Multiple-correction and continued fraction approximation (II)

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Abstract

The main aim of this paper is to further develop the multiple-correction method that formulated in our previous works [7, 8]. As its applications, we establish a kind of hybrid-type finite continued fraction approximations related to BBP-type series of the constant $\pi$ and other classical constants, such as Catalan constant, $\pi^2$, etc.

1 Introduction

In the theory of mathematical constants (for example, $\pi$, Euler-Mascheroni constant $\gamma$, Catalan constant $G$, $\ln 2$, etc.), it is very important to construct new sequences which converge to these fundamental constants with increasingly high speed. See e.g. the survey paper of Bailey, Borwein, Mattingly, and Wightwick [3] and references therein, and the books of Brent and Zimmermann [6], Graham, Knuth and Patashnik [10], Ifrah [11], and Wilf [17]. In a celebrated paper of Bailey-Borwein-Plouffe [4], they proposed the following fast series

$$\pi = \sum_{m=0}^{\infty} \frac{1}{16^m} \left( \frac{4}{8m+1} - \frac{2}{8m+4} - \frac{1}{8m+5} - \frac{1}{8m+6} \right) := \sum_{m=0}^{\infty} \frac{\rho(m)}{16^m}. $$

(1.1)

This formula has the remarkable property that permits one to directly calculate binary digits of $\pi$, beginning at an arbitrary position $d$, without needing to calculate any of the first $d - 1$ digits. Since this discovery in 1997, many BBP-type formulas for various mathematical constants have been discovered with the general form

$$\alpha = \sum_{m=0}^{\infty} \frac{1}{b^m q(m)}.$$

(1.2)

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where $\alpha$ is the constant, $p$ and $q$ are polynomials in integer coefficients, and $b \geq 2$ is an integer numerical base.

Motivated by the important work of Mortici [13], in this paper we will continue our previous works [7, 8], and apply the *multiple-correction method* to construct some new sequences from a BBP-type series, which have faster rate of convergence. We give some examples to illustrate this method, such as $\pi$, Catalan constant $G$, $\pi^2$, etc. Moreover, we establish sharp bounds for the related error terms. It should be stressed that the investigation of the error terms in the approximations generated by BBP-type series is very important topic, because these error estimates can be used to study the irrationality, transcendentally involved the constants. For example, $e$ (for more details, see Aigner and Ziegler [1]), Apréy constant $\zeta(3)$ (see [2]), etc.

The paper is organized as follows. In Section 2, we explain how to find a finite continued fraction approximation by using the *multiple-correction method*. In Section 3, Section 4 and Section 5, we discuss $\pi$, Catalan constant and $\pi^2$, respectively. In the last section, we give three further results, and analyze the related perspective of research in this direction.

**Notation.** Throughout the paper, the notation $P_k(x)$ (or $Q_k(x)$) denotes a polynomial of degree $k$ in $x$. The notation $\Psi(k;x)$ means a polynomial of degree $k$ in $x$ with all of its non-zero coefficients positive, which may be different at each occurrence. While, $\Phi(k;x)$ denotes a polynomial of degree $k$ in $x$ with the leading coefficient equals one, which may be different at different section. To save space, we also use the shorthand notation to write a continued fraction

$$ \frac{a_0}{b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots \frac{a_k}{b_k}}}} = \frac{K}{K_k} \frac{a_k}{b_k}. \quad (1.3) $$

2 The multiple-correction method

Let a series $\sum_{m=0}^{\infty} t_m$ converge to constant $\alpha$. If we use the finite sum $\sum_{m=1}^{n-1} t_m$ to approximate or compute constant $\alpha$ for some “comparative large” positive integer $n$, the error term $E(n)$ equals to $\sum_{m=n}^{\infty} t_m$. To evaluate it more accurately, in general, we need to “separate” extra main-term $MC(n)$ from $E(n)$ such that the new error term $E(n) - MC(n)$ has a faster rate of convergence than $E(n)$ when $n$ tends to infinity. The idea of the *multiple-correction method* is that we can achieve it in some cases by looking for the proper structure of $MC(n)$, where $MC(n)$ is a finite continued fraction (see [8]) or a *Hyper-power expansion* (see [7]) in $n$. Hence, in some senses, we can view it as a rational function approximation problem of the error term $E(n)$. In fact, the *multiple-correction method* is a recursive algorithm, and one of its advantages is that by repeating correction process we always can accelerate the convergence. To describe this method clearly, we will give some definitions as follows.

**Definition 1.** We call the integer $l - m$ to be the *degree* of a rational function $R(k) = \frac{P(k)}{Q(k)}$ in $k$, and write $\text{deg } R(k) = l - m$.

**Definition 2.** Let a series $\sum_{m=0}^{\infty} t_m$ be convergent. A function $t_m$ is said to be a *proper BBP-type term* if it can be written in the form

$$ t_m = R(m) \frac{\prod_{i=1}^{m} (a_i + b_i) \cdot 1}{\prod_{j=1}^{m} (b_j + d_j) \cdot q^m} \cdot \frac{a_0}{b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots \frac{a_k}{b_k}}}}. \quad (2.1) $$
in which $q \in (0, +\infty)$ is a specific constant, and
1. $R(m)$ is a rational function in $m$,
2. the $a_i, c_i, b_j$ and $d_j$ are specific integers with $a_i > 0, b_j > 0$, and
3. the quantities $uu$ and $vv$ are finite, nonnegative, specific integers.

Throughout the paper, we always assume that the $t_m$ is a proper BBP-type term and $q \neq 1$. Now we can describe the multiple-correction method as the following steps:

(Step 1) Simplify the ratio $\frac{t_{m+1}}{t_m}$ to bring the form $\frac{P_r(m)}{Q_s(m)}$, where $P, Q$ are polynomials.
(Step 2) We begin from $k = 0$, and in turn find the finite continued approximation solution $MC_k(m)$ of the difference equation

\begin{equation}
y(m) - \frac{P_r(m)}{Q_s(m)}y(m + 1) - R(m) = 0,
\end{equation}

until some suitable $k = k^*$ you want.
(Step 3) Substitute the above $k$-th correction function $MC_k(m)$ into the left-hand side of (2.2) to find the constant $C_k$ and positive integer $K_0$ such that

\begin{equation}
MC_k(m) - \frac{P_r(m)}{Q_s(m)}MC_k(m + 1) - R(m) + \frac{C_k}{m^{K_0}} = O\left(\frac{1}{m^{K_0+1}}\right),
\end{equation}

(Step 4) Consider the new proper BBP-type term appearing in (Step 3)

\begin{equation}
tt_m = \frac{1}{mK_0} \prod_{i=1}^{uu} (a_i m)! \frac{1}{\prod_{j=1}^{vv} (b_j m)!} q^m,
\end{equation}

then repeat (Step 1) to (Step 3), here it should be noted that it is often suffice for us to obtain some weak results in these cases.
(Step 5) Define the $k$-th correction error term $E_k(n)$ as

\begin{equation}
E_k(n) := \alpha - \sum_{m=0}^{n-1} t_m - \frac{\prod_{i=1}^{uu} (a_i n + c_i)!}{\prod_{j=1}^{vv} (b_j n + d_j)!} q^n MC_k(n).
\end{equation}

Prove the rate of convergence of the $k$-th correction error term $E_k(n)$ when $n$ tends to infinity.
(Step 6) Based on (Step 5), we further prove sharp double-sides inequalities of $E_k(n)$ for as possible as smaller $n$.

Here it should be worth remarking that (Step 2) plays an important role in the multiple-correction method. The idea of the above algorithm is originated from Mortici [13] and Gosper’s Algorithm(see Chapter 5 of Petkovsek, Wilf and Zeilberger[14]).

Now we explain how to look for all the related coefficients in $MC_k(m)$. The initial-correction function $MC_0(m)$ is vital. Let $\deg MC_0(m) = -\kappa_0 \in \mathbb{Z}$, and denote its first coefficient by $\lambda_0 \neq 0$. It is not difficult to obtain $\kappa_0$ and $\lambda_0$, which satisfy the following condition:

\begin{equation}
\min_{\kappa, \lambda} \deg \left( \frac{\lambda}{m^\kappa} - \frac{P_r(m)}{Q_s(m)} \frac{\lambda}{(m + 1)^\kappa} - R(m) \right).
\end{equation}
If $\kappa_0 > 0$, then $MC_0(m)$ has the form $\frac{\lambda_0}{\Phi(\kappa_0; m)}$. Otherwise, it is a polynomial of degree $-\kappa_0$ with the leading coefficient $\lambda_0$. Next, just did as our previous paper [7, 8] (also see (2.7) below), we can find other coefficients in $MC_0(m)$ by solving a linear equation in turn.

Once one determines the initial-correction function $MC_0(m)$, other correction functions $MC_k(m)$ for $k \geq 1$ will become easy. Actually, one may apply two approaches to treat them. One method is power series expansion, another is that putting the whole thing over a common denominator such that

\[
\deg \left( MC_k(m) - \frac{P_k(m)}{Q_3(m)}MC_k(m+1) - R(m) \right)
\]

is a strictly decreasing function of $k$.

Next, we explain how to do (Step 5). First, by multiplying the formula (2.3) by $\prod_{i=1}^{uu} (a_i + c_i)!$ $q^{1}$, then by adding these formulas from $m = n$ to $m = \infty$, finally by checking

\[
\lim_{n \to \infty} \prod_{i=1}^{uu} (a_i n + c_i)! q^{n} MC_k(n) = 0,
\]

in this way it is not difficult to get the desired results for the rate of convergence of the $k$-th correction error term $E_k(n)$.

Finally, there doesn’t exist the general method to treat (Step 6), which needs many delicate estimations for the involved series.

Since $MC_k(m)$ and other constants need a huge of computations, we often use an appropriate symbolic computation software. In addition, the exact expression at each occurrence also takes a lot of space. Hence, in this paper we omit some related details for space limitation. For interested readers can see our previous papers [7, 8, 18].

**Remark 1.** If $q = 1$, and $uu = vv = 0$, we may replace equation (2.2) by $y(m) - y(m+1) - R(m) = 0$, the above method is still efficient.

**An example.** We would like to give an example to show how to manipulate (Step 1) to (Step 3). It is well-known that

\[
\frac{1}{\pi} = \frac{1}{16} \sum_{m=0}^{\infty} \frac{(2m)!^3}{(m!)^6} \frac{42m + 5}{4096^m} := \frac{1}{16} \sum_{m=0}^{\infty} t_m,
\]

which is proposed by Srinivasa Ramanujan [15], also see (1.4) of Mortici [13]. We take $R(m) = 42m + 5, q = 4096, uu = 3$ and $vv = 6$ in Definition 2, hence it is a proper BBP-type term.

(Step 1) It is easy to check

\[
t_{m+1} = \frac{(2m + 2)^3(2m + 1)^3}{4096(m + 1)^6}.
\]

(Step 2) We choose $k^* = 6$. Consider the difference equation

\[
y(m) - \frac{(2m + 2)^3(2m + 1)^3}{4096(m + 1)^6} y(m + 1) - R(m) = 0.
\]
By using Mathematica software, it is not difficult to find

\[
MC_0(m) = \frac{128}{3}m + \frac{128}{27}, \quad MC_k(m) = MC_0(m) + \frac{k}{m} \frac{a_j}{m + b_j}, \quad (k \geq 1),
\]

where

\[
a_1 = \frac{32}{81}, \quad b_1 = \frac{10}{9},
\]

\[
a_2 = -\frac{7}{324}, \quad b_2 = \frac{27}{7},
\]

\[
a_3 = \frac{3969}{19856}, \quad b_3 = -\frac{145795}{156366},
\]

\[
a_4 = \frac{1396171}{462043}, \quad b_4 = \frac{15549372115}{4455381114},
\]

\[
a_5 = \frac{818973874600}{3222301435929}, \quad b_5 = \frac{24496617933181}{3948754138854},
\]

\[
a_6 = \frac{767641960475706}{881904503553129}, \quad b_6 = \frac{535521415681420831}{571477212182467206}.
\]

(Step 3) By using Mathematica software again, we easily find

\[
K_0 = 2k + 1 \quad \text{and} \quad C_0 = \frac{7}{2}, \quad C_1 = \frac{2482}{59049}, \quad C_2 = \frac{7292342}{219839427}, \quad C_3 = \frac{9239028400}{2861932547871}, \quad C_4 = \frac{52785664805293}{100647847362777935517},
\]

which satisfy

\[
MC_k(m) = MC_k(m + 1) - R(m) + \frac{C_k}{m^{2k+1}} = O\left(\frac{1}{m^{2k+2}}\right).
\]

Now we let

\[
E_k(n) := \frac{1}{\pi} - \frac{1}{16} \sum_{m=0}^{n-1} \frac{(2m)!^3}{(m!)^6} \frac{42m + 5}{4096m} - \frac{(2n)!^3}{16(n!)^6} MC_k(n) \frac{4096^n}{4096^n}.
\]

Then for \(0 \leq k \leq 6\), we may prove

\[
\lim_{n \to \infty} \frac{(n!)^6}{((2n)!)^3} 4096^n n^{2k+1} E_k(n) := \frac{4C_k}{63}.
\]

### 3 The results for \(\pi\)

In order to illustrate the so-called the multiple-correction method formulated in previous section, first we will prove the following theorem.

**Theorem 1.** Let \(\rho(m)\) be defined as \([1, 1]\). For every integer \(k \geq 0\), the \(k\)-th correction \(MC_k(n)\) is defined by

\[
MC_0(n) = \frac{1}{(n + \frac{7}{16})^2 - \frac{73}{256}}, \quad MC_k(n) = \frac{1}{(n + \frac{7}{16})^2 - \frac{73}{256}} \frac{k}{j=1} \frac{a_j}{n + b_j}, \quad (k \geq 1),
\]
where

\[
\begin{align*}
a_1 &= \frac{21}{64}, \quad b_1 = \frac{15}{7}, \\
a_2 &= -\frac{265}{392}, \quad b_2 = \frac{9299}{2968}, \\
a_3 &= -2809, \quad b_3 = \frac{20517}{4876}, \\
a_4 &= -\frac{8464}{3260043}, \quad b_4 = \frac{21896}{25408967}, \\
a_5 &= -\frac{453152}{3740382415}, \quad b_5 = \frac{7496524}{48248423355}, \\
a_6 &= -\frac{496062002}{435259601465}, \quad b_6 = \frac{48248423355}{72002790104}, \\
a_7 &= -\frac{326597391169}{18170745077870217}, \quad b_7 = \frac{133863589556959}{4859799720860}, \\
a_8 &= \frac{36157137144200}{118418827901493239625}, \quad b_8 = \frac{554909761792537711960915}{5291704918098810592904}, \\
a_9 &= -\frac{399390489791710771232}{1685165872287169019904}, \quad b_9 = \frac{599390489791710771232}{1685165872287169019904}.
\end{align*}
\]

Let the \( k \)-th correction error term \( E_k(n) \) be defined as

\[
E_k(n) := \pi - \sum_{m=0}^{n-1} \frac{\rho(m)}{16^m} - \frac{1}{16^n} MC_k(n),
\]

Then for all integers \( 0 \leq k \leq 9 \), we have

\[
\lim_{n \to \infty} 16^n n^{2k+5} E_k(n) = \frac{16C_k}{15},
\]

where

\[
\begin{align*}
C_0 &= -\frac{315}{4096}, \quad C_1 = -\frac{11925}{4096}, \quad C_2 = -\frac{108675}{1223070}, \quad C_3 = -\frac{1686825}{410798570}, \quad C_4 = -\frac{28792525}{528448749568}, \quad C_5 = -\frac{4009999971375}{16083165872287169019904}, \\
C_6 &= -\frac{277092923351793}{496062002}, \quad C_7 = \frac{58005312596590975}{111139030101184}, \quad C_8 = -\frac{287025525}{305585164092673588}, \quad C_9 = -\frac{45728068681917702268125}{305585164092673588}.
\end{align*}
\]

**Lemma 1.** Under the same notation of Theorem 1, when \( m \) tends to \( \infty \), we have for \( 0 \leq k \leq 9 \)

\[
MC_k(m) - \frac{1}{16} MC_k(m + 1) - \rho(m) + \frac{C_k}{m^{2k+5}} = O\left(\frac{1}{m^{2k+6}}\right).
\]

**Proof.** By using Mathematica software, we expand \( MC_k(m) - \frac{1}{16} MC_k(m + 1) - \rho(m) \) as the power series in terms of \( m^{-1} \), then after some simplifications we can prove \( \text{(3.4)} \).

**Lemma 2.** Let

\[
\begin{align*}
u(n) &= \frac{1}{15} n^{23} + \frac{23}{16} n^{22}, \quad v(n) = \frac{1}{15} n^{23} + \frac{23}{16} n^{22} - \frac{4183}{240} n^{21},
\end{align*}
\]
Then for \( n \geq 4 \), we have

\[
\frac{1}{16^n} u(n) < \sum_{m=n}^{\infty} \frac{1}{m^{23}16^m} < \frac{1}{16^n} v(n).
\]

**Proof.** We can check for \( m \geq 4 \)

\[
u(m) - \frac{1}{16} v(m + 1) - \frac{1}{m^{23}} = -\frac{\Psi_1(22; m)}{m^{23}(23 + 15m)\Psi_2(23; m)} < 0,
\]

\[
u(m) - \frac{1}{16} v(m + 1) - \frac{1}{m^{23}} = \frac{\Psi_3(21; m)(m - 4) + 4593130341153628118305}{m^{23}(225m + 1245)(m - 4)817}(\Psi_4(22; m)(m - 4) + 1519680023193359375) > 0.
\]

By multiplying (3.7) and (3.8) by \( 16^{-m} \), we obtain the telescoping inequalities

\[
u(m) - \frac{1}{16} v(m + 1) - \frac{1}{m^{23}} = -\frac{1}{16^n} u(n) < \frac{1}{16^n} v(n) - \frac{1}{m^{23}} v(m + 1).
\]

Now by adding the above inequalities from \( m = n \) to \( m = \infty \), we can obtain (3.6) at once.

**The proof of Theorem 1.** First, by multiplying \( (3.4) \) by \( 16^{-m} \), we have

\[
u(m) - \frac{1}{16} v(m + 1) - \frac{1}{m^{23}} M_{C_{k}}(m) - \frac{1}{16 m + 1} M_{C_{k}}(m + 1) - \frac{\rho(m)}{16^m} + \frac{C_k}{16^m m^{2k+5}} = O \left( \frac{1}{16^m m^{2k+6}} \right).
\]

Then, by adding these formulas from \( m = n \) to \( m = \infty \), we get

\[
u(m) - \frac{1}{16} v(m + 1) - \frac{1}{m^{23}} E_k(n) = \sum_{m=n}^{\infty} \frac{\rho(m)}{16^m} - \frac{M_{C_{k}}(n)}{16^n} = \sum_{m=n}^{\infty} \frac{C_k}{16^m m^{2k+5}} + O \left( \sum_{m=n}^{\infty} \frac{1}{16^m m^{2k+6}} \right).
\]

It is easy to prove

\[
u(m) - \frac{1}{16} v(m + 1) - \frac{1}{m^{23}} \sum_{m=n}^{\infty} \frac{1}{16^m m^{2k+6}} = O \left( \frac{1}{n^{2k+6}16^n} \right).
\]

Combining (3.10), (3.11) and (3.6) completes the proof of Theorem 1 in case of \( k = 9 \). For \( 0 \leq k \leq 8 \), we may prove the theorem in the same approach.

The following theorem tells us how to improve (3.3).

**Theorem 2.** Under the same notation of Theorem 3, we have for \( n \geq 88 \)

\[
u(m) - \frac{1}{16} v(m + 1) - \frac{1}{m^{23}} = \frac{16 C_9}{15 \cdot 16^n (n + 1)^{23}} < E_k(n) < \frac{16 C_9}{15 \cdot 16^n (n + 5)^{23}}.
\]
Lemma 3. Let \( f(m) = MC_9(m) - \frac{1}{16}MC_9(m + 1) - \rho(m) \). Then we have for \( m \geq 41 \)
\[
-\frac{D_{10}}{(m + \frac{55}{32})^{24}} < f(m) + \frac{C_9}{m^{23}} < -\frac{D_{10}}{(m + \frac{71}{32})^{24}},
\]
where \( D_{10} = \frac{2892876339921176287296777111194638413125}{240903642185365962212633496131584} \).

Proof. We can check for \( m \geq 41 \)
\[
f(m) + \frac{C_9}{m^{23}} + \frac{D_{10}}{(m + \frac{55}{32})^{24}} = \Psi_5(47; m)(m - 41) + 702881 \cdots 544759
\]
\[
\frac{1}{301 \cdots 448m^{23}(1 + 2m)(3 + 4m)(1 + 8m)(5 + 8m)(55 + 32m)^{24}} \Psi_6(22; m) > 0.
\]
Similarly, we can check for \( m \geq 1 \)
\[
f(m) + \frac{C_9}{m^{23}} + \frac{D_{10}}{(m + \frac{71}{32})^{24}} = -\frac{162820783125}{301 \cdots 448m^{23}(1 + 2m)(3 + 4m)(1 + 8m)(5 + 8m)(71 + 32m)^{24}} \Psi_8(22; m) < 0.
\]
This completes the proof of Lemma 3.

Lemma 4. We let
\[
\frac{1}{16}u_1(n) = \frac{1}{16(n + \frac{63}{32})^{24}}, \quad v_1(n) = \frac{1}{16(n + \frac{55}{32})^{24}},
\]
\[
\frac{1}{16}u_2(n) = \frac{1}{16(n + \frac{79}{32})^{24}}, \quad v_2(n) = \frac{1}{16(n + \frac{71}{32})^{24}}.
\]
Then for all positive integer \( n \)
\[
\frac{1}{16^n}u_1(n) < \sum_{m=n}^{\infty} \frac{1}{(m + \frac{55}{32})^{24}16^m} < \frac{1}{16^n}v_1(n),
\]
\[
\frac{1}{16^n}u_2(n) < \sum_{m=n}^{\infty} \frac{1}{(m + \frac{71}{32})^{24}16^m} < \frac{1}{16^n}v_2(n).
\]
Proof. Similar to the proof of (3.11), we can prove the inequalities of right-hand sides in both (3.18) and (3.19) trivially. By using Mathematica software, it is not difficult to prove
\[
u_1(m) - \frac{1}{16}u_1(m + 1) - \frac{1}{(m + \frac{55}{32})^{24}} = -\frac{8507 \cdots 2864\Psi_1(47; m)}{15(55 + 32m)^{24}(63 + 32m)^{24}(95 + 32m)^{24}} < 0,
\]
\[
u_2(m) - \frac{1}{16}u_2(m + 1) - \frac{1}{(m + \frac{71}{32})^{24}} = -\frac{8507 \cdots 2864\Psi_2(47; m)}{15(71 + 32m)^{24}(79 + 32m)^{24}(111 + 32m)^{24}} < 0,
\]
Now by multiplying the above two inequalities by $16^{-m}$, then by adding these formulas from $m = n$ to $m = \infty$, we can prove the other two inequalities in Lemma 4 immediately.

The proof of Theorem 2. By multiplying (3.13) by $16^{-m}$, then by adding these formulas from $m = n$ to $m = \infty$, we have

\[
E_0(n) = \sum_{m=n}^{\infty} \frac{C_9}{16^m m^{23}} + \sum_{m=n}^{\infty} \frac{D_{10}}{16^m (m + \frac{71}{32})^{24}} < \sum_{m=n}^{\infty} \frac{C_9}{16^m m^{23}} + \sum_{m=n}^{\infty} \frac{D_{10}}{16^m (m + \frac{39}{32})^{24}}.
\]

By using Mathematica software, it is not difficult to prove for $n \geq 88$

\[
\Psi_3(45; n)(n - 88) + 147532 \cdot \cdots \cdot 759049 \geq 0.
\]

By (3.23), Lemma 2 and Lemma 4, we have

\[
C_9 \cdot 16^a u(n) + D_{10} \cdot 16^a v_1(n) < \sum_{m=n}^{\infty} \frac{C_9}{16^m m^{23}} + \sum_{m=n}^{\infty} \frac{D_{10}}{16^m (m + \frac{55}{32})^{24}} < C_9 \cdot 16^a u(n) + D_{10} \cdot 16^a v_2(n) < \sum_{m=n}^{\infty} \frac{C_9}{16^m m^{23}} + \sum_{m=n}^{\infty} \frac{D_{10}}{16^m (m + \frac{39}{32})^{24}}.
\]

Similarly, we also have from Lemma 2 and Lemma 4

\[
C_9 \cdot 16^a v(n) + D_{10} \cdot 16^a u_2(n) > \sum_{m=n}^{\infty} \frac{C_9}{16^m m^{23}} + \sum_{m=n}^{\infty} \frac{D_{10}}{16^m (m + \frac{71}{32})^{24}} > C_9 \cdot 16^a v(n) + D_{10} \cdot 16^a u_2(n) > \sum_{m=n}^{\infty} \frac{C_9}{16^m m^{23}} + \sum_{m=n}^{\infty} \frac{D_{10}}{16^m (m + \frac{39}{32})^{24}}.
\]

Here we note for $n \geq 41$

\[
\Psi_4(45; n)(n - 41) + 58459 \cdot \cdots \cdot 68269 < 0.
\]

Finally, Theorem 2 follows from (3.22), (3.24) and (3.25) at once.
4 Catalan constant

Catalan constant can be defined as

\[ G = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = 0.915965594 \ldots \]  

which is arguably the most basic constant whose irrationality and transcendence remain unproven. The most economical BBP-type series for computing Catalan constant may be

\[ G = \frac{1}{4096} \sum_{m=0}^{\infty} \frac{\nu(m)}{4096^m} \]  

Theorem 3. For every integer \( k \geq 0 \), the \( k \)-th correction function \( MC_k(n) \) is defined by

\[ MC_0(n) = \frac{3}{(n + \frac{13}{72})^2 + \frac{41}{432}}, \quad MC_k(n) = \frac{3}{(n + \frac{13}{72})^2 + \frac{41}{432}} + \sum_{j=1}^{k} a_j n + b_j, \quad (k \geq 1), \]

where

\[ a_1 = \frac{-517}{23328}, \quad b_1 = \frac{156655}{148896}, \]
\[ a_2 = \frac{36682315}{82111808}, \quad b_2 = \frac{-73939238279831}{16385572930720}, \]
\[ a_3 = \frac{-9809376428771254025}{15188280405677908798218890885299115}, \quad b_3 = \frac{-19324063406187162986288867667}{299122227350085279497481360}, \]
\[ a_4 = \frac{24627466973909279332577879325543168}{10414320422851149518238529301402392329619615007}, \quad b_4 = \frac{1125439555781241535752796061860324225756510608}{1125439555781241535752796061860324225756510608}. \]

We define the \( k \)-th correction error term \( E_k(n) \) as

\[ E_k(n) := G - \frac{1}{4096} \sum_{m=0}^{n-1} \frac{\nu(m)}{4096^m} - \frac{1}{4096^{n+1}} MC_k(n). \]
Then for all integers $0 \leq k \leq 4$, we have

$$
\lim_{n \to \infty} 4096^n n^{2k+5} E_k(n) = \frac{C_k}{4095}, 
$$

(4.5) where $C_0 = -\frac{235235}{59816943195790029038974755834408642761775}$, $C_1 = \frac{16690468325}{408829190128418608}$, $C_2 = -\frac{171770824494197747}{1270540662269045733824044641}$, $C_3 = \frac{1883922668487810936804537501055}{191377827608729835340924450733024044641}$, $C_4 = -\frac{59816943195790029038974755834408642761775}{191377827608729835340924450733024044641}$.

Lemma 5. Let $0 \leq k \leq 4$. Under the same notation of Theorem 3, when $m$ tends to infinity we have

$$
MC_k(m) - \frac{1}{4096} MC_k(m + 1) - \nu(m) + \frac{C_k}{m^{2k+3}} = O\left(\frac{1}{m^{2k+6}}\right). 
$$

(4.6)

Proof. First, by using Mathematica software we expand $MC_k(m) - \frac{1}{4096} MC_k(m + 1) - \nu(m)$ as the power series in terms of $m^{-1}$, then after some simplifications we can prove (4.6).

Lemma 6. We let

$$
u(n) := \frac{1}{4096^{m/13}} + \frac{1}{4096^{12/13}}, \quad v(n) := \frac{1}{4096^{m/13}} + \frac{1}{4096^{12/13}} - \frac{1333}{645120} n^{11/13}.
$$

(4.7)

Then for all positive integers $n$, we have

$$
\frac{1}{4096^n} u(n) < \sum_{m=n}^{\infty} \frac{1}{m^{13}4096^m} < \frac{1}{4096^n} v(n).
$$

(4.8)

Proof. By applying Mathematica software, we can prove for $m \geq 1$

$$
u(m) - \frac{1}{4096} \nu(m + 1) - \frac{1}{m^{13}} = -\frac{\Psi_1(12; m)}{13m^{13}(1 + m)^{12}(1 + 315m)(316 + 315m)} < 0,
$$

(4.9)

and

$$
u(m) - \frac{1}{4096} \nu(m + 1) - \frac{1}{m^{13}} = \frac{\Psi_2(12; m)}{m^{13}(1 + m)^{11}(-28666 + 4095m + 1289925m^2)(1265354 + 2583945m + 1289925m^2)} > 0.
$$

(4.10)

We multiply (4.9) by $\frac{1}{4096^m}$ to obtain the telescoping inequality

$$
\frac{1}{4096^m} u(m) - \frac{1}{4096^{m+1}} u(m + 1) - \frac{1}{m^{13}4096^m} < 0.
$$

Then, by adding these inequalities from $m = n$ to $m = \infty$, we have for all integers $n \geq 1$

$$
\frac{1}{4096^n} u(n) - \sum_{m=n}^{\infty} \frac{1}{m^{13}4096^m} < 0.
$$

(4.11)
Similarly, we multiply (4.10) by \(\frac{1}{4096^m}\) to get the telescoping inequality
\[
\frac{1}{4096^m}v(m) - \frac{1}{4096^{m+1}}v(m + 1) - \frac{1}{m^{13} 4096^m} > 0.
\]
Then, by adding these inequalities from \(m = n\) to \(m = \infty\), we have for all integer \(n \geq 1\)
\[
(4.12) \quad \frac{1}{4096^n}v(n) - \sum_{m=n}^{\infty} \frac{1}{m^{13} 4096^m} > 0.
\]
Finally, combining (4.11) and (4.12) completes the proof of Lemma 6.

The proof of Theorem 3. We only give the proof of Theorem 1 in the case of \(k = 4\), the other can be proved similarly. By multiplying (4.6) by \(\frac{1}{4096^m}\), we get the telescoping estimate
\[
\frac{1}{4096^m}MC_4(m) - \frac{1}{4096^{m+1}}MC_4(m + 1) - \frac{\nu(m)}{4096^m} + \frac{C_4}{4096^m m^{13}} = O \left(\frac{1}{4096^m m^{14}}\right).
\]
Then, by adding these formulas from \(m = n\) to \(m = \infty\), we have
\[
(4.13) \quad \frac{1}{4096^n}MC_4(n) - \sum_{m=n}^{\infty} \frac{\nu(m)}{4096^m} + \sum_{m=n}^{\infty} \frac{C_4}{4096^m m^{13}} = O \left(\frac{1}{n^{14} \sum_{m=n}^{\infty} \frac{1}{4096^m}}\right).
\]
It is not difficult to check that
\[
\sum_{m=n}^{\infty} \frac{1}{m^{14} 4096^m} = O \left(\frac{1}{n^{14} \sum_{m=n}^{\infty} \frac{1}{4096^m}}\right) = O \left(\frac{1}{n^{14} 4096^n}\right).
\]
Thus from (4.13)
\[
(4.14) \quad \sum_{m=n}^{\infty} \frac{\nu(m)}{4096^m} - \frac{1}{4096^n}MC_4(n) = \sum_{m=n}^{\infty} \frac{C_4}{4096^m m^{13}} + O \left(\frac{1}{4096^n n^{14}}\right)
\]
\[
= C_4 \sum_{m=n}^{\infty} \frac{1}{4096^m m^{13}} + O \left(\frac{1}{4096^n n^{14}}\right).
\]
It follows from (4.12), (4.13), (4.14)
\[
(4.15) \quad E_4(n) = \frac{1}{4096} \sum_{m=n}^{\infty} \frac{\nu(m)}{4096^m} - \frac{1}{4096^{m+1}}MC_4(n)
\]
\[
= \frac{C_4}{4096} \sum_{m=n}^{\infty} \frac{1}{4096^m m^{13}} + O \left(\frac{1}{4096^n n^{14}}\right).
\]
Now combining (4.15) and Lemma 6 finishes the proof of Theorem 3.
**Theorem 4.** Under the same notation of Theorem 3, we have the following double-sides inequalities for \( n \geq 12 \)

\[
\frac{C_4}{4095 \cdot 4096^n n^{13}} < E_4(n) < \frac{C_4}{4095 \cdot 4096^n (n + 5)^{13}}.
\]

**Remark 2.** In fact, by applying the same method as the proof Theorem 4, in the cases of \( 0 \leq k \leq 3 \) we can get analogous estimates for \( E_k(n) \). Here we leave these for readers to check.

**Lemma 7.** Let \( f(n) = MC_4(m) - \frac{1}{4096} MC_4(m + 1) - \nu(m) \). Then we have for \( m \geq 2 \)

\[
f(m) + \frac{C_4}{m^{13}} + \frac{D_5}{(m + \frac{1}{4})^{14}} < \frac{1}{4095 \cdot 4096^{m-1} (n + \frac{1}{4})^{14}} < \frac{1}{4095 \cdot 4096^{m-1} (n + \frac{1}{4})^{14}}.
\]

**Proof.** By using Mathematica software, we can prove for \( m > 0 \)

\[
f(m) + \frac{C_4}{m^{13}} + \frac{D_5}{(m + \frac{1}{4})^{14}} = -\frac{5\Psi_5(56; m)}{21394742 \cdot 74768128 \Psi_2(71; m)} < 0.
\]

This completes the proof of right-hand side inequality of (4.17). Similarly, one has

\[
f(m) + \frac{C_4}{m^{13}} + \frac{D_5}{(m + \frac{1}{4})^{14}} = \frac{5\Psi_5(53; m)(m - 2) + 261503 \cdot 734375}{21394742 \cdot 74768128 m^{13}} \Psi_4(56; m) > 0.
\]

Hence the left-hand side inequality of (4.17) holds for \( m \geq 2 \). This completes the proof of Lemma 7.

**Lemma 8.** For \( n \geq 1 \), we have

\[
\frac{1}{4095 \cdot 4096^{n-1} (n + \frac{1}{4})^{14}} < \sum_{m=n}^{\infty} \frac{1}{4096^m (m + \frac{1}{4})^{14}} < \frac{1}{4095 \cdot 4096^{n-1} (n + \frac{1}{4})^{14}}.
\]

\[
\frac{1}{4095 \cdot 4096^{n-1} (n + 1)^{14}} < \sum_{m=n}^{\infty} \frac{1}{4096^m (m + \frac{3}{4})^{14}} < \frac{1}{4095 \cdot 4096^{n-1} (n + \frac{3}{4})^{14}}.
\]

**Proof.** We note that both upper bounds in the lemma are trivial. Let

\[
r_1(m) = \frac{4096}{4095 \cdot \frac{1}{(m + \frac{1}{4})^{14}}}, \quad r_2(m) = \frac{4096}{4095 \cdot \frac{1}{(m + 1)^{14}}},
\]

\[
s_1(m) = \frac{1}{(m + \frac{1}{4})^{14}}, \quad s_2(m) = \frac{1}{(m + \frac{3}{4})^{14}}.
\]

It is not difficult to check

\[
r_1(m) - \frac{1}{4096} r_1(m + 1) - s_1(m) = -\frac{16384\Psi_5(27; m)}{4095(1 + 2m)^{14}(3 + 2m)^{14}(1 + 4m)^{14}} < 0,
\]

\[
r_2(m) - \frac{1}{4096} r_2(m + 1) - s_2(m) = -\frac{\Psi_6(27; m)}{4095(1 + m)^{14}(2 + m)^{14}(3 + 4m)^{14}} < 0.
\]
Now by multiplying the above two inequalities by $4096^{-m}$, then by adding these formulas from $m = n$ to $m = \infty$, we can prove the other two inequalities in Lemma 8 immediately.

The proof of Theorem 4. Similar to the proof of (4.15), from Lemma 7 we have

\[(4.24) \quad \frac{D_5}{4096} \sum_{m=n}^{\infty} \frac{1}{4096^m(m + \frac{3}{4})^{14}} < E_4(n) - \frac{C_4}{4096} \sum_{m=n}^{\infty} \frac{1}{4096^m m^{13}} < \frac{D_5}{4096} \sum_{m=n}^{\infty} \frac{1}{4096^m(m + \frac{3}{4})^{14}}.\]

From Lemma 8 and Lemma 6, we have

\[(4.25) \quad E_4(n) < \frac{C_4}{4096} \frac{1}{4096^n u(n)} + \frac{D_5}{4096} \frac{1}{4096^5 \cdot 4096^{n-1}(n + \frac{1}{4})^{14}} \]
\[\quad = \frac{C_4}{4096 \cdot 4096^n} \left( \frac{1}{n^{13} + \frac{1}{315} n^{12}} + \frac{D_5}{C_4} \frac{1}{(n + \frac{1}{4})^{14}} \right) \]
\[\quad < \frac{C_4}{4095 \cdot 4096^n (n + 5)^{13}} \quad (n \geq 12).\]

Here we use \(\frac{(n+5)^{13}}{n^{13} + \frac{1}{315} n^{12}} + \frac{D_5}{C_4} \frac{(n+5)^{13}}{(n+\frac{1}{4})^{14}} - 1 = \frac{\Psi_6(13; n)(n-12)+542244\cdots+288881}{102878\cdots-80445n^{14}(1+4n)^{14}(1+315n)} > 0 \) for \(n \geq 12\).

Similarly, we can check that for \(n > 2\)

\[n^{13} \left( \frac{1}{n^{13} + \frac{1}{315} n^{12} - \frac{28666}{1289925}} + \frac{D_5}{C_4} \frac{1}{(n + 1)^{14}} \right) - 1 = - \frac{\Psi_6(14; n)(n-2)+932326\cdots+839936}{164605\cdots-287120(1+n)^{14}(-28666 + 4095n + 1289925n^2)} < 0.\]

Hence for \(n \geq 2\)

\[(4.26) \quad E_4(n) > \frac{C_4}{4096} \frac{1}{4096^n v(n)} + \frac{D_5}{4096} \frac{1}{4096^5 \cdot 4096^{n-1}(n + \frac{1}{4})^{14}} \]
\[\quad = \frac{C_4}{4095 \cdot 4096^n} \left( \frac{1}{n^{13} + \frac{1}{315} n^{12} - \frac{28666}{1289925}} + \frac{D_5}{C_4} \frac{1}{(n + 1)^{14}} \right) \]
\[\quad > \frac{C_4}{4095 \cdot 4096^n n^{13}}.\]

Finally, combining (4.25) and (4.26) completes the proof of Theorem 4.
5 The results for $\pi^2$

The following BBP-type formula is taken from (18) in Bailey, Borwein, Mattingly and Wightwick [3]

\[
\pi^2 = \frac{2}{27} \sum_{m=0}^{\infty} \frac{1}{729^m} \left( \frac{243}{(12m+1)^2} - \frac{405}{(12m+2)^2} - \frac{81}{(12m+4)^2} - \frac{27}{(12m+5)^2} \right.
\]

\[
\left. - \frac{72}{(12m+6)^2} - \frac{9}{(12m+7)^2} - \frac{9}{(12m+8)^2} - \frac{5}{(12m+10)^2} + \frac{1}{(12m+11)^2} \right) \] (5.1)

\[
\sum_{m=0}^{\infty} \frac{1}{729^m}.
\]

**Theorem 5.** For every integer $k \geq 0$, the $k$-th correction $MC_k(n)$ is defined by

\[
MC_0(n) := \frac{-10935}{(n + \frac{3473}{10920})^2 + \frac{508433}{1324960}}, \quad MC_k(n) := \frac{-10935}{(n + \frac{3473}{10920})^2 + \frac{508433}{1324960} + K \sum_{j=1}^{k} a_j n + b_j}, \quad (k \geq 1),
\]

where

\[
a_1 = \frac{1704001969}{54257112000}, \quad b_1 = \frac{2133779424499}{12405134334320},
\]

\[
a_2 = -\frac{2237771146927854765858675}{55399448826908967430750464}, \quad b_2 = \frac{7838462085871364023219390913487412021}{6662364404905290370545187619443579824},
\]

\[
a_3 = -\frac{33815588448062847387677263213305773122041909570}{518071383229948104130947807715226040921415380062488629146343414684409},
\]

\[
b_3 = \frac{25863297368006753161082557162074173516368660284099249450182379596650}{25863297368006753161082557162074173516368660284099249450182379596650}.
\]

We define the $k$-th correction error term $E_k(n)$ as

\[
E_k(n) := \pi^2 - \frac{2}{27} \sum_{m=0}^{n-1} \frac{\rho(m)}{729^m} - \frac{2}{27} \cdot 729^n MC_k(n).
\]

Then for $0 \leq k \leq 3$, we have

\[
\lim_{n \to \infty} 729^n n^{2k+5} E_k(n) := \frac{27 \cdot C_k}{364}.
\]

where

\[
C_0 = \frac{1704001969}{28937126400}, \quad C_1 = \frac{895108458771141906343547}{37631431943365237081767936},
\]

\[
C_2 = \frac{33074676617409163665475129038532721493305}{272827314097968502831375688476265805833927168},
\]

\[
C_3 = \frac{5178229083132302650865202336606730861855228893902257466379094191}{25645536505061295272046603371156784994696908444725810442196683325440}.
\]
Lemma 9. Let $0 \leq k \leq 3$. When $m$ tends to $\infty$, we have

\begin{equation}
MC_k(m) - \frac{1}{729} MC_k(m+1) - \varrho(m) + \frac{C_k}{m^{2k+5}} = O\left(\frac{1}{m^{2k+6}}\right).
\end{equation}

Proof. First, by using Mathematica software we expand $MC_k(m) - \frac{1}{729} MC_k(m+1) - \varrho(m)$ as the power series in terms of $m^{-1}$, then after some simplifications we can prove Lemma 9.

Lemma 10. Let

\begin{align}
\tag{5.6}
u(n) &= \frac{1}{729} n^{11} + \frac{1}{729} n^{10}, \\
v(n) &= \frac{1}{729} n^{11} + \frac{11}{729} n^{10} - \frac{48059}{530712} n^9.
\end{align}

Then for all positive integers $n$

\begin{equation}
\frac{1}{729^n} u(n) < \sum_{m=n}^{\infty} \frac{1}{m! 729^m} < \frac{1}{729^n} v(n).
\end{equation}

Proof. By manipulating Mathematica software, it is not difficult to check

\begin{align}
\tag{5.8} u(n) - \frac{1}{729} u(n+1) - \frac{1}{n^{11}} &= -\frac{\Psi_1(10;n)}{n^{11}(1+n)^{10}(11+728n)(739+728n)} < 0, \\
\tag{5.9} v(n) - \frac{1}{729} v(n+1) - \frac{1}{n^{11}} &= \frac{\Psi_2(10;n)}{(529984n + 537992)(n-1) + 489933)(489933 + 1067976n + 529984n^2)} > 0.
\end{align}

By multiplying the above inequalities by $\frac{1}{729^n}$, then adding these telescoping estimates from $m = n$ to $m = \infty$, we can finish the proof of Lemma 10.

The proof of Theorem 5. Just as the proof of Theorem 1, Theorem 5 can be proved similarly by Lemma 9 and 10. Here we omit the detail.

Lemma 11. Let $g(m) = MC_3(m) - \frac{1}{729} MC_3(m+1) - \varrho(m)$. We have for $m \geq 1$

\begin{equation}
\frac{-D_4}{(m + \frac{1}{2})^{12}} < g(m) + \frac{C_3}{m^{11}} < -\frac{D_4}{m^{12}},
\end{equation}

where

\begin{align*}
D_4 &= -124280353667510106220979748750667909695624573786666800 \\
&\quad + 114069359390500727018449872352585153675322995125291007697/
\quad 741003093304537143754634300103897398471054081282472499619 \\
&\quad + 4805971487253427817067830381380462371074531080221491200.
\end{align*}
Proof. We can check by using \textit{Mathematica} software
\[
g(m) + \frac{C_3}{m^{11}} + \frac{D_4}{(m + \frac{1}{2})^{12}} = \frac{\Psi_3(28; m)}{2315 \cdots 1600 \cdot m^{11}(1 + 2m)^{12}(1 + 3m)^2(1 + 6m)^2(1 + 12m)^2\Psi_4(10; m)} > 0,
\]
\[
g(m) + \frac{C_3}{m^{11}} + \frac{D_4}{m^{12}} = -\frac{\Psi_5(17; m)}{7410 \cdots 1200 \cdot m^{12}(1 + 3m)^2(1 + 6m)^2(1 + 12m)^2\Psi_6(10; m)} < 0,
\]
and this completes the proof of the lemma.

Lemma 12. Let
\[
(5.11) \quad u_1(n) = \frac{1}{729^{n^{12}}} + \frac{1}{1234n^{11}}, \quad v_1(n) = \frac{1}{729^{n^{12}}},
\]
\[
(5.12) \quad u_2(n) = \frac{1}{729^{(n + \frac{3}{4})^{12}}}, \quad v_2(n) = \frac{1}{729^{n^{12}} + \frac{1600}{243}n^{11}}.
\]

Then
\[
(5.13) \quad \frac{1}{729^n}u_1(n) < \sum_{m=n}^{\infty} \frac{1}{m^{12}729^{m}} < \frac{1}{729^n}v_1(n),
\]
\[
(5.14) \quad \frac{1}{729^n}u_2(n) < \sum_{m=n}^{\infty} \frac{1}{(m + \frac{1}{2})^{12}729^{m}} < \frac{1}{729^n}v_2(n).
\]

Proof. The upper bound in the first inequalities is trivial. By applying \textit{Mathematica} software, it isn’t difficult to check
\[
(5.15) \quad u_1(n) - \frac{1}{729}u_1(n + 1) - \frac{1}{n^{12}} = -\frac{\Psi_4(11; n)}{4n^{12}(1 + n)^{11}(3 + 182n)(185 + 182n)} < 0,
\]
\[
(5.16) \quad u_2(n) - \frac{1}{729}u_2(n + 1) - \frac{1}{(n + \frac{1}{2})^{12}} = -\frac{4096\Psi_2(23; n)}{91(1 + 2n)^{12}(3 + 4n)^{12}(7 + 4n)^{12}} < 0,
\]
\[
(5.17) \quad v_2(n) - \frac{1}{729}v_2(n + 1) - \frac{1}{(n + \frac{1}{2})^{12}} = \frac{\Psi_3(22; n)}{4n^{11}(1 + n)^{11}(1 + 2n)^{12}(1095 + 182n)(1277 + 182n)} > 0.
\]

Now by multiplying the above inequalities by $\frac{1}{729^n}$, then adding these telescoping estimates from $m = n$ to $m = \infty$, we can finish the proof of Lemma 12.

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Theorem 6. For \( n \geq 15 \), we have the following inequalities

\[
\frac{27C_3}{364} \frac{1}{729^n(n + \frac{3}{2})^{11}} < E_3(n) < \frac{27C_3}{364} \frac{1}{729^n(n + \frac{1}{2})^{11}}.
\]

Proof. It follows from (5.3)

\[
E_3(n) = \frac{2}{27} \left( \sum_{m=n}^{\infty} \frac{g(m)}{729^m} - MC_3(n) \right).
\]

By multiplying the inequalities (5.10) by \( \frac{1}{729^m} \), then adding these telescoping estimates from \( m = n \) to \( m = \infty \), we get

\[
\frac{2}{27} \left( C_3 \sum_{m=n}^{\infty} \frac{1}{729^m} + D_4 \sum_{m=n}^{\infty} \frac{1}{729^m} \right) < E_3(n) < \frac{2}{27} \left( C_3 \sum_{m=n}^{\infty} \frac{1}{729^m} + D_4 \sum_{m=n}^{\infty} \frac{1}{729^m} \right).
\]

It follows from Lemma 10 and Lemma 12

\[
E_3(n) < \frac{2C_3}{27} \frac{1}{729^n} \left( \frac{728}{729} n^{11} + \frac{11}{729} n^{10} - \frac{48059}{307212} n^9 + \frac{1}{729} (n + \frac{3}{2})^{12} \right).
\]

By using Mathematica software again, it is not difficult to verify for \( n \geq 15 \)

\[
\frac{728}{729} n^{11} + \frac{11}{729} n^{10} - \frac{48059}{307212} n^9 + \frac{1}{729} (n + \frac{3}{2})^{12} = -\frac{1944 \cdot (\Psi_4(11; n)(n-1) + 2368 \cdots 3535)}{1418 \cdots 2655 \cdot n^9(1 + 2n)^{11}(3 + 4n)^{12}((529984 n + 537992)(n-1) + 489933)} < 0.
\]

This completes the proof of the right-hand side inequality in Theorem 6. Similarly, we have for \( n \geq 12 \)

\[
E_3(n) > \frac{2C_3}{27} \frac{1}{729^n} \left( \frac{1}{729} n^{11} + \frac{11}{729} n^{10} + \frac{D_4}{C_3} \frac{1}{729} (n + \frac{3}{2})^{11} \right)
\]

\[
> \frac{27C_3}{364} \frac{1}{729^n(n + \frac{3}{2})^{11}},
\]

here we use

\[
\frac{728}{729} n^{11} + \frac{11}{729} n^{10} + \frac{D_4}{C_3} \frac{1}{729} (n + \frac{3}{2})^{11} \left( \Psi_5(11; n)(n-12) + 2864 \cdots 5343 \right) = \frac{243 (\Psi_5(11; n)(n-12) + 2864 \cdots 5343)}{3630 \cdots 9680 n^{12}(3 + 2n)^{11}(11 + 728n)} > 0.
\]

This finishes the proof of Theorem 6.
6 Conclusions

Our method may be used to establish similar results for many series with a proper BBP-type term. For example, such kind of series can be founded in [5, 9, 16, 19, 20]. In this paper, we don’t do any computation for these mathematical constants. However, for this question, we would like to pointed out that the computations of two main terms in our method (e.g. the second and third member of right hand side of (3.2) in Theorem 1) should play the same role, i.e. their computations should be “matched”.

In what follows, we give three examples to illustrate that the \( k \)-th correction function \( MC_k(n) \) may be established occasionally by a precise expression.

**Example 1** One has the following simple formula for Catalan constant (see Entry 22 in [16])

\[
G = \frac{1}{2} \sum_{n=0}^{\infty} \frac{4^n n!^2}{(2n)! (2n+1)^2}.
\]

By using Mathematica software, one may check that the \( k \)-th correction \( MC_k(n) \) satisfies

\[
MC_0(n) = \frac{1}{n + \frac{1}{6}}, \quad MC_k(n) = \frac{1}{n + \frac{1}{6} + \sum_{j=1}^{k} a_j n + b_j}, (k \geq 1),
\]

where for \( 1 \leq k \leq 20 \)

\[
a_k = \frac{2k^3(2k-1)^3}{(4k+1)(4k-1)^2(4k-3)}, \quad b_k = \frac{4k^2 + 2k - 1}{2(4k-1)(4k+3)}.
\]

**Example 2** In 1668, Nicolas Mercator [12] proved the following classical formula

\[
\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}.
\]

One may check that the \( k \)-th correction \( MC_k(n) \) has the form

\[
MC_0(n) = \frac{1}{n + \frac{1}{2}}, \quad MC_k(n) = \frac{1}{n + \frac{1}{2} + \sum_{j=1}^{k} a_j n + b_j}, (k \geq 1),
\]

where for \( 1 \leq k \leq 20 \)

\[
a_k = -2 \cdot k^2, \quad b_k = 3k - 2.
\]

**Example 3** For the series \( \sum_{m=1}^{\infty} \frac{1}{(4m+1)^2} \), we can check

\[
MC_0(n) = \frac{1}{n + \frac{1}{4}}, \quad MC_k(n) = \frac{1}{n + \frac{1}{4} + \sum_{j=1}^{k} a_j n + \frac{1}{4}}, (k \geq 1),
\]

where for \( 1 \leq k \leq 20 \)

\[
a_k = \frac{k^4}{4(2k-1)(2k+1)}.
\]

Finally, we conjecture the above results should be true for all \( k \).
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