Chaos and Shadowing Lemma for Autonomous Systems of Infinite Dimensions

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January 12, 2022

*This work is partially supported by the Guggenheim Fellowship. MSC: 35 46
Abstract

For finite-dimensional maps and periodic systems, Palmer rigorously proved Smale horseshoe theorem using shadowing lemma in 1988 [18]. For infinite-dimensional maps and periodic systems, such a proof was completed by Steinlein and Walther in 1990 [23], and Henry in 1994 [9]. For finite-dimensional autonomous systems, such a proof was accomplished by Palmer in 1996 [16]. For infinite-dimensional autonomous systems, the current article offers such a proof. First we prove an Inclination Lemma to set up a coordinate system around a pseudo-orbit. Then we utilize graph transform and the concept of persistence of invariant manifold, to prove the existence of a shadowing orbit.
1 Introduction

Since the invention of Shadowing Lemma by Anosov in 1967, it has been applied in a variety of situations. Starting from 1984 [17], Palmer had been trying to use shadowing lemma to rigorously prove Smale horseshoe theorem. In 1988, he successfully completed such a proof [18]. This proof works for finite-dimensional maps and periodic systems. Since then, such an application of shadowing lemma had been amplified in different situations including infinite dimensions, non-invertible maps etc. [3] [2] [8] [1] [22] [25] [26]. For infinite-dimensional maps and periodic systems, rigorous proof of Smale horseshoe theorem using shadowing lemma was completed by Steinlein and Walther in 1990 [23], and Henry in 1994 [9]. Such a proof for autonomous systems had been elusive for a long time. For finite-dimensional autonomous systems, such a proof was completed by Palmer in 1996 [16]. Applications of such shadowing lemma for finite-dimensional autonomous systems had been amplified [4] [5]. Symbolic labeling of orbits for finite-dimensional autonomous systems had been investigated by Silnikov [20]. For infinite-dimensional autonomous systems, the current article offers a proof of Smale horseshoe theorem using shadowing lemma. We first set up a pseudo-orbit, then prove an Inclination Lemma to set up a proper coordinate system around the pseudo-orbit. Finally we use graph transform and the concept of persistence of invariant manifold of Fenichel [7], to prove the existence of a shadowing orbit.

Our interest in such a proof of Smale horseshoe theorem for infinite-dimensional autonomous systems, lies in the development of the theory of chaos in partial differential equations [13] [12] [10] [11]. Indeed, we have presented as an example, a derivative nonlinear Schrödinger equation.

The article is organized as follows: Section 1 is the Introduction. Section 2 is the Setup of Assumptions. In Section 3, we set up the pseudo-orbits. In Section 4, we prove an Inclination Lemma, and set up a proper coordinate system around a pseudo-orbit. In Section 5, we prove a Shadowing Lemma. In Section 6, we prove Smale Horseshoe Theorem for a Poincaré return map. In Section 7, we present an example: A Derivative Nonlinear Schrödinger Equation. In Section 8, we present an example of periodic systems: A Periodically Perturbed Sine-Gordon Equation.

2 The Setup

Let $B$ be a Banach space on which an autonomous flow is defined. We set up the assumptions as follows.

- **Assumption (A1):** There exist a hyperbolic limit cycle $S$ and a transversal homoclinic orbit $\xi$ asymptotic to $S$. As curves, $S$ and $\xi$ are $C^3$.

- **Assumption (A2):** The Fenichel fiber theorem is valid at $S$. That is, there exist a family of unstable Fenichel fibers $\{F^u(q) : q \in S\}$ and a family of stable Fenichel fibers $\{F^s(q) : q \in S\}$. For each fixed $q \in S$, $F^u(q)$ and $F^s(q)$ are $C^3$ submanifolds. $F^u(q)$ and $F^s(q)$ are $C^2$ in $q, \forall q \in S$. The unions $\bigcup_{q \in S} F^u(q)$ and $\bigcup_{q \in S} F^s(q)$ are the unstable and stable manifolds of $S$. Both families are invariant, i.e.

$$F^t(F^u(q)) \subset F^u(F^t(q)), \forall t \leq 0, q \in S,$$
Figure 3.1: An Illustration of Loop-0.

\[ F^t(F^s(q)) \subset F^s(F^t(q)), \forall \ t \geq 0, q \in S, \]

where \( F^t \) is the evolution operator. There are positive constants \( \kappa \) and \( \hat{C} \) such that \( \forall q \in S, \forall q^- \in F^u(q) \) and \( \forall q^+ \in F^s(q) \),

\[
\| F^t(q^-) - F^t(q) \| \leq \hat{C} e^{\kappa t} \| q^- - q \|, \forall \ t \leq 0 ,
\]

\[
\| F^t(q^+) - F^t(q) \| \leq \hat{C} e^{-\kappa t} \| q^+ - q \|, \forall \ t \geq 0 .
\]

- **Assumption (A3):** \( F^t(q) \) is \( C^0 \) in \( t \), for \( t \in (-\infty, \infty) \), \( q \in B \). For any fixed \( t \in (-\infty, \infty) \), \( F^t(q) \) is a \( C^2 \) diffeomorphism on \( B \).

**Remark 2.1** Notice that we do not assume that as functions of time, \( S \) and \( \xi \) are \( C^3 \), and we only assume that as curves, \( S \) and \( \xi \) are \( C^3 \).

### 3 The Pseudo-Orbits

The building blocks of the pseudo-orbits are what we call Loop-0 and Loop-1.

**Definition 1** Loop-0, denoted by \( \eta_0 \), is defined to be the \( m \)-times circulation of the limit cycle \( S \), where \( m \) is to be determined. See Figure 3.1 for an illustration.

To define Loop-1, choose points \( p_s, p_c, \) and \( p_u \) on \( S \), such that the arc-lengths \( \hat{p}_s \hat{p}_c \) and \( p_u \hat{p}_u \) are equal to \( \delta \), where \( \delta \) is a small parameter to be determined. Let \( \hat{p}_u \) be one of the points of intersection \( \xi \cap F^u(p_u) \) such that \( \| \hat{p}_u - p_u \| \sim O(\delta^\nu) \) as \( \delta \to 0 \), and \( \hat{p}_s \) be one of the points of intersection \( \xi \cap F^s(p_s) \) such that \( \| \hat{p}_s - p_s \| \sim O(\delta^\nu) \) as \( \delta \to 0 \), where \( \nu \geq 4 \) (\( \nu \) will be determined again later). Let \( \zeta = \zeta_a \cup \zeta_s \) be a curve connecting \( \hat{p}_s, p_c, \) and \( \hat{p}_u \). \( \zeta_s \) lies in the stable manifold \( W^s(S) \) of \( S \), and connects \( \hat{p}_s \) and \( p_c \). \( \zeta_a \) lies in the unstable manifold \( W^u(S) \) of \( S \), and connects \( p_c \) and \( \hat{p}_u \). Let \( \xi \) be the connected arc-portion of \( \xi \), which links \( \hat{p}_u \) and \( \hat{p}_s \). Then the union \( \eta_1 = \hat{\xi} \cup \zeta \) is a loop. We choose \( \zeta \) such that \( \eta_1 \) is \( C^3 \), \( p_u \hat{p}_u \cup \zeta_a \) is \( C^3 \), and \( \zeta_s \cup \hat{p}_s \hat{p}_u \) is \( C^3 \). See Figure 3.2 for an illustration.

**Definition 2** Loop-1 is defined to be \( \eta_1 = \hat{\xi} \cup \zeta \).

To define the pseudo-orbits, first we introduce sequences of symbols.
Figure 3.2: An Illustration of Loop-1.

**Definition 3** Let \( \Sigma \) be a set that consists of elements of the doubly infinite sequence form:
\[
a = (\cdots a_{-2} a_{-1} a_0, a_1 a_2 \cdots),
\]
where \( a_k \in \{0, 1\}, \ k \in \mathbb{Z} \). We introduce a topology in \( \Sigma \) by taking as neighborhood basis of
\[
a^* = (\cdots a^*_{-2} a^*_{-1} a^*_0, a^*_1 a^*_2 \cdots),
\]
the set
\[
B_j = \{ a \in \Sigma \mid a_k = a^*_k (|k| < j) \}
\]
for \( j = 1, 2, \ldots \). This makes \( \Sigma \) a topological space. The Bernoulli shift automorphism \( \chi \) is defined on \( \Sigma \) by
\[
\chi : \Sigma \mapsto \Sigma, \quad \forall a \in \Sigma, \ \chi(a) = b, \ \text{where} \ b_k = a_{k+1}.
\]
The Bernoulli shift automorphism \( \chi \) exhibits sensitive dependence on initial conditions, which is a hallmark of chaos.

To each \( a_k \in \{0, 1\} \), we associate the loop-\( a_k \), \( \eta_{a_k} \). Then each doubly infinite sequence
\[
a = (\cdots a_{-2} a_{-1} a_0, a_1 a_2 \cdots)
\]
is associated with a \( \delta \)-pseudo-obit
\[
\eta_a = (\cdots \eta_{a_{-2}} \eta_{a_{-1}} \eta_{a_0}, \eta_{a_1} \eta_{a_2} \cdots).
\]
Since \( \eta_a \) is a \( C^3 \) curve, we choose a parametrization of \( \eta_a \): \( \eta_a = \eta_a(\tau), \ \tau \in R \), such that \( \eta_a(\tau) \) is a \( C^3 \) function of \( \tau \).

### 4 Coordinates along the Pseudo-Orbits and Inclination Lemma

We define bundles along the limit cycle \( S \) and the homoclinic orbit \( \xi \) as follows:

**Definition 4** The bundles \( E^u \), \( E^c \), and \( E^s \) along \( S \) are defined by
\[
E^u(q) = T_q F^u(q), \quad E^c(q) = T_q S, \quad E^s(q) = T_q F^s(q), \quad q \in S,
\]
where \( T_q \) indicates the tangent space at \( q \).
See Figure 4.1 for an illustration. $E^u$ and $E^s$ provide a coordinate system for a tubular neighborhood of $S$, that is, any point in this neighborhood has a unique coordinate

$$(\tilde{v}^s, \tilde{\theta}, \tilde{v}^u), \quad \tilde{v}^s \in E^s(\tilde{\theta}), \quad \tilde{v}^u \in E^u(\tilde{\theta}), \quad \tilde{\theta} \in S.$$ 

Fenichel fibers provide another coordinate system for the tubular neighborhood of $S$. For any $\theta \in S$, the Fenichel fibers $F^s(\theta)$ and $F^u(\theta)$ have the expressions

$$\begin{align*}
\tilde{v}^s &= v^s, \\
\tilde{\theta} &= \theta + \Theta_s(v^s, \theta), \\
\tilde{v}^u &= V_s(v^s, \theta),
\end{align*}$$

and

$$\begin{align*}
\tilde{v}^u &= v^u, \\
\tilde{\theta} &= \theta + \Theta_u(v^u, \theta), \\
\tilde{v}^s &= V_u(v^u, \theta),
\end{align*}$$

where $v^s$ and $v^u$ are the parameters parametrizing $F^s(\theta)$ and $F^u(\theta)$,

$$\Theta_z(0, \theta) = \frac{\partial}{\partial v_z} \Theta(0, \theta) = V_z(0, \theta) = \frac{\partial}{\partial v_z} V_z(0, \theta) = 0, \quad z = u, s;$$

and $\Theta_z(v^z, \theta)$ and $V_z(v^z, \theta)$ ($z = u, s$) are $C^3$ in $v^z$ and $C^2$ in $\theta$. The coordinate transformation from $(v^s, \theta, v^u)$ to $(\tilde{v}^s, \tilde{\theta}, \tilde{v}^u)$

$$\begin{align*}
\tilde{v}^s &= v^s + V_u(v^u, \theta), \\
\tilde{\theta} &= \theta + \Theta_u(v^u, \theta) + \Theta_s(v^s, \theta), \\
\tilde{v}^u &= v^u + V_s(v^s, \theta),
\end{align*}$$

is a $C^2$ diffeomorphism. In terms of the Fenichel coordinate $(v^s, \theta, v^u)$, the Fenichel fibers coincide with their tangent spaces. From now on, we always work with the Fenichel coordinate $(v^s, \theta, v^u)$.

**Definition 5** Let $q^u, q^s \in \xi$, and $q \in S$, such that $q^z \in F^z(q)$, $z = u, s$. The bundles $E^u$, $E^c$, and $E^s$ along $\xi$ are defined by

$$E^u(F^t(q^u)) = DF^t(q^u)(\mathcal{T}_{q^u}F^u(q)), \quad t \in (-\infty, \infty),$$

$$E^c(p) = \mathcal{T}_p \xi, \quad p \in \xi,$$

$$E^s(F^t(q^s)) = DF^t(q^s)(\mathcal{T}_{q^s}F^s(q)), \quad t \in (-\infty, \infty).$$

See Figure 4.2 for an illustration. We have the following inclination lemma.
Lemma 4.1 (Inclination Lemma) For $\delta$ small enough, there exists a $\nu_0 > 0$ ($\nu_0$ depends upon $\delta$), such that for any $q \in S$, let $q^u \in \xi \cap \mathcal{F}^u(q)$ and $q^s \in \xi \cap \mathcal{F}^s(q)$,

1. When $\|q^s - q\| \sim O(\delta^\nu)$, $\nu \geq \nu_0$, $E^u(q^s) \oplus E^c(q^s)$ is $\delta^3$-close to $E^u(q) \oplus E^c(q)$.

2. When $\|q^u - q\| \sim O(\delta^\nu)$, $\nu \geq \nu_0$, $E^s(q^u) \oplus E^c(q^u)$ is $\delta^3$-close to $E^s(q) \oplus E^c(q)$.

Proof: Let $q_1^s \in \xi \cap \mathcal{F}^s(q)$ such that $\|q_1^s - q\| \sim O(\delta^\nu)$. Notice that $\mathcal{F}^s(q) = E^s(q)$. Let $v_1 \in E^u(q_1^s) \oplus E^c(q_1^s)$, $\|v_1\| = 1$. We represent $v_1$ in the frame $(E^s(q), E^u(q) \oplus E^c(q))$,

$v_1 = (v_1^s, v_1^{uc})$,

where $\|v_1^{uc}\| \neq 0$, since $\xi$ is a transversal homoclinic orbit. Let $\lambda_1 = \|v_1^s\|/\|v_1^{uc}\|$. The transversality of $\xi$ implies that $\lambda_1$ has an upper bound for all $v_1$ [9]. Since $S$ is compact, $\lambda_1$ has an upper bound for all $q \in S$. Let $q_n^s$ be the consecutive intersection points between $\xi$ and $\mathcal{F}^s(q) = E^s(q)$, $q_n^s = F^{\tau_s}(q_{n-1}^s)$, where $\tau_s$ is the period of the limit cycle $S$. Let

$$(\hat{v}_2^s, \hat{v}_2^{uc}) = DF^{4m\tau_s}(q_1^s)v_1 = DF^{4m\tau_s}(q_1^s)(v_1^s, 0) + DF^{4m\tau_s}(q)(0, v_1^{uc})$$

$$+ [DF^{4m\tau_s}(q_1^s) - DF^{4m\tau_s}(q)](0, v_1^{uc}).$$

Also let

$$(r^s, r^{uc}) = [DF^{4m\tau_s}(q_1^s) - DF^{4m\tau_s}(q)](0, v_1^{uc}).$$

Then

$$\hat{v}_2^s = DF^{4m\tau_s}(q_1^s)v_1^s + r^s, \quad \hat{v}_2^{uc} = DF^{4m\tau_s}(q)v_1^{uc} + r^{uc}.$$ 

We choose $m$ large enough such that the constant $\hat{C}$ in Assumption (A2) satisfies $\hat{C} \leq e^{\kappa m\tau_s}$, and

$$DF^{4m\tau_s}(q)v_1^{uc} \geq 2e^{-\kappa m\tau_s}\|v_1^{uc}\|.$$ 

For such fixed $m$, choosing $\delta$ small enough, we have

$$\|DF^{4m\tau_s}(q_1^s) - DF^{4m\tau_s}(q)\| \leq \delta^3 e^{-\kappa m\tau_s}.$$ 

Then

$$\|\hat{v}_2^s\| \leq e^{-3\kappa m\tau_s}\|v_1^s\| + \delta^3 e^{-\kappa m\tau_s}\|v_1^{uc}\|,$$

$$\|\hat{v}_2^{uc}\| \geq 2e^{-\kappa m\tau_s}\|v_1^{uc}\| - \delta^3 e^{-\kappa m\tau_s}\|v_1^{uc}\| \geq e^{-\kappa m\tau_s}\|v_1^{uc}\|.$$

Figure 4.2: An illustration of the bundles along the homoclinic orbit $\xi$. 
Thus

\[ \lambda_2 = \left\| \dot{\upsilon}_2^s \right\| / \left\| \dot{\upsilon}_2^{uc} \right\| \leq e^{-2\kappa m T_s} \lambda_1 + \delta^3. \]

Iterating the argument, we obtain

\[ \lambda_N \leq e^{-2\kappa(N-1)m T_s} \lambda_1 + \delta^3 \sum_{l=0}^{N-2} e^{-2\kappa l m T_s}. \]

For such fixed \( \delta \), when \( N \) is large enough,

\[ \lambda_N \leq 4\delta^3. \]

There exists \( \nu_0 > 0 \) such that

\[ \left\| F^{4(N-1)m T_s} (q_s^1) - q_1 \right\| \sim \mathcal{O}(\delta^\nu), \quad \nu \geq \nu_0. \]

Similarly for the case of \( q^u \). The proof is completed. \( \square \)

**Remark 4.1** Inclination lemmas have been utilized in proving many significant theorems [15] [24]. Here we show how to use inclination lemma to prove shadowing lemma in the autonomous case.

Based up the fact in Lemma 4.1, \( C^2 \) bundles can be constructed along the pseudo-orbit \( \eta_a \) as follows (cf. Figure 3.2): Since \( E^u(p_\hat{s}) \oplus E^c(p_\hat{s}) \) is \( \delta^3 \)-close to \( E^u(p_s) \oplus E^c(p_s) \), we can construct \( C^2 \) bundle \( E^u(p) \oplus E^c(p) \), \( p \in \hat{p}_s p_c \), along the curve \( \hat{p}_s p_c \) such that \( E^u(p) \oplus E^c(p) \) is \( C^2 \) along both the curve \( p \in \hat{p}_s p_c \) and the curve \( p \in \hat{p}_s p_c \). \( E^c(p) \) is the tangent space \( \mathcal{T}_{\hat{p}_s p_c} \). For any \( p \in \hat{p}_s p_c \), \( p \in F^s(q), q \in S \), we define \( E^s(p) = \mathcal{T}_{\hat{p}_s p_c} \). Similarly, we can define such bundles along the curve \( p_c \hat{p}_s \). Thus we obtain \( C^2 \) bundles \( E^u(p) \oplus E^c(p) \) and \( E^s(p) \) along \( \eta_a \). We also need to rectify \( E^u(p) \) and \( E^s(p) \) along Loop-1. For any two subspaces \( E_1 \) and \( E_2 \), one can define the angle \( \vartheta(E_1, E_2) \) as follows [9]:

\[ \vartheta(E_1, E_2) = \inf_{v_1, v_2} \left\{ \left\| v_1 - v_2 \right\| \mid v_1 \in E_1, v_2 \in E_2, \left\| v_1 \right\| = \left\| v_2 \right\| = 1 \right\}. \]

Since \( E^u(p) \) (\( p \in S \)) is a transversal bundle along \( S \), and \( S \) is compact, \( \vartheta(E^u(p), E^c(p)) \geq \hat{\vartheta} > 0 \) for all \( p \in S \). Let \( p_n^s \) be the consecutive intersection points of \( \xi \) with \( F^s(p) \). Then \( E^c(p_n^s) \to E^c(p) \) as \( n \to \infty \). Thus, there exists \( n_1 \), \( \vartheta(E^u(p), E^c(p_n^s)) \geq \frac{1}{3} \hat{\vartheta} \) for all \( n \geq n_1 \), and all \( p \in S \), by the compactness of \( S \). Let \( p^+ \in \xi \), such that when \( \delta \) is sufficiently small,

\[ \vartheta(E^u(p), E^c(p)) \geq \frac{1}{3} \hat{\vartheta}, \quad \text{for all } \hat{p} \in p^c p_c, \quad \hat{p} \in F^s(p), \quad p \in S. \]

See Figure 4.3 for an illustration. We can choose \( p^+ \) close enough to \( p_0 \in S \), \( p^+ \in F^s(p_0) \), such that \( E^u(q^+) \oplus E^c(q^+) \) is \( \mathcal{O}(\frac{1}{300}) \) close to \( E^u(q_0) \oplus E^c(q_0) \), \( q^+ \in p^c p_c, q^+ \in F^s(q_0), q_0 \in S \). For any \( \tilde{v}^u \in E^u(q_0), \left\| \tilde{v}^u \right\| = 1 \), \( \tilde{v}^u \) has the representation in the frame \( (E^s(q^+), E^u(q^+) \oplus E^c(q^+)) \)

\[ \tilde{v}^u = v^s + v^{uc}. \]
Alternatively, $v^{uc}$ has the representation

$$v^{uc} = -v^s + \tilde{v}^u$$

in the frame $(E^s(q_0), E^u(q_0) \oplus E^c(q_0))$, since $E^s(q_0) = E^s(q^+)$. Thus

$$\|v^s\| \leq \frac{1}{300} \hat{v} \|v^u\| = \frac{1}{300} \hat{v},$$

and

$$1 - \frac{1}{300} \hat{v} \leq \|v^{uc}\| \leq 1 + \frac{1}{300} \hat{v}.$$

Let $v^c \in E^c(q^+)$, $\|v^c\| = 1$, we have

$$\frac{1}{3} \hat{v} \leq \|\tilde{v}^u - v^c\| = \|v^s + v^{uc} - v^u/\|v^{uc}\| + v^{uc}/\|v^{uc}\| - v^c\|$$

$$\leq \|v^s\| + \left|1 - \frac{1}{\|v^{uc}\|}\right| \|v^{uc}/\|v^{uc}\| - v^c\|$$

$$\leq \frac{1}{300} \hat{v} + \frac{1}{300} \hat{v} + \|v^{uc}/\|v^{uc}\| - v^c\|.$$  

Thus we have

$$\left\|v^{uc}/\|v^{uc}\| - v^c\right\| \geq \frac{98}{300} \hat{v}.$$

In fact, $v^{uc}$ is the projection of $\tilde{v}^u$ onto $E^u(q^+) \oplus E^c(q^+)$. All such $v^{uc}$'s span the projection $\tilde{E}^u(q^+)$ of $E^u(q_0)$ onto $E^u(q^+) \oplus E^c(q^+)$. Thus

$$\vartheta(\tilde{E}^u(q^+), E^c(q^+)) \geq \frac{98}{300} \hat{v}, \quad q^+ \in \tilde{p}^+p_c.$$

At $q^+ \in p^+p_c$, we replace $E^u(q^+)$ by $\tilde{E}^u(q^+)$, and we use the same notion $E^u(q^+)$. $E^u(q^+)$ is still $C^2$ along both $p^+_t p_d \tilde{p}_u$ and $p^+_u p_d p_u$. Similarly we can choose $p^- \in \xi$, construct $\tilde{E}^s(q^-)$ ($q^- \in p^-p_c$), replace $E^s(q^-)$ by $\tilde{E}^s(q^-)$, and use the same notation $E^s(q^-)$. Let $T_0 > 0$ be the time such that $p^+_t = F^T_0(p^-)$. Let $\hat{\xi}$ be the portion of $\hat{\xi}$, $\hat{\xi} = U_0 \cup T_0 F^t(p^-)$. Inside $E^u(q) \oplus E^c(q)$ ($q \in \xi$), we choose a $C^2$ transversal bundle $\tilde{E}^u(q)$ along $\xi$, replace $E^u(q)$ by $\tilde{E}^u(q)$, and use the same notation $E^u(q)$. Similarly, inside $E^s(q) \oplus E^c(q)$ ($q \in \xi$), we choose a $C^2$ transversal bundle $\tilde{E}^s(q)$ along $\xi$, replace $E^s(q)$ by $\tilde{E}^s(q)$, and use the same notation $E^s(q)$. This way, we construct $C^2$ transversal bundles $E^u(q)$ and $E^s(q)$ along $\eta_0$.  

Figure 4.3: A rectification of the transversal bundle along Loop-1.
5 Shadowing Lemma

We will use graph transform and the concept of persistence of invariant manifold, to establish a shadowing orbit [7]. In [7], the estimates in the current case are not given in details. We will remedy that below.

Denote by \( \hat{E} \) the transversal bundle

\[
\hat{E} = \{(q, E^u(q), E^s(q)) \mid q \in \eta_a\},
\]

which serves as a coordinate system around \( \eta_a \). Using the parametrization \( \eta_a = \eta_a(\tau), \tau \in R \), we can introduce the coordinate in a neighborhood of \( \eta_a \):

\[
(\tau, x^u, x^s), \quad \text{where} \quad \tau \in R, x^u \in E^u(\eta_a(\tau)), x^s \in E^s(\eta_a(\tau)).
\]

In this coordinate system, the evolution operator \( F^T \) has the representation:

\[
F^T(\tau, x^u, x^s) = (f(\tau, x^u, x^s), g^u(\tau, x^u, x^s), g^s(\tau, x^u, x^s)),
\]

where \( T > 0 \) is a large time. First we define Lipschitz sections over \( \eta_a \).

**Definition 6** Let \( \Gamma_\epsilon \) be the space of sections of \( \hat{E} \):

\[
\Gamma_\epsilon = \{ \sigma \mid \sigma(\tau) = (\tau, x^u(\tau), x^s(\tau)), \tau \in R, \|x^u(\tau)\| \leq \epsilon, \|x^s(\tau)\| \leq \epsilon \}.
\]

We define the \( C^0 \) norm of \( \sigma \in \Gamma_\epsilon \) as

\[
\|\sigma\|_{C^0} = \max \{ \sup_{\tau \in R} \|x^u(\tau)\|, \sup_{\tau \in R} \|x^s(\tau)\| \}.
\]

Then we define a Lipschitz semi-norm on \( \Gamma_\epsilon \):

\[
\text{Lip} \{ \sigma \} = \max \left\{ \sup_{|\tau_1 - \tau_2| \leq \Delta} \frac{\|x^u(\tau_1) - x^u(\tau_2)\|}{\|\tau_1 - \tau_2\|}, \sup_{|\tau_1 - \tau_2| \leq \Delta} \frac{\|x^s(\tau_1) - x^s(\tau_2)\|}{\|\tau_1 - \tau_2\|} \right\}
\]

for some small fixed \( \Delta > 0 \). Let \( \Gamma_{\epsilon, \gamma} \) be a subset of \( \Gamma_\epsilon \),

\[
\Gamma_{\epsilon, \gamma} = \{ \sigma \in \Gamma_\epsilon \mid \text{Lip} \{ \sigma \} \leq \gamma \}.
\]

**Lemma 5.1** \( \Gamma_{\epsilon, \gamma} \) is closed under the \( C^0 \) norm.

Proof: Assume that \( \{ \sigma_j \}_{j=1,2,\ldots} \) is a Cauchy sequence in \( \Gamma_{\epsilon, \gamma} \) under the \( C^0 \) norm. Then \( \forall \tau \in R, x^z_j(\tau)(z = u, s) \) is a Cauchy sequence, which has a limit \( x^z(\tau) \). Define a new section \( \sigma \) by

\[
\sigma(\tau) = (\tau, x^u(\tau), x^s(\tau)).
\]

First we want to show that \( \sigma \in \Gamma_\epsilon, \forall \tau \in R, \forall j = 1, 2, \ldots \),

\[
\|x^z_j(\tau)\| \leq \epsilon, \quad (z = u, s).
\]

Then \( \forall \tau \in R \),

\[
\|x^z(\tau)\| \leq \epsilon, \quad (z = u, s).
\]
Thus $\sigma \in \Gamma$. Next, we want to show that $\sigma \in \Gamma_{\epsilon, \gamma}$. We know that for $\tau_1$ and $\tau_2$ such that $|\tau_1 - \tau_2| \leq \Delta$, and any $j = 1, 2, \ldots$,

$$\frac{\|x_j^z(\tau_1) - x_j^z(\tau_2)\|}{\|\tau_1 - \tau_2\|} \leq \gamma, \quad (z = u, s).$$

Then

$$\frac{x^z(\tau_1) - x^z(\tau_2)}{\|\tau_1 - \tau_2\|} = \lim_{j \to \infty} \frac{\|x_j^z(\tau_1) - x_j^z(\tau_2)\|}{\|\tau_1 - \tau_2\|} \leq \gamma, \quad (z = u, s).$$

Thus

$$\text{Lip } \{\sigma\} = \max \left\{ \sup_{|\tau_1 - \tau_2| \leq \Delta} \frac{\|x^u(\tau_1) - x^u(\tau_2)\|}{\|\tau_1 - \tau_2\|}, \sup_{|\tau_1 - \tau_2| \leq \Delta} \frac{\|x^s(\tau_1) - x^s(\tau_2)\|}{\|\tau_1 - \tau_2\|} \right\} \leq \gamma.$$

The proof is completed. \(\Box\)

For any $\sigma \in \Gamma_{\epsilon, \gamma}$,

$$\sigma(\tau) = (\tau, x^u(\tau), x^s(\tau)), \quad \tau \in \mathbb{R},$$

let $T > 0$ be a large time, we define the graph transform $G$ as follows:

$$(G\sigma)(\tau) = (\tau, x^u_t(\tau), x^s_t(\tau)), \quad (5.1)$$

where

$$f(\tau^-, x^u(\tau^-), x^s(\tau^-)) = \tau,$$
$$g^s(\tau^-, x^u(\tau^-), x^s(\tau^-)) = x^s_t(\tau),$$
$$f(\tau, x^u_1(\tau), x^s(\tau)) = \tau^+,$$
$$g^u(\tau, x^u_1(\tau), x^s(\tau)) = x^u(\tau^+).$$

See Figure 5.1 for an illustration.

First we shall prove the existence of a fixed point of $G$ in $\Gamma_{\epsilon, \gamma}$. Then we will show that the graph of the fixed point is an orbit. Thereby, we establish the existence of an orbit that $\epsilon$-shadows the pseudo-orbit. The following preliminary lemmas are quoted from [6](pp.155 and pp.186).
**Lemma 5.2 (Mean Value Theorem)** Let $E_1$ and $E_2$ be two Banach spaces, $F$ a continuous mapping from a neighborhood of a segment $\ell$ joining two points $q_0, q_0 + q_1$ of $E_1$, into $E_2$. If $F$ is differentiable at every point of $\ell$, then

$$\|F(q_0 + q_1) - F(q_0)\| \leq \|q_1\| \sup_{0 \leq \alpha \leq 1} \|D F(q_0 + \alpha q_1)\|. \quad (5.2)$$

**Lemma 5.3 (Taylor’s Formula)** Let $E_1$ and $E_2$ be two Banach spaces, $\Omega$ an open subset of $E_1$, $F$ a $n$-times continuously differentiable mapping of $\Omega$ into $E_2$. Then, if the segment joining $q_0$ and $q_0 + q_1$ is in $\Omega$, we have

$$F(q_0 + q_1) = F(q_0) + D F(q_0) \circ q_1 + \frac{1}{2!} D^2 F(q_0) \circ q_1^{(2)} + \cdots + \frac{1}{(n-1)!} D^{n-1} F(q_0) \circ q_1^{(n-1)} + \left( \int_0^1 \frac{(1 - \alpha)^{n-1}}{(n-1)!} D^n F(q_0 + \alpha q_1) d\alpha \right) \circ q_1^{(n)}, \quad (5.3)$$

where $q_1^{(k)}$ stands for $(q_1, \ldots, q_1)$ ($k$-times).

Now we set up a tubular neighborhood of the closure $\xi \cup S$ of the homoclinic orbit $\xi$. For any $T$ such that $0 < T < \infty$, $F^T$ is a $C^2$ diffeomorphism. $\forall q \in \xi \cup S$, choose $r > 0$ such that

$$\|D^\ell F^{\pm T}(q_1) - D^\ell F^{\pm T}(q)\| < 1, \quad (\ell = 1, 2) \quad (5.4)$$

for any $q_1 \in B_q(r_q)$, the ball centered at $q$ with radius $r_q$, i.e.

$$B_q(r_q) = \{ q_1 \in B \mid \|q_1 - q\| < r_q \}. \quad (5.5)$$

Then $\bigcup_{q \in \xi \cup S} B_q(r_q)$ is an open covering of $\xi \cup S$. Since $\xi \cup S$ is compact, there is a finite subcovering $\bigcup_{q \in \xi \cup S(1 \leq j \leq N)} B_{q_j}(r_{q_j})$. For simplicity, we denote $B_{q_j}(r_{q_j})$ and $r_{q_j}$ by $B_j$ and $r_j$ respectively ($1 \leq j \leq N$). Denote by $\mathcal{B}$ the collection

$$\mathcal{B} = \{ B_j, (1 \leq j \leq N) \} \quad (5.5)$$

which is referred as the tubular neighborhood of $\xi \cup S$. See Figure 5.2 for an illustration. $\forall q \in \xi \cup S$, we define

$$d_q = \text{dist}(q, \partial \mathcal{B}) = \max_{j \in B_j} \{ \text{dist}(q, \partial B_{j_k}), \ q \in B_{j_k} \}. \quad (5.6)$$

Then we define

$$d = \text{dist}(\xi \cup S, \partial \mathcal{B}) = \inf_{q \in \xi \cup S} \{ \text{dist}(q, \partial \mathcal{B}) \}. \quad (5.6)$$

**Lemma 5.4** $d > 0$.

Proof: $\forall q \in B_j \cap (\xi \cup S), (1 \leq j \leq N)$, let $B_q^j$ be a ball centered at $q$ with radius equal to $\frac{1}{2} \text{dist}(q, \partial B_j)$. Then $\bigcup_{1 \leq j \leq N} \bigcup_{q \in B_j \cap (\xi \cup S)} B_q^j$ is an open covering of $\xi \cup S$; thus there is a finite subcovering $\bigcup_{k=1}^K \bigcup_{\ell=1}^{L_k} B_{q_{\ell}}^{j_k}$ for some positive integers $K$ and $L_k$. Let $r_* > 0$ be the smallest
Lemma 5.6

Let

\[ \Lambda_\ell = \max_{+-} \sup_{q \in B} \| D^\ell F^{\pm T}(q) \| = \max_{+-} \max_{1 \leq j \leq N} \sup_{q \in B_j} \| D^\ell F^{\pm T}(q) \|, \quad (\ell = 1, 2). \]  

(5.7)

Then (5.4) and (5.5) imply that \( \Lambda_\ell < \infty \) (\( \ell = 1, 2 \)).

Lemma 5.5

\( \forall \mu > 0 \), fix a \( T \) large enough, and fix a \( \epsilon \) small enough, if \( \delta \) is small enough, then

\[ (A_1)^k \Pi_3^s < \frac{1}{2}, \quad (0 \leq k \leq 2), \quad \Pi_\ell^s < \mu, \quad (\ell = 1, 2), \]

\[ (A_1)^k \Pi_2^u < \frac{1}{2}, \quad (0 \leq k \leq 2), \quad \Pi_\ell^u < \mu, \quad (\ell = 1, 3), \]

where \( \| x^u \| \leq \epsilon, \| x^s \| \leq \epsilon, D_1 = D_r, D_2 = D_x^u, D_3 = D_x^s, \) and

\[ \Pi_\ell^s = \sup_{\tau, x^u, x^s} \| D_\ell g^s(\tau, x^u, x^s) \|, \quad (\ell = 1, 2, 3), \]

\[ \Pi_\ell^u = \sup_{\tau, x^u, x^s} \| D_\ell g^u(\tau, x^u, x^s) \|, \quad (\ell = 1, 2, 3), \]

\[ \Pi_2^u = \sup_{\tau, x^u, x^s} \| \{ D_2 g^u(\tau, x^u, x^s) \}^{-1} \|. \]

Proof: Since \( E^s(q) \) and \( E^u(q) \) (\( q \in \eta_a \)) are transversal bundles, the two inequalities involving \( \Pi_3^s \) and \( \Pi_2^u \) follow from standard arguments. Notice that \( E^s(q) \oplus E^c(q) \) and \( E^u(q) \oplus E^c(q) \) along Loop-0 and Loop-1 (except the small portion \( \zeta \)), are invariant bundles under \( DF^t \). When \( \delta \) is small enough, the inequalities for \( R_s^s (\ell = 1, 2) \) and \( R_s^u (\ell = 1, 3) \) follow. □

Lemma 5.6

\( G : \Gamma_{\epsilon, \gamma} \mapsto \Gamma_{\epsilon, \gamma} \)

Proof: First we show that \( G : \Gamma_{\epsilon, \gamma} \mapsto \Gamma_{\epsilon} \).

\[ \| x_1^s(\tau) \| = \| g^s(\tau^-, x^u(\tau^-), x^s(\tau^-)) \|. \]

We use the Taylor formula (5.3) in some \( B_j \),

\[ g^s(\tau^-, x^u(\tau^-), x^s(\tau^-)) = g^s(\tau^-, 0, 0) + D_2 g^s(\tau^-, 0, 0)x^u(\tau^-) + D_3 g^s(\tau^-, 0, 0)x^s(\tau^-) \]

\[ + \left( \int_0^1 \frac{(1 - \alpha)}{1!} D^2 g^s(\tau^-, \alpha x^u(\tau^-), \alpha x^s(\tau^-)) d\alpha \right) (x^u(\tau^-), x^s(\tau^-))(2). \]

(5.8)
From (5.7), we have
\[
\| \int_0^1 \frac{(1 - \alpha)}{1!} D^2 g^s(\tau^-, \alpha x^u(\tau^-), \alpha x^s(\tau^-)) d\alpha \| \leq \frac{1}{2} A_2. \tag{5.9}
\]
From Lemma 5.5,
\[
\| D_2 g^s(\tau^-, 0, 0) \| < \mu, \quad \| D_3 g^s(\tau^-, 0, 0) \| < \frac{1}{2}. \tag{5.10}
\]
For each fixed $T$, if $\delta$ is sufficiently small, then
\[
g^s(\tau^-, 0, 0) \sim O(\delta), \quad \forall \tau^- \in R. \tag{5.11}
\]
Thus, by (5.8)-(5.11), if $\epsilon$ is small enough and, for each $\epsilon$, $\delta$ is sufficiently small, then
\[
\| x^s_1(\tau) \| \leq \frac{9}{10} \epsilon, \quad \forall \tau \in R. \tag{5.12}
\]
Next we estimate $\| x^u_1(\tau) \|$. We start with considering $g^u(\tau, x^u, x^s(\tau))$ where $\| x^u \| \leq \epsilon$. We use the Taylor formula (5.3) in some $B_j$,
\[
g^u(\tau, x^u, x^s(\tau)) = g^u(\tau, 0, 0) + D_2 g^u(\tau, 0, 0) x^u + D_3 g^u(\tau, 0, 0) x^s(\tau) + O(\epsilon^2). \tag{5.13}
\]
From Lemma 5.5,
\[
\| D_3 g^u(\tau, 0, 0) \| < \mu, \quad \| \{ D_2 g^u(\tau, 0, 0) \}^{-1} \| < \frac{1}{2}. \tag{5.14}
\]
For each fixed $T$, if $\delta$ is sufficiently small, then
\[
g^u(\tau, 0, 0) \sim O(\delta), \quad \forall \tau \in R. \tag{5.15}
\]
From (5.13),
\[
x^u = \{ D_2 g^u(\tau, 0, 0) \}^{-1}\{ g^u(\tau, x^u, x^s(\tau)) - g^u(\tau, 0, 0) - D_3 g^u(\tau, 0, 0) x^s(\tau) + O(\epsilon^2) \}.
\]
Then by (5.14) and (5.15), if $\epsilon$ is small enough and for each $\epsilon$, $\delta$ is sufficiently small,
\[
\| x^u \| \leq \frac{1}{2} \left( \| g^u(\tau, x^u, x^s(\tau)) \| + \frac{1}{2} \epsilon \right).
\]
If we take $x^u = x^u_1(\tau)$, we have
\[
\| x^u_1(\tau) \| \leq \frac{3}{4} \epsilon. \tag{5.16}
\]
Next we show that $G: \Gamma_{\epsilon, \gamma} \mapsto \Gamma_{\epsilon, \gamma}$,
\[
(G\sigma)(\tau_\ell) = (\tau_\ell, x^u_1(\tau_\ell), x^s_1(\tau_\ell)), \quad (\ell = 1, 2),
\]
where
\[
\tau_\ell = f(\tau_\ell^-, x^u(\tau_\ell^-), x^s(\tau_\ell^-)),
\]
\[
x^s_1(\tau_\ell) = g^s(\tau_\ell^-, x^u(\tau_\ell^-), x^s(\tau_\ell^-)),
\]
\[
f(\tau_\ell, x^u_1(\tau_\ell), x^s_1(\tau_\ell)) = \tau_\ell^+,
\]
\[
g^u(\tau_\ell, x^u_1(\tau_\ell), x^s_1(\tau_\ell)) = x^u(\tau_\ell^+).
\]
We have
\[
\|x^s_1(\tau_1) - x^s_1(\tau_2)\| = \|g^s(\tau_1^-, x^u(\tau_1^-), x^s(\tau_1^-)) - g^s(\tau_2^-, x^u(\tau_2^-), x^s(\tau_2^-))\|
\leq \|g^s(\tau_1^-, x^u(\tau_1^-), x^s(\tau_1^-)) - g^s(\tau_2^-, x^u(\tau_1^-), x^s(\tau_1^-))\| + \|g^s(\tau_2^-, x^u(\tau_2^-), x^s(\tau_1^-)) - g^s(\tau_2^-, x^u(\tau_2^-), x^s(\tau_2^-))\|
\leq \Pi_1^s|\tau_1^- - \tau_2^-| + \Pi_2^s\gamma|\tau_1^- - \tau_2^-| + \Pi_3^s|\tau_1^- - \tau_2^-|,
\]
by (5.2). That is,
\[
\|x^s_1(\tau_1) - x^s_1(\tau_2)\| \leq (\Pi_1^s + \Pi_2^s\gamma + \Pi_3^s\gamma)|\tau_1^- - \tau_2^-|. \tag{5.17}
\]
Next we need to estimate $|\tau_1^- - \tau_2^-|$ in terms of $|\tau_1 - \tau_2|$. We consider the map
\[
\varphi(\tau^-) = f(\tau^-, x^u(\tau_1^-), x^s(\tau_1^-)), \quad \tau_1^- \text{ fixed},
\]
which is 1-to-1. Then
\[
\tau^- = \varphi^{-1}(f(\tau^-, x^u(\tau_1^-), x^s(\tau_1^-))).
\]
Thus
\[
D\varphi^{-1}(f(\tau^-, x^u(\tau_1^-), x^s(\tau_1^-))) = \{D_1 f(\tau^-, x^u(\tau_1^-), x^s(\tau_1^-))\}^{-1}.
\]
By (5.2),
\[
|\tau_1^- - \tau_2^-| \leq \Lambda_1 |f(\tau_1^-, x^u(\tau_1^-), x^s(\tau_1^-)) - f(\tau_2^-, x^u(\tau_2^-), x^s(\tau_2^-))| \tag{5.19}
\]
Then we have from (5.18) and (5.19),
\[
|\tau_1 - \tau_2| \geq (\Lambda_1^{-1} - 2\Lambda_1\gamma)|\tau_1^- - \tau_2^-| \tag{5.20}
\]
Thus from (5.17) and (5.20),
\[
\|x^s_1(\tau_1) - x^s_1(\tau_2)\| \leq (\Pi_1^s + \Pi_2^s\gamma + \Pi_3^s\gamma)(\Lambda_1^{-1} - 2\Lambda_1\gamma)^{-1}|\tau_1 - \tau_2|
= \frac{\Lambda_1\Pi_1^s + \Lambda_1\Pi_2^s\gamma + \Lambda_1\Pi_3^s\gamma}{1 - 2\Lambda_1^2\gamma}|\tau_1 - \tau_2|. \tag{5.21}
\]
Then by Lemma 5.5,
\[
\|x^s_1(\tau_1) - x^s_1(\tau_2)\| \leq \frac{\Lambda_1\mu(1 + \gamma) + \frac{1}{2}\gamma}{1 - 2\Lambda_1^2\gamma}|\tau_1 - \tau_2|. \tag{5.22}
\]
First we choose $\gamma$ small enough such that $1 - 2\Lambda_2^2 \gamma > \frac{3}{4}$. Then for each $\gamma$ we choose $\mu$ small enough such that

$$\mu < \frac{7}{40 \Lambda_1(1 + \gamma)}.$$ 

With these choices, we have

$$\|x_1^u(\tau_1) - x_1^u(\tau_2)\| < \frac{9}{10} \gamma |\tau_1 - \tau_2|. \quad (5.23)$$

Next we estimate $\|x_1^u(\tau_1) - x_1^u(\tau_2)\|$.

$$\|x_1^u(\tau_1^+) - x_1^u(\tau_2^+)\| = \left(\left(1 + 2\Lambda_2^2 \gamma + \Pi_{\Delta 2}^u\right)^{-1} - \Pi_{\Delta 2}\right)\|g^u(\tau_1, x_1^u(\tau_1), x^s(\tau_1)) - g^u(\tau_2, x_1^u(\tau_2), x^s(\tau_2))\| \leq \hat{\Pi}_{\Delta 2}^u \left(\|\gamma\| + (\Pi_{\Delta 1}^u + \Pi_{\Delta 2}^u)\|\tau_1 - \tau_2\|\right). \quad (5.26)$$

Next we estimate $|\tau_1^+ - \tau_2^+|$ in terms of $|\tau_1 - \tau_2|$.

$$|\tau_1^+ - \tau_2^+| = |f(\tau_1, x_1^u(\tau_1), x^s(\tau_1)) - f(\tau_2, x_1^u(\tau_2), x^s(\tau_2))| \leq \Lambda_1 |\tau_1 - \tau_2| + \|x_1^u(\tau_1) - x_1^u(\tau_2)\| + \gamma |\tau_1 - \tau_2|. \quad (5.27)$$

From (5.26) and (5.27),

$$\|x_1^u(\tau_1) - x_1^u(\tau_2)\| \leq (1 - \Lambda_1 \hat{\Pi}_{\Delta 2}^u \gamma)^{-1} \left[\Lambda_1 \hat{\Pi}_{\Delta 2}^u (1 + \gamma) + \Pi_{\Delta 2}^u (\Pi_{\Delta 1}^u + \Pi_{\Delta 2}^u)\right] |\tau_1 - \tau_2|. \quad (5.28)$$

By Lemma 5.5,

$$\|x_1^u(\tau_1) - x_1^u(\tau_2)\| \leq \left(1 - \frac{1}{2} \gamma\right)^{-1} \left[\frac{1}{2} (1 + \gamma) + \frac{1}{2} \mu (1 + \gamma)\right] |\tau_1 - \tau_2|. \quad (5.29)$$

First we choose $\gamma$ small enough such that $1 - \frac{1}{2} \gamma > \frac{15}{16}$. Then for each $\gamma$ we choose $\mu$ small enough such that

$$\mu < \frac{3}{16} (1 + \gamma)^{-1} \gamma.$$ 

With these choices, we have

$$\|x_1^u(\tau_1) - x_1^u(\tau_2)\| < \frac{9}{10} \gamma |\tau_1 - \tau_2|. \quad (5.30)$$

The proof of the lemma is completed. □
Lemma 5.7 $G$ is a contraction on $\Gamma_{\epsilon, \gamma}$ in $C^0$ norm.

Proof: Let $\sigma^{(\ell)} (\ell = 1, 2)$ be any two sections in $\Gamma_{\epsilon, \gamma}$, and let

$$\sigma^{(\ell)} (\tau) = (\tau, x^{(u,\ell)}(\tau), x^{(s,\ell)}(\tau)), \quad (\ell = 1, 2).$$

Let

$$(G\sigma^{(\ell)})(\tau) = (\tau, x^{(u,\ell)}_1(\tau), x^{(s,\ell)}_1(\tau)), \quad (\ell = 1, 2),$$

where

$$\tau = f(\tau^{(-,\ell)}, x^{(u,\ell)}(\tau^{(-,\ell)}), x^{(s,\ell)}(\tau^{(-,\ell)})), \quad (\ell = 1, 2),$$

$$x^{(s,\ell)}_1(\tau) = g^s(\tau^{(-,\ell)}, x^{(u,\ell)}(\tau^{(-,\ell)}), x^{(s,\ell)}(\tau^{(-,\ell)})), \quad (\ell = 1, 2),$$

$$f(\tau, x^{(u,\ell)}_1(\tau), x^{(s,\ell)}(\tau)) = \tau^{(+,\ell)}, \quad (\ell = 1, 2)$$

g^u(\tau, x^{(u,\ell)}_1(\tau), x^{(s,\ell)}(\tau)) = x^{(u,\ell)}(\tau^{(+,\ell)}), \quad (\ell = 1, 2).$$

First we estimate $\|x^{(s,1)}_1(\tau) - x^{(s,2)}_1(\tau)\|$

$$= \|g^s(\tau^{(-1)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)})) - g^s(\tau^{(-2)}, x^{(u,2)}(\tau^{(-2)}), x^{(s,2)}(\tau^{(-2)}))\|
\leq \|g^s(\tau^{(-1)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)})) - g^s(\tau^{(-2)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)}))\|
+ \|g^s(\tau^{(-2)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)})) - g^s(\tau^{(-2)}, x^{(u,2)}(\tau^{(-2)}), x^{(s,2)}(\tau^{(-2)}))\|
+ \|g^s(\tau^{(-2)}, x^{(u,2)}(\tau^{(-2)}), x^{(s,2)}(\tau^{(-1)})) - g^s(\tau^{(-2)}, x^{(u,2)}(\tau^{(-2)}), x^{(s,2)}(\tau^{(-2)}))\|
\leq \Pi_1^s|\tau^{(-1)} - \tau^{(-2)}| + \Pi_2^s\|x^{(u,1)}(\tau^{(-1)}) - x^{(u,2)}(\tau^{(-1)})\|
+ \Pi_2^s\|x^{(s,1)}(\tau^{(-1)}) - x^{(s,2)}(\tau^{(-1)})\||\tau^{(-1)} - \tau^{(-2)}|
= (\Pi_1^s + \Pi_2^s + \Pi_3^s)|\tau^{(-1)} - \tau^{(-2)}| + \Pi_2^s\|x^{(u,1)}(\tau^{(-1)}) - x^{(u,2)}(\tau^{(-1)})\|
+ \Pi_2^s\|x^{(s,1)}(\tau^{(-1)}) - x^{(s,2)}(\tau^{(-1)})\||\tau^{(-1)} - \tau^{(-2)}|.

Next we estimate $|\tau^{(-1)} - \tau^{(-2)}|$.

Notice that,

$$\|f(\tau^{(-1)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)})) - f(\tau^{(-2)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)}))\|
\leq \Lambda_1|\tau^{(-1)} - \tau^{(-2)}|,$$

where

$$\|f(\tau^{(-1)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)})) - f(\tau^{(-2)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)}))\|
\geq \Lambda_1^{-1}|\tau^{(-1)} - \tau^{(-2)}|,$$

and

$$\|f(\tau^{(-2)}, x^{(u,2)}(\tau^{(-2)}), x^{(s,2)}(\tau^{(-2)})) - f(\tau^{(-2)}, x^{(u,1)}(\tau^{(-1)}), x^{(s,1)}(\tau^{(-1)}))\|
\leq \Lambda_1 (2\gamma|\tau^{(-1)} - \tau^{(-2)}| + \|x^{(u,1)}(\tau^{(-1)}) - x^{(u,2)}(\tau^{(-1)})\|
+ \|x^{(s,1)}(\tau^{(-1)}) - x^{(s,2)}(\tau^{(-1)})\|).$$
Then
\[
|\tau^{-1} - \tau^{+1}| \leq \frac{\Lambda_1}{\Lambda_1^{-1} - 2\gamma \Lambda_1} \left[ \|x^{(u,1)}(\tau^{-1}) - x^{(u,2)}(\tau^{-1})\| + \|x^{(s,1)}(\tau^{-1}) - x^{(s,2)}(\tau^{-1})\| \right].
\]
(5.32)

From (5.31) and (5.32),
\[
\|x^{(s,1)}(\tau) - x^{(s,2)}(\tau)\| \leq c_s \|x^{(s,1)}(\tau^{-1}) - x^{(s,2)}(\tau^{-1})\| + c_u \|x^{(u,1)}(\tau^{-1}) - x^{(u,2)}(\tau^{-1})\|,
\]
where
\[
c_s = \frac{1}{2} + \frac{\mu(1 + \gamma)\Lambda_1^2 + \frac{1}{2}\gamma \Lambda_1}{1 - 2\gamma \Lambda_1^2}, \quad c_u = \mu + \frac{\mu(1 + \gamma)\Lambda_1^2 + \frac{1}{2}\gamma \Lambda_1}{1 - 2\gamma \Lambda_1^2}.
\]

If \(\gamma\) and \(\mu\) are small enough, then
\[
\|x^{(s,1)}(\tau) - x^{(s,2)}(\tau)\| \leq \frac{9}{10} \|\sigma^{(1)} - \sigma^{(2)}\|_0.
\]
(5.33)

Next we estimate \(\|x^{(u,1)}(\tau) - x^{(u,2)}(\tau)\|\).
\[
\|x^{(u,1)}(\tau^{-1}) - x^{(u,2)}(\tau^{-1})\| = \|g^u(\tau, x^{(u,1)}(\tau), x^{(s,1)}(\tau)) - g^u(\tau, x^{(u,2)}(\tau), x^{(s,2)}(\tau))\| \\
\geq \|g^u(\tau, x^{(u,1)}(\tau), x^{(s,1)}(\tau)) - g^u(\tau, x^{(u,2)}(\tau), x^{(s,1)}(\tau))\| \\
- \|g^u(\tau, x^{(u,2)}(\tau), x^{(s,1)}(\tau)) - g^u(\tau, x^{(u,2)}(\tau), x^{(s,2)}(\tau))\| \\
\geq (\Pi^u_2)^{-1} \|x^{(u,1)}(\tau) - x^{(u,2)}(\tau)\| - \Pi^u_3 \|x^{(s,1)}(\tau) - x^{(s,2)}(\tau)\|.
\]
Then
\[
\|x^{(u,1)}(\tau) - x^{(u,2)}(\tau)\| \leq \hat{\Pi}_2^u \left[ \|x^{(u,1)}(\tau^{-1}) - x^{(u,2)}(\tau^{-1})\| + \Pi^u_3 \|x^{(s,1)}(\tau) - x^{(s,2)}(\tau)\| \right].
\]
(5.34)

Next we estimate \(|\tau^{(1)} - \tau^{(2)}|\).
\[
|\tau^{(1)} - \tau^{(2)}| = \|f(\tau, x^{(u,1)}(\tau), x^{(s,1)}(\tau)) - f(\tau, x^{(u,2)}(\tau), x^{(s,2)}(\tau))\| \\
\leq \Lambda_1 \left[ \|x^{(u,1)}(\tau) - x^{(u,2)}(\tau)\| + \|x^{(s,1)}(\tau) - x^{(s,2)}(\tau)\| \right].
\]
(5.36)

From (5.35) and (5.36),
\[
\|x^{(u,1)}(\tau) - x^{(u,2)}(\tau)\| \leq \frac{\hat{\Pi}^u_2}{1 - \gamma \Lambda_1 \hat{\Pi}^u_2} \left[ \|x^{(u,1)}(\tau^{(1)}) - x^{(u,2)}(\tau^{(1)})\| \\
+ \Pi^u_3 \|x^{(s,1)}(\tau) - x^{(s,2)}(\tau)\| \right].
\]
\[
+(\Pi_3^u + \gamma \Lambda_1) \|x^{(s,1)}(\tau) - x^{(s,2)}(\tau)\|
\leq \frac{1}{1 - \frac{1}{2} \gamma} \left[ \|x^{(u,1)}((\tau^{(1,1)})) - x^{(u,2)}((\tau^{(1,1)}))\| + (\mu + \gamma \Lambda_1)\|x^{(s,1)}(\tau) - x^{(s,2)}(\tau)\| \right].
\]

If \(\gamma\) and \(\mu\) are small enough, then
\[
\|x^{(u,1)}(\tau) - x^{(u,2)}(\tau)\| \leq \frac{9}{10} \|\sigma^{(1)} - \sigma^{(2)}\|_{C^0}. \tag{5.37}
\]

By (5.33) and (5.37), we have
\[
\|G\sigma^{(1)} - G\sigma^{(2)}\|_{C^0} \leq \frac{9}{10} \|\sigma^{(1)} - \sigma^{(2)}\|_{C^0}. \tag{5.38}
\]

The proof of the lemma is completed. \(\Box\)

**Theorem 5.1** The graph transform \(G\) has a unique fixed point \(\sigma^*\) in \(\Gamma_{\epsilon, \gamma}\). Graph \(\sigma^*\) is an orbit that \(\epsilon\)-shadows the \(\delta\)-pseudo-orbit \(\eta_a\).

**Proof:** By Lemmas 5.1, 5.6, and 5.7, \(G\) has a unique fixed point \(\sigma^*\) in \(\Gamma_{\epsilon, \gamma}\). Assume that \(\sigma^*\) has the representation
\[
\sigma^*(\tau) = (\tau, x^u(\tau), x^s(\tau)), \quad \tau \in R.
\]

Then
\[
(G\sigma^*)(\tau) = (\tau, x^u(\tau), x^s(\tau)), \quad \tau \in R,
\]

where
\[
\begin{align*}
  f(\tau^-, x^u(\tau^-), x^s(\tau^-)) &= \tau, \tag{5.39} \\
  g^s(\tau^-, x^u(\tau^-), x^s(\tau^-)) &= x^s(\tau), \tag{5.40} \\
  f(\tau, x^u(\tau), x^s(\tau)) &= \tau^+, \tag{5.41} \\
  g^u(\tau, x^u(\tau), x^s(\tau)) &= x^u(\tau^+). \tag{5.42}
\end{align*}
\]

Replacing \(\tau\) by \(\tau^-\) in (5.41) and (5.42), we have
\[
\begin{align*}
  f(\tau^-, x^u(\tau^-), x^s(\tau^-)) &= \tau, \tag{5.43} \\
  g^u(\tau^-, x^u(\tau^-), x^s(\tau^-)) &= x^u(\tau). \tag{5.44}
\end{align*}
\]

From (5.39), (5.40), (5.43), and (5.44), \(\forall (\tau^-, x^u(\tau^-), x^s(\tau^-)) \in \sigma^*\), there exists a unique \((\tau, x^u(\tau), x^s(\tau)) \in \sigma^*\) such that
\[
F^T(\tau^-, x^u(\tau^-), x^s(\tau^-)) = (\tau, x^u(\tau), x^s(\tau)),
\]

i.e.
\[
F^T(\text{Graph } \sigma^*) \subset \text{Graph } \sigma^*;
\]

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\(\forall (\tau, x^u(\tau), x^s(\tau)) \in \sigma^*,\) there exists a unique \((\tau^-, x^u(\tau^-), x^s(\tau^-)) \in \sigma^*\) such that

\[
(\tau, x^u(\tau), x^s(\tau)) = F^T(\tau^-, x^u(\tau^-), x^s(\tau^-)),
\]
i.e.

\[
\text{Graph } \sigma^* \subset F^T(\text{Graph } \sigma^*).
\]

Thus

\[
F^T(\text{Graph } \sigma^*) = \text{Graph } \sigma^*. \tag{5.45}
\]

Let \(\sigma\) be a section in \(\Gamma_{\epsilon, \gamma}\) such that

\[
F^T(\text{Graph } \sigma) = \text{Graph } \sigma. \tag{5.46}
\]

We denote \(\sigma\) as

\[
\sigma(\tau) = (\tau, x^u(\tau), x^s(\tau)), \quad \tau \in R.
\]

Then \(G\sigma\) is given in (5.1). By (5.46),

\[
f(\tau^-, x^u(\tau^-), x^s(\tau^-)) = \tau,
g^s(\tau^- , x^u(\tau^-), x^s(\tau^-)) = x^s(\tau),
f(\hat{\tau}^-, x^u_1(\hat{\tau}^-), x^s(\hat{\tau}^-)) = \tau = f(\tau^-, x^u(\tau^-), x^s(\tau^-)), \tag{5.47}
g^u(\hat{\tau}^- , x^u_1(\hat{\tau}^-), x^s(\hat{\tau}^-)) = x^u(\tau) = g^u(\tau^-, x^u(\tau^-), x^s(\tau^-)). \tag{5.48}
\]

Now we will show that \(\hat{\tau}^- = \tau^-, \) \(x^u_1(\hat{\tau}^-) = x^u(\tau^-).\) From (5.48),

\[
0 = \|g^u(\hat{\tau}^-, x^u_1(\hat{\tau}^-), x^s(\hat{\tau}^-)) - g^u(\tau^-, x^u(\tau^-), x^s(\tau^-))\|
\geq (\Pi_\gamma^u)^{-1}\|x^u_1(\tau^-) - x^u(\tau^-)\| - \Lambda_1 \gamma |\hat{\tau}^- - \tau^-| - \Pi_\gamma^u |\hat{\tau}^--\tau^-| - \Pi_\gamma^u |\hat{\tau}^--\tau^-| \tag{5.49}.
\]

From (5.47),

\[
0 = |f(\hat{\tau}^-, x^u_1(\hat{\tau}^-), x^s(\hat{\tau}^-)) - f(\tau^-, x^u(\tau^-), x^s(\tau^-))| \geq \Lambda_1 |\hat{\tau}^- - \tau^-| - \Lambda_1 (\|x^u_1(\tau^-) - x^u(\tau^-)\| + 2\gamma |\hat{\tau}^- - \tau^-|).
\]

Then,

\[
|\hat{\tau}^- - \tau^-| \leq \frac{\Lambda_1}{\Lambda_1 - 2\gamma \Lambda_1} \|x^u_1(\tau^-) - x^u(\tau^-)\|. \tag{5.50}
\]

Thus, from (5.49) and (5.50),

\[
\left[1 - \frac{1}{2} \frac{\Lambda_1 \gamma + \mu (1 + \gamma)}{\Lambda_1 - 2\gamma \Lambda_1}\right] \|x^u_1(\tau^-) - x^u(\tau^-)\| \leq 0. \tag{5.51}
\]

When \(\gamma\) and \(\mu\) are sufficiently small, (5.51) implies that

\[
\|x^u_1(\tau^-) - x^u(\tau^-)\| = 0,
\]

which in turn implies that, by (5.50),

\[
|\hat{\tau}^- - \tau^-| = 0.
\]
Thus, $\sigma$ is a fixed point of $G$. To summarize, we have $\sigma$ is a fixed point of $G$, i.e. $G\sigma = \sigma$, if and only if equation (5.46) holds. For $t \in [-t_0, t_0]$, $t_0 > 0$, we define $F^t(\text{Graph } \sigma^*)$ to be the graph of a certain section $\sigma_t^*$. Since $G\sigma^* = \sigma^*$, by (5.12), (5.16), (5.23), and (5.30),

$$||\sigma^*||_{C^0} \leq \frac{9}{10} \epsilon, \quad \text{Lip } \{\sigma^*\} \leq \frac{9}{10} \gamma.$$ 

Thus there exists a small $t_0 > 0$, such that

$$||\sigma_t^*||_{C^0} \leq \epsilon, \quad \text{Lip } \{\sigma_t^*\} \leq \gamma, \quad \forall t \in [-t_0, t_0],$$

i.e. $\sigma_t^* \in \Gamma_{\epsilon, \gamma}$. From (5.45),

$$F^t(\text{Graph } \sigma^*) = F^t F^T(\text{Graph } \sigma^*) = F^T F^t(\text{Graph } \sigma^*). \quad (5.52)$$

(5.52) is equivalent to $G\sigma_t^* = \sigma_t^*$. Then by the uniqueness of the fixed point of $G$ in $\Gamma_{\epsilon, \gamma}$,

$$\sigma_t^* = \sigma^*, \quad \forall t \in [-t_0, t_0].$$

Thus,

$$F^t(\text{Graph } \sigma^*) = \text{Graph } \sigma^*, \quad \forall t \in [-t_0, t_0]. \quad (5.53)$$

Iteration of (5.53) leads to

$$F^t(\text{Graph } \sigma^*) = \text{Graph } \sigma^*, \quad \forall t \in (-\infty, \infty).$$

That is, $\text{Graph } \sigma^*$ is an orbit that $\epsilon$-shadows the $\delta$-pseudo-orbit $\eta_a$. The proof of the theorem is completed. □

**Remark 5.1** As curves, the shadowing orbits are Lipschitz, and can be $C^k$ smooth for some $k > 0$. But this does not mean that the shadowing orbits are Lipschitz in time.

## 6 Chaos

First we will define a return map $P$. We will use notations from Section 3. Pick a point $p_*$ on $S$, which is $O(1)$ away from $p_c$ in $\delta$. At $p_*$, we set up a transversal section $\Xi$ to $S$. For any pseudo-orbit $\eta_a$, denote by $h_{a_0}$ the portion of the shadowing orbit, that shadows the portion Loop-$a_0$ of the pseudo-orbit. Let $q_a$ be the first intersection point of $h_{a_0}$ with $\Xi$. Let $\Lambda$ be the set consisting of $q_a$ for all doubly infinite sequences $a \in \Sigma$. We define the return map $P : \Lambda \mapsto \Lambda$ as follows: For any $q_a \in \Lambda$, $P(q_a) = q_{\chi(a)}$.

**Theorem 6.1 (Chaos Theorem)** The subset $\Lambda \subset \Sigma$ is invariant under the return map $P$.

The action of $P$ on $\Lambda$ is topologically conjugate to the action of the shift automorphism $\chi$ on $\Sigma$. That is, there exists a homeomorphism $\phi : \Sigma \mapsto \Lambda$ such that the following diagram commutes:

$$\begin{array}{ccc}
\Sigma & \xrightarrow{\phi} & \Lambda \\
\chi \downarrow & & \downarrow P \\
\Sigma & \xrightarrow{\phi} & \Lambda
\end{array}$$

Proof: The invariance of $\Lambda$ under $P$ follows from the definitions of $\Lambda$ and $P$. We define $\phi : \Sigma \mapsto \Lambda$ as follows: For any $a \in \Sigma$, $\phi(a) = q_a$. It is straightforward to show that $\phi$ is a homeomorphism, and $P$ and $\chi$ are topologically conjugate. □
7 An Example: A Derivative Nonlinear Schrödinger Equation

Consider the derivative nonlinear Schrödinger equation,

\[ i q_t = q_{xx} + 2|q|^2 q + i\epsilon \left( \left( \frac{9}{16} - |q|^2 \right) q + \mu |\hat{\partial}_x q|^2 \bar{q} \right), \]

(7.1)

where \( q \) is a complex-valued function of two real variables \( t \) and \( x \), \( \epsilon > 0 \) is the perturbation parameter, \( \mu \) is a real constant, and \( \hat{\partial}_x \) is a bounded Fourier multiplier,

\[ \hat{\partial}_x q = -\sum_{k=1}^{K} k\bar{q}_k \sin kx, \]

for \( q = \sum_{k=0}^{\infty} \bar{q}_k \cos kx \),

and some fixed \( K \) (cf: [12]). Periodic boundary condition and even constraint are imposed,

\[ q(t, x + 2\pi) = q(t, x), \quad q(t, -x) = q(t, x). \]

**Theorem 7.1 (Transversal Homoclinic Orbit Theorem)** There exists a \( \epsilon_0 > 0 \), such that for any \( \epsilon \in (0, \epsilon_0) \), and \( |\mu| > 5.8 \), there exist two transversal homoclinic orbits asymptotic to the limit cycle \( q_c = \frac{3}{4} \exp\left\{ -i\frac{9}{8} t + \gamma \right\} \).

Proof: Denote by \( q_c \) the limit cycle,

\[ q_c = \frac{3}{4} \exp\left\{ -i\frac{9}{8} t + \gamma \right\}. \]

The eigenvalue of this limit cycle is,

\[ \lambda = -\epsilon \frac{9}{16} \pm \sqrt{k^2 \left( \frac{9}{4} - k^2 \right) + \epsilon^2 \left( \frac{3}{4} \right)^2}, \quad \text{for} \quad k = 0, 1, \ldots. \]

There is only one unstable mode. The same argument as in [12] [11] shows that the size of the stable manifold of the limit cycle is \( O(\sqrt{\epsilon}) \). Also the same argument as in [14] [12] [11] shows that the Fenichel’s persistent invariant manifold theorem and fiber theorem are true. As a result, there exist codimension 1 center-stable and center-unstable manifolds, codimension 2 center manifold, together with stable and unstable fibrations. Thus if the Melnikov measurement is successful, that is, there exists an orbit in the intersection of the unstable manifold of the limit cycle and the center-stable manifold, then the orbit will be a homoclinic orbit asymptotic to the limit cycle, due to the fact that the size of the stable manifold of the limit cycle is \( O(\sqrt{\epsilon}) \). The Melnikov function is given as,

\[ M = \int_{-\infty}^{\infty} \int_{0}^{2\pi} \left\{ \frac{\delta F_1}{\delta q} \left[ \left( \frac{9}{16} - |q|^2 \right) q + \mu |\hat{\partial}_x q|^2 \bar{q} \right] \right. \]

\[ + \left. \frac{\delta F_1}{\delta \bar{q}} \left[ \left( \frac{9}{16} - |q|^2 \right) \bar{q} + \mu |\hat{\partial}_x \bar{q}|^2 q \right] \right\} dx \ d\tau, \]

where \( F_1 \) is defined in [12] [11],

\[ \left( \frac{\delta F_1}{\delta \bar{q}} \right)^2 \sim (|u_1|^2 + |u_2|^2)^{-2} \left( \frac{q_c u_1^2}{-q_c u_2^2} \right), \]
and
\begin{align*}
u_1 &= \cosh \frac{\tau}{2} \cos z - i \sinh \frac{\tau}{2} \sin z , \\
u_2 &= -\sinh \frac{\tau}{2} \cos(z - \vartheta_0) + i \cosh \frac{\tau}{2} \sin(z - \vartheta_0) ,
\end{align*}
and
\begin{align*}q &= q_c \left[ 1 + \sin \vartheta_0 \sech \tau \cos x \right]^{-1} \\
&= q_c \left[ \cos 2\vartheta_0 - i \sin 2\vartheta_0 \tanh \tau \right. \\
&\left. \left. - \sin \vartheta_0 \sech \tau \cos x \right] \right. \\
\tau &= \frac{\sqrt{5}}{2} \mu - \rho , \quad \vartheta_0 = \arctan \frac{\sqrt{5}}{2} , \quad z = \frac{x}{2} + \frac{1}{2}(\arctan \frac{\sqrt{5}}{2} - \pi) ,
\end{align*}
where \( \rho \) is the Bäcklund parameter. \( q_c \) can be rewritten as
\[q_c = \frac{3}{4} \exp \left\{-i \left[ \frac{9}{4\sqrt{5}} \tau - \tilde{\gamma} \right] \right\},\]
where
\[\tilde{\gamma} = -\left( \gamma + \frac{9}{4\sqrt{5}} \rho \right) .\]
The solutions for \( M = 0 \) are given by,
\[\cos 2\tilde{\gamma} = \frac{5.8}{\mu} .\]
This completes the proof. \( \square \)

**Theorem 7.2 (Chaos Theorem)** There exists a \( \epsilon_0 > 0 \), such that for any \( \epsilon \in (0, \epsilon_0) \), and \( |\mu| > 5.8 \), Theorem 6.1 holds for the derivative nonlinear Schrödinger equation (7.1).

Proof: Arguments as in [11] show that the transversal homoclinic orbit is a classical solution. Thus, Assumption (A1) is valid. Assumption (A2) follows from the standard arguments in [14] [12] [11]. Since the perturbation in (7.1) is bounded, Assumption (A3) follows from standard arguments. \( \square \)

## 8 Appendix: Chaos in Non-Autonomous Perturbed Soliton Equations

Consider the periodically perturbed sine-Gordon equation,
\[u_{tt} = c^2 u_{xx} + \sin u + \epsilon [-au_t + u^3 \chi(\|u\|) \cos t] , \tag{8.1}\]
where
\[\chi(\|u\|) = \begin{cases} 1, & \|u\| \leq M, \\ 0, & \|u\| \geq 2M , \end{cases}\]
for $M < \|u\| < 2M$, $\chi(\|u\|)$ is a smooth bump function (see Figure 8.1), under odd periodic boundary condition,

$$u(x + 2\pi, t) = u(x, t), \quad u(x, t) = -u(x, t),$$

$\frac{1}{4} < c^2 < 1$, $a > 0$, $\epsilon$ is a small perturbation parameter.

**Theorem 8.1** ([12], [19]) There exists an interval $I \subset \mathbb{R}^+$ such that for any $a \in I$, there exists a transversal homoclinic orbit $u = \xi(x, t)$ asymptotic to 0 in $H^1$.

Denote by $P$ the time-$2\pi$ Poincaré map of the system (8.1). Then $P$ is a $C^2$-diffeomorphism on $H^1$ [12] (and references thereof). Under $P$, the transversal homoclinic orbit $u = \xi(x, t)$ changes into the transversal homoclinic orbit $\{\xi_j(x)\}_{j \in \mathbb{Z}}$ asymptotic to 0. Using shadowing lemma, Bernoulli shift dynamics can be established in the neighborhood of the transversal homoclinic orbit. This has been done by H. Steinlein and H. O. Walther [22, 23] and D. Henry [9] in infinite dimensions. The theorem stated specifically for the perturbed sine-Gordon system (8.1) can be described as follows.

**Definition 7** Denote by $\Sigma_m$ ($m \geq 2$) the set of doubly infinite sequences

$$k = (\ldots, k_{-1}, k_0, k_1, \ldots)$$

where $k_j \in \{1, 2, \ldots, m\}$. So $\Sigma_m = \{1, 2, \ldots, m\}^\mathbb{Z}$.

We give the set $\{1, 2, \ldots, m\}$ the discrete topology and $\Sigma_m$ the product topology. The Bernoulli shift $\beta : \Sigma_m \to \Sigma_m$ is the homeomorphism defined by

$$[\beta(k)]_j = k_{j+1}.$$

**Theorem 8.2** There is an integer $\ell$ and a homeomorphism $\phi$ of $\Sigma_m$ onto a compact Cantor subset $\Lambda$ of $H^1$. $\Lambda$ is invariant under the time-$2\pi$ Poincaré map $P$ of the perturbed sine-Gordon equation (8.1). The action of $P^\ell$ on $\Lambda$ is topologically conjugate to the action of $\beta$ on $\Sigma_m$: $P^\ell \circ \phi = \phi \circ \beta$. That is, the following diagram commutes:

$$\begin{array}{ccc}
\Sigma_m & \xrightarrow{\phi} & \Lambda \\
\downarrow{\beta} & & \downarrow{P} \\
\Sigma_m & \xrightarrow{\phi} & \Lambda
\end{array}$$

**Remark 8.1** In finite dimensions, this theorem was proved by K. Palmer using shadowing lemmas [17, 18]. This is the well-known Smale horseshoe theorem [21]. See also related works [25, 26]. In infinite dimensions, the above theorem was proved by H. Steinlein and H.O. Walther [22, 23] and D. Henry [9]. See also related works [8, 3, 2, 1].
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