Induced modules of support $\tau$-tilting modules and extending modules of semibricks over blocks of finite groups\textsuperscript{*}\textsuperscript{†}

Ryotaro KOSHIO        Yuta KOZAKAI

September 15, 2022

Abstract

In this article we study support $\tau$-tilting modules, semibricks and more over blocks of group algebras. Let $k$ be an algebraically closed field of characteristic $p > 0$, $\tilde{G}$ a finite group and $G$ a normal subgroup of $\tilde{G}$. Moreover, let $\tilde{B}$ be a block of $k\tilde{G}$ and $B$ a block of $kG$ covered by $\tilde{B}$. We show that, under certain conditions for the factor group $\tilde{G}/G$ and $B$, induced modules and extending modules of support $\tau$-tilting modules and semibricks over $\tilde{B}$ are also the ones over $B$, respectively.

Contents

1 Introduction 2

2 Preliminary results of $\tau$-tilting theory 5
  2.1 Functorially finiteness of torsion classes and torsion-free classes 5
  2.2 Support $\tau$-tilting modules 7
  2.3 Bricks and semibricks 10
  2.4 Silting complexes and simple-minded collections 16

3 Preliminaries of modular representation theory of finite groups 19
  3.1 Restriction functors and induction functors 19
  3.2 Indecomposable modules, bricks and simple modules satisfying the stable conditions 20
  3.3 Blocks of group algebras 25
  3.4 Clifford’s theory for blocks of normal subgroups 26

4 The main results and their applications 28
  4.1 Main theorems and their proof 28
  4.2 Some applications of main theorems 34

\textsuperscript{*}Mathematics Subject Classification (2020). 20C20, 16G10.
\textsuperscript{†}Keywords. Support $\tau$-tilting modules, Semibricks, two-term tilting complexes, two-term simple-minded collections, Blocks of finite groups.
1 Introduction

The study of derived equivalences of blocks of finite groups has been motivated and inspired by “Broué’s conjecture”, which can be conceived of as a local-global principle in the modular representation theory of finite groups. In [21], the solution to the problem of determining the equivalence of two given algebras was reduced to the problem of finding an appropriate tilting complex. Therefore, abundant constructions of tilting complexes over blocks enables us to find the algebras which are derived equivalent to the blocks. Of course, it is very hard to construct appropriate tilting complexes over blocks and to determine all tilting complexes over blocks. The classes of tilting complexes called two-term tilting complexes are considered to be non-trivial and a bit easier to handle because it is showed that there exists a one-to-one correspondence between the two-term tilting complexes and the support \( \tau \)-tilting modules over symmetric algebras in [2]. Abundant constructions of two-term tilting complexes over blocks are also useful for plenty of constructions of general tilting complexes over blocks by using the tilting mutations introduced in [4]. Therefore, we focus on support \( \tau \)-tilting modules, over blocks and their corresponding representation-theoretic objects, such as semibricks, functorially finite torsion classes of module categories, two-term simple-minded collections and more, which are also useful to study of derived equivalences of blocks [7, 12]. Finally, we got some results which work effectively for the purpose stated above.

In order to describe these, we set notation as follows. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), \( \bar{G} \) a finite group, \( G \) a normal subgroup of \( \bar{G} \), \( B \) a block of \( kG \) and \( \bar{B} \) a block of \( k\bar{G} \) covering \( B \), that is, \( 1_B 1_{\bar{B}} \neq 0 \), where \( 1_B \) and \( 1_{\bar{B}} \) mean the respective unit elements of \( B \) and \( \bar{B} \). In this setting, there are some useful properties about the restriction functor \( \operatorname{Res}^{\bar{G}}_G \) and the induction functor \( \operatorname{Ind}^{\bar{G}}_G \) between the category of \( B \)-modules and the one of \( \bar{B} \)-modules. We denote the inertial group of the block \( B \) in \( \bar{G} \) by \( I^{\bar{G}}_G(B) \) and the second group cohomology of the factor group \( I^{\bar{G}}_G(B)/G \) with coefficients in the unit group \( k^\times \) of the field \( k \) with trivial \( G \)-action by \( H^2(I^{\bar{G}}_G(B)/G, k^\times) \). We say that a \( B \)-module \( U \) is \( I^{\bar{G}}_G(B) \)-stable if \( xU \cong U \) as \( B \)-modules for any \( x \in I^{\bar{G}}_G(B) \). Furthermore, we use the following notation:

- \( \tau \)-tilt \( B \) (or \( \tau \)-tilt \( \bar{B} \)) means the set of isomorphism classes of basic support \( \tau \)-tilting modules over \( B \) (or \( \bar{B} \), respectively),
- \( \text{fL-sbrick} \) \( B \) (or \( \text{fL-sbrick} \) \( \bar{B} \)) means the set of isomorphism classes of basic left finite semibricks over \( B \) (or \( \bar{B} \), respectively),
- 2-tilt \( B \) (or 2-tilt \( \bar{B} \)) means the set of isomorphism classes of basic two-term tilting complexes in \( K^b(B\text{-proj}) \) (or \( K^b(\bar{B}\text{-proj}) \), respectively),
- 2-smc \( B \) (or 2-smc \( \bar{B} \)) means the set of isomorphism classes of two-term simple-minded collections in \( D^b(B\text{-mod}) \) (or \( D^b(\bar{B}\text{-mod}) \), respectively).

The following result contributes to abundant constructions of support \( \tau \)-tilting \( \bar{B} \)-modules and two-term tilting complexes in \( K^b(\bar{B}\text{-proj}) \).

**Main Theorem 1.1** (see Theorem 4.3). Under the above notation, we assume the following conditions hold:
(1) Any left finite brick in $B$-mod is $I\tilde{G}(B)$-stable.

(2) $H^2(I\tilde{G}(B)/G, k^\times) = 1$.

(3) $k[I\tilde{G}(B)/G]$ is basic as a $k$-algebra.

Then the maps

\[ \text{st-tilt } B \longrightarrow \text{st-tilt } \tilde{B} \quad (1.0.1) \]

defined by $\text{st-tilt } B \ni M \mapsto \tilde{B}\text{Ind}_{I\tilde{G}(B)}^G M \in \text{st-tilt } \tilde{B}$ and

\[ 2\text{-tilt } B \longrightarrow 2\text{-tilt } \tilde{B} \quad (1.0.2) \]

defined by $2\text{-tilt } B \ni T \mapsto \tilde{B}\text{Ind}_{I\tilde{G}(B)}^G T \in 2\text{-tilt } \tilde{B}$ are well-defined and injective. Moreover, we get the following commutative diagram:

\[
\begin{array}{ccc}
\text{st-tilt } B & \xrightarrow{(1.0.1)} & \text{st-tilt } \tilde{B} \\
\text{[2] for } B & \downarrow & \text{[2] for } \tilde{B} \\
2\text{-tilt } B & \xrightarrow{(1.0.2)} & 2\text{-tilt } \tilde{B}.
\end{array}
\]

One of our interests is, for finite group $\tilde{G}$ and its normal subgroup $G$ with a cyclic Sylow $p$-subgroup and of $p$-power index in $\tilde{G}$, whether the representation theoretical objects over $\tilde{B}$ can be obtained from those over $B$, where $B$ is a block of $kG$ and $\tilde{B}$ is that of $k\tilde{G}$ covering $B$ (for example see [23, 15]). In [18], the authors showed that the induction functor $\text{Ind}_{I\tilde{G}(B)}^G$, induces a poset isomorphism between st-tilt $B$ and st-tilt $\tilde{B}$ and between 2-tilt $B$ and 2-tilt $\tilde{B}$ in the above setting. On the other hand, there must be an explicit correspondence between $f_L$-sbrick $B$ and $f_L$-sbrick $\tilde{B}$ and that between 2-smc $B$ and 2-smc $\tilde{B}$, but it was not made clear that how they correspond in that paper. Therefore, one of our goals is clarifying the correspondences, and we get the following theorems as positive answers which can be applied to our interested situation.

Main Theorem 1.2 (see Theorem 4.4). With the same assumptions in Main Theorem 1.1, the following hold:

(1) Let $e$ be the number of isomorphism classes of simple $k[I\tilde{G}(B)/\tilde{G}]$-modules. Then for any left finite semibrick $S$ in $B$-mod and any indecomposable direct summand $S_i$ of $S$, there exist exactly $e$ isomorphism classes of bricks $\tilde{S}_i^{(1)}, \ldots, \tilde{S}_i^{(e)}$ in $kI\tilde{G}(B)$-mod satisfying $\text{Res}^G_{I\tilde{G}(B)}(\tilde{S}_i^{(j)}) \cong S_i$ for all $i, j$.

(2) The map

\[ f_L\text{-sbrick } B \longrightarrow f_L\text{-sbrick } \tilde{B} \quad (1.0.3) \]

defined by $S \cong \bigoplus_{i=1}^{n_S} S_i \mapsto \tilde{B}\text{Ind}_{I\tilde{G}(B)}^G \left( \bigoplus_{i=1}^{n_S} \bigoplus_{j=1}^{n_{S_i}} \tilde{S}_i^{(j)} \right)$, where $S \cong \bigoplus_{i=1}^{n_S} S_i$ is a direct sum decomposition into bricks, is well-defined and injective.

3
(3) We get the following commutative diagram:

\[
\begin{array}{ccc}
\text{st-tilt } B & \xrightarrow{(1.0.1)} & \text{st-tilt } \tilde{B} \\
\downarrow \text{[7] for } B & & \downarrow \text{[7] for } \tilde{B} \\
\text{f}_L\text{-sbrick } B & \xrightarrow{(1.0.3)} & \text{f}_L\text{-sbrick } \tilde{B}.
\end{array}
\]

In addition to the above theorem, in our setting, we get an explicit map

\[
\text{2-smc } B \longrightarrow \text{2-smc } \tilde{B}
\]

induced by (1.0.3) (see Corollary 4.7 for the detail) and get the following result.

**Main Theorem 1.3** (see Theorem 4.8). With the same assumptions in Main Theorem 1.1, the following diagram is commutative:

\[
\begin{array}{ccc}
\text{2-smc } B & \xrightarrow{(1.0.4)} & \text{2-smc } \tilde{B} \\
\downarrow \text{[7, 12] for } B & & \downarrow \text{[7, 12] for } \tilde{B} \\
\text{2-tilt } B & \xrightarrow{(1.0.2)} & \text{2-tilt } \tilde{B} \\
\downarrow \text{[2] for } B & & \downarrow \text{[2] for } \tilde{B} \\
\text{st-tilt } B & \xrightarrow{(1.0.1)} & \text{st-tilt } \tilde{B}.
\end{array}
\]

At a glance, the assumption in Main Theorem 1.1, which is also required in Main Theorems 1.2 and 1.3, seems strong, but in fact it can be applied to many situations, including the one we are interested in. In the setting of following Main Theorem 1.4, the conditions of Main Theorem 1.1 are satisfied automatically. In that sense, Main Theorem 1.1 is a generalization of the main theorem in [18], and Main Theorems 1.2 and 1.3 which require the conditions of Main Theorem 1.1 bring more representation theoretical information on the covering blocks including the classes dealt with in [18].

**Main Theorem 1.4** (see Corollaries 4.12 and 4.14 and Examples 4.15 and 4.16). Let \( G \) be a normal subgroup of a finite group \( \tilde{G} \), \( B \) a block of \( kG \) and \( \tilde{B} \) a block of \( k\tilde{G} \) covering \( B \) satisfying one of the following conditions, then the assumptions of Main Theorem 1.1 hold. In particular, in the situation (1), the all horizontal maps in (1.0.5) are bijective and all bricks over \( \tilde{B} \) can be obtained by the extensions of those over \( B \).

1. \( G \) has a cyclic Sylow \( p \)-subgroup and the quotient group \( \tilde{G}/G \) is a \( p \)-group.
2. \( G \) has a cyclic Sylow \( p \)-subgroup, the quotient group \( \tilde{G}/G \) is a cyclic group or isomorphic to the dihedral group \( D_{2p} \) of order \( 2p \) and \( B \) is the principal block \( B_0(kG) \) or a block of \( kG \) with distinct dimensional simple \( B \)-modules to each other.
3. \( G = A_5 \) (the alternating group of degree 5) and \( \tilde{G} = S_5 \) (the symmetric group of degree 5) where \( p = 5 \).
(4) $G$ is an arbitrary finite group and $\tilde{G} = G \times H$, where $H$ is a $p$-group, a cyclic group or the dihedral group of order $2p$.

In this paper, we use the following notation and convention. Modules mean finitely generated left modules and complexes mean cochain complexes. Let $\Lambda$ be a finite dimensional algebra over a field $k$. For a $\Lambda$-module $U$, we denote by $\text{Rad}(U)$ the Jacobson radical of $U$, by $\text{Soc}(U)$ the socle of $U$, by $\text{P}(U)$ the projective cover of $U$, by $\text{I}(U)$ the injective envelope of $U$, by $\text{Ω}(U)$ the syzygy of $U$, by $\text{Ω}^{-1}(U)$ the cosyzygy of $U$ and by $\tau U$ the Auslander–Reiten translate of $U$. We denote by $\text{Λ}-\text{mod}$ the module category of $\Lambda$, by $\text{Kb}(\text{Λ-proj})$ the homotopy category consisting of bounded complexes of projective $\Lambda$-modules and by $\text{Db}(\text{Λ-mod})$ the bounded derived category consisting of complexes of $\Lambda$-modules. For an object $X$ of $\text{Λ-mod}$ (of $\text{Kb}(\text{Λ-proj})$, of $\text{Db}(\text{Λ-mod})$), we denote by $\text{add}X$ the full subcategory of $\text{Λ-mod}$ (of $\text{Kb}(\text{Λ-proj})$, of $\text{Db}(\text{Λ-mod})$ respectively) whose objects are isomorphic to finite direct sums of direct summands of $X$. For $\Lambda$-modules $U$ and $U'$, we denote by $\text{Rad}_{\text{Λ-mod}}(U,U')$ the Jacobson radical of $\text{Hom}_\Lambda(U,U')$. We say that an object $X$ of $\text{Λ-mod}$, $\text{Kb}(\text{Λ-proj})$ or $\text{Db}(\text{Λ-mod})$ is basic if any two indecomposable direct summands of $X$ are non-isomorphic. We denote by $\nu_\Lambda$ the Nakayama functor of $\text{Λ-mod}$ which maps any projective $\Lambda$-module to injective $\Lambda$-module.

2 Preliminary results of $\tau$-tilting theory

In this section, $k$ means an algebraically closed field and $\Lambda$ means a finite dimensional $k$-algebra.

2.1 Functorially finiteness of torsion classes and torsion-free classes

Let $\mathcal{C}$ be a full subcategory of the module category $\Lambda$-mod. We say that $\mathcal{C}$ is contravariantly finite in $\Lambda$-mod if any object in $\Lambda$-mod has a right $\mathcal{C}$-approximation, that is, for every object $M$ of $\Lambda$-mod there exist an object $C$ of $\mathcal{C}$ and a morphism $f: C \to M$ such that the sequence of functors from $\mathcal{C}$ to $k$-mod

$$\text{Hom}_\Lambda(-,C)|_\mathcal{C} \xrightarrow{f\bullet} \text{Hom}_\Lambda(-,M)|_\mathcal{C} \longrightarrow 0$$

is exact. Dually, we say that $\mathcal{C}$ is covariantly finite in $\Lambda$-mod if any object in $\Lambda$-mod has a left $\mathcal{C}$-approximation, that is, for every object $M$ of $\Lambda$-mod there exist an object $C$ of $\mathcal{C}$ and a morphism $g: M \to C$ such that the sequence of functors from $\mathcal{C}$ to $k$-mod

$$\text{Hom}_\Lambda(C,-)|_\mathcal{C} \xrightarrow{\bullet g} \text{Hom}_\Lambda(M,-)|_\mathcal{C} \longrightarrow 0$$

is exact. We say that $\mathcal{C}$ is functorially finite if $\mathcal{C}$ is both contravariantly finite and covariantly finite in $\Lambda$-mod. We denote the right perpendicular subcategory of $\mathcal{C}$ by

$$\mathcal{C}^\perp := \{ X \in \Lambda$-mod $| \forall C \in \mathcal{C}, \text{Hom}_\Lambda(C,X) = 0 \}$$

and the left perpendicular subcategory of $\mathcal{C}$ by

$$\mathcal{C}^{\perp} := \{ X \in \Lambda$-mod $| \forall C \in \mathcal{C}, \text{Hom}_\Lambda(X,C) = 0 \}.$$
We denote by $\text{Fac}(\mathcal{C})$ the full subcategory of $\Lambda$-mod consisting of all factor modules of finite direct sums of objects in $\mathcal{C}$. Dually, we denote by $\text{Sub}(\mathcal{C})$ the full subcategory of $\Lambda$-mod consisting of all submodules of finite direct sums of objects in $\mathcal{C}$. We denote by $\text{Filt}(\mathcal{C})$ the full subcategory of $\Lambda$-mod consisting of all modules having a finite $\text{add}\, \mathcal{C}$-filtration, that is,

$$\text{Filt}(\mathcal{C}) := \left\{ M \in \Lambda\text{-mod} \mid \begin{array}{l}
\exists l \in \mathbb{N} \text{ and a sequence } \\
0 = M_0 \subset M_1 \subset \cdots \subset M_{l-1} \subset M_l = M \\
of \Lambda\text{-modules with } M_i/M_{i-1} \in \text{add}\, \mathcal{C} \text{ for } \\
all i = 1, \ldots, l.
\end{array} \right\}.$$ 

Let $\mathcal{T}$ and $\mathcal{F}$ be full subcategories of the module category $\Lambda$-mod. We say that $\mathcal{T}$ is a torsion class if $\mathcal{T}$ is closed under taking factor modules, direct sums and extensions. Dually, we say that $\mathcal{F}$ is a torsion-free class if $\mathcal{F}$ is closed under taking submodules, direct sums and extensions. We use the following notation:

- $\text{tors} \, \Lambda$ means the set of torsion classes in $\Lambda$-mod,
- $\text{torf} \, \Lambda$ means the set of torsion-free classes in $\Lambda$-mod,
- $\text{f-tors} \, \Lambda$ means the set of functorially finite torsion classes in $\Lambda$-mod,
- $\text{f-torf} \, \Lambda$ means the set of functorially finite torsion-free classes in $\Lambda$-mod.

These sets are ordered by inclusion. Let $\mathcal{C}$ be a full subcategory of the module category $\Lambda$-mod. We define $T(\mathcal{C})$ (or $F(\mathcal{C})$) to be the smallest torsion class (or torsion-free class, respectively) containing $\mathcal{C}$. For a $\Lambda$-module $U$, we abbreviate $\text{Fac}(\text{add}\, U)$, $\text{Sub}(\text{add}\, U)$, $\text{T}(\text{add}\, U)$ and $\text{F}(\text{add}\, U)$ as $\text{Fac}(U)$, $\text{Sub}(U)$, $\text{T}(U)$ and $\text{F}(U)$, respectively.

The following assertion is obvious, but plays an important role, so we give its proof briefly.

**Proposition 2.1 ([19, Lemma 3.1]).** Let $\mathcal{C}$ be a full subcategory of the module category $\Lambda$-mod. Then we have $T(\mathcal{C}) = \text{Filt}(\text{Fac}(\mathcal{C}))$ and $F(\mathcal{C}) = \text{Filt}(\text{Sub}(\mathcal{C}))$.

**Proof.** We only prove that $T(\mathcal{C}) = \text{Filt}(\text{Fac}(\mathcal{C}))$; the other statement $F(\mathcal{C}) = \text{Filt}(\text{Sub}(\mathcal{C}))$ follows similarly. First, we show that the subcategory $\text{Filt}(\text{Fac}(\mathcal{C}))$ of $\Lambda$-mod is a torsion class. It is obvious that $\text{Filt}(\text{Fac}(\mathcal{C}))$ is closed under direct sums and extensions. Let $U$ be an arbitrary object of $\text{Filt}(\text{Fac}(\mathcal{C}))$ and $U \xrightarrow{f} V$ an arbitrary epimorphism from $U$ to a $\Lambda$-module $V$. Then we can take a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{l-1} \subset U_l = U$$

of $U$ satisfying $U_i/U_{i-1} \in \text{add}\, \text{Fac}(\mathcal{C}) = \text{Fac}(\mathcal{C})$ for all $i = 1, \ldots, l$. Then we have the filtration

$$0 = f[U_0] \subset f[U_1] \subset \cdots \subset f[U_{l-1}] \subset f[U_l] = V$$

of $V$. For any $i = 1, \ldots, l$, the epimorphism $f$ induces an epimorphism

$$U_i/U_{i-1} \to f[U_i]/f[U_{i-1}].$$

Since $\text{Fac}(\mathcal{C})$ is closed under taking factor modules, we have $f[U_i]/f[U_{i-1}] \in \text{Fac}(\mathcal{C})$ and $V \in \text{Filt}(\text{Fac}(\mathcal{C}))$. Hence, we have that $\text{Filt}(\text{Fac}(\mathcal{C}))$ is closed under taking factor modules. Thus, we get that $\text{Filt}(\text{Fac}(\mathcal{C}))$ is a torsion class in $\Lambda$-mod.
Next we show that $\text{Filt}(\text{Fac}(\mathcal{C})) = \mathcal{T}(\mathcal{C})$. Since $\text{Filt}(\text{Fac}(\mathcal{C}))$ is a torsion class and $\text{Filt}(\text{Fac}(\mathcal{C})) \supset \mathcal{C}$, we get $\text{Filt}(\text{Fac}(\mathcal{C})) \supset \mathcal{T}(\mathcal{C})$ by minimality of $\mathcal{T}(\mathcal{C})$. Moreover, the inclusion $\mathcal{C} \subset \mathcal{T}(\mathcal{C})$ and the definitions of torsion classes imply that $\text{Filt}(\text{Fac}(\mathcal{C})) \subset \mathcal{T}(\mathcal{C})$.

The following proposition which gives the connection between torsion classes and torsion free classes is crucial.

**Proposition 2.2** (for example, see [9]). The following maps are mutually inverse isomorphisms of partially ordered sets:

\[
\begin{align*}
tors \Lambda & \longrightarrow (\text{torf } \Lambda)^{op} \\
\mathcal{T} & \longleftrightarrow \perp \mathcal{T}, \\
torf \Lambda & \longrightarrow (\text{tors } \Lambda)^{op} \\
\mathcal{F} & \longleftrightarrow \mathcal{F}^{\perp}.
\end{align*}
\]

Moreover, these isomorphisms restrict to the following isomorphisms of partially ordered sets, respectively:

\[
\begin{align*}
f-tors \Lambda & \longrightarrow (f-\text{tors } \Lambda)^{op} \\
\mathcal{T} & \longleftrightarrow \perp \mathcal{T}, \\
f-\text{torf } \Lambda & \longrightarrow (f-\text{tors } \Lambda)^{op} \\
\mathcal{F} & \longleftrightarrow \mathcal{F}^{\perp}.
\end{align*}
\]

2.2 Support $\tau$-tilting modules

We recall the definitions and basic properties of support $\tau$-tilting modules and support $\tau^{-1}$-tilting modules which are dual notion of support $\tau$-tilting modules. For a $\Lambda$-module $M$, we denote by $|M|$ the number of isomorphism classes of indecomposable direct summands of $M$. In particular, $|\Lambda| := |\text{End } \Lambda|$ means the number of isomorphism classes of simple $\Lambda$-modules. Also, we denote by $s(M)$ the number of isomorphism classes of simple modules appearing as composition factors of $M$.

**Definition 2.3** ([2, Definition 0.1]). Let $M$ be a $\Lambda$-module.

(1) We say that $M$ is $\tau$-rigid if $\text{Hom}_{\Lambda}(M, \tau M) = 0$.

(2) We say that $M$ is $\tau$-tilting if $M$ is a $\tau$-rigid module and $|M| = |\Lambda|$.

(3) We say that $M$ is support $\tau$-tilting if there exists an idempotent $e$ of $\Lambda$ such that $M$ is a $\tau$-tilting $\Lambda/e\Lambda$-module.

**Definition 2.4** (The dual of [2, Definition 0.1]). Let $N$ be a $\Lambda$-module.

(1) We say that $N$ is $\tau^{-1}$-rigid if $\text{Hom}_{\Lambda}(\tau^{-1} N, N) = 0$.

(2) We say that $N$ is $\tau^{-1}$-tilting if $N$ is a $\tau^{-1}$-rigid module and $|N| = |\Lambda|$.
(3) We say that $N$ is support $\tau^{-1}$-tilting if there exists an idempotent $e$ of $\Lambda$ such that $N$ is a $\tau^{-1}$-tilting $\Lambda/e\Lambda$-module.

**Remark 2.5** ([3, Proposition 2.3 (a), (b)], [1, Proposition 1.8]). Since $e = 0$ is an idempotent of $\Lambda$ and $\Lambda/e\Lambda = \Lambda$, any $\tau$-tilting module (or any $\tau^{-1}$-tilting module) is a support $\tau$-tilting module (or a support $\tau^{-1}$-tilting module, respectively). Moreover, for any $\tau$-rigid $\Lambda$-module $M$, the following conditions are equivalent:

1. $M$ is support $\tau$-tilting module.
2. There exist a projective $\Lambda$-module $P$ satisfying that $\text{Hom}_\Lambda(P, M) = 0$ and that $|M| + |P| = |\Lambda|$.
3. $|M| = s(M)$.

We use the following notation:

- $s\tau$-tilt $\Lambda$ means the set of isomorphism classes of basic support $\tau$-tilting $\Lambda$-modules,
- $s\tau^{-1}$-tilt $\Lambda$ means the set of isomorphism classes of basic support $\tau^{-1}$-tilting $\Lambda$-modules,
- indec. $\tau$-rigid $\Lambda$ means the set of isomorphism classes of indecomposable $\tau$-rigid $\Lambda$-modules,
- indec. $\tau^{-1}$-rigid $\Lambda$ means the set of isomorphism classes of indecomposable $\tau^{-1}$-rigid $\Lambda$-modules.

**Proposition 2.6** ([2, Theorem 2.7, Theorem 2.15]). With the above notation, the following maps give bijections:

$$
\begin{align*}
\text{sr-tilt } \Lambda & \longrightarrow \text{f-tors } \Lambda \\
M & \longmapsto \text{Fac } M,
\end{align*}
$$

$$
\begin{align*}
\text{sr^{-1}-tilt } \Lambda & \longrightarrow \text{f-torf } \Lambda \\
N & \longmapsto \text{Sub } N.
\end{align*}
$$

We can give $\text{sr-tilt } \Lambda$ and $\text{sr^{-1}-tilt } \Lambda$ partially ordered set structures by the above bijections and inclusions of f-tors $\Lambda$ and f-tors $\Lambda$.

**Definition 2.7.** For $M, M' \in \text{sr-tilt } \Lambda$, we write $M \geq M'$ if $\text{Fac } M \supset \text{Fac } M'$, or equivalently there exist a positive integer $r$ and an epimorphism

$$
M^\oplus_r \xrightarrow{\varphi} M'.
$$

Dually, for $N', N \in \text{sr^{-1}-tilt } \Lambda$, we write $N' \leq N$ if $\text{Sub } N' \subset \text{Sub } N$, or equivalently there exist a positive integer $r$ and a monomorphism

$$
N' \xleftarrow{\psi} N^\oplus_r.
$$

Based on [2, Theorem 2.33], we define support $\tau$-tilting mutations and support $\tau^{-1}$-tilting mutations as follows.
Definition 2.8 ([2, Theorem 2.33]). Let $M$ and $M'$ be support $\tau$-tilting $\Lambda$-modules. We say that $M'$ is a support $\tau$-tilting left mutation of $M$ (or a support $\tau$-tilting right mutation) if $M > M'$ holds and if there is no support $\tau$-tilting $\Lambda$-module $L$ such that $M > L > M'$ (if $M < M'$ holds and if there is no support $\tau$-tilting $\Lambda$-module $L$ such that $M < L < M'$, respectively).

Definition 2.9 (Dual assertion of [2, Theorem 2.33]). Let $N$ and $N'$ be support $\tau^{-1}$-tilting $\Lambda$-modules. We say that $N'$ is a support $\tau^{-1}$-tilting right mutation (or a support $\tau^{-1}$-tilting left mutation) if $N < N'$ holds and if there is no support $\tau^{-1}$-tilting $\Lambda$-module $L$ such that $N < L < N'$ (or if $N > N'$ holds and if there is no support $\tau^{-1}$-tilting $\Lambda$-module $L$ such that $N > L > N'$, respectively).

We recall some fundamental properties of support $\tau$-tilting modules and support $\tau^{-1}$-tilting modules.

Proposition 2.10 ([2, below Theorem 2.15 and Proposition 2.27]). The following hold:

1. The following map gives an isomorphism as partially ordered sets:

$$
\begin{align*}
\text{sr-tilt } \Lambda & \longrightarrow (\text{sr}^{-1}\text{-tilt } \Lambda)^{\text{op}} \\
M & \longrightarrow \tau M \oplus \nu P,
\end{align*}
$$

(2.2.3)

here $P$ is a basic projective $\Lambda$-module satisfying that $\text{Hom}_\Lambda(P, M) = 0$ and that $|M| + |P| = |\Lambda|$.

2. The above maps make the following diagram of partially ordered sets commutative:

$$
\begin{array}{ccc}
\text{sr-tilt } \Lambda & \longrightarrow & (\text{sr}^{-1}\text{-tilt } \Lambda)^{\text{op}} \\
\downarrow & & \downarrow \\
\text{f-tors } \Lambda & \longrightarrow & (\text{f-torf } \Lambda)^{\text{op}}
\end{array}
$$

(2.2.1)

(2.2.3)

(2.2.2)

(2.1.1)

Proposition 2.11 ([10, Theorem 5.10], [12, Lemma 4.3, Lemma 4,4]). Let $V$ be an indecomposable $\Lambda$-module. Then the following hold:

1. If $V$ is a $\tau$-rigid module, then Fac $V$ is a torsion class of $\Lambda$-mod and $\tau V$ is a $\tau^{-1}$-rigid module.

2. If $V$ is a $\tau^{-1}$-rigid module, then Sub $V$ is a torsion-free class of $\Lambda$-mod and $\tau^{-1}V$ is a $\tau$-rigid module.

3. The following map gives an injection:

$$
\begin{align*}
\text{indec. } \tau\text{-rigid } \Lambda & \longrightarrow \text{f-tors } \Lambda \\
X & \hookrightarrow \text{Fac } X.
\end{align*}
$$

(2.2.4)

4. The following map gives a bijection:

$$
\begin{align*}
\text{indec. } \tau\text{-rigid } \Lambda & \longrightarrow \text{indec. } \tau^{-1}\text{-rigid } \Lambda \\
X & \longleftrightarrow \begin{cases} 
\tau X & \text{(if } X \text{ is non-projective),} \\
\nu X & \text{(if } X \text{ is projective).}
\end{cases}
\end{align*}
$$

(2.2.5)
(5) The following map is well-defined and injective:

\[
\begin{array}{ccc}
\text{indec. } \tau\text{-rigid } \Lambda & \longrightarrow & \text{s}\tau\text{-tilt } \Lambda \\
X & \longrightarrow & M_X,
\end{array}
\quad (2.2.6)
\]

here \(M_X\) is the support \(\tau\)-tilting \(\Lambda\)-module satisfying \(\text{Fac} M_X = \text{Fac} X\).

(6) The following map gives an injection:

\[
\begin{array}{ccc}
\text{indec. } \tau^{-1}\text{-rigid } \Lambda & \longrightarrow & \text{f-tors } \Lambda \\
Y & \longrightarrow & \text{Sub} Y,
\end{array}
\quad (2.2.7)
\]

(7) The following map is well-defined and injective:

\[
\begin{array}{ccc}
\text{indec. } \tau^{-1}\text{-rigid } \Lambda & \longrightarrow & \text{s}\tau^{-1}\text{-tilt } \Lambda \\
Y & \longrightarrow & N_Y,
\end{array}
\quad (2.2.8)
\]

here \(N_Y\) is the support \(\tau^{-1}\)-tilting \(\Lambda\)-module satisfying \(\text{Sub} N_Y = \text{Sub} Y\).

(8) The above maps make the following diagram commutative:

\[
\begin{array}{ccc}
\text{s}\tau\text{-tilt } \Lambda & \longrightarrow & \text{indec. } \tau\text{-rigid } \Lambda \\
\text{indec. } \tau^{-1}\text{-rigid } \Lambda & \longrightarrow & \text{f-tors } \Lambda \\
\text{s}\tau^{-1}\text{-tilt } \Lambda & \longrightarrow & \text{indec. } \tau^{-1}\text{-rigid } \Lambda
\end{array}
\]

2.3 Bricks and semibricks

We recall the definitions and basic properties of bricks and semibricks.

**Definition 2.12.** Let \(S\) be \(\Lambda\)-module.

(1) We say that a module \(S\) is a brick in \(\Lambda\text{-mod}\) if \(\text{End}_\Lambda(S) \cong k\).

(2) We say that a module \(S\) is a semibrick in \(\Lambda\text{-mod}\) if \(S\) is isomorphic to a direct sum of bricks \(S_1, \ldots, S_l\) in \(\Lambda\text{-mod}\) which satisfy that \(\text{Hom}_\Lambda(S_i, S_j) = 0\) if \(S_i \not\cong S_j\).

**Definition 2.13.** We say that a semibrick \(S\) in \(\Lambda\text{-mod}\) is left finite (or right finite) if the torsion class \(T(S)\), which is the smallest torsion class containing \(S\) (or if the torsion-free class \(F(S)\), which is the smallest torsion-free class containing \(S\), respectively), is functorially finite.

We use the following notation:
• \(\text{sbrick} \Lambda\) means the set of isomorphism classes of basic semibricks in \(\Lambda\text{-mod}\),

• \(\text{brick} \Lambda\) means the set of isomorphism classes of bricks in \(\Lambda\text{-mod}\),

• \(f_L\text{-sbrick} \Lambda\) means the set of isomorphism classes of basic left finite semibricks in \(\Lambda\text{-mod}\),

• \(f_R\text{-sbrick} \Lambda\) means the set of isomorphism classes of basic right finite semibricks in \(\Lambda\text{-mod}\),

• \(f_L\text{-brick} \Lambda\) means the set of isomorphism classes of left finite bricks in \(\Lambda\text{-mod}\),

• \(f_R\text{-brick} \Lambda\) means the set of isomorphism classes of right finite bricks in \(\Lambda\text{-mod}\).

For \(\Lambda\text{-modules} U\) and \(V\), we denote by \(R(U,V)\) the following submodule of \(V\):

\[
\sum_{f \in \text{Rad} \Lambda\text{-mod}(U,V)} \text{Im} \ f.
\]

We denote by \(S(U,V)\) the following submodule of \(U\):

\[
\bigcap_{f \in \text{Rad} \Lambda\text{-mod}(U,V)} \text{Ker} \ f.
\]

**Remark 2.14.** It is easy to check that any semisimple module is a left finite semibrick and a right finite semibrick.

**Theorem 2.15** ([7, Lemma 2.5, Proposition 2.13], [12, Theorem 4.1]). Let \(M\) be a basic support \(\tau\)-tilting \(\Lambda\)-module and \(X\) an indecomposable direct summand of \(M\). Then the following hold:

1. The module \(X/R(X,X)\) is a left finite brick over \(\Lambda\).
2. The module \(X/R(M,X)\) is a brick or zero module.
3. The following conditions are equivalent:
   
   (a) The module \(X/R(M,X)\) is nonzero.

   (b) The module \(X/R(M,X)\) is a brick.

   (c) \(X \notin \text{Fac}(M/X)\).

Let \(M\) be a basic support \(\tau\)-tilting \(\Lambda\)-module. We use the notation \(\mathcal{LM}(M)\) to denote the following set:

\[
\left\{ X \in \text{indec.} \tau\text{-rigid} \Lambda \mid \begin{array}{c}
X \text{ is a direct summand of } M \\
X \notin \text{Fac}(M/X)
\end{array} \right\}.
\]

We remark that \(M/R(M,M) \cong \bigoplus_{X \in \mathcal{LM}(M)} X/R(M,X)\).

**Theorem 2.16** ([7, Theorem 2.3, Proposition 2.13]). Let \(M\) be a basic support \(\tau\)-tilting \(\Lambda\)-module. Then the following hold:
(1) The following map gives an injection:

\[ \text{LM}(M) \longrightarrow \text{brick} \Lambda \\
X \longmapsto X/R(M,X). \]  

(2.3.1)

(2) The following map is well-defined and bijective:

\[ \text{LM}(M) \longrightarrow \{ M' \in s\tau \text{-tilt} \Lambda \mid M' \text{ is a left mutation of } M \} \\
X \longmapsto \mu_X(M), \]  

(2.3.2)

where \( \mu_X(M) \) is a unique support \( \tau \)-tilting module having \( M/X \) as a direct summand and being not \( M \).

**Corollary 2.17.** By Theorem 2.16, we get the injective map

\[ \{ M' \in s\tau \text{-tilt} \Lambda \mid M' \text{ is a left mutation of } M \} \longrightarrow \text{brick} \Lambda \]  

(2.3.3)

which makes the following diagram commutative:

\[ \begin{array}{ccc}
\{ M' \in s\tau \text{-tilt} \Lambda \mid M' \text{ is a left mutation of } M \} & \overset{(2.3.2)}{\longrightarrow} & \text{brick} \Lambda \\
\text{LM}(M) & \overset{(2.3.1)}{\longrightarrow} & \text{LM}(M).
\end{array} \]

**Theorem 2.18** ([7, Theorem 2.3], [12, Theorem 4.1, Lemma 4.3]). Let \( M \) be a basic support \( \tau \)-tilting \( \Lambda \)-module. Then the following hold:

(1) The module \( M/R(M,M) \) is a left finite semibrick.

(2) The following map gives a bijection:

\[ \text{s} \tau \text{-tilt} \Lambda \longrightarrow \text{f}_L\text{-sbrick} \Lambda \\
M \longrightarrow M/R(M,M). \]  

(2.3.4)

(3) The following map gives a bijection:

\[ \text{f}_L\text{-sbrick} \Lambda \longrightarrow \text{f-tors} \Lambda \\
S \longrightarrow T(S). \]  

(2.3.5)

(4) The following map gives a bijection:

\[ \text{indec.} \, \tau \text{-rigid} \Lambda \longrightarrow \text{f}_L\text{-brick} \Lambda \\
X \longrightarrow X/R(X,X). \]  

(2.3.6)

(5) The following map gives an injection:

\[ \text{f}_L\text{-brick} \Lambda \longrightarrow \text{f-tors} \Lambda \\
S \longrightarrow T(S). \]  

(2.3.7)
The above maps make the following diagram commutative:

\[
\begin{array}{ccc}
\text{indec. } \tau\text{-rigid } & \quad & \text{f}_t\text{-brick } \\
\downarrow \text{(2.3.6)} & & \downarrow \text{(2.3.7)} \\
\text{st}\text{-tilt } & \quad & \text{f}_t\text{-sbrick } \\
\downarrow \text{(2.3.4)} & & \downarrow \text{(2.3.5)} \\
\text{f-tors } & \quad & \text{f-tors }.
\end{array}
\]

The dual assertions are also true.

**Theorem 2.19** (The dual assertion of Theorem 2.15). Let \( N \) be a basic support \( \tau^{-1}\)-tilting \( \Lambda \)-module and \( Y \) an indecomposable direct summand of \( N \). Then the following hold:

(1) The module \( S(Y, Y) \) is a right finite brick over \( \Lambda \).

(2) The module \( S(Y, N) \) is a brick or zero module.

(3) The following conditions are equivalent:

(a) The module \( S(Y, N) \) is nonzero.
(b) The module \( S(Y, N) \) is a brick.
(c) \( Y \notin \text{Sub}(N/Y) \).

Let \( N \) be a basic support \( \tau^{-1}\)-tilting module. We use the notation \( R(M)(N) \) to denote the following set:

\[
\left\{ Y \in \text{indec. } \tau^{-1}\text{-rigid } \Lambda \left| \begin{array}{c}
Y \text{ is a direct summand of } N \\
Y \notin \text{Sub}(N/Y)
\end{array} \right. \right\}.
\]

We remark that \( S(N, N) \equiv \bigoplus_{Y \in R(M)(N)} S(Y, N) \).

**Theorem 2.20** (The dual assertion of Theorem 2.16). Let \( N \) be a basic support \( \tau^{-1}\)-tilting \( \Lambda \)-module. Then the following hold:

(1) The following map gives a bijection:

\[
\begin{array}{ccc}
R(M)(N) & \quad & \text{brick } \Lambda \\
\downarrow \text{(2.3.8)} & & \downarrow \\
Y & \quad & S(Y, N)
\end{array}
\]

(2) The following map gives a bijection:

\[
\begin{array}{ccc}
R(M)(N) & \quad & \{ N' \in \text{sr}^{-1}\text{-tilt } \Lambda \left| N' \text{ is a right mutation of } N \right. \} \\
\downarrow \text{(2.3.9)} & & \downarrow \\
Y & \quad & \mu_Y(N),
\end{array}
\]

here \( \mu_Y(N) \) is a unique support \( \tau^{-1}\)-tilting \( \Lambda \)-module having \( N/Y \) as a direct summand and being not \( N \).
Corollary 2.21. By Theorem 2.20, we get the injective map
\[
\{ N' \in s^\tau \text{-tilt } \Lambda \mid N' \text{ is a right mutation of } N \} \rightarrow \text{brick } \Lambda
\] (2.3.10)
which make the following diagram commutative:
\[
\begin{array}{c}
\{ N' \in s^\tau \text{-tilt } \Lambda \mid N' \text{ is a right mutation of } N \} \\
\uparrow_{(2.3.9)} \\
\mathcal{R} \mathcal{M}(N).
\end{array}
\]

Theorem 2.22 (Dual assertion of Theorem 2.18). Let $N$ be a basic support $\tau^{-1}$ tilting $\Lambda$-module. Then the following hold:

(1) The module $S(N,N)$ is a right finite semibrick.

(2) The following map gives a bijection:
\[
s^\tau \text{-tilt } \Lambda \rightarrow f_{\mathbb{R}} \text{-sbrick } \Lambda
\]
\[
N \rightarrow S(N,N).
\] (2.3.11)

(3) The following map gives a bijection:
\[
f_{\mathbb{R}} \text{-sbrick } \Lambda \rightarrow f \text{-torf } \Lambda
\]
\[
S \rightarrow F(S).
\] (2.3.12)

(4) The following map gives a bijection:
\[
\text{indec. } \tau^{-1} \text{-rigid } \Lambda \rightarrow f_{\mathbb{R}} \text{-brick } \Lambda
\]
\[
Y \rightarrow S(Y,Y).
\] (2.3.13)

(5) The following map gives an injection:
\[
f_{\mathbb{R}} \text{-brick } \Lambda \rightarrow f \text{-torf } \Lambda
\]
\[
S \rightarrow F(S).
\] (2.3.14)

(6) The above maps make the following diagram commutative:
Corollary 2.23. By Propositions 2.10 and 2.11 and Theorems 2.18 and 2.22, we get the bijective maps
\[
\begin{align*}
  f_L\text{-sbrick } \Lambda & \longrightarrow f_R\text{-sbrick } \Lambda, \\
  f_L\text{-brick } \Lambda & \longrightarrow f_R\text{-brick } \Lambda,
\end{align*}
\] (2.3.15) (2.3.16)
which make the following diagram commutative:

Definition 2.24 ([2, Definition 2.29], [7, Definition 2.14]). We define the support \( \tau \)-tilting Hasse quiver \( \mathcal{H}(s\tau\text{-tilt } \Lambda) \) labeled with brick over \( \Lambda \) as follows:
- The set of vertices is \( s\tau\text{-tilt } \Lambda \).
- We draw an arrow from \( M \) to each support \( \tau \)-tilting left mutation \( M' \) of \( M \), and we label this arrow with the corresponding brick to \( M' \) under (2.3.3) for \( M \).

Definition 2.25. We define the support \( \tau^{-1} \)-tilting Hasse quiver \( \mathcal{H}(s\tau^{-1}\text{-tilt } \Lambda) \) labeled with bricks over \( \Lambda \) as follows:
- The set of vertices is \( s\tau^{-1}\text{-tilt } \Lambda \).
- We draw an arrow from \( N \) to each support \( \tau \)-tilting right mutation \( N' \) of \( N \), and we label this arrow with the corresponding brick to \( N' \) under (2.3.10) for \( N \).

Proposition 2.26 ([7, Lemma 2.16, Proposition 2.17]). Let \( M \) be a support \( \tau \)-tilting module and \( M' \) be a support \( \tau \)-tilting left mutation of \( M \). Furthermore, \( N \) and \( N' \) be support \( \tau^{-1} \)-tilting module corresponding to \( M \) and \( M' \) under (2.2.3), respectively. Then there exist a unique brick in the subcategory \( \text{Fac } M \cap \text{Sub } N' \), and it is isomorphic to the brick corresponding to \( M' \) under (2.3.1) for \( M \).

Theorem 2.27 ([7, Theorem 2.15, Lemma 2.16, Proposition 2.17]). Let \( M \) be a support \( \tau \)-tilting \( \Lambda \)-module, \( M' \) a support \( \tau \)-tilting left mutation of \( M \), \( S \) a brick on the arrow \( M \rightarrow M' \) in \( \mathcal{H}(s\tau\text{-tilt } \Lambda) \). Moreover, \( N \) and \( N' \) be support \( \tau^{-1} \)-tilting \( \Lambda \)-modules corresponding to \( M \) and \( M' \) under (2.2.3), respectively. Then \( N \) is a right mutation of \( N' \) and \( S \) is also the brick labeled with the arrow \( N' \rightarrow N \) in \( \mathcal{H}(s\tau^{-1}\text{-tilt } \Lambda) \).
2.4 Silting complexes and simple-minded collections

We recall the definition of silting complexes, which is a generalization of tilting complexes. The concept of silting complexes is originated from [16], and recently there have been many papers on silting complexes and silting mutations starting with [4]. In particular, in [2], it is shown that there is a one-to-one correspondence between the two-term silting complexes and the support $\tau$-tilting modules.

**Definition 2.28.** Let $T$ be a complex in $K^b(\Lambda\text{-proj})$.

(1) We say that $T$ is presilting (or pretilting) if

$$\text{Hom}_{K^b(\Lambda\text{-proj})}(T, T[i]) = 0$$

for any $i > 0$ (or for any $i \neq 0$, respectively).

(2) We say that $T$ is silting (or tilting) if it is presilting (or pretilting, respectively) and satisfies $\text{thick} T = K^b(\Lambda\text{-proj})$, where $\text{thick} T$ means the smallest triangulated subcategory of $K^b(\Lambda\text{-proj})$ containing $\text{add} T$.

We say that a complex $T \in K^b(\Lambda\text{-proj})$ is two-term if $T_i = 0$ for all $i \neq 0, -1$.

Moreover, we use the following notation:

- $\text{silt} \Lambda$ means the set of isomorphism classes of basic silting complexes in $K^b(\Lambda\text{-proj})$,
- $\text{tilt} \Lambda$ means the set of isomorphism classes of basic tilting complexes in $K^b(\Lambda\text{-proj})$,
- $2\text{-silt} \Lambda$ means the set of isomorphism classes of basic two-term silting complexes in $K^b(\Lambda\text{-proj})$,
- $2\text{-tilt} \Lambda$ means the set of isomorphism classes of basic two-term tilting complexes in $K^b(\Lambda\text{-proj})$.

We can define a partially ordered set structure on $\text{silt} \Lambda$ as follows.

**Definition 2.29** ([4, Definition 2.10, Theorem 2.11]). For $T, T' \in \text{silt} \Lambda$, we write $T \geq T'$ if

$$\text{Hom}_{K^b(\Lambda\text{-proj})}(T, T'[i]) = 0$$

for any $i > 0$.

**Theorem 2.30** ([2, Theorem 3.2 and Corollary 3.9]). The following map gives an isomorphism of partially ordered sets:

$$s\tau\text{-tilt} \Lambda \xrightarrow{\text{2-silt} \Lambda} M \xrightarrow{\text{2-tilt} \Lambda} (P_1 \oplus P \xrightarrow{(f_1, 0)} P_0)$$

(2.4.1)

here $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ is a minimal projective presentation of $M$ and $P$ is a basic projective module satisfying that $\text{Hom}_\Lambda(P, M) = 0$ and that $|M| + |P| = |\Lambda|$ (see [2, Proposition 2.3 (b)]).

We remark that the correspondence above commutes with support $\tau$-tilting mutations and silting mutations [2, Corollary 3.9].
Proposition 2.31 ([4, Example 2.8]). If \( \Lambda \) is a finite dimensional symmetric \( k \)-algebra, then any silting complex in \( K^b(\Lambda\text{-proj}) \) is a tilting complex.

Definition 2.32. We say that a set \( \mathcal{X} \) of isomorphism classes of objects in \( D^b(\Lambda\text{-mod}) \) is a simple-minded collection in \( D^b(\Lambda\text{-mod}) \) if it satisfies the following conditions:

1. For any \( X \in \mathcal{X} \), we have
   \[ \operatorname{End}_{D^b(\Lambda\text{-mod})}(X) \cong k. \]
2. For any \( X_1, X_2 \in \mathcal{X} \) with \( X_1 \neq X_2 \), we have
   \[ \operatorname{Hom}_{D^b(\Lambda\text{-mod})}(X_1, X_2) = 0. \]
3. For any \( X_1, X_2 \in \mathcal{X} \) and \( t < 0 \), we have
   \[ \operatorname{Hom}_{D^b(\Lambda\text{-mod})}(X_1, X_2[\ell]) = 0. \]
4. The triangulated subcategory \( \text{thick}\mathcal{X} \) coincides with \( D^b(\Lambda\text{-mod}) \).

We say that a simple-minded collection \( \mathcal{X} \) in \( D^b(\Lambda\text{-mod}) \) is two-term if \( \mathcal{X} \) satisfies the condition that \( H^i(X) = 0 \) for any \( i \neq -1, 0 \) and any \( X \in \mathcal{X} \). Moreover, we use the following notation:

- \( \text{smc} \Lambda \) means the set of simple-minded collections in \( D^b(\Lambda\text{-mod}) \),
- \( \text{2-smc} \Lambda \) means the set of two-term simple-minded collections in \( D^b(\Lambda\text{-mod}) \).

Proposition 2.33 ([11, Remark 4.11]). For any two-term simple-minded collection \( \mathcal{X} \) in \( D^b(\Lambda\text{-mod}) \), every \( X \in \mathcal{X} \) belongs to either \( \Lambda\text{-mod} \) or \( (\Lambda\text{-mod})[1] \) up to isomorphisms in \( D^b(\Lambda\text{-mod}) \).

Proposition 2.34 ([7, Theorem 3.3]). The following maps give bijections:

\begin{align*}
\text{2-smc} \Lambda & \longrightarrow f_L\text{-sbrick} \Lambda \\
\mathcal{X} & \longrightarrow \bigoplus_i S_i, \quad (2.4.2) \\
\text{2-smc} \Lambda & \longrightarrow f_R\text{-sbrick} \Lambda \\
\mathcal{X} & \longrightarrow \bigoplus_i R_i, \quad (2.4.3)
\end{align*}

where \( S_i \) are objects in \( \mathcal{X} \cap \Lambda\text{-mod} \) and \( R_i \) are objects in \( \mathcal{X}[1] \cap \Lambda\text{-mod} \).

Now we recall the construction of silting complexes from simple-minded collections based on [17, 22]. Let \( \mathcal{X} = \{X_1, \ldots, X_r\} \) be a simple-minded collection in \( D^b(\Lambda\text{-mod}) \). By induction on \( n \), we shall construct sequences

\[ X_i^{(0)} \longrightarrow X_i^{(1)} \longrightarrow X_i^{(2)} \longrightarrow \cdots \longrightarrow X_i^{(n)} \longrightarrow X_i^{(n+1)} \longrightarrow \cdots \]
of objects and morphisms in $D^b(\Lambda\text{-Mod})$ for $1 \leq i \leq r$. Set $X_i^{(0)} := X_i$. Suppose we have constructed $X_i^{(n-1)}$. For each $1 \leq j \leq r$ and $t < 0$, choose a basis $B^{(n)}(j, t, i)$ of $\text{Hom}_{D^b(\Lambda\text{-mod})}(X_j[t], X_i^{(n-1)})$. Put

$$Z_i^{(n-1)} = \bigoplus_{t < 0} \bigoplus_{j=1}^r \bigoplus_{f \in B^{(n)}(j, t, i)} X_j[t]$$

and let $\alpha_i^{(n-1)} : Z_i^{(n-1)} \to X_i^{(n-1)}$ be the map whose restriction to the component indexed by $t < 0, j = 1, \ldots, r, f \in B^{(n)}(j, t, i)$ is exactly $f$. Now define $X_i^{(n)}$ together with morphism $X_i^{(n-1)} \xrightarrow{\beta_i^{(n-1)}} X_i^{(n)}$ by forming the distinguished triangle

$$Z_i^{(n-1)} \xrightarrow{\alpha_i^{(n-1)}} X_i^{(n-1)} \xrightarrow{\beta_i^{(n-1)}} X_i^{(n)} \to Z_i^{(n)}[1].$$

Let $C_i$ be the homotopy colimit of the sequence

$$X_i^{(0)} \xrightarrow{\beta_i^{(0)}} X_i^{(1)} \xrightarrow{\beta_i^{(1)}} X_i^{(2)} \to \cdots \xrightarrow{\beta_i^{(n-1)}} X_i^{(n)} \to \cdots$$

and put $T^X = \nu^{-1}(\bigoplus_i C_i)$. This construction induces a well-defined bijection between $\text{smc } \Lambda$ and $\text{silt } \Lambda$.

**Theorem 2.35** ([17, Proposition 5.9, Theorem 6.1], [11, Corollary 4.3]). The following map gives a bijection:

$$\text{smc } \Lambda \quad \xrightarrow{\text{bijection}} \quad \text{silt } \Lambda$$

This bijection restricts to a bijection:

$$\text{2-smc } \Lambda \quad \xrightarrow{\text{bijection}} \quad \text{2-silt } \Lambda.$$  \hfill (2.4.4)

As can be seen from the definition, it is difficult in general to calculate $T^X$ from $X$. However, if we restrict the discussions to two-term simple-minded collections and two-term silting complexes, the following theorem make the above computation easier via the correspondence between support $\tau$-tilting modules and left finite semibricks.

**Theorem 2.36** ([7, Theorem 3.3]). The following diagram is commutative:

$$\text{2-smc } \Lambda \quad \xrightarrow{\text{(2.4.4)}} \quad \text{2-silt } \Lambda$$

$$\xrightarrow{\text{f}_L\text{-sbrick } \Lambda} \quad \xrightarrow{\text{(2.4.1)}} \quad \text{2-silt } \Lambda.$$  \hfill (2.4.2)

The following theorem enables us to calculate the corresponding two-term simple-minded collections from left finite semibricks (or right finite semibricks).

**Theorem 2.37** ([7, Theorem 3.3]). The following diagram is commutative:

$$\text{2-smc } \Lambda \quad \xrightarrow{\text{(2.4.3)}} \quad \text{f}_L\text{-sbrick } \Lambda$$

$$\xrightarrow{\text{(2.4.2)}} \quad \xrightarrow{\text{f}_R\text{-sbrick } \Lambda} \quad \text{f}_L\text{-sbrick } \Lambda.$$  \hfill (2.3.15)
3 Preliminaries of modular representation theory of finite groups

In this section, let $k$ be an algebraically closed field of characteristic $p > 0$. For any finite group $G$, the field $k$ can always be regarded as a $kG$-module by defining $gx = x$ for any $g \in G$ and $x \in k$. This module is called the trivial module and is denoted by $kG$. For $kG$-modules $U$ and $V$, the $k$-module $U \otimes V = U \otimes_k V$ has a $kG$-module structure given by $g(u \otimes v) = gu \otimes gv$ for all $g \in G$, $u \in U$ and $v \in V$.

3.1 Restriction functors and induction functors

Let $G$ be a finite group and $H$ a subgroup of $G$. We denote by $\text{Res}_G^H$ the restriction functor from $kG$-mod to $kH$-mod and $\text{Ind}_G^H := kG \otimes_{kH} \bullet$ the induction functor from $kH$-mod to $kG$-mod. The functors $\text{Res}_G^H$ and $\text{Ind}_G^H$ are exact functors and have the following properties.

Proposition 3.1 (see [5, Lemma 8.5, Lemma 8.6]). Let $G$ be a finite group, $K$ a subgroup of $G$, $H$ a subgroup of $K$, $U$ a $kG$-module and $V$ a $kH$-module. Then the following hold:

1. $\text{Res}_K^G \text{Res}_H^K \cong \text{Res}_G^H$.
2. $\text{Ind}_K^G \text{Ind}_H^K \cong \text{Ind}_G^H$.
3. The functors $\text{Res}_G^H$ and $\text{Ind}_G^H$ are left and right adjoint to each other.
4. The functors $\text{Res}_G^H$ and $\text{Ind}_G^H$ send projective modules to projective modules.
5. $U \otimes kG \cong U$.
6. $\text{Ind}_G^H(\text{Res}_G^H U \otimes V) \cong U \otimes \text{Ind}_G^H V$.

Let $\tilde{G}$ be a finite group, $G$ a normal subgroup of $\tilde{G}$ and $U$ a $kG$-module. For $\tilde{g} \in \tilde{G}$, we define a $k\tilde{G}$-module $\tilde{g}U$ consisting of symbols $\tilde{g}u$ as a set, where $u \in U$ and its $k\tilde{G}$-module structure is given by $\tilde{g}(\tilde{u} + \tilde{u}') := \tilde{g}(\tilde{u} + \tilde{u}')$, $\lambda(\tilde{g}u) := \tilde{g}(\lambda u)$ and $g(\tilde{gu}) := \tilde{g}(\tilde{g}^{-1}\tilde{g}u)$ for any $u, u' \in U$, $\lambda \in k$ and $g \in G$.

Theorem 3.2 (Mackey’s decomposition formula for normal subgroups). Let $\tilde{G}$ be a finite group and $G$ a normal subgroup of $\tilde{G}$, and $U$ a $kG$-module. Then we have

$$\text{Res}_{\tilde{G}}^{\tilde{G}} \text{Ind}_{G}^{\tilde{G}} U \cong \bigoplus_{t \in [\tilde{G}/G]} tU,$$

where $[\tilde{G}/G]$ is a set of representatives of the factor group $\tilde{G}/G$.

For a $kG$-module $U$, we denote by $I_{\tilde{G}}(U)$ the inertial group of $U$ in $\tilde{G}$, that is

$$I_{\tilde{G}}(U) := \left\{ \tilde{g} \in \tilde{G} \bigg| \tilde{g}U \cong U \text{ as } kG\text{-modules} \right\}.$$
Theorem 3.3 (Clifford’s Theorem for simple modules). Let \( \hat{G} \) be a finite group, \( G \) a normal subgroup of \( \hat{G} \), \( S \) a simple \( k\hat{G} \)-module and \( S' \) a simple \( kG \)-submodule of \( \text{Res}_N^G S \). Then we have a \( kG \)-module isomorphism
\[
\text{Res}_G^\hat{G} S \cong \bigoplus_{\hat{g} \in [\hat{G}/G(S')]} \hat{g}S'^{\oplus r}
\]
for some integer \( r \), which is called the ramification index of \( S \) in \( \hat{G} \).

3.2 Indecomposable modules, bricks and simple modules satisfying the stable conditions

Let \( \hat{G} \) be a finite group, \( G \) a normal subgroup of \( \hat{G} \) and \( U \) a \( kG \)-module. If \( I_{\hat{G}}(U) = \hat{G} \), we say that \( U \) is \( \hat{G} \)-stable, that is, for any \( \hat{g} \in \hat{G} \) the \( kG \)-module \( \hat{g}U \) is isomorphic to \( U \). For a \( k\hat{G} \)-module \( \hat{U} \), we say that \( \hat{U} \) is \( k\hat{G} \)-module of \( U \) if \( \text{Res}_G^\hat{G} \hat{U} \cong U \).

Remark 3.4. We remark that any extending \( k\hat{G} \)-module of indecomposable \( kG \)-modules (respectively, of bricks in \( kG \)-mod, of simple \( kG \)-modules) is also an indecomposable \( kG \)-module (respectively, a brick in \( k\hat{G} \)-mod and a simple \( k\hat{G} \)-module).

Lemma 3.5. Let \( \hat{G} \) be a finite group, \( G \) a normal subgroup, \( U \) a \( kG \)-module and \( \hat{U} \) an extending \( k\hat{G} \)-module of \( U \). Then the following hold:

1. \( \text{Ind}_G^\hat{G} U \cong \hat{U} \otimes k[\hat{G}/G] \).

2. \( T(\text{Ind}_G^\hat{G} U) = T\left( \bigoplus V \left( \hat{U} \otimes V \right) \right) \), where \( V \) runs through representatives of the isomorphism classes of simple \( k[\hat{G}/G] \)-modules.

3. \( F(\text{Ind}_G^\hat{G} U) = F\left( \bigoplus V \left( \hat{U} \otimes V \right) \right) \), where \( V \) runs through representatives of the isomorphism classes of simple \( k[\hat{G}/G] \)-modules.

Proof. The assertion (1) follows from the following isomorphisms:
\[
\text{Ind}_G^\hat{G} U \cong \text{Ind}_G^\hat{G} (\text{Res}_G^\hat{G} \hat{U} \otimes k_G) \cong \hat{U} \otimes \text{Ind}_G^\hat{G} k_G \cong \hat{U} \otimes k[\hat{G}/G],
\]
here the first and second isomorphisms come from Proposition 3.1.

We prove the assertion (2). By taking a composition series
\[
0 = W_0 \subset W_1 \subset \cdots \subset W_{e-1} \subset W_e = \text{k}\hat{G} k[\hat{G}/G]
\]
of the module \( \text{k}\hat{G} k[\hat{G}/G] \), we have the filtration
\[
0 = \hat{U} \otimes W_0 \subset \hat{U} \otimes W_1 \subset \cdots \subset \hat{U} \otimes W_{e-1} \subset \hat{U} \otimes W_e = \hat{U} \otimes \text{k}\hat{G} k[\hat{G}/G]
\]
of \( \hat{U} \otimes k[\hat{G}/G] \cong \text{Ind}_G^\hat{G} U \). Therefore, we get \( T(\text{Ind}_G^\hat{G} U) = T\left( \bigoplus V \left( \hat{U} \otimes V \right) \right) \) by Proposition 2.1.

The dual arguments show that the assertion (3) is also true. \( \square \)
The same arguments of [20, Theorems 3.5.7, Theorem 3.5.8 and Corollary 3.5.9] work for bricks. Hence, we have the following.

**Proposition 3.6** (Brick version of [20, Corollary 3.5.9]). Let $\tilde{G}$ be a finite group, $G$ a normal subgroup of $\tilde{G}$ and $S$ a brick in $kG$-mod. Moreover, let $e$ be the number of isomorphism classes of 1-dimensional $k[\tilde{G}/G]$-modules. Assume the following conditions hold:

- $S$ is $\tilde{G}$-stable,
- $H^2(\tilde{G}/G, k^\times) = 1$.

Then there exist $e$ isomorphism classes of extending $k\tilde{G}$-modules $\tilde{S}^{(1)}, \ldots, \tilde{S}^{(e)}$ of $S$. Moreover, for $\tilde{S}^{(i)}$ and $\tilde{S}^{(j)}$, there exists a unique 1-dimensional $k[\tilde{G}/G]$-module $V$ such that $\tilde{S}^{(i)} \otimes V \cong \tilde{S}^{(j)}$ up to isomorphisms.

**Proof.** Let $\xi_S : G \to GL_k(S)$ be the linear representation of $G$ corresponding to $kG$-module $S$ and $X$ a representative of $\tilde{G}$ containing the unit element $1_{\tilde{G}}$ of $\tilde{G}$. For any $1_{\tilde{G}} \neq x \in X$, we can take an isomorphism $\psi_x : xS \to S$ as $kG$-modules, and we define $\psi_{1_{\tilde{G}}} = \text{Id}_S$. For any $x \in X$, we define the $k$-linear map $\varphi_x \in GL_k(S)$ of $S$ by $s \mapsto \varphi_x(s) := \psi_x(xs)$, then we get that $\xi_S(g)\varphi_x = \varphi_x\xi_S(g^x)$ for any $x \in X$ and $g \in G$, here we mean that $g^x = x^{-1}gx$.

Since every element $\tilde{g}$ of $\tilde{G}$ can be expressed uniquely as $\tilde{g} = xg$ with $g \in G$ and $x \in X$, it is possible to define the map $\tilde{\xi} : \tilde{G} \to GL_k(S)$ by $xg \mapsto \varphi_x\xi_S(g)$, which is extending map of $\xi_S$. For any $g, h \in G$ and $x \in X$, we have the following:

$$
\xi_S(h)\tilde{\xi}(xg) = \xi_S(h)\varphi_x\xi_S(g)
= \varphi_x\xi_S(h^x)\xi_S(g)
= \varphi_x\xi_S(h^xg)
= \varphi_x\xi_S(x^{-1}hxg)
= \varphi_x\xi_S(gg^{-1}x^{-1}hxg)
= \varphi_x\xi_S(gx^g)
= \varphi_x\xi_S(g)\xi_S(x^g)
= \tilde{\xi}(xg)\xi_S(h^x).
$$

Therefore, we get

$$
\xi_S(h)\tilde{\xi}(\tilde{g}) = \tilde{\xi}(\tilde{g})\xi_S(h^\tilde{g})
$$

for any $\tilde{g} \in \tilde{G}$ and $h \in G$. Hence, for any $\tilde{g}, \tilde{h} \in \tilde{G}$, the $k$-linear map $\tilde{\xi}(\tilde{g})\tilde{\xi}(\tilde{h})\xi_S(g)^{-1}$ of $S$ is an endomorphism of $kG$-module $S$ since it holds that

$$
\tilde{\xi}(\tilde{g})\tilde{\xi}(\tilde{h})\xi_S(g)^{-1}\xi_S(g) = \tilde{\xi}(\tilde{g})\tilde{\xi}(\tilde{h})\xi_S(g^\tilde{h})\xi_S(g)^{-1}
= \tilde{\xi}(\tilde{g})\xi_S(g^\tilde{h})\xi_S(\tilde{h})\xi_S(g)^{-1}
= \xi_S(g)\tilde{\xi}(\tilde{g})\tilde{\xi}(\tilde{h})\xi_S(g^\tilde{h})^{-1}
$$

for any $g \in G$. Now the $kG$-module $S$ is a brick in $kG$-mod, hence we can take a scalar $\alpha(\tilde{g}, \tilde{h}) \in k^\times$ such that

$$
\tilde{\xi}(\tilde{g})\tilde{\xi}(\tilde{h})\xi_S(g^\tilde{h})^{-1} = \alpha(\tilde{g}, \tilde{h})\text{Id}_S,
$$

21
We define $\tilde{\delta}(1_G, 1_G) = 1$. Moreover, for any $g, h \in G$ and $x, y \in X$, we get $\alpha(gx, hy) = \alpha(x, y)$ as follows:

$$\alpha(xg, yh)\text{Id}_S = \tilde{\xi}(xyg)\tilde{\xi}(yh)^{-1} \tilde{\xi}(xy)\alpha(x, y)\text{Id}_S = \varphi_x\xi_S(g)\varphi_y\xi_S(h)\tilde{\xi}(xyg)\tilde{\xi}(yh)^{-1} \tilde{\xi}(xy)\alpha(x, y)\text{Id}_S = \varphi_x\varphi_y\xi_S(g)\xi_S(h)\varphi_x\xi_S(fg)\varphi_y\xi_S(h)\tilde{\xi}(xyg)\tilde{\xi}(yh)^{-1} \tilde{\xi}(xy)\alpha(x, y)\text{Id}_S = \varphi_x\varphi_y\xi_S(f)\tilde{\xi}(xyg)\tilde{\xi}(yh)^{-1} \tilde{\xi}(xy)\alpha(x, y)\text{Id}_S = \tilde{\xi}(x)\xi(y)\tilde{\xi}(xy)^{-1} \tilde{\xi}(x)\alpha(x, y)\text{Id}_S = \alpha(x, y)\text{Id}_S.$$

We can check that the map $\alpha: \tilde{G} \times \tilde{G} \to k^\times$ satisfies the 2-cocycle condition easily (for example, see [20]). Hence, $\alpha$ induces the map $\bar{\alpha}: \tilde{G}/G \times \tilde{G}/G \to k^\times$ satisfying the 2-cocycle condition. Since $H^2(\tilde{G}/G, k^\times)$ is trivial, there exists a map $\beta: \tilde{G}/G \to k^\times$ satisfying that for any $\tilde{g}, \tilde{h} \in \tilde{G}$,

$$\beta(\tilde{g}\tilde{h})\beta(\tilde{g}\tilde{h})^{-1}(\tilde{g}G) = \bar{\alpha}(\tilde{g}G, \tilde{h}G) = \alpha(\tilde{g}, \tilde{h}).$$

We define $\tilde{\xi}: \tilde{G} \to GL_k(S)$ by $\tilde{\xi}(\tilde{g}) = \beta(\tilde{g}G)\tilde{\xi}(\tilde{g})$. Then $\tilde{\xi}$ is a group homomorphism and an extension of $\xi_S$ since $\beta(1_G) = 1$. Hence, the corresponding $k\tilde{G}$-module $\tilde{S}$ to $\tilde{\xi}$ is an extending $k\tilde{G}$-module of the $kG$-module $S$. Let $W$ be another extending $k\tilde{G}$-module of $S$ and $\xi_W: \tilde{G} \to GL_k(W)$ the linear representation of $\tilde{G}$ corresponding to $W$. Then there exists a $kG$-isomorphism $\nu: \text{Res}_{\tilde{G}}W \to S$, that is, the following equation holds for any $g \in G$:

$$\nu\xi_W(g) = \xi_S(g)\nu. \quad (3.2.1)$$

For $\tilde{g} \in \tilde{G}$, we define the following linear map of $S$:

$$\gamma(\tilde{g}) := \nu\xi_W(\tilde{g})\nu^{-1}\tilde{\xi}(\tilde{g})^{-1}.$$

If $g \in G$ then $\gamma(g)$ is the identity map by (3.2.1), and we can show that $\gamma(\tilde{g})$ is an endomorphism of $k\tilde{G}$-module $S$ as follows:

$$\gamma(\tilde{g})\xi_S(g) = \nu\xi_W(\tilde{g})\nu^{-1}\tilde{\xi}(\tilde{g})^{-1} \xi_S(g)$$

$$= \nu\xi_W(\tilde{g})\nu^{-1}\tilde{\xi}(\tilde{g})^{-1} \xi_S(g)\tilde{\xi}(\tilde{g})\tilde{\xi}(\tilde{g})^{-1}$$

$$= \nu\xi_W(\tilde{g})\nu^{-1}\tilde{\xi}(\tilde{g})^{-1} \xi_S(g)\tilde{\xi}(\tilde{g})^{-1}$$

$$= \nu\xi_W(\tilde{g})\xi_S(g)\nu^{-1}\tilde{\xi}(\tilde{g})^{-1}$$

$$= \xi_S(g)\nu\xi_W(\tilde{g})\nu^{-1}\tilde{\xi}(\tilde{g})^{-1}$$

$$= \xi_S(g)\gamma(\tilde{g}).$$

Thus, there exists $\delta(\tilde{g}) \in k^\times$ such that $\gamma(\tilde{g}) = \delta(\tilde{g})\text{Id}_S$, so we get $\delta(\tilde{g})\tilde{\xi}(\tilde{g})\nu = \nu\xi_W(\tilde{g})$ for any $\tilde{g} \in \tilde{G}$. It is easy to show that $\delta: \tilde{G} \to k^\times$ is a group homomorphism and induces the group homomorphism $\delta: \tilde{G}/G \to k^\times$. Let $V$ be the $k[\tilde{G}/G]$-module corresponding to $\delta$. Then we have $V \otimes \tilde{S} \cong W$. \qed
Proposition 3.7. Let \( \tilde{G} \) be a finite group and \( G \) a normal subgroup of \( \tilde{G} \). \( S \) a semibrick in \( kG \)-mod and \( S \cong \bigoplus_{i=1}^{n_S} S_i \) a direct sum decomposition into bricks. Assume the following conditions hold:

- \( S_i \) is \( \tilde{G} \)-stable for any \( i = 1, \ldots, n_S \),
- \( H^2(\tilde{G}/G, k^\times) = 1 \),
- \( k[\tilde{G}/G] \) is basic as a \( k \)-algebra.

Then the following hold:

1. For each indecomposable direct summand \( S_i \) of \( S \), there exist exactly \( e := |k[\tilde{G}/G]| = 1 \) isomorphism classes of extending \( \tilde{G} \)-modules \( \tilde{S}_i^{(1)}, \ldots, \tilde{S}_i^{(e)} \) of \( S_i \).

2. The direct sum \( \bigoplus_{i=1}^{n_S} \bigoplus_{j=1}^{e} \tilde{S}_i^{(j)} \) is a semibrick in \( k\tilde{G} \)-mod.

Proof. The assertion (1) immediately follows from Proposition 3.6. In order to prove the assertion (2), we only have to show that a direct sum \( \bigoplus_{j=1}^{e} \tilde{S}_i^{(j)} \) is also a semibrick by Remark 3.4. Since the group algebra \( k[\tilde{G}/G] \) is basic, by Proposition 3.6, this module is isomorphic to

\[
\bigoplus_V \left( \tilde{S}_i^{(1)} \otimes V \right) \cong \tilde{S}_i^{(1)} \otimes \text{Soc}(k[\tilde{G}/G]),
\]

here \( V \) runs through representatives of the isomorphism classes of simple \( k[\tilde{G}/G] \)-modules. Now we have that

\[
\tilde{S}_i^{(1)} \otimes \text{Soc}(k[\tilde{G}/G]) \subset \tilde{S}_i^{(1)} \otimes k[\tilde{G}/G] \cong \text{Ind}_{\tilde{G}}^G S_i.
\]

Hence, for any simple \( k[\tilde{G}/G] \)-module \( V \), we get that

\[
\text{Hom}_{k\tilde{G}}(\tilde{S}_i^{(1)} \otimes V, \tilde{S}_i^{(1)} \otimes V) \cong \bigoplus_{V' \neq V} \text{Hom}_{k\tilde{G}}(\tilde{S}_i^{(1)} \otimes V, \tilde{S}_i^{(1)} \otimes V')
\]

\[
\cong \text{Hom}_{k\tilde{G}}(\tilde{S}_i^{(1)} \otimes V, \bigoplus_{V'} \tilde{S}_i^{(1)} \otimes V')
\]

\[
\cong \text{Hom}_{kG}(\text{Res}_{\tilde{G}}^G(\tilde{S}_i^{(1)} \otimes V), S_i)
\]

\[
\cong \text{Hom}_{kG}(S_i, S_i).
\]

Therefore, we get \( \bigoplus_{V' \neq V} \text{Hom}(\tilde{S}_i^{(1)} \otimes V, \tilde{S}_i^{(1)} \otimes V') = 0 \) since

\[
\dim_k \text{Hom}_{k\tilde{G}}(\tilde{S}_i^{(1)} \otimes V, \tilde{S}_i^{(1)} \otimes V) = \dim_k \text{Hom}_{kG}(S_i, S_i) = 1,
\]

which shows that \( \bigoplus_V (\tilde{S}_i^{(1)} \otimes V) \cong \bigoplus_{j=1}^{e} \tilde{S}_i^{(j)} \) is a semibrick in \( k\tilde{G} \)-mod. \( \Box \)
Remark 3.8. With the same assumption in Proposition 3.7, if \( e \) is prime to \( p \), then we have

\[
\text{Ind}^{\tilde{G}}_G S_i \cong \bigoplus_{j=1}^{e} \tilde{S}^{(j)}_{i}
\]

by Lemma 3.5 and Proposition 3.6.

Theorem 3.9. Assume that the cohomology group \( H^2(\tilde{G}/G, k) \) is the trivial group and the group algebra \( k[\tilde{G}/G] \) is basic as a \( k \)-algebra. Then for any indecomposable \( \tilde{G} \)-stable \( kG \)-module \( U \), the number of indecomposable direct summands of \( \text{Ind}^{\tilde{G}}_G U \) is equal to \( |k[\tilde{G}/G]| \) and any two direct summands of \( \text{Ind}^{\tilde{G}}_G U \) are not isomorphic. In particular, we have that \( |\text{Ind}^{\tilde{G}}_G U| = |k[\tilde{G}/G]| \).

Proof. The direct sum decompositions of the induced module \( \text{Ind}^{\tilde{G}}_G U \) into indecomposable \( \tilde{G} \)-modules correspond to orthogonal primitive idempotents decompositions of the unit element of the endomorphism \( k \)-algebra \( \text{End}_{kG}(\text{Ind}^{\tilde{G}}_G U) \). By the \( \tilde{G} \)-stability of \( U \), the algebra \( \text{End}_{k\tilde{G}}(\text{Ind}^{\tilde{G}}_G U) \) has a strongly \( \tilde{G}/G \)-graded algebra structure with the two-sided ideal

\[
\text{Rad}(\text{End}_{kG}(\text{Ind}^{\tilde{G}}_G U)) \subset \text{Rad}(\text{End}_{k\tilde{G}}(\text{Ind}^{\tilde{G}}_G U)).
\]

(3.2.3)

The algebra \( \text{End}_{k\tilde{G}}(\text{Ind}^{\tilde{G}}_G U) \) is local and \( k \) is an algebraically closed field, hence the algebra (3.2.3) isomorphic to a twisted group algebra \( k_{\alpha}[\tilde{G}/G] \), where \( \alpha \) is a Schur multiplier. However, the assumption \( H^2(\tilde{G}/G, k) = 1 \) implies that \( k_{\alpha}[\tilde{G}/G] \) is isomorphic to the group algebra \( k[\tilde{G}/G] \) which is a basic \( k \)-algebra by our assumptions. Therefore, the number of indecomposable direct summand of \( \text{Ind}^{\tilde{G}}_G U \) is equal to \( |k[\tilde{G}/G]| \) and any two direct summands of \( \text{Ind}^{\tilde{G}}_G U \) are not isomorphic.

Lemma 3.10. Let \( G \) be a normal subgroup of a finite group \( \tilde{G} \) and \( V \) an indecomposable \( k\tilde{G} \)-module. If \( V \) is \( \tilde{G} \)-stable, then its projective cover \( P(V) \), syzygy \( \Omega(V) \), injective envelope \( I(V) \) and cosyzygy \( \Omega^{-1}(V) \) are also \( \tilde{G} \)-stable.

Proof. For any \( \tilde{g} \in \tilde{G} \), there exists an isomorphism \( \phi : \tilde{g}V \to V \) by the \( \tilde{G} \)-stability of \( V \). We consider the following commutative diagram in \( k\tilde{G} \)-mod with exact rows:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{g}\Omega(V) & \longrightarrow & \tilde{g}P(V) & \longrightarrow & \tilde{g}V & \longrightarrow & 0 \\
& & \downarrow \phi' & & \downarrow \phi' & & \downarrow \phi & \\
0 & \longrightarrow & \Omega(V) & \longrightarrow & P(V) & \longrightarrow & V & \longrightarrow & 0.
\end{array}
\]
Since $\mu$ is an essential epimorphism and $\phi$ is an isomorphism, the vertical morphisms $\phi'$ and $\phi''$ are isomorphisms. Moreover, by using the dual arguments, we get that $I(V)$ and $\Omega^{-1}(V)$ are also $\tilde{G}$-stable.

**Lemma 3.11.** Let $G$ be a normal subgroup of a finite group $\tilde{G}$ satisfying that $H^2(\tilde{G}/G, k^\times) = 1$ and that $k[\tilde{G}/G]$ is basic as a $k$-algebra. For any indecomposable $\tilde{G}$-stable $kG$-module $V$, the following hold:

1. $P(\text{Ind}_{\tilde{G}}^G V) \cong \text{Ind}_{\tilde{G}}^G P(V)$,
2. $\Omega(\text{Ind}_{\tilde{G}}^G V) \cong \text{Ind}_{\tilde{G}}^G \Omega(V)$,
3. $\tau(V) \cong \Omega\Omega(V)$.

**Proof.** We have the following commutative diagram in $k\tilde{G}$-mod with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ind}_{\tilde{G}}^G \Omega(V) & \longrightarrow & \text{Ind}_{\tilde{G}}^G P(V) & \longrightarrow & \text{Ind}_{\tilde{G}}^G V & \longrightarrow & 0 \\
& & \downarrow \phi' & & \downarrow \phi & & \downarrow \text{Ind}_{\tilde{G}}^G \phi & \quad & \\
0 & \longrightarrow & \Omega(\text{Ind}_{\tilde{G}}^G V) & \longrightarrow & P(\text{Ind}_{\tilde{G}}^G V) & \longrightarrow & \text{Ind}_{\tilde{G}}^G V & \longrightarrow & 0.
\end{array}
\]

Since $\mu$ is an essential epimorphism, the vertical morphisms $\phi$ and $\phi'$ are split epimorphisms, and there exists a direct summand $Q$ of $\text{Ind}_{\tilde{G}}^G P(V)$ such that $Q \oplus P(\text{Ind}_{\tilde{G}}^G V) \cong \text{Ind}_{\tilde{G}}^G P(V)$ and $Q \oplus \Omega(\text{Ind}_{\tilde{G}}^G V) \cong \text{Ind}_{\tilde{G}}^G \Omega(V)$. We recall that $\Omega V$ is indecomposable and $\tilde{G}$-stable by Lemma 3.10. The numbers of indecomposable direct summands of $\text{Ind}_{\tilde{G}}^G \Omega(V)$ and $\Omega(\text{Ind}_{\tilde{G}}^G V)$ are equal to $|k[\tilde{G}/G]|$ by Theorem 3.9. Hence, we get $Q = 0$ and the proofs of assertion (1) and assertion (2) are completed.

Finally, we prove the assertion (3). Since $k\tilde{G}$ and $kG$ are symmetric $k$-algebras, it holds that $\tau V \cong \Omega V$ and $\tau(\text{Ind}_{\tilde{G}}^G V) \cong \Omega \Omega(V)$ for any $kG$-module $V$. Therefore, we complete the proof of assertion (3).

The dual argument gives the following.

**Lemma 3.12.** Let $G$ be a normal subgroup of a finite group $\tilde{G}$ satisfying that $H^2(\tilde{G}/G, k^\times) = 1$ and that $k[\tilde{G}/G]$ is basic as a $k$-algebra. For any indecomposable $\tilde{G}$-stable $kG$-module $V$, the following hold:

1. $I(\text{Ind}_{\tilde{G}}^G V) \cong \text{Ind}_{\tilde{G}}^G I(V)$,
2. $\Omega^{-1}(\text{Ind}_{\tilde{G}}^G V) \cong \text{Ind}_{\tilde{G}}^G \Omega^{-1}(V)$,
3. $\tau^{-1}(\text{Ind}_{\tilde{G}}^G V) \cong \text{Ind}_{\tilde{G}}^G \tau^{-1}V$.

### 3.3 Blocks of group algebras

We recall the definition of blocks of group algebras. Let $G$ be a finite group. The group algebra $kG$ has a unique decomposition

\[
kG = B_0 \times \cdots \times B_l
\]  

(3.3.1)
into the direct product of indecomposable \( k \)-algebras \( B_i \). We call each indecomposable direct product component \( B_i \) a block of \( kG \) and the above decomposition the block decomposition. We remark that any block \( B_i \) is a two-sided ideal of \( kG \).

For any indecomposable \( kG \)-module \( U \), there exists a unique block \( B_i \) of \( kG \) such that \( U = B_i U \) and \( B_j U = 0 \) for all \( j \neq i \). Then we say that \( U \) lies in the block \( B_i \) or simply \( U \) is a \( B_i \)-module. We denote by \( B_0(kG) \) the principal block of \( kG \), in which the trivial \( kG \)-module lies.

**Remark 3.13.** We remark that the block decomposition (3.3.1) induces the following isomorphism of partially ordered sets:

\[
\text{sr-tilt}(kG) \longrightarrow \text{sr-tilt}(B_0) \times \cdots \times \text{sr-tilt}(B_l)
\]

\[
M \longrightarrow (B_0 M, \ldots, B_l M).
\]

Now we recall the definition and basic properties of defect groups of blocks.

**Definition 3.14.** Let \( B \) be a block of \( kG \). A defect group \( D \) of \( B \) is a minimal subgroup of \( G \) satisfying the following condition: the \( B \)-bimodule morphism

\[
B \otimes_{kD} B \xrightarrow{\mu_D} B
\]

\[
\beta_1 \otimes \beta_2 \xleftarrow{\;} \beta_1 \beta_2
\]

is a split epimorphism.

**Proposition 3.15** (see [5, Chapter 4, 5]). Let \( B \) be a block of \( kG \) and \( D \) a defect group of \( B \). Then the following hold:

1. \( D \) is a \( p \)-subgroup of \( G \) and the set of all defect groups of \( B \) forms the conjugacy class of \( D \) in \( G \).

2. \( D \) is a cyclic group if and only if the algebra \( B \) is finite representation type.

3. If \( B \) is the principal block of \( kG \), then \( D \) is a Sylow \( p \)-subgroup of \( G \).

**Theorem 3.16** (see [5, Corollary 14.6, Theorem 17.1 and proof of Lemma 19.3]). Let \( B \) be a block of \( kG \) and \( D \) a defect group of \( B \).

1. \( D \) is the trivial group if and only if \( B \) is a simple algebra.

2. \( D \) is a non-trivial cyclic group if and only if \( B \) is a Brauer tree algebra with \( e \) edges and multiplicity \((|D| - 1)/e\), where \( e \) is a divisor of \( p - 1 \).

### 3.4 Clifford’s theory for blocks of normal subgroups

Let \( G \) be a finite group, \( \hat{G} \) a finite group containing \( G \) as a normal subgroup, \( B \) a block of \( kG \) and \( \hat{B} \) a block of \( k\hat{G} \). We say that \( \hat{B} \) covers \( B \) if \( 1_B1_{\hat{B}} \neq 0 \). We denote by \( I_{\hat{G}}(B) \) the inertial group of \( B \) in \( \hat{G} \), that is \( I_{\hat{G}}(B) := \{ x \in \hat{G} \mid xBx^{-1} = B \} \).

**Remark 3.17** (see [5, Theorem 15.1, Lemma 15.3]). With the above notation, the following are equivalent:
(1) The block $\tilde{B}$ covers $B$.

(2) There exists a non-zero $\tilde{B}$-module $U$ such that $\text{Res}^{\tilde{G}}_G U$ has a non-zero direct summand lying in $B$.

(3) For any non-zero $\tilde{B}$-module $U$, there exists a non-zero direct summand of $\text{Res}^{\tilde{G}}_G U$ lying in $B$.

**Remark 3.18.** The principal block $B_0(kG)$ of $kG$ is covered by the principal block $B_0(k\tilde{G})$ of $k\tilde{G}$ and $I_{\tilde{G}}(B_0(kG)) = \tilde{G}$ since the trivial $kG$-module $kG$ is $\tilde{G}$-stable $\text{Res}^{\tilde{G}}_G kG \cong kG$.

**Theorem 3.19** (Clifford’s Theorem for blocks [5, Theorem 15.1, Lemma 15.3]). Let $\tilde{G}$ be a finite group, $G$ a normal subgroup of $\tilde{G}$, $B$ a block of $kG$, $\tilde{B}$ a block of $k\tilde{G}$ covering $B$ and $U$ a $\tilde{B}$-module. Then the following hold:

1. The set of blocks of $kG$ covered by $\tilde{B}$ equals to the conjugacy class of $B$ in $\tilde{G}$:

   $$\{ B' \mid B' \text{ is a block of } kG \text{ covered by } \tilde{B} \} = \{ xBx^{-1} \mid x \in \tilde{G} \}.$$

2. We get the following isomorphism of $kG$-modules:

   $$\text{Res}^{\tilde{G}}_G U \cong \bigoplus_{x \in [\tilde{G}/I_{\tilde{G}}(B)]} xBU.$$

**Proposition 3.20** (see [20, Theorem 5.5.10, Theorem 5.5.12]). Let $G$ be a normal subgroup of a finite group $\tilde{G}$ and $B$ a block of $kG$, $\beta$ a block of $kI_{\tilde{G}}(B)$ covering $B$. Then the following hold:

1. For any $B$-module $V$, the induced module $\text{Ind}^{I_{\tilde{G}}(B)}_{\tilde{G}} V$ is a direct sum of $kI_{\tilde{G}}(B)$-module lying blocks covering $B$.

2. There exists a unique block $\tilde{B}$ of $k\tilde{G}$ covering $B$ such that the induction functor $\text{Ind}^{I_{\tilde{G}}(B)}_{\tilde{G}} : kI_{\tilde{G}}(B)\text{-mod} \to k\tilde{G}\text{-mod}$ restricts to a Morita equivalence

   $$\text{Ind}^{I_{\tilde{G}}(B)}_{\tilde{G}} : \beta\text{-mod} \to \tilde{B}\text{-mod}.$$

**Proposition 3.21** ([20, Corollary 5.5.6, Theorem 5.5.13, Lemma 5.5.14]). Let $G$ be a normal subgroup of $\tilde{G}$ and $B$ a block of $kG$, then the following conditions hold:

1. If $\tilde{G}/G$ is a $p$-group, then there exists a unique block of $k\tilde{G}$ covering $B$.

2. If a defect group $D$ of $B$ satisfies $C_{\tilde{G}}(D) \subset G$, then there exists a unique block of $k\tilde{G}$ covering $B$.

**Lemma 3.22.** Let $G$ be a normal subgroup of a finite group $\tilde{G}$ satisfying that $H^2(\tilde{G}/G,k^\times) = 1$ and that $k[\tilde{G}/G]$ is basic as a $k$-algebra, $B$ a $\tilde{G}$-stable block of $k\tilde{G}$ and $\tilde{A}$ the direct product of the all blocks of $k\tilde{G}$ covering $B$. Assume that any simple $B$-module is $\tilde{G}$-stable. Then we have $|\tilde{A}| = |k[\tilde{G}/G]| \cdot |B|$.
Proof. By Remark 3.4 and Remark 3.17, any extending $k\tilde{G}$-module of a simple $B$-module is also a simple $A$-module, and we get that $|A| \geq |k[\tilde{G}/G]| \cdot |B|$. On the other hand, for an arbitrary simple $A$-module $S'$, there exists a simple $B$-submodule $S$ of $\text{Res}_{\tilde{G}}^G S'$ such that $\text{Res}_{\tilde{G}}^G S' \cong S^{\oplus r}$ for some positive integer $r$ by Theorem 3.3 and Remark 3.17. We denote an extending $k\tilde{G}$-module of $S$ by $\tilde{S}$, which is also a simple $A$-module. By Lemma 3.5, we get

$$S' \subset \text{Ind}_G^\tilde{G} \text{Res}_{\tilde{G}}^G S' \cong \text{Ind}_G^\tilde{G} S^{\oplus r} \cong (\tilde{S} \otimes k[\tilde{G}/G])^{\oplus r}.$$ 

Therefore, we get a 1-dimensional $k[\tilde{G}/G]$-module $V$ such that $S' \cong \tilde{S} \otimes V$ by the Jordan–Hölder theorem. Therefore, $S'$ is also an extending $k\tilde{G}$-module of $S$. Hence, we get that $|A| \leq |k[\tilde{G}/G]| \cdot |B|$ by Proposition 3.6.

4 The main results and their applications

In this section, we give lemmas and prove the main results. After that, we give some applications and examples of our main results.

4.1 Main theorems and their proof

The following assertions are related to the assumptions of our main results.

Lemma 4.1. Let $G$ be a normal subgroup of $\tilde{G}$ and $B$ a $\tilde{G}$-stable block of $kG$. Then the following are equivalent:

(1) Any left finite brick in $B$-mod is $\tilde{G}$-stable.

(2) Any indecomposable $\tau$-rigid $B$-module is $\tilde{G}$-stable.

(3) Any right finite brick in $B$-mod is $\tilde{G}$-stable.

(4) Any indecomposable $\tau^{-1}$-rigid $B$-module is $\tilde{G}$-stable.

Proof. Let $S$ be a right finite brick over $B$ and $U$ the indecomposable $\tau$-rigid module corresponding to $S$ in the bijection (2.3.6). We can easily check that $\tilde{g} \text{Fac}(U) = \text{Fac} (\tilde{g}U)$ and $\tilde{g} \text{T}(S) = \text{T}(\tilde{g}S)$ for any $\tilde{g} \in \tilde{G}$, which implies the equivalence of (1) and (2) by Theorem 2.18. The similar argument show the equivalence of (3) and (4). We prove the equivalence of (2) and (4). For any indecomposable non-projective module $V$, the $\tilde{G}$-stability of $V$ means those of $\Omega V$, $\Omega^{-1} V$, $\tau V$ and $\tau^{-1} V$ by Lemma 3.10. Hence, the conclusion follows from Proposition 2.11.

The following is a slight generalization of [18, Lemma 3.22].

Lemma 4.2 ([18, Lemma 3.22]). Let $G$ be a normal subgroup of a finite group $\tilde{G}$ and $B$ a $\tilde{G}$-stable block of $kG$ with a cyclic defect group. Then the following hold:

(1) If any simple $B$-module is $\tilde{G}$-stable, then any indecomposable $B$-module is also $\tilde{G}$-stable.

(2) If $\tilde{G}/G$ is a $p$-group, then any indecomposable $B$-module is $\tilde{G}$-stable.
Proof. We prove the assertion (1) by using similar way as [18, Lemma 3.22]. We prove that \( \tilde{I}_G(V) = \tilde{G} \) for indecomposable \( B \)-module \( V \) by induction on the composition length of \( V \). If \( V \) is simple or indecomposable projective, there is nothing to show by the assumption and Lemma 3.10. We assume that the composition length of \( V \) is two or more and that \( V \) is not projective. We remark that any indecomposable non-projective \( B \)-module is a string module (for example, see [24]). Hence, we can take a simple \( B \)-module \( S \) and an indecomposable \( B \)-module \( V' \) which satisfy at least one of the following conditions:

1. There exists a non-split exact sequence
   \[
   0 \rightarrow S \xrightarrow{\mu} V \xrightarrow{\nu} V' \rightarrow 0.
   \]

2. There exists a non-split exact sequence
   \[
   0 \rightarrow V' \xrightarrow{\nu'} V \xrightarrow{\mu'} S \rightarrow 0.
   \]

It suffices to prove \( I_G(V) = \tilde{G} \) under the assumption that there exists the first exact sequence; the other case can be proved similarly. Since \( \text{Ext}^1_B(V', S) \) is 1-dimensional over \( k \) (see [5, Proposition 21.7]), we can prove the conclusion by induction on the composition length of \( V \).

By Theorem 3.16 and [15, Lemma 2.2], any simple \( B \)-module \( S \) is \( \tilde{G} \)-stable, we have the assertion (2) from the first one.

Now we give proofs of the main theorems. First, we state the main result again, which are stated in the introduction.

**Theorem 4.3.** Let \( \tilde{G} \) be a finite group, \( G \) a normal subgroup of \( \tilde{G} \), \( B \) be a block of \( kG \) and \( \tilde{B} \) be a block of \( k\tilde{G} \) covering \( B \). We assume the following conditions hold:

- Any left finite bricks in \( B \)-mod is \( I_{\tilde{G}}(B) \)-stable.
- \( H^2(I_{\tilde{G}}(B)/G, k^\times) = 1 \).
- \( k[I_{\tilde{G}}(B)/G] \) is basic as a \( k \)-algebra.

Then the maps

\[
\begin{align*}
\text{s}\tau\text{-tilt } B & \rightarrow \text{s}\tau\text{-tilt } \tilde{B}, \\
2\text{-tilt } B & \rightarrow 2\text{-tilt } \tilde{B}
\end{align*}
\]

defined by \( \text{s}\tau\text{-tilt } B \ni M \mapsto \tilde{B}\text{Ind}_{\tilde{G}}^{G}M \in \text{s}\tau\text{-tilt } \tilde{B} \) and

\[
\begin{align*}
\text{2-tilt } B & \rightarrow 2\text{-tilt } \tilde{B}, \\
\end{align*}
\]

defined by \( 2\text{-tilt } B \ni T \mapsto \tilde{B}\text{Ind}_{\tilde{G}}^{G}T \in 2\text{-tilt } \tilde{B} \) are well-defined and injective. Moreover, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{s}\tau\text{-tilt } B & \xrightarrow{(4.1.1)} & \text{s}\tau\text{-tilt } \tilde{B} \\
2\text{-tilt } B & \xrightarrow{(4.1.2)} & 2\text{-tilt } \tilde{B} \\
\end{array}
\]

(2.4.1) for \( B \)\
(2.4.1) for \( \tilde{B} \)\

(4.1.3)
Proof. By Proposition 3.1 and Proposition 3.20, we may assume that $B$ is $\hat{G}$-stable, that is, $\hat{G} = I_{\hat{G}}(B)$. For $M \in \mathfrak{sr}$-tilt $B$, let

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a minimal projective presentation of $M$ and $P$ a projective module such that $\text{Hom}_B(P, M) = 0$ and $|P| + |M| = |B|$. Let $A$ be the direct product algebra of all blocks of $k\hat{G}$ covering $B$. In order to show that the maps (4.1.1) and (4.1.2) are well-defined and that the diagram (4.1.3) is commutative, we only have to show the following:

1. $\text{Ind}_{\hat{G}}^G M$ is a $\tau$-rigid $k\hat{G}$-module.
2. $\text{Hom}_A(\text{Ind}_{\hat{G}}^G P, \text{Ind}_{\hat{G}}^G M) = 0$.
3. $|\text{Ind}_{\hat{G}}^G M| + |\text{Ind}_{\hat{G}}^G P| = |A|$.
4. The sequence

$$\text{Ind}_{\hat{G}}^G P_1 \xrightarrow{\text{Ind}_{\hat{G}}^G f_1} \text{Ind}_{\hat{G}}^G P_0 \xrightarrow{\text{Ind}_{\hat{G}}^G f_0} \text{Ind}_{\hat{G}}^G M \to 0$$

is also a minimal projective presentation of $k\hat{G}$-module $\text{Ind}_{\hat{G}}^G M$.

We show (1), that is, $\text{Ind}_{\hat{G}}^G M$ is also a $\tau$-rigid $k\hat{G}$-module as follows:

$$\text{Hom}_{k\hat{G}}(\text{Ind}_{\hat{G}}^G M, \tau \text{Ind}_{\hat{G}}^G M) \cong \text{Hom}_{k\hat{G}}(\text{Ind}_{\hat{G}}^G M, \text{Ind}_{\hat{G}}^G \tau M)$$

$$\cong \text{Hom}_{k\hat{G}}(\bigoplus_{x \in [\hat{G}/G]} xM, \tau M)$$

$$\cong \bigoplus_{x \in [\hat{G}/G]} \text{Hom}_B(M, \tau M)$$

$$= 0,$$

here the first isomorphism comes from Lemma 3.11, the second isomorphism comes from Proposition 3.1, the third isomorphism comes from Theorem 3.2 and the fourth isomorphism comes from our assumption. We show (2) as follows:

$$\text{Hom}_A(\text{Ind}_{\hat{G}}^G P, \text{Ind}_{\hat{G}}^G M) \cong \text{Hom}_{k\hat{G}}(\text{Res}_{\hat{G}}^G \text{Ind}_{\hat{G}}^G P, M)$$

$$\cong \text{Hom}_{k\hat{G}}(\bigoplus_{x \in [\hat{G}/G]} xP, M)$$

$$\cong \bigoplus_{x \in [\hat{G}/G]} \text{Hom}_B(P, M)$$

$$\cong 0,$$

here the first isomorphism comes from Proposition 3.1, the second isomorphism comes from Theorem 3.2 and the third isomorphism comes from our assumption. The assertion (3) is followed by Lemma 3.22. By Remark 2.5, we have that the map (4.1.1) is well-defined. The assertion (4) is followed by Lemma 3.11. Therefore, we get the map (4.1.2) is well-defined, and the diagram (4.1.3) is commutative. The injectivities of (4.1.1) and (4.1.2) are obvious. \qed
Theorem 4.4. With the same assumptions in Theorem 4.3, the following hold:

1. For any left finite semibrick $S$ in $B$-mod and its indecomposable direct summand $S_i$ of $S$, there exist exactly $e := |k[I_G(B)/G]|$ isomorphism classes of extending $k[I_G(B)]$-modules $\tilde{S}_i(1), \ldots, \tilde{S}_i(e)$ of $S_i$.

2. Then the map
$$f_L\text{-sbrick } B \longrightarrow f_L\text{-sbrick } \tilde{B} \quad (4.1.4)$$
defined by $S \cong \bigoplus_{i=1}^e S_i \rightarrow \tilde{B} \text{Ind}_{G}^{\tilde{G}}(\bigoplus_{i=1}^e \tilde{S}_i^{(j)})$ is well-defined and injective, where $S \cong \bigoplus_{i=1}^e S_i$ is a direct sum decomposition into bricks.

3. We get the following commutative diagram:

$$\begin{array}{ccc}
\text{str-tilt } B & \xrightarrow{(4.1.1)} & \text{str-tilt } \tilde{B} \\
\xrightarrow{(2.3.4) \text{ for } B} & & \xleftarrow{(2.3.4) \text{ for } \tilde{B}} \\
f_L\text{-sbrick } B & \rightleftharpoons & f_L\text{-sbrick } \tilde{B}. \\
\end{array}$$

Proof. By Proposition 3.1 and Proposition 3.20, we may assume that $B$ is $\tilde{G}$-stable. In order to apply Proposition 3.7, we show that any brick $S_i$ appearing as a direct summand of the left finite semibrick $S$ in $B$-mod is $\tilde{G}$-stable. Let $M$ be the support $\tau$-tilting $B$-module corresponding to $S$ under (2.3.4) and $M'$ the support $\tau$-tilting $B$-module corresponding to $S_i$ under (2.3.1) for $M$. By Proposition 2.26, the subcategory $\text{Fac } M \cap \text{Sub } N'$ contains a unique brick, which isomorphic to $S_i$, where $N'$ is the support $\tau^{-1}$-tilting $B$-module corresponding to $M'$ under (2.2.3). For any $\tilde{g} \in \tilde{G}$, we have
$$\tilde{g}(\text{Fac } M \cap \text{Sub } N') = \text{Fac } M \cap \text{Sub } N' \quad (4.1.5)$$
by Lemma 4.1. On the other hand, the subcategory in left-hand side of (4.1.5) contains the brick $\tilde{g}S$, which means that the subcategory $\text{Fac } M \cap \text{Sub } N'$ contains $\tilde{g}S$. Hence, $\tilde{g}S \cong S$ by the uniqueness of the brick in $\text{Fac } M \cap \text{Sub } N'$. Therefore, we have shown the well-definedness of (4.1.4). The injectivity of (4.1.4) is obvious.

Let $M$ and $S$ be corresponding support $\tau$-tilting $B$-module and semibrick in $B$-mod under (2.3.4) for $B$, that is, it holds $\text{Fac } M = T(S)$. To show that the diagram in (3) is commutative, we only have to show that $\text{Fac}(\text{Ind}_{G}^{\tilde{G}}M) = T(\text{Ind}_{G}^{\tilde{G}}S)$ by Lemma 3.5 because we know that $\text{Ind}_{G}^{\tilde{G}}M$ is a support $\tau$-tilting module by Theorem 4.3 and because $\bigoplus_{i=1}^e \tilde{S}_i^{(j)}$ is a semibrick by Proposition 3.7.

Since $M/R(M, M) \cong S$, $\text{Ind}_{G}^{\tilde{G}}S$ is a homomorphic image of $\text{Ind}_{G}^{\tilde{G}}M$ by the exactness of $\text{Ind}_{G}^{\tilde{G}}$. Therefore, $\text{Fac}(\text{Ind}_{G}^{\tilde{G}}M) \supset T(\text{Ind}_{G}^{\tilde{G}}S)$ holds. We show the reverse inclusion. Since $M \in T(S) = \text{Filt}(\text{Fac}(S))$, we get a filtration of $M$:
$$0 = M_0 \subset M_1 \subset \cdots \subset M_{l-1} \subset M_l = V,$$
such that $M_{l+1}/M_l \in \text{Fac}(S)$. By the exactness of $\text{Ind}_{G}^{\tilde{G}}$ again, we get a filtration
$$0 = \text{Ind}_{G}^{\tilde{G}}M_0 \subset \text{Ind}_{G}^{\tilde{G}}M_1 \subset \cdots \subset \text{Ind}_{G}^{\tilde{G}}M_{l-1} \subset \text{Ind}_{G}^{\tilde{G}}M_l = \text{Ind}_{G}^{\tilde{G}}V.$$
of $\text{Ind}_{G/G}^G M$ such that $\text{Ind}_{G/G}^G M_{i+1}/\text{Ind}_{G/G}^G M_i \in \text{Fac}(\text{Ind}_{G/G}^G S)$. Therefore, we get that $\text{Fac}(\text{Ind}_{G/G}^G M) \subset \text{Fac}(\text{Filt}(\text{Ind}_{G/G}^G S)) = T(\text{Ind}_{G/G}^G S)$. 

We can prove the following theorem by the dual argument on Theorems 4.3 and 4.4.

**Theorem 4.5.** With the same assumptions in Theorem 4.3, the following hold:

1. The map

$$s\tau^{-1}\text{-tilt } B \longrightarrow s\tau^{-1}\text{-tilt } \tilde{B}$$

defined by $s\tau^{-1}\text{-tilt } B \ni N \mapsto \tilde{B}\text{Ind}_{G/G}^G N \in s\tau^{-1}\text{-tilt } \tilde{B}$ is well-defined and injective.

2. For any right finite brick $S_i$ in $B$-mod, there exist exactly $e := |k[\tilde{G}/G]|$ isomorphism classes of extending $k\tilde{G}$-modules $\tilde{S}_i^{(1)}, \ldots, \tilde{S}_i^{(e)}$ of $S_i$.

3. Then the map

$$f_{R}\text{-sbrick } B \longrightarrow f_{R}\text{-sbrick } \tilde{B}$$

defined by $S \cong \bigoplus_{i=1}^{n_S} S_i \mapsto \tilde{B}\text{Ind}_{G/G}^G \left( \bigoplus_{i=1}^{n_S} \bigoplus_{j=1}^{e} \tilde{S}_i^{(j)} \right)$ is well-defined and injective, here $S \cong \bigoplus_{i=1}^{n_S} S_i$ is a direct sum decomposition into bricks.

4. We get the following commutative diagram:

$$
\begin{array}{ccc}
\text{s}\tau^{-1}\text{-tilt } B & \longrightarrow & \text{s}\tau^{-1}\text{-tilt } \tilde{B} \\
\downarrow \text{(4.1.6)} & & \downarrow \text{(4.1.7)} \\
\text{f}_{R}\text{-sbrick } B & \longrightarrow & \text{f}_{R}\text{-sbrick } \tilde{B} \\
\end{array}
$$

The next theorem is followed by Theorem 4.4 and Theorem 4.5.

**Theorem 4.6.** With the same assumptions in Theorem 4.3, the following diagram is commutative:

$$
\begin{array}{ccc}
\text{s}\tau^{-1}\text{-tilt } B & \longrightarrow & \text{s}\tau^{-1}\text{-tilt } \tilde{B} \\
\downarrow \text{(4.1.6)} & & \downarrow \text{(4.1.1)} \\
\text{f}_{R}\text{-sbrick } B & \longrightarrow & \text{f}_{R}\text{-sbrick } \tilde{B} \\
\downarrow \text{(2.3.11) for } B & & \downarrow \text{(2.3.15) for } \tilde{B} \\
\text{s}\tau^{-1}\text{-tilt } B & \longrightarrow & \text{s}\tau^{-1}\text{-tilt } \tilde{B} \\
\uparrow \text{(2.3.16) for } B & & \uparrow \text{(2.2.3) for } \tilde{B} \\
\text{f}_L\text{-sbrick } B & \longrightarrow & \text{f}_L\text{-sbrick } \tilde{B} \\
\uparrow \text{(2.3.4) for } B & & \uparrow \text{(2.3.4) for } \tilde{B} \\
\text{s}\tau^{-1}\text{-tilt } B & \longrightarrow & \text{s}\tau^{-1}\text{-tilt } \tilde{B} \\
\end{array}
$$

**Proof.** By Corollary 2.23, Theorem 4.4 and Theorem 4.5, we only have to show that the following diagram is commutative:

$$
\begin{array}{ccc}
\text{s}\tau^{-1}\text{-tilt } B & \longrightarrow & \text{s}\tau^{-1}\text{-tilt } \tilde{B} \\
\downarrow \text{(2.2.3) for } B & & \downarrow \text{(2.2.3) for } \tilde{B} \\
\text{s}\tau\text{-tilt } B & \longrightarrow & \text{s}\tau\text{-tilt } \tilde{B} \\
\end{array}
$$

32
Let $M$ be an arbitrary support $\tau$-tilting $B$-module and $P$ a projective $B$-module satisfying that $\text{Hom}_B(P, M) = 0$ and that $|M| + |P| = |B|$. Then the corresponding support $\tau^{-1}$-tilting module of $M$ in the bijection (2.2.3) is $\tau M \oplus P$. Let $A$ be the direct product of the all blocks of $k\tilde{G}$ covering $B$. By Theorem 3.9 and Lemma 3.22, we get $|\text{Ind}_{\tilde{G}}^G M| + |\text{Ind}_{\tilde{G}}^G P| = |A|$. Therefore, the corresponding support $\tau^{-1}$-tilting module of $\text{Ind}_{\tilde{G}}^G M$ with respect to (2.2.3) is $\tau \text{Ind}_{\tilde{G}}^G M \oplus \text{Ind}_{\tilde{G}}^G P$, which is isomorphic to the induced module $\text{Ind}_{\tilde{G}}^G (\tau M \oplus P)$ by Lemma 3.11.

**Corollary 4.7.** With the same assumptions in Theorem 4.3, we get the injective map

$$2\text{-smc } B \longrightarrow 2\text{-smc } \tilde{B}$$

which makes the following diagram commutative:

$$\begin{array}{c}
\text{f}_R\text{-sbrick } B & \xrightarrow{(4.1.7)} & \text{f}_R\text{-sbrick } \tilde{B} \\
\downarrow{(2.3.15) \text{ for } B} & & \downarrow{(2.3.15) \text{ for } \tilde{B}} \\
\text{f}_L\text{-sbrick } B & \xrightarrow{(4.1.4)} & \text{f}_L\text{-sbrick } \tilde{B}.
\end{array}$$

The following theorem summaries Theorem 4.3, Theorem 4.4 and Corollary 4.7.

**Theorem 4.8.** Let $\tilde{G}$ be a finite group, $G$ a normal subgroup of $\tilde{G}$, $B$ a block of $kG$ and $\tilde{B}$ a block of $k\tilde{G}$ covering $B$. We assume that the following conditions hold:

1. Any left finite bricks in $B$-mod is $I_{\tilde{G}}(B)$-stable.
2. $H^2(I_{\tilde{G}}(B)/G, k^\times) = 1$.
3. $k[I_{\tilde{G}}(B)/G]$ is basic as a $k$-algebra.

Then the following diagram is commutative:

$$\begin{array}{c}
2\text{-smc } B & \xrightarrow{(4.1.8)} & 2\text{-smc } \tilde{B} \\
\downarrow{(2.4.4) \text{ for } B} & & \downarrow{(2.4.4) \text{ for } \tilde{B}} \\
2\text{-tilt } B & \xrightarrow{(4.1.2)} & 2\text{-tilt } \tilde{B} \\
\downarrow{(2.4.2) \text{ for } B} & & \downarrow{(2.4.2) \text{ for } \tilde{B}} \\
\text{f}_L\text{-sbrick } B & \xrightarrow{(4.1.4)} & \text{f}_L\text{-sbrick } \tilde{B} \\
\downarrow{(2.3.4) \text{ for } B} & & \downarrow{(2.3.4) \text{ for } \tilde{B}} \\
\text{sr}\text{-tilt } B & \xrightarrow{(4.1.1)} & \text{sr}\text{-tilt } \tilde{B}.
\end{array}$$

33
4.2 Some applications of main theorems

We give some applications and examples of main theorems in this section. The following arguments provide sufficient conditions for the assumptions of our main results.

**Proposition 4.9** ([20, Lemma 3.5.4, Theorem 3.5.11], [13, Theorem 5.3, Theorem 5.9]). Let $\tilde{G}$ be a finite group and $G$ a normal subgroup of $\tilde{G}$. Then we have the following:

1. If $\tilde{G}/G$ is a $p$-group, then we have that $H^2(\tilde{G}/G, k^\times) = 1$ and that the group algebra $k[\tilde{G}/G]$ is basic.
2. If $\tilde{G}/G$ is cyclic, then we have that $H^2(\tilde{G}/G, k^\times) = 1$ and that the group algebra $k[\tilde{G}/G]$ is basic.
3. If $\tilde{G}/G$ is isomorphic to the dihedral group $D_{2p}$ of order $2p$, then we have that $H^2(\tilde{G}/G, k^\times) = 1$ and that the group algebra $k[\tilde{G}/G]$ is basic.
4. If $\tilde{G}/G$ is abelian, then we have that the group algebra $k[\tilde{G}/G]$ is basic.

The following argument describes the Hasse quiver of support $\tau$-tilting modules labeled by bricks and will be used to prove the refinement of the main result of [18].

**Proposition 4.10.** With the same assumptions in Theorem 4.3, let $M$ be a support $\tau$-tilting $B$-module and $M'$ a support $\tau$-tilting left mutation of $M$ and $S$ the brick in $B$-mod corresponding to the left mutation $M'$ in the bijection (2.3.3) for $M$. If $\tilde{B}\text{Ind}_{\tilde{G}}G M$ is a support $\tau$-tilting left mutation of $\tilde{B}\text{Ind}_{\tilde{G}}G M$, then there exists only one sort of extending $k\tilde{G}$-module $\tilde{S}$ of $S$ lying in $\tilde{B}$ up to isomorphisms. In particular, the brick labeling the arrow from $\tilde{B}\text{Ind}_{\tilde{G}}G M$ to $\tilde{B}\text{Ind}_{\tilde{G}}G M'$ in $\mathcal{H}(\tau\text{-tilt } B)$ is isomorphic to $\tilde{S}$.

**Proof.** Let $N$ and $N'$ be the support $\tau^{-1}$-tilting $B$-modules corresponding to $M$ and $M'$ respectively in the bijection (2.2.3). By Proposition 2.26, we have $S \in \text{Fac} M \cap \text{Sub} N'$. Therefore, we get $\tilde{B}\text{Ind}_{\tilde{G}}G S \in \text{Fac} \tilde{B}\text{Ind}_{\tilde{G}}G M \cap \text{Sub} \tilde{B}\text{Ind}_{\tilde{G}}G N'$. By the assumption, the subcategory $\text{Fac} \tilde{B}\text{Ind}_{\tilde{G}}G M \cap \text{Sub} \tilde{B}\text{Ind}_{\tilde{G}}G N'$ of $B$-mod contain a unique brick in $B$-mod by Proposition 2.26. Hence, we get that there exists only one sort of extending $kG$-module of $S$ lying in $B$ up to isomorphisms. Therefore, the extending $k\tilde{G}$-module of $S$ lying in $\tilde{B}$ is isomorphic to $\tilde{S}$. 

Now we state some consequences of our main results stated in section 4.1. The following result is a more precise result than [18, Theorem 4.2].

**Theorem 4.11.** Let $\tilde{G}$ be a finite group, $G$ a normal subgroup of $\tilde{G}$ such that the quotient group $\tilde{G}/G$ is a $p$-group, $\tilde{B}$ a block of $k\tilde{G}$ and $B$ a block of $kG$ covered by $\tilde{B}$. Assume that any left finite brick in $B$-mod is $I_{\tilde{G}}(B)$-stable. Then we have the following:

1. The maps (4.1.1) and (4.1.4) induce the embedding of quiver with labels:

   $$\mathcal{H}(\tau\text{-tilt } B) \longrightarrow \mathcal{H}(\tau\text{-tilt } \tilde{B}).$$

   Moreover, any connected component of $\mathcal{H}(\tau\text{-tilt } B)$ is embedded as a connected component of $\mathcal{H}(\tau\text{-tilt } \tilde{B})$. 

34
(2) If $B$ is a support $\tau$-tilting finite block, then the map (4.1.1) is an isomorphism from $s\tau$-tilt $B$ to $s\tau$-tilt $\tilde{B}$ as partially ordered sets. Moreover, all the maps appearing in the commutative diagram in Theorem 4.8 are bijective. In particular, the maps (4.1.1) and (4.1.4) induce an isomorphism from $\mathcal{H}(s\tau$-tilt $B)$ to $\mathcal{H}(s\tau$-tilt $\tilde{B})$ as labeled quivers.

Proof. By Proposition 3.21, we get $\text{Ind}_{\tilde{G}}^G U = \tilde{B}\text{Ind}_{\tilde{G}}^G U$ for any $B$-module $U$. By the same arguments of [18, Theorem 4.2], the induction functor $\text{Ind}_{\tilde{G}}^G$ preserves support $\tau$-tilting left mutations. Hence, we get conclusions by Proposition 4.10.

Corollary 4.12. Let $\tilde{G}$ be a finite group and $G$ be a normal subgroup with cyclic Sylow $p$-subgroup such that the quotient group $\tilde{G}/G$ is a $p$-group. Then the induction functor $\text{Ind}_{\tilde{G}}^G$ induces the following isomorphism as partially ordered sets:

$$\begin{align*}
s\tau\text{-tilt } kG & \longrightarrow s\tau\text{-tilt } k\tilde{G} \\
M & \longmapsto \text{Ind}_{\tilde{G}}^G M.
\end{align*}$$

Proof. Since any defect group of a block of $kG$ is contained in a Sylow $p$-subgroup of $G$, any block has a cyclic defect group. Hence, any block of $kG$ is $\tau$-tilting finite. Thus, the conclusion follows from Theorem 4.11 for all blocks of $kG$. $\Box$

Proposition 4.13. Let $G$ be a normal subgroup of a finite group $\tilde{G}$, $\tilde{B}$ a block of $k\tilde{G}$ and $B$ a cyclic defect block of $kG$ covered by $\tilde{B}$ satisfying one of the following conditions:

1. There is an $I_{\tilde{G}}(B)$-stable simple $B$-module $S$ whose corresponding edge is a terminal edge of the Brauer tree of $B$.
2. There is a simple $B$-module $S$ whose corresponding edge of the Brauer tree of $B$ is a terminal edge and the dimension of $S$ is distinct to that of any other simple $B$-module.
3. Any two simple $B$-modules have distinct dimensions.

If the quotient group $\tilde{G}/G$ is a cyclic group or isomorphic to the dihedral group $D_{2p}$ of order $2p$, then Main Theorems 1.1 to 1.3 hold.

Proof. We can assume that $I_{\tilde{G}}(B) = \tilde{G}$ by Proposition 3.20. We enough to show that the three conditions in Theorem 4.3 are satisfied in the situations (1),(2) and (3). By Proposition 4.9, we have that the second and third conditions are satisfied in our situation. Assume that the block $B$ of $kG$ satisfies the condition (1) and let $S$ be an $I_{\tilde{G}}(B)$-stable simple $B$-module whose corresponding edge is a terminal edge of the Brauer tree of $B$. Then, since there exists a unique simple $B$-module $T$ such that $\text{Ext}^1_B(S, T) \cong k$ and that $\text{Ext}^1_B(S, T') = 0$ for any distinct simple $B$-module $T'$ to $T$, we have that $\text{Ext}^1_B(S, xT) \cong \text{Ext}^1_B(xS, xT) \cong \text{Ext}^1_B(S, T) \cong k$.

Hence, we have $xT \cong T$ as $B$-modules for any $x \in I_{\tilde{G}}(B)$ by the uniqueness of $T$ again. Also, since there exists a unique simple $B$-module $U$ distinct to $S$ such
that Ext$^1_B(T, U) \cong k$ and Ext$^1_B(T, U') = 0$ for any distinct simple $B$-module $U'$ to $U$ and $S$, we have that

$$\text{Ext}^1_B(T, xU) \cong \text{Ext}^1_B(xT, xU) \cong \text{Ext}^1_B(T, U) \cong k,$$

which implies that $xU \cong U$ as $B$-modules for any $x \in I_G(B)$. By repeating this argument, we have that any simple $B$-module is $I_G(B)$-stable. Therefore, we have that any $B$-module is $I_G(B)$-stable by Lemma 4.2 (1). In particular, the block $B$ satisfies the first condition in Theorem 4.3.

Next, assume that the block $B$ of $kG$ satisfies the condition (2). Then a simple $B$-module $S$ whose corresponding edge is a terminal edge of the Brauer tree of $B$ is $I_G(B)$-stable because $xS$ is a simple $B$-module with the same dimension as $S$ for any $x \in I_G(B)$. Therefore, by (1), the block $B$ satisfies the first condition in Theorem 4.3. The statement for (3) follows from that for (2) immediately.

**Corollary 4.14.** Let $G$ be a finite group with a cyclic Sylow $p$-group and $\tilde{G}$ a finite group having $G$ as a normal subgroup. If the quotient group $\tilde{G}/G$ is a cyclic group, then the principal block $B_0(k\tilde{G})$ satisfies the three conditions in Theorem 4.3. Therefore, Main Theorems 1.1 to 1.3 hold for the principal blocks $B_0(kG)$ and $B_0(k\tilde{G})$.

**Proof.** The trivial $kG$-module $k_G$ is $\tilde{G}$-stable. Moreover, the trivial $kG$-module corresponds to the terminal edge in the Brauer tree of the principal block $B_0(kG)$ (for example, see [14, section 1.1]). Hence, it concludes the proof by Proposition 4.13.

**Example 4.15.** Let $G := A_5$ be the alternating group of degree 5, $\tilde{G} := \mathfrak{S}_5$ the symmetric group of degree 5 and $P$ a Sylow 5-subgroup of $G$. Since $P$ is cyclic and the centralizer $C_G(P) = P$ is contained in $G$, the only block covering the principal block $B_0(kG)$ is the principal block $B_0(k\tilde{G})$ of $k\tilde{G}$ by Remark 3.18 and [5, Theorem 4.15.1 (5)]. Also, the number $\#sr$-tilt $B_0(k\tilde{G})$ of support $\tau$-tilting $B_0(k\tilde{G})$-modules is equal to $\binom{4}{2} = 6$ by [6, 8]. So it is difficult to classify support $\tau$-tilting $B_0(k\tilde{G})$-modules. On the other hand, the classification of support $\tau$-tilting $B_0(kG)$-modules is easy because the number $\#sr$-tilt $B_0(kG)$ of support $\tau$-tilting $B_0(kG)$-modules is equals to $\binom{4}{2} = 6$ by [6] or [8] again. Hence, we can easily construct six support $\tau$-tilting $B_0(k\tilde{G})$-modules, six semibricks over $B_0(k\tilde{G})$, six two-term tilting complexes in $K^b(B_0(k\tilde{G})$-$\text{proj})$ and six two-term simple-minded collections in $D^b(B_0(k\tilde{G})$-$\text{mod})$ from $sr$-tilt $B_0(kG)$ by using our main theorems.

**Example 4.16.** Let $k$ be an algebraically closed field of characteristic $p$, $G$ an arbitrary finite group and $H$ a finite group satisfying that $H^2(H, k^*) = 1$ and that $kG_2$ is basic as a $k$-algebra. For example, we can take a $p$-group, a cyclic group or the dihedral group $D_{2p}$ of order $2p$ by Proposition 4.9. Also, it is clear that $M \cong xM$ for any $kG$-module $M$ and for any $x \in G \times H$. Hence, for the direct product group $G \times H$, we can apply our main theorem. The induction functor $\text{Ind}_G^{G \times H}$ induces the injective map $sr$-tilt $kG \rightarrow sr$-tilt $k[G \times H]$.

**Acknowledgements**

The authors are grateful to Naoko Kunugi for giving valuable comments. The first author would like to thank Yuta Katayama for useful advice about the group cohomology of the dihedral groups.
References

[1] T. Adachi. The classification of \(\tau\)-tilting modules over Nakayama algebras. \textit{J. Algebra}, 452:227–262, 2016. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2015.12.013. URL https://doi.org/10.1016/j.jalgebra.2015.12.013.

[2] T. Adachi, O. Iyama, and I. Reiten. \(\tau\)-tilting theory. \textit{Compos. Math.}, 150(3):415–452, 2014. ISSN 0010-437X. doi: 10.1112/S0010437X13007422. URL https://doi.org/10.1112/S0010437X13007422.

[3] T. Adachi, T. Aihara, and A. Chan. Classification of two-term tilting complexes over Brauer graph algebras. \textit{Math. Z.}, 290(1-2):1–36, 2018. ISSN 0025-5874. doi: 10.1007/s00209-017-1906-9. URL https://doi.org/10.1007/s00209-017-1906-9.

[4] T. Aihara and O. Iyama. Silting mutation in triangulated categories. \textit{J. Lond. Math. Soc. (2)}, 85(3):633–668, 2012. ISSN 0024-6107. doi: 10.1112/jlms/jdr055. URL https://doi.org/10.1112/jlms/jdr055.

[5] J. L. Alperin. \textit{Local representation theory}, volume 11 of \textit{Cambridge Studies in Advanced Mathematics}. Cambridge University Press, Cambridge, 1986. ISBN 0-521-30660-4; 0-521-44926-X. doi: 10.1017/CBO9780511623592. URL https://doi.org/10.1017/CBO9780511623592.

[6] T. Aoki. Classifying torsion classes for algebras with radical square zero via sign decomposition, 2019. URL https://arxiv.org/abs/1803.03795v1.

[7] S. Asai. Semibricks. \textit{Int. Math. Res. Not. IMRN}, 2020(16):4993–5054, 2018. ISSN 1073-7928. doi: 10.1093/imrn/rny150. URL https://doi.org/10.1093/imrn/rny150.

[8] H. Asashiba, Y. Mizuno, and K. Nakashima. Simplicial complexes and tilting theory for Brauer tree algebras. \textit{Journal of Algebra}, 551:119 – 153, 2020. ISSN 0021-8693. doi: https://doi.org/10.1016/j.jalgebra.2019.12.020. URL http://www.sciencedirect.com/science/article/pii/S0021869320300284.

[9] I. Assem, D. Simson, and A. Skowroński. \textit{Elements of the representation theory of associative algebras. Vol. 1}, volume 65 of \textit{London Mathematical Society Student Texts}. Cambridge University Press, Cambridge, 2006. ISBN 978-0-521-58423-4; 978-0-521-58631-3; 0-521-58631-3. doi: 10.1017/CBO9780511614309. URL https://doi.org/10.1017/CBO9780511614309. Techniques of representation theory.

[10] M. Auslander and S. O. Smalø. Almost split sequences in subcategories. \textit{J. Algebra}, 69(2):426–454, 1981. ISSN 0021-8693. doi: 10.1016/0021-8693(81)90214-3. URL https://doi.org/10.1016/0021-8693(81)90214-3.

[11] T. Brüstle and D. Yang. Ordered exchange graphs. In \textit{Advances in representation theory of algebras}, EMS Ser. Congr. Rep., pages 135–193. Eur. Math. Soc., Zürich, 2013.
[12] L. Demonet, O. Iyama, and G. Jasso. \(\tau\)-tilting finite algebras, bricks, and \(g\)-vectors. *Int. Math. Res. Not. IMRN*, 2019(3):852–892, 2019. ISSN 1073-7928. doi: 10.1093/imrn/rnx135. URL https://doi.org/10.1093/imrn/rnx135.

[13] D. Handel. On products in the cohomology of the dihedral groups. *Tohoku Math. J. (2)*, 45(1):13–42, 1993. ISSN 0040-8735. doi: 10.2748/tmj/1178225952. URL https://doi.org/10.2748/tmj/1178225952.

[14] G. Hiss and K. Lux. *Brauer trees of sporadic groups*. Oxford: Clarendon Press, 1989. ISBN 0-19-853381-0.

[15] M. Holloway, S. Koshitani, and N. Kunugi. Blocks with nonabelian defect groups which have cyclic subgroups of index \(p\). *Arch. Math. (Basel)*, 94(2):101–116, 2010. ISSN 0003-889X. doi: 10.1007/s00013-009-0075-7. URL https://doi.org/10.1007/s00013-009-0075-7.

[16] B. Keller and D. Vossieck. Aisles in derived categories. *Bull. Soc. Math. Belg. Sér. A*, 40(2):239–253, 1988.

[17] S. Koenig and D. Yang. Silting objects, simple-minded collections, \(t\)-structures and co-\(t\)-structures for finite-dimensional algebras. *Doc. Math.*, 19:403–438, 2014. ISSN 1431-0635.

[18] R. Koshio and Y. Kozakai. On support \(\tau\)-tilting modules over blocks covering cyclic blocks. *J. Algebra*, 580:84–103, 2021. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2021.03.021. URL https://doi.org/10.1016/j.jalgebra.2021.03.021.

[19] F. Marks and J. Šťovíček. Torsion classes, wide subcategories and localisations. *Bull. Lond. Math. Soc.*, 49(3):405–416, 2017. ISSN 0024-6093. doi: 10.1112/blms.12033. URL https://doi.org/10.1112/blms.12033.

[20] H. Nagao and Y. Tsushima. *Representations of finite groups*. Academic Press, Inc., Boston, MA, 1989. ISBN 0-12-513660-9.

[21] J. Rickard. Morita theory for derived categories. *J. London Math. Soc. (2)*, 39(3):436–456, 1989. ISSN 0024-6107. doi: 10.1112/jlms/s2-39.3.436. URL https://doi.org/10.1112/jlms/s2-39.3.436.

[22] J. Rickard. Equivalences of derived categories for symmetric algebras. *J. Algebra*, 257(2):460–481, 2002. ISSN 0021-8693. doi: 10.1016/S0021-8693(02)00520-3. URL https://doi.org/10.1016/S0021-8693(02)00520-3.

[23] R. Rouquier. Block theory via stable and Rickard equivalences. In *Modular representation theory of finite groups (Charlottesville, VA, 1998)*, pages 101–146. de Gruyter, Berlin, 2001.

[24] S. Schroll. Brauer graph algebras: a survey on Brauer graph algebras, associated gentle algebras and their connections to cluster theory. In *Homological methods, representation theory, and cluster algebras*, CRM Short Courses, pages 177–223. Springer, Cham, 2018.
Ryotaro KOSHIO (Corresponding author)
Department of Mathematics, Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo, 162-8601, Japan
E-mail: 1120702@ed.tus.ac.jp

Yuta KOZAKAI
Department of Mathematics, Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo, 162-8601, Japan
E-mail: kozakai@rs.tus.ac.jp