Mathematical simulation of stability of steady-state conditions for a road train

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Abstract. The article is devoted to the application of the parameter continuation method for solving the problem of maneuverability of a three-trailer road train. The mathematical model of a three-trailer road train with a wheelless intermediate link is worked out. The analysis of the equilibrium conditions of the system is performed. The structural instability of the model of the road train in the neighbourhood of the linear motion is presented. The presence of stable circular steady-state conditions is proved when moving along circles of a sufficiently small radius. It is established that for a given longitudinal velocity there is a certain range of changes in the rotation angle of the steering wheels of the tractor unit, within which stable steady-state conditions may be. The analysis of the stability of the steady-state conditions at $\theta = 0.38$ rad.

For the steady-state conditions, the configuration of the road train is drawn, where it can be seen that the without bearing link can only be oriented to the outer side of the turn. Also in the process of numerical integration, it is found that the resulting steady-state conditions are asymptotically stable.

1. Introduction

The mathematical model of the canonical road train is an object of research by many authors [1-3]. The results of the analysis are realized mainly for linear models [4-6]. The stability of circular steady-state conditions of motion of road train has not been studied sufficiently. Usually, a vehicular movement along the trajectory of major radius is considered [5, 7]. In this paper the software – computing base is developed. The base makes it possible to research the variety of the steady-state conditions of the road train model system control parameters such as the longitudinal velocity of freight traffic (the discrete parameter) and the wheel turning angle of the road train (the continuous parameter) are changed.

The paper is aimed at studying the nonlinear mathematical model of the three-trailer road train and shows possible restructurings of the configuration of the road train when the initial perturbation of the phase variable is changed. It requires the non-standard analysis.

It is well known that a wide class of problem which arises in several branches of pure and applied science can be studied in the general framework of the nonlinear equations. Due to their importance,
several methods have been suggested and analyzed under certain conditions. It is tabular integration method, iteration method and parameter continuation method.

The object of the paper is a study the mobility and stability of the circular steady-state conditions of a model of a three-trailer road train and detection the set of steady-state conditions of the freight traffics by the parameter continuation method. The picture of the configuration of the road train is investigated when various traffic conditions.

We apply the parameter continuation method in order to study maneuverability of a three-trailer road train and periodic motions of stability, to find the bifurcation points. Technique allows finding out solutions step by step for each value of continuation parameter passing by the transitional process. It also opens the possibility to easily distinguish the instability zones and determine the bifurcation points. These methods show a good agreement of the results. In general, many authors have obtained bifurcation diagrams by time integration of the dynamics.

Tabular integration is an alternative to traditional integration by parts. The benefit of tabular integration is that it can save you a ton of time compared to integration by parts [8, 9]. This method requires that one of the functions be differentiable until it is zero. We have to integrate the other function every time differentiate the first function.

Iterative method uses successive approximations to obtain more accurate solutions to a linear system at each step. This method is the only choice for nonlinear equations [10-12].

2. The mathematical model of motion of a three-trailer road train

The mathematical model of motion of a road train is worked out. State variables are chosen for the most complete description and research of possible steady-state conditions of a three-trailer road train with without bearing intermediate member with rigid steering control (Figure 1) and without bearing semitrailer (Figure 2). Where x-y is fixed coordinate system; x₀-y₀ is moving coordinates, which is connected with the center of mass of a tractor unit; x₁-y₁ is moving coordinates, which is connected with the center of mass of a semi-trailer; x₂-y₂ is moving coordinates, which is connected with the center of mass of a trailer. The mathematical model helps to explain a system of motion of a road train and to study the effects of different components (variables), and to make predictions about behavior and stability of steady-state conditions.

![Figure 1. Basic design diagram of a three-trailer road train.](image1)

![Figure 2. Basic design diagram of a three-trailer road train with without bearing semitrailer.](image2)

The front axle of a tractor unit turns by an angle θ. The connection between the links is carried out by cylindrical hinges, which allow free relative angulation of the links in the plane of their movement.
The position of each link is given by the coordinates $x_i, y_i$ of its center of mass $C_i$ and a course angle $\psi_i$ (it is enclosed between the direct axis of the corresponding link and the $x$-axis of the fixed coordinate system).

Parameters of the system are the following:

- $v$ is longitudinal velocity component of the center of mass of the tractor unit
- $a$ is distance from the center of mass of the tractor unit to the points of attachment of the front axle of a tractor unit
- $b$ is distance from the center of mass of the tractor unit to the points of attachment of the rear axle of a tractor unit
- $c$ is distance from the center of mass of the tractor unit to the hitch point with a rear link
- $d_i$ is distance from the center of mass of a semi-trailer to the hitch point
- $2K$ is overall width of the road train
- $k_i$ is friction coefficient
- $k_1, k_2$ and $k_3$ are skid coefficient of axe (first and second and third axes)
- $\chi_1, \chi_2$ and $\chi_3$ are coefficient of friction to determining the force of lateral skid (first and second and third axes)
- $\theta$ is wheel turning angle of subordinate module
- $Y_1, Y_2$ and $Y_3$ are reduced lateral reaction of a roadbed on bearing axle skid (first and second and third axes).

The following values are set in the mathematical model:

- $C, C_1$ and $C_2$ are centers of mass of the tractor unit, the semi-trailer and the trailer
- $m, m_1$ and $m_2$ are masses of tractor unit, the semi-trailer and the trailer
- $I$ is central moment of inertia about vertical axis of the tractor unit
- $I_1$ is central moment of inertia about vertical axis of the semi-trailer
- $I_2$ is central moment of inertia about vertical axis of the trailer
- $\phi$ is folding angle is formed between the longitudinal axes of the tractor unit and semi-trailer
- $\phi_1$ is folding angle is formed between the longitudinal axes of the tractor unit and trailer
- $\omega, \omega_1$ and $\omega_2 = \omega_{\phi_1}$ are absolute angular velocities of the driving and driven links

Absolute angular velocities of links are the following:

$$\omega_1 = \omega - \phi \quad \text{and} \quad \omega_2 = \omega_1 - \phi_1$$

We set the absolute velocity of particles $C, C_i$ by resolutions in the unit vectors of the corresponding bases:

$$v_c = i_0 \cdot v + j_0 \cdot u$$
$$v = \dot{x} \cdot \cos \psi + \dot{y} \cdot \sin \psi$$
$$v_1 = v \cdot \cos \phi - (u - \omega \cdot c) \cdot \sin \phi$$
$$v_2 = v_1 \cdot \cos \phi_1 - (u_1 - \omega_1 \cdot b_1) \cdot \sin \phi_1$$
$$v_{c1} = i_1 \cdot v_1 + j_1 \cdot u_1$$
$$u = -\dot{x} \cdot \sin \psi + \dot{y} \cdot \cos \psi$$
$$u_1 = v \cdot \sin \phi + (u - \omega \cdot c) \cdot \cos \phi - \omega_1 \cdot d_1$$
$$u_2 = v_1 \cdot \sin \phi_1 + (u_1 - \omega_1 \cdot b_1) \cdot \cos \phi_1 - \omega_2 \cdot d_2$$

The differential equation system of motion of a road train describes the change in the phase variables $(u, \omega, \psi, \phi_1, \Phi, \Phi_1)$. Where $u$ is lateral velocity of center of mass of the tractor unit (quasi-velocity); $U$ is derivative of $u$ in the moving coordinates; $\Omega$ is angular acceleration relative to the vertical axis; $\Phi$ – velocity of folding angle $\phi$; $\Phi_1$ – velocity of folding angle $\phi_1$; $PP$ – angular
acceleration of a semi-trailer relative to intrinsic vertical axis; $\PP_1$ – angular acceleration of a trailer relative to intrinsic vertical axis.

There are a lot of theories of rolling of elastically deformed wheels. The axiomatic is the most widespread, which asserts that a lateral reaction $Y_i$ of a roadbed is applied at the tooth bearing center of the rolling elastic wheel. It is a function of wheel slip angle $\delta_i$.

The reduced angles of lateral skid of wheel axles are determined by expressions:

$$
\delta_1 = \theta - \arctg \left( \frac{u+a \cdot \omega}{v} \right)
$$

$$
\delta_2 = \arctg \left( \frac{-u+b \cdot \omega}{v} \right)
$$

$$
\delta_3 = \arctg \left( \frac{-u_1 + b_1 \cdot \omega_1}{v_1} \right)
$$

Dependences of forces of lateral skid possess an empirical origin \[13\]. It is determined by expressions:

$$
Y_i = \frac{k_i \cdot \delta_i}{\sqrt{1 + (k_i \cdot \delta_i / X_i \cdot Z_i)^2}}
$$

Where $Z_i$ is reaction of footprint on axes.

Values $Z_1$, $Z_2$, $Z_3$ are determined by expressions:

$$
Z_1 = \frac{1}{l} \left[ m \cdot g \cdot b - (m_2 + m_h) \cdot g \cdot \frac{b_i}{L_i} \cdot (c - b) \right]
$$

$$
Z_2 = \frac{1}{l} \left[ m \cdot g \cdot a + (m_2 + m_h) \cdot g \cdot \frac{b_i}{L_i} \cdot (c + a) \right]
$$

$$
Z_3 = (m_2 + m_h) \cdot g \cdot \frac{d_i}{L_i}; \; l = a + b_i, \; L_i = d_i + b_i + d_2 + b_2
$$

The motion equations of tractor unit are formed as:

$$
m \cdot (V - u \cdot \omega) - X_{01} + Y_1 \cdot \sin(\theta) = 0
$$

$$
m \cdot (U + v \cdot \omega) - Y_{01} - Y_1 \cdot \cos(\theta) - Y_2 = 0
$$

$$
I \cdot \Omega + Y_{01} \cdot c - Y_1 \cdot a \cdot \cos(\theta) + Y_2 \cdot b = 0
$$

The motion equations of semi-trailer are formed as:

$$
m_1 \cdot (V_1 - u_1 \cdot \omega_1) + X_{01} \cdot \cos(\phi) - Y_{01} \cdot \sin(\phi) - X_{12} = 0
$$

$$
m_1 \cdot (U_1 + v_1 \cdot \omega_1) + X_{01} \cdot \sin(\phi) + Y_{01} \cdot \cos(\phi) - Y_{12} - Y_3 = 0
$$

$$
I_1 \cdot \Omega_1 + X_{01} \cdot d_i \cdot \sin(\phi) + Y_{01} \cdot d_1 \cdot \cos(\phi) + Y_{12} \cdot b_1 + Y_3 \cdot (b_1 - c_1) = 0
$$

The motion equations of without bearing semitrailer are formed as:

$$
m_1 (V_1 - u_1 \omega_1) + X_{01} \cos(\phi) - Y_{01} \sin(\phi) - X_{12} = 0,
$$

$$
m_1 (U_1 + v_1 \omega_1) + X_{01} \sin(\phi) + Y_{01} \cos(\phi) - Y_{12} = 0,
$$

$$
J_1 \Omega_1 + X_{01} \cdot d_1 \sin(\phi) + Y_{01} \cdot d_1 \cos(\phi) + Y_{12} \cdot b_1 = 0.
$$
The motion equations of trailer are formed as:

\[ \begin{align*}
\dot{m}_2 \cdot (V_2 - u_2 \cdot \omega_2) + X_{12} \cdot \cos(\phi_1) - Y_{12} \cdot \sin(\phi_1) &= 0 \\
\dot{m}_1 \cdot (U_1 + v_1 \cdot \omega_1) + X_{12} \cdot \sin(\phi_1) + Y_{12} \cdot \cos(\phi_1) - Y_4 &= 0 \\
I_2 \cdot \Omega_2 + X_{12} \cdot d_2 \cdot \sin(\phi_1) + Y_{12} \cdot d_2 \cdot \cos(\phi_1) + Y_4 \cdot b_2 &= 0
\end{align*} \] (8)

The motion equations of without bearing semitrailer are formed as:

\[ \begin{align*}
\dot{m}_2 (V_2 - u_2 \omega_2) + X_{12} \cos(\phi_1) - Y_{12} \sin(\phi) &= 0, \\
\dot{m}_1 (U_1 + v_1 \omega_1) + X_{12} \sin(\phi_1) + Y_{12} \cos(\phi_1) - Y_3 &= 0, \\
J_2 \Omega_1 + X_{12} d_2 \sin(\phi_1) + Y_{12} d_2 \cos(\phi_1) + Y_3 b_2 &= 0.
\end{align*} \] (9)

Internal forces \( Y_{01}, X_{01}, Y_{12}, X_{12} \) are eliminated from Equations (5), (6), (7), (8) and (9). The system of nonlinear differential equation is got more generally by variable values \( v, u, \omega, \phi, \phi_1 \):

\[ f(v, u, \omega, \phi, \phi_1, V, U, \Omega, \Phi, \Phi_1, PP, PP_1, \theta) = 0 \] (10)

With steady-state condition of motion, the values of all phase variables become constant. Thus, the set of steady-state conditions is assessed by system (10), where \( v = \text{const} \) and \( V = U = \Omega = \Phi = \Phi_1 = PP = PP_1 = 0 \).

The system is got more general view:

by variable value

\[ \begin{align*}
u & f(u, \omega, \phi, \phi_1, \theta) = 0 \\
\omega & f(u, \omega, \phi, \phi_1, \theta) = 0 \\
\phi & f(u, \omega, \phi, \phi_1, \theta) = 0 \\
\phi_1 & f(u, \omega, \phi, \phi_1, \theta) = 0
\end{align*} \] (11)

3. Analysis of stability of a three-trailer road train with bearing semitrailer

The system is analyzed for the presence of circular steady-state conditions and to assess of the road train.

An example is reduced for curve construction of real solutions of the quadratic equation when one parameter \( x \) changes, using the parameter continuation method. The equation is formed as:

\[ x^2 + y^2 - 1 = 0 \] (12)

Solution real set of the equation is presented in the form of implicit function \( G(x, y) = 0 \). In plane \((x, y)\) it is presented some curve, which passes across origin of coordinates (obvious solution of equation \( \{x = 0, y = 0\} \)). The curve is parameterized by natural parameter. It is arc length

\[ s \left( ds = \sqrt{dx^2 + dy^2} \right). \]

The system of differential equations is obtained by the parameter continuation method \( x \) [14]:

\[ \frac{dx}{ds} = -\frac{\partial G}{\partial y}, \quad \frac{dy}{ds} = \frac{\partial G}{\partial x} \]
Cubic parabola of Equation (12) is drawn using the parameter continuation method (Figure 3).

\[
\frac{dy}{ds} = -\frac{\frac{\partial G}{\partial x}}{\sqrt{\left(\frac{\partial G}{\partial y}\right)^2 + \left(\frac{\partial G}{\partial x}\right)^2}}
\]

**Figure 3.** Cubic parabola (horizontal axis is the parameter \(x\) of the Equation (12); vertical axis is the parameter \(y\) of the Equation (12)).

4. **Results of numerical analysis of the mathematical model of the road train with without bearing semitrailer**

The system is examined for the presence of circular steady-state conditions and the configuration of the road train is assessed.

The system of Equations (11) is used to determine the steady-state conditions by the iterative method. System of Equations (10) is used for numerical integration. It is necessary to prove the instability of this model in the neighborhood of the linear motion. Jacobi matrix of Equations (11) is equated by phase variables:

\[
\begin{bmatrix}
    k_1 + k_2 + k_3, \quad m_1 v^2 + m_3 v^2 + k_1 a - k_3 c - k_3 d_1 - k_3 c_1 - k_3 d_2 - k_2 b - k_2 b + m v^2, k_3, k_3
\end{bmatrix}
\]
\[
\begin{align*}
\begin{bmatrix}
k_3c - k_3a + k_3b & v \\
-k_3cd_1 - k_2b^2 - k_3e^2 - k_1a^2 - ck_3c_1 + cm_1\nu^2 + cm_2\nu^2 - k_3d_2 - k_3b_2c & -k_3c, -k_3c
\end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
\begin{bmatrix}
k_3(1 + c_1) & v \\
-k_3d_1 - k_3c_1, k_3(1 - c_1) & v
\end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
\begin{bmatrix}
k_3(1 + c_2) & v \\
-k_3d_2 - k_3c_2, k_3(1 - c_2) & v
\end{bmatrix}
\end{align*}
\]

The determinant of the matrix is equal to the free term of the characteristic equation. A necessary condition for stability according to A. Stodol is that the free term of the characteristic equation must be greater than zero. The value of the longitudinal velocity \( v \) is critical, which determines the determinant of the matrix to zero (when this velocity is reached, a loss of stability occurs). Thus, if determinant is calculated the analytically, equated it to zero, and expressed \( v \) from the resulting equality, we obtain the expression of the critical velocity in the analytic form. Calculations are shown, that the determinant of the matrix is identically zero, and therefore it does not depend on \( v \). So the model is structurally unstable in the neighbourhood of the rectilinear mode, and the value of the critical velocity is zero. Attempts to determine stable steady-state conditions when moving along circles of a sufficiently large radius also revealed structural instability.

Circular steady-state conditions are determined when the road train moves along circles of a sufficiently small radius. For the received steady-state conditions, the characteristic configuration of the road train is used.

The following parameters are defined:

\[a_{t} = 0.8,\ k_{1} = 300000 \text{H},\ k_{2} = 800000 \text{H},\ k_{3} = 1000000 \text{H},\]
\[m = 6250 \text{kg},\ m_{2} = 6160 \text{kg},\ a = 1.27 \text{m},\ b = 2.33 \text{m},\ c = 1.6 \text{m},\ b_{1} = 2.8 \text{m},\ d_{1} = 4 \text{m},\ v = 5 \text{m/s},\ c_{1} = 3.79 \text{m};\]
\[\theta = 0.38 \text{rad} ;\ J_{1} = 10000 \text{kg \cdot m}^2,\ J_{2} = 25000 \text{kg \cdot m}^2,\ J_{2} = 200 \text{kg \cdot m}^2\]

These parameters are corresponded to the steady circular steady-state conditions:

\[a^* = 1.04 \text{Im/s},\ \omega^* = 0.51 \text{rad/s},\ \varphi^* = -7.898 \text{rad},\ \varphi_1^* = -4.676 \text{rad}\]

The stability of the steady-state conditions is confirmed the eigenvalue spectrum:

\[
e_{1} = -0.005282860944 \quad + 0.6250834805 \quad I,\quad -14.17459637 + 1.386043145 \quad I,
\]
\[
-33.64465075,\quad -123.1874797,\quad -14.17459637 - 1.386043145 \quad I,
\]
\[
-0.005282860944 - 0.6250834805 \quad I
\]

The roots of the performance equation of the system in variations have negative real parts. According to the Lyapunov theoreum, the circular traffic condition is asymptotically stable. Besides, the configuration of the road train has the essential singularity. This wheelless link is oriented to the outer side of the center of mass of the tractor unit (Figure 4). The ordinate of point \( C \) is determined the radius of the circle on which the center of mass of the tractor unit is moved \((Y_c = 9.99 \text{ m})\), the rotation center is the coordinate origin. In this case, the velocity vector of the center of mass of the tractor unit is parallel to the abscissa axis, so radius of rotation of the tractor unit is perpendicular to the same axis. This result is expected, since, with a different configuration of the road train, it is not possible to balance the centrifugal force of the without bearing semitrailer. The stability of these conditions is asymptotic. This is confirmed by the results obtained by the numerical integration of the differential equations of the road train movement (Figure 5).

The Figure 5a is illustrated the process of entering into steady-state conditions of the road train, whereupon the radius of curvature of the trajectory remains almost constant \( R = 10 \text{ m} \), but there is the displacement in the rotation center. This fact is confirmed by the geometric construction of the
trajectory in the steady-state conditions by the iterative method and the radius value in this case is $R = 9.986$ m.

**Figure 4.** Configuration of road train in the road plane [m] when steady-state condition: 1 – longitudinal axis of the tractor; 2 – longitudinal axis of the semi-trailer, 3 – longitudinal axis of the bearing bogie; C – center of mass of the tractor unit.

**Figure 5.** The trajectory of the center of mass of the tractor unit in the road plane [m]; a) by math software Maple; b) by geometrically.
Graphs of phase dependence of phase variable on time are presented in the Figure 6. The figure fully is confirmed the asymptotic stability in the steady-state conditions. The Figures 6a, 6b and 6c are shown phase trajectories that have a vibrational damped character. The Figure 6d is shown that the folding angle of the without bearing link is less than zero. This is corroborated the orientation of the semitrailer to the outside of the turn.

**Figure 6.** Phase trajectory: a) folging angle of the semitrailer; b) folging angle of the s bearing bogie; c) transverse velocity; d) angular velocity.

The change of the configuration of the road train at different values of the steering wheel turning angle of the tractor unit is analysed. The road train configurations are presented in Figure 7. The construction is based on the values of the steady-state conditions by the iterative method.
The Figure 7 is showed that the configurations are not significantly different, the radius of rotation of the road train is changed (it is the ordinates of points C) and the folding angles is changed slightly. 

When analyzing the mathematical model, stable steady-state conditions are realized in the range of $0.1\text{ rad} < \theta < 0.9\text{ rad}.$

5. Conclusions
The mathematical model of a three-trailer road train with a wheelless intermediate link is worked out. The analysis of the equilibrium conditions of the system is performed. The structural instability of the model of the road train in the neighbourhood of the linear motion is presented. The presence of stable circular steady-state conditions is proved when moving along circles of a sufficiently small radius. It is established that for a given longitudinal velocity there is a certain range of changes in the rotation angle of the steering wheels of the tractor unit, within which stable steady-state conditions may be. The analysis of the stability of the steady-state conditions at $\theta = 0.38\text{ rad.}$ For the steady-state conditions, the configuration of the road train is drawn, where it can be seen that the without bearing link can only be oriented to the outer side of the turn. Also in the process of numerical integration, it is found that the resulting steady-state conditions are asymptotically stable.

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