Repo convexity

Paul McCloud
Department of Mathematics, University College London
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Abstract

There is an observed basis between repo discounting, implied from market repo rates, and bond discounting, stripped from the market prices of the underlying bonds. Here, this basis is explained as a convexity effect arising from the decorrelation between the discount rates for derivatives and bonds.

Using a Hull-White model for the discount basis, expressions are derived that can be used to interpolate the repo rates of bonds with different maturities and to extrapolate the repo curve for discounting bond-collateralised derivatives.

Keywords: Repo pricing; repo convexity; bond curve; bond-derivative basis; bond collateral; bond-collateralised derivatives; FVA.
1 Securities and derivatives

No-arbitrage assumptions imply the fair value of a payoff is determined from a risk-neutral pricing measure $E$ and a predictable numeraire $q_t$ via the martingale condition:

$$\frac{a_t}{q_t} = E_t[\frac{a_T}{q_T}]$$  \hspace{1cm} (1)

While this model appears to be sensitive to the numeraire, in practice the only property required of the numeraire is predictability. Securities with observable market prices are marked-to-market, while the inclusion of funding flows implies the discounting of derivatives follows the discounting of the securities used to fund them. This argument is considered in more detail below.

Consider a security with observable market price $\bar{p}_t$. The discount rate $\bar{r}_t$ associated with the security is defined to be the risk-neutral expected return on the security:

$$\bar{r}_t \, dt = \frac{E_t[d\bar{p}_t]}{\bar{p}_t}$$  \hspace{1cm} (2)

Dividends, which are excluded in this model for the security, can be added as discrete terms in the expected return. The expression for the discount rate is integrated to the martingale property:

$$\bar{p}_t = E_t[exp[-\int_{\tau=t}^{T} \bar{r}_\tau \, d\tau] \bar{p}_T]$$  \hspace{1cm} (3)

allowing the security price to be modelled in terms of the terminal payoff and the security discount rate $\bar{r}_t$.

Consider a derivative with price $a_t$ that can be funded with any of a range of securities with prices $\bar{p}_i^t$. Funding with the $i$th security, the return to the seller of the derivative over the interval $dt$ is $(a_t/\bar{p}_i^t)d\bar{p}_i^t - da_t$. The seller chooses to invest the proceeds of the sale in the funding security that maximises the return value. The fair value model then implies the price expression:

$$0 = \max_i E_t[(a_t/\bar{p}_i^t)d\bar{p}_i^t - da_t]$$  \hspace{1cm} (4)

$$= a_t \max_i [\bar{r}_i^t] \, dt - E_t[da_t]$$  \hspace{1cm} (5)

where predictability has been used to take the numeraire outside the expectation. The numeraire can now be cancelled, and the expression integrated to generate the martingale property:

$$a_t = E_t[exp[-\int_{\tau=t}^{T} r_\tau \, d\tau] a_T]$$  \hspace{1cm} (6)

allowing the derivative price to be modelled in terms of the terminal payoff and the derivative discount rate $r_t$:

$$r_t = \max_i [\bar{r}_i^t]$$  \hspace{1cm} (7)
This argument assumes that the funding security can be switched in its entirety at any time, which is not typically the case. More generally, the relationship between the derivative discount rate and the discount rates of the securities that fund it is heavily path-dependent.

As a special case, the derivative discount factor for maturity $T$ is:

$$p_t^T = E_t[\exp[- \int_{\tau=t}^T r_\tau d\tau]]$$

being the price of the derivative with unit payoff at maturity.

The martingale expressions for the security and derivative prices involve different discount rates but are otherwise the same. In the following, the volatility of the difference between these discount rates is used to explain the observed basis between the repo discount factors implied from observed repo rates and the discount factors stripped from the bond curve.

### 2 Repo convexity

The bond market for an issuer is assumed to comprise discount bonds with price $\bar{p}_t^T$ for each maturity $T$. Ignoring the possibility of default, the bond satisfies the boundary condition $\bar{p}_T^T = 1$, and the martingale property for the bond price becomes:

$$\bar{p}_t^T = E_t[\exp[- \int_{\tau=t}^T \bar{r}_\tau^T d\tau]]$$

Common features among the family of bonds that derive from macro-economic considerations and the conditions of the issuer are encapsulated in the curve contribution $\bar{r}_t^T$ to the bond discount rate, with the residual contribution $\bar{z}_t^T$ for the individual bond reflecting liquidity and investor preference. The bond discount rate $\bar{r}_t^T$ then decomposes as:

$$\bar{r}_t^T = \bar{r}_t + \bar{z}_t^T$$

There is a degree of arbitrariness in this decomposition, and expert knowledge is required to separate curve and liquidity contributions to the discount rate.

Consider a forward-starting repo that sets at time $t$ over the period starting at time $s$ and ending at time $e$ on the bond maturing at time $T$, where $t \leq s < e \leq T$. At time $s$, the unit cashflow is exchanged for $(1/\bar{p}_s^T)$ units of the bond, a price-neutral exchange. At time $e$, the bonds are returned in exchange for the cashflow $(1 + f_{ts}^T \delta)$, where $f_{ts}^T$ is the repo rate and $\delta$ is the daycount. Haircuts and bond coupons are not considered in this construction, though both features are straightforward to add albeit at the cost of additional complexity in the expression for the repo rate.

By construction this has zero price at time $t$, leading to the price expression:

$$0 = E_t[\exp[- \int_{\tau=t}^e r_\tau d\tau] \frac{\bar{p}_t^T}{\bar{p}_s^T} \left(1 + f_{ts}^T \delta\right)]$$
Repo convexity arises from the dual-discounted nature of this construction. The repo price depends on the bond discount rate \( \tilde{r}_t^T \) that determines the bond price at settlement and the derivative discount rate \( r_t \) used for discounting. Decorrelation between these discount rates leads to a convexity adjustment for the repo rate.

The repo price expression can be re-arranged to identify the convexity adjustment for the repo rate. Define the discounting basis \( b_t^T \) and the liquidity basis \( \tilde{s}_t^T \):

\[
b_t^T = r_t - \tilde{r}_t^T \\
\tilde{s}_t^T = z_t^T - \tilde{z}_t^e
\]

First note that:

\[
\mathbb{E}_t[\exp[- \int_{\tau=t}^e r_{\tau} \, d\tau / \tilde{p}_s^T]] = \mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau / \tilde{p}_s^T]] \\
\mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau / \tilde{p}_s^T] \mathbb{E}_t[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T]] = \mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau / \tilde{p}_s^T] \mathbb{E}_t[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T]] \\
= \mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau] \mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau]] \mathbb{E}_t[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T] \mathbb{E}_t[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T]] \\
= p_t^e \frac{\tilde{p}_s^T}{\tilde{p}_s^T} \exp[-L_t^T] \exp[C_{ts}^T]
\]

The liquidity adjustment \( L_t^T \) and convexity adjustment \( C_{ts}^T \) in this expression are:

\[
L_t^T = \log\frac{\mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau]]}{\mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau]]} \\
C_{ts}^T = \log\frac{\mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau] \mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau] \mathbb{E}_t[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T] \mathbb{E}_t[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T]]]}{\mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau] \mathbb{E}_t[\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau] \mathbb{E}_t[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T] \mathbb{E}_t[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T]]]}
\]

The recurrence of the integral kernel:

\[
\exp[- \int_{\tau=t}^e \tilde{r}_\tau^T \, d\tau]
\]

in these expressions suggest the switch to the equivalent measure \( \mathbb{E}^{c_T} \) related to the risk-neutral measure \( \mathbb{E} \) by the Radon-Nikodym derivative:

\[
\frac{d\mathbb{E}^{c_T}}{d\mathbb{E}} = \exp[- \int_{\tau=0}^e \tilde{r}_\tau^T \, d\tau] / \mathbb{E}[\exp[- \int_{\tau=0}^e \tilde{r}_\tau^T \, d\tau]]
\]

In this measure, the liquidity and convexity adjustments simplify:

\[
L_t^{c_T} = \log[\mathbb{E}_t^{c_T}[\int_{\tau=t}^e \tilde{s}_t^T \, d\tau]] \\
C_{ts}^{c_T} = \log[\mathbb{E}_t^{c_T}[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T] \mathbb{E}_t^{c_T}[\exp[- \int_{\tau=t}^e b_r^T \, d\tau / \tilde{p}_s^T]]]
\]
demonstrating that $L^e_T$ is driven by the liquidity basis and $C^e_{ts}$ is driven by the covariance between the bond discount rate and the discount basis.

Further decompose the convexity adjustment into the maturity adjustment $M^e_T$ and the forwardness adjustment $F^e_T$:

$$M^e_T = C^e_{tt}$$
$$F^e_T = C^e_{ts} - (C^e_{tt} - C^e_{st})$$

so that:

$$C^e_{ts} = (M^e_T - M^e_{st}) + F^e_{ts}$$

The expression for the repo rate is then:

$$f^e_T = \frac{1}{\delta} \left( \frac{\hat{p}^e_T}{\hat{p}^e_t} \exp[F^e_{ts}] - 1 \right)$$

where the repo discount factor $\hat{p}^e_T$ is defined by:

$$\hat{p}^e_T = \bar{p}_e^T \exp[-L^e_T - M^e_T]$$

This shows that the repo rate follows the standard formula in terms of the bond discount factors, with liquidity and maturity adjustments applied to the discount factors and forwardness adjustment applied to the rate.

The liquidity adjustment $L^e_T$ is the adjustment applied to the repo discount factor $\hat{p}^e_T$ to account for the liquidity spread between the bond maturing at time $e$ and the bond maturing at time $T$. The liquidity adjustment depends on the mean and variance of the liquidity basis:

$$L^e_T = \log[\bar{E}[\exp[S]]] \approx \mu_S + \frac{1}{2} \sigma_S^2$$

where:

$$S = \int_{\tau=1}^e \bar{c}^T d\tau$$

The approximation is exact when the variable $S$ is normal in the measure $\bar{E} \equiv \bar{E}^e_T$.

The maturity adjustment $M^e_T$ is the convexity adjustment applied to the repo discount factor $\hat{p}^e_T$ to account for the delay between the settlement of the repo and the maturity of the bond. This adjustment satisfies the boundary condition $M^e_1 = 0$, so that there is no maturity adjustment in the repo-to-maturity case $e = T$. The forwardness adjustment $F^e_{ts}$ is the convexity adjustment applied to the repo rate $f^e_T$ to account for the delay between the fixing of the repo rate and the start of the repo period. This adjustment satisfies the boundary condition $F^e_{tt} = 0$, so there is no forwardness adjustment in the spot-starting case $t = s$. The convexity adjustment depends on the covariance between the discount basis and the bond price:

$$C^e_{ts} = \log[\frac{\bar{E}[\exp[-B + P]]}{\bar{E}[\exp[-B]|\bar{E}[\exp[P]]]}] \approx -\rho_{BP} \sigma_B \sigma_P$$
where:

\[ B = \int_{\tau=t}^{e} b_{\tau}^{T} d\tau \]  \hspace{1cm} (24)

\[ P = \log \left[ \frac{\bar{p}_{T}}{\bar{p}_{s}} \right] \]

The approximation is exact when the variables \( B \) and \( P \) are joint normal in the measure \( \bar{E} \equiv \bar{E}^{\mathbb{E}_{t}} \).

3 Hull-White model for repo convexity

In this section, the liquidity contribution to the bond discount rate is taken to be zero, and a model for the convexity adjustment is constructed using correlated Hull-White models for the bond discount rate and the discount basis. The convexity adjustment depends on the correlations between the variables:

\[ R = \int_{\tau=t}^{e} \bar{r}_{\tau} d\tau \]  \hspace{1cm} (25)

\[ B = \int_{\tau=t}^{e} b_{\tau} d\tau \]

\[ P = \log \left[ \frac{\bar{p}_{T}}{\bar{p}_{s}} \right] \]

When these variables are joint normal in the risk-neutral measure \( \bar{E} \) the convexity adjustment becomes:

\[ C_{ts}^{\mathbb{E}_{t}} = -\rho_{BP} \sigma_{B} \sigma_{P} \]  \hspace{1cm} (26)

where \( \sigma_{B} \) and \( \sigma_{P} \) are the standard deviations of \( B \) and \( P \) and \( \rho_{BP} \) is the correlation between them.

In order to generate an expression for the convexity adjustment, consider the Hull-White model for the bond discount rate and the discount basis:

\[ d\bar{r}_{t} = \theta(\bar{r}_{t}^{*} - \bar{r}_{t}) dt + \sigma dx_{t} \]  \hspace{1cm} (27)

\[ db_{t} = \kappa(b_{t}^{*} - b_{t}) dt + \varepsilon dy_{t} \]

where \( \sigma \) and \( \varepsilon \) are the normal volatilities and \( \theta \) and \( \kappa \) are the mean reversion rates of the bond discount rate and discount basis, and the Brownian processes \( x_{t} \) and \( y_{t} \), driftless in the risk-neutral measure, are correlated:

\[ dx_{t} dy_{t} = \rho dt \]  \hspace{1cm} (28)

The mean reversion levels \( \bar{r}_{t}^{*} \) and \( b_{t}^{*} \) are calibrated to the initial bond and derivative discount factors.
The core variables whose covariance generates the convexity are normal in this model. Integrating the model leads to:

\[
\log[p_T] = -\sigma \int_{\tau=t}^e e^{-\theta(e)} dx_T + \text{drift}
\]

\[
\log[p_e] = -\sigma \int_{\tau=t}^e e^{-\theta(e)} dx_T + \text{drift}
\]

\[
\int_{\tau=t}^e b_T d\tau = \frac{\sigma}{\kappa} \int_{\tau=t}^e (1 - e^{-\kappa(e)}) dy_T + \text{drift}
\]

The covariance that generates the convexity adjustment is then:

\[
CeT = \frac{\rho \sigma \varepsilon}{\theta \kappa} \int_{\tau=t}^e e^{-\theta(e)} (1 - e^{-\kappa(e)}) d\tau
\]

The convexity depends on the three time intervals:

\[
\tau = s - t
\]

\[
\delta = e - s
\]

\[
\mu = T - e
\]

where \(\tau\) is the forwardness of the repo, \(\delta\) is the length of the repo period, and \(\mu\) is the time-to-maturity from the end of the repo of the reference discount bond. The convexity can then be expressed as:

\[
CeT = \frac{\rho \sigma \varepsilon}{\theta \kappa} B[s - t, e - s, T - e; \theta, \kappa]
\]

where:

\[
B[\tau, \delta, \mu; \theta, \kappa] = \frac{1}{\theta \kappa} (1 - e^{-\theta \mu}) \left( \frac{1}{\theta}(1 - e^{-\theta(\tau + \delta)}) - \frac{1}{\theta + \kappa}(1 - e^{-(\theta + \kappa)(\tau + \delta)}) \right)
\]

This function has finite limits as \(\theta\) and \(\kappa\) tend to zero. The maturity and forwardness adjustments are then:

\[
M_{ts}^e = \frac{\rho \sigma \varepsilon}{\theta \kappa} (1 - e^{-\theta(T - e)})(\frac{1}{\theta}(1 - e^{-\theta(e - t)}) - \frac{1}{\theta + \kappa}(1 - e^{-(\theta + \kappa)(e - t)}))
\]

\[
F_{ts}^e = -\frac{\rho \sigma \varepsilon}{\theta \kappa(\theta + \kappa)} (1 - e^{-\theta(T - s)})(1 - e^{-\kappa(e - s)})(1 - e^{-(\theta + \kappa)(s - t)})
\]

4 Calibration to repo discount factors

The convexity adjustment satisfies the boundary condition \(C_{es} = 0\) in the spot-starting repo-to-maturity case, in which case the repo rate is:

\[
f_{es} = \frac{1}{\delta} \left( \frac{1}{p_s} - 1 \right)
\]
More generally, the repo rate is impacted by the convexity adjustment $C_{ts}^T$ due to the time-to-maturity $\mu = T - e$ of the reference bond and the forwardness $\tau = s - t$ of the repo.

Consider first the convexity adjustment arising from the bond maturity. In the spot-starting infinite maturity case the repo rate is given by:

$$f_{ss}^\infty = \frac{1}{\delta} \left( \frac{1}{\hat{p}_e} - 1 \right)$$  (36)

where the repo discount factor $\hat{p}_e$ is defined by:

$$\hat{p}_e = \bar{p}_e \exp\left[ -\frac{\rho\sigma\varepsilon}{\theta\kappa} \left( \frac{1}{\theta} (1 - e^{-\theta(e-t)}) \right) - \frac{1}{\theta + \kappa} (1 - e^{-(\theta + \kappa)(e-t)}) \right]$$  (37)

This expression defines the repo discount factor in terms of the bond discount factor and the model parameters. The instantaneous forward rates $\hat{f}_t^e$ and $\bar{f}_t^e$ for the bond and repo are then related by:

$$\hat{f}_t^e = \bar{f}_t^e + \frac{\rho\sigma\varepsilon}{\theta\kappa} e^{-\theta(e-t)}(1 - e^{-\kappa(e-t)})$$  (38)

If the repo forward rates are observed in the market up to some finite maturity $E$, this model can be used to extrapolate the repo curve using the bond forward rates as reference:

$$\hat{f}_t^e = \hat{f}_t^e + (\hat{f}_E^e - \bar{f}_E^e) e^{-\theta(e-E)} \frac{1 - e^{-\kappa(e-t)}}{1 - e^{-\kappa(E-t)}}$$  (39)

Practical applications of this expression include the extrapolation of the repo curve for use in discounting bond-collateralised derivatives.

Extending to the spot-starting finite maturity case, the repo rate is:

$$f_{ss}^T = \frac{1}{\delta} \left( \frac{1}{\hat{p}_s^T} - 1 \right)$$  (40)

where the repo discount factor $\hat{p}_s^T$ geographically interpolates between the bond discount factor $\bar{p}_s^T$ and the repo discount factor $\hat{p}_s^T$:

$$\hat{p}_s^T = (\hat{p}_s^{\infty})^{\exp[-\theta(T-e)]} (\hat{p}_s) \exp[-\theta(T-e)]$$  (41)

The convexity adjustment for the spot-starting repo is absorbed in the definition of the repo discount factors. The only model parameter that appears in this expression is the mean reversion rate for the bond discount rate, which determines the speed of interpolation between the bond and repo discount factors as the maturity increases.

The convexity adjustment arising from the forwardness of the repo cannot be absorbed as an adjustment to the repo discount factors. Including the forwardness adjustment, the general expression for the repo rate is:

$$f_{ts}^T = \frac{1}{\delta} \left( \frac{\hat{p}_s^T}{\hat{p}_t^e} \exp\left[ -\frac{\rho\sigma\varepsilon}{\theta\kappa(\theta + \kappa)} (1 - e^{-\theta(T-s)}) (1 - e^{-\kappa(e-s)}) (1 - e^{-(\theta + \kappa)(s-t)}) \right] - 1 \right)$$  (42)
This model geometrically interpolates between the zero and infinite forwardness cases:

\[ 1 + f_{ts}^T \delta = (1 + f_{ss}^T \delta)^{\exp[-(\theta+\kappa)(s-t)]}(1 + f_{-\infty s}^T \delta)^{1-\exp[-(\theta+\kappa)(s-t)]} \]  

(43)

The contribution from forwardness is implemented as a convexity adjustment to the ratio of repo discount factors. This convexity adjustment is nonzero even in the repo-to-maturity case when the repo is forward-starting.

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