ANALYTIC LIGHTCURVES FOR PLANETARY TRANSIT SEARCHES
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ABSTRACT

We present exact analytic formulae for the eclipse of a star described by quadratic or nonlinear limb darkening. In the limit that the planet radius is less than a tenth of the stellar radius, we show that the exact lightcurve can be well approximated by assuming the region of the star blocked by the planet has constant surface brightness. We apply these results to the HST observations of HD 209458, showing that the ratio of the planetary to stellar radii is 0.1207 ± 0.0003. These formulæ give a fast and accurate means of computing lightcurves using limb-darkening coefficients from model atmospheres which should aid in the detection, simulation, and parameter fitting of planetary transits.

Subject headings: eclipses — occultations — stars: binaries: eclipsing — stars: planetary systems

1. INTRODUCTION

The eclipse of the star HD 209458 by an orbiting planet was recently used to measure the size and mass of the planet, which had been found with velocity measurements (Charbonneau et al. 2000; Henry et al. 2000). With this landmark discovery, the planetary transit tool was added to the planet-finder’s toolbox, already yielding several planetary candidates (Udalski et al. 2002a; 2002b). Lightcurve fits to transit events may be used to characterize the planet and star, yielding important constraints on planet formation (Cody & Sasselov 2002; Hubbard et al. 2001; Seager & Mallen-Ornélas 2002). The data require an accurate description of limb-darkening as demonstrated by Hubble Space Telescope observations of HD 209458 of such high quality that a quadratic limb-darkening law was needed to fit the transit lightcurve rather than the usual linear limb-darkening law (Brown et al. 2001). The limb-darkening of main-sequence stars is represented by functions of \(\mu = \cos \theta\), where \(\theta\) is the angle between the normal to the stellar surface and the line of sight to the observer (Figure 1a). Claret (2000) has found that the most accurate limb-darkening functions are the quadratic law in \(\mu\) and the “nonlinear” law which is a Taylor series to fourth order in \(\mu^{1/2}\); the latter conserves flux to better than 0.05%.

In this paper, we compute analytic functions for transit lightcurves for the quadratic and nonlinear limb-darkening laws, and make available our codes for the community (§ 7). For treatment of subtler effects during planetary transits see Seager et al. (2000); Seager & Sasselov (2000); Hubbard et al. (2001); Hui & Seager (2002). In section 2 we review the lightcurve of a uniform spherical source. In section 3 we derive the lightcurve for eclipses of nonlinear limb-darkened stars. In section 4 we give a simpler form in the limit of a quadratic limb-darkening law. In section 5 we give an approximation for the lightcurve in the case \(p \lesssim 0.1\) which is very fast to compute and is fairly accurate. In section 6 we apply the results to some example cases, and in section 7 we conclude.

2. UNIFORM SOURCE

We model the transit as an eclipse of a spherical star by an opaque, dark sphere. In what follows, \(d\) is the center-to-center distance between the star and the planet, \(r_p\) is the radius of the planet, \(r_s\) is the stellar radius, \(z = d/r_s\) is the normalized separation of the centers, and \(p = r_p/r_s\) is the size ratio (Figure 1b). The flux relative to the unobsured flux is \(F\).

For a uniform source, the ratio of obscured to unobscured flux is \(F^*(p,z) = 1 - \lambda^*(p,z)\) where

\[
\lambda^*(p,z) = \begin{cases} 
0 & 1 + p < z \\
\frac{1}{\pi} \left[ \frac{p^2 \kappa_0 + \kappa_1 - \sqrt{4p^2 - (1 + z^2 - p^2)^2}}{4} \right] & |1 - p| < z \leq 1 + p \\
\frac{p^2}{1} & z \leq 1 - p \\
\frac{1}{z - p - 1} & z \geq 1 - p,
\end{cases}
\]

and \(\kappa_1 = \cos^{-1}[(1 - p^2 + z^2)/2z], \quad \kappa_0 = \cos^{-1}[(p^2 + z^2 - 1)/2pz]\).

We next consider the effects of limb-darkening.

3. NON-LINEAR LIMB DARKENING

Limb darkening causes a star to be more centrally peaked in brightness compared to a uniform source. This leads to more significant dimming during eclipse and creates curvature in the trough. Thus, including limb-darkening is important for computing accurate eclipse lightcurves. Claret (2000) proposed a nonlinear limb-darkening law that fits well a wide range of stellar models and observational bands, \(I(r) = 1 - \sum_{n=1}^{4} c_n (1 - \mu^{n/2})\), where \(\mu = \cos \theta = \sqrt{1 - r^2}\), \(0 \leq r \leq 1\) is the normalized radial coordinate on the disk of the star and \(I(r)\) is the specific intensity as a function of \(r\) or \(\mu\) with \(I(0) = 1\). Figure 1(a) shows the geometry of lensing and the definition of \(\mu\). The lightcurve in the limb-darkened case is given by

\[
F(p,z) = \int_0^1 dr [2rI(r)]^{-1} \int_0^1 dr d \left[ \frac{F^*(p/r,z/r)}{dr} \right],
\]

where \(F^*(p,z)\) is the lightcurve of a uniform source defined in the previous section.

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In Case VII the edge of the planet’s disk touches the stellar center, but does not cover the stellar center. We define
\[ N = \frac{(1-a)^{(n+4)/4}}{(b-a)^{-1/2}} \left[ \frac{\pi}{\frac{\pi}{12}} \right] \left( a \right) F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, \frac{n+10}{a}, -\frac{1}{b-a} \right) \]
and
\[ L = p^2 \left( \frac{1-p^2}{2}-\zeta^2 \right). \]
Then the relative flux is given by
\[ F = 1 - (4\Omega)^{-1} \left[ C_0 p^2 + \frac{3}{4} \sum_{m=0}^{n} M c_m (n+4)^{-1} + c_1 L \right]. \]
This case requires the planet to be less than half of the size of the star. For Case V the edge of the planet touches the center of the stellar disk, and the planet lies entirely within the stellar disk. The relative flux is
\[ F_V = \frac{1}{2} + (2\Omega)^{-1} \sum_{m=0}^{n+4} c_m (n+4)^{-1} \left( \frac{1}{2}, -\frac{3}{4}, 1, \frac{1}{2}, \frac{n+10}{a}, -\frac{1}{b-a} \right) F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, \frac{n+10}{a}, -\frac{1}{b-a} \right) \]
and
\[ F = 1 + \frac{1}{2} \sqrt{\frac{\pi}{\Omega}} \sum_{m=0}^{n+4} \frac{c_m}{n+4} \gamma \left( \frac{3}{2} + \frac{n}{2} \right) / \gamma (2+\frac{n}{2}). \]
In Case VII the edge of the planet’s disk touches the stellar center, but the planet is not entirely contained inside the area of the stellar disk. The relative flux is
\[ F = \frac{1}{2} + \frac{1}{4} \pi \Omega \sum_{m=0}^{n+4} \frac{c_m}{n+4} B \left( \frac{1}{2}, \frac{n+10}{a}, \frac{1}{2}, \frac{n+10}{a} \right) \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]
and
\[ F = \frac{1}{2} + \frac{1}{4} \pi \Omega \sum_{m=0}^{n+4} \frac{c_m}{n+4} B \left( \frac{1}{2}, \frac{n+10}{a}, \frac{1}{2}, \frac{n+10}{a} \right) \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

\[ \eta_b = \left( \frac{2}{b} \right)^{-1} \left[ \kappa_1 + 2n_2 n_0 - \frac{1}{2} \left( 1 + 5p^2 + \zeta^2 \right) \sqrt{\left( 1-a \right) \left( b-1 \right)} \right]. \]
where \(k = \sqrt{(1-a)/(4xp)}\) and \(q = p^2 - z^2\). Here \(\Pi(n,k)\) is the complete elliptic integral of the third kind with the sign convention of Gradshteyn and Ryzhik (1994). For linear limb-darkening, \(\gamma_2 = 0\), Merrill (1950) presents an equivalent analytic expression in terms of an “eclipse function,” \(\alpha\). The expressions here require fewer evaluations of the elliptic integrals, which decreases computation time, and include quadratic limb darkening. Our expression for eclipse with quadratic limb-darkening decreases computation time by more than an order of magnitude compared to evaluating the expressions in the previous section or numerical integration of the unocculted flux.

5. SMALL PLANETS

For a small planet, \(p \ll 0.1\), the interior of the lightcurve, \(z < 1-p\), can be approximated by assuming the surface-brightness of the star is constant under the disk of the planet, so that \(F = 1 - \frac{I^*(z)}{4\Omega}\), and \(I^*(z) \approx (4z^3p)^{-1} \int_{-p}^{z+p} I(r)2rdr\). If one knows the limb-darkening coefficients of the star in question (from, say, spectral information), and if the semi-major axis is much larger than the size of the star so that the orbit can be approximated by a straight line, then the shape of the eclipse for \(p \ll 0.1\) is simply determined by the smallest impact parameter, \(z_0 = a_p \cos i/r_s\), where \(a_p\) is the semi-major axis and \(i\) is the inclination angle. For example, at the midpoint of the eclipse, \(z = z_0\), while at the 1/4 and 3/4 phases of the eclipse, \(z = z_{1/4} = (1+3z_0^2)^{1/2}/2\). Taking the ratio of the depth of the eclipse at these points yields \(R = (1 - F(z_0))/(1 - F(z_{1/4})) = I(z_0)/I(z_{1/4}).\) This then determines an equation for \(z_0\), and the resulting \(z_0\) can then be used to determine \(p = \sqrt{4\Omega(1 - F(z_0))/I(z_0)}\). When \(1-p < z < 1+p\) and \(p \ll 0.1\) an approximation to the lightcurve is

\[
F = 1 - \frac{I^*(z)}{4\Omega} \left[p^2 \cos^{-1} \left(\frac{z-1}{p}\right) - (z-1) \sqrt{p^2 - (z-1)^2}\right]. \tag{8}
\]

where \(I^*(z) = (1-a)^{-1} \int_{-p}^{p} I(r)2rdr\), which is accurate to better than 2% of \(1-F(0)\) for \(p = 0.1\) and \(\Sigma_{n=1}^{\infty} c_n \leq 1\) (see Figure 2). This is a very fast means of computing transit lightcurves with reasonable accuracy, and may be used for any limb-darkening function, generalizing the approach of Deeg et al. (2001).

6. DISCUSSION

Figure 2 shows five lightcurves, the first of which has \(c_n = 0, \{n = 1, 4\}\), while the other four have \(c_n = 1, c_0 = 0, \{m \neq n\}\) for \(p = 0.1\); these may be thought of as a basis set for any nonlinear limb-darkening. Note that the higher order functions have flux which is concentrated more strongly toward the center of the star, and thus have a more gradual ingress and a deeper minimum as more flux is blocked at the center than the edge. All of the curves cross near \(z \sim 0.7\), which means that accurate observations are required near minimum and egress/ingress to constrain the coefficients of the various basis functions. If the inclination is large enough that \(z \gtrsim 0.7\) for the entire transit, then it may be difficult to constrain the \(c_n\)'s.

As Claret (2000) claims that the nonlinear limb-darkening law is the most accurate, we have compared the transit with \(p = 0.1\) for the nonlinear and quadratic laws. Of the entire grid of models computed by Claret (2000) which covers 2000 K < \(T_{eff}\) < 50000 K, 0 < \(\log g\) < 5, -5 < \([\text{M/H}]\) < 1, and filters u, b, v, y, U, B, V, R, I, J, H, K, the largest difference between the quadratic and nonlinear models for \(p = 0.1\) is 3% of the maximum of \(1-F\). Thus, the quadratic law should be sufficient for main-sequence stars when an accuracy of less than 3% is required; indeed, the average difference for the entire grid of models is about 1% of the maximum value of \(1-F\) for each lightcurve. In absolute terms, this is about \(10^{-4}(p/0.1)^2\) of the total flux, an accuracy which can be achieved from space.

So far we have only presented the lightcurve as a function of \(z\) and \(p\). To determine \(z\) as a function of time requires the planetary orbital parameters, which for zero eccentricity is given by \(z = a_p \cos i \left[ (\sin \omega t)^2 + (\cos \omega t)\right]^{1/2}\), where \(\omega\) is the orbital frequency, while \(t\) is the time measured from the center of the transit. Contribution of flux from the planetary companion or other companions may be added to the lightcurve, reducing the transit depth.

To illustrate the utility of our formulae, we have fit the nonlinear limb-darkened lightcurve to the HST-STIS data of HD 209458 (Brown et al. 2001). The best-fit parameters in this case are \(p = 0.12070 \pm 0.00027, \ i = 86.591^\circ \pm 0.055, \ a_p/r_s = 8.779 \pm 0.032, \ c_1 = 0.701, \ c_2 = 0.149, \ c_3 = 0.277, \ c_4 = -0.297,\) with a reduced \(\chi^2 = 1.046\). The \(c_n\)'s are poorly constrained given the small differences in the basis functions relative to the observed errors and the large impact parameter for this system. The errors on the other parameters are marginalized over the \(c_n\)'s. Limiting the limb-darkening to quadratic, we find \(\gamma_2 = 0.296 \pm 0.025\) and \(\gamma_3 = 0.34 \pm 0.04\), consistent with the values derived by Brown et al. (2001) and with stellar atmosphere predictions. The value for \(p\) in the quadratic case is consistent with the nonlinear case, indicating that the fit is independent of the assumed limb-darkening law. For a stellar mass of 1.1 \pm 0.1 M\(_{\odot}\) (Mazeh et al. 2000) and period of \(T = 3.5248\) days (Brown et al. 2001), we find \(r_p = 1.145 \pm 0.035 R_{\odot}\) and \(r_p = 1.376 \pm 0.043 R_{\text{Jup}}\).

We apply the small-planet approximation described in the previous section to HD 209458, assuming quadratic limb-darkening. Cody & Sasselov (2002) determined the effective temperature and surface gravity for this star, which imply
\[ \gamma_1 = 0.292 \text{ and } \gamma_2 = 0.35 \text{ for the I-band flux (close to the effective wavelength of the HST STIS data) from the models of Claret (2000). From the lightcurve, one finds } F(z_0) = 0.9835 \text{ and } F(z_{1/4}) = 0.9847. \text{ Solving the equations from Section 5 gives } z_0 = 0.546 \text{ and } p = 0.12, \text{ very similar to the parameters derived by a fit to the entire lightcurve. This technique may be used for finding initial parameters for lightcurve fitting.}

7. CONCLUSIONS

We have derived analytic expressions for an eclipse including quadratic limb-darkening and nonlinear limb-darkening. The nonlinear law (Section 3) provides an accurate fit to realistic stellar limb-darkening, while the quadratic fit (Section 4) provides a fast means of obtaining a relatively accurate lightcurve. For an extremely fast and fairly accurate approximation for any limb-darkening law, the equations in Section 5 may be used to derive lightcurves. If the limb-darkening law is known from the spectral type of the star, then one can use the formulae in Section 5 to analytically estimate both the minimum impact parameter (in units of stellar radius) and the ratio of the planetary radius to the stellar radius. We have written a code which takes the properties of a host star, finds the limb-darkening coefficients in the tables of Claret (2000), and computes lightcurves for the parameters of a given planetary transit. This code will be useful for simulating planetary transit searches (Gaudi 2000; Defay et al. 2001; Jenkins et al. 2002; Remund et al. 2002; Jenkins 2002; Pepper & Gould 2002), searching for planetary transit signals in lightcurves collected by a given search, and for fitting and measuring the errors of the parameters of detected planetary transit events. Planetary searches suffer from two important backgrounds: grazing eclipsing binaries and triple systems in which two stars eclipse while the flux from the third reduces the depth of the eclipse. Using the appropriate limb-darkening coefficients for each star’s spectral type will help to distinguish these contaminants from true planetary transits, which can be accomplished using the formulae presented here. These routines are made available by download from http://www.pha.jhu.edu/~agol/.

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