Supersymmetric $SO(N_c)$ Gauge Theory and Matrix Model

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Abstract

By applying the method of Dijkgraaf-Vafa, we study matrix model related to supersymmetric $SO(N_c)$ gauge theory with $N_f$ flavors of quarks in the vector representation found by Intriligator-Seiberg. By performing the matrix integral over tree level superpotential characterized by light meson fields (mass deformation) in electric theory, we reproduce the exact effective superpotential in the gauge theory side. Moreover, we do similar analysis in magnetic theory. It turns out the matrix descriptions of both electric and magnetic theories are the same: Seiberg duality in the gauge theory side.
1 Introduction

Recently, Dijkgraaf and Vafa [1] have proposed a technique for calculating the effective superpotential for the glueball superfield in an $\mathcal{N} = 1$ gauge theory through the planar diagrams of matrix model. Several tests have been performed for this proposal in [2]-[50]. In particular, the theories with matter fields in the fundamental representation have been found in [15, 17, 19, 21, 22, 27, 29, 30, 43, 46, 47, 48, 50].

In this letter, we compute the matrix integral over tree level superpotential characterized by light meson fields (mass deformation) in electric $\mathcal{N} = 1 \text{SO}(N_c)$ gauge theory with $N_f$ flavors of quarks in the vector (fundamental) representation found by Intriligator-Seiberg [51] and we reproduce the exact effective superpotential in the gauge theory side. Moreover, we do similar analysis in magnetic theory. The matrix descriptions of both electric and magnetic theories are coincident with each other implying Seiberg duality in the gauge theory side. In [9, 37, 40, 41, 42, 45], there are some relevant works on the gauge group $\text{SO}(N_c)$ in the view point of matrix model. For $U(N_c)$ gauge theory with $N_f$ flavors of quarks in the fundamental and anti-fundamental representations, the matrix descriptions of both electric and magnetic theories have been found in [29, 30].

2 Matrix model description of electric theory

Let us deform our $\mathcal{N} = 1$ supersymmetric $\text{SO}(N_c)$ gauge theory with $N_f$ flavors of quarks $Q^a_j (j = 1, 2, \cdots, N_f, a = 1, 2, \cdots, N_c)$ in the vector (fundamental) representation ( $N_c \geq 4, N_f \leq N_c - 5$) [51], by adding the mass terms of gauge invariant meson superfields $M^{ij} = Q^i \cdot Q^j$

$$W_{\text{tree}} = \frac{1}{2} \text{Tr} \ m M = \frac{1}{2} \sum_{j=1}^{N_f} m_j \sum_{a=1}^{N_c} Q^j_a Q^j_a$$

where it is understood that quark superfields are represented in an $N_f \times N_c$ matrix form in terms of the flavor and color indices. In the gauge theory side, one can make this superfield matrix diagonal by using the gauge and global rotation.

The idea is to use this tree level superpotential as the potential for the matrix model. At first, let us consider $\text{SO}(N)$ matrix model at large $N$ by replacing the gauge theory fields with $N_f \times N$ matrices in order to calculate the contributions to the free energy. Then the partition function can be written as

$$Z = \frac{1}{\text{Vol}(\text{SO}(N))} \left( \frac{\Lambda}{2\pi g_s} \right)^{\frac{1}{2} N_f N} \int \prod_{j=1}^{N_f} [dQ^j_a] e^{-\frac{1}{2g_s} \sum_{j=1}^{N_f} m_j \sum_{a=1}^{N_c} Q^j_a Q^j_a}$$
where we put the factor \( \left( \frac{\Lambda}{2\pi g_s} \right)^{\frac{1}{4}N_f N} \) in order to make the integration measure dimensionless\(^1\) and \( \Lambda \) is the scale of \( SO(N_c) \) gauge theory and \( m_j \) is the mass of \( j \)-th quark. In this gauge theory there is no difference between left- and right-handed quarks which is different from \( SU(N_c) \) gauge theory where the right-handed quark is the left-handed anti-quark. Recall that the superpotential should have \( R \)-charge 2 and dimension 3.

The large \( N \)-limit behavior of the volume for \( SO(N) \) group, the normalization of the matrix path integral, can be read off from \([52, 53, 41]\)

\[
\log \text{Vol}(SO(N)) = \log \frac{\sqrt{2} (2\pi)^{N^2/4}}{(N - 3)! (N - 5)! \cdots 3! (N/2 - 1)!} = -\frac{1}{4} N^2 \log N + \frac{1}{4} N^2 \left( \frac{3}{2} + \log \pi + \log 2 \right) + \frac{N}{4} \log N
\]

\[
+ \frac{1}{24} \log N + \frac{N}{4} (-1 + \log 2 - \log \pi) + O(1).
\]

The usual matrix integral for flavor part is

\[
\int \prod_{j=1}^{N_f} [dQ^j_a] e^{-\frac{1}{2g_s} \sum_{j=1}^{N_f} m_j \sum_{a=1}^{N} Q^j_a Q^j_a} = \left( \frac{\pi^{N_f} g_s^{N_f}}{\det m} \right)^{N/2}.
\]

In the large \( N \)-limit we are interested in, the glueball superfield \( S \) can be identified with \( g_s N \) as a second step and one can write the log of the partition function as follows:

\[
\log Z = \frac{1}{4} N^2 \log \left( \frac{N}{2\pi e^{3/2}} \right) + \frac{N}{2} N_f \log \Lambda - \frac{N}{2} \log \det m
\]

\[
= \frac{S^2}{4g_s^2} \log \left( \frac{S}{2\pi e^{3/2} g_s} \right) + \frac{S}{2g_s} N_f \log \Lambda - \frac{S}{2g_s} \log \det m
\]

\[
\equiv -\frac{1}{g_s^2} F_2 - \frac{1}{g_s} F_1.
\]

Note that all the effects of the matter is in \( F_1 \).

Then the effective superpotential for the glueball field \( S \) can be computed as the derivative of the contribution to free energy plus the contribution from flavors \([1, 15]\)

\[
W = (N_c - 2) \frac{\partial F_2}{\partial S} + F_1
\]

\[
= \frac{1}{2} (N_c - 2) \left( S - S \log \left( \frac{S}{\Lambda^3} \right) - \frac{S}{N_c - 2} N_f \log \Lambda + \frac{S}{N_c - 2} \log \det m \right)
\]

\[
= \frac{1}{2} (N_c - 2) \left( S - S \log \left( \frac{S}{\Lambda^3(N_c-2)-N_f \det m} \right)^{1/(N_c-2)} \right). \quad (2.2)
\]

\(^1\)In fact we can put any dimensionful term with the right dimension inside the parenthesis, but this particular form gives the right form of the superpotential for the gauge theory.
Here we have used the results that the derivative of $F_2$ gives the Veneziano-Yankielowicz superpotential [1].

Solving the F-flatness condition $\partial_S W = 0$ (minimizing $W$ with respect to a glueball superfield $S$) one gets, by the scale normalization $\Lambda^{3(N_c-2)-N_f} = 16\Lambda^{3(N_c-2)-N_f}_{N_c,N_f}$,

$$S = \left(16\Lambda^{3(N_c-2)-N_f}_{N_c,N_f} \det m\right)^{1/(N_c-2)} \epsilon_{N_c-2}, \quad \epsilon_{N_c-2} = e^{2\pi i k/(N_c-2)}, \quad k = 1, \cdots, (N_c-2)$$

with the phase factor $e^{2\pi i k/(N_c-2)}, k = 1, \cdots, (N_c-2)$ reflecting the $(N_c-2)$ supersymmetric vacua but physically equivalent vacua of the theory coming from the spontaneous breaking of a discrete symmetry. Then the exact superpotential by plugging $S$ into (2.2) (integrating out the $S$) leads to

$$W = \frac{1}{2} (N_c - 2) \epsilon_{N_c-2} \left(16\Lambda^{3(N_c-2)-N_f}_{N_c,N_f} \det m\right)^{1/(N_c-2)}$$

$$= \frac{1}{2} (N_c - 2) \epsilon_{N_c-2} \Lambda^{3}_{N_c-N_f,0}$$

(2.3)

where $\Lambda^{3}_{N_c-N_f,0} = \left(16\Lambda^{3(N_c-2)-N_f}_{N_c,N_f} \det m\right)^{1/(N_c-2)}$ is the low energy scale [51] for the $SO(N_c-N_f)$ Yang-Mills theory. This is nothing but ADS superpotential below for pure $SO(N_c)$ Yang-Mills gauge theory. In the low energy effective theory, the classical vacuum degeneracy was lifted by quantum effects which is represented by a dynamically generated superpotential for the light meson fields $M^{ij}$. The superpotential [51] generated by gaugino condensation leads to

$$W_{ADS} = \frac{1}{2} (N_c - N_f - 2) \epsilon_{N_c-N_f-2} \left(16\Lambda^{3(N_c-2)-N_f}_{N_c,N_f} \det m\right)^{1/(N_c-N_f-2)}$$

where $\epsilon_{N_c-N_f-2}$ is the $(N_c - N_f - 2)$-th root of unity. This theory has no vacuum but by adding the mass terms (2.1) to the superpotential $W_{ADS}$, the theory has $(N_c - 2)$ supersymmetric vacua. If not all of the matter fields are massive, in the gauge theory side, one can integrate out massive quarks and get the effective superpotential at low energy for the massless ones. It is the same form as the above $W_{ADS}$ but with the scale replaced by the low energy one.

### 3 Matrix model description of magnetic theory

In the IR theory of electric theory in previous section, the magnetic theory is described by an $SO(N_f = N_f + 4)$ gauge theory $(N_f > N_c, N_c \geq 4)$ with $N_f$ flavors of dual quarks $q^a_j (j = 1, \cdots, N_f, a = 1, \cdots, \widetilde{N}_c)$ and the additional gauge singlet fields $M^{ij}$ which is an elementary field of dimension 1 at the UV-fixed point [51]. The matter field variables in the magnetic theory are the original electric variables $M^{ij}$ and magnetic quarks $q_i$ with the superpotential...
together with mass term
\[ W = \frac{1}{2\mu} M^{ij} q_i \cdot q_j + \frac{1}{2} \text{Tr} \, mM. \]

After integrating \( X \) first, the partition function can be written as a delta function along the line of [22]:
\[
Z = \frac{1}{\text{Vol}(SO(N))} \left( \frac{\tilde{\Lambda}}{2\pi g_s} \right)^{\frac{1}{2}N_f \tilde{N}} \int [dX] \prod_{j=1}^{N_f} [dq_j] \delta \left( X^{ij} - M^{ij} \right) e^{-\frac{1}{2} \sum_{i,j=1}^{N_f} \left( X^{ij}q_i \cdot q_j \right)} = 1
\]
\[
\times \prod_{j=1}^{N_f} [dq_j] \delta \left( \mu m + q_i \cdot q_j \right).
\]

In the gauge theory side, this behavior of delta function is equivalent to the \( M^{ij} \) equations of motion led by \( < q_i \cdot q_j >= -\mu m \). The constrained matrix integral over \( N_f \) flavors of length \( \tilde{N} \) can be obtained Wishart random matrices and the result [22, 54] for this is
\[
\int \prod_{j=1}^{N_f} [dq_j] \delta \left( \mu m + q_i \cdot q_j \right) = c \times (\text{det} (-\mu m))^{(\tilde{N} - N_f - 1)/2}
\]
where the coefficient \( c \) behaves like as
\[
e^{-\frac{1}{2}N_f \tilde{N} \log \tilde{\Lambda}^2}
\]
for large \( \tilde{N} \)-limit.

As we did for electric case, the effective superpotential from the log of partition function can be expressed as
\[
W = \frac{1}{2} \left( \tilde{N}_c - 2 \right) \left( S - S \log \frac{S}{\tilde{\Lambda}^2} \right) - \frac{1}{2} N_f \left( S - S \log \frac{S}{\tilde{\Lambda}^2} \right) + S N_f \log \tilde{\Lambda} - \frac{1}{2} S \log \text{det} (-\mu m)
\]
\[
= \frac{1}{2} \left( \tilde{N}_c - N_f - 2 \right) \left( S - S \log S + S \log \left( \frac{\tilde{\Lambda}^{3(\tilde{N}_c - 2) - N_f}}{\text{det} (-\mu m)} \right)^{1/(\tilde{N}_c - N_f - 2)} \right).
\]

Solving the F-flatness condition \( \partial S W = 0 \) one gets, with the phase factor \( e^{2\pi ik/(\tilde{N}_c - N_f - 2)}, k = 1, \cdots, (\tilde{N}_c - N_f - 2) \) reflecting the \( (\tilde{N}_c - N_f - 2) \) supersymmetric vacua,
\[
S = \left( \frac{16 \tilde{\Lambda}^{3(\tilde{N}_c - 2) - N_f}}{\text{det} (-\mu m)} \right)^{1/(\tilde{N}_c - N_f - 2)} \tilde{\Lambda}^{3(\tilde{N}_c - 2) - N_f} = 16 \tilde{\Lambda}^{3(\tilde{N}_c - 2) - N_f} N_f^{N_f - N_c + 4N_f}.\]
Then the exact superpotential by plugging this $S$ into (3.1) leads to

$$W = \frac{1}{2} \left( \tilde{N}_c - N_f - 2 \right) \epsilon_{\tilde{N}_c-N_f-2} \left( \frac{-3(\tilde{N}_c-2)-N_f}{16\hat{\Lambda}_{N_f-N_c+4,N_f} \det(-\mu m)} \right)^{1/(\tilde{N}_c-N_f-2)}$$

$$= \frac{1}{2} (-N_c + 2) \epsilon_{N_c-2} \left( \frac{16\hat{\Lambda}_{N_f-N_c+4,N_f}}{(-1)^{N_f}\mu^{N_f}\det m} \right)^{-1/(N_c-2)}$$

$$= \frac{1}{2} (N_c - 2) \epsilon_{N_c-2} \left( 16\hat{\Lambda}_{N_c,N_f} \det m \right)^{1/(N_c-2)}$$

$$= \frac{1}{2} (N_c - 2) \epsilon_{N_c-2} \tilde{\Lambda}^3_{N_c-N_f,0}$$

(3.2)

which is exactly the same as the one in (2.3). Here we used the fact that $\tilde{N}_c = N_f - N_c + 4$, $\det(-\mu m) = (-1)^{N_f}\mu^{N_f}\det m$ and $\epsilon_{\tilde{N}_c-N_f-2} = \epsilon_{N_c+2} = \epsilon_{N_c-2}$. The scale of the magnetic theory in the gauge theory side was related to that of electric theory by [51]

$$2^8 \Lambda^{3(N_c-2)-N_f} \Lambda^{3(N_f-N_c+2)-N_f}_{N_f-N_c+4,N_f} = (-1)^{N_f-N_c} \mu^{N_f}$$

where the normalization factor $1/2^8$ was chosen to get the consistent low energy behavior under large mass deformation and along the flat directions. Note that the factor $(-1)^{N_f/(N_c-2)} = -1$ in (3.2) is cancelled exactly by the overall $-1$ factor. If not all of the matter fields are massive, the remaining low energy magnetic theory is a magnetic $SO(N_f - N_c + 4 - M)$ gauge theory with $(N_f - M)$ flavors and the low energy superpotential. This low energy magnetic theory is dual to the low energy $SO(N_c)$ gauge theory with $(N_f - M)$ massless quarks. One can arrive at the scale relation which connects the $SO(N_c)$ electric theory with $(N_f - M)$ flavors to the $SO(N_f - N_c + 4 - M)$ magnetic theory with $(N_f - M)$ flavors.

4 Discussions

In this paper, we apply the matrix model for fundamental flavors without any adjoint matter to check the Seiberg duality when we deform mass terms. By explicit matrix path integral in both electric and magnetic theories, we have found the complete agreement with the field theory result.

One can consider and generalize degenerate mass deformation in which some eigenvalues of the mass matrix $m$ are vanishing. We expect to have the following superpotential, effectively $SO(N_c)$ theory with $K$ flavors,

$$W = \frac{1}{2} (N_c - K - 2) \epsilon_{N_c-K-2} \left( \frac{16\hat{\Lambda}_{N_c,K} \det m}{\det M} \right)^{1/(N_c-K-2)}$$
where $M^{ij}(i, j = 1, 2, \cdots, K)$ are the meson from the massless flavors and the index $i$ of mass matrix eigenvalues $m_i$ runs $i = K + 1, K + 2, \cdots, N_f$. We have to add the terms from the delta function constraint for massless flavor part. For $K = 0$, we can reproduce the result of section 2 and 3.
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