GAUSSIAN FLUCTUATIONS FOR $\beta$ ENSEMBLES.

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Abstract. We study the Circular and Jacobi $\beta$-Ensembles and prove Gaussian fluctuations for the number of points in one or more intervals in the macroscopic scaling limit.

1. Introduction

The circular $\beta$ ensemble with $n$ points, or $C\beta E_n$, was introduced by Dyson, \[7\], as a simplification of previous random matrix ensembles. It is a random process of $n$ points distributed on the unit circle in $\mathbb{C}$ so that

$$
\mathbb{E}_n^\beta(f) = \frac{1}{Z_{n,\beta}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(e^{i\phi_1}, \ldots, e^{i\phi_n}) |\Delta(e^{i\phi_1}, \ldots, e^{i\phi_n})|^{\beta \frac{d\phi_1}{2\pi} \cdots \frac{d\phi_n}{2\pi}}
$$

(1.1)

for any symmetric function $f$. Here, $\Delta$ denotes the Vandermonde determinant,

$$
\Delta(z_1, \ldots, z_n) = \prod_{1 \leq j < k \leq n} (z_k - z_j),
$$

(1.2)

and the partition function is given by

$$
Z_{n,\beta} = \frac{\Gamma(\frac{1}{2} \beta n + 1)}{[\Gamma(\frac{1}{2} \beta + 1)]^n};
$$

(1.3)

as shown in \[9, 31\]. This law also arises as the Gibbs measure for $n$ identical charged particles confined to lie on the circle and interacting via the 2-dimensional Coulomb law. For this reason, it is also known as the log-gas. An important example of (1.1) is as the distribution of eigenvalues of an $n \times n$ unitary matrix chosen at random according to Haar measure; this corresponds to $\beta = 2$.

The second point process we will discuss is known as the Jacobi $\beta$-ensemble with $n$-points. (We will abbreviate this name to $J\beta E_n$.) In this case, there are $n$ points on the interval $[-2, 2]$ with joint probability density

$$
p_n^\beta(x_1, \ldots, x_n) = \frac{1}{Z} |\Delta(x_1, \ldots, x_n)|^{\beta} \prod_{j=1}^{n} (2 - x_j)^{a-1}(2 + x_j)^{b-1},
$$

(1.4)

where the parameters $a, b$ are positive real numbers. The partition function, $Z$, was determined by Selberg, \[19\]. The name Jacobi ensemble derives from the relation to the classical orthogonal polynomials of this name.

In this paper, we determine the statistical behaviour of the number of particles in one or more intervals as $n \to \infty$ with the length of the intervals remaining fixed as $n \to \infty$. This is what may be termed macroscopic statistics; the microscopic regime, where the length of the intervals scale as $\frac{1}{n}$, is much more difficult and will not be addressed.

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The statement for the Jacobi ensemble is the cleaner of the two:

**Theorem 1.1.** Given a sample from $J\beta E_n$, let $N_n(\theta)$ denote the number of points that lie in the interval $[2\cos(\theta), 2]$. Then for any distinct $\theta_1, \ldots, \theta_J \in (0, \pi)$,

$$\sqrt{\frac{\pi^2 \beta}{\log(n)}}[N_n(\theta_j) - n\theta_j], \quad j \in \{1, \ldots, J\}$$

converge to independent Gaussian random variables with mean 0 and variance 1 as $n \to \infty$.

For the circular ensemble, we need one preliminary. Let us write $\Psi$ for the zero-mean Gaussian process on $(-\pi, \pi)$ with covariance

$$E\{\Psi(\theta_1)\Psi(\theta_2)\} = \begin{cases} 1 : \theta_1 = \theta_2 \\ 0 : \theta_1 \neq \theta_2 \end{cases}$$

that is, $\Psi(\theta)$ represents an independent Gaussian associated to each point in this arc.

**Theorem 1.2.** Let us fix $\theta_1 < \theta_2 < \cdots < \theta_J \in (-\pi, \pi)$ and write $N(a, b)$ for the number of points in a sample from $C\beta E_n$ that lie in the arc between $a$ and $b$. Then the joint distribution of

$$\sqrt{\frac{\pi^2 \beta}{2 \log(n)}}[N_n(\theta_j, \theta_l) - n\frac{\theta_l - \theta_j}{2\pi}], \quad j < l$$

converges to that of $\Psi(\theta_l) - \Psi(\theta_j)$ as $n \to \infty$.

In the special case of a single arc, we see that

$$\sqrt{\frac{\pi^2 \beta}{2 \log(n)}}[N_n(0, \theta) - n\frac{\theta}{2\pi}]$$

converges to Gaussian random variable of mean zero and variance one.

For $\beta \in \{1, 2, 4\}$, Theorem 1.2 was proved by Costin and Lebowitz, [2]. At these three temperatures, the model becomes exactly soluble in the sense that there are explicit determinantal formulae for the correlation functions; see [8] or [16]. There are many other works on Gaussian fluctuations at these three temperatures; in particular, we would like to draw the reader’s attention to [3, 4, 11, 13, 22, 23, 24, 30]. One of the questions addressed in these papers are the laws of other linear statistics. Given a point process $\{x_j\}$, say on $\mathbb{R}$, and a function $f : \mathbb{R} \to \mathbb{R}$, the associated linear statistic is

$$X_f := \sum_j f(x_j).$$

In particular, the number of points in an interval $I$ corresponds to $f = \chi_I$.

For the circular problem with $\beta = 2$, the behaviour of linear statistics can be recast as a question about the asymptotics of Toeplitz determinants. In this case, we may attribute the proof of asymptotically Gaussian fluctuations to Szegő, [25], at least for $f \in C^{1+\epsilon}$. Of the numerous papers devoted to Szegő’s Theorem in the last half-century, one stands out for its adaptability to the case of general $\beta$, namely [10]. This paper proves asymptotically Gaussian fluctuations for $C\beta E$ with $f \in C^{1+\epsilon}$. A subsequent paper, [11], covers ensembles on the real line.

The approach taken here is to use matrix models for $C\beta E_n$ and $J\beta E_n$ that where described in [14]. This paper extended work of Dumitriu and Edelman, [5], who discovered tri-diagonal matrix models the $\beta$-Hermite and $\beta$-Laguerre ensembles. By ‘matrix model’ we mean an ensemble of random matrices whose eigenvalues
follow the desired law. We should also mention [27] as the progenitor of both these papers.

In [15], Dumitriu and Edelman proved Gaussian linear statistics for polynomials by studying traces of powers of their matrix models.

In [6], the circular matrix models were studied in the microscopic scaling limit using a Prüfer variables approach. Having finished that paper, it dawned upon me that the use of Prüfer variables leads to a very simple treatment of the problem of the number of particles in an interval. This is what is presented here.

The virtue of using Prüfer variables to treat the microscopic scaling limit of the matrix models was discovered independently by Valkó and Virág, [29]. Specifically, they study the eigenvalue statistics of the $\beta$-Hermite ensemble in a neighbourhood of a reference energy lying in the bulk. Earlier, [18] studied the statistics in a neighbourhood of the edge using the Riccati transformation.

Notation. While dealing with the Jacobi ensembles, we will write $X \lesssim Y$ to indicate that $X \leq CY$ for some constant $C$. In all cases, the implicit constant will depend only on $\beta, a$, and $b$.

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2. Background

2.1. A change of variables. In this subsection, we recount the results of [14]. This paper describes probability distributions on discrete measures with the property that the marginal distribution of the location of the mass points happen to have the laws that interest us. The virtue of the random measures described in [14] appears when one looks at them from the point of view of orthogonal polynomials.

Given a probability measure $d\mu$ on the unit circle in $\mathbb{C}$ that is supported at exactly $N$ points, we may write

$$\int f \, d\mu = \sum \mu_j f(e^{i\theta_j}). \quad (2.1)$$

Applying the Gram–Schmidt procedure to $\{1, z, \ldots, z^{N-1}\}$ leads to an orthogonal basis for $L^2(d\mu)$ built of monic polynomials. We write $\Phi_0(z) \equiv 1, \ldots, \Phi_{N-1}(z)$ for these polynomials.

As discovered by Szegő, these polynomials obey a recurrence relation,

$$\begin{align*}
\Phi_{k+1}(z) &= z\Phi_k(z) - \bar{\alpha}_k \Phi_k^*(z) \\
\Phi_{k+1}^*(z) &= \Phi_k^*(z) - \alpha_k z \Phi_k(z)
\end{align*} \quad (2.2)$$

with $\Phi_0^* = \Phi_0 = 1$. Here $\Phi_k^*$ denotes the reversed polynomial,

$$\Phi_k^*(z) = z^k \overline{\Phi_k(\bar{z}^{-1})}, \quad (2.3)$$

and $\alpha_0, \ldots, \alpha_{N-2}$ are recurrence coefficients belonging to $\mathbb{D}$. Expanding $z^N$ in this basis shows that

$$z\Phi_{N-1}(z) - e^{-i\eta} \Phi_{N-1}^*(z) = 0 \quad \text{in } L^2(d\mu) \quad (2.4)$$

for a unique $\eta \in [0, 2\pi)$. A basic fact in the theory of orthogonal polynomials is that the mapping

$$\{(\theta_1, \mu_1), \ldots, (\theta_N, \mu_N)\} \mapsto (\alpha_0, \ldots, \alpha_{N-2}, e^{-i\eta})$$
is a bijection. For a proof of this (and the other claims in this subsection), see\[20\] or \[26\]. As in the former reference, we will refer to $\alpha_0, \ldots, \alpha_{N-2}, e^{-i\eta}$ as the Verblunsky parameters of $d\mu$.

It will be expedient for us to encode (2.2) in a different way. As the zeros of the orthogonal polynomials lie inside the unit disk, $B_k(z) = z\Phi_k(z)/\Phi_k^*(z)$ is a Blaschke product of degree $k+1$; moreover,

$$B_0(z) = z, \quad B_{k+1}(z) = z B_k(z) \frac{1 - \bar{\alpha}_k B(z)}{1 - \alpha_k B(z)}. \tag{2.5}$$

Lastly, by (2.4), we have $\text{supp}(d\mu) = \{ z : B_{N-1}(z) = e^{-i\eta} \}$.

We have two systems of coordinates for the set of probability measures on $S^1$ with exactly $N$ mass points, on the one hand, the location and weight of the masses, on the other, the Verblunsky coefficients. Choosing random Verblunsky coefficients gives rise to a random measure and thus to a random subset of the unit circle. The good news is that they will be chosen independently. In the model for the the circular ensembles, they will follow a $\Theta$ distribution:

**Definition 2.1.** A complex random variable, $X$, with values in the unit disk, $\mathbb{D}$, is $\Theta_\nu$-distributed (for $\nu > 1$) if

$$\mathbb{E}\{ f(X) \} = \frac{\nu^n-1}{\nu^n} \int_D f(z) (1-|z|^2)^{(\nu-3)/2} \, d^2z. \tag{2.6}$$

Simple computations show $\mathbb{E}\{ X \} = 0$, $\mathbb{E}\{ |X|^2 \} = \frac{2}{\nu+1}$ and $\mathbb{E}\{ |X|^4 \} = \frac{8}{(\nu+1)(\nu+3)}$.

In the Jacobi ensemble, they will be Beta-distributed:

**Definition 2.2.** A real-valued random variable $X$ is said to be Beta-distributed with parameters $s, t > 0$, which we denote by $X \sim B(s, t)$, if

$$\mathbb{E}\{ f(X) \} = \frac{2^{1-s-t}\Gamma(s+t)}{\Gamma(s)\Gamma(t)} \int_{-1}^1 f(x)(1-x)^{s-1}(1+x)^{t-1} \, dx. \tag{2.7}$$

Note that $\mathbb{E}\{ X \} = \frac{s}{s+t}$ and $\mathbb{E}\{ X^2 \} = \frac{(s+t)^2 + (s+t)}{(s+t)(s+t+1)}$.

The combined content of Theorem 1.2 and Proposition B.2 from \[14\] is:

**Theorem 2.3.** Given $\beta > 0$, let $\alpha_k \sim \Theta_{\beta(k+1)+1}$ be independent random variables and let $e^{i\eta}$ be independent and uniformly distributed on $S^1$. If one forms $B_{n-1}(z)$ according to (2.3), then

$$\{ z : B_{n-1}(z) = e^{-i\eta} \}$$

is distributed according to the $\text{CBE}_n$ ensemble.

The result for the Jacobi model is similar; however, we need one extra ingredient. A probability measure on $S^1$ is invariant under complex conjugation,

$$\int f(z) d\mu(z) = \int f(\bar{z}) d\mu(z),$$

if and only if it has real Verblunsky coefficients. From Theorem 1.5 and Proposition B.2 of \[14\], we have

**Theorem 2.4.** Given $\beta > 0$, let $\alpha_k$ be independent and distributed as follows

$$\alpha_k \sim \begin{cases} B\left(\frac{k}{4}\beta + a, \frac{k}{4}\beta + b\right) & k \text{ even}, \\ B\left(\frac{k+1}{4}\beta + a + b, \frac{k}{4}\beta\right) & k \text{ odd}. \end{cases} \tag{2.8}$$
Then the $n$ points
\[ \{ z + \bar{z} : B_{2n-1}(z) = -1 \} \tag{2.9} \]
are distributed according to the Jacobi ensemble \([1.4]\).

2.2. The basic processes. In the previous subsection, we showed how one could encode the $\beta$ ensembles as random Blaschke products built from independent random variables through a simple recurrence, \([2.5]\). In this subsection, we will take matters a few steps further; a similar path was followed in \([15]\).

From Theorems \([2.3]\) and \([2.4]\) we see that we need only study the Blaschke products on the boundary of the unit circle. To this end, let us introduce random continuous functions $\psi_k : (-\pi, \pi) \to \mathbb{R}$ via $B_k(e^{i\theta}) = e^{i\psi_k(\theta)}$. We break the $2\pi\mathbb{Z}$ ambiguity by choosing a branch for the logarithm in the recurrence relation \([2.5]\). Specifically, we define
\[ \psi_0(\theta) = \theta, \quad \psi_{k+1}(\theta) = \psi_k(\theta) + \theta + \Upsilon(\psi_k(\theta), \alpha_k), \tag{2.10} \]
where
\[ \Upsilon(\psi, \alpha) := -2 \Im \log[1 - \alpha e^{i\psi}] = \Im \sum_{\ell=1}^{\infty} 2\ell e^{i\ell\psi} \alpha^\ell. \tag{2.11} \]
As the argument of a Blaschke product, $\psi_k$ is an increasing function of $\theta$.

Readers wishing to compare this with \([15]\) should note that here $\psi_k(\theta)$ denotes the true Prüfer phase rather than the relative Prüfer phase.

To motivate what follows, let us give a quick description of how we will prove Gaussian fluctuations in the circular case. The Jacobi case is analogous. If we choose $\eta, \alpha_k$ to be distributed as in Theorem \([2.3]\) and use these to produce a sequence of increasing functions $\psi_k(\theta)$ as in \([2.10]\), then the set of points
\[ \{ e^{i\theta} : \psi_{n-1}(\theta) \in 2\pi\mathbb{Z} + \eta \} \]
will be distributed according to the $C\beta E_n$. In particular, the number of points lying in the arc $[a, b] \subset (-\pi, \pi)$ is approximately $\frac{1}{2\pi} [\psi_{n-1}(b) - \psi_{n-1}(a)]$; indeed the error is plus or minus one. In this way, it suffices to show that asymptotically, $\psi_{n-1}(b)$ and $\psi_{n-1}(a)$ follow a joint normal law. The error of $\pm 1$ will drop out in the limit when we divide by the square root of the variance.

In the circular case, $\psi_k(\theta) - (k+1)\theta$ is a sum of independent random variables (because $\alpha_k$ have rotationally invariant laws) and so it is easy to demonstrate that it has Gaussian behaviour. To study the joint distribution of $\psi_{n-1}$ at several values of $\theta$ takes a little more care, but can be dealt with using the Central Limit Theorem for Martingales. Implementing this argument requires a few estimates on $\Upsilon$, which we record here.

**Lemma 2.5.** Suppose $\phi, \psi \in \mathbb{R}$ and $\alpha \sim \Theta_L$. If $\tilde{\Upsilon}(\psi, \alpha) = 2 \Im \{ e^{i\psi} \alpha \}$, then
\[ \mathbb{E}\{ \Upsilon(\psi, \alpha) \} = \mathbb{E}\{ \tilde{\Upsilon}(\psi, \alpha) \} = 0, \tag{2.12} \]
\[ \mathbb{E}\{ \tilde{\Upsilon}(\psi, \alpha) \tilde{\Upsilon}(\phi, \alpha) \} = \frac{4}{\nu + 1} \cos(\psi - \phi), \tag{2.13} \]
\[ \mathbb{E}\{ \tilde{\Upsilon}(\psi, \alpha)^4 \} = \frac{48}{(\nu + 1)(\nu + 3)}, \tag{2.14} \]
\[ \mathbb{E}\{ | \Upsilon(\psi, \alpha) - \tilde{\Upsilon}(\psi, \alpha) |^2 \} \leq \frac{16}{(\nu + 1)(\nu + 3)}. \tag{2.15} \]
Lastly, combining (2.13) and (2.15) shows
\[ E\{|\Upsilon(\psi, \alpha)|^2\} \leq \frac{8}{(\nu+1)}. \]  
(2.16)

**Proof.** The vanishing of \( E\{|\Upsilon(\psi, \alpha)|\} \) merely uses the fact that \( \alpha \) follows a rotationally invariant law. Specifically, for any \( 0 \leq r < 1 \),
\[ \int_0^{2\pi} \log[1 - re^{i\theta + i\psi}] \frac{d\theta}{2\pi} = 0 \]
by the Mean Value Principle for harmonic functions. The second identity in (2.12) follows in the same manner.

To prove (2.13) we simply compute:
\[ E\{\tilde{\Upsilon}(\psi, \alpha) \tilde{\Upsilon}(\phi, \alpha)\} = 4(\nu - 1)^2 \pi \int_0^{2\pi} r^2 \sin(\theta + \psi)(1 - r^2)^{(\nu-3)/2} r \, dr \, d\theta \]
\[ = \frac{4}{\nu+1} \cos(\psi - \phi). \]

Similarly,
\[ E\{\tilde{\Upsilon}(\psi, \alpha)^4\} = \frac{16(\nu - 1)^2}{2\pi} \int_0^{2\pi} r^4 \sin^4(\theta + \psi)(1 - r^2)^{(\nu-3)/2} r \, dr \, d\theta \]
\[ = \frac{48}{(\nu+1)(\nu+3)}. \]  
(2.17)

Applying Plancharel’s theorem to the power series formula for \( \Upsilon \) gives
\[ E\{|\Upsilon(\psi, \alpha) - \tilde{\Upsilon}(\psi, \alpha)|^2\} = \sum_{\ell=2}^{\infty} \frac{2}{\ell^2} E\{|\alpha|^{2\ell}\} \leq 2\left(\frac{\pi^2}{6} - 1\right) E\{|\alpha|^4\}. \]  
(2.19)

Noting that \( \frac{\pi^2}{6} \leq 2 \) and \( E\{|\alpha|^4\} = \frac{8}{(\nu+1)(\nu+3)} \) proves (2.15). \( \square \)

The analogue of Lemma 2.5 for the Jacobi case is

**Lemma 2.6.** Let \( \tilde{\Upsilon}(\psi, \alpha) = 2(\alpha - E(\alpha)) \sin(\psi) \) with \( \alpha \sim B(s, t) \). Then
\[ E\{\tilde{\Upsilon}(\psi, \alpha)\} = 0 \]  
(2.20)
\[ E\{\tilde{\Upsilon}(\psi, \alpha) \tilde{\Upsilon}(\phi, \alpha)\} = \frac{8st}{s+t+1} \left[\cos(\psi - \phi) - \cos(\psi + \phi)\right] \]  
(2.21)
for any \( \phi, \psi \in \mathbb{R} \). Moreover, if \( \alpha_k \) is distributed as in (2.8), then
\[ E\{\tilde{\Upsilon}(\psi, \alpha_k)^4\} \lesssim (k+1)^{-2} \]  
(2.22)
\[ E\{|\Upsilon(\psi, \alpha_k) - 2\alpha_k \sin(\psi) - \alpha_k^2 \sin(2\psi)|^2\} \lesssim (k+1)^{-2} \]  
(2.23)
\[ E\{|\Upsilon(\psi, \alpha_k)|^2\} \lesssim (k+1)^{-1/2} \]  
(2.24)
where the implicit constant does not depend on \( k \), merely \( \beta, a, \) and \( b \).

**Proof.** Equation (2.20) is immediate from the definition, while (2.21) follows from (A.24) and \( 2\sin(\psi)^2 \sin(\psi) = \cos(\psi - \phi) - \cos(\psi + \phi) \).

To obtain (2.22), we can estimate rather crudely:
\[ E\{\tilde{\Upsilon}(\psi, \alpha_k)^4\} \leq 2^8 E\{\alpha_k^4\} \lesssim (k^2 + 1)^{-1} \]
using (A.4). Next we prove (2.23). It follows from combining

\[
\text{LHS}(2.23) = \mathbb{E}\left\{ \sum_{l=3}^{\infty} \frac{2}{l} \sin(l\psi) \alpha_l \right\} \leq \left| \mathbb{E}\{\alpha^3\} \right| + \mathbb{E}\left\{ \sum_{m=2}^{\infty} \left[ \frac{2}{m} + \frac{2}{m+1} \right] \alpha_{2m} \right\} \\
\leq \left| \mathbb{E}\{\alpha^3\} \right| + 2 \mathbb{E}\left\{ -\alpha^2 \log|1 - \alpha^2| \right\}.
\]

with (A.3) and Lemma A.1.

The last estimate follows from (2.23) and (A.2) by applying the triangle and Cauchy–Schwarz inequalities. □

On several occasions we will make use of

Lemma 2.7. Given real valued sequences \( \epsilon_k, X_k, Y_k \) with \( X_{k+1} = X_k + \delta + Y_k \) for some \( \delta \in (0, 2\pi) \),

\[
\left| \sum_{k=0}^{n-1} \epsilon_k e^{iX_k} \right| \leq 2 \left\| \epsilon_k \right\| \ell_{\infty} + \left\| \epsilon_k - \epsilon_{k-1} \right\| \ell_1 + \left\| \epsilon_k Y_k \right\| \ell_1.
\]

Proof. Using summation by parts and other elementary manipulations,

\[
\begin{align*}
\sum_{k=0}^{n-1} \epsilon_k e^{iX_k} &= \sum_{k=0}^{n-1} \epsilon_k \frac{e^{iX_k} - e^{i(X_k + \delta)}}{1 - e^{i\delta}} \\
&= \sum_{k=0}^{n-1} \epsilon_k e^{iX_k} - \sum_{k=0}^{n-1} \epsilon_k e^{iX_{k+1}} + \sum_{k=0}^{n-1} \epsilon_k \frac{e^{iX_{k+1}} - e^{i(X_k + \delta)}}{1 - e^{i\delta}} \\
&= \frac{e_n e^{iX_n} - e_0 e^{iX_0}}{1 - e^{i\delta}} + \sum_{k=1}^{n-1} \epsilon_k - \epsilon_{k-1} - \frac{e^{i(X_k + \delta)} [e^{iY_k} - 1]}{1 - e^{i\delta}}.
\end{align*}
\]

This implies (2.25) in a most obvious way. □

2.3. Martingale Central Limit Theorem. In both the Circular and Jacobi cases, we will approximate \( \psi_{m-1}(\theta) \) by

\[
S(m, \theta) = \sum_{k=0}^{m-1} \tilde{\Upsilon}(\psi_k(\theta), \alpha_k).
\]

The recursive definition of \( \psi_k \) and the fact that \( \mathbb{E}\{\tilde{\Upsilon}\} = 0 \) shows that this is a martingale for the sigma algebras

\[
\mathcal{M}_m := \sigma(\alpha_0, \ldots, \alpha_{m-1}),
\]

that is, the smallest sigma algebra generated by these random variables. In the circular case, there is no need to pass from \( \tilde{\Upsilon} \) to \( \tilde{\Upsilon} \) in order to obtain a martingale; nevertheless, doing so simplifies the proof just a little.
Proposition 2.8. Fix $\theta_1, \ldots, \theta_J$ distinct. Suppose that as $n \to \infty$
\[
\frac{1}{\log(n)} \sum_{k=0}^{n-1} \mathbb{E} \left\{ \tilde{Y}(\psi_k(\theta_j), \alpha_k) \tilde{Y}(\psi_k(\theta_l), \alpha_k) \right\} \to \sigma^2 \delta_{jl} \quad \text{in } L^1 \text{ sense} \quad (2.27)
\]
\[
\frac{1}{\log^2(n)} \sum_{k=0}^{n-1} \mathbb{E} \left| \tilde{Y}(\psi_k(\theta_j), \alpha_k) \right|^4 \to 0 \quad (2.28)
\]
\[
\frac{1}{\sqrt{\log(n)}} \mathbb{E} \left\{ \left| \sum_k Y(\psi_k(\theta_j), \alpha_k) - \tilde{Y}(\psi_k(\theta_j), \alpha_k) \right| \right\} \to 0 \quad (2.29)
\]
for all $1 \leq j, l \leq J$. Then in the random variables
\[
\frac{\psi_n^{-1}(\theta_j) - n\theta_j}{\sqrt{\log(n)}}, \quad j \in \{1, 2, \ldots, J\}, \quad (2.30)
\]
converge to independent Gaussians of mean zero and variance $\sigma^2$.

Proof. The first two conditions are far more than is really needed to apply the
Martingale Central Limit Theorem to the sequence of random vectors
\[
\frac{1}{\sqrt{\log(n)}} (S(n, \theta_1), \ldots, S(n, \theta_J));
\]
see [17, 28], for example. These references present the proof for scalar martingales
— a small extension of the usual Central Limit Theorem — which can then be
applied to any linear combination of the components of the vector. In this way, we
see that
\[
\frac{1}{\sqrt{\log(n)}} (S(n, \theta_1), \ldots, S(n, \theta_J)) \to N(0, \sigma^2 \delta_{jl})
\]
in distribution. This extends to (2.30) because (2.29) says that the difference be-
tween these to vectors converges to 0 in $L^1$ sense. \qed

3. Circular case

In this section, we prove Theorem 1.2. The discussion in the previous section
explains why this amounts to checking the hypotheses in Proposition 2.8. We begin
with a recap.

Let $\alpha_k \sim \Theta_{\beta(k+1)+1}$ be chosen independently (as in Theorem 2.3) and then form
the process $\psi_n^{-1}(\theta)$ by solving the recurrence (2.10). If $\eta$ is chosen independently
from $[0, 2\pi)$ according to the uniform distribution, then
\[
\{ e^{i\phi} : \psi_n^{-1}(\phi) + \eta \in 2\pi \mathbb{Z} \}
\]
is distributed according to $C\beta E_n$. Notice that the number of points in the arc
$[a, b] \subset (-\pi, \pi)$ differs from $\frac{1}{2\pi} [\psi_n^{-1}(b) - \psi_n^{-1}(a)]$ by at most $\pm 1$. Thus Theorem 1.2
will follow once we show that
\[
\frac{1}{\sqrt{\log(n)}} [\psi_n^{-1}(\theta) - n\theta] \to \sqrt{\frac{1}{2\pi}} \Psi(\theta)
\]
in distribution. This in turn follows from Proposition 2.8 once we check its hyp-
otheses. The two lemmas that follow verify these conditions in the order they
appear there.
Lemma 3.1. Given $\theta_1, \theta_2 \in (-\pi, \pi)$,
\[
\frac{1}{\log(n)} \sum_{k=0}^{n-1} \mathbb{E}\left\{ \tilde{\Upsilon}^2(\psi_k(\theta_1), \alpha_k) \right\} \to \begin{cases} 
\frac{4}{\beta} : \theta_1 = \theta_2 \\
0 : \theta_1 \neq \theta_2
\end{cases}
\tag{3.1}
\]
in $L^1$ sense.

Proof. By (2.13),
\[
\text{LHS (3.1)} = \frac{1}{\log(n)} \sum_{k=0}^{n-1} \frac{4}{\beta(k+1)+2} \cos(\psi_k(\theta_1) - \psi_k(\theta_2)),
\tag{3.2}
\]
which immediately settles the case $\theta_1 = \theta_2$. By symmetry, this leaves us only to treat the case $\theta_1 > \theta_2$. We do this using Lemma 2.7 with
\[
X_k = \psi_k(\theta_1) - \psi_k(\theta_2), \quad Y_k = \Upsilon(\psi_k(\theta_1), \alpha_k) - \Upsilon(\psi_k(\theta_2), \alpha_k), \quad \epsilon_k = \frac{4}{\beta(k+1)+2},
\]
and $\delta = \theta_1 - \theta_2$. Indeed, we may deduce
\[
\mathbb{E}\left\{ |\text{RHS (3.2)}| \right\} = O(1/\log(n))
\]
since by (2.13), $\mathbb{E}\{|Y_k|\} \leq 8(\beta(k+1) + 2)^{-1/2}$. □

Lemma 3.2. Given $\theta \in (-\pi, \pi)$,
\[
\frac{1}{\log^2(n)} \sum_{k=0}^{n-1} \mathbb{E}\{|\tilde{\Upsilon}(\psi_k(\theta), \alpha_k)|^4\} \to 0 \tag{3.3}
\]
\[
\frac{1}{\log(n)} \mathbb{E}\left\{ \left( \sum_{k=0}^{n-1} |\Upsilon(\psi_k(\theta), \alpha_k) - \tilde{\Upsilon}(\psi_k(\theta), \alpha_k)| \right)^2 \right\} \to 0 \tag{3.4}
\]
as $n \to \infty$.

Proof. From (2.14), we see that not only does LHS (3.3) converge to zero, it is $O(1/\log^2(n))$. We turn now to (3.4). By (2.12) and (2.15) we have
\[
\text{LHS (3.4)} = \frac{1}{\log(n)} \sum_{k=0}^{n-1} \mathbb{E}\{|\tilde{\Upsilon}^2(\psi_k(\theta), \alpha_k) - \tilde{\Upsilon}(\psi_k(\theta), \alpha_k)|^2\}
\leq \frac{1}{\log(n)} \sum_{k=0}^{n-1} \frac{16}{\beta(k+1) + 2|\beta(k+1) + 4|},
\]
which is $O(1/\log(n))$. Note that (3.4) implies (2.20) via the Cauchy–Schwarz inequality. □

4. Jacobi case

In the Jacobi case we choose $\alpha_k$ to be distributed as in Theorem 2.4. For then
\[
\{2 \cos(\theta) : \psi_{n-1}(\theta) - \pi \in 2\pi \mathbb{Z}, \theta \in (0, \pi)\}
\]
is distributed according to $J\beta E_n$ with parameters $a$ and $b$. As the Verblunsky coefficients are real-valued, $\psi_k(0) \equiv 0$. Thus
\[
|N_n(\theta) - \frac{1}{n} \psi_{n-1}(\theta)| \leq \frac{1}{2}
\]
with probability one. Recall from the introduction that $N_n(\theta)$ denotes the number of particles in the interval $[2 \cos(\theta), 2]$. 

In light of this discussion, we see that in order to prove Theorem 1.1 we need only show that for any distinct $\theta_1, \ldots, \theta_j \in (0, \pi)$,

$$\sqrt{\frac{n}{4 \log(n)}} \left[ \psi_{n-1}(\theta_j) - n\theta_j \right]$$

converge to independent Gaussian random variables with mean 0 and variance 1 as $n \to \infty$. This in turn can be effected via Proposition 2.8 provided we verify its hypotheses. That is precisely what we will do.

**Lemma 4.1.** Given $\theta_1, \theta_2 \in (0, \pi)$,

$$\frac{1}{\log(n)} \sum_{k=0}^{n-1} \mathbb{E} \left\{ \widetilde{\Upsilon}(\psi_k(\theta_1), \alpha_k)\widetilde{\Upsilon}(\psi_k(\theta_2), \alpha_k) \right\} \rightarrow \begin{cases} \frac{4}{\beta} : \theta_1 = \theta_2 \\ 0 : \theta_1 \neq \theta_2 \end{cases}$$

(4.1)
in $L^1$ sense.

**Proof.** By (2.21),

$$\text{LHS} (4.1) = \frac{1}{\log(n)} \sum_{k=0}^{n-1} \epsilon_k \left[ \cos(\psi_k(\theta_1) - \psi_k(\theta_2)) - \cos(\psi_k(\theta_1) + \psi_k(\theta_2)) \right]$$

(4.2)

where

$$\epsilon_k = \begin{cases} \frac{4(\kappa_1 + 2a + b \kappa_1 + 2a + 2b + 2)}{\kappa_1 + 2a + 2b} & : k \text{ even} \\ \frac{4(\kappa_1 - 2a + 2b + 2 \kappa_1 + 2a + 2b + 2)}{\kappa_1 + 2a + 2b} & : k \text{ odd} \end{cases}$$

Applying Lemma 2.7 with $X_k = \psi_k(\theta_1) + \psi_k(\theta_2)$, $Y_k = \widetilde{\Upsilon}(\psi_k(\theta_1), \alpha_k) + \widetilde{\Upsilon}(\psi_k(\theta_2), \alpha_k)$, and $\delta = \theta_1 + \theta_2$, then using (2.24), shows that

$$\text{LHS} (4.1) = \frac{1}{\log(n)} \sum_{k=0}^{n-1} \epsilon_k \cos(\psi_k(\theta_1) - \psi_k(\theta_2)) + O(1/\log(n)).$$

(4.3)

This renders (4.1) in the same manner as in Lemma 3.1. \qed

**Lemma 4.2.** Given $\theta \in (0, \pi)$,

$$\frac{1}{\log^2(n)} \sum_{k=0}^{n-1} \mathbb{E} \left\{ |\widetilde{\Upsilon}(\psi_k(\theta), \alpha_k)|^4 \right\} \rightarrow 0$$

(4.4)

$$\frac{1}{\sqrt{\log(n)}} \mathbb{E} \left\{ \left| \sum_{k=0}^{n-1} \Upsilon(\psi_k(\theta), \alpha_k) - \widetilde{\Upsilon}(\psi_k(\theta), \alpha_k) \right| \right\} \rightarrow 0$$

(4.5)
as $n \to \infty$.

**Proof.** By (2.22), not only does LHS (4.4) converge to zero, it is $O(1/\log^2(n))$.

We turn now to (4.5). By (2.23),

$$\text{LHS} (4.5) = \frac{1}{\sqrt{\log(n)}} \mathbb{E} \left\{ \left| \sum_{k=0}^{n-1} 2 \mathbb{E} \left\{ \alpha_k \right\} \sin(\psi_k(\theta)) - \alpha_k^2 \sin(2\psi_k(\theta)) \right| \right\} + O\left( \frac{1}{\sqrt{\log(n)}} \right).$$

Lemma 2.7 with $\delta = \theta$, $X_k = \psi_k(\theta)$, and $\epsilon_k = \mathbb{E} \left\{ \alpha_k \right\}$ shows that

$$\mathbb{E} \left\{ \left| \sum_{k=0}^{n-1} 2 \mathbb{E} \left\{ \alpha_k \right\} \sin(\psi_k(\theta)) \right| \right\} = O(1)$$
and in a similar way,
\[
E\left\{ \left| \sum_{k=0}^{n-1} E\{\alpha_k^2\} \sin(2\psi_k(\theta)) \right| \right\} = O(1).
\]

In view of these estimates, (4.5) follows from
\[
E\left\{ \left| \sum_{k=0}^{n-1} \left[ \alpha_k^2 - E\{\alpha_k^2\} \right] \sin(2\psi_k(\theta)) \right| \right\}^2 = \sum_{k=0}^{n-1} E\left\{ \left[ \alpha_k^2 - E\{\alpha_k^2\} \right]^2 \right\} \sin^2(2\psi_k(\theta)) \leq \sum_{k=0}^{n-1} E\{\alpha_k^4\} = O(1).
\]

The last deduction is based on (A.4). □

**Appendix A. Some integrals of Beta random variables**

As seen in (2.7) above,
\[
I(s,t) := \int_{-1}^{1} (1-x)^{s-1}(1+x)^{t-1} \, dx = \frac{2\Gamma(s)\Gamma(t)}{\Gamma(s+t)};
\]
indeed this is the famous Beta integral of Euler. In this appendix, we record a few simple computations that were needed in the text; nothing is novel.

By the binomial theorem,
\[
x^k = 2^{-k} \sum_{m=0}^{k} (-1)^m \binom{k}{m} (1-x)^m (1+x)^{k-m}
\]
and so if \(X \sim B(s,t)\), then
\[
E\{X^k\} = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \frac{I(s+m,t+k-m)}{I(s,t)}.
\]

With a little algebra one then obtains
\[
E\{X\} = \frac{t-s}{t+s} \quad (A.1)
\]
\[
E\{X^2\} = \frac{(t-s)^2 + (t+s)}{(t+s)(t+s+1)} \quad (A.2)
\]
\[
E\{X^3\} = \frac{(t-s)[(t-s)^2 + 3(t+s) + 2]}{(s+t)(1+s+t)(s+t+2)} \quad (A.3)
\]
\[
E\{X^4\} = \frac{(t-s)^2[(t-s)^2 + 6(s+t) + 8] + 3(t+s)^2 + 6(t+s)}{(s+t)(1+s+t)(s+t+2)(s+t+3)} \quad (A.4)
\]

In particular,
\[
E\{(X - E\{X\})^2\} = \frac{4st}{(t+s)^2(t+s+1)} \quad (A.5)
\]
Differentiating $I(s, t)$ with respect to $s$ or $t$ gives access to expectations of logarithms. For example,

$$
E\{-X^2 \log[1 - X^2]\} = \frac{(\partial_s + \partial_t)[I(s + 1, t + 1) + I(s, t)]}{I(s, t)}
$$

$$
= \frac{(s - t)^2 + s + t}{(s + t)(1 + s + t)} [2 \Psi(s + t) - \Psi(t) - \Psi(s) - 2 \log(2)]
$$

$$
+ \frac{(s + t)(s - t)^2 + t^2 + s^2}{(s + t)^2(1 + s + t)^2}.
$$

Where $\Psi(x) := \partial_x \log \Gamma(x)$ is the digamma function.

**Lemma A.1.** Given $s, t \in (\epsilon, \infty)$ obeying $|s - t| \leq \delta$ and $X \sim B(s, t)$,

$$
E\{-X^2 \log[1 - X^2]\} \lesssim (s + t)^{-2}
$$

where the implicit constant depends only on $\epsilon$ and $\delta$.

**Proof.** In light of the calculation above, we need only prove

$$
|2 \Psi(s + t) - \Psi(t) - \Psi(s) - 2 \log(2)| \lesssim (s + t)^{-1}.
$$

This in turn can be deduced from $\Psi(x) = \log(x) + O(x^{-1})$ as $x \to \infty$ — a very weak form of Stirling’s formula. Indeed,

$$
2 \log(s + t) - \log(t) - \log(s) - 2 \log(2) = \log\left[\frac{(s+t)^2}{4st}\right] = \log\left[1 + \frac{(s-t)^2}{4st}\right]
$$

which is $\lesssim (s + t)^{-2}$. \qed

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