Quantum properties of a single beam splitter

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Abstract

When a single beam-splitter receives two beams of bosons described by Fock states (Bose-Einstein condensates at very low temperatures), interesting generalizations of the two-photon Hong-Ou-Mandel effect take place for larger number of particles. The distributions of particles at two detectors behind the beam splitter can be understood as resulting from the combination of two effects, the spontaneous phase appearing during quantum measurement, and the quantum angle. The latter introduces quantum “population oscillations”, which can be seen as a generalized Hong-Ou-Mandel effect, although they do not always correspond to even-odd oscillations.

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Beam splitters are an essential component of many experiments designed to observe quantum effects. They are involved in experimental and theoretical schemes that both Helmut Rauch and Daniel Greenberger have studied. Indeed, the famous neutron experiments of H. Rauch and colleagues \cite{1,2} were made possible by the realization of an appropriate device allowing neutron beams to be split into two coherent beams, which can then be recombined and give rise to various interesting quantum interference effects. The observation of equally famous quantum GHZ (Greenberger, Horne and Zeilinger) violations of local realism \cite{3,4} may also require the use of photon beam splitters \cite{5}. Still another example is given by the entanglement swapping effect, which requires indistinguishable photons to be measured at the output ports of a beam splitter \cite{6}. This list is, of course, non-exhaustive.

Here, we come back to the basic properties of a single beam splitter and show that, as simple as it may look, it already exhibits strong quantum properties. Previous studies of the quantum properties of beam splitters include Refs. \cite{7,8,9}. Generally, beam splitters are used in conditions where they receive particles one by one. Here we generalize the discussion and consider the case where a beam splitter receives groups of particles in its two input beams, described by Fock states of bosons. We will study the effects of the “quantum angle”, which was introduced in the context of more elaborate interferometry experiments involving several beam splitters, and leads to violations of local realist BCHSH and GHZ inequalities \cite{10,11}. Holland and Burnett have studied the quantum limits on the detection of small phase shifts with interferometers involving two beam splitters \cite{12}; for this purpose they also study the distribution of the relative phase of the two output beams at a single beam splitter, assuming that the two incoming Fock states have equal populations (twin states). Here we release this assumption and study the distributions of the number of particles at the outputs.

The production of Fock states with photons is not an easy task, if not impossible, except for a small number of photons; see for instance Ref. \cite{13} for a description of an experiment with two states containing two photons. Coherent states are, of course, much easier to produce, even with a small average number of photons (very small intensities), but they remain fundamentally very different from
Fock states. Fortunately, the phenomenon of Bose-Einstein condensation in ultra-cold gases provides us with a method to produce condensates thermally and, when repulsive interactions between the atoms stabilize the condensate, there are good reasons to believe that its state is well described by a Fock state. Thermal excitations are of course always present, but they can be reduced very efficiently by reducing the temperature. Moreover, the technique of Bragg scattering of atoms from standing laser waves \[14\] can be used to obtain efficient atomic beam splitters \[15\], and even observe interferences with Bose-Einstein condensates in interferometers with macroscopic arm separation \[16\].

1 Classical and quantum calculation

The situation we consider is shown schematically in Fig. 1. We will perform a quantum calculation but, as a point of comparison, we start with a simple classical calculation.

![Figure 1: N_α, N_β bosons proceed from the sources to a beam splitter, followed by two detectors 1 and 2, where m_1 and m_2 particles are detected. A phase shift θ is inserted in the α-arm for generality, but turns out to play no role in the results.](image)

1.1 Classical model

In classical optics, if two beams with equal intensities \(I_0\) are sent to a beam splitter with a relative phase \(λ\), the output intensities \(I_1\) and \(I_2\) at the output detectors 1 and 2 are proportional to \([1 + \cos (λ - π/2)]\) and \([1 - \cos (λ - π/2)]\) (the \(π/2\) arises because a phase shift occurs at a reflection, but not at a transmission). If the phase \(λ\) is completely unknown, these expressions have to be summed over \(λ\) between \(-π\) and \(+π\); a well-known classical calculation then shows that the distribution \(P(I)\) of the random variables \(I_{1,2}\) is given by:

\[
P_{\text{class.}}(I) \sim \frac{1}{\sqrt{4(I_0 - I)}}
\]

If the input intensities are different, \(I_\alpha\) and \(I_\beta = x^2I_\alpha\), this calculation can easily be generalized. The two intensities are now proportional to \([1 + r \cos (λ - π/2)]\) and \([1 - r \cos (λ - π/2)]\), where:

\[
r = \frac{2x}{1 + x^2} = \frac{2\sqrt{I_\alpha I_\beta}}{I_\alpha + I_\beta} \leq 1
\]
and (1) becomes:

\[ P_{\text{class}}(I) = \frac{1}{\pi \sqrt{\left[I - I_{\alpha} (1 - x)^2 / 2\right] \left[-I + I_{\alpha} (1 + x)^2 / 2\right]}} \]  

(3)

These expressions result from purely classical wave theory.

Semi-classical expressions can be obtained by considering a flux of classical particles reaching independently the beam splitter, each having a probability \(1 + r \cos (\lambda - \pi/2)\)/2 to go to detector 1, and a probability \(1 - r \cos (\lambda - \pi/2)\)/2 to go to detector 2. The probability that, among a total of \(N\) particles, \(m_1\) will go to detector 1 and \(m_2\) to detector 2 (with \(m_1 + m_2 = N\)) is then given by:

\[ P_{\text{semi-class.}}(m_1, m_2) = \frac{N!}{2^N m_1! m_2!} [1 + r \cos (\lambda - \pi/2)]^{m_1} [1 - r \cos (\lambda - \pi/2)]^{m_2} \]  

(4)

For Fock states, we expect that the relative phase \(\lambda\) should be completely random, so that this expression becomes:

\[ P_{\text{semi-class.}}^{\text{Fock}}(m_1, m_2) = \frac{N!}{2^N m_1! m_2!} \int_{-\pi}^{+\pi} \frac{d\lambda}{2\pi} [1 + r \cos \lambda]^{m_1} [1 - r \cos \lambda]^{m_2} \]  

(5)

Figure 2-a shows an example of such a distribution for equal intensities of the incoming beams, which reproduces the shape of the classical distribution (1), with a minimum at the center and maxima at the edges. Figure 2-b shows another example, assuming now that the intensities of the two incoming beams are different, and that their ratio is 6/44. Because the interference effect between the two beams can no longer be completely destructive, the distribution tends to concentrate more towards the center or the curve (medium values of \(m_1\)).

Figure 2: The left part (a) shows the classical (dotted line) and semiclassical (full line) distributions as a function of \(m_1\) when the intensities of the input beams are equal \((x = r = 1)\) and when \(m_1 + m_2 = 50\). Values of \(m_1\) near the maximum and the minimum are more likely to occur with this distribution. The right part (b) shows the same distributions for the same total number of particles, but when the intensities of the input beams are different (their ratio is 6/44).

An equivalent summation formulation more suitable for computations is found by expanding the binomials \((1 \pm r \cos \lambda)^{m_1}\) and integrating term by term. The result is:

\[ P_{\text{semi-class.}}^{\text{Fock}} = K \left[ \sum_{p=0}^{m_1} \sum_{q=0}^{m_2} \frac{(-1)^q r^{p+q} (p+q)!}{2^{p+q} p! (m_1 - p)! q! (m_2 - q)! \left[ \left( \frac{2r}{1^2} \right)^q \right]^2} \right] \]  

(6)

where \(K\) is a normalization factor. This formula was used, for instance, to produce Fig. 2 and the dotted line of Fig. 2b.
1.2 Quantum calculation

We give two calculations, for input beams described either by Fock states or by coherent states.

1.2.1 Fock states

Before the beams of bosons cross the beam splitter, they are described by the quantum state:

\[ |N_\alpha, N_\beta\rangle = \frac{1}{\sqrt{N_\alpha!N_\beta!}} a_\alpha^{N_\alpha} a_\beta^{N_\beta} |0\rangle \]  

(7)

Our calculation that is essentially the same as that of [10] and [11]. The destruction operators associated with the two output beams (and detectors) are:

\[ a_1 = \frac{1}{\sqrt{2}} (e^{i\theta} a_\alpha + ia_\beta) \quad a_2 = \frac{1}{\sqrt{2}} (ie^{i\theta} a_\alpha + a_\beta) \]  

(8)

The amplitude for finding \( m_1, m_2 \) particles in the detectors given sources with \( N_\alpha, N_\beta \) particles in the sources is:

\[ C_{m_1,m_2}(N_\alpha, N_\beta) = \frac{1}{\sqrt{m_1!m_2!N_\alpha!N_\beta!}} \langle 0 | a_1^{m_1} a_2^{m_2} a_\alpha^{N_\alpha} a_\beta^{N_\beta} | 0 \rangle \]

\[ = \frac{e^{i\chi}}{\sqrt{N_\alpha!N_\beta!}} \frac{1}{\sqrt{m_1!m_2!}} \sum_{p,q} i^{q-p} m_1! m_2! p!q!(m_1-p)!q!(2)^{m_1+m_2-p-q} \delta_{p+q,N_\alpha} \delta_{m_1+m_2-p-q,N_\beta} \]  

(9)

where \( \chi \) is a phase factor without physical relevance. Two methods of calculation are now possible.

It is possible to replace the second \( \delta \)-function in Eq. (9) by:

\[ \delta_{m_1+m_2-p-q,N_\beta} = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i\varphi(m_1+m_2-p-q-N_\beta)} \]  

(10)

to obtain:

\[ C_{m_1,m_2}(N_\alpha, N_\beta) = \frac{e^{i\chi}}{2\pi \sqrt{N_\alpha!N_\beta!}} \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-iN_\beta\varphi} \]

\[ \times \left( e^{i\theta} + ie^{i\varphi} \right)^{m_1} \left( ie^{i\theta} + e^{i\varphi} \right)^{m_2} \]  

(11)

The square the modulus of this expression contains an integral over two variables \( \varphi \) and \( \varphi' \); if we make the changes of variables:

\[ \lambda = \frac{\varphi + \varphi' + \pi}{2} - \theta \quad \Lambda = \frac{\varphi - \varphi'}{2} \]  

(12)

we find for the probability the expression:

\[ P(m_1, m_2) = \frac{N_\alpha!N_\beta!}{m_1!m_2!} \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} \int_{-\pi}^{\pi} \frac{d\Lambda}{2\pi} \cos \left( (N_\alpha - N_\beta) \Lambda \right) \]

\[ \cos \lambda \cos \lambda^m \]  

(13)

(note that the phase shift \( \theta \) has disappeared from this result). Assume for a moment that the \( \Lambda \) can be replaced by 0 in the three cosines that contain it. Then \( \Lambda \) disappears, and we are left with an expression that is identical to (5) with \( r = 1 \), except for normalization factors. We therefore see that \( \lambda \) (or, more
precisely, \( \lambda + \pi/2 \) plays the role of the classical relative phase of the two sources; this phase is averaged over \( 2\pi \), which is normal since the phase in a Fock state is completely undetermined. For this reason, we will call \( \lambda \) the classical phase angle, and \( \Lambda \) the quantum angle; we study in more detail below how the non-zero values of \( \Lambda \) introduce quantum effects.

Another method is to use the \( \delta \)-functions to eliminate the summation variable \( q \) in Eq. (9) and then square the result. We then find:

\[
P(m_1, m_2) = \frac{m_1!m_2!N_\alpha!N_\beta!}{2^N} \left[ \sum_{p=0}^{m_1} \frac{(-1)^p}{p!(m_1-p)!(N_\alpha-p)!(p + m_2 - N_\alpha)!} \right]^2
\] (14)

This expression is more convenient than (13) for accurate numerical calculations.

### 1.2.2 Coherent states

We replace the ket (7) by a product of coherent input states:

\[
|\Psi\rangle \sim \sum_{n_\alpha} \sum_{n_\beta} \frac{1}{\sqrt{n_\alpha!n_\beta!}} \left[ E_\alpha a_\alpha^\dagger \right]^{n_\alpha} \left[ E_\beta a_\beta^\dagger \right]^{n_\beta} |0\rangle = \exp \left[ E_\alpha a_\alpha^\dagger + E_\beta a_\beta^\dagger \right] |0\rangle
\] (15)

where \( E_\alpha \) and \( E_\beta \) are complex numbers defining the intensity \( I_{\alpha, \beta} = |E_{\alpha, \beta}|^2 \) and the phases \( \varphi_{\alpha, \beta} \) of the incoming fields. In this expression, we can replace the creation operators by their expressions obtained from (8) and obtain:

\[
\exp \left[ E_\alpha a_\alpha^\dagger + E_\beta a_\beta^\dagger \right] |0\rangle = \exp \left[ \frac{E_\alpha e^{i\theta} + iE_\beta}{\sqrt{2}} a_\alpha^\dagger + \frac{iE_\alpha e^{i\theta} + E_\beta}{\sqrt{2}} a_\beta^\dagger \right] |0\rangle
\] (16)

Therefore, the output state is a product of coherent states as well, with amplitudes of the fields given by \((E_\alpha + iE_\beta) / \sqrt{2}\) and \((iE_\alpha + E_\beta) / \sqrt{2}\), which correspond exactly to the classical formulas. The operators contained in the exponential commute. By expanding it into a series as in (15), we obtain the probability to measure \( m_1 \) bosons at output 1 and \( m_2 \) at output 2 as a product of Poissonian distributions:

\[
P(m_1, m_2) \sim \left( \frac{m_1!m_2!}{(m_1+m_2)!} \right) \left( \frac{I_\alpha + iI_\beta}{\sqrt{2}} \right)^{m_1} \left( \frac{iI_\alpha e^{i\theta} + I_\beta}{\sqrt{2}} \right)^{m_2} [1 + r \cos (\lambda - \pi/2)]^{m_1} [1 - r \cos (\lambda - \pi/2)]^{m_2}
\] (17)

where \( \lambda = \varphi_\alpha - \varphi_\beta + \theta \). The result is therefore very similar to Eq. (4), as well as to (5) if we assume that the phases of the incoming coherent beams are random.

### 1.3 The generalized beam splitter theorem

The properties of the distributions obtained in (13) and (14) yield the generalized Hong-Ou-Mandel theorem as we see next.

#### 1.3.1 Calculation

The second line in Eq. (11) can be factored to produce:

\[
(e^{i\theta} + ie^{i\phi})^{m_1} (ie^{i\theta} + e^{i\phi})^{m_2} = (i)^{m_1} \left( 2^N e^{i\phi N/2} Q(\phi) \right)
\] (18)

where:

\[
Q(\phi) = \left( \cos \frac{\phi}{2} \right)^{m_1} \left( \sin \frac{\phi}{2} \right)^{m_2}
\] (19)
with:
\[ \tilde{\phi} \equiv \phi - \theta + \frac{\pi}{2} \] (20)

An exact result then is:
\[
C_{m_1m_2}(N_\alpha N_\beta) = 2^{N/2} \frac{N_\alpha!N_\beta!}{m_1!m_2!} \int_{-\pi}^{\pi} d\tilde{\phi} e^{i(N_\alpha - N_\beta)\tilde{\phi}/2} \]
\[
\times \left( \cos \frac{\tilde{\phi}}{2} \right)^{m_1} \left( \sin \frac{\tilde{\phi}}{2} \right)^{m_2} \] (21)

If \( N_\alpha = N_\beta = 1 \), we are in the situation of the Hong-Ou-Mandel effect [17]. When two photons fall on a beam splitter from two symmetrical directions, it is known that a quantum interference effect prevents them from leaving the beam splitter separately; they always leave together (in the same direction). Here we obtain a direct generalization of this theorem: if an equal number of particles approaches from each side to meet at the beam splitter, an even number must emerge from each side. This is because, if \( N_\alpha = N_\beta \) in the probability amplitude (21), we have \( N = m_1 + m_2 \) even, in which case \( m_1 \) and \( m_2 \) are both even or both odd; but, if \( m_2 \) is odd, the integral is over an odd function and therefore vanishes.

1.3.2 A Gaussian fit

A plot of the second line in Eq. (21) is shown in Fig. 3. We see from the figure that a (double) Gaussian fit to \( Q(\tilde{\phi}) \) is likely to be a good approximation. We set the first derivative of the logarithm of \( Q(\tilde{\phi}) \) to zero, which gives the maxima positions at \( \pm \phi_0 \) where:
\[
\phi_0 = 2 \arccos \left( \frac{m_1}{N} \right) \] (22)

and also find that the second derivative there is \(-2N\). The peak at \(-\phi_0\) is negative if \( m_2 \) is odd. Hence we get the approximation:
\[
Q(\tilde{\phi}) = \left( \frac{m_1}{N} \right)^{m_1/2} \left( \frac{m_2}{N} \right)^{m_2/2} \left[ e^{-\frac{N}{4}(\tilde{\phi} - \phi_0)^2} + (-1)^{m_2} e^{-\frac{N}{4}(\tilde{\phi} + \phi_0)^2} \right] \] (23)

where the prefactor to the Gaussians comes from \( Q(\phi_0) \) upon use of \( \cos^2 \phi_0/2 = m_1/M \) and \( \sin^2 \phi_0/2 = m_2/M \). In Eq. (21) we then Fourier transform the Gaussians; doing these integrals and squaring gives
the probability:

\[
P_{m_1 m_2}(N_\alpha N_\beta) = \frac{2^{N+2} N_\alpha! N_\beta!}{\pi N^{N+1}} m_1^{m_1} m_2^{m_2} e^{-\frac{1}{2\pi} (N_\alpha - N_\beta)^2} \\
\times \begin{cases} 
\cos \left( \frac{(N_\alpha - N_\beta) \phi_0}{2} \right)^2 & \text{for } m_2 \text{ even} \\
\sin \left( \frac{(N_\alpha - N_\beta) \phi_0}{2} \right)^2 & \text{for } m_2 \text{ odd}
\end{cases}
\]  

(24)

An interesting further approximation uses the Stirling formula for the \( m_i! \). We have

\[
m_i! \simeq \sqrt{2\pi m_i}^{m_i + \frac{1}{2}} e^{-m_i}
\]  

(25)

The factor of 1/2 in the exponent is usually dropped, but is actually important in our case in giving a characteristic shape to the probability curves. Moreover it makes the Stirling formula accurate to within a few percent for \( m_i > 2 \). The result is then:

\[
P_{m_1 m_2}(N_\alpha N_\beta) = \frac{2^{N+1} N_\alpha! N_\beta!}{\pi^2 N^{N+1}} \frac{e^{-N}}{\sqrt{m_1 m_2}} e^{-\frac{1}{2\pi} (N_\alpha - N_\beta)^2} \\
\times \begin{cases} 
\cos \left( \frac{(N_\alpha - N_\beta) \phi_0}{2} \right)^2 & \text{for } m_2 \text{ even} \\
\sin \left( \frac{(N_\alpha - N_\beta) \phi_0}{2} \right)^2 & \text{for } m_2 \text{ odd}
\end{cases}
\]  

(26)

Note the characteristic \( \sqrt{m_1 m_2} \) form in the denominator stemming here from the Stirling formula. This formula matches the exact result in Eq. (14) very accurately for \( m_i > 2 \).

2 Physical discussion

2.1 Few bosons

We now study in more detail the consequences of the rule according to which even numbers of particles emerge from the beam splitter if an equal number impinge on each side. For this purpose, we use Eq. (14) to calculate the distribution corresponding to the various possible numbers of particles detected at 1 and 2.

![Figure 4: The probability for \( N_\alpha = 4, N_\beta = 4 \) illustrating the rule that, if an even number of particles enters each side of the beam splitter, an even number must emerge from each side.](image)
Fig. 4 shows four particles entering each side of the beam splitter. As expected, only an even number can emerge on each side, which explains the zeroes of the curve. Comparing with Fig. 2 immediately indicates that these cancellations superimpose strong quantum oscillations onto the classical intensity distribution. We have a sort of combination of a classical average over a phase $\lambda$ with rapid variations created by the Hong-Ou-Mandel effect.

![Graph showing probability distribution](image)

Figure 5: The probability for $N_\alpha = 4$, $N_\beta = 5$. The probability no longer vanishes for odd $m_1$ but oscillations remain. Now, of course, odd $m_1$ implies even $m_2$ and vice versa.

For $N_\alpha = 4$, $N_\beta = 5$, the rule no longer applies, but the calculation of the distribution can still be done. The result is shown in Fig. 5 which again contains odd-even variations and oscillations, even if the probability does not vanish for any value of the $m_1$'s. In this case also, we have the superposition of an average curve, which can be understood in terms of a classical phase, and of an oscillation that can be seen as a generalized Hong-Ou-Mandel effect.

2.2 More particles

For larger values of $N$, the even rule is shown in Fig. 6 where the variation of the probability with $m_1$ is plotted for $N_\alpha = N_\beta = 25$. The characteristic probability variation $1/\sqrt{m_1 m_2}$ given in Eq. (26) is visible, but again with strong quantum oscillations. In this case in Eq. (26) the factor $\sin \left[ (N_\alpha - N_\beta) \phi_0 / 2 \right]^2$ vanishes to satisfy the even rule.

Suppose now we have slightly different source populations, $N_\alpha = 26$ and $N_\beta = 25$. The result is shown in Fig. 7. When $N_\alpha - N_\beta = 1$ we can find explicitly that:

\[
\cos \left( \frac{(N_\alpha - N_\beta) \phi_0}{2} \right)^2 = \cos \left( \frac{\phi_0}{2} \right)^2 = \frac{m_1}{N}
\]

\[
\sin \left( \frac{(N_\alpha - N_\beta) \phi_0}{2} \right)^2 = \sin \left( \frac{\phi_0}{2} \right)^2 = \frac{N - m_1}{N}
\]

The probability oscillates between these two values, modulated by the $1/\sqrt{m_1 m_2}$ factor, and changes over from maxima at even to odd at $m_1 = 25$. It is interesting to see that, in this case, the quantum oscillations vanish at the center of the distribution, but remain very pronounced at both sides.

Larger population imbalances in the sources, at constant sum $N = 50$, result in even more complicated
Figure 6: The probability for $N_\alpha = N_\beta = 25$. The probability vanishes for odd $m_1$. The graph has been cut off at $m_1 = 0, 50$ where it is about twice as high. The variation with $1/\sqrt{m_1m_2}$ is evident. The dotted line shows the corresponding semi-classical distribution.

Figure 7: The probability for $N_\alpha = 26, N_\beta = 25$.

behavior. What happens for instance if $N_\alpha - N_\beta = 2$ is shown in Fig. 8. We can then show that:

$$\cos \left( \frac{(N_\alpha - N_\beta) \phi_0}{2} \right)^2 = \cos (\phi_0)^2 = \left( \frac{2m_1}{N} - 1 \right)^2$$

$$\sin \left( \frac{(N_\alpha - N_\beta) \phi_0}{2} \right)^2 = \sin (\phi_0)^2 = 4 \frac{m_1(N - m_1)}{N^2}$$

(28)

The probability oscillates between these two curves with “nodes” at $m_1 = 7$ and $43$ corresponding to the crossing of the two quadratic curves. The nodes are actually a consequence of the discrete character of $m_1$: if $m_1$ is replaced by a continuous variable in Eq. (21), then the probability distribution becomes an oscillating function with a slowly varying amplitude. If the maxima and minima occur near integer values of $m_1$, they remain very visible in the discrete version of the distribution, resulting in antinodes; but, if they occur near half integer values, the oscillations disappear in the discrete version, resulting in nodes. This “stroboscopic effect” also explains the minimum of oscillations at the center of Fig. 7.

The “beating wavelength” becomes shorter as $N_\alpha - N_\beta$ becomes larger. The case $N_\alpha = 28, N_\beta = 22$ is shown in Fig. 8-a; the case $N_\alpha = 44, N_\beta = 6$ is shown in Fig. 8-b. In this case, the fast oscillations of the generalized Hong-Ou-Mandel type have disappeared and have become significantly slower, which presumably makes them easier to observe experimentally.
2.3 Role of the quantum angle

If $N_{\alpha} = N_{\beta}$, we can compare the semi-classical expression, Eq. (5) with $r = 1$, to the quantum result Eq. (13). We see that, instead of classical probabilities $|1 \pm \cos \lambda|$, the quantum expression contains quasi-probabilities $|\cos \Lambda \pm \cos \lambda|$, which can take negative values when the quantum angle $\Lambda$ does not vanish. This introduces quantum effects in a way that is reminiscent of quantum effects arising from negative values of the Wigner transform. The quantum angle is responsible for the population oscillations introduced by the beam splitter.

When $N_{\alpha} \neq N_{\beta}$, Eq. (13) shows that the quantum angle also controls the effect of population imbalance. The quantum formula, instead of including in the quasi-probabilities a factor:

$$r = \frac{2\sqrt{N_{\alpha}N_{\beta}}}{N_{\alpha} + N_{\beta}}$$

contains inside the integral an oscillating function $\cos [(N_{\alpha} - N_{\beta}) \Lambda]$. Figs. 8 and 9 illustrate the effects of population imbalance.

To see what happens when the effect of $\Lambda$ is cancelled, let us set $\Lambda = 0$ in Eq. (13). We write $|1 \pm \cos \lambda|$ in terms of sine and cosine of the half angle, expand these in exponentials, expand the binomials, and...
integrate. The result is:

\[
P_{\lambda=0}(m_1, m_2) = \frac{N!(2m_1)!(2m_2)!}{4^N m_1! m_2!} \left[ \sum_{p=0}^{\min(m_1,m_2)} \frac{(-1)^{m_1+p}}{p!(2m_1-p)!(N-p)!(p-2m_1+N)!} \right]^{m_1} \]

(30)

No oscillation then takes place; for instance, the case of \(N_\alpha = N_\beta = 25\) was already shown in Fig. 2.

2.4 Pair-probability formulation

When \(N_\alpha = N_\beta = 1\), the Hong-Ou-Mandel result is that the probabilities of having \(n\) particles in the detector 1 and \(2-n\) in detector 2 is:

\[
P_n = \frac{1}{2} [1 + (-1)^n]
\]

(31)

with \(n = 0, 1, 2\). Now, a natural question is: can we consider that the distributions obtained above can be interpreted as those that one would be obtained by repeating the Hong-Ou-Mandel experiment a sufficient number of times, and accumulating the counts in each detector?

First consider the case when \(N_\alpha = N_\beta = N/2\). We pick the first pair, and it produces either 2 particles on the left or none. Then the second pair does the same. We continue until we have \(m_1/2\) pairs on the left and \(m_2/2\) pairs on the right. We consider that, on the left we have \(m_1/2\) filled pair slots and \(m_2/2\) empty pair slots, which could have been selected in any order. We interchange the slots, while not counting interchanges of the empty slots among themselves and the filled among themselves. We have then \((N/2)!/(m_1/2)!(m_2/2)!\) different ways for getting the \((m_1, m_2)\) probability, which is:

\[
P_{\text{pair}}^{m_1,m_2} \sim \frac{(N/2)!}{(m_1/2)!(m_2/2)!} [(1 + (-1)^{m_1})]
\]

(32)

where the last factor ensures that there are an even number of particles on each side. For 50 particles the result is shown in Fig. [10]

![Figure 10: The pair probability for \(N_\alpha = 25\), \(N_\beta = 25\).](image)

We see that the distribution obtained in this way still has the expected odd-even behavior (actually almost by construction), but that it does not reproduce the characteristic \(1/\sqrt{m_1 m_2}\) shape for the envelope. Instead, it tends to concentrate the most likely results around \(m_1 \approx m_2 \approx N/2\), which is natural: if we have independent scattering events of pairs into either channel, chosen randomly, one expects that the most likely values will be equal in both outputs.
By contrast, the curve of Fig. 6 has its maxima at $m_1 = 0$ and $m_1 = N$, which is a completely different behavior, and indicates the effect of bosonic statistics (bunching into one channel). The distribution is no longer peaked at the center but spreads towards both sides. As Fig. 2 shows, this behavior can be explained if we add a new ingredient, a relative phase. But, since Fock sources do not have any initial phase, this can be understood as a result of a spontaneous choice of a relative phase by the two sources under the effect of quantum measurement [11]; since this phase is completely unknown, an average over all possible values is taken.

Now, if we have an excess of particles on one side, for example, $N_\alpha = N_\beta + \mathcal{N}$, then we can assume each $\beta$ particle is paired with an $\alpha$ particle and the $\mathcal{N}$ extras appear anywhere in the sequence of selections as singles. On the left side, we have $N_\beta$ pair slots with $f$ of them filled and $e$ of them empty. Of the single slots on the left, $s$ are filled and $o$ are open. Then we have

$$N_\beta = f + e$$
$$N_\alpha - N_\beta = s + o$$
$$m_1 = 2f + s$$

These can be solved to give:

$$f = \frac{1}{2}(m_1 - s)$$
$$o = N_\alpha - N_\beta - s$$
$$e = N_\beta - \frac{1}{2}(m_1 - s)$$

Then among the pairs and singles on the left we can rearrange in $f + e + s + o = N_\alpha$ total ways with rearrangements among the same kind of slots not counting to give a probability

$$P_{m_1 m_2} \sim \sum_{s=0}^{N_\alpha - N_\beta} \frac{(N_\alpha)!}{(m_1 - s)! (N_\alpha - N_\beta - s)! s! (N_\beta - \frac{1}{2}(m_1 - s))!} \left[ (1 + (-1)^{m_1 - s}) \right]$$

The probability vanishes if $m_1 - s$ is odd. The result for one extra particle is shown in Fig. 11. The results are then very different from those of Fig. 7. If there are two excess particles the steps on the side of the peak are completely smoothed out. The conclusion of that, in this case, the model of independent and repeated Hong-Ou-Mandel scatterings does not provide a good representation of the phenomenon at all.

![Figure 11: The pair probability for $N_\alpha = 26$, $N_\beta = 25$.](image)

The conclusion of that, in this case, the model of independent and repeated Hong-Ou-Mandel scatterings does not provide a good representation of the phenomenon at all.
3 Conclusion

A single beam splitter cannot exhibit quantum non-local effects; violating local realism requires the combination of several such devices to form interferometers [11]. Nevertheless, here we have seen one beam splitter is sufficient to obtain interesting quantum effects, provided it receives Fock states at its two inputs: these effects are similar to the “population oscillations” predicted in more elaborate cases [18,19]. The oscillations are related to the Hong-Ou-Mandel effect, but they cannot be understood as a simple juxtaposition of many separate two-photon-experiments. Actually, many-boson effects take place as a consequence of quantum statistics, which can be understood as a consequence of the tendency of two Fock states to acquire a relative phase under the effect of quantum measurement. Since this phase is completely unknown, the characteristic dependence shown in Fig. 2 results, onto which quantum oscillations are superimposed. Experimentally, the major difficulty for observing these effects is the production of Fock states with well-defined populations. Nevertheless, the experimental techniques that have been developed for Bose-Einstein condensates in ultra-cold gases seem well suited to planning experiments with input states that contain for instance a few tens of bosons.

Note added in proofs: the authors have recently become aware of Ref. [20], which gives a study of the localization of phase obtained by measurements of particles at the output of a single beam splitter. The theoretical treatment is similar to ours but this reference does not assume that all particles are detected; the oscillations we have discussed (generalization of the Hong-Ou-Mandel effect) do not appear.

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