1. Introduction

This note discusses the computation and use of a direction of negative curvature in the regularized sequential quadratic programming primal-dual augmented Lagrangian method (pdSQP) of Gill and Robinson [7], [8] for the purpose of ensuring convergence towards second-order optimal points. Section 2 discusses how to compute a direction of negative curvature using appropriate matrix factorizations. Section 3 discusses the specific relevant changes to the algorithm. Section 4 discusses the changes in the convergence results established by Gill and Robinson [8], showing that the desired convergence results continue to hold. Section 5 discusses global convergence to points satisfying the second-order necessary optimality conditions.

2. Direction of negative curvature

2.1. The active-set estimate

An index set $W_k$ is maintained that consists of the variable indices that estimate which components of $x$ on their bounds. This set determines the the space in which to calculate the directions of negative curvature. The tolerance for an index to be in $W_k$ must converge to zero. A test such as $i \in W_k$ if $[x_k]_i \leq \min\{\mu_k, \epsilon_a\}$, would be appropriate for the purpose of forming a $W_k$ for convexification, initializing the QP, and obtaining a direction of negative curvature. Otherwise, it would be necessary to use three different factorizations.

2.2. Calculating the direction

Recall that in pdSQP, the QP must use a Lagrangian Hessian $\tilde{H}$ such that $\tilde{H} + \frac{1}{\mu} J^T J$ is positive definite. The process for forming the requisite $\tilde{H}$, as well as calculating a direction of negative curvature begins with the inertia-controlling factorization of the KKT matrix (see Forsgren [4]). Consider the KKT matrix,

$$\begin{pmatrix} H_F & J_F^T \\ J_F & -\mu I_{|F|} \end{pmatrix},$$

with $F$ the set of estimated free variables (those not in $W_k$), and $I_{|F|}$ the identity matrix with $|F|$ rows and columns.

The algorithm begins an LBL$^T$ factorization of the KKT matrix, where $L$ is lower triangular and $B$ is a symmetric diagonal with $1 \times 1$ and $2 \times 2$ diagonal blocks. Standard pivoting strategies are described in the literature (see Bunch and Parlett.
Let the lower-right block be defined as \( D = -\mu I_{[F]} \).

At step \( k \) of the factorization, let the partially factorized matrix have the following structure:

\[
\begin{pmatrix}
L_1 & 0 \\
L_2 & I
\end{pmatrix}
\begin{pmatrix}
B & 0 \\
0 & A
\end{pmatrix}
\begin{pmatrix}
L_1^T & L_2^T \\
0 & I
\end{pmatrix},
\]

with \( L_1 \) being lower triangular, \( I \) the identity of appropriate size, and \( A \) the matrix remaining to be factorized. Let \( A \) be partitioned as \( A = \begin{pmatrix} a & b^T \\ b & C \end{pmatrix} \). If the top left element is chosen as a \( 1 \times 1 \) pivot, at the next step,

\[
\begin{pmatrix}
L_1 & 0 & 0 \\
L_3 & 1 & 0 \\
L_4 & a^{-1}b & I
\end{pmatrix}
\begin{pmatrix}
B & 0 & 0 \\
0 & a & 0 \\
0 & 0 & C - ba^{-1}b^T
\end{pmatrix}
\begin{pmatrix}
L_1^T & L_3^T & L_4^T \\
0 & 1 & a^{-1}b^T \\
0 & 0 & I
\end{pmatrix}.
\]

Let \( S = C - ba^{-1}b \) be the Schur complement of the factorization. The matrix \( S \) is factorized at the next step.

For inertia control, this factorization has two stages. In the first stage, we restrict the factorization to allow only for pivots of type \( H^+, D^- \), or \( HD \). This means that an element \( (i, j) \) of \( H \) is selected such that \( H_{ij} > 0 \), a diagonal element of \( D \) is selected, or \( (i_1, i_2, j_1, j_2) \) is selected such that \( (i_1, j_1) \) is an element of \( H \), \( (i_2, j_2) \) is an element of \( D \) and \( S_k[(i_1, i_2), (j_1, j_2)] \) has mixed eigenvalues. This procedure is continued until there are no such remaining pivots.

The KKT matrix can be partitioned as

\[
\begin{pmatrix}
H_{11} & H_{12} & J_1^T \\
H_{21} & H_{22} & J_2^T \\
J_1 & J_2 & -\mu I
\end{pmatrix},
\]

where, all of the pivots have come from the rows and columns of \( H_{11}, J_1, \) and \( -\mu I \).

At the end of the first stage, the factorization can be written as:

\[
\begin{pmatrix}
L_1 & 0 \\
L_2 & I
\end{pmatrix}
\begin{pmatrix}
B & 0 \\
0 & H_{22} - K_{21}K_{11}^{-1}K_{12}
\end{pmatrix}
\begin{pmatrix}
L_1^T & L_2^T \\
0 & I
\end{pmatrix}.
\]

Let \( S = H_{22} - K_{21}K_{11}^{-1}K_{12} \). Proposition 3 of Forsgren [4] shows that if \( \delta I \) is added to \( H_{22} \) such that \( \delta > \|S\| \) then \( K_F \) has the correct inertia. In practice this \( \delta \) is excessively large for the purpose of constructing the appropriate matrix with the required eigenvalues, but this result does indicate that such a constant exists.

Instead of proceeding to the second phase of this factorization, the procedure of Lemma 2.4 in Forsgren et al. [6] is applied to \( S \) to compute \( \hat{u} \), a direction of negative curvature for \( S \). The procedure to calculate this \( \hat{u} \) is as follows:

Let \( \rho = \max_{i,j} |S_{ij}| \) with \( |S_{qr}| = \rho \). Define \( \hat{u} \) as the solution to:

\[
\begin{pmatrix}
L_1 \\
L_2
\end{pmatrix}
\begin{pmatrix}
0 \\
I
\end{pmatrix} \hat{u} = \sqrt{\rho} h,
\]

\[ (2.3) \]
where
\[
  h = \begin{cases} 
  e_q & \text{if } q = r, \\
  \frac{1}{\sqrt{2}}(e_q - \text{sgn}(b_{qr})e_r) & \text{otherwise.}
  \end{cases}
\]

This \( \hat{u} \) satisfies \( \hat{u}^T S \hat{u} \leq \gamma \lambda_{\min}(S) ||\hat{u}||^2 \), with \( \gamma \) independent of \( S \).

The following bounds are important for the subsequent second-order convergence theory.

**Lemma 2.1.** Let \( \hat{u} \) be defined as in (2.3), \( S \) be the Schur complement of the partially factorized matrix (2.2), \( J_F \) and \( H_F \) defined as in (2.1), and \( Z \) a matrix consisting of columns for the basis of the null-space of \( J_F \), then
\[
  \frac{\hat{u}^T S \hat{u}}{\gamma ||\hat{u}||^2} \leq \lambda_{\min}(S) \leq \lambda_{\min}(H_F + \frac{1}{\mu} J_F^T J_F) \leq \lambda_{\min}(Z^T H_F Z).
\]

**Proof.** Lemma 2.4 in Forsgren et al. [6] directly implies that \( \frac{\hat{u}^T S \hat{u}/\gamma ||\hat{u}||^2}{\lambda_{\min}(S)} \).

The proof that \( \lambda_{\min}(S) \leq \lambda_{\min}(H_F + \frac{1}{\mu} J_F^T J_F) \) is given in the proof of Theorem 4.5 in Forsgren and Gill [5]. For the final inequality, let \( w = Z v \), with \( Z^T H_F Z v = \lambda_{\min}(Z^T H_F Z) v \) and \( ||v|| = 1 \). Then
\[
  \lambda_{\min}(H_F + \frac{1}{\mu} J_F^T J_F) \leq \frac{w^T (H_F + \frac{1}{\mu} J_F^T J_F) w}{w^T w} = w^T H_F w = v^T Z^T H_F Z v = \lambda_{\min}(Z^T H_F Z).
\]

3. Implementing Directions of Negative Curvature

3.1. Step of negative curvature

Several changes must be made to the algorithm of Gill and Robinson [8]. In order to minimize the number of factorizations, the computation of the direction of negative curvature should be followed by a test of second-order optimality. In addition, it is necessary that the direction of negative curvature is bounded, and a feasible direction with respect to both the linearized equalities and the bound constraints. Finally, the line-search must be extended to allow for this additional step of negative curvature.

In the description below, the subscript \( k \) denoting the step number in the sequence of iterations is suppressed.

The following procedure satisfies these requirements.

1. The first step computes the direction of negative curvature for the free KKT-matrix as described in Section 2, denoted as \( \hat{\tilde{u}}_F \), then defines \( \tilde{u} \) to be \( [\tilde{\tilde{u}}]_F = \tilde{\tilde{u}}_F \) and \( [\tilde{\tilde{u}}]_A = 0 \). If no such direction of negative curvature exists, then \( \tilde{u} \) is set to zero.

2. The second step uses \( \tilde{u} \) in a test of second-order optimality. This is described in Section 3.2.
3. The corresponding change in the multipliers corresponding to the definition for \( \tilde{w} \) is defined as \( \tilde{w} = -\frac{1}{\mu} J \tilde{u} \). This ensures that the linearized equality constraints are satisfied, i.e.,

\[
0 = Jp + c + \mu q = J(p + \tilde{u}) + c + \mu (q - \frac{1}{\mu} J \tilde{u}).
\]

The final resulting \((u, w)\) is shown below in Section 3.3 to be a direction of negative curvature for \( \nabla^2 M \).

4. Since both \((\tilde{u}, \tilde{w})\) and 
\(- (\tilde{u}, \tilde{w})\) are directions of negative curvature, the sign is chosen so that the step is a descent direction for \( \nabla M \), i.e.,

\[
\nabla M^T \left( \frac{\tilde{u}}{\tilde{w}} \right) \leq 0.
\]

5. Compute \( \Delta v = (p, q) \), the solution of the convex QP.

6. The direction of negative curvature is scaled so that it is both bounded by

\[
\max(u_{\text{max}}, 2||p||)
\]

and, in conjunction with the QP step, satisfies the bound constraints \( x \geq 0 \).

Specifically, \( u \) and \( w \) are set as

\[
u = \beta \tilde{u} \quad \text{and} \quad w = \beta \tilde{w},
\]

where

\[
\beta = \left\{ \max \beta \mid x + p + \beta \tilde{u} \geq 0, \| \beta \tilde{u} \| \leq \max(u_{\text{max}}, 2||p||) \right\}.
\]

Note that this implies that if \([x + p]_i = 0\) and \([u]_i < 0\), then \( u \) is set to zero.

### 3.2. Optimality measures

Recall that in Gill and Robinson [8], with

\[
\phi_S(v) = \eta(x) + 10^{-5} \omega(v) \quad \text{and} \quad \phi_L(v) = 10^{-5} \eta(x) + \omega(v),
\]

where

\[
\eta(x) = \| c(x) \| \quad \text{and} \quad \omega(x, y) = \| \min \left( x, g(x) - J(x)^T y \right) \|,
\]

an iterate is an S-iterate if \( \phi_S(v) \leq \frac{1}{2} \phi_{S}^{\text{max}} \) and an L-iterate if \( \phi_L(v) \leq \frac{1}{2} \phi_{L}^{\text{max}} \).

Otherwise, an iterate is an M-iterate if

\[
\| \nabla_y M(v_{k+1}; y_k^E, \mu_k^R) \| \leq \tau_k \quad \text{and} \quad \| \nabla_x M^\nu(v_{k+1}; y_k^E, \mu_k^R) \| \leq \tau_k.
\]

If none of these conditions hold, then an iterate \( v_k \) is an F-iterate.

In order to force convergence to a second-order optimal point, it is necessary to change the function \( \omega(x, y) \) that appears in \( \phi_S \) and \( \phi_L \), as well as the test for an iteration being an M-iterate.

Ideally, the minimum eigenvalue of \( H \) in the null-space for \( J_F \) should be found, as well as the minimum eigenvalue of \( \nabla_{xx}^2 M \). However, this would require extensive computation. Instead, these quantities are estimated based on the value of the negative curvature. Recall that

\[
\frac{\tilde{u}^T (H + \frac{1}{\mu} J^T J) \tilde{u}}{\gamma \| \tilde{u} \|} \leq \lambda_{\text{min}}(H + \frac{1}{\mu} J^T J),
\]

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where we suppress the suffix $F$. Since $\gamma$ is bounded from below and above, if $\hat{u}^T(H + \frac{1}{\mu} J^T J)\hat{u} / ||\hat{u}||^2 \to 0$, the estimate for $\hat{u}$ implies $\lim \lambda_{\text{min}}(H + \frac{1}{\mu} J^T J) \geq 0$. Hence, the test for M-iterate optimality is changed to:

$$
||\nabla_y M(v_{k+1}; y_k^E, \mu_k^R) || \leq \tau_k
$$
and

$$
|| \min(x_{k+1}, \nabla_x M^v(v_{k+1}; y_k^E, \mu_k^R)) || \leq \tau_k
$$
and

$$
\frac{\hat{u}_{k+1}^T(H + \frac{1}{\mu} J^T J)\hat{u}_{k+1}}{||\hat{u}_{k+1}||^2} \geq \tau_k.
$$

Similarly, for the filter functions,

$$
\phi_S(v) = \eta(x) + 10^{-5} \omega(v) \quad \text{and} \quad \phi_L(v) = 10^{-5} \eta(x) + \omega(v)
$$

the optimality tests become

$$
\eta(x) = ||c(x)|| \quad \text{and} \quad \omega(x, y) = \min(|| \min(x, g(x) - J(x)^T y), -\frac{\hat{u}_{k+1}^T(H + \frac{1}{\mu} J^T J)\hat{u}_{k+1}}{||\hat{u}_{k+1}||^2} ||)
$$

### 3.3. Merit function

The line-search must also be changed to include the direction of negative curvature. First, it will be shown that the full primal-dual step is a step of negative curvature for the merit function Hessian.

**Lemma 3.1.** The vector $(u, w)$ defined as in 3.1 is a direction of negative curvature for $\nabla^2 M$.

**Proof.** Consider the calculation of $\begin{pmatrix} u \\ w \end{pmatrix}^T \nabla^2 M \begin{pmatrix} u \\ w \end{pmatrix}$.

$$
\begin{pmatrix} u \\ w \end{pmatrix}^T \begin{pmatrix} H + \frac{1}{\mu}(1 + \nu) J^T J & \nu J^T \\ \nu J & \nu \mu I \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u \\ w \end{pmatrix}^T \begin{pmatrix} Hu + \frac{1}{\mu}(1 + \nu) J^T J u + \nu J^T w \\ \nu J u + \nu \mu w \end{pmatrix} = u^T H u + \frac{1}{\mu}(1 + \nu) u^T J^T J u + 2\nu u^T J^T w + \nu \mu ||w||^2.
$$

From the definition above, $u = \beta \hat{u}$ and $\hat{u}^T(H + \frac{1}{\mu} J^T J)\hat{u} \leq \gamma \lambda_{\text{min}}(H + \frac{1}{\mu} J^T J)||\hat{u}||^2$, so multiplying both sides by $\beta^2$, the expression becomes $u^T(H + \frac{1}{\mu} J^T J)u \leq \gamma \lambda_{\text{min}}(H + \frac{1}{\mu} J^T J)||u||^2$. Let $\tilde{\gamma} = \gamma \lambda_{\text{min}}(H + \frac{1}{\mu} J^T J)$.

Using $w \triangleq -\frac{1}{\mu} J u$,

$$
u u^T H u + \frac{1}{\mu}(1 + \nu) u^T J^T J u + 2\nu u^T J^T w + \nu \mu ||w||^2 \leq -\tilde{\gamma} ||u||^2 - 2\nu ||u^T J^T J u + \frac{1}{\mu} ||J u||^2 \\
= -\tilde{\gamma} ||u||^2 - \frac{\nu}{\mu} ||J u||^2 \\
\leq -\tilde{\gamma} ||u||^2 - \nu \mu ||w||^2.
$$


For the line-search, let $R_k \triangleq u_k^T \nabla^2 \mathcal{M}^\nu(v_k; y_k^E, \mu_k^R) u_k \leq 0$. Define $\alpha_k = 2^{-j}$ such that
\[
\mathcal{M}^\nu(v_k + \alpha_k u_k + \alpha^2_k \Delta v_k; y_k^E, \mu_k^E) \leq \mathcal{M}^\nu + \alpha^2_k \eta S N_k + \alpha_k \eta S R_k.
\tag{3.1}
\]
Letting $\bar{\alpha} \triangleq \min(\alpha_{\min}, \alpha_k)$ and $\bar{\mu} \triangleq \max\left(\frac{1}{2} \mu_k, \mu_{k+1}\right)$, the update for the penalty parameter becomes:
\[
\mu_{k+1} = \begin{cases}
\mu_k, & \mathcal{M}^\nu(v_{k+1}; y_k^E, \mu_k) \leq \mathcal{M}^\nu(v_k; y_k^E, \mu_k) + \bar{\alpha} \eta S R_k + \bar{\alpha}^2 \eta S N_k \\
\bar{\mu}, & \text{otherwise},
\end{cases}
\tag{3.2}
\]

4. Consistency with established convergence theory

In their first-order analysis, Gill and Robinson [8] make the following assumptions:

**Assumption 4.1.** Each $\bar{H}(x_k, y_k)$ is chosen so that the sequence $\{\bar{H}(x_k, y_k)\}_{k \geq 0}$ is bounded, with $\{\bar{H}(x_k, y_k) + (1/\mu_k^R) J(x_k)^T J(x_k)\}_{k \geq 0}$ uniformly positive definite.

**Assumption 4.2.** The functions $f$ and $c$ are twice continuously differentiable.

**Assumption 4.3.** The sequence $\{x_k\}_{k \geq 0}$ is contained in a compact set.

Since $\nabla \mathcal{M}^\nu$ does not involve any term involving the objective or constraint Hessians, much of the first-order convergence theory holds. Incorporating the direction of negative curvature, Theorem 4.1 changes to:

**Theorem 4.1.** If there exists an integer $\hat{k}$ such that $\mu_{\hat{k}}^R \equiv \mu^R > 0$ and $k$ is an $F$-iterate for all $k \geq \hat{k}$, then the following hold:

1. $\{||\Delta v_k|| + ||u_k||\}_{k \geq \hat{k}}$ is bounded away from zero

2. There exists an $\epsilon$ such that for all $k \geq \hat{k}$, it holds that
\[
\nabla \mathcal{M}^\nu(v_k, y_k^E, \mu_k^R)^T \Delta v_k \leq -\epsilon \text{ or } u_k^T \nabla^2 \mathcal{M}^\nu(v_k; y_k^E, \mu_k^R) u_k \leq -\epsilon.
\]

**Proof.** If all iterates $k \geq \hat{k}$ are $F$-iterates, then,
\[
\tau_k \equiv \tau > 0, \quad \mu_k^R = \mu^R, \text{ and } y_k^E = y^E \text{ for all } k \geq \hat{k}
\]

Proof of the first result: Assume the contrary, i.e., there exists a subsequence $S_1 \subset \{k | k \geq \hat{k}\}$ such that $\lim_{k \in S_1} \Delta v_k = 0$ and $\lim_{k \in S_1} u_k = 0$. The solution $\Delta v_k$ to the QP subproblem satisfies
\[
\begin{pmatrix}
z_k \\
0
\end{pmatrix} = H_M^\nu(v_k; \mu_k^R) \Delta v_k + \nabla \mathcal{M}^\nu(v_k; y_k^E, \mu_k^R) \text{ and } 0 = \min(x_k + p_k, z_k).
\]

As $H_M^\nu$ is uniformly bounded, eventually for some $k \in S_1$ sufficiently large, $\Delta v_k$ satisfies the first-order conditions of an M-iterate, i.e.,
\[
||\nabla y \mathcal{M}^\nu(v_k; y_k^E, \mu_k^R)|| \leq \tau_k \text{ and } ||\min(x_{k+1}, \nabla x \mathcal{M}^\nu(v_{k+1}; y_k^E, \mu_k^R))|| \leq \tau_k.
\]

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In the construction of $u_k$, recall that $||u||$ is the largest possible value, subject to an upper bound, that is feasible. This implies that if $\lim u_k \rightarrow 0$, then eventually, $u$ is constrained by feasibility, or set to zero.

In the first case, i.e. the limiting upper bound constraint on $u_k$ must be $x_k + p_k + u_k \geq 0$, eventually since $u_k \rightarrow 0$ and $p_k \rightarrow 0$, if $i$ is a blocking bound for $u_k$, $x_i \leq \min(\mu, \epsilon_i)$ and $i \in W_k$, which implies that $[u]_i \equiv 0$. Hence, by construction and the fact that the set of possible indices is finite, $u_k$ is eventually identically zero. This implies that the second-order conditions of an M-iterate are also satisfied trivially, i.e.,

$$\frac{\tilde{u}_{k+1}^T (H + \frac{1}{\mu} J^T J) \tilde{u}_{k+1}}{||\tilde{u}_{k+1}||^2} \geq \tau_k,$$

and $\mu_k^R$ is decreased. This contradicts the assumption that $\mu_k^R$ is held fixed at $\mu_k^R \equiv \mu_k^R$ for all $k \geq \hat{k}$.

Proof of part (2): Assume, to the contrary, that there exists a subsequence $S_2$ of $\{k : k \geq \hat{k}\}$ such that

$$\lim_{k \in S_2} \nabla M'(v_k; y^E, \mu^R)^T \Delta v_k = 0 \quad (4.1)$$

and

$$\lim_{k \in S_2} u_k^T \nabla^2 M'(v_k; y^E, \mu^R) u_k = 0.$$

Consider the matrix

$$L_k = \left( \begin{array}{cc} I & 0 \\ \frac{1}{\mu R} J_k & I \end{array} \right).$$

Since the $\Delta v = 0$ is feasible and $\Delta v_k$ a solution for the convex problem, it follows that

$$-\nabla M'(v_k; y^E, \mu^R)^T \Delta v_k \geq \frac{1}{2} \Delta v_k^T H'_M(v_k; \mu^R) \Delta v_k$$

$$= \frac{1}{2} \Delta v_k^T L_k^{-1} L_k^T H'_M(v_k; \mu^R) L_k L_k^{-1} \Delta v_k$$

$$= \left( \begin{array}{c} p_k \\ q_k + \frac{1}{\mu R} J_k p_k \end{array} \right)^T \left( \bar{H}_k + \frac{1}{\mu R} J_k^T J_k 0 0 \nu \mu R \right) \left( \begin{array}{c} p_k \\ q_k + \frac{1}{\mu R} J_k p_k \end{array} \right)$$

Since $H'_M$ is bounded,

$$\Delta v_k^T L_k^{-1} L_k^T H'_M(v_k; \mu^R) L_k L_k^{-1} \Delta v_k \geq \bar{\lambda}_{\text{min}} ||p_k||^2 + \nu \mu R ||q_k + (1/\mu R) J_k p_k||^2;$$

for some $\bar{\lambda}_{\text{min}} > 0$. Combining this with (4.1) it follows that

$$\lim_{k \in S_2} p_k = \lim_{k \in S_2} \left( q_k + \frac{1}{\mu R} J_k p_k \right) = 0,$$

in which case $\lim_{k \in S_2} q_k = 0$. Hence $\Delta v_k \in S_2 \rightarrow 0$. 

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Since \( \lim_{k \in S} u_k^T \nabla^2 \mathcal{M}^\nu(x_k, y_k^E, \mu) u_k = 0 \), there exists a \( \hat{k}_2 \), such that for all \( k \geq \hat{k}_2 \), \( u_k^T \nabla^2 \mathcal{M}^\nu(x_k, y_k^E, \mu) u_k/\|u_k\|^2 > -\tau \) or \( u_k \to 0 \). The former, by the same argument as for part (1), together with \( \Delta v_k \to 0 \), implies that eventually \( k \) is an M-iterate. The latter, together with \( \lim \Delta v_k = 0 \), contradicts the statement of part (1) of the theorem, so part (3) must hold.

The proofs of the first result of Theorem 4.1 and Theorem 4.2 of Gill and Robinson \[8\] do not change.

5. Global convergence to second-order optimal points

5.1. Filter Convergence

**Definition 5.1.** The Weak Constant Rank (WCR) condition holds at \( x \) if there is a neighborhood \( M(x) \) for which the rank of \( \begin{pmatrix} J(z) \\ E_A^T \end{pmatrix} \) is constant for all \( z \in M(x) \), where \( E_A \) is the columns of the identity corresponding to the indices of \( x \) active at \( x \) (as in \( i \in A \) if \( x_i = 0 \)).

**Theorem 5.1.** Assume there is a subsequence \( v_k \) of S- and L-iterates converging to \( v^* \), with \( v^* = (x^*, y^*) \) satisfying the first-order KKT conditions. Furthermore, assume that MFCQ and WCR hold at \( v^* \). Then \( v^* \) satisfies the necessary second-order necessary optimality conditions.

**Proof.** Let \( d \in T(x^*) \equiv \{ d \mid J(x^*) d = 0 \text{ and } E_A^T d = 0 \} \) with \( \|d\| = 1 \). By Lemma 3.1 of Andreani et al. \[9\] there exists \( \{ d_k \} \) such that \( d_k \in T(x_k) \) and \( d_k \to d \), where

\[
T(x_k) = \{ d \mid J(x_k) d = 0 \text{ and } E_A^T d = 0 \}.
\]

Without loss of generality, we may let \( \|d_k\| = 1 \). Since \( x_k \to x^* \), eventually \( \mathcal{W}_k = \mathcal{A}^* \), where \( \mathcal{A}^* \) is the active set at \( x^* \). Then, by the definition of the S- and L-iterates, and Lemma 2.1, \( d_k^T (\nabla^2 f(x_k) + \sum y_k \nabla^2 c(x_k)) d_k > \lambda_{\min}(Z_k H_k Z_k) > -\xi_k \), where \( 0 < \xi_k \to 0 \). Taking limits, it follows that \( d^T (\nabla^2 f(x_k) + \sum y^* \nabla^2 c(x^*)) d \geq 0 \).

**References**

[1] J. R. Bunch and L. Kaufman. A computational method for the indefinite quadratic programming problem. *Linear Algebra Appl.*, 34:341–370, 1980.

[2] J. R. Bunch and B. N. Parlett. Direct methods for solving symmetric indefinite systems of linear equations. *SIAM J. Numer. Anal.*, 8:639–655, 1971.

[3] Roger Fletcher. Factorizing symmetric indefinite matrices. *Linear Algebra Appl.*, 14:257–272, 1976.

[4] Anders Forsgren. Inertia-controlling factorizations for optimization algorithms. *Appl. Num. Math.*, 43:91–107, 2002.

[5] Anders Forsgren and Philip E. Gill. Primal-dual interior methods for nonconvex nonlinear programming. *SIAM J. Optim.*, 8:1132–1152, 1998.
[6] Anders Forsgren, Philip E. Gill, and Walter Murray. Computing modified Newton directions using a partial Cholesky factorization. Report TRITA-MAT-1993-9, Division of Optimization and Systems Theory, Department of Mathematics, Royal Institute of Technology, 1993.

[7] Philip E. Gill and Daniel P. Robinson. A primal-dual augmented Lagrangian. *Computational Optimization and Applications*, pages 1–25, 2010.

[8] Philip E. Gill and Daniel P. Robinson. Regularized sequential quadratic programming methods. Numerical Analysis Report 11-02, Department of Mathematics, University of California, San Diego, La Jolla, CA, 2011.

[9] R. Andreani J.M. Martinez and M.L. Schuverdt. On second-order optimality conditions for nonlinear programming. 2010.