New Bogomolny Equations in Some Submodels of The Generalized Skyrme Model

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Abstract

We use the BPS Lagrangian method, developed in [1] and its extension procedure shown in [2], to derive some known and new Bogomolny equations of submodels in the generalized Skyrme model. We consider the submodels that consist of two terms under a particular ansatz. We are able to reproduce Bogomolny equations of the known BPS submodels. In addition there are four new submodels with non-vanishing BPS Lagrangian density and they share same feature that the BPS Lagrangian densities are not boundary terms. It turns out that only one of our newfound Bogomolny equations has nice non-trivial solutions in the form of compacton.

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I. INTRODUCTION

Skyrme model [3–5] is a type of nonlinear sigma model whose solutions are topological solitons named Skyrmions. This model was proposed for an alternative theory of hadrons and has been considered as a low energy limit of QCD [6–8]. Its static energy has lower bound proportional to a topological degree $B$, which is identified as baryon number. This is known from employing the original Bogomolny method [9, 10]. Unfortunately, the only solutions satisfying the Bogomolny equations are trivial one, $B = 0$, [11]. For a comprehensive review, see [12, 13].

Bogomolny equations has extensively used in some theories especially with topological solitons. First implemented to nonabelian monopoles and dyons, the Bogomolny method [9], by arranging the terms to have the form of squared terms plus some boundary terms, is able to produce Bogomolny equations which are first order and satisfy the exact solutions of monopoles and dyons, or known as BPS monopoles/dyons, found by Prasad and Sommerfield [14]. Total energy of solutions of Bogomolny equations saturate its lowest energy bound, which turn out to be proportional to its topological degree. We sometimes called the Bogomolny equations that has non-trivial solutions as BPS (Bogomolny-Prasad-Sommerfield) equations.

Other than monopoles and dyons, the Nielsen-Olesen magnetic vortices [15] are also known to have BPS equations. There are some generalizations that also has BPS equations, for instance when the form of kinetic term is similar to Dirac-Born-Infeld for vortices [16] and for nonabelian monopole and dyon [17]. Since the Bogomolny method is harder to implement to more general models, there has been many proposals for refinement, for instance the first-order formalism [18, 19], the concept of strong necessary conditions [20–23], the on-shell method [24, 25], and the BPS Lagrangian method [1, 2].

In Skryme model, since the Bogomolny equations from the original one has no nontrivial solution, there has been proposed at least two modified models, i.e. a sextic term in first derivative and a potential [26–28], or known as BPS Skyrme model, and a quartic term in first derivative and a potential [29]. Recently, it has been found that the Skyrme model has two submodels [30]. One of them had been found to contain two subsubmodels [31].

In this paper, we will use the BPS Lagrangian method proposed in [1] to find Bogomolny equations in some submodels of the generalized Skryme model. This method initially had
been used for some models of vortices and it had also been used for nonabelian magnetic monopoles and dyons \[32, 33\]. This paper is organized as follows. In the next section, we revisit the Skyrme model just until the effective Lagrangian from a given ansatz. The method is given in the following section with a subsection for explanations on modifications for this model. Then in the following sections, we implement the method to some known submodels and some new ones.

II. THE SKYRME MODEL REVISITED

The model starts from employing a map \( U \) that maps from \( \mathbb{R}^3 \) to the SU(2) group space. Skyrme has determined the form \( U \) to be

\[
U = \exp(i\xi \hat{n} \cdot \vec{\tau}) = \cos \xi \mathbb{1} + \sin \xi (\hat{n} \cdot \vec{\tau}).
\]

The second equality is due to the real-valued unit vector \( \hat{n} \) and the properties of Pauli matrices \( \vec{\tau} \) (commutation relation \([\tau_a, \tau_b] = 2i\epsilon_{abc}\tau_c\) and trace identity \(T r(\tau_a\tau_b) = 2\delta_{ab}\)). Here \( \xi \) is a real-valued function and \( \mathbb{1} \) is just \( 2 \times 2 \) identity matrix.

It is usual to use the stereographic projection projected from the north pole of \( S^2 \) for the unit vector

\[
\hat{n} = \frac{1}{1 + u \bar{u}} (u + \bar{u}, -i(u - \bar{u}), 1 - u\bar{u}),
\]

where \( u \) is a complex function and \( \bar{u} \) is its complex conjugate. For compactness, it is a common practice to use the following notation

\[
L_\mu = U^\dagger \partial_\mu U,
\]

called the left-invariant one-form whose value lie in the algebra of \( su(2) \). The symmetry property have the known topological degree \( B = \int d^3 x \mathcal{B}^0 \) from the following definition of baryon current in terms of \( \xi, u, \) and \( \bar{u} \) \[28\]

\[
\mathcal{B}^\mu = \frac{1}{24\pi^2 \sqrt{-\det g_{\mu\nu}}} \epsilon^{\mu\nu\rho\sigma} T r(L_\nu L_\rho L_\sigma) = \frac{(-i)}{\pi^2} \frac{\sin^2 \xi}{(1 + u\bar{u})^2} \sqrt{-\det g_{\mu\nu}} \epsilon^{\mu\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma.
\]

The subscript for \( \xi, u, \) and \( \bar{u} \) means derivative with respect to the coordinates, e.g. \( \xi_\mu \equiv \partial \xi / \partial x^\mu \). Throughout this paper, we use metric signature \((+, -, -, -)\). The generalized Skyrme model has the following Lagrangian density

\[
\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_0,
\]

(5)
Here we omit the coupling constants \( f_\pi \) and \( e \) in which they can be reintroduced by rescaling the length and energy units. In this article, we shall use the following ansatz, in spherical coordinates,

\[
\xi = \xi(r), \quad u(\theta, \varphi) = f(\theta) \exp(i g(\varphi)),
\]

in which the metric is given by \( g_{\mu\nu} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta) \). Using this ansatz, we obtain

\[
\mathcal{L}_2 = -\frac{1}{2} g^{\mu\nu} \text{Tr}(L_\mu L_\nu) = \xi_\mu \xi^\mu + \frac{4 \sin^2 \xi}{(1 + u \bar{u})} u_\mu \bar{u}^\mu, \quad (6)
\]

\[
\mathcal{L}_4 = \frac{1}{16} g^{\mu\nu} g^{\rho\sigma} \text{Tr}([L_\mu, L_\nu][L_\rho, L_\sigma]) = -\frac{4 \sin^2 \xi}{(1 + u \bar{u})^2} \left( \xi_\mu \xi^\mu u_\nu \bar{u}^\nu - \xi_\mu \xi^\nu u_\nu \bar{u}^\mu + \frac{\sin^2 \xi}{(1 + u \bar{u})^2} ((u_\nu \bar{u}^\nu)^2 - u_\mu u^\mu \bar{u}^\nu \bar{u}^\nu) \right), \quad (7)
\]

\[
\mathcal{L}_6 = -\lambda^2 \pi^4 g_{\mu\nu} B^\mu B^\nu = \lambda^2 \frac{\sin^4 \xi}{(1 + u \bar{u})^4} g_{\mu\nu} \left( \frac{\epsilon^{\mu\rho\sigma}}{\sqrt{-\det g_{\mu\nu}}} \xi_\rho \xi_\sigma \right) \left( \frac{\epsilon^{\nu\alpha\beta}}{\sqrt{-\det g_{\mu\nu}}} \xi_\alpha \xi_\beta \xi_\gamma \right), \quad (8)
\]

\[
\mathcal{L}_0 = -\mu^2 V(\text{Tr}(U)) = -\mu^2 V(\xi). \quad (9)
\]

Here we omit the coupling constants \( f_\pi \) and \( e \) in which they can be reintroduced by rescaling the length and energy units. In this article, we shall use the following ansatz, in spherical coordinates,

\[
\xi = \xi(r), \quad u(\theta, \varphi) = f(\theta) \exp(i g(\varphi)),
\]

in which the metric is given by \( g_{\mu\nu} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta) \). Using this ansatz, we obtain

\[
\mathcal{B}^0 = -\frac{2}{\pi^2} \frac{f \sin^2 \xi}{(1 + f^2)^2 r^2 \sin \theta} (\xi_r f_\theta g_{\varphi}), \quad \mathcal{B}^m = 0, \quad (11)
\]

\[
\mathcal{L}_2 = \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + \mathcal{L}_2^{(3)},
\]

\[
= -\frac{4 \sin^2 \xi}{(1 + f^2)r^2 f_\theta} - \frac{4 f^2 \sin^2 \xi}{(1 + f^2) r^2 \sin^2 \theta} g_{\varphi}^2 - \xi_r^2, \quad (12)
\]

\[
\mathcal{L}_4 = \mathcal{L}_4^{(1)} + \mathcal{L}_4^{(2)} + \mathcal{L}_4^{(3)},
\]

\[
= -\frac{4 f^2 \sin^2 \xi}{(1 + f^2)^2 r^2 \sin^2 \theta} (\xi_r g_{\varphi})^2 - \frac{4 \sin^2 \xi}{(1 + f^2)^2 r^2} (\xi_r f_\theta)^2 - \frac{16 f^2 \sin^4 \xi}{(1 + f^2)^4 r^4 \sin^2 \theta} (f_\theta g_{\varphi})^2, \quad (13)
\]

\[
\mathcal{L}_6 = -\lambda^2 \frac{4 f^2 \sin^4 \xi}{(1 + f^2)^4 r^4 \sin^2 \theta} (\xi_r f_\theta g_{\varphi})^2, \quad (14)
\]

where \( \xi_r \equiv \frac{\partial \xi}{\partial r}, \ f_\theta \equiv \frac{\partial f}{\partial \theta}, \) and \( g_{\varphi} \equiv \frac{\partial g}{\partial \varphi} \). The boundary conditions \( \xi(r \to 0) = \pi \) and \( \xi(r \to \infty) = 0 \) must be satisfied to ensure that \( U \) is well defined at the origin required by topology and reaches the vacuum \( U = 1 \) near the boundary. For a particular ansatz, \( f = \tan(\theta/2) \) and \( g = n \varphi \), it can be shown that an integer \( n \) is simply the topological degree \( B = \int d^3 x \mathcal{B}^0 \).
III. THE BPS LAGRANGIAN METHOD

In this section we will review the BPS Lagrangian method in [1]. Using static energy 
\[ E = - \int d^3x \mathcal{L} \] of any static system, if the Lagrangian density has Bogomolny equations, one obtains
\[ \mathcal{L} = \text{(Squared terms)} + \mathcal{L}_{BPS}, \tag{15} \]
where BPS Lagrangian density \( \mathcal{L}_{BPS} \) supposedly contains (only) boundary terms. Suppose the Lagrangian density \( \mathcal{L} \) of \( N \) fields \( \phi_1, \ldots, \phi_N \) has at most square of first derivative of the fields, \( \partial \phi_i \ (i = 1, \ldots, N) \), as such the squared terms is taking the following form\(^1\)
\[ \text{(Squared terms)} \propto \sum_{i=1}^{N} (\partial \phi_i - f_i(\phi_1, \ldots, \phi_N; \vec{x}))^2, \tag{16} \]
where \( f_i \) may depend explicitly on coordinates \( \vec{x} \). In the BPS limit,
\[ \partial \phi_i = f_i(\phi_1, \ldots, \phi_N; \vec{x}) \tag{17} \]
and \( \mathcal{L} - \mathcal{L}_{BPS} \to 0 \). These first-order equations are called Bogomolny equations, which satisfy the Euler-Lagrange equations derived from \( E \). Following procedure in BPS Lagrangian method [1], these Bogomolny equations can be obtained by subtracting the effective Lagrangian density with the BPS Lagrangian density and setting it to zero, \( \mathcal{L} - \mathcal{L}_{BPS} = 0 \).

With the squared terms taking the form of (16), we may consider it as a (mutually) quadratic equation of first derivative of the fields, \( \partial \phi_i \) with \( i = 1, \ldots, N \). Solving it and equating the (two) solutions for every \( \partial \phi_i \) give us the Bogomolny equations (17) and some constraint equations.

Being boundary terms, the BPS Lagrangian can be recasted into
\[ \int d^3x \sqrt{\det(g_{mn})} \mathcal{L}_{BPS} = \int d^3x \partial_m J^m \ (m, n = 1, 2, 3), \tag{18} \]
where we have used the Einstein summation index with \( m \) and \( n \) are the spatial indices. With a suitable set of ansatz, we may set \( J^m \) to be explicitly independent of the spatial coordinates, and thus \( \partial_m J^m = (\partial J^m / \partial \phi_i) \partial_m \phi_i \). In many cases, \( \sqrt{\det(g_{mn})} \mathcal{L}_{BPS} \) turns out to depend effectively on one coordinate, e.g. radial coordinate in the spherical coordinates, while dependency over the remaining coordinates appear as numerical factors.

\(^1\) In more general situation \( f_i \) may also depend on first derivative of other fields \( \partial \phi_j \), where \( j \neq i \).
Now we extend the recipe a little more. At first we observe that the BPS Lagrangian density, with previously defined $J^m$, can be rewritten as

$$L_{BPS} = \frac{1}{\sqrt{\det(g_{mn})}} \frac{\partial J^m(\phi_1, \ldots, \phi_N)}{\partial \phi_i} \partial_m \phi_i$$  \hspace{1cm} (19)$$

which is proportional to first power of $\partial \phi_i$. Since the Lagrangian density $L$ contains also square of $\partial \phi_i$, we could ask a question if there are other possible boundary terms that proportional to higher power of $\partial \phi_i$. These possible boundary terms has been studied in [21] and as an example in our case, with $N = 3$, it is given by

$$L_{BPS} = \frac{1}{\sqrt{\det(g_{mn})}} J^{[lmn]}(\phi_1, \phi_2, \phi_3) \partial_l \phi_i \partial_m \phi_j \partial_n \phi_k,$$ \hspace{1cm} (20)$$

where indices inside $[\cdots]$ are totally antisymmetric. In general we can write the BPS Lagrangian density as polynomial function of $\partial \phi_i$ in which the “constants” are functions of $\phi_i$, or also explicitly of coordinates $\vec{x}$. The boundary terms are defined as the BPS Lagrangian density that gives trivial the Euler-Lagrange equations. As an example for the BPS Lagrangian density with only first power of $\partial \phi_i$, we can write

$$L_{BPS} = \frac{1}{\sqrt{\det(g_{mn})}} Q^m_i(\phi_1, \ldots, \phi_N) \partial_m \phi_i.$$ \hspace{1cm} (21)$$

By imposing that the BPS Lagrangian density should produce trivial Euler-Lagrange equations, one can simply show that $Q^m_i = \frac{\partial J^m}{\partial \phi_i}$ with $J^m(\phi_1, \ldots, \phi_N)$. This generalization rises a question if we are allowed to add terms into the BPS Lagrangian density, which are not boundary terms. Adding more terms that are not boundary terms into the BPS Lagrangian density will produce more constraint equations which are basically the Euler-Lagrange equations of these terms. An interesting feature of these non-boundary terms is that the resulting Bogomolny equations can be shown to produce non-zero pressure density, for more details see [2].

A. BPS Lagrangian Density for The Generalized Skyrme Model

We see that the Lagrangian density (5), in spherical coordinates with ansatz (10), contains at most squared of $\xi_r$, $f_\theta$, and $g_\phi$. Notice that the BPS Lagrangian density with boundary terms, as disscussed in the previous section, contains at most first power of $\xi_r$, $f_\theta$, and $g_\phi$. 
Thus general form of the BPS Lagrangian density with boundary terms at least contain the following terms

\[ \mathcal{L}_{\text{BPS}} \propto Q_\xi \xi_r + Q_f f_\theta + Q_g g_\varphi + Q_{\xi f} \xi_r f_\theta + Q_{\xi g} \xi_r g_\varphi + Q_{f g} f_\theta g_\varphi, \tag{22} \]

where all \( Q \)s are functions of \( \xi, f, \) and \( g, \) with subscript indices denote first derivative of the fields appearing in the corresponding term. We will not use this general form of BPS Lagrangian density in deriving Bogomolny equations for each submodels of the generalized Skyrme model because it will take too much workload. Instead we only pick some terms in (22) as the BPS Lagrangian density by two steps. First, for each term in a submodel that contains square of total first derivative of fields (\( \xi_r, f_\theta, \) and \( g_\varphi \)), we can pick a term in (22) that contains first power of the corresponding total first derivative of the fields. Next, we also pick term in (22) that its total derivative of fields equal to combinations of total derivative of fields of terms in the initial BPS Lagrangian density constructed in the first step. The second step can actually include the terms containing quadratic power of first derivative of the fields, which are not listed in the (22). We will show in more detail how to implement these two steps in constructing BPS Lagrangian density of some known submodels of the generalized Skyrme model.

IV. SOME KNOWN SUBMODELS

A. The First BPS Submodel

There are two BPS submodels that have been identified to have Bogomolny equations, with nontrivial solutions, in the Skyrme model [30]. One of the BPS submodels can actually be decomposed into two BPS subsubmodels derived using the concept of strong necessary condition [22]. Here we will show that Bogomolny equations of these BPS subsubmodels can also be derived using the BPS Lagrangian method.

1. The First BPS subsubmodel

The effective Lagrangian density has the form

\[ \mathcal{L} = \mathcal{L}_2^{(1)} + \mathcal{L}_4^{(1)} = -\frac{4 \sin^2(\xi)}{r^2 (1 + f^2)^2} \left( f_\theta^2 + \frac{f^2}{\sin^2(\theta)} \xi_r g_\varphi^2 \right). \tag{23} \]
The corresponding BPS Lagrangian density has this form

\[ \mathcal{L}_{\text{BPS}} = -\frac{Q_f}{r^2 \sin(\theta)} f_\theta - \frac{Q_{\xi g}}{r^2 \sin(\theta)} \xi_r g_\varphi - \frac{Q_{\xi f g}}{r^2 \sin(\theta)} f_\varphi, \]

(24)

First two terms in the BPS Lagrangian density are obtained by the first step as described in previous section, while the second step results in last term of the BPS Lagrangian density.

To find the Bogomolny equations, we first consider \( \mathcal{L} - \mathcal{L}_{\text{BPS}} = 0 \) as quadratic equation of \( f_\theta \) in which its solutions are

\[ f_{\theta \pm} = \frac{1}{8} \csc(\theta) \csc^2(\xi) \left\{ (f^2 + 1)^2 (Q_{\xi f g} \xi_r g_\varphi + Q_f) \pm \csc(\theta) \sqrt{\sin^2(\theta) D_1} \right\}, \]

(25)

with

\[ D_1 = \left( f^2 + 1 \right)^4 (Q_{\xi f g} \xi_r g_\varphi + Q_f)^2 + 16 \xi_r g_\varphi \sin^2(\xi) \left\{ 2f^2 \xi_r g_\varphi (\cos(2\xi) - 1) + (f^2 + 1)^2 Q_{\xi f g} \sin(\theta) \right\} \]

(26)

must be zero for two solutions to be equal. Taking \( D_1 = 0 \) as a quadratic equation of \( \xi_r g_\varphi \), we find two solutions

\[ (\xi_r g_\varphi)_{\pm} = \frac{-(f^2 + 1)^4 Q_f Q_{\xi f g} - 8 (f^2 + 1)^2 Q_{\xi g} \sin(\theta) \sin^2(\xi) \pm 4 \sqrt{(f^2 + 1)^4 \sin^2(\xi) D_2}}{(f^2 + 1)^4 Q_{\xi f g}^2 - 64 f^2 \sin^4(\xi)}, \]

(27)

with

\[ D_2 = \left( -2f^2 Q_{\xi g}^2 \cos(2\xi) + 2f^2 Q_{\xi g}^2 + (f^2 + 1)^2 Q_f Q_{\xi f g} Q_{\xi g} \sin(\theta) + 4 Q_{\xi g}^2 \sin^2(\theta) \sin^2(\xi) \right) \]

(28)

again must be zero for two solutions to be equal. We must find all \( Qs \) that give \( D_2 = 0 \) everywhere; for any values of \( r, \theta, \) and \( \phi \). Notice that there are terms in \( D_2 \) that depend on spatial coordinates explicitly, and so a natural way to find these \( Qs \) are by taking each terms in \( D_2 \) to be zero everywhere. Hence we find the only non-trivial solution is if \( Q_{\xi g} = Q_f = 0 \).

If there is any of the \( Qs \) is zero then it is suggested to repeat the BPS Lagrangian method from beginning after eliminating all the zero \( Qs \) in the BPS Lagrangian density. This is because our previous assumption about the form of BPS Lagrangian density is invalid and this may lead to non-existence of some Bogomolny equations. As an example above inserting \( Q_{\xi g} = Q_f = 0 \) into (27) will give trivial solutions, but this is not correct since the Bogomolny equation (27) does not actually exist because it is not non-trivial solutions of \( D_1 = 0 \), with \( Q_{\xi g} = Q_f = 0 \).
After eliminating terms containing $Q_{\xi g}$ and $Q_f$ in the BPS Lagrangian density, we set again $\mathcal{L} - \mathcal{L}_{\text{BPS}} = 0$. From it, we find two solutions for $(\xi_r g_\varphi)$ as follows

$$(\xi_r g_\varphi)^\pm = \frac{\csc^2(\xi)}{8 f^2} \left( (f^2 + 1)^2 (f_\theta) Q_{\xi f g} \sin(\theta) \pm \sqrt{(f_\theta)^2 \sin^2(\theta) D_1} \right),$$  \hspace{1cm} (29)

where

$$D_1 = (f^8 Q_{\xi f g}^2 + 4 f^6 Q_{\xi f g}^2 + 6 f^4 Q_{\xi f g}^2 + 32 f^2 \cos(2 \xi) - 8 f^2 \cos(4 \xi) + 4 f^2 Q_{\xi f g}^2 - 24 f^2 + Q_{\xi f g}^2).$$  \hspace{1cm} (30)

Two solutions will be equal if $D_1 = 0$ which then give us

$$Q_{\xi f g} = \pm \frac{8 f \sin^2(\xi)}{(f^2 + 1)^2}. \hspace{1cm} (31)$$

Since $D_1$ do not contain $f_\theta$, we left $f_\theta$ undetermined. Substituting this into the solutions of $\xi_r g_\varphi$, we obtain the Bogomolny equation

$$\xi_r g_\varphi = \pm \frac{f_\theta \sin(\theta)}{f}. \hspace{1cm} (32)$$

It is easy to prove that the BPS Lagrangian density is a boundary term and so the Bogomolny equation satisfies the Euler-Lagrange equations.

2. The Second BPS subsubmodel

The effective Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}_2^{(2)} + \mathcal{L}_4^{(2)} = -\frac{4 \sin^2(\xi)}{(f^2 + 1)^2} \left( \frac{f^2}{r^2 \sin^2(\theta)} g_\varphi^2 + \frac{(\xi_r f_\theta)^2}{r^2} \right).$$  \hspace{1cm} (33)

The corresponding BPS Lagrangian density for this subsubmodel is

$$\mathcal{L}_{\text{BPS}} = -\frac{Q_g}{r^2 \sin(\theta)} g_\varphi - \frac{Q_{\xi f}}{r^2 \sin(\theta)} \xi_r f_\theta - \frac{Q_{\xi f g}}{r^2 \sin(\theta)} \xi_r f_\theta g_\varphi.$$$  \hspace{1cm} (34)

Solving $\mathcal{L} - \mathcal{L}_{\text{BPS}} = 0$, two solutions for $g_\varphi$ are

$$g_\varphi^\pm = \frac{(f^2 + 1)^2 \sin(\theta) (Q_g + Q_{\xi f g} \xi_r f_\theta) \pm \sqrt{\sin(\theta) D_1}}{8 f^2},$$  \hspace{1cm} (35)

where

$$D_1 = (f^2 + 1)^4 \sin(\theta) (Q_g + Q_{\xi f g} \xi_r f_\theta)^2$$

$$+ 16 \sin^2(\xi) \left( (f^3 + f)^2 Q_{\xi f} - 4 f^2 \sin(\theta) \sin^2(\xi) \xi_r f_\theta \right) \xi_r f_\theta.$$  \hspace{1cm} (36)
Setting $D_1 = 0$, we obtain two solutions for $\xi_r f_\theta$,

$$
(\xi_r f_\theta)_\pm = -\frac{(f^2 + 1)^4 Q_g Q_{\xi f g} - 4 \csc(\theta) \left( 2 f^2 (f^2 + 1)^2 Q_{\xi f} \sin^2(\xi) \pm f (f^2 + 1)^2 \sin(\xi) \sqrt{D_2} \right)}{(f^2 + 1)^4 Q_{\xi f g}^2 - 64 f^2 \sin^4(\xi)},
$$

(37)

with

$$
D_2 = \left( (f^2 + 1)^2 Q_g Q_{\xi f} Q_{\xi f g} \sin(\theta) + 4 f^2 Q_{\xi f}^2 \sin^2(\xi) + 4 Q_g^2 \sin^2(\theta) \sin^2(\xi) \right).
$$

(38)

Similarly as in the case of previous subsubmodel, the only non trivial solution for $D_2 = 0$ is if $Q_{\xi f} = Q_g = 0$.

Again after removing the zero $Q$s in the BPS Lagrangian density and setting again $\mathcal{L} - \mathcal{L}_{BPS} = 0$, we find two solutions for $g_\varphi$ as follows

$$
g_\varphi = \frac{1}{8 f^2} \csc^2(\xi) \sin(\theta) \xi_r f_\theta \left( (f^2 + 1)^2 Q_{\xi f g} \pm \sqrt{D_1} \right),
$$

(39)

with

$$
D_1 = \left( f^8 Q_{\xi f g}^2 + 4 f^6 Q_{\xi f g}^2 + 6 f^4 Q_{\xi f g}^2 + 32 f^2 \cos(2\xi) - 8 f^2 \cos(4\xi) + 4 f^2 Q_{\xi f g}^2 - 24 f^2 + Q_{\xi f g}^2 \right).
$$

(40)

With only $Q_{\xi f g}$ in $D_1$, the solution to $D_1 = 0$ is equal to (31). This then imply the Bogomolny equation

$$
g_\varphi = \pm \frac{\xi_r f_\theta \sin \theta}{f}.
$$

(41)

As in the previous subsubmodel, the BPS Lagrangian density is a boundary term and so this Bogomolny equation satisfies the Euler-Lagrange equations.

The Bogomolny equations of (32) and (41) imply that solutions for $g(\varphi)$ and $\xi(r)$ are linear functions of $\varphi$ and $r$, respectively. With the right boundary condition for $f$,

$$
f(\theta = 0) = 0, \quad f(\theta = \pi) \to \infty,
$$

(42)

solutions for $f$ are

$$
f(\theta) \propto \left( \tan \left( \frac{\theta}{2} \right) \right)^{C_{g\xi}},
$$

(43)

where $C_{g\xi} > 0$ is a constant related to constant slopes in the linear solutions of $g(\varphi)$ and $\xi(r)$. Explicit solutions were given in [31] and they are known as compacton [30].
B. The Second BPS Submodel

Although the second BPS submodel in [30] was not derived using the concept of strong necessary condition, here we would like to show that its Bogomolny equation can also be derived using the BPS Lagrangian method. The effective Lagrangian density of the second BPS submodel is

\[ L = L^{(3)}_2 + L^{(3)}_4 = -\xi^2 - \frac{16f^2\sin^4 \xi}{(1 + f^2)^4 r^4 \sin^2 \theta} (f_\theta g_\varphi)^2, \]  

and the corresponding BPS Lagrangian density is

\[ L_{\text{BPS}} = -\frac{Q_\xi}{r^2 \sin \theta} \xi r - \frac{Q_{f_\theta g_\varphi}}{r^2 \sin \theta} f_\theta g_\varphi - \frac{Q_{\xi f_\theta g_\varphi}}{r^2 \sin \theta} f_\theta g_\varphi. \]  

Equating both, we obtain two solutions for \( \xi_r \),

\[ \xi_{r\pm} = \frac{\csc(\theta)}{2 (f^2 + 1)^2 r^2} \left( (f^2 + 1)^2 (Q_\xi + Q_{\xi f_\theta g_\varphi}) \pm \csc(\theta) \sin(\theta) \sqrt{D_1} \right) \]  

with

\[ D_1 = (f^2 + 1)^4 (f_\theta g_\varphi)^2 Q_{\xi f_\theta g_\varphi} + Q_\xi^2 + 4(f_\theta g_\varphi) \left( (f^2 + 1)^4 Q_{f_\theta g_\varphi} r^2 \sin(\theta) - 2f^2(f_\theta g_\varphi)(\cos(4\xi) - 4 \cos(2\xi)) \right) + 2 (f^2 + 1)^4 (f_\theta g_\varphi) Q_\xi Q_{\xi f_\theta g_\varphi} + f^2 \left( (f^6 + 4f^4 + 6f^2 + 4) Q_\xi^2 - 24(f_\theta g_\varphi)^2 \right). \]  

From setting \( D_1 = 0 \), in order for the two solutions to be equal, we then obtain two solutions for \( f_\theta g_\varphi \),

\[ (f_\theta g_\varphi)_{\pm} = \frac{-2 (f^2 + 1)^4 Q_{f_\theta g_\varphi} r^2 \sin(\theta) - (f^2 + 1)^4 Q_\xi Q_{\xi f_\theta g_\varphi} \pm 2\sqrt{D_2}}{-8f^2(\cos(4\xi) - 4 \cos(2\xi)) + (f^2 + 1)^4 Q_{\xi f_\theta g_\varphi}^2 - 24f^2}, \]  

with

\[ D_2 = (f^2 + 1)^8 Q_{f_\theta g_\varphi}^2 r^2 \sin^2(\theta) + (f^2 + 1)^8 Q_{f_\theta g_\varphi} Q_\xi Q_{\xi f_\theta g_\varphi} r^2 \sin(\theta) + 16f^2 (f^2 + 1)^4 Q_\xi^2 \sin^4(\xi). \]  

Then we set \( D_2 = 0 \) to make the two solutions to be equal. We have to solve \( D_2 = 0 \) for whole value of spherical coordinates. Expanding it first as a series with respect to \( r \) and then solving it for each term in the series. The terms with \( r^0 \) and \( r^1 \) give \( Q_\xi = 0 \) and \( Q_{f_\theta g_\varphi} = 0 \), respectively, leaving \( Q_{\xi f_\theta g_\varphi} \) undetermined, which are enough to solve it.

Repeating again the BPS Lagrangian method, with only \( Q_{\xi f_\theta g_\varphi} \) is nonzero in the BPS Lagrangian density, then we get two solutions for \( \xi_r \),

\[ \xi_{r\pm} = \frac{\csc(\theta)}{2r^2 (f^2 + 1)^2} f_\theta g_\varphi \left( Q_{\xi f_\theta g_\varphi} (f^2 + 1)^2 \pm \csc(\theta) \sin(\theta) \sqrt{D_1} \right), \]  

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with
\[ D_1 = \left(-8f^2(\cos(4\xi) - 4\cos(2\xi)) + (f^2 + 1)^4 Q_{\xi fg}^2 - 24f^2\right). \] (51)

This two solutions will be equal if \( D_1 = 0 \) which is solved by the same \( Q_{\xi fg} \) given in (31).

Substituting (31) into \( \xi_r \) we obtain the Bogomolny equation
\[ \xi_r = \pm 4f \csc(\theta) \sin^2(\xi) \frac{Q_{\xi fg}}{r^2 (f^2 + 1)^2} f_\theta g_\phi. \] (52)

We also check that this Bogomolny equation indeed satisfies the Euler-Lagrange equations and the solutions have been discussed in [30].

C. The BPS Skyrme model

The Lagrangian density has this form
\[ \mathcal{L} = \mathcal{L}_6 + \mathcal{L}_0 = -\lambda^2 \frac{4f^2 \sin^4(\xi)}{(1 + f^2)^4} \sin^2(\theta) (\xi_r f_\theta g_\phi)^2 - \mu^2 V(\xi). \] (53)

We can see that the only first derivative field term involved is \((\xi_r f_\theta g_\phi)\) hence we use
\[ \mathcal{L}_{BPS} = -\frac{Q_{\xi fg}}{r^2 \sin(\theta)} (\xi_r f_\theta g_\phi). \] (54)

So solutions of \((\xi_r f_\theta g_\phi)\) from \( \mathcal{L} - \mathcal{L}_{BPS} = 0 \) are
\[ (\xi_r f_\theta g_\phi)_\pm = \frac{\csc^4(\xi)}{8f^2 \lambda^2} (f^2 + 1)^2 r^2 \sin(\theta) \left((f^2 + 1)^2 Q_{\xi fg} \pm \sqrt{D}\right), \] (55)

with
\[ D = f^8 Q_{\xi fg}^2 + 4f^6 Q_{\xi fg}^2 + 6f^4 Q_{\xi fg}^2 + 4f^2 Q_{\xi fg}^2 + 8f^2 \lambda^2 f_\mu^2 V \cos(2\xi) - 2f^2 \lambda^2 f_\mu^2 V \cos(4\xi) - 6f^2 \lambda^2 f_\mu^2 V + Q_{\xi fg}^2. \] (56)

The solutions will be equal if \( D = 0 \), which then gives us
\[ Q_{\xi fg} = \pm \frac{4f \lambda \mu \sqrt{V} \sin^2(\xi)}{(f^2 + 1)^2}. \] (57)

So we obtain the Bogomolny equation
\[ (\xi_r f_\theta g_\phi) = \pm \frac{(f^2 + 1)^2 \mu r^2 \sqrt{V} \sin(\theta) \csc^2(\xi)}{2f \lambda}. \] (58)

This is in agreement with the result in [26] when we substitute \( g = n\phi \) and \( f = \tan(\theta/2) \).
V. NEW SUBMODELS WITH TWO TERMS

In this section we will show that using the ansatz (10) we could find other Bogomolny equations in other submodels of the generalized Skyrme model. The submodels that we consider here are combinations of two terms of the generalized Skyrme model and what we present below are all possible submodels, other than the ones discussed in the previous section, that possess Bogomolny equations.

A. First submodel

The effective Lagrangian density is

\[ L = L^{(1)}_4 + L^{(2)}_4 = -\frac{4 \sin^2(\xi)}{(f^2 + 1)^2} \left( \frac{(\xi_r f_\theta)^2}{r^2} + \frac{f^2 (\xi_r g_\phi)^2}{r^2 \sin^2(\theta)} \right). \]  

For this submodel, the BPS Lagrangian density has this form

\[ L_{\text{BPS}} = -\frac{Q_{\xi f}}{r^2 \sin(\theta)} \xi_r f_\theta g_\phi - \frac{Q_{\xi g}}{r^2 \sin(\theta)} \xi_r g_\phi - \frac{Q_{\xi \xi f g}}{r^2 \sin(\theta)} \xi_r f_\theta g_\phi. \]

It follows that we obtain the following Bogomolny equations

\[
(\xi_r g_\phi) = \frac{\csc^2(\xi)}{8f^2} (f^2 + 1)^2 \sin(\theta) (Q_{\xi g} + (\xi_r f_\theta) Q_{\xi \xi f g}), \\
(\xi_r f_\theta) = \frac{8 \csc(\theta) (f^3 + f)^2 Q_{\xi f} \sin^2(\xi) + (f^2 + 1)^4 Q_{\xi g} Q_{\xi \xi f g}}{-(f^2 + 1)^4 Q_{\xi \xi f g} + 64f^2 \sin^4(\xi)},
\]

with a constraint equation

\[ f^2 (f^2 + 1)^4 \sin^2(\xi) \left( 4 \sin^2(\xi) (f^2 Q_{\xi f}^2 + Q_{\xi g}^2 \sin^2(\theta)) + (f^2 + 1)^2 Q_{\xi f} Q_{\xi g} Q_{\xi \xi f g} \sin(\theta) \right) = 0. \]

The only non-trivial solution for this constraint equation, that is valid in the whole space, is \( Q_{\xi f} = Q_{\xi g} = 0 \).

Again repeating the derivation by setting \( Q_{\xi f} = Q_{\xi g} = 0 \) in the BPS Lagrangian density above, we obtain two solutions

\[
(\xi_r g_\phi)_\pm = \frac{\xi_r f_\theta \sin(\theta) \csc^2(\xi)}{8f^2} \left( (f^2 + 1)^2 Q_{\xi \xi f g} \pm \sqrt{D_{\xi g}} \right),
\]

with

\[ D_{\xi g} = \left( -8f^2 (\cos(4\xi) - 4 \cos(2\xi)) + (f^2 + 1)^4 Q_{\xi \xi f g}^2 - 24f^2 \right). \]
Solution for $D_{\xi\theta} = 0$ is given by

$$Q_{\xi f\varphi} = \pm \frac{8 f \sin^2(\xi)}{(f^2 + 1)^2},$$  

(66)

which then implies Bogomolny equation

$$g_{\varphi} = \pm \frac{f_{\theta} \sin(\theta)}{f}.$$  

(67)

Unlike previous submodels, here Euler-Lagrange equations of the BPS Lagrangian density are not all trivial. It turns out the Euler-Lagrange equation for $\xi$ is non-trivial, which then becomes a constraint equation,

$$\xi_{r r} \sin(\xi) + \xi_r^2 \cos(\xi) = 0.$$  

(68)

The Bogomolny equation will satisfy the Euler-Lagrange equations derived from the effective Lagrangian if $\xi$ satisfies (68), which has solutions

$$\xi(r) = \pm \cos^{-1} \left( c_1 (r) - c_2 c_1 \right),$$  

(69)

where $c_1$ and $c_2$ are constants of integration. Since $\xi$ is a real valued function, the solutions above are valid within a range of $| (c_1 (r) - c_2 c_1) | \leq 1$. Considering boundary conditions for $\xi$, possible solutions for $\xi$ are

$$\xi = \begin{cases} 
\cos^{-1} \left( \frac{r}{R} - 1 \right), & 0 \leq r \leq 2R \\
0, & r > 2R 
\end{cases},$$  

(70)

with $R > 0$ is an arbitrary constant related to size of the soliton. These solutions are compacton similar to the one obtained in [30], but here size of the compacton is not fixed and it depends on the constant $R$.

**B. Second submodel**

The effective Lagrangian density here contains only

$$\mathcal{L} = \mathcal{L}^{(1)}_2 + \mathcal{L}^{(2)}_2 = -\frac{4 \sin^2(\xi)}{(f^2 + 1)^2} \left( \frac{f^2 g^2_{\varphi}}{r^2 \sin^2(\theta)} + \frac{f_{\varphi}^2}{r^2} \right).$$  

(71)

Here we use the following BPS Lagrangian density

$$\mathcal{L}_\text{BPS} = -\frac{Q_f}{r^2 \sin(\theta)} f_{\theta} - \frac{Q_g}{r^2 \sin(\theta)} g_{\varphi} - \frac{Q_{fg}}{r^2 \sin(\theta)} f_{\theta} g_{\varphi}.$$  

(72)
Similarly, as in the previous cases, we conclude that \( Q_g = Q_f = 0 \). The equation \( \mathcal{L} - \mathcal{L}_{\text{BPS}} = 0 \), with \( Q_g = Q_f = 0 \), has two solutions

\[
f_{\theta \pm} = \frac{g^2}{8} \csc(\theta) \csc^2(\xi) \left( (f^2 + 1)^2 Q_{fg} \pm \csc(\theta) \sin(\theta) \sqrt{D_1} \right),
\]

with

\[
D_1 = \left( f^8 Q_{fg}^2 + 4 f^6 Q_{fg}^2 + 6 f^4 Q_{fg}^2 + 32 f^2 \cos(2\xi) - 8 f^2 \cos(4\xi) + 4 f^2 Q_{fg}^2 - 24 f^2 + Q_{fg}^2 \right).
\]

Setting \( D_1 = 0 \) gives us

\[
Q_{fg} = \pm \frac{8 f \sin^2(\xi)}{(f^2 + 1)^2}.
\]

Substituting this \( Q_{fg} \) into the solutions, we obtain Bogomolny equation

\[
f_{\theta} = \pm f \csc(\theta) g_{\varphi}.
\]

Again Euler-Lagrange equations of the BPS Lagrangian density are not all trivial, and so implies a constraint equation \( \sin(2\xi) = 0 \). Solutions to the constraint equation are constant values of \( \xi \),

\[
\xi = n \frac{\pi}{2},
\]

with \( n \) is an integer. Therefore it is identified as vacuum solution of the Skyrme model, with \( n = 0 \).

### C. Third submodel

Here the effective Lagrangian density is

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{(3)}^0 = -\mu^2 V - \frac{16 f^2 \sin^4(\xi)}{(1 + f^2)^4 r^4 \sin^2(\theta)} (f_{\theta} g_{\varphi})^2.
\]

The corresponding BPS Lagrangian density turns out to be

\[
\mathcal{L}_{\text{BPS}} = -\frac{Q_{fg}}{r^2 \sin(\theta)} f_{\theta} g_{\varphi},
\]

where

\[
Q_{fg} = \pm \frac{8 f \mu \sqrt{V} \sin^2(\xi)}{(f^2 + 1)^2}.
\]

The resulting Bogomolny equation is then

\[
f_{\theta} g_{\varphi} = \pm \frac{(f^2 + 1)^2 \mu^2 \sqrt{V} \sin(\theta) \csc^2(\xi)}{4f}.
\]
and with additional constraint equation
\[
\frac{\partial (\sin^2(\xi) \sqrt{V})}{\partial \xi} = 0. \tag{82}
\]

Solution to this constraint equation is
\[
V = (c_0)^2 \csc^4 \xi, \tag{83}
\]
where \(c_0\) is a constant of integration. With nonzero potential, \(c_0 \neq 0\), the vacuum is different from the standard Skyrme potential. Here the vacuum is at constant value of \(\xi = \pi/2\) in which \(\frac{\partial V}{\partial \xi} = 0\). Therefore the boundary conditions for \(\xi\) now becomes \(\xi(r = 0) = \pi\) at the origin and \(\xi(r \to \infty) = \pi/2\) near the boundary.

From the Bogomolny equation above, we can infer solution for \(\xi\) is
\[
\xi = \sin^{-1}\left(\frac{c_0 r^2}{c_1}\right)^{1/4}, \tag{84}
\]
where \(c_1\) is a nonzero constant. Unfortunately this solution of \(\xi\) is only well defined within the range of \(0 \leq r \leq \left|\frac{c_1}{c_0}\right|^{1/2}\). One could widen the range by taking \(c_0 \ll \) and \(c_1 \gg \) which correspond to very small potential and very large winding number, respectively. However it is still problematic since outside this range \(\xi\) takes a complex value which is not allowed.

**D. Fourth submodel**

Here the effective Lagrangian density is
\[
\mathcal{L} = \mathcal{L}_2^{(3)} + \mathcal{L}_0 = -\xi_r^2 - \lambda^2 \frac{4 f^2 \sin^4 \xi}{(1 + f^2)^4 r^4 \sin^2 \theta} (\xi_r f_{\gamma \varphi})^2. \tag{85}
\]

The resulting BPS Lagrangian has this form
\[
\mathcal{L}_{\text{BPS}} = -\frac{Q_{\xi f g}}{r^2 \sin(\theta)} \xi_r^2 f_{\theta \varphi}, \tag{86}
\]
with
\[
Q_{\xi f g} = \pm \frac{4 f \lambda \sin^2(\xi)}{(f^2 + 1)^2}, \tag{87}
\]
which imply the Bogomolny equation
\[
2 f \lambda \csc(\theta) \sin^2(\xi) f_{\theta \varphi} = \pm (f^2 + 1)^2 r^2. \tag{88}
\]
There is a constraint equation which is equal to (68), which then could lead to the compacton solution (70). However, unlike case of the (new) first submodel, the Bogomolny equation here contains $\xi$ and coordinate $r$ and so implies solution for $\xi$,

$$\xi = \pm \sin^{-1}(\sqrt{c_1}r) + \pi c_2$$  \hspace{1cm} (89)

where $c_1$ a positive-valued constant and $c_2$ is an integer. Unfortunately, this solution satisfies the constraint equation (68) only if $c_1 = 0$. Here $\xi$ is a constant and thus indentified, with $c_2 = 0$, as vacuum solution of the Skyrme model.

VI. DISCUSSION

We have shown how to use the BPS Lagrangian method to obtain Bogomolny equations of submodels in the generalized Skyrme model for particular ansatz (10) in spherical coordinates. We limit our self to submodels that consist of two terms in the generalized Skyrme model with the aforementioned ansatz and able to reproduce the Bogomolny equations of two BPS subsubmodels, fractions of the first BPS submodel, that were consider in [31] using the concept of strong necessary conditions, and also other BPS submodels such as the Second BPS submodel and the BPS Skyrme model. Those BPS (sub)models have one common feature in the BPS Lagrangian method that their BPS Lagrangian densities are boundary terms by means that Euler-Lagrange equations of the BPS Lagrangian densities are trivial. On the other hand, we also employed the BPS Lagrangian method to other possible BPS submodels, that consist of two terms under the ansatz (10), in the generalized Skyrme model. The resulting (new) submodels have non-zero BPS Lagrangian density that are not boundary terms, and hence implied additional constraint equations. We found only one of the four new submodels that has nice non-trivial solutions. It is possible to consider submodels with more than two terms. However, the calculations will be more complicated since the BPS Lagrangian density involved in the process is much larger than in the previous submodels with two terms.

Here we have used a particular ansatz (10), similar to the natural (hedgehog) ansatz in [26], in deriving the Bogomolny equations of the submodels in the generalized Skyrme model. We can try to use a different ansatz and perhaps obtain different possible BPS submodels, other than the ones obtained here, using the BPS Lagrangian method. We can
also loosen up the requirement for terms in the BPS Lagrangian density such that the \( Q_s \) could depend on the spatial coordinates explicitly. However, this will make the analysis to become more complicated.

The BPS Lagrangian method opens many possibilities in deriving Bogomolny equations. It allows terms to be included in the BPS Lagrangian density that are not boundary terms as shown in the new submodels, while the concept of strong necessary conditions, or the FOEL method, only allows adding the boundary terms in the Lagrangian. So far the BPS Lagrangian method works in the effective Lagrangian description and it would be nice to have this method to works in more general description, without a priori imposing an ansatz, such that we could connect the resulting Bogomolny equations with the BPS equations in the context of supersymmetric theory.

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