THE ERDŐS-SZEKERES PROBLEM FOR NON-CROSSING CONVEX SETS

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ABSTRACT. We show an equivalence between a conjecture of Bisztriczky and Fejes Tóth convexity properties of arrangements of planar convex bodies to a conjecture of Goodman and Pollack about convexity properties of point sets in topological affine planes. As a corollary of this equivalence we improve the upper bound of Pach and Tóth on the Erdős-Szekeres theorem for disjoint convex bodies, as well as the recent upper bound obtained by Fox, Pach, Sudakov and Suk, on the Erdős-Szekeres theorem for non-crossing convex bodies. Our methods also imply improvements on the positive fraction Erdős-Szekeres theorem for disjoint (and non-crossing) convex bodies, as well as a generalization of the partitioned Erdős-Szekeres theorem of Pór and Valtr to arrangements of non-crossing convex bodies.

1. Introduction

1.1. The happy ending theorem. In 1935, Erdős and Szekeres proved the following foundational result in combinatorial geometry and Ramsey theory.

Theorem (Erdős-Szekeres [8]). For every integer \( n \geq 3 \) there exists a minimal positive integer \( f(n) \) such that any set of \( f(n) \) points in the Euclidean plane, in which every triple is convexly independent, contains \( n \) points which are convexly independent.

Here convexly independent means that no point is contained in the convex hull of the others. Determining the precise growth of the function \( f(n) \) is one of the longest-standing open problems of combinatorial geometry, and has generated a considerable amount of research. For history and details, see [1, 24] and the references therein. Two proofs are given in [8], one of which shows that \( f(n) \leq \left(2n-4\right)/\left(n-2\right)+1 \), and in [9] Erdős and Szekeres give a construction showing that \( f(n) \geq 2n-2+1 \).

Conjecture (Erdős-Szekeres). \( f(n) = 2^{n-2} + 1 \).

This conjecture has been verified for \( n \leq 6 \) [8, 31], while for \( n > 6 \) the best known upper bound is \( f(n) \leq \left(2n-5\right)/\left(n-2\right)+1 \sim 4^n/\sqrt{n} \), which is due to Tóth and Valtr [33]. Asymptotically this is the same as the bound given by Erdős and Szekeres in their seminal paper.

1.2. Generalized configurations. It was observed by Goodman and Pollack that the Erdős-Szekeres theorem extends to so-called generalized configurations, i.e. point sets in a topological affine plane [18, 20]. One may consider this as a finite set of points in the plane where each pair of points are connected by a pseudoline, such that the set of all connecting pseudolines form a pseudoline arrangement [14]. Generalized configurations also have a purely combinatorial characterization and are equivalent to what are called uniform rank 3 acyclic oriented matroids [9] or CC-systems [23].

Theorem (Goodman-Pollack [16]). For every integer \( n \geq 3 \) there exists a minimal positive integer \( g(n) \) such that any generalized configuration of size \( g(n) \), in which every triple is convexly independent, contains \( n \) points which are convexly independent.

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1Paul Erdős colloquially referred to this as the “happy ending theorem” as it led to the meeting of George Szekeres and Esther Klein, who went on to get married and live happily ever after . . .
It should be noted that this is a proper generalization of the Erdős-Szekeres theorem as there are substantially more combinatorially distinct point sets in topological planes than there are in the Euclidean plane \[10\] \[17\]. By containment it follows that \( f(n) \leq g(n) \).

**Conjecture** (Goodman-Pollack). \( f(n) = g(n) \).

It is an easy exercise to extend the proof of Tóth and Valtr to generalized configurations, as their proof uses no metric properties. We therefore have \( g(n) \leq (\binom{2n-5}{n-2} + 1 \) for \( n \geq 7 \). Also, Szekeres and Peters \[31\] confirmed that \( g(n) = f(n) = 2^{n-2} + 1 \) for all \( n \leq 6 \).

### 1.3. Mutually disjoint convex bodies.

In a different direction, initiated by Bisztriczky and Fejes Tóth, the Erdős-Szekeres theorem was generalized to arrangements of compact convex sets in the plane (which we call bodies for brevity). An arrangement of bodies is *convexly independent* if no member is contained in the convex hull of the others.

**Theorem** (Bisztriczky-Fejes Tóth \[3\]). For any integer \( n \geq 3 \) there exists a minimal positive integer \( h_0(n) \) such that any arrangement of \( h_0(n) \) mutually disjoint bodies in the Euclidean plane, in which every triple is convexly independent, contains an \( n \)-tuple which is convexly independent.

This reduces to the Erdős-Szekeres theorem when the bodies are points, but was somewhat more complicated to establish in general. The added complexity is reflected in the original upper bound \( h_0(n) \leq t_n(t_{n-1}(\ldots t_1(cn)\ldots)) \), where \( t_n \) is the \( n \)-th tower function. The upper bound was later reduced to \( 16^n/n \) by Pach and Tóth in \[26\]. By containment we have \( f(n) \leq h_0(n) \).

**Conjecture** (Bisztriczky-Fejes Tóth). \( f(n) = h_0(n) \).

Before this work the only known exact values are \( h_0(4) = 5 \) and \( h_0(5) = 9 \) which were established in \[4\].

### 1.4. Non-crossing convex bodies.

The disjointness hypothesis was relaxed by Pach and Tóth, who showed that an Erdős-Szekeres theorem also holds for arrangements of *non-crossing* bodies, which means that for any pair of bodies \( A \) and \( B \), the set \( A \setminus B \) is simply connected.

**Theorem** (Pach-Tóth \[27\]). For any integer \( n \geq 3 \) there exists a minimal positive integer \( h_1(n) \) such that any arrangement of \( h_1(n) \) non-crossing bodies in the Euclidean plane, in which every triple is convexly independent, contains an \( n \)-tuple which is convexly independent.

The original upper bound on \( h_1(n) \) was improved to a doubly exponential function in \[22\]. Recently Fox, Pach, Sudakov, and Suk \[12\] obtained the upper bound \( h_1(n) \leq 2^{O(n^2 \log n)} \). See also \[3\] \[92\] for related work.

The known bounds are summarized as follows.

\[
2^{n-2} + 1 \leq f(n) \leq g(n) \leq \binom{2n-5}{n-1} + 1 \quad \text{(for } n \geq 7 \text{)}
\]

\[
f(n) \leq h_0(n) \leq \binom{2n-4}{n-2}^2
\]

\[
h_0(n) \leq h_1(n) \leq 2^{O(n^2 \log n)}
\]

\[
2^{n-2} + 1 = f(n) = g(n) \quad \text{(for } n \leq 6 \text{)}
\]

\[
2^{n-2} + 1 = h_0(n) \quad \text{(for } n \leq 5 \text{)}
\]
1.5. Our results. In this paper we make considerable improvements on $h_0(n)$ and $h_1(n)$ by establishing the following.

**Theorem 1.** The Erdős-Szekeres problems for generalized configurations and for arrangements of non-crossing bodies are equivalent. In other words, $g(n) = h_1(n)$.

Here is the idea of the proof. For the lower bound we use the fact that a generalized configuration has a dual representation as a marked pseudoline arrangement, i.e. a wiring diagram [11, 13]. Using this representation we show that every generalized configuration can be represented by an arrangement of bodies in the Euclidean plane. This shows that $f(n) \leq h_1(n)$.

To establish the reverse inequality we start with an arrangement of bodies and consider its dual system of support curves drawn on the cylinder $S^1 \times \mathbb{R}^1$. This system of curves induces a cell complex which encodes the convexity properties of the arrangement. We show how to modify this complex by elementary operations, similar to those of Habert and Pocchiola [21] and Ringel [30], while maintaining control of the convexity properties of the arrangement. The process ends with a complex induced by an arrangement representing a generalized configuration. The details are given in section 2.

In view of Theorem 1 we obtain the following bounds.

$$2^{n-2} + 1 \leq f(n) \leq h_0(n) \leq h_1(n) = g(n) \leq \binom{2n-5}{n-2} + 1 \quad \text{for } n \geq 7$$

$$2^{n-2} + 1 = f(n) = h_0(n) = h_1(n) = g(n) \quad \text{for } n \leq 6$$

Our proof actually provides a general procedure for reducing non-crossing arrangements to generalized configurations, and can therefore be applied to the multitude of Erdős-Szekeres-type results previously proven separately for point sets, then for arrangements of bodies. (See for instance [2, 25, 28, 29].) In particular we obtain the positive fraction version and the partitioned version of the Erdős-Szekeres theorem for non-crossing arrangements. This will be discussed in section 3.

2. Proof of Theorem 1

2.1. Preliminaries. We call a compact convex subset of $\mathbb{R}^2$ a body, and a finite collection of at least three bodies an arrangement. For a body $A$, recall its support function $h_A : S^1 \to \mathbb{R}^1$ on the unit circle

$$h_A(\theta) := \max_{x \in A} \langle \theta, x \rangle$$

The dual of a body is the graph of its support function drawn on the cylinder $S^1 \times \mathbb{R}^1$, i.e.

$$A^* := \{(\theta, h_A(\theta)) : \theta \in S^1\}$$

We use the term system when referring to a finite collection of at least three curves on $S^1 \times \mathbb{R}^1$ which are graphs of continuous functions $\gamma : S^1 \to \mathbb{R}^1$. In this way every arrangement $\mathcal{A}$ is associated with its dual system $\mathcal{A}^*$. Notice that $\mathcal{A}^*$ determines $\mathcal{A}$, and that the dual of a convexly independent arrangement is a system in which every curve appears on the upper envelope. An arrangement is generic if the following hold.

- No triple of bodies share a common supporting tangent.
- For any pair of bodies $A_1$ and $A_2$ with common supporting tangent $\ell$ the intersection $A_1 \cap A_2 \cap \ell$ is empty.

A standard perturbation argument shows that the optimal values for $h_1(n)$ are attained by generic arrangements, so hereby all arrangements are assumed to be generic. Notice that when $\mathcal{A}$ is generic the curves of $\mathcal{A}^*$ intersect transversally.
A pair of bodies $A_1, A_2$ is non-crossing if $A_1 \setminus A_2$ is connected, or equivalently, if $A_1$ and $A_2$ have precisely two common supporting tangents. A triple of bodies $A_1, A_2, A_3$ is orientable if every pair is non-crossing and $\text{conv}(A_i \cup A_j) \setminus A_k$ is simply connected for all choices of distinct $i, j, k$, or equivalently, the convex hull of $A_1 \cup A_2 \cup A_3$ is supported by exactly three of the common supporting tangents determined by the pairs $A_i$ and $A_j$. A non-crossing arrangement is one in which each pair is non-crossing, and an orientable arrangement is one in which each triple is orientable.

Each member of an orientable triple contributes a single connected arc to the boundary of its convex hull, so traversing the boundary of its convex hull in the counter-clockwise direction will impose a cyclic ordering of the triple. It is easily verified that the set of cyclic orderings of all triples of the arrangement form a rank 3 uniform oriented matroid $\mathcal{M}$.

### 2.2. Representing generalized configurations by orientable arrangements.

**Lemma 2.** For every generalized configuration $\mathcal{P}$, in which every triple is convexly independent, there is an orientable arrangement $\mathcal{A}$, such that $\mathcal{P}$ and $\mathcal{A}$ induce the same oriented matroid. In particular we have $g(n) \leq h_1(n)$.

**Proof.** Let $\mathcal{P}$ be a generalized configuration. It is well-known that $\mathcal{P}$ has a dual representation given by a wiring diagram $\mathcal{W}$ which encodes the allowable sequence of $\mathcal{P}$ [13, 15]. We can view $\mathcal{W}$ as a system of curves drawn on the Möbius strip, each pair crossing once with all crossing points distinct. Extending $\mathcal{W}$ to its double cover, we obtain a system of curves $\mathcal{F}$ on the cylinder $S^1 \times \mathbb{R}^1$, each pair crossing twice, with all crossing points distinct (see Figure 1). Notice that the oriented matroid of $\mathcal{P}$ is encoded by $\mathcal{F}$ in the following way. For every triple $\mathcal{T} \subset \mathcal{F}$ the cyclic order in which the curves appear on the upper envelope of $\mathcal{T}$ agrees with the cyclic order of the corresponding triple of $\mathcal{P}$. This implies that a sub-configuration $\mathcal{P}' \subset \mathcal{P}$ is convexly independent if and only if every curve of the corresponding sub-system $\mathcal{F}' \subset \mathcal{F}$ appears on the upper envelope of $\mathcal{F}'$.

![Figure 1](image)

**Figure 1.** Top left: A point configuration $\mathcal{P}$; Top right: The dual wiring diagram $\mathcal{W}$ encoding the allowable sequence of $\mathcal{P}$; Bottom: Extension to the double cover resulting in the system $\mathcal{F}$ represented by smooth $2\pi$-periodic functions.

Each curve of $\mathcal{F}$ can be taken to be the graph of a smooth function $f : S^1 \to \mathbb{R}^1$. Blaschke showed that a smooth function $f : S^1 \to \mathbb{R}^1$ is the support function of a convex body if and only if $f + f'' > 0$ (see [19]). This implies that there exists a constant $c_0$ such that $f + c$ is the support function of a body for any $c > c_0$. Therefore there is a common constant we can add to each of the curves making $\mathcal{F}$ the dual system of an orientable arrangement. $\square$

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$^2$Grünbaum implicitly makes this observation in his discussion on planar arrangements of simple curves. We refer the reader to Section 3.3 of [20].
2.3. **Weak maps.** Let \( A \) and \( B \) be arrangements of \( n \) bodies. A bijection \( \varphi : A \rightarrow B \) is a *weak map* if \( \varphi^{-1}(B') \) is convexly independent for all convexly independent sub-configurations \( B' \subset B \). The inequality \( h_1(n) \leq g(n) \) is a consequence of the following.

**Lemma 3.** For every non-crossing arrangement \( A \), in which every triple is convexly independent, there exists a weak map \( \varphi : A \rightarrow B \) where \( B \) is an orientable arrangement.

The dual arrangement \( A^* \) induces a cell complex \( C(A) \), homeomorphic to \( S^1 \times [0,1] \). The weak map \( \varphi \) will be defined in terms of elementary operations on \( C(A) \). Since every triple of \( A \) is convexly independent, there are two types of triples to consider: The *orientable* and the *non-orientable* ones. The dual of a non-orientable triple \( T \) is characterized by one of its support curves occurring two distinct times on the upper envelope of \( T^* \). In \( C(T) \) this corresponds to a pair of disjoint triangular cells whose top edges are both contained in the same support curve (see Figure 2). Notice that in the non-orientable case these are the only triangular cells, while in the orientable case every cell is triangular.

![Figure 2](image2.png)

**Figure 2.** Left: An orientable triple with its dual system below. Right: A non-orientable triple with its dual system below.

A non-orientable triple \( T \) is related to an orientable one by an elementary operation called a *triangle flip*, which is defined by “flipping” the orientation of one of the two triangular cells of \( C(T) \) (see Figure 3). Notice that a triangle flip defines a weak map from a non-orientable triple to an orientable one.

![Figure 3](image3.png)

**Figure 3.** Above: A non-orientable triple. Below: The orientable triple obtained after applying a triangle flip.

We deduce Lemma 3 from the following.

**Lemma 4.** If \( A \) is not orientable, then \( C(A) \) contains a triangular cell bounded by the support curves of a non-orientable triple.

**Proof of Lemma 3.** If \( A \) is not orientable, Lemma 4 implies that we can apply a triangle flip to \( C(A) \) obtaining a new cell complex \( C' \). Clearly we may assume \( C' \) is induced by a system of smooth curves, which is therefore the dual system of an arrangement \( A' \) (as in the proof of Lemma 2). This induces a weak map \( \varphi' : A \rightarrow A' \). Since a triangle flip reduces the number of non-orientable triples, Lemma 3 follows by induction. \( \square \)
A few technical terms are needed for proving Lemma 4. Let $T$ be a non-orientable triple. The top edges of the two triangular cells of $C(T)$ belong to the same support curve, called the top curve, which appears twice on the upper envelope of $T^*$. When $T$ belongs to a larger system $F$, the triangular cells of $C(T)$ may no longer be cells in $C(F)$, so instead we refer to these open triangular regions as the zones of $T^*$. When we say that $T^*$ bounds a zone, it is implicit that $T$ is non-orientable. A zone is called empty if no curve of the system intersects its interior, and is called free if no curve intersects its top edge (see Figure 4). In other words, we want to show:

If $A$ contains non-orientable triples, then $C(A)$ contains an empty zone.

For the proof we will consider a minimal counter-example. It is, however, much easier to handle free zones rather than empty ones, so we first establish the following.

Claim 5. If $C(A)$ contains an free zone, then $C(A)$ contains an empty zone.

![Figure 4](image-url) The triple $T^* = \{a, b, c\}$ bounds two zones (shaded) and the top curve is $b$ (red). Neither of the zones of $T^*$ are empty, but the left one is free.

Proof of Claim 5 We assume without loss of generality that $Z_0$ is a free zone bounded by $a, b, c$ where $b$ is the top curve. Let $w_1, \ldots, w_k$ denote curves that intersect $Z_0$. The fact that every triple of $A$ is convexly independent has the following consequences (whose proofs we leave to the reader).

1. Each $w_i$ intersects $Z_0$ in a single connected arc. We may assume $w_i$ enters $Z_0$ by crossing curve $c$ and exits $Z_0$ by crossing curve $a$.
2. The triangular region in $Z_0$ bounded by $a, w_i, c$ is a zone.
3. Distinct curves $w_i$ and $w_j$ cross at most once inside $Z_0$.

Of course, the zones appearing inside $Z_0$ are not necessarily free, so we also need the following observation concerning zones that are not free.

Observation 6. Let $Z$ be a zone bounded by $a, b, c$ where $b$ is the top curve. Suppose $w$ enters $Z$ by crossing $c$ and exits $Z$ by crossing $b$, then proceeds to cross $a$. Then one of the triples $a, w, b$ or $w, b, c$ bound a zone. (See Figure 5)

For the proof of Claim 5 we proceed by induction on $k$, the number of curves which intersect $Z_0$. If $k = 0$, then $Z_0$ is an empty zone, so assume $k > 0$. Start at the top left corner of $Z_0$ at the crossing between $b$ and $c$. Move on the boundary of $Z_0$ along $c$ and stop at the first crossing we encounter. Assume that this is the crossing between $c$ and $w_k$. This crossing is the top corner of a zone $Z_k \subset Z_0$ (bounded by $a, w_k, c$ by (2) above). Move into the interior of $Z_0$ along curve $w_k$ (the top edge of $Z_k$) and stop at the first crossing we encounter. If this is the crossing between $w_k$ and $a$, we may apply the induction step, since then $Z_k$ is a free zone with less than $k$ intersecting curves. So assume that this is a crossing between $w_k$ and $w_{k-1}$. By Observation 6 this is the top vertex of a zone $Z_{k-1}$. If $Z_{k-1} \subset Z_k$ (i.e bounded by $w_{k-1}, w_k, c$), then $Z_{k-1}$ is free and we are done by induction, so assume $Z_{k-1}$ is bounded by paths $a, w_{k-1}, w_k$. Proceed along path $w_{k-1}$ (the top edge of $Z_{k-1}$) and repeat the process. Eventually a free zone which is crossed by less than $k$ curves is reached, completing the proof of Claim 5 (see Figure 6). □

We are in position to complete the proof of Lemma 4.
Figure 5. **Top:** The zone $Z$ is bounded by $a, b, c$ (shaded). After $w$ leaves $Z$ and crosses $a$ it enters a digon bounded by curves $a$ and $b$, so it must cross one of them again before crossing $c$. **Bottom left:** If the next crossing of $w$ is with $a$, then $a, w, b$ bound a zone (shaded). **Bottom right:** If the next crossing of $w$ is with $b$ then $w, b, c$ bound a zone (shaded).

Figure 6. Starting at top left corner of $Z_0$ (light shade) move along the boundary until we meet the first crossing. This is the top corner of a zone bounded by $a, w_k, c$. Proceed along $w_k$ until we meet the next crossing. By Observation 6 one of the two dark shaded regions must be a zone.

**Proof of Lemma 4.** Suppose $A$ is a minimal counter-example. Then $C(A)$ contains zones, but no empty ones, and any proper sub-arrangement $A' \subset A$ is either orientable or the complex $C(A')$ has at least one empty zone. We will reach a contradiction by showing that $C(A)$ contains a free zone.

Assume first that *any* curve we delete from the lower envelope of $A^*$ destroys all non-orientable triples. Then the lower envelope consists of exactly two curves $a$ and $c$, and a triple is non-orientable if and only if it includes both of these curves. To see this, note that if there were three curves on the lower envelope, then these form an orientable triple, so for any non-orientable triple there is a curve on the lower envelope not belonging to it. Let $Z$ be a zone bounded by $a, b, c$. Some curve $w$ should intersect the top edge of $Z$ or else it is free, in which case $a, b, c$ or $b, c, w$ is non-orientable, contradicting our initial assumption.

We may therefore assume that there is a curve $w$ appearing on the lower envelope, and a triple $a, b, c$ which bound a zone $Z$ where $b$ is the top curve, and $w$ is the only curve which meets the interior of $Z$. Furthermore $w$ must cross the top edge of $Z$ (if not $Z$ is free, contradicting the assumption that $A$ was a counter-example). Now we use the fact that $w$ was on the lower envelope. This implies that $w$ must also cross one of the other edges of $Z$. Up to symmetry there are then two cases that can occur (see Figure 7).

1. $w$ is on the lower envelope, crosses $a$, then $c$ (entering $Z$), and then $b$ (leaving $Z$).
2. $w$ is on the lower envelope, crosses $c$, then $a$ (entering $Z$), and then $b$ (leaving $Z$).

In both cases we consider the order in which $w$ intersects the other curves after leaving $Z$. In case (1) it results in a zone bounded by $w, b, c$ contained in $Z$. This must be empty, since $w$ is
the only curve that intersects \( Z \). In case (2) this results in a zone bounded by \( w, a, b \) where \( a \) is the top curve. It is adjacent to \( Z \) along the curve \( a \), which implies that it is a free zone since \( w \) is the only curve intersecting the interior of \( Z \). This completes the proof. \( \square \)

![Figure 7](image_url)

**Figure 7.** Consider \( w \) after it leaves \( Z \). Case (1), left: If \( w \) crosses \( b \) before \( c \), then \( w, b, c \) bound an empty zone contained in \( Z \). If \( w \) crosses \( c \) before \( b \), then \( w, a, c \) is not in convex position. Case (2), right: If \( w \) crosses \( a \) before \( c \), then \( w, a, c \) bound a free zone below \( Z \). If \( w \) crosses \( c \) before \( a \), then \( b \) intersects \( w \) again after its two crossings with \( a \), which implies that \( w, a, b \) is not in convex position.

### 3. Further generalizations

A **convex \( n \)-clustering** is a disjoint union of point sets \( S_1, S_2, \ldots, S_n \) of equal size such that all \( n \)-tuples \((p_1, p_2, \ldots, p_n)\) with \( p_i \in S_i \) are convexly independent. The cardinality of the \( S_i \) is the **size** of the clustering. This notion naturally extends to arrangements of bodies as well.

#### 3.1. The positive fraction version

Bárány and Valtr gave the following generalization of the Erdős-Szekeres theorem, known as the **positive fraction Erdős-Szekeres theorem**.

**Theorem (Bárány-Valtr [2]).** For every integer \( n > 3 \) there exists a constant \( c_n > 0 \) such that any finite set \( S \) in the Euclidean plane, in which every triple is convexly independent, contains a convex \( n \)-clustering of size \( c_n |S| \).

The current best value for \( c_n \) is due to Pór and Valtr [28], and shows that \( c_n \geq n \cdot 2^{-32n} \). Their argument can be repeated verbatim to hold for generalized configurations as well. By Lemmas 2 and 3 the positive fraction version and for generalized configurations is equivalent to the positive fraction version for non-crossing arrangements of bodies, and therefore holds with the same bound on \( c_n \).

**Theorem 7.** For every integer \( n > 3 \) there exists a constant \( c_n > 0 \) such that any non-crossing arrangement \( A \), in which every triple is convexly independent, contains a convex \( n \)-clustering of size \( c_n |A| \).

#### 3.2. The partitioned version

Answering a question of Kalai, the positive fraction version was further generalized by Pór and Valtr [28] with what is called the **partitioned Erdős-Szekeres theorem**.

**Theorem (Pór-Valtr [28]).** For every \( n \geq 3 \) there exist constants \( p = p_n \) and \( r = r_n \) such that for any finite set \( S \) the Euclidean plane, in which every triple is convexly independent, there is a sub-arrangement \( S' \) of size at most \( r \) such that \( S \setminus S' \) can be partitioned into at most \( p \) convex \( n \)-clusterings.

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The positive fraction Erdős-Szekeres theorem was first established for arrangements of mutually disjoint bodies by Pach and Solymosi [23], and their method was subsequently improved by Pór and Valtr [29]. Our methods imply a substantial quantitative improvement, as well as relaxing the disjointness assumption.
Extending this theorem to generalized configurations can be done in a more or less a routine way. Essentially one needs to modify the proofs of Claims 1 – 3 in [28]. This can be done by replacing any “distance arguments” by “continuous sweep arguments” (see for instance [7, 18]). The remaining parts of the proof of Pór and Valtr are combinatorial, and do not need further modification. By applying Lemmas 2 and 3 it follows that the fractional versions for generalized configurations and for non-crossing arrangements are equivalent. We therefore obtain the following.

Theorem 8. For every \( n \geq 3 \) there exist constants \( p = p_n \) and \( r = r_n \) such that the following holds. For every non-crossing arrangement \( A \) in which every triple is convexly independent there is a sub-arrangement \( A' \) of size at most \( r \) such that \( A \setminus A' \) can be partitioned into at most \( p \) convex \( n \)-clusterings.

Remark 9. It is natural to ask whether the non-crossing condition can be further relaxed. In a subsequent paper we show that this is indeed the case, confirming a conjecture of Pach and Tóth [27].

4. Acknowledgments

Dobbins was supported by NRF grant 2011-0030044 funded by the government of South Korea (SRC-GAIA) and BK21. Holmsen was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021048). Hubard was supported by Fondation Sciences Mathématiques de Paris. Hubard would like to thank KAIST for their hospitality and support during his visit.

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