Parameterized Bilinear Matrix Inequality Techniques in $\mathcal{H}_\infty$ Fuzzy PID Control Design

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Abstract—Proportional-integral-derivative (PID) structured controller is the most popular class of industrial control but still could not be appropriately exploited in fuzzy systems. To gain the practicability and tractability of fuzzy systems, this paper develops a parameterized bilinear matrix inequality characterization for the $\mathcal{H}_\infty$ fuzzy PID control design, which is then relaxed into a bilinear matrix inequality optimization problem of nonconvex optimization. Several computational procedures are then developed for its solution. The merit of the developed algorithms is shown through the benchmark examples.

Index Terms—Tagaki-Sugeno (T-S) fuzzy system, $\mathcal{H}_\infty$ fuzzy proportional-integral-derivative (PID) control, parameterized bilinear matrix inequality (PBMI), bilinear matrix inequality (BMI), nonconvex optimization techniques.

I. INTRODUCTION

Tagaki-Sugeno (T-S) fuzzy model [1] has proved as one of the most practical tools for representing complex nonlinear systems by gain-scheduling systems, which are easily implemented online. Treating T-S fuzzy models as gain-scheduling systems allows the application of advanced gain-scheduling control techniques in tackling state feedback and output feedback stabilization of nonlinear systems [2], [3]. Until now, most of the gain-scheduling controllers are assumed structure-free and full-rank to admit computationally tractable parameterized linear matrix inequality (PLMI) or linear matrix inequality (LMI) formulations [2]–[4].

Meanwhile, proportional-integral-derivative (PID) structured controller is the indispensable component of industrial control so that PID control theory is still the subject of recent research [5]–[10], mainly concerning with linear time-invariant systems in the frequency domain. PID controller for fuzzy systems has been considered in [11]. Reference [12] proposed an LMI based iterative algorithm for a proportional-integral (PI) controller in T-S systems under the specific structure of both system and controller. A recent work [13] transformed the fuzzy diagonal PID controller into a static output feedback problem with the dimension of controller dramatically increased. That is why all its testing examples are restricted on single input and single output systems with two states.

This paper is concerned with the PID parallel distribution compensation (PDC) for T-S fuzzy models. The control design problem is formulated as a parameterized bilinear matrix inequality (PBMI) optimization problem that is in contrast to the PLMI formulation for the structure-free PDC design [2]. This is quite expected because the PID controller design for linear time-invariant systems is already nonconvex, which is equivalent to a BMI optimization problem in the state space. In our approach, PBMI is then relaxed to a bilinear matrix inequality (BMI) for more tractable computation. It should be noted that BMI optimization constitutes one of the most computational challenging problems, for which there is no efficient computational methodology. The state-of-the-art BMI solvers [14], [15] in addressing the structure-constrained stabilizing controllers for linear time-invariant systems must initialize from a feasible controller and then move within a convex feasibility subset containing this initialized point. Usually their convergence is very slow [15]. Furthermore, finding a feasible structure-constrained stabilizing controller is still a NP-hard problem [16]. The most efficient method to find such a feasible controller is via the so-called spectral abscissa optimization [17], which seeks a controller such that the state matrix of the closed loop system has only eigenvalues with negative real parts. This spectral abscissa optimization-based approach cannot be extended to gain-scheduling systems, whose stability does not quite depend on the spectrum of the time-varying state matrix. The main contribution of the present paper is to develop efficient computational procedures for the BMI arisen from the PBMI optimization, which generate a sequence of unstabilizing controllers that rapidly converges to the optimal stabilizing controller.

The rest of this paper is organized as follows. Section II is devoted to formulating the $\mathcal{H}_\infty$ fuzzy PID control in T-S system by a PLMI, which is then relaxed by a system of BMIs. Several nonconvex optimization techniques for addressing this BMI system are developed in Section III. Simulation for benchmark systems is provided in Section IV to support the solution development of the previous sections. Section V concludes the paper.

Notation. Notation used in this paper is standard. Particularly, $X \preceq 0$, $X \succeq 0$, $X \preceq 0$ and $X \preceq 0$ mean that a symmetric matrix $X$ is positive semi-definite, positive definite, negative semi-definite and negative definite, respectively. Trace($X$) represents the trace of $X$, while $||X||_2^2$ = Trace($XX^T$) is its square norm. In symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for terms that are induced by symmetry, e.g.,

$$K \left[ \begin{array}{cc} S + (\ast) & M \\ M & Q \end{array} \right] \ast = K \left[ \begin{array}{cc} S + S^T & M^T \\ M & Q \end{array} \right] K^T.$$  

All matrix variables are boldfaced. Denote by $I_n$ the identity matrix of dimension $n \times n$ and by $0_{n \times m}$ the zero matrix of dimension $n \times m$. The subscript $n \times m$ is omitted when it is either not important or is clear in context.
II. $H_{\infty}$ FUZZY PID PDS FOR T-S SYSTEMS

Suppose that $x$ is the state vector with dimension $n_x$, $u$ is the control input with dimension $n_u$, $y$ is the measurement output with dimension $n_y$, $w$ and $z$ are the disturbance and controlled output of the system with the same dimension $n_{\infty}$, and $L$ denotes the number of IF-THEN rules. In T-S fuzzy modeling, each $i$-th plant rule is the form

IF $z_1(t)$ is $N_{i1}$ and ... $z_p(t)$ is $N_{ip}$

THEN $\begin{bmatrix} x \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_{12} \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}$. (1)

Here $z_i$ are premise variables, which are assumed independent of the control $u$, and $N_{ij}$ are fuzzy sets. Denoting by $N_{ij}(z_i(t))$ the grade of membership of $z_i(t)$ in $N_{ij}$, the weight $w_i(t) = \prod_{j=1}^p N_{ij}(z_i(t))$ of each $i$-th IF-THEN rule is then normalized by

$$\alpha_i(t) = \frac{w_i(t)}{\sum_{j=1}^L w_j(t)} \geq 0, \quad i = 1, 2, \ldots, L$$

$$\Rightarrow \alpha(t) = (\alpha_1(t), \ldots, \alpha_L(t)) \in \Gamma,$$

with $\Gamma := \{ \alpha \in \mathbb{R}^L : \sum_{i=1}^L \alpha_i = 1, \alpha_i \geq 0 \}$. (3)

In the state space, the T-S model is thus represented by the following gain-scheduling system

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\alpha(t)) & B_1(\alpha(t)) & B_{21}(\alpha(t)) \\ C_1(\alpha(t)) & D_{11}(\alpha(t)) & D_{12}(\alpha(t)) \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}$$

where

$$\begin{bmatrix} A(\alpha(t)) & B_1(\alpha(t)) & B_{21}(\alpha(t)) \\ C_1(\alpha(t)) & D_{11}(\alpha(t)) & D_{12}(\alpha(t)) \\ C_2 & D_{21} & 0 \end{bmatrix} = \sum_{i=1}^L \alpha_i(t) \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & D_{11i} & D_{12i} \\ C_{2i} & D_{21i} & 0 \end{bmatrix}. \quad (5)$$

In this paper, we seek the output feedback controller in the class of PID PDC with each i-th plant rule inferred by

IF $z_1(t)$ is $N_{i1}$ and ... $z_p(t)$ is $N_{ip}$

THEN $\begin{bmatrix} x_K \\ u \end{bmatrix} = \begin{bmatrix} 0_{n_x \times n_x} & 0_{n_x \times n_x} \\ 0_{n_x \times n_x} & -T_n \end{bmatrix} \begin{bmatrix} R_{i1} \\ R_{i2} \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix}$

for a given $\tau > 0$, where $R_{x,j} \in \mathbb{R}^{n_x \times n_y}, x \in \{ I, D, P \}$.

Note that the transfer function of this $i$-th plant rule is

$$K_i(s) = \frac{R_{p,i}}{s} + \frac{R_{d,i}}{s + \tau}$$

and $\varepsilon = 1/\tau$. It is clear from (8) that $K_{p,i}$, $K_{f,i}$ and $K_{d,i}$ respectively are the proportional, integral and derivative gain matrices, while $\epsilon$ is a small tuning scalar which determines how close the last term in (5) comes to a pure derivative action (8). In other words, (8) is the state-space representation of multi-input multi-output PID structured controllers.

The PID PDC with rule set by (6) and the membership function defined by (3) is thus represented by the following gain-scheduling PID controller

$$\begin{bmatrix} x_K \\ u \end{bmatrix} = \left( \sum_{i=1}^L \alpha_i(t)K_i(R_i) \right) \begin{bmatrix} x_K \\ y \end{bmatrix} \quad (9)$$

where

$$K_i(R_i) = \begin{bmatrix} R_{i1} \\ R_{i2} \end{bmatrix} \in \mathbb{R}^{(n_x \times n_u)}, i = 1, \ldots, L,$$

for

$$R_i = \begin{bmatrix} R_{i1} \\ R_{i2} \\ R_{p,i} \end{bmatrix} \in \mathbb{R}^{(3n_u \times n_y)}, i = 1, \ldots, L.$$
Using the quadratic Lyapunov function $V(t) := x^T(t)Xx(t)$, $0 < X \in \mathbb{R}^{(m + 2n_u) \times (m + 2n_u)}$ to make $[11c]$ fulfilled by forcing

$$
\dot{V}(t) + \gamma^{-1}\|z(t)\|^2 - \gamma\|w(t)\|^2 \leq 0,
$$

one can easily see that $[11c]$ is fulfilled by the following parameterized matrix inequality

$$
\begin{bmatrix}
(A_0(x) + B_0(k(x))C_0 + \alpha \gamma I) & (B_0 + B_0(k(x))C_0 + \alpha \gamma I) \\
C_0(k(x)) + D_1(k(x))C_2 & D_1(k(x)) + D_2(k(x))C_2 \\
\end{bmatrix} \prec 0,
$$

which is a BMI optimization in the decision variables $X$.

III. NONCONVEX SPECTRAL OPTIMIZATION TECHNIQUES FOR SOLVING BMIs

The sparse structure of matrix $C$ in [14] suggests that [14] be a sparse nonlinear constraint in the sense that there are not so many nonlinear terms in its right hand side. Indeed, by partitioning

$$
\begin{bmatrix}
\alpha_{11} & \cdots & \alpha_{1L} \\
\vdots & \ddots & \vdots \\
\alpha_{L1} & \cdots & \alpha_{LL} \\
\end{bmatrix}
\begin{bmatrix}
R_{11} & \cdots & R_{1L} \\
\vdots & \ddots & \vdots \\
R_{L1} & \cdots & R_{LL} \\
\end{bmatrix}
\begin{bmatrix}
X_{11} & \cdots & X_{13} \\
\vdots & \ddots & \vdots \\
X_{L1} & \cdots & X_{L3} \\
\end{bmatrix},
$$

with $X_{ii}$ symmetric, it can be checked that

$$
K_j((R_j)C)X = \begin{bmatrix}
R_{1j}C_2X_{11} & -\tau X_{12}^T + R_{2j}C_2X_{11} & \cdots & -\tau X_{13} + R_{3j}C_2X_{13} \\
R_{Lj}C_2X_{11} & -\tau X_{21}^T + R_{2j}C_2X_{11} & \cdots & -\tau X_{23} + R_{3j}C_2X_{13} \\
\vdots & \vdots & \ddots & \vdots \\
R_{1j}C_2X_{11} & -\tau X_{L1}^T + R_{2j}C_2X_{11} & \cdots & -\tau X_{L3} + R_{3j}C_2X_{13} \\
\end{bmatrix},
$$

for

$$
X_{11} = \begin{bmatrix}
X_{11} & \cdots & X_{13} \\
\vdots & \ddots & \vdots \\
X_{L1} & \cdots & X_{L3} \\
\end{bmatrix} \in \mathbb{R}^{(m + 2n_u) \times (m + 2n_u)}.
$$

Therefore, the bilinear constraints [14] are expressed by the linear constraints

$$
W_j = \begin{bmatrix}
0 & \cdots & 0 \\
-\tau X_{13}^T & \cdots & -\tau X_{L3} \\
X_{12}^T + X_{13}^T & \cdots & X_{22} + X_{23} + X_{33} \\
\end{bmatrix},
$$

plus the bilinear constraints

$$
Y_j = R_jC_2X_1, j = 1, \ldots, L.
$$

In other words, the BMI feasibility problem [20] in $X, R$ and $W$ is now equivalently transformed to the following BMI feasibility problem in $X, R, W$ and $Y := (Y_1, \ldots, Y_L)$:

$$
[13b], [14], [17], [18], [24], [25],
$$

where [13b], [17] and [18] are linear matrix inequality (LMI) constraints, while [24] is linear constraints. The difficulty is
now concentrated at $L$ bilinear constraints in (25), in which only $X_1$ is considered as a complicating variable that makes $L$ constraints in (25) nonlinear. Based on this observation, our strategy is to decouple this complicating variable $X_1$ from (25) for a better treatment. Let us recall an auxiliary result.

**Lemma 1:** [19] For given matrix $W_{12}, W_{22}$ of sizes $n \times n$ and $m \times m$ with $W_{22} \succeq 0$, one has

$$
\begin{pmatrix}
0 & W_{12} \\
W_{12}^T & W_{22}
\end{pmatrix} \succeq 0
$$

(27)

if and only if $W_{22} = 0$.

Using the above Lemma, we are now in a position to state the following result, which is a cornerstone in handling bilinear constraints like (25), which share a common complicating variable.

**Theorem 1:** $L$ bilinear constraints in (25) are equivalently expressed by the following $L$ LMI constraints

$$
\begin{pmatrix}
W_{11,j} & Y_{j} & R_j \\
Y_{j}^T & W_{22} & X_j^T C_2^T \\
R_j^T & C_2 X_j & I_{n_y}
\end{pmatrix} \succeq 0, \quad j = 1, \ldots, L,
$$

(28)

plus the single bilinear constraint

$$
W_{22} = X_j^T C_2^T C_2 X_1.
$$

(29)

**Proof.** It can be easily seen that those $Y_j, R_j$ and $X_1$ that are constrained by (25) together with $W_{11,j} = R_j(R_j)^T$ and $W_{22} = C_2 X_1 (C_2 X_1)^T$ are feasible for (28) and (29), showing the implication (25) $\Rightarrow$ (28) & (29).

On the other hand, by Shur’s complement, it follows from (28) that

$$
0 \preceq \begin{pmatrix}
W_{11,j} & Y_{j} & R_j \\
Y_{j}^T & W_{22} & X_j^T C_2^T \\
R_j^T & C_2 X_j & I_{n_y}
\end{pmatrix} - \begin{pmatrix}
R_j & X_j^T C_2 \end{pmatrix} \begin{pmatrix}
R_j & C_2 X_1 \end{pmatrix} = \begin{pmatrix}
W_{11,k} & Y_k - R_k C_2 X_1 \\
Y_k^T - X_k^T C_2 R_k^T & W_{22} - X_k^T C_2^T C_2 X_1 \\
Y_k & -R_k C_2 X_1 & 0
\end{pmatrix},
$$

(30)

where we also used (29) in obtaining the last equality (30). Then applying Lemma 1 yields (25), showing the implication (28) & (29) $\Rightarrow$ (25).

Now, the problem’s nonconvexity is concentrated on the single constraint (29) that involves only $X_1$.

**Theorem 2:** Under LMI constraints (25), the bilinear constraint (29) is equivalent to any from the two following constraints:

(i) The matrix rank constraint

$$
\text{rank}(Q) = n_y,
$$

(31)

for

$$
Q := \begin{pmatrix}
W_{22} & X_1^T C_2^T \\
C_2 X_1 & I_{n_y}
\end{pmatrix};
$$

(32)

(ii) The quadratic constraint

$$
\text{Trace}(W_{22}) = ||C_2 X_1||^2
$$

(33)

Proof. Note that (28) implies

$$
Q \succeq 0
$$

(34)

which also yields

$$
W_{22} \succeq X_1^T C_2^T C_2 X_1
$$

(35)

by Shur’s complement. Also,

$$
\text{rank}(Q) = \text{rank}(I_{n_y}) + \text{rank}(W_{22} - X_1^T C_2^T C_2 X_1)
$$

$$
= n_y + \text{rank}(W_{22} - X_1^T C_2^T C_2 X_1),
$$

so (31) holds true if and only if $\text{rank}(W_{22} - X_1^T C_2^T C_2 X_1) = 0$, which is (29).

Next, it follows from (35) that (29) holds true if and only if

$$
\text{Trace}(W_{22} - X_1^T C_2^T C_2 X_1) = 0
$$

$$
\Leftrightarrow \quad \text{Trace}(W_{22}) - ||C_2 X_1||^2 = 0
$$

$$
\Leftrightarrow \quad (33).
$$

This completes the proof of Theorem 2.

The rank constraint (31) is discrete and absolutely intractable in general. However, under condition (25), this rank constraint is equivalent to the following continuous matrix-spectral constraint

$$
\text{Trace}(Q) - \lambda_{[n_y]}(Q) = 0,
$$

(36)

where $\lambda_{[n_y]}(Q)$ is the summation of the $n_y$ largest eigenvalues of $Q$. Indeed, $\text{rank}(Q) \geq n_y$ but (36) means $Q$ has at most $n_y$ nonzero eigenvalues so its rank is $n_y$.

On the other hand, as

$$
\text{Trace}(Q) - \lambda_{[n_y]}(Q) \geq 0,
$$

it follows from (36) that

$$
\text{Trace}(Q) = \lambda_{[n_y]}(Q)
$$

(37)

can be used to measure the degree of satisfaction of the rank constraint (31). Instead of handling the nonconvex constraint (36) we incorporate it into the objective, resulting in the following alternative formulation to (26)

$$
\min_{x, w, r, y} \quad \text{subject to} \begin{cases}
F(Q) := \text{Trace}(Q) - \lambda_{[n_y]}(Q) & \text{(38a)} \\
(33b), \quad (17b), \quad (18b), \quad (24b), \quad (25b). \quad \text{(38b)}.
\end{cases}
$$

Suppose $X_1^{(\kappa)}$ and $W_{22}^{(\kappa)}$ are feasible for (38). Set

$$
Q^{(\kappa)} := \begin{pmatrix}
W_{22}^{(\kappa)} & X_1^{(\kappa)}^T C_2^T \\
C_2 X_1^{(\kappa)} & I_{n_y}
\end{pmatrix};
$$

Function $\lambda_{[n_y]}(Q)$ is nonsmooth but is lower bounded by the linear function

$$
\sum_{i=1}^{n_y} (w^{(\kappa)}_i)^T Q w^{(\kappa)}_i,
$$

(39)

where $w^{(\kappa)}_1, \ldots, w^{(\kappa)}_{n_y}$ are the normalized eigenvectors corresponding to $n_y$ largest eigenvalues of $Q^{(\kappa)}$. Thus, the following convex optimization problem provides an upper bound for the nonconvex optimization problem (38),

$$
\min_{x, w, r, y} \quad F^{(\kappa)}(Q) := \text{Trace}(Q) - \sum_{i=1}^{n_y} (w^{(\kappa)}_i)^T Q w^{(\kappa)}_i
$$

(40)

s.t. (38b).
Suppose that \((X_1^{(\kappa+1)}, W_{22}^{(\kappa+1)})\) is the optimal solution of (40) and 

\[
Q^{(\kappa+1)} := \begin{pmatrix}
X_1^{(\kappa+1)} & (X_1^{(\kappa+1)})^T \cdot C_2^T \\
C_2 \cdot X_1^{(\kappa+1)} & I_{n_y}
\end{pmatrix}
\]

Then 

\[
F(Q^{(\kappa+1)}) \leq F(Q^{(\kappa)})
\]

as far as \(Q^{(\kappa+1)} \neq Q^{(\kappa)}\), implying that \(Q^{(\kappa+1)}\) is better than \(Q^{(\kappa)}\) towards optimizing (40). Similarly to [20], we establish the following result.

**Proposition 1:** Initialized by any feasible point \(Q^{(0)}\) for the convex constraints (38b), \(\{Q^{(\kappa)}\}\) is a sequence of improved feasible points of the nonconvex optimization problem (38), which converges to a point satisfying the first-order necessary optimality conditions.

In Algorithm 1 we propose a convex programming based computational procedure for the nonconvex optimization problem (38).

So far, in solving (38) we are based on (37) as the satisfaction degree of the rank constraint (31) and thus of the bilinear constraint (29). For larger value of \(n_y\), Algorithm 1 may converge slowly. We now use

\[
1 - \frac{||C_2 X_1||^2}{\text{Trace}(W_{22})}
\]

as an alternative degree for satisfaction of the bilinear constraint (29) because according to (33), (43) is positive and by (33), it is zero if and only if the bilinear constraint (29) is satisfied. Accordingly, instead of (38) we use the following optimization problem:

\[
\min_{X, W, R, Y} - \frac{||C_2 X_1||^2}{\text{Trace}(W_{22})} \quad \text{s.t.} \quad (38b)
\]

**Algorithm 2** Fractional Optimization Algorithm for Solving BMI feasibility

1: **Initialization.** Set \(\kappa := 0\) and solve the LMI (38b) to find a feasible point \((X^{(\kappa)}, W^{(\kappa)}, R^{(\kappa)}, Y^{(\kappa)})\). Given computational tolerance \(\epsilon > 0\), stop the algorithm and accept \((X^{(0)}, W^{(0)}, R^{(0)}, Y^{(0)})\) as the solution of BMI (20) if

\[
1 - g(X_1^{(\kappa)}, W_{22}^{(\kappa)}) \leq \epsilon.
\]

2: **repeat**

3: Solve the convex optimization problem (40), to find the optimal solution \((X^{(\kappa+1)}, W^{(\kappa+1)}, R^{(\kappa+1)}, Y^{(\kappa+1)})\)

4: Set \(\kappa := \kappa + 1\).

5: **until**

\[
\frac{F(Q^{(\kappa-1)}) - F(Q^{(\kappa)})}{F(Q^{(\kappa-1)})} \leq \epsilon.
\]

6: Accept \((X^{(\kappa)}, W^{(\kappa)}, R^{(\kappa)}, Y^{(\kappa)})\) as the solution of (38). Accept \((X^{(\kappa)}, W^{(\kappa)}, R^{(\kappa)}, Y^{(\kappa)})\) as the solution of BMI (20) if \(F(Q^{(\kappa)}) \leq \epsilon\). Otherwise declare that BMI (20) is infeasible.

Note that function \(g(X_1, W_{22}) := ||C_2 X_1||^2/\text{Trace}(W_{22})\) is convex in \(X_1\) and \(W_{22} \geq 0\) [21], so

\[
g(X_1, W_{22}) \geq g(X_1^{(\kappa)}, W_{22}^{(\kappa)}) + \langle \nabla g(X_1^{(\kappa)}, W_{22}^{(\kappa)}), (X_1, W_{22}) - (X_1^{(\kappa)}, W_{22}^{(\kappa)}) \rangle
\]

\[
= -2 \frac{\text{Trace}((X_1^{(\kappa)})^T \cdot C_2^T \cdot C_2 \cdot X_1)}{\text{Trace}(W_{22})} - \frac{||C_2 X_1^{(\kappa)}||^2}{\text{Trace}(W_{22})}.
\]

Thus, instead of (40), we solve the following convex optimization problem, which is an upper bound for the nonconvex optimization problem (44), to generate \((X^{(\kappa+1)}, W^{(\kappa+1)}, R^{(\kappa+1)}, Y^{(\kappa+1)})\) at the \(\kappa\)-th iteration:

\[
\min_{X, W, R, Y} - \frac{2 \text{Trace}((X_1^{(\kappa)})^T \cdot C_2^T \cdot C_2 \cdot X_1)}{\text{Trace}(W_{22})} - \frac{||C_2 X_1^{(\kappa)}||^2}{\text{Trace}(W_{22})} \quad \text{s.t.} \quad (38b)
\]

A pseudo-code for the computational procedure, which is based on computation for (45) at each iteration, is described by Algorithm 2.

**IV. SIMULATION RESULTS**

An important step is to check if there is a controller (6) to stabilize system (4). Define the block \((1,1)\) in (15) as

\[
\tilde{M}_{ij}(X, R_j, W_j) := (A_0 X + B_j W_j) + (*).
\]
Then the existing of a stabilizing controller (6) is guaranteed by the feasibility of the system consisting of (13b), (14) and
\[
\frac{1}{L-1} \tilde{M}_{ii}(X, R_i, W_i) < 0, \quad i = 1, \ldots, L, \quad (48)
\]
\[
\frac{1}{L-1} \tilde{M}_{ii}(X, R_i, W_i) + \frac{1}{2} (\tilde{M}_{ij}(X, R_i, W_j) + \tilde{M}_{ji}(X, R_i, W_j)) < 0, \quad 1 \leq i \neq j \leq L. \quad (49)
\]
Thus, we can use Algorithm 1 or Algorithm 2 to check its feasibility, which invokes either the convex optimization problem
\[
\min_{X, W, R, Y} \text{Trace}(Q) - \sum_{k=0}^{n_w} (u_i^{(k)})^T Q u_i^{(k)} \quad (50)
\]
s.t. \(\square \) or the convex optimization problem
\[
\min_{X, W, R, Y} \frac{-2 \text{Trace}(X_i^{(k)})^T C_2 X_i^{(k)}}{\text{Trace}(W_{22}))^2} + \frac{\|C_2 X_i^{(k)})\|^2}{\text{Trace}(W_{22}))^2} \quad (51)
\]
s.t. \(\square \) instead of \(\square \) or \(\square \) at the \(\kappa\)-th iteration to generate the next iterative point \((X^{(\kappa+1)}, W^{(\kappa+1)}, R^{(\kappa+1)}, Y^{(\kappa+1)})\). Whenever a feasible point \((X^{(k)}, W^{(k)}, R^{(k)}, Y^{(k)})\) of (13b), (14), (48) and (49) is found, we solve the following convex optimization problem to determine the initial \(\gamma_i\) for the bisection procedure:

\[
\min_{\gamma_i} \quad \text{s.t. } M_{ii}(X^{(k)}, R_i^{(k)}, W_i^{(k)}, \gamma) < 0,
\]
\[
\frac{1}{L-1} M_{ii}(X^{(k)}, R_i^{(k)}, W_i^{(k)}, \gamma) + \frac{1}{2} (M_{ij}(X^{(k)}, R_j^{(k)}, W_j^{(k)}, \gamma) + M_{ji}(X^{(k)}, R_j^{(k)}, W_j^{(k)}, \gamma)) < 0,
\]
\[
1 \leq i \neq j \leq L.
\]

A. Inverted pendulum control

The motion of an inverted pendulum system with a point mass \(m = 2\) kg, a rigid rod of the length \(\ell = 0.5m\) and a cart of mass \(M = 8\) kg can be described by (5) with \(L = 2\) and

\[
A_1 = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 12.6305 & 0 \\ 0.1 & -0.1765 \end{bmatrix}, \quad B_{11} = B_{12} = 0,
\]
\[
B_{21} = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix}, \quad C_{11} = C_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 3 & 0 \end{bmatrix},
\]
\[
D_{11,i} = 0.1, \quad D_{12,i} = 0, \quad D_{21} = 0.
\]
The system state is \(x = (x_1, x_2)^T\), where \(x_1\) is the angle measured from the inverted equilibrium position (angular position) and \(x_2\) is the angular velocity. The membership functions in (2) are

\[
\alpha_1(t) = \begin{cases} 1 - (1 + e^{-7(x_1(t) - \pi/4)})^{-1}, \\ 1 + e^{-7(x_1(t) + \pi/4)} \end{cases}, \quad \alpha_2(t) = 1 - \alpha_1(t), \quad x_1(t) \in [-\pi/3, \pi/3]. \quad (52)
\]

Based on the measured output \(y = x_1(t)\) the task of the PID control is to minimize the effect of the disturbance in stabilizing the system. Therefore, the controlled output is set as \(z = x_1 + x_2\).

In this example, \(\tau = 6\) is set for (6). The minimal \(\gamma = 0.12\) is obtained by using the bisection procedure. At \(\gamma = 0.12\), Algorithm 1 needs 4 iterations to arrive the following numerical values for implementing PID PDC (6): \(R_{P1} = 72.3777, R_{P2} = 99.2379, R_{I1} = 0.1449, R_{I2} = 0.1028, R_{D1} = 5.0864\) and \(R_{D2} = 8.8573\). Figs. 1, 2 respectively show the behavior of the system state and control with disturbance \(w = 3 \sin(5\pi t)\) and with no disturbance. The initial state is \(x(0) = (\pi/4, -\pi/4)^T\). The obtained PID PDC stabilizes the inverted pendulum system well in both scenarios. The system state motion and control load are very smooth compared with [13] Fig. 2.
The membership functions in (2) are with control input $u$ different form of fuzzy systems for this oscillation. Without $z$ as $x(0) = (0.1, 0)^T$. Fig. 6 depicts the behavior of the state and PID PDC. Again the PID PDC stabilizes the Duffing forced-oscillation system well.

Meanwhile Algorithm 2 achieves worse $\gamma = 1.4$ and needs 20 iterations for converge for this value of $\gamma$. Fig. 7 shows the convergence behaviour of Algorithm 1 (for $\gamma = 1.1$) and Algorithm 2 (for $\gamma = 1.4$). Their convergence is dependent on initial points. Algorithm 1 converges not rapidly until the seventh iteration, while Algorithm 1 converges rapidly after the first iteration.

C. TORA

By [24] and [2], the eccentric rotational proof mass actuator (TORA) system can be represented by T-S model (5) with $L = 4$, $\alpha = 0.99$, $\phi = 0.1$, $c = 4$.
Algorithm 2 for the Duffing forced-oscillation system

\[ A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2\phi/\pi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \phi & 0 \\ 0 & 0 & 0 & 1 \\ \phi/(1 - \phi^2) & 0 & -\phi^2/(1 - \phi^2) & 0 \end{bmatrix}, \]

\[ A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \phi & 0 \\ 0 & 0 & 0 & 1 \\ \phi/(1 - \phi^2) & 0 & -\phi^2(1 - \phi^2)/(1 - \phi^2) & 0 \end{bmatrix}, \]

\[ B_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{1 - \phi^2} \end{bmatrix}, \]

\[ B_{23} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{24} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{1 - \phi^2} \end{bmatrix}, \]

\[ C_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \]

The membership functions in (2) are

\[ \alpha_1(t) = \frac{x_1(t)}{b \sin(x_3(t)) - x_3(t) \sin(t)} \]

\[ \alpha_2(t) = \frac{1}{2} - \alpha_1(t), \quad \alpha_3(t) = \frac{b \sin(x_3(t)) - x_3(t) \sin(t)}{x_3(t)(b - \sin(b))}, \quad \alpha_4(t) = \frac{1}{2} - \alpha_3(t), \]

with \( a = 0.8, b = 0.6, \) and \( x_1(t) \in [-a, a] \) and \( x_3(t) \in [-b, b]. \) The system state is \( x = (x_1, x_2, x_3, x_4), \) where \( x_3 = \theta \) and \( x_4 = \dot{\theta} \) are the angular position and angular velocity of the rotational proof mass, and \( x_1 = \bar{x}_1 + \epsilon \sin x_3, \) \( x_2 = \bar{x}_2 + \epsilon x_4 \cos x_3 \) with \( \bar{x}_1 = q \) and \( \bar{x}_2 = \dot{q} \) the translational position and velocity of the cart. In this application, only the translation position and angular position are measurable so \( y = (x_1, x_3)^T. \)

The main task is to minimize the effect of the disturbance \( w \) in regulating the translation and angular positions to the equilibrium so the controlled output is set as \( z = (x_1, x_3)^T. \)

We set \( \tau = 1 \) for (6) in this example. The minimal \( \gamma = 9.9 \) is obtained by the bisection procedure. For this value of \( \gamma, \) Algorithm [2] needs 4 iterations to arrive the following numerical values for implementing PID PDC (6):

\[ R_{P1} = [-7.1101, -16.1981, 11.42817], \]

\[ R_{P2} = [-5.5390, -11.9724, 8.5207], \]

\[ R_{P3} = [-5.7119, -12.9499, 9.0553], \]

\[ R_{P4} = [-5.7189, -12.9240, 9.0397], \]

\[ R_{I1} = [-0.3471, -1.0139, 0.6820], \]

\[ R_{I2} = [-0.3450, -1.01858, 0.6830], \]

\[ R_{I3} = [-0.4091, -1.1184, 0.7660], \]

\[ R_{I4} = [-0.4337, -1.1552, 0.7969], \]

\[ R_{D1} = [0.8938, 2.0883, -1.3215], \]

\[ R_{D2} = [0.6084, 1.3740, -0.7847], \]

\[ R_{D3} = [0.5537, 1.4268, -0.7742], \]

\[ R_{D4} = [0.5048, 1.4486, -0.7513]. \]
Figs. 8-9 respectively show the behavior of system state and control with disturbance $w = 10\sin(\pi t)$ and with no disturbance. The initial state condition is $x(0) = (0, 0, 0.5, 0)^T$. The TORA system is smoothly stabilized well by PID PDC.

Algorithm 1 achieves worse $\gamma = 10.3$ and needs 11 iterations for converge for this value of $\gamma$. Fig. 3 shows the convergence behaviour of Algorithm 1 (for $\gamma = 10.3$) and Algorithm 2 (for $\gamma = 9.9$).

V. CONCLUSION

This paper has addressed the problem of designing $H_\infty$ PID PDC for T-S systems based on a parameterized bilinear matrix inequality (PLMI), which is a system of infinitely many bilinear matrix inequalities. Efficient computational procedures for this PLMI have been developed. Their merit has been analysed through the benchmark examples. In the end, the effectiveness of PID PDC in smoothly stabilizing nonlinear systems has been confirmed.

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