Number of $h$-cycles in the Internet at the Autonomous System Level

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We present here a study of the clustering and cycles present in the graph of Internet at the Autonomous Systems level. Even if the whole structure is changing with time, we present some evidence that the statistical distributions of cycles of order 3,4,5 remain stable during the evolution. This could suggest that cycles are among the characteristic motifs of the Internet. Furthermore, we compare data with the results obtained for growing network models aimed to reproduce the Internet evolution. Namely the fitness model, the Generalized Network Growth model and the Bosonic Network model. We are able to find some qualitative agreement with the experimental situation even if the actual number of cycles seems to be larger in the data than in any proposed growing network model. The task to capture this feature of the Internet represent one of the challenges in the future Internet modeling.

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Internet is a beautiful example of a complex system with many degrees of freedom resulting in global scaling properties. It has been shown [1, 2] that the Internet belongs to the wide class of scale-free networks [3, 4, 5, 6]. Indeed, it can be described as a network, with nodes and links representing respectively Autonomous Systems (AS) and physical lines connecting them; moreover, its degree distribution follows a power-law behavior.

Different topological quantities have also been measured beside the degree distribution exponent. Among those, the clustering coefficient $C(k)$ and the average nearest neighbor degree $k_{nn}(k)$ of a node as a function of its degree $k$ [7, 8, 9]. In particular, measurements in Internet yield $C(k) \sim k^{-0.75}$ [10] and $k_{nn} \sim k^{-\nu}$ with $\nu \simeq 0.5$ [10]. A two-vertices degree anti-correlation has also been measured [12]. Accordingly, Internet is said to display disassortative mixing [11], because nodes prefer to be linked to peers with different rather than similar degree. Moreover, the modularity of the Internet due to the national patterns has been studied by measuring the slow decaying modes of a diffusion process defined on it [13].

Recently, more attention has been devoted to network motifs [14, 15], i.e. subgraphs that recur with a higher frequency than in maximally random graphs with the same degree distribution. Among those, the most natural class includes cycles [14, 15], closed paths of various lengths that visit each node only once. Cycles (or loops) are interesting because they account for the multiplicity of paths between any two nodes. Therefore, they encode the redundant information in the network structure. Following the arguments of [14], it can be shown that the number $N_h$ of cycle of size $h$, in a equilibrium undirected scale-free network of $N$ nodes with a power-law degree distribution $P(k) \sim k^{-\gamma}$, is

$$N_h(N) \sim N^\xi(h)$$  \hspace{1cm} (1) \hspace{1cm} \text{with} \hspace{1cm} 

$$\xi(h) = \begin{cases} 
1 & \text{for } \gamma \leq 2 \\
3 - \gamma & \text{for } 2 < \gamma \leq 3 \\
0 & \text{for } \gamma \geq 3 
\end{cases} \hspace{1cm} (2)$$

In other words, $N_h(N)$ is an algebraic function of the system size with an exponent $\xi$ independent of the length $h$ of the cycle.

In contrast, the only analytical result [10] for off-equilibrium, scale-free networks refers to the Barabási-Albert model [18], and reads

$$N_h(N) \sim \left( \frac{m}{2} \log(N) \right)^{\psi(h)}, \hspace{1cm} (3)$$

with $\psi(h) = h$.

To measure the actual scaling in Internet at the AS level, we considered its symmetrical adjacency matrix $\{a_{ij}\}$, with $a_{ij} = 1$ if $i$ and $j$ are connected and $a_{ij} = 0$ otherwise. We assume that no self-loop is present, i.e. $a_{ii} = 0$ for all $i$. In this case, for $h = 3$ we simply have [16]

$$N_3 = \frac{1}{6} \sum_i (a^3)_{ii}. \hspace{1cm} (4)$$

For $h = 4$ and $h = 5$, by simple arguments it is possible to show that

$$N_4 = \frac{1}{8} \left[ \sum_i (a^4)_{ii} - 2 \sum_i (a^2)_{ii} (a^2)_{ii} + \sum_i (a^2)_{ii} \right] \hspace{1cm} (5)$$

and that

$$N_5 = \frac{1}{10} \left[ \sum_i (a^5)_{ii} - 5 \sum_i (a^3)_{ii} (a^2)_{ii} + 5 \sum_i (a^3)_{ii} \right]. \hspace{1cm} (6)$$

The data of the Internet at the Autonomous System level are collected by the University of Oregon Route
Each node has a fitness, or ability that a node acquires a new link. The resulting network is a scale-free one with $\gamma = 2.255$. It has also been found that $C(k)$ and $k_{na}(k)$ are in qualitative agreement with Internet data.

As a second instance, we compare the Internet data to the recently proposed Generalized Network Growth Model (GNG) [22]. According to its definition, at each time step

1. either a node is added and linked with vertex $i$ with probability

$$p \frac{k_i}{\sum_{j=1,N} k_j}. \quad (8)$$

2. or a link is added (if absent) between nodes $i$ and $j$ already present, with probability

$$\left(1 - p\right) \frac{k_i}{\sum_{k=1,N} k_k \sum_{k\neq i=1,N} |k_i - k_k|}. \quad (9)$$

The resulting network is a scale-free one, with $\gamma(p) = 2 + \frac{\beta}{2 - p}$. Besides, it displays the non trivial features of the degree correlations as measured in Internet.

Finally, we considered the Bosonic Network (BN), where each node $i$ is assigned an innate quality in the spirit of Ref. [20], represented by a random ‘energy’ $\epsilon_i$ drawn from the probability distribution $p(\epsilon_i)$. The attractiveness of each node $i$ is then determined jointly by its connectivity $k_i$ and its energy $\epsilon_i$. Namely, the probability that node $i$ acquires a link at time $t$ is given by

$$\Pi_i = \frac{e^{-\beta \epsilon_i} k_i(t)}{\sum_j e^{-\beta \epsilon_j} k_j(t)}, \quad (10)$$

i.e. low energy, high degree nodes are more likely to acquire new links. The parameter $\beta = 1/T$ in $\Pi_i$ tunes the relevance of the quality with respect to the degree in the acquisition probability of new links. Indeed, for $T \to \infty$ the probability $\Pi_i$ does not depend any more on the energy $\epsilon_i$, and the BN model reduces to the Barabási-Albert (BA) model, based only on preferential attachment.

On the other hand, in the limit $T \to 0$ only the lowest energy node has non-zero probability to acquire new links. In Ref. [21], it has been shown that the connectivity distribution in this network model can be mapped into the occupation numbers of a Bose gas. Accordingly, one would expect a corresponding phase transition for the topology of the network at some temperature value $T_c$. In fact, such a critical value is observed for energy distributions where $p(\epsilon) \to 0$ for $\epsilon \to 0$. For $T > T_c$ the system is in the “fit-get-rich” (FGR) phase, where low-energy nodes acquire links at a higher rate than high-energy ones, while for $T < T_c$ a “Bose-Einstein condensate” (BEC) or “winner-takes-all” phase emerges, where a single node grabs a finite fraction of all the links. We simulated this model assuming

$$p(\epsilon) = (\theta + 1) e^\theta \quad \text{and} \quad \epsilon \in (0, 1) \quad (11)$$

Views Project and made available by the NLANR (National Laboratory of Applied Network Research). The subset we used in this manuscript are mirrored at COSIN web page [http://www.cosin.org]. We considered 13 snapshots of the Internet network at the AS level at different times starting from November 1997 (when $N = 3015$) toward January 2001 ($N = 9048$). Throughout this period, the degree distribution is a power-law with a nearly constant exponent $\gamma \simeq 2.22(1)$. Using relations (4), (5), (6), we measure $N_h(t)$ for $h = 3, 4, 5$ in the Internet at different times, corresponding to different network size.

We observe in figure 1 that the data follow a scaling of the type $h^\xi$, as predicted by [14] for maximally random (equilibrium) scale-free networks. Unfortunately, the exponents $\xi(h)$ strongly depend on $h$, as reports table I, and significantly exceed the predicted value (Eq. (2)) for equilibrium scale-free networks with same $\gamma$, that is, $\xi = 0.78$.

So, we can state that loops up to size 5 are much more frequent in Internet than in a random scale-free networks with a similar degree distribution. $\xi(h)$ and $N_h$ are large even when compared with off-equilibrium networks inspired by the Internet growth. The models we consider here reproduce the most accurately the Internet behavior as regards the degree, clustering and centrality probability distributions.

The fitness model [12], for example, is a growing network model where, at each time step, a new node is added to the network and connected by $m$ links to existing ones. Each node has a fitness $\eta_i$, randomly drawn from a uniform distribution in $[0, 1]$, which enters into the probability that a node acquires a new link,

$$\Pi_i = \frac{\eta_i k_i(t)}{\sum_j \eta_j k_j(t)}. \quad (7)$$

The fitness represents an intrinsic ability of a node in the acquisition of new links.
where \( \theta = 0.5 \). Varying \( T \), one observes a change in the behavior of \( N_h \) in the bosonic network from a scaling of the type \( (3) \), shown to be exact in the \( \beta = 0 \) limit for the BA network model \( (10) \), to a scaling of the type \( (4) \), valid in the low-temperature limit. In reference \( (16) \), we claim that this change occurs right at the Bose-Einstein condensation temperature \( T_c \). A careful analysis of the transition shows in fact that the transition is rather smooth at \( T_c \).

In order to compare networks with a similar mean degree \( <k> \approx 3.5 \) for the Internet\), we consider the fitness model with \( m = 2 \) \( (11) \) and the GNG model with parameter \( p = 0.5 \) \( (12) \), shown to be exact in the \( \beta = 0 \) limit for the BA network model \( (10) \). In the GNG network with \( p = 0.5, 0.6 \) one numerically finds \( \gamma = 2.5(2) \) \( (13) \).

In figure 3 we show the scaling of \( N_h \) as a function of the system size for the fitness model with \( m = 2 \) and the GNG model with \( p = 0.5, p = 0.6 \). For large \( N \), \( N_h(N) \) is a power-law as in the real Internet, yet with much smaller exponents, as shown in Table 1.

| System      | \( \xi(3) \)         | \( \xi(4) \)         | \( \xi(5) \)         |
|-------------|----------------------|----------------------|----------------------|
| AS          | 1.45 ± 0.07          | 2.07 ± 0.01          | 2.45 ± 0.01          |
| Fitness     | 0.59 ± 0.02          | 0.86 ± 0.02          | 1.10 ± 0.02          |
| GNG (p=0.5) | 0.53 ± 0.03          | 0.72 ± 0.03          | 0.96 ± 0.02          |
| GNG (p=0.6) | 0.53 ± 0.03          | 0.74 ± 0.03          | 0.99 ± 0.02          |

TABLE 1: The exponent \( \xi(n) \) for \( n = 3, 4, 5 \) as defined in equation \( (11) \) for real data and network models.

When considering the bosonic network model, the picture is more complicated. The loops number behavior depends strongly on the temperature parameter.

We can distinguish a high-temperature phase, where \( N_h(N) \) is better fitted by \( (3) \) - FGR phase- and a low-temperature phase, where \( N_h(N) \) scales as \( (10) \) - BEC phase. Even when one decreases the temperature, \( \xi(h) \) remains always far from the real network exponents, as it is shown in figure 3 so that also the bosonic network fails in reproducing correctly such feature. Furthermore, no significant sign for a ‘winner’ node are found in the Internet data in which the most connected node has a fraction of links \( k/N = 2042/9048 = 0.22 \) for the January 2001 AS data.

Following \( (17) \), we also measured the clustering coefficients \( c_{3,i} \) and \( c_{4,i} \) as a function of the connectivity \( k_i \) of node \( i \) for all \( i \)'s. In particular, \( c_{3,i} \) is the usual clustering coefficient \( C_i \), i.e. the number of triangles including node \( i \) divided by the number of possible triangles \( k_i(k_i-1)/2 \).

Similarly, \( c_{4,i} \) measures the number of quadrilaterals passing through node \( i \) divided by the number of possible quadrilaterals \( Z_i \). This last quantity is the sum of all possible primary quadrilaterals \( Z_i^1 \) (where all vertices are nearest neighbors of node \( i \)) and all possible secondary quadrilaterals \( Z_i^2 \) (where one of the vertices is a second neighbor of node \( i \)). If node \( i \) has \( k_i^{NN} \) second neighbors, \( Z_i^1 = k_i(k_i-1)(k_i-2)/2 \) and \( Z_i^2 = k_i^{NN}k_i(k_i-1)/2 \). In Fig. 4 (a) we plot \( c_3(k) \) and \( c_4(k) \) for the Internet data at three different times (November 1997, January 1999 and January 2001) showing that the behavior of \( c_3(k) \) and \( c_4(k) \) is invariant with time and scales as

\[
c_h(k) \sim k^{-\delta(h)}
\]

with \( \delta(3) = 0.7(1) \) and \( \delta(4) = 1.1(1) \).

In Fig. 4 we compare the behavior of \( c_3(k) \) and \( c_4(k) \) in real Internet data and in the Internet models. We found a similar behavior in the three networks model and in the Internet with the \( c_3(k) \) and \( c_4(k) \) of the Internet models scaling as \( (11) \). Exponents, however, vary significantly, as shown in Table 1.

The fitness model reproduces the best the Internet clustering scaling pattern. Nevertheless, we observe that the number of triangles and quadrilaterals in real data is much larger than in the fitness network. Indeed, we have \( c_3(10^3) \sim 10^{-2} \) and \( c_4(10^3) \sim 10^{-4} \) in the AS network, while in the fitness model \( c_3(10^3) \sim 10^{-3} \) and \( c_4(10^3) \sim 10^{-5} \).

In conclusion, we computed the number \( N_h(t) \) of \( h \)-loops of size \( h = 3, 4, 5 \) in the Internet at the Autonomous System level and we have identified them as proper motifs of the Internet. We have then compared the actual
data with the behavior of $N_h(N)$ in the fitness model, in the GNG model and in the Bosonic network, chosen as the most accurate Internet model developed to our best knowledge. Aside, the generalized clustering coefficients around individual nodes have been investigated as a function of nodes degrees. We have observed that, although some qualitative feature of the loop scaling and of the clustering coefficient are captured by models, the much larger number of cycles observed in the real network invoke for improvement of the theory.

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![Graph](image)

FIG. 4: The clustering coefficients $c_2(k)$ and $c_4(k)$ in Internet (graph(a)) and in the fitness model (graph(b)), the GNG model (graph(c)) with $p = 0.5$ (circles), $p = 0.6$ (triangles) and the bosonic network model with $\beta = 2.5$ (graph(d)). Empty (filled) symbols refer to $c_2(k)$ ($c_4(k)$). Graph(a) shows data as obtained in November '97 (circles), January '99 (squares) and the data taken in January '01 (triangles). Solid lines refer to power law fittings, whose exponents are reported in table II.

| System     | $\delta(3)$ | $\delta(4)$ |
|------------|-------------|-------------|
| AS         | $0.7 \pm 0.1$ | $1.1 \pm 0.01$ |
| Fitness    | $0.67 \pm 0.01$ | $0.99 \pm 0.01$ |
| GNG ($p=0.5$) | $0.32 \pm 0.02$ | $1.68 \pm 0.03$ |
| GNG ($p=0.6$) | $0.27 \pm 0.02$ | $0.93 \pm 0.01$ |
| Bosonic ($\beta = 2.5$) | $0.91 \pm 0.04$ | $1.07 \pm 0.07$ |

TABLE II: The exponent of the clustering coefficient $c_2(k)$ and $c_4(k)$ as measured from Internet data and from simulations of network models.

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