MOMENTS OF RIEZ MEASURES ON POINCARÉ DISK AND HOMOGENEOUS TREE – A COMPARATIVE STUDY

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Abstract. One of the purposes of this paper is to clarify the strong analogy between potential theory on the open unit disk and the homogeneous tree, to which we dedicate an introductory section. We then exemplify this analogy by a study of Riesz measures. Starting from interesting work by Favorov and Golinskii [10], we consider subharmonic functions on the open unit disk, resp. on the homogeneous tree. Supposing that we can control the way how those functions may tend to infinity at the boundary, we derive moment type conditions for the Riesz measures. One one hand, we generalise the previous results of [10] for the disk, and on the other hand, we show how to obtain analogous results in the discrete setting of the tree.

1. Introduction

The homogeneous tree $T = T_q$ with degree $q + 1$ is in many respects a discrete analogue of the hyperbolic plane. These are the two basic examples of Gromov-hyperbolic metric spaces. In the Poincaré metric, the hyperbolic plane is the open unit disk $\mathbb{D}$ as a topological space. Its natural geometric compactification is obtained by passing from the hyperbolic to the Euclidean metric and taking the closure, i.e., the closed unit disk. Analogously, the end compactification of $T$ is obtained by passing from the original graph metric to a new (bounded) metric and taking the completion.

Various objects, formulas, properties, theorems, etc., of geometric, algebraic, analytic, potential theoretic, or stochastic nature on $\mathbb{D}$ have counterparts on $T$ and vice versa. It is not always immediately apparent that looking at $\mathbb{D}$ both with Euclidean and with hyperbolic eyeglasses may provide additional insight. But this is true when one wants to understand the analogies between $T$ and $\mathbb{D}$. The purpose of this note is to exhibit some potential theoretic aspects of that correspondence. The starting point is a classical theorem of Blaschke [5]:

A set $\{z_k : k \in \mathbb{N}\} \subset \mathbb{D}$ is the set of zeroes of a bounded analytic function $f$ on $\mathbb{D}$ if and only if
$$\sum_k (1 - |z_k|) < \infty.$$ 

This also allows for the case where each $z_k$ is counted according to its multiplicity $\text{mult}(z_k)$ as a zero of $f$. We interpret this theorem in terms of the subharmonic function $u : \mathbb{D} \to [-\infty, \infty)$ given by $u(z) = \log |f(z)|$. We let $\mu^u$ be the Riesz measure of $u$ in its Riesz decomposition (see below for details). Being bounded above, $u$ has a harmonic majorant, which leads to finiteness of the “moment”

$$\int_{\mathbb{D}} (1 - |z|) \, d\mu^u(z) < \infty.$$ 

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Since $\mu^u = \sum_k \text{mult}(z_k) \cdot \delta_{z_k}$, this just means finiteness of $\sum_k (1 - |z_k|) \text{mult}(z_k)$, so that the Blaschke condition takes the form (1.1).

A change of the viewpoint is now suggestive. We start directly with a subharmonic function $u$, and instead of assuming that it is bounded above, we admit that it tends to $\infty$ in some controlled way when approaching a subset $E \subset \partial \mathbb{D} = \mathbb{S}$, the boundary of the disk (the unit circle). From properties of $E$ and the way how $u$ tends to infinity at $E$, we then want to deduce properties of the Riesz measure $\mu^u$. This approach was undertaken in two substantial papers by Favorov and Golinskii [10], [11], which were the main inspiration for the present note. We shall provide more general versions of some of their results on the Riesz measure of subharmonic functions on the disk.

On the homogeneous tree $\mathbb{T}$, the geometrical habit is converse as compared to the disk: on one hand, one is used to look at the Euclidean unit disk $\mathbb{D}$ and its closure, which in the spirit of the present note arises by a change from the “original” hyperbolic metric to the “new” Euclidean metric which is the one of its compactification. On the other hand, one is used to look at the tree with its habitual integer-valued graph metric – this is our hyperbolic object, and when we introduce the end compactification, we pass to a suitable new, maybe less habitual metric which is the one that corresponds to the Euclidean metric of $\mathbb{D}$.

(Sub)harmonic functions on $\mathbb{T}$ are defined via the discrete Laplacian $P - I$ (or $I - P$, if one desires a positive semidefinite operator), where $P$ is the transition matrix of the simple random walk. Of course, here we also have the Riesz decomposition theorem. We shall see that once we understand the correspondence between tree and disk completely, we can obtain the same type of moment condition for the Riesz measure of a subharmonic function as in (1.1): we need to realise that the term $1 - |z|$ in (1.1) is the distance from $z$ to the boundary in the metric of the respective compactification.

In the next Section 2, we provide an expository description of the basic potential theoretic features of $\mathbb{D}$ and $\mathbb{T}$. On purpose slightly beyond the scope of the subsequent section, it aims at providing a good understanding of part of the many common features of those two structures. Subsequently, in Section 3 we present and investigate our basic moment conditions for subharmonic functions on those two spaces.

2. Basic potential theory on disk and tree

A. Euclidean and hyperbolic disk

The Euclidean unit disk

$$\mathbb{D} = \{z = x + iy \in \mathbb{C} : |z| = \sqrt{x^2 + y^2} < 1\}.$$ 

carries the Euclidean metric $d_{\mathbb{D}}$, induced by the absolute value, resp. length element

$$d_{\mathbb{D}}(z, w) = |z - w| \quad \text{and} \quad d_{\mathbb{D}}s = \sqrt{dx^2 + dy^2}.$$ 

The standard measure is Lebesgue measure – for which here we sometimes write $m_{\mathbb{D}}$ – with area element $d_{\mathbb{D}}z = dz = dx dy$. The Euclidean Laplace operator is

$$\Delta_{\mathbb{D}} = \partial_x^2 + \partial_y^2.$$
A \textit{harmonic function} is a real-valued function \( h \in C^2(\mathbb{D}) \) such that \( \Delta h = 0 \). For the definition of a \textit{subharmonic function}, see e.g. \textsc{Helms} [16] p.58, who rather considers superharmonic functions: the correspondence is just by a change of the sign. A function \( u : \mathbb{D} \rightarrow [-\infty, +\infty) \) is subharmonic on \( \mathbb{D} \) if it is upper semicontinuous, and for every \( z \in \mathbb{D} \) and \( r < 1 - |z| \), one has \( A_r u(z) \geq u(z) \), where
\[
A_r u(z) = A_r^D u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) \, dt
\]
is the (Euclidean) average of \( u \) over the circle with radius \( r \) centred at \( z \). In addition, we require that the set \( \{ z : u(z) = -\infty \} \) has Lebesgue measure 0. It is well known that \( u \in C^2(\mathbb{D}) \) is subharmonic if and only if \( \Delta_D u \geq 0 \), see [16] Thm. 4.8. If \( u \) is not smooth, then \( \Delta_D u \) is defined in the sense of distributions. Subharmonicity means that this is a non-negative Radon measure. The \textit{Riesz measure} associated with \( u \) is then
\[
\mu^u = \frac{1}{2\pi} \Delta_D u,
\]
that is,
\[
\int_{\mathbb{D}} f \, d\mu^u = \frac{1}{2\pi} \int_{\mathbb{D}} u(z) \, \Delta_D f(z) \, d\mu_D z
\]
for every \( C^\infty \)-function \( f \) on \( \mathbb{D} \) with compact support in \( \mathbb{D} \). If \( u \in C^2(\mathbb{D}) \) then the ordinary function \( \frac{1}{2\pi} \Delta_D u \) is the density of \( \mu^u \) with respect to Lebesgue measure \( \mu_D \). Furthermore, \( h \in C^2(\mathbb{D}) \) is harmonic if and only if \( A_r h(z) = h(z) \) for every \( z \in \mathbb{D} \) and \( r < 1 - |z| \).

The \textit{Green function} of \( \Delta_D \) is
\[
G_D(z,w) = \log \frac{|1 - z\bar{w}|}{|z - w|}, \quad z, w \in \mathbb{D}.
\]
For any non-negative measure \( \mu \) on \( \mathbb{D} \), the function \( G_{\mathbb{D}} \mu \) on \( \mathbb{D} \) defined by
\[
G_{\mathbb{D}} \mu(z) = \int_{\mathbb{D}} G_D(z,w) \, d\mu(w),
\]
is called the \textit{potential} of \( \mu \) if the integral is finite at some (\( \iff \) almost every) \( z \in \mathbb{D} \). Then \( -G_{\mathbb{D}} \mu \) is is a subharmonic function. If \( u \) is subharmonic and, in addition, posseses some harmonic majorant on \( \mathbb{D} \), then it possesses its smallest harmonic majorant \( h \). In this case, the \textit{Riesz decomposition} of \( u \) has the form
\[
u = h - G_{\mathbb{D}} \mu^u.
\]
See e.g. \textsc{Ransford} [17] Thm. 4.5.4. In absence of a harmonic majorant, for the general Riesz decomposition theorem see [17] Thm. 3.7.9 or [16] Thm. 6.18.

We now consider the hyperbolic plane \( \mathbb{H} \). Basic hyperbolic potential theory appears rather to be “common knowledge” than being accessible in a comprehensive treatise, with the exception of \textsc{Stoll} [19]. See also the introductory chapter of \textsc{Helgason} [15]. We use the \textit{Poincaré disk model}; see e.g. \textsc{Beardon} [4] Chapter 7. \( \mathbb{H} \) coincides with \( \mathbb{D} \) as a set and topologically, but the hyperbolic length element and metric are
\[
d_{\mathbb{H}} s = \frac{2\sqrt{dx^2 + dy^2}}{1 - |z|^2} \quad \text{and} \quad \rho_{\mathbb{H}}(z,w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}.
\]
The hyperbolic measure \( \mu_{\mathbb{H}} \) has area element
\[
d\mu_{\mathbb{H}}(z) = d_{\mathbb{H}} z = \frac{4dz}{(1 - |z|^2)^2} = 4 \cosh^4 \frac{\rho_{\mathbb{H}}(z,0)}{2} \, dz.
\]
This means conversely that we can express Lebesgue measure \( m_\mathbb{H} \) on \( \mathbb{H} \) as
\[
(2.7) \quad dm_\mathbb{H}(z) = \frac{1}{4 \cosh^4(\rho_\mathbb{H}(z,0)/2)} dm_\mathbb{D}(z) \approx e^{-2\rho_\mathbb{H}(z,0)} dm_\mathbb{H}(z), \quad \text{as} \ \rho_\mathbb{H}(z,0) \to \infty.
\]

The hyperbolic Laplace operator in the variable \( z = x + iy \) is
\[
(2.8) \quad \Delta_\mathbb{H} = \frac{(1 - |z|^2)^2}{4} \Delta_\mathbb{D}.
\]
In particular, its harmonic functions are the same as the \( \Delta_\mathbb{D} \)-harmonic functions. Above, we defined the Euclidean average over a circle in \( \mathbb{D} \). Now, we let \( r > 0 \) and \( z \in \mathbb{H} \) and consider the hyperbolic circle \( C^\mathbb{H}(z,r) = \{ w \in \mathbb{H} : \rho_\mathbb{H}(z,w) = r \} \). This is also a Euclidean circle: \( C^\mathbb{H}(z,r) = C^\mathbb{D}(z',r') \), where
\[
z' = \frac{1 - \tanh^2(r/2)}{1 - |z|^2 \tanh^2(r/2)} 1 - |z|^2 \tanh^2(r/2), \quad \text{and} \quad r' = \frac{1 - |z|^2}{1 - |z|^2 \tanh^2(r/2)} \tanh(r/2).
\]
Its hyperbolic length is \( 2\pi \sinh r \), see [4, page 132].

Now, a function \( u : \mathbb{H} \to [-\infty, +\infty) \) is subharmonic on \( \mathbb{H} \) if it is lower semicontinuous, \( m_\mathbb{H}(\{ z : u(z) = -\infty \}) = 0 \), and for every \( z \in \mathbb{H} \) and \( r > 0 \), one has \( A^\mathbb{H}_r u(z) \geq u(z) \), where
\[
(2.9) \quad \text{Lemma.} \quad \text{A function } u \text{ is hyperbolically superharmonic if and only if it is superharmonic on } \mathbb{D} \text{ in the Euclidean sense.}
\]
The Green function of \( \Delta_\mathbb{H} \) is the same as the one for \( \Delta_\mathbb{D} \) given in (2.3), and will henceforth also be denoted by \( G^\mathbb{H}(\cdot, \cdot) \). Using the hyperbolic metric,
\[
(2.10) \quad G^\mathbb{H}(z,w) = -\log \tanh(\rho^\mathbb{H}(z,w)/2).
\]
Consequently, the hyperbolic Riesz decomposition and the Riesz measure of a superharmonic function \( u \) are the same as the Euclidean one.

The natural hyperbolic compactification \( \bar{\mathbb{H}} \) of \( \mathbb{H} \) arises from the identification of \( \mathbb{H} \) with \( \mathbb{D} \) and taking the Euclidean closure. The boundary at infinity \( \partial \mathbb{H} \) of \( \mathbb{H} \) is then the unit circle \( \mathbb{S} \). It is instructive to interpret this as follows: we first transform the metric \( \rho^\mathbb{H} \) of the hyperbolic plane into a new metric, namely the Euclidean metric. For use in the subsection on trees, note that on the large scale, the change of the metric is quantified by
\[
(2.11) \quad d_\mathbb{D}(z,\mathbb{S}) = 1 - |z| = \frac{2}{1 + e^{\rho^\mathbb{H}(z,0)}} \approx 2e^{-\rho^\mathbb{H}(z,0)} \text{ as } |z| \to 1, \text{ or equivalently, as } \rho^\mathbb{H}(z,0) \to \infty.
\]

In order to get used to the two geometric views on the same object, we can freely switch back and forth: \( \mathbb{D} \leftrightarrow \mathbb{H} \) and \( \mathbb{S} \leftrightarrow \partial \mathbb{H} \).

The Poisson kernel on \( \mathbb{H} \times \partial \mathbb{H} = \mathbb{D} \times \mathbb{S} \) is defined for \( z \in \mathbb{H}, \xi \in \mathbb{S} \) as
\[
(2.12) \quad P(z,\xi) = \frac{1 - |z|^2}{|\xi - z|^2} = \lim_{w \to \xi} \frac{G^\mathbb{H}(z,w)}{G^\mathbb{H}(0,w)} = e^{-b^\mathbb{H}(z,\xi)}.
\]
with the Busemann function

\begin{equation}
\text{b}_H(z, \xi) = \lim_{w \to \xi} \left( \rho_H(w, z) - \rho_H(w, 0) \right).
\end{equation}

It also has a probabilistic interpretation: we start Euclidean Brownian motion (BM) at \( z \in \mathbb{D} \) and consider its hitting distribution \( \nu_z \) on the boundary \( \mathbb{S} \). That is, if \( B \subset \mathbb{S} \) is a Borel set, then \( \nu_z(B) \) is the probability that the first visit of BM to \( \mathbb{S} \) occurs in a point of \( B \). Denoting by \( \lambda_\mathbb{S} \) the normalized Lebesgue arc measure on the unit circle, we have

\begin{equation}
\frac{d\nu_z}{d\lambda_\mathbb{S}}(\xi) = P(z, \xi), \quad \xi \in \mathbb{S}.
\end{equation}

Note that \( \nu_0 = \lambda_\mathbb{S} \).

**Theorem.** (a) For every \( \xi \in \mathbb{S} \), the function \( z \mapsto P(z, \xi) \) is harmonic on \( \mathbb{D} \equiv H \).

(b) [Poisson representation] For every positive harmonic function \( h \) on \( \mathbb{D} \equiv H \), there is a unique Borel measure \( \nu^h \) on \( \mathbb{S} \equiv \partial H \) such that

\begin{equation}
h(z) = \int_\mathbb{S} P(z, \cdot) d\nu^h.
\end{equation}

(c) For every continuous function \( \varphi \) on \( \mathbb{S} \equiv \partial H \),

\begin{equation}
h(z) = \int_\mathbb{S} P(z, \cdot) \varphi d\lambda_\mathbb{S}
\end{equation}

is the unique harmonic function \( h \) on \( \mathbb{D} \equiv H \) such that

\begin{equation}
\lim_{z \to \xi} h(z) = \varphi(\xi) \quad \text{for every } \xi \in \mathbb{S}.
\end{equation}

**B. Homogeneous tree**

We think of a graph as a set of vertices, equipped with a symmetric neighbourhood relation \( \sim \). An edge is a pair (usually considered un-oriented) \( e = [x, y] \) with \( x \sim y \). Now, we consider the homogeneous tree \( T = T_q \), where every vertex has \( q + 1 \geq 3 \) neighbours. The discrete Laplacian \( \Delta_T \) acts on functions \( f : T \to \mathbb{R} \) by

\begin{equation}
\Delta_T f(x) = \frac{1}{q + 1} \sum_{y \sim x} (f(y) - f(x)).
\end{equation}

It is related with simple random walk (SRW) on \( T \) in the same way as the above Laplacians are the infinitesimal generators of Euclidean and hyperbolic Brownian motion. (Hyperbolic BM is Euclidean BM slowed down close to the boundary of the hyperbolic disk.) SRW is the Markov chain \( (Z_n)_{n \geq 0} \) on \( T \) where \( Z_n \) is the random position at discrete time \( n \) of the particle, which moves from the current vertex \( x \) to any of its neighbours \( y \) with equal probability \( p(x, y) = 1/(q + 1) \), while \( p(x, y) = 0 \) if \( x \not\sim y \). This gives rise to the transition operator \( Pf(x) = \sum_y p(x, y)f(y) \), and \( \Delta_T = P - I \), where \( I \) is the identity operator. For potential theory on trees, see e.g. Woess [20].

For any pair of vertices \( x, y \), there is a geodesic path \( \pi(x, y) \) from \( x \) to \( y \) without repetitions. The number of edges of that path is the graph distance

\begin{equation}
d_T(x, y).
\end{equation}
The standard measure on (the vertex set of) \( T \) is the counting measure \( m_T(A) = |A| \) \((A \subset T)\). In comparing \( T \) with \( \mathbb{H} \), \( d_T \) and \( m_T \) correspond to the hyperbolic distance and area element \( \rho_\mathbb{H} \) and \( m_\mathbb{H} \) on \( \mathbb{H} \), respectively.

Functions \( h, u : T \to \mathbb{R} \) are harmonic, resp. subharmonic, if \( \Delta_T h = 0 \), resp. \( \Delta_T u \geq 0 \). Equivalently, \( Ph = h \), resp. \( Pu \geq u \), a definition in terms of the arithmetic averages over spheres with radius 1. Here, it makes no sense to allow for value \(-\infty\), since sets of \( m_T \)-measure 0 are empty.

The Green function of \( \Delta_T \) is
\[
G_T(x, y) = \frac{q}{q-1} q^{-d_T(x,y)}, \quad x, y \in T.
\]
The potential of a non-negative real function \( f \) on \( T \) is
\[
G_T f(x) = \sum_{y \in T} G_T(x, y) f(y).
\]
Since \( T \) is countable, measures on \( T \) are defined by their atoms, that is, they can be identified with non-negative functions. Thus, the Riesz measure of a subharmonic function \( u \) can be identified with the function
\[
\mu^u = \Delta_T u = Pu - u.
\]
More precisely, the function \( Pu - u \) should be understood as the density of the Riesz measure \( \mu^u \) with respect to the counting measure \( m_T \). If \( u \) has a harmonic majorant, then its Riesz decomposition reads
\[
u = h - G_T \mu^u, \quad \text{where} \quad h(x) = \lim_{n \to \infty} P^n u(x)
\]
is the smallest harmonic majorant of \( u \). Again, if there is no harmonic majorant, then the statement of the Riesz decomposition theorem is a little bit more involved. In the Markov chain (= Discrete Potential Theory) literature, the only source for the latter seems to be \([21]\), but it will not be needed here.

We next describe the end compactification of \( T \). A ray or geodesic ray is a one-sided infinite sequence \( \pi = [x_0, x_1, x_2, \ldots] \) of vertices such that \( x_n \sim x_{n-1} \) and \( x_n \neq x_m \) for all \( n, m, n \neq m \). Two rays are called equivalent, if they differ only by finite initial pieces. An end of \( T \) is an equivalence class of rays. The set of all ends is the boundary \( \partial T \). We set \( \tilde{T} = T \cup \partial T \). For every vertex \( x \in T \) and every \( \xi, \eta \in \partial T \), there is precisely one geodesic ray \( \pi(x, \xi) \) starting at \( x \) that represents \( \xi \). Analogously, for any two distinct ends \( \xi, \eta \), there is a unique two-sided geodesic \( \pi(\xi, \eta) = [\ldots, -x_2, -x_1, x_0, x_1, x_2, \ldots] \) such that \([x_k, x_{k+1} \ldots \] and \([x_0, x_1, \ldots \) are rays representing \( \xi \) and \( \eta \), respectively.

We now pursue the line followed above by an exponential change of the metric of \( \mathbb{H} \), see \((2.11)\). A natural choice is as follows. We fix a root vertex \( o \in T \). For \( z \in \tilde{T} \), we denote \(|z| = d_T(o, z)\), with value \( \infty \) if \( z \in \partial T \). For \( w, z \in \tilde{T} \), we define their confluent \( w \land z \) with respect to \( o \) as the last common element on the geodesics \( \pi(o, w) \) and \( \pi(o, z) \). This is a vertex, unless \( z = w \in \partial T \). We let
\[
p_T(w, z) = \begin{cases} q^{-|w\land z|}, & \text{if } z \neq w, \\ 0, & \text{if } z = w. \end{cases}
\]
This is an ultra-metric. In the induced topology, $\hat{T}$ is compact, and $T$ is discrete and dense. Convergence in this topology is as follows: if $\xi \in \partial T$ then a sequence $(z_n)$ in $\hat{T}$ converges to $\xi$ if and only if $|\xi \wedge z_n| \to \infty$.

At this point, we underline that in the “translation” from disk to tree, the graph metric $d_T$ corresponds to the hyperbolic metric $\rho_H$, while the metric $\rho_T$ is the one that may be interpreted to correspond to the Euclidean metric $d_D$. The next identity should be compared with (2.11).

\[(2.21)\quad \rho_T(x, \partial T) = q^{-|x|} \text{ for } x \in T.\]

The Martin kernel on $T \times \partial T$ is defined for $x \in T$, $\xi \in \partial T$ as

\[(2.22)\quad K(x, \xi) = \lim_{y \to \xi} \frac{G_T(x, y)}{G_T(o, y)} = q^{-h_T(x, \xi)}\]

with the Busemann function

\[(2.23)\quad h_T(x, \xi) = \lim_{y \to \xi} (d_T(y, x) - d_T(y, o)) = d_T(x \wedge \xi, x) - d_T(x \wedge \xi, o).\]

Again, we have a probabilistic interpretation. It is a well-known exercise to show that SRW on $T$ converges almost surely in the topology of $\hat{T}$ to a limit random variable $Z_\infty$ that takes its values in $\partial T$. Let $\nu_x$ be the distribution of $Z_\infty$, when SRW starts at vertex $x$. Then $\nu_o = \lambda_{\partial T}$ is the tree-analogue of the normalized Lebesgue measure $\lambda_S$ on the unit circle: $\lambda_{\partial T}$ is the unique probability measure on $\partial T$ which is invariant under “rotations” of $T$, that is, self-isometries of the graph $T$ which fix the root vertex $o$. Connectedness of $T$ implies that $\nu_x$ is absolutely continuous with respect to $\lambda_{\partial T}$, and the Radon-Nikodym-derivative is (realised by) the Martin kernel:

\[(2.24)\quad \frac{d\nu_x}{d\lambda_{\partial T}}(\xi) = K(x, \xi).\]

We have a perfect analogy with Theorem 2.15.

**Theorem.** (a) For every $\xi \in \partial T$, the function $x \mapsto K(x, \xi)$ is harmonic on $T$.

(b) For every positive harmonic function $h$ on $T$, there is a unique Borel measure $\nu^h$ on $\partial T$ such that

\[h(x) = \int_{\partial T} K(x, \cdot) \, d\nu^h.\]

(c) [Solution of the Dirichlet problem] For every continuous function $\varphi$ on $\partial T$,

\[h(x) = \int_{\partial T} \varphi \, d\nu_x = \int_{\partial T} K(x, \cdot) \varphi \, d\lambda_{\partial T}.\]

is the unique harmonic function $h$ on $T$ such that

\[\lim_{x \to \xi} h(x) = \varphi(\xi) \quad \text{for every } \xi \in \partial T.\]

**C. A table of correspondences**

As a general “rule” of translation from $\mathbb{H}$ to $T$, we note that base $e$ (Eulerian number) has to be replaced by base $q$ (branching number of the tree).
There are many further analogies between analysis, probability, group actions, etc. on \( \mathbb{D} \) and \( \mathbb{T} \). The present introduction is not intended to cover all those aspects. For further tips of the iceberg, see e.g. Casadio Tarabusi, Cohen, Korányi and Picardello [7], Rigoli, Salvatori and Vignati [18], Cohen, Colonna and Singman [9], Atanasi and Picardello [3] or Casadio Tarabusi and Figà-Talamanca [8], and the references given there.

3. Moment conditions and harmonic majorants

Let \( X = \mathbb{D} \) or \( X = \mathbb{T} \), with respective boundary \( \partial X \) and compactification \( \hat{X} \) (see the above table). The boundary carries the metric \( \text{dist} \) and measure \( \lambda \), where \( \text{dist} = d_\mathbb{D} \) and \( \lambda = \lambda_S \) in case of the disk, while \( \text{dist} = \rho_\mathbb{T} \) and \( \lambda = \lambda_\mathbb{T} \) in case of the tree. Given a subharmonic function \( u \) on \( X \) and its Riesz measure \( \mu^u \), we are interested in finiteness of its \textit{first} (boundary) moment

\[
\int_X \text{dist}(x, \partial X) \, d\mu^u(x)
\]

and variants thereof. One principal tool is the following lemma.

\textbf{(3.2) Lemma.} The subharmonic function \( u \) has a harmonic majorant on \( X \) if and only if \( \mu^u \) has finite first moment (3.1).

\textit{Proof.} Our function \( u \) has a harmonic majorant if and only if \( G_X \mu^u \) is a potential, that is, it is finite at some \( x \in X \).

If \( X = \mathbb{D} \) then Armitage and Gardiner [2] Thms. 4.2.4 and 4.2.5 show that \( G_X \mu^u \) is a potential if and only if (3.1) holds.
If $X = \mathbb{T}$ then by (2.18), $G_\tau(x, y) \leq q^{-|x|}G_\tau(o, y)$ for all $x, y$, so that $G_\tau\mu^u$ is finite at $x \in \mathbb{T}$ if and only if $G_\tau\mu^u(o) < \infty$. Now
\[
G_\tau\mu^u(o) = \sum_{x \in \mathbb{T}} \frac{q}{q-1} q^{-|x|}\mu^u(x) = \frac{q}{q-1} \int_\mathbb{T} \rho_\tau(x, \partial\mathbb{T}) d\mu^u(x).
\]
\[\square\]

So in fact what we are going to do is to exhibit a sufficient condition for a subharmonic function on $X = \mathbb{D}$, resp. $X = \mathbb{T}$, to possess a (global or restricted) harmonic majorant, even if it is not bounded above.

(3.3) Theorem. Let $u$ be a subharmonic function on $X$ and consider the closed set
\[
E = \left\{ \xi \in \partial X : \limsup_{x \to \xi} u(x) = \infty \right\}.
\]
Suppose that $\Psi : [0, \text{diam}(X)] \to [0, \infty]$ is a continuous, decreasing function with
\[
\Psi(t) = \infty \iff t = 0 \quad \text{and} \quad \lim_{t \to 0} \Psi(t) = \infty,
\]
and that
\[
u(x) \leq \Psi(\text{dist}(x, E)) \quad \text{for all} \ x \in X.
\]
If
\[
(3.4) \quad \int_{\partial X} \Psi(\text{dist}(\xi, E)) \, d\lambda(\xi) < \infty,
\]
then $u$ has a finite harmonic majorant, and the Riesz measure $\mu^u$ has finite first boundary moment.

We note that for condition (3.4) it is necessary that $\lambda(E) = 0$. For the proof of the theorem, we shall work with the function
\[
(3.5) \quad h = \int_{\partial X} K_X(\cdot, \xi) \Psi(\text{dist}(\xi, E)) \, d\lambda(\xi),
\]
where $K_X$ is the Poisson kernel (2.12) when $X = \mathbb{D}$, and the Martin kernel (2.22) when $X = \mathbb{T}$. Since for fixed $x \in X$, the function $\xi \mapsto K_X(x, \xi)$ is continuous on $\partial X$ (whence bounded), the function $h$ is finite and harmonic on $X$ under condition (3.4).

We need some preparations. We let $0 < t \leq \max\{\text{dist}(x, E) : x \in X\}$ and consider the sets
\[
E^{(t)} = \{ \xi \in \partial X : \text{dist}(\xi, E) \leq t \} \quad \text{and} \quad E^{(t)}_x = \{ \xi \in \partial X : \text{dist}(\xi, E) > t \},
\]
and, for $0 < t < 1$, the set $X^{(t)}$ which is the component of the origin of the set $\{ x \in X : \text{dist}(x, E) > t \}$.

Disk case: $\mathbb{D}^{(t)}$ (denoted $\Omega_t$ in [10]) is an open domain, and its boundary is
\[
\partial \mathbb{D}^{(t)} = \partial_\infty \mathbb{D}^{(t)} \cup \Gamma^{(t)}, \quad \text{where} \quad \partial_\infty \mathbb{D}^{(t)} \subset E^{(t)}_* \quad \text{and} \quad \Gamma^{(t)} = \Gamma^{(t)}_{\mathbb{D}} = \{ z \in \mathbb{D} : d_\mathbb{D}(z, E) = t \}.
\]
The sets $E^{(t)}$ and $\partial_\infty \mathbb{D}^{(t)}$ are both unions of finitely many closed arcs on $S$ and meet at finitely many endpoints of those arcs. $\partial_\infty \mathbb{D}^{(t)}$ may be a strict subset of the closure of $E^{(t)}_*$, because some arcs of the latter set can be the boundary of a different component of $\{ z \in \mathbb{D} : d_\mathbb{D}(z, E) > t \}$. (The latter can arise as “triangular” regions bounded by an arc of $S$ and of arcs of two intersecting circles $\{ z : |z - \zeta_j| = t \}$, where $\zeta_j \in E$, $j = 1, 2$.)
Tree case: The origin is of course the root vertex of $T$. The metric $\text{dist} = \rho_T$ takes only the countably many values $q^{-k}$, $k \geq 0$ (integer). For $0 < t < 1$ let $k = k(t)$ be the integer such that

\[(3.6) \quad q^{-k} \leq t < q^{-(k-1)}, \quad k = k(t).\]

For any vertex $y \in T$, we consider the branch of $T$ at $y$. This is the subtree (induced by)

\[T_y = \{ u \in T : y \in \pi(o, u) \}.\]

Its boundary $\partial T_y \subset \partial T$ consists of those ends which are represented by geodesics that lie entirely within $T_y$. Note that the open-compact sets $\partial T_y$, $y \in T$, are a basis of the topology of $\partial T$. Given $t$, let $k = k(t)$ and consider the set

\[\Gamma^{(t)} = \Gamma^{(t)}_T = \{ y \in T : |y| = k, \partial T_y \cap E \neq \emptyset \}.\]

We have

\[E^{(t)} = E^{(t)}_T = \bigcup_{y \in \Gamma^{(t)}} \partial T_y.\]

For small $t \equiv k = k(t)$, only few vertices $y$ with $|y| = k$ belong to $\Gamma^{(t)}$: as $t \to 0 \equiv k \to \infty$, we have

\[\frac{|\Gamma^{(t)}|}{|\{ y \in T : |y| = k(t) \}|} = \lambda_{\partial T}(E^{(t)}) \to \lambda_{\partial T}(E) = 0.\]

When $X = T$, the set $T^{(t)}$ is the subtree of $T$ obtained by chopping off each branch $T_y$, $y \in \Gamma^{(t)}$, that is,

\[T^{(t)} = T \setminus \bigcup_{y \in \Gamma^{(t)}} T_y.\]

The boundary of this truncated tree is

\[\partial T^{(t)} = \partial_{\infty} T^{(t)} \cup \Gamma^{(t)}_T, \quad \text{where} \quad \partial_{\infty} T^{(t)} = E^{(t)}_T,\]

while $\Gamma^{(t)}$ is the outer vertex boundary of $T^{(t)}$: it consists of those vertices in the complement that have a neighbour (here: precisely one neighbour) in $T^{(t)}$. In the topology of $\widehat{T}$, we have the compact subspaces $\widehat{T}^{(t)} = T^{(t)} \cup \partial T^{(t)}$ and the boundary $\partial T^{(t)}$.

We shall need the following simple estimate.

(3.7) Lemma. For $x \in X$, consider the harmonic measure $\nu_x$ on $\partial X$; see \((2.14), \text{resp.} (2.24)\). Then

\[\text{for } y \in \Gamma^{(t)}, \quad \nu_y(E^{(t)}) \geq 1/c_X = \begin{cases} 1/3, & \text{if } X = \mathbb{D}, \\ q/(q+1), & \text{if } X = T. \end{cases}\]

Proof. A. Disk case. For $y \in \Gamma^{(t)}_D$ there is $\zeta = \zeta_y \in E$ such that $|y - \zeta| = d(y, E) = t$. Consider the arc $\gamma_{\zeta} = \{ \xi \in S : |\xi - \zeta| \leq t \} \subset E^{(t)}$, as well as the circle $\{ z \in \mathbb{C} : |z - \zeta| = t \}$. At any of the two intersection points of that circle with $S$, the angle $\alpha$ between the tangents to the two circles is such that $\pi/2 > \alpha > \pi/3$, as $0 < t < 1$. By \cite{13} p. 13, Fig.1.1, $\nu_y(\gamma_{\zeta}) = \alpha/\pi > 1/3$. (In \cite{10}, the lower estimate 1/6 is used, but apparently also 1/3 works.)

B. Tree case. For $y \in \Gamma^{(t)}_T$, we have that $\partial T_y \subset E^{(t)}$. We note that $\nu_y$ gives equal
mass to the boundaries of each of the $q + 1$ branches of $T$ that are emanating from $y$. Among those, $q$ branches are part of $T_y$, that is, $\nu_y(\partial T_y) = q/(q + 1)$, providing the lower bound.

**Proof of Theorem 3.3.** Consider the function $\psi(t) = \min\{\Psi(t), \Psi(\text{dist}(\xi, E))\}$ on $\partial X$ and the harmonic function

$$
h(t)(x) = \int_{\partial X} K(x, \cdot) \psi(t) \, d\lambda = \int_{\partial X} \psi(t) \, d\nu_x.
$$

We know from theorems 2.15 resp. 2.25 that it is the solution of the Dirichlet problem on $X$ with continuous boundary function $\psi(t)$. We have $\psi(t)(\xi) = \Psi(t)$ on $E(t)$, while $\psi(t)(\xi) = \Psi(t)$ on $E(t) \supset \partial_{\infty} X(t)$. Thus,

$$
(3.8) \quad h(t)(x) = \int_{E(t)} \Psi(\text{dist}(\cdot, E)) \, d\nu_x + \Psi(t) \, \nu_x(E(t)).
$$

Taking boundary limits for points $x$ within $\tilde{X}(t)$, and using Lemma 3.7

$$
\lim_{x \to \xi} h(t)(x) = \Psi\left(\text{dist}(\xi, E)\right), \quad \text{for } \xi \in \partial_{\infty} X(t), \quad \text{and}
$$

$$
(3.9) \quad \lim_{x \to y} h(t)(x) = h(t)(y) \geq \Psi(t) \nu_y(E(t)) \geq \Psi(t)/c_X \quad \text{for } y \in \Gamma(t).
$$

(In the tree case, since $y$ is an isolated point, the last limit just means stabilisation at $y$.) On the other hand, by assumption our subharmonic function $u$ satisfies

$$
\limsup_{x \to \xi} u(x) \leq \Psi(\text{dist}(\xi, E)) \quad \text{for } \xi \in \partial_{\infty} X(t), \quad \text{and}
$$

$$
(3.10) \quad \limsup_{x \to y} u(y) \leq \Psi(t) \quad \text{for } y \in \Gamma(t).
$$

Therefore, again taking boundary limits within $\tilde{X}(t)$,

$$
\limsup_{x \to \eta} \left( u(x) - c_X h(t)(x) \right) \leq 0 \quad \text{for every } \eta \in \partial X(t).
$$

Thus, by the maximum principle (which also holds on the tree because $T(t)$ is a connected graph, a simple exercise),

$$
(3.11) \quad u(x) \leq c_X h(t)(x) \quad \text{for every } x \in X(t).
$$

Having this, we obtain the proposed first moment: let $h(x) = \int_{\partial X} K(x, \cdot) \Psi(\text{dist}(\cdot, E)) \, d\lambda$ be the harmonic function proposed in (3.5). Then $h(t) \leq h$ on $X(t)$ for any $t$. Given any $x \in X$, we can choose $t < \text{dist}(x, E)$ to see that $c_X \cdot h$ is a (finite) harmonic majorant for our subharmonic function $u$. \hfill \square

**Proposition.** Let $u$ be a subharmonic function on $X$, and let $E$ and $\Psi$ be as in Theorem 3.3. If

$$
u \geq \Psi(\text{dist}(x, E)) \quad \text{for all } x \in X
$$

and

$$
(3.13) \quad \int_{\partial X} \Psi(\text{dist}(\xi, E)) \, d\lambda(\xi) = \infty
$$

There is a simple converse to Theorem 3.3.
then $u$ has no harmonic majorant on $X$, and the first moment of $\mu^u$ is infinite.

**Proof.** We give a combined proof for $X = \mathbb{D}$ and $X = \mathbb{T}$. Suppose that the first moment of $\mu^u$ is finite. Then by Lemma 3.2, $u$ has a (finite) harmonic majorant $h$. Consider the continuous function $\Psi_M = \min\{\Psi, M\}$. Then for all $x \in X$,

$$h(x) \geq u(x) \geq \Psi_M(\text{dist}(x, E))$$

The function

$$g_M(x) = \int_{\partial X} K_X(\cdot, \xi) \psi_M(\text{dist}(\xi, E)) \, d\lambda(\xi),$$

defined analogously to (3.5), provides the solution of the Dirichlet problem on $X$ with boundary data $\psi_M(\text{dist}(\xi, E))$. We have

$$\liminf_{x \to \xi}(h(x) - g_M(x)) \geq 0 \quad \text{for every } \xi \in \partial X.$$

By the minimum principle, $h \geq g_M$ on $X$, and in particular, $h(o) \geq g_M(o)$. Letting $M \to \infty$, monotone convergence yields $h(o) = \infty$, contradicting finiteness of $h$. \qed

Next, in a similar spirit to [10], we want to extend Theorem 3.3 to a situation where the integral in (3.14) is infinite. For that purpose, we shall need an estimate of the Green function $G_{X(t)}(x, y) = G_{X(t)}(y, x)$ of $X(t)$. On the disk, this function is of course well described in the classical potential theory literature.

On the tree, for $x, y \in \mathbb{T}(t)$, it is the expected number of visits to $y$ of the random walk starting at $x$ before it hits $\Gamma(t)$. It is natural to define $G_{T(t)}(x, y) = 0$ when one of $x, y$ lies in $\Gamma(t)$ and the other in $\mathbb{T}(t)$. In potential theoretic terms, $f = G_{T(t)}(\cdot, y)$ is the smallest non-negative function on $\mathbb{T}(t) \cup \Gamma(t)$ satisfying $\Delta_* f(x) = -\delta_y(x)$ for $x \in \mathbb{T}(k)$. This corresponds directly to the disk situation.

(3.14) **Theorem.** Define $r = r_X$, $a = a_X$ and $b = b_X$ for $X = \mathbb{D}$ or $X = \mathbb{T}$ by

$$r_D = 7 \quad \text{and} \quad a_D = b_D = 18, \quad \text{resp.} \quad r_T = 1, \quad a_T = q/(q - 1) \quad \text{and} \quad b_T = 1.$$  

Let $0 < t < 1/r$. Then for any $x \in X^{(rt)}$, we have

$$G_X(x, o) \geq G_{X(t)}(x, o) \geq \frac{1}{a} G_X(x, o) \geq \frac{1}{b} \text{dist}(x, \partial X),$$

where $o$ is the origin (root) of $X$.

**Proof.** The first inequality is clear in both cases. The third inequality is also clear, and it is an equality in the tree case. We need to prove the second inequality separately for tree and disk, and begin this time with the tree.

**A. Tree case.** Let $\nu_k(x)$ be the harmonic measure of $\mathbb{T}(t)$ on its boundary. In particular, for $y \in \Gamma(t)$, the probability that the random walk starting at $x$ first hits $\Gamma(t)$ in $y$ is $\nu_k(x,y)$. The function $g_k(x) = G_T(x, o) - G_{T(t)}(x, o)$ is positive harmonic on $\mathbb{T}(t)$. We have

$$\lim_{x \to \xi} g_k(x) = 0 \quad \text{for } \xi \in \partial \infty \mathbb{T}(t) \quad \text{(because this holds for } G_T(x, o)),$$

and $g_k(x) = G_T(y, o)$ for $y \in \Gamma(t)$. Since the Dirichlet problem on $\mathbb{T}(t)$ admits solution (a straightforward
adaptation of [6, Thm.4], including in that argument vertices which are boundary points), we get that
\[ g^{(t)}(x) = \sum_{y \in \Gamma^{(t)}} G_{\Gamma^{(t)}}(y, o) \nu_{x}^{(t)}(y) = \frac{q}{q-1} q^{-k} \nu_{x}^{(t)}(\Gamma^{(t)}), \]
where \( k = k(t) \), as defined in (3.6). In the last identity (which can of course also be derived probabilistically), (2.18) was used. Now let \( x \in \Gamma^{(t)} \) and let \( x_0 \) be the last point on the geodesic \( \pi(o, x) \) that lies on some \( \pi(o, y) \) with \( y \in \Gamma^{(t)} \). Note that \( |x_0| \leq k - 1 \). In order to reach \( \Gamma^{(t)} \), the random walk starting at \( x \) needs to pass through \( x_0 \). Unless \( x = x_0 \), this is unrestricted random walk on \( T \) before the first visit in \( x_0 \), because up to that time it evolves on a branch of \( T \) that contains no element of \( \Gamma^{(t)} \). It is well known and easy to see that
\[ \Pr[\exists n: Z_n = x_0 \mid Z_0 = x] = G(x, x_0)/G(x_0, x_0), \]
see e.g. [20, Thm.1.38]. Thus (compare with [20, Prop.9.23]),
\[\nu_x^{(t)}(\Gamma^{(t)}) = \Pr[\exists n: Z_n = x_0 \mid Z_0 = x] \nu_{x_0}^{(t)}(\Gamma^{(t)}) \leq q^{-d_T(x, x_0)} = q^{\left|x_0|-|x|\right|} \leq q^{k-1-|x|}.\]
We infer that
\[ g_k(x) \leq \frac{q}{q-1} q^{-k} q^{k-1-|x|} = \frac{1}{q-1} q^{-|x|} \]
Consequently,
\[ G_{\Gamma^{(t)}}(x, o) = G_T(x, o) - g_k(x) = \frac{q}{q-1} q^{-|x|} - g_k(x) \geq q^{-|x|}, \]
and in view of (2.18), the proposed estimate is proved for the tree.

**B. Disk case.** The proof follows [10], but we re-elaborate it to get the constant \( a_D = 7 \) and to have \( G_D(z, 0) \) in the lower bound. As before, we prefer to write \( z \) instead of \( x \) for the elements of \( D \). We start in the same way as for the tree. We know that \( G_D(z, 0) = \log \frac{1}{|z|} \), and we can decompose
\[ G_{\Gamma^{(t)}}(z, 0) = G_D(z, 0) - g^{(t)}(z), \quad z \in \Gamma^{(t)}, \]
where \( g^{(t)} \) is harmonic on \( \Gamma^{(t)} \) with boundary values 0 at \( \partial_{\infty} \Gamma^{(t)} \). For \( z \in \Gamma^{(t)} \), there is \( \zeta \in E \) with \( |z - \zeta| = t \), whence \( |z| \geq 1 - t \). Thus, using (3.7),
\[ g^{(t)}(z) = G_{\Gamma^{(t)}}(z, 0) \leq \log \frac{1}{1 - t} \leq 3 \log \frac{1}{1 - t} \nu_x(E^{(t)}). \]
The right hand side is a harmonic function of \( z \) on the whole of \( D \). By the maximum principle, (3.15) holds on all of \( \Gamma^{(t)} \).

We now choose real parameters \( r > s > 1 \) with \( r - s > 1 \). We assume that \( t < 1/r \). Let \( z \in D \).

**Case 1.** Let \( |z| < (1 - t)^s \). Then \( g^{(t)}(z) \leq \log \frac{1}{1 - t} \leq \frac{1}{s} \log \frac{1}{|z|} \), and
\[ G_{\Gamma^{(t)}}(z, 0) \geq \frac{s - 1}{s} G_D(z, 0). \]
Case 2. Let \( z \in \mathbb{D}^{(rt)} \) with \( |z| \geq (1 - t)^s \). By the Bernoulli inequality, \( |z| \geq 1 - st \).

Following [10], we write \( z = |z|e^{i\theta} \) and

\[
\nu_z(E^{(t)}) = \int_{E^{(t)}} P(z, \xi) \, d\lambda_{\mathbb{D}}(\xi) = (1 - |z|^2) \frac{1}{2\pi} \int_{\{\varphi : e^{i\varphi} \in E^{(t)}\}} \frac{d\varphi}{(1 - |z|)^2 + 4|z| \sin^2 \frac{\varphi - \theta}{2}}.
\]

Then for \( \varphi \in (-\pi, \pi) \) with \( e^{i\varphi} \in E^{(t)} \), using \( rt \leq \text{dist}(z, E) \leq 1 - |z| + \text{dist}(e^{i\theta}, E) \),

\[
\pi \geq |\phi - \theta| \geq 2 \sin \frac{\varphi - \theta}{2} = |e^{i\theta} - e^{i\varphi}| \geq \text{dist}(e^{i\theta}, E) - t \geq rt - (1 - |z|) - t \geq \tau t,
\]

where \( \tau = r - s - 1 \). Combining these estimates with (3.13),

\[
g^{(t)}(z) \leq 3 \left( \log \frac{1}{1 - t} \right) (1 - |z|^2) \frac{1}{2\pi} \int_{\{\varphi : \tau t \leq |\varphi - \theta| \leq \pi\}} \frac{d\varphi}{(1 - |z|)^2 + 4|z| \sin^2 \frac{\varphi - \theta}{2}}
\]

\[
= \frac{6}{\pi} \left( \log \frac{1}{1 - t} \right) (1 - |z|^2) \int_{\tau t/2}^{\pi/2} \frac{d\varphi}{(1 - |z|)^2 + 4|z| \sin^2 \varphi}
\]

\[
= \frac{6}{\pi} \left( \log \frac{1}{1 - t} \right) \arctan \left( \frac{1 - |z|}{1 + |z|} \cot \left( \frac{\tau t}{2} \right) \right)
\]

\[
\leq \frac{6}{\pi} \left( \log \frac{1}{1 - t} \right) \left( \cot \left( \frac{\tau t}{2} \right) \right) (1 - |z|).
\]

Since \( rt < 1 < \pi/3 \), we have \( \tau t/2 < \pi/6 \), whence \( \cot(\tau t/2) \leq 2\pi/(3\tau t) \). Also, for \( 0 < t < 1/r \), we have \( \log 1/(1 - t) \leq rt/(r - 1) \). Therefore

\[
g^{(t)}(z) \leq \frac{4}{\tau t} \left( \log \frac{1}{1 - t} \right) (1 - |z|) \leq \frac{4r}{(r - 1)(r - s - 1)} \log \frac{1}{|z|}
\]

Thus, in Case 2,

\[
G_{\mathbb{D}^{(t)}}(z, 0) \geq \left( 1 - \frac{4r}{(r - 1)(r - s - 1)} \right) G_{\mathbb{D}}(z, 0).
\]

Choosing \( r = 7 \) and \( s = 18/17 \), we get the proposed estimate. \( \square \)

At the cost of increasing \( r \), one can get a better (bigger) lower bound on the disk. For our purpose, smaller \( r_{\mathbb{D}} \) will be better. The proof allows to take any number \( r > (7 + \sqrt{41})/2 \).

With \( u \) and \( \Psi \) as in Theorem 3.3, we would like to have a more general type of boundary moment to be finite, even when the integral in (3.4) is infinite. To this end, we consider a continuous, increasing function \( \Phi : [0, \text{diam}(X)] \to [0, \infty) \) with \( \Phi(0) = 0 \). With \( \Phi \) as well as with \( \Psi \), we associate the continuous, non-negative measures \( d\Phi \) and \( d\Psi \) on \( [0, \text{diam}(X)] \) which give mass \( \Phi(b) - \Phi(a) \), resp. \( \Psi(a) - \Psi(b) \) to any interval \( (a, b) \subset (0, \text{diam}(X)] \). Furthermore, we consider the decreasing, continuous function

\[
(3.16) \quad \Upsilon : [0, \text{diam}(X)] \to [0, \infty], \quad \Upsilon(t) = \int_t^{\text{diam}(X)} \Phi(s) \, d\Psi(s).
\]

It will (typically) occur that \( \Upsilon(0) = \infty \). We should consider \( \Upsilon \) as a downscaling of \( \Psi \); indeed, \( \Upsilon(t) \leq \|\Phi\|_{\infty} \Psi(t) \). If \( \Psi \) is differentiable on \( (0, \text{diam}(X)) \), then \( d\Psi(t) = -\Upsilon'(t) \, dt \), and \( \Upsilon'(t) = \Phi(t) \, \Psi'(t) \). The case considered in [10] is the one where \( \Psi(t) = t^{-q} \) and \( \Phi(t) = t^{\alpha} \), where \( 0 < \alpha < q \), so that \( \Upsilon(t) \asymp t^{\alpha-q} \).
(3.17) Theorem. Let the subharmonic function $u$ on $X$, the "singular" set $E \subset \partial X$ and the function $\Psi$ be as in Theorem 3.13 but with infinite integral in (3.14). For continuous, increasing $\Phi : [0, \text{diam}(X)] \to [0, \infty)$ with $\Phi(0) = 0$ and the associated function $\Upsilon(t)$ according to (3.16), suppose that

$$\int_{\partial X} \Upsilon(\text{dist}(\xi, E)) \, d\lambda(\xi) < \infty.$$ 

Then the Riesz measure $\mu^u$ satisfies the extended boundary moment condition

$$\int_X \text{dist}(x, \partial X) \Phi(\text{dist}(x, E)/R) \, d\mu^u(x) < \infty,$$

where $R = R_X$ is given by $R_\mathbb{D} = 14$, resp. $R_T = 1$.

For the disk case, when $\Psi(t) = t^{-q}$ and $\Phi(t) = t^\alpha$ ($0 < \alpha < q$), this boils down to Theorem 1-ii)-(7) of [10].

In typical instances, $\Phi$ will have the doubling property $\Phi(t/2) \geq C \cdot \Phi(t)$ for a fixed $C > 0$. In this case, division by $R$ can be omitted in (3.18) even on the disk.

(3.19) Corollary. Consider the disk. Under the assumptions of Theorem 3.17, if $1/\Psi$ is doubling and

$$\int_S \Psi(\text{dist}(\xi, E))^{1-\varepsilon} \, d\lambda_S(\xi) < \infty,$$

then

$$\int_S \text{dist}(x, S) \Psi(\text{dist}(x, E))^{-\varepsilon} \, d\mu^u(x) < \infty.$$

Proof of Theorem 3.17. Once again, the proof works in similar ways on disk and tree. We should keep in mind that on the tree, integrals with respect to the Riesz measure are infinite sums.

For most of the proof, we assume that $u(o)$ is finite. On the tree, this is always required, but on the disk, one may have $u(z) = -\infty$ on a set of measure 0. We shall briefly explain at the end how to handle the case $u(0) = -\infty$.

We take up the thread from the end of the proof of Theorem 3.3 in particular (3.11). That inequality tells us that $u$ has $c_X h^{(t)}$ as a harmonic majorant on $X^{(t)}$. Thus, it has its least harmonic majorant $v^{(t)}$ on that set, and we have the Riesz decomposition

$$u(x) = v^{(t)}(x) - G^{(t)}_{X^{(t)}} \mu^u(x), \quad x \in X^{(t)}.$$

We have $G^{(t)} (z, 0) \geq 1 - |z| = d_{\mathbb{D}}(z, S)$ on the disk, and $G^{(t)} (x, o) = b_T \rho_T(x, \partial T)$. Using Theorem 3.14 we get for $0 < t < 1/r$ ($r = r_X$)

$$\int_{X^{(t)}} \text{dist}(x, \partial X) \, d\mu^u(x) \leq b_X G^{(t)}_{X^{(t)}} \mu^u(o)$$

$$= b_X (v^{(t)}(o) - u(o)) \leq b_X c_X h^{(t)}(o) - b_X u(o)$$

$$= b_X c_X \int_{E^{(t)}} \Psi(\text{dist}(\cdot, E)) \, d\lambda + b_X c_X \Psi(t) \lambda(E^{(t)}) - b_X u(o).$$

(In the disk case, $o$ stands once more for the origin.) For the next computation, we note that $\max\{\text{dist}(x, E) : x \in X\}$ has value 1 for the tree, but may be between 1 and 2 for the
disk. Tacitly using continuity of the involved measures, and using monotonicity of $\Psi$, for $0 < t < 1$

\[ \int_{E_s(t)} \Psi(\text{dist}(\xi, E)) \, d\lambda(\xi) = \int_{E_s(t) \cap E_s(\xi)} \Psi(\text{dist}(\xi, E)) \, d\lambda(\xi) + \int_{E_s(t)} \Psi(\text{dist}(\xi, E)) \, d\lambda(\xi) \]

\[ \leq \int_{E_s(t) \cap E_s(\xi)} \int_0^1 \, d\Psi(s) \, d\lambda(\xi) + \Psi(1) \lambda(E_s(t) \cap E_s(\xi)) = \int_0^1 \lambda(\{\xi \in \partial \mathbb{D} : t < \text{dist}(\xi, E) \leq s\}) \, d\Psi(s) + \Psi(1) \lambda(E_s(\xi)) \]

\[ = \int_0^1 \lambda(E_s(\xi)) \, d\Psi(s) - \lambda(E_s(t)) \Psi(t) + \Psi(1) . \]

Combining this with the previous inequality, we get for $0 < t < 1$

\[ (3.20) \quad \int_{x \in \mathbb{T}(t)} \text{dist}(x, \partial \mathbb{X}) \, d\mu^u(x) \leq b^\mathbb{X} c^\mathbb{X} \int_{t/r}^1 \lambda(E_s(\xi)) \, d\Psi(s) + C_1 , \]

where $C_1 = b_\mathbb{X} c_\mathbb{X} \Psi(1) - b_\mathbb{X} u(o)$. Because of several smaller subtleties, we now conclude the proofs separately.

**A. Tree case.** Recalling that $b_\mathbb{T} = r_\mathbb{T} = R_\mathbb{T} = 1$,

\[ \sum_{x \in \mathbb{T}} \rho_\mathbb{T}(x, \partial \mathbb{T}) \Phi(\rho_\mathbb{T}(x, E)) \mu^u(x) = \sum_{x \in \mathbb{T}} \rho_\mathbb{T}(x, \partial \mathbb{T}) \int_0^{\rho_\mathbb{T}(x, E)} \, d\Phi(t) \mu^u(x) \]

\[ = \int_0^1 \left( \sum_{x \in \mathbb{T}} \rho_\mathbb{T}(x, \partial \mathbb{T}) \mu^u(x) \right) \, d\Phi(t) \]

\[ \leq c_\mathbb{T} \int_0^1 \int_t^1 \lambda_\mathbb{T}(E_s(\xi)) \, d\Psi(s) \, d\Phi(t) + C_1 \, \Phi(1) \]

\[ [\text{Fubini}] \quad = c_\mathbb{T} \int_0^1 \lambda_\mathbb{T}(E_s(\xi)) \, d\Psi(s) + C_2 = c_\mathbb{T} \int_{\partial \mathbb{T}} \gamma(\rho_\mathbb{T}(\xi, E)) \, d\lambda_\mathbb{T}(\xi) + C_2 , \]

which is finite by assumption.

**B. Disk case.** Note that the maximum possible value of $d_\mathbb{D}(z, E)$ is 2. We refer to a simple observation of [10]: if $0 < t < 2$ then for every $z \in \mathbb{D}$ and $\alpha \in [0, 1]$, we have $d_\mathbb{D}(z, E) \leq 2d_\mathbb{D}(\alpha z, E)$. In particular, if $d_\mathbb{D}(z, E) > t$ then $d_\mathbb{D}(\alpha z, E) > t/2$, so that $z$ lies in the component of 0 of the set $\{w \in \mathbb{D} : d_\mathbb{D}(w, E) > t/2\}$. This means that

\[ (3.21) \quad \{z \in \mathbb{D} : d_\mathbb{D}(z, E) > t\} \subset \mathbb{D}^{(t/2)} . \]
Using this, we now compute
\[ \int_{\mathbb{D}} d_{\mathbb{D}}(z, S) \Phi(d_{\mathbb{D}}(z, E)/14) \, d\mu^u(z) = \int_{\mathbb{D}} \int_{0}^{d_{\mathbb{D}}(z,E)/14} d_{\mathbb{D}}(z, S) \, d\Phi(t) \, d\mu^u(z) \]
\[ = \int_{0}^{1/7} \int_{\{z \in \mathbb{D} : d_{\mathbb{D}}(z,E) > 14t\}} d_{\mathbb{D}}(z, S) \, d\mu^u(z) \, d\Phi(t) \]
\[ \leq \int_{0}^{1/7} \int_{\mathbb{D}(7t)} d_{\mathbb{D}}(z, S) \, d\mu^u(z) \, d\Phi(t) \]
\[ \leq b_X c_X \int_{0}^{1} \int_{0}^{1} \lambda(E(s)) \, d\Psi(s) + C_1 \Phi(1), \]
which is seen to be finite by the same calculation as in the tree case.

The case when \( u(0) = -\infty \) can be treated exactly as in [10, p.43] (where the subharmonic function is denoted \( v \)) and is omitted here. \( \square \)

Finally, we want to prove a converse to Theorem 3.17 analogous to Proposition 3.12.

**Theorem.** Let the set \( E \subset \partial X \) and the function \( \Psi \) be as in Theorem 3.3, but with infinite integral in (3.4). Let \( \Phi : [0, 1] \to [0, \infty) \) be continuous and increasing with \( \Phi(0) = 0 \) and \( \Phi(t) > 0 \) for \( t > 0 \). For the associated function \( \Upsilon(t) \) according to (3.16), suppose that
\[ \int_{\partial X} \Upsilon(\text{dist}(\xi, E)) \, d\lambda(\xi) = \infty. \]
If \( u \) is a subharmonic function on \( X \) such that
\[ u(x) \geq \Psi(\text{dist}(x, E)) \]
then the Riesz measure \( \mu^u \) is such that
\[ (3.23) \quad \int_{X} \text{dist}(x, \partial X) \Psi(\text{dist}(x, E)) \, d\mu^u(x) = \infty. \]

**Proof.** First of all, we note that (3.23) hold if and only if
\[ (3.24) \quad \int_{X} G(x, o) \Phi(\text{dist}(x, E)) \, d\mu^u(x) = \infty. \]
On the tree, this is obvious, because \( G_T(x, o) = \frac{q}{q-1} \rho_T(x, \partial T) \). On the disk, it is clear that (3.23) implies (3.24). Conversely,
\[ \int_{|z|<1/2} G(z, 0) \Phi(d_{\mathbb{D}}(z, E)) \, d\mu^u(z) \leq \|\Phi\|_{\infty} \int_{|z|<1/2} G(z, 0) \, d\mu^u(z) < \infty, \]
while for \( |z| \geq 1/2 \), we have \( G(z, 0) = \log \frac{1}{|z|} \leq (2 \log 2)(1 - |z|) \), so that (3.24) implies
\[ 2 \log 2 \int_{|z|\geq1/2} (1 - |z|) \Phi(d_{\mathbb{D}}(z, E)) \, d\mu^u(z) \geq \int_{|z|\geq1/2} G(z, 0) \Phi(d_{\mathbb{D}}(z, E)) \, d\mu^u(z) = \infty. \]
Case 1. Suppose that there is \( t \in (0, 1) \) such that \( u \) has no harmonic majorant on the set \( X(t) \). Then \( G_{X(t)} \mu^u \) is infinite on that set. Thus,
\[
\int_X G(x, o) \Phi(\text{dist}(x, E)) \, d\mu^u(x) \geq \int_{X(t)} G_{X(t)}(x, o) \Phi(\text{dist}(x, E)) \, d\mu^u(x) \\
\geq \Phi(t) G_{X(t)} \mu^u(o) = \infty,
\]
and the equivalence of (3.23) with (3.24) implies the result.

Case 2. We are left with the case when for each \( t \in (0, 1) \) there is the (finite) least harmonic majorant \( v(t) \) of \( u \) on \( X(t) \). Recall the function \( h(t) \) of (3.8). Then for every \( \eta \in \partial X(t) \),
\[
\limsup_{x \to \eta} v(t)(x) \geq \limsup_{x \to \eta} u(x) \geq \Psi(\text{dist}(\eta, E)) = \lim_{x \to \eta} h(t)(x).
\]
By the minimum principle, applied to the harmonic function \( v(t) - h(t) \), we have \( v(t) \geq h(t) \) on \( X(t) \). Now we can replace the computations of the proof of Theorem 3.17 with similar inequalities in the reverse direction.

\[
\int_{X(t)} G(x, o) \, d\mu^u(x) \geq G_{X(t)} \mu^u(o) = v(t)(o) - u(o) \geq h(t)(o) - u(o)
\]
\[
= \int_{E(t) \cap E(t)} \Psi(\text{dist}(\cdot, E)) \, d\lambda + \int_{E(t)} \Psi(\text{dist}(\cdot, E)) \, d\lambda + \Psi(t) \lambda(E(t)) - u(o)
\]
\[
\geq \int_{t}^{1} \lambda(E^{(s)} \setminus E^{(t)}) \, d\Psi(s) + \Psi(1) \lambda(E^{(1)} \setminus E^{(t)}) + \Psi(1) \lambda(E^{(1)} \setminus E^{(t)}) + \Psi(t) \lambda(E^{(t)}) - u(o)
\]
\[
= \int_{t}^{1} \lambda(E^{(s)}) \, d\Psi(s) + C_3, \quad \text{where} \quad C_3 = \Psi(1) - u(o).
\]
Now let \( 0 < \varepsilon < 1 \). Let \( \Phi_\varepsilon(s) = \max\{\Phi(s) - \Phi(\varepsilon), 0\} \). Since \( u \) has a harmonic majorant on \( X(\varepsilon) \), the first integral in the following computation is finite. The above estimate is used in the third line.

\[
\int_{X(\varepsilon)} G(x, o) \Phi(\text{dist}(x, E)) \, d\mu^u(x) \geq \int_{X(\varepsilon)} G(x, o) \int_{\varepsilon}^{\text{dist}(x, E)} d\Phi(t) \, d\mu^u(x)
\]
\[
\geq \int_{\varepsilon}^{1} \int_{X(t)} G(x, o) \, d\mu^u(x) \, d\Phi(t)
\]
\[
\geq \int_{\varepsilon}^{1} \int_{t}^{1} \lambda(E^{(s)}) \, d\Psi(s) \, d\Phi(t) + (1 - \varepsilon)C_3
\]
\[
= \int_{\varepsilon}^{1} \lambda(E^{(s)}) \int_{\varepsilon}^{1} \, d\Phi(t) \, d\Psi(s) + (1 - \varepsilon)C_3
\]
\[
= \int_{0}^{1} \int_{\xi \in \partial X : \text{dist}(\xi, E) \leq s} d\lambda(\xi) \Phi_\varepsilon(s) \, d\Psi(s) + (1 - \varepsilon)C_3
\]
\[
= \int_{E(t)} \int_{\text{dist}(\xi, E)} \Phi_\varepsilon(s) \, d\Psi(s) \, d\lambda(\xi) + (1 - \varepsilon)C_3
\]
As \( \varepsilon \to 0 \), by monotone convergence, the double integral in the last line tends to
\[
\int_{E(t)} \left( \Upsilon(\text{dist}(\xi, E)) - \Upsilon(1) \right) \, d\lambda(\xi),
\]
which is infinite by assumption.

\( \square \)

\((3.25)\) **Remarks.** (a) *[Hyperbolic versus Euclidean.]* In the introduction and in Section 2 we insisted on a hyperbolic “spirit” inherent in the material presented here. After all, this was not dominant in most of our computations. Not only on the disk, we always used the Euclidean metric \( d_D \), but also on the tree, the dominant role was played by the metric \( \rho_T \) which is the tree-analogue of the Euclidean metric. One point is that to see the latter analogy, one should first understand that the graph metric on the tree corresponds to the hyperbolic one on the disk.

One result where hyperbolicity is strongly present is Theorem 3.14. The proof in the tree case relies directly on the fact that the tree with its graph metric is \( \delta \)-hyperbolic in the sense of GROMOV [14], with \( \delta = 0 \): every vertex is a cut-point (it disconnects the tree). Analogously, one might try to prove that theorem in the disk case using \( \delta \)-hyperbolicity with \( \delta = \log(1 + \sqrt{2}) \). Indeed, this is related with the inequalities of ANCONA [1] which say that the Green kernel of the open disk is almost submultiplicative along hyperbolic geodesics. (For the disk, this can be seen by direct inspection via the explicit formulas for the Green kernel.) Now, for points \( z \in \mathbb{D}(rt) \) and \( \xi \in E(t) \), the hyperbolic geodesic from \( z \) to \( \xi \) must be at bounded hyperbolic distance from the origin (depending on \( r \) and \( t \)), similarly to the (simpler) tree case. However, this idea is more vague than the down-to-earth proof following [10].

(b) In view of the equivalence \((3.23) \iff (3.24)\), in all the results presented here, one can replace the distance to the boundary \( \text{dist}(x, \partial X) \) with the Green kernel \( G(x, o) \).

(c) Among the common features of disk and tree which allowed us to formulate and prove the results in very similar ways, the key facts are

- comparability of \( G(x, o) \) with \( \text{dist}(x, \partial X) \) (the metric is “intrinsic” in this sense),
- solvability of the Dirichlet problem for continuous functions on \( \partial X \), and in particular, vanishing of the Green kernel at the boundary, and
- the Green kernel estimate of Theorem 3.14.

\((3.26)\) **An extension for trees.** Instead of the homogeneous tree, we can take an arbitrary locally finite tree \( T \) and equip its edges with conductances \( a(x, y) = a(y, x) > 0 \iff x \sim y \). Letting \( m(x) = \sum_y a(x, y) \), the transition probabilities \( p(x, y) = a(x, y)/m(y) \) give rise to a nearest neighbour random walk \( (Z_n)_{n \geq 0} \) and to the associated Laplacian

\[
\Delta_T f(x) = \sum_{y \sim x} p(x, y)(f(y) - f(x)).
\]

We assume the following.

(i) Strong irreducibility: \( 0 < m_0 \leq m(x) \leq M_0 < \infty \) and \( a(x, y) \geq a_0 > 0 \) for all \( x \) and all \( y \sim x \).

(ii) Strong transience: \( F(x, y) \leq \delta < 1 \) for all \( x \) and all \( y \sim x \), where for arbitrary \( x, y \in T \),

\[
F(x, y) = \Pr[\exists n \geq 0 : Z_n = y \mid Z_0 = x].
\]
The associated Green kernel

\[ G(x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y), \quad \text{where} \quad p^{(n)}(x, y) = \Pr[Z_n = y \mid Z_0 = x], \quad x, y \in X \]

is finite and tends to 0 at infinity by assumption (ii). Note that in our notation, \( G(x, y) = F(x, y)G(y, y) \).

We can adapt all the above results regarding the homogenous tree to this more general situation. The main issue is to define a suitable metric on the compactification \( \hat{T} \) in the right way: for \( z, w \in \hat{T} \),

\[
\rho_T(w, z) = \begin{cases} 
F(w \land z, o), & \text{if } z \neq w, \\
0, & \text{if } z = w.
\end{cases}
\]

[For simple random walk on the homogeneous tree, as considered above, this is just the metric of (2.20).]

In this setting, the tree-versions of theorems 3.3, 3.17 and 3.22 remain true. This applies, in particular, to arbitrary symmetric nearest neighbour random walks on the free group (≡ homogeneous tree with even degree).

In conclusion, we remark that the very recent note by Favorov and Radchenko [12] was written in parallel to the present article without mutual knowledge. The results of [12] concern the disk case and are a bit less general than ours. We want to point out that here, our main focus has been on elaborating some aspects of the very strong analogies of the potential theory on disk and tree, respectively, via focussing on properties of Riesz measures.

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