Magnetoexciton Superfluidity in Graphene-Dielectric-Graphene Structures

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Abstract. Superfluid state of a magnetoexciton gas in bilayers is studied with reference to graphene-dielectric-graphene structures subjected by a perpendicular to graphene layers magnetic field \( B \). We find that in difference with quantum Hall bilayers with the total filling factor \( \nu_T = 1 \), an imbalance of filling factors of graphene layers is required. An imbalance can be created by an electrostatic field \( E \) applied perpendicular to graphene layers. We determine the range of \( B \) and \( E \) where magnetoexciton superfluidity can be realized. The dependence of critical temperature and critical current on magnetic field is computed. It is found that the maximum critical temperature is reached at \( B \sim 0.5\phi_0/\pi d^2 \), where \( \phi_0 \) is the magnetic flux quantum, and \( d \) is the interlayer distance. It is shown that the interaction of electrons with impurities reduces the critical temperature. The critical concentration of impurities is determined. Stationary waves in a superfluid magnetoexciton gas are considered. The waves are induced by counterpropagating electrical currents that flow in a bilayer with a point obstacle. It is found that the stationary wave pattern is modified qualitatively under variation of \( B \).

1. Introduction
Considerable attention to the Bose-Einstein condensation and superfluidity of excitons is motivated in an essential part by expectations that the critical temperature for such systems is larger than for other known superfluids (except nuclear matter). Quantum Hall bilayers (QHB) with the total filling factor \( \nu_T = 1 \) are considered as promising systems for a realization of exciton superfluidity [1, 2]. In QHB excitons are formed by electrons and holes (empty states in the lowest Landau level) that belong to different layers. Spatially indirect excitons can provide counterflow superconductivity - a flow of equal in modulus and oppositely directed electrical supercurrents in the adjacent layers. QHB are one of main objects for the experimental study of exciton superfluidity. Temperature dependence of the counterflow conductivity in QHB has been measured in a number of independent experiments [3, 4, 5], and an exponential increase of the conductivity under lowering of temperature was observed.

To demonstrate magnetoexciton superfluidity QHB should satisfy two conditions: \( d \lesssim \ell \), and \( \ell \lesssim a_B^* \), where \( d \) is the interlayer distance, \( \ell = \sqrt{\hbar c/eB} \) is the magnetic length, \( a_B^* = \varepsilon h^2/m^*e^2 \) is the effective Bohr radius, \( \varepsilon \) is the dielectric constant of the matrix, and \( m^* \) is the effective electron mass. The first one is the stability condition for the state with electron-hole pairing. The second one is the condition for the Coulomb energy \( \varepsilon^2/\varepsilon \ell \) to be smaller than the energy distance between Landau levels. In the form given above the latter condition is applied to the systems with nonzero effective mass of carriers. For GaAs heterostructures \( a_B^* \approx 10 \) nm and the condition
The order parameter for the component \( E \) Landau levels in graphene are given by the expression

\[
\beta
\]

Hamiltonian, phase diagram and collective mode spectrum

structure. We also determine in what range of parameters the maximum critical temperature of allowed interlayer distances and dielectric constants depends on a geometry of the bilayer we investigate how the critical parameters for the magnetoexciton superfluidity as well as the range of allowed interlayer distances and dielectric constants depends on a geometry of the bilayer structure. We also determine in what range of parameters the maximum critical temperature can be achieved and evaluate the effect of its reduction caused by impurities.

2. Hamiltonian, phase diagram and collective mode spectrum

Landau levels in graphene are given by the expression 

\[
E_{\pm N} = \frac{h v_F}{\ell} \sqrt{2|N|},
\]

where \( v_F \approx 10^6 \text{ m/s} \) is the Fermi velocity. There is an additional four-fold spin-valley degeneracy of Landau levels. We consider electron-hole pairing in zero Landau level in graphene. The Hamiltonian of Coulomb interaction in zero Landau level approximation reads as

\[
H_C = \frac{1}{2S} \sum_{i,i'} \sum_{X,X',\beta,\beta'} \sum_{q} V_{ii'}(q) e^{-i q (X'-X)} c^+_{i\beta X} e^{i q_2 c^+_{i'\beta' X'}} e^{-i q_1 c_{i'\beta' X'}}, \quad (1)
\]

where \( V_{ii'}(q) \) is the Fourier-component of the Coulomb interaction, \( c^+_{i\beta X} \), \( c_{i\beta X} \) are the creation and annihilation electron operators, \( i, i' = 1, 2 \) is the layer index, \( \beta = 1, 2, 3, 4 \) is the spin-valley component index, \( X \) is the guiding center coordinate, and \( S \) is the area of the system. Let the system is subjected by an electrostatic field \( E \) directed perpendicular to the layers and this fields creates an interlayer external gate voltage \( V_g \). The Hamiltonian of interaction with the external field is

\[
H_g = -\frac{e V_g}{2} \sum_{X,\beta} \left( \epsilon_{1\beta X}^\dagger \epsilon_{1\beta X}^+ - \epsilon_{2\beta X}^\dagger \epsilon_{2\beta X}^+ \right).
\]

We consider the probe wave function that describes a state with electron-hole pairing in each component (an analog of the BCS wave function, see, for instance, [13])

\[
|\Psi\rangle = \prod_X \prod_\beta \left( \cos \frac{\theta_\beta}{2} \epsilon_{1\beta X}^+ + e^{i \phi_\beta} \sin \frac{\theta_\beta}{2} \epsilon_{2\beta X}^+ \right) |0\rangle, \quad (2)
\]

The order parameter for the component \( \beta \) is \( \Delta_\beta = e^{i \phi_\beta} \sin \frac{\theta_\beta}{2} \). The energy in the state (2) is

\[
E_{mf} = \frac{S}{8 \pi \ell^2} \left( W \sum_{\beta,\beta'} \cos \theta_\beta \cos \theta_{\beta'} - \left( \frac{J_{11} + J_{22}}{2} - J_{12} \right) \sum_{\beta} \cos^2 \theta_\beta - (2 e V_g + J_z) \sum_{\beta} \cos \theta_\beta \right), \quad (3)
\]

where \( W = e^2 d / \ell^2 \) is the energy of the direct Coulomb interaction (\( \varepsilon \) is the dielectric constant for the dielectric that separates two graphene layers), \( J_{1k} = \frac{1}{2 \pi} \int_0^\infty q V_{ik}(q) e^{-q \varepsilon^2 / 2} dq \) are the exchange energies, and \( J_z = J_{11} - J_{22} \).

One finds that at \( V_g = 0 \) the minimum of (3) is reached at \( \theta_1 = \theta_2 = 0, \theta_3 = \theta_4 = \pi \). It indicates the absence of electron-hole pairing in balanced bilayer graphene systems. The situation is similar to ones that takes place in \( \nu_T = 2 \) quantum Hall bilayers [14].

If \( V_g \) belongs to one of the intervals \( nW + J_{22} - J_{12} < e V_g < (n+2)W - J_{11} + J_{12} \), where \( n = -4, -2, 0, 2 \), the energy minimum is reached at \( \theta_\beta = \theta_\alpha \neq 0, \pi \) for one of the components.
We will call such a component the active one. The maximum order parameter (and maximum critical temperature) corresponds to \( \theta_a = \pi/2 \). In this paper we concentrate the main attention on the \( \theta_a = \pi/2 \) case. It is realized at \( eV_g = -\frac{\ell}{2} \pm W \) and \( eV_g = -\frac{\ell}{2} \pm 3W \). To keep \( \theta_a = \pi/2 \) the gate voltage should be varied synchronically with \( B \).

A component that belongs completely to one layer does not take part in the pairing. In what follows we consider the dynamics of only active component. We take into account small fluctuations of the amplitude and the phase of the order parameter and specify a state with nonzero uniform supercurrent

\[
|\Psi\rangle = \prod_X \left( \cos \frac{\theta_X}{2} e^{i1,X+Q_y\epsilon^2/2} + e^{i(Q_x X + \varphi X)} \sin \frac{\theta_X}{2} e^{i2,X-Q_y\epsilon^2/2} \right) |0\rangle. \tag{4}
\]

One can show that the vector \( Q \) has the sense of the gradient of the phase of the order parameter.

The energy in the state (4) can be decomposed as \( E = E_0 + E_2 + \ldots \). The zero-order in fluctuations term is equal to \( E_0 = \frac{s}{8\pi^2} \left( [W - F_S(|q|)] \cos^2 \theta_a - F_D(Q) \sin^2 \theta_a \right) \), where \( F_S(q) = \frac{1}{4\pi} \int_0^\infty pJ_0(pV^2)[V_{11}(p) + V_{22}(p)]e^{-\epsilon^2}\frac{p^2}{2} dp \) and \( F_D(q) = \frac{1}{2\pi} \int_0^\infty pJ_0(pV^2)V_{12}(p)e^{-\epsilon^2}\frac{p^2}{2} dp \). The quadratic in fluctuations term can be presented in the diagonal form

\[
E_2 = \sum_q [m_z(-q)K_{zz}(q)m_z(q) + \frac{1}{4}\varphi(-q)K_{\varphi\varphi}(q)\varphi(q) - \frac{1}{2}(im_z(-q)K_{z\varphi}(q)\varphi(q) + c.c.)]. \tag{5}
\]

where \( m_z(q) = \frac{1}{2}\sqrt{2\pi^2} \sum_X (\cos \theta_X - \cos \theta_a) e^{-i\theta_X}, \varphi(q) = \sqrt{\frac{2\pi^2}{s}} \sum_X \varphi(X) e^{-i\theta_X}. \)

Eqs. (5) yields the energy of fluctuations with the wave vector, directed along the \( x \) axis. We present the components of the matrix \( K \) in form independent of the choice of the direction of the coordinate axes: \( K_{zz}(q, Q) = H(q, Q) - F_S(|q|) + F_D(|Q|) + \Xi(q, Q) \cot^2 \theta_a, \ K_{\varphi\varphi}(q, Q) = \sin^2 \theta_a \Xi(q, Q), \ K_{z\varphi}(q, Q) = -\cos \theta_a \left[ F_D(q + Q) - F_D(|q - Q|) \right] /2, \) where \( H(q, Q) = \frac{1}{2\pi^2} \left[ V_{11}(q + \frac{Q}{2}) + V_{12}(q + \frac{Q}{2}) \cos \left( |q \times Q|/2 \right) \right] e^{-\frac{q^2}{2\ell^2}} \) and \( \Xi(q, Q) = \left[ F_D(|Q|) - F_D(|q + Q|) + F_D(|q - Q|) \right] /2. \) In (5) \( K_{\alpha\beta}(q) = K_{\alpha\beta}(q, Q) \big|_{q=0}. \)

The quantities \( m_z(q) \) and \( \varphi(q) \) are the conjugated variables. They satisfy the canonical equations of motion. The collective mode spectrum is \( \Omega(q, Q) = \sqrt{K_{\varphi\varphi}(q, Q)K_{zz}(q, Q) + K_{z\varphi}(q, Q)}. \tag{6} \)

We analyze three heterostructures: a free standing graphene-dielectric-graphene sandwich (A-type), a sandwich on a dielectric substrate (B-type) - a structure used in [12], and a sandwich embedded in a dielectric matrix (C-type) (Fig. 1). We consider the same dielectric constants \( \varepsilon \) for the substrate, for the dielectric matrix, and for the dielectric that separates graphene layers.

Let us first consider the case \( Q = 0 \). The condition for the dynamical stability of the state (4) is \( \text{Im}(\Omega(q, 0)) = 0 \) at all \( q \). This condition yields the restriction on the interlayer distance: \( 0 < d/\ell < d_c(\varepsilon) \). The dependence \( d_c(\varepsilon) \) at zero imbalance of the active component (\( \theta_a = \pi/2 \)) is shown in Fig. 2. One can see that in systems with one and two open graphene layers the critical interlayer distance increases under increase of the dielectric constant. The largest \( d_c \) is for the A-type structure. Our approach is applied if the Coulomb energy is smaller than the distance between Landau levels. Since we study the pairing in \( N = 0 \) Landau level we should compare the Coulomb energy with the energy distance between \( N = 0 \) and \( N = 1 \) levels \( \omega_F = \sqrt{2}h v_F/\ell \). We have four parameters that characterize the Coulomb energy: \( W, J_{11}, J_{22} \) and \( J_{12} \). At \( d/\ell = d < d_c \) the largest of them is \( J_{11} \) (the intralayer exchange interaction in
the open layer). Therefore, it is natural to consider the condition \( J_{11} < \omega_c \) as an additional restriction on the parameters of bilayer graphene systems. This inequality can be rewritten as 
\[ \varepsilon > \varepsilon_c(d) \]
where \( \varepsilon_c \) is the critical dielectric constant shown in Fig. 2.

\[ \delta E \]

\[ \text{Figure 1. A-, B- and C-type heterostructures with two graphene layers.} \]

\[ \text{Figure 2. Critical interlayer distance (solid lines) and critical dielectric constant (dashed lines).} \]

3. Critical current and critical temperature

In bilayer graphene heterostructures magnetoexciton superfluidity can be realized in a wide range of magnetic field. The variation of \( B \) at fixed gate voltage results in a change of the imbalance of the active component. Simultaneous tuning of \( V_g \) allows to keep zero imbalance and maximum order parameter under variation of \( B \). Below we find the dependence of critical parameter on \( B \) assuming such a tuning.

Electrical current in the layers is given by the variation of the energy caused by the variation of the vector-potential \( \delta E = -\frac{1}{2} \int d^2r \sum_i j_i \delta A_i \), where \( A_i \) is the in-plane component of the vector-potential in the layer \( i \). The vector potential enters into the gauge-invariant phase of the order parameter in the following way \( \varphi_X = QX - \frac{e}{\hbar c} (A_{x1} - A_{x2}) X \) (we specify the case \( \mathbf{Q} \parallel \mathbf{x} \)). From these relations one finds the densities of the currents \( j_1(Q) = -j_2(Q) = -\frac{e}{\hbar} \frac{\sin^2 \theta_a}{\Omega(Q)} \frac{dF_D(Q)}{dQ} \).

Under increase of \( Q \) the spectrum becomes negative at some \( q \) (the Landau instability) and, then, complex valued (the dynamical instability). The Landau instability regime can be realized only at nonzero imbalance of the active component (\( \theta_a \neq \pi/2 \)). At zero imbalance the system switches directly to the dynamical instability regime at \( Q = Q_c \) [15]. The critical phase gradient \( Q_c \) determines the critical current \( j_c = |j_1(Q_c)| \) at \( \theta_a = \pi/2 \). In Fig. 3 we present the dependence of the critical current on magnetic field. One can see that critical current is larger for the structures with open graphene layers. Critical current goes to zero at \( B \to 0 \) and \( B \to B_c \). The critical magnetic field is determined by the relation \( B_c = \frac{d^2c}{\hbar c ed^2} \) (it is the field for which \( d/\ell = d_c \)). The maximum critical current is reached at \( B \approx 0.5 B_c \).

Superfluid transition temperature is given by the Kostelits-Thouless equation \( T_c = \frac{\pi}{2} \rho_s(T_c) \), where \( \rho_s(T) \) is the superfluid stiffness at finite temperature. The superfluid stiffness is the coefficient in the expansion of the free energy in phase gradient \( F = F_0 + \int d^2r \rho_s(\nabla \varphi)^2/2 \). The second derivative of the energy \( E_0 \) with respect to \( Q \) yields the superfluid stiffness for the magnetoexciton gas at \( T = 0 \): \( \rho_s(0) = \frac{2 \sin^2 \theta_a}{32 \pi^2} \int_0^\infty p^3 V_{12}(p) e^{-\frac{p^2}{2}} dp \). Taking into account linear excitations we present the free energy in the form \( F = E_0 + T \sum_q \ln \left( 1 - e^{-\frac{\Omega(q, Q)}{T}} \right) \). It yields the following expression for the superfluid stiffness

\[ \rho_s(T) = \rho_s(0) + \frac{1}{S} \sum_q \left( \frac{d^2 \Omega(q, Q)}{dQ^2} \right)_{Q=0} N_q - \frac{1}{T} N_q \left( 1 + N_q \right) \left( \frac{d \Omega(q, Q)}{dQ} \right)_{Q=0}^2, \]

(7)
where \( N_q = 1/(e^{\Omega/T} - 1) \) is the Bose distribution function. Note that for \( \theta_a = \pi/2 \) the second term in the sum is equal to zero and only the first term describes the effect of reduction of the superfluid stiffness.

The dependence of the critical temperature on the magnetic field is shown in Fig. 4. One can see that for all the structures considered the maximum critical temperature is reached approximately for the same magnetic field \( B \approx 0.5B_d \), where \( B_d = \hbar c/ed^2 \).

4. The effect of electron-impurity interaction

The Hamiltonian for the interaction of the active component with the impurities has the form

\[
H_{\text{imp}} = \frac{1}{S} \sum_q \sum_i \frac{U_i(q)}{S} \sum_X \left[ c^+_i X a q + \frac{\Delta}{2} c_i X a \right] \exp \left( -i q X - \frac{q^2 a^2}{2} \right),
\]

where \( U_i(q) \) is the Fourier-component of the impurity potential. The interaction (8) induces spatial fluctuations of the density and the phase of the order parameter. Their values can be obtained from the Euler-Lagrange equations \( \delta E / \delta \dot{x} = 0 \), \( \delta E / \delta \phi = 0 \), where \( E \) is the energy of the system with the Hamiltonian \( H = H_C + H_q + H_{\text{imp}} \) in the state (4). These equations can be solved in linear in impurity potential approximation. At \( \theta_a = \pi/2 \) we obtain \( m_X = \frac{1}{S} \sum_q e^{i q X} \frac{U_i(q)c_q}{K_{\text{zz}}(q)} - \frac{2a^2}{\ell^2} \) (where \( U_z = U_1 - U_2 \)), and \( \phi_X = 0 \).

We specify the case of neutral impurities located in graphene layers. The Fourier-component of the impurity potential can be presented in the form \( U_i(q) = \sum_{x,a} e^{i q x} v_{x,a} u_0 \), where \( x,a \) are the impurity coordinates. Computing the dependence of the current on \( Q \) at \( Q \ell \ll 1 \) one finds \( j_1 = -j_2 = \frac{e}{2\pi \rho_0(1 - \langle m_X^2 \rangle)} Q \). The coefficient \( \rho_0(1 - \langle m_X^2 \rangle) \) can be understood as the superfluid stiffness renormalized due to electron-impurity interaction. Summing the contribution of the fluctuations with all wave vectors one obtains the following expression for the renormalized superfluid density \( \rho_{s_{\text{imp}}} = \rho_0 \left( 1 - \frac{2n_{\text{imp}} a^2}{S} \sum_q \frac{e^{i q X}}{K_{\text{zz}}(q)} \right) \), where \( n_{\text{imp}} \) is the impurity concentration in the layer. Renormalization of the superfluid stiffness results in a lowering of the critical temperature

\[
\frac{\Delta T_c}{T_c} = -\frac{2n_{\text{imp}} a^2}{S} \sum_q \frac{e^{i q X}}{K_{\text{zz}}(q,0)} = f(B)n_{\text{imp}} a^2 \left( \frac{e^2}{4\pi^2} \right),
\]

where \( f(B) \) is the numerical factor that depends on \( B/B_d \). For \( B \approx 0.5B_d \) this factor is evaluated as \( f(B) \approx 10 \). We define the critical impurity density as density at which \( |\Delta T_c/T_c| = 1 \). Evaluating \( u_0 \) by the expression \( u_0 = c^2 a \) (where \( a \) is the carbon-carbon distance in graphene) we obtain for \( B = 0.5B_d \) and \( \varepsilon = 4 \) the following critical concentration \( n_{\text{imp}}^c \approx 10^{13} \text{ cm}^{-2} \).
5. **Stationary waves in the magnetoexciton gas in graphene bilayers**

In the Landau instability regime a point obstacle embedded in a superfluid system induces stationary waves. Recently, the observation of such waves was used as an independent test for an exciton-polariton superfluidity [16]. The same phenomenon takes place in the magnetoexciton gas [15]. The conditions for the observation of stationary waves are the following. 1. An imbalance of the active component ($\theta_a \neq \pi/2$) is required. 2. The system should not be in the dynamical instability regime ($\text{Im}(\Omega(q,Q)) = 0$ for all $q$). The latter condition means that the range of currents at which the phenomenon can be observed is restricted not only from below, but from above as well. We find that at all $B < B_c$ stationary waves are excited outside the Mach cone, in similarity with the case of weakly nonideal Bose gas [17]. The specifics is that under increase of $B$ another family of stationary waves emerges inside the Mach cone, and at $B \to B_c$ cusps appear at the wave crests. Typical wave crests patterns are shown in Fig. 5.

![Wave crests patterns](image-url)

**Figure 5.** Stationary wave crests for the B-type structure at $\theta_a = \pi/3$ for $B = 0.2B_c$ (a), $B = 0.42B_c$ (b), and $B = 0.6B_c$ (c). Dashed curves correspond to crests located outside the Mach cone, solid curves - the crests located inside the cone. $\varepsilon = 4$. The spatial scale is $1 = \ell$.

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