Irreversibility of time for quasi-isolated systems

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A physical system is called quasi-isolated if it subject to small random uncontrollable perturbations. Such a system is, in general, stochastically unstable. Moreover, its phase-space volume at asymptotically large time expands. This can be described by considering the local expansion exponent. Several examples illustrate that the stability indices and expansion exponents of quasi-isolated systems are not only asymptotically positive but are asymptotically increasing. This means that the divergence of dynamical trajectories and the expansion of phase volume at large time occurs with acceleration. Such a strongly irreversible evolution of quasi-isolated systems explains the irreversibility of time.

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The evolution of all real physical systems in nature is, as is well known, irreversible. But the majority of physical laws is formulated as time-reversible dynamical equations. The problem of obtaining irreversible evolution of macroscopic systems from reversible deterministic dynamics goes back to Boltzmann [1] who connected irreversibility with monotonic growth of entropy in an isolated system evolving toward equilibrium. The irreversibility of time in quantum systems was attributed by von Neumann [2] to the measurement process. In the frame of a short communication, it is impossible to give a detailed description of all approaches to and nuances of the old and widely discussed problem of time irreversibility. A good survey of the problem can be found in recent reviews [3,4] and a thorough description of its history in book [5].

The approach, relating the irreversibility of the evolution laws of nature with the increase of entropy in isolated systems tending to equilibrium, has some weak points. First of all, there is no effective definition of increasing entropy for strongly nonequilibrium systems [6]. Then, internal chaotic dynamics in a real isolated system often does not provide finite relaxation time to equilibrium or fast decay of fluctuations, but instead displays a property of persistence of nonequilibrium [4]. Also, the irreversibility of natural evolution laws is not the privilege of only statistical systems characterized by entropy, even if the latter could be well defined. But it is rather a general law pertinent to all realistic systems, whether statistical or mechanical.

An important fact is that isolated systems as such are merely an abstraction. No real system can be absolutely isolated from its surroundings [7–12]. Even more, from the practical point of view, the concept of an isolated system is logically self-contradictory, since to realize the isolation, one needs to employ technical devices acting on the system; and to ensure that the latter is kept isolated, one must apply measuring instruments perturbing the system [13,14]. All real physical systems can be not more than quasi-isolated, that is, almost isolated but, anyway, subject to the action of uncontrollable random noise from their environment. Quasi-isolated systems are, in other words, quasi-open and as such, even being in a seemingly steady state, can experience, during their lifetime, large nonequilibrium fluctuations of entropy, density or temperature [15–17].

An accurate mathematical treatment of quasi-isolated systems has recently been advanced [18], from where it follows that these systems are, in general, stochastically unstable. In the present communication, this result is essentially strengthened in several aspects. In addition to the stability index, describing the divergence of trajectories of quasi-isolated systems [18], we introduce here the expansion exponent, for stochastic systems, characterizing the expansion (or contraction) of the average phase-space volume. The phase-volume expansion is, generally, a stronger characteristic than the trajectory divergence. A system may be unstable, though its phase volume be conserved, as it happens, e.g., for Hamiltonian systems [19]. But if the phase volume expands, the system is certainly unstable. It turns out that in quasi-isolated systems at large time not only trajectories diverge but also the phase volume expands. Moreover, the stability index and the expansion exponent not merely become positive at large time but even are increasing, which implies that the trajectory divergence and phase-volume expansion proceed with acceleration. This essentially irreversible asymptotic behaviour of quasi-isolated systems fixes the direction of time arrow. The increase of time may be connected with the increase of the expansion exponent. The strongly irreversible asymptotic evolution of quasi-isolated systems, and hence of all real systems, can be interpreted as the cause of time irreversibility.

In order to avoid misunderstanding, it is worth emphasizing the principal points of the approach, presented in this paper, which distinguishes it from all previous publications treating the irreversibility of time. A detailed discussion of the standard ways of considering the arrow of time, related to thermodynamic notions, can be found in Refs. [3–6,17–19].

1. First of all, the consideration below is based on studying the behaviour of dynamical systems but not that of statistical or thermodynamic systems. For the former, contrary to the latter, there is no need of dealing with entropy, entropy production, and other thermodynamic characteristics, as well as with the second law of thermodynamics. This different point of view looks more natural and general, since the irreversibility of time is the property common not solely for thermodynamic systems but for any real systems of this world.

2. A pivotal concept, employed in the paper, is that of quasi-isolated systems. Such a system is subject to the influence of infinitesimally weak uncontrollable random perturbations, so that at each instant of time the zero-noise limit yields a time-reversible system. The irreversibility comes into play not merely because of negligibly small stochastic perturbations but due to the noncommutativity of the zero-noise and large-time limits.

3. An important notion, the approach is based on, is the local expansion exponent \( X(t) \). For deterministic systems, there exists the limit

\[
\lim_{t \to \infty} \frac{d}{dt} X(t) = \sum_{n} \lambda_n ,
\]

in which the right-hand side is the sum of all Lyapunov exponents. However, for stochastic systems, such a relation in general does not exist, and even more, the Lyapunov exponents as such often are not well defined [23]. In addition,
for our purpose of analysing the noncommutativity of different limits, we need to consider the local temporal values of $X(t)$, but not simply the large-time limit. This is why the usage of the local expansion exponent is of principal importance.

The evolution of physical systems is usually described by partial differential equations. Let $x \in \mathbb{D} \subset \mathbb{R}^d$ be a $d$-dimensional set of continuous variables on a domain $\mathbb{D}$ and $t \in \mathbb{R}_+ \equiv [0, \infty)$ denote time. Let a discrete index $i \in \mathbb{N}_+ \equiv \{1, 2, \ldots\}$ enumerate dynamical states. In what follows, we employ the matrix notation [18,20] treating the pair $\{i, x\}$ as a point in the label space $\mathbb{N}_+ \times \mathbb{D}$. Then the stochastic field $\xi(t) = [\xi_i(x, t)]$ is a column with respect to the multi-index $\{i, x\}$. The dynamical state $y(\xi, t) = [y_i(x, \xi, t)]$ is also a column with respect to $\{i, x\}$, as is the vector field $v(y, \xi, t) = [v_i(x, y, \xi, t)]$. The evolution of a dynamical system, subject to the action of noise, is presented by a stochastic differential equation

$$\frac{d}{dt} y(\xi, t) = v(y, \xi, t),$$

(1)

with an initial condition $y(\xi, 0) = y(0)$ and given boundary conditions. Here and everywhere below, the stochastic differential equations are understood in the sense of Stratonovich [21]. As the main measurable quantity, we consider the average trajectory

$$y(t) = [y_i(x, t)] \equiv \ll y(\xi, t) \gg,$$

(2)

with the averaging accomplished over the stochastic field.

Important characteristics of motion are the stochastic multiplier matrix $\hat{M}(\xi, t) = [M_{ij}(x, x', \xi, t)]$, with the elements

$$M_{ij}(x, x', \xi, t) \equiv \frac{\delta y_i(x, \xi, t)}{\delta y_j(x', 0)},$$

(3)

and the stochastic Jacobian matrix $\hat{J}(\xi, t) = [J_{ij}(x, x', \xi, t)]$, whose elements are

$$J_{ij}(x, x', \xi, t) \equiv \frac{\delta v_i(x, y, \xi, t)}{\delta y_j(x', \xi, t)}.$$

(4)

These matrices are connected through the equation

$$\frac{d}{dt} \hat{M}(\xi, t) = \hat{J}(\xi, t) \hat{M}(\xi, t),$$

(5)

following from the variation of Eq. (1). Equation (5) is to be complemented with the initial condition $\hat{M}(\xi, 0) = \hat{1} = [\delta_{ij} \delta(x - x')]$ and the boundary conditions resulting from the variation of those for the dynamical state (see details in Ref. [18]).

The stability of the system is characterized by considering the behaviour of the trajectory deviation $||\delta y(t)||$ caused by an infinitesimal variation of initial conditions. Here the Hermitian vector norm

$$||y(t)|| = \left[ \sum_i \int y_i^*(x, t) y_i(x, t) \, dx \right]^{1/2}$$

is assumed. The local stability index is defined [18] as

$$\sigma(t) \equiv \ln \sup_{\delta y(0)} \frac{||\delta y(t)||}{||\delta y(0)||}.$$

(6)

This shows the trajectory deviation $||\delta y(t)|| \sim ||\delta y(0)||^{\sigma(t)}$ at a given time $t$. The stability index (6) can be expressed [18] through the multiplier matrix as

$$\sigma(t) = \ln \|| \ll \hat{M}(\xi, t) \gg ||,$$

(7)

where the Hermitian norm of the matrix is meant.

To describe the behaviour of the phase-space volume, we need, first, to define [22] the elementary phase volume

$$\delta \Gamma(t) = \prod_i \prod_x \delta y_i(x, t),$$

(8)
where the continuous product over $x$ is specified in Ref. [15]. Now, we introduce the local expansion exponent

$$ X(t) \equiv \ln \left| \frac{\delta \Gamma(t)}{\delta \Gamma(0)} \right|, \quad (9) $$

which determines the temporal behaviour of the phase volume $||\delta \Gamma(t)|| \sim ||\delta \Gamma(0)|| e^{X(t)}$. When $X(t) < 0$, the phase volume at time $t$ contracts; if $X(t) = 0$, the volume is preserved; and when $X(t) > 0$, it expands. The expansion exponent (9) can be related to the multiplier matrix as $X(t) = \text{Tr} \hat{L}(t)$, with

$$ \hat{L}(t) = \ln | \ll M_{ij}(x, x', \xi, t) \gg |. $$

For a matrix $\hat{A}$, we have the identity $\text{Tr} \ln \hat{A} = \ln \text{det} \hat{A}$. Employing the latter, we obtain

$$ X(t) = \ln | \det \ll \hat{M}(\xi, t) \gg |. \quad (10) $$

Equations (7) and (10) can be simplified invoking the diagonal representation for the multiplier matrix, when

$$ M_{mn}(\xi, t) = \delta_{mn} \mu_n(\xi, t), \quad (11) $$

with $m$ and $n$ being the appropriate multi-indices [18].

Quasi-isolated systems are those that are subject to the action of a weak stochastic noise modelling the random uncontrollable perturbations from the environment [18]. To have the possibility for varying the noise amplitude, it is convenient to affix to the stochastic field $\xi(t)$ an explicit factor $\alpha$ whose value could be regulated, for which we change $\xi(t)$ by $\alpha \xi(t)$. Then the stability index (7) and expansion exponent (10) acquire the dependence on $\alpha$. For instance, in the diagonal representation (11), we get the stability index

$$ \sigma_\alpha(t) = \sup_n \ln | \ll \mu_n(\alpha \xi, t) \gg | \quad (12) $$

and the expansion exponent

$$ X_\alpha(t) = \sum_n \ln | \ll \mu_n(\alpha \xi, t) \gg |. \quad (13) $$

To illustrate explicitly the similarities and differences between the stability index and expansion exponent, let us consider several examples of stochastic systems, for which the multiplier matrix can be calculated exactly. To simplify calculations, we treat the stochastic field $\xi(t)$ as a scalar white noise centered at zero, with the averages

$$ \ll \xi(t) \gg = 0, \quad \ll \xi(t) \xi(t') \gg = 2\gamma \delta(t-t'), \quad (14) $$

where $\gamma > 0$. The technique of such calculations was explained in detail in Ref. [18]. Therefore, here we will not write down intermediate manipulations but shall present only the results.

An oscillatory process in stochastic background is described by the equation

$$ \frac{dy}{dt} = (i\omega + \alpha \xi) y, \quad (15) $$

where $y = y(\alpha \xi, t)$, $\xi = \xi(t)$, and $\omega$ is a real frequency. If $\alpha$ is real-valued, then the stochastic term corresponds to the attenuation-generation noise. For the local multiplier, we get

$$ \mu(\alpha \xi, t) = \exp \left\{ i\omega t + \alpha \int_0^t \xi(t') dt' \right\}, $$

which yields

$$ \sigma_\alpha(t) = X_\alpha(t) = \alpha^2 \gamma t. \quad (16) $$

The divergence of trajectories for stochastic systems is not necessarily exponential but can be of algebraic law [23]. For this to occur, one has to consider slightly more complicated equations or a coloured noise. Here we give an explicit example of the arising algebraic equation with white noise. Consider the process
\[
\frac{dy}{dt} = v_0 + \frac{\alpha \xi}{\sqrt{1+t}} y, 
\]
where \(v_0 = v_0(t)\) does not include \(y\) and \(\alpha\) is again real. The local multiplier is

\[
\mu(\alpha \xi, t) = \exp \left\{ \alpha \int_0^t \frac{\xi(t')}{\sqrt{1+t'}} \, dt' \right\} .
\]

The stochastic averaging results in the power law

\[
\ll \mu(\alpha \xi, t) \gg = (1 + t)^{\alpha^2 \gamma},
\]
characterizing the algebraic divergence of the system trajectories. The stability index and expansion exponent become

\[
\sigma_\alpha(t) = X_\alpha(t) = \alpha^2 \gamma \ln(1 + t).
\]

In the above examples, the stability index and expansion exponent coincide because of the one-dimensionality of the considered dynamical systems. The situation is different for higher-dimensional systems. As an example, let us examine the stochastic Schrödinger equation

\[
\frac{\partial \psi}{\partial t} = (-iH + \alpha \xi) \psi,
\]
in which \(\psi = \psi(r, \alpha \xi, t); r \in \mathbb{R}^3; H = H(r)\) is a Hamiltonian, and the Planck constant \(\hbar \equiv 1\). The stochastic term, with a real-valued \(\alpha\), imitates an attenuation-generation influenced by a random surrounding. The multiplier matrix, in the diagonal representation (11), possesses the elements

\[
\mu_n(\alpha \xi, t) = \exp \left\{ -iE_n t + \alpha \int_0^t \xi(t') \, dt' \right\},
\]
where \(E_n\) are the eigenvalues of the Hamiltonian, defined by the stationary problem \(H \psi_n = E_n \psi_n\), with a multi-index \(n\) enumerating quantum states. Denoting the total number of states as \(N = \sum_n 1\), we find the stability index and expansion exponent,

\[
\sigma_\alpha(t) = \alpha^2 \gamma t, \quad X_\alpha(t) = \alpha^2 N \gamma t,
\]
which differ by the factor \(N\).

As another case of an infinite-dimensional dynamical system, let us treat the stochastic diffusion equation

\[
\frac{\partial y}{\partial t} = (D + \alpha \xi) \frac{\partial^2 y}{\partial x^2},
\]
in which \(y = y(x, \alpha \xi, t); x \in [0, 1]; D > 0\). Since the diffusion constant \(D\) plays the role of an attenuation parameter, the stochastic term with a real \(\alpha\) again models the relaxation-generation impact of a random environment. Besides an initial condition, Eq. (22) is to be supplemented by the boundary conditions, which we take in the form \(y(0, \alpha \xi, t) = b_0\) and \(y(1, \alpha \xi, t) = b_1\), where \(b_0\) and \(b_1\) are constants. The diagonal representation for the multiplier matrix gives

\[
\mu_n(\alpha \xi, t) = \exp \left\{ -D(\pi n)^2 t - \alpha(\pi n)^2 \int_0^t \xi(t') \, dt' \right\},
\]
where \(n = 1, 2, \ldots, N\), with \(N \to \infty\). From here, we find the stability index (12) and expansion exponent (13),

\[
\sigma_\alpha(t) \simeq -\delta_{\alpha 0} D \pi^2 t + (1 - \delta_{\alpha 0}) \alpha^2 (\pi N)^4 \gamma t,
\]
\[
X_\alpha(t) \simeq -\frac{\pi^2}{3} N^3 Dt + \alpha^2 \frac{\pi^4}{5} N^5 \gamma t,
\]
whose forms are rather different.

Analysing the cases, considered above, resulting in Eqs. (16), (19), (21), and (23), we see the following. The limits \(\alpha \to 0\) and \(t \to \infty\) do not commute for the stability index \(\sigma_\alpha(t)\), hence these quasi-isolated systems are stochastically
unstable, as discussed in Ref. [18]. In addition to this, we find that these limits do not commute also for the expansion exponent $X_\alpha(t)$. Moreover, for the commutator of the limits, we have

$$[\lim_{\alpha \to 0}, \lim_{t \to \infty}] \sigma_\alpha(t) = \infty, \quad [\lim_{\alpha \to 0}, \lim_{t \to \infty}] X_\alpha(t) = \infty.$$  

(24)

This happens because the deterministic systems, obtained when switching off stochastic terms by setting $\alpha \to 0$, are either stable and phase-volume contracting or neutrally stable and phase-volume preserving, while in the presence of arbitrary small stochastic terms, the quasi-isolated systems become unstable and phase-volume expanding, such that

$$\sigma_\alpha(t) \to \infty, \quad X_\alpha(t) \to \infty \quad (\alpha \neq 0, \ t \to \infty).$$  

(25)

Repeating the same arguments as in Ref. [18], it can be inferred that the asymptotic property (25) holds true for quasi-isolated systems in general.

It is important to emphasize that such an instability, as is described above, certainly happens not for any weak noise. One could easily find examples when a weak external noise, vice versa, would suppress chaotic behaviour, thus, stabilizing the considered system. This may occur, e.g., for some cases of additive noise. As is explained in Ref. [18], the existence of multiplicative noise is a paramount requirement for the appearance of stochastic instability. It is also clear that not any multiplicative noise would produce instability. But we must always keep in mind that the central feature of a quasi-isolated system is the presence of noise that, though being weak, is also uncontrollable. The latter implies that, analysing the stochastic stability of a given system, we are obliged to check the stability with respect to all admissible types of noise and to select among those the worst situation, that is, the noise which results in maximal instability. This maximization of instability is what is meant when terming the noise as uncontrollable. And the existence of infinitesimally weak uncontrollable perturbations is in the heart of the definition of quasi-isolated systems [18]. This principal feature has always been silently assumed when choosing the type of perturbations for the examples of the present paper, so that the most unstable case has always been treated. By definition, uncontrollable includes the worst possible, which in the present context means the least stable.

In this way, quasi-isolated systems not only are stochastically unstable but their phase-space volume asymptotically expands. What is more, both the stability index and the expansion exponent are increasing at large time. This means that the divergence of trajectories and phase-volume expansion occur with acceleration. To imagine this picture, one may keep in mind the accelerated expansion of Universe [24,25]. The asymptotic increase of the stability index and expansion exponent for quasi-isolated systems shows that the evolution of the latter is essentially irreversible. Actually, the increase of the stability index, expansion exponent, and time occurs, according to the law (25), simultaneously. Since all real-world systems are not more than quasi-isolated, the irreversibility of time is a general law of nature.

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