Heavy quarkonia description from an energy dependent quark-antiquark potential.

P. González
Departamento de Física Teórica, Universidad de Valencia (UV) and IFIC (UV-CSIC), Valencia, Spain.
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We model the spectral effect of open flavor meson-meson thresholds in heavy quarkonia. The proposed energy dependent quark-antiquark static potential tries to incorporate in a quark model scheme the results from unquenched lattice calculations. A good qualitative and a reasonable quantitative description of electrically neutral charmonium and bottomonium, including the “new charmonium-like” states, is obtained.

I. INTRODUCTION

The discovery of $X(3872)$ \cite{1} opened a new era in heavy quarkonia spectroscopy since its properties could not be reasonably reproduced with any conventional $c\bar{c}$ quark model employed until then for the description of charmonium. Since its mass is about the $D^0(1865)\bar{D}^0$ (2007)\textsuperscript{0} threshold energy alternative descriptions based on tetraquark or meson-meson models have been proposed to explain its nature. In this regard it should be pointed out that the $J^{PC}$ quantum number assignment to $X(3872)$ is still an experimental issue. Other resonances such as $X(4260)$, $X(4360)$ ... that were discovered later are considered unconventional $c\bar{c}$ states as well (see \cite{1} and references therein).

Taking for granted that these resonances are not pure conventional $c\bar{c}$ states and realizing that all of them are close to open flavor meson-meson channels it is important to analyze the effect of these meson-meson thresholds on the quark-antiquark interaction and the role they could play in the description of such non conventional states. This effect has been studied in a general form in lattice QCD by considering the static quark-antiquark ($Q\bar{Q}$) ground state energy when an open flavor meson-meson system is taken into account. The results are summarized in Fig. 22 of reference \cite{2} that we reproduce here for completeness as Fig. 1. In this figure $E(r)$, in the $Y$ axis, is the quark-antiquark ground state energy in terms of $r$, the quark-antiquark distance represented in the $X$ axis. The two thin curved lines following lattice data (circles and pentagons) represent the calculated $E(r) - 2m_B$ when only one static-light two meson threshold $B\bar{B}$ is present whereas the three thick lines correspond to an educated guess for the case of two thresholds $B\bar{B}$ and $B_s\bar{B}_s$. Let us centre on the two threshold case. Then $E(r) - 2m_B$ is given below the $B\bar{B}$ threshold by the lower thick line, in between the $B\bar{B}$ and $B_s\bar{B}_s$ thresholds by the intermediate thick line and above the $B_s\bar{B}_s$ threshold by the upper thick line. So the form of $E(r)$ is different in these three energy regions. It should be realized though that these three different forms of $E(r)$ correspond, when not close to any threshold, to the same three-parameter fit $\sigma r - \frac{\pi}{r} + E_0$.

We use these results from lattice to build in Section III a non relativistic quark model based on a quark-antiquark static potential. As this model extends the conventional quark model to incorporate open flavor meson-meson threshold effects we call it Extended Quark Model (EQM). In Section III we detail the application of the EQM to calculate the spectra of heavy quarkonia which are presented in Section XV where an analysis of the structure of the spectral states is also carried out. Sections XV, VI and VII are dedicated to a qualitative discussion of possible decay modes. Then in Sections VIII and IX a state by state study of charmonium and bottomonium is completed. Finally in Section X our main results and conclusions are summarized.

II. EXTENDED QUARK MODEL (EQM)

Aiming at a description of the heavy quarkonia ($Q = b, c$) spectra and properties we shall rely on a non relativistic quark potential model. We expect this to be valid for bottomonium and to a certain extent (up to relativistic corrections) also for charmonium. Heavy quarkonia states will be identified with the $Q\bar{Q}$ bound
states obtained by solving the Schrödinger equation for the quark-antiquark potential. We shall assume that the effectiveness of the parameters (quark masses and parameters of the potential fitted to reproduce the spectra) may be appropriately taking into account, at least in part, kinetic and potential energy corrections.

A. Energy Dependent Potential

In the non relativistic static approximation the quark-antiquark ground state energy \( E(r) \) may be identified with the sum of the masses of the quark \( (m_Q) \) and the antiquark \( (m_{\bar{Q}}) \) plus the static quark-antiquark potential \( V(r) \), this is \( V(r) = E(r) - m_Q - m_{\bar{Q}} \).

Let us first consider the two threshold case. Let us name the first (second) threshold as \( T_1 \) (\( T_2 \)) with mass \( M_{T_1} \) \( (M_{T_2}) \) (in Fig. 1 \( T_1 = \overline{BB} \) \( T_2 = \overline{B, B} \) with \( M_{T_1} = 2m_B \) \( M_{T_2} = 2m_B \)). As the forms of \( E(r) \) are different below \( M_{T_1} \), in between \( M_{T_1} \) and \( M_{T_2} \), and above \( M_{T_2} \), the potential \( V(r) \) has different forms in these energy regions. In this sense \( V(r) \) is an energy dependent potential. In practice this means that heavy quarkonia \( (Q\overline{Q}) \) bound states with masses \( M_{Q\overline{Q}} \) belonging for example to the energy region \( 0 < E_{Q\overline{Q}} < M_{T_1} \) will be obtained by solving the Schrödinger equation with the form of the potential corresponding to this energy region and so on. Henceforth we shall rename \( V(r) \) as \( V_{E_{Q\overline{Q}}}(r) \) to make explicit the energy dependence.

More precisely, in the first energy region defined by \( 0 < E_{Q\overline{Q}} < M_{T_1} \), this is for \( E_{Q\overline{Q}} \in [M_{T_0}, M_{T_1}] \) where we have defined \( M_{T_0} = 0 \) in order to unify the notation (let us realize that \( T_0 \) does not correspond to any real meson-meson threshold), the form of the potential \( V_{E_{Q\overline{Q}}}(r) \) will be called \( V_{[M_{T_0}, M_{T_1}]}(r) \). This form is given by \( V_{[M_{T_0}, M_{T_1}]}(r) = E(r) - m_Q - m_{\bar{Q}} \) with \( E(r) \) corresponding to the lower thick line in Fig. 1. According to our previous discussion about the form of \( E(r) \) when not close to threshold, this potential has at short distances the form \( (\sigma r - \frac{\chi}{r} + V_0) \). We shall include the constant \( V_0 \) in the definition of the quark and antiquark masses so that we shall write the potential as \( \sigma r - \frac{\chi}{r} \). As can be checked from Fig. 1 this form maintains up to a distance close below \( r_{T_1} \), then it rises until getting for a distance close above \( r_{T_1} \), the form \( \sigma r - \frac{\chi}{r} \). This form is maintained up to a distance close below the crossing distance \( r_{T_2} \) defined from \( V_{[M_{T_1}, M_{T_2}]}(r_{T_2}) = \sigma r_{T_2} - \frac{\chi}{r_{T_2}} = M_{T_2} - m_Q - m_{\bar{Q}} \) where \( V_{[M_{T_1}, M_{T_2}]}(r) \) starts to flatten approaching its asymptotic value \( M_{T_2} - m_Q - m_{\bar{Q}} \).

This analysis of the two threshold case can be easily generalized to the general many threshold case by assuming that in between any two thresholds the potential form is similar to \( V_{[M_{T_i}, M_{T_{i+1}}]}(r) \) but substituting the corresponding thresholds. For the sake of simplicity we shall reduce the size of the transition regions to the flat potentials just to the crossing points \( r_{T_j} \). Thus the Extended Quark Model (EQM) potential \( V_{E_{Q\overline{Q}}}(r) \) is defined as:

\[
V_{E_{Q\overline{Q}}}(r) = V_{[M_{T_{i-1}}, M_{T_i}]}(r) \quad \text{if} \quad M_{T_{i-1}} < E_{Q\overline{Q}} \leq M_{T_i}
\]

with \( i \geq 1 \), and where the forms of the potential in the different spectral regions are:

\[
V_{[M_{T_0}, M_{T_1}]}(r) = \begin{cases} \sigma r - \frac{\chi}{r} & r \leq r_{T_1} \\ M_{T_1} - m_Q - m_{\bar{Q}} & r > r_{T_1} \end{cases}
\]

and

\[
V_{[M_{T_{j-1}}, M_{T_j}]}(r) = \begin{cases} M_{T_{j-1}} - m_Q - m_{\bar{Q}} & r \leq r_{T_{j-1}} \\ \sigma r - \frac{\chi}{r} & r_{T_{j-1}} \leq r \leq r_{T_j} \\ M_{T_j} - m_Q - m_{\bar{Q}} & r > r_{T_j} \end{cases}
\]

for \( j > 1 \) with the crossing distances \( r_{T_{j-1}} \) defined by

\[
\sigma r_{T_{j-1}} - \frac{\chi}{r_{T_{j-1}}} = M_{T_{j-1}} - m_Q - m_{\bar{Q}}
\]

For instance the EQM potential \( V_{E_{Q\overline{Q}}}(r) \) for \( b\bar{b} \) states with \( J^{PC} = 0^+(0^{++}) \) quantum numbers, whose first threshold is \( B\overline{B} \), is drawn in Fig. 2 for the first and second energy regions.

Let us remark that the EQM potential \( V_{E_{Q\overline{Q}}}(r) \) incorporates open flavor meson-meson threshold effects on the \( Q\overline{Q} \) interaction. If no thresholds were considered the \( Q\overline{Q} \) interaction would be described by the quenched non energy dependent Cornell potential (see for example [8] for a derivation of this form from lattice)

\[
V_{C_{\text{op}}}(r) \equiv \sigma r - \frac{\chi}{r} \quad r : 0 \rightarrow \infty
\]

As threshold effects are related to the presence of sea quark pairs \( (q\overline{q}) \), the EQM potential \( V_{E_{Q\overline{Q}}}(r) \) corresponds to an unquenched quark-antiquark potential.

Henceforth we shall refer to open flavor meson-meson thresholds simply as thresholds.
III. HEAVY QUARKONIA

Certainly the EQM just defined might be too simplistic for spectroscopic purposes. First, the quenched potential form used when not close to any threshold, \( \sigma r - \frac{\chi}{r} \), does not contain spin dependent terms that, apart from relativistic corrections, we know may be significant for the lower spectral states. Second, only open flavor thresholds have been considered and no threshold widths have been taken into account. Third, the effect of any threshold has been approximated by an abrupt (instead of a physically soft) change in the quark-antiquark potential at the crossing radii. Moreover i) the same effect from thresholds with \( s\bar{s} \), \( u\bar{c}, \bar{u}c \) or \( d\bar{d} \) content has been considered but it could be different for thresholds with \( s\bar{s} \) content and ii) the effect of a threshold with several possible values of angular momentum \( J \) could be different for each of these values.

Anyhow, keeping in mind these possible shortcomings, we think it is worthwhile to examine the physical consequences deriving from this simple dynamical model for heavy quarkonia to try to learn from them possible avenues for future progress.

A. Parameters of the Potential

Aiming at a joint description of charmonium (\( c\bar{c} \)) and bottomonium (\( b\bar{b} \)) we shall use for both the same values for the parameters \( \sigma \) and \( \chi \) of the potential. Let us realize that in the first spectral region \([M_{T_0}, M_{T_1}]\), for energies far below the threshold, we hardly expect any threshold effect. In other words the Cornell potential \( V_{Cor}(r) \equiv \sigma r - \frac{\chi}{r} \) (\( r : 0 \to \infty \)) should describe reasonably well this part of the spectrum. Actually this is the case. It turns out that for a value of the Coulomb strength \( \chi = 100 \text{ MeV.fm} \) corresponding to a strong quark-gluon coupling \( \alpha_s = \frac{2\pi}{\hbar c} \approx 0.38 \) (in agreement with the value derived from QCD from the hyperfine splitting of \( 1p \) states in bottomonium [2] and also with the value obtained from the fine structure splitting of \( 1p \) states in charmonium [10]), one can choose correlated values of \( \sigma \) and \( m_Q \) to get such description. In this regard, as we are dealing with a spin independent potential, we may compare as usual the calculated \( s^- \) wave states with spin-triplets, the \( p^- \) wave states with the centroids obtained from data and the \( d^- \) wave states with the only existing experimental candidates. Indeed it would be better a comparison with the centroids for all states but the dearth of spin singlet data makes this unfeasible.

We shall fix \( \sigma = 850 \text{ MeV.fm} \), \( m_b = 4793 \text{ MeV} \) and \( m_c = 1348.5 \text{ MeV} \) so that the differences from the calculated Cornell masses to data below the first corresponding thresholds are less than 30 MeV in bottomonium and 60 MeV in charmonium (these differences could be reduced when comparing with the centroids for all states). For \( m_c \) its value has been fine tuned to get an EQM mass for \( X(3872) \) within its experimental energy range (see below).

The set of parameters that will be used henceforth is then

\[
\begin{align*}
\sigma &= 850 \text{ MeV.fm} \\
\chi &= 100 \text{ MeV.fm} \\
m_b &= 4793 \text{ MeV} \\
m_c &= 1348.5 \text{ MeV}
\end{align*}
\]

B. \( 0(J^{++}) \) Thresholds

In order to apply the EQM to a particular set of heavy quarkonia states with definite \( I(J^{PC}) \) we have to look for meson (\( Q\bar{Q} \)) - meson (\( Q\bar{q} \)) thresholds \((q : u,d,s)\) coupling to these quantum numbers. We consider the two mesons to be in a relative \( S^- \) wave so that the threshold mass corresponds to the sum of the masses of the constituent mesons.

The lower \( 0(J^{++}) \) thresholds for charmonium and bottomonium are listed in Tables I and II (the study of \( 0(1^{--}) \) thresholds deserves special attention and will be done later on).

It is important to remark that we have used isospin symmetry to construct thresholds with well defined isospin. This means that we are neglecting the mass differences between the electrically neutral and charged members of the same isospin multiplet, for example \( D^0 \)
TABLE I: Lower open flavor meson-meson thresholds for 0(++), 0(1+), and 0(2+) states. Threshold masses ($M_{T_i}$) obtained from the charmed and charmed strange mesons quoted in [11]. Crossing distances ($r_{T_i}$) calculated from (4).

| $I(J^{PC})$ | $T_i$ | Charmonium Thresholds | $M_{T_i}$ (MeV) | $r_{T_i}$ (fm) |
|------------|------|----------------------|----------------|-------------|
| 0(0++)     |      | $D^0 B^0 - D^+ D^-$  | 3730           | 1.31        |
|            |      | $D^+_s D^-_s$       | 3937           | 1.54        |
| 0(1++)     |      | $(D^0 B^0 (2007)^- - D^+ D^* (2010)^-)$ | 3872 | 1.46 |
|            |      | $(D^+ (2007)^0 D^* (2007)^- - D^* (2010)^+ D^* (2010)^-)$ | 4014 | 1.62 |
| 0(2++)     |      | $(D^+ (2007)^0 D^* (2007)^- - D^* (2010)^+ D^* (2010)^-)$ | 4014 | 1.62 |
|            |      | $D^*_s + D^*_s$      | 4224           | 1.86        |

TABLE II: Lower open flavor meson-meson thresholds for 0(J++) b B states. Threshold masses ($M_{T_i}$) obtained from the bottom and bottom strange meson masses quoted in [11]. Crossing distances ($r_{T_i}$) calculated from (4).

| $I(J^{PC})$ | $T_i$ | Bottomonium Thresholds | $M_{T_i}$ (MeV) | $r_{T_i}$ (fm) |
|------------|------|----------------------|----------------|-------------|
| 0(0++)     |      | $B^0 B^0 - B^+ B^-$  | 10558          | 1.24        |
|            |      | $B^0 B^0 - B^{++} B^-$ | 10650 | 1.34 |
|            |      | $B^0 B^- - B^{0*} B^*$ | 10734 | 1.43 |
| 0(1++)     |      | $(B^0 B^- - B^{0*} B^*)$ - c.c. | 10604 | 1.29 |
|            |      | $(B^0 B^- - B^{0*} B^*)$ | 10650 | 1.34 |
| 0(2++)     |      | $(B^0 B^- - B^{0*} B^*)$ | 10650 | 1.34 |
|            |      | $B^- B^+ - B^{0*} B^*$ | 10830 | 1.54 |

Regarding C parity we rely on the quark model assignment of C parity values to charmed and bottom mesons (see for example the Quark Model Section in [11]) to obtain the C parity value for the threshold as the product of the C parities of the component mesons. Thus the sign preceding the charge conjugate term (c.c.) is determined from the required threshold C parity.

C. 0+(J++) Spectral States

Heavy quarkonia bound states are obtained by solving the Schrödinger equation for the EQM potential $V_{EQM}(r)$. In the energy region $[M_{T_{r_1}}, M_{T_1}]$ they satisfy

$$\left( T + V_{[M_{T_{r_1}}, M_{T_1}]} \right) \left( \langle Q\overline{Q} \rangle_k |_{[r_{T_1}, r_1]} \right) = M_k |_{[r_{T_1}, r_1]} \left( \langle Q\overline{Q} \rangle_k |_{[r_{T_1}, r_1]} \right)$$

where $T$ stands for the kinetic energy operator, $\langle Q\overline{Q} \rangle_k |_{[r_{T_1}, r_1]}$ for the bound state and $M_k |_{[r_{T_1}, r_1]}$ for its mass. As we have a radial potential we use the spectroscopic notation $\kappa \equiv nl$, in terms of the radial, $n$, and orbital angular momentum, $l$, quantum numbers of the $Q\overline{Q}$ system.

To fix the ideas let us consider for example the spectral states for 0+(0++) $b B$.

In the first energy region the potential $V_{[M_{T_0}, M_{T_1}]}(r)$, given by (2), reads (solid line in Fig. 2)

$$V_{[0,10558]}(r) = \begin{cases} \sigma r - \frac{\sigma}{r} & r \leq 1.24 \text{ fm} \\ 972 \text{ MeV} & r \geq 1.24 \text{ fm} \end{cases}$$

where $M_{T_1}$ and $r_{T_1}$ have been taken from Table I and the values of the parameters ($\sigma, \chi, m_b$) are given by (5).

By solving the Schrödinger equation for $V_{[0,10558]}(r)$ we get the EQM spectrum in $[M_{T_0}, M_{T_1}]$. It has only three bound states, $1p_{[T_0,T_1]}, 2p_{[T_0,T_1]}$, and $3p_{[T_0,T_1]}$, whose masses are listed in Table II.

In the second energy region the potential, $V_{[M_{T_1}, M_{T_2}]}(r)$, reads (dashed line in Fig. 2)

$$V_{[10558,10650]}(r) = \begin{cases} 972 \text{ MeV} & r \leq 1.24 \text{ fm} \\ \sigma r - \frac{\sigma}{r} & 1.24 \text{ fm} \leq r \leq 1.34 \text{ fm} \\ 1064 \text{ MeV} & r \geq 1.34 \text{ fm} \end{cases}$$

and $D^\pm$ with PDG quoted masses [11] 1864.91 ± 0.17 and 1869.5 ± 0.4 respectively or $D^*(2007)^0$ and $D^*(2010)^-$ with quoted masses 2006.98 ± 0.15 and 2010.21 ± 0.13 respectively. Indeed we have calculated the threshold masses by taking the lower mass value in any isospin multiplet (1865 MeV and 2007 MeV in the examples just mentioned). We shall comment later on the consequences deriving from isospin breaking.
TABLE III: Calculated masses from $V_{E_{Q\overline{Q}}}(r)$: $M_{EQM}$. Calculated masses from $V_{Corn}(r)$: $M_{Corn}$. Results for $0^+(0^{++})$ $\bar{b}b$ and $0^+(1^{++})$ $\sigma$ are shown. Conventional spectroscopic notation has been used to denote the Cornell states.

| $Q\overline{Q}$ ($j^{PC}$) | $k_{[T_{i-1},T_i]}$ | $M_{EQM}$ (MeV) | $M_{Corn}$ (MeV) | Cornell States |
|---------------------------|-------------------|-----------------|-----------------|----------------|
| $\bar{b}b$ ($0^{++}$)    | $1p_{[T_0,T_1]}$  | 9920            | 9920            | $1p$           |
|                           | $2p_{[T_0,T_1]}$  | 10259           | 10259           | $2p$           |
|                           | $3p_{[T_0,T_1]}$  | 10521           | 10531           | $3p$           |
|                           | $1p_{[T_1,T_2]}$  | 10620           |                 |                |
|                           | $1p_{[T_2,T_3]}$  | 10704           |                 |                |
|                           | $1p_{[T_3,T_4]}$  | 10784           |                 |                |
| $\sigma$ ($1^{++}$)      | $1p_{[T_0,T_1]}$  | 3456            | 3456            | $1p$           |
|                           | $2p_{[T_0,T_1]}$  | 3871.6          | 3911            | $2p$           |
|                           | $1p_{[T_1,T_2]}$  | 4003            |                 |                |
|                           | $1p_{[T_2,T_4]}$  | 4190            |                 |                |

where the threshold masses and crossing radii are taken from Table III. The spectrum has only one bound state $1p_{[T_1,T_2]}$ whose mass is listed in Table III.

By proceeding in the same way for other energy regions we can get the complete EQM bound state spectrum. In Table III the calculated masses for $0^+(0^{++})$ $\bar{b}b$ and $0^+(1^{++})$ $\sigma$ states up to the fifth energy region are listed (see Sections VIII and IX for a complete list of the thresholds used). For the sake of comparison we have also listed the masses for the states of the Cornell spectrum.

### D. Threshold Effects

A look at Table III makes clear that the more significant spectral effect from the unquenched energy dependent EQM potential $V_{E_{Q\overline{Q}}}(r)$ is the bigger number of spectral states as compared to the quenched Cornell potential $V_{Corn}(r)$ case. Thus, for $0^+(0^{++})$ $\bar{b}b$ there are four EQM bound states with masses between 10500 MeV and 10800 MeV for only two Cornell states in this energy interval. This denser spectral pattern, caused by thresholds, will be able to accommodate conventional and unconventional heavy quarkonia resonances as we shall see.

In order to analyze in detail the spectral effect of introducing a threshold we shall first consider no thresholds at all and then we shall introduce only one threshold. If no thresholds were present the $Q\overline{Q}$ spectrum would be the Cornell spectrum listed in the fourth and fifth columns of Table III. If we had only the threshold $T_1$ then the potential would be

$$V_{Q\overline{Q}}^{ONE}(r) = \begin{cases} 
V_{[M_{T_0},M_{T_1}]}(r) & \text{if } M_{T_0} \leq E_{Q\overline{Q}} \leq M_{T_1} \\
V_{[M_{T_1},M_{T_\infty}]}(r) & \text{if } M_{T_1} < E_{Q\overline{Q}} \leq M_{T_\infty}
\end{cases}$$

where the form $V_{[M_{T_0},M_{T_1}]}(r)$ is given by (2) and the form $V_{[M_{T_1},M_{T_\infty}]}(r)$ is defined as the linearly confining potential

$$V_{[M_{T_1},M_{T_\infty}]}(r) = \begin{cases} 
M_{T_1} - m_Q - m_{Q\overline{Q}} & r \leq r_{T_1} \\
\sigma r - \frac{\chi}{r} & r \geq r_{T_1}
\end{cases}$$

Note that in the first energy region $[M_{T_0}, M_{T_1}]$ the potential $V_{Q\overline{Q}}^{ONE}(r)$ (7) is identical to $V_{E_{Q\overline{Q}}}(r)$ (8). Therefore the spectrum from $V_{E_{Q\overline{Q}}}(r)$ contains for $0^+(0^{++})$ $\bar{b}b$ the three EQM states $1p_{[T_0,T_1]}$, $2p_{[T_0,T_1]}$, $3p_{[T_0,T_1]}$ and for $0^+(1^{++})$ $\sigma$ the two EQM states $1p_{[T_0,T_1]}$, $2p_{[T_0,T_1]}$ listed in Table III.

By comparing the EQM state $np_{[T_0,T_1]}$ with the corresponding state $np$ from the (non threshold) Cornell potential it turns out that one can not distinguish them if their masses are far below the threshold $T_1$ (for example the $1p_{[T_0,T_1]}$ state is identical to the $1p$). In general the presence of $T_1$ gives rise to attraction in the sense that the mass of the $np_{[T_0,T_1]}$ state is lower than the mass of the $np$ state (see Table III). Indeed the closer the $np$ mass to $M_{T_1}$ the bigger the attraction. Therefore we can interpret the EQM state $k_{[T_0,T_1]}$ as resulting from the attraction produced by the threshold $T_1$ on the Cornell state $k$. From this interpretation, to which we shall come back in Section IV, we shall refer to a Cornell state corresponding to a $k_{[T_0,T_1]}$ EQM state as its generator.

It is also worthwhile to realize that for $0^+(1^{++})$ $\sigma$ the attraction produced by the threshold $T_1 \equiv D^0\bar{D}^*(2007)$ with $M_{T_1} = 3872$ MeV makes the $2p_{[T_0,T_1]}$ state to be just below threshold whereas the $2p$ Cornell state is above it. We shall see later on how this $2p_{[T_0,T_1]}$ state may well correspond to the unconventional $X$ (3872) resonance.

The spectrum from $V_{E_{Q\overline{Q}}}(r)$ is completed with the bound states in the energy region $[M_{T_1}, M_{T_\infty}]$ which
are obtained by solving the Schrödinger equation for \( V_{[M_{T_1}, M_{T_\infty}]}(r) \). We shall denote these states as \( np_{[T_i, T_\infty]} \).

To go a step further in the detailed analysis of threshold effects let us now consider that we had only the two thresholds \( T_1 \) and \( T_2 \). Then the potential would be

\[
V_{E_{\text{EQM}}}^{\text{TWO}}(r) = \begin{cases} 
V_{[M_{T_0}, M_{T_1}]}(r) & \text{if } M_{T_0} \leq E_{\text{EQM}} \leq M_{T_1} \\
V_{[M_{T_1}, M_{T_2}]}(r) & \text{if } M_{T_1} < E_{\text{EQM}} \leq M_{T_2} \\
M_{T_2} - E_{\text{EQM}} & \text{if } M_{T_2} < E_{\text{EQM}} \leq M_{T_\infty} 
\end{cases}
\]

with \( V_{[M_{T_0}, M_{T_1}]}(r) \) given by (2), \( V_{[M_{T_1}, M_{T_2}]}(r) \) by (3) and

\[
V_{[M_{T_2}, M_{T_\infty}]}(r) = \begin{cases} 
M_{T_2} - m_Q - m_{\text{EQM}} & r \leq r_{T_2} \\
\sigma r - \frac{1}{r} & r \geq r_{T_2}
\end{cases}
\]

(9)

By comparing \( V_{E_{\text{EQM}}}^{\text{TWO}}(r) \) and \( V_{E_{\text{EQM}}}^{\text{ONE}}(r) \) it is clear that the opening of the potential in the first energy region \( [M_{T_0}, M_{T_1}] \) say the potentials \( V_{E_{\text{EQM}}}^{\text{TWO}}(r) \) and \( V_{E_{\text{EQM}}}^{\text{ONE}}(r) \) are identical below \( M_{T_1} \), so that their spectra are identical in this energy region. Thus for \( 0^+(0^{++}) \) \( b \bar{b} \) there are three bound states \( 1p_{[T_0, T_1]}, 2p_{[T_0, T_1]}, 3p_{[T_0, T_1]} \).

On the other hand, in the second energy region \( [M_{T_1}, M_{T_2}] \), the potentials \( V_{E_{\text{EQM}}}^{\text{TWO}}(r) \) and \( V_{E_{\text{EQM}}}^{\text{ONE}}(r) \) are identical and so it is with the spectrum. For example for \( 0^+(0^{++}) \) \( b \bar{b} \) it contains only the state \( 1p_{[T_1, T_2]} \) listed in Table [III]

To analyze the spectral effect produced by \( T_2 \) we compare this state \( 1p_{[T_1, T_2]} \) from \( V_{E_{\text{EQM}}}^{\text{TWO}}(r) \) with the corresponding spectral state when \( T_2 \) is not present, this is with \( 1p_{[T_1, T_\infty]} \) from \( V_{E_{\text{EQM}}}^{\text{ONE}}(r) \). The result is shown in Table [IV] from which it can be checked that the introduction of \( T_2 \) gives rise to attraction in the sense that the mass of the \( 1p_{[T_1, T_2]} \) state is lower than the mass of the \( 1p_{[T_1, T_\infty]} \) state.

This comparison procedure can be generalized to any interthreshold region by substituting \( (T_1, T_2) \to (T_i, T_{i+1}) \) and by defining \( V_{[M_{T_i}, M_{T_\infty}]}(r) \) for \( i \geq 1 \) as

\[
V_{[M_{T_i}, M_{T_\infty}]}(r) = \begin{cases} 
M_{T_i} - m_Q - m_{\text{EQM}} & r \leq r_{T_i} \\
\sigma r - \frac{1}{r} & r \geq r_{T_i}
\end{cases}
\]

(10)

### Table IV: Calculated masses for interthreshold states

| \( b \bar{b} \) (0^{++}) | \( 0p_{[T_1, T_2]} \) | 10620 | 10628 | 1p_{[T_1, T_\infty]} |
|-----------------|-----------------|------|------|-----------------|
|                 | 1p_{[T_2, T_3]} | 10704 | 10711 | 1p_{[T_2, T_\infty]} |
|                 | 1p_{[T_3, T_4]} | 10784 | 10789 | 1p_{[T_3, T_\infty]} |
| \( \sigma \) (1^{++}) | 1p_{[T_1, T_2]} | 4003 | 4029 | 1p_{[T_1, T_\infty]} |
|                 | 1p_{[T_3, T_4]} | 4190 | 4206 | 1p_{[T_3, T_\infty]} |

The results obtained for \( 0^+(0^{++}) \) \( b \bar{b} \) and \( 0^+(1^{++}) \) \( \sigma \) are shown in Table [V] from which it is clear that the presence of the threshold \( T_{i+1} \) gives rise to attraction in the sense that the mass of the EQM bound state below \( M_{T_1} \) is lower than the mass of the \( np_{[T_1, T_\infty]} \) state. We have also checked for other cases with more than one EQM bound state between \( T_i \) and \( T_{i+1} \) that the closer the \( np_{[T_1, T_\infty]} \) mass to \( M_{T_{i+1}} \) the bigger the attraction so that for a \( np_{[T_1, T_\infty]} \) state far below \( T_{i+1} \) there is no difference with \( np_{[T_1, T_\infty]} \).

By generalizing the results obtained to any other set of quantum numbers we conclude that any EQM bound state \( k_{[T_{i-1}, T_i]} \) solution of (6), can be considered as the result of the attraction produced by the threshold \( T_i \) on the so called generator state \( k_{[T_{i-1}, T_\infty]} \). This generator satisfies the Schrödinger equation

\[
(T + V_{[M_{T_{i-1}}, M_{T_\infty}]}) (Q\overline{Q}) k_{[T_{i-1}, T_\infty]} = M_{k_{[T_{i-1}, T_\infty]}} (Q\overline{Q}) k_{[T_{i-1}, T_\infty]}
\]

(11)

(note that the states \( k_{[T_0, T_\infty]} \) should be identified with the Cornell states \( k \)).

### E. Effective Thresholds

Until now we have considered non overlapping thresholds in the sense that \( T_{i+1} \) has no effect on the form of the potential above \( M_{T_{i+1}} \) (see 11, 2, 3).
In other words $T_{i+1}$ does not produce any attraction below $M_{T_i}$ and $T_i$ does not produce any attraction above $M_{T_{i+1}}$. However, we do not expect this to be realistic when we have two thresholds very close in mass. Instead we expect an accumulative attractive effect from both thresholds.

To be more precise let us assume that there are two very close thresholds that we call $T^1_n$ and $T^2_n$ with masses $M_{T^1_n} \lesssim M_{T^2_n}$. A simple way to incorporate their joint attraction is to substitute the two thresholds $T^1_n$ and $T^2_n$ by only one effective threshold $T_n$ with mass $M_{T_n}$ satisfying 

$$M_{T_n} < M_{T^1_n} \lesssim M_{T^2_n}$$

By proceeding in this manner it can be easily checked that the masses for bound states from $V_{[M_{T_{n-1}}, M_{T_n}]}(r)$ are lower than the corresponding masses from $V_{[M_{T_{n-1}}, M_{T^1_n}]}(r)$ and the masses for bound states from $V_{[M_{T_n}, M_{T_{n+1}}]}(r)$ are lower than the corresponding masses from $V_{[M_{T^2_n}, M_{T_{n+1}}]}(r)$ as they should be from the joint attraction, see Fig. 3

Let us note that a resulting bound state from $V_{[M_{T_n}, M_{T_{n+1}}]}(r)$ may have a mass lower than $M_{T^2_n}$ as it occurs in Fig. 3. As we shall see this may explain the existence of the $\sigma$ state $X(4260)$ with a mass below the close thresholds $D^0\bar{D}_s(2420)^0$ and $D^0\bar{D}_s(2430)^0$.

Therefore the substitution of any set of very close thresholds (two or more) by one effective threshold whose unknown mass can be considered as a parameter to be fixed from data allows for a complete study of the spectrum with the simple EQM potential $V_{[E_{CM}]}(r)$ previously defined. This will be the case for $0^-(1^{--})$ states which we examine next.

### F. $0^-(1^{--})$ Thresholds

Almost degenerate thresholds are present for $0^-(1^{--})$ $\sigma\tau$ and $b\bar{b}$ states as can be checked from Tables VI and VII where the lower thresholds for charmonium and bottomonium are listed.
\[ I(J^{PC}) \quad T_i \quad \text{Bottomonium Thresholds} \quad M_{T_i} \quad r_{T_i} \quad (\text{MeV}) \quad (\text{fm}) \]

\begin{align*}
0(1^{--}) & \quad (B^0 B_1(5721)^0 - B^+ B_1(5721)^-) + \text{c.c.} \quad < 11003 < 1.73 \\
 & \quad (B^0 B_1(?)^0 - B^+ B_1(?)^-) - \text{c.c.} \\
T_2 & \quad (B^0 B_0^*(5732)^0 - B^{++} B_5^*(5732)^-) + \text{c.c.} \quad 11023 \quad 1.76 \\
 & \quad (B^0 B_1(5721)^0 - B^{++} B_1(5721)^-) - \text{c.c.} \quad < 11049 < 1.79 \\
 & \quad (B^0 B_3(?)^0 - B^{++} B_1(?)^-) + \text{c.c.} \\
T_3 & \quad (B^0 B_2^*(5747)^0 - B^{++} B_2^*(5747)^-) + \text{c.c.} \quad 11072 \quad 1.81
\end{align*}

TABLE VI: Lower open flavor meson-meson thresholds for 0^{--} (1^{--}) \( B\bar{B} \) states. Threshold masses \( (M_{T_1}) \) calculated from the bottom and bottom strange meson masses quoted in [1]. Crossing distances \( (r_{T_1}) \) calculated from [1]. For \( B_1^*(5732) \) with quoted mass 5691 MeV we have assumed \( J = 0 \). A question mark has been used for the mass of the unknown meson and inequalities for the masses and crossing distances of possible effective thresholds merging from almost degenerate thresholds.

As it was the case for 0 \((J^{++})\) thresholds isospin symmetry has been used to construct the 0^{--} (1^{--}) thresholds. In this case a significant deviation from this symmetry can come from \( D^+_s(2400)^0 \) and \( D^+_0(2400)^- \) with quoted masses 2318 \pm 29 MeV and 2403 \pm 14 \pm 35 MeV respectively. We have calculated the threshold masses by taking the lower mass value in any isospin multiplet (2318 MeV for the case just mentioned).

Regarding \( C \) parity we have fixed it by following the same procedure employed for 0 \((J^{++})\) thresholds (see Section III B).

Unfortunately not all thresholds are experimentally well known. For example in Table VI there is one threshold with \( D^0 D_1(2430)^0 \), which contains a \((3P_1 - 1P_1)\) mixing state \( D_1(2430) \) with a quite uncertain mass 2427 \pm 26 \pm 25 MeV. This threshold is expected to be almost degenerate to the similar one containing the other state of the \((1P_1 - 3P_1)\) mixing \( D_1(2420) \). We have assumed they overlap and have substituted this pair of overlapping thresholds by an effective one with a mass smaller than the well established mass in the pair \( M_{D^0 D_1(2420)} = 4287 \) MeV.

Another effective threshold has been used for the multiple overlap of \( D^*(2007)^0 D_1(2420)^0 \), \( D^*(2007)^0 D_1(2430)^0 \), \( D_s^+ D_s(2460)^- \) and \( D_s^+ D_s^*(2317)^- \) with masses about 4429 MeV.

A quite similar or even more uncertain situation appears in bottomonium, see Table VII, where there is a known \( (1P_1 - 1P_1) \) mixing state \( B_1(5721)^0 \) but its \((3P_1 - 1P_1)\) partner \( B_1(?) \) is unknown. We expect this missing state to have a mass close to that of \( B_1(5721)^0 \) giving rise to an effective threshold. Besides there is a \( B_1^*(5732) \) with quantum numbers not established yet that we have tentatively assigned to \( J^P = 0^+ \).

G. 0^- (1^{--}) Spectral States

Following exactly the same procedure explained for 0^+ \((J^{++})\) states we get the EQM spectrum for 0^- (1^{--}) \( b\bar{b} \) and \( c\bar{c} \) states. We should note though that in this case the presence of almost degenerate thresholds requires the consideration of effective thresholds whose masses are not determined. We fix them phenomenologically so as to reasonably reproduce the charmonium and bottomonium spectra.

To fix the ideas let us consider for example 0^- (1^{--}) \( c\bar{c} \) states. By substituting the two almost degenerate thresholds \( T_1^1 = D^0 D_1(2420)^0 \) and \( T_1^2 = D^0 D_1(2430)^0 \) with masses about 4287 MeV by one effective threshold \( T_1 \) with mass \( M_{T_1} = 4237 \) MeV \( (r_{T_1} = 1.87 \text{ fm}) \) and the four almost degenerate thresholds with masses about 4429 MeV by another effective threshold \( T_3 \) with mass \( M_{T_3} = 4379 \) MeV \( (r_{T_3} = 2.04 \text{ fm}) \) we get the lower spectral states shown in Table VII (see Section VIII for a more detailed discussion).

We should emphasize again the denser EQM spectral pattern obtained as compared to the quenched Cornell spectrum. In particular there appear new states \( 1s[T_1, T_2] \) and \( 1s[T_2, T_3] \) which may well correspond, as we shall see, to the unconventional \( X(4260) \) and \( X(4360) \) resonances.

IV. EQM SPECTRUM

Once the spectral calculation procedure has been established for 0^+ \((J^{++})\) as well as for 0^- (1^{--}) states the \( b\bar{b} \) and \( c\bar{c} \) spectra can be evaluated. The complete
heavy quarkonia spectra from the EQM are shown in Tables VIII and IX for charmonium and in Tables X and XI for bottomonium. For a complete list of the thresholds employed see Sections VIII and IX. Calculated masses are compared to data from [11]. For the sake of completeness they are also compared to the results from the quenched Cornell potential $V_{C\text{or}}$ used in conventional quark models.

A sound assignment of the calculated EQM bound states to experimental resonances requires to go beyond the simple comparison of masses. In particular for the $Q\bar{Q}$ system the analysis of leptonic widths, radiative transition rates and strong decay modes, for which there are available data, may allow for an unambiguous assignment of EQM states to heavy quarkonia resonances. Therefore we postpone the detailed analysis of the spectra to Sections VIII and IX after a qualitative study of decays is done in Sections VII and VII and centre now in the compositeness of the EQM states.

A. Compositeness

For the sake of comparison with existing conventional models it is interesting to analyze the structure of the EQM (quark-antiquark) bound states in terms of a conventional description involving linearly confined quark-antiquark states as well as molecular states.

As explained in Section III any EQM bound state $k_{[T_{i-1}, T_i]}$ can be considered as the result of the attraction produced by the threshold $T_i$ on the generator state $k_{[T_{i-1}, T_{\infty}]}$. Hence, apart from the generator component, there should be a meson-meson hadronic molecule component in the EQM state coming out from the generator-threshold interaction (see for example [12]).

Let us consider for a definite set of quantum numbers $I(J^{PC})$ a EQM bound state \(|Q\bar{Q}(k_{[T_{i-1}, T_i]})\rangle\) with generator state \(|Q\bar{Q}(k_{[T_{i-1}, T_i]})\rangle\) and let be $M_{IT_i}$ and $M_{2T_i}$ the two mesons forming the threshold $T_i$. Then we can write

\[
\langle Q\bar{Q}(k_{[T_{i-1}, T_i]}) | \alpha_{k_{[T_{i-1}, T_i]}} \langle Q\bar{Q}(k_{[T_{i-1}, T_{\infty}]}) + \beta_{k_{[T_{i-1}, T_i]}} | M_{IT_i} M_{2T_i} \rangle = \sum_{\alpha_{k_{[T_{i-1}, T_i]}} \beta_{k_{[T_{i-1}, T_i]}}}
\]

where $|M_{IT_i} M_{2T_i}\rangle$ stands for the corresponding molecular state and $\alpha_{k_{[T_{i-1}, T_i]}}$ and $\beta_{k_{[T_{i-1}, T_i]}}$ are constants satisfying

\[
\alpha_{k_{[T_{i-1}, T_i]}}^2 + \beta_{k_{[T_{i-1}, T_i]}}^2 = 1
\]

since the linearly confined state $\langle Q\bar{Q}(k_{[T_{i-1}, T_{\infty}]}$ and the molecular state $|M_{IT_i} M_{2T_i}\rangle$ are mutually orthogonal.

From [13] we can write the constants in a convenient way in terms of a mixing angle $0 \leq \theta_{k_{[T_{i-1}, T_i}}} \leq \frac{\pi}{2}$ in the form

\[
\alpha_{k_{[T_{i-1}, T_i]}} \equiv \cos \theta_{k_{[T_{i-1}, T_i]}},
\]

\[
\beta_{k_{[T_{i-1}, T_i]}} \equiv \sin \theta_{k_{[T_{i-1}, T_i]}}
\]
indicating a very significant probability of 44% for the \( \alpha \) (see below) one gets

\[
\text{TABLE IX: Calculated J_{\text{EQM}} states from } V_{\text{EQM}}(r) : M_{\text{EQM}}. \text{ Masses for experimental resonances, } M_{\text{PDG}}, \text{ have been taken from [11]. Masses from the Cornell potential, } M_{\text{Cor}}, \text{ are also shown for comparison.}
\]

| \( J^P \) | EQM States \( k_{[r_1 \rightarrow r_2]} \) | \( M_{\text{EQM}} \) | \( M_{\text{PDG}} \) | \( M_{\text{Cor}} \) |
|---|---|---|---|---|
| 0++ | \( |1p_{[r_0 \rightarrow r_1]} \rangle \) | 3456 | 3414.75 ± 0.31 | 3456 |
| 1++ | \( |1p_{[r_0 \rightarrow r_1]} \rangle \) | 3456 | 3510.66 ± 0.07 | 3456 |
| 2++ | \( |1p_{[r_0 \rightarrow r_1]} \rangle \) | 3456 | 3556.20 ± 0.09 | 3456 |
| 1++ | \( |2p_{[r_0 \rightarrow r_1]} \rangle \) | 3871.6 | 3871.56 ± 0.22 | 3911 |
| 0++ | \( |1p_{[r_1 \rightarrow r_2]} \rangle \) | 3898 | 3915.5 ± 0.22 | 3911 |
| 2++ | \( |2p_{[r_0 \rightarrow r_1]} \rangle \) | 3903 | 3929 ± 5 ± 2 | 3911 |
| \( ? : 0^{--} \) | | 3942.7 ± 6 |

\[
\text{TABLE X: Calculated } 1^{--} \text{ bottomonium masses from } V_{\text{EQM}}(r) : M_{\text{EQM}}. \text{ Masses for experimental resonances, } M_{\text{PDG}}, \text{ have been taken from [11]. Masses from the Cornell potential, } M_{\text{Cor}}, \text{ are also shown for comparison.}
\]

| \( J^P \) | EQM States \( k_{[r_1 \rightarrow r_2]} \) | \( M_{\text{EQM}} \) | \( M_{\text{PDG}} \) | \( M_{\text{Cor}} \) |
|---|---|---|---|---|
| 1-- | \( |1s_{[r_0 \rightarrow r_1]} \rangle \) | 9459 | 9460.30 ± 0.26 | 9459 |
| 2s | \( |1s_{[r_0 \rightarrow r_1]} \rangle \) | 10012 | 10023.06 ± 0.31 | 10012 |
| 1d | \( |1d_{[r_0 \rightarrow r_1]} \rangle \) | 10157 | 10163.7 ± 1.4 | 10157 |
| 3s | \( |3s_{[r_0 \rightarrow r_1]} \rangle \) | 10342 | 10355.2 ± 0.5 | 10342 |
| 2d | \( |2d_{[r_0 \rightarrow r_1]} \rangle \) | 10438 | 10438 | |
| 4s | \( |4s_{[r_0 \rightarrow r_1]} \rangle \) | 10608 | 10579.4 ± 1.2 | 10608 |
| 3d | \( |3d_{[r_0 \rightarrow r_1]} \rangle \) | 10682 | 10682 | |
| 5s | \( |5s_{[r_0 \rightarrow r_1]} \rangle \) | 10840 | 10841 | |
| 4d | \( |4d_{[r_0 \rightarrow r_1]} \rangle \) | 10898 | 10902 | |
| 1s \( |1s_{[r_1 \rightarrow r_2]} \rangle \) | 10995 | 11019 ± 8 | |
| 1s \( |1s_{[r_2 \rightarrow r_3]} \rangle \) | 11039 | 11053 | |
| 1s \( |1s_{[r_3 \rightarrow r_4]} \rangle \) | 11090 | 11105 | |
| 1d \( |1d_{[r_3 \rightarrow r_4]} \rangle \) | 11130 | | |
| 2s \( |2s_{[r_3 \rightarrow r_4]} \rangle \) | 11140 | | |

configuration space can be extracted from the EQM and generator ones by assuming that \( r \), the quark-antiquark distance, corresponds also to the meson-meson distance.

V. LEPTONIC WIDTHS

Let be the EQM bound state \( (Q\bar{Q})_{k_{[r_1 \rightarrow r_2]}} \). Leptonic transitions \( (Q\bar{Q})_{k_{[r_1 \rightarrow r_2]}} \rightarrow e^+e^- \) take place through a virtual photon. For \( ^3S_1 \) heavy quarkonia states with quantum numbers \( 0^-(1^{--}) \), to which we shall restrict our attention, the transition probability is determined by the wave function at the origin that we shall write as \( \Phi_{nS_{[r_1 \rightarrow r_2]}}(0) \). Dimensionally the leptonic width depends on \( |\Phi_{nS_{[r_1 \rightarrow r_2]}}(0)|^2 \) divided by a square mass factor.

In order to guess this mass factor we can take into account that far below the first \( 0(1^{--}) \) threshold the \( ns_{[r_0 \rightarrow r_1]} \) EQM states are identical to the \( ns \) Cornell states as we have seen. Theoretical estimations of leptonic widths for the Cornell states \( (Q\bar{Q})_{ns} \) are based on the

The coefficient \( \alpha_k_{[r_1 \rightarrow r_2]} \) can be calculated in a straightforward manner as

\[
\alpha_k_{[r_1 \rightarrow r_2]} = \langle (Q\bar{Q})_{k_{[r_1 \rightarrow r_2]}} \rangle \langle (Q\bar{Q})_{k_{[r_1 \rightarrow r_2]}} \rangle \tag{15}
\]

Then from (14) one gets \( \theta_k_{[r_1 \rightarrow r_2]} \). For instance for the \( 2p \) EQM state of \( \sigma^- (0^+(1^{++})) \) in the first energy region listed in Table IX we shall assign to \( X(3872) \) (see below) one gets \( \alpha = 0.75 \) (\( \beta = 0.66 \)) and \( \theta = 41.4^o \) indicating a very significant probability of 44% for the \( D^0\bar{D}^0(2007)^+ \) molecular component.

Notice that (12) can be also interpreted as the definition of the molecular component whose wave function in
TABLE XI: Calculated \( J^{++} \) bottomonium masses from \( V_{\text{EQM}}(r) \) : \( M_{\text{EQM}} \). Masses for experimental resonances, \( M_{\text{PDG}} \), have been taken from [11]. For \( p \) waves we quote separately the \( np_0, np_1 \) and \( np_2 \) states. Masses from the Cornell potential, \( M_{\text{Cor}} \), are also shown for comparison.

| \( J^{PC} \) | \( k \) | \( M_{\text{EQM}} \) | \( M_{\text{PDG}} \) | \( M_{\text{Cor}} \) |
|-------------|------|----------|----------|----------|
| \( 0^{++} \) | 1\( p_1[T_0,T_1] \) | 9920 | 9859.44 \( \pm \) 0.42 \( \pm \) 0.31 | 9920 |
| \( 1^{++} \) | 1\( p_1[T_0,T_1] \) | 9920 | 9892.78 \( \pm \) 0.26 \( \pm \) 0.31 | 9920 |
| \( 2^{++} \) | 1\( p_1[T_0,T_1] \) | 9920 | 9912.21 \( \pm \) 0.26 \( \pm \) 0.31 | 9920 |
| \( 0^{++} \) | 2\( p_1[T_0,T_1] \) | 10259 | 10232.5 \( \pm \) 0.4 \( \pm \) 0.5 | 10259 |
| \( 1^{++} \) | 2\( p_1[T_0,T_1] \) | 10259 | 10255.46 \( \pm \) 0.22 \( \pm \) 0.50 | 10259 |
| \( 2^{++} \) | 2\( p_1[T_0,T_1] \) | 10259 | 10268.65 \( \pm \) 0.22 \( \pm \) 0.50 | 10259 |
| \( 0^{++} \) | 3\( p_1[T_0,T_1] \) | 10521 |  | 10531 |
| \( 1^{++} \) | 3\( p_1[T_0,T_1] \) | 10526 | | 10531 |
| | | | \( \pm 10 \) | |
| \( 2^{++} \) | 3\( p_1[T_0,T_1] \) | 10528 | | 10531 |
| \( 0^{++} \) | 1\( p_1[T_1,T_2] \) | 10620 | | |
| \( 0^{++} \) | 1\( p_1[T_1,T_2] \) | 10704 | | |
| \( 1^{++} \) | 1\( p_1[T_1,T_2] \) | | \( - \) | |
| \( 1^{++} \) | 1\( p_1[T_1,T_2] \) | 10708 | | |
| \( 2^{++} \) | 1\( p_1[T_1,T_2] \) | 10710 | | |
| \( 0^{++} \) | 1\( p_1[T_2,T_4] \) | 10784 | | |
| \( 2^{++} \) | 2\( p_1[T_1,T_2] \) | 10815 | | |
| \( 1^{++} \) | 1\( p_1[T_3,T_4] \) | 10822 | | |

\[
\Gamma_{(Q\bar{Q})n_s[T_0,T_1]} \to e^+e^- \simeq \frac{16\pi\alpha^2(1 - \frac{16\alpha_s}{3\pi})}{M_{\text{EQM}}^2} |\Phi_{n_s[T_0,T_1]}(0)|^2
\]

where \( \alpha_Q \) is the quark charge (\( e_c = 2/3, e_b = -1/3 \)), \( \alpha \simeq \frac{1}{137} \) is the electromagnetic coupling constant and \( \alpha_s \) the strong coupling strength. \( \Phi_{n_s} \) and \( M_{n_s} \) stand for the wave function and the mass of the Cornell state \( (Q\bar{Q})_{n_s} \).

It should be pointed out that this formula works well when comparing ratios of leptonic widths with data. However, it does not reproduce precisely absolute leptonic widths what would require the consideration of uncontrolled QCD and relativistic corrections.

From [10] it seems natural to propose for the \( n_s[T_0,T_1] \) \( J^{++} \) bottomonium masses the expression

\[
\Gamma_{(Q\bar{Q})n_s[T_0,T_1]} \to e^+e^- \simeq 16\pi\alpha^2(1 - \frac{16\alpha_s}{3\pi}) |\Phi_{n_s[T_0,T_1]}(0)|^2 M_{n_s[T_0,T_1]}^2
\]

In order to generalize this expression to the interthreshold energy regions \( [T_{j-1}, T_{j}] \) with \( j > 1 \) we can repeat the reasoning employed in the first energy region in the sense that far below the upper threshold \( T_j \) the EQM states \( n_s[T_{j-1}, T_j] \) are identical to their generator states \( n_s[T_{j-1}, T_j] \). There are two mass dimensional terms for the generator state problem, the mass of the state \( M_{n_s[T_{j-1}, T_j]} \) (equal to \( M_{n_s[T_{j-1}, T_j]} \) far below \( T_j \)) and the mass of the threshold \( M_{T_{j-1}} \) so that the unknown mass factor in the leptonic width could involve both.

Taking into account these considerations for \( n_s[T_{j-1}, T_j] \) EQM states with \( j > 1 \) and the expression assumed above for the \( n_s[T_0,T_1] \) case we shall adopt for the \( (Q\bar{Q})_{n_s[T_0,T_1]} \) leptonic width \( (i \geq 1) \) the general ansatz:

\[
\Gamma_{(Q\bar{Q})n_s[T_0,T_1]} \to e^+e^- \simeq 16\pi\alpha^2(1 - \frac{16\alpha_s}{3\pi}) |\Phi_{n_s[T_0,T_1]}(0)|^2 M_{n_s[T_0,T_1]}^2
\]

where the square mass factor \( \tilde{M}^2 \) must satisfy \( \tilde{M}^2(M_{n_s[T_0,T_1]}/M_{n_s[T_{j-1}, T_j]}) = M^2_{T_{j-1}} \).

We shall make use of this ansatz to try to fix the specific form of \( \tilde{M}^2(M_{n_s[T_{j-1}, T_j]}, M_{n_s[T_0,T_1]}) \) from phenomenology. In this respect one should keep in mind, according to our previous comments, the convenience of comparing predicted ratios of leptonic widths with data (when available) instead of comparing absolute leptonic widths.

VI. STRONG DECAYS

A consistent study of heavy quarkonia strong decay processes within our quark model framework should imply not only the EQM description of the heavy quarkonia initial states but also the EQM description of the final state mesons involving which involve light quarks. This would require further refinements of the EQM model and it is out of the scope of our current analysis. As an alternative for a qualitative understanding of the physical mechanisms underlying the decay and of the dominant decay modes we shall use the decomposition of the EQM states in their generator and molecular components [12] and apply the lore on strong decays from conventional quark and molecular models.
In what follows we shall consider, for a definite set of quantum numbers $I(J^{PC})$, a EQM bound state $|\bar{Q}Q\rangle_{k_{[T_1,\tau_1]}}$ with mass $M_{k_{[T_1,\tau_1]}}$ above the first absolute threshold (note that this threshold may correspond to other $J^{PC}$ quantum numbers, for instance for $\sigma$ the first absolute threshold is $D^0\bar{D}^0$ with $0^+$, see Table XIII below).

**A. Generator Component Decay Modes**

Regarding the generator component $|\bar{Q}Q\rangle_{k_{[T_1,\tau_1]}}$ we shall assume the dominant decay modes correspond to transitions to kinematically allowed open-flavor two meson final states.

Actually this is an usual assumption when evaluating decays in conventional quark models (note that for $i = 1$ the component $|\bar{Q}Q\rangle_{k_{[T_0,\tau_0]}}$ is a conventional Cornell state). The physical mechanism underlying these decays is the creation of light quark - light antiquark pairs $q\bar{q}$ (think for example of a $\Delta F = 0$ model). Notice though that the quantitative implementations of this mechanism carried out until now have not allowed for a precise computation of strong widths (see for instance [13 14]).

Let us consider separately the $i > 1$ and the $i = 1$ cases.

1. $i > 1$ Case

The dominant decay modes will be then the $S-$ wave open flavor meson-meson channels with mass closer below $M_{k_{[T_1,\tau_1]}}$. Let us remind that $M_{T_1-1} < M_{k_{[T_1,\tau_1]}} < M_{T_1}$. Then if $T_{i-1}$ is a regular (non effective) threshold the dominant mode will be the $S-$ wave $M_{1T_{i-1}}M_{2T_{i-1}}$ channel:

$$|\bar{Q}Q\rangle_{k_{[T_1,\tau_1]}} \rightarrow M_{1T_{i-1}}M_{2T_{i-1}}$$

Else if $T_{i-1}$ is an effective threshold, resulting from the two thresholds $T_{i-1}^1$ and $T_{i-1}^2$, and $M_{T_{i-1}} \approx M_{T_{i-1}^1} < M_{k_{[T_1,\tau_1]}}$, then the dominant decay modes will be the $S-$ wave $M_{1T_{i-1}^1}M_{2T_{i-1}^1}$ and $M_{1T_{i-1}^2}M_{2T_{i-1}^2}$ channels:

$$|\bar{Q}Q\rangle_{k_{[T_1,\tau_1]}} \rightarrow M_{1T_{i-1}^1}M_{2T_{i-1}^1}$$

$$|\bar{Q}Q\rangle_{k_{[T_1,\tau_1]}} \rightarrow M_{1T_{i-1}^2}M_{2T_{i-1}^2}$$

Else if $T_{i-1}$ is an effective threshold and $M_{k_{[T_1,\tau_1]}} < M_{T_{i-1}}$, $M_{T_{i-1}}^1$ then the dominant decay mode will be the $S-$ wave open flavor meson-meson channel with mass closer below $M_{k_{[T_1,\tau_1]}}$. Notice though that $P$ or $D-$ waves could also contribute if there is no available $S-$ wave channel or it is far below threshold. We shall generically denote this meson-meson channel as $\mathcal{M}_{12} < \mathcal{M}_{12}$. Furthermore let us realize that in this case even virtual $S-$ wave $M_{1T_{i-1}^1}M_{2T_{i-1}^2}$ and $M_{1T_{i-1}^2}M_{2T_{i-1}^2}$ channels could be active if $M_{k_{[T_1,\tau_1]}} \approx M_{T_{i-1}} \approx M_{T_{i-1}}^2$. Hence we have the decay modes:

$$|\bar{Q}Q\rangle_{k_{[T_1,\tau_1]}} \rightarrow (\mathcal{M}_{12} \mathcal{M}_{12})_{\text{virtual}}$$

2. $i = 1$ Case

As $0 < M_{k_{[T_1,\tau_1]}} < M_{T_1}$ the dominant decay mode will be a $P$ (or $D$) wave open flavor meson-meson channel with mass closer below $M_{k_{[T_1,\tau_1]}}$. We shall generically call this meson-meson channel $\mathcal{M}_{12}^1 \mathcal{M}_{12}^2$. Of particular interest will be the case when $T_1$ is a non effective threshold and $M_{k_{[T_1,\tau_1]}} \approx M_{T_1}$. Then the dominant decay may proceed through the virtual $S-$ wave $M_{1T_1}M_{2T_1}$ channel:

$$|\bar{Q}Q\rangle_{k_{[T_0,\tau_0]}} \rightarrow (\mathcal{M}_{1T_1} \mathcal{M}_{2T_1})_{\text{virtual}}$$

**B. Molecular Component Decay Modes**

As for the molecular component for a regular (non effective) threshold $|\mathcal{M}_{1T_1} \mathcal{M}_{2T_1}|$ we shall assume the leading order interaction between $\mathcal{M}_{1T_1}$, with structure $|\bar{Q}Q\rangle$, and $\mathcal{M}_{2T_1}$, with structure $|\bar{q}q\rangle$, involves quark exchange between the mesons, giving rise to a heavy-heavy meson with structure $|\bar{Q}Q\rangle$ and to a light-light meson with structure $|\bar{q}q\rangle$.

Actually this is the proposed decay mechanism for molecular states in reference [3] based on the analysis of the color structure of the quark-antiquark interaction (whose spectroscopic contribution is implicit in the values of the parameters we have used for the potential).

Therefore we shall consider that the $|\mathcal{M}_{1T_1} \mathcal{M}_{2T_1}|$ decay proceeds through the kinematically allowed (heavy-heavy + light-light) two meson channels. In particular $S-$ wave two meson channels with mass close below $M_{k_{[T_1,\tau_1]}}$...
can be expected to be favored (again $P$ or $D$—waves could also contribute if there were not $S$—wave channel close below threshold; in this case even a virtual $S$—wave channel at about threshold could be active if $M_{T_{l=1}} \simeq M_{T_{l}}$). Moreover, as the heavy quark $(Q)$ mass is much bigger than the light quark $(q)$ one we may assign the orbital angular momentum to the light quarks so that the formation of an $l = 0$ heavy-heavy meson state may be expected to prevail.

By adopting this decay mechanism for effective thresholds as well we shall assume for the molecular component $|\mathbf{M}_{1T},\mathbf{M}_{2T}\rangle$ that kinematically allowed ($|(Q\bar{q})_{1}=0 + (\bar{q}q)|$) two meson $S$—wave channels are the dominant decay modes.

By denoting the two mesons as $M_{(Q\bar{q})_{1}=0}$ and $M_{(\bar{q}q)}$ we may schematically represent

$$|\mathbf{M}_{1T},\mathbf{M}_{2T}\rangle \rightarrow M_{(Q\bar{q})_{1}=0} M_{(\bar{q}q)}$$

VII. E1 TRANSITIONS

The decomposition the EQM states in their generator and molecular components (12) is also useful to have a qualitative understanding of the physical mechanisms underlying E1 transitions $(Q\bar{q})_{initial} \rightarrow \gamma(Q\bar{q})_{final}$. So in conventional quark models the decay is assumed to take place through the Elementary Emission (EE) of the photon by the quark or the antiquark (in the nonrelativistic, zero recoil and dipole approximations) whereas in molecular models Vector Meson Dominance (VMD) the photon comes out through $p$ and $\omega$ conversion is usually employed. This drives us to assume for the E1 electromagnetic decays of the EQM states an Extended Vector Domiance (EVMD) mechanism (13). This is an EE mechanism for the generator component and a VMD mechanism for the molecular one.

Although we do not develop here the quantitative aspects of the EVMD mechanism which will be the subject of a future work we shall use this mechanism to analyze some results from conventional quark and molecular models in the description of $X(3872)$ (see below).

VIII. CHARMONIUM

Charmonium masses from the EQM have been listed in Tables VII and IX where they are compared to data from [11] and to Cornell masses.

A list of the thresholds used in the calculation of the charmonium spectrum (the end of the list is imposed by the current partial or total lack of knowledge of further open flavor mesons) appears in Table XII where a simplified notation with respect to the one used in Tables I and

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{JPC} & \textbf{T} & \textbf{Meson1} – \textbf{Meson2} & \textbf{(J}_1^\text{PC},\textbf{J}_2^\text{PC}) & \textbf{MT}\_1 \\
\hline
1\textsuperscript{--} & T_1 & D^0\bar{D}^0 & (0^-,1^-) & ? : 4237 \\
 & T_2 & D^* (2007)\bar{D}^* (2420) & (1^-,0^+) & 4325 \\
 & T_3 & D^*(2007)\bar{D}_1 (2430) & (1^-,1^+) & ? : 4379 \\
 & T_4 & D^*(2007)\bar{D}_2 (2450) & (1^-,2^+) & 4470 \\
 & T_5 & D_{1+}^* D_{1-} (2535) & (0^-,1^+) & 4504 \\
 & T_6 & D_{2+}^* D_{1-} (2536) & (1^-,1^+) & 4572 \\
 & T_7 & D_{2+}^* D_{2-} (2536) & (1^-,2^+) & 4648 \\
 & T_8 & D_{2+}^* D_{3a} (2573) & (1^-,2^+) & 4685 \\
\hline
0\textsuperscript{++} & T_1 & D^0\bar{D}^0 & (0^-,0^-) & 3730 \\
 & T_2 & D_{1-}^* D_{1-} & (0^-,0^-) & 3937 \\
 & T_3 & D^* (2007)\bar{D} (2007) & (1^-,1^-) & 4014 \\
 & T_4 & D_{1+}^* D_{1-} & (1^-,1^-) & 4224 \\
 & T_5 & D^0\bar{D} (2550) & (0^-,0^-) & 4405 \\
\hline
1\textsuperscript{++} & T_1 & D^0\bar{D}^* (2007) & (0^-,1^-) & 3872 \\
 & T_2 & D^* (2007)\bar{D} (2007) & (1^-,1^-) & 4014 \\
 & T_3 & D_{1+}^* D_{1-} & (0^-,1^-) & 4080 \\
 & T_4 & D_{2+}^* D_{1-} & (1^-,1^-) & 4224 \\
\hline
2\textsuperscript{++} & T_1 & D^* (2007)\bar{D} (2007) & (1^-,1^-) & 4014 \\
 & T_2 & D_{1+}^* D_{1-} & (1^-,1^-) & 4224 \\
\hline
\end{tabular}
\caption{Meson-meson thresholds for $I = 0$ $\bar{s}d$ states. $J^P_{1}$ and $J^P_{2}$ stand for the angular momenta of the mesons entering in the threshold. Threshold masses $(M_{T_{l}})$ obtained from the charmed and charmed strange meson masses quoted in [11]. A question mark followed by a colon precedes the chosen mass for an effective threshold.}
\end{table}

\[1\] has been employed: a threshold has been denoted by the first meson-meson term entering in the $I = 0$ linear combination which defines its content. Thus, the first $0(0^{++})$ threshold in Table I $(D^0\bar{D}^0 - D^+ D^-)$ is specified as $D^0\bar{D}^0$ and so on.

As mentioned before the masses of the effective thresholds have to be fixed phenomenologically. We have denoted these masses by a question mark followed by a
colon and by our guessed value which have been chosen so as to reproduce the charmonium spectrum. Thus we have considered two effective thresholds with masses 50 MeV smaller than the known masses $M_{D^0(D_{1}(2420))}$ and $M_{D^*(2007)^0(D_{1}(2420))}$ respectively. Let us note that the second effective threshold substitutes four overlapping thresholds whilst the first one substitutes only two. However it is plausible that the effect from thresholds containing $u\bar{u}$ and $d\bar{d}$ is bigger than for thresholds with $s\bar{s}$ content. This may justify our choice of two effective thresholds with the same difference in mass with respect to the overlapping ones. In any case we have checked that decreasing or increasing the mass of the second effective threshold by less than 30 MeV would not alter the resulting spectral pattern. Notice also that we could increase the mass of the first effective threshold by 30 MeV or even more and still reproduce the known charmonium pattern but the mass for $X(4260)$ would be badly overestimated (on the contrary a decrease in the mass of the first effective threshold would improve the mass description).

A glance at Tables [VIII] and [XII] makes clear that mixing is in general (an exception is the $X(3872)$) more relevant for states above the first threshold (notice that $\theta = 45^\circ$ corresponds to equal probability for the generator and molecular components). This is also manifest from the comparison of the masses and rms radii for the EQM and generator states: the masses $M_{EQM}$ are smaller than the corresponding $M_{GEN}$ whereas the rms radii $\langle r^2 \rangle^{1/2}_{EQM}$ are bigger than $\langle r^2 \rangle^{1/2}_{GEN}$. The higher the mixing the larger these differences.

In order to establish the dominant strong decay modes of the EQM states from the molecular component we need to know, following Section ??, apart from their masses and related thresholds, the masses for the $S$-wave $(|x\rangle_{L=0}^+)$ light-light meson decay channels coupling to the same quantum numbers. These masses are tabulated in Table [XV].

| $J^{PC}$ | $[T_{i-1}, T_i]$ | $M_{T_{i-1}, T_i}$ MeV | $k$ | $M_{EQM}$ MeV | $M_{GEN}$ MeV | $\langle r^2 \rangle^{1/2}_{EQM}$ (fm) | $\langle r^2 \rangle^{1/2}_{GEN}$ (fm) | $\theta(\circ)$ |
|---------|-----------------|---------------------|-----|--------------|--------------|-------------------------------|-------------------------------|----------|
| 1$^{--}$ | $[T_0, T_1]$ | [0, 4237] | 1s | 3046 | 3046 | (0.5) | (0.5) | 0 |
|         |                 |                     |     | 2s | 3632 | 3632 | (0.9) | (0.9) | 0 |
|         |                 |                     |     | 1d | 3743 | 3743 | (0.9) | (0.9) | 0.2 |
|         |                 |                     |     | 3s | 4061 | 4066 | (1.2) | (1.2) | 5 |
|         |                 |                     |     | 2d | 4137 | 4143 | (1.3) | (1.3) | 6 |
|         | $[T_1, T_2]$ | [4237, 4325] | 1s | 4280 | 4290 | (1.5) | (1.2) | 15 |
|         | $[T_2, T_3]$ | [4325, 4379] | 1s | 4360 | 4374 | (1.8) | 1.3 | 23 |
|         | $[T_3, T_4]$ | [4379, 4470] | 1s | 4418 | 4425 | 1.5 | 1.3 | 12 |
|         | $[T_4, T_5]$ | [4470, 4504] | 1s | 4496 | 4513 | (2.3) | (1.4) | 33 |
|         | $[T_5, T_6]$ | [4504, 4572] | 1s | 4537 | 4545 | (1.7) | (1.4) | 16 |
|         | $[T_6, T_7]$ | [4572, 4648] | 1s | 4604 | 4611 | (1.7) | (1.5) | 13 |
|         | $[T_7, T_8]$ | [4648, 4685] | 1s | 4673 | 4684 | (2.2) | (1.5) | 26 |

TABLE XIII: Calculated mixing angles $\theta_{[T_{i-1}, T_i]}$, generically called $\theta$, for the charmonium $1^{--}$ EQM states. The corresponding spectral energy regions, $[M_{T_{i-1}, T_i}]$, are shown. For the sake of completeness the EQM masses, $M_{EQM}$, and the rms radii, $\langle r^2 \rangle^{1/2}_{EQM}$, are also listed and compared to the generator masses, $M_{GEN}$, and rms radii, $\langle r^2 \rangle^{1/2}_{GEN}$.

We shall consider the resonances $f$, $\omega$ and $\phi$ as mixed components containing $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ components. Therefore molecular components formed by charmed mesons as well as by charmed strange mesons can decay into two body channels containing these resonances. Notice though that when the kinematically dominant decay mode involves prominently a light quark flavor not present in the molecular component some alternative decay could be favored. For instance the $J/\psi \phi$ mode involves prominently $s\bar{s}$ while the molecular component of the $(0^+)(0^{++})$, $1p_{[T_{i}, T_{j}]}$ state corresponding to $D^0\bar{D}(2550)^0$ (see Table [XIII]) involves $u\bar{u}$ and $d\bar{d}$. So an alternative decay to $J/\psi \omega$ might be favored.

Once we know the masses for the EQM states, their
Thus in Tables XVI and XVII these modes for EQM states above the first absolute threshold $D^0\bar D^0$ are listed.

| $J^{PC}$ | $[M_{T_{i-1}}, M_{T_i}]$ (MeV) | $k$ | $M_{EQM}$ (MeV) | $M_{GEN}$ (MeV) | $\theta(\circ)$ |
|----------|-----------------|-----|----------------|----------------|-------------|
| 0++      | $[T_0, T_1]$    | 1p  | 3456           | 3456           | 0.6         |
|          |                  |     | (0.7)          | (0.7)          |             |
|          | $[T_1, T_2]$    | 1p  | 3898           | 3916           | 16          |
|          |                  |     | (1.3)          | (1.1)          |             |
|          | $[T_2, T_3]$    | 1p  |                | 4083           |             |
|          |                  |     | (1.3)          |                |             |
|          | $[T_3, T_4]$    | 1p  | 4140           | 4149           | 9           |
|          |                  |     | (1.4)          | (1.3)          |             |
|          | $[T_4, T_5]$    | 1p  | 4325           | 4333           | 9           |
|          |                  |     | (1.5)          | (1.4)          |             |
| 1++      | $[T_0, T_1]$    | 1p  | 3456           | 3456           | 0.6         |
|          |                  |     | (0.7)          | (0.7)          |             |
|          |                  | 2p  | 3871.6         | 3911           | 41          |
|          |                  |     | (3.3)          | (1.1)          |             |
|          | $[T_1, T_2]$    | 1p  | 4003           | 4029           | 27          |
|          |                  |     | (1.7)          | (1.2)          |             |
|          | $[T_2, T_3]$    | 1p  |                | 4149           |             |
|          |                  |     | (1.3)          |                |             |
|          | $[T_3, T_4]$    | 1p  | 4190           | 4206           | 17          |
|          |                  |     | (1.6)          | (1.3)          |             |
| 2++      | $[T_0, T_1]$    | 1p  | 3456           | 3456           | 0.3         |
|          |                  |     | (0.7)          | (0.7)          |             |
|          |                  | 2p  | 3903           | 3911           | 7           |
|          |                  |     | (1.2)          | (1.1)          |             |
|          | $[T_1, T_2]$    | 1p  | 4140           | 4149           | 9           |
|          |                  |     | (1.4)          | (1.3)          |             |

TABLE XIV: Calculated mixing angles $\theta_k[T_{i-1}, T_i]$, generically called $\theta$, for the charmonium $J^{++}$ EQM states. The corresponding spectral energy regions, $[M_{T_{i-1}}, M_{T_i}]$, are shown. For the sake of completeness the EQM masses, $M_{EQM}$, and the rms radii, $\langle r^2 \rangle_{EQM}$, are also listed and compared to the generator masses, $M_{GEN}$, and rms radii, $\langle r^2 \rangle_{GEN}$. For $(0^{++}, 1^{++})$ the insufficient threshold attraction prevents the generation of a EQM state in the $[T_2, T_3]$ region.

related thresholds and the masses for the $S$– wave $((c\bar c)_{l=0} +$ light-light meson) channels we can establish, following Section VII the dominant strong decay modes. Thus in Tables XVI and XVII these modes for EQM states above the first absolute threshold $D^0\bar D^0$ are listed.

The comparison of masses and decay modes with data makes feasible a sensible identification of EQM states with charmonium resonances. As a matter of fact this assignment has been implicitly used in Tables VIII and IX to locate any EQM state and its assigned resonance in the same row.

It is worth to realize that relativistic corrections to the calculated masses are expected to be much more important for states in the first energy region than for interthreshold energy states. The calculation of their masses, $M_{CH}$, the quoted masses for $(c\bar c)_{l=0}$ and light-light mesons from [11] have been used. When the mass of some light-light meson, $f_0(500)$ and $f_0(1370)$, is quite uncertain we have put a symbol $\sim$ in front of its nominal value.

The comparison of masses and decay modes with data makes feasible a sensible identification of EQM states with charmonium resonances. As a matter of fact this assignment has been implicitly used in Tables VIII and IX to locate any EQM state and its assigned resonance in the same row.

Let us analyze more in detail the resulting spectrum.

A. First Energy Region : $M_{EQM} \in [M_{T_0}, M_{T_1}]$

In the first energy region most states are very predominantly Cornell like states (let us recall that in this region the generators are eigenstates of the Cornell potential).

TABLE XV: Two-body decay channels including $l = 0$ hidden charm states. For the calculation of their masses, $M_{CH}$, the quoted masses for $(c\bar c)_{l=0}$ and light-light mesons from [11] have been used. When the mass of some light-light meson, $f_0(500)$ and $f_0(1370)$, is quite uncertain we have put a symbol $\sim$ in front of its nominal value.
TABLE XVI: Dominant strong decays (subindex $P$ for required $P$-wave) from the generator and molecular components for $1^{--}$ EQM charmonium states with masses $M_{EQM}$ above the absolute threshold $\sqrt{s}$ at 3730 MeV. $f_{01.2}(\sim 1300)$ and $f_{01.2}(\sim 1400)$ stand for $f_0(\sim 1370)$ together with $(f_1(1285), f_2(1270))$ and $(f_1(1420), f_2(1430))$ respectively. The final products result from the decays of the second members of the decay channels (excepting $D^{*}$'s and $D_s^*$'s) as given in [11].

| $J^{PC}$ | State Notation | Strong Decay Modes | Final Products |
|----------|----------------|-------------------|---------------|
| $1^{--}$ | $D_{s}D_{s1}(2536)$ | $D_{s}D_{s}^*$  | $D_{s}D_{s}^*$ |
|          | $(\psi(3s)f_{0}(\sim 500))$ | $(\psi(3s)f_{0}(\sim 500))$ | $(\psi(3s)f_{0}(\sim 500))$ |
|          | $1s_{[76, T_8]}$ | $D_{s}D_{s1}(2460)$ | $D_{s}D_{s1}(2460)$ |
|          | $(\psi(3s)f_{0}(\sim 500))$ | $(\psi(3s)f_{0}(\sim 500))$ | $(\psi(3s)f_{0}(\sim 500))$ |
|          | $2d_{[76, T_8]}$ | $(D_{s}D_{s}^*, D_s^*D_{s}^*, D_{s}^*D_{s}^*)_p$ | $D_{s}D_{s}^*$  |
|          | $3s_{[76, T_8]}$ | $(D_{s}^*D_{s}^*, D_{s}^*D_{s}^*, D_{s}^*D_{s}^*)_p$ | $D_{s}D_{s}^*$  |
|          | $1s_{[76, T_8]}$ | $(D_{s}D_{s1}(2460), D_{s}D_{s1}(2460))$ | $D_{s}D_{s1}(2460)$ |
|          | $1s_{[76, T_8]}$ | $(D_{s}D_{s1}(2460), D_{s}D_{s1}(2460))$ | $D_{s}D_{s1}(2460)$ |
|          | $0^{++}$ | $2d_{[76, T_8]}$ | $(D_{s}D_{s}^*, D_s^*D_{s}^*, D_{s}^*D_{s}^*)_p$ |
|          | $1s_{[76, T_8]}$ | $(D_{s}D_{s}^*, D_s^*D_{s}^*, D_{s}^*D_{s}^*)_p$ | $D_{s}D_{s}^*$  |
|          | $1s_{[76, T_8]}$ | $(D_{s}D_{s}^*, D_s^*D_{s}^*, D_{s}^*D_{s}^*)_p$ | $D_{s}D_{s}^*$  |
|          | $1s_{[76, T_8]}$ | $(D_{s}D_{s}^*, D_s^*D_{s}^*, D_{s}^*D_{s}^*)_p$ | $D_{s}D_{s}^*$  |

We shall assume that the implementation of non considered relativistic and spin dependent corrections in the EQM may give, to a large extent, proper account of the experimental masses and properties of resonances in this region. In this regard we expect mass corrections to be more significant for the lower energy states.

Mixing angles $\theta$ in this region, see Tables XIII and XIV have values below $8^\circ$ except for the more prominent non-dominant Cornell resonance, the $2_{P[T_6, T_8]}$ state with quantum numbers $1^{++}$ that we assign to $X(3872)$ as explained below.

More precisely for $J^{PC} = 1^{--}$ the $1s_{[76, T_8]}(3046)$, $2s_{[76, T_8]}(3632)$ and $1d_{[76, T_8]}(3743)$ EQM states can be
considered as pure or almost pure Cornell states ($\theta \simeq 0^\circ$), see Table XIV. They are assigned to the $J/\psi (1s), \psi (2s)$ and $\psi (3770)$ PDG resonances.

For $3s_{[T_0,T_1]}(4061)$ and $2d_{[T_0,T_1]}(4137)$, assigned to $\psi (4040)$ and $\psi (4160)$ respectively, there is a small mixing: $\theta_{3s} = 5^\circ$ and $\theta_{2d} = 6^\circ$. From it we expect for $3s_{[T_0,T_1]}$ and $2d_{[T_0,T_1]}$ some increase of the $J/\psi \pi \pi$ decay width with respect to their corresponding Cornell generators since this mode is dominant for the molecular component (see Table XVII). Unfortunately we have only an experimental upper bound for the width $\Gamma(J/\psi) < 320$ KeV (see Table 12 in [6] and references therein) so that we can not extract any definite conclusion about this expectation.

Regarding $J^{PC} = J^{++}$ the $1p_{[T_0,T_1]}$ EQM states for $J = 0, 1, 2$, all with a mass of 3456 MeV, have very small mixing ($0.6^\circ, 0.6^\circ$ and $0.3^\circ$ respectively (see Table XIV). These three states are assigned to $\chi_{c1}(1p)$, $\chi_{c2}(1p)$ and $\chi_{c0}(1p)$ respectively.

On the other hand the $2p_{[T_0,T_1]}(3903)$ EQM state for $J^{PC} = 2^{++}$, which is assigned to $\chi_{c2}(2p)$ with a mass of 3929 MeV, should show some effect from mixing since $\theta_{2p(3903)} = 7^\circ$ (see Table XIV). In particular as the EQM mass is above that of $\omega J/\psi$ we expect an increase of the decay width to $\omega J/\psi$ (with respect to the Cornell decay width) caused by the molecular component (see Table XVII).

As for the $2p$ EQM state for $J^{PC} = 1^{++}$ the mixing is most relevant as we analyze next.

1. $X(3872)$

In the EQM the $(1^{++}, (2p)_{[T_0,T_1]})$ with mass 3871.6 MeV, which we identify with $X(3872)$, results from the attraction produced by the $D^0 D^{*}(2007)$ threshold on the $(1^{++}, (2p)_{[T_0,T_\infty]})$ Cornell generator state with mass 3911 MeV. As pointed out before the precise coincidence of the EQM mass with the experimental one has been required to fine tune $m_c$. As a matter of fact we could get any mass as close (below) as we wanted to the threshold by slightly changing $m_c$.

From Table XIV we have $\theta_{2p(3903)}^{1^{++}} \simeq 41^\circ$ so that the state can be decomposed as

$$X(3872) \rightarrow 0.75 \zeta_{c1}(2p) + 0.66 \Psi_{1^{++}}(D D^*)$$

where we have used a more specific notation than in Section IV A. Thus $\zeta$ and $\Psi$ stand for the generator (Cornell) and molecular components respectively. Note that although $X(3872)$ is still predominantly a quark-antiquark conventional state (56%) the molecular component is very significant (44%). This is correlated to the fact that the root mean square radius (r.m.s.) for $X(3872) \equiv (1^{++}, (2p)_{[T_0,T_1]})$ is 3.3 fm, much larger than the r.m.s. for the generator Cornell state $(1^{++}, 2p_{[T_0,T_\infty]})$ which is 1.1 fm, see Table XIV.

Let us consider first the decays to $\gamma J/\psi$ and $\gamma \psi(2s)$. As explained in Section VII electromagnetic decays can also be analyzed from the compositeness of the EQM state. Thus, the absolute value of the amplitude for $\Psi_{1^{++}}(D D^*) \rightarrow \gamma \psi(2s)$ is expected to be very small since $(\omega, \rho)$ vector meson dominance is kinematically suppressed. On the other hand the absolute value of the amplitude for $\Psi_{1^{++}}(D D^*) \rightarrow \gamma J/\psi$ could be significantly smaller than for $\zeta_{c1}(2p) \rightarrow \gamma J/\psi$ as it is the case for the molecular model considered in [4] (see Table 9 of this reference). If this were the case

$$\Gamma(X(3872) \rightarrow \gamma J/\psi) \simeq 0.56 \Gamma(\zeta_{c1}(2p) \rightarrow \gamma J/\psi)$$

then

$$\Gamma(X(3872) \rightarrow \gamma \psi(2s)) \simeq \frac{\Gamma(\zeta_{c1}(2p) \rightarrow \gamma \psi(2s))}{\Gamma(\zeta_{c1}(2p) \rightarrow \gamma J/\psi)}$$

Unfortunately the left hand side ratio is not well measured experimentally. Indeed measurements from different groups seem to be not compatible (see Table 12 in [6] and references therein).

However, recalling the experimental result for 1p states

$$\frac{\Gamma(\chi_{c1}(1p) \rightarrow \gamma \psi(2s))}{\Gamma(\chi_{c1}(1p) \rightarrow \gamma J/\psi)} \simeq \frac{\Gamma(\chi_{b1}(1p) \rightarrow \gamma \Upsilon(2s))}{\Gamma(\chi_{b1}(1p) \rightarrow \gamma \Upsilon(1s))}$$

and taking into account that $\chi_{c1}(1p)$ and $\chi_{b1}(1p)$ are pure Cornell states (see Section VIII A and IX A), $\chi_{c1}(1p) \simeq \zeta_{c1}(1p)$ and $\chi_{b1}(1p) \simeq \zeta_{b1}(1p)$, we may tentatively assume a similar ratio for 2p states:

$$\frac{\Gamma(\zeta_{c1}(2p) \rightarrow \gamma \psi(2s))}{\Gamma(\zeta_{c1}(2p) \rightarrow \gamma J/\psi)} \simeq \frac{\Gamma(\zeta_{b1}(2p) \rightarrow \gamma \Upsilon(2s))}{\Gamma(\zeta_{b1}(2p) \rightarrow \gamma \Upsilon(1s))}$$

As the bottomonium state $\chi_{b1}(2p)$ is also an almost pure Cornell state (see Section IX A), $\chi_{b1}(2p) \simeq \zeta_{b1}(2p)$ we may substitute

$$\frac{\Gamma(\zeta_{b1}(2p) \rightarrow \gamma \Upsilon(2s))}{\Gamma(\zeta_{b1}(2p) \rightarrow \gamma \Upsilon(1s))} \simeq \frac{\Gamma(\chi_{b1}(2p) \rightarrow \gamma \Upsilon(2s))}{\Gamma(\chi_{b1}(2p) \rightarrow \gamma \Upsilon(1s))}$$

where the right hand side is well measured experimentally

$$\frac{\Gamma(\chi_{b1}(2p) \rightarrow \gamma \Upsilon(2s))}{\Gamma(\chi_{b1}(2p) \rightarrow \gamma \Upsilon(1s))} \simeq \frac{0.199 \pm 0.019}{0.092 \pm 0.008} = 2.2 \pm 0.4$$


Putting together (20), (21), (22) and (23) we may conclude that
\[ \frac{\Gamma(X(3872) \to \gamma J/\psi(2s))}{\Gamma(\Psi(1S) \to \gamma J/\psi)} \simeq 2.2 \pm 0.4 \]
although this result could be modified by the molecular contribution to the \( \gamma J/\psi \) decay.

Let us consider now strong decays. Let us take for example \( X(3872) \to \omega J/\psi \). As the \( \chi_b(2p) \to \omega \Upsilon(1S) \) decay has a branching ratio of 1.63% there may also be a comparable ratio for \( \zeta_{1}(2p) \to \omega J/\psi \). From the estimated partial width for \( \chi_b(2p) \to \omega \Upsilon(1S) : 1.56 \pm 0.46 \) KeV (see Table 8 in reference [3]) we may reasonably expect, using the parallelism between charmonium and bottomonium results, the partial width for \( \zeta_{1}(2p) \to \omega J/\psi \) to be of the same order of magnitude.

We may also have a \( \Psi_{1++} (D\overline{D}^*) \to \omega J/\psi \) decay due to the isospin breaking driven by the difference in mass between the \( D^{+}\Upsilon^{*}(2010)^{-} \) and \( D^{0}\Upsilon^{*}(2007)^{0} \) components of the threshold. Actually the \( D^{+}\Upsilon^{*}(2010)^{-} \) mass (3880 MeV) coincides with \( M_{\omega} + M_{J/\psi} \) making feasible the decay \( \Gamma (\Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-}\pi^{0} J/\psi) \) through an intermediate \( \omega \). For the same reason (let us recall that \( D^{+}\Upsilon^{*}(2010)^{-} \) is a mixing of \( I = 0 \) and \( I = 1 \) states) it is also possible the \( \Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-} J/\psi \) decay through an intermediate \( \rho \).

The width \( \Gamma (\Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-}\pi^{0} J/\psi) \) has been estimated with the molecular model of [4] (see Table 8 of this reference) to be smaller than \( \Gamma (\Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-} J/\psi) \). Although these quantitative estimations (720 KeV vs 1290 KeV) can not be taken for granted (see below) they seem to suggest that our estimated value for \( \Gamma (\zeta_{1}(2p) \to \omega J/\psi) \) of the order of a few KeV could be much smaller than \( \Gamma (\Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-}\pi^{0} J/\psi) \). Therefore we may tentatively conclude that
\[ \Gamma (X(3872) \to \omega J/\psi) \sim 0.44 \Gamma (\Psi_{1++} (D\overline{D}^*) \to \omega J/\psi) \]
(24)

A similar argument can be applied to \( \zeta_{1}(2p) \to \pi^{+}\pi^{-} J/\psi \) as compared to \( \Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-} J/\psi \) so that
\[ \Gamma (X(3872) \to \pi^{+}\pi^{-} J/\psi) \sim 0.44 \Gamma (\Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-} J/\psi) \]
(25)
From (24) and (25) we get
\[ \frac{\Gamma(X(3872) \to \omega J/\psi)}{\Gamma(X(3872) \to \pi^{+}\pi^{-} J/\psi)} \sim \frac{\Gamma(\Psi_{1++} (D\overline{D}^*) \to \omega J/\psi)}{\Gamma(\Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-} J/\psi)} \]
(26)
The right hand side of this expression may be estimated from the molecular model of [4] (see Table 8 of this reference) to be about \( 0.5 - 0.6 \) what may correspond to data within the experimental uncertainty (see Table 12 in [3]).

Furthermore from (19) and (25) we can establish the approximate relation
\[ \frac{\Gamma(X(3872) \to \gamma J/\psi)}{\Gamma(X(3872) \to \pi^{+}\pi^{-} J/\psi)} \sim \frac{0.56 \Gamma(\zeta_{1}(2p) \to \gamma J/\psi)}{0.44 \Gamma(\Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-} J/\psi)} \]
Then we can use the rate (see Table 12 in [3])
\[ \frac{\Gamma(X(3872) \to \gamma J/\psi)}{\Gamma(X(3872) \to \pi^{+}\pi^{-} J/\psi)} \sim 0.26 \pm 0.1 \]
(28)
to conclude that
\[ \Gamma (\zeta_{1}(2p) \to \gamma J/\psi) \sim 0.2 \Gamma (\Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-} J/\psi) \]
(29)
although this result could be modified by the molecular contribution to the \( \gamma J/\psi \) decay.

Finally let us examine the \( X(3872) \to D^{0}\overline{D}^{*}(2007)^{0} \) process. On the one hand the molecular model of reference [3] tells us that the \( \Psi_{1++} (D\overline{D}^*) \to D^{0}\overline{D}^{*}(2007)^{0} \) width could be quite smaller than the \( \Psi_{1++} (D\overline{D}^*) \to \pi^{+}\pi^{-} J/\psi \) width. On the other hand data tell us that the \( X(3872) \to D^{0}\overline{D}^{*}(2007)^{0} \) decay is an order of magnitude more prevalent than \( X(3872) \to \pi^{+}\pi^{-} J/\psi \) (see [6] and references therein). Therefore we may conclude that \( X(3872) \to D^{0}\overline{D}^{*}(2007)^{0} \) proceeds mainly through \( \zeta_{1}(2p) \to D^{0}\overline{D}^{*}(2007)^{0} \) in accord with our decay mechanism since \( D^{0}\overline{D}^{*}(2007)^{0} \) is the dominant mode for \( \zeta_{1}(2p) \):
\[ \Gamma \left( X(3872) \to D^{0}\overline{D}^{*}(2007)^{0} \right) \sim 0.56 \Gamma (\zeta_{1}(2p) \to D^{0}\overline{D}^{*}(2007)^{0}) \]
Taking into account the experimental value for the total width of the \( X(3872) \), \( \Gamma_{total} = 1.3 \pm 0.6 \) MeV, this would imply
\[ \Gamma (\zeta_{1}(2p) \to D^{0}\overline{D}^{*}(2007)^{0}) < 3.4 \) MeV
(30)
what is far below the quantitative result from some model calculations [13, 14] but it might be compatible with the Cornell coupled-channel model of reference [15] once the mass for the \( \zeta_{1}(2p) \) state is fixed at 3871.6 MeV (see also Table 4 in [4] for a comparative study of several models).

Putting together all the previous results we may guess some quantitative intervals of values for the partial
widths:

\[ \Gamma(X(3872) \to \gamma J/\psi) : 10 - 40 \text{ KeV} \]
\[ \Gamma(X(3872) \to \gamma \psi(2s)) : 20 - 80 \text{ KeV} \]
\[ \Gamma(X(3872) \to \pi^+ \pi^- J/\psi) : 40 - 160 \text{ KeV} \]
\[ \Gamma(X(3872) \to \omega J/\psi) : 20 - 80 \text{ KeV} \]

0.3 MeV < \Gamma \left( X(3872) \to D^0 \overline{D}^*(2007)^0 \right) < 1.9 \text{ MeV}

It should be remarked that the guessed intervals for electromagnetic transitions are quite compatible with the values calculated in reference \[17\] whereas the open flavor strong decay width may be obtained from the Cornell coupled-channel model of reference \[16\] as explained above. Moreover the rate \[\Gamma(X(3872) \to \omega J/\psi)/\Gamma(X(3872) \to \pi^+ \pi^- J/\psi)\] is consistent with the one coming out from the molecular model of reference \[4\].

B. Inter Threshold Resonances (ITR):

\[ M_{EQM} \in [M_{T_{j-1}}, M_{T_j}] \]

In the energy regions between two neighbor thresholds \([M_{T_{j-1}}, M_{T_j}]\) with \(j > 1\), the generator state of most EQM resonances, that we shall call Inter Threshold Resonances or ITR, is the lower energy eigenstate of the generator potential \(V_{[M_{T_{j-1}}, M_{T_j}]}(r)\). The mass of this generator can be bigger or smaller than \(M_{T_j}\).

Sometimes, as it is the case in the \([M_{T_2} - M_{T_3}]\) region for \(0^{++}\) and \(1^{++}\), the generator state is far above the upper threshold \((T_3)\) so that the attraction is not sufficient to generate any \((1p)_{[T_2,T_3]}\) ITR (see Tables \[X\] and \[XIV\]). The same mechanism explains the\( shocking\) absence of \(d^-\) wave \(1^-\) resonances from 4200 MeV to 4700 MeV. Despite this and contrary to conventional quark models the EQM predicts more ITR than experimentally extracted resonances until now.

Following a PDG like notation \[11\] we shall name the missing EQM charmonium states \(0^- (1^-)\) as \(\bar{\psi}(M_{EQM})\) and the \(0^+(J^{++})\) as \(\bar{\chi}_{cJ}(M_{EQM})\).

In particular three \(1^-\) missing resonances, \(\bar{\psi}(4496), \bar{\psi}(4537)\) and \(\bar{\psi}(4604)\), and two \(1^{++}\) missing resonances, \(\bar{\chi}_{c1}(4003)\) and \(\bar{\chi}_{c1} (4190)\) are predicted. It should be emphasized that all these states except \(\bar{\chi}_{c1}(4003)\) involve at least one threshold containing \(sT\). If the effect of this type of threshold were reduced against the effect from thresholds containing \(uT\) or \(d\overline{d}\) then we might expect \(\bar{\chi}_{c1}(4003)\) to be the better candidate for its experimental extraction.

With respect to the PDG cataloged neutral unconventional \(X\) resonances all of them, excepting the \(X(3872)\) which lies in the first energy region as shown before and the \(X(3940)\), can be reasonably assigned to calculated \(1^-\) and \(J^{++}\) ITR’s.

Let us examine in more detail these assignments. For this purpose we shall make use of the dominant strong decay modes listed in Tables \[XVI\] and \[XVII\].

1. \(X(3915)\)

The mass \((3915.5 \text{ MeV})\) and the observed decay mode \(J/\psi \omega\) of the \(X(3915)\) suggest its identification with the \((0^+(0^{++}), 1p)_{[T_1,T_2]}\) ITR at 3898 MeV. Alternatively \(X(3915)\) could be the same resonance as \(\chi_{c2}(2p)\) (mass 3929 MeV) to which we have assigned the \((0^+(2^{++}), 2p)_{[T_0,T_1]}\) state at 3903 MeV in the first energy region. If this alternative were correct then the \((0^+(0^{++}), 1p)_{[T_1,T_2]}\) ITR would correspond to a missing resonance \(\bar{\chi}_{c0}(3898)\). Therefore it is plausible the identification of \(X(3915)\) with either a \(\bar{\chi}_{c0}\) or a \(\chi_{c2}\) state.

2. \(X(3940)\)

The decay mode for this PDG resonance \((\overline{D}\overline{D}^*\text{ seen}, \overline{D}\overline{D}^*\text{ not seen}, J/\psi \omega \text{ not seen})\) does not fit with any of the calculated ITR. The fact that \(X(3940)\) is produced in double charm production as well as \(\eta_c(1s)\) and \(\eta_c(2s)\) suggests its identification with a \(\eta_c\) resonance with quantum numbers \(0^+(0^+)\). From conventional quark models the mass of \(\eta_c(3s)\) is expected to be about 40 MeV smaller than the mass of \(\psi(4040)\) \[13\] : moreover additional relativistic corrections might push down this value \[15\]. In the EQM the first \(S^-\) wave \(0^+(0^-)\) threshold for \(\eta_c, D^0 \overline{D}^*_0(2400)^0\), at 4183 MeV would be an additional (although weak) source of attraction. Besides there is a \(0(1^-)\) \(D\overline{D}^*\) threshold at 3872 MeV (differing from that in Table \[I\] in the sign in front of the charge conjugate component (c.c.)) so that the dominant EQM decay mode of this \(\eta_c\) would be \((D\overline{D}^* )_p\) whereas \(D\overline{D}^*\) and \(J/\psi \omega\) could not be reached by quantum numbers. Putting all these results together it seems quite plausible the identification of \(X(3940)\) with \(\eta_c(3940)\).

3. \(X(4140), X(4160)\)

The PDG mass \((4143 \text{ MeV})\) and decay mode \((J/\psi \phi)\) of \(X(4140)\) suggest its assignment either to \((0^+(0^{++}), 1p)_{[T_1,T_2]}\) or to \((0^+(2^{++}), 1p)_{[T_0,T_2]}\). Exactly the same situation occurs for \(X(4160)\) (PDG mass 4156 MeV). Therefore the two ITR can be assigned to these two resonances although their precise one to one identification is not possible yet since it requires the experimental determination of the total angular momentum of at least one of the two cataloged resonances.
4. \(X(4350)\)

The lack of predictions for EQM \(0^+(1^{++}, 2^{++})\) states above their last well known threshold at 4224 MeV does not allow for a clear assignment of an ITR to \(X(4350)\) (PDG mass 4350.6 MeV) in spite of the fact that there is a \((0^+(0^{++}), 1p)\) ITR at 4325 MeV with a dominant decay to \(J/\psi\phi\) as experimentally observed. Indeed the presence of this ITR suggests that \(0^+(1^{++}, 2^{++})\) ITR’s from still unknown thresholds may also be close in energy. Hence we prefer to leave open the assignment until more data on thresholds becomes available.

5. \(\widetilde{X}_c(4003)\) and \(\widetilde{X}_c(4190)\)

The analysis carried out until now allows for a reasonable assignment of EQM states to the PDG listed neutral \(J^0\) resonances up to 4350 MeV. But the EQM predicts the existence of two more ITR in this energy region, which should be assigned to missing resonances, the \(\widetilde{X}_c(4003) \equiv \{0^+(1^{++}), 1p\}[T_1, T_2]\) with predicted decay modes, \(D\bar{D}\bar{D}\) and \(J/\psi\omega\) and the \(\widetilde{X}_c(4190) \equiv \{0^+(1^{++}), 1p\}[T_1, T_2]\) decaying to \(D\bar{D}\bar{D}\) and \(J/\psi\phi\).

As point out before \(\widetilde{X}_c(4003)\) is an ITR between two thresholds containing \(s\bar{s}\) or \(d\bar{d}\) whilst \(\widetilde{X}_c(4190)\) lies in between two thresholds containing \(s\bar{s}\). Hence the experimental extraction of \(\widetilde{X}_c(4003)\) could be easier.

6. \(X(4260)\) and \(\psi(4415)\)

In the EQM the existence of the \(0^-(1^{--}, 1s)[T_1, T_2]\) ITR with a mass of 4280 MeV, below the mass of first well known regular threshold \(DD_1(2420)\) (4287 MeV), relies on the presence of the effective threshold \(T_1\) at 4237 MeV resulting from the overlap of \(DD_1(2420)\) and \(DD_1(2430)\). This ITR may be assigned to \(X(4260)\) (actually the choice of a lower mass for the effective threshold \(T_1\) could fit precisely the PDG mass 4263 MeV). Indeed the seen decays \(J/\psi(\pi\pi), K\bar{K}\) correspond to dominant ITR modes through \(J/\psi f_0(980)\). Notice that these modes come out from molecular component \(D^+D^0 \equiv D_1^*(2400)^0\). Concerning the predicted modes \((D^*_sD^0_s, D_sD^*_s, D^*D^\pi)\) from the generator component the experimental situation is much more uncertain.

The other ITR involving an effective threshold is the \(0^+(1^{++}, 1s)[T_1, T_2]\) state at 4418 MeV which may be assigned to \(\psi(4415)\) (mass 4421 MeV). Although the \(S\)–wave EQM dominant mode from the generator component is, from Table \(\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\), the \(D^+D_1(2400)\) channel, this mode lies quite below the EQM mass which may do the \(D\)–wave decay channel \((DD_1(2400))_D\) to be the dominant mode as experimentally observed. Regarding the molecular component the expected dominant decays are to \(J/\psi(\pi\pi, 4\pi)\).

The \(\psi(4415) \rightarrow e^+e^-\) width has also been measured to be \(0.58 \pm 0.07\) KeV. Following Section \(\Box\Box\Box\Box\Box\) we shall approximate the width by

\[
\Gamma_{\psi(4415)\rightarrow e^+e^-} \approx 4e^2a^2(1 - \frac{M_{\pi\pi}}{M_{\psi}})^2, \tag{31}
\]

where \(R_{1s[T_1, T_2]}(0)\) stands for the EQM radial wave function at the origin.

From (31) we can get a close value (0.69 KeV) to the experimental measurement by choosing

\[
\tilde{M}^2(M_{1s[T_1, T_2]}M_{T_3}) \sim M^2_{1s[T_1, T_2]} - M^2_{T_3}
\]

Unfortunately we have not more leptonic width data for interthreshold resonances to check the possible validity of the generalization of this prescription:

\[
\tilde{M}^2(M_{ns[T_{j-1}, T_j]}M_{T_{j-1}}) \sim M^2_{ns[T_{j-1}, T_j]} - M^2_{T_{j-1}}
\]

7. \(X(4360)\)

The calculated EQM mass (4360 MeV) and decay modes \((D^+D\pi, \psi(2s)\pi\pi)\) of the ITR \((0^-(1^{--}), 1s)[T_1, T_2]\) strongly suggest its assignment to \(X(4360)\) (PDG mass 4361 MeV and observed decays to \(D^0D^{*-}\pi^+\) and \(\psi(2s)\pi^+\pi^-\)).

Notice that for the lower threshold \(T_2 \equiv D^0D_0^*(2400)\) there may be a significant isospin breaking since \(D^0_5(2400)^0\) and \(D_0^*(2400)^\pm\) with quoted PDG masses 2318 \(\pm\) 29 MeV and 2403 \(\pm\) 14 \(\pm\) 35 MeV respectively may significantly differ in mass. Actually the observed decay \(D^0\) points out to a decay through \(\psi(2s)\) and \(D^0D_0^*\) what could be indicating that the mass of this channel is pretty close below the mass of the ITR. This would mean that the mass of the \(D^0_5(2400)^\pm\) would be close to the lower value of its experimental mass interval (2354 MeV).

8. \(\tilde{\psi}(4496), \tilde{\psi}(4537)\) and \(\tilde{\psi}(4604)\)

Three missing resonances, \(\tilde{\psi}(4496), \tilde{\psi}(4537)\) and \(\tilde{\psi}(4604)\) are predicted in between the experimentally known \(\psi(4155)\) and \(X(4600)\). They correspond to the \((0^-(1^{--}), 1s)[T_{j-1}, T_j]\) ITR’s with \(j = 5, 6, 7\).

For \(\tilde{\psi}(4604)\) the predicted mass and dominant decays may be altered by the presence of two still unknown thresholds \(D_0D_1(2750)^0\) since two \(D_1\) mixing states \((3P_{1-1} - P_{j})\) with masses around 2800 MeV can be expected (there is a \(D(2750)\) with non established quantum numbers that could well be a 0\(^+-\) state).

As mentioned before the experimental extraction of these resonances, involving thresholds containing \(s\bar{s}\), may
be difficult. For $\bar{\psi}(4496)$ and $\bar{\psi}(4537)$ their vicinity may add difficulty to their experimental disentanglement. In this respect their differentiated dominant decays could be of some help.

9. $X(4660)$

The $X(4660)$ (PDG mass 4664 MeV) with seen decay to $\psi(2S)\pi^+\pi^-$ is assigned to the $0^-(1^{--}, 1_s^0 [T_2, T_0])$ ITR at 4673 MeV although its cataloged decay to $D^{0}\bar{D}^{*}\pi^+$ does not correspond in principle to the predicted dominant decay ($D_s^*D^*K$) of the ITR. However, as explained for $\bar{\psi}(4664)$ there may be an effective threshold from $D^0\bar{D}_1(?)$ with mass close to that of $D_s^*D_{s1}(2536)^-$. If this occurred the ITR mass could still be within the experimental mass interval for $X(4660)$ but with a dominant decay to $DD^*\pi$ as observed.

IX. BOTTOMONIUM

The bottomonium masses from the EQM have been listed in Tables X and XI where they are compared to data from [11] and to Cornell masses.

The list of thresholds employed in the calculation of the bottomonium spectrum appears in Table XVIII. The lack of knowledge about further thresholds prevents extending the list to higher energies.

As it was the case for charmonium there may be two possible effective thresholds corresponding to the pairs of close thresholds \( \left( B_0^0\bar{B}_1^-(5721)^0, B_0^0\bar{B}_1(?) \right) \) and \( \left( B_0^0\bar{B}_1^-(5721)^0, B_0^0\bar{B}_1(?) \right) \). Taking them for granted we have chosen their masses 25 MeV smaller than \( M_{B_0^0\bar{B}_1(5721)} \) and \( M_{B_0^0\bar{B}_1(5721)} \) respectively in order to reasonably reproduce the data. As a result the effective threshold for \( \left( B_0^0\bar{B}_1^-(5721)^0, B_0^0\bar{B}_1(?) \right) \) is located at about the same energy that the regular threshold \( B_0^0\bar{B}_1^-(5732)^0 \) (11023 MeV), see Table XVIII. We should note though that this regular threshold is not so close in mass to the almost degenerate thresholds \( \left( B_0^0\bar{B}_1^-(5721)^0, B_0^0\bar{B}_1(?) \right) \) with masses about 11049 MeV. Therefore we do not expect any overlap between the regular threshold and the almost degenerate ones. In practice this means that we shall only consider one threshold at 11023 MeV.

At difference with charmonium unconventional resonances have not been clearly identified experimentally in bottomonium until now. According to the calculated EQM spectrum the only PDG cataloged neutral resonance that may correspond to an ITR is $\Upsilon(11020)$, see $J_{PC} T_1 \, Meson_1 - \, Meson_2 \, (J_{1}^P, J_{2}^P) \, \, M_{T_1}$

| $1^{--}$ | \( T_1 \, B_0^0\bar{B}_1^-(5721)^0 \, B_0^0\bar{B}_1(?) \) | \( 0^-, 1^+ \) | : \( 10978 \) |
|----------|------------------------------------------------|----------------|--------|
|          | \( B_0^0\bar{B}_1^-(5732)^0 \) | \( 1^-, 0^+ \) | \( 11023 \) |
| $T_2$    | \( B_0^0\bar{B}_1^-(5721)^0 \) | \( 1^-, 1^+ \) | \( 11072 \) |
|          | \( B_0^0\bar{B}_1(?) \) | \( 1^-, 1^+ \) | : \( 11072 \) |

| $0^{++}$ | \( T_1 \, B_0^0\bar{B}_1^-(5721)^0 \, B_0^0\bar{B}_1(?) \) | \( 0^+, 0^- \) | \( 10558 \) |
|----------|------------------------------------------------|----------------|--------|
|          | \( T_2 \, B_0^0\bar{B}_1^-(5721)^0 \) | \( 1^-, 1^- \) | \( 10650 \) |
|          | \( T_3 \, B_0^0\bar{B}_1^-(5721)^0 \) | \( 0^-, 0^- \) | \( 10734 \) |
|          | \( T_4 \, B_0^0\bar{B}_1^-(5721)^0 \) | \( 1^-, 1^- \) | \( 10830 \) |

| $1^{++}$ | \( T_1 \, B_0^0\bar{B}_1^-(5721)^0 \, B_0^0\bar{B}_1(?) \) | \( 0^-, 1^- \) | \( 10604 \) |
|----------|------------------------------------------------|----------------|--------|
|          | \( T_2 \, B_0^0\bar{B}_1^-(5721)^0 \) | \( 1^-, 1^- \) | \( 10650 \) |
|          | \( T_3 \, B_0^0\bar{B}_1^-(5721)^0 \) | \( 0^-, 1^- \) | \( 10782 \) |
|          | \( T_4 \, B_0^0\bar{B}_1^-(5721)^0 \) | \( 1^-, 1^- \) | \( 10830 \) |

| $2^{++}$ | \( T_1 \, B_0^0\bar{B}_1^-(5721)^0 \, B_0^0\bar{B}_1(?) \) | \( 1^-, 1^- \) | \( 10650 \) |
|----------|------------------------------------------------|----------------|--------|
|          | \( T_2 \, B_0^0\bar{B}_1^-(5721)^0 \) | \( 1^-, 1^- \) | \( 10830 \) |

TABLE XVIII: Meson-meson thresholds for $I = 0 \, s\bar{s}$ states. $J_{1}^P$ and $J_{2}^P$ stand for the angular momenta of the mesons entering in the threshold. Threshold masses ($M_{T_1}$) obtained from the bottom and bottom strange meson masses quoted in [11]. A question mark followed by a colon preceeds the chosen mass for an effective threshold.

For the rest of PDG cataloged resonances an unambiguous parallelism with calculated EQM states can be established (although some cataloged $1^{--}$ resonances could involve $s$ and $d$ state mixing).

The generator masses and mixing angles for the EQM states appear in Tables XX and XXI. The same comments done in charmonium about the bigger relevance of mixing above the first threshold can be traslated to bottomonium.

In order to assign the EQM states to PDG resonances we proceed to an identification of masses and
dominant decay modes. For this purpose the $S-$ wave
\((|\bar b\bar b\rangle=0)\) light-light meson possible decay channels from
the molecular component are listed in Table XXI.

We shall consider the resonances $f$, $\omega$ and $\phi$ as mixed
states containing $u\bar u$, $d\bar d$ and $s\bar s$ components in order to
establish the dominant decay mode of an EQM state. Therefore
molecular components formed from bottomed mesons as well as from
bottomed strange mesons can decay into two body channels containing
these resonances. Notice though that when the kinematically
dominant decay mode involves prominently a light quark flavor not
TABLE XIX: Calculated mixing angles $\theta_{kT_{i-1}-T_i}$, generically
called $\theta$, for the $1^{--}$ EQM bottomonium states. The corre-
spending interthreshold energy intervals, $[M_{T_{i-1}}, M_{T_i}]$, are
shown. For the sake of completeness the EQM masses $M_{EQM}$
and the rms radii $\langle r^2 \rangle_{EQM}$ are also listed and compared to
the generator masses $M_{GEN}$ and rms radii $\langle r^2 \rangle_{GEN}$.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
j^{PC} & [T_{i-1}, T_i] & k & M_{EQM} & M_{GEN} & \langle r^2 \rangle_{EQM} & \langle r^2 \rangle_{GEN} \\
& [M_{T_{i-1}}, M_{T_i}] & (\text{MeV}) & (\text{MeV}) & (\text{fm}) & (\text{fm}) & \theta(\circ) \\
\hline
1^{--} & [T_0, T_1] & 1s & 9459 & 9459 & 0 & 0.2 & 0 \\
& & 2s & 10012 & 10012 & 0 & 0.5 & 0 \\
& & 1d & 10157 & 10157 & 0 & 0.6 & 0 \\
& & 3s & 10342 & 10342 & 0 & 0.8 & 0 \\
& & 2d & 10438 & 10438 & 0 & 0.8 & 0 \\
& & 4s & 10608 & 10608 & 0.3 & 1.0 & 0.4 \\
& & 3d & 10682 & 10682 & 0.4 & 1.0 & 3 \\
& & 5s & 10840 & 10840 & 5 & 1.2 & 13 \\
& & 4d & 10898 & 10902 & 11 & 1.2 & 13 \\
& [T_1, T_2] & 1s & 10995 & 10998 & 3 & 1.2 & 13 \\
& [T_2, T_3] & 1s & 11039 & 11042 & 3 & 1.2 & 11 \\
& [T_3, T_4] & 1s & 11090 & 11090 & 6 & 1.1 & 3 \\
& & 1d & 11130 & 11133 & 6 & 1.4 & 1 \\
& & 2s & 11140 & 11144 & 9 & 1.3 & 1 \\
\hline
& [T_0, T_1] & 1p & 9920 & 9920 & 0 & 0.4 & 0 \\
& & 2p & 10259 & 10259 & 0 & 0.7 & 0.4 \\
& & 3p & 10521 & 10531 & 13 & 1.0 & 0.9 \\
& [T_1, T_2] & 1p & 10620 & 10628 & 14 & 1.1 & 0.9 \\
& [T_2, T_3] & 1p & 10704 & 10711 & 14 & 1.1 & 0.9 \\
& [T_3, T_4] & 1p & 10784 & 10789 & 10 & 1.2 & 1 \\
\hline
1^{++} & [T_0, T_1] & 1p & 9920 & 9920 & 0 & 0.4 & 0 \\
& & 2p & 10259 & 10259 & 0 & 0.7 & 0.3 \\
& & 3p & 10526 & 10531 & 7 & 1.0 & 0.9 \\
& [T_1, T_2] & 1p & 10604 & 10669 & - & 1.0 & - \\
& [T_2, T_3] & 1p & 10708 & 10711 & 6 & 1.1 & 0.9 \\
& [T_3, T_4] & 1p & 10822 & 10834 & 26 & 1.5 & 1.1 \\
\hline
2^{++} & [T_0, T_1] & 1p & 9920 & 9920 & 0 & 0.4 & 0 \\
& & 2p & 10259 & 10259 & 0 & 0.7 & 0 \\
& & 3p & 10528 & 10531 & 4 & 0.9 & 0.9 \\
& [T_1, T_2] & 1p & 10710 & 10711 & 3 & 1.0 & 0.9 \\
& [T_0, T_1] & 1p & 10815 & 10826 & 20 & 1.3 & 1.0 \\
\hline
\end{array}
\]

TABLE XX: Calculated mixing angles $\theta_{kT_{i-1}-T_i}$, generically
called $\theta$, for the $J^{++}$ EQM bottomonium states. The corre-
spending interthreshold energy intervals, $[M_{T_{i-1}}, M_{T_i}]$, are
shown. For the sake of completeness the EQM masses $M_{EQM}$
and the rms radii $\langle r^2 \rangle_{EQM}$ are also listed and compared to
the generator masses $M_{GEN}$ and rms radii $\langle r^2 \rangle_{GEN}$.

For $1^{++}$ the insufficient threshold attraction prevents the generation of
an EQM state in the $[T_1, T_2]$ region.
| Decay Channel | $I^G(J^{PC})$ | ((b\bar{b})_{I=0} + light-light meson) | $M_{Ch}$ (MeV) |
|---------------|-------------|-----------------------------------|---------------|
| $0^-(1^{--})$ | $\Upsilon(1s)f_0(\sim 500)$ | $\sim 9960$ | $\Upsilon(1s)f_0(980)$ | $10450$ | $\Upsilon(1s)f_0(10523)$ | $10735$ | $\Upsilon(1s)f_0(\sim 1370)$ | $10830$ | $\Upsilon(3s)f_0(\sim 500)$ | $10855$ | $\Upsilon(1s)f_1(1420)$ | $10886$ | $\Upsilon(1s)f_1(1430)$ | $10890$ | $\Upsilon(1s)f_0(1500)$ | $10695$ | $\Upsilon(1s)f_1(1510)$ | $10978$ | $\Upsilon(1s)f_2(1525)$ | $10985$ | $\Upsilon(2s)f_0(980)$ | $11013$ | $\Upsilon(1s)f_1(1565)$ | $11022$ | $\Upsilon(4s)f_0(\sim 500)$ | $11079$ | $\Upsilon(1s)f_3(1640)$ | $11099$ |

| $0^+(0, 1, 2^{++})$ | $\Upsilon(1s)\omega(782)$ | $10243$ | $\Upsilon(1s)\phi(1020)$ | $10480$ | $\Upsilon(2s)\omega(782)$ | $10806$ | $\Upsilon(1s)\omega(\sim 1430)$ | $10880$ | $\Upsilon(2s)\phi(1020)$ | $11043$ | $\Upsilon(1s)\phi(1680)$ | $11140$ |

| $1^{[\pi\pi, \pi\pi]}$ | $1^{++}$ | $4s_{[\pi\pi, \pi\pi]}$ | $10608$ | $\Upsilon(2s)f_0(\sim 500)$ | $10682$ | $\Upsilon(1s)f_0(980)$ | $10695$ | $\Upsilon(1s)f_1(1500)$ | $10978$ | $\Upsilon(1s)f_2(1525)$ | $10985$ | $\Upsilon(2s)f_0(980)$ | $11013$ | $\Upsilon(1s)f_1(1565)$ | $11022$ | $\Upsilon(4s)f_0(\sim 500)$ | $11079$ | $\Upsilon(1s)f_3(1640)$ | $11099$ |

| $1^{[\pi\pi, \pi\pi]}$ | $1^{++}$ | $3d_{[\pi\pi, \pi\pi]}$ | $10608$ | $\Upsilon(2s)f_0(\sim 500)$ | $10682$ | $\Upsilon(1s)f_0(980)$ | $10695$ | $\Upsilon(1s)f_1(1500)$ | $10978$ | $\Upsilon(1s)f_2(1525)$ | $10985$ | $\Upsilon(2s)f_0(980)$ | $11013$ | $\Upsilon(1s)f_1(1565)$ | $11022$ | $\Upsilon(4s)f_0(\sim 500)$ | $11079$ | $\Upsilon(1s)f_3(1640)$ | $11099$ |

| $1^{[\pi\pi, \pi\pi]}$ | $1^{++}$ | $5s_{[\pi\pi, \pi\pi]}$ | $10840$ | $\Upsilon(1s)f_0(\sim 1300)$ | $10840$ | $\Upsilon(1s)f_0(\sim 500)$ | $10595$ | $\Upsilon(1s)f_1(1500)$ | $10978$ | $\Upsilon(1s)f_2(1525)$ | $10985$ | $\Upsilon(2s)f_0(980)$ | $11013$ | $\Upsilon(1s)f_1(1565)$ | $11022$ | $\Upsilon(4s)f_0(\sim 500)$ | $11079$ | $\Upsilon(1s)f_3(1640)$ | $11099$ |

TABLE XXI: Two-body decay channels including I = 0 hidden bottom states. For the calculation of their masses, $M_{Ch}$, the quoted masses for $((b\bar{b})_{I=0} + light-light meson)$ and $\omega(1420)$ are quite uncertain we have put a symbol of approximation (~) in front of its nominal value.

Once we know the masses for the EQM states, their related thresholds and the masses for the S-wave $((b\bar{b})_{I=0} + light-light meson)$ channels we can establish, following Section V, the dominant decay modes. Thus in Tables XXII and XXIII the dominant strong decay modes for EQM states above the first absolute threshold $B^s\overline{B}^s$ are listed.

As it happened in charmonium the calculated values of $v/c$ for the $1^{--}$ EQM states are much bigger in the first energy region (0.3) than in the interthreshold ones (0.05) where the effect of relativistic corrections gets reduced. Therefore we may expect the calculated masses present in the molecular component some alternative decay could be favored. For instance the $\Upsilon(1s)\phi$ mode involves prominently an $s\bar{s}$ while the molecular component of the $(0^+(0^{++}, 1p)[T_1, T_2]$ state given by $B^s\overline{B}^s$ involves $\omega\pi$ and $u\bar{d}$. So we have also listed an alternative decay to $\Upsilon(1s)\omega$.

| Strong Decay Modes | Final Products |
|-------------------|---------------|
| $B^s\overline{B}^s$ | $B^s\overline{B}^s$ |
| $B^s\overline{B}^s$ | $B^s\overline{B}^s$ |
| $B^s\overline{B}^s$ | $B^s\overline{B}^s$ |

TABLE XXII: Dominant strong decay channels (subindex P for required P-wave) from the generator and molecular components for $1^{--}$ EQM bottomonium states with masses, $M_{EQM}$, above the absolute threshold $B^s\overline{B}^s$ at 10558 MeV. $B_i(5721, ?)$ stands for $B_i(5721)$ and $B_i(?, f_0(1300, 1400)$ for $f_0(\sim 1300)$ with $(f_0(1285, f_2(1270))$ and $(f_2(1270), f_0(1420), f_2(1430))$ respectively, and $f_0(1300, 1500)$ for $f_0(\sim 1500)$ with $(f_0(1510), f_2(1525))$. The final products result from the decays of the second members of the decay channels (excepting $B^s$’s and $B^s$’s) as given in [11].
More precisely, the $J^{++}$ $1p_{[T_0,T_1]}$ EQM states with mass 9920 MeV, which are assigned to the PDG $\chi_b(0,1.2)(1p)$ have no mixing at all ($\theta = 0^\circ$) (see Table XX).

The $J^{++}$ $2p_{[T_0,T_1]}$ EQM states with mass 10259 MeV, assigned to $\chi_b(0,1.2)(2p)$ have a very small mixing ($\theta : 0 - 0.4^\circ$).

The $(0,1,2)^{++} 3p_{[T_0,T_1]}$ EQM states with masses 10521 MeV, 10526 MeV and 10528 MeV respectively, which are assigned to $\chi_b(0,1.2)(3p)$, present some mixing, particularly the $J = 0$ state ($\theta_{3p}(0^{++}) = 13^\circ$). Therefore we expect for these EQM states an increase of the $\omega \Upsilon(1s)$ decay width with respect to their corresponding Cornell generators due to the molecular component.

For $J^{PC} = 1^{--}$ the $1s_{[T_0,T_1]}$ (9459), $2s_{[T_0,T_1]}$ (10012), $1d_{[T_0,T_1]}$ (10157) and $3s_{[T_0,T_1]}$ (10342) EQM states, assigned to $\Upsilon(1s)$, $\Upsilon (2s)$, $\Upsilon (1d)$ and $\Upsilon (3s)$ respectively, have no mixing at all as well as for the still missing $2d_{[T_0,T_1]}$ (10438) (see Table XIX). They correspond to pure Cornell states.

The $4s_{[T_0,T_1]}$ (10608), assigned to $\Upsilon (4s)$, and the still missing $3d_{[T_0,T_1]}$ (10682) have very small mixing (0.3$^\circ$, 0.4$^\circ$) being almost pure Cornell states.

The $5s_{[T_0,T_1]}$ (10840) assigned to $\Upsilon (10860)$ and the still missing $4d_{[T_0,T_1]}$ (10898) have small mixings ($\theta_{5s} \simeq 3^\circ, \theta_{4d} = 0^\circ$) which can be however relevant for strong decays. A look at Tables X and XXII tells us that in particular the $5s_{[T_0,T_1]}$ and $4d_{[T_0,T_1]}$ decay widths to $\Upsilon (1s)(\pi\pi)$ and $\Upsilon (3s)(\pi\pi)$ can be significantly enhanced with respect to the Cornell calculation since the dominant two body decay channels $\Upsilon (1s)f_{0}(1.2)$($\sim 1300, \sim 1400$) and $\Upsilon (3s)f_{0}(\sim 500)$ have masses pretty close to the EQM masses. This may help to explain the huge signals observed for these decays around 10870 MeV (see Table 36 in [5] and references therein). We shall comment below on the even higher signal to $\Upsilon (2s)(\pi\pi)$.

### B. Inter Threshold Resonances (ITR):

In the energy regions between two neighbor thresholds $[M_{T_{j-1}}, M_{T_j}]$ with $j > 1$, sometimes the attraction caused on the generator state by the upper threshold is not sufficient to generate an ITR as it occurred in charmonium. This explains the absence of a $1^{++}$ ITR in between $T_1$ and $T_2$ in Table X as well as the absence of $d$--wave $1^{--}$ ITR's between $T_{j-1}$ and $T_j$ for $j = 2,3$ in Table X.
For $J^{PC} = 1^{--}$ the EQM predicts five ITR states between 10990 MeV and 11190 MeV, see Table X. The first two of them, $1s_{[T_1, T_2]}$ at 10995 MeV and $1s_{[T_3, T_2]}$ at 11039 MeV, might be hidden within $\Upsilon(11020)$ (see below). The other three, $1s_{[T_2, T_2]}$ at 11090 MeV, $1d_{[T_3, T_2]}$ at 11130 MeV and $2s_{[T_2, T_2]}$ at 11140 MeV are missing resonances whose experimental extraction may be difficult given their proximity to each other and the fact that the corresponding upper threshold for all of them has $s\bar{s}$ content what may reduce their formation probability (as compared to those resonances from thresholds containing $u\bar{u}$ or $d\bar{d}$). Actually this energy region has been experimentally scanned [19] not finding any clear evidence of these resonances. A thorough analysis of data should be of interest to draw more definite conclusions about the presence of these missing states.

Additionally the EQM predicts a $0^{++} 1p_{[T_1, T_2]}$ ITR at 10620 MeV and 0, 1, $2^{++}$ ITR's around central mass values of 10707 MeV and 10813 MeV, see Table X. From them the $0^{++} 1p(10620)$ is well isolated and involves thresholds with $u\bar{u}$ or $d\bar{d}$ content what could favor its experimental extraction.

Following a PDG like notation [11] we shall name the missing EQM bottomonium states $0^-(1^{--})$ as $\tilde{\Upsilon}(M_{EQM})$ and the $0^+(J^{++})$ as $\tilde{\chi}_{bJ}(M_{EQM})$.

Let us analyze in more detail these predicted ITR and their comparison to currently existing PDG resonances. For this purpose we shall make use of the dominant strong decay modes listed in Tables XXII and XXIII.

1. $\Upsilon(11020)$

The quoted mass of this PDG resonance (11019 $\pm$ 8) MeV lies in between the $(0^-(1^{--}), 1s)_{[T_1, T_2]}$ ITR with a mass of 10995 MeV and the $(0^-(1^{--}), 1s)_{[T_2, T_2]}$ ITR with a mass of 11039 MeV. Although these ITR masses are related to a somewhat arbitrary choice of the masses for the two effective thresholds it is for sure in the EQM the presence of at least two ITR, one below the well established threshold at 11023 MeV and other between this threshold and the well established one at 11072 MeV. This suggests that $\Upsilon(11020)$ could actually come out from the overlapping of these two ITR $\tilde{\Upsilon}(10995)$ and $\tilde{\Upsilon}(11039)$. Indeed the BaBar collaboration has reported a resonance with extracted mass 10996 $\pm$ 2 MeV [13] which might be assigned to $\tilde{\Upsilon}(10995)$. Moreover a dominant decay mode from the molecular component of this ITR is $(\Upsilon(2s) f_0(980))_{\text{virtual}}$ (notice that the $f_0(980)$ mass uncertainty, 990 $\pm$ 20 MeV allows for this decay) what may contribute to explain the huge $\Upsilon(2s)\pi\pi$ signal observed around 10870 MeV. In this regard the $\tilde{\Upsilon}(11039)$ could be also contributing to the signal. Therefore we may conclude that there are several indications pointing out the validity of these EQM predictions although no definite conclusion should be extracted until the experimental confirmation or refutation of the existence of $\tilde{\Upsilon}(11039)$ as a differenicated resonance from $\tilde{\Upsilon}(10995)$.

2. $\tilde{\Upsilon}(11090)$, $\tilde{\Upsilon}(11130)$ and $\tilde{\Upsilon}(11140)$

The EQM predicts three $1^{--}$ missing resonances between 10990 MeV and 11190 MeV. These are $\tilde{\Upsilon}(11090) \equiv (0^-(1^{--}), 1s)_{[T_1, T_2]}$, $\tilde{\Upsilon}(11130) \equiv (0^-(1^{--}), 1d)_{[T_2, T_2]}$ and $\tilde{\Upsilon}(11140) \equiv (0^-(1^{--}), 2s)_{[T_2, T_2]}$. As mentioned before all of them involve thresholds with $s\bar{s}$ content. Notice that the $\tilde{\Upsilon}(11090)$ would be the best candidate for its isolated experimental observation. For it we expect dominant decays to $B^*B\pi$, $B^*B\pi$ and to $\Upsilon(4s)\pi\pi$ and $\Upsilon(1s)(\omega\omega, 4\pi, K\bar{K})$.

3. $\tilde{\chi}_{b0}(10620)$

In the $J^{++}$ sector the EQM predicts a $\tilde{\chi}_{b0}(10620) \equiv (0^+(0^{++}), 1p)_{[T_1, T_2]}$. As explained before it involves thresholds with $u\bar{u}$ or $d\bar{d}$ content. Moreover its well isolated character may facilitate its experimental extraction through the dominant decay modes $BB$ and $\Upsilon(1s)\omega$.

4. $\tilde{\chi}_{b(0,1,2)}(10704, 10708, 10710)$ and $\tilde{\chi}_{b(0,1,2)}(10784, 10815, 10822)$

Two mass triplets of $\tilde{\chi}_{bJ}$ missing resonances around 10707 MeV and 10813 MeV respectively are predicted. This has to do with the presence of common thresholds in the 0, 1, $2^{++}$ channels. Nonetheless both triplets are different in the sense that in the first triplet the three states share the same dominant decay mode $(B^*B\bar{B})$ from the generator component whereas in the second one they do not, being the dominant decay mode different for each member of the triplet $(B_sB_s, B_s\bar{B}_s$ and $B^*B^{*}\bar{B}$ respectively) what might help to their experimental disentanglement.

X. SUMMARY

From an educated guess for the ground state energy of a static quark - static antiquark system based on lattice calculations we have proposed an energy dependent quark-antiquark potential for calculating the spectra of heavy quarkonia within a nonrelativistic quark model framework. This potential, incorporating the interaction with open flavor meson-meson channels, allows for an unquenching of the conventional quark model (based on the quenched Cornell potential) that we call Extended Quark Model (EQM).
The EQM has been applied to the calculation of heavy quarkonia spectra. Regarding the level of precision of the calculated EQM masses we may distinguish the first energy region (below the first corresponding threshold) from the upper ones since the form of the potential makes the inter threshold resonances (ITR) to become truly non relativistic states. Therefore we may expect an improvement of the precision for the calculated ITR masses as compared to most masses calculated in the first energy region.

In order to make a sensible assignment of calculated EQM states to experimental resonances we have analyzed leptonic widths and we have examined for strong decays and E1 transitions the physical mechanisms underlying these processes. For this examination we have made use of the compositeness of the EQM states in terms of a linearly confined and a molecular components. This has allowed us to get a qualitative understanding of the dominant decay modes for any EQM state.

More specifically each PDG cataloged $c\bar{c}$ and $b\bar{b}$ neutral meson has been reasonably assigned to a EQM state with calculated mass close to the experimental one and with dominant decay modes fully compatible with the observed decay channels. In charmonium the predicted absence of $d-$ wave $1^{--}$ resonances from 4200 MeV to 4700 MeV should also be emphasized.

Besides the EQM predicts the existence of non cataloged $c\bar{c}$ and $b\bar{b}$ neutral resonances whose discovery would give definite support to the model. Incidentally most of these missing states are related to at least one threshold with $s\bar{s}$ content what could imply a reduction of the formation probability for these states. The only exceptions to this rule are the so called $\chi_c(4003)$ and $\chi_b(10620)$ which are a priori the ideal candidates to check the EQM.

Certainly the model relies on threshold states whose meson components should be also consistently described by the EQM. In this respect the radial form of the quenched Cornell potential that we have taken as the base for threshold unquenching could not be sufficient for an approximate description of these mesons containing light quarks and some refinements might be needed. Anyway there are some clear indications that a refined EQM could provide us with an appropriate description at all meson sectors. For instance in the charmed strange sector the mesons $D_{s0}(2317)$ and $D_{s1}(2460)$ have unexpected low masses as compared to most conventional quark model calculations. In the EQM $D_{s0}(2317)$ is a $I(J^P) = 0(0^+)$ state with mass below the first $0(0^+)$ threshold $DK$ $(M_{DK} = 2360$ MeV) whereas $D_{s1}(2460)$ is a $0(1^+)$ state with mass below the first $0(1^+)$ threshold $D^*K$ $(M_{D^*K} = 2502$ MeV). It is then plausible that these two states result from the attraction caused by the respective thresholds on the corresponding (conventional) generator states. In other meson sectors like the light unflavored one the EQM may also play a very relevant role in the understanding of the puzzle concerning $\chi^3P_0$ states such as $f_0(980)$ lying close below the $K\bar{K}$ threshold.

Furthermore the EQM generalization to baryons may provide a general scheme for the solution of endemic problems related to the description of some baryonic resonances as $N^*(1440)$, $\Lambda(1405)$, $\Delta_{5/2}^-(1930)$. As a matter of fact a perturbative evaluation of threshold effects in light baryons, closely connected to our EQM treatment has allowed for an explanation of light baryon spectral anomalies [20].

In conclusion we have developed an unquenched quark model that may allow for a general description of hadrons as made of constituent quarks and antiquarks (quark-antiquark for a meson and three quarks for a baryon). The results obtained for heavy quarkonia with the simplest version of this model seem to point out that it incorporates essential physical ingredients needed for an accurate study of the hadronic structure.

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