OPTIMAL POLYNOMIAL DECAY OF FUNCTIONS AND OPERATOR SEMIGROUPS

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Abstract. We characterize the polynomial decay of orbits of Hilbert space $C_0$-semigroups in resolvent terms. We also show that results of the same type for general Banach space semigroups and functions obtained recently in [6] are sharp. This settles a conjecture posed in [6].

1. Introduction

One of the main issues in the theory of partial differential equations is to determine whether the solutions to these equations approach an equilibrium and if yes then how fast do the solutions approach it. Recently, an essential progress was achieved in treating such asymptotic problems by operator-theoretical methods involving $C_0$-semigroups. For different accounts of developments, highlights, and techniques of asymptotic theory of $C_0$-semigroups see [2], [5], [10], and [24].

In particular, the following result was proved in [4, p. 803], see also [5, p.41-42]. (The result is implicitly contained already in [1].)

Theorem 1.1. Let $(T(t))_{t \geq 0}$ be a bounded $C_0$-semigroup on a Banach space $X$ with generator $A$. Suppose that $i\mathbb{R}$ is contained in the resolvent set $\varrho(A)$ of $A$. Then

\begin{equation}
\|T(t)A^{-1}\| \to 0, \quad t \to \infty.
\end{equation}

In other words, all classical solutions to the abstract Cauchy problem

\begin{equation}
\begin{cases}
\dot{x}(t) = Ax(t), & t \geq 0, \\
x(0) = x_0, & x_0 \in X,
\end{cases}
\end{equation}

given by $x(t) = T(t)x_0$, $t \geq 0$, $x_0 \in D(A)$, converge uniformly (on the unit ball of $D(A)$ with the graph norm) to zero at infinity if $A$ satisfies the conditions of Theorem 1.1.

2000 Mathematics Subject Classification. Primary 47D06; Secondary 34D05, 46B20.

Key words and phrases. bounded $C_0$-semigroup, orbit, resolvent, rate of decay, Cauchy transform, Laplace transform.

The authors were partially supported by the Marie Curie "Transfer of Knowledge" programme, project "TODEQ". The first author was also partially supported by the ANR project DYNOP. The second author was also partially supported by a MNiSzW grant Nr. N201384834.
In general, without any additional assumptions, the decay in (1.1) can be arbitrarily slow. However, in a number of special situations involving concrete PDE’s, e.g. damped wave equations, the rate of decay in (1.1) corresponds to the rate of decay of the energy of the system described by \( (T(t))_{t \geq 0} \), and it is of interest to determine whether this rate of decay can be achieved.

By rewriting equations in the abstract form (1.2), the rates of the decay of sufficiently smooth orbits for the corresponding semigroup \((T(t))_{t \geq 0}\) (and equivalently of solutions to (1.2)) can be associated with the size of the resolvent \( R(\lambda, A) = (\lambda - A)^{-1} \) of \( A \) on the imaginary axis. This approach was initiated in [19] and later pursued, in particular, in [7], [8], [13], [20]. However, with a few exceptions, the issue of optimality or (non-optimality) of the rates of growth has not been studied so far.

The above applications (mainly in the abstract set-up) motivated a thorough study of the decay rates on \( \|T(t)A^{-1}\| \) for bounded \( C_0 \)-semigroups \((T(t))_{t \geq 0}\) on Banach spaces in [3], [21], and, most recently, in [6]. In the latter paper, the authors developed a unified and simplified approach for estimating the decay rates for \( \|T(t)A^{-1}\| \) in terms of the growth of \( R(is, A) \), \( s \in \mathbb{R} \), using the contour integrals technique by Newman–Korevaar. In particular, the following theorem is proved there.

For \( A \) as in Theorem 1.1 we define a continuous non-decreasing function

\[
M(\eta) = \max_{t \in [-\eta, \eta]} \|R(it, A)\|, \quad \eta \geq 0,
\]

and the associated function

\[
M^{\log}(\eta) := M(\eta)(\log(1 + M(\eta)) + \log(1 + \eta)), \quad \eta \geq 0.
\]

Let \( M^{\log}_{-1} \) be the inverse of \( M^{\log} \) defined on \( [M^{\log}(0), +\infty) \).

**Theorem 1.2** (Batty, Duyckaerts). Let \((T(t))_{t \geq 0}\) be a bounded \( C_0 \)-semigroup on a Banach space \( X \) with generator \( A \), such that \( i\mathbb{R} \subset \rho(A) \). Let the functions \( M \) and \( M^{\log} \) be defined by (1.3) and (1.4). Then there exist \( C, B > 0 \) such that

\[
\|T(t)A^{-1}\| \leq \frac{C}{M^{\log}_{-1}(t/C)}, \quad t \geq B.
\]

Note that in the case \( \alpha > 0 \), \( M(\eta) \leq C(1 + \eta^\alpha) \), \( \eta \geq 0 \), the Batty-Duyckaerts result gives

\[
\|T(t)A^{-1}\| \leq C\left(\frac{\log t}{t}\right)^{\frac{1}{\alpha}}, \quad t \geq B.
\]

It was conjectured in [6] that Theorem 1.2 can be improved by removing the logarithmic factor in (1.6) in the case when \( X \) is a Hilbert space, and that this factor is necessary if \( X \) is merely a Banach space, see [6, Remark 9]. (See also the introduction in [3] and the comments after Theorem 3.5 therein.)
In our paper, we address the problem of optimality of the rate of decay in (1.3) and confirm the conjecture from [6] for the case of polynomially growing $M$. We show that the logarithmic factor can be dropped in (1.3) if $X$ is a Hilbert space. Thus, various results on polynomial decay of solutions of PDE’s, e.g. in [3], [8], [21], [22] can be improved to sharp formulations or shown to be sharp. See also [6] and references therein.

On the other hand, we show that Theorem 1.2 is essentially sharp (see Theorems 3.1 and 4.1 below). This is done by a function-theoretical construction which may be interesting for its own sake and may be useful in other instances related to $C_0$-semigroups as well. We prove, in particular, that given $\alpha > 0$ there exists a Banach space $X_\alpha$ and a bounded $C_0$-semigroup $(T(t))_{t\geq 0}$ in $X_\alpha$ with generator $A$ such that

$$
\|R(is,A)\| = O(|s|^\alpha), \quad |s| \to \infty,
$$

and

$$
\limsup_{t\to\infty} \left( \frac{t}{\log t} \right)^{1/\alpha} \|T(t)A^{-1}\| > 0.
$$

The classical problem of estimating the local energy for solutions to wave equations leads to the study of decay rates for functions of the form $\|T_1 T(t)T_2\|$, where $T_1, T_2$ are bounded operators on $X$, so that the assumptions are imposed on the cut-off resolvent $F(\lambda) = T_1 R(\lambda,A)T_2$ rather than on the resolvent itself, see e.g. [6], [7], [13], [27]. Due to the lack of the Neumann series expansions for $F$ we have to assume that $F$ extends analytically to the region $\Omega$ of known shape and satisfies certain growth restrictions there. The domain $\Omega$ and the growth of $F$ in $\Omega$ are not in general related to each other. Moreover, the operator-theoretical approach can hardly be used to deal with $F$ since the resolvent identity is not available as well. Thus, it is natural to put the problem into the framework of decay estimates for $L^\infty(\mathbb{R}_+,X)$ functions given that their Laplace transforms extend to the specific $\Omega$ with, in our case, polynomial estimates. In this direction, we obtain a result on the polynomial rates of decay for bounded functions which partially generalizes [6, Theorem 10]. The result has its version for the rates of decay of $\|T_1 T(t)T_2\|$ thus improving [6, Corollary 11].

We use standard notations. Given a closed linear operator $A$ we denote by $\sigma(A)$, $\rho(A)$, $D(A)$, and $\text{Im}(A)$ the spectrum of $A$, the resolvent set of $A$, the domain of $A$ and the image of $A$ respectively. By $C$, $C_1$ etc. we denote generic constants which may change from line to line.

The plan of the paper is as follows. In the section 2 we characterize the rate of polynomial decay of the semigroup orbits in the Hilbert space via the growth rate of the resolvent on the imaginary axis. Examples of functions constructed in the section 3 show that the version of Theorem 1.2 for functions given in [6] is sharp. These examples are used in the section 4 to show that Theorem 1.2 is itself sharp.

**Acknowledgment** The authors are grateful to the referees for useful comments and suggestions which led to an improvement of the paper.
2. Decay of Hilbert space semigroups

We start with recalling two simple and essentially known statements on \(C_0\)-semigroups. The first one is a version of the well-known criterion on the generators of bounded Hilbert space \(C_0\)-semigroups, \([14], [25]\). In our case, the proof is particularly easy and, to make our presentation self-contained, we give an easy argument. Let \(\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Re } z > 0\}\).

**Lemma 2.1.** Let \((T(t))_{t \geq 0}\) be a \(C_0\)-semigroup on a Hilbert space \(H\) with generator \(A\). Then \((T(t))_{t \geq 0}\) is bounded if and only if
\[
\mathbb{C}^+ \subset \sigma(A), \quad \text{and}
\sup_{\xi > 0} \xi \int_{\mathbb{R}} \left( \| R(\xi + i\eta, A)x \|^2 + \| R(\xi + i\eta, A^*)x \|^2 \right) d\eta < \infty
\]
for every \(x \in H\).

**Remark 2.2.** By the closed graph theorem, in the conditions of Lemma 2.1, for \(x \in H\) we have
\[
\sup_{\xi > 0} \xi \int_{\mathbb{R}} \left( \| R(\xi + i\eta, A)x \|^2 + \| R(\xi + i\eta, A^*)x \|^2 \right) d\eta \leq C\|x\|^2.
\]

**Proof of Lemma 2.1.** The necessity is a direct consequence of the Plancherel theorem applied to the families \(\{e^{-\xi t}T(t)x : x \in H, \xi > 0\}, \{e^{-\xi t}T^*(t)x : x \in H, \xi > 0\}\). The sufficiency follows from the representation
\[
\langle T(t)x, x^* \rangle = \frac{1}{2\pi i} \int_{\frac{1}{t} - i\infty}^{\frac{1}{t} + i\infty} e^{\lambda t} \langle R^2(\lambda, A)x, x^* \rangle d\lambda, \quad t > 0,
\]
where the integral converges absolutely by the Hölder inequality and our assumptions, see e.g. \([14], [25], [9]\). \qed

The second statement allows us to cancel the growth of the resolvent by restricting it to sufficiently smooth elements of \(H\).

**Lemma 2.3.** Let \((T(t))_{t \geq 0}\) be a bounded \(C_0\)-semigroup on a Hilbert space \(H\) with generator \(A\), such that \(i\mathbb{R} \subset \sigma(A)\). Then for a fixed \(\alpha > 0\)
\[
\| R(\lambda, A)(-A)^{-\alpha} \| \leq C, \quad \text{Re } \lambda > 0,
\]
if and only if
\[
\| R(is, A) \| = O(|s|^{\alpha}), \quad s \to \infty.
\]

**Proof.** The lemma is proved in \([18\text{ Lemma 3.2}], [15\text{ Lemma 1.1}]\) in a version saying, in particular, that the condition
\[
\| R(\lambda, A) \| \leq C(1 + |\lambda|^{\alpha}), \quad 0 < \text{Re } \lambda < 1,
\]
is equivalent to
\[
\| R(\lambda, A)(-A)^{-\alpha} \| \leq C_1, \quad 0 < \text{Re } \lambda < 1.
\]
To get our version of the assertion it suffices to apply the maximum principle to the function \( F(\lambda) = R(\lambda, A)\lambda^{-\alpha}(1 - \lambda^2) \) on the domain \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq 0, 1 \leq |\lambda| \leq B \} \) for large \( B \), and to use the estimate
\[
\| R(\lambda, A) \| \leq \frac{C}{\text{Re} \lambda}, \quad \text{Re} \lambda > 0,
\]
for \( \lambda \) with \( |\lambda| = B \). Then (2.1) implies
\[
\| R(\lambda, A) \| \leq C(1 + |\lambda|^\alpha), \quad \text{Re} \lambda > 0.
\]
Since \( R(\lambda, A) \) is bounded in any halfplane strictly included in \( \mathbb{C}_+ \), Lemma 2.3 is reduced to its strip version mentioned above.

The next theorem is one of the main results of the paper. Its proof is based on a trick: we pass to a matrix semigroup whose boundedness gives the required rate of decay of \( \| T(t)(-A)^{-\alpha} \| \) (and of \( \| T(t)A^{-1} \| \)) “for free”.

**Theorem 2.4.** Let \( (T(t))_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on a Hilbert space \( H \) with generator \( A \) such that \( i\mathbb{R} \subset \sigma(A) \). Then for a fixed \( \alpha > 0 \) the following conditions are equivalent:

(i)
\[
\| R(is, A) \| = O(|s|^\alpha), \quad s \to \infty.
\]

(ii)
\[
\| T(t)(-A)^{-\alpha} \| = O(t^{-1}), \quad t \to \infty.
\]

(iii)
\[
\| T(t)(-A)^{-\alpha}x \| = o(t^{-1}), \quad t \to \infty, \ x \in H.
\]

(iv)
\[
\| T(t)A^{-1} \| = O(t^{-1/\alpha}), \quad t \to \infty.
\]

(v)
\[
\| T(t)A^{-1}x \| = o(t^{-1/\alpha}), \quad t \to \infty, \ x \in H.
\]

**Proof.** The implication (ii) \( \Rightarrow \) (i) was proved in [6]; the implication (iii) \( \Rightarrow \) (ii) is a consequence of the uniform boundedness principle. Moreover, the equivalence (iv) \( \iff \) (ii) was obtained in [3, Proposition 3.1] as a consequence of the moment inequalities for \( A \). Its ‘o’-counterpart (iii) \( \iff \) (v) can be obtained by the same argument. Thus, it remains to prove that (i) \( \Rightarrow \) (iii).

(i) \( \Rightarrow \) (iii): Let \( \mathcal{H} = H \oplus H \) be the direct sum of two copies of \( H \). Consider the operator \( \mathcal{A} \) on \( \mathcal{H} \) given by the operator matrix
\[
\mathcal{A} = \begin{pmatrix} A & (-A)^{-\alpha} \\ O & A \end{pmatrix}
\]
with the diagonal domain \( D(A) = D(A) \oplus D(A) \). Then \( \sigma(A) = \sigma(A) \), and the resolvent \( R(\lambda, A) \) of \( A \) is of the form
\[
R(\lambda, A) = \begin{pmatrix}
R(\lambda, A) & R^2(\lambda, A)(-A)^{-\alpha} \\
O & R(\lambda, A)
\end{pmatrix}, \quad \lambda \in \rho(A).
\]
The operator \( A \) is the generator of the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on \( H \) defined by
\[
T(t) = \begin{pmatrix}
T(t) & tT(t)(-A)^{-\alpha} \\
O & T(t)
\end{pmatrix},
\]
because the resolvents of \( A \) and of the generator of \( (T(t))_{t \geq 0} \) coincide. By (2.2) and Lemma 2.3,
\[
\|R(\lambda, A)(-A)^{-\alpha}\| \leq C, \quad \Re \lambda > 0.
\]
Hence, for every \( x = (x_1, x_2) \in H \) and \( \lambda \in \mathbb{C}_+ \),
\[
\|R(\lambda, A)x\|^2 \leq C \left( \|R(\lambda, A)x_1\|^2 + \|R(\lambda, A)x_2\|^2 \right),
\]
and similarly
\[
\|R(\lambda, A^*)x\|^2 \leq C \left( \|R(\lambda, A^*)x_1\|^2 + \|R(\lambda, A^*)x_2\|^2 \right).
\]
By Lemma 2.1,
\[
\sup_{\xi > 0} \xi \int_{\mathbb{R}} \left( \|R(\xi + i\eta, A)x\|^2 + \|R(\xi + i\eta, A^*)x\|^2 \right) d\eta < \infty
\]
for every \( x \in H \), so that
\[
\sup_{\xi > 0} \xi \int_{\mathbb{R}} \left( \|R(\xi + i\eta, A)x\|^2 + \|R(\xi + i\eta, A^*)x\|^2 \right) d\eta < \infty
\]
for every \( x \in H \). Then again by Lemma 2.1, \( (T(t))_{t \geq 0} \) is bounded on \( H \). Since \( (T(t))_{t \geq 0} \) is bounded, the definition of \( (T(t))_{t \geq 0} \) implies that
\[
\sup_{t \geq 0} \|tT(t)(-A)^{-\alpha}\| < \infty.
\]
Furthermore, \( i\mathbb{R} \subset \rho(A) \). Then by Theorem 1.1
\[
T(t)x \rightarrow 0, \quad t \rightarrow \infty, \quad \text{for every } x \in H,
\]
since \( D(A) = \text{Im} (A^{-1}) \) is dense in \( H \). Again by Theorem 1.1 (2.9) implies that
\[
\|tT(t)(-A)^{-\alpha}x\| = o(1), \quad t \rightarrow \infty, \quad x \in H.
\]

Remark 2.5. If one is merely interested in the proof of the implication (i) \( \Rightarrow \) (ii), then there is no need to invoke Theorem 1.1 as the argument above shows.

There is another argument for (i) \( \Rightarrow \) (iii), which does not use Theorem 1.1 too. Let \( n \geq 1 + \alpha \) be an integer. By the resolvent identity, \( R(\cdot, A)x \in \mathbb{C} \) is
$L^2(i\mathbb{R}, H)$, and, moreover, $R(\cdot, A)x \in H^2(\mathbb{C}_+, H)$ (the Hardy class in the right half-plane) for every $x$ from the dense set $\text{Im } (-A)^{-n}$. Then,

$$\lim_{\xi \to -0+} \xi \int_{\mathbb{R}} \|R(\xi + i\eta, A)x\|^2 \, d\eta = 0, \quad x \in \text{Im } (-A)^{-n},$$

and since $(T(t))_{t \geq 0}$ is bounded, the last relation holds for all $x \in H$. By the simple integral resolvent stability criterion from [26, Theorem 3.1], this means that $(T(t))_{t \geq 0}$ satisfies (2.9). Using the estimates (2.7), (2.8), we conclude that $R(\xi + i\eta, A)x$, $\xi > 0$, satisfies an analog of (2.10) for every $x \in H$. Since the semigroup $(T(t))_{t \geq 0}$ is bounded, by the same stability criterion, $(T(t))_{t \geq 0}$ is stable. Therefore, in particular,

$$\|T(t)(-A)^{-\alpha}x\| = o\left(\frac{1}{t}\right), \quad t \to \infty, \, x \in H.$$

**Remark 2.6.** Consider the elementary example of a multiplication $C_0$-semigroup $(T(t))_{t \geq 0}$,

$$(T(t)f)(z) = e^{tz}f(z), \quad t \geq 0,$$

on $L^2(S, \mu)$, where $S := \{z \in \mathbb{C} : \text{Re } z < -1/(1+|\text{Im } z|)^\alpha\}$ and $\mu$ is Lebesgue measure on $S$.

The operator $Af(z) = zf(z)$ with maximal domain is the generator of $(T(t))_{t \geq 0}$, and for $f, g \in L^2(S, \mu)$ we have

$$\langle R(\lambda, A)f, g \rangle = \int_S \frac{f(\zeta)g(\zeta) \, d\mu(\zeta)}{\lambda - \zeta}, \quad \lambda \in \mathbb{C} \setminus S,$$

$$\langle T(t)(-A)^{-\alpha}f, g \rangle = \int_S e^{\xi\zeta}f(\zeta)g(\zeta) \, d\mu(\zeta) \frac{(-\zeta)^\alpha}{(\xi)^\alpha}, \quad t \geq 0.$$

Straightforward estimates give that

$$\|R(is, A)\| = O(|s|^{\alpha}), \quad |s| \to \infty, \quad \text{and} \quad \lim_{t \to +\infty} t\|T(t)(-A)^{-\alpha}\| = 1,$$

which demonstrates the optimality of the rates of decay in Theorem 2.4.

A similar fact is true for the decay rate of $\|T(t)A^{-1}x\|$, $x \in H$, see, for example, [3, Proposition 4.1].

**Remark 2.7.** An advantage of the construction in the proof of Theorem 2.4 is that it reduces the problem of finding optimal polynomial rates in (2.4) under resolvent growth conditions to proving the boundedness of $(T(t))_{t \geq 0}$ under the same kind of conditions. While there is a criterion for boundedness of Hilbert space semigroups in terms of boundary behavior of certain resolvent means, see e.g. [14], [25], a similar criterion for semigroups on Banach spaces is yet to be found. Note that the Banach space semigroups $(T(t))_{t \geq 0}$ satisfying $\|T(t)\| \leq w(t)$, where $w : \mathbb{R}_+ \to \mathbb{R}_+$ is submultiplicative, can be characterized by Hille-Yosida type conditions, see [12, Theorem 5.1] and the comments following it. However, we do not yet know how to verify such conditions for $(T(t))_{t \geq 0}$ and for the weight $w(t) = C \ln(e + t)$ to conclude that $\|T(t)\| \leq C \ln(e + t)$, $t \geq 0$, recovering thus Theorem 1.2 for polynomially growing functions $M$ in the Banach space setting.
3. Decay of functions

Let $M$ be a continuous non-decreasing function, and let $M_{\log}$ be defined by (1.4).

The following result is an analog of Theorem 1.2 for the decay of functions $f \in L^\infty(\mathbb{R}_+, X)$ (see also [6, Theorem 10]). Given $f \in L^\infty(\mathbb{R}_+, X)$, its Laplace transform is defined by

$$\hat{f}(z) = \int_0^\infty e^{-zt} f(t) \, dt.$$ 

**Theorem 3.1** (Batty, Duyckaerts). Let $f \in L^\infty(\mathbb{R}_+, X)$ be such that

a) $\hat{f}$ extends analytically to the domain
\[ \Omega := \{ z \in \mathbb{C} : \Re z > -1/M(|\Im z|) \}, \]
and

b) $\| \hat{f}(z) \| \leq M(|\Im z|), \quad z \in \Omega.$

Then there exist $C, C_1 > 0$ (depending on $\|f\|$ and $M$) such that

$$\| \hat{f}(0) - \int_0^t f(s) \, ds \| \leq C_{1} \log(t/C), \quad t \geq t_0.$$ 

If we are interested only in polynomial rate of growth of $\hat{f}$ and in (possibly different) polynomial rate of narrowing of $\Omega \setminus \mathbb{C}_+$, we can formulate the following variant of [6, Theorem 10].

**Proposition 3.2.** Let $\alpha, \beta, c_1, c_2 > 0$, $\Omega := \{ z \in \mathbb{C} : \Re z > -c_1(1 + |\Im z|)^{-\alpha} \}$. Suppose that $f \in L^\infty(\mathbb{R}_+, X)$ is such that $\hat{f}$ admits an analytic extension to $\Omega$ and

$$\| \hat{f}(z) \| \leq c_2(1 + |\Im z|)^\beta, \quad z \in \Omega.$$

Then

$$\| \hat{f}(0) - \int_0^t f(s) \, ds \| \leq C \left( \frac{\log t}{t} \right)^{1/\alpha}, \quad t \geq 2,$$

with $C$ depending only on $c_1, c_2, \|f\|_{\infty}, \alpha, \beta$.

**Remark 3.3.** Thus, given polynomial growth of $\hat{f}$ in $\Omega$, the rate of polynomial decay of $(\hat{f}(0) - \int_0^t f(s) \, ds)$ is determined only by the shape of $\Omega$. This is in sharp contrast to the semigroup case where the growth of $R(\lambda, A)$ in $\Omega$ and the size of $\Omega$ are related to each other due to the Neumann series expansions of the resolvent.

Without loss of generality we assume that $\hat{f}$ is continuous up to $\partial \Omega$.

**Lemma 3.4.** Under the conditions of Proposition 3.2, for any $\varepsilon > 0$ and for some $C_1, C_2$ depending only on $c_1, c_2, \|f\|_{\infty}, \alpha, \beta, \varepsilon$ we have

$$\| \hat{f}(z) \| \leq C_2(1 + |\Im z|)^{\alpha + \varepsilon}, \quad z \in \Omega',$$

where $\Omega' := \{ z \in \mathbb{C} : \Re z > -c_1(1 + |\Im z|)^{-\alpha} \}$. 

Proof. Suppose that $\beta > \alpha + \varepsilon$ (otherwise, there is nothing to prove). We use the fact that the function $\log \| \hat{f} \|$ is subharmonic in $\Omega$. Fix $A > 2\beta/\varepsilon$ and $y > 1$ (we deal with the case $y < -1$ by symmetry), and set

$$Q_y = \{ z : -c_1(y + 1)^{-\alpha} < \text{Re} \, z < \frac{c_1}{A} (y + 1)^{-\alpha}, \ y - 1 < \text{Im} \, z < y + 1 \},$$

$$E_1 = \partial Q_y \cap \left( \frac{c_1}{A} (y + 1)^{-\alpha} + i\mathbb{R} \right),$$

$$E_2 = \partial Q_y \setminus E_1.$$ 

Next we use that

$$\log \| \hat{f}(z) \| \leq \log c_2 + \beta \log(y + 1), \quad z \in E_2,$$

$$\log \| \hat{f}(z) \| \leq \log \frac{\| f \|_{\infty} \cdot A}{c_1} + \alpha \log(y + 1), \quad z \in E_1,$$

and an estimate of harmonic measure in the thin rectangle $Q_y$,

$$\omega\left( -\frac{c_1}{A} (y + 1)^{-\alpha} + iy, E_2, Q_y \right) < \frac{2}{A},$$

for large $y$; here $\omega(z, E, U)$ is harmonic measure of $E \subset \partial U$ with respect to $z \in U$. By the theorem on two constants (see, for example, [16, VII B1]) we obtain

$$\log \| \hat{f}(z) \| \leq \omega(-\frac{c_1}{A} (y + 1)^{-\alpha} + iy, E_2, Q_y) \sup_{E_2} \log \| \hat{f} \| + \omega(-\frac{c_1}{A} (y + 1)^{-\alpha} + iy, E_1, Q_y) \sup_{E_1} \log \| \hat{f} \|.$$ 

Hence there exists $y_0$ (depending on $c_1, c_2, \alpha, \beta, A$, and on $\| f \|_{\infty}$) such that

$$\log \| \hat{f}(z) \| \leq (\alpha + \varepsilon) \log(y + 1), \quad y > y_0.$$ 

To get (3.1) we could just observe that on $Q_y$ we have

$$\omega(\cdot, E_1, Q_y) \geq \omega(\cdot, E_1, Q) - \omega(\cdot, E', Q),$$

where $Q = \{ z : -c_1(y + 1)^{-\alpha} < \text{Re} \, z < \frac{c_1}{A} (y + 1)^{-\alpha} \}$ and $E' = \partial Q \setminus \partial Q_y$, and use elementary estimates for $\omega(\cdot, E_1, Q)$ and $\omega(\cdot, E', Q)$ obtained via conformal mapping of the strip $Q$ onto the half-plane.

Proof of Proposition 3.2. By Lemma 3.4, we can assume that $\beta \leq 2\alpha$.

Next, we follow the argument from [6] (which goes back to [17] and [23]). By the Cauchy integral formula, for any sufficiently small contour $\gamma$ around 0 and for any $R > 1$ we have

$$(3.2) \quad \hat{f}(0) = \frac{1}{2\pi i} \int_0^t e^{zs} f(s) \, ds = \frac{1}{2\pi i} \int_0^t e^{zs} f(s) \, ds + e^t \frac{dz}{z}.$$
Let $\gamma_1 = RT \cap \mathbb{C}_+$, $\gamma_2 = RT \cap (\Omega \setminus \mathbb{C}_+)$, $\gamma_3 = RT \setminus \mathbb{C}_+$, $\gamma_4 = \partial \Omega \cap R D$, where $D$ is the unit disc of $\mathbb{C}$, and $T = \partial D$. Then

$$2\pi \| \hat{f}(0) - \int_0^t f(s) ds \| \leq \left\| \int_{\gamma_1} \left( 1 + \frac{z^2}{R^2} \right) \left( \hat{f}(z) - \int_0^t e^{-zs} f(s) ds \right) e^{zt} \frac{dz}{z} \right\|$$

$$+ \left\| \int_{\gamma_2} \left( 1 + \frac{z^2}{R^2} \right) \hat{f}(z) e^{zt} \frac{dz}{z} \right\| + \left\| \int_{\gamma_3} \left( 1 + \frac{z^2}{R^2} \right) \left( \int_0^t e^{-zs} f(s) ds \right) e^{zt} \frac{dz}{z} \right\|$$

$$+ \left\| \int_{\gamma_4} \left( 1 + \frac{z^2}{R^2} \right) \hat{f}(z) e^{zt} \frac{dz}{z} \right\| = I_1 + I_2 + I_3 + I_4.$$

Now

$$I_1 \leq c \int_{-\pi/2}^{\pi/2} \left( \int_0^\infty e^{-s R \cos \theta} \| f(s + t) \| ds \right) \cos \theta \, d\theta \leq \frac{c \cdot \| f \|_{L^\infty(\mathbb{R}_+, X)}}{R},$$

$$I_2 \leq c \int_0^{c_1(R+1)^{-\alpha}} \frac{c_2(R+1)^{\beta} y dy}{R^2} \leq cc_1^2 c_2(R + 1)^{\beta - 2\alpha - 2},$$

$$I_3 \leq c \int_{-\pi/2}^{\pi/2} \left( \int_0^t e^{-s R \cos \theta} \| f(t - s) \| ds \right) \cos \theta \, d\theta \leq \frac{c \cdot \| f \|_{L^\infty(\mathbb{R}_+, X)}}{R},$$

$$I_4 \leq c \int_0^R c_2(s + 1)^{\beta} e^{-c_1 t(s+1)^{-\alpha}} \frac{ds}{s+1}.$$

Setting $R = c(c_1, \alpha, \beta)(t/\log t)^{1/\alpha}$, we obtain

$$I_1 + I_2 + I_3 + I_4 \leq C(c_1, c_2, \| f \|_{L^\infty}, \alpha, \beta) \left( \frac{\log t}{t} \right)^{1/\alpha}, \quad t \geq 2.$$

\[\square\]

Remark 3.5. We can avoid using Lemma 3.4 by replacing the term $(1 + \frac{z^2}{R^2})$ in the right hand side of (3.2) by $(1 + \frac{z^2}{R^2})^N$ for a suitable $N$. However, Lemma 3.4 provides some additional information on the growth of $\hat{f}$ close to the imaginary axis.

The statement of Proposition 3.2 can be sharpened if $f$ belongs to the space of bounded uniformly continuous $X$-valued functions $BUC(\mathbb{R}_+, X)$.

**Proposition 3.6.** Suppose that $f \in BUC(\mathbb{R}_+, X)$ satisfies the assumptions of Proposition 3.2. Then

$$\| \hat{f}(0) - \int_0^t f(s) ds \|^\alpha = o\left( \frac{\log t}{t} \right), \quad t \to \infty.$$

**Proof.** By [2] Corollary 4.4.6, our hypothesis on $f$ imply that

$$\| f(t) \| = o(1), \quad t \to \infty.$$
Arguing as in the proof of Proposition 3.2 and using (3.4) we obtain
\[ I_1 \leq c \sup_{s \geq 0} \| f(s + t) \| = o \left( \frac{1}{R} \right), \quad t \to \infty, \]
\[ I_3 \leq c \sup_{-\pi/2 < \theta < \pi/2} \cos \theta \int_0^t e^{-sR \cos \theta} \| f(t - s) \| \, ds = o \left( \frac{1}{R} \right), \quad t \to \infty, \]
and (3.3) follows.

The following result is a version of Proposition 3.2 written in the semi-group language. Let \( \alpha, \beta > 0 \) and the domain \( \Omega \) be as above.

**Corollary 3.7.** Let \( (T(t))_{t \geq 0} \) be a \( C_0 \)-semigroup on a Banach space \( X \) with generator \( A \), such that \( \sup_{t \geq 0} \| T(t) \| := M < \infty \), and let \( T_1 \) and \( T_2 \) be bounded operators on \( X \). Suppose that \( F(z) = T_1 R(z, A) T_2 \) admits a holomorphic extension to the domain \( \Omega \), and suppose that
\[ \| F(z) \| \leq M_1 (1 + | \text{Im} \, z |^\beta), \quad z \in \Omega. \]
Then there exist \( c = c(M, M_1, \alpha, \beta) > 0 \) such that
\[ (3.5) \quad \| T_1 T(t)(I - A)^{-1} T_2 \| \leq c \left( \frac{\log t}{t} \right)^{1/\alpha}, \quad t \geq 2. \]

**Proof.** As in the proof of [6 Corollary 11] we consider the function
\[ f(t) = \frac{d}{dt} (T_1 T(t)(I - A)^{-1} T_2) = T_1 T(t) A (I - A)^{-1} T_2 \]
\[ = -T_1 T(t) T_2 + T_1 T(t)(I - A)^{-1} T_2. \]
Its Laplace transform \( \hat{f} \) extends analytically to \( \Omega \) with the same estimates as \( F \) (up to a constant multiple). Since
\[ \hat{f}(0) - \int_0^t f(s) \, ds = -T_1 T(t)(I - A)^{-1} T_2, \]
Proposition 3.2 yields the claim.

The next result shows that Proposition 3.6 is sharp, at least for \( \beta > \alpha/2 \). It suffices to consider just scalar-valued functions.

**Theorem 3.8.** Given \( \alpha > 0 \), \( \beta > \alpha/2 \), and a positive function \( \gamma \in C_0(\mathbb{R}_+) \), there exists a function \( f \in C_0(\mathbb{R}_+) \) such that
a) \( \hat{f} \) admits an analytic extension to the region \( \Omega := \{ z \in \mathbb{C} : \text{Re} \, z > -1/(1 + | \text{Im} \, z |) \} \) and
\[ \hat{f}(z)(1 + | \text{Im} \, z |)^{\beta} \to 0, \quad |z| \to \infty, \quad z \in \Omega, \]
and
b) \[ \limsup_{t \to \infty} \frac{t}{\gamma(t) \log t} \left| \hat{f}(0) - \int_0^t f(s) \, ds \right|^\alpha > 0. \]
To prove Theorem 3.8 we imitate the multiplication semigroup example described in Remark 2.6. However, we choose an infinite charge \(\mu\) so that while formal integral expressions for the resolvent, the semigroup and its orbits remain the same as in Remark 2.6, the corresponding size estimates behave in a different way due to the lack of absolute convergence of the integrals.

The proof of Theorem 3.8 is based on the following lemma. Denote by \(\chi_A\) the characteristic function of a set \(A \subset \mathbb{C}\).

**Lemma 3.9.** Given \(Q > 0\) and \(\varepsilon > 0\), there exist an integer \(k > Q\) and a complex measure \(\mu\) with compact support in \(\mathbb{C} \setminus \Omega\) such that for some \(B = B(\alpha, \beta)\), we have

\[
\left| \int_{\mathbb{C} \setminus \Omega} \frac{d\mu(\zeta)}{\zeta - z} \right| \leq B(1 + |\text{Im } z|)^{\beta} \cdot \chi_{\{z: |z| > Q\}} + \varepsilon, \quad z \in \Omega,
\]

\[
\left| \int_{\mathbb{C} \setminus \Omega} e^{ik\zeta} d\mu(\zeta) \right| \leq B \chi_{\{t:t > Q\}} + \varepsilon, \quad t \geq 0,
\]

\[
1/B \leq \left| \int_{\mathbb{C} \setminus \Omega} e^{ik\zeta} d\mu(\zeta) \right| \leq B,
\]

\[
\left| \int_{\mathbb{C} \setminus \Omega} e^{ik\zeta} d\mu(\zeta) \right|^\alpha \leq B \frac{\log t}{t} \cdot \chi_{\{t:Q < t < 2k\}} + \frac{\varepsilon}{t + 1}, \quad t \geq 0,
\]

and

\[
\left| \int_{\mathbb{C} \setminus \Omega} e^{ik\zeta} d\mu(\zeta) \right|^\alpha \geq \frac{\log k}{Bk}.
\]

**Proof.** Choose \(H > Q\) large enough, and for an integer \(k \geq 2\), such that

\[H^\alpha \leq k \leq H^{3\alpha/2},\]

define

\[
A := 2k \log k, \\
\tau := A^{k-1}/\sqrt{k}, \\
q := e^{2\pi i/k}, \\
w := iH - 1.
\]

We define also a finite measure

\[
\mu := \tau \sum_{1 \leq s \leq k} q^s (1 + q^s/(Aw)) \delta_{w + q^s/A},
\]

where \(\delta_x\) is the unit mass at \(x\).

Observe that

\[
\sum_{1 \leq s \leq k} \frac{q^s}{x - q^s} = \frac{k}{x^k - 1}, \quad \sum_{1 \leq s \leq k} \frac{q^{2s}}{x - q^s} = \frac{kx}{x^k - 1}.
\]
Indeed, to prove the first identity we use that 
\[ \sum_{1 \leq s \leq k} \frac{q^s}{x - q^s} = \frac{P(x)}{x^k - 1} \]
for some polynomial \( P \) with \( \deg P < k \) such that \( P(x) = P(qx) \) so that \( P(x) = \text{const} \), and then \( P(x) = P(0) = k \). The second equality can be proved in a similar way by using that if 
\[ \sum_{1 \leq s \leq k} \frac{q^{2s}}{x - q^s} = \frac{P(x)}{x^k - 1} \]
then \( P(qx) = qP(x) \) which yields \( P(x) = kx \).

Now using (3.11) we get 
\[ C_{\mu}(z) = \int_{C \setminus \Omega} \frac{d\mu(\zeta)}{z - \zeta} = \tau \sum_{1 \leq s \leq k} \frac{q^s(1 + q^s/(Aw))}{(z - w) - q^s/A} \]
\[ = \tau \sum_{1 \leq s \leq k} \left[ \frac{Aq^s}{A(z - w) - q^s} + \frac{q^{2s}/w}{A(z - w) - q^s} \right] \]
\[ = \tau A \frac{k}{A^k(z - w)^k - 1} + \frac{\tau kA(z - w)}{w A^k(z - w)^k - 1} \]
\[ = \frac{z}{w A^k(z - w)^k - 1}. \]

If \( z \in \Omega \), \( H > 1 \), then 
\[ |z - w| \geq 1 - \frac{1}{H^\alpha}. \]
Indeed, if \( \text{Im } z < H - 1 \), then \( |z - w| > 1 \), and otherwise, \( \text{Re } z \geq -H^{-\alpha} \).

Now if \( H/2 \leq \text{Im } z \leq |z| \leq 2H \), \( z \in \Omega \), then we use that \( A|z - w| > 2 \), 
\( |A^k(z - w)^k - 1| > A^k e^{-k/H^\alpha}/2 \), to obtain that 
\[ \left| \frac{z}{w A^k(z - w)^k - 1} \right| \leq c\tau kA^{1 - k/H^\alpha} = c\sqrt{k} e^{k/H^\alpha}. \]
From now on we assume that \( k \) satisfies the condition 
(3.12) \( \sqrt{k} e^{k/H^\alpha} \leq H^3 \).

Then 
(3.13) \( |C_{\mu}(z)| \leq cH^3, \quad H/2 \leq \text{Im } z \leq |z| \leq 2H, \ z \in \Omega. \)

If \( z \in \Omega \) and \( |\text{Im } z - H| + |z - H| > H/2 \), then \( |z - w| > c \max(|z|, H) \), and under condition (3.12) we have for large \( H \):
\[ |C_{\mu}(z)| \leq \frac{c_1 |z| \sqrt{k}}{H(c \max(|z|, H))^k} \leq \frac{c_1 |z| \sqrt{k}}{H(c \max(|z|, H))^{\alpha + 1}} \leq \varepsilon. \]
This, together with (3.13), proves (3.6).
Next, \[ \mathcal{L}_\mu(t) \overset{\text{def}}{=} \int_{C \setminus \Omega} e^{\zeta^t} d\mu(\zeta) = \tau \sum_{1 \leq s \leq k} q^s(1 + q^s/(Aw)) e^{(tw+q^s/A)}, \]

and

\[ |\mathcal{L}_\mu(t)| = \tau e^{-t} \left| \sum_{1 \leq s \leq k} q^s(1 + q^s/(Aw)) e^{q^s t/A} \right|. \] (3.14)

Furthermore, we have

\[
\sum_{1 \leq s \leq k} q^s(1 + q^s/(Aw)) e^{q^s t/A} = k \sum_{m \geq 1} \left[ \frac{tk^{m-1}}{A^{km-1}} \cdot \frac{1}{(km-1)!} + \frac{tk^{m-2}}{A^{km-2}} \cdot \frac{1}{(km-2)!} \cdot \frac{1}{Aw} \right] = \frac{kt^{k-1}}{A^{k-1}(k-1)!} \sum_{m \geq 1} \left( \frac{tk}{A^k} \right)^{m-1} \frac{(k-1)!}{(km-1)!} \left( 1 + \frac{km-1}{tw} \right),
\]

and

\[
|\mathcal{L}_\mu(t)| = \frac{k^{3/2}tk^{k-1}e^{-t}}{k!} \sum_{m \geq 1} \left( \frac{tk}{A^k} \right)^{m-1} \frac{(k-1)!}{(km-1)!} \left( 1 + \frac{km-1}{tw} \right). \]

Thus, for some constants \(c, c_1, c_2, c_3\) we have

\[
|\mathcal{L}_\mu(t)| \leq c \frac{k^{3/2}tk^{k-2}e^{-t}}{k!} \sum_{m \geq 1} \left( \frac{k}{HA} \right)^{k(m-1)} \frac{(k-1)!}{(km-1)!} \frac{km}{H} \leq c_1 e^{-tk^{5/2}k^{k-2}/(k!H)}, \quad 0 \leq t \leq k/H,
\]

and

\[
c_2 e^{-tk^{3/2}k^{-1}/k!} \leq |\mathcal{L}_\mu(t)| \leq c_3 e^{-tk^{3/2}k^{-1}/k!}, \quad k/H \leq t \leq A.
\]

If \(t \geq A\), then by (3.14),

\[
|\mathcal{L}_\mu(t)| \leq 2\tau e^{-tk^{1/A}} = 2\sqrt{kA^{k-1}} e^{-t(1/A)}.
\]

The function \(t \mapsto e^{-tk^{k-2}}\) attains its maximum on \([0, k/H]\) at \(t = k/H\); the function \(t \mapsto e^{-tk^{k-1}}\) attains its maximum on \([k/H, A]\) at \(t = k - 1\); the function \(t \mapsto e^{-t(1/A)}\) attains its maximum on \([A, +\infty)\) at \(t = A\). Using
that \( A = 2k \log k \), by Stirling’s formula, we obtain that
\[
0 < c_1 \leq |\mathcal{L}_\mu(k)| \leq c_2 |\mathcal{L}_\mu(k - 1)| \leq c_3,
\]
\[
\max_{[k/H, k/2]} |\mathcal{L}_\mu| = o(1), \quad H \to \infty,
\]
\[
e^{-k/H}k^{5/2}(k/H)^{k-2}/(k!) \to 0, \quad H \to \infty,
\]
\[
\sqrt{k}A^{k-1}e^{-A} \to 0, \quad H \to \infty.
\]
Hence,
\[
0 < c_1 \leq |\mathcal{L}_\mu(k)| \leq c_2 \max_{\mathbb{R}_+} |\mathcal{L}_\mu| \leq c_3,
\]
\[
\max_{[0, k/2]} \cup [A, +\infty) |\mathcal{L}_\mu| = o(1), \quad H \to \infty,
\]
and (3.7) and (3.8) follow for large \( H \).

Finally,
\[
\mathcal{N}_\mu(t) \overset{\text{def}}{=} \int_{C \setminus \Omega} e^{t\zeta} \frac{d\mu(\zeta)}{\zeta} = \frac{\tau e^{-t+Ht}}{w} \sum_{1 \leq s \leq k} q^s e^{q^st/A}.
\]

Here we use the formula
\[
\sum_{1 \leq s \leq k} q^s e^{q^st/A} = \sum_{1 \leq s \leq k} \sum_{n \geq 0} q^s (q^st/A)^n \frac{1}{n!}
\]
\[
= k \sum_{m \geq 1} \frac{t^{km-1}}{A^{km-1}} \cdot \frac{1}{(km-1)!}
\]
\[
= \frac{kt^{k-1}}{A^{k-1}(k-1)!} \sum_{m \geq 1} \left( \frac{t^k}{A^k} \right)^{m-1} \frac{(k-1)!}{(km-1)!}
\]
\[
= (1 + o(1)) \frac{kt^{k-1}}{A^{k-1}(k-1)!}, \quad 0 \leq t \leq A, \quad H \to \infty.
\]

Therefore, by Stirling’s formula, we have for large \( H \):
\[
(3.15) \quad |\mathcal{N}_\mu(k)| \geq c_1 \frac{\tau e^{-k}}{H} \cdot \frac{k^k}{A^{k-1}(k-1)!} \geq c_2 \frac{k^k}{H}.
\]

Moreover,
\[
(3.16) \quad t^{1/\alpha} |\mathcal{N}_\mu(t)| \leq \frac{c_1 k}{tH} \cdot t^k e^{k-t^{1/\alpha}}
\]
\[
\leq \varepsilon + c_1 \frac{k^{1/\alpha}}{H} \cdot \chi_{\{t; k/2 < t < 2k\}}, \quad 0 \leq t \leq A,
\]
and
\[
(3.17) \quad t^{1/\alpha} |\mathcal{N}_\mu(t)| \leq \frac{c_t e^{-t}}{H} \cdot k^{1/\alpha} e^{t^{1/\alpha}} \leq c_1 \sqrt{k} e^{-A} \leq \varepsilon, \quad t \geq A.
\]

Now we fix \( 0 < \psi < \beta - \frac{\alpha}{2} \) and \( k = \psi H^\alpha \log H \) in such a way that \( k \in \mathbb{N} \). Then (3.12) is satisfied for large \( H \), (3.15) implies (3.10), and (3.16), (3.17) imply (3.9). \( \square \)
Proof of Theorem 3.8 Without loss of generality, we can assume that $\gamma$ is non-increasing.

Our function $f$ will be defined by an inductive construction. Set $Q_1 = 1$. On step $n \geq 1$ we use Lemma 3.9 with $Q = Q_n$, $\varepsilon = 2^{-n}$ to find $\mu_n$, $k_n$ satisfying (3.6)–(3.10). Since
\[
\lim_{|z| \to \infty, z \in \Omega} C \mu_n(z) = 0,
\]
\[
\lim_{t \to \infty} L \mu_n(t) = 0,
\]
we can find $Q_{n+1} > k_n$ such that
\[
|C \mu_n(z)| \leq \varepsilon, \quad |z| > Q_{n+1}, z \in \Omega,
\]
\[
|L \mu_n(t)| \leq \varepsilon, \quad t > Q_{n+1},
\]
which completes the induction step.

Finally, for some numbers $\phi_n \in \mathbb{C}$, $|\phi_n| = 1$, to be chosen iteratively later on, we set
\[
f = \sum_{n \geq 1} \phi_n \gamma(k_n)^{1/\alpha} L \mu_n.
\]
The series above converges absolutely by the choice of $\mu_n$, and therefore $f \in C_0(\mathbb{R}_+)$. Similarly, the function
\[
\hat{f} = \sum_{n \geq 1} \phi_n \gamma(k_n)^{1/\alpha} C \mu_n
\]
extends analytically to $\Omega$, and
\[
\hat{f}(z)(1 + |\text{Im } z|)^{-\beta} \to 0, \quad |z| \to \infty, z \in \Omega.
\]

Finally, for $m \geq 1$ we have
\[
\frac{k_m}{\gamma(k_m) \log k_m} \left| \hat{f}(0) - \int_0^{k_m} f(s) \, ds \right|^\alpha
\]
\[
= \frac{k_m}{\gamma(k_m) \log k_m} \left| \sum_{n \geq 1} \phi_n \gamma(k_n)^{1/\alpha} N \mu_n(k_m) \right|^\alpha
\]
\[
\geq \frac{c k_m}{\gamma(k_m) \log k_m} \left( \left| \sum_{1 \leq n \leq m} \phi_n \gamma(k_n)^{1/\alpha} N \mu_n(k_m) \right|^\alpha - \left| \sum_{n > m} \phi_n \gamma(k_n)^{1/\alpha} N \mu_n(k_m) \right|^\alpha \right).
\]

On step $m \geq 1$ we choose $\phi_m$ in such a way that
\[
\left| \sum_{1 \leq n \leq m} \phi_n \gamma(k_n)^{1/\alpha} N \mu_n(k_m) \right|
\]
\[
= \left| \gamma(k_m)^{1/\alpha} L \mu_m(k_m) \right| + \left| \sum_{1 \leq n < m} \phi_n \gamma(k_n)^{1/\alpha} N \mu_n(k_m) \right|.
\]
Then
\[
\frac{k_m}{\gamma(k_m) \log k_m} \left( \left| \sum_{1 \leq n \leq m} \phi_n \gamma(k_n)^{1/\alpha} \mathcal{N}_{\mu_n}(k_m) \right|^\alpha - \left| \sum_{n > m} \phi_n \gamma(k_n)^{1/\alpha} \mathcal{N}_{\mu_n}(k_m) \right|^\alpha \right)
\geq \frac{k_m}{\gamma(k_m) \log k_m} \left( \gamma(k_m) \mathcal{N}_{\mu_m}(k_m) \right)^\alpha - \left( \sum_{n > m} \phi_n \gamma(k_n)^{1/\alpha} \mathcal{N}_{\mu_n}(k_m) \right)^\alpha
\geq c_1 - \frac{c_2}{\log k_m} \left( \sum_{n > m} 2^{-n/\alpha} \right)^\alpha \geq c_3 > 0.
\]

\[\square\]

Remark 3.10. A variant of our construction works with \(\gamma = 1\) in Theorem 3.8, if one looks just for \(f \in L^\infty(\mathbb{R}_+)\).

4. Decay of Banach space semigroups

Using the construction of Theorem 3.8 we show next that the analogue of Theorem 1.2 for \(C_0\)-semigroups on Banach spaces, Theorem 3.1, is also sharp. Estimates for the local resolvents of group generators similar to the ones used below has been also employed, in particular, in [2] and [11, Section II.4.6].

**Theorem 4.1.** Given \(\alpha > 0\), there exist a Banach space \(X_\alpha\) and a bounded \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(X_\alpha\) with generator \(A\) such that
\[
\text{(a)} \quad \|R(is, A)\| = O(|s|^\alpha), \quad |s| \to \infty,
\]
and
\[
\text{(b)} \quad \limsup_{t \to \infty} \left( \frac{t}{\log t} \right)^{\frac{\alpha}{\alpha + 1}} \|T(t)A^{-1}\| > 0.
\]

**Proof.** Let \((S(t))_{t \geq 0}\) be the left shift semigroup on \(\text{BUC}(\mathbb{R}_+)\), let \(\Omega := \{\lambda \in \mathbb{C} : \text{Re} \lambda > -1/(1 + |\text{Im} \lambda|^\alpha)\}\), and let \(\Omega_0 := \Omega \cap \{\lambda \in \mathbb{C} : |\text{Re} \lambda| < 1\}\). Furthermore, let \(X_\alpha\) be the space of functions \(f \in \text{BUC}(\mathbb{R}_+)\) such that the Laplace transform \(\hat{f}\) extends to an analytic function in \(\Omega_0\), and
\[
|\hat{f}(\lambda)|(1 + |\text{Im} \lambda|)^{-\alpha} \to 0, \quad \lambda \to \infty, \lambda \in \Omega_0.
\]
Then \(X_\alpha\) equipped with the norm
\[
\|f\|_{X_\alpha} := \|f\|_{\infty} + \|f\|_\alpha := \|f\|_{\infty} + \sup_{\lambda \in \Omega_0} |\hat{f}(\lambda)|(1 + |\text{Im} \lambda|)^{-\alpha},
\]
is a Banach space. Moreover, \(S(t)X_\alpha \subset X_\alpha, t \geq 0,\) and the restriction \((T(t))_{t \geq 0}\) of \((S(t))_{t \geq 0}\) to \(X_\alpha\) is also a \(C_0\)-semigroup. To prove this assertion
it suffices to observe that
\[
\begin{align*}
\hat{f}(\lambda) - T(t) & f(\lambda) \\
& = \int_0^\infty e^{-\lambda s} f(s) \, ds - \int_0^\infty e^{-\lambda s} f(t + s) \, ds \\
& = (1 - e^{\lambda t}) \hat{f}(\lambda) + e^{\lambda t} \int_0^t e^{-\lambda s} f(s) \, ds, \quad \text{Re } \lambda > 0,
\end{align*}
\]
and the same equality holds on \( \Omega_0 \). By the definition of \( X_\alpha \),
\[
\| T(t) f - f \|_{X_\alpha} \to 0, \quad t \to 0^+,
\]
and then
\[
\| T(t) f - f \|_{X_\alpha} \to 0, \quad t \to 0^+.
\]

Let \( A \) stand for the generator of \( (T(t))_{t \geq 0} \).

Next we prove that for every \( f \in X_\alpha \) the local resolvent \( R(\lambda, A) f \) satisfies the estimate

\[
(4.1) \quad \| R(\lambda, A) f \|_{X_\alpha} \leq C (1 + |\text{Im } \lambda|)^\alpha \| f \|_{X_\alpha}, \quad 0 < \text{Re } \lambda < 1.
\]

We will estimate the quantities \( \| R(\lambda, A) f \|_\infty \) and \( \| R(\lambda, A) f \|_\alpha \) separately.

Observe first that for every \( t \in \mathbb{R}_+ \) and every \( \lambda \in \mathbb{C}_+ \) one has
\[
(R(\lambda, A) f)(t) = \int_0^\infty e^{-\lambda s} f(t + s) \, ds = e^{\lambda t} \hat{f}(\lambda) - \int_0^t e^{\lambda(t-s)} f(s) \, ds.
\]
It follows that for every fixed \( t \in \mathbb{R}_+ \) the function \( \lambda \mapsto (R(\lambda, A) f)(t) \) extends to an analytic function on \( \Omega_0 \), and moreover
\[
|(R(\lambda, A) f)(t)| \leq \begin{cases} 
\frac{\| f \|_\infty}{|\text{Re } \lambda|} & \text{if } \text{Re } \lambda > 0, \\
\frac{\| f \|_\infty}{|\text{Re } \lambda|} + |\hat{f}(\lambda)| & \text{if } \text{Re } \lambda < 0.
\end{cases}
\]

Applying Levinson’s log-log theorem (see, for example, [16, VII D7]) or, rather, its polynomial growth version [2, Lemma 4.6.6] to \( (R(\lambda, A) f)(t) \) in the squares \( \{ \lambda : |\text{Re } \lambda| < (s + 2)^{-\alpha}, |s - \text{Im } \lambda| < (s + 2)^{-\alpha} \} \), we conclude that

\[
(4.2) \quad \| R(\lambda, A) f \|_\infty \leq C(1 + |\text{Im } \lambda|)^\alpha (\| f \|_\infty + \| f \|_\alpha), \quad 0 < \text{Re } \lambda < 1.
\]

Fix \( \lambda \) with \( \text{Re } \lambda \in (0, 1) \). To estimate \( \| R(\lambda, A) f \|_\alpha \) note that
\[
(R(\lambda, A) f)(\mu) = \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} f(t + s) \, ds \, dt = \int_0^\infty e^{(\lambda - \mu) t} \int_t^\infty e^{-\lambda s} f(s) \, ds \, dt = -\frac{1}{\lambda - \mu} \int_0^\infty e^{-\lambda s} f(s) \, ds + \frac{1}{\lambda - \mu} \int_0^\infty e^{-\mu t} f(t) \, dt = -\frac{\hat{f}(\lambda) - \hat{f}(\mu)}{\lambda - \mu}, \quad \text{Re } \mu > 1.
\]
Therefore, $R(\lambda,A)f$ extends analytically to $\Omega_0$, and
\[
(R(\lambda,A)f)(\mu) = \begin{cases} 
\frac{f(\lambda)-f(\mu)}{\lambda-\mu}, & \lambda \neq \mu, \mu \in \Omega_0, \\
-\hat{f}'(\mu), & \lambda = \mu.
\end{cases}
\]

Now, if $|\lambda-\mu| \geq 1$, $\mu \in \Omega_0$, then
\[
|\hat{(R(\lambda,A)f)(\mu)}| \leq |\hat{f}(\lambda)| + |\hat{f}(\mu)| \leq c(1 + |\text{Im } \mu|)^\alpha (1 + |\lambda|)^\alpha \|f\|_\alpha.
\]

Furthermore, if
\[
1 > |\lambda - \mu| \geq \frac{1}{2(1 + |\text{Im } \lambda|)^\alpha}, \quad \mu \in \Omega_0,
\]
then we have
\[
|\hat{(R(\lambda,A)f)(\mu)}| \leq 2(1 + |\text{Im } \lambda|)^\alpha (|\hat{f}(\lambda)| + |\hat{f}(\mu)|) \leq c(1 + |\text{Im } \lambda|)^\alpha (1 + |\text{Im } \mu|)^\alpha \|f\|_\alpha.
\]

Finally, if
\[
|\lambda - \mu| \leq \frac{1}{2(1 + |\text{Im } \lambda|)^\alpha},
\]
then, applying Cauchy’s formula on the circle
\[
C_\lambda := \{ z \in \mathbb{C} : |z - \lambda| = \frac{2}{3}(1 + |\text{Im } \lambda|)^{-\alpha} \},
\]
we obtain that
\[
|\hat{(R(\lambda,A)f)(\mu)}| \leq c(1 + |\text{Im } \mu|)^\alpha (1 + |\lambda|)^\alpha \|f\|_\alpha.
\]

Thus,
\[
(4.3) \quad \|R(\lambda,A)f\|_\alpha \leq C(1 + |\lambda|)^\alpha \|f\|_\alpha, \quad 0 < \text{Re } \lambda < 1.
\]

The estimates (4.2) and (4.3) together give us (4.1). Since
\[
\|R(\lambda,A)\|_{X_\alpha} \geq \frac{1}{\text{dist } (\lambda, \sigma(A))},
\]
the estimate (4.1) implies that $i\mathbb{R} \subset \mathbb{C} \setminus \sigma(A)$, and that
\[
(4.4) \quad \|R(\lambda,A)\|_{X_\alpha} \leq C(1 + |\lambda|)^\alpha, \quad \lambda \in i\mathbb{R}.
\]

Since $\sigma(A) \cap i\mathbb{R} = \emptyset$, by Theorem [1.1] we obtain
\[
A^{-1}f = \lim_{t \to -\infty} (A^{-1}f - T(t)A^{-1}f) = -\int_{-\infty}^0 T(t)f dt,
\]
for every $f \in X_\alpha$.

By Lemma [3.3] there exist $f_n \in X_\alpha$, $f_n = \mathcal{L}\mu_n$, and $k_n \to \infty$ as $n \to \infty$, such that
\[
\|f_n\|_{X_\alpha} \leq 1, \quad |N\mu_n(k_n)| \geq C(\alpha) \left( \frac{\log k_n}{k_n} \right)^{1/\alpha}, \quad n \geq 1.
\]
Therefore,
\[
\|T(k_n)A^{-1}\|_{X^\alpha} \geq \|T(k_n)A^{-1}f_n\|_{X^\alpha} \geq \left\| \int_0^\infty f_n(\cdot + k_n + r)\, dr \right\|_{X^\alpha} \\
\geq \left| \int_{k_n}^\infty f_n(r)\, dr \right| \geq \left| \hat{f}_n(0) - \int_0^{k_n} f_n(s)\, ds \right| = \left| \mathcal{N}\mu_n(k_n) \right| \\
\geq C(\alpha) \left( \frac{\log k_n}{k_n} \right)^{1/\alpha}.
\]

\[\square\]

5. Acknowledgments

We would like to thank C. J. K. Batty for sending us a preliminary version of [6] prior to its publication and A. M. Gomilko for helpful discussions and remarks.

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