Thermodynamic Lattice Study for Preconformal Dynamics in Strongly Flavored Gauge Theory

Kohtaroh Miura
INFN - Laboratori Nazionali di Frascati Via Enrico Fermi, n.40 Frascati Rome Italy
E-mail: Kohtaroh.Miura@lnf.infn.it

Abstract. By using the lattice Monte-Carlo simulation, we investigate the finite temperature chiral phase transition in color SU(3) gauge theories with various species of fundamental fermions, and discuss the signals of the (pre-)conformality at large \( N_f \) (number of flavors) via their comparisons. With increasing \( N_f \), we confirm stronger fermion screening which results from a larger fermion multiplicity. We investigate a finite \( T \) step-scaling which is attributed to the uniqueness of the critical temperature \( (T_c) \) at each \( N_f \), then the vanishing step-scaling signals the emergence of the conformality around \( N_f^* \sim 10 - 12 \). Further, motivated by the recent functional renormalization group analyses, we examine the \( N_f \) dependence of \( T_c \), whose vanishing behavior indicates that the conformal phase sets in around \( N_f^* \sim 9 - 10 \).

1. Introduction
Conformal invariance is anticipated to emerge in asymptotically free non-Abelian gauge theories when the number of fermion species, \( N_f \), exceeds a critical value \( N_f = N_f^* \) [1,2]. The approach to the conformality from below is in principle associated with a pre-conformal (walking) behavior of the running coupling, which has been advocated as a basis for strongly interacting mechanisms of electroweak symmetry breaking [3].

Recent lattice studies [4] focused on the computation of \( N_f^* \) and the analysis of the conformal window itself, either with fundamental fermions or other representations. Among the many interesting results with fundamental fermions, we single out the observation that the color SU(3) gauge theory with \( N_f = 8 \) is still in the hadronic phase [5,6], while \( N_f = 12 \) seems to be close to the critical number of flavors, with some groups favoring conformality [6–9], and others chiral symmetry breaking [10].

In order to attack the walking and the conformal dynamics, it is more informative beyond a fixed \( N_f \) to investigate the vanishing or reducing chiral dynamics with increasing \( N_f \). To this end, we investigate the \( N_f \) dependence of the chiral phase transition at finite temperature \( (T) \) based on our recent work [5,11]: The vanishing (reducing) finite \( T \) step-scaling which attributes to the uniqueness of \( T_c \) at each \( N_f \) signals the emergence of the (pre-)conformality. Further, motivated by the recent functional renormalization group (FRG) studies [12], we examine the \( N_f \) dependence of \( T_c \), whose vanishing (decreasing) behavior indicates the (pre-)conformality. This thermodynamic lattice study for the large \( N_f \) non-Abelian gauge theory has played a crucial role to extract a notion of more strongly interacting non-Abelian plasma [13], and it is expected to provide a new connection between the lattice and the Gauge/Gravity duality [14].
2. Simulation setups

Simulations have been performed by utilizing the publicly available MILC code [15]. We use an improved version of the staggered action, the Asqatd action, with a one-loop Symanzik [16] and tadpole [17] improved gauge action. The tadpole factor $u_0$ is determined by performing zero temperature simulations on the $12^4$ lattice, and used as an input for finite temperature simulations. To generate configurations with mass degenerate dynamical flavors, we have used the rational hybrid Monte Carlo algorithm (RHMC) [18].

We evaluate the thermalized ensemble averages of the chiral condensate (PBP) and Polyakov loop (PLOOP) for various lattice couplings $\beta_L$ of the staggered action, the Asqtad action, with a one-loop Symanzik [16] and tadpole [17] improved gauge action. The tadpole factor $u_0$ is determined by performing zero temperature simulations on the $12^4$ lattice, and used as an input for finite temperature simulations. To generate configurations with mass degenerate dynamical flavors, we have used the rational hybrid Monte Carlo algorithm (RHMC) [18].

Our observables are the chiral condensate $\langle \bar{\psi}\psi \rangle$ and the Polyakov loop $U_{4,tx}$.

\[
\langle \bar{\psi}\psi \rangle = \frac{N_f}{4N_f^2N_t} \text{Tr}[M^{-1}], \quad L = \frac{1}{N_cN_f^2} \sum_x \text{Re} \left( \text{tr}_c \prod_{t=1}^{N_t} U_{4,tx} \right),
\]

where $N_s$ ($N_t$) represents the number of lattice sites in the spatial (temporal) direction and $U_{4,tx}$ is the temporal link variable. The other important observable is the ratio of a scalar and a pseudo-scalar susceptibility [19],

\[
R_\pi = \frac{\chi_\sigma}{\chi_\pi} = \frac{a^2 \langle \bar{\psi}\psi \rangle / \partial m}{\langle \bar{\psi}\psi \rangle / m} = \frac{\chi_{\text{conn}} + \chi_{\text{disc}}}{\langle \bar{\psi}\psi \rangle / m},
\]

where $a^2 \chi_{\text{conn}} = -N_f \langle \text{Tr}[(MM)^{-1}] \rangle / (4N_f^2N_t)$ and $a^2 \chi_{\text{disc}} = N_t^2 \langle \text{Tr}[M^{-1}]^2 \rangle - \langle \text{Tr}M^{-1} \rangle^2 / (16N_f^3N_t)$. Here, $R_\pi \sim \mathcal{O}(1)$ indicates the scalar and pseudo-scalar degeneracy attributed to the approximate chiral restoration, while $R_\pi \ll 1$ indicates the chiral symmetry breaking [5, 19].

3. Results

We evaluate the thermalized ensemble averages of the chiral condensate (PBP) and Polyakov loop (PLOOP) for various lattice couplings $\beta_L$, lattice sizes with the finite $T$ set up $N_s \gg N_t$, and the number of flavors $N_f$. All results have been obtained by using a single value for a lattice bare fermion mass $am = 0.02$. Then we locate the lattice bare coupling $\beta_L^c$ associated with the chiral crossover which is signaled by the drastic decrease (increase) of PBP (PLOOP) as a function of $\beta_L$. In practice, the ratio of the scalar and pseudo-scalar susceptibility $R_\pi$ gives a stronger signal of the chiral crossover, owing to its renormalization invariant property. In Table 1, we summarize the obtained critical lattice couplings as a function of $(N_f, N_t)$. We have confirmed the (approximate) asymptotic scaling for the normalized critical temperature $T_c/\Lambda_L$ varying $N_t$ at each $N_f$, where $\Lambda_L$ is so-called lattice Lambda. This indicates that our $\beta_L^c$ have been determined near the continuum limit [20].

Table 1. Summary of the (pseudo) critical lattice couplings $\beta_L^c$ for the theories with $N_f = 0, 4, 6, 8, am = 0.02$ and varying $N_t = 4, 6, 8, 12$. All results are obtained by using the action with the same level of improvements.

| $N_f \backslash N_t$ | 4        | 6        | 8        | 12       |
|---------------------|----------|----------|----------|----------|
| 0                   | 7.35 ± 0.1 | 7.88 ± 0.05 | 8.20 ± 0.1 | –        |
| 4                   | 5.60 ± 0.1 | 5.89 ± 0.05 | 6.10 ± 0.1 | –        |
| 6                   | 4.65 ± 0.05 | 5.05 ± 0.05 | 5.2 ± 0.05 | 5.55 ± 0.1 |
| 8                   | –        | 4.1125 ± 0.0125 | 4.20 ± 0.1 | 4.34 ± 0.04 |

We shall now discuss the emergence of the (pre-)conformality by using our critical lattice $g_L^c = \sqrt{10/\beta_L^c}$ collection. The uniqueness of a critical temperature at each $N_f$, $T_c^{-1} =$
To this end, we plot the (pseudo) critical lattice coupling of \(N_f\) thermal step (or equivalently the asymptotic) scalings for our fermion screening effects due to the larger number of fermion species. We have observed the flavors satisfying \(\Delta N_f\) and now we have completed it. To this end, we plot the (pseudo) critical lattice coupling \(g_L^c = \sqrt{10/\beta_L^c}\) as a function of \(N_f\) in Fig. 1, which gives an extension of Miransky-Yamawaki diagram [21] to finite \(T\) cases.

Let us first pick up the lattice critical couplings for \(N_f = 6\) and 8, and consider a “constant \(N_f\)” line. As shown in the left panel of Fig. 1, \(N_f = 6\) and 12 lines join at \((g_L^c, N_f) = (1.825\pm 0.02, 11.57\pm 0.17)\), and \(N_f = 6\) and 12 lines at \((g_L^c, N_f) = (1.753\pm 0.02, 10.715\pm 0.17)\), indicating an infra-red fixed point with vanishing thermal step scalings. Next we shall investigate the critical lattice couplings at \(N_f = 6\) and 8 for the whole range of \(N_f = 0\) – 8. They can be well fitted by assuming the functional expression \(N_f (g_L^c) = A \cdot \log [B \cdot (g_L^c - g_L^0|N_f=0) + 1]\) giving \((g_L^c, N_f) = (1.88\pm 0.09, 11.39\pm 0.78)\) (right panel of Fig. 1). Thus, the thermal step-scaling with the use of our lattice critical couplings supports the emergence of conformal window near the 12-flavor system, whose (pre-)conformality is now under debate in the recent lattice studies [4].

**Figure 1.** (Pseudo) critical values of the lattice coupling \(g_L^c = \sqrt{10/\beta_L^c}\) for theories with \(N_f = 0\), 4, 6, 8 and for several values of \(N_t\) in the Miransky-Yamawaki phase diagram. Left: We have picked up \(g_L^c = \sqrt{10/\beta_L^c}\) for \(N_f = 6\) and 8, and consider the “constant \(N_f\)” line with \(N_t = 6\), 8, 12. Candidates for the IRFP have been estimated by the crossing points of \(N_t = 12\) line with \(N_f = 6\) or \(N_f = 8\) line. Right: The dashed line is a fit with the ansatz \(N_f (g_L^c) = A \cdot \log [B \cdot (g_L^c - g_L^0|N_f=0) + 1]\) for \(N_f = 6\) and \(N_t = 8\) results.

As indicated by the constant \(N_t\) line in Fig. 1, the critical coupling is an increasing function of \(N_f\) for a fixed lattice temporal extension. This behavior is the direct consequence of enhanced fermion screening effects due to the larger number of fermion species. We have observed the thermal step (or equivalently the asymptotic) scalings for our \(\beta_L^c\), thereby, the enhancement of the screening effects gets to a physical significance. The test of the asymptotic scaling has been the historical homework since the pioneering finite \(T\) study at large \(N_f\) by J. Kogut and collaborators [22] and now we have completed it.

We shall now investigate the \(N_f\) dependence of the critical temperature. In order to compare a physical quantity such as a critical temperature among different theories with a different \(N_f\), it is necessary to introduce a \(N_f\) independent reference scale by hand. To set the reference scale, ideally speaking, we would like to measure Wilson loops of various sizes at various numbers of flavors in Monte-Carlo simulations to obtain the running couplings \(\bar{g}\) in the wide
range of scales at each number of flavors. Instead of performing such a massive simulation, we
approximately construct the renormalization flow via the integral of the two-loop beta-function
\[ \beta(g) = -g^3(b_0 + b_1 g^2) \]
\[ \frac{T_c}{M(g_L^\text{ref})} = \frac{1}{N_f} \frac{a^{-1}(g_L^c)}{M(g_L^\text{ref})} = \frac{1}{N_f} \int_{g_L^\text{ref}}^{g_L^c} \frac{dg}{-g^3(b_0 + b_1 g^2)}. \]  
(3)
To specify the reference scale \( M(g_L^\text{ref}) \), we utilize our plaquette (tad-pole factor \( u_0 \)) data shown in the left panel of Fig. 2. Note that the plaqettes can be regarded as a kind of renormalized couplings. Let us consider a constant \( u_0 \) without \( N_f \) dependence, for instance \( u_0 = 0.9 \) in figure, and read-off the corresponding bare lattice couplings at each \( N_f \). The obtained \( g_L(N_f) = \sqrt{10/\beta_0(N_f)} \) is used as a reference coupling \( g_L^\text{ref} \) in Eq. (3). This procedure imitates the scale setting in the potential scheme renormalization, and the use of \( N_f \) independent \( u_0 \) is motivated by the FRG scale setting method [12].
To be analogous to the FRG study, we should choose \( u_0 \) so as to get a UV \( M(g_L) \) free from the chiral dynamics. The middle panel of Fig. 2 displays the \( N_f \) dependence of \( T_c/M(g_L^\text{ref}) \) defined by Eq. (3) with \( u_0 = 0.9 \). Fitting \( T_c/M(g_L^\text{ref}) \) with the FRG motivated ansatz \( T_c = K|N_f^*-N_f|^\left(-2b_0/b_1\right)(N_f^*) \), we now read-off the lower edge of conformal window \( N_f^* \sim 9.47 \pm 0.02 \), which is somewhat smaller value comparing to those obtained by the vanishing thermal scale settings. In the middle panel of Fig. 2, we find \( T_c/M(g_L^\text{ref}) \ll 1 \), indicating the UV nature of the reference scale \( M(g_L^\text{ref}) \).
To get more transparent view for the UV reference scale, we here consider the particular reference coupling \( g_L^\text{ref} \) - the thermal critical coupling \( g_T^c \) which makes the reference scale \( M(g_L^\text{ref} = g_T^c) \) equivalent to \( T_c \) in Eq. (3): \( T_c/M(g_T^c) = 1 \), giving a typical interaction strength at \( T_c \). As shown in the right panel of Fig. 2, the increasing nature of \( g_T^c \) indicates a realization of more strongly interacting non-Abelian plasma at larger \( N_f \) as discussed in Ref. [13]. The criterion to set the UV reference scale \( M(g_L^\text{ref}) \) at every \( N_f \) would be given by the condition \( g_L^\text{ref} \ll g_T^c(N_f) \) for all \( N_f \). We find that \( u_0 \geq 0.84 \) meets the requirement, while the use of too large \( u_0 \) suffers from the strong discritization errors. In practice, we find that the number of flavors giving the vanishing \( T_c/M(g_L^\text{ref}) \) is relatively stable within the range \( 0.84 \leq u_0 \leq 0.94 \) which results in \( 9.85 \geq N_f^* \geq 9.17 \).

Figure 2. Left: The \( \beta_L \) dependences of the tadpole factor \( u_0 \) at zero temperature with the use of \( 12^4 \) lattice. Specifying a constant \( u_0 \) (e.g. \( u_0 = 0.9 \) in figure), we read off the corresponding lattice couplings \( \beta_L \), which are used to define the scale \( M \) at each theory with \( N_f \). Middle: The \( N_f \) dependence of \( T_c/M \) where \( M \) is the UV scale with \( u_0 = 0.9 \) at each theory with \( N_f \). The dashed line represents the fit for data by using the FRG motivated ansatz \( T_c = K|N_f^*-N_f|^\left(-2b_0/b_1\right)(N_f^*) \)
Right: The thermal critical coupling with increasing \( N_f \).
4. Summary
We have investigated the (pre-)conformal dynamics in color SU(3) gauge theories with multi-species of fundamental fermions by using the lattice Monte-Carlo simulation. In order to study the conformality beyond the fixed number of flavors $N_f$, we have focused on a reducing chiral dynamics at finite $T$ as a function of increasing $N_f$. We have observed stronger fermion screenings resulting from a larger fermion multiplicity at larger $N_f$. We have investigated a finite $T$ step-scaling which follows from the uniqueness of critical temperature ($T_c$) at each $N_f$, then the vanishing step-scaling signals the conformal dynamics at $N_f^\ast \sim 10 - 12$. Further, motivated by the recent FRG based studies [12], we have examined the $N_f$ dependence of $T_c$, by introducing a UV $N_f$ independent reference scale $M(g^\text{ref}_L)$. We have used the thermal critical coupling $g^c_T$ as a criterion to insure the UV nature of $M(g^\text{ref}_L)$ by imposing the condition $g^\text{ref}_L \ll g^c_T(N_f)$ for all $N_f$. We have found that the number of flavors giving a vanishing $T_c/M(g^\text{ref}_L)$ is relatively stable within the range $0.84 \leq u_0 \leq 0.94$, which results in $9.85 \geq N_f^\ast \geq 9.17$.

As a future perspective, we should measure Wilson loops of various sizes at various numbers of flavors, and perform more rigorous scale settings in the potential scheme. It is also mandatory to investigate the chiral limit and the thermodynamic limit at large $N_f$. This, together with a more extended set of flavor numbers, will allow a quantitative analysis of the critical behavior in the vicinity of the conformal IR fixed point.

Acknowledgments
The author thanks Maria Paola Lombardo and Elisabetta Pallante for continuous discussions. He thanks Edward Shuryak for fruitful discussion during the xQCD workshop. This work was in part based on the MILC Collaboration’s public lattice gauge theory code [15]. The numerical calculations were carried out on the IBM-SP6 and BG/P at CINECA, Italian-Grid-Infrastructures in Italy, and the Hitachi SR-16000 at YITP, Kyoto University in Japan.

References
[1] Caswell W E 1974 Phys. Rev. Lett. 33 244.
[2] Banks T and Zaks A 1982 Nucl. Phys. B 196 189.
[3] For a recent review, see Sannino F 2009 Acta Phys. Polon. B 40 3533.
[4] For recent reviews, see Del Debbio L 2010 PoS LAT2010 004; Pallante E 2009 PoS LAT2009 015.
[5] Deuzeman A, Lombardo M P and Pallante E 2008 Phys. Lett. B 670 41.
[6] Appelquist T, Fleming G T and Neil E T 2008 Phys. Rev. Lett. 100 171607 [Erratum-ibid. 2009 102 149902]; 2009 Phys. Rev. D 79 076010.
[7] Deuzeman A, Lombardo M P and Pallante E 2010 Phys. Rev. D 82 074503.
[8] Appelquist T, Fleming G T, Lin M F, Neil E T and Schaich D A 2011 Phys. Rev. D 84 054501.
[9] Hasenfratz A 2010 Phys. Rev. D 82 014506.
[10] Fodor Z, Holland K, Kuti J, Nogradi D and Schroeder C 2011 Phys. Lett. B 703 348-58.
[11] Miura K, Lombardo M P and Pallante E 2012 Phys. Lett. B 710 670.
[12] Braun J, Fisher C S and Gies H 2011 Phys. Rev. D 84 034045.
[13] Braun and H. Gies, 2010 JHEP 1005 060; 2006 0606 024.
[14] Liao J, Shuryak E 2012 Phys. Rev. Lett. 109 152001.
[15] For a recent development, see a review, Gursoy U, Kiritsis E, Mazzanti L, Michalogiorgakis G and Nitti F 2011 Lect. Notes Phys. 828 79.
[16] MILC Collaboration, http://www.physics.indiana.edu/~sg/milc.html
[17] Lepage G P and Mackenzie P B 1993 Phys. Rev. D 48 2250.
[18] Clark M A 2006 PoS LAT2006 004.
[19] Kocić A, Kogut J B and Lombardo M P 1993 Nucl. Phys. B 398 376.
[20] Gupta S 2001 Phys. Rev. D 64 034507.
[21] Miransky V A and Yamawaki K 1997 Phys. Rev. D 55 5051 [Erratum-ibid. 1997 D 56 3768].
[22] Kogut J B, Polonyi J, Wyld H W and Sinclair D K 1985 Phys. Rev. Lett. 54 1475.