EINSTEIN-YANG-MILLS SOLITONS: TOWARDS NEW DEGREES OF FREEDOM

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A recent progress in obtaining non-spherical and non-static solitons in the four-dimensional Einstein–Yang–Mills (EYM) theory is discussed, and a non-perturbative formulation of the stationary axisymmetric problem is attempted. First a 2D dilaton gravity model is derived for the spherically symmetric time-dependent configurations. Then a similar Euclidean representation is constructed for the stationary axisymmetric non-circular SU(2) EYM system using the (2+1)+1 reduction scheme suggested by Maeda, Sasaki, Nakamura and Miyama. The crucial role in this reduction is played by the extra terms entering the reduced Yang–Mills and Kaluza–Klein two-forms similarly to Chern–Simons terms in the theories with higher rank antisymmetric tensor fields. We also derive a simple 2D action describing static axisymmetric magnetic EYM configurations and discuss a possibility of existence of cylindrical EYM sphalerons.

1 Introduction

Investigation of classical solutions to 4D gravity coupled non-Abelian gauge theories, inspired by the discovery of particle-like solutions to the Einstein-Yang-Mills (EYM) equations by Bartnik and McKinnon, turned out to be a fruitful field of research which revealed many unexpected features both in the black hole and soliton physics. EYM black holes gave rise to critical revision of some conceptual foundations of the black hole theory, especially of such folkloric beliefs as the no-hair and uniqueness conjectures. A more

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*a* Extended version of a talk given at the International Workshop “Mathematical cosmology”, Potsdam, March 30 – April 4, 1998.

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recent progress in this area was due to investigation of particle-like and black hole solutions without spherical symmetry. It was shown that the celebrated Israel’s theorems, implying that static vacuum and electrovacuum black holes are spherically symmetric, are no longer true for non-Abelian gravity coupled theories. Namely, in the EYM theory the static solitons and black holes with higher winding numbers are only axially symmetric, like multimonopole and multispaleron solutions in the flat space. In more sophisticated models even axial symmetry no longer holds for static black holes.

A substantial progress was also achieved recently in understanding the nature of rotating solutions. It was shown by Volkov and Straumann that electrically charged SU(2) black holes, prohibited in the static spherically symmetric case by a ‘non-abelian baldness’ theorem, still may exist, but they are non-static and only axially symmetric. Stationary EYM black holes are likely to increase the number of characterizing them independent parameters (a mass and a node number in the static spherically symmetric case) by two: an angular momentum and an electric charge. In contrary, for the regular rotating EYM sphalerons these two parameters were found to be related, thus the electric charge has to be regarded rather as an induced one. It was also claimed that in the theories including Higgs fields no rotating particle-like solutions may exist at all, perhaps unless the angular momentum takes discrete values. The corresponding rotating black holes necessarily acquire electric charges.

These intriguing results for the stationary axisymmetric non-Abelian self-gravitating solutions were obtained, however, only perturbatively. In the existing literature no fully non-linear analysis is available for the stationary axisymmetric EYM system (apart from some preliminary considerations in), and even the corresponding ansatz for the YM field is not exhibited yet. The essential complication in the non-Abelian case with respect to various Abelian theories is that now the circularity assumption for the spacetime metric should be avoided, as was argued by Heusler and Straumann. Therefore a general analysis of the EYM system for stationary axisymmetric non-circular spacetime is necessary for further progress. Here we present such an analysis in terms of the Kaluza-Klein reduction, which ends up in a derivation of an Euclidean 2D dilaton gravity model coupled to non-Abelian scalar and vector fields inherited from the 4D SU(2) YM field, as well as to Kaluza-Klein two-forms and scalar moduli. We also derive a simple representation for the static axially symmetric magnetic configurations and discuss one curious cylindrically symmetric truncation.
2 Spherical EYM system as 2D dilaton gravity

Throughout the paper we will be dealing with the 4D SU(2) Einstein-Yang-Mills (EYM) system described by the action (up to a boundary term)

\[ S_{EYM} = (16\pi)^{-1} \int \{ R_4 + 2 \text{tr} F_{\mu\nu} F^{\mu\nu} \} \sqrt{-g_4} \, d^4x, \tag{1} \]

where an antihermitean representation of the algebra is understood, and the gravitational and gauge constants are removed by an appropriate coordinate rescaling. As a warming up exercise, let us consider the dimensional reduction of (1) for the spherically symmetric time-dependent spacetimes. We will adopt the following parameterization of the spacetime metric

\[ ds^2 = g_{ab} dx^a dx^b + e^{-2\psi} (d\vartheta^2 + \sin^2 \vartheta d\phi^2), \tag{2} \]

where \( g_{ab}, \psi \) are functions of \( x^a = t, r \). The Yang–Mills potential one-form can be decomposed correspondingly as \( A = a + \Phi \), where

\[ a = a_b \tau_r dx^b \tag{3} \]

is a ‘dynamical’ part, parameterized by a 2D real Abelian one-form \( a_b \) depending on \( x^a \), and

\[ \Phi = \text{Re} \left\{ (w - 1)(\tau_\varphi - i\tau_\theta)(d\vartheta - i \sin \vartheta d\varphi) \right\} \tag{4} \]

is an effective Higgs field parameterized by a complex-valued function \( w(x^a) \). We use the rotated basis

\[ \tau_r = \sin \vartheta (\cos \varphi \tau_1 + \sin \varphi \tau_2) + \cos \vartheta \tau_3, \quad \tau_\theta = \partial_\theta \tau_r, \quad \tau_\varphi = \frac{1}{\sin \vartheta} \partial_\varphi \tau_r, \tag{5} \]

and \( \tau_k = \sigma_k/(2i), k = 1, 2, 3 \), with \( \sigma_k \) being the Pauli matrices.

As it is well-known, the dimensional reduction of the Einstein-Hilbert action under an assumption of spherical symmetry leads to the 2D dilaton gravity action

\[ S_G = \frac{1}{4} \int \left\{ e^{-2\psi} \left( R + 2(\partial \psi)^2 \right) + 2 \right\} \sqrt{|g|} \, d^2x, \tag{6} \]

where \( d^2x = drdt \) and \( R \) is the 2D scalar curvature. For the reduced YM field one gets the dilaton coupled version of the 2D scalar electrodynamics, in which \( w \) enters as a complex scalar, while \( a_b \) — as a U(1) gauge field:

\[ S_{YM} = - \int \left\{ \frac{1}{4} e^{-2\psi} f_{ab} f^{ab} + |\hat{D}w|^2 + \frac{1}{2} e^{2\psi}(1 - |w|^2)^2 \right\} \sqrt{|g|} \, d^2x, \tag{7} \]
where
\[ f_{ab} = \partial_a a_b - \partial_b a_a, \quad \hat{D}_b w = \partial_b w - i a_b w. \] (8)

The sum of the actions \( S_G + S_{YM} \) describes the dynamics of the moving self-gravitating spherical shells of the YM field. The system is thus equivalent to the 2D Abelian Higgs model coupled to a Jackiw-Teitelboim type theory. Note that no cosmological term is present in (6), instead there is a constant term not multiplied by the dilaton. The action is invariant under the 2D diffeomorphisms and has a U(1) gauge symmetry
\[ w \rightarrow e^{i\alpha} w, \quad a_b \rightarrow a_b + \partial_b \alpha, \] (9)

where \( \alpha(x^a) \) is a gauge parameter. In 2D dilaton gravity it is usual to use a diffeomorphism invariance in order to impose a conformal gauge \( g_{ab} = \exp(2\rho) \eta_{ab} \), but here we prefer to leave the metric coefficients unspecified (for reduction of the spherical EYM system in a different fixed gauge see [3]).

The total energy-momentum tensor including the dilaton part is
\[ T_{ab} = e^{-2\psi} (\partial_a \psi \partial_b \psi - \psi_{;ab}) + T_{YM}^{YM} + \text{trace terms}, \] (10)

where \( T_{YM}^{YM} \) corresponds to a variation of (5). Since the 2D Einstein tensor vanishes, 2D Einstein’s equations take the form
\[ T_{ab} = 0, \] (11)

encoding the \((t, r)\) components of the 4D Einstein’s equations, the remaining part of which is contained in the equation for the ‘dilaton’. The matter equations can be obtained by a variation of the action, after that one can fix the YM and gravitational gauge using the three-parameter gauge group described above.

3 (2+1)+1 reduction for non-circular spacetime

Our purpose is to derive a two-dimensional representation for the stationary axisymmetric EYM system. The corresponding spacetime admits two commuting Killing vectors, which may be written in appropriate coordinates as
\[ K = \partial_t, \quad \tilde{K} = \partial_{\phi}. \] (12)

For most of known explicitly stationary axisymmetric solutions in general relativity these Killing vectors satisfy Frobenius conditions
\[ K_{[\mu} \tilde{K}_{\nu} \tilde{K}_{\lambda];\tau} = 0, \quad \tilde{K}_{[\mu} K_{\nu} K_{\lambda];\tau} = 0, \] (13)
which mean that they are *hypersurface orthogonal*. In this case the metric admits a particularly simple Lewis-Papapetrou parameterization by three real-valued functions of two coordinates.

The *circularity theorem*\(^\text{13,14}\) asserts that the necessary and sufficient conditions for Eqs. (13) to hold are provided by the *Ricci-circularity conditions* (for a more recent discussion see\(^\text{15}\)):

\[
K_{[\mu} \bar{K}_{\nu]} R^{\tau}_{\lambda] \bar{K}_{\tau}} = 0, \quad \bar{K}_{[\mu} K_{\nu]} R^{\tau}_{\lambda] \bar{K}_{\tau}} = 0.
\]  

(14)

Since the Ricci tensor \(R_{\tau}^{\lambda}\) is involved here, the validity of (13) depends strongly on the matter system. In particular, these conditions are satisfied in the Einstein–Maxwell theory provided the Maxwell field is also stationary and axially symmetric. For the EYM system, as was discussed by Heusler and Straumann\(^\text{11}\) this is not the case, however. In order to fulfill the Ricci conditions, the YM field strength should have the components \(\Gamma_{t\varphi}, \bar{\Gamma}_{t\varphi}\) vanishing, but this is not true already for a spherically symmetric configuration \(\text{3-4}\). Therefore, a sufficiently general EYM system with two commuting Killing vectors cannot be described by the Lewis-Papapetrou ansatz.

Here we show that an appropriate dimensional reduction of the stationary axisymmetric non-circular EYM system leads to some 2D dilaton gravity model which, though being more complicated than that in the spherical case, still looks tractable. We will follow the (2+1)+1 approach suggested by Maeda, Sasaki, Nakamura and Miyama\(^\text{16}\) consisting in a two-step dimensional reduction of the metric with respect to two commuting Killing vector fields. In their case the first reduction was performed with respect to the spacelike Killing field, what was suitable for treatment of an axisymmetric gravitational collapse. An alternative reduction, first in the timelike direction, was employed by Gourgoulhon and Bonazzola\(^\text{17}\) aiming to attack the problem of a non-circular magnetized star. In both cases the dimensional reduction was effected at the level of Einstein *equations*. The resulting systems are still too complicated to be used with the YM matter source. We will see that a considerable simplification may be achieved if one performs dimensional reduction at the level of the *action*. An additional advantage of our procedure is that no gauge fixing is necessary at any stage of the computation, so one is free to use different gauge choices to further simplify the final system.

As the first step let us perform reduction with respect to \(K\), representing the 4D metric \((x^\mu = t, x^i)\) as

\[
ds^2 = -e^\psi (dt + v_i dx^i)^2 + e^{-\psi} dt^2_3, \quad dl^2_3 = h_{ij} dx^i dx^j.
\]

(15)

This reduction for the EYM and EYMH systems was described recently by
Brodbeck and Heusler[3]. The decomposition of the Einstein term is standard:

\[ \sqrt{-g} R = \sqrt{h} \left( R_3 - \frac{1}{2} (\partial \psi)^2 + \frac{1}{2} e^{2\psi} \Omega_{ij} \Omega^{ij} \right) + \text{div}, \tag{16} \]

where \( R_3 \) is the three-dimensional Ricci scalar, ‘div’ stands for a total divergence term, and \( \Omega_{ij} \) is the Kaluza-Klein two-form

\[ \Omega_{ij} = \partial_i v_j - \partial_j v_i. \tag{17} \]

To reduce the material part it is convenient to use for the matrix-valued YM potential a standard Kaluza-Klein split

\[ A = A_\mu dx^\mu = \Phi (dt + v_i dx^i) + A_i dx^i, \tag{18} \]

where \( A \) is a purely three-dimensional gauge field one-form, while \( \Phi \) plays a role of a three-dimensional Higgs field. It could be expected that, once the potential is split with respect to some Killing symmetry, one should demand an independence of its components on the coordinates along the Killing orbits, as it is usual for toroidal reduction in the Abelian case. For a non-Abelian field, however, this might be in general too restrictive. In fact, the symmetry of the gauge field under a spacetime isometry means that the action of an isometry can be compensated by a suitable gauge transformation, what at the infinitesimal level reads

\[ \mathcal{L}_K A_\mu = D_\mu W_m, \tag{19} \]

with \( W \) being a Lie-algebra valued gauge function. In the case of a unique timelike Killing vector one can always choose \( W = 0 \), and hence \( A \) and \( \Phi \) in [13] may be regarded as functions of \( x^i \) only. But for the second Killing vector \( \tilde{K} \) we will prefer to avoid such a choice and allow for more general gauges.

Evaluating the field strength corresponding to (18) one finds

\[ F = dA + A \wedge A = \mathcal{F} + D\Phi \wedge (dt + v_i dx^i), \tag{20} \]

where the first term represents the three-dimensional components of the full tensor

\[ \mathcal{F}_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] + \Phi \Omega_{ij}, \tag{21} \]

while the second — mixed components, which take the form of a 3D covariant derivative of the scalar \( \Phi \) with respect to the connection \( A \):

\[ D_i \Phi = \partial_i \Phi + [A_i, \Phi]. \tag{22} \]

An effective 3D field strength (21) contains an extra term proportional to the Kaluza-Klein two-form (17), which arises similarly to Chern-Simons terms in
the non-diagonal reduction of higher rank antisymmetric forms in Kaluza-Klein supergravities.

Now it is simple to find the following relation between four and three-dimensional YM lagrangians:

\[ \sqrt{-g_4} \text{tr} F_{\mu\nu} F^{\mu\nu} = \sqrt{h} \text{tr} \left( e^{\psi} F_{ij} F^{ij} - e^{-\psi} D_i \Phi D^i \Phi \right), \]  
(23)

where it is understood that all 4D indices are raised by the 4D metric, while the 3D ones — by the inverse to \( h_{ij} \). Therefore the resulting 3D EYM action reads (an overall coefficient being omitted):

\[ S = \int \left\{ R_3 - \frac{1}{2} (\partial \psi)^2 + \frac{1}{2} e^{2\psi} \Omega^{ij} \Omega_{ij} + 2 \text{tr} \left( e^{\psi} F_{ij} F^{ij} - e^{-\psi} D_i \Phi D^i \Phi \right) \right\} \sqrt{h} d^3 x. \]  
(24)

It describes a Euclidean system of 3D Yang-Mills and Higgs fields (without potential), a dilaton and an Abelian two-form coupled to gravity. The equation for the Kaluza-Klein field strength following from this action assumes the form of the 3D Maxwell equations

\[ \partial_i \left( e^{2\psi} \sqrt{h} \Omega^{ij} + 4 e^{\psi} \sqrt{h} \text{tr} (F^{ij} \Phi) \right) = 0, \]  
(25)

which can be solved by introducing the twist potential

\[ \Omega^{ij} + 4 e^{-\psi} \text{tr} (F^{ij} \Phi) = e^{\psi} \frac{\epsilon^{ijk}}{\sqrt{h}} \partial_k \chi. \]  
(26)

We will not, however, pursue this direction further, but rather pass to the second step of reduction in terms of the Kaluza-Klein two-form \( \Omega_{ij} \) itself.

By virtue of an axial symmetry, the 3D metric can now be parameterized as follows:

\[ ds_3^2 = e^{2\phi} (d\varphi + k_a dx^a)^2 + g_{ab} dx^a dx^b, \]  
(27)

where a new two-dimensional Kaluza-Klein one-form \( k_a \) and the 2D metric \( g_{ab} \) depend only on \( x^a, a = 1, 2 \). The one-form \( k_a \) generates a field strength

\[ \kappa_{ab} = \partial_a k_b - \partial_b k_a, \]  
(28)

which is actually a scalar, \( \kappa_{ab} = \kappa \epsilon_{ab} \). The 3D curvature scalar decomposes as

\[ \sqrt{h} R_3 = \sqrt{g e^\phi} \left( R - \frac{1}{2} \kappa_{ab} \kappa^{ab} e^{2\phi} \right) + \text{div}, \]  
(29)
where $R$ is the 2D Ricci scalar. The action has the form of the Euclidean Jackiw-Teitelboim action without cosmological constant. In order to reduce the $\Omega^2_{ij}$ term in (16), first one has to decompose the Kaluza-Klein one-form

$$v_i dx^i = \omega (d\varphi + k_a dx^a) + \nu_a dx^a,$$

(30)

where a 2D scalar $\omega$ and a one-form $\nu_a$ are $\varphi$-independent. In the reduced action the 2D field strength corresponding to $\nu_a$ acquires an extra term proportional to $\kappa_{ab}$:

$$\omega_{ab} = \partial_a \nu_b - \partial_b \nu_a + \omega_{\kappa_{ab}},$$

(31)

so that

$$\sqrt{h} \Omega_{ij} \Omega^{ij} = \sqrt{g} \left( e^{-\Phi} \partial_a \omega \partial^a \omega + e^\Phi \omega_{\kappa_{ab}} \omega_{\kappa_{ab}} \right).$$

(32)

To further reduce the YM part of the 3D action one has to split the matrix-valued YM one-form in a way similar to (18):

$$A = \Psi (d\varphi + k_a dx^a) + a_b dx^b.$$

(33)

Now we will no more assume the 2D matrix-valued scalar $\Psi$ and the one-form $a_b$ to be $\varphi$-independent, but rather denote

$$\partial_\varphi \Psi = \Psi', \quad \partial_\varphi a_b = a'_b,$$

(34)

these derivatives will enter the final formulas explicitly. Similarly to (20), we obtain the following decomposition of the three-dimensional field strength

$$F = f + (D'_a \Phi + \partial_a \omega \Phi - a'_b) dx^a \wedge (d\varphi + k_a dx^a),$$

(35)

where an effective 2D field strength is

$$f_{ab} = \partial_a a_b - \partial_b a_a + [a_a, a_b] + a'_b k_a - a'_a k_b + \Phi \omega_{ab} + \Psi \kappa_{ab}.$$

(36)

A primed covariant derivative is defined as follows:

$$D'_b \Psi = D_b \Psi - k_b \Psi',$$

(37)

the first term being the usual 2D covariant derivative with respect to the connection $a_b$

$$D_b = \partial_b + [a_b, \cdot].$$

(38)

An extended field strength (36) now contains extra terms proportional to the Kaluza-Klein two-forms and also gauge-dependent terms whose emergence is due to an allowed additional dependence of the gauge potential on $\varphi$. This
gauge dependence, however, should not cause any trouble once the final stationary axisymmetric ansatz is fixed, but it will just give more flexibility in the choice of a gauge for this ansatz (the final choice is outside the scope of the present paper).

Finally, the 3D gauge covariant derivative $D_i \Phi$ splits into the corresponding 2D derivative (primed) and a commutator term $[\Phi, \Psi]$ combined with $\Phi'$. Collecting all terms one obtains the following 2D dilaton gravity model:

$$S_2 = \int e^\phi (R + L_m) \sqrt{g} d^2 x,$$

with $L_m$ accounting both for YM and Kaluza-Klein variables:

$$L_m = 2 \text{tr} \left\{ [f_{ab} f^{ab} + e^{-2\phi} (D'_a \Psi + \Phi \partial_a \omega - a'_a) (D'^a \Psi + \Phi \partial^a \omega - a'^a)] e^\psi 
- [D'_a \Phi D'^{a} \Phi + e^{-2\phi} (\Phi' + [\Phi, \Psi])^2] e^{-\psi} 
+ \frac{1}{2} \left( e^{2\psi} \omega_{ab} \omega^{ab} - e^{2\phi} \kappa_{ab} \kappa^{ab} + e^{2(\psi - \phi)} \partial_a \omega \partial^a \omega - \partial_a \psi \partial^a \psi \right) \right\},$$

a primed covariant derivative of $\Phi$ being defined in the same way as (37).

Thus the stationary axially symmetric non-circular 4D EYM system gives rise to an Euclidean 2D dilaton gravity model, in which the matter (apart form the ‘dilaton’ $\phi$) includes the 2D Yang-Mills field, two Higgs fields (now with a quartic interaction term), two scalar moduli $\psi, \omega$, and two Kaluza-Klein two-forms $\omega_{ab}, \kappa_{ab}$. (Note that, although no derivatives of $\phi$ appear explicitly, they will be produced after elimination of second derivatives entering the scalar curvature.) Again, by virtue of the 2D Einstein equations, the corresponding 2D energy-momentum tensor should vanish, $T_{ab} = 0$, the rest of the field equations being obtainable by variations of the action over the ‘matter’ fields. Purely gravitational degrees of freedom (the third line in (40)) include three scalars, two 2D two-forms (one degree of freedom each) and a 2D metric which remains unspecified. It is instructive to write down the entire 4D metric in terms of the variables introduced:

$$ds^2 = -e^\nu [dt + \omega d\varphi + (\nu_a + \omega k_a) dx^a]^2 + e^{2\phi - \psi} (d\varphi + k_a dx^a)^2 + e^{-\psi} g_{ab} dx^a dx^b.$$  

The decomposition of the full 4D YM connection reads:

$$A = \Phi [dt + \omega d\varphi + (\nu_a + \omega k_a) dx^a] + \Psi (d\varphi + k_a dx^a) + a_b dx^b.$$  

(42)
4 Static case: axial and cylindrical symmetry

A specification of the stationary axisymmetric YM ansatz requires a study of transformations of the matrix-valued quantities under spacetime isometries. This will be presented elsewhere, while here we will concentrate on a simpler case of the static configurations. Let us construct a 2D dilaton gravity model for a purely magnetic 4D ansatz used by Kleihaus and Kunz in the numerical investigations of static solutions with multiple winding numbers \( \nu \). In our notation this corresponds to \( \Phi = 0 \) and

\[
a_b = a_b \tau_\varphi, \quad \Psi = \text{Re} \ w \tau_\rho + (\text{Im} \ w - \nu) \tau_z,
\]

where \( w \) is a complex-valued function of \( \rho, z \),

\[
\tau_\rho = \tau_x \cos \nu \varphi + \tau_y \sin \nu \varphi, \quad \tau_\varphi = -\tau_x \sin \nu \varphi + \tau_y \cos \nu \varphi,
\]

and we have denoted an Abelian one-form \( a_b = a_b(\rho, z) \) by the same symbol.

In the static case one has \( k_a = \nu_a = \omega = 0 \) implying \( \omega_{ab} = 0 = \kappa_{ab} \), and the 2D dilaton gravity action simplifies dramatically:

\[
S = \int \left\{ e^\phi \left( R - \frac{1}{2} \partial_a \psi \partial^a \psi \right) + e^{(\psi - \phi)} D_a w \bar{D}^a \bar{w} + e^{(\psi + \phi)} f_{ab} f^{ab} \right\} \sqrt{g} d^2 x,
\]

where \( f_{ab} = \partial_a a_b - \partial_b a_a \). (Note that for an ansatz \( a'_{ab} \) and \( \Psi' \) are non-zero.) Contrary to the spherical case, now there is no self-interaction term for \( w \), so we deal with the linear scalar electrodynamics in Euclidean space interacting with three extra scalar fields.

A variation over the metric gives Einstein equations \( T_{ab} = 0 \), where the total energy momentum tensor can be read off from (45). The remaining matter equations (including those for the scalar fields) may then be written in the conformal gauge

\[
g_{ab} dx^a dx^b = e^\eta (d\rho^2 + dz^2),
\]

Instead of writing down the equations, we just give the matter action in this fixed gauge, in which it reduces to a simple flat space action

\[
S_m = \int \left\{ e^{\phi} \partial \phi \cdot \partial \eta - \frac{1}{2} (\partial \psi)^2 + e^{(\psi - \phi)} |Dw|^2 + e^{(\psi + \phi - \eta)} f_{ab}^2 \right\} d^2 x,
\]

where now all contractions correspond to the metric \( \delta_{ab} \). Note that the total number of the four-dimensional gravitational degrees of freedom is three
\[(\psi, \phi, \eta)\]. The total number of the YM degrees of freedom is also three: four real functions minus one gauge.

A similarity with the scalar electrodynamics suggests the existence of a cylindrical YM ansatz which could simulate the Nielsen-Olesen vortex. This is indeed the case: just choose the gauge \(\text{Re } w = 0\) and set \(a_\rho = 0\), the remaining functions \(a_z = R(\rho)\), and \(\text{Im } w = P(\rho)\) will do the job. If one omits all gravitational variables, the equations of motion in the flat space will exactly coincide with the Nielsen-Olesen equations, but now with the self-interaction term switched off. Recall that the BPS limit for an Abelian vortex corresponds to a non-zero value of the self-interaction constant. For zero coupling no finite energy solutions exist, indeed, we are dealing with the flat space Yang–Mills theory, in which classical glueballs are prohibited. But still one can hope that gravity will glue the vortex as it does for the spherical EYM sphaleron, perhaps at the expense of loosing an asymptotic flatness. Note that our cylindrical ansatz is not strictly two–dimensional: \(z\)-component of the vector-potential is non-zero, hence another no-go result prohibiting the 2+1 gravitating glueballs does not apply.

In the cylindrical case one can not ensure the conformal gauge for \(g_{ab}\), but one can always set \(g_{\rho\rho} = e^\psi\) by a coordinate transformation. Denoting
\[
\begin{align*}
g_{zz} &= e^\xi g_{\rho\rho}, \\
\lambda &= 2\phi - \psi,
\end{align*}
\]
we obtain from (45) the following one-dimensional action:
\[
S = \int e^{(\psi + \xi + \lambda)/2} \left( \psi'\xi' + \xi'\lambda' + \lambda'\psi' + e^{-\xi}R'^2 + e^{-\lambda}P'^2 + e^{-(\xi+\lambda)}R^2P^2 \right) d\rho.
\]
This corresponds to the 4D quantities
\[
\begin{align*}
ds^2 &= -e^\psi dt^2 + d\rho^2 + e^\lambda d\varphi^2 + e^\xi dz^2, \\
A &= R \tau_\varphi dz + (P - \nu) \tau_\varphi d\varphi,
\end{align*}
\]
where \(\tau_\varphi\) is given by (44) and primes denote derivatives with respect to \(\rho\). Switching gravity off \((\psi = \xi = 0, \lambda = \ln \rho^2)\) leads exactly to the Nielsen-Olesen system with zero scalar self-coupling constant.

We conclude this section with a following remark. A cylindrical geometry does not prohibit the possibility of saddle points on the energy surface in the configuration space. The reduced space dimensionality can be compensated in the minimax argument by passing from non-contractible loops to non-contractible spheres. Such cylindrical sphalerons were found indeed in the electroweak theory.
5 Conclusions

We have shown the stationary axisymmetric non-circular 4D SU(2) EYM system admits a representation in terms of Euclidean 2D dilaton gravity coupled to a set of matter fields. The ansatz for the YM connection is specified only by usual Kaluza-Klein toroidal assumptions suitably adapted to the non-Abelian case. New representation is aimed to facilitate further consistent truncations of the model for different particular problems. As an application, a static cylindrical symmetric ansatz is suggested as a candidate for a cylindrical EYM sphaleron. Our results may also be useful in the analysis of non-circular stationary axisymmetric spacetimes with other matter sources.

Acknowledgments

The author would like to thank the Organizing Committee for an invitation and a stimulating atmosphere of the Workshop. He also thanks the Yukawa Institute for Theoretical Physics for hospitality while the final version of the paper was written. Fruitful discussions with K. Maeda, T. Nakamura and M. Sasaki and useful correspondence with M. Volkov are gratefully acknowledged. The research was supported in part by the RFBR Grant 96–02–18899.

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