Iordanskii Force and the Gravitational Aharonov-Bohm effect for a Moving Vortex

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Abstract

I discuss the scattering of phonons by a vortex moving with respect to a superfluid condensate. This allows us to test the compatibility of the scattering-theory derivation of the Iordanskii force with the galilean invariance of the underlying fluid dynamics. In order to obtain the correct result we must retain \( O(v_s^2) \) terms in the sound-wave equation, and this reinforces the interpretation, due to Volovik, of the Iordanskii force as an analogue of the gravitational Bohm-Aharonov effect.

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I. INTRODUCTION

The problem of computing the transverse force acting on a vortex in a superfluid has recently engendered a certain amount of controversy. If the vortex moves at a velocity $v_L$ while the superfluid and normal components have asymptotic velocities $v_s$ and $v_n$ respectively, then the most general form of the transverse force per unit length that is consistent with galilean invariance (i.e. depends only on velocity differences) can be written

$$\mathbf{F} = A\kappa \hat{z} \times (v_L - v_s) + B\kappa \hat{z} \times (v_L - v_n).$$

Here $\kappa$ is the magnitude of the quantum of circulation about the vortex, $\kappa = \frac{\hbar}{m}$, with $m$ the mass of a helium atom, and $\hat{z}$ a unit tangent to the vortex line in the direction of the circulation. In the absence of any normal component, elementary fluid mechanics shows that the momentum flux into the vortex core is

$$\mathbf{F}_M = \rho_{tot} \kappa \hat{z} \times (v_L - v_s),$$

where $\rho_{tot}$ is the total mass density. This is the Magnus force. Once a normal component is present, however, a variety of different expressions for the coefficients $A$ and $B$ have been given in the literature.

It is generally accepted that the coefficient $A$ is $\rho_s$ [3,4]. An appealing thermodynamic argument for this has recently been given by Wexler [1]. The controversy stems from the claim of Wexler [1], and Thouless et al. [5] that the coefficient of $v_L$ in $\mathbf{F}$, is also equal to $\rho_s$. Since this coefficient is $A + B$, their claim, if true, would force $B$ to be zero — thus ruling out the existence of the second term, the Iordanskii force, which is supposed to originate in a left/right asymmetry in the scattering of quasi-particles by the vortex line [6]. Sonin [7,8], on the other hand, has presented a detailed review of the scattering of phonons by a vortex line at rest with respect to the superflow. His analysis (which has been challenged by Wexler and Thouless [9,10], but which I believe to be correct) shows that the asymmetry
arises from a fluid dynamical analog [11] of the Bohm-Aharonov effect [12], and gives the coefficient of $-v_n$ as $\rho_n$. Thus $B = \rho_n$. Combining this value of $B$ with the accepted value for $A$ gives the transverse part of the force per unit length as

$$F = \rho_s \kappa \hat{z} \times (v_L - v_s) + \rho_n \kappa \hat{z} \times (v_L - v_n).$$

(1.4)

This is the most commonly accepted expression for the force. When it is written in this form the first term is usually refered to as the superfluid Magnus force and the second term as the Iordanskii force.

Since the total density is $\rho_{tot} = \rho_s + \rho_n$, equation (1.4) may equally well be written

$$F = \rho_{tot} \kappa \hat{z} \times (v_L - v_s) + \rho_n \kappa \hat{z} \times (v_s - v_n).$$

(1.5)

The first part of (1.5) is the momentum transfer to the vortex due to the condensate motion (and possibly this should be called the superfluid Magnus force), so the second term must be the force on the vortex due to phonon scattering. Part of the phonon force is responsible for reducing the coefficient of $v_s$ from $\rho_{tot}$ to $\rho_s$. Notice that this expression for the phonon force does not depend on the motion of the vortex line relative to either component of the fluid. Although Sonin, who works in the frame $v_L = 0$, writes $\rho_n (v_s - v_n)$ in his expression for the phonon force, his analysis of the scattering process is restricted to the situation where $v_s = 0$, i.e to the case where there is no relative motion between the vortex line and the condensate. The $v_s$ part of the force is inferred from the thermodynamic and galilean invariance argument given above. A direct demonstration that the phonon force is independent of the relative velocity of the vortex and the condensate, and hence that coefficient of $v_s$ in the phonon force is indeed equal to $\rho_n$, would provide a useful consistency test of the conventional scattering-theory derivation of (1.4), because the galilean invariance that went into deducing this coefficient is not manifest in the linearized sound wave equation. In this paper I will provide such a demonstration. In obtaining the result we will find it useful to consider the analogy, first pointed out by Volovik [13], of phonon vortex scattering with the gravitational Bohm-Aharonov effect where particles are scattered by a spinning cosmic string.
In the next section I will review the acoustic Bohm-Aharonov effect and rederive Sonin’s results for the phonon force in the case that the vortex is at rest with respect to the condensate. In section three I will extend these results to the case in which the vortex moves relative to the condensate, and relate the momentum given to the phonons by the vortex to the time delay of signals passing on either side of a cosmic string. In the last section I will discuss the apparent conflict between our results and the claims of Thouless et al., and some possible resolutions.

II. THE ACOUSTIC BOHM-AHARONOV EFFECT

The scattering of phonons by a superfluid vortex was first studied by Fetter [14]. The wave equation used by most recent authors [7–10] is

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi - 2(v \cdot \nabla) \frac{\partial \phi}{\partial t}. \quad (2.1)$$

Here $\phi$ is the velocity potential, $v = v_v$ is the velocity field of the vortex

$$((v_v)_x, (v_v)_y) = \frac{\kappa}{2\pi} \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right), \quad (2.2)$$

and $c$ the speed of sound.

When the sound field $\phi$ has harmonic time dependence, $\phi(r, t) = e^{-i\omega t}\phi(r)$, equation (2.1) becomes

$$-\omega^2 \phi = c^2 \nabla^2 \phi + 2i\omega (v_v \cdot \nabla) \phi. \quad (2.3)$$

We will be interested in effects only to first order in the circulation $\kappa$, therefore it is natural to add harmless $O(v_v^2)$ terms to (2.3) so that it becomes the Schrödinger equation for unit charge particles minimally coupled to a gauge field $A = \frac{\omega}{c^2} v_v$, viz.

$$-\omega^2 \phi = c^2 \left( \nabla + i\frac{\omega}{c^2} v_v \right)^2 \phi. \quad (2.4)$$

Notice that this rewriting requires $\nabla \cdot v_v = 0$. 

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Equation (2.4) describes the Bohm-Aharonov interaction of particles with a threadlike tube of magnetic flux in the gauge where $\nabla \cdot \mathbf{A} = 0$. The total flux in the tube is

$$\Phi = \oint \mathbf{A} \cdot d\mathbf{r} = \frac{\omega}{c^2} \oint \mathbf{v} \cdot d\mathbf{r} = \frac{\omega}{c^2} \kappa. \quad (2.5)$$

Here the integration contour surrounds the vortex, which we have taken to be at the origin of our coordinate system. We will use the symbol $\alpha$ to denote the ratio of this flux to the Dirac flux quantum, $\Phi_0 = 2\pi$.

In their original paper [12] Aharonov and Bohm provided a partial-wave series expansion for scattering of a plane wave by the flux tube. Figures [1] and [2] are numerical plots of the sum of the first forty terms in this expansion for the cases where a plane wave is incident from the right on flux tubes with $\alpha$ equal to 0.25 and 0.5 respectively. These plots should be compared to the ripple tank photographs of surface waves interacting with a “bathtub” vortex in [11]. The most noticeable feature in both the photographs and the plots is the “seam” or “tear” in the wavefunction downstream of the flux tube. The incident plane wave is cut in two by the flux tube and, apart from diffraction effects, the upper and lower halves of the incident wave propagate parallel to each other but with a relative phase shift of a fraction $\alpha$ of a wavelength. It is in the region of the seam that the transverse momentum imparted to the beam by the flux resides. Indeed in Fig [1] one can plainly see that the wavefronts are directed slightly upwards in this region.

The time-average momentum density in the sound wave is $\langle \rho(1) \mathbf{v}(1) \rangle$ where $\mathbf{v}(1) = \nabla \phi$ is the fluid velocity due to the sound wave and

$$\rho(1) = -\frac{\rho(0)}{c^2} \{ \dot{\phi} + \mathbf{v} \cdot \nabla \phi \}. \quad (2.6)$$

The angular brackets denote a time average. (See the appendix for a derivation the expression for $\rho_{(1)}$.) For a plane wave

$$\phi(r, t) = \text{Re} \left\{ \phi_0 e^{ikr-i\omega t} \right\}, \quad (2.7)$$

and with the background flow $\mathbf{v}$ vanishing, we have
\begin{align}
\langle \rho(1) v(1) \rangle &= \frac{1}{2} \rho(0) \frac{\omega}{c^2} |\phi_0|^2 k. 
\end{align}

More generally we find

\begin{align}
\langle \rho(1) v(1) \rangle &= \frac{1}{4i} \rho(0) \frac{\omega_r}{c^2} (\phi^* \nabla \phi - (\nabla \phi^*) \phi),
\end{align}

where $\omega_r$ is the frequency of the wave relative to the fluid. (Notice that (2.9) is not the “gauge invariant” form of the current for our minimally coupled Schr"odinger equation.)

Once we are out of the region where $v$ is significant, we can write the momentum density as

\begin{align}
\langle \rho(1) v(1) \rangle &= \frac{1}{2} \rho(0) \frac{\omega}{c^2} |\phi_0|^2 \nabla \chi,
\end{align}

where $\phi(x) = |\phi_0| e^{i\chi(x)}$. If we temporarily ignore the reduction in the amplitude of the sound wave in the seam region, we can find the total transverse momentum by integrating the $y$ component of this momentum from one side of the seam to the other along a line parallel to the $y$ axis. The total transverse momentum per unit length at abscissa $x$ is therefore

\begin{align}
\langle p_y \rangle &= \frac{1}{2} \rho(0) \frac{\omega}{c^2} |\phi_0|^2 \Delta \chi(x),
\end{align}

where $\Delta \chi(x)$, the phase difference across the seam, is zero long before the sound wave interacts with the vortex, and

\begin{align}
\Delta \chi(x) &= 2\pi \alpha = \frac{\omega}{c^2} \kappa
\end{align}

well after the sound waves have passed the vortex. In this manner the transverse momentum is found by examining the wave at large (but not infinite) impact parameter, and the result is insensitive to details such as diffraction effects.

The transverse momentum per unit length of the seam can also be written

\begin{align}
\left( \frac{1}{2} \rho(0) \frac{\omega}{c^2} |\phi_0|^2 k \right) \cdot \frac{1}{k} \cdot \frac{\omega}{c^2} \kappa &= \langle j_{ph} \rangle \cdot \frac{1}{k} \cdot \frac{\omega}{c^2} \kappa,
\end{align}

where $\langle j_{ph} \rangle$ is the mass current, or momentum density in the unperturbed sound wave. If a finite pulse of sound is sent past the vortex then a length of seam equal to the group velocity
of the waves (here \( c \)) is created every second. The transverse force is the rate of creation of transverse momentum and this is

\[
\langle \dot{P}_\perp \rangle = \langle j_{\text{ph}} \rangle \cdot \frac{1}{k} \cdot \frac{\omega}{c^2} \kappa \cdot c = \langle j_{\text{ph}} \rangle \kappa
\] (2.14)

in agreement with ref [8].

The mass flux due to phonons in the two fluid model is

\[\langle j_{\text{ph}} \rangle = \rho_n (v_n - v_s),\] (2.15)

and we can find the thermal average of the phonon force by substituting this in (2.14). Since we have so far assumed that \( v_s \) is zero we have, however, only established the \( \rho_n \kappa v_n \) part of the Iordanskii force.

A more rigorous approach to computing the momentum given to the phonons exploits the momentum flux tensor \( \Pi_{ij}^{\text{phon}} \). The relevant terms are

\[\Pi_{xy}^{\text{phon}} = \rho(0) \langle v(1)i v(1)j \rangle + \langle \rho(1) v(1)i \rangle + \langle \rho(1) v(1)j \rangle,\] (2.16)

The only terms contributing to \( \Pi_{xy}^{\text{phon}} = \Pi_{yx}^{\text{phon}} \) to first order in \( v_v \) turn out to be

\[\Pi_{xy}^{\text{phon}} = \left( \Pi_{xy}^{\text{phon}} \right)_{v_v = 0} + \frac{c}{k} \left( \partial_y \chi + \frac{k}{c} (v_v)_y \right) \langle j_{\text{ph}} \rangle \chi,\] (2.17)

while \( \Pi_{yy}^{\text{phon}} \) is of at least second order in \( v_v \) and can be neglected.

For most of the \( x, y \) plane we may use the eikonal approximation for the phase \( \chi \),

\[\chi(x, y) = kx - \frac{\omega}{c^2} \int_{-\infty}^{x} (v_v)_x \, dx'.\] (2.18)

Here the integral is taken along the line from the point \((-\infty, y)\) to \((x, y)\). (The eikonal approximation will be described further in the next section.) From the formulae for \((v_v)_x, (v_v)_y\) in (2.2) we find

\[\int_{-\infty}^{x} (v_v)_x \, dx = -\frac{\kappa}{2\pi} \frac{y}{|y|} \left( \tan^{-1} \frac{x}{|y|} + \frac{\pi}{2} \right).\] (2.19)

This expression is continuous across the negative \( x \) axis, but jumps discontinuously by \( \kappa \) across the positive \( x \) axis. We therefore find that the eikonal phase has the expected \( \omega \kappa / c^2 \)
discontinuity across the seam. The physical wave fronts, of course, smoothly interpolate the phase across the seam. We immediately see that outside this interpolating zone, in the region where the eikonal approximation to the phase is valid, we have

$$\partial_y \chi + \frac{k}{c}(v_y)_y = 0.$$  

(2.20)

This means that $\Pi_{xy}$ is zero outside the seam region. Thus the flux of $p_y$ through any curve vanishes unless it intersects the seam. Indeed we find that the only regions that have a net transverse momentum flux out of them are those that include the vortex. For these the momentum flux is entirely due to the interpolating phase and comes out to be $\langle (j_{\text{ph}})_x \rangle \kappa$ as found by the previous, more intuitive, method.

The transverse momentum flux tensor vanishes because the transverse component of $k$ acquired by interaction with the flow is cancelled by the transverse component of $v_v$ when they are combined to form the group velocity

$$\frac{\partial \omega}{\partial k} = \frac{c}{k}k + v_v.$$  

(2.21)

The classical phonon trajectories therefore remain straight, and, just as in the ordinary Bohm-Aharonov effect, the transverse momentum is consequence of the wave-particle duality.

A more detailed analysis would take into account the reduction of the amplitude of the sound wave in the seam region. It is well understood from the theory of Bohm-Aharovov scattering [13] that the effect of this is to replace $\Phi$ by $\sin \Phi$ in the force equation. (Inspection of Fig 2 where $\Phi = \pi$ shows that that the sound wave amplitude is exactly zero in the seam).

Using $c = 230 \text{ms}^{-1}$ for the speed of sound in liquid helium, we find

$$\alpha = \frac{\omega \hbar}{c^2 m} = 0.035 E_{\text{phon}}$$  

(2.22)

where $E_{\text{phon}}$ is the energy of the phonon measured in degrees Kelvin. We see that $\Phi$ is small at temperatures below 0.2 K where phonons dominate the scattering process, so the correction will be unimportant.
III. MOVING THE VORTEX

So far we have merely reproduced the results of [8]. We now extend our analysis to the case in which the vortex moves with respect to the condensate. So as to retain a time independent equation we will keep the vortex fixed at the origin, but allow a non-zero asymptotic $v_s$. For simplicity of description we will initially consider only the case where the uniform $v_s$ is in the direction of propagation of the sound wave, which as before we take to be the $+x$ direction. We will write $v_s = U \hat{x}$. In this case the length of seam created per second is $c + U$. We need to find the phase shift between two halves of the wavefront to complete the computation.

It is tempting to simply replace the $v_v$ in (2.4) by $v = U \hat{x} + v_v$, but this will not serve to give the correct answer. Because of the Doppler shift, the frequency is now related to the wavelength by $\omega = (c + U)k$, so for a wave with the same $k$ the frequency, and hence the effective flux

$$\Phi = \frac{\omega}{c^2} \oint v_v \cdot dr$$

(3.1)

is increased, and this makes the phase shift larger. This is not what we expect, and is incorrect. The terms added to the sound-wave equation to make it into the minimally coupled Schrödinger equation are no longer harmless. This is because even when we work only to $O(v_v)$ accuracy, we must not neglect $O(U^2)$ terms.

We require a more accurate equation. In the appendix we show that the relevant equation is that given by Unruh [16,17]

$$\left( \frac{\partial}{\partial t} + \nabla \cdot v \right) \rho \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \phi = \nabla (\rho \nabla \phi),$$

(3.2)

who interprets his equation as that of a scalar field propagating in a curved space-time background.

We can find the phase offset by working at large impact parameter where $\phi$ is essentially independent of $y$, and where also $\partial_x v_v$ is negligible. We can therefore set...
\[ \phi = \varphi(x)e^{ikx-i\omega t} \]  
(3.3)

with \( \omega = (c + U)k \), and expect \( \varphi \) to be slowly varying. The equation for \( \phi \) becomes

\[ -\omega^2 \phi - 2i\omega v_x \partial_x \phi + v_x^2 \partial^2_{xx} \phi = c^2 \partial^2_{xx} \phi \]  
(3.4)

where \( v_x = (v_v)_x + U \). Since \( \varphi(x) \) is slowly varying we can ignore \( \partial^2_{xx} \varphi(x) \). Doing so we find that (3.4) reduces to

\[ (v_v)_x k \varphi - i(U + c) \partial_x \varphi = 0. \]  
(3.5)

(I have ignored terms containing \( v_v \) in the coefficient of \( \partial_x \varphi \) because they will not affect our result to \( O(\kappa) \).) This gives

\[ \varphi(x) = e^{-i\chi(x)} \]  
(3.6)

with

\[ \chi(x) = \frac{k}{U + c} \int_{-\infty}^{x} (v_v)_x dx'. \]  
(3.7)

We see that as \( x \to +\infty \) the phase offset of the two wavefronts becomes

\[ \Delta \chi = \frac{k}{U + c} \oint v_v \cdot dr = \frac{k}{U + c} \kappa. \]  
(3.8)

The factor of \( U + c \) cancels against the length of seam being created per second to give

\[ \langle \dot{P}_{\perp} \rangle = \langle j_{ph} \rangle \kappa \]  
(3.9)

as before.

This result can be confirmed by examining the momentum flux tensor. The \( O(v_v) \) part of \( \Pi_{xy} \) is now

\[ \Pi_{xy} = \left( (c + U) \frac{1}{k} \partial_y \chi + (v_v)_y \right) \langle (j_{ph})_x \rangle, \]  
(3.10)

where we have included a non-zero contribution from \( U \langle \rho_{(1)}(v_{(1)} y) \rangle \). Once again we see that outside the seam region the gradient of \( \chi \) cancels the \( (v_v)_y \) advective term, and that the discontinuity across the seam provides the momentum flux obtained in the previous paragraph.
The wavefront offset can also be calculated from the time delay between phonons passing on other side of the vortex. Since the phonon trajectories with large impact parameters are hardly deflected, we can find them as the null geodesics of the simplified form of the Unruh metric (A17)

\[ ds^2 = \frac{\rho}{c} \left\{ - (dx - (v + c)dt)(dx - (v - c)dt) - dy^2 - dz^2 \right\}. \]  (3.11)

The null geodesics are given by

\[ \frac{dx}{dt} = v \pm c. \]  (3.12)

In our present case, the time of arrival of a signal at the point \((x, y)\) is

\[ t(x, y) = \int_{-\infty}^{(x,y)} \frac{dx}{U + c + (v_v)_x} = \text{const.} - \frac{1}{(U + c)^2} \int_{-\infty}^{(x,y)} (v_v)_x dx + O(v_v^2). \]  (3.13)

We convert the time delay to a phase shift by multiplying by \(\omega = (U + c)k\). Again we find that the relative phase shift between waves that pass above and below the vortex to be

\[ \Delta \chi = \frac{k}{U + c} \kappa. \]  (3.14)

As described by Volovik [13] this separation in the time of arrival for signals passing abitarily far from the vortex is characteristic of a “spinning cosmic string”.

Matters become slightly more complicated when the uniform background superflow \(v_s\) is not oriented parallel (or anti-parallel) to the incident phonon flux. After a little work we find that the eikonal equation becomes

\[ (U^g \cdot \nabla) \chi + k \cdot v_v = 0, \]  (3.15)

where \(U^g = \hat{c}k + v_s\). The phase discontinuity seam no longer lies exactly along the \(x\) axis, but instead points in the direction of \(U^g\). The rate of transverse momentum production does depend on the angle this vector makes with the \(x\) axis, but the effects are \(O(U^2/c^2)\). They do not seem to be worth working out in detail since at this order we should also include compressibility effects and the dependence of \(\rho_n\) and \(\rho_s\) on \((v_n - v_s)^2/c^2\).
IV. DISCUSSION

We have computed the force on a vortex due to the scattering of phonons in the case where the vortex is moving with respect to the condensate. We have shown that to first order in $|v_s - v_L|/c$ the rate of transverse momentum production is independent of the relative velocity of the vortex and the condensate, and that the Bohm-Aharonov phase shift of the phonon wavefront passing on either side of the vortex leads to a force

$$F_{\text{phon}} = \rho_n \kappa \hat{z} \times (v_s - v_n),$$

(4.1)

which is consistent with galilean invariance. This supports the view that the Bohm-Aharonov analysis of the phonon force is correct.

The expression we have obtained for the phonon force leads to the commonly accepted form of the Iordanskii force and so disagrees with the claims of Thouless et al. that it vanishes identically. Their claim is based on earlier work by Thouless, Ao and Niu (TAN) [18] which establishes a general theorem relating the force on a vortex to the circulation of momentum at infinity. Since there are no impurities present, this theorem should hold here and we have a puzzle that needs to be resolved.

In response to criticism of their claims [19,20], Thouless et al. [21] have given two possible explanations for the discrepancy. The first suggests that the computation of the cross section asymmetry is mathematically flawed because a conditionally convergent series is mistreated. If this were correct, their objection would also apply to the case where $v_L = v_n$, but we have shown here that the scattering asymmetry correctly predicts the reduction of the coefficient of $v_s$ from $\rho_{tot}$ to $\rho_s$. This explanation seems unlikely therefore.

The second possible explanation is more subtle. In the calculations presented here, and in all earlier work on phonon scattering, the incident flux of phonons is calculated by assuming a thermal phonon distribution derived from the asymptotic $v_n$ and $v_s$, i.e. a distribution that does not seem to take into account the effect of the local vortex flow field $v_v$. Because the assumed phonon flux to the left and right of the vortex is the same, it appears that the
phonons make an equal reduction in the fluid momentum on either side of the vortex. In other words they appear not to change the value of the total momentum circulation

\[ K_{\text{tot}} = \oint \langle \rho v \rangle \cdot dr, \]  

(4.2)

so that it remains \( \rho_{\text{tot}} h/m \) instead of being reduced to \( \rho_s h/m \). If this were correct there would be circulation in the normal fluid as well as in the superfluid component. If we include this unphysical circulation in the TAN formula, then it agrees with the calculated Iordanskii force.

The problem with this explanation is that the motion of the phonons can be derived from a hamiltonian, \( H = c|k| + v_v \cdot k \). Consequently Liouville’s theorem applies to the distribution function. Even in the absence of phonon-phonon interactions, a phonon distribution function that is in thermal equilibrium will remain in equilibrium and so phonons arriving from infinity will have their local distribution modified by the flow field so as to correctly bring about the expected reduction in the circulation. Although there are an equal number number of phonons passing to the right or to the left of the vortex, those moving against the superflow are going slower, so their number density is higher. This means that the phonon momentum opposing the \( \rho_{\text{tot}} v_v \) is left-right asymmetric as it should be.

There remains a contradiction, therefore.

What are we to conclude? The TAN arguments implicitly make use of the change in the total momentum of the medium outside the vortex. In fluid mechanics it is well known that the total momentum associated with a system of vortices is ill-defined, being given by a conditionally convergent integral. This problem is usually dealt with defining the impulse \[ 2 \] of the vortex system. The impulse is defined in terms the velocity potential in the vicinity of the vortex system, and does not have contributions from the effects of distant boundaries (the ultimate origin of the ill-defined momentum). Perhaps this is at the root of the problem. On the other hand the present calculation might be also described as naïve. We are not distinguishing between the true newtonian momentum and the pseudo-momentum \[ 22 \] possessed by the phonons. However pseudo-momentum is usually exactly
what is needed for computing forces on immersed objects. Clearly, more work is needed to resolve the paradox.

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APPENDIX A: THE UNRUH EQUATION

The homentropic potential flow of a fluid is derivable from the action

\[ S = \int d^4x \left\{ \rho \dot{\phi} + \frac{1}{2} \rho (\nabla \phi)^2 + V(\rho) \right\}. \]

(A1)

Here \( \rho \) is the mass density and \( \nabla \phi = v \), the fluid velocity. Varying \( S \) with respect to \( \phi \) gives the continuity equation

\[ \dot{\rho} + \nabla \cdot (\rho \nabla \phi) = 0, \]

(A2)

while varying with respect to \( \rho \) gives Bernoulli’s equation

\[ \dot{\phi} + \frac{1}{2} (\nabla \phi)^2 + \mu(\rho) = 0, \]

(A3)

with \( \mu(\rho) = dV/d\rho \).

In order to consider the propagation of sound waves in the background flow, set

\[ \phi = \phi(0) + \phi(1) \]

\[ \rho = \rho(0) + \rho(1) + \cdots \]

(A4)

where \( \phi(0) \) and \( \rho(0) \) obey the equations of motion, and \( \phi(1) \) and \( \rho(1) \) are small amplitude perturbations. Expanding \( S \) to quadratic order in the perturbations gives
\[ S = S_0 + \int d^4x \left\{ \rho(1) \dot{\phi}(1) + \frac{1}{2} \left( \frac{c^2}{\rho(0)} \right) \rho^2(1) + \frac{1}{2} \rho(0) (\nabla \phi(1))^2 + \rho(1) \mathbf{v} \cdot \nabla \phi(1) \right\}. \] (A5)

(The terms linear in the perturbations vanish because of our assumption that the zeroth order variables obey the equation of motion.) Here \( \mathbf{v} \equiv \mathbf{v}(0) = \nabla \phi(0) \). The speed of sound, \( c \), is defined by

\[ \frac{c^2}{\rho(0)} = \frac{d\mu}{d\rho} \bigg|_{\rho(0)}, \] (A6)

or more familiarly

\[ c^2 = \frac{dP}{d\rho}. \] (A7)

Since the new action is quadratic in \( \rho(1) \), we can eliminate it via its equation of motion

\[ \rho(1) = -\frac{\rho(0)}{c^2} \{ \dot{\phi}(1) + \mathbf{v} \cdot \nabla \phi(1) \}. \] (A8)

We find the effective action for the sound waves in the background flow \( \mathbf{v} \) to be

\[ S_{(2)} = \int d^4x \left\{ \frac{1}{2} \rho(0)(\nabla \phi(1))^2 - \frac{\rho(0)}{2c^2} (\dot{\phi}(1) + \mathbf{v} \cdot \nabla \phi(1))^2 \right\}. \] (A9)

After changing an overall sign for convenience, we can write this as

\[ S = \int d^4x \frac{1}{2} g^{-\mu\nu} \partial_\mu \phi(1) \partial_\nu \phi(1), \] (A10)

where

\[ g^{-\mu\nu} = \frac{\rho(0)}{c^2} \begin{pmatrix} 1, & \mathbf{v}^T \\ \mathbf{v}, & \mathbf{vv}^T - c^2 \mathbf{1} \end{pmatrix}. \] (A11)

(We use the convention that greek letters run over all four space-time indices with 0 \( \equiv t \), while roman indices refer to the spatial components.)

The resultant equation of motion

\[ \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \phi(1) = 0, \] (A12)

is
\[
\left( \frac{\partial}{\partial t} + \nabla \cdot \mathbf{v} \right) \frac{\rho(0)}{c^2} \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \phi(1) = \nabla (\rho(0) \nabla \phi(1)). \tag{A13}
\]

This equation and its interpretation as the wave equation for a scalar field propagating in the background space-time metric (A11) is due to Unruh [16,17].

In four dimensions we have \(\sqrt{-g} = \rho^2(0)/c\) and

\[
g_{\mu\nu} = \frac{\rho(0)}{c} \begin{pmatrix} c^2 - v^2, & \mathbf{v}^T \\ \mathbf{v}, & -1 \end{pmatrix}. \tag{A14}
\]

The associated space-time interval is therefore

\[
ds^2 = \frac{\rho(0)}{c} \left\{ c^2 dt^2 - \delta_{ij}(dx^i - v^i dt)(dx^j - v^j dt) \right\}. \tag{A15}
\]

Up to the overall conformal factor \(\frac{\rho(0)}{c}\) we see that \(c\) and \(-v^i\) play the role of the lapse function and shift vector appearing in the Arnowitt-Deser-Misner (ADM) formalism of general relativity [23]. A conformal factor does not affect null geodesics, and so variations in \(\rho(0)\) do not influence the ray tracing for the sound waves.

It is also sometimes convenient to write

\[
ds^2 = \frac{\rho(0)}{c} \left\{ (c^2 - v^2) \left( dt + \frac{v^i dx^i}{c^2 - v^2} \right)^2 - \left( \delta_{ij} + \frac{v^i v^j}{c^2 - v^2} \right) dx^i dx^j \right\}. \tag{A16}
\]

When \(\mathbf{v}\) is in the \(x\) direction only, we can also rewrite \(ds^2\) as

\[
ds^2 = \frac{\rho(0)}{c} \left\{ - (dx - (v + c) dt)(dx - (v - c) dt) - dy^2 - dz^2 \right\}. \tag{A17}
\]

This shows that the \(x - t\) plane null geodesics coincide with the expected characteristics of the wave equation in the background flow.
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FIG. 1. Bohm-Aharonov scattering for $\alpha = 0.25$
FIG. 2. Bohm-Aharonov scattering for $\alpha = 0.5$