Some constructions of complete decomposition of dihedral group

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Abstract. Let $G$ be a finite non-abelian group. For integer $k \geq 2$, we let $A_1, \ldots, A_k$ be non-empty subsets of $G$. If $A_1, \ldots, A_k$ are pairwise disjoint and if the subset product $A_1 \cdots A_k = \{a_1 \cdots a_k | a_i \in A_i, j = 1, \ldots, k\}$ coincides with $G$, where the $A_i$ are all distinct and $\{A_1, \ldots, A_k\} = \{\{A_1, \ldots, A_k\}\}$, then $(A_1, \ldots, A_k)$ is called a $(k,|A_1|,\ldots,|A_k|)$-complete decomposition of $G$. For integer $n \geq 3$, let $D_{2n}$ be the dihedral group of order $n$. Let $A, B$ be the subset of $D_{2n}$. In this paper, we show some constructions, namely $(2,|A|,|B|)$-complete decomposition of $D_{2n}$ where $|B| \in \{2, 4, \ldots, 2 \left\lfloor \frac{n}{2} \right\rfloor, n-1, n\}$ and $|A| = 2n - |B|$.

1. Introduction

In recent years, research on finding covering for finite group received a lot of attention. In [1], the connection between maximal sets of pairwise non-commuting elements and coverings of the finite group by proper subgroups was established. Also, $p$-groups without $p+2$ pairwise non-commuting elements were classified. In year 2012, Berchenko [2] determined the minimum cardinality of the product set $AB$ where $A$ and $B$ are subsets of $G$ such that $|A| = r$ and $|B| = s$ for some integers $r$ and $s$. The groups such as the non-abelian group of order $pq$ where $p$ and $q$ are prime and finite nilpotent groups are considered in [2]. A spectrum of $G$ is the set of elements orders of a finite group $G$ and is denoted as $w(G)$. A group $G$ is recognizable by spectrum among covers if $w(G) \neq w(H)$ for any proper cover of $H$. In [3], Grechkoseeva studied the property of being recognizable among covers for finite non-abelian simple groups. Besides, he showed that the conjecture where all finite non-abelian simple groups share this property except for some Lie-type groups of low lie rank is true. The large Davenport constant of groups written multiplicatively with a cyclic, index 2 subgroup and general upper bound was studied in [4, 5]. The factorization of groups discussed in [6] which related to abelian group written additively is somewhat analogous to complete decomposition we introduce in this paper, the main difference is that the groups we discuss are nonabelian multiplicative group. The most recently related results were shown in [7]. Chin and Chen [7] determine the existence of complete decomposition for cyclic group $\mathbb{Z}_n$ and then extend to finite abelian group. We modify the definition of complete decomposition defined in [7] to suit our case that the group $G$ is written multiplicatively.

Let $n \geq 3$ be an integer. For the rest of this paper, the non-abelian group $G$ we consider will be the dihedral group $D_{2n}$ of order $n$ with presentation $\langle r, s | r^n = s^2 = 1, sr = r^{-1}s\rangle$. We show some constructions of $(2,|A|,|B|)$-complete decomposition of dihedral groups for two
subsets $A$ and $B$. We first note that $D_{2n} = \langle r \rangle \cup \langle r \rangle s$, where $\langle r \rangle$ is the cyclic subgroup of $D_{2n}$ generated by $r$. To justify our later constructions, we first let $A' = \langle r \rangle$ and $B' = \langle r \rangle s$. Then $A'B' = \langle r \rangle \langle r \rangle s = n\langle r \rangle s$. Clearly $A'B' \neq D_{2n}$, which implies that $(A', B')$ is not a $(2, |A'|, |B'|)$-complete decomposition of $D_{2n}$. Thus, it motivated us to modify the subsets $A'$ and $B'$ such that there exists a complete decomposition of $D_{2n}$.

Throughout this paper, we let $P, Q, A$ and $B$ be the subsets of $D_{2n}$ and define as follows:

$P = \{r^i | 1 \leq i \leq v \}$,

$Q = \{r^j s | 0 \leq j \leq v - 1 \}$,

$A = (\langle r \rangle \cup \langle r \rangle s) \setminus (P \cup Q)$ and

$B = P \cup Q$.

Clearly that $P \subset \langle r \rangle$, $Q \subset \langle r \rangle s$ and $A \cap B = \emptyset$. We shall write $|A|$ for the number of distinct elements in subset $A$. Since $|P| = |Q| = v$ where $v \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \}$, it follows that $|B| \in \{2, 4, \ldots, 2 \left\lfloor \frac{n}{2} \right\rfloor, n - 1, n \}$ and $|A| = 2n - |B|$.

For convenience, we state the definition of the complete decomposition [7] as follows:

**Definition 1.1** Let $G$ be a finite non-abelian group. For integer $k \geq 2$, we let $A_1, \ldots, A_k$ be non-empty subsets of $G$. If $A_1, \ldots, A_k$ are pairwise disjoint and if the subset product $A_1 \cdots A_k = \{a_1 \cdots a_k | a_i \in A_i, j = 1, \ldots, k \}$ coincides with $G$, where the $A_i$ are all distinct and $\{A_1, \ldots, A_k\} = \{A_1, \ldots, A_k\}$, then $(A_1, \ldots, A_k)$ is called a $(k, |A_1|, \ldots, |A_k|)$-complete decomposition of $G$.

### 2. Preliminary Results

A multiset is a collection of elements in which elements are allowed to repeat. The number of times an element occurs in a multiset is called its multiplicity, refer to [8] for more details about multiset. Throughout this paper, we present our set as a multiset. In order to simplify the notation, we indicate the multiplicity of the elements from the set $\{2 \cdot r^2, 1 \cdot r^3 s, 4 \cdot 1\}$ as $\{2r^2, r^3 s, 4\}$.

Let $E = \{e_1, e_2, \ldots, e_i\}$ and $F = \{f_1, f_2, \ldots, f_j\}$. We perform the left and right multiplications of the subsets $E$ and $F$ as follows:

$EF = \{e_1f_1, e_1f_2, \ldots, e_1f_j, e_2f_1, e_2f_2, \ldots, e_2f_j, \ldots, e_if_1, e_if_2, \ldots, e_if_j\}$,

$FE = \{f_1e_1, f_1e_2, \ldots, f_1e_i, f_2e_1, f_2e_2, \ldots, f_2e_i, \ldots, f_je_1, f_je_2, \ldots, f_je_i\}$.

For some elements $r^j \in D_{2n}$, the multiplication between the set $E$ and the element $r^j$ are defined as follows:

$r^j E = \{r^je_1, r^je_2, \ldots, r^je_i\}$,

$Er^j = \{e_1r^j, e_2r^j, \ldots, e_ir^j\}$.

In addition, the multiplication between a positive integer $c$ and the set $E$ are performed as follows:

$cE = \{ce_1, ce_2, \ldots, ce_i\}$ for some positive integers $c$.

In this section, we generalize the products $PP$, $QQ$, $PQ$ and $QP$. In addition, we show some simple constructions of complete decomposition of $D_{2n}$.

**Lemma 2.1** Suppose $P = \{r^i | 1 \leq i \leq v \}$ and $Q = \{r^j s | 0 \leq j \leq v - 1 \}$ are subsets of $D_{2n}$ where $1 \leq v \leq n - 1$. Then the following holds:

(i) \(PP = \{r^2, 2r^3, \ldots, (v - 2)r^{v - 1}, (v - 1)r^v, vr^{v + 1}, (v - 1)r^{v + 2}, (v - 2)r^{v + 3}, \ldots, 2r^{2v - 1}, r^{2v}\}\),

(ii) \(QQ = \{r^v, 2r^{v + 1}, \ldots, (v - 2)r^{2v - 2}, (v - 1)r^{2v - 1}, v, (v - 1)r, (v - 2)r^2, \ldots, 2r^{v - 2}, r^{v - 1}\}\),

(iii) \(PQ = \{sr^3, 2r^3s, \ldots, (v - 2)r^{v - 2}s, (v - 1)r^{v - 1}s, vr^v, (v - 1)r^{v + 1}s, (v - 2)r^{v + 2}s, \ldots, 2r^{2v - 2}s, r^{2v - 1}s\}\),

(iv) \(QP = \{r^{-v}s, 2r^{-v + 1}s, \ldots, (v - 2)r^{-v - 3}s, (v - 1)r^{-v - 2}s, vr^{-1}s, (v - 1)r^{v - 1}s, (v - 2)r^{v + 2}s, \ldots, 2r^{2v - 2}s, r^{2v - 1}s\}\).
**Proof** We prove this lemma by using induction on $v$. The basis step is trivial and thus is ignored. First, we prove that $PP$ is true for all $v \in \{1, 2, \ldots, n-1\}$. Let $k \in \mathbb{N}$, suppose that $PP$ is true for $v = k$, then

$$PP = \{r^2, 2r^3, \ldots, (k-2)r^{k-1}, (k-1)r^k, kr^{k+1}, (k-1)r^{k+2}, (k-2)r^{k+3}, \ldots, 2r^{2k-1}, r^{2k}\}.$$  

Now, we show that the result is also true for $v = k+1$. Let $P' = P \cup \{r^{k+1}\}$, we compute

$$P'P' = (P \cup \{r^{k+1}\})(P \cup \{r^{k+1}\})$$

$$= PP \cup P\{r^{k+1}\} \cup \{r^{k+1}\}P \cup \{r^{2k+2}\}$$

$$= PP \cup 2P\{r^{k+1}\} \cup \{r^{2k+2}\}$$

$$= PP \cup 2\{r^{k+2}, r^{k+3}, \ldots, r^{2k+1}\} \cup \{r^{2k+2}\}$$

$$= \{r^2, 2r^3, \ldots, (k-1)r^k, kr^{k+1}, (k+1)r^{k+2}, kr^{k+3}, (k-1)r^{k+4}, \ldots, 2r^{2k+1}, r^{2(k+1)}\},$$

which clearly implies that the result is also true for $v = k+1$. Therefore, by mathematical induction, we obtain the result in part (i).

Next, we show that part (iv) is true, that is, we show that $QP$ is true for all $v \in \{1, 2, \ldots, n-1\}$. Clearly, $QP$ is true when $v = 1$, however the computation $QP$ when $v = 1$ is trivial and thus is ignored. By assuming that $QP$ is true for an arbitrary positive integer $k$, we obtain

$$QP = \{r^{-k}s, 2r^{-k+1}s, \ldots, (k-2)r^{-3}s, (k-1)r^{-2}s, kr^{-1}s, (k-1)s, (k-2)s, 2r^{-k-3}s, r^{-2k-2}s\}.$$  

Then, we show that the result is also true when $v = k+1$. Let $P' = P \cup \{r^{k+1}\}$ and $Q' = Q \cup \{r^k\}$, we compute $Q'P'$ as follows:

$$Q'P' = (Q \cup \{r^k\})(P \cup \{r^{k+1}\})$$

$$= QP \cup Q\{r^{k+1}\} \cup \{r^k\}P \cup \{r^{-1}s\}$$

$$= QP \cup \{r^{-k-1}s, r^{-k}s, \ldots, r^{-3}s, r^{-2}s\} \cup \{r^{k-1}s, r^{k-2}s, \ldots, rs, s\} \cup \{r^{-1}s\}$$

$$= \{r^{-k-1}s, 2r^{-k}s, \ldots, (k-1)r^{-3}s, kr^{-2}s, (k+1)r^{-1}s, ks, \ldots, 2r^{-k-2}s, r^{-k-1}s\}.$$  

This implies that the result is true for $v = k+1$. By mathematical induction, we obtain the result in part (iv).

The proof for $QQ$ and $PQ$ are not included here as the proof is similar to $PP$ and $QP$. 

By combining $PP, QQ, PQ$ and $QP$, we obtain the following remark.

**Remark 2.2** For $1 \leq v \leq n-1$,

$$PP \cup QQ = \{r^{-v+1}, 2r^{-v+2}, \ldots, (v-2)r^{-2}, (v-1)r^{-1}\} \cup \{v\} \cup$$

$$(v-1)\{r, r^2, \ldots, r^v\} \cup \{vr^{v+1}, (v-1)r^{v+2}, \ldots, r^{2v}\},$$

$$PQ \cup QP = \{r^{-v}s, 2r^{-v+1}s, \ldots, vr^{-1}s\} \cup \{v-1\}\{s, rs, \ldots, r^{v-1}s\} \cup$$

$$\{vr^v, (v-1)r^{v+1}s, \ldots, r^{2v-1}s\}.$$  

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Let $A = (\langle r \rangle \cup \langle r \rangle s) \setminus (P \cup Q)$ and $B = P \cup Q$ where $|P| = |Q| = v$. We compute the product of subsets $AB$ of $D_{2n}$ as follows:

$$AB = [(\langle r \rangle \cup \langle r \rangle s) \setminus (P \cup Q)](P \cup Q)$$
$$= (\langle r \rangle P \cup (r)Q \cup (r)sP \cup (r)sQ) \setminus (PP \cup QQ \cup PQ \cupQP)$$
$$= (v(r) \cup v(r) \cup v(r) \cup v(r)) \setminus (PP \cup QQ \cup PQ \cupQP)$$
$$= (2v(r) \cup 2v(r)s) \setminus (PP \cup QQ \cup PQ \cupQP),$$

which extensively use in later proof.

Before further discussion, we first show a numerical example.

**Example 2.3** Let $n = 4$ and $v = 2$, we have $P = \{r, r^2\}$, $Q = \{s, rs\}$, $A = \{1, r^3, r^2s, r^3s\}$ and $B = \{r, r^2, s, rs\}$. Then $AB = \{1, 3r, 3r^2, r^3, 3s, 3rs, r^2s, r^3s\}$. Clearly $|AB| = 8$, it follows that $AB = D_8$, and hence $(A, B)$ is a $(2, 4, 4)$-complete decomposition of $D_8$.

The following propositions show the existence of $(2, |A|, |B|)$-complete decomposition of $D_{2n}$ for the case when $v = 1$ and $v = 2$.

**Proposition 2.4** There exists a $(2, 2n - 2, 2)$-complete decomposition of $D_{2n}$ for all $n \geq 3$.

**Proof** By Lemma II.1, we have $PP = \{r^2\}$, $QQ = \{1\}$, $PQ = \{rs\}$ and $QP = \{r^{-1}s\}$ when $v = 1$. Then, by Equation (1) we have $AB = (2\langle r \rangle \cup 2\langle r \rangle s) \setminus \{1, r^2, r^{-1}s, rs\}$. Since the elements $1 \neq r^2$ and $r^{-1}s \neq rs$ for all $n \geq 3$, it follows that $|AB| = 2n$. Hence $(A, B)$ is a $(2, 2n - 2, 2)$-complete decomposition of $D_{2n}$.

**Proposition 2.5** There exists a $(2, 2n - 4, 4)$-complete decomposition of $D_{2n}$ for all $n \geq 3$.

**Proof** When $v = 2$, by Lemma II.1, we have $PP, QQ, PQ,QP$ as follows:

$$PP = \{r^2, 2r^3, r^4\},$$
$$QQ = \{r^{-1}, 2, r\},$$
$$PQ = \{rs, 2r^2s, r^3s\},$$
$$QP = \{r^{-2}s, 2r^{-1}s, s\}.$$

Then, by Equation (1) we have

$AB = (4\langle r \rangle \cup 4\langle r \rangle s) \setminus \{r^{-1}, 2, r, r^2, 2r^3, r^4, r^{-2}s, 2r^{-1}s, s, r^2s, r^3s\}$. To ensure that $(A, B)$ form a complete decomposition of $D_{2n}$, we first check the multiplicity of all the elements in $\langle r \rangle$ of $AB$. The element $r^{-1}$ might be equal to $r^2$, $r^3$ or $r^4$ depending on the value of $n$. The element $r^3$ has the highest multiplicity which is 2, if $r^{-1} = r^3$, then we have $3r^3$. Since $4 - 3 = 1 \geq 1$, it follows that $|4\langle r \rangle \setminus \{r^{-1}, 2, r, r^2, 2r^3, r^4\}| = n$.

It remains to show that $|4\langle r \rangle s \setminus \{r^{-2}s, 2r^{-1}s, s, rs, 2r^2s, r^3s\}| = n$. Firstly, we note that the element $r^{-2}s \neq r^{-1}s$ for all $n \geq 3$. The element $r^2s$ has the highest multiplicity which is 2. If $r^{-1}s$ or $r^{-2}s$ is equal to $r^2s$, then we have $3r^2s$. Since $4 - 3 = 1 \geq 1$, we then obtain $|4\langle r \rangle s \setminus \{r^{-1}s, rs, 2r^2s, r^3s\}| = n$. Hence, $(A, B)$ is a $(2, 2n - 4, 4)$-complete decomposition of $D_{2n}$.
3. Some Constructions of Complete Decomposition of Dihedral Groups

In this section, we first show the constructions of \((2, n + 1, n - 1)\)-complete decomposition of \(D_{2n}\) for integer \(n \geq 3\).

**Lemma 3.1** There exists a \((2, n + 1, n - 1)\)-complete decomposition of \(D_{2n}\) for odd integer \(n \geq 3\).

**Proof** By Remark II.2, we have \(PP \cup QQ\) and \(PQ \cup QP\) for \(v = \frac{n-1}{2}\) as follows:

\[
PP \cup QQ = \left\{ \frac{n-3}{2} \right\} \cup \left\{ \frac{n-1}{2}, r, \ldots, r^{n-2}, r^{n-1} \right\} \cup \\
\left\{ \frac{n-3}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, r^{n-2}, r^{n-1} \right\}.
\]

\[
PQ \cup QP = \left\{ \frac{n-3}{2} \right\} \cup \left\{ s, rs, \ldots, r^{n-2}, r^{n-1} \right\} \cup \\
\left\{ \frac{n-3}{2}, r, rs, \ldots, r^{n-2}, r^{n-1} \right\}.
\]

By Equation (1) we have

\[
AB = (2v(r) \cup 2v(r)s) \setminus (PP \cup QQ \cup PQ \cup QP).
\]

We can see that the highest multiplicity of the elements in \(PP \cup QQ\) and \(PQ \cup QP\) is \(\frac{n-1}{2}\). Since \(2v = n - 1\), it follows that the minimum multiplicity of each element in \(AB\) is \(n - 1 - \left( \frac{n-1}{2} \right) = \frac{n-1}{2} \geq 1\). Thus \(|AB| = 2n\), and hence \((A, B)\) is a \((2, n + 1, n - 1)\)-complete decomposition of \(D_{2n}\). \(\blacksquare\)

Let \(n\) be an integer. In order to construct a \((2, n + 1, n - 1)\)-complete decomposition of \(D_{2n}\) for even \(n \geq 4\), we modify the subsets \(A\) and \(B\) to become subsets \(R\) and \(S\). The construction is shown in lemma below.

**Lemma 3.2** Let \(R = \{1, r^{\frac{n+1}{2}}, r^2, \ldots, r^{n-1}, s, r^{\frac{n+1}{2}s}, r^{2s}, \ldots, r^{n-1}s\}\) and \(S = \{r, r^2, \ldots, r^2, rs, r^2s, \ldots, r^{n-1}s\}\) be the subsets of \(D_{2n}\). Then there exists a \((2, n + 1, n - 1)\)-complete decomposition of \(D_{2n}\) for even integer \(n \geq 4\).

**Proof** To obtain the explicit form of product of subsets \(RS\), we compute

\[
Rr = \{r, r^{\frac{n+3}{2}}, r^2, \ldots, 1, r^{n-1}s, r^{\frac{n+1}{2}s}, r^2s, \ldots, r^{n-2}s\},
\]

\[
Rr^2 = \{r^2, r^{\frac{n+3}{2}}, r^4, \ldots, r^{n-2}s, r^{\frac{n+1}{2}s}, r^4s, \ldots, r^{n-3}s\},
\]

\[
\vdots
\]

\[
Rr^{\frac{n}{2}} = \{r^{\frac{n}{2}}, r^2, \ldots, r^{\frac{n}{2}+1}, r^{\frac{n}{2}+3}s, rs, \ldots, r^{\frac{n}{2}-1}s\},
\]

\[
Rrs = \{rs, r^{\frac{n}{2}+2}s, r^{\frac{n}{2}+4}s, \ldots, s, r^{n-1}, r^{\frac{n}{2}+1}, r^3, \ldots, r^{n-2}\},
\]

\[
Rr^2s = \{r^2s, r^{\frac{n}{2}+3}s, r^{\frac{n}{2}+5}s, \ldots, rs, r^{n-2}, r^{\frac{n}{2}+2}, r^4, \ldots, r^{n-3}\},
\]

\[
\vdots
\]

\[
Rr^{\frac{n-1}{2}}s = \{r^{\frac{n-1}{2}s}, r^{\frac{n-3}{2}s}, r^{\frac{n-1}{2}}, r^2, \ldots, r^{\frac{n}{2}}\}.
\]

It follows that
Lemma 3.4
Let $A$ modify the subsets $B$.

Proof
Theorem 3.3
There exists a $(2, n + 1, n - 1)$-complete decomposition of $D_{2n}$ for integer $n \geq 3$.

Proof
Lemmas III.1 and III.2 complete the proof for Theorem 3.3.

In order to construct a $(2, n, n)$-complete decomposition of $D_{2n}$ for odd integer $n \geq 3$, we modify the subsets $A$ and $B$ to subsets $U$ and $V$.

Lemma 3.4
Let $U = \{1, r, r^2, \ldots, r^{\frac{n-1}{2}}, s, rs, \ldots, r^{\frac{n-3}{2}}s\}$ and $V = \{r^{\frac{n+1}{2}}, r^{\frac{n+3}{2}}, \ldots, r^{n-1}, r^{\frac{n-1}{2}}s, r^{\frac{n+1}{2}}s, \ldots, r^{n-1}s\}$ be the subsets of $D_{2n}$. Then there exists a $(2, n, n)$-complete decomposition of $D_{2n}$ for odd integer $n \geq 3$.

Proof
In order to obtain the explicit form of product of subsets $UV$, we first compute the following:

\[
U^{\frac{n+1}{2}} = \{r^{\frac{n+1}{2}}, r^{\frac{n+3}{2}}, \ldots, 1, r^{\frac{n-1}{2}}s, r^{\frac{n+1}{2}}, \ldots, r^{-2}\},
\]
\[
U^{\frac{n+3}{2}} = \{r^{\frac{n+3}{2}}, r^{\frac{n+5}{2}}, \ldots, r^{\frac{n-3}{2}}s, r^{\frac{n+1}{2}}s, \ldots, r^{-3}\},
\]
\[\vdots\]
\[
U^{n-1} = \{r^{n-1}, 1, r^{3n-3}, rs, srs, \ldots, r^{n-1}\},
\]
\[
U^{\frac{n-1}{2}} = \{r^{\frac{n-1}{2}}, r^{\frac{n+1}{2}}, \ldots, r^{n-1}, r^{\frac{n+1}{2}}, \ldots, r^{-1}\},
\]
\[
U^{\frac{n+1}{2}} = \{r^{\frac{n+1}{2}}, r^{\frac{n+3}{2}}, \ldots, r^{n-1}, r^{\frac{n+1}{2}}, \ldots, r^{-2}\},
\]
\[\vdots\]
\[
U^{n-1} = \{r^{n-1}, s, \ldots, r^{3n-3}, s, r^2, \ldots, r^{-n-1}\}.
\]

Then we can express $UV$ explicitly as

\[
UV = \left(\frac{n-1}{2}\right)\{1, r, \ldots, r^{\frac{n-1}{2}}\} \cup \left(\frac{n+1}{2}\right)\{r^{\frac{n+1}{2}}, r^{\frac{n+3}{2}}, \ldots, r^{n-1}\} \cup \left(\frac{n-1}{2}\right)\{s, rs, \ldots, r^{\frac{n-3}{2}}s\} \cup \left(\frac{n+1}{2}\right)\{r^{\frac{n-1}{2}}s, r^{\frac{n+1}{2}}s, \ldots, r^{n-1}s\}.
\]

Clearly that the minimum multiplicity of the elements in the product of subsets $UV$ is $\frac{n-1}{2}$. Since $n \geq 3$, $\frac{n-1}{2} \geq 1$. Thus $|UV| = 2n$, and hence the result follows.

Lemma 3.5
There exists a $(2, n, n)$-complete decomposition of $D_{2n}$ for even integer $n \geq 4$. 


Proof By Remark II.2, we have $PP \cup QQ$ and $PQ \cup QP$ for $v = \frac{n}{2}$ as follows:

$$PP \cup QQ = \{\frac{n}{2} + 1\} \cup (\frac{n}{2} - 1)\{r, r^2, r^3, \ldots, r^{\frac{n}{2} - 2}, r^{\frac{n}{2} - 1}, r^{\frac{n}{2}}\} \cup \left(\frac{n}{2} + 1\right)\{r^{\frac{n}{2} + 1}, r^{\frac{n}{2} + 2}, \ldots, r^{n - 1}\}$$

$$PQ \cup QP = (\frac{n}{2} - 1)\{s, rs, \ldots, r^{\frac{n}{2} - 2}s, r^{\frac{n}{2} - 1}s\} \cup \left(\frac{n}{2} + 1\right)\{r^{\frac{n}{2}s}, r^{\frac{n}{2}s + 1}, \ldots, r^{n - 2}s, r^{n - 1}s\}.$$ 

By Equation (1) we have

$$AB = (2v(r) \cup 2v(r)s) \setminus (PP \cup QQ \cup PQ \cup QP).$$

The highest multiplicity of the elements in $PP \cup QQ$ and $PQ \cup QP$ are $\frac{n}{2} + 1$. It follows that the minimum multiplicity of each element in $(r)$ and $(r)s$ of $AB$ is $2v - (\frac{n}{2} + 1) = 2(\frac{n}{2}) - (\frac{n}{2} + 1) = \frac{n}{2} - 1 \geq 1$, which implies that $|AB| = 2n$. This completes the proof. □

Lemmas III.4 and III.5 immediately imply the following.

Theorem 3.6 There exists a $(2, n, n)$-complete decomposition of $D_{2n}$ for integer $n \geq 3$.

Proof Lemmas III.4 and III.5 complete the proof for Theorem 3.6. □

The following theorem shows the construction of $(2, 2n - 2v, 2v)$-complete decomposition of $D_{2n}$ for $v \in \{1, 2, \ldots, \left\lfloor \frac{n}{3} \right\rfloor\}$. In order to ensure that the elements in $PP \cup QQ$ and $PQ \cup QP$ are not repeating, we set a smaller upper bound for $v$ which is $\left\lceil \frac{n}{3} \right\rceil$.

Theorem 3.7 Let $n$ be an integer $\geq 3$. There exists a $(2, 2n - 2v, 2v)$-complete decomposition of $D_{2n}$ for $v \in \{1, 2, \ldots, \left\lceil \frac{n}{3} \right\rceil\}$.

Proof By Remark II.2,

$$PP \cup QQ = \{r^{-v+1}, 2r^{-v+2}, \ldots, (v - 2)r^{-2}\} \cup (v - 1)r^{-1} \cup \{v\} \cup (v - 1)\{r^i | 1 \leq i \leq v\} \cup \{vr^{v+1}, (v - 1)r^{v+2}, \ldots, r^{2v}\},$$

$$PQ \cup QP = \{r^{-v}s, 2r^{-v+1}s, \ldots, vr^{-1}s\} \cup (v - 1)\{r^js | 0 \leq j \leq v - 1\} \cup \{vr^vs, (v - 1)r^{v+1}s, \ldots, r^{2v-1}s\}.$$ 

By Equation (1) we have

$$AB = (2v(r) \cup 2v(r)s) \setminus (PP \cup QQ \cup PQ \cup QP).$$

It remains to show that $|(2v(r) \cup 2v(r)s) \setminus (PP \cup QQ \cup PQ \cup QP)| = 2n$. Observe that the highest multiplicity of the elements in both $PP \cup QQ$ and $PQ \cup QP$ is $v$. Since $v \in \{1, 2, \ldots, \left\lceil \frac{n}{3} \right\rceil\}$, it follows that the minimum multiplicity of each element in $(r)$ and $(r)s$ of $AB$ is $2v - v = v \geq 1$. Hence, $(A, B)$ is a $(2, 2n - 2v, 2v)$-complete decomposition of $D_{2n}$ for $v \in \{1, 2, \ldots, \left\lceil \frac{n}{3} \right\rceil\}$. □
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