Clauser-Horne-Bell inequality for three three-dimensional systems

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In this paper we show a Bell inequality of Clauser-Horne type for three three-dimensional systems (qudits). Violation of the inequality by quantum mechanics is shown for the case in which each of the three observers measures two non-commuting observables, defined by the so called unbiased symmetric six port beamsplitters, on a maximally entangled state of three qudits. The strength of the violation of the inequality agrees with the numerical results presented in Kaszlikowski et. al., quant-ph//0202019.

PACS numbers:
Local realistic (classical) description of the above experiment is equivalent to the existence of a joint probability distribution from which the quantum probabilities \( W_{QM}(a^i_1, b^j_m, c^k_n) \) can be derived as the marginals. Let us denote this hypothetical joint probability distribution by \( W_{LR}(a^i_1, b^j_m, c^k_n) \). Thus, a local realistic description of the experiment exists if and only if the following marginals

\[
W_{LR}(a^i_1, b^j_m, c^k_n) = \sum_{l+i=1}^{3} \sum_{m+j=1}^{3} \sum_{n+k=1}^{3} W_{LR}(a^i_1, a^j_2, b^k_m, c^l_n, c^{l'}_n, c^{l''}_n) \tag{2}
\]

are equal to the quantum probabilities, i.e.,

\[
W_{LR}(a^i_1, b^j_m, c^k_n) = W_{QM}(a^i_1, b^j_m, c^k_n)
\]

where the addition on the indices is computed using modulo 2 arithmetics. Owing to (2), \( W_{LR}(a^i_1, b^j_m, c^k_n) \) must obey the following inequality

\[
-\Gamma_{221} - \Gamma_{111} + 2\Gamma_{122} + \Gamma_{121} - \Gamma_{212} + \Gamma_{211} + \Gamma_{222} + \Gamma_{112} \leq 3, \tag{3}
\]

and

\[
\Gamma_{ijk} = \sum_{l+m+n=1 \mod 3} W_{LR}(a^i_1, b^j_m, c^k_n),
\]

\[
\Gamma'_{i'j'k'} = \sum_{l+m+n=0 \mod 3} W_{LR}(a^{i'}_1, b^{j'}_m, c^{k'}_n),
\]

\[
\Gamma''_{i''j''k''} = \sum_{l+m+n=2 \mod 3} W_{LR}(a^{i''}_1, b^{j''}_m, c^{k''}_n), \tag{4}
\]

and \((i,j,k) = (221,111,122), (i',j',k') = (121,212), (i'',j'',k'') = (211,222,112)\). This is the Clauser-Horne-Bell inequality for three qutrits. It must be obeyed by any local realistic theory that claims to reproduce the correlations generated by three qutrits.

To prove the inequality in (4), we first replace the marginals in the left hand side of the inequality in (4) by the appropriate sums of joint probabilities given in (4). Naturally, we get an extremely long expression (this is why we do not show it here) in which the joint probabilities \( W_{LR}(a^i_1, a^j_2, b^k_m, b^{l'}_m, b^{l''}_m, c^l_n, c^{l'}_n, c^{l''}_n) \) appear only with coefficients -3, 0 or 3 and nothing else. Since the sum of all joint probabilities adds to one, i.e.,

\[
\sum_{a^i_1,a^j_2,b^k_m,b^{l'}_m,b^{l''}_m,c^l_n,c^{l'}_n,c^{l''}_n} W_{LR}(a^i_1, a^j_2, b^k_m, b^{l'}_m, b^{l''}_m, c^l_n, c^{l'}_n, c^{l''}_n) = 1,
\]

it follows immediately that the entire expression is less than or equal to 3.

We should stress at this point that the above inequality is a member of the set of inequalities that can obtained from (4) by permutations of indices enumerating the outcomes of the measurements as well as the permutations of indices enumerating the observables.

Consider a gedanken experiment in which Alice, Bob and Celine measure observables defined by unbiased symmetric six-port beamsplitters \(^8\) on the maximally entangled state of three qutrits \(|\psi\rangle = \frac{1}{\sqrt{3}}(|111\rangle + |222\rangle + |333\rangle\).

The unbiased symmetric six-port beamsplitters is an optical device with three input and three output ports. In front of every input port there is a phase shifter that changes the phase of the photon entering the given port. If a phase shifter in some input port is set to zero and a photon enters the device through this port then it has an equal chance of leaving the device through any output port. The phase shifters can be changed by the observers; they represent the local macroscopic parameters available to the observers.

The matrix elements of an unbiased symmetric six-port beamsplitter are given by \( U_{kl}(\bar{\phi}) = \frac{1}{\sqrt{3}} |\alpha^{(k-l)(l-1)} \rangle \exp(i\pi) \langle \bar{\phi}| \), where \( \bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \) and \( \bar{\phi}_l \) \((k, l = 1, 2, 3)\) are the settings of the appropriate phase shifters (for convenience we denote them as a three dimensional vector \( \bar{\phi} \) and \( \alpha = \exp(\frac{2\pi}{3}) \).

The observables measured by Alice, Bob and Celine are now defined as follows. The set of projectors for Alice’s \( i \)-th experiment is given by \( P^i_l = U^i_A(\bar{\phi}_i)\langle l|\langle l|U^i_A(\bar{\phi}_i) \) \((l = 1, 2, 3)\), where \( U^i_A(\bar{\phi}_i) \) is the matrix of Alice’s unbiased symmetric six-port beamsplitter defined by the set of phases \( \bar{\phi}_i = (\bar{\phi}_1^i, \bar{\phi}_2^i, \bar{\phi}_3^i) \). Bob’s set of projectors \( j \)-th experiment is given by \( Q^j_m = U^j_B(\bar{\psi}_j)|m\rangle\langle m|U^j_B(\bar{\psi}_j) \) where \( \bar{\psi}_j = (\bar{\psi}_1^j, \bar{\psi}_2^j, \bar{\psi}_3^j) \) is a set of Bob’s phases defining his unbiased symmetric six-port beamsplitter, whereas Celine’s projectors in the \( k \)-th experiment is given by \( R^k_n = U^k_C(\bar{\delta}_k)|n\rangle\langle n|U^k_C(\bar{\delta}_k) \) where \( \bar{\delta}_k = (\bar{\delta}_1^k, \bar{\delta}_2^k, \bar{\delta}_3^k) \) is a set of Celine’s phases defining her unbiased symmetric six-port beamsplitter.
To each result of the measurement of the projectors $P_{i}^{a}, P_{j}^{b}, P_{k}^{c}$ for any $i,j,k$ we assign the complex numbers $\alpha^{n}$ ($n = 1, 2, 3$), namely $a_{1}^{i}, a_{2}^{i}, b_{1}^{j}, \ldots$ have been assigned the values $\alpha^{1}, \alpha^{2}, \alpha^{m_{1}}, \ldots$ respectively. This special assignment was first used in Ref. to generalize the Bell experiment for higher dimensions.

In this way, the probability of getting the set of three numbers $(a_{1}^{i}, b_{1}^{j}, c_{nk}^{k})$ can now be computed using the formula (1). Because we are not going to use the explicit form of these probabilities, interested readers are kindly referred to Ref. . However, note that we need to use the following property regarding these probabilities. All the probabilities $W_{QM}(a_{1}^{i}, b_{1}^{j}, c_{nk}^{k})$ can be sorted into three groups consisting of nine equal probabilities. The first group consists of the probabilities for which $l_{i} + m_{j} + n_{k} = 1 \mod 3$, the second one consists of the probabilities for which $l_{i} + m_{j} + n_{k} = 2 \mod 3$ and the third one consists of the probabilities for which $l_{i} + m_{j} + n_{k} = 0 \mod 3$. Let us denote each probability (they are equal, so it suffices to take an arbitrary one as a representative of the whole group) from the first group by $W_{1}^{1}(ijk)$, from the second one by $W_{2}^{2}(ijk)$ and from the third one by $W_{3}^{3}(ijk)$. It is obvious that we have $W_{1}^{1}(ijk) + W_{2}^{2}(ijk) + W_{3}^{3}(ijk) = \frac{1}{3}$ for any triple $i, j, k$.

Let us now define the following correlation function (for details see ) for each triple of experiments that we denote by $Q_{ijk}$

$$Q_{ijk} = \sum_{l_{i}, m_{j}, n_{k}=1}^{3} \alpha^{l_{i}+m_{j}+n_{k}}W_{QM}(a_{l_{i}}^{i}, b_{m_{j}}^{j}, c_{n_{k}}^{k}).$$

Using the explicit form of the probabilities, it can be shown easily that such correlation function acquires the following symmetric form

$$W_{1}^{3}(ijk) = \frac{1}{9}W_{1}^{1}(ijk) - W_{2}^{2}(ijk).$$

Putting the probabilities expressed by the equations into the Clauser-Horne-Bell inequality, we obtain the following inequality (which is totally equivalent to (1) in the case considered here)

$$\left|\Re\{Q_{121} - Q_{212} + \alpha(Q_{112} + Q_{221} + Q_{222}) + \alpha^{2}(2Q_{122} - Q_{111} - Q_{221})\}\right| \leq 3$$

We will show that this inequality is not satisfied by quantum mechanics for appropriate choice of the phase shifts for Alice, Bob and Celine. The phase shifts are as follows $\phi_{1} = (0, 0, \frac{2\pi}{3}), \phi_{2} = (0, 0, 0), \psi_{1} = (0, 0, \pi), \psi_{2} = (0, 0, \frac{2\pi}{3}), \delta_{1} = (0, \frac{\pi}{3}, 0), \delta_{2} = (0, 0, \pi), 0$. The values of the correlation function computed using the above phase shifts read $Q_{111} = \frac{1}{3}(1 + \alpha^{2}), Q_{112} = \frac{1}{3}\alpha^{2}, Q_{121} = \frac{2}{3}, Q_{122} = -\frac{2}{3}(1 + \alpha^{2}), Q_{211} = \frac{2}{3}\alpha^{2}, Q_{221} = -\frac{1}{3}, Q_{222} = \frac{1}{3}\alpha^{2}$. Putting them into the left hand side of the inequality in we arrive at a violation of the inequality in which the left hand side is equal to 5.

In Ref. , a proposal was made to measure the strength of violation of local realism by the minimal amount of noise that must be added to the system in order to hide the non-classical character of the observed correlations. This is equivalent to a replacement of the pure state $|\psi\rangle\langle\psi|$ by the mixed state $\rho(F)$ of the form $\rho(F) = (1 - F)|\psi\rangle\langle\psi| + F I \otimes I \otimes I$, where $I$ is an identity matrix and where $F$ $(0 \leq F \leq 1)$ is the amount of noise present in the system.
It can be checked immediately that such addition of the noise in the gedenken experiment considered here changes the correlation function $Q_{ijk}$ to $Q_{ijk}^F = (1 - F)Q_{ijk}$. Therefore, the minimal amount of noise $F_{\text{min}}$ that must be added to the system to conceal the non-classicality of quantum correlations is $F_{\text{min}} = \frac{1}{10}$, which is consistent with the numerical results presented in Ref. [5].

In conclusion, we have found the Clauser-Horne-Bell type inequality (we name it in this way because the inequality involves probabilities and not a correlation function of some sort) for three qutrits that gives us the necessary conditions for the existence of a local realistic (classical) description of quantum correlations. We have shown that the violation of this inequality for the gedenken experiment with the maximally entangled state in which two observables defined by unbiased symmetric six-port beamsplitters are measured at three sites. The strength of the violation agrees with the numerical results obtained in Ref. [5]. Moreover, the numerical method presented in Ref. [4] gives necessary and sufficient conditions for the violation of local realism which strongly suggests that the inequality found has similar property.

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