Novel Special Affine Wavelet Transform and Associated Uncertainty Inequalities

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Abstract. Due to the extra degrees of freedom, special affine Fourier transform (SAFT) has achieved a respectable status within a short span and got versatile applicability in the areas of signal processing, image processing, sampling theory, quantum mechanics. However, due to its global kernel, SAFT fails to obtain local information of non-transient signals. To overcome this, we in this paper introduce the concept of novel special affine wavelet transform (NSAWT) and extend key harmonic analysis results to NSAWT analogous to those for the wavelet transform. We first establish some fundamental properties including Moyal’s principle, Inversion formula and the range theorem. Some Heisenberg type inequalities and Pitt’s inequality are established for SAFT and consequently Heisenberg uncertainty principle is derived for NSAWT.

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1. Introduction

The special affine wavelet transform (SAFT), which was introduced in [1], is a six-parameter class of linear integral transformation which generalizes several well known unitary transformations including the Fourier transform, the fractional Fourier transform, the linear canonical transform (LCT) and the Fresnel transform [3, 12, 13]. The SAFT can be regarded as a time-shifted and frequency modulated version of the well known linear canonical transform [9, 6]. Let \( f^* \) denote the complex-conjugate of \( f \) and \( \{f, g\} = \int f(x)g^*(x)dx \) be the standard \( L^2 \) inner product. The SAFT is a mapping \( F_{SAFT} : f \rightarrow \hat{f}_\Lambda \) and is defined as

\[
F_{SAFT}[f](\omega) = \hat{f}_\Lambda(\omega) = \begin{cases} 
\langle f(x) K_\Lambda(x, \omega) \rangle, & B \neq 0 \\
\sqrt{D} \exp \left\{ \frac{1}{2} \left( CD(\omega - p)^2 + 2\omega q \right) \right\} f(D(\omega - p)), & B = 0,
\end{cases}
\]

where \( K_\Lambda(x, \omega) \) denotes the kernel of the SAFT given by

\[
K_\Lambda(x, \omega) = K_B^* \exp \left\{ \frac{i}{2B} \left( Ax^2 + 2x(p - \omega) - 2\omega(Dp - Bq) + D\omega^2 \right) \right\}, \quad K_B = \frac{1}{\sqrt{2\pi B}}
\]

and \( \Lambda\textsuperscript{(2x3)} \) denotes the augmented SAFT parameter matrix, which is of the form

\[
\Lambda\textsuperscript{(2x3)} = [\Lambda| \lambda],
\]
which in turn is obtained by LCT matrix \( \Lambda = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) and an offset vector \( \lambda = \begin{bmatrix} p \\ q \end{bmatrix} \).

This is the reason that the SAFT is also called as the offset linear canonical transform. Moreover, we shall only consider the case \( B \neq 0 \), since the SAFT is just a chirp multiplication operation in case \( B = 0 \). We also note that the phase-space transform (1.1) is lossless if and only if the matrix \( \Lambda \) is unimodular, that is; \( AD - BC = 1 \) and for this reason, SAFT is also known as the inhomogeneous canonical transform [2]. By virtue of the additive property of SAFT, the inverse SAFT corresponding to (1.1) is defined by

\[
f(x) = \langle \hat{f}_{A_\Lambda}(\omega) \mathcal{K}_{\Lambda_{\text{inv}}}^{A_\Lambda}(\omega, x) \rangle (1.3)
\]

where

\[
\Lambda_{\text{inv}}^{A_\Lambda} = \begin{bmatrix} D & -B : -Bq - Dp \\ -C & A : Cp - Aq \end{bmatrix}
\]

The Parseval’s formula for the special affine Fourier transform reads as follows

\[
\langle f(x), g(x) \rangle_{L^2(\mathbb{R})} = \langle \hat{f}_{A_\Lambda}(\omega), \hat{g}_{A_\Lambda}(\omega) \rangle_{L^2(\mathbb{R})}, \quad \forall f, g \in L^2(\mathbb{R}).
\]

The theory of wavelet transforms have emanated as a broadly used tool in various disciplines of science and engineering including image processing, spectrometry, machine learning, turbulence, computer graphics, telecommunications, DNA sequence analysis, quantum physics, solution of differential equations. For any \( f \in L^2(\mathbb{R}) \), the continuous wavelet transform (CWT) is denoted by \( \mathcal{W}_\psi[f](t, \zeta) \) and is defined as

\[
\mathcal{W}_\psi[f](t, \zeta) = \frac{1}{\sqrt{\zeta}} \int_{\mathbb{R}} f(x) \psi \left( \frac{x - t}{\zeta} \right) dx, \quad t \in \mathbb{R}, \ \zeta \in \mathbb{R}^+.
\]

where \( \zeta \) is the scaling parameter and \( t \) is the translation parameter. Shah et.al [17] introduced an amalgam of CWT and SAFT namely special affine wavelet transform (SAWT) which provides a joint time and frequency localization of signals. Convolution plays a pivotal role as far as applications of integral transforms are concerned. SAFT does not work well with the standard convolution operation. Xiang and Qin [21] introduced a convolution which works well for the SAFT and by which the SAFT of the convolution of two functions is the product of their SAFT’s and a phase factor but their convolution structure does not work well with the inverse transform. Bhandari and Zayed [6], introduced a new convolution in the special affine Fourier domain that works well with both the SAFT and its inverse leading to an analogue of the convolution and product formulas for the Fourier transform. They also introduced a second convolution that eliminates the phase factor in the convolution proposed by Xiang and Qin [21].

Uncertainty principles are mathematical results that give limitations on the simultaneous concentration of a function and its quaternion Fourier transform. They have implications in two main areas: quantum physics and signal analysis [8, 13, 10, 5]. In quantum physics, they tell us that a particle’s speed and position cannot both be measured with infinite precision. In signal analysis, they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. There are many ways to get the statement about concentration precise. This principle has been extended to different setups by various researchers [4, 5, 7, 11, 15, 16, 19, 20].
Motivated and inspired by the above work, we introduce a new special affine wavelet transform based on the novel convolution introduced in \cite{Zayed} and we call it novel special affine wavelet transform (NSAWT). We first establish some fundamental properties including Moyal’s principle, inversion formula and the range theorem. Some Heisenberg type inequalities and Pitt’s inequality are established for SAFT and Heisenberg uncertainty principle is also derived for NSAWT.

The rest of the paper is tailored as follows. In section 2, we introduce a notion of novel special affine wavelet transform (NSAWT) and establish a relationship between special affine Fourier transform (SAFT) and the proposed NSAWT. Section 3 is dedicated to the key harmonic analysis results related to novel special affine wavelet transform (NSAWT). Some generalizations of Heisenberg type inequalities are established in section 4.

2. Novel Special Affine Wavelet Transform

In this section, we first notion of novel special affine wavelet transform which is based on novel convolution. Furthermore we establish a relationship between special affine Fourier transform (SAFT) and the proposed novel special affine wavelet transform (NSAWT).

Firstly, we recall the following definition of novel convolution and the corresponding convolution theorem given by Bhandari and Zayed \cite{Bhandari2018}.

**Definition 2.1. (Chirp Modulation)** Let $\Lambda_S$ be the augmented SAFT matrix. We define the modulation function $m_{\Lambda_S}$ as follows

$$m_{\Lambda_S}(x) = \exp \left\{ \frac{iAx^2}{2B} \right\}.$$  \hspace{1cm} (2.1)

Then, for a given $f \in L^2(\mathbb{R})$, the chirp modulated functions associated with the augmented SAFT matrix $\Lambda_S$ and inverse SAFT matrix $\Lambda_S^{inv}$ are defined in the following manner:

$$\hat{\bar{f}}(x) = m_{\Lambda_S}(x) f(x) \quad ; \quad \hat{\bar{f}}(x) = m_{\Lambda_S}^*(x) f(x).$$  \hspace{1cm} (2.2)

and

$$\check{\bar{f}}(x) = m_{\Lambda_S^{inv}}(x) f(x) \quad ; \quad \check{\bar{f}}(x) = m_{\Lambda_S^{inv}}^*(x) f(x).$$  \hspace{1cm} (2.3)

**Definition 2.2. (SAFT Convolution)** Let $f, g \in L^2(\mathbb{R})$ be two given functions. The special affine convolution $\ast_{\Lambda_S}$ is defined as

$$h(t) = (f \ast_{\Lambda_S} g)(t) = K_B m_{\Lambda_S}^*(t) \left( \hat{\bar{f}}(x) \ast \hat{\bar{g}}(t) \right).$$  \hspace{1cm} (2.4)

where $\ast$ denote the usual convolution operator.

**Lemma 2.3. (SAFT Convolution Theorem)** Let $f$ and $g$ be two functions such that $h(t) = (f \ast_{\Lambda_S} g)(t)$ exists, then

$$\hat{\bar{h}}_{\Lambda_S}(\omega) = \Phi_{\Lambda_S}(\omega) \hat{\bar{f}}_{\Lambda_S}(\omega) \hat{\bar{g}}_{\Lambda_S}(\omega),$$  \hspace{1cm} (2.5)
where
\[
\Phi_{\Lambda S}(\omega) = \exp \left\{ \frac{i\omega(Dp - Bq)}{B} \right\} \exp \left\{ -iD\frac{\omega^2}{2B} \right\}. \tag{2.6}
\]

On the basis of SAFT convolution defined in Definition 2.1, we shall introduce the notion of novel special affine wavelet transform (NSAWT).

**Definition 2.4.** For any finite energy signal \( f \in L^2(\mathbb{R}) \), the continuous novel special affine wavelet transform of \( f \) with respect to the wavelet \( \psi \in L^2(\mathbb{R}) \) is defined by
\[
\mathcal{A}_{\Lambda S}^\psi [f](t, \zeta) = \int_{\mathbb{R}} f(x) \psi_{t,\zeta}^* \left( \frac{x - t}{\zeta} \right) \exp \left\{ \frac{iAx(t - x)}{B} \right\} dx, \quad t \in \mathbb{R}, \ \zeta \in \mathbb{R}^+, \tag{2.7}
\]
where \( \psi_{t,\zeta}^* (x) \) is given as follows
\[
\psi_{t,\zeta}^* (x) = K_B \sqrt{\zeta} \psi \left( \frac{x - t}{\zeta} \right) \exp \left\{ \frac{iAx(t - x)}{B} \right\}. \tag{2.8}
\]

It is worth noting that the NSAWT boils down to some existing integral transforms as well as gives birth to some new time-frequency transforms as mentioned below:

(i) For \( \Lambda_S = (A, B, C, D : 0, 0) \), we get a novel linear canonical wavelet transform.

(ii) For \( \Lambda_S = (\cos \theta, \sin \theta, -\sin \theta, \cos \theta : p, q) \), \( \theta \neq n\pi \), we get a novel fractional wavelet transform defined by
\[
\left( \mathcal{A}_{\Lambda S}^\psi f \right)(t, \zeta) = \frac{1}{\sqrt{2\pi \xi \sin \theta}} \int_{\mathbb{R}} f(x) \psi_{t,\zeta}^* \left( \frac{x - t}{\zeta} \right) \exp \left\{ (ix^2 - itx) \cot \theta \right\} dx.
\]

(iii) For \( \Lambda_S = (1, B, 0, 1 : p, q) \), \( B \neq 0 \), we obtain a novel Fresnel-wavelet transform:
\[
\left( \mathcal{A}_{\Lambda S}^\psi f \right)(t, \zeta) = \frac{K_B}{\sqrt{\zeta}} \int_{\mathbb{R}} f(x) \psi_{t,\zeta}^* \left( \frac{x - t}{\zeta} \right) \exp \left\{ \frac{(ix^2 - itx)}{B} \right\} dx.
\]

Now, we proceed to establish a fundamental relationship between the special affine Fourier transform given by (1.1) and the proposed novel special affine wavelet transform defined in Definition 2.4.

**Theorem 2.5.** Let \( \mathcal{A}_{\Lambda S}^\psi [f](t, \zeta) \) and \( \mathcal{F}_{SAFT}[f](\omega) \) be the continuous novel special affine wavelet transform and the special affine Fourier transform of any finite energy signal \( f \in L^2(\mathbb{R}) \). Then, we have
\[
\mathcal{F}_{SAFT} \left[ \left( \mathcal{A}_{\Lambda S}^\psi f \right)(t, \zeta) \right](\omega) = \sqrt{\zeta} \exp \left\{ \frac{i}{2B} \left( 2\omega(\xi(Dp - Bq) - D\omega^2 \xi^2) \right) \right\} \hat{f}_{\Lambda S}(\omega) \tilde{\Psi}_{\Lambda S}(\zeta \omega, \zeta), \tag{2.9}
\]
where
\[
\Psi(x, \zeta) = \exp \left\{ \frac{i}{2B} \left( Ax^2(\xi^2 - 1) + 2xp(\xi - 1) \right) \right\} \tilde{\psi}(x). \tag{2.10}
\]
Proof. By the definition of special affine Fourier transform, we have

\[
\mathcal{F}_{SAFT} \left[ (A_{\Lambda_S}^\psi f)(t, \zeta) \right](\omega) = \mathcal{F}_{SAFT} \left[ \frac{1}{\sqrt{\zeta}} \psi^* \left( \frac{-x}{\zeta} \right) \exp \left\{ -\frac{iAx^2}{2B} \right\} \right](\omega)
\]

\[
= \exp \left\{ \frac{i\omega(Dp - Bq)}{B} \right\} \exp \left\{ -\frac{iD\omega^2}{2B} \right\} (\omega)
\times \mathcal{F}_{SAFT} \left[ f(x) \right](\omega) \mathcal{F}_{SAFT} \left[ \psi^\ast \left( \frac{-x}{\zeta} \right) \right](\omega), \quad (2.11)
\]

Further, we have

\[
\mathcal{F}_{SAFT} \left[ \frac{1}{\sqrt{\zeta}} \psi^* \left( \frac{-x}{\zeta} \right) \exp \left\{ -\frac{iAx^2}{2B} \right\} \right](\omega)
\]

\[
= K_B \sqrt{\zeta} \int_{\mathbb{R}} \psi^* \left( \frac{-x}{\zeta} \right) \exp \left\{ \frac{i}{2B} \left( Ax^2 + D\omega^2 + 2x(p - \omega) - 2\omega(Dp - Bq) \right) \right\} dx
\]

\[
= K_B \sqrt{\zeta} \int_{\mathbb{R}} \psi^* (-z) \exp \left\{ \frac{i}{2B} \left( Az^2 \zeta^2 + D\omega^2 - 2\omega(Dp - Bq) \right) \right\} \exp \left\{ \frac{i\zeta(p - \omega)}{B} \right\} dz
\]

\[
= K_B \sqrt{\zeta} \exp \left\{ \frac{i}{2B} \left( D\omega^2 - 2\omega(Dp - Bq) - D(\zeta\omega)^2 + 2\zeta\omega(Dp - Bq) \right) \right\}
\times \int_{\mathbb{R}} \psi^* (-z) \exp \left\{ \frac{i}{2B} \left( A\zeta^2 z^2 + 2z\zeta p - A z^2 - 2zp \right) \right\} \frac{K_{\Lambda_S}(z, \zeta \omega)}{K_B} dz
\]

\[
= \sqrt{\zeta} \exp \left\{ \frac{i}{2B} \left( D\omega^2(1 - \zeta^2) + 2\omega(Dp - Bq)(\zeta - 1) \right) \right\}
\times \int_{\mathbb{R}} \psi (-z) \exp \left\{ \frac{i}{2B} \left( A\zeta^2 (\zeta^2 - 1) + 2zp(\zeta - 1) \right) \right\} K_{\Lambda_S}(z, \zeta \omega) dz
\]

\[
= \sqrt{\zeta} \exp \left\{ \frac{i}{2B} \left( D\omega^2(1 - \zeta^2) + 2\omega(Dp - Bq)(\zeta - 1) \right) \right\}
\times \hat{f}_{\Lambda_S}(\omega) \hat{\Psi}_{\Lambda_S}(\zeta\omega, \zeta),
\]

where

\[
\Psi(x, \zeta) = \exp \left\{ \frac{i}{2B} \left( Ax^2(\zeta^2 - 1) + 2xp(\zeta - 1) \right) \right\} \psi^* (-x).
\]

(2.12)

From the equation (2.11), we obtain the required result as

\[
\mathcal{F}_{SAFT} \left[ (A_{\Lambda_S}^\psi f)(t, \zeta) \right](\omega) = \sqrt{\zeta} \exp \left\{ \frac{i}{2B} \left( D\omega^2(1 - \zeta^2) + 2\omega(Dp - Bq)(\zeta - 1) \right) \right\}
\times \exp \left\{ \frac{i\omega(Dp - Bq)}{B} - \frac{iD\omega^2}{2B} \right\} \hat{f}_{\Lambda_S}(\omega) \hat{\Psi}_{\Lambda_S}(\zeta\omega, \zeta)
\]

\[
= \sqrt{\zeta} \exp \left\{ \frac{i}{2B} \left( 2\omega\zeta(Dp - Bq) - D\omega^2\zeta^2 \right) \right\} \hat{f}_{\Lambda_S}(\omega) \hat{\Psi}_{\Lambda_S}(\zeta\omega, \zeta).
\]
This completes the proof.

From the Theorem 2.5, we conclude that if the analyzing functions $\psi_{t,\zeta}^\Lambda(x)$ are supported in the time-domain or the special affine Fourier domain, then the proposed transform $\mathcal{W}_\psi^M [f](a,b)$ is accordingly supported in the respective domains. This implies that the special affine wavelet transform is capable of providing the simultaneous information of the time and the special affine frequency in the time-frequency domain. To be more specific, suppose that $\psi(t)$ is the window with centre $E_\psi$ and radius $\Delta_\psi$ in the time domain. Then, the centre and radii of the time-domain window function $\psi_{t,\zeta}(x)$ of the proposed transform (2.7) is given by

$$E\left[\psi_{t,\zeta}^\Lambda(x)\right] = \frac{\int_{-\infty}^{\infty} x\left|\psi_{t,\zeta}^\Lambda(x)\right|^2 dx}{\int_{-\infty}^{\infty} \left|\psi_{t,\zeta}^\Lambda(x)\right|^2 dx}$$

$$= \frac{\int_{-\infty}^{\infty} x\left|\psi_{t,\zeta}(x)\right|^2 dx}{\int_{-\infty}^{\infty} \left|\psi_{t,\zeta}(x)\right|^2 dx}$$

$$= E\left[\psi_{t,\zeta}(x)\right] = t + \zeta E_\psi$$

and

$$\Delta\left[\psi_{t,\zeta}^\Lambda(x)\right] = \left\{ \frac{\int_{-\infty}^{\infty} \left(x - (t + \zeta E_\psi)\right)\left|\psi_{t,\zeta}^\Lambda(x)\right|^2 dx}{\int_{-\infty}^{\infty} \left|\psi_{t,\zeta}^\Lambda(x)\right|^2 dx} \right\}^{1/2}$$

$$= \left\{ \frac{\int_{-\infty}^{\infty} \left(x - t - \zeta E_\psi\right)\left|\psi_{t,\zeta}(x)\right|^2 dx}{\int_{-\infty}^{\infty} \left|\psi_{t,\zeta}(x)\right|^2 dt} \right\}^{1/2}$$

$$= \Delta\left[\psi_{t,\zeta}(x)\right] = \zeta \Delta_\psi,$$

respectively. Let $\Gamma(\omega)$ be the window function in the special affine Fourier domain (SAFD) given by

$$\Gamma(\omega) = \mathcal{F}_{SAFT} \left\{ \exp \left\{ \frac{i}{2B} \left( 2x\zeta p - 2xp - Ax^2 \right) \right\} \psi^*(-x) \right\} (\omega).$$

Then, we can derive the center and radius of the special affine Fourier domain (SAFD) window function

$$\Gamma(\zeta\omega) = \mathcal{F}_{SAFT} \left\{ \exp \left\{ \frac{i}{2B} \left( 2x\zeta p - 2xp - Ax^2 \right) \right\} \psi^*(-x) \right\} (\zeta\omega).$$
appearing in (2.12) as
\[ E[\Gamma(\zeta\omega)] = \frac{\int_{-\infty}^{\infty} (a\omega)|\Gamma(\zeta\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\Gamma(\zeta\omega)|^2 d\omega} = \zeta E_\Gamma, \]
and
\[ \Delta[\Gamma(\zeta\omega)] = \zeta \Delta_\Gamma. \]
Thus, the Q-factor of the proposed transform (2.7) is given by
\[ Q_{NSAWT} = \frac{\text{width of the window function}}{\text{centre of the window function}} = \frac{\Delta[\Gamma(\zeta\omega)]}{E[\Gamma(\zeta\omega)]} = \frac{\Delta_\Gamma}{E_\Gamma} = \text{constant}, \]
which is independent of the uni-modular matrix \( A_S \) and the scaling parameter \( \zeta \). Therefore, the localized time and frequency characteristics of the novel special affine wavelet transform (NSAWT) are given in the time and frequency windows
\[ \left[ t + \zeta E_\psi - \zeta \Delta_\psi, t + \zeta E_\psi + \zeta \Delta_\psi \right] \quad \text{and} \quad \left[ a E_\Gamma - \zeta \Delta_\Gamma, \zeta E_\Gamma + \zeta \Delta_\Gamma \right], \]
respectively. Hence, the joint resolution of the continuous novel special affine wavelet transform (NSAWT) in the time-frequency domain is described by a flexible window \( \psi \) having a total spread \( 4\Delta_\psi \Delta_\Gamma \) and is given by
\[ \left[ t + \zeta E_\psi - \zeta \Delta_\psi, t + \zeta E_\psi + \zeta \Delta_\psi \right] \times \left[ a E_\Gamma - \zeta \Delta_\Gamma, \zeta E_\Gamma + \zeta \Delta_\Gamma \right]. \]

3. Basic Properties of Novel Special Affine Wavelet Transform

In this section, we establish fundamental properties of the novel special affine wavelet transform (NSAWT). Some well known harmonic analysis results namely Moyal’s principle, inversion formula, characterization of the range of the novel special affine wavelet transform are derived.

Now, we proceed to state some fundamental properties of the novel special affine wavelet transform (NSAWT) defined in Definition 2.4.

**Theorem 3.1.** For any functions \( f, g \in L^2(\mathbb{R}) \) and \( \alpha, \beta, \gamma \in \mathbb{R}, \mu \in \mathbb{R}^+ \), the continuous novel special affine wavelet transform satisfies the following properties:

(i) **Linearity:** \( A_{\psi A_S}^\psi [\alpha f + \beta g](t, \zeta) = \alpha A_{A_S}^\psi [f](t, \zeta) + \beta A_{A_S}^\psi [g](t, \zeta). \)

(ii) **Translation:** \( A_{\psi A_S}^\psi [f(x-\gamma)](t, \zeta) = \exp\left\{-\frac{iA\gamma(t-\gamma)}{B}\right\} A_{A_S}^\psi \left\{ \exp\left\{\frac{iA\gamma}{B}\right\} f(y) \right\}(t-\gamma, \zeta). \)

(iii) **Scaling:** \( A_{A_S}^\psi [f(\gamma x)](t, \zeta) = \frac{1}{\sqrt{\gamma}} A_{A_S}^\psi [f](\gamma t, \gamma \zeta), \) where
\[ A_S = \begin{bmatrix} A & \gamma^2B : p \\ C & D : q \end{bmatrix} \]

**Proof.** These properties are obvious, therefore we omit the proofs.

Now, we shall study some important theorems including the Moyal’s theorem, inversion formula and range theorem pertaining to the novel special affine wavelet transform defined in
Definition 2.4. Firstly, we shall derive the admissibility condition associated with the novel special affine wavelet transform.

**Theorem 3.2 (Admissibility Condition).** Let \( \psi \in L^2(\mathbb{R}) \) be a given function, then \( \psi \) is said to be admissible if

\[
C_\psi = \int_{\mathbb{R}^+} \left| \frac{\hat{\Psi}_{A_\psi} (\omega, \zeta)}{\zeta} \right|^2 d\zeta < \infty, \quad a.e. \quad \omega \in \mathbb{R}
\]

where \( \Psi \) is given by (2.10).

*Proof.* For any \( f \in L^2(\mathbb{R}) \), we have

\[
\int_{\mathbb{R} \times \mathbb{R}^+} \left| \langle f, \psi_{\zeta, \zeta} \rangle \right|^2 dt d\zeta = \int_{\mathbb{R} \times \mathbb{R}^+} \left| (f(x) *_{A_\psi} \frac{1}{\sqrt{\zeta}} \psi^* \left( \frac{-x}{\zeta} \right) \exp \left\{ \frac{-iAx^2}{2B} \right\} ) (t) \right|^2 dt d\zeta
\]

\[
= \int_{\mathbb{R} \times \mathbb{R}^+} \left| \mathcal{F}_{SAFT} \left[ (f(x) *_{A_\psi} \frac{1}{\sqrt{\zeta}} \psi^* \left( \frac{-x}{\zeta} \right) \exp \left\{ \frac{-iAx^2}{2B} \right\} ) \right] (\omega) \right|^2 d\omega d\zeta
\]

\[
= \int_{\mathbb{R} \times \mathbb{R}^+} \left| \hat{f}_{A_\psi} (\omega) \right|^2 \left| \frac{\hat{\Psi}_{A_\psi} (\omega, \zeta)}{\zeta} \right|^2 d\omega d\zeta
\]

\[
= \int_{\mathbb{R}} \left| \hat{f}_{A_\psi} (\omega) \right|^2 \left\{ \int_{\mathbb{R}^+} \left| \frac{\hat{\Psi}_{A_\psi} (\omega, \zeta)}{\zeta} \right|^2 d\zeta \right\} d\omega.
\]

On putting \( f = \psi \), (3.2) reduces to

\[
\int_{\mathbb{R} \times \mathbb{R}^+} \left| \langle \psi, \psi_{\zeta, \zeta} \rangle \right|^2 dt d\zeta = \int_{\mathbb{R}} \left| \hat{\psi}_{A_\psi} (\omega) \right|^2 \left\{ \int_{\mathbb{R}^+} \left| \frac{\hat{\Psi}_{A_\psi} (\omega, \zeta)}{\zeta} \right|^2 d\zeta \right\} d\omega.
\]

(3.3)

Since \( \psi \in L^2(\mathbb{R}) \), therefore we conclude that the R.H.S of (3.3) is finite provided

\[
C_\psi = \int_{\mathbb{R}^+} \left| \frac{\hat{\Psi}_{A_\psi} (\omega, \zeta)}{\zeta} \right|^2 d\zeta < \infty, \quad a.e. \quad \omega \in \mathbb{R}. \quad \square
\]

The following is the Moyal’s principle for the novel special affine wavelet transform (NSAWT).

**Theorem 3.3 (Moyal’s Principle).** Let \( A_{A_\psi} [f] (t, \zeta) \) and \( A_{A_\psi} [g] (t, \zeta) \) be the novel special affine wavelet transforms of \( f \) and \( g \) belonging to \( L^2(\mathbb{R}) \), respectively. Then, we have

\[
\int_{\mathbb{R} \times \mathbb{R}^+} A_{A_\psi} [f] (t, \zeta) A_{A_\psi} [g] (t, \zeta) \frac{dt d\zeta}{\zeta^2} = C_\psi \left\langle f, g \right\rangle_{L^2(\mathbb{R})},
\]

(3.4)

where \( C_\psi \) is given by (3.1).

*Proof.* Applying Theorem 2.5, we have for any pair of square integrable functions \( f \) and \( g \)

\[
A_{A_\psi} [f] (t, \zeta) = \sqrt{\zeta} \int_{\mathbb{R}} \exp \left\{ \frac{i}{2B} (2\omega \zeta (Dp - Bq) - D\omega^2 \zeta^2) \right\} \hat{f}_{A_\psi} (\omega) \hat{\Psi}_{A_\psi} (\omega, \zeta) K_{A_\psi} (\omega, t) d\omega
\]
and
\[ A_{\Lambda S}^\psi [g](t, \zeta) = \sqrt{\zeta} \int_\mathbb{R} \exp \left\{ \frac{i}{2B} \left( 2\eta \zeta (Dp - Bq) - D\eta^2 \zeta^2 \right) \right\} \hat{g}_{\Lambda S}(\eta) \hat{\Psi}_{\Lambda S}(\zeta \eta, \zeta) \mathcal{K}_{\Lambda S}^{\text{inv}}(\eta, t) \, d\eta, \]

where \( \Psi \) is given by (2.10), respectively. Further, we have
\[
\int_{\mathbb{R} \times \mathbb{R}^+} A_{\Lambda S}^\psi [f](t, \zeta) A_{\Lambda S}^\psi [g](t, \zeta) \frac{dt \, d\zeta}{\zeta^2}
\]
\[
= \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+} \exp \left\{ \frac{i}{2B} \left( 2\zeta (Dp - Bq)(\omega - \eta) - D\zeta^2(\omega^2 - \eta^2) \right) \right\}
\]
\[
\times \hat{f}_{\Lambda S}(\omega) \hat{g}_{\Lambda S}(\eta) \hat{\Psi}_{\Lambda S}(\zeta \omega, \zeta) \hat{\Psi}_{\Lambda S}(\zeta \eta, \zeta) \mathcal{K}_{\Lambda S}^{\text{inv}}(\omega, t) \mathcal{K}_{\Lambda S}^{\text{inv}}(\eta, t) \frac{dt \, d\omega \, d\eta \, d\zeta}{\zeta}
\]
\[
= \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+} \exp \left\{ \frac{i}{2B} \left( 2\zeta (Dp - Bq)(\omega - \eta) - D\zeta^2(\omega^2 - \eta^2) \right) \right\}
\]
\[
\times \hat{f}_{\Lambda S}(\omega) \hat{g}_{\Lambda S}(\eta) \hat{\Psi}_{\Lambda S}(\zeta \omega, \zeta) \hat{\Psi}_{\Lambda S}(\zeta \eta, \zeta) \delta(\omega - \eta) \frac{d\omega \, d\eta \, d\zeta}{\zeta}
\]
\[
= \int_{\mathbb{R} \times \mathbb{R}^+} \hat{f}_{\Lambda S}(\omega) \hat{g}_{\Lambda S}(\omega) \left| \hat{\Psi}_{\Lambda S}(\zeta \omega, \zeta) \right|^2 \frac{d\omega \, d\zeta}{\zeta}
\]
\[
= \int_{\mathbb{R}} \hat{f}_{\Lambda S}(\omega) \hat{g}_{\Lambda S}(\omega) \left\{ \int_{\mathbb{R}^+} \frac{\left| \hat{\Psi}_{\Lambda S}(\zeta \omega, \zeta) \right|^2}{\zeta} \, d\zeta \right\} \, d\omega
\]
\[
= \mathcal{C}_\psi \langle \hat{f}_{\Lambda S}, \hat{\Psi}_{\Lambda S} \rangle_{L^2(\mathbb{R})}
\]
\[
= \mathcal{C}_\psi \langle f, g \rangle_{L^2(\mathbb{R})}.
\]

This completes the proof. \( \blacksquare \)

It is worth to mention that for \( f = g \), the above theorem reduces to:
\[
\int_{\mathbb{R} \times \mathbb{R}^+} \left| A_{\Lambda S}^\psi [f](t, \zeta) \right|^2 \frac{dt \, d\zeta}{\zeta^2} = \mathcal{C}_\psi \| f \|_2^2.
\]
This is energy preserving theorem for the novel special affine wavelet transform.

Now, the following theorem is the inversion formula for the novel special affine wavelet transform $A_{\Lambda S}^\psi[f](t, \zeta)$.

**Theorem 3.4 (Inversion Formula).** Let $f \in L^2(\mathbb{R})$ be a given function and $\psi$ is admissible. If $A_{\Lambda S}^\psi[f](t, \zeta)$ is the novel special affine wavelet transform of $f$, then $f$ can be reconstructed as

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} A_{\Lambda S}^\psi[f](t, \zeta) \psi_{t,\zeta}^A(x) \frac{dt \, d\zeta}{\zeta^2}, \quad \text{a.e.} \quad (3.5)$$

**Proof.** By virtue of Moyal’s principle, we have

$$\left\langle f, g \right\rangle = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} A_{\Lambda S}^\psi[f](t, \zeta) A_{\Lambda S}^{*\psi}[g](t, \zeta) \frac{dt \, d\zeta}{\zeta^2}$$

$$= \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} A_{\Lambda S}^\psi[f](t, \zeta) \left\{ \int_{\mathbb{R}} g^*(x) \psi_{t,\zeta}(x) \, dx \right\} \frac{dt \, d\zeta}{\zeta^2}$$

$$= \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+} A_{\Lambda S}^\psi[f](t, \zeta) \psi_{t,\zeta}^A(x) g^*(x) \frac{dt \, d\zeta}{\zeta^2}$$

$$= \frac{1}{C_\psi} \left\langle \int_{\mathbb{R} \times \mathbb{R}^+} A_{\Lambda S}^\psi[f](t, \zeta) \psi_{t,\zeta}^A(x) \frac{dt \, d\zeta}{\zeta^2}, g(x) \right\rangle.$$

Since $g$ is chosen arbitrarily from $L^2(\mathbb{R})$, therefore we obtain

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} A_{\Lambda S}^\psi[f](t, \zeta) \psi_{t,\zeta}^A(x) \frac{dt \, d\zeta}{\zeta^2} \quad \text{a.e.}$$

Thus the proof is completed.

The following theorem provides a complete characterization of the range of the novel special affine wavelet transform $A_{\Lambda S}^\psi$.

**Theorem 3.5 (Characterization of Range of $A_{\Lambda S}^\psi$).** If $f \in L^2(\mathbb{R} \times \mathbb{R}^+)$ and $\psi$ is admissible wavelet, then $f$ belongs to the range of $A_{\Lambda S}^\psi$ if and only if

$$f(t', \zeta') = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} f(t, \zeta) \left\langle \psi_{t,\zeta}^A, \psi_{t',\zeta'}^A \right\rangle \frac{dt \, d\zeta}{\zeta^2}.$$  \quad (3.6)

**Proof.** Let $f$ belongs to range of $A_{\Lambda S}^\psi$. Then, there exists a square integrable function $g$, such
that $A_{\Lambda_S}^\psi g = f$. In order to show that $f$ satisfies (3.6), we proceed as

$$f(t', \zeta') = A_{\Lambda_S}^\psi [g](t', \zeta')$$

$$= \int_{\mathbb{R}} g(x) \psi^*_{t', \zeta'}(x) \, dx$$

$$= \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} \left\{ \int_{\mathbb{R} \times \mathbb{R}^+} A_{\Lambda_S}^\psi [g](t, \zeta) \psi^*_{t', \zeta'}(x) \, dt \frac{d\zeta}{\zeta^2} \right\} \psi^*_{t', \zeta'}(x) \, dx$$

$$= \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} A_{\Lambda_S}^\psi [g](t, \zeta) \left\{ \int_{\mathbb{R}} \psi_{t, \zeta}^s(x) \psi^*_{t', \zeta'}(x) \, dx \right\} \frac{dt \, d\zeta}{\zeta^2}$$

$$= \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} f(t, \zeta) \left\langle \psi_{t, \zeta}^s, \psi^*_{t', \zeta'} \right\rangle_2 \frac{dt \, d\zeta}{\zeta^2},$$

which proves the necessary part. Conversely, suppose that a square integrable function $f$ satisfies (3.6). In order to prove that $f$ belongs to range of $A_{\Lambda_S}^\psi$, we need a function $g \in L^2(\mathbb{R})$ satisfying

$$A_{\Lambda_S}^\psi g = f.$$ This required function $g$ is constructed as follows

$$g(x) = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} f(t, \zeta) \psi_{t, \zeta}^s(x) \frac{dt \, d\zeta}{\zeta^2}. \quad (3.7)$$

It is clear that $\|g\|_2 \leq \|f\|_2 < \infty$; that is $g \in L^2(\mathbb{R})$. Also, we have

$$A_{\Lambda_S}^\psi [g](t', \zeta') = \int_{\mathbb{R}} g(x) \psi^*_{t', \zeta'}(x) \, dx$$

$$= \frac{1}{C_\psi} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R} \times \mathbb{R}^+} f(t, \zeta) \psi_{t, \zeta}^s(x) \frac{dt \, d\zeta}{\zeta^2} \right\} \psi^*_{t', \zeta'}(x) \, dx$$

$$= \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}^+} f(t, \zeta) \left\langle \psi_{t, \zeta}^s, \psi^*_{t', \zeta'} \right\rangle_2 \frac{dt \, d\zeta}{\zeta^2}$$

$$= f(t', \zeta').$$

This completes the proof.

Corollary 3.6 (Reproducing Kernel Hilbert Space). For any admissible wavelet $\psi \in L^2(\mathbb{R})$, the range of $A_{\Lambda_S}^\psi$ is a reproducing kernel Hilbert space embedded as a subspace in $L^2(\mathbb{R} \times \mathbb{R}^+)$ with the kernel given by

$$K_{\Lambda_S}^\psi (t, \zeta; t', \zeta') = \left\langle \psi_{t, \zeta}^s, \psi_{t', \zeta'}^s \right\rangle. \quad (3.8)$$
4. Uncertainty Principles Associated with SAFT and NSAWT

The Pitt’s inequality in the Fourier domain expresses a fundamental relationship between a sufficiently smooth function and the corresponding Fourier transform \[5\]. For every \(f \in \mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})\), the inequality states that

\[
\int_{\mathbb{R}} |\omega|^{-\alpha} |\mathcal{F}[f](\omega)|^2 \, d\omega \leq C_\alpha \int_{\mathbb{R}} |x|^{\alpha} |f(x)|^2 \, dx, \quad 0 \leq \alpha < 1
\]

where

\[
C_\alpha = \pi^\alpha \left[ \Gamma \left( \frac{1 - \alpha}{4} \right) / \Gamma \left( \frac{1 + \alpha}{4} \right) \right]^2,
\]

and \(\Gamma(\cdot)\) denotes the well known Euler’s gamma function. The Schwartz class in \(L^2(\mathbb{R})\) is defined by

\[
\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{t \in \mathbb{R}} t^\beta \, \text{D}^\gamma f(t) < \infty \right\},
\]

where \(C^\infty(\mathbb{R})\) is the class of smooth functions, \(\beta, \gamma\) are non-negative integers, and \(\text{D}\) denotes the usual differential operator.

For any \(f \in L^2(\mathbb{R})\), the Heisenberg’s uncertainty inequality in the special affine Fourier domain is given by \[19\]

\[
\left\{ \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right\}^{1/2} \left\{ \int_{\mathbb{R}} \omega^2 \left| \mathcal{F}_A[\omega](\omega) \right|^2 \, d\omega \right\}^{1/2} \geq \frac{|B|}{2} \left\{ \int_{\mathbb{R}} |f(x)|^2 \, dx \right\},
\]

with equality if and only if \(f\) is a multiple of a suitable Gaussian function.

By adopting the strategy analogous to Wilcock \[20\] and Cowling and Price \[7\], we establish a generalization of the uncertainty principle given by (4.1). Furthermore, we derive an uncertainty inequality comparing the localization of the special affine Fourier transform (SAFT) of a function \(f\) with the novel special affine wavelet transform (NSAWT) \(A^{\Lambda}_A[f](t, \zeta)\), regarded as a function of the time variable \(t\).

**Theorem 4.1.** For any \(f \in L^2(\mathbb{R})\), the generalized uncertainty inequality for the special affine Fourier transform (1.1) is given by:

\[
\left\{ \int_{\mathbb{R}} |x|^p |f(x)|^p \, dx \right\}^{1/p} \left\{ \int_{\mathbb{R}} |\omega|^p \left| \mathcal{F}_A[\omega](\omega) \right|^p \, d\omega \right\}^{1/p} \geq \frac{|B|^{(p+2)/2p}}{2} \left\| f \right\|_2^2, \quad 1 \leq p \leq 2.
\]

**Proof.** For any \(f \in L^2(\mathbb{R})\), the generalized uncertainty inequality in the classical Fourier domain is given by

\[
\left\{ \int_{\mathbb{R}} |x|^p |f(x)|^p \, dx \right\}^{1/p} \left\{ \int_{\mathbb{R}} |\omega|^p |\mathcal{F}[f](\omega)|^p \, d\omega \right\}^{1/p} \geq \frac{1}{2} \left\{ \int_{\mathbb{R}} |f(x)|^2 \, dx \right\}, \quad 1 \leq p \leq 2
\]

where \(\mathcal{F}[f]\) denotes the classical Fourier transform of \(f\). We rewrite the definition of the special affine Fourier transform (1.1) as

\[
\mathcal{F}_A[\omega](\omega) = K_\Lambda \int_{\mathbb{R}} g(x) \exp \left\{ \frac{i}{2B} \left( D \omega^2 - 2x \omega - 2\omega(Dp - Bq) \right) \right\} \, dx,
\]

\[4.4\]
where \( g(x) = f(x) \exp \left\{ i(Ax^2 + 2xp)/2B \right\} \). From equation (4.4), it is quite evident that

\[
\hat{f}_{A_S}(\omega) = \frac{1}{\sqrt{|B|}} \exp \left\{ \frac{i}{2B} \left(D\omega^2 - 2\omega(Dp - Bq)\right) \right\} \mathcal{F}[g] \left( \frac{\omega}{B} \right),
\]

so that, \( \sqrt{|B|} |\hat{f}_{A_S}(B\omega)| = |\mathcal{F}[g](\omega)| \). Invoking (4.3) for the function \( g \), we obtain the generalized uncertainty inequality for the special affine Fourier transform:

\[
\left\{ \int_{\mathbb{R}} |x|^p |f(x)|^p dx \right\}^{1/p} \left\{ \int_{\mathbb{R}} |\omega|^p |\hat{f}_{A_S}(\omega)|^p d\omega \right\}^{1/p} \geq \frac{|B|^{(p+2)/2p}}{2} ||f||_2^2, \quad 1 \leq p \leq 2.
\]

Remark: For \( p = 2 \), the generalized uncertainty principle (4.2) boils down to the classical Heisenberg’s uncertainty principle for the special affine Fourier transform.

In view of classical Pitt’s inequality and the relationship between the classical Fourier and the special affine Fourier transform given by (4.5), one clears obtain the following Pitt’s inequality for SAFT.

**Theorem 4.2 (Pitt’s Inequality).** For any \( f \in \mathcal{S}(\mathbb{R}) \), the Pitt’s inequality for the special affine Fourier transform (1.1) is given by:

\[
|B|^\alpha \int_{\mathbb{R}} |\omega|^{-\alpha} |\hat{f}_{A_S}(\omega)|^2 d\omega \leq C_\alpha \int_{\mathbb{R}} |x|^\alpha |f(x)|^2 dx, \quad 0 \leq \alpha < 1.
\]

Now, we shall derive an uncertainty inequality governing the simultaneous localization of \( \mathcal{F}_{SAFT}[f](\omega) \) and \( A^\psi_{A_S}[f](t, \cdot) \).

**Theorem 4.3.** If \( A^\psi_{A_S}[f](t, \cdot) \) is the novel special affine wavelet transform of any nontrivial function \( f \in L^2(\mathbb{R}) \), then the following uncertainty inequality holds:

\[
\left\{ \int_{\mathbb{R}} t^2 |A^\psi_{A_S}[f](t, \cdot)|^2 \frac{d\xi dt}{\xi^2} \right\}^{1/2} \left\{ \int_{\mathbb{R}} \omega^2 \left| \hat{f}_{A_S}(\omega) \right|^2 d\omega \right\}^{1/2} \geq \frac{\sqrt{C_\psi} |B|}{2} ||f||_2^2. \quad (4.7)
\]

**Proof.** The classical Heisenberg-Pauli-Weyl inequality in the SAFT domain is given by

\[
\left\{ \int_{\mathbb{R}} t^2 |f(t)|^2 dt \right\}^{1/2} \left\{ \int_{\mathbb{R}} \omega^2 |\mathcal{F}_{SAFT}[f](\omega)|^2 d\omega \right\}^{1/2} \geq \frac{|B|}{2} \left\{ \int_{\mathbb{R}} |f(t)|^2 dt \right\}. \quad (4.8)
\]

Identifying \( A^\psi_{A_S}[f](t, \cdot) \) as a function of the time variable \( t \) and invoking (4.8), so that

\[
\left\{ \int_{\mathbb{R}} t^2 |A^\psi_{A_S}[f](t, \cdot)|^2 dt \right\}^{1/2} \times \left\{ \int_{\mathbb{R}} \omega^2 |\mathcal{F}_{SAFT}[A^\psi_{A_S}[f](t, \cdot)](\omega)|^2 d\omega \right\}^{1/2} \geq \frac{|B|}{2} \left\{ \int_{\mathbb{R}} |A^\psi_{A_S}[f](t, \cdot)|^2 dt \right\}. \quad (4.9)
\]
Integrating (4.9) with respect to the $d\zeta/\zeta^2$, we obtain
\[
\int_{\mathbb{R}^+} \left\{ \int_{\mathbb{R}} t^2 |A_{\Lambda_S}^\psi [f](t, \zeta)|^2 dt \right\}^{1/2} \times \left\{ \int_{\mathbb{R}} \omega^2 |F_{SAFT} [A_{\Lambda_S}^\psi [f](t, \zeta)](\omega)|^2 d\omega \right\}^{1/2} \frac{d\zeta}{\zeta^2} \\
\geq \frac{|B|}{2} \left\{ \int_{\mathbb{R} \times \mathbb{R}^+} |A_{\Lambda_S}^\psi [f](t, \zeta)|^2 \frac{dt \, d\zeta}{\zeta^2} \right\}.
\] (4.10)

By virtue of the Cauchy-Schwartz's inequality and Fubini theorem we can express (4.10) as
\[
\left\{ \int_{\mathbb{R} \times \mathbb{R}^+} t^2 |A_{\Lambda_S}^\psi [f](t, \zeta)|^2 \frac{d\zeta \, dt}{\zeta^2} \right\}^{1/2} \times \left\{ \int_{\mathbb{R} \times \mathbb{R}^+} \omega^2 |F_{SAFT} [A_{\Lambda_S}^\psi [f](t, \zeta)](\omega)|^2 \frac{d\zeta \, d\omega}{\zeta^2} \right\}^{1/2} \\
\geq \frac{C_\psi |B|}{2} \left\| f \right\|^2.
\]

Using Theorem 2.5, the above inequality can be written as
\[
\left\{ \int_{\mathbb{R} \times \mathbb{R}^+} t^2 |A_{\Lambda_S}^\psi [f](t, \zeta)|^2 \frac{d\zeta \, dt}{\zeta^2} \right\}^{1/2} \\
\times \left\{ \int_{\mathbb{R}} \omega^2 \left| \hat{f}_{\Lambda_S}(\omega) \right|^2 \left\{ \int_{\mathbb{R}^+} \frac{\left| \hat{\Psi}_{\Lambda_S}(\zeta \omega, \zeta) \right|^2}{\zeta} \frac{d\zeta}{\zeta} \right\} \frac{d\omega}{\zeta} \right\}^{1/2} \\
\geq \frac{C_\psi |B|}{2} \left\| f \right\|^2.
\]

Applying Theorem 2.5, we obtain the desired result:
\[
\left\{ \int_{\mathbb{R} \times \mathbb{R}^+} t^2 |A_{\Lambda_S}^\psi [f](t, \zeta)|^2 \frac{d\zeta \, dt}{\zeta^2} \right\}^{1/2} \left\{ \int_{\mathbb{R}} \omega^2 \left| \hat{f}_{\Lambda_S}(\omega) \right|^2 \frac{d\omega}{\zeta} \right\}^{1/2} \geq \frac{\sqrt{C_\psi} |B|}{2} \left\| f \right\|^2.
\]

This completes the proof. \( \square \)

Remark: It should be noted that by choosing an appropriate matrix $\Lambda_S$ yields the respective uncertainty inequalities for the various novel integral transforms.

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