Bounding the Mostar index

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Abstract

Došlić et al. defined the Mostar index of a graph $G$ as
$$ Mo(G) = \sum_{uv \in E(G)} |n_G(u, v) - n_G(v, u)|, $$
where, for an edge $uv$ of $G$, the term $n_G(u, v)$ denotes the number of vertices of $G$ that have a smaller distance in $G$ to $u$ than to $v$. They conjectured that $Mo(G) \leq \frac{148}{27} n^3$ for every graph $G$ of order $n$. As a natural upper bound on the Mostar index, Geneson and Tsai implicitly consider the parameter $Mo^*(G) = \sum_{uv \in E(G)} (n - \min\{d_G(u), d_G(v)\})$. For a graph $G$ of order $n$, they show that $Mo^*(G) \leq \frac{5}{27}(1 + o(1))n^3$.

We improve this bound to $Mo^*(G) \leq \left(\frac{2}{\sqrt{3}} - 1\right)n^3$, which is best possible up to terms of lower order. Furthermore, we show that $Mo^*(G) \leq \left(\frac{2}{\sqrt{\Delta}} + \left(\frac{\Delta}{n}\right)^2\right)n^3$ provided that $G$ has maximum degree $\Delta$.

Keywords: Mostar index; distance unbalance

1 Introduction

Došlić et al. [9] defined the Mostar index $Mo(G)$ of a (finite and simple) graph $G$ as
$$ Mo(G) = \sum_{uv \in E(G)} |n_G(u, v) - n_G(v, u)|, $$
where, for an edge $uv$ of $G$, the term $n_G(u, v)$ denotes the number of vertices of $G$ that have a smaller distance in $G$ to $u$ than to $v$. Since its introduction in 2018 the Mostar index has already incited a lot of research, mostly concerning sparse graphs and trees [3,7,16,17,20], chemical graphs [5,6,14,15,21], and hypercube-related graphs [12,19], see also the recent survey [2].

Došlić et al. [9] conjectured that $S_{n/3,2n/3}$ has maximum Mostar index among all graphs of order $n$ (cf. [9, Conjecture 20]), where $n$ is a multiple of 3, and $S_{k,n-k}$ denotes the split graph that arises from the disjoint union of a clique $C$ of order $k$ and an independent set $I$ of order $n-k$ by adding all possible edges between $C$ and $I$. Note that
$$ Mo(S_{n/3,2n/3}) = \frac{4}{27}(1 - o(1))n^3 = 0.148(1 - o(1))n^3. $$

As observed in [9] the Mostar index of a graph $G$ of order $n$ is less than $\frac{n^3}{2}$, each of its less than $\frac{n^2}{2}$ edges contributes less than $n$ to $Mo(G)$. Geneson and Tsai [13] improved this trivial upper bound to
\[ \frac{2}{3^n}(1 + o(1))n^3 \approx 0.2083(1 + o(1))n^3. \] They actually show this upper bound for the parameter

\[ Mo^*(G) = \sum_{uv \in E(G)} (n - \min\{d_G(u), d_G(v)\}), \]

where \( d_G(u) \) denotes the degree of a vertex \( u \) in \( G \). The parameter \( Mo^*(G) \) is a natural upper bound on \( Mo(G) \): If \( uv \) is an edge of \( G \) with \( n_G(u, v) \geq n_G(v, u) \), then \( n_G(v, u) \geq |\{v\}| = 1 \) and \( n_G(u, v) \leq |V(G) \setminus (N_G[v] \setminus \{u\})| = n - d_G(v) \), where \( N_G[v] \) denotes the closed neighborhood of \( v \) in \( G \). We obtain

\[ |n_G(u, v) - n_G(v, u)| = n_G(u, v) - n_G(v, u) \leq n - 1 - d_G(v) < n - \min\{d_G(u), d_G(v)\}, \]

and, hence,

\[ Mo(G) \leq Mo^*(G) \text{ for every graph } G. \]

As our first main result, we prove the following.

**Theorem 1.** If \( G \) is a graph of order \( n \), then

\[ Mo^*(G) \leq \left( \frac{2}{\sqrt{3}} - 1 \right) n^3 \leq 0.1548n^3. \]

Theorem 1 is best possible up to terms of lower order, which means that we cannot prove Conjecture 20 from [9] by considering only \( Mo^*(G) \): Starting with a complete bipartite graph whose smaller partite set contains about a \( \gamma = \frac{\sqrt{3} - 1}{2} \approx 0.366 \) fraction of all vertices, and recursively inserting in that smaller partite set a further complete bipartite graph whose smaller partite set contains about a \( \gamma \) fraction of its vertices yields a recursive construction of a graph \( G \) of order \( n \) with \( Mo^*(G) = \left( \frac{2}{\sqrt{3}} - 1 \right) (1 - o(1))n^3 \). Note that the complement of the constructed graph is the disjoint union of cliques of approximate orders \( \gamma^i(1 - \gamma)n \) for \( i = 0, 1, 2, 3, \ldots \). Inspecting the proof of Theorem 1 reveals that the above recursive construction of the (approximately) extremal graphs is quite natural for \( Mo^* \), which is a rather unusual and mathematically pleasing feature of this new parameter.

Our second main result relies on a linear programming approach that we introduced in [18], where we determined essentially best possible upper bounds on the Mostar index of bipartite graphs and split graphs.

**Theorem 2.** If \( G \) is a graph of order \( n \) and maximum degree \( \Delta \), then

\[ Mo^*(G) \leq \left( 2 \left( \frac{\Delta}{n} \right)^2 + \left( \frac{\Delta}{n} \right) - 2 \left( \frac{\Delta}{n} \right)^2 \right) n^3. \]

The bound in Theorem 2 is increasing in \( \Delta/n \) and improves Theorem 1 only for \( \Delta \) up to about 0.725n. Before we proceed to Section 2 where we prove our two theorems, we discuss further results and possible approaches.

It is easy to see that \( Mo(G) = irr(G) \) for graphs \( G \) of order \( n \) and diameter at most 2, where

\[ irr(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)| \]

is the *irregularity* introduced by Albertson [11]. As he showed that \( irr(G) \leq \frac{3}{\pi} n^3 \), Conjecture 20 from [9] follows immediately for graphs of diameter at most 2. This suggests the approach to show that, for every order \( n \), some graph \( G \) maximizing the Mostar index among all graphs of order \( n \) has a universal vertex. This, in turn, suggests considering the effect on the Mostar index of adding missing
edges to some given graph. Unfortunately, adding edges can have considerable non-local effects on the contribution of individual edges to the Mostar index. Nevertheless, examples suggest that suitable missing edges whose addition to a given graph $G$ might have a controllable effect on the Mostar index of $G$ can be identified using the following partial orientation of $G$:

For every edge $uv$ of $G$ with $n_G(u, v) > n_G(v, u)$, orient $uv$ from $v$ to $u$.

This orientation is acyclic in the following sense: As observed in [14][10], we have $n_G(u, v) - n_G(v, u) = \sigma_G(v) - \sigma_G(u)$ for every edge $uv$ of $G$, where $\sigma_G(x) = \sum_{y \in V(G)} \text{dist}_G(x, y)$, and $\text{dist}_G(x, y)$ denotes the distance in $G$ between the vertices $x$ and $y$. Now, if $C : u_1u_2 \ldots u_{\ell}u_1$ is a cycle in $G$ such that the edge $u_\ell u_1$ is oriented from $u_\ell$ to $u_1$ and, for every $i \in [\ell - 1]$, the edge $u_iu_{i+1}$ is either oriented from $u_i$ to $u_{i+1}$ or is not oriented at all, then $\sum_{uv \in E(C)} |n_G(u, v) - n_G(v, u)|$ should be strictly positive, yet

$$\sum_{uv \in E(C)} |n_G(u, v) - n_G(v, u)| = \sum_{i=1}^{\ell-1} \left( n_G(u_{i+1}, u_i) - n_G(u_i, u_{i+1}) \right) + \left( n_G(u_1, u_\ell) - n_G(u_\ell, u_1) \right) = \sum_{i=1}^{\ell-1} \left( \sigma_G(u_i) - \sigma_G(u_{i+1}) \right) + \left( \sigma_G(u_1) - \sigma_G(u_\ell) \right) = 0,$$

that is, no such cycle exists in $G$. We believe that adding missing edges between vertices of zero outdegree and vertices of zero indegree in this partial orientation might have a controllable effect on the Mostar index of $G$. Unfortunately, we have not been able to quantify this intuition sufficiently well.

The approach of Geneson and Tsai [13] actually allows to obtain upper bounds on $\text{Mo}^*(G)$ depending on the degree sequence of $G$: Let the graph $G$ have $n$ vertices, $m$ edges, and vertex degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. Let $V(G) = \{u_1, \ldots, u_n\}$ be such that $d_G(u_i) = d_i$ for every $i \in \{1, 2, \ldots, n\}$. Furthermore, for every such $i$, let $e_i$ be the number of neighbors of $u_i$ in $\{u_{i+1}, \ldots, u_n\}$. Now, $m = \sum_{i=1}^n e_i$, and $\text{Mo}^*(G) = \sum_{i=1}^n e_i(n - d_i) = nm - \sum_{i=1}^n e_i d_i$. Clearly, for every $i$, we have

$$\max\{0, d_i - i + 1\} =: e_i^- \leq e_i^+ := \min\{d_i, n - i\},$$

which easily implies $\sum_{i=1}^n e_i d_i \geq s := \sum_{i=1}^{k-1} e_i^+ d_i + \sum_{i=k}^n e_i^- d_i$, where $k$ is the smallest integer with $m \leq \sum_{i=1}^k e_i^+ + \sum_{i=k+1}^n e_i^-$. Altogether, we obtain $\text{Mo}^*(G) \leq nm - s$, where $n$, $m$, and $s$ only depend on the degree sequence of $G$. The Mostar index of trees of given degree sequences has been studied in [7].

## 2 Proofs of Theorems 1 and 2

In the present section we prove our two main results.

**Proof of Theorem 1** The proof is by induction in $n$. For $n = 1$, the graph $G$ has no edge, $\text{Mo}^*(G) = 0$, and the statement is trivial. Now, let $n > 1$, and let the graph $G$ of order $n$ be chosen in such a way that

(i) $\text{Mo}^*(G)$ is as large as possible,
(ii) subject to (i), the graph $G$ has as many edges as possible, and

(iii) subject to (i) and (ii), the term $\sum_{u \in V(G)} d_G^2(u)$ is as large as possible.

For a linear ordering $\pi : u_1, u_2, \ldots, u_n$ of the vertices of $G$, an edge $u_i u_j$ with $i < j$ is called a forward edge at $u_i$. For $i \in [n]$, let $d^+_i$ be the number of forward edges at $u_i$. Note that $d^+_i$ depends on the specific choice of $\pi$.

Now, choose $\pi : u_1, u_2, \ldots, u_n$ such that

(iv) $d_G(u_1) \leq d_G(u_2) \leq \ldots \leq d_G(u_n)$, and

(v) subject to (iv), the term $w(\pi) = \sum_{i=1}^{n} (n-i)d^+_i$ is as large as possible.

Claim 1. If $d_G(u_i) = d_G(u_{i+1})$ for some $i \in [n-1]$, then $d^+_i \geq d^+_{i+1}$.

Proof of Claim 1 Suppose, for a contradiction, that $d^+_i < d^+_{i+1}$. Let the linear ordering $\pi'$ arise from $\pi$ by exchanging $u_i$ and $u_{i+1}$. If $u_i$ and $u_{i+1}$ are not adjacent, then

$$w(\pi') - w(\pi) = (n-i)d^+_{i+1} + (n-i-1)d^+_i - (n-i)d^+_{i+1} - (n-i-1)d^+_i > 0,$$

and, if $u_i$ and $u_{i+1}$ are adjacent, then

$$w(\pi') - w(\pi) = (n-i)(d^+_{i+1} + 1) + (n-i-1)(d^+_i - 1) - (n-i)d^+_{i+1} - (n-i-1)d^+_i = d^+_{i+1} - d^+_i + 1 > 0.$$

Since $\pi'$ satisfies (iv), we obtain a contradiction to condition (v) in the choice of $\pi$. □

Claim 2. If $d_G(u_i) < d_G(u_{i+1})$ for some $i \in [n-1]$, then $d^+_i \geq d^+_{i+1}$.

Proof of Claim 2 Suppose, for a contradiction, that $d^+_i < d^+_{i+1}$. This implies the existence of a forward edge $u_{i+1} u_j$ at $u_{i+1}$ for which $u_i$ is not adjacent to $u_j$.

Let $G' = G - u_{i+1} u_j + u_i u_j$.

In order to lower bound $Mo^*(G') - Mo^*(G)$, we consider the contributions of the different edges.

- The edge $u_{i+1} u_j$ of $G$ contributes $n - d_G(u_{i+1})$ to $Mo^*(G)$.

- The edge $u_i u_j$ of $G'$ contributes $n - (d_G(u_i) + 1)$ to $Mo^*(G')$.

- Each of the $d^+_{i+1} - 1$ forward edges of $G$ at $u_{i+1}$ that are distinct from $u_{i+1} u_j$ contributes one more to $Mo^*(G')$ than to $Mo^*(G)$.

- Each of the $d^+_i$ forward edges of $G$ at $u_i$ contributes at most one less to $Mo^*(G')$ than to $Mo^*(G)$.

Note that, if $u_i$ and $u_{i+1}$ are adjacent, then the edge between them is one of these forward edges.

- All remaining edges contribute at least as much to $Mo^*(G')$ as to $Mo^*(G)$.
Since $\Mo^*(G') \leq \Mo^*(G)$ by the choice of $G$, these observations imply

\[
0 \geq \Mo^*(G') - \Mo^*(G) \\
\geq - \left( n - d_G(u_{i+1}) \right) + \left( n - \left( d_G(u_i) + 1 \right) \right) + (d_{i+1}^+ - 1) - d_i^+ \\
= \left( d_G(u_{i+1}) - d_G(u_i) - 1 \right) + (d_{i+1}^+ - d_i^+ - 1) \geq 0,
\]
and, hence,

\[
d_G(u_{i+1}) = d_G(u_i) + 1 \quad \text{and} \quad d_{i+1}^+ = d_i^+ + 1.
\]

If $u_i$ and $u_{i+1}$ are adjacent, then, by \ref{ef0a}, the forward edge $u_iu_{i+1}$ at $u_i$ contributes the same to $\Mo^*(G')$ as to $\Mo^*(G)$, which implies the contradiction $\Mo^*(G') - \Mo^*(G) > 0$. Hence, the vertices $u_i$ and $u_{i+1}$ are not adjacent.

Let $G^+ = G + u_iu_j$ and $G^- = G - u_{i+1}u_j$.

In order to lower bound $\Mo^*(G^+) - \Mo^*(G)$, we consider the contributions of the different edges.

- The edge $u_iu_j$ of $G^+$ contributes $n - (d_G(u_i) + 1)$ to $\Mo^*(G^+)$.
- Each of the $d_i^+$ forward edges of $G$ at $u_i$ contributes one less to $\Mo^*(G^+)$ than to $\Mo^*(G)$.
- Each of the $d_j^+$ forward edges of $G$ at $u_j$ contributes at most one less to $\Mo^*(G^+)$ than to $\Mo^*(G)$.
- All remaining edges contribute at least as much to $\Mo^*(G^+)$ as to $\Mo^*(G)$.

Together, these observations imply

\[
\Mo^*(G^+) - \Mo^*(G) \geq n - d_G(u_i) - 1 - d_i^+ - d_j^+. \tag{3} \label{ef1}
\]

In order to upper bound $\Mo^*(G) - \Mo^*(G^-)$, we consider the contributions of the different edges.

- The edge $u_{i+1}u_j$ of $G$ contributes $n - d_G(u_{i+1})$ to $\Mo^*(G)$.
- Each of the $d_{i+1}^+ - 1 \geq d_i^+$ forward edges of $G$ at $u_{i+1}$ that are distinct from $u_{i+1}u_j$ contributes one less to $\Mo^*(G)$ than to $\Mo^*(G^-)$.
- Each of the $d_j^+$ forward edges of $G$ at $u_j$ contributes one less to $\Mo^*(G)$ than to $\Mo^*(G^-)$.
- All remaining edges contribute at most as much to $\Mo^*(G)$ as to $\Mo^*(G^-)$.

Together, these observations imply

\[
\Mo^*(G) - \Mo^*(G^-) \leq n - d_G(u_i) - 1 - d_i^+ - d_j^+. \tag{4} \label{ef2}
\]

Combining \ref{ef1} and \ref{ef2}, and using the specific choice of $G$, we obtain the contradiction

\[
0 > \Mo^*(G^+) - \Mo^*(G) \geq \Mo^*(G) - \Mo^*(G^-) \geq 0.
\]
By Claims 1 and 2 we have \( d_i^+ \geq d_2^+ \geq \ldots \geq d_n^+ \). \{claim3\}

**Claim 3.** If \( u_i \) and \( u_j \) are adjacent for some \( 1 \leq i < j \leq n-1 \), then \( u_i \) is adjacent to \( u_{j+1}, \ldots, u_n \).

**Proof of Claim 3** Suppose, for a contradiction, that \( u_i \) is adjacent to \( u_j \) but not to \( u_{j+1} \) for some \( 1 \leq i < j \leq n-1 \).

Let \( G' = G - u_iu_j + u_iu_{j+1} \).

In order to lower bound \( Mo^*(G') - Mo^*(G) \), we consider the contributions of the different edges.

- Each of the \( d_j^+ \) forward edges of \( G \) at \( u_j \) contributes one more to \( Mo^*(G') \) than to \( Mo^*(G) \).
- Each of the \( d_{j+1}^+ \) forward edges of \( G \) at \( u_{j+1} \) contributes at most one less to \( Mo^*(G') \) than to \( Mo^*(G) \).
- All remaining edges contribute at least as much to \( Mo^*(G') \) as to \( Mo^*(G) \).

Since \( Mo^*(G') \leq Mo^*(G) \) by the choice of \( G \), these observations imply

\[
0 \geq Mo^*(G') - Mo^*(G) \geq d_j^+ - d_{j+1}^+ \geq 0,
\]

that is, we have \( Mo^*(G') = Mo^*(G) \). Note that \( G' \) has the same number of edges as \( G \) but that

\[
\sum_{u \in V(G)} d_{G'}^2(u) - \sum_{u \in V(G)} d_G^2(u) = (d_G(u_j) - 1)^2 + (d_G(u_{j+1}) + 1)^2 - d_G(u_j)^2 - d_G(u_{j+1})^2
\]

\[
= 2(d_G(u_{j+1}) - d_G(u_j)) + 2 > 0,
\]

which implies a contradiction to condition (iii) in the choice of \( G \). \( \square \)

Let \( \delta = d_G(u_1) \). By Claim 2 the neighborhood of \( u_1 \) in \( G \) is \( V(G) \setminus I \), where \( I = \{ u_1, \ldots, u_{n-\delta} \} \). If \( u_iu_j \) is an edge of \( G \) for \( 1 \leq i < j \leq n-\delta \), then Claim 2 implies \( d_i^+ \geq n-j+1 \geq \delta + 1 \), which implies the contradiction \( d_G(u_1) = d_i^+ \geq d_j^+ \geq \delta + 1 \). Hence, the set \( I \) is independent. Since each vertex in \( I \) has degree at least \( \delta \), and \( V \setminus I \) contains exactly \( \delta \) vertices, it follows that each vertex in \( I \) has degree exactly \( \delta \), and that there are all possible \( \delta(n-\delta) \) edges in \( G \) between \( I \) and \( V \setminus I \).

Let \( H = G - I \) and \( x = \frac{\delta}{n} \).

Using

\[
\delta - \min\{d_H(u), d_H(v)\} = n - \min\{d_G(u), d_G(v)\}
\]

for every edge \( uv \) of \( H \),

induction, and

\[
\max \left\{ x(1-x)^2 + \left( \frac{2}{\sqrt{3}} - 1 \right) x^3 : x \in [0,1] \right\} = \frac{2}{\sqrt{3}} - 1,
\]

we obtain

\[
Mo^*(G) = \delta(n-\delta)^2 + Mo^*(H)
\]

\[
\leq \delta(n-\delta)^2 + \left( \frac{2}{\sqrt{3}} - 1 \right) \delta^3
\]

\[
= \left( x(1-x)^2 + \left( \frac{2}{\sqrt{3}} - 1 \right) x^3 \right) n^3
\]

\[
\leq \left( \frac{2}{\sqrt{3}} - 1 \right) n^3,
\]

which completes the proof. \( \square \)
We proceed to the proof of Theorem \[2\] Let \( G \) be a graph of order \( n \) and maximum degree at most \( \Delta \) for positive integers \( n \) and \( \Delta \) with \( \Delta \leq n - 1 \).

Let \( I = \{0, 1, \ldots, \Delta\} \), and let \( G \) have

- \( x_i, n \) vertices of degree \( i \) for every \( i \in I \), and
- \( y_{i,j}, n^2 \) edges between vertices of degree \( i \) and vertices of degree \( j \) for every \( i, j \in I \) with \( i \leq j \).

Double-counting the edges incident with vertices of degree \( i \) in \( G \) implies

\[
x_i n = 2y_{i,i} n^2 + \sum_{j \in I, j < i} y_{j,i} n^2 + \sum_{j \in I, j < j} y_{i,j} n^2.
\]

We obtain

\[
Mo^*(G) \leq \sum_{i,j \in I, i \leq j} (n - i) y_{i,j} n^2 \leq \text{OPT}(P) n^3,
\]

where \( \text{OPT}(P) \) denotes the optimal value of the following linear program \((P)\):

\[
\max \quad \sum_{i,j \in I, i \leq j} (1 - \frac{i}{n}) y_{i,j}
\]

\[
\text{s.t.h.} \quad \sum_{i \in I} x_i = 1, \quad 2y_{i,i} + \sum_{j \in I, j < i} y_{j,i} + \sum_{j \in I, j < j} y_{i,j} - \frac{i}{n} x_i = 0 \quad \text{for every} \ i \in I, \quad \text{and} \ x_{i,i}, y_{i,j} \geq 0 \quad \text{for every} \ i, j \in I \text{ with } i \leq j.
\]

The dual of \((P)\) is the following linear program \((D)\):

\[
\min \quad p
\]

\[
\text{s.t.h.} \quad q_i + q_j \geq 1 - \frac{i}{n} \quad \text{for every} \ i, j \in I \text{ with } i \leq j, \quad p \geq \frac{i}{n}q_i \quad \text{for every} \ i \in I, \quad \text{and} \ p, q_i \in \mathbb{R} \quad \text{for every} \ i \in I.
\]

For our argument, we actually only need the weak duality inequality chain for \((P)\) and \((D)\), which holds for all pairs of feasible solutions of \((P)\) and \((D)\):

\[
\sum_{i,j \in I, i \leq j} (1 - \frac{i}{n}) y_{i,j} \leq \sum_{i,j \in I, i \leq j} (q_i + q_j) y_{i,j} + \sum_{i \in I} \left( p - \frac{i}{n}q_i \right) x_i
\]

\[
= \sum_{i \in I} x_i p + \sum_{i \in I} q_i \left( 2y_{i,i} + \sum_{j \in I, j < i} y_{j,i} + \sum_{j \in I, j < j} y_{i,j} - \frac{i}{n} x_j \right)
\]

\[
= p.
\]

Theorem \[2\] follows by combining \((5)\), weak duality \( \text{OPT}(P) \leq \text{OPT}(D) \), and the following lemma.

**Lemma 3.** \( \text{OPT}(D) \leq p_{\Delta} \) for \( p_{\Delta} = 2 \left( \frac{\Delta}{n} \right)^2 + \left( \frac{\Delta}{n} \right) - 2 \left( \frac{\Delta}{n} \right) \sqrt{\left( \frac{\Delta}{n} \right)^2 + \left( \frac{\Delta}{n} \right)} \).

**Proof.** Since \((D)\) is a minimization problem, it suffices to provide a feasible solution with objective function value \( p_{\Delta} \). Therefore, let \( p = p_{\Delta}, \ q_0 = 1, \) and \( q_i = \frac{i}{n}p \) for every \( i \in I \setminus \{0\} \). Note that this ensures the dual constraint \( p \geq \frac{i}{n}q_i \) for every \( i \in I \). Since \( q_i \) is decreasing for \( i \geq 1 \), and \( \frac{i}{n} \in [0, \frac{\Delta}{n}] \),
we have $q_j \geq \frac{\Delta}{n} p$ for $j \in I$, and the dual constraint

$$q_i + q_j \geq 1 - \frac{i}{n} \text{ for every } i, j \in I \text{ with } i \leq j$$

holds provided that

$$\frac{p}{x} + x + \frac{\Delta}{n} p \geq 1 \text{ for } x \in (0, \frac{\Delta}{n}].$$

The function $x \mapsto \frac{p}{x} + x$ for $x \in (0, \infty)$ assumes its minimum of $2\sqrt{p}$ at $\sqrt{p}$. Hence, (7) holds provided that $2\sqrt{p} + \frac{\Delta}{n} p = 1$. It is easy to verify that this holds indeed for $p = p\Delta$, which completes the proof. \qed

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