Statistical Inference in Parametric Preferential Attachment Trees

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Abstract

The preferential attachment (PA) model is a popular way of modelling dynamic social networks, such as collaboration networks. Assuming that the PA function takes a parametric form, we propose and study the maximum likelihood estimator of the parameter. Using a supercritical continuous-time branching process framework, we prove the almost sure consistency and asymptotic normality of this estimator. We also provide an estimator that only depends on the final snapshot of the network and prove its consistency, and its asymptotic normality under general conditions. We compare the performance of the estimators to a nonparametric estimator in a small simulation study.

1 Introduction and Notation

We study the preferential attachment (PA) model—a dynamic network model which in our setup evolves from an initial stage consisting of a single node of degree one (a root node with a loose edge or dead parent) by recursively adding at each step a single node and edge. The incoming node connects to a node in the existing network with probability proportional to a non-decreasing function of its degree. The term preferential attachment reflects that nodes of higher degrees (“the rich”) inspire more incoming connections (“get richer”), thus leading to “the-rich-get-richer” effect, or the so-called Matthew effect.

For a precise description, define $[n] = \{1, \ldots, n\}$, and denote the nodes at time $n$ by $\{v_i\}_{i \in [n]}$. If the corresponding degrees are $\{d_i(n)\}_{i \in [n]}$, then the node $v_{n+1}$ connects to the existing node $v_i \in \{v_i\}_{i \in [n]}$ with probability proportional to $f(d_i(n))$, for a given function $f: \mathbb{N}_+ \to \mathbb{R}_+$, i.e., with probability

$$
\frac{f(d_i(n))}{\sum_{j=1}^{n} f(d_j(n))}.
$$

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We refer to \( f \) as the \emph{preferential attachment function}, to \( f(k) \) as the \emph{preference} for a node of degree \( k \), and to the denominator in the display as the \emph{total preference} at time \( n \). After the incoming node \( v_{n+1} \) has made its choice, the scheme repeats itself at time \( n+1 \) with the set of existing nodes \( \{ v_i \}_{i \in \lfloor n+1 \rfloor} \), their updated degrees \( \{ d_i(n+1) \}_{i \in \lfloor n+1 \rfloor} \), and the incoming node \( v_{n+2} \). The model may evolve to reach any number of nodes, whence we obtain a random graph that evolves over time.

The PA model received its modern conception and fame in connection to the prevalence of so-called \emph{scale-free} networks in the real world, as shown by Albert Barabási and his co-authors (Barabási and Albert (1999); Barabási, Albert-László, Albert and Jeong (1999, 2000)), and in many subsequent scientific studies in different disciplines. A simple version of the PA model appeared in Barabási and Albert (1999) as a possible explanation of the emergence of the scale-free property and henceforth the PA model has remained one of the few \emph{dynamic} models to produce scale-freeness.

Scale-freeness is usually defined in terms of polynomial decay of the empirical degree distribution \( P_k(n) \), which is the proportion of nodes of degree \( k \) at time \( n \):

\[
P_k(n) = \frac{1}{n} \sum_{i \in [n]} 1_{\{ d_i(n) = k \}} =: \frac{1}{n} N_k(n).
\]

In the case that the PA function \( f \) is affine with \( f(k) = k + \alpha \) with \( \alpha > -1 \), it is known that \( P_k(n) \to p_k \) almost surely, as \( n \to \infty \), for any fixed \( k \), where the limit \( (p_k)_{k=1}^{\infty} \) is a proper probability distribution with atoms proportional to \( k^{-(3+\alpha)} \), as \( k \to \infty \) (up to a slow-varying factor, see Móri (2002); van der Hofstad (2017)). The Barabási–Albert model is the special case with \( f(k) = k \) and gives decrease proportional to \( k^{-3} \).

Two different scenarios arise when we no longer restrict attention to the affine PA functions—superlinear and sublinear. Roughly speaking, in the superlinear case the PA function \( f \) grows faster than any linear function and the resulting PA tree looks like a star with one dominating node (with high probability, the first or the oldest one) connecting to virtually almost every other node, see Oliveira and Spencer (2005) and references therein. In the sublinear case \( f \) grows more slowly than any linear function, yielding more interesting PA trees. The slower growth of strictly sublinear \( f \) yields less preference towards high-degree nodes, rendering the emergence of high-degree nodes less likely, and leading to limiting degree distributions with in general lighter tails than power laws, which have fine and subtle details.

In this paper we consider the statistical estimation of the PA function from an observed network. We adopt a parametric specification of a sublinear PA function and consider statistical inference on the parameter. A prototype of such a model is \( f(k) = (k + \alpha)^{\beta} \), for parameters \( \alpha \) and \( \beta \leq 1 \). This includes the submodel \( f(k) = k^{\beta} \), for \( \beta \leq 1 \), considered in (Barabási, 2016, Section 5.8), and the affine linear model \( f(k) = k + \alpha \). The latter model was considered in Gao and van der Vaart (2017) by relatively direct arguments and martingale methods, which break down for more general models. The main contribution of the present paper is to exploit the framework of supercritical Malthusian branching processes due to Jagers (1975) and Nerman (1981), and applied to derive the limiting degree distribution in PA models by Rudas, Tóth and Valkó (2007), to analyze general
parametric models. The branching process framework allows to study the likelihood function and to prove the consistency and asymptotic normality of the maximum likelihood estimator.

We also propose a Wald-type test to test the null hypothesis that the PA function is affine, thus allowing a test for scale-freeness versus a well-defined alternative. In the case that only a final snapshot and not the evolution history is observed, we propose a history-free remedy to the likelihood function and obtain a pseudo maximum likelihood estimator, which is shown to be consistent in general and asymptotically normal under general conditions, and those conditions are verified for the case where the PA function becomes constant after reaching a certain degree.

An alternative to parametric estimation is the nonparametric approach of Gao et al. (2017), who introduce an empirical estimator and show that this is consistent for general PA functions \( f \). It is unknown whether this empirical estimator is asymptotically normal. We show by simulation that the maximum likelihood estimator of the present paper is significantly more efficient if the parametric model is correctly specified.

1.1 Outline of the paper

The paper is organized as follows. We present the likelihood and the maximum likelihood estimator in Section 2. Section 3 introduces supercritical Malthusian branching processes and deduce the results needed for the present paper (with a more formal introduction in the appendix, Section A). In Section 4, we prove that the maximum likelihood estimator is consistent and in Section 5 that it is asymptotically normal. To overcome the problem of relying on the entire evolution history of the network, we propose in Section 6 a pseudo maximum likelihood estimator, which depends only on the final snapshot of the tree, and give conditions for its asymptotic normality. Section 7 gives a new perspective on the empirical estimator, presenting this as a nonparametric version of the pseudo maximum likelihood estimator. In Section 8 we present simulation results that demonstrate the performance of the maximum likelihood estimator and pseudo maximum likelihood estimator, and compare this to the empirical estimator. Section 9 collects all proofs to the results. In a second section B of the appendix we prove the asymptotic normality of the empirical degree distribution in a new case of special interest, and give a new proof in the affine case.

1.2 Notation

Write \( \mathbb{N}_+ \) for the set of positive natural numbers \( \{1, 2, \ldots\} \) and \( \mathbb{N} \) for the set of natural numbers including zero. Write \( \mathbb{R}_+ = (0, \infty) \). For a sequence \( (a_k)_{k=1}^\infty \), define \( a_{>k} = \sum_{l>k} a_l \). Let \( N_k(t) \) be the number of nodes of degree \( k \) in the network at time \( t \), and \( P_k(t) = N_k(t)/t \) the proportion of such nodes. For a given function \( h: \mathbb{N}_+ \to \mathbb{R} \) set \( S_h(t) = \sum_{k=1}^\infty h(k) N_k(t) \). The superscript \( (0) \) stresses that a quantity, such as the limiting degree distribution \( p^{(0)}_k \), is considered under the true parameter \( \theta_0 \). For a vector \( v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d \), let \( \|v\|_1 = \|v\|_\infty = \max_{1 \leq i \leq d} |v_i| \); and for a matrix \( A \in \mathbb{R}^{m \times n} \), set \( \|A\|_1 = \max_{i,j} |A_{ij}| \). Define the diagonal matrix \( \text{diag}(a_1, \ldots, a_d) \) by \( (\text{diag}(a_1, \ldots, a_d))_{ij} = 1_{(i=j)}a_i \). Define \( u_+ = \)}
max(0, −u) for u ∈ ℜ. We write a ∧ b = min(a, b) for a, b ∈ ℜ. When cb_n ≤ a_n ≤ Cb_n for some positive constants c and C, we write a_n ∼ b_n.

1.3 Model

Throughout the paper \{f_θ: θ ∈ Θ\} is a collection of non-decreasing functions \( f_θ: \mathbb{N}_+ \rightarrow \mathbb{R}_+ \) indexed by a subset \( Θ \subset \mathbb{R}^d \). It is assumed that every element of the family satisfies one of the two possibilities:

(i) \( f_θ(k) \leq Ck^β \) for every \( k \in \mathbb{N} \) for some positive constants \( C \) and \( β < 1 \).

(ii) \( f_θ(k) = k + α \) for every \( k \in \mathbb{N} \), for some \( α > −1 \).

It is also assumed that \( θ \mapsto f_θ(k) \) is twice continuously differentiable with derivatives denoted by \( \dot{f}_θ(k) \) and \( \ddot{f}_θ(k) \), which are a vector (gradient) and a matrix (Hessian) in the case of a multidimensional parameter. Differentiation with respect to the parameter \( θ \) is also denoted by a dot in general.

2 Construction of the Maximum Likelihood Estimator

Let \( N_k(t) \) be the number of nodes of degree \( k \in \mathbb{N} \) in the graph with nodes \( v_1, \ldots, v_t \). Initially there is a single node \( v_1 \) with degree one and hence \( N_1(1) = 1 \), and we can set \( N_k(1) = 0 \), for every \( k \geq 2 \), to define a complete degree sequence. If \( D_t \) denotes the degree of the node to which the node \( v_t \) is attached, then, for \( t \geq 2 \),

\[
N_k(t) = N_k(t − 1) + 1_{\{D_t=k−1\}} − 1_{\{D_t=k\}} + 1_{\{k=1\}}.
\]

(2.1)

The random graph evolves as a Markov process, and hence the likelihood factorizes as the conditional likelihoods of the new node given the current tree. Any of the existing nodes may be chosen to attach the new node, but only the degree \( D_t \) of this node is important for the value of the likelihood, which takes the form

\[
L_n(f_θ) = \prod_{t=2}^{n} \frac{f_θ(D_t)N_{D_t}(t−1)}{S_{f_θ}(t−1)},
\]

where the norming “constant” \( S_{f_θ}(t−1) = \sum_{k=1}^{∞} f_θ(k)N_k(t−1) \) is the total preference in the graph with nodes \( v_1, \ldots, v_{t−1} \) given the PA function \( f_θ \). The total preference can be computed recursively by the rule \( S_{f}(t) = S_{f}(t−1) + f(D_t) + f(1) \), for \( t \geq 2 \), with the initialization \( S_{f}(1) = f(1) \). In particular, the total preference at stage \( t \) can be expressed in the degree sequence up to time \( t−1 \), and the full likelihood depends on the data only through \( D^{(n)}_t = (D_t)_{t=2}^{n} \). The normalized log-likelihood up to the term \( \sum_{t=2}^{n} \log N_{D_t}(t−1) \) is given by

\[
\ell_n(f_θ) = \frac{1}{n} \sum_{k=1}^{∞} \log f_θ(k) \prod_{t=2}^{n} 1_{\{D_t=k\}} − \frac{1}{n} \sum_{t=2}^{n} \log S_{f_θ}(t−1)
\]

\[
= \sum_{k=1}^{∞} \log f_θ(k)P_{>k}(n) − \frac{1}{n} \sum_{t=2}^{n} \log S_{f_θ}(t−1).
\]

(2.2)
In the last step we use the identity $\sum_{t=2}^{n} 1_{\{D_t=k\}} = N_{>k}(n)$, which is essentially (Gao et al., 2017, Lemma 1) and results from the fact that any node of degree strictly large $k$ in the tree at time $n$ must have been chosen for attachment while it had degree $k$, exactly once up until this time (namely when its degree went up from $k$ to $k+1$).

The derivative of the log-likelihood is
\[
\ell_n(f_0) = \frac{\partial}{\partial \theta} \log L_n(f_0) = \sum_{t=2}^{n} \left[ \frac{\dot{f}_\theta(D_t)}{f_\theta} - \frac{S_{f_\theta}(t-1)}{S_{f_\theta}(t-1)} \right]
\]
\[
= \sum_{t=2}^{n} \left[ \frac{\dot{f}_\theta(D_t)}{f_\theta} - \mathbb{E}_{\theta} \left[ \frac{\dot{f}_\theta(D_t)}{f_\theta} | \mathcal{F}_{t-1} \right] \right],
\]
where $(\mathcal{F}_t)_{t \geq 1}$ is the filtration generated by the stochastic process of the graph’s evolution. The last expression follows from the fact that $\mathbb{P}(D_t = k | \mathcal{F}_{t-1}) = f_\theta(k)N_k(t-1)/S_{f_\theta}(t-1)$, and shows that the score process is a martingale under the true parameter $\theta$, as usual, which can also be seen by readily verifying $\mathbb{E}[\ell_n(f_0) | \mathcal{F}_{n-1}] = \ell_{n-1}(f_0)$. Dividing this martingale by the number $n$ of nodes in the network and rewriting as in (2.2) gives
\[
i_n(f_0) = \frac{1}{n} \ell_n(f_0) = \sum_{k=1}^{\infty} \frac{\dot{f}_\theta(k)P_{>k}(n)}{\int f_\theta(k)P_{>k}(n)} - \frac{1}{n} \sum_{t=2}^{n} \frac{S_{f_\theta}(t-1)}{S_{f_\theta}(t-1)}
\]
\[
= \sum_{k=1}^{\infty} \frac{\dot{f}_\theta(k)P_{>k}(n)}{\int f_\theta(k)P_{>k}(n)} - \frac{1}{n} \sum_{t=2}^{n} \sum_{k=1}^{\infty} \frac{P_k(t-1)\dot{f}_\theta(k)}{\sum_{k=1}^{\infty} P_k(t-1)f_\theta(k)}. \tag{2.4}
\]

The maximum likelihood estimator $\hat{\theta}_n$ can be defined to be either the maximizer of the log-likelihood (2.3) or a solution to the equation $i_n(f_0) = 0$.

To understand the behavior of the maximum likelihood estimator, we need to study the rescaled log-likelihood in (2.2) or its derivative (2.4). However, the quantities in these equations are anything but easy—they are functionals of the entire evolution history of a complex Markov process, where the influence of the past persists in the likelihood. Simple and straightforward approaches such as martingale methods (cf. (van der Hofstad, 2017, Chapter 8)) are no longer meaningful. Instead, we employ the theory of supercritical Malthusian branching processes, which will be introduced in the section 3.

Adapting such a powerful framework, we will be able to assert the utility of the maximum likelihood estimator by showing its consistency and asymptotic normality, respectively. In particular, in both Theorems 4.1 (from the viewpoint of M-estimator) and 4.6 (from the viewpoint of Z-estimator), we show under some mild assumptions on the parametric family that $\hat{\theta} \to \theta_0$ in an almost sure sense as the number of nodes $n \to \infty$. Theorem 5.1 illuminates that $\sqrt{n}(\hat{\theta} - \theta_0) \to N(0, \sigma_{\hat{\theta}}^2)$ for some properly defined $\sigma_{\hat{\theta}}^2$ such that constructing confidence sets and testing particular well-specified hypotheses are possible.

3 The continuous random tree model

Before stating the main results formally, it is necessary to adopt a continuous-time framework, where nodes are added after exponentially distributed waiting times. To set this
up we equip every node \( v \) with a pure birth process \( \xi_v \), whose events correspond to new, future nodes being attached to this particular node and which in calendar time starts at its own birth, i.e. when it is added to the tree. These birth processes are i.i.d. across nodes and a typical birth process \( \xi = (\xi(t): t \geq 0) \) has birth rate equal to \( f(\xi(t) + 1) \). Thus, \( \xi \) is a continuous-time Markov process with state space \( \mathbb{N}_+ \), initial value \( \xi(0) = 0 \), and with the only possible transitions stepwise increases \( k - 1 \to k \), determined by

\[
\mathbb{P}(\xi(t+dt) = k \mid \xi(t) = k - 1) = f(k) \, dt + o(dt). \tag{3.1}
\]

Every birth corresponds to a new node attached to the existing tree at the node whose birth process produced the event. A node whose birth process has had \( k - 1 \) births will have \( k - 1 \) children and one parent in the tree and hence possess degree \( k \). It will produce a new child with rate \( f(k) \), explaining the right side of the display. At a given calendar time \( t \) every node in the current tree \( \Upsilon_t \) will have a corresponding, active birth process. The total rate of all active birth processes will be \( S(t) = \sum_{v \in \Upsilon_t} f(d_v(t)) \), where \( d_v(t) \) is the degree of \( v \in \Upsilon_t \), and a new node will be attached to node \( v \in \Upsilon_t \) with probability \( f(d_v(t))/S(t) \) after an exponential waiting time with mean \( 1/S(t) \). The process \( \Upsilon_t \) starts at time 0 with a single node that is understood to have degree 1 and hence the first birth will be after an exponential time with mean \( f(1) \). For a more formal setup in the language of general branching processes, see Section A.

Thus, we obtain a continuous-time branching process \( \Upsilon_t \) that contains the discrete-time process as a skeleton. We define \( T_i \) as the total number of births in the continuous process up until time \( t \), and \( \tau_n = \inf\{t > 0: T_i \geq n - 1\} \), for \( n = 1, 2, \ldots \), as the time of the \( n \)th birth (where \( \tau_1 = 0 \)). When evaluated at the stopping times \( \tau_1, \tau_2, \ldots \), the continuous-time process gives a sequence of trees \( \Upsilon_{\tau_1}, \Upsilon_{\tau_2}, \ldots \), that is equivalent to the PA model.

The advantage of the continuous-time setup is that the results on branching processes become more straightforward. To every node we may attach besides a birth process a second continuous-time process, called a characteristic, also starting at the birth of the node. Just as the birth processes, the characteristics are assumed identically distributed, and the birth process and characteristic attached to a single node may be dependent, but every time a new node is added, a pair of a birth process and a characteristic are created that evolve independently of the processes attached to the nodes that appeared earlier in the history of the tree. For a given characteristic \( \varphi \) we consider the process \( (Z^\varphi_t: t \geq 0) \) given by

\[
Z^\varphi_t = \sum_{v \in \Upsilon_t} \varphi_v(t - \sigma_v).
\]

Here \( \sigma_v \) is the calendar time at which node \( v \) is added to the tree, so that \( t - \sigma_v \) is the lifetime of the node since its birth. The characteristic of node \( v \), denoted by \( \varphi_v \), has \( \varphi_v(t) \) interpreted as its value at age \( t \) and hence \( \varphi_v(t - \sigma_v) \) as its value at calendar time \( t \). The variable \( Z^\varphi_t \) gives the sum of the characteristics of all individuals in the tree at time \( t \). In the supercritical case the processes \( Z^\varphi_t \) grow exponentially in time at a rate \( e^{\lambda^* t} \), where \( \lambda^* \) is the so-called Malthusian parameter, and \( e^{-\lambda^* t}Z^\varphi_t \) tends to a (random) limit as \( t \to \infty \). We shall employ these limit theorems with appropriate choices of characteristics to derive the asymptotics of the likelihood function.
A key element is the Laplace transform of the reproduction function $\mu(t) = \mathbb{E}[\xi(t)]$, the mean number of births of a single node at age $t$, which in our case can be expressed in the PA function as (see Rudas, Tóth and Valkó (2007) or the proof in Section 9)

$$\rho_f(\lambda) := \int_0^\infty e^{-\lambda t} \mu(dt) = \sum_{l=1}^\infty \prod_{k=1}^l \frac{f(k)}{\lambda + f(k)}.$$ (3.2)

The function $\rho_f$ is convex and decreasing on its domain (the set where it is finite), which is an interval $(\Delta, \infty)$ or $[\Delta, \infty)$ in the positive half line, and tends to zero as $\lambda \to \infty$. The Malthusian parameter $\lambda^*$ is the solution of the equation $\rho_f(\lambda^*) = 1$.

In the case of a strictly sublinear PA function $f$, we have $\lambda = 0$ and $\rho_f(\lambda) \uparrow \infty$ as $\lambda \downarrow 0$, while for the PA function $f(k) = k + \alpha$ we have $\Delta = 1$ and the exact form $\rho_f(\lambda) = (1 + \alpha)/(\lambda - 1)$ is known (see Gao et al. (2017); Rudas, Tóth and Valkó (2007) or the proof of Proposition 3.1). In both cases the range of $\rho_f$ contains the point 1 as an interior point and the Malthusian parameter exists. If the sublinearity assumption is violated in the sense that neither of the two sublinear conditions holds, the existence of the Malthusian parameter is not guaranteed, and the associated branching process could explode or behave irregularly, whence the branching process framework and the ensuing may results fail.

Furthermore, under the sublinear assumptions of the PA function, we have the following limit theorem.

**Proposition 3.1.** Suppose that the range of $\rho_f$ contains an open neighborhood of 1 and $\varphi_1$ and $\varphi_2$ be monotone increasing characteristics such that, for some constants $C > 0$ and $\gamma \geq 0$ and every $t > 0$,

$$\varphi_i(t) \leq C \xi(t)^{\gamma}, \quad i = 1, 2, \text{ almost surely.}$$ (3.3)

If $f: \mathbb{N}_+ \rightarrow \mathbb{R}_+$ is monotone with $f(k) \leq Ck^\beta$ for some constants $C$ and $\beta < 1$, then, as $t \to \infty$,

$$\frac{Z_t^{\varphi_1}}{Z_t^{\varphi_2}} \xrightarrow{a.s.} \frac{\int_0^\infty e^{-\lambda t}\mathbb{E}[\varphi_1(t)]\,dt}{\int_0^\infty e^{-\lambda t}\mathbb{E}[\varphi_2(t)]\,dt}.$$ (3.4)

The same is true if $f(k) = k + \alpha$, for some $\alpha > -1$, provided that for some $r < 2 + \alpha$ with $r \leq 2$,

$$\varphi_i(t) \leq C \xi(t)^r, \quad i = 1, 2, \text{ almost surely.}$$ (3.5)

For a given PA function $f$ with Malthusian parameter $\lambda^*$, define

$$p_k = \frac{\lambda^*}{\lambda^* + f(k)} \prod_{j=1}^{k-1} \frac{f(j)}{\lambda^* + f(j)}, \quad k \in \mathbb{N}_+, \quad \text{as} \quad k \to \infty.$$ (3.6)

where the empty product is defined to be 1, so that $p_1 = \lambda^*/(\lambda^* + f(1))$. The definition of $\lambda^*$ may be used to show that $(p_k)_{k=1}^\infty$ is a probability distribution on $\mathbb{N}_+$. In fact, it is the limit of the empirical degree distribution $(P_k(n))_{k=1}^\infty$ of the PA network, as shown by Rudas, Tóth and Valkó (2007). More generally, we have the following limiting result.
Corollary 3.2. If \( f: \mathbb{N}_+ \to \mathbb{R}_+ \) is monotone increasing and satisfies \( f(k) \leq Ck^\beta \), for some \( \beta < 1 \), and \( h: \mathbb{N}_+ \to \mathbb{R}_+ \) satisfies \( h(k) \leq Ck^2 \), for a constant \( C \), and every \( k \), then the empirical degrees \( P_k(n) \) in the model with PA function \( f \) satisfy, as \( n \to \infty \),

\[
\sum_{k=1}^{\infty} h(k)P_k(n) \xrightarrow{a.s.} \sum_{k=1}^{\infty} h(k)p_k.
\] (3.7)

The same is true for the PA function given by \( f(k) = k + \alpha \), for some \( \alpha > -1 \), and every \( h: \mathbb{N}_+ \to \mathbb{R}_+ \) satisfying \( h(k) \leq Ck^r \), for some \( r < 2 + \alpha \) with \( r \leq 2 \).

Choosing \( h \) equal to the indicator of the set \( \{k\} \) for a given \( k \), we recover the convergence \( P_k(n) \to p_k \) of the empirical degrees to the limit \( p_k \), first obtained in Rudas, Tóth and Valkó (2007).

It is worth noting that the tail of the distribution \((p_k)_{k=1}^{\infty}\) (of which the dependence on the PA function \( f \) is suppressed from the notation), as \( k \to \infty \) is heaviest among sublinear \( f \) when \( f \) is affine with \( f: k \mapsto k + \alpha \), and it corresponds to the limiting (asymptotic) power law with exponent \( 3 + \alpha \).

The following lemma records two useful identities, which readily follow from the definition (3.6) of \( p_k \), and the definition of \( \lambda^* \) in terms of the Laplace transform (3.2).

Lemma 3.3. Suppose \((p_k)_{k=1}^{\infty}\) is the limiting degree distribution specified in (3.6) for the PA function \( f \) with Malthusian parameter \( \lambda^* \). Then \( \lambda^* = \sum_{j=1}^{\infty} f(j)p_j \) and, for all \( k \geq 1 \),

\[
p_{>k} = \frac{f(k)p_k}{\sum_{j=1}^{\infty} f(j)p_j}.
\] (3.8)

4 Consistency

The following theorem shows that the maximum likelihood estimator in the model introduced in Section 1.3 is consistent. We assume that the true parameter \( \theta_0 \in \Theta \) is identifiable in the model \( \{f_\theta : \theta \in \Theta\} \) in the sense that \( f_\theta(k) = cf_\theta^0(k) \) for every \( k \in \mathbb{N}_+ \) and some constant \( c \) if and only if \( \theta = \theta_0 \).

Theorem 4.1. In the model \( \{f_\theta : \theta \in \Theta\} \) stated in Section 1.3 with compact parameter space \( \Theta \subset \mathbb{R}^d \) and PA functions satisfying \( f_\theta(k) \leq Ck^\beta \) for some constants \( C \) and \( \beta < 1 \) or \( f_\theta(k) = k + \alpha \) for some constant \( \alpha > -1 \), for every \( k \) and every \( \theta \in \Theta \), the maximum likelihood estimator \( \hat{\theta}_n \) satisfies \( \hat{\theta}_n \to \theta_0 \) almost surely under \( \theta_0 \).

4.1 Identifiability From Score Equation

For computational ease the maximum likelihood estimator may be characterized as a solution to the likelihood equations \( i_n(f_\theta) = 0 \), for \( i_n \) given in (2.4). The asymptotic version of this function is

\[
i(f_\theta) = \sum_{k=1}^{\infty} \frac{\tilde{f}_\theta^0(k) p_{>k}^{(0)}}{\tilde{f}_\theta^0(k) p_{>k} - \sum_{k=1}^{\infty} p_k^{(0)} \tilde{f}_\theta^0(k) \tilde{f}_\theta^0(k)}.
\] (4.1)
It follows from (3.8) that the true parameter $\theta_0$ solves the equation $i(f_\theta) = 0$. The following proposition shows that the processes $\hat{\epsilon}_n$ tend uniformly to $i$. The proof is similar to the proof of Theorem 4.1 and will be omitted.

**Proposition 4.2.** In the model $\{f_\theta: \theta \in \Theta\}$ stated in Section 1.3 with compact parameter space $\Theta \subset \mathbb{R}^d$ and PA functions satisfying $f_\theta(k) \leq Ck^\beta$ for some constants $C$ and $\beta < 1$ or $f_\theta(k) = k + \alpha$ for some constant $\alpha > -1$, for every $k$ and every $\theta \in \Theta$, assume that, for some constants $C$ and $\gamma$,

$$
\left\| \hat{f}_\theta(k) \right\| \leq Ck \log^\gamma k,
$$

$$
\left\| \hat{f}_\theta(k) \right\| \leq C \log^\gamma k.
$$

Then $\sup_{\theta \in \Theta} |i_n(f_\theta) - i(f_\theta)| \to 0$, almost surely, as $n \to \infty$.

It follows that the maximum likelihood estimator is asymptotically the unique solution to the likelihood equations in compact subsets of the parameter space in which $\theta_0$ is the unique zero of $\theta \mapsto i(f_\theta)$ (compare Theorem 5.9 in van der Vaart (2000)). However, in general proving global uniqueness turns out to be difficult. We present the following partial results, starting with two useful lemmas.

**Lemma 4.3.** Suppose $(v_k)_{k=1}^\infty$ is strictly decreasing with respect to $k$. If $(p_k)_{k=1}^\infty$ and $(q_k)_{k=1}^\infty$ are probability distributions on $\mathbb{N}_+$ such that $p_k \leq q_k$ for $k \leq K$ and $p_k > q_k$ for $k > K$, then

$$
\sum_{k=1}^\infty p_k v_k > \sum_{k=1}^\infty q_k v_k.
$$

In case $(v_k)_{k=1}^\infty$ is strictly increasing, the inequality is true in the opposite direction.

**Lemma 4.4.** For a probability distribution $(p_k)_{k=1}^\infty$ and nonnegative sequence $(w_k)_{k=1}^\infty$ such that $\sum_{k=1}^\infty p_k w_k < \infty$, define $(q_k)_{k=1}^\infty$ by $q_k = p_k w_k / \sum_j p_j w_j$. If $(w_k)_{k=1}^\infty$ is strictly increasing, then there exists a $K$ such that $p_k \geq q_k$ for $k \leq K$ and $p_k < q_k$ for $k > K$. If $(w_k)_{k=1}^\infty$ is strictly decreasing, then there exists a $K$ such that $p_k \leq q_k$ for $k \leq K$ and $p_k > q_k$ for $k > K$.

We say that changing from $\theta_0$ to $\theta$ induces monotonicity if $f_\theta(k)/f_{\theta_0}(k)$ is either strictly increasing or strictly decreasing in $k$.

**Lemma 4.5.** If changing from $\theta_0$ to $\theta$ induces monotonicity on $f_\theta/f_{\theta_0}$ for every $\theta$ in a subset $\Theta' \subset \Theta$, then $i(f_\theta) \neq 0$, for every $\theta \in \Theta'$.

For illustrative purposes, we next study in detail different variants of the parametric form $k \mapsto (k + \alpha)^\beta$ in the sections 4.1.1–4.1.3. However, we point out our results work in much more general capacity, as studied in section 4.1.5. For instance, one could easily work out that our results apply to the parametric specification $k \mapsto (\log(k + \alpha))^\beta$. We provide a toy example in Section 4.1.4 for such a parametric form besides $k \mapsto (k + \alpha)^\beta$ and their variants.
4.1.1 The model $f_{\alpha,\beta}(k) = (k + \alpha)^{\beta}$

In the case that $f_{\alpha,\beta}(k) = (k + \alpha)^{\beta}$, changing the parameter $(\alpha_0, \beta_0)$ to another parameter $(\alpha, \beta)$ does not always induce monotonicity on $f_{\alpha,\beta}/f_{\alpha_0,\beta_0}$. An analysis of the derivative of the function $g(x) = (x + \alpha)^{\beta}/(x + \alpha_0)^{\beta_0}$ yields that $f_{\alpha,\beta}/f_{\alpha_0,\beta_0}(k)$ is increasing in $k$ on $\{(\alpha, \beta); \beta - \beta_0 + \beta\alpha_0 - \beta_0\alpha \geq 0, \beta \geq \beta_0\}$ and is decreasing on the set $\{(\alpha, \beta); \beta - \beta_0 + \beta\alpha_0 - \beta_0\alpha \geq 0, \beta \leq \beta_0\}$. The preceding technique does prove that $(\alpha_0, \beta_0)$ is a unique root of $\bar{i}(f_{\alpha,\beta}) = 0$ in these sets, but this does not exhaust the full parameter set.

However, the lemmas 4.4 and 4.3 may be used to prove local uniqueness. The Hessian matrix of $i(f_{\alpha,\beta})$ evaluated at $(\alpha_0, \beta_0)$ can be calculated as, with the shorthand notation $f_0 := f_{\alpha_0,\beta_0}$,

$$i(f_{\alpha_0,\beta_0}) = -\frac{1}{a^2} \begin{pmatrix} ab - c^2 & af - ce \\ ad - ce & af - c^2 \end{pmatrix},$$  

(4.2)

where the quantities $a, b, c, d, e, f$ are defined as

\[
\begin{align*}
    a &= \sum_{k=1}^{\infty} p_k^{(0)} f_0(k), \\
    b &= \sum_{k=1}^{\infty} p_k^{(0)} f_0(k) \frac{\beta_0}{(k + \alpha_0)^2}, \\
    c &= \sum_{k=1}^{\infty} p_k^{(0)} f_0(k) \frac{\beta_0}{k + \alpha_0}, \\
    d &= \sum_{k=1}^{\infty} p_k^{(0)} f_0(k) \frac{\beta_0}{k + \alpha_0} \log(k + \alpha_0), \\
    e &= \sum_{k=1}^{\infty} p_k^{(0)} f_0(k) \log(k + \alpha_0), \\
    f &= \sum_{k=1}^{\infty} p_k^{(0)} f_0(k) \log^2(k + \alpha_0).
\end{align*}
\]

It follows that $ab - c^2$ is strictly positive, since by the Cauchy–Schwarz inequality, where $K$ follows the law $(p_k^{(0)})_{k=1}^{\infty}$,

$$\left( \mathbb{E}_{p_0} \left[f_0(K) \frac{\beta_0}{K + \alpha_0} \right] \right)^2 < \mathbb{E}_{p_0} \left[f_0(K)\right] \mathbb{E}_{p_0} \left[f_0(K) \frac{\beta_0^2}{(K + \alpha_0)^2} \right].$$

The same arguments work to prove that $af - c^2 > 0$ and $bf - d^2 > 0$. The determinant of the Hessian matrix is given by

$$|i(f_{\alpha_0,\beta_0})| = a^2 b f + c^2 e^2 - abe^2 - af c^2 - a^2 d^2 - c^2 e^2 + 2acde$$

$$= a^2 (bf - d^2) - ae(be - cd) - ac(cf - ed).$$

This can be shown to be strictly positive by showing that both $be - cd < 0$ and $cf - ed < 0$. We shall prove $be < cd$; the proof that also $cf < ed$ is similar. Define $x_k = p_k^{(0)} f_0(k) \beta_0/(k + \alpha_0)$ and $u_k = (k + \alpha_0) \log(k + \alpha_0)/\beta_0$, $y_k = x_k u_k = p_k^{(0)} f_0(k) \log(k + \alpha_0)$. Define $p_x(k) = x_k / \sum_j x_j$ and $p_y(k) = y_k / \sum_j y_j$. Since $u_k$ is strictly monotone decreasing, an application of Lemma 4.4 tells us that there exists a $K$ such that $p_x(k) \geq p_y(k)$ for $k \leq K$ and $p_x(k) < p_y(k)$ for $k > K$. Applying Lemma 4.3 with $w_k = \beta_0/(k + \alpha_0)$, we see that $be < cd$.

We conclude that the Hessian matrix is negative definite, so that $(\alpha_0, \beta_0)$ is a unique root of $\bar{i}(f_{\alpha,\beta})$ in a neighborhood around $(\alpha_0, \beta_0)$.
4.1.2 Global concavity of the model $f_\beta(k) = (k + \alpha_0)^\beta$ with known $\alpha_0$

The single-parameter model $f(k) = k^\beta$ is a main example of sublinear PA, treated in (Barabási, 2016, Section 5.8). We allow in addition a nonzero offset $\alpha_0$ and consider $f_\beta(k) = (k + \alpha_0)^\beta$, with $\beta$ the only parameter. The limit function (4.1) reduces to

$$i(f_\beta) = \sum_{k=1}^{\infty} \log(k + \alpha_0) p_{>k}^{(0)} - \frac{\sum_{k=1}^{\infty} p_k^{(0)} (k + \alpha_0)^\beta \log(k + \alpha_0)}{\sum_{k=1}^{\infty} p_k^{(0)} (k + \alpha_0)^\beta}.$$

The second order derivative can be computed as

$$\ddot{i}(f_\beta) = -\frac{\sum_{k=1}^{\infty} p_k^{(0)} (k + \alpha_0)^\beta \log^2(k + \alpha_0)}{\sum_{k=1}^{\infty} p_k^{(0)} (k + \alpha_0)^\beta} + \left(\frac{\sum_{k=1}^{\infty} p_k^{(0)} (k + \alpha_0)^\beta \log(k + \alpha_0)}{\sum_{k=1}^{\infty} p_k^{(0)} (k + \alpha_0)^\beta}\right)^2.$$

This can be seen to be strictly negative for any $\beta \in [0, 1]$, as a consequence of the Cauchy–Schwarz inequality, with $K$ following the law $(p_k^{(0)})_{k=1}^{\infty}$,

$$\left(\mathbb{E}_{p_0}[\log(K + \alpha_0)]\right)^2 < \mathbb{E}_{p_0}[\log^2(K + \alpha_0)]\mathbb{E}_{p_0}(K + \alpha_0)^\beta.$$

Thus, the limiting log likelihood is concave and the root of the limiting score function is unique. Another perspective is that moving the parameter $\beta$ away from $\beta_0$ induces monotonicity on $f_\beta/f_{\beta_0}(k) = (k + \alpha_0)^{\beta - \beta_0}$.

4.1.3 Almost-global uniqueness in case of $f(\alpha) = (k + \alpha)^{\beta_0}$ with known $\beta_0$

For any $\alpha \neq \alpha_0$, the function $k \mapsto (k + \alpha)^{\beta_0}/(k + \alpha_0)^{\beta_0}$ is monotone increasing (when $\alpha < \alpha_0$) or decreasing (when $\alpha > \alpha_0$). Applying Lemma 4.5, we conclude that the root is unique in every bounded domain.

4.1.4 Global concavity of the model $f(\beta) = \log^\beta(k)$

For further illustrative purposes, we study the toy case of the PA function being $k \mapsto \log^\beta(k)$, where $\beta$ is the only parameter. We easily calculate the score function and its derivative as follows

$$i(f_\beta) = \sum_{k=1}^{\infty} \log \log(k) p_{>k}^{(0)} - \frac{\sum_{k=1}^{\infty} p_k^{(0)} \log^\beta(k) \log \log(k)}{\sum_{k=1}^{\infty} p_k^{(0)} \log^\beta(k)},$$

$$\ddot{i}(f_\beta) = -\frac{\sum_{k=1}^{\infty} p_k^{(0)} \log^\beta(k) (\log \log(k))^2}{\sum_{k=1}^{\infty} p_k^{(0)} \log^\beta(k)} + \left(\frac{\sum_{k=1}^{\infty} p_k^{(0)} \log^\beta(k) \log \log(k)}{\sum_{k=1}^{\infty} p_k^{(0)} \log^\beta(k)}\right)^2.$$

Similar to the analysis in Section 4.1.2, $\ddot{i}(f_\beta)$ here is strictly negative for any $\beta > 0$ by the Cauchy–Schwarz inequality

$$\left(\mathbb{E}_{p_0}[\log^\beta(K) \log \log(K)]\right)^2 < \mathbb{E}_{p_0}[\log^\beta(K)(\log \log(K))^2]\mathbb{E}_{p_0}[\log^\beta(K)],$$

where $K \sim (p_k^{(0)})_{k=1}^{\infty}$ is an auxiliary random variable. As such, the limiting score function is monotone decreasing with respect to the parameter $\beta$, and has a unique zero at $\beta = \beta_0$. Furthermore, the limiting log-likelihood is concave with a unique maximizer.
4.1.5 The case of general $f_\theta$

In practice, it could happen that the score function (2.4) has multiple roots, particularly if its limit (4.1) has multiple roots. In such cases, we may employ the empirical estimators in Gao et al. (2017) to identify the correct one. These are defined as follows (see their Equation (2)):

$$\hat{r}_k(n) = \frac{N_{>k}(n)}{N_k(n)}.$$ 

In Gao et al. (2017) these estimators are shown to converge to $f_{\theta_0}(k)/\left(\sum_{j=1}^{\infty} f_{\theta_0}(j)p_j^{(0)}\right)$, almost surely. This suggests an empirical estimator for the parameter $\theta \in \Theta \subset \mathbb{R}^d$ as the solution of the system of equations:

$$\frac{f_\theta(k)}{f_\theta(1)} = \frac{\hat{r}_k(n)}{\hat{r}_1(n)}, \quad k = 2, 3, \ldots, d + 1. \tag{4.3}$$

Two computational strategies suggest themselves. If finding the set of solutions to the likelihood equations $i_n(f_\theta) = 0$ is easier than solving (4.3), then we may select from this set the solution that minimizes

$$\sum_{k=2}^{d+1} \frac{|f_\theta(k)|}{f_\theta(1)} - \frac{\hat{r}_k(n)}{\hat{r}_1(n)}.$$

On the other hand, if (4.3) is easier to solve than (2.4), then we may find the solution of likelihood equations in a neighborhood of the solution of (4.3), possibly by an iterative scheme such as Newton’s algorithm.

In both cases the resulting estimator will be consistent.

**Theorem 4.6.** Under the conditions of Theorem 4.1, the solution of the likelihood equation resulting from either of the two indicated procedures is almost surely consistent for $\theta_0$.

5 Asymptotic Normality

We prove asymptotic normality of consistent solutions to the likelihood equations, as follows. Recall that for $\theta \in \Theta \subset \mathbb{R}^d$ and every $k \in \mathbb{N}_+$, $\hat{f}_\theta(k) \in \mathbb{R}^d$ and $\hat{f}_\theta(k) \in \mathbb{R}^{d \times d}$ are the gradient and the Hessian matrix of $f_\theta(k)$ with respect to $\theta$, respectively.

**Theorem 5.1.** In the model $\{f_\theta; \theta \in \Theta\}$ stated in Section 1.3 with compact parameter space $\Theta \subset \mathbb{R}^d$ and PA functions satisfying $f_\theta(k) \leq Ck^\beta$ for some constants $C$ and $\beta < 1$ or $f_\theta(k) = k + \alpha$ for some constant $\alpha > -1$, for every $k$, and every $\theta \in \Theta$, assume in addition that the PA functions satisfy, for some constants $\gamma > 0$ and $C > 0$,

$$\|\hat{f}_\theta(k)\| + \|\hat{f}_\theta(k)\| \leq Ck \log^\gamma k, \tag{5.1}$$

$$\|\hat{f}_\theta(k)\| + \|\hat{f}_\theta(k)\| \leq Ck \log^\gamma k. \tag{5.2}$$
Then as $n \to \infty$, for any consistent sequence of solutions $\hat{\theta}_n$ to the likelihood equations $i_n(f_0) = 0$, the sequence $\sqrt{n}(\hat{\theta}_n - \theta_0)$ converges in distribution to the $N(0, V_0^{-1})$ distribution, for $V_0$ the $(d \times d)$ matrix given by

$$V_0 = \sum_{k=1}^{\infty} \frac{\hat{f}_{\theta_0}^T(k)p_{>k}^{(0)}}{\hat{f}_{\theta_0}(k)} - \left(\sum_{k=1}^{\infty} \frac{\hat{f}_{\theta_0}(k)p_{>k}^{(0)}}{\hat{f}_{\theta_0}(k)}\right)^2.
$$

(5.3)

In particular this is true for the maximum likelihood estimator if $\theta_0$ is identifiable and interior to $\Theta$.

**Corollary 5.2.** For the parametric family $f_{\alpha,\beta}(k) = (k + \alpha)^\beta$ with true parameter $(\alpha_0, \beta_0)$ in the interior of the parameter set $\Theta_\varepsilon = [-1 + \varepsilon, \varepsilon] \times [0, 1]$ for some small $\varepsilon > 0$, the maximum likelihood estimator $(\hat{\alpha}_n, \hat{\beta}_n)$ satisfies, as $n \to \infty$,

$$\sqrt{n}\left(\begin{bmatrix} \hat{\alpha}_n \\ \hat{\beta}_n \end{bmatrix} - \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}\right) \rightsquigarrow N(0, V_{\alpha,\beta}^{-1}),$$

where $V_{\alpha,\beta}$ is defined as the negative of $\ddot{i}(f_{\alpha,\beta})$ in (4.2).

**Theorem 5.3.** For the parametric family $f_{\alpha,\beta}(k) = (k + \alpha)^\beta$ with true parameter $(\alpha_0, 1)$ in the parameter set $\Theta_\varepsilon = (-1 + \varepsilon, \varepsilon^{-1}) \times [0, 1]$ for some small $\varepsilon > 0$, the sequence $T_n$ tends in distribution to $Z_1Z_{\leq 0}$, for $Z$ a standard normal variable.

### 6 A Remedy to the History Problem

A practical problem with the maximum likelihood estimator is that the log likelihood function (2.2) and its derivative (2.4) depend on the history of the network evolution. In many real-world applications observing the entire history is impossible or too costly, and only the final snapshot at time $n$ is available. For instance, when building a social network model, we may observe the final network, but recovering how it exactly evolved into its current shape is difficult—we would need to check with everyone when s/he became friends with everyone else and establish a strict time order.

This problem is solvable. The log likelihood (2.2) or its derivative (2.4) consist of two terms, and the history problem only arises in the second term, which results from the norming constant to the likelihood. The first terms in (2.2) or (2.4) depend on the network only through $P_{>k}(n)$ and hence are available from the final snapshot. The second terms
are the Césaro averages of \( \log S_{f_0}(t-1) \) and its derivative \( S_{f_0}(t-1)/S_{f_0}(t-1) \), respectively. Because \( S_h(t)/t \) tends to the limit \( \sum_{k=1}^{\infty} h(k)p_k \) almost surely, as \( t \to \infty \), these Césaro averages are asymptotically actually very close to \( \log S_{f_0}(n)/n \) and \( S_{f_0}(n)/S_{f_0}(n) \), which do depend only on the snapshot of the network at time \( n \). The remedy is to replace (2.2) or (2.4) by

\[
\dot{i}_n(f_0) = \sum_{k=1}^{\infty} \log f_0(k) P_{>k}(n) - \log S_{f_0}(n), \tag{6.1}
\]

\[
\dot{i}_n(\theta) = \sum_{k=1}^{\infty} \frac{\dot{f}_\theta(k)}{f_\theta(k)} P_{>k}(n) - \frac{S_{f_0}(n)}{S_{f_0}(n)}. \tag{6.2}
\]

Define a pseudo maximum likelihood estimator \( \tilde{\theta}_n \) as the maximizer of the first function or a zero of the second. Inspection of the proof of Theorem 4.1 readily shows that this pseudo maximum likelihood estimator is consistent under the same conditions as the maximum likelihood estimator.

**Theorem 6.1.** Under the conditions of Theorem 4.1, the pseudo maximum likelihood estimator \( \tilde{\theta}_n \) is consistent, i.e., as \( n \to \infty \), \( \tilde{\theta}_n \to \theta_0 \) almost surely, under \( \theta_0 \).

We have no proof of the asymptotic normality of the pseudo maximum likelihood estimator in the same generality as for the maximum likelihood estimator, but we note the following general theorem, and its corollary.

Let \( F_\theta(k) = \left( \sum_{j=1}^{k} \dot{f}_\theta(j), \dot{f}_\theta(k), f_\theta(k) \right)^T \).

**Theorem 6.2.** Assume the conditions of Theorem 4.1, and in addition assume that the sequence of random vectors \( \sum_{k=1}^{\infty} F_{\theta_0}(k) \sqrt{n} (P_k(n) - p_{k}^{(0)}) \) is asymptotically normal with mean zero and covariance \( W_0 \) and that \( \theta_0 \) is interior to \( \Theta \). Then the sequence of pseudo maximum likelihood estimators \( \tilde{\theta}_n \) satisfies \( \sqrt{n}(\tilde{\theta}_n - \theta_0) \sim N(0, V_0^{-1}VV_0^{-1}) \), under \( \theta_0 \) as \( n \to \infty \), where \( V \) is given in the proof below and \( V_0 \) is given in (5.3).

It is plausible that the sequence of empirical degrees \( \sqrt{n}(P_k(n) - p_{k}^{(0)}) \) is asymptotically normally distributed in some generality, but this has been established only for the affine PA function (see Móri (2002) and Resnick and Samorodnitsky (2016), or Proposition B.2). The preceding theorem requires that certain linear combinations of the variables over \( k \) are asymptotically normal, where the coefficients typically tend to infinity with \( k \) (e.g. at the order \( k \log k \)). Although this convergence requires additional bounds for large \( k \), the condition of the theorem seems plausible in general, for sublinear PA models.

In the appendix we verify the condition for the interesting case of PA functions that are eventually constant, for which the limiting degree distribution follows a power law with exponential cut-off (Rudas, Tóth and Valkó (2007)). This leads to the following corollary.

**Corollary 6.3.** In the model with PA functions \( f_\theta \) satisfying \( f_\theta(k) = f_\theta(k \land K) \), for some given \( K \in \mathbb{N}_+ \), for which the eigenvalue condition \( \text{Re } \lambda_2(A_K) < \lambda_1(A_K)/2 \) (as in the appendix) holds at \( \theta_0 \), the pseudo maximum likelihood estimator \( \tilde{\theta}_n \) satisfies \( \sqrt{n}(\tilde{\theta}_n - \theta_0) \sim N(0, V_0^{-1}VV_0^{-1}) \), provided that it is consistent at \( \theta_0 \).
The limiting covariance in the preceding theorem and corollary may be complicated. To get around this, we propose the following bootstrap procedure. Suppose that we observe the final snapshot of the network $G_n$ with $n$ nodes.

1. Obtain the pseudo maximum likelihood estimator $\tilde{\theta}_n$ based on $G_n$.
2. Given $\tilde{\theta}_n$, simulate PA networks $(G_m^{(i)})_{i=1}^s$ with PA function $f_{\tilde{\theta}_n}$, each with $m$ nodes.
3. Obtain the pseudo maximum likelihood estimator $\tilde{\theta}_m^{(i)}$ based on $G_m^{(i)}$, for $i \in [s]$.
4. Approximate the limit variance of $\sqrt{n}(\tilde{\theta}_n - \theta_0)$ by
   \[
   \tilde{\Sigma}_{m,s}(f_{\tilde{\theta}_n}) := m \left\{ \frac{1}{s} \sum_{i=1}^s \tilde{\theta}_m^{(i)} (\tilde{\theta}_m^{(i)})^T - \left( \frac{1}{s} \sum_{i=1}^s \tilde{\theta}_m^{(i)} \right) \left( \frac{1}{s} \sum_{i=1}^s \tilde{\theta}_m^{(i)} \right)^T \right\}.
   \]

This procedure will be consistent under a mild continuity condition on the model $\theta \mapsto f_\theta$.

Section 7 of Gao and van der Vaart (2017) on the affine PA model addressed a different history problem—the history of the initial degrees. In the present paper we study the case of fixed initial degree 1, but need to know how the incoming nodes connect, a problem that did not arise in the affine model. The empirical estimator considered in Gao et al. (2017) is curiously free of the history problem.

7 Connecting the empirical estimator and the pseudo maximum likelihood estimator

Write $f(k)$ as $\theta_k$ and suppose that we are interested in estimating the infinite-dimensional vector $(\theta_k)_{k=1}^\infty$. The pseudo log-likelihood function (6.1) then takes the form

\[
\hat{\tau}_n(\theta) = \sum_{k=1}^\infty (\log \theta_k) P_{>k}(n) - \log S_\theta(n),
\]

where $S_\theta(t) = \sum_{k=1}^\infty \theta_k N_k(t)$ is the total preference. Taking the derivative with respect to $\theta_k$, we obtain

\[
\frac{\partial \hat{\tau}_n(\theta)}{\partial \theta_k} = \frac{P_{>k}(n)}{\theta_k} - \frac{N_k(n)}{\sum_{j=1}^\infty \theta_j N_j(n)}.
\]

Setting this equation to zero for every $k$, we obtain the system of equations (for simplicity assume that $N_k(n) > 0$ for every $k$)

\[
\frac{\theta_k}{\sum_{j=1}^\infty \theta_j P_j(n)} = \frac{N_{>k}(n)}{N_k(n)}, \quad k \in \mathbb{N}_+.
\]

The right side is the aforementioned empirical estimator $\hat{r}_k(n)$, defined in Gao et al. (2017). The PA function $f$ or parameter $\theta_k$ is identifiable up to a scale factor only. We conclude that the empirical estimator is the pseudo maximum likelihood estimator if we do not impose any parametric assumption and wish to estimate $\theta_k$ individually for any $k$. 

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8 Numerical Illustrations

In this section we numerically study the performance of the maximum likelihood estimator, the pseudo maximum likelihood estimator, the empirical estimator, the Wald test and the bootstrap estimator. We simulated data using the following three examples of a PA function, each of the type \( f(k) = (k + \alpha)^\beta \):

\[
\begin{align*}
\text{f}^{(2)}(k) &= k^{2/3}, & \alpha &= 0, \beta = 2/3, \\
\text{f}^{(4)}(k) &= (k + 4)^{4/5}, & \alpha &= 4, \beta = 4/5, \\
\text{f}^{(5)}(k) &= k + 2, & \alpha &= 2, \beta = 1.
\end{align*}
\]

In every setting we conducted \( N = 1000 \) repetitions of the experiment in which we simulated a PA tree of \( n \) nodes, for varying \( n \), and computed the estimators and/or test. For every estimator \( \hat{\theta}_i \) of \( \theta = (\alpha, \beta) \) we computed the sample mean difference \((1/N) \sum_{i=1}^{N} (\hat{\theta}_i - \theta_0)\) and the rescaled sample covariance matrix \((n/N) \sum_{i=1}^{N} (\hat{\theta}_i - \theta_0)(\hat{\theta}_i - \theta_0)^T\). According to our theory, for large \( n \) the first should be close to zero, and the second should be close to the deterministic matrix \( V_0^{-1} \), for \( V_0 \) given in (5.3), which depends on the PA function.

8.1 MLE

The limiting covariance matrix \( V_0^{-1} \) of \( \sqrt{n}(\hat{\theta}_n - \theta_0) \) under the PA function \( f^{(2)} \) is computed to be

\[
\begin{pmatrix}
169.30 & 47.56 \\
47.56 & 14.94
\end{pmatrix}.
\]

Table 1 gives the results of our simulation experiments, where we simulated PA trees of three different sizes: \( n = 10^4 \), \( n = 10^5 \) or \( n = 10^6 \) nodes. As expected the sample mean difference decreases with \( n \), while the sample covariance swings around the expected limit.

| # of nodes | Sample Mean Difference | Sample Covariance |
|------------|-----------------------|-------------------|
| 10^4       | 7.29e-03  3.24e-05    | 173.19  48.14     |
|            |                       | 48.14  15.08      |
| 10^5       | 5.71e-04  -7.03e-05   | 167.89  47.23     |
|            |                       | 47.23  15.03      |
| 10^6       | 1.71e-04  -2.59e-05   | 163.02  46.60     |
|            |                       | 46.60  14.97      |

Table 1: Sample mean difference and covariance of the maximum likelihood estimator of the parameters \( \alpha \) and \( \beta \) of the PA function \( f(k) = (k + \alpha)^\beta \) for trees of three sizes of node sets \( n \) generated according to PA function \( f^{(2)} \).

We conducted the same experiment with PA functions \( f^{(4)} \) and \( f^{(5)} \), but for trees of a single size of \( n = 10^6 \) nodes. The results presented in Table 2 again confirm the theory.
Table 2: Sample mean difference and covariance of the maximum likelihood estimator of the parameters $\alpha$ and $\beta$ of the PA function $f(k) = (k + \alpha)^\beta$ for trees of $n = 10^6$ nodes generated according to the PA functions $f^{(4)}$ and $f^{(5)}$. The last column gives the covariance matrix $V_0^{-1}$.

| PA Function | Sample Mean Difference | Sample Covariance | Limit Variance |
|-------------|------------------------|-------------------|---------------|
| $f^{(4)}$   | -2.43e-02              | 42764.75          | 47249.33      |
|             | -2.72e-03              | 4743.46           | 4716.76       |
| $f^{(5)}$   | -5.35e-04              | 1817.94           | 1762.05       |
|             | -1.01e-04              | 325.66            | 316.58        |

8.2 Comparing the estimators

We compared the performance of the empirical estimator (4.3), the pseudo maximum likelihood estimator and the maximum likelihood estimator on samples of trees of $n = 10^6$ nodes generated using the PA function $f^{(2)}$. Table 3 gives numerical results, while Figure 1 shows QQ-plots of the estimators. The first line of Table 3 repeats the relevant (third) line of Table 1.

The comparison is particularly interesting as the theoretical variances of the empirical estimator and the pseudo maximum likelihood estimator are unknown. In our experiment the sample covariance of the pseudo maximum likelihood estimator is only twice bigger than that of the maximum likelihood estimator. In contrast, the sample covariance of the empirical estimator is larger than that of the other two estimators by an order of magnitude. We conclude that it helps to use a parametric model, if this can be correctly specified.

As is evident from Figure 1, the maximum likelihood estimator and pseudo maximum likelihood estimator are asymptotically normal. The same seems to be true for the empirical estimator, with the largest visible possible deviation in the left tail of its distribution.

8.3 The Wald-type test for affinity

We applied the Wald-type test for affinity given in Corollary 5.3 on PA trees generated using $f^{(4)}$ and $f^{(5)}$. The nominal size of the tests was set at 0.05. In both cases we registered the proportion of repetitions in which the null hypothesis was rejected. As $f^{(5)}$ is affine, rejection constitutes a type-I error in this case, while for $f^{(4)}$ the proportion of rejections is a measure of the test’s power. The results are summarized in Table 4. The expected proportion of type-I errors for $f^{(5)}$ was 5.2%, close to the nominal value. The power of the test at $f^{(4)}$ was outright 1, resulting from the fact that the number of nodes was large and $f^{(4)}$ is far from affine.
Table 3: Comparison of the maximum likelihood estimator, the empirical estimator and the pseudo maximum likelihood estimator for estimating the parameters $\alpha$ and $\beta$ of the PA function $f(k) = (k + \alpha)^\beta$ based on trees with $n = 10^6$ nodes generated according to PA function $f^{(2)}$.

| Estimator | Sample Mean Difference | Sample Covariance |
|-----------|------------------------|-------------------|
| MLE       | 1.71e-04 -2.59e-05     | 163.02 46.60      |
| EE        | 1.02e-03 2.58e-04      | 6840.42 2965.18   |
| PMLE      | -1.05e-03 -3.79e-04    | 297.11 85.40      |

(a) Estimators of $\alpha$

(b) Estimators of $\beta$

Figure 1: QQ-plots of 1000 realizations of the maximum likelihood estimator, the empirical estimator and the pseudo maximum likelihood estimator of the parameters $\alpha$ and $\beta$ of the PA function $f(k) = (k + \alpha)^\beta$ based on trees with $n = 10^6$ nodes generated according to PA function $f^{(2)}$. 

18
The pseudo maximum likelihood estimator with bootstrapped variance

We conducted two experiments to illuminate the bootstrap procedure for the pseudo maximum likelihood estimator in Section 6, applied to the model \( f(k) = (k + \alpha)^\beta \). In both cases the data was generated under the PA function \( f^{(2)} \), corresponding to \( \alpha = 0 \) and \( \beta = 2/3 \). Furthermore, the bootstrap sample size in (6.3) was set to \( m = 10^5 \) and the number of bootstrap replicates to \( s = 10^3 \).

In the first experiment we performed Wald type tests for the hypotheses \( H_0: \alpha = 0 \) and \( H_0: \beta = 2/3 \) using the relevant coordinate of the pseudo maximum likelihood estimator \((\tilde{\alpha}_n, \tilde{\beta}_n)\) with its variance estimated by the bootstrap estimator (6.3). In the case of the hypothesis \( H_0: \alpha = 0 \) this entails the test statistic

\[
T_{n,m,s} = \frac{n^{1/2} \tilde{\alpha}_n}{\sqrt{\Sigma_{m,s}(f_{0,\tilde{\beta}_n})_{1,1}}}.
\]

The test for \( H_0: \beta = 2/3 \) is similarly based on a standardized version of \( \tilde{\beta}_n \). We rejected the null hypothesis when \( |T_{n,m,s}| > z_{0.025} \), corresponding to the working hypothesis that \( T_{n,m,s} \sim N(0, 1) \) under the null hypothesis and nominal size 0.05. Table 5 gives the proportions of rejections in \( N = 1000 \) repetitions of the experiment.

| \( H_0 \)  | Type-I error |
|---------|-------------|
| \( \alpha = 0 \) | 0.047       |
| \( \beta = 2/3 \) | 0.063       |

Table 5: Proportions of rejections of the size-0.05 Wald-type tests with bootstrapped variance of the null hypotheses \( H_0: \alpha = 0 \) and \( H_0: \beta = 2/3 \) on the PA function \( f(k) = (k + \alpha)^\beta \) for trees with \( n = 10^9 \) nodes generated according to PA function \( f^{(2)} \). The bootstrap sample size was set to \( m = 10^5 \) and the number of bootstrap replicates to \( s = 10^3 \).

In the second experiment we simulated \( N = 1000 \) replicates of the normalized and projected pseudo maximum likelihood estimator

\[
d_n = r_n^T \sqrt{n(\Sigma_{m,s}(f_{\tilde{\alpha}_n,\tilde{\beta}_n}))^{-1/2} (\tilde{\alpha}_n, \tilde{\beta}_n)}.
\]
where \( r_n \) are i.i.d. random vectors from the unit circle. Figure 2 shows a QQ-plot of these 1000 values against the standard normal distribution.

Figure 2: QQ-plot of 1000 realizations of the randomly projected pseudo maximum likelihood estimators in the model \( f(k) = (k + \alpha)^\beta \) normalized by their bootstrapped variances based on trees with \( n = 10^6 \) nodes generated according to \( f(2) \). The bootstrap sample size was set to \( m = 10^5 \) and the number of bootstrap replicates to \( s = 10^3 \).

Both experiments support the practical use of asymptotic normal limit theory for the pseudo maximum likelihood estimator and validate the bootstrap procedure for estimating the asymptotic variance.

9 Proofs of main results

Proof of Proposition 3.1. We apply the general results due to Jagers (1975) and Nerman (1981), as summarized in the appendix, Section A.

The events of the pure birth process (3.1) can be represented as \( T_1 < T_1 + T_2 < T_1 + T_2 + T_3 < \cdots \), for independent exponential random variables \((T_k)_{k=1}^{\infty}\) with rates \((f(k))_{k=1}^{\infty}\). The total number of births \( \xi(t) = \int 1_{(0,t]}(u) \xi(du) \) at time \( t \) is equal to \( \sum_{i=1}^{\infty} 1_{(0,t]}(T_1 + \cdots + T_i) \), which tends to infinity almost surely as \( t \to \infty \) by the assumption that the PA function is bounded from below by \( f(1) > 0 \). The birth times are clearly not restricted to any lattice, and the functions \( t \mapsto \mathbb{E}[\varphi_i(t)] \) are continuous almost everywhere in view of their monotonicity. Thus, it suffices to show that the Malthusian parameter exists and to verify conditions (A.9) and (A.10).
Since \( f e^{-\lambda u} \xi(du) = \sum_{l=1}^{\infty} e^{-\lambda(T_{1}+\cdots+T_{l})} \),

\[
\mathbb{E}\left[ \int_{0}^{\infty} e^{-\lambda u} \xi(du) \right] = \mathbb{E}\left[ \sum_{l=1}^{\infty} e^{-\lambda(T_{1}+\cdots+T_{l})} \right] = \sum_{l=1}^{\infty} \prod_{i=1}^{l} \frac{f(i)}{\lambda + f(i)}.
\]

\[
\mathbb{E}\left[ \int_{0}^{\infty} e^{-\lambda u} \xi(du) \right]^{2} = \mathbb{E}\left[ \sum_{l=1}^{\infty} e^{-\lambda(T_{1}+\cdots+T_{l})} \right]^{2} = \sum_{k=1}^{\infty} \sum_{l=1}^{k} \mathbb{E}\left[ e^{-\lambda(T_{1}+\cdots+T_{k})-\lambda(T_{1}+\cdots+T_{l})} \right]
\]

\[
= \rho_{f}(2\lambda) + 2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{k} \prod_{i=k+1}^{l} \frac{f(i)}{2\lambda + f(i)} \prod_{i=k+1}^{l} \frac{f(i)}{\lambda + f(i)}.
\]

The left side of the first formula is \( f_{0}^{\infty} e^{-\lambda u} \mu(du) = \rho_{f}(\lambda) \), and hence this formula verifies equation (3.2).

First consider the strictly sublinear case, where \( f(k) \leq Ck^{\beta} \). Because both expressions in the display are monotone in \( f \), we obtain upper bounds on these expressions by evaluating their right sides for \( f(k) = Ck^{\beta} \). Because \( \log(1 + x) \geq x/2 \) for \( x \in [0, 1] \), we have \( f(i)/(\lambda + f(i)) \leq \exp(-\lambda/(2f(i))) \), for \( i \geq I_{\lambda} := (\lambda/C)^{1/\beta} \). Since the quotients and their products are also bounded by 1,

\[
\sum_{l=1}^{\infty} \prod_{i=1}^{l} \frac{f(i)}{\lambda + f(i)} \leq I_{\lambda} + \sum_{l>I_{\lambda}} e^{-\sum_{i=1}^{l} \lambda/(2C_{i}^{\beta})}.
\]

Because \( \beta < 1 \), the sum in the exponent is of the order \( l^{1-\beta} \) as \( l \to \infty \), and hence the series is finite for every \( \lambda > 0 \). The function \( \rho_{f} \) is monotone decreasing and tends to zero as \( \lambda \to \infty \), by the dominated convergence theorem, and to infinity as \( \lambda \downarrow 0 \), by the monotone convergence theorem. It follows that the Malthusian parameter \( \lambda^{\ast} \) exists and is contained in \((0, \infty) \). We also conclude that (A.9) is satisfied, for any \( \lambda \in (0, \lambda^{\ast}) \).

The second moment of \( f_{0}^{\infty} e^{-\lambda u} \xi(du) \) is also finite for every \( \lambda > 0 \). To see this, it suffices to bound the double sum on the right. By the same arguments, for \( k > I_{\lambda} \),

\[
\sum_{l=k+1}^{\infty} \prod_{i=k+1}^{l} \frac{f(i)}{\lambda + f(i)} \leq \sum_{l=k+1}^{\infty} e^{-\sum_{i=k+1}^{l} \lambda/(2C_{i}^{\beta})} \leq \sum_{l=k+1}^{\infty} e^{-(\lambda/2C(1-\beta))(l^{1-\beta}-(k+1)^{1-\beta})}.
\]

This is bounded from above by a constant for every \( k \) (in fact by a multiple of \( k^{-\beta} \)). Inserting this bound in the double sum, we are left with a single sum of the same type as above, except that \( \lambda \) is replaced by \( 2\lambda \). We conclude that the second moment of \( f_{0}^{\infty} e^{-\lambda u} \xi(du) \) is finite for every \( \lambda > 0 \) as well.

We have \( e^{-\lambda t} \xi(t) = f_{0}^{t} e^{-\lambda t} \xi(du) \leq f_{0}^{t} e^{-\lambda u} \xi(du) \), for every \( t \) and \( \lambda > 0 \). Combined with the assumption \( \phi_{i}(t) \leq C \xi(t)^{2} \), we see that \( e^{-\lambda t} \phi_{i}(t) \leq C \left[ f_{0}^{t} e^{-\lambda u/2} \xi(du) \right]^{2} \). It follows that

\[
\mathbb{E}\sup_{t>0} \left( e^{-\lambda t} \phi_{i}(t) \right) \leq C \mathbb{E} \left[ f_{0}^{\infty} e^{-\lambda u/2} \xi(du) \right]^{2} < \infty, \text{ for every } \lambda > 0, \text{ thus verifying (A.10)}.
\]

In the case that \( f(k) = k + \alpha, \text{ for } \alpha > -1 \), the function \( \rho_{f} \) takes the form \( \rho_{f}(\lambda) = (1 + \alpha)/(\lambda - 1) \) (see Rudas, Tóth and Valkó (2007)) and hence the Malthusian parameter is \( \lambda^{\ast} = 2 + \alpha \) and the function \( \rho_{f} \) is finite for \( \lambda > 1 \). We conclude that (A.9) is satisfied. We show below that the second moment of \( f_{0}^{\infty} e^{-\lambda u} \xi(du) \) is finite for every \( \lambda > 1 \) as well. If \( \phi_{i}(t) \leq C \xi(t)^{r}, \text{ then } \sup_{t>0} e^{-\lambda t} \phi_{i}(t) \leq C \left[ f_{0}^{\infty} e^{-\lambda u} \xi(du) \right]^{r}, \text{ which then has a} \)
finite moment for any \( \lambda \) such that \( \lambda/r > 1 \) for some \( r \leq 2 \). Since \( r \) can be chosen such that \( r \leq 2 \) and \( r < 2 + \alpha \), where \( 2 + \alpha > 1 \), there exists \( \lambda \in (1, \lambda^*) \) that satisfies these restrictions, and hence (A.10) follows.

For \( i \to \infty \), we have \( f(i)/(\lambda + f(i)) = \exp(-\log(1 + \lambda/(i + \alpha)) \asymp \exp(-\lambda/(i + \alpha)) \).

Hence, for \( k \to \infty \), and \( \lambda > 1 \),

\[
\sum_{l=k+1}^{\infty} \prod_{i=k+1}^{l} \frac{f(i)}{\lambda + f(i)} \asymp \sum_{l=k+1}^{\infty} e^{-\lambda \log \frac{l+\alpha}{l+\alpha}} = \sum_{l=k+1}^{\infty} \left( \frac{k + \alpha}{l + \alpha} \right)^{\lambda} \asymp k,
\]

\[
\prod_{i=1}^{k} \frac{f(i)}{2\lambda + f(i)} \asymp e^{-2\lambda \log(k+\alpha)} \asymp k^{-2\lambda}.
\]

Therefore, the double sum in the second moment of \( f_0^{\infty} e^{-\lambda u} \xi(du) \) is bounded by a multiple of \( \sum_{k=1}^{\infty} k^{-(2\lambda-1)} \), which is finite for \( \lambda > 1 \).

**Proof of Corollary 3.2.** The random characteristics \( \varphi_1(t) = h(\xi(t) + 1) \) and \( \varphi_2(t) = 1_{\{t \geq 0\}} \) satisfy the conditions of Proposition 3.1, and hence \( Z_t^{\varphi_1}/Z_t^{\varphi_2} \) converges almost surely to the limit given in the proposition, as \( t \to \infty \). If \( \tau_n \) is the first time that the PA tree \( T_n \) possesses \( n \) nodes, then \( \tau_n \to \infty \) almost surely, as the total number of individuals at any given \( t \) is finite, almost surely. (At any given (finite) time, every individual has a finite number of offspring and there can be at most finitely many generations, since a new generation can be formed not faster than an exponential waiting time with mean \( 1/f(1) \).

Hence, the sequence \( Z_{\tau_n}^{\varphi_1}/Z_{\tau_n}^{\varphi_2} \) converges to the same limit, as \( n \to \infty \).

The process \( Z_{\tau_n}^{\varphi_2} \) simply counts the number of nodes \( v \) in the tree at time \( t \), and hence \( Z_{\tau_n}^{\varphi_2} = n \). Since \( \xi(1) = 1 \) is the degree of node \( v \) at time \( t \), the quotient can be rewritten as

\[
\frac{Z_{\tau_n}^{\varphi_1}}{Z_{\tau_n}^{\varphi_2}} = \frac{1}{n} \sum_{v \in T_n} h(\deg(v, T_n)) = \frac{1}{n} \sum_{k=1}^{\infty} h(k) N_k(n) = \sum_{k=1}^{\infty} h(k) P_k(n).
\]

This is the left side of the corollary, and it remains to identify the limit given in Proposition 3.1 as its right side. The latter is equal to

\[
\frac{\int_{0}^{\infty} e^{-\lambda t} \Pi h(\xi(t) + 1) \, dt}{\int_{0}^{\infty} e^{-\lambda t} \Pi e_{\{t \geq 0\}} \, dt} = \frac{\sum_{k=1}^{\infty} h(k) \lambda \int_{0}^{\infty} e^{-\lambda t} \Pi (\xi(t) + 1 = k) \, dt}{\lambda \int_{0}^{\infty} e^{-\lambda t} \Pi \, dt}.
\]

The denominator is simply 1 and the numerator is \( \sum_{k=1}^{\infty} h(k) p_k \) by identifying \( p_k \) from Equation (17) in Gao et al. (2017).

**Proof of Theorem 4.1.** Recall the notation \( \iota_n(f_\theta) \) for the scaled log likelihood in (2.2). We show below that \( \sup_{\theta} |\iota_n(f_\theta) - \iota(f_\theta) - c_n| \to 0 \), almost surely, for \( c_n = n^{-1} \sum_{t=2}^{n} \log(t - 1) \) and

\[
\iota(f_\theta) = \sum_{k=1}^{\infty} \left( \log f_\theta(k) \right) p_{\theta,k}^{(0)} - \log \left( \sum_{k=1}^{\infty} f_\theta(k) p_{\theta,k}^{(0)} \right).
\]

Because \( \hat{\theta}_n \) maximizes \( \theta \mapsto \iota_n(f_\theta) \), it next suffices to show that the limit function \( \theta \mapsto \iota(f_\theta) \) possesses \( \theta_0 \) as a well-separated point of maximum, in the sense of van der Vaart (2000),
Theorem 5.7. Since the function $\theta \mapsto \iota(f_\theta)$ is continuous on the compact set $\Theta$, this is equivalent to showing that $\theta_0$ is a unique point of maximum.

Consider the probability distributions $q^\theta = (q^\theta_k)_{k=1}^\infty$ on $\mathbb{N}_+$ defined by

$$q^\theta_k = \frac{1}{c(\theta)} f_{\theta_0}(k) p^\theta_{>k},$$

(9.1)

where the norming constant satisfies

$$c(\theta) = \sum_k f_{\theta_0}(k) p^\theta_{>k} = \sum_k f_{\theta}(k) p^\theta_{>k},$$

by (3.8). In particular $c(\theta_0) = 1$ and hence $q^\theta_{k} = p^\theta_{>k}$. The Kullback–Leibler divergence of $q^\theta$ relative to $q^{\theta_0}$ can be seen to be equal to $\iota(f_{\theta_0}) - \iota(f_\theta)$, and is strictly positive unless $q^\theta = q^{\theta_0}$. The latter is equivalent to $f_\theta \propto f_{\theta_0}$, which is excluded by the identifiability assumption.

We finish by proving the uniform convergence, where we show that the two components of $\iota_n(f_\theta)$ converge to the two components of $\iota(f_\theta)$. By Fubini’s theorem the difference of the first components satisfies

$$\left| \sum_{k=1}^n \log f_{\theta}(k)(P_{>k}(n) - p^\theta_{>k}) \right| = \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} \log f_{\theta}(k)(P_{j}(n) - p^\theta_{j}) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} \log f_{\theta}(k) \left| p^\theta_{j} - P_{j}(n) \right|.$$

Because $\log f_{\theta}(j) \leq C \log k$ and $|u| = u + 2u_-$, for every $u \in \mathbb{R}$, this is bounded above by a multiple of

$$\sum_{j=1}^{\infty} j \log j \left( p^\theta_{j} - P_{j}(n) \right) + 2 \sum_{j=1}^{\infty} j \log j \left( p^\theta_{j} - P_{j}(n) \right)_-.$$

The first series on the left tends to zero almost surely by Corollary 3.2 applied with the function $h: j \mapsto j \log j$. The terms of the second series are bounded from above by $j \log j \left( p^\theta_{j} - P_{j}(n) \right)$ and tend to zero almost surely for every $j$ as $n \to \infty$ by Corollary 3.2 applied with $h: k \mapsto 1_{\{k\}}$. Furthermore, the series $\sum_j j \log j p^\theta_{j}$ converges. Hence, the second series tends to zero almost surely by the dominated convergence theorem. The two series give an upper bound independent of $\theta$ and hence the supremum over $\theta$ of the left side of the second last equation tends to zero almost surely.

For the second component of $\iota_n(f_\theta)$, we first note that for some constant $C$ not depending on $\theta$,

$$\left| \frac{S_{f_{\theta}}(n)}{n} \right| = \sum_{k=1}^{\infty} f_{\theta}(k) p^\theta_{k} \leq C \sum_{k=1}^{\infty} k \left| p^\theta_{k} - P_{k}(n) \right|.$$

The right side tends to zero almost surely as $n \to \infty$, as in preceding paragraph. It follows that the supremum over $\theta$ of the left side tends to zero almost surely. The limit
\[ \sum_k f_\theta(k)p_k^{(0)} \] is a continuous, positive function and hence is bounded away from 0. By the continuous mapping theorem,

\[
\sup_\theta \left| \log \frac{S_n(t)}{n} - \log \left( \sum_{k=1}^{\infty} f_\theta(k)p_k^{(0)} \right) \right| \xrightarrow{n \to \infty} 0.
\]

By Lemma 9.1 the Cesàro’s sums \( n^{-1} \sum_{t=2}^{n} \log(S_f(t-1)/(t-1)) = n^{-1} \sum_{t=1}^{n} \log S_f(t) - c_n \) have the same limit. □

**Lemma 9.1** (Uniform Cesàro convergence for processes). If \( Z_\theta(1), Z_\theta(2), \ldots \) are bounded stochastic processes such that \( \sup_\theta |Z_\theta(n) - Z_\theta| \to 0 \) almost surely, for some process \( Z_\theta \), then the Cesàro means \( \bar{Z}_\theta(n) = n^{-1} \sum_{t=1}^{n} Z_\theta(t) \) of \( Z_\theta(n) \) satisfy \( \sup_\theta |\bar{Z}_\theta(n) - Z_\theta| \to 0 \), almost surely.

**Proof of Lemma 9.1.** For every \( m \) the difference \( |\bar{Z}_\theta(n) - Z_\theta| \) is bounded from above by

\[
\frac{1}{n} \sum_{t=m+1}^{n} |Z_\theta(t) - Z_\theta|.
\]

For every \( \varepsilon > 0 \) there exists \( m \) so that every term in the second sum is bounded from above by \( \varepsilon \) and hence this sum divided by \( n \) is bounded by \( \varepsilon \). The first term tends to zero as \( n \to \infty \), for every fixed \( m \). This argument is true also after taking the supremum over \( \theta \) across.

**Proof of Lemma 4.3.** Since \( \sum_{k=1}^{\infty} p_k = 1 = \sum_{k=1}^{\infty} q_k \), we have \( \sum_{k=1}^{K} (q_k - p_k) = 0 \), where the terms of the sums are nonnegative by assumption. The strict monotonicity of \( v_k \) gives \( \sum_{k=1}^{K} (q_k - p_k)v_k < \sum_{k=K+1}^{\infty} (p_k - q_k)v_k \). Rearranging the terms gives the desired result. □

**Proof of Lemma 4.4.** Since both sequences sum to 1, it is impossible that \( p_k > q_k \) for every \( k \in \mathbb{N}_+ \). If \( (w_k)_{k=1}^{\infty} \) is strictly increasing and \( p_j \leq q_j \), then

\[
\frac{q_{j+1}}{p_{j+1}} = \frac{w_{j+1}}{\sum_{k=1}^{\infty} p_k w_k} < \frac{w_j}{\sum_{k=1}^{\infty} p_k w_k} = \frac{q_j}{p_j} \geq 1.
\]

By mathematical induction, \( p_k < q_k \) for every \( k > j \). Then \( p_k < q_k \) for any \( k > K \) and \( K+1 \) the smallest value \( j \) with \( p_j < q_j \) (which cannot be \( j = 1 \)).

In the case that \( (w_k)_{k=1}^{\infty} \) is strictly decreasing, the sequence \( w_k^{-1} \) is strictly increasing, and we apply the preceding argument with \( p_k = q_k w_k^{-1} / \sum_{j} q_j w_j^{-1} \) and the roles of \( p_k \) and \( q_k \) swapped. □

**Proof of Lemma 4.5.** A more illustrative view of (4.1) is as follows:

\[
\begin{align*}
i(f_\theta) &= \sum_{k=1}^{\infty} \frac{f_\theta(k)}{f_\theta} p^{(0)}_{k} - \sum_{k=1}^{\infty} \frac{p^{(0)}_{k} f_\theta(k)}{\sum_{1}^{\infty} p^{(0)}_{j} f_\theta(j)} \frac{f_\theta(k)}{f_\theta} \\
&= \sum_{k=1}^{\infty} \frac{p^{(0)}_{k} f_\theta(k)}{f_\theta} - \sum_{k=1}^{\infty} \frac{q^{(0,\theta)} f_\theta(k)}{f_\theta},
\end{align*}
\]

(9.2)
where }q^{(0,\theta)}_k \propto p^{(0)}_k f_\theta(k)/f_{0\theta}(k)\text{ is the probability distribution generated by reweighting } (p^{(0)}_k)_{k=1}^\infty \text{ with } (f_\theta/f_{0\theta}(k))_{k=1}^\infty.

Fix any } \theta \in \Theta' \text{ and assume that } \theta \text{ renders } f_\theta/f_{0\theta} \text{ decreasing. By Lemma 4.4 applied with weights } w_k = f_\theta/f_{0\theta}(k), \text{ there exists } K \text{ such that } p^{(0)}_{>k} \leq q^{(0,\theta)}_k \text{ for } k \leq K \text{ and } p^{(0)}_{>k} > q^{(0,\theta)}_k \text{ for } k > K. \text{ Then Lemma 4.3 with } v_k = f_\theta/f_{0\theta}(k) \text{ (understood component-wise) and the probability distributions } p^{(0)}_k \text{ and } q^{(0,\theta)}_k, \text{ shows that } i(f_{\theta}) < 0. \text{ If } f_\theta/f_{0\theta}(k) \text{ is increasing, then the same argument applies, but we find that } i(f_{\theta}) > 0. \quad \blacksquare

For reference, we state the martingale central limit theorem—a version of Theorem 3.2 of Hall and Heyde (2014).

**Proposition 9.2.** Suppose that } X_t \text{ is a martingale difference series relative to the filtration } F_t. \text{ If as } n \to \infty, n^{-1} \sum_{i=1}^n E[X_i^2|F_{t-1}] \xrightarrow{P} v \text{ for a positive constant } v \text{ and } n^{-1} \sum_{i=1}^n E[X_i^2 I_{|X_i|>\sqrt{n}}|F_{t-1}] \xrightarrow{P} 0 \text{ for every } \varepsilon \in \mathbb{R}, \text{ then } \sqrt{n}X_n \xrightarrow{D} N(0, v).

**Proof of Theorem 5.1.** By Theorem 4.1 the maximum likelihood estimator } \hat{\theta}_n \text{ is consistent if } \theta_0 \text{ is identifiable. The } \hat{\theta}_n \text{ will eventually be interior to the parameter set if } \theta_0 \text{ is interior, and hence satisfy the system of likelihood equations } i_n(f_{\theta_n}) = 0. \text{ Thus, the second assertion of the theorem follows from the first.}

By a Taylor expansion of the } i^\text{th} \text{ of the likelihood equations, it can be expanded as } 0 = i_n(f_{\theta_n})_i + i_n(f_{}\theta_n)_i(\hat{\theta}_n - \theta_0), \text{ for } \theta'_{n,i} \text{ on the line segment between } \hat{\theta}_n \text{ and } \theta_0 \text{ and } i_n(f_{\theta}), \text{ the } i^\text{th} \text{ row of the second derivative matrix } i_n(f_{\theta}). \text{ Thus, } i_n(f_{\theta_n})(\hat{\theta}_n - \theta_0) = -i_n(f_{\theta_0}), \text{ where } \theta'_{n,i} \text{ is understood to be the } (d \times d) \text{ matrix with } i^\text{th} \text{ row } \theta'_{n,i}, \text{ even though the vector } \theta'_{n,i} \text{ may be different for different } i. \text{ The proof can be concluded by showing that } \sqrt{n}i_n(f_{\theta_n}) \xrightarrow{D} N(0, V_0) \text{ and that } i_n(f_{\theta_n}) \to -V_0 \text{ in probability.}

As noted in Section 2, the sequence } n_i(f_{\theta_0}) \text{ is a (vector-valued) martingale. The asymptotic normality can be obtained from the martingale central limit theorem (see Proposition 9.2). Because } P_{\theta}(D_t = k|F_{t-1}) = f_{\theta}(k)N_k(t-1)/S_{\theta}(t-1), \text{ the martingale differences } (\hat{f}_{\theta}/f_{\theta_0})(D_t - E_{\theta_0}((\hat{f}_{\theta}/f_{\theta_0})(D_t)|F_{t-1}) \text{ possess conditional covariances }

\[ \Sigma_t := \sum_{k=1}^{\infty} \left( \frac{\hat{f}_{\theta}}{f_{\theta_0}}(k) \right) \left( \frac{\hat{f}_{\theta}}{f_{\theta_0}}(k) \right)^T \frac{f_{\theta_0}(k)P_k(t-1)}{S_{\theta_0}(t-1)/(t-1)} - \left( \sum_{k=1}^{\infty} \frac{\hat{f}_{\theta}}{f_{\theta_0}}(k) \right) \left( \sum_{k=1}^{\infty} \frac{\hat{f}_{\theta}}{f_{\theta_0}}(k) \right)^T \frac{f_{\theta_0}(k)P_k(t-1)}{S_{\theta_0}(t-1)/(t-1)}. \]

As seen in the proof of Theorem 4.1, the sequence } S_{\theta_0}(t)/t \text{ tends almost surely to } \sum_j f_{\theta_0}(j)p_{j}^{(0)}, \text{ as } t \to \infty. \text{ Corollary 3.2 applied with } h \text{ equal to the entries of the matrix } \hat{f}_{\theta}/f_{\theta_0} \text{ or the vector } \hat{f}_{\theta}, \text{ shows that the preceding display tends almost surely to}

\[ \sum_{k=1}^{\infty} \frac{\hat{f}_{\theta}}{f_{\theta_0}}(k) \frac{p_{k}^{(0)}}{\sum_j f_{\theta_0}(j)p_{j}^{(0)}} - \left( \sum_{k=1}^{\infty} \frac{\hat{f}_{\theta}}{f_{\theta_0}}(k) \right) \left( \sum_{k=1}^{\infty} \frac{\hat{f}_{\theta}}{f_{\theta_0}}(k) \right)^T \frac{\sum_j f_{\theta_0}(j)p_{j}^{(0)}}{\sum_j f_{\theta_0}(j)p_{j}^{(0)}}. \]

In view of equation (3.8), this is equal to the matrix } V_0. \text{ The averages } n^{-1} \sum_{t=2}^n \Sigma_t \text{ of the conditional covariances tend to the same limit, by Lemma 9.1.
Because $D_t \leq t$, we bound, using (5.2),

$$\left\| \frac{\dot{f}_{\theta_0}}{f_{\theta_0}} (D_t) \right\| \leq C \log^\gamma t \leq C \log^\gamma n, \quad t \leq n.$$ 

As this is smaller than $\epsilon \sqrt{n}$, eventually for every $\epsilon > 0$, the conditional Lindeberg condition is trivially satisfied. We conclude that $\sqrt{n} i_n(f_{\theta_0}) \rightsquigarrow N(0, V_0)$, by the martingale central limit theorem, for instance, Proposition 9.2.

The Hessian matrix $i_n(\theta)$ takes the form

$$i_n(f_{\theta}) = \sum_{k=1}^{\infty} \left( \frac{\dot{f}_{\theta}}{f_{\theta}} - \frac{\dot{f}_{\theta} f_{\theta}^T}{f_{\theta}^2} \right)(k)P_{\theta_k}(n) - \frac{1}{n} \sum_{t=2}^{n} \left( \frac{S_{f_{\theta}}}{S_{f_{\theta}}^2 - S_{f_{\theta}} f_{\theta} f_{\theta}^T} \right)(t - 1)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} \left( \frac{\dot{f}_{\theta}}{f_{\theta}} - \frac{\dot{f}_{\theta} f_{\theta}^T}{f_{\theta}^2} \right)(k)P_{\theta_j}(n) - \frac{1}{n} \sum_{t=2}^{n} \left( \frac{S_{f_{\theta}}}{S_{f_{\theta}}^2 - S_{f_{\theta}} f_{\theta} f_{\theta}^T} \right)(t - 1). \quad (9.3)$$

Using Corollary 3.2, the first term in the second line can be shown to converge as $n \to \infty$ to the expression obtained by replacing $P_j(n)$ by $P_j^{(0)}(n)$, or equivalently replacing $P_{\theta_k}(n)$ by $P_{\theta_k}^{(0)}(n)$ in the first line. For the second term we first note that $S_f(t)/t$ tends almost surely to $\sum_j f(j)p^{(0)}_j$, as $t \to \infty$, for $f$ equal to $\dot{f}_{\theta}$, $\ddot{f}_{\theta}$ or $f_{\theta}$. By the continuous mapping theorem the terms of the sum converge to the corresponding limit. The second term then converges almost surely to the same limit, in view of Lemma 9.1, still uniformly in $\theta$. By arguments similar to those in the proof of Theorem 4.1, the convergences of both terms can be seen to be uniform in $\theta$.

Finally, the continuity of the limit and consistency of $\hat{\theta}_n$ for $\theta_0$ give that the $i_n(f_{\theta_n})$ tends to the limit evaluated at $\theta_0$. This can be seen to be equal to the matrix $-V_0$ with the help of (3.8), where the two terms involving $\ddot{f}_{\theta}$ cancel each other. \[\square\]

**Proof of Theorem 5.3.** Because the true parameter is on the boundary of the parameter set, the maximum likelihood estimator may not solve the likelihood equations. Instead, we use its characterization as the maximizer of the log likelihood. The log likelihood evaluated at the parameter $\theta_0 + h/\sqrt{n}$ satisfies

$$\ell_n(f_{\theta_0 + h/\sqrt{n}}) - \ell_n(f_{\theta_0}) = h^T \sqrt{n} i_n(f_{\theta_0}) + \frac{1}{2} h^T i_n(f_{\theta_n(h)}) h,$$

where $\theta_n(h)$ is on the line segment between $\theta_0$ and $\theta_0 + h/\sqrt{n}$. The rescaled maximum likelihood estimator $\hat{\theta}_n = \sqrt{n}(\hat{\theta}_n - \theta_0)$ maximizes this process over the set $H_n$ of all $h = \sqrt{n}(\theta - \theta_0)$ such that $\theta = (\alpha, \beta)$ belongs to the parameter set $\{(\alpha, \beta) : \alpha \in (-1 + \epsilon, \epsilon), \beta \leq 1\}$. Since $\hat{\theta}_n$ is consistent for $\theta_0$, by Theorem 4.1, the set $H_n$ can be further reduced to a set such that $\|h\| < \sqrt{n} \delta_n$, for some $\delta_n \to 0$. By the arguments in the proof of Theorem 5.1, we have $\sup_{h \in H_n} |\hat{i}_n(f_{\theta_n(h)}) - i(f_{\theta_0})| \to 0$, in probability. Combined with the continuity of $\theta \mapsto i(f_{\theta})$, we see that $\sup_{h \in H_n} |\hat{i}_n(f_{\theta_n(h)}) - i(f_{\theta_0})| \to 0$, almost surely. From the nonsingularity and negative definiteness of $-V_0 = i(f_{\theta_0})$, we then see that $h^T \hat{i}_n(f_{\theta_n(h)}) h < -c\|h\|^2$, for every $h \in H_n$ and some $c > 0$, with probability tending to one. Using that $0 \in H_n$ and $h^T \sqrt{n} i_n(f_{\theta_0}) \leq \|h\|O_P(1)$, we conclude that $\hat{\theta}_n = O_P(1)$.
Next the argmax continuous mapping theorem (e.g. Corollary 5.58 and Lemma 7.13 in van der Vaart (2000)) shows that

\[ \hat{h}_n = \arg\max_{h \in H_n} \left( h^T \sqrt{n} i_n(f_{\theta_0}) + \frac{1}{2} h^T \dot{i}_n(f_{\theta_0}(h)) h \right) \rightsquigarrow \arg\max_{h \in H} (h^T Z_0 - \frac{1}{2} h^T V_0 h), \]

for \( H = \{(h_1, h_2); h_2 \leq 0\} \) the limit of the sequence of sets \( H_n \) and \( Z_0 \sim N(0, V_0) \) the limit in distribution of the sequence \( \sqrt{n} i_n(f_{\theta_0}) \). The right side has the claimed distribution, by Lemma 9.3 (where we set \( V = V_0) \). 

**Lemma 9.3.** If \( \hat{h} = \arg\max_{h:a^T h \leq 0} (2h^T Z - h^T V_0 h) \) for \( a \in \mathbb{R}^d \) and a random variable \( Z \sim N_d(0, V) \) for positive semi-definite matrices \( V \) and \( V_0 \), then \( a^T \hat{h} \sim W_{1W \leq 0} \) for \( W \sim N(0, a^T V_0^{-1} V_0^{-1} a) \).

**Proof of Lemma 9.3.** Since \( 2h^T Z - h^T V_0 h = -\|V_0^{-1/2} h - V_0^{-1/2} Z\|^2 + Z^T V_0^{-1} Z \), the variable \( \hat{h} \) can be seen to be equal to \( V_0^{-1/2} \Pi_G(V_0^{-1/2} Z) \), for \( \Pi_G \) the projection onto the half space \( G: = V_0^{1/2} \{h: a^T h \leq 0\} = \{g: b^T g \leq 0\} \), for \( b = V_0^{-1/2} a \). Therefore, \( a^T \hat{h} = b^T \Pi_G(\hat{W}) = 1_{b^T \hat{W} \leq 0} b^T \hat{W} \), for \( \hat{W} = V_0^{-1/2} Z \sim N_d(0, V_0^{-1/2} V_0^{-1/2}) \). The proof is complete by setting \( W = b^T \hat{W} = a^T V_0^{-1} Z \).

**Proof of Theorem 6.2.** Since \( \sum_{k=1}^{\infty} a_k P_{>k}(n) = \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} a_k P_j(n) \), we can write \( i_n(f_{\theta_0}) = \phi(S_n) \), for \( S_n = \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} F_{\theta_0}(k) P_j(n) \), and \( \phi: \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) given by \( \phi(s_1, s_2, s_3) = s_1 - \frac{1}{s_3} s_2 \). Similarly, it follows from (3.8) that \( \phi(S^{(0)}) = 0 \), for \( S^{(0)} = \sum_{k=1}^{\infty} F_{\theta_0}(k) P_k^{(0)} \). The assumption on \( S_n \) and the delta-method then give that the sequence \( \sqrt{n} i_n(f_{\theta_0}) - \sqrt{n} \left( \phi(S_n) - \phi(S^{(0)}) \right) \) tends in \( \mathbb{R}^d \) in distribution to a normal distribution with mean zero and covariance matrix \( V = (I, -I/S_3^{(0)}, S_2^{(0)}/(S_3^{(0)})^2) W_0 (I, -I/S_3^{(0)}, S_2^{(0)}/(S_3^{(0)})^2)^T \).

The difference between the pseudo score function \( i(f_{\theta_0}) \) and \( i(f_{\theta_0}) \) is only in their second terms, which are \( S_{\hat{f}_n}/S_{\hat{f}_n}(n) \) for the first and the Césaro averages of these for the second. The Hessian matrices \( \dot{i}(f_{\theta_0}) \) and \( \dot{i}(f_{\theta_0}) \) also only differ by the derivatives of the second terms. In the proof of Theorem 5.1 it was noted that the latter are again Césaro averages (see (9.3)) and they were shown to converge by showing convergence of the individual terms. Thus, this proof applies also in the present situation, and the full proof can be finished as the proof of Theorem 5.1.

**Proof of Corollary 6.3.** In view of Theorem 6.2 it suffices to verify the asymptotic normality of the sequence \( \sum_{k=1}^{\infty} F_{\theta_0}(k) \sqrt{n} (P_k(n) - P_k^{(0)}) \), for \( F_{\theta_0} \) as given. Because \( f_{\theta_0}(k) \) is constant in \( k \geq K \), so is \( F_{\theta_0}(k) \) and hence \( \sum_k F_{\theta_0}(k) P_k(n) = \sum_{k \leq K} F_{\theta_0}(k) P_k(n) + F_{\theta_0}(K) P_{>K}(n) \), and the same for \( P_k^{(0)} \) instead of \( P_k(n) \). The desired convergence therefore follows from Proposition B.4 and the continuous mapping theorem.
A Branching Processes and Rooted Ordered Trees

A rooted ordered tree is a tree in which one node is designated as the root and the other nodes can be oriented in parent-child relations in reference to their distance to this root node. In a dynamic setup the root is the initial ancestor who is responsible for giving births directly or indirectly to every other node. In the Ulam–Harris labelling notation for branching processes, the root is denoted by $\emptyset$ and every other node has the form $(i_1,\ldots,i_l)$, for positive natural numbers $i_j \in \mathbb{N}_+$ with $l \in \mathbb{N}_+$. The node $x = (i)$ is the $i$-th child of the root, and more generally the node $x = (i_1,\ldots,i_k)$ is the $i_k$-th child of $\tilde{x} = (i_1,\ldots,i_{k-1})$. By induction, the set of all possible individuals is

$$\mathcal{I} = \{\emptyset\} \cup \left(\bigcup_{k=1}^{\infty} \mathbb{N}_+^k\right).$$

The root is the zero-th generation and the $k$-th generation consists of all $x \in \mathbb{N}_+^k$. For $x = (x_1,\ldots,x_k)$ and $y = (y_1,\ldots,y_l)$ the notation $xy$ is shorthand for the concatenation $(x_1,\ldots,x_k,y_1,\ldots,y_l)$, and, in particular, $xl = (x_1,\ldots,x_k,l)$. The labelling of the nodes contains all parental information and a rooted ordered tree is defined to be a subset $G \subset \mathcal{I}$ such that if $x = (x_1,\ldots,x_k) \in G$, then both $(x_1,\ldots,x_{k-1}) \in G$ and $(x_1,\ldots,x_k-1) \in G$ in case $x_k \geq 2$. A corresponding graphical representation is obtained by drawing a node for every $x \in G$ and connecting two nodes by an edge if they are in a parent-child relationship. The degree of node $x \in G$ is

$$\text{deg}(x,G) = |\{l \in \mathbb{N}_+ \mid xl \in G\}| + 1,$$

(A.1)

where the extra one is for the parent of the node.

To set up a stochastic branching process, each individual $x \in \mathcal{I}$ is associated with a stochastic variable $\lambda_x$ and two stochastic processes $\xi_x$ and $\varphi_x$, called the life length, the reproduction process and the characteristic of $x$. The triples $(\lambda_x,\xi_x,\varphi_x)$ are taken IID across the nodes $x \in \mathcal{I}$. Formally they may be defined as copies on a product probability space

$$(\Omega,\mathcal{B},P) = \prod_{x \in \mathcal{I}} (\Omega_x,\mathcal{B}_x,P_x),$$

where each $(\Omega_x,\mathcal{B}_x,P_x)$ is a copy of a probability space $(\Omega_0,\mathcal{B}_0,P_0)$. For a given measurable map $(\lambda,\xi,\varphi)$ defined on $(\Omega_0,\mathcal{B}_0,P_0)$, we then define $(\lambda_x,\xi_x,\varphi_x)(\omega) = (\lambda,\xi,\varphi)(\omega_x)$ if $\omega = (\omega_x)_{x \in \mathcal{I}} \in \Omega$. The life length $\lambda$ is a nonnegative random variable, which in our case we take identically $\infty$ (no node will die). The reproduction process $\xi = (\xi(t) : t \geq 0)$ will be a counting process starting with $\xi(0) = 0$ and increasing by steps of size 1 at random times. We identify $\xi$ with a random $\mathbb{N}_+$-valued measure through $\xi([0,t]) = \xi(t)$ and denote by $\mu : t \mapsto \mathbb{E}[\xi(t)]$ its mean (or intensity) measure, which is often called the reproduction function in this context. The characteristic will also be a stochastic process $\varphi = (\varphi(t) : t \geq 0)$, where we may set $\varphi(t) = 0$ for $t < 0$.

The random point process $\xi_x$ models the birth times of the children of individual $x$ relative to the birth time $\sigma_x$ of $x$. The latter birth times are formally defined recursively
by setting the birth time of the root $∅$ at $t = 0$ (hence $σ_∅ = 0$), and next the birth time $σ_y$ of $y$ in calendar time by

$$σ_y = σ_x + \inf\{t \geq 0 : ξ_x(t) \geq l\}, \quad \text{if } y = xl.$$ 

The calendar time is the evolution time of the branching process, as opposed to the local time scales of the processes $ξ_x$ and $ϕ_x$, of which the local zero time is interpreted to be $σ_x$ in calendar time. The variable $ϕ_x(t)$ is interpreted as the characteristic of individual $x$ when $x$ has age $t$. Calendar time is also different from the discrete time steps used to describe the evolution of a PA network.

For a given characteristic we define the process

$$Z_t^ϕ = \sum_{x ∈ S : σ_x \leq t} ϕ_x(t − σ_x).$$

The variable $t − σ_x$ is the time since birth of individual $x$ and hence $ϕ_x(t − σ_x)$ can be interpreted as the characteristic of individual $x$ at calendar time $t$. The variable $Z_t^ϕ$ is the sum of all such characteristics over the individuals that are alive at time $t$.

A branching process is supercritical and Malthusian if its reproduction function $µ$ does not concentrate on any lattice $\{0, h, 2h, \ldots\}$, for some $h > 0$, and there exists a number $λ^* > 0$ such that

$$\int_0^∞ e^{−λ^*t} µ(dt) = 1.$$ (A.2)

We shall also assume the integrability assumption

$$\int_0^∞ t^2 e^{−λ^*t} µ(dt) < ∞.$$ (A.3)

Existence of a solution to equation (A.2) is called the Malthusian assumption, and $λ^*$ is called the Malthusian parameter.

The following proposition can be obtained by combining Theorem 3.1, Corollary 3.4 and Theorem 6.3 of Nerman (1981). Define $λ_ξ(t) = ∫_0^t e^{−λu} ξ(du)$.

**Proposition A.1.** Assume that the reproduction function $µ(t) = E[ξ(t)]$ satisfies conditions (A.2) and (A.3) and does not concentrate on any lattice. Assume that $t ↦ E[ϕ(t)]$ is continuous almost everywhere with respect to the Lebesgue measure and the following conditions hold:

$$\sum_{k=0}^∞ \sup_{k≤t≤k+1} \left(e^{−λ^*t} E[ϕ(t)]\right) < ∞, \quad \text{(A.4)}$$

$$E[\sup_{s≤t} ϕ(s)] < ∞ \quad \text{for all } t < ∞.$$ (A.5)

Then there exists a random variable $Y_∞$ depending only on the reproduction process $ξ(t)$ such that, as $t → ∞$,

$$e^{−λ^*t} Z_t^ϕ \xrightarrow{P} Y_∞ m^ϕ_∞, \quad \text{(A.6)}$$

where $m^ϕ_∞$ is defined as

$$m^ϕ_∞ = \int_0^∞ e^{−λ^*t} E[ϕ(t)] dt \int_0^∞ te^{−λ^*t} dµ(t).$$
The convergence in (A.6) also holds in the $L_1$ sense if
\[ \mathbb{E}[\lambda \xi(\infty) \log^+ \lambda \xi(\infty)] < \infty. \] (A.7)

Suppose that the reproduction process $\xi$ satisfies (A.7), and both $\varphi_1$ and $\varphi_2$ satisfy the conditions (A.4) and (A.5). Define $T_t$ as total number of births up to and including time $t$. Then, on the event $\{T_t \to \infty\}$, as $t \to \infty$.
\[ \frac{Z_t^{\varphi_1}}{Z_t^{\varphi_2}} \to \frac{m_\varphi^{\infty}}{m_\varphi^{\infty}} = \int_0^\infty e^{-\lambda t} \mathbb{E}[\varphi_1(t)] \, dt \] (A.8)

If $\varphi_1$ and $\varphi_2$ have càdlàg paths and there exists a $\lambda < \lambda^*$ such that
\[ \mathbb{E}[\lambda \xi(\infty)] < \infty, \] (A.9)
\[ \mathbb{E}\left[ \sup_t e^{-\lambda t} \varphi_i(t) \right] < \infty, \quad i = 1, 2, \] (A.10)
then, on $\{T_t \to \infty\}$, the convergence in (A.8) is also in the almost sure sense.

\section*{B Asymptotic normality of Empirical Degrees}

In this section we derive the asymptotic normality of the empirical degrees in some cases of the preferential attachment model, using an urn process studied by Janson (2004).

The urn process consists of vectors $X_n = (X_{n,1}, \ldots, X_{n,q})^T$ in $[0, \infty)^q$, of which the $i$-th coordinate represents the quantity in the $i$-th urn at time $n$. Given are, for each $i \in [q]$, an ‘activity’ $a_i \geq 0$ and a vector $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,q})^T \in \mathbb{R}^q$ with $\xi_{i,j} \geq 0$ for $j \neq i$ and $\xi_{ii} \geq -1$. The process $(X_n)_{n=0}^\infty$ evolves as a Markov process, with transitions determined by: given $X_{n-1},$

1. pick an urn $i \in [q]$ with probability $a_i X_{n-1,i} / \sum_{j=1}^q a_j X_{n-1,j};$
2. set $X_n := X_{n-1} + \xi_i.$

It is assumed that $\sum_{j=1}^q \xi_{i,j} \geq 0$, for every $i \in [q]$, with strict inequality for some $i$, so that the total content $\sum_{i=1}^q X_{n,i}$ of the urns is nondecreasing. (Actually, Janson (2004) allows the vectors $\xi_i$ to be random, but deterministic vectors suffice in our situation, and allow a simpler statement of the main result.)

Define the ‘transfer’ matrix $A \in \mathbb{R}^{q \times q}$ by
\[ A_{ij} = a_j \xi_{j,i}. \] (B.1)

We assume that $A$ is irreducible. By an application of the Perron–Frobenius theorem (to the nonnegative matrix $A + \alpha I_q$, for sufficiently large $\alpha$), it can be seen that the eigenvalue of $A$ with the largest real value is real, and the eigenvalues can be ranked by their real parts as $\lambda_1 > \text{Re} \lambda_2 \geq \text{Re} \lambda_3 \geq \cdots$. Furthermore, $\lambda_1 > 0$, has multiplicity one, and the associated eigenvector $v_1$ has all positive coordinates. We normalize $v_1$ such that $a^T v_1 = 1$, where $a = (a_1, \ldots, a_q)^T$. 

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In this setup, conditions (A1)-(A6) in Janson (2004) are satisfied (see his Lemma 2.1). A crucial further assumption in the following proposition, which restates Theorems 3.21-3.22 of Janson (2004), is that Re$\lambda_2 < \lambda_1/2$.

Define the following quantities:

\[ B_i = \sum_{i=1}^{q} v_i a_i \xi_i^T, \]  
\[ \varphi(s, A) = \sum_{n=1}^{\infty} \frac{s^n}{n!} A^{n-1} = \int_{0}^{s} e^{tA} dt, \]  
\[ \psi(s, A) = e^{sA} - \lambda_1 v_1 a^T \varphi(s, A). \]

**Proposition B.1.** Under the preceding conditions, as $n \to \infty$,

\[ n^{-1} X_n \xrightarrow{a.s.} \lambda_1 v_1. \]

If, moreover, Re$\lambda_2 < \lambda_1/2$, then, as $n \to \infty$,

\[ n^{1/2}(n^{-1} X_n - \lambda_1 v_1) \xrightarrow{d} N(0, \Sigma), \]

where the covariance matrix $\Sigma$ is defined as (with quantities defined in (B.2)-(B.4))

\[ \Sigma = \int_{0}^{\infty} \psi(s, A) B \psi(s, A)^T e^{-\lambda_1 s} ds - \lambda_1^2 v_1 v_1^T. \]  

The preferential attachment model would be naturally described using an infinite number of urns, with the content $X_{n,i}$ of the $i$-th urn corresponding to the number of nodes of degree $i$ at time $n$. The activities $a_i$ can then be set equal to the preferences $f(i)$ and the vectors $\xi_i$ defined by their coordinates

\[ \xi_{i,j} = -1_{\{j=i\}} + 1_{\{j=i+1\}} + 1_{\{j=1\}}. \]  

The last term $1_{\{j=1\}}$ corresponds to the new node, which has degree 1, and is counted in the first urn, while the first two terms on the right describe the decrease and increase by 1 of the numbers of nodes of degrees $j$ and $j + 1$, respectively, if the new node is attached to an existing node of degree $j$. There would then be infinitely many vectors ($i \in \mathbb{N}_+$) each with infinitely many coordinates ($j \in \mathbb{N}_+$), but, unfortunately, Proposition B.1 allows a fixed, finite number of urns only. In the following, we consider two examples of PA models in which the infinite process can be reduced to a finite number: the case of an affine PA function, and the case that the PA function is constant from a fixed degree onward.

In the affine case with PA function $f(k) = k + \alpha$, we study the degree distribution up to some given degree $\kappa$ by gathering all nodes of degree strictly bigger than $\kappa$ in a single urn, labelled $\kappa + 1 =: q$. To accommodate that the latter nodes have different preferences, we define the vectors $\xi_i \in \mathbb{R}^{\kappa+1}$ for $i = 1, \ldots, \kappa - 1$ as in (B.6) with coordinates restricted to $j \in [\kappa + 1]$, but redefine $\xi_\kappa$ and $\xi_{\kappa+1}$ by

\[ \xi_{\kappa,j} = -1_{\{j=\kappa\}} + 1_{\{j=\kappa+1\}}(\kappa + 1 + \alpha) + 1_{\{j=1\}}, \]  
\[ \xi_{\kappa+1,j} = 1_{\{j=\kappa+1\}} + 1_{\{j=1\}}. \]
We combine this with the vector of activities \(a_\kappa = (1 + \alpha, 2 + \alpha, \ldots, \kappa + \alpha, 1)^T\). Thus, urns \(1, \ldots, \kappa\) have activities equal to the preferential attachment function, but urn \(\kappa + 1\) has activity 1. With these definitions, for \(i = 1, \ldots, \kappa\) the variable \(X_{n,i}\) corresponds to the numbers of nodes of degree \(i\), but \(X_{n,\kappa + 1}\) is set to correspond to the total preference of all nodes of degree bigger than \(\kappa\). Indeed, a choice of an urn \(i = 1, \ldots, \kappa - 1\) follows the scheme described before, with the transition given by (B.6). Second, a choice of urn \(\kappa\) corresponds to choosing a node of degree \(\kappa\); by (B.7) the count of urn \(\kappa\) is then decreased by one, and \(\kappa + 1 + \alpha\) balls are added to urn \(\kappa + 1\), each weighted by activity \(a_{\kappa + 1} = 1\), thus giving the correct increase of total preference of the nodes of degree bigger than \(\kappa\). Third, a choice of urn \(\kappa + 1\) corresponds to choosing a node of some degree bigger than \(\kappa\); this node is replaced by a node of degree one bigger, resulting in an increase by 1 of the total preference of the nodes of degree bigger than \(\kappa\). Thus, (B.8) correctly changes the total preferences of the nodes of degree bigger than \(\kappa\). In both (B.7) and (B.8) the term on the far right corresponds to the new node of degree 1, counted in \(X_{n,1}\).

Asymptotic normality of the empirical degrees \(P_k(n)\) in the affine case was first proved in Móri (2002) and Resnick and Samorodnitsky (2016). We deduce it here by a simple argument based on the preceding proposition,

**Proposition B.2.** In the PA model with PA function \(f(k) = k + \alpha\), the centered and rescaled empirical degree distribution \(\sqrt{n}(P_k(n) - p_k), k = 1, 2, \ldots\) converges in distribution in \(\mathbb{R}^{N+}\) to a centered Gaussian process.

**A novel proof.** We fix arbitrary \(\kappa \in \mathbb{N}\), and define activities \(a_i\) and update vectors \(\xi_i\), for \(i \in [\kappa + 1]\), as indicated in (B.6)–(B.8), and initial vector \(X_0 = (1,0,\ldots,0)^T\). For every \(n\), the vector \((X_{n,1}, \ldots, X_{n,\kappa}, X_{n,\kappa + 1})\) is then identically distributed to the vector \((N_1(n), \ldots, N_\kappa(n), \sum_{j>\kappa} N_j(n) (j + \alpha))\). Since (see, e.g., Billingsley (2013, Example 2.4) or van der Vaart and Wellner (1996, Theorem 1.6.1)) weak convergence in \(\mathbb{R}^{N+}\) is the same as convergence of all finite marginals, the first assertion is proved if we can prove convergence of these vectors for every fixed \(\kappa\). For this we apply Proposition B.2.

The transfer matrix \(A = A_\kappa\) is given by

\[
A_\kappa = \begin{pmatrix}
0 & 2 + \alpha & 3 + \alpha & \cdots & \kappa - 1 + \alpha & \kappa + \alpha & 1 \\
1 + \alpha & -2 - \alpha & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 + \alpha & -3 - \alpha & \cdots & 0 & 0 & 0 \\
0 & 0 & 3 + \alpha & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\kappa + 1 - \alpha & 0 & 0 \\
0 & 0 & 0 & \cdots & \kappa - 1 + \alpha & -\kappa - \alpha & 0 \\
0 & 0 & 0 & \cdots & 0 & (\kappa + \alpha)(\kappa + 1 + \alpha) & 1
\end{pmatrix}.
\]

We can calculate that \(\det(\lambda I - A_2) = (\lambda - 2 - \alpha)(\lambda + 1 + \alpha)(\lambda + 2 + \alpha)\). Furthermore, by subtracting \((\kappa + \alpha)\) times the \((\kappa + 1)\)-th column from the \(\kappa\)-th column, next pulling out the factor \(\lambda + \kappa + \alpha\) from the \(\kappa\)-th column and finally adding \((\kappa - 1 + \alpha)\) times the \(\kappa\)-th column to the \((\kappa - 1)\)-th column, we can see that \(\det(\lambda I - A_\kappa) = (\lambda + \kappa + \alpha) \det(\lambda I - A_{\kappa - 1})\).
Therefore, by mathematical induction
\[
\det(A_\kappa - \lambda I) = (\lambda - (2 + \alpha)) \prod_{l=1}^{\kappa} (\lambda + (l + \alpha)).
\]

We conclude that all eigenvalues are real, and that the only positive eigenvalue is \(\lambda_{1,\kappa} = 2 + \alpha\), so that certainly \(\text{Re} \lambda_{2,\kappa} < \lambda_{1,\kappa}/2\). The transfer matrix \(A_\kappa\) is irreducible.

Thus, the conditions of Proposition B.1 are verified. It suffices to identify the limiting mean.

By (8.6.12) of van der Hofstad (2017), the limiting degree distribution for PA trees with the PA function \(f: k \mapsto k + \alpha\) has coordinates
\[
p_k = (2 + \alpha) \frac{\Gamma(k + \alpha) \Gamma(3 + 2\alpha)}{\Gamma(k + 3 + 2\alpha) \Gamma(1 + \alpha)}.
\]

This corresponds to the recursion (with \(p_1 = (2 + \alpha)/(3 + 2\alpha)\))
\[
p_k = \frac{k - 1 + \alpha}{k + 2 + 2\alpha} p_{k-1}. \tag{B.9}
\]

We now verify that \(v_{1,\kappa} := (2 + \alpha)^{-1}(p_1, p_2, \ldots, p_\kappa, \sum_{l=\kappa}^{\infty} p_l(l + \alpha))\) is an eigenvector associated with the eigenvalue \(2 + \alpha\) with \(a^T v_{1,\kappa} = 1\), and hence obtain that \(n^{-1}X_n \to \lambda_{1,\kappa} v_{1,\kappa} = (p_1, p_2, \ldots, p_\kappa, \sum_{l=\kappa}^{\infty} p_l(l + \alpha))\), almost surely.

The first coordinate of \(A_\kappa(2 + \alpha)v_{1,\kappa}\) is \(\sum_{l>1} p_l(l + \alpha) = (2 + \alpha) - p_1(1 + \alpha) = (2 + \alpha)p_1\), by Lemma B.3. The second to \(\kappa\)-th coordinates are equal to \((2 + \alpha)p_k\), for \(k = 2, \ldots, \kappa\), by the relation (B.9). The \((\kappa + 1)\)-st coordinate of \(A_\kappa(2 + \alpha)v_{1,\kappa}\) is \(\sum_{l>\kappa} p_l(l + \alpha) + (\kappa + \alpha)(\kappa + 1 + \alpha)p_\kappa\), which coincides with \((2 + \alpha)\sum_{l>\kappa} p_l(l + \alpha)\), again by Lemma B.3.

The covariance function of the limiting Gaussian vector can be obtained from (B.5) by somewhat tedious calculations, which we omit. We refer to (4.28) of (Resnick and Samorodnitsky, 2016, page 18) for its exact form.

By the simple linear relation (with drift) \(P_{>k}(n) = 1 - \sum_{j=1}^{k} P_j(n)\), and the continuous mapping theorem, it follows from the preceding theorem that, for any \(\kappa \in \mathbb{N}_+\),
\[
\sqrt{n}(P_{>k}(n) - p_{>k}); k = 1, \ldots, \kappa \, \sim \, N(0, R_\kappa),
\]

The covariance matrix \(R_\kappa\) in this limit takes a simple form, first pointed out in Móri (2002), given by
\[
(R_\kappa)_{ij} = 1_{i=j}p_i(1 - p_i) - 1_{i \neq j}p_ip_j. \tag{B.10}
\]

**Lemma B.3.** The limiting degree distribution \((p_k)_{k=1}^{\infty}\) in the affine PA model with the PA function \(f(k) = k + \alpha\), satisfies, for \(k \in \mathbb{N}_+\),
\[
\sum_{l>k} p_k(k + \alpha) = \frac{(k + \alpha)(k + 1 + \alpha)}{1 + \alpha} p_k.
\]
Proof. For \( k = 1 \) the left side of the lemma is equal to
\[
\sum_{l=1}^{\infty} p_l(l + \alpha) - p_1(1 + \alpha) = 2 + \alpha - \frac{(2 + \alpha)(1 + \alpha)}{(3 + 2\alpha)} = \frac{(1 + \alpha)(2 + \alpha)}{1 + \alpha} p_1.
\]
This proves the claim for \( k = 1 \). We proceed by mathematical induction. If the statement is true for any integer up to \( k - 1 \), then the left side of the lemma is equal to
\[
\sum_{l>k-1} p_l(l + \alpha) - p_k(k + \alpha) = \frac{(k - 1 + \alpha)(k + \alpha)}{1 + \alpha} p_k \frac{k + 2 + 2\alpha}{k - 1 + \alpha} - p_k(k + \alpha),
\]
by the induction hypothesis and the relation (B.9) between \( p_k \) and \( p_{k-1} \). The right side can be reduced to the right side of the lemma.

As a second application of Proposition B.1, we obtain the asymptotic normality of the empirical degrees in PA models with PA function that is constant eventually. From our numerical experiments, we infer that the key eigenvalue condition \( \Re \lambda_2(A) < \lambda_1(A)/2 \) is generally satisfied when the PA function is sublinear, but we do not know general conditions for this.

To apply Proposition B.1 to a PA model with eventually constant PA function, we simply gather all nodes of degrees higher than a cut-off \( \kappa \) after which the PA function is constant (not necessarily the smallest such value) in a single urn, the \((\kappa + 1)\)-th one. Since the preferences of the corresponding nodes are equal, it is not necessary to keep track of the different degrees of the nodes inside this bin when studying the lower degrees. The evolution of the empirical degrees \( (P_1(n), \ldots, P_{\kappa}(n), P_{>\kappa}(n)) \) will be the same as the evolution of the vectors \( (X_{n,1}, \ldots, X_{n,\kappa}, X_{n,\kappa+1}) \), if we define the preferences as \( a_i = f(i \wedge \kappa) = f(i) \), and the transition vectors \( \xi_i \) for \( i \in [\kappa] \) by (B.6)–(B.8) with \( j \) restricted to coordinates \( j \in [\kappa + 1] \) and \( \xi_{\kappa+1} = (1, 0, 0, \ldots, 0)^T \). The last definition corresponds to adding a ball to urn 1 (counting the added node of degree 1) and moving a ball within the \((\kappa + 1)\)-st urn (for attaching this node to a node of degree \( j > \kappa \)), so not changing \( X_{n,\kappa+1} = N_{>\kappa}(n) \).

**Proposition B.4.** In the PA model with PA function \( f \) that is constant on \([\kappa, \infty)\) such that the matrix \( A_\kappa \) defined in (B.11) satisfies \( \Re \lambda_2(A_\kappa) < \lambda_1(A_\kappa)/2 \), there exist a probability distribution \( (p_k) \) such that the sequence \( \sqrt{n}(P_1(n) - p_1, \ldots, P_{\kappa}(n) - p_{\kappa}, P_{>\kappa}(n) - p_{>\kappa}) \) tends to a centered normal distribution. If the PA function depends continuously on a parameter \( \theta \), pointwise, then \( p_k \) and the covariance matrix depend continuously on \( \theta \) as well. If the condition on \( A_\kappa \) holds for every sufficiently large \( \kappa \), then the sequence \( \sqrt{n}(P_k(n) - p_k; k = 1, 2, \ldots) \) converges in \( \mathbb{R}^\infty \).

**Proof.** For the given \( \kappa \) define an urn process as indicated preceding the proposition. The
matrix $A_\kappa$ is given by

$$A_\kappa = \begin{pmatrix}
0 & f(2) & f(3) & f(4) & \cdots & f(\kappa) & f(\kappa + 1) \\
-f(2) & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & f(3) & -f(4) & 0 & \cdots & 0 & 0 \\
0 & 0 & f(4) & -f(5) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -f(\kappa) & 0 \\
0 & 0 & 0 & 0 & \cdots & f(\kappa) & 0
\end{pmatrix}. \tag{B.11}$$

The matrix $A_\kappa$ is irreducible, and its eigenvalues satisfy $\text{Re}\, \lambda_2 < \lambda_1/2$, by assumption. Thus, the vector of empirical degrees $\left( P_1(n), \ldots, P_\kappa(n), P_{>\kappa}(n) \right)$, suitably centered and scaled, is asymptotically normal by Proposition B.1. If this is true for every sufficiently large $\kappa \in \mathbb{N}_+$, the infinite sequence in the final assertion of the proposition converges as well.

It remains to sort out the continuity of the asymptotic mean and covariance if the PA function depends continuously on a parameter. Denote the parameter by $\theta$ and set $q = \kappa + 1$. By its definition the map $\theta \mapsto A_\kappa(\theta)$ inherits the continuity from the PA function. Employing the min-max formula (Horn and Johnson (2012, Corollary 8.1.31)),

$$\lambda_1(\theta) = \max_{x > 0} \min_{1 \leq i \leq q} \frac{1}{x_i} \sum_{j=1}^{q} (A_\kappa(\theta))_{ij} x_j,$$

where $x > 0$ is understood component-wise. By the maximum theorem (e.g. Ok (2011, page 229)), $\theta \mapsto \lambda_1(\theta)$ is continuous. The corresponding eigenvector can be obtained as $\text{adj}(C(\theta))e_1$, for $\text{adj}(C)$ the adjugate matrix of $C = A_\kappa - \lambda_1 I_q$. This follows, because $C \text{adj}(C) = (\text{det} C) I_q = 0$ and $\text{adj}(C)e_1 \neq 0$. The latter can be seen from the fact that the range of $C$, which has dimension $q - 1$, is the null space of $\text{adj}(C)$, as $\text{adj}(C)C = 0$, and $e_1$ is not in the range of $C$. Since the adjugate matrix depends continuously on $A$ and $\lambda_1$, so does the eigenvector $\text{adj}(C(\theta))e_1$, and this remains valid after scaling. Inspecting the quantities in (B.2)–(B.4) and (B.5), we see that the asymptotic covariance matrix is continuous.

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