HYERS-ULAM STABILITY OF THE SPHERICAL FUNCTIONS

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Abstract. In [10] the authors obtained the Hyers-Ulam stability of the functional equation

\[ \int_K \int_G f(xtk \cdot y)d\mu(t)dk = f(x)g(y), \quad x, y \in G, \]

where \(G\) is a Hausdorff locally compact topological group, \(K\) is a compact subgroup of morphisms of \(G\), \(\mu\) is a \(K\)-invariant complex measure with compact support, provided that the continuous function \(f\) satisfies some Kannappan Type condition. The purpose of this paper is to remove this restriction.

1. Introduction

The stability problem of functional equations was posed for the first time by S. M. Ulam [58] in the year 1940. Ulam stated the problem as follows:

Given a group \(G_1\), a metric group \((G_2, d)\), a number \(\varepsilon > 0\) and a mapping \(f : G_1 \rightarrow G_2\) which satisfies the inequality \(d(f(xy), f(x)f(y)) < \varepsilon\) for all \(x, y \in G_1\), does there exist an homomorphism \(h : G_1 \rightarrow G_2\) and a constant \(k > 0\), depending only on \(G_1\) and \(G_2\) such that \(d(f(x), h(x)) \leq k\varepsilon\) for all \(x\) in \(G_1\)?

The first affirmative answer was given by D. H. Hyers [25], under the assumption that \(G_1\) and \(G_2\) are Banach spaces.

In 1978, Th. M. Rassias [43] gave a remarkable generalization of the Hyers’s result which allows the Cauchy difference to be unbounded, as follows:

**Theorem 1.1.** [16] Let \(f : V \rightarrow X\) be a mapping between Banach spaces and let \(p < 1\) be fixed. If \(f\) satisfies the inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|x\|^p) \]

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for some $\theta \geq 0$ and for all $x, y \in V \ (x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique additive mapping $T : V \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{|2 - 2p|} \|x\|^p$$

for all $x \in V \ (x \in V \setminus \{0\}$ if $p < 0$).

If, in addition, $f(tx)$ is continuous in $t$ for each fixed $x$, then $T$ is linear.

Several papers have been published in this subject and some interesting variants of Ulam’s problem have been also investigated by a number of mathematicians. We refer the reader to the following references [11], [13], [20], [21], [22], [26]–[46].

The stability of functional equations highlighted a new phenomenon which is now usually called superstability. Consider the functional equation $E(f) = 0$ and assume we are in a framework where the notion of boundedness of $f$ and of $E(f)$ makes sense. We say that the equation $E(f) = 0$ is superstable if the boundedness of $E(f)$ implies that either $f$ is bounded or $f$ is a solution of $E(f) = 0$. This property was first observed when the following theorem was proved by J. Baker, J. Lawrence, and F. Zorzitto [9].

**Theorem 1.2.** Let $V$ be a vector space. If a function $f : V \rightarrow \mathbb{R}$ satisfies the inequality

$$|f(x + y) - f(x)f(y)| \leq \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in V$, then either $f$ is bounded on $V$ or $f(x + y) = f(x)f(y)$ for all $x, y \in V$.

The result was generalized by J. A. Baker [8], by replacing $V$ by a semigroup and $\mathbb{R}$ by a normed algebra $E$, in which the norm is multiplicative, i.e. $\|uv\| = \|u\|\|v\|$, for all $u, v \in E$, by R. Ger, P. Semrl [24], where $E$ is an arbitrary commutative complex semisimple Banach algebra and by J. Lawrence [33] in the case where $E$ is the algebra of all $n \times n$ matrices. A different generalization of the result of Baker, Lawrence and Zorzitto was given by L. Székelyhidi [55], [56], [57]. It involves an interesting generalization of the class of bounded function on a group or semigroup. For other superstability results, we can see for example [17], [12], [23], [31], [32] and [48].

Let $G$ be a Hausdorff locally compact group, $e$ its identity element. Let $K$ be a compact subgroup of the group $\text{Mor}(G)$ of all mappings $k$ of $G$ onto itself that are either automorphisms and homeomorphisms ($k \in K^+$), or antiautomorphisms and homeomorphisms ($k \in K^-$). The
action of \( k \in K \) on \( x \in G \) will be denoted by \( k \cdot x \). Let \( \mu \) be a complex bounded measure on \( G \) with compact support (i.e, \( \mu \) is an element of the topological dual of the Banach spaces of continuous functions vanishing at infinity on \( G \)). \( \mu \) is assumed to be a \( K \)-invariant measure that is, \( \int_G f(k \cdot t)d\mu(t) = \int_G f(t)d\mu(t) \), for all \( k \in K \) and for all continuous complex valued function \( f \) on \( G \).

The main purpose of this paper is to investigate the Hyers-Ulam stability of the functional equations

\[
\int_G \int_K f(xtk \cdot y)d\mu(t)dk = f(x)g(y), \quad x, y \in G.
\]

Indeed we prove the superstability theorem of the functional equation

\[
\int_G \int_K f(xtk \cdot y)d\mu(t)dk = f(x)f(y), \quad x, y \in G.
\]

The functional equation (1.1) is a generalization of many functional equations. The functional equation (1.2) with \( \mu = \delta_e \): Dirac measure concentrated on the identity element of \( G \) reduce to \( K \)-spherical functions:

\[
\int_K f(xk \cdot y)dk = f(x)f(y), \quad x, y \in G.
\]

The \( K \)-spherical functions and related equations has been widely studied by H. Stetkær see for example [53] and [54]. The bounded solutions of \( K \)-spherical functions in an abelian group are obtained by W. Chojnacki [16] and later by Badora [5] while Stetkær [50], [49] studied unbounded solutions. In [47] H. Shin’ya described all continuous solutions of (1.3) for abelian group. The functional equation (1.2) is considered in [18], [19] and [5]. The functional equation

\[
\int_K f(xk \cdot y)dk = f(x)g(y), \quad x, y \in G
\]

has been examined in special cases by many mathematicians. These cases for example include the cosine equation or d’Alembert’s functional equations (cf. [11], [51], [52]...)

\[
f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in G,
\]

Wilson’s functional equation

\[
f(x + y) + f(x - y) = 2f(x)g(y), \quad x, y \in G,
\]

where \( K = \{Id, -Id\} \) and the Cauchy’s equation

\[
f(x + y) = f(x)f(y), \quad x, y \in G,
\]
During the last three decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability of a special case of the functional equation (1.3) and its generalization (1.4). In [6] R. Badora obtained the Hyers-Ulam stability of equation (1.4), where $G$ is abelian and $K \subseteq \text{Aut}(G)$: the group of automorphisms of $G$.

We note here that the results of R. Badora [6] are also corrects in the case where $G$ is not necessarily abelian and $K \subseteq \text{Aut}(G)$. Other results of stability of functional equations related to $K$-spherical functions were studied in [2], [3], [4], [14] and [15].

In [10] B. Bouikhalene and E. Elqorachi obtained the Hyers-Ulam stability of the functional equation (1.1), provided that the continuous function $f$ satisfies the Kannappan type condition:

$$\int_G \int_G f(ztysx)d\mu(t)d\mu(t) = \int_G \int_G f(ztysx)d\mu(t)d\mu(t)$$

for all $x, y, z \in G$. The purpose of this paper is to remove this restriction.

Throughout this paper, $G$ is a locally compact group (not necessarily abelian) $K$ is a compact subgroup of morphisms of $G$ and $\mu$ is a complex measure with compact support and which is $K$-invariant.

2. HYERS ULAM STABILITY OF EQUATION (1.1)

In this section, we will investigate the Hyers Ulam stability of equation (1.1). The following lemma will be helpful in the sequel.

**Lemma 2.1.** Let $f: G \rightarrow \mathbb{C}$ be a continuous function. Let $\mu$ be a complex measure with compact support and which is $K$-invariant. Then (2.1)

$$\int_G \int_K \int_G f(zt\cdot(k\cdot ysx))d\mu(t)d\mu(s) + \int_G \int_K \int_G f(zt\cdot(xsk\cdot y))d\mu(t)d\mu(s)dk$$

$$= \int_G \int_K \int_G f(ztk\cdot ysh\cdot x)d\mu(t)d\mu(s) + \int_G \int_K \int_G f(zt\cdot xsk\cdot y)d\mu(t)d\mu(s)dk$$

for all $x, y, z \in G$.

**Proof.**

$$\int_G \int_K \int_G f(zt\cdot(k\cdot ysx))d\mu(t)d\mu(s) + \int_G \int_K \int_G f(zt\cdot(xsk\cdot y))d\mu(t)d\mu(s)dk$$
Theorem 2.2. Let □ + □ = \int x, y for all (2.4)

\[ g(\cdot) = 2g(x)g(y), \quad x, y \in G \]
or iii) $g$ is unbounded, $f$ satisfies (1.7) (if $f \neq 0$, then $g$ satisfies equation (2.3)).

Proof. Assume that $f, g$ are continuous and satisfy inequality (2.4). In the first case, we suppose that $f$ is unbounded. Then from (2.4) we get

$$\left| \int_G \int_K \int_K J_G f(z)(xsk \cdot y) d\mu(t) dh d\mu(s) - f(z) \int_G \int_K J_G g(xsk \cdot y) d\mu(t) dh d\mu(s) \right|$$

$$\leq \int_G \int_K \int_K J_G f(z)(xsk \cdot y) d\mu(t) dh - f(z) \int_G \int_K J_G g(xsk \cdot y) d\mu(t) dh \leq \delta \|\mu\|$$

Therefore,

$$\left| \int_G \int_K \int_K J_G f(z)(xsk \cdot y) d\mu(t) dh d\mu(s) - f(z) \int_G \int_K J_G g(xsk \cdot y) d\mu(t) dh d\mu(s) \right| \leq \delta \|\mu\|.$$ 

So, by using lemma 2.1, the triangle inequality, we obtain

$$|f(z)||2g(x)g(y) - \int_G \int_K g(xtk \cdot y) d\mu(t) dk - \int_G \int_K g(k \cdot ytx) d\mu(t) dk|$$

$$\leq |\int_G \int_K \int_K J_G f(z)(xsk \cdot y) d\mu(t) dh d\mu(s) - f(z) \int_G \int_K J_G g(xsk \cdot y) d\mu(t) dh d\mu(s)|$$

$$+ |\int_G \int_K \int_K J_G f(z)(xsk \cdot y) d\mu(t) dh d\mu(s) - f(z) \int_G \int_K J_G g(xsk \cdot y) d\mu(t) dh d\mu(s)|$$

Therefore,

$$|f(z)||2g(x)g(y) - \int_G \int_K g(xtk \cdot y) d\mu(t) dk - \int_G \int_K g(k \cdot ytx) d\mu(t) dk|$$

$$\leq \delta \|\mu\| + \delta \|\mu\| + \delta \|\mu\| + \delta \|\mu\| + |g(y)| \delta + |g(x)| \delta.$$ 

Since $f$ is assumed to be unbounded, then we get

$$2g(x)g(y) - \int_G \int_K g(xtk \cdot y) d\mu(t) dk - \int_G \int_K g(k \cdot ytx) d\mu(t) dk = 0$$

for all $x, y \in G$. This proves the case ii). Now, assume that $g$ is unbounded. It’s easily verified that $f = 0$ satisfies equation (1.1). For latter, we suppose that $f \neq 0$. From inequality (2.4) and the triangle
inequality, we conclude that $f$ is also unbounded, then from the case ii) the function $g$ satisfies equation (2.5). For all $x, y, z \in G$, we have
\[ |g(z)| \int_G \int_K f(xtk \cdot y) d\mu(t) dk - f(x)g(y)| 
\leq |\int_G \int_K \int_G \int_K f(xtk \cdot ysk \cdot z) d\mu(t) d\mu(s) dhdk - \int_G \int_K f(xtk \cdot y) d\mu(t) dk g(z)| 
+ |\int_G \int_K \int_G \int_K f(xth \cdot ysk \cdot z) d\mu(t) d\mu(s) dhdk - f(x) \int_G \int_K g(ysk \cdot z) d\mu(s) dk| 
+ |\int_G \int_K \int_G \int_K f(xth \cdot k \cdot zsy) d\mu(t) d\mu(s) dhdk - f(x) \int_G \int_K g(k \cdot zsy) d\mu(s) dk| 
+ |\int_G \int_K \int_G \int_K f(xtk \cdot ysh \cdot z) d\mu(t) d\mu(s) dhdk - \int_G \int_K f(xtk \cdot z) d\mu(t) dk g(y)| 
+ |g(y)| |\int_G \int_K f(xtk \cdot z) d\mu(t) dk - f(x)g(z)| 
+ |f(x)||\int_G \int_K g(ysk \cdot z) d\mu(s) dk + \int_G \int_K g(k \cdot zsy) d\mu(s) dk - 2g(y)g(z)| 
\leq 4\delta\|\mu\| + |g(y)|\delta + |f(x)| \times 0 = 4\delta\|\mu\| + |g(y)|\delta.
Since $g$ is unbounded, then $f$ satisfies equation (1.1). This completes the proof. \hfill \square

By using the above result, we get the following corollary.

**Corollary 2.3.** (Superstability of equation (1.2)) Let $\delta > 0$. If a continuous function $f: G \rightarrow \mathbb{C}$ satisfies the inequality

\[ |\int_G \int_K f(xtk \cdot y) d\mu(t) dk - f(x)f(y)| \leq \delta, \quad x, y \in G. \]

Then either

\[ |f(x)| \leq \frac{\|\mu\| + \sqrt{\|\mu\|^2 + 4\delta}}{2}, \quad x \in G \]

or

\[ \int_G \int_K f(xtk \cdot y) d\mu(t) dk = f(x)f(y), \quad x, y \in G. \]

**Corollary 2.4.** (Superstability of the classical d’Alembert’s functional equation) Let $\mu = \delta_e$. Let $\delta > 0$. Let $\sigma: G \rightarrow G$ be an involution of $G$ ( $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$). Let $K = \{I, \sigma\}$. If a function $f: G \rightarrow \mathbb{C}$ satisfies the inequality

\[ |f(xy) + f(x\sigma(y)) - 2f(x)f(y)| \leq \delta, \quad x, y \in G. \]
Then either

\[ |f(x)| \leq \frac{1 + \sqrt{1 + 2\delta}}{2}, \quad x \in G \]

or

\[ f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G. \]

The following general corollary holds on any group and for \( K \subseteq Mor(G) \). It’s a generalization of the result obtained by Badora in [6].

**Corollary 2.5.** Let \( \mu = \delta_e \). Let \( \delta > 0 \). Suppose that the continuous function \( f: G \rightarrow \mathbb{C} \) satisfy the inequality

\[ |\int_K f(xk \cdot y)dk - f(x)g(y)| < \delta, \quad x, y \in G. \]

Then,

i) \( f, g \) are bounded or

ii) \( f \) is unbounded and \( g \) satisfies the functional equation

\[ \int_K g(xk \cdot y)dk + \int_K g(k \cdot x)dk = 2g(x)g(y), \quad x, y \in G \]

or iii) \( g \) is unbounded, \( f \) satisfies (1.4) (if \( f \neq 0 \), then \( g \) satisfies equation (2.13)).

**Corollary 2.6.** [6] Let \( \mu = \delta_e, K \subseteq \text{Aut}(G) \). Let \( \delta > 0 \). Suppose that the continuous functions \( f, g: G \rightarrow \mathbb{C} \) satisfy the inequality

\[ |\int_K f(xk \cdot y)dk - f(x)g(y)| < \delta, \quad x, y \in G. \]

Then,

i) \( f, g \) are bounded or

ii) \( f \) is unbounded and \( g \) satisfies the functional equation

\[ \int_K g(xk \cdot y)dk = g(x)g(y), \quad x, y \in G \]

or iii) \( g \) is unbounded, \( f \) satisfies (1.4) (if \( f \neq 0 \), then \( g \) satisfies equation (2.13)).

**Corollary 2.7.** Let \( \delta > 0 \). Let \( K = \{\text{Id}, \sigma\} \), where \( \sigma \) is an involution of \( G \). \( \mu \) is a complex measure with compact support and which is \( \sigma \)-invariant. Suppose that the continuous functions \( f, g : G \rightarrow \mathbb{C} \) satisfy the inequality

\[ |\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)| < \delta, \quad x, y \in G. \]
Then,  

i) if, g are bounded or

ii) f is unbounded and g satisfies the functional equation

\[ (2.17) \int_G g(xty) d\mu(t) + \int_G g(ytx) d\mu(t) + \int_G g(\sigma(y)tx) d\mu(t) = 4g(x)g(y) \]

or iii) g is unbounded, f satisfies

\[ (2.18) \int_G f(xty) d\mu(t) + \int_G f(xt\sigma(y)) d\mu(t) = 2f(x)g(y), x, y \in G \]

(if \( f \neq 0 \), then g satisfies equation \( (2.17) \)).

The following corollary is a generalization of the result obtained by E. Elqorachi and M. Akkouchi in [17] under the condition that f satisfies the Kannappan type condition or \( \mu \) is a generalized Gelfand measure.

**Corollary 2.8.** Let \( \delta > 0 \). Let \( K = \{\text{Id}, \sigma\} \), where \( \sigma \) is an involution of \( G \). \( \mu \) is a complex measure with compact support and which is \( \sigma \)-invariant. Suppose that the continuous functions \( f, g : G \rightarrow \mathbb{C} \) satisfy the inequality

\[ (2.19) \left| \int_G f(xty) d\mu(t) + \int_G f(xt\sigma(y)) d\mu(t) - 2f(x)f(y) \right| < \delta, x, y \in G. \]

Then either

\[ (2.20) |f(x)| \leq \frac{\|\mu\| + \sqrt{\|\mu\|^2 + 2\delta}}{2}, x \in G \]

or

\[ (2.21) \int_G f(xty) d\mu(t) + \int_G f(xt\sigma(y)) d\mu(t) = 2f(x)f(y), x, y \in G \]

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