Super-Penrose process with charged particles near naked singularity

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We consider the Penrose process near the naked singularity in the Reissner-Nordström metric. Particle 0 falls from infinity and decays to two fragments at some point $r_0$. We show that the energy extraction due to this process can be indefinitely large in the limit $r_0 \to 0$. In doing so, the value of the particle charge can remain bounded, in contrast to the previously known examples of the Penrose process in the electric field with unbounded energy extraction. The effect persists even in the limit of the flat space-time.

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I. INTRODUCTION

The Penrose process (PP) is one of remarkable physical effects typical of general relativity and other theories of gravity. Let us suppose that in a space-time there exists a region (called ergosphere) where the Killing energy $E$ measured at infinity can be negative. If a particle 0 within the ergosphere splits to two fragments 1 and 2, one of them can have $E_1 < 0$. Then, the conservation of energy entails that the second fragment has $E_2 > E_0$. This is just PP. Originally, it was found in the vicinity of rotating black holes. However, it was understood later that the similar effect should occur in the background of static charged black holes [2]. It was investigated further in detail [3] - [6].

Moreover, there exists a limit to flat space-time when the effect under discussion persists [7], [8]. If the energy of a new particle 2 can be arbitrarily large, such a kind of the PP is

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called super-Penrose process (SPP).

In recent decade, a new venue for the PP became popular after discovering the so-called Bañados-Silk-West (BSW) effect \[^9\]. It consists in acceleration of particles to high energies due to collisions near black holes. If two particles collide near the horizon, under certain conditions their energy \(E_{c.m.}\) in the center of mass frame can become indefinitely large. However, one should not mix \(E_{c.m.}\) and \(E\). It turns out that for collisions of neutral particles near black holes PP does indeed occur (it is called collisional Penrose process) but SPP is forbidden (see, e.g. \[^10\] and references therein). However, SPP is indeed possible for the charged black holes, even for pure radial particle motion in the Reissner-Nordstrom (RN) background \[^11\], \[^12\]. (More general scenarios can include both the electric charge and rotation \[^13\].)

In the present work, we show that there exists one more type of the SPP. It is realized in the background of naked singularities. This is different from high energy processes near the RN naked singularity considered earlier \[^14\], \[^15\]. In aforementioned papers (i) collisions of shells were studied, (ii) the high energy process implies high \(E_{c.m.}\). Meanwhile, we show that (i) the effect under discussion is valid for test particles, (ii) it involves ultra-high energies \(E\), (iii) there are no collisions at all, the process represents a standard PP, not a collisional one.

The examples of the PP, known before, share the common feature. If one wants to gain large energy in PP, the electric charge is also should be large. This concerns both the standard and collisional PP as well as the confined one \[^16\]. However, the electric charge of elementary particles, atoms or nuclei cannot be arbitrary large \[^11\], \[^17\]. Also, there exist similar restrictions for macroscopic bodies \[^12\]. Meanwhile, we demonstrate that for naked singularities not only PP but also SPP does exist for a finite value of a particle charge.

As is known, the electric charge of astrophysical objects is rather small. (Although it can, in principle, lead to observable effects \[^18\].) However, in some aspect the electrical charge can model what happens in more complicated realistic astrophysical systems with rotation. Meanwhile, the RN metric is much easier than, say, the Kerr metric describing the vacuum solution of the Einstein equations with rotation. Anyway, the PP is one of most universal and nontrivial processes in gravitating systems, so it is necessary to study all its potential manifestations.

We use the geometric system of units in which fundamental constants \(G = c = 1\).
II. BASIC EQUATIONS

We consider the Reissner-Nordström metric

\[ ds^2 = -dt^2 f + \frac{dr^2}{f} r^2 d\omega^2, \]  

where \( d\omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \),

\[ f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \]

Here \( M \) and \( Q \) are the mass and electric charge, respectively. We take \( Q > 0 \). We assume that \( M < Q \), so there is a naked singularity at \( r = 0 \) in this space-time.

Now, we consider motion of test particles. We restrict ourselves by pure radial motion. This is sufficient to demonstrate that the effect under discussion does exist. Then, equations of motion read

\[ m \dot{t} = \frac{X}{f}, \]  
\[ X = E - q\varphi, \]
\[ m \dot{r} = \sigma P, \quad P = m \sqrt{U}, \quad U = \frac{X^2}{m^2} - f, \]

\( \sigma = \pm 1 \) depending on the direction of motion, dot denotes differentiation with respect to the proper time \( \tau \). Here, \( q \) is the particle’s electric charge, \( m \) being its mass, \( E \) the energy.

The forward-in-time condition \( \dot{t} > 0 \) entails

\[ X > 0. \]

The electric Coulomb potential

\[ \varphi = \frac{Q}{r}. \]

Hereafter, we use notations \( \varepsilon = \frac{E}{m}, \tilde{q} = \frac{q}{m} \). Then,

\[ U(r) = (\varepsilon - \tilde{q}Q/r)^2 - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right). \]

Then, the possible turning points \( r_t \) can be found from the condition \( P = 0 \), whence

\[ r_t^\pm = \frac{1}{\varepsilon^2 - 1}(\varepsilon \tilde{q}Q - M \pm \sqrt{C}), \]
\[ C = (M - \varepsilon \tilde{q}Q)^2 + (1 - \tilde{q}^2)Q^2(\varepsilon^2 - 1). \]
III. SCENARIO OF DECAY

In what follows, we will be mainly interested in the situation when decay occurs near the singularity, so the point of decay \( r_0 \to 0 \). Then, for fixed \( q_0 \) and \( E_0 \), the forward-in-time condition (6) requires \( q_0 < 0 \). Assuming also that particle moves from infinity, so \( \varepsilon_0 > 1 \), we see from (9) that no more than one turning point \( r_i^+ \) can exist, provided \( |\tilde{q}_0| < 1 \).

Let particle 0 with \( \varepsilon > 1 \) fall from infinity. In some point \( r_0 \) it decays to two new fragments 1 and 2. We assume the conservation of the energy and electric charge in the point of decay, so

\[
E_0 = E_1 + E_2,
\]
\[
q_0 = q_1 + q_2.
\]

The necessary condition that makes decay possible is

\[
m_0 \geq m_1 + m_2.
\]

For given characteristics \( E_0, m_0, q_0 \) of particle 0, one can solve (11), (12) with (5) taken into account. We can take advantage of already obtained results - see eqs. (19) - (25) of Ref. [19]. Only minimum changes are required: (i) instead of indices 3, 4 we use here 1,2, (ii) the quantity \( X \) is defined according to (5) instead of eq. (5) of [19].

Then,

\[
X_1 = \frac{1}{2m_0^2} \left( X_0 \Delta_+ + P_0 \delta \sqrt{D} \right),
\]
\[
X_2 = \frac{1}{2m_0^2} \left( X_0 \Delta_- - P_0 \delta \sqrt{D} \right),
\]
\[
P_1 = \left| \frac{P_0 \Delta_+ + \delta X_0 \sqrt{D}}{2m_0^2} \right|,
\]
\[
P_2 = \left| \frac{P_0 \Delta_- - \delta X_0 \sqrt{D}}{2m_0^2} \right|,
\]

where \( \delta = \pm 1 \),

\[
\Delta_\pm = m_0^2 \pm (m_1^2 - m_2^2),
\]
\[
D = \Delta_+^2 - 4m_0^2m_1^2 = \Delta_-^2 - 4m_0^2m_2^2.
\]

It follows from (13) that \( D \geq 0 \). The equality holds if \( m_0 = m_1 + m_2 \) only.
The solutions (14) - (19) are classified according to 4 quantities \((\sigma_2, h_2, h_1, \delta)\). The corresponding allowed combinations are listed in eq. (30) of [19]. (A reader should bear in mind that the role of particle 3 in [19] is played now by particle 2). Here,

\[ h_1 = \text{sgn}H_1, \quad H_1 = \Delta_+ \sqrt{f} - 2m_1X_0, \]  
\[ h_2 = \text{sgn}H_2, \quad H_2 = \Delta_- \sqrt{f} - 2m_2X_0. \]  

We are interested in the situation when particle 2 escapes. If it moves after decay immediately to infinity, \(\sigma_2 = +1\). We will consider this type of scenario first. (Afterwards, we will also discuss an alternative scenario when particle 2 bounces back from the potential barrier.)

Then, according to [19], there are only two possibilities: 1(\(++,+,+,-\)) and 2(\(+,+,-,-\)). Thus \(\delta = -1\) and we should also have

\[ H_2 > 0 \]  

while \(H_1\) can have any sign.

IV. ENERGY EXTRACTION

The Penrose process implies that particle 1 moves towards the center with \(E_1 < 0\), whereas particle 2 escapes to infinity with \(E_2 > 0\). In doing so, \(\sigma_1 = -1\) and \(\sigma_2 = +1\). Our goal is to obtain the energy extraction as large as possible. According to (4),

\[ E_2 = X_2 + \frac{q_2Q}{r_0}. \]  

Taking into account (6), we see that if \(q_2 > 0\) and \(r_0 \to 0\), the energy \(E_2\) is unbounded. Is it possible to achieve this goal in our scenario?

When \(r_0 \to 0\), condition (6) requires \(q_0 < 0\). Thus we should have

\[ q_0 < 0, \quad q_2 > 0. \]  

Further, we should consider two different cases depending on whether or not there is a turning point for particle 0.
A. No turning point

This case is realized if $|\tilde{q}_0| > 1$ since both roots $|q_0|$ become negative. Then, we can take the limit $r_0 \to 0$ directly. In this limit, it follows from (1), (3), (7) that

$$X_0 \approx \frac{|q_0| Q}{r_0},$$ (25)

$$P_0 \approx \frac{Q}{r_0} \sqrt{q_0^2 - m_0^2}.$$ (26)

Now, using (15), (23), (25) and (26), one obtains in the main approximation

$$E_2 \approx \frac{Q}{r_0} \left[g_2 + \frac{1}{2m_0^2}(|q_0| \Delta + \sqrt{q_0^2 - m_0^2 \sqrt{D}})\right]$$ (27)

and $E_2 \to \infty$ when $r_0 \to 0$.

Now,

$$H_2 \approx \frac{Q}{r_0} (\Delta - 2m_2 |q_0|).$$ (28)

Eq. (22) is valid, if

$$|q_0| < \frac{\Delta}{2m_2}.$$ (29)

It is consistent with $|\tilde{q}_0| > 1$.

This is not the end of story. We must check that particle 2 does escape, so it does not have a new turning point for all $r > r_0$, $U > 0$. We should verify that $U_2(r) > U_2(r_0)$ for any $r > r_0$. Using (5), it is easy to find that

$$U_2(r) - U_2(r_0) \approx \frac{(r_2 - r_0)}{r_0^2} B,$$ (30)

$$B = Q^2 [1 + \tilde{q}_2^2 + \frac{r_0(1 - \tilde{q}_2^2)}{r} + \frac{q_2}{m_0^2} (|q_0| \Delta + \sqrt{q_0^2 - m_0^2 \sqrt{D}})] + 2 \left(\frac{E_0 q_2 Q}{m_2^2} - M\right)r_0.$$ (31)

Obviously, if $r_0 \to 0$, $B > 0$ for any $r > r_0$. Thus,

$$U_2(r) > U_2(r_0) > 0$$ (32)

and there are no additional turning points, so particle 2 escapes to infinity freely.

B. Decay in the turning point

Let us suppose now that the turning point for particle 0 does exist. As we try to obtain the maximum possible $E_2$, it makes sense to choose (for given values of other parameters) the
minimum possible value of \( r_0 \). To this end, we put \( r_0 = r_i^{(+)} \). We assumed (as is explained above) that \( q_0 < 0 \). Then, the existence of a turning point requires \( |\tilde{q}_0| \leq 1 \). In this point we have \( P_0 = 0 \) by definition, so it follows from (14) - (17) that

\[
X_1 = \frac{X_0 \Delta_+}{2m_0^2},
\]

\[
X_2 = \frac{X_0 \Delta_-}{2m_0^2},
\]

\[
P_1 = P_2 = \frac{X_0}{2m_0} \sqrt{D},
\]

where now

\[
X_0 = E_0 + \frac{|q_0| Q}{r_0},
\]

\[
E_2 = \frac{E_0 \Delta_-}{2m_0^2} + \frac{Q}{r_0} (q_2 + |q_0| \frac{\Delta_-}{2m_0^2}).
\]

In doing so, \( X_0 > 0 \) for any \( r_0 \) due to \( q_0 < 0 \), so condition (6) holds. As we want to minimize \( r_0 \), we choose

\[
|\tilde{q}_0| = 1 - \beta, \beta \ll 1.
\]

Then, it follows from (9) that

\[
r_0 \approx \frac{Q^2 \beta}{M + \varepsilon Q}.
\]

can be made as small as one likes. As a result, we have from (37) that

\[
E_2 \approx \frac{(M + \varepsilon Q)}{Q \beta} (q_2 + |q_0| \frac{\Delta_-}{2m_0^2}).
\]

When \( \beta \to 0 \), \( E_2 \to \infty \), so the SPP does exist.

Eq. (32) is valid in the case under consideration as well. It is worth noting that if \( q = 0 \), the expression (9) coincides with eq. (13) of [14]. However, we saw that in both versions of the scenario under consideration (with a turning point or without it) it is essential for the SPP that \( q_0 \neq 0 \). Thus, this process is possible for charged particles and is absent for neutral ones.

V. ALTERNATIVE TYPE OF SCENARIO

For completeness, we must consider the case when particle 2 after decay moves in the same direction as particle 1, so \( r \) continues to decrease. However, immediately after decay
particle 2 bounces back from the potential barrier. This means that \( r_0 \) is the turning point for particle 2, so \( P_2 = 0 \) and, therefore,

\[
X_2 = m_2 \sqrt{f}. \tag{41}
\]

When \( r_0 \to 0 \),

\[
X_2 \approx \frac{m_2 Q}{r_0}. \tag{42}
\]

According to (17), we must take \( \delta = +1 \) and

\[
P_0 \Delta_+ = X_0 \sqrt{D}. \tag{43}
\]

It is easy to check that this is equivalent to \( H_2 = 0 \) in (21), so

\[
\Delta_+ \sqrt{f} = 2m_2 X_0. \tag{44}
\]

If \( q_0 < 0 \), there is no turning point for particle 0. In the limit \( r_0 \to 0 \), \( X_0 \approx \frac{|q_0| Q}{r_0}, \sqrt{f} \approx \frac{Q}{r_0} \)
and we obtain from (44)

\[
|q_0| \approx \frac{\Delta_+}{2m_2}. \tag{45}
\]

If \( q_0 > 0 \), choosing \( r_0 \) to be a turning point for particle 0 as well, we have \( X_0 = m_0 \sqrt{f} \), so it follows from (14) that \( \Delta_+ = 2m_0m_2 \), whence \( m_0 = m_1 + m_2 \). In both cases, according to (12),

\[
E_2 \approx \frac{Q(m_2 + q_2)}{r_0}. \tag{46}
\]

Here, we should take \( q_2 > -m_2 \). Thus the SPP does exist in this case also.

VI. FLAT SPACE-TIME LIMIT

Decay in the case of the flat space-time is of special interest. More precisely, we put \( M = Q = 0 \) in the metric thus neglecting the influence of the electromagnetic field on space-time. However, we take into account the electric charge in equations of motion. Again, let particle 0 decay in the point \( r_0 \). Then, eqs. (14), (15) are now valid with \( f = 1 \). Now we will show that the SPP is still possible for a finite value of \( |q_0| \) (the corresponding scenarios were overlooked in our previous paper [8]).

If the turning point \( P_0 = 0 \) exists, its coordinate is given by

\[
r_0 = \frac{q_0 Q}{E_0 - m_0}. \tag{47}
\]

Here, it is assumed that \( E_0 > m_0 \). Now there are two different cases.
A. No turning point

Let $q_0 < 0$. Then, the turning point for particle 0 is absent. We want particle 2 to escape to infinity, so the turning point for particle 2 should be absent as well, $q_2 = -|q_2| < 0$. Proceeding along the same lines as before, we obtain

$$E_2 \approx E_0 + \frac{Q}{r_0} \left[ \frac{|q_0|}{2m_0} (\Delta_+ + \sqrt{D}) - |q_2| \right]$$ (48)

This expression differs from (27) since in (27) it was implied that $f \sim \frac{Q^2}{r_0} \to \infty$, whereas now $f = 1$.

The positivity of $E_2$ for small $r_0$ requires

$$|q_2| < \frac{|q_0|}{2m_0} (\Delta_+ + \sqrt{D}).$$ (49)

Formally, (48) grows indefinitely when $r_0 \to 0$. Actually, as we neglected the corresponding term $\frac{Q^2}{r_0}$ in the RN metric, there is an additional constraint on $r_0$. Restoring dimensionality, we have $r_0 \gg \sqrt{GQc}$, so there is restriction $E_2 \ll \frac{c^2}{\sqrt{G}} \left[ \frac{|q_0|}{2m_0} (\Delta_+ + \sqrt{D}) - |q_2| \right]$ that is rather weak in the case $G \to 0$. Now,

$$H_2 = \Delta_- - 2m_2 (E_0 + \frac{|q_0| Q}{r_0}).$$ (50)

It is seen that for a fixed $|q_0|$ and $r_0 \to 0$ eq. (22) cannot be fulfilled. It means that the scenario under discussion cannot be realized, so particle 2 falls in the center along with particle 1 and does not escape. The situation changes if we take $|q_0| = \alpha r_0$, where $\alpha = O(1)$. Then, we have from (50) that

$$\alpha < Q^{-1} (\frac{\Delta_-}{2m_2} - E_0).$$ (51)

B. Decay in the turning point

Now, $P_0 = 0$, so eqs. (33) - (36) apply. As a particle falls from infinity, $E_0 > m_0$. According to (47), we must also take $q_0 > 0$. For particle 2 we take $q_2 < 0$ to exclude a possible turning point after decay. As we want $r_0 \to 0$, it is seen from (47) that we must assume $q_0 \to 0$. Then, in eq. (37) the main contribution becomes negative, so the SPP is absent in this case.
C. Alternative scenario

Is it possible the alternative scenario in the flat case? It implies that particle 2 bounces back from the turning point. Then, \( P_2 = 0 \), so

\[ X_2 = m_2. \tag{52} \]

Now, one should put \( f = 1 \) in eq. (44) typical of a turning point. Then, putting there \( X_0 \approx \frac{|q_0|Q}{r_0} \), we obtain

\[ |q_0| \approx \frac{r_0 \Delta}{Q 2m_2}. \tag{53} \]

Thus for a fixed \( q_0 \) we have

\[ E_2 = m + \frac{q_2Q}{r_0}, \tag{54} \]

where we took into account (52). Then,

\[ E_2 \approx \frac{q_2 \Delta}{2m_2 |q_0| Q}, \tag{55} \]

where it is assumed \( q_2 > 0 \). If \( q_0 \to 0 \), \( E_2 \to \infty \), so the SPP occurs. As now \( X_2 \) is monotonically increasing function of \( r \), there are no other turning points and particle 2 escapes.

VII. CONCLUSIONS

Thus we found one more type of systems for which the SPP is possible. It exists in almost all scenarios considered above. It is instructive to compare the case under discussion with the collisional PP relevant in the context of the BSW effect near black holes \([11]\). To gain large energy \( E \) in the latter case, one is led to prepare a particle in the fine-tuned state (so-called critical one). But there is a bound on \( q < Z e \), where \( e \) is the value of an elementary charge and \( Z = 170 \) comes from quantum electrodynamics. This places restriction on the fractional enhancement of energy in collisions. As the energy of a particle produced in collision turns out to be proportional to the charge \( q \) \([11]\), this gives rise to the restriction of the efficiency of the PP. Meanwhile, in the configuration considered in the present paper (i) the SPP can occur without fine-tuning at all and (ii) to gain large \( E \), there is no need to have large \( q \). More precisely, fine-tuning is required for the subcase with the turning point \([38]\) only but is absent if such a point is missing. One more difference between the present
version of SPP and that near a black hole consists in that now the original particle needs not be ultrarelativistic having any finite value \( \varepsilon > 1 \) (cf. Sec. IV C3 in [17]). In this sense, the present version of the SPP is less restrictive than the previous one typical of the BSW effect. We would also like to stress that the effect under discussion concerns indefinitely large Killing energy \( E \), whereas \( E_{c.m.} \) is irrelevant since we considered particle decay, not collision.

In combination with the previous results [2], [3], [11], [7], [8], this means that any type of the Reissner-Nordström space-time (black hole, flat space-time, naked singularity) is pertinent to the SPP. A separate interesting question is whether and how the naked singularity can produce the SPP for neutral rotating naked singularities.

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