Fully non-linear elliptic equations on compact almost Hermitian manifolds

Jianchun Chu · Liding Huang · Jiaogen Zhang

Received: 11 August 2022 / Accepted: 30 January 2023 / Published online: 24 February 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
In this paper, we establish a priori estimates for solutions of a general class of fully non-linear equations on compact almost Hermitian manifolds. As an application, we solve the complex Hessian equation and the Monge–Ampère equation for \((n-1)\)-plurisubharmonic functions in the almost Hermitian setting.

Mathematics Subject Classification 58J05 · 32Q60 · 35J60

1 Introduction
Let \((M, \chi, J)\) be a compact almost Hermitian manifold of real dimension \(2n\). Suppose that \(\omega\) is a real \((1, 1)\)-form on \((M, J)\). For \(u \in C^2(M)\), we write
\[
\omega_u := \omega + \sqrt{-1}d\bar{\partial}u := \omega + \frac{1}{2}(dJdu)^{(1,1)}
\]
and let \( \mu(u) = (\mu_1(u), \ldots, \mu_n(u)) \) be the eigenvalues of \( \omega_u \) with respect to \( \chi \). For notational convenience, we sometimes denote \( \mu_i(u) \) by \( \mu_i \) when no confusion will arise. We consider the following PDE:

\[
F(\omega_u) := f(\mu_1, \ldots, \mu_n) = h,
\]

where \( h \in C^\infty(M) \) and \( f \) is a smooth symmetric function.

The real version of the equation (1.1) has been studied extensively. In the pioneering work [4], Caffarelli–Nirenberg–Spruck considered the Dirichlet problem for domains of \( \mathbb{R}^n \). In [15], Guan studied such equations on Riemannian manifolds, introduced a kind of subsolution and derived \( C^2 \) estimate if subsolution exists. Later, Székelyhidi [28] focused on the complex setting, proposed another kind of subsolution (i.e. \( C^\alpha \)-subsolution) and established \( C^2,\alpha \) estimate when \( C^\alpha \)-subsolution exists. In [7], Chu-McCleerey derived the real Hessian estimate independent of \( \inf_M h \), which can be applied to the degenerate case of (1.1).

Following the setting of Székelyhidi [28], we assume that \( f \) is defined on an open symmetric cone \( \Gamma \subseteq \mathbb{R}^n \) with vertex at the origin containing the positive orthant \( \Gamma_n = \{ \mu \in \mathbb{R}^n : \mu_i > 0 \text{ for } i = 1, \ldots, n \} \). Furthermore, suppose that

(i) \( f_i = \frac{\partial f}{\partial \mu_i} > 0 \text{ for all } i \) and \( f \) is concave,
(ii) \( \sup_{\partial\Gamma} f < \inf_M h \),
(iii) For any \( \sigma < \sup_{\Gamma} f \) and \( \mu \in \Gamma \), we have \( \lim_{t \to \infty} f(t\mu) > \sigma \),

where

\[
\sup_{\partial\Gamma} f = \sup_{\lambda' \in \partial\Gamma} \lim_{\lambda \to \lambda'} f(\lambda).
\]

Many geometric equations are of the form (1.1), such as complex Monge–Ampère equation, complex Hessian equation, complex Hessian quotient equation and the Monge–Ampère equation for \((n-1)\)-plurisubharmonic functions.

For the complex Monge–Ampère equation, when \((M, \omega)\) is Kähler, Yau [34] proved the existence of solution and solved Calabi’s conjecture (see [5]). When \((M, \omega)\) is Hermitian, the complex Monge–Ampère equation has been solved under some assumptions on \( \omega \) (see [6, 16, 18, 30, 37]). The general Hermitian case was solved by Tosatti-Weinkove [31]. More generally, in the almost Hermitian setting, analogous results were obtained by Chu-Tosatti-Weinkove [8].

When \((M, \omega)\) is Kähler, using a priori estimates of Hou [20] and Hou–Ma–Wu [21], Dinew-Kołodziej [9] solved the complex Hessian equation. This result was generalized to general Hermitian case by Székelyhidi [28] and Zhang [35] independently.

For the complex Hessian quotient equation (i.e. complex \( \sigma_k/\sigma_l \) equation where \( 1 \leq l < k \leq n \)), when \((M, \omega)\) is Kähler, \( k = n, l = n - 1 \) and \( h \) is constant, Song–Weinkove [23] obtained the necessary and sufficient condition for the existence of solution. This condition is equivalent to the existence of \( \mathcal{C} \)-subsolution. This result was generalized by Fang–Lai–Ma [11] to \( k = n \) and general \( l \), and by Székelyhidi [28] to general \( k \) and \( l \). When \( \omega \) is Hermitian and \( h \) is not constant, analogous results was obtained by Sun [25–27] (see also [17, 22]). When \( k = n \) and \( 1 \leq l \leq n - 1 \), the third author [36] proved a priori estimates and gave a sufficient condition for the existence of solution in the almost Hermitian setting.

The Monge–Ampère equation for \((n-1)\)-plurisubharmonic functions was introduced and studied by Fu–Wang–Wu [12, 13], which is a kind of Monge–Ampe type equation as follows:

\[
\left( \eta + \frac{1}{n-1} \left( (\Delta^C u)\chi - \sqrt{-1}\partial\bar{\partial} u \right) \right)^n = e^h \chi^n,
\]
where $\eta$ is a Hermitian metric, $\Delta^{C}$ denotes the canonical Laplacian operator of $\chi$. When $\chi$ is a Kähler metric, the equation (1.2) was solved by Tosatti-Weinkove [32]. They later generalized this result to general Hermitian metric $\chi$ in [33].

Our main result is the following estimate:

**Theorem 1.1** Let $(M, \chi, J)$ be a compact almost Hermitian manifold of real dimension $2n$ and $u$ is a $C$-subsolution (see Definition 2.1) of (1.1). Suppose that $u$ is a smooth solution of (1.1). Then for any $\alpha \in (0, 1)$, we have the following estimate

$$
\|u\|_{C^{2,\alpha}(M, \chi)} \leq C,
$$

where $C$ is a constant depending only on $\alpha, u, h, \omega, f, \Gamma$ and $(M, \chi, J)$.

Using Theorem 1.1, we can solve the complex Hessian equation and the Monge–Ampère equation for $(n-1)$-plurisubharmonic equations on compact almost Hermitian manifolds. For the definitions of $k$-positivity and $\Gamma_k(M, \chi)$, we refer the reader to Sect. 6.

**Theorem 1.2** Let $(M, \chi, J)$ be a compact almost Hermitian manifold of real dimension $2n$ and $\omega$ be a smooth $k$-positive real $(1,1)$-form. For any integer $1 \leq k \leq n$, there exists a unique pair $(u, c) \in C^\infty(M) \times \mathbb{R}$ such that

$$
\begin{align*}
\omega^k \wedge \chi^{n-k} &= e^{h+c} \chi^n, \\
\omega_u &\in \Gamma_k(M, \chi), \\
\sup_M u &= 0.
\end{align*}
$$

**Theorem 1.3** Let $(M, \chi, J)$ be a compact almost Hermitian manifold of real dimension $2n$ and $\eta$ be an almost Hermitian metric. There exists a unique pair $(u, c) \in C^\infty(M) \times \mathbb{R}$ such that

$$
\begin{align*}
\left(\eta + \frac{1}{n-1}((\Delta^{C} u)\chi - \sqrt{-1} \partial \overline{\partial} u)^n\right) &= e^{h+c} \chi^n, \\
\eta + \frac{1}{n-1}((\Delta^{C} u)\chi - \sqrt{-1} \partial \overline{\partial} u) &> 0, \\
\sup_M u &= 0.
\end{align*}
$$

We now discuss the proof of Theorem 1.1. The zero order estimate can be proved by adapting the arguments of [28, Proposition 11] and [8, Proposition 3.1] which are based on the method of Błocki [2, 3]. For the second order estimate, we first show that the real Hessian can be controlled by the gradient quadratically as follows:

$$
\sup_M |\nabla^2 u|_{\chi} \leq C \sup_M |\partial u|^2_{\chi} + C.
$$

Then the second order estimate follows from the blowup argument and Liouville type theorem [28, Theorem 20].

To prove (1.5), a possible method is to follow the arguments of [8, Proposition 5.1] or [7, Theorem 4.1]. However, these previous arguments do not seem to work for (1.1) directly in the almost Hermitian setting. Precisely, on one hand, although [8] investigated the complex Monge–Ampère equation on compact almost Hermitian manifolds, the argument of [8, Proposition 5.1] depends heavily on the special structure of Monge–Ampère operator. It is hard to generalize it to (1.1) directly. On the other hand, the estimate (1.5) for the general equation (1.1) was considered in [7]. But the underlying manifold in [7] is Hermitian (stronger than almost Hermitian condition), and the real Hessian estimate [7, Theorem 4.1] is built on the a priori estimates (including complex Hessian estimate) of [28]. These make
the situation considered in [7] simpler than the almost Hermitian setting. Hence, to prove (1.5), some essential modifications of the arguments in [8, Proposition 5.1] and [7, Theorem 4.1] are needed.

The proof of (1.5) is the heart of this paper. We apply the maximum principle to a quantity involving the largest eigenvalue of \( \nabla^2 u \) as in [7, 8]. The main task is to control negative third order terms by positive third order terms. We will use the ideas of [8, Proposition 5.1] to deal with the terms arising from the non-integrability of the almost complex structure, and refine the techniques of [7, Theorem 4.1] to control negative terms. More precisely, we modify the definitions of index sets \( S \) and \( I \) in [7, Section 4.2], and introduce a new index set \( J \).

According to the index sets \( J \) and \( S \), we divide the proof of (1.5) into three cases. The third case \((J, S \neq \emptyset)\) is the most difficult case. We split the bad third order term \( B \) (see (4.16)) into three terms \( B_1, B_2, B_3 \) (see (4.33)). For the terms \( B_1 \) and \( B_2 \), since the constant \( C \) in (1.5) should be independent of \( \sup_M |\partial u|_\chi^2 \), we may not apply the argument of [7, Section 4.2.1] directly. Thanks to the definition of modified set \( I \) and new set \( J \), terms \( B_1 \) and \( B_2 \) can be controlled by some lower order terms (see Lemma 4.6). For the main negative term \( B_3 \), we first derive a lower bound of \( \omega_u \), which depends on \( \sup_M |\partial u|_\chi \) quadratically (see Lemma 4.8). Combining this lower bound and some delicate calculations, we control \( B_3 \) by positive terms \( G_1, G_2, G_3 \) (see (4.15), (4.16) and Lemma 4.7). Here \( G_2 \) comes from the concavity of equation (1.1), and \( G_1, G_3 \) are provided by \( \log \lambda_1, \xi(|\rho|^2) \) in the quantity \( Q \), respectively.

The paper is organized as follows. In Sect. 2, we will introduce some notations, recall definition and an important property of \( C \)-subsolution. The zero order estimate will be established in Sect. 3. We will derive key inequality (1.5) in Sect. 4, and complete the proof of Theorem 1.1 in Sect. 5. In Sect. 6, we will prove Theorem 1.2 and 1.3.

2 Preliminaries

2.1 Notations

Recall that \((M, \chi, J)\) is an almost Hermitian manifold of real dimension \(2n\). Using the almost complex structure \( J \), we can define \((p, q)\)-form and operators \( \partial, \bar{\partial} \) (see e.g. [8, p. 1954]). Write

\[
A^{1,1}(M) = \{ \alpha : \alpha \text{ is a smooth real (1,1)-forms on}(M, J) \}.
\]

For any \( u \in C^\infty(M) \), we see that (see e.g. [8, p. 1954])

\[
\sqrt{-1} \partial \bar{\partial} u = \frac{1}{2} (dJ du)^{(1,1)}
\]

is a real (1, 1)-form in \( A^{1,1}(M) \).

For any point \( x_0 \in M \), let \( \{e_i\}_{i=1}^n \) be a local unitary \((1, 0)\)-frame with respect to \( \chi \) near \( x_0 \). Denote its dual coframe by \( \{\theta^i\}_{i=1}^n \). Then we have

\[
\chi = \sqrt{-1} \delta_{ij} \theta^j \wedge \bar{\theta}^i.
\]

Suppose that

\[
\omega = \sqrt{-1} g_{ij} \theta^i \wedge \bar{\theta}^j, \quad \omega_u = \sqrt{-1} \tilde{g}_{ij} \theta^i \wedge \bar{\theta}^j.
\]
Then we have (see e.g. [19, (2.5)])
\[ \tilde{g}_{ij} = g_{ij} + \left( \partial \bar{\partial} u \right) (e_i, \bar{e}_j) = g_{ij} + e_i \bar{e}_j(u) - [e_i, \bar{e}_j]^{(0,1)}(u), \]
where \([e_i, \bar{e}_j]^{(0,1)}\) is the \((0,1)\)-part of the Lie bracket \([e_i, \bar{e}_j]\). Define
\[ F_{ij} = \frac{\partial F}{\partial \tilde{g}_{ij}}, \quad F_{ijkl} = \frac{\partial^2 F}{\partial \tilde{g}_{ij} \partial \tilde{g}_{kl}}. \]

After making a unitary transformation, we may assume that \(\tilde{g}_{ij}(x_0) = \delta_{ij} \tilde{g}_{ii}(x_0)\). We denote \(\tilde{g}_{ii}(x_0)\) by \(\mu_i\). It is useful to order \(\{\mu_i\}\) such that
\[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n. \tag{2.1} \]

At \(x_0\), we have the expressions of \(F_{ij}\) and \(F_{ijkl}\) (see e.g. [1, 14, 24])
\[ F_{ij} = \delta_{ij} f_i, \quad F_{ijkl} = f_{ik} \delta_{ij} \delta_{kl} + \frac{f_i - f_j}{\mu_i - \mu_j} (1 - \delta_{ij}) \delta_{il} \delta_{jk}, \tag{2.2} \]
where the quotient is interpreted as a limit if \(\mu_i = \mu_j\). Using (2.1), we obtain (see e.g. [10, 24])
\[ F_{11} \leq F_{22} \leq \cdots \leq F_{nn}. \tag{2.3} \]

On the other hand, the linearized operator of equation (1.1) is given by
\[ L := \sum_{i,j} F_{ij} (e_i \bar{e}_j - [e_i, \bar{e}_j]^{(0,1)}). \tag{2.4} \]

Note that \([e_i, \bar{e}_j]^{(0,1)}\) is a first order differential operator, and so \(L\) is a second order elliptic operator.

### 2.2 C-subsolution

**Definition 2.1** (Definition 1 of [28]) We say that a smooth function \(u : M \to \mathbb{R}\) is a \(C\)-subsolution of (1.1) if at each point \(x \in M\), the set
\[ \{ \mu \in \Gamma : f(\mu) = h(x) \text{ and } \mu - \mu(u) \in \Gamma_1 \} \]
is bounded, where \(\mu(u) = (\mu_1(u), \ldots, \mu_n(u))\) denote the eigenvalues of \(\omega + \sqrt{-1} \partial \bar{\partial} u\) with respect to \(\chi\).

For \(\sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f)\), we denote
\[ \Gamma^\sigma = \{ \mu \in \Gamma : f(\mu) > \sigma \}. \tag{2.5} \]

By Definition 2.1, for any \(C\)-subsolution \(u\), there are constants \(\delta, R > 0\) depending only on \(u, (M, \chi, J), f\) and \(\Gamma\) such that at each \(x \in M\) we have
\[ (\mu(u) - \delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^{h(x)} \subset B_R(0), \tag{2.6} \]
where \(\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n\) and \(B_R(0) \subset \mathbb{R}^n\) denotes the Euclidean ball with radius \(R\) and center \(0\).

The following proposition follows from [28, Lemma 9] and [28, Proposition 6] (which is a refinement of [15, Theorem 2.18]). It will play an important role in the proof of Theorem 4.1.
Proposition 2.2 Let $\sigma \in [\inf_M h, \sup_M h]$ and $A$ be a Hermitian matrix with eigenvalues $\mu(A) \in \partial \Gamma^\sigma$.

1. There exists a constant $\tau$ depending only on $f$, $\Gamma$ and $\sigma$ such that

\[ F(A) := \sum_i F^{ii}(A) > \tau. \]

2. For $\delta, R > 0$, there exists $\theta > 0$ depending only on $f$, $\Gamma$, $h$, $\delta$, $R$ such that the following holds. If $B$ is a Hermitian matrix satisfying

\[ (\mu(B) - 2\delta 1 + \Gamma_n) \cap \partial \Gamma^\sigma \subset B_R(0), \]

then we have either

\[ \sum_{p \neq q} F^{pq}(A) [B_{pq} - A_{pq}] > \theta \sum_p F^{pp}(A) \]

or

\[ F^{ii}(A) > \theta \sum_p F^{pp}(A), \text{ for all } i. \]

Proof (1) is [28, Lemma 9 (b)]. For (2), when $|\mu(A)| > R$, the conclusion follows from [28, Proposition 6]. When $|\mu(A)| \leq R$, we consider the set

\[ S_{R,\sigma} = \left\{ N \in \mathcal{H}_n : \mu(N) \in \overline{B_R(0)} \cap \partial \Gamma^\sigma \right\}, \]

where $\mathcal{H}_n$ denotes the set of Hermitian $n \times n$ matrices. It is clear that $S_{R,\sigma}$ is compact, and then there exists a constant $C > 0$ such that for $A \in S_{R,\sigma}$,

\[ C^{-1} \leq F^{ii}(A) \leq C, \text{ for all } i. \]

Decreasing $\theta$ if necessary,

\[ F^{ii}(A) > \theta \sum_p F^{pp}(A), \text{ for all } i. \]

Remark 2.3 In the proof of Proposition 2.2 (2), we use the compactness of the set $S_{R,\sigma}$. Then the constant $\theta$ depends on $\inf_M h$, and so Proposition 2.2 is only applicable in the non-degenerate setting. For the analogous result in the degenerate setting, we refer the reader to [7, Section 3].

3 Zero order estimate

The proof of zero order estimate will be given in this section. We follow the arguments of [28, Proposition 11] and [8, Proposition 3.1], which are generalizations of the arguments in [2, 3]. We begin with the following proposition.

Proposition 3.1 (Proposition 2.3 of [8]) Let $(M, \chi, J)$ be a compact almost Hermitian manifold. Suppose that $\varphi$ satisfies

\[ \sup_M \varphi = 0, \quad \Delta^C \varphi \geq -B \]
for some constant $B$, where $\Delta^C$ denotes the canonical Laplacian operator of $\chi$. Then there exists a constant $C$ depending only on $B$ and $(M, \chi, J)$ such that

$$\int_M (-\varphi) \chi^n \leq C.$$ 

**Proof** Actually, Proposition 3.1 is slightly stronger than [8, Proposition 2.3]. Precisely, the assumption $\chi + \sqrt{-1} \delta \bar{\delta} \varphi > 0$ in [8, Proposition 2.3] is replaced by $\Delta^C \varphi \geq -B$. Fortunately, as the reader may check, under this weaker assumption, the argument of [8, Proposition 2.3] still works. \qed

**Proposition 3.2** Suppose that $u$ is a $C$-subsolution of (1.1). Let $u$ be a smooth solution of (1.1) with $\sup_M (u - \bar{u}) = 0$. Then there exists a constant $C$ depending on $\|u\|_{C^2}, \|h\|_{C^0}, \|\omega\|_{C^0}, f, \Gamma$ and $(M, \chi, J)$ such that

$$\|u\|_{L^\infty} \leq C.$$ 

**Proof** Replacing $\omega$ by $\omega_u$, we may assume that $\bar{u} = 0$. It then suffices to establish the lower bound of the infimum $I = \inf_M u$. Assume that $I$ is attained at $x_0$. Choose a local coordinate chart $(x^1, \ldots, x^{2n})$ in a neighborhood of $x_0$ containing the unit ball $B_1(0) \subset \mathbb{R}^{2n}$ such that the point $x_0$ corresponds the origin $0 \in \mathbb{R}^{2n}$.

Consider the function $v = u + \varepsilon \sum_{i=1}^{2n} (x^i)^2$ for a small $\varepsilon > 0$ determined later. It is clear that

$$v(0) = I, \quad v \geq I + \varepsilon \quad \text{on} \quad \partial B_1(0).$$

We define the set $P$ by

$$P = \left\{ x \in B_1(0) : |Dv(x)| \leq \frac{\varepsilon}{2}, v(y) \geq v(x) + Dv(x) \cdot (y-x), \forall y \in B_1(0) \right\}. \quad (3.1)$$

Thanks to [28, Proposition 10], we have

$$c_0 \varepsilon^{2n} \leq \int_P \det(D^2v), \quad (3.2)$$

where $c_0$ is a constant depending only on $n$.

Let $(D^2u)^J$ be the $J$-invariant part of $D^2u$, i.e.

$$(D^2u)^J = \frac{1}{2}(D^2u + J^T \cdot D^2u \cdot J), \quad (3.3)$$

where $J^T$ denotes the transpose of $J$. The definition of $P$ (3.1) shows $D^2v \geq 0$ on $P$. Then

$$(D^2u)^J(x) \geq (D^2v)^J(x) - C \varepsilon \text{Id} \geq -C \varepsilon \text{Id} \quad \text{on} \quad P. \quad (3.4)$$

Consider the bilinear form $H(u)(X, Y) = \sqrt{-1} \delta \bar{\delta} u(X, JY)$ on $B_1(0)$. Direct calculation shows (see e.g. [29, p. 443])

$$H(u)(X, Y)(x) = \frac{1}{2} (D^2u)^J(x) + E(u)(x),$$
where $E(u)(x)$ is an error matrix which depends linearly on $Du(x)$. On $P$, we have $|Dv| \leq \frac{\varepsilon}{2}$ and so $|Du| \leq \frac{5\varepsilon}{2}$. Combining this with (3.4),

$$H(u) \geq -C\varepsilon \Id.$$  

It then follows that

$$\omega - \omega = \sqrt{-1} \partial \bar{\partial} u \geq -C\varepsilon \chi.$$  

Choosing $\varepsilon$ sufficiently small such that $C\varepsilon \leq \delta$, we see that

$$\mu(u) \in \mu(0) - \delta \Id + \Gamma_n.$$  

Since $u$ solves the equation (1.1), then $\mu(u) \in \partial \Gamma^h$. By (2.6),

$$\mu(u) \in (\mu(0) - \delta \Id + \Gamma_n) \cap \partial \Gamma^h \subset B_R(0)$$  

for some $R > 0$. This gives upper bound for $H(u)$. We define $H(v)$ similarly and obtain

$$(D^2v)^J = 2H(v) - 2E(v) \leq 2H(u) + C\varepsilon \Id \leq C \Id.$$  

(3.5)

Note that $\det(A + B) \geq \det(A) + \det(B)$ for non-negative definite Hermitian matrices $A$, $B$. On $P$, we have $D^2v \geq 0$ and so

$$\det((D^2v)^J) = 2^{-2n} \det(D^2v + J^T \cdot D^2v \cdot J) \geq 2^{-2n+1} \det(D^2v).$$  

Combining this with (3.5),

$$\det(D^2v) \leq 2^{2n-1} \det((D^2v)^J) \leq C.$$  

Plugging this into (3.2), we obtain

$$c_0\varepsilon^2n \leq C|P|.$$  

(3.6)

For each $x \in P$, choosing $y = 0$ in (3.1), we have

$$I = v(0) \geq v(x) - |Dv(x)| \cdot |x| \geq v(x) - \frac{\varepsilon}{2}.$$  

We assume without loss of generality that $I + \varepsilon \leq 0$. It then follows that

$$|I + \varepsilon| \leq -v \text{ on } P.$$  

Integrating both sides and using (3.6), we obtain

$$c_0\varepsilon^{2n} \leq |P| \leq \frac{\int_P (-v) \chi^n}{|I + \varepsilon|}.$$  

On the other hand, the assumptions of $\Gamma$ shows (see [4, (4)] or [28, (44)])

$$\Gamma \subset \Gamma_1 := \{(w_1, \ldots, w_n) : \sum_i w_i > 0\}.$$  

Then we have $\tr \chi \omega_u > 0$ and so

$$\Delta^C u = \tr \chi \omega_u - \tr \chi \omega \geq -C.$$  

(3.7)

Using Proposition 3.1,

$$c_0\varepsilon^{2n} |I + \varepsilon| \leq \int_P (-v) \chi^n \leq \int_P (-u) \chi^n + C\varepsilon \leq C.$$  

This gives the required estimate of $I$.  

$\square$
4 Second order estimate

In this section, we prove the following second order estimate.

**Theorem 4.1** Suppose that $u$ is a $C$-subsolution of (1.1). Let $u$ be a smooth solution of (1.1) with $\sup_M (u-u) = 0$. Then there exists a constant $C$ depending only on $\|u\|_{C^4}, \|h\|_{C^2}, \|\omega\|_{C^2}, f, \Gamma$ and $(M, \chi, J)$ such that

$$\sup_M |\nabla^2 u| \chi \leq C \sup_M |\partial u|^2 \chi + C, \tag{4.1}$$

where $\nabla$ denotes the Levi-Civita connection with respect to $\chi$.

Replacing $\omega$ by $\omega_u$ and $u$ by $u-1$, we may assume $u = 0$ and $\sup_M u = -1$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n}$ be the eigenvalues of $\nabla^2 u$ with respect to $\chi$. For notational convenience, we write

$$|\cdot| = |\cdot|_\chi.$$

Combining $\Delta^C u \geq -C$ (see (3.7)) with [8, (5.2)],

$$\sum_{\alpha=1}^{2n} \lambda_{\alpha} \geq 2\Delta^C u - C \sup_M |\partial u| \geq -C \sup_M |\partial u| - C,$$

which implies

$$|\nabla^2 u| \leq C\lambda_1 + C \sup_M |\partial u| + C. \tag{4.2}$$

Set:

$$K = \sup_M |\partial u|^2 + 1, \quad N = \sup_M |\nabla^2 u| + 1, \quad \rho = \nabla^2 u + N \chi.$$

Let us remark that $\rho > 0$ by definition. We consider the quantity

$$Q = \log \lambda_1 + \xi(|\rho|^2) + \eta(|\partial u|^2) + e^{-Au}$$

on $\Omega := \{\lambda_1 > 0\} \subset M$, where

$$\xi(s) = -\frac{1}{4} \log(5N^2 - s), \quad \eta(s) = -\frac{1}{4} \log(2K - s)$$

and $A > 0$ is a large constant to be determined later. It is clear that

$$\xi'' = 4(\xi')^2, \quad \frac{1}{20N^2} \leq \xi' \leq \frac{1}{4N^2},$$

$$\eta'' = 4(\eta')^2, \quad \frac{1}{8K} \leq \eta' \leq \frac{1}{4K}. \tag{4.3}$$

We may assume $\Omega$ is a nonempty open set (otherwise we are done by (4.2)). Let $x_0$ be a maximum point of $Q$. Note that $Q(z) \to -\infty$ as $z$ approaches to $\partial \Omega$. Then $x_0$ must be an interior point of $\Omega$. To prove Theorem 4.1, it suffices to show

$$\lambda_1(x_0) \leq CK. \tag{4.4}$$

Indeed, by the definition of $Q$ and Proposition 3.2,

$$Q(x_0) \leq \log \lambda_1(x_0) - \frac{1}{4} \log N^2 - \frac{1}{4} \log K + e^{CA}. \tag{4.5}$$
Let $y_0$ be the maximum point of $|\nabla^2 u|$. Then (4.2) implies
\[
\sup_M |\nabla^2 u| + 1 = N \leq C\lambda_1(y_0) + C \sup_M |\partial u| + C \leq \lambda_1(y_0) + C\sqrt{K}.
\]
Without loss of generality, we assume that $N \geq 2C\sqrt{K}$. Then $\lambda_1(y_0) \geq \frac{N}{2C} > 0$ and so
\[
Q(y_0) \geq \log \frac{\lambda_1(y_0)}{N} - \frac{1}{4} \log(5N^2) - \frac{1}{4} \log(2K)
\geq \log \frac{N}{2C} - \frac{1}{2} \log N - \frac{1}{4} \log K - C \tag{4.6}
\geq \frac{1}{2} \log N - \frac{1}{4} \log K - C.
\]
Recalling that $x_0$ is the maximum point of $Q$, we have $Q(y_0) \leq Q(x_0)$. Combining this with (4.5) and (4.6),
\[
N \leq C_A \lambda_1(x_0) \tag{4.7}
\]
for some constant $C_A$ depending on $A$. This shows (4.4) implies Theorem 4.1.

Near $x_0$, we choose a local unitary frame $\{e_i\}_{i=1}^n$ with respect to $\chi$ such that at $x_0$, $x_{ij} = \delta_{ij}$, $\tilde{g}_{i\overline{j}} = \delta_{ij} \delta_{i\overline{j}}$, $\tilde{g}_{1\overline{1}} \geq \tilde{g}_{2\overline{2}} \geq \cdots \geq \tilde{g}_{n\overline{n}}$. \tag{4.8}
Note that $\tilde{g}_{i\overline{j}}$ is defined by $\omega_u = \sqrt{-1} \tilde{g}_{i\overline{j}} \theta^i \wedge \overline{\theta^j}$, where $\{\theta^i\}_{i=1}^n$ denotes the dual coframe of $\{e_i\}_{i=1}^n$. By (2.3), at $x_0$, we have
\[
F^{1\overline{1}} \leq F^{2\overline{2}} \leq \cdots \leq F^{n\overline{n}}. \tag{4.9}
\]
In addition, since $(M, \chi, J)$ is almost Hermitian, then $\chi$ and $J$ are compatible, and so there exists a coordinate system $(U, \{x^\alpha\}_{\alpha=1}^{2n})$ in a neighborhood of $x_0$ such that at $x_0$,
\[
e_i = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^{2i-1}} - \sqrt{-1} \frac{\partial}{\partial x^{2i}} \right), \quad \text{for } i = 1, 2, \ldots, n, \tag{4.10}\]
and
\[
\frac{\partial \chi_{\alpha\beta}}{\partial x^\gamma} = 0, \quad \text{for } \alpha, \beta, \gamma = 1, 2, \ldots, 2n, \tag{4.11}\]
where $\chi_{\alpha\beta} = \chi(\partial_\alpha, \partial_\beta)$ and $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$. Write $u_{\alpha\beta} = (\nabla^2 u)(\partial_\alpha, \partial_\beta)$ and define
\[
\Phi^\alpha_{\beta} := \chi^{\alpha\gamma} u_{\gamma\beta},
\]
where $\chi^{\alpha\gamma}$ denotes the inverse of the matrix $(\chi_{\alpha\gamma})$. Recall that $\lambda_\alpha$ denote the eigenvalues of $\nabla^2 u$ with respect to $\chi$. Then $\lambda_\alpha$ are also the eigenvalues of $\Phi$. Let $V_1, \ldots, V_{2n}$ be the $\chi$-unit eigenvectors for $\Phi$ at $x_0$, corresponding to eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ respectively. Define $V^\beta_\alpha$ by $V_\alpha = V^\beta_\alpha \partial_\beta$ at $x_0$, and extend $V_\alpha$ to be vector fields near $x_0$ by taking the components to be constants. Using the perturbation argument (see [8, p. 1965]), we may assume that $\lambda_1 > \lambda_2$ at $x_0$, and so $\lambda_1$ is smooth near $x_0$.

At $x_0$, the maximum principle shows
\[
\frac{(\lambda_1)_i}{\lambda_1} = -\xi e_i(|\rho|^2) - \eta' e_i(|\partial u|^2) + Ae^{-Au} u_i, \quad \text{for } i = 1, 2, \ldots, n, \tag{4.12}\]
and
\[
0 \geq L(Q) = \frac{L(\lambda_1)}{\lambda_1} - \frac{\mu^i |(\lambda_1)i|^2}{\lambda_1^2} + \xi' L(|\rho|^2) + \xi'' \mu^i |e_i|(|\rho|^2)|^2
\]
(4.13)
\[+ \eta' L(|\partial u|^2) + \eta'' \mu^i |e_i|(|\partial u|^2)|^2 - A e^{-Au} L(u) + A^2 e^{-Au} \mu^i |u|^2,\]
where the operator \( L \) is defined in (2.4).

From now on, all the calculations are done at \( x_0 \). We will use the Einstein summation convention. We usually use \( C \) to denote a constant depending only on \( \|h\|_{C^2}, \|w\|_{C^2}, f, \Gamma, (M, \chi, J) \), and \( C_A \) to denote a constant depending only on \( A, \|h\|_{C^2}, \|w\|_{C^2}, f, \Gamma, (M, \chi, J) \).

In the following argument, we always assume without loss of generality that \( \lambda_1 \geq CK \) for some \( C \), and \( \lambda_1 \geq C_A K \) for some \( C_A \).

### 4.1 Lower bound for \( L(Q) \)

**Proposition 4.2** For \( \varepsilon \in (0, \frac{1}{3}] \), at \( x_0 \), we have
\[
0 \geq L(Q) \geq G_1 + G_2 + G_3 - B + \xi'' \mu^i |e_i|(|\rho|^2)|^2
\]
\[+ \frac{3\eta'}{4} \sum_{i,j} \mu^i |e_i| e_j |u|^2 + \eta'' \mu^i |e_i|(|\partial u|^2)|^2
\]
(4.14)
\[- A e^{-Au} L(u) + A^2 e^{-Au} \mu^i |u|^2 - \frac{C}{\varepsilon} \mathcal{F},\]
where
\[
G_1 := (2 - \varepsilon) \sum_{\alpha > 1} \frac{\mu^i |e_i| (uv_1 v_\alpha)|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)}, \quad G_2 := -\frac{1}{\lambda_1} \mu^{i,j} \nu^i_1 (\tilde{g}_{ij}) \nu^j_1 (\tilde{g}_{ij}), \]
(4.15)
\[
G_3 := \sum_{\alpha, \beta} \frac{\mu^i |e_i| (u_{\alpha\beta})|^2}{C_A \lambda_1^2}, \quad B := (1 + \varepsilon) \frac{\mu^i |(\lambda_1)|^2}{\lambda_1^2}, \quad \mathcal{F} := \sum_i \mu^i. \]
(4.16)

To prove Proposition 4.2, we need to estimate the lower bounds of \( L(\lambda_1), L(|\rho|^2) \) and \( L(|\partial u|^2) \) respectively.

#### 4.1.1 Lower bound of \( L(\lambda_1) \)

**Lemma 4.3** For \( \varepsilon \in (0, \frac{1}{3}) \), at \( x_0 \), we have
\[
L(\lambda_1) \geq (2 - \varepsilon) \sum_{\alpha > 1} \frac{\mu^i |e_i| (uv_1 v_\alpha)|^2}{\lambda_1 - \lambda_\alpha} - \mu^{i,j} \nu^i_1 (\tilde{g}_{ij}) \nu^j_1 (\tilde{g}_{ij}) - \varepsilon \frac{\mu^i |(\lambda_1)|^2}{\lambda_1} - \frac{C}{\varepsilon} \lambda_1 \mathcal{F}.
\]

**Proof** The following formulas are well-known (see e.g. [8, Lemma 5.2]):
\[
\frac{\partial \lambda_1}{\partial \Phi^\alpha} = V^\alpha_1 V^\beta_1,
\]
\[
\frac{\partial^2 \lambda_1}{\partial \Phi^\alpha \partial \Phi^\beta} = \sum_{\mu > 1} \frac{V^\alpha_1 V^\beta_1 V^\gamma_1 V^\delta_1 + V^\alpha_1 V^\beta_1 V^\gamma_1 V^\delta_1}{\lambda_1 - \lambda_\mu}.
\]
Then we compute

\[
L(\lambda_1) = F^{ij} \frac{\partial^2 \lambda_1}{\partial \Phi_{\delta}^{\alpha} \partial \Phi_{\delta}^{\beta}} e_i(\Phi^{\alpha}_{\delta})\tilde{e}_i(\Phi^{\beta}_{\delta}) + F^{ij} \frac{\partial \lambda_1}{\partial \Phi_{\beta}^{\alpha}} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0,1)})(\Phi^{\alpha}_{\beta})
\]

\[
= F^{ij} \frac{\partial^2 \lambda_1}{\partial \Phi_{\delta}^{\alpha} \partial \Phi_{\delta}^{\beta}} e_i(u_{\gamma \delta})\tilde{e}_i(u_{\alpha \beta}) + F^{ij} \frac{\partial \lambda_1}{\partial \Phi_{\beta}^{\alpha}} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0,1)})(u_{\alpha \beta})
\]

\[
+ F^{ij} \frac{\partial \lambda_1}{\partial \Phi_{\beta}^{\alpha}} u_{\gamma \beta} e_i \tilde{e}_i(\chi^{\alpha \gamma})
\]

\[
\geq 2 \sum_{\alpha > 1} \frac{F^{ii} |e_i(u_{V_1 V_\alpha})|^2}{\lambda_1 - \lambda_\alpha} + F^{ii} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0,1)})(u_{V_1 V_1}) - C\lambda_1 F,
\]

where \((\chi^{\alpha \beta})\) denotes the inverse of the matrix \((\chi_{\alpha \beta})\). Note that \(F \geq \tau\) has uniform positive lower bound thanks to Proposition 2.2.

\[\square\]

**Claim 1** At \(x_0\), we have

\[
F^{ij} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0,1)})(u_{V_1 V_1}) \geq - F^{ik, jl} V_1(\tilde{g}_{ik}) V_1(\tilde{g}_{jl}) - C\lambda_1 F - 2T,
\]

where

\[
T := F^{ii} \{ [V_1, \tilde{e}_i] V_1 e_i(u) + [V_1, e_i] V_1 \tilde{e}_i(u) \}.
\]

**Proof of Claim 1** It is clear that

\[
F^{ij} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0,1)})(u_{V_1 V_1})
\]

\[
= F^{ij} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0,1)})(V_1 V_1(u) - (\nabla V_1 V_1)(u))
\]

\[
\geq F^{ij} e_i \tilde{e}_i V_1 V_1(u) - F^{ij} e_i \tilde{e}_i (\nabla V_1 V_1)(u) - F^{ij} [e_i, \tilde{e}_i]^{(0,1)} V_1 V_1(u) - C\lambda_1 F.
\]

Set \(W = \nabla V_1 V_1\). Then

\[
e_i \tilde{e}_i W(u) = e_i W \tilde{e}_i(u) + e_i [\tilde{e}_i, W](u)
\]

\[
= W e_i \tilde{e}_i(u) + [e_i, W] \tilde{e}_i(u) + e_i [\tilde{e}_i, W](u)
\]

\[
= W(\tilde{g}_{ij}) - W(g_{ij}) + W[e_i, \tilde{e}_i]^{(0,1)}(u) + [e_i, W] \tilde{e}_i(u) + e_i [\tilde{e}_i, W](u).
\]

Applying \(W\) to the equation (1.1),

\[
F^{ii} W(\tilde{g}_{ij}) = W(h).
\]

Substituting this into (4.19),

\[
|F^{ij} e_i \tilde{e}_i (\nabla V_1 V_1)(u)| = |F^{ii} e_i \tilde{e}_i W(u)| \leq C\lambda_1 F.
\]

Combining this with (4.18),

\[
F^{ij} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0,1)})(u_{V_1 V_1})
\]

\[
\geq F^{ii} \{ e_i \tilde{e}_i V_1 V_1(u) - [e_i, \tilde{e}_i]^{(0,1)} V_1 V_1(u) \} - C\lambda_1 F.
\]
Let $O(\lambda_1)$ denote a term satisfying $|O(\lambda_1)| \leq C\lambda_1$ for some uniform constant $C$. Commuting the derivatives and absorbing the lower order commutator terms into $T$ and the error term $O(\lambda_1)\mathcal{F}$, we compute

\[
F_i^i \{ e_i \tilde{e}_i V_1 V_1(u) - [e_i, \tilde{e}_i]^{(0,1)} V_1 V_1(u) \}
\]

\[
= F_i^i \{ V_1 e_i \tilde{e}_i(u) - V_1 V_1 e_i [V_1, e_i \tilde{e}_i(u)] - V_1 e_i [V_1, \tilde{e}_i V_1(u) - e_i [V_1, \tilde{e}_i] V_1(u)] \} - F_i^i \{ V_1 e_i [V_1, e_i \tilde{e}_i]^{(0,1)}(u) - V_1 V_1 [e_i, \tilde{e}_i]^{(0,1)}(u) - [V_1, [e_i, \tilde{e}_i]^{(0,1)}] V_1(u) \} \tag{4.21}
\]

\[
= F_i^i V_1 e_i \tilde{e}_i(u) - [e_i, \tilde{e}_i]^{(0,1)}(u) + O(\lambda_1)\mathcal{F} - 2T
\]

Substituting (4.21) into (4.20),

\[
F_i^i (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0,1)})(u V_1 V_1) \geq F_i^i V_1 (\tilde{g}_{ii}) - C\lambda_1\mathcal{F} - 2T. \tag{4.22}
\]

To deal with the first term, we apply $V_1 V_1$ to the equation (1.1) and obtain

\[
F_i^i V_1 (\tilde{g}_{ii}) = - F_i^i k_{ij} V_1 (\tilde{g}_{ik}) V_1 (\tilde{g}_{jl}) + V_1 V_1 (h). \tag{4.23}
\]

Then Claim 1 follows from (4.22) and (4.23).

\[\Box\]

**Claim 2** For $\varepsilon \in (0, \frac{1}{2}]$, at $x_0$, we have

\[
2T \leq \varepsilon \frac{F_i^i (\lambda_1 i)^2}{\lambda_1} + \varepsilon \sum_{\alpha > 1} \frac{F_i^i |e_i(u V_1 V_1)|^2}{\lambda_1 - \lambda_\alpha} + \frac{C}{\varepsilon} \lambda_1 \mathcal{F},
\]

assuming without loss of generality that $\lambda_1 \geq K$.

**Proof of Claim 2** At $x_0$, we write

\[
[V_1, e_i] = \sum_\beta \tau_{i\beta} V_\beta, \quad [V_1, \tilde{e}_i] = \sum_\beta \bar{\tau}_{i\beta} V_\beta,
\]

where $\tau_{i\beta} \in \mathbb{C}$ are uniformly bounded constants. Then

\[
2T = 2 F_i^i \{ [V_1, e_i] V_1 e_i(u) + [V_1, e_i] V_1 \tilde{e}_i(u) \} \leq C \sum_\alpha F_i^i |V_\alpha V_1 e_i(u)|.
\]

Using $\lambda_1 \geq K$, we compute

\[
|V_\alpha V_1 e_i(u)| = |e_i V_\alpha V_1(u) + V_\alpha [V_1, e_i] V_1(u)|
\]

\[
= |e_i (u V_\alpha V_1) + e_i (\nabla V_1 V_\alpha) V_1(u) + V_\alpha [V_1, e_i] V_1(u)|
\]

\[
\leq |e_i (u V_\alpha V_1)| + C\lambda_1.
\]

It then follows that

\[
2T \leq C \sum_\alpha F_i^i |e_i (u V_\alpha V_1)| + C\lambda_1 \mathcal{F}
\]

\[
= C F_i^i |e_i (u V_1 V_1)| + C \sum_{\alpha > 1} F_i^i |e_i (u V_1 V_\alpha)| + C\lambda_1 \mathcal{F}
\]

\[
\leq \varepsilon \frac{F_i^i |e_i (u V_1 V_1)|^2}{\lambda_1} + \frac{C}{\varepsilon} \lambda_1 \mathcal{F} + \varepsilon \sum_{\alpha > 1} \frac{F_i^i |e_i (u V_1 V_\alpha)|^2}{\lambda_1 - \lambda_\alpha} + \frac{C}{\varepsilon} \sum_{\alpha > 1} (\lambda_1 - \lambda_\alpha) \mathcal{F} + C\lambda_1 \mathcal{F}
\]

\[
\leq \varepsilon \frac{F_i^i |(\lambda_1 i)|^2}{\lambda_1} + \varepsilon \sum_{\alpha > 1} \frac{F_i^i |e_i (u V_1 V_\alpha)|^2}{\lambda_1 - \lambda_\alpha} + \frac{C}{\varepsilon} \lambda_1 \mathcal{F}.
\]
where we used \((\lambda_1)_i = e_i(u_{V_i^1})\) and (4.2) in the last inequality.

Combining (4.17), Claim 1 and 2, we obtain Lemma 4.3.

4.1.2 Lower bound of \(L(|\rho|^2)\)

**Lemma 4.4** For \(\varepsilon \in (0, \frac{1}{3}]\), at \(x_0\), we have

\[
L(|\rho|^2) \geq (2-\varepsilon) \sum_{\alpha, \beta} F^{\tilde{i}} |e_i(u_{\alpha\beta})|^2 - \frac{C}{\varepsilon} N^2 \mathcal{F}.
\]

**Proof** Applying \(\partial_\alpha \partial_\beta\) to the equation (1.1),

\[
F^{\tilde{i}} \partial_\alpha \partial_\beta (\tilde{g}_{i\tilde{i}}) = - F^{ik, jl} \partial_\alpha (\tilde{g}_{ik}) \partial_\beta (\tilde{g}_{jl}) + \partial_\alpha \partial_\beta (h).
\]

By the similar calculation of (4.22), we obtain

\[
F^{\tilde{i}} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{[0,1]})(u_{\alpha\beta})
\]

\[
= F^{\tilde{i}} \partial_\alpha \partial_\beta (\tilde{g}_{i\tilde{i}}) - 2 F^{\tilde{i}} \text{Re}([\partial_\beta, e_i] \partial_\alpha \tilde{e}_i u) - 2 F^{\tilde{i}} \text{Re}([\partial_\alpha, e_i] \partial_\beta \tilde{e}_i u) + O(\lambda_1) \mathcal{F}
\]

\[
= - F^{ik, jl} \partial_\alpha (\tilde{g}_{ik}) \partial_\beta (\tilde{g}_{jl}) - 2 F^{\tilde{i}} \text{Re}([\partial_\beta, e_i] \partial_\alpha \tilde{e}_i u) - 2 F^{\tilde{i}} \text{Re}([\partial_\alpha, e_i] \partial_\beta \tilde{e}_i u) + O(\lambda_1) \mathcal{F}.
\]

Using (2.2), the concavity of \(f\) and \(\rho > 0\), we have

\[
-2 F^{ij, lpq} \rho_{ij} \partial_\alpha (\tilde{g}_{ij}) \partial_\beta (\tilde{g}_{pq}) \geq 0,
\]

and so

\[
L(|\rho|^2) \geq 2 \sum_{\alpha, \beta} F^{\tilde{i}} |e_i(u_{\alpha\beta})|^2 + 2 \sum_{\alpha, \beta} F^{\tilde{i}} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{[0,1]})(u_{\alpha\beta}) \rho_{\alpha\beta}
\]

\[
+ \sum_{\alpha, \beta, \gamma, \delta} F^{\tilde{i}} e_i e_\gamma (\chi^{\alpha\gamma} \chi^{\delta}) \rho_{\alpha\beta} \rho_{\gamma\delta}
\]

\[
\geq 2 \sum_{\alpha, \beta} F^{\tilde{i}} |e_i(u_{\alpha\beta})|^2 - 2 \sum_{\alpha, \beta} \rho_{\alpha\beta} F^{\tilde{i}} \text{Re}([\partial_\beta, e_i] \partial_\alpha \tilde{e}_i u)
\]

\[
- 2 \sum_{\alpha, \beta} \rho_{\alpha\beta} F^{\tilde{i}} \text{Re}([\partial_\alpha, e_i] \partial_\beta \tilde{e}_i u) - C N^2 \mathcal{F}.
\]

Using the similar argument of Claim 2 in Lemma 4.3,

\[
2 \sum_{\alpha, \beta} \rho_{\alpha\beta} F^{\tilde{i}} \text{Re}([\partial_\beta, e_i] \partial_\alpha \tilde{e}_i u) + 2 \sum_{\alpha, \beta} \rho_{\alpha\beta} F^{\tilde{i}} \text{Re}([\partial_\alpha, e_i] \partial_\beta \tilde{e}_i u)
\]

\[
\leq \varepsilon \sum_{\alpha, \beta} F^{\tilde{i}} |e_i(u_{\alpha\beta})|^2 + \frac{C}{\varepsilon} N^2 \mathcal{F}.
\]

Then we obtain Lemma 4.4.

4.1.3 Lower bound of \(L(|\partial u|^2)\)

**Lemma 4.5** At \(x_0\), we have

\[
L(|\partial u|^2) \geq \frac{3}{4} \sum_{i, j} F^{\tilde{i}} |e_i e_j u|^2 + |e_i \tilde{e}_j u|^2 - CK \mathcal{F}.
\]
**Proof** Direct calculation shows

\[ L(\|\partial u\|^2) = F^{ij}(e_i e_j (\|\partial u\|^2) - [e_i, \bar{e}_i]^{(0,1)}(\|\partial u\|^2)) = I_1 + I_2 + I_3, \]

where

\[ I_1 := F^{ij}(e_i \bar{e}_i e_j u - [e_i, \bar{e}_i]^{(0,1)} e_j u), \]
\[ I_2 := F^{ij}(e_i \bar{e}_i e_j u - [e_i, \bar{e}_i]^{(0,1)} e_j u), \]
\[ I_3 := F^{ij}([e_i e_j u]^2 + |e_i e_j u|^2). \]

Applying \( e_j \) to the equation (1.1),

\[ F^{ij}(e_j e_i \bar{e}_i u - e_j [e_i, \bar{e}_i]^{(0,1)} u) = h_j. \]

Note that

\[ F^{ij}(e_i \bar{e}_i e_j u - [e_i, \bar{e}_i]^{(0,1)} e_j u) \]
\[ = F^{ij}(e_j e_i \bar{e}_i u + e_i \bar{e}_i, e_j u + [e_i, e_j] \bar{e}_i u - [e_i, \bar{e}_i]^{(0,1)} e_j u) \]
\[ = h_j + F^{ij} e_j [e_i, \bar{e}_i]^{(0,1)} u + F^{ij} (e_i \bar{e}_i, e_j u + [e_i, e_j] \bar{e}_i u - [e_i, \bar{e}_i]^{(0,1)} e_j u) \]
\[ = h_j + F^{ij} \{ e_i [\bar{e}_i, e_j] u + \bar{e}_i [e_i, e_j] u + [e_i, e_j] \bar{e}_i u - [e_i, \bar{e}_i]^{(0,1)} e_j u \}. \]

Similarly,

\[ F^{ij}(e_i \bar{e}_i e_j u - [e_i, \bar{e}_i]^{(0,1)} e_j u) \]
\[ = h_j + F^{ij} \{ e_i [\bar{e}_i, e_j] u + \bar{e}_i [e_i, e_j] u + [e_i, e_j] \bar{e}_i u - [e_i, \bar{e}_i]^{(0,1)} e_j u \}. \]

By the Cauchy–Schwarz inequality,

\[ I_1 + I_2 \]
\[ \geq 2 \Re \left( \sum_j h_j u_j \right) - C \|\partial u\| \sum_{i,j} F^{ij}([e_i e_j u] + |e_i \bar{e}_i u|) - C \|\partial u\|^2 \mathcal{F} \]
\[ \geq - C \|\partial u\| - \frac{1}{4} \sum_j F^{ij}([e_i e_j u]^2 + |e_i \bar{e}_i u|^2) - C \|\partial u\|^2 \mathcal{F}. \]

(4.25)

Then

\[ L(\|\partial u\|^2) = I_1 + I_2 + I_3 \]
\[ \geq \frac{3}{4} \sum_{i,j} F^{ij}([e_i e_j u]^2 + |e_i \bar{e}_i u|^2) - C(\|\partial u\| + \|\partial u\|^2) \mathcal{F} \]
\[ \geq \frac{3}{4} \sum_{i,j} F^{ij}([e_i e_j u]^2 + |e_i \bar{e}_i u|^2) - CK \mathcal{F}. \]

\[ \square \]

We will use the above computations to prove Proposition 4.2.
Proof of Proposition 4.2 Combining (4.13), Lemma 4.3, 4.4 and 4.5, we obtain

\[
0 \geq (2 - \varepsilon) \sum_{\alpha > 1} F^{i \bar{i}} |e_i (u_{V_{\alpha} V_1})|^2 - \frac{1}{\lambda_1} F^{i \bar{k}, j \bar{j}} V_1 (\tilde{g}_{i \bar{k}}) V_1 (\tilde{g}_{j \bar{j}}) \\
+ (2 - \varepsilon) \xi' \sum_{\alpha, \beta} F^{i \bar{i}} |e_i (u_{\alpha \beta})|^2 - (1 + \varepsilon) \frac{F^{i \bar{i}} |(\lambda_{i \bar{i}})|^2}{\lambda_1^2} + \xi'' F^{i \bar{i}} |e_i (|\rho|^2)|^2 \\
+ \frac{3\eta'}{4} \sum_{i, j} F^{i \bar{i}} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + \eta'' F^{i \bar{i}} |e_i (|\partial u|^2)|^2 \\
- Ae^{-\lambda u} L(u) + A^2 e^{-\lambda u} F^{i \bar{i}} |u_i|^2 - \frac{C}{\varepsilon} (1 + \xi' N^2 + \eta' K) F.
\]

The first, second and fourth term are \(G_1, G_2\) and \(B\) respectively. It suffices to deal with the third and last term. For the third term, using (4.3) and (4.7),

\[
(2 - \varepsilon) \xi' \sum_{\alpha, \beta} F^{i \bar{i}} |e_i (u_{\alpha \beta})|^2 \geq \sum_{\alpha, \beta} F^{i \bar{i}} |e_i (u_{\alpha \beta})|^2 \geq \sum_{\alpha, \beta} \frac{F^{i \bar{i}} |e_i (u_{\alpha \beta})|^2}{20 N^2} = G_3.
\]

For the last term, using (4.3) again,

\[
- \frac{C}{\varepsilon} (1 + \xi' N^2 + \eta' K) F \geq - \frac{C}{\varepsilon} F.
\]

Combining the above inequalities, we obtain Proposition 4.2. \(\square\)

4.2 Proof of Theorem 4.1

To prove Theorem 4.1, we define the index set:

\[
J := \left\{ 1 \leq k \leq n : \frac{\eta'}{2} \sum_j (|e_k e_j u|^2 + |e_k \bar{e}_j u|^2) \geq A^{5n} e^{-5nu} K \text{ at } x_0 \right\}.
\]

If \(J = \emptyset\), then we obtain Theorem 4.1 directly. So we assume \(J \neq \emptyset\) and let \(j_0\) be the maximal element of \(J\). If \(j_0 < n\), then we define another index set:

\[
S := \left\{ j_0 \leq i \leq n - 1 : F^{i \bar{i}} \leq A^{-2} e^{2\lambda u} F^{i+1 \bar{i+1}} \text{ at } x_0 \right\}.
\]

According to the index sets \(J\) and \(S\), the proof of Theorem 4.1 can be divided into three cases:

**Case 1** \(j_0 = n\).

**Case 2** \(j_0 < n\) and \(S = \emptyset\).

**Case 3** \(j_0 < n\) and \(S \neq \emptyset\).

Actually, Case 3 is the most difficult case. Case 1 and 2 are relatively easy, and their arguments are very similar.

4.2.1 Proofs of Case 1 and 2

In Case 1, we choose \(\varepsilon = \frac{1}{3}\). By (4.12) and the elementary inequality

\[
|a + b + c|^2 \leq 3|a|^2 + 3|b|^2 + 3|c|^2,
\]

\(\square\) Springer
we get
\[ B = -(1 + \varepsilon) \frac{F^{ij}_{ii} (\lambda_1)_i^2}{\lambda_1^2} \]
\[ \geq -4KA^2e^{-2\Lambda u} F^i - 4(\xi')^2 F^{ii} |e_i(\rho)|^2 - 4(\eta')^2 F^{ii} |ei(\partial u)|^2. \]

Substituting this into (4.14), dropping the non-negative terms \( G_i \) \( (i = 1, 2, 3) \) and \( A^2e^{-\Lambda u} F^{ii}_i |u_i|^2 \), and using (4.3), we obtain
\[ 0 \geq \frac{3\eta'}{4} \sum_{i, j} F^{ii} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \left( C + 4KA^2e^{-2\Lambda u} \right) F - Ae^{-\Lambda u} L(u). \quad (4.28) \]

The assumption of Case 1 shows \( n = j_0 \in J \) and so
\[ \frac{\eta'}{2} \sum_{i, j} F^{ii} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \geq \frac{\eta'}{2} F^{nn} \sum_{j} (|e_n e_j u|^2 + |e_n \bar{e}_j u|^2) \]
\[ \geq A^{5n}e^{-5nu} K F^{nn} \geq A^{5n}e^{-5nu} K F, \quad (4.29) \]
where we used (4.9) in the last inequality. It then follows that
\[ 0 \geq \frac{\eta'}{4} \sum_{i, j} F^{ii} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \left( C + 4KA^2e^{-2\Lambda u} \right) F \]
\[ + \frac{1}{n} A^{5n}e^{-5nu} K F - Ae^{-\Lambda u} L(u). \]

For the last term, by the Cauchy-Schwarz inequality, we obtain
\[ L(u) = \sum \limits_i F^{ii} (e_i \bar{e}_i u - [e_i, \bar{e}_i]^{(0, 1)} u) \leq \frac{\eta'}{4} F^{ii} |e_i \bar{e}_i u|^2 + CK F \quad (4.30) \]
and so
\[ 0 \geq \frac{1}{n} A^{5n}e^{-5nu} K F - (4A^2e^{-2\Lambda u} + C) K F. \quad (4.31) \]

Recalling \( \sup_{M} u = -1 \) and increasing \( A \) if necessary, this yields a contradiction.

In Case 2, using \( S = \emptyset \), we see that
\[ F^{j_0_0} \geq A^{-2}e^{2\Lambda u(x_0)} F^{j_0+1} + 1 \geq \cdots \geq A^{-2n}e^{2n\Lambda u(x_0)} F^{nn}. \]
Combining this with (4.9),
\[ \frac{\eta'}{2} \sum_{i, j} F^{ii} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \geq \frac{\eta'}{2} F^{j_0 j_0} \sum_{j} (|e_j e_j u|^2 + |e_j \bar{e}_j u|^2) \]
\[ \geq A^{5n}e^{-5nu} K F^{j_0 j_0} \geq A^{3n}e^{-3nu} K F^{nn} \geq \frac{1}{n} A^{3n}e^{-3nu} K F. \quad (4.32) \]
Replacing (4.29) by (4.32), and using the similar argument of Case 1, we complete the proof of Case 2.
4.2.2 Proof of Case 3

In Case 3, we have \( S \neq \emptyset \). Let \( i_0 \) be the minimal element of \( S \). Using \( i_0 \), we define another nonempty set

\[
I = \{i_0 + 1, \cdots, n\}.
\]

Now we decompose the term \( B \) into three terms based on \( I \):

\[
B = (1 + \varepsilon) \sum_i F^{ii} |(\lambda_1)_i|^2 \frac{1}{\lambda_1^2} = (1 + \varepsilon) \sum_{i \not\in I} F^{ii} |(\lambda_1)_i|^2 \frac{1}{\lambda_1^2} + 3\varepsilon \sum_{i \in I} F^{ii} |(\lambda_1)_i|^2 \frac{1}{\lambda_1^2} + (1 - 2\varepsilon) \sum_{i \in I} F^{ii} |(\lambda_1)_i|^2
\]

\[
=: B_1 + B_2 + B_3.
\]

- Terms \( B_1 \) and \( B_2 \) We first deal with the terms \( B_1 \) and \( B_2 \).

**Lemma 4.6** At \( x_0 \), we have

\[
B_1 + B_2 \leq \frac{\eta'}{4} \sum_j F^{ii} \left( |e_i e_j u|^2 + |e_i \tilde{e}_j u|^2 \right) + \varepsilon'' F^{ii} |e_i (|\rho|^2)|^2 + \eta'' F^{ii} |e_i (|\partial u|^2)|^2 + 9\varepsilon A^2 e^{-2\Lambda u} F^{ii} |u_i|^2.
\]

**Proof** Using (4.12), the elementary inequality (4.27) and \( \varepsilon \in (0, \frac{1}{3}] \), we obtain

\[
B_1 = (1 + \varepsilon) \sum_{i \not\in I} F^{ii} \left| -\xi' e_i (|\rho|^2) - \eta' e_i (|\partial u|^2) + A e^{-\Lambda u} u_i \right|^2 \\
\leq 4(\xi')^2 \sum_{i \not\in I} F^{ii} |e_i (|\rho|^2)|^2 + 4(\eta')^2 \sum_{i \not\in I} F^{ii} |e_i (|\partial u|^2)|^2 + 4A^2 e^{-2\Lambda u} \sum_{i \not\in I} F^{ii} |u_i|^2 \\
\leq \varepsilon'' \sum_{i \not\in I} F^{ii} |e_i (|\rho|^2)|^2 + \eta'' \sum_{i \not\in I} F^{ii} |e_i (|\partial u|^2)|^2 + 4A^2 e^{-2\Lambda u} K \sum_{i \not\in I} F^{ii},
\]

where we used (4.3) in the last line. Similarly,

\[
B_2 = 3\varepsilon \sum_{i \in I} F^{ii} \left| -\xi' e_i (|\rho|^2) - \eta' e_i (|\partial u|^2) + A e^{-\Lambda u} u_i \right|^2 \\
\leq 9\varepsilon(\xi')^2 \sum_{i \in I} F^{ii} |e_i (|\rho|^2)|^2 + 9\varepsilon(\eta')^2 \sum_{i \in I} F^{ii} |e_i (|\partial u|^2)|^2 + 9\varepsilon A^2 e^{-2\Lambda u} \sum_{i \in I} F^{ii} |u_i|^2 \\
\leq \varepsilon'' \sum_{i \in I} F^{ii} |e_i (|\rho|^2)|^2 + \eta'' \sum_{i \in I} F^{ii} |e_i (|\partial u|^2)|^2 + 9\varepsilon A^2 e^{-2\Lambda u} F^{ii} |u_i|^2.
\]

It then follows that

\[
B_1 + B_2 \leq 4A^2 e^{-2\Lambda u} K \sum_{i \not\in I} F^{ii} + \varepsilon'' F^{ii} |e_i (|\rho|^2)|^2 + \eta'' F^{ii} |e_i (|\partial u|^2)|^2 + 9\varepsilon A^2 e^{-2\Lambda u} F^{ii} |u_i|^2.
\]
Since $I = \{i_0 + 1, \ldots, n\}$, then for each $i \notin I$, we have $i \leq i_0$ and $F^{ii} \leq F^{i_0 i_0}$ (see (4.9)). This shows
\begin{equation}
B_1 + B_2 \leq 4nA^2 e^{-2Au} K F^{i_0 i_0} + \xi'' F^{ii} |e_i(\rho)^2| + \eta'' F^{ii} |e_i(\partial u|^2|2 + 9AE^2 e^{-2Au} F^{ii} |u_i|^2. \tag{4.34}
\end{equation}

On the other hand, we assert that
\begin{equation}
F^{j_0 j_0} > A^{-2n} e^{2nAu} F^{i_0 i_0}. \tag{4.35}
\end{equation}

The definitions of $i_0$ and $j_0$ show $j_0 \leq i_0$. If $j_0 = i_0$, the above is trivial. If $j_0 \leq i_0 - 1$, since $i_0$ is the minimal element of $S$, then we obtain $j_0, \ldots, i_0 - 1 \notin S$ and
\begin{equation*}
F^{j_0 j_0} > A^{-2n} e^{2nAu} F^{j_0 j_0} + \cdots > A^{-2(n_0 - j_0)} e^{2(i_0 - j_0)Au} F^{i_0 i_0} \leq A^{-2n} e^{2nAu} F^{i_0 i_0}.
\end{equation*}

Using $j_0 \in J$ and (4.35),
\begin{equation*}
\frac{\eta'}{4} \sum_{i,j} F^{ii} (|e_i e_j u|^2 + |e_i e_j u|^2) \geq \frac{\eta'}{4} F^{j_0 j_0} \sum_j (|e_{j_0} e_j u|^2 + |e_{j_0} e_j u|^2) \geq \frac{1}{2} A^3 e^{-3nAU} K F^{j_0 j_0} \geq \frac{1}{2} A^3 e^{-3nAU} K F^{i_0 i_0}. \tag{4.36}
\end{equation*}

Combining (4.34) and (4.36), and increasing $A$ if necessary, we are done. \hfill \Box

\textbf{Term $B_3$}

We will use $G_1$, $G_2$ and $G_3$ to control the term $B_3$.

\textbf{Lemma 4.7} If $e = \frac{e^{Au(x_0)}}{\rho}$, then at $x_0$, we have
\begin{equation*}
B_3 \leq G_1 + G_2 + G_3 + \frac{C}{\varepsilon} \mathcal{F}.
\end{equation*}

We first define
\begin{equation*}
W_1 = \frac{1}{\sqrt{2}} (V_1 - \sqrt{-1} JV_1), \quad W_1 = \sum_q v_q e_q, \quad JV_1 = \sum_{\alpha > 1} \varsigma_\alpha V_\alpha, \tag{4.37}
\end{equation*}

where we used $V_1$ is orthogonal to $JV_1$. At $x_0$, since $V_1$ and $e_q$ are $\chi$-unit, then
\begin{equation*}
\sum_q |v_q|^2 = 1, \quad \sum_{\alpha > 1} \varsigma_\alpha^2 = 1.
\end{equation*}

\textbf{Lemma 4.8} At $x_0$, we have
\begin{enumerate}
\item $\omega_u \geq -C_A K \chi$,
\item $|v_i| \leq \frac{C_A K}{\varepsilon}$, for $i \in I$.
\end{enumerate}

\textbf{Proof} For (1), recall that the maximal element of $J$ is $j_0$, and the minimal element of $I$ is $i_0 + 1$ satisfying $i_0 + 1 > j_0$. This shows $I \cap J = \emptyset$, i.e. for any $i \in I$, we have $i \notin J$. Thus,
\begin{equation*}
\frac{\eta'}{4} \sum_j (|e_i e_j u|^2 + |e_i e_j u|^2) \leq \frac{1}{2} A^3 e^{-3nAU} K, \quad \text{for } i \in I. \tag{4.38}
\end{equation*}
Since \( n \in I \), then (4.38) implies \( e_n \bar{e}_n u \geq -C_A K \) and so

\[
\tilde{g}_{n\pi} = g_{n\pi} + e_n \bar{e}_n u - [e_n, \bar{e}_n]^{(0,1)} u \geq e_n \bar{e}_n u - C K \geq -C_A K.
\]

Combining this with (4.8), we obtain \( \omega_u \geq -C_A K \chi \).

For (2), use (4.38) again, we see that

\[
\sum_{\gamma=2i_0+1}^{2n} \sum_{\beta=1}^{2n} |u_{\gamma\beta}| \leq C_A K.
\]

It then follows that

\[
|\Phi^\gamma_{\beta}| = |u_{\beta\gamma}| \leq C_A K, \quad \text{for} 2i_0 + 1 \leq \gamma \leq 2n, 1 \leq \beta \leq 2n.
\]

Recalling \( \Phi(V_1) = \lambda_1 V_1 \), we obtain

\[
|V_1^\gamma| = \left| \frac{1}{\lambda_1} \sum_{\beta} \Phi^\gamma_{\beta} V_1^\beta \right| \leq \frac{C_A K}{\lambda_1}, \quad \text{for} 2i_0 + 1 \leq \gamma \leq 2n,
\]

which implies

\[
|\nu_i| \leq |V_1^{2i-1}| + |V_1^{2i}| \leq \frac{C_A K}{\lambda_1}, \quad \text{for} i \in I.
\]

\( \square \)

The definition of \( W_1 \) (4.37) gives \( V_1 = -\sqrt{-1} J V_1 + \sqrt{2} \overline{W_1} \). Then direct calculation shows

\[
e_i(u_{V_1 V_1}) = -\sqrt{-1} e_i(u_{V_1 J V_1}) + \sqrt{2} e_i(u_{V_1 \overline{W}_1})
\]

\[
= -\sqrt{-1} \sum_{\alpha > 1} \zeta_\alpha e_i(u_{V_1 V_\alpha}) + \sqrt{2} \sum_q \psi_q V_1 e_i \overline{q}_u + O(\lambda_1)
\]

\[
= -\sqrt{-1} \sum_{\alpha > 1} \zeta_\alpha e_i(u_{V_1 V_\alpha}) + \sqrt{2} \sum_{\psi \notin I} \psi_q V_1 (\tilde{g}_{i\psi}) + \sqrt{2} \sum_{\psi \in I} \psi_q V_1 e_i \overline{q}_u + O(\lambda_1).
\]

Combining this with the Cauchy–Schwarz inequality and Lemma 4.8,

\[
B_3 = (1 - 2\varepsilon) \sum_{i \in I} \frac{F_{i\bar{i}} |e_i(u_{V_1 V_1})|^2}{\lambda_1^2}
\]

\[
\leq (1 - \varepsilon) \sum_{i \in I} \frac{F_{i\bar{i}}}{\lambda_1^2} \left| -\sqrt{-1} \sum_{\alpha > 1} \zeta_\alpha e_i(u_{V_1 V_\alpha}) + \sqrt{2} \sum_{\psi \notin I} \psi_q V_1 (\tilde{g}_{i\psi}) \right|^2
\]

\[
+ \frac{C_A K^2}{\varepsilon \lambda_1^2} \sum_{i \in I} \sum_{q \in I} \frac{F_{i\bar{i}} |V_1 e_i \overline{q}_u|^2}{\lambda_1^2} + \frac{C F}{\varepsilon}.
\]
For any $\gamma > 0$, using the Cauchy-Schwarz inequality again,

$$B_3 \leq (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \sum_{i \in \mathcal{I}} \frac{F_i^T}{\lambda_i^2} \left( \sum_{\alpha > 1} \zeta_{\alpha} \epsilon_i(u_{V_V}) \right)^2$$

$$+ (1 - \varepsilon)(1 + \gamma) \frac{2 F_i^T}{\lambda_i^2} \left( \sum_{q \not\in \mathcal{I}} \frac{\gamma q}{V_i(\tilde{g}_i \tilde{q})} \right)^2$$

$$+ \frac{C_A K^2}{\varepsilon \lambda_1^2} \sum_{i \in \mathcal{I}} \left( \sum_{\alpha > 1} \frac{F_i^T |V_1 e_i q u|^2}{\lambda_1^2} + \frac{C \mathcal{F}}{\varepsilon} \right)$$

$$=: B_{31} + B_{32} + B_{33} + \frac{C \mathcal{F}}{\varepsilon}.$$

Now we give the proof of Lemma 4.7.

**Proof of Lemma 4.7** We first deal with $B_{33}$. It is clear that

$$|V_1 e_i q u| \leq C \sum_{\alpha, \beta} |e_i(u_{\alpha \beta})| + C \lambda_1$$

and so

$$B_{33} = \frac{C_A K^2}{\varepsilon \lambda_1^2} \sum_{i \in \mathcal{I}} \sum_{q \in \mathcal{I}} \frac{F_i^T |V_1 e_i q u|^2}{\lambda_1^2} \leq \frac{C_A K^2}{\varepsilon \lambda_1^2} \sum_{\alpha, \beta} \frac{F_i^T |e_i(u_{\alpha \beta})|^2}{\lambda_1^2} + \frac{C_A K^2}{\varepsilon \lambda_1^2} \mathcal{F}.$$

Without loss of generality, we assume that $\lambda_1 \geq \frac{C_A K}{\varepsilon}$, which implies

$$B_{33} \leq \sum_{\alpha, \beta} \frac{F_i^T |e_i(u_{\alpha \beta})|^2}{C_A \lambda_1^2} + \mathcal{F} = G_3 + \mathcal{F}.$$

To prove Lemma 4.7, it suffices to show

$$B_{31} + B_{32} \leq G_1 + G_2. \quad (4.39)$$

The process of proving (4.39) will be fairly long and involve some further cases. For the term $B_{31}$, by the Cauchy–Schwarz inequality,

$$B_{31} \leq (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \sum_{i \in \mathcal{I}} \frac{F_i^T}{\lambda_i^2} \left( \sum_{\alpha > 1} \frac{(\lambda_1 - \lambda_{\alpha}) s_{\alpha}^2}{\lambda_1 - \lambda_{\alpha}} \right) \left( \sum_{\alpha > 1} |e_i(u_{V_V})|^2 \right)$$

$$= (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \sum_{i \in \mathcal{I}} \frac{F_i^T}{\lambda_i^2} \left( \lambda_1 - \sum_{\alpha > 1} \lambda_{\alpha} s_{\alpha}^2 \right) \left( \sum_{\alpha > 1} |e_i(u_{V_V})|^2 \right) \quad (4.40)$$

$$\leq \frac{1 - \varepsilon}{(2 - \varepsilon) \lambda_1} \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_{\alpha} s_{\alpha}^2 \right) G_1.$$

旮 Springer
For the term $B_{32}$, using the Cauchy-Schwarz inequality again,

$$B_{32} = (1 - \varepsilon)(1 + \gamma) \sum_{i \in I} \frac{2F^{ii}}{\lambda_i^2} \left| \sum_{q \not\in I} \nu_q V_1(\tilde{g}_{i\pi}) \right|^2$$

$$\leq (1 - \varepsilon)(1 + \gamma) \sum_{i \in I} \frac{2F^{ii}}{\lambda_i^2} \left( \sum_{q \not\in I} (\tilde{g}_{q\pi} - \tilde{g}_{i\pi})|\nu_q|^2 \right) \left( \sum_{q \not\in I} (F^{ii} - F^{q\pi})|V_1(\tilde{g}_{i\pi})|^2 \right).$$

The denominators in the above are not zero. Indeed, for $i \in I$ and $q \not\in I$, by definition of the index set $I$,

$$F^{q\pi} \leq F^{i0\pi} \leq A^{-2} e^{2Au} F^{i0} + 1 \leq A^{-2} e^{2Au} F^{ii}.$$ 

Increasing $A$ if necessary, we have $F^{ii} \neq F^{q\pi}$ and so $\tilde{g}_{q\pi} \neq \tilde{g}_{i\pi}$. Recalling $\omega_u \geq -CAK \chi$ (see Lemma 4.8), for $i \in I$ and $q \not\in I$, we have

$$0 \leq \frac{(\tilde{g}_{q\pi} - \tilde{g}_{i\pi})|\nu_q|^2}{F^{ii} - F^{q\pi}} \leq \frac{\tilde{g}_{q\pi}|\nu_q|^2 - \tilde{g}_{i\pi}|\nu_q|^2}{(1 - A^{-2} e^{2Au})F^{ii}} \leq \frac{\tilde{g}_{q\pi}|\nu_q|^2 + CAK}{(1 - A^{-2} e^{2Au})F^{ii}}. \quad (4.41)$$

It then follows that

$$B_{32} \leq (1 - \varepsilon)(1 + \gamma) \sum_{i \in I} \frac{2F^{ii}}{\lambda_i^2} \left( \sum_{q \not\in I} \tilde{g}_{q\pi}|\nu_q|^2 + CAK \right) \sum_{i \in I} \frac{2}{\lambda_i^2} \left( \sum_{q \not\in I} (F^{ii} - F^{q\pi})|V_1(\tilde{g}_{i\pi})|^2 \right).$$

Using (2.2) and the concavity of $f$, we have

$$G_2 = -\frac{1}{\lambda_1} F^{ik,j\pi} V_1(\tilde{g}_{ik}) V_1(\tilde{g}_{j\pi}) \geq 2 \sum_{i \in I} \sum_{q \not\in I} \frac{(F^{ii} - F^{q\pi})|V_1(\tilde{g}_{i\pi})|^2}{\tilde{g}_{q\pi} - \tilde{g}_{i\pi}}$$

and so

$$B_{32} \leq \frac{(1 - \varepsilon)(1 + \gamma)}{\lambda_1(1 - A^{-2} e^{2Au})} \left( \sum_{q \not\in I} \tilde{g}_{q\pi}|\nu_q|^2 + CAK \right) G_2.$$ 

Increasing $A$ if necessary, $\varepsilon = \frac{A \nu_{u(i)}}{\chi}$ implies

$$\frac{(1 - \varepsilon)(1 + \gamma)}{\lambda_1(1 - A^{-2} e^{2Au})} \leq \left( 1 - \frac{\varepsilon}{2} \right) \left( 1 + \frac{1 + \gamma}{\lambda_1} \right).$$

Then

$$B_{32} \leq \left( 1 - \frac{\varepsilon}{2} \right) \left( 1 + \frac{1 + \gamma}{\lambda_1} \right) \left( \sum_{q \not\in I} \tilde{g}_{q\pi}|\nu_q|^2 + CAK \right) G_2. \quad (4.42)$$
Combining (4.40) and (4.42),
\[ B_{31} + B_{32} \leq \frac{1 - \varepsilon}{(2 - \varepsilon)\lambda_1} \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1,} \lambda_\alpha s_\alpha^2 \right) G_1 + \left( 1 - \frac{\varepsilon}{2} \right) \left( \frac{1 + \gamma}{\lambda_1} \right) \left( \sum_{q \notin I} \tilde{g}_{q\bar{q}} |v_q|^2 + C_A K \right) G_2. \] (4.43)

Thanks to (4.41), increasing \( C_A \) if necessary, we have
\[ \sum_{q \notin I} \tilde{g}_{q\bar{q}} |v_q|^2 + C_A K > 0. \] (4.44)

We split the proof of (4.39) into two cases.

Case (i) \( \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2 \right) > (1 - \frac{\varepsilon}{2}) \left( \sum_{q \notin I} \tilde{g}_{q\bar{q}} |v_q|^2 + C_A K \right). \)

By the assumption of Case (i) and (4.44),
\[ \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2 \right) > \left( 1 - \frac{\varepsilon}{2} \right) \left( \sum_{q \notin I} \tilde{g}_{q\bar{q}} |v_q|^2 + C_A K \right) > 0. \]

Since \( \lambda_1 > \lambda_2 \) at \( x_0 \), we can choose
\[ \gamma := \frac{\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2}{\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2} > 0. \]

Then, by (4.43),
\[ B_{31} + B_{32} \leq \frac{1}{2\lambda_1} \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2 \right) G_1 + \frac{1 + \gamma}{2\lambda_1} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2 \right) G_2 = G_1 + G_2. \]

Case (ii) \( \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2 \right) \leq (1 - \frac{\varepsilon}{2}) \left( \sum_{q \notin I} \tilde{g}_{q\bar{q}} |v_q|^2 + C_A K \right). \)

It is clear that
\[ \tilde{g}(W_1, \bar{W}_1) = g(W_1, \bar{W}_1) + (\bar{\partial} \bar{u})(W_1, \bar{W}_1) = W_1 \bar{W}_1(u) + O(K) \]
\[ = \frac{1}{2} (u_W v_1 + u_J v_1 J v_1) + O(K) = \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2 \right) + O(K), \] (4.45)

where \( O(K) \) denotes a term satisfying \( |O(K)| \leq CK \) for some uniform constant \( C \). Thanks to Lemma 4.8, we have \( \omega_u \geq -C_A K \chi \), and then
\[ \sum_{q \notin I} \tilde{g}_{q\bar{q}} |v_q|^2 + C_A K \leq \sum_q \tilde{g}_{q\bar{q}} |v_q|^2 + C_A K \]
\[ = \tilde{g}(W_1, \bar{W}_1) + C_A K \leq \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha s_\alpha^2 \right) + C_A K, \]

where we used (4.37) in the last inequality. Combining this with the assumption of Case (ii), we see that
\[ \sum_{q \notin I} \tilde{g}_{q\bar{q}} |v_q|^2 + C_A K \leq \frac{C_A K}{\varepsilon}. \] (4.46)
Using $\omega_u \geq -CAK\chi$ and (4.45),

$$\frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha S_\alpha^2 \right) \geq \tilde{g}(W_1, \tilde{W}_1) - CK = \sum_q \tilde{g}_{q\bar{q}}|v_q|^2 - CK \geq -CAK.$$ 

It then follows that

$$0 < \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha S_\alpha^2 \leq 2\lambda_1 + CAK \leq (2 + 2\epsilon^2)\lambda_1,$$

where we assume without loss of generality that $\lambda_1 \geq \frac{CAK}{\epsilon^2}$. Choosing $\gamma = \frac{1}{\epsilon^2}$ and using (4.43) and (4.46),

$$B_{31} + B_{32} \leq \frac{2 - 2\epsilon}{2 - \epsilon} (1 + \epsilon^2)G_1 + \frac{CAK}{\epsilon^2 \lambda_1} G_2.$$

Recalling $\epsilon = \frac{e^{A(u(x_0))}}{9}$ and increasing $A$ if necessary, we may assume that

$$\frac{2 - 2\epsilon}{2 - \epsilon} (1 + \epsilon^2)^2 \leq 1.$$

Then $B_{31} + B_{32} \leq G_1 + G_2$. \hfill \Box

Now we complete the proof of Case 3 of Theorem 4.1.

**Proof** (Finishing the Proof of Case 3) Combining Proposition 4.2, Lemma 4.6 and 4.7, we have

$$0 \geq (A^2 e^{-Au} - 9\epsilon A^2 e^{-2Au}) F^{ji} |ui|^2 - \frac{C}{\epsilon} \mathcal{F} + \frac{\eta'}{4} \sum_{i,j} F^{ji} (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2) - Ae^{-Au} L(u).$$

Recalling that $\epsilon$ has been chosen as $\frac{e^{A(u(x_0))}}{9}$ in Lemma 4.7, we obtain

$$0 \geq - \frac{C}{\epsilon} \mathcal{F} + \frac{\eta'}{4} \sum_{i,j} F^{ji} (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2) - Ae^{-Au} L(u). \quad (4.47)$$

We now choose $A = \frac{9C + 1}{\theta}$, where $\theta$ is the constant in Proposition 2.2. There are two subcases:

**Subcase 3.1** $-L(u) \geq \theta \mathcal{F}$.

In this subcase, (4.47) implies

$$0 \geq \left( A \theta e^{-Au} - \frac{C}{\epsilon} \right) \mathcal{F} + \frac{\eta'}{4} \sum_{i,j} F^{ji} (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2).$$

Using $A = \frac{9C + 1}{\theta}$, we have

$$A \theta e^{-Au} - \frac{C}{\epsilon} = A \theta e^{-Au} - 9Ce^{-Au} \geq e^{-Au}.$$

It then follows that

$$0 \geq e^{-Au} \mathcal{F} + \frac{\eta'}{4} \sum_{i,j} F^{ji} (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2).$$
This yields a contradiction.

**Subcase 3.2** $F_{ii}^{i} \geq \theta F$ for $i = 1, 2, \ldots, n$.

By the Cauchy-Schwarz inequality,

$$Ae^{-Au} L(u) = Ae^{-Au} \sum_{i} F_{ii}^{i} (e_{i} \bar{e}_{i} u - [e_{i}, \bar{e}_{i}]^{(0,1)} u)$$

$$\leq Ae^{-Au} \sum_{i} |e_{i} \bar{e}_{i} u| + C Ae^{-Au} K F$$

$$\leq \frac{\theta \eta'}{8} \sum_{i} |e_{i} \bar{e}_{i} u|^{2} + C A K F.$$

Substituting this into (4.47),

$$\frac{\theta \eta'}{8} \sum_{i,j} (|e_{i} e_{j} u|^{2} + |e_{i} \bar{e}_{j} u|^{2}) \leq C A K F.$$  

It then follows that

$$\sum_{i,j} (|e_{i} e_{j} u|^{2} + |e_{i} \bar{e}_{j} u|^{2}) \leq C A K^{2}$$

and so $\lambda_{1} \leq C A K$. This completes the proof of Subcase 3.2.  

\[\Box\]

## 5 Proof of Theorem 1.1

In this section, we will prove the $C^{2}$ estimate by adapting the argument of [28, Section 6] in the almost Hermitian setting, and then give the proof of Theorem 1.1.

### 5.1 $C^{2}$ estimate

**Proposition 5.1** Let $u$ be a smooth solution of the equation (1.1) with $\sup_{M} (u - u) = 0$. Then there exists a constant $C$ depending on $u, h, \omega, f, \Gamma$ and $(M, \chi, J)$ such that

$$\sup_{M} |\nabla^{2} u|_{\chi} \leq C.$$  

**Proof** By Theorem 4.1, it suffices to prove

$$\sup_{M} |\partial u|_{\chi}^{2} \leq C.$$  

We follow the argument of [28, Section 6]. For convenience, we denote $|\cdot|_{\chi}$ by $|\cdot|$. To prove (5.1), we argue by contradiction. Suppose that there exist sequences of smooth functions $u_{k}$ and $h_{k}$ such that

1. $\sup_{\Gamma} f < h_{k} \leq \sup_{\Gamma} f - C_{0}^{-1}$ and $\|h_{k}\|_{C^{2}} \leq C_{0}$,
2. $F(\omega_{u_{k}}) = h_{k}$ and it admits a $C$-subsolution $u_{k}$ with $\sup_{M} (u_{k} - u_{k}) = 0$ and $\|u_{k}\|_{C^{4}} \leq C_{0}$,
3. $\sup_{M} |\partial u_{k}|^{2} =: \Lambda_{k}^{2} \to \infty$,

where $C_{0}$ is a positive constant independent of $k$. Let $p_{k}$ be the maximum point of $|\partial u_{k}|^{2}$, i.e.

$$|\partial u_{k}|^{2}(p_{k}) = \sup_{M} |\partial u_{k}|^{2} = \Lambda_{k}^{2} \to \infty.$$
Passing to a subsequence, we may assume that $p_k$ converges to $p_\infty$. Near $p_\infty$, we choose the coordinate system $(U, \{x^\alpha\}_{\alpha=1}^{2n})$ such that at $p_\infty$,

$$\chi_{\alpha\beta} = \chi \left( \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) = \delta_{\alpha\beta}$$

and

$$J \left( \frac{\partial}{\partial x^{2i-1}} \right) = \frac{\partial}{\partial x^{2i}}, \quad \text{for } i = 1, 2, \ldots, n.$$ 

For convenience, we assume that $U$ contains the Euclidean ball of radius 2 and write $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ for $\alpha = 1, 2, \ldots, 2n$. Let $\{e_i\}_{i=1}^{n}$ be a local $\chi$-unitary frame of $(1, 0)$-vectors with respect to $J$ such that at $p_\infty$,

$$e_i = \frac{1}{\sqrt{2}} \left( \partial_{2i-1} - \sqrt{-1} \partial_{2i} \right), \quad \text{for } i = 1, 2, \ldots, n.$$ 

Let $Z_i$ be the vector field defined by

$$Z_i = \frac{1}{\sqrt{2}} \left( \partial_{2i-1} - \sqrt{-1} \partial_{2i} \right), \quad \text{for } i = 1, 2, \ldots, n.$$ 

Note that $e_i(p_\infty) = Z_i(p_\infty)$, but they may be different outside $p_\infty$. Let $g_E$ and $J_E$ be the standard metric and complex structure of the Euclidean space. Then $\{Z_i\}_{i=1}^{n}$ is the standard $g_E$-unitary frame of $(1, 0)$-vectors with respect to $J_E$. Note that $Z_i$ may be not $(1, 0)$-vector field with respect to $J$. To characterize the difference of $e_i$ and $Z_i$, we write

$$Z_i = e_i + A_i^\alpha \partial_\alpha.$$ 

Since $e_i(p_\infty) = Z_i(p_\infty)$, then for any $i$ and $\alpha$, we have the following limit of the coefficient $A_i^\alpha$:

$$\lim_{p \to p_\infty} A_i^\alpha (p) = 0.$$ 

Next, we compute

$$Z_i Z_j u_k = (e_i + A_i^\alpha \partial_\alpha) \left( \bar{e}_j + \bar{A}_j^\beta \partial_\beta \right) u_k$$

$$= e_i \bar{e}_j u_k + e_i (\bar{A}_j^\beta \partial_\beta) u_k + A_i^\alpha \partial_\alpha \bar{A}_j^\beta \partial_\beta u_k + (A_i^\alpha \partial_\alpha) (\bar{A}_j^\beta \partial_\beta) u_k$$

$$= e_i \bar{e}_j u_k + (e_i \bar{A}_j^\beta) (\partial_\beta u_k) + \bar{A}_j^\beta e_i \partial_\beta u_k + A_i^\alpha \partial_\alpha \bar{A}_j^\beta u_k + A_i^\alpha (\partial_\alpha \bar{A}_j^\beta) (\partial_\beta u_k) + A_i^\alpha \bar{A}_j^\beta \partial_\alpha \partial_\beta u_k.$$ 

Thanks to Theorem 4.1, we have

$$\sup_M |\nabla^2 u_k| \leq C \sup_M |\nabla u_k|^2 + C \leq C \Lambda_k^2$$

(5.2)

and so

$$Z_i Z_j u_k = e_i \bar{e}_j u_k + O(\Lambda_k) + o_E \cdot \Lambda_k^2$$

$$= \omega_{u_k}(e_i, \bar{e}_j) - \omega(e_i, \bar{e}_j) + [e_i, \bar{e}_j]^{(0,1)} u_k + O(\Lambda_k) + o_E \cdot \Lambda_k^2$$

(5.3)

$$= \omega_{u_k}(e_i, \bar{e}_j) + O(\Lambda_k) + o_E \cdot \Lambda_k^2,$$

where $O(\Lambda_k)$ denotes a term satisfying $|O(\Lambda_k)| \leq C \Lambda_k$, and $o_E$ denotes a term satisfying $\lim_{p \to p_\infty} o_E = 0$. 

$\square$ Springer
Define the function \( v_k : B_{\Lambda_k}(0) \to \mathbb{R} \) by \[
v_k(z) := u_k \left( p_k + \frac{z}{\Lambda_k} \right),
\]

By Proposition 3.2 and (5.2), we obtain

1. \[ \| v_k \|_{C^2(B_{\Lambda_k}(0))} \leq C, \]
2. \[ |\partial v_k|_0 = 1 + o(1), \text{ where } o(1) \text{ denotes a term satisfying } \lim_{k \to \infty} o(1) = 0. \]

Passing to a subsequence, we may assume that \( v_k \) converges to \( v_\infty \) in \( C^{1,1}_{\text{loc}}(\mathbb{C}^n) \), and so \[
\| v_\infty \|_{C^{1,1}(\mathbb{C}^n)} \leq C, \quad |\partial v_\infty|_0 = 1.
\]

We have the following claim:

**Claim** The function \( v_\infty \) is a \( \Gamma \)-solution, i.e.,

1. If \( \varphi \) be a \( C^2 \) function such that \( \varphi \geq v_\infty \) near \( z_0 \) and \( \varphi(z_0) = v_\infty(z_0) \), then we have
   \[
   \lambda \left[ \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}(z_1) \right] \in \Gamma,
   \]
2. If \( \varphi \) be a \( C^2 \) function such that \( \varphi \leq v_\infty \) near \( z_0 \) and \( \varphi(z_0) = v_\infty(z_0) \), then we have
   \[
   \lambda \left[ \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}(z_0) \right] \in \mathbb{R}^n \setminus \Gamma.
   \]

**Proof of Claim** (1) For any \( \varepsilon > 0 \), we choose sufficiently large \( k \) such that

\[
\| v_k - v_\infty \|_{C^0(B_1(z_0))} \leq \varepsilon^2.
\]

Then there exist \( a \in \mathbb{R} \) with \( |a| \leq \varepsilon \) and \( z_1 \in B_\varepsilon(z_0) \) satisfying

\[
\varphi + \varepsilon |z - z_0|^2 + a \geq v_k \quad \text{on } B_\varepsilon(z_0)
\]

and

\[
\varphi(z_1) + \varepsilon |z_1 - z_0|^2 + a = v_k(z_1).
\]

It follows that

\[
\frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}(z_1) + \varepsilon \delta_{ij} \geq \frac{\partial^2 v_k}{\partial z^i \partial \bar{z}^j}(z_1).
\]

On the other hand, by the definitions of \( v_k, Z_i \) and (5.3),

\[
\frac{\partial^2 v_k}{\partial z^i \partial \bar{z}^j}(z_1) = \frac{1}{\Lambda_k^2} (Z_i \bar{Z}_j u_k) \left( p_\infty + \frac{z_1}{\Lambda_k} \right)
\]

\[
= \frac{1}{\Lambda_k^2} \omega_{ik} (e_i, \bar{e}_j) \left( p_\infty + \frac{z_1}{\Lambda_k} \right) + o(1)
\]

\[
= \frac{1}{\Lambda_k^2} \varepsilon \Gamma + o(1) = \Gamma + o(1).
\]

(5.4)

Increasing \( k \) if necessary,

\[
\lambda \left[ \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}(z_1) \right] \in \Gamma - 2\varepsilon \mathbf{1}.
\]
Letting $\varepsilon \to 0$, we obtain

$$\lambda \left[ \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}(z_0) \right] \in \Gamma.$$  

(2) For any $\varepsilon > 0$, we choose sufficiently large $k$ such that

$$\|v_k - v_{\infty}\|_{C^0(B_1(z_0))} \leq \varepsilon^2.$$  

Then there exist $a \in \mathbb{R}$ with $|a| \leq \varepsilon$ and $z_1 \in B_\varepsilon(z_0)$ satisfying

$$\varphi - \varepsilon |z - z_0|^2 + a \leq v_k \text{ on } B_\varepsilon(z_0)$$  

and

$$\varphi(z_1) - \varepsilon |z_1 - z_0|^2 + a = v_k(z_1).$$  

It follows that

$$\frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}(z_1) - \varepsilon \delta_{ij} \leq \frac{\partial^2 v_k}{\partial z^i \partial \overline{z}^j}(z_1).$$  

Suppose that

$$\lambda \left[ \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}(z_1) \right] \in \Gamma + 3 \varepsilon \mathbf{1}. \quad (5.5)$$

Then

$$\lambda \left[ \frac{\partial^2 v_k}{\partial z^i \partial \overline{z}^j}(z_1) \right] \in \Gamma + 2 \varepsilon \mathbf{1}.$$  

Combining this with (5.4),

$$\frac{1}{\Lambda_k^2} \cdot \lambda \left[ \omega_{u_k}(e_i, \overline{e}_j) \left( p_{\infty} + \frac{z_1}{\Lambda_k} \right) \right] = \lambda \left[ \frac{\partial^2 v_k}{\partial z^i \partial \overline{z}^j}(z_1) \right] + o(1) \in \Gamma + \varepsilon \mathbf{1}.$$  

This shows

$$\lambda \left[ \omega_{u_k}(e_i, \overline{e}_j) \left( p_{\infty} + \frac{z_1}{\Lambda_k} \right) \right] \in \Lambda_k^2 \Gamma + \varepsilon \Lambda_k^2 \mathbf{1} = \Gamma + \varepsilon \Lambda_k^2 \mathbf{1}.$$  

By $f(\lambda(\omega_{u_k})) = F(\omega_{u_k}) = h_k$ and [28, Lemma 9 (a)], we obtain

$$h_k \left( p_{\infty} + \frac{z_1}{\Lambda_k} \right) = f(\lambda \left[ \omega_{u_k}(e_i, \overline{e}_j) \left( p_{\infty} + \frac{z_1}{\Lambda_k} \right) \right]) \to \sup \Gamma f \text{ as } k \to \infty.$$  

If $\sup \Gamma f = \infty$, then the above contradicts with $\|h_k\|_{C^2} \leq C_0$. If $\sup \Gamma f < \infty$, then the above contradicts with

$$h_k \leq \sup \Gamma f - C_0^{-1}.$$  

This implies (5.5) is impossible and so

$$\lambda \left[ \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}(z_1) \right] \notin \Gamma + 3 \varepsilon \mathbf{1}.$$  

Letting $\varepsilon \to 0$, we obtain
\[ \lambda \left[ \frac{\partial^2 \varphi}{\partial z^i \partial z^j}(z_0) \right] \in \mathbb{R}^n \setminus \Gamma. \]

\[ \square \]

Combining the above claim and \( \|v_\infty\|_{C^{1,1}(\mathbb{C}^n)} \leq C \) with [28, Theorem 20], the function \( v_\infty \) is constant, which contradicts with \( |\partial v_\infty|(0) = 1 \). Then we complete the proof of Theorem 5.1.

### 5.2 \( C^{2,\alpha} \) estimate

**Proof of Theorem 1.1** For any \( p \in M \), we choose a coordinate system \((U, \{x^\alpha\}_{\alpha=1}^{2n})\) such that at \( p \),

\[ \chi_{\alpha\beta} = \chi \left( \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) = \delta_{\alpha\beta} \]

and

\[ J \left( \frac{\partial}{\partial x^{2i-1}} \right) = \frac{\partial}{\partial x^{2i}}, \text{ for } i = 1, 2, \ldots, n. \]

We further assume that \( U \) contains \( B_1(0) \subset \mathbb{R}^{2n} \). In \( B_1(0) \), we define the \((1, 0)\)-vector field \( Z_i \) by

\[ Z_i = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^i} - \sqrt{-1} J \left( \frac{\partial}{\partial x^i} \right) \right), \text{ for } i = 1, 2, \ldots, n. \]

It is clear that \( \{Z_i(0)\} \) is a basis for \( T_p^{(1,0)}M \). Shrinking \( B_1(0) \) if necessary, we may assume that \( \{Z_i\} \) is also a \((1, 0)\)-frame of \( T_{\mathbb{C}}M \) on \( B_1(0) \). Let \( \theta^i \) be the dual frame of \( \{Z_i\}_{i=1}^n \). Write

\[ \chi = \sqrt{-1} \chi_{ij} \theta^i \wedge \bar{\theta}^j, \quad \omega = \sqrt{-1} g_{ij} \theta^i \wedge \bar{\theta}^j, \quad \omega_u = \sqrt{-1} g_{ij} \theta^i \wedge \bar{\theta}^j. \]

In order to apply [29, Theorem 1.2], we first define \( T, S \) and \( \mathcal{G} \) (since the letter \( F \) has already been used in (1.1), here we use \( \mathcal{G} \) instead). Let \( \text{Sym}(2n) \) be the space of symmetric \( 2n \times 2n \) matrices with real entries. Define the map \( T : \text{Sym}(2n) \times B_1(0) \to \text{Sym}(2n) \) by

\[ T(N, x) = \frac{1}{4} \left( N + J^T(x) \cdot N \cdot J(x) \right). \]

Let \( H(u) \) be the symmetric bilinear form given by

\[ H(u)(X, Y) = (dJ du)^{(1,1)}(X, JY) = 2\sqrt{-1} \bar{\partial}u(X, JY), \]

By [29, p. 443], we have

\[ H(u)(x) = 2T(D^2u(x), x) + E(u)(x), \]

where \( E(u)(x) \) is an error matrix which depends linearly on \( Du(x) \). Define the map \( S : B_1(0) \to \text{Sym}(2n) \) by

\[ S(x) = g(x) + \frac{1}{2} E(u)(x). \]

Then

\[ S(x) + T(D^2u(x), x) = g(x) + \frac{1}{2} H(u)(x) \]
and
\[
[S(x) + T(D^2u(x), x)](Z_i, \overline{Z}_j) = g_{ij} + \frac{1}{2}H(u)(Z_i, \overline{Z}_j)
\]
(5.6)
\[
= g_{ij} + (\sqrt{-1}\partial\overline{\partial}u)(Z_i, J\overline{Z}_j) = g_{ij} + (\partial\overline{\partial}u)(Z_i, \overline{Z}_j) = \tilde{g}_{ij}.
\]

For any \( N \in \text{Sym}(2n) \), we write
\[
N_{ij}(x) = N(Z_i(x), \overline{Z}_j(x)).
\]
Then \( (N_{ij}(x)) \) is a Hermitian matrix. Let \( \mu(N, x) \) be the eigenvalues of \( (N_{ij}(x)) \) with respect to \( \chi(x) \). Consider the set
\[
\mathcal{E} = \{ N \in \text{Sym}(2n) \mid \mu(N, 0) \in \Gamma^\sigma \cap B_R(0) \},
\]
where \( \sigma \) and \( R > 0 \) are two constants determined later. By the assumptions of \( f \) and \( \Gamma \), we see that \( \mathcal{E} \) is compact and convex.

After shrinking \( B_1(0) \), we may assume that \( \chi(x) \) is close to \( \chi(0) \), which implies \( \mu(N, x) \) is close to \( \mu(N, 0) \). Thanks to Proposition 5.1, choosing \( \sigma \) and \( R \) appropriately, we have
\[
S(x) + T(D^2u(x), x) \in \mathcal{E}, \quad \text{on } B_1(0).
\]
Furthermore, there exists a neighborhood \( \mathcal{U} \) of \( \mathcal{E} \) such that \( \mu(N, x) \in \Gamma \) for any \( (N, x) \in \mathcal{U} \times B_1(0) \). Define the function \( \mathcal{G} : \mathcal{U} \times B_1(0) \to \mathbb{R} \) by
\[
\mathcal{G}(N, x) := f(\mu(N, x)).
\]
Extend it smoothly to \( \text{Sym}(2n) \times B_1(0) \), and still denote it by \( \mathcal{G} \) for convenience. Combining (5.6) and the definition of \( \mathcal{G} \), it is clear that
\[
\mathcal{G}(S(x) + T(D^2u(x), x), x) = F(\omega_u).
\]
Then on \( B_1(0) \), the equation (1.1) can be written as
\[
\mathcal{G}(S(x) + T(N, x), x) = h(x).
\]
The assumptions of \( f \) and \( (M, \chi, J) \) show \( \mathcal{G}, T \) satisfies [29, H1, H2]. In the definition of \( S \), the term \( E(u) \) depends on \( Du \) linearly. Proposition 5.1 gives a uniform \( C^1 \) bound of this term. Then \( S \) satisfies [29, H3]. By [29, Theorem 1.2] and a covering argument, we obtain the required \( C^{2,\alpha} \) estimate. \( \square \)

6 Proofs of Theorem 1.2 and 1.3

6.1 Complex Hessian equation

For \( 1 \leq k \leq n \), let \( \sigma_k \) and \( \Gamma_k \) be the \( k \)-th elementary symmetric polynomial and the \( k \)-th Gårding cone on \( \mathbb{R}^n \), respectively. Namely, for any \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \in \mathbb{R}^n \), we have
\[
\sigma_k(\mu) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}
\]
and
\[
\Gamma_k = \{ \mu \in \mathbb{R}^n : \sigma_i(\mu) > 0 \text{ for } i = 1, 2, \cdots, k \}.
\]
The above definitions can be extended to almost Hermitian manifold \((M, \chi, J)\) as follows:

\[
\sigma_k(\alpha) = \binom{n}{k} \frac{\alpha^k \wedge \chi^{n-k}}{\chi^n}, \quad \text{for any } \alpha \in A^{1,1}(M),
\]

and

\[
\Gamma_k(M, \chi) = \{ \alpha \in A^{1,1}(M) : \sigma_i(\alpha) > 0 \text{ for } i = 1, 2, \ldots, k \}.
\]

We say that \(\alpha \in A^{1,1}(M)\) is \(k\)-positive if \(\alpha \in \Gamma_k(M, \chi)\).

Now we are in position to prove Theorem 1.2.

**Proof of Theorem 1.2** The uniqueness of solution follows from the maximum principle (see e.g. [8, p. 1980–1981]). For the existence of solution, we consider the family of equations

\[
\begin{align*}
\omega^k u_t \wedge \chi^{n-k} &= e^{th+(1-t)h_0+c_t \chi^n}, \\
\omega u_t &\in \Gamma_k(\chi), \\
\sup_M u_t &= 0,
\end{align*}
\]

where \(\{c_t\}\) are constants and

\[
h_0 = \log \frac{\omega^k \wedge \chi^{n-k}}{\chi^n}.
\]

Let us define

\[
S = \{ t \in [0, 1] : \text{there exists a pair} (u_t, c_t) \in C^\infty(M) \times \mathbb{R} \text{ solving } (\ast)_t \}.
\]

Note that \((0, 0)\) solves \((\ast)_0\) and hence \(S \neq \emptyset\). To prove the existence of solution for (1.3), it suffices to prove that \(S\) is closed and open.

**Step 1** \(S\) is closed.

We first show that \(\{c_t\}\) is uniformly bounded. Suppose that \(u_t\) achieves its maximum at the point \(p_t \in M\). Then the maximum principle shows \(\sqrt{-1} \partial \bar{\partial} u_t\) is non-positive at \(p_t\). Combining this with \((\ast)_t\), we obtain the upper bound of \(c_t\):

\[
c_t \leq -th(p_t) + th_0(p_t) \leq C,
\]

for some \(C\) depending only on \(h, \omega\) and \(\chi\). The lower bound of \(c_t\) can be proved similarly.

Define

\[
f = \log \sigma_k, \quad \Gamma = \Gamma_k.
\]

As pointed out in [28, p. 368–369], one can verify that the above setting satisfies the structural assumptions in Sect. 1. Furthermore, the \(k\)-positivity of \(\omega\) shows \(u = 0\) is a \(C\)-subsolution of \((\ast)_t\). Then \(C^\infty\) a priori estimates of \(u_t\) follows from Theorem 1.1 and the standard bootstrapping argument. Combining this with the Arzelà-Ascoli theorem, \(S\) is closed.

**Step 2** \(S\) is open.

Suppose there exists a pair \((u_t, c_t)\) satisfies \((\ast)_t\). We hope to show that when \(t\) is close to \(\hat{t}\), there exists a pair \((u_t, c_t) \in C^\infty(M) \times \mathbb{R}\) solving \((\ast)_t\).

We first consider the linearized operator of \((\ast)_t\) at \(u_t\):

\[
L_{u_t}(\psi) := k \frac{\sqrt{-1} \partial \bar{\partial} \psi \wedge \omega_{u_t}^{k-1} \wedge \chi^{n-k}}{\omega_{u_t}^k \wedge \chi^{n-k}}, \quad \text{for } \psi \in C^2(M).
\]
The operator $L_{u_t}$ is elliptic and then the index is zero. By the maximum principle,

$$\text{Ker}(L_{u_t}) = \{\text{constant}\}.$$  \hfill (6.1)

Denote $L_{u_t}^*$ by the $L^2$-adjoint operator of $L_{u_t}$ with respect to the volume form

$$d\text{vol}_k = \omega_{u_t}^k \wedge \chi^{n-k}.$$  

By the Fredholm alternative, there is a non-negative function $\xi$ such that

$$\text{Ker}(L_{u_t}^*) = \text{Span}\{\xi\}.$$  \hfill (6.2)

It follows from the strong maximum principle that $\xi > 0$. Up to a constant, we may and do assume

$$\int_M \xi \, d\text{vol}_k = 1.$$  

To prove the openness of $S$, we define the space by

$$U^{2,\alpha} := \{\phi \in C^{2,\alpha}(M) : \omega_{\phi} \in \Gamma_k(\chi) \text{ and } \int_M \phi \cdot \xi \, d\text{vol}_k = 0\}.$$  

Then the tangent space of $U^{2,\alpha}$ at $u_t$ is given by

$$T_{u_t}U^{2,\alpha} := \{\psi \in C^{2,\alpha}(M) : \int_M \psi \cdot \xi \, d\text{vol}_k = 0\}.$$  

Let us consider the map

$$G(\phi, c) = \log \sigma_k(\omega_{\phi}) - c,$$

which maps $U^{2,\alpha} \times \mathbb{R}$ to $C^\alpha(M)$. It is clear that the linearized operator of $G$ at $(u_t, \hat{t})$ is given by

$$(L_{u_t} - c) : T_{u_t}U^{2,\alpha} \times \mathbb{R} \longrightarrow C^\alpha(M).$$  \hfill (6.3)

On the one hand, for any $h \in C^\alpha(M)$, there exists a constant $c$ such that

$$\int_M (h + c) \cdot \xi \, d\text{vol}_k = 0.$$  

By (6.2) and the Fredholm alternative, we can find a real function $\psi$ on $M$ such that

$$L_{u_t}(\psi) = h + c.$$  

Hence, the map $L_{u_t} - c$ is surjective. On the other hand, suppose that there are two pairs $(\psi_1, c_1), (\psi_2, c_2) \in T_{u_t}U^{2,\alpha} \times \mathbb{R}$ such that

$$L_{u_t}(\psi_1) - c_1 = L_{u_t}(\psi_2) - c_2.$$  

It then follows that

$$L_{u_t}(\psi_1 - \psi_2) = c_1 - c_2.$$  

Applying the maximum principle twice, we obtain $c_1 = c_2$ and $\psi_1 = \psi_2$. Then $L_{u_t} - c$ is injective.

Now we conclude that $L_{u_t} - c$ is bijective. By the inverse function theorem, when $t$ is close to $\hat{t}$, there exists a pair $(u_t, c_t) \in U^{2,\alpha} \times \mathbb{R}$ satisfying

$$G(u_t, c_t) = th + (1 - t)h_0.$$  

@Springer
The standard elliptic theory shows that $u_t \in C^\infty(M)$. Then $S$ is open. □

6.2 The Monge–Ampère equation for $(n - 1)$ plurisubharmonic functions

As in [28, pp. 371–372], let $T$ be the linear map given by

$$T(\mu) = (T(\mu)_1, \ldots, T(\mu)_n), \quad T(\mu)_k = \frac{1}{n-1} \sum_{i \neq k} \mu_i, \quad \text{for } \mu \in \mathbb{R}^n.$$ 

Define

$$f = \log \sigma_n(T), \quad \Gamma = T^{-1}(\Gamma_n).$$

It is straightforward to verify that the above setting satisfies the assumptions (i)-(iii) in Sect. 1.

Write

$$\omega = (\text{tr}_\chi \eta) \chi - (n - 1) \eta,$$

and so the equation in (1.4) can be written as

$$F(\omega_n) = h + c.$$ 

Then Theorem 1.3 can be proved by the similar argument of Theorem 1.2.

Date availability There is no associated data in this paper.

References

1. Andrews, B.: Contraction of convex hypersurfaces in Euclidean space. Calc. Var. Partial Differ. Equ. 2(2), 151–171 (1994)
2. Błocki, Z.: On uniform estimate in Calabi–Yau theorem. Sci. China Ser. A Suppl. 48, 244–247 (2005)
3. Błocki, Z.: On the uniform estimate in the Calabi–Yau theorem, II. Sci. China Math. 54(7), 1375–1377 (2011)
4. Caffarelli, L., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. Acta Math. 155(3–4), 261–301 (1985)
5. Calabi, E.: On Kähler Manifolds with Vanishing Canonical Class. Algebraic Geometry and Topology. A Symposium in Honor of S. Lefschetz, pp. 78–89. Princeton University Press, Princeton (1957)
6. Cherrier, P.: Équations de Monge-Ampère sur les variétés Hermitiennes compactes. Bull. Sci. Math. 2(111), 343–385 (1987)
7. Chu, J., McCleerey, N.: Fully non-linear degenerate elliptic equations in complex geometry. J. Funct. Anal. 281(9), Paper No. 109176 (2021)
8. Chu, J., Tosatti, V., Weinkove, B.: The Monge–Ampère equation for non-integrable almost complex structures. J. Eur. Math. Soc. (JEMS) 21(7), 1949–1984 (2019)
9. Dinew, S., Kołodziej, S.: Liouville and Calabi-Yau type theorems for complex Hessian equations. Am. J. Math. 139(2), 403–415 (2017)
10. Ecker, K., Huisken, G.: Immersed hypersurfaces with constant Weingarten curvature. Math. Ann. 283(2), 329–332 (1999)
11. Fang, H., Lai, M., Ma, X.-N.: On a class of fully nonlinear flows in Kähler geometry. J. Reine Angew. Math. 653, 189–220 (2011)
12. Fu, J., Wang, Z., Wu, D.: Form-type Calabi–Yau equations. Math. Res. Lett. 17(5), 887–903 (2010)
13. Fu, J., Wang, Z., Wu, D.: Form-type Calabi–Yau equations on Kähler manifolds of nonnegative orthogonal bisectional curvature. Calc. Var. Partial Differ. Equ. 52(1–2), 327–344 (2015)
14. Gerhardt, C.: Closed Weingarten hypersurfaces in Riemannian manifolds. J. Differ. Geom. 43(3), 612–641 (1996)
15. Güan, B.: Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. Duke Math. J. 163(8), 1491–1524 (2014)
16. Guan, B., Li, Q.: Complex Monge–Ampère equations and totally real submanifolds. Adv. Math. 225(3), 1185–1223 (2010)
17. Guan, B., Sun, W.: On a class of fully nonlinear elliptic equations on Hermitian manifolds. Calc. Var. Partial Differ. Equ. 54(1), 901–916 (2015)
18. Hanani, A.: Équations du type de Monge-Ampère sur les variétés hermitiennes compactes. J. Funct. Anal. 137(1), 49–75 (1996)
19. Harvey, F.R., Lawson, H.B.: Potential theory on almost complex manifolds. Ann. Inst. Fourier (Grenoble) 65(1), 171–210 (2015)
20. Hou, Z.: Complex Hessian equation on Kähler manifold. Int. Math. Res. Not. IMRN 16, 3098–3111 (2009)
21. Hou, Z., Ma, X.-N., Wu, D.: A second order estimate for complex Hessian equations on a compact Kähler manifold. Math. Res. Lett. 17(3), 547–561 (2010)
22. Li, Y.: A priori estimates for Donaldson’s equation over compact Hermitian manifolds. Calc. Var. Partial Differ. Equ. 50(3–4), 867–882 (2014)
23. Song, J., Weinkove, B.: On the convergence and singularities of the $J$-flow with applications to the Mabuchi energy. Commun. Pure Appl. Math. 61(2), 210–229 (2008)
24. Spruck, J.: Geometric Aspects of the Theory of Fully Nonlinear Elliptic Equations, Global Theory of Minimal Surfaces, pp. 283–309. American Mathematical Society, Providence (2005)
25. Sun, W.: On a class of fully nonlinear elliptic equations on closed Hermitian manifolds. J. Geom. Anal. 26(3), 2459–2473 (2016)
26. Sun, W.: On a class of fully nonlinear elliptic equations on closed Hermitian manifolds II: $L^\infty$ estimate. Commun. Pure Appl. Math. 70(1), 172–199 (2017)
27. Sun, W.: On uniform estimate of complex elliptic equations on closed Hermitian manifolds. Commun. Pure Appl. Anal. 16(5), 1553–1570 (2017)
28. Székelyhidi, G.: Fully non-linear elliptic equations on compact Hermitian manifolds. J. Differ. Geom. 109(2), 337–378 (2018)
29. Tosatti, V., Wang, Y., Weinkove, B., Yang, X.: $C^{2,\alpha}$ estimates for nonlinear elliptic equations in complex and almost complex geometry. Calc. Var. Partial Differ. Equ. 54(1), 431–453 (2015)
30. Tosatti, V., Weinkove, B.: Estimates for the complex Monge–Ampère equation on Hermitian and balanced manifolds. Asian J. Math. 14(1), 19–40 (2010)
31. Tosatti, V., Weinkove, B.: The complex Monge–Ampère equation on compact Hermitian manifolds. J. Am. Math. Soc. 23(4), 1187–1195 (2010)
32. Tosatti, V., Weinkove, B.: The Monge–Ampère equation for $(n-1)$-plurisubharmonic functions on a compact Kähler manifold. J. Am. Math. Soc. 30(2), 311–346 (2017)
33. Tosatti, V., Weinkove, B.: Hermitian metrics, $(n-1, n-1)$ forms and Monge–Ampère equations. J. Reine Angew. Math. 755, 67–101 (2019)
34. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I. Commun. Pure Appl. Math. 31(3), 339–411 (1978)
35. Zhang, D.: Hessian equations on closed Hermitian manifolds. Pac. J. Math. 291(2), 485–510 (2017)
36. Zhang, J.: Monge–Ampère type equations on almost Hermitian manifolds. Preprint arXiv:2101.00380
37. Zhang, X., Zhang, X.: Regularity estimates of solutions to complex Monge–Ampère equations on Hermitian manifolds. J. Funct. Anal. 260(7), 2004–2026 (2011)