ON TOPOLOGICAL GROUPS CONTAINING
A FRÉCHET-URYSOHN FAN

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ABSTRACT. Suppose $G$ is a topological group containing a (closed) topological copy of the Fréchet-Urysohn fan. If $G$ is a perfectly normal sequential space (a normal $k$-space) then every closed metrizable subset in $G$ is locally compact. Applying this result to topological groups whose underlying topological space can be written as a direct limit of a sequence of closed metrizable subsets, we get that every such a group either is metrizable or is homeomorphic to the product of a $k_\omega$-space and a discrete space.

The present investigation was stimulated by the paper [Pe] of E. Pentsak who studied the topology of the direct limit $X^\infty = \lim_\leftarrow X^n$ of the sequence

$$X \subset X \times X \subset X \times X \times X \subset \ldots,$$

where $(X, \ast)$ was a “nice” pointed space and $X^n$ was identified with the subspace $X^n \times \{\ast\}$ of $X^{n+1}$. In particular, in [Pe] the topology of the direct limit $l_2^\infty$, where $l_2$ is the separable Hilbert space, was characterized. To characterize the space $l_2^\infty$, it was necessary to glue together maps into $l_2^\infty$ and at this point it turned out that the equiconnected function generated by the natural convex structure on $l_2^\infty$ was discontinuous. The same concerned the addition operation on $l_2^\infty$ — it was discontinuous.

So question arose: is $l_2^\infty$ homeomorphic to a topological group or a convex set in a linear topological space?

We pose this problem more generally: find simple conditions on a topological space $X$ under which $X$ does not support certain algebraic structure.

In order to answer this question, we will define spaces $K$, $V$, and $W$, called test spaces, and will prove that an existence in $X$ (closed) subspaces homeomorphic to one (or several) of the spaces $K$, $V$, $W$ forbids $X$ to carry certain algebraic structures.

Now we define two of three test spaces.

1) The space $K$. Let

$$K = \{(0,0)\} \cup \{(\frac{1}{n}, \frac{1}{n-m}) \mid n, m \in \mathbb{N}\} \subset \mathbb{R}^2.$$
The space $K$ is metrizable and not locally compact. Moreover, $K$ is a minimal space with these properties in the sense that each metrizable non-locally compact space contains a closed copy of $K$. In sake of simplicity of denotations in the sequel, put $x_0 = (0, 0)$ and $x_{n,m} = (\frac{1}{n}, \frac{1}{nm})$, $n, m \in \mathbb{N}$. Thus $K = \{ x_0, x_{n,m} \mid n, m \in \mathbb{N} \}$.

2) The space $V$ (the Fréchet-Urysohn fan). Let $S_0 = \{ 0 \} \cup \{ \frac{1}{n} \mid n \in \mathbb{N} \}$ denote the convergent sequence and let

$$V = \mathbb{N} \times S_0 / \mathbb{N} \times \{ 0 \}.$$ 

Denote by $\pi_V : \mathbb{N} \times S_0 \to V$ the quotient map. Let $y_{n,m} = \pi_V(n, \frac{1}{m})$, $n, m \in \mathbb{N}$, and $y_0$ be the (unique) non-isolated point of $V$. So $V = \{ y_0, y_{n,m} \mid n, m \in \mathbb{N} \}$. Evidently, for every $n \in \mathbb{N}$ the sequence $\{ y_{n,m} \}_{m=1}^{\infty}$ converges to $y_0$. For each $k \in \mathbb{N}$ let $V_k = \{ y_0, y_{n,m} \mid n \leq k, m \in \mathbb{N} \}$. It is easy to see that $V$ has the direct limit topology with respect to the sequence $V_1 \subset V_2 \subset \ldots$ (that is a set $U \subset V$ is open if and only if the intersection $U \cap V_n$ is open in $V_n$ for every $n \in \mathbb{N}$).

A space $X$ contains a closed copy of $V$, provided $X$ can be written as a direct limit of a sequence

$$X_1 \subset X_2 \subset \ldots,$$

where each $X_n$ is a closed metrizable subset of $X$, nowhere dense in $X_{n+1}$. In particular, the space $l_2^{\infty}$ contains a topological copy of $V$.

We call a subset $A$ of a topological group $G$ multiplicative if for every $a, b \in A$ we have $a \ast b \in A$ (here $\ast$ stands for the group operation on $G$). The following theorem implies that $l_2^{\infty}$ carries no topological group structure.

**Theorem 1.** A normal $k$-space $X$ containing closed copies of the test spaces $K$ and $V$ is homeomorphic to a) no closed multiplicative subset of a topological group and b) no closed convex set in a linear topological space.

**Proof.** Assume the converse and let $f : K \times V \to X$ be the map defined for $(x, y) \in K \times V \subset X \times X$ by $f(x, y) = x \ast y$ if $X$ is a closed multiplicative subset of a topological group with $\ast$ standing for the group operation, or by $f(x, y) = \frac{1}{2} x + \frac{1}{2} y$ if $X$ is a closed convex set in a linear topological space. It is easily verified that the map $f$ has the following properties:

1) the map $(pr_K, f) : K \times V \to K \times X$ is a closed embedding;

2) the map $f_{y_0} : K \to X$ defined by $f_{y_0} : x \mapsto f(x, y_0)$, $x \in K$, is a closed embedding.

Denote by $\text{conv}(K) = \{ (0, 0) \} \cup \{ (x, y) \mid 0 < y \leq x \leq 1 \}$ the convex hull of $K$ in $\mathbb{R}^2$ and let $h : X \to \text{conv}(K)$ be a continuous extension of the map $f_{y_0}^{-1} : f_{y_0}(K) \to K$, see [Hu, p.63].

For $n, m \in \mathbb{N}$ let $\varepsilon_{n,m} = \frac{1}{2nm(m+1)}$ and set

$$O_{n,m} = \text{conv}(K) \cap \left( (\frac{1}{n} - \varepsilon_{n,m}, \frac{1}{n} + \varepsilon_{n,m}) \times (\frac{1}{nm} - \varepsilon_{n,m}, \frac{1}{nm} + \varepsilon_{n,m}) \right).$$

One can check that $O_{n,m}, n, m \in \mathbb{N}$, is a collection of pairwise disjoint neighborhoods of the points $x_{n,m}$ in $\text{conv}(K)$. Since $\lim_{m \to \infty} y_{n,m} = y_0$ and $h \circ f(x_{n,m}, y_0) = x_{n,m} = (\frac{1}{n}, \frac{1}{nm})$, for every $n, m \in \mathbb{N}$, we may find a number $k(n, m) \in \mathbb{N}$ such that
Let \( \tilde{y} \) be any extension of the map \( f \) to \( V \). Since \( f \) is an embedding, we claim that \( Z = \{ f(x_{n,m}, y_{n,k(m)}) \mid n, m \in \mathbb{N} \} \). Therefore \( Z \not\ni f(x_0, y_0) \) is a closed set in \( X \). Using continuity of \( f \), find neighborhoods \( U(x_0) \subset K \) and \( V(y_0) \subset V \) of \( x_0 \) and \( y_0 \) such that \( f(U(x_0) \times V(y_0)) \cap Z = \emptyset \). Fix \( n \) such that \( x_{n,m} \in U(x_0) \) for every \( m \). Since the sequence \( \{ y_{n,m} \}_{m=1}^{\infty} \) converges to \( y_0 \) and the sequence \( \{ k(n,m) \}_{m=1}^{\infty} \) is increasing we may find \( m \) such that \( y_{n,k(m,m)} \in V(y_0) \). Then \( f(U(x_0) \times V(y_0)) \cap Z \ni f(x_{n,m}, y_{n,k(m,m)}) \) is not empty, a contradiction. \( \square \)

In light of Theorem 1 the following Question arises.

**Question.** Is \( l_2^\infty \) homeomorphic a) to a multiplicative subset of a topological group, b) to a convex set in a linear topological space?

We will give a negative answer to the question a) under the additional assumption that the multiplicative subset contains an idempotent (the unity of the group). This follows from topological homogeneity of \( l_2^\infty \) and the following theorem.

**Theorem 2.** A perfectly normal sequential space \( X \) containing a closed copy of \( K \) and a copy of \( V \) is homeomorphic to a) no closed convex set in a linear topological space, b) no closed multiplicative subset of a topological group, c) no multiplicative subset of a topological group such that the nonisolated point \( y_0 \) of \( V \subset X \) is an idempotent.

**Proof.** Assume the converse and similarly as in the previous proof define the map \( f : K \times V \to X \). Observe that the map \( f \) has the following properties:

1') the map \( (pr_k, f) : K \to V \to K \times X \) is an embedding;

2') the map \( f_{y_0} : K \to X \) is a closed embedding (in the case (c) \( f_{y_0} \) is the identity embedding because \( y_0 \) is the unity of the group).

Let \( h : X \to \text{conv}(K) \) be a continuous extension of the map \( f_{y_0}^{-1} : f_{y_0}(K) \to K \) such that \( h^{-1}(x_0) = f(x_0, y_0) \). (The map \( h \) can be constructed as follows. Using the perfect normality of \( X \), fix a map \( \lambda : X \to [0, 1] \) such that \( \lambda^{-1}(0) = f_{y_0}(K) \). Let \( h : X \to \text{conv}(K) \) be any extension of the map \( f_{y_0}^{-1} : f_{y_0}(K) \to K \) and define a map \( h : X \to \text{conv}(K) \) letting \( h(x) = \lambda(x) \cdot x_{1,1} + (1 - \lambda(x))h(x) \) for \( x \in K \).
Similarly as in the previous proof define neighborhoods \( O_{n,m} \) and the set \( Z \not
 f(x_0, y_0) \). As in the proof of Theorem 1, to get a contradiction, it suffices to show that the set \( Z \) is closed in \( X \). Since the space \( X \) is sequential, it is enough to verify that for every convergent sequence \( S \subset X \) the intersection \( S \cap Z \) is closed in \( S \). We shall show that \( S \cap Z \) is always finite. Assume on the contrary, \( S \cap Z \) is infinite.

If \( f(x_0, y_0) \) is not a limit point of \( S \) then \( S \cap Z \) is finite because the collection \( \{h^{-1}(O_{n,m})\}_{n,m \in \mathbb{N}} \) is discrete in \( X \setminus f(x_0, y_0) \) and \( S \setminus f(x_0, y_0) \) is compact. So assume \( f(x_0, y_0) \) is a limit point of \( S \). Enumerate \( S \cap Z = \{z_i\}_{i=1}^\infty \). Evidently, the sequence \( \{z_i\}_{i=1}^\infty \) converges to \( f(x_0, y_0) \). For every \( i \in \mathbb{N} \) find (unique) \( n_i, m_i \) such that \( z_i = f(x_{n_i,m_i}, y_{n_i,k(n_i,m_i)}) \). Observe that the sequence \( \{x_{n_i,m_i}\}_{i=1}^\infty \) converges to \( x_0 \). Then the sequence \( \{x_{n_i,m_i}, z_i\}_{i=1}^\infty \) converges to \( (x_0, f(x_0, y_0)) \) and lies (together with its limit) in \( (\operatorname{pr}_K, f)(K, V) \). Since \( (\operatorname{pr}_K, f) \) is an embedding and the projection \( \operatorname{pr}_V : K \times V \to V \) is continuous, we get the set

\[
C_2 = \{y_0\} \cup \{y_{n_i,k(n_i,m_i)} \mid i \in \mathbb{N} \} = \operatorname{pr}_V \circ (\operatorname{pr}_K, f)^{-1}(\{(x_0, f(x_0, y_0))\} \cup \{(x_{n_i,m_i}, z_i) \mid i \in \mathbb{N}\}) \subset V
\]

is compact. Then \( C_2 \subset V_{n_0} \) for some \( n_0 \) and thus the sequence \( \{n_i\} \) is bounded, a contradiction with the convergence of the sequence \( \{x_{n_i,m_i}\} \).

Therefore both Theorems 1 and 2 give us that \( l_2^\infty \) is homeomorphic to no topological group. And what about its powers \( (l_2^\infty)^n \)? Do they admit a compatible group structure? It turns out that the answer here is negative too. Observe that Theorems 1 or 2 are not applicable because the powers of \( l_2^\infty \) are not \( k \)-spaces. So we must think out something new.

3. The test space \( W \). We let \( W \) be the direct limit of a sequence \( W_0 \subset W_1 \subset \ldots \), where the spaces \( W_n \subset K \times V \) are defined as follows. In \( K \times V \) let us consider the points: \( z_0 = (x_0, y_0) \), \( z_{n,m} = (x_{n,m}, y_0) \), and \( z_{n,m,p,q} = (x_{n,m}, y_{p,q}) \), \( n, m, p, q \in \mathbb{N} \). Let \( W_0 = \{z_0, z_{n,m} \mid n, m \in \mathbb{N} \} \) and \( W_p = W_{p-1} \cup \{z_{n,m,p,q} \mid n, m, p, q \in \mathbb{N} \} \) for \( p \geq 1 \). It is easy to see that for every \( p \geq 1 \) \( W_p \) is a closed subspace of \( K \times V \) and \( W_0 \) is a nowhere dense closed copy of \( K \) in \( W_0 \cup (W_p \setminus W_{p-1}) \). On the union \( W = \bigcup_{p=0}^\infty W_p \) consider the topology of the direct limit of \( W_p \), allowing a subset \( U \subset W \) to be open if and only if \( U \cap W_p \) is open in \( W_p \) for every \( p \). Observe that a space \( X \) contains a closed copy of \( W \), provided \( X \) can be written as the direct limit of a sequence \( X_0 \subset X_1 \subset \ldots \), where each \( X_n \) is a closed metrizable subset of \( X \), \( X_n \) is nowhere dense in \( X_{n+1} \), and \( X_0 \) is not locally compact. Since the space \( l_2^\infty \) admits such a representation, it contains a copy of \( W \).

Remark that each direct limit \( X \) of a sequence of metrizable spaces satisfies the following property:

\((\mathcal{M})\) for every map \( f : Y \to X \) of a metrizable space \( Y \), every point \( y \in Y \) has a neighborhood \( U \subset Y \) such that \( f(U) \) admits a countable neighborhood base at \( f(y) \).

Observe that a finite product of spaces with the property \((\mathcal{M})\) enjoys this property too. Since the space \( l_2^\infty \) has the property \((\mathcal{M})\) and contains a copy of the test space \( W \), the following theorem implies that for every \( 1 \leq n \leq \omega \) the power \((l_2^\infty)^n\) does not admit a compatible group operation.
Theorem 3. A topological group containing a copy of the test space $W$ can not be embedded into a countable product of spaces satisfying the property $(\mathcal{M})$.

Proof. Suppose $W \subset X \subset \prod_{n=1}^{\infty} X_n$, where each $X_n$ has the property $(\mathcal{M})$. Suppose $X$ is a topological group and denote by $\pi$ the group operation and by $e$ the unity of $X$. For each $k \in \mathbb{N}$ let $Y_k = \prod_{i=1}^{k} X_i$ and denote by $\pi_k : X \to Y_k$ the projection onto the first $k$-coordinates.

By induction on $k$ we shall construct increasing number sequences $\{n(k)\}_{k=1}^{\infty}$, $\{q(k,m)\}_{m=1}^{\infty}$, $k \in \mathbb{N}$, such that for every $k$ the sequence

$$\{\pi_k(z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)})\}_{m=1}^{\infty}$$

converges in $Y_k$.

Let $n(0) = 0$ and suppose that for $k - 1$, the number $n(k-1)$ is known. Define a map $f_k : W_0 \times W_k \to Y_k$ letting $f_k(x,y) = \pi_k(x^{-1} * y)$ for $(x,y) \in W_0 \times W_k$. Since the space $W_0 \times W_k$ is metrizable and the space $Y_k$, being a finite product of the spaces $X_i$’s, has the property $(\mathcal{M})$, the point $(z_0, z_0)$ has a neighborhood $U_1 \times U_2 \subset W_0 \times W_k$ such that $f_k(U_1 \times U_2)$ has a countable neighborhood base $\{O_m\}_{m=1}^{\infty}$ at $f_k(z_0, z_0) = \pi_k(e)$. Pick $n(k) > n(k-1)$ so that $z_{n(k),m} \in U_1$ for every $m$. Since for every $m$ the sequence $\{z_{n(k),m,k,q}\}_{q=1}^{\infty}$ converges to $z_{n(k),m}$, we get the sequence $\{\pi_k(z_{n(k),m}^{-1} * z_{n(k),m,k,q})\}_{q=1}^{\infty}$ converges to $\pi_k(e)$. Thus, inductively, for every $m$ we can find a number $q(k,m) > q(k,m-1)$ such that $\pi_k(z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)}) \in O_m$. The inductive step is complete.

Consider the set

$$Z = \{z_{n(k),m,k,q(k,m)} \mid k, m \in \mathbb{N}\} \subset W$$

and notice that $Z$ is closed in $W$. Since $z_0 \notin Z$, we may find neighborhoods $U(z_0), U(e) \subset X$ of $z_0$ and $e$ such that $(U(z_0) * U(e)) \cap Z = \emptyset$. Let $k$ be such that $U(e) \supset \pi_k^{-1}(O)$ for some neighborhood $O \subset Y_k$ of $\pi_k(e)$. We may assume the number $k$ to be so great that $z_{n(k),m} \in U(z_0)$ for every $m$. Find finally $m$ such that

$$\pi_k(z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)}) \in O.$$

Then $z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)} \in U(e)$ and hence the intersection $(U(z_0) * U(e)) \cap Z \ni z_{n(k),m}^{-1} * (z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)}) = z_{n(k),m,k,q(k,m)}$ is not empty, a contradiction. $\square$

Now let us consider some applications of the obtained results.

**Structure of topological groups that are $\mathcal{M}_\omega$-spaces**

Recall that a topological space $X$ is called a $k_\omega$-space if $X$ contains a countable collection $\mathcal{K}$ of compact subsets of $X$ such that a subset $U$ of $X$ is open in $X$ if and only if the intersection $U \cap K$ is closed in $K$ for every $K \in \mathcal{K}$ (equivalently, $X$ is a $k_\omega$-space, provided $X$ is the direct limit of a sequence of its compact subsets).
We define a topological space $X$ to be an $\mathcal{M}_\omega$-space if $X$ contains a countable collection $\mathcal{M}$ of closed metrizable subsets of $X$ such that a subset $U$ of $X$ is open in $X$ if and only if the intersection $U \cap M$ is closed in $M$ for every $M \in \mathcal{M}$ (equivalently, $X$ is an $\mathcal{M}_\omega$-space, if $X$ is the direct limit of a sequence of its closed metrizable subsets).

It turns out that an existence of a compatible group structure imposes very strict restrictions on the topology of $\mathcal{M}_\omega$-spaces.

**Theorem 4.** Suppose a topological group $X$ is an $\mathcal{M}_\omega$-space. If $X$ is not metrizable, then

1. $X$ contains a closed copy of the Fréchet-Urysohn fan;
2. each closed metrizable subset of $X$ is locally compact;
3. $X$ contains an open subgroup $H$ that is a $k_\omega$-space;
4. $X$ is homeomorphic to a product of a $k_\omega$-space and a discrete space;
5. $X$ is homeomorphic to an open subset of a $k_\omega$-space.

**Proof.** Suppose $X$ is not metrizable and let $e$ denote the unity of the group $X$. Write $X = \lim X_n$ be the direct limit of a sequence $\{e\} = X_0 \subset X_1 \subset X_2 \subset \ldots$ consisting of closed metrizable subsets of $X$. To prove 1) we will show that for every $n$ there is $m$ such that $e$ is a limit point of the set $X_m \setminus X_n$ in $X_m$. Fix $n$ and a decreasing neighborhood base $\{U_i\}_{i=1}^\infty$ of $e$ in $X_n$. Since $X$ is not metrizable, each $U_i$ is not open in $X$, and thus $U_i$ is not open in some $X_{m(i)}$. Consequently, there is a sequence $\{y_{ij}\}_{j=1}^\infty \subset X_{m(i)} \setminus X_n$ convergent to a point $x_i \in U_i$. Let $k(i) = \min\{k \in \mathbb{N} \mid \forall j_0 \in \mathbb{N} \exists j \geq j_0$ such that $y_{ij} \in X_k\}$. Passing to a subsequence, if necessary, we may assume that $\{y_{ij}\}_{j=1}^\infty \subset X_{k(i)} \setminus X_{k(i)-1}$.

If $m = \sup\{k(i) \mid i \in \mathbb{N}\} < \infty$ then all the points $y_{ij}$, $i, j \in \mathbb{N}$ lie in the set $X_m \setminus X_n$. Since $X_m$ is metrizable and the sequence $\{x_i\}$ tends to $e$, we may choose a subsequence $\{z_j\}_{j=1}^\infty \subset \{y_{ij} \mid i, j \in \mathbb{N}\}$ convergent to $e$. Thus $e$ is a limit point of the set $X_m \setminus X_n$ and we are done.

Now suppose $\sup\{k(i) \mid i \in \mathbb{N}\} = \infty$. Using the continuity of the multiplication $\ast$, find $p \in \mathbb{N}$ such that $U_p \ast U_p \subset X_k$ for some $k$. Let $i$ be such that $k(i) > k$ and $i \geq p$. Obviously, the sequence $\{x_i^{-1} \ast y_{ij}\}_{j=1}^\infty$ converges to $e$. We claim that there exists $j_0 \in \mathbb{N}$ such that $x_i^{-1} \ast y_{ij} \notin X_n$ for all $j \geq j_0$. Assuming the converse we would find $j$ such that $x_i^{-1} \ast y_{ij} \in U_p \subset X_n$. Then $y_{ij} \in x_i \ast U_p \subset U_p \ast U_p \subset X_k$, a contradiction with $k(i) > k$ and $y_{ij} \in X_{k(i)} \setminus X_{k(i)-1}$. Thus we have proven that $\{x_i^{-1} \ast y_{ij}\}_{j \geq j_0} \subset X \setminus X_n$ for some $j_0$. Since this sequence converges to $e$, it is contained in some $X_m$.

Now we are ready to construct a closed copy of $V$ in $X$. Applying the statement proved above, we may construct inductively an increasing number sequence $\{m(i)\}_{i=1}^\infty$ and sequences $\{y_{ij}\}_{j=1}^\infty; i \in \mathbb{N}$, such that

$$\lim_{j \to \infty} y_{ij} = e; y_{ij} \in X_{m(i)} \setminus X_{m(i-1)}, j \in \mathbb{N}.$$

Evidently, the set $\{e\} \cup \{y_{ij} \mid i, j \in \mathbb{N}\}$ is a closed copy of $V$ in $X$. Hence a) is proven.

To prove 2), notice that $X$, being a direct limit of a sequence of metrizable spaces, is a perfectly normal sequential space. Since $X$ contains a copy of the test
space $V$, by Theorems 1 and 2, $X$ contains no closed copy of the test space $K$. Because every metrizable non-locally compact space contains a closed copy of $K$, every closed metrizable subset in $X$ must be locally compact.

To prove 3), let us firstly construct an open separable subset $U \subset X$. By 2), each $X_n$ is locally compact. Thus, we may choose inductively open neighborhoods $U_n$ of $e$ in $X_n$ so that the closure $\bar{U}_n$ is compact and $\bar{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$. Since each $\bar{U}_n$ is a (separable) metric compactum, the union $U = \bigcup_{n \in \mathbb{N}} U_n$ is an open separable subset of $X$. Then its span $H = \text{span}(U)$ is an open separable subgroup of $X$. Using the separability of $H$, one can easily show that for every $n$ the locally compact space $H_n = H \cap X_n$ is separable, and thus $H_n$ is a $k_\omega$-space. Then $H = \lim_{\leftarrow} H_n$ is a $k_\omega$-space too. Since the subgroup $H$ is open in $X$ the decomposition of $X$ onto right residue classes of $H$ just provides us with a homeomorphism of $X$ onto the product $H \times D$ for some discrete space $D$. Let $\alpha D$ be the one-point compactification. Evidently, $H \times D$ is an open subset of the $k_\omega$-space $H \times \alpha D$, thus (5) follows. □

References

[Pe] E. Pentsak, *On manifolds modeled on direct limits of $C$-universal ANR’s*, Matematychni Studii 5, 107–116.