Representing subalgebras as retracts of finite subdirect powers

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Abstract. We prove that if $A$ is an algebra that is supernilpotent with respect to the 2-term higher commutator, and $B$ is a subalgebra of $A$, then $B$ is representable as a retract of a finite subdirect power of $A$.

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1. Introduction

The paper [11] by Peter M. Neumann opens with the remark that, in conversations with other group theorists, the following question arose several times: Must a formation of finite nilpotent groups be closed under subgroups? Since the formation generated by a finite group $A$ is the class of groups that are representable as homomorphic images of finite subdirect powers of $A$, this question is equivalent to the following one: If $A$ is a finite nilpotent group and $B$ is a subgroup of $A$, must $B$ be representable as a homomorphic image of a finite subdirect power of $A$?

Neumann proves that the answer is Yes. His proof relies on the nonobvious result of Michael Vaughan-Lee that any characteristic subgroup of a relatively free class-$c$ nilpotent group of rank $k$ must be fully invariant, provided $k > c$ (cf. [12]).

Using different ideas, our paper extends Neumann’s result from finite groups to arbitrary algebraic structures. Where Neumann focuses on finite groups that are nilpotent, we focus instead on possibly-infinite algebras that
are supernilpotent with respect to the 2-term higher commutator. The present
result is an extension of the older one, since nilpotent groups are supernilpotent
with respect to the 2-term higher commutator.

We close the paper by describing an algebra \( A \) that shows that the
assumption of ‘supernilpotence’ in our theorem cannot be weakened to ‘nilpo-
tence’. Our algebra has the following properties:

1. \( A \) is an expansion of a finite group.
2. \( A \) is 2-step nilpotent as an algebraic structure, but it is not supernilpotent.
3. \( A \) has a subalgebra \( B \) that is not representable as a homomorphic image
   of a finite subdirect power of \( A \).

2. Discussion

Our main theorem is

**Theorem 2.1.** If \( A \) is an algebra that is supernilpotent with respect to the 2-
term higher commutator, and \( B \) is a subalgebra of \( A \), then \( B \) is representable
as a retract of a finite subdirect power of \( A \).

In the Introduction we described the result in a slightly weaker form, namely that

**Theorem 2.2.** If \( A \) is an algebra that is supernilpotent with respect to the 2-
term higher commutator, and \( B \) is a subalgebra of \( A \), then \( B \) is representable
as a quotient of a finite subdirect power of \( A \).

The difference in the two forms is that \( B \) is a *quotient* of \( D \) if there is a
surjective homomorphism \( \mu : D \to B \), while \( B \) is a *retract* of \( D \) if there is a
surjective homomorphism \( \mu : D \to B \) which has a homomorphism \( \nu : B \to D \)
that is a right inverse (\( \mu \circ \nu = \text{id}_B \)).

The main result can be phrased as a theorem about formations of alge-
bras.

2.1. Classes of algebras

We use the following terminology for classes of algebras of the same type.

1. A *variety* is a class of algebras definable by a set of identities.
2. (Provisional) A *pseudovariety* is a class of finite algebras closed under the
   construction of homomorphic images, subalgebras, and finite products.
3. (Provisional) A *formation* is a class of finite algebras closed under the
   construction of homomorphic images and finite subdirect products.
4. A class of algebras is *hereditary* if it is closed under subalgebras.
5. A class of algebras is *axiomatic* if it first-order axiomatizable.

In this paper we shall allow all classes to contain infinite algebras, so we set
aside the two provisional definitions above (formation, pseudovariety) and
adopt these definitions instead: a *formation* is any class of similar algebras
closed under the construction of homomorphic images and finite subdirect
products, while a pseudovariety is any class of similar algebras closed under
the construction of homomorphic images, subalgebras, and finite products.
Under either set of definitions, a class is a pseudovariety if and only if it is a hereditary formation. Moreover, under the adopted definitions, we have:

**Theorem 2.3.** (Theorem 1.1 [5]) Let \( \mathcal{V} \) be a class of similar algebraic structures. The following conditions are equivalent:

1. \( \mathcal{V} \) is a variety.
2. \( \mathcal{V} \) is closed under homomorphic images, subalgebras and products.
3. \( \mathcal{V} \) is closed under homomorphic images and subdirect products.
4. \( \mathcal{V} \) is closed under homomorphic images, subalgebras, finite products, and \( \mathcal{V} \) is axiomatizable.
5. \( \mathcal{V} \) is an axiomatic formation that is closed under subalgebras.

Hence \( \mathcal{V} \) is a variety if and only if it is an axiomatizable pseudovariety if and only if it is an axiomatizable hereditary formation.

It is not hard to produce formations that are not hereditary, that is, are not closed under the construction of subalgebras. For example the formation generated by the alternating group \( A_5 \) consists of the groups isomorphic to finite powers of \( A_5 \), so the proper nontrivial subgroups of \( A_5 \) are in the pseudovariety generated by \( A_5 \) but not in the formation generated by \( A_5 \).

Using the terminology of formations, the most obvious corollary of our main theorem is the following:

**Corollary 2.4.** Let \( \mathcal{F} \) be a formation of algebras whose members are supernilpotent in the sense of the 2-term higher commutator. Then

1. \( \mathcal{F} \) is hereditary. (Equivalently, \( \mathcal{F} \) is a pseudovariety.)
2. \( \mathcal{F} \) is a variety if and only if \( \mathcal{F} \) is axiomatic.

An easily-derived corollary of Theorem 2.2, which is also best stated in terms of classes of algebras is the following:

**Corollary 2.5.** Let \( \mathcal{V} \) be a variety of algebras whose members are supernilpotent in the sense of the 2-term higher commutator. If \( X \in \mathcal{V} \) is finite, then any finite member of the variety generated by \( X \) is representable as a quotient of a finite subdirect power of \( X \).

[Idea of proof:] Any finite algebra \( Y \in \mathcal{V}(X) \) is a quotient of a finite, relatively free algebra \( F_{\mathcal{V}(X)}(|Y|) \). Since \( F_{\mathcal{V}(X)}(|Y|) \) is isomorphic to a subalgebra of \( X^{[X]|Y|} \), we can apply Theorem 2.2 to \( A = X^{[X]|Y|} \) and \( B = F_{\mathcal{V}(X)}(|Y|) \) to obtain that \( F_{\mathcal{V}(X)}(|Y|) \) is isomorphic to a quotient of a finite subdirect power of \( X \). By taking a further quotient, we get that \( Y \) is isomorphic to a quotient of a finite subdirect power of \( X \).

We shall derive Theorem 2.1 from a slightly more general statement, Theorem 3.1. Before proving that result we clarify some of the language used in Theorem 2.1.

### 2.2. 2-term commutator versus ordinary commutator

The theory of the ordinary 1-term binary commutator, \([\alpha, \beta]\), is developed for arbitrary algebraic structures in Chapter 3 of [3] by Hobby and McKenzie. The
2-term binary commutator, $[\alpha, \beta]_2$, was introduced by Emil Kiss in [8]. Kiss proves in [8] that the 2-term binary commutator is larger or equal to the ordinary binary commutator ($[\alpha, \beta] \leq [\alpha, \beta]_2$), and that the two are equal on any algebra in a congruence modular variety. In [6], the equality of these and other commutators is proved to hold for any variety satisfying a nontrivial idempotent Maltsev condition. These concepts (2-term commutator versus ordinary commutator) for the binary commutator have analogues for higher arity commutators. The comparability of the $k$-ary ordinary higher commutator with the $k$-ary 2-term higher commutator ($[\alpha, \beta, \gamma, \ldots, \omega] \leq [\alpha, \beta, \gamma, \ldots, \omega]_2$) may be proved in the same way it was proved for the binary commutator. The equality of the ordinary higher commutator and the 2-term higher commutator for congruence modular varieties was proved by Andrew Moorhead in [9, Theorem 6.4]. Moorhead has recently extended this result to show (in [10]) that the ordinary higher commutator agrees with the 2-term higher commutator in any variety satisfying a nontrivial idempotent Maltsev condition. Hence we can rephrase Theorem 2.1 to eliminate the “2-term” part, provided we restrict the scope of the theorem to algebras satisfying a nontrivial idempotent Maltsev condition. The theorem would then read:

Corollary 2.6. Assume that $A$ is an algebra satisfying a nontrivial idempotent Maltsev condition. If $A$ is supernilpotent, and $B$ is a subalgebra of $A$, then $B$ is representable as a retract of a finite subdirect power of $A$.

Hence, if $\mathcal{F}$ is a formation of supernilpotent algebras in which every member satisfies a nontrivial idempotent Maltsev condition, then $\mathcal{F}$ is a pseudovariety.

2.3. Supernilpotence versus ordinary nilpotence.

Supernilpotence is a form of nilpotence defined in terms of the higher commutator. It is proved in [7] that supernilpotence implies nilpotence for finite algebras. From the results of [1,4], the exact degree to which supernilpotence is stronger than nilpotence is well understood for finite algebras satisfying nontrivial idempotent Maltsev conditions. This allows us to rewrite Corollary 2.6 in a way that refers to nilpotence instead of supernilpotence:

Corollary 2.7. Assume that $A$ is a finite algebra satisfying a nontrivial idempotent Maltsev condition. If $A$ is a product of nilpotent algebras of prime power cardinality and $B$ is a subalgebra of $A$, then $B$ is representable as a retract of a finite subdirect power of $A$.

Hence, if $\mathcal{F}$ is a formation of finite nilpotent algebras in which every member satisfies a nontrivial idempotent Maltsev condition, and every directly indecomposable member of $\mathcal{F}$ has prime power cardinality, then $\mathcal{F}$ is a pseudovariety.

This consequence of Theorem 2.1 includes Neumann’s result, since (i) any group satisfies a nontrivial idempotent Maltsev condition, and (ii) a finite, directly indecomposable, nilpotent group has prime power cardinality.
2.4. When $A$ satisfies no nontrivial idempotent Maltsev condition.

It would be wrong to treat Corollaries 2.6 and 2.7 as if they expressed the essential content of Theorem 2.1. If $A$ is a finite nilpotent algebra, and $A$ satisfies a nontrivial idempotent Maltsev condition, then it can be shown that $A$ must have a Maltsev term operation $p(x, y, z)$, i.e. a term operation for which $A \models p(x, x, y) \approx y \approx p(y, x, x)$. By Corollary 7.4 of [2], the nilpotence of $A$ forces $p$ to be invertible in its first and last variables, so this term operation behaves much like the group term operation $p(x, y, z) = xy^{-1}z$ which controls most of a group’s properties. (The term operations of any group are generated by $p(x, y, z) = xy^{-1}z$ along with the constant 1.) In this circumstance $A$ is a “group-like” algebra. In this context, Corollaries 2.6, 2.7 are only slight extensions of what was already known. The real interest in Theorem 2.1 should be in what it says about formations with members that do not satisfy any nontrivial idempotent Maltsev condition, as in the following corollary.

Corollary 2.8. Assume that $A$ is an algebra which has a finite bound on the essential arity of its term operations. If $B$ is a subalgebra of $A$, then $B$ is representable as a retract of a finite subdirect power of $A$.

Hence, if $F$ is a formation of algebras and every member of $F$ has a finite bound on the essential arity of its term operations, then $F$ is a pseudovariety.

This corollary is a consequence of Theorem 2.1, since any algebra of essential arity at most $k$ will be supernilpotent of class at most $k$ in the sense of the 2-term higher commutator.

Even the following consequence of Corollary 2.8 seems to be new.

Corollary 2.9. Any formation of unary algebras is a pseudovariety.

3. The proof of Theorem 2.1

We begin this section by defining what it means for a congruence to be supernilpotent of class $c = 2$ with respect to the 2-term higher commutator, and then we indicate how to modify the definition for other values of $c$.

Let $\theta$ be a congruence on an algebra $A$. We shall define a subalgebra $M(\theta, \theta, \theta)$ of $A^{2^3}$ by indicating its generating $2^3$-tuples. The coordinates of a $2^3$-tuple are the elements of the set $2^3 = \{0,1\}^3$. Since a coordinate of a tuple $t \in A^{2^3}$ is itself a tuple $(a, b, c) \in 2^3$, and since we dread the prospect of referring to “coordinates of coordinates”, we shall refer to an element $(a, b, c) \in 2^3$ as an “address of a coordinate” if we are treating it as a string of 0’s and 1’s, and we wish to refer to different places in the string.

We arrange the coordinates in the shape of a 3-dimensional cube by stipulating that two coordinates are adjacent if the Hamming distance of their addresses is 1. See Figure 1.

We have labeled the first, second, and third spatial axes of the cube with the letters $x$, $y$, and $z$. If we move from one address to another in the $x$-direction, then the address changes only in its first place, i.e., addresses of the form $(0, b, c)$ change to $(1, b, c)$. Similarly, if we move from one address to
another in the $y$- or $z$-directions, then the address changes only in its second or third places. The \textit{last coordinate} among all $2^3$ coordinates will be the coordinate with address $(1,1,1)$. The \textit{earlier coordinates} will be all coordinates that are not the last coordinate.

The \textit{standard generators of $M(\theta, \theta, \theta)$ in the $x$-direction} are all tuples in $A^{2^3}$ which have the form indicated in Figure 2, where $(u,v) \in \theta$.

That is, for each pair $(u,v) \in \theta$ there is a standard generator of $M(\theta, \theta, \theta)$ in the $x$-direction, which is a $2^3$-tuple $t$, where at every address $\sigma = (x,y,z)$ with $x = 0$ we have coordinate value $t_\sigma = u$ and at every address $\sigma$ with $x = 1$ we have coordinate value $t_\sigma = v$. (Visually, $t_\sigma = u$ on the “$x = 0$ hyperface” of the cube, while $t_\sigma = v$ on the “$x = 1$ hyperface”.) Define standard generators in the $y$- and $z$-directions analogously.

$M(\theta, \theta, \theta)$ is defined to be the subalgebra of $A^{2^3}$ that is generated by the standard generators in all directions. This subalgebra contains the diagonal of $A^{2^3}$, since the constant tuples $( = \text{diagonal elements})$ of $A^{2^3}$ are standard generators in each direction.

A congruence $\theta$ is called \textit{supernilpotent} (of class 2) with respect to the 2-term higher commutator if whenever two elements $s, t \in M(\theta, \theta, \theta)$ agree at all earlier coordinates, then they also agree at the last coordinate. This can be expressed in many different ways, e.g.:

(1) (The definition) If $s, t \in M(\theta, \theta, \theta)$ and $s_\sigma = t_\sigma$ for all $\sigma \in \{0,1\}^3$ other than $\sigma = (1,1,1)$, then we also have $s_{(1,1,1)} = t_{(1,1,1)}$. 

Figure 1. Eight coordinates, arranged by adjacency of address

Figure 2. A standard generator in the $x$-direction
(2) $M(\theta, \theta, \theta)$ is the graph of a function whose domain is the projection of $M(\theta, \theta, \theta)$ onto the earlier coordinates and whose range is the projection of $M(\theta, \theta, \theta)$ onto the last coordinate.

(3) The projection of $M(\theta, \theta, \theta)$ onto its seven earlier coordinates is a bijection.

(4) (Since $M(\theta, \theta, \theta)$ is a subalgebra of $A^{23}$): $M(\theta, \theta, \theta)$ is the graph of a partial homomorphism $\mu : A^7 \rightarrow A$. I.e., $M(\theta, \theta, \theta)$ is the graph of a homomorphism $\mu : D \rightarrow A$ for $D (\leq A^7)$ equal to the projection of $A^{23}$ to its seven earlier coordinates.

In terms of the ternary higher commutator, these conditions may be expressed by writing $[\theta, \theta, \theta]_2 = 0$.

The only difference between supernilpotence of class 2 and supernilpotence of class $c$ is that for class $c$ we use $(c+1)$-dimensional hypercubes in place of 3-dimensional cubes. The vertices of these cubes have addresses in $\{0, 1\}^{c+1}$, $x_1$- through $x_{c+1}$-spatial directions, $M(\theta, \ldots, \theta)$ is generated by standard generators in all of the directions, and $\theta$ is supernilpotent of class $c$ with respect to the 2-term higher commutator if the last coordinate of any tuple in $M(\theta, \ldots, \theta)$ depends functionally on the earlier coordinates.

For more about supernilpotence and higher commutator theory see [1, 9].

We need one more definition before proceeding. If $A$ is an algebra, $B \subseteq A$ is a subset, and $\theta \in \text{Con}(A)$ is a congruence, then

$$B^\theta = \bigcup_{b \in B} b/\theta = \{a \in A \mid (a, b) \in \theta \text{ for some } b \in B\}$$

is the saturation of $B$ by $\theta$. The saturation of $B$ by $\theta$ is the least subset of $A$ containing $B$ that is a union of $\theta$-classes. If $B$ is a subuniverse of $A$, then it can be shown that $B^\theta$ is also a subuniverse.

**Theorem 3.1.** Let $A$ be an algebra, and let $B$ be a subalgebra of $A$ and $\theta \in \text{Con}(A)$. If

1. $B^\theta = A$,
2. $\theta$ is supernilpotent with respect to the 2-term higher commutator,

then $B$ is a retract of a finite subdirect power of $A$.

**Proof.** We shall draw pictures as if we work in three spatial dimensions, but it will be clear that the arguments we give work in dimension $c + 1$ for any supernilpotence class $c$.

Let $\Gamma$ be the subset of the standard generators of $M(\theta, \ldots, \theta)$ consisting of only those standard generators (in all directions) where the value in the last coordinate lies in the subalgebra $B$, as indicated in Figure 3.

**Claim 3.2.**

1. Every tuple in $\Gamma$ has last coordinate in $B$.
2. For any $b \in B$, $\Gamma$ contains a tuple whose last coordinate is $b$.
3. For any $a \in A$ and any earlier coordinate $\sigma$, $\Gamma$ contains a tuple $t$ such that $t_\sigma = a$. 
Item (1) of Claim 3.2 is part of the definition of $\Gamma$.

Item (2) of Claim 3.2 follows from the fact that $\Gamma$ contains all of the diagonal tuples with diagonal value in $B$.

For Item (3) of Claim 3.2, fix any $a \in A$ and any coordinate $\sigma$ other than the last coordinate. Since $B^\theta = A$, there exists some $b \in B$ such that $(a, b) \in \theta$. We explain why there is a tuple $t \in \Gamma$ which has $b$ in the last coordinate and $a$ in coordinate $\sigma$.

Split the hypercube into two parallel hyperfaces in a way the puts the last coordinate in a different hyperface than coordinate $\sigma$. Let $t$ be the standard generator whose coordinate value is $b$ in all coordinates of the hyperface containing the last coordinate, and is $a$ in all coordinates of the hyperface containing coordinate $\sigma$. (See Figure 4.) This $t \in \Gamma$ satisfies the condition in Item (3) of Claim 3.2.

Now we conclude the proof of the theorem. Let $\mu = \langle \Gamma \rangle$ be the subalgebra of $A^{2c+1}$ that is generated by $\Gamma$. Because $\Gamma \subseteq M(\theta, \ldots, \theta)$, we get that $\mu$ is a subalgebra of $M(\theta, \ldots, \theta)$. Since $M(\theta, \ldots, \theta)$ is a functional relation from its first $2c+1 - 1$ coordinates to its last coordinate, $\mu$ is also a functional relation.

Items (1) and (2) of Claim 3.2 guarantee that every element of $B$ and only elements of $B$ appear in last coordinates of tuples of $\mu$. If $D \leq A^{2c+1-1}$ is the projection of $\mu$ onto its first $2c+1 - 1$ coordinates, Item (3) of Claim 3.2 guarantees that $D \leq A^{2c+1-1}$ is a subdirect product representation of $D$. Then $\mu : D \to B$ is a representation of $B$ as a homomorphic image of a finite subdirect power of $A$. This homomorphism has a right inverse $\nu : B \to D$ which maps
b ∈ B to the diagonal tuple in \( D \) with diagonal value \( b \). This represents \( B \) as a retract of finite subdirect power of \( A \).

**Proof of Theorem 2.1.** Apply Theorem 3.1 in the setting where \( \theta = 1_A \) is the universal congruence on \( A \).

**4. An example**

Let \( A = \langle A; +, s, c \rangle \) be an algebra of signature \( \langle 2, 1, 0 \rangle \) whose universe is \( \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\} \) = integers modulo 6. The operations on \( A \) are defined by the following tables.

\[
\begin{array}{c|ccccc}
+^A & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 2 & 3 & 4 & 5 & 0 \\
2 & 2 & 3 & 4 & 5 & 0 & 1 \\
3 & 3 & 4 & 5 & 0 & 1 & 2 \\
4 & 4 & 5 & 0 & 1 & 2 & 3 \\
5 & 5 & 0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{c|ccccc}
s^A & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 3 & 3 & 0 & 3 & 3 \\
\end{array}
\]

\[ c^A = 3. \]

The \( + \)-operation is addition modulo 6. We shall use the symbol 0 to denote the constant term \( c + c \).

The subalgebra lattice and the congruence lattice of \( A \) are depicted in Figure 5. For each congruence we indicate the partition into congruence classes. We also label the congruence lattice with indices of covering pairs.

The only proper subuniverse of \( A \) is \( B = \{0, 3\} \), and the subalgebra supported by this set is \( B = \langle \{0, 3\}; +, s, c \rangle \) where the operations are given by the tables

\[
\begin{array}{c|cc}
+^B & 0 & 3 \\
\hline
0 & 0 & 3 \\
3 & 3 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c}
s^B & 0 & 3 \\
\hline
0 & 0 \\
\end{array}
\]

\[ c^B = 3. \]

Our goal in the section is to prove the following.

**Theorem 4.1.**

1. \( A \) is an expansion of a finite group.
2. \( A \) is 2-step nilpotent as an algebraic structure, but it is not supernilpotent.
(3) The subalgebra $B \leq A$ is not representable as a homomorphic image of a finite subdirect power of $A$.

The proof of this theorem spans the rest of this section.

Item (1) of this theorem means that the term operations of $A$ include those of a finite group. Those term operations are $x + y := x +_A y$, $-x := x +_A x +_A x +_A x +_A x = 5x$, and $0 := c +_A c$.

The assertion in Item (2) that $A$ is 2-step nilpotent means that $[1, 1] > [1, [1, 1]] = 0$ where $[x, y]$ is the binary commutator. Since, by (1), $A$ generates a congruence modular variety, $[x, y] = [x, y]_2$, so it does not matter whether we are referring to the 1-term or 2-term commutator. To establish $(\leq 2)$-step nilpotence it will be enough to check that $[1, 1] \leq \theta$ and $[1, \theta] = 0$.

The algebra $A/\theta$ is term equivalent to the abelian group $\mathbb{Z}_3$, which is an abelian algebra, and this is enough to prove that $[1, 1] \leq \theta$. To see that $[1, \theta] = 0$ it is enough to exhibit a congruence on $A(\theta) = \theta \leq A^2$ which has the diagonal of $A^2$ as a single class. That congruence is: the equivalence relation $\Delta$ on $A(\theta)$ with exactly two classes, the diagonal and the off-diagonal. The reason that this is a congruence on $A(\theta)$ is that $\Delta$ is an equivalence relation that is compatible with the operations of the pointed abelian group $\langle A; +, c \rangle$, since $\theta$ is a congruence of this reduct and this reduct is abelian. Finally, $\Delta$ is compatible with $s$ since $s$ maps all of $A(\theta)$ into the diagonal, which is a single $\Delta$-class.

We already mentioned in Corollary 2.7 that a finite supernilpotent algebra with a Maltsev operation must factor into a product of prime power cardinality nilpotent algebras. Since $A$ does not have prime power cardinality, and its congruence lattice indicates that it has no nontrivial direct factorization, we derive that $A$ cannot be supernilpotent. It therefore cannot be abelian, hence its nilpotence class must be exactly 2. This completes the proof of Item (2).

Now we focus all of our attention on the key element of Theorem 4.1, namely that the subalgebra $B \leq A$ is not representable as a homomorphic image of a finite subdirect power of $A$.

The language of $A$ has $c$ and $0 := s(c) = c + c$ as constant terms, and in a given member of the variety generated by $A$ these terms may or may not interpret as the same constant. In both $A$ and $B$ these constants are interpreted differently, which means that the algebras $A$ and $B$ fail to satisfy the identity $c = 0$. We shall prove Item (3) of Theorem 4.1 by showing that any 2-element algebra that is representable as a homomorphic image of a finite subdirect power of $A$ must satisfy the identity $c = 0$.

Since $A$ is an expansion of an abelian group, we may use concepts from group theory/ring theory. If $X$ is any member of the variety generated by $A$, then by an ideal of $X$ we mean a congruence class containing 0. (I.e., $I = 0/\alpha$ for some $\alpha \in \text{Con}(X)$.) The index of one ideal in another is computed as one would compute the index of the underlying additive group of one in the other. By a Sylow $p$-subgroup of $X$ we mean a Sylow $p$-subgroup of the underlying additive group. We write the Sylow $p$-subgroup of $X$ as $V_p(X)$.

Suppose that $T$ is any 2-element algebra that is representable as a homomorphic image of some finite subdirect power $D$ of $A$. Suppose that $I$ is an
ideal of $D$ for which $D/I \cong \mathbb{T}$. Since $|T| = 2$, it must be that $[D : I] = 2$, hence $V_3(D) \subseteq I$. We shall argue that if

1. $D \leq \mathbb{A}^n$ is subdirect,
2. $I$ is an ideal of $D$, and
3. $V_3(D) \subseteq I$,

then

4. $c^D \in I$.

Hence $D/I \cong \mathbb{T}$ satisfies the identity $c = 0$. This will show that $B \not\cong \mathbb{T}$. Since $T$ was an arbitrary 2-element quotient of a finite subdirect power of $A$, this will prove Theorem 4.1 (3).

**Claim 4.2.** If $D \leq \mathbb{A}^n$ is a representation of $D$ as a subdirect power of the algebra $\mathbb{A}$, then $V_3(D) \leq V_3(\mathbb{A}^n) = V_3(\mathbb{A})^n$ is a representation of the group $V_3(D)$ as a subdirect power of the group $V_3(\mathbb{A})$.

Choose any $i \in \{1, \ldots, n\}$ and any $a \in V_3(\mathbb{A})$. Use the fact that $D \leq \mathbb{A}^n$ is subdirect to find a tuple $d \in D$ such that $d_i = a$. The underlying group of $\mathbb{A}$ has an idempotent unary term operation $e(x) = 4x$ for which $e(\mathbb{A}) = V_3(\mathbb{A})$. Now $e(d) = (e(d_1), \ldots, e(d_n))$ has the properties that

(i) $e(d)_i = e(d_i) = e(a) = a$, and
(ii) $e(d) \in e(D) \subseteq D \cap e(\mathbb{A}^n) = D \cap V_3(\mathbb{A}^n) = V_3(D)$.

This shows that the group $V_3(D)$ contains a tuple $e(d)$ whose $i$-th coordinate is $a$. Since $i \in \{1, \ldots, n\}$ and $a \in V_3(\mathbb{A})$ were arbitrary, this is enough to show that $V_3(D) \leq V_3(\mathbb{A})^n$ is subdirect.

The proof of the following claim completes the proof of Theorem 4.1 (3).

**Claim 4.3.** If $D \leq \mathbb{A}^n$ is a subdirect product representation of $D$, and $I$ is an ideal of $D$ containing $V_3(D)$, then $c^D \in I$.

Partition $V_3(D)$ into disjoint sets, $\{0\}, P, -P$ in any way desired, subject to the condition $(d \in P) \iff (-d \in -P)$. This is possible, since the group of permutations of $V_3(D)$ consists of the identity function $x \mapsto x$ and the negation function $x \mapsto -x$ acts on $V_3(D)$ in a way that partitions this set into a single 1-element orbit $\{0\}$ and a family of 2-element orbits $\{d, -d\}$, $d \neq 0$, and one may create suitable $P$ and $-P$ by choosing one element from each 2-element orbit for membership in $P$ and defining $-P = V_3(D) - (P \cup \{0\})$.

**Subclaim 4.4.** $\sum_{d \in P} s(d) = c^D = (c^A, \ldots, c^A)$.

We will show this by examining each coordinate of $\sum_{d \in P} s(d)$ separately, and showing that the result is always $c^A$. Let $i$ be an arbitrary coordinate. Since $V_3(D)$ is subdirect in $V_3(\mathbb{A})^n$, the projection $\pi_i$ onto the $i$-th coordinate is a nonconstant linear functional which maps the $F_3$-vector space $V_3(D)$ onto $V_3(\mathbb{A}) \cong F_3$. The size of the domain of $\pi_i$ is $|V_3(D)| = 3^k$, where $k = \dim_{F_3}(V_3(D))$. The size of the (codimension-1) kernel of $\pi_i$ is therefore $|V_3(D)|/3 = 3^{k-1}$, and so

$$|V_3(D) \setminus \ker(\pi_i)| = 3^k - 3^{k-1} = 2 \cdot 3^{k-1}. \quad (4.1)$$
If $d \in \ker(\pi_i)$, then $d_i = 0$, so $s(d)_i = s(0) = 0$. If $d \in V_3(\mathbb{D}) \setminus \ker(\pi_i)$, then $s(d)_i = s(d_i) = c^A$. If $d \in V_3(\mathbb{D}) \setminus \ker(\pi_i)$, then $-d \in V_3(\mathbb{D}) \setminus \ker(\pi_i)$, and exactly one of $\{d, -d\}$ belongs to $P$, so exactly half of the elements in $V_3(\mathbb{D}) \setminus \ker(\pi_i)$ belong to $P$. Hence,

$$\pi_i \left( \sum_{d \in P} s(d) \right) = \sum_{d \in P} \pi_i(s(d)) = \frac{1}{2} |V_3(\mathbb{D}) \setminus \ker(\pi_i)| \cdot c^A \overset{4.1}{=} 3^{k-1} \cdot c^A = c^A.$$ 

(The last equality uses the facts that the additive order of $c^A$ is 2 and $3^{k-1}$ is odd.) Since $\pi_i(\sum_{d \in P} s(d)) = c^A$ for every $i$, we have $\sum_{d \in P} s(d) = c^D$.

Now we complete the proof of Claim 4.3. Given that $P \subseteq V_3(\mathbb{D}) \subseteq I$, we have that $d \in P$ implies $d \in I$, and the latter may be written as $d \equiv 0 \pmod{I}$. Hence $s(d) \equiv s(0) = 0 \pmod{I}$, since $I$ is a congruence class. By Subclaim 4.4 we have $c^D = \sum_{d \in P} s(d) \equiv 0 \pmod{I}$. This yields $c^D \in I$, which is the conclusion to be drawn in Claim 4.3.

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