Charged Boson Stars in AdS and a Zero Temperature Phase Transition

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We numerically construct charged boson stars in asymptotically AdS spacetime. We find an intricate phase diagram dominated by solutions whose main matter contribution are alternately provided by the scalar field or by the gauge field.

Introduction

The study of boson stars dates back to the late 1960’s with the work of Kaup [1] (see also [2]) who, inspired by ideas of Wheeler, constructed Klein-Gordon geons. These objects and their generalizations have found numerous applications in and beyond general relativity. Some of the classical reviews in the subject include: [3–7].

In this letter we construct charged boson stars in asymptotically Anti-de-Sitter (AdS) spacetimes and investigate their properties. Namely, we consider particle-like solutions of a complex scalar field coupled to gravity and a Maxwell field in the presence of a negative cosmological constant. We have three main motivations to study these objects.

First, it is intrinsically interesting to understand particle-like solutions in asymptotically AdS space times to enhance and test our intuition of highly symmetric solutions of Einstein gravity with a negative cosmological constant. For example, the role of boundary conditions necessary to define dynamics in AdS is one aspect that constantly challenges our Minkowski-based intuition.

Our second motivation is also linked to understanding dynamics in AdS but it is more concretely related to gravitational collapse in AdS. It has recently been established that AdS is unstable under arbitrarily small perturbations with respect to black hole formation [8]. This result has been discussed and elaborated upon for massless fields [9–13]. Given previous history with critical collapse of massive [14] and Yang-Mills fields [15], it is possible that the phase diagram of critical collapse in asymptotically AdS spacetimes gets modified by the introduction of mass and other fields. In particular, the existence of boson stars might prevent a direct channel to black hole formation.

Thirdly, solutions of a scalar field in Einstein-Maxwell gravity in asymptotically AdS spacetimes have proven a fruitful ground for applications of the AdS/CFT correspondence to various situations in condensed matter physics (see, for example, [16–18]).

Charged boson stars

We consider a massive charged complex scalar field, $\phi$, interacting with electromagnetism and minimally coupled to Einstein gravity with a negative cosmological constant,

$$ S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R - \Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (D_\mu \phi)(D^\mu \phi^*) - m^2 \phi^* \phi \right], \quad (1) $$

$$ D_\mu = \nabla_\mu - i q A_\mu, \quad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad (2) $$

where $G$ is Newton’s constant. Since we are interested in stationary, spherically symmetric solutions, we write the metric in Schwarzschild form

$$ ds^2 = -e^{2\sigma} dt^2 + e^{2\nu} d\rho^2 + \rho^2 d\Omega^2. \quad (3) $$

Here $\rho$ is the radial coordinate, and we take $u = u(\rho)$ and $v = v(\rho)$. We then take the scalar to be time-harmonic

$$ \phi = \frac{1}{\sqrt{2}} e^{i\omega t} \sigma(\rho), \quad (4) $$

where $\sigma(\rho)$ is a real function of $\rho$. Since $\phi$ is electrically charged, it will source the electromagnetic field, and hence we turn on a scalar potential, $A_t = A_t(\rho)$.

At this point, a few quick comments are in order. Firstly, the Maxwell equation arising from (1) takes the form

$$ D_\mu \phi \to \phi D_\mu \phi^* + \phi^* D_\mu \phi, \quad (5) $$

is the conserved particle number current. As a result, the total charge of the boson star is given by $Q = qN$ where $N$ is the conserved particle number. Secondly, the time-dependence of $\phi$ may be removed by a gauge transformation of the form $A_t \to A_t - \omega/q$ along with $\phi \to e^{-i\omega t} \phi$, so only the combination $\omega - qA_t$ is physical. This is in contrast with the standard (ungauged) boson stars, where $\omega$ has an intrinsic meaning.

It is straightforward to derive the coupled equations of motion corresponding to the above spherically symmetric ansatz. When developing the numerical solutions, we take as input the mass $m$ and charge $q$ of the scalar field as well as the cosmological constant $\Lambda$, and scale by Newton’s constant $G$ when appropriate in order to work with dimensionless quantities. We also introduce the gauge invariant combination

$$ \hat{A}(\rho) = (\omega - qA_t(\rho))/(\omega - qA_t(0)). \quad (6) $$
To define our boundary value problem, we first demand regularity near the origin, $\rho = 0$. This leads to the following conditions:

$$
\begin{align*}
    u(0) &= u_0, \quad v(0) = 0, \quad \sigma(0) = \sigma_0, \quad \sigma'(0) = 0, \\
    \hat{A}(0) &= 1, \quad \hat{A}'(0) = 0.
\end{align*}
$$

At asymptotic infinity, $\rho \to \infty$, AdS boundary conditions imply that the matter fields behave as

$$
\begin{align*}
    \sigma(\rho) &= \sigma_1 \rho^{-\Delta} + \sigma_2 \rho^{\Delta-3}, \quad \Delta = \frac{3}{2} - \sqrt{\frac{9}{4} + (mL)^2}, \\
    A_4(\rho) &= a_0 + a_1 \rho^{-1}.
\end{align*}
$$

Our task is to look for solutions with normalizable modes at infinity, namely $\sigma_2 = 0$. Therefore we have a boundary value problem which we solve numerically using shooting techniques. We determine $\sigma_2 = 0$ with a precision of $10^{-20}$. In our minimizing algorithm we shoot by changing $\sigma_0$ of the initial data for a given $u_0$; we use $(mL) = 10$ throughout.

Properties of the solutions

In the absence of a cosmological constant, there is a critical charge, $q_{\text{crit}}$, above which the star does not exist in flat space. In the Newtonian limit the value of $q_{\text{crit}}$ is obtained by comparing the gravitational attraction with the electrostatic repulsion of an elementary particle interacting with the star; the result is $Gm^2 = \frac{q_{\text{crit}}^2}{4\pi}$. In the context of general relativity this result is only slightly modified. However, for boson stars in AdS, the value of $q_{\text{crit}}$ above which a star does not exist is substantially larger. Moreover, the phase space of solutions is quite different.

Intuitively we can understand the increase of $q_{\text{crit}}$ in the presence of a negative cosmological constant as follows. Once the electrostatic force has overcome the gravitational pull, the only force standing in the way of charged particles flying away is the pull generated by the cosmological constant. Therefore for large enough charges the main mechanism supporting the star is the balance between the gauge forces and the cosmological constant.

For small charge $q$, the boson stars in AdS resemble those in flat space. The regular zero-node solutions have a smooth scalar profile, with maximum particle density at the core, and a gradual fall off as a function of $\rho$. Since the scalar field dominates the energy density, we denote these regular solutions as _scalar dominated_. For sufficiently large charge $q$, on the other hand, we find a new type of solution where the gauge field contribution dominates the total energy of the system. We naturally denote these as _gauge field dominated_.

An example of the regular and new solutions is given in Fig. 1. The regular scalar dominated solution is shown on the left, while the new gauge field dominated solution is on the right. There are three key properties that distinguish the new solution from the regular one: (i) The new solution has a sharper surface, as defined by the profile of the scalar field; (ii) The gauge field is concentrated near the surface of the star, instead of distributed in the interior as in the standard case; (iii) The contribution to the total mass of the star is dominated by the gauge field.

We now consider the mass of the boson star as a function of its core density $\sigma(0)$ and as a function of the particle number $N$. Since the interplay between regular and new solutions depends on the charge $q$, we define two critical charges, $q_1$ and $q_2$. For $q < q_1$ only the scalar dominated solution is possible. For $q_1 < q < q_2$, we enter a transition region where both the scalar dominated and gauge field dominated solutions exist as distinct branches. Finally, for $q > q_2$, the scalar and gauge field dominated solutions merge.

The mass as a function of $\sigma(0)$ and as a function of $N$ for an intermediate value of the charge, $q_1 < q < q_2$, is shown in Fig. 2. The green curve represents the regular scalar dominated solution, while the red curve corresponds to the gauge field dominated solution. We have also shown the one-node solution as the blue curve. While this is ordinarily discarded as being an excited state, here we wish to note its proximity to the gauge field dominated solution. For each of these solutions, the right panel shows three curves of the same color. These curves correspond to the total mass and the separate scalar and gauge field contributions to the mass. For the scalar dominated (green) curves, the scalar contribution to the mass is the larger one, while for the gauge field dominated (red) curves, the gauge field contribution to the mass is larger. Note that there is a gap on the right panel of Fig. 2. This suggests some sort of transition between scalar dominance on the left and gauge field dominance on the right.
dominance on the right.

When the charge $q$ is increased beyond the critical value $q_2$ we observe a very different behavior of the mass as a function of $\sigma(0)$ and as a function of $N$, as shown in Fig. 3. There is no longer a gap in the right panel and we see clearly the change in the contribution to the total energy coming from the scalar-dominated region prevalent at small values of $N$ and the gauge-dominated region at large values of $N$. The blue curve again corresponds to a one-node solution. This figure should be read as a merging of the previous one in the cases where the charge has increased, that is, increasing the charge narrows the gap between the two kind of solutions which is evident in the region of $q_1 < q < q_2$.

The three green curves on the right panel of Fig. 3 are total energy, the contribution from the scalar field and the contribution from the gauge field respectively. This graph should be read as a merging of branches: now the green curve is a mixture of the normal branch for low $N$ and gauge-dominated branch for large $N$; the red curves are only in the areas of overlap in the previous graph. The most prominent feature of Fig. 3 is the crossing of the scalar and gauge contributions to the total energy.

In order to interpret Fig. 3 we note that, for a given scalar charge $q$, boson stars exist for a range values of $M$. What we plot is the critical mass, namely the largest mass for the scalar dominated branch, and the minimum and the maximum masses for the gauge field dominated branch. To be more graphic, the right panel in Fig. 2 contains three distinguished points: the largest mass for the green branch and the smallest and largest mass for the red branch. These are precisely the three points in the phase diagram in the region $q_1 < q < q_2$. For $q < q_1$ there is only one type of solution and we simply plot its maximum mass. Similarly for $q > q_2$, we plot only the maximum mass. In this work we are not going to be concerned with the existence of a maximum charge, $q_{\text{max}}$, above which no regular solution exists. However, preliminary investigations suggest that such a point exists, and we will discuss its determination elsewhere.

A Zero Temperature Phase Transition

The main property of a quantum phase transition [19] is the existence of a value of a coupling $g_c$ at which there can be a level-crossing where an excited level becomes the ground state. Usually this creates a point of non-analyticity of the ground state energy as a function of $g$. Our previous graphs in Figs. 2 and 3 show this non-analyticity and level crossing behavior. To emphasize this point, we plot the critical mass of a boson star as a function of charge in Fig. 4; note the previously defined $q_1$ and $q_2$ are the two inflexion points in this graph. Allowed values for boson star masses lie below and to the right of the curve.

From a holographic point of view, a plot of mass versus the time component of the Maxwell field represents the
free energy as a function of chemical potential. While we use the gauge invariant $\hat{A}$ defined in (6), our asymptotic boundary conditions relate this to the value of $\omega$. We thus plot the mass versus $\omega$ in Fig. 5 for solutions in the intermediate and large charge regimes.

A very important role in phase transitions is played by the order parameter. In this case we identify it as the value of the scalar at asymptotic infinity, $\sigma_1$. In particular, in the case of the holographic superconductors [20], $\sigma_1$ corresponds to the expectation value of an operator of mass dimension $\Delta$. In Fig. 6 we plot $\sigma_1$ for critical mass boson stars (those following the curve of Fig. 4), and we see a sharp drop around the value of $q_2$.

**FIG. 4:** The phase diagram of charged boson stars in AdS.

**FIG. 5:** The Mass as a function of the frequency $\omega$, which is a holographic proxy for chemical potential. The left panel corresponds to $q_1 < q < q_2$, while the right panel corresponds to $q > q_2$.

**FIG. 6:** The behavior of the order parameter $\sigma_1$ as a function of the charge $q$.

**Conclusions**

In the context of a charged scalar field minimally coupled to Einstein-Maxwell gravity with a negative cosmological constant, we have constructed explicit solutions and established a phase diagram of boson stars. The new type of solutions contain a gauge-field-dominated branch that represents a sort of “Geon” as originally envisioned by Wheeler, that is, a particle-like solution from mostly the smooth, classical fields of electromagnetism coupled to general relativity.

Let us conclude by commenting on some open questions stemming from our work. The main difference in gravitational collapse in asymptotically AdS spaces is the presence of a timelike boundary at spatial and null infinities. Under these conditions, the question of stability of our configurations is a particularly important one. In the phase space of initial conditions for gravitational collapse in AdS it is important to consider that a small perturbation does not escape to spatial infinity and actually returns to the center of AdS. We postpone such detailed study to a separate publication but will speculate on the role that boson stars might play.

Already in the context of asymptotically Minkowski gravitational collapse, boson stars [21] play a particularly important role; they are the basis for a different type of critical collapse [14]. It is natural to expect that in asymptotically AdS spaces they will figure prominently. It will be interesting to pursue the construction and investigation of properties of real boson stars, similar to those of asymptotically Minkowski spacetimes [21], to asymptotically AdS. One would expect that real boson stars, if they exist, get similarly destroyed after a certain number of bounces of the scalar field at spatial infinity. Recently, however, the conjecture that boson stars might be non-linearly stable has been advanced in [22] based on the observation that a mass scale might prevent the resonant turbulent mechanism of [8] from being realized.

Scalar boson stars have been previously considered in asymptotically AdS spacetimes by [24–25]. Our work has concentrated on charged objects but a systematic anal-
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