On list chromatic numbers of 2-colorable hypergraphs

Danila Cherkashin\textsuperscript{a,b}, Alexey Gordeev\textsuperscript{c,d}

a. Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O., 29B, Saint Petersburg 199178 Russia
b. Moscow Institute of Physics and Technology, Laboratory of Combinatorial and Geometric Structures,
c. St. Petersburg Department of V. A. Steklov Institute of Mathematics of the Russian Academy of Sciences
d. The Euler International Mathematical Institute, St. Petersburg, Russia

Abstract

We give an upper bound on the list chromatic number of a 2-colorable hypergraph which generalizes the bound of Schauz on $k$-partite $k$-uniform hypergraphs. It makes sense for sparse hypergraphs: in particular we show that a $k$-uniform $k$-regular hypergraph has the list chromatic number 2 for $k \geq 4$. Also we obtain both lower and upper bound on the list chromatic number of a complete $s$-uniform 2-colorable hypergraph in the vein of Erdős–Rubin–Tayler theorem.

1 Introduction

A hypergraph $H$ is a pair of sets $(V, E)$, where $V$ is finite and $E \subseteq 2^V$; $V$ is called the set of vertices and $E$ the set of edges. A hypergraph is said to be $n$-uniform if all of its edges have size $n$ (further we call them $n$-graphs). Note that a graph is a 2-uniform hypergraph.

A hypergraph $H$ is said to be proper if each edge $e \in E$ contains two vertices $v_1, v_2 \in e$ such that $f(v_1) \neq f(v_2)$. The minimal number $r$ for which there exists a proper $r$-coloring of $H$ is called the chromatic number $\chi(H)$ of the hypergraph $H$.

Consider a 2-colorable hypergraph $H$. Its proper 2-coloring gives a partition of $V$ into two sets $A \cup B = V$, $A \cap B = \emptyset$ such that every edge of $H$ intersects both $A$ and $B$. Further we call such sets $A$ and $B$ parts of $H$. For a subset of a part $T \subset A$ define its neighborhood

$$N_H(T) = \bigcup_{v \in E, e \in T \neq \emptyset} e \setminus A.$$

For convenience define also $N_H(v) = N_H(\{v\})$.

Define a complete 2-colorable hypergraph $K^{s}_{n,m}$ as $s$-graph with sizes of parts $n$, $m$ and all possible edges of size $s$ intersecting both parts.

An orientation of a hypergraph $H = (V, E)$ is a map $\varphi : E \rightarrow V$ which satisfies $\varphi(e) \in e$ for any $e \in E$. For orientation $\varphi$ define a degree function $d_{\varphi}$ on the set of vertices as follows: $d_{\varphi}(v) = |\varphi^{-1}(v)|$.

To each vertex $v$ we assign a list $L(v)$ of colors which can be used for $v$. Given a map $f : V \rightarrow \mathbb{N}$, we say that a hypergraph $H$ is $f$-list-colorable ($f$-choosable) if for any set of lists with lengths $|L(v)| = f(v)$ there exists a proper coloring of $H$. Hypergraph is $k$-choosable if it is $f$-choosable, where $f(v) = k$ for each $v$. The list chromatic number $\text{ch}(H)$ of the hypergraph $H$ is the minimum number $k$ such that $H$ is $k$-choosable.

List colorings of graphs and hypergraphs were independently introduced by Vizing and by Erdős, Rubin, and Taylor. Clearly, $\text{ch}(H) \geq \chi(H)$, since all lists can be taken equal to $\{1, \ldots, \text{ch}(H)\}$. At the same time, the chromatic number and the list chromatic number are different, for example, for the complete bipartite graph $K_{3,3}$, for which $\text{ch}(K_{3,3}) = 3$ (equality is attained at the lists $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ assigned to the vertices of each part).

Sparse case. For a hypergraph $H = (V, E)$ denote

$$L(H) = \max_{\emptyset \neq E' \subseteq E} \frac{|E'|}{|\bigcup_{e \in E'} e|}.$$

One of the methods of estimating the list chromatic number of a graph is the Alon–Tarsi method [3]: if there exists an orientation $\varphi$ of a graph $G$ with certain structural properties, then $G$ is $(d_{\varphi} + 1)$-choosable. In the case of bipartite (2-colorable) graphs any orientation has the desired properties, which gives two following theorems.

**Theorem 1.** If a bipartite graph $G = (V, E)$ has an orientation $\varphi$, then $G$ is $(d_{\varphi} + 1)$-choosable.

**Theorem 2** (Theorem 3.2 in [3]). For any bipartite graph $G$, $\text{ch}(G) \leq [L(G)] + 1$. 




In 2010 Schauz [10] generalized these results to \(k\)-partite \(k\)-uniform hypergraphs, i.e. \(k\)-uniform hypergraphs whose set of vertices can be partitioned into \(k\) sets \(V_1, \ldots, V_k\) such that each edge has exactly one vertex in each of \(V_i\) (bipartite graphs correspond to the case \(k = 2\)). We show that in fact the same results hold in even more general case of 2-colorable hypergraphs.

**Dense case.** Let \(N(n, r)\) denote the minimum number of vertices in an \(n\)-partite (\(n\)-colorable) graph with the list chromatic number larger than \(r\). The following classic theorem express the asymptotics of \(N(2, r)\) in terms of the minimal number of edges in an \(n\)-uniform hypergraph without a proper 2-coloring. The latter quantity is denoted by \(m(n)\).

**Theorem 3** (Erdős – Rubin – Taylor [5]). For any \(r\)

\[
m(r) \leq N(2, r) \leq 2m(r).
\]

The problem of finding \(m(n)\) is well-known, the best current bounds are

\[
e^2 \sqrt{\frac{n}{\ln n}} 2^n \leq m(n) \leq (1 + o(1)) \frac{e \ln 2}{4} n^2 2^n.
\]

For details see survey [9].

Kostochka extended Theorem 3 in two ways. Let \(Q(n, r)\) be the minimal number of edges in an \(n\)-uniform \(n\)-partite hypergraph with the chromatic number greater than \(r\); \(m(n, r)\) be the minimal number of edges in an \(n\)-uniform hypergraph with the chromatic number greater than \(r\), and finally \(p(n, r)\) be the minimal number of edges in an \(n\)-uniform hypergraph without an \(r\)-coloring such that every edge meets every color.

**Theorem 4** (Kostochka [8]). For every \(n, r \geq 2\) the following inequalities hold

\[
m(r, n) \leq Q(n, r) \leq nm(r, n);
\]

\[
p(r, n) \leq N(n, r) \leq np(r, n).
\]

Upper and lower bounds on the quantities \(m(n, r)\) and \(p(n, r)\) heavily depend on the relations between \(n\) and \(r\). The picture is collected in survey [9] (see also more recent paper [1]).

## 2 Sparse case

We give an alternative, more direct proof of the next theorem in the Appendix, since it is rather concise.

**Theorem 5.** If a 2-colorable hypergraph \(H = (V, E)\) has an orientation \(\varphi\), then \(H\) is \((d_\varphi + 1)\)-choosable.

**Proof.** Since \(H\) is 2-colorable, we can choose a vertex \(u \in e \in E\) such that \(\varphi(e)\) and \(u\) belong to different parts of \(H\).

Consider a bipartite graph \(B\) on the set of vertices \(V\) with the set of edges \(\cup_{e \in E} \{\varphi(e), u_e\}\). Consider the following orientation of \(B\): \(\varphi'((\varphi(e), u_e)) = \varphi(e)\). Note that \(d_{\varphi'}(v) = d_\varphi(v)\) for any \(v \in V\). Then, by Theorem 1, \(B\) is \((d_\varphi + 1)\)-choosable. Since any proper coloring of \(B\) is a proper coloring of \(H\), it follows that \(H\) is \((d_\varphi + 1)\)-choosable.

The following lemma first appeared in [10] (see Lemma 3.2). We give a proof of it here for the sake of completeness.

**Lemma 1.** For any hypergraph \(G = (V, E)\) there exists an orientation \(\varphi\) of \(H\) with \(d_\varphi \leq \lceil L(H) \rceil\).

**Proof.** Consider a bipartite graph \(G\) with parts \(V \times \{1, \ldots, \lceil L(H) \rceil\}\) and \(E\), in which \((v, i)\) and \(e\) are connected if and only if \(v\) and \(e\) are incident in \(H\). The statement of lemma is true if and only if there is a matching in \(G\) which covers \(E\).

Let \(\emptyset \neq T \subseteq E\) be an arbitrary subset of edges of \(H\). We estimate the size of the neighborhood of \(T\) in \(G\):

\[
|N_G(T)| = |\cup_{e \in T} e| \cdot \lceil L(H) \rceil \geq |\cup_{e \in T} e| \cdot \frac{|T|}{|\cup_{e \in T} e|} = |T|.
\]

By Hall’s marriage theorem it follows that there exists a matching in \(G\) which covers \(E\).

**Theorem 6.** For any 2-colorable hypergraph \(H\), \(\text{ch}(H) \leq \lceil L(H) \rceil + 1\).

The exact value of \(L(H)\) can be difficult to calculate, so we provide a bound in simpler terms. For a hypergraph \(H\) denote the maximum degree of a vertex in \(H\) by \(\Delta(H)\) and the minimum size of an edge in \(H\) by \(s(H)\).

2
Corollary 1. Let \( H = (V, E) \) be a 2-colorable hypergraph. Then
\[
\text{ch}(H) \leq \left\lceil \frac{\Delta(H)}{s(H)} \right\rceil + 1.
\]

Proof. For any \( E' \subseteq E \) we have
\[
|E'| \cdot s(H) \leq \sum_{e \in E'} |e| \leq |\cup_{e \in E'} e| \cdot \Delta(H),
\]
so
\[
L(H) \leq \frac{\Delta(H)}{s(H)}.
\]

For an arbitrary (not necessarily 2-colorable) hypergraph a weaker bound with an additional factor of 2 holds. The next theorem was essentially proved by Gravin and Karpov [6], though they were not considering their results in the context of list colorings. In fact, the theorem in [6] has the second part in which \( s+1 \) is removed under some assumptions; it can be also generalized on the list chromatic numbers.

Theorem 7. Let \( H = (V, E) \) be a hypergraph. Then
\[
\text{ch}(H) \leq \left\lceil \frac{2\Delta(H)}{s(H)} \right\rceil + 1.
\]

Proof. Denote
\[
k = \left\lceil \frac{2\Delta(H)}{s(H)} \right\rceil.
\]
Let \( G \) be the incidence graph of \( H \), i.e., a bipartite graph with parts \( V, E \) and with \( v \in V, e \in E \) connected in \( G \) if and only if \( v \) and \( e \) are incident in \( H \). We want to delete some edges from \( G \) to obtain a graph \( K \) with the following conditions on degrees:
\[
d_K(e) = 2 \text{ for any } e \in E; \quad \max_{v \in V} d_K(v) \leq k.
\]
If there exists such \( K \), denote \( N_K(e) = \{v_1, u_1\} \) and consider a graph \( B \) on a set of vertices \( V \) with the set of edges
\[
\cup_{e \in E} (\{v_1, u_1\}).
\]
One can color vertices of \( B \) in any order, each time using any color not yet taken by adjacent vertices, to obtain a bound \( \text{ch}(B) \leq \Delta(B) + 1 \leq k + 1 \). Since any proper coloring of \( B \) is a proper coloring of \( H \), it follows that \( \text{ch}(H) \leq k + 1 \).

If there is no \( K \) with desired properties, then we choose such \( K = (V, E_K) \) that \( d_K(e) = 2 \) for any \( e \in E \) and
\[
p(K) = \sum_{e \in V} \max(0, d_K(v) - k)
\]
is as small as possible. Denote by \( S \subseteq V \) the set of vertices with \( d_K(v) > k \). Define an augmenting path as such a sequence of vertices and edges \( v_1, e_1, \ldots, v_m, e_m, v_{m+1} \) such that \( v_i \in V, e_i \in E, (v_i, e_i) \in E_K, (v_{i+1}, e_i) \notin E_K, e_i \) are pairwise distinct, \( v_1 \in S \). Let \( U \subseteq V \) be the subset of vertices of \( V \) reachable by an augmenting path. Note that:

- For any \( v \in U \) the inequality \( d_K(v) \geq k \) holds (otherwise we could flip the status of edges on the corresponding augmenting path and decrease \( p(K) \)).
- For any \( e \in E \) if \( N_K(e) \cap U \neq \emptyset \) then \( |N_G(e) \cap U| \leq 1 \). Indeed, suppose that \( v \in N_K(e) \cap U \), then any \( w \in N_G(e) \setminus N_K(e) \) must lie in \( U \), because there is an augmenting path ending in \( w \); it follows that the only vertex of \( N_G(e) \) which can possibly not lie in \( U \) is the unique vertex from \( N_K(e) \setminus \{v\} \).

Let \( T \subseteq N_K(U) \) be a set of \( e \in N_K(U) \) such that \( |N_G(e) \cap U| = 1 \). Since \( d_K(v) \geq k \) for any \( v \in U \) and \( d_K(v) > k \) for any \( v \in S \), we get the inequality
\[
|N_K(U) \setminus T| > \frac{k|U| - |T|}{2}.
\]
Now we obtain the lower bound on the sum of degrees \( d_G \) over vertices of \( U \) by summing up \( |N_G(e) \cap U| \) over \( e \in N_K(U) \):
\[
\sum_{v \in U} d_G(v) > |T|(s(H) - 1) + \frac{k|U| - |T|}{2} s(H) \geq \Delta(H)|U| + |T| \left( s(H) - 1 - \frac{s(H)}{2} \right) \geq \Delta(H)|U|.
\]
It follows that there exists such \( v \in U \) with \( d_G(v) > \Delta(H) \), which is a contradiction. \( \square \)
3 Dense case

**Theorem 8.** Let $H$ be an $s$-uniform 2-colorable hypergraph on $t$ vertices and

$$t < \frac{1}{4}(1 + s^{1/l})^l.$$

Then

$$\text{ch}(H) \leq l.$$

**Proof.** Consider the set of lists as a hypergraph $F$. Then $t = |E(F)|$; we still refer to edges of $F$ as lists to avoid the confusion with edges of $H$. We split the palette $V(F)$ randomly into three parts: blue colors, red colors and neutral colors in the following way. Every vertex of $V(F)$ is red or blue uniformly and independently with probability $\frac{1 - p}{2}$, so with probability $p$ it is neutral; the value of $p$ will be defined later. Then the expectation of monochromatic lists is equal to

$$A = 2 \left(\frac{1 - p}{2}\right)^l |E(F)|.$$

We call a list *dangerous*, if it has no blue or no red vertices. The expectation of dangerous lists equals to

$$B = 2 \left(\frac{1 + p}{2}\right)^l |E(F)|.$$

In the case $A < 1/2$ and $B < s/2$ Markov inequality implies that with positive probability $F$ has no monochromatic edges and the number of dangerous lists is smaller than $s$. Recall that $H$ is 2-colorable; under the assumption one can color vertices of one part of $H$ in blue or neutral colors, vertices of another part of $H$ in red or neutral colors, in such a way that vertices of $H$ corresponding to dangerous lists have neutral colors. Since we have less than $s$ dangerous lists, and every other edge of $H$ has a red and a blue vertex, we construct a proper coloring of $H$.

Now we show that the following choice of $p$ satisfies the desired inequalities. Put

$$p = \frac{s^{1/l} - 1}{1 + s^{1/l}}, \quad \text{then} \quad \frac{B}{A} = \left(\frac{1 + p}{1 - p}\right)^l = s.$$

On the other hand,

$$\left(\frac{2}{1 - p}\right)^l = \left(1 + s^{1/l}\right)^l.$$

Summing up,

$$A = \frac{2|E(F)|}{(1 + s^{1/l})^l} = \frac{s^l}{(1 + s^{1/l})^l} < \frac{1}{2},$$

hence

$$B = A \left(\frac{1 + p}{1 - p}\right)^l = As < \frac{s}{2},$$

so the condition on $t$ implies $\text{ch}(H) \leq l$. \qed

Obviously, $\text{ch}(K^2_{t/2,t/2}) \leq \text{ch}(K^2_{t/2,t/2}) = (1 + o(1)) \log_2 t$ by Erdős–Rubin–Taylor theorem. Note that for $s = t^\alpha$, where $\alpha$ is a constant, the bound in Theorem 8 is constant times better.

**Corollary 2.** Suppose that

$$t \leq \sqrt{s} \cdot 2^{l-2}.$$

Then

$$\text{ch}(K^s_{k,t-k}) \leq l$$

for every $k$ from 1 to $t - 1$.

**Proof.** Indeed,

$$1 + s^{1/l} \geq 2s^{1/(2l)}.$$ 

Hence

$$(1 + s^{1/l})^l \geq \sqrt{s} 2^l$$

and we are done by Theorem 8. \qed

**Theorem 9.** Suppose that

$$t = \Omega \left((\log s + \log l) \cdot l^2(1 + s^{1/l})^l\right).$$

Then

$$\text{ch}(K^s_{t/2,t/2}) > l.$$
Proof. Denote $H = K_{t/2,t/2}$.

Put $v = t^2$ and consider a set of random (independent and uniform) lists with size $l$ over the left part of $H$, and its copy over the right part. Let $F$ be the resulting random hypergraph of colors; we refer to its edges as lists to avoid the confusion with the edges of $H$.

Suppose the contrary: in particular it means that for every $F$ hypergraph $H = K_{t/2,t/2}$ has proper coloring in colors of $F$. Consider an arbitrary such proper coloring $\pi_H$. We call a color (a vertex of $F$) poor, if it appears in both parts of $H$; so a poor color appears at most $s - 1$ times. Define a coloring $\pi_F$ on vertices of $F$ as follows: if a color appears only in the left part of $H$ then the corresponding vertex of $F$ is blue, if a color appears only in the right part of $H$ then the corresponding vertex of $F$ is red, finally if a color is poor then the corresponding vertex has no color. Note that $\pi_F$ contains no monochromatic lists; indeed if a list is blue then one cannot choose a color from a copy of this list over the vertex in the right part of $H$, which contradicts with the existence of $\pi_H$. We show that with positive probability such a coloring $\pi_F$ cannot exist.

Consider an arbitrary vertex coloring $\pi$ of $F$ in two colors in which some vertices remain colorless. We call a list dangerous if it has no blue or no red vertices. It turns out that with positive probability every $\pi$ has a monochromatic list or at least $s - 1$ dangerous lists. Then the probability such a coloring $\pi$ appears only in the left part of $H$ is $1 - p_1(\pi)$. Thus for a random list coloring $\pi$, the probability that a random list is colorless is smaller than $p_2(\pi)$ the probability that it is dangerous. By definition

$$p_1 = \left(1 - \frac{1}{2}\right)^l, \quad p_2 = \left(1 + \frac{1}{2}\right)^l.$$

Then the probability that we have at most $(s - 1)$ dangerous lists is smaller than the probability that there are at most $sl^2$ dangerous lists. The latter is evaluated by

$$\sum_{i=1}^{sl^2} \binom{t/2}{i} p_2^i (1 - p_2)^{t/2 - i} \leq \sum_{i=1}^{sl^2} \left(\frac{tp_2}{2}\right)^i e^{-p_2(t/2 - sl^2)} \leq \sum_{i=1}^{sl^2} \left(\frac{tp_2}{2}\right)^i e^{-p_2(t/2 - sl^2)}.$$

Consider the following substitution $c = \frac{s}{2(s/l + 1)}$ and put $T = \frac{t}{2(s/l + 1)}$. We got

$$p_2 = \left(1 + \frac{1}{2}\right)^l = \left(\frac{s}{s/l + 1}\right)^l = \frac{s}{(s/l + 1)^l}.$$

Then $\frac{tp_2}{2} = sT$ and

$$\sum_{i=1}^{sl^2} \left(\frac{tp_2}{2}\right)^i e^{-p_2(t/2 - sl^2)} \leq sl^2 \left(\frac{tp_2}{2}\right)^{sl^2} e^{-p_2(t/2 - sl^2)}.$$

One has

$$e^{-p_2(t/2 - sl^2)} \leq e^{-p_2(t/2 - sl^2)} = e^{-sT + sl^2}.$$

Also

$$sl^2 \left(\frac{tp_2}{2}\right)^{sl^2} = (sT + o(1))^{sl^2} = e^{(1 + o(1)) \log(sT) sl^2}.$$

Summing up, under the conditions of theorem

$$(1 + o(1))^{l^2 s \log(sT)} - sT + sl^2 < 0$$

and the event that the number of dangerous lists is smaller than $sl^2$ has probability smaller than $\frac{1}{2}$.

On the other hand

$$(1 - p_1)^{l^2/2} \leq e^{-p_1 l^2/2} \leq e^{-\frac{t}{2(s/l + 1)^l}} = e^{-T} < \frac{1}{2}.$$

So for $c = \frac{s}{2(s/l + 1)}$ a random $l$-graph of lists with positive probability has both a monochromatic list and at least $sl^2$ dangerous lists in every 2-coloring. From a combinatorial argument for smaller $c$ a random $l$-graph of lists with positive probability has a monochromatic list in every 2-coloring, and for bigger $c$ with positive probability has at least $sl^2$ dangerous lists in every 2-coloring.

\[\square\]

4 Applications and discussion

1. A hypergraph is $k$-regular if the degree of every vertex is equal to $k$. It is known that a $k$-uniform $k$-regular hypergraph is 2-colorable for $k \geq 4$ [11, 7]. Thus Theorem 6 gives that such graph has the list chromatic number 2. So $k$-uniform $k$-regular hypergraphs are chromatic-choosable for $k \geq 4$. 

\[\square\]
2. It turns out that there is a large gap between the bounds in dense and sparse cases; the same holds even for 2-graphs. As far as we are aware, the best known general bounds on the choice number of \( d \)-regular bipartite graph \( G \) are the following (see [4])

\[
\left( \frac{1}{2} - o(1) \right) \log d \leq \text{ch}(G) \leq c \frac{d}{\log d}.
\]

Note that Erdős – Rubin – Taylor gives a tight bound for a complete bipartite graph

\[
\text{ch}(K^2_{t,t}) = (1 + o(1)) \log t.
\]

Acknowledgments. The research of Danila Cherkashin is supported by «Native towns», a social investment program of PJSC «Gazprom Neft». The research of Alexey Gordeev was funded by RFBR, project number 19-31-90081.

References

[1] Margarita Akhmejanova, József Balogh, and Dmitry Shabanov. Chain method for panchromatic colorings of hypergraphs. arXiv preprint arXiv:2008.03827, 2020.

[2] N. Alon. Combinatorial Nullstellensatz. Combinatorics, Probability and Computing, 8(1-2):7–29, 1999.

[3] N. Alon and M. Tarsi. Colorings and orientations of graphs. Combinatorica, 12(2):125–134, 1992.

[4] Noga Alon. Degrees and choice numbers. Random Structures & Algorithms, 16(4):364–368, 2000.

[5] Paul Erdős. Some old and new problems in various branches of combinatorics. In Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing. Congressus Numerantium XXIII, pages 19–37. Winnipeg: Utilitas Mathematica, 1979.

[6] N. V. Gravin and D. V. Karpov. On proper colorings of hypergraphs. Zap. Nauchn. Sem. POMI, 391:79–89, 2011.

[7] Michael A. Henning and Anders Yeo. 2-colorings in \( k \)-regular \( k \)-uniform hypergraphs. European Journal of Combinatorics, 34(7):1192–1202, 2013.

[8] Alexandr Kostochka. On a theorem of Erdős, Rubin, and Taylor on choosability of complete bipartite graphs. The Electronic Journal of Combinatorics, 9(9):1, 2002.

[9] Andrei Mikhailovich Raigorodskii and Danila Dmitrievich Cherkashin. Extremal problems in hypergraph colourings. Russian Mathematical Surveys, 75(1):89, 2020.

[10] Uwe Schauz. A paintability version of the combinatorial Nullstellensatz, and list colorings of \( k \)-partite \( k \)-uniform hypergraphs. The Electronic Journal of Combinatorics, page R176, 2010.

[11] Carsten Thomassen. The even cycle problem for directed graphs. Journal of the American Mathematical Society, 5(2):217–229, 1992.

Appendix. Alternative proof of Theorem 5

Proof. Denote parts of \( H \) as \( A, B \subset V \); every edge of \( H \) intersects both \( A \) and \( B \). For each edge \( e \) of \( H \) choose a spanning tree \( T_e \) on its set of vertices such that each edge of \( T_e \) connects a vertex from part \( A \) with a vertex from part \( B \). Consider the following polynomial on variables \( \{x_a\}_{a \in A}, \{y_b\}_{b \in B} \):

\[
F_H(x, y) = \prod_{e \in E} \left( \sum_{(a, b) \in T_e} (x_a - y_b) \right).
\]

Note that if \( F_H(x, y) \neq 0 \) then the values of \( x, y \) form a proper coloring of \( H \) (the reverse is not necessarily true). By Combinatorial Nullstellensatz (see [2]), if the coefficient

\[
\left[ \prod_{a \in A} x_a^{d_x(a)} \prod_{b \in B} y_b^{d_y(b)} \right] F_H \neq 0,
\]

then \( H \) is \((d_x + 1)\)-choosable. Consider now

\[
F_H^*(x, y) = F_H(x, -y) = \prod_{e \in E} \left( \sum_{(a, b) \in T_e} (x_a + y_b) \right).
\]
Note that
\[
\left[ \prod_{a \in A} x_a^{d_a(a)} \prod_{b \in B} y_b^{d_b(b)} \right] F_H = \pm \left[ \prod_{a \in A} x_a^{d_a(a)} \prod_{b \in B} y_b^{d_b(b)} \right] F_H^*,
\]
hence one can study coefficients of \( F_H^* \) instead of \( F_H \). The coefficient of \( F_H^* \) in question is nonzero, because no summands will cancel out after opening the brackets, and orientation \( \varphi \) corresponds to at least one summand with the desired coefficient.