Quantum cohomology of the infinite dimensional generalized flag manifolds

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Abstract

Consider the infinite dimensional flag manifold $LK/T$ corresponding to the simple Lie group $K$ of rank $l$ and with maximal torus $T$. We show that, for $K$ of type $A$, $B$ or $C$, if we endow the space $H^*(LK/T) \otimes \mathbb{R}[q_1, \ldots, q_{l+1}]$ (where $q_1, \ldots, q_{l+1}$ are multiplicative variables) with an $\mathbb{R}[[q_j]]$-bilinear product satisfying some simple properties analogous to the quantum product on $QH^*(K/T)$, then the isomorphism type of the resulting ring is determined by the integrals of motion of a certain periodic Toda lattice system, in exactly the same way as the isomorphism type of $QH^*(K/T)$ is determined by the integrals of motion of the non-periodic Toda lattice (see Kim [9]). This is an infinite dimensional extension of the main result of [11] and at the same time a generalization of [6].

1 Introduction

Let $K$ be a compact, connected, simply connected Lie group, $T \subset K$ a maximal torus and $LK$ the group of loops in $K$ (i.e. smooth maps from the circle $S^1$ to $K$). The evaluation of a loop at the point 1 defines a topologically trivial bundle $p : LK/T \rightarrow K/T$ of fiber $LK/K = \Omega(K)$, where $\Omega(K)$ is the space of based loops in $K$. Consequently $LK/T$ is homeomorphic to $K/T \times \Omega(K)$. Hence a simple description in terms of generators and relations of the cohomology ring with real coefficients of $LK/T$ can be deduced from the fact — which is a direct consequence of a classical result of Serre — that the loop space $\Omega(K)$ has the rational homotopy type of a direct product of spaces of the type $\Omega(S^m)$, for certain $m \geq 2$ (recall that $H^*(\Omega(S^m)) = \mathbb{R}[x]$, where $\deg x = m - 1$) and from Borel’s presentation of $H^*(K/T)$. More precisely, if $t$ denotes the Lie algebra of $T$, we can identify

$$H_2(K/T, \mathbb{Z}) = \pi_2(K/T) = \pi_1(T) = \ker(\exp : t \rightarrow T)$$

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the latter being just the coroot lattice in $\mathfrak{t}$. This leads to the identification of the basis of $H^2(K/T)$ consisting of the degree 2 Schubert classes with the fundamental weights $\lambda_1, \ldots, \lambda_l \in \mathfrak{t}^*$ associated to a simple root system. We have

$$H^*(K/T) = \mathbb{R}[\lambda_1, \ldots, \lambda_l]/\langle u_k(\lambda_1, \ldots, \lambda_l) \mid 1 \leq k \leq l \rangle$$

where $u_k(\lambda_1, \ldots, \lambda_l), 1 \leq k \leq l$ are the fundamental homogeneous generators of the subring of all $W$-invariant polynomials in $S(\mathfrak{t}^*) = \mathbb{R}[\lambda_1, \ldots, \lambda_l]$. We consider $p^*(\lambda_1), \ldots, p^*(\lambda_l) \in H^2(LK/T)$, where $p^*: H^*(K/T) \to H^*(LK/T)$ is the (injective) map induced by $p$ at the level of the cohomology rings. If $\mathcal{H}$ denotes $H^*(LK/T)$, we deduce that

$$\mathcal{H} = H^*(LK/T) = (\mathbb{R}[p^*(\lambda_1), \ldots, p^*(\lambda_l)] \otimes H^*(\Omega(K)))/\langle u_k(p^*(\lambda_1), \ldots, p^*(\lambda_l)) \mid 1 \leq k \leq l \rangle$$

where $H^*(\Omega(K))$ is just a polynomial ring (see above) whose generators are not involved in the ideal of relations.

Fix a system of simple roots $\alpha_1, \ldots, \alpha_l \in \mathfrak{t}^*$ corresponding to the root system of $K$ and let $-\alpha_{l+1}$ be the highest root (the convention concerning the minus sign comes from Goodman and Wallach’s definition [4] of the periodic Toda lattice; see also section 2 of our paper). Then $\alpha_1^\vee, \ldots, \alpha_l^\vee \in \mathfrak{t}$ are a system of simple roots of the coroot system. The strictly positive integers $m_1, \ldots, m_l$ determined by

$$-\alpha_{l+1}^\vee = m_1\alpha_1^\vee + \ldots + m_l\alpha_l^\vee$$

will be important objects in our paper. Consider the formal multiplicative variables $q_1, \ldots, q_{l+1}$ and the space $\mathcal{H} \otimes \mathbb{R}[q_1, \ldots, q_{l+1}]$ which consists of all expressions of the type

$$a = \sum_{d=(d_1, \ldots, d_{l+1}) \geq 0} a_d q^d,$$

where $q^d$ denotes $q_1^{d_1} \ldots q_{l+1}^{d_{l+1}}$. We would like to stress that for any $a \in \mathcal{H} \otimes \mathbb{R}[q_1, \ldots, q_{l+1}]$, we denote by $a_d \in \mathcal{H}$ the coefficient of $q^d$.

The main result of our paper is:

**Theorem 1.1** Suppose that the Lie group $K$ is simple of type $A$, $B$ or $C$. Let $\bullet$ be an $\mathbb{R}[\{q_i\}]$-linear product on $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$ with the following properties:

(i) $\bullet$ preserves the grading induced by the usual grading of $H^*(LK/T)$ combined with $\deg q_j = 4, 1 \leq j \leq l + 1$
• is a deformation of the usual product, in the sense that if we formally replace all $q_j$ by 0, we obtain the usual product on $H$.

(ii) is commutative

(iii) is associative

(v) $p^*(\lambda_i) \cdot p^*(\lambda_j) = p^*(\lambda_i)p^*(\lambda_j) + \delta_{ij}q_j + m_im_jq_{l+1}$, $1 \leq i, j \leq l$

(vi) $(d_i - m_id_{l+1})(p^*(\lambda_j) \cdot a)_d = (d_j - m_jd_{l+1})(p^*(\lambda_i) \cdot a)_d$, for any $d = (d_1, \ldots, d_{l+1}) \geq 0$, $a \in H$, $1 \leq i, j \leq l$

(vii) For $i \in \{1, \ldots, l\}$, $d = (d_1, \ldots, d_{l+1}) \neq 0$, if the coefficient $(p^*(\lambda_i) \cdot a)_d$ of $q^d$ in $p^*(\lambda_i) \cdot a$ is nonzero, then $d_i - m_id_{l+1} \neq 0$.

Then the ring $H \otimes \mathbb{R}[\{q_j\}]$ is generated by $p^*(\lambda_1), \ldots, p^*(\lambda_l)$ and $H^*(\Omega(K))$, subject to the relations

$$F_k(-\langle \alpha^\vee_1, \alpha^\vee_1 \rangle q_1, \ldots, -\langle \alpha^\vee_{l+1}, \alpha^\vee_{l+1} \rangle q_{l+1}, p^*(\lambda_1) \cdot, \ldots, p^*(\lambda_l) \cdot) = 0$$

for $1 \leq k \leq l$, where $F_k$ are the integrals of motion of the periodic Toda lattice (see Theorem 1.2 below) corresponding to the coroot system of $K$.

The Toda lattice we are referring to in our theorem is the Hamiltonian system which consists of the standard symplectic manifold $(\mathbb{R}^{2l}, \sum_{i=1}^{l} dr_i \wedge ds_i)$ with the Hamiltonian function

$$E = \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle r_ir_j + \sum_{j=1}^{l+1} e^{2s_j},$$

where by definition

$$s_{l+1} = -m_1s_1 - \ldots - m_ls_l.$$  

The following result, proved by Goodman and Wallach in [4] and [5], gives details concerning the complete integrability of this system (for more details, see section 2):

**Theorem 1.2** (see [4], [5]) There exist $l$ functionally independent functions

$$E = F_1, F_2, \ldots, F_l : \mathbb{R}^{2l} \to \mathbb{R},$$

such that for every $k \in \{1, \ldots, l\}$ we have:

(i) $F_k$ is a homogeneous polynomial in variables $e^{2s_1}, \ldots, e^{2s_l}, e^{2s_{l+1}}, r_1, \ldots, r_l$. 

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(ii) \( \{ F_k, E \} = 0 \),

(iii) \( F_k(0, \ldots, 0, \lambda_1, \ldots, \lambda_l) = u_k(\lambda_1, \ldots, \lambda_l) \), where \( u_k \) is the \( k \)-th fundamental \( \mathcal{W} \)-invariant polynomial (see above).

Remarks

1. So far there exists no rigorous definition of a product \( \bullet \) on \( \mathcal{H} \otimes \mathbb{R}\{q_j\} \) with the properties (i)-(vii) stated in Theorem 1.1. We expect that Gromov-Witten invariants and a quantum product on \( \mathcal{H} \otimes \mathbb{R}\{q_i\} \) which satisfies those properties can be defined by using the same arguments as in the case of the finite dimensional flag manifold \( K/T \) (see for instance [2]), and the following two facts:

   a) the natural complex structure of \( LK/T \) (see section 4)

   b) the fact that — as in the finite dimensional situation — the “first Chern class” of the tangent bundle of \( LK/T \) (in the sense defined by Freed [1]), equals

\[
2 \sum_{j=1}^{l+1} \tilde{\lambda}_j,
\]

where \( \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{l+1} \) denote the degree 2 Schubert classes in \( LK/T \) (as in the finite dimensional situation, we identify the corresponding Schubert varieties with the fundamental weights of the corresponding affine Lie algebra); this seems to indicate that the moduli space of pointed holomorphic maps of given multidegree \( d \) from \( \mathbb{C}P^1 \) to \( LK/T \) is finite dimensional, of dimension

\[
4 \sum_{j=1}^{l+1} d_j.
\]

2. Readers who are familiar with the quantum cohomology of \( K/T \) will certainly find the requirements (i)-(iv) from Theorem 1.1 quite natural. A few explanations concerning (v), (vi) and (vii) are needed, though. A crucial relation is (see Proposition 3.1):

\[
p^*(\lambda_i) = \tilde{\lambda}_i - m_i \tilde{\lambda}_{l+1},
\]

1 \( \leq i \leq l \). Hence (v) would follow from

\[
\tilde{\lambda}_i \bullet \tilde{\lambda}_j = \tilde{\lambda}_i \tilde{\lambda}_j + \delta_{ij} q_j, \quad 1 \leq i, j \leq l + 1
\]
and one may expect that a quantum product on $H^*(LK/T)$ satisfies this equation (for a simple proof of the same property in the finite dimensional case, see [9] or [10]). In order to explain why should (vi) be satisfied, too, let us consider the coefficient $(p^*(\lambda_i) \cdot a)_d \in H^*(LK/T)$ of $q^d$ in $p^*(\lambda_i) \cdot a$ and $\beta$ an arbitrary element in $H_*(LK/T)$, and look at
\begin{equation}
\langle (p^*(\lambda_i) \cdot a)_d, \beta \rangle = \langle \tilde{\lambda}_i \cdot a, \beta \rangle - m_i \langle \tilde{\lambda}_{i+1} \cdot a, \beta \rangle.
\end{equation}
One would expect the “Gromov-Witten invariants”
\[ \langle \tilde{\lambda}_k \cdot a, \beta \rangle = \langle \tilde{\lambda}_k | a | \beta \rangle_d \]
to satisfy the “divisor property”
\[ \langle \tilde{\lambda}_k | a | \beta \rangle_d = d_k \langle a, \beta \rangle_d, \quad 1 \leq k \leq l + 1 \]
so that the right hand side of (5) is
\begin{equation}
(d_i - m_i d_{i+1}) \langle a, \beta \rangle_d
\end{equation}
and property (vi) becomes clear. Property (vii) follows immediately from (5) and (6) by taking $\beta \in H_*(LK/T)$ with the property that the left hand side of (5) is non-zero.

3. Even though the functionally independent integrals of motion $F_1, \ldots, F_l$ exist for any simple Lie group $K$, (note that Theorem 1.2 contains no restriction on $K$), the methods of our paper enable us to prove the main result only for $K$ of type $A, B$ or $C$ (see Theorem 1.1). More precisely, our proof relies on the relationship between the polynomials $F_k$ and the commutator of a certain Laplacian in the universal enveloping algebra of the corresponding $ax+b$-algebra: it is only for $K$ of one of the types mentioned before that the structure of this commutator has been determined by Goodman and Wallach in [4] and [5] (for more details, see section 2). In fact, this is the only piece of information which is lacking: whenever we have elements of the universal enveloping algebra which commute with the Laplacian, Theorem 5.1 produces relations from them.

4. Theorem 1.1 is an extension of the theorem of Kim [9] to the affine case. The point of view we are adopting here (i.e. an axiomatic approach to the isomorphism type of the quantum cohomology ring) is the same as in [11]. At the same time, Theorem 1.1 is a generalization of the main result of Guest and Otofuji [6], where the situation of the periodic flag manifold $F^{(n)} = L(SU(n))/T$ (i.e. when $K$ is of type $A$) has been considered.

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2 Periodic Toda lattices according to Goodman and Wallach

The goal of this section is to present some details about the mechanical system of periodic Toda lattice type, as defined by Goodman and Wallach [4].

Definition 2.1 (see [4]) The \((ax + b)\)-algebra corresponding to the extension by \(\alpha_{l+1}^\vee\) of the Dynkin diagram corresponding to the coroot system of \(K\) is the Lie algebra

\[(b = t^* \oplus u, [~, ~]),\]

where the Lie bracket \([~, ~]\) is defined by:

- \(t^*\) and \(u\) are abelian
- \(u\) has a basis \(X_1, \ldots, X_{l+1}\) such that
  \[\{\lambda, X_j\} = \lambda (\alpha_j^\vee) X_j, \quad \lambda \in t^*, 1 \leq j \leq l + 1. \tag{7}\]

The set \(S(b)\) of polynomial functions on \(b^*\) becomes a Poisson algebra and by (7) we have

\[\{\lambda_i, \lambda_j\} = 0, \quad \{\lambda_i, X_j\} = \lambda_i (\alpha_j^\vee) X_j, \quad \{X_{j_1}, X_{j_2}\} = 0,\]

for any \(1 \leq i_1, i_2, i \leq l, 1 \leq j, j_1, j_2 \leq l + 1\).

On the other hand, one can easily see that the Poisson bracket of functions on the standard symplectic manifold \(\mathbb{R}^{2l}, \sum_{i=1}^l dr_i \wedge ds_i\) satisfies

\[\{r_{i_1}, r_{i_2}\} = 0, \quad \{r_i, e^{s_j}\} = \lambda_i (\alpha_j^\vee) e^{s_j}, \quad \{e^{s_{j_1}}, e^{s_{j_2}}\} = 0,\]

for \(1 \leq i_1, i_2, i \leq l, 1 \leq j, j_1, j_2 \leq l + 1\) (for \(j = l + 1\), we have used the assumption that \(s_{l+1} = -\sum_{i=1}^l m_i s_i\) (see [4]) and the fact that \(\lambda_i (\alpha_{l+1}^\vee) = -m_i\)). Consequently the map

\[X_j \mapsto e^{s_j}, \quad \lambda_i \mapsto r_i,\]

\(1 \leq i \leq l\) is a homomorphism of Poisson algebras from \(S(b)\) to the Poisson subalgebra \(\mathbb{R}[e^{s_1}, \ldots, e^{s_{l+1}}, r_1, \ldots, r_l]\) of \(C^\infty(\mathbb{R}^{2l})\). In this way, integrals of motion of the Hamiltonian system determined by (3) can be obtained from elements of the space \(S(b)^{\{, ~\}}\) consisting of all elements of \(S(b)\) which \{, \}-commute with the polynomial function on \(b^*\) given by

\[\sum_{i,j=1}^l \langle \alpha_i^\vee, \alpha_j^\vee \rangle \lambda_i \lambda_j + \sum_{j=1}^{l+1} X_j^2. \tag{8}\]
Now let us consider the universal enveloping algebra
\[ U(b) = T(b)/\langle x \otimes y - y \otimes x - [x, y], x, y \in b \rangle \]
and the isomorphism
\[ J : S(b) \to U(b) \]
induced by the symmetrization map followed by the canonical projection (see [7, Corollary E, §17.3]). Since \( t^* \) and \( u \) are abelian, the element of \( S(b) \) described by (8) is mapped by \( J \) to
\[ \Omega := \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \lambda_i \lambda_j + \sum_{j=1}^{l+1} X_j^2, \]
the right hand side being regarded this time as an element of \( U(b) \).

The complete integrability of the Toda lattice follows from the following two theorems of Goodman and Wallach:

**Theorem 2.2** (see [4, Theorem 6.4]) If \( K \) is of classical type, then the Poisson bracket commutator \( S(b) \{,\} \) is mapped by \( J \) isomorphically onto the space \( U(b)^{[l]} \) of all \( f \in U(b) \) with the property that \( [f, \Omega] = 0 \).

**Theorem 2.3** (see [5, Theorem 3.3]) Suppose that \( K \) is of type \( A, B \) or \( C \). Let \( \mu : U(b) \to U(t^*) = S(t^*) \) be the map induced by the natural Lie algebra homomorphism \( b \to t^* \). Then
\[ \mu(U(b)^{[l]}) \subset S(t^*)^W. \]

Moreover, there exist \( \Omega = \Omega_1, \ldots, \Omega_l \in U(b) \), such that for each \( k \in \{1, \ldots, l\} \) we have:

1. \([\Omega_k, \Omega] = 0\),
2. \( \mu(\Omega_k) = u_k \) and \( \deg \Omega_k = \deg u_k \), where \( u_k \) is the \( k \)-th fundamental generator of \( S(t^*)^W \).

Each \( \Omega_k \) is contained in the subring of \( U(b) \) which is spanned by elements of the form \( X^{2l} \lambda^j \).

Note that \( U(b)^{[l]} \) is generated as an algebra by the elements \( \Omega_1, \ldots, \Omega_l \) described in the previous theorem, plus \( X_1^{m_1} \cdots X_l^{m_l} X_{l+1} \).

**Remark.** The integrals of motion of the Toda lattice mentioned in Theorem 1.2 are the polynomials \( F_k \) obtained from \( J^{-1}(\Omega_k) \) by the transformations [7].
3 The degree 2 cohomology modules of $K/T$ and $LK/T$

As in the introduction, $K$ is a compact, connected, simply connected and simple Lie group and $T \subset K$ a maximal torus. The goal of this section is to point out a relation (see Proposition 3.1 below) between the cohomology of the complex flag manifold $K/T$ and the cohomology of the infinite dimensional version of it, $LK/T$.

Consider again a simple root system $\{\alpha_1, \ldots, \alpha_l\} \subset t^*$ and $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\} \subset t$ the corresponding simple coroot system. As pointed out in the introduction, there is a natural isomorphism $\phi$ between $H^2(K/T, \mathbb{Z})$ and the coroot lattice in $t$, the latter being the same as the integral lattice. More precisely, $\phi$ is the composition of the Hurewicz isomorphism $H^2(K/T, \mathbb{Z}) \cong \pi_2(K/T)$ with the boundary map $\pi_2(K/T) \to \pi_1(T)$ from the long exact sequence of the bundle $T \to K \to K/T$. Consequently, $H^2(K/T, \mathbb{Z})$ can be identified with the weight lattice, i.e. the lattice generated by the elements $\lambda_i$ of $t^*$, uniquely determined by $\lambda_i(\alpha_j^\vee) = \delta_{ij}$, $1 \leq i, j \leq l$.

Similar considerations can be made for $LK/T$, but only after obtaining a special presentation of it, as follows: Take first a central extension $\mathbb{S}^1 \longrightarrow \tilde{L}K \longrightarrow LK$, in the sense of [12, Chapter 4]. If $T \subset LK$ is the set of constant loops in $T$, then we get the central extension $S^1 \longrightarrow \pi^{-1}(T) \longrightarrow T$.

But the only central extension of the torus $T$ is the $S^1 \times T$, hence $\pi^{-1}(T) = S^1 \times T$. The space $LK/T$ can be identified with $\tilde{L}(K)/\pi^{-1}(T)$ via the map $\tilde{l}\pi^{-1}(T) \mapsto \pi(l)T$.

And now, exactly as before for $K/T$, we take the composition of the Hurewicz isomorphism $H_2(\tilde{L}K/\pi^{-1}(T), \mathbb{Z}) \cong \pi_2(\tilde{L}K/\pi^{-1}(T))$ with the boundary map $\pi_2(\tilde{L}K/\pi^{-1}(T)) \to \pi_1(\pi^{-1}(T))$ from the long exact sequence of the bundle $\pi^{-1}(T) \to \tilde{L}K \to \tilde{L}K/\pi^{-1}(T)$, and denote the resulting isomorphism by

$$\tilde{\phi} : H_2(\tilde{L}K/\pi^{-1}(T), \mathbb{Z}) \to \pi_1(\pi^{-1}(T)).$$ (9)

Again the integral lattice $\pi_1(\pi^{-1}(T))$ is the same as the coroot lattice. The latter has a basis consisting of the simple coroots

$$\tilde{\alpha}_1^\vee = (\alpha_1^\vee, 0), \ldots, \tilde{\alpha}_l^\vee = (\alpha_l^\vee, 0), \tilde{\alpha}_{l+1}^\vee = (\alpha_{l+1}^\vee, -\frac{1}{2}(\alpha_{l+1}^\vee, \alpha_{l+1}^\vee)),$$ inside the Lie algebra $t + \mathbb{R}$ of $\pi^{-1}(T)$, where $\alpha_{l+1}$ is the highest root (for more details, see [12, Chapter 4]). So the isomorphism $\tilde{\phi} : H_2(\tilde{L}K/\pi^{-1}(T), \mathbb{Z}) \to \pi_1(\pi^{-1}(T))$ assigns to any
coroot $\alpha^\vee$ the loop $\exp(t\alpha^\vee)$, $t \in [0, 1]$. Consequently, $H^2(LK/T)$ can be identified with the weight lattice, i.e. the lattice generated by the elements $\tilde{\lambda}_i$ of $(t + \mathbb{R})^*$, uniquely determined by $\tilde{\lambda}_i(\tilde{\alpha}_j^\vee) = \delta_{ij}$, $1 \leq i, j \leq l + 1$.

We are interested in the relationship between

$$H_2(K/T, \mathbb{R}) = t, \quad H^2(K/T, \mathbb{R}) = t^*$$

on the one hand and

$$H_2(LK/T, \mathbb{R}) = t + \mathbb{R}, \quad H^2(LK/T) = (t + \mathbb{R})^*$$

on the other. Consider first the inclusion map $I : K/T \to LK/T$, which induces the maps $I_*$ and $I^*$ by functoriality. One can easily see that:

(i) $I_*$ is the inclusion map of $t$ into $t + \mathbb{R}$;

(ii) $\tilde{\lambda}_i|_t = \lambda_i$, $1 \leq i \leq l$ and $\tilde{\lambda}_{l+1}|_t = 0$, hence $I^* : H^2(LK/T) \to H^2(K/T)$ maps $\tilde{\lambda}_i$ to $\lambda_i$, $1 \leq i \leq l$ and $\tilde{\lambda}_{l+1}$ to zero.

The main result of the section concerns the imbedding of $H^2(K/T)$ into $H^2(LK/T)$ induced by the map $p : LK/T \to K/T$,

$$p(\gamma T) := \gamma(1)T.$$  

**Proposition 3.1** We have that

$$p^*(\lambda_i) = \tilde{\lambda}_i - m_i\tilde{\lambda}_{l+1}, \quad 1 \leq i \leq l,$$

where $m_i$ are given by $-\alpha_i^\vee = \sum_{i=1}^{l} m_i \alpha_i^\vee$.

**Proof.** From $p \circ I = \text{id}$ follows that $I^* \circ p^* = \text{id}$, hence $p^*(\lambda_i)$ must be of the form $\tilde{\lambda}_i + k\tilde{\lambda}_{l+1}$, $k \in \mathbb{Z}$. In turn, $k$ can be obtained as follows:

$$k = p^*(\lambda_i)(\tilde{\alpha}_{l+1}^\vee) = \lambda_i(p_*((\alpha_{l+1}^\vee, -\frac{1}{2}\langle \alpha_{l+1}^\vee, \alpha_{l+1}^\vee \rangle)))$$

$$= \lambda_i(-\alpha_{l+1}^\vee) = -m_i,$$

where we have used the following property of $p$:

$$p_*((0, \frac{1}{2}\langle \alpha_{l+1}^\vee, \alpha_{l+1}^\vee \rangle)) = 0.$$  

(10)
In order to prove (10), we consider the map $P : \tilde{L}K \to K$,

$$P(\tilde{l}) := \pi(\tilde{l})(1),$$

where $\pi$ is the central extension (see above). The pair $(P, p)$ is a morphism between the bundles $(\tilde{L}K, \tilde{L}K/\pi^{-1}T)$ and $(K, K/T)$. By the functoriality of the maps $\phi$ and $\tilde{\phi}$, we have the following commutative diagram:

$$
\begin{array}{ccc}
\pi_1(\pi^{-1}(T)) & \overset{P^\#}{\longrightarrow} & \pi_1(T) \\
\tilde{\phi} \uparrow & \quad & \uparrow \phi \\
H_2(LK/T) & \overset{p^*}{\longrightarrow} & H_2(K/T)
\end{array}
$$

The loop

$$\exp(t(0, \frac{1}{2}(\alpha_{i+1}^\vee, \alpha_{i+1}^\vee))), \quad t \in [0, 1]$$

in $\pi^{-1}(T)$ corresponds to the coroot $(0, \frac{1}{2}(\alpha_{i+1}^\vee, \alpha_{i+1}^\vee))$ in $t + \mathbb{R}$. This loop is obviously mapped to the constant loop $e$ by $\pi$, hence by $P$. So (10) is true. $\square$

4 The Schubert basis of $H^*(LK/T)$

The goal of this section is to describe the Schubert varieties in $LK/T$, by following the construction outlined in [12, section 8.7].

Let $G$ be the complexification of $K$ and $B_0 \subset G$ a Borel subgroup. Consider the following subgroups of $LG$:

a) $L^+G$, the group of loops in $G$ which extend holomorphically to the unit disc;

b) $B^+$, the subgroup of $L^+G$ consisting of loops $\gamma$ with $\gamma(0) \in B_0$.

Then we have the diffeomorphism

$$LK/T \simeq LG/B^+,$$

which induces an action of $B^+$ on $LK/T$. 

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The affine Weyl group $W_{\text{aff}}$ is defined as the semidirect product of

$$W = N_K(T)/T$$

with the lattice in $t$ generated by the coroots. One can easily see that $W_{\text{aff}}$ is generated by the reflections $s_1, \ldots, s_l, s_{l+1}$ in the walls $\ker \alpha_1, \ldots, \ker \alpha_l$ and $\ker \alpha_{l+1}$ of the fundamental chamber in $t$. In particular, to any $w \in W_{\text{aff}}$ corresponds a length $l(w)$, which is the minimal number of factors in a decomposition of $w$ as a product of $s_j$’s. Since $K$ is simply connected, the coroot lattice coincides with the integral lattice $\tilde{T} = \text{Hom}(S^1, T)$. Consequently we have that

$$W_{\text{aff}} = (N_K(T) \cdot \tilde{T})/T \subset LK/T.$$  

As in the finite dimensional situation, for any $w \in W_{\text{aff}}$, the $B^+$ orbit

$$C_w = B^+.w$$

is a complex cell of dimension $l(w)$, called the Bruhat cell. The set

$$\{[C_w] \mid w \in W_{\text{aff}}\}$$

consisting of the fundamental cycles of the closures of Bruhat cells is a basis of $H_\ast(LK/T)$. We denote by $\sigma_w \in H^\ast(LK/T)$ the dual of $[C_w]$ with respect to the evaluation pairing $\langle \cdot, \cdot \rangle$. More precisely, $\sigma_w$ is defined by

$$\langle \sigma_w, [C_v] \rangle = \delta_{vw}, \quad v, w \in W_{\text{aff}}$$

where $\delta_{vw}$ denotes the Kronecker delta. The basis $\{\sigma_w \mid w \in W_{\text{aff}}\}$ is the Schubert basis of $H^\ast(LK/T)$.

The set $\{[C_{s_j}] \mid 1 \leq j \leq l + 1\}$ is a basis of $H_2(LK/T, \mathbb{Z})$. The isomorphism $\tilde{\phi}$ defined in section 3 (see equation (9)) maps any $[C_{s_j}]$ to the coroot $\tilde{\alpha}_j$, $1 \leq j \leq l + 1$. This can be proved by using the $SU(2)$ embedding in $\tilde{L}(K)$ associated to the root $\alpha_j$ (see [12, section 5.2]) and the naturality of $\tilde{\phi}$. Consequently the transpose of $\tilde{\phi}$ maps $\sigma_{s_j}$ to the fundamental weight $\tilde{\lambda}_j$, $1 \leq j \leq l + 1$, and we often use this isomorphism as an identification.

### 5 Relations in the ring $(\mathcal{H} \otimes \mathbb{R}[\{q_j\}], \bullet)$: proof of the main result

Let us consider the space

$$\mathbb{R}[[t_i]][[e^{t_j/2}]]$$
of all formal series of the type
\[ g = \sum_{d=(d_1, \ldots, d_{l+1}) \geq 0} g_d e^{t_d} \]
where \( g_d \in \mathbb{R}[[t_i]] \) is a polynomial and \( e^{t_d} \) means \( e^{t_1 d_1} \ldots e^{t_{l+1} d_{l+1}} \). This space has an obvious structure of a commutative and associative algebra with unit. For any \( 1 \leq i \leq l \), we consider the derivative-type operator \( \partial_i \) on \( \mathbb{R}[[t_i]][[[e^{t_j/2}]]] \), which is \( \mathbb{R} \)-linear and satisfies
\[ \partial_i (g e^{t_d/2}) := \frac{\partial}{\partial t_i} (g) e^{t_d/2} + \frac{1}{2} (d_i - m_i d_{l+1}) g e^{t_d/2}, \]
for any \( g \in \mathbb{R}[t_1, \ldots, t_l] \) and any \( d = (d_1, \ldots, d_{l+1}) \geq 0 \). The facts that \( \partial_i \) is \( \mathbb{R} \)-linear and satisfies the Leibniz rule can be easily verified.

Now let us consider the following assignment \( \rho \):
\[ \rho(\lambda_i) := 2 \partial_i, \quad \rho(X_j) = \frac{2\sqrt{-1}}{h} \sqrt{\langle \alpha_j^\vee, \alpha_j^\vee \rangle} e^{t_j/2}, \]
for \( 1 \leq i \leq l, 1 \leq j \leq l + 1 \), where \( h \) is a nonzero real parameter. One can easily see that \( \rho \) is actually a representation of the Lie algebra \( \mathfrak{b} \) on the space \( \mathbb{R}[[t_i]][[[e^{t_j/2}]]] \), i.e. it satisfies
\[ \rho[\lambda_i, X_j] = [\rho(\lambda_i), \rho(X_j)], \quad 1 \leq i \leq l, 1 \leq j \leq l + 1. \]

Consider \( \Omega_k \) defined in Theorem 2.3 and set
\[ D_k = h^{\deg \Omega_k} \rho(\Omega_k), 1 \leq k \leq l. \]
By the last property mentioned in Theorem 2.3, \( D_k \) leaves the subspace \( \mathbb{R}[[t_i]][[[e^{t_j}]]] \) of \( \mathbb{R}[[t_i]][[[e^{t_j/2}]]] \) invariant. Only the action on \( \mathbb{R}[[t_i]][[[e^{t_j}]]] \) will be used later, and on this space, \( \partial_i \) is defined by
\[ \partial_i (g e^{t_d}) := \frac{\partial}{\partial t_i} (g) e^{t_d} + (d_i - m_i d_{l+1}) g e^{t_d}, \quad 1 \leq i \leq l. \quad (11) \]
Since \( F_k \) is homogeneous in variables \( e^{t_j}, r_i, 1 \leq j \leq l + 1, 1 \leq i \leq l \), it follows that \( \Omega_k \) — being essentially the same as \( J(F_k) \) — has a presentation as a homogeneous, symmetric polynomial in the variables \( X_j, \lambda_i \). We use the commutation relations (7) in order to express \( \Omega_k \) as a linear combination of elements of the form \( X_j^2 \lambda^J \) (see Theorem 2.3). The polynomial expression we obtain in this way appears as
\[ \Omega_k = F_k(X_i^2, \lambda_i) + f_k(X_i^2, \lambda_i) \]
where
\[ \deg f_k < \deg F_k. \]
Consequently \( D_k \) appears as a polynomial expression \( D_k(e^{t_1}, \ldots, e^{t_{l+1}}, h \frac{\partial}{\partial t_1}, \ldots, h \frac{\partial}{\partial t_l}, h) \), the last “variable”, \( h \), being due to the possible occurrence of \( f_k \).

Let us consider the variables \( Q_1, \ldots, Q_{l+1}, \Lambda_1, \ldots, \Lambda_l \). To each polynomial \( D \in \mathbb{R}[Q_j, \Lambda_i, h] \) we assign the differential operator \( D(e^{t_j}, h \partial_i, h) \) obtained from \( D \) as follows: first write \( D \) as a sum of monomials of the type \( Q_I \Lambda_J h^m \), and then replace \( Q_j \rightarrow e^{t_j}, \Lambda_i \rightarrow h \partial_i, 1 \leq j \leq l+1, 1 \leq i \leq l \). For instance, the differential operator
\[ \frac{1}{4}D_1 = \frac{1}{4}h^2 \rho(\Omega) = \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle h^2 \partial_i \partial_j - \sum_{j=1}^{l+1} \langle \alpha_j^\vee, \alpha_j^\vee \rangle e^{t_j} \]
arises as \( H(e^{t_j}, h \partial_i, h) \), where
\[ H = \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \Lambda_i \Lambda_j - \sum_{j=1}^{l+1} \langle \alpha_j^\vee, \alpha_j^\vee \rangle Q_j. \]

The main result of this section is:

**Theorem 5.1** Assume that the polynomial \( D \in \mathbb{R}[Q_1, \ldots, Q_{l+1}, \Lambda_1, \ldots, \Lambda_l, h] \) satisfies
\[ \deg \{Q_j^{1/2}, \Lambda_i\}(D) \leq 2 \sum_{i=1}^{l} m_i. \tag{12} \]

If
(a) \([D(e^{t_j}, h \partial_i, h), H(e^{t_j}, h \partial_i, h)] = 0 \) for any \( h \neq 0 \),
(b) the polynomial \( D(0, \ldots, 0, \Lambda_1, \ldots, \Lambda_l, h) \) does not depend on \( h \),
(c) \( D(0, \ldots, 0, \lambda_1, \ldots, \lambda_l, 0) \in S(t^*)^W, \)
then the relation
\[ D(q_j, p^*(\lambda_i) \bullet, 0) = 0 \]
holds in \( \mathcal{H} \otimes \{[q_j]\} \).
Proof of Theorem 1.1: In fact, we only have to prove that the equation (2) holds, for any \( 1 \leq k \leq l \): by a result of Siebert and Tian (see [13]), these generate the entire ideal of relations. We apply Theorem 5.1 for the polynomial \( D_k \). The only thing which still has to be checked is the degree condition (12). By the definition of \( D_k \), its degree with respect to \( Q_j^{1/2} \), \( \Lambda_i \) is the same as the degree of \( \Omega_k \) hence, by Theorem 2.3 (ii), it is equal to the degree of \( u_k \). Now, the inequality
\[
\deg(u_k) \leq 2 \sum_{i=1}^{l} m_i
\]
can be easily verified by checking usual tables: see for instance [8] for \( \max(\deg(u_k)) \), the maximal degree of the fundamental generators, and [4] for the coefficients \( m_i \) from
\[
-\alpha_{l+1}^\vee = \sum_{i=1}^{l} m_i \alpha_i^\vee.
\]
We summarize the results in the following table. Note that the inequality (5) is true for any type of \( K \), even though we need it only for the types \( A, B \) and \( C \) (see Remark 3 in the introduction).

| Type | \( \max(\deg(u_k)) \) | \( \sum m_i \) |
|------|-----------------|---------|
| \( A_l \) | \( l + 1 \) | \( l \) |
| \( B_l \) | \( 2l \) | \( 2l - 2 \) |
| \( C_l \) | \( 2l \) | \( l \) |
| \( D_l \) | \( 2l - 2 \) | \( 2l - 3 \) |
| \( E_6 \) | 12 | 11 |
| \( E_7 \) | 18 | 17 |
| \( E_8 \) | 30 | 29 |
| \( F_4 \) | 12 | 8 |
| \( G_2 \) | 6 | 5 |

\[ \square \]

6 The proof of Theorem 5.1

In this section, \( q_j \) is always regarded as \( e^{t_j} \), \( 1 \leq j \leq l + 1 \). The basic ingredient of the proof are the endomorphisms \( p^*(\lambda_i) \), \( 1 \leq i \leq l \), of \( H^*(LK/T) \) and their duals with respect to the
intersection pairing
\[ \langle \sigma_w, C_v \rangle = \delta_{vw}, \quad v, w \in W_{aff}, \]
(see section 4). More precisely, we will denote by \( A_i \) the endomorphism of \( H_\ast(LK/T) \) defined by
\[ \langle p^\ast(\lambda_i) \bullet \sigma_w, C_v \rangle = \langle \sigma_w, A_i(C_v) \rangle, \quad v, w \in W_{aff}. \]
Let us consider an ordering of \( W_{aff} \) compatible with the length; this will induce an ordering of the basis \( \{ \sigma_w \mid w \in W_\circ \} \) of \( H_\ast(LK/T) \) and also of the basis \( \{ [\bar{C}_w] \mid w \in W_{aff} \} \) of \( H_* (LK/T) \). The matrix of \( A_i \) with respect to the latter basis is the transpose of the matrix of \( p^\ast(\lambda_i) \bullet \) with respect to the former one.

The basis described above induces a natural identification of \( H_\ast(LK/T) \) with the space \( \mathbb{R}_0^\infty \) of all sequences with finite support. The endomorphisms of \( H_\ast(LK/T) \) will be represented by matrices of the form
\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots \\
a_{21} & a_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]
where each column is in \( \mathbb{R}_0^\infty \). Let us denote the space of these matrices by \( M_{0}^{\infty \times \infty} (\mathbb{R}) \).

The matrices \( A_i, 1 \leq i \leq l \) have the following features:

a) \( A_i \) commutes with \( A_j \), for any two \( i, j \);

b) \( \partial_t A_j = \partial_j A_i, 1 \leq i, j \leq l \);

c) for any \( i \in \{1, \ldots, l\} \), we have that
\[ A_i = A'_i(e^{tj}) + A''_i \]
where \( A'_i \) is strictly lower triangular, and its coefficients are linear combinations of \( e^{td} = e^{t_1 d_1} \ldots e^{t_{l+1} d_{l+1}}, d = (d_1, \ldots, d_{l+1}) \geq 0 \), where
\[ d_i - m_i d_{l+1} \neq 0; \]
\( A''_i \) is strictly upper triangular and its coefficients do not depend on \( t \) (in particular, the diagonal of \( A_i \) is identically zero).

Property a) follows from the associativity of \( \bullet \). As for b) and c), they are direct consequences of the requirements (vi) and (vii) from Theorem [1].

The following result will be needed later:
Proposition 6.1 Let $\mathcal{R}$ be a commutative, associative real algebra with unit and $A_1, \ldots, A_l \in M^{\infty\times\infty}_0(\mathcal{R}[e^{t_j}])$ be matrices which satisfy the properties a), b), c) from above. Consider the following system of differential equations:

\[(*) \partial_i g = A_i g, \quad 1 \leq i \leq l\]

where

\[g = \sum_{d=(d_1, \ldots, d_{l+1}) \geq 0} g_d(t_1, \ldots, t_l) e^{td} \in \mathcal{R}_0^\infty[[t_1]][[e^{t_j}]].\]

The solutions of $(*)$ are uniquely determined by those $g_0^d$ (the degree zero term of the polynomial $g_d$) where $d$ satisfies

\[d_1 = m_1d_{l+1}, \ldots, d_l = m_l d_{l+1}.\]

More precisely, the solution of $(*)$ can be obtained by the following recurrence relations:

\[g_d = \begin{cases} G_d(g_0^d, g_{d'}) | d' < d), & \text{if } d_1 = m_1d_{l+1}, \ldots, d_l = m_l d_{l+1} \\ G_d(g_{d'} | d' < d), & \text{if contrary} \end{cases} \tag{13}\]

where the maps $G_d$ are “$\mathcal{R}$-linear in coefficients” (by this I mean that the coefficients of the polynomial $g_d$ are of the form $C_0 g_0^d + \sum_j C_j v_j$ or $\sum_j C_j v_j$ where $C_j \in M^{\infty\times\infty}_0(\mathcal{R})$ and $v_j \in \mathcal{R}_0^\infty$ are coefficients of certain polynomials $g_{d'}$ with $d' < d$ (i.e. $d' \leq d$ and $d' \neq d$).

In order to prove Proposition 6.1 we will need the following lemma:

Lemma 6.2 Let $A \in M^{\infty\times\infty}_0(\mathcal{R})$ be a matrix of the type described above and $g \in \mathcal{R}_0^\infty[t]$ a polynomial. Consider the differential equation:

\[\frac{df}{dt} = Af + g,\]

where $f$ is in $\mathcal{R}_0^\infty[t]$.

(i) If $A - I$ is strictly upper triangular, then we have a unique solution, which depends $\mathcal{R}$-linearly on the coefficients of $g$.

(ii) If $A$ is strictly upper triangular, then the solution is uniquely determined by the degree zero term $f_0 \in \mathcal{R}_0^\infty$ of $f$. In fact, the solution depends $\mathcal{R}$-linearly on $f_0$ and the coefficients of $g$. 

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Proof. Put \( g = \sum_{k=0}^{p} g_k t^k \). Look for \( f \) of the form \( \sum_{k=0}^{m} f_k t^k \). We must have

\[
f_1 + 2f_2 t + \ldots + mf_m t^{m-1} = Af_0 + Af_1 t + \ldots + Af_m t^m + g_0 + g_1 t + \ldots + g_p t^p,
\]
hence:

\[
f_1 = Af_0 + g_0 \\
2f_2 = Af_1 + g_1 = A^2 f_0 + Ag_0 + g_1 \\
\vdots \\
mf_m = Af_{m-1} + g_{m-1} = c_mA^m f_0 + c_mA^{m-1} g_0 + \ldots + c_2 A g_{m-2} + g_{m-1},
\]
where \( c_j \) are certain nonzero coefficients which arise in an obvious way and \( g_r := 0 \) for \( r \geq p \) (consequently, we have that \( m \geq p \)).

If \( A \) is the sum of \( I \) with a strictly upper triangular matrix, then \( m \) equals \( p \). The coefficients \( f_j \) are uniquely determined by the recursion formulae (14) and the requirement

\[
Af_p + g_p = 0,
\]
which determines the initial term \( f_0 \).

If \( A \) is strictly upper triangular, then we consider the following relations:

\[
(k + p + 1)f_{k+p+1} = c_{k+p+1} A^{k+p+1} f_0 + c_{k+p+1} A^{k+p} g_0 + \ldots + c_{k+1} A^k g_p, \quad k \geq 0.
\]
If \( k \geq 0 \) is minimal with the property that \( c_{k+p+1} A^{k+p+1} f_0 + c_{k+p+1} A^{k+p} g_0 + \ldots + c_{k+1} A^k g_p = 0 \), then \( m \) must be \( k + p \) and the coefficients \( f_j \) are uniquely determined by \( f_0 \).

Proof of Proposition 6.1 We will prove this result by induction on \( l \geq 1 \). Take first \( l = 1 \). We must solve the equation

\[
\partial_1 g = A_1 g,
\]
where \( g = \sum_{d=(d_1,d_2)\geq 0} g_d e^{td_1} \), \( g_d \in \mathcal{R}_0^\infty[t_1] \). Decompose \( A_1 \) as \( \sum_{d \geq 0} (A_1)^d e^{td} \), where \( (A_1)^d \in M_0^{\infty \times \infty}(\mathcal{R}) \). We identify the coefficients of \( e^{td} \) and write the differential equation piecewise as:

\[
\frac{d g_d}{dt_1} + (d_1 - m_1 d_2) g_d = \sum_{d' + d'' = d} (A_1)^{d''} g_{d'}.
\]
We apply Lemma 6.2 bearing in mind that \( (A_1)^0 \) is strictly upper triangular.
The induction step from $l-1$ to $l$ now follows. We write the $g$ we are looking for as

$$g = \sum_{d=(d_1,\ldots,d_{l+1}) \geq 0} g_d t_1^{d_1} \cdots t_l^{d_l} t_{l+1}^{d_{l+1}} = \sum_{r=(r_1,\ldots,r_l) \geq 0} h_{r} t_1^{r_1} \cdots t_l^{r_l} t_{l+1}^{r_{l+1}},$$

where $g_d \in R^\infty_0[t_1,\ldots,t_l]$ and $h_r \in R^\infty_0[t_l][[e^{t_l}]]t_1,\ldots,t_{l-1}$.

Recall that the relation between $g_d$ and $h_r$ is

$$h_r = \sum_k g_{r_1,\ldots,r_{l-1},k} r_{l+1} e^{kt_l}. $$

A more detailed formula will give the coefficient $h_r^{n_1,\ldots,n_{l-1}}$ of $t_1^{n_1} \cdots t_{l-1}^{n_{l-1}}$ in $h_r$ as:

$$h_r^{n_1,\ldots,n_{l-1}} = \sum_k \sum_m g_{r_1,\ldots,r_{l-1},k}^{n_1,\ldots,n_{l-1},m} t_{l-1}^{m} e^{kt_l}. \quad (15)$$

If $h_r^{n_1,\ldots,n_{l-1}}$ are in $(R[t_l][[e^{t_l}]]^\infty_0$, then all $g_{r_1,\ldots,r_{l-1},k} r_{l+1}$ are in $R^\infty_0$ (but not conversely).

Now we put $S = R[t_l][[e^{t_l}]]$ and solve the system of differential equations

$$\partial_i h = A_i h, \ 1 \leq i \leq l-1,$$

where $h \in S^\infty_0[t_1,\ldots,t_{l-1}][[e^{t_1},\ldots,e^{t_{l-1}},e^{t_l}]]$. If we regard $A_1,\ldots,A_{l-1}$ as elements of $M^\infty_0 \times \infty_0(S[e^{t_1},\ldots,e^{t_{l-1}},e^{t_l}]),$ we can easily see that the conditions a), b) and c) are still satisfied. By the induction hypothesis, we know that the latter system can be solved by formulae of the following type:

$$h_r = \begin{cases} H_r(h_{r'}; h_{r'}) | r' < r, & \text{if } r_1 = m_1 r_{l+1}, \ldots, r_{l-1} = m_{l-1} r_{l+1} \\ H_r(h_{r'}; | r' < r, & \text{otherwise.} \end{cases}$$

We take first the situation when at least one of the conditions

$$r_1 = m_1 r_{l+1}, \ldots, r_{l-1} = m_{l-1} r_{l+1}$$

is not satisfied and show how can we obtain all polynomials of the form $g_{r_1,\ldots,r_{l-1},k} r_{l+1}, k \geq 0$ from

$$h_r = H_r(h_{r'}; | r' < r).$$

We only have to take into account and the fact that $H_r$ is a $S = R[t_l][[e^{t_l}]]$-linear function of the coefficients of $h_{r'}, r' < r$. From

$$h_r^{n_1,\ldots,n_{l-1}} = \sum_{r' < r} B_{r'} h_{r'}^{n_1,\ldots,n_{l-1}},$$

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with $B_v \in M_0^\infty(\mathcal{R}[t_1][[e^{t_1}]])$ we deduce easily, by equating the coefficients of $t_i^m e^{k t_i}$, that $g_{r_1,\ldots,r_{l-1},k,r_{l+1}}^{m_1,\ldots,m_l}$ is a $\mathcal{R}$-linear function of the coefficients of $g_{r_1',\ldots,r_{l-1}',k',r_{l+1}'}$, with $r' < r$ and $k' < k$.

If $r_1 = m_1 r_{l+1}, \ldots, r_{l-1} = m_{l-1} r_{l+1}$, we choose $h_0^\nu \in S_0^\infty$ to be a solution of:

$$\partial_t(h_0^\nu e^{t_1 r_1 + \cdots + t_{l-1} r_{l-1} + t_{l+1} r_{l+1}}) = \sum_{r' + r'' = r} (A_r^\nu h_0^\nu) e^{t_1 r_1 + \cdots + t_{l-1} r_{l-1} + t_{l+1} r_{l+1}},$$

where the matrices $A_r^\nu \in M_0^{\infty \times \infty}(\mathcal{R}[e^{t_1}])$ come from the decomposition

$$A_r^\nu = \sum_{r = (r'_1, \ldots, r'_{l-1}, r_{l+1}) \geq 0} A_{r_1}^\nu e^{t_1 r_1 + \cdots + t_{l-1} r_{l-1} + t_{l+1} r_{l+1}}.$$

In order to be more precise, we write

$$h_0^\nu = \sum_{k \geq 0} f_k(t_i) e^{k t_i}$$

and then we identify the coefficients of $e^{t_1 r_1 + \cdots + t_{l-1} r_{l-1} + t_{l+1} r_{l+1}}$ in both sides of (16). One obtains the following sequence of differential equations:

$$\frac{d f_k}{d t_i} + (k - m_i r_{l+1}) f_k = (A_i^0 h_0^0)_k + b = \sum_{u+v=k} (A_i^0)_u f_v + b $$

where $(A_i^0 h_0^0)_k$ denotes the coefficient of $e^{k t_i}$ in $A_i^0 h_0^0$ and $(A_i^0)_u \in M_0^{\infty \times \infty}(\mathcal{R})$ is the coefficient of $e^{u t_i}$ in $A_i^0 \in M_0^{\infty \times \infty}(\mathcal{R}[e^{t_1}])$. Here $b \in \mathcal{R}_0^\infty[t_i]$ is obtained by adding the coefficients of $e^{k t_i}$ from all expressions $A_r^\nu h_0^0$, where $r' + r'' = r$, $r' \neq 0$; hence it depends $\mathcal{R}$-linearly on the polynomials $g_{r_1',\ldots,r_{l-1}',k',r_{l+1}'}$, where $r'' < r$ and $k' \leq k$.

We solve the sequence of differential equations (17) using Lemma 6.2. First we write (17) as:

$$\frac{d f_k}{d t_i} + (k - m_i r_{l+1}) f_k = (A_i^0)_k f_k + c + b,$$

where $c \in \mathcal{R}_0^\infty[t_i]$ depends $\mathcal{R}$-linearly on $f_0, \ldots, f_{k-1}$. The matrix $(A_i^0)_k \in M_0^{\infty \times \infty}(\mathcal{R})$ is obviously $A_i^\nu$ (see condition c)), hence it is strictly upper triangular.

We can easily see that the coefficients of $f_k$ depend $\mathcal{R}$-linearly on:

- the coefficients of $g_{r_1',\ldots,r_{l-1}',k',r_{l+1}'}$, with $r' < r$ and $k' \leq k$, if $k \neq m_i r_{l+1};$
• $g^0_{r_1,\ldots,r_{l-1},k,r_{l+1}}$ and the coefficients of $g^r_{r_1,\ldots,r_{l-1},k',r_{l+1}'}$, with $r < r'$ and $k' \leq k$, if $k = m_r r_{l+1}$.

As in the previous case, we take

$$h_r = H_r^0(h_r^0, h_r^r | r' \neq r)$$

and obtain expressions for the coefficients of $g_{r_1,\ldots,r_{l-1},k,r_{l+1}}$ via \[15\] and the structure of $f_k$ described above. More precisely, we take the equality

$$h_{r_1}^{n_1,\ldots,n_l-1} = C_0 h_r^0 + \sum_{r' < r} B_{r'} h_{r'}^r,$$

with $C_0, B_{r'} \in M_{\infty \times \infty}(\mathcal{R}[t_1][[e_t]])$ and identify the coefficients of $t_1^m e_t k$. We deduce that $g_{r_1,\ldots,r_{l-1},k,r_{l+1}}$ is

• a $\mathcal{R}$-linear function of the coefficients of $g_{r_1',\ldots,r_{l-1}',k',r_{l+1}'}$, with $r < r'$ and $k' \leq k$, if $k \neq m_r r_{l+1}$;

• a $\mathcal{R}$-linear function of $g_{r_1,\ldots,r_{l-1},k,r_{l+1}}$ and the coefficients of $g_{r_1',\ldots,r_{l-1}',k',r_{l+1}'}$, with $r < r'$ and $k' \leq k$, if $k = m_r r_{l+1}$.

The only thing which remains to be proved is that the $g$ we have just constructed satisfies

$$\partial_l g = A_l g.$$ \hspace{1cm} (18)

To this end, we notice first that

$$\partial_i(\partial_l g - A_l g) = A_i(\partial_l g - A_l g),$$

for all $1 \leq i \leq l - 1$. Also $\partial_l g - A_l g$ can be written as

$$\partial_l g - A_l g = \sum_{r=(r_1,\ldots,r_{l-1},r_{l+1})} q_r e^{t_1 r_1 + \ldots + t_{l-1} r_{l-1} + t_{l+1} r_{l+1}} ,$$

where $q_r \in \mathcal{S}_{\infty}^r[t_1,\ldots,t_{l-1}]$, $r \geq 0$ (we have in mind that $\partial_l$ maps $\mathcal{S}_{\infty}^0[t_1,\ldots,t_{l-1}][e^{t_1},\ldots,e^{t_{l-1}},e^{t_{l+1}}]$ onto itself). The degree zero term $q^0_r$ of $q_r$ is the coefficient of $e^{t_1 r_1 + \ldots + t_{l-1} r_{l-1} + t_{l+1} r_{l+1}}$ in

$$\partial_l(h_r^0 e^{t_1 r_1 + \ldots + t_{l-1} r_{l-1} + t_{l+1} r_{l+1}}) - \sum_{r' + r'' = r} A^r_{r'} h_r^{r'} e^{t_1 r_1 + \ldots + t_{l-1} r_{l-1} + t_{l+1} r_{l+1}}.$$

Whenever $r_1 = m_1 r_{l+1}, \ldots, r_{l-1} = m_{l-1} r_{l+1}$ happens, from the choice of $h_r^0$ it follows that $q^0_r = 0$. But these $q^0_r$’s determine $\partial_l g - A_l g$ uniquely, hence the latter is zero. \qed

We use Proposition 6.1 with $A_i$ replaced by $\frac{1}{n_i} A_i$ and obtain:

20
**Corollary 6.3** For any \( w \in W_{\text{aff}} \) one can find at least one \( s_w \in \mathcal{H} \otimes \mathbb{R}[\{t_i\}][\{e^j\}] \) with the properties
\[
\begin{align*}
h \partial_i s_w &= A_i s_w, \quad 1 \leq i \leq l \\
(s_w)_0^0 &= [\bar{C}_w].
\end{align*}
\]

The following lemma will make clear the importance of the formal series \( s_w, w \in W_{\text{aff}} \), in proving Theorem 5.1.

**Lemma 6.4** (i) Let \( D \in \mathbb{R}[Q_j, \Lambda, h] \). Suppose that there exists \( M \) a positive integer with
\[
\deg_{Q_j/2, \Lambda_i}(D) \leq 2M
\]
and
\[
(D(e^j, h \partial_i, h)) \langle 1, s_w \rangle_d = 0,
\]
for any \( w \in W_{\text{aff}} \), any multi-index \( d \) with \(|d| \leq M\) and any \( h \neq 0 \). Then we have that
\[
D(e^{t_1}, \ldots, e^{t_{l+1}}, p^*(\lambda_1)\bullet, \ldots, p^*(\lambda_l)\bullet, 0) = 0.
\]

(ii) We have that
\[
H(e^j, h \partial_i, h) \langle 1, s_w \rangle = 0.
\]

**Proof.** (i) For any \( f \in \mathcal{H} \otimes \mathbb{R}[e^{t_1}, \ldots, e^{t_{l+1}}] \) we have that:
\[
\begin{align*}
h \partial_i \langle f, s_w \rangle &= \langle h \partial_i f, s_w \rangle + \langle f, h \partial_i s_w \rangle = \langle h \partial_i f, s_w \rangle + \langle f, A_i s_w \rangle \\
&= \langle h \partial_i f, s_w \rangle + \langle p^*(\lambda_i)\bullet f, s_w \rangle = \langle (h \partial_i + p^*(\lambda_i)\bullet) f, s_w \rangle.
\end{align*}
\]

We deduce that
\[
D(e^j, h \partial_i, h) \langle f, s_w \rangle = \langle D(e^{t_1}, \ldots, e^{t_{l+1}}, p^*(\lambda_1)\bullet + h \partial_1, \ldots, p^*(\lambda_l)\bullet + h \partial_i, h)f, s_w \rangle. \tag{19}
\]

Replacing \( f \) by 1 and denoting \( \mathcal{D} := D(e^j, p^*(\lambda_i)\bullet + h \partial_i, h)1 \), we obtain:
\[
\langle \mathcal{D}, s_w \rangle_d = 0, \quad \text{if } |d| \leq M. \tag{20}
\]

Notice now that
\[
\deg_{e^j} \mathcal{D} \leq \deg_{e^j} D(e^j, p^*(\lambda_i)\bullet, 0) \leq M.
\]
To justify the last inequality, notice that if $Q^v \Lambda^u$ is a monomial from $D(Q_j, \Lambda_i, 0)$, with $u \in \mathbb{N}^l, v \in \mathbb{N}^{l+1}$, then we have:

$$\deg_{e^t} \lambda^u e^w \leq \frac{1}{2} |u| + |v| = \frac{1}{2}(|u| + 2|v|) = \frac{1}{2} \deg_{Q_{j/2} \Lambda_i} (Q^v \Lambda^u) \leq \frac{1}{2} \cdot 2M = M.$$ 

For the rest of the proof, “degree” and “degree zero term” will refer to the “variables” $e^{t_1}, \ldots, e^{t_{l+1}}$. Decompose $D$ as

$$D = D_0 + D_1 + \ldots + D_m$$

where $D_k \in \mathcal{H} \otimes \mathbb{R}[e^{t_1}, \ldots, e^{t_{l+1}}]$ denotes the sum of all terms of degree $k$, $0 \leq k \leq m$. Recall that the degree zero term of $s_w$ is $(s_w)_0 \in H_s(LK/T) \otimes \mathbb{R}[t_1, \ldots, t_l]$ with $(s_w)_0 = [\bar{C}_w]$. The degree zero term of $\langle D, s_w \rangle$ is $\langle D_0, (s_w)_0 \rangle$. From the vanishing of the latter we obtain that

$$\langle D_0, (s_w)_0 \rangle = \langle D_0, [\bar{C}_w] \rangle = 0,$$

for all $w \in W_{aff}$, hence $D_0 = 0$.

Also the sum of the terms of degree 1 in $\langle D, s_w \rangle$ is 0, hence we have

$$\langle D_1, (s_w)_0 \rangle = 0.$$

Exactly as before, this implies $D_1 = 0$. Since $m \leq M$, we can continue the algorithm until we obtain $D_m = 0$, hence

$$D = 0.$$

Now we let $h$ vary and deduce that

$$D(e^{t_j}, p^*(\lambda_i) \bullet, 0) = 0.$$ 

(ii) When computing $H(e^{t_j}, h\partial_t, h)\langle 1, s_w \rangle$, only the following two relations will be used:

$$h\partial_t \langle 1, s_w \rangle = \langle p^*(\lambda_i), s_w \rangle$$

and

$$h^2 \partial_j \partial_t \langle 1, s_w \rangle = \langle p^*(\lambda_j) \bullet p^*(\lambda_i), s_w \rangle.$$

Hence we have to show that:

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle p^*(\lambda_i) \bullet p^*(\lambda_j) - \sum_{j=1}^{l+1} \langle \alpha_j^\vee, \alpha_j^\vee \rangle q_j = 0.$$
This follows immediately from property (v), Theorem 1.1 and the fact that
\[ \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle m_i m_j = \sum_{i,j=1}^{l} \langle m_i \alpha_i^\vee, m_j \alpha_j^\vee \rangle = \langle \alpha_{l+1}^\vee, \alpha_{l+1}^\vee \rangle. \]
\[ \square \]

Another important step will be made by the following version of Kim’s Lemma (see [9] or [3]):

**Lemma 6.5** Let
\[ g = g_0 + \sum_{d>0} g_d e^{td} \in \mathbb{R}\{\{t_i\}\}[\{\{e^t\}\}] \]
be a formal series which satisfies
\[ g_0 = 0 \quad \text{and} \quad H(e^{t_j}, h\partial_i, h)g = 0. \]
Then \( g_d = 0 \) for any \( d \) with \( |d| \leq \sum_{i=1}^{l} m_i. \)

**Proof.** Take \( d \in \mathbb{N}^{l+1} \) with \( g_d \neq 0 \) and \( |d| := \sum_{i=1}^{l+1} d_i > 0 \) minimal. From \( H(e^{t_j}, h\partial_i, h)g = 0 \) it follows that
\[ \left( \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \partial_{ij}^2 \right)(g_d e^{td}) = 0 \]
and then
\[ \left( \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \partial_{ij}^2 \right)(e^{td}) = 0. \]

On the other hand, we have that
\[ (\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \partial_{ij}^2)(e^{td}) = \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle (d_i - m_i d_{l+1}) (d_j - m_j d_{l+1}) e^{td} = \| \sum_{j=1}^{l} (d_j - m_j d_{l+1}) \alpha_j^\vee \|^2 e^{td}, \]
hence we must have \( d_i = m_i d_{l+1}, \) for all \( 1 \leq i \leq l. \) From the fact that \( d \neq 0, \) it follows that \( d_{l+1} \geq 1 \) and \( d_i \geq m_i \) for any \( 1 \leq i \leq l \) and finally that
\[ |d| \geq 1 + \sum_{i=1}^{l} m_i. \]
The assertion stated in the lemma follows from the minimality of \( |d|. \) \[ \square \]
Proof of Theorem 5.1. We show that
\[ g := D(e^{t_j}, h\partial_i, h)(1, s_w) = \sum_{d \geq 0} g_d e^{td} \]
vanishes. Put first \( f = 1 \) in (19) and obtain
\[ g = \langle D(e^{t_j}, . . . e^{t_{l+1}}, p^*(\lambda_1) \bullet + h\partial_1, . . . , p^*(\lambda_l) \bullet + h\partial_l, h)1, s_w \rangle = \langle D(0, . . . , 0, p^*(\lambda_1) \bullet, . . . , p^*(\lambda_l) \bullet, h) + R, s_w \rangle, \]
where \( R \equiv 0 \mod \{e^{t_j}\} \). Hence \( g_0 \in \mathbb{R}[t_1, . . . , t_l] \) will be the same as
\[ \langle D(0, . . . , 0, p^*(\lambda_1), . . . , p^*(\lambda_l), h), s_w \rangle_0 \]
where, as usual, the subscript 0 indicates the degree zero term with respect to \( e^{t_1}, . . . , e^{t_{l+1}} \). But
\[ D(0, . . . , 0, p^*(\lambda_1), . . . , p^*(\lambda_l), h) = p^*(D(0, . . . , 0, \lambda_1, . . . , \lambda_l, h)) = 0, \]
hence \( g_0 = 0 \). If
\[ M := \sum_{i=1}^{l} m_i, \]
then \( g_d = 0 \), for \( |d| \leq M \) (we take into account that \( [D(e^{t_j}, h\partial_i, h), H(e^{t_j}, h\partial_i, h)] = 0, \) and also that \( H(e^{t_j}, h\partial_i, h)(1, s_w) = 0, \) which imply that \( H(e^{t_j}, h\partial_i, h)g = 0 \) and we apply Lemma 6.5). Finally, from Lemma 6.4 it follows that
\[ D(e^{t_j}, \lambda_i \bullet, 0) = 0 \]
and the proof is complete. \( \Box \)

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