GLOBAL EXISTENCE AND BOUNDEDNESS OF SOLUTION OF A PARABOLIC–PARABOLIC–ODE CHEMOTAXIS–HAPTOTAXIS MODEL WITH (GENERALIZED) LOGISTIC SOURCE

LING LIU

1Department of Basic Science
Jilin Jianzhu University, Changchun 130118, China

JIASHAN ZHENGa,b

aSchool of Information
Renmin University of China, Beijing, 100872, China
bSchool of Mathematics and Statistics Science
Ludong University, Yantai 264025, China

(Communicated by Michael Winkler)

ABSTRACT. In this paper, we study the following chemotaxis–haptotaxis system with (generalized) logistic source

\begin{align}
  &u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + u(a - \mu u^{r-1} - w), \\
  &v_t = \Delta v - v + u, \\
  &w_t = -vw, \\
  &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
  &u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), \quad x \in \Omega,
\end{align}

in a smooth bounded domain \( \mathbb{R}^N \) \((N \geq 1)\), with parameter \( r > 1 \). The parameters \( a \in \mathbb{R}, \mu > 0, \chi > 0 \). It is shown that when \( r > 2 \), or

\[ \mu > \mu^* = \frac{(N-2)+}{N} (\chi + C_\beta) \frac{1}{N+1}, \text{ if } r = 2, \]

the considered problem possesses a global classical solution which is bounded, where \( C \frac{1}{N+1} \) is a positive constant which is corresponding to the maximal sobolev regularity. Here \( C_\beta \) is a positive constant which depends on \( \xi, \|u_0\|_{C(\bar{\Omega})}, \|v_0\|_{W^{1,\infty}(\Omega)} \text{ and } \|w_0\|_{L^{\infty}(\Omega)}. \) This result improves or extends previous results of several authors.

1. Introduction. Chemotaxis is the oriented cell movement along concentration gradients of a chemical signal produced by the cells themselves. In 1970s, a well-known chemotaxis model was proposed by Keller and Segel ([13]), which describes the aggregation processes of the cellular slime mold Dictyostelium discoideum. Since then, a number of variations of the Keller–Segel model have attracted the attention of many mathematicians, and the focused issue was the boundedness or blow-up of the solutions ([5, 7, 9, 10, 40, 20]). The striking feature of Keller–Segel models is the possibility of blow-up of solutions in a finite (or infinite) time (see, e.g.,

2010 Mathematics Subject Classification. 92C17, 35K55, 35K59, 35K20.
Key words and phrases. Boundedness, chemotaxis–haptotaxis, (generalized) logistic source.
Corresponding author: Jiashan Zheng.
[1, 9, 18, 40]), which strongly depends on the space dimension. We also refer the reader to Winkler [39, 42, 41] (and the references therein) for some other works on the finite-time blow up of solutions of the variants of Keller–Segel models. Moreover, some recent studies have shown that the blow-up of solutions can be inhibited by the nonlinear diffusion (see Ishida et al. [11] Winkler et al. [1, 27, 36, 43]) and the (generalized) logistic damping (see Li and Xiang [14], Tello and Winkler [31], Wang et al. [33], Zheng et al. [51]).

In order to describe the cancer invasion mechanism, in 2005, Chaplain and Lolas ([3]) extended the classical Keller–Segel model where, in addition to random diffusion, cancer cells bias their movement towards a gradient of a diffusible matrix-degrading enzyme (MDE) secreted by themselves, as well as a gradient of a static tissue, referred to as extracellular matrix (ECM), by detecting matrix molecules such as vitronectin adhered therein. The latter type of directed migration of cancer cells is usually referred to as haptotaxis (see Chaplain and Lolas [4]). According to the model proposed in [3, 4, 8], in this paper, we consider the chemotaxis–haptotaxis system with (generalized) logistic source

\[\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + u(1 - u^{r-1} - w), \quad x \in \Omega, t > 0, \\
  v_t &= \Delta v + u - v, \quad x \in \Omega, t > 0, \\
  w_t &= -vw, \quad x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
  u(x, 0) &= u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), \quad x \in \Omega,
\end{align*}\]

(1.1)

where \(r > 1, \Omega \subset \mathbb{R}^N (N \geq 1)\) is a bounded domain with smooth boundary, \(\frac{\partial}{\partial \nu}\) denotes the outward normal derivative on \(\partial \Omega\), the three variables \(u, v, w\) represent the cancer cell density, the MDE concentration and the ECM density, respectively. The parameters \(\chi, \xi\) and \(\mu\) are positive which measure the chemotactic, haptotactic sensitivities and the proliferation rate of the cells, respectively. As is pointed out by [1] (see also Tao and Winkler [26], Winkler [39], Zheng [52]), in this modeling context the cancer cells are also usually assumed to follow a generalized logistic growth \(u(1 - u^{r-1} - w)\) \((r > 1)\), which denotes the proliferation rate of the cells and competing for space with healthy tissue. And the initial data \((u_0, v_0, w_0)\) supposed to be satisfied the following conditions

\[\begin{align*}
  &u_0 \in C(\bar{\Omega}) \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \neq 0, \\
  &v_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 \geq 0 \text{ in } \Omega, \\
  &w_0 \in C^{2+\vartheta}(\bar{\Omega}) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega} \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial \Omega
\end{align*}\]

(1.2)

with some \(\vartheta \in (0, 1)\).

In order to better understand model (1.1), let us mention the following quasilinear chemotaxis–haptotaxis system, which is a closely related variant of (1.1)

\[\begin{align*}
  u_t &= \nabla \cdot (\phi(u) \nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u^{r-1} - w), \quad x \in \Omega, t > 0, \\
  \tau v_t &= \Delta v + u - v, \quad x \in \Omega, t > 0, \\
  w_t &= -vw, \quad x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
  u(x, 0) &= u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), \quad x \in \Omega.
\end{align*}\]

(1.3)
where \( \mu \geq 0, \tau \in \{0, 1\} \), the function \( \phi(u) \) fulfills

\[
\phi \in C^2((0, \infty))
\]

and there exist constants \( m \geq 1 \) and \( C_\phi \) such that

\[
\phi(u) \geq C_\phi (u + 1)^{m-1} \quad \text{for all } u \geq 0.
\]

When \( w \equiv 0 \), (1.3) is reduced to the chemotaxis-only system with (generalized) logistic source (see Xiang [44], Zheng et al. [47, 48, 50, 56, 54]). And global existence, boundedness and asymptotic behavior of solution were studied in [15, 17, 32, 49]. Going beyond the above statements, we should mention the papers [40] and [42] (and references therein), which deal with the blow-up of solutions to parabolic-elliptic versions of (1.3) (with \( w \equiv 0 \)). For example, when \( D(u) \equiv 1 \) and \( N \geq 3 \), it is demonstrated in [42] that a superlinear growth condition on logistic source may be insufficient to prevent finite time blow-ups for a parabolic-elliptic system of (1.3), which is the first rigorous detection of blow-up in a superlinearly dampened of Keller-Segel system in three-dimensional case. From a theoretical point of view, due to the fact that the chemotaxis and haptotaxis terms require different \( L^p \)-estimate techniques, the problem related to the chemotaxis–haptotaxis models of cancer invasion presents an important mathematical challenging. There are only few results on the mathematical analysis of this (quasilinear) chemotaxis–haptotaxis system (1.3) (Cao [2], Zheng et al. [16], Tao et al. [23, 24, 25, 26, 28, 29, 30], Wang et al. [16, 34, 35, 56]). Indeed, if MDEs diffuses much faster than cells (see [12, 29]), (\( \tau = 0 \) in the second equation of (1.3)), (1.3) is reduced to the parabolic–ODE–elliptic chemotaxis–haptotaxis system the (generalized) logistic source. To the best of our knowledge, there exist some boundedness and stabilization results on the simplified parabolic–elliptic–ODE chemotaxis–haptotaxis model [25, 29, 28].

When, \( r = 2 \) in the first equation of (1.3), the global boundedness of solutions to the chemotaxis–haptotaxis system with the standard logistic source has been proved for any \( \mu > 0 \) in two dimensions and for large \( \mu \) (compared to the chemotactic sensitivity \( \chi \)) in three dimensions (see Tao and Wang [25]). In [29], Tao and Winkler studied the global boundedness for model (1.3) under the condition \( \mu > \frac{(N-2)^r}{N^{r-1}} \chi \), moreover, in additional explicit smallness on \( w_0 \), they gave the exponential decay of \( w \) in the large time limit. While if \( r > 1 \) (the (generalized) logistic source), one can see [56].

As for parabolic–ODE–parabolic system (1.3), if \( r = 2 \), there has been some progress made in two or three dimensions (see Cao [2], Tao and Winkler [22, 24, 29]). In fact, when \( \phi \equiv 1 \), Cao ([2]) and Tao ([23]) proved that (1.3) admits a unique, smooth and bounded solution if \( \mu > 0 \) on \( N = 2 \) and \( \mu \) is large enough on \( N = 3 \). Recently, assume that \( \mu \) is large enough and \( 3 \leq N \leq 8 \), the boundedness of the global solution of system (1.3) are obtained by Wang and Ke in [35]. However, they did not give the lower bound estimation for the logistic source. Note that the global existence and boundedness of solutions to (1.3) is still open in three dimensions for small \( \mu \) > 0 and in higher dimensions.

The main object of the present paper is to address the boundedness to solutions of (1.1) without any restriction on the space dimension. Our main result is the following.

**Theorem 1.1.** Assume that the initial data \( (u_0, v_0, w_0) \) fulfills (1.2). For any \( N \geq 1 \), if one of the following cases holds:

(i) \( r > 2 \);
(ii) 
\[ \mu > \mu^* = \frac{(N-2)+}{N}\left(\chi + C_\beta\right)C_1^{\frac{1}{N^2+1}}, \text{ if } r = 2, \]  
then there exists a triple \((u, v, w) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))\) which solves (1.1) in the classical sense, where \(C_1^{\frac{1}{N^2+1}}\) is a positive constant which is corresponding to the maximal sobolev regularity. Here \(C_\beta\) are positive constants which depends on \(\xi, \|u_0\|_{C(\bar{\Omega})}, \|v_0\|_{W^{1,\infty}(\Omega)}\) and \(\|w_0\|_{L^\infty(\Omega)}\). Moreover, \(u, v\) and \(w\) are bounded in \(\Omega \times (0, \infty)\).

Remark 1.1. (i) From Theorem 1.1, we derive that the global boundedness of the solution for the complete parabolic–parabolic and parabolic–elliptic models, which need a coefficient of the logistic source to keep the same (except a constant).

(ii) Obviously, if \(r = 2\) and \(\mu > \frac{(N-2)+}{N}\left(\chi + C_\beta\right)C_1^{\frac{1}{N^2+1}}\), thus, Theorem 1.1 extends the results of Theorem 1.1 of Cao ([2]), who proved the boundedness in the case \(N = 3, r = 2\) and \(\mu\) is appropriately large.

(iii) Obviously, if \(r = 2\) and \(\mu > \frac{(N-2)+}{N}\left(\chi + C_\beta\right)C_1^{\frac{1}{N^2+1}}\), hence Theorem 1.1 extends the results of Theorem 1.1 of Wang and Ke ([35]), who proved the boundedness of the solutions in the case \(3 \leq N \leq 8, r = 2\) and \(\mu\) is appropriately large.

(iv) Obviously, if \(r > 2\), then, \(2 < \frac{N+2}{2}\), therefore, Theorem 1.1 (partly) extends the results of Theorem 1.1 of Zheng ([52]), who showed the boundedness of the solutions in the cases \(r > \frac{N+2}{2}\).

(v) If \(w \equiv 0\), (the PDE system (1.1) is reduced to the chemotaxis-only system), it is not difficult to obtain that the solutions under the conditions of Theorem 1.1 are uniformly bounded when \(r = 2\) and \(\mu > \frac{(N-2)+}{N}\left(\chi + C_\beta\right)C_1^{\frac{1}{N^2+1}}\), which extends and coincides with the results of Winkler (see Theorem 0.1 of [37]) and the result of Osaki et al. ([19]), respectively.

(vi) From Theorem 1.1, we derive that solutions of model (1.1) are global and bounded for any \(r = 2, \mu > 0\) and \(N \leq 2\), which coincides with the result of Tao ([23]).

(vii) With the help of precise estimation, the ideas of our paper can also be used to deal with the three-dimensional chemotaxis-fluid system with (generalized) logistic source.

If \(\phi\) is a nonlinear function of \(u\), then (1.3) becomes a quasilinear parabolic–ODE–parabolic chemotaxis–haptotaxis system. There are only few results on the mathematical analysis of this quasilinear parabolic–ODE–parabolic chemotaxis–haptotaxis system with the standard logistic source (\(r = 2\) in the first equation of (1.3)). In fact, if \(N = 2\), Zheng et al. ([57]) mainly studied the global boundedness for model (1.3) with \(\phi\) satisfies (1.4)–(1.5) and \(m > 1\). While, Tao and Winkler ([26]) proved that model (1.3) possesses at least one nonnegative \textbf{global} classical solution when \(\phi\) satisfy (1.4)–(1.5) with \(m > \max\{1, \bar{m}\}\) and

\[ \bar{m} := \begin{cases} \frac{2N^2+4N-4}{N(N+4)} & \text{if } N \leq 8, \\ \frac{2N^2+3N+2-\sqrt{8N(N+1)}}{N(N+20)} & \text{if } N \geq 9. \end{cases} \]  

(1.7)
Further, by using the boundedness of $\int_\Omega |\nabla v|^l (1 \leq l < \frac{N}{N-1})$, assuming that $m > 2 - \frac{2}{N}$, Wang ([34]) obtained the boundedness of the global solutions to (1.3). Recently, with the help of the boundedness of $\int_\Omega |\nabla v|^2$, Zheng ([53]) extends the results of [34] when $m > 2 - \frac{2}{N}$. Very recently, it is asserted that if $\phi$ satisfies (1.4)–(1.5) and $m > 2 - \frac{2}{N} + 2$ (if $1 < r < N + 2$, $N + 2 \geq r \geq N + 2 - \frac{2}{N}$, $\geq 1$ if $r > N + 2$), (1.8)

we ([52]) proved that the unique nonnegative classical solution of quasilinear parabolic–ODE–parabolic chemotaxis–haptotaxis system with generalized logistic source ($r > 1$ in the first equation of (1.3)) which is global in time and bounded, however, we have to leave open here the question of how far the above hypothesis (1.8) is sharp.

It is worth to remark the main idea underlying the proof of our results. The proof of theorem 1.1 is based on an iterative $L^p$ estimation argument involving the maximal Sobolev regularity and the Moser-type limit procedure. Indeed, with the help of

$$
\min_{y > 0} (y + \frac{1}{q_0 + 1})^{-q_0} \left( \frac{q_0 - 1}{q_0} \right)^{q_0 + 1} y^{-q_0} C_{q_0 + 1} = \frac{(q_0 - 1)}{q_0} C_{q_0 + 1}^\frac{1}{q_0 + 1}
$$

and

$$
\min_{y > 0} (y + \frac{1}{q_0 + 1})^{-q_0} \left( \frac{q_0 - 1}{q_0} \right)^{q_0 + 1} y^{-q_0} C_{q_0 + 1} C_{\beta} = \frac{(q_0 - 1)}{q_0} C_{q_0 + 1}^\frac{1}{q_0 + 1} C_{\beta}
$$

(see Lemma 3.3), we can obtain a subtle combination of entropy like estimates for $\int_\Omega u^{q_0}$ for some $q_0 > \frac{N}{2}$.

Then we shall involve the variation-of-constants formula for variable $v$ to gain

$$
\int_\Omega |\nabla v|^q \leq C \text{ for all and } q \in [1, \frac{N q_0}{N - q_0} + 1],
$$

(1.10)

and thereby establish the a priori estimates of the functional

$$
\int_\Omega u^p \leq C \text{ for all and } p > 1.
$$

(1.11)

Finally, in light of the Moser iteration method (see e.g. Lemma A.1 of [27]) and the standard estimate for Neumann semigroup, we established the $L^\infty$ bound of $u$ (see the proof of Theorem 1.1).

2. Preliminaries and main results. Firstly, we recall some preliminary lemmas, which play essential roles in our subsequent analysis. To this end, firstly, let us collect some basic solution properties which essentially have already been used in [10] (see also Winkler [38], Zhang and Li [45]).

Lemma 2.1. ([10]) For $p \in (1, \infty)$, let $A := A_p$ denote the sectorial operator defined by

$$
A_p u := -\Delta u \text{ for all } u \in D(A_p) := \{ \varphi \in W^{2,p}(\Omega) \mid \frac{\partial \varphi}{\partial v} |_{\partial \Omega} = 0 \}.
$$

(2.1)

The operator $A + 1$ possesses fractional powers $(A + 1)^\alpha (\alpha \geq 0)$, the domains of which have the embedding properties

$$
D((A + 1)^\alpha) \hookrightarrow W^{1,p}(\Omega) \text{ if } \alpha > \frac{1}{2},
$$

(2.2)
If \( m \in \{0, 1\} \), \( p \in [1, \infty] \) and \( q \in (1, \infty) \) with \( m - \frac{N}{p} < 2\alpha - \frac{N}{q} \), then we have
\[
|u|_{W^{m,p}(\Omega)} \leq C \|(A + 1)^\alpha u\|_{L^q(\Omega)} \text{ for all } u \in D((A + 1)^\alpha),
\]
where \( C \) is a positive constant. The fact that the spectrum of \( A \) is \( p \)-independent countable set of positive real numbers \( 0 = \mu_0 < \mu_1 < \mu_2 < \cdots \) entails the following consequences: For all \( 1 \leq p \leq q < \infty \) and \( u \in L^p(\Omega) \) the general \( L^p-L^q \) estimate
\[
\|(A + 1)^\alpha e^{-tA}u\|_{L^q(\Omega)} \leq ct^{-\frac{\mu}{2} + (\frac{1}{p} - \frac{1}{q})} e^{(1-\mu)t} \|u\|_{L^p(\Omega)},
\]
for any \( t > 0 \) and \( \alpha \geq 0 \) with some \( \mu > 0 \).

For convenience, we state the well-known Gagliardo-Nirenberg inequality:

**Lemma 2.2.** ([6, 11]) Let \( s \geq 1 \) and \( q \geq 1 \). Assume that \( p > 0 \) and \( a \in (0,1) \)
\[
\frac{1}{2} - \frac{p}{N} = (1 - a)\frac{q}{s} + a\left(\frac{1}{2} - \frac{1}{N}\right) \text{ and } p \leq a.
\]
Then there exist \( c_0, c'_0 > 0 \) such that for all \( u \in W^{1,2}(\Omega) \cap L^\frac{q}{2}(\Omega) \),
\[
\|u\|_{W^{1,2}(\Omega)} \leq c_0 \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^\frac{q}{2}(\Omega)}^{1-a} + c'_0 \|u\|_{L^\frac{q}{2}(\Omega)}.
\]

As an essential ingredient of the proof of our main result, we will use the following Maximal Sobolev regularity property.

**Lemma 2.3.** ([2, 55]) Suppose \( \gamma \in (1, +\infty) \). Consider the following evolution equation
\[
\begin{cases}
  v_t - \Delta v = g, \quad (x, t) \in \Omega \times (0, T), \\
  \frac{\partial v}{\partial \nu} = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
  v(x, 0) = v_0(x), \quad (x, t) \in \Omega.
\end{cases}
\]
For each \( v_0 \in W^{2,\gamma}(\Omega) \) such that \( \frac{\partial v_0}{\partial \nu} = 0 \) and any \( g \in L^\gamma((0, T); L^\gamma(\Omega)) \), there exists a unique solution \( v \in W^{1,\gamma}((0, T); L^\gamma(\Omega)) \cap L^\gamma((0, T); W^{2,\gamma}(\Omega)) \). Moreover, there exists a positive constant \( \delta_0 \) such that
\[
\int_0^T \|v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma dt + \int_0^T \|v_1(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma dt + \int_0^T \|\Delta v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma dt \\
\leq \delta_0 \left( \int_0^T \|g(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma dt + \|v_0(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma + \|\Delta v_0(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma \right). 
\]

On the other hand, consider the following evolution equation
\[
\begin{cases}
  v_t - \Delta v + v = g, \quad (x, t) \in \Omega \times (0, T), \\
  \frac{\partial v}{\partial \nu} = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
  v(x, 0) = v_0(x), \quad (x, t) \in \Omega.
\end{cases}
\]
Then there exists a positive constant \( C_\gamma := C_{\gamma,|\Omega|} \) such that if \( s_0 \in [0, T] \), \( v(\cdot, s_0) \in W^{2,\gamma}(\Omega)(\gamma > N) \) with \( \frac{\partial v(\cdot, s_0)}{\partial \nu} = 0 \), then
\[
\int_{s_0}^T e^{\gamma s}(\|v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma + \|\Delta v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma) ds \\
\leq C_\gamma \left( \int_{s_0}^T e^{\gamma s}(\|g(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma + \|v_0(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma + \|\Delta v_0(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma) dt \right). 
\]
The local solvability and extendibility of the system (1.1) are well established by using a suitable fixed point argument and standard parabolic regularity theory; see, for example, [26, 16].

**Lemma 2.4.** Assume that the nonnegative functions $u_0, v_0$, and $w_0$ satisfies (1.2) for some $\vartheta \in (0, 1)$. Then there exists a maximal existence time $T_{\text{max}} \in (0, \infty)$ and a triple of nonnegative functions
\[
\begin{cases}
u \in C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})), \\
v \in C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})), \\
w \in C^{2,1}(\bar{\Omega} \times [0, T_{\text{max}}]),
\end{cases}
\]
which solves (1.1) classically and satisfies $w \leq \|w_0\|_{L^\infty(\Omega)}$ in $\Omega \times (0, T_{\text{max}})$. In particular, if $T_{\text{max}} < +\infty$, then
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \text{ as } t \nearrow T_{\text{max}}. \tag{2.9}
\]

Firstly, by Lemma 2.4, we can choose $s_0 \in (0, T_{\text{max}})$, $s_0 \leq 1$ and $\beta > 0$ such that for all $\tau \in [0, s_0]$,
\[
\|u(\cdot, \tau)\|_{L^\infty(\Omega)} \leq \beta, \quad \|v(\cdot, \tau)\|_{W^{1,\infty}(\Omega)} \leq \beta \quad \text{and} \quad \|w(\cdot, \tau)\|_{W^{2,\infty}(\Omega)} \leq \beta. \tag{2.10}
\]

### 3. The proof of Theorem 1.1.

In this section, we derive the claimed boundedness result via the maximum Sobolev regularity with a Moser-type iteration and the standard estimate for Neumann semigroup. To achieve this, firstly, following the idea from Lemma 3.1 in [35] (see also Lemma 2.1 of [37]), we recall some basic knowledge on systems of type (1.1).

**Lemma 3.1.** Assume that the conditions in Theorem 1.1 is fulfilled. Then there exists a positive constant $C$ such that the solution of (1.1) satisfies
\[
\int_{\Omega} u(x, t) + \int_{\Omega} v^2(x, t) + \int_{\Omega} |v(x, t)|^2 \leq C \text{ for all } t \in (0, T_{\text{max}}). \tag{3.1}
\]
Moreover, for each $T \in (0, T_{\text{max}})$, there exists a positive constant $C$ such that
\[
\int_{0}^{T} \int_{\Omega} (\|v\|^2 + u^r + |\Delta v|^2) \leq C. \tag{3.2}
\]

Now, employing almost exactly the same arguments as in the proof of Lemma 3.2 of [34] we derive the following Lemma:

**Lemma 3.2.** Let $(u, v, w)$ be a solution to (1.1) on $(0, T_{\text{max}})$. Under the assumptions in Theorem 1.1, then for any $k > 1$, there exists a positive constant $C_\beta := C(\xi, \|w_0\|_{L^\infty(\Omega)}, \beta)$ which depends on $\xi, \|w_0\|_{L^\infty(\Omega)}$ and $\beta$ such that
\[
-\xi \int_{\Omega} u^{k-1} \nabla \cdot (u \nabla v) \leq C_\beta \left( \frac{k - 1}{k} \int_{\Omega} u^k + 1 \right) + k \int_{\Omega} u^{k-1} |\nabla u|, \tag{3.3}
\]
where $\beta$ is the same as (2.10).

**Proof.** Here and throughout the proof of Lemma 3.2, we shall denote by $M_i (i \in N)$ several positive constants independent of $k$. Firstly, observing that the third equation of (1.1) is an ODE, we derive that
\[
w(x, t) = w(x, s_0) e^{- \int_{0}^{s} v(x, s) ds}, \quad (x, t) \in \Omega \times (0, T_{\text{max}}). \tag{3.4}
\]
Hence, in view of a basic calculation, we derive that
\[
\nabla w(x, t) = \nabla w(x, s_0)e^{-\int_0^t v(x, s)ds} - w(x, s_0)e^{-\int_0^t v(x, s)ds} \int_0^t \nabla v(x, s)ds, \quad (x, t) \in \Omega \times (0, T_{\text{max}})
\]
and
\[
\Delta w(x, t) \geq \Delta w(x, s_0)e^{-\int_0^t v(x, s)ds} - 2\nabla w(x, s_0) \cdot \int_0^t \nabla v(x, s)ds e^{-\int_0^t v(x, s)ds} - w(x, s_0)e^{-\int_0^t \Delta v(x, s)ds} \int_0^t \Delta v(x, s)ds.
\]

On the other hand, for any \( k \geq 1 \), integrating by parts yields
\[
-\xi \int_{\Omega} u^{k-1} \nabla \cdot (u \nabla w)
= -\frac{k-1}{k} \int_{\Omega} u^k \Delta w
\leq \frac{k-1}{k} \int_{\Omega} u^k (\Delta w(x, s_0)e^{-\int_0^t v(x, s)ds} + 2\nabla w(x, s_0) \cdot \int_0^t \nabla v(x, s)ds e^{-\int_0^t v(x, s)ds} + \int_0^t \Delta v(x, s)ds)
:= J_1.
\]

Now, with the help of \( v \geq 0 \) and the Young inequality, we conclude that
\[
J_1 \leq -\frac{k-1}{k} \int_{\Omega} u^k \Delta w(x, s_0)e^{-\int_0^t v(x, s)ds}
+ \frac{k-1}{k} \int_{\Omega} u^k w(x, s_0)e^{-\int_0^t v(x, s)ds} \int_0^t \Delta v(x, s)ds
+ \frac{2(k-1)}{k} \int_{\Omega} u^k \nabla w(x, s_0) \cdot \int_0^t \nabla v(x, s)ds e^{-\int_0^t v(x, s)ds}
= -\frac{k-1}{k} \int_{\Omega} u^k \Delta w(x, s_0)e^{-\int_0^t v(x, s)ds}
+ \frac{k-1}{k} \int_{\Omega} u^k w(x, s_0)e^{-\int_0^t v(x, s)ds} \int_0^t \Delta v(x, s)ds
+ \frac{2(k-1)}{k} \int_{\Omega} u^k \nabla w(x, s_0) \cdot \int_0^t \nabla v(x, s)ds e^{-\int_0^t v(x, s)ds}
\leq \xi \beta \int_{\Omega} u^k + \frac{k-1}{k} \int_{\Omega} u^k w(x, s_0)e^{-\int_0^t v(x, s)ds} \int_0^t \Delta v(x, s)ds
+ 2(\xi(k-1)) \int_{\Omega} u^{k-1} \nabla u \cdot \nabla w(x, s_0)e^{-\int_0^t v(x, s)ds}
+ \frac{2(k-1)}{k} \int_{\Omega} u^k \Delta w(x, s_0)e^{-\int_0^t v(x, s)ds}
\leq M_1 \int_{\Omega} u^k + \frac{k-1}{k} \int_{\Omega} u^k w(x, s_0)e^{-\int_0^t v(x, s)ds} \int_0^t \Delta v(x, s)ds
+ M_2(k) \int_{\Omega} u^{k-1} |\nabla u| + \int_{\Omega} u^k,
\]
where \( \beta = \frac{\xi}{\xi + \Delta} \), \( \xi > 0 \), and \( \Delta \) is a positive constant.
where \( M_1 = \xi \beta \) and \( M_2 = \max\{2 \xi \sup_{x \in \Omega} |\nabla w(x, s_0)|, 2 \sup_{x \in \Omega} |\Delta w(x, s_0)|\} \). Next, by using the second equality of (1.1), we conclude that

\[
\frac{\xi - 1}{k} \int \Omega u^k w(x, s_0)e^{-\int_0^s v(x,s)ds} \int_0^t \Delta v(x,s)ds = \frac{\xi - 1}{k} \int \Omega u^k w(x, s_0)e^{-\int_0^s v(x,s)ds} \int_0^t (v_s(x,s) + v(x,s) - u(x,s))ds.
\]

Here we have use the fact that \( u \geq 0 \) and \( \frac{\xi}{r} \leq 1 \) (for all \( t \geq 0 \)). Collecting (3.8) with (3.9), we can get the result. \( \square \)

As one more tool from elementary analysis needed in Lemma 3.4, let us separately state the following lemma:

**Lemma 3.3.** Let

\[
A_1 = \frac{1}{\delta + 1} \left[ \frac{\delta + 1}{\delta} \right]^{-\delta} \left( \frac{\delta - 1}{\delta} \right)^{\delta + 1},
\]

\( H(y) = y + A_1 y^{-\delta} x^{\delta + 1} C_{\delta + 1} \) and \( \bar{H}(y) = y + A_1 y^{-\delta} x^{\delta + 1} C_{\delta + 1} \) for \( y > 0 \). For any fixed \( \delta \geq 1, \chi, \beta, C_{\delta + 1} > 0 \), Then

\[
\min_{y > 0} H(y) = \frac{\delta - 1}{\delta} C_{\delta + 1} \chi.
\]

and

\[
\min_{y > 0} \bar{H}(y) = \frac{\delta - 1}{\delta} C_{\delta + 1} \beta.
\]

**Proof.** A straightforward computation shows that

\[
H'(y) = 1 - A_1 \delta C_{\delta + 1} \left( \frac{\chi}{y} \right)^{\delta + 1}.
\]

Let \( H'(y) = 0 \), we have

\[
y = (A_1 C_{\delta + 1})^{1/\delta} \chi.
\]

Next, in view of \( \lim_{y \to 0^+} H(y) = +\infty \) and \( \lim_{y \to +\infty} H(y) = +\infty \), we have

\[
\min_{y > 0} H(y) = H[(A_1 C_{\delta + 1})^{1/\delta} \chi] = \left( A_1 C_{\delta + 1} \right)^{1/\delta} \chi = \frac{(\delta - 1)}{\delta} C_{\delta + 1}^{1/\delta} \chi.
\]

(3.11) can be proved very similarly, therefore, we omit it. \( \square \)

**Lemma 3.4.** Assume that \( r = 2 \) and \((u,v,w)\) is a solution to (1.1) on \((0,T_{\text{max}})\). If

\[
\mu > \frac{(N - 2)_+}{N} (\chi + C_{\beta}) C_{\frac{N + 1}{2} + 1},
\]

then for all \( p > 1 \), there exists a positive constant \( C := C(p, |\Omega|, \mu, \chi, \xi, \beta) \) such that

\[
\int_{\Omega} u^p(x,t)dx \leq C \text{ for all } t \in (0,T_{\text{max}}).
\]
Proof. Due to \( \mu > \frac{(N-2)\chi}{N} (\chi + C_\beta) \), we can choose \( q_0 > \frac{N}{2} \) such that

\[
\mu > \frac{q_0 - 1}{q_0} (C_\beta + \chi) C_{q_0+1}. \tag{3.15}
\]

Let \( l = q_0 \). Multiplying the first equation of (1.1) by \( u^{l-1} \) and integrating over \( \Omega \), we have

\[
\frac{1}{l} \frac{d}{dt} \|u\|_{L^l(\Omega)}^l + (l-1) \int_\Omega u^{l-2} |\nabla u|^2 dx = -\chi \int_\Omega \nabla \cdot (u \nabla v) u^{l-1} dx - \xi \int_\Omega \nabla \cdot (u \nabla w) u^{l-1} dx + \int_\Omega u^{l-1} (au - \mu u^2) dx, \tag{3.16}
\]

that is,

\[
\frac{1}{l} \frac{d}{dt} \|u\|_{L^l(\Omega)}^l + (l-1) \int_\Omega u^{l-2} |\nabla u|^2 dx \leq \frac{l+1}{l} \int_\Omega u^l dx - \chi \int_\Omega \nabla \cdot (u \nabla v) u^{l-1} dx - \xi \int_\Omega \nabla \cdot (u \nabla w) u^{l-1} dx + \int_\Omega \left( \frac{l+1}{l} u^l + u^{l-1} (au - \mu u^2) \right) dx. \tag{3.17}
\]

Hence, in light of the Young inequality, it reads that

\[
\int_\Omega \left( \frac{l+1}{l} u^l + u^{l-1} (au - \mu u^2) \right) dx \leq \frac{l+1}{l} \int_\Omega u^l dx + a \int_\Omega u^{l-1} dx - \mu \int_\Omega u^{l+1} dx \leq (\varepsilon_1 - \mu) \int_\Omega u^{l+1} dx + C_1(\varepsilon_1, l), \tag{3.18}
\]

where

\[
\varepsilon_1 = \frac{1}{4} \left( \mu - \frac{q_0 - 1}{q_0} (C_\beta + \chi) C_{q_0+1} \right) > 0
\]

and

\[
C_1(\varepsilon_1, l) = \frac{1}{l+1} \left( \frac{\varepsilon_1}{l+1} \right)^{-l} \left( \frac{l+1}{l+1} + a \right)^{l+1} |\Omega|.
\]

Next, integrating by parts to the first term on the right hand side of (3.16), we obtain

\[
-\chi \int_\Omega \nabla \cdot (u \nabla v) u^{l-1} dx = \left( l - 1 \right) \chi \int_\Omega u^{l-1} \nabla u \cdot \nabla v dx \leq \frac{l-1}{l} \chi \int_\Omega u^{l-1} \Delta v dx. \tag{3.19}
\]
Next, due to (3.3) and the Young inequality, we derive that there exist positive constants $C_2 := \left(\frac{1}{2} l \beta C_\beta^2 + C_\beta\right)$ and $C_3 := \frac{1}{l+1} (\varepsilon_3 \frac{l+1}{l})^{-1} C_2^{l+1}$ such that

\[-\xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{l-1} dx \leq C_\beta \left(\frac{l-1}{l} \int_{\Omega} u^l (v + 1) + l \int_{\Omega} u^{l-1} |\nabla u| \right) \leq \frac{l-1}{2} \int_{\Omega} u^{l-2} |\nabla u|^2 + \frac{l-1}{l} C_\beta \int_{\Omega} u^l + C_\beta^{\frac{l-1}{l}} \int_{\Omega} u^l v \leq \frac{l-1}{2} \int_{\Omega} u^{l-2} |\nabla u|^2 + C_2 \int_{\Omega} u^l + C_\beta \int_{\Omega} u^{l-1} \int_{\Omega} u^l v \leq \frac{l-1}{2} \int_{\Omega} u^{l-2} |\nabla u|^2 + (\varepsilon_2 + \varepsilon_3) \int_{\Omega} u^{l+1} + C_3,\]

where $\varepsilon_2 := \tilde{\lambda}_0$, $\varepsilon_3 = \frac{1}{4} (\mu - 2(\mu - (\chi + C_\beta) C_{\beta_0}^{\frac{1}{q_0+1}})) > 0$ and

\[\tilde{\lambda}_0 := (A_1 C_{l+1})^{\frac{1}{l+1}} C_\beta.\] (3.21)

Here $A_1$ is given by (3.10).

Now, let

\[\lambda_0 := (A_1 C_{l+1})^{\frac{1}{l+1}} \chi.\] (3.22)

While from (3.19) and the Young inequality, we conclude that

\[-\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{l-1} dx \leq \lambda_0 \int_{\Omega} u^{l+1} dx + \frac{l}{l+1} \left[ \lambda_0 \frac{l+1}{l} \right]^{l-1} \left(\frac{l-1}{l} \chi\right)^{l+1} \int_{\Omega} |\Delta v|^{l+1} dx \leq \lambda_0 \int_{\Omega} u^{l+1} dx + A_1 \lambda_0^{-1} \chi^{l+1} \int_{\Omega} |\Delta v|^{l+1} dx,\] (3.23)

where $A_1$ is given by (3.10). Thus, inserting (3.18), (3.20) and (3.23) into (3.17), we get

\[\frac{1}{l+1} \int_{\Omega} u^l \frac{d}{dt} \|u\|_{L^1(\Omega)} \leq \frac{l-1}{2} \int_{\Omega} u^{l-2} |\nabla u|^2 \leq (\varepsilon_1 + \tilde{\lambda}_0 + \varepsilon_3 + \lambda_0 - \mu) \int_{\Omega} u^{l+1} dx - \frac{l+1}{l} \int_{\Omega} u^l dx + A_1 \lambda_0^{-1} \chi^{l+1} \int_{\Omega} |\Delta v|^{l+1} dx + A_1 \lambda_0^{-1} C_2^{l+1} \int_{\Omega} v^{l+1} + C_1 + C_3.\]
For any \( t \in (s_0, T_{max}) \), involving the variation-of-constants formula to the above inequality, we obtain

\[
\begin{align*}
&\frac{1}{t} \| u(\cdot, t) \|_{L^1(\Omega)}^t \\
\leq &\frac{1}{t} e^{-((t+1)(t-s_0))} \| u(s_0) \|_{L^1(\Omega)}^t + (\varepsilon_1 + \tilde{\lambda}_0 + \varepsilon_3 + \lambda_0 - \mu) \int_{s_0}^t e^{-(l+1)(t-s)} \int_{\Omega} u^{l+1} dx ds \\
+ &A_1 \lambda_0^{-l} \chi^{l+1} \int_{s_0}^t e^{-(l+1)(t-s)} \int_{\Omega} |\Delta v|^{l+1} dx ds \\
+ & (C_1 + C_3) \int_{s_0}^t e^{-(l+1)(t-s)} ds + A_1 \tilde{\lambda}_0^{-l} C_{\beta}^{l+1} \int_{s_0}^t e^{-(l+1)(t-s)} \int_{\Omega} v^{l+1} dx ds \\
\leq & (\varepsilon_1 + \tilde{\lambda}_0 + \varepsilon_3 + \lambda_0 - \mu) \int_{s_0}^t e^{-(l+1)(t-s)} \int_{\Omega} u^{l+1} dx ds \\
+ &A_1 \lambda_0^{-l} \chi^{l+1} \int_{s_0}^t e^{-(l+1)(t-s)} \int_{\Omega} |\Delta v|^{l+1} dx ds \\
+ &A_1 \tilde{\lambda}_0^{-l} C_{\beta}^{l+1} \int_{s_0}^t e^{-(l+1)(t-s)} \int_{\Omega} v^{l+1} dx ds + C_4
\end{align*}
\]

(3.24)

with

\[
C_4 := C_4(\varepsilon_1, \varepsilon_3, l) = \frac{1}{t} \| u(\cdot, s_0) \|_{L^1(\Omega)}^t + (C_1 + C_3) \int_{s_0}^t e^{-(l+1)(t-s)} ds.
\]

Now, by (3.21), Lemma 2.3 and the second equation of (1.1), we have

\[
\begin{align*}
A_1 \lambda_0^{-l} \chi^{l+1} \int_{s_0}^t e^{-(l+1)(t-s)} \int_{\Omega} |\Delta v|^{l+1} dx ds \\
= & A_1 \lambda_0^{-l} \chi^{l+1} e^{-(l+1)t} \int_{s_0}^t e^{(l+1)s} \int_{\Omega} |\Delta v|^{l+1} dx ds
\end{align*}
\]

(3.25)

\[
\leq A_1 \lambda_0^{-l} \chi^{l+1} e^{-(l+1)t} C_{l+1} \left( \int_{s_0}^t \int_{\Omega} e^{(l+1)s} u^{l+1} dx ds + e^{(l+1)s_0} \| v(s_0, t) \|_{W^{2,l+1}}^{l+1} \right)
\]

and

\[
\begin{align*}
A_1 \tilde{\lambda}_0^{-l} C_{\beta}^{l+1} \int_{s_0}^t e^{-(l+1)(t-s)} \int_{\Omega} v^{l+1} dx ds \\
= & A_1 \tilde{\lambda}_0^{-l} C_{\beta}^{l+1} e^{-(l+1)t} \int_{s_0}^t e^{(l+1)s} \int_{\Omega} v^{l+1} dx ds \\
\leq & A_1 \tilde{\lambda}_0^{-l} C_{\beta}^{l+1} e^{-(l+1)t} C_{l+1} \left( \int_{s_0}^t \int_{\Omega} e^{(l+1)s} u^{l+1} dx ds + e^{(l+1)s_0} \| v(s_0, t) \|_{W^{2,l+1}}^{l+1} \right)
\end{align*}
\]

(3.26)
Thus, due to (3.30) and (3.32), we have

\[
\int_{s_0}^t \frac{1}{C_{l+1}} u_{l+1} \, ds \leq \left( \varepsilon_1 + \lambda_0 + A_1 \lambda_0^{-l} C_{l+1} C_{l+1} + \lambda_0 + A_1 \lambda_0^{-l} C_{l+1} C_{l+1} - \mu \right)
\]

\[
+ A_1 \lambda_0^{-l} \chi^{l+1} + \lambda_0^{-l} C_{l+1} e^{-(l+1)(t-s)} C_{l+1} \| v(s_0, t) \|_{W_{l+1}}^{l+1} + C_4
\]

by using (3.21) and Lemma 3.3. Since \( l = q_0 \), therefore,

\[
\int_{s_0}^t \frac{l-1}{C_{l+1}} C_{l+1} C_{l+1} + \frac{l-1}{C_{l+1} C_{l+1}} \chi - \mu = \frac{q_0 - 1}{q_0} (C_{l+1} + \chi) C_{\frac{q_0}{q_0+1}} - \mu,
\]

so that,

\[
0 < \varepsilon_1 + \varepsilon_3 = \frac{l-1}{C_{l+1} C_{l+1}} + \frac{l-1}{C_{l+1} C_{l+1}} \chi - \mu = \frac{q_0 - 1}{q_0} (C_{l+1} + \chi) C_{\frac{q_0}{q_0+1}} - \mu. \tag{3.28}
\]

Collecting (3.28) and (3.27), we derive that there exists a positive constant \( C_5 \) such that

\[
\int_\Omega u^{q_0}(x,t) dx \leq C_5 \text{ for all } t \in (s_0, T_{max}). \tag{3.29}
\]

Next, we fix \( q < \frac{N q_0}{(N - q_0)^+} \) and choose some \( \alpha > \frac{1}{2} \) such that

\[
q < \frac{1}{q_0} - \frac{1}{N} + \frac{2}{N} (\alpha - \frac{1}{2}) \leq \frac{N q_0}{(N - q_0)^+}. \tag{3.30}
\]

Now, in light of the variation-of-constants formula for \( v \), we have

\[
v(t) = e^{-t(A+1)} v(s_0) + \int_{s_0}^t e^{-(t-s)(A+1)} u(s) ds, \quad t \in (s_0, T_{max}). \tag{3.31}
\]

Hence, by virtue of (2.10) and (3.31), we conclude that

\[
\|(A+1)^\alpha v(\cdot, t)\|_{L^q(\Omega)} \leq C_6 \int_{s_0}^t (t-s)^{-\frac{\alpha}{2}} \frac{1}{(N - q_0)^+} e^{-\mu(s-t)} \| u(\cdot, s) \|_{L^q(\Omega)} ds
\]

\[
+ C_6 s_0^{-\frac{\alpha}{2}} \beta \| v(\cdot, s_0) \|_{L^q(\Omega)}
\]

\[
\leq C_6 \int_0^\infty \sigma^{-\frac{\alpha}{2}} \frac{1}{(N - q_0)^+} e^{-\mu \sigma} d\sigma + C_6 s_0^{-\frac{\alpha}{2}} \beta \tag{3.32}
\]

Thus, due to (3.30) and (3.32), we have

\[
\int_\Omega |\nabla v(\cdot, t)|^q \leq C_7 \text{ for all } t \in (s_0, T_{max}) \tag{3.33}
\]

and \( q \in [1, \frac{N q_0}{(N - q_0)^+}] \). Finally, in view of (2.10) and (3.33), we can get

\[
\int_\Omega |\nabla v(\cdot, t)|^q \leq C_8 \text{ for all } t \in (0, T_{max}) \text{ and } q \in [1, \frac{N q_0}{(N - q_0)^+}] \tag{3.34}
\]

with some positive constant \( C_8 \).
Testing the first equation in (1.1) by $u^{p-1}$, integrating over $\Omega$ and integrating by parts, we arrive at
\[
\frac{1}{p} \int \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + (p-1) \int \Omega u^{p-2} |\nabla u|^2 \, dx
= -\chi \int \nabla \cdot (u \nabla v) u^{p-1} \, dx - \xi \int \nabla \cdot (u \nabla w) u^{p-1} \, dx + \int \Omega u^{p-1} (au - \mu u^2) \, dx
= \chi (p-1) \int \Omega u^{p-1} \nabla u \cdot \nabla v \, dx + \xi (p-1) \int \Omega u^{p-1} \nabla u \cdot \nabla w \, dx
+ \int \Omega u^{p-1} (au - \mu u^2) \, dx,
\]
which combined with the Young inequality and (3.3) implies that
\[
\frac{1}{p} \int \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + (p-1) \int \Omega u^{p-2} |\nabla u|^2 \, dx
\leq \frac{p-1}{2} \int \Omega u^{p-2} |\nabla u|^2 \, dx + \frac{\chi^2 (p-1)}{2} \int \Omega u^p |\nabla v|^2 \, dx + C_9 \int \Omega v^{p+1} \frac{1}{2} \int \Omega u^{p+1} \, dx + C_{10},
\]
for some positive constants $C_9$ and $C_{10}$. Now, in light of $q_0 > \frac{N}{2}$ and (3.34), we derive that there exists a positive constant $C_{11}$ such that
\[
C_9 \int \Omega v^{p+1}(x, t) \leq C_{11} \text{ for all } t \in (0, T_{max}) \text{ and } p > 1
\]
by using the Sobolev imbedding theorem. Since, $q_0 > \frac{N}{2}$ yields \(q_0 < \frac{Nq_0}{2(N-q_0)}\), in view of the H"{o}lder inequality, (2.10) and (3.34), we arrive at
\[
\frac{\chi^2 (p-1)}{2} \int \Omega u^p |\nabla v|^2 \, dx \leq \frac{\chi^2 (p-1)}{2} \left( \int \Omega u^{\frac{q_0}{20-\nu}} \right)^{\frac{20-\nu}{m}} \left( \int \Omega |\nabla v|^{2q_0} \right)^{\frac{1}{m}} \leq C_{12} \|u^\frac{p}{2}\|_{L^{\frac{2q_0}{m-1}}(\Omega)},
\]
where $C_{10}$ is a positive constant. Observe $q_0 > \frac{N}{2}$ and $p > q_0 - 1$, we have
\[
\frac{q_0}{p} \leq \frac{q_0}{q_0 - 1} \leq \frac{N}{N - 2},
\]
which together with the Gagliardo–Nirenberg inequality (see [46]) yields that
\[
C_{12} \|u^\frac{p}{2}\|_{L^{\frac{2q_0}{m-1}}(\Omega)} \leq C_{13} (\|u^\frac{p}{2}\|_{L^{\frac{2q_0}{m-1}}(\Omega)} 1^{\frac{1}{2} - \frac{\mu_1}{2q_0}} + \|u^\frac{p}{2}\|_{L^{\frac{2q_0}{m-1}}(\Omega)} ^2)^{\frac{1}{2}} \leq C_{14} (\|u^\frac{p}{2}\|_{L^{\frac{2q_0}{m-1}}(\Omega)} ^2 + 1)^{\frac{1}{2}} \leq C_{15} (\|u^\frac{p}{2}\|_{L^{\frac{2q_0}{m-1}}(\Omega)} ^2 + 1)
\]
with some positive constants $C_{13}, C_{14}$ and
\[
\mu_1 = \frac{Np}{2q_0} - \frac{Np}{2q_0 - p} = \frac{Np}{2q_0} - \frac{Np}{2q_0 - p} \in (0, 1).
\]
Now, in view of the Young inequality, we derive that
\[
\frac{\chi^2 (p-1)}{2} \int \Omega u^p |\nabla v|^2 \, dx \leq \frac{p-1}{4} \int \Omega u^{p-2} |\nabla u|^2 \, dx + C_{15}.
\]
for some positive constant $C_{15}$. Inserting (3.40) into (3.36), we arrive at
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + \frac{p-1}{4} \int_{\Omega} u^{p-2} |\nabla u|^2 \, dx + \frac{\mu}{2} \int_{\Omega} u^{p+1} \, dx \leq C_{16}. \tag{3.41}
\]
Finally, integrating the above inequality with respect to $t$ yields
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_{17} \text{ for all } p \geq 1 \text{ and } t \in (0, T_{\max}) \tag{3.42}
\]
by using the Hölder inequality. The proof Lemma 3.4 is complete. □

**Lemma 3.5.** Let $r > 2$ and $(u, v, w)$ be a solution to (1.1) on $(0, T_{\max})$. Then for all $p > 1$, there exists a positive constant $C := C(p, |\Omega|, r, \mu, \xi, \chi, \beta)$ such that
\[
\int_{\Omega} u^p(x, t) \, dx \leq C \text{ for all } t \in (0, T_{\max}). \tag{3.43}
\]

**Proof.** Firstly, testing the first equation of (1.1) by $u^{l-1}$ and integrating over $\Omega$, we get
\[
\frac{1}{l} \frac{d}{dt} \|u\|_{L^l(\Omega)}^l + (l-1) \int_{\Omega} u^{l-2} |\nabla u|^2 \, dx
\]
\[
= -\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{l-1} \, dx - \xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{l-1} \, dx + \int_{\Omega} u^{l-1} (au - \mu u^r) \, dx. \tag{3.44}
\]
With the help of the Young inequality and $r > 2$, it reads that there exists a positive constant $C_1$ such that
\[
\int_{\Omega} \left( \frac{l+1}{l} u^l + u^{l-1} (au - \mu u^r) \right) \, dx
\]
\[
\leq \frac{l}{l+1} \int_{\Omega} u^l \, dx + a \int_{\Omega} u^l \, dx + \mu \int_{\Omega} u^{l+r-1} \, dx \tag{3.45}
\]
\[
\leq -\frac{l \mu}{8} \int_{\Omega} u^{l+r-1} \, dx + C_1.
\]
Next, integrating by parts to the first term on the right hand side of (3.44) and using the Young inequality, we obtain
\[
-\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{l-1} \, dx
\]
\[
\leq \frac{l-1}{l} \chi \int_{\Omega} u^l |\Delta v| \, dx
\]
\[
\leq \frac{\mu}{8} \int_{\Omega} u^{l+r-1} \, dx + C_2 \int_{\Omega} |\Delta v|^\frac{l+r-1}{l} \, dx
\]
\[
\leq \frac{\mu}{8} \int_{\Omega} u^{l+r-1} \, dx + \int_{\Omega} |\Delta v|^{l+1} \, dx + C_3. \tag{3.46}
\]
Next, in view of (3.3) and the Young inequality, we derive that there exist positive constants $C_4, C_5$ and $C_6$ such that
\[
-\xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{l-1} \, dx
\]
\[
\leq C_{\beta} \left( \frac{l-1}{l} \int_{\Omega} u^l (v + 1) + l \int_{\Omega} u^{l-1} |\nabla u| \right)
\]
\[
\leq C_4 \left( \int_{\Omega} u^l (v + 1) + l \int_{\Omega} u^{l-1} |\nabla u| \right)
\]
\[
\leq \frac{l-1}{2} \int_{\Omega} u^{l-2} |\nabla u|^2 + C_5 \int_{\Omega} u^l + C_5 \int_{\Omega} u^r \, dx
\]
\[
\leq \frac{l-1}{2} \int_{\Omega} u^{l-2} |\nabla u|^2 + \frac{\mu}{8} \int_{\Omega} u^{l+r-1} + \int_{\Omega} u^{l+1} + C_6. \tag{3.47}
\]
Thus, inserting (3.45)–(3.47) into (3.44), we get
\[
\frac{1}{t} \frac{d}{dt} ||u||_{L^t(\Omega)} + \frac{t-1}{2} \int_{\Omega} u^{t-2} |\nabla u|^2 \leq -\frac{5\mu}{8} \int_{\Omega} u^{l+r-1} dx - \frac{l+1}{t} \int_{\Omega} u^l dx + \int_{\Omega} |\Delta u|^{t+1} dx + \int_{\Omega} v^{l+1} + C_7. \tag{3.48}
\]

For any \( t \in (s_0, T_{max}) \), employing the variation-of-constants formula to (3.48), we conclude that
\[
\frac{1}{t} ||u(t)||_{L^t(\Omega)} \leq \frac{1}{t} e^{-(l+1)(t-s_0)} ||u(s_0)||_{L^t(\Omega)} - \frac{5\mu}{8} \int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} u^{l+r-1} dx ds \\
+ \int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} |\Delta u|^{t+1} dx ds \\
+ C_7 \int_{s_0}^{t} e^{-(l+1)(t-s)} ds + \int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} v^{l+1} dx ds \\
\leq -\frac{5\mu}{8} \int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} u^{l+r-1} dx ds \\
+ \int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} (|\Delta u|^{t+1} + v^{l+1}) dx ds + C_8,
\]
where
\[C_8 := \frac{1}{t} ||u(s_0)||_{L^t(\Omega)} + C_7 \int_{s_0}^{t} e^{-(l+1)(t-s)} ds.\]

Now, in light of Lemma 2.3, we conclude that for all \( t \in (s_0, T_{max}) \),
\[
\int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} (|\Delta u|^{t+1} + v^{l+1}) dx ds \\
= e^{-(l+1)t} \int_{s_0}^{t} e^{(l+1)s} \int_{\Omega} (|\Delta u|^{t+1} + v^{l+1}) dx ds \\
\leq e^{-(l+1)t} C_{t+1} \int_{s_0}^{t} \int_{\Omega} e^{(l+1)s} u^{l+1} dx ds + e^{(l+1)s_0} ||v(\cdot, s_0)||_{W^{l+1}_{2,t+1}}. \tag{3.50}
\]

Collecting (3.50) and (3.49) and using the Young inequality, we get
\[
\frac{1}{t} ||u(t)||_{L^t(\Omega)} \leq -\frac{5\mu}{8} \int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} u^{l+r-1} dx ds + C_{t+1} \int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} u^{l+1} dx ds \\
+ e^{-(l+1)(t-s_0)} C_{t+1} ||v(s_0, t)||_{W^{l+1}_{2,t+1}} + C_8 \\
\leq -\frac{\mu}{2} \int_{s_0}^{t} e^{-(l+1)(t-s)} \int_{\Omega} u^{l+r-1} dx ds + C_9 \tag{3.51}
\]
with
\[
C_9 = e^{-(l+1)(t-s_0)} C_{t+1} ||v(s_0, t)||_{W^{l+1}_{2,t+1}} \\
+ \frac{r-2}{r+l-1} \left( \frac{\mu r + l - 1}{8} \right)^{\frac{-l-1}{l+1}} C_{t+1} \frac{\mu}{8} \int_{s_0}^{t} e^{-(l+1)(t-s)} ds + C_8.
\]

Therefore, integrating (3.51) respect to \( t \) and using (2.10) yields
\[
||u(\cdot, t)||_{L^t(\Omega)} \leq C_{11} \text{ for all } l \geq 1 \text{ and } t \in (0, T_{max}) \tag{3.52}
\]
by the Hölder inequality. The proof of Lemma 3.5 is complete.

With Lemmata 3.4–3.5 and Lemma 2.4 at hand, we are now in the position to prove the main result by using a Moser-type iteration (see Lemma A.1 in [27]) and the standard estimate for Neumann semigroup. Indeed, we are going to prove Theorem 1.1 by three steps. To achieve this, in view of the standard estimate for Neumann semigroup, we first apply Lemmata 3.4–3.5 and Lemma 2.4 to improve the regularity of \( \nabla v \) from \( L^2(\Omega) \) to \( L^\infty(\Omega) \).

**Lemma 3.6.** Under the assumptions in Theorem 1.1, there is \( C > 0 \) such that

\[
\| \nabla v(\cdot, t) \|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}).
\]

**Proof.** Firstly, we derive that there exist positive constants \( q_0 > N \) and \( C_1 \) such that

\[
\| u(\cdot, t) \|_{L^{q_0}(\Omega)} \leq C_1 \text{ for all } t \in (0, T_{\text{max}})
\]

by Lemmata 3.4–3.5. Next, involving the standard estimate for Neumann semigroup provides \( C_2 \) and \( C_3 > 0 \) such that

\[
\| \nabla v(\cdot, t) \|_{L^\infty(\Omega)} \leq C_2 \int_{t_0}^{t} + \infty (t-s)^{-\alpha - \frac{N}{q_0}} e^{-\mu(t-s)} \| u(\cdot, s) \|_{L^{q_0}(\Omega)} ds + C_2 s_0^{-\alpha} \| v(\cdot, s_0) \|_{L^\infty(\Omega)}
\]

\[
\leq C_2 \int_{0}^{+\infty} (t-s)^{-\alpha - \frac{N}{q_0}} e^{-\mu(t-s)} ds + C_2 s_0^{-\alpha} \beta
\]

\[
\leq C_3 \text{ for all } t \in (0, T_{\text{max}})
\]

by (2.10).

Now we are in a position to prove boundedness of \( u \) by using a Moser-type iteration.

**Lemma 3.7.** Assume that the conditions in Theorem 1.1 is fulfilled. Then there exists a positive constant \( C \) such that the solution \( (u, v, w) \) of (1.1) satisfies

\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}).
\]

**Proof.** Testing the first equation in (1.1) by \( u^{p-1} \), integrating over \( \Omega \) and integrating by parts, we conclude that

\[
\frac{1}{p} \frac{d}{dt} \| u \|_{L^p(\Omega)}^p + (p-1) \int_{\Omega} u^{p-2} | \nabla u |^2 dx
\]

\[
= \chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v dx - \xi \int_{\Omega} \nabla \cdot (u \nabla w) u^{p-1} dx + \int_{\Omega} u^{p-1} (au - \mu w) dx.
\]

(3.55)

Due to (3.3) and (3.54) and the Young inequality, we derive that there exist positive constants \( C_1, C_2, C_3 \) and \( C_4 \) independent of \( p \) such that

\[
\chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v dx \leq \chi(p-1) C_1 \int_{\Omega} u^{p-1} | \nabla u | dx
\]

\[
\leq \frac{p-1}{4} \int_{\Omega} u^{p-2} | \nabla u |^2 + C_2 p \int_{\Omega} u^p
\]

(3.56)

and

\[
-\xi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla w) \leq C_3 \int_{\Omega} u^p (v+1) + p \int_{\Omega} u^{p-1} | \nabla u |)
\]

\[
\leq \frac{p-1}{4} \int_{\Omega} u^{p-2} | \nabla u |^2 + C_4 p \int_{\Omega} u^p.
\]

(3.57)
Hence, we may choose $\lambda$ large by the Young inequality. Here $s$ satisfying $C$ positive constants.

Collecting (3.59) and (3.58), we find that

$3374 \text{ LING LIU AND JIASHAN ZENG}$

In what follows, let $C_i (i \in \mathbb{N})$ denote some different constants, which are independent of $p$, and if no special explanation, they depend at most on $\Omega, \chi, \xi, \mu, u_0, v_0$ and $w_0$. Next, the Gagliardo–Nirenberg inequality ensure that there are $C_7$ and $C_8$ satisfying

$$C_0 p^2 \int_\Omega \frac{d}{dt} \|u\|_p^p + \int_\Omega u^p \leq C_0 p^2 \int_\Omega u^p.$$ (3.58)

by the Young inequality. Here

$$0 < \varsigma_1 = \frac{N - \frac{N}{2}}{1 - \frac{N}{2} + N} = \frac{N}{N + 2} < 1.$$ (3.59)

Collecting (3.59) and (3.58), we find that

$$\frac{d}{dt} \|u\|_{L^p(\Omega)}^p + \int_\Omega u^p \leq C_0 p^{2+N} \|u\|_{L^p(\Omega)}^2.$$ (3.60)

Now, picking $p_i = 2^i$ and putting $M_i = \max\{1, \sup_{t \in (0, T)} \int_\Omega \frac{\partial u}{\partial t}\}$ for $T \in (0, T_{\max})$ and $i = 1, 2, \cdots$, therefore, (3.60) easily entails that

$$\frac{d}{dt} \|u\|_{L^{p_i}(\Omega)}^{p_i} + \int_\Omega u^{p_i} \leq C_0 p^{2+N} M_i^2(T).$$ (3.61)

The comparison theorem for the above ODE yields that there exists a $\lambda > 1$ independent of $i$ such that

$$M_i(T) \leq \max\{\lambda^i M_i^2(T), \|\Omega\||u_0|_{L^\infty(\Omega)}^{p_i}\}. \quad (3.62)$$

Now, if $\lambda^i M_i^2(T) \leq \|\Omega\||u_0|_{L^\infty(\Omega)}^{p_i}$ for infinitely many $i \geq 1$, we get

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_0 \text{ for all } t \in (0, T),$$ (3.63)

where $C_0 = \|u_0\|_{L^\infty(\Omega)}$. Otherwise, if $\lambda^i M_i^2(T) > \|\Omega\||u_0|_{L^\infty(\Omega)}^{p_i}$ for all sufficiently large $i$, then by (3.62), we derive that

$$M_i(T) \leq \lambda^i M_i^2(T) \text{ for all sufficiently large } i.$$ (3.64)

Hence, we may choose $\lambda$ large enough such that

$$M_i(T) \leq \lambda^i M_i^2(T) \text{ for all } i \geq 1.$$ (3.65)
Thus, in light of a straightforward induction (see Lemma 3.12 of [30]) we have
\begin{align}
M_i(T) & \leq \lambda^i(\lambda^{i-1}M_{i-2}^2)^2 \\
& = \lambda^{i+2(i-1)}M_{i-2}^{2^2} \\
& \leq \lambda^{i+\Sigma_{j=1}^{i}(j-1)}M_0^{2^i}.
\end{align}
(3.66)

Taking $p_i$-th root on both sides leads to
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{10} \text{ for all } t \in (0, T_{\text{max}}) \]  
(3.67)
by taking $T \nearrow T_{\text{max}}$.

\textbf{The proof of Theorem 1.1} Firstly, with the above estimate in hand (see Lemma 3.6), we may establish
\[ \|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}) \]  
(3.68)
by using (3.5). Next, we see that the boundedness of $u$ as well as $v$ and $w$ follow from Lemmata 3.6–3.7 and (3.68). Thereupon the assertion of Theorem 1.1 is immediately obtained from Lemma 2.4.

\textbf{Acknowledgments.} This work is partially supported by the National Natural Science Foundation of China (No. 11601215), Shandong Provincial Science Foundation for Outstanding Youth (No. ZR2018JL005), Shandong Provincial Natural Science Foundation, China (No. ZR2016AQ17) and the Doctor Start-up Funding of Ludong University (No. LA2016006).

\textbf{REFERENCES}

[1] N. Bellomo, A. Belloquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.
[2] X. Cao, Boundedness in a three-dimensional chemotaxis–haptotaxis model, Z. Angew. Math. Phys., 67 (2015), Art. 11, 13 pp.
[3] M. A. J. Chaplain and G. Lolas, Mathematical modelling of cancer invasion of tissue: The role of the urokinase plasminogen activation system, Math. Models Methods Appl. Sci., 11 (2005), 1685–1734.
[4] M. A. J. Chaplain and G. Lolas, Mathematical modelling of cancer invasion of tissue: Dynamic heterogeneity, Net. Hetero. Med., 1 (2006), 399–439.
[5] T. Cieślak and C. Stinner, Finite-time blowup and global-in-time unbounded solutions to a parabolic–parabolic quasilinear Keller–Segel system in higher dimensions, J. Diff. Eqns., 252 (2012), 5832–5851.
[6] H. Hajaeiæj, L. Molinet, T. Ozawa and B. Wang, Necessary and sufficient conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier–Stokes and generalized boson equations, in: Harmonic Analysis and Nonlinear Partial Differential Equations, in: RIMS Kôkyûroku Bessatsu, Res. Inst. Math. Sci. (RIMS), Kyoto, 26 (2011), 159–175.
[7] T. Hillen and K. J. Painter, A use’s guide to PDE models for chemotaxis, J. Math. Biol., 58 (2009), 183–217.
[8] T. Hillen, K. J. Painter and M. Winkler, Convergence of a cancer invasion model to a logistic chemotaxis model, Math. Models Methods Appl. Sci., 23 (2013), 165–198.
[9] D. Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences, I. Jahresbericht der Deutschen Mathematiker-Vereinigung, 105 (2003), 103–165.
[10] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Diff. Eqns., 215 (2005), 52–107.
[11] S. Ishida, K. Seki and T. Yokota, Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains, J. Diff. Eqns., 256 (2014), 2993–3010.
[12] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Am. Math. Soc., 329 (1992), 819–824.
[13] E. Keller and L. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., 26 (1970), 399–415.
[14] X. Li and Z. Xiang, Boundedness in quasilinear Keller-Segel equations with nonlinear sensitivity and logistic source, Discrete Contin. Dyn. Syst., 35 (2015), 3503–3531.
[15] G. Litcanu and C. Morales-Rodrigo, Asymptotic behavior of global solutions to a model of cell invasion, Math. Models Methods Appl. Sci., 20 (2010), 1721–1758.
[16] J. Liu, J. Zheng and Y. Wang, Boundedness in a quasilinear chemotaxis-haptotaxis system with logistic source, Z. Angew. Math. Phys., 67 (2016), Art. 21, 33 pp.
[17] A. Marciniak-Czochra and M. Ptashnyk, Boundedness of solutions of a haptotaxis model, Math. Models Methods Appl. Sci., 20 (2010), 449–476.
[18] V. Nanjundiah, Chemotaxis, signal relaying and aggregation morpholog, J. Theor. Biol., 42 (1973), 63–105.
[19] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, Exponential attractor for a chemotaxis growth system of equations, Nonlinear Anal. TMA., 51 (2002), 119–144.
[20] J. Simon, Compact sets in the space $L^p(O,T;B)$, Annali di Matematica Pura ed Applicata, 146 (1986), 65–96.
[21] Y. Tao, Global existence of classical solutions to a combined chemotaxis–haptotaxis model with logistic source, J. Diff. Eqns., 252 (2012), 692–715.
[22] Y. Tao and M. Winkler, Boundedness and stabilization in a multi-dimensional chemotaxis–haptotaxis model, Proceedings of the Royal Society of Edinburgh, 144 (2014), 1067–1084.
[23] Y. Tao and M. Winkler, Domination of chemotaxis in a chemotaxis–haptotaxis model, Nonlinearity, 27 (2014), 1225–1239.
[24] Y. Wang, Boundedness in the higher-dimensional chemotaxis-haptotaxis model with nonlinear diffusion, J. Diff. Eqns., 260 (2016), 1975–1989.
[25] Y. Wang and Y. Ke, Large time behavior of solution to a fully parabolic chemotaxis-haptotaxis model in higher dimensions, J. Diff. Eqns., 260 (2016), 6960–6988.
GLOBAL EXISTENCE AND BOUNDEDNESS OF SOLUTION OF...

[41] M. Winkler, A critical blow-up exponent in a chemotaxis system with nonlinear signal production, *Nonlinearity*, 31 (2018), 2031–2056.

[42] M. Winkler, Finite-time blow-up in low-dimensional Keller–Segel systems with logistic-type superlinear degradation, *Z. Angew. Math. Phys.*, 69 (2018), Art. 69, 40 pp.

[43] M. Winkler and K. C. Djie, Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect, *Nonlinear Anal. TMA.*, 72 (2010), 1044–1064.

[44] T. Xiang, Boundedness and global existence in the higher-dimensional parabolic–parabolic chemotaxis system with/without growth source, *J. Diff. Eqns.*, 258 (2015), 4275–4323.

[45] Q. Zhang and Y. Li, Global boundedness of solutions to a two-species chemotaxis system, *Z. Angew. Math. Phys.*, 66 (2015), 83–93.

[46] J. Zheng, Optimal controls of multi-dimensional modified Swift-Hohenberg equation, *International Journal of Control*, 88 (2015), 2117–2125.

[47] J. Zheng, Boundedness of solutions to a quasilinear parabolic–elliptic Keller–Segel system with logistic source, *J. Diff. Eqns.*, 259 (2015), 120–140.

[48] J. Zheng, Boundedness of solutions to a quasilinear parabolic–parabolic Keller–Segel system with logistic source, *J. Math. Anal. Appl.*, 431 (2015), 867–888.

[49] J. Zheng, Boundedness and global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with nonlinear a logistic source, *J. Math. Anal. Appl.*, 450 (2017), 1047–1061.

[50] J. Zheng, A note on boundedness of solutions to a higher-dimensional quasi–linear chemotaxis system with logistic source, *Zeitschrift für Angewandte Mathematik und Mechanik*, 97 (2017), 414–421.

[51] J. Zheng, Boundedness in a two-species quasi-linear chemotaxis system with two chemicals, *Topological Methods in Nonlinear Analysis*, 49 (2017), 463–480.

[52] J. Zheng, Boundedness of solution of a higher-dimensional parabolic–ODE–parabolic chemotaxis–haptotaxis model with generalized logistic source, *Nonlinearity*, 30 (2017), 1987–2009.

[53] J. Zheng, Boundedness of solutions to a quasilinear higher-dimensional chemotaxis–haptotaxis model with nonlinear diffusion, *Discrete and Continuous Dynamical Systems*, 37 (2017), 627–643.

[54] J. Zheng, Global weak solutions in a three-dimensional Keller-Segel-Navier-Stokes system with nonlinear diffusion, *J. Diff. Eqns.*, 263 (2017), 2606–2629.

[55] J. Zheng, A new result for global existence and boundedness of solutions to a parabolic–parabolic Keller–Segel system with logistic source, *J. Math. Anal. Appl.*, 431 (2015), 867–888, arXiv:1712.00906, 2017.

[56] J. Zheng and Y. Wang, Boundedness of solutions to a quasilinear chemotaxis–haptotaxis model, *Comput. Math. Appl.*, 71 (2016), 1898–1909.

[57] P. Zheng, C. Mu and X. Song, On the boundedness and decay of solutions for a chemotaxis-haptotaxis system with nonlinear diffusion, *Disc. Cont. Dyna. Syst.*, 36 (2015), 1737–1757.

Received March 2018; 1st revision August 2018; 2nd revision August 2018.

E-mail address: liuling2004@sohu.com

E-mail address: zhengjiashan2008@163.com