Trees, valuations and the Green-Lazarsfeld set.

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1 Introduction.

The aim of this paper is the study of the relationship between two objects, the Green-Lazarsfeld set and the Bieri Neumann Strebel invariant, which appear simultaneously in 1987 ([GL], [BNS]). Let us recall some basic definitions.

Let Γ be a finitely generated group, and K be a field. A 1-character χ is an homomorphism from Γ to K∗; in this article we will only consider 1-characters, and call them characters. A character χ is called exceptional if H₁(Γ, χ) ≠ 0, or more geometrically if χ can be realized as the linear part of a fixed point free affine action of Γ on a K-line.

The set of exceptional characters, E¹(Γ, K) is a subset of the abelian group Hom(Γ, K∗), and our aim is to understand its geometry, in particular if Γ is the fundamental group of a Kähler manifold.

Motivated by the pioneering work of M. Green and R. Lazarsfeld, algebraic geometers studied the case where K = C is the field of complex numbers, and Γ = π₁(X) is the fundamental group of a projective or more generally a Kähler manifold. In this case, the geometry of Hom(Γ, K∗) is well understood: it is the union of a finite set, made up with torsion characters, and a finite set of translates of subtori. This result has been conjectured by A. Beauville and F. Catanese, ([Be]) proved by C. Simpson [Si 2]) for projective manifolds, and extended by F. Campana to the Kähler case (see [Ca] for a detailed introduction). The main tools used by C. Simpson were the flat hyper-Kähler structure of Hom(Γ, C∗) and the Schneider-Lang theorem in transcendence theory. Another proof, model theoretic, has been proposed by R. Pink and D. Roessler [PR].

The definition of a exceptional class in the sense of Bieri Neumann Strebel is easier to explain in the case of an integral cohomology class (an element of H¹(Γ, Z)). Such a class is exceptional if it can be realized as the translation class of a parabolic, non loxodromic action of Γ in some tree.

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The link between these two notions, explained in the next paragraph, can be sketched as follows. Let \( \chi \) be a Green-Lazarsfeld character of \( \Gamma \). Suppose \( \chi(\Gamma) \) is not contained in the ring of algebraic integers of \( K \). There exists a discrete non archimedian valuation on \( K \) such that \( v \circ \chi \) is a non trivial homomorphism to \( \mathbb{Z} \). It appears that \( v \circ \chi \) is an exceptional class in the sense of Bieri Neumann Strebel. More precisely, on can find a parabolic action of \( \Gamma \) on the Bruhat Tits tree of \( K_v \), with translation length \( v \circ \chi \).

Due to the work of C. Simpson [Si 3], M. Gromov and R. Shoen [GS], exceptional cohomology classes on Kähler manifold are well understood (see also [De] for a detailed study of the BNS invariant of a Kähler group). Let \( X \) be a Kähler manifold, and \( \omega \) an exceptional class; there exists a holomorphic map \( F \) from \( X \) to a hyperbolic Riemann orbifold \( \Sigma \) such that \( \omega \) belongs to \( F^*H^1(\Sigma, \mathbb{Z}) \). Recall that a complex 2-orbifold \( \Sigma \) is a Riemann surface \( S \) marked by a finite set of marked points \( \{(q_1, m_1), \ldots, (q_n, m_n)\} \), where the \( m_i \)'s are integers \( \geq 2 \). A map \( F : X \to \Sigma \) is called holomorphic if it is holomorphic in the usual sense, and for every \( q_i \) the multiplicity of the fiber \( F^{-1}(q_i) \) is divisible by \( m_i \).

The main result of this paper is a description of the (generalized) Green Lazarsfeld set of \( \pi_1(X) \) in terms of the finite list of its fibrations on hyperbolic 2-orbifolds.

**Theorem** Let \( \Gamma \) be the fundamental group of a Kähler manifold \( X \), \( (\Sigma_i)_{1 \leq i \leq n} \) the family of fibration of \( X \) over hyperbolic 2-orbifolds. Let \( K \) be a field of characteristic \( p \) (if \( p = 0, K = \mathbb{C} \)), \( F_p \subset K \) the algebraic closure of \( F_p \) in \( K \). Then \( E^1(\Gamma, K) \) is the union of a finite set of torsion characters (contained in \( E^1(\Gamma, \bar{F}_p) \) if \( p > 0 \)) and the union \( \bigcup_{1 \leq i \leq n} F_i^*E^1(\pi_1^{orb}(\Sigma_i), K^*) \).

**Remarks**

a) Let \( \Sigma = (S; (q_i, m_i))_{1 \leq i \leq n} \) a hyperbolic 2-orbifold, and \( \Gamma = \pi_1^{orb}(\Sigma) \) its fundamental group. Then, by a simple computation (see prop.22), one checks that \( E^1(\pi_1^{orb}(\Sigma), K^*) = \text{Hom}(\pi_1^{orb}(\Sigma), K^*) = (K^*)^{2g} \times \Phi \), where \( \Phi \) is a finite abelian group, unless \( g = 1 \) and for all \( i, m_i \neq 0(\text{char } K) \). If \( g = 1 \) and for all \( i, m_i \neq 0(\text{char } K) \), \( E^1(\pi_1^{orb}(\Sigma), K^*) \) is finite, made of torsion characters.

In every cases, the Green Lazarsfeld set is the union of a finite set of torsion characters and a finite set of abelian groups which are translates of tori; this is our generalization of Simpson’s theorem.

b) The main tool used by Simpson to prove his theorem [Si 2] was the study of algebraic triple tori; if \( \text{char } K \neq 0 \) no such a structure is available. Our proof furnishes a geometric (i.e. non arithmetic) alternative to Simpson’s proof in the case of characteristic 0. In fact, in this case (\( \text{car } K = 0 \)) our method proves that \( E^1(\Gamma, K) \) is made with a finite set of integral characters (in the sense of Bass [Ba]), and the union \( \bigcup_{1 \leq i \leq n} F_i^*\text{Hom}(\pi_1^{orb}(\Sigma_i), K^*) \); the conclusion follows from the study of the the absolute value \( |\chi| \) of exceptional characters, which was already done by A. Beauville [Be].

In a recent preprint, [CS], C. Simpson and K. Corlette study the variety of characters of a Kähler group \( \Gamma, \text{ Hom}^{ab}(\Gamma, \text{PSL}(2, \mathbb{C}))/\text{PSL}(2, \mathbb{C}) \) from a very similar point of view; they prove in particular that a Zariski dense representation...
of a Kähler group which is not integral in the sense of Bass factorizes through a fibration over a hyperbolic 2-orbifold. Their proof is based on the same idea as ours: if a representation $\rho$ is not integral, there exists a valuation on the field generated by $\rho(\Gamma)$ such that the action of $\Gamma$ on the Bruhat Tits building is non elementary. The conclusion follows by applying the theory of Gromov Shoen on harmonic maps with value in a tree. Using Simpson’s work on Higgs bundles they prove further a rigid representation come from a complex variation of Hodge structure.

In paragraph 2, we explain the relationship between the Green-Lazarsfeld and Bieri-Neumann-Strebel invariants; in paragraph 3 we study the Green-Lazarsfeld set of a metabelian group: a finiteness result on this set is established. These two paragraphs are purely group theoretic, and no Kähler structure is mentioned. In the paragraph 4 we prove the main result.

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2 From an affine action on a line to a parabolic action on a tree.

2.1 Affine action on the line: the Green-Lazarsfeld set

Let $K$ be a field. The affine group of transformations of a $K$-line, $\text{Aff}_1(K)$, is isomorphic to $K^* \rtimes K$. We identify this group with the set of upper triangular $(2,2)$ matrices $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ with values in $K$.

Let $\Gamma$ be a finitely generated group. An affine action of $\Gamma$ on the line is a morphism $\rho: \Gamma \to \text{Aff}_1(K)$. One can write $\rho(g) = \begin{pmatrix} \chi(g) & \theta(g) \\ 0 & 1 \end{pmatrix}$. The linear part of $\rho$ is an homomorphism $\chi: \Gamma \to K^*$. Its translation part $\theta: \Gamma \to K$ is a 1-cocycle of $\Gamma$ with value in $\chi$, i.e. a function which satisfies $\theta(gh) = \theta(g) + \chi(g)\theta(h)$. The representation $\rho$ is conjugate to a diagonal representation if and only if $\rho(\Gamma)$ fixes a point $\mu \in K$, or equivalently if and only if there exists a $\mu \in K$ such that $\theta(g) = \mu(-1 + \chi(g))$ is a coboundary.

**Definition 1** A character $\chi \in \text{Hom}(\Gamma, K^*)$ is exceptional if it can be realized as the linear part of a fixed point free affine action of $\Gamma$ on the line, i.e. if $H^1(\Gamma, \chi) \neq 0$. The set of exceptional characters $E^1(\Gamma, K)$ is called the Green-Lazarsfeld set of $\Gamma$.

2.2 Parabolic action on a tree: the Bieri Neumann Strebel invariant.

Let $T$ be a simplicial tree. We endow $T$ with its natural simplicial metric, and think of $T$ as a complete geodesic space. Let us recall the definitions of
the boundary of \( T \), and of the Busemann cocyle associated to a point in this boundary.

A ray in \( T \) is an isometric map \( r : [a, +\infty] \to T \). Two rays \( r : [a, +\infty] \to T \), \( s : [b, +\infty] \to T \) are equivalent (or asymptotic) if they coincide after a certain time: there exists \( a', b' \) s.t. for all \( t \geq 0 \), \( r(a' + t) = s(b' + t) \). The boundary of \( T \), denoted \( \partial T \), is the set of equivalence classes of rays. If \( \alpha \in \partial T \) and \( r : [a, +\infty] \to T \) represents \( \alpha \), for every point \( x \), the function \( t \to d(x, r(t)) - t \) is eventually constant. Its limit \( b_r(x) \) is called the Busemann function of \( r \). If \( s \) is equivalent to \( r \), the difference \( b_r - b_s \) is a constant.

**Definition 2** (Busemann cocyle). Let \( \Gamma \) be a group acting on \( T \), and \( \alpha \in \partial T \). If \( \Gamma \) fixes \( \alpha \), one define an homomorphism, the Busemann cocyle, by the formula:

\[
\omega_\alpha : \Gamma \to \mathbb{Z}
\]

\[
\omega_\alpha(g) = b_r \circ g - b_r.
\]

**Definition 3** (Exceptional classes) The action of \( \Gamma \) is called parabolic if it fixes some point at infinity. It is called exceptional if fixes a unique point at infinity, and if the associated Busemann cocyle is not trivial. A class \( \omega \in H^1(\Gamma, \mathbb{Z}) \) is exceptional if it can be realized as the Busemann cocycle of an exceptional action of \( \Gamma \) in some tree. The set of exceptional classes is denoted \( E^1(\Gamma, \mathbb{Z}) \).

**Remark 1** A topological definition of a exceptional class can also be given, in the case where \( \Gamma \) is finitely presented. Let \( \Gamma = \pi_1(X) \), where \( X \) is a compact manifold, and let \( \omega \) be some class in \( H^1(\Gamma, \mathbb{Z}) \). One represents \( \omega \) by a closed 1-form \( w \) on \( X \) and considered a primitive \( F : \tilde{X} \to \mathbb{R} \) of the lift of \( w \) to the universal cover of \( X \). Then \( \omega \) is exceptional iff \( F \geq 0 \) has several components on which \( F \) is unbounded (see [Bi], [Le], [Bro]).

**Remark 2** The notion of an exceptional class, defined by Bieri Neumann Strebel and studied by several authors, in particular [Bro], [Le], is more general: it concerns homomorphism with value in \( \mathbb{R} \) and can be defined along the same lines, using \( \mathbb{R} \)-trees instead of combinatorial trees. Our point of view is that of Brown; it is interesting to remark that [Bro], [BNS] and [GL] are published in the same issue of the same journal, but apparently nobody remarked that [Bro] and [GL] studied the same object from a different point of view. This remark justify the choice of our title.

### 2.3 Discrete valuations and Bruhat-Tits trees.

In this paragraph we fix a field \( K \). Let \( v : K^* \to \mathbb{Z} \) be a discrete non archimedian valuation on \( K \). Bruhat and Tits [BT] constructed a tree \( T_v \) with an action of \( \text{PGL}(2, K) \). One should think of the action of \( \text{PGL}(2, K) \) of \( T_v \) as an analogue
of the action of \( \text{PGL}(2, \mathbb{C}) \) on the hyperbolic space of dimension 3; we recall below some basic facts about this action (see [Se] for a detailed study).

Let \( O_v \subset K \) denote the valuation ring \( v \geq 0 \). The vertices of \( T_v \) are the homothety classes of \( O_v \)-lattices, i.e., free \( O_v \)-modules of rank 2, in \( K^2 \). The boundary of this tree is the projective line \( P^1(K_v) \) over the \( v \)-completion of \( K \).

By the general theory of lattices, if \( \Lambda, \Lambda' \) are two lattices, one can find a \( O_v \)-base of \( \Lambda \) such that, in this base, \( \Lambda' \) is generated by \((t^a,0)\) and \((0,t^b)\) for some \( t \) with \( v(t) = 1 \); hence up to homothety by \((1,0)\) and \((0,t^n)\), for \( n = b - a \). Then the distance between \( \Lambda \) and \( \Lambda' \) is \(|n|\), and the segment between \( \Lambda \) and \( \Lambda' \) is the set of lattices generated by \((1,0)\) and \((0,t^k)\), \( k = 1, n \). More generally if \( l, l' \) are two different lines in \( K^2 \), considered as points in \( \partial T_v \), the geodesic from \( l \) to \( l' \) is the set of product of lattices in \( l \) and \( l' \).

The matrix \( g_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \) fixes the lattice \( \Lambda_n \) generated by \((1,0)\) and \((0,t^n)\) for \( n \leq v(u) \). The matrix \( g_u = \begin{pmatrix} t^n & u \\ 0 & 1 \end{pmatrix} \) transforms \( \Lambda_m \) to \( \Lambda_{m+n} \) if \( m+n \leq v(u) \).

Acting on \( T_v \) the Borel sub-group \( \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \) is parabolic: it fixes an end of \( T_v \) (namely the line generated by the first basis vector), but neither a point of \( T_v \) nor a pair of points of \( \partial T_v \).

The Busemann cocycle of this parabolic subgroup is \( b \left( \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \right) = v(\alpha) \).

The relation between the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant is now simple to explain.

**Proposition 1** Let \( \chi \in H^1(\Gamma, K^*) \). Suppose that \( \chi \in E^1(\Gamma, K^*) \) and let \( \theta \in H^1(\Gamma, \chi) \neq 0 \). Let \( \rho : \Gamma \to \text{Gl}(2, K) \) be defined by \( \rho(g) = \begin{pmatrix} \chi(g) & \theta(g) \\ 0 & 1 \end{pmatrix} \). If \( v \circ \chi \in H^1(\Gamma, \mathbb{Z}) \) is not 0, \( \rho \) is an exceptional action on \( T_v \).

**Proof.** By construction the action of \( \Gamma \) on \( T_v \) fixes a point at infinity. It contains an hyperbolic element as \( v \circ \chi \neq 0 \), but the action cannot fix a line: the other point in the boundary \( P^1(K_v) \) would be fixed by the group \( \Gamma \), and \( \rho \) would be conjugate to diagonalizable action. The orbit of any point of \( \Gamma \) is therefore a minimal tree which is not a line. \( \square \)

### 3 Metabelian groups

If \( \Gamma \) is a group, let \( \Gamma' = [\Gamma, \Gamma] \) its derived group. Recall that a group is *metabelian* if \( \Gamma' \) is abelian, or \( \Gamma^2 = (\Gamma')' \) is trivial. If \( \Gamma \) is a f.g. group, \( \Gamma/\Gamma^2 \) is metabelian.

#### 3.1 The Green-Lazarsfeld set of a metabelian group.

If \( K \) is a field, the Green-Lazarsfeld set \( E^1(\Gamma, K) \) of the group \( \Gamma \) only depends on its metabelianized \( \Gamma/\Gamma^2 \) as it only depends of the set of representation of \( \Gamma \).
in the metabelian group $\text{Aff}_1(K) = K^* \ltimes K$.

Let $\Gamma$ be a metabelian group. We write $1 \to [\Gamma, \Gamma] \to \Gamma \to Q \to 1$, where $Q = \Gamma/[\Gamma, \Gamma]$ is the abelianized group, and $[\Gamma, \Gamma]$ is abelian. As an abelian group, $M = [\Gamma, \Gamma]$ is not necessary f.g, however we can let $Q$ acts on $[\Gamma, \Gamma]$ by conjugation, so that $M$ can be promoted as a $\mathbb{Z}Q$ module. The following fact is basic and well-known.

**Lemma 1** The module $M$ is finitely generated as a $\mathbb{Z}Q$ module.

If $g_1, \ldots, g_r$ are generators of $\Gamma$, the commutators $h_{ij} = [g_i, g_j]$ generate $[\Gamma, \Gamma]$ as a $\mathbb{Z}Q$ module : if $[g, h]$ if $h = ab$ we have $[g, h] = [g, ab] = gag^{-1}abg^{-1}b^{-1}a^{-1} = [g, a]a[g, b]a^{-1} = [g, a][g, b]$, and the result follows by induction. □

**Theorem 1** Let $\Gamma$ be a finitely generated group. Given a prime number $p$ ($p$ might be 0), there exists a finite number of fields $K_\nu$ of characteristic $p$ and of finite transcendence degree over $F_p$ (if $p = 0$, set $F_p = \mathbb{Q}$) and characters $\xi_\nu : \Gamma \to K_\nu^*$ such that :

1. $H^1(\Gamma, \xi_\nu) \neq 0$, i.e. $\xi_\nu \in E^1(\Gamma, K_\nu)$
2. If $K$ is a field of characteristic $p$ and $\chi \in E^1(\Gamma, K)$ a Green-Lazarsfeld character, then there exists an index $\nu$ s.t. $\ker \chi \supset \ker \xi_\nu$.

**Proof** Let $F_p$ be the field with $p$ elements and $F_p[Q]$ the group ring of $Q$ with $F_p$ coefficients. Let $M_p = [\Gamma, \Gamma] \otimes F_p, J \subset F_p[Q]$ the annihilator of $M_p$, and $A = F_p[Q]/J$. As $Q$ is a finitely generated abelian group, isomorphic to $\mathbb{Z}^r \times \Phi$, with $\Phi$ finite abelian, $A$ is a noetherian ring. Thus $A$ admits a finite number of minimal prime ideals $(p_\nu)_1 \leq \nu \leq \nu_1$. Let $k_i$ be the field of fraction of $A/p_i$, and $\xi_i$ be the natural character $\Gamma \to Q \to A/p_i \to k_i$. Up to re-ordering the list of these ideals, we may assume that for $1 \leq i \leq \nu_1, H^1(\Gamma, \xi_i) \neq 0$.

The theorem 8 is a consequence of the following :

**Lemma 2** Let $\chi \in E^1(\Gamma, K)$ be an exceptional character, $\chi \neq 1$, and let $p$ be a minimal prime ideal contained in $\ker \chi$. Then, the character $\xi_p$ belongs to $E^1(\Gamma, k_\nu^*)$, i.e. $H^1(\Gamma, \xi) \neq 0$.

Let $M_p = M \otimes A_p$, and $M_0 = M \otimes A K = M_p/pM_p$. Note that $M_0$ is a finitely generated $k_p$ vector space, on which $\Gamma$ acts by homotheties : the action of $g$ is the homothety of ratio $\xi(g)$. Let $\pi : [\Gamma, \Gamma] \to M_0$ the canonical map. We shall prove that $H^1(\Gamma, M) \neq 0$.

For some $g_0 \in \Gamma, \xi(g_0)$ is not 1 (as an element of $k_p$) : if not $\Gamma = \ker \xi_p$ so $\chi = 1$.

The map $\Gamma \to M_0$ defined by $c(g) = \pi(g_0g_0^{-1}g^{-1})$ satisfies $c(gh) = \pi(g_0gh_0^{-1}h^{-1}g^{-1}) = \pi(g_0g_0^{-1}g^{-1}) + \pi(g_0h_0^{-1}h^{-1}g^{-1}) = c(g) + \xi(g)\pi(g_0h_0^{-1}h^{-1}) = c(g) + \xi(g)c(h)$.

Therefore $c$ is a 1-cocycle of $\Gamma$ with value in $M$.

Let us prove, by contradiction, that the cohomology class of $c$ is not 0.
For every $m \in M_0$, $c(m) = (\xi(g_0)m - m) = (\xi(g_0) - 1)m$. If $c = 0$, as $\xi(g_0) \neq 1$, then $M_0 = 0$. But if $M_0 = 0, M_p/pM_p = 0$, i.e. $pM_p = M_p$, and $M = pM$. But $p \subset \ker \chi$, so this would implies that $M \otimes_A K = 0$ and $H^1(\Gamma, \chi) = 0$.

If this cocyle is a coboundary we could find some $m \in M_0$ s.t. $c(g) = (1 - \xi(g))m$, but $c(g_0) = 0$, and $\xi(g_0) \neq 1$, so $c$ would be 0.

In order to prove lemma 9, we see that, for every linear map $l = M_0 \to K$, $l \circ c$ is a non trivial $1 -$ cocycle. This proves theorem 8.

\[ \square \]

Remark 3 The previous proof is a combination of arguments by [BG] and [Bre]. In their remarkable paper R. Bieri and J. Groves describe the BNS invariant of a metabelian group in terms of the finite set of field $k_\nu$ and characters $\xi_\nu$ for a finite set of primes $p$ (the primes $p$ for which $[\Gamma, \Gamma]$ has $p$-torsion). For every such a field and every valuation on it, $v \circ \xi_\nu$ is exceptional. This provide a map from the cone of valuations on $k_\nu$ to the BNS set. This set turns out to be the union of the images of these cones. In [Bre], Breuillard proves along the same lines, that a metabelian not virtually nilpotent group admits a non trivial affine action.

4 Fundamental groups of Kähler manifolds.

4.1 Fibering a Kähler manifold.

For the general study of orbifolds and their fundamental groups, we refer to W. Thurston [Th] chap. 13. Complex 2-orbifolds are 2-orbifolds with singularities modeled on the quotient of the unit disk by the action of $\mathbb{Z}/n\mathbb{Z}$. The usefulness of this notion in our context of (fibering complex manifolds to Riemann surfaces) has been pointed out by C. Simpson [Si1].

Definition 4 Complex 2-orbifold, and holomorphic maps. A complex 2-orbifold $\Sigma$ is a Riemann surface $S$ marked by a finite set of marked points $\{(q_1, m_1), \ldots, (q_n, m_n)\}$, where the $m_i$'s are integers $\geq 2$.

Let $X$ be a complex manifold, $f : X \to \Sigma$ a map. Let $x \in X, q = f(x)$. Let $m \in \mathbb{N}^*$ be the multiplicity of $q$, so that there exists an holomorphic map $u : D(0, r) \subset \mathbb{C} \to (\Sigma, q)$ which is a ramified cover of order $m$ of a neighborhood of $q$. Then, $f$ is called holomorphic at $x$, if there exists a neighborhood $U$ of $q$ and a lift $\tilde{f} : U \to D$, holomorphic at $x$ such that $f = u \circ \tilde{f}$.

Definition 5 Fundamental group. Let $\Sigma = (S; \{(q_1, m_1), \ldots, (q_n, m_n)\})$ be a 2-orbifold. Let $q \in S \setminus \{(q_1, m_1), \ldots, (q_n, m_n)\}$. The fundamental group -in the sense of orbifolds- of $\Sigma$ at the point $p$ is the quotient $\pi_1^{\text{orb}}(\Sigma, p) = \pi_1(S \setminus \{q_1, \ldots, q_n\})/ \ll$
\( \gamma_i^{m_i} \gg \), where \( \gamma_i \) is the class of homotopy (well defined up to conjugacy) of a small circle turning once around \( q_i \), and \( \ll \gamma_i^{m_i} \gg \) is the normal subgroup generated by all the conjugates of \( \gamma_i^{m_i} \).

**Example 1** (This is the main example, see [Th] chap. 13) Let \( \Gamma \subset \text{PSL}(2, \mathbb{R}) \) be a uniform (discrete co-compact) lattice. The quotient \( S = D/\Gamma \) of the unit disk by the action of \( \Gamma \) is a Riemann surface. If \( p \in D \), its stabilizer is a finite hence cyclic subgroup of \( \text{PSL}(2, \mathbb{R}) \). Modulo the action of \( \Gamma \) there are only a finite set of points \( \{q_1, \ldots, q_n\} \) with non trivial stabilizers of order \( m_i \). The quotient orbifold is \( \Sigma = (S; \{(q_1, m_1), \ldots, (q_n, m_n)\}) \). One proves that \( \Gamma = \pi\text{orb}^1(\Sigma) \). An orbifold is called hyperbolic if it is obtained in this way; an orbifold is hyperbolic if and only if its Euler characteristic \( \chi(\Sigma) = \chi(S) - \Sigma_1 \leq i \leq n (1 - \frac{1}{m_i}) \) is non positive.

The following definition is useful to understand the structure of Kähler groups (see [ABCKT]).

**Definition 6** A Kähler manifold \( X \) fibers if there exists a pair \((\Sigma, F)\) where \( \Sigma = (S; \{(q_1, m_1), \ldots, (q_n, m_n)\})\) is a hyperbolic 2-orbifold, and \( F : X \to \Sigma \) an holomorphic map with connected fibers. Two such maps \( F : X \to \Sigma, F' : X' \to \Sigma' \) are equivalent if the fibers of \( F \) and \( F' \) are the same and images in \( \Sigma \) and \( \Sigma' \) of singular fibers have same order. In this case there exists an holomorphic isomorphism from \( S \) to \( S' \) which maps singular points of \( S \) to singular points of \( S' \) preserving the multiplicity.

Let \( \pi : X \to S \) be an holomorphic map from a compact complex surface to a curve. If \( q \in S \) is a singular value of \( \pi \), the analytic set \( \pi^{-1}(q) \) can be decomposed in a finite union of irreducible sets, \( (D_i) \). Away from a set of complex dimension \( n - 2 \) in \( D_i \), hence of complex codimension 2 in \( X \), the map \( p \) can by written \( \pi(z_1, \ldots, z_n) = z_1^{d_1} \), where \( d_i \) is the multiplicity of \( D_i \). The multiplicity of the fiber \( \pi^{-1}(q) \) is by definition \( m = \text{pgcd}(d_i) \). Let \( \Sigma \) be the orbifold whose underlying space is \( S \), singular points are singular values of \( \pi \) with corresponding multiplicity.

**Lemma 3** \( \pi : X \to \Sigma \) is holomorphic.

By construction, locally in the neighborhood of a point of \( \pi^{-1}(q) \), \( \pi(x) = f_1^{d_1} \cdots f_k^{d_k} + \text{cte} \), with \( m|\text{pgcd}(d_i) \).

The following finiteness theorem is well-known in the smooth case, and implicit in the literature at several places; we give below a short proof based on the hyperbolic geometry of hyperbolic orbifolds.

**Theorem 2** Let \( X \) be a compact complex manifold. There exists, up to equivalence, a finite set of pair \((\Sigma_i, F_i)\) where \( \Sigma_i \) is a complex hyperbolic 2-orbifold, \( F_i : X \to \Sigma_i \) is holomorphic with connected fibers.
Let us give a proof of this (well known) fact based on the Kobayashi-hyperbolicity of a hyperbolic 2 orbifold: there exist no holomorphic map from \( \mathbb{C} \) to an hyperbolic 2-orbifold as there exists no holomorphic map from \( \mathbb{C} \) to the unit disk. Thus, by the Bloch principle, as \( X \) is compact there exists a uniform bound on the differential of an holomorphic map \( F : X \to \Sigma \). Therefore the set of pairs \( (F, \Sigma) \) is compact (two such orbifold are \( \varepsilon \)-close if they are close for the Gromov-Hausdorff topology, i.e. there exists a map between them which is isometric up to an error of \( \varepsilon \)). But this compact space has only isolated points: if \( F_1 : X \to \Sigma_1 \) is given, and, the (Gromov-Hausdorff) distance of \( F \) to \( F_1 \) is smaller than the diameter of \( \Sigma_1 \) (for instance \( \leq \frac{1}{2} \text{diam}(X) \)) where \( X \) is endowed the Kobayashi pseudo-metric) all the fibers of \( F_1 \) are send by \( F \) inside a disk (or an annulus in the case of the Margulis constant) therefore to a constant by the maximum principle; in other words \( F \) factorizes through \( F_1 \) and induces an isomorphism between \( \Sigma \) and \( \Sigma_1 \).

\[ \square \]

Remark 4 This proof shows that the number of pairs \( (F, \Sigma) \) for a given complex manifold \( X \) can be bounded by the Kobayashi diameter of \( X \).

The following is well known (see [Si 1] [CKO]).

Theorem 3 Let \( F : X \to S \) by an holomorphic map with connected fibers from the complex manifold \( X \) to a complex curve \( S \). Let \( \Sigma \) be the orbifold whose singular points are the singular values of \( p \) and multiplicity the multiplicity of the corresponding fiber. Let \( Y = F^{-1}(b) \) be the fiber of a non singular point of \( S \). Let \( \pi_1(Y) \) the image in \( \pi_1(X) \) of \( \pi_1(Y) \). One has the exact sequence

\[ 1 \to \pi_1(Y) \to \pi_1(X) \to \pi_1^{orb}(\Sigma) \to 1 \]

in particular the kernel of \( \pi_* : \pi_1(X) \to \pi_1^{orb}(\Sigma) \) is finitely generated. \( \square \)

4.2 Valuations.

The next result is a reformulation of a fibration theorem of Gromov-Shoen [GS] and Simpson [Si 3] in terms of the exceptional set in the sense of Bieri Neumann Strebel; see also [De] for a more general study of the BNS invariant of a Kähler group, where \( \omega \in H^1(\Gamma, \mathbb{C}) \) rather than \( H^1(\Gamma, \mathbb{Z}) \).

Theorem 4 Let \( \omega \in H^1(\Gamma, \mathbb{Z}) \). Then \( \omega \) is exceptional iff there exist a hyperbolic orbifold \( \Sigma \), an holomorphic map \( F : X \to \Sigma \) such that \( \omega \in F^*H^1(\Sigma, \mathbb{Z}) \).

Let \( \eta \) be a closed holomorphic \((1, 0)\) form whose real part is the harmonic representative of \( \omega \). Let \( \tilde{X} \) the universal cover of \( X \), and \( F : \tilde{X} \to \mathbb{R} \) a primitive of \( \text{Re}\eta \). From the definition (Remark 5) of \( \mathcal{E}^1 \) we know that \( F \geq 0 \) is not connected; [Si 3] applies. One can also apply the proof of corollary 9.2 of [GS] to the foliation defined by the complex valued closed \((1, 0)\) form whose real part is the harmonic representative of \( \omega \).

To prove the converse (which will not be used), one remarks that for every \( w \in H^1(\Sigma, \mathbb{Z}) \), its pull back to \( H^1(\Sigma, \mathbb{Z}) \) is exceptional, as \( \pi_1^{orb}(\Sigma) \) is hyperbolic, and the kernel of \( \pi_1^{orb}(\Sigma) \to \mathbb{Z} \) cannot be finitely generated. \( \square \)
4.3 The Green-Lazarsfeld set of a Kähler group.

Let $K$ be a field. Recall that a character $\chi : \Gamma \to K^*$ is called integral in the sense of Bass [Ba] if $\chi(\Gamma) \subset O$, the ring of algebraic integers of $K$.

**Proposition 2** Let $X$ be a Kähler manifold, $\chi \in E^1(\Gamma, K^*)$ be a character. If $\chi$ is not integral, $X$ fibers over a 2-orbifold $\Sigma$ such that $\chi \in F^*E^1(\pi_1^{\text{orb}}(\Sigma), K^*)$.

Proof. Let $v$ be some valuation such that $w = v \circ \chi \neq 0$. Let $\Gamma$ acts on $T_v$. By prop.6 this action is exceptional. Applying Thm. 18 we get a pair $F, \Sigma$ such that $w \in F^*H^1(\Sigma, \mathbb{Z})$. From the exact sequence of Theorem 17, we see that $\pi_1'(Y)$ is a finitely generated normal subgroup of $\Gamma$ made up with elliptic elements. As $\pi_1'(Y)$ is finitely generated, the subtree of $T_v$ made up with fixed points of $\pi_1'(F)$ is not empty. As $\pi_1'(Y)$ is normal, it is invariant by the action of $\Gamma$. Therefore the boundary of this tree contains at least 3 distinct elements. Thus acting on $P^1(K)$ $\pi_1'(Y)$ fixes three different points and is the identity : $\pi_1'(Y) \subset \ker \rho$, and $\rho$ descends to some character on $\pi_1^{\text{orb}}(\Sigma)$. □

The following proposition is a reformulation of a result by Beauville [Be] (Cor 3.6), it will be used to study the cohomology class of $v \circ \chi$, for the archimedian valuation $v(z) = \ln |z|$ an $\chi : \Gamma \to \mathbb{C}^*$ a character.

**Proposition 3** Let $X$ be a Kähler manifold, $\chi \in E^1(\Gamma, \mathbb{C}^*)$ be character. If $|\chi| \neq 1$, there exist an holomorphic map $F : X \to \Sigma$ from $X$ to a 2-orbifold $\Sigma$ such that $\chi \in F^*E^1(\pi_1^{\text{orb}}(\Sigma), K^*)$.

Combining propositions 19 and 20, we get the description of the GL set of a Kähler manifold in terms of its fibering over hyperbolic 2-orbifolds. It generalizes results by M. Green R. Lazarsfeld [GL], A. Beauville [Be], C. Simpson [Si], F. Campana [Ca], R. Pink D. Roessler [PR], who studied the case where the field $K$ is the field of complex numbers.

**Theorem 5** Let $\Gamma$ be the fundamental group of a Kähler manifold $X$, $(F_i, \Sigma_i)_{1 \leq i \leq n}$ the family of fibration of $X$ over hyperbolic 2-orbifolds. Let $K$ be a field of characteristic $p$ (if $p = 0$, $K = \mathbb{C}$), $\bar{F}_p \subset K$ the algebraic closure of $F_p$ in $K$. Then $E^1(\Gamma, K)$ is made with a finite set of torsion characters (contained in $E^1(\Gamma, \bar{F}_p)$ if $p > 0$) and the union of $F_i^* \text{Hom}(\pi_1^{\text{orb}}(\Sigma_i), K^*)$.

**Proof** We shall prove that a character $\chi$ which is not in the union $\bigcup F_i^* \text{Hom}(\pi_1^{\text{orb}}(\Sigma_i), K^*)$ must be a torsion character of bounded order. Let us fix such a character $\chi$.

From theorem 8, we know that there exists a finite number of fields $K_\nu$ and characters $\xi_\nu$ such that $H^1(\Gamma, \xi_\nu) \neq 0$, and for every $\chi \in E^1(\Gamma, K)$ there exists an index $\nu$ for which $\ker \xi_\nu \subset \ker \chi$. If $\xi_\nu$ is not integral, there exists a 2-orbifold $\Sigma$ and a holomorphic map $F : X \to \Sigma$ such that $\ker F_\nu \supset \ker \xi_\nu$ : therefore $\ker F_\nu \supset \ker \chi$ and $\chi \in F^*E^1(\pi_1^{\text{orb}}(\Sigma))$.

Thus, as $\chi \notin \bigcup F_i^* \text{Hom}(\pi_1^{\text{orb}}(\Sigma_i), K^*)$ $\chi$ is integral.

Let us first discuss the case of positive characteristic. If $\xi_\nu$ is integral, then $\xi_\nu(\Gamma)$ is made with roots of unity of $K_\nu$. But $K_\nu$ is of finite transcendence
degree over $F_p$ so admits only a finite number of roots of unity of degree $d_\nu$ (see [Ba] for instance). Therefore, $\chi$ is a torsion character of order $d$ dividing $d_\nu$.

Suppose now that $\text{char } K = 0$, and $\xi_\nu$ is integral. Thus $K_\nu$ is a number field, and $\xi_\nu(\Gamma)$ is contained in the ring $O_\nu$ of integers of $\xi_\nu$. If $[\xi_\nu] \neq 1$, or if one of its conjugates $\sigma(\xi_\nu)$ has $|\sigma(\xi_\nu)| \neq 1$, as $H^1(\Gamma, \xi_\nu) \neq 0$ we know (prop. 20) that there exists a 2-orbifold $\Sigma$ and a holomorphic map $F : X \to \Sigma$ such that $\ker F_* \supset \ker \xi_\nu$; the previous argument apply and proves that $\chi \in F^* E^1(\pi_1(\Sigma))$. Therefore, $\chi$ must be a root of unity, by a theorem of Kronecker, of bounded degree $d$, as the degree of the $n$-th cyclotomic polynomial goes to infinity with $n$, and as $d$ divides the degree of $K_\nu$. The rest of the argument is unchanged.

Thus, the theorem 21 reduces the computation of $E^1(\Gamma, K^*)$ to the case where $\Gamma$ is the fundamental group of a 2-orbifold.

**Proposition 4** Let $\Gamma = \pi_1(\Sigma)$, for $\Sigma = (S; (q_i, m_i)_{i \in S_n})$ a hyperbolic 2-orbifold then,

$$E^1(\pi_1^{\text{orb}}(\Sigma), K^*) = \text{Hom}(\pi_1^{\text{orb}}(\Sigma), K^*)$$

unless $g = 1$ and for all $i$, $m_i \neq 0(\text{char } K)$.

If $g = 1$ and for all $i$ $m_i \neq 0(\text{char } K)$, $E^1(\pi_1^{\text{orb}}(\Sigma), K^*)$ is finite, made of torsion characters.

Let $\chi : \pi_1^{\text{orb}}(\Sigma) \to K^*$ be a representation. If $\chi = 1$, $H^1(\pi_1^{\text{orb}}(\Sigma), K^*) = \text{Hom}(\pi_1^{\text{orb}}(\Sigma), K^*) \neq 0$. If $g > 1$, consider a simple closed curves on $S$ such that $c$ are homologous to 0, which separated $S$ in two compact surface of positive genus $S_1, S_2$, with common boundary $c$ and such that all singular points are in $S_2$; if $g = 1$ consider a curve $c$, which bounds a disk $D$ on $S$ containing all singular points $q_i$, and let $S_1 = S \setminus \text{int}(D)$ be the other component. One consider a representation $\chi : \pi_1^{\text{orb}}(\Sigma) \to K^*$, and note that $\chi(c) = 1$ as $c$ is homologous to 0. We think of $\chi$ as a local system on $\Sigma$ and we will use a Mayer Vietoris exact sequence.

First note that if $\chi|_{\pi_1(S_1)}$ and $\chi|_{\pi_1^{\text{orb}}(S_2)}$ are not 1, then $H^1(\pi_1^{\text{orb}}(\Sigma), K^*) \neq 0$:

- let $x_0 \in K$, there exists a unique 1-cocyle $c$ such that $c(g) = x_0(1 - \chi(g))$ is $g \in \pi_1(S_1)$, $c(g) = 0$ if $g \in S_2$.

- If $\chi|_{S_1} = 1$, as $H^1(S_1, \partial S_1, K) = K^2$, one can find a 1-cocyle $c$ whose restriction on $S_2$ or $D$ is 0, and restriction on $S_1$ is not trivial.

We are left to the case $\chi|_{S_2}$ or $\chi|_D = 1$. If $g(S_2) > 0H^1(\pi_1^{\text{orb}}(\Sigma_2), C, K) \to K^{2g}$ and the previous argument apply.

The remaining case is $g = 1$, $\chi|_{\pi_1^{\text{orb}}(D)} = 1$, $\chi|_{\pi_1(S_1)} \neq 1$. Note that in this case $\chi$ is a torsion character. Furthermore, $H^1(\pi_1^{\text{orb}}(\Sigma_2), K) = \{(z_1, \ldots, z_n) \in K/m_i z_i = 0\}$. This space is 0 unless $m_i \equiv 0$ (char $K$) for some $i$. On the other hand, if $\rho|_{\pi_1(S_1)} \neq 0$ the homomorphism $H^1(\pi_1(\Sigma_1), \rho) \to K$ which sends $\theta$ to $\theta(c)$ is an isomorphism. Using the exact sequence of Mayer Vietoris, we see that $H^1(\pi_1^{\text{orb}}(\Sigma_1), \chi) \neq 0$ if $g > 1$ or $g = 1$ and for some $m_i$, $m_i$ divides the characteristic of $K$. □
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