GOLDMAN FLOWS ON A NONORIENTABLE SURFACE

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Abstract. Given an embedded cylinder in an arbitrary surface, we give a gauge theoretic definition of the associated Goldman flow, which is a circle action on a dense open subset of the moduli space of equivalence classes of flat $SU(2)$-connections over the surface. A cylinder in a compact nonorientable surface lifts to two cylinders in the orientable double cover, and the composite flow is the composition of one of the associated flows with the inverse flow of the other. Providing explicit descriptions, we relate the flow on the moduli space of the nonorientable surface with the composite flow on the moduli space of the double cover. We prove that the composite flow preserves a certain Lagrangian submanifold.

1. Introduction

We generalize the Goldman flow of L. Jeffrey and J. Weitsman to the moduli space of an arbitrary (possible nonorientable) surface: given an embedded oriented cylinder in the surface, we define an associated circle action on a dense open subset of the $SU(2)$ moduli space, which coincides with action of L. Jeffrey and J. Weitsman when the surface is compact and oriented. Here, if $G$ is a Lie group then the moduli space of a surface $S$ is the quotient $M(S) = \mathcal{A}_{flat}(S)/\mathcal{B}(S)$ of the space of flat connections on the trivial principal $G$-bundle by the group of gauge transformations.

We restrict our attention to a compact nonorientable surface. An embedded cylinder lifts to two disjoint cylinders in the orientable double cover, and the two associated Goldman flows commute; composing one of these flows with the inverse flow of the other produces a circle action on the moduli space of the double cover, which we shall call the composite flow. Using convenient generators of the fundamental group of the nonorientable surface and their preimages in the double cover, we give explicit descriptions both of the Goldman flow on the moduli space of the surface, and of the composite flow on the moduli space of the double cover. The pullback of the deck transformation induces an involution on the moduli space of the double cover, and the fixed point set of this involution has been shown by N.-K. Ho to be a Lagrangian submanifold. We prove that the composite flow preserves this Lagrangian submanifold. The pullback of the covering map induces a map from the moduli space of the surface to the moduli space of the double cover, and we prove that the image of this map is also preserved by the composite flow.

This paper was inspired by the work of W. Goldman. The moduli space of a surface $S$ may be identified with the space $\text{Hom}(\pi_1(S), G)/G$ of conjugacy classes of homomorphisms from the fundamental group into the Lie group. In [3], starting with a simple closed curve in a compact Riemann surface $S$ and a conjugation invariant function on a rather general Lie group $G$, W. Goldman defines an associated $\mathbb{R}$-action on $\text{Hom}(\pi_1(S), G)/G$. After using the invariant function and the simple closed curve to produce a function on the symplectic space $\text{Hom}(\pi_1(S), G)/G$, this $\mathbb{R}$-action is the flow of the associated Hamiltonian vector field.

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The typical example of an invariant function on a Lie group is the trace function, \( \text{tr}(g) \). The Goldman flow for an arbitrary invariant function is periodic when restricted to any one orbit, but the periods generally differ from orbit to orbit. In [8], working with a Riemann surface and the group \( G = SU(2) \), L. Jeffrey and J. Weitsman consider the invariant function \( \text{Arccos}(\text{tr}(g)/2) \). Although the associated Goldman flow is defined only on a dense open subset of the moduli space, it has single period for each orbit and thus defines a circle action on its domain. This circle action has been studied by various other authors; see, for instance, [2] or [10].

1.1. Outline of the paper. In Section 2 we use the language of gauge theory to define the Goldman flow associated to an embedded oriented cylinder in an arbitrary surface; it is a circle action on an open dense subset of the \( SU(2) \) moduli space of the surface. Specifically, we define two \( \mathbb{R} \)-actions on the space of flat connections, one corresponding to the left half of the cylinder and one corresponding to the right, both of which cover the Goldman flow on the moduli space. The key step in defining these two \( \mathbb{R} \)-actions is to use Lemma 2.3 from [8], which says that a flat connection on the surface can be adapted to the cylinder. When the surface is compact and oriented, our circle action coincides with the circle action of L. Jeffrey and J. Weitsman given in [8].

In Section 3 we consider a compact nonorientable surface with an embedded cylinder. In this paper we assume that the cylinder does not separate the surface into two pieces. With minor modifications, the technique used may treat the case where the cylinder divides the surface in two.

In Sections 3.1 and 3.2 we explore the topology of the nonorientable surface and its oriented double cover. We choose convenient generators of the fundamental group of the surface, and view the surface as a polygon with edge identifications. Lifting these generators to the double cover, we view the double cover as two polygons with edge identifications. The interior of each of the polygons representing the double cover is mapped diffeomorphically by the covering map onto the interior of the polygon representing the surface.

Section 3.3 is devoted to identifying the moduli spaces of the surface and of the double cover with spaces that are much easier to work with. A well known construction allows us to identify the moduli space of flat connections modulo based gauge transformations with a subset \( \mathcal{R} \) of the direct sum of the same number of copies of \( G \) as there are generators; the set \( \mathcal{R} \) may also be identified with the space \( \text{Hom}(\pi, G) \) of homomorphisms from the fundamental group of the surface to \( G \). The Lie group acts on \( \mathcal{R} \) by conjugation on each factor, and the quotient is identified with the moduli space of the nonorientable surface. The preimage under the covering map of the base point of the fundamental group is two points, and lifting the generators of the fundamental group produces twice as many curves in the double cover as there are generators; some of these lifted curves are loops, and some are paths from one preimage of the base point to the other. Utilizing a construction of N.-K. Ho that appears in [6], we use the lifts of the generators to define a subset \( \tilde{\mathcal{R}} \) of the direct sum of twice as many copies of \( G \) as there are generators, and we equip \( \tilde{\mathcal{R}} \) with an action of \( G \times G \) so that the quotient is identified with the moduli space of the double cover.

The deck transformation of the double cover induces an involution on the moduli space of the double cover, and the fixed point set of this involution is shown in [6] to be a Lagrangian submanifold. In Section 3.5 we adapt a lemma from [6] that expresses this fixed point set as a union of more manageable sets.
W. Goldman defines the flow on \( \text{Hom}(\pi_1(S), G)/G \) for a compact oriented surface \( S \) as the projection of a certain flow on \( \text{Hom}(\pi_1(S), G) \); see \([3]\), \([4]\), and \([5]\). Analogously, in Section 3.6 we give an explicit definition the Goldman flow on the moduli space \( \mathcal{R}/G \) of the nonorientable surface by describing a lift of the flow to \( \mathcal{R} \). The cylinder in the nonorientable surface lifts to two disjoint cylinders in the double cover, and in Section 3.6 we give explicit descriptions of the two associated Goldman flows on the moduli space \( \tilde{\mathcal{R}}/(G \times G) \) by defining two lifted flows on \( \tilde{\mathcal{R}} \). Composing one of these flows with the inverse of the other gives the composite flow. Note that when dealing with the composite flow on the moduli space of the double cover, we can’t use the analogy with homomorphisms from fundamental group to \( G \) because the base point in the nonorientable surface forces us to consider two base points in the double cover. The gauge theoretic definition of the Goldman flow developed in Section 2 easily handles the two base points.

In Section 3.7 we prove that the composite flow on the moduli space of the double cover preserves both the fixed point set of the involution induced by the deck transformation, and the image of the map between moduli spaces induced by the covering map.

2. The \( SU(2) \) Goldman flow

2.1. Notation and conventions. Suppose \( \Sigma \) is a real 2-dimensional smooth manifold. The surface \( \Sigma \) may be nonorientable or noncompact. Let \( \mathcal{A}_{\text{flat}}(\Sigma) \subset \Omega^1(\Sigma) \otimes \mathfrak{su}(2) \) be the space of flat connections on the trivial principal \( SU(2) \)-bundle over \( \Sigma \). Our convention for principal bundles is to use the left action induced by left multiplication in the Lie group; thus, for instance, the curvature of a connection \( A \) is \( dA - \frac{1}{2}[A \wedge A] \). Here’s a useful formula: if \( \sigma \in C^\infty([0,1], \Sigma) \) is a curve and if \( A \) is a connection that takes values in an abelian Lie subalgebra of \( \mathfrak{su}(2) \) when restricted to \( \sigma \) then

\[
\text{Hol}_\sigma(A) = \exp\left(-\int_0^1 \sigma^* A \right). \tag{2.1}
\]

The gauge group is \( \mathcal{G}(\Sigma) = C^\infty(\Sigma, SU(2)) \), and the moduli space is \( \mathcal{M}(\Sigma) = \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma) \). Let \( \mathcal{G} = SU(2) \) and let \( \mathfrak{g} = \mathfrak{su}(2) \).

2.2. The cylinder. Let \( U = S^1 \times [-1,1] \subset \Sigma \) be an embedded cylinder with coordinates \((\theta, s)\) and orientation \( d\theta \wedge ds \); here, we are viewing the circle as \( \mathbb{R}/2\pi \mathbb{Z} \). If \( \Sigma \) is oriented then we assume that the embedding is orientation preserving. Let \( \gamma \) be the central curve \( S^1 \times \{0\} \) with orientation \( d\theta \) and base point \( p = (0,0) \); see Figure 1.

![Figure 1](image-url)

**Figure 1.** The cylinder \( U \), the curve \( \gamma \), and the point \( p \).
2.3. **The function.** Let \( G = SU(2) \) and consider the \( \text{Ad}_G \)-invariant inner product \( \langle \zeta, \eta \rangle = -\frac{1}{2} \text{tr}(\zeta \eta) \) on the Lie algebra \( \mathfrak{g} = \mathfrak{su}(2) \). Let \( f \) be the smooth \( \mathbb{R} \)-valued conjugation invariant function on \( G \setminus \{ \pm 1 \} \) defined as follows:

\[
f(g) = \arccos\left(\frac{1}{2} \text{tr}(g)\right).
\]

Define a *logarithm* map \( \ell \) on \( G \setminus \{ \pm 1 \} \) by requiring that \( \exp(\ell(g)) = g \) and \( \langle \ell(g), \ell(g) \rangle < \pi^2 \). The *variation* of \( f \) with respect to \( \langle \cdot, \cdot \rangle \) is the \( \mathfrak{g} \)-valued function \( F \) on \( G \setminus \{ \pm 1 \} \) defined by setting \( \langle F(g), \xi \rangle = d_f(\ell^1)(\xi) = d_f(\ell^R)(\xi) \), for all \( \xi \in \mathfrak{g} \). Explicitly, if \( g \in G \setminus \{ \pm 1 \} \), let \( \alpha = f(g) \in (0, \pi) \) and write \( g = x \left( e^{i\alpha} e^{-i\alpha} \right) x^{-1} \) for some \( x \in G \); then

\[
\ell(g) = x \left( i \alpha e^{-i\alpha} \right) x^{-1},
\]

and

\[
F(g) = \frac{1}{\sqrt{1 - (\text{tr}(g))^2}} (g - g^{-1}) = \frac{1}{\sqrt{(\ell(g), \ell(g))}} \ell(g) = x \left( i \alpha e^{-i\alpha} \right) x^{-1}.
\]

We shall eventually need the following fact: if \( g \in G \setminus \{ \pm 1 \}, x \in G \), and \( xgx^{-1} = g \) then

\[
\text{Ad}_x F(g) = F(g).
\]

2.4. **The Goldman flow.** Let \( \mathcal{S}_\gamma = \{ A \in \mathcal{A}_\text{flat}(\Sigma) : \text{Hol}_\gamma A \neq \pm 1 \} \), and let \( \mathcal{M}_\gamma = \mathcal{S}_\gamma/\mathcal{G}(\Sigma) \subset \mathcal{M}(\Sigma) \). Use the curve \( \gamma \) to define an \( \mathbb{R} \)-valued function \( f_\gamma \) on \( \mathcal{S}_\gamma \),

\[
f_\gamma(A) = f(\text{Hol}_\gamma A).
\]

The Goldman flow associated to the cylinder \( U \), which we define in the proof of the following theorem, is a periodic \( \mathbb{R} \)-action \( \{ \Xi_t \}_{t \in \mathbb{R}} \) on the dense open subset \( \mathcal{M}_\gamma \) of the moduli space.

**Theorem 1.** There are \( \mathbb{R} \)-actions \( \{ \Xi^+_t \}_{t \in \mathbb{R}} \) and \( \{ \Xi^-_t \}_{t \in \mathbb{R}} \) on \( \mathcal{S}_\gamma \) satisfying the following conditions:

(i) The \( \mathbb{R} \)-actions \( \Xi^+_t \) have “support” in \( U \) in the following sense: \( \Xi^-_t(A) = A \) outside of some compact subset of \( S^1 \times (-1, 0) \subset U \), and \( \Xi^+_t(A) = A \) outside of some compact subset of \( S^1 \times (0, 1) \subset U \).

(ii) If \( d(f_\gamma)_A \) is the tangent map of \( f_\gamma \) at \( A \) then

\[
d(f_\gamma)_A(B) = \int_U \langle \left( \frac{d}{dt} \bigg|_{t=0} \Xi^+_t \right) A \rangle B \rangle,
\]

for \( B \in T_A \mathcal{S}_\gamma = T_A \mathcal{A}_\text{flat}(\Sigma) \).

(iii) \( \Xi^+_t \) are \( \mathcal{G} \) equivariant: if \( A \in \mathcal{A}_\text{flat} \) and \( \psi \in \mathcal{G} \) then \( \Xi^+_t(\psi_A) = \psi(\Xi^+_t(A)) \).

(iv) If \( A \in \mathcal{S}_\gamma \) and \( t \in \mathbb{R} \) then there exists \( \psi \in \mathcal{G} \) such that \( \psi . \Xi^-_t(A) = \Xi^-_t(A) \).

The \( \mathbb{R} \)-actions \( \Xi^+_t \) and \( \Xi^-_t \) on \( \mathcal{S}_\gamma \) thus define a common \( \mathbb{R} \)-action \( \{ \Xi_t \}_{t \in \mathbb{R}} \) on \( \mathcal{M}_\gamma \). The action \( \Xi_t \) is periodic, with period \( \pi \) if \( \Sigma \setminus \gamma \) is disconnected and period \( 2\pi \) if \( \Sigma \setminus \gamma \) is connected. The \( S^1 \)-action on \( \mathcal{M}_\gamma \) defined by \( \Xi_t \) is called the Goldman flow associated to the cylinder \( U \).

**Proof.** We shall construct the maps \( \Xi^+_t \), and leave the proofs of the above statements as an exercise for the reader. Let \( \eta_-(s), \eta_+(s) \in C^\infty(\mathbb{R}) \) be smooth bump functions with compact support in either \((-1, 0) \) or \((0, 1) \) that have integral \( 1 \); see Figure\textsuperscript{[2]} We use the following variation of Lemma 2.3 from \textsuperscript{[3]}: if \( A \in \mathcal{S}_\gamma \) then \( \text{Hol}_\gamma A = \exp(\ell(\text{Hol}_\gamma A)) \), and there is a unique gauge transformation \( u \in \mathcal{G}(U) \) on \( U \) with \( u(p) = 1 \) such that

\[
u . (A|_U) = -\frac{d\theta}{2\pi} \otimes \ell(\text{Hol}_\gamma A).
\]
Here, we say that \( u(A|_U) \) is adapted to the cylinder \( U \), or that \( u \) adapts \( A \) to \( U \). For \( t \in \mathbb{R} \) define

\[
\Xi^\pm_t(A) = A + \text{Ad}_{u^{-1}}(\eta_\pm(s)ds \otimes tF(\text{Hol}_\gamma A)),
\]

(2.9)

where the second term \( \text{Ad}_{u^{-1}}(\eta_\pm(s)ds \otimes tF(\text{Hol}_\gamma A)) \) is supported on a compact subset of \( S^1 \times (-1, 1) \subset U \) and therefore extends by 0 to a flat connection on \( \Sigma \).

To prove (ii), use the infinitesimal version of equation (2.8): if \( A \in S_\gamma \) and \( B \in T_A A_{\text{flat}}(\Sigma) \) then there exists a Lie algebra-valued function \( \varphi \in C^\infty(U) \otimes g \) on the cylinder such that

\[
(B|_U) + dA\varphi = -\frac{d\theta}{2\pi} \otimes \zeta,
\]

(2.10)

for some \( \zeta \in g \) satisfying \([\zeta, F(\text{Hol}_\gamma A)] = 0\). \( \square \)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bump_functions.png}
\caption{The bump functions \( \eta_\pm(s) \in C^\infty(\mathbb{R}) \), with area \( \int_{-\infty}^{\infty} \eta_\pm(s)ds = 1. \)}
\end{figure}

The function \( f_\gamma \) is \( G \)-invariant and thus defines a function on \( M_\gamma \), (which we still call \( f_\gamma \)). When \( \Sigma \) is oriented and compact, there is a well-known symplectic structure on \( M(\Sigma) \) given by

\[
\omega_B[[B], [C]] = \int_\Sigma \langle B \wedge C \rangle, \quad \text{for } B, C \in T_A A_{\text{flat}}; \]

(2.11)

see, for instance, [1] or [9]. The following corollary is an immediate consequence of item (ii) in Theorem 1.

**Corollary 1.** If \( \Sigma \) is oriented and compact then the Goldman flow \( \Xi_t \) is the flow of the Hamiltonian vector field on \( M_\gamma \) with Hamilton function \( f_\gamma \).

2.5. Holonomy. We shall eventually choose generators of the fundamental group, and identify the moduli space with a subset \( \mathcal{R}/G \) of \((G \times \cdots \times G)/G \) by taking holonomies along the generators. The following theorem, which describes how the holonomy along certain types of curves is effected by the Goldman flow, will allow us to work with the Goldman flow on \( \mathcal{R}/G \).

**Theorem 2.** Let \( \sigma : [0, 1] \to \Sigma \) be a curve in \( \Sigma \), with one endpoint at \( p \) and the other endpoint either at \( p \) or in \( \Sigma \setminus U \), and suppose that \( \sigma \) does not otherwise intersect \( \gamma \). If \( A \in S_\gamma \), let

\[
\zeta_t = \zeta_t(A) = \exp(tF(\text{Hol}_\gamma A)),
\]

(2.12)

where \( F \) is defined in (2.4); then the holonomy of \( \Xi^\pm_t(A) \) along such a curve \( \sigma \) is given in the table appearing in Figure 3.
Figure 3. The holonomy of $\Xi_t^\pm(A)$ along $\sigma$; see Theorem 2 and Remark 1.

Remark 1. Since the holonomy of a flat connection along a curve doesn’t change if the curve is deformed by a homotopy that fixes its endpoints, in Theorem 2 we only need to consider how the curve $\sigma$ behaves near its endpoints relative to the point $p$, the curve $\gamma$, and the cylinder $U$. The first column of the table in Figure 3 shows the eight possible behaviours.

Proof of Theorem Fix $A \in S_\gamma$, and let $u.(A|_U)$ be adapted to $U$ as in equation (2.8). Let $\Upsilon_-(s)$ and $\Upsilon_+(s)$ be the compactly supported jump functions on $\mathbb{R} \setminus \{0\}$ obtained by integrating the bump functions $\eta_-(s)$ and $\eta_+(s)$; see Figure 4. The two gauge transformations

$$
\psi^\pm_t = u^{-1} \exp(t\Upsilon^\pm(s) \otimes F(\text{Hol}_\gamma A))u
$$

on $U \setminus \gamma$ extend by 1 to gauge transformations $\psi^\pm_t \in G(\Sigma \setminus \gamma)$, which satisfy

$$
\psi^\pm_t(A|_{\Sigma \setminus \gamma}) = (\Xi_t^\pm A)|_{\Sigma \setminus \gamma}.
$$

To complete the proof, use the fact that if $\psi$ is a gauge transformation on $\Sigma$ then the holonomies of $A$ and $\psi(A)$ along $\sigma$ are related by

$$
\text{Hol}_\sigma(\psi(A)) = (\psi(\sigma_0))(\text{Hol}_\sigma A)(\psi(\sigma_1))^{-1},
$$

Figure 4. The jump functions $\Upsilon_\pm(s) \in C^\infty(\mathbb{R} \setminus \{0\})$, with $\frac{d}{ds} \Upsilon_\pm(s) = \eta_\pm(s)$. 

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To complete the proof, use the fact that if $\psi$ is a gauge transformation on $\Sigma$ then the holonomies of $A$ and $\psi(A)$ along $\sigma$ are related by

$$
\text{Hol}_\sigma(\psi(A)) = (\psi(\sigma_0))(\text{Hol}_\sigma A)(\psi(\sigma_1))^{-1},
$$

Figure 4. The jump functions $\Upsilon_\pm(s) \in C^\infty(\mathbb{R} \setminus \{0\})$, with $\frac{d}{ds} \Upsilon_\pm(s) = \eta_\pm(s)$.
and consider the limits of $\psi_t^\pm(p_0)$ as $p_0$ tends to $p$ in $U \setminus \gamma$ from the left and from the right. \hfill \square

3. The flow for a nonorientable surface

For the remainder of this paper, let $\Sigma$ be a compact connected nonorientable surface with an embedded oriented cylinder $U$. As in Figure 1, view $U$ as a tubular neighbourhood of the oriented curve $\gamma$ with base point $p$.

We shall assume that the surface $\Sigma \setminus \gamma$ is connected; the method used easily adapts to the case where $\Sigma \setminus \gamma$ is disconnected, and the interested reader may find this a worthwhile exercise. Note that $\Sigma$ cannot be $\mathbb{R}P^2$, because a simple closed curve in $\mathbb{R}P^2$ either is contractible (and bounds a disk) or else has no orientable tubular neighbourhood.

![Figure 5](image)

Figure 5. The surface $\Sigma_+$ is either orientable or nonorientable.

3.1. The topology of $\Sigma$. Since $\Sigma \setminus \gamma$ is connected, we may choose a simple closed curve $\beta$ that intersects $\gamma$ exactly once, transitively at $p$. Recall that the cylinder $U$ has coordinates $(s, \theta)$, and orient $\beta$ so that it leaves $p$ in the positive $s$ direction and approaches $p$ from the negative $s$ direction. Two things may occur: either $\beta$ possesses an orientable tubular neighbourhood in $\Sigma$, or it does not. A sketch of $U$ and a tubular neighbourhood of $\beta$ appears in the right half of Figure 5. Let $\Sigma_+$ be the surface with boundary pictured in the left half of Figure 5. The surface $\Sigma_+$ is either orientable or nonorientable, and in either case it has a
single boundary component, which we call \( \delta \). Viewed as an element of \( \pi_1(\Sigma_+, p) \),

\[
\delta = \begin{cases} 
\beta^{-1}\gamma\beta^{-1} & \text{for } \Sigma_+ \text{ orientable,} \\
\beta^{-1}\gamma^{-1}\beta^{-1} & \text{for } \Sigma_+ \text{ nonorientable.}
\end{cases}
\]  

(3.1)

Let \( \Sigma_- \) be the surface obtained by deleting the interior of \( \Sigma_+ \) from \( \Sigma \). The surface \( \Sigma_- \) has a single boundary component, the curve \( \delta \), and gluing together \( \Sigma_+ \) and \( \Sigma_- \) along \( \delta \) recovers \( \Sigma \). The surface \( \Sigma_- \) is either nonorientable or orientable; since our original surface \( \Sigma \) is nonorientable, the only restriction is that \( \Sigma_+ \) and \( \Sigma_- \) cannot both be orientable. We must thus consider the following three cases:

\[
\begin{array}{|l|}
\hline
\text{case (i): } \Sigma_- \text{ nonorientable, } \Sigma_+ \text{ orientable} \\
\text{case (ii): } \Sigma_- \text{ nonorientable, } \Sigma_+ \text{ nonorientable} \\
\text{case (iii): } \Sigma_- \text{ orientable, } \Sigma_+ \text{ nonorientable} \\
\hline
\end{array}
\]  

(3.2)

If \( \Sigma_- \) is nonorientable then it is diffeomorphic to a disk with \( k > 0 \) Möbius strips attached, and we choose generators \( \alpha_1, \ldots, \alpha_k \) of the rank \( k \) free group \( \pi_1(\Sigma_-, p) \) that satisfy

\[
\delta = \alpha_1^2 \cdots \alpha_k^2 ;
\]  

(3.3)

this situation is pictured in the bottom half of Figure 6. If \( \Sigma_- \) is orientable then it is diffeomorphic to a disk with \( k \geq 0 \) handles attached, and we choose generators \( \alpha_1, \ldots, \alpha_{2k} \) of the rank \( 2k \) free group \( \pi_1(\Sigma_-, p) \) that satisfy

\[
\delta = [\alpha_1, \alpha_2] \cdots [\alpha_{2k-1}, \alpha_{2k}] .
\]  

(3.4)

The fundamental group of \( \Sigma \) is as follows:

- case (i), \( \pi_1(\Sigma, p) = \langle \gamma, \beta, \alpha_1, \ldots, \alpha_k | \beta^{-1}\gamma\beta^{-1} = \alpha_1^2 \cdots \alpha_k^2 \rangle ; \)
- case (ii), \( \pi_1(\Sigma, p) = \langle \gamma, \beta, \alpha_1, \ldots, \alpha_k | \beta^{-1}\gamma^{-1}\beta^{-1} = \alpha_1^2 \cdots \alpha_k^2 \rangle ; \)
- case (iii), \( \pi_1(\Sigma, p) = \langle \gamma, \beta, \alpha_1, \ldots, \alpha_{2k} | \beta^{-1}\gamma^{-1}\beta^{-1} = [\alpha_1, \alpha_2] \cdots [\alpha_{2k-1}, \alpha_{2k}] \rangle . \)  

(3.5)

**Remark 2.** The number of generators appearing in our presentation of \( \pi_1(\Sigma, p) \) can be expressed in terms of the Euler characteristic of \( \Sigma \). If \( \Sigma_- \) is nonorientable then \( \chi(\Sigma_-) = 1 - k \), and if \( \Sigma_- \) is orientable then \( \chi(\Sigma_-) = 1 - 2k \). In all three cases, \( \chi(\Sigma_+) = -1 \) and \( \chi(\Sigma) = \chi(\Sigma_+) + \chi(\Sigma_-) \). Therefore, the number of generators in equation (3.5) is

\[
2 - \chi(\Sigma) = \begin{cases} 
2 + k, & \text{cases (i) and (ii)}, \\
2 + 2k, & \text{case (iii)}.
\end{cases}
\]  

(3.6)

3.2. **The double cover \( \tilde{\Sigma} \).** Choose an orientation on the orientable double cover \( \tilde{\Sigma} \) of \( \Sigma \). The preimage in \( \tilde{\Sigma} \) of the cylinder \( U \) under the covering map is two cylinders \( V \) and \( \overline{V} \), both of which we orient using the fixed orientation on \( \Sigma \), that are distinguished by requiring the covering map to be orientation preserving on \( V \) and orientation reversing on \( \overline{V} \). The base point \( p \) lifts to two points \( P \in V \) and \( \overline{P} \in \overline{V} \), and the oriented curve \( \gamma \) lifts to two oriented curves \( \Gamma \subset V \) and \( \overline{\Gamma} \subset \overline{V} \).

Let \( B \) and \( \overline{B} \) be the lifts of \( \beta \), and for each \( j \) let \( A_j \) and \( \overline{A}_j \) be the lifts of \( \alpha_j \); here, the lifts \( B \) and \( A_j \) start at \( P \), and the lifts \( \overline{B} \) and \( \overline{A}_j \) start at \( \overline{P} \). If a loop based at \( p \) in \( \Sigma \) possesses an orientable tubular neighbourhood then it lifts to two loops in \( \tilde{\Sigma} \), one based at \( P \) and one
Figure 6. The double cover \( \tilde{\Sigma} \) for \( \Sigma \) nonorientable.

Based at \( \overline{P} \). If, on the other hand, a loop in \( \Sigma \) possesses no orientable tubular neighbourhood then it lifts to two paths, one from \( P \) to \( \overline{P} \) and one from \( \overline{P} \) to \( P \).

Let the subsets \( \tilde{\Sigma}_- \) and \( \tilde{\Sigma}_+ \) of \( \tilde{\Sigma} \) be the double covers of \( \Sigma_- \) and \( \Sigma_+ \), respectively, and let \( \Delta \) and \( \overline{\Delta} \) be the lifts of the curve \( \delta \) with \( P \in \Delta \) and \( \overline{P} \in \overline{\Delta} \). The boundary of both \( \tilde{\Sigma}_- \) and \( \tilde{\Sigma}_+ \) is the (disjoint) union of \( \Delta \) and \( \overline{\Delta} \), and \( \tilde{\Sigma} \) is obtained by gluing together \( \tilde{\Sigma}_- \) and \( \tilde{\Sigma}_+ \) along this common boundary.

If \( \Sigma_+ \) is orientable then its double cover \( \tilde{\Sigma}_+ \) is the disjoint union of the following two components: a copy of \( \Sigma_+ \) with \( \gamma, \beta, \delta, \) and \( p \) relabeled as \( \Gamma, B, \Delta, \) and \( P \); a copy of the mirror image of \( \Sigma_+ \) with \( \gamma, \beta, \delta, \) and \( p \) relabeled as \( \overline{\Gamma}, \overline{B}, \overline{\Delta}, \) and \( \overline{P} \). When \( \Sigma_+ \) is nonorientable, its double cover \( \tilde{\Sigma}_+ \) is pictured in Figure 7 where the deck transformation is a 180° rotary-reflection (i.e. a 180° rotation followed by a reflection that fixes the axis of rotation). The case where \( \Sigma_- \) is nonorientable is shown in the bottom half of Figure 6; its double cover \( \tilde{\Sigma}_- \) appears in the top half of the figure, where the deck transformation is the obvious reflection. If \( \Sigma_- \) is orientable then its double cover \( \tilde{\Sigma}_- \) is the disjoint union of the following two components: a copy of \( \Sigma_- \) with \( \delta \) and \( p \) relabeled as \( \Delta \) and \( P \), and with each \( \alpha_j \) relabeled as \( A_j \); a copy of the mirror image of \( \Sigma_- \) with \( \delta \) and \( p \) relabeled as \( \overline{\Delta} \) and \( \overline{P} \), and with each \( \alpha_j \) relabeled as \( \overline{A}_j \).

Remark 3. Since the presentation of \( \pi_1(\Sigma, p) \) given in equation (3.5) has a single relation, which results from setting \( \delta \in \pi_1(\Sigma_-, p) \) equal to \( \delta \in \pi_1(\Sigma_+, p) \), the surface \( \Sigma \) can be realized
as a polygon with the following (clockwise) boundary:

- case (i), \( \gamma \beta^{-1} \gamma^{-1} \beta \alpha_1^2 \cdots \alpha_k^2 \);
- case (ii), \( \gamma \beta^{-1} \gamma \alpha_1^2 \cdots \alpha_k^2 \);
- case (iii), \( \gamma \beta^{-1} \gamma [\alpha_1, \alpha_2] \cdots [\alpha_{2k-1}, \alpha_{2k}] \).

We can repeat this process in \( \tilde{\Sigma} \). View \( \Delta \) as an element of both \( \pi_1(\tilde{\Sigma}_-, P) \) and \( \pi_1(\tilde{\Sigma}_+, P) \), and and view \( \Delta \) as an element of both \( \pi_1(\tilde{\Sigma}_-, P) \) and \( \pi_1(\tilde{\Sigma}_+, P) \). The surface \( \tilde{\Sigma} \) can be realized as two polygons with the following boundaries:

\[
\begin{align*}
\text{case (i),} & \quad \text{clockwise } \Gamma^{-1} B A_1 A_1 \cdots A_k A_k; \\
& \quad \text{counterclockwise } \Gamma B^{-1} B A_1 A_1 \cdots A_k A_k; \\
\text{case (ii),} & \quad \text{clockwise } \Gamma^{-1} B A_1 A_1 \cdots A_k A_k; \\
& \quad \text{counterclockwise } \Gamma B^{-1} B A_1 A_1 \cdots A_k A_k; \\
\text{case (iii),} & \quad \text{clockwise } \Gamma^{-1} B [A_1, A_2] \cdots [A_{2k-1}, A_{2k}]; \\
& \quad \text{counterclockwise } \Gamma B^{-1} B [A_1, A_2] \cdots [A_{2k-1}, A_{2k}]. \\
\end{align*}
\] (3.7)

3.3. The identifications. For the rest of this paragraph let \( K = -\chi(\Sigma) \), and recall from Remark 2 that the presentation of \( \pi_1(\Sigma, p) \) appearing in equation (3.5) has \( 2 + K \) generators. Introduce the following notation:

\[
(c, b, \underline{a}) = (c, b, a_1, \ldots, a_K) \in G^{2-\chi(\Sigma)},
(c, b, \underline{a}, \underline{\overline{a}}, \underline{\overline{\overline{a}}}) = (c, b, a_1, \ldots, a_K, \underline{a}, \underline{\overline{a}}, \underline{\overline{\overline{a}}}) \in G^{2(2-\chi(\Sigma))}.
\]

We can left and right multiply as follows: if \( x, y \in G \) then, for instance,

\[ x \underline{a} y = (xa_1 y, \ldots, xa_K y). \]

Consider the following subset \( \mathcal{R} \) of \( G^{2-\chi(\Sigma)} \):

\[
\begin{align*}
\text{case (i),} & \quad \mathcal{R} = \{ (c, b, \underline{a}) \in G^{2+k} \mid b^{-1} c b^{-1} = a_1^2 \cdots a_k^2 \}; \\
\text{case (ii),} & \quad \mathcal{R} = \{ (c, b, \underline{a}) \in G^{2+k} \mid b^{-1} c b^{-1} = a_1^2 \cdots a_k^2 \}; \\
\text{case (iii),} & \quad \mathcal{R} = \{ (c, b, \underline{a}) \in G^{2+2k} \mid b^{-1} c^{-1} b^{-1} = [a_1, a_2] \cdots [a_{2k-1}, a_{2k}] \}.
\end{align*}
\] (3.8)
Let $G$ act on $\mathcal{R}$, and on $G^{2-\chi(\Sigma)}$, as follows:

$$g.(c, b, A) = (gcg^{-1}, gbg^{-1}, gA_g^{-1}).$$

(3.9)

Denote by $\mathcal{G}(\Sigma, p)$ the set of gauge transformations on $\Sigma$ that evaluate to 1 at $p$. We identify $\mathcal{R}$ with $\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma, p)$ by taking holonomies around the generators of $\pi_1(\Sigma, p)$ appearing in equation (3.5), and we further identify $\mathcal{M} = \mathcal{R}/G$ with the moduli space $\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma)$:

$$\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma, p) \leftrightarrow \mathcal{R},$$

$$\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma) \leftrightarrow \mathcal{M} = \mathcal{R}/G.$$  

(3.10)

This identification is quite well known; see, for instance, [7]. Note that $\mathcal{R}$ may also be identified with the set of homomorphisms from $\pi_1(\Sigma, p)$ to $G$.

Using a construction from [6], let $G \times G$ act on the following subset $\tilde{\mathcal{R}}$ of $G^{2(2-\chi(\Sigma))}$: case(i),

$$\tilde{\mathcal{R}} = \left\{ (c, b, A, \tilde{c}, \tilde{b}, \tilde{A}) \in G^{2(2+k)} \mid \begin{array}{l}
    b^{-1}c\tilde{c}^{-1} = a_1a_1 \cdots a_ka_k, \\
    \tilde{b}^{-1}c\tilde{c}^{-1} = \tilde{a}_1a_1 \cdots \tilde{a}_ka_k
\end{array} \right\},$$

(3.11)

$$\quad (g, h).(c, b, A, \tilde{c}, \tilde{b}, \tilde{A}) = (gcg^{-1}, gbg^{-1}, gA_g^{-1}, h\tilde{c}h^{-1}, h\tilde{b}h^{-1}, h\tilde{A}g^{-1});$$

case(ii),

$$\tilde{\mathcal{R}} = \left\{ (c, b, A, \tilde{c}, \tilde{b}, \tilde{A}) \in G^{2(2+k)} \mid \begin{array}{l}
    \tilde{b}^{-1}c\tilde{c}^{-1} = a_1a_1 \cdots a_ka_k, \\
    \tilde{b}^{-1}c\tilde{c}^{-1} = \tilde{a}_1a_1 \cdots \tilde{a}_ka_k
\end{array} \right\},$$

(3.12)

$$\quad (g, h).(c, b, A, \tilde{c}, \tilde{b}, \tilde{A}) = (gcg^{-1}, gbg^{-1}, gA_g^{-1}, h\tilde{c}h^{-1}, h\tilde{b}h^{-1}, h\tilde{A}g^{-1});$$

case(iii),

$$\tilde{\mathcal{R}} = \left\{ (c, b, A, \tilde{c}, \tilde{b}, \tilde{A}) \in G^{2(2+2k)} \mid \begin{array}{l}
    \tilde{b}^{-1}c\tilde{c}^{-1} = [a_1, a_2] \cdots [a_{2k-1}, a_{2k}], \\
    \tilde{b}^{-1}c\tilde{c}^{-1} = [\tilde{a}_1, \tilde{a}_2] \cdots [\tilde{a}_{2k-1}, \tilde{a}_{2k}]
\end{array} \right\},$$

(3.13)

$$\quad (g, h).(c, b, A, \tilde{c}, \tilde{b}, \tilde{A}) = (gcg^{-1}, gbg^{-1}, gA_g^{-1}, h\tilde{c}h^{-1}, h\tilde{b}h^{-1}, h\tilde{A}g^{-1}).$$

Denote by $\mathcal{G}(\tilde{\Sigma}, \{P, \overline{P}\})$ the set of gauge transformations on $\tilde{\Sigma}$ that evaluate to 1 at $P$ and at $\overline{P}$. Each of the generators of $\pi_1(\Sigma, p)$ appearing in equation (3.5) lifts to two curves in $\tilde{\Sigma}$. By taking holonomies along the lifts of the generators, we identify $\tilde{\mathcal{R}}$ with $\mathcal{A}_{\text{flat}}(\tilde{\Sigma})/\mathcal{G}(\tilde{\Sigma}, \{P, \overline{P}\})$, and we further identify $\tilde{\mathcal{M}} = \tilde{\mathcal{R}}/(G \times G)$ with the moduli space $\mathcal{A}_{\text{flat}}(\tilde{\Sigma})/\mathcal{G}(\tilde{\Sigma})$:

$$\mathcal{A}_{\text{flat}}(\tilde{\Sigma})/\mathcal{G}(\tilde{\Sigma}, \{P, \overline{P}\}) \leftrightarrow \tilde{\mathcal{R}},$$

$$\mathcal{A}_{\text{flat}}(\tilde{\Sigma})/\mathcal{G}(\tilde{\Sigma}) \leftrightarrow \tilde{\mathcal{M}} = \tilde{\mathcal{R}}/(G \times G).$$  

(3.14)

Compare the definition of $\tilde{\mathcal{R}}$ with equation (3.7) in Remark 3. To understand the action of $G \times G$ on $\tilde{\mathcal{R}}$, consider how the holonomies along the lifts of the generators are effected by a gauge transformation on $\tilde{\Sigma}$ that evaluates to $g$ at $P$ and to $h$ at $\overline{P}$. 


3.4. **Some induced maps.** Let $\rho$ and $\tilde{\rho}$ be the natural quotient maps,

$$\rho : \mathbb{R} \rightarrow \mathcal{M} = \mathbb{R}/G,$$

$$\tilde{\rho} : \tilde{\mathbb{R}} \rightarrow \tilde{\mathcal{M}} = \tilde{\mathbb{R}}/(G \times G).$$

The pullback of the covering map from $\tilde{\Sigma}$ to $\Sigma$ takes a Lie algebra valued 1-form on $\Sigma$ and produces one on $\tilde{\Sigma}$, and this pullback induces a (well defined) map from $A_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma, p)$ to $A_{\text{flat}}(\tilde{\Sigma})/\mathcal{G}(\tilde{\Sigma}, \{P, \overline{P}\})$. Using the identifications appearing in equations (3.10) and (3.14), the pullback of the covering map induces a map $I : \mathbb{R} \rightarrow \tilde{\mathbb{R}}$ given by

$$I(c, b, A) = (c, b, c, b, A). \quad (3.15)$$

The map $I$ satisfies $I(g.(c, b, A)) = (g, g).(I(c, b, A))$, and thus descends to a well-defined map $\iota : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$.

Similarly, the pullback of the nontrivial deck transformation of $\tilde{\Sigma}$ induces an involution

$$T(c, b, A, \tau, \overline{b}, \overline{A}) = (\tau, \overline{b}, \overline{A}, c, b, A). \quad (3.16)$$

The map $T$ satisfies $T((g, h).(c, b, A, \tau, \overline{b}, \overline{A})) = (h, g).(T(c, b, A, \tau, \overline{b}, \overline{A}))$, and thus descends to a well-defined involution $\tau : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$.

The following commutative diagrams attempt to summarize these definitions:

$$\begin{array}{ccc}
\tilde{\mathbb{R}} & \xrightarrow{T} & \tilde{\mathbb{R}} \\
\downarrow{\tilde{\rho}} & & \downarrow{\tilde{\rho}} \\
\tilde{\mathcal{M}} & \xrightarrow{\tau} & \tilde{\mathcal{M}}
\end{array} \quad (3.17)
$$

$$\begin{array}{ccc}
\mathbb{R} & \xrightarrow{I} & \tilde{\mathbb{R}} \\
\downarrow{\rho} & & \downarrow{\tilde{\rho}} \\
\mathcal{M} & \xrightarrow{\iota} & \tilde{\mathcal{M}}
\end{array} \quad (3.18)
$$

**Remark 4.** Recall that $G = SU(2)$. Although $I$ is injective, the map $\iota$ is not injective. If $\rho(c, b, \Lambda) \in \mathcal{M}$ and

$$\rho(f, e, \mathbb{D}) = \begin{cases} 
\rho(c, b, -\Lambda) & \text{case (i)}, \\
\rho(c, -b, -\Lambda) & \text{case (ii)}, \\
\rho(c, -b, \Lambda) & \text{case (iii)}
\end{cases} \quad (3.19)$$

then $\iota(\rho(c, b, \Lambda)) = \iota(\rho(f, e, \mathbb{D}))$. It can be shown that these two points of $\mathcal{M}$ are always distinct if $\chi(\Sigma)$ is odd, and “usually” distinct if $\chi(\Sigma)$ is even: in case (i), for instance, if $\text{tr}(a_1) \neq 0$ then

$$\rho(c, b, \Lambda) \neq \rho(c, b, -\Lambda) \quad (3.20)$$

because $a_1$ is not conjugate to $-a_1$. 

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3.5. The fixed point set of $\tau$. Let $\widetilde{M}^\tau$ be the fixed point set of $\tau$. In [6], this fixed point set is shown to be a Lagrangian submanifold of the moduli space $\widetilde{M}$ with respect to the symplectic form appearing in equation [2.11]. The following proposition, adapted from Section 3.1 of [6], allows us to write $\widetilde{M}^\tau$ as a union of more manageable sets.

**Proposition 1** (N.-K. Ho). For each $x \in G$, define a subset $N_x \subset \tilde{R}$ as follows:

- **case (i),** $N_x = \{(c, b, A, c, b, A) \in \tilde{R} \mid x.(c, b, A) = (c, b, A)\};$
- **case (ii),** $N_x = \{(c, b, A, c, b, xA) \in \tilde{R} \mid x.(c, b, A) = (c, b, A)\};$ (3.21)
- **case (iii),** $N_x = \{(c, b, A, c, b, xA) \in \tilde{R} \mid x.(c, b, A) = (c, b, A)\}.$

The image of $\iota$ is $\iota(M) = \tilde{\rho}(N_1)$, and the fixed point set of $\tau$ is $\widetilde{M}^\tau = \bigcup_{x \in G} \tilde{\rho}(N_x)$.

**Proof.** To prove the first statement, note that $N_1 = \{(c, b, A, c, b, A) : (c, b, A) \in \mathcal{R}\}$, and use diagram (3.18) to conclude that $\tilde{\rho}(N_1) = \iota(M)$.

We'll only prove the second statement for case (i), since the proofs are nearly identical for each of the three cases. First note that

\[ \tilde{\rho}(c, b, A, \overline{c}, \overline{b}, \overline{A}) \in \tilde{M}^\tau \]
\[ \iff \exists g, h \in G \text{ such that } T(c, b, A, \overline{c}, \overline{b}, \overline{A}) = (g, h).(c, b, A, \overline{c}, \overline{b}, \overline{A}) \]
\[ \iff (\overline{c}, \overline{b}, \overline{A}, c, b, A) = (gcg^{-1}, gbh^{-1}, gAh^{-1}, \overline{c}b\overline{h}^{-1}, \overline{b}\overline{h}^{-1}, \overline{A}g^{-1}) \]
\[ \iff \exists g, h \in G \text{ such that } \begin{cases} \overline{c} = gcg^{-1} \\ \overline{b} = gbh^{-1} \\ \overline{A} = gAh^{-1} \\ (hg)^{-1}(c)(hg) = c \\ (hg)^{-1}(b)(hg) = b \\ (hg)^{-1}(A)(hg) = Ag. \end{cases} \]

(\subseteq) If $\tilde{\rho}(c, b, A, \overline{c}, \overline{b}, \overline{A}) \in \tilde{M}^\tau$ then equation (3.24) implies that

\[ (1, g^{-1}).(c, b, A, \overline{c}, \overline{b}, \overline{A}) = (c, b, Ag, c, b, Ah^{-1}) = (c, b, Ag, c, b, (Ag)(hg)^{-1}) \in N_{(hg)^{-1}}. \]

(\supseteq) If $(c, b, A, c, b, A) \in N_x$ then

\[ T(c, b, A, c, b, A) = (c, b, A, x^{-1}cx, x^{-1}bx, x^{-1}A) \]
\[ = (1, x^{-1}).(c, b, A, c, b, A). \]

**Remark 5.** An element $(c, b, A, \overline{c}, \overline{b}, \overline{A})$ of $\tilde{R}$ lies in $N_x$ if and only if

\[ T(c, b, A, \overline{c}, \overline{b}, \overline{A}) = (1, x^{-1}).(c, b, A, \overline{c}, \overline{b}, \overline{A}). \]
Remark 6. Recall that $G = SU(2)$. In [6], an element $\tilde{\rho}(c, b, A) \in \tilde{\mathcal{M}}$ is called \textit{generic} when the stabilizer of $(c, b, A, \tilde{A}, \tilde{b}, \tilde{A}) \in \tilde{\mathcal{R}}$ is $\{(1, 1), (-1, -1)\}$. Since an element of $N_x$ is stabilized by $(x, x)$, the fixed point set $\tilde{\mathcal{M}}$ is generically $\tilde{\rho}(N_1) \cup \tilde{\rho}(N_{-1})$, and a straightforward computation shows that $\tilde{\rho}(N_1)$ and $\tilde{\rho}(N_{-1})$ are generically disjoint. We shall content ourselves with the description of $\tilde{\mathcal{M}}$ appearing in Proposition 1; arguing as in [6], however, it is possible to show that $\tilde{\mathcal{M}} = \tilde{\rho}(N_1) \cup \tilde{\rho}(N_{-1})$.

Figure 8. The cylinder $U \subset \Sigma$.

Figure 9. The cylinders $\nabla, V \subset \tilde{\Sigma}$.

3.6. The Goldman flows. The cylindrical neighbourhood $U$ of $\gamma$ appears in Figure 8 and the cylindrical neighbourhoods $V$ of $\Gamma$ and $\nabla$ of $\tilde{\Gamma}$ are shown in Figure 9. Recall from Section 3.2 that both $V$ and $\nabla$ are embedded in $\tilde{\Sigma}$ in an orientation preserving way, and that the covering map is orientation preserving on $V$ and orientation reversing on $\nabla$. In cases (i) and (ii), each $A_j$ is a path from $P$ to $T$, and each $A_j$ is a path from $P$ to $\tilde{T}$. In case (iii), each $A_j$ is a loop based at $P$, and each $\tilde{A}_j$ is a loop based at $\tilde{P}$.

Under the identifications appearing in lines (3.10) and (3.14), if $(c, b, A) \in \mathcal{R}$ then $c$ corresponds with the holonomy along $\gamma$, and if $(c, b, \tilde{A}, \tilde{b}, \tilde{A}) \in \tilde{\mathcal{R}}$ then $c$ and $\tilde{c}$ correspond
respectively with the holonomies along $\Gamma$ and $\Gamma$. For $t \in \mathbb{R}$ and $g \in G \setminus \{\pm 1\}$, let
\[ \zeta_t(g) = \exp(tF(g)), \] (3.27)
where $F$ is the the variation of $f$ defined in equation (2.34). By comparing Figure 8 and Figure 9 with the table in Figure 13 we can now describe the Goldman flows associated to each of the three cylinders $U$, $V$ and $W$.

**Definition 1.** Define an $\mathbb{R}$-action $\{\Xi_t^+\}_{t \in \mathbb{R}}$ on the following subset $\mathcal{R}_\gamma$ of $\mathcal{R}$:
\[ \mathcal{R}_\gamma = \{(c, b, A) \in \mathcal{R} \mid c \neq \pm 1\}, \]
\[ \Xi_t^+(c, b, A) = (c, (\zeta_t(c))^{-1}b, A). \] (3.28)
The $\mathbb{R}$-action $\Xi_t^+$ covers the $2\pi$-periodic Goldman flow $\Xi_t$ on $\mathcal{M}_\gamma = \mathcal{R}_\gamma/G$ associated to the cylinder $U$.

**Definition 2.** Define an $\mathbb{R}$-action $\{\Phi_t^+\}_{t \in \mathbb{R}}$ on the following subset $\mathcal{R}_\Gamma$ of $\mathcal{R}$:
\[ \mathcal{R}_\Gamma = \{(c, b, A, \overline{c}, \overline{b}, \overline{A}) \in \mathcal{R} \mid c \neq \pm 1\}, \]
\[ \Phi_t^+(c, b, A, \overline{c}, \overline{b}, \overline{A}) = (c, (\zeta_t(c))^{-1}b, \overline{c}, \overline{b}, \overline{A}). \] (3.29)
The $\mathbb{R}$-action $\Phi_t^+$ covers the $2\pi$-periodic Goldman flow $\Phi_t$ on $\mathcal{M}_\Gamma = \mathcal{R}_\Gamma/(G \times G)$ associated to the cylinder $V$.

**Definition 3.** Define an $\mathbb{R}$-action $\{\Psi_t^-\}_{t \in \mathbb{R}}$ on the following subset $\mathcal{R}_\mathcal{T}$ of $\mathcal{R}$:
\[ \mathcal{R}_\mathcal{T} = \{(c, b, A, \overline{c}, \overline{b}, \overline{A}) \in \mathcal{R} \mid \overline{c} \neq \pm 1\}, \]
\[ \Psi_t^-(c, b, A, \overline{c}, \overline{b}, \overline{A}) = (c, b, A, \overline{c}, (\zeta_t(\overline{c}))^{-1}\overline{b}, \overline{A}). \] (3.30)
The $\mathbb{R}$-action $\Psi_t^-$ covers the $2\pi$-periodic Goldman flow $\Psi_t$ on $\mathcal{M}_\mathcal{T} = \mathcal{R}_\mathcal{T}/(G \times G)$ associated to the cylinder $W$.

**Remark 7.** The actions $\Phi_t^+$ and $\Psi_t^-$ commute because the cylinders $V$ and $W$ are disjoint. The composition $\Phi_t^+ \circ \Psi_t^-$ defines an $\mathbb{R}$-action on $\mathcal{R}_\Gamma \cap \mathcal{R}_\mathcal{T}$,
\[ (\Phi_t^+ \circ \Psi_t^-)(c, b, A, \overline{c}, \overline{b}, \overline{A}) = (c, (\zeta_t(c))^{-1}b, A, \overline{c}, (\zeta_t(\overline{c}))^{-1}\overline{b}, \overline{A}), \] (3.31)
which covers a $2\pi$-periodic $\mathbb{R}$-action $\Phi_t \circ \Psi_t$ on $\mathcal{M}_\Gamma \cap \mathcal{M}_\mathcal{T}$. In the introduction, this flow was referred to as the *composite flow*.

### 3.7. Proof of the main theorem.
In this section we prove that the composite flow preserves not only the Langrangian submanifold $\tilde{\mathcal{M}}^\tau$, but also the image of the map $\iota$.

**Proposition 2.** $(\Phi_t \circ \Psi_{-t}) \circ \tau = \tau \circ (\Phi_t \circ \Psi_{-t})$.

*Proof.** It suffices to show that $(\Phi_t^+ \circ \Psi_{-t}) \circ T = T \circ (\Phi_t^+ \circ \Psi_{-t})$.
\[ ((\Phi_t^+ \circ \Psi_{-t}) \circ T)(c, b, A, \overline{c}, \overline{b}, \overline{A}) = (\Phi_t^+ \circ \Psi_{-t})(\overline{c}, \overline{b}, \overline{A}, c, b, A) \]
\[ = (\overline{c}, (\zeta_t(\overline{c}))^{-1}\overline{b}, \overline{A}, c, (\zeta_t(c))^{-1}b, A) \]
\[ = T(c, (\zeta_t(c))^{-1}b, A, \overline{c}, (\zeta_t(\overline{c}))^{-1}\overline{b}, \overline{A}) \]
\[ = (T \circ (\Phi_t^+ \circ \Psi_{-t}))(c, b, A, \overline{c}, \overline{b}, \overline{A}). \square \]

**Proposition 3.** $(\Phi_t \circ \Psi_{-t}) \circ \iota = \iota \circ \Xi_t$.
Proof. It suffices to show that \((\Phi_t^+ \circ \Psi_{-t}) \circ I = I \circ \Xi_t^+\).

\[
\begin{align*}
((\Phi_t^+ \circ \Psi_{-t}) \circ I)(c, b, A) &= (\Phi_t^+ \circ \Psi_{-t})(c, b, A, c, b, A) \\
&= (c, (\zeta_t(c))^{-1}b, A, c, (\zeta_t(c))^{-1}b, A) \\
&= I(c, (\zeta_t(c))^{-1}b, A) \\
&= (I \circ \Xi_t^+)(c, b, A).
\end{align*}
\] (3.33)

Lemma 1. Recall that \(\iota(\mathcal{M}) = \tilde{\rho}(\mathcal{N}_1)\) and \(\tilde{\mathcal{M}}^r = \bigcup_{x \in G} \tilde{\rho}(\mathcal{N}_x)\). The flow \(\Phi_t \circ \Psi_{-t}\) on \(\tilde{\mathcal{M}}_\Gamma \cap \tilde{\mathcal{M}}_\Gamma^r\) preserves \(\tilde{\rho}(\mathcal{N}_x)\).

Proof. It suffices to show that the flow \(\Phi_t^+ \circ \Psi_{-t}\) on \(\tilde{\mathcal{R}}_\Gamma \cap \tilde{\mathcal{R}}_\Gamma^r\) preserves \(\mathcal{N}_x\). We’ll only give the proof for case (i), since the proofs are nearly identical for each of the three cases. Suppose \((c, b, A, \overline{\tau}, \overline{\rho}, \overline{\Lambda}) = (c, b, A, c, b, A x) \in \mathcal{N}_x \cap (\tilde{\mathcal{R}}_\Gamma \cap \tilde{\mathcal{R}}_\Gamma^r)\), and note that

\[
(\Phi_t^+ \circ \Psi_{-t})(c, b, A, c, b, A x) = (c, (\zeta_t(c))^{-1}b, A, c, (\zeta_t(c))^{-1}b, A x).
\] (3.34)

Since \(x\) commutes with both \(c\) and \(b\), equation (2.5) implies that \(\text{Ad}_x F(c) = F(c)\),

\[
\Rightarrow x(\zeta_t(c))x^{-1} = \zeta_t(c) \\
\Rightarrow x(\zeta_t(c))^{-1}bx^{-1} = (\zeta_t(c))^{-1}b.
\] (3.35)

Thus \((c, (\zeta_t(c))^{-1}b, A, c, (\zeta_t(c))^{-1}b, A x) \in \mathcal{N}_x\), and the proof is complete. \(\square\)

Theorem 3 (Main Theorem). Consider the Goldman flow \(\Xi_t\) on \(\mathcal{M}_\gamma\) and the composite flow \(\Phi_t \circ \Psi_{-t}\) on \(\tilde{\mathcal{M}}_\Gamma \cap \tilde{\mathcal{M}}_\Gamma^r\).

(i) \(\Phi_t \circ \Psi_{-t}\) preserves \(\tilde{\mathcal{M}}^r\),
(ii) \(\Phi_t \circ \Psi_{-t}\) preserves \(\iota(\mathcal{M})\),
(iii) \((\Phi_t \circ \Psi_{-t}) \circ \tau = \tau \circ (\Phi_t \circ \Psi_{-t})\),
(iv) \((\Phi_t \circ \Psi_{-t}) \circ \iota = \iota \circ \Xi_t\).

Proof. We have already proved parts (iii) and (iv). Part (i) follows either from part (iii) or from Lemma 1. Part (ii) follows either from part (iv) or from Lemma 1. \(\square\)

Remark 8. If the nonorientable surface \(\Sigma\) is noncompact then Theorem 3 is still true, and the gauge theoretic description of the Goldman flow given in Section 2 can be used to produce a proof. Let \(\eta(s)\) be a bump function with support in \((0, 1)\). As in the proof of Theorem 1 use \(\eta(s)\) to define both the flow \(\Xi_t^+\) associated to \(U \subset \Sigma\) and the flow \(\Phi_t^+\) associated to \(V \subset \tilde{\Sigma}\), and use \(\eta(-s)\) to define the flow \(\psi_t^-\) associated to \(\overline{V} \subset \tilde{\Sigma}\). Recall from the proof of Theorem 2 that if \(A\) is a flat connection on \(\Sigma\) then there are gauge transformations \(\xi_t\) on \(\Sigma \setminus \gamma\) such that

\[
\xi_t(A|_{\Sigma \setminus \gamma}) = (\Xi_t^+ A)|_{\Sigma \setminus \gamma}.
\] (3.36)

Similarly, if \(A\) is a flat connection on \(\tilde{\Sigma}\) then there are gauge transformations \(\varphi_t \in \mathcal{G}(\tilde{\Sigma} \setminus \Gamma)\) and \(\psi_t \in \mathcal{G}(\tilde{\Sigma} \setminus \overline{\Gamma})\) such that

\[
\varphi_t(A|_{\Sigma \setminus \Gamma}) = (\Phi_t^+ A)|_{\Sigma \setminus \Gamma},
\psi_t(A|_{\Sigma \setminus \overline{\Gamma}}) = (\Psi_t^- A)|_{\Sigma \setminus \overline{\Gamma}}.
\] (3.37)

The gauge transformations \(\xi_t\), \(\varphi_t\), and \(\psi_t\) can be used to prove Proposition 2 and Proposition 3.
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