INFINITE DIMENSIONAL GEOMETRY AND
QUANTUM FIELD THEORY OF STRINGS
II. INFINITE DIMENSIONAL NONCOMMUTATIVE
GEOMETRY OF A SELF–INTERACTING STRING FIELD

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Abstract. A geometric interpretation of quantum self–interacting string field theory is given. Relations between various approaches to the second quantization of an interacting string are described in terms of the geometric quantization. An algorithm to construct a quantum nonperturbative interacting string field theory in the quantum group formalism is proposed. Problems of a metric background (in)dependence are discussed.

This is the second part of the paper devoted to various structures of an infinite dimensional geometry, appearing in quantum field theory of (closed) strings; the objects connected with the second quantization of a free string were described in the first part [1], whereas an analogous material for a self–interacting string field will be discussed now.

It is proposed to continue the investigation of an infinite dimensional geometry related to the quantum field theory of strings in the following two parts (parts III,IV). The third part is devoted to an infinite dimensional $W$–geometry of a second quantized free string [2]; the fourth part should contain materials on infinite dimensional geometry of a self–interacting $W$–string field.

In the whole paper we follow a general ideology of string theory presented in [3]. All four parts of the publication maybe considered as a sequel of previous one [4] devoted to geometric aspects of quantum conformal field theory: the transition from the 2D quantum conformal field theory to the self–interacting string field theory maybe considered as one from the abstract geometry of noncommutative Riemann

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surfaces (in spirit of [4]) to aspects of, roughly speaking, their "imbeddings" into target spaces. Such geometric picture seems to underlie a very powerful machinery for explorations of geometry of those spaces (see [5] for some topological parallels as well as [6] on so-called "quantum cohomology" as an essential part of them). The naturality of the framework of the infinite dimensional geometry for the subject was clearly explained in [7,4]. Here it should be only specially mentioned that in the second quantized formalism the imbeddings of algebraic curves into target spaces are described by algebraic structures on the space of their germs at initial point, which encode all global topological or geometric information on the intrinsic as well as the extrinsic geometry of a world-sheet (cf. with similar situation in the theory of univalent functions, where one predicts global behaviour of a conformal mapping by coefficients of its Taylor expansion [8]); it allows to account the nonperturbative effects, which seems to be essential for differential geometry in contrast to topology (in the first quantized approach such effects maybe described only by something cumbersome as imbeddings of Riemann surfaces of infinite genus). Thus, a transition from the first quantized formalism to the second quantized one means an enlargement of the substantially finite-dimensional algebro-geometric picture [9] by infinite dimensional geometric one. It produce a lot of questions, which should be attributed to the functional analysis, nevertheless, we try to avoid them below if they do not explicate straightforwardly the underlying infinite dimensional geometry (it does not mean that they are regarded as less important mathematically, but only as "second order" ones for our rather complicated geometric picture).

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1. Infinite dimensional noncommutative geometry and 2D quantum conformal field theory

In this chapter the material of [4] is exposed with a lot of new details. The main idea is that all principal structures of 2D QCFT maybe related to some faces of the noncommutative geometry [10,11]. The categoric (Kontsevich–Segal) approach is described in par.1.1. The approach based on operator algebras is described in par.1.2. as an infinitesimal counterpart of the first. Par.1.3. is devoted to a description of the renormalization of pointwise product in operator algebras and, therefore, an imbedding of both approaches into a framework of the noncommutative geometry. Relations of the noncommutative geometrical formulation of 2D QCFT with the Krichever–Novikov formalism (in its renormalized version) are also considered.

1.1. The Lie algebra Vect(S^1) of vector fields on a circle S^1, the group Diff+(S^1) of diffeomorphisms of a circle S^1, the Virasoro algebra vir, the Virasoro–Bott group Vir and the Neretin semigroup Ner (the mantle Mantle(Diff+(S^1)) of the group of diffeomorphisms of a circle); the semigroup Voile(Diff+(S^1)) – the voile of the group of diffeomorphisms of a circle, manifolds Trinion(Diff+(S^1)) and Polynion(Diff+(S^1)) of trinions and polynions, the Kontsevich–Segal category Train(Diff+(S^1)) – the train of the group of diffeomorphisms of a circle, the modular functor. Let us briefly expose some facts and constructions following [1,4]. Let Diff(S^1) be the group of analytic diffeomorphisms of a circle S^1, it consists of two connected components – the normal subgroup Diff+(S^1) of diffeomorphisms preserving an orienta-
tion on a circle $S^1$ and the coset $\text{Diff}_-(S^1)$ of ones changing it; a Lie algebra of the connected group $\text{Diff}_+(S^1)$ is identified with the Lie algebra $\text{Vect}(S^1)$ of analytic vector fields $v(t)d/dt$ on a circle $S^1$; the structure constants of the complexification $\mathbb{C}\text{Vect}(S^1)$ of the Lie algebra $\text{Vect}(S^1)$ have the form $c'_{jk} = (j-k)\delta_{j+k}$ in the basis $e_k := i \cdot \exp(ikt)d/dt$; in 1968 I.M.Gelfand and D.B.Fuchs discovered a non-trivial central extension of Lie algebra $\text{Vect}(S^1)$, the corresponding 2–cocycle may be written as $c(u,v) = \int u'(t)dv'(t)$; independently this central extension was discovered in 1969 by M.Virasoro and was called later the Virasoro algebra vir (the same name belongs to the complexification $\mathbb{C}\text{vir}$ of this algebra), the Virasoro algebra $\mathbb{C}\text{vir}$ is generated by the vectors $e_k$ and the central element $c$, the commutation relations in it have the form $[e_j, e_k] = (j-k)e_{j+k} + \delta(j+k) \cdot \frac{i^j k^j}{12} \cdot c$; one may correspond an infinite dimensional group Vir to the Lie algebra $\text{vir}$ which is a central extension of the group $\text{Diff}_+(S^1)$, the corresponding 2–cocycle was calculated by R.Bott in 1977 $c(f, g) = \int \log(g(f))'d\log f'$, the group Vir is called the Virasoro–Bott group. There are no any infinite dimensional groups corresponding to the Lie algebras $\mathbb{C}\text{Vect}(S^1)$ and $\mathbb{C}\text{vir}$ but it is useful to consider the following construction, which is attributed to Yu.Neretin and was developed by M.Kontsevich: let us consider accordingly to Yu.Neretin a local group $\mathcal{L}\text{Diff}_+(S^1)$ of all analytic mappings $g$ from $S^1$ to $\mathbb{C}\setminus\{0\}$ with a Jordan image $g(S^1)$ homotopical to $S^1$ as an oriented contour in $\mathbb{C}\setminus\{0\}$ such that $g'(e^{it})$ is not equal to 0 anywhere; the Neretin semigroup $\text{Ner}$ is just the semigroup of all elements $g$ of $\mathcal{L}\text{Diff}_+(S^1)$ such that $|g(\exp(it))| < 1$, it is evident that $\text{Ner}$ is a local semigroup, the globalization was performed by Yu.Neretin (a brief description of Yu.Neretin construction one may find in [1,7], the original and more expanded version is contained in [12,13]). As it was shown by M.L.Kontsevich the elements of the Neretin semigroup $\text{Ner}$ maybe identified with triples $(K,p,q)$, where $K$ is a Riemann surface with a boundary biholomorphically equivalent to a ring, $p$ and $q$ are fixed analytic parametrizations of the two components of the boundary $\partial K$ of the surface $K$, so that $p$ is an input parametrization and $q$ is an output parametrization; the product of two elements of the Neretin semigroup is just the gluing of Riemann surfaces. The Neretin semigroup $\text{Ner}$ is called the mantle of the group $\text{Diff}_+(S^1)$ of diffeomorphisms of a circle and is denoted also by $\text{Mantle}(\text{Diff}_+(S^1))$; the Neretin semigroup $\text{Ner}$ admits a central extension $\tilde{\text{Ner}}$, the corresponding 2–cocycle was calculated by Yu.Neretin in 1989.

Following the way of [14,App.3] it is reasonable to consider a certain generalization of the mantle $\text{Mantle}(\text{Diff}_+(S^1))$ — the voile $\text{Voile}(\text{Diff}_+(S^1))$ of the group of diffeomorphisms of a circle. The elements of the voile $\text{Voile}(\text{Diff}_+(S^1))$ are triples $(K,p,q)$, where $K$ is a Riemann surface of arbitrary genus with two component boundary, $p$ and $q$ are fixed analytic parametrizations of these components, $p$ is an input parametrization and $q$ is an output parametrization; the product of two elements of the semigroup $\text{Voile}(\text{Diff}_+(S^1))$ is the sewing of Riemann surfaces. The genus of Riemann surfaces defines a system of characters of the voile $\text{Voile}(\text{Diff}_+(S^1))$ as well as its grading. As it was marked in [14,App.3] it is very interesting to consider the “fractal” completion of the voile $\text{Voile}(\text{Diff}_+(S^1))$ including surfaces of infinite genus, which maybe considered as a “fluctuating” exponent of the Lie algebra $\mathbb{C}\text{Vect}(S^1)$ describing an evolution of virtual particles’ clothes.

The semigroup constructions (related to the mantle $\text{Mantle}(\text{Diff}_+(S^1))$ and the voile $\text{Voile}(\text{Diff}_+(S^1))$ of the group of diffeomorphisms of a circle) maybe generalized to the category constructions (related to the train $\text{Train}(\text{Diff}_+(S^1))$ of the
group of diffeomorphisms of a circle) following to G.Segal [15]. Nevertheless, for the following purposes it is rather reasonable to consider constructions of manifolds Trinion(Diff+(S1)) and Polynion(Diff+(S1)) of trinions and polynomials and their representations before the general exposition of ones of the Kontsevich–Segal category Train(Diff+(S1)) and its representations (modular functors) by an analogy to [14, App.3].

First of all let’s briefly remind a geometric way to construct the highest weight representations over the Virasoro algebra CVir [16] and their extensions to projective representations of the Neretin semigroup Ner [12,13] based on the infinite dimensional geometry of the flag manifold for the Virasoro–Bott group [17,16,7]. The flag manifold M(Vir) for the Virasoro–Bott group is the homogeneous space Diff+(S1)/S1; there exist several realizations of this manifold: the realization of M(Vir) as an infinite dimensional homogeneous space Diff+(S1)/S1 is called algebraic (in this realization M(Vir) maybe identified also with the quotient of the Neretin semigroup Ner by its subsemigroup Ner0 consisting of elements g ∈ Ner, which admit an analytic extension to D− = {z ∈ C : |z| ≥ 1}); in the probabilistic realization the group Diff+(S1) acts on the space of all probabilistic measures u(t) dt on S1 with an analytic positive density u(t) in a natural way: in the orbital realization the space M(Vir) is identified with coadjoint orbits of the groups Diff+(S1) and Vir (such realization provides M(Vir) by a two–parametric family of symplectic structures ωh,c), in the analytic realization the space M(Vir) is identified with the class S of functions f(z) analytic and univalent in the closed unit disc D+ normalized by the conditions f(0) = 0, f′(0) = 1, f′(exp(it)) ≠ 0 by the Kirillov construction (the Taylor coefficients c1, c2, c3, . . . ck, . . . of a function f(z) = z + c1z2 + c2z3 + . . . + ckzk+1 + . . . determine a coordinate system on S; necessary and sufficient conditions for univalency of a function f(z) maybe found in [8]; the action of CVect(S1) on M(Vir) has the form Lw f(z) = −if2(z) f′(z) f′(w) f(w) f(z) dw (f ∈ S, v ∈ CVect(S1)). The symplectic structures ωh,c coupled with complex structure on M(Vir) form the two–parametric family of (pseudo)–Kähler metrics wh,c; a geometric way to construct the Verma modules over the Virasoro algebra is based on the following facts: (1) To each Diff+(S1)–invariant Kähler metric wh,c wh,c on the space M(Vir) one should correspond the linear holomorphic bundle Eh,c over M(Vir) with the following properties: (a) Eh,c is the Hermite bundle with metric exp(−Uh,c) dλ dλ, where λ is a coordinate in a fiber, Kh,c = exp(Uh,c) is the Bergman kernel function, the exponential of the Kähler potential of the metric wh,c; (b) algebra CVir holomorphically acts in the prescribed bundle by covariant derivatives with respect to the hermitean connection with the curvature form being equal to 2πiωh,c; (2) let O(Eh,c) be the space of all polynomial (in some natural trivialization) germs of sections of the bundle Eh,c (the action of CVir in its Z+–graded module O(Eh,c) (deg(ck) = k) is defined by the formulas Lp = −∂p ηp + ∑k≥1(k + 1)ck e−ηp ηk ηp p > 0, L0 = ∑k≥1kc k ηp ηp, L−1 = ∑k≥1((k + 2)c k + 1−2ck c1) ηk ηp 2h c1, L−2 = ∑k≥1((k + 3)c k + 2−(4c2−c1)ck− bk(c1, . . . c k+2) ηk ηp + h(4c2−c1 2) + 2(2c2−c1) ηp ηp k− 1ηp 2ηp) η2−2 L−1 · L−2 (n > 0)); let us fix the basis e a1,...an = e a1,...an in O(Eh,c), and let O∗(Eh,c) be the space of all linear functionals p on O(Eh,c), which obey the property: if p(x) = 0 then deg(x) ≤ Np (the space O∗(Eh,c) is called the Fock space of the pair (M(Vir),Eh,c)) and is denoted by F(Eh,c)); the Verma module Vh,c over CVir is realized in the Fock space F(Eh,c) and if one fix the basis e a1,...an = e a1,...an in F(Eh,c) such
that \( < e_{a_1 \ldots a_n}, e^{b_1 \ldots b_m} > = a_1! \ldots a_n! \delta^{b_1}_{a_1} \ldots \delta^{b_m}_{a_n} \) the action of the Virasoro algebra in such basis will be defined by the formulas \( L_p = c_p + \sum_{k \geq 1} c_{k+p} \frac{\alpha_{k+p}}{\alpha_k} \) (\( p > 0 \)), \( L_0 = \sum_{k \geq 1} k c_k \frac{\alpha_k}{\alpha_{k+1}} + h \), \( L_1 = \sum_{k \geq 1} c_k \left( k + 2 \right) \frac{\alpha_k}{\alpha_{k+1}} - 2 \frac{\alpha_k}{\alpha_{k+1}} \) + \( 2 h \), \( L_2 = \sum_{k \geq 1} c_k \left( k + 3 \right) \frac{\alpha_k}{\alpha_{k+2}} - (4 \frac{\alpha_k}{\alpha_{k+2}} - \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^2) \) + \( b_k \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^2 \) + \( \frac{(\alpha_k}{\alpha_{k+1}})^2 \), \( L_n = \frac{\alpha_n}{\alpha_{n-1}} \) \( \text{ad}^{n-2} L_1 \cdot L_2 \) (\( n > 2 \)). Such action of \( \text{Vir} \) may exponentiated to a projective representation of Ner.

Points of a manifold \( \text{Trinion}(\text{Diff}_+(S^1)) \) are \textit{trinions}, the triples \((K, p_1, p_2, q)\), where \( K \) is a Riemann surface of genus 0 with a boundary \( \partial K = \partial K^{\text{in}} \cup \partial K^{\text{out}} \) (\( \partial K^{\text{in}} \sim \partial K^{\text{out}} \sim S^1 \)), \( p_i \) are input parametrizations of \( \partial K^{\text{in}} \) and \( q \) is an output parametrization of \( \partial K^{\text{out}} \). The three copies of the Neretin semigroup Ner act on \( \text{Trinion}(\text{Diff}_+(S^1)) \); the corresponding infinitesimal action of \( \mathbb{C} \text{Vect}(S^1) + \mathbb{C} \text{Vect}(S^1) + \mathbb{C} \text{Vect}(S^1) \) on \( \text{Trinion}(\text{Diff}_+(S^1)) \) is transitive. Let \( \pi \) be the projective representation of Ner \( \times \) Ner \( \times \) Ner in the space \( \mathbb{P}(V) \), the (Ner \( \times \) Ner \( \times \) Ner)–equivariant (or infinitesimally \( \mathbb{C} \text{Vect}(S^1) + \mathbb{C} \text{Vect}(S^1) + \mathbb{C} \text{Vect}(S^1) \)–equivariant) mapping of \( \text{Trinion}(\text{Diff}_+(S^1)) \) into \( \mathbb{P}(V) \) is called the projective representation of the manifold of trinions \( \text{Trinion}(\text{Diff}_+(S^1)) \). The next fact (which should be attributed to folklore, I think, and which is convenient to formulate as a proposition) plays a crucial role.

**Proposition 1.** The manifold \( \text{Trinion}(\text{Diff}_+(S^1)) \) of trinions for the group \( \text{Diff}_+(S^1) \) of diffeomorphisms of a circle admits one and only one projective representation in the space \( \text{Hom}(V_{h_1,c} \otimes V_{h_2,c}, V_{h_3,c}) \) for each central charge \( c \) and for each triple of weights \((h_1, h_2, h_3)\).

The manifold of trinions \( \text{Trinion}(\text{Diff}_+(S^1)) \) will be also denoted by \( \text{Trinion}^{\dagger}(\text{Diff}_+(S^1)) \), whereas \( \text{Trinion}^{-}(\text{Diff}_+(S^1)) \) will denote the space of \textit{anti-trinions}, the triples \((K, p, q_1, q_2)\), where \( K \) is a Riemann surface of genus 0 with a boundary \( \partial K = \partial K^{\text{in}} \cup \partial K^{\text{out}} \) (\( \partial K^{\text{in}} \sim \partial K^{\text{out}} \sim S^1 \)), \( p \) is an input parametrization of \( \partial K^{\text{in}} \) and \( q_1, q_2 \) are output parametrizations of \( \partial K^{\text{out}} \). The three copies of the Neretin semigroup Ner act on \( \text{Trinion}^{-}(\text{Diff}_+(S^1)) \) and the corresponding infinitesimal action of \( \mathbb{C} \text{Vect}(S^1) + \mathbb{C} \text{Vect}(S^1) + \mathbb{C} \text{Vect}(S^1) \) on \( \text{Trinion}^{-}(\text{Diff}_+(S^1)) \) is transitive. The definition of the projective representation of the manifold \( \text{Trinion}^{-}(\text{Diff}_+(S^1)) \) is the same as for \( \text{Trinion}^{\dagger}(\text{Diff}_+(S^1)) \). The following analog of Proposition 1 holds: the manifold \( \text{Trinion}^{-}(\text{Diff}_+(S^1)) \) of \textit{anti-trinions} for the group \( \text{Diff}_+(S^1) \) of diffeomorphisms of a circle admits one and only one projective representation in the space \( \text{Hom}(V_{h_1,c}, V_{h_2,c} \otimes V_{h_3,c}) \) for each central charge \( c \) and for each triple of weights \((h_1, h_2, h_3)\).

It should be mentioned that trinions maybe glued with each other. On this way we receive a structure, which underlies self-interacting string field theory on tree level. Namely, the points of the manifold of \textit{polynomials} \( \text{Polynomial}(\text{Diff}_+(S^1)) \) are data \((K, p_1, \ldots, p_n, q)\), where \( K \) is a Riemann surface of genus 0 with a boundary \( \partial K = \bigsqcup_{i=1}^n \partial K^{\text{in}} i \cup \partial K^{\text{out}} \) (\( \partial K^{\text{in}} \sim \partial K^{\text{out}} \sim S^1 \)), \( p_i \) are input parametrizations of \( \partial K^{\text{in}} \) and \( q \) is an output parametrization of \( \partial K^{\text{out}} \). The manifold \( \text{Polynomial}(\text{Diff}_+(S^1)) \) is \( \mathbb{Z}_+ \)–graded, the connected component \( \text{Polynomial}_{n}(\text{Diff}_+(S^1)) \) of degree \( n \) consists of data \((K, p_1, \ldots, p_{n-1}, q)\); it should be mentioned that \( \text{Polynomial}_{0}(\text{Diff}_+(S^1)) = \text{Mantle}(\text{Diff}_+(S^1)) \) and \( \text{Polynomial}_{1}(\text{Diff}_+(S^1)) = \text{Trinion}(\text{Diff}_+(S^1)) \). The \((n + 1) \) copies of the Neretin semigroup Ner act on \( \text{Polynomial}_{n}(\text{Diff}_+(S^1)) \) and the corresponding infinitesimal action of \((n + 1) \mathbb{C} \text{Vect}(S^1) \) is transitive. One may define a projective representation of the manifold \( \text{Polynomial}_{n}(\text{Diff}_+(S^1)) \) in the same way as
for Trinion(Diff+ (S^1)) and to formulate an analog of Proposition 1 (namely, that the manifold Polynion_n (Diff+ (S^1)) of polynions for the group Diff+ (S^1) of diffeomorphisms of a circle admits one and only one projective representation in the space Hom(⨁_{i=1}^n V_{k,i}, V_{h_{i+1},i+1})). But the whole space Polynion_n (Diff+ (S^1)) admits a subsidiary structure defined by gluing. Namely, two elements (K^{(1)}, p_1^{(1)}, . . . p_n^{(1)}, q^{(1)}) and (K^{(2)}, p_1^{(2)}, . . . p_n^{(2)}, q^{(2)}) maybe glued in m different ways into the third element (K^{(3)}, p_1^{(3)}, . . . p_{n+m−1}^{(3)}, q^{(3)}) as follows K^{(3)} = K^{(1)} ∪_{q^{(1)}(e^{(1)})=p_i^{(2)}(e^{(2)}), q^{(2)}} K^{(2)} (i = 1 . . . m), p_j^{(3)} = p_j^{(2)} if j = 1 . . . i − 1, p_{j−i+1}^{(1)} if j = m . . . i + m − 1 and p_{j−n+1}^{(2)} if j = i + m . . . m + n − 1, q^{(3)} = q^{(2)}. It should be mentioned that the gluing is compatible with the grading and that Polynion_n (Diff+ (S^1)) (n > 1) are generated by Trinion(Diff+ (S^1)) = Polynion_1 (Diff+ (S^1)) via gluing. Now one may define the projective representation of Polynion(Diff+ (S^1)) as the set of representations of Polynion_n (Diff+ (S^1)) in Hom(H⊗(n+1), H), which correspond the composition of operators to the gluing.

Here some subsidiary comments are needed. First of all, the definition of polynions may be formulated also for anti-polynions (the set of which maybe denoted by Polynion− (Diff+ (S^1)) whereas the set of polynions maybe denoted also by Polynion+ (Diff+ (S^1))). Second, the operation of the gluing maybe generalized to the partial operation of the sewing: which correspond to two polynions their gluing but with the extracting (if it is possible) the element of the mantle from the input (or output) of the gluing. The partial operation of the sewing maybe extended to the more large manifold than one of polynions (see f.e. [18]). Thirth, it should be mentioned that the definition of representation of Trinion(Diff+ (S^1)) do not deal with operators from the space Hom(·, ·), the definition of the representation of polynions maybe formulated in analogous manner, it allows, for example, to consider arbitrary linear relations instead of linear operators (it is effective in infinite dimensional case [11]), etc; but below we shall work with unbounded linear operators and linear relations as with ordinary operators to avoid the "second–order" technical details of functional analysis hidden in our infinite–dimensional geometric picture. Fourth, the important class of representations of Polynion(Diff+ (S^1)) is formed by the permutation–invariant ones; namely, the representation of Polynion(Diff+ (S^1)) is called permutation–invariant, if it is equivariant with respect to the actions of symmetric groups S_n on the inputs of elements of Polynion_{n−1} (Diff+ (S^1)) and the multiples in tensor products H⊗n. This is a natural class of representations but not unique interesting one, for instance the permutation–invariant representations maybe generalized to braided ones (in which permutations of multiples in tensor product H⊗n should be accompanied by some transforms in them).

The definition of the space Polynion(Diff+ (S^1)) and its representations maybe generalized to the notion of the Kontsevich–Segal category Train(Diff+ (S^1)), the train of the group Diff+ (S^1) of diffeomorphisms of a circle, and its representations (modular functors) [15,19] (see also [12,13]). Objects A, B, C, . . . of the Kontsevich–Segal category are ordered finite sets (which are represented by disjoint ordered unions of circles); morphisms from Mor(A, B) are data (K, p_1, . . . p_n, q_1, . . . q_m), where K is arbitrary Riemann surface with a boundary ∂K = ∪_{i=1}^n ∂K_{in} ∪_{j=1}^m ∂K_{out} (n = #A, m = #B), p_i are output parametrizations of ∂K_{in} and q_j are output parametrizations of ∂K_{out}. The (n + m) copies of the Neretin semigroup Ner act on Mor(A, B) (n = #A, m = #B) but the infinitesimal action of (n + m) CVect(S^1) is
not transitive; nevertheless, the representations of $\text{Mor}(\mathcal{A}, \mathcal{B})$ maybe correctly defined in the same way as for $\text{Polynion}(\text{Diff}_+(S^1))$. The scope of representations of $\text{Mor}(\mathcal{A}, \mathcal{B})$ forms a representation of the whole category $\text{Train}(\text{Diff}_+(S^1))$ (or modular functor) if (1) they transform the composition of morphisms (the gluing) into the composition of operators, (2) if $K = K_1 \sqcup K_2$ then the representation operator of morphism corresponding to $K$ is a tensor product of ones corresponding to $K_i$.

Some remarks should be done. First, one may include $\varnothing$ into the class of objects of $\text{Train}(\text{Diff}_+(S^1))$ (see e.g. [15,19] but we prefer not to do it). Second, the permutation–invariant modular functors maybe defined in a way similar to the described above one for polynomials.

1.2. Infinitesimal objects for representations of $\text{Polynion}^\pm(\text{Diff}_+(S^1))$: QCFT–operator algebras and coalgebras. Infinitesimal objects for permutation–invariant representations of $\text{Polynion}^\pm(\text{Diff}_+(S^1))$: vertex operator algebras and coalgebras. Infinitesimal objects for modular functors: QCFT–operator crossing–algebras and vertex operator crossing–algebras. The infinitesimal counterparts of objects of par. 1.1. will be constructed below following ideas of [14,App.3] (see also [18]).

First, let describe the operation of the vertex insertion into the element of the $\text{Mantle}(\text{Diff}_+(S^1))$ (the Neretin semigroup $\text{Ner}$). Namely, let us consider an arbitrary projective representation of the manifold $\text{Trinion}^+(\text{Diff}_+(S^1))$ in $\text{Hom}(V_{h_1,c} \otimes V_{h_2,c}, V_{h_3,c})$: the operator corresponding to the trinion $(K, p_1, p_2, q)$ will be denoted by $Y_{(K,p_1,p_2,q)}(\cdot, \cdot)$ or $Y_{(K,p_1,p_2,q)}(\cdot)$, where the second argument should stand out of brackets. Let us consider an arbitrary element $(K, p, q)$ of $\text{Mantle}(\text{Diff}_+(S^1))$ and a point $z$ on $K$: the operator $\lim_{p'(v'\to z)} Y_{(K,p',p,q)}(v_h)$ will be considered as a result of an insertion of the vertex $v_h$ into $(K, p, q)$ at point $z$, where $v_h$ is the highest vector in $V_{h_1}$ and limit is considered up to a multiple (i.e. in the projective space); it will be denoted by $Y_{(K,p,q)}(v_h; z)$ (it should be mentioned that this operator is defined up to a number multiple). Second, one may define a vertex itself as $\lim_{K \to \mathbb{S}^1, p(\nu(\nu)) \to v(\nu)} Y_{(K,p,q)}(v; z)$, the vertex (which is defined up to a multiple) will be denoted by $Y(v_h; z)$. In this construction $z \in \mathbb{S}^1$, but $Y(v_h; z)$ is a well–defined operator if $|z| \ll 1$ as a rule (but this circumstance should be considered as "second order" one).

It should be mentioned that the change of $\text{Trinion}^+(\text{Diff}_+(S^1))$ on $\text{Trinion}^-(\text{Diff}_+(S^1))$ leads to the notion of a co–vertex and the co–vertex insertion. So the vertex will be also denoted by $Y^+(v; z)$ and the result of its insertion by $Y^+_{(K,p,q)}(v; z)$ whereas their co–counterparts will be denoted by $Y^-(v; z)$ and $Y^-_{(K,p,q)}(v; z)$.

Now let us consider the situation, when a representation of $\text{Trinion}^\pm(\text{Diff}_+(S^1))$ is extended to a representation of $\text{Polynion}^\pm(\text{Diff}_+(S^1))$. It is natural that the gluing operation in the least should induce some algebraic structure on vertices (co–vertices). Indeed, this case vertices (co–vertices) form a closed operator algebra (co–algebra) in the sense specified below. Some definitions are necessary here [20-22].

Definition 1.

A QFT–operator algebra (operator algebra of quantum field theory) is a pair $(H, t^i_{\alpha \beta}(\vec{x}))$, where $H$ is a vector space and $t^i_{\alpha \beta}(\vec{x})$ is a $H$–valued tensor field on $\mathbb{R}^n$ or $\mathbb{C}^n$ such that $t^i_{\alpha \gamma}(\vec{x})t^j_{\beta \gamma}(\vec{y}) = t^i_{\alpha \beta}(\vec{x} - \vec{y})t^j_{\gamma}(\vec{y})$.
B. A QFT–operator coalgebra (operator coalgebra of quantum field theory) is a pair \((H, t^\alpha_{\gamma}(\vec{x}))\), where \(H\) is a vector space and \(t^\alpha_{\gamma}(\vec{x})\) is a \(H\)-valued tensor field on \(\mathbb{R}^n\) or \(\mathbb{C}^n\) such that \(t^\beta_{\alpha}(\vec{x})t^\gamma_{\delta}(\vec{y}) = t^\delta_{\alpha}(\vec{y})t^\gamma_{\beta}(\vec{x} - \vec{y})\).

It should be mentioned that if \(H\) is a QFT–operator algebra then \(H^*\) is QFT–operator coalgebra and vice versa. So below we shall be interested presumably in QFT–operator algebras.

Let us define the operators \(l_\vec{x}(e_\alpha)\) \((l_\vec{x}(e_\alpha)e_\beta = t^\gamma_{\alpha\beta}(\vec{x})e_\gamma)\) in QFT–operator algebras, then the following identities will hold \(l_\vec{x}(e_\alpha)l_\vec{y}(e_\beta) = t^\gamma_{\alpha\beta}(\vec{x} - \vec{y})l_\vec{y}(e_\beta)\) (operator product expansion) and \(l_\vec{x}(a)l_\vec{y}(b) = l_\vec{y}(l_\vec{x}^{-}(\vec{y})a)b\) (duality). On the other hand one may introduce the multiplication operation \(m_\vec{x} : H \otimes H \mapsto H\) as \(m_\vec{x}(a, b) = l_\vec{x}(ab)\), then \(m_\vec{x}(\text{id} \otimes m_\vec{y}) = m_\vec{y}(m_\vec{x} \otimes \text{id})\). Analogously, one may introduce the comultiplication operation \(\Delta_\vec{x} : H \mapsto H \otimes H\) as \(\Delta_\vec{x}(e_\alpha) = t^\gamma_{\alpha\beta}(\vec{x})e_\beta \otimes e_\gamma\) in the QFT–operator coalgebra, then \((\text{id} \otimes \Delta_\vec{x})\Delta_\vec{x} = (\Delta_\vec{x} - \vec{y} \otimes \text{id})\Delta_\vec{x}\).

Below we shall need in a specific class of QFT–operator (co)algebras [20-22].

**Definition 2.** A QFT–operator algebra \((H, l^\alpha_{\alpha\beta}(u); u \in \mathbb{C})\) is called a QFT–operator algebra (operator algebra of the quantum projective field theory) iff \(1\) \(H\) is a direct sum of Verma modules \(V^a\) over Lie algebra \(\mathfrak{sl}(2, \mathbb{C})\) with highest vectors \(v^a\) of highest weights \(h^a\), (2) the operator fields \(l_\vec{x}(v^a)\) are \(\mathfrak{sl}(2, \mathbb{C})\)–primary (quasi–primary in terminology of [23]) of weight \(h^a: [L_k, l_\vec{x}(v^a)] = (-u)^k(u_+^a + (k + 1)h^a)l_\vec{x}(v^a)\) \(([L_i, L_j] = (i - j)L_{i+j})\), (3) the rule of descendant generation holds: \(l_\vec{x}(L_{-1}\Phi) = L_{-1}l_\vec{x}(\Phi)\). A QFT–operator algebra is called a derived QPFT–operator algebra if conditions \((1),(2)\) and the derived rule of descendant generation \((l_\vec{x}(L_{-1}\Phi) = [L_{-1}, l_\vec{x}(\Phi)] = \frac{\partial}{\partial u}l_\vec{x}(\Phi))\) hold.

As it was shown in [21] the categories of QPFT–operator algebras and derived QPFT–operator algebras are equivalent. The equivalency is realised in a rather simple manner described in [21]. Below we shall not distinguish both types of algebras considering them as two faces of one object.

**Definition 3.**

A. A highest vector \(T\) in the QPFT–operator algebra is called the conformal stress–energy tensor iff the operator field \(T(u) = l_u(T)\) has an expansion \(T(u) = \sum_{k \in \mathbb{Z}} L_ku^{-k-2}\) where \(L_k\) form the Virasoro algebra \(\mathcal{C}_{\text{vir}}:\ [L_i, L_j] = (i - j)L_{i+j} + \delta(i + j)\frac{L^2}{12} \cdot 1\).

B. A QPFT–operator algebra with fixed conformal stress–energy tensor is called a QCF–operator algebra (operator algebra of the quantum conformal field theory) in a wide sense.

C. A QCF–operator algebra in a wide sense is called a QCF–operator algebra in a narrow sense iff its space \(H\) is a sum of the Virasoro highest weight modules \(V^a\), which highest vectors \(v^a\) are \(\mathcal{C}_{\text{vir}}\)–primary (i.e. the identity \((2)\) of def.2 holds for all generators of \(\mathcal{C}_{\text{vir}}\)).

It should be mentioned that an arbitrary QCF–operator algebra in a narrow sense admits a strict representation by matrices with coefficient in the algebra \(\text{Vert}(\mathcal{C}_{\text{vir}}; c)\) of vertex operators for the Virasoro algebra [20].

It should be mentioned that an analog of Def.2 may be formulated for QPFT–coalgebras. Also one should mention that def.3C may be formulated without supposition that generators of the Virasoro algebra form the conformal stress–energy tensor (see f.e.[20]), so one may define QCF–operator coalgebra in such manner.
Let us now define several main objects of this paragraph.

**Definition 4.**

A. A crossing–algebra is a triple \((H, m, \Delta)\), where the mappings \(m : H \otimes H \mapsto H\) and \(\Delta : H \mapsto H \otimes H\) define structures of associative algebra and associative coalgebra on \(H\) such that \((\text{id} \otimes m)(\Delta \otimes \text{id}) = \Delta m = (m \otimes \text{id})(\text{id} \otimes \Delta)\), such identity should also hold after a change of \(m\) on \(m'(a, b) = m(b, a)\).

B. A QFT–operator crossing–algebra is the triple \((H, m_{\vec{x}}, \Delta_{\vec{x}})\), where the mappings \(m_{\vec{x}} : H \otimes H \mapsto H\) and \(\Delta_{\vec{x}} : H \mapsto H \otimes H\) define structures of QPFT–operator algebra and coalgebra on \(H\) such that \(\Delta_{\vec{x}}^{-g} m_{\vec{x}} = (\text{id} \otimes m_{\vec{x}})(\Delta_{\vec{x}} \otimes \text{id})\), \(\Delta_{\vec{x}} m_{\vec{x}}^{-g} = (\text{id} \otimes m_{\vec{x}})(\Delta_{\vec{x}} \otimes \text{id})\), such identity should also hold after a change of \(m\) on \(m'(a, b) = m(b, a)\).

C. A QPFT–operator crossing–algebra is a QFT–operator crossing–algebra built from QPFT–operator algebra and QPFT–operator coalgebra; a QCFT–operator crossing–algebra is a QPFT–operator crossing algebra built from QCFT–operator algebra and QCFT–operator coalgebra.

**Remark.** A crossing algebra \(H\) supplied by commutator operations is a Lie bialgebra.

Proposition 2.

A. The space \(H\) of any representation of the manifold \(\text{Polynion}^+(\text{Diff}_+^{+}(S^1))\) is supplied by a structure of a QCFT–operator algebra.

B. The space \(H\) of any representation of the manifold \(\text{Polynion}^-(\text{Diff}_+^{-}(S^1))\) is supplied by a structure of a QCFT–operator coalgebra.

C. The space \(H\) of any representation of the category \(\text{Train}(\text{Diff}_+^{+}(S^1))\), the train of the group \(\text{Diff}_+^{+}(S^1)\) of diffeomorphisms of a circle, is supplied by a structure of QCFT–operator crossing–algebra (in a narrow sense).

Namely, the vertices and co–vertices generate the structure of such algebra. This fact on a tree level should be attributed to folklore. The compatibility conditions for the multiplication and the co–multiplication in the QCFT–operator crossing–algebra is a natural sequence of the cutting–gluing conditions for representations of \(\text{Train}(\text{Diff}_+^{+}(S^1))\).

**Definition 5.** QCFT–operator algebra is called a vertex operator algebra iff operator fields \(l_u(\Phi)\) are mutually local i.e. \([l_u(\Phi), l_v(\Psi)] = 0\) if \(u \neq v\).

The more formal definition maybe found in [24,23,17] (there are claimed also that (1) all operator product expansions are meromorphic, (2) the weights of all elements are integral, (3) the spaces of a fixed weight are finite–dimensional and empty for a sufficiently small weights, but for our purposes these conditions are excessive). The locality condition maybe rewritten as \(m_u(m_v \otimes \text{id}) = m_v(m_u \otimes \text{id})\) if \(u \neq v\). So one may define a vertex operator coalgebra as QCFT–operator coalgebra such as \((\Delta_u \otimes \text{id})\Delta_v = \Delta_v(\Delta_u \otimes \text{id})\) and a vertex operator crossing–algebra as a QCFT–operator crossing–algebra, which as QCFT–operator (co)algebra is a vertex operator (co)algebra.

The second main proposition of this paragraph maybe formulated as follows.

Proposition 3.
A. The space $H$ of any permutation–invariant representation of the manifold $Polynion^+ (\text{Diff}_+ (S^1))$ is supplied by a structure of a vertex operator algebra.

B. The space $H$ of any permutation–invariant representation of the manifold $Polynion^- (\text{Diff}_+ (S^1))$ is supplied by a structure of a vertex operator coalgebra.

C. The space $H$ of any permutation–invariant representation of the category $\text{Train}(\text{Diff}_+ (S^1))$, the train of the group $\text{Diff}_+ (S^1)$ of diffeomorphisms of a circle, is supplied by a structure of vertex operator crossing–algebra.

It is partially (on a tree level) contained in [18] (in a slight different language based rather on the concept of the sewing than on one of the gluing).

Some remarks are necessary. First, the vertex operator algebras were recently described in the new–fashion operadic terms [25], it will be very interesting to give analogous interpretations for other objects of this paragraph. Second, it is rather interesting to investigate finite–dimensional ordinary (not operator) crossing–algebras — the simplest examples related to matrix algebras and group rings for finite groups seem to be very interesting. Third, the generalization of the representation theory of vertex operator algebras [26] on other objects of the paragraph maybe rather interesting. Fourth, one may extend the category $\text{Train}(\text{Diff}_+ (S^1))$ by supplying its morphisms by additional structures (f.e. by the fixed polarization in (co)homologies, see [13]), in this case the crossing–identities are broken and it is interesting to explicite their reduced form. Fifth, one may generalize Def.5 on arbitrary QFT–operator algebras to obtain the object called vertex algebra (see f.e. [18]), similarly one may define vertex coalgebras and vertex crossing–algebras. Sixth, it should be mentioned that in papers [13,20,21] the term ”vertex operator algebra” is used in the sense of ”QFT–operator algebra”. Seventh, one may define vertex superalgebras and vertex operator superalgebras if changes the claim of locality (commutativity of operators $l_x$) on its superanalag.

1.3. Renormalization of pointwise product in QCFT–operator algebras: local conformal field algebras (LCFA). The renormalized Krichever–Novikov functor. Let us now imbed our constructions into a framework of the noncommutative geometry following [27,4]. First, we shall formally describe the result of the renormalization of pointwise product in QCFT–operator algebras; second, we shall indicate a concrete renormalization procedure. It should be mentioned that all our constructions maybe considered as based on the initial idea of E.Witten [28] adapted to QCFT (for some hints see f.e. [29]).

In general, the approach is based on the following considerations [4]. The classical conformal field is a tensorial quantity on a complex curve; the set of such fields can be regarded as a quantity on a covering of this curve, and quantization means the transition to noncommutative coverings. Every curve has inner degrees of freedom (modules) and therefore has to be considered with all its deformations. It is natural to assume that a deformation of a noncommutative object is noncommutative, and this means that we have deal with quantum deformations — noncommutative spaces ”projected” onto the noncommutative bases. The separate fibers cannot be isolated from each other and so have to be considered collectively. Thus, we do not deal with individual fields but with families of conformal quantum fields. Below the curve is a unit complex disc.

Some constructions are necessary [4,7]. A model of the Verma modules over the Virasoro algebra $C_{\text{vir}}$ is a direct integral of the Verma modules $V_{h,c}$ with the fixed central charge $c$ over this algebra. The simplest realisation of the model is that
of I.N. Berbstein, I.M. Gelfand and S.I. Gelfand: the model space is the Fock space over the fundamental affine space for Vir, the universal covering of the fundamental affine space \( \mathcal{A}(\text{Vir}) \) for Vir. The fundamental affine space \( \mathcal{A}(\text{Vir}) \) is stratified over the flag manifold \( M(\text{Vir}) \) with fibre \( \mathbb{C}^n \), so the model space can be identified with the space of analytic functions of an infinite set of variables \( t, c_1, c_2, c_3, \ldots, c_n, \ldots \) (there are admitted arbitrary degrees of \( t \), the action of the Virasoro algebra in the model is defined by the following formulas \( L_p = c_p + \sum_{k \geq 1} c_{k+p} \frac{\partial}{\partial t} \) \((p > 0)\), \( L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial t} - t \frac{\partial}{\partial t} \), \( L_1 = \sum_{k \geq 1} c_k ((k + 2) \frac{\partial}{\partial t} + \frac{2}{\sigma_{k+1}} \partial c_k) - t \frac{\partial}{\partial t} \), \( L_2 = \sum_{k \geq 1} c_k ((k + 3) \frac{\partial}{\partial t} + (4 \frac{\partial}{\partial t} - (\frac{\partial}{\partial t})^2) \frac{\partial c_k}{\partial t} - b_k (\sigma_{k+1}, \ldots, \sigma_{k+2}) - \frac{1}{\sigma_{k+1}} \frac{\partial}{\partial t} (4 \frac{\partial}{\partial t} - (\frac{\partial}{\partial t})^2) + \frac{1}{\sigma_k} (\frac{\partial c_k}{\partial t} - (\frac{\partial}{\partial t})^2)) \), \( L_n = (\frac{\partial}{\partial t})^n \text{ad}^{-2} L_1 \cdot L_2 \) \((n > 2)\); the highest vector \( v_h \) of weight \( h \) has the form \( t^{-h} G(h, c; t c_1, t^2 c_2, \ldots) \) \((c > 0)\), where \( G(h, c; u_1, u_2, \ldots) \) is a certain system of differential equations \([7, 27]\), which allows to consider it as the model, such that the Verma modules over the Virasoro algebra, realized in the Fock space over the universal deformation of a complex disc \([7, 27]\). Namely, if \( f \) is the function from class \( S \) then the function \( f(1 - wt)^{-1} \) has the same property iff \( w^{-1} \) is not contained in the image of \( f \). The set of such points forms a domain \( \mathbb{C} \backslash \{ f(D_+) \}^{-1} \) in the complex plane \( \mathbb{C} \). The union of the pairs \((f, w)\), where \( f \in S \) and \( w^{-1} \mathcal{C} f(D+) \) is the space \( \mathcal{A}(\text{Vir}) \) of the universal deformation of a complex disc; the map \((f, w) \mapsto f \) is the projection onto the basis, which coincides with the flag space \( M(\text{Vir}) \) for the Virasoro–Bott group \( V \). The model space is the same as in the BGG–realisation; the action of the Virasoro algebra is given by the formulas \( L_p = c_p + \sum_{k \geq 1} c_{k+p} \frac{\partial}{\partial t} \) \((p > 0)\), \( L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial t} - t \frac{\partial}{\partial t} \), \( L_1 = \sum_{k \geq 1} c_k ((k + 2) \frac{\partial}{\partial t} + \frac{2}{\sigma_{k+1}} \partial c_k) - t \frac{\partial}{\partial t} \), \( L_2 = \sum_{k \geq 1} c_k ((k + 3) \frac{\partial}{\partial t} + (4 \frac{\partial}{\partial t} - (\frac{\partial}{\partial t})^2) \frac{\partial c_k}{\partial t} - b_k (\sigma_{k+1}, \ldots, \sigma_{k+2}) - \frac{1}{\sigma_{k+1}} \frac{\partial}{\partial t} (4 \frac{\partial}{\partial t} - (\frac{\partial}{\partial t})^2) + \frac{1}{\sigma_k} (\frac{\partial c_k}{\partial t} - (\frac{\partial}{\partial t})^2)) \), \( L_n = (\frac{\partial}{\partial t})^n \text{ad}^{-2} L_1 \cdot L_2 \) \((n > 2)\); the highest vector \( v_h \) of the weight \( h \) has the form \( t^{-h} G(h, c; t c_1, t^2 c_2, \ldots) \) \((c > 0)\), where \( G(h, c; u_1, u_2, \ldots) \) is the confluent hypergeometric function in the sense of Gelfand et al, corresponding to the double fibration of the universal deformation of the complex disc, which bases are the complex disc and the flag space of the Virasoro–Bott group.

It turns out that the model of the Verma modules over the Lie algebra \( \mathfrak{g}_{\text{vir}} \) is equipped with a richer structure than manifested one \([27, 4]\).

Let’s remind that if \( R \) is an associative algebra with identity over a field \( k \) and \( g \) is the Lie subalgebra of the algebra \( \text{Der}(R) \) of derivations of \( R \) then an associative algebra \( A \) with identity over \( k \) is called an \( L \)-algebra over the pair \((R, g)\) iff \( A \) is right \( R \)-module, the mapping \( r \mapsto r \cdot 1 \) from \( R \) to \( A \) is a ring morphism, and, therefore, \( A \) is \( R \)-bimodule, \( A \) is a \( g \)-module such that the \( g \)-module structure is compatible with the left \( R \)-module structure.

Let \( O = O(\mathbb{C}^n) \) and \( \mathfrak{g}_{\text{vir}} = \text{span}(L_p, p = -1, 0, 1, 2, \ldots) \). The model of the Verma modules over the Virasoro algebra, realized in the Fock space over the universal deformation of a complex disc, admits a natural right \( O \)-module structure via multiplication by functions of a single variable \( t \). As it was shown in \([27]\) the model of the Verma modules over the Lie algebra \( \mathfrak{g}_{\text{vir}} \), realized in the Fock space over the universal deformation of the complex disc, possesses exactly one structure of \( L \)-algebra over \((O, \mathfrak{g}_{\text{vir}})\), compatible with the right \( O \)-module structure in the model, such that \( L_p T(\Phi) = T(L_p \Phi) \) \((p = -1, -2, -3, -4, \ldots) \); \( T \) is the operator of the left multiplication in the \( L \)-algebra for every element \( \Phi \) of the model. The operator \( T \) is defined as \( T(c_p) = L_{-p}, T(t) = \sum_{k \geq 0} t^{1-k} P_k (\sigma_{k+1}, \ldots, \sigma_k) \cdot (-1)^k \), where the polynomials \( P_k \) have the form \( P_0 = 0, P_1(u_1) = u_1, P_2(u_1, u_2) = u_2 - u_1^2 \).
is its right inverse, i.e. $P$ these data determine a structure of an LCFA in $\mathcal{L}$ simultaneously with the Kontsevich–Segal approach. If the initial Q CFT–operator point imbedding $P, \phi$ and $f, \phi$ cation on $L\phi$, arbitrary element of $R\phi$ pair $(L\phi, g\phi)$, $L\phi$ is called $a$ local field algebra with the algebra of primary fields $B\phi$ and the algebra of singularities maybe defined if or an arbitrary LCFA. It allows to define the quantum universal deformations of noncommutative coverings of a complex disc $[4]$. The representation of elements of $L(\mathcal{C}\phi)$ by elements of the model of the Verma modules over $\mathcal{C}\phi$, realized in the Fock space over the universal deformation of a complex disc, is called a direct recording. The direct recording of an element $\Phi$ will be denoted by $[\Phi]_d$. Let’s also introduce a reverse recording $[27,4]$; we say that $t_1 c_1^{k_1} \ldots c_n^{k_n}$ is the reverse recording of the element $\Phi$ of the L-algebra $L(\mathcal{C}\phi)$ and we write $[\Phi]_r = t^\lambda c_1^{k_1} \ldots c_n^{k_n}$ if $\Phi = T(t)^\lambda c_1^{k_1} \ldots c_n^{k_n}$. As it was shown in [27] the action of the Virasoro algebra $\mathcal{C}\phi$ on the L-algebra $L(\mathcal{C}\phi)$ with the reverse recording of elements has the form $L_{-p} = c_p + \sum_{k \geq 1} c_k p k^p (-t)^{-1-p} \frac{\partial}{\partial t} (p > 0), L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial t} - t \frac{\partial}{\partial t}, L_1 = \sum_{k \geq 1} c_k ((k+2) \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial t^2} + t^2 \frac{\partial}{\partial t}), L_2 = \sum_{k \geq 1} c_k ((k+3) \frac{\partial}{\partial t^2} - 4 \frac{\partial}{\partial t} - \frac{\partial}{\partial t^2}) \frac{\partial}{\partial t} b_k (\frac{\partial}{\partial t}, \ldots, \frac{\partial}{\partial t}) + (\frac{\partial}{\partial t} - \frac{\partial}{\partial t^2}) - t^3 \frac{\partial}{\partial t}, L_n = (\frac{\partial}{\partial t})^n \frac{\partial}{\partial t} L_1 \cdot L_2 (n > 2).$

Several additional definitions are needed $[27,4]$. Namely, an L-algebra $B\phi$ over the pair $(\mathcal{R}, g\phi)$ is called an $L^\phi$-algebra over $(\mathcal{R}, g\phi)$ if $g\phi \subseteq \text{Der}(B\phi)$. Let $A$ be an L-algebra over $(\mathcal{R}, g\phi)$ and $B\phi$ be an L$^\phi$-algebra over the same pair. An L-algebra $C\phi$ over $(\mathcal{R}, g\phi)$ is called a local field algebra with the algebra of primary fields $B\phi$ and the structure algebra $A\phi$ if (1) $C\phi$ is a left $A\phi$-module, (2) $C\phi$ is an L-algebra over the pair $(B\phi, g\phi)$. If $\mathcal{R} = O\phi, g\phi = \mathcal{C}\phi_{\text{reg}}$ and $A = L(\mathcal{C}\phi)$ then $C\phi$ is called a local conformal field algebra (LCFA). The local conformal field algebras maybe regarded as structure rings of the quantum universal deformations of noncommutative coverings of a complex disc according to [4]. It should be marked that the analogs of direct and reverse recordings maybe defined ifor an arbitrary LCFA. It allows to define a canonical connection in an LCFA $C\phi$, namely, if $C\phi$ is identified with $\mathbb{C}[c_1, c_2, \ldots, c_n, \ldots] \otimes B\phi$ via the reverse recording then $\nabla = 1 \otimes l_{-1},$ where $l_{-1}$ is the action of the element $e_{-1}$ of the Virasoro algebra $\mathcal{C}\phi$ in $B\phi$.

On the other hand, LCFA$\phi$s can be regarded as the result of the renormalization of a pointwise product of fields in the corresponding field theories. Let’s describe the concrete renormalization procedure.

Namely, let $H\phi$ be an arbitrary QCFT–operator algebra. Let’s denote $\mathcal{O} \otimes H\phi$ by $\mathcal{A}(H\phi)$. The structure of the LCFA in $\mathcal{A}(H\phi)$ is defined as follows. Let $\phi$ be an arbitrary element of $H\phi$, let’s define $\varphi(\phi)$ as $\text{res}\{f(u)l_u(\phi) \frac{\partial}{\partial t}\} = \lim_{n \to 0} \{f(u)l_u(\phi) - \text{singularities}\}$ $(f \in \mathcal{O}); \varphi(\phi)$ maybe considered as an operator of the left multiplication on $(f, \varphi)$ in $\mathcal{A}(H\phi)$. The Virasoro algebra $\mathcal{C}\phi$ naturally acts in $\mathcal{A}(H\phi)$. All these data determine a structure of an LCFA in $\mathcal{A}(H\phi)$.

Below some additional structures on the LCFA$\phi$s will be needed. They are point projectors $P_i : \mathcal{A}(H\phi) \mapsto H\phi$ and point imbeddings $I_i : H\phi \mapsto \mathcal{A}(H\phi)$. The point projector $P_i$ correspond the element $f(t)\phi$ to the pair $(f, \phi)$. The point imbedding is its right inverse, i.e. $P_i I_i = \text{id}$. The additional condition, which determines the point imbedding $I_i$ is that $\nabla I_i = 0$, where $\nabla$ is the canonical connection in the LCFA.

Let’s very shortly describe a renormalized analog of the Krichever–Novikov construction of operator product expansions on Riemann surfaces $[30]$ combining it simultaneously with the Kontsevich–Segal approach. If the initial QCFT–operator algebra is meromorphic (i.e. all operators $l_u(\phi)$ are meromorphic) then one may
construct a sheaf $A_\Xi$ on a Riemann surface $R_\Xi$ (representing a morphism $\Xi$ from $\text{Train}(\text{Diff}_+(S^1))$) from the algebra $A(H)$. So one corresponds to a Riemann surface $R_\Xi$ its noncommutative covering, which structural ring consists of global sections of $A_\Xi$ over $R_\Xi$. This correspondence maybe considered as a renormalized Krichever–Novikov functor. This construction maybe generalized on QCFT–operator algebras with half–integer spectrum of primary fields by the supplying the Riemann surfaces by a spinor structure. One may also consider a general case with some minor negligence of the full mathematical rigor.

2. INFINITE DIMENSIONAL GEOMETRY AND TOPOLOGICAL CONFORMAL FIELD THEORY

One of the main features of a consistent string field theory, which extracts it from all 2D QCFT, is that it is a topological conformal field theory (TCFT), i.e. roughly speaking, it contains a "correctly" defined BRST–operator; these chapter is devoted to an adaptation of our infinite dimensional geometric picture to such theories — it should be mentioned that from the point of view of the noncommutative geometry topological conformal field theories are naturally appeared in the framework of quantum conformal field theories as their "noncommutative de Rham complexes" (cf. [10,11]). There exist several approaches to TCFT and they are, in general, similar to ones described in the first chapter. But now a reverse order of a presentation is more preferable: BRST–operators and formal cohomological machinery in a general framework of the quantum projective field theory will be described following [31,par.2.1] in par.2.1; these concepts are adapted to QCFT in par.2.2.; at least, the infinitesimal picture is "exponentiated" to the concept of a string background in par.2.3.

2.1. Differential and topological QFT–operator algebras and their cohomology; topological QPFT–operator algebras; stress–energy tensors, currents and their charges, BRST–currents and ghost fields in QPFT–operator algebras. Below we shall consider QFT–operator algebras with identity (i.e. an element 1 such that $l_\vec{x}(1) = \text{id}$). There is defined a vector–operator $\vec{L}$ as $\vec{L}\Phi = \left. \frac{d}{dx} l_\vec{x}(\Phi) \right|_{\vec{x}=0}$. It belongs to $\text{Der}(A)$, i.e. $[\vec{L}, l_\vec{x}(\Phi)] = l_\vec{x}(\vec{L}\Phi)$.

Definition 6.

A. The $\mathbb{Z}$–graded QFT–operator algebra $A$ is called a differential QFT–operator algebra if there is defined an element $Q$ (called a BRST–operator) of degree 1 in $\text{Der}(A)$ such that $Q^2 = 0$.

B. The differential QFT–operator algebra $A$ is called a topological QFT–operator algebra if there exist a vector–operator $\vec{B}$ such that $[Q, \vec{B}] = \vec{L}$ (here and below $[, , ]$ is a supercommutator).

C. The QFT–operator algebra is called topological vertex superalgebra if it is a topological QFT–operator algebra and vertex superalgebra simultaneously.

Topological vertex superalgebras were considered in [32]. Now let’s formulate the first main proposition of this paragraph.

Proposition 4.

A. The cohomology $H^*(A)$ of a differential QFT–operator algebra forms a QFT–operator algebra.

B. The cohomology $H^*(A)$ of a topological QFT–operator algebra forms an associative algebra.
C. The cohomology \(H^*(\mathfrak{A})\) of a topological vertex superalgebra forms an associative supercommutative algebra.

The demonstrations from [33] maybe straightforwardly adapted to our case.

Let’s now consider the case of QPFT–operator algebras. Differential QPFT–operator algebras maybe defined as QPFT–operator algebras, which are differential QFT–operator algebras. If there exist an operator \(B\) in the differential QPFT–operator algebra such that \(L_{-1} = [Q, B]\) (it should be mentioned that in the derived QPFT–operator algebras \(L = L_{-1}\) ) then the sl(2, \(\mathbb{C}\))–module generated from \(B\) by an adjoint action of \(Q\) admits an epimorphism onto the adjoint module. We shall claim that this epimorphism maybe split, i.e. one may find three operators \(B_{-i}\) \((i = -1, 0, 1; B_{-1} = B)\), which transforms under sl(2, \(\mathbb{C}\)) as elements of an adjoint module, i.e. \([L_i, B_j] = (i - j)B_{i+j}\). Differential QPFT–operator algebra with additional data \((Q, B_i)\) will be called topological QPFT–operator algebra. It should be mentioned that we do not know the commutation relations of \(B_i\); they should be determined in each concrete case, so one may say that \(B_i\) generate hidden symmetries, which will be called hidden ghost symmetries. It is a rather important problem to set them free in the concrete case (cf. [34]).

Let’s formulate the second main proposition of this paragraph.

**Proposition 5.** Operator \(B_0\) in the topological QPFT–operator algebra \(\mathfrak{A}\) defines an operator \(\hat{B}_0\) in its cohomology \(H^*(\mathfrak{A})\) such that \(\hat{B}^2_0 = 0\). If \(\mathfrak{A}\) is also a topological vertex superalgebra then \(H^*(\mathfrak{A})\) is a Batalin–Vilkovisky algebra.

The proof of this fact maybe obtained by a straightforward adaptation of arguments from [33]. One should see also [35] for a definition of Batalin–Vilkovisky algebras and [32] for their operadic formulations.

Some remarks are convenient. First, it is very interesting to apply the described cohomological machinery to the concrete QPFT — the \(q_R\)–conformal and super–\(q_R\)–conformal field theories [31,par.2.2.2.3]. Second, Prop.5 has its co– and crossing counterparts; nevertheless, the corresponding co– and crossing analogs of Batalin–Vilkovisky algebras were not investigated. Third, it is rather interesting, to unravel a Batalin–Vilkovisky–like structure in a general case of topological (non–local) QPFT–operator algebras.

Now let us call the meromorphic primary operator field of weight 1 in the QPFT–operator algebra by a current (see [31,par.2.1]). Each current has charge — the coefficient in the Laurent expansion by \(u^{-1}\); charges of currents form a Lie algebra [31,par.2.1]. Let us call the primary operator field of weight 2 by a stress–energy tensor iff its three components form the Lie algebra sl(2, \(\mathbb{C}\)); it should be mentioned that in an arbitrary QPFT–operator algebra the stress–energy tensor is not obligatory local (see f.e. [31,par.2.2] for \(q_R\)–conformal field theories as examples).

**Remark.**

A. In each QPFT–operator algebra the generators of the Lie algebra sl(2, \(\mathbb{C}\)) uniquely define a stress–energy tensor.

B. In each differential QPFT–operator algebra the BRST–operator uniquely defines a corresponding current (a BRST–current).

C. In each topological QPFT–operator algebra the generators of the hidden ghost symmetries uniquely define a primary operator field of weight 2 (ghost field).

These statements are sequences of the main structural theorem for QPFT–operator algebras identifying them with subalgebras of Mat\(_n\)(Vert(sl(2, \(\mathbb{C}\)))),
Vert(sl(2, $\mathbb{C}$)) is a special QPFT–operator algebra — the algebra of vertex operators for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ [31,22] and the explicit formulas for primary fields in Vert(sl(2, $\mathbb{C}$)) [36].

In general, neither stress–energy tensor nor a BRST–current or a ghost field do not correspond to any elements of a QPFT–operator algebra. Nevertheless, we can add them so below it will be supposed that stress-energy tensor, BRST–current and ghost field belong to the considered QPFT–operator algebra.

2.2. Topological conformal field theories and $N=2$ superconformal field theories. Let’s now adapted the general picture of par.2.1. to quantum conformal field theories. In this case the stress–energy tensor is local and it seems to be natural to claim the BRST–current and the ghost field to be local, too. We also suppose that there exist the ghost number counting current $J(z)$. All such claims allow to transform the topological conformal field theory into the $N=2$ superconformal one by the twist $L(z) \rightarrow L(z) + \frac{1}{2} \partial J(z)$ (here $L(z)$ is a stress–energy tensor). The ghost field and the BRST–current are transformed into the fermionic fields $G^\pm(z)$ of spin $\frac{3}{2}$ such that the commutation relations of the components hold:

\[
\begin{align*}
[L_n, L_m] &= (n - m) L_{n+m} + \frac{1}{4} d(n^3 - n) \delta(n + m) \\
[L_n, G^+_m] &= \frac{1}{2} (n - m) G^+_{n+m} \\
[L_n, J_m] &= - m J_{m+n} \\
[G^+_n, G^-_m] &= L_{n+m} + \frac{1}{2} (n - m) J_{n+m} + \frac{1}{2} d(n^2 - \frac{1}{4}) \delta(n + m) \\
[J_n, J_m] &= d n \delta(n + m) \\
[J_n, G^+_m] &= \pm G^+_n \\
[G^+_n, G^-_m] &= 0.
\end{align*}
\]

It should be also mentioned that topological conformal field theories (or equivalently $N = 2$ superconformal ones) possess additional algebraic structures related to so–called homotopy Lie algebras (see [37-39]).

Let’s consider the renormalization of pointwise product of fields in the QCFT–operator algebras of the topological conformal field theory. Let $\mathcal{A} = \mathcal{O}(\bar{\mathbb{C}}^*, H)$ be the corresponding LCFA.

**Proposition 6.** The LCFA $\mathcal{A} = \mathcal{O}(\bar{\mathbb{C}}^*, H)$ ($H$ is the space of the operator algebra of the topological conformal field theory) maybe enlarged to the complex $\mathcal{A}' = \Omega (\bar{\mathbb{C}}^*, H)$ with the differential $D = d + Q$ (an enlarged BRST–operator), where $d$ is a natural differential in $\Omega (\bar{\mathbb{C}}^*)$ and $Q$ is the BRST–operator in $H$.

**Proof.** One should introduce a new variable $dt$ and postulate the commutation relations between $dt$ and $\xi \in \mathcal{A}$ of the form: $[dt, \xi] = D([t, \xi]) - [t, Q \xi]$.

$\mathcal{A}'$ maybe considered as a noncommutative de Rham complex (cf.[10,11]). It gives a nice description of topological conformal field theories by the LCFA, which are simultaneously such complexes. The ghost field corresponds to “internal derivatives” in the complex.

2.3. String backgrounds. Let’s define string backgrounds according to [19,38]. Namely, the string background is the set of representations of a chain complex
C. Mor(\mathcal{A}, \mathcal{B}) in the spaces Hom(H^{\otimes A}, H^{\otimes B})$, where $H$ is a complex (a graded vector space with a differential $Q$: $Q^2 = 0$). These representations are morphisms of complexes and the gluings are transformed into the compositions of operators.

As it was shown in [19,38] a string background defines a structure of a topological QPFT–operator algebra in the space $H$.

In view of the equivalency of the topological conformal field theories and $N = 2$ superconformal ones a string background maybe transformed into the representation of the category Train(\mathcal{NS}) of $N = 2$ superconformal Riemann surfaces, the train related to the Neveu–Schwarz Lie superalgebra (it was marked in [38]).
3. Infinite dimensional noncommutative geometry of a self–interacting string field

In this chapter we shall work with the following objects [1]:

\( \mathcal{Q} \) (or the dual \( \mathcal{Q}^* \)) — the space of external degrees of a freedom of a string. The coordinates \( x_n^{\mu} \) on \( \mathcal{Q} \) are the Taylor coefficients of functions \( x^\mu(z) \), which determines a world-sheet of a string in a complexified target space.

\( M(\text{Vir}) \) — the space of internal degrees of a freedom of a string, which is identified via Kirillov construction with the class \( S \) of the univalent functions \( f(z) \); the natural coordinates on \( S \) are coefficients \( c_k \) of the Taylor expansion of an univalent function \( f(z) \): 
\[
    f(z) = z + c_1 z^2 + c_2 z^3 + c_3 z^4 + \ldots + c_n z^{n+1} + \ldots.
\]

\( \mathcal{C} \) — the universal deformation of a complex disc with \( M(\text{Vir}) \) as a base and with fibers isomorphic to \( D_z \); the coordinates on \( \mathcal{C} \) are \( z, c_1, c_2, \ldots, c_n, \ldots \), where \( c_k \) are coordinates on the base and \( z \) is a coordinate in the fibers.

\( M(\text{Vir}) \times \mathcal{Q}^* \) — the space of both external and internal degrees of freedom of a string, the same as the bundle over \( M(\text{Vir}) \) associated with \( p: \mathcal{C} \rightarrow M(\text{Vir}) \), which fibers are \( \text{Map}(\mathcal{C}/M(\text{Vir}); \mathbb{C}^n)^* \) — linear spaces dual to ones of mappings of fibers of \( p: \mathcal{C} \rightarrow M(\text{Vir}) \) into \( \mathbb{C}^n \).

\( \Omega^\text{SI}_{\text{BP}}(E_{h,c}) \) — the space of the Banks–Peskin differential forms, they are some geometric objects on \( M(\text{Vir}) \times \mathcal{Q}^* \).

\( \Omega^\text{SI}_{\text{BP}}(E_{h,c})^* \) — the space of the Siegel string fields with the (pseudo)hermitean metric \( (\cdot|\cdot) \).

\( \mathcal{Q}^* \) — the Kato–Ogawa BRST–operator in the space of Siegel string fields, the conjugate to \( Q \); it defines a new (pseudo)hermitean metric \( (\cdot|\cdot) = (\cdot|\mathcal{Q}^* \cdot) \) in \( \Omega^\text{SI}_{\text{BP}}(E_{h,c})^* \).

\( \text{FG}_{h,c}(M(\text{Vir})) \) — the Fock–plus–ghost bundle over \( M(\text{Vir}) \), its sections are just the Banks–Peskin differential forms.

\( \nabla^\text{GM} \) — the Gauss–Manin string connection in \( \text{FG}_{h,c}(M(\text{Vir})) \); the covariantly constant sections of which are the Bowick–Rajeev vacua.

\( D_{\nabla^\text{GM}} \) — the covariant differential with respect to the Gauss–Manin string connection.

\( \Omega^\text{SI}_{\text{BP}}(E_{h,c})^*_{\text{GI}} \) — the space of the gauge–invariant Siegel string fields, it is just the dual to the space of the Bowick–Rajeev vacua; this space possess a (pseudo)hermitean metric \( (\cdot|\cdot)_0 \), which is a restriction of the metric \( (\cdot|\cdot) \).

\( \mathcal{Q}^*_{0} \) — the Kato–Ogawa BRST–operator in the space of gauge–invariant Siegel string fields \( \mathcal{Q}^* = D_{\nabla^\text{GM}} + \mathcal{Q}^*_{0} \); the (pseudo)hermitean metric \( (\cdot|\cdot) = (\cdot|\mathcal{Q}^*_{0} \cdot) \) is just the restriction of \( (\cdot|\cdot) \) on \( \Omega^\text{SI}_{\text{BP}}(E_{h,c})^*_{\text{GI}} \).

It should be mentioned that the spaces of the Banks–Peskin differential forms, the Siegel string fields, the gauge–invariant Siegel string fields, the Bowick–Rajeev vacua are indeed superspaces and various objects on them are (odd or even) super-objects, but the prefix ‘super’ will be omitted everywhere.

The action of the Virasoro algebra \( \mathcal{C}_\text{vir} \) in the space of the Banks–Peskin differential forms in the flat background have the form [1]

\[
    L_p = \frac{\partial}{\partial \xi_p} + \sum_{k \geq 1} (k + 1)c_k \frac{\partial}{\partial c_{k+p}} - \sum_k (p + 2k)\xi_k \frac{\partial}{\partial \xi_k} \quad (p > 0),
\]
\[ L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial c_k} + \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} - 2 \sum_{k} k \xi_k \frac{\partial}{\partial \xi_k} + h, \]

\[ L_{-1} = \sum_{k \geq 1} ((k + 2)c_{k+1} - 2c_1c_k) \frac{\partial}{\partial c_k} + 2c_1 \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} + \]

\[ \sum_{k \geq 1} (k + 1)x_k^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k} (1 - 2k) \xi_{k-1} \frac{\partial}{\partial \xi_k} + 2hc_1, \]

\[ L_{-2} = \sum_{k \geq 1} ((k + 3)c_{k+2} - (4c_2 - c_1^2)c_k - b_k(c_1, \ldots, c_{k+2})) \frac{\partial}{\partial c_k} + \]

\[ (4c_2 - c_1^2) \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} + 3c_1 \sum_{k \geq 1} x_{k+1}^\mu \frac{\partial}{\partial x_k^\mu} + \]

\[ \sum_{k \geq 1} (k + 2)x_{k+2}^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k} 2(1 - k) \xi_{k-2} \frac{\partial}{\partial \xi_k} + \]

\[ \frac{x_k^2}{2} + h(4c_2 - c_1^2) + \frac{c}{2}(c_2 - c_1^2), \]

\[ L_{-n} = \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 2), \]

(here \(e_\mu = 0\).

The action of the Virasoro algebra \(\mathcal{C}_{\text{vir}}\) in the space of the Siegel string fields in the flat background have the form [1]

\[ L_{-p} = c_p + \sum_{k \geq 1} (k + 1)c_{k+p} \frac{\partial}{\partial c_k} - \sum_{k} (k + p) \xi_{k-p} \frac{\partial}{\partial \xi_k} \quad (p > 0), \]

\[ L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial c_k} + \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} - \sum_{k} \xi_k \frac{\partial}{\partial \xi_k} + h, \]

\[ L_1 = \sum_{k \geq 1} c_k ((k + 2) \frac{\partial}{\partial c_{k+1}} - 2 \frac{\partial}{\partial c_1} \frac{\partial}{\partial c_k}) + 2 \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} + \]

\[ \sum_{k \geq 1} (k + 1)x_k^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k} (1 - k) \xi_{k+1} \frac{\partial}{\partial \xi_k} + 2hc_1, \]

\[ L_2 = \sum_{k \geq 1} c_k ((k + 3) \frac{\partial}{\partial c_{k+2}} - (4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) \frac{\partial}{\partial c_k} - b_k(\frac{\partial}{\partial c_1}, \ldots, \frac{\partial}{\partial c_{k+2}})) + \]

\[ \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2 + 3 \sum_{k \geq 1} (k + 1)x_k^\mu \frac{\partial}{\partial x_k^\mu} + \]

\[ \sum_{k \geq 1} (k + 2)x_{k+2}^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k} (2 - k) \xi_{k+2} \frac{\partial}{\partial \xi_k} + \]

\[ \frac{1}{2} \frac{\partial^2}{\partial x_1^2} + h(4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) + \frac{c}{2}(\frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2), \]

\[ L_n = \frac{(-1)^n}{(n-2)!} \text{ad}^{n-2} L_1 \cdot L_2 \quad (n > 2). \]

Here \(\xi_k^*\) and \(\xi_k\) are ghosts and antighosts, respectively.

The formulas for a curved background maybe received from par.2.2. of [1].
3.1. Remarks on infinite dimensional geometry of a free string field: Bowick–Rajeev string instantons on curved backgrounds, Kähler structures and Poisson brackets on instanton spaces, background (in)dependence. It should be mentioned that the gauge–invariant Siegel string fields (as well as the Bowick–Rajeev vacua) exist if and only if the Gauss–Manin connection $\nabla^{GM}$ is flat (more precisely, flat on a sufficiently large subbundle — see [1]). The conditions of a flatness of the Gauss–Manin connection pick out the critical dimension and produce a certain equation on the background metric field (string Einstein equations). So the formalism based on the Bowick–Rajeev vacua is essentially set on the moduli space of solutions of the string Einstein equations. Indeed, it is necessary to consider a string field theory on arbitrary metric backgrounds for the applications to the differential geometry (of course, taking an importance of string Einstein equations in account). Similar situation is appeared in ordinary differential geometry, where all constructions have a general meaning, but Einstein manifolds also play a considerable role.

A generalisation maybe performed in a standard ”instantonic” way. Namely, one can consider minima of a certain functional (“action”) instead of Bowick–Rajeev vacua. Namely, such string instanton action maybe chosen of the form $S = S \left( \frac{D_{\mathcal{C}}^{GM} \Phi, D_{\mathcal{C}}^{GM} \Phi}{(\Phi, \Phi)} \right)$, $S \geq 0$, $S(0) = 0$ (it is clamed the metric $(\cdot|\cdot)$ to be hermitean), where $\Phi$ is a Siegel string field. A minimum of the action $S$ will be called the Bowick–Rajeev instanton. We shall suppose that such minima exist.

Let us mentioned that the space $\mathcal{I}_{BR} = \mathcal{I}_{BR}(g_{\mu\nu}; h, c)$ ($g_{\mu\nu}$ is a background metric and $h, c$ are parameters of $\Omega_{SP}(E^{*}_{h,c})^{*}$ of the Bowick–Rajeev instantons is a curved CR–space in contrast to a case of the Bowick–Rajeev vacua. Nevertheless, one may consider a restriction of the hermitean metric $(\cdot|\cdot)$ on $\mathcal{I}_{BR}$ supplying it by a CR-(pseudo)-Kähler form $w_{g_{\mu\nu}; h, c}$.

Some remarks are necessary. First, it should be mentioned that one needs rather in the Poisson brackets than in the CR-(pseudo)-Kähler structure for a quantization. If the CR-(pseudo)-Kähler metric $w_{g_{\mu\nu}; h, c}$ is nondegenerate (it claims, in particular, that the tangent space to $\mathcal{I}_{BR}$ does not contain a solution of the equation $Q^{*} \Phi = 0$) then a transition to Poisson brackets is straightforward but it is not so in general. Second, the Bowick–Rajeev instantons are certain Siegel string fields, whereas in a flat background the correct Poisson brackets are defined in functionals on the space of the Banks–Peskin differential forms. With respect to the 2nd remark one should mentioned that the metric $(\cdot, \cdot)$ is nondegenerate so the spaces of Siegel string fields and the Banks–Peskin differential forms maybe identified. With respect to the 1st remark one may use the following rather standard construction. Namely, let us consider the isotropic foliation $\mathcal{F}_{tot}$ on $\mathcal{I}_{BR}$ and a relative cotangent bundle, which will be called the $\Pi$–instanton space (to emphasize an analogy with $\Pi$–spaces considered in [1]) and denoted by $\Pi_{BR}$. The additional coordinates on $\Pi_{BR}$ maybe considered as the extraghosts. Then the $\Pi$–instanton space is symplectic and therefore the Poisson brackets in functionals on it are defined.

There are two possibilities to get rid of extraghosts. First, one may exclude them considering a flat connection in the relative cotangent bundle and performing a hamiltonian reduction [40]. But it seems that hamiltonian reduction maybe performed only in degenerate cases in view of nontrivial topology of $\mathcal{I}_{BR}$. Second, one may suppose that free string field theory on a curved background have no a quasiclassical counterpart and is analogous to minimal models in quantum confor-
mal field theory. In this case one is willing to perform something analogous to the Alekseev–Shatashvili construction [41] (see also comments in par.1.4. of [1]) and here an analogy between \( \Pi \)-instanton spaces and \( \Pi \)-spaces will be crucial. This is a more reasonable but only hypothetical way and, therefore, we have to work with extraghosts in free string field theory.

The topological properties of \( \Pi_{\text{BR}} \) (the topology of the \( \Pi \)-instanton space \( \Pi_{\text{BR}} \) is determined, first, by a topology of the Bowick–Rajeev instanton space \( \textbf{I}_{\text{BR}} \) itself, second, by a topology of the isotropic foliation \( F_{\text{isotr}} \) on it) are essential characteristics of a metric background. If they differ it implies the non–equivalency of free string field theories for considered backgrounds. It should be marked that free string field theories are determined by pair \( (\Pi_{\text{BR}}, \{\cdot,\cdot\}) \), where \( \{\cdot,\cdot\} \) are Poisson brackets in \( \mathcal{O}(\Pi_{\text{BR}}) \). If such pairs are the same the theories are equivalent. If the considered backgrounds are solutions of string Einsteins equations then topological equivalence of vacua spaces implies the equivalency of theories. So the theories from one connected component of the moduli space of solutions of the string Einsteins equations are equivalent, i.e. an infinitesimal background independence holds. Both facts (an infinitesimal background independence and that topological equivalence implies equivalence of theories) do not hold, in general, for arbitrary backgrounds. Indeed, nondegenerate Poisson brackets \( \{\cdot,\cdot\} \) define a CR–Kähler form \( \Pi_{\text{BR}} \), which is interesting as an element of \( H^2(\Pi_{\text{BR}}) \). If cohomology \( H^2(\Pi_{\text{BR}}) \) of the \( \Pi \)-instanton space \( \Pi_{\text{BR}} \) is not trivial then the cohomological classes are invariants of free string field theories. Therefore, in general, background dependence is described by the CR–differential topology of the \( \Pi \)-instanton space \( \Pi_{\text{BR}} \) and the cohomological class from \( H^2(\Pi_{\text{BR}}) \).

It should be marked, however, that a special choice of the instanton action \( S \) may essentially simplify a picture. For example, if one is able to choose \( S(t) = t \) then \( \textbf{I}_{\text{BR}} \) is a linear subspace of \( \Omega^\text{SI}_{\text{BP}}(E_h,c) \) (or \( \Omega^\text{SI}_{\text{BP}}(E^*_h,c)^* \)), therefore \( \Pi_{\text{BR}} \) as well as \( \textbf{I}_{\text{BR}} \) are contractible, moreover \( \Pi_{\text{BR}} \) is a flat CR–space, which carries a constant Poisson brackets. Hence, two theories are equivalent iff the corresponding pairs \( (\Pi_{\text{BR}}, F_{\text{isotr}}) \) are isomorphic as pairs of topological vector spaces. If one prefers to deal with them as Hilbert spaces then the full metric background independence of free string field theory will be received.

3.2. String field algebras, Poisson brackets on duals to them and their quantization (flat background). Above we have deal with a chiral case but we need in a real one below. They differs by minor ”second order” details, so we will not repeat all constructions again. In this paragraph we shall consider a flat background.

First of all, it should be mentioned that the space of the Siegel string fields do not admit a structure of local conformal field algebra because it contains Verma modules over the Virasoro algebra \( C_{\text{vir}} \) only of a discrete spectrum of weights. To get rid of this difficulty let’s enlarge the spaces of the Banks–Peskin differential forms and the Siegel string fields. The space of enlarged Banks–Peskin differential forms will be defined as the space \( \Omega_{\text{BP};enl} = \Omega(\tilde{\mathcal{C}}^*, \Omega^\text{SI}_{\text{BP}}(E_h,c)) \). Let \( t \) be a coordinate on \( \tilde{\mathcal{C}}^* \) then the action of the Virasoro algebra generators in \( \Omega_{\text{BP};enl} \) \((i = 1, 2)\) will be defined by the formulas

\[
L_p = \frac{\partial}{\partial c_p} + \sum_{k \geq 1} (k+1)c_k \frac{\partial}{\partial c_{k+p}} - \sum_k (p+2k)\xi_k \frac{\partial}{\partial \xi_k} \quad (p > 0),
\]
The space of enlarged Siegel string fields will be defined in analogous way \( \Omega_{\text{sf,enl}} \) will be defined by the formulas (\( \Omega_{\text{sf,enl}}(E^*_{h,c}) \)). The action of the Virasoro algebra generators in \( \Omega_{\text{sf,enl}} \) will be defined by the formulas

\[
L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial c_k} + \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} - 2 \sum_{k} k \xi_k \frac{\partial}{\partial \xi_k} - t \frac{\partial}{\partial t} - i + h,
\]

\[
L_{-1} = \sum_{k \geq 1} ((k + 2)c_{k+1} - 2c_1c_k) \frac{\partial}{\partial c_k} + 2c_1 \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k \geq 1} (k + 1)x_{k+1}^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k} (1 - 2k) \xi_{k-1} \frac{\partial}{\partial \xi_k} + t^2 \frac{\partial}{\partial t} + 2it + 2hc_1,
\]

\[
L_{-2} = \sum_{k \geq 1} ((k + 3)c_{k+2} - (4c_2 - c_1^2)c_k - b_k(\xi_1, \ldots, c_{k+2})) \frac{\partial}{\partial c_k} + (4c_2 - c_1^2) \sum_{k \geq 1} k x_k^\mu \frac{\partial}{\partial x_k^\mu} + 3c_1 \sum_{k \geq 1} x_{k+1}^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k \geq 1} (k + 2)x_{k+2}^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k} 2(1 - k) \xi_{k-2} \frac{\partial}{\partial \xi_k} - 3t^2 \frac{\partial}{\partial t} + \frac{x_1^2}{2} + 3it^2 + h(4c_2 - c_1^2) + \frac{c}{2}(c_2 - c_1^2),
\]

\[
L_{-n} = \frac{1}{(n - 2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 2).
\]

One may also construct the enlarged BRST–operators \( Q_{\text{enl}} \) and \( Q^*_{\text{enl}} \) as exterior differentials in \( \Omega^*_{\text{BP,enl}} \) and \( \Omega^*_{\text{sf,enl}} \) from old BRST–operators \( Q \) and \( Q^* \).
Proposition 7. The space $\Omega_{sf;enl}$ admits a structure of a BRST–invariant LCFA, covariant with respect to the Gauss–Manin connection $\nabla_{GM}$.

Lemma. The space $\Omega_{sf;enl}^0$ admits a structure of a LCFA, covariant with respect to the Gauss–Manin connection.

Proof of lemma. One should apply the procedure of the renormalization of a pointwise product of par.1.3. to the standard vertex operator algebra (of the first quantized string on a flat background) in the space $\Omega_{SI,BP}^*(E^*_h,c)^*_{GI}$.

To obtain a structure of a BRST–invariant LCFA, covariant with respect to the Gauss–Manin connection $\nabla_{GM}$ in the whole space $\Omega_{sf;enl}$ one should use a construction of the Prop.6.

Therefore, the space $\Omega_{sf;enl}$ maybe considered as a noncommutative de Rham complex with respect to the enlarged BRST–operators. Such complex will be called the enlarged string field algebra.

Now one may apply the construction of par.1.3. to receive a structure of the non–associative string field algebra in the space of Siegel string fields (cf. [42]).

It should be mentioned that elements of $\Omega_{sf;enl}^0$ form a Lie algebra under the imaginary part of $(\cdot|\cdot)$. If one consider the space of connections on $\hat{C}^*$ valued in $\Omega_{SI,BP}^*(E^*_h,c)^*$, i.e. gauge fields on $\hat{C}^*$ valued in Siegel string fields, then elements of $\Omega_{sf;enl}^0$ will realize their infinitesimal gauge–transformations. These gauge–transformations are closed (as in Witten string field theory [28]) so the corresponding Lie algebra will be called Witten string Lie algebra and will be denoted by $\mathfrak{wit}$ (the circled arrow $\circ$ is a code for ”string”). The space of $\nabla_{GM}$–covariant elements of $\mathfrak{wit}$ will be denoted by $\mathfrak{wit}_{\nabla_{GM}}$ and also called by the Witten string Lie algebra.

There are defined canonical (Lie–Berezin) Poisson brackets in the space $\mathfrak{wit}^*$ (or $\mathfrak{wit}_{\nabla_{GM}}^*$) dual to the Witten string algebra $\mathfrak{wit}$ (or $\mathfrak{wit}_{\nabla_{GM}}$, which may be quantized as such.

Let’s mention that the point projectors and the point imbeddings allow to perform a hamiltonian reduction of these Poisson brackets and to receive the non-polynomial Poisson brackets in the space of functionals on the Banks–Peskin differential forms (or Bowick–Rajeev vacua). These Poisson brackets generate a Lie quasi(pseudo)algebra (quasialgebra in terminology of [43] and pseudoalgebra in terminology of [40]) of non–polynomial infinitesimal gauge–transformations. The non–polynomial transformations in the space of Bowick–Rajeev vacua were considered in [44;38]. The Lie quasi(pseudo)algebra generated by them will be called the Zwiebach string Lie quasi(pseudo)algebra and will be denoted by $\mathfrak{zwe}_{\nabla_{GM}}$, whereas the corresponding Lie quasi(pseudo)algebra in the space of Banks–Peskin differential forms will be denoted by $\mathfrak{zwe}$. The nonpolynomial Poisson brackets are defined in functionals on the dual $\mathfrak{zwe}^*$ (or $\mathfrak{zwe}_{\nabla_{GM}}^*$) to the Lie quasi(pseudo)algebra $\mathfrak{zwe}$ (or $\mathfrak{zwe}_{\nabla_{GM}}$). It should be mentioned that the central charge of the Witten string Lie algebra is transformed by hamiltonian reduction into the inverse $\gamma^{-1}$ of the coupling constant $\gamma$; this fact enlight a non–perturbative nature of a coupling constant $\gamma$. Also it should be marked that the Zwiebach string Lie quasi(pseudo)algebra maybe received from the non–associative string field algebra.
as its "commutator" algebra. More precisely, the higher operations of Sabinin–Mikheev multialgebra \cite{45} constructed by the Zwiebach string Lie algebra are just higher commutators in the non–associative string field algebra.

Thus, the nonpolynomial string field theory maybe received from Witten-type string field theory in enlarged space by a hamiltonian reduction.

Some remarks on quantization are necessary. There are two possibilities to quantize the nonpolynomial Poisson brackets on $\mathcal{C}$. First, one may quantized them as nonlinear brackets asymptotically \cite{40}. Second, one may perform a quantum reduction of quantized brackets on $\mathcal{C}$ (i.e. to receive the corresponding algebra of observables as a certain quantum reduction of $\mathcal{U}(\mathcal{C})$).

It is necessary to mark that the objects, which we were constructed describe the interacting string field theory on a tree level (or "classical interacting string field theory" \cite{37}). To describe this theory completely in the nonperturbative mode one may use the following result.

**Proposition 8.** The Witten string Lie algebra $\mathcal{C}$ admits a structure of a Lie bialgebra.

Indeed, the Siegel string field possess a structure of a string background \cite{38} so the enlarged string field algebra is a crossing–algebra (it holds also for the subalgebra of $\nabla$–covariant elements in it).

Therefore, on the quantum level one have to consider a quantum universal enveloping algebra $\mathcal{U}(\mathcal{C})$ (or $\mathcal{U}(\mathcal{C})$) (cf.\cite{46}) and its reductions. To construct explicitly all these objects is a problem.

### 3.3. Remarks on classical interacting string field theory in curved backgrounds; background (in)dependence.

The main idea how to formulate the interacting string field theory in curved background is to use the fact that the complex $\Omega^\ast_	ext{sf} E_{\hbar,c}$ does not depend on a background, so one may consider the enlarged string field algebra as universal for all backgrounds. That means that string interactions (vertices and covertices) after the accounting of the internal degrees of freedom of a string do not depend on metric background. Therefore, the Witten string Lie algebra $\mathcal{C}$ as well as the Zwiebach string Lie quasi(pseudo)algebra $\mathcal{C}$ are also universal so all information on the metric background is encoded in the Gauss–Manin connection $\nabla$. For a general background this connection is not flat in any sense so the algebras $\mathcal{C}$ and $\mathcal{C}$ do not exist. And if they exist (f.e. for solutions of string Einsteinequtions) they are not, in general, identical.

Let’s combine the approaches of par.3.1. and par.3.2. for an arbitrary metric background.

Let’ consider an arbitrary (pseudo)–Kähler symplectic leaf $\mathcal{O}$ (parametrizes such leaves) of the non–polynomial brackets on $\Omega^\ast_{\text{BR}} E_{\hbar,c}$ (so $\mathcal{O}$ is naively a "coadjoint orbit" of the Zwiebach string Lie algebra $\mathcal{C}$ and $\mathcal{P}$ is the space of such orbits). Also let us consider the intersection $\mathcal{I} \cap \mathcal{O}$ of this leaf with the Bowick–Rajeev instanton space $\mathcal{I}$; the (pseudo)–Kähler structure on $\mathcal{O}$ defines a CR–(pseudo)–Kähler metric on $\mathcal{I}$. One may now use a construction of par.3.1. and build the $\Pi$–instanton space $\Pi \mathcal{I}$ and Poisson brackets in functionals on it. So the classical interacting string field theory is determined by the following data:

- the $\Pi$–instanton space $\Pi \mathcal{I}$;
– the Poisson brackets (or (pseudo)-Kähler metric) on it.

The Π-instanton space \( \Pi_p \) is constructed from
– the Bowick–Rajeev instanton space \( I_{\text{BR}}(p) \);
– the isotropic foliation \( F_{\text{iso}} \) on it.

The Bowick–Rajeev instanton space \( I_{\text{BR}}(p) \) is an intersection of
– the fixed (pseudo)-Kähler symplectic leaf \( \mathcal{O}_p \) of the background independent Poisson brackets on \( \Omega_{\text{SP}}(E_{h,c}) \);
– the Bowick–Rajeev instanton space \( I_{\text{BR}} \), which does not depend on the coupling constant \( \gamma \).

The isotropic foliation \( F_{\text{iso}} \) depends as on background as on \( \gamma \).

So the independent parameters of classical interacting string field theory are (1) the background metric \( g_{\mu\nu} \), (2) the parameters of string fields \( h \) and \( c = 26 \), (3) the coupling constant \( \gamma \), (4) the value of coordinate \( p \) on the space \( P \) of (pseudo)-Kähler symplectic leaves of the non-polynomial Poisson brackets on the space of Banks–Peskin differential forms.

It was proved in [46] that the classical interacting string field theory is infinitesimally independent on the metric background on the space of solutions of string Einshstein equations. That means that in this case infinitesimal variations of the Gauss–Manin connection does not influence on the resulting string field theory.

### 3.4. Open problems of the nonperturbative approach on a quantum level.

There are three main open problem to construct the nonperturbative quantum interacting string field theory:

1. to quantize the space of parameters; i.e. to extract for a fixed \( h \) and \( \gamma \) the set \( P_{\text{integral}} = \{ p \in P : \text{the CR-\text{\texttt{(pseudo)-\texttt{\texttt{Kähler}}}} metric on } \Pi_{\text{BR}}(p) \text{ belongs to } H^2(\Pi_{\text{BR}}(p); \mathbb{Z}) \}; \)

2. to find the selection rules, which extract triples \( (h, \gamma, p) \) for which the theories are unitarizable, i.e. a hermitean metric in a Fock space over \( \Pi_{\text{BR}}(p) \) is non-negatively definite;

3. to find the quantum Clebsch–Gordan coefficients, which define tensor products of these Fock spaces for different \( p \) and maybe \( h \) (\( \gamma \) are fixed).

Also an important problem is a quantum metric background (in)dependence in the case of arbitrary curved backgrounds. Such infinitesimal independence for the metrics obeying string Einshstein equations was proved in [48] in context of the Lagrangian formulation of the perturbative quantum interacting string field theory.

Another problem is to compare the results of perturbative and nonperturbative approaches and to estimate the nonperturbative effects.

### Conclusions

So an infinite dimensional geometric interpretation of a self-interacting string field theory was given and relations between various approaches to the second quantization of an interacting string were described in terms of the geometric quantization. A certain algorithm to construct a quantum nonperturbative interacting string field theory in the quantum group formalism was proposed; the main problems were formulated in geometric terms.

Their more mathematically rigorous and detailed discussion will be contained in the forthcoming papers. Applications of the developped machinery to problems of differential geometry will be also discussed there.
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