Research Article

Stability Analysis of a Dynamical Model for Malware Propagation with Generic Nonlinear Countermeasure and Infection Probabilities

Xulong Zhang and Xiaoxia Song

School of Computer and Network Engineering, Shanxi Datong University, Datong 037009, China

Correspondence should be addressed to Xulong Zhang; zxl-095@163.com

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The dissemination of countermeasures is widely recognized as one of the most effective strategies of inhibiting malware propagation, and the study of general countermeasure and infection has an important and practical significance. On this point, a dynamical model incorporating generic nonlinear countermeasure and infection probabilities is proposed. Theoretical analysis shows that the model has a unique equilibrium which is globally asymptotically stable. Accordingly, a real network based on the model assumptions is constructed, and some numerical simulations are conducted on it. Simulations not only illustrate theoretical results but also demonstrate the reasonability of general countermeasure and infection.

1. Introduction and Model Formulation

Human society has been subjected to great financial losses since malware constantly emerged (e.g., [1, 2]). The study of modeling and understanding malware spreading has attracted a lot of attention in the past three decades or so, and a multitude of propagation models capturing the behaviors of malware have been proposed. Specifically, SIS (susceptible-infected-susceptible) models (e.g., [3, 4]), SIRS (susceptible-infected-recovered-susceptible) models (e.g., [5, 6]), SLBS (susceptible-latent-breaking-susceptible) models (e.g., [7, 8]), SICS (susceptible-infected-countermeasured-susceptible) models (e.g., [9–11]), and SDIRS (susceptible-delitescent-infected-recovered-susceptible) model (e.g., [12]).

In the field of malware, countermeasures such as software patches or warnings can supply a valid approach to helping individuals and organizations avert malware infection problems (e.g., [13, 14]). In 2004, the CMC (Countermeasure Competing) strategy is proposed by Chen and Carley [15]. Their results reveal that the CMC strategy is more effective than previous strategies by the empirical malware data.

Inspired by this work and in order to macroscopically describe the mixing transmission of malware and countermeasures, Zhu et al. [9] presented a compartment model. The dynamics of the model was performed. Later, Yang and Yang [10] simply extended this model by incorporating the impacts of infected removable storage media and external nodes (e.g., computers). However, these two models both neglect two important facts. On the one hand, they ignore the fact that the linear infection probability is a well fit for the real-world situations only when the infected nodes are few. On the other hand, they overlook the fact that countermeasures may propagate through networks at different rates. Thus, the assumptions of linear infection and countermeasure probabilities are unreasonable.

To remedy these flaws and considering the impacts of general countermeasure and infection on the spread of malware, this paper studies a new dynamical model (see Figure 1), which incorporates generic countermeasure and infection probabilities. Here, $S(t), I(t),$ and $C(t)$ ($S, I,$ and $C,$ for short) denote the average numbers of susceptible, infected, and countermeasured internal nodes (i.e., nodes on the network) at time $t,$ respectively. Their entering rates are
\( \mu_1 > 0, \mu_2 > 0, \) and \( \mu_3 > 0 \), respectively. Besides, the following basic hypotheses of the model are made:

(H1) Each internal node leaves the network with probability \( \delta > 0 \).

(H2) At time \( t \), each susceptible internal node gets infected by infected internal nodes with probability \( \beta(I(t)) \), where \( \beta \) is twice continuously differentiable, \( \beta' > 0, \beta'' < 0, \) and \( \beta(0) = 0 \). The concavity hypothesis fits well with the saturation property of the infection probability.

(H3) At time \( t \), each infected or susceptible internal node obtains the newest countermeasure with probability \( \gamma_1(C(t)) \), where \( \gamma_1 \) is twice continuously differentiable, \( \gamma_1' > 0, \gamma_1'' < 0, \) and \( \gamma_1(0) = 0 \).

(H4) By reinstalling the operating system, each infected (or countermeasured) internal node becomes susceptible with probability \( \gamma_2 > 0 \) (or \( \alpha > 0 \)).

Combining the above hypotheses, the new proposed model can be represented by the following system:

\[
\begin{align*}
\frac{dS}{dt} &= \mu_1 - \beta(I)S - \gamma_1(C)S + \gamma_2 I + \alpha C - \delta S, \\
\frac{dI}{dt} &= \mu_2 + \beta(I)S - \gamma_1(C)I - \gamma_2 I - \delta I, \\
\frac{dC}{dt} &= \mu_3 + \gamma_1(C)S + \gamma_1(C)I - \alpha C - \delta C,
\end{align*}
\]

with initial condition \((S(0), I(0), C(0)) \in \mathbb{R}_+^3\).

The globally asymptotic stability of the unique (viral) equilibrium of model (1) is proved and illustrated completely. Additionally, a new network is constructed based on the above assumptions, on which some numerical simulations are examined.

The paper is organized in this fashion. Section 2 determines the (viral) equilibrium and investigates its local and global stabilities. Experimental analysis is presented in Section 3. Finally, some conclusions and outlooks are given in Section 4.

2. Model Analysis

Let \( N = S + I + C, \) and \( \mu = \mu_1 + \mu_2 + \mu_3 \). Adding up the three equations of system (1), one can easily obtain that \( \lim_{t \to \infty} N = \mu / \delta \). It follows by the asymptotically autonomous system theory [16] that system (1) is equivalent to the following reduced limiting system:

\[
\begin{align*}
\frac{dI}{dt} &= \mu_2 + \beta(I)\left(\frac{\mu}{\delta} - I - C\right) - \gamma_1(C)I - \gamma_2 I - \delta I, \\
\frac{dC}{dt} &= \mu_3 + \gamma_1(C)\left(\frac{\mu}{\delta} - C\right) - (\alpha + \delta)C,
\end{align*}
\]  

(2)

with initial condition \((I(0),C(0)) \in \Omega, \) where

\[ \Omega = \left\{(I,C) \in \mathbb{R}_+^2 : I + C \leq \frac{\mu}{\delta} \right\}, \]

and \( \Omega \) is positively invariant for system (2).

In the following sections, we just need to investigate the dynamical behavior of system (2).

2.1. Equilibrium

**Theorem 1.** System (2) has a unique (viral) equilibrium \( E^* = (I^*, C^*) \), where \( E^* \) is the unique positive solution to the following system:

\[
\begin{align*}
\mu_2 + \beta(x)\left(\frac{\mu}{\delta} - x - y\right) - \gamma_1(y)x - \gamma_2 x - \delta x &= 0, \\
\mu_3 + \gamma_1(y)\left(\frac{\mu}{\delta} - y\right) - (\alpha + \delta)y &= 0,
\end{align*}
\]

(4)

with the initial condition \((x(0), y(0)) \in \Omega. \)

**Proof.** Let us assume that \( E^* = (I^*, C^*) \) is an equilibrium of system (2). Clearly, \( E^* \) satisfies system (4).

Firstly, let us prove that the second equation of system (4) has a unique positive root. Let

\[ f(y) = \mu_3 + \gamma_1(y)\left(\frac{\mu}{\delta} - y\right) - (\alpha + \delta)y, \quad y \in \left[0, \frac{\mu}{\delta}\right]. \]

(5)
As \( f (0) = \mu_3 > 0 \) and \( f (\mu/\delta) = -\alpha \mu/\delta - (\mu_1 + \mu_2) < 0 \), it follows that \( f \) has a zero located in the interval \((0, \mu/\delta)\). Furthermore, note that

\[
f'\left(\frac{\mu}{\delta}\right) = -\gamma_1\left(\frac{\mu}{\delta}\right) - (\alpha + \delta) < 0, \quad f''(y) = \gamma_1''(y)\left(\frac{\mu}{\delta} - y\right) - 2\gamma_1'(y) < 0.
\]

We shall proceed by distinguishing two possibilities depending on whether \( f'(0) \) is positive or negative.

**Case 1:** \( f'(0) > 0 \). Let

\[
g(x) = \mu_2 + \beta(x)\left(\frac{\mu}{\delta} - C^* - x\right) - (\gamma_2 + \delta + \gamma_1(C^*))x, \quad x \in \left[0, \frac{\mu}{\delta} - C^*\right].
\]

As \( g(0) = \mu_2 > 0 \) and \( g(\mu/\delta - C^*) = -\mu_1 - \alpha C^* - \gamma_2(\mu/\delta - C^*) < 0 \), \( g \) does have a (positive) zero located in the interval \((0, \mu/\delta - C^*)\). Besides, notice that

\[
g'\left(\frac{\mu}{\delta} - C^*\right) = -\beta\left(\frac{\mu}{\delta} - C^*\right) - (\gamma + \delta + \gamma_1(C^*)) < 0,
\]

\[
g''(x) = \beta''(x)\left(\frac{\mu}{\delta} - C^* - x\right) - 2\beta'(x) < 0.
\]

We shall also proceed by distinguishing two possibilities depending on whether \( g'(0) \) is positive or negative.

**Case 1:** \( g'(0) > 0 \). Let

\[
\bar{x} = \max\left\{ x \in \left[0, \frac{\mu}{\delta} - C^*\right] : g'(x) > 0 \right\}.
\]

and its two eigenvalues are

\[
\lambda_1 = g'(I^*) - \gamma_1(C^*) - \gamma_2 - \delta = g'(I^*) < 0,
\]

\[
\lambda_2 = \gamma_1(C^*)\left(\frac{\mu}{\delta} - C^*\right) - \gamma_1(C^*) - (\alpha + \delta) = f'(C^*) < 0.
\]

Thus, the claimed result follows from the Lyapunov stability theorem [17].

**2.3. Global Stability**

**Lemma 1.** System (2) admits no periodic orbit.

**Theorem 2.** \( E^* \) is locally asymptotically stable with respect to \( \Omega \).

**Proof.** Let \( S^* = \mu/\delta - I^* - C^* \). The corresponding Jacobian matrix of system (2) at \( E^* \) is given as follows:

\[
\begin{pmatrix}
\beta'(I^*)S^* - \beta(I^*) - \gamma_1(C^*) - \gamma_2 - \delta \\
0
\end{pmatrix}
\]

and its two eigenvalues are

\[
\lambda_1 = \beta'(I^*)S^* - \beta(I^*) - \gamma_1(C^*) - \gamma_2 - \delta = g'(I^*) < 0,
\]

\[
\lambda_2 = \gamma_1(C^*)\left(\frac{\mu}{\delta} - C^*\right) - \gamma_1(C^*) - (\alpha + \delta) = f'(C^*) < 0.
\]

Thus, the claimed result follows from the Lyapunov stability theorem [17].

In the interior of \( \Omega \), it is easily obtained that
Let 
\[ k_1(x) = \beta'(x)x - \beta(x). \] 
As \( k_1(0) = 0 \) and \( k_1'(x) = \beta''(x)x < 0 \) for all \( x > 0 \), \( k_1(I) < 0 \).

Let 
\[ k_2(x) = y_1(x)x - y_2(x). \] 
As \( k_2(0) = 0 \) and \( k_2'(x) = y''(x)x < 0 \) for all \( x > 0 \), \( k_2(C) < 0 \). Thus, we have \( \partial(Dh_1)/\partial I + \partial(Dh_2)/\partial C < 0 \).

Hence, it follows from the Bendixson–Dulac criterion [17] that system (2) admits no periodic orbit in the interior of \( \Omega \).

On the boundary of \( \Omega \), let \( (\bar{I}, \bar{C}) \) denote an arbitrary point. Thus, three possibilities can be considered.

Case 1: \( 0 < \bar{C} < \mu \delta, \bar{I} = 0 \). Then, \( dI/dt|_{(\bar{I}, \bar{C})} = \mu_I > 0 \).

Case 2: \( 0 < \bar{I} < \mu \delta, \bar{C} = 0 \). Then, \( dC/dt|_{(\bar{I}, \bar{C})} = \mu_C > 0 \).

Case 3: \( \bar{I} + \bar{C} = \mu \delta, \bar{C} \neq 0, \bar{I} \neq 0 \). Thus,

\[ \frac{d(I + C)}{dt}|_{(\bar{I}, \bar{C})} = -\mu_I - y_2I - a\bar{C} < 0. \] 

Hence, system (2) has no periodic orbit across \( (\bar{I}, \bar{C}) \). In conclusion, the claimed result is proved.

By Theorems 1 and 2, Lemma 1, and the generalized Poincaré–Bendixson theorem [17], we can easily obtain the main result of this paper as follows.

\[ s_i(k + 1) = \begin{cases} 
\text{susceptible} & \text{with probability } 1 - \beta(I_i(k)) - y_1(C_i(k)), \\
\text{infected} & \text{with probability } \beta(I_i(k)), \\
\text{countermeasured} & \text{with probability } y_1(C_i(k)). 
\end{cases} \] 

\[ s_i(k + 1) = \begin{cases} 
\text{susceptible} & \text{with probability } \gamma_2, \\
\text{infected} & \text{with probability } 1 - y_2 - y_1(C_i(k)), \\
\text{countermeasured} & \text{with probability } y_1(C_i(k)). 
\end{cases} \] 

\[ s_i(k + 1) = \begin{cases} 
\text{susceptible} & \text{with probability } \alpha, \\
\text{countermeasured} & \text{with probability } 1 - \alpha. 
\end{cases} \]
Example 1. Consider system (1) with $\mu_1 = 11.01$, $\mu_2 = 7.02$, $\mu_3 = 5.11$, $\gamma_2 = 0.021$, $\alpha = 0.012$, $\delta = 0.048$, $\beta(I) = 0.115^{0.552}$, and $\gamma_1(C) = 0.015^{0.498}$. Three initial conditions are $(S(0), I(0), C(0)) = (325, 55, 10)$, $(S(0), I(0), C(0)) = (225, 200, 50)$, and $(S(0), I(0), C(0)) = (80, 135, 15)$, respectively. Figure 4 shows that the results of theoretical prediction quite agree with the experimental ones.

Example 2. Consider three sets of parameters for system (1): (a) $\mu_1 = 3.51$, $\mu_2 = 1.52$, $\mu_3 = 1.03$, $\gamma_2 = 0.0201$, $\alpha = 0.112$, $\delta = 0.012$, $\beta(I) = 0.0423^{0.355}$, and $\gamma_1(C) = 0.029^{0.153}$, (b) $\mu_1 = 11.01$, $\mu_2 = 7.02$, $\mu_3 = 5.11$, $\gamma_2 = 0.021$, $\alpha = 0.012$, $\delta = 0.048$, $\beta(I) = 0.115^{0.552}$, and $\gamma_1(C) = 0.015^{0.498}$; (c) $\mu_1 = 9.2$, $\mu_2 = 4.5$, $\mu_3 = 2.9$, $\gamma_2 = 0.0263$, $\alpha = 0.0153$, $\delta = 0.0171$, $\beta(I) = 0.0981^{0.472}$, and $\gamma_1(C) = 0.0456^{0.398}$. The common initial condition is $(S(0), I(0), C(0)) = (325, 55, 10)$. Figure 5 reveals that the results of experimental prediction quite agree with the theoretical ones.

Example 3. Consider two systems induced by system (1) with $\mu_1 = 11.01$, $\mu_2 = 7.02$, $\mu_3 = 5.11$, $\gamma_2 = 0.021$, $\alpha = 0.012$, and $\delta = 0.048$, where one system is with $\beta(I) = 0.115^{0.552}$ and $\gamma_1(C) = 0.015^{0.498}$ and the other with $\beta(I) = 0.115^{0.552}$ and $\gamma_1(C) = 0.015^{0.498}$. The common initial condition is $(S(0), I(0), C(0)) = (325, 55, 10)$. Figure 6 demonstrates that the new model with nonlinear infection and countermeasured probabilities is more reasonable than the original model [9] because malware would be always there and would not go extinct.
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