Universal Multiparty Data Exchange and Secret Key Agreement

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Abstract

Multiple parties observing correlated data seek to recover each other’s data and attain omniscience. To that end, they communicate interactively over a noiseless broadcast channel: Each bit transmitted over this channel is received by all the parties. We give a universal interactive communication protocol for omniscience which requires communication of rate only $O(n^{-1/2} \sqrt{\log n})$ more than the optimal rate for every independent and identically distributed (in time) sequence of data. Drawing on the duality between omniscience and secret key agreement due to Csiszár and Narayan, as a by-product, we obtain a universal protocol for generating a multiparty secret key of rate at most $O(n^{-1/2} \sqrt{\log n})$ less than the maximum rate possible.

I. INTRODUCTION

An $m$ party omniscience protocol is an interactive communication protocol that enables $m$ parties to recover each other’s data. The communication is error-free and is in a broadcast mode wherein the transmission of each party is received by all the other parties. Such protocols were first considered in [11] in a two-party setup, where bounds for the number of bits communicated on average and in the worst-case were derived for the case when no error is allowed. The $m$ party version, and the omniscience terminology, was proposed in [9] where the collective observations of the parties was assumed to be an independent and identically distributed (IID) sequence generated from a known distribution $\mathbb{P}_{X_1 \ldots X_m}$. It was shown in [9] that a simultaneous communication protocol based on sending random hash bits of appropriate rates is sum-rate optimal. A common feature of these prior works is that the protocol relies

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1Throughout we shall restrict to finite random variables and use the phrase probability distribution interchangeably with probability mass function (pmf).
on the knowledge of the underlying distribution $P_{X_1\cdots X_m}$. Our goal in this paper is to give a protocol for omniscience that is *universal*, *i.e.*, it does not rely on the knowledge of the underlying IID distribution. Note that the protocol proposed in [9] relies on the classic multiterminal source coding scheme given in [7], [12]. Thus, it inherits the following universality feature from that scheme: If for $1 \leq i \leq m$ the $i$th party communicates rate $R_i$, the protocol attains omniscience for any source distribution $P_{X_1\cdots X_m}$ for which the rate vector $(R_1, \ldots, R_m)$ lies in the omniscience rate-region corresponding to $P_{X_1\cdots X_m}$. Nevertheless, this provides no guarantee of universal optimality for the sum-rate $(R_1 + \cdots + R_m)$ for an arbitrary source $P_{X_1\cdots X_m}$. A naive solution entails using first $n'$ samples to estimate the joint distribution and then applying the optimal protocol of [9] for the estimated distribution. However, this approach will result in an excess communication rate of $O\left(n^{-1/3} \sqrt{\log n}\right)$ over the optimal rate. In this paper, we present a universal protocol for omniscience that requires an excess rate of only $O\left(n^{-1/2} \sqrt{\log n}\right)$ and which, we believe, is a stepping-stone towards a practical protocol for the multiparty data exchange problem.

It was shown in [26] that interaction enables an ACK–NACK based universal variable-length coding scheme for the Slepian-Wolf problem, where only party 1 needs to send its data to party 2. Our protocol, too, is interactive in a similar spirit, but it relies on carefully increasing the rate of communication for each party. Note that while for $m = 2$ a simple extension of the protocol in [26] works for the data exchange problem as well, this is not the case when $m > 2$. For $m > 2$, the order in which the parties communicate must be carefully chosen. We give a very simple criterion for choosing this order of communication and show that the resulting protocol is universally rate-optimal. Specifically, in our protocol each party first communicates its empirical type to all the other parties. The parties then communicate in the order of the entropies of their empirical types, with the highest entropy party communicating first, followed by the next highest entropy party, and so on. The delay in communication between the parties is chosen to ensure that for every pair of communicating parties, the difference of their rates of communication, at any instance, is equal to the difference of their estimated entropies. We follow this policy and increase the rate in steps until a subset of parties can attain local omniscience, *i.e.*, recover each other’s data. A key feature of our protocol is its recursive structure wherein whenever a subset of parties attains local omniscience, the rate vector appears as if the parties were together to begin with. Furthermore, any subset $A$ attaining local omniscience does so by using only the communication from the parties in $A$ and using the minimum possible rate of communication. The proof of this recursive feature is the main technical contribution of this paper. We build upon this result to show that our protocol is sum-rate optimal with an excess rate of roughly $O\left(n^{-1/2} \sqrt{\log n}\right)$. In fact, we provide a performance-guarantee for individual sequences and obtain the claimed results in the limit as $n$ goes to infinity. Note that even for the case
when the underlying distribution is known, the optimal rate can only be achieved asymptotically and an excess rate is often needed. In particular, for $m = 2$, the precise leading asymptotic term in excess worst-case rate was established in [23] and was shown to be $O(n^{-1/2})$. Furthermore, the best-known average excess rate of $O(n^{-3/4})$ follows from [26].

An interesting application of our omniscience protocol appears in secret key (SK) agreement [15], [1], [9]. Specifically, Csiszár and Narayan showed in [9] that an optimum-rate SK can be generated by first attaining omniscience and then extracting secure bits from the recovered data. We follow the same procedure here with our universal protocol in place of the omniscience protocol of [9] and obtain a universal SK of rate at most $O(n^{-1/2} \sqrt{\log n})$ less than the optimal average and the worst-case rate. Note that for the case $m = 2$ with known distribution, the precise leading asymptotic term in the gap to optimal worst-case rate was established in [13] and was shown to be $O(n^{-1/2})$. Therefore, for multiparty data exchange as well as SK agreement our protocols can roughly attain the worst-case performance for the case of known distributions, without requiring the knowledge of the distribution. Also, for average rate, the universal $O(n^{-1/2} \sqrt{\log n})$ gap to optimal rates attained by our protocols is to our knowledge the best-known.

There is a rich literature relating to the problems considered here. Following the seminal work of Slepian and Wolf [21], which introduced fixed-length distributed source coding for two parties, universal error-exponents for the multiparty extension of this problem were considered in [8], [6], [18]. For the case of two parties, universal variable length protocols with optimal average rate were proposed in [10], [26]. In particular, the protocol used in [26] has excess rate less than $O(n^{-1/2})$, which is the best-known. A related protocol was used in [23] in a single-shot setup which, when applied to IID observations with a known distribution, was shown to be of optimal worst-case length even up to the second-order asymptotic term. A slight variant of the omniscience problem, which assumes the data of the parties to be elements of a finite field and requires exact recovery using linear communication, has been considered in [20], [22], [5], [16], [17]. While our proposed protocol doesn’t directly relate to these works, we propose it as an alternative approach for ensuring data exchange in these settings.

The remainder of this paper is organized as follows: The next section contains the formal description of the omniscience problem. We first describe an idealized version of our protocol in Section III where we

\[ \text{For } m > 2, \text{ a variant of our protocol can be used to obtain an excess rate of roughly } O(n^{-1/2}) \text{ when the distribution is known, for both average and worst-case rates.} \]

\[ \text{For } m = 2 \text{ even our omniscience protocol has excess rate less than } O(n^{-1/2}). \text{ The extra } O(\sqrt{\log n}) \text{ factor for a general } m \text{ appears since the optimal sum-rate may not be a concave function of } P_{X,M} \text{ for } m > 2, \text{ and we take recourse to a Taylor approximation of the sum-rate function.} \]
assume that the rates can be continuously increased and an ideal decoder is available. We also illustrate the working of our ideal protocol with examples. The ideal assumptions are removed in the subsequent section and our complete protocol is described; this section also contains our main technical results on the performance of the proposed protocol. The SK agreement problem and our universal SK agreement protocol are described in Section V. All the proofs are given in Section VI. Our proofs rely on technical properties of the formula for minimum communication for omniscience. Some of these properties are new and maybe of independent interest.

**Notations.** We start by recalling the standard notations: We consider discrete random variables $X$ taking values in a finite set $\mathcal{X}$ and with pmf $P_X$. For random variables $(X_i : i \in \mathcal{M})$ and $A \subseteq \mathcal{M}$, $X_A$ denotes the collection $(X_i : i \in A)$. Also, $X^n_A$ denotes the sequence of IID random variables $\{X_A,t\}_{t=1}^n$, where $X_{A,t} = (X_{i,t} : i \in A)$. Similarly, $X_A$ denotes the product set $\prod_{i \in A} X_i$ and $X^n = X_1 \times \cdots \times X_n$. For given distributions $P$ and $Q$, their variational distance is denoted by $\|P - Q\| = \frac{1}{2} \sum_x |P(x) - Q(x)|$. While our protocols are conceptually simple, the analysis is notationally heavy and relies on some bespoke notations. For easy reference, we summarize all nonstandard notations used in this paper in Table I. We often need to think of a subset of parties as a single party and use natural extensions of our notations to indicate such cases. For instance, for a partition $\sigma$ of $A$, the notation $(R^*_\sigma (A_\sigma) : 1 \leq i \leq |\sigma|)$ extends $R^*_i (A)$ given in Table I and denotes the solution $(R_1, \ldots, R_{|\sigma|})$ for equations

$$\sum_{j \neq i} R_j = H(X_A|X_{\sigma_i}), \quad \forall 1 \leq i \leq |\sigma|.$$

**II. Omniscience**

Parties in a set $\mathcal{M} = \{1, \ldots, m\}$ observe an IID sequence $X^n_M = (X_{M1}, \ldots, X_{Mn})$, with the $i$th party observing $\{X_{it}\}_{t=1}^n$ and $X_{Mt} = (X_{it} : i \in \mathcal{M}) \sim P_{X_M}$ denoting the collective data at the $t$th time instance. The parties have access to shared public randomness (public coins) $U$ such that $U$ is independent jointly of $X^n_M$. Furthermore, the $i$th party, $i \in \mathcal{M}$, has access to private randomness (private coins) $U_i$ such that $U_M$, $U$, and $X^n_M$ are mutually independent. Thus, the $i$th party observes $(X^n_i, U_i, U)$.

For simplicity, we restrict our exposition to **tree-protocols** (cf. [14]) described below. A tree-protocol $\pi$ for $\mathcal{M}$ consists of a binary tree, termed the *protocol-tree*, with the vertices labeled by the elements of $\mathcal{M}$. The protocol starts at the root and proceeds towards the leaves. When the protocol is at vertex $v$ with label $i_v$, party $i_v$ communicates a bit $b_v$ based on its local observations $(X^n_{i_v}, U_{i_v}, U)$. The protocol proceeds to the left- or the right-child of $v$, respectively, if $b_v$ is 0 or 1. The protocol terminates when it reaches a leaf, at which point each party produces an output based on its local observations and the
TABLE I: Summary of notations used in the paper.

| Notation | Description |
|----------|-------------|
| $\Sigma(A)$ | Set of all nontrivial partitions of $A$ |
| $|\sigma|$ | Number of parts in the partition $\sigma$ |
| $\sigma_f(A)$ | The finest partition $\{\{i\} : i \in A\}$ of $A$ |
| $\sigma_B(A), B \subset A$ | The partition $\{\{A \setminus B\}, \{i\} : i \in B\}$ of $A$ |
| $R_A$ | Sum rate $\sum_{i \in A} R_i$ |
| $\mathcal{R}_{CD}(A)$ | Set of all vectors $(R_i : i \in A)$ s.t. $R_B \geq H(X_B|X_{A\setminus B}), \forall B \subset A$ |
| $\mathcal{R}_{\Delta CD}(A)$ | Set of all vectors $(R_i : i \in A)$ s.t. $R_B \geq H(X_B|X_{A\setminus B}) + |B|\Delta, \forall B \subset A$ |
| $R_{CD}(A)$ | Minimum of $R_A$ over all $R \in \mathcal{R}_{CD}(A)$ |
| $H_\sigma(A), \sigma \in \Sigma(A)$ | $\frac{1}{|\sigma|-1} \sum_{i=1}^{|\sigma|} H(X_A|X_{\sigma_i})$ |
| $R_i^*(A), i \in A$ | Solution of $\sum_{j \neq i} R_j = H(X_A|X_i), \forall i \in A$ |
| $A_\sigma, \sigma \in \Sigma(A)$ | $\{A_{\sigma_i} : 1 \leq i \leq |\sigma|\}$, where $A_{\sigma_i} = \sigma_i$ treated as a single party |
| $|\pi|$ | Maximum number of bits communicated in any execution of the protocol $\pi$ |
| $|\pi|_{av}$ | Expected value of the number of bits communicated in an execution of the protocol $\pi$ |

The (worst-case) length $|\pi|$ of a protocol $\pi$ is the maximum number of bits that are transmitted in any execution of the protocol and equals the depth of the protocol-tree. Also, the average length $|\pi|_{av}$ is given by the expected value of the number of bits transmitted in an execution of the protocol $\pi$.

In the omniscience problem, the parties engage in interactive communication to recover each other’s data. A protocol $\pi$ constitutes an $\epsilon$-omniscience protocol if, at the end of the protocol, the $i$th party can
output an estimate \( \hat{X}_i = \hat{X}_i(X^n_i, U_i, U, \Pi) \in X^n_M \) such that
\[
P \left( \hat{X}_i = X^n_M : i \in M \right) \geq 1 - \epsilon.
\]

**Definition 1** (Communication for omniscience). Given IID observations with a common distribution \( P_{X_M} \) as above, for \( 0 \leq \epsilon < 1 \), a rate \( R \geq 0 \) is an \( \epsilon \)-achievable omniscience rate if there exists an \( \epsilon \)-omniscience protocol \( \pi \) with length \( |\pi| \) less than \( nR \), for all \( n \) sufficiently large. The infimum over all \( \epsilon \)-achievable omniscience rates is denoted by \( R_\epsilon(P_{X_M}) \). The **minimum rate of communication for omniscience** \( R(P_{X_M}) \) is given by
\[
R(P_{X_M}) = \lim_{\epsilon \to 0} R_\epsilon(P_{X_M}).
\]

The **minimum average rate of communication for omniscience** \( R_{av}(P_{X_M}) \) is defined similarly by replacing length \( |\pi| \) with average length \( |\pi|_{av} \).

A simple characterization of \( R(P_{X_M}) \) as a linear program was derived in \([9]\). Here we recall an alternative characterization in terms of partitions of \( M \). This form was obtained as a lower bound for \( R(P_{X_M}) \) in \([9]\), which was shown to be tight for \( m = 3 \). Later, it was shown in \([3]\) that, in fact, this lower bound is tight. Our universal protocol for omniscience directly achieves this rate, thereby providing an alternative proof for the tightness of this lower bound.

**Theorem 1** (\([9], [3]\)). Given a distribution \( P_{X_M} \),
\[
R(P_{X_M}) = R_{av}(P_{X_M}) = \max_{\sigma \in \Sigma(M)} H_\sigma(M|P_{X_M}),
\]
where \( \Sigma(M) \) denotes the set of partitions of \( M \), and, for each \( \sigma \in \Sigma(M) \),
\[
H_\sigma(M|P_{X_M}) := \frac{1}{|\sigma|-1} \sum_{i=1}^{|\sigma|} H(X_M|X_{\sigma_i}).
\]

**Remark 1** (Rate-region of communication for omniscience). The right-side of (1) is equal to the minimum sum \( \sum_{i=1}^m R_i \) of rates \( \mathbf{R} = (R_1, \ldots, R_m) \) such that
\[
\sum_{i \in B} R_i \geq H(X_B|X_{B^c}), \quad \forall B \subseteq M.
\]
Following \([9]\), the collection of all rate vectors \( \mathbf{R} \) satisfying the constraints above, termed the CO region, will be denoted by \( \mathcal{R}_{CO}(M|P_{X_M}) \), and the minimum sum rate by \( R_{CO}(M|P_{X_M}) \). When the distribution \( P_{X_M} \) is clear from the context, we shall omit it from the notation and simply use \( \mathcal{R}_{CO}(M) \) and \( R_{CO}(M) \).
While there can be several maximizers of $\mathbb{H}_\sigma$, there exists a maximizing partition which is a further partition of any other maximizing partition [2, Theorem 5.2], the finest maximizing partition; we shall call this finest maximizer of $\mathbb{H}_\sigma$ in (1) the dominant partition. The finest partition $\sigma_f(\mathcal{M}) := \{\{i\}, i \in \mathcal{M}\}$ plays a particularly important role in our protocol. Note that when the finest partition is dominant, the optimal rate assignment is uniquely given by the solution $R^* = (R^*_1, \ldots, R^*_m)$ of

$$\sum_{i \in \mathcal{M}\setminus\{j\}} R_i = H(X_M|X_j), \quad j = 1, \ldots, m. \quad (3)$$

While the result in [9] was shown to hold only for $R(P_{X_M})$, the characterization of $R_{av}(P_{X_M})$ given above holds since the prefix-free set of distinct transcripts of a tree protocol $\pi$ satisfies Kraft’s inequality and thereby $H(\Pi) \leq |\pi|_{av}$. The converse proof can be completed by proceeding exactly as in [9]. The achievability follows upon bounding the rate of communication by $\log |X_M|$ under the error event and taking the limit $\epsilon_n \to 0$ for the error probability $\epsilon_n$ in the omniscience protocol given in [9]. An alternative proof of achievability is provided by the analysis of our universal protocol below. In fact, we show that our proposed protocol can attain omniscience without the knowledge of the distribution $P_{X_M}$, using an average rate of communication equal to $R_{av}(P_{X_M})$.

### III. Universal Protocol for Omniscience Under Ideal Assumptions

We give a universal protocol for omniscience, which, when $x_M$ is observed, will transmit communication of rate no more than $R_{CO}(\mathcal{M}|P_{X_M})$. Since the empirical distribution approaches the true distribution in the limit as $n$ goes to $\infty$, and $R_{CO}(\mathcal{M}|P_{X_M})$ is continuous in $P_{X_M}$, we achieve the optimal rate universally. To present the main idea behind our protocol, we first describe it under some idealistic conditions.

Specifically, we make two assumptions:

(a) **Continuous rate assumption**: Rates can be continuously increased as a function of time; and

(b) **Ideal decoder assumption**: We assume the availability of an error-free, ideal decoder $DEC_{id}$ which correctly decodes a sequence once sufficient communication has been sent and declares a NACK otherwise.

Clearly, this does not hold in practice since the rate of communication corresponds to the number of bits transmitted per $n$ and can only be increased in discrete steps. The continuous rate assumption allows us to examine, loosely speaking, the “fluid limit” behavior of our protocol.

In analysis of the ideal protocol, we do not account for the rate needed to send NACKs. In practice, each NACK symbol counts for a bit of communication and the size $\Delta$ of discrete increments must be carefully chosen to render the rate needed to send NACKs negligible.
A standard universal decoder used in source coding is the minimum entropy decoder which, given side-information $y$ and an $nR$-bit random hash of $X^n$, searches for the unique sequence $x$ such that the joint type $P_{XY} = P_{XY}$ satisfies $H(X | Y) \leq R$ and the hash of $x$ matches the received hash bits. The decoder that we prescribe in the next section works on a similar principle except that it searches for any possible subset of sequences it can decode with the current rate. To avoid the additional complications due to decoding error, we first assume the availability of an ideal decoder $\text{DEC}_{\text{id}}$ which enables omniscience for all parties $j \in A$ as soon as the rate received from the parties in $A$ is sufficient. That is, the ideal decoder guarantees that each party $i \in A$ can recover the correct sequence $x_A$ if the rates of communication $R = (R_i : i \in A)$ satisfy $R \in \mathcal{R}_{\text{CO}}(A | P_{X_A})$. Furthermore, if $R \notin \mathcal{R}_{\text{CO}}(A | P_{X_A})$, the ideal decoder does not mistakenly output a wrong sequence $x_A'$, but declares a NACK instead. Protocol 1 summarizes our assumed ideal decoder.

**Protocol 1: Ideal decoder $\text{DEC}_{\text{id}}(j, \sigma, R)$**

**Input:** An index $1 \leq j \leq m$, a partition $\sigma \in \Sigma(M)$, a rate vector $R = (R_1, \ldots, R_m)$.

**Output:** An ACK message $(\text{ACK}, A)$ or a NACK message.

1) For $\sigma_i$ such that $j \in \sigma_i$, search for the maximal set $A \subseteq M$ such that $\sigma_i \subsetneq A$, $(R_i : i \in A) \in \mathcal{R}_{\text{CO}}(A | P_{X_A})$ and reveal $x_A$ to party $j$.

2) **if** such an $A$ was found in Step 1 **then**
   
   return $(\text{ACK}, A)$.

   **else**

   return NACK.

With this ideal decoder at our disposal, under the continuous rates assumption, finding a universal protocol is tantamount to finding a policy for increasing the rates $(R_1, \ldots, R_m)$ such that when the rate vector enters $\mathcal{R}_{\text{CO}}(M | P_{X_M})$ for the first time, the sum rate is $R_{\text{CO}}(M | P_{X_M})$. Note that initially the marginal types $P_{x_i}$ are available to each party and can be transmitted using $O(\log n)$ bits, since there are only polynomially many types. Also, if a subset $A$ attains local omniscience in the middle of the protocol, any $j \in A$ upon recovering $x_A$ can transmit $P_{X_A}$ in $O(\log n)$ bits to all the parties, who in turn can use it to compute $H(P_{X_A})$.

As an illustration, consider the simple case when $m = 2$. Parties first share $P_{X_1}$ and $P_{X_2}$; suppose $H(P_{X_1}) \geq H(P_{X_2})$. Then, party 1 starts communicating and increases its rate $R_1$ at slope\(^7\) 1. When $nR$ is required to be an integer. When this is not the case, we simply use $\lceil nR \rceil$ bits in place of $nR$. This convention will be used throughout this paper and will be accounted for in our analysis.

\(^6\) The slope is defined as the derivative of rate with respect to the time (blocklength) under the continuous rate assumption.
Fig. 2: Illustration of protocol for \( m = 2 \). The transition point \( t_1 \) depends only on the marginal types \( P_{x_1} \) and \( P_{x_2} \). The rate \( R_1 \) reaches \( H(P_{x_1}) - H(P_{x_2}) \), party 2 starts communicating at slope 1 as well. Throughout the protocol, each party is trying to decode the other using the ideal decoder \( \text{DEC}_{i,d} \) and they keep on communicating as long as the ideal decoders output NACKs. The parties will decode each other as soon as \((R_1, R_2)\) enters \( \mathcal{R}_{\mathcal{C}_0} \{1, 2\}|P_{x_1}, x_2 \}, \text{i.e.}, when

\[
R_1 \geq H(X_1|X_2) \text{ and } R_2 \geq H(X_2|X_1),
\]

where \((X_1, X_2) \sim P_{x_1}, x_2 \). Note that once both parties start communicating, the difference \( R_1 - R_2 \) is maintained as \( H(X_1) - H(X_2) \). Thus, when \((R_1, R_2)\) enters \( \mathcal{R}_{\mathcal{C}_0} \{1, 2\} \), it holds that

\[
R_1 = H(X_1|X_2) \text{ and } R_2 = H(X_2|X_1);
\]

the red line in Figure 2 illustrates this evolution of rates.

Our protocol extends the idea above to a general \( m \). We design our protocol so that the first subset \( A \) which attains local omniscience does so by using communication only from the parties in \( A \) and of sum rate

\[
R_A = \mathbb{H}_{\sigma_j(A)}(A|P_{x_A}) = \sum_{i \in A} R^*_i(A);
\]

see (9) in Lemma 8 given in Section VI below for the second equality. To that end, we note (see Lemma 8 for a proof) that for every \( A \)

\[
R^*_i(A) - R^*_j(A) = H(X_i) - H(X_j).
\]

A key point here is that for \( P_{x,M} \) this difference can be computed using only the marginal types \( P_{x_i} \) and \( P_{x_j} \). Our protocol ensures that for every pair \((i, j)\) of communicating parties, the rate of communication

\[
R_i - R^*_i(A) = R_j - R^*_j(A),
\]

\(^8\)It is also possible to proceed along the blue line for the \( m = 2 \) case. However, its extension to a general \( m \) is not clear.
which in turn can be ensured if the \textit{constant difference} property, namely

$$R_i - R_j = H(\mathbf{X}_i) - H(\mathbf{X}_j),$$ \hspace{1cm} (4)

is maintained throughout the protocol for every pair of communicating parties. Thus, all communicating parties \(i\) reach the rate \(R^*_i(A)\) at the same time. Specifically, we first arrange parties in decreasing order of the entropy of the empirical distribution of their local observations, which are shared in \(O(\log n)\)-bits. Assuming \(H(P_{x_1}) \geq H(P_{x_2}) \geq \cdots \geq H(P_{x_m})\), party 1 starts communicating, and the \(i\)th party starts communicating when \(R_1 \geq H(P_{x_1}) - H(P_{x_i})\). This ensures the constant difference property (4) for every pair \((i, j)\) of communicating parties. When a subset \(A\) attains local omniscience, we decrease the rate-slope for each party \(i \in A\) to \(1/|A|\), thereby ensuring that collectively parties in \(A\) increase the rate of communication \(R_A\) at slope 1. Note that since parties in \(A\) have recovered \(x_A\), any one party \(i \in A\) can compute the type \(P_{X_A}\) and transmit it using \(O(\log n)\) bits. Our main observation is that at this point the rates appear as if the parties in \(A\) were collocated to begin with and have been executing the protocol as a single party. In particular, \(R_A - R_j = H(\mathbf{X}_A) - H(\mathbf{X}_j)\) for any communicating party \(j \notin A\). The second crucial observation is that for the first subset \(A\) which attains local omniscience, \((R^*_i(A) : i \in A) \in R_{\text{CD}}(A)\). Since \(\sum_{i \in A} R^*_i(A)\) is a lower bound for \(R_{\text{CD}}(A)\), whereby the parties cannot attain omniscience before they communicate at sum-rate \(\sum_{i \in A} R^*_i(A)\), and our protocol ensures that all parties in \(A\) reach the rate \(R^*_i(A)\) at the same time, we can show that the parties in \(A\) must have communicated at sum rate

$$R_A = \sum_{i \in A} R^*_i(A) = \mathbb{H}_{\sigma_f(A)}(A|P_{X_A})$$

when they attain local omniscience. Thus, as the protocol proceeds, subsets of parties keep attaining local omniscience and start behaving as a single party. Proceeding recursively, it follows that when all parties attain omniscience, the rate of communication must equal \(\mathbb{H}_\sigma(\mathcal{M}|P_{X_M})\) for some \(\sigma \in \Sigma(\mathcal{M})\), which in view of Theorem 1 is no more than \(R_{\text{CD}}(\mathcal{M}|P_{X_M})\) and must be optimal in the limit as \(n \to \infty\).

To help the reader build heuristics for our complete protocol and its analysis, we provide a sketch of the analysis for the ideal situation. The formal proofs for the ideal case closely follow those for the results for the actual protocol and have been omitted. As mentioned, our ideal protocol proceeds recursively by increasing the rates with fixed slopes until a subset attains omniscience, at which point the slopes are changed so that the parties in an omniscience attaining subset behave as if they are collocated. We describe the one-step omniscience protocol \(\text{OMN}_{\text{id}}\) in Protocol 2. The protocol takes as input a partition \(\sigma\) such that parties in any one part are behaving as collocated parties, a vector \(\mathbf{H} = (H_\sigma, 1 \leq i \leq |\sigma|)\)
Protocol 2: OMN_{1d}(\sigma, H, R)

**Input:** A partition \( \sigma \in \Sigma(M) \) with \(|\sigma| = k \), an entropy estimate vector \( H = (H_{\sigma_1}, \ldots, H_{\sigma_k}) \), a rate vector \( R = (R_1, \ldots, R_m) \); we assume that \( H \) is sorted, i.e., \( H_{\sigma_1} \geq H_{\sigma_2} \geq \cdots \geq H_{\sigma_k} \).

**Output:** A rate vector \( R^{out} \), a family of subsets \( O \) that have attained omniscience.

1. Initialize \( s := \max\{ j : R_j \geq 0 \} \).
2. All parties \( j \) such that \( j \in \sigma_i \) for some \( 1 \leq i \leq k \) increase their rates \( R_j \) at slope \( 1/|\sigma_i| \).
3. **if** There exists \( i > s \) such that \( R_{\sigma_i} \geq H_{\sigma_i} - H_{\sigma_i} \) **then**
   - set \( R_j = 0 \) for all \( j \in \sigma_i \), and set \( s = \max\{ j : R_j \geq 0 \} \).

4. **for** all \( j \) such that \( j \in \sigma_i \) for some \( 1 \leq i \leq s \), execute DEC_{1d}(j, \sigma, R), which outputs NACK or (ACK, A_j).

5. **if** All parties send a NACK **then**
   - return to Step 2.
   **else**
   - Identify the omniscience family
     \[ O = \{ B \subset M : \text{ all } j \in B \text{ returned } (\text{ACK}, B) \}. \]
   - Set \( R^{out} = R \) and return \((R, O)\).

consisting of estimates of entropy for marginal distribution of parties in any part of \( \sigma \), and a rate vector \( R = (R_1, \ldots, R_m) \) of rates of communication sent by all the parties up to this point.

Note that a “valid” rate vector should reflect that parties in any one part have communicated enough to attain local omniscience. Also, since we shall recursively call OMN_{1d}, the only rate vectors OMN_{1d} encounters are those which can arise by increasing the rates in the manner of our protocol. We call the collection of rate vectors satisfying the two conditions above \((\sigma, H)\)-valid. Formally,

**Definition 2.** For \( \sigma \in \Sigma(M) \) with \(|\sigma| = k \) and \( H = (H_{\sigma_1}, \ldots, H_{\sigma_k}) \) with \( H_{\sigma_1} \geq H_{\sigma_2} \geq \cdots \geq H_{\sigma_k} \), a rate vector \((R_1, \ldots, R_m)\) is \((\sigma, H)\)-valid if

\[
(R_j, j \in \sigma_i) \in R_{\text{C0}}(\sigma_i), \quad \forall i \text{ s.t. } |\sigma_i| \geq 2,
\]

and \((R_{\sigma_i}, 1 \leq i \leq k)\) can be obtained by starting with \((0, -1, -1, \ldots, -1)\) and incrementing the rates as in Protocol 2 when the parties in each part \( \sigma_i \) are collocated, i.e., each part \( \sigma_i \) starts increasing its rate at slope 1 once \( R_{\sigma_i} \geq H_{\sigma_i} - H_{\sigma_i} \).

The result below shows a recursive property of OMN_{1d} that facilitates our optimal rate universal protocol. Specifically, it shows that if \( R \) is \((\sigma, H)\)-valid then, when OMN_{1d}(\sigma, H, R) terminates, the output rate vector is \((\sigma^{out}, H^{out})\)-valid where \( \sigma^{out} \) is obtained by combining the omniscience achieving parts in \( \sigma \) and \( H^{out} \) is the corresponding estimate for entropies of the marginals of parts of \( \sigma^{out} \).

Furthermore, for every set \( A \) that attains local omniscience, the sum rate \( R_A \) at the end of OMN_{1d} is
exactly $H_{\sigma_f(A_\sigma)}(A_\sigma)$.\footnote{When $A = \cup_{i=1}^{c} \sigma_i$, by our convention $H_{\sigma_f(A_\sigma)}(A_\sigma) = \mathbb{H}_{(\sigma_1 \cdots |\sigma_c)}(A|P_{x,A})$.}

**Theorem 2.** For $\sigma \in \Sigma(M)$ with $|\sigma| = k$ and $H = (H_{\sigma_1}, \ldots, H_{\sigma_k})$ with $H_{\sigma_1} \geq H_{\sigma_2} \geq \cdots H_{\sigma_k}$, let $R^{\text{in}} = (R_{1}^{\text{in}}, \ldots, R_{m}^{\text{in}})$ be $(\sigma, H)$-valid. Then, if $\text{OMN}_{1d}(\sigma, H, R^{\text{in}})$ is executed, the final rates $R^{\text{out}}$ and the omniscience family $\mathcal{O}$ satisfy the following:

1) Every $A \in \mathcal{O}$ comprises parts of $\sigma$, i.e.,
$$A = \bigcup_{l=1}^{c} \sigma_{i_l}$$
for some $\{i_1, \ldots, i_c\} \subseteq \{1, \ldots, |\sigma|\}$, and the sum-rate $R^{\text{out}}_A$ satisfies
$$R^{\text{out}}_A = \mathbb{H}_{(\sigma_{i_1} | \cdots | \sigma_{i_c})}(A|P_{x,A}).$$

2) Let $\sigma^{\text{out}} \in \Sigma(M)$ be the partition obtained by combining the parts in $\sigma$ that belong to the same $A$ in $\mathcal{O}$. Let $H^{\text{out}}_{\sigma^{i}}$ denote the entropy of the type of $x_{\sigma^{i}}$. Then, with $H^{\text{out}} = (H^{\text{out}}_{\sigma_1}, 1 \leq i \leq |\sigma^{\text{out}}|)$, $R^{\text{out}}$ is $(\sigma^{\text{out}}, H^{\text{out}})$-valid.

The proof of Theorem 2 follows closely that of Theorem 3 and has been omitted.

Thus, if we proceed by recursively calling $\text{OMN}_{1d}$, each time with $(\sigma^{\text{out}}, H^{\text{out}}, R^{\text{out}})$ obtained from the previous call, we shall ultimately attain omniscience using the sum rate $\mathbb{H}_{\sigma}(M)$ for some partition $\sigma$. Since $\mathbb{H}_{\sigma}(M)$ is a lower bound for $R_{\text{CO}}(M)$, this rate must be optimal by Theorem 1

We summarize the overall ideal protocol in Protocol 3.

**Protocol 3: A universal omniscience protocol under ideal conditions**

1) Initialize $\sigma = \sigma_f(M)$, $R = (0, -1, -1, \ldots, -1)$, $k = |\sigma|$.
2) while $k > 1$ do
   (i) For $1 \leq i \leq k$, a party $j \in \sigma_i$ computes $P_{x_{\sigma_i}}$ and broadcasts it.
       Each party computes $H_{\sigma_i} = H(P_{x_{\sigma_i}}), 1 \leq i \leq k$.
   (ii) Let $H$ be the sorted version of $(H_{\sigma_i} : 1 \leq i \leq k)$, i.e., assume $H_{\sigma_1} \geq H_{\sigma_2} \geq \cdots \geq H_{\sigma_k}$.
       Call $\text{OMN}_{1d}(\sigma, H, R)$.
       Let $(R^{\text{out}}, \mathcal{O})$ be its output.
   (iii) Let $\sigma^{\text{out}} = \{\sigma_i : \sigma_i \in \sigma \text{ s.t. } \sigma_i \not\subset A \forall A \in \mathcal{O}\} \bigcup \{A : A \in \mathcal{O}\}$.
       Update $R = R^{\text{out}}$, $\sigma = \sigma^{\text{out}}$, and $k = |\sigma^{\text{out}}|$.

**Remark 2.** It has been shown in [4] that if a set $A$ corresponds to a part in the partition that attains
Fig. 3: Illustration of $R_{\text{CO}}(\{1, 2, 3\})$ for Example 1.

the maximum in (1), then omniscience can be attained in such a manner that the parties in $A$ can attain omniscience along the way from the communication of the parties in $A$. Our protocol explicitly has this feature and attains omniscience for each part of the maximizing partition along the way.

We conclude this section with a few illustrative examples to demonstrate the working of the ideal protocol. The first example is for $m = 3$ and exhibits a case where $\sigma_f(M)$ is dominant.

**Example 1.** Let $X_1 \sim \text{Ber}(1/2)$, $X_3 \sim \text{Ber}(q)$, and $X_2 = X_1 \oplus X_3$. In this case, $R_{\text{CO}}(\{1, 2, 3\})$ is given by rate vectors satisfying the following linear constraints:

- $R_1 + R_2 \geq 1$,
- $R_2 + R_3 \geq h(q)$,
- $R_1 + R_3 \geq h(q)$.

When $\frac{1}{2} < h(q) \leq 1$, the finest partition is dominant, and

$$R_{\text{CO}}(\{1, 2, 3\}) = h_{\{1\}|\{2,3\}} = \frac{1 + 2h(q)}{2}.$$  

The CO region is depicted in Figure 3. As can be seen from the figure, $R_{\text{CO}}(\{1, 2, 3\})$ is achieved by the unique rate assignment $R^* = (1/2, 1/2, (2h(q) − 1)/2)$. In our protocol, parties 1 and 2 communicate first and increase their rates at slope 1 until $R_1 = R_2 = H(X_1) − H(X_3) = H(X_2) − H(X_3) = 1 − h(q)$. At this point, party 3 starts communicating and all the parties increase their rates at slope 1. Owing to the initial lead of $R_1$ and $R_2$ over $R_3$, all the parties reach $R^*$ simultaneously.

When $h_{\sigma}$ is maximized by a partition $\sigma$ other than the finest partition $\sigma_f(M)$, as our protocol proceeds, the parties in parts of $\sigma$ attain local omniscience, along the way, before all the parties attain omniscience.
Consider the following example, again for $m = 3$.

**Example 2.** Let $W_1, W_2 \sim \text{Ber}(1/2)$ and $V_1, V_2 \sim \text{Ber}(q)$ for some $0 < q < \frac{1}{2}$, and let $X_1 = (W_1, W_2)$, $X_2 = (W_1 \oplus V_1, W_2)$, and $X_3 = W_2 \oplus V_2$. In this case, $H\{12|3\} = 1 + 3h(q)$ is dominant, and our protocol proceeds as follows: Parties 1 and 2 start increasing their rates at slope 1. When their rates reach $h(q)$, they attain local omniscience. At this point they start increasing their rates at slope $1/2$ and continue doing so until $R_1 + R_2$ reaches $H(X_1, X_2) - H(X_3) = 1 + h(q)$. Now, party 3 starts communicating at slope 1. When all the parties reach $((1 + 2h(q))/2, (1 + 2h(q))/2, h(q))$, they attain omniscience.

Note that $\{1, 2\}$ attain local omniscience even before 3 starts communicating, illustrating the recursive nature of our protocol wherein a subset attaining local omniscience start behaving as if the parties in it were collocated to begin with. In fact, this recursive property holds even when only a subset of communicating parties attains omniscience, as our final example with $m = 4$ illustrates.

**Example 3.** Let $W_1, W_2, W_3 \sim \text{Ber}(1/2)$ and $V_1, V_2 \sim \text{Ber}(q)$ for some $0 < q < \frac{1}{2}$, and let $X_1 = (W_1, W_2)$, $X_2 = (W_1 \oplus V_1, W_2)$, $X_3 = W_2 \oplus V_2$, and $X_4 = W_3$. Note that the observations of subset $\{1, 2, 3\}$ are exactly as in Example 2. In this case, $H\{123|4\} = 3 + 2h(q)$ is dominant, and our protocol proceeds as in Figure 4. At $t_1$, parties 1 and 2 attain local omniscience and change the slopes of $R_1$ and $R_2$ to $1/2$. At $t_2$, parties 3 and 4 start communicating. At $t_3$, parties in $\{1, 2, 3\}$ attain local omniscience and change their slope to $1/3$. Note that up to $t_3$, the evolution of $(R_1, R_2, R_3)$ is exactly the same as that in Example 2. Also, at $t_3$ the rate difference $(R_1 + R_2 + R_3 - R_4)$ equals $H(X_1, X_2, X_3) - H(X_4) = 1 + 2h(q)$. Thus, after $t_3$ the rate pair $(R_1 + R_2 + R_3, R_4)$ behaves as if the parties in $\{1, 2, 3\}$ were collocated to begin with. Finally, all parties attain omniscience at $t_4$. 

Fig. 4: The evolution of rates for Example 3.
IV. UNIVERSAL PROTOCOL FOR OMNISCIENCE: FULL DESCRIPTION

Moving now to the real world, rates must be increased in discrete increments and a positive decoding error probability must be tolerated. To that end, the parties incrementally transmit independent hash bits, \( n\Delta \) at a time. The ideal decoder of the previous section is replaced with a typical decoder \( \text{DEC}(j, \sigma, R) \) which searches for the maximal set \( A \) such that there exists a unique sequence \( x_A \) that contains the current rate vector in its CO region and is consistent with the local observation and the received hash values. In fact, instead of working with the original CO region \( R_{\text{CO}}(A) \), we use the more restrictive region \( R_{\Delta \text{CO}}(A) \) consisting of vectors \( (R_i, i \in A) \) such that

\[
R_B \geq H(X_B | X_{A \setminus B}) + |B| \Delta, \quad \forall B \subsetneq A.
\]

The complete decoder is described in Protocol 4.

**Protocol 4: DEC(\( j, \sigma, R \))**

**Input:** An index \( 1 \leq j \leq m \), a partition \( \sigma \in \Sigma(M) \), a rate vector \( R = (R_1, \ldots, R_m) \)

**Output:** A NACK message, an ACK message \((\text{ACK}, A)\), or an error message ERR.

1. For \( \sigma_i \) such that \( j \in \sigma_i \), find the maximal set \( A \subseteq M \) such that \( \sigma_i \subsetneq A \) and there exists a joint type \( P_{X_A} \) for \( X_A \) satisfying the following:
   - (i) \( P_{X_A} = P_{X_{\sigma_i}} \),
   - (ii) \( (R_i : i \in A) \in R_{\Delta \text{CO}}(A | P_{X_A}) \), and
   - (iii) there is a unique \( \hat{x}_A \) of joint type \( P_{X_A} \) such that the hashes of \( \hat{x}_A \) match all the previously received hashes from parties in \( A \setminus \{j\} \).

2. **if** If there is a unique maximal \( A \) found in Step 1 **then**
   - return \((\text{ACK}, A)\).
   **else if** If there is no sequence found in Step 1(iii) **then**
   - return NACK.
   **else**
   - Return ERR.

Note that the decoder declares \((\text{ACK}, A)\) if it can find a unique maximal set \( A \) and a unique sequence \( x_A \), declares NACK if it finds no such set, or an ERR otherwise. In fact, an error may occur even when it is not detected, *i.e.*, when ERR is not transmitted. However, we can identify an event \( \mathcal{E} \) (described formally in Section VI-B) of small probability such that under \( \mathcal{E}^c \) the real decoder \( \text{DEC} \) behaves exactly like \( \text{DEC}_{\text{id}} \), but with \( R_{\text{CO}}(A) \) replaced with \( R_{\Delta \text{CO}}(A) \). Therefore, omniscience can be achieved in a similar manner as the ideal protocol of the previous section.

The main component of our universal protocol is the one step omniscience protocol OMN described in Protocol 5 which uses DEC for decoding. Protocol OMN proceeds very much like the ideal protocol...
Protocol 5: OMN(σ, α, H, R)

Input: A partition σ ∈ Σ(M) with |σ| = k, an α ∈ ℕ, an entropy estimate vector

\[ H = (H_{σ_i} : 1 \leq i \leq k), \] a rate vector \( R = (R_1, \ldots, R_m) \); we assume that \( H \) is sorted, i.e.,

\[ H_{σ_1} \geq H_{σ_2} \geq \cdots \geq H_{σ_k} \]

Output: A rate vector \( R^{\text{out}} \), a family of subsets \( O \) that have attained omniscience.

1) Initialize \( s := \max \{ j : R_j \geq 0 \} \).
2) All parties \( j \) such that \( j \in σ_i \) for some \( 1 \leq i \leq k \) send \( \lceil n\Delta/|σ_i| \rceil \) random hash bits.
   Update \( R_j \rightarrow R_j + \Delta/|σ_i| \).
3) if There exists \( i > s \) such that \( R_{σ_i} \geq H_{σ_1} - H_{σ_i} + α\Delta \) then
   set \( R_j = 0 \) for all \( j \in σ_i \), and set \( s = \max \{ j : R_j \geq 0 \} \).
4) For all \( j \) such that \( j \in σ_i \) for some \( 1 \leq i \leq s \), execute DEC(\( j, σ, R \)), which outputs \( \text{NACK}, (\text{ACK}, A_j) \), or \( \text{ERR} \).
5) if All parties send a \( \text{NACK} \) then
   return to Step 2.
   else if No party declares an \( \text{ERR} \) and some parties declare an \( \text{ACK} \), then
   Identify the omniscience family
   \[ O = \{ B \subset M : \text{ all } j \in B \text{ returned } (\text{ACK}, B) \} \].
   if \( O \) is nonempty then
   Set \( R^{\text{out}} = R \), and return \( (R, O) \).
   else
   \( \text{Declare an error.} \)
   else
   \( \text{Declare an error.} \)
except that a new party \( i \) starts communicating when \( R_1 \geq H(P_{ξ_1}) - H(P_{ξ_i}) + α\Delta \), where \( α \in ℕ \) is an increasing threshold parameter which is updated as the protocol proceeds. Throughout the protocol, a rate \( R_i = -1 \) indicates that the \( i \)th party is not yet transmitting and only parties with \( R_i \geq 0 \) communicate.

The decoder tries to attain omniscience only among the communicating parties.

The ideal protocol of the previous section works due to its recursive structure whereby when a subset \( A \) attains local omniscience, the rate vector appears as if the parties in \( A \) have been collocated from the start. Moreover, the first subset to attain local omniscience does so by using a communication of rate \( H_{σ_1}(A) \). Both these properties were captured by Theorem 2. The result below establishes a similar recursive property of OMN. However, the definition of “validity” needs to be modified from Definition 2 – in place of the operational definition in the ideal case, we use the more technical definition below which captures all the key features that we need.

Definition 3. For \( α \in ℕ \), \( σ \in Σ(M) \) with \( |σ| = k \) and \( H = (H_{σ_1}, \ldots, H_{σ_k}) \), a rate vector \( (R_1, \ldots, R_m) \) is \( (σ, H, α) \)-valid if, for \( s = \max \{ i : R_{σ_i} \geq 0 \} \), the following conditions hold:
(i) (Constant difference) For $1 \leq i, j \leq s$,

$$R_{\sigma_i} - R_{\sigma_j} \leq H_{\sigma_i} - H_{\sigma_j} + \alpha \Delta;$$

(ii) (Noncommunicating parties)

$$R_{\sigma_1} < H_{\sigma_1} - H_{\sigma_{s+1}} + \alpha \Delta; \quad (5)$$

(iii) (Combined parties) $\forall 1 \leq i \leq k$ such that $|\sigma_i| \geq 2$,

$$(R_j : j \in \sigma_i) \in \mathcal{R}_{\text{CO}}^\Delta (\sigma_i); \quad (6)$$

(iv) (Separate parts) for all $A \subseteq \{1, \ldots, k\}$ with $|A| \geq 2$,

$$(R_j : j \in \sigma_i, i \in A) \notin \mathcal{R}_{\text{CO}}^\Delta \left( \bigcup_{i \in A} \sigma_i \right).$$

The following theorem captures our key observation about OMN; its proof is given in Section VI-B.

**Theorem 3.** For $\alpha \in \mathbb{N}$, $\sigma \in \Sigma(M)$ with $|\sigma| = k$ and $H = (H_{\sigma_1}, \ldots, H_{\sigma_k})$ with $H_{\sigma_1} \geq H_{\sigma_2} \geq \cdots H_{\sigma_k}$, let $\mathbf{R}^{\text{in}} = (R_1^{\text{in}}, \ldots, R_m^{\text{in}})$ be $(\sigma, H, \alpha)$-valid. Then, if OMN$(\sigma, \alpha, H, \mathbf{R}^{\text{in}})$ is executed and error $E$ (defined in Section VI-B) does not occur, the final rates $\mathbf{R}^{\text{out}}$ and the omniscience family $\mathcal{O}$ satisfy the following:

(I) For every $A \in \mathcal{O}$, it holds that

a) $A$ comprises parts of $\sigma$, i.e.,

$$A = \bigcup_{l=1}^c \sigma_{i_l}$$

for some $\{i_1, \ldots, i_c\}$, and

b) denoting by $A_\sigma$ the set $\{\sigma_{i_1}, \ldots, \sigma_{i_c}\}$, we have

$$R_{\sigma_{i_l}}^* (A_\sigma) - 2\alpha \Delta \leq R_{\sigma_{i_l}}^{\text{out}} \leq R_{\sigma_{i_l}}^* (A_\sigma) + (m + 2\alpha)\Delta, \quad 1 \leq l \leq c.$$  

(II) Let $\sigma^{\text{out}} \in \Sigma(M)$ be the partition obtained by combining the parts in $\sigma$ that belong to the same $A$ in $\mathcal{O}$. Let $H_{\sigma^{\text{out}}}$ denote the entropy of the type of $x_{\sigma^{\text{out}}}$.

Then, with $\mathbf{H}^{\text{out}} = (H_{\sigma^{\text{out}}_1}, 1 \leq i \leq |\sigma^{\text{out}}|)$,

$\mathbf{R}^{\text{out}}$ is $(\sigma^{\text{out}}, \mathbf{H}^{\text{out}}, c'_m(\alpha))$-valid, where $c'_m$ is a constant depending only on $m$.

We are now in a position to describe our universal protocol for omniscience. We begin by calling OMN with $\sigma = \sigma_f(M)$, $\alpha = 1$, the sorted entropy estimates $\mathbf{H}$ computed from marginal empirical
distributions $P_{x_i}$, and the rate vector $R = (0, -1, \ldots, -1)$ indicating that party 1 starts communicating and every one else remains quiet. Note that $R$ is $(\sigma, H, 1)$-valid. A new party $i$ starts communicating when $R_1 \geq H_1 - H_i + \Delta$. If no error occurs, OMN will terminate when a subset $A$ attains omniscience. In view of Theorem 3 at this point $R_A$ should be close to $H^{|A|}$ and the rates will be $(\sigma^{out}, H^{out}, c'_m \alpha)$-valid. Thus, we are in a similar situation as the first call to OMN except that $\alpha$ must be replaced by $c'_m \alpha$ and the parties in a single part of $\sigma^{out}$ are behaving as if they are collocated. The protocol proceeds by calling OMN again with these updated parameters. Note that under $E^c$, any party $j \in A$ for $A \in O$ can correctly compute $P_{x_A}$ and transmit it using $O(\log n)$ bits. Proceeding recursively in this manner, the protocol stops when parties in $M$ attain omniscience, which by Theorem 3 can only happen when the sum-rate $R_M$ is close to $H_{\sigma}(A|P_{x_M})$ for some partition $\sigma$ of $M$. Thus, omniscience will be attained in communication of rate roughly less than $R_{CO}(M|P_{x_M})$. We formally state our overall protocol in Protocol 6 and summarise its performance in Theorem 4.

**Protocol 6**: A universal omniscience protocol

1) Initialize $\sigma = \sigma_f(M)$, $R = (0, -1, -1, \ldots, -1)$, $k = |\sigma|$, $\alpha = 1$.

2) while $k > 1$ do

   (i) For $1 \leq i \leq k$, a party $j \in \sigma_i$ computes $P_{x_{\sigma_i}}$ and broadcasts it.

   Each party computes $H_{\sigma_i} = H(P_{x_{\sigma_i}})$, $1 \leq i \leq k$.

   (ii) Let $H$ be the sorted version of $(H_{\sigma_i} : 1 \leq i \leq k)$, i.e., assume $H_{\sigma_1} \geq H_{\sigma_2} \geq \cdots \geq H_{\sigma_k}$.

   Call OMN$(\sigma, \alpha, H, R)$.

   if There is no error declared then

   let $(R^{out}, O)$ be its output.

   else

   Terminate.

   (iii) Let $\sigma^{out} = \{\sigma_i : \sigma_i \in \sigma \text{ s.t. } \sigma_i \nsubseteq A \forall A \in O\} \cup \{A : A \in O\}$.

   Update $R = R^{out}$, $\sigma = \sigma^{out}$, $k = |\sigma^{out}|$, and $\alpha \rightarrow c'_m \alpha$.



We close with the following result claiming the universal rate optimality of our protocol for every IID distribution. Proof is a simple consequence of Theorem 3 and is given in Section VI. Note that while Protocol 6 is a variable length protocol, its fixed length variant can be obtained simply by aborting the protocol once the total number of bits communicated crosses $nR$.

**Theorem 4.** There exist constants $C_i > 0$, $i = 1, \ldots, 4$ depending only on $m$ and a polynomial $p(n)$ depending on $X_i$, $i \in M$, such that for every $\Delta > 0$ and every sequence $x_M$, the probability of error...
for Protocol 6 is bounded above by

\[ C_1 \left( \frac{\log |\mathcal{X}_M|}{\Delta} + m \right) p(n)2^{-n\Delta}. \]

Furthermore, if an error does not occur, the number of bits communicated by the protocol for input \( x_M \) is bounded above by

\[ nR_{CO}(\mathcal{M}|P_{X_M}) + nC_2\Delta + C_3 \left( \frac{\log |\mathcal{X}_M|}{\Delta} + m \right) + C_4 \log n. \] (7)

**Corollary 5.** For \( \Delta = \frac{1}{\sqrt{R}} \) and every distribution \( P_{X_M} \), Protocol 6 has a probability of error \( \epsilon_n \) vanishing to 0 as \( n \to \infty \) and average length \( |\pi|_{av} \) less than\(^\text{[10]}\)

\[ nR_{CO}(\mathcal{M}|P_{X_M}) + O(\sqrt{n \log n}). \]

Furthermore, for a fixed \( R > 0 \), the fixed-length variant of Protocol 6 has probability of error \( \epsilon_n \) vanishing to 0 as \( n \to \infty \) for all distributions \( P_{X_M} \) that satisfy

\[ R > R_{CO}(\mathcal{M}|P_{X_M}) + O\left( \sqrt{n^{-1} \log n} \right). \]

**V. Universal Secret Key Agreement**

Closely related to the omniscience problem is the SK agreement problem where the parties seek to generate shared random bits which are almost independent of the communication used to generate them. Specifically, an \((\epsilon, \delta)\)-SK agreement protocol consists of an interactive communication protocol \( \pi \) with public randomness \( U \), private randomness \( U_i \) at Party \( i \), and with the output of the \( i \)th party \( K_i = K_i(X^n_i, U_i, U, \Pi) \) such that there exists a \( \mathcal{K} \)-valued random variable \( K \) satisfying the **recoverability condition**

\[ P(K_i = K, \forall i \in \mathcal{M}) \geq 1 - \epsilon, \]

and the **secrecy condition\(^\text{[11]}\)**

\[ \|P_{K\Pi U} - P_{\text{unif}} \times P_{\Pi U}\| \leq \delta, \]

where \( P_{\text{unif}} \) denotes the uniform distribution on \( \mathcal{K} \).

\(^{10}\)The constant implied by \( O(\sqrt{n \log n}) \) depends on \( P_{X_M} \); see \( \text{[39]} \) below.

\(^{11}\)We assume that the public randomness \( U \) is available to the eavesdropper.
Definition 4 (Secret key capacity). For $\epsilon, \delta \in [0, 1)$, a rate $R \geq 0$ is an $(\epsilon, \delta)$-achievable SK rate if there exists a $K^{(n)}$-valued $(\epsilon, \delta)$-SK with $\log |K^{(n)}| \geq nR$ for all $n$ sufficiently large. The supremum over all $(\epsilon, \delta)$-achievable SK rates is called the $(\epsilon, \delta)$-SK capacity, denoted $C_{\epsilon, \delta}(\mathcal{M}|P_{X_M})$. The SK capacity for $P_{X_M}$ is given by

$$C(\mathcal{M}|P_{X_M}) = \lim_{\epsilon + \delta \to 0} C_{\epsilon, \delta}(\mathcal{M}|P_{X_M}).$$

Theorem 6 ([9]). Given a distribution $P_{X_M}$,

$$C(\mathcal{M}|P_{X_M}) = H(X_M) - R_{\text{CO}}(\mathcal{M}|P_{X_M}).$$

In fact, it was shown in [24], [25] that a strong converse holds and $C_{\epsilon, \delta}(\mathcal{M}|P_{X_M}) = C(\mathcal{M}|P_{X_M})$ for all $\epsilon + \delta < 1$.

The achievability of rate $H(X_M) - R_{\text{CO}}(\mathcal{M}|P_{X_M})$ was shown in [9] by establishing a connection between SK agreement and omniscience. In particular, a SK achieving capacity was generated by first communicating at rate $R_{\text{CO}}(\mathcal{M}|P_{X_M})$ to attain omniscience, and then extracting a SK from $X^n_M$ which is almost independent of the communication used for omniscience. Following the same methodology, we provide a universal SK agreement protocol which builds upon the universal omniscience protocol of the previous section.

We consider a slight generalization of the definition of SK above, which admits variable length SKs. An $(\epsilon, \delta)$-SK $K$ and its estimates $K_1, \ldots, K_m$ now take values in $K = \{0, 1\}^*$, the set of finite length binary sequences. The recoverability condition remains as before. However, the secrecy condition needs to be modified. Specifically, denoting by $T$ the random length of $K$, which we assume to be available to the eavesdropper, the secrecy condition now requires

$$\sum_t P_T(t) \| P_{K|U|T=t} - P_{\text{unif}|T=t} \times P_{\Pi|U|T=t} \| \leq \delta,$$

where $P_{\text{unif}|T=t}$ denotes the uniform distribution on $\{0, 1\}^t$. The average achievable rate and average SK capacity are defined as above with the worst-case length $\log |K|$ replaced by the average length $\mathbb{E}[T]$. Instead of introducing a new notation for average SK capacity, we note that it equals $C(\mathcal{M}|P_{X_M})$ and, with an abuse of notation, use $C(\mathcal{M}|P_{X_M})$ to denote both the SK capacity and the average SK capacity. Indeed, the achievability is the same as above since a fixed length SK constitutes a variable length SK. For the converse, denoting

$$\epsilon_t := 1 - \mathbb{P}(K = K_i, i \in \mathcal{M}|T = t) \text{ and } \delta_t := \| P_{K|U|T=t} - P_{\text{unif}|T=t} \times P_{\Pi|U|T=t} \|,$$
it follows by applying the converse proof of [9] for each fixed value $T = t$ that

\[ \frac{t}{n} \leq C(\mathcal{M}|P_{X_n}) + g_1(\epsilon_t) + g_2(\delta_t), \]

where $g_1$ and $g_2$ are concave, increasing functions satisfying $g_i(x) \to 0$ as $x \to 0$. Thus,

\[ \frac{E[T]}{n} \leq C(\mathcal{M}|P_{X_n}) + E[g_1(\epsilon_T) + g_2(\delta_T)] \]

\[ \leq C(\mathcal{M}|P_{X_n}) + g_1(E[\epsilon_T]) + g_2(E[\delta_T]) \]

\[ \leq C(\mathcal{M}|P_{X_n}) + g_1(\epsilon) + g_2(\delta), \]

where the last two inequalities hold since $g_i, i = 1, 2$, are concave and increasing.

We present a universal SK agreement protocol that generates a SK of average length $nC(\mathcal{M}|P_{X_n}) - O(\sqrt{n} \log n)$ without the knowledge of the underlying distribution $P_{X_n}$. Specifically, first the parties use Protocol 6 with $\Delta = 1/\sqrt{n}$ to recover $X^n_{\mathcal{M}}$. If no error occurs and the recovered sequence is $x_{\mathcal{M}}$, by Theorem 4 the number of bits communicated is no more than

\[ l(x_{\mathcal{M}}) = nR_{\text{CD}}(\mathcal{M}|P_{X_n}) + O(\sqrt{n}). \]

We extract a SK from recovered $x^n_{\mathcal{M}}$ by randomly hashing\(^\text{12}\) it to roughly $nH(P_{X_n}) - l(x_{\mathcal{M}})$ values. Formal description of the protocol is given in Protocol 7; the length of the SK is tuned to the secrecy parameter $\delta$.

**Protocol 7:** A universal SK agreement protocol

**Input:** Step size parameter $\Delta$ and secrecy parameter $\delta$

1) Parties execute Protocol 6 with step-size $\Delta$.

2) **if** Protocol 6 **completes without declaring an error**

   Protocol 6.

   Each party $i \in \mathcal{M}$ forms an estimate $K_i$ of the SK as follows:

   (i) Denoting by $P^{(i)}$ the type of the estimate $x^{(i)}_{\mathcal{M}}$ of $X^n_{\mathcal{M}}$ at Party $i$ and by $\epsilon_n$ the maximum error probability of Protocol 6, set $l(P^{(i)})$ to be the quantity in (7) for $P^{(i)}$ and

   \[ k(P^{(i)}) = nH(P^{(i)}) - l(P^{(i)}) - |X_{\mathcal{M}}| \log(n + 1) - 2 \log \frac{1}{\delta - 2\epsilon_n} + 2; \]

   (ii) generate $K_i$ by randomly hashing $x^{(i)}_{\mathcal{M}}$ to $k(P^{(i)})$ bits.

else

   Declare an error.

\(^{12}\)The random hash can be replaced by a randomly selected member of a 2-universal hash family.
Theorem 7. For $\Delta = \frac{1}{\sqrt{n}}$, $0 < \delta < 1$, and every distribution $P_{X_M}$, Protocol 7 generates a variable length $(\epsilon_n, \delta)$-SK with $\epsilon_n$ vanishing to 0 as $n \to \infty$ and average length greater than

$$nC(M|P_{X_M}) - O(\sqrt{n \log n}).$$

(8)

VI. TECHNICAL RESULTS AND PROOFS

This section contains the proofs of our results. We begin by noting a few properties of the mathematical quantities involved in our proofs.

A. Properties of CO region and related quantities

With a general subset $A \subseteq M$ in the role of $M$, recall the notations $R^*_i(A)$ and $H_\sigma(A)$, $\sigma \in \Sigma(A)$, given in (3) and (2), respectively. Our first lemma notes some simple properties of $H_\sigma(A)$ and $R^*_i(A)$.

Lemma 8. For $A \subseteq M$ and $\sigma \in \Sigma(A)$, the following relations hold between $R^*_i(A)$ and $H_\sigma(A)$:

$$\sum_{i \in A} R^*_i(A) = H_{\sigma_i}(A);$$

$$R^*_i(A) = H_{\sigma_i}(A) - H(X_A|X_i), \quad \forall i \in A;$$

$$\sum_{i \in B} R^*_i(A) - H(X_A|X_{A\setminus B}) = |B| \left[ H_\sigma(A) - H_{\sigma_B}(A) \right];$$

where the final equality holds for every $B \subseteq A$, with the shorthand $\sigma_B$ for the partition $\sigma_B(A) \in \Sigma(A)$ given by $\{\{A \setminus B\}, \{i\} : i \in B\}$.

Furthermore, $R^*_i(A)$ satisfies the following properties:

$$\sum_{j \in A} R^*_j(A) - R^*_i(A) = H(X_A|X_i), \quad \forall i \in A;$$

$$R^*_i(A) - R^*_j(A) = H(X_i) - H(X_j), \quad \forall i, j \in A.$$  

Finally, for $A \subseteq M$ and $\sigma \in \Sigma(A)$, similar results holds for $R^*_{\sigma_i}(A_{\sigma_i})$, $1 \leq i \leq |\sigma|$, with $A_\sigma$ in place of $A$.

Proof: Since $(R^*_i(A) : i \in A)$ is the solution of

$$\sum_{j \in A} R_j = H(X_A|X_i), \quad i \in A,$$
by taking the summation of all the constraints and by dividing by \(|A| - 1\), we have (9). Then, by subtracting the constraint for \(i\) from (9), we have (10). From (10), for every \(B \subsetneq A\) it holds that

\[
\sum_{i \in B} R^*_i(A) = |B| \mathbb{H}_{\sigma_i}(A) - \sum_{i \in B} H(X_A|X_i).
\]

(14)

Also,

\[
\mathbb{H}_{\sigma_B}(A) = \frac{1}{|B|} \left[ \sum_{i \in B} H(X_A|X_i) + H(X_A|X_{A \setminus B}) \right],
\]

which is equivalent to

\[
\sum_{i \in B} H(X_A|X_i) = |B| \mathbb{H}_{\sigma_B}(A) - H(X_A|X_{A \setminus B}).
\]

Combining this with (14), we have (11).

By taking the difference of (9) and (10), we have (12); (13) also follows from (10). The final statement is proved exactly in the same manner by regarding \(X_{\sigma_i}\) as a single random variable.

Next, we prove another useful relation between \(\mathbb{H}\) and \(R^*_i\) showing that the difference \(\sum_{i \in B} R^*_i(A) - \mathbb{H}_{\sigma_i}(B)\) must have the same sign as \(\mathbb{H}_{\sigma_\pi}(A) - \mathbb{H}_{\sigma_i}(A)\), where \(\overline{B}\) denotes \(A \setminus B\) and, as before, we have used the shorthand \(\sigma_B\) for the partition \(\sigma_B(A)\) of \(A\).

**Lemma 9.** For every \(B \subsetneq A \subseteq \mathcal{M}\) with \(\overline{B} = A \setminus B\),

\[
\sum_{i \in B} R^*_i(A) = \mathbb{H}_{\sigma_i}(B) + \frac{|B||\overline{B}|}{|B| - 1} \left[ \mathbb{H}_{\sigma_\pi}(A) - \mathbb{H}_{\sigma_i}(A) \right].
\]

(15)

For \(A \subseteq \mathcal{M}\) and \(\sigma \in \Sigma(A)\), similar results holds for \(R^*_\sigma(A_\sigma)\), \(1 \leq i \leq |\sigma|\), with \(B \subsetneq A_\sigma\) in place of \(B \subsetneq A\).

**Proof:** First we have

\[
(|B| - 1) \left[ R^*_B(A) - \mathbb{H}_{\sigma_i}(B) \right] = (|B| - 1) R^*_B(A) - \sum_{i \in B} H(X_B|X_i)
\]

\[
= (|B| - 1) \left| B \right| \mathbb{H}_{\sigma_i}(A) - \sum_{i \in B} H(X_A|X_i) - \sum_{i \in B} H(X_B|X_i)
\]

\[
= \frac{|B|}{|A| - 1} \left[ \sum_{i \in A} H(X_A|X_i) - (|A| - 1) \sum_{i \in B} H(X_A|X_i) \right] - \sum_{i \in B} H(X_B|X_i)
\]
\[
\begin{align*}
&= \left(\frac{|B|}{|A|} - 1\right) \sum_{i \in B} H(X_A | X_i) + \left(\frac{|B|}{|A|} - 1\right) \sum_{i \in B} H(X_B | X_i) \\
&\quad + \left(\frac{|B|}{|A|} - 1\right) \sum_{i \in B} \left(H(X_B | X_i) + H(X_A | X_B)\right),
\end{align*}
\]  
(16)

where we used (10) in the second equality. On the other hand, we have

\[
\begin{align*}
&\frac{|B||B|}{|A| - 1} \left[ H_{\sigma} (A) - H_{\sigma_i} (A) \right] \\
&= \frac{|B|}{|A| - 1} \left[ \left(\frac{|B|}{|A|} - 1\right) \sum_{i \in B} H(X_A | X_i) + \left(\frac{|B|}{|A|} - 1\right) H(X_A | X_B) - \frac{|B|}{|A|} \sum_{i \in A} H(X_A | X_i) \right] \\
&= \frac{|B|}{|A| - 1} \left[ \left(\frac{|B|}{|A|} - 1\right) \sum_{i \in B} H(X_A | X_i) + \left(\frac{|B|}{|A|} - 1\right) H(X_A | X_B) - \frac{|B|}{|A|} \sum_{i \in B} \left(H(X_B | X_i) + H(X_A | X_B)\right) \right] \\
&= \frac{|B|}{|A| - 1} \left(\frac{|B|}{|A|} - 1\right) \sum_{i \in B} H(X_A | X_i) - \frac{|B|}{|A| - 1} \sum_{i \in B} H(X_B | X_i) \\
&\quad + \frac{|B|}{|A| - 1} \left(\frac{|B|}{|A|} - 1\right) H(X_A | X_B),
\end{align*}
\]  
(17)

where we used $|B| = |A| - |B|$ in the second equality. We can verify that the coefficient of each term in (16) and (17) coincides. Thus, we have (15).

The second statement is proved exactly in the same manner by regarding $X_{\sigma_i}$ as a single random variable.

As our protocol proceeds, subsets of parties that have attained local omniscience start behaving as one. In the next recursive step of the protocol such sets of parties behaving as one attain omniscience. The next lemma ensures that when the rate is sufficient for these sets of parties to attain omniscience, it is also sufficient for the individual members of these sets to attain omniscience.

**Lemma 10.** For a subset $A \subseteq \mathcal{M}$ and a partition $\sigma \in \Sigma(A)$ with $|\sigma| = k$, suppose that for every $1 \leq i \leq k$

\[
(R_j : j \in \sigma_i) \in \mathcal{R}_{\sigma_i}^A (\sigma_i),
\]  
(18)

and

\[
(R_{\sigma_i} : 1 \leq i \leq k) \in \mathcal{R}_{\sigma}^A (A_{\sigma}),
\]  
(19)

where the elements of the set $A_{\sigma}$ consist of parts of the partition $\sigma$ (each part treated as a single element).
Then, it holds that

\[(R_i : i \in A) \in \mathcal{R}_{\Delta}^A(A).\]

**Proof:** We prove that for any \( B \subseteq A, \)

\[R_B \geq H(X_B|X_{A\setminus B}) + |B|\Delta.\]

Without loss of generality, we can assume

\[B = \left( \bigcup_{i=1}^{k'} B_i \right) \cup \left( \bigcup_{i=k'+1}^k B_i \right)\]

for some \( 1 \leq k' \leq k, \) where \( B_i \subsetneq \sigma_i \) for \( 1 \leq i \leq k' \) and \( B_i = \sigma_i \) for \( k' + 1 \leq i \leq k \) (\( B_i \) may be empty set for \( 1 \leq i \leq k' \)). Then, from (18) and (19), we have

\[
R_B = \sum_{i=1}^{k'} R_{B_i} + \sum_{i=k'+1}^k R_{B_i} \\
\geq \sum_{i=1}^{k'} H(X_{B_i}|X_{\sigma_i \setminus B_i}) + H(X_{\sigma_{k'+1}} \cup \cdots \cup X_{\sigma_k} | X_{A \setminus (\sigma_{k'+1} \cup \cdots \cup \sigma_k)}) + |B|\Delta \\
= \sum_{i=1}^{k'} H(X_{B_i}|X_{\sigma_i \setminus B_i}) + H(X_{B_{k'+1}} \cup \cdots \cup X_{B_k} | X_{A \setminus (\sigma_{k'+1} \cup \cdots \cup \sigma_k)}) + |B|\Delta \\
\geq \sum_{i=1}^{k'} H(X_{B_i}|X_{A \setminus B_i}) + H(X_{B_{k'+1}} \cup \cdots \cup X_{B_k} | X_{A \setminus (\sigma_{k'+1} \cup \cdots \cup \sigma_k)}) + |B|\Delta \\
= H(X_B|X_{A\setminus B}) + |B|\Delta.
\]

The next observation helps us to relax the assumptions of the previous lemma by showing that the collocated parts of a partition will attain local omniscience even if a collection of (nonempty) subsets of each part attains local omniscience.

**Lemma 11.** For a subset \( A \subseteq \mathcal{M} \) and a partition \( \sigma \in \Sigma(A) \) with \( |\sigma| = k, \) let \( B_i \subseteq \sigma_i \) be nonempty for \( 1 \leq i \leq k. \) Suppose that for every \( 1 \leq i \leq k \)

\[(R_j : j \in \sigma_i) \in \mathcal{R}_{\Delta}^A(\sigma_i), \quad (20)\]
and
\[(R_j : j \in B_i, 1 \leq i \leq k) \in \mathcal{R}_{\mathcal{O}_0}^\Delta \left( \bigcup_{i=1}^{k} B_i \right).\] (21)

Then, it holds that
\[(R_{\sigma_i} : 1 \leq i \leq k) \in \mathcal{R}_{\mathcal{O}_0}^\Delta (A_{\sigma}).\]

Proof: It suffices to show that for any \(C = \bigcup_{i=1}^{c} \sigma_i\) with \(1 \leq c \leq k\)
\[R_C \geq H(X_C|X_{A_{\sigma}\setminus C}) + |C|\Delta.\] (22)

To that end, we have from (20) and (21) that
\[R_C = \sum_{i=1}^{c} R_{\sigma_i} \]
\[= \sum_{i=1}^{c} R_{\sigma_i \setminus B_i} + \sum_{i=1}^{c} R_{B_i} \]
\[\geq \sum_{i=1}^{c} H(X_{\sigma_i \setminus B_i}|X_{B_i}) + H(X_{B_1}, \ldots, X_{B_i}|X_{B_{i+1}}, \ldots, X_{B_k}) + |C|\Delta \]
\[\geq \sum_{i=1}^{c} H(X_{\sigma_i \setminus B_i}|X_{\bigcup_{j=1}^{i-1} (\sigma_j \setminus B_j)}, X_{B_1}, \ldots, X_{B_i}, X_{\sigma_{i+1}}, \ldots, X_{\sigma_k}) \]
\[+ H(X_{B_1}, \ldots, X_{B_i}|X_{\sigma_{i+1}}, \ldots, X_{\sigma_k}) + |C|\Delta \]
\[= H(X_C|X_{A_{\sigma}\setminus C}) + |C|\Delta.\]

We need to show that when each call to our recursive step OMN terminates, which happens when a subset \(A\) attains local omniscience, the rate of communication used for each party \(i \in A\) is \(R_i^*(A)\) (or if OMN was called with a partition \(\sigma\) then the same property holds with \(i\) and \(A\) replaced by \(\sigma_i\) and \(A_{\sigma}\), where \(A_{\sigma}\) is the set of parts that comprise \(A\)). Recall that our recursive step OMN ensures that for each communicating party (or a set consisting of collocated parties) the difference \(R_i - R_i^*(A)\) is maintained for every \(A \subseteq \mathcal{M}\). Therefore, all the parties in \(A\) will reach \(R_i^*(A)\) at the same time, and, since before reaching this rate their sum-rate will not be sufficient for omniscience, it suffices to show that the rate vector \((R_i^*(A) : i \in A)\) lies the omniscience region for \(A\). The next technical lemma provides a crucial step towards proving this claim.
Lemma 12. For a partition $\sigma \in \Sigma(\mathcal{M})$ and $A \subseteq \mathcal{M}$ such that

$$A = \bigcup_{i=1}^{c} \sigma_{i}$$

there exists $B \subseteq \{1, \ldots, c\}$ with $|B| \geq 2$ such that

$$(R_{\sigma_{i_{l}}}^{\ast}(A_{\sigma}) + |\sigma_{i_{l}}|\Delta : l \in B) \in \mathcal{R}_{\mathcal{C}^{0}}(\{\sigma_{i_{l}} : l \in B\}),$$

(23)

where $A_{\sigma}$ is the set of parts $\sigma_{i}$ that comprise $A$, with each part treated as a single element.

Proof: Since (23) is equivalent to

$$(R_{\sigma_{i_{l}}}^{\ast}(A_{\sigma}) : l \in B) \in \mathcal{R}_{\mathcal{C}^{0}}(\{\sigma_{i_{l}} : l \in B\}),$$

we prove the claim for $\Delta = 0$. We proceed by induction on $c$. For $c = 2$, since

$$R_{\sigma_{i_{1}}}^{\ast}(A_{\sigma}) = H(X_{\sigma_{i_{1}}}|X_{\sigma_{i_{2}}}),$$
$$R_{\sigma_{i_{2}}}^{\ast}(A_{\sigma}) = H(X_{\sigma_{i_{2}}}|X_{\sigma_{i_{1}}}),$$

$B = \{1, 2\}$ satisfies the claim. Suppose that the claim holds for all $c \leq b$. For $c = b + 1$, if

$$(R_{\sigma_{i_{l}}}^{\ast}(A_{\sigma}) : 1 \leq l \leq c) \in \mathcal{R}_{\mathcal{C}^{0}}(A_{\sigma}),$$

then $B = \{1, \ldots, c\}$ satisfies the claim. Otherwise, there exists $\overline{C} \subseteq \{1, \ldots, c\}$ with $\overline{C} \neq \emptyset$ such that, with $C = \{1, \ldots, c\}\setminus\overline{C}$,

$$\sum_{l \in \overline{C}} R_{\sigma_{i_{l}}}^{\ast}(A_{\sigma}) < H(X_{\cup_{l \in \overline{C}}\sigma_{i_{l}}}|X_{\cup_{l \in C}\sigma_{i_{l}}}).$$

Then, by Lemma 9 and (11) of Lemma 8 it holds that

$$\sum_{l \in C} R_{\sigma_{i_{l}}}^{\ast}(A_{\sigma}) > \mathbb{H}_{\sigma_{f}}(\{\sigma_{i_{j}} : j \in C\})$$
$$\quad = \sum_{l \in C} R_{\sigma_{i_{l}}}^{\ast}(\{\sigma_{i_{j}} : j \in C\}).$$

(24)

Since

$$R_{\sigma_{i_{l}}}^{\ast}(A_{\sigma}) - R_{\sigma_{i_{l'}}}^{\ast}(A_{\sigma}) = H(X_{\sigma_{i_{l}}}) - H(X_{\sigma_{i_{l'}}})$$
$$= R_{\sigma_{i_{l}}}(\{\sigma_{i_{j}} : j \in C\}) - R_{\sigma_{i_{l'}}}(\{\sigma_{i_{j}} : j \in C\})$$

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for every $l \neq l'$ (cf. (13) of Lemma 8), (24) implies

$$R_{\sigma_i}^*(A_\sigma) > R_{\sigma_i}^*(\{\sigma_{ij} : j \in C\}), \forall l \in C.$$  \hspace{1cm} (25)

Since $|C| \leq b$, by the induction hypothesis, there exists $B \subseteq C$ such that

$$(R_{\sigma_i}^*(\{\sigma_{ij} : j \in C\}) : l \in B) \in \mathcal{R}_{C_0}^\Delta(\{\sigma_{ii} : l \in B\}),$$

which together with (25) implies that $B$ satisfies the claim for $c = b + 1$. \hfill \blacksquare

We are now in a position to prove our main results.

**B. Proof of Theorem 3**

Before going to the proof of Theorem 3 let us formally define the error event $\mathcal{E}$. We shall show that this event will happen with vanishing probability.

Let $L$ denote the maximum number of rounds, over all possible realizations, of communication needed for the protocol to terminate. Since the protocol terminates either correctly or erroneously once the rate vector enters the omniscience region, and since

$$\left( \log |X_{\sigma_i}| + |\sigma_i| \Delta : 1 \leq i \leq |\sigma| \right) \in \mathcal{R}_{C_0}^\Delta(M_\sigma|P_x)$$

holds for any $\sigma \in \Sigma(M)$ and any sequence $x$, the maximum number of rounds of communication $L$ is bounded above as

$$L \leq \max_{\sigma \in \Sigma(M)} \max_{1 \leq i \leq |\sigma|} \frac{\log |X_{\sigma_i}|}{\Delta} + |\sigma_i|$$

$$\leq \frac{\log |X_M|}{\Delta} + m. \hspace{1cm} (26)$$

For a fixed $x \in \mathcal{X}^n_M$, let $L(x)$ be the maximum number of rounds of communication when $x$ is observed by the parties. With a slight abuse of notation, denote by $R(l)$ the rate of communication after $l$ rounds, $1 \leq l \leq L$, if the protocol does not declare an error till then. Also, let $h_l(x_i)$ denote the random hash bits sent by the $i$th party (observing $x_i$) in the $l$th round.

For $B \subseteq A \subseteq M$ and $1 \leq l \leq L(x)$, let

$$\mathcal{T}^B_l(x) = \{x'_A : (R_i(l) : i \in A) \in \mathcal{R}_{C_0}^\Delta(A|P_{x'_A}) \text{ and } \{i \in A : x'_i \neq x_i\} = B\}.$$
Note that
\[
|T^B_l(x)| \leq p(n) \max_{p_{x_A} \in \mathcal{P}_n(x_A) : \langle R_i : i \in A \rangle \in \mathcal{R}_A^n(A)} \{x'_A : p_{x'_A} = p_{x_A}, \{i \in A : x'_i \neq x_i\} = B\}
\]
\[
\leq p(n) \max_{p_{x_A} \in \mathcal{P}_n(x_A) : \langle R_i : i \in A \rangle \in \mathcal{R}_A^n(A)} 2^n H(x_A|x_A;B)
\]
\[
\leq p(n) 2^{R_B(l) - n |B|/\Delta}, \tag{27}
\]
where \(\mathcal{P}_n(x_A)\) is the set of all types on \(x_A\) and \(p(n)\) is the number of types and is polynomial in \(n\).

For \(B \subseteq A\), denote by \(E_A(l, B)\) the error event
\[
E_A(l, B) = \{\exists x'_A \in T^B_l(x) \text{ s.t. } h_k(x_j) = h_k(x'_j) \ \forall j \in B, \forall 1 \leq k \leq l\}.
\]

Finally, let
\[
E_A(l) = \bigcup \{E_A(l, B) : B \neq \emptyset, B \subseteq A\},
\]
\[
\mathcal{E} = \bigcup \{E_A(l) : 1 \leq l \leq L(x), A \subseteq \mathcal{M}\}.
\]

**Lemma 13.** There exists a constant \(C\) depending only on \(m\) such that, for every sequence \(x \in \mathcal{X}_M^n\) the probability of the error event \(\mathcal{E} = \mathcal{E}(x)\) defined above is bounded by \(C L p(n) 2^{-n \Delta}\).

**Remark 3.** Suppose for a sequence \(x \in \mathcal{X}_M^n\), error \(\mathcal{E}\) does not occur. By definitions of \(\mathcal{E}\) and DEC, when OMN terminates, a set \(A\) belongs to \(O\) if and only if the final rates of communication \(R\) satisfy \((R_i : i \in A) \in \mathcal{R}_A^n(A)\).

**Proof:** By noting the bound in (27), we have
\[
\Pr(\mathcal{E}_A(l, B) \mid X^n_M = x) \leq \frac{1}{2^{n R_B(l)}} |T^B_l(x)| \leq p(n) 2^{-n \Delta}.
\]
Thus,
\[
\Pr(\mathcal{E} \mid X^n_M = x) \leq 2^m \cdot L \cdot \max_{A \subseteq \mathcal{M}, 1 \leq l \leq L(x)} \Pr(\mathcal{E}_A(l) \mid X^n_M = x)
\]
\[
\leq 4^m \cdot L \cdot \max_{A \subseteq \mathcal{M}, 1 \leq l \leq L(x), B \subseteq A} \Pr(\mathcal{E}_A(l, B) \mid X^n_M = x)
\]
\[
\leq 4^m \cdot L \cdot p(n) \cdot 2^{-n \Delta}.
\]
Proof of Theorem 3. We prove each statement of Theorem 3 separately.

Proof of (Ia): Denoting $B_i = A \cap \sigma_i$, let $\{i_1, \ldots, i_c\}$ be those indices $i \in \{1, \ldots, k\}$ for which $B_i \neq \emptyset$. Since $R_{in}$ is $(\sigma, H, \alpha)$-valid, it satisfies

$$(R_{in}^j : j \in \sigma_i) \in R_{CO}^\Delta(\sigma_i), \forall 1 \leq i \leq k \text{ s.t. } |\sigma_i| \geq 2,$$

and therefore, so does $R_{out}$, i.e.,

$$(R_{out}^j : j \in \sigma_i) \in R_{CO}^\Delta(\sigma_i), \forall 1 \leq i \leq k \text{ s.t. } |\sigma_i| \geq 2.$$

Furthermore, since an error does not occur and the parties in $A$ attain omniscience, by Remark 3

$$(R_{out}^j : j \in A) \in R_{CO}^\Delta(A).$$

Thus, by Lemma 11 and Lemma 10

$$(R_{out}^j : j \in \sigma_i, 1 \leq l \leq c) \in R_{CO}^\Delta(\bigcup_{l=1}^c \sigma_{il}).$$

Therefore, since no error has occurred, by Remark 3 the parties in $\bigcup_{l=1}^c \sigma_{il}$ must attain omniscience. But by the definition of $O$ the set $A$ is a maximal set attaining omniscience and $A \subseteq \bigcup_{l=1}^c \sigma_{il}$. Hence, $A$ must be $\bigcup_{l=1}^c \sigma_{il}$.

Proof of (Ib): As a preparation of our proof, we first show that for $1 \leq j, l \leq c$, the difference between $(R_{out}^{\sigma_{ij}} - R_{in}^{\sigma_{ij}}(A_{\sigma}))$ and $(R_{out}^{\sigma_{il}} - R_{in}^{\sigma_{il}}(A_{\sigma}))$ is bounded above by $(2\alpha + 1)\Delta$. Indeed, for $1 \leq j, l \leq c$, the parties in $\sigma_{ij}$ and $\sigma_{il}$ are communicating when OMN terminates. In fact, defining

$$i_l \leq s_{in} := \max\{j : R_{in}^j \geq 0\},$$

the parties in $\sigma_{il}$ were communicating even when OMN was initiated, and therefore,

$$R_{out}^{\sigma_{il}} - R_{in}^{\sigma_{il}} = R_{in}^{\sigma_{il}} - R_{out}^{\sigma_{il}},$$

which by the assumption that $R_{in}$ is $(\sigma, H, \alpha)$-valid yields

$$H_{\sigma_{il}} - H_{\sigma_{ili}} - \alpha \Delta \leq R_{out}^{\sigma_{il}} - R_{in}^{\sigma_{il}} \leq H_{\sigma_{il}} - H_{\sigma_{ili}} + \alpha \Delta. \quad (28)$$
On the other hand, if \( i_l > s^{\text{in}} \), then the parties in \( \sigma_{i_l} \) start communicating when

\[
R_{\sigma_1} = \lfloor H_{\sigma_1} - H_{\sigma_{i_l}} + \alpha \Delta \rfloor \Delta,
\]

where \( \lceil a \rceil \Delta := \min \{ i \Delta : i \in \mathbb{N}, i \Delta \geq a \} \). Thereafter, the parties in \( \sigma_1 \) as well as \( \sigma_{i_l} \) communicate at sum rate \( \Delta \) per round. Thus, in this case,

\[
R^\text{out}_{\sigma_1} - R^\text{out}_{\sigma_{i_l}} = \lfloor H_{\sigma_1} - H_{\sigma_{i_l}} + \alpha \Delta \rfloor \Delta. \tag{29}
\]

Upon combining (28) and (29), we get that for every \( 1 \leq j, l \leq c \),

\[
R^\text{out}_{\sigma_{j_l}} - R^\text{out}_{\sigma_{i_l}} \leq H_{\sigma_{j_l}} - H_{\sigma_{i_l}} + (2\alpha + 1) \Delta
= R^*_\sigma_{j_l} (A_\sigma) - R^*_\sigma_{i_l} (A_\sigma) + (2\alpha + 1) \Delta, \tag{30}
\]

where the previous equation is by (13).

Now, we prove the lower bound in (1b). Suppose that there exists a \( j \in \{1, \ldots, c\} \) such that

\[
R^\text{out}_{\sigma_{j_l}} < R^*_\sigma_{j_l} (A_\sigma) + (|\sigma_{i_l}| - 2\alpha - 1) \Delta.
\]

It follows from (30) that

\[
\sum_{l=1}^{c} R^\text{out}_{\sigma_{i_l}} < \sum_{l=1}^{c} R^*_\sigma_{i_l} (A_\sigma) + \sum_{l=1}^{c} |\sigma_{i_l}| \Delta
= \sum_{l=1}^{c} R^*_\sigma_{i_l} (A_\sigma) + |A| \Delta
= H_{\sigma_j} (A_\sigma) + |A| \Delta, \tag{31}
\]

where the previous equation is by (9). Also, since no error occurs and parties in \( A \) attain omniscience, by Remark 3,

\[
(R^\text{out}_j : j \in A) \in \mathcal{R}^\Delta_{\text{CD}} (A),
\]

which in turn implies that

\[
\sum_{l=1}^{c} R^\text{out}_{\sigma_{i_l}} = \frac{1}{c - 1} \sum_{l=1}^{c} \sum_{j=1 \atop j \neq l}^{c} R^\text{out}_{\sigma_{j_l}}
\geq \frac{1}{c - 1} \sum_{l=1}^{c} \left[ H(X_A | X_{\sigma_{i_l}}) + (|A| - |\sigma_{i_l}|) \Delta \right]
\]
\[
\mathbb{H}_{\sigma} (A_\sigma) + |A| \Delta,
\]
which contradicts (31). Thus, for every \(1 \leq l \leq c\),
\[
R_{\sigma_i}^{\text{out}} \geq R_{\sigma_i}^{*} (A_{\sigma}) + (|\sigma_i| - 2\alpha - 1) \Delta \\
\geq R_{\sigma_i}^{*} (A_{\sigma}) - 2\alpha \Delta.
\]

Moving to the proof of the upper bound in (Ib), suppose that there exists an \(l\) such that
\[
R_{\sigma_i}^{\text{out}} > R_{\sigma_i}^{*} (A_{\sigma}) + (m + 2\alpha + 1).
\]
(32)

From Lemma 12 there exists \(B \subseteq \{1, \ldots, c\}\) with \(|B| \geq 2\) such that
\[
(R_{\sigma_i}^{*} (A_{\sigma}) + |\sigma_i| \Delta : l \in B) \in \mathcal{R}_{\Delta}^{\Delta} (\{\sigma_i : l \in B\}).
\]
(33)

Then, (30), (32) and (35) imply that (by noting \(|\sigma_i| < m\))
\[
(R_{\sigma_i}^{\text{out}} - \Delta : l \in B) \in \mathcal{R}_{\Delta}^{\Delta} (\{\sigma_i : l \in B\}).
\]
(34)

Also, note that for every \(j \in \sigma_i\), \(1 \leq l \leq c\) with \(|\sigma_i| \geq 2\),
\[
R_{\sigma_i}^{\text{out}} - R_{\sigma_i}^{\text{in}} \geq \frac{\Delta}{|\sigma_i|},
\]
(35)

since otherwise there is no communication in the execution of OMN, which in turn by Remark 3 contradicts the assumption that \(R_{\sigma_i}^{\text{in}}\) is \((\sigma, H, \alpha)\)-valid. Upon combining (35) with (6), we get
\[
\left( R_{\sigma_i}^{\text{out}} - \frac{\Delta}{|\sigma_i|} : j \in \sigma_i \right) \in \mathcal{R}_{\Delta}^{\Delta} (\sigma_i),
\]
for every \(1 \leq l \leq c\) with \(|\sigma_i| \geq 2\), which together with (34) and Lemma 10 yields
\[
\left( R_{\sigma_i}^{\text{out}} - \frac{\Delta}{|\sigma_i|} : j \in \sigma_i, l \in B \right) \in \mathcal{R}_{\Delta}^{\Delta} \left( \bigcup_{l \in B} \sigma_i \right).
\]

But then by Remark 3 the parties in \(\bigcup_{l \in B} \sigma_i\) attain omniscience one round before OMN terminates, which is a contradiction since no error has occurred and OMN must terminate as soon as a subset in \(O\) is recognized.

**Proof of (II):** For each \(\sigma_i^{\text{out}} \in \sigma^{\text{out}}\) either \(\sigma_i^{\text{out}} \in \sigma\) or \(\sigma_i^{\text{out}} \in O\); in the latter case, by (Ia), \(\sigma_i^{\text{out}}\) must equal a union of parts of \(\sigma\). Note that, by the argument leading to (30), for every \(\sigma_i, \sigma_j \in \sigma\) such
that $R_{\sigma_i}^{\text{out}} \geq 0$ and $R_{\sigma_j}^{\text{out}} \geq 0$

$$R_{\sigma_i}^{\text{out}} - R_{\sigma_j}^{\text{out}} \leq H_{\sigma_{\cdot i}} - H_{\sigma_{\cdot j}} + (2\alpha + 1)\Delta. \tag{36}$$

Also, for $\sigma_i^{\text{out}} = \cup_{l=1}^c \sigma_{i_l} \in \mathcal{O}$, note that\footnote{We show the argument for $\sum_{l=2}^c R_{\sigma_{i_l}}^{\text{out}}$; the same argument extends to $\sum_{l=1, l \neq i}^c R_{\sigma_{i_l}}^{\text{out}}$ for every $i \in \{2, \ldots, c\}$.} by (1b)

$$\sum_{l=2}^c R_{\sigma_{i_l}}^{\text{out}}(\{\sigma_{i_1}, \ldots, \sigma_{i_c}\}) - 2\alpha c \Delta \leq \sum_{l=2}^c R_{\sigma_{i_l}}^{\text{out}}$$

$$\leq \sum_{l=2}^c R_{\sigma_{i_l}}^{\text{out}}(\{\sigma_{i_1}, \ldots, \sigma_{i_c}\}) + (mc + 2\alpha c)\Delta,$$

which by (12) is the same as

$$H(\mathbb{X}_{\sigma_{\cdot i}^{\text{out}}} | \mathbb{X}_{\sigma_{i_1}^{\text{out}}}) - 2\alpha c \Delta \leq \sum_{l=2}^c R_{\sigma_{i_l}}^{\text{out}}$$

$$\leq H(\mathbb{X}_{\sigma_{\cdot i}^{\text{out}}} | \mathbb{X}_{\sigma_{i_1}^{\text{out}}}) + (mc + 2\alpha c)\Delta. \tag{37}$$

To prove condition (i) in the definition of a valid rate vector (cf. Definition 3), consider $\sigma_i^{\text{out}}$ and $\sigma_j^{\text{out}}$ such that $R_{\sigma_i^{\text{out}}} \geq 0$ and $R_{\sigma_j^{\text{out}}} \geq 0$. The following three cases are possible:

- Case $\sigma_i^{\text{out}}, \sigma_j^{\text{out}} \in \sigma$: In this case, the claim follows from (36).
- Case $\sigma_i^{\text{out}} \in \mathcal{O}, \sigma_j^{\text{out}} \in \sigma$: Let $\sigma_i^{\text{out}} = \cup_{l=1}^c \sigma_{i_l}$. Then, by (36) and (37)

$$R_{\sigma_i^{\text{out}}}^{\text{out}} - R_{\sigma_j^{\text{out}}}^{\text{out}} = \sum_{l=2}^c R_{\sigma_{i_l}}^{\text{out}} + R_{\sigma_{i_1}}^{\text{out}} - R_{\sigma_j^{\text{out}}}^{\text{out}}$$

$$\leq H(\mathbb{X}_{\sigma_{\cdot i}^{\text{out}}} | \mathbb{X}_{\sigma_{i_1}^{\text{out}}}) + H(\mathbb{X}_{\sigma_{\cdot i}^{\text{out}}}) - H(\mathbb{X}_{\sigma_{i_1}^{\text{out}}}) + (mc + 2\alpha c + 2\alpha + 1)\Delta$$

$$= H(\mathbb{X}_{\sigma_{\cdot i}^{\text{out}}}) - H(\mathbb{X}_{\sigma_{\cdot j}^{\text{out}}}) + (mc + 2\alpha c + 2\alpha + 1)\Delta,$$

and similarly,

$$R_{\sigma_j^{\text{out}}}^{\text{out}} - R_{\sigma_i^{\text{out}}}^{\text{out}} \leq H(\mathbb{X}_{\sigma_{\cdot j}^{\text{out}}}) - H(\mathbb{X}_{\sigma_{\cdot i}^{\text{out}}}) + (2\alpha c + 2\alpha + 1)\Delta.$$

- Case $\sigma_i^{\text{out}}, \sigma_j^{\text{out}} \in \mathcal{O}$: Using argument similar to the previous case, we can show

$$R_{\sigma_i^{\text{out}}}^{\text{out}} - R_{\sigma_j^{\text{out}}}^{\text{out}} \leq H(\mathbb{X}_{\sigma_{\cdot i}^{\text{out}}}) - H(\mathbb{X}_{\sigma_{\cdot j}^{\text{out}}}) + (mc + 4\alpha c + 2\alpha + 1)\Delta.$$

Condition (ii) can be proved similarly by considering two cases: $\sigma_1^{\text{out}} \in \sigma$ and $\sigma_1^{\text{out}} \in \mathcal{O}$. Specifically,
let \( s' = \max\{i : R_{\sigma_i}^{\text{out}} \geq 0\} \). If \( \sigma_{1}^{\text{out}} \in \sigma \), then \( \sigma_{1}^{\text{out}} = \sigma_{1} \) and condition \( \text{(iii)} \) holds since the party \( \sigma_{s' + 1}^{\text{out}} \) did not start communicating. On the other hand, if \( \sigma_{j}^{\text{out}} = \bigcup_{i=1}^{s} \sigma_{i} \in \mathcal{O} \), then by Remark 3 \( \mathcal{R}_{\sigma_{s'}^{\text{out}}}^{\text{co}} \) where the strict inequality is by (36) and (37), since the party \( \sigma_{s'} \) did not start communicating. Proofs of Theorem 4 and Corollary 5.

For condition \( \text{(iv)} \), if \( \sigma_{s'}^{\text{out}} \) equals to a part of \( \sigma \), then \( (R_{j}^{\text{out}} : j \in \sigma_{s'}^{\text{out}}) \in \mathcal{R}_{\sigma_{s'}^{\text{out}}}^{\text{co}}(\sigma_{s'}^{\text{out}}) \) since \( (R_{j}^{\text{in}} : j \in \sigma_{s'}^{\text{out}}) \in \mathcal{R}_{\sigma_{s'}^{\text{out}}}^{\text{co}}(\sigma_{s'}^{\text{out}}) \) and \( R_{j}^{\text{out}} \geq R_{j}^{\text{in}} \) for every \( 1 \leq j \leq m \). On the other hand, if \( \sigma_{s'}^{\text{out}} \) belongs to \( \mathcal{O} \), then \( (R_{j}^{\text{out}} : j \in \sigma_{s'}^{\text{out}}) \in \mathcal{R}_{\sigma_{s'}^{\text{out}}}^{\text{co}}(\sigma_{s'}^{\text{out}}) \) by Remark 3.

Finally, for condition \( \text{(iv)} \), if there exists \( A \subseteq \{1, \ldots, |\sigma^{\text{out}}|\} \), \( |A| \geq 2 \), such that

\[
(R_{j}^{\text{out}} : j \in \sigma_{i}^{\text{out}}, i \in A) \in \mathcal{R}_{\sigma_{i}^{\text{out}}}^{\text{co}} \left( \bigcup_{i \in A} \sigma_{i}^{\text{out}} \right),
\]

then by Remark 3 \( \bigcup_{i \in A} \sigma_{i}^{\text{out}} \in \mathcal{O} \), which further implies that \( \bigcup_{i \in A} \sigma_{i}^{\text{out}} \in \mathcal{O} \) is a part of \( \sigma^{\text{out}} \), a contradiction. Thus, condition \( \text{(iv)} \) must hold for \( R^{\text{out}} \).

\[\square\]

C. Proofs of Theorem 4 and Corollary 5

For Theorem 4 by Lemma 13 the probability of the error event \( \mathcal{E} = \mathcal{E}(x) \) is bounded above by \( C_{1}Lp(n)2^{-n\Delta} \) for some constant \( C_{1} \), where \( L \) is the maximum number of rounds and is bounded above by \( \frac{\log |X_{M}|}{\Delta} + m \) using (20). Under the assumption that the error event \( \mathcal{E} \) did not occur, at the end of the \( j \)th call to OMN with input partition \( \sigma \), Theorem 3 guarantees that the total number of bits sent by each subset \( A \in \mathcal{O} \) is bounded above by \( 15 \)

\[
n \mathbb{H}_{\sigma, (A_{x})}(A_{x} | P_{X_{A}}) + nc(m + 2\alpha_{j})\Delta + C_{3}L + C_{4}\log n,
\]

for some constants \( C_{3}, C_{4} > 0 \), where \( \alpha_{j} \) is recursively defined by setting \( \alpha_{1} = 1 \) and \( \alpha_{j+1} = c'm\alpha_{j} \) with \( c'm \) given in Theorem 3-1(II). Since the size of partition \( \sigma \) strictly decreases in each execution of OMN,

\[14\]It can be seen that the parts of \( \sigma \) which did not start communicating must be singleton.

\[15\]The \( \log n \) term corresponds to the bits communicated to share types of the locally recovered observations. Additional \( C_{3}L \) bits are added to account for the overhead arising from rounding-off the required number of bits to an integer and ACK/NACK bits for each round.
the number of calls to OMN is at most \( m \) and \( \alpha_j \) remains bounded above by a constant that depends only on \( m \). Theorem 4 follows upon using (38) for \( A = \mathcal{M} \), and noting that

\[
\mathbb{H}_{\sigma_j(\mathcal{M}_s)}(\mathcal{M}_s|\mathcal{P}_{\mathcal{X}_s}) = \mathbb{H}_{\sigma}(\mathcal{M}|\mathcal{P}_{\mathcal{X}_s})
\]

\[
\quad \leq R_{\text{co}}(\mathcal{M}|\mathcal{P}_{\mathcal{X}_s}),
\]

where the inequality follows from Theorem 1.

Corollary 5 is obtained as a consequence of Theorem 4 as follows. First, note that under the error event \( \mathcal{E} \), which occur with probability less than \( C_1 p(n) L_2^{-n\Delta} \), the number of communicated bits is bounded above by \( C_5 n \) for some constant \( C_5 > 0 \). Next, by the Taylor approximation of the entropy function around \( \mathcal{P}_{\mathcal{X}_s} \), for \( \mathcal{Q}_{\mathcal{X}_s} \) satisfying \( \|\mathcal{P}_{\mathcal{X}_s} - \mathcal{Q}_{\mathcal{X}_s}\| \leq \delta \) and \( \text{supp}(\mathcal{Q}_{\mathcal{X}_s}) \subset \text{supp}(\mathcal{P}_{\mathcal{X}_s}) \), we have

\[
|R_{\text{co}}(\mathcal{M}|\mathcal{P}_{\mathcal{X}_s}) - R_{\text{co}}(\mathcal{M}|\mathcal{Q}_{\mathcal{X}_s})| \leq C_6 \delta
\]

for a sufficiently small \( \delta \), where \( C_6 > 0 \) is a constant that depends \(^{16}\) on \( \mathcal{P}_{\mathcal{X}_s} \). Denoting

\[
B_{\delta}(\mathcal{P}_{\mathcal{X}_s}) := \{ \mathcal{Q}_{\mathcal{X}_s} : \|\mathcal{P}_{\mathcal{X}_s} - \mathcal{Q}_{\mathcal{X}_s}\| \leq \delta, \text{supp}(\mathcal{Q}_{\mathcal{X}_s}) \subset \text{supp}(\mathcal{P}_{\mathcal{X}_s}) \},
\]

Theorem 4 implies that, when \( \mathcal{E} \) does not occur, the number of bits communicated is no more than

\[
nR_{\text{co}}(\mathcal{M}|\mathcal{P}_{\mathcal{X}_s}) + C_6 n \delta + \Pr\left( \text{type}(X^n_{\mathcal{X}_s}) \notin B_{\delta}(\mathcal{P}_{\mathcal{X}_s}) \right) n \log |\mathcal{X}_s| + C_2 n \Delta + C_3 L + C_4 \log n
\]

\[
\leq nR_{\text{co}}(\mathcal{M}|\mathcal{P}_{\mathcal{X}_s}) + C_6 n \delta + 2|\mathcal{X}_s| \exp(-2n\delta^2) n \log |\mathcal{X}_s| + C_2 n \Delta + C_3 L + C_4 \log n,
\]

where the inequality uses the Hoeffding bound

\[
\Pr\left( \text{type}(X^n_{\mathcal{X}_s}) \notin B_{\delta}(\mathcal{P}_{\mathcal{X}_s}) \right) \leq 2|\mathcal{X}_s| \exp(-2n\delta^2).
\]

The claimed upper bound for the expected number of bits communicated follows by combining the bounds under \( \mathcal{E} \) and \( \mathcal{E}^c \) and setting \( \delta = \frac{\log n}{n}, \Delta = \frac{1}{\sqrt{n}} \).

\[\square\]

\[\text{D. Proof of Theorem 7}\]

We first recall the leftover hash lemma (\textit{cf.} [19]); a proof of the version stated below is given in, for instance, [13, Appendix B].

\(^{16}\)The dependence of \( C_6 \) on \( \mathcal{P}_{\mathcal{X}_s} \) can be omitted by replacing \( \delta \) with \( \delta \log \frac{|\mathcal{X}_s|}{\delta} \).
Lemma 14 (Leftover Hash). Consider random variables $X$ and $V$ taking values in finite sets $X$ and $V$, respectively. Let $S$ be a random seed such that $f_S$ is uniformly distributed over a 2-universal hash family. Then, for $K = f_S(X)$, we have
\[
\|P_{KVS} - P_{unif} \times P_V P_S\|_1 \leq \frac{1}{2}\sqrt{|V|2^{-H_{\min}(P_X)}},
\]
where $P_{unif}$ is the uniform distribution on $K$ and
\[
H_{\min}(P_X) = -\log \max_x P_X(x).
\]

We assume that the public randomness $U$ used in Protocol 6 is available to the eavesdropper. Denote by $E$ the error event of Protocol 6, which is determined by $(X^n_{M}, U)$, and by $\Pi'$ an expurgated transcript defined as
\[
\Pi' = \begin{cases} 
\Pi, & \text{if } (X^n_{M}, U) \notin E, \\
\text{constant,} & \text{otherwise.}
\end{cases}
\]

Our security analysis will show that $\Pi'$ reveals negligible information about the SK and then use the large probability of agreement between $\Pi$ and $\Pi'$ to claim the security of the SK. Note that when the joint type of $X^n_{M}$ is $P_{X_{M}}$ and an error did not occur in Protocol 6 the length of the transcript $\Pi$ is bounded by $l(P_{X_{M}})$; thereby the length of $\Pi'$ is bounded by $l(P_{X_{M}})$ as well.

For each realization $X^n_{M} = x$, we generate a SK of length $k(P_X)$ by randomly hashing $x$ to $k(P_X)$ bits\footnote{Specifically, we use a seeded extractor for each fixed joint type $P_X$. For ease of presentation, we omit the dependence on seed from our notation.}. Clearly, the recoverability condition is satisfied with $1 - \epsilon_n$, where $\epsilon_n$ is the error probability of Protocol 6. Without loss of generality, we assume that the eavesdropper has access to the joint type $P_X$ of $x$. Note that such an eavesdropper has potentially more information than that available to the actual eavesdropper in our protocol. Thus, security against this stronger eavesdropper implies security against the actual eavesdropper. Denoting by $T = t$ a fixed realization of the random type, triangular inequality yields
\[
\sum_{t \in P_n(X_M)} P_T(t)\|P_{KIU|T=t} - P_{unif,t} \times P_{IU|T=t}\|
\leq \sum_{t \in P_n(X_M)} P_T(t)\left[\|P_{KIU|T=t} - P_{KIV|T=t}\| + \|P_{IV|U|T=t} - P_{IU|T=t}\|\right]
\]
+ \| P_{K|U|T=t} - P_{\text{unif},t} \times P_{\Pi|U|T=t} \|, \\
where $P_{\text{unif},t}$ is the uniform distribution on $\{0, 1\}^{k(t)}$. The first two terms on the right-side above are each bounded above by $\Pr(\Pi \neq \Pi')$. Also, by Lemma 14 applied for each fixed $(t, u)$, the third term is bounded above by 
\[
\sum_{t \in \mathcal{P}_n(X_M)} P_T(t) \frac{1}{2} \sqrt{2^{l(t)} + k(t)(n + 1)|X_M|2^{-H(t)}},
\]
where we have used the independence of $U$ and $X^u_M$ and the observation that 
\[
H_{\min}(P_{X^u_M|T=t, U=u}) = H_{\min}(P_{X^u_M|T=t}) \\
\geq nH(t) - |X_M|\log(n + 1).
\]
Thus, by combining the bounds above, we get
\[
\sum_{t \in \mathcal{P}_n(X_M)} P_T(t) \| P_{K|U|T=t} - P_{\text{unif},t} \times P_{\Pi|U|T=t} \| \\
\leq 2 \Pr(\Pi \neq \Pi') + \sum_{t \in \mathcal{P}_n(X_M)} P_T(t) \frac{1}{2} \sqrt{2^{l(t)} + k(t)(n + 1)|X_M|2^{-H(t)}} \\
\leq \delta,
\]
where the previous inequality uses $\Pr(\Pi \neq \Pi') \leq \epsilon_n$ and the definitions of $l(t)$, $k(t)$, and $\delta$. The average length $\sum_{t \in \mathcal{P}_n(X_M)} P_T(t)k(t)$ is lower bounded by (5) using Theorem 6 in a similar manner as the proof of Corollary 5.

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