REAL HAMILTONIAN FORMS OF AFFINE TODA FIELD THEORIES: SPECTRAL ASPECTS

V. S. Gerdjikov∗†, G. G. Grahovski‡, and A. A. Stefanov∗§

The paper is devoted to real Hamiltonian forms of 2-dimensional Toda field theories related to exceptional simple Lie algebras, and to the spectral theory of the associated Lax operators. Real Hamiltonian forms are a special type of “reductions” of Hamiltonian systems, similar to real forms of semisimple Lie algebras. Examples of real Hamiltonian forms of affine Toda field theories related to exceptional complex untwisted affine Kac–Moody algebras are studied. Along with the associated Lax representations, we also formulate the relevant Riemann–Hilbert problems and derive the minimal sets of scattering data that uniquely determine the scattering matrices and the potentials of the Lax operators.

Keywords: real Hamiltonian form, 2-dimensional Toda field theory, exceptional Lie algebras, spectral properties of Lax operators

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1. Introduction

Affine Toda field theories [1]–[4] received considerable attention of the mathematical physics community in the past three decades and are one of the best understood integrable massive field theories at classical and quantum levels in 1 + 1 dimensions [5]. The interest in Toda field theories was inspired by the work of Zamolodchikov [6] on integrability-preserving deformations of conformal field theories. It was shown in [6] that the resultant theory is characterized by eight masses related to the Cartan matrix of $E_8$ and by integrals of motion with spins given by the exponents of $E_8$ modulo its Coxeter number.

Affine Toda field theories (ATFTs) are integrable models of a real scalar field $q = (q_1, \ldots, q_n)$ in one spatial dimension with exponential interactions. The Lagrangian of the theory is given by

$$\mathcal{L}[q] = \frac{1}{2} (\partial_\mu q \cdot \partial^\mu q) - \frac{m^2}{\beta^2} \sum_{j=0}^n n_j (e^{\beta(\alpha_j, q)} - 1),$$

(1)
where the field $\mathbf{q}(x,t)$ is an $n$-dimensional vector. This corresponds to the equations of motion

$$\frac{\partial^2 \mathbf{q}}{\partial x \partial t} = \sum_{j=0}^{n} n_j \alpha_j e^{-(\alpha_j, \mathbf{q}(x,t))}.$$  \hfill (2)

Each ATFT is associated with a (finite) simple Lie algebra $\mathfrak{g}$ [5], [7]. Here, $n$ is the rank of $\mathfrak{g}$, the $\alpha_k$ ($k = 1, \ldots, n$) are the simple roots of $\mathfrak{g}$, and $\alpha_0$ is the minimal root. Thus, Lagrangian (1) is described by the extended Dynkin diagram of the associated affine algebra $\hat{\mathfrak{g}}$. The fields $q_k$ can be rescaled such that $\beta$ only appears in $L$ through a common factor $1/\beta^2$ (expanding in powers of $\beta^2$ is equivalent in quantum theory to expanding in $\hbar$) [2].

The ATFTs associated with the $B_n$, $C_n$, $F_4$, and $G_2$ algebras follow by “folding” the corresponding Dynkin diagrams. The same process allows obtaining solutions for the $B_n$, $C_n$, $F_4$, and $G_2$ theories from the classical solutions with special symmetries of the parent $A_n$, $D_n$, and $E_n$ theories.

An important feature of affine Toda field theories is their integrability by the inverse scattering method (ISM) [8], [9]. The starting point is the existence of the so-called Lax operator (see (3) below). The interpretation of the ISM as a generalized Fourier transform [10]–[12] allows studying all the fundamental properties of the corresponding nonlinear evolutionary equations (NLEEs):

1. the description of the class of NLEEs related to a given Lax operator $L(\lambda)$ and solvable by the ISM;
2. an infinite family of the integrals of motion;
3. the hierarchy of Hamiltonian structures [13];
4. the description of gauge-equivalent systems [14]–[16].

Real Hamiltonian forms (RHF)s are another type of “reductions” of Hamiltonian systems. The extraction of RHF$s is similar to obtaining a real form of a semisimple Lie algebra. The Killing form for the latter is indefinite in general (it is negative-definite for compact real forms). It is therefore unsurprising that RHF$s then follow with an indefinite kinetic energy quadratic form. Of course, this is an obstruction to their quantization.

The purpose of this paper is to outline the spectral theory of the Lax operators of real Hamiltonian forms of affine Toda field theories related to complex untwisted exceptional affine Lie algebras.

The structure of the paper is as follows. In Sec. 2, we give a brief summary of the Lax representation of ATFTs and the necessary Lie algebraic background knowledge. As regards the root systems of exceptional algebras, we are following the conventions in [17]. In Sec. 3, we first describe the general method for constructing RHF$s and then briefly describe the sets of admissible roots of the exceptional Lie algebras and the explicit constructions of their RHF$s. Section 4 is devoted to the spectral theory of the Lax operators of ATFTs with $\mathbb{Z}_h$-reductions, where $h$ is the Coxeter number of $\mathfrak{g}$. We first provide the general construction of the fundamental analytic solutions (FAS$s) \xi_\nu(x,t,\lambda)$ of the Lax operator and show their relevance to the Riemann–Hilbert problem (RHP) on a set of $2h_g$ rays $l_\nu$ closing the angles $\pi/h_g$. Then we introduce the asymptotics of the FAS $\xi_\nu(x,t,\lambda)$ for $x \to \pm \infty$ and $\lambda \in l_\nu$, which determine the scattering data of $L$. That section ends with a theorem specifying the minimal sets of scattering data for a generic choice of the simple Lie algebra $\mathfrak{g}$. Section 5 contains the specific data concerning each of the exceptional Lie algebra needed to determine the minimal set of scattering data in each case. We conclude in Sec. 6.

2. Affine Toda field theories: preliminaries

To each simple Lie algebra $\mathfrak{g}$, one can relate a Toda field theory in $1 + 1$ dimensions. It admits a Lax representation in a zero-curvature form $[L, M] = 0$, where $L$ and $M$ are first-order ordinary differential
(Lax) operators,

\[
L \psi \equiv \left( i \frac{d}{dx} - i q(x,t) - \lambda J_0 \right) \psi(x,t,\lambda) = 0,
\]

\[
M \psi \equiv \left( i \frac{d}{dt} - \frac{1}{\lambda} I(x,t) \right) \psi(x,t,\lambda) = 0,
\]

whose potentials take values in \( \mathfrak{g} \). Here, \( q(x,t) \in \mathfrak{h} \) is the Cartan subalgebra of \( \mathfrak{g} \), and \( q(x,t) = (q_1, \ldots, q_r) \) is its dual \( r \)-component vector (\( r = \text{rank} \mathfrak{g} \)). The potentials of the Lax operators are chosen as

\[
J_0 = \sum_{\alpha \in \pi} E_\alpha, \quad I(x,t) = \sum_{\alpha \in \pi} e^{-(\alpha, q(x,t))} E_{-\alpha},
\]

where \( \pi_\mathfrak{g} \) stands for the set of admissible roots of \( \mathfrak{g} \), i.e., \( \pi_\mathfrak{g} = \{\alpha_0, \alpha_1, \ldots, \alpha_r\} \), with \( \alpha_1, \ldots, \alpha_r \) being the simple roots of \( \mathfrak{g} \) and \( \alpha_0 \) being the minimal root of \( \mathfrak{g} \). The corresponding Toda field theory is known as an affine Toda field theory (ATFT). The Dynkin graph corresponding to the set of admissible roots \( \pi_\mathfrak{g} = \{\alpha_0, \alpha_1, \ldots, \alpha_r\} \) of \( \mathfrak{g} \) is called the extended Dynkin diagram (EDD). The equations of motion are of the form

\[
\frac{\partial^2 q}{\partial x \partial t} = \sum_{j=0}^r n_j \alpha_j e^{-(\alpha_j, q(x,t))},
\]

where \( n_1, \ldots, n_r \) are the minimal positive integer coefficients that provide the decomposition of the \( \alpha_0 \) with respect to the simple roots of \( \mathfrak{g} \):

\[
-\alpha_0 = \sum_{j=1}^r n_j \alpha_j.
\]

It is well known that ATFT models are infinite-dimensional Hamiltonian systems. The (canonical) Hamiltonian structure is given by

\[
H_\mathfrak{g} = \int_{-\infty}^{\infty} dx \mathcal{H}_\mathfrak{g}(x,t), \quad \mathcal{H}_\mathfrak{g}(x,t) = \frac{1}{2} \langle \mathbf{p}(x,t), \mathbf{p}(x,t) \rangle + \sum_{k=0}^r n_k (e^{-(q(x,t), \alpha_k)} - 1),
\]

\[
\Omega_\mathfrak{g} = \int_{-\infty}^{\infty} dx \omega_\mathfrak{g}(x,t), \quad \omega_\mathfrak{g}(x,t) = (\delta \mathbf{p}(x,t) \wedge \delta \mathbf{q}(x,t)),
\]

where \( H_\mathfrak{g} \) is the canonical Hamilton function and \( \Omega_\mathfrak{g} \) is the canonical symplectic structure. Here also \( \mathbf{p} = dq/dt \) are the canonical momenta and their coordinate variables satisfying the canonical Poisson bracket relations

\[
\{q_k(x,t), p_j(y,t)\} = \delta_{jk} \delta(x-y).
\]

The infinite-dimensional phase space \( \mathcal{M} = \{\mathbf{q}(x,t), \mathbf{p}(x,t)\} \) is spanned by the canonical coordinates and momenta.

### 3. Real Hamiltonian forms and affine Toda field theories

The Lax representations of ATFT models widely discussed in the literature (see, e.g., [1], [3], [18], [19] and the references therein) are related mostly to the normal real forms of Lie algebras \( \mathfrak{g} \), see [20]. Here, we study real Hamiltonian forms of the ATFT models. The notion of an RHF was introduced in [21] and used to study reductions of ATFTs in [22]. After a brief outline of the basic theory (following [21]), in this Section we describe the RHFs for ATFT models related to exceptional untwisted complex Kac–Moody algebras [23]–[25].
The densities of the corresponding Hamiltonian and symplectic form are
\[ q^C = q^0 + iq^1, \quad p^C = p^0 + ip^1. \] (9)

Next, we introduce an involution \( C \) acting on the phase space \( \mathcal{M} \equiv \{ q_k(x), p_k(x) \}_{k=1}^{r} \) as
\[
\begin{align*}
C(F(p_k, q_k)) &= F(C(p_k), C(q_k)), \quad (10a) \\
C(\{F(p_k, q_k), G(p_k, q_k)\}) &= \{C(F), C(G)\}, \quad (10b) \\
C(H(p_k, q_k)) &= H(p_k, q_k). \quad (10c)
\end{align*}
\]

Here, \( F(p_k, q_k), G(p_k, q_k) \), and the Hamiltonian \( H(p_k, q_k) \) are functionals on \( \mathcal{M} \) depending analytically on the fields \( q_k(x, t) \) and \( p_k(x, t) \).

The complexification of the ATFT is rather straightforward. The resulting complex ATFT (CATFT) can be written as a standard Hamiltonian system with twice as many fields \( q^a(x, t), p^a(x, t), a = 0, 1 \):
\[
\begin{align*}
p^C(x, t) &= p^0(x, t) + ip^1(x, t), \quad q^C(x, t) = q^0(x, t) + iq^1(x, t), \\
\{q^0_k(x, t), p^0_j(y, t)\} &= -\{q^1_k(x, t), p^1_j(y, t)\} = \delta_{kj}(x - y). \quad (11)
\end{align*}
\]

The densities of the corresponding Hamiltonian and symplectic form are
\[
\begin{align*}
\mathcal{H}^C_{\text{ATFT}} &= \text{Re} \mathcal{H}^C_{\text{ATFT}}(p^0 + ip^1, q^0 + iq^1) = \frac{1}{2}(p^0 \cdot p^0) - \frac{1}{2}(p^1 \cdot p^1) + \sum_{k=1}^{r} e^{-(q^0, \alpha_k)} \cos((q^1, \alpha_k)), \\
\omega^C &= (dp^0 \wedge dq^0) - (dp^1 \wedge dq^1). \quad (12)
\end{align*}
\]

A family of RHFs is then obtained from the CATFT by imposing an invariance condition under the involution \( \tilde{C} \equiv C \circ * \), where \( * \) denotes complex conjugation. The involution \( \tilde{C} \) splits the phase space \( \mathcal{M}^C \) into a direct sum \( \mathcal{M}^C = \mathcal{M}^C_0 \oplus \mathcal{M}^C_1 \), where
\[
\begin{align*}
\mathcal{M}^C_+ &= \mathcal{M}_0 \oplus i \mathcal{M}_1, \quad \mathcal{M}^C_- = \mathcal{M}_0 \oplus -i \mathcal{M}_1, \\
C(q^+ + iq^-) &= q^+ - iq^-, \quad C(p^+ + ip^-) = p^+ - ip^- \quad (13)
\end{align*}
\]

Each involution \( C \) also induces an involution (involutive automorphism) \( C^\# \) of \( g \). In what follows, we choose \( C^\# \) in a special way related to the Coxeter automorphism defined in its dihedral form in Eq. (21) below (see [17], [20], [26]). Indeed, we define \( C^\# \) using the reflection \( S_1 \) with respect to the set of black roots of the Dynkin diagram (see Fig. 1). More precisely,
\[
\begin{align*}
C^\# \alpha_j = \alpha_j \quad \text{for} \quad \alpha_j \in W_g, \\
C^\# \beta_j = -\beta_j \quad \text{for} \quad \beta_j \in B_g, \\
p^0(x, t) &= \sum_{\alpha_j \in W_g} p_j(x, t) \alpha_j, \quad p^1(x, t) = \sum_{\beta_j \in B_g} p_j(x, t) \beta_j, \\
q^0(x, t) &= \sum_{\alpha_j \in W_g} q_j(x, t) \alpha_j, \quad q^1(x, t) = \sum_{\beta_j \in B_g} q_j(x, t) \beta_j, \quad (14)
\end{align*}
\]

where \( W_g \) (resp. \( B_g \)) is the set of white roots (resp. black roots) of the Dynkin diagram.

To each involution \( C \), we can thus relate an RHF of the ATFT. We note that \( C^\# \) preserves the system of admissible roots of \( g \) and the EDDs of \( g \) studied in [19]. Indeed, it follows from (10c) that
\[
(C(q, \alpha) = (q, C^\#(\alpha)), \quad \alpha \in \pi_g, \quad (15)
\]
and therefore we must have $C(\pi_g) = \pi_g$. As a result,

$$
\begin{align*}
C(p^0(x, t), \alpha_j) &= (p^0(x, t), \alpha_j), & C(p^1(x, t), \beta_j) &= -(p^1(x, t), \beta_j), \\
(p^0(x, t), \beta_j) &= 0, & (p^1(x, t), \alpha_j) &= 0,
\end{align*}
$$

(16)

where $\alpha_j \in W_g$ and $\beta_j \in B_g$. Using the ideas in [18], we then obtain the following result for the RHF of the ATFT related to $g$:

$$
\begin{align*}
\mathcal{H}_g^\mathbb{R} &= \frac{1}{2}(p^+, p^+) - \frac{1}{2}(p^-, p^-) + \sum_{\alpha_k \in W_g} n_k'(e^{-(q^+, \alpha_k)} - 1) + \sum_{\beta_k \in B_g} n_k''(\cos(q^-, \beta_k) - 1), \\
\omega_g^\mathbb{R} &= (\delta p^+ \wedge \delta q^+) - (\delta p^- \wedge \delta q^-),
\end{align*}
$$

(17)

The Hamiltonian, along with the related to the simple roots, also contains the term corresponding to the minimal root $-\alpha_0$. It is well known that the maximal root $\alpha_0$ can be decomposed into a sum of simple roots with integer nonnegative coefficients:

$$
\alpha_0 = \sum_{j=1}^r n_j \alpha_j = \sum_{\alpha_k \in W_g} n_k' \alpha_k + \sum_{\beta_j \in B_g} n_k'' \beta_j.
$$

(18)

In the second expression above, we separated the terms with the white and black roots.

The RHF's of ATFT are more general integrable systems than the models described in [19], [27]–[29], which involve only the fields $q^+$ and $p^+$ invariant under $C$.

3.2. Affine Toda field theories related to $E_6^{(1)}$. The set of admissible roots for this algebra is

$$
\begin{align*}
\alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), & \alpha_2 &= e_1 + e_2, \\
\alpha_3 &= e_2 - e_1, & \alpha_4 &= e_3 - e_2, & \alpha_5 &= e_4 - e_3, & \alpha_6 &= e_5 - e_4, \\
\alpha_0 &= -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8).
\end{align*}
$$

(19)

Here, $\alpha_1, \ldots, \alpha_6$ form the set of simple roots of $E_6$ and $\alpha_0$ is the minimal root of the algebra. The EDD of $E_6^{(1)}$ is shown in Fig. 1. This is the standard definition of the root system of $E_6$ embedded into the 8-dimensional Euclidean space $\mathbb{E}^8$. The root space $\mathbb{E}_6$ of $E_6$ is the 6-dimensional subspace of $\mathbb{E}^8$ orthogonal

![Fig. 1. Extended Dynkin diagram of the complex untwisted affine Kac–Moody algebra $E_6^{(1)}$. The white roots are invariant under the automorphism $S_2$ in (21).](image-url)

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to the vectors $e_7 + e_8$ and $e_6 + e_7 + 2e_8$. Therefore, any vector $q$ belonging to $\mathbb{E}_6$ has only six independent coordinates and can be written as

$$q = \sum_{k=1}^{5} q_k e_k + q_6 e_6', \quad e_6' = \frac{1}{\sqrt{3}}(e_6 + e_7 - e_8). \quad (20)$$

The fundamental weights of $E_6$ are

$$\omega_1 = \frac{2}{3}(e_8 - e_7 - e_6), \quad \omega_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8),$$
$$\omega_3 = \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + \frac{5}{6}(e_8 - e_7 - e_6), \quad \omega_4 = e_3 + e_4 + e_5 - e_6 - e_7 + e_8,$$
$$\omega_5 = e_4 + e_5 + \frac{2}{3}(e_8 - e_7 - e_6), \quad \omega_6 = e_5 + \frac{1}{3}(e_8 - e_7 - e_6).$$

We use the dihedral realization of the Coxeter automorphism for $E_6^{(1)}$:

$$\text{Cox}(E_6^{(1)}) = S_1 \circ S_2, \quad S_1 = S_{\alpha_2} \circ S_{\alpha_3} \circ S_{\alpha_5}, \quad S_2 = S_{\alpha_1} \circ S_{\alpha_4} \circ S_{\alpha_6}. \quad (21)$$

If we require the invariance of the Hamiltonian and the symplectic form (17) under $S_1$ and restrict to the set of admissible roots $\beta_6$, then we obtain an RHF of the $E_6^{(1)}$ ATFT, described by

$$H_{E_6}^R = \frac{1}{2}(p^+, p^+) - \frac{1}{2}(p^-, p^-) + \sum_{k=0,1,4,6} n'_k (e^{-q^{+, \alpha_k}} - 1) + \sum_{k=2,3,5} n''_k (\cos(q^-, \beta_k) - 1), \quad (22)$$

where $n'_1 = n'_5 = 1$, $n'_3 = 3$, $n''_2 = n''_5 = n''_6 = 2 [17]$, and

$$\omega_{E_6}^R = (\delta p^+ \wedge \delta q^+) - (\delta p^- \wedge \delta q^-). \quad (23)$$

3.3. Affine Toda field theories related to $E_7^{(1)}$. The extended root system of this algebra is

$$\begin{align*}
\alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \quad \alpha_2 = e_1 + e_2, \\
\alpha_3 &= e_2 - e_1, \quad \alpha_4 = e_3 - e_2, \quad \alpha_5 = e_4 - e_3, \\
\alpha_6 &= e_5 - e_4, \quad \alpha_7 = e_6 - e_5, \quad \alpha_0 = e_7 - e_8,
\end{align*} \quad (24)$$

where $\alpha_1, \ldots, \alpha_7$ form the set of simple roots of $E_7$ and $\alpha_0$ is the minimal root of the algebra. The EDD of $E_7^{(1)}$ is shown in Fig. 2. This is the standard definition of the root system of $E_7$ embedded into the 8-dimensional Euclidean space $\mathbb{E}_8$. The $E_7$ root space $\mathbb{E}_7$ is the 7-dimensional subspace of $\mathbb{E}_8$ orthogonal

![Fig. 2.](image)

The white roots are invariant under the automorphism $S_1$ in (26).
to the vector \( e_7 + e_8 \). Therefore, any vector \( q \) belonging to \( E_7 \) has 7 independent coordinates and can be written as

\[
q = \sum_{k=1}^{6} q_k e_k + q_7 e'_7, \quad e'_7 = \frac{1}{\sqrt{2}}(e_7 - e_8).
\]

Using the dihedral realization of the Coxeter automorphism for \( E_7^{(1)} \),

\[
\text{Cox}(E_7^{(1)}) = S_1 \circ S_2, \quad S_1 = S_{\alpha_1} \circ S_{\alpha_4} \circ S_{\alpha_6}, \quad S_2 = S_{\alpha_2} \circ S_{\alpha_3} \circ S_{\alpha_5} \circ S_{\alpha_7},
\]

we require that Hamiltonian (17) be invariant under the automorphism \( S_1 \) in (26). Then the root space \( E_7 \) splits into a direct sum

\[
E_7 = E_{7,+} \oplus E_{7,-}, \quad \text{with} \quad \alpha_2, \alpha_3, \alpha_5, \alpha_7 \in E_{7,+}, \quad \alpha_1, \alpha_4, \alpha_6 \in E_{7,-}.
\]

If we again require the invariance of the Hamiltonian and symplectic form (17) under \( S_1 \) in (26), and restrict to the set of admissible roots \( \beta_k \), then we obtain an RHF of the \( E_7^{(1)} \) ATFT, described by

\[
H_{E_7}^R = \frac{1}{2}(p^+, p^+) - \frac{1}{2}(p^-, p^-) + \sum_{k=0,2,3,5,7} n'_k(e^{-q^+ \cdot \alpha_k} - 1) + \sum_{k=1,4,6} n''_k(\cos(q^-, \beta_k) - 1),
\]

where \( n'_1 = n'_6 = 2, n'_4 = 4, n''_2 = 2, n''_3 = n''_5 = 3, n''_7 = 1 \), and

\[
\omega_{E_7^{(1)}} = (\delta p^+ \wedge \delta q^+) - (\delta p^- \wedge \delta q^-).
\]

### 3.4. Affine Toda field theories related to \( E_8^{(1)} \)

The set of admissible roots for this algebra is

\[
\begin{align*}
\alpha_0 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8), \\
\alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \\
\alpha_2 &= e_1 + e_2, \quad \alpha_3 = e_2 - e_1, \quad \alpha_4 = e_3 - e_2, \\
\alpha_5 &= e_4 - e_3, \quad \alpha_6 = e_5 - e_4, \quad \alpha_7 = e_6 - e_5, \quad \alpha_8 = e_7 - e_6,
\end{align*}
\]

where \( \alpha_1, \ldots, \alpha_8 \) form the set of simple roots of \( E_8 \) and \( \alpha_0 \) is the minimal root of the algebra. If we exclude \( \alpha_8 \) from (29), we obtain the set of positive \( E_7 \) roots (24). The EDD of \( E_8^{(1)} \) is shown in Fig. 3.

Fig. 3. Extended Dynkin diagram of the complex untwisted affine Kac–Moody algebra \( E_8^{(1)} \). The white roots are invariant under the automorphism \( S_2 \) in (30).

The fundamental weights of \( E_8 \) are

\[
\begin{align*}
\omega_1 &= 2e_8, \quad \omega_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + 5e_8), \\
\omega_3 &= \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + 7e_8), \quad \omega_4 = e_3 + e_4 + e_5 + e_6 + e_7 + 5e_8, \\
\omega_5 &= e_4 + e_5 + e_6 + e_7 + 4e_8, \quad \omega_6 = e_5 + e_6 + e_7 + 3e_8, \\
\omega_7 &= e_6 + e_7 + 2e_8, \quad \omega_8 = e_7 + e_8 = -\alpha_0.
\end{align*}
\]
We use the dihedral realization of the Coxeter automorphism for $E_8^{(1)}$:
\[
\text{Cox}(E_8^{(1)}) = S_1 \circ S_2, \quad S_1 = S_{\alpha_4} \circ S_{\alpha_5} \circ S_{\alpha_7} \circ S_{\alpha_6}, \quad S_2 = S_{\alpha_2} \circ S_{\alpha_3} \circ S_{\alpha_5} \circ S_{\alpha_7}.
\] 

We require that Hamiltonian (17) be invariant under the automorphism $S_1$ in (26). Then the root space $E_8$ splits into a direct sum $E_8 = E_{8,+} \oplus E_{8,-}$, with
\[
E_8 = E_{8,+} \oplus E_{8,-}, \quad \text{with} \quad \alpha_2, \alpha_3, \alpha_5, \alpha_7 \in E_{8,+}, \quad \alpha_1, \alpha_4, \alpha_6, \alpha_8 \in E_{8,-}.
\]

Requiring the invariance of the Hamiltonian and symplectic form (17) under $S_1$ in (30) and restricting to the set of admissible roots $\beta_k$ results in an RHF of the $E_8^{(1)}$ ATFT, described by
\[
\mathcal{H}_E^{R} = \frac{1}{2}(p^+ p^-) - \frac{1}{2}(p^- p^+), \quad \sum_{k=0,2,4} n_k'(e^{-q^+,\alpha_k} - 1) + \sum_{k=1,3,6} n_k''(\cos(q^-,\beta_k) - 1),
\]
where $n_0' = n_0'' = 2, n_2' = 6, n_4' = 4, n_6' = 3, n_3'' = 4, n_5'' = 5$ [17], and
\[
\omega_{E_8^{(1)}} = (\delta p^+ \wedge \delta q^+) - (\delta p^- \wedge \delta q^-).
\]

### 3.5. Affine Toda field theories related to $F_4^{(1)}$

The extended roots of this algebra are
\[
\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4,
\]
\[
\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \quad \alpha_0 = -e_1 - e_2,
\]
with $\alpha_0$ being the minimal root. The EDD of $F_4^{(1)}$ is shown in Fig. 4.

![Extended Dynkin diagram of the complex untwisted affine Kac–Moody algebra $F_4^{(1)}$.](image)

The white roots are invariant under the automorphism $S_2$ in (35).

The fundamental weights of $F_4$ are
\[
\omega_1 = e_1 + e_2, \quad \omega_2 = 2e_1 + e_2 + e_3,
\]
\[
\omega_3 = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4), \quad \omega_4 = e_1.
\]

Using the dihedral realization of the Coxeter automorphism for $F_4^{(1)}$,
\[
\text{Cox}(F_4^{(1)}) = S_1 \circ S_2, \quad S_1 = S_{\alpha_4} \circ S_{\alpha_5} \circ S_{\alpha_7}, \quad S_2 = S_{\alpha_2} \circ S_{\alpha_3} \circ S_{\alpha_5},
\]
and imposing the invariance of Hamiltonian (17) under the automorphism $S_1$ in (35), we obtain an RHF of the $F_4^{(1)}$ ATFT in the form
\[
\mathcal{H}_E^{R} = \frac{1}{2} \sum_{k=1}^{4} ((p_k^+)^2 - (p_k^-)^2) + \sum_{k=0,2,4} n_k'(e^{-q^+,\alpha_k} - 1) + \sum_{k=1,3} n_k''(\cos(q^-,\beta_k) - 1),
\]
where $n_0' = 3, n_2' = 2, n_4' = 2, n_6' = 4$. In addition,
\[
\omega_{F_4^{(1)}} = (\delta p^+ \wedge \delta q^+) - (\delta p^- \wedge \delta q^-).
\]

Then the root space $E_4$ splits into a direct sum:
\[
E_4 = E_{4,+} \oplus E_{4,-}, \quad \text{with} \quad \alpha_2, \alpha_4 \in E_{4,+}, \quad \alpha_1, \alpha_3 \in E_{4,-}.
\]
3.6. **Affine Toda field theories related to** $G^{(1)}_{2}$. The set of admissible roots for this algebra is given by

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = -e_1 + e_2 + e_3, \quad \alpha_0 = -e_1 - e_2 + 2e_3. \quad (38)$$

The fundamental weights of $G_{2}$ are

$$\omega_1 = e_1 - e_2 + 2e_3, \quad \omega_2 = -e_1 - e_2 + 2e_3.$$ 

The EDD of $G^{(1)}_{2}$ is shown in Fig. 5.

**Fig. 5.** Extended Dynkin diagram of the complex untwisted affine Kac–Moody algebra $G^{(1)}_{2}$.

We set $C_{1} = S_{\alpha_2}$. The invariance of Hamiltonian (17) under the automorphism $C_{1}$ gives an RHF of the $G^{(1)}_{2}$ ATFT described by

$$H_{G^{(1)}_{2}}^{R} = \frac{1}{2} \sum_{k=1}^{r} ((p_{k}^{+})^2 - (p_{k}^{-})^2) + (e^{-\langle q^{+}, \alpha_0 \rangle} - 1) + 3(e^{-\langle q^{+}, \alpha_1 \rangle} - 1) + 2(\cos(q^{-}, \alpha_2) - 1), \quad (39)$$

$$\omega_{G^{(1)}_{2}}^{R} = (\delta p^{+} \wedge \delta q^{+}) - (\delta p^{-} \wedge \delta q^{-}).$$

The root space $E_2$ splits into a direct sum:

$$E_2 = E_{2,+} \oplus E_{2,-}, \text{ with } \alpha_0, \alpha_1 \in E_{2,+}, \quad \alpha_2 \in E_{2,-}.$$

4. **On the spectral properties of the Lax operators with $\mathbb{Z}_n$-reduction**

4.1. **General theory.** The operator $L$ (3) is very convenient for deriving a 2-dimensional TFT. However, this formulation is not convenient for constructing the FAS of $L$. The first step we take here is to subject $L$ to the similarity transformation

$$\tilde{L} \dot{\psi} = g_{0}^{-1} L g_{0} \dot{\psi} = i \frac{\partial \psi}{\partial x} - \left( i \sum_{k=0}^{6} q_{k} E_{k} + \lambda \tilde{J}_{0} \right) \dot{\psi} = 0 \quad (40)$$

where $\tilde{J}_{0} \in \mathfrak{h}$, which means that $\tilde{J}_{0}$ must be a linear combination

$$\tilde{J}_{0} = \sum_{j=1}^{6} y_{j} H_{\alpha_{j}} = \sum_{k=1}^{6} z_{j} H_{\omega_{j}},$$

where $H_{\alpha_{j}}$ and $H_{\omega_{j}}$ are dual to the simple roots $\alpha_{j}$ or the fundamental weights $\omega_{j}$, and $y_{j}$ and $z_{j}$ must be expressed in terms of $\omega = e^{2\pi i / h}$.

We somewhat simplify the notation for the two Lax operators

$$L \psi \equiv \frac{\partial \psi}{\partial x} + (\bar{q}_{x} - \lambda \tilde{J}_{0}) \psi(x, t, \lambda) = 0,$$

$$\tilde{L} \dot{\psi} \equiv \frac{\partial \dot{\psi}}{\partial x} + (Q(x, t) - \lambda \tilde{J}_{0}) \dot{\psi}(x, t, \lambda) = 0, \quad (41)$$

where $J_{0} = \sum_{\alpha \in \delta_{0}} E_{\alpha}$, $\delta_{0}$ is the set of simple roots $\alpha_{j}$ and the minimal root $\alpha_{0}$, and $\tilde{J}_{0} \in \mathfrak{h}$.

The spectral theory of the Lax operators with Mikhailov’s $\mathbb{Z}_n$-reduction groups [18] for the classical series of simple Lie algebras are by now well developed, see [12], [30], [31]. Here, we formulate them taking the peculiarities of exceptional algebras into account whenever necessary.
In [32], the problem of the interrelation between \(L\) and \(\hat{L}\) was solved by using the Chevalley basis for \(A_5^{(1)}\) [33]. For the classical series, the Chevalley bases for typical representations are well known, and it is therefore not difficult to diagonalize \(J_0\) and then evaluate \(\alpha_j(J_0)\). The same procedure for the exceptional algebras (except \(G_2\)) is somewhat more complicated. Indeed, the bases for the typical representations of \(F_4\), \(E_6\), and \(E_7\) are well known (see [34], where one can find all root vectors of \(F_4\) as \(26 \times 26\) matrices). For \(E_6\) and \(E_7\), the simple roots are given in [34]; it is possible that all root vectors for all exceptional algebras have been evaluated in the MAGMA project framework. In general, one can therefore find \(J_0\) for the typical representations of all exceptional algebras. Then the construction of \(\hat{J}_0\) becomes the next task. The eigenvalues of \(J_0\) and the eigenvectors can be calculated, although the eigenvectors would be rather involved expressions. The next challenge would be to construct a properly normalized eigenvector \(g_0\) and to ensure that it is an element of the corresponding exceptional group.

The easiest way to obtain a realization of the Coxeter automorphism \(C_g\) is to represent it as an element of the Cartan subgroup,

\[
C_g = \exp\left(\frac{2\pi i}{h_g} \sum_{j=1}^{r} H_{\omega_j}\right),
\]

where \(h_g\) is the Coxeter number and \(\omega_j\) are the fundamental weights of the algebra \(g\). Indeed, using the Cartan–Weyl commutation relations [20], we can easily verify that

\[
C_g J_0 C_g^{-1} = \omega J_0, \quad \omega = \exp\left(\frac{2\pi i}{h_g}\right).
\]

This construction is compatible with the Lax operator \(L\) because \(C_g\) commutes with \(\tilde{q}_x\) and therefore

\[
C_g L(\lambda) C_g^{-1} = L(\omega \lambda). \quad (44)
\]

Dealing with the operator \(\hat{L}\) requires the use of the Coxeter automorphism \(\tilde{C}_g\) realized as an element of the Weyl group \(W_g\). We already mentioned the difficulties involved with the standard approach based on the use of the Chevalley basis. Therefore, in discussing exceptional algebras, we use a different approach. Our idea is to work only in the Euclidean space \(E_r\) into which the root systems are embedded. There, we can construct \(\tilde{C}_g\) using the dihedral form [17], [20], [26],

\[
\tilde{C}_g = S_w \circ S_B, \quad S_w = \prod_{\alpha \in W_g} S_{\alpha}, \quad S_B = \prod_{\beta \in B_g} S_{\beta},
\]

where \(W_g\) (respectively \(B_g\)) are the sets of “white” (resp. “black”) roots of the Dynkin diagram of \(g\). We note that all elements of \(W_g\) (resp. \(B_g\)) are orthogonal, and hence \(S_w\) and \(S_B\) are Weyl reflections, i.e., \(S_w^2 = \mathbb{1}\) and \(S_B^2 = \mathbb{1}\). Then \(C_g^{h_g} = \mathbb{1}\). Thus, we find a realization of the Coxeter automorphism as an element of the Weyl group \(W_g\) of the algebra.

The next step should be to construct the Cartan subalgebra element \(\tilde{J}_0\). In fact, we bypass this step, and instead of constructing \(\tilde{J}_0\), we construct the vector \(\tilde{v}_g \in E_r\) that is dual to \(\tilde{J}_0\). Then we use the fact that \(\alpha_j(\tilde{J}_0) = (\alpha_j, \tilde{v}_g)\). This is all we need to construct the continuous spectrum and the spectral data for \(\hat{L}\). On the other hand, because \(\hat{L}\) is related to \(L\) by the similarity transformation \(\hat{L} = g_0 L g_0^{-1}\), both operators have the same spectral data.

The construction of the vector \(\tilde{v}_g\) is based on its basic property of invariance under Mikhailov’s \(Z_h\)-reduction group; as in (44), we must have

\[
\tilde{C}_g \hat{L}(\lambda) \tilde{C}_g^{-1} = \hat{L}(\omega \lambda), \quad \text{i.e.} \quad \tilde{C}_g(\tilde{v}_g) = \omega \tilde{v}_g.
\]

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Such invariant vectors can be obtained by taking weighted average of the action by $\tilde{C}_g$:

$$\vec{v}_0 = \sum_{s=0}^{h_g-1} \omega^{-s} \tilde{C}_s^g(\vec{v}_g).$$

(47)

Obviously, there is an arbitrariness in the choice of $\vec{v}_g$: it can be a root or a fundamental weight of $g$. At the same time, we can use the fact that the spectral parameter $\lambda$ can be redefined, to account for this arbitrariness.

4.2. The FAS of the Lax operators $L$. Here, we outline the procedure for constructing the FASs for $L$ [12], [30], [35]. First, we have to determine the analyticity regions. For smooth potentials $Q(x)$ that fall off fast enough as $x \to \pm \infty$, these regions are the $2h$ sectors $\Omega_\nu$ separated by the rays $l_\nu$ on which $\Re \lambda(\alpha, \vec{v}_g) = 0$, where $\alpha$ is a root of $g$ (see Fig. 6). The rays $l_\nu$ and the sectors $\Omega_\nu$ are given by

$$l_\nu: \arg(\lambda) = \frac{\pi(\nu - 1)}{h_g}, \quad \Omega_\nu: \frac{\pi(\nu - 1)}{h_g} \leq \arg(\lambda) \leq \frac{\nu}{h_g}, \quad \nu = 1, \ldots, 2h,$$

(48)

and close angles are equal to $\pi/h$. The main result is that in each of the sectors $\Omega_\nu$, we can construct an FAS $\xi_\nu(x, t, \lambda)$ for the Lax operator $L$:

$$L_{\xi_\nu}(x, t, \lambda) \equiv i\frac{\partial \xi_\nu}{\partial x} + Q_x \xi_\nu(x, \lambda) - \lambda [\tilde{J}, \xi_\nu] = 0,$$

$$Q_x = \sum_{j=1}^{r} q_{j,x} O_j, \quad O_j = \sum_{s=0}^{h_g-1} C_s^g(E_{\alpha_j}).$$

(49)

Fig. 6. Continuous spectrum of the Lax operators related to the algebras $E_6$ and $F_4$, whose Coxeter numbers equal 12. Obviously, the continuous spectra for $E_8$, $E_7$, and $G_2$ have different numbers of rays, respectively equal to 60, 36, and 12.
The asymptotics of $\xi^\nu(x, \lambda)$ and $\xi^{\nu-1}(x, \lambda)$ along the ray $l_\nu$ can be written in the form [12], [36]

$$
\begin{align*}
\lim_{x \to +\infty} e^{-\lambda J_x} \xi^\nu(x, \lambda e^{i0}) e^{\lambda J_x} &= S^+_\nu(\lambda), \\
\lim_{x \to -\infty} e^{-\lambda J_x} \xi^{\nu-1}(x, \lambda e^{-i0}) e^{\lambda J_x} &= S^-_\nu(\lambda), \\
\lim_{x \to +\infty} e^{-\lambda J_x} \xi^\nu(x, \lambda e^{i0}) e^{\lambda J_x} &= T^+_\nu D^+_\nu(\lambda), \\
\lim_{x \to -\infty} e^{-\lambda J_x} \xi^{\nu-1}(x, \lambda e^{-i0}) e^{\lambda J_x} &= T^-_\nu D^-_\nu(\lambda),
\end{align*}
$$

where the matrices $S^+_\nu$ and $T^+_\nu$ (resp. $S^-_\nu$ and $T^-_\nu$) are upper-triangular (resp. lower-triangular) with respect to the $\nu$-ordering [12], [30], [31], [37]. They provide the Gauss decomposition of the scattering matrix with respect to the $\nu$-ordering:

$$
T_\nu(\lambda) = \begin{cases} 
T^-_\nu(\lambda) D^-_\nu(\lambda) \tilde{S}^+_\nu(\lambda), & \lambda \in l_\nu e^{i0}, \\
T^+_\nu(\lambda) D^+_\nu(\lambda) \tilde{S}^-_\nu(\lambda), & \lambda \in l_\nu e^{-i0}.
\end{cases}
$$

A more careful analysis shows [12] that in fact $T_\nu(\lambda)$ belongs to a subgroup $G_\nu$ of $SL(n, \mathbb{C})$. In particular, we can choose $\lambda$ such that the simple roots of $\mathfrak{g}$ are related to the rays $l_0$ and $l_1$; more precisely, the black roots are related to $l_0$ and the white roots are related to $l_1$. In particular, this means that $l_0$ and $l_1$ can be associated with subalgebras of $\mathfrak{g}$ that are direct sums of several copies of $sl(2)$. In addition, we can introduce the analogues of the reflection coefficients $\rho^\pm_{\nu,j}$ and $\tau^\pm_{\nu,j}$ with $\nu = 0, 1$ as

$$
\begin{align*}
S^+_0(\lambda) &= \exp\left( \sum_{\alpha \in B_0} \tau^+_0(\lambda) E_{\pm\alpha} \right), & T^+_0(\lambda) &= \exp\left( \sum_{\alpha \in B_0} \rho^+_0(\lambda) E_{\mp\alpha} \right), \\
S^+_1(\lambda) &= \exp\left( \sum_{\alpha \in W_1} \tau^+_1(\lambda) E_{\pm\alpha} \right), & T^+_1(\lambda) &= \exp\left( \sum_{\alpha \in W_1} \rho^+_1(\lambda) E_{\mp\alpha} \right), \\
D^+_0(\lambda) &= \exp\left( \sum_{j=1}^r \frac{2d^+_0(\lambda)}{(\alpha_j, \alpha_j)} H_{\alpha_j} \right), & D^+_1(\lambda) &= \exp\left( \sum_{j=1}^r \frac{2d^+_1(\lambda)}{(\alpha_j, \alpha_j)} H_{\alpha_j} \right).
\end{align*}
$$

Next, the $\mathbb{Z}_h$ symmetry allows extending these scattering data to the other rays. Indeed, applying the Coxeter automorphism, we first obtain relations between the FASs,

$$
\begin{align*}
C_\nu^\nu \xi^\nu(x, \lambda) &= \xi^{\nu+2}(x, \lambda \omega), & C_\nu^\nu (T^\pm_\nu(\lambda)) &= T^{\nu+2}_\nu(\lambda \omega), \\
C_\nu^\nu (S^\pm_\nu(\lambda)) &= S^\pm_{\nu+2}(\lambda \omega), & C_\nu^\nu (D^\pm_\nu(\lambda)) &= D^\pm_{\nu+2}(\lambda \omega),
\end{align*}
$$

where the index $\nu + 2$ is to be taken modulo $2h$. Consequently, we only the data on two rays, e.g., $l_1$ and $l_0$, can be considered independent; all the rest will be recovered using (53).

### 4.3. The inverse scattering problem and the Riemann–Hilbert problem.

The next important step is the possibility to reduce the solution of the inverse scattering problem (ISP) for the generalized Zakharov–Shabat system to a (local) RHP. More precisely, we have

$$
\begin{align*}
\xi^\nu(x, t, \lambda) &= \xi^{\nu-1}(x, t, \lambda) G_\nu(x, t, \lambda), & \lambda \in l_\nu, \\
G_\nu(x, t, \lambda) &= e^{\lambda J_x - \lambda^{-1} I_t} G_{0,\nu}(\lambda) e^{-\lambda J_x + \lambda^{-1} I_t}, & G_{0,\nu}(\lambda) &= \tilde{S}^-_\nu S^+_\nu(\lambda) |_{t=0},
\end{align*}
$$

where $\tilde{I} = g_{0}^{-1} I(t) g_0$ and $\tilde{X} \equiv X^{-1}$. The collection of all relations (54) for $\nu = 1, \ldots, 2h$ together with

$$
\lim_{\lambda \to \infty} \xi^\nu(x, t, \lambda) = 1,
$$

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can be viewed as a local RHP posed on the collection of rays \( \Sigma \equiv \{ \ell_{\nu} \}_{\nu=1}^{2h} \) with canonical normalization. Rather straightforwardly, we can prove that if \( \xi^\nu(x, \lambda) \) is a solution of RHP (54), (55), then \( \chi^\nu(x, \lambda) = \xi^\nu(x, \lambda)e^{-M_{11}x} \) is an FAS of \( L \) with the potential

\[
Q_x(x, t) = \lim_{\lambda \to \infty} \lambda(\tilde{J} - \xi^\nu(x, t, \lambda)\tilde{J}\xi^\nu(x, t, \lambda)).
\]  

(56)

For the classical Lie algebras, the analyticity properties of \( d^\pm_{\nu,j}(\lambda) \) allow reconstructing them from the matching function \( G(\lambda) \) (54) and from the locations of their zeroes and poles \([12],[13]\). The idea was to consider relations (51) in the \( j \)th fundamental representation of \( g \) and take the matrix elements between the highest-weight vectors \( |\omega_j^+\rangle \) (resp. lowest-weight vectors \( |\omega_j^-\rangle \)). This can also be done for exceptional groups with the result

\[
d^+_{\nu,j}(\lambda) - d^-_{\nu,j}(\lambda) = -\ln(\omega_j^+|\hat{T}_\nu^+\hat{T}_\nu^-(\lambda)|\omega_j^+\rangle) = -\ln(\omega_j^-|\hat{S}_\nu^-\hat{S}_\nu^+(\lambda)|\omega_j^-\rangle).
\]  

(57)

If \( g \) is one of the classical Lie algebras, then the right-hand sides of (57) can be related to the principal upper and lower minors of the scattering matrix \( T(\lambda) \) in a typical representation. For the exceptional algebras, the fundamental representations cannot be related so easy to the typical (or lowest-dimension) representations. Therefore, recovering the generating functionals of the integrals of motion from their analyticity properties is an open problem.

5. The minimal sets of scattering data

5.1. The minimal sets of scattering data from the RHP. As a consequence of the results in Sec. 4.3, we conclude that the following theorem holds.

**Theorem 1.** Assume that the Lax operator \( \mathcal{L} \) (49) is constructed following the ideas in Sec. 4, i.e., it is related to a simple Lie algebra \( g \) and has the \( \mathbb{Z}_{h_g} \) symmetry, where \( h_g \) is the Coxeter number of \( g \). Also assume that its potential \( Q_x \) is such that \( \mathcal{L} \) has no discrete eigenvalues. Then its FAS \( \xi^\nu(x, t, \lambda) \) in the sector \( \Omega_\nu \) provides a regular solution of RHP (54).

Let \( \delta_0 \) and \( \delta_1 \) denote the subsets of roots of \( g \) that satisfy the equations

\[
\alpha \in \begin{cases} 
\delta_0 & \text{if } \text{Im}\lambda(\alpha, \bar{v}_g) = 0 \text{ for } \lambda \in \ell_0, \\
\delta_1 & \text{if } \text{Im}\lambda(\alpha, \bar{v}_g) = 0 \text{ for } \lambda \in \ell_1.
\end{cases}
\]  

(58)

Let also

\[
\mathcal{T}_1 = \{ \tau_{\alpha,0}^\pm(\lambda) \mid \alpha \in \delta_0, \lambda \in \ell_0 \} \cup \{ \tau_{\alpha,1}^\pm(\lambda) \mid \alpha \in \delta_1, \lambda \in \ell_1 \}, \\
\mathcal{T}_2 = \{ \rho_{\alpha,0}^\pm(\lambda) \mid \alpha \in \delta_0, \lambda \in \ell_0 \} \cup \{ \rho_{\alpha,1}^\pm(\lambda) \mid \alpha \in \delta_1, \lambda \in \ell_1 \}.
\]  

(59)

Then

1. \( \mathcal{T}_1 \) (resp. \( \mathcal{T}_2 \)) provide a minimal set of scattering data that allow recovering
   a) the Gauss factors \( S_{\nu}^\pm(\lambda) \) for \( \nu = 0,1 \) (resp. \( T_{\nu}^\pm(\lambda) \) for \( \nu = 0,1 \));
   b) the Gauss factors \( D_{\nu}^\pm(\lambda) \) for \( \nu = 0,1 \) (resp. \( T_{\nu}^\pm(\lambda) \) for \( \nu = 0,1 \));
   c) the matching functions \( G_{\nu}(x,t,\lambda) \) and the scattering matrices \( T_{\nu}(\lambda) \) for each ray \( \ell_{\nu} \), \( \nu = 0, \ldots, 2h_g - 1 \).

2. \( \mathcal{T}_1 \) (resp. \( \mathcal{T}_2 \)) allows constructing a regular solution \( \xi^\nu(x,t,\lambda) \) of the RHP.

3. \( \mathcal{T}_1 \) (resp. \( \mathcal{T}_2 \)) allows recovering the potential \( Q_x \) of the Lax operator.
Proof. 1. Given $T_1$ (respectively $T_2$) and using (53), we recover the Gauss factors, i.e., the group-valued functions $S^\pm_\nu(\lambda,t)$ (respectively $T^\pm_\nu(\lambda,t)$) for $\nu = 0, 1$. Next, using the $\mathbb{Z}_h$ symmetry, we construct $S^\pm_\nu(\lambda,t)$ (respectively $T^\pm_\nu(\lambda,t)$) for $\nu = 2, \ldots, 2h - 1$, see Eq. (53).

The next step consists in recovering the functionals of the integrals of motion $d^\pm_{\nu,\beta}(\lambda)$. Relations (57) provide this possibility in principle. Indeed, given the sets $T_1$ and $T_2$, we know the Gauss factors $S^\pm_\nu(\lambda,t)$ and $T^\pm_\nu(\lambda,t)$ not only in the typical representations but also in all fundamental representations. Of course, there are serious difficulties in evaluating the right-hand sides of (57). However, there is no doubt that they are determined uniquely by any of the sets $T_1$ and $T_2$. As a result, we have determined all Gauss factors in Eq. (52), i.e., we obtained all matching functions $G_\nu(x,t,\lambda)$ and the scattering matrices $T_\nu(t,\lambda)$.

2. Because the operator $\mathcal{L}$ has no discrete eigenvalues, the corresponding FASs satisfy a regular RHP. Therefore, the matching functions $G_\nu(x,t,\lambda)$ uniquely determine the solution $\xi^\pm_\nu(\lambda,t)$ of the RHP.

3. Because the solution $\xi^\pm_\nu(\lambda,t)$ of the RHP is uniquely determined, the corresponding potential of $\mathcal{L}$ is determined from Eq. (56).

5.2. Minimal sets of scattering data for the exceptional algebras. Here, we formulate the results about the spectral data for the Lax operators related to the exceptional algebras. The calculations were performed by Maple. They allowed us to calculate the corresponding vectors $\vec{e}_g$ and the scalar products $\langle \vec{e}_g, \alpha_j \rangle$ and to establish the results in Sec. 4.2 for each exceptional algebra. In particular, we found that the subsets of roots $\delta_0$ and $\delta_1$ are given by the sets of black and white roots, i.e., $\delta_0 \equiv B_g, \delta_1 \equiv W_g$. Below, besides the sets of roots $\delta_0$ and $\delta_1$, we also list Coxeter automorphisms as linear operators (matrices) acting on the Euclidean space $\mathbb{E}^r$.

5.2.1. Case $g \simeq E_6^{(1)}$. Let $\varepsilon_6 = (-e_6 - e_7 + e_8)/\sqrt{3}$. Then the set of admissible roots (19) can be rewritten as

\[
\alpha_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + \sqrt{3}\varepsilon_6), \quad \alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 + \sqrt{3}\varepsilon_6),
\]

\[
\alpha_2 = e_1 + e_2, \quad \alpha_3 = e_2 - e_1, \quad \alpha_4 = e_3 - e_2, \quad \alpha_5 = e_4 - e_3, \quad \alpha_6 = e_5 - e_4.
\]

There are two rays (48) corresponding to the Coxeter automorphism of $E_6^{(1)}$. The roots related to the rays $l_0$ and $l_1$ are

\[
l_0: \quad \arg \lambda = 0, \quad B_g \equiv \{\alpha_0, \alpha_4, \alpha_6\},
\]

\[
l_1: \quad \arg \lambda = \frac{\pi}{12}, \quad W_g \equiv \{-\alpha_2, -\alpha_3, -\alpha_5\}.
\]

The Coxeter number of $E_6^{(1)}$ is $h = 12$ and the exponents are 1, 4, 5, 7, 8, 11. The Coxeter automorphism as a linear operator on the dual Euclidean space $\mathbb{E}_6$ is given by

\[
C_{E_6} = \frac{1}{4}
\begin{pmatrix}
3 & -1 & -1 & -1 & -1 & \sqrt{3} \\
-1 & 1 & -3 & 1 & 1 & -\sqrt{3} \\
1 & -1 & -1 & -1 & 3 & \sqrt{3} \\
1 & 3 & -1 & -1 & -1 & \sqrt{3} \\
1 & -1 & -1 & 3 & -1 & \sqrt{3} \\
-\sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} & 1
\end{pmatrix}, \quad C_{E_6}^{12} = \mathbb{I}_6, \quad \det C_{E_6} = 1.
\]

Here, $e_j, j = 1, \ldots, 5$, and $\varepsilon_6$ is a basis of $\mathbb{E}_6$. The characteristic polynomial of $C_{E_6}$ is

\[
P_{E_6} = z^6 + z^5 - z^3 + z + 1.
\]
5.2.2. Case $g \simeq E_7^{(1)}$. Introducing the notation $\varepsilon_7 = -(e_7 - e_8)/\sqrt{2}$, we can rewrite the set of admissible roots (24) as

$$
\alpha_0 = -\sqrt{2}\varepsilon_7 = e_7 - e_8, \quad \alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + \sqrt{2}\varepsilon_7),
$$

$$
\alpha_2 = e_1 + e_2, \quad \alpha_2 = e_1 + e_2, \quad \alpha_3 = e_2 - e_1, \quad \alpha_4 = e_3 - e_2, \quad \alpha_5 = e_4 - e_3, \quad \alpha_6 = e_5 - e_4, \quad \alpha_3 = e_6 - e_5.
$$

The roots related to the rays $l_0$ and $l_1$ are respectively given by

$$
l_0: \arg \lambda = 0, \quad B_g \equiv \{\alpha_1, \alpha_4, \alpha_6\},
$$

$$
l_1: \arg \lambda = \frac{\pi}{18}, \quad W_g \equiv \{-\alpha_2, -\alpha_3, -\alpha_5, -\alpha_7\}.
$$

The Coxeter number of $E_7^{(1)}$ is $h = 18$ and the exponents are 1, 5, 7, 9, 11, 13, 17. The Coxeter automorphism acts as a linear operator on the space $E_7$ as

$$
C_{E_7} = \frac{1}{4}
\begin{pmatrix}
-3 & -1 & 1 & 1 & 1 & 1 & \sqrt{2} \\
-1 & 1 & -1 & 3 & -1 & -1 & -\sqrt{2} \\
-1 & -3 & -1 & -1 & -1 & -1 & -\sqrt{2} \\
-1 & 1 & 1 & -1 & 3 & -1 & -\sqrt{2} \\
-1 & 1 & -1 & -1 & 3 & -1 & -\sqrt{2} \\
\end{pmatrix}, \quad C_{E_7}^9 = -I_7, \quad \det C_{E_7} = -1.
$$

Here, $e_j, j = 1, \ldots, 6$, and $\varepsilon_7$ forms a basis in the dual Euclidean space $E_7$. The characteristic polynomial of $C_{E_7}$ is

$$
P_{E_7} = z^7 + z^8 - z^4 - z^3 + z + 1.
$$

5.2.3. Case $g \simeq E_8^{(1)}$. The set of admissible roots is given by (29). The roots related to the rays $l_0$ and $l_1$ are

$$
l_0: \arg \lambda = 0, \quad B_g \equiv \{\alpha_1, \alpha_4, \alpha_6, \alpha_8\},
$$

$$
l_1: \arg \lambda = \frac{\pi}{30}, \quad W_g \equiv \{-\alpha_2, -\alpha_3, -\alpha_5, -\alpha_7\}.
$$

The Coxeter number of $E_8^{(1)}$ is $h = 30$ and the exponents are 1, 7, 11, 13, 17, 19, 23, 29. The action of the Coxeter automorphism as a linear operator on the dual Euclidean space $E_7$ is given by

$$
C_{E_8} = \frac{1}{4}
\begin{pmatrix}
-3 & -1 & 1 & 1 & 1 & 1 & -1 \\
-1 & 1 & -1 & 3 & -1 & -1 & 1 \\
-1 & -3 & -1 & -1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 3 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 3 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 3 \\
\end{pmatrix}, \quad C_{E_8}^{15} = -I_8, \quad \det C_{E_8} = -1.
$$

Here, $e_j, j = 1, \ldots, 6$, and $\varepsilon_7$ form a basis in $E_7$. The characteristic polynomial of $C_{E_8}$ is

$$
P_{E_8} = z^8 + z^7 - z^5 - z^3 + z + 1.
$$
5.2.4. Case $g \simeq F_4^{(1)}$. The set of admissible roots is given by (33). The roots related to the rays $l_0$ and $l_1$ are
\[ l_0: \arg \lambda = 0, \quad B_g \equiv \{ \alpha_1, \alpha_3 \}, \]
\[ l_1: \arg \lambda = \frac{\pi}{12}, \quad W_g \equiv \{-\alpha_2, -\alpha_4\}. \] (71)
The Coxeter number of $F_4^{(1)}$ is $h = 12$ and the exponents are 1, 5, 7, 11. The Coxeter automorphism acts as a linear operator on the dual space $E_4$ via
\[ C_{F_4} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C_{F_4}^6 = -I_4, \quad \det C_{F_4} = 1. \] (72)

Here, $e_j$, $j = 1, \ldots, 4$, is a basis of $E_4$. The characteristic polynomial of $C_{F_4}$ is
\[ P_{F_4} = z^4 - z^2 + 1. \] (73)

5.2.5. Case $g \simeq G_2^{(1)}$. This is the simplest and simultaneously the most peculiar case. The peculiarity is that although the rank of $G_2$ is 2, its root system is traditionally written in a 3-dimensional space $E_3$ as a set of vectors that are all orthogonal to the vector $e_1 + e_2 + e_3$. The Coxeter automorphism here is the composition of the Weyl reflection $C_{G_2} = S_{\alpha_1}S_{\alpha_2}$. It can be verified that $C_{G_2}$ is expressed as the following linear operator on $E_3$:
\[ C_{G_2} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}, \] (74)
with the properties
\[ C_{G_2}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_{G_2}^4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_{G_2}^6 = I_3. \] (75)
The characteristic polynomial is
\[ P_{G_2} = z^3 - 2z^2 + 2z - 1 = (z - 1)(z^2 - z + 1). \] (76)

Another way to approach the problem is to project the $G_2$ root system onto the 2-dimensional plane orthogonal to the vector $e_1 + e_2 + e_3$. Indeed, let
\[ \varepsilon_1 = \frac{1}{\sqrt{2}}(e_2 - e_1), \quad \varepsilon_2 = \frac{1}{\sqrt{2}}(e_3 - e_2). \]

Then the set of admissible roots (38) can be rewritten as
\[ \alpha_0 = \sqrt{2}(\varepsilon_1 + 2\varepsilon_2) = -e_1 - e_2 + 2e_3, \quad \alpha_1 = \sqrt{2}\varepsilon_1 = e_2 - e_1, \]
\[ \alpha_2 = \sqrt{2}(\varepsilon_2 - \varepsilon_1) = e_1 - 2e_2 + e_3. \] (77)
The roots related to the rays $l_0$ and $l_1$ are
\[ l_0: \arg \lambda = 0, \quad B_g \equiv \{ \alpha_1 \}, \]
\[ l_1: \arg \lambda = \frac{\pi}{12}, \quad W_g \equiv \{-\alpha_2\}. \] (78)
The Coxeter number for $G_2^{(1)}$ is $h = 6$ and the exponents are $1, 5$. This Coxeter automorphism induces an action on the dual space $E_2$ with the matrix

$$C_{G_2}' = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad C_{G_2}'^3 = -\mathbb{I}_2, \quad \det C_{G_2}' = 1. \quad (79)$$

Here, $\{\varepsilon_1, \varepsilon_2\}$ form a basis in $E_2$. Finally, the characteristic polynomial of $C_{G_2}'$ is

$$P_{G_2}' = z^2 - z + 1. \quad (80)$$

6. Conclusions

We presented real Hamiltonian forms of affine Toda field theories related exceptional untwisted complex Kac–Moody algebras. We established that the special properties of these models allow relating the construction of the RHF to the study of $Z_h$ symmetries of the EDD (with $h$ being the Coxeter number of $g$). The general construction is illustrated by several examples of such models related to the exceptional Kac–Moody algebras $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $D_4^{(1)}$, and $G_2^{(1)}$. We used the “dihedral realization” of the Coxeter automorphism $\text{Cox}(g) = S_1 \circ S_2$, where $S_1$ and $S_2$ are $Z_2$ automorphisms of $g$. One of these automorphisms extracts the RHF and the other acts as an additional $Z_2$-reduction on it.

Each of the real Hamiltonian forms obtained above has its dual one. Indeed, we could define the involution $\mathcal{C}$ using the Weyl reflection $S_2$. The consideration is quite analogous: we simply interchange the black roots with the white ones. The reformulation of all the above results is rather obvious. We note that the dual real Hamiltonian forms of 2D TFT are not equivalent to the ones obtained above. Indeed, it is easy to verify that in all cases discussed above, the number of white roots is different from the number of black roots.

Some additional problems arise as natural extensions of the results presented here.

- The complete classification of all nonequivalent RHFs of an ATFT.
- The description of the hierarchy of Hamiltonian structures of an RHF of ATFT (for a review of the infinite-dimensional cases, see, e.g., [33], [36] and the references therein). It is also an open problem to construct the RHFs for ATFTs using some of the higher Hamiltonian structures.
- The extension of the Zakharov–Shabat dressing method [38] to the above classes of Lax operators is also an open problem. One of the difficulties is that the $Z_h$ reductions require dressing factors with $2h$ pole singularities [39], [40]. This makes the relevant linear algebraic equations rather involved [41]–[46].
- Another open problem is to study types of boundary conditions and boundary effects of ATFTs and their RHFs [47], [48]. This includes types of boundary defects [49].
- The last and technically more involved problem is to solve the inverse scattering problem for the Lax operator and thus prove the complete integrability of all these models. The ideas in [10], [13] about the interpretation of the inverse scattering method as a generalized Fourier transform also holds for the $Z_h$ reduced Lax operators [12], [30], [31], [37], [50], [51]. This may allow deriving the action–angle variables for these classes of NLEEs.

Conflicts of interest. The authors declare no conflicts of interest.
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