ON THE HEIGHT OF SOLUTIONS TO NORM FORM EQUATIONS

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Abstract. Let $k$ be a number field. We consider norm form equations associated to a full $O_k$-module contained in a finite extension field $l$. It is known that the set of solutions is naturally a union of disjoint equivalence classes of solutions. We prove that each nonempty equivalence class of solutions contains a representative with Weil height bounded by an expression that depends on parameters defining the norm form equation.

1. Introduction

Classically norm form equations are defined over the field of rational numbers. Let $\omega_1, \omega_2, \ldots, \omega_N$, be points in $\mathbb{Q}$ that are $\mathbb{Q}$-linearly independent, and let $K = \mathbb{Q}(\omega_1, \omega_2, \ldots, \omega_N)$ be the algebraic number field that they generate. We assume that $[K : \mathbb{Q}] = d$, and we write $\sigma_1, \sigma_2, \ldots, \sigma_d$, for the distinct embeddings of $K$ into $\overline{\mathbb{Q}}$. Using $\omega_1, \omega_2, \ldots, \omega_N$, we define a homogeneous polynomial in a vector variable $x$ having $N$ independent coordinates $x_1, x_2, \ldots, x_N$, by

$$G(x) = \prod_{i=1}^{d} \left\{ \sum_{n=1}^{N} \sigma_i(\omega_n)x_n \right\}. \tag{1.1}$$

It is easy to verify that $G(x)$ has rational coefficients and, as $\omega_1, \omega_2, \ldots, \omega_N$, are $\mathbb{Q}$-linearly independent, $G(x)$ is not identically zero. The homogeneous polynomial $G(x)$ is called a norm form, because if $\xi$ is a nonzero point with rational integer coordinates $\xi_1, \xi_2, \ldots, \xi_N$, then

$$G(\xi) = \text{Norm}_{K/\mathbb{Q}}(\omega_1\xi_1 + \omega_2\xi_2 + \cdots + \omega_N\xi_N),$$

where

$$\text{Norm}_{K/\mathbb{Q}} : K^\times \to \mathbb{Q}^\times$$

is the norm homomorphism. In [13] Schmidt proved his fundamental result, that a norm form equation $G(x) = b$, where $b \in \mathbb{Q}$, has only finitely many solutions if $G$ satisfies some natural non-degeneracy condition. Later, in another breakthrough work [14], Schmidt dealt also with the case that $G$ is degenerate and showed that in that case, the set of solutions of the norm form equation can be partitioned in a natural way into families, and is the union of finitely many such families. This was soon followed by an analogous $p$-adic result due to Schlickewei [12]. Schmidt’s results have been generalized in different interesting ways (e.g. [6, 7, 8]), in particular

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Laurent, in [9], considered norm form equations into $k$, a finite algebraic extension of $\mathbb{Q}$.

Let $k$ and $l$ be algebraic number fields such that

$$\mathbb{Q} \subseteq k \subseteq l \subseteq \overline{\mathbb{Q}},$$

where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. We write $k^\times$ and $l^\times$ for the multiplicative group of nonzero elements in $k$ and $l$, respectively, and

$$(1.2) \quad \text{Norm}_{l/k} : l^\times \to k^\times$$

for the norm homomorphism. We also write $O_k$ for the ring of algebraic integers in $k$, $O_k^\times$ for the multiplicative group of units in $O_k$, and Tor($O_k^\times$) for the finite group of roots of unity in $O_k^\times$. Then $O_l$, $O_l^\times$, and Tor($O_l^\times$) are the analogous subsets in $l$.

Let $\omega_1, \omega_2, \ldots, \omega_e$, be $k$-linearly independent elements of $l$ that form a basis for $l$ as a $k$-vector space, and let $\sigma_1, \sigma_2, \ldots, \sigma_e$, be the collection of distinct embeddings of $l$ into $\overline{\mathbb{Q}}$ that fix the subfield $k$. It follows that

$$(1.3) \quad F(x) = \prod_{i=1}^e (\sigma_i(\omega_1)x_1 + \sigma_i(\omega_2)x_2 + \cdots + \sigma_i(\omega_e)x_e)$$

is a homogeneous polynomial of degree $e$ in independent variables $x_1, x_2, \ldots, x_e$, and the coefficients of $F$ belong to the field $k$. The homogeneous polynomial $F(x)$ defined by (1.3) is an example of a norm form. For $\beta \neq 0$ in $k$, we consider the norm form equation

$$(1.4) \quad F(x) = \zeta \beta, \quad \text{where } \zeta \in \text{Tor}(O_k^\times),$$

and we seek to describe the solutions in $(O_k)^e$.

Rather than working with the polynomial $F$ defined by (1.3), we will work instead with the full $O_k$-module

$$(1.5) \quad \mathfrak{M} = \{ \omega_1 \nu_1 + \omega_2 \nu_2 + \cdots + \omega_e \nu_e : \nu_i \in O_k \text{ for } i = 1, 2, \ldots, e \}$$

generated by the basis $\omega_1, \omega_2, \ldots, \omega_e$. If $\nu = (\nu_i)$ is a nonzero point in $(O_k)^e$, we have

$$F(\nu) = \text{Norm}_{l/k}(\mu),$$

where

$$\mu = \omega_1 \nu_1 + \omega_2 \nu_2 + \cdots + \omega_e \nu_e$$

belongs to the full $O_k$-module $\mathfrak{M}$. Thus for $\beta \neq 0$ in $k$, we wish to describe the set of solutions

$$(1.6) \quad \{ \mu \in \mathfrak{M} : \text{Norm}_{l/k}(\mu) \in \text{Tor}(O_k^\times) \beta \}.$$

There is a natural equivalence relation in $\mathfrak{M} \setminus \{0\}$, such that the set (1.6) is either empty, or it is a disjoint union of finitely many equivalence classes.

If $\alpha \neq 0$ belongs to $l$, then $\omega_1 \alpha, \omega_2 \alpha, \ldots, \omega_e \alpha$, is also a basis for $l$ as a $k$-vector space. This second basis generates the full $O_k$-module

$$\alpha \mathfrak{M} = \{ \alpha \omega_1 \nu_1 + \alpha \omega_2 \nu_2 + \cdots + \alpha \omega_e \nu_e : \nu_i \in O_k \text{ for } i = 1, 2, \ldots, e \}.$$ 

We say that the $O_k$-modules $\mathfrak{M}$ and $\alpha \mathfrak{M}$ are proportional. It is obvious that proportionality is an equivalence relation in the collection of all full $O_k$-modules contained in $l$. As

$$\text{Norm}_{l/k}(\alpha \mu) = \text{Norm}_{l/k}(\alpha) \text{Norm}_{l/k}(\mu),$$
the problem of describing the solution set (1.6) changes insignificantly if the \( O \)-module \( \mathfrak{M} \) is replaced by a proportional \( O_k \)-module \( \alpha \mathfrak{M} \). Each proportionality class plainly contains a representative that is a subset of \( O_l \). Therefore in the remainder of this paper we assume that \( \mathfrak{M} \subseteq O_l \). With this assumption we can restrict our attention to solution sets (1.6) such that \( \beta \neq 0 \) also belongs to \( O_k \).

The coefficient ring associated to the full module \( \mathfrak{M} \) is the subset

\[
O_{\mathfrak{M}} = \{ \alpha \in l : \alpha \mathfrak{M} \subseteq \mathfrak{M} \}.
\]

It is easy to check that proportional \( O_k \)-modules contained in \( l \) have the same coefficient ring. Let \( \psi_1, \psi_2, \ldots, \psi_f \), be an integral basis for \( O_k \). It follows that

\[
eq [l : k], \quad f = [k : \mathbb{Q}],
\]

and that

\[
\{ \omega_i \psi_j : i = 1, 2, \ldots, e, \text{ and } j = 1, 2, \ldots, f \}
\]

is a basis for \( \mathfrak{M} \) as a full \( \mathbb{Z} \)-module in \( l \). Therefore we can appeal to classical results such as [3, Chap. 2, Sec. 2, Theorem 3], and conclude that the coefficient ring \( O_{\mathfrak{M}} \) is an order in \( l \). We recall (see [10, Chapter 5, section 1]) that \( O_l \) is the maximal order in \( l \), so that

\[
O_{\mathfrak{M}} \subseteq O_l.
\]

Let \( r(l) \) be the rank of the group \( O_l^\times \) of units in \( O_l \), and let \( r(k) \) be the rank of \( O_k^\times \). By the extension of Dirichlet’s unit theorem to orders, the subgroup

\[
O_{\mathfrak{M}}^\times = O_{\mathfrak{M}} \cap O_l^\times
\]

of units in \( O_{\mathfrak{M}} \) has rank \( r(l) \), and therefore the index \( [O_l^\times : O_{\mathfrak{M}}^\times] \) is finite. And it follows from (1.7) that

\[
O_{\mathfrak{M}}^\times = \{ \alpha \in l : \alpha \mathfrak{M} = \mathfrak{M} \}.
\]

Hence the group \( O_{\mathfrak{M}}^\times \) acts on the module \( \mathfrak{M} \) by multiplication. Let

\[
\mathcal{E}_{l/k}(\mathfrak{M}) = \{ \alpha \in O_{\mathfrak{M}}^\times : \text{Norm}_{l/k}(\alpha) \in \text{Tor}(O_k^\times) \}
\]

be the subgroup of relative units in the coefficient ring \( O_{\mathfrak{M}} \). In Lemma 2.1 we show that the subgroup \( \mathcal{E}_{l/k}(\mathfrak{M}) \) has rank

\[
r(l/k) = r(l) - r(k).
\]

Now suppose that \( \beta \neq 0 \) belongs to \( O_k \), and \( \mu \in \mathfrak{M} \) satisfies

\[
\text{Norm}_{l/k}(\mu) = \zeta \beta, \quad \text{where } \zeta \in \text{Tor}(O_k^\times).
\]

If \( \gamma \) belongs to the group \( \mathcal{E}_{l/k}(\mathfrak{M}) \) of relative units in \( O_{\mathfrak{M}} \), then (1.9) implies that \( \gamma \mu \) belongs to \( \mathfrak{M} \). And it follows from (1.10) that

\[
\text{Norm}_{l/k}(\gamma \mu) = \text{Norm}_{l/k}(\gamma) \zeta \beta = \zeta' \beta, \quad \text{where } \zeta' \in \text{Tor}(O_k^\times).
\]

We say that two nonzero elements \( \mu_1 \) and \( \mu_2 \) in \( \mathfrak{M} \) are equivalent if there exists an element \( \gamma \) in the group \( \mathcal{E}_{l/k}(\mathfrak{M}) \) such that \( \gamma \mu_1 = \mu_2 \). It is trivial that this is an equivalence relation in \( \mathfrak{M} \setminus \{0\} \). Indeed, each equivalence class is also a coset in the quotient group \( l^\times / \mathcal{E}_{l/k}(\mathfrak{M}) \). It follows from (1.11) and (1.12) that for each \( \beta \neq 0 \) in \( O_k \), the set (1.6) is a disjoint union of equivalence classes. It is known that (1.6) is a disjoint union of finitely many such equivalence classes (see [9, 14]). A finiteness result of this sort also follows from Northcott’s theorem [11] (see also [2, Theorem 1.6.8]) and the following inequality. Here we write \( \alpha \mapsto h(\alpha) \) for the Weil height of an algebraic number \( \alpha \neq 0 \), and we define this explicitly in (5.2).
Theorem 1.1. Let the full $O_k$-module $\mathfrak{M} \subseteq O_l$ be defined by (1.5), and assume that the rank $r(l/k)$ of the group $E_{l/k}(\mathfrak{M})$ of relative units is positive. Let
\begin{equation}
\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r(l/k),
\end{equation}
be multiplicatively independent units in the subgroup $E_{l/k}(\mathfrak{M})$. Assume that $\beta \neq 0$ is a point in $O_k$, and $\mu \neq 0$ is a point in $\mathfrak{M}$, such that
\begin{equation}
\text{Norm}_{l/k}(\mu) = \zeta \beta, \quad \text{where } \zeta \in \text{Tor}(O_k^\times).
\end{equation}
Then there exists an element $\gamma$ in $E_{l/k}(\mathfrak{M})$, such that $\gamma \mu$ belongs to $\mathfrak{M}$,
\begin{equation}
\text{Norm}_{l/k}(\gamma \mu) = \zeta' \beta, \quad \text{where } \zeta' \in \text{Tor}(O_k^\times),
\end{equation}
and
\begin{equation}
\begin{split}
2. \text{ The rank of the group of relative units}
\end{split}
\end{equation}

Following Costa and Friedman \cite{4} and \cite{5}, the subgroup of relative units in $O_l^\times$ with respect to the subfield $k$, is defined by
\begin{equation}
E_{l/k} = \{ \alpha \in O_l^\times : \text{Norm}_{l/k}(\alpha) \in \text{Tor}(O_k^\times) \}.
\end{equation}
Hence the subgroup of relative units in $O_{2\mathfrak{M}}$ is
\begin{equation}
E_{l/k}(\mathfrak{M}) = E_{l/k} \cap O_{2\mathfrak{M}}^\times.
\end{equation}
Costa and Friedman show that $E_{l/k}$ has rank $r(l) - r(k)$ (see also \cite{4} section 3). Here we show that the subgroup $E_{l/k}(\mathfrak{M})$ also has rank $r(l) - r(k)$.

Lemma 2.1. Let the full $O_k$-module $\mathfrak{M} \subseteq O_l$ be defined by (1.5), and let the subgroup $E_{l/k}(\mathfrak{M})$ of relative units in $O_{2\mathfrak{M}}$ be defined by (1.10). Then the rank of $E_{l/k}(\mathfrak{M})$ is
\begin{equation}
r(l/k) = r(l) - r(k).
\end{equation}

Proof. As $O_{2\mathfrak{M}}$ is an order in $l$, it follows from the extension of Dirichlet’s unit theorem to orders (see \cite{3} Chap. 2, Sec. 4, Theorem 5) that the group of units $O_{2\mathfrak{M}}^\times$ has rank $r(l)$. The norm (1.2) restricted to $O_{2\mathfrak{M}}^\times$ is a homomorphism
\begin{equation}
\text{Norm}_{l/k} : O_{2\mathfrak{M}}^\times \to O_k^\times,
\end{equation}
and the norm restricted to the torsion subgroup $\text{Tor}(O_{2\mathfrak{M}}^\times)$ is a homomorphism
\begin{equation}
\text{Norm}_{l/k} : \text{Tor}(O_{2\mathfrak{M}}^\times) \to \text{Tor}(O_k^\times).
Hence we get a well defined homomorphism, which we write as
\[ \text{norm}_{l/k} : O_{2\mathfrak{M}}^\times / \text{Tor}(O_{2\mathfrak{M}}^\times) \to O_k^\times / \text{Tor}(O_k^\times), \]
by setting
\[ \text{norm}_{l/k}(\alpha \text{Tor}(O_{2\mathfrak{M}}^\times)) = \text{Norm}_{l/k}(\alpha) \text{Tor}(O_k^\times). \]
To simplify notation we write
\[ F_{2\mathfrak{M}} = O_{2\mathfrak{M}}^\times / \text{Tor}(O_{2\mathfrak{M}}^\times), \]
and
\[ F_k = O_k^\times / \text{Tor}(O_k^\times), \]
and we use coset representatives rather than cosets for points in \( F_{2\mathfrak{M}} \) and \( F_k \). It is clear that \( F_{2\mathfrak{M}} \) and \( F_k \) are free groups such that
\[ \text{rank } F_{2\mathfrak{M}} = \text{rank } O_{2\mathfrak{M}}^\times = r(l), \quad \text{rank } F_k = \text{rank } O_k^\times = r(k), \]
and
\[ \text{norm}_{l/k} : F_{2\mathfrak{M}} \to F_k. \]
We note that the image of the subgroup
\[ E_{l/k}(\mathfrak{M}) = \{ \alpha \in O_{2\mathfrak{M}}^\times : \text{Norm}_{l/k}(\alpha) \in \text{Tor}(O_k^\times) \} \]
of relative units in the group \( F_{2\mathfrak{M}} \), is the kernel
\[ \{ \alpha \in F_{2\mathfrak{M}} : \text{norm}_{l/k}(\alpha) = 1 \} \]
of the homomorphism \( \text{norm}_{l/k} \). Thus it suffice to show that the kernel (2.3) has rank \( r(l) - r(k) \).

Let \( \varphi_1, \varphi_2, \ldots, \varphi_{r(k)} \) be multiplicatively independent elements in the group \( O_k^\times \).
As \( O_k^\times \subseteq O_l^\times \) and the index \( [O_l^\times : O_k^\times] \) is finite, there exist positive integers \( m_1, m_2, \ldots, m_{r(k)} \) such that
\[ \varphi_1^{m_1}, \varphi_2^{m_2}, \ldots, \varphi_{r(k)}^{m_{r(k)}} \]
are multiplicatively independent elements in
\[ O_{2\mathfrak{M}}^\times \cap O_k^\times. \]
For each \( j = 1, 2, \ldots, r(k) \) we have
\[ \text{norm}_{l/k}(\varphi_j^{m_j}) = \varphi_j^{em_j}, \]
where \( e = [l : k] \). Hence the image of the homomorphism \( \text{norm}_{l/k} \) in \( O_k^\times \) has rank \( r(k) \). It follows that the kernel of the homomorphism \( \text{norm}_{l/k} \) has rank \( r(l) - r(k) \).

We have already noted that this is the rank of \( E_{l/k}(\mathfrak{M}) \), and so the proof of the lemma is complete. \( \square \)

3. INEQUALITIES FOR RELATIVE UNITS

At each place \( w \) of \( l \) we write \( l_w \) for the completion of \( l \) at \( w \), so that \( l_w \) is a local field. We select two absolute values \( || ||_w \) and \( | |_w \) from the place \( w \). The absolute value \( || ||_w \) extends the usual archimedean or nonarchimedean absolute value on the subfield \( \mathbb{Q} \). Then \( | |_w \) must be a power of \( || ||_w \), and we set
\[ | |_w = || ||_w^{d_w/d}, \]
where \( d_w = [l_w : \mathbb{Q}_w] \) is the local degree of the extension, and \( d = [l : \mathbb{Q}] \) is the global degree. With these normalizations the Weil height (or simply the height) is a function
\[ h : l^\times \to [0, \infty) \]
defined at each algebraic number $\alpha$ that belongs to $l^\times$, by

$$h(\alpha) = \sum_w \log^+ |\alpha|_w = \frac{1}{2} \sum_w \log |\alpha|_w.$$  

(3.2)

Each sum in (3.2) is over the set of all places $w$ of $l$, and the equality between the two sums follows from the product formula. Then $h(\alpha)$ depends on the algebraic number $\alpha \neq 0$, but it does not depend on the number field $l$ that contains $\alpha$. It is often useful to recall that the height is constant on each coset of the quotient group $l^\times / \text{Tor}(l^\times)$, and therefore we have $h(\zeta \alpha) = h(\alpha)$ for each element $\alpha$ in $l^\times$, and each root of unity $\zeta$ in $\text{Tor}(l^\times)$. Elementary properties of the height (see [2] for further details) imply that the map $(\alpha, \beta) \mapsto h(\alpha \beta^{-1})$ defines a metric on the group $l^\times / \text{Tor}(l^\times)$. If $\alpha$ belongs to the subgroup $k^\times$, we have

$$h(\alpha) = \frac{1}{2} \sum_w \log |\alpha|_w = \frac{1}{2} \sum_v \log |\alpha|_v,$$  

(3.3)

where the sum on the right of (3.3) is over the set of all places $v$ of $k$, and the absolute values $| \cdot |_v$ are normalized with respect to $k$. We write $v$ for a place of $k$, and use $w$ or $x$ for a place of $l$. Additional properties of the Weil height on groups are discussed in [1], and [15].

For each place $v$ of $k$ we write

$$W_v(l/k) = \{ w : w \text{ is a place of } l \text{ and } w|v \}.$$  

The set $W_\infty(l/Q)$ of archimedean (or infinite) places of $l$ has cardinality $r(l) + 1$, and similarly the set $W_\infty(k/Q)$ has cardinality $r(k) + 1$. Let $\mathbb{R}^{r(l)+1}$ denote the real vector space of (column) vectors $\xi = (\xi_w)$ with coordinates indexed by places $w$ in $W_\infty(l/Q)$. We define

$${\mathcal D}_{r(l/k)} = \left\{ \xi \in \mathbb{R}^{r(l)+1} : \sum_{w|v} \xi_w = 0 \text{ for each } v \text{ in } W_\infty(k/Q) \right\},$$

(3.4)

so that $${\mathcal D}_{r(l/k)}$$ is a subspace of dimension $$(r(l) + 1) - (r(k) + 1) = r(l) - r(k) = r(l/k),$$

contained in $\mathbb{R}^{r(l)+1}$. Let $\eta_1, \eta_2, \ldots, \eta_{r(l/k)}$, be a fundamental system of units for the subgroup $\mathcal E_{l/k}(\mathfrak{M})$ of relative units in $O_{l/k}^\times$, so that

$$\mathcal E_{l/k}(\mathfrak{M}) = \text{Tor}(\mathcal E_{l/k}(\mathfrak{M})) \otimes \langle \eta_1, \eta_2, \ldots, \eta_{r(l/k)} \rangle.$$

Then let $L$ denote the $(r(l) + 1) \times r(l/k)$ real matrix

$$L = (\log |\eta_j|_w),$$

(3.5)

where $w$ in $W_\infty(l/Q)$ indexes rows, and $j = 1, 2, \ldots, r(l/k)$, indexes columns. Because the relative regulator does not vanish (see [4] and [3]), it follows that the matrix $L$ has $\mathbb{R}$-rank equal to $r(l/k)$. Then using the product formula we find that

$$y \mapsto Ly = \left( \sum_{j=1}^s y_j \log |\eta_j|_v \right)$$

(3.6)

is a linear map from the $\mathbb{R}$-linear space

$$\mathbb{R}^{r(l/k)} = \{ y = (y_j) : j = 1, 2, \ldots, r(l/k), \text{ and } y_j \in \mathbb{R} \}$$

(3.7)

onto the subspace $${\mathcal D}_{r(l/k)}.$$
Lemma 3.1. Let \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r(l/k)} \) be a collection of multiplicatively independent elements in the group \( \mathcal{E}_{l/k}(\mathcal{M}) \) of relative units, and write

\[
\mathcal{E} = \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r(l/k)} \rangle \subseteq \mathcal{E}_{l/k}(\mathcal{M})
\]

for the subgroup they generate. Let \( z = (z_w) \) be a vector in the subspace \( \mathcal{D}_{r(l/k)} \). Then there exists a point \( \gamma \) in \( \mathcal{E} \) such that

\[
\sum_{w|\infty} \left| \log |\gamma|_w - z_w \right| \leq r(l/k) \sum_{j=1}^{r(l/k)} h(\varepsilon_j).
\]

Proof. Let \( M \) be the \((r(l) + 1) \times r(l/k)\) real matrix

\[
M = (\log |\varepsilon_j|_w),
\]

where \( w \) in \( W_\infty(l/Q) \) indexes rows, and \( j = 1, 2, \ldots, r(l/k) \), indexes columns. Because \( \eta_1, \eta_2, \ldots, \eta_{r(l/k)} \) is a basis for the group \( E_{l/k} \), there exists an \( r(l/k) \times r(l/k) \) matrix \( A = (a_{ij}) \) with integer entries such that

\[
\log |\varepsilon_j|_w = \sum_{i=1}^{r(l/k)} a_{ij} \log |\eta_i|_w
\]

for each place \( w \) in \( W_\infty(l/Q) \) and each \( j = 1, 2, \ldots, r(l/k) \). Alternatively, we have the matrix equation

\[
M = LA.
\]

By hypothesis \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r(l/k)} \), are multiplicatively independent elements of \( E_{l/k} \). It follows that \( A \) is nonsingular, and \( M \) has rank \( r(l/k) \). Using (3.6) we conclude that

\[
y \mapsto Ay \mapsto LAy = My = \left( \sum_{j=1}^{r(l/k)} y_j \log |\varepsilon_j|_w \right)
\]

is a linear map from the \( \mathbb{R} \)-linear space \( (3.7) \) onto the subspace \( \mathcal{D}_{r(l/k)} \). In particular, there exists a unique point \( u = (u_j) \) in \( (3.7) \) such that

\[
z_w = \sum_{j=1}^{r(l/k)} u_j \log |\varepsilon_j|_w
\]

at each place \( w \) in \( W_\infty(l/Q) \). Let \( m = (m_j) \) in \( \mathbb{Z}^{r(l/k)} \) satisfy

\[
|m_j - u_j| \leq \frac{1}{2}, \quad \text{for each } j = 1, 2, \ldots, r(l/k).
\]

Then write

\[
\gamma = \varepsilon_1^{m_1} \varepsilon_2^{m_2} \cdots \varepsilon_s^{m_s}, \quad \text{where } s = r(l/k),
\]
so that \(\gamma\) belongs to the subgroup \(\mathcal{E}\). Using (3.2), (3.10), (3.11), and (3.12), we find that

\[
\sum_{w|\infty} |\log |\gamma|_w - z_w| = \sum_{w|\infty} \left| \sum_{i=1}^{r(l/k)} m_i \log |\varepsilon_i|_w - \sum_{j=1}^{r(l/k)} u_j \log |\varepsilon_j|_w \right|
\]

\[
\leq \sum_{w|\infty} \sum_{j=1}^{r(l/k)} |m_j - u_j| \log |\varepsilon_j|_w
\]

\[
\leq \frac{1}{2} \sum_{w|\infty} \sum_{j=1}^{r(l/k)} \log |\varepsilon_j|_w
\]

\[
= \sum_{j=1}^{r(l/k)} h(\varepsilon_j).
\]

This proves the lemma.

\[\square\]

**Lemma 3.2.** Let \(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r(l/k)}\), be a collection of multiplicatively independent elements in the group \(E_{l/k}(\mathbb{M})\) of relative units, and write

\[
\mathcal{E} = \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r(l/k)} \rangle \subseteq E_{l/k}(\mathbb{M})
\]

for the subgroup they generate. If \(\mu\) belongs to \(k^*\), then there exists \(\gamma\) in \(\mathcal{E}\) such that

\[
\sum_{v|\infty \atop v|w} |\log |\gamma\mu|_w - |W_v(l/k)|^{-1} \sum_{x|v} \log |\gamma\mu|x| \leq \sum_{j=1}^{r(l/k)} h(\varepsilon_j).
\]

**Proof.** Let \(z = (z_w)\) be the vector in \(\mathbb{R}^{r(l)+1}\) defined at each place \(w\) in \(W_v(l/k)\) by

\[
z_w = |W_v(l/k)|^{-1} \sum_{x|v} \log |\mu|x| - \log |\mu|_w.
\]

It follows that at each place \(v\) in \(W_{\infty}(k/Q)\) we have

\[
\sum_{w|v} z_w = \sum_{w|v} \left( |W_v(l/k)|^{-1} \sum_{x|v} \log |\mu|x| - \log |\mu|_w \right)
\]

\[
= \sum_{x|v} \log |\mu|x| - \sum_{w|v} \log |\mu|_w
\]

\[
= 0.
\]

Therefore \(z = (z_w)\) belongs to the subspace \(D_{r(l/k)}\). By Lemma 3.1 there exists an element \(\gamma\) in \(\mathcal{E}\) such that

\[
\sum_{w|\infty} |\log |\gamma|_w - z_w| \leq \sum_{j=1}^{r(l/k)} h(\varepsilon_j).
\]

If \(w|v\) then using (3.14) and (3.14), we find that

\[
|\log |\gamma|_w - z_w| = |\log |\gamma\mu|_w - |W_v(l/k)|^{-1} \sum_{x|v} \log |\mu|x| |
\]

\[
= |\log |\gamma\mu|_w - |W_v(l/k)|^{-1} \sum_{x|v} \log |\gamma\mu|x| |.
\]
The inequality (3.13) follows by combining (3.15) and (3.16). □

4. PROOF OF THEOREM 1.1 AND THEOREM 1.2

We suppose that the full $O_k$-module $M \subseteq O_l$ is defined as in (1.5), and that the rank $r(l/k)$ of the group $\mathcal{E}_{l/k}(\mathfrak{M})$ of relative units is positive. Let

$$\mathfrak{E} = \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r(l/k)} \rangle \subseteq \mathcal{E}_{l/k}(\mathfrak{M})$$

be the subgroup generated by the multiplicatively independent units (1.13), and assume that $\beta \neq 0$ in $O_k$, and $\mu \neq 0$ in $\mathfrak{M}$, satisfy (1.14).

Let $\gamma$ be a point in $\mathfrak{E}$ such that the inequality (3.13) holds. Then at each place $v$ of $k$ we have

$$[l : k] \sum_{w|v} \log |\gamma\mu|_w = \log |\text{Norm}_{l/k}(\gamma\mu)|_v$$

(4.1)

$$= \log |\text{Norm}_{l/k}(\mu)|_v$$

$$= \log |\beta|_v.$$  

We also have

$$2[l : k] h(\gamma \mu) = [k : l] \sum_{w|\infty} \log |\gamma\mu|_w + [k : l] \sum_{w|\infty} \left| \log |\gamma\mu|_w \right|.$$  

(4.2)

Using (3.13) and (1.1) we estimate the first sum on the right of (4.2) by

$$[l : k] \sum_{w|\infty} \left| \log |\gamma\mu|_w \right| \leq [l : k] \sum_{v|\infty} \sum_{w|v} \log |\gamma\mu|_w - \sum_{x|v} \sum_{w|v} \left| \log |\gamma\mu|_x \right|$$

$$+ [l : k] \sum_{v|\infty} \left| \log |\gamma\mu|_v \right|$$

(4.3)

$$\leq [l : k] \sum_{j=1}^{r(l/k)} h(\varepsilon_j) + \sum_{v|\infty} \left| \log |\text{Norm}_{l/k}(\gamma\mu)|_v \right|$$

$$= [l : k] \sum_{j=1}^{r(l/k)} h(\varepsilon_j) + \sum_{v|\infty} \left| \log |\beta|_v \right|.$$  

As $\mu$ and $\gamma\mu$ belong to $O_l$, we get

$$\log |\gamma\mu|_w \leq 0$$

at each finite place $w$ of $l$. Hence the second sum on the right of (4.2) is

$$[l : k] \sum_{w|\infty} \left| \log |\gamma\mu|_w \right| = - [l : k] \sum_{v|\infty} \sum_{w|v} \log |\gamma\mu|_w$$

(4.4)

$$= - \sum_{v|\infty} \left| \log |\text{Norm}_{l/k}(\gamma\mu)|_v \right|$$

$$= \sum_{v|\infty} \left| \log |\beta|_v \right|.$$
By combining (4.2), (4.3), and (4.4), we find that
\[
2[l : k]h(\gamma \mu) \leq [l : k] \sum_{j=1}^{r(l/k)} h(\varepsilon_j) + \sum_{v} |\log |\beta|_v| \\
= [l : k] \sum_{j=1}^{r(l/k)} h(\varepsilon_j) + 2h(\beta).
\]
(4.5)

The inequality (4.5) is also (1.16) in the statement of Theorem 1.1.

Next we prove Theorem 1.2, where we assume that the rank of \(E_{l/k}(M)\) is zero. That is, we assume that the rank \(r(k)\) is equal to the rank \(r(l)\). In general we have \(r(k) \leq r(l)\), and we recall (see [10, Proposition 3.20]) that \(r(k) = r(l)\) if and only if \(l\) is a CM-field, and \(k\) is the maximal totally real subfield of \(l\). Assume that \(\beta \neq 0\) in \(O_k\), and \(\mu \neq 0\) in \(M\), satisfy (1.17). As in (4.1) we have
\[
[l : k] \sum_{w|v} \log |\mu|_{w} = \log |\text{Norm}_{k/l}(\mu)|_{v} = \log |\beta|_{v}
\]
(4.6)

at each place \(v\) of \(k\). Because \(l\) is a CM-field and \(k\) is the maximal totally real subfield of \(l\), for each archimedean place \(v\) of \(k\) the set \(W_v(l/k)\) contains exactly one place of \(l\). If for each archimedean place \(v\) of \(k\) we write
\[
W_v(l/k) = \{w_v\},
\]
then (4.6) asserts that
\[
[l : k] \log |\mu|_{w_v} = \log |\beta|_{v}.
\]
(4.6)

In particular, at each archimedean place \(v\) of \(k\) we get
\[
[l : k] \log^{+} |\mu|_{w_v} = \log^{+} |\beta|_{v}.
\]
(4.8)

As \(\beta \neq 0\) and \(\mu \neq 0\) are algebraic integers, we have
\[
\log |\beta|_{v} \leq 0, \quad \text{and} \quad \log |\mu|_{w} \leq 0,
\]
at each nonarchimedean place \(v\) of \(k\), and each nonarchimedean place \(w\) of \(l\). Now (4.8) and (4.9) imply that
\[
[l : k]h(\mu) = [l : k] \sum_{v|\infty} \log^{+} |\mu|_{w_v} = \sum_{v|\infty} \log^{+} |\beta|_{v} = h(\beta).
\]
(4.10)

This verifies the identity (1.18).

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