ON \((p, q)\)-EIGENVALUES OF SUBELLIPTIC OPERATORS ON NILPOTENT LIE GROUPS

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Abstract. In the article we study the Dirichlet \((p, q)\)-eigenvalue problem for subelliptic non-commutative operators on nilpotent Lie groups. We prove solvability of this eigenvalue problem and existence of the minimizer of the corresponding variational problem.

1. Introduction

In this article we consider the Dirichlet \((p, q)\)-eigenvalue problem, \(1 < p < \nu, 1 < q < \nu^* = \nu p/(\nu - p)\) for subelliptic non-commutative operators

\[
- \text{div}_H (|\nabla_H u|^{p-2}\nabla_H u) = \lambda \|u\|^{p_q}_{L^q(\Omega)} |u|^{q-2} u \quad \text{in} \; \Omega, \; \ u = 0 \; \text{on} \; \partial \Omega,
\]

where \(\Omega\) is a bounded domain on a stratified nilpotent Lie group \(G\).

In the commutative case \(G = \mathbb{R}^n\) the eigenvalue problem \((p = q = 2)\) arises to works by Lord Rayleigh \cite{23} where the author established the variational formulation of this problem which is based on the Dirichlet integral

\[
\|u\|^2_{W^{1,2}_0(\Omega)} = \int_{\Omega} |\nabla u|^2 \, dx.
\]

We note classical works \cite{21, 22} devoted to eigenvalues of linear elliptic operators and its connections with problems of continuum mechanics.

The non-linear commutative case \(p = q \neq 2\) was investigated by many authors as a typical non-linear eigenvalue problem, see, for example, \cite{1}, for extensive references we refer to \cite{15}. The case \(p \neq q\) was considered in \cite{11, 13, 19}.

Subelliptic eigenvalue problems were considered first in \cite{10}. In the recent time eigenvalue problems for \(p\)-sub-Laplace operators were intensively studied, for example, in \cite{3, 17, 26}.

In the present work we consider the Dirichlet \((p, q)\)-eigenvalue problem \((1.1)\) in the weak formulation: a function \(u\) solves the eigenvalue problem iff \(u \in W^{1,p}_0(\Omega)\) and

\[
\int_{\Omega} |\nabla_H u|^{p-2}\nabla_H u \nabla_H v \, dx = \lambda \|u\|^{p_q}_{L^q(\Omega)} \int_{\Omega} |u|^{q-2} uv \, dx
\]

for all \(v \in W^{1,p}_0(\Omega)\).

In this case we refer to \(\lambda\) as an eigenvalue and \(u\) as the corresponding eigenfunction.

\[\text{Key words and phrases: Subelliptic operators, Eigenvalue problem, Carnot groups.}\]

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We prove solvability of the Dirichlet \((p, q)\)-eigenvalue problem \((1.1)\), see Theorem 3.2 and Theorem 3.3. Indeed, in Theorem 3.3, we have considered the following minimizing problem given by

\[
\lambda = \inf_{u \in W^{1,p}_0(\Omega): \|u\|_{L^q(\Omega)} = 1} \left\{ \int_{\Omega} |\nabla H u|^p \, dx \right\}
\]

and proved existence of a function \(v \in W^{1,p}_0(\Omega), \|v\|_{L^q(\Omega)} = 1\), such that

\[
\lambda = \int_{\Omega} |\nabla H v|^p \, dx.
\]

Moreover, we observe that \(v\) is an eigenfunction corresponding to \(\lambda\) and its associated eigenfunctions are precisely the scalar multiple of those vectors at which \(\lambda\) is reached. Finally, in Theorem 3.4 we establish some qualitative properties of the eigenfunctions of \((1.1)\).

2. Nilpotent Lie groups and Sobolev spaces

Recall that a stratified homogeneous group \([12]\), or, in another terminology, a Carnot group \([20]\) is a connected simply connected nilpotent Lie group \(G\) whose Lie algebra \(V\) is decomposed into the direct sum \(V_1 \oplus \cdots \oplus V_m\) of vector spaces such that \(\dim V_1 \geq 2, [V_1, V_i] = V_{i+1}\) for \(1 \leq i \leq m-1\) and \([V_1, V_m] = \{0\}\). Let \(X_{11}, \ldots, X_{1n_1}\) be left-invariant basis vector fields of \(V_1\). Since they generate \(V\), for each \(i, 1 < i \leq m\), one can choose a basis \(X_{ik}\) in \(V_i\), \(1 \leq k \leq n_i = \dim V_i\), consisting of commutators of order \(i-1\) of fields \(X_{ik}\) \(\in V_1\). We identify elements \(g\) of \(G\) with vectors \(x \in \mathbb{R}^N, \, N = \sum_{i=1}^m n_i, \, x = (x_{ik}), \, 1 \leq i \leq m, \, 1 \leq k \leq n_i\) by means of exponential map \(\exp(\sum x_{ik}X_{ik}) = g\). Dilations \(\delta_t\) defined by the formula

\[
\delta_t x = ((t^i x_{ik})_{1 \leq i \leq m, 1 \leq k \leq n_j}
\]

\[
= (tx_{11}, \ldots, tx_{1n_1}, t^2x_{21}, \ldots, t^m x_{mn_m}, \ldots, t^{m-1}x_{m1}, \ldots, t^{m}x_{mm_m}),
\]

are automorphisms of \(G\) for each \(t > 0\). Lebesgue measure \(dx\) on \(\mathbb{R}^N\) is the bi-invariant Haar measure on \(G\) (which is generated by the Lebesgue measure by means of the exponential map), and \(d(\delta_t x) = t^\nu \, dx\), where the number \(\nu = \sum_{i=1}^m n_i \) is called the homogeneous dimension of the group \(G\). The measure \(|E|\) of a measurable subset \(E\) of \(G\) is defined by \(|E| = \int_E \, dx\).

Recall that a continuous map \(\gamma : [a, b] \to G\) is called a continuous curve on \(G\). This continuous curve is rectifiable if

\[
\sup \left\{ \sum_{k=1}^m \left| (\gamma(t_k))^{-1} \gamma(t_{k+1}) \right| \right\} < \infty,
\]

where the supremum is taken over all partitions \(a = t_1 < t_2 < \ldots < t_m = b\) of the segment \([a, b]\). The length \(l(\gamma)\) of a rectifiable curve \(\gamma : [a, b] \to G\) can be calculated by the formula

\[
l(\gamma) = \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_0^\frac{1}{2} \, dt = \int_a^b \left( \sum_{i=1}^n |a_i(t)|^2 \right)^\frac{1}{2} \, dt
\]

where \(\langle \cdot, \cdot \rangle_0\) is the inner product on \(V_1\). The result of [3] implies that one can connect two arbitrary points \(x, y \in G\) by a rectifiable curve. The Carnot-Carathéodory
distance $d(x, y)$ is the infimum of the lengths over all rectifiable curves with endpoints $x$ and $y$ in $G$. The Hausdorff dimension of the metric space $(G, d)$ coincides with the homogeneous dimension $\nu$ of the group $G$.

2.1. Sobolev spaces on Carnot groups. Let $G$ be a Carnot group with one-parameter dilatation group $\delta_t$, $t > 0$, and a homogeneous norm $\rho$, and let $E$ be a measurable subset of $G$. The Lebesgue space $L^p(E)$, $p \in [1, \infty]$, is the space of $p$th-power integrable functions $f : E \to \mathbb{R}$ with the standard norm:

$$
\|f\|_{L^p(E)} = \left( \int_E |f(x)|^p \, dx \right)^{\frac{1}{p}},
$$

and $\|f\|_{L^\infty(E)} = \text{ess}\sup_E |f(x)|$ for $p = \infty$. We denote by $L^p_{loc}(E)$ the space of functions $f : E \to \mathbb{R}$ such that $f \in L^p(F)$ for each compact subset $F$ of $E$.

Let $\Omega$ be an open set in $G$. The (horizontal) Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \nu$, consists of the functions $f : \Omega \to \mathbb{R}$ which are locally integrable in $\Omega$, having the weak derivatives $X_i f$ along the horizontal vector fields $X_i$, $i = 1, \ldots, n_1$, and the finite norm

$$
\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla_H f\|_{L^p(\Omega)},
$$

where $\nabla_H f = (X_1 f, \ldots, X_n f)$ is the horizontal subgradient of $f$. If $f \in W^{1,p}(U)$ for each bounded open set $U$ such that $\overline{U} \subset \Omega$ then we say that $f$ belongs to the class $W^{1,p}_{loc}(\Omega)$.

The Sobolev space $W^{1,p}_0(\Omega)$ is defined to be the closure of $C_c^{\infty}(\Omega)$ under the norm

$$
\|f\|_{W^{1,p}_0(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla_H f\|_{L^p(\Omega)}.
$$

For the following result, refer to [11, 24, 25, 27].

**Lemma 2.1.** The space $W^{1,p}_0(\Omega)$ is a real separable and uniformly convex Banach space.

The following embedding result follows from [7] (2.8) and [11, 14, Theorem 8.1], see also [2, Theorem 2.3].

**Lemma 2.2.** Let $\Omega \subset G$ be a bounded domain and $1 < p < \nu$. Then $W^{1,p}_0(\Omega)$ is continuously embedded in $L^q(\Omega)$ for every $1 \leq q \leq \nu^*$ where $\nu^* = \nu p / (\nu - p)$. Moreover, the embedding is compact for every $1 \leq q < \nu^*$.

Hence, in the case $1 < p < \nu$ we can consider the Sobolev space $W^{1,p}_0(\Omega)$ with the norm

$$
\|f\|_{W^{1,p}_0(\Omega)} = \|\nabla_H f\|_{L^p(\Omega)}.
$$

3. Dirichlet $(p, q)$-eigenvalue problems

The system of basis vectors $X_1, X_2, \ldots, X_n$ of the space $V_1$ (here and throughout we set $n_1 = n$ and $X_1, X_i = X_i$, where $i = 1, \ldots, n$) satisfies the Hörmander’s hypoellipticity condition. We study the non-linear eigenvalue problem in the geometry of the vector fields satisfying the Hörmander’s hypoellipticity condition.

Let $1 < p < \nu$, $\lambda \in \mathbb{R}$ and consider the following subelliptic equation

$$
(3.1) \quad - \text{div}_H(\nabla_H u |^{p-2} \nabla_H u) = \lambda \|u\|_{L^p(\Omega)}^{p-q} |u|^{q-2} u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,
$$
where $1 < q < \nu^* = \frac{2p}{p-q}$. We say that $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega) \setminus \{0\}$ is an eigenpair of (3.1) if for every $v \in W_0^{1,p}(\Omega)$, we have

$$\int_{\Omega} |\nabla_H u|^{p-2} |\nabla_H u| v \, dx = \lambda \|u\|^{p-q}_{L^q(\Omega)} \int_{\Omega} |u|^{q-2} u v \, dx. \quad (3.2)$$

Moreover, we refer to $\lambda$ as an eigenvalue and $u$ as the corresponding eigenfunction.

Let us consider the following minimizing problem given by

$$\lambda := \inf \left\{ \int_{\Omega} |\nabla_H u|^p \, dx : v \in W_0^{1,p}(\Omega) \cap SL^q(\Omega) \right\}, \quad (3.3)$$

where $SL^q(\Omega) := \{ v \in L^q(\Omega) : \|v\|_{L^q(\Omega)} = 1 \}$.

Let us define

$$(U, \| \cdot \|_U) = (W_0^{1,p}(\Omega), \| \cdot \|_{W_0^{1,p}(\Omega)}) \quad (3.5)$$

and

$$(V, \| \cdot \|_V) = (L^q(\Omega), \| \cdot \|_{L^q(\Omega)}) \quad (3.6)$$

and we denote by $SV = SL^q(\Omega)$, where $SL^q(\Omega)$ is defined in (3.4). By Lemma 2.1 and Lemma 2.2, it follows that $(U, \| \cdot \|_U)$ is a uniformly convex Banach space, which is compactly embedded in the Banach space $(V, \| \cdot \|_V)$. Define the operators

$$A : U \to U^* \text{ by } \langle Av, w \rangle = \langle \text{div} \left( |\nabla_H v|^{p-2} \nabla_H v \right), w \rangle = \int_{\Omega} |\nabla_H v|^{p-2} \nabla_H v \nabla_H w \, dx$$

and

$$B : V \to V^* \text{ by } \langle B(v), w \rangle = \int_{\Omega} |v|^{q-2} v w \, dx. \quad (3.8)$$

Below, we prove the following properties of $A$ and $B$.

**Lemma 3.1.** (H1) $A(tv) = |t|^{p-2} t A(v)$ $\forall t \in \mathbb{R}$ and $\forall v \in U$,

(H2) $B(tv) = |t|^{q-2} t B(v)$ $\forall t \in \mathbb{R}$ and $\forall v \in V$,

(H3) $\langle A(v), w \rangle \leq \|v\|_{U}^{p-1}\|w\|_{U}$ for all $v, w \in U$, where the equality holds if and only if $v = 0$ or $w = 0$ or $v = tw$ for some $t > 0$.

(H4) $\langle B(v), w \rangle \leq \|v\|_{V}^{q-1}\|w\|_{V}$ for all $v, w \in V$, where the equality holds if and only if $v = 0$ or $w = 0$ or $v = tw$ for some $t \geq 0$.

(H5) For every $w \in V \setminus \{0\}$ there exists $u \in U \setminus \{0\}$ such that

$$\langle A(u), v \rangle = \langle B(v), v \rangle \quad \forall \ v \in U.$$

Moreover, $A$ and $B$ are continuous.

By the property (H5) in Lemma 3.1 as in [9] page 579 and page 584 – 585 for every $w_0 \in SV$ there exists a sequence $\{w_n\}_{n \in \mathbb{N}} \subset U \cap SV$ such that

$$\langle A(w_{n+1}), v \rangle = \mu_n \langle B(w_n), v \rangle \quad \forall \ v \in U,$$

where $\mu_n \geq \lambda$, with $\lambda$ given by (3.1).
3.1. Main results. The main results in this section are stated as follows:

**Theorem 3.2.** Let \(1 < p < \nu\) and \(1 < q < \nu^*\). Then, the sequences \(\{\mu_n\}_{n \in \mathbb{N}}\) and \(\{\|w_{n+1}\|_U^p\}_{n \in \mathbb{N}}\) given by (3.9) are nonincreasing and converge to the same limit \(\mu\), which is bounded below by \(\lambda\). Moreover, \(\mu\) is an eigenvalue of (3.1) and there exists a minimizing sequence \(\{n_j\}_{j \in \mathbb{N}}\) such that both \(\{w_{n_j}\}_{j \in \mathbb{N}}\) and \(\{w_{n_j+1}\}_{j \in \mathbb{N}}\) converges in \(U\) to the same limit \(\nu \in U \cap SV\), which is an eigenvector corresponding to \(\mu\).

**Theorem 3.3.** Let \(1 < p < \nu\) and \(1 < q < \nu^*\). Suppose \(\{u_n\}_{n \in \mathbb{N}} \subset U \cap SV\) is a minimizing sequence for \(\lambda\), that is \(\|u_n\|_V = 1\) and \(\|u_n\|_U^{\nu} \to \lambda\). Then there exists a subsequence \(\{u_{n_j}\}_{j \in \mathbb{N}}\) which converges weakly in \(U\) to \(u \in U \cap SV\) such that

\[
\lambda = \int_\Omega |\nabla_H u|^p \, dx.
\]

Moreover, \(u\) is an eigenfunction corresponding to \(\lambda\) and its associated eigenfunctions are precisely the scalar multiple of those vectors at which \(\lambda\) is reached.

Our final main result concerns the following qualitative properties of the eigenfunctions of (3.1).

**Theorem 3.4.** Let \(1 < p < \nu\) and \(1 < q < \nu^*\). Assume that \(\lambda > 0\) is an eigenvalue of the problem (3.1) and \(u \in U \cap SV\) is a corresponding eigenfunction. Then (i) \(u \in L^\infty(\Omega)\). (ii) Moreover, if \(u \in U \cap SV\) is nonnegative in \(\Omega\), then \(u > 0\) in \(\Omega\).

Further, for every \(\omega \in \Omega\) there exists a positive constant \(c\) depending on \(\omega\) such that \(u \geq c > 0\) in \(\omega\).

To prove our main results above, first we prove Lemma 3.1 below.

**Proof of Lemma 3.1.**

\((H_1)\) Follows by the definition of \(A\).

\((H_2)\) Follows by the definition of \(B\).

\((H_3)\) First, using Cauchy-Schwartz inequality and then by Hölder’s inequality with exponents \(\frac{1}{p-1}\) and \(p\), we obtain

\[
\langle Av, w \rangle = \int_\Omega |\nabla_H v|^{p-2}\nabla_H v \nabla_H w \, dx \leq \int_\Omega |\nabla_H v|^{p-1} |\nabla_H w| \, dx \leq \|v\|_U^{p-1} \|w\|_U.
\]

If \(v = 0\) or \(w = 0\) then the equality \(\langle Av, w \rangle = \|v\|_U^{p-1} \|w\|_U\) holds. So we assume this equality such that both \(v \neq 0\) and \(w \neq 0\). Then the equality in Cauchy-Schwartz and Hölder’s inequality hold simultaneously. That is, at one end (due to the case of the equality in Cauchy-Schwartz inequality) we get

\[
\int_\Omega |\nabla_H v|^{p-2}\nabla_H v \nabla_H w \, dx = \int_\Omega |\nabla_H v|^{p-1} |\nabla_H w| \, dx,
\]

which gives \(|\nabla_H v|^{p-2}\nabla_H v \nabla_H w = |\nabla_H v|^{p-1} |\nabla_H w|\) and hence, \(\nabla_H v(x) = c(x) \nabla_H w(x)\) for almost every \(x \in \Omega\) for some \(c(x) \geq 0\). Also, due to the equality in Hölder’s inequality, we have

\[
\int_\Omega |\nabla_H v|^{p-2}\nabla_H v \nabla_H w \, dx = \|v\|_U^{p-1} \|w\|_U,
\]

which gives \(|\nabla_H v| = t|\nabla_H w|\) in \(\Omega\) for some constant \(t > 0\). Therefore, \(c(x) = t\) in \(\Omega\). Hence \(\nabla_H v = t\nabla_H w\) in \(\Omega\) and therefore \(v = tw\) in \(\Omega\) for some \(t > 0\). Thus, \((H_3)\) holds.
This property can be verified similarly as in \((H_3)\).

\((H_5)\) We prove it by applying Theorem 4.2 as follows:

**Boundedness:** From Hölder’s inequality, we have
\[
\|Av\|_{U^*} = \sup_{\|w\|_U \leq 1} |\langle Av, w \rangle| \leq \|v\|_{U^*} \|w\|_U \leq \|v\|_{U^*}.
\]

Thus, \(A\) is bounded.

**Continuity:** Suppose \(v_n \in U\) such that \(v_n \to v\) in the norm of \(U\). Thus, up to a subsequence \(\nabla_H v_{n_j} \to \nabla_H v\) pointwise almost everywhere in \(\Omega\). We observe that
\[
\|\nabla_H v_{n_j}\|_{L^{p-1}(\Omega)} \leq \|\nabla_H v_{n_j}\|^{p-1} \leq C,
\]
for some uniform constant \(C > 0\), which is independent of \(n\). Thus,
\[
\nabla_H v_{n_j} \rightharpoonup \nabla_H v \text{ weakly in } L^{p-1}(\Omega).
\]
Since, the weak limit is independent of the choice of the subsequence, it follows that
\[
\nabla_H v_{n_j} \rightharpoonup \nabla_H v \text{ weakly in } L^{p-1}(\Omega).
\]
As a consequence, we have
\[
\langle Av_n, w \rangle \to \langle Av, w \rangle
\]
for every \(w \in U\). Thus \(A\) is continuous. Similarly, we obtain \(B\) is continuous.

**Coercivity:** We observe that
\[
\langle Av, v \rangle = \int_\Omega |\nabla_H v|^p \, dx.
\]
Since \(p > 1\), the operator \(A\) is coercive.

**Monotonicity:** By Lemma 4.1 we have
\[
\langle Av - Aw, v - w \rangle = \int_\Omega (|\nabla_H v|^{p-2}\nabla_H v - |\nabla_H w|^{p-2}\nabla_H w, \nabla_H (v - w)) \, dx \geq 0,
\]
for every \(v, w \in U\). Thus, \(A\) is monotone.

Now, by the continuous embedding of \(U\) in \(V\) from Lemma 2.2, we observe that \(B(w) \in U^*\) for every \(w \in V \setminus \{0\}\). Note that by Lemma 2.1, it follows that \(U\) is a separable and reflexive Banach space. Therefore, by Theorem 4.2 there exists \(u \in U \setminus \{0\}\) such that
\[
\langle A(u), v \rangle = \langle B(w), v \rangle \quad \forall v \in U.
\]
Hence the hypothesis \((H_5)\) holds. This completes the proof. \(\square\)

The next result is useful to prove boundedness of the eigenfunctions of \((\mathbf{K})\).

**Lemma 3.5.** Let \(\Omega \subset \mathbb{G}\) be such that \(|\Omega| < \infty\) and \(1 < p < \nu, 1 < l < \nu^* = \frac{\nu}{\nu - l}\).

Then for every \(u \in W^{1,p}_0(\Omega)\), there exists a positive constant \(C = C(p,l,\nu)\) such that
\[
\left(\int_\Omega |u|^l \, dx\right)^{\frac{1}{l}} \leq C|\Omega|^\frac{1}{l} \left(\int_\Omega |\nabla_H u|^p \, dx\right)^{\frac{1}{p}}.
\]
Proof. Proceeding as in [10] Corollary 1.57, we set
\[
s = \begin{cases} 
1, & \text{if } l \nu \leq \nu + l \\
\frac{\nu}{\nu + l}, & \text{if } l \nu > \nu + l.
\end{cases}
\]
Then \(1 \leq s \leq p, s < \nu \) and \( s^* = \frac{s \nu}{\nu - s} \geq l\). Using Hölder’s inequality along with Lemma 2.2, we obtain
\[
\|u\|_{L^p(\Omega)} \leq \|u\|_{L^{s^*}(\Omega)}|\Omega|^\frac{1}{s^*} \leq C\|\nabla Hu\|_{L^{s^*}(\Omega)}|\Omega|^\frac{s}{s^*} + \frac{s - 1}{s^*}
\]
\[
\leq C\|\nabla Hu\|_{L^{s^*}(\Omega)}|\Omega|^\frac{s}{s^*} + \frac{s - 1}{s^*}.
\]
Hence the proof.

3.2. Proof of the main results: Proof of Theorem 3.2. The proof follows by Lemma 3.1 and [9, Theorem 1].

Proof of Theorem 3.3. The proof follows by Lemma 3.1 and [9, Proposition 2].

Proof of Theorem 3.4. Due to the homogeneity of the equation (3.1), without loss of generality, we assume that \(\|u\|_V = 1\). Let \(k \geq 1\) and set \(A(k) := \{x \in \Omega : u(x) > k\}\). Choosing \(v = (u - k)^+\) as a test function in (3.2), we obtain
\[
\int_{A(k)} |\nabla Hu|^p dx = \lambda \int_{A(k)} |u|^{q-2} u(u - k) dx \leq \lambda \int_{A(k)} |u|^{q-1}(u - k) dx.
\]
We prove the result in the following two cases:

Case 1. Let \(q \leq p\), then since \(k \geq 1\), over the set \(A(k)\), we have \(|u|^{q-1} \leq |u|^{p-1}\). Therefore, from (3.12) we have
\[
\int_{A(k)} |\nabla Hu|^p dx = \lambda \int_{A(k)} |u|^{q-2} u(u - k) dx \leq \lambda \int_{A(k)} |u|^{p-1}(u - k) dx.
\]
where to obtain the last inequality above, we have used the inequality \((a + b)^{p-1} \leq 2^{p-1}(a^{p-1} + b^{p-1})\) for \(a, b \geq 0\). Using the Sobolev inequality (3.10) with \(l = p\) in (3.13) we obtain
\[
(1 - S\lambda^2)^p |A(k)|^\frac{p}{p - 1} \int_{A(k)} (u - k)^p dx \leq \lambda S2^{p-1}k^{p-1} |A(k)|^\frac{p}{p - 1} \int_{A(k)} (u - k) dx,
\]
where \(S > 0\) is the Sobolev constant. Note that \(\|u\|_{L^p(\Omega)} \geq k|A(k)|\) and therefore for every \(k \geq k_0 = (2^p S\lambda)^\frac{p}{p - 1} |u|_{L^p(\Omega)}\), we have \(S\lambda^2k^{p-1} |A(k)|^\frac{p}{p - 1} \leq \frac{1}{2}\). Using this fact in (3.14), for every \(k \geq \max\{k_0, 1\}\), we get
\[
\int_{A(k)} (u - k)^p dx \leq \lambda S2^p k^{p-1} |A(k)|^\frac{p}{p - 1} \int_{A(k)} (u - k) dx.
\]
Using Hölder’s inequality and the estimate (3.15) we obtain
\[
\int_{A(k)} (u - k) dx \leq (\lambda S2^p)^\frac{p}{p - 1} k |A(k)|^\frac{1}{1 + \frac{p}{p - 1}}.
\]
By Hölder’s inequality and the estimate (3.21) we arrive at

\[ \text{Lemma 4.1.} \]

\[ \text{Case II. Let } q > p, \text{ then using the inequality } (a + b)^{q-1} \leq 2^{q-1}(a^{q-1} + b^{q-1}) \text{ for } a, b \geq 0 \text{ in (3.12), we get} \]

\[ (3.17) \quad \int_{A(k)} |\nabla_H u|^p \, dx \leq \lambda \int_{A(k)} (2^{q-1}(u - k)^q + 2^{q-1}k^{q-1}(u - k)) \, dx. \]

Now, using the Sobolev inequality (3.10) with \( l = q \) in the estimate (3.17) we obtain

\[ (3.18) \quad \left( \int_{A(k)} (u - k)^q \, dx \right)^{\frac{p}{q}} \leq S\lambda |A(k)|^p(\frac{1}{q} - \frac{1}{p} + \frac{1}{q}) \int_{A(k)} (2^{q-1}(u - k)^q + 2^{q-1}k^{q-1}(u - k)) \, dx, \]

where \( S > 0 \) is the Sobolev constant. Since \( \int_{A(k)} (u - k)^q \, dx \leq \|u\|^q_{L^q(\Omega)} = 1 \) and \( q > p \), the quantity in the left side of (3.18) can be estimated from below as

\[ (3.19) \quad \left( \int_{A(k)} (u - k)^q \, dx \right)^{\frac{p}{q}} = \left( \int_{A(k)} (u - k)^q \, dx \right)^{\frac{p}{q} - 1} \geq \int_{A(k)} (u - k)^q \, dx. \]

Using (3.19) in (3.18) we get

\[ (3.20) \quad \left( 1 - S\lambda 2^{q-1}|A(k)|^p\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{q}\right) \right) \int_{A(k)} (u - k)^q \, dx \leq S\lambda 2^{q-1}k^{q-1}|A(k)|^p(\frac{1}{q} - \frac{1}{p} + \frac{1}{q}) \int_{A(k)} (u - k) \, dx. \]

Let \( \alpha = p\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{q}\right) \), which is positive, since \( 1 < q < \nu^* \). Choosing \( k_1 = (S\lambda 2^q)^{\frac{1}{p}} \|u\|_{L^q(\Omega)} \), due to the fact that \( k|A(k)| \leq \|u\|_{L^1(\Omega)} \), we obtain for every \( k \geq k_1 \) that \( S\lambda 2^{q-1}|A(k)|^\alpha \leq \frac{1}{2} \). Using this property in (3.20), we have

\[ (3.21) \quad \int_{A(k)} (u - k)^q \, dx \leq S\lambda 2^{q-1}k^{q-1}|A(k)|^\alpha \int_{A(k)} (u - k) \, dx. \]

By Hölder’s inequality and the estimate (3.21) we arrive at

\[ (3.22) \quad \int_{A(k)} (u - k) \, dx \leq (\lambda S2^q)^{\frac{1}{1+\frac{1}{q}}} k|A(k)|^{1+\frac{1}{q}}. \]

Noting (3.22), by [18] Lemma 5.1, we get \( u \in L^\infty(\Omega) \).

(ii) By [24] Theorem 5, the result follows. \( \square \)

4. Coercive operators in Banach spaces

In this section, we state some auxiliary results. The first one is the following algebraic inequality from [1] Lemma 2.1.

**Lemma 4.1.** Let \( 1 < p < \infty \). Then for any \( a, b \in \mathbb{R}^N \), there exists a constant \( C = C(p) > 0 \) such that

\[ (4.1) \quad (|a|^{p-2}a - |b|^{p-2}b, a - b) \geq C(|a| + |b|)^{p-2}|a - b|^2. \]

Next, we state the following result, which follows from [5] Theorem 9.14.
Theorem 4.2. Let $V$ be a real separable reflexive Banach space and $V^*$ be the dual of $V$. Assume that $A : V \to V^*$ is a bounded, continuous, coercive and monotone operator. Then $A$ is surjective, i.e., given any $f \in V^*$, there exists $u \in V$ such that $A(u) = f$. If $A$ is strictly monotone, then $A$ is also injective.

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