A SPATIALLY LOCALIZED $L \log L$ ESTIMATE ON THE VORTICITY IN THE 3D NSE

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ABSTRACT. The purpose of this note is to present a spatially localized $L \log L$ bound on the vorticity in the 3D Navier-Stokes equations, assuming a very mild, purely geometric condition. This yields an extra-log decay of the distribution function of the vorticity, which in turn implies breaking the criticality in a physically, numerically, and mathematical analysis-motivated criticality scenario based on vortex stretching and anisotropic diffusion.

1. INTRODUCTION

Three-dimensional Navier-Stokes equations (3D NSE) – describing a flow of 3D incompressible viscous fluid – read

$$u_t + (u \cdot \nabla)u = -\nabla p + \Delta u,$$

supplemented with the incompressibility condition $\text{div} \, u = 0$, where $u$ is the velocity of the fluid, and $p$ is the pressure (here, the viscosity is set to 1). Applying the curl operator yields the vorticity formulation,

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \Delta \omega,$$

where $\omega = \text{curl} \, u$ is the vorticity of the fluid. Taking the spatial domain to be the whole space, the velocity can be recovered from the vorticity via the Biot-Savart Law,

$$u(x) = c \int \nabla \frac{1}{|x-y|} \times \omega(y) \, dy;$$

this makes (1) a closed (non-local) system for the vorticity field alone.

An a priori $L^1$-bound on the evolution of the vorticity in the 3D NSE was obtained by Constantin in [4]. This is a vorticity analogue of Leray’s a priori $L^2$-bound on the velocity, and both quantities scale in the same fashion. So far, there has been no a priori estimate on the weak solutions to the 3D NSE breaking this scaling.

The goal of this short article is to show that a very mild, purely geometric assumption yields a uniform-in-time $L \log L$ bound on the vorticity, effectively breaking the aforementioned scaling. More precisely, the assumption is a uniform-in-time boundedness of the localized vorticity direction in the weighted $\text{bmo}$-space $\widetilde{\text{bmo}}_{|\log r|}^{-1}$. An interesting feature of this space (cf. [11]) is that it allows…
for discontinuous functions exhibiting singularities of, e.g., \( \sin \log |\log(\text{something algebraic})| \)-type. In particular, the vorticity direction can blow-up in a geometrically spectacular fashion – every point on the unit sphere being a limit point – and the \( L \log L \) bound will still hold.

Besides being of an independent interest, the \( L \log L \) bound on the vorticity implies an extra-log decay of the distribution function. This is significant as it transforms a recently exposed \([9, 5]\) large-data criticality scenario for the 3D NSE into a no blow-up scenario. Shortly, creation and persistence (in the sense of the time-average) of the axial filamentary scales comparable to the macro-scale, paired with the \( L^1 \)-induced decay of the volume of the suitably defined region of intense vorticity, leads to creation and persistence of the transversal micro-scales comparable to the scale of local, anisotropic linear sparseness, enabling the anisotropic diffusion to equalize the nonlinear effects. The extra log-decay of the volume transforms the equalizing scenario into the anisotropic diffusion-win scenario.

The present result is, in a way, complementary to the results obtained by the authors in \([1]\). The class of conditions leading to an \( L \log L \)-bound presented in \([1]\) can be characterized as ‘wild in time’ with a uniform spatial (e.g., algebraic) structure, while the condition presented here can be characterized as ‘wild in space’ and uniform in time. As in \([1]\), the proof is based on an adaptation of the method exposed in \([4]\), the novel component being utilization of analytic cancelations in the vortex-stretching term via the Hardy space-version of the Div-Curl Lemma \([2]\), the local version \([8]\) of \( \mathcal{H}^1 - \text{BMO} \) duality \([6, 7]\), and the intimate connection between the \( \text{BMO} \)-norm and the logarithm. While the argument in \([1]\) relied on the structure of the evolution of the scalar components and the result (cf. \([14, 16]\)) stating that the \( \text{BMO} \)-norm of the logarithm of a polynomial is bounded independently of the coefficients, the present argument relies on sharp pointwise multiplier theorem in \( \text{bmo} \) \([11, 13, 12]\) and Coifman-Rochberg’s \( \text{BMO} \)-estimate on the logarithm of the maximal function of a locally integrable function \([3]\), the estimate being fully independent of the function and depending only on the dimension of the space.

2. AN EXCURSION TO HARMONIC ANALYSIS

In this section, we compile several results from harmonic analysis that will prove useful.

Hardy spaces \( \mathcal{H}^1 \) and \( \mathfrak{h}^1 \)

The maximal function of a distribution \( f \) is defined as,
\[
M_h f(x) = \sup_{t > 0} |f \ast h_t(x)|, \quad x \in \mathbb{R}^n,
\]
where \( h \) is a fixed, normalized (\( \int h \, dx = 1 \)) test function supported in the unit ball, and \( h_t \) denotes \( t^{-n}h(\cdot/t) \).

The distribution \( f \) is in the Hardy space \( \mathcal{H}^1 \) if \( \|f\|_{\mathcal{H}^1} = \|M_h f\|_1 < \infty \).

The local maximal function is defined as,
\[
m_h f(x) = \sup_{0 < t < 1} |f \ast h_t(x)|, \quad x \in \mathbb{R}^n,
\]
and the distribution \( f \) is in the local Hardy space \( \mathfrak{h}^1 \) if \( \|f\|_{\mathfrak{h}^1} = \|m_h f\|_1 < \infty \).
Div-Curl Lemma (Coifman, Lions, Meyer, Semmes [2])

Suppose that $E$ and $B$ are $L^2$-vector fields satisfying $\text{div} \ E = \text{curl} \ B = 0$ (in the sense of distribution). Then,

$$
\|E \cdot B\|_{H^1} \leq c(n) \|E\|_{L^2} \|B\|_{L^2}.
$$

BMO and weighted $bmo$ spaces

The classical space of bounded mean oscillations, $BMO$ is defined as follows,

$$
BMO = \left\{ f \in L^1_{\text{loc}} : \sup_{x \in \mathbb{R}^3, r > 0} \Omega(f, I(x, r)) < \infty \right\},
$$

where

$$
\Omega(f, I(x, r)) = \frac{1}{|I(x, r)|} \int_{I(x, r)} |f(x) - f_I| \, dx
$$

is the mean oscillation of the function $f$ with respect to its mean $f_I = \frac{1}{|I(x, r)|} \int_{I(x, r)} f(x) \, dx$, over the cube $I(x, r)$ centered at $x$ with the side-length $r$.

A local version of $BMO$, usually denoted by $bmo$, is defined by finiteness of the following expression,

$$
\|f\|_{bmo} = \sup_{x \in \mathbb{R}^3, 0 < r < \delta} \Omega(f, I(x, r)) + \sup_{x \in \mathbb{R}^3, r \geq \delta} \frac{1}{|I(x, r)|} \int_{I(x, r)} |f(y)| \, dy,
$$

for some positive $\delta$.

If $f \in L^1$, we can focus on small scales, e.g., $0 < r < \frac{1}{2}$. Let $\phi$ be a positive, non-decreasing function on $(0, \frac{1}{2})$, and consider the following version of local weighted spaces of bounded mean oscillations,

$$
\|f\|_{\widetilde{bmo}_{\phi}} = \|f\|_{L^1} + \sup_{x \in \mathbb{R}^3, 0 < r < \frac{1}{2}} \frac{\Omega(f, I(x, r))}{\phi(r)}
$$

(cf. [12]).

Of special interest will be the spaces $\widetilde{bmo} = \widetilde{bmo}_1$, and $\widetilde{bmo}_{\frac{1}{\log r}}$.

$H^1 - BMO$ and $h^1 - bmo$ duality (Fefferman [6, 7], Goldberg [8])

$(H^1)^* = BMO$ and $(h^1)^* = bmo$; the duality is realized via integration of one object against the other.

Pointwise multipliers in $\widetilde{bmo}$ ([11, 13, 12])

A sharp pointwise multiplier theorem. ([12]) Let $h$ be in $\widetilde{bmo}$, and $g$ in $L^\infty \cap \widetilde{bmo}_{\frac{1}{\log r}}$. Then,

$$
\|g h\|_{\widetilde{bmo}} \leq c(n) \left( \|g\|_{\infty} + \|g\|_{\widetilde{bmo}_{\frac{1}{\log r}}} \right) \|h\|_{\widetilde{bmo}}.
$$

More precisely, the space of pointwise $\widetilde{bmo}$ multipliers coincides with $L^\infty \cap \widetilde{bmo}_{\frac{1}{\log r}}$. 


Coifman and Rochberg’s estimate on $\|\log Mf\|_{BMO}$ (3)

Let $M$ denote the Hardy-Littlewood maximal operator. Coifman and Rochberg [3] obtained a characterization of $BMO$ in terms of images of the logarithm of the maximal function of non-negative locally integrable functions (plus a bounded part). The main ingredient in demonstrating one direction is the following estimate,

$$\|\log Mf\|_{BMO} \leq c(n),$$

for any locally integrable function $f$. (The bound is completely independent of $f$.)

This estimate remains valid if we replace $Mf$ with $Mf = (M\sqrt{|f|})^2$ (cf. [10]); the advantage of working with $M$ is that the $L^2$-maximal theorem implies the following estimate,

$$(2) \quad \|Mf\|_1 \leq c(n)\|f\|_1,$$

a bound that does not hold for the original maximal operator $M$.

3. Setting and the Result

Consider a weak (distributional) Leray solution $u$ on $\mathbb{R}^3$. The vorticity analogue of the Leray’s a priori bound on the energy was presented in [4]: assuming that the initial vorticity is in $L^1$ (or, more generally, a bounded measure), the $L^1$-norm of the vorticity remains bounded on any finite time-interval.

Our goal is to obtain a spatially localized $L\log L$ bound on the vorticity under a suitable assumption on the structural blow-up of the vorticity direction $\xi = \frac{\omega}{|\omega|}$.

Fix a spatial ball $B(0, R_0)$, and consider a test function $\psi$ supported in $B = B(0, 2R_0)$ such that $\psi = 1$ on $B(0, R_0)$, and $|\nabla \psi(x)| \leq c\frac{1}{R_0}\psi^\delta(x)$ for some $\delta > 0$.

Let $w = \sqrt{1 + |\omega|^2}$. We aim to control the evolution of $\psi w \log w$; by the Stein’s lemma [15], this is essentially equivalent to controlling the $L^1$-norm of $Mw$.

For simplicity of the exposition, we assume that the initial vorticity is also in $L^2$, and that $T > 0$ is the first (possible) blow-up time. This way, the solution in view is smooth on $(0, T)$, and we can focus on obtaining a $\sup_{t \in (0, T)}$-bound. Alternatively, one can employ the retarded mollifiers.

**Theorem 1.** Let $u$ be a Leray solution to the 3D NSE. Assume that the initial vorticity $\omega_0$ is in $L^1 \cap L^2$, and that $T > 0$ is the first (possible) blow-up time. Suppose that

$$\sup_{t \in (0, T)} \|(\psi \xi)(\cdot, t)\|_{bmo} \frac{1}{|\log r|} < \infty.$$

Then,

$$\sup_{t \in (0, T)} \int \psi(x) w(x, t) \log w(x, t) \, dx < \infty.$$

**Remark 2.** Since $\omega_0$ is in $L^1$, in addition to the Leray’s a priori bounds on $u$,

$$\sup_t \|u(\cdot, t)\|_{L^2} < \infty \quad \text{and} \quad \int_t^T \int_x |\nabla u(x, t)|^2 \, dx \, dt < \infty,$$
the following *a priori* bounds on $\omega$ are also at our disposal \cite{4},

$$
sup_t \| \omega(\cdot, t) \|_{L^1} < \infty \quad \text{and} \quad \int_t^\infty \int_x |\nabla \omega(x, t)|^{4/(3+\epsilon)} \, dx \, dt < \infty.
$$

**Proof.** Setting $q(y) = \sqrt{1 + |y|^2}$, the evolution of $w = \sqrt{1 + |\omega|^2}$ satisfies the following partial differential inequality \cite{4},

\begin{equation}
\partial_t w - \Delta w + (u \cdot \nabla)w \leq \omega \cdot \nabla u \cdot \frac{\omega}{w}.
\end{equation}

Since our goal is to control the evolution of $\psi w \log w$, it will prove convenient to multiply (3) by $\psi (1 + \log w)$. Here is the calculus corresponding to each of the four terms.

**time-derivative**

$$
\partial_t w \times \psi (1 + \log w) = \partial_t (\psi w \log w).
$$

**Laplacian**

$$
-\Delta w \times \psi (1 + \log w) \\
= -\Delta (\psi w \log w) + \Delta \psi w \log w \\
+ \psi \frac{1}{w} \sum_i (\partial_i w)^2 + 2 \sum_i \partial_i \psi \partial_i w (1 + \log w).
$$

**advection**

$$
(u \cdot \nabla)w \times \psi (1 + \log w) \\
= \sum_i u_i \partial_i w \psi (1 + \log w) \\
= \sum_i (\partial_i (u_i w \psi) (1 + \log w) - u_i w \partial_i \psi (1 + \log w) - u_i \psi \partial_i w) \\
= \sum_i (\partial_i (u_i w \psi) (1 + \log w) - u_i w \partial_i \psi (1 + \log w) - \partial_i (u_i \psi w) + (u_i \partial_i \psi w)).
$$

**vortex-stretching**

$$
\omega \cdot \nabla u \cdot \frac{\omega}{w} \times \psi (1 + \log w) = \omega \cdot \nabla u \cdot \psi \frac{\omega}{|\omega|} (1 + \log w) + \omega \cdot \nabla u \cdot \psi \left( \frac{\omega}{w} - \frac{\omega}{|\omega|} \right) (1 + \log w).
$$
Integrating over the space-time, the above representation yields (dropping the zero and the positive terms, and estimating the remaining terms in the straightforward fashion via Hölder and Sobolev),

\[
I(\tau) \equiv \int \psi(x) w(x, \tau) \log w(x, \tau) \, dx \leq I(0) + c \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log w \, dx \, dt + a \text{ priori} \text{ bounded,}
\]

for any \( \tau \) in \([0, T)\).

In order to take the advantage of the Coifman-Rochberg’s estimate, we decompose the logarithmic factor as

\[
\log w = \log \frac{w}{\mathcal{M}w} + \log \mathcal{M}w.
\]

Denoting \( \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log w \, dx \, dt \) by \( J \), this yields \( J = J_1 + J_2 \) where

\[
J_1 = \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log \frac{w}{\mathcal{M}w} \, dx \, dt
\]

and

\[
J_2 = \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log \mathcal{M}w \, dx \, dt.
\]

For \( J_1 \), we use the pointwise inequality

\[
w \log \frac{w}{\mathcal{M}w} \leq \mathcal{M}w - w
\]

(a consequence of the pointwise inequality \( \mathcal{M}f \geq f \), and the inequality \( e^{x-1} \geq x \) for \( x \geq 1 \)).

This leads to

\[
J_1 \leq \int_0^\tau \int_x |\nabla u| \left( \mathcal{M}w - w \right) \psi \, dx \, dt
\]

which is \textit{a priori} bounded by the Cauchy-Schwarz and the \( L^2 \)-maximal theorem.

For \( J_2 \), we have the following string of inequalities,
\begin{align*}
J_2 & \leq c \int_0^T \| \omega \cdot \nabla u \|_{b^1} \| \psi \xi \log Mw \|_{b^{\infty}} \, dt \\
& \leq c \int_0^T \| \omega \cdot \nabla u \|_{B^{1}} \| \psi \xi \log Mw \|_{b^{\infty}} \, dt \\
& \leq c \int_0^T \| \omega \|_2 \| \nabla u \|_2 \left( \| \psi \xi \|_{\infty} + \| \psi \xi \|_{b^{\infty}} \frac{1}{|\log r|} \right) \left( \| \log Mw \|_{B^{0}} + \| \log Mw \|_{1} \right) \, dt \\
& \leq c \sup_{t \in (0, T)} \left\{ 1 + \| \psi \xi \|_{b^{\infty}} \frac{1}{|\log r|} \right\} \left( \| \log Mw \|_{B^{0}} + \| \log Mw \|_{1} \right) \int_t^\infty \int_x |\nabla u|^2 \, dx \, dt \\
& \leq c \left( 1 + \sup_{t \in (0, T)} \| \omega \|_1 \right) \left( 1 + \sup_{t \in (0, T)} \| \psi \xi \|_{b^{\infty}} \frac{1}{|\log r|} \right) \int_t^\infty \int_x |\nabla u|^2 \, dx \, dt
\end{align*}

by $b^1 - b^{\infty}$ duality, the Div-Curl Lemma, the pointwise $b^{\infty}$-multiplier theorem, the Coifman-Rochberg’s estimate, and the bound (2) combined with a couple of elementary inequalities. This completes the proof of the $L \log L$ estimate.

\[ \square \]

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