Local uncontrollability for affine control systems with jumps†

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ABSTRACT
This paper investigates affine control systems with jumps for which the ideal \( I(f, g_1, \ldots, g_m) \) generated by the drift vector field \( f \) in the Lie algebra \( L(f, g_1, \ldots, g_m) \) can be imbedded as a kernel of a linear first-order partial differential equation. It will lead us to uncontrollable affine control systems with jumps for which the corresponding reachable sets are included in explicitly described differentiable manifolds.

1. Introduction

Algebraic–geometric methods are successfully applied in control systems theory with relevant results regarding controllability, observability, robust stability, etc. In this paper, the analysis is focused on two special features involving nonlinear affine control systems with jumps associated with uncontrollability. The main theoretical ingredients are presented in Vârsan (1999), where the analysis includes as applications, among others, the control systems and their gradient flow representation, without considering any jumps. A special attention is paid to geometric methods applied to affine control systems which, in a way, can be taken as non-holonomic constraints appearing in mechanics and, therefore, directly connected with hyperbolic differential equations analyzed in the first part of Vârsan (1999).

Motivated by the analysis, with remarkable results, performed in Vârsan (1999) regarding gradient flow representations for some solutions, Lie algebras, gradient systems in a Lie algebra, algebraic representation of gradient systems and their integral manifolds, this paper develops and improves more results in the current literature. The techniques and mathematical tools used in this paper (see also Doroftei & Treanță, 2012; Parveen & Akram, 2012; Treanță & Vârsan, 2013b, 2014; Vârsan, 1994) lead us to a nice gradient flow representation of solutions satisfying the affine control system with jumps considered in the present work.

The standard form of a control system includes trajectories without any jumps. In our non-standard model, we consider piecewise constant control functions. It induces a finite set of jumps for each trajectory and the integral form of a trajectory uses new bounded variation controls as integration variables. In other words, the analysis of non-standard control systems includes a finite set of jumps which influence the trajectories of the control systems. Any affine control system with jumps has the property that its trajectories are contained in the kernel of a linear first-order partial differential equation (PDE). It should be noted that in our case the trajectories are piecewise smooth bounded variation functions. This explains the non-standard writing used in the present work. The main result can be easily connected with so-called unreachability of an affine control system. When controllability or uncontrollability properties are involved, we need a special analysis and our paper presents the details of the uncontrollability properties connected with complete affine control systems under jumps conditions.

The purpose of this paper is to formulate and prove Lie-algebraic sufficient conditions which involve the local uncontrollability of an affine control system with jumps. In the main result of this paper, the uncontrollability conditions are not influenced when a non-standard affine control system is replaced by a standard one. In addition, at each fixed time, the range of affine control system is contained in a shifted manifold using the flow of the drift vector field, while in a driftless control system, the range of the control system is contained in the same \( k \)-dimensional manifold (\( k < n \)). It is to be mentioned that, for driftless affine control systems, the \( k \)-dimensional differentiable manifold is constructed in Parveen and Akram (2012). Also, the reachable set of affine control...
systems with jumps, at any time, can be included in the values of a given smooth mapping. This part, under more restrictive assumptions, can be associated with the analysis performed in Vârsan (1994). For other different but related viewpoints on this subject (discrete-time piecewise linear delay systems and continuous-time Markovian jump systems with time-varying delay and deficient transition descriptions), the reader is directed to Qiu, Feng, and Yang (2009) and Qiu, Wei, and Karimi (2015). As well, by using the concepts of graph completion and generalised solution, some impulsive control systems (where the evolution equation depends linearly on the time derivative of the control function) have been also studied in a series of works (see Bressan, 1987; Bressan & Rampazzo, 1988, 1991, 1994). In the case of measurable control functions, various definitions of generalised trajectories have been introduced in the literature by considering suitable limits of classical solutions, under key assumptions involving either the total variation of the control function or the commutativity of the vector fields. A basic assumption in Bressan and Rampazzo (1991) is that the vector fields \( g_\alpha := (\tilde{g}_\alpha, e_\alpha) \), \( \alpha = 1, m \), commute, i.e. their Lie brackets vanish identically. By the commutativity hypothesis, the authors construct a transformation which reduces the initial control system to a control system in the usual form, with absolutely continuous trajectories. A generalised solution, for the control system studied in Bressan and Rampazzo (1991), is introduced as a trajectory \( t \to x(u, t) \) for which there exists a sequence of controls \( v_k \in C^1([0, T], R^n) \) such that \( v_k(0) = u(0) = \tilde{x} \), \( v_k \to u \) in the \( L^1 \)-norm, and the corresponding trajectories \( x(v_k, \cdot) \) have uniformly bounded values and tend to \( x(u, \cdot) \) in the \( L^1 \)-norm. The case when the vector fields \( g_i \), \( i = 1, m \), do not commute is analysed in Bressan and Rampazzo (1994) by considering a quotient control system. In the present paper, applying the standard Picard’s iterative procedure (used for ordinary differential equations (ODEs)), we get a unique solution for our non-standard control system by combining a smooth mapping (an adequate composition of flows) with a solution satisfying an auxiliary affine control system in the space of parameters.

The paper is organised as follows: in Section 2, we present the notations, definitions and the preliminary results to be used in the sequel; in Section 3, a constructive proof for the problem proposed is presented, and Section 4 formulates the conclusion of this work.

**2. Preliminaries**

We begin this section by introducing some notations, definitions and preliminary results which will be used further.

**Definition 2.1:** The space \( C^1_b(B(x_0, 2\rho \subseteq R^n; R) \) consists of all first-order continuously differentiable functions \( h(x) : B(x_0, 2\rho) \to R \) (see \( B(x_0, 2\rho) \) as the ball centred in \( x_0 \in R^n \) and radius \( 2\rho \), with \( \rho > 0 \) fixed) for which

\[
\partial_i h(x) := \frac{\partial h}{\partial x_i}(x), \quad x \in B(x_0, 2\rho), \quad i \in \{1, \ldots, n\},
\]

are bounded functions.

**Definition 2.2:** A bounded variation function \( v \in BV([0, T]; R) \) is called piecewise smooth (see \( v \in PSBV([0, T]; R) \)) if there is a partition

\[
0 = \tau_0 < \cdots < \tau_N = T,
\]

depending on \( v \), such that \( v \in C^1((\tau_j, \tau_{j+1}); R), \quad j = 0, N - 1 \).

**Definition 2.3:** The set of admissible controls \( V_C \) consists of all \( v \in PSBV([0, T]; R^n) \) satisfying

\[
v(t) = \int_0^t u(s)ds + \hat{v}(t), \quad t \in [0, T], \quad \hat{v}(0) = 0,
\]

where \( \hat{v} \in BV([0, T]; R^n) \) is piecewise constant and \( u(t) : [0, T] \to R^n \) is piecewise continuous such that

\[
|u(t)| \leq C, \quad |\hat{v}(t)| \leq C, \quad t \in [0, T], \quad C > 0.
\]

Consider a finite set of smooth vector fields

\[
\{f, g_1, \ldots, g_m\} \subseteq C^\infty(R^n; R^n)
\]

and associate an affine control system with jumps

\[
\begin{align*}
d_t x &= f(x)dt + \sum_{i=1}^m g_i(x)[u^i(t)dt + d\hat{v}^i(t)] \\
x(0) &= x_0, \quad t \in [0, T], \quad x \in B(x_0, 2\rho) \subseteq R^n,
\end{align*}
\]

where

\[
v(t) = (v^1(t), \ldots, v^m(t))
\]

\[
= \left( \int_0^t u^1(s)ds + \hat{v}^1(t), \ldots, \int_0^t u^m(s)ds + \hat{v}^m(t) \right)
\]

is an admissible control \( v \in V_C \) (see Definition 2.3).

**Definition 2.4:** We say that \( T > 0 \) and \( C > 0 \) are sufficiently small, if the following conditions are fulfilled:

\[
TK_1 + CK_1 \leq 2\rho, \quad TL_1 + CL_2 \in [0, 1),
\]
where

\[
K_1 = \max\{|f(x)| : x \in B(x_0, 2\rho)\},
\]
\[
K_2 = \max\left\{ \sum_{i=1}^{m} |g_i(x)| : x \in B(x_0, 2\rho) \right\},
\]
\[
L_1 = \max\{|\partial_x f(x)| : x \in B(x_0, 2\rho)\},
\]
\[
L_2 = \max\left\{ \sum_{i=1}^{m} |\partial_x g_i(x)| : x \in B(x_0, 2\rho) \right\}.
\]

Applying the standard Picard’s iterative procedure (used for ODEs), we get a unique solution

\[
\{ x_v(t, x_0) \in B(x_0, 2\rho) : t \in [0, T]\},
\]
\[
x_v \in PSBV([0, T]; R^n)
\]
satisfying the affine control system (2), i.e.

\[
d_t x_v(t, x_0) = f(x_v(t, x_0)) dt + \sum_{i=1}^{m} g_i(x_v(t, x_0)) u^i(t) dt
\]
\[
+ \sum_{i=1}^{m} g_i(x_v(t, x_0)) d\delta^i(t),
\]
\[
x_v(0, x_0) = x_0,
\]
\[
t \in [0, T], \text{ for each } v \in V_C, \text{ provided } T > 0 \text{ and } C > 0 \text{ are sufficiently small} \text{ (see Definition 2.4).}
\]

**Comment 2.1:** Let us establish the form of equation given in (2) on each interval of partition (see Definition 2.2). Consider the first interval \([0, \tau_1]\) in our partition, associated with \(v \in V_C\) (see Definition 2.3), where \(t = \tau_1\) is a jump of the control function. In this situation, the equation given in (2) will be written as follows:

\[
\frac{dx_v}{dt}(t, x_0) = f(x_v(t, x_0))
\]
\[
+ \sum_{i=1}^{m} g_i(x_v(t, x_0)) u^i(t),
\]
\[
x_v(\tau_1, x_0) = x_v(\tau_1-, x_0)
\]
\[
+ \sum_{i=1}^{m} g_i(x_v(\tau_1-, x_0)) \left( \delta^i(\tau_1+) - \delta^i(\tau_1-) \right),
\]
\[
t = \tau_1,
\]
\[
\text{for each } v \in V_C, \text{ provided } T > 0 \text{ and } C > 0 \text{ are sufficiently small} \text{ (see Definition 2.4).}
\]

Repeat this procedure on each interval \([\tau_j, \tau_{j+1}]\), \(j = 0, N - 1\). In addition, the partition associated with \(x_v \in PSBV([0, T]; R^n)\) coincides with that of \(v \in V_C\) and the jumps

\[
\Delta x_v(\tau_j, x_0) := x_v(\tau_j+, x_0) - x_v(\tau_j-, x_0),
\]
\[
\Delta \delta^i(\tau_j) := \delta^i(\tau_j+) - \delta^i(\tau_j-)
\]
satisfy the algebraic equations

\[
\Delta x_v(\tau_j, x_0) = \sum_{i=1}^{m} g_i(x_v(\tau_j-, x_0)) \Delta \delta^i(\tau_j),
\]
\[
\quad j = 0, N.
\]

Assuming Definition 2.4 and having in mind the Banach fixed-point theorem, the following result can be formulated.

**Lemma 2.1:** Let \(B(x_0, 2\rho) \subset R^n\) be a fixed ball and take \(T > 0\), \(C > 0\) such that Definition 2.4 is fulfilled. Then, for each \(v \in V_C\), there exists a unique solution

\[
\{ x_v(t, x_0) \in B(x_0, 2\rho) : t \in [0, T]\},
\]
\[
x_v \in PSBV([0, T]; R^n)
\]
satisfying the affine control system with jumps in (2).

**Definition 2.5:** The affine control system (2) is locally uncontrollable if there exist \(\bar{T} > 0\), \(\bar{C} > 0\) and a \(k\)-dimensional \((k < n)\) differentiable manifold, \(M_{x_0} \subset B(x_0, \rho)\), containing \(x_0 \in R^n\), such that the unique solutions

\[
\{ x_v(t, x_0) \in B(x_0, 2\rho) : t \in [0, \bar{T}], \, v \in V_C \}
\]
of (2) satisfy

\[
x_v(t, x_0) \in \{ F(t)[\lambda] : \lambda \in M_{x_0}, \, t \in [0, \bar{T}] \},
\]
\[
\text{for any } v \in V_C. \text{ Here}
\]
\[
\{ F(\tau)[\lambda] : \tau \in [-\bar{T}, \bar{T}], \, \lambda \in B(x_0, \rho) \}
\]
is the local solution of ODEs \(\frac{dF(\tau)[\lambda]}{d\tau} = f(F(\tau)[\lambda])\), \(\tau \in [-\bar{T}, \bar{T}]\), with initial value \(F(0)[\lambda] = \lambda \in B(x_0, \rho)\).

**Definition 2.6:** Let \(\{g_1, \ldots, g_m\} \subset C^\infty(R^n; R^n)\) be a finite set of smooth vector fields. Define the Lie algebra generated by these vector fields, denoted \(L = L(g_1, \ldots, g_m) \subset C^\infty(R^n; R^n)\), as being algebra which contains Lie products (of arbitrary length) composed by elements of \(\{g_1, \ldots, g_m\}\) and of their linear span.

**Definition 2.7:** Let \(L\) be a Lie algebra. A Lie ideal of \(L\) is a linear subspace \(I \subset L\) such that \([f; g] \in I\), whenever \(f \in L\) and \(g \in I\).

**Definition 2.8:** The ideal \(I_f(g_1, \ldots, g_m) \subset L(f, g_1, \ldots, g_m)\) generated by \(f\) in the Lie algebra \(L(f, g_1, \ldots, g_m)\) is obtained as the Lie subalgebra generated by an infinite set of smooth vector fields

\[
\{ad_f^j(g_i) : i \in \{1, \ldots, m\}, \, j \geq 0\}.
\]
that is,

\[ I_f (g_1, \ldots, g_m) := \{ ad_f (g_i) : i \in \{ 1, \ldots, m \}, \ j \geq 0 \} . \]

Also, define \( \dim I_f (g_1, \ldots, g_m) (x_0), \ x_0 \in \mathbb{R}^n \), as the dimension of the vector space spanned by \( I_f (g_1, \ldots, g_m) (x_0) \).

**Definition 2.9:** The ideal \( I_f (g_1, \ldots, g_m) \) is called locally of finite type if there exists a system of generators \( \{ X_1, \ldots, X_M \} \subseteq I_f (g_1, \ldots, g_m) \) such that any \( X \in I_f (g_1, \ldots, g_m) \) can be written as

\[ X(x) = \sum_{j=1}^{M} \alpha_j (x) X_j (x), \ x \in B(x_0, r) \subseteq \mathbb{R}^n, \]

using smooth scalar functions \( \{ \alpha_1, \ldots, \alpha_M \} \subseteq C^\infty (B(x_0, r); \mathbb{R}) \) depending on \( X \), where \( r > 0 \) is fixed.

**Comment 2.2:** A simple example when \( L(f, g_1, \ldots, g_m) \) and \( I_f (g_1, \ldots, g_m) \) are of finite type is based on linear vector fields \( f(x) = Ax, \ g_1 (x) = B_1 x, \ i = 1, m, \) where \( \{ A, B_1, \ldots, B_m \} \subseteq \mathbb{M}_{n \times n} \) generates the finite-dimensional Lie subalgebras \( L(A, B_1, \ldots, B_m) \) and \( I_A (B_1, \ldots, B_m) \).

### 3. Main result

Our main goal is to stipulate Lie-algebraic sufficient conditions leading us to the local uncontrollability of the affine control system with jumps given in (2). More precisely, we shall solve the following:

**Main problem. Under the following conditions:**

(i) The ideal \( I_f (g_1, \ldots, g_m) \subseteq L(f, g_1, \ldots, g_m) \) generated by \( f \) in the Lie algebra \( L(f, g_1, \ldots, g_m) \) is locally of finite type (see Definitions 2.8 and 2.9);

(ii)

\[ \dim I_f (g_1, \ldots, g_m) (x_0) = k < n, \]

prove that (2) is locally uncontrollable.

The main result (see Theorem 3.1) can be easily connected with so-called unreachable problem of an affine control system. This indicates that the reachable set \( R(t, x_0) := \{ x \in B(x_0, 2\rho) : x = x_0 (t, x_0), \ \rho \in \mathbb{R}_+ \} \) satisfies \( \text{int}_R \mathbb{R}(t, x_0) = \emptyset \), for any \( t \in [0, \hat{T}] \), even if the largest set of controls \( \rho \in \mathbb{R}_+ \) is used. This set of admissible controls, \( \rho \in \mathbb{R}_+ \), generates analytic solution in the auxiliary control system, provided the assumptions in Vărsan (1994) (which are more restrictive) will be used.

The algebraic-geometric methods used here lead us to a nice gradient flow representation which is based, this time, on a complete smooth mapping including the flow generated by the drift \( \{ f \} \) (see Parveen & Akram, 2012, for driftless affine control systems). On the other hand, the reduced smooth mapping (as a second component of the main mapping) is generated by the corresponding infinitesimal generators in the ideal \( I_f (g_1, \ldots, g_m) \) and the evolution of the original dynamical system will be described using an auxiliary affine control system with jumps. In addition, the Lie subalgebra \( I_f (g_1, \ldots, g_m) \), under the hypotheses stipulated in **Main problem**, can be imbedded as a kernel of a linear first-order PDE. It is to be mentioned that, for driftless affine control systems, the k-dimensional differentiable manifold is constructed in Parveen and Akram (2012), provided \( \dim L(g_1, \ldots, g_m) (x_0) = k \) is assumed. In particular, under the singularity assumption, \( \dim I_f (g_1, \ldots, g_m) (x_0) = k < n \), we can extend the auxiliary results (see Lemmas 3.1 and 3.2) involving a larger class of admissible controls described by piecewise smooth bounded variation functions (see \( v \in PSBV ([0, T]; \mathbb{R}^m) \)). Taking into account the computation complexity of the derived results, certain calculations will be omitted in our presentation. For more details, there will be indicated bibliographic references.

The algorithm for **Main problem** relies on the gradient flow representation of the solutions \( \{ x_c (t, x_0) : t \in [0, \hat{T}] \}, \ v \in \mathbb{V}_c \) satisfying the affine control system with jumps in (2). In this respect, fixing a system of generators \( \{ g_1, \ldots, g_m, h_{m+1}, \ldots, h_M \} \subseteq I_f (g_1, \ldots, g_m) \), we define an adequate composition of flows:

\[ y = G(t, p; x_0) = F(t) \circ G_1 (t_1) \circ \cdots \circ G_m (t_m) \]

\[ \circ H_{m+1} (t_{m+1}) \circ \cdots \circ H_M (t_M) [x_0], \]

\[ t \in [-\hat{T}, 0], \ p = (t_1, \ldots, t_m) \in \hat{D}_M = \prod_{i=1}^{M} [\hat{a}_i, \hat{a}_i]. \]

Here

\[ \{ F(t)[z] : t \in [-\hat{T}, 0] \}, \ \{ G_i (t_i)[z] : t_i \in [-\hat{a}_i, \hat{a}_i] \}, \]

\[ \{ H_j (t_j)[z] : t_j \in [-\hat{a}_j, \hat{a}_j] \} \]

are the local solutions satisfying ODEs driven by the vector fields \( f, g_i \) and \( h_j \) correspondingly, with the initial conditions

\[ F(0)[z] = z, \ G_i (0)[z] = z, \ H_j (0)[z] = z, \]

\[ i = 1, m, j = m+1, M. \]

In addition, \( \hat{T} > 0, \hat{a}_k > 0, k = 1, M, \) are sufficiently small such that
\[ |F(t)[z] - z| \leq \rho/(M + 1), \quad |G_i(t)z - z| \leq \rho/(M+1), \]
\[ |H_j(t)z - z| \leq \rho/(M + 1), \]

for any \( z \in B(x_0, \rho), \ t \in [-\hat{T}, \hat{T}], \ t_i \in [-\hat{a}_i, \hat{a}_i] \) and \( t_j \in [-\hat{a}_j, \hat{a}_j], \ i = 1, m, \ j = m+1,M. \) With these minor arrangements, we notice that the smooth mapping in (5) satisfies
\[ y = G(t; p; x_0) \in B(x_0, 2\rho) \subseteq R^n, \]
\[ t \in [-\hat{T}, \hat{T}], \ p \in \hat{D}_M. \] \hfill (6)

The main goal of the so-called gradient flow representation in (5) is to make sure that each infinitesimal generator
\[ \partial_0 y := Y_k(t, t_1, \ldots, t_{k-1}; y), \]
\[ k = 1, M, \quad (Y_0 = f, \ t_0 = t) \] \hfill (7)

can be represented with respect to the fixed system of generators
\[ \{g_1, \ldots, g_m, h_{m+1}, \ldots, h_M\} \subseteq I_f(g_1, \ldots, g_m) \]
as follows:
\[ Y_k(t, t_1, \ldots, t_{k-1}; y) = \sum_{i=1}^m \alpha_i^k(t; p)g_i(y) \]
\[ + \sum_{j=m+1}^M \alpha_j^k(t; p)h_j(y), \] \hfill (8)

where the smooth scalar functions \( \alpha_j^k \in C^\infty([-\hat{T}, \hat{T}] \times \hat{D}_M; R), \ j = 1, M, \) depend on \( Y_k, \ k = 1, M. \)

**Definition 3.1:** Let \( Z, Z_j : D \subseteq R^n \rightarrow R^n, \ j = 1, M, \) be some vector fields. We shall denote by
\[ \{Z_1(y), \ldots, Z_M(y)\}, \quad \{Z_1, \ldots, Z_M\}(y) \]
a \((n \times M)\) matrix whose column \( j \) is the vector \( Z_j(y), \ j = 1, M, \) and also \( adZ\{Z_1, \ldots, Z_M\}(y) \) is defined as \( \{adZ\{Z_1, \ldots, Z_M\}\}(y) \), with \( adZ\{Z_1, \ldots, Z_M\}\) \( = [Z_1, Z_j](x) = \frac{\partial Z_j}{\partial x}(x)Z_1(x) - \frac{\partial Z_1}{\partial x}(x)Z_j(x). \) \vspace{0.2cm}

Finally, the algebraic representation in (8) must be
\[ \{Y_1(t; y), Y_2(t, t_1; y), \ldots, Y_M(t, t_1, \ldots, t_{M-1}; y)\} \]
\[ = \{g_1, \ldots, g_m, h_{m+1}, \ldots, h_M\}(y)A(t, p), \]
\[ (y = G(t, p; x_0)), \] \hfill (9)

where the \((M \times M)\) smooth matrix \( A(t, p) \) is a non-singular one for any \( t \in [0, \hat{T}], \ p \in \hat{D}_M, \ A(0, 0) = I_M \) (for more details, see Váršan, 1999).

Both representations (see (5) and (9)) are helpful for extending our considerations, but the non-singularity of the algebraic equations in (9) allows us to introduce an auxiliary affine control system in the space of parameters \( p \in \hat{D}_M \subseteq R^M, \)
\[ d_i p = \sum_{i=1}^m q_i(t, p)dt_i(t), \quad t \in [0, \hat{T}], \ p \in \hat{D}_M, \ v \in V_\hat{C}, \]
\[ p(0) = 0. \] \hfill (10)

Here, the smooth vector fields \( \{q_1, \ldots, q_m\} \subseteq C^\infty([0, \hat{T}] \times \hat{D}_M; R^M) \) are found as the unique solution of the algebraic equations
\[ A(t, p)q_i(t, p) = e_i, \quad i = 1, M, \ t \in [0, \hat{T}], \ p \in \hat{D}_M, \] \hfill (11)

where the non-singular matrix \( A(t, p) \) is given in (9) and \( \{e_1, \ldots, e_M\} \subseteq R^M \) stands for the canonical basis. Notice that \( 0 < \hat{T} < T \) and \( 0 < \hat{C} \leq C \) will be taken sufficiently small such that each solution of (10) satisfies
\[ \{p_0(t) \in B(0, \delta) \subseteq \hat{D}_M : t \in [0, \hat{T}]\}, \]

(see \( \delta > 0 \) fixed) for any admissible control \( v \in V_\hat{C}. \) Combining the smooth mapping \( y = G(t, p; x_0) \) (see (5)) with a solution \( p_0 \in PSBV([0, \hat{T}] ; B(0, \delta) \subseteq \hat{D}_M), \) satisfying the auxiliary control system (10), we get (by a straight computation)
\[ x_0(t, x_0) := G(t, p_0(t); x_0) \in B(x_0, 2\rho), \quad t \in [0, \hat{T}] \] \hfill (12)

as the unique solution of the original affine control system (2), for any \( v \in V_\hat{C}. \)

The last step in our algorithm is to construct a \( k \)-dimensional manifold \( M_{x_0} \subseteq B(x_0, \rho) \subseteq R^n (k < n, \text{see (ii) of Main problem}) \) such that
\[ \hat{G}(p_0(t); x_0) \in M_{x_0}, \quad t \in [0, \hat{T}], \ v \in V_\hat{C}. \] \hfill (13)

where \( \{p_0(t) \in B(0, \delta) \subseteq \hat{D}_M : t \in [0, \hat{T}]\} \) satisfy (10) and the smooth mapping \( \hat{G}(p; x_0) \in B(x_0, \rho), \ p \in \hat{D}_M, \) is given by
\[ \hat{G}(p; x_0) := G_1(t_1) \circ \cdots \circ G_m(t_m) \circ H_{m+1}(t_{m+1}) \cdots \circ H_M(t_M)[x_0]. \] \hfill (14)
Notice that the complete mapping \( G(t, p; x_0) \) defined in (5) is the composition of the flow \( F(t)[z] \) (generated by the vector field \( f \)) with the reduced smooth mapping \( \hat{G}(p; x_0) \) described in (14). In addition, the reduced smooth mapping \( \hat{G}(p; x_0) \) is generated by a fixed system of generators

\[
\left\{ g_1, \ldots, g_m, h_{m+1}, \ldots, h_M \right\} \subseteq I_f(g_1, \ldots, g_m) \tag{15}
\]

with respect to which the non-singular algebraic representation (9) is valid.

**Definition 3.2:** A system of generators \( \{X_1, \ldots, X_M\} \) for the (locally) of finite-type Lie algebra \( I_f(g_1, \ldots, g_m) \) is called \((k, x_0)\)-minimal system of generators, if

\[
\{X_1(x_0), \ldots, X_k(x_0)\} \subseteq \mathbb{R}^n
\]

are linearly independent and \( X_j(x_0) = 0 \) for \( j = k + 1, M \), where \( k = \dim I_f(g_1, \ldots, g_m)(x_0). \)

To get (13), we need to use a \((k, x_0)\)-minimal system of generators

\[
\{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\} \subseteq I_f(g_1, \ldots, g_m).
\]

Consequently (see Definition 3.2),

\[
\{X_1(x_0), \ldots, X_k(x_0)\} \subseteq \mathbb{R}^n \tag{16}
\]

are linearly independent, and \( X_j(x_0) = 0 \) for \( j = k + 1, M \).

A \((k, x_0)\)-minimal system of generators, \( \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\} \), replacing \( \{g_1, \ldots, g_m, h_{m+1}, \ldots, h_M\} \) in (15), will be found from the original system of generators \( \{g_1, \ldots, g_m, h_{m+1}, \ldots, h_M\} \subseteq I_f(g_1, \ldots, g_m) \) multiplied on the right-hand side by a non-singular \((M \times M)\) constant matrix \( K \), i.e.

\[
\{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\} = \{g_1, \ldots, g_m, h_{m+1}, \ldots, h_M\}K. \tag{17}
\]

Making an abuse, we shall preserve the notation introduced for the reduced mapping \( \{\hat{G}(p; x_0) : p \in \hat{D}_M\} \) (see (14)) and rewrite it as

\[
\hat{G}(p; x_0) = G_1(t_1) \circ \cdots \circ G_M(t_M)[x_0], \quad p = (t_1, \ldots, t_M) \in \hat{D}_M, \tag{18}
\]

where \( \{G_j(t)[z] : t \in [-\hat{a}_j, \hat{a}_j]\} \) is the solution of ODEs

\[
\frac{dx}{dt} = X_j(x), \quad t \in [-\hat{a}_j, \hat{a}_j], \quad j = 1, M \tag{19}
\]

\( x(0) = z. \)

According to (15), this time, \( \{\hat{G}(p; x_0) : p \in \hat{D}_M\} \) in (18) has a reduced form

\[
\hat{G}(p; x_0) = \hat{G}(\hat{p}; x_0) = G_1(t_1) \circ \cdots \circ G_k(t_k)[x_0], \quad \hat{p} \in \hat{D}_k, \tag{20}
\]

which is more suitable to describe it as a \(k\)-dimensional manifold \( M_{x_0} \subseteq B(x_0, \rho) \). Though the smooth mapping in (5) will be replaced by

\[
y = G(t, p; x_0) = F(t) \circ \hat{G}(p; x_0), \quad t \in [0, \hat{T}], \quad p \in \hat{D}_M, \tag{21}
\]

using \( \{\hat{G}(p; x_0) : p \in \hat{D}_M\} \) defined in (18), the final algebraic non-singular representation in (9) will be written according to the original system of generators as follows:

\[
\{Y_1(t; y), Y_2(t, t_1; y), \ldots, Y_M(t, t_1, \ldots, t_{M-1}; y)\} = \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\} \hat{A}(t, p)
\]

\[
\{g_1, \ldots, g_m, h_{m+1}, \ldots, h_M\}(y)A(t, p), \quad t \in [0, \hat{T}], \quad p \in \hat{D}_M, \tag{22}
\]

where \( A(t, p) := K\hat{A}(t, p), \quad t \in [0, \hat{T}], \quad p \in \hat{D}_M, \) is the non-singular \((M \times M)\) matrix used for constructing the auxiliary control system in (10).

The \(k\)-dimensional manifold \( M_{x_0} \subseteq B(x_0, \rho) \) associated with the reduced mapping \( \hat{G}(p; x_0) = \hat{G}(\hat{p}; x_0) \in B(x_0, \rho) \) (see (18) and (20)) will be obtained using the same theoretical ingredients as in Parveen and Akram (2012) (see Lemma 2.1). We shall formulate it as the following lemma:

**Lemma 3.1:** Assume that the conditions (i) and (ii) of Main problem are satisfied and define the reduced mapping

\[
\hat{G}(p; x_0) = \hat{G}(\hat{p}; x_0) \in B(x_0, \rho), \quad p \in \hat{D}_M, \quad \hat{p} \in \hat{D}_k
\]

(see (18) and (20)). Then there exist \( 0 < \hat{\rho} \leq \rho \) and non-constant smooth scalar functions \( \{\lambda_{k+1}, \ldots, \lambda_n\} \subseteq C^1(B(x_0, \hat{\rho}) \subseteq \mathbb{R}^n; R) \) such that

\[
\lambda_j(\hat{G}(p; x_0)) = \lambda_j(x_0), \quad p \in B(0, \delta) \subseteq \hat{D}_M, \quad j = k + 1, n. \tag{23}
\]

The proof of this lemma is a direct application of the implicit functions theorem. On the other hand, there are some differences between the non-singular algebraic representation in (22) and that which was proved in Lemma 2 of Parveen and Akram (2012). The main difference
involves the smooth mapping
\[
y = G(t, p; x_0) T := F(t) \circ \hat{G}(p; x_0), \quad p \in \hat{D}_M
\]
along which the non-singular algebraic representation in (22) is performed. Here, apart from the smooth mapping \( \hat{G}(p; x_0), \quad p \in \hat{D}_M \), generated by the \( (k, x_0) \)-minimal system of generators \( \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\} \), we use the group of diffeomorphisms \( \{F(t)[z] : t \in [-\hat{T}, \hat{T}], \quad z \in B(x_0, \rho)\} \) obtained as local solutions satisfying ODEs
\[
\frac{dx}{dt} = f(x), \quad t \in [-\hat{T}, \hat{T}]
\]
x(0) = z \in B(x_0, \rho).

Consider the smooth mappings
\[
y = G(t, p; x_0) := F(t) \circ \hat{G}(p; x_0)
\]
\[
\hat{y} := \hat{G}(p; x_0), \quad p = (t_1, \ldots, t_M) \in \hat{D}_M, \quad t \in [0, \hat{T}],
\]
where
\[
\hat{y} = \hat{G}(p; x_0) := G_1(t_1) \circ \cdots \circ G_M(t_M)[x_0], \quad p \in \hat{D}_M
\]
is generated by the \( (k, x_0) \)-minimal system of generators
\[
\{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\} \subseteq I_f(g_1, \ldots, g_m)
\]
(see (18)). The Lie subalgebra \( I_f(g_1, \ldots, g_m) \) fulfills the conditions of Lemma 2 in Parveen and Akram (2012) and it allows us to write the following non-singular algebraic representation associated with \( \hat{y} = \hat{G}(p; x_0) \), \( p \in \hat{D}_M \),
\[
\{\partial_1 \hat{y}, \partial_2 \hat{y}, \ldots, \partial_m \hat{y}\}
\]
\[
= \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\} \hat{y} B(p), \quad p \in \hat{D}_M,
\]
where the \( (M \times M) \) matrix \( B(p) \) is a non-singular one (see \( B(0) = I_M \)). On the other hand, the computation of the infinitesimal generators associated with the complete smooth mapping \( y = G(t, p; x_0) \) in (24) leads us directly to (see (26) and \( \hat{y} = F(-t)[y] \))
\[
\{\partial_1 y; \partial_2 y; \ldots, \partial_m y\}
\]
\[
= (\partial_1 F(-t)[y])^{-1} \{\partial_1 \hat{y}; \partial_2 \hat{y}; \ldots, \partial_m \hat{y}\}
\]
\[
= (\partial_2 F(-t)[y])^{-1} \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\}
\]
\[
(F(-t)[y]) B(p),
\]
for any \( t \in [0, \hat{T}] \) and \( p = (t_1, \ldots, t_m) \in \hat{D}_M \).

For each \( t \in (0, \hat{T}] \) and \( s \in [0, t] \), denote
\[
N(s; t, p) = (\partial_2 F(-s)[y])^{-1} \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\} (F(-s)[y])
\]
(see \( y = G(t, p; x_0) \) in (24)) and compute \( \frac{d}{ds} N(s; t, p) \) as follows:
\[
\frac{dN(s; t, p)}{ds} = N(s; t, p) D(t - s, p), \quad s \in [0, t]
\]
\[
N(0; t, p) = \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_M\}(y).
\]

Here, the \( (M \times M) \) matrix \( D(t - s, p) \) stands for the algebraic representation associated with the linear map (see \( ad_f : I_f(g_1, \ldots, g_m) \rightarrow I_f(g_1, \ldots, g_m) \))
\[
ad_f \{X_1, \ldots, X_M\} (F(-s)[y] \circ \hat{G}(p, x_0))
\]
\[
= \{X_1, \ldots, X_M\} (F(-s)[y]) D(t - s, p).
\]

Applying the standard Picard’s iterative procedure to the linear ODEs (29), we get the unique solution
\[
N(s; t, p) = \{X_1, \ldots, X_M\}(y) B_1(s; t, p), \quad s \in [0, t], \quad p \in \hat{D}_M,
\]
where the non-singular \( (M \times M) \) matrix \( B_1 \) satisfies the linear matrix differential equations
\[
\frac{dB_1(s; t, p)}{ds} = B_1(s; t, p) D(t - s, p), \quad s \in [0, t]
\]
\[
B_1(0; t, p) = I_M.
\]

In particular, denote \( A_1(t, p) = B_1(t; t, p) \) and write (27) as follows (see \( A(t, p) = KA_1(t, p)B(p) \) and (16)):
\[
\{\partial_1 y; \partial_2 y; \ldots, \partial_m y\} = \{X_1, \ldots, X_M\}(y) A_1(t, p) B(p)
\]
\[
= \{g_1, \ldots, g_m, h_{m+1}, \ldots, h_M\}(y) A(t, p),
\]
\[
t \in [0, \hat{T}], \quad p \in \hat{D}_M.
\]

where the \( (M \times M) \) matrix \( A(t, p) \) is a non-singular one (see \( A(0, 0) = K \)).

The previous computation will be stated as follows:

**Lemma 3.2**: Assume that the hypotheses (i) and (ii) are satisfied. Consider the smooth mapping \( y = G(t, p; x_0) \in B(x_0, 2\rho) \) as in (24) and compute the infinitesimal generators
\[
\{\partial_1 y, \partial_2 y; \ldots, \partial_m y\}
\]
\[
= \{Y_1(t; y), Y_2(t, t_1; y), \ldots, Y_M(t, t_1, \ldots, t_{M-1}; y)\}.
\]
Then the non-singular algebraic representation in (30) is valid (see \(A(0, 0) = K\)).

Now, we shall formulate and prove the main result of the present paper. This indicates that each reachable set \(R(t, x_0)\) is contained in a shifted \(k\)-dimensional differentiable manifold \((k < n)\), or \(\int_{\mathbb{R}} R(t, x_0) = \emptyset\), provided \(\dim I_f(g_1, \ldots, g_m)(x_0) = k\) and \(I_f(g_1, \ldots, g_m)\) is of finite type. This reflects the possibility that the corresponding Lie algebra

\[
I_f(g_1, \ldots, g_m) := \{ ad^f_i(g_i) : i \in \{1, \ldots, m\}, j \geq 0 \}
\]

contains a finite set of generators.

**Theorem 3.1:** Let the smooth vector fields \([f, g_1, \ldots, g_m] \subseteq C^\infty(\mathbb{R}^n; \mathbb{R}^n)\) be given such that (i) and (ii) are satisfied. Consider \(\hat{T} > 0\) and \(\hat{C} > 0\) sufficiently small (see Lemma 2.1) such that the unique solution \((p_\nu(t) : t \in [0, \hat{T}])\) of (10) satisfies \((p_\nu(t) \in B(0, \delta) \subseteq \hat{D}_M : t \in [0, \hat{T}], \nu \in \hat{V}_C)\), for each \(\nu \in \hat{V}_C\). Then (2) is locally uncontrollable, where

\[
\begin{align*}
x_\nu(t, x_0) &:= G(t, p_\nu(t); x_0) \\
&= F(t) \circ \hat{G}(p_\nu(t); x_0), \quad t \in [0, \hat{T}], \nu \in \hat{V}_C
\end{align*}
\]

satisfies the control system (2) and

\[
\begin{align*}
y_\nu(t, x_0) &:= \hat{G}(p_\nu(t); x_0) \in M_{x_0}, \quad t \in [0, \hat{T}], \nu \in \hat{V}_C,
\end{align*}
\]

where the \(k\)-dimensional manifold \(M_{x_0} \subseteq B(x_0, \rho) (k < n)\) is constructed in Lemma 3.1.

**Proof:** By hypothesis, the conditions of Lemmas 2.1, 3.1 and 3.2 are satisfied such that the smooth mappings

\[
\hat{y} = \hat{G}(p; x_0)
\]

\[
\hat{y} := G_1(t_1) \circ \cdots \circ G_M(t_M)[x_0] \in B(x_0, \rho), \quad p \in \hat{D}_M,
\]

defined in Lemma 3.1, and

\[
y = G(t; p; x_0)
\]

\[
y := F(t) \circ \hat{G}(p; x_0) \in B(x_0, 2\rho), \quad p \in \hat{D}_M, \quad t \in [0, \hat{T}],
\]

described in Lemma 3.2, can be used to represent the solutions

\[
\{ x_\nu(t, x_0) : t \in [0, \hat{T}], \nu \in \hat{V}_C \}
\]

satisfying affine control system with jumps in (2). Using the non-singular algebraic representation given in Lemma 3.2, associated with the infinitesimal generators \([\delta_i y, \ldots, \delta_{i_a} y]\), define the smooth vector fields \([q_1, \ldots, q_m] \subseteq C^\infty([0, \hat{T}] \times \hat{D}_M; \hat{R}^m)\) such that

\[
A(t, p)q_i(t, p) = e_i, \quad p \in \hat{D}_M, \quad t \in [0, \hat{T}], \quad i = 1, m,
\]

where \([e_1, \ldots, e_M] \subseteq \hat{R}^m\) is the canonical basis.

Define the auxiliary control system in (10) and assume that \(\hat{T}, \hat{C} > 0\) are sufficiently small (see Lemma 2.1) such that each unique solution of (10) satisfies

\[
p_\nu(t) \in B(0, \delta) \subseteq \hat{D}_M, \quad t \in [0, \hat{T}], \nu \in \hat{V}_C.
\]

Define

\[
x_\nu(t, x_0) := G(t, p_\nu(t); x_0)
\]

and

\[
y_\nu(t, x_0) := \hat{G}(p_\nu(t); x_0), \nu \in \hat{V}_C, \quad t \in [0, \hat{T}].
\]

By a straight computation, we get that

\[
\{ x_\nu(t, x_0) \in B(x_0, 2\rho) : t \in [0, \hat{T}], \nu \in \hat{V}_C \}
\]

satisfies the affine control system (2). In addition, \(y_\nu(t, x_0) \in M_{x_0} \subseteq B(x_0, \rho), \quad t \in [0, \hat{T}], \nu \in \hat{V}_C\), which lead us to \(x_\nu(t, x_0) \in \{ F(t)[\lambda] : \lambda \in M_{x_0} \}\), for any \(\nu \in \hat{V}_C\). The proof is now complete.

### 3.1 Application to non-holonomic systems

To highlight how the reasonings were designed in this work, we consider the following application. Let us consider a non-holonomic system defined as a non-integrable system of Pfaff forms (Doroftei, 2001)

\[
\sum_{j=1}^{n} a_{i_j}(t, x)dx_j + a_{i_0}(t, x)dt = 0,
\]

\[
i \in \{1, 2, \ldots, k\}, \quad k < n,
\]

where the scalar functions \(a_{i_j}(t, x), a_{i_0}(t, x) : I \times G \to R\) are of \(C^\infty\)-class and \(I \subseteq R, G \subseteq \mathbb{R}^n\) are open sets. Making some assumptions and notations, the foregoing system (31) can be written as follows:

\[
dy = \sum_{j=0}^{m} \hat{g}_j(\tau, y)d\tau_j,
\]

where \(dy = col(dx_1, \ldots, dx_k)\), with the variable \(y \in R^k, k < n\), as function depending on \((t, x_{k+1}, \ldots, x_n) := \tau := (\tau_0, \ldots, \tau_m) \in \hat{D}_{m+1} := \prod_{j=0}^{m}(\tau_j^0 - a_{j}, \tau_j^0 + a_{j})\).
$a_j > 0$, $m := n - k$; also, $\hat{g}_j \in C^\infty (\tilde{D}_{m+1} \times \tilde{D}_k := \tilde{D}_{n+1}; R^k)$, $\tilde{D}_k := \prod_{j=1}^k \{ x_j^0 - \alpha_j, x_j^0 + \alpha_j \}$, $\alpha_j > 0$ and $(\tau_0^0, x_0^0, \ldots, x_k^0, \tau_1^0, \ldots, \tau_m^0) := (\tau^0, x^0) \in I \times G$ is fixed.

The previous two systems with differentials, (31) and (32), are equivalent relative to the set of solutions. The system (31) is non-integrable because the system (32) does not satisfy the Frobenius integrability condition associated with the gradient system

$$\frac{\partial y}{\partial \tau_0} = \hat{g}_0(\tau, y), \ldots, \frac{\partial y}{\partial \tau_m} = \hat{g}_m(\tau, y)$$

for $i \in \mathbb{Z}^+ : [0, \delta]$.

A weak equivalence of the systems (31) and (32) is obtained if the notion of solution for (31) and (32) is relaxed. In this regard, let us consider the set of curves $\tau(s) := (-\delta, \delta) \rightarrow \tilde{D}_{n+1}$, of $C^1$-class, with $\tau(0) := \tau^0 := (\tau_0^0, \tau_1^0, \ldots, \tau_m^0)$, denoted $\Upsilon$. By definition, a solution for the system (32) is a $C^1$-class curve $\gamma(s) := (-\delta, \delta) \rightarrow \tilde{D}_k$, satisfying the following system:

$$\frac{dy}{ds}(s) = \sum_{j=0}^m \hat{g}_j(\tau(s), y(s)) \frac{d\tau_j}{ds}(s), \quad s \in I_5 := [0, \delta]$$

$$y(0) = x^0 := (x_1^0, \ldots, x_k^0)$$

for an arbitrary curve $\tau \in \Upsilon$.

For $z := (\tau, y) \in R^{n+1}$, we rewrite (33) as a (linear) control system

$$\frac{dz}{ds} = \sum_{j=0}^m u_j(s) g_j(z), \quad z \in \tilde{D}_{n+1}, \quad s \in I_6$$

$$z(0) = (\tau^0, x^0) := z_0$$

$$= (\tau_0^0, x_0^0, \tau_1^0, \ldots, \tau_m^0) \in R^{n+1}, \quad (34)$$

where $u(s) = (u_0(s), \ldots, u_m(s)) : I_6 \rightarrow R^{m+1}$ is continuous and $g_j(z) := (e_j, \hat{g}_j(\tau, y)) \in R^{n+1}$, with $\{e_0, \ldots, e_m\} \in R^{m+1}$ the canonical basis.

The geometric structure for the set of all $C^1$-class trajectories $(t(s), x(s)) : I_6 \rightarrow I \times G$, which satisfy the non-holonomic system (31) and $(\tau(0), x(0)) := (\tau_0^0, x^0)$, can and will be achieved by association of a minimal integral manifold for (34). Compared to the control system (with jumps) analysed in this paper, this time, the system (34) is linear in control.

For the control system (34), we define the real Lie algebra $L(g_0, \ldots, g_m) \subseteq C^\infty (\tilde{D}_{n+1}; R^{m+1})$ determined by the $C^\infty$-class vector fields $\{g_0, \ldots, g_m\}$. The following result occurs (the integral manifold associated with a locally of finite-type Lie algebra).

**Proposition 3.1:** Let us suppose that $L(g_0, \ldots, g_m)$ is locally of finite type and $\dim L(g_0, \ldots, g_m)(z_0) = n_0$, with $n_0 < n + 1$. Then there exists a smooth manifold $M_{z_0} \subseteq \tilde{D}_{n+1}$, of dimension $n_0$, associated with (34), such that

$$M_{z_0} := \left\{ z \in \tilde{D}_{n+1} : z = z(p), \quad p \in D_{n_0} := \prod_{i=1}^{n_0} (-a_i, a_i) \right\},$$

where $z(p) = F_t(t_1) \circ \cdots \circ F_{n_0}(t_{n_0})(z_0), \quad p = (t_1, \ldots, t_{n_0}), \quad F_t(\tau, \lambda)$ is the local flow generated by the vector field $Y_t(z)$, and $\{Y_1, \ldots, Y_{M}\} \subseteq L(g_0, \ldots, g_m)$ (with $M \geq n_0$) is a $n_0$-minimal system of generators.

- for every $z''(s), \quad s \in I_5, \quad u \in C(I_5; B(0, \rho_0) \subset R^{m+1})$, solution in (34), there exists $p''(s) \in D_{n_0}, \quad s \in I_5$, of $C^1$-class such that

$$z''(s) = z(p''(s)), \quad s \in I_5$$

and

$$\frac{dp''}{ds}(s) = \sum_{i=0}^m u_i(s) q_i(p''(s)), \quad p''(0) = 0,$$

where $q_i \in C^\infty (D_{n_0}; R^{m+1})$ satisfies $\frac{d}{dp}(p) q_i(p) = g_i(z(p))$ and

$$\dim L(q_0, \ldots, q_m)(p) = n_0, \quad \forall p \in D_{n_0}.$$

**4. Conclusion**

The standard form of a control system includes trajectories without any jumps. In our non-standard model, we started with piecewise constant control functions, replacing the usual bounded and measurable functions. In this paper, we have investigated affine control systems...
with jumps for which the ideal \( I_f(g_1, \ldots, g_m) \), generated by the drift \( f \) in the Lie algebra \( L(f, g_1, \ldots, g_m) \), fulfils a degeneracy condition \( \dim I_f(g_1, \ldots, g_m)(x_0) = k < n \), for some \( x_0 \in \mathbb{R}^n \). It led us to uncontrollable affine control systems with jumps, where the main (complete) smooth mapping includes the flow generated by the drift \( f \) combined with a reduced smooth mapping determined by a system of generators in the ideal \( I_f(g_1, \ldots, g_m) \subseteq L(f, g_1, \ldots, g_m) \). This reduced mapping is a finite composition of flows starting from a fixed point \( x_0 \in \mathbb{R}^n \) and its values can be restricted to a \( k \)-dimensional differentiable manifold \( (k < n) \) provided we assume \( \dim I_f(g_1, \ldots, g_m)(x_0) = k \). The main result (see Theorem 3.1) indicates that each reachable set \( R(t, x_0) \) is contained in a shifted \( k \)-dimensional differentiable manifold \( (k < n) \), or \( \text{int}_{\mathbb{R}^n} R(t, x_0) = \emptyset \), provided \( \dim I_f(g_1, \ldots, g_m)(x_0) = k \) and \( I_f(g_1, \ldots, g_m) \) is of finite type. For other different but related viewpoints on this subject, the reader is directed to Akram and Lomadze (2009), Jurdevic and Sussmann (1972), Malybaev (2003), Polderman and Willems (1998), Pomet (1999), Qiu et al. (2009), Qiu et al. (2015), Sonntag (1990), Sussmann (1983), Treanţă and Udrişte (2013a) and Treanţă (2014).

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