The structure of spaces with Bakry–Émery Ricci curvature bounded below

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Abstract. We explore the structure of limit spaces of sequences of Riemannian manifolds with Bakry–Émery Ricci curvature bounded below in the Gromov–Hausdorff topology. By extending the techniques established by Cheeger and Colding for Riemannian manifolds with Ricci curvature bounded below, we prove that each tangent space at a point of the limit space is a metric cone. We also analyze the singular structure of the limit space as in a paper of Cheeger, Colding and Tian. Our results will be applied to study the limit spaces for a sequence of Kähler metrics arising from solutions of certain complex Monge–Ampère equations for the existence problem of Kähler–Ricci solitons on a Fano manifold via the continuity method.

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0. Introduction

In a series of papers [5–7], Cheeger and Colding studied the singular structure of limit spaces of sequences of Riemannian manifolds with Ricci curvature bounded below in the

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Gromov–Hausdorff topology. As a fundamental result, they proved the existence of metric cone structure for the tangent cone on the limit space [6]. Namely:

**Theorem 0.1 ([6]).** Let \((M_i, g_i; p_i)\) be a sequence of \(n\)-dimensional Riemannian manifolds which satisfy

\[
\text{Ric}_{M_i}(g_i) \geq -(n - 1)A^2 g_i \quad \text{and} \quad \text{vol}_{g_i}(B_{p_i}(1)) \geq v > 0.
\]

Then \((M_i, g_i; p_i)\) converges to a metric space \((Y; p_\infty)\) in the pointed Gromov–Hausdorff topology. Moreover, for any \(y \in Y\), each tangent cone \(T_y Y\) is a metric cone over another metric space whose diameter is less than \(\pi\).

Based on the above theorem, Cheeger and Colding introduced a notion of \(S_k\)-type (with \(k \leq n - 1\)) singularities of the limit space \(Y\) as follows.

**Definition 0.2.** Let \((Y; p_\infty)\) be the limit of \((M_i, g_i; p_i)\) as in Theorem 0.1. We call \(y \in (Y; p_\infty)\) an \(S_k\)-type singular point if there exists a tangent cone at \(y\) which can be split out a Euclidean space \(\mathbb{R}^k\) isometrically with dimension at most \(k\).

As an application of Theorem 0.1 to appropriate tangent cone spaces \(T_y Y\), Cheeger and Colding [6] showed that the dimension of set \(S_k\) is less than \(k\). After that, the important work was made by Cheeger, Colding and Tian [8], and Cheeger [4] to determine which kind of singularities can be excluded in the limit space \(Y\) under certain curvature condition for the sequence of \((M_i, g_i)\). For example, Cheeger, Colding and Tian proved the following:

**Theorem 0.3 ([8]).** Let \((M_i, g_i; p_i)\) be a sequence of \(n\)-dimensional manifolds and \((Y, p_\infty)\) its limit as in Theorem 0.1. Suppose that the integrals of sectional curvature

\[
\frac{1}{\text{vol}_{g_i}(B_{p_i}(1))} \int_{B_{p_i}(1)} |\text{Rm}|^p
\]

are uniformly bounded. Then for any \(\epsilon > 0\), the following is true:

\[
\dim(B_{p_\infty}(1) \setminus R_\epsilon) \leq n - 4 \quad \text{if} \quad p = 2
\]

and

\[
\mathcal{H}^{n-2p}(B_{p_\infty}(1) \setminus R_\epsilon) < \infty \quad \text{if} \quad 1 \leq p < 2,
\]

where \(R_\epsilon\) consists of points \(y\) in \(Y\) which satisfy \(\text{dist}_{\text{GH}}(B_y(1), B_0(1)) \leq \epsilon\) for the unit ball \(B_0(1)\) in \(\mathbb{R}^n\) and a unit distance ball \(B_y(1)\) in some tangent cone \(T_y Y\).

The purpose of the present paper is to extend the above Cheeger–Colding theory in the Bakry–Émery geometry. Specifically, we analyze the structure of Gromov–Hausdorff limit for a sequence of \(n\)-dimensional Riemannian manifolds with Bakry–Émery Ricci curvature bounded below in the class \(\mathcal{M}(A, \Lambda, v)\) defined by

\[
\mathcal{M}(A, \Lambda, v) = \{(M, g; p) : M \text{ is an } n\text{-dimensional complete Riemannian manifold which satisfies } \text{Ric}_M(g) + \text{hess}(f) \geq -(n - 1)A^2 g, \text{vol}_g(B_p(1)) \geq v > 0 \text{ and } |\nabla f|_g \leq A\}.
\]
Here \( f \) is a smooth function on \( M \) and \( \text{hess}(f) \) denotes the Hessian tensor of \( f \) with respect to \( g \). Recall that \( \text{Ric}_M(g) + \text{hess}(f) \) is called the Bakry–Émery Ricci curvature associated to \( f \). For simplicity, we denote it by \( \text{Ric}^f_{M, g} \) or just \( \text{Ric}^f_g \). The notion of this curvature has appeared in the work on diffusion processes by Bakry and Émery [1] (also see the more early papers [19, 20] by Lichnerowicz). Clearly any compact Ricci soliton is contained in one of the classes \( \mathcal{M}(A, \Lambda, v) \) (see [15, 24]). We show that both Theorem 0.1 and Theorem 0.3 still hold for a sequence in \( \mathcal{M}(A, \Lambda, v) \) (cf. Sections 4 and 5).

As in [5], we shall establish various integral comparison theorems of the gradients and Hessians between appropriate functions and coordinate functions or distance functions on a Riemannian manifold with Bakry–Émery Ricci curvature bounded below. We will use \( f \)-harmonic functions to construct these appropriate functions instead of harmonic functions (cf. Section 2). Another technique is to generalize the segment inequality lemmas in [5] to our case of weighted volume form (cf. Lemma 3.3, Lemma 3.4, Lemma 3.5) so that the triangle lemmas in [3] are still true on a Riemannian manifold with almost flat Bakry–Émery Ricci curvature (cf. Lemma 3.2, Lemma 4.4). These triangle lemmas are crucial in the proofs of the splitting theorem and the metric cone theorem (cf. Theorem 3.1, Theorem 4.3). We shall point out that various versions of such triangle lemmas were used by Colding, Cheeger and Colding in earlier papers to study the rigidity of Riemannian metrics [5, 9, 10].

Another motivation of this paper is to study the limit space for a sequence of Kähler metrics \( g_t \) arising from solutions of certain complex Monge–Ampère equations for the existence problem of Kähler–Ricci soliton via the continuity method [25, 26]. We show that such metrics naturally belong to \( \mathcal{M}(A, v, \Lambda) \). As a consequence, for any sequence \( \{g_t\} \) there exists a subsequence which converges to a metric space with complex codimension of singularities at least one in the Gromov–Hausdorff topology (cf. Theorem 6.2). Furthermore, in case of weak almost Kähler–Ricci solitons (cf. Theorem 6.8), we prove:

**Theorem 0.4.** Suppose that \( \{g_t\} \) is a sequence of weak almost Kähler–Ricci solitons on a Fano manifold \( M \). Then there exists a subsequence of \( \{g_t\} \) which converges to a metric space \( (M_\infty, g_\infty) \) in the Gromov–Hausdorff topology. The complex codimension of singularities of \( M_\infty \) is at least two.

As a corollary of Theorem 0.4, we show that there exists a sequence of weak almost Kähler–Ricci solitons on \( M \) which converges to a metric space \( (M_\infty, g_\infty) \) with complex codimension of the singular set of \( (M_\infty, g_\infty) \) at least two in the Gromov–Hausdorff topology if the modified Mabuchi \( K \)-energy defined in [25] is bounded below. In a sequel of papers [17, 27], we will further confirm that the regular part of \( (M_\infty, g_\infty) \) is in fact a Kähler–Ricci soliton by using the Ricci flow as done in [23], and prove the Tian’s partial \( C^0 \)-conjecture for almost Kähler–Ricci solitons.

The organization of paper is as follows. In Section 1, we first recall an \( f \)-Laplace comparison lemma of Wei and Wylie for distance functions (cf. Lemma 1.1). Then, as applications of Lemma 1.1, we prove several useful lemmas under the condition of Bakry–Émery Ricci curvature bounded below. In Section 2, we give various integral estimates for gradients and Hessians of \( f \)-harmonic functions. In Section 3 and Section 4, we prove the splitting theorem (cf. Theorem 3.1) and the metric cone theorem (cf. Theorem 4.3), respectively. In Section 5, we give a generalization of Cheeger–Colding–Tian’s Theorem 0.3 in the setting of Bakry–Émery geometry (cf. Theorem 5.4). In Section 6, we prove Theorem 6.2 and Theorem 6.8. Finally,
in Appendix A we explain how to use the technique of conformal transformation in [24] to give another proofs of Theorem 6.2 and Theorem 6.3, and we prove (6.6) of Section 6 in Appendix B.

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After the paper was posted in the spring of 2013, there are other related developments in this area, for instance, [12, 16, 29, 30], etc. We are grateful to many friends for their interests, in particular to the referees for valuable comments.

1. Distance function comparison and other comparison lemmas

In the view of conformal geometry, one can introduce a weighted volume form and an $f$-Laplace operator associated to $f$ on $(M, g)$ as follows:

$$d\nu_f = e^{-f} d\nu$$

and

$$\Delta^f = \Delta - \langle \nabla f, \nabla \rangle.$$

Then $\Delta^f$ is a self-adjoint elliptic operator under the following weighted inner product:

$$(u, v) = \int_M uv \, d\nu_f \text{ for all } u, v \in L^2(M).$$

That is,

$$\int_M \Delta^f uv \, d\nu_f = \int_M \langle \nabla u, \nabla v \rangle \, d\nu_f = \int_M \Delta^f u \, v \, d\nu_f.$$ 

The divergence theorem with respect to $\Delta^f$ is

$$\int_{\partial \Omega} \langle \nabla u, n \rangle e^{-f} \, d\sigma,$$

where $\Omega$ is a domain in $M$ with piece-wisely smooth boundary, $n$ denotes the outer unit normal vector field on $\partial \Omega$ and $d\sigma$ is an induced area form of $g$ on $\partial \Omega$.

Let $r = r(x) = \text{dist}(p, x)$ be a distance function on $(M, g)$. In [28], Wei and Wylie computed the $f$-Laplacian for $r$ and got the following comparison result under the Bakry–Émery Ricci curvature condition.

**Lemma 1.1** ([28]). Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold which satisfies

$$\text{Ric}^f_g \geq -(n - 1) \Lambda^2 g.$$ 

Then

$$\Delta^f r \leq (n - 1 + 4A) \Lambda \coth \Lambda r \text{ if } |f| \leq A$$

and

$$\Delta^f r \leq (n - 1) \Lambda \coth \Lambda r + A \text{ if } |\nabla f| \leq A.$$ 

As an application of Lemma 1.1, Wei and Wylie proved the following weighted volume comparison theorem.
Theorem 1.2 ([28]). Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold which satisfies (1.1). Then for any \(0 < r \leq R\),

\[
\frac{\text{vol}^f(B_p(r))}{\text{vol}^f(B_p(R))} \geq \frac{\text{vol}_{\Lambda}^{n+4A}(B(r))}{\text{vol}_{\Lambda}^{n+4A}(B(R))} \quad \text{if} \ |f| \leq A \tag{1.4}
\]

and

\[
\frac{\text{vol}^f(B_p(r))}{\text{vol}^f(B_p(R))} \geq e^{-AR} \frac{\text{vol}_{\Lambda}^n(B(r))}{\text{vol}_{\Lambda}^n(B(R))} \quad \text{if} \ |\nabla f| \leq A, \tag{1.5}
\]

where \(\text{vol}_{\Lambda}^n(B(r))\) denotes the volume of geodesic ball \(B(r)\) with radius \(r\) in \(n\)-dimensional space form with constant curvature \(-\Lambda\).

The proof of Theorem 1.2 depends on a monotonicity formula for the weighted volume form as follows.

By choosing a polar coordinate with the origin at \(p\), we write

\[ e^{-f} d\nu = A^f(s, \theta) ds \wedge d\theta. \]

Then

\[ \frac{d}{ds} A^f(s, \theta) = A^f(s, \theta) \Delta^f r. \]

In case that \(|\nabla f| \leq A\), it follows from (1.3) that

\[ \frac{d}{ds} A^f(s, \theta) \leq A^f(s, \theta) l_{\Lambda, A}(r), \tag{1.6} \]

where \(l_{\Lambda, A}(r) = (n - 1) \Lambda \coth \Lambda r + A\). Thus if we put

\[ L_{\Lambda, A}(r) = e^{Ar} \left( \frac{\sinh \Lambda r}{\Lambda} \right)^{n-1}, \tag{1.7} \]

which is a solution of the equation

\[ \frac{L'_{\Lambda, A}}{L_{\Lambda, A}} = l_{\Lambda, A}, \quad \frac{L_{\Lambda, A}(r)}{r^{n-1}} \to 1 \quad \text{as} \ r \to 0, \]

then (1.6) is equivalent to the following monotonic formula:

\[ \frac{A^f(b, \theta)}{A^f(a, \theta)} \leq \frac{L_{\Lambda, A}(b)}{L_{\Lambda, A}(a)} \quad \text{for all} \ b \geq a. \tag{1.8} \]

By a simple computation, we get (1.5) from (1.8). Similarly, we can prove (1.4).

Another application of Lemma 1.1 is the following weighted Poincaré inequality.

Lemma 1.3. Let \((M, g)\) be a complete Riemannian manifold which satisfies

\[ \text{Ric}^f_g \geq -(n - 1) \Lambda^2 g \quad \text{and} \quad |\nabla f| \leq A. \tag{1.9} \]

Let \(A_p(a, b) = B_p(b) \setminus B_p(a)\) be an annulus in \(M\). Then for any Lipschitz function \(h\) in the annulus \(A_p(a, b)\) with \(h|_{\partial A_p(a, b)} = 0\), one has

\[ \int_{A_p(a, b)} h^2 e^{-f} \, d\nu \leq c(a, b, A, \Lambda) \int_{A_p(a, b)} |\nabla h|^2 e^{-f} \, d\nu. \tag{1.10} \]
Proof. By (1.3), it is easy to see that
\[ \Delta^f r^{-k} \geq -kr^{-k-1}l_{\Lambda,A}(r) + k(k+1)r^{-k-2} = kr^{-k-1}\left(-l_{\Lambda,A}(r) + \frac{k+1}{r}\right), \]
where \( k \) is a positive real number. Putting \( \frac{k+1}{b} = l_{\Lambda,A}(a) + 1 \), we have
\[ \Delta^f r^{-k} \geq c(a,b,\Lambda,A) > 0. \]
Thus for \( h \) with zero boundary value, we get
\[ c(a,b,\Lambda,A) \int_{A_p(a,b)} h^2 e^{-f} \, dv \leq \int_{A_p(a,b)} (\Delta^f r^{-k}) h^2 e^{-f} \, dv \]
\[ = -2 \int_{A_p(a,b)} \nabla h, \nabla (r^{-k}) e^{-f} \, dv \]
\[ \leq 2k \int_{A_p(a,b)} |\nabla h|^2 e^{-f} \, dv \]
\[ \leq 2k \left( \int_{A_p(a,b)} h^2 e^{-f} \, dv \right)^{1/2} \left( \int_{A_p(a,b)} |\nabla h|^2 e^{-f} \, dv \right)^{1/2}. \]
Hence, (1.10) follows from the above immediately.

For the \( f \)-Laplace operator, we have the following Bochner-type identity:
\[ \frac{1}{2} \Delta^f |\nabla u|^2 = |\text{hess } u|^2 + \langle \nabla u, \nabla \Delta^f u \rangle + \text{Ric}_g (\nabla u, \nabla u), \quad u \in C^\infty(M). \]
By formula (1.11) and Lemma 1.1, we derive the following Li–Yau-type gradient estimate for \( f \)-harmonic functions on \((M, g)\).

**Proposition 1.4.** Let \((M, g)\) be a complete Riemannian manifold which satisfies (1.9). Let \( u > 0 \) be an \( f \)-harmonic function defined on the unit distance ball \( B_p(1) \subset (M, g) \), i.e.
\[ \Delta^f u = 0 \text{ in } B_p(1). \]
Then
\[ |\nabla u|^2 \leq (C_1 \Lambda + C_2 A^2 + C_3) u^2 \text{ in } B_p\left(\frac{1}{2}\right), \]
where the constants \( C_i, 1 \leq i \leq 3, \) depend only on \( n \).

Proof. The proof is standard as in the case \( f = 0 \) for a harmonic function (cf. [22]). In general, let \( v = \ln u \). Then
\[ \Delta^f v = \Delta v - \langle \nabla f, \nabla v \rangle = \nabla \left( \frac{\nabla u}{u} \right) - \nabla \left( \frac{v f}{u} \right) \]
\[ = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} - \left( \nabla f, \frac{\nabla u}{u} \right) = -\frac{|\nabla u|^2}{u^2}. \]
Note that
\[ |\text{hess } v|^2 \geq \frac{|\Delta u|^2}{n}. \]
and

$$|\Delta v|^2 \geq \frac{|\Delta f v|^2}{2} - C_1 A^2 Q,$$

where $Q = |\nabla v|^2$. Thus applying (1.11) to $v$, we get

$$(1.12) \quad \frac{1}{2} \Delta^f Q \geq \frac{Q^2}{2n} - \frac{1}{n} C_1 A^2 Q + \langle \nabla v, \nabla Q \rangle - \Lambda^2 Q.$$

Choose a decreasing cut-off function $\eta(t)$ on $t \in [0, \infty]$ such that

$$\eta(t) = 1 \quad \text{if } t \leq \frac{1}{2}, \quad \phi = 0 \quad \text{if } t \geq 1,$$

$$-C_2 \eta^\frac{1}{2} \leq \eta' \quad \text{if } t \geq \frac{1}{2},$$

$$|\eta''| \leq C_2.$$

Then if $\phi = \eta(r(\cdot, p))$,

$$|\nabla \phi|^2 \phi^{-1} \leq C_2^2,$$

and by Lemma 1.1,

$$\Delta^f \phi = \Delta^f \eta(r) \geq -C_3 (A + \Lambda).$$

Hence, by (1.12), we obtain

$$(1.13) \quad \Delta^f (\phi Q) = \Delta (\phi Q) + \langle \nabla f, \nabla (\phi Q) \rangle$$

$$= \phi \Delta^f Q + Q \Delta^f \phi + 2 \langle \nabla Q, \nabla \phi \rangle$$

$$\geq \phi \left( \frac{Q^2}{2} - \left( \frac{2}{n} C_1 A^2 + 2 \Lambda^2 \right) Q \right) - C_3 (A + \Lambda) Q$$

$$+ 2 \langle \nabla v, \nabla Q \rangle + 2 \langle \nabla Q, \nabla \phi \rangle.$$

Suppose that $(Q \phi)(q) = \max_M (Q \phi)$ for some $q \in M$. Then at this point, $\nabla (Q \phi) = 0$. It follows that

$$\nabla Q = -\frac{Q \nabla \phi}{\phi}$$

and

$$|\langle \nabla Q, \nabla \phi \rangle| = \frac{Q}{\phi} |\nabla \phi|^2 \leq C_2^2 Q.$$

Also

$$|\langle \nabla Q, \nabla v \rangle| \leq Q^\frac{3}{2} \frac{|\nabla \phi|}{\phi} \leq C_2 Q^\frac{3}{2} \phi^{-\frac{1}{2}}.$$

Therefore, by applying the maximum principle to $\phi Q$ at the point $q$, we get from (1.13) that

$$0 \geq \phi \left( \frac{Q^2}{n} - \frac{2}{n} C_1 A^2 Q - 2 C_2 Q^\frac{3}{2} \phi^{-\frac{1}{2}} \right) - C_3 (\Lambda Q + A) - 2 C_2 Q.$$

As a consequence, we derive

$$\phi Q \leq (\phi Q)(q) \leq C_4 \Lambda + C_5 A^2 + C_6 \quad \text{in } B_p(1).$$

This proves the proposition. \hfill \Box
As an application of Proposition 1.4, we are able to construct a cut-off function with the bounded gradient and \( f \)-Laplace. Such a function will be used in the next section.

**Lemma 1.5.** Under condition (1.9) in Lemma 1.3, there exists a cut-off function \( \phi \) supported in \( B_p(2) \) such that

1. \( \phi \equiv 1 \) in \( B_p(1) \).
2. \( |\nabla \phi|, |\Delta^f \phi| \leq C(n, \Lambda, A) \).

**Proof.** We will use an argument from [5, Theorem 6.33]. First we consider a solution of the ODE

\[
G'' + G' l_{\Lambda, A} = 1 \quad \text{on } [1, 2]
\]

with \( G(1) = a \) and \( G(2) = 0 \). It is easy to see that there is a number \( a = a(n, \Lambda, A) \) such that \( G' < 0 \). Then by (1.3), we have

\[
\Delta^f G(d(p, \cdot)) \geq 1.
\]

Let \( w \) be a solution of equation,

\[
\Delta^f w = \frac{1}{a} \quad \text{on } B_p(2) \setminus \overline{B_p(1)},
\]

with \( w = 1 \) on \( \partial B_p(1) \) and \( w = 0 \) on \( \partial B_p(2) \). Thus by the maximum principle, we get

\[
w \geq \frac{G(d(\cdot, p))}{a}.
\]

Secondly, we choose another function \( H \) with \( H' > 0 \) which is a solution of the ODE

\[
H'' + H' l_{\Lambda, A} = 1 \quad \text{on } [0, \infty)
\]

with \( H(0) = 0 \). Then by (1.3), we have

\[
\Delta^f H(d(x, \cdot)) \leq 1 \quad \text{for any fixed point } x.
\]

Thus by the maximum principle, we get

\[
w(y) - \frac{H(d(x, y))}{a} \leq \max \left\{ 1 - \frac{H(d(x, p) - 1)}{a}, 0 \right\}
\]

for any \( y \) in the annulus \( A_p(1, 2) = B_p(2) \setminus \overline{B_p(1)} \). It follows

\[
w(x) \leq \max \left\{ 1 - \frac{H(d(x, p) - 1)}{a}, 0 \right\} \quad \text{for all } x \in A_p(1, 2).
\]

Now we choose a number \( \eta(n, \Lambda, A) \) such that \( \frac{G(1 + \eta)}{a} > 1 - \frac{H(1 - \eta)}{a} \) and we define a function \( \psi(x) \) on \([0, 1]\) with bounded derivative up to second order, which satisfies

\[
\psi(x) = 1 \quad \text{if } x \geq \frac{G(1 + \eta)}{a}
\]

and

\[
\psi(x) = 0 \quad \text{if } x \leq \max \left\{ 1 - \frac{H(1 - \eta)}{a}, 0 \right\}.
\]
It is clear that $\phi = \psi \circ w$ is constant near the boundary of $A_p(1, 2)$. So we can extend $\phi$ inside $B_p(1)$ by setting $\phi = 1$. By Proposition 1.4, one sees that $|\nabla \phi|$ is bounded by a constant $C(n, \Lambda, A)$ in $B_2(p)$. Since

$$\Delta^f \phi = \psi'' |\nabla w|^2 + \psi' \Delta^f w,$$

we also derive that $|\Delta^f \phi| \leq C(n, \Lambda, A)$. 

2. $L^2$-Integral estimates for Hessians of functions

In this section, we establish various integral comparisons of gradient and Hessian between appropriate $f$-harmonic functions and coordinate functions or distance functions. We start with a basic lemma about a distance function along a long approximate line in a manifold.

**Lemma 2.1.** Let $(M, g)$ be a complete Riemannian manifold which satisfies

$$\text{Ric}_g \geq -\frac{n-1}{R^2} g \quad \text{and} \quad |f| \leq A.$$

Suppose that there are three points $p, q^+, q^-$ in $M$ which satisfy

$$d(p, q^+) + d(p, q^-) - d(q^+, q^-) < \epsilon$$

and

$$d(p, q^+), d(p, q^-) > R.$$

Then for any $q \in B_p(1)$, the following holds:

$$E(q) := d(q, q^+) + d(q, q^-) - d(q^+, q^-) < \Psi\left(\epsilon, \frac{1}{R}; A, n\right),$$

where the quantity $\Psi(\epsilon, \frac{1}{R}; A, n)$ means that it goes to zero as $\epsilon, \frac{1}{R}$ go to zero while $A, n$ are fixed.

**Proof.** Let

$$\bar{l}(s) = (n - 1 + 4A)\frac{1}{R} \coth \frac{s}{R}.$$

For given $t > 0$, we construct a function $G = G_t(s)$ on $[0, t]$ which satisfies the ODE

$$G'' + \bar{l}(s)G' = 1, \quad G'(s) < 0$$

with $G(0) = +\infty$ and $G(t) = 0$. Then $G(s) \sim s^{2-n-4A}$ as $s \to 0$. Furthermore, by (1.2) in Lemma 1.1, we have

$$\Delta^f G(d(x, \cdot)) = G' \Delta^f d(x, \cdot) + G'' \geq G'' + G' \bar{l}(s) = 1.$$

By Lemma 1.1,

$$\Delta^f E(q) \leq \frac{10(n - 1 + A)}{R} := b.$$
Claim 2.2. For any \(0 < c < 1\),
\[
E(q) \leq 2c + bG_2(c) + \epsilon.
\]
Suppose that the claim is not true. Then there exists point \(q_0 \in B_p(1)\) such that for some \(c\),
\[
E(q_0) > 2c + bG_2(c) + \epsilon.
\]
We consider \(u(x) = bG_2(d(q_0, x)) - E(x)\) in the annulus \(A_{q_0}(c, 2)\). Clearly,
\[
\Delta^f u \geq 0.
\]
Note that we may assume that \(p \in A_{q_0}(c, 1)\). Otherwise \(d(q_0, p) < c\) and \(E(q_0) \leq E(p) + 2c\), so the claim is true and the proof is complete. On the other hand, it is easy to see that on the inner boundary \(\partial B_{q_0}(c)\),
\[
u(x) = bG_2(c) - E(x) \leq bG_2(c) - E(q_0) - 2c \leq -\epsilon,
\]
and on the outer boundary \(\partial B_{q_0}(2)\),
\[
u(x) = -E(x) \leq 0.
\]
Thus applying the maximum principle, it follows that \(u(p) \leq 0\). However,
\[
u(p) = bG_2(d(p, q_0)) - E(p) \geq bG_2(1) - \epsilon,
\]
which is impossible. Therefore, the claim is true.

By choosing \(c\) with the order \(\frac{1}{\pi - \frac{1}{1+\epsilon}}\) in Claim 2.2, we prove Lemma 2.1.

Let \(b^+(x) = d(q^+, x) - d(q^+, p)\) and let \(h^+\) be an \(f\)-harmonic function which satisfies
\[
\Delta^f h^+ = 0 \quad \text{in } B_p(1)
\]
with \(h^+ = b^+\) on \(\partial B_p(1)\). Then:

Lemma 2.3. Under the conditions in Lemma 2.1 with \(|\nabla f| \leq A\), we have
\[
\|h^+ - b^+\|_{C^0(B_p(1))} < \Psi\left(\frac{1}{R}, \epsilon; A\right),
\]
\[
\frac{1}{\operatorname{vol}(B_p(1))} \int_{B_p(1)} |\nabla h^+ - \nabla b^+|^2 e^{-f} \, dv < \Psi\left(\frac{1}{R}, \epsilon; A\right),
\]
\[
\frac{1}{\operatorname{vol}(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} |\operatorname{hess} h^+|^2 e^{-f} \, dv < \Psi\left(\frac{1}{R}, \epsilon; A\right).
\]

Proof. Choose a point \(q\) in \(\partial B_p(2)\) and let \(g = \varphi(d(q, \cdot))\), where \(\varphi\) is a solution of (2.1) restricted on the interval \([1, 3]\). Then
\[
\Delta^f g = \varphi' \Delta^f r + \varphi'' \geq \varphi'^2 + \varphi'' = 1 \quad \text{in } B_p(1).
\]
It follows that
\[
\Delta^f (h^+ - b^+ + \Psi\left(\frac{1}{R}, \epsilon; A\right)g) > 0 \quad \text{in } B_p(1).
\]
Thus by the maximum principle, we get
\[
h^+ - b^+ < \Psi\left(\frac{1}{R}, \epsilon; A\right).
\]
On the other hand, we have
\[ \Delta f (-b^--h^+ + \Psi(1/R, \epsilon; A) g) > 0 \quad \text{in } B_p(1), \]
where \( b^- = d(q^-, x) - d(p, q^-) \). Since \( b^+ + b^- \) is small as long as \( 1/R \) and \( \epsilon \) are small by Lemma 2.1, by the maximum principle, we also get
\[ h^+ - b^+ > -(b^+ + b^-) - \Psi(1/R, \epsilon; A) > -\Psi(1/R, \epsilon; A). \]

For the second estimate (2.3), we see
\[
\int_{B_p(1)} |\nabla h^+ - \nabla b^+|^2 e^{-f} \, dv \leq \int_{B_p(1)} (h^+ - b^+)(\Delta f b^+ - \Delta f h^+) e^{-f} \, dv
\]
\[ < \Psi(1/R, \epsilon; A) \int_{B_p(1)} |\Delta f b^+| e^{-f} \, dv \]
and
\[
\int_{B_p(1)} |\Delta f b^+| e^{-f} \, dv \leq \left| \int_{B_p(1)} \Delta f b^+ e^{-f} \, dv \right| + 2e^A \sup_{B_p(1)} (\Delta f b^+) \text{vol}(B_p(1))
\]
\[ \leq e^A \text{vol}(\partial B_p(1)) + C(A) \text{vol}(B_p(1)) \]
\[ \leq C'(A) \text{vol}(B_p(1)). \]

Here we used (1.8) at the last inequality. Then (2.3) follows.

To get estimate (2.4), we choose a cut-off function \( \phi \) supported in \( B_p(1) \) as constructed in Lemma 1.5. Since
\[ \Delta f (|\nabla h^+|^2 - |\nabla b^+|^2) = |\text{hess } h^+|^2 + \text{Ric}_g f (\nabla h^+, \nabla h^+), \]
multiplying both sides of the above by \( \phi e^{-f} \, dv \) and using integration by parts, we get
\[
\int_{B_p(1)} \phi |\text{hess } h^+|^2 e^{-f} \, dv \leq \int_{B_p(1)} \Delta f \phi (|\nabla h^+|^2 - |\nabla b^+|^2) e^{-f} \, dv
\]
\[ + \frac{n-1}{R^2} \int_{B_p(1)} \phi |\nabla h^+|^2 e^{-f} \, dv. \]

Note that \( |\nabla h^+| \) is locally bounded by Proposition 1.4, we derive (2.4) from (2.3).

Next, we construct an approximate function to compare the square of a distance function with asymptotic integral gradient and Hessian estimates. Such estimates are crucial in the proof of the metric-cone theorem in Section 4.

Let \( q \in M \) and \( h \) be a solution of the following equation:
\[ \Delta f h = n \quad \text{in } B_q(b) \setminus \overline{B_q(a)}, \quad h|\partial B_q(b) = \frac{b^2}{2}, \quad h|\partial B_q(a) = \frac{a^2}{2}. \]
Let \( p = \frac{r(q; \cdot)^2}{2} \). Then:

**Lemma 2.4.** Let \( (M, g) \) be a complete Riemannian manifold which satisfies
\[ \text{Ric}_g f \geq -(n-1)\epsilon^2 \Lambda^2 g \quad \text{and} \quad |\nabla f| \leq \epsilon A. \]
Let \( a < b \). Suppose that
\[ \frac{\text{vol}(\partial B_q(b))}{\text{vol}(\partial B_q(a))} \geq (1-\omega) \frac{L_{\epsilon \Lambda, \epsilon A}(b)}{L_{\epsilon \Lambda, \epsilon A}(a)}. \]
for some \( \omega > 0 \), where \( L_{\epsilon \Lambda, \epsilon A}(r) \) is the function defined by (1.7) with respect to constants \( \epsilon \Lambda \) and \( \epsilon A \). Then

\[
\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} |\nabla p - \nabla h|^2 e^{-f} d\nu < \Psi(\omega, \epsilon; \Lambda, A, a, b). \tag{2.7}
\]

Moreover,

\[
\|h - p\|_{C^0(B_q(a', b'))} < \Psi(\omega, \epsilon; \Lambda, A, a, b, a', b'), \tag{2.8}
\]

where \( a < a' < b' < b \).

**Proof.** Since

\[
\Delta^f r \leq (n-1) \epsilon \Lambda \coth(\epsilon \Lambda r) + \epsilon A = l_{\epsilon \Lambda, \epsilon A},
\]

we have

\[
\Delta^f p = p'' + p' \Delta^f r < n + \Psi(\epsilon; \Lambda, A, a, b) \quad \text{in} \, A(a, b). \tag{2.9}
\]

Thus we get

\[
\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \Delta^f p e^{-f} d\nu < e^{-f(0)}(n + \Psi(\epsilon; \Lambda, A, a, b)). \tag{2.10}
\]

On the other hand, by the monotonicity formula (1.8), we have

\[
\int_a^b \frac{L_{\epsilon \Lambda, \epsilon A}(s)}{L_{\epsilon \Lambda, \epsilon A}(b)} \text{vol}^f(\partial B_q(b)) \leq \text{vol}^f(A_q(a, b)) \leq \frac{\int_a^b L_{\epsilon \Lambda, \epsilon A}(s)}{L_{\epsilon \Lambda, \epsilon A}(a)} \text{vol}^f(\partial B_q(a)).
\]

By (2.6), it follows that

\[
\text{vol}^f(A_q(a, b)) \leq (1 - \omega)^{-1} \frac{\int_a^b L_{\epsilon \Lambda, \epsilon A}(s)}{L_{\epsilon \Lambda, \epsilon A}(b)} \text{vol}^f(\partial B_q(b)).
\]

Since

\[
\int_{A_q(a, b)} \Delta^f p e^{-f} d\nu = b \text{vol}^f(\partial B_q(b)) - a \text{vol}^f(\partial B_q(a)),
\]

we get

\[
\frac{1}{\text{vol}^f(A_q(a, b))} \int_{A(a, b)} \Delta^f p e^{-f} d\nu \geq (1 - \omega) \frac{L_{\epsilon \Lambda, \epsilon A}(b)}{\int_a^b L_{\epsilon \Lambda, \epsilon A}(s)} \left( b - a \frac{\text{vol}^f(\partial B_q(a))}{\text{vol}^f(\partial B_q(b))} \right).
\]

Observe that \( \text{vol}^f \) is close to \( e^{-f(0)} \) and \( \frac{L_{\epsilon \Lambda, \epsilon A}(s)}{s^{n-1}} \) is close to a constant as \( \epsilon \) is small. Hence we derive immediately

\[
\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \Delta^f p e^{-f} d\nu > e^{-f(0)}(n + \Psi(\omega, \epsilon; \Lambda, A, a, b)). \tag{2.11}
\]

By (2.10) and (2.11), we have

\[
\left| \int_{A_q(a, b)} (\Delta^f p - n) e^{-f} d\nu \right| < \text{vol}(A_q(a, b)) \Psi(\omega, \epsilon; \Lambda, A, a, b).
\]

Then one can follow the argument for estimate (2.3) in Lemma 2.3 to obtain (2.7).
Applying Lemma 1.3 to the function $p - h$ together with estimate (2.7), we see that

$$\frac{1}{\text{vol}^f(A_q(a, b))} \int_{A_q(a, b)} |p - h|^2 e^{-f} \, dv < \Psi(\omega, \epsilon; \Lambda, A, a, b).$$

Then for any point $x \in A_q(a', b')$, by (1.5), there is a point $y \in B_x(\eta)$ such that

$$|p(y) - h(y)|^2 \leq \frac{\text{vol}^f(A_q(a, b))}{\text{vol}^f(B_x(\eta))} \frac{1}{\text{vol}^f(A_q(a, b))} \int_{A_q(a, b)} |p - h|^2 e^{-f} \, dv$$

$$< \frac{\text{vol}^f(A_q(a, b))}{\text{vol}^f(B_x(\eta))} \frac{1}{\text{vol}^f(A_q(a, b))} \int_{A_q(a, b)} |p - h|^2 e^{-f} \, dv$$

$$< C(\Lambda, A, b) \frac{\text{vol}^f(A_q(a, b))}{\text{vol}^f(B_x(\eta))} \Psi(\omega, \epsilon; \Lambda, A, a, b).$$

On the other hand, by Proposition 1.4, we have

$$|(p(x) - h(x)) - (p(y) - h(y))| \leq (\|\nabla h\|_{C^0(A_q(a' - \eta, b' + \eta))} + 1) \text{dist}(x, y)$$

$$\leq C(\Lambda, A, a, b, a' - \eta, b' + \eta) \eta.$$ 

Thus we derive

$$|p(x) - h(x)| < C(\Lambda, A, b) \frac{\text{vol}^f(A_q(a, b))}{\text{vol}^f(B_x(\eta))} \Psi(\omega, \epsilon; \Lambda, A, a, b) + C(\Lambda, A, a, b, a' - \eta, b' + \eta) \eta.$$ 

Choosing $\eta = \Psi^{\frac{1}{n+1}}$, we prove (2.8). \hfill \Box

**Lemma 2.5.** Under the condition in Lemma 2.4, one has

$$\frac{1}{\text{vol}(A_q(a_2, b_2))} \int_{A_q(a_2, b_2)} |\text{hess } h - g|^2 e^{-f} \, dv < \Psi(\omega, \epsilon; \Lambda, A, a_1, b_1, a_2, b_2, a, b),$$

where $a < a_1 < a_2 < b_2 < b_1 < b$.

**Proof.** First observe that

$$\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} |\text{hess } h - g|^2 e^{-f} \, dv = \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} |\text{hess } h|^2 e^{-f} \, dv$$

$$+ \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} (n - 2\Delta h)e^{-f} \, dv.$$

Let $\varphi$ be a cut-off function of $A_q(a, b)$ with support in $A_q(a_1, b_1)$ as constructed in Lemma 1.5 which satisfies

(i) $\varphi \equiv 1$ in $A_q(a_2, b_2),$

(ii) $|\nabla \varphi|, |\Delta^f \varphi|$ are bounded in $A_q(a, b)$.

Then

$$\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi |\text{hess } h - g|^2 e^{-f} \, dv$$

$$= \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi |\text{hess } h|^2 e^{-f} \, dv$$

$$+ \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi (n - 2\Delta h)e^{-f} \, dv.$$
By the Bochner formula (1.11), we have
\[
\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi |\text{hess} h|^2 e^{-f} \, dv \\
< \frac{1}{2 \cdot \text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi \Delta f |\nabla h|^2 e^{-f} \, dv + \Psi(\epsilon; \Lambda, A, a_1, b_1, a_2, b_2, a, b).
\]
It follows by Lemma 2.4 that
\[
\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi |\text{hess} h|^2 e^{-f} \, dv \\
< \frac{1}{2 \cdot \text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi \Delta f |\nabla p|^2 e^{-f} \, dv + \Psi(\epsilon, \omega; \Lambda, A, a_1, b_1, a_2, b_2, a, b) \\
= \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi \Delta f p e^{-f} \, dv + \Psi(\epsilon, \omega; \Lambda, A, a_1, b_1, a_2, b_2, a, b).
\]
On the other hand,
\[
\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi (n - 2\Delta h) e^{-f} \, dv \\
= \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi (-n - 2 \langle \nabla f, \nabla h \rangle) e^{-f} \, dv \\
= \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} -n \varphi e^{-f} \, dv + \Psi(\epsilon, \omega; \Lambda, A, a_1, b_1, a, b).
\]
Hence we derive from (2.12) that
\[
\frac{1}{\text{vol}(A_q(a_2, b_2))} \int_{A_q(a_2, b_2)} |\text{hess} h - g|^2 e^{-f} \, dv \\
\leq \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi |\text{hess} h - g|^2 e^{-f} \, dv \\
< \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b)} \varphi (\Delta f p - n) e^{-f} \, dv + \Psi(\epsilon, \omega; \Lambda, A, a_1, b_1, a_2, b_2, a, b) \\
< \Psi(\epsilon, \omega; \Lambda, A, a_1, b_1, a_2, b_2, a, b).
\]
Here we used (2.9) at the last inequality.

3. A splitting theorem

In this section, we prove the splitting theorem of Cheeger–Colding in the Bakry–Émery geometry [5]. Recall that \( \gamma(t) \) (\( t \in (-\infty, \infty) \)) is a line in a metric space \( Y \) if
\[
\text{dist}(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2| \quad \text{for all} \quad t_1, t_2 \in (-\infty, \infty).
\]

**Theorem 3.1.** Let \( (M_i, g_i; p_i) \) be a sequence of Riemannian manifolds which satisfy
\[
\text{Ric}_{f_i}^{M_i, g_i} \geq -\epsilon_i^2 g_i, \quad |f_i|, |\nabla f_i| \leq A.
\]
Let \( (Y; y) \) be a limit metric space of \( (M_i, g_i; p_i) \) in the pointed Gromov–Hausdorff topology as \( \epsilon_i \to 0 \). Suppose that \( Y \) contains a line passing \( y \). Then \( Y = \mathbb{R} \times X \) for some metric space \( X \).
We will follow the argument in [5] to prove Theorem 3.1. The proof depends on the following triangle lemma in terms of small integral Hessian of appropriate function.

**Lemma 3.2.** Let \( x, y, z \) be three points in a complete Riemannian manifold \( M \). Let \( \gamma(s) (s \in [0, a], a = d(x, y)) \) be a geodesic curve connecting \( x, y \) and let \( \gamma_s(t) (s \in [0, l(s)]) \), \( l(s) = d(z, \gamma(s)) \) be a family of geodesic curves connecting \( z \) and \( \gamma(s) \). Suppose that \( h \) is a smooth function on \( M \) which satisfies

\[
\begin{align*}
&\text{(i)} \quad |h(z) - h(x)| < \delta \ll 1, \\
&\text{(ii)} \quad \int_{[0,a]} |\nabla h(\gamma(s)) - \gamma'(s)| < \delta \ll 1, \\
&\text{(iii)} \quad \int_{[0,a]} \int_{[0,l(s)]} |\text{hess } h(\gamma_s(t))| \ dt \ ds < \delta \ll 1.
\end{align*}
\]

Then

\[
|d(z, x)^2 + d(x, y)^2 - d(y, z)^2| < \epsilon(\delta) \ll 1.
\]

**Proof.** The proof below comes essentially from [3, Lemma 9.16]. First by condition (ii), we have

\[
|h(\gamma(s)) - h(\gamma(0)) - s| = \left| \int_0^s ((\nabla h(\gamma(s)) - \gamma'(s), \gamma'(s)) \right| \leq \delta.
\]

Then

\[
s = h(\gamma(s)) - h(x) + o(1).
\]

By condition (i), it follows that

\[
\frac{1}{2} d(x, y)^2 = \int_0^a s \ ds
\]

\[
= \int_0^a (h(\gamma(s)) - h(x)) \ ds + o(1)
\]

\[
= \int_0^a (h(\gamma_s(l(s)))) - h(\gamma_s(0))) \ ds + o(1)
\]

\[
= \int_0^{l(s)} \int_0^a (\nabla h(\gamma_s(t)), \gamma'_s(t)) \ dt \ ds + o(1).
\]

On the other hand,

\[
|\langle \nabla h(\gamma_s(t)), \gamma'_s(t) \rangle - (\nabla h(\gamma_s(l(s))), \gamma'_s(l(s)))| = \left| \int_{l(s)}^{l(s)} \text{hess } h(\gamma'_s(\tau), \gamma'_s(\tau)) \ d\tau \right|
\]

\[
\leq \int_0^{l(s)} |\text{hess } h(\gamma'_s(t), \gamma'_s(t))| \ dt.
\]

Hence from condition (iii), we get

\[
\frac{1}{2} d(x, y)^2 = \int_0^{l(s)} \int_0^a (\nabla h(\gamma_s(l(s))), \gamma'_s(l(s))) \ dt \ ds + o(1)
\]

\[
= \int_0^a (\nabla h(\gamma_s(l(s))), \gamma'_s(l(s)))l(s) \ ds + o(1)
\]

\[
= \int_0^a (\nabla h(\gamma(s)), \gamma'_s(l(s)))l(s) \ ds + o(1).
\]
Secondly, by the first variation formula of geodesic curve, we see that
\[ l'(s) = \langle y'_s(l(s)), y'(s) \rangle. \]
Then by condition (ii), we obtain
\[
\int_0^a \langle \nabla h(y(s)), y'_s(l(s)) \rangle l(s) \, ds = \int_0^a l'(s) l(s) \, ds + o(1) = \frac{1}{2} (d(y, z)^2 - d(z, x)^2).
\]
Therefore, combining (3.2), we derive (3.1).

In order to get the above configuration in Lemma 3.2, we need a segment inequality lemma in terms of the Bakry–Émery Ricci curvature. In the following, we will always assume that the manifold \((M, g)\) satisfies
\[
\text{Ric}^f_g \geq -(n-1) \Lambda^2 g, \quad |f|, |\nabla f| \leq A.
\]
and the volume form \(dv\) is replaced by \(dv^f = e^{-f} dv\).

**Lemma 3.3.** Let \(A_1, A_2\) be two subsets of \(M\) and let \(W\) be another subset of \(M\) such that \(\bigcup_{y_1 \in A_1, y_2 \in A_2} \gamma_{y_1, y_2} \subseteq W\), where \(\gamma_{y_1, y_2}\) is a minimal geodesic curve connecting \(y_1\) and \(y_2\) in \(M\). Let
\[
D = \sup \{ d(y_1, y_2) : y_1 \in A_1, y_2 \in A_2 \}.
\]
Then for any smooth function \(e\) on \(W\), one has
\[
\int_{A_1 \times A_2} \int_0^{d(y_1, y_2)} e(\gamma_{y_1, y_2}(s)) \, ds \, dy_1 \, dy_2 \leq c(n, \Lambda, A) D [\text{vol}^f(A_1) + \text{vol}^f(A_2)] \int_W e,
\]
where
\[
c(n, \Lambda, A) = \sup_{s, u} \left\{ \frac{L_{\Lambda, A}(s)}{L_{\Lambda, A}(u)} : 0 < \frac{s}{2} \leq u \leq s \right\}.
\]

**Proof.** Note that
\[
\int_{A_1 \times A_2} \int_0^{d(y_1, y_2)} e(\gamma_{y_1, y_2}(s)) \, ds \, dy_1 \, dy_2 = \int_{A_1} dy_1 \int_{A_2} \int_0^{d(y_1, y_2)} e(\gamma_{y_1, y_2}(s)) \, ds \, dy_2 \, dy_1 \\
+ \int_{A_1} dy_1 \int_{A_2} \int_0^{d(y_1, y_2)} e(\gamma_{y_1, y_2}(s)) \, ds \, dy_1.
\]
On the other hand, for a fixed \(y_1 \in A_1\), by using the monotonicity formula (1.8), we have
\[
\int_{A_2} \int_0^{d(y_1, y_2)} e(\gamma_{y_1, y_2}(s)) \, ds \, dy_2 = \int_{A_2} \int_0^r e(\gamma_{y_1, y_2}(s)) A^f(r, \theta) \, dr \, d\theta \, ds \leq c(n, \Lambda, A) \int_{A_2} \int_0^r e(\gamma_{y_1, y_2}(s)) A^f(s, \theta) \, dr \, d\theta \, ds \leq c(n, \Lambda, A) D \int_W e.
\]
Similarly,
\[ \int_{A_1} \int \frac{d(y_1, y_2)}{d(y_1, y_2)} e(\gamma_{y_1y_2}(s)) \, ds \, dy_1 \leq c(n, \Lambda, A) D \int_W e. \]

Then (3.4) follows from the above two inequalities. \qed

Using the same argument above, we can prove the following:

**Lemma 3.4.** Given two points \( q, q' \) with \( d(q, q') \geq 10 \) and a smooth function \( e \) with support in \( B_p(1) \), the following inequality holds:
\[ \int_{B_{q}(r)} dy \int_0 \frac{d(q, y)}{d(q, y)} e(\gamma_{y-y}(s)) \, ds \leq c(\Lambda, A) \int_{B_{p}(1)} e(y) \, dy. \]

Combining Lemma 3.1 and Lemma 3.4, we get another segment inequality lemma as follows.

**Lemma 3.5.** Let \( b^+(q) = d(q, q^+) - d(p, q^+) \) for any \( q \) with \( d(q, q^+) \geq 10 \). Let \( h^+ \) be a smooth function which satisfies
\[ \int_{B_p(1)} |\nabla h^+ - \nabla b^+| \leq \epsilon \text{vol}^f(B_p(1)) \]
and
\[ \int_{B_p(1)} |\text{hess} h^+| \leq \epsilon \text{vol}^f(B_p(1)). \]

We assume that Lemma 3.3 and Lemma 3.4 are true. Then for any two points \( q, q' \in B_p(1/\eta) \) and any small number \( \eta > 0 \), there exist \( y^*, z^* \) with \( d(y^*, q) < \eta, d(z^*, q') < \eta \), and a minimal geodesic line \( \gamma(t) (0 \leq t \leq l(y^*)) \) from \( y^* \) to \( q^- \) with \( \gamma(0) = y^*, \gamma(l(y^*)) \in \partial B_p(1/\eta) \) such that the following is true:

\[ \int_0^l \gamma(y(s)) |\nabla h^+(s) - \gamma'(s)| \, ds \leq \epsilon \frac{\text{vol}^f(B_q(2))}{\text{vol}^f(B_q(\eta))} \]

and
\[ \int_0^l \gamma(y(s)) \, ds \int_0^l d(z^*, \gamma(s)) |\text{hess} h^+(\gamma_s(t))| \, dt \leq \epsilon \left( \frac{\text{vol}^f(B_q(2))}{\text{vol}^f(B_q(\eta))} \right)^2. \]

where \( \gamma_s(t) \) is the minimal geodesic curve connecting \( \gamma(s) \) and \( q' \).

**Proof.** Choose a cut-off function \( \phi = \phi(\text{dist}(p, \cdot)) \) with support in \( B_p(1) \). Let
\[ e = \phi |\nabla h^+ - \nabla b^+|, \quad e_1 = \phi |\text{hess} h^+|, \quad e_2(y) = \int_{B_{q'}(\eta)} dz \int_0 d(y, z) e_1(\gamma_{yz}(s)) \, ds. \]

Then by Lemma 3.4, we have
\[ \int_{B_{q}(\eta)} \int \frac{d(q^-, y)}{d(q^-, y)} e(\gamma_{q^-y}(s)) \, ds \, dy \leq c(A, \Lambda) \int_{B_{p}(1)} e(y) \, dy. \]
On the other hand, by Lemma 3.3, one sees
\[
\int_{B_p(1)} e_2(y) \, dy = \int_{B_p(1)} dy \int_{B_{q'}(\eta)} dz \int_0^{d(y,z)} e_1(\gamma_{yz}) (s) \, ds \\
\leq c_1(\Lambda, A) \text{vol}^f(B_p(1)) \int_{B_p(1)} e_1(y) \, dy.
\]
Thus by Lemma 3.4, we get
\[
(3.8) \quad \int_{B_q(\eta)} \int_0^{d(q',y)} e_2(\gamma_{q'y} (s)) \, ds \, dy 
\leq c_2(\Lambda, A) \int_{B_p(1)} e_2(y) \, dy \\
\leq \text{vol}^f (B_p(1)) c_3(\Lambda, A) \int_{B_p(1)} e_1(y) \, dy.
\]
Observe that the left-hand side of (3.8) is equal to
\[
\int_{B_q(\eta)} dy \int_{B_{q'}(\eta)} dz \int_0^{d(q',y)} \int_0^{d(\gamma_{q'y}, z)} e_1(\tilde{\gamma}_s(t)) \, dt \, ds,
\]
where \( \tilde{\gamma}_s(t) \) is the minimal geodesic from \( z \) to \( \gamma_{q'y} (s) \) with arc-length parameter \( t \). Combining (3.7) and (3.8), we find two points \( y^*, z^* \) such that both (3.5) and (3.6) are satisfied.

Now we apply Lemma 3.5 to prove a local version of Theorem 3.1.

**Proposition 3.6.** Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold which satisfies
\[
\text{Ric}^f_g \geq -\frac{n-1}{R^2}, \quad |f|, |\nabla f| \leq A.
\]
Suppose that there exist three points \( p, q^+, q^- \) such that
\[
d(p, q^+) + d(p, q^-) - d(q^+, q^-) < \epsilon
\]
and
\[
(3.9) \quad d(p, q^+) \geq R, \quad d(p, q^-) > R.
\]
Then there exists a map
\[
u : B_p(\frac{1}{3}) \to B_{(0,x)}(\frac{1}{3})
\]
as a \( \Psi(1, R; \epsilon; A, n) \)-Gromov–Hausdorff approximation, where \( B_{(0,x)}(\frac{1}{3}) \subset \mathbb{R} \times X \) is a \( \frac{1}{3} \)-radius ball centered at \((0, x) \in \mathbb{R} \times X \) and \( X \) is given by the level set \((h^+)^{-1}(0)\) as a metric space measured in \( B_p(1) \).

**Proof.** For simplicity, we denote the terms on the right-hand side of (2.2), (2.3) and (2.4) in Lemma 2.3 by \( \delta = \delta(\epsilon, \frac{1}{R}) \). Define a map \( u \) on \( B_p(1) \) by \( u(q) = (x_q, h^+(q)) \), where \( x_q \) is the nearest point to \( q \) in \( X \). We are going to prove that \( u \) is a \( \Psi(1, \frac{1}{R}; \epsilon; A) \)-Gromov–Hausdorff approximation. Since \( |\nabla h^+| \leq c = c(A) \) in \( B_p(\frac{1}{2}) \), we have
\[
h^+(y) \leq 0 \quad \text{for all } y \in B_q(\eta) \quad \text{if } h^+(q) < -c\eta,
\]
where \( \eta \) is an appropriate small number and it will be determined later. We call the area of \( h^+(q) < -c\eta \) the upper region, the area of \( h^+(q) > c\eta \) the lower region and the rest the middle region, respectively.
Case 1: $q_1$ and $q_2$ in the upper region (we may assume that $h^+(q_1) > h^+(q_2)$). Let $q$ be a point in the upper region. Then by applying Lemma 3.5 to $x_q$, we get a geodesic from a point $y$ near $q$ to $q^-$ whose direction is almost the same as $\nabla h^+$. Thus this geodesic must intersect $h^+ = 0$. Applying Triangle Lemma 3.2, we see that the intersection is near $x_q$. Hence for $q_1$ and $q_2$, we can find $y_1$ and $y_2$ nearby $q_1$ and $q_2$, respectively, such that two geodesics from $y_1$ and $y_2$ to $q^-$ intersect $X$ with points $x_1$ and $x_2$, respectively. Denote the geodesic from $x_2$ to $y_2$ by $\gamma(s): \gamma(0) = x_2, \gamma(h^+(y_2)) = y_2$. Applying Triangle Lemma 3.2 to triples $\{y_1, y_2, \gamma(h^+(y_1))\}, \{x_2, y_1, \gamma(h^+(y_1))\}$ and $\{x_1, x_2, y_1\}$, respectively, we get

$$
|d(y_1, y_2)^2 - |h^+(y_2) - h^+(y_1)|^2 - d(y_1, \gamma(h^+(y_1)))^2| \leq c(n, A) \frac{\delta}{\eta^n},
$$

$$
|d(y_1, x_2)^2 - d(y_1, \gamma(h^+(y_1)))^2 - h^+(y_1)^2| \leq c(n, A) \frac{\delta}{\eta^n},
$$

$$
|d(y_1, x_2)^2 - d(x_1, x_2)^2 - h^+(y_1)^2| \leq c(n, A) \frac{\delta}{\eta^n}.
$$

Combining the above three relations, we derive, as $\delta = o(\eta^n)$,

$$
(3.10) \quad |d(q_1, q_2) - d(u(q_1), u(q_2))| \leq c(n, A) \frac{\delta}{\eta^n} \ll 1.
$$

Case 2: $q_1$ is in the middle region and $q_2$ is in the upper region. Note that $x_q$ is near $q$ if $q$ is in the middle region. Then we can find two points $y_1$ and $y_2$ near $q_1$ and $q_2$, respectively, such that Triangle Lemma 3.2 holds for the triple $\{y_1, y_2, x_q\}$. Hence for such two points $q_1$ and $q_2$, we get (3.10) immediately.

Case 3: $q_1$ is in the lower region and $q_2$ is in the upper region. As in Case 1, we can get one geodesic from $q^+$ to a point near $q_1$ and another geodesic from $q^-$ to a point near $q_2$, respectively. Thus we can use same argument in Case 1 to obtain (3.10). Similarly, we can settle down another two cases, both $q_1$ and $q_2$ in the lower region and both $q_1$ and $q_2$ in the middle region.

Proof of Theorem 3.1. Let us suppose that the line in $Y$ is $\gamma(t)$ and $\gamma(0) = y$. Define a Busemann function $b$ along $\gamma$ by

$$
b(y) = \lim_{t \to +\infty} (d(y, \gamma(t)) - t).
$$

Since $d_{GH}(B_{p_i}(j), B_Y(j)) \to 0$ as $i \to \infty$ for any given integer number $j > 0$, we may assume that

$$
d_{GH}(B_{p_i}(j), B_Y(j)) < \frac{1}{j}, \quad \epsilon_i < \frac{n - 1}{j^2} \quad \text{for } i = i(j) \text{ large enough}.
$$

Choose a Gromov–Hausdorff approximation from $B_Y(j)$ to $B_{p_i}(j)$ so that the images of the endpoints $\gamma(j)$ and $\gamma(−j)$ of the line in $B_Y(j)$ together with $p_i$ satisfy condition (3.9) in Proposition 3.6. Then we see that there exist a metric space $X_j$ and a Gromov–Hausdorff approximation $u_j : B_{p_i}(1) \to B_{0 \times X_j}(1)$ such that

$$
d_{GH}(B_{p_i}(1), u_j(B_{p_i}(1))) < \Psi(\frac{1}{j}).
$$

As a consequence, there exists a map $\hat{u}_j : B_Y(1) \to B_{0 \times X_j}(1)$ such that

$$
d_{GH}(B_Y(1), \hat{u}_j(B_Y(1))) < \Psi.
This implies that all the projection of $\mathbb{R}$ component from the space $\mathbb{R} \times X_j$ are close to the Busemann function $b$ along the given line in $Y$ for $j \gg 1$, so they are almost the same. Hence, $\{X_j\}$ is a Cauchy sequence in Gromov–Hausdorff topology with a limit $X$. It follows that $B_x(1) = B_{0xX}(1)$, where $x$ is the limit point of $\{x_j\}$ in $X$. Since the number 1 can be replaced by any positive number, we finish the proof of theorem. 

\section{Existence of metric cone}

In this section, we prove the existence of metric cone of a tangent space on the limit space of a sequence in $\mathcal{M}(A, v, \Lambda)$. Recall:

\textbf{Definition 4.1.} Let $(Y, d)$ be a metric space. We call the limit of $(Y, \epsilon_i^{-2}d; y)$ in the Gromov–Hausdorff topology as $\epsilon_i \to 0$ a tangent cone of $Y$ at $y$ (if it exists). We denote it as $T_yY$.

\textbf{Definition 4.2.} Given a metric space $X$, the space $\mathbb{R}^+ \times X$ with the metric defined by

\[
  d((r_1, x_1), (r_2, x_2)) = \begin{cases} 
    \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos d(x_1, x_2)} & \text{if } d(x_1, x_2) \leq \pi, \\
    r_1 + r_2 & \text{if } d(x_1, x_2) \geq \pi,
  \end{cases}
\]

is called a metric cone over $X$. We usually denote it by $C(X)$ with the metric $\mathbb{R}^+ \times_r X$.

The main theorem of this section can be stated as follows.

\textbf{Theorem 4.3.} Let $\{((M_i, g_i; p_i))\}$ be a sequence of manifolds in $\mathcal{M}(A, v, \Lambda)$. Then there exists a subsequence of $\{((M_i, g_i; p_i))\}$ that converges to a metric space $(Y; y)$ in the pointed Gromov–Hausdorff topology. Moreover, for each $z \in (Y; y)$, each tangent cone $T_zY$ is a metric cone over another metric space whose diameter is less than $\pi$.

The proof of Theorem 4.3 is similar to one of Splitting Theorem 3.1. We need another triangle lemma to estimate the distance.

\textbf{Lemma 4.4.} Let $x, y$ be two points in a minimal geodesic curve from $p$ and denote the part of the geodesic curve from $x$ to $y$ by $\gamma(t)$. Let $\gamma_t(t)$ be a family of geodesic curves connecting $z$ and $\gamma(s)$ as in Lemma 3.2. Suppose that there is a smooth function $h$ on $M$ which satisfies

(i) \[ |h(z) - h(x) - r(z)^2 - r(x)^2| < \delta \ll 1, \]

(ii) \[ \int_{[0, a]} |\nabla h(\gamma(s)) - r(\gamma(s))\gamma'(s)| < \delta \ll 1, \]

(iii) \[ \int_{[0, a]} \int_{[0, l(s)]} |\text{hess } h - g| \, dt \, ds < \delta \ll 1. \]

Here $r(\cdot) = \text{dist}(p, \cdot)$. Then

\[ d(z, y)^2r(x) - d(x, z)^2r(y) + r(z)^2(r(y) - r(x)) - r(x)r(y)(r(y) - r(x)) < \epsilon(\delta). \]
Proof. The proof is similar to one of Lemma 3.2. First, we have
\[
\left| h(\gamma(s)) - h(\gamma(0)) - \frac{(s + r(x))^2}{2} + \frac{r^2(x)}{2} \right| = \left| \int_0^s \langle \nabla h(\gamma(s)) - (s + r(x))\gamma'(s), \gamma'(s) \rangle \right| \leq \delta.
\]
Then
\[
h(\gamma_s(l(s))) = h(\gamma(s)) = h(x) + \frac{(s + r(x))^2}{2} - \frac{r^2(x)}{2} + o(1).
\]
Since
\[
l(s)h'(\gamma_s(0)) = h(\gamma_s(l(s))) - h(z) - \frac{l^2(s)}{2} - \int_0^a \int_0^{l(s)} (\text{hess } h(\gamma'_s(t), \gamma'_s(t))) dt ds,
\]
from condition (iii) and (i), we get
\[
l(s)h'(\gamma_s(0)) = \frac{(s + r(x))^2}{2} - \frac{r^2(z)}{2} + h(x) - h(z) - \frac{l^2(s)}{2} + o(1)
\]
\[
= \frac{(s + r(x))^2}{2} - \frac{r^2(z)}{2} - \frac{l^2(s)}{2} + o(1).
\]
Consequently, we obtain
\[
l(s)h'(\gamma_s(l(s))) = \frac{(r(x) + s)^2 - r^2(z)}{2} + \frac{l^2(s)}{2}
\]
\[
+ l(s) \int_0^{l(s)} (\text{hess } h(\gamma'_s(t), \gamma'_s(t))) dt + o(1).
\]
Hence we derive
\[
(4.2) \quad \int_0^a \left( \frac{2l(s)h(\gamma'_s(l(s)))}{(s + r(x))^2} - \frac{l^2(s)}{s + r(x)} \right) \, ds = a + \frac{r^2(z)}{r(x)} - \frac{r^2(z)}{r(x)} + o(1).
\]
Secondly, by the first variation formula,
\[
l'(s) = \langle \gamma'_s(l(s)), \gamma'(s) \rangle,
\]
we get from condition (ii) that
\[
\int_0^a \left( \frac{l^2(s)}{s + r(x)} \right) \, ds = \int_0^a \left( \frac{2l(s)l'(s)}{s + r(x)} - \frac{l^2(s)}{(s + r(x))^2} \right) \, ds
\]
\[
= \int_0^a \left( \frac{2l(s)(s + r(x))\langle \gamma'_s(l(s)), \gamma'(s) \rangle}{(s + r(x))^2} - \frac{l^2(s)}{(s + r(x))^2} \right) \, ds
\]
\[
= \int_0^a \left( \frac{2l(s)\langle \gamma'_s(l(s)), \nabla h(\gamma(s)) \rangle}{(s + r(x))^2} - \frac{l^2(s)}{(s + r(x))^2} \right) \, ds + o(1)
\]
\[
= \int_0^a \left( \frac{2l(s)h'(\gamma_s(l(s)))}{(s + r(x))^2} - \frac{l^2(s)}{(s + r(x))^2} \right) \, ds + o(1).
\]
Therefore, by combining (4.2), we get (4.1) immediately. □
It is easy to see that the left-hand side of (4.1) is zero in a metric cone \( C(X) \) if \( x, y \) lie in a radial direction. We need a few lemmas more to prove Theorem 4.3.

**Lemma 4.5.** Given \( \eta > 0 \), there exists \( \omega = \omega(a, b, \eta, A, \Lambda) \) such that the following is true: if

\[
\text{Ric}^f_g \geq -(n-1)\Lambda^2 g \quad \text{and} \quad |\nabla f| \leq A \tag{4.3}
\]

and

\[
\frac{\text{vol}^f(\partial B_p(b))}{\text{vol}^f(\partial B_p(a))} \geq (1 - \omega) \frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a)}, \tag{4.4}
\]

then for any point \( q \) on \( \partial B_p(a) \), there exists \( q' \) on \( \partial B_p(b) \) such that

\[d(q, q') \leq b - a + \eta.\]

**Proof.** Suppose that the conclusion fails to hold for some \( \eta \) and \( q_1 \in \partial B_p(a) \). Then for any point in \( B_{q_1}(\eta/3) \), there is no point \( q \) on \( \partial B_p(b) \) such that \( d(q_1, q) \leq b - a + \eta/3 \). Thus for any \( r < \eta/3 \), any minimal geodesic from \( p \) to \( \partial B_p(b) \) does not intersect with \( B_{q_1}(\eta/3) \cap \partial B_p(a + r) \).

Since

\[\text{vol}^f(B_{q_1}(\eta/3)) \geq \frac{L_{\Lambda,A}(\eta/3)}{L_{\Lambda,A}(2b)} \text{vol}^f(A_p(a, b)),\]

by the coarea formula, there exists some \( \eta/4 < r < \eta/3 \) such that

\[\text{vol}^f(B_{q_1}(\eta/3) \cap \partial B_p(a + r)) \geq \frac{1}{\eta} \frac{L_{\Lambda,A}(\eta/3)}{L_{\Lambda,A}(2b)} \text{vol}^f(A_p(a, b)).\]

Using the monotonicity formula (1.8), we get

\[
\text{vol}^f(\partial B_p(b)) \leq \text{vol}^f(\partial B_p(a + r) \setminus B_{q_1}(\eta/3)) \frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a + r)}
\]

\[
\leq \left( \text{vol}^f(\partial B_p(a + r)) - \frac{1}{\eta} \frac{L_{\Lambda,A}(\eta/3)}{L_{\Lambda,A}(2b)} \text{vol}^f(A_p(a, b)) \right) \frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a + r)}. \tag{4.5}
\]

It follows

\[
\text{vol}^f(\partial B_p(b)) \leq (1 + \delta'(\eta, b, a))^{-1} \text{vol}^f(\partial B_p(a + r)) \frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a + r)}
\]

\[
\leq (1 + \delta'(\eta, b, a))^{-1} \text{vol}^f(\partial B_p(a)) \frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a)}. \tag{4.6}
\]

But this is a contradiction to (4.4) as \( \omega < \frac{1}{2} \delta'(\eta, b, a) \). Therefore, the lemma is proved. \( \square \)

By applying [5, Theorem 3.6] with the help of Lemma 4.4 and Lemma 4.5, we have the following proposition.

**Proposition 4.6.** Given \( \eta > 0 \), there exist \( \omega = \omega(a, b, \eta) \) and \( \delta = \delta(\eta) \) such that if (4.3) and (4.4) are satisfied, then there is a length space \( X \) such that

\[d_{GH}(A_p(a, b), (a, b) \times_r X) < \eta,\]

where \( (a, b) \times_r X \) is an annulus in \( C(X) \) and the metric of \( A_p(a, b) \) is measured in a slightly bigger annulus in \( M \).
Proof. It suffices to verify the condition for distance function in [5, Theorem 3.6]. Let \( x, y, z, w \) be four points in the annulus \( A_p(a, b) \) such that both pairs \( \{ x, y \} \) and \( \{ z, w \} \) are in the radial direction from \( p \). Then by applying the segment inequality of Lemma 3.5 to the function \( h \) in Lemma 2.4 and Lemma 2.5, we can find another four points \( x_1, y_1, z_1, w_1 \) near the four points respectively such that Triangular Lemma 4.4 holds for two triples \( \{ x_1, y_1, z_1 \} \) and \( \{ y_1, z_1, w_1 \} \). Now we choose four points \( x_2, y_2, z_2, w_2 \) in the plane \( \mathbb{R}^2 \) such that both triples \( \{ O, x_2, y_2 \} \) and \( \{ O, z_2, w_2 \} \) are co-linear. Moreover, we can require that

\[
r(x_2) = r(x_1), \quad r(y_2) = r(y_1), \quad r(z_2) = r(z_1), \quad r(w_2) = r(w_1)
\]

and

\[
d(x_1, z_1) = d(x_2, z_2).
\]

Thus by using Triangle Lemma 4.4 to \( \{ x_1, y_1, z_1 \} \), it is easy to see that

\[
|d(y_2, z_2) - d(y_1, z_1)| < \Psi. \tag{4.5}
\]

Applying Triangle Lemma 4.4 to \( \{ y_1, z_1, w_1 \} \), we have

\[
|d(y_1, z_1)^2 r(w_1) + r(w_1) r(z_1)(r(w_1) - r(z_1)) - d(y_1, w_1)^2 r(z_1) - r(y_1)^2 d(z_1, w_1)| < \Psi. \tag{4.6}
\]

Note that the left-hand side of estimate (4.6) is zero when the triple \( \{ y_1, z_1, w_1 \} \) is replaced by \( \{ y_2, z_2, w_2 \} \) in the plane. Since

\[
|d(z_1, w_1) - (r(w_1) - r(z_1))| < \Psi,
\]

we get from (4.5) and (4.6) that

\[
|d(y_1, w_1) - d(y_2, w_2)| < \Psi.
\]

On the other hand, \( d(y_2, w_2) \) can be written as the following function:

\[
d(y_2, w_2) = Q(r(x_2), r(y_2), r(z_2), r(w_2), d(x_2, z_2)).
\]

Therefore

\[
|d(y_1, w_1) - Q(r(x_1), r(y_1), r(z_1), r(w_1), d(x_1, z_1))| < \Psi.
\]

It follows that

\[
|d(y, w) - Q(r(x), r(y), r(z), r(w), d(x, z))| < \Psi. \tag{4.7}
\]

Estimate (4.7) is just the condition for distance function in [5, Theorem 3.6].

By (4.7) and Lemma 4.5 we see that two conditions in [5, Theorem 3.6] are satisfied. Hence as a consequence of this theorem, we obtain Proposition 4.6. In fact, \( X \) is a level set of \( r^{-1}(a) \) with a \( \chi \)-intrinsic metric defined by

\[
\ell^X(x, y) = \frac{1}{a} \inf \sum_{i=1}^{n} d(x_{i-1}, x_i),
\]

where the infimum is taken among all the sequences \( \{ x_i \} \in X \) which satisfy \( x_0 = x, x_n = y \) and \( d(x_{i-1}, x_i) \leq \chi \). \( \square \)
It remains to verify condition (4.4) in Lemma 4.5.

**Lemma 4.7.** Given $a$ and $b$ such that $0 < a < b = a\Omega$, $\Omega > 0$, there exists an integer $N = N(n, \Omega, \Lambda, v, A)$ such that for any sequence of $r_i$ (1 ≤ $i$ ≤ $N$) with $\Omega r_{i+1} \leq r_i \leq \frac{b}{\Omega}$, the volume condition (4.4) for any manifold $(M, g) \in M(\Lambda, v, A)$ in Lemma 4.5 holds for some annulus $A_p(\eta r_k, \eta br_k) \subset M (1 \leq k \leq N)$ with rescaling metric $\hat{g} = \frac{g}{r^2}$.

**Proof.** We only need to give an upper bound of $N$ in case that the inequality

$$
(4.8) \quad \frac{\text{vol}_g^f(\partial B_p(br_k))}{L_{r_k A, r_k A}(br_k)} \geq e^{-\omega} \frac{\text{vol}_g^f(\partial B_p(\eta r_k))}{L_{r_k A, r_k A}(\eta r_k)}
$$

does not hold for any $1 \leq k \leq N$. Then by the monotonicity formula (1.8), we know that

$$
\frac{\text{vol}_g^f(\partial B_p(br_N))}{L_{r_k A, r_k A}(br_N)} \leq e^{-N \omega} \frac{\text{vol}_g^f(\partial B_p(br_1))}{L_{r_k A, r_k A}(br_1)}.
$$

Thus by the non-collapsing condition the left-hand side has a lower bound $c_1(n, \Lambda, v, A)$, and by Theorem 1.2 the right-hand side is not greater than $e^{-N \omega} c_2(n, \Lambda, v, A)$. Thus this helps us to get an upper bound of $N$. Hence, if $N$ is larger than this bound, there must be some $k$ such that (4.8) holds. The lemma is proved.

**Proof of Theorem 4.3.** Without loss of generality, we may assume that $z = y$ since each point in $(Y, d; y)$ is a limit of sequence of volume non-collapsing points in $M_i$. Also we note that the tangent cone $T_y Y$ always exists in our case by Gromov’s theorem [14]. By the contradiction argument, we suppose that $T_y Y$ is not a metric cone. Then it is easy to see that there exist numbers $0 < a < b, \eta_0 > 0$ and a sequence $\{r_i\}$, which tends to 0, such that for any length space $X$ annulus $A_y(\eta r_i, \eta br_i) \subset (Y, d; y)$ satisfy

$$
(4.9) \quad d_{GH}(A_y(\eta r_i, \eta br_i), (\eta r_i, \eta br_i) \times_X X) > 3r_i \eta_0.
$$

By taking a subsequence, we may assume that $\Omega r_{i+1} \leq r_i$ ($\Omega = \frac{b}{2}$) and $r_i$ is smaller than $\delta$ in Lemma 4.5. On the other hand, since $Y$ is the limit of $M_i$, we can find an increasing sequence $m_i$ such that for every $j \geq m_i$

$$
(4.10) \quad d_{GH}(A_y(\eta r_i, \eta br_i), A_{\eta p_j}(\eta r_i, \eta br_i)) < r_i \eta_0.
$$

Let $\omega$ be a small number as chosen in Proposition 4.6 and $N$ an integer such that Lemma 4.7 is true for $\omega > 0$. Thus by (4.10), we see that there exist a subsequence $\{r_{i_k}\} \to 0$ and a sequence $\{j_k\} \to \infty$ such that

$$
d_{GH}(A_y(\eta r_{i_k}, \eta br_{i_k}), A_{\eta p_{j_k}}(\eta r_{i_k}, \eta br_{i_k})) < r_{i_k} \eta_0.
$$

where the annuli $A_{\eta p_{j_k}}(\eta r_{i_k}, \eta br_{i_k})$ are chosen as in Lemma 4.7. Now we can apply Proposition 4.6 to show that for each large $k$ there exists a length space $X$ such that

$$
d_{GH}(A_{\eta p_{j_k}}(\eta r_{i_k}, \eta br_{i_k}), (\eta r_{i_k}, \eta br_{i_k}) \times_X X) < r_{i_k} \eta_0.
$$

But this is impossible by (4.9). Therefore, $T_y Y$ must be a metric cone.
The diameter estimate follows from Splitting Theorem 3.1. In fact, if \( \text{diam}(X) > \pi \), there will be two points \( p \) and \( q \) in \( X \) such that \( d(p, q) = \pi \). By Theorem 3.1, it follows that 
\[
C(X) = \mathbb{R} \times Y_1, \text{ where } Y_1 \text{ is also a metric cone, i.e. } Y_1 = C(X_1).
\]
It is clear that \( \text{diam}(X_1) > \pi \) since \( \text{diam}(X) > \pi \). Thus we can continue to apply Theorem 3.1 to split off \( X_1 \). By induction, \( C(X) \) should be a Euclidean space, and consequently \( X \) is a standard sphere. But this is impossible by the assumption that \( \text{diam}(X) > \pi \).

Following the argument in the proofs of Theorem 4.3 and Proposition 4.6, we actually prove the following strong approximation of Gromov–Hausdorff to the flat space.

**Corollary 4.8.** For all \( \epsilon > 0 \), there exist \( \delta = \delta(n, \epsilon) \) and \( \eta = \eta(n, \epsilon) \) such that if

\[
\text{Ric}^f_g \geq -(n-1)\delta^2 g, \quad |\nabla f| \leq \eta
\]

and

\[
e^{-f(0)} \text{vol}_f(B_p(1)) \geq (1-\delta) \text{vol}(B_0(1))
\]

are satisfied, then

\[
d_{GH}(B_p(1), B_0(1)) < \epsilon.
\]

**Proof.** Suppose that conclusion (4.13) is not true. Then there exist sequences of \( \{\delta_i\} \) and \( \{\eta_i\} \) which tend to 0 both, and a sequence of manifolds \( \{(M_i, g_i)\} \) with conditions (4.11) and (4.12) such that

\[
d_{GH}(B_p(1), B_0(1)) \geq \epsilon_0 > 0,
\]

where \( B_{p_i}(1) \subset M_i \). Then by following the argument in the proofs of Theorem 4.3 and Proposition 4.6, it is no hard to show that the \( B_{p_i}(1) \) converge to a limit \( B_x(1) \) which is a metric ball with radius 1 in a metric-cone \( (C(X), d) \) with vertex \( x \). Since the blowing-up space of \( B_x(1) \) at \( x \) is \( C(X) \) itself, we see that there are subsequences \( \{j\} \) and \( \{i_j\} \), both of which tend to infinity, such that

\[
(B_{p_{i_j}}(j), j^2 g_{i_j}, q_{i_j}) \to (C(X), d, x).
\]

For any \( y \in X \), we choose a sequence of points \( q_{i_j} \in B_{p_{i_j}}(j) \subset (M_{i_j}, j^2 g_{i_j}) \) which tends to \( y \). Then for any given \( R > 0 \), we have

\[
B_{q_{i_j}}(R) (\subset (M_{i_j}, j^2 g_{i_j})) \to B_y(R).
\]

Since the volume condition (4.12) implies

\[
e^{-f(0)} \text{vol}_f(B_{q_{i_j}}(R)) \to \text{vol}(B_0(R)),
\]

by the above argument, \( B_y(R) \) is in fact a metric ball with radius \( R \) in a metric cone \( C(Y) \) with vertex \( y \). Note that \( R \) is arbitrary. We prove that \( C(X) \) is also a cone with vertex at \( y \). This shows that there exists a line connecting \( x \) and \( y \) in \( C(X) \). By Splitting Theorem 3.1, \( C(X) \) can split off a line along the direction \( xy \). Since \( y \in X \) can be taken in any direction, \( C(X) \) must be a Euclidean space. But this is impossible according to (4.14). The corollary is proved.
Remark 4.9. Corollary 4.8 is a generalization of [3, Theorem 9.69] in the Bakry–Émery geometry. It will be used in Section 5 and Section 6 for the blowing-up analysis. We also note that $e^{-f(0)} \text{vol}^f(B_p(1))$ is close to $\text{vol}(B_p(1))$ since $|\nabla f|$ is small enough. Thus the volume condition (4.12) can be replaced by
\[
\text{vol}(B_p(1)) \geq (1 - \delta) \text{vol}(B_0(1)).
\]

For the rest of this section, we prove Colding’s volume convergence theorem in the Bakry–Émery geometry by using the Hessian estimates in [11, Section 2].

Theorem 4.10. Let $(M^n_i, g_i)$ be a sequence of Riemannian manifolds which satisfy the conditions in (4.3). Suppose that $M_i$ converge to an $n$-dimensional compact manifold $M$ in the Gromov–Hausdorff topology. Then
\[
\lim_{i \to \infty} \text{vol}(M_i, g_i) = \text{vol}(M).
\]

We first prove a local version of Theorem 4.10 as follows.

Lemma 4.11. Given $\epsilon > 0$, there exist $R = R(\epsilon, \Lambda, A, n) > 1$ and $\delta = \delta(\epsilon, \Lambda, A, n)$ such that if
\[
\text{Ric}_M^f \geq -(n - 1) \frac{\Lambda^2}{R^2} g, \quad |\nabla f| \leq \frac{A}{R}
\]
and
\[
d_{GH}(B_p(R), B_0(R)) < \delta,
\]
then we have
\[
\text{vol}(B_p(1)) \geq (1 - \epsilon) \text{vol}(B_0(1)).
\]

Proof. We need to construct a Gromov–Hausdorff approximation map by using $f$-harmonic functions constructed in Section 2. Choose $n$ points $q_i$ in $B_p(R)$ which are close to $Re_i$ in $B_0(R)$, respectively. Let $l_i(q) = d(q, q_i) - d(q_i, p)$ and $h_i$ a solution of
\[
\Delta^f h_i = 0 \quad \text{in } B_1(p)
\]
with $h_i = l_i$ on $\partial B_1(p)$. Then by Lemma 2.3, we have
\[
\frac{1}{\text{vol}(B_p(1))} \int_{B_p(1)} |\text{hess } h_i|^2 < \Psi(\frac{1}{R}, \delta; A).
\]
By using an argument in [11, Lemma 2.9], it follows that
\[
(4.15) \quad \frac{1}{\text{vol}(B_p(1))} \int_{B_p(1)} |(\nabla h_i, \nabla h_j) - \delta_{ij}| < \Psi(\frac{1}{R}, \delta; A).
\]
Define a map by $h = (h_1, h_2, \ldots, h_n)$. It is easy to see that the map $h$ is a $\Psi(\frac{1}{R}, \delta; \Lambda)$-Gromov–Hausdorff approximation to $B_p(1)$ by using estimate (2.2) in Lemma 2.3. Since $h$ maps $\partial B_p(1)$ nearby $\partial B_0(1)$ with distance less than $\Psi$, by a small modification to $h$ we may assume that
\[
h : (B_p(1), \partial B_p(1)) \to (B_0(1 - \Psi), \partial B_0(1 - \Psi)).
\]
Next we use a degree argument in [3] to show that the image of \( h \) contains \( B_0(1 - \Psi) \). By using the Vitali Covering Lemma, there exists a point \( x \) in \( B_p\left(\frac{1}{8}\right) \) such that for any \( r \) less than \( \frac{1}{8} \) it holds

\[
\frac{1}{\text{vol}(B_x(r))} \int_{B_x(r)} |\text{hess} \, h_i| < \Psi
\]

and

\[
\frac{1}{\text{vol}(B_x(r))} \int_{B_x(r)} |(\nabla h_i, \nabla h_j) - \delta_{ij}| < \Psi.
\]

Let \( \eta = \Psi^{\frac{n}{n+1}} \). For any \( y \) with \( d(x, y) = r < \frac{1}{8} \), applying Lemma 3.4 to \( A_1 = B_x(\eta r) \), \( A_2 = B_y(\eta r) \), \( e = |\text{hess} \, h_i| \), we get from (4.16) that

\[
\int_{B_x(\eta r) \times B_y(\eta r)} \int_{y_{zw}} |\text{hess} \, h_i(y', y')| < r (\text{vol}(B_x(\eta r)) + \text{vol}(B_y(\eta r))) \text{vol}(B_x(r)) \Psi.
\]

It follows that

\[
\int_{B_x(\eta r)} \left( Q(r, \eta) \int_{B_y(\eta r)} \int_{y_{zw}} \sum_{i=1}^n |\text{hess} \, h_i(y', y')| + |(\nabla h_i, \nabla h_j) - \delta_{ij}| \right) < \text{vol}(B_x(\eta r)) \Psi.
\]

where

\[
Q(r, \eta) = \frac{\text{vol}(B_x(\eta r))}{r (\text{vol}(B_x(\eta r)) + \text{vol}(B_y(\eta r))) \text{vol}(B_x(r))}.
\]

Consider

\[
Q(r, \eta) \int_{B_y(\eta r)} \int_{y_{zw}} \sum_{i=1}^n |\text{hess} \, h_i(y', y')| + |(\nabla h_i, \nabla h_j) - \delta_{ij}|
\]

as a function of \( z \in B_x(\eta r) \). Then one sees that there exists a point \( x^* \in B_x(\eta r) \) such that

\[
|\langle \nabla h_i, \nabla h_j \rangle (x^*) - \delta_{ij}| < \Psi
\]

and

\[
\sum_{i=1}^n \int_{B_y(\eta r)} \int_{y_{x^* w}} |\text{hess} \, h_i(y', y')| < r \text{vol}(B_x(r)) \eta^{-n} \Psi.
\]

Here at the last inequality, we used the volume comparison (1.5). Moreover, by (4.18), we can find a point \( y^* \in B_y(\eta r) \) such that

\[
\sum_{i=1}^n \int_{y_{x^* w}} |\text{hess} \, h_i(y', y')| < \eta r.
\]

By a direct calculation with help of (4.17) and (4.19), we get

\[
(h(x^*) - h(y^*))^2 = (1 + \Psi^{\frac{1}{n+1}})^2 r^2.
\]

This shows that \( h(x) \neq h(y) \) for any \( y \) with \( d(y, x) = \frac{1}{8} \). On the other hand, for any \( y \) with \( d(y, x) \geq \frac{1}{8} \), it is clear that \( h(x) \neq h(y) \) since \( h \) is a \( \Psi \)-Gromov–Hausdorff approximation. Thus we prove that the pre-image of \( h(x) \) is unique. Therefore the degree of \( h \) is 1, and consequently, \( B_0(1 - \Psi) \subset h(B_p(1)) \). The lemma is proved because the volume of \( B_p(1) \) is almost same to one of \( h(B_p(1)) \) by the fact (4.15). \( \square \)
Proof of Theorem 4.10. Choose finite balls $B_{q_i}(r_i)$ to cover $M$ with $r_i$ small enough to make all balls close to Euclidean balls so that
\[ \sum_i \text{vol}(B(q_i, r_i)) < (1 + \epsilon) \text{vol}(M) \]
for any given $\epsilon > 0$. Then for $j$ sufficiently large, $M_j$ can be covered by balls $B_{q_{ji}}(r_{ji})$ with $r_{ji} \leq (1 + \epsilon) r_i$. Thus by the volume comparison (1.5), we have
\[ \text{vol}(M_j) \leq \sum_i \text{vol}(B(q_{ji}, r_{ji})) < (1 + \Psi(\delta : \Lambda, A)) \sum_i \text{vol}(B(q_i, r_i)). \]
Here $\delta = \max\{r_i\}$. Hence we get
\[ \limsup_{j \to \infty} \text{vol}(M_j) \leq \text{vol}(M). \]
On the other hand, for any $\epsilon > 0$, we choose numbers $R, \delta$ as in Lemma 4.11. Then there are disjoint small balls $B_{q_i}(r_i)$ in $M$ such that
\[ d_{\text{GH}}(B_{q_i}(Rr_i), B_0(Rr_i)) \leq \delta r_i \]
and
\[ (1 + \epsilon) \sum_i r^n_i \text{vol}(B_0(1)) \geq \sum_i \text{vol}(B(q_i, r_i)) \geq (1 - \epsilon) \text{vol}(M). \]
For $q_{ij}$ converging to $q_i$, applying Lemma 4.11 to each ball $B_{q_{ij}}(Rr_i)$ with rescaling metric $\frac{g_j}{r_i}$, we have
\[ \text{vol}(B_{q_{ij}}(r_i)) \geq (1 - \epsilon) r^n_i \text{vol}(B_0(1)). \]
As a consequence,
\[ \text{vol}(M_j) \geq \sum_i \text{vol}(B_{q_{ij}}(r_i)) \geq (1 - \epsilon) \sum_i r^n_i \text{vol}(B_0(1)). \]
By (4.20), it follows that
\[ (1 + \epsilon) \text{vol}(M_j) > (1 - \epsilon)^2 \text{vol}(M). \]
Taking $\epsilon$ to 0, we get
\[ \liminf_{j \to \infty} \text{vol}(M_j) \geq \text{vol}(M). \]
The theorem is proved. \hfill \square

5. Structure of singular set I: Case of Riemannian metrics

According to Theorem 4.3, we may introduce a notion of $\delta_k$-type singular point $y$ in the limit space $(Y, d_\infty; p_\infty)$ of a sequence of Riemannian manifolds $\{(M_i, g_i; p_i)\}$ in $\mathcal{M}(A, v, \Lambda)$ as Definition 0.2 if there exists a tangent cone at $y$ which can be split out a Euclidean space $\mathbb{R}^k$ isometrically with dimension at most $k$. By applying Metric Cone Theorem 4.3 to appropriate the tangent cone spaces $T_y Y$, we can follow the argument in [6] to show that dimension of $\delta_k$ is less than $k$. Moreover, $\delta = \delta(Y) = \delta_{n-2}$, where $\delta(Y) = \bigcup_{i=0}^{n-1} \delta_i$. The latter is equivalent to that any tangent cone cannot be the upper half space, which can be proved by using a topological argument as in the case of Ricci curvature bounded below (cf. [6, Theorem 6.2]). Thus we have the following theorem.
Theorem 5.1. Let \((M_i, g_i; p_i)\) be a sequence of Riemannian manifolds in \(\mathcal{M}(A, v, \Lambda)\) and let \((Y, d_\infty; p_\infty)\) be its limit in the Gromov–Hausdorff topology. Then \(\dim S_k \leq k\) and \(S(Y) = S_{n-2}\).

Remark 5.2. By Theorem 5.1, one sees that \(\mathcal{H}^n(S) = 0\). Thus by Theorem 4.10,

\[
\lim_{i \to \infty} \text{vol}(M_i) = \mathcal{H}^n(Y).
\]

Moreover, if \(B_i(r) \subset M_i\) converge to \(B_\infty(r) \subset Y\),

\[
\lim_{i \to \infty} \text{vol}(B_i(r)) = \mathcal{H}^n(B_\infty(r)),
\]

where \(B_i(r)\) and \(B_\infty(r)\) are radius \(r\)-balls in \(M_i\) and \(Y\), respectively.

We define \(\delta\)-regular points in \(Y\).

Definition 5.3. We say that \(y \in (Y; p_\infty)\) is an \(\delta\)-regular point if there exist an \(\epsilon\) and a sequence \(\{r_i\}\) such that

\[
\text{dist}_{GH}\left(\left(B_y(1), \frac{1}{r_i}d_\infty\right), B_0(1)\right) < \epsilon \quad \text{as} \quad i \to \infty.
\]

Here \(B_0(1)\) is the unit ball in \(\mathbb{R}^n\). We denote the set of those points by \(\mathcal{R}_\delta\).

In this section, our main purpose is to generalize \([8, \text{Theorem 1.15}]\) to analyze the singular structure \(S\) under an additional assumption of \(L^p\)-integral bounded curvature.

Theorem 5.4. Let \((M_i, g_i; p_i)\) be a sequence in \(\mathcal{M}(A, v, \Lambda)\) and \((Y; p_\infty)\) its limit as in Theorem 5.1. Suppose that

\[
\int_{B_{p_i}(2)} |Rm|^p < C.
\]

Then for any \(\epsilon > 0\), the following is true:

\[
\mathcal{H}^{n-2p}(B_{p_\infty}(1) \setminus \mathcal{R}_{2\delta}) < \infty \quad \text{if} \quad 1 \leq p < 2,
\]

\[
\dim(B_{p_\infty}(1) \setminus \mathcal{R}_{2\delta}) \leq n - 4 \quad \text{if} \quad p = 2.
\]

The theorem is a consequence of following result of \(\delta\)-regularity.

Proposition 5.5. For any positive \(v\) and \(\epsilon\), there exist three small numbers \(\delta = \delta(v, \epsilon, n)\), \(\eta = \eta(v, \epsilon, n)\), \(\tau = \tau(v, \epsilon, n)\) and a big number \(l = l(v, \epsilon, n)\) such that if \((M^n, g)\) satisfies

\[
\text{Ric}^f_M > -(n-1)\tau^2, \quad |\nabla f| < \tau, \quad \text{vol}(B_p(1)) \geq v,
\]

\[
\frac{1}{\text{vol}(B_p(3))} \int_{B_p(3)} |Rm| < \delta,
\]

and for some metric space \(X\),

\[
d_{GH}(B_p(l), B_{(0,x)}(l)) < \eta
\]

holds for \(k = 2\) or \(3\), where \((0, x)\) is the vertex in \(\mathbb{R}^{n-k} \times C(X)\), then

\[
d_{GH}(B_p(1), B_0(1)) < \epsilon.
\]
To prove Proposition 5.5, it is sufficient to prove that \( \text{vol}(B_p(1)) \) is close to \( \text{vol}(B_0(1)) \) according to Corollary 4.8. The latter is equivalent to showing that \( \text{vol}(B_0(1)) \) is close to \( \text{vol}(B_{\alpha,x}(1)) \) by Remark 5.2. Thus we shall estimate the volume of the section \( X \). In the following, we will use the idea in [8] to turn into estimating the volume of a pre-image of \( X \) by constructing a Gromov–Hausdorff approximation.

Let \( h_i (i = 1, \ldots, n-k) \) be \( f \)-harmonic functions on \( B_p(5) \) with appropriate boundary values as constructed in the proof of Splitting Theorem 3.1 (cf. Proposition 3.6) and \( h \) an approximation of \( \frac{r^2}{2} \) as constructed in the proof of Metric Cone Theorem 4.3 (cf. Lemma 2.4, Lemma 2.5), which is a solution of

\[
\Delta^f h = n \quad \text{in} \quad B_p(5), \quad h|_{\partial(B_p(5))} = \frac{25}{2}.
\]

Let

\[
w_0 = 2h - \sum_j h_j^2.
\]

Define \( w \) to be a solution of

\[
\Delta^f w = 2k, \quad w|_{\partial B_p(4)} = w_0.
\]

Then \( w \) is almost positive, so it can be transformed to be positive by adding a small number. Set

\[
u^2 = w + \Psi > 0.
\]

We recall some estimates for functions \( h_i, h \) and \( w \):

\[
(5.8) \quad \frac{1}{\text{vol}(B_p(3))} \int_{B_p(3)} \sum_i |\text{hess } h_i|^2 + \sum_{i \neq j} |\langle \nabla h_i, \nabla h_j \rangle| + \frac{1}{\text{vol}(B_p(3))} \int_{B_p(3)} (|\nabla h_i| - 1)^2 < \Psi,
\]

\[
(5.9) \quad \frac{1}{\text{vol}(B_p(3))} \int_{B_p(3)} (|\nabla h - \nabla r|^2 + |\text{hess } h - g|^2) < \Psi,
\]

\[
(5.10) \quad \frac{1}{\text{vol}(B_p(3))} \int_{B_p(3)} |\text{hess } w_0 - \text{hess } w|^2 < \Psi
\]

and

\[
\frac{1}{\text{vol } B_p(3)} \int_{B_p(3)} |\nabla w_0 - \nabla w|^2 < \Psi.
\]

The first two estimates are proved in Section 2 (cf. Lemma 2.3, Lemma 2.4, Lemma 2.5). We note that condition (2.6) in both Lemma 2.4 and Lemma 2.5 is satisfied by (5.7) according to (5.1) in Remark 5.2. The others can also be obtained in a similar way.

We define maps \( \Phi \) and \( \Gamma \) respectively by

\[
\Phi = (h_j) : B_p(4) \rightarrow \mathbb{R}^{n-k}, \quad \Gamma = (h_j, u) : B_p(4) \rightarrow \mathbb{R}^{n-k+1}.
\]

Let

\[
V_{\Phi, u}(z) = \text{vol}(\Phi^{-1}(z) \cap U_u),
\]

where \( U_u = \Gamma^{-1}(B_{0}^{n-k}(1) \times [0, u]) \) for \( u \leq 2 \). Then the following lemma holds.
Lemma 5.6. One has
\begin{equation}
\frac{1}{\text{vol}(B_0^{n-k}(1))} \int_{B_0^{n-k}(1)} \left| V_{\Phi,u}(z) - \frac{u^k}{k} \text{vol}(X) \right| < \Psi.
\end{equation}

Proof. Set
\[ v_\Phi = \nabla h_1 \wedge \cdots \wedge \nabla h_{n-k}. \]
Then \( v_\Phi \) is the Jacobian of \( \Phi \) in \( B_0^{n-k}(1) \). By (5.8), one can show that it is almost 1 almost everywhere in \( B_0^{n-k}(1) \). In fact, the proof is the same to one of (4.15). Hence by the coarea formula, we get
\begin{equation}
\frac{1}{\text{vol}(B_0^{n-k}(1))} \int_{B_0^{n-k}(1)} V_{\Phi,u}(z) = \frac{1}{\text{vol}(B_0^{n-k}(1))} \int_{U} |v_\Phi| = \frac{\text{vol}(U)}{\text{vol}(B_0^{n-k}(1))} + \Psi.
\end{equation}

To compute the variation of \( V_{\Phi,u}(z) \), we modify \( V_{\Phi,u}(z) \) to
\[ J_{\Phi,u,\delta} = \int_{\Phi^{-1}(z)} \chi_\epsilon(|v_\Phi|^2) \psi_{u,\delta}, \]
where \( \psi_{\delta,u} = \xi(u^2) \) with a cut-off function \( \xi \) which satisfies
\[ \xi(t) = \begin{cases} 1 & \text{for } t \in [0, ((1-2\delta)u)^2], \\ 0 & \text{for } t \in [[(1-\delta)u]^2, u^2], \end{cases} \]
and \( \chi_\epsilon(t) \) is another cut-off function which satisfies
\[ \chi_\epsilon(t) = \begin{cases} 0 & \text{for } t \in [0, \epsilon], \\ (1-\epsilon)t & \text{for } t \in [2\epsilon, 1-\epsilon], \\ 1 & \text{for } t \geq 1, \end{cases} \]
and
\[ |\chi_\epsilon'(t)| \leq 3. \]

A direct computation shows that
\begin{equation}
\frac{\partial J_{\Phi,u,\delta}}{\partial z_j} = \int_{\Phi^{-1}(z) \cap U} \chi_\epsilon(|v_\Phi|^2) \sum_i a_{i,j} \nabla h_i(|v_\Phi|^2) \psi_{u,\delta},
\end{equation}
\[ \quad + \int_{\Phi^{-1}(z) \cap U} \chi_\epsilon(|v_\Phi|^2) \sum_i a_{i,j} \text{tr} (\text{hess} h_i) \psi_{u,\delta}, \]
\[ \quad + \int_{\Phi^{-1}(z) \cap U} \chi_\epsilon(|v_\Phi|^2) \sum_i a_{i,j} \langle \nabla \psi_{\delta}, \nabla h_i \rangle. \]

Here \( a_{i,j} \) is the inverse of \( \langle \nabla h_i, \nabla h_j \rangle \) so that \( \Phi_*(\sum_i a_{i,j} \nabla h_i) = \frac{\partial}{\partial z_j} \), and \( \text{tr}(\text{hess} h_i) \) denotes the trace restricted to \( \Phi^{-1}(z) \). By using the coarea formula, the integrations of the first two terms on the right side of (5.13) in \( B_0^{n-k}(1) \) can be controlled by the Hessian estimate in (5.8). Moreover, similar to (4.15), by (5.8) and (5.10), one can show that
\[ \frac{1}{\text{vol}(B_p(1))} \int_{B_p(1)} |\langle \nabla u^2, \nabla h_j \rangle| < \Psi. \]
Thus the integration of the third term on the right side of (5.13) in $B^{n-k}(1)$ is also small. Hence we get
\[
\frac{1}{\text{vol}(B_0^{n-k}(1))} \int_{B_0^{n-k}(1)} |\nabla J_{\Phi, u, \delta}| < \Psi.
\]
On the other hand, by (5.1), it is easy to see
\[
\left| \frac{\text{vol}(U_u)}{\text{vol}(B_0^{n-k}(1))} - \frac{u^k}{k \text{ vol}(X)} \right| < \Psi.
\]
Therefore, we derive (5.11) from (5.12).

Similar to (5.11), by using the above argument for the map $\Gamma$, one can also obtain the following estimate:
\[
\frac{1}{\text{vol}(B^{n-k}(1) \times [0, 1])} \int_{B^{n-k}(1) \times [0, 1]} |V_\Gamma(z, u) - u^{k-1} \text{ vol}(X)| < \Psi,
\]
where $V_\Gamma(z, u) = \text{vol}(\Gamma^{-1}(z, u))$. A similar proof can be also found in [8, Theorem 2.63], so we omit it. Thus we see:

**Lemma 5.7.** There exists a subset of $D_{\epsilon, l} \subseteq B^{n-k}(1) \times [0, 1]$ which depends only on $\epsilon, l$ such that
\[
\text{vol}(D_{\epsilon, l}) > (1 - \Psi) \text{ vol}(B^{n-k}(1) \times [0, 1])
\]
and
\[
(5.14) \quad |V_\Gamma(z, u) - u^{k-1} \text{ vol}(X)| < \Psi \quad \text{for all } (z, u) \in D_{\epsilon, l}.
\]

Next, we use the Bochner identity in terms of the Bakry–Émery Ricci curvature to estimate the second fundamental forms of the pre-images of $\Phi$ and $\Gamma$. Let $v_1, v_2, \ldots, v_m$ be $m$ smooth vector fields. Put $v = v_1 \wedge v_2 \wedge \cdots \wedge v_m$. We compute
\[
\Delta^f |v|^2 = 2\langle \Delta^f v, v \rangle + 2|\nabla v|^2
\]
and
\[
\Delta^f (|v|^2 + \eta)^{1/2} = (|v|^2 + \eta)^{-1/2} \left( |\nabla v|^2 - \frac{\langle \nabla v, v \rangle^2}{|v|^2 + \eta} \right) + (|v|^2 + \eta)^{-1/2} \langle \Delta^f v, v \rangle, \quad \eta > 0.
\]
It follows that
\[
(5.15) \quad (|v|^2 + \eta)^{-1/2} |\pi(\nabla v)|^2 \leq -\frac{|v|}{(|v|^2 + \eta)^{3/2}} (I - \pi) \Delta^f v + \Delta^f (|v|^2 + \eta)^{1/2},
\]
where $\pi : \wedge^m TM \to v^\perp$ is the complement of the orthogonal projection to $v$. On the other hand, if we choose $v_j = \nabla l_j$ and take the map $F = (l_1, \ldots, l_m)$ and $v = v_F$, then
\[
(5.16) \quad |v_F||\Phi_{F^{-1}(c)}|^{-1} \leq |v_F|^{-1} |\pi(\nabla v_F)|^2,
\]
where $\Phi_{F^{-1}(c)}$ denotes the second fundamental form of the level set $F^{-1}(c)$ in $M$. Hence the quantity $(I - \pi) \Delta^f v_F$ in (5.15) gives us an estimate for the second fundamental form of the map $F$. 
To estimate \((I - \pi)\Delta^f v_F\), we use the following formula:

\[
\Delta^f \nabla l_i = \nabla \Delta^f l_i + \text{Ric}^f \left( \nabla l_i, \cdot \right).
\]

Note that in our case \(\Delta^f l_i\) is constant for the map \(F = \Phi\) or \(F = \Gamma = (\Phi, u^2)\). Then it is easy to see

\[
(I - \pi)\Delta^f v_F = 2(I - \pi) \left( \sum_{j_1 < j_2} \nabla l_{1j} \wedge \cdots \wedge \nabla_{e_i} \nabla l_{j_1} \wedge \cdots \wedge \nabla_{e_i} \nabla l_{j_2} \wedge \cdots \wedge \nabla l_m \right) + \text{tr}(\text{Ric}^f),
\]

where \(\text{tr}(\text{Ric}^f)\) is the trace over the space spanning by \(\nabla l_i\).

**Lemma 5.8.** There exists a subset \(E_{\epsilon,l} \subseteq B^{n-k}(1) \times [0, 1]\) which depends only on \(\epsilon, l\) and satisfies

\[
\text{vol}(E_{\epsilon,l}) \geq (1 - \Psi) \text{vol}(B^{n-k}(1) \times [0, 1])
\]
such that for any \((z, u) \in E_{\epsilon,l}\) one has

\[
\frac{1}{V_{\Phi,u}(z)} \int_{\Phi^{-1}(z) \cap U_u} |\Pi_{\Phi^{-1}}|^2 < \Psi,
\]

\[
\frac{1}{V_{\Gamma}(z, u)} \int_{\Gamma^{-1}(z, u)} |\Pi_{\Gamma^{-1}}|^2 < \Psi,
\]

\[
\frac{1}{V_{\Gamma}(z, u)} \int_{\Gamma^{-1}(z, u)} |\Pi_{\Gamma^{-1}}| - u^{-1} g_{\Gamma^{-1}(z, u)} \otimes \nabla u|^2 < \Psi.
\]

**Proof.** Let \(\phi\) be a cut-off function with support in \(B_p(3)\) as constructed in Lemma 1.5. Note that \(v_\Phi\) is almost 1 almost everywhere in \(U_u\) by the Hessian estimates in (5.8). Then by (5.17), we have

\[
\int_{U_u} (|v|^2 + \eta) - \frac{1}{2} |\pi(\nabla v)|^2 e^{-f} d\nu \leq \int_{B_p(3)} |v_\Phi||(I - \pi)\Delta^f v_\Phi| e^{-f} d\nu + \int_{B_p(3)} \phi \Delta^f ((|v_\Phi|^2 + \eta)^{\frac{1}{2}} - 1) e^{-f} d\nu
\]

\[
< \Psi + \int_{B_p(3)} |\Delta^f \phi||(v_\Phi|^2 + \eta)^{\frac{1}{2}} - 1| e^{-f} d\nu.
\]

By (5.16), it follows that

\[
\int_{U_u} |v_\Phi||\Pi_{\Phi^{-1}(z)}|^2 e^{-f} d\nu \leq \lim_{\eta \to 0} \int_{U_u} (|v|^2 + \eta)^{-\frac{1}{2}} |\pi(\nabla v)|^2 e^{-f} d\nu < \Psi.
\]

On the other hand, by the coarea formula, we have

\[
\int_{B^{n-k}(1)} \int_{\Phi^{-1}(z) \cap U_u} |\Pi_{\Phi^{-1}(z)}|^2 e^{-f} d\nu = \int_{U_u} |v_\Phi||\Pi_{\Phi^{-1}(z)}|^2 e^{-f} d\nu.
\]

Thus (5.18) follows from (5.21) immediately. Again by the coarea formula, we get (5.19) from (5.18). Estimate (5.20) can also be obtained by using the same argument above to the map \(\Gamma\) (cf. [8, Theorem 3.7]).
Completion of Proof of Proposition 5.5. We will finish the proof of Proposition 5.5 by applying the Gauss–Bonnet formula to an appropriate level set of \( \Gamma \). For \( k = 2 \), by Lemma 5.7, we see that there exists \((z, u) \) (\( u \) is close to 1) such that
\[
\left| 2\pi t - \frac{1}{u} V_\Gamma(z, u) \right| < \Psi,
\]
where \( t \) is the radius of \( X \). Note that \( X \) is a circle here. On the other hand, by applying the Gauss–Bonnet formula to \( \Phi^{-1}(z) \cap U_u \), we have
\[
\int_{\Gamma^{-1}(z,u)} H + \int_{\Phi^{-1}(z) \cap U_u} K = 2\pi \chi(\Phi^{-1}(z) \cap U_u),
\]
where \( K \) and \( H \) are Gauss curvature and mean curvature of \( \Phi^{-1}(z) \cap U_u \) and \( \Gamma^{-1}(z,u) \), respectively. By (5.18) and (5.6) together with the Gauss–Coddazzi equation, we see that
\[
\left| \int_{\Phi^{-1}(z) \cap U_u} K \right| < \Psi.
\]
Also we get from (5.20) that
\[
\left| \int_{\Gamma^{-1}(z,u)} H - \frac{1}{u} V_\Gamma(z, u) \right| < \Psi.
\]
Thus \( t \) is close to \( \chi(\Phi^{-1}(z) \cap U_u) \) which is an integer. The non-collapsing condition implies that \( \chi(\Phi^{-1}(z) \cap U_u) \) is not zero. So \( t > 1 - \Psi \). As a consequence, the volume of the ball \( B(1) \subset \mathbb{R}^{n-1} \times C(X) \) is close to the one of a unit flat ball. Hence by Remark 5.2, we see that \( \text{vol}(B_p(1)) \) is close to \( \text{vol}(B_0(1)) \). Therefore, we prove that \( B_p(1) \) is close to \( B_0(1) \) by Corollary 4.8.

In case \( k = 3 \), we see that there exists \((z, u) \) in Lemma 5.7 such that
\[
|V_\Gamma(z, u) - \text{vol}(X)| < \Psi, \tag{5.22}
\]
as \( u \) is close to 1. On the other hand, by the Gauss–Bonnet formula, we have
\[
\int_{\Gamma^{-1}(z,u)} K = 2\pi \chi(\Gamma^{-1}(z,u)). \tag{5.23}
\]
Since by (5.19) and (5.6) together with the Gauss–Coddazzi equation,
\[
\left| \int_{\Gamma^{-1}(z,u)} R_{\Phi^{-1}(z)} \right| < \Psi,
\]
where \( R_{\Phi^{-1}(z)} \) is the curvature tensor of the submanifold \( \Phi^{-1}(z) \), it follows from (5.20) that
\[
\left| \int_{\Gamma^{-1}(z,u)} K - V_\Gamma(z, u) \right| < \Psi.
\]
By (5.23), we have
\[
V_\Gamma(z, u) > 4\pi - \Psi,
\]
since the Euler number is even. Thus by (5.22), we get
\[
\text{vol}(B_p(1)) > \text{vol}(B(1)) - \Psi > \text{vol}(B_0(1)) - \Psi.
\]
As a consequence, the volume of \( B_p(1) \) is close to the one of \( B_0(1) \). Therefore, we also prove that \( B_p(1) \) is close to \( B_0(1) \) by Corollary 4.8.
\[\square\]
Proof of Theorem 5.4. Let $\tau, \delta$ be as in Proposition 5.5. For $0 < \rho \leq \theta \leq \tau$, we define a subset in $B_{p_j}(2) \subset M_j$ by

$$Q_j(\theta, \rho) = \left\{ q : \text{there exists } \rho < s \leq \theta \text{ such that } \frac{1}{\vol(B_q(s))} \int_{B_q(s)} |Rm| \geq \delta s^{-2} \right\}. $$

Denote the limit of $Q_j(\theta, \rho)$ in $Y$ by $Q(\theta, \rho)$ and set

$$Q(\theta) = \bigcup_{\rho \leq \theta} Q(\theta, \rho).$$

Claim 5.9. One has

$$B_{y}(1) \subseteq R_{3\epsilon} \cup Q(\theta) \cup S_{n-4}. $$

Suppose that the claim is not true. Then there exists a point $z \notin R_{3\epsilon} \cup Q(\theta) \cup S_{n-4}$ and a tangent cone $T_z Y$ which is $\mathbb{R}^{n-k} \times C(X)$ for $k = 2, 3$ and $d_{GH}(B_{z_{\infty}}(1), B_0(1)) > 3\epsilon$. Thus, there is a sequence $r_i$ approaching 0 such that $(Y, \frac{d}{r_i}; z) \to T_z Y$ in the Gromov–Hausdorff topology. Hence, as $i$ large enough, we have

$$d_{GH}(B_z(r_i), B_0(r_i)) \geq 3\epsilon r_i$$

and

$$d_{GH}(B_z(lr_i), B_{(0, x)}(lr_i)) \leq \frac{1}{2} r_i \eta,$$

where $\eta = \eta(\epsilon, n) \ll 1$ and $l = l(n, \epsilon) \gg 1$ are both determined in Proposition 5.5. For fixed $i$ in the above inequalities, we take $j$ large enough and $q_j \in M_j \to z$ such that

$$d_{GH}(B_{q_j}(r_i), B_0(r_i)) \geq 2\epsilon r_i$$

and

$$d_{GH}(B_{q_j}(lr_i), B_{(0, x)}(lr_i)) \leq r_i \eta.$$

Also we have

$$\frac{1}{\vol(B_{q_j}(3r_i))} \int_{B_{q_j}(3r_i)} |Rm| \leq \delta(3r_i)^{-2} \leq \delta(r_i)^{-2} \text{ for all } j \gg 1.$$

Thus applying Proposition 5.5 to the manifolds $M_j$ with rescaled metrics $\frac{g_j}{r_i}$ together with conditions (5.24), (5.25) and (5.26), we get

$$d_{GH}(B_{q_j}(r_i), B_0(r_i)) \leq \epsilon r_i.$$

But this is impossible by (5.24). The claim is proved.

Putting

$$Q = \bigcap_{\theta > 0} Q(\theta),$$

we have

$$B_{y}(1) \subseteq R_{3\epsilon} \cup Q \cup S_{n-4}. $$

We need to estimate $\mathcal{H}^{n-2P}(Q)$. By the Vitali Covering Lemma, there is a collection of disjoint balls $B_{x_{k,j}}(s_{k,j})$ with $x_{k,j} \in Q_j(\theta, \rho)$ such that

$$\bigcup B_{x_{k,j}}(5s_{k,j}) \supseteq Q_j(\theta, \rho).$$
and
\[
(5.28) \quad \frac{1}{\text{vol}(B_{x_{k,j}}(s_{k,j}))} \int_{B_{x_{k,j}}(s_{k,j})} |Rm| \geq \delta s_{k,j}^{-2}.
\]
Applying the Hölder inequality to (5.28) together with (5.2), we get
\[
\sum s_{k,j}^{n-2p} \leq \frac{c(n,v,C)}{\delta p}.
\]
By definition, for any set \(A\), we see that
\[
\mathcal{H}^{n-2p}_{\theta} (A) = \omega_{n-2p} \inf \left\{ \sum_{k} s_{k,i}^{n-2p} : \bigcup B_{x_{i}}(s_{i}) \supseteq A \text{ with } s_{i} \leq \theta \right\}
\]
and
\[
\mathcal{H}^{n-2p} (A) = \lim_{\theta \to 0} \mathcal{H}^{n-2p}_{\theta} (A),
\]
where \(\omega_{n-2p}\) is the volume of the unit ball in \(\mathbb{R}^{n-2p}\). Then by (5.27), for the set \(Q_j(\theta, \rho)\), we have
\[
\mathcal{H}^{n-2p}_{5\theta} (Q_j(\theta, \rho)) \leq \frac{c(n,v,C)}{\delta p}.
\]
Since \(s_{k,j} \geq \rho > 0\), it follows that the balls \(B_{x_{k,j}}(s_{k,j})\) are uniformly finite. Thus by taking the limit of \(B_{x_{k,j}}(s_{k,j})\), we see that there are balls \(B_{x_k}(s_k) \subset Y\) such that
\[
\bigcup B_{x_k}(5s_k) \supseteq Q(\theta, \rho)
\]
and
\[
\mathcal{H}^{n-2p}_{6\theta} (Q(\theta, \rho)) \leq \frac{c(n,v,C)}{\delta p}.
\]
Note that \(Q(\theta) = \bigcup_{\rho \leq \theta} Q(\theta, \rho)\). Hence
\[
\mathcal{H}^{n-2p}_{\theta} (Q(\theta)) \leq \frac{c(n,v,C)}{\delta p}.
\]
Consequently,
\[
\mathcal{H}^{n-2p}_{6\theta} (Q) \leq \mathcal{H}^{n-2p}_{6\theta} (Q(\theta)) \leq \frac{c(n,v,C)}{\delta p},
\]
and so
\[
\mathcal{H}^{n-2p}(Q) \leq C'.
\]
Estimates (5.3) and (5.4) follow from the above immediately since \(\mathcal{H}^{n-2p}(\delta_{n-2}) = 0\) for any \(p < 2\).

6. Structure of singular set II: Case of Kähler metrics

In this section, we will study the limit space for a sequence of Kähler metrics arising from solutions of certain complex Monge–Ampère equations for the existence problem of Kähler–Ricci soliton on a Fano manifold via the continuity method [25, 26]. We assume that \((M, g)\) is a compact Kähler manifold with positive first Chern class \(c_1(M) > 0\) (namely, \(M\) is Fano), and \(\omega_g\) is the Kähler form of \(g\) in \(2\pi c_1(M)\). Then there exists a Ricci potential \(h\) of the metric \(g\) such that
\[
\text{Ric}(g) - \omega_g = \sqrt{-1} \partial \bar{\partial} h, \quad \int_{M} e^{h} \omega_{g}^{n} = \int_{M} \omega^{n} = V.
\]
In [25], Tian and Zhu considered a family of complex Monge–Ampère equations for Kähler potentials \( \phi \) on \( M \),

\[
\det(g_{i\overline{j}} + \phi_{i\overline{j}}) = \det(g_{i\overline{j}}) e^{h - \theta_X - X(\phi) - t\phi},
\]
where \( t \in [0, 1] \) is a parameter and \( \theta_X \) is a real-valued potential of a reductive holomorphic vector field on \( M \) which is defined

\[
\tilde{\partial} \theta_X = i_X \omega_g, \quad \int_M e^{\theta_X} \omega_g^n = V,
\]
according to the choice of \( g \) with \( K_X \)-invariant. Equation (6.1) is equal to

\[
\text{Ric}(\omega_\phi) - L_X \omega_\phi = t \omega_\phi + (1 - t) \omega_g.
\]
Thus \( \omega_\phi \) will define a Kähler–Ricci soliton if \( \phi \) is a solution of (6.1) at \( t = 1 \). It was proved that the set \( I \) of \( t \) for which (6.1) is solvable is open [25]. In other words, there exists \( T \geq 1 \) such that \( I = [0, T) \). Equation (6.2) implies

\[
\text{Ric}(\omega_\phi) + \sqrt{-1} \tilde{\partial} \tilde{\partial}(-\theta_X(\phi)) \geq t \omega_\phi,
\]
where

\[
\theta_X(\phi) = \theta_X + X(\phi)
\]
is a potential of \( X \) associated to \( \omega_\phi \), which is uniformly bounded [31].

**Lemma 6.1.** Both \( |\tilde{\partial}((\theta_X + X(\phi)))| = |X|_{\omega_\phi} \) and \( \Delta_\tilde{\partial}(\theta_X(\phi)) \) are uniformly bound by \( C(M, \omega, X) \), where \( \Delta_\tilde{\partial} = \frac{1}{2} \Delta \) is a \( \tilde{\partial} \)-Laplace operator associated to \( \omega_\phi \).

**Proof.** We will use the maximum principle to prove the lemma. First we recall that \( \theta_X(\phi) \) satisfies the identity (see [13])

\[
\Delta_\tilde{\partial}[\theta_X(\phi)] + \theta_X(\phi) + X(h) = 0,
\]
where \( h \) is a Ricci potential of Kähler form \( \omega_\phi \) at \( t \). Note that

\[
h = \theta_X(\phi) + (t - 1)\phi
\]
byme (6.2). Thus \( \theta_X(\phi) \) satisfies

\[
\Delta_\tilde{\partial}[\theta_X(\phi)] + |\overline{\partial} \theta_X(\phi)|^2 + \theta_X(\phi) = (1 - t)X(\phi).
\]
By the Bochner formula, one sees

\[
\Delta_\tilde{\partial}(|\overline{\partial} \theta_X(\phi)|^2) = |\nabla^2 \theta_X(\phi)|^2 + 2 \text{Re}((\overline{\partial} \theta_X(\phi), \overline{\partial} \Delta_\tilde{\partial} \theta_X(\phi)) + \text{Ric}(\overline{\partial} \theta_X(\phi), \overline{\partial} \theta_X(\phi)).
\]
It follows that

\[
(\Delta_\tilde{\partial} + X)(|\overline{\partial} \theta_X(\phi)|^2) = |\nabla^2 \theta_X(\phi)|^2 + 2 \text{Re}((\overline{\partial} \theta_X(\phi), \overline{\partial} (\Delta_\tilde{\partial} \theta_X(\phi) + |\overline{\partial} \theta_X(\phi)|^2)))
\]

\[
+ (\text{Ric} - \nabla^2 \theta_X(\phi))(|\overline{\partial} \theta_X(\phi), \overline{\partial} \theta_X(\phi)).
\]
Thus, by (6.3), we get

\[
(\Delta_\tilde{\partial} + X)(|\overline{\partial} \theta_X(\phi)|^2) = |\nabla^2 \theta_X(\phi)|^2 - t|\overline{\partial} \theta_X(\phi)|^2 - (1 - t)|X|^2_g.
\]
Note that
\[ |\nabla \nabla \theta_X(\phi)|^2 \geq \frac{(\Delta \theta_X(\phi))^2}{n} \geq \frac{(\bar{\partial} \theta_X(\phi))^2 - C_1)^2}{n}, \]
where \( C_1 = \max_M \{|\theta_X(\phi) - (1-t)X(\phi)|\}. \) Applying the maximum principle to \( |\bar{\partial} \theta_X(\phi)\|^2 \) in (6.4), we derive at a maximal point of \( |\bar{\partial} \theta_X(\phi)|^2 \),
\[ 0 \geq \frac{1}{n}((\bar{\partial} \theta_X(\phi))^2 - C_1)^2 - t|\bar{\partial} \theta_X(\phi)|^2 - C_2. \]
Therefore, the gradient estimate of \( \theta_X(\phi) \) follows from the above inequality immediately. By (6.3), we also get the \( \bar{\partial} \)-Laplace estimate of \( \theta_X(\phi) \).

By Lemma 6.1 and Theorem 5.1, we prove the following:

**Theorem 6.2.** For any sequence of Kähler metrics \( g_{t_i} \) associated to solutions \( \phi_{t_i} \) of equation (6.1) at \( t = t_i \in I \), there exists a subsequence which converges to a limit metric space \( Y \) in the Gromov–Hausdorff topology. Moreover, \( \delta(Y) = \delta_{2n-2} \). In particular, the complex codimension of singularities of \( Y \) is at least 1.

**Proof.** It suffices to verify that
\[ \text{vol}_{g_t}(B_p(1)) \geq v > 0 \quad \text{for all} \quad p \in M. \]
But this is just a consequence of an application of Volume Comparison Theorem 1.2 since the diameter of \( g_t \) is uniformly bounded by a result of Mabuchi [21].

In a special case \( t_i \to 1 \) when \( I = [0, 1) \) in Theorem 6.2, we can strengthen Theorem 6.2 as follows.

**Theorem 6.3.** Let \( g_{t_i} \) be a sequence of Kähler metrics in Theorem 6.2 with \( t_i \to 1 \). Then \( \delta(Y) = \delta_{2n-4} \). In particular, the complex codimension of singularities of \( Y \) is at least 2.

The interval \( I = [0, 1) \) can be guaranteed when the modified Mabuchi \( K \)-energy is bounded below and \( X \) is an extremal holomorphic vector field which determined by the modified Futaki invariant [26]. This can be proved following an argument by Futaki for the study of an almost Kähler–Einstein metric under the assumption that the Mabuchi \( K \)-energy is bounded below on a Fano manifold [13]. Thus as a corollary of Theorem 6.3, we have the following:

**Corollary 6.4.** Suppose that the modified \( K \)-energy is bounded below on a Fano manifold. There exists a subsequence of weak almost Kähler–Ricci solitons on \( M \) which converges to a limit metric space \( Y \) in the Gromov–Hausdorff topology. Moreover, the complex codimension of singularities of \( Y \) is at least 2.

**Remark 6.5.** In case that \( X = 0 \), the modified Mabuchi \( K \)-energy is just the Mabuchi \( K \)-energy. In this case, the \( K \)-energy is bounded from below if and only if the Fano manifold is \( K \)-semistable by a recent work of Li [18].

It may be useful to introduce a more general sequence of Kähler metrics than the one in Theorem 6.3 inspired by a recent work of Tian and Wang [23].
Definition 6.6. Let \( (M_i, J_i, g_i) \) be a sequence of Kähler metrics. We call \( (M_i, J_i, g_i) \) weak almost Kähler–Ricci solitons if there are uniform constants \( \Lambda \) and \( A \) such that

1. \( \text{Ric}(g_i) + \nabla \nabla f_i \geq -\Lambda^2 g_i, \nabla \nabla f_i = 0, \)
2. \( \| \bar{\partial} f_i \|_{g_i} \leq A, \)
3. \( \lim_{i \to \infty} \| \text{Ric}(g_i) - g_i + \nabla \nabla f_i \|_{L^1(g_i)} = 0. \)

Here \( f_i \) are some smooth functions and they define reductive holomorphic vector fields on the Fano manifolds \( M_i. \)

Lemma 6.7. Let \( \{g_{t_i}\} \) be a sequence of Kähler metrics in Theorem 6.2 with \( t_i \to 1. \) Then \( \{g_{t_i}\} \) is a sequence of weak almost Kähler–Ricci solitons on \( M. \)

Proof. By Lemma 6.1, it suffice to check condition (iii) in Definition 6.6. In fact, we have

\[
\int_M |\text{Ric}(\omega_\phi) - \sqrt{-1} \bar{\partial} \partial \chi(\phi) - \omega_\phi| \\
\leq \int_M |\text{Ric}(\omega_\phi) - \sqrt{-1} \bar{\partial} \partial \chi(\phi) - t \omega_\phi| + n(1-t) \text{vol}(M) \\
= \int_M (\text{Ric}(\omega_\phi) - \sqrt{-1} \bar{\partial} \partial \chi(\phi) - t \omega_\phi) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} + n(1-t) \text{vol}(M) \\
= 2n(1-t) \text{vol}(M) \to 0,
\]

as desired. \( \square \)

We now begin to prove the following main result in this section.

Theorem 6.8. Let \( (M_i, g_i) \) be a sequence of weak almost Kähler–Ricci solitons. Suppose that there exists a point \( p_i \) at each \( M_i \) such that

\[ \text{vol}_{M_i}(B_{p_i}(1)) \geq v > 0. \]

Then there exists a subsequence of \( (M_i, g_i; p_i) \) which converges to a limit metric space \( Y \) in the pointed Gromov–Hausdorff topology. Moreover, \( \delta(Y) = \delta_{2n-4}. \) In particular, the complex codimension of singularities of \( Y \) is at least 2.

We need a lemma of \( \epsilon \)-regularity for the tangent cone to prove Theorem 6.8.

Lemma 6.9. For any positive \( \mu_0 \) and \( \epsilon, \) there exist three small numbers \( \delta = \delta(v, \epsilon, n), \eta = \eta(v, \epsilon, n), \tau = \tau(v, \epsilon, n) \) and a big number \( l = l(v, \epsilon, n) \) such that if a Kähler manifold \( (M^n, g) \) satisfies

1. \( \text{Ric}_M^f(g) > -(n-1)\tau^2 g, \nabla \nabla f = 0, \)
2. \( \text{vol}_g(B_p(1)) \geq \mu_0, \)
3. \( |\nabla f| < \tau, \)
4. \( \frac{1}{\text{vol}(B_p(2))} \int_{B_p(2)} |\text{Ric}(g) + \nabla \nabla f| \, dV_g < \delta, \)
5. \( d_{GH}(B_p(l), B_{(0,x)}(l)) < \eta. \)
where \( B_{(0,x)}(l) \) is an \( l \)-radius ball in the cone \( \mathbb{R}^{2n-2} \times C(X) \) centered at the vertex \((0,x)\) for some metric space \( X \), then
\[
\text{d}_{\text{GH}}(B_p(1), B(1)) < \epsilon.
\]

**Proof.** The proof of Lemma 6.9 is a modification of that of Proposition 5.5. Note that \( X \) is a circle of radius \( t \) in the present case. It suffices to show that \( t \) is close to \( \frac{1}{2} \) by Lemma 4.8. Let \( \Phi = (h_1, \ldots, h_{2n-2}) \) and \( \Gamma = (\Phi, \mathfrak{u}) \) be the two maps constructed in Proposition 5.5. By Proposition B.4 in Appendix B, we may also assume
\[
\int_{B_p(3)} |\nabla h_{n-1+i} - \mathcal{J} \nabla h_i|^2 < \Psi(t, \epsilon, \frac{1}{2}; v).
\]
We compute the differential characteristic \( \widehat{c}_{1,\nabla} \) of the tangent bundle \((TM, \nabla)\) restricted on \( \Gamma^{-1}(z, u) = \Phi^{-1}(z) \cap U_u \) with fixed \( z \) (cf. [4]), where \( \nabla \) is the Levi-Civita connection on \( TM \) and \((z, u)\) is a regular point of \( \Gamma \) such that both Lemma 5.7 and Lemma 5.8 hold. It is easy to see that by the coarea formula and condition (iv), the set
\[
D = \left\{ z : \Phi^{-1}(z) \cap U_u \text{ is a regular surface in } M \text{ and } \int_{\Phi^{-1}(z) \cap U_u} |\text{Ric}(g) + \nabla^2 f| < c \delta \right\}
\]
has positive volume in \( \mathbb{R}^{2n-2} \) for some constant \( c \) which depends only on \( n \). For each \( z \in D \), we have the estimate
\[
\left| \int_{\Phi^{-1}(z) \cap U_u} \text{Ric}(g) \right| \leq \int_{\Phi^{-1}(z) \cap U_u} |\text{Ric}(g) + \nabla^2 f| + \left| \int_{\Phi^{-1}(z) \cap U_u} \sqrt{-1} \partial \bar{\partial} f \right| \\
\leq c \delta + \int_{\Gamma^{-1}(z, u)} |\nabla f| \leq c \delta + \text{vol}(\Gamma^{-1}(z, u)) \tau.
\]
Since
\[
\int_{\Gamma^{-1}(z,u)} \widehat{c}_{1,\nabla} = \int_{\Phi^{-1}(z) \cap U_u} \text{Ric}(g) \text{ mod } \mathbb{Z},
\]
we get
\[
(6.7) \quad \int_{\Gamma^{-1}(z,u)} \widehat{c}_{1,\nabla} = \Psi \text{ mod } \mathbb{Z}.
\]
To compute the left term of equation (6.7), we will decompose the tangent bundle \((TM, \nabla)\) over \( \Gamma^{-1}(z, u) \) as follows. By our construction of the map \( \Gamma \), using the coarea formula, we may assume that
\[
(C1) \quad \int_{\Gamma^{-1}(z, u)} |(\nabla h_i, \nabla h_j) - \delta_{ij}| < \Psi,
\]
\[
(C2) \quad \int_{\Gamma^{-1}(z, u)} |\text{hess } h_i| < \Psi,
\]
\[
(C3) \quad \int_{\Gamma^{-1}(z, u)} |(\nabla \mathfrak{u}^2, \nabla h_j)| < \Psi,
\]
\[
(C4) \quad \int_{\Gamma^{-1}(z, u)} |\nabla (\nabla \mathfrak{u}^2, \nabla h_j)| < \Psi.
\]
Since $\Gamma^{-1}(z, u)$ is a one-dimensional manifold with bounded length, conditions (C1)–(C4) imply that $|\langle \nabla h_i, \nabla h_j \rangle - \delta_{ij}|$ and $|\nabla \mathbf{u}^2, \nabla h_j| |$ are both small on $\Gamma^{-1}(z, u)$, respectively. Moreover, applying the coarea formula to (6.6) together with condition (C2), we also get

$$|\nabla h_{n^{-1}+i} - J \nabla h_i| < \Psi.$$ 

Hence by using the Gram–Schmidt process, we obtain $2n - 1$ orthogonal sections of $TM$ over $\Gamma^{-1}(z, u)$, namely, $e_i, J(e_i)$ ($1 \leq i \leq n - 1$), $\mathbf{N}$, from the sections $\nabla h_i$ ($1 \leq i \leq n - 1$), $\nabla \mathbf{u}$. Define $\mathbb{E}$ to be the sub-bundle spanning by $e_i, J(e_i)$ and decompose $TM$ into

$$TM = \mathbb{E} \oplus \mathbb{E}^\perp,$$

where $\mathbb{E}^\perp$ is the orthogonal complement of $\mathbb{E}$. We introduce a Whitney sum connection $\nabla'$ on $TM$ over $\Gamma^{-1}(z, u)$ by combining two projection connections on $\mathbb{E}$ and $\mathbb{E}^\perp$, which are both induced by $\nabla$. Then by condition (C2), it is easy to show

$$\int_{\Gamma^{-1}(z, u)} |\nabla' - \nabla'| < \Psi,$$

(6.8)

where $\nabla - \nabla'$ is regarded as a 1-form on $\text{End}(TM)$. Also we can introduce another connection $\nabla''$ which is flat on $\mathbb{E}$. Namely, $\nabla''$ satisfies

$$\nabla''(e_i) = \nabla''(J(e_i)) = 0.$$ 

Similar to (6.8), we have

$$\int_{\Gamma^{-1}(z, u)} |\nabla'' - \nabla'| < \Psi,$$

(6.9)

Therefore, combining (6.8) and (6.9), we derive

$$|\langle c_1, \nabla'' - c_1, \nabla \rangle(\Gamma^{-1}(z, u))| < 1.$$ 

On the other hand, by the flatness of $\nabla''$ on $\mathbb{E}$ over $\Gamma^{-1}(z, u)$, the quantity $2\pi c_1, \nabla''(\Gamma^{-1}(z, u))$ is just equal to the holonomy of the connection around $\Gamma^{-1}(z, u)$ (measured by angle),

$$2\pi c_1, \nabla''(\Gamma^{-1}(z, u)) = \int_{\Gamma^{-1}(z, u)} \langle \nabla''_X N, JN \rangle,$$

where $X$ is the unit tangent vector of $\Gamma^{-1}(z, u)$. Thus by the choice of $\mathbf{N}$ together with (6.8), (6.9) and (5.20), we see that the angle is close to the length of $\Gamma^{-1}(z, u)$. By (6.7), it follows that $\frac{1}{2\pi} \text{vol}(\Gamma^{-1}(z, u))$ is close to zero modulo integers. Hence, the non-collapsing of $B_{(0, \infty)}(1)$ implies that $\text{vol}(\Gamma^{-1}(z, u))$ is close to $2\pi$. Consequently, we prove that $t$ is close to $2\pi$ by (5.14) in Lemma 5.7.

\textbf{Proof of Theorem 6.8.} By Volume Comparison Theorem 1.2, for any $r \leq 1$, we have

$$\text{vol}_{g_i}(\text{vol}(B_p(r)) \geq \lambda_0 r^n \quad \text{for all } p \in M_i,$$

where $\lambda_0$ depends only on the three constants $\Lambda, A, v$ in Definition 6.6. Thus by Gromov’s compactness theorem [14], there exists a subsequence of $(M_i, g_i; p_i)$ which converges to
a metric space \( Y_\infty \) in the pointed Gromov–Hausdorff topology. In the remaining, we show that \( \delta(Y_\infty) = \delta_{S_2 n-4} \). We will prove it by contradiction. On the contrary, for a ball \( B_\gamma(1) \subset Y \), by Proposition B.5 in Appendix B, there exist a point \( z \in S \cap B_\gamma(1) \) and a sequence \( \{r_i\} \) \((r_i \to 0)\) such that the sequence \((Y, \frac{d}{r_i^2}, z)\) converges a tangent cone \( T_z Y = \mathbb{R}^{2n-2} \times C(X) \).

This implies that there exists an \( \epsilon > 0 \) such that the unit metric ball \( B_{z_\infty}(1) \subset T_z Y \) centered at \( z_\infty \equiv z \) satisfies

\[
\text{d}_{GH}(B_{z_\infty}(1), B(1)) > 2\epsilon,
\]

and for any \( l \gg 1 \) and \( \epsilon \ll 1 \) one can choose sufficiently large numbers \( i \) and \( k \) such that

\[
\text{d}_{GH}(\hat{B}_{z_k}(1), B(1)) > \epsilon, \quad \text{d}_{GH}(\hat{B}_{z_k}(l), B(0,\gamma)(l)) < \eta,
\]

where \( z_k \in M_k \to z \in Y \) as \( k \to \infty \), and \( \hat{B}_{z_k}(1) \) and \( \hat{B}_{z_k}(l) \) are two balls with radius 1 and \( l \), respectively, in \( (M_k, \hat{g}_k/r_i^2) = (M_k, \hat{g}_k) \). On the other hand, by using Volume Comparison Theorem 1.2, for fixed \( i \), we can choose a sufficiently large \( k \) such that

\[
\frac{r_i^2}{\text{vol}(B_{z_k}(2r_i))} \int_{B_{z_k}(2r_i)} |\text{Ric}(\hat{g}_k) - \hat{g}_k + \nabla \nabla f_k| \, d\nu_{\hat{g}_k} < \frac{1}{2} \delta.
\]

Since

\[
\frac{r_i^2}{\text{vol}(B_{z_k}(2r_i))} \int_{B_{z_k}(2r_i)} |\hat{g}_k| \, d\nu_{\hat{g}_k} \leq c(n, C) r_i^2,
\]

we have

\[
\frac{1}{\text{vol}(\hat{B}_{z_k}(2))} \int_{\hat{B}_{z_k}(2)} |\text{Ric}(\hat{\hat{g}}_k) + \nabla \nabla f_k| \, d\nu_{\hat{\hat{g}}_k} < \delta.
\]

Hence, for large \( k \), \( (M_k, \hat{g}_k) \) satisfies conditions (i)–(v) in the statement of Lemma 6.9, and consequently, we get

\[
\text{d}_{GH}(\hat{B}_{z_k}(1), B(1)) < \epsilon,
\]

which is a contradiction to (6.10). The theorem is proved.

Theorem 6.3 follows from Theorem 6.8 with the help of Lemma 6.7 and relation (6.5).

A. Appendix 1

This appendix is a discussion about how to use the technique of conformal transformation as in [24] to prove Theorem 6.2 and Theorem 6.3 in Section 6.

First, Theorem 6.2 can be proved by using the conformal technique. In fact, by the formula of Ricci curvature for conformal metric \( e^{2u}g \),

\[
\text{Ric}(e^{2u}g) = \text{Ric}(g) - (n - 2)(\text{hess } u - du \otimes du) + (\Delta u + (n - 2)|\nabla u|^2)g,
\]

the condition \( \text{Ric}_{M}^f(g) \geq -C \) implies that the Ricci curvature \( \text{Ric}(e^{-\frac{2f}{n-2}}g) \) of the conformal metric \( e^{-\frac{2f}{n-2}}g \) is bounded below if both \( \nabla f \) and \( \Delta f \) are bounded. Thus by Lemma 6.1, we see that \( \text{Ric}(e^{\frac{2\partial_f}{n-2} g_1}) \) is uniformly bounded below. Hence, Theorem 6.2 follows from [6, Theorem 6.2] immediately.
Secondly, following [4, proof of Theorem 5.4], Lemma 6.9 with the additional condition
(vi) $|\Delta f| < \tau$
can be proved by using the conformal change of the bundle metric. We note that condition (vi)
can be guaranteed for the Kähler manifolds $(M, g_t)$ in Theorem 6.3 with blowing-up metrics.
Thus by (A.1), the Ricci curvature of blowing-up metric of $e^{-\frac{2f}{n-2}}g_t$ is almost positive.

For a Kähler manifold $(M, g, J)$, the $(1,0)$-type Hermitian connection $\nabla$ on the holomorphic
bundle $(TM, h)$ is the same as the Levi-Civita connection, where $h$ is the Hermitian metric corresponding to $g$. Then $c_1, \nabla$ of $(TM, h)$ is the same as the Ricci form of $g$. If we choose a Hermitian metric $e^{\psi}g$ for a smooth function $\psi$, then

$$\tilde{\nabla} = \nabla + \partial \psi$$

is the corresponding $(1, 0)$-type Hermitian connection. It follows that

$$F^{\tilde{\nabla}} = F^{\nabla} + d\partial \psi$$

and

(A.2) \[ \sqrt{-1} \text{tr}(F^{\tilde{\nabla}}) = \sqrt{-1} \text{tr}(F^{\nabla}) - n \sqrt{-1} \partial \bar{\partial} \psi, \]

where $F^{\nabla}$ ($F^{\tilde{\nabla}}$) denotes the curvature of the connection $\nabla$ ($\tilde{\nabla}$) on $TM$. Thus by putting $\psi = -\frac{2\pi}{n} f$ and using (A.2), we have

(A.3) \[ c^{\tilde{\nabla}}(\Gamma^{-1}(z, u)) = \int_{\Gamma^{-1}(z, u)} |\text{Ric}(\omega_g) + \sqrt{-1} \partial \bar{\partial} f| \mod \mathbb{Z}, \]

where the map $\Gamma$ is defined as in Section 5 and Section 6 for the conformal metric $\tilde{g} = e^{-\frac{2f}{n-2}}g$. Thus $c^{\tilde{\nabla}}(\Gamma^{-1}(z, u))$ is small modulo integers. Moreover, by [8, Theorem 3.7] (compared to Lemma 5.8 in Section 5), we have

(A.4) \[ \frac{1}{V^{\Gamma}(z, u)} \int_{\Gamma^{-1}(z, u)} |\Pi_{\Gamma^{-1}(z, u)} - u^{-1} \tilde{g}_{\Gamma^{-1}(z, u)} \otimes \nabla u|^2 < \Psi. \]

On the other hand, since the Ricci curvature of $\tilde{g}$ is almost positive, for the connection $\tilde{\nabla}$, we can follow the argument in [4, proof of Theorem 5.4] to show that the quantity $2\pi c^{\tilde{\nabla}}(\Gamma^{-1}(z, u))$ is close to a holonomy of another perturbation connection $\tilde{\nabla}''$ of $\tilde{\nabla}$ around $\Gamma^{-1}(z, u)$ (see also the argument in the proof of Lemma 6.9). The latter is close to

$$\int_{\Gamma^{-1}(z, u)} \Pi_{\Gamma^{-1}(z, u)}.$$

Thus combining (A.3) and (A.4), we get

$$\left| c^{\tilde{\nabla}}(\Gamma^{-1}(z, u)) - \frac{\text{vol}(\Gamma^{-1}(z, u))}{2\pi} \right| < \Psi.$$

It follows that the diameter of the section $X$ in the two-dimensional cone $C(X)$ with rescaled cone metric is close to $2\pi$. Thus the Gromov–Hausdorff distance between $B_{p}(1)$ and $B_{(0, x)}(1)$ both with rescaled metrics is close to zero. By [11, Theorem 9.69], we prove Lemma 6.9 with the additional condition (vi). Theorem 6.3 follows from applying Lemma 6.9 to the sequence $\{(M, g_t)\} (t \to 1)$ with blowing-up metrics; for details, see the proof of Theorem 6.8 in the end of Section 6.
B. Appendix 2

In this appendix, we prove (6.6) in Section 6. We need several lemmas. First, as an application of Lemma 2.5, we have the following:

Lemma B.1. Under the conditions of Lemma 2.4, for a vector field $X$ on $A_p(a, b)$ which satisfies

\[ |X|_{C^0(A_p(a, b))} \leq D, \quad \frac{1}{\text{vol}^f(A_p(a, b))} \int_{A_p(a, b)} |\nabla X|^2 \, dv^f < \delta, \]

there exists an $f$-harmonic function $\theta$ defined in $A_p(a_2, b_2)$ such that

\[ \frac{1}{\text{vol}^f(A_p(a_2, b_2))} \int_{A_p(a_2, b_2)} |\nabla \theta - X|^2 \, dv^f < \Psi(\epsilon, \omega, \delta; A, a_1, b_1, a_2, a, b) \]

and

\[ \frac{1}{\text{vol}^f(A_p(a_3, b_3))} \int_{A_p(a_3, b_3)} |\text{hess} \theta|^2 \, dv^f < \Psi(\epsilon, \omega, \delta; A, a_1, b_1, a_2, b_2, a_3, b_3, a, b), \]

where $A_p(a_3, b_3)$ is an even smaller annulus in $A_p(a_2, b_2)$.

Proof. Let $h$ be the $f$-harmonic function constructed in (2.5) and let $\theta_1 = \langle X, \nabla h \rangle$. Then $\nabla \theta_1 = \langle \nabla X, \nabla h \rangle + \langle X, \text{hess} h \rangle$. It follows that

\[ \int_{A_p(a_2, b_2)} |\nabla \theta_1 - X|^2 \, dv^f \leq 2 \int_{A_p(a_2, b_2)} (|\nabla X|^2 + \langle X, \text{hess} h - g \rangle^2) \, dv^f. \]

Thus by (B.1) and Lemma 2.5, we get

\[ \frac{1}{\text{vol}^f(A_p(a_2, b_2))} \int_{A_p(a_2, b_2)} |\nabla \theta_1 - X|^2 \, dv^f < \Psi. \]

Let $\theta$ be a solution of the equation

\[ \Delta^f \theta = 0 \quad \text{in } A_p(a_2, b_2) \]

with $\theta = \theta_1$ on $\partial A_p(a_2, b_2)$. Then

\[ \int_{A_p(a_2, b_2)} (|\nabla \theta - \nabla \theta_1, X| + (\theta - \theta_1) \text{div} X) \, dv^f = \int_{A_p(a_2, b_2)} \text{div}((\theta - \theta_1)X) \, dv^f = \int_{A_p(a_2, b_2)} (\theta - \theta_1) \langle \nabla f, X \rangle \, dv^f. \]

It follows that

\[ \int_{A_p(a_2, b_2)} \langle \nabla \theta_1 - \nabla \theta, X \rangle \, dv^f < \Psi. \]

On the other hand, since

\[ \int_{A_p(a_2, b_2)} \langle \nabla \theta_1, \nabla \theta \rangle \, dv^f = \int_{A_p(a_2, b_2)} (\theta - \theta_1) \Delta^f \theta \, dv^f = 0, \]
we have
\[ \int_{A_p(a_2,b_2)} |\nabla \theta|^2 d\nu^f = \int_{A_p(a_2,b_2)} (\nabla \theta, \nabla \theta) d\nu^f. \]
By the Hölder inequality, we get
\[ \int_{A_p(a_2,b_2)} |\nabla \theta|^2 d\nu^f \leq \int_{A_p(a_2,b_2)} |\nabla \theta_1|^2 d\nu^f < C. \]
Hence,
\[ \int_{A_p(a_2,b_2)} (\nabla \theta - X)^2 d\nu^f \]
\[ = \int_{A_p(a_2,b_2)} (|\nabla \theta| + |X|^2 - 2(\nabla \theta, X)) d\nu^f \]
\[ = \int_{A_p(a_2,b_2)} ((\nabla \theta, \nabla \theta_1) + |X|^2 - 2(\nabla \theta, X)) d\nu^f \]
\[ = \int_{A_p(a_2,b_2)} ((\nabla \theta_1 - X, \nabla \theta) + (X, X - \nabla \theta_1) + (X, \nabla \theta_1 - \nabla \theta)) d\nu^f. \]
Therefore, combining (B.1) and (B.4), we derive (B.2) immediately.

To get (B.3), we choose a cut-off function \( \phi \) which is supported in \( A_p(a_2, b_2) \) with bounded gradient and \( f \)-Laplace as in Lemma 1.5. Then by the Bochner identity, we have
\[ \int_{A_p(a_2,b_2)} \frac{1}{2} \phi \Delta f |\nabla \theta|^2 d\nu^f = \int_{A_p(a_2,b_2)} \phi(|\text{hess } \theta|^2 + \text{Ric}(\nabla \theta, \nabla \theta)) d\nu^f. \]
Since
\[ \int_{A_p(a_2,b_2)} \frac{1}{2} \phi \Delta f |X|^2 d\nu^f = -\int_{A_p(a_2,b_2)} \langle \nabla \phi, (X, \nabla X) \rangle d\nu^f, \]
we obtain
\[ \int_{A_p(a_2,b_2)} \phi(|\text{hess } \theta|^2 d\nu^f < \int_{A_p(a_2,b_2)} \frac{1}{2} \phi \Delta f (|\nabla \theta|^2 - |X|^2) d\nu^f \]
\[ + \Psi(\epsilon, \omega, \delta; A, a_1, b_1, a_2, b_2, a_3, b_3, a, b). \]
Therefore, using integration by parts, we derive (B.3) from (B.2).

Next, we generalize Proposition 3.6 to the case without the assumption of the existence of an almost line.

**Lemma B.2.** Let \((M, g)\) be a Riemannian manifold which satisfies (3.3). Let \( h^+ \) be an \( f \)-harmonic function which satisfies
\[
|\nabla h^+| \leq c(n, \Lambda, A),
\]
\[
\left| \frac{1}{\text{vol}(B_p(1))} \int_{B_p(1)} |\nabla h^+|^2 - 1 \right| d\nu^f < \delta,
\]
\[
\frac{1}{\text{vol}(B_p(1))} \int_{B_p(1)} |\text{hess } h^+|^2 d\nu^f < \delta.
\]
Then there exists a \( \Psi(\delta; \Lambda, n) \)-Gromov–Hausdorff approximation from the ball \( B_p(\frac{1}{8}) \) to the ball \( B_{(0, \infty)}(\frac{1}{8}) \subset \mathbb{R} \times X. \)
The proof of Lemma B.2 depends on the following fundamental lemma which is in fact a consequence of [2, Theorem 16.32 and Lemma 8.17].

**Lemma B.3.** Under condition (3.3), for an \( f \)-harmonic function \( h^+ \) which satisfies (B.5)–(B.7) in \( B_p(1) \), there exists a Lipschitz function \( \rho \) in \( B_p(\frac{1}{2}) \) such that \( |h^+ - \rho| < \Psi \) and

\[
|\rho(z) - t| - d(z, \rho^{-1}(t))| < \Psi.
\]

**Proof.** First, we note that the following Poincaré inequality holds for any \( C^1 \)-function \( h \),

\[
\frac{1}{\text{vol}^f(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} |h(x) - a|^2 \, d\nu \leq c(n, \Lambda, A) \frac{1}{\text{vol}^f(B_p(1))} \int_{B_p(1)} |\nabla h|^2 \, d\nu,
\]

where

\[
a = \frac{1}{\text{vol}^f(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} h \, d\nu^f.
\]

This is in fact a consequence of Lemma 3.4 by applying the function \( e \) to \( |\nabla h|^2 \), because

\[
\frac{1}{\text{vol}^f(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} |h(x) - a|^2 \, d\nu
\]

\[
= \frac{1}{\text{vol}^f(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} d\nu \left( \frac{1}{\text{vol}^f(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} (h(x) - h(y)) \, d\nu^f \right)^2
\]

\[
\leq \frac{1}{\text{vol}^f(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} \frac{1}{\text{vol}^f(B_p(1))} \int_{B_p(\frac{1}{2})} (h(x) - h(y))^2 \, d\nu^f \, d\nu^f
\]

\[
\leq \frac{1}{\text{vol}^f(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} \frac{1}{\text{vol}^f(B_p(1))} \int_{B_p(\frac{1}{2})} \int_{0}^{d(x,y)} |\nabla h((y(s)))|^2 \, d\nu^f \, d\nu^f
\]

\[
\leq c(n, \Lambda, A) \frac{1}{\text{vol}^f(B_p(1))} \int_{B_p(1)} |\nabla h|^2 \, d\nu.
\]

Thus by taking \( h = |\nabla h|^2 \), we get from (B.5)–(B.7) that

\[
(B.9) \quad \frac{1}{\text{vol}^f(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} |\nabla h|^2 - 1 \, d\nu^f < \Psi.
\]

Next we apply [2, Theorem 16.32] to \( h^+ \) with conditions (B.5), (B.6) and (B.9). It is sufficient to check a doubling condition for the measure \( d\nu^f \) and an \( (\epsilon, \delta) \)-inequality. The \( (\epsilon, \delta) \)-inequality says that, for any \( \epsilon, \delta > 0 \) and two points \( x, y \in M \) with \( d(x, y) = r \), there exist \( C_{\epsilon, \delta} \) and another two points \( x', y' \) with \( d(x', x) \leq \delta r \) and \( d(y', y) \leq \delta r \), respectively, such that

\[
F_{\phi, \epsilon}(z', z_1) \leq \frac{C_{\epsilon, \delta} r}{\text{vol}^f(B_{z_1}((1 + \delta)(1 + 2\epsilon)r))} \int_{B_{z_1}((1 + \delta)(1 + 2\epsilon)r)} \phi \, d\nu^f,
\]

where

\[
F_{\phi, \epsilon}(x, y) = \inf \int_{0}^{l} \phi(c(s)) \, ds \quad \text{for all } \phi(\geq 0) \in C^0(M),
\]
and the infimum takes among all curves from $x$ to $y$ with length $l \leq (1 + \epsilon)d(x, y)$. The doubling condition follows from Volume Comparison Theorem 1.2, and the $(\epsilon, \delta)$-inequality follows from Volume Comparison Theorem 1.2 and the segment inequality in Lemma 3.3. Thus we can construct a Lipschitz function $\rho$ from $h^+$ such that

$$|h^+ - \rho| \leq \Psi.$$

Moreover, by [2, Lemma 8.17], we get (B.8).

Proof of Lemma B.2. As in the proof of Proposition 3.6, we define $X = (h^+)^{-1}(0)$ and the map $u$ by

$$u(q) = (h^+(q), x_q),$$

where $x_q$ is the nearest point in $X$ to $q$. To show that $u$ is a Gromov–Hausdorff approximation, we shall use Lemma 3.2. In fact, by (B.8) in Lemma B.3, we see

$$\|h^+(z) - t - d(z, (h^+)^{-1}(t))\| < \Psi.$$  \hspace{1cm} (B.10)

Then instead of (3.1) by (B.10), Lemma 3.2 is still true since (B.7) holds [3]. Hence the proof of Proposition 3.6 works for Lemma B.2.

Now we begin to prove (6.6). Let $(M, g)$ be a Kähler manifold which satisfies (5.5). Let $B_p(l) \subset M$ and $B_{(0 \times X)}(l) \subset \mathbb{R}^{2n-2} \times X$ be two $l$-radius distance balls as in Section 6. Then the following proposition holds:

**Proposition B.4.** Suppose that

$$d_{GH}(B_p(l), B_{(0 \times X)}(l)) < \eta.$$

Then either $B_p(\frac{1}{2})$ is close to a Euclidean ball in the Gromov–Hausdorff topology or for a suitable choice of the orthogonal coordinates in $\mathbb{R}^{2n-2}$, the map $\Phi = (h_1, \ldots, h_{2n-2})$ constructed in Section 5 satisfies

$$\frac{1}{\text{vol}^f B_p(1)} \int_{B_p(1)} |\nabla h_{n-1+i} - J \nabla h_i|^2 d\nu \leq \Psi(\tau, \eta, \frac{1}{2}; \nu).$$ \hspace{1cm} (B.11)

Proof. Roughly speaking, if the space spanned by $\nabla h_i$ is not almost $J$-invariant, we can find a vector field nearly perpendicular to these $\nabla h_i$, and it satisfies condition (B.1) in Lemma B.1. Then by Lemma B.2, $B_p(1)$ will be almost split off along a new line. This implies that $B_p(\frac{1}{2})$ is close to a Euclidean ball.

Let $V$ be a $(4n - 4)$-dimensional line space spanned by $\nabla h_i, J \nabla h_i$ with the $L^2$-inner product

$$(b_i, b_j)_{L^2} = \int_{B_p(1)} \langle b_i, b_j \rangle d\nu.$$  \hspace{1cm} \text{Then } J \text{ induces an complex structure on } V \text{ such that the inner product is } J \text{-invariant. We introduce a distance in the Grassmannian } G(2n, k) \text{ as follows:}$$

$$d(\Lambda_1, \Lambda_2)^2 = \sum_j \|\text{pr}_{\Lambda_2}^+ (e_j)\|_{L^2}^2$$
for any two $k$-dimensional subspaces $\Lambda_1, \Lambda_2$ in $\mathbb{R}^{2n}$, where the vectors $e_i$ form a unit orthogonal basis of $\Lambda_1$ and $\text{pr}^\perp_{\Lambda_2}$ is the complement of the orthogonal projection to $\Lambda_2$. First we suppose that
\[ d(W, JW)^2 < \Psi, \]
where
\[ W = \text{span}\{\nabla h_i : i = 1, 2, \ldots, 2n - 2\}. \]
Then by the Gram–Schmidt process, one can find a unit orthogonal basis $\{w_i\}$ of $W$ such that
\[ \|Jw_i - w_{n-1+i}\|_{L^2} < \Psi. \]
This is equivalent to the existence of a matrix $a_{ij} \in \text{GL}(2n - 2, \mathbb{R})$ which is nearly orthogonal such that
\[ w_i = \sum_j a_{ij} \nabla h_j. \]
Thus by changing the orthogonal basis in $\mathbb{R}^{2n-2}$, (B.11) will be true.

Secondly, we suppose that
\[ d(W, JW) > \delta_0. \]
This implies that there exists some $j$ such that
\[ \|\text{pr}^\perp_W (J\nabla h_i)\|_{L^2} = \|J\nabla h_i - \text{pr}^\perp_W (J\nabla h_i)\|_{L^2} > \frac{\delta_0}{2n}. \]
Let
\[ X = \frac{\text{pr}^\perp_W (J\nabla h_i)}{\|\text{pr}^\perp_W (J\nabla h_i)\|_{L^2}}. \]
Then $\text{pr}^\perp_W (J\nabla h_i)$ is perpendicular to $W$ with
\[ \|\text{pr}^\perp_W (J\nabla h_i)\|_{L^2} = 1 \]
and it satisfies condition (B.1) in Lemma B.1. Thus we see that there exists an $f$-harmonic function $\theta$ which satisfies conditions (B.5), (B.6) and (B.7) in Lemma B.2. As a consequence, $B_p(\frac{1}{8})$ will almost split off along a new line associated to the coordinate function $\theta$. Since $X \in W^\perp$, it follows that $B_p(\frac{1}{8})$ in fact splits off $\mathbb{R}^{2n-1}$ almost. But the latter implies that $B_p(\frac{1}{8})$ is close to a Euclidean ball in the Gromov–Hausdorff topology by using a topological argument as in [6, Theorem 6.2] or by Proposition B.5 below for Kähler manifolds.

By using the similar argument in Proposition B.4, we prove the following:

**Proposition B.5.** Let $Y$ be a limit space of a sequence of Kähler manifolds in Theorem 5.1. Then
\[ S(Y) = S_{2k+1} = S_{2k}. \]

**Proof.** It is sufficient to show that if a tangent cone $T_y Y$ at a point $y \in Y$ can split off $\mathbb{R}^{2k+1}$, then $T_y Y$ can split off $\mathbb{R}^{2k+2}$. Let $h_i, i = 1, \ldots, 2k + 1$, be $f$-harmonic functions which approximate $2k + 1$ distance functions with different directions as constructed in
Section 2 and Section 3. Then as in the proof of Proposition B.4, we consider a linear space \( V = \text{span}\{\nabla h_i, \mathbf{J}\nabla h_i\} \) with \( L^2 \)-inner product. Since the dimension of \( W = \text{span}\{\nabla h_i\} \) is odd, we have \( d(W, \mathbf{J}W) \geq 1 \). Thus \( T_y Y \) will split off a new line. The proposition is proved. \( \square \)

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