. Let $X$ be a Banach space. A separating polynomial on $X$ is a polynomial $q$ on $X$ such that $0 = q(0) < \inf\{|q(x)| : x \in S_X\}$. It is known [FPWZ] that if $X$ is superreflexive and admits a $C^\infty$-smooth bump function then $X$ admits a separating polynomial. The following lemma makes precise, observations of Kurzweil in [K].

**Lemma 1.** Let $X$ be a Banach space with a separating polynomial of degree $n$. Then there exist $K > 1$ and a $2n$ degree homogeneous separating polynomial $q : X \to [0, +\infty)$ such that $\|x\|^{2n} \leq q(x) \leq K\|x\|^{2n}$ for all $x \in X$.

**Proof.** We may suppose that $p = \sum_{i=1}^n p_i$, where $p_i$ is $i$-homogeneous for $1 \leq i \leq n$. Define $q = p_1^{2n} + p_2^{2n/2} + \cdots + p_n^2$. Note that $q$ is $2n$-homogeneous. As $p$ is separating, there is some $\eta > 0$ such that $q(x) \geq \eta$ for all $x \in S_X$. By scaling, we may assume that $\eta = 1$. Then from the $2n$-homogeneity, $q(x) \geq \|x\|^{2n}$. Now, as $q$ is continuous at 0, there is $\delta > 0$ with $\|x\| \leq \delta$ implies $q(x) \leq 1$, so again from $2n$-homogeneity, $1 \geq q\left(\frac{\|x\|}{\|x\|^{2n}}\right) = q(x) \frac{\delta^{2n}}{\|x\|^{2n}}$, and we are done. \hfill \square

Note that a separating polynomial, even though is Lipschitz on every bounded set, is never Lipschitz on all of $X$. In the proof of our main results we will require the existence of a globally Lipschitz, real analytic separating function on $X$. The following lemma provides us with such separating functions for every Banach space having a separating polynomial.

**Lemma 2.** Let $X$ be a Banach space with a separating polynomial $q$ as in the previous Lemma. Then the function $Q : X \to [0, +\infty)$ defined by

$$Q(x) = (1 + q(x))^{\frac{1}{2n}} - 1$$

is real analytic, satisfies $\inf_{\|x\| \geq 1} Q(x) > 0 = Q(0)$, and is Lipschitz on all of $X$. Moreover, there is some $\delta_Q > 0$ such that $Q$ has an holomorphic
extension $\tilde{Q}$ to the open strip $W_{\delta Q} := \{ x + z : x \in X, z \in \tilde{X}, \|z\|_{\tilde{X}} < \delta Q \}$ in $\tilde{X}$ such that $\tilde{Q}$ is still Lipschitz on $W_{\delta Q}$. Finally, we have that

$$Q(x) < 4r \implies \|x\| < 8r, \text{ for all } r \geq 1, x \in X.$$

In particular $Q(x) \geq \frac{1}{2}\|x\|$ for $\|x\| \geq 8$ and $Q$ is separating.

**Proof.** Since the function $w \mapsto w^{1/2n}$ is well defined (taking the usual branch of log) and holomorphic on the half-plane $\text{Re } w \geq 1/2$ in $\mathbb{C}$, it is clear that $\tilde{Q}(x + z) = (1 + \tilde{q}(x + z))^{1/2m}$ will be well defined and holomorphic on such a strip $W_{\delta Q}$ as soon as we find $\delta = \delta Q > 0$ small enough so that $\text{Re } (1 + \tilde{q}(x + z)) \geq 1/2$ whenever $x \in X, z \in \tilde{X}, \|z\| \leq \delta$.

Let $A$ (resp. $\tilde{A}$) denote a $2n$-multilinear form on $X$ (resp. $\tilde{X}$) such that $q(x) = A(x, ..., x)$ (resp. $\tilde{q}(w) = \tilde{A}(w, ..., w)$). We have, for $x \in X$ and $z \in \tilde{X}$ with $\|z\| \leq \delta$,

$$\text{Re } \tilde{q}(x + z) = \text{Re } A(x, x + z) = A(x, x, ..., x) + \text{ terms of the form } \text{Re } A(y_1, ..., y_{2n}),$$

where $y_j \in \{x, z\}$ for all $j$, and $y_k = z$ for at least one $k$

$$\geq A(x, ..., x) - \sum_{j=1}^{2n} \|A\| \binom{2n}{j} \|x\|^{2n-j} \|z\|^j$$

$$\geq \|x\|^{2n} - \|A\| \sum_{j=1}^{2n} \binom{2n}{j} \|x\|^{2n-j} \delta^j$$

$$\geq \|x\|^{2n} - \delta \|A\| \sum_{j=1}^{2n} \binom{2n}{j} \|x\|^{2n-j}$$

$$\geq \|x\|^{2n} - \delta \|A\| (\|x\| + 1)^{2n}.$$

Hence

$$\text{Re } (1 + \tilde{q}(x + z)) \geq \frac{1}{2} + \frac{1}{2} + \|x\|^{2n} - \delta \|A\| (\|x\| + 1)^{2n}.$$

Since the real function

$$f(t) = \frac{1}{2} + t^{2n}$$

is continuous, positive and satisfies $\lim_{|t| \to \infty} f(t) = 1$, there exists $\alpha \in (0, 1)$ such that $f(t) \geq \alpha$, and therefore $1 + t^{2n} - \alpha(|t| + 1)^{2n} \geq 1/2$ for all $t \in \mathbb{R}$. If we set

$$\delta = \frac{1}{2} \min\{1, \alpha/\|A\|\},$$
this implies that \(1 + \|x\|^{2n} - \delta \|A\| (\|x\| + 1)^{2n} \geq 1/2\) for all \(x \in X\), and therefore
\[
\text{Re} \left( 1 + \tilde{g}(x + z) \right) \geq \frac{1}{2}, \quad \text{for all } x + z \in W_{\delta},
\]
and the function \(\tilde{Q}\) is holomorphic on \(W_{\delta}\).

Now let us check that \(\tilde{Q}\) is Lipschitz on \(W_{\delta}\). By the same estimation as above, for \(x \in X\), \(z \in \tilde{X}\) with \(\|z\| \leq \delta\), we have
\[
|1 + \tilde{g}(x + z)| \geq \text{Re} \left( 1 + \tilde{g}(x + z) \right) \geq 1 + \|x\|^{2n} - \delta \|A\| (\|x\| + 1)^{2n} \geq 1/2,
\]
and on the other hand, the derivative of \(\tilde{g}\) being a \((2n - 1)\)-homogeneous polynomial,
\[
|\tilde{g}'(x + z)| \leq \|\tilde{g}'\| (\|x\| + \|z\|)^{2n-1} \leq \|\tilde{g}'\| (\|x\| + \|z\|)^{2n-1}.
\]
By combining these two inequalities we get, for \(x + z \in W_{\delta}\),
\[
\|\tilde{Q}'(x + z)\| = \frac{\|\tilde{g}'(x + z)\|}{2n (1 + \tilde{g}(x + z))^{1 - \frac{1}{2n}}} \leq \frac{\|\tilde{g}'\| (\|x\| + \|z\|)^{2n-1}}{(1 + \|x\|^{2n} - \delta \|A\| (\|x\| + 1)^{2n})^{\frac{2n-1}{2n}}}.
\]
Since the function
\[
\mathbb{R} \ni t \mapsto \frac{\|\tilde{g}'\| (|t| + \delta)^{2n-1}}{(1 + |t|^{2n} - \delta \|A\| (|t| + 1)^{2n})^{\frac{2n-1}{2n}}} \in \mathbb{R}
\]
is bounded on \(\mathbb{R}\), it follows that the derivative \(\tilde{Q}'\) is bounded on the convex set \(W_{\delta}\), and therefore the function \(\tilde{Q}\) is Lipschitz on \(W_{\delta}\).

Finally, assume that \(\|x\| \geq 8r\), \(r \geq 1\). Since \(\tilde{g}(x) \geq \|x\|^{2n}\) and the function \(h(t) = (1 + t^{2n})^{1/2n} - 1\) is increasing, we have
\[
Q(x) \geq h(\|x\|) \geq h(8r).
\]
Now, for \(r \geq 1\) we have
\[
(1 + 4r)^{2n} \leq 2^{2n}(4r)^{2n} \leq 1 + (8r)^{2n},
\]
from which it follows that
\[
h(8r) \geq 4r,
\]
and therefore \(Q(x) \geq 4r\).

5. The proof of the main result

In the sequel we will be using the following notation: for the real analytic and Lipschitz separating function \(Q\) constructed in Lemma 2, and for every \(x \in X\), \(r > 0\), we define the \(Q\)-bodies
\[
D_Q(x, r) = \{y \in X : Q(y - x) < r\}.
\]
5.1. Real analytic sup-partitions of unity. The following Lemma provides us, for any given $r > 0, \varepsilon \in (0, 1)$, with a real-analytic analogue of sup-partitions of unity (see [F2, HJ]) which, one could say, are $\varepsilon$-subordinated to a covering of the form $X = \bigcup_{j=1}^{\infty} D_{Q}(x_{j}, 4r)$.

Lemma 3. Let $\tilde{V} = W_{\delta_{Q}}$ be an open strip around $X$ in $\tilde{X}$ in which the function $\overline{Q}$ of Lemma 3 is defined. Given $r, K, \eta > 0, \varepsilon \in (0, 1)$, and a covering of the form $X = \bigcup_{j=1}^{\infty} D_{Q}(x_{j}, r)$, there exists a sequence of holomorphic functions $\varphi_{n} = \varphi_{n, r, \varepsilon} : \tilde{V} \to \mathbb{C}$, whose restrictions to $X$ we denote by $\varphi_{n} = \varphi_{n, r, \varepsilon}$, with the following properties:

1. The collection $\{\varphi_{n, r, \varepsilon} : X \to [0, 2] \mid n \in \mathbb{N}\}$ is equi-Lipschitz on $X$, with Lipschitz constant $L = 2\text{Lip}(Q)/r$ (independent of $\varepsilon$).
2. $0 \leq \varphi_{n, r, \varepsilon}(x) \leq 1 + \varepsilon$ for all $x \in X$.
3. For each $x \in X$ there exists $m = m_{x, r} \in \mathbb{N}$ (independent of $\varepsilon$) with $\varphi_{m, r, \varepsilon}(x) > 1$. Besides, $m_{x, s} \geq m_{x, r}$ whenever $r \geq s$.
4. $0 \leq \varphi_{n, r, \varepsilon}(x) \leq \varepsilon$ for all $x \in X \setminus D_{Q}(x_{n}, 4r)$.
4'. $|\varphi_{n, r, \varepsilon}(x)| \leq \varepsilon$ for all $x \in X \setminus D_{Q}(x_{n}, 5r)$.
5. For each $x \in X$ there exist $\delta_{x, r} > 0, a_{x, r} > 0$, and $n_{x, r} \in \mathbb{N}$ (independent of $\varepsilon$) such that

$$|\varphi_{n, r, \varepsilon}(x + z)| < \frac{1}{n\lambda_{r}^{2}M_{n}} \text{ for } n > n_{x, r}, z \in \tilde{X} \text{ with } \|z\|_{\tilde{X}} < \delta_{x, r},$$

where $M_{n} := e^{2K^{2}n}(1 + \|x_{n}\|)$.

6. For each $x \in X$ there exists $\overline{\delta}_{x, r} > 0$ (independent of $\varepsilon$) and $n_{x, r, \varepsilon} \in \mathbb{N}$ such that for $\|z\|_{\tilde{X}} < \overline{\delta}_{x, r}$ and $n > n_{x, r, \varepsilon}$ we have $|\varphi_{n, r, \varepsilon}(x + z)| < \varepsilon$.
7. For each $x \in X$ there exists $\delta_{x, r, \varepsilon}$ such that

$$|\varphi_{n, r, \varepsilon}(x + z)| \leq 1 + 2\varepsilon \text{ for } n \in \mathbb{N}, r \geq 1, z \in \tilde{X} \text{ with } \|z\|_{\tilde{X}} \leq \delta_{x, r, \varepsilon}.$$

Moreover, in the case when $r \geq 1$, the numbers $\delta_{x, r}, a_{x, r}, n_{x, r}, \overline{\delta}_{x, r, \varepsilon}$ and $n_{x, r, \varepsilon}$ can be assumed to be independent of $r$.

Proof. Define subsets $A_{1, r} = \{y_{1} \in \mathbb{R} : -1 \leq y_{1} \leq 4r\}$, and, for $n \geq 2$,

$A_{n, r} = \{y = \{y_{j}\}_{j=1}^{n} \in \ell_{\infty}^{n} : -1 - r \leq y_{n} \leq 4r, 2r \leq y_{j} \leq M_{n, r} + 2r \text{ for } 1 \leq j \leq n-1\},$

$A'_{n, r} = \{y = \{y_{j}\}_{j=1}^{n} \in \ell_{\infty}^{n} : -1 \leq y_{n} \leq 3r, 3r \leq y_{j} \leq M_{n, r} + r \text{ for } 1 \leq j \leq n-1\},$

where $M_{n, r} = \sup\{Q(x - x_{j}) : x \in D_{Q}(x_{n}, 4r), 1 \leq j \leq n\}$.

Let $b_{n} = b_{n, r, \varepsilon} : \ell_{\infty}^{n} \to [0, 2]$ be the function defined by

$$b_{n}(y) = (1 + \varepsilon) \max\{0, 1 - \frac{1}{r}\text{dist}_{\infty}(y, A'_{n})\},$$

where $\text{dist}_{\infty}(y, A) = \inf\{\|y - a\|_{\infty} : a \in A\}$.

In the sequel, in order not to be too burdened by notation, will omit the subscripts $r, \varepsilon$ whenever they do not play a relevant role locally in the argument.
It is clear that \( \text{support}(b_n) = A_n \), that \( b_n = 1 + \varepsilon \) on \( A'_n \), and that \( b_n \) is \((2/r)\)-Lipschitz (note in particular that the Lipschitz constant of \( b_n \) does not depend on \( n \in \mathbb{N} \) or \( \varepsilon \in (0, 1) \)).

Since the function \( b_n = b_{n, r, \varepsilon} \) is \( \frac{2}{r} \)-Lipschitz and bounded by 2 on \( \mathbb{R}^n \), it is a standard fact that the normalized integral convolutions of \( b_n \) with the Gaussian-like kernels \( y \mapsto G_n(y) := e^{-\kappa \sum_{j=1}^{\infty} 2^{-j} y_j^2} \),

\[
x \mapsto \frac{1}{T_n} b_n * G_n(x) = \frac{1}{\int_{\mathbb{R}^n} e^{-\kappa \sum_{j=1}^{n} 2^{-j} y_j^2} dy} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa \sum_{j=1}^{n} 2^{-j} (x_j - y_j)^2} dy,
\]

where \( T_n = T_{\kappa, n} = \int_{\mathbb{R}^n} e^{-\kappa \sum_{j=1}^{n} 2^{-j} y_j^2} dy \), converge to \( b_n \) uniformly on \( \mathbb{R}^n \) as \( \kappa \to +\infty \). Moreover, since \( b_n = b'_n = 0 \) on the open set \( \ell^m_{\infty} \setminus A_n \), we also have that

\[
\lim_{\kappa \to +\infty} \left( \frac{1}{T_{\kappa, n}} b_{n, r, \varepsilon} * G_n \right)'(x) = b'_{n, r, \varepsilon}(x)
\]

uniformly in \( x \in \{ y \in \ell^m_{\infty} : \text{dist}_\infty(y, A_n) \geq r \} \). Therefore, for each \( n \in \mathbb{N} \) we can find \( \kappa_n = \kappa_{n, r, \varepsilon} > 0 \) large enough so that

\[
|b_{n, r, \varepsilon}(x) - \frac{1}{T_{\kappa, n}} b_{n, r, \varepsilon} * G_{\kappa_n, \varepsilon}(x)| \leq \frac{\varepsilon}{2} \text{ for all } x \in \mathbb{R}^n \text{ and } (*)
\]

\[
||b'_{n}(x) - (b_n * G_{\kappa_n})'(x)|| \leq \frac{\varepsilon}{2} \text{ whenever } \text{dist}_\infty(x, A_n) \geq r. \text{ (**)}
\]

For reasons that will become apparent later on, we shall also take each \( \kappa_{n, r, \varepsilon} \) to be large enough so that

\[
\kappa_n \geq 2(\sqrt{2})^n (n!)^2. \text{ (***)}
\]

Also, note that when \( r \geq 1 \) then the functions \( b_{n, r, \varepsilon} \) are all 2-Lipschitz and therefore \( \kappa_{n, r, \varepsilon} \) can be assumed to be independent of \( r \).

Let us also define \( \widetilde{T}_n = \int_{\mathbb{R}^n} e^{-\sum_{j=1}^{n} 2^{-j} y_j^2} dy \), and

\[
T_n := T_{\kappa, n} = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} y_j^2} dy = \frac{1}{\kappa_n^{n/2}} \widetilde{T}_n,
\]

and observe that by a change of variables

\[
T_n = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} y_j^2} dy = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (A_j - y_j)^2} dy
\]

for any \( A_j \in \mathbb{R} \).

Now define \( \nu_n : \ell^m_{\infty} \to \mathbb{R} \) by

\[
\nu_n(x) := \frac{1}{T_{\kappa_n}} b_n * G_{\kappa_n}(x) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (x_j - y_j)^2} dy.
\]

Let us note that

\[
\nu_n(x) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (x_j - y_j)^2} dy = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(x-y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} y_j^2} dy,
\]
Lemma 2, calling them \( \tilde{\nu} \).

\[ \nu_n (x) = \nu_n (x') = \frac{1}{T_n} \int_{\mathbb{R}^n} (b_n (x - y) - b_n (x' - y)) e^{-\kappa n \sum_{j=1}^{n} 2^{-j} y_j^2} dy \]

\[ \leq \frac{1}{T_n} \int_{\mathbb{R}^n} |b_n (x - y) - b_n (x' - y)| e^{-\kappa n \sum_{j=1}^{n} 2^{-j} y_j^2} dy \]

\[ \leq \frac{2}{r} \| x - x' \|_\infty \frac{1}{T_n} \int_{\mathbb{R}^n} e^{-\kappa n \sum_{j=1}^{n} 2^{-j} y_j^2} dy \]

\[ = \frac{2}{r} \| x - x' \|_\infty . \]

Hence, \( \nu_n \) is \( \frac{2}{r} \)-Lipschitz. Note also that \( \| \nu_n \|_\infty \leq \| b_n \|_\infty \leq 2 \).

Next, consider the map \( \lambda_n : X \rightarrow l_n^\infty \) given by

\[ \lambda_n (x) = (Q (x - x_1), \ldots , Q (x - x_n)) , \]

where \( Q \) is the real analytic and Lipschitz separating function constructed in Lemma 2.

Then for \( n \geq 1 \) we define (real) analytic maps \( \varphi_n = \varphi_{n,r,\varepsilon} : X \rightarrow \mathbb{R} \) by

\[ \varphi_n (x) = \nu_n (\lambda_n (x)) = \nu_n \left( \{ Q (x - x_j) \}_{j=1}^{n} \right) \]

\[ = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n (y) e^{-\kappa n \sum_{j=1}^{n} 2^{-j} (Q(x-x_j)-y_j)^2} dy. \]

Since \( Q \) is Lipschitz on \( X \), we have that

\[ |\varphi_n (x) - \varphi_n (x')| = |\nu_n (\lambda_n (x)) - \nu_n (\lambda_n (x'))| \leq \frac{2}{r} \| \lambda_n (x) - \lambda_n (x') \|_\infty \]

\[ = \frac{2}{r} \| \{ Q (x - x_j) - Q (x' - x_j) \}_{j=1}^{n} \|_\infty \leq \frac{2}{r} \text{Lip}(Q) \| x - x' \|_X , \]

hence the collection \( \{ \varphi_{n,r,\varepsilon} : n \in \mathbb{N} \} \) is uniformly Lipschitz on \( X \), with constant \( \frac{2}{r} \text{Lip}(Q) \).

We can extend the maps \( \varphi_{n,r,\varepsilon} \) to complex valued maps defined on \( W_{\delta_Q} \) (see Lemma 2), calling them \( \tilde{\varphi}_{n,r,\varepsilon} \). Namely (where \( x \in X, z \in \tilde{X} \)),

\[ \tilde{\varphi}_n (x + z) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n (y) e^{-\kappa n \sum_{j=1}^{n} 2^{-j} (\tilde{Q}(x-x_j+z)-y_j)^2} dy. \]

Note that the \( \tilde{\varphi}_n \) are well defined (as the \( b_n \) have compact supports) and are holomorphic where \( \tilde{Q} \) is (namely on \( \tilde{V} \)). Hence the above calculation establishes (1) as \( \tilde{\varphi}_n |_X = \varphi_n \). Bearing in mind that \( \| \nu_n \|_\infty \leq \| b_n \|_\infty \), it is also clear that \( 0 \leq \varphi_n (x) \leq 1 + \varepsilon \) for all \( x \in X, n \in \mathbb{N} \), which proves (2).

Let us show (3). For each fixed \( x \in X \), there exists \( m = m_{x,r} \) with \( x \in D_Q(x_n,3r) \) but with \( x \notin D_Q(x_i,3r) \) for \( i < m \). Observe that if \( r \geq s \) then \( x \notin D_Q(x_i,3s) \) for \( i < m_{x,r} \), so we necessarily have \( m_{x,s} \geq m_{x,r} \) for all \( r \geq s \).
This implies that the point \((Q(x-x_1), Q(x-x_2), ..., Q(x-x_m))\) belongs to \(A_{r,\epsilon}',\) where the function \(b_{m,\epsilon} \) takes the value 1 + \(\epsilon\). According to (\(\ast\)), we have
\[
|1 + \epsilon - \varphi_{m}(x)| = \frac{1}{T_{\kappa_n}} b_{m} (Q(x-x_1), Q(x-x_2), ..., Q(x-x_m)) \leq \epsilon/2,
\]
which yields \(\varphi_{m}(x) \geq 1 + \epsilon - \epsilon/2 > 1\).

Properties (4) and (4') are shown similarly: if \(Q(x-x_n) \geq 4r\) then the point \((Q(x-x_1), ..., Q(x-x_n))\) lies in a region of \(\mathbb{R}^n\) where the function \(b_n\) takes the value 0, and (\(\ast\)) immediately gives us \(\varphi_{n}(x) \leq \epsilon/2\). And if \(Q(x-x_n) \geq 5r\) then \((Q(x-x_1), ..., Q(x-x_n)) \in \{y \in T_{\kappa_n}: \text{dist}_{\infty}(y, A_n) \geq r\}\), so (\(\ast\ast\)) implies that \(\|\varphi'_{n}(x)\| \leq \epsilon/2\).

We finally show the more delicate properties (5), (6) and (7). For \(x \in X\) and \(z \in \tilde{X}\) with \(\|z\| < \delta_Q\), according to Lemma 2 we have
\[
\tilde{Q}(x-x_j + z) = Q(x-x_j) + Z_j,
\]
where \(Z_j \in \mathbb{C}\) with \(|Z_j| \leq C\|z\|_{\tilde{X}},\)
\(\tilde{C}\) being the Lipschitz constant of \(\tilde{Q}\) on the strip \(W_{\delta_Q}\).

Now
\[
\left(\tilde{Q}(x-x_j + z) - y_j\right)^2 = (Q(x-x_j) - y_j + Z_j)^2
\]
\[
= (Q(x-x_j) - y_j)^2 + 2(Q(x-x_j) - y_j)Z_j + Z_j^2.
\]
Hence, for \(\|z\|_{\tilde{X}} < \delta_Q\) we have
\[
\Re\left(\tilde{Q}(x-x_j + z) - y_j\right)^2
\]
\[
= (Q(x-x_j) - y_j)^2 + 2(Q(x-x_j) - y_j)\Re Z_j + \Re(Z_j^2)
\]
\[
= (Q(x-x_j) - y_j + \Re Z_j)^2 - (\Re Z_j)^2 + \Re(Z_j^2)
\]
\[
\geq (Q(x-x_j) - y_j + \Re Z_j)^2 - 2C^2\|z\|_{\tilde{X}}^2.
\]

Next, for each \(x \in X\) there exists \(n_{x,r}\) so that \(Q(x-x_{n_{x,r}}) < r\) (note that \(n_{x,r}\) can be assumed to be equal to \(n_{x,1}\) in the case \(r \geq 1\)), and also for \(n > n_{x,r}\) and \(y \in A_n = \text{support}(b_n)\) we have \(y_{n_{x,r}} \geq 2r\). Hence, \(y_{n_{x,r}} - Q(x-x_{n_{x,r}}) > 2r - r = r\). Thus, for \(\|z\|_{\tilde{X}} < \min\{r/2C, \delta_Q\}, n > n_{x,r}\) and \(y \in A_n\) we have
\[
(Q(x-x_{n_{x,r}}) - y_{n_{x,r}} + \Re(Z_{n_{x,r}}))^2 = (y_{n_{x,r}} - Q(x-x_{n_{x,r}}) - \Re(Z_{n_{x,r}}))^2
\]
\[
\geq (y_{n_{x,r}} - Q(x-x_{n_{x,r}}) - C\|z\|_{\tilde{X}})^2 \geq (r - C\|z\|_{\tilde{X}})^2
\]
\[
\geq \left(\frac{r}{2}\right)^2 = \frac{r^2}{4}.
\]
It follows that for $\|z\|_{\tilde{X}} < \min\{r/2C, \delta_{Q}\}$, $n > n_{x,r}$, and $y \in A_n$,

$$
\sum_{j=1}^{n} 2^{-j} \text{Re} \left( \tilde{Q} (x - x_j + z) - y_j \right)^2
$$

$$
\geq \sum_{j=1}^{n} 2^{-j} (Q(x - x_j) - y_j + \text{Re} Z_j)^2 - 2C^2 \sum_{j=1}^{n} 2^{-j} \|z\|^2_{\tilde{X}}
$$

$$
= \frac{1}{2} \sum_{j=1}^{n} 2^{-j} (Q(x - x_j) - y_j + \text{Re} Z_j)^2 + \frac{1}{2} \sum_{j=1}^{n} 2^{-j} (Q(x - x_j) - y_j + \text{Re} Z_j)^2 - 2C^2 \sum_{j=1}^{n} 2^{-j} \|z\|^2_{\tilde{X}}
$$

$$
\geq \frac{1}{2} \sum_{j=1}^{n} 2^{-j} (Q(x - x_j) - y_j + \text{Re} Z_j)^2 + \frac{1}{2} 2^{-n_{x,r} - 3} r^2 - 2C^2 \|z\|^2_{\tilde{X}}.
$$

Define $\delta_{x,r} = r \sqrt{\frac{2^{-n_{x,r} - 4}}{C^2}}$, $a_{x,r} = r^2 2^{-n_{x,r} - 4}$ for $0 < r \leq 1$, and put $a_{x,1}$ and $\delta_{x,1} := \delta_{x,1}$ if $r \geq 1$. For every $n > n_{x,r}$, $y \in A_n$, and $z \in \tilde{X}$ with $\|z\|_{\tilde{X}} \leq \delta_{x,r}$, we have

$$
\sum_{j=1}^{n} 2^{-j} \text{Re} \left( \tilde{Q} (x - x_j + z) - y_j \right)^2
$$

$$
\geq \frac{1}{2} \sum_{j=1}^{n} 2^{-j} (Q(x - x_j) - y_j + \text{Re} Z_j)^2 + 2^{-n_{x,r} - 3} r^2 - 2C^2 \delta_{x,r}^2
$$

$$
= \frac{1}{2} \sum_{j=1}^{n} 2^{-j} (Q(x - x_j) - y_j + \text{Re} Z_j)^2 + a_{x,r}.
$$

Therefore, for every $n > n_{x,r}$ and $z \in \tilde{X}$ with $\|z\|_{\tilde{X}} \leq \delta_{x,r}$ we can estimate

$$
|\varphi_{n,r,\varepsilon} (x + z)| = \left| \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (\tilde{Q}(x_0 - x_j + z) - y_j)^2} \, dy \right|
$$

$$
= \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \text{Re} \sum_{j=1}^{n} 2^{-j} (\tilde{Q}(x_0 - x_j + z) - y_j)^2} \, dy
$$

$$
\leq \frac{2}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n a_{x,r} - \frac{1}{2} \kappa_n \sum_{j=1}^{n} 2^{-j} (Q(x - x_j) + \text{Re}(Z_j) - y_j)^2} \, dy
$$

$$
\leq \frac{2e^{-\kappa_n a_{x,r}}}{T_n} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \kappa_n \sum_{j=1}^{n} 2^{-j} (Q(x - x_j) + \text{Re}(Z_j) - y_j)^2} \, dy
$$

$$
= \frac{2e^{-\kappa_n a_{x,r}}}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} u_j^2} (\sqrt{2})^n \, du
$$

$$
= 2(\sqrt{2})^n e^{-\kappa_n a_{x,r}}.
$$
Now, by choice of $\kappa_n = \kappa_{n,x,r}$ (see (***) above), we have

$$2(\sqrt{2})^n e^{-\kappa_n a_{x,r}} \leq \frac{2(\sqrt{2})^n n!}{\kappa_n^2 a_{x,r}^2} \leq \frac{1}{a_{x,r} n! M_n}.$$ 

Hence

$$|\tilde{\varphi}_{n,r,\varepsilon}(x + z)| \leq \frac{1}{n! a_{x,r} M_n}$$

for all $n > n_{x,r}$, $\varepsilon \in (0, 1)$, and $z \in \tilde{X}$ with $\|z\| \leq \delta_{x,r}$. This establishes (5).

In particular by taking $n_{x,r,\varepsilon} > n_{x,r}$ sufficiently large, we can guarantee that for $n > n_{x,r,\varepsilon}$ and $\|z\|_{\tilde{X}} < \delta_{x,r}$ we have $|\tilde{\varphi}_n(x_0 + z)| < \varepsilon$, which proves (6).

As for (7), we can write, as above,

$$\tilde{Q} (x - x_j + z) = Q(x - x_j) + Z_j,$$

where $Z_j \in C$ with $|Z_j| \leq C \|z\|_{\tilde{X}} \leq C \delta_{x,r}$, and we have

$$\text{Re} \left(\tilde{Q} (x - x_j + z) - y_j\right)^2 \geq (Q(x - x_j) - y_j + \text{Re} Z_j)^2 - 2C^2 \delta_{x,r}^2,$$

hence

$$|\tilde{\varphi}_{n,r,\varepsilon}(x + z)| = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \text{Re} \sum_{j=1}^n 2^{-j}(Q(x_0 - x_j + z) - y_j)} dy$$

\[ \leq \frac{1 + \varepsilon}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j}(Q(x - x_j) + \text{Re} Z_j - y_j)} dy \]

\[ = \frac{1 + \varepsilon}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j}(Q(x - x_j) + \text{Re} Z_j - y_j)} dy \]

\[ = (1 + \varepsilon)^e \leq C \delta_{x,r}.$$

This is true for every $n \in \mathbb{N}$, $z \in \tilde{X}$ with $\|z\|_{\tilde{X}} \leq \delta_{x,r}$. On the other hand, for $n > n_{x,r}$ we already know that

$$|\tilde{\varphi}_{n,r,\varepsilon}(x + z)| \leq \frac{1}{a_{x,r} n!}$$

if $\|z\|_{\tilde{X}} \leq \delta_{x,r}$, and since this sequence converges to 0 we can find $m_{x,r,\varepsilon}$ such that

$$|\tilde{\varphi}_{n,r,\varepsilon}(x + z)| \leq \frac{1}{a_{x,r} n!} < 1 + 2\varepsilon$$

for $n \geq m_{x,r,\varepsilon}$, $\|z\|_{\tilde{X}} \leq \delta_{x,r}$. Then we can define

$$\delta_{x,r,\varepsilon} = \min \left\{ \delta_{x,r}, \sqrt{\frac{1}{2\kappa_{x,r} C^2} \log \left( \frac{1 + 2\varepsilon}{1 + \varepsilon} \right)}, \ldots, \sqrt{\frac{1}{2\kappa_{m_{x,r,\varepsilon},r,\varepsilon} C^2} \log \left( \frac{1 + 2\varepsilon}{1 + \varepsilon} \right)} \right\},$$

so we get

$$|\tilde{\varphi}_{n,r,\varepsilon}(x + z)| \leq 1 + 2\varepsilon$$

if $\|z\|_{\tilde{X}} \leq \delta_{x,r,\varepsilon}$, for all $n \in \mathbb{N}$ and $r > 0$. In the case $r \geq 1$ the numbers $\kappa_{n,r,\varepsilon}, \delta_{x,r}, a_{x,r}$ are independent of $r$, hence so are $m_{x,r,\varepsilon}$ and $\delta_{x,r,\varepsilon}$. \qed
Remark 1. In this paper we will only use the above Lemma in the case $r \geq 1$, and we will not need property \((4')\). The $M_n$ can also be dropped from property \((5)\). However, we shall use the full power of this Lemma in \([AFK]\).

5.2. A refinement of the implicit function theorem for holomorphic mappings on complex Banach spaces. We shall use the following refinement of \([CHP]\) Theorem 1.1, which provides a lower bound on the size of the domains of existence of the solutions to a family of complex analytic implicit equations, in a uniform manner with respect to the equation, whenever appropriate bounds are available for the functions defining the equations and its first derivatives with respect to the dependent variable.

Proposition 1. Let $\psi : E \times \mathbb{C} \times P \to \mathbb{C}$ be a function, where $E$ is a complex Banach space and $P$ is a set of parameters. Let $(x_0, \mu_0) \in E \times \mathbb{C}$, $R > 0$, and suppose that for each $p \in P$ the function

\[ (x, \mu) \mapsto \psi_p(x, \mu) = \psi(x, \mu, p) \]

is (complex) analytic on $B = \{(x, \mu) : \|x - x_0\| \leq R, |\mu - \mu_0| \leq R\}$, and that there exist $M, A > 0$ such that

\[ \frac{\partial \psi_p}{\partial \mu}(x_0, \mu_0) \geq A, \quad \text{and} \quad |\psi_p(x, \mu)| \leq M \quad \text{for all} \quad (x, \mu) \in B, p \in P. \]

Then there exist $0 < r = r(A, R, M) < R$ and $s = s(A, R, M) > 0$ so that for each $p \in P$ there exists a unique analytic function $\mu = \mu_p(x)$, defined on the ball $B(x_0, s)$, such that $\psi_p(x, \mu) = 0$ for $(x, \mu) \in B(x_0, s) \times B(\mu_0, r)$ if and only if $\mu = \mu_p(x)$ for $x \in B(x_0, s)$.

Proof. This is essentially a restatement of Theorem 1.1 in \([CHP]\), where the result is proved (with estimates for $r$ and $s$) in the case when $P$ is a singleton and $E = \mathbb{C}^n$. The part of the proof in \([CHP]\) that does not follow directly from the implicit function theorem for holomorphic mappings combines an estimation of the Taylor series of $\psi_p$ at $(x_0, \mu_0)$ with respect to the variable $\mu \in \mathbb{C}$, and an application of Rouche’s theorem to the functions $\mathbb{C} \ni \mu \mapsto \psi_p(x_0, \mu) \in \mathbb{C}$ and $\mathbb{C} \ni \mu \mapsto \psi_p(x, \mu) \in \mathbb{C}$. In these two steps the variable $x \in E$ can be regarded as a parameter, so the same argument applies, with obvious changes, to the present situation. \qed

5.3. Proof of the main result in the case of a 1-Lipschitz function taking values in $[0, 1]$. Let us define

\[ A = \{\alpha = \{\alpha_n\}_{n=1}^{\infty} : 1 \leq \alpha_n \leq 1001 \text{ for all } n \in \mathbb{N}\}. \]

For every $\alpha \in A$ and $z = \{z_n\}_{n=1}^{\infty} \in \ell_0 \setminus \{0\}$, let us denote $\alpha z := \{\alpha_n z_n\}_{n=1}^{\infty}$, and observe that $\alpha z \in \ell_0 \setminus \{0\}$. In fact the mapping $\alpha : \ell_0 \to \ell_0$, defined by $\alpha(z) = \alpha z$ is a linear isomorphism satisfying $\|z\|_\infty \leq \|\alpha z\|_\infty \leq 1001\|z\|_\infty$. 


Lemma 4. Let \( r \geq 1 \), and let \( \varphi_n = \varphi_{n,r,\varepsilon}, \tilde{\varphi}_n = \tilde{\varphi}_{n,r,\varepsilon} \) be the collections of functions defined in Lemma 3. We have that:

1. There exists an open neighborhood \( \tilde{V}_1 \subset \tilde{V} \) of \( X \) in \( \tilde{X} \) such that the mapping \( \tilde{\Phi}_{\alpha,r,\varepsilon} = \tilde{\Phi} : \tilde{V}_1 \to \tilde{c}_0 \) defined by

\[
\tilde{\Phi}(z) = \left\{ \alpha_n \tilde{\varphi}_n(z) \right\}_{n=1}^\infty
\]

is holomorphic, for any \( \alpha \in A, \ r \geq 1, \ \varepsilon \in (0,1) \).

2. Fix \( \varepsilon \in (0,1) \). Then there exists an open neighborhood \( \tilde{W}_\varepsilon \) of \( X \) in \( \tilde{X} \) such that for every \( \alpha \in A, \ r \geq 1 \), there is a holomorphic extension \( \tilde{F}_{\alpha,r,\varepsilon} \) of \( \lambda \circ \tilde{\Phi}_{\alpha,r,\varepsilon} \), defined on \( \tilde{W}_\varepsilon \), where \( \lambda \) is the Preiss norm from section 4.1, and such that

\[
|\tilde{F}_{\alpha,r,\varepsilon}(x+iy) - \lambda \circ \tilde{\Phi}_{\alpha,r,\varepsilon}(x)| \leq \varepsilon
\]

for every \( x, y \in X \) with \( x+iy \in \tilde{W}_\varepsilon \).

Proof. Since \( r \geq 1 \) we have \( a_{x,r} = a_{x,1} \) and \( \delta_{x,r} = \delta_{x,1} \) in Lemma 3, so we will simply denote these numbers by \( a_x \) and \( \delta_x \), respectively. Because the numerical series \( \sum_{n=1}^\infty 1/n!a_x^n \) is convergent, and bearing in mind property (5) of Lemma 3, we have that the series of functions

\[
\sum_{n=1}^\infty |\alpha_n \tilde{\varphi}_n(z) |
\]

is uniformly convergent on the ball \( B_{\tilde{X}}(x, \delta_x) \), which clearly implies that the series

\[
\sum_{n=1}^\infty \alpha_n \tilde{\varphi}_n(z) e_n
\]

is uniformly convergent for \( z \in B_{\tilde{X}}(x, \delta_x) \), for any \( \{a_n\}_{n=1}^\infty \in \ell_\infty \). If we set \( \tilde{V}_1 = \bigcup_{x \in X} B_{\tilde{X}}(x, \delta_x) \) and note that for each \( n \in \mathbb{N} \) the mapping

\[
\tilde{V} \ni z \mapsto \varphi_n(z) e_n \in \tilde{c}_0
\]

is holomorphic, it is clear that \( \tilde{\Phi}_{\alpha,r,\varepsilon} = \sum_{n=1}^\infty \tilde{\varphi}_{n,r,\varepsilon} e_n \), being a series of holomorphic mappings which converges locally uniformly in \( \tilde{V}_1 \), defines a holomorphic mapping from \( \tilde{V}_1 \) into \( \tilde{c}_0 \). Note that \( \tilde{V}_1 \) is independent of \( \alpha, r, \varepsilon \) because so is \( \delta_x \).

Let us now prove (2). Define the function \( \psi_{\alpha,r,\varepsilon} : \tilde{V}_1 \times \mathbb{C} \setminus \{0\} \to \mathbb{C} \) by

\[
\tilde{\psi}_{\alpha,r,\varepsilon}(w, \mu) = \sum_{n=0}^\infty \left( \frac{\alpha_n \tilde{\varphi}_{n,r,\varepsilon}(w)}{\mu} \right)^{2n} - 1.
\]

The function \( \tilde{\psi}_{\alpha,r,\varepsilon} \) is holomorphic on \( \tilde{V}_1 \), because it is the composition of \( \tilde{\Phi}_{\alpha,r,\varepsilon} : \tilde{V}_1 \to \tilde{c}_0 \) with the function \( \tilde{C} : \tilde{c}_0 \times \mathbb{C} \setminus \{0\} \to \mathbb{C} \) defined by

\[
\tilde{C}(\{z_n\}) = \sum_{n=1}^\infty z_n^{2n} - 1,
\]

which is also holomorphic on \( \tilde{c}_0 \times \mathbb{C} \setminus \{0\} \).
For \( \varepsilon > 0 \) fixed and \( x \in X \) given, we are going to apply the preceding Proposition to the family of functions \( \tilde{\psi}_{\alpha,r,\varepsilon} \) indexed by \( P = \{ (\alpha, r) : \alpha \in A, r \geq 1 \} \). So we have to find appropriate bounds for \( \tilde{\psi}_{\alpha,r,\varepsilon} \) and \( \partial_s \tilde{\psi}_{\alpha,r,\varepsilon} / \partial \mu \).

For our given \( x \in X \), we can apply properties (5) and (7) of Lemma 3 to find \( n_x, a_x, \delta_x > 0 \) (independent of \( r, \varepsilon \)) and \( \delta_{x,\varepsilon} > 0 \) (independent of \( r \)) such that

\[
|\tilde{\psi}_{n,r,\varepsilon}(w)| < \frac{1}{n! a_x^n} \quad \text{for } n > n_x, r \geq 1, w \in \tilde{X} \text{ with } \|w - x\|_{\tilde{X}} < \delta_x.
\]

and

\[
|\tilde{\psi}_{n,r,\varepsilon}(w)| \leq 1 + 2\varepsilon \quad \text{for } n \in \mathbb{N}, r \geq 1, w \in \tilde{X} \text{ with } \|w - x\|_{\tilde{X}} \leq \delta_{x,\varepsilon}.
\]

Let us fix \( 0 < R_{x,\varepsilon} < \min\{ \varepsilon, 1/6, \delta_x, \delta_{x,\varepsilon} \} \), and observe that \( \tilde{V}_1 \supset B_{\tilde{X}}(x, R_{x,\varepsilon}) \).

Note that \( \lambda(\Phi_{\alpha,r,\varepsilon}(x)) \geq \|\{\alpha_n \tilde{\psi}_n(x)\}_{n=1}^{\infty}\|_{\infty} > 1 \) by property (3) of Lemma 3 so in particular \( |\mu| \geq \lambda(\Phi_{\alpha,r,\varepsilon}(x)) - R_{x,\varepsilon} \geq 1 - 1/6 = 5/6 \) whenever \( |\mu - \lambda(\Phi_{\alpha,r,\varepsilon}(x))| \leq R_{x,\varepsilon} \).

Then we have, for every \( (\alpha, r) \in P \), for \( \|w - x\|_{\tilde{X}} \leq R_{x,\varepsilon} \), and for \( |\mu - \lambda(\Phi_{\alpha,r,\varepsilon}(x))| \leq R_{x,\varepsilon} \),

\[
|\tilde{\psi}_{\alpha,r,\varepsilon}(w, \mu)| \leq \left| \sum_{n=1}^{\infty} \frac{\alpha_n \tilde{\psi}_n(w)}{\mu} \right|^{2n} - 1 \leq 1 + \sum_{n=1}^{n_x} \frac{1001(1 + 2\varepsilon)}{5/6} 2^n + \sum_{n=n_x+1}^{\infty} \frac{1}{n! a_x^n} = M_{x,\varepsilon}
\]

(observe that \( M_{x,\varepsilon} \) is independent of \( (r, \alpha) \in P \).

On the other hand we can apply property (3) of Lemma 3 to find, for every \( r \geq 1 \), numbers

\( m_{x,r} \leq m_{x,1} \) such that \( \varphi_{m_{x,r},r,\varepsilon}(x) > 1 \).

Since \( \lambda \leq \| \cdot \|_{\infty} \) on \( c_0 \), and using property (2) of Lemma 3 we have

\[
\lambda(\Phi_{\alpha,r,\varepsilon}(x)) \leq 2 \|\Phi_{\alpha,r,\varepsilon}(x)\|_{\infty} \leq 2002 \|\Phi_{1,r,\varepsilon}(x)\|_{\infty} \leq 4004.
\]

So, if we set

\[
A_x := \frac{1}{4004^{2m_{x,1} + 1}}
\]

(clearly independent of \( (r, \alpha) \in P \)), and we use the fact that \( m_{x,r} \leq m_{x,1} \) for all \( r \geq 1 \), we can estimate

\[
\left| \frac{\partial \tilde{\psi}_{\alpha,r,\varepsilon}(x, \lambda(\Phi_{\alpha,r,\varepsilon}(x)))}{\partial \mu} \right| = \sum_{n=1}^{\infty} 2n \left( \frac{\alpha_n \varphi_n(x)}{\lambda(\Phi_{\alpha,r,\varepsilon}(x))} \right)^{2n} \frac{1}{\lambda(\Phi_{\alpha,r,\varepsilon}(x))} \geq \sum_{n=1}^{\infty} 2n \left( \frac{\varphi_n(x)}{4004} \right)^{2n} \frac{1}{4004} \geq \frac{m_{x,r}}{4004} \left( \frac{1}{4004} \right)^{2m_{x,r}} \geq \frac{1}{4004} \left( \frac{1}{4004} \right)^{2m_{x,1}} = A_x.
\]
Therefore, according to the preceding Proposition, we can find a number 
\( s_x = s_{x, \varepsilon} > 0 \) (independent of \( \alpha, r \)) such that there is a holomorphic solution 
\( \tilde{\mu} = \tilde{F}_{\alpha, r, \varepsilon}(w) \) to the implicit equation 
\( \tilde{\psi}_{\alpha, r, \varepsilon}(w, \mu) = 0 \), defined on the ball 
\( B_{\tilde{\lambda}}(x, s_x) \), for every \( \alpha \in A, r \geq 1 \). Since the solution is locally unique, two 
holomorphic functions which coincide on a neighborhood of a point are equal 
to one another in the connected component of that point, and the function 
w \mapsto \tilde{\lambda}(\tilde{\Phi}_{\alpha, r, \varepsilon}(w)) also solves the implicit equation 
\( \tilde{\psi}_{\alpha, r, \varepsilon}(w, \mu) = 0 \) for \( \mu \) in terms of \( w \) in a neighborhood of \( x \), one easily deduces that 
\( \tilde{F}_{\alpha, r, \varepsilon} \) can be defined on all of 
\[ \tilde{W}_\varepsilon := \bigcup_{x \in X} B_{\tilde{\lambda}}(x, s_{x, \varepsilon}), \]
and that \( \tilde{F}_{\alpha, r, \varepsilon} \) is a holomorphic extension of \( \lambda \circ \Phi_{\alpha, r, \varepsilon}. \)

Besides, by the same Proposition, we also have that for every \( \alpha \in A, r \geq 1 \) the function 
\( \tilde{F}_{\alpha, r, \varepsilon} \) maps the ball \( B_{\tilde{\lambda}}(x, s_x) \) into a disc \( B_{\tilde{\lambda}}(\tilde{\Phi}_{\alpha, r, \varepsilon}(x), r_x) \), 
where \( 0 < r_x = r_{x, \varepsilon} < R_{x, \varepsilon} \leq \varepsilon \). Hence we have that 
\[ |\tilde{F}_{\alpha, r, \varepsilon}(x + iy) - \lambda \circ \Phi_{\alpha, r, \varepsilon}(x)| \leq \varepsilon \]
for every \( x, y \in X \) with \( x + iy \in \tilde{W}_\varepsilon \). \( \square \)

Now we turn to the proof of Theorem 4. We shall first consider the case 
when \( f : X \to [1, 1001] \) is \( L \)-Lipschitz with \( 0 < L \leq 1 \). Define \( r = 1/L \), so 
r \geq 1, rename \( \varepsilon = \delta \in (0, 10^{-4}) \), and apply Lemma 3 to find a corresponding 
}\( \varphi_n = \varphi_n \circ \frac{1}{\lambda}, n \in \mathbb{N} \). The functions \( \varphi_n \) are thus \( 2C/r \)-
Lipschitz, where we denote \( C = \text{Lip}(Q) \).

Define a function \( g : X \to \mathbb{R} \) by 
\[ g(x) = \frac{\lambda(\{f(x)\varphi_n(x)\}_{n=1}^{\infty})}{\lambda(\{\varphi_n(x)\}_{n=1}^{\infty})} = \frac{\lambda \circ \Phi_{\alpha, r, \delta}(x)}{\lambda \circ \Phi_{1, r, \delta}(x)} \]
where here we denote \( \alpha = \{f(x_n)\}_{n=1}^{\infty} \in A \), and also \( 1 = (1, 1, 1, ...) \in A \subset \ell_\infty \).

The function \( g \) is well defined, is real analytic, and has a holomorphic 
extension \( \tilde{g} \) defined on the neighborhood \( \tilde{U} := \tilde{W}_\delta \) of \( X \) in \( \tilde{X} \) provided by 
Lemma 4 given by 
\[ \tilde{g}(w) = \frac{\tilde{F}_{\alpha, r, \delta}(w)}{\tilde{F}_{1, r, \delta}(w)}. \]

Since the functions \( \varphi_n \) are \( 2C/r \)-Lipschitz, the norm \( \lambda : c_0 \to \mathbb{R} \) is \( 2 \)-
Lipschitz (with respect to the usual norm \( \| \cdot \|_\infty \) of \( c_0 \)), and \( 1 \leq f \leq 1001 \), we have 
\[ |\lambda(\{f(x)\varphi_n(x)\}_{n=1}^{\infty}) - \lambda(\{f(x)\varphi_n(y)\}_{n=1}^{\infty})| \leq \]
\[ 2\|\{f(x)(\varphi_n(x) - \varphi_n(y))\}_{n=1}^{\infty}\|_\infty \leq 2002\|\{\varphi_n(x) - \varphi_n(y)\}_{n=1}^{\infty}\|_\infty \leq \frac{4004C}{r}\|x - y\|, \]
that is the function $\lambda \circ \Phi_{a,r,\delta}$ is $4004C/r$-Lipschitz on $X$, and is bounded by 2002. Similarly, since the function $t \mapsto 1/t$ is 1-Lipschitz on $(1, \infty)$ and $\lambda \circ \Phi_{1,r,\delta}$ is bounded below by 1, we have that the function $1/\lambda \circ \Phi_{1,r,\delta}$ is also $4004C/r$-Lipschitz on $X$, and bounded above by 1. Therefore the product satisfies
\[
\text{Lip}(g) \leq 2002 \times \frac{4004C}{r} + 1 \times \frac{4004C}{r} = \frac{8020012C}{r} = 8020012C \text{Lip}(f).
\]
On the other hand, we have
\[
|g(x) - f(x)| = \left| \frac{\lambda(\{f(x)\varphi_j(x)\})}{\lambda(\{\varphi_j(x)\})} - f(x) \right|
\]
\[
= \left| \frac{\lambda(\{f(x)\varphi_j(x)\})}{\lambda(\{\varphi_j(x)\})} - f(x) \cdot \frac{\lambda(\{\varphi_j(x)\})}{\lambda(\{\varphi_j(x)\})} \right|
\]
\[
= \frac{1}{\lambda(\{\varphi_j(x)\})} |\lambda(\{f(x)\varphi_j(x)\}) - \lambda(\{f(x)\varphi_j(x)\})|
\]
\[
\leq \frac{1}{\lambda(\{\varphi_j(x)\})} \lambda(\{f(x)\varphi_j(x)\} - \{f(x)\varphi_j(x)\})
\]
\[
= \frac{1}{\lambda(\{\varphi_j(x)\})} \lambda(\{\varphi_j(x)(f(x) - f(x))\}).
\]
Now recall that
\[
\lambda(\{\varphi_j(x)(f(x) - f(x))\}) \leq 2 \left\| \{\varphi_j(x)(f(x) - f(x))\} \right\|_{\infty}
\]
\[
= 2 \max_j \{\varphi_j(x)|f(x_j) - f(x)|\}.
\]
Set $J = \{j : x \in DQ(x_j, 4r)\}$. For $j \in J$, according to Lemma 2, we have $\|x - x_j\|_X < 8r$ and so, because $\text{Lip}(f) = 1/r$,
\[
\varphi_j(x)|f(x_j) - f(x)| \leq \varphi_j(x) 8.
\]
It follows that for $j \in J$
\[
\frac{\varphi_j(x)|f(x_j) - f(x)|}{\lambda(\{\varphi_j(x)\})} \leq \frac{\varphi_j(x) 8}{\left\| \{\varphi_j(x)\} \right\|_{\infty}} \leq 8.
\]
On the other hand, for $j \notin J$ we have by part (4) of Lemma 3 that
\[
\varphi_j(x)|f(x_j) - f(x)| \leq 2002\varphi_j(x) \leq 2002\delta < 1.
\]
Hence, given that $\lambda(\{\varphi_j(x)\}) \geq 1$, we have for $j \notin J$
\[
\frac{\varphi_j(x)|f(x_j) - f(x)|}{\lambda(\{\varphi_j(x)\})} \leq 1.
It follows that
\[ |g(x) - f(x)| \leq 8. \]

If we reset \( C \) to 8020012C, this argument proves Theorem 1 in the case when \( \varepsilon = 8, f : X \to [1, 1001], \) Lip\((f) \leq 1.\)

Moreover, according to property (2) of Lemma 4, we have
\[ |\tilde{F}_{\alpha,r,\delta}(x + iy) - \lambda(\Phi_{\alpha,r,\delta}(x))| \leq \delta \]
for every \( x, y \in X \) with \( x + iy \in \tilde{W}_\delta, \alpha \in A. \) Therefore, taking into account that \( \lambda(\Phi_{\alpha,r,\delta}(x)) \leq 2002 \) and \( 1 \leq \lambda(\Phi_{1,r,\delta}(x)) \leq 2, \) we have
\[
|\tilde{g}(x + iy) - g(x)| \leq \left| \frac{\tilde{F}_{\alpha,r,\delta}(x + iy) - \lambda(\Phi_{\alpha,r,\delta}(x))}{\lambda(\Phi_{1,r,\delta}(x))} \right| \cdot |\lambda(\Phi_{1,r,\delta}(x))\left(\tilde{F}_{\alpha,r,\delta}(x + iy) - \lambda(\Phi_{\alpha,r,\delta}(x))\right)| + \lambda(\Phi_{\alpha,r,\delta}(x))\left(\lambda(\Phi_{1,r,\delta}(x)) - \tilde{F}_{1,r,\delta}(x + iy)\right) \leq
\]
\[ \frac{1}{1 - \delta}(2\delta + 2002\delta) = \frac{\delta}{1 - \delta}2004 \]
for every \( x, y \in X \) with \( x + iy \in \tilde{W}_\delta, \alpha \in A, r \geq 1. \)

Up to scaling and subtracting appropriate constants we have thus proved the following intermediate result.

**Proposition 2.** Let \( X \) be a Banach space having a separating polynomial, and let \( \eta \in (0, 1/2). \) Then there exists \( C_\eta \geq 1 \) (depending only on \( X \) and \( \eta \)) such that, for every \( \delta > 0 \) there is an open neighborhood \( \tilde{U}_{\delta,\eta} \) of \( X \) in \( \tilde{X} \) such that, for every Lipschitz function \( f : X \to [0, 1] \) with Lip\((f) \leq 1, \) there exists a real analytic function \( g : X \to \mathbb{R}, \) with holomorphic extension \( \tilde{g} : \tilde{U}_{\delta,\eta} \to \mathbb{C}, \) such that

1. \( |f(x) - g(x)| \leq \eta \) for all \( x \in X. \)
2. \( g \) is Lipschitz, with Lip\((g) \leq C_\eta \text{Lip}(f). \)
3. \( |\tilde{g}(x + iy) - g(x)| \leq \delta \) for all \( z = x + iy \in \tilde{U}_{\delta,\eta}. \)

Next we are going to see that this result remains true for all bounded, nonnegative functions \( f \) with Lip\((f) \leq 1 \) if we allow \( C \) to be slightly larger and we replace \( \eta \) by 1.

For a 1-Lipschitz, bounded function \( f : X \to [0, +\infty) \) we define, for \( n \in \mathbb{N}, \) the functions
\[
f_n(x) = \begin{cases} 
    f(x) - n + 1 & \text{if } n - 1 \leq f(x) \leq n, \\
    0 & \text{if } f(x) \leq n - 1, \\
    1 & \text{if } n \leq f(x).
\end{cases}
\]
Note that since \( f \) is bounded there exists \( N \in \mathbb{N} \) such that \( f_n = 0 \) for all \( n \geq N. \) The functions \( f_n \) are clearly 1-Lipschitz and take values in the
Clearly the path interval \([0, 24] \) is used as follows: since
\[ |f_n - f| \leq \eta, \text{ and } |g(x + iy) - g(x)| \leq \delta \]
for all \( z = x + iy \in \bar{U}_{\delta, \eta} \). Since \( f_n = 0 \) for all \( n \geq N \), we may obviously assume that \( g_n = 0 \) for all \( n \geq N \), as well.

Now define a path \( \beta : [0, +\infty) \to \ell_\infty \) by,
\[
\beta(t) = \left( 1, \ldots, 1, t - n + 1, 0, 0, \ldots \right) = \sum_{j=1}^{n-1} e_j + (t - n + 1)e_n \text{ if } n - 1 \leq t \leq n.
\]

Clearly the path \( \beta \) is a 1-Lipschitz injection of \([0, +\infty)\) into \( \ell_\infty \), with a uniformly continuous (but not Lipschitz) inverse \( \beta^{-1} : \beta([0, +\infty)) \to [0, +\infty) \).

Define a uniformly continuous (not Lipschitz) function \( h \) on the path \( \beta \) by \( h(\beta(t)) = t \) for all \( t \geq 0 \), that is \( h(y) = \beta^{-1}(y) \) for \( y \in \beta([0, +\infty)) \). Then we have \( f(x) = h(\{f_n(x)\}_{n=1}^\infty) \).

5.4. The gluing tube function. Now we are going to construct an open tube of radius \( 2\eta \) (with respect to the supremum norm \( \| \cdot \|_\infty \)) around the path \( \beta \) in \( \ell_\infty \), and a real-analytic approximate extension (with bounded derivative) \( H \) of the function \( h \) defined on this tube. This construction is meant to be used as follows: since \( |g_n - f_n| \leq \eta \) and \( \{f_n(x)\}_{n=1}^\infty \) takes values in the path \( \beta \), then \( \{g_n(x)\}_{n=1}^\infty \) will take values in this tube, and therefore \( g(x) := H(\{g_n(x)\}_{n=1}^\infty) \) will approximate \( H(\{f_n(x)\}_{n=1}^\infty) \), which in turn approximates \( h(\{f_n(x)\}_{n=1}^\infty) = f(x) \). Besides, since \( H \) has a bounded derivative on the tube and the functions \( g_n \) are \( C_\eta \)-Lipschitz then \( g \) will be \( C_\eta M \)-Lipschitz, where \( M \) is an upper bound of \( DH \) on the tube.

**Lemma 5.** For every \( \varepsilon \in (0, 1] \) there exist \( r, \delta \in (0, \varepsilon/64) \) and a real-analytic mapping \( G : \ell_\infty \to \ell_\infty \) with holomorphic extension \( \tilde{G} : \ell_\infty \to \ell_\infty \) such that:

1. \( G \) diffeomorphically maps the straight tube \( S \) defined by \( \{x \in \ell_\infty : |x_n| < r \text{ for all } n \geq 2, x_1 > -r \} \) onto a twisted tube \( T \) around the path \( \beta \) in such a way that
\[
\{x \in \ell_\infty : dist(x, \beta([0, \infty))) < r/2 \} \subseteq T \subseteq \{x \in \ell_\infty : dist(x, \beta([0, \infty))) < 2r \}.
\]

2. We have \( \|x - \beta(t)\|_\infty \leq r/2 \implies |e^*_tG^{-1}(x) - t| \leq \varepsilon \).

(2') There is a function \( \alpha : \mathbb{R} \to \mathbb{R} \) with a holomorphic extension \( \tilde{\alpha} : \mathbb{C} \to \mathbb{C} \) such that the function \( H := \alpha \circ e^*_t \circ G^{-1} : T \to \mathbb{R} \) satisfies
\[
\|DH(x)\| \leq (1 + \varepsilon), \text{ and}
\]
\[
\|x - \beta(t)\|_\infty < r/2 \implies |H(x) - t| \leq \varepsilon.
\]
(3) The derivatives of the maps $G|_{S}$ and $(G|_{S})^{-1}$ are bounded by $(2 + \varepsilon)$ on the sets $S$ and $T$ respectively.

(4) $\|G(x + iy) - G(x)\|_{\infty} \leq \varepsilon$ for all $x, y \in S$ with $\|y\|_{\infty} \leq \delta$.

(5) A holomorphic extension $\tilde{G}^{-1}$ of $G^{-1}$ is defined from $\tilde{T}_{\delta} := \{u + iv : u \in T, v \in \ell_{\infty}, \|v\|_{\infty} < \delta\}$ into $\ell_{\infty}$, with the properties that

$$\|\tilde{G}^{-1}(u + iv) - G^{-1}(u)\|_{\infty} \leq \varepsilon,$$

and

$$|\tilde{H}(u + iv)| = |\tilde{\alpha}(\tilde{e}_{1}^{*}(\tilde{G}^{-1}(u + iv)))| \leq 2(H(u) + 1)$$

for all $x \in S, u \in T$ and $v \in \ell_{\infty}$ such that $u + iv \in \tilde{T}_{\delta}$.

Proof. Let $r \in (0, 1/8)$, and define $S = [0, 2] \times [-r, r], T = [0, 1 + r] \times [-r, r] \cup [1 - r, 1 + r] \times [-r, 1]$. Consider the mapping $F : S \rightarrow T$ defined by

$$F(x, y) = \begin{cases} (x - yx, y) & \text{if } 0 \leq x \leq 1, -r \leq y \leq r, \\ (1 - y, x - 1 + y(2 - x)) & \text{if } 1 \leq x \leq 2, -r \leq y \leq r. \end{cases}$$

It is clear that $F$ is a Lipschitz homeomorphism from $S$ onto $T$, with inverse

$$F^{-1}(u, v) = \begin{cases} \left(\frac{u}{1 - v}, v\right) & \text{if } (u, v) \in [0, 1 + r] \times [-r, r], u + v \leq 1, \\ \left(\frac{2u + v - 1}{1 - u}, 1 - u\right) & \text{if } (u, v) \in [1 - r, 1 + r] \times [-r, 1], u + v \geq 1. \end{cases}$$

Moreover, from these expressions it is immediately checked that $F$ is $C^{\infty}$ smooth except on the set $\{x = 1\}$, that $F^{-1}$ is also $C^{\infty}$ smooth except on the set $\{u + v = 1\}$, and that $F, F^{-1} \in W^{1, \infty}$. In fact, using the norm $\|\cdot\|_{\infty}$ in $\mathbb{R}^{2}$, the derivatives $DF$ and $DF^{-1}$ satisfy that

$$\|DF\| \leq 2 + r \text{ a. e. on } S, \text{ and } \|DF^{-1}\| \leq \frac{2}{(1 - r)^{2}} \text{ a. e. on } S.$$ 

This implies that $F : S \rightarrow T$ is $(2 + r)$-Lipschitz (because $S$ is convex), and that $F^{-1}$ is locally $2/(1 - r)^{2}$-Lipschitz (because $T$ is locally convex). Given $\varepsilon \in (0, 1/8)$, by using convolution with mollifiers and observing that $\lim_{r \rightarrow 0} \frac{2}{(1 - r)^{2}} = 2 = \lim_{r \rightarrow 0} (2 + r)$, for a given $\varepsilon \in (0, 1)$, we can find $r = \varepsilon/64$ and $\overline{F}, \overline{F} \in C^{\infty}(\mathbb{R}^{2}, \mathbb{R}^{2})$ satisfying

$$\|\overline{F}|_{S} - F\|_{\infty} < \frac{\varepsilon}{10} \text{ on } S, \|\overline{F}|_{S} - F\|_{\infty} < \frac{\varepsilon}{10} \text{ on } T,$$

and such that $\overline{F}$ is a diffeomorphism from $T$ onto $\overline{F}(T)$ with inverse $\overline{G}|_{\overline{F}(T)}$, and $\overline{F}$ and $\overline{G}$ are locally $(2 + \varepsilon/4)$-Lipschitz (with respect to the norm $\|\cdot\|_{\infty}$). Next, by scaling, translating and appropriately gluing with the isometries $(x, y)$ and $(1 - y, x - 1)$ (allowing $r$ to be smaller if necessary), one can obtain a $C^{\infty}$ mapping $\overline{\Phi}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ such that

(i) $\overline{\Phi}$ is the identity on the rectangle $[-2r, 1 - \varepsilon] \times [-2r, 2r]$;

(ii) $\overline{\Phi}$ maps isometrically the rectangle $[1 + \varepsilon, 2] \times [-2r, 2r]$ onto the rectangle $[1 - 2r, 1 + 2r] \times [\varepsilon, 1]$, and $D\overline{\Phi}(s, t)(1, 0) = (0, 1)$ for all $(s, t) \in [1 + \varepsilon, 2] \times [-2r, 2r]$;

(iii) $\overline{\Phi}$ is a diffeomorphism from $[0, 2] \times [-2r, 2r]$ onto $\overline{\Phi}([0, 2] \times [-2r, 2r])$;
(iv) considering, by abusing notation, that \( \beta : [0, 2] \to \mathbb{R}^2 \subset \ell_\infty \), we have
\[
\{ (u, v) \in \mathbb{R}^2 : \text{dist}((u, v), \beta([0, 2])) < \frac{r}{2} + \frac{r}{64} \} \subseteq \\
2 \mathbb{R}([0, 2] \times [-r, r]) \subseteq \{(s, t) \in \mathbb{R}^2 : \text{dist}((s, t), \beta([0, 2])) < r - \frac{r}{64} \},
\]
and, for \( n \geq 1 \), define \( \overline{\Phi}_{(n+1,n+2)} : \ell_\infty \to \ell_\infty \) by
\[
\overline{\Phi}_{(n+1,n+2)}(x) = \\
\sum_{j=1}^n (1 - x_j) e_j + \overline{\varphi}(x_1 - n, x_{n+2})e_{n+1} + \overline{\psi}(x_1 - n, x_{n+2})e_{n+2} + \sum_{k \geq n+3} x_k e_k.
\]
\( \overline{\Phi} \) is \((2 + \varepsilon/2)\)-Lipschitz on the set \([0, 2] \times [-r, r] \), and \( \overline{\Phi}^{-1} \) is \((2 + \varepsilon/2)\)-locally Lipschitz on the set \( \{(u, v) \in \mathbb{R}^2 : \text{dist}((u, v), \beta([0, 2])) < r \}\).

Observe that such a mapping \( \overline{\Phi} \) also satisfies
\[(vii) \| e_1 \circ \overline{\Phi}^{-1}(x) - t \| \leq 3 \varepsilon \text{ whenever } \| x - \beta(t) \| \leq r, t \in [0, 2].\]

One can check that the mappings \( \overline{\Phi}_{(n,n+1)} \) are \(C^\infty\) smooth on \( \ell_\infty \), satisfy
\( \overline{\Phi}_{(n,n+1)}(x) = \overline{\Phi}_{(n+1,n+2)}(x) \) whenever \( n+1/4 \leq x_1 \leq n+3/4 \) and \( |x_j| < 1/6 \) for all \( j \geq 2 \). Moreover, there exists \( N > 0 \) such that
\( \| \overline{\Phi}_{(n,n+1)}(x) \| \leq N \) for all \( x \in S, n \in \mathbb{N} \)
and
\( \| D^k \overline{\Phi}_{(n,n+1)}(x) \| \leq N \) for all \( x \in \ell_\infty, n \in \mathbb{N}, k \in \{1, 2\} \).

Then define \( \overline{G} : S \subset \ell_\infty \to \ell_\infty \) by
\[
\overline{G}(x) = \sum_{n=0}^\infty \overline{\theta}(x_1 - n) \overline{\Phi}_{(n+1,n+2)}(x),
\]
where \( \overline{\theta} : \mathbb{R} \to \mathbb{R} \) is a \(C^\infty\) function such that
\[
\overline{\theta}(t) = 0 \text{ if and only if } t \in (-\infty, -1/2] \cup [3/4, +\infty); \\
\overline{\theta}(t) = 1 \text{ if and only if } t \in [-1/4, 1/2].
\]
\[ \mathcal{G}(t) = 1 - \mathcal{G}(t - 1) \text{ if } t \in (1/2, 3/4). \]

Note in particular that the collection of functions \( t \mapsto \mathcal{G}(t - n), n \in \mathbb{N}, \) form a \( C^\infty \) partition of unity on \( \mathbb{R} \). Observe also that, for every \( x \in \ell_\infty \) such that \( |x_j| \leq 2r \) for \( j \geq 2 \), there exist \( n_x \in \mathbb{N} \) and a neighborhood \( U_x \) of \( x \) so that \( \mathcal{G}(y) = \Phi_{nx,nx+1}(y) \) for all \( y \in U_x \).

It is not difficult to check that the mapping \( \mathcal{G} : \mathcal{S} \subset \ell_\infty \to \ell_\infty \) has properties (1), (2) and (3) of the statement (with the slightly sharper bound \( 1/2 + 1/8 \), instead of \( 3/4 \), in (2)). In particular \( \mathcal{G}^{-1} \) is uniformly continuous and there exists \( m > 0 \) such that

\[ ||D\mathcal{G}(x)(h)|| \geq m||h|| \text{ for all } x, h \in \ell_\infty. \]

But of course \( \mathcal{G} \) is not real analytic. In order to obtain a required real-analytic function \( G \) we shall substitute \( \varphi, \psi, \theta \) with real-analytic approximations of these functions defined by appropriate integral convolutions with Gaussian kernels in \( \mathbb{R}^2 \) and \( \mathbb{R} \). Namely, let us define

\[ \varphi(s,t) = a_\kappa \int_{\mathbb{R}^2} \varphi(u,v) \exp \left( -\kappa \left[ (s-u)^2 + (t-v)^2 \right] \right) dudv, \]
\[ \psi(s,t) = a_\kappa \int_{\mathbb{R}^2} \psi(u,v) \exp \left( -\kappa \left[ (s-u)^2 + (t-v)^2 \right] \right) dudv, \]
\[ \theta(t) = b_\kappa \int_{-\infty}^{+\infty} \theta(u) e^{-\kappa(t-u)^2} du, \]
\[ \Phi_{(n, n+1)}(x) = \sum_{j=1}^{n} (1 - x_j) e_j + \varphi(x_1-n, x_{n+2}) e_{n+1} + \psi(x_1-n, x_{n+2}) e_{n+2} + \sum_{k \geq n+3} x_k e_k. \]

where

\[ a_\kappa = 1/ \int_{\mathbb{R}^2} \exp \left( -\kappa \left[ u^2 + v^2 \right] \right) dudv, \text{ and} \]
\[ b_\kappa = 1/ \int_{-\infty}^{+\infty} e^{-\kappa v^2} dv. \]

By taking \( \kappa \) large enough we may assume that the functions \( \varphi, \psi, \theta \) and their first and second derivatives are as close to \( \varphi, \psi, \theta \) and their first and second derivatives, respectively, as we want. Therefore the mappings \( \Phi_{(n, n+1)} \) and their first and second derivatives can be taken as close as needed to \( \Phi_{(n, n+1)} \) and their first and second derivatives, say

\[ ||D^{(k)} \Phi_{(n, n+1)}(x) - D^{(k)} \Phi_{(n, n+1)}(x)||_\infty \leq \varepsilon/2 \]

for all \( x \in \ell_\infty, k = 0, 1, 2 \). Moreover, it is easily checked that the holomorphic extensions \( \varphi, \psi \) of \( \varphi, \psi \) to \( \mathbb{C}^2 \), as well as the holomorphic extension of \( \theta \) to \( \mathbb{C} \) (defined by the same formulae by letting \( s, t \in \mathbb{C} \)) satisfy

\[ |\varphi(s + ir, t + iw)| \leq e^{\kappa(r^2 + w^2)} |\varphi(s, t)|, \]
\[ |\psi(s + ir, t + iw)| \leq e^{\kappa(r^2 + w^2)} |\psi(s, t)|, \]
\[ |	heta(a + ib)| \leq e^{\kappa b^2} |	heta(a)| \]

and in particular we have

\[ |\varphi(s + ir, t + iw)| \leq (1 + \varepsilon)|\varphi(s, t)|, \]
\[ |\psi(s + ir, t + iw)| \leq (1 + \varepsilon)|\psi(s, t)|, \]
\[ |	heta(a + ib)| \leq (1 + \varepsilon)|\theta(a)| \]
provided that $\max\{|r|^2 + |w|^2, |b|^2\} < \frac{\log(1+\varepsilon)}{\kappa}$. It follows that the corresponding holomorphic extension $\tilde{\Phi}_{(n,n+1)}$ of $\Phi_{(n,n+1)}$ to $\ell_\infty$ (defined by a similar formula just replacing $\varphi, \psi$ with $\tilde{\varphi}, \tilde{\psi}$) satisfies

$$\|\tilde{\Phi}_{(n,n+1)}(x + iz)\|_\infty \leq (1 + \varepsilon)\|\Phi_{(n,n+1)}(x)\|_\infty$$

for all $x, z \in \ell_\infty$ such that $\|z\|_\infty \leq \sqrt{\frac{\log(1+\varepsilon)}{2\kappa}}$.

Now define

$$G(x) = \sum_{n=0}^{\infty} \theta(x_1 - n)\Phi_{(n+1,n+2)}(x).$$

Let us see that $G$ is real analytic and has a holomorphic extension $\tilde{G}$ to $\tilde{\ell}_\infty$ defined by

$$\tilde{G}(x + iz) = \sum_{n=0}^{\infty} \tilde{\theta}(x_1 - n + iz_1)\tilde{\Phi}_{(n+1,n+2)}(x + iz).$$

It is enough to check that the series of holomorphic functions defining $\tilde{G}$ is locally uniformly absolutely convergent. Taking into account that the mappings $\tilde{\Phi}_{(n,n+1)}$ are clearly uniformly bounded on bounded sets (meaning that for every $R > 0$ there exists $K > 0$ such that $\|\tilde{\Phi}_{(n,n+1)}(x + iz)\| \leq K$ for all $x, z \in B(0, R)$ and all $n \in \mathbb{N}$), this amounts to showing that

$$\sum_{n=0}^{\infty} |\tilde{\theta}(t - n + is)| < +\infty$$

locally uniformly for $t + is \in \mathbb{C}$. Assume $|t|, |s| < R$, $u \in [-1/2, 3/4]$, then we have $-\kappa|t - n|^2 + 2\kappa(t-n)u \leq -\kappa(n-R)^2 + \frac{2\kappa}{2}(n+R)$, hence

$$0 \leq \theta(t-n) = \frac{1}{\int_{-\infty}^{\infty} e^{-\kappa u^2} dv} \int_{-\infty}^{\infty} \tilde{\theta}(u)e^{-\kappa(t-n-u)^2} du \leq \frac{1}{\int_{-\infty}^{\infty} e^{-\kappa v^2} dv} e^{-\kappa[\frac{3}{2}(n+R)]} \int_{-1/2}^{3/4} e^{-\kappa u^2} du \leq e^{-\kappa[\frac{3}{2}(n+R)]},$$

and consequently

$$|\tilde{\theta}(t - n + is)| \leq \theta(t - n)e^{\kappa s^2} \leq e^{-\kappa[\frac{3}{2}(n+R) - R^2]}.$$

Since

$$\sum_{n \geq R} e^{-\kappa[\frac{3}{2}(n+R) - R^2]} < +\infty,$$

it is then clear that the series $\sum_{n=0}^{\infty} |\tilde{\theta}(t - n + is)|$ is bounded on $\{t + is : |t|, |s| < R\}$ by an absolutely convergent numerical series, and therefore it is locally uniformly convergent.
Now let us show that, given \( \varepsilon > 0 \), \( G \) satisfies \( \| G(x) - \overline{G}(x) \|_{\infty} \leq \varepsilon \) for all \( x \in S \), provided \( \kappa > 0 \) is large enough. Let us first observe that we can take \( \kappa > 0 \) sufficiently large so that

\[
\sum_{n=0}^{\infty} |\theta(t - n) - \overline{\theta}(t - n)| \leq \frac{\varepsilon}{2(N + 1)}.
\]

Indeed, on the one hand, if \( |t - n| \geq 2 \) we have \( \overline{\theta}(t - n) = 0 \), and \( |t - n|^2 - \frac{3}{2} |t - n| = |t - n| (|t - n| - \frac{3}{2}) \geq \frac{1}{2} |t - n| \), hence

\[
|\theta(t - n) - \overline{\theta}(t - n)| = \theta(t - n) = \frac{1}{\int_{-\infty}^{\infty} e^{-\kappa u^2} du} \int_{-\infty}^{\infty} \overline{\theta}(u) e^{-\kappa(t-n-u)^2} du \leq \frac{1}{\int_{-\infty}^{\infty} e^{-\kappa u^2} du} e^{-\kappa |t-n|^2} \int_{-\frac{3}{4}}^{3/4} e^{-\kappa u^2} du \leq e^{-\kappa |t-n|/2},
\]

consequently

\[
\sum_{|t-n| \geq 2} |\theta(t - n) - \overline{\theta}(t - n)| \leq \sum_{|t-n| \geq 2} e^{-\kappa |t-n|/2} \leq 2 \sum_{n=2}^{\infty} e^{-\kappa n/2}.
\]

and because \( 2 \sum_{n=2}^{\infty} e^{-\kappa n/2} \to 0 \) as \( \kappa \to +\infty \), we may assume \( \kappa \) is large enough so that

\[
\sum_{|t-n| \geq 2} |\theta(t - n) - \overline{\theta}(t - n)| \leq \frac{\varepsilon}{4(N + 1)}.
\]

On the other hand, we may also assume \( \kappa > 0 \) is large enough so that \( |\theta(u) - \overline{\theta}(u)| \leq \varepsilon / 16(N + 1) \) for all \( u \in \mathbb{R} \), and clearly there are at most four integers \( n \) with \( |t - n| < 2 \), so we have

\[
\sum_{|t-n| < 2} |\theta(t - n) - \overline{\theta}(t - n)| \leq 4 \frac{\varepsilon}{16(N + 1)} = \frac{\varepsilon}{4(N + 1)}.
\]

By combining the two last inequalities we get

\[
\sum_{n=0}^{\infty} |\theta(t - n) - \overline{\theta}(t - n)| \leq \frac{\varepsilon}{2(N + 1)}
\]

as required.
Now, for every \( x \in \mathcal{S} \) we can estimate
\[
\| G(x) - \bar{G}(x) \|_\infty \leq \bigg\| \sum_{n=0}^{\infty} \left( \theta(x_1 - n) - \bar{\theta}(x_1 - n) \right) \Phi_{n+1,n+2}(x) \bigg\| + \\
+ \bigg\| \sum_{n=0}^{\infty} \bar{\theta}(x_1 - n) \left( \Phi_{n+1,n+2}(x) - \bar{\Phi}_{n+1,n+2}(x) \right) \bigg\|
\]
\[
\leq \sum_{n=0}^{\infty} \| \Phi_{n+1,n+2}(x) \| \| \theta(x_1 - n) - \bar{\theta}(x_1 - n) \| + \\
+ \sum_{n=0}^{\infty} \bar{\theta}(x_1 - n) \| \Phi_{n+1,n+2}(x) - \bar{\Phi}_{n+1,n+2}(x) \| \\
\leq (N + 1) \sum_{n=0}^{\infty} \| \theta(x_1 - n) - \bar{\theta}(x_1 - n) \| + \frac{\varepsilon}{2} \sum_{n=0}^{\infty} \| \theta(x_1 - n) \| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

By using the facts that the derivatives \( \theta', \theta'' \) of \( \theta \) decrease exponentially at infinity and approximate \( \bar{\theta}', \bar{\theta}'' \), and by performing similar calculations, one can also show that
\[
\| DG(x) - D\bar{G}(x) \| \leq \varepsilon, \quad \text{and} \quad \| D^2G(x) - D^2\bar{G}(x) \| \leq \varepsilon, \quad \text{for all} \ x \in \mathcal{S}
\]
provided \( \kappa > 0 \) is large enough.

Let us now see that \( G \) is a diffeomorphism from \( \mathcal{S} \) onto \( \mathcal{T} := G(\mathcal{S}) \). We know that \( \bar{G} \) is of class \( C^\infty \) with bounded derivatives of all orders, in particular \( \bar{G} \) is Lipschitz and has a Lipschitz derivative. Moreover \( DG(x) \) is a linear isomorphism for all \( x \in \mathcal{S} \) and the mapping \( x \mapsto \| DG(x)^{-1} \| \) is bounded above on \( \mathcal{S} \) by a number \( M > 0 \). This implies that if \( L : \ell_\infty \to \ell_\infty \) is a linear mapping such that \( \| L \| < 1/M \) then \( DG(x) + L \) is a linear isomorphism as well, for every \( x \in \mathcal{S} \). Since we may assume that \( \| DG(x) - DG(x) \| < 1/M \) for all \( x \in \mathcal{S} \), we have that \( DG(x) \) is a linear isomorphism for all \( x \in \mathcal{S} \).

Now we recall that a standard proof of the inverse mapping theorem (the one that uses the fixed point theorem for contractive mappings) shows that if a function \( F \) (defined on an open set \( U \) of a Banach space \( X \) and taking values in \( X \)) has the property that \( DF(x) \) is a linear isomorphism for all \( x \), the mappings \( x \mapsto DF(x) \) and \( x \mapsto DF(x)^{-1} \) are bounded, and the mapping \( x \mapsto DF(x) \) is Lipschitz, then there exist uniform lower bounds \( r, s > 0 \) (depending only on \( \text{Lip}(DF) \) and the bounds for \( DF, (DF)^{-1} \)) such that \( F \) maps diffeomorphically each open ball \( B(x, r) \) contained in \( U \) onto an open subset of \( X \) containing the ball \( B(F(x), s) \); see for instance [AMR, Proposition 2.5.6]. Since \( D\bar{G}, D^2\bar{G} \) are bounded, and \( DG, D^2G \) are being assumed to be close enough to these functions for large \( \kappa \), it follows that \( DG, D^2G \) are uniformly bounded as well for all \( \kappa > 0 \) large enough, and we can apply the mentioned fact with \( F = G \) to conclude that there exists \( r > 0 \) such that, for all \( \kappa > 0 \) large enough, \( G \) is a diffeomorphism from
each open ball \( B(x,r) \subset S \) onto an open subset of \( \ell_\infty \), and in particular \( G \) is uniformly locally injective (at distances less that \( r \)). The same property is of course true of \( \overline{G} \).

Thus, in order to prove that \( G \) maps \( S \) diffeomorphically onto its image it only remains to be seen that \( G \) is globally injective as well (at distances greater than \( r \), at least for \( \kappa > 0 \) large enough). Clearly we have \( 0 < \inf \{ \| \overline{G}(v) - \overline{G}(w) \|_\infty : v, w \in S, \| v - w \|_\infty \geq \varepsilon \} \), so we can take \( \varepsilon \) with

\[
0 < \varepsilon < \frac{1}{4} \inf \{ \| \overline{G}(v) - \overline{G}(w) \|_\infty : v, w \in S, \| v - w \|_\infty \geq \varepsilon \},
\]

and of course we may assume \( \kappa > 0 \) is large enough so that \( \| G(x) - \overline{G}(x) \|_\infty < \varepsilon \) for all \( x \in \mathcal{S} \). Now take \( x, y \in \mathcal{S} \) with \( x \neq y \). If \( G(x) = G(y) \) then necessarily \( \| x - y \|_\infty \geq \varepsilon \), and we have

\[
\| \overline{G}(x) - \overline{G}(y) \|_\infty = \| \overline{G}(x) - G(x) + G(x) - G(y) + G(y) - \overline{G}(y) \|_\infty \leq \| \overline{G}(x) - G(x) \|_\infty + \| G(y) - \overline{G}(y) \|_\infty \\
\leq 2 \varepsilon < \inf \{ \| \overline{G}(v) - \overline{G}(w) \|_\infty : v, w \in S, \| v - w \| \geq \varepsilon \};
\]
a contradiction.

This proves the first part of (1). The second part of (1) follows easily from the definitions of \( \overline{\mathcal{F}} \), \( \overline{G} \) and the fact that \( G \) approximates \( \overline{G} \) (we may assume \( 0 < \varepsilon < 1/64 \) and \( \kappa > 0 \) large enough so that \( \| G(x) - \overline{G}(x) \| \leq \varepsilon \) for all \( x \in \mathcal{S} \)).

Now define, for \( t \in \mathbb{R}, z \in \mathbb{C} \), the functions

\[
\overline{\alpha}(z) = \frac{\int_{\mathbb{R}} \overline{\alpha}(s) e^{-\kappa(z-s)^2} ds}{\int_{\mathbb{R}} e^{-\kappa s^2} ds},
\]

and put \( \alpha = \overline{\alpha}|_{[1,2]} \). Observe that the functions \( \overline{\alpha} \) and \( \alpha \) are \( \frac{1}{1-4\varepsilon} \)-Lipschitz.

Also, we have \( |\overline{\alpha}(t) - t| \leq 2 \varepsilon \), and hence \( |\alpha(t) - t| \leq 3 \varepsilon \) for all \( t \geq 0 \), if \( \kappa \) is large enough. Besides \( \overline{\alpha}(t) = 0 \) whenever \( |t - n| < 2 \varepsilon \), hence we can also assume \( \kappa \) is large enough so that \( |\alpha'(t)| \leq \varepsilon/4 \) whenever \( |t - n| \leq 3 \varepsilon/2 \).

Set \( \overline{\mathcal{P}} := \alpha \circ e_1^* \circ \overline{\mathcal{P}}^{-1} \), and \( H = \alpha \circ e_1^* \circ G^{-1} \). From property (vii) above we get

\[
|e_1^* \circ \overline{\mathcal{P}}^{-1}(x) - t| \leq 4 \varepsilon \text{ whenever } \| x - \beta(t) \| \leq r/2,
\]

which in particular shows (2). Hence we have, for \( \| x - \beta(t) \|_\infty \leq r/2 \),

\[
|\overline{\mathcal{P}}(x) - t| \leq |\alpha(e_1^*(\overline{\mathcal{P}}^{-1}(x)))-e_1^*(G^{-1}(x))|+|e_1^*(\mathcal{P}^{-1}(x))| \leq 3 \varepsilon + 4 \varepsilon = 7 \varepsilon,
\]
By using the fact that $G^{-1}$ approximates $\overline{G}^{-1}$ for $\kappa > 0$ large enough\(^1\), we also deduce

$$|H(x) - t| = |\alpha(e^*_1(G^{-1}(x))) - t| \leq 8\varepsilon, \text{ whenever } \|x - \beta(t)\|_\infty \leq r/2.$$  

Up to a change of $\varepsilon$ and $r$, this shows the second part of property (2').

Next, the bounds in (3) can be obtained from similar bounds for $D\overline{G}$, $(D\overline{G})^{-1}$, and the facts that these derivatives are uniformly approximated by $DG$, $(DG)^{-1}$, respectively, for $\kappa > 0$ large enough\(^2\). Then we can also show the first part of (2'): If $|e^*_1G^{-1}(x) - n| \leq 3\varepsilon/2$ for some $n \in \mathbb{N}$ then

$$\|DH(x)\| \leq |\alpha'(e^*_1G^{-1}(x))|\|DG^{-1}(x)\| \leq \frac{\varepsilon}{4}(2 + \varepsilon) \leq \varepsilon,$$

while, if $|e^*_1G^{-1}(x) - n| \geq 3\varepsilon/2$ for all $n \in \mathbb{N}$ then we have that $\overline{G}^{-1}$ is an affine isometry on a neighborhood of $x$, hence

$$\|DG^{-1}(x)\| \leq \|D\overline{G}^{-1}(x)\| + \varepsilon = 1 + \varepsilon,$$

which implies that $\|DH(x)\| \leq \frac{1+\varepsilon}{1-\varepsilon}$. Since $\lim_{\varepsilon \to 0} \frac{1+\varepsilon}{1-\varepsilon} = 1$, the first part of (2') follows up to a change of $\varepsilon$ and $r$.

As for property (4), we shall first check that $D\overline{G}$ is bounded on a neighborhood of $S$ in $\ell_\infty$ of the form $\{x + iz : x \in S, z \in \ell_\infty, \|z\| \leq 1\}$, so $\overline{G}$ is Lipschitz on this set. Indeed, on the one hand, if $t, s \in \mathbb{R}, |s| \leq 1$, we have

$$\left|\frac{1}{\int_{-\infty}^{\infty} e^{-\kappa u^2} dv} \int_{-\infty}^{\infty} \tilde{\theta}(u)2(t + is - n - u)e^{-\kappa(t+i(s-n-u))^2} du\right| \leq 2e^\kappa (|t - n| + 2) e^{-\kappa(|t-n|^2 + \frac{1}{2}|t-n|)} \int_{-1/2}^{3/4} e^{-\kappa u^2} du \leq 2e^\kappa (|t - n| + 2) e^{-\kappa(|t-n|^2 + \frac{1}{2}|t-n|)}.$$

For $|t - n| < 2$ this expression is bounded by $8e^{4\kappa}$, and since there are at most four integers $n$ with $|t - n| < 2$ we have

$$\sum_{|t-n|<2} |(\tilde{\theta})'(t + is - n)| \leq 32e^{4\kappa}.$$

\(^1\)This is a consequence of the following exercise: if $f_k$ is a sequence of diffeomorphisms which uniformly converges to a diffeomorphism $f$ with a uniformly continuous inverse $f^{-1}$, then $f_k^{-1}$ uniformly converges to $f^{-1}$.

\(^2\)This is a consequence of the following (easy to prove) fact: if $f_k$ is a sequence of diffeomorphisms which uniformly converges to a diffeomorphism $f$ such that there exists $m > 0$ with $\|Df(x)(h)\| \geq m\|h\|$ for all $x, h$, then the sequence of derivatives $D(f_k^{-1})$ converges to $D(f^{-1})$.  

And for $|t - n| \geq 2$ we can estimate
\[
\sum_{|t - n| \geq 2} |\tilde{\theta}'(t + is - n)| \leq \sum_{|t - n| \geq 2} 2e^\kappa (|t - n| + 2) e^{-\kappa(|t - n|^2 - \frac{1}{2}|t - n|)} \leq 2e^\kappa \sum_{m=1}^{\infty} (m + 2)e^{-\kappa m^2 / 2} < +\infty.
\]
Therefore there exists $0 < C := 32e^{4\kappa} + 2e^\kappa \sum_{m=1}^{\infty} (m + 2)e^{-\kappa m^2 / 2} < +\infty$ such that
\[
\sum_{n=0}^{\infty} |(\tilde{\theta})'(t + is - n)| \leq C
\]
for all $t, s \in \mathbb{R}$ with $|s| \leq 1$. We also have
\[
\sum_{n=0}^{\infty} |\tilde{\theta}(t + is - n)| \leq C' \text{ for all } t, s \in \mathbb{R}, |s| \leq 1
\]
for some $C' > 0$.

On the other hand, it is easy to see that there exists $C'' > 0$ such that
\[
\|D\tilde{\Phi}(x + iz)\| \leq C'' \text{ and } \|\tilde{\Phi}(x + iz)\| \leq C''
\]
for all $x \in S, z \in \ell_\infty, \|z\| \leq 1$. Therefore we get
\[
\|D\tilde{G}(x + iz)\| \leq \sum_{n=0}^{\infty} |(\tilde{\theta}')'(x_1 + iz_1 - n)| \|D\tilde{\Phi}_{(n+1,n+2)}(x + iz)\| + \\
+ \sum_{n=0}^{\infty} |\tilde{\theta}(x_1 + iz_1 - n)| \|D\tilde{\Phi}_{(n+1,n+2)}(x + iz)\| \leq \\
\sum_{n=0}^{\infty} |(\tilde{\theta}')'(x_1 + iz_1 - n)| C'' + \sum_{n=0}^{\infty} |\tilde{\theta}(x_1 + iz_1 - n)| C'' \leq C''(C + C')
\]
for all $x \in S, z \in \ell_\infty, \|z\| \leq 1$.

Since $D\tilde{G}$ is bounded on the convex set \(\{x + iz : x \in S, z \in \ell_\infty, \|z\| \leq 1\}\), it immediately follows from the mean value theorem that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in S, z \in \ell_\infty$ and $\|z\| \leq \delta$ then $\|G(x + iz) - G(x)\| \leq \varepsilon$. This shows (4).

Similar calculations show that $D^2\tilde{G}$ is also bounded on \(\{x + iz : x \in S, z \in \ell_\infty, \|z\| \leq 1\}\). And we already know that $DG$ and $(DG)^{-1}$ are bounded on $S$. Then, according to Proposition 2.5.6 of [AMR] (or rather, by an analogous statement for holomorphic mappings, which can be proved in the same fashion by refining a standard proof of the inverse mapping theorem for holomorphic mappings), there exist $r', s' > 0$ (depending only on the bounds for $DG$ and $(DG)^{-1}$ on $S$ and on the bound of $D^2\tilde{G}$ on $\{x + iz : x \in S, z \in \ell_\infty, \|z\| \leq 1\}$) such that, for every $x \in S$, the mapping $G$ is a holomorphic diffeomorphism from the ball $B_{\ell_\infty}(x, r')$ onto an open
subset of \( \ell_\infty \) which contains the ball \( B_{\ell_\infty}(G(x), s') \). In particular, for every \( y = G(x) \in \mathcal{T} \) there exists a holomorphic extension \( (G^{-1})_y \) of \( G^{-1} \) defined on the ball \( B_{\ell_\infty}(G(x), s') \) and which maps this ball diffeomorphically within the ball \( B_{\ell_\infty}(x, r') \).

Now define \( \tilde{G}^{-1}(u + iv) = (G^{-1})_y(u + iv) \) if \( u + iv \in B_{\ell_\infty}(y, s') \) for some \( y \in \mathcal{T} \). This mapping is well defined. For, if \( w = u + iv \in B_{\ell_\infty}(y_1, s') \cap B_{\ell_\infty}(y_2, s') \) with \( y_1, y_2 \in \mathcal{T}, y_1 \neq y_2 \), then there exist \( y_3 = G(x_3) \in \mathcal{T} \) and \( t \in (0, s') \) such that \( B_{\ell_\infty}(y_3, t) \subset B_{\ell_\infty}(y_1, s') \cap B_{\ell_\infty}(y_2, s') \), and since \( \tilde{G} \) maps diffeomorphically the ball \( B_{\ell_\infty}(x_3, r') \) onto an open set containing \( B_{\ell_\infty}(y_3, t) \), and the mappings \( (G^{-1})_y, (G^{-1})_y, (G^{-1})_{y_3} \) are local inverses of \( \tilde{G} \) defined on \( B_{\ell_\infty}(y_3, t) \) and taking values in \( B_{\ell_\infty}(x_3, r') \), we necessarily have \( (G^{-1})_y = (G^{-1})_{y_2} = (G^{-1})_{y_3} \) on the open ball \( B_{\ell_\infty}(y_3, t) \), by uniqueness of the inverse. But then, by the identity theorem for holomorphic mappings, we must have \( (G^{-1})_y = (G^{-1})_{y_2} \) on the open connected set \( B_{\ell_\infty}(y_1, s') \cap B_{\ell_\infty}(y_2, s') \), and in particular \( (G^{-1})_y(w) = (G^{-1})_{y_2}(w) \). Therefore \( G^{-1} \) is a holomorphic extension of \( G^{-1} \) to the open neighborhood \( \tilde{T}_v := \{ u + iv : y \in \mathcal{T}, v \in \ell_\infty, \|v\| < s' \} \) of \( \mathcal{T} \) which maps \( \tilde{T}_v \) into \( \tilde{S}_v := \{ x + iz : x \in \mathcal{S}, z \in \ell_\infty, \|z\| < r' \} \). Hence

\[
\|\tilde{G}^{-1}(u + iv) - G^{-1}(u)\| < r',
\]

for all \( u \in \mathcal{T}, v \in \ell_\infty \) with \( \|v\| < s' \). Obviously we can assume \( r' < \varepsilon \), so the first part of (5) is proved. Finally, in order to check the second part of (5), observe that, for all \( t, \nu \in \mathbb{R} \), we have

\[
|\tilde{\alpha}(t + iv)| \leq e^{\kappa \nu^2} \alpha(t).
\]

Therefore, for all \( t + s + iv \in \mathbb{C} \), with \( |s| \leq 1/\text{Lip}(\alpha) \) we have

\[
|\tilde{\alpha}(t + s + iv)| \leq e^{\kappa \nu^2} \alpha(t + s) \leq e^{\kappa \nu^2} (\alpha(t) + \text{Lip}(\alpha)|s|) \leq e^{\kappa \nu^2} (\alpha(t) + 1).
\]

We can also assume \( 0 < r' \leq \min\{\frac{1}{\text{Lip}(\alpha)}, \sqrt{\frac{2 \log 2}{\kappa}}\} \), so we get

\[
|\tilde{\alpha}(t + z)| \leq 2(\alpha(t) + 1) \text{ if } t \in \mathbb{R}, z \in \mathbb{C}, |z| \leq r'.
\]

By combining this estimation with (\star) we obtain

\[
|\tilde{\alpha}(e_1 \tilde{G}^{-1}(u + iv))| \leq 2(\alpha(e_1(G^{-1}(u))) + 1)
\]

for all \( u \in \mathcal{T}, v \in \ell_\infty \) with \( \|v\| < s' \), which establishes the second part of (5).

5.5. Proof of the main result in the case of a 1-Lipschitz, bounded function. Let us continue with the proof of our main result in the case of a nonnegative, bounded, 1-Lipschitz function. Fixing \( \varepsilon = 1 \), there exist \( \delta_1, r_1 > 0 \) so that properties (4) and (5) of the preceding Lemma hold. Now, given a bounded 1-Lipschitz function \( f : X \to \mathbb{R} \), we may assume (up to the addition of a constant) that \( f \) takes values in the interval \([0, +\infty)\). Let
Let $N \in \mathbb{N}$ be such that $N \geq f(x) \geq 0$ for all $x \in X$. By Proposition 2 applied with $\eta = r_1/2$ and this $\delta_1$, we can find real analytic functions $g_1, \ldots, g_N$, with holomorphic extensions $\tilde{g}_1, \ldots, \tilde{g}_N$ defined on $\tilde{U} := \tilde{U}_{\delta_1, r_1/2} \supset X$, such that

1. $|f_i(x) - g_i(x)| \leq r_1/2$ for all $x \in X$.
2. $g_i$ is $C_{r_1/2}$-Lipschitz.
3. $|\tilde{g}_i(x + iy) - g_i(x)| \leq \delta_1$ for all $z = x + iy \in \tilde{U}$.

Now let us define $g$ by $g = \alpha \circ \epsilon_i \circ G^{-1} \circ \{g_i\}_{i=1}^{\infty} = H \circ \{g_i\}_{i=1}^{\infty}$, where we understand $g_i = 0$ for all $i > N$.

Note that, in spite of $G^{-1}$ having a bounded derivative, because $T$ is not convex we cannot deduce that $G^{-1}$ is Lipschitz. As a matter of fact this mapping is not Lipschitz, though it is uniformly continuous. Nevertheless, since $X$ is convex and the derivative of the function $H$ is bounded by $1 + \varepsilon = 2$, we do have that $g$ is Lipschitz. And since the mapping

$$X \ni x \mapsto (g_1(x), \ldots, g_N(x), 0, 0, \ldots) \in \ell_\infty$$

is obviously $C_{r_1/2}$-Lipschitz, it follows that $g = H \circ \{g_i\}_{i=1}^{\infty}$ is $2C_{r_1/2}$-Lipschitz.

Because $(f_1(x), \ldots, f_N(x), 0, 0, \ldots) \in \beta([0, N])$, and also $|f_i - g_i| \leq r_1/2$, we have $(g_1(x), \ldots, g_N(x), 0, 0, \ldots) \in T$ and, by property (2) of the preceding Lemma,

$$|g(x) - f(x)| = |H(\{g_n(x)\}_{n=1}^{\infty}) - h(\{f_n(x)\}_{n=1}^{\infty})| \leq \varepsilon = 1.$$

The function $g$ is clearly real analytic, with holomorphic extension $\tilde{g}(z) = (\tilde{\alpha} \circ \tilde{\epsilon}_i \circ \tilde{G}^{-1})(\tilde{g}_1(z), \tilde{g}_2(z), \ldots, \tilde{g}_N(z), 0, 0, \ldots)$ defined on $\tilde{U}$. And, because $|\tilde{g}_i(x + iy) - g_i(x)| \leq \delta$ for all $z = x + iy \in \tilde{U}$, we have, using property (5) of the preceding Lemma, that $|\tilde{g}(x + iy)| \leq 2(|g(x)| + 1)$ for all $x + iy \in \tilde{U}$.

By resetting $C_{r_1/2}$ to $C := 2C_{r_1/2}$, we have thus proved the following version of our main result for bounded functions.

**Proposition 3.** Let $X$ be a Banach space having a separating polynomial. Then there exist $C \geq 1$ and an open neighborhood $\tilde{U}$ of $X$ in $\tilde{X}$ such that, for every $1$-Lipschitz, bounded function $f : X \to \mathbb{R}$, there exists a real analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\tilde{g} : \tilde{U} \to \mathbb{C}$, such that

1. $|f(x) - g(x)| \leq 1$ for all $x \in X$.
2. $g$ is $C$-Lipschitz.
3. $|\tilde{g}(x + iy)| \leq 2(|g(x)| + 1)$ for all $z = x + iy \in \tilde{U}$.

### 5.6. Proof of the main result in the case of an unbounded Lipschitz function.

Let $\varepsilon \in (0, 1)$ be fixed, and let $f : X \to \mathbb{R}$ be an unbounded 1-Lipschitz function. For $n \in \mathbb{N}, n \geq 2$, let us define the crowns

$$C_n = C_n^\varepsilon := \{x \in X : \frac{2^n - 1}{\varepsilon} \leq Q(x) \leq \frac{2^n + 1}{\varepsilon}\}.$$
and for $n = 1$ set $C_1 = C_1' = \{x \in X : Q(x) \leq 4/\varepsilon\}$. Let $f_n$ denote a bounded, 1-Lipschitz extension of $f|_{\overline{C}_n}$ to $X$ (defined for instance by $x \mapsto \max\{-\|f|_{\overline{C}_n}\|, \min\{\|f|_{\overline{C}_n}\|, \inf_{y \in C_n}\{f(y) + \|x - y\|\}\})$. According to the preceding Proposition there exist an open neighborhood $\widetilde{U} = \widetilde{U}_\varepsilon$ of $X$ in $\overline{X}$, and real analytic functions $g_n : X \to \mathbb{R}$, with holomorphic extensions $\tilde{g}_n : \widetilde{U} \to \mathbb{C}$, such that

1. $|f_n(x) - g_n(x)| \leq 1$ for all $x \in X$.
2. $g_n$ is $C$-Lipschitz.
3. $|\tilde{g}_n(x + iy)| \leq 2|g_n(x)| + 1$ for all $z = x + iy \in \widetilde{U}$.

For $n = 1$, let $\overline{\vartheta}_1 : \mathbb{R} \to [0, 1]$ be a $C^\infty$ function such that

1. $\overline{\vartheta}_1(t) = 1 \iff t \in [0, 2/\varepsilon]$
2. $\overline{\vartheta}_1(t) > 0 \iff t \in (-1/\varepsilon, 4/\varepsilon)$
3. $\overline{\vartheta}_1(t) < 0 \iff t \in (2/\varepsilon, 4/\varepsilon)$
4. $\text{Lip}(\overline{\vartheta}_1) \leq 1$

Now let $\overline{\vartheta}_n : \mathbb{R} \to [0, 1], n \in \mathbb{N}, n \geq 2$, be a sequence of $C^\infty$ functions with the following properties:

1. $\overline{\vartheta}_n(t) > 0 \iff t \in (2^{n-1}/\varepsilon, 2^n/\varepsilon)$
2. $\overline{\vartheta}_n(t) > 0 \iff t \in (2^{n-1}/\varepsilon, 2^n/\varepsilon)$
3. $\overline{\vartheta}_n(t) < 0 \iff t \in (2^n/\varepsilon, 2^{n+1}/\varepsilon)$
4. $\overline{\vartheta}_n(2^n/\varepsilon) = 1$
5. $\overline{\vartheta}_n(t) = 1 - \overline{\vartheta}_{n-1}(t)$ whenever $t \in (2^{n-1}/\varepsilon, 2^n/\varepsilon)$
6. $\text{Lip}(\overline{\vartheta}_n) \leq \varepsilon/2^{n-2}$

The functions $\overline{\vartheta}_n$ form a partition of unity on $\mathbb{R}$, and

$$\sum_{n=1}^{\infty} \text{Lip}(\overline{\vartheta}_n) \leq 3\varepsilon.$$ 

Then the functions $x \mapsto \overline{\vartheta}_n(Q(x)), n \in \mathbb{N}$ also form a partition of unity subordinated to the covering $\bigcup_{n \in \mathbb{N}} C_n = X$, and the sum of the Lipschitz constants of these functions is bounded by $3\varepsilon \text{Lip}(Q)$.

Define real analytic functions $\vartheta_n : \mathbb{R} \to [0, 1]$ by

$$\vartheta(t) = a_n \int_{\mathbb{R}} \overline{\vartheta}_n(s)e^{-\kappa_n(t-s)^2}ds,$$

where $a_n := \int_{\mathbb{R}} e^{-\kappa_n s^2}ds$, and $\kappa_n$ is large enough so that

1. $e^{-\kappa_n} \leq \varepsilon/2^{n+3}(1 + \|g_n\|_\infty)$
2. $\vartheta_n(t) - \overline{\vartheta}_n(t) \leq \varepsilon/2^{n+3}(1 + \|g_n\|_\infty)$ for all $t \in \mathbb{R}$
3. $|\vartheta_n(t) - \overline{\vartheta}_n(t)| \leq \varepsilon/2^{n+3} \text{Lip}(Q)(1 + \|g_n\|_\infty)$ for all $t \in \mathbb{R}$.
Also, $\text{Lip}(\theta_n) = \text{Lip}(\tilde{\theta}_n)$, and hence $\sum_{n=1}^{\infty} \text{Lip}(\theta_n) \leq 3\varepsilon$. Let us define $g : X \to \mathbb{R}$ and $\tilde{g} : \tilde{U} \to \mathbb{C}$ by

$$g(x) = \sum_{n=1}^{\infty} \theta_n(Q(x))g_n(x), \quad \text{and} \quad \tilde{g}(z) = \sum_{n=1}^{\infty} \tilde{\theta}_n(\tilde{Q}(z))\tilde{g}_n(z),$$

where $\tilde{\theta}(u + iv) = a_n \int_{\mathbb{R}} \theta_n(s)e^{-\kappa_n(u+iv-s)^2} \, ds$.

In order to see that $g$ is well defined and real-analytic, with holomorphic extension $\tilde{g}$ defined on $\tilde{U}$, it is enough to show that the series of holomorphic mappings defining $\tilde{g}$ is locally uniformly and absolutely convergent on $\tilde{U}$.

And, since $|\tilde{g}_n(z)| \leq 2(\|g_n\|_{\infty} + 1)$ for all $z \in \tilde{U}$, it is sufficient to check that

$$\sum_{n=1}^{\infty} |\tilde{\theta}_n(\tilde{Q}(z))|(1 + \|g_n\|_{\infty}) < +\infty$$

locally uniformly on $\tilde{U}$.

We may assume that $\text{Lip}(\tilde{Q}) \leq C$. Then, for each $x \in X$, according to Lemma 2 we can write

$$\tilde{Q}(x + z) = Q(x) + Z_x, \quad \text{where} \quad Z_x \in \mathbb{C}, \quad |Z_x| \leq C\|z\|_X.$$

Fix $x \in X$. There exists $n_x \in \mathbb{N}$ so that $x \in C_{n_x}$ and in particular $Q(x) \leq 2^{n_x+1}/\varepsilon$.

Now, if $n \geq n_x + 3$ and $s \in \text{support}(\tilde{\theta}_n) = [2^{n-1}/\varepsilon, 2^{n+1}/\varepsilon]$ we have $s \geq 2^{n_x+2}/\varepsilon$ and therefore, for $\|z\|_X \leq 1/2C$,

\[
\text{Re} \left( \tilde{Q}(x + z) - s \right)^2 \\
= (Q(x) - s)^2 + 2(Q(x) - s)\text{Re} Z_x + \text{Re}(Z_x^2) \\
\geq (Q(x) - s + \text{Re} Z_x)^2 - (\text{Re} Z_x)^2 + \text{Re}(Z_x^2) \\
\geq (Q(x) - s + \text{Re} Z_x)^2 - \frac{1}{2} \\
\geq \frac{1}{2}(s - Q(x) - \text{Re} Z_x)^2 + \frac{1}{2}(2^{n_x+2}/\varepsilon)^2 - \frac{2^{n_x+1}}{\varepsilon} - \frac{1}{2} - \frac{1}{2} \\
\geq \frac{1}{2}(s - Q(x) - \text{Re} Z_x)^2 + 1.
\]
Hence, for $n \geq n_x + 3$ and $\|z\|_X \leq 1/2C$, we have, according to the choice of $\kappa_n$, that

$$|\tilde{\theta}_n(\tilde{Q}(x + z))| \leq a_n \int_{\mathbb{R}} \tilde{\theta}_n(s)e^{-\kappa_n \text{Re}(\tilde{Q}(x + z) - s)^2} ds \leq a_n \int_{\mathbb{R}} e^{-\kappa_n \left(\frac{1}{2}(s - Q(x) - \text{Re} Z_x)^2 + 1\right)} ds = e^{-\kappa_n a_n} \int_{\mathbb{R}} e^{-\kappa_n u^2} \sqrt{2} du = \sqrt{2} e^{-\kappa_n} \leq \sqrt{2} \frac{1}{2^n (1 + \|g_n\|_{\infty})}.$$ 

Then it is clear that the series

$$\sum_{n=1}^{\infty} |\tilde{\theta}_n(\tilde{Q}(x + z))| (1 + \|g_n\|_{\infty})$$

is uniformly convergent for $\|z\|_X \leq 1/2C$. Since we can obviously assume $\tilde{U} \subset \{x + iy : x \in X, \|y\| \leq 1/2C\}$ (and in fact this is always the case, see the proof of Lemma 3), this argument shows that the series

$$\sum_{n=1}^{\infty} \tilde{\theta}_n(\tilde{Q}(z)) \tilde{g}_n(z)$$

defines a holomorphic function $\tilde{g}$ on $\tilde{U}$.

Next let us show that the real analytic function $g$ approximates $f$ on $X$. Let us define an auxiliary $C^\infty$ function $\overline{g}$ by

$$\overline{g}(x) = \sum_{n=1}^{\infty} \theta_n(Q(x))g_n(x).$$

Since the functions $\theta_n(Q(x))$ form a partition of unity subordinated to the covering $\bigcup_{n \in \mathbb{N}} C_n = X$, $|g_n - f_n| \leq 1$, and $f_n = f$ on $C_n$, it is immediately checked that

$$|\overline{g}(x) - f(x)| \leq 1 \text{ for all } x \in X.$$ 

On the other hand we have

$$|g(x) - \overline{g}(x)| = |\sum_{n=1}^{\infty} (\theta_n(Q(x)) - \theta_n(Q(x)))g_n(x)| \leq \sum_{n=1}^{\infty} \|g_n\|_{\infty} |\theta_n(Q(x)) - \theta_n(Q(x))| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

By combining the last two estimations we get

$$|g(x) - f(x)| \leq 2 \text{ for all } x \in X.$$
In order to estimate \( \text{Lip}(g) \), observe first that, because support(\( \theta_n \)) = \( C_n \) and \( f_n = f \) on \( C_n \), we can write, for every \( x, y \in X \),

\[
f(x) = \sum_{n=1}^{\infty} f(x)\theta_n(Q(y)) = \sum_{n=1}^{\infty} f_n(x)\theta_n(Q(y)).
\]

On the one hand, for every \( x, y \in X \),

\[
\overline{g}(x) - \overline{g}(y) = \sum_{n=1}^{\infty} \theta_n(Q(x))g_n(x) - \sum_{n=1}^{\infty} \theta_n(Q(y))g_n(y) = \\
\sum_{n=1}^{\infty} (g_n(x) - f_n(x)) (\theta_n(Q(x)) - \theta_n(Q(y))) + \sum_{n=1}^{\infty} (g_n(x) - g_n(y))\theta_n(Q(y)) \\
\leq \sum_{n=1}^{\infty} \text{Lip}(\theta_n)\text{Lip}(Q)\|x - y\| + \sum_{n=1}^{\infty} C\|x - y\|\theta_n(Q(y)) \\
\leq (3\varepsilon\text{Lip}(Q) + C')\|x - y\|,
\]

so \( \overline{g} \) is \( (C + 3\varepsilon\text{Lip}(Q)) \)-Lipschitz. And on the other hand,

\[
\text{Lip}(g - \overline{g}) \leq \sum_{n=1}^{\infty} \text{Lip}(\theta_n - \overline{\theta}_n)\|g_n\|_{\infty} + \sum_{n=1}^{\infty} \|\theta_n - \overline{\theta}_n\|_{\infty}2 \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

which implies

\[
\text{Lip}(g) \leq \text{Lip}(\overline{g}) + \varepsilon \leq C + 4\varepsilon\text{Lip}(Q).
\]

Up to a replacement of \( 4\varepsilon\text{Lip}(Q) \) with \( \varepsilon \), we have thus shown that for every \( \varepsilon > 0 \) and every 1-Lipschitz function \( f : X \to \mathbb{R} \) there exists a \( (C + \varepsilon) \)-Lipschitz, real analytic function \( g : X \to \mathbb{R} \) such that \( |f - g| \leq 2 \).

Finally, for arbitrary \( \varepsilon \in (0, 2) \) and \( \text{Lip}(f) := L \in (0, \infty) \), we consider the function \( F : X \to \mathbb{R} \) defined by \( F(x) = \frac{\varepsilon}{2} f(\frac{\varepsilon}{2L} x) \), which is 1-Lipschitz so, by what we have just proved there exists a \( (C + \varepsilon) \)-Lipschitz, real analytic function \( G : X \to \mathbb{R} \) such that \( |F - G| \leq 2 \). If we define \( g(x) = \frac{\varepsilon}{2} G(\frac{2\varepsilon}{L} x) \), we get a real analytic function \( g : X \to \mathbb{R} \) with \( \text{Lip}(g) \leq (C + \varepsilon)\text{Lip}(f) \), and such that \( |g - f| \leq \varepsilon \). This concludes the proof of Theorem 4. \( \square \)

**Remark 2.** However, note that the neighborhood of \( X \) in \( \widetilde{X} \) on which a holomorphic extension \( \widetilde{g} \) of \( g \) is defined is \( \widetilde{U}_{L,\varepsilon} := \frac{\varepsilon}{2L} \widetilde{U} \). In particular, as \( L = \text{Lip}(f) \) increases to \( \infty \) (or as \( \varepsilon \) decreases to 0), the neighborhood \( \widetilde{U}_{L,\varepsilon} \) of \( X \) where a holomorphic extension of the approximation \( g \) of \( f \) is defined shrinks to \( X \).

This unfortunate dependence of \( \widetilde{U}_{L,\varepsilon} \) on \( L,\varepsilon \) (which, according to the Cauchy-Riemann equations, is an inherent disadvantage of any conceivable scaling procedure, not just the one we have used) prevents our going further and combining our main result with a real analytic refinement of Moulis’ techniques [M] in order to prove the following conjecture: that every \( C^1 \)
function defined on a Banach space having a separating polynomial can be $C^1$-finely approximated by real analytic functions.

It is no use, either, trying to avoid scaling by choosing $r$ of the order of $\varepsilon/L$ and reworking the statements and the proofs of the results following Lemma 3 in order to allow $r \in (0,1)$, because in such case the neighborhood $\tilde{V}$ where the holomorphic extensions $\tilde{\varphi}_n$ of the functions $\varphi_n$ are defined and conveniently bounded will be contained in $\tilde{V}_r = \{x + iy : x, y \in X, ||y|| < r\}$, which also shrinks to $X$ as $r$ decreases to 0.

Despite this difficulty, the real analytic sup partitions of unity constructed in Lemma 3 can be used, together with Theorem 1, to prove Theorem 3, see \[AFK\]. In fact, an analysis of the proof of \[AFK\] (in view of Proposition 1 and following the lines of Lemma 4 above) shows that the domain where a holomorphic extension of the function $g$ in Theorem 3 is defined and $\varepsilon$-close to $g$ only depends on $||f||_\infty$, on $\varepsilon$, and on the modulus of continuity of $f'$.

Namely, we have the following sharp version of Theorem 3.

**Theorem 5.** Let $X$ be a separable Banach space with a separating polynomial, and let $M,K,\varepsilon > 0$ be given. Then there exists an open neighborhood $\tilde{U}$ of $X$ in $\overline{X}$, depending only on $M,K,\varepsilon$, such that for every function $f \in C^{1,1}(X)$ with $||f||_\infty \leq M$ and $\text{Lip}(f') \leq K$ there exists a real analytic function $g : X \rightarrow \mathbb{R}$, with holomorphic extension $\tilde{g} : \tilde{U} \rightarrow \mathbb{C}$, such that

1. $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$.
2. $|f'(x) - g'(x)| \leq \varepsilon$ for all $x \in X$.
3. $|\tilde{g}(x+iy) - g(x)| \leq \varepsilon$ for all $z = x + iy \in \tilde{U}$.

### 6. Proof of Theorem 1

Finally, by combining Theorem 5 with the Lasry-Lions sup-inf convolution regularization technique [LL] and with the above gluing tube function and crown arguments, we will prove Theorem 1.

We start considering the special case when $f$ is bounded.

#### 6.1. Proof of Theorem 1 in the case of a function $f : X \rightarrow [0,M]$

Given an $L$-Lipschitz function $f : X \rightarrow [0,M]$ defined on a separable Hilbert space $X$, set

$$f_\lambda(x) = \inf \{f(u) + \frac{1}{2\lambda}|x-u|^2 : u \in X\} = \inf \{f(x-u) + \frac{1}{2\lambda}|u|^2 : u \in X\}$$

$$f^{\mu}(x) = \sup \{f(u) - \frac{1}{2\lambda}|x-u|^2 : u \in X\} = \sup \{f(x-u) - \frac{1}{2\lambda}|u|^2 : u \in X\}.$$ 

Since the supremum (and the infimum) of a family of $L$-Lipschitz functions is $L$-Lipschitz, it is clear that $f_\lambda$ and $f^{\mu}$ are $L$-Lipschitz.

Now, since $f$ is bounded and uniformly continuous, according to [LL], the function

$$g_{\lambda,\mu}(x) := (f_\lambda)^{\mu}(x) = \sup_{z \in X} \inf_{y \in X} \left\{f(y) + \frac{1}{2\lambda}|z-y|^2 - \frac{1}{2\mu}|x-z|^2\right\}$$
is well defined and has a Lipschitz derivative on $X$ satisfying
\[
\text{Lip}(g_{\lambda,\mu}') \leq \max\left\{ \frac{1}{\mu}, \frac{1}{\lambda - \mu} \right\},
\]
for all $0 < \mu < \lambda$ small enough, and converges to $f(x)$, uniformly on $X$, as $0 < \mu < \lambda \to 0$. In fact, as noted in [11], the rate of convergence of $g_{\lambda,\mu}$ to $f$ only depends on $\text{Lip}(f)$, so for every $\varepsilon > 0$ there exists $\lambda_0 > 0$ (only depending on $\varepsilon$ and $L$) so that $|g_{\lambda,\mu}(x) - f(x)| \leq \varepsilon/2$ for all $x \in X$, $0 < \mu < \lambda \leq \lambda_0$.

Also, according to the above observations, this function is $L$-Lipschitz. Therefore we have
\[
\|g_{\lambda,\mu}(x)\| \leq L, \text{ and } |f(x) - g_{\lambda,\mu}(x)| \leq \frac{\varepsilon}{2}
\]
for all $x \in X$, for some $0 < \mu < \lambda$ small enough. Now fix $\lambda, \mu$ with
\[
0 < \lambda < \lambda_0, \quad \mu := \frac{\lambda}{2},
\]
and apply Theorem 3 to obtain a real analytic function $g : X \to \mathbb{R}$ such that
\[
|g_{\lambda,\mu}(x) - g(x)| \leq \frac{\varepsilon}{2} \text{ and } |g_{\lambda,\mu}'(x) - g'(x)| \leq \frac{\varepsilon}{2}
\]
for all $x \in X$. By combining the last two inequalities we get $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$ and $\text{Lip}(g) \leq L + \varepsilon$. Moreover $g_{\lambda,\mu}$ has a holomorphic extension to a neighborhood $\tilde{U}$ of $X$ in $\tilde{X}$ which only depends on $L$, on $\text{Lip}(g_{\lambda,\mu}')$, on $M$, and on $\varepsilon$. Since $\text{Lip}(g_{\lambda,\mu}') \leq \max\{1/\mu, 1/(\lambda - \mu)\} = 2/\lambda$ and in turn $\lambda$ only depends on $\varepsilon$ and on $\text{Lip}(f) \leq L$, we have thus proved the following.

**Proposition 4.** Let $X$ be a separable Hilbert space. For every $L, M, \varepsilon > 0$ there exists a neighborhood $\tilde{U} := \tilde{U}_{L,M,\varepsilon}$ of $X$ in $\tilde{X}$ such that, for every $L$-Lipschitz function $f : X \to [0, M]$ there exists a real analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\tilde{g} : \tilde{U} \to \mathbb{C}$, such that

1. $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$.
2. $g$ is $(L + \varepsilon)$-Lipschitz.
3. $|\tilde{g}(x + iy) - g(x)| \leq \varepsilon$ for all $z = x + iy \in \tilde{U}$.

In particular this establishes Theorem 4 in the case when $f$ is bounded.

Now, fix $L = M = 1$ and $\varepsilon > 0$, and let $r, \delta \in (0, \varepsilon/64)$ be as in Lemma 3. Given a bounded $1$-Lipschitz function $f : X \to \mathbb{R}$, we may assume that $f$ takes values in the interval $[0, +\infty)$. Let $N \in \mathbb{N}$ be such that $N \geq f(x) \geq 0$ for all $x \in X$. By Proposition 4 we can find real analytic functions $g_1, ..., g_N$, with holomorphic extensions $\tilde{g}_1, ..., \tilde{g}_N$ defined on $\tilde{U} := \tilde{U}_{1,1,\min\{r,2\delta\}} \supset X$, such that

1. $|f_i(x) - g_i(x)| < r/2$ for all $x \in X$.
2. $g_i$ is $(1 + \varepsilon)$-Lipschitz.
3. $|\tilde{g}_i(x + iy) - g_i(x)| \leq \delta$ for all $z = x + iy \in \tilde{U}$.
Let us define \( g = e_1^* \circ G^{-1} \circ \{ g_i \}_{i=1}^{\infty} = H \circ \{ g_i \}_{i=1}^{\infty} \), where we understand \( g_i = 0 \) for all \( i > N \).

Since \( \text{Lip}(g_i) \leq 1 + \varepsilon \) for all \( i \), and \( DH \) is bounded by \( 1 + \varepsilon \), it follows that \( g = H \circ \{ g_i \}_{i=1}^{\infty} \) is \((1 + \varepsilon)^2\)-Lipschitz.

Because \( (f_1(x), \cdots, f_N(x), 0, 0, \cdots) \in \beta([0, N]) \), and also \( |f_i - g_i| < r/2 \), we have \( (g_1(x), \cdots, g_N(x), 0, 0, \cdots) \in \mathcal{T} \) and, by property (2) of the preceding Lemma,

\[
|g(x) - f(x)| = |H(\{g_n(x)\}_{n=1}^{\infty}) - h(\{f_n(x)\}_{n=1}^{\infty})| \leq \varepsilon.
\]

The function \( g \) is clearly real analytic, with holomorphic extension \( \widetilde{g}(z) = (e_1^* \circ G^{-1})(\tilde{g}_1(z), \tilde{g}_2(z), \ldots, \tilde{g}_N(z), 0, 0, \ldots) \) defined on \( \tilde{U} \). And, because \( |\tilde{g}_i(x + iy) - g_i(x)| \leq \delta \) for all \( z = x + iy \in \tilde{U} \), we have, using property (5) of the Lemma, that \( |\tilde{g}(x + iy)| \leq 2(|g(x)| + 1) \) for all \( x + iy \in \tilde{U} \).

Up to a change of \( \varepsilon \) we have thus proved the following.

**Proposition 5.** Let \( X \) be a separable Hilbert space. For every \( \varepsilon > 0 \) there exists an open neighborhood \( U = U_\varepsilon \) of \( X \) in \( \tilde{X} \) such that, for every \( 1 \)-Lipschitz, bounded function \( f : X \to \mathbb{R} \), there exists a real analytic function \( g : X \to \mathbb{R} \), with holomorphic extension \( \tilde{g} : \tilde{U} \to \mathbb{C} \), such that

1. \( |f(x) - g(x)| \leq \varepsilon \) for all \( x \in X \).
2. \( g \) is \((1 + \varepsilon)^2\)-Lipschitz.
3. \( |\tilde{g}(x + iy)| \leq 2(|g(x)| + 1) \) for all \( z = x + iy \in \tilde{U} \).

Then, by using exactly the same argument as in section 5.6 above (just noting that now we are lucky to have \( C = 1 + \varepsilon \)), one immediately obtains Theorem 4 in full generality.

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