Stability of non-topological string
in supersymmetric $SU(2) \times U(1)$ gauge theory

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Abstract

We construct a non-topological string solution for a supersymmetric gauge theory with $SU(2) \times U(1)$ gauge symmetry which is spontaneously broken to $U(1)$ by developing the vacuum expectation value of two doublet Higgses. It is a supersymmetric extension of the electroweak string while supersymmetry is unbroken. We discuss the classical stability of the non-topological string by perturbations. We show that the classical stability are determined only by two parameters, and that the allowed region becomes essentially the same as in the electroweak string.
I. INTRODUCTION

Since the gravitational waves from black hole mergers were observed at LIGO\(^1\), examining various models on elementary particle physics by primordial gravitational waves have been actively discussed\(^2\text{–}^5\). One of them is to obtain evidence of symmetry breaking by observing gravitational waves from cosmic strings\(^4\text{,}^5\), which are produced during symmetry breaking\(^6\text{,}^7\). Recently, NANOGrav experiment reported the candidate of gravitational waves\(^8\), which is consistent with the signal of gravitational waves from cosmic strings\(^9\text{–}^11\). This signal suggests the symmetry breaking scale is around 10\(^{14\text{–}16}\) GeV\(^11\text{,}^12\). Therefore, cosmic strings research becomes increasingly important.

However, most of studies on cosmic strings are concerned with topological strings, such as the Nielsen-Olesen strings\(^15\) and \(Z_2\) strings\(^16\), whose stability is guaranteed by topology of the vacuum and few are concerned with non-topological strings\(^17\text{,}^18\) which are quantumly unstable. One of the most commonly studied non-topological strings is the electroweak string\(^17\), which had been discussed to be produced in breaking the electroweak symmetry \(SU(2)_L \times U(1)_Y\) in the standard model (SM). The idea of the electroweak string has been proposed in Ref. \(^19\) by Nambu, and the realistic solution as a cosmic string in the electroweak phase transition has been shown in Ref. \(^20\) by Vachaspati. The classical stability is determined only by two parameters, the Weinberg angle \(\sin \theta_W\) and the ratio of Higgs mass to \(Z\) boson mass \(\beta \equiv m_H^2/m_Z^2\), and the parameter region, in which the electroweak string becomes classically stable, has been numerically calculated in Ref. \(^21\text{,}^22\). Unfortunately, the electroweak string becomes unstable even classically with the measured parameters in the SM. However, this is not the case for the models beyond the SM, in which the parameters have not been measured yet.

In this paper, we examine a non-topological string in the supersymmetric (SUSY) \(SU(2) \times U(1)\) gauge theory. This is nothing but a SUSY extension of the electroweak string, while the SUSY breaking is neglected because we consider the physics at much higher scale than the electroweak scale such as the grand unified theories with \(SO(10)\)\(^23\text{–}^26\) or \(E_6\)\(^27\text{–}^30\) unified group. We just embed the electroweak string configuration into the SUSY \(SU(2) \times U(1)\) gauge theory which has two doublet Higgs multiplets. We will show that the stability of this non-topological string can be determined only by two parameters, which are essentially the *The tension between NANOGrav and PPTA observation\(^13\) is known in the discussion of gravitational waves from simple topological string. Several ideas to avoid this tension have been discussed\(^10\text{,}^14\).*
same as in the electroweak string. It is obvious that the dangerous mode, which determine
the allowed parameter region in the electroweak string, exists also in this SUSY extension of
the electroweak string. Therefore, there is no possibility to expand the allowed parameter
region. The question is whether the allowed region becomes smaller or not. We will show
that the allowed region of this non-topological string is the same as that of the electroweak
string. Actually, we will show that all other modes do not destabilize the non-topological
string. This is the main result of this paper.

In section 2, we review the electroweak string which can be constructed by embedding
the Nielsen-Olesen string into the $SU(2) \times U(1)$ gauge theory. The classical stability con-
ditions are reminded. In section 3, we construct the non-topological string by imposing the
similar ansatz to the electroweak string on the SUSY $SU(2) \times U(1)$ gauge theory with two
doublet Higgs multiplets. In section 4, we study the classical stability conditions of the
non-topological string. The summary and discussion are devoted in section 5.

II. REVIEW OF ELECTROWEAK STRING

First, we will review the electroweak string since we will discuss the SUSY extension
of it. The electroweak string can be constructed in a gauge theory with $SU(2)_L \times U(1)_Y$
 gauge symmetry which is spontaneously broken by developing the vacuum expectation value
(VEV) of a doublet Higgs. This theory is similar to the standard model.

A. Brief review of Nielsen-Olesen string

First of all, we review the Nielsen-Olesen string[15] which is constructed in an $U(1)$ gauge
theory because the electroweak string can be constructed by embedding the Nielsen-Olesen
string into $SU(2)_L \times U(1)_Y$ gauge theory.

Let us consider an $U(1)$ gauge theory with a charged complex scalar $\phi(x)$ whose La-
grangian $\mathcal{L}$ is given as

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + (D_\mu \phi)^* D_\mu \phi - V(\phi),$$

where $F_{\mu \nu}$ is the field strength of the $U(1)$ gauge field $A_\mu(x)$ and

$$D_\mu \phi \equiv \partial_\mu \phi - ie A_\mu \phi.$$
Here, $e$ is the $U(1)$ gauge coupling constant and the potential $V(\phi)$ is written by

$$V(\phi) = \lambda \left( \phi^* \phi - \frac{v^2}{2} \right)^2 \quad (\lambda > 0, v > 0).$$

Without loss of generality, the global minimum of the potential is given as

$$\phi_v(x) = \frac{v}{\sqrt{2}},$$

which becomes classical solution of the equations of motion with

$$A_{\nu\mu}(x) = 0.$$

These give the minimum value of the energy of this system. Actually, the energy for static solutions is given as

$$E = \int d^3x \left[ \frac{1}{4} F_{ij} F_{ij} + (D_i \phi)^* D_i \phi + V(\phi), \right]$$

and the above solution give the minimum energy $E = 0$.

Let us explain another stable and static classical solution with $E > 0$, which was found by Nielsen and Olesen [15]. The solutions with the translational symmetry in $z$ direction are

$$\phi_s(x) = f(r)e^{in\theta} \quad (n \in \mathbb{Z}\{0\})$$

$$\vec{A}_s(x) = \frac{na(r)}{r} \vec{e}_\theta,$$

where $f(r)$ and $a(r)$ are real valued functions of $r$ with the following boundary conditions:

$$f(0) = a(0) = 0$$

$$f(\infty) = \frac{v}{\sqrt{2}}, \quad a(\infty) = \frac{1}{e}.$$

Here, we use cylindrical coordinates $(r, \theta, z)$ whose unit vectors are $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$. The equations of motion are given by

$$f''(r) + \frac{f'(r)}{r} - n^2 f(r) \left(1 - ea(r)\right)^2 + 2\lambda \left( \frac{v^2}{2} - f^2(r) \right) f(r) = 0,$$

$$a''(r) - \frac{a'(r)}{r} + 2enf^2(r) \left(1 - ea(r)\right) = 0,$$

where $f'(r) \equiv \frac{df}{dr} f(r)$. Note that satisfying the equations of motion is a necessary condition to obtain stable solutions. These are the Nielsen-Olesen string solutions.
It is guaranteed by topological feature of the moduli space that the Nielsen-Olesen string solutions do not decay to the global minimum. Generally, if the first homotopy group to the moduli of the breaking $G \to H$, $\pi_1(G/H)$, is non-trivial, a stable string solution appears. The first homotopy group of the breaking $U(1) \to \times$ becomes non-trivial as $\pi_1(U(1)) = \mathbb{Z}$. A string whose stability is guaranteed by topology is called a topological string, while the stability of a non-topological string is not guaranteed by topology, i.e., the first homotopy group to the moduli becomes trivial as $\pi_1(G/H) = 1$.

B. Electroweak string

One of the most interesting examples of non-topological strings is the electroweak string, which may appear in breaking $SU(2) \times U(1) \to U(1)$ by a doublet Higgs, although any topological strings cannot appear, i.e., $\pi_1(SU(2) \times U(1)/U(1)) = 1$. The stability of the electroweak string has been discussed in Ref. [21, 22], and unfortunately, for the measured parameters in the SM, the electroweak string becomes unstable.

However, this may not the case for the models beyond the standard model. Therefore, the produced string produces gravitational waves which may be observed in near future.

In this subsection, we will review the electroweak string briefly. First, let us consider the $SU(2)_L \times U(1)_Y$ gauge theory with a doublet Higgs with $U(1)_Y$ charge $1/2$, whose Lagrangian is given as

$$\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu H)^\dagger D^\mu H + \mu^2 H^\dagger H - \lambda (H^\dagger H)^2,$$

(13)

where $W^a_{\mu\nu}$ and $F_{\mu\nu}$ are the field strengths of $SU(2)_L$ and $U(1)_Y$, respectively. The moduli space of $H$ becomes

$$\mathcal{V}_{EW} = \left\{ H \left| H^\dagger H = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \right. \right\},$$

(14)

where $v > 0$. Since $\pi_1(\mathcal{V}_{EW})$ is trivial, a topological string does not appear in this model.

When the VEV of $H$ becomes

$$H = \left( \begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array} \right),$$

(15)
the kinetic term of the Higgs field gives the mass terms of the gauge fields as

\[
(D_\mu H)^\dagger D^\mu H \supset \left( \frac{v}{\sqrt{2}} \right) \left( g W_\mu^a \frac{\sigma^a}{2} + g' B_\mu \frac{1}{2} \right)^2 \left( \frac{v}{\sqrt{2}} \right) \\
= \frac{v^2}{2} \left[ \frac{g^2}{4} W_\mu^a W^{a\mu} - \frac{gg'}{2} W_\mu^3 B^\mu + \frac{g'^2}{4} B_\mu B^\mu \right] \\
= \frac{g^2 v^2}{8} W_\mu^a W^{a\mu} + \frac{\alpha^2 v^2}{8} \left( \frac{g}{\alpha} W_\mu^3 - \frac{g'}{\alpha} B_\mu \right) \left( \frac{g}{\alpha} W^{3\mu} - \frac{g'}{\alpha} B^\mu \right) \quad (\bar{a} = 1, 2),
\]

where \( g \) and \( g' \) are the gauge coupling constants of \( SU(2)_L \) and \( U(1)_Y \), respectively, \( \alpha \equiv \sqrt{g^2 + g'^2} \), and \( \sigma^a/2 \) are generators of \( SU(2)_L \). The \( W \) boson and the \( Z \) boson as \( Z \equiv \cos \theta_W W_\mu^3 - \sin \theta_W B_\mu \) obtain the masses \( gv/2 \) and \( \alpha v/2 \), respectively. Here \( \cos \theta_W = g/\alpha \).

On the other hand, the photon, \( A_\mu = \sin \theta_W W_\mu^3 + \cos \theta_W B_\mu \), remains massless, which is the gauge field corresponding to unbroken \( U(1)_{EM} \).

Let us consider \( U(1)_L \) whose generator is \( \sigma_3/2 \), instead of \( SU(2)_L \). The VEV of Higgs breaks \( U(1)_L \times U(1)_Y = U(1)_Z \times U(1)_{EM} \) into \( U(1)_{EM} \). Since the first homotopy group for this breaking is non-trivial, a topological string appears. This is nothing but the Nielsen-Olesen string for \( U(1)_Z \) gauge theory with gauge coupling constant \(-\alpha/2\). When this Nielsen-Olesen string solution is embedded in the \( SU(2)_L \times U(1)_Y \) gauge theory as

\[
H(x) = \begin{pmatrix} 0 \\ \phi_s(x) \end{pmatrix}, \quad Z_\mu(x) = A_{s\mu}(x) \\
A_\mu(x) = W^\bar{a}_\mu(x) = 0,
\]

these satisfy the equations of motion. This is the electroweak string solution.

\[ \text{(17)} \]

\[ \text{C. Stability of electroweak string} \]

Although the electroweak string becomes a classical solution, it is not obvious that this solution is classically stable or not. The stability can be determined by the presence or absence of perturbation modes which make the energy lower.

Let us consider the perturbations from the \( n = 1 \) electroweak string

\[
H(x) = \begin{pmatrix} 0 \\ f(r)e^{i\theta} \end{pmatrix}, \quad Z_0(x) = 0, \quad \bar{Z}(x) = -\frac{z(r)}{r \bar{\varepsilon}_\theta} \\
A_\mu(x) = W^\bar{a}_\mu(x) = 0
\]

\[ \text{(18)} \]
as
\[ H(x) = \begin{pmatrix} h(x) \\ f(r) e^{i\theta} + \delta \phi(x) \end{pmatrix}, \quad Z_0(x) = \delta Z_0(x), \quad \vec{Z}(x) = -\frac{z(r)}{r} \vec{e}_\theta + \delta \vec{Z}(x) \]
\[ A_\mu(x) = A_\mu(x), \quad W_\mu^a(x) = W_\mu^a(x). \] (19)

Since the \( t \) and \( z \) dependence of the perturbations and non-vanishing \( t \) and \( z \) component gauge fields increases the energy, we take the perturbations independent of \( t \) and \( z \), and we ignore the \( t \) and \( z \) component gauge fields. Thus, the energy linear density (string tension) \( \mu \) becomes
\[ \mu = \int r dr d\theta \left[ \frac{1}{4} (W^a_{ij})^2 + \frac{1}{4} (F_{ij})^2 + (D_t H)^\dagger D_t H - \mu^2 H^\dagger H + \lambda (H^\dagger H)^2 \right] \] (\( i, j = 1, 2 \)). (20)

Substituting the electroweak string solution [18] into (20), the string tension becomes
\[ \mu_0 = \int r dr d\theta \left[ \frac{z'}{2r^2} + f^2 + \left( 1 - \frac{\alpha}{2} z \right) \frac{f^2}{r^2} + \lambda \left( f^2 - \frac{v^2}{2} \right)^2 \right], \] (21)
which is the tension of the electroweak string. Since the electroweak string solution satisfies the classical equations of motion, the leading terms of the variation of the string tension are quadratic terms of these perturbation modes \( \delta \phi(x), \delta Z_\mu(x), h(x), A_\mu(x), W_\mu^a(x) \). Because of conservation of \( U(1)_{EM} \) charge, the quadratic terms of neutral part \( \delta \mu, \delta Z, \delta A \) and charged part \( \delta \mu, \delta Z, \delta A \) are separated, i.e., \( \delta \mu = \mu - \mu_0 \sim \delta \mu, \delta Z, \delta A \) + \( \delta \mu, \delta Z, \delta A \). The neutral part \( \delta \mu, \delta Z, \delta A \) must be non-negative because this part is the same as the Nielsen-Olesen string which is stable. It has been also shown numerically in Ref. [31]. Therefore, we have to examine whether \( \delta \mu, \delta Z, \delta A \)
\[ \delta \mu, \delta Z, \delta A = \int r dr d\theta \left[ \left( \partial_i - i \frac{\alpha \cos 2\theta_w}{2} Z_i \right) h \right]^2 + 2\lambda \left( f^2 - \frac{v^2}{2} \right) h^\dagger h \\
+ i \frac{\alpha \cos \theta_w}{2} W_i^a \left( H^\dagger \sigma^a d_i H - (d_i H)^\dagger \sigma^a H \right) \\
+ \frac{g^2}{4} f^2 \vec{W}^a \cdot \vec{W}^a + \alpha \cos^2 \theta_w \left( \vec{W}^1 \times \vec{W}^2 \right) \cdot \left( \nabla \times \left( -\frac{z(r)}{r} \vec{e}_\theta \right) \right) \\
+ \frac{1}{2} \left| \nabla \times \vec{W}^1 + \alpha \cos^2 \theta_w \vec{W} \vec{W}^2 \times \left( -\frac{z(r)}{r} \vec{e}_\theta \right) \right|^2 \\
+ \frac{1}{2} \left| \nabla \times \vec{W}^2 - \alpha \cos^2 \theta_w \vec{W} \vec{W}^1 \times \left( -\frac{z(r)}{r} \vec{e}_\theta \right) \right|^2 \right] \] (22)
Thus, the variation of string tension by these deformations becomes

\[
\delta \mu_{h,W} = 2\pi \int r dr \sum_{m} [\varepsilon_{\chi,m} + \varepsilon_{c,m} + \varepsilon_{W,m}] \quad (26)
\]

\[
\varepsilon_{\chi,m} = |\chi'|^2 + \left\{ \frac{1}{r^2} \left( m + \frac{\alpha \cos 2\theta_W}{2} \right)^2 + 2\lambda \left( f^2 - \frac{v^2}{2} \right) \right\} |\chi_m|^2 \quad (27)
\]

\[
\varepsilon_{c,m} = \frac{\alpha \cos \theta_W}{4} \times \left\{ (\chi^*_m f' - \chi^*_m f) \left( iW^1_{|1-m|} + W^2_{|1-m|} + \left( -W^1_{|1-m|} + iW^2_{|1-m|} \right) \text{sgn}(1-m) \right) 
+ (\chi_m f' - \chi_m f) \left( -iW^1_{|m-1|} + W^2_{|m-1|} + \left( W^1_{|m-1|} + iW^2_{|m-1|} \right) \text{sgn}(m-1) \right) \right\} \quad (28)
\]

\[
\varepsilon_{W,m} = \frac{g^2 f^2}{8} \left\{ (W^a_m)^2 + \frac{1}{r^2} (\omega^a_m)^2 \right\} + \frac{\alpha \cos^2 \theta_W z'}{2r^2} (W^2_m \omega^1_m - W^1_m \omega^2_m) 
+ \frac{1}{4r^2} \left\{ \left( mW^1_m - \omega^1_m + \alpha \cos^2 \theta_W z W^2_m \right)^2 + \left( mW^2_m + \omega^2_m + \alpha \cos^2 \theta_W W^1_m \right)^2 \right\} 
+ \left( W^1_m \rightarrow \overline{W^1}_m, W^2_m \rightarrow -\overline{W^2}_m, \omega^1_m \rightarrow -\overline{\omega^1}_m, \omega^2_m \rightarrow \overline{\omega^2}_m \right). \quad (29)
\]

In the above equations, for sets of modes \( \chi_m, W^a_m, |\omega^a_m|, |\overline{W^a}_m|, |\overline{\omega^a}_m| \) (m \( \in \mathbb{Z} \)), the sets with different \( m \) are independent of each other. Among these sets, the set with \( m = 0 \) must includes the most dangerous modes for stability because the main negative contribution from the second term in eq. (27) becomes positive around \( r \sim 0 \) unless \( m \neq 0 \). Therefore,
we fix $m = 0$ in the following discussion, and for simplicity, we take $\chi \equiv \chi_0$, $W^a \equiv W_1^a$, $\omega^a \equiv \omega_1^a$, $W^d \equiv W_1^d$, $\omega^d \equiv \omega_1^d$.

Moreover, it is seen that

$$
\delta \mu_{h,W} \equiv \mu_{\text{even}}(\text{Re} \chi, W^a, \omega^a) + \mu_{\text{odd}}(\text{Im} \chi, \overline{W^a}, \overline{\omega^a}),
$$

(30)

where $\text{Re} \chi, W^a, \omega^a$ are CP even modes while $\text{Im} \chi, \overline{W^a}, \overline{\omega^a}$ are CP odd modes. Since $\mu_{\text{even}}$ is the same functional as $\mu_{\text{odd}}$, it is sufficient to study one of the two functionals, for example, $\mu_{\text{even}}$, in order to check the stability of this string solution.

$\mu_{\text{even}}$ is given as

$$
\mu_{\text{even}} = 2\pi \int r dr \left[ \chi_R^2 + \left\{ \frac{(1 + \frac{\alpha}{2} \cos 2\theta_W z)^2}{r^2} + 2\lambda \left( f^2 - \frac{\nu^2}{2} \right) \right\} \chi_R^2 
+ \frac{\alpha \cos \theta_W}{2} (\chi_R f' - \chi_R f) (W^2 - W^1) 
- \frac{\alpha \cos \theta_W}{4r^2} (1 - \alpha \sin^2 \theta_W z) f \chi_R (\omega^1 + \omega^2) 
+ \frac{g^2 f^2}{8} \left\{ (W^a)^2 + \frac{1}{r^2} (\omega^a)^2 \right\} + \frac{\alpha \cos^2 \theta_W z'}{2r^2} (W^2 \omega^1 - W^1 \omega^2) 
+ \frac{1}{4r^2} \left\{ (W^1 - \omega^1 + \alpha \cos \theta_W z W^2)^2 
+ (W^2 + \omega^2 + \alpha \cos \theta_W z W^1)^2 \right\} \right],
$$

(31)

where we write $\chi_R(r) \text{ instead of } \text{Re} \chi$. By completing the square for variables $W^a$, we can neglect these variables by taking the squares vanishing. As the result, we have only three variables, $\chi_R, \xi_\pm \equiv (\omega^2 \pm \omega^1)/2$. And $\mu_{\text{even}}$ is again divided into two parts as

$$
\mu_{\text{even}} = \mu_{\xi_-} + \mu_{\chi, \xi_+}
$$

(32)

$$
\mu_{\xi_-} = \pi \int \frac{dr}{r} \left[ \frac{\alpha^2 \cos^2 \theta_W z f^2}{2P_-} \xi_-^2 + \left( \frac{\alpha^2 \cos^2 \theta_W f^2}{2} - \frac{\gamma^2 \theta_W z^2}{P_-} \right) \xi_-^2 - r \frac{d}{dr} \left( \gamma z' (1 + \gamma z) \right) \right] 
$$

(33)
\[ \mu_{\chi, \xi} = 2\pi \int r dr \left[ \frac{(1 - \gamma z)^2}{P_+} \chi R'^2 \right. \\
+ \left. \left\{ \left( 1 + \frac{\alpha}{2} \cos 2\theta W z \right)^2 \right\} + 2\lambda \left( f^2 - \frac{v^2}{2} \right) \right. \\
+ \left. \alpha^2 \cos^2 \theta W r^2 \left( \frac{f'^2 - ff'}{2P_+} \right) \right\} \chi R^2 \\
+ \alpha^2 \cos^2 \theta W r^2 \frac{f'^2}{4P_+} \zeta'^2 + \left\{ \frac{\alpha^2 \cos^2 \theta W f^2}{r^2} - \frac{\gamma z^2}{2r^2P_+} - \frac{\gamma' z(1 - \gamma z)}{2r^2P_+} \right\} \xi'^2 \\
- \alpha \cos \theta W \left\{ \frac{1 - \alpha \sin^2 \theta W z}{r^2} f\chi R \zeta + \frac{\chi R f' - \chi R' f}{P_+} \right\} \right] \], \quad (34)

where \( \gamma \equiv \alpha \cos^2 \theta W \) and

\[ P_+ \equiv \left( 1 \mp \alpha \cos^2 \theta W z \right)^2 + \frac{\alpha^2 \cos^2 \theta W r^2 f^2}{2}. \quad (35) \]

It is confirmed by numerical calculation that \( \mu_{\xi_-} \) does not include negative contribution \[22\]. Since one of the linear combination of \( \chi R(r), \xi_+(r) \) is just the gauge transformation, the physical perturbation becomes

\[ \zeta(r) = (1 - \gamma z) \chi R(r) + \frac{\alpha \cos \theta W f_+}{2} \xi_+(r). \quad (36) \]

In summary, the classical stability of this electroweak string can be determined by the following energy variation as

\[ \mu_{\xi} = 2\pi \int r dr \zeta O \zeta \quad (37) \]

\[ O \equiv -\frac{1}{r} \frac{d}{dr} \left( \frac{r d}{P_+ dr} \right) + \left\{ \frac{2S_+}{g^2 r^2 f^2} + \frac{f'^2}{f^2 P_+} + \frac{1}{r} \frac{d}{dr} \left( \frac{rf'}{f P_+} \right) \right\} \quad (38) \]

\[ S_+ \equiv \frac{g^2 f^2}{2} - \frac{\gamma z^2}{P_+} + \frac{d}{dr} \left\{ \frac{\gamma' z(1 - \gamma z)}{r P_+} \right\}. \quad (39) \]

It is important to know whether the eigenfunction \( \zeta(r) \) with negative eigenvalue for the operator \( O \). This numerical calculation has been done, and the allowed region in parameter space \((\beta \equiv m_H^2/m_Z^2, \cos^2 \theta_W)\) has been obtained in Ref. \[21, 22\].

III. NON-TOPOLOGICAL STRING ON SUPERSYMMETRIC \( SU(2) \times U(1) \) MODEL

We consider the supersymmetric \( SU(2) \times U(1) \) gauge theory with the Higgs fields \( H_1 \) and \( H_2 \), whose quantum numbers are \((2, \frac{1}{2})\) and \((2, -\frac{1}{2})\) under \( SU(2) \times U(1) \), respectively, and a gauge singlet \( S \). The superpotential \( W \) is

\[ W = yS \left( H_1^2 (i\sigma^2) H_1 - u^2 \right), \quad (40) \]
where $i\sigma^2$ is the $2 \times 2$ antisymmetric matrix. We take parameters $y$ and $u$ real for simplicity. Although there are fermions as supersymmetric partner, we do not consider that the fermions develop the non-vanishing VEVs. Thus, we consider the scalar components $h_1$, $h_2$ and $s$ for chiral multiplets $H_1$, $H_2$ and $S$. The potential $V(h_1, h_2, s)$ becomes

$$V(h_1, h_2, s) = y^2 |h_2^2(i\sigma^2)h_1 - u^2|^2 + y^2 |s|^2 \left( |h_1|^2 + |h_2|^2 \right) + \frac{g_2^2 + g_1^2}{8} \left( |h_1|^2 - |h_2|^2 \right)^2 + \frac{g_2^2}{2} \left| h_2^1 h_1^2 \right|^2,$$

(41)

where $g_1$ and $g_2$ are the gauge couplings of $U(1)$ and $SU(2)$, respectively. The Lagrangian for scalar and gauge fields is

$$\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu h_1|^2 + |D_\mu h_2|^2 + |\partial_\mu s|^2 - V(h_1, h_2, s),$$

(42)

where the notation is the same as in section 2 for the $SU(2) \times U(1)$ gauge fields. We take the same ansatz of solutions as in Ref. [32], in which the electroweak string in general two Higgs doublet models (2HDMs) are constructed [32]. The string solutions are

$$h_1(x) = f_1(r) e^{i m \theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h_2(x) = f_2(r) e^{-i m \theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{Z}(x) = -m \frac{z(r)}{r} e_\theta$$

(43)

where $m$ is integer and non-zero. $f_1(r), f_2(r)$ and $z(r)$ are real-valued functions and satisfy the boundary conditions:

$$f_1(0) = f_2(0) = z(0) = 0,$$

(44)

$$f_1(\infty) = f_2(\infty) = u, \quad z(\infty) = \frac{2}{\alpha}.$$  

(45)

Therefore we obtain the string solutions to be aligned the $z$ axis.

We consider F- and D-flatness conditions [34] for eq. (43). First, F-flatness conditions give

$$f_1(r) f_2(r) - u^2 = 0,$$

$$s = 0.$$  

(46)

† Topological strings in 2HDMs has been actively researched [33].
With the boundary conditions given in (44), \( f_1(r) \) and \( f_2(r) \) do not satisfy F-flatness condition. This indicates that supersymmetry is broken inside the string. On the outside the string, eq. (46) is almost satisfied. Next, D-flatness condition gives

\[
f_2^2(r) - f_2^2(r) = 0. 
\]

(47)

For the string solutions to satisfy eq. (47), we require \( f_1(r) = f_2(r) \equiv f(r) \). Therefore the string solutions are

\[
h_1(x) = f(r)e^{im\theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h_2(x) = f(r)e^{-im\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{Z}(x) = -m \frac{z(r)}{r} \vec{e}_\theta. \]

(48)

On substituting eq. (48) into the equations of motion on Lagrangian given in (42), they reduce to

\[
z'' - \frac{z'}{r} + 2m\alpha \left[ \left( 1 - \frac{\alpha}{2} z \right) f^2 \right] = 0 \]

\[
f'' + \frac{f'}{r} - m^2 \left( 1 - \frac{\alpha}{2} z \right)^2 \frac{f}{r^2} - y^2 \left( f^2 - u^2 \right) f = 0. \]

(49)

where \( \alpha \equiv \sqrt{g_1^2 + g_2^2} \). The equations in eq. (49) are the same as eq. (11) and eq. (12) by replacing \((f(r), z(r), \alpha, y^2, u)\) to \((f(r)/\sqrt{2}, a(r), 2e, 4\lambda, v/2)\). It indicates that \( f(r) \) and \( z(r) \) in this electroweak string are essentially equivalent to \( f(r) \) and \( a(r) \) in the Nielsen-Olesen string.

**IV. STABILITY OF NON-TOPOLOGICAL STRING**

Because the first homotopy group of the Higgs vacuum is trivial, the string solutions given in eq. (48) is not topologically stable. We shall investigate the stability of the string solutions by considering infinitesimal perturbations around them and finding whether the energy decreases or not. The energy functional for static and \( z \)-independent fields is denoted by \( E = \int dz \mu \), where \( \mu \) is

\[
\mu = \int r dr d\theta \left[ \frac{1}{4} W_{ij}^a W_{ij}^a + \frac{1}{4} F_{ij}^a F_{ij}^a + |D_i h_1|^2 + |D_i h_2|^2 + |\partial_i s|^2 + V(h_1, h_2, s) \right]. \]

(50)

Substituting eq. (48) into eq. (50), the string tension is obtained as

\[
\mu = \int r dr d\theta \left[ m^2 \frac{z'^2}{r^2} + 2f'^2 + \frac{2m^2}{r^2} \left( 1 - \frac{\alpha}{2} z \right) f^2 + y^2 \left( f^2 - u^2 \right)^2 \right]. \]

(51)
We shall set \( m = 1 \) in the followings.

Before the detailed calculation, we note that we can ignore the \( z \)-dependent perturbations and the \( z \)-components of the vector fields. The gradient energy terms, which are related with these perturbations, are given by

\[
\frac{1}{2} W_{iz}^a W_{iz}^a + \frac{1}{2} F_{iz} F_{iz} + |D_z h_1|^2 + |D_z h_2|^2 + |\partial_z s|^2,
\]

which are separated from other variations and become positive. In addition, if the variation of potential energy made by the \( z \)-dependent perturbations is denoted as \( \int dz \delta \mu(z) \), it always becomes larger than \( \int dz \mu(z_{\text{min}}) \) where \( z_{\text{min}} \) minimizes the \( \mu(z) \). For this reason, we ignore the \( z \)-dependent perturbations and the \( z \)-component of the gauge fields. Similar discussion is possible for \( t \)-dependent perturbations and \( t \) component gauge fields. Thus, it is sufficient to check whether the string tension decreases or not by perturbations instead of the string energy.

Because of unbroken \( U(1) \) gauge symmetry, the quadratic variations separate to the neutral part \( \mu_n \) and the charged part \( \mu_c \) as in the calculation in section 2. The neutral part of the string tension is given by

\[
\int r dr d\theta \left[ \frac{1}{2} \left( \nabla \times \vec{Z} \right)^2 + \left| \left( \partial_i + i \frac{\alpha}{2} Z_i \right) \varphi_1 \right|^2 + \left| \left( \partial_i - i \frac{\alpha}{2} Z_i \right) \varphi_2 \right|^2 + |\partial_1 s|^2 + y^2 \left| \varphi_1 \varphi_2 - u^2 \right|^2
\]

\[
+ y^2 s \left( |\varphi_1|^2 + |\varphi_2|^2 \right) + \frac{\alpha^2}{8} \left( |\varphi_1|^2 - |\varphi_2|^2 \right)^2 + \frac{1}{2} \left( \nabla \times \vec{A} \right)^2 \right],
\]

where \( \varphi_1 \) and \( \varphi_2 \) are neutral components of \( h_1 \) and \( h_2 \), respectively. In eq. (53), we denote the original string solution and the perturbations together as \( \varphi_1, \varphi_2, s, Z_i, A_i \). The fourth, sixth and eighth terms are non negative and it can be zero by setting \( s = 0 \) and \( \nabla \times \vec{A} = 0 \). Thus, we can ignore these terms. We have to check whether the perturbations of \( \varphi_1, \varphi_2, Z_i \) can make the string tension lower or not. However, this problem is equivalent to checking the stability of the Nielsen-Olesen string, which appears when an \( U(1) \) is spontaneously broken by developing the VEVs of two Higgs bosons. This has been discussed in Ref.\[35\]. Because the Nielsen-Olesen string with two Higgs bosons is also a topological string, it is stable and does not decay. Because the string tension (53) with vanishing \( s \) and \( \nabla \times \vec{A} \) is nothing but the string tension of the Nielsen-Olesen string with two Higgs bosons, we conclude that the neutral perturbations does not reduce the string tension.

Let us consider whether the unbroken \( U(1) \) charged perturbations destabilize the string
solution or not. We redefine the Higgs fields $\phi_1$ and $\phi_2$ as

\[
\begin{pmatrix}
\phi_1(x) \\
\phi_2(x)
\end{pmatrix}
\equiv \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
h_1(x) \\
-i\sigma_2(h_2)^*(x)
\end{pmatrix},
\]

(54)

where $\phi_1$ and $\phi_2$ has the same charge. On the $SU(2) \times U(1) \rightarrow U(1)$, $\phi_1$ develops a non-vanishing VEV, $\langle \phi_1 \rangle = \sqrt{2}u$, while $\phi_2$ has vanishing VEV. The Higgs potential can be rewritten as

\[
\tilde{V}(\phi_1, \phi_2) = \frac{y^2}{4} \left( |\phi_1|^2 - |\phi_2|^2 - 2u^2 \right)^2 + \frac{y^2}{4} \left| \phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1 \right|^2 + \frac{\alpha^2}{8} \left( \phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1 \right)^2
\]

\[+ \frac{g_2^2}{2} \left[ |\phi_1|^2 |\phi_2|^2 - (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1) \right]. \tag{55}\]

We consider the charged perturbations of $\phi_1$ and $\phi_2$ as

\[
\begin{align*}
\phi_1(x) &= \begin{pmatrix} \eta_1(x) \\ \sqrt{2}f(r)e^{i\theta} \end{pmatrix}, \\
\phi_2(x) &= \begin{pmatrix} \eta_2(x) \\ 0 \end{pmatrix}.
\end{align*}
\]

(56)

Substituting eq. (56) to eq. (55) and expanding the potential up to the second order of $\eta_1$ and $\eta_2$, we obtain

\[
\tilde{V}(\phi_1, \phi_2) \sim \frac{y^2}{4} (u^2 - f^2)^2 - y^2 (u^2 - f^2)|\eta_1|^2 + y^2 (u^2 - f^2)|\eta_2|^2 + g_2^2 f^2 |\eta_2|^2. \tag{57}\]

In eq. (57), the terms which include $|\eta_2|^2$ give positive contributions to the potential. In addition, the gradient energy for $\eta_2$ is positive because the kinetic term of $\phi_2$ at $\eta_2 = 0$ gives the minimum kinetic energy, $E_{\text{min}} = 0$. Therefore we conclude that $\eta_2$ does not destabilize the string solution.

The perturbations which may decrease the string tension are $\eta_1$ and the perturbations of $W_i^8$. However, these modes are nothing but the dangerous modes in the electroweak string, which is discussed in section 2. When we take $\phi_2 = 0$ on eq. (42), the Lagrangian becomes the same as the Lagrangian of the electroweak string in eq. (13) by corresponding the parameters $\lambda$ and $v$ to $y^2/4$ and $2u$, respectively. It indicates that the perturbations $\eta_1$ and $W_i^8$ for the string solution (48) is equivalent to the charged perturbations of the electroweak string. Therefore we conclude that the stability of the string solution (48) is the same as the stability of the electroweak string.

The parameter region in the parameter plane ($\sqrt{\beta} = m_H/m_Z, \cos^2 \theta_W$) for the classically stable electroweak string is given in Ref. [22]. This parameter region can be applied to the parameter region for the classically stable non-topological string discussed in this paper.
V. DISCUSSION AND SUMMARY

We have extended the electroweak string to the non-topological string in the SUSY $SU(2) \times U(1)$ gauge theory and have seen that its classical stability conditions are the same as those for the electroweak string, with appropriate replacement of some parameters. Since the configuration of the electroweak string is embedded in the SUSY gauge theory, the most dangerous mode which makes the electroweak string unstable appears also in the non-topological string in the SUSY gauge theory. Therefore, the classical stability conditions cannot be weaker than those of the electroweak string. The question is whether the classical stability conditions become more severe or not. We have concluded that the classical stability conditions become the same as those in the electroweak string. We have shown that the classical stability is determined only by two parameters, $(\cos \theta_W, \beta)$, as in the electroweak string, and the other modes than the above dangerous mode do not destabilize the non-topological string configurations.

The stability region in the $(\sin^2 \theta_W, \sqrt{\beta})$ for the electroweak string has been shown in Ref. [21, 22] by James, Perivolaropoulos, and Vachaspati. They have shown that the electroweak string becomes classically stable only in the limited region where $\cos^2 \theta_W < 0.1$ and $\beta \leq 1$. Therefore, not only the electroweak string in the SM ($\sqrt{\beta} = 1.4, \sin^2 \theta_W = 0.23$) but also the non-topological string in the SUSY $SU(2)_R \times U(1)_{B-L}$ gauge theory, which appears in $SO(10)$ grand unified theory ($\sin^2 \theta_W = 0.6$), becomes classically unstable.

We think that the study for the classical stability of non-topological string becomes important to test the models beyond the SM via gravitational wave detection. We hope that this work contributes to this interesting subject.

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