ANALYSIS OF THE $\Lambda_c(2860)$, $\Lambda_c(2880)$, $\Xi_c(3055)$ AND $\Xi_c(3080)$ AS D-WAVE BARYON STATES WITH QCD SUM RULES

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Abstract

In this article, we tentatively assign the $\Lambda_c(2860)$, $\Lambda_c(2880)$, $\Xi_c(3055)$ and $\Xi_c(3080)$ to be the D-wave baryon states with the spin-parity $J^P = \frac{3}{2}^+$, $\frac{5}{2}^+$, $\frac{7}{2}^+$ and $\frac{9}{2}^+$, respectively, and study their masses and pole residues with the QCD sum rules in a systematic way by constructing three-types interpolating currents with the quantum numbers $(L_{\rho}, L_{\lambda}) = (0, 2), (2, 0)$ and $(1, 1)$, respectively. The present predictions favor assigning the $\Lambda_c(2860)$, $\Lambda_c(2880)$, $\Xi_c(3055)$ and $\Xi_c(3080)$ to be the D-wave baryon states with the quantum numbers $(L_{\rho}, L_{\lambda}) = (0, 2)$ and $J^P = \frac{3}{2}^+$, $\frac{5}{2}^+$, $\frac{7}{2}^+$ and $\frac{9}{2}^+$, respectively. While the predictions for the masses of the $(L_{\rho}, L_{\lambda}) = (2, 0)$ and $(1, 1)$ D-wave $\Lambda_c$ and $\Xi_c$ states can be confronted to the experimental data in the future.

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1 Introduction

Recently, the LHCb collaboration studied the mass spectrum of excited $\Lambda_c^+$ states that decay into $D^0p$, and observed a new resonance $\Lambda_c(2860)^+$ near threshold \[^1\]. The measured masses, widths and quantum numbers of the $\Lambda_c(2860)^+$, $\Lambda_c(2880)^+$ and $\Lambda_c(2940)^+$ states are

$\Lambda_c(2860) : M = 2856.1^{+2.0}_{-1.7} \pm 0.5^{+1.1}_{-0.6} \text{ MeV}, \Gamma = 67.6^{+10.1}_{-8.1} \pm 1.4^{+5.9}_{-20.0} \text{ MeV}, J^P = \frac{3}{2}^+$,

$\Lambda_c(2880) : M = 2881.75 \pm 0.29 \pm 0.07^{+0.14}_{-0.20} \text{ MeV}, \Gamma = 5.43^{+0.77}_{-0.71} \pm 0.29^{+0.75}_{-0.00} \text{ MeV}, J^P = \frac{5}{2}^+$,

$\Lambda_c(2940) : M = 2944.8^{+3.5}_{-2.5} \pm 0.4^{+0.1}_{-0.6} \text{ MeV}, \Gamma = 27.7^{+8.2}_{-6.6} \pm 0.9^{+5.2}_{-10.4} \text{ MeV}, J^P = \frac{3}{2}^-$,

but other assignments with the spins $J = \frac{1}{2}$ to $\frac{7}{2}$ are not excluded for the $\Lambda_c(2940)^+$ \[^1\]. The $\Lambda_c(2880)^+$ was first observed by the CLEO collaboration in the $\Lambda_c^+ \pi^+ \pi^-$ channel \[^2\], confirmed by the BaBar collaboration in the $D^0p$ channel \[^3\] and the Belle collaboration in the $\Sigma_c(2455/2520)^{0,+} \pi^+ \pi^-$ channels \[^4\]. The available experimental analysis indicates that the $\Lambda_c(2880)^+$ has the spin-parity $J^P = \frac{5}{2}^+$. The theoretical predictions for the masses of the D-wave $\Lambda_c^+$ baryon states with $J^P = \frac{3}{2}^+$ and $\frac{5}{2}^+$ are about (2.85 − 2.90) GeV \[^5\] \[^6\] \[^7\] \[^8\] \[^9\] \[^10\]. The $\Lambda_c(2860)^+$ and $\Lambda_c(2880)^+$ can be assigned to be the D-wave charmed baryon states.

Their strange cousins $\Xi_c(3055)^+$ and $\Xi_c(3080)^+$ were observed in the channel $\Lambda_c^+ K^- \pi^+$ by the Belle collaboration \[^11\] and in the channels $\Sigma_c(2455/2520)^{0,+} K^-$ by the BaBar collaboration \[^12\]. In 2016, the $\Xi_c(3055)^+$ and $\Xi_c(3055)^0$ were first observed by the Belle

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collaboration in the $D^+\Lambda$ and $D^0\Lambda$ channels, respectively [13], the measured masses and widths were

$$\Xi_c(3055)^+: M = 3055.8 \pm 0.4 \pm 0.2\,\text{MeV}, \quad \Gamma = 7.8 \pm 1.2 \pm 1.5\,\text{MeV},$$

$$\Xi_c(3055)^0: M = 3059.0 \pm 0.5 \pm 0.6\,\text{MeV}, \quad \Gamma = 6.4 \pm 2.1 \pm 1.1\,\text{MeV},$$

(2)

furthermore, the Belle collaboration observed the first evidence for the $\Xi_c(3080)^+$ with the estimated mass $3077.9 \pm 0.9\,\text{MeV}$ and width $3.0 \pm 0.7 \pm 0.4\,\text{MeV}$. The theoretical predictions of the masses of the D-wave $\Xi_c$ baryon states with $J^P = \frac{3}{2}^+$ and $\frac{5}{2}^+$ are about $(3.05 - 3.10)\,\text{GeV}$ [5, 6, 7, 8, 9, 10], the $\Xi_c(3055)^+$, $\Xi_c(3055)^0$ and $\Xi_c(3080)^+$ can be assigned to be the D-wave charmed baryon states.

In this article, we tentatively assign the $\Lambda_c(2860)$, $\Lambda_c(2880)$, $\Xi_c(3055)$ and $\Xi_c(3080)$ to be the D-wave charmed baryon states with the spin-parity $J^P = \frac{3}{2}^+$, $\frac{5}{2}^+$, $\frac{3}{2}^+$ and $\frac{5}{2}^+$, respectively, and study their masses and pole residues with the QCD sum rules in a systematic way. The QCD sum rules is a powerful theoretical approach in studying the ground state mass spectrum of the heavy baryon states, and has given many successful descriptions [8, 14, 15, 16, 17, 18, 19].

We can construct the interpolating currents without introducing the relative P-wave to study the negative parity heavy, doubly-heavy and triply-heavy baryon states [14, 15, 16], or introducing the relative P-wave explicitly to study the negative parity heavy, doubly-heavy and triply-heavy baryon states [18, 19]. For the D-wave heavy baryon states, it is better to introduce the relative D-wave explicitly to study them with the QCD sum rules [8]. In Ref.[8], Chen et al study the mass spectrum of the D-wave heavy baryon states with the QCD sum rules combined with the heavy quark effective theory in a systematic way. In this article, we study the $\Lambda_c(2860)$, $\Lambda_c(2880)$, $\Xi_c(3055)$ and $\Xi_c(3080)$ as the D-wave heavy baryon states with the full QCD sum rules by introducing the relative D-wave explicitly in constructing the interpolating currents, which differ from the currents constructed in Ref.[8].

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the D-wave $\frac{3}{2}^+$ and $\frac{5}{2}^+$ charmed baryon states in Sect.2; in Sect.3, we present the numerical results and discussions; and Sect.4 is reserved for our conclusion.

2 QCD sum rules for the D-wave $\frac{3}{2}^+$ and $\frac{5}{2}^+$ charmed baryon states

Firstly, we write down the two-point correlation functions $\Pi_{\alpha\beta}(p)$ and $\Pi_{\alpha\beta\mu\nu}(p)$ in the QCD sum rules,

$$\Pi_{\alpha\beta}(p) = i \int d^4x e^{ip \cdot x} \langle 0| T \{ J/\eta_\alpha(x) \bar{J}/\bar{\eta}_\beta(0) \} |0\rangle,$$

$$\Pi_{\alpha\beta\mu\nu}(p) = i \int d^4x e^{ip \cdot x} \langle 0| T \{ J/\eta_{\alpha\beta}(x) \bar{J}/\bar{\eta}_{\mu\nu}(0) \} |0\rangle,$$

(3)
where $J/\eta_\alpha(x) = J/\eta_\alpha^1(x), J/\eta_\alpha^2(x), J/\eta_{\alpha\beta}(x) = J/\eta_{\alpha\beta}^1(x), J/\eta_{\alpha\beta}^2(x), J/\eta_{\alpha\beta}^3(x),$}

\begin{align*}
J_{\alpha}^1(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 s_j(x) + \partial^\mu q_i^T(x) C\gamma_5 \partial^\nu s_j(x) + \partial^\nu q_i^T(x) C\gamma_5 \partial^\mu s_j(x) \\
&\quad + q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu s_j(x) \right] \Gamma_{\mu\nu\alpha} c_k(x), \\
J_{\alpha}^2(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 s_j(x) - \partial^\mu q_i^T(x) C\gamma_5 \partial^\nu s_j(x) - \partial^\nu q_i^T(x) C\gamma_5 \partial^\mu s_j(x) \\
&\quad + q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu s_j(x) \right] \Gamma_{\mu\nu\alpha} c_k(x), \\
J_{\alpha}^3(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 s_j(x) - q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu s_j(x) \right] \Gamma_{\mu\nu\alpha} c_k(x),
\end{align*}

(4)

\begin{align*}
\eta_{\alpha}^1(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 q_j^T(x) + \partial^\mu q_i^T(x) C\gamma_5 \partial^\nu q_j^T(x) + \partial^\nu q_i^T(x) C\gamma_5 \partial^\mu q_j^T(x) \\
&\quad + q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu q_j^T(x) \right] \Gamma_{\mu\nu\alpha} c_k(x), \\
\eta_{\alpha}^2(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 q_j^T(x) - \partial^\mu q_i^T(x) C\gamma_5 \partial^\nu q_j^T(x) - \partial^\nu q_i^T(x) C\gamma_5 \partial^\mu q_j^T(x) \\
&\quad + q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu q_j^T(x) \right] \Gamma_{\mu\nu\alpha} c_k(x), \\
\eta_{\alpha}^3(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 q_j^T(x) - q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu q_j^T(x) \right] \Gamma_{\mu\nu\alpha} c_k(x),
\end{align*}

(5)

\begin{align*}
J_{\alpha\beta}^1(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 s_j(x) + \partial^\mu q_i^T(x) C\gamma_5 \partial^\nu s_j(x) + \partial^\nu q_i^T(x) C\gamma_5 \partial^\mu s_j(x) \\
&\quad + q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu s_j(x) \right] \Gamma_{\mu\nu\alpha\beta} c_k(x), \\
J_{\alpha\beta}^2(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 s_j(x) - \partial^\mu q_i^T(x) C\gamma_5 \partial^\nu s_j(x) - \partial^\nu q_i^T(x) C\gamma_5 \partial^\mu s_j(x) \\
&\quad + q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu s_j(x) \right] \Gamma_{\mu\nu\alpha\beta} c_k(x), \\
J_{\alpha\beta}^3(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 s_j(x) - q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu s_j(x) \right] \Gamma_{\mu\nu\alpha\beta} c_k(x),
\end{align*}

(6)

\begin{align*}
\eta_{\alpha\beta}^1(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 q_j^T(x) + \partial^\mu q_i^T(x) C\gamma_5 \partial^\nu q_j^T(x) + \partial^\nu q_i^T(x) C\gamma_5 \partial^\mu q_j^T(x) \\
&\quad + q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu q_j^T(x) \right] \Gamma_{\mu\nu\alpha\beta} c_k(x), \\
\eta_{\alpha\beta}^2(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 q_j^T(x) - \partial^\mu q_i^T(x) C\gamma_5 \partial^\nu q_j^T(x) - \partial^\nu q_i^T(x) C\gamma_5 \partial^\mu q_j^T(x) \\
&\quad + q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu q_j^T(x) \right] \Gamma_{\mu\nu\alpha\beta} c_k(x), \\
\eta_{\alpha\beta}^3(x) &= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q_i^T(x) C\gamma_5 q_j^T(x) - q_i^T(x) C\gamma_5 \partial^\mu \partial^\nu q_j^T(x) \right] \Gamma_{\mu\nu\alpha\beta} c_k(x),
\end{align*}

(7)

with

\begin{align*}
\Gamma_{\mu\nu\alpha} &= \left( g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - \frac{1}{2} g_{\mu\nu} g_{\alpha\beta} \right) \gamma^\beta \gamma_5, \\
\Gamma_{\mu\nu\alpha\beta} &= g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - \frac{1}{6} g_{\mu\nu} g_{\alpha\beta} - \frac{1}{4} g_{\mu\alpha} \gamma_\nu \gamma_\beta - \frac{1}{4} g_{\mu\beta} \gamma_\nu \gamma_\alpha - \frac{1}{4} g_{\nu\alpha} \gamma_\mu \gamma_\beta - \frac{1}{4} g_{\nu\beta} \gamma_\mu \gamma_\alpha \\
&\quad + \frac{1}{24} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta + \frac{1}{24} \gamma_\mu \gamma_\beta \gamma_\nu \gamma_\alpha + \frac{1}{24} \gamma_\nu \gamma_\alpha \gamma_\mu \gamma_\beta + \frac{1}{24} \gamma_\nu \gamma_\beta \gamma_\mu \gamma_\alpha,
\end{align*}

(8)

$q, q' = u, d$, the $i, j, k$ are color indices, the $C$ is the charge conjugation matrix. The currents satisfy the relations $\gamma^\alpha J_{\alpha}^i(x) = \gamma^\alpha \eta_{\alpha}^i(x) = 0$, $\gamma^\alpha J_{\alpha\beta}^i(x) = \gamma^\alpha \eta_{\alpha\beta}^i(x) = 0$, 3
\( \gamma^\beta J^\alpha_{i\beta}(x) = \gamma^\beta \eta^i_{\alpha\beta}(x) = 0 \), where \( i = 1, 2, 3 \). We choose the currents \( J/\eta^i_{\alpha\beta}(x) \) and \( J/\eta^i_{\alpha\beta}(x) \) to interpolate the \( J^P = \frac{3}{2}^+ \) and \( \frac{5}{2}^+ \) charmed baryon states, respectively. In this article, we tentatively assign the \( \Lambda_c(2860) \), \( \Lambda_c(2880) \), \( \Xi_c(3055) \) and \( \Xi_c(3080) \) to be the D-wave charmed baryon states with the spin-parity \( J^P = \frac{3}{2}^+ , \frac{5}{2}^+ , \frac{3}{2}^+ \) and \( \frac{5}{2}^+ \), respectively, the currents \( J_\alpha(x) \), \( J_{\alpha\beta}(x) \), \( \eta_\alpha(x) \) and \( \eta_{\alpha\beta}(x) \) may couple potentially to the \( \Lambda_c(2860) \), \( \Lambda_c(2880) \), \( \Xi_c(3055) \) and \( \Xi_c(3080) \), respectively.

Now we take a short digression to illustrate how to construct the currents. The attractive interaction of one-gluon exchange favors formation of the diquarks in color antitriplet \( \overline{3}_c \) [20]. The color antitriplet diquarks \( \varepsilon_{ijk} q_i^T C \Gamma d_j^\dagger \) have five structures in Dirac spinor space, where \( C \Gamma = C \gamma_5 \), \( C \gamma_{\mu} \gamma_5 \), \( C \gamma_{\mu} \) and \( C \sigma_{\mu\nu} \) for the scalar, pseudoscalar, vector, axialvector and tensor diquarks, respectively. The structures \( C \gamma_{\mu} \) and \( C \sigma_{\mu\nu} \) are symmetric, while the structures \( C \gamma_5 \), \( C \gamma_{\mu} \gamma_5 \) and \( C \gamma_{\mu} \) are antisymmetric. The calculations based on the QCD sum rules indicate that the favored configurations are the \( C \gamma_5 \) and \( C \gamma_{\mu} \) diquark states, while the most favored configurations are the \( C \gamma_5 \) diquark states [21].

We usually construct the heavy baryon states according to the light-diquark-heavy-quark model. In the diquark-quark models, the angular momentum between the two light quarks is denoted by \( L_\rho \), while the angular momentum between the light diquark and the heavy quark is denoted by \( L_\Lambda \). If the two light quarks in the diquark are in relative S-wave or \( L_\rho = 0 \), then the baryons with the \( J^P = 0^+ \) and \( 1^+ \) diquarks (the ground state diquarks) are called \( \Lambda \)-type and \( \Sigma \)-type baryons, respectively [22]. We can denote the \( C \gamma_5 \) and \( C \gamma_{\mu} \) diquarks as \( J^P_\rho = 0^+_\rho \) and \( 1^+_\rho \), respectively, the relative P-wave and D-wave as \( J^P_{\rho/\lambda} = L^P_{\rho/\lambda} = 1^-_{\rho/\lambda} \) and \( 2^+_{\rho/\lambda} \), respectively, the c-quark as \( J^P_c = \frac{1}{2}^c \), then we construct the D-wave baryon states according to the routines,

\[
\begin{align*}
0^+_d \otimes 2^+_p/\lambda \otimes 1^+_c &= 2^+_p \oplus \frac{1}{2}^+_c = \frac{3}{2}^+ \oplus \frac{5}{2}^+, \\
0^+_d \otimes 1^-_p \otimes 1^-_c \otimes \frac{1}{2}^+_c &= [0^+ \oplus 1^+ \oplus 2^+] \otimes \frac{1}{2}^+_c = \frac{1}{2}^+ \oplus \left[ \frac{1}{2}^+ \oplus \frac{3}{2}^+ \right] \oplus \left[ \frac{3}{2}^+ \oplus \frac{5}{2}^+ \right], \\
1^+_d \otimes 2^+_p/\lambda \otimes \frac{1}{2}^+_c &= [1^+ \oplus 2^+ \oplus 3^+] \otimes \frac{1}{2}^+_c = \left[ \frac{1}{2}^+ \oplus \frac{3}{2}^+ \right] \oplus \left[ \frac{3}{2}^+ \oplus \frac{5}{2}^+ \right] \oplus \left[ \frac{5}{2}^+ \oplus \frac{7}{2}^+ \right], \\
1^+_d \otimes 1^-_p \otimes 1^-_c \otimes \frac{1}{2}^+_c &= \left[ 0^- \oplus 1^- \oplus 2^- \right] \otimes \frac{1}{2}^+_c = \left[ 0^- \oplus \left[ 0^+ \oplus 1^+ \oplus 2^+ \right] \oplus \left[ 1^+ \oplus 2^+ \oplus 3^+ \right] \right] \otimes \frac{1}{2}^+_c \\
&\quad = \left[ \frac{1}{2}^+ \oplus \frac{3}{2}^+ \right] \oplus \frac{1}{2}^+ \oplus \left[ \frac{1}{2}^+ \oplus \frac{3}{2}^+ \right] \oplus \left[ \frac{3}{2}^+ \oplus \frac{5}{2}^+ \right] \oplus \left[ 1^+ \oplus \frac{3}{2}^+ \right] \\
&\quad \oplus \left[ \frac{3}{2}^+ \oplus \frac{5}{2}^+ \right] \oplus \left[ \frac{5}{2}^+ \oplus \frac{7}{2}^+ \right].
\end{align*}
\] (9)

(10)

(11)

It is difficult or impossible to construct currents to interpolate all the D-wave baryon states with \( J^P = \frac{1}{2}^+ , \frac{3}{2}^+ , \frac{5}{2}^+ \) and \( \frac{7}{2}^+ \) in a systematic way. In this article, we study the underlined D-wave baryon states with \( J^P = \frac{3}{2}^+ \) and \( \frac{5}{2}^+ \) in details based on the most favored configurations \( C \gamma_5 \) [21]. Experimentally, the measured quantum numbers of the \( \Lambda_c(2860)^+ \) and \( \Lambda_c(2880)^+ \) are \( J^P = \frac{3}{2}^+ \) and \( \frac{5}{2}^+ \) respectively from the LHCb collaboration [1], while the masses of the \( \Xi_c(3055)^+ \), \( \Xi_c(3055)^0 \) and \( \Xi_c(3080)^+ \) are consistent with the
theoretical predictions of the D-wave $\Xi_c$ baryon states with $J^P = \frac{3}{2}^+$ and $\frac{5}{2}^+$ [9, 10].

We can choose either the partial derivative $\partial_\mu$ or the covariant derivative $D_\mu$ to construct the interpolating currents. The currents with the covariant derivative $D_\mu$ are gauge invariant, but blur the physical interpretation of the angular momentum. In Ref. [23], we study the masses and decay constants of the heavy tensor mesons $D_\mu(x)$, $D_\mu(x)$, $B_\mu(x)$, and $B_\mu(x)$ with the QCD sum rules. In calculations, we observe that the predictions based on the currents with the partial derivative and covariant derivative differ from each other about 1%, if the same parameters are chosen. If we refit the Borel parameters and threshold parameters, the differences about 1% can be reduced remarkably, so the currents with the partial derivative work well. In this article, we choose the partial derivative $\partial_\mu$ to construct the interpolating currents. Furthermore, from the Table 1 in Section 3, we can see that the dominant contributions come from the perturbative terms, so neglecting the contributions originate from the gluons in the covariant derivative $D_\mu$ cannot change the conclusion.

For $L_\rho = 1$ and $L_\lambda = 0$, the light diquark state with $J^P = 1^-$ can be written as

$$\varepsilon^{ijk} \left[ \partial^\mu q^T_i(x) C\gamma_5 q^T_j(x) - q^T_i(x) C\gamma_5 \partial^\mu q^T_j(x) \right],$$

then we introduce an additional P-wave between the two quarks $q$ and $q'$, and obtain the light diquark state with $L_\rho = 2$, $L_\lambda = 0$ and $J^P = 2^+$,

$$\varepsilon^{ijk} \left\{ \left[ \partial^\mu \partial^\nu q^T_i(x) C\gamma_5 q^T_j(x) - \partial^\nu q^T_i(x) C\gamma_5 \partial^\mu q^T_j(x) \right] - \left[ \partial^\mu q^T_i(x) C\gamma_5 \partial^\nu q^T_j(x) - \partial^\nu q^T_i(x) C\gamma_5 \partial^\mu q^T_j(x) \right] \right\}, \quad (13)$$

In the heavy quark limit, the $c$-quark is static, the $\partial_\mu$ is reduced to $\partial_\mu$ when operating on the $c$-quark field. For $L_\rho = 0$ and $L_\lambda = 2$, the light diquark state with $J^P = 2^+$ can be written as

$$\partial^\mu \partial^\nu \left[ \varepsilon^{ijk} q^T_i(x) C\gamma_5 q^T_j(x) \right] = \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q^T_i(x) C\gamma_5 q^T_j(x) + \partial^\mu q^T_i(x) C\gamma_5 \partial^\nu q^T_j(x) \right]$$

$$+ \partial^\nu q^T_i(x) C\gamma_5 \partial^\mu q^T_j(x) + q^T_i(x) C\gamma_5 \partial^\mu \partial^\nu q^T_j(x) \right] \right]. \quad (14)$$

For $L_\rho = 1$ and $L_\lambda = 1$, the light diquark state with $J^P = 2^+$ can be written as

$$\partial^\mu \varepsilon^{ijk} \left[ \partial^\nu q^T_i(x) C\gamma_5 q^T_j(x) - q^T_i(x) C\gamma_5 \partial^\nu q^T_j(x) \right]$$

$$= \varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q^T_i(x) C\gamma_5 q^T_j(x) + \partial^\mu q^T_i(x) C\gamma_5 \partial^\nu q^T_j(x) - \partial^\nu q^T_i(x) C\gamma_5 \partial^\mu q^T_j(x) \right]$$

$$- q^T_i(x) C\gamma_5 \partial^\mu \partial^\nu q^T_j(x) \right]. \quad (15)$$

We symmetrize the Lorentz indexes $\mu$ and $\nu$, and obtain the light diquark state with $L_\rho = 1$ and $L_\lambda = 1$ in a more simple form,

$$\varepsilon^{ijk} \left[ \partial^\mu \partial^\nu q^T_i(x) C\gamma_5 q^T_j(x) - q^T_i(x) C\gamma_5 \partial^\mu \partial^\nu q^T_j(x) \right]. \quad (16)$$

The light diquark states with $J^P = 2^+$ then combine with the $c$-quark to form $J^P = \frac{3}{2}^+$ or $\frac{5}{2}^+$ baryon states, see Eqs.(9-10).
The interpolating currents can be classified by

\[
\begin{align*}
(L_\rho, L_\lambda) & = (0,2) \quad \text{for} \quad J/\eta_\alpha^1(x), \ J/\eta_{\alpha\beta}^1(x), \\
(L_\rho, L_\lambda) & = (2,0) \quad \text{for} \quad J/\eta_\alpha^2(x), \ J/\eta_{\alpha\beta}^2(x), \\
(L_\rho, L_\lambda) & = (1,1) \quad \text{for} \quad J/\eta_\alpha^3(x), \ J/\eta_{\alpha\beta}^3(x).
\end{align*}
\]

The currents \( J/\eta_\alpha(0) \) and \( J/\eta_{\alpha\beta}(0) \) couple potentially to the \( 3^\pm/2 \) and \( 5^\pm/2 \) charmed baryon states \( B^\pm_{3/2} \) and \( B^\pm_{5/2} \), respectively [17] [24] [25], which are supposed to be the excited \( \Lambda_c \) or \( \Xi_c \) states,

\[
\begin{align*}
\langle 0| J/\eta_\alpha(0)| B^+_2(p) \rangle & = \lambda^{+2}_{3/2} U^+_\alpha(p, s), \\
\langle 0| J/\eta_{\alpha\beta}(0)| B^+_2(p) \rangle & = \lambda^{+2}_{5/2} U^+_{\alpha\beta}(p, s), \\
\langle 0| J/\eta_\alpha(0)| B^-_2(p) \rangle & = \lambda^{-2}_2 i\gamma_5 U^-_\alpha(p, s), \\
\langle 0| J/\eta_{\alpha\beta}(0)| B^-_2(p) \rangle & = \lambda^{-2}_2 i\gamma_5 U^-_{\alpha\beta}(p, s),
\end{align*}
\]

where the \( \lambda^{\pm}_{3/2} \) and \( \lambda^{\pm}_{5/2} \) are the pole residues or the current-baryon coupling constants, the spinors \( U^+_\alpha(p, s) \) and \( U^+_{\alpha\beta}(p, s) \) satisfy the Rarita-Schwinger equations \( (p^2 - M_\pm)U^\pm_\alpha(p, s) = 0 \) and \( (p^2 - M_\pm)U^\pm_{\alpha\beta}(p, s) = 0 \), and the relations \( \gamma^\alpha U^\pm_\alpha(p, s) = 0, \ p^\alpha U^\pm_\alpha(p, s) = 0, \gamma^\alpha U^\pm_{\alpha\beta}(p, s) = 0, \ p^\alpha U^\pm_{\alpha\beta}(p, s) = 0, \) \( U^\pm_\alpha(p, s) = U^\pm_\beta(p, s) \), which are consistent with relations \( \gamma^\alpha J^i_\alpha(0) = \gamma^\alpha \eta^i_\alpha(0) = 0, \gamma^\alpha J^i_{\alpha\beta}(0) = \gamma^\alpha \eta^i_{\alpha\beta}(0) = 0, \gamma^\beta J^i_{\alpha\beta}(0) = \gamma^\beta \eta^i_{\alpha\beta}(0) = 0. \)

At the hadron side, we insert a complete set of intermediate charmed baryon states with the same quantum numbers as the current operators \( J/\eta_\alpha(x), i\gamma_5 J/\eta_\alpha(x), J/\eta_{\alpha\beta}(x) \) and \( i\gamma_5 J/\eta_{\alpha\beta}(x) \) into the correlation functions \( \Pi_{\alpha\beta}(p) \) and \( \Pi_{\alpha\beta\mu\nu}(p) \) to obtain the hadronic representation [26] [27]. We isolate the pole terms of the lowest charmed baryon states with positive parity and negative parity, and obtain the results:

\[
\Pi_{\alpha\beta}(p) = \lambda^{+2}_{3/2} \frac{\not{p} + M_+}{M_+^2 - p^2} \left( -g_{\alpha\beta} + \frac{\gamma_\alpha \gamma_\beta}{3} + \frac{2p_\alpha p_\beta}{3p^2} - \frac{p_\alpha g_{\gamma\beta} - p_\beta g_{\gamma\alpha}}{3\sqrt{p^2}} \right) + \lambda^{-2}_{5/2} \frac{\not{p} - M_-}{M_-^2 - p^2} \left( -g_{\alpha\beta} + \frac{\gamma_\alpha \gamma_\beta}{3} + \frac{2p_\alpha p_\beta}{3p^2} - \frac{p_\alpha g_{\gamma\beta} - p_\beta g_{\gamma\alpha}}{3\sqrt{p^2}} \right) + \cdots ,
\]

\[
= \Pi^{+}_{\frac{3}{2}}(p^2) (-g_{\alpha\beta}) + \cdots ,
\]

where \( g_{\alpha\beta} \) is the energy-momentum tensor.
\[ \Pi_{\alpha\beta\mu}(p) = \frac{\lambda^2}{2} \frac{\not{p} + M_+}{M_+^2 - p^2} \left[ \frac{\bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} + \bar{g}_{\mu\beta} \bar{g}_{\nu\alpha}}{2} - \frac{\bar{g}_{\mu\nu} \bar{g}_{\alpha\beta}}{5} - \frac{1}{10} \left( \gamma_{\alpha\gamma} \gamma_{\mu\nu} - \frac{\gamma_{\alpha\mu} p_{\nu} - p_{\alpha\mu} p_{\nu}}{p^2} \right) \bar{g}_{\nu\beta} \right] \\
- \frac{1}{10} \left( \gamma_{\alpha\gamma} \gamma_{\mu\nu} + \frac{\gamma_{\alpha\mu} p_{\nu} - p_{\alpha\mu} p_{\nu}}{p^2} \right) \bar{g}_{\mu\beta} + \ldots \right] + \ldots, \]

where \( \bar{g}_{\mu\nu} = g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \). The currents \( J/\eta_\alpha(0) \), \( J/\eta_{\alpha\beta}(0) \) also have non-vanishing couplings with the spin \( J = \frac{1}{2} \) and \( J = \frac{1}{2}, \frac{3}{2} \) charmed baryon states, respectively, we choose the tensor structures \( g_{\alpha\beta} \) and \( g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} \) for analysis, the baryon states with the spin \( \frac{1}{2} \) and \( \frac{3}{2} \) have no contaminations [25].

In calculations, we have used two summations over the polarizations \( s \) in the spinors \( U_{\alpha}^\pm(p, s) \) and \( U_{\alpha\beta}^\pm(p, s) \) [28],

\[ \sum_s U_{\alpha} U_{\beta} = (\not{p} + M_+) \left( -g_{\alpha\beta} + \frac{\gamma_{\alpha\beta}}{3} + \frac{2p_{\alpha} p_{\beta}}{3p^2} - \frac{p_{\alpha} p_{\beta} - p_{\beta} p_{\alpha}}{3 \sqrt{p^2}} \right), \]

\[ \sum_s U_{\mu\nu} U_{\alpha\beta} = (\not{p} + M_+) \left\{ \frac{\bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} + \bar{g}_{\mu\beta} \bar{g}_{\nu\alpha}}{2} - \frac{\bar{g}_{\mu\nu} \bar{g}_{\alpha\beta}}{5} - \frac{1}{10} \left( \gamma_{\alpha\gamma} \gamma_{\mu\nu} + \frac{\gamma_{\alpha\mu} p_{\nu} - p_{\alpha\mu} p_{\nu}}{p^2} \right) \bar{g}_{\nu\beta} \right\} - \frac{1}{10} \left( \gamma_{\nu\gamma} \gamma_{\alpha\beta} + \frac{\gamma_{\nu\alpha} p_{\beta} - p_{\nu\alpha} p_{\beta}}{p^2} \right) \bar{g}_{\mu\beta} - \frac{1}{10} \left( \gamma_{\nu\gamma} \gamma_{\beta\nu} + \frac{\gamma_{\nu\beta} p_{\gamma} - p_{\nu\beta} p_{\gamma}}{p^2} \right) \bar{g}_{\mu\alpha} \right\}, \]

and \( p^2 = M_+^2 \) on mass-shell.

We obtain the hadronic spectral densities at the hadron side through dispersion relation,

\[ \frac{\text{Im} \Pi_j(s)}{\pi} = \int \frac{d^4q}{(2\pi)^4} \frac{\lambda_j^2}{2} \frac{\delta(s - M_j^2)}{M_j^2} \left[ M_+ \lambda_j^2 \delta(s - M_j^2) - M_- \lambda_j^{-2} \delta(s - M_j^2) \right], \]

\[ = p \rho_{j, H}^0(s) + \rho_{j, H}^0(s), \]

where \( j = \frac{3}{2}, \frac{5}{2} \), the subscript \( H \) denotes the hadron side, then we introduce the exponential function \( \exp(-\frac{s}{T^2}) \) to depress the continuum state contributions to obtain the QCD sum rules at the hadron side,

\[ \int_{m_0^2}^{s_0} ds \left[ \sqrt{s} \rho_{j, H}^0(s) + \rho_{j, H}^0(s) \right] \exp(-\frac{s}{T^2}) = 2M_+ \lambda_j^2 \exp\left(-\frac{M_j^2}{T^2}\right), \]

where the \( s_0 \) are the continuum thresholds and the \( T^2 \) are the Borel parameters [25]. From Eq.(25), we can see that the \( \frac{3}{2}^- \) and \( \frac{5}{2}^- \) charmed baryon states have no contaminations.
according to the special combination $\sqrt{s}\rho^1_{j,H}(s) + \rho^0_{j,H}(s)$. On the other hand, we can obtain the QCD sum rules for the charmed baryon states with negative parity,

$$
\int_{m_c^2}^{s_0} ds \left[ \sqrt{s}\rho^1_{j,H}(s) - \rho^0_{j,H}(s) \right] \exp \left( -\frac{s}{T^2} \right) = 2M_-\lambda_j^{-2} \exp \left( -\frac{M^2}{T^2} \right).
$$

(26)

The contributions of the $\frac{3}{2}^-$ and $\frac{5}{2}^+$ charmed baryon states can be separated unambiguously. In this article, we will focus on the $\Lambda_c$ and $\Xi_c$ states with positive parity.

At the QCD side, we calculate the light quark parts of the correlation functions $\Pi_{\alpha\beta}(p)$ and $\Pi_{\alpha\beta\mu\nu}(p)$ with the full light quark propagators $S_{ij}(x)$ in the coordinate space

$$
S_{ij}(x) = \frac{i\delta_{ij} x^\mu - \delta_{ij}m_q}{2\pi^2 x^2} + \frac{\delta_{ij} \langle \bar{q}q \rangle}{12} + \frac{i\delta_{ij} \bar{m}_q \langle \bar{q}q \rangle}{48} + \frac{\delta_{ij} x^2 \langle \bar{q}g_s\sigma Gq \rangle}{192} + \frac{i\delta_{ij} x^2 \bar{m}_q \langle \bar{q}g_s\sigma Gq \rangle}{1152} - \frac{i\bar{g}_s G^{\alpha}_{\alpha\beta} x_{ij} \langle \bar{q}x\sigma^\alpha + \sigma^\alpha \bar{x} \rangle}{32\pi^2 x^2} - \frac{1}{8} \langle \bar{q}j_{\mu\nu}q \rangle \sigma_{\mu\nu} + \cdots,
$$

(27)

and take the full $c$-quark propagator $C_{ij}(x)$ in the momentum space,

$$
C_{ij}(x) = \frac{i}{(2\pi)^4} \int d^4k e^{-ik\cdot x} \left\{ \frac{\delta_{ij} - g_s G^{\alpha}_{\alpha\beta} t^{\alpha}_{ij} \bar{\sigma}(k - m_c)(k + m_c) + (k + m_c)\sigma^\alpha}{4(k^2 - m_c^2)^2} \right\} - \frac{g^2_s (t^{a\beta} b)_{ij} G^{a}_{\alpha\beta} G^{b}_{\mu\nu} (f^{\alpha\beta\mu\nu} + f^{a\mu\beta\nu} + f^{a\mu\beta\nu})}{4(k^2 - m_c^2)^5} + \cdots,
$$

(28)

$$
f^{\alpha\beta\mu\nu} = (k + m_c)_{\gamma} (k + m_c)_{\gamma'} (k + m_c)_{\gamma} (k + m_c)_{\gamma'},
$$

(29)

$q = u, d, s$, $t^a = \lambda^a/2$, the $\lambda^a$ is the Gell-Mann matrix [27]. In Eq.(27), we retain the term $\langle \bar{q}j_{\mu\nu}q \rangle$ originates from the Fierz re-arrangement of the $\langle \bar{q}q \rangle$ to absorb the gluons emitted from the other quark lines to form $\langle \bar{q}j_{\mu\nu}q \rangle$. Then we compute the integrals both in the coordinate space and momentum space to obtain the correlation functions $\Pi_j(p^2)$, and obtain the QCD spectral densities through dispersion relation,

$$
\frac{\text{Im}\Pi_j(s)}{\pi} = \left( \pi \rho^1_{j,\text{QCD}}(s) + \rho^0_{j,\text{QCD}}(s) \right),
$$

(30)

where $j = \frac{3}{2}, \frac{5}{2}$, the explicit expressions of the QCD spectral densities $\rho^1_{j,\text{QCD}}(s)$ and $\rho^0_{j,\text{QCD}}(s)$ are shown in the Appendix. In this article, we carry out the operator product expansion up to the vacuum condensates of dimension 10 and take into account the condensates, which are vacuum expectations of the operators of order $O(\alpha_s^k)$ with $k \leq 1$, in a consistent way. In calculations, we observe that only the vacuum condensates $\langle \bar{q}q \rangle$, $\langle \bar{s}s \rangle$, $\langle \bar{s}aG G \rangle$, $\langle \bar{q}g_s\sigma Gq \rangle$, $\langle \bar{q}g_s\sigma Gs \rangle$, $\langle \bar{q}g_s\sigma Gq \rangle$, $\langle \bar{q}g_s\sigma Gs \rangle$ have contributions.

Once the analytical expressions of the QCD spectral densities $\rho^1_{j,\text{QCD}}(s)$ and $\rho^0_{j,\text{QCD}}(s)$ are obtained, we take the quark-hadron duality below the continuum thresholds $s_0$ and introduce the exponential function $\exp \left( -\frac{M^2}{T^2} \right)$ to depress the continuum state contributions to obtain the QCD sum rules:

$$
2M_+\lambda_j^{-2} \exp \left( -\frac{M^2}{T^2} \right) = \int_{m_c^2}^{s_0} ds \sqrt{s}\rho^1_{j,\text{QCD}}(s) + \rho^0_{j,\text{QCD}}(s) \exp \left( -\frac{s}{T^2} \right).
$$

(31)
We derive Eq.(31) with respect to \( \frac{d}{d(1/T^2)} \) and obtain the QCD sum rules for the masses of the charmed baryon states with \( J^P = \frac{3}{2}^+ \) and \( \frac{5}{2}^+ \),

\[
M_+^2 = -\frac{\int ds}{\pi} \frac{\exp(-\frac{\mu}{\Lambda})}{\pi} \int ds \left[ \sqrt{s} \rho^1_{J,QCD}(s) + \rho^0_{J,QCD}(s) \right] \exp\left(-\frac{\mu}{\Lambda} \right).
\]

3 Numerical results and discussions

The input parameters at the QCD side are taken to be the standard values \( \langle \bar{q}q \rangle = -(0.24 \pm 0.01 \text{ GeV})^2 \), \( \langle ss \rangle = (0.8 \pm 0.1) \langle \bar{q}q \rangle \), \( \langle \bar{q}g_s \sigma G q \rangle = m_0^2 \langle \bar{q}q \rangle \), \( \langle s g_s \sigma G s \rangle = m_0^2 \langle ss \rangle \), \( m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2 \), \( \langle \alpha_s^2 \rangle = (0.33 \text{ GeV})^2 \) at the energy scale \( \mu = 1 \text{ GeV} \) \cite{20, 27, 29}, \( m_c(m_c) = (1.275 \pm 0.025) \text{ GeV} \) and \( m_s(\mu = 2 \text{ GeV}) = (0.095 \pm 0.005) \text{ GeV} \) from the Particle Data Group \cite{30}. Furthermore, we set \( m_u = m_d = 0 \) due to the small current quark masses. We take into account the energy-scale dependence of the input parameters from the renormalization group equation,

\[
\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(Q) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{\Delta}{2}},
\]

\[
\langle ss \rangle(\mu) = \langle ss \rangle(Q) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{\Delta}{2}},
\]

\[
\langle \bar{q}g_s \sigma G q \rangle(\mu) = \langle \bar{q}g_s \sigma G q \rangle(Q) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{\Delta}{2}},
\]

\[
\langle s g_s \sigma G s \rangle(\mu) = \langle s g_s \sigma G s \rangle(Q) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{\Delta}{2}},
\]

\[
m_c(\mu) = m_c(m_c) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{42}{29}},
\]

\[
m_s(\mu) = m_s(2 \text{GeV}) \left[ \frac{\alpha_s(\mu)}{\alpha_s(2 \text{GeV})} \right]^{\frac{4}{7}},
\]

\[
\alpha_s(\mu) = \frac{1}{b_0 t} \left[ 1 - b_1 \log \frac{t}{b_0^2} + b_2 (log^2 t - \log t - 1) + b_0 b_2 \right],
\]

where \( t = \log \frac{\mu^2}{\Lambda^2} \), \( b_0 = \frac{33-2n_f}{12\pi}, b_1 = \frac{153-19n_f}{24\pi}, b_2 = \frac{2857}{128\pi^4}n_f + \frac{325}{24\pi^2} \), \( \Lambda = 213 \text{ MeV}, 296 \text{ MeV} \) and \( 339 \text{ MeV} \) for the flavors \( n_f = 5, 4 \) and \( 3 \), respectively \cite{30}, and evolve all the input parameters to the optimal energy scales \( \mu \) to extract the masses of the charmed baryon states.

In the heavy quark limit, the \( Q \)-quark serves as a static well potential and combines with a light quark \( q \) to form a heavy diquark in color antitriplet, or combines with a light antiquark \( \bar{q} \) to form a heavy meson in color singlet (meson-like state in color octet), or combines with a light diquark \( \varepsilon^{ijk}q_i q_j q_k \) to form a heavy baryon in color singlet (triquark
The energy scale formula works well for the to study the hidden-charm pentaquark states \[25\] and charmed baryon states \[19, 34\].

\[
q^i + Q^k \rightarrow \varepsilon^{ijk} q^i Q^k,
\]

\[
\bar{q}^i + Q^k \rightarrow \bar{q}^j \delta_{jk} Q^k (\bar{q}^j \lambda^a Q^k),
\]

\[
\varepsilon^{ijl} q^i q^j + Q^k \rightarrow \varepsilon^{ijl} q^i q^j \delta_{lk} Q^k (\varepsilon^{lkm} \varepsilon^{ijl} q^i q^j Q^k),
\]

where the \( i, j, k, l, m \) are color indexes, the \( \lambda^a \) is Gell-Mann matrix. The \( \bar{Q} \)-quark serves as another static well potential and has similar property. Then

\[
\varepsilon^{ijk} q^i Q^k + \varepsilon^{imn} \bar{q}^m \bar{Q}^n \rightarrow \text{compact tetraquark states},
\]

\[
\varepsilon^{ijk} q^i q^j Q^k + \varepsilon^{imn} \bar{q}^m \bar{Q}^n \rightarrow \text{compact pentaquark states},
\]

\[
\varepsilon^{ijk} q^i q^j Q^k + \bar{Q} q^l \rightarrow \text{loose molecular states},
\]

\[
\bar{q} \lambda^a Q + \bar{Q} \lambda^a q' \rightarrow \text{molecule-like states}.
\]

The three-quark systems \( q q' Q \), four-quark systems \( q q' Q \bar{Q} \), five-quark systems \( q q' q'' Q \bar{Q} \) are characterized by the effective heavy quark masses \( M_Q \) (or constituent quark masses) and the virtuality \( V = \sqrt{M_B^2 - M_Q^2}, \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2}, \sqrt{M_P^2 - (2M_Q)^2} \) (or bound energy not as robust), where the \( B \) denotes the conventional baryon states, the \( X, Y, Z \) denote the hidden-charm (bottom) tetraquark quark states, molecular states or molecule-like states, the \( P \) denotes the (molecular) pentaquark states. It is natural to take the energy scales of the QCD spectral densities to be \( \mu = V \).

The effective \( Q \)-quark masses \( M_Q \) have three universal values, which correspond to

1. the diquark-quark type baryon states \( \varepsilon^{ijk} q^i q^j Q^k \),
2. the diquark-antidiquark type tetraquark states \( \varepsilon^{ijk} \varepsilon^{imn} q^i q^j Q^k \bar{Q}^n \),
3. the diquark-diquark-antiquark type pentaquark states \( \varepsilon^{ijk} \varepsilon^{imn} q^i q^j Q^k \bar{Q}^n \bar{Q} \).
4. the meson-meson type molecular states \( \bar{q} Q \bar{Q} q' \),
5. the molecule-like states \( \bar{q} \lambda^a Q \bar{Q} \lambda^a q' \),

shown in Eqs. (34-35), respectively, and embody the net effects of the complex dynamics \[25, 31, 32, 33\].

We fit the effective \( Q \)-quark masses \( M_Q \) to reproduce the experimental values \( M_{Z_c(3900)} \) and \( M_{Z_b(10610)} \) in the scenario of tetraquark states \[31\], then we take the \( M_c \) and \( M_b \) as input parameters, and use the energy scale formula \( \mu = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2}, \sqrt{M_P^2 - (2M_Q)^2} \),

\[
\sqrt{M_B^2 - M_Q^2}
\]

to study the hidden-charm (hidden-bottom) tetraquark states, hidden-charm pentaquark states and charmed baryon states. We call the energy scale formula empirical, because the energy scale formula was used to study the hidden-charm (hidden-bottom) tetraquark states and molecular states first \[31, 32\], then it was extended to study the hidden-charm pentaquark states \[25\] and charmed baryon states \[19, 31\].

The energy scale formula works well for the \( X(3872), Z_c(3885/3900), X^*(3860), Y(3915), Z_{c}(4020/4025), Z(4430), X(4500), Y(4630/4660), X(4700), Z_b(10610), Z_b(10650) \), \[35\]

\( P_c(4380), P_b(4450) \), \[25\], \( \Lambda_c(2625), \Xi_c(2815) \), \[19\], \( \Omega_c(3050), \Omega_c(3066), \Omega_c(3090) \) and \( \Omega_c(3119) \) \[31\].

\[2\] All the relevant references can be found in Ref. \[35\].
In this article, we use the empirical formula $\mu = \sqrt{M_B^2 - M_c^2}$ to determine the ideal energy scales of the QCD spectral densities. If we take the updated value of the effective $c$-quark mass $M_c = 1.82$ GeV \cite{33}, then the optimal energy scales are $\mu = 2.2$ GeV, 2.2 GeV, 2.5 GeV and 2.5 GeV for the $\Lambda_c(2860)$, $\Lambda_c(2880)$, $\Xi_c(3055)$ and $\Xi_c(3080)$, respectively. In calculations, we observe that if the charmed baryon states $\Lambda_c(2860)$, $\Lambda_c(2880)$, $\Xi_c(3055)$ and $\Xi_c(3080)$ have the quantum numbers $(L_\rho, L_\lambda) = (0, 2)$, the experimental values of the masses $M_{\Lambda_c}/M_c$ can be reproduced approximately. The currents with the quantum numbers $(L_\rho, L_\lambda) = (2, 0)$ and $(L_\rho, L_\lambda) = (1, 1)$ couple potentially to the D-wave charmed baryon states having larger masses than the corresponding charmed baryon states $\Lambda_c(2860)/2880$ and $\Xi_c(3055/3080)$, so their QCD spectral densities should be calculated at larger energy scales according to the virtuality $V = \sqrt{M_B^2 - M_c^2}$, the empirical energy scale formula $\mu = \sqrt{M_B^2 - M_c^2}$ serves as a powerful constraint to satisfy. In Fig.1, we plot the masses of the charmed baryon states $\Xi_c(0, 2; \frac{3}{2})$, $\Xi_c(0, 2; \frac{3}{2})$, $\Lambda_c(0, 2; \frac{3}{2})$ and $\Lambda_c(0, 2; \frac{3}{2})$ with variations of the energy scale $\mu$ for the central values of the Borel parameters and threshold parameters shown in Table 1. From the figure, we can see that the predicted masses depend on the energy scale $\mu$ slightly, the acceptable ranges of the energy scale are rather large, the constraint $\mu = \sqrt{M_B^2 - M_c^2}$ is not difficult to satisfy in the present case. On the other hand, the pole residues increase monotonously and quickly with increase of the energy scale, it is important to choose the ideal energy scales.

We search for the ideal Borel parameters $T^2$ and continuum threshold parameters $s_0$ according to the four criteria:

1. Pole dominance at the hadron side, the pole contributions are about (50 – 80)%;
2. Convergence of the operator product expansion, the dominant contributions come from the perturbative terms;
3. Appearance of the Borel platforms, the uncertainties $\delta M/M$ originate from the Borel parameters are about (2 – 5)% in the Borel windows;
4. Satisfying the energy scale formula.

by try and error, and present the optimal energy scales $\mu$, ideal Borel parameters $T^2$, continuum threshold parameters $s_0$, pole contributions and perturbative contributions in Table 1. In the QCD sum rules for the baryon states, the predicted masses usually increase monotonously but slowly with increase of the Borel parameters \cite{37}, there cannot appear platforms as flat as that appear in the case of the conventional mesons and tetraquark states \cite{29,31}. In this article, we observe that the predicted masses also increase with increase of the Borel parameters, so we constrain the uncertainties $\delta M/M$ originate from the Borel parameters will not exceed 5% in the Borel windows.

From Table 1, we can see that the pole dominance at the hadron side is well satisfied and the operator product expansion is well convergent, the criteria 1 and 2 (the basic criteria of the QCD sum rules) are satisfied, so we expect to make reliable predictions. In Ref.\cite{8}, Chen et al study the D-wave heavy baryon states with the QCD sum rules combined with the heavy quark effective theory, and extract the masses with the pole contributions $\leq 20\%$, while in the present work, the pole contributions are about (50 – 80)%. The QCD spectral densities have the terms $m_s \langle \bar{q}q \rangle$, $m_s \langle ss \rangle$, $m_s \langle \bar{q}g_s \sigma G \rangle$, $m_s \langle \bar{s}g_s \sigma G \rangle$, which are greatly depressed by the small $s$-quark mass and are of minor importance, the dominant contributions come from the perturbative terms.
Table 1: The optimal energy scales $\mu$, Borel parameters $T^2$, continuum threshold parameters $s_0$, pole contributions (pole) and perturbative contributions (perturbative) for the D-wave charmed baryon states.

| $(L_\rho, L_\lambda)$ | $J^P$ | $\mu$(GeV) | $T^2$(GeV$^2$) | $\sqrt{s_0}$(GeV) | pole | perturbative |
|-----------------------|------|------------|---------------|-----------------|------|-------------|
| $\Xi_c$               | (0,2) | $3^+$      | 2.5           | 1.8 - 2.2       | 3.7 ± 0.1 | (46 - 76)%  | (87 - 92)%  |
| $\Xi_c$               | (0,2) | $3^+$      | 2.5           | 1.5 - 1.9       | 3.6 ± 0.1 | (48 - 81)%  | (96 - 99)%  |
| $\Lambda_c$           | (0,2) | $3^+$      | 2.2           | 1.5 - 1.9       | 3.6 ± 0.1 | (53 - 86)%  | (76 - 90)%  |
| $\Lambda_c$           | (0,2) | $3^+$      | 2.2           | 1.2 - 1.6       | 3.4 ± 0.1 | (49 - 87)%  | (88 - 99)%  |
| $\Xi_c$               | (2,0) | $3^+$      | 2.7           | 1.8 - 2.2       | 3.8 ± 0.1 | (47 - 77)%  | (97 - 98)%  |
| $\Lambda_c$           | (2,0) | $3^+$      | 2.7           | 1.7 - 2.1       | 3.8 ± 0.1 | (51 - 81)%  | (98 - 99)%  |
| $\Xi_c$               | (2,0) | $3^+$      | 2.7           | 1.7 - 2.1       | 3.8 ± 0.1 | (52 - 81)%  | (95 - 97)%  |
| $\Lambda_c$           | (2,0) | $3^+$      | 2.7           | 1.7 - 2.1       | 3.8 ± 0.1 | (51 - 81)%  | (96 - 97)%  |
| $\Xi_c$               | (1,1) | $3^+$      | 2.7           | 1.8 - 2.2       | 3.8 ± 0.1 | (50 - 79)%  | (97 - 98)%  |
| $\Xi_c$               | (1,1) | $3^+$      | 2.7           | 1.6 - 2.0       | 3.8 ± 0.1 | (55 - 84)%  | (101 - 102)%|
| $\Lambda_c$           | (1,1) | $3^+$      | 2.7           | 1.8 - 2.2       | 3.8 ± 0.1 | (50 - 79)%  | (96 - 97)%  |
| $\Lambda_c$           | (1,1) | $3^+$      | 2.7           | 1.6 - 2.0       | 3.8 ± 0.1 | (55 - 84)%  | (100 - 100)%|

Table 2: The masses and pole residues of the D-wave charmed baryon states, the masses are compared with the experimental data and other QCD sum rules predictions.

| $(L_\rho, L_\lambda)$ | $J^P$ | $M$(GeV) | $\lambda$(GeV$^3$) | (expt) (MeV) | $S$ (GeV) |
|-----------------------|------|--------|-------------------|-------------|---------|
| $\Xi_c$               | (0,2) | $3^+$  | 3.09$^{+0.13}_{-0.15}$ | 3.73$^{+0.89}_{-0.85} \times 10^{-2}$ | 3076.94/3079.9 | 3.05$^{+0.15}_{-0.16}$ |
| $\Xi_c$               | (0,2) | $3^+$  | 3.06$^{+0.11}_{-0.13}$ | 1.47$^{+0.37}_{-0.35} \times 10^{-1}$ | 3055.1 | 3.04$^{+0.15}_{-0.15}$ |
| $\Lambda_c$           | (0,2) | $3^+$  | 2.88$^{+0.18}_{-0.29}$ | 2.47$^{+0.89}_{-0.92} \times 10^{-2}$ | 2881.5 | 2.84$^{+0.37}_{-0.29}$ |
| $\Lambda_c$           | (0,2) | $3^+$  | 2.83$^{+0.15}_{-0.24}$ | 0.84$^{+0.32}_{-0.33} \times 10^{-1}$ | 2856.1 | 2.81$^{+0.33}_{-0.18}$ |
| $\Xi_c$               | (2,0) | $3^+$  | 3.25$^{+0.10}_{-0.11}$ | 1.42$^{+0.34}_{-0.27} \times 10^{-1}$ | 3.26$^{+0.17}_{-0.15}$ |
| $\Xi_c$               | (2,0) | $3^+$  | 3.23$^{+0.10}_{-0.11}$ | 2.50$^{+0.56}_{-0.50} \times 10^{-1}$ | 3.25$^{+0.16}_{-0.14}$ |
| $\Lambda_c$           | (2,0) | $3^+$  | 3.22$^{+0.10}_{-0.12}$ | 1.37$^{+0.38}_{-0.28} \times 10^{-1}$ | 3.28$^{+0.18}_{-0.30}$ |
| $\Lambda_c$           | (2,0) | $3^+$  | 3.29$^{+0.11}_{-0.11}$ | 2.50$^{+0.56}_{-0.51} \times 10^{-1}$ | 3.25$^{+1.72}_{-0.28}$ |
| $\Xi_c$               | (1,1) | $3^+$  | 3.23$^{+0.11}_{-0.11}$ | 6.02$^{+1.22}_{-1.09} \times 10^{-2}$ | 3.25$^{+0.51}_{-0.31} \times 10^{-1}$ |
| $\Xi_c$               | (1,1) | $3^+$  | 3.23$^{+0.10}_{-0.11}$ | 1.53$^{+0.35}_{-0.31} \times 10^{-1}$ | 3.25$^{+0.11}_{-0.11}$ |
| $\Lambda_c$           | (1,1) | $3^+$  | 3.23$^{+0.10}_{-0.11}$ | 6.01$^{+1.22}_{-1.09} \times 10^{-2}$ | 3.25$^{+0.51}_{-0.31} \times 10^{-1}$ |
| $\Lambda_c$           | (1,1) | $3^+$  | 3.21$^{+0.11}_{-0.11}$ | 1.53$^{+0.35}_{-0.32} \times 10^{-1}$ | 3.25$^{+0.11}_{-0.11}$ |
Figure 1: The masses of the charmed baryon states with variations of the energy scale $\mu$ for the central values of the Borel parameters and threshold parameters shown in Table 1, where the $A$, $B$, $C$ and $D$ correspond to the charmed baryon states $\Xi_c (0, 2; \frac{5}{2})$, $\Xi_c (0, 2; \frac{3}{2})$, $\Lambda_c (0, 2; \frac{5}{2})$ and $\Lambda_c (0, 2; \frac{3}{2})$, respectively.

Figure 2: The masses of the charmed baryon states with variations of the Borel parameters $T^2$, where the $A$, $B$, $C$ and $D$ correspond to the charmed baryon states $\Xi_c (0, 2; \frac{5}{2})$, $\Xi_c (0, 2; \frac{3}{2})$, $\Lambda_c (0, 2; \frac{5}{2})$ and $\Lambda_c (0, 2; \frac{3}{2})$, respectively, the Expt value denotes the experimental values.
Figure 3: The masses of the charmed baryon states with variations of the Borel parameters $T^2$, where the $A$, $B$, $C$ and $D$ correspond to the charmed baryon states $\Xi_c \left( 2, 0; \frac{5}{2} \right)$, $\Xi_c \left( 2, 0; \frac{3}{2} \right)$, $\Lambda_c \left( 2, 0; \frac{5}{2} \right)$ and $\Lambda_c \left( 2, 0; \frac{3}{2} \right)$, respectively.
Figure 4: The masses of the charmed baryon states with variations of the Borel parameters $T^2$, where the $A$, $B$, $C$ and $D$ correspond to the charmed baryon states $\Xi_c (1, 1; 5/2)$, $\Xi_c (1, 1; 3/2)$, $\Lambda_c (1, 1; 5/2)$ and $\Lambda_c (1, 1; 3/2)$, respectively.
Figure 5: The pole residues of the charmed baryon states with variations of the Borel parameters $T^2$, where the $A$, $B$, $C$ and $D$ correspond to the charmed baryon states $\Xi_c(0, 2; \frac{5}{2})$, $\Xi_c(0, 2; \frac{3}{2})$, $\Lambda_c(0, 2; \frac{5}{2})$ and $\Lambda_c(0, 2; \frac{3}{2})$, respectively.
Figure 6: The pole residues of the charmed baryon states with variations of the Borel parameters $T^2$, where the $A$, $B$, $C$ and $D$ correspond to the charmed baryon states $\Xi_c (2,0; \frac{5}{2})$, $\Xi_c (2,0; \frac{3}{2})$, $\Lambda_c (2,0; \frac{5}{2})$ and $\Lambda_c (2,0; \frac{3}{2})$, respectively.
Figure 7: The pole residues of the charmed baryon states with variations of the Borel parameters $T^2$, where the $A$, $B$, $C$ and $D$ correspond to the charmed baryon states $\Xi_c (1, 1; \frac{5}{2})$, $\Xi_c (1, 1; \frac{3}{2})$, $\Lambda_c (1, 1; \frac{5}{2})$ and $\Lambda_c (1, 1; \frac{3}{2})$, respectively.
We take into account all uncertainties of the input parameters, and obtain the masses and pole residues of the D-wave charmed baryon states $\Lambda_c$ and $\Xi_c$, which are shown explicitly in Figs.2-7 and Table 2. In Figs.2-7, we plot the masses and pole residues with variations of the Borel parameters at much larger intervals than the Borel windows shown in Table 1. In the Borel windows, the uncertainties $\delta M/M$ originate from the Borel parameters are very small, about $(2-5)\%$, the Borel platforms exist approximately. Furthermore, the energy scale formula $\mu = \sqrt{M_B^2 - M_c^2}$ is well satisfied. The criteria 3 and 4 are satisfied, now the four criteria are all satisfied.

In Fig.2 and Table 2, we also present the experimental values \cite{1, 30} and predictions from the QCD sum rules combined with the heavy quark effective theory \cite{8}. The present predictions are consistent with the experimental values \cite{1, 30} and other QCD sum rules calculations \cite{8}, and support assigning the $\Lambda_c(2860), \Lambda_c(2880), \Xi_c(3055)$ and $\Xi_c(3080)$ to be the D-wave charmed baryon states with the quantum numbers $(L_\rho, L_\lambda) = (0, 2)$ and $J^P = \frac{3}{2}^+, \frac{5}{2}^+, \frac{7}{2}^+$ and $\frac{9}{2}^+$, respectively. The predictions for the $(L_\rho, L_\lambda) = (2, 0)$ and $(L_\rho, L_\lambda) = (1, 1)$ D-wave $\Lambda_c$ and $\Xi_c$ states can be confronted to the experimental data in the future.

4 Conclusion

In this article, we tentatively assign the $\Lambda_c(2860), \Lambda_c(2880), \Xi_c(3055)$ and $\Xi_c(3080)$ to be the D-wave charmed baryon states with $J^P = \frac{3}{2}^+, \frac{5}{2}^+, \frac{7}{2}^+$ and $\frac{9}{2}^+$, respectively, and study their masses and pole residues with the QCD sum rules in a systematic way by constructing three-types interpolating currents with the quantum numbers $(L_\rho, L_\lambda) = (0, 2), (2, 0)$ and $(1, 1)$, respectively. As the currents couple potentially to both the positive parity and negative parity baryon states, we separate the contributions of the $\frac{3}{2}^+$ and $\frac{5}{2}^+$ charmed baryon states unambiguously, and the QCD sum rules do not suffer from the contaminations of the charmed baryons states with negative parity. We carry out the operator product expansion up to the vacuum condensates of dimension 10 in a consistent way, and use the empirical energy scale formula to determine the optimal energy scales of the QCD spectral densities to extract the hadron masses. The present predictions support assigning the $\Lambda_c(2860), \Lambda_c(2880), \Xi_c(3055)$ and $\Xi_c(3080)$ to be the D-wave baryon states with the quantum numbers $(L_\rho, L_\lambda) = (0, 2)$ and $J^P = \frac{3}{2}^+, \frac{5}{2}^+, \frac{7}{2}^+$ and $\frac{9}{2}^+$, respectively. The predictions for the masses of the $(L_\rho, L_\lambda) = (2, 0)$ and $(1, 1)$ D-wave $\Lambda_c$ and $\Xi_c$ states can be confronted to the experimental data in the future.

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Appendix

The explicit expressions of the QCD spectral densities $\rho_{j,QCD}^0(s)$ and $\rho_{j,QCD}^1(s)$,

\[
\rho_{j,QCD}^0(s) = m_c \rho_{j,L^0}^0(s),
\rho_{j,QCD}^1(s) = \rho_{j,L^1}^1(s),
\]

(36)

\[
\rho_{\frac{1}{2},2,0}^0(s) = \frac{1}{69120\pi^4} \int_{x_i}^1 dx \left( 9x^2 + 34x + 132 \right)(1-x)^4(s - \bar{m}_c^2)^4
+ \frac{5m_s \langle \bar{s}s \rangle - 2m_s \langle \bar{q}q \rangle}{96\pi^2} \int_{x_i}^1 dx \left( 9x^2 + 34x + 132 \right)(1-x)^2(s - \bar{m}_c^2)^2
+ \frac{m_s \langle \bar{q}g_s \sigma Gq \rangle}{36\pi^2} \int_{x_i}^1 dx x(3x - 4)(1-x)(s - \bar{m}_c^2)
+ \frac{m_s \langle \bar{g}s \sigma Gs \rangle}{216\pi^2} \int_{x_i}^1 dx x(31 - 27x)(1-x)(s - \bar{m}_c^2)
- \frac{m_c^2}{51840\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \left( 9x^2 + 34x + 132 \right)(1-x)^4(s - \bar{m}_c^2)
+ \frac{1}{34560\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \left( 9x^2 + 34x + 132 \right)(1-x)^4(s - \bar{m}_c^2)^2
+ \frac{1}{6912\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx (9x^2 + 20x + 46)(1-x)^2(s - \bar{m}_c^2)^2
+ \frac{\langle \bar{q}g_s \sigma Gq \rangle \langle \bar{g}s \sigma Gs \rangle}{72} \delta(s - \bar{m}_c^2),
\]

(37)

\[
\rho_{\frac{1}{2},2,0}^1(s) = \frac{1}{69120\pi^4} \int_{x_i}^1 dx x(27x^2 + 55x + 128)(1-x)^4(s - \bar{m}_c^2)^4
+ \frac{5m_s \langle \bar{s}s \rangle - 2m_s \langle \bar{q}q \rangle}{288\pi^2} \int_{x_i}^1 dx x^2(9x - 1)(1-x)^2(s - \bar{m}_c^2)^2
+ \frac{m_s \langle \bar{q}g_s \sigma Gq \rangle}{72\pi^2} \int_{x_i}^1 dx x(18x^2 - 17x + 1)(1-x)(s - \bar{m}_c^2)
+ \frac{m_s \langle \bar{g}s \sigma Gs \rangle}{216\pi^2} \int_{x_i}^1 dx x(-81x^2 + 70x - 4)(1-x)(s - \bar{m}_c^2)
- \frac{m_c^2}{51840\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \left( 27x^2 + 55x + 128 \right)(1-x)^4(s - \bar{m}_c^2)
+ \frac{1}{6912\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx x(27x^2 + 23x + 40)(1-x)^2(s - \bar{m}_c^2)^2
+ \frac{5\langle \bar{q}g_s \sigma Gq \rangle \langle \bar{g}s \sigma Gs \rangle}{432} \delta(s - \bar{m}_c^2),
\]

(38)
\[ \rho_{0;2}^{\Xi_c}(s) = \frac{1}{4608\pi^4} \int_{x_i}^1 dx \, x(3x-2)(1-x)^4(s-\bar{m}_c^2)^4 + \frac{m_s(\bar{s}s) - 2m_s(\bar{q}q)}{96\pi^2} \int_{x_i}^1 dx \, x^2(1-x)^2(s-\bar{m}_c^2)^2 + \frac{m_c^2}{3456\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \, \frac{(2-3x)(1-x)^4}{x^2} \delta(s-\bar{m}_c^2) + \frac{1}{2304\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \, \frac{(3x-2)(1-x)^4}{x} \delta(s-\bar{m}_c^2) + \frac{1}{768\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \, x^2(1-x)^2(s-\bar{m}_c^2)^2 + \frac{\langle \bar{q}g_s \sigma Gq \rangle \langle s_g \sigma Gs \rangle}{72} \delta(s-m_c^2), \tag{39} \]

\[ \rho_{0;1,1}^{\Xi_c}(s) = \frac{1}{4608\pi^4} \int_{x_i}^1 dx \, x^2(9x+1)(1-x)^4(s-\bar{m}_c^2)^4 + \frac{m_s(\bar{s}s) - 2m_s(\bar{q}q)}{288\pi^2} \int_{x_i}^1 dx \, x^2(9x-1)(1-x)^2(s-\bar{m}_c^2)^2 \]

\[ - \frac{m_c^2}{3456\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \, \frac{(9x+1)(1-x)^4}{x} \delta(s-\bar{m}_c^2) + \frac{1}{2304\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \, x^2(9x-1)(1-x)^2(s-\bar{m}_c^2)^2 + \frac{5\langle \bar{q}g_s \sigma Gq \rangle \langle s_g \sigma Gs \rangle}{432} \delta(s-m_c^2), \tag{40} \]

\[ \rho_{0;2,1,1}^{\Xi_c}(s) = \frac{1}{13824\pi^4} \int_{x_i}^1 dx \, (3x^2 + 8x - 3)(1-x)^4(s-\bar{m}_c^2)^4 + \frac{3m_s(\bar{s}s) - 2m_s(\bar{q}q)}{96\pi^2} \int_{x_i}^1 dx \, x^2(1-x)^2(s-\bar{m}_c^2)^2 + \frac{m_s(\bar{q}g_s \sigma Gq)}{48\pi^2} \int_{x_i}^1 dx \, x(2x-1)(1-x)(s-\bar{m}_c^2) - \frac{5m_s(\bar{s}g_s \sigma Gs)}{432\pi^2} \int_{x_i}^1 dx \, x(3x-1)(1-x)(s-\bar{m}_c^2) \]

\[ - \frac{m_c^2}{10368\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \, \frac{(3x^2 + 8x - 3)(1-x)^4}{x^3} \delta(s-\bar{m}_c^2) + \frac{1}{6912\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \, \frac{(3x^2 + 8x - 3)(1-x)^4}{x^2} \delta(s-\bar{m}_c^2)^2 + \frac{1}{4608\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{x_i}^1 dx \, (6x^2 + 10x - 1)(1-x)^2(s-\bar{m}_c^2)^2, \tag{41} \]
\[ \rho_{\mathbf{1}, \mathbf{0}, 1, 1}^{1.\Xi, c}(s) = \frac{1}{13824\pi^4} \int_{x_i}^{1} dx \, x(9x^2 + 14x + 2)(1 - x)^4(s - \bar{m}_c^2)^4 \\
+ \frac{3m_s \langle ss \rangle - 2m_s \langle \bar{q}q \rangle}{288\pi^2} \int_{x_i}^{1} dx \, x^2(9x - 1)(1 - x)^2(s - \bar{m}_c^2)^2 \\
+ \frac{m_s \langle \bar{q}g_s\sigma Gq \rangle}{288\pi^2} \int_{x_i}^{1} dx \, x(36x^2 - 21x + 1)(1 - x)(s - \bar{m}_c^2) \\
+ \frac{m_s \langle g_s\sigma Gs \rangle}{432\pi^2} \int_{x_i}^{1} dx \, x(-45x^2 + 23x - 1)(1 - x)(s - \bar{m}_c^2) \\
- \frac{m^2_c}{10368\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{x_i}^{1} dx \, \frac{(9x^2 + 14x + 2)(1 - x)^4}{x^2}(s - \bar{m}_c^2) \\
+ \frac{1}{4608\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{x_i}^{1} dx \, x^2(18x + 11)(1 - x)^2(s - \bar{m}_c^2)^2, \quad (42) \]

\[ \rho_{\mathbf{3}, \mathbf{0}, 0}^{0.\Xi, c}(s) = \frac{1}{3072\pi^4} \int_{x_i}^{1} dx \, (4x + 33)(1 - x)^4(s - \bar{m}_c^2)^4 \\
+ \frac{7m_s \langle g_s\sigma Gs \rangle - 6m_s \langle \bar{q}g_s\sigma Gq \rangle}{24\pi^2} \int_{x_i}^{1} dx \, x(1 - x)(s - \bar{m}_c^2) \\
- \frac{m^2_c}{2304\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{x_i}^{1} dx \, \frac{(4x + 33)(1 - x)^4}{x^3}(s - \bar{m}_c^2) \\
+ \frac{1}{1536\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{x_i}^{1} dx \, \frac{(4x + 33)(1 - x)^4}{x^2}(s - \bar{m}_c^2)^2 \\
+ \frac{1}{768\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{x_i}^{1} dx \, (8x + 31)(1 - x)^2(s - \bar{m}_c^2)^2 \\
+ \frac{3 \langle \bar{q}g_s\sigma Gq \rangle \langle g_s\sigma Gs \rangle}{32} \delta(s - m^2_c), \quad (43) \]
\[
\rho_{1/2,0}^{0,\Xi_c}(s) = \frac{5}{3072\pi^4} \int_{x_i}^1 dx \left(4x + 9\right)\left(1 - x\right)^4 (s - \tilde{m}_c^2)^4 \\
+ \frac{25m_s \langle \bar{s}s \rangle - 10m_s \langle \bar{q}q \rangle}{16\pi^2} \int_{x_i}^1 dx \left(1 - x\right)^2 (s - \tilde{m}_c^2)^2 \\
+ \frac{m_s \langle \bar{q}g_s \sigma G_q \rangle}{4\pi^2} \int_{x_i}^1 dx \left(7x - 4\right)(1 - x)(s - \tilde{m}_c^2) \\
+ \frac{m_s \langle \bar{s}g_s \sigma G_s \rangle}{24\pi^2} \int_{x_i}^1 dx \left(33 - 64x\right)(1 - x)(s - \tilde{m}_c^2) \\
- \frac{5m_c^2}{2304\pi^2} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^1 dx \frac{(4x + 9)(1 - x)^4}{x^2} (s - \tilde{m}_c^2) \\
+ \frac{1}{768\pi^2} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^1 dx \left(34x + 35\right)(1 - x)^2 (s - \tilde{m}_c^2)^2 \\
+ \frac{5\langle \bar{q}g_s \sigma G_q \rangle \langle \bar{s}g_s \sigma G_s \rangle}{96} \delta(s - m_c^2),
\]

(44)

\[
\rho_{1/2,0}^{0,\Xi_c}(s) = \frac{1}{1024\pi^4} \int_{x_i}^1 dx \left(3 - 4x\right)(1 - x)^4 (s - \tilde{m}_c^2)^4 \\
- \frac{m_c^2}{768\pi^2} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^1 dx \frac{\left(3 - 4x\right)(1 - x)^4}{x^3} (s - \tilde{m}_c^2) \\
+ \frac{1}{512\pi^2} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^1 dx \frac{\left(3 - 4x\right)(1 - x)^4}{x^2} (s - \tilde{m}_c^2)^2 \\
+ \frac{3\langle \bar{q}g_s \sigma G_q \rangle \langle \bar{s}g_s \sigma G_s \rangle}{32} \delta(s - m_c^2),
\]

(45)

\[
\rho_{1/2,0}^{1,\Xi_c}(s) = \frac{7}{1024\pi^4} \int_{x_i}^1 dx \left(4x + 1\right)(1 - x)^4 (s - \tilde{m}_c^2)^4 \\
+ \frac{5m_s \langle \bar{s}s \rangle - 10m_s \langle \bar{q}q \rangle}{16\pi^2} \int_{x_i}^1 dx \left(1 - x\right)^2 (s - \tilde{m}_c^2)^2 \\
- \frac{7m_c^2}{768\pi^2} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^1 dx \frac{(4x + 1)(1 - x)^4}{x^2} (s - \tilde{m}_c^2) \\
+ \frac{5}{128\pi^2} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^1 dx \left(4x + 1\right)(1 - x)^2 (s - \tilde{m}_c^2)^2 \\
+ \frac{5\langle \bar{q}g_s \sigma G_q \rangle \langle \bar{s}g_s \sigma G_s \rangle}{96} \delta(s - m_c^2),
\]

(46)
\[
\rho^{0;\Xi_c}_{\frac{3}{2},1,1}(s) = \frac{1}{768\pi^4} \int_{x_i}^{1} dx (x - 3)(1 - x)^4(s - \bar{m}_c^2)^4 \\
- \frac{m_s\langle \bar{q}g_s\sigma Gs \rangle}{48\pi^2} \int_{x_i}^{1} dx x(1 - x)(s - \bar{m}_c^2) \\
- \frac{m_s^2\langle \alpha_s G\sigma G\rangle}{576\pi^2} \int_{x_i}^{1} dx \frac{(x - 3)(1 - x)^4}{x^3} (s - \bar{m}_c^2) \\
+ \frac{1}{384\pi^2} \langle \alpha_s G\sigma G\rangle \int_{x_i}^{1} dx \frac{(x - 3)(1 - x)^4}{x^2} (s - \bar{m}_c^2)^2 \\
+ \frac{1}{1024\pi^2} \langle \alpha_s G\sigma G\rangle \int_{x_i}^{1} dx (8x - 5)(1 - x)^2(s - \bar{m}_c^2)^2, \tag{47}
\]

\[
\rho^{1;\Xi_c}_{\frac{3}{2},1,1}(s) = \frac{1}{768\pi^4} \int_{x_i}^{1} dx (8x + 7)(1 - x)^4(s - \bar{m}_c^2)^4 \\
+ \frac{15m_s\langle \bar{q}s\rangle - 10m_s\langle \bar{q}q\rangle}{16\pi^2} \int_{x_i}^{1} dx x^2(1 - x)(s - \bar{m}_c^2)^2 \\
+ \frac{5m_s\langle \bar{q}g_s\sigma Gq \rangle}{16\pi^2} \int_{x_i}^{1} dx x(3x - 1)(1 - x)(s - \bar{m}_c^2) \\
+ \frac{m_s\langle \bar{q}g_s\sigma Gs \rangle}{48\pi^2} \int_{x_i}^{1} dx x(11 - 38x)(1 - x)(s - \bar{m}_c^2) \\
- \frac{m_s^2\langle \bar{q}g_s\sigma Gq \rangle}{576\pi^2} \int_{x_i}^{1} dx \frac{(x + 7)(1 - x)^4}{x^2} (s - \bar{m}_c^2) \\
+ \frac{1}{1024\pi^2} \langle \alpha_s G\sigma G\rangle \int_{x_i}^{1} dx (44x + 15)(1 - x)^2(s - \bar{m}_c^2)^2, \tag{48}
\]

\[
\rho^{0;\Lambda_c}_{\frac{3}{2},2,0}(s) = \rho^{0;\Xi_c}_{\frac{3}{2},2,0}(s) \bigg|_{m_s \to 0}, \langle \bar{q}s\rangle \to \langle \bar{q}q \rangle, \langle \bar{q}g_s\sigma Gs \rangle \to \langle \bar{q}g_s\sigma Gq \rangle, \\
\rho^{0;\Lambda_c}_{\frac{3}{2},0,2}(s) = \rho^{0;\Xi_c}_{\frac{3}{2},0,2}(s) \bigg|_{m_s \to 0}, \langle \bar{q}s\rangle \to \langle \bar{q}q \rangle, \langle \bar{q}g_s\sigma Gs \rangle \to \langle \bar{q}g_s\sigma Gq \rangle, \\
\rho^{0;\Lambda_c}_{\frac{3}{2},1,1}(s) = \rho^{0;\Xi_c}_{\frac{3}{2},1,1}(s) \bigg|_{m_s \to 0}, \langle \bar{q}s\rangle \to \langle \bar{q}q \rangle, \langle \bar{q}g_s\sigma Gs \rangle \to \langle \bar{q}g_s\sigma Gq \rangle, \\
\rho^{1;\Lambda_c}_{\frac{3}{2},2,0}(s) = \rho^{1;\Xi_c}_{\frac{3}{2},2,0}(s) \bigg|_{m_s \to 0}, \langle \bar{q}s\rangle \to \langle \bar{q}q \rangle, \langle \bar{q}g_s\sigma Gs \rangle \to \langle \bar{q}g_s\sigma Gq \rangle, \\
\rho^{1;\Lambda_c}_{\frac{3}{2},0,2}(s) = \rho^{1;\Xi_c}_{\frac{3}{2},0,2}(s) \bigg|_{m_s \to 0}, \langle \bar{q}s\rangle \to \langle \bar{q}q \rangle, \langle \bar{q}g_s\sigma Gs \rangle \to \langle \bar{q}g_s\sigma Gq \rangle, \\
\rho^{1;\Lambda_c}_{\frac{3}{2},1,1}(s) = \rho^{1;\Xi_c}_{\frac{3}{2},1,1}(s) \bigg|_{m_s \to 0}, \langle \bar{q}s\rangle \to \langle \bar{q}q \rangle, \langle \bar{q}g_s\sigma Gs \rangle \to \langle \bar{q}g_s\sigma Gq \rangle, \tag{49}
\]
\[
\rho_{\frac{1}{2},0}^{0;\Lambda_c}(s) = \rho_{\frac{1}{2},0}^{0;\Xi_c}(s) \left| m_s \to 0, \langle \bar{s}s \rangle \to \langle \bar{q}q \rangle, \langle \bar{s}g \sigma G s \rangle \to \langle \bar{q}g \sigma G q \rangle \right. \\
\rho_{\frac{1}{2},0}^{0;\Lambda_c}(s) = \rho_{\frac{1}{2},0}^{0;\Xi_c}(s) \left| m_s \to 0, \langle \bar{s}s \rangle \to \langle \bar{q}q \rangle, \langle \bar{s}g \sigma G s \rangle \to \langle \bar{q}g \sigma G q \rangle \right. \\
\rho_{\frac{1}{2},1,1}^{0;\Lambda_c}(s) = \rho_{\frac{1}{2},1,1}^{0;\Xi_c}(s) \left| m_s \to 0, \langle \bar{s}s \rangle \to \langle \bar{q}q \rangle, \langle \bar{s}g \sigma G s \rangle \to \langle \bar{q}g \sigma G q \rangle \right. \\
\rho_{\frac{1}{2},2,0}^{1;\Lambda_c}(s) = \rho_{\frac{1}{2},2,0}^{1;\Xi_c}(s) \left| m_s \to 0, \langle \bar{s}s \rangle \to \langle \bar{q}q \rangle, \langle \bar{s}g \sigma G s \rangle \to \langle \bar{q}g \sigma G q \rangle \right. \\
\rho_{\frac{1}{2},0,2}^{1;\Lambda_c}(s) = \rho_{\frac{1}{2},0,2}^{1;\Xi_c}(s) \left| m_s \to 0, \langle \bar{s}s \rangle \to \langle \bar{q}q \rangle, \langle \bar{s}g \sigma G s \rangle \to \langle \bar{q}g \sigma G q \rangle \right. \\
\rho_{\frac{1}{2},1,1}^{1;\Lambda_c}(s) = \rho_{\frac{1}{2},1,1}^{1;\Xi_c}(s) \left| m_s \to 0, \langle \bar{s}s \rangle \to \langle \bar{q}q \rangle, \langle \bar{s}g \sigma G s \rangle \to \langle \bar{q}g \sigma G q \rangle \right. \\
\langle \bar{s}s \rangle \to \langle \bar{q}q \rangle, \langle \bar{s}g \sigma G s \rangle \to \langle \bar{q}g \sigma G q \rangle \\
\end{align*}

(50)

\[ \tilde{m}_c^2 = \frac{m_c^2}{s}, \quad x_i = \frac{m_i^2}{s}, \]

References

[1] R. Aaij et al, JHEP 1705 (2017) 030.
[2] M. Artuso et al, Phys. Rev. Lett. 86 (2001) 4479.
[3] B. Aubert et al, Phys. Rev. Lett. 98 (2007) 012001.
[4] R. Mizuk et al, Phys. Rev. Lett. 98 (2007) 262001.
[5] W. Roberts and M. Pervin, Int. J. Mod. Phys. A23 (2008) 2817.
[6] D. Ebert, R. N. Faustov and V. O. Galkin, Phys. Rev. D84 (2011) 014025.
[7] Z. Shah, K. Thakkar, A. K. Rai and P. C. Vinodkumar, Eur. Phys. J. A52 (2016) 313.
[8] H. X. Chen, Q. Mao, A. Hosaka, X. Liu and S. L. Zhu, Phys. Rev. D94 (2016) 114016.
[9] B. Chen, K. W. Wei, X. Liu and T. Matsuki, Eur. Phys. J. C77 (2017) 154.
[10] B. Chen, X. Liu and A. Zhang, Phys. Rev. D95 (2017) 074022.
[11] R. Chistov et al, Phys. Rev. Lett. 97 (2006) 162001.
[12] B. Aubert et al, Phys. Rev. D77 (2008) 012002.
[13] Y. Kato et al, Phys. Rev. D94 (2016) 032002.
[14] Z. G. Wang, Eur. Phys. J. A47 (2011) 81.
[15] Z. G. Wang, Commun. Theor. Phys. 58 (2012) 723.
[16] T. M. Aliev, K. Azizi and M. Savci, J. Phys. G40 (2013) 065003; K. Azizi and H. Sundu, Eur. Phys. J. Plus 132 (2017) 22.
[17] Z. G. Wang, Phys. Lett. B685 (2010) 59; Z. G. Wang, Eur. Phys. J. C68 (2010) 459; Z. G. Wang, Eur. Phys. J. A45 (2010) 267.

[18] H. X. Chen, W. Chen, Q. Mao, A. Hosaka, X. Liu and S. L. Zhu, Phys. Rev. D91 (2015) 054034; Q. Mao, H. X. Chen, W. Chen, A. Hosaka, X. Liu and S. L. Zhu, Phys. Rev. D92 (2015) 114007.

[19] Z. G. Wang, Eur. Phys. J. C75 (2015) 359.

[20] A. De Rujula, H. Georgi and S. L. Glashow, Phys. Rev. D12 (1975) 147; T. DeGrand, R. L. Jaffe, K. Johnson and J. E. Kiskis, Phys. Rev. D12 (1975) 2060.

[21] Z. G. Wang, Commun. Theor. Phys. 59 (2013) 451.

[22] J. G. Korner, M. Kramer and D. Pirjol, Prog. Part. Nucl. Phys. 33 (1994) 787.

[23] Z. G. Wang and Z. Y. Di, Eur. Phys. J. A50 (2014) 143.

[24] Y. Chung, H. G. Dosch, M. Kremer and D. Schall, Nucl. Phys. B197 (1982) 55; D. Jido, N. Kodama and M. Oka, Phys. Rev. D54 (1996) 4532.

[25] Z. G. Wang, Eur. Phys. J. C76 (2016) 70.

[26] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147 (1979) 385, 448.

[27] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rept. 127 (1985) 1.

[28] Shi-Zhong Huang, "Free particles and fields of high spins" (in chinese), Anhui peoples Publishing House, 2006.

[29] P. Colangelo and A. Khodjamirian, [hep-ph/0010175]

[30] C. Patrignani et al, Chin. Phys. C40 (2016) 100001.

[31] Z. G. Wang and T. Huang, Phys. Rev. D89 (2014) 054019; Z. G. Wang, Eur. Phys. J. C74 (2014) 2874; Z. G. Wang and T. Huang, Nucl. Phys. A930 (2014) 63.

[32] Z. G. Wang and T. Huang, Eur. Phys. J. C74 (2014) 2891; Z. G. Wang, Eur. Phys. J. C74 (2014) 2963.

[33] Z. G. Wang, Int. J. Mod. Phys. A30 (2015) 1550168.

[34] Z. G. Wang, Eur. Phys. J. C77 (2017) 325.

[35] Z. G. Wang, Eur. Phys. J. C77 (2017) 174; Z. G. Wang, Eur. Phys. J. A53 (2017) 192.

[36] Z. G. Wang, Eur. Phys. J. C76 (2016) 387.

[37] Y. J. Xu, Y. L. Liu and M. Q. Huang, Commun. Theor. Phys. 63 (2015) 209.