THE SQUARE INTEGRABLE REPRESENTATIONS ON GENERALIZED WEYL-HEISENBERG GROUPS

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ABSTRACT. This paper presents the square integrable representations of generalized Weyl-Heisenberg group. We investigate the quasi regular representation of generalized Weyl-Heisenberg group. Moreover, we obtain a concrete form for admissible vector of this representation. Finally, we provide some examples to support our technical considerations.

1. INTRODUCTION

Wavelet transform has rich theoretical structures and is extremely useful as tools for building signal transforms, adapted to various signal geometries, quantum mechanics, etc. Continuous wavelet transform admits a generalization to locally compact groups. Such a unified approach seems to be useful, since it emphasizes on a clear way to basic features of continuous wavelet transform and includes all important cases for applications [2, 3, 5]. It should be mentioned that, the Weyl Heisenberg group plays a significant designations in various aspects of the connections between the classical harmonic analysis and concrete applications of numerical harmonic analysis.

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This paper contains 4 sections. Section 2 includes the definition of semi-direct product of two locally compact groups and generalized Weyl Heisenberg group. In Section 3, we study the square integrable representations on the generalized Weyl Heisenberg group and then we obtain the necessary and sufficient conditions for admissible wavelet on this group. In Section 4, some examples are proved as application of our results.

2. Preliminaries and notation

Let $H$ and $K$ be two locally compact groups with the identity elements $e_H$ and $e_K$, respectively and let $\tau : H \rightarrow Aut(K)$ be a homomorphism such that the map $(h, k) \mapsto \tau_h(k)$ is continuous from $H \times K$ onto $K$, where $H \times K$ eips with the product topology. The semi-direct product topological group $G_\tau = H \rtimes \tau K$ is the locally compact topological space $H \times K$ under the product topology, with the group operations:

$$(h_1, k_1) \times_{\tau} (h_2, k_2) = (h_1 h_2, k_1 \tau_{h_1}(k_2)),$$

$$(h, k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1})).$$

It is worth to note that $K_1 = \{(e_H, k); k \in K\}$ is a closed normal subgroup and $H_1 = \{(h, e_K); h \in H\}$ is a closed subgroup of $G_\tau$ such that $G_\tau = HK$. Moreover, the left Haar measure of the locally compact group $G_\tau$ is

$$d\mu_{G_\tau}(h, k) = \delta_H(h)d\mu_H(h)d\mu_K(k),$$

in which $d\mu_H, d\mu_K$ are the left Haar measures on $H$ and $K$, respectively and $\delta_H : H \rightarrow (0, \infty)$ is a positive continuous homomorphism that satisfies

$$d\mu_K(k) = \delta_H(h)d\mu(\tau_h(k)).$$
for $h \in H, k \in K$. Moreover, the modular function $\Delta_{G_\tau}$ is

$$\Delta_{G_\tau} = \delta_H(h)\Delta_H(h)\Delta_K(k),$$

where $\Delta_H, \Delta_K$ are the modular functions of $H, K$, respectively.

When $K$ is also abelian, one can define $\hat{\tau} : H \to Aut(\hat{K})$ via $h \mapsto \hat{\tau}_h$ where

$$\hat{\tau}_h(\omega) = \omega \circ \tau_{h^{-1}},$$

for all $\omega \in \hat{K}$. We usually denote $\omega \circ \tau_{h^{-1}}$ by $\omega_h$. With this notation, it is easy to see

$$\omega_{h_1h_2} = (\omega_{h_2})_{h_1},$$

where $h_1, h_2 \in H$ and $\omega \in \hat{K}$. The semi-direct product $G_\tau = H \times_\tau \hat{K}$ is a locally compact group with the left Haar measure

$$d\mu_{\hat{G}}(h, \omega) = \delta_H(h)^{-1}d\mu_H(h)d\mu_{\hat{K}}(\omega),$$

where $d\mu_{\hat{K}}$ is the Haar measure on $\hat{K}$. Also, for all $h \in H$,

$$d\mu_{\hat{K}}(\omega_h) = \delta_H(h)d\mu_{\hat{K}}(\omega),$$

for $\omega \in \hat{K}$, (see more details in [4, 1, 3].)

Let $G_\tau = H \times_\tau K$, and define $\theta : G_\tau \to Aut(\hat{K} \times \mathbb{T})$ via

$$(h, k) \mapsto \theta_{(h,k)}(\omega, z) = (\hat{\tau}_h(\omega), \hat{\tau}_h(\omega)(k)z) = (\omega_h, \omega_h(k)z),$$

for all $(h, k) \in H \times_\tau K$ and $(\omega, z) \in \hat{K} \times \mathbb{T}$. The mapping $\theta$ is a continuous homomorphism. Thus the semi-direct product

$$G_\tau \times_\theta (\hat{K} \times \mathbb{T}) = (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T}),$$

is a locally compact group and it is called the generalized Weyl Heisenberg group associated with the semi direct product group $G_\tau = H \times_\tau K$, and
denoted by $\mathbb{H}(G_\tau)$. It is easy to see that the group operations of $\mathbb{H}(G_\tau)$ are

$$(h_1, k_1, \omega_1, z_1).(h_2, k_2, \omega_2, z_2) = (h_1h_2, k_1\tau_{h_1}(k_2), \omega_1\omega_{2h_1}, \omega_{2h_1}(k)z_1z_2),$$

$$(h_1, k_1, \omega_1, z_1)^{-1} = (h_1^{-1}, \tau_{h_1}^{-1}(k^{-1}), \omega_{h_1}^{-1}, \omega_{h_1}^{-1}(\tau_{h_1}^{-1}(k^{-1}))z^{-1}),$$

for $(h_1, k_1, \omega_1, z_1), (h_2, k_2, \omega_2, z_2) \in \mathbb{H}(G_\tau)$ (see [4]) and the left Haar measure of $\mathbb{H}(G_\tau)$ is:

$$d\mu_{\mathbb{H}(G_\tau)}(h, k, \omega, z) = d\mu_H(h)d\mu_K(k)d\mu_{\hat{K}}(\omega)d\mu_{\mathbb{T}}(z).$$

### 3. The Square Integrable Representation of $\mathbb{H}(G_\tau)$

Throughout this section, we assume that $H$ and $K$ are locally compact topological groups and that $K$ is abelian, too. We denote the left Haar measures of $H$ and $K$ by $d\mu_H, d\mu_K$, respectively. Suppose that $h \mapsto \tau_h$ from $H$ to $\text{Aut}(K)$ is a homomorphism such that $(h, k) \mapsto \tau_h(k)$ from $H \times K$ into $K$ is continuous. $G_\tau = H \times_\tau K$ is the semi-direct product of $H$ and $K$ that is a locally compact topology group with the left Haar measure $d\mu_{G_\tau}(h, k) = \delta_H(h)d\mu_H((h)d\mu_K(k)$, where $\delta_H : H \mapsto (0, \infty)$ is a continuous homomorphism. Consider the homomorphism $\theta : G_\tau \rightarrow \text{Aut}(\hat{K} \times \mathbb{T})$ is defined by

$$(((h, k), (\omega, z)) \mapsto \theta_{(h, k)}(\omega, z),$$

where $\theta_{(h, k)}(\omega, z) = (\omega \circ \tau_{h^{-1}}, \omega \circ \tau_{h^{-1}}(k).z)$. This makes $\mathbb{H}(G_\tau) = G_\tau \times_\theta (\hat{K} \times \mathbb{T})$ a locally compact topological group where $\mathbb{H}(G_\tau)$ is equipped with the product topology and the group operations as

$$(h_1, k_1, \omega_1, z_1).(h_2, k_2, \omega_2, z_2) = (h_1h_2, k_1\tau_{h_1}(k_2), \omega_1\omega_{2h_1}, \omega_{2h_1}(k)z_1z_2),$$

$$(h_1, k_1, \omega_1, z_1)^{-1} = (h_1^{-1}, \tau_{h_1}^{-1}(k^{-1}), \omega_{h_1}^{-1}, \omega_{h_1}^{-1}(\tau_{h_1}^{-1}(k^{-1}))z^{-1}),$$
for \((h_1, k_1, \omega_1, z_1), (h_2, k_2, \omega_2, z_2) \in \mathbb{H}(G_r)\). The left Haar measure of \(\mathbb{H}(G_r)\) is

\[
d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z) = d\mu_H(h)d\mu_K(k)d\mu_\tau(\omega)d\mu_T(z).
\]

Now, we are going to define a square integrable representation on \(\mathbb{H}(G_r)\). With the above notations define \(\pi: \mathbb{H}(G_r) \to U(L^2(\hat{K}))\) by

\[
(3.1) \quad \pi(h, k, \omega, z)f(\xi) = \delta^{-1/2}_H(h)z\xi(k)\overline{\omega(k)f((\xi \omega)_{h^{-1}})},
\]

then \(\pi\) is a homomorphism. Indeed,

\[
\pi((h_1, k_1, \omega_1, z_1)(h_2, k_2, \omega_2, z_2))f(\xi) = \pi(h_1 h_2, k_1 \tau_{h_1}(k_2), \omega_1(\omega_2)_{h_1}, (\omega_2)_{h_1}(k_1)z_1 z_2)f(\xi)
\]

\[
= \delta^{-1/2}_H(h_1 h_2)(\omega_2)_{h_1}(k_1)z_1 z_2 \xi(k_1 \tau_{h_1}(k_2))\overline{\omega_1(\omega_2)_{h_1}(k_1 \tau_{h_1}(k_2))f((\xi \omega)_{h_1}(\omega_1)_{h_2^{-1}})}
\]

\[
= \delta^{-1/2}_H(h_1 h_2)(\omega_2)_{h_1}(k_1)z_1 z_2 \xi(k_1)\xi_{h_1^{-1}}(k_2)\overline{\omega_1(\omega_1)_{h_2^{-1}}(k_2)\omega_2(k_2)f(\xi_{h_2^{-1}}(\omega_1)_{h_2^{-1}})}
\]

Also,

\[
\pi(h_1, k_1, \omega_1, z_1)\pi(h_2, k_2, \omega_2, z_2)f(\xi)
\]

\[
= \delta^{-1/2}_H(h_1)z_1 \xi(k_1)\overline{\omega_1(k_1)\pi(h_2, k_2, \omega_2, z_2)f((\xi \omega_{h_1}^{-1})_{h_2}^{-1})}
\]

\[
= \delta^{-1/2}_H(h_1)\delta^{-1/2}_H(h_2)z_1 z_2 \xi(k_1)\overline{\omega_1(k_1)\omega_2(k_2)(\xi \omega_{h_1}^{-1})(k_2)f((\xi \omega_{h_1}^{-1} \omega_{h_2}^{-1})_{h_2}^{-1})}
\]

\[
= \delta^{-1/2}_H(h_1 h_2)z_1 z_2 \xi(k_1)\xi_{h_1^{-1}}(k_2)\overline{\omega_1(k_1)\omega_2(k_2)f(\xi_{h_2^{-1}})(\omega_{h_2}^{-1})_{h_2}^{-1}}
\]

Moreover, \(\pi\) is unitary. In fact we have,

\[
\|\pi(h, k, \omega, z)f\|^2_2 = \int_{\hat{K}} |\pi(h, k, \omega, z)f(\xi)|^2d\mu_\hat{K}(\xi)
\]

\[
= \int_{\hat{K}} \delta^{-1}_H(h)|f((\xi \omega)_{h^{-1}})|^2d\mu_\hat{K}(\xi)
\]

\[
= \int_{\hat{K}} \delta^{-1}_H(h)|f((\xi)_{h^{-1}}|^2d\mu_\hat{K}(\xi)
\]

\[
= \|f\|^2_2.
\]

And it is easy to check that \(\pi\) is continuous and onto. So, \(\pi\) is a continuous unitary representation of group \(\mathbb{H}(G_r)\) to the Hilbert space \(L^2(\hat{K})\). In the
sequel, we show that \( \pi \) is irreducible when \( H \) is compact. Furthermore, it is also shown that \( \pi \) is square integrable if and only if \( H \) is compact. Note that when \( H \) is a compact group, we normalize the Haar measure \( \mu_H \) such that \( \mu_H(H) = 1 \).

**Theorem 3.1.** Let \( \mathbb{H}(G_{\tau}) = (H \times_{\tau} K) \times_{\theta} (\hat{K} \times \mathbb{T}) \) where \( H \) is a locally compact group and \( K \) is a locally compact abelian group. Then for \( \varphi, \psi \) in \( L^2(\hat{K}) \),

\[
(3.2) \quad \int_{\mathbb{H}(G_{\tau})} | \left< \varphi, \pi(h, k, \omega, z) \psi \right> |^2 d\mu_{\mathbb{H}(G_{\tau})}(h, k, \omega, z) = \| \varphi \|_2^2 \| \psi \|_2^2.
\]

if and only if \( H \) is compact.
Proof. For $\varphi, \psi$ in $L^2(\hat{K})$ we first consider the following observations:

$$\int_{G(G_r)} |< \varphi, \pi(h, k, \omega, z) \psi > |^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

$$= \int_{H(G_r)} | \int_{K} \varphi(\xi) \pi(h, k, \omega, z) \psi(\xi) d\mu_{\mathbb{H}(G_r)}(\xi)|^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

$$= \int_{H(G_r)} | \int_{K} \varphi(\xi) \delta_{H}^{-1/2}(h) \overline{z\xi(k) \omega(k) \psi(\xi) \omega^{-1}} d\mu_{\mathbb{H}(G_r)}(\xi)|^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

$$= \int_{H(G_r)} | \int_{K} \varphi(\xi) \delta_{H}^{-1/2}(h) \overline{z\xi(k) \psi(\xi) \omega^{-1}} d\mu_{\mathbb{H}(G_r)}(\xi)|^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

$$= \int_{H(G_r)} | \int_{K} R_{\omega} \varphi(\xi) \delta_{H}^{-1/2}(h) \overline{z\xi(k) \psi(\xi) \omega^{-1}} d\mu_{\mathbb{H}(G_r)}(\xi)|^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

$$= \int_{H(G_r)} \delta_{H}(h) | \int_{K} R_{\omega} \varphi(\xi) \delta_{H}^{-1/2}(h) \overline{z\xi(k) \psi(\xi) \omega^{-1}} d\mu_{\mathbb{H}(G_r)}(\xi)|^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

$$= \int_{H(G_r)} \delta_{H}(h) | \int_{K} |(R_{\omega} \varphi(\xi) \delta_{H}^{-1/2}(h) \overline{z\xi(k) \psi(\xi) \omega^{-1}})|^2 d\mu_{\mathbb{H}(G_r)}(\xi)|^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

$$= \int_{H(G_r)} \delta_{H}(h) | \int_{K} |(R_{\omega} \varphi(\xi) \delta_{H}^{-1/2}(h) \overline{z\xi(k) \psi(\xi) \omega^{-1}})|^2 d\mu_{\mathbb{H}(G_r)}(\xi)|^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

$$= \int_{H(G_r)} \delta_{H}(h) | \int_{K} |(R_{\omega} \varphi(\xi) \delta_{H}^{-1/2}(h) \overline{z\xi(k) \psi(\xi) \omega^{-1}})|^2 d\mu_{\mathbb{H}(G_r)}(\xi)|^2 d\mu_{\mathbb{H}(G_r)}(h, k, \omega, z)$$

Now, if $H$ is compact, then $\mu_{H}(H) = 1$. So, (3.2) holds. Conversely, if (3.2) holds, the above observation implies that $\mu_{H}(H) = 1$. So, we can conclude that $H$ is compact. \qed

Corollary 3.2. With notation as above, the representation $\pi$ of $\mathbb{H}(G_r)$ on $L^2(\hat{K})$ is irreducible if $H$ is compact.
Proof. If \( H \) is compact, then (3.2) in Theorem 3.1 holds. Now, suppose that \( M \) is a closed subspace of the Hilbert space \( L^2(\hat{K}) \) that is invariant under \( \pi \). Then for any \( \varphi \in M \) we have,

\[
\{ \pi(h, k, \omega, z)\varphi; (h, k, \omega, z) \in \mathbb{H}(G_{\tau}) \} \subseteq M.
\]

Let \( \psi \in L^2(\hat{K}) \) be orthogonal to \( M \), that is \( \langle \psi, \pi(h, k, \omega, z)\varphi \rangle = 0 \), for all \( (h, k, \omega, z) \in \mathbb{H}(G_{\tau}) \). Thus by (3.2), \( \|\varphi\|_2\|\psi\|_2 = 0 \), and hence \( \psi = 0 \). So, \( M^\perp = \{0\} \), that is, \( M = L^2(\hat{K}) \). Namely, \( \pi \) is irreducible. \( \square \)

We remind the reader that, an irreducible representation \( \pi \) of \( \mathbb{H}(G_{\tau}) \) on \( L^2(\hat{K}) \) is called square integrable if there exists a non zero element \( \psi \) in \( L^2(\hat{K}) \) such that

(3.3) \( \langle \pi(.,.,.,.)\psi, f \rangle \in L^2(\mathbb{H}(G_{\tau})) \),

for all \( f \in L^2(\hat{K}) \). A unit vector \( \psi \) satisfying (3.3) is said to be an admissible wavelet for \( \pi \), and the constant

\[
c_\psi = \int_{\mathbb{H}(G_{\tau})} |\langle \pi(h, k, \omega, z)\psi, \psi \rangle|^2 d\mu_{\mathbb{H}(G_{\tau})},
\]

is called the wavelet constant associated to the admissible wavelet \( \psi \).

Also, for the wavelet vector \( \psi \), the continuous wavelet transform is defined by

\[
W_\psi f(h, k, \omega, z) = \langle f, \pi(h, k, \omega, z)\psi \rangle.
\]

It is easy to see that \( (h, k, \omega, z) \mapsto W_\psi f(h, k, \omega, z) \) is a continuous function on \( \mathbb{H}(G_{\tau}) \). Moreover, \( W_\psi \) intertwines \( \pi \) and the left regular representation on \( \mathbb{H}(G_{\tau}) \).

**Corollary 3.3.** The representation \( \pi \) of the GWH group \( \mathbb{H}(G_{\tau}) = (H \times_{\tau} K) \ltimes_\theta (\hat{K} \times \mathbb{T}) \) on \( L^2(\hat{K}) \) is square integrable if and only if \( H \) is compact.
Proof. If $H$ is compact, then by Theorem 3.1 and Corollary 3.2, $\pi$ is square integrable. For the inverse, if $\pi$ is square integrable, then there exists a non-zero element $\varphi \in L^2(\hat{K})$ such that

$$\langle \pi(\ldots,\cdot)\varphi, \psi \rangle \in L^2(\mathbb{H}(G_\tau)),$$

for all $\psi \in L^2(\hat{K})$. On the other hand,

$$\int_{\mathbb{H}(G_\tau)} |\langle \varphi, \pi(h, k, \omega, z)\psi \rangle|^2 d\mu_{\mathbb{H}(G_\tau)}(h, k, \omega, z) = \|\varphi\|_2^2 \|\psi\|_2^2 \mu_H(H).$$

So $\mu_H(H) < \infty$. That is $H$ is compact. \qed

Remark 3.4. There is another irreducible representation of $\mathbb{H}(G_\tau)$ on Hilbert space $L^2(K)$. Indeed, consider

$$\tilde{\pi} : \mathbb{H}(G_\tau) \to U(L^2(K)), \quad \tilde{\pi}(h, k, \omega, z)f(k') = \delta_H(h)^{1/2}z\omega(k')f(\tau_{h^{-1}}(k'k)),$$

for all $(h, k, \omega, z) \in \mathbb{H}(G_\tau)$, $f \in L^2(K)$. $\tilde{\pi}$ is homomorphism and unitary.

In fact we have

$$\tilde{\pi}((h_1, k_1, \omega_1, z_1)(h_2, k_2, \omega_2, z_2))f(k')$$

$$= \tilde{\pi}(h_1h_2, k_1\tau_{h_1}(k_2), \omega_1(\omega_2)_{h_1}, (\omega_2)_{h_1}(k_1)z_1z_2)f(k')$$

$$= \delta_H^{1/2} (h_1h_2)_{h_1}(k_1)z_1z_2\omega_1(\omega_2)_{h_1}(k')f(\tau_{h_1h_2}^{-1}((k'k)_{h_1}h_1(k_2)))$$

$$= \delta_H^{1/2} (h_1h_2)z_1z_2\omega_1(k')(\omega_2)_{h_1}(k'k_1)f(\tau_{h_2^{-1}h_1}^{-1}(k'k_1)\tau_{h_2}^{-1}(k_2))$$

$$= \delta_H^{1/2} (h_1h_2)z_1z_2\omega_1(k')(\omega_2)_{h_1}(k'k_1)f(\tau_{h_2^{-1}h_1}^{-1}(k'k_1)\tau_{h_2}^{-1}(k_2)),$$

and

$$\tilde{\pi}(h_1, k_1, \omega_1, z_1)\tilde{\pi}(h_2, k_2, \omega_2, z_2)f(k')$$

$$= \delta_H^{1/2} (h_1)z_1\omega_1(k')\tilde{\pi}(h_2, k_2, \omega_2, z_2)f(\tau_{h_1}^{-1}(k'k_1))$$

$$= \delta_H^{1/2} (h_1)\delta_H^{1/2} (h_2)z_1z_2\omega_1(k')\omega_2(\tau_{h_1}^{-1}(k'k_1))f(\tau_{h_2^{-1}h_1}^{-1}(k'k_1)k_2)$$

$$= \delta_H^{1/2} (h_1h_2)z_1z_2\omega_1(k')(\omega_2)_{h_1}(k'k_1)f(\tau_{h_2^{-1}h_1}^{-1}(k'k_1)\tau_{h_2}^{-1}(k_2)).$$

Also,
\[ \| \tilde{\pi}(h, k, \omega, z) f \|_2^2 = \int_K |\tilde{\pi}(h, k, \omega, z) f(k')|^2 d\mu_K(k') \]
\[ = \int_K \delta_H(h) |f(\tau_h^{-1}(k')k)|^2 d\mu_K(k') \]
\[ = \int_K \delta_H(h) |f(k')|^2 d\mu_K(\tau_h(k')) . \]
\[ = \int_K |f(k')|^2 d\mu_K(k') \]
\[ = |f|^2_2. \]

Using the Plancherel theorem, \( \pi, \tilde{\pi} \) are unitarily equivalent. So, \( \tilde{\pi} \) is square integrable if and only if \( \pi \) is square integrable.

**Remark 3.5.** The inverse of Corollary 3.2 does not hold, generally. An obvious example is when \( H \) is a non compact group and \( K \) is the trivial group \( \{e\} \). Then the representation \( \pi : \mathbb{H}(H \times \tau \{e\}) \to U(\mathbb{C}) \) is an irreducible representation. Here we give a non trivial example in which \( \pi \) is an irreducible representation, but \( H \) is not compact. Let \( H = \mathbb{R}^+, K = \mathbb{R} \). Define the representation \( \pi \) of \( \mathbb{H}(\mathbb{R}^+ \times \mathbb{R}) \) as follows:

\[ \pi : \mathbb{H}(\mathbb{R}^+ \times \mathbb{R}) \to U(L^2(\mathbb{R})); \quad \pi(a, x, \omega, z) f(\xi) = a^{1/2} z e^{2\pi i x (\xi - \omega)} f((\xi \omega)_{a^{-1}}), \]

in which \( (\xi \omega)_{a^{-1}} = (\xi \omega) \circ \tau_a, \tau_a(x) = a.x \) and \( \delta_H(a) = a^{-1} \). This representation is irreducible. Indeed, let \( M \) be a closed invariant subspace of \( L^2(\mathbb{R}) \) under \( \pi \). Then for any \( f \in M \), we have \( \pi(h, k, \omega, z) f \in M \). Consider \( 0 \neq g \in M^\perp \), so that \( \prec g, \pi(h, k, \omega, z) f \succ = 0 \). Then

\[ 0 = \int_{\mathbb{R}} g(\xi) e^{-2\pi i x \xi} \tilde{f}((\xi \omega)_{a^{-1}}) d\xi = \int_{\mathbb{R}} g(\xi \omega) e^{-2\pi i x \xi \omega} \tilde{f}(\xi) d\xi. \]

Thus, \( g(\xi \omega) \tilde{f}(\xi) = 0 \), for almost all \( \xi \in \mathbb{R} \). Suppose that \( \tilde{f}(\xi) \neq 0 \), for all \( \xi \) in a set \( A \) with positive measure. Then for all \( \xi \in A \), \( g(\xi \omega) = 0 \), for all \( \omega \in \mathbb{R}, a \in \mathbb{R}^+ \). Thus \( g = 0 \). This is a contradiction. So, \( \pi \) is an irreducible representation, but \( H \) is not compact.
In the sequel, we define the quasi regular representation and we obtain a concrete form for an admissible vector. Note that \( \mathbb{H}(G_r) \) acts on the Hilbert space \( L^2(\hat{K} \times \mathbb{T}) \) and this action induces the quasi regular representation \( \{ \rho, L^2(\hat{K} \times \mathbb{T}) \} \) as follows:

\[
(3.4) \quad \rho : (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T}) \to U(L^2(\hat{K} \times \mathbb{T})),
\]

where

\[
\rho(h, k, \omega, z) f(\xi, t) = \delta^{-1/2}_{H \times_\tau K}(h, k) f(\theta_{(h, k)^{-1}}(\xi, t)(\omega, z)^{-1})
\]

\[
= \delta^{-1/2}_{H}(h) f(\theta_{(h^{-1}, k^{-1})}(\xi\omega, tz^{-1}))
\]

\[
= \delta^{-1/2}_{H}(h) f((\xi\omega)_{h^{-1}}, (\xi\omega)_{h^{-1}}(k^{-1}))(tz^{-1}).
\]

Note that \( \delta_{H \times_\tau K}(h, k) = \delta_{H}(h)^{-1} \). (see Corollary 3.3 in [4])

A type of the Fourier transform of the quasi regular representation \( \rho \) obtains as follows:

\[
\rho(h, k, \omega, z) f(k', n') = \int_{\hat{K} \times \mathbb{T}} \rho(h, k, \omega, z) f(\xi, t)(k', n')(\xi, t)d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)
\]

\[
= \delta_{H}(h)^{-1/2} \int_{\hat{K} \times \mathbb{T}} f((\xi\omega)_{h^{-1}}, (\xi\omega)_{h^{-1}}(k^{-1}))(tz^{-1})(k')d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)
\]

\[
= \delta_{H}(h)^{-1/2} \int_{\hat{K} \times \mathbb{T}} f((\xi\omega)_{h^{-1}}, (\xi\omega)_{h^{-1}}(k^{-1}))(\xi\omega)(k')d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)
\]

\[
= \delta_{H}(h)^{-1/2} \int_{\hat{K} \times \mathbb{T}} f(\theta_{(h^{-1}, k^{-1})}(\xi, t)(k')d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)
\]

\[
= \delta_{H}(h)^{-1/2} \int_{\hat{K} \times \mathbb{T}} f(\theta_{(h^{-1}, k^{-1})}(\xi, t))(\xi)(k')d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)
\]

\[
= \delta_{H}(h)^{-1/2} \int_{\hat{K} \times \mathbb{T}} f(\theta_{(h^{-1}, k^{-1})}(\xi, t))(k', n')(\xi, t)d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)
\]

\[
= \delta_{H}(h)^{-1/2} \int_{\hat{K} \times \mathbb{T}} f(\theta_{(h^{-1}, k^{-1})}(\xi, t))(k', n')(\xi, t)d\mu_{\hat{K}}(\xi)d\mu_{\mathbb{T}}(t)
\]

for all \( (k', n') \in K \times \mathbb{Z} = (\hat{K} \times \mathbb{T}) \).

So,

\[
(3.5) \quad \rho(h, k, \omega, z) f(k', n') = \delta_{H}(h)^{-1/2} \tilde{\omega}(k')(f \circ \theta_{(h, k)^{-1}})(k', n').
\]
Theorem 3.6. With the notation as above, let $\rho$ be the quasi regular representation on $\mathbb{H}(G_\tau)$, and $\psi, f \in L^2(\hat{K} \times T)$.

(i) If $\psi$ is a wavelet vector, then

$$W_\psi f(h, k, \omega, z) = \delta_H^{-1/2}(h) \int_K \sum_{n' \in \mathbb{Z}} \hat{f}(k', n') z^{n'} \omega(k') (\hat{\psi} \circ \hat{\theta})(h, k) - 1(k', n') d\mu_K(k').$$

(ii) The vector $\psi$ is wavelet if

$$\left\| \hat{\psi}(k', n') \circ \theta_{(h, k)}^{-1} \right\|^2 d\mu_{H \times K}(h, k) < \infty.$$

Proof. For $(k', n') \in K \times \mathbb{Z},$

(i) By the Plancherel’s theorem and (3.5), we have

$$W_\psi f(h, k, \omega, z) = \langle f, \rho(h, k, \omega, z) \psi \rangle$$

$$= \langle \hat{f}, \rho(h, k, \omega, z) \psi \rangle$$

$$= \delta_H^{-1/2}(h) \int_K \sum_{n' \in \mathbb{Z}} \hat{f}(k', n') z^{n'} \omega(k') (\psi \circ \theta)(h, k) - 1(k', n') d\mu_K(k').$$

(ii) By applying the part (i), for $f \in L^2(\hat{K} \times T)$, we get

$$\int_{K \times T} |W_\psi f(h, k, \omega, z)|^2 d\mu_{K \times T}(\omega, z)$$

$$= \int_{K \times T} W_\psi f(h, k, \omega, z) \overline{W_\psi f(h, k, \omega, z)} d\mu_{K \times T}(\omega, z)$$

$$= \delta_H^{-1}(h) \int_{K \times T} \left( \int_K \sum_{n' \in \mathbb{Z}} \hat{f}(k', n') z^{n'} \omega(k')(\psi \circ \theta)(h, k) - 1(k', n') d\mu_K(k') \right)$$

$$\times \left( \int_K \sum_{n'' \in \mathbb{Z}} \hat{\psi}(k'', n'') \omega(k'')(\psi \circ \theta)(h, k) - 1(k'', n'') d\mu_K(k'') \right)$$

$$= \delta_H^{-1}(h) \int_{K \times T} |F(\omega, z)|^2 d\mu_{K \times T}$$

$$= \delta_H^{-1}(h) \int_{K \times T} |F(k', n')|^2 d\mu_{K \times Z}$$

$$= \delta_H^{-1}(h) \int_K \sum_{n' \in \mathbb{Z}} |\hat{f}(k', n')|^2 |(\psi \circ \theta)(k', n')|^2 d\mu_K(k').$$
where $\hat{F} = \hat{f}(\psi \circ \theta) \in L^1(K \times \mathbb{Z})$. It is easy to see that

$$
(\hat{\psi} \circ \theta)(k', n') = \delta_H^{-1}(h) \hat{\psi}(k', n') \circ \theta_{(h,k)}^{-1}.
$$

Then

$$
\int_{K \times \mathbb{T}} |W_\psi f(h, k, \omega, z)|^2 d\mu_K \times \mathbb{T}(\omega, z) = \delta_H^{-1}(h) \int_K \sum_{n' \in \mathbb{Z}} |\hat{f}(k', n')|^2 |(\hat{\psi}(k', n') \circ \theta_{(h,k)}^{-1})|^2 d\mu_K(k').
$$

Now, by using (3.6) we have

$$
\|W_\psi f\|_2^2 = \int_{\mathbb{H}(G_\tau)} |W_\psi f(h, k, \omega, z)|^2 d\mu_{\mathbb{H}(G_\tau)}(h, k, \omega, z)
= \int_{H \times K} \int_{K \times \mathbb{T}} |W_\psi f(h, k, \omega, z)|^2 \delta_H^{-1}(h) d\mu_K \times \mathbb{T}(\omega, z) d\mu_{H \times K}(h, k)
= \int_{H \times K} \sum_{n' \in \mathbb{Z}} |\hat{f}(k', n')|^2 |(\hat{\psi}(k', n') \circ \theta_{(h,k)}^{-1})|^2 d\mu_K(k') d\mu_{H \times K}(h, k),
$$

and then the proof of part (ii) is complete. $\square$

4. Examples and applications

**Example 4.1.** Let $K$ be an abelian locally compact group and $H = \{e\}$ (the trivial group). In this case the generalized weyl Heisenberg group $\mathbb{H}(G_\tau)$ coincides with the standard weyl Heisenberg group $G := K \times \theta (\hat{K} \times \mathbb{T})$. In this case the square integrable representation of $G = K \times \theta (\hat{K} \times \mathbb{T})$ on $L^2(\hat{K})$ is as follows:

$$
\pi(k, \omega, z) f(\xi) = z\xi(k)\omega(k) f(\xi z).
$$

**Example 4.2.** Let $E(n)$ be the Euclidean group which is the semi-direct product of $So(n) \times_{\tau} \mathbb{R}^n$ where the continuous homomorphism $\tau : So(n) \to Aut(\mathbb{R}^n)$ given by $\sigma \mapsto \tau_\sigma$ via $\tau_\sigma(x) = \sigma x$, for all $x \in \mathbb{R}^n$. The group operation for $E(n)$ is

$$(\sigma_1, x_1) \times_{\tau} (\sigma_2, x_2) = (\sigma_1\sigma_2, x_1 + \sigma_1 x_2).$$
Consider the continuous homomorphism \( \hat{\tau} : So(n) \rightarrow Aut(\mathbb{R}^n) \) via \( \sigma \mapsto \hat{\tau}_\sigma \) which is given by \( \tau_\sigma(\omega) = \omega \circ \tau_\sigma^{-1} \). Thus the generalized Weyl Heisenberg group of \( E(n) \), is the set \( \mathbb{H}(E(n)) = (So(n) \times \mathbb{R}^n) \times_\theta (\mathbb{R}^n \times \mathbb{T}) \) with the group operation

\[
(\sigma_1, x_1, \omega_1, z_1)(\sigma_2, x_2, \omega_2, z_2) = (\sigma_2, x_1 + \sigma_1 x_2, \omega_1(\omega_2)\sigma_1, (\omega_2)\sigma_1(x_1)z_1z_2),
\]

for all \( (\sigma_1, x_1, \omega_1, z_1)(\sigma_2, x_2, \omega_2, z_2) \in \mathbb{H}(E(n)) \) and with the product topology. Then the square integrable representation \( \pi \) of \( \mathbb{H}(E(n)) \) onto \( L^2(\mathbb{R}^n) \) is

\[
\pi(\sigma, x, \omega, z)f(\xi) = e^{2\pi i x(\xi - \omega)}f((\xi - \omega)_{\sigma^{-1}}).
\]

Note that \( H \) is compact and \( \delta_H(h) = 1 \).

**Example 4.3.** Let \( \mathbb{H}(\mathbb{R}^n) = \mathbb{R}^n \times_\theta (\mathbb{R}^n \times \mathbb{T}) \) be the classical Heisenberg group on \( \mathbb{R}^n \), in which the continuous homomorphism \( x \mapsto \theta_x \) from \( \mathbb{R}^n \) into \( Aut(\mathbb{R}^n \times \mathbb{T}) \) is defined by \( \theta_x(y, z) = (y, ze^{2\pi i x y}) \). Then the square integrable representation \( \pi \) of \( \mathbb{H}(\mathbb{R}^n) \) onto \( L^2(\mathbb{R}^n) \) is

\[
\pi(x, \omega, z)f(\xi) = ze^{2\pi i x(\xi - \omega)}f(\xi - \omega).
\]

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