TOWARD AN OPTIMAL THEORY OF INTEGRATION FOR QUASI-BANACH-SPACE-VALUED FUNCTIONS

JOSÉ L. ANSORENA AND GLENIER BELLO

Abstract. We present a new approach to define a suitable integral for functions with values in quasi-Banach spaces. The integrals of Bochner and Riemann have deficiencies in the non-locally convex setting. The study of an integral for \( p \)-Banach spaces initiated by Vogt is neither totally satisfactory, since there are quasi-Banach spaces which are \( p \)-convex for all \( 0 < p < 1 \), so it is not always possible to choose an optimal \( p \) to develop the integration. Our method puts the emphasis on the \( \text{galb} \) of the space, which permits a precise definition of its convexity. The integration works for all spaces of \( \text{galb} \) known in the literature. We finish with a fundamental theorem of calculus for our integral.

1. Introduction

If \( X \) is a non-locally convex space, it is easy to construct a sequence of simple functions

\[
s_n : [0, 1] \rightarrow X, \quad s_n(t) = \sum_{m=1}^{n} \chi_{A_{m,n}}(t)x_{m,n},
\]

where \((A_{m,n})_{m=1}^{n}\) is a partition of the interval \([0, 1]\) for each \( n \in \mathbb{N} \), and \( \chi \) denotes the characteristic function, such that

\[
\sup_{1 \leq m \leq n} \|x_{m,n}\| \rightarrow 0, \quad \sum_{m=1}^{n} \mu(A_{m,n})x_{m,n} \not\rightarrow 0,
\]

as \( n \) goes to infinity, where \( \mu \) denotes the Lebesgue measure (cf. [26, pp. 121-123]). Therefore, Bochner-Lebesgue integration cannot be extended to non-locally convex spaces. On the other hand, the definition
of the Riemann integral extends verbatim for functions defined on an interval \([a, b]\) with values in an \(F\)-space \(X\). However, it has some problems in the non-locally convex setting. For example, Mazur and Orlicz [23] proved that the \(F\)-space \(X\) is non-locally convex if and only if there is a continuous function \(f : [0, 1] \to X\) which is not Riemann integrable. But the main drawback is that the Riemann integral operator \(I_R\), acting from the set of \(X\)-valued simple functions \(S([a, b], X)\) to \(X\) by

\[
I_R \left( \sum_{j=1}^{n} x_j \chi_{[t_{j-1}, t_j]} \right) = \sum_{j=1}^{n} (t_j - t_{j-1}) x_j,
\]

is not continuous when \(X\) is not locally convex (see [1, Theorem 2.3]).

An important attempt (somehow missed in the literature) to develop a theory of integration based on operators for functions with values in a quasi-Banach-space (i.e. a locally bounded \(F\)-space) was initiated by Vogt [34]. A remarkable theorem of Aoki and Rolewicz [5, 25] says that any quasi-normed space is \(p\)-convex for some \(0 < p \leq 1\). The idea of Vogt was the following. Given a quasi-Banach space \(X\), let \(0 < p \leq 1\) be such that \(X\) is \(p\)-convex. For this fixed \(p\), he developed a theory of integration based on an identification of tensor spaces with function spaces (see [34, Satz 4]). Among the papers that approach integration of quasi-Banach-valued functions from Vogt’s point of view we highlight [22].

The main advantage of Vogt’s integration with respect other approaches to integration in the non locally convex setting is that it provides a bounded operator from the space of integrable functions into the target quasi-Banach space. Regarding the limitations, its main drawback is that it depends heavily on the convexity parameter \(p\) chosen, and for some spaces there is no optimal choice of \(p\). Take, for instance, the weak Lorentz space \(L_{1,\infty} = L_{1,\infty}(\mathbb{R})\). This classical space, despite not being locally convex, is \(p\)-convex for any \(0 < p < 1\) (see [15, (2.3) and (2.6)]).

The concept that permits a precise definition of the convexity of a space was introduced and developed by Turpin in a series of papers (cf. [31, 32]) and a monograph ([33]) in the early 1970’s. Given an \(F\)-space \(X\), its galb, denoted by \(\mathcal{G}(X)\), is the vector space of all sequences \((a_n)_{n=1}^{\infty}\) of scalars such that whenever \((x_n)_{n=1}^{\infty}\) is a sequence in \(X\) with \(\lim x_n = 0\), the series \(\sum_{n=1}^{\infty} a_n x_n\) converges in \(X\). We say that a sequence space \(\mathcal{Y}\) galbs \(X\) if \(\mathcal{Y} \subseteq \mathcal{G}(X)\). With this terminology, \(X\) is \(p\)-convex if and only if \(\ell_p \subseteq \mathcal{G}(X)\).

The galb of certain classical spaces is known. Turpin [31] computed the galb of locally bounded, non-locally convex Orlicz function spaces.
\( L_\phi(\mu) \), where \( \mu \) is either a nonatomic measure or the counting measure, and showed that the result is an Orlicz sequence space \( \ell_\phi \) modeled after a different Orlicz function \( \phi \). Hernández [12–14] continued the study initiated by Turpin and computed, in particular, the galb of certain vector-valued Orlicz spaces. The study of the convexity of Lorentz spaces took a different route. Before Turpin invented the notion of galb, Stein and Weiss [30] proved that the Orlicz sequence space \( \ell \log \ell \) for \( 0 < q < \infty \) was studied (see [9, 29]). In [8], general weighted Lorentz spaces were considered.

The geometry of spaces of galbs is quite unknown, however. Probably, the most significant advance in this direction since seminal Turpin work was made in [16]. Solving a question raised in [33], Kalton proved that if \( X \) is \( p \)-convex and is not \( q \)-convex for any \( q > p \), then \( \mathcal{G}(X) = \ell_p \).

In this paper, we use galbs to develop a theory of integration for quasi-Banach-space-valued functions in the spirit of Vogt that fits as well as possible the convexity of the target space. Our construction is closely related to tensor products, and to carry out it we construct topological tensor products adapted to our needs. More precisely, for an appropriate function quasi-norm \( \lambda \) over \( \mathbb{N} \) we define the tensor product space \( X \otimes_\lambda L_1(\mu) \) so that there are bounded linear canonical maps

\[
J: X \otimes_\lambda L_1(\mu) \to L_1(\mu, X), \quad x \otimes f \mapsto xf, \quad \text{and}
\]

\[
I: X \otimes_\lambda L_1(\mu) \to X, \quad x \otimes f \mapsto x \int_\Omega f \, d\mu.
\]

If \( I \) factors through \( J \), that is, there is a map \( \mathcal{I} \) (defined on the range of \( J \)) such that the diagram

\[
\begin{array}{c}
X \otimes_\lambda L_1(\mu) \\
\downarrow J \\
L_1^\lambda(\mu, X) := J(X \otimes_\lambda L_1(\mu)) \rightarrow \\
\rightarrow X
\end{array}
\]

commutes, then \( \mathcal{I} \) defines a suitable integral for functions in \( L_1^\lambda(X) \). Thus, we say that \((\lambda, X)\) is amenable if \( \lambda \) galbs \( X \) (i.e., \( (a_n)_{n=1}^\infty \in \mathcal{G}(X) \) whenever \( \lambda((a_n)_{n=1}^\infty) < \infty \)) and \( I \) factors through \( J \).

There is a tight connection between the existence of the integral \( \mathcal{I} \) and the injectivity of \( J \). In fact, we will prove that if \((\lambda, X)\) is amenable, then \( J \) is one-to-one (see Theorem 6.5). This connection leads us to
study the injectivity of $J$. More generally, we consider the map

$$J : X \otimes_\lambda L_\rho \to L_\rho(X)$$

associated with the quasi-Banach space $X$, the function quasi-norm $\lambda$ and a function quasi-norm $\rho$ over $(\Omega, \Sigma, \mu)$, and we obtain results that generalize those previously obtained for Lebesgue spaces $L_q(\mu)$ and tensor quasi-norms in the sense of $\ell_p$, $0 < p \leq q \leq \infty$ (see [34, Satz 4]).

With the terminology of this paper, Vogt proved that if $X$ is a $p$-Banach space, $0 < p \leq 1$, then $(\ell_p, X)$ is amenable. So, in order to exhibit the applicability of the theory of integration developed within this paper, we must exhibit new examples of amenable pairs. Since the space of galbs of the quasi-Banach space $X$ arises from a function quasi-norm on $N$, say $\lambda_X$, the question of whether the pair $(\lambda_X, X)$ is amenable arises. For answering it, one first need to know whether the space of galbs $G(X)$ is always 1-concave as a quasi-Banach lattice or not. See Questions 6.6 and 4.15. As long as there is no general answer to these questions, we focus on the spaces of galbs that have appeared in the literature. In Theorem 6.7 we prove that for all of them Question 6.6 has a positive answer.

Once the theory is built, the first goal should be the study of its integration properties. By construction, our integral behaves linearly and has suitable convergence properties. Hence, we finish with a fundamental theorem of calculus for our integral (see Theorem 7.1).

The paper is organized as follows. In Section 2 we introduce the terminology and notation that will be employed. The theory of function norms (i.e., the locally convex setting) has been deeply developed (cf. [6, 21]). However, a systematic study in the non-locally convex setting is missing. For that reason, in Section 3 we do a brief survey on function quasi-norms covering the most relevant aspects, and all the results that we need. Section 4 is devoted to galbs. In Section 5 we briefly collect some results on tensor products. In Section 6 we present our main results of integration for quasi-Banach-space-valued functions. Finally, in Section 7 we give a fundamental theorem of calculus that improves [1, Theorem 5.2].

2. Terminology

We use standard terminology and notation in Banach space theory as can be found, e.g., in [2]. The unfamiliar reader will find general information about quasi-Banach spaces in [20]. We next gather the notation on quasi-Banach spaces that we will use.
A quasi-normed space will be a vector space over the real or complex field $\mathbb{F}$ endowed with a quasi-norm, i.e., a map $\| \cdot \|: X \to [0, \infty)$ satisfying

(Q.1) $\|x\| = 0$ if and only if $x = 0$;
(Q.2) $\|tx\| = |t|\|x\|$ for $t \in \mathbb{F}$ and $x \in X$; and
(Q.3) there is a constant $\kappa \geq 1$ so that for all $x$ and $y$ in $X$ we have

$$\|x + y\| \leq \kappa(\|x\| + \|y\|).$$

The smallest number $\kappa$ in (Q.3) will be called the modulus of concavity of the quasi-norm. If it is possible to take $\kappa = 1$ we obtain a norm.

A quasi-norm clearly defines a metrizable vector topology on $X$ whose base of neighborhoods of zero is given by sets of the form $\{x \in X: \|x\| < 1/n\}$, $n \in \mathbb{N}$. Given $0 < p \leq 1$, a quasi-normed space is said to be $p$-convex if it has an absolutely $p$-convex neighborhood of the origin. A quasi-normed space $X$ is $p$-convex if and only if there is a constant $C$ such that

$$\left\| \sum_{j=1}^{n} x_j \right\|^p \leq C \sum_{j=1}^{n} \|x_j\|^p, \quad n \in \mathbb{N}, x_j \in X. \quad (2.1)$$

If, besides (Q.1) and (Q.2), (2.1) holds with $C = 1$ we say that $\| \cdot \|$ is a $p$-norm. Any $p$-norm is a quasi-norm with modulus of concavity at most $2^{1/p-1}$. A $p$-normed space is a quasi-normed space endowed with a $p$-norm. By the Aoki-Rolewicz theorem [5,25] any quasi-normed space is $p$-convex for some $0 < p \leq 1$. In turn, any $p$-convex quasi-normed space can be equipped with an equivalent $p$-norm. Hence, any quasi-normed space becomes, for some $0 < p \leq 1$, a $p$-normed space under suitable renorming.

A $p$-Banach (resp. quasi-Banach) space is a complete $p$-normed (resp. quasi-normed) space. It is known that a $p$-convex quasi-normed space is complete if and only if for every sequence $(x_n)_{n=1}^{\infty}$ in $X$ such that $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ the series $\sum_{n=1}^{\infty} x_n$ converges.

A semi-quasi-norm on a vector space $X$ is a map $\| \cdot \|: X \to [0, \infty)$ satisfying (Q.2) and (Q.3). A standard procedure, to which we refer as the completion method allow us to manufacture a quasi-Banach from a semi-quasi-norm (see e.g. [3, §2.2]).

As the Hahn-Banach Theorem depends heavily on convexity, it does not pass through general quasi-Banach spaces. In fact, there are quasi-Banach spaces as $L_p([0,1])$ for $0 < p < 1$ whose dual space is null (see [10]). Following [20], we say that the quasi-Banach space $X$ has point separation property if for every $f \in X \setminus \{0\}$ there is $f^* \in X^*$ such that $f^*(f) \neq 0$. 
For any subset $A$ of a quasi-Banach space we denote by $[A]$ its closed linear span.

Given a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ and a quasi-Banach space $X$, we denote by $L_0^+(\mu)$ the set consisting of all measurable functions from $\Omega$ into $[0, \infty]$, and by $L_0(\mu, X)$ the vector space consisting of all measurable functions from $\Omega$ into $X$. As usual, we identify almost everywhere (a.e. for short) coincident functions. We set $L_0(\mu) = L_0(\mu, \mathbb{F})$ and

$$\Sigma(\mu) = \{ A \in \Sigma : \mu(A) < \infty \}.$$  

We denote by $\mathcal{S}(\mu, X)$ the vector space consisting of all integrable $X$-valued simple functions. That is,

$$\mathcal{S}(\mu, X) = \{ x\chi_E : E \in \Sigma(\mu), x \in X \}.$$  

We say that $(\Omega, \Sigma, \mu)$ is infinite-dimensional if $\mathcal{S}(\mu) = \mathcal{S}(\mu, \mathbb{F})$ is.

Given $f \in L_0^+(\mu)$ we set

$$\Omega_f(s) = \{ \omega \in \Omega : f(\omega) > s \} \text{ and } \rho_f(s) = \rho(\chi_{\Omega_f(s)}), \quad s \in [0, \infty).$$  

Set also $\Omega_f(\infty) = \{ \omega \in \Omega : f(\omega) = \infty \}$ and $\rho_f(\infty) = \rho(\chi_{\Omega_f(\infty)})$. If $\rho$ is the function quasi-norm associated with $L_1(\mu)$, then $\mu_f := \rho_f$ is the distribution function of $f$. We say $f$ has a finite distribution function if $\mu_f(s) < \infty$ for all $s > 0$.

An order ideal in $L_0(\mu)$ will be a (linear) subspace $L$ of $L_0(\mu)$ such that $\overline{f} \in L$ whenever $f \in L$, and $\max\{f, g\} \in L$ whenever $f$ and $g$ are real-valued functions in $L$. A cone in $L_0^+(\mu)$ will be a subset $C$ of $L_0^+(\mu)$ such that for all $f, g \in C$ and all $\alpha, \beta \geq 0$ we have $f < \infty$ a.e., $\alpha f + \beta g \in C$, and $\max\{f, g\} \in C$. It is immediate that if $L$ is an order ideal in $L_0(\mu)$, then

$$L^+ := L \cap L_0^+(\mu)$$

is a cone in $L_0^+(\mu)$; and reciprocally, if $C$ is a cone in $L_0^+(\mu)$, there is a unique order ideal $L$ with $L^+ = C$. Namely,

$$L = \{ f \in L_0(\mu) : |f| \leq g \text{ for some } g \in C \}.$$

Given a quasi-Banach space $X$, we say that a quasi-Banach space $\mathbb{U}$ is complemented in $X$ via a map $S : \mathbb{U} \to X$ if there is a map $P : X \to \mathbb{U}$ such that $P \circ S = \text{Id}_X$.

The unit vector system is the sequence $(e_k)_{k=1}^\infty$ in $\mathbb{F}^\mathbb{N}$ defined by $e_k = (\delta_{k,n})_{n=1}^\infty$, where $\delta_{k,n} = 1$ if $k = n$ and $\delta_{k,n} = 0$ otherwise. A block basis sequence with respect to the unit vector system is a sequence $(f_k)_{k=1}^\infty$ such that

$$f_k = \sum_{n=1+n_k-1}^{n_k} a_n e_n, \quad k \in \mathbb{N}.$$
for some sequence \((a_n)_{n=1}^{\infty}\) in \(F^N\) and some increasing sequence \((n_k)_{k=0}^{\infty}\) of non-negative scalars with \(n_0 = 0\).

### 3. Function quasi-norms

As mentioned in the Introduction, in contrast to the theory of function norms, there is no systematic study in the non-locally convex setting. In this section we try to go one step forward in that direction. We begin with the basic properties of function quasi-norms. Here, we do not impose them to satisfy a Fatou property (something that Bennett and Sharpley [6] do for function norms). We devote a subsection to the study of this property. Then we study the properties of absolute continuity and domination for function quasi-norms, as well as Minkowski-type inequalities. We also discuss the use of conditional expectation (via the notion of leveling function quasi-norms), which will be relevant for the proof of Theorem 6.5. We conclude the section with some comments on function quasi-norms over \(N\) endowed with the counting measure, a specially important particular case.

**Definition 3.1.** A function quasi-norm over a \(\sigma\)-finite measure space \((\Omega, \Sigma, \mu)\) is a mapping \(\rho: L^+_0(\mu) \to [0, \infty]\) such that

\[(F.1)\] \(\rho(tf) = t\rho(f)\) for all \(t \geq 0\) and \(f \in L^+_0(\mu)\);

\[(F.2)\] if \(f \leq g\) a.e., then \(\rho(f) \leq \rho(g)\);

\[(F.3)\] if \(E \in \Sigma(\mu)\), then \(\rho(\chi_E) < \infty\);

\[(F.4)\] for every \(E \in \Sigma(\mu)\) and every \(\varepsilon > 0\), there is \(\delta > 0\) such that \(\mu(A) \leq \varepsilon\) whenever \(A \in \Sigma\) satisfies \(A \subseteq E\) and \(\rho(\chi_A) \leq \delta\); and

\[(F.5)\] there is a constant \(\kappa\) such that \(\rho(f + g) \leq \kappa(\rho(f) + \rho(g))\) for all \(f, g \in L^+_0(\mu)\).

The optimal \(\kappa\) in (F.5) is called the modulus of concavity of \(\rho\).

Notice that (F.4) implies that \(\rho(\chi_E) > 0\) for all \(E \in \Sigma\) with \(\mu(E) > 0\).

**Definition 3.2.** A function norm is a function quasi-norm with modulus of concavity 1. More generally, given \(0 < p \leq 1\), a function \(p\)-norm is a function \(\rho: L^+_0(\mu) \to [0, \infty]\) which satisfies (F.1)–(F.4), and

\[(F.6)\] \(\rho^p(f + g) \leq \rho^p(f) + \rho^p(g)\) for all \(f, g \in L^+_0(\mu)\).

The inequality \(a^p + b^p \leq 2^{1-p}(a+b)^p\) for all \(a, b \in [0, \infty]\) and \(p \in (0, 1]\) yields that any function \(p\)-norm is a function quasi-norm with modulus of concavity at most \(2^{1/p-1}\).

This generalization of the notion of a function norm follows ideas from [6] and [21]. Asides (F.5), the main differences between our definition and that adopted by Luxemburg and Zaanen in [21] lie in restricting ourselves to \(\sigma\)-finite spaces, and in imposing condition (F.3),
which, on the one hand, prevents from existing non null sets $E$ on which $\rho$ is trivial (in the sense that if $f \in L^+_0(\mu)$ is null outside $E$ then $\rho(f)$ is either 0 or $\infty$) and, on the other hand, guarantees the existence of enough functions with finite quasi-norm. Regarding the approach in [6], we point out that Bennet and Sharpley imposed a function norm to satisfy

(F.7) for every $E \in \Sigma(\mu)$ there is a constant $C = C_E$ such that

$$\int_E f \, d\mu \leq C_E \rho(f), \quad f \in L^+_0(\mu). \quad (3.1)$$

The most natural examples of functions quasi-norms are $L_p$-quasi-norms, $0 < p < \infty$, defined by

$$f \mapsto \left( \int_{\Omega} f^p \, d\mu \right)^{1/p}, \quad f \in L^+_0(\mu).$$

To avoid introducing cumbrous notations, sometimes the symbol $L_p(\mu)$ will mean the function quasi-norm defining the space $L_p(\mu)$ instead of the space itself, and the same convention will be used for Lorentz and Orlicz spaces. Since, if $\mu$ is not purely atomic and $0 < p < 1$, $L_p(\mu)$ does not satisfy (F.7), imposing this condition to all function quasi-norms is somewhat nonsense in the non-locally convex setting. Thus we impose its natural substitute (F.4) instead. Also, unlike Bennet and Sharpley, we do not a priori impose $\rho$ to satisfy Fatou property (see Section 3.1).

**Definition 3.3.** We say that a function quasi-norm $\rho$ is **rearrangement invariant** if every function $f \in L^+_0(\mu)$ with $\rho(f) < \infty$ has a finite distribution function, and $\rho(f) = \rho(g)$ whenever $\mu_f = \mu_g$.

The proof of the following lemma is based on the elementary inequality

$$s \rho_f(s) \leq \rho(f), \quad f \in L^+_0(\mu), \quad s \in [0, \infty].$$

**Lemma 3.4.** Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$.

(i) If $f \in S(\mu)$, then $\rho(|f|) < \infty$.
(ii) If $f \in L^+_0(\mu)$ satisfies $\rho(f) < \infty$, then $f < \infty$ a.e.
(iii) If $f \in L^+_0(\mu)$ satisfies $\rho(f) = 0$, then $f = 0$ a.e.
(iv) Let $E \in \Sigma(\mu)$, $s > 0$, and $\epsilon > 0$. Then there is $\delta > 0$ such that for all $f \in L^+_0(\mu)$ with $\rho(f) \leq \delta$ we have

$$\mu(\{\omega \in E : f(\omega) > s\}) \leq \epsilon.$$
Proof. Statement (i) is clear. Now let \( f \in L^+_0(\mu) \). If \( \rho(f) \) is finite, then \( \rho(f(\infty)) = 0 \) and (ii) follows. If \( \rho(f) = 0 \), then \( \rho(f(s)) = 0 \) for all \( s > 0 \). Since \( \Omega_f(0) = \bigcup_{n=1}^{\infty} \Omega_f(2^{-n}) \), we obtain (iii). Finally, let \( E \in \Sigma(\mu) \), \( s > 0 \), and \( \varepsilon > 0 \). By (F.4), there is \( \delta > 0 \) such that if \( A \subseteq E \) with \( \rho(\chi_A) \leq \delta \), then \( \mu(A) \leq \varepsilon \). Take \( \delta := s\delta \), and let \( f \in L^+_0(\mu) \) with \( \rho(f) \leq \delta \). Set \( A := \{ \omega \in E : f(\omega) > s \} \). Since \( \rho(\chi_A) \leq \rho(f)/s \leq \delta \), we obtain (iv). \( \square \)

**Definition 3.5.** A function quasi-norm \( \rho \) is said to be \( p \)-convex if there is a constant \( C \) such that

\[
\rho^p \left( \sum_{j=1}^{n} f_j \right) \leq C \sum_{j=1}^{n} \rho^p(f_j), \quad n \in \mathbb{N}, \ f_j \in L^+_0(\mu).
\]

**Proposition 3.6** (Aoki-Rolewicz Theorem for function quasi-norms). Any function quasi-norm is \( p \)-convex for some \( 0 < p \leq 1 \). Indeed, if \( \kappa \) is the modulus of concavity we can choose \( p \) such that \( 2^{1/p - 1} = \kappa \).

**Proof.** It goes over the lines of the proof of the Aoki-Rolewicz Theorem (see e.g. [20, Lemma 1.1]). So, we omit it. \( \square \)

**Definition 3.7.** Given two function quasi-norms \( \rho \) and \( \lambda \) over a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \), we say that \( \rho \) dominates \( \lambda \) if there is a constant \( C \) such that \( \lambda(f) \leq C\rho(f) \) for all \( f \in L^+_0(\mu) \). If \( \rho \) dominates and is dominated by \( \lambda \), we say that \( \rho \) and \( \lambda \) are equivalent.

**Lemma 3.8.** Let \( 0 < p \leq 1 \), and let \( \rho \) be a function quasi-norm. Then \( \rho \) is equivalent to a function \( p \)-norm if and only if it is \( p \)-convex.

**Proof.** It is clear that any function \( p \)-norm is \( p \)-convex, and \( p \)-convexity is inherited by passing to an equivalent function quasi norm. Reciprocally, if \( \rho \) is a \( p \)-convex function quasi-norm over a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \), then it is immediate that the map map \( \lambda: L^+_0(\mu) \to [0, \infty] \) given by

\[
\lambda(f) = \inf \left\{ \left( \frac{\sum_{j=1}^{n} \rho^p(f_j)}{n} \right)^{1/p} : n \in \mathbb{N}, \ f = \sum_{j=1}^{n} f_j \right\}
\]

is a function \( p \)-norm equivalent to \( \rho \). \( \square \)

**Corollary 3.9.** Any function quasi-norm is equivalent to a function \( p \)-norm for some \( 0 < p \leq 1 \).

**Proof.** It follows from Proposition 3.6 and Lemma 3.8. \( \square \)

In light of Corollary 3.9, it is natural, and convenient in some situations, to restrict ourselves to function quasi-norms that are function \( p \)-norms for some \( p \). However, we emphasize that some \( p \)-convex
spaces arising naturally in Mathematical Analysis are given by a function quasi-norm that is not a $p$-norm. Take, for instance the 1-convex (i.e., locally convex) function space $L_{r,\infty}$, $r > 1$. So, when working in the general framework of non-locally convex spaces, it is convenient to know whether a given property pass to equivalent function quasi-norms.

**Definition 3.10.** Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, and let $\mathbb{X}$ be a quasi-Banach space. The space
\[
L_\rho(\mathbb{X}) = \{ f \in L_0(\mu, \mathbb{X}): \|f\|_\rho := \rho(\|f\|) < \infty \}
\]
endowed with the gauge $\| \cdot \|_\rho$ will be called the vector-valued Köthe space associated with $\rho$ and $\mathbb{X}$. The space $L_\rho = L_\rho(\mathbb{F})$ will be called the Köthe space associated with $\rho$.

Note that we do not impose the functions in $L_\rho(\mathbb{X})$ to be strongly measurable. If $\rho$ is the function quasi-norm associated to the Lebesgue space $L^p(\mu)$, $0 < p < \infty$, we set $L^p(\mu, \mathbb{X}) := L_\rho(\mathbb{X})$. If $A \in \Sigma$, we set $L^p(A, \mu, \mathbb{X}) := L^p(\mu|_A, \mathbb{X})$, where $\mu|_A$ is the restriction of $\mu$ to $\Sigma \cap P(A)$.

In general, if $\rho|_A$ is the function quasi-norm defined by $\rho|_A(f) = \rho(\tilde{f})$, where
\[
\tilde{f}(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in A, \\ 0 & \text{otherwise,} \end{cases}
\]
we set $L_\rho(A, \mathbb{X}) = L_{\rho|_A}(\mathbb{X})$.

It is clear that $L_\rho$ is an order ideal in $L_0(\mu)$. By Lemma 3.4 (ii), its cone is given by
\[
L_\rho^* = \{ f \in L_0^*(\mu): \rho(f) < \infty \}.
\]

** Lemma 3.11.** Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ and $\mathbb{X}$ be a quasi-Banach space.

(i) $L_\rho(\mathbb{X})$ is a quasi-normed space.

(ii) $S(\mu, \mathbb{X}) \subseteq L_\rho(\mathbb{X})$.

(iii) If we endow $L_0(\mu, \mathbb{X})$ with the vector topology of the local convergence in measure, then $L_\rho(\mathbb{X}) \subseteq L_0(\mu, \mathbb{X})$ continuously.

(iv) If $\mathcal{K}$ is a closed subset of $\mathbb{X}$, then $L_\rho(\mathcal{K}) := \{ f \in L_\rho(\mathbb{X}): f(\omega) \in \mathcal{K} \text{ a.e. } \omega \in \Omega \}$ is closed in $L_\rho(\mathbb{X})$.

**Proof.** Statements (i), (ii), and (iii) are straightforward from the very definition of function quasi-norm and Lemma 3.4. Now let $\mathcal{K}$ be a closed subset of $\mathbb{X}$, and let $x$ be a function in $L_\rho(\mathbb{X}) \setminus L_\rho(\mathcal{K})$ (assuming that this set is non-empty). There is $\varepsilon > 0$ and $A \subseteq \Sigma$ with $\mu(A) > 0$ such that $\|x(a) - k\| \geq \varepsilon$ for all $a \in A$ and all $k \in \mathcal{K}$. Therefore $\|x - y\|_\rho \geq \varepsilon \rho(\chi_A) > 0$ for all $y \in L_\rho(\mathcal{K})$, and we obtain (iv). \qed
Lemma 3.12. Let $\rho$ be a function quasi-norm, and let $X$ be a Banach space. If a sequence $(x_n)_{n=1}^\infty$ converges to $x$ in $L_\rho(X)$, then $(\|x_n\|)_{n=1}^\infty$ converges to $\|x\|$ in $L_\rho$.

Proof. It follows from the inequality $\|x_n\| - \|x\| \leq \|x_n - x\|$ for all $n \in \mathbb{N}$.

Proposition 3.13. Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, let $X$ be a quasi-Banach space, and let $(x_n)_{n=1}^\infty$ be a sequence in $L_0(\mu, X)$ such that $\lim_n \|x_n - x\|_\rho = 0$ for some $x \in L_0(\mu, X)$. Then, there is a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $\lim_n y_n(\omega) = x(\omega)$ a.e. $\omega \in A$.

Proof. Let $(A_j)_{j=1}^\infty$ be an increasing sequence of finite-measure sets such that $x_n$ is null outside $A = \bigcup_{j=1}^\infty A_j$ for all $n \in \mathbb{N}$. Then $\rho(\|x\|\chi_{\Omega\setminus A}) = 0$. Therefore $x(\omega) = 0$ a.e. $\omega \in \Omega \setminus A$. By Lemma 3.11 (iii), for each $j \in \mathbb{N}$ there is an increasing sequence $(n_k)_{k=1}^\infty$ such that $\lim_k x_{n_k}(\omega) = x(\omega)$ a.e. $\omega \in A_j$. The Cantor diagonal technique yields a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $\lim_n y_n(\omega) = x(\omega)$ a.e. $\omega \in A$.

3.1. The Fatou property.

Definition 3.14. Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. We say that $\rho$ has the rough Fatou property if there is a constant $C$ such that $\rho(\lim_n f_n) \leq C \lim_n \rho(f_n)$ whenever $(f_n)_{n=1}^\infty$ is a non-decreasing sequence in $L_0^+(\mu)$. If the above holds with $C = 1$ we say that $\rho$ has the Fatou property. We say that $\rho$ has the weak Fatou property if $\rho(\lim_n f_n) < \infty$ whenever the non-decreasing sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$ satisfies $\lim_n \rho(f_n) < \infty$.

Fatou property does not pass to equivalent function quasi-norms. In contrast, both rough and weak Fatou property are preserved. In fact, these two notions are equivalent.

Proposition 3.15 (cf. [4, Lemma]). If $\rho$ is a function quasi-norm with the weak Fatou property, then it also has the rough Fatou property.

Proof. Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. By Corollary 3.9, we can assume without loss of generality that it is a function $p$-norm for some $0 < p \leq 1$. Suppose that $\rho$ does not have the rough Fatou property. Then, for each $k \in \mathbb{N}$ there is a non-decreasing sequence $(f_{k,n})_{n=1}^\infty$ in $L_0^+(\mu)$ with $\sup_n \rho(f_{k,n}) \leq 1$ and $\rho(\lim_n f_{k,n}) > 2^{2k/p}$. The sequence $(g_n)_{n=1}^\infty$ defined by

$$g_n = \sum_{k=1}^n 2^{-k/p} f_{k,n}, \quad n \in \mathbb{N},$$
is non-decreasing, and we have
\[ 2^{-k/p} f_{k,n} \leq g := \lim_{n} g_n, \quad k \leq n. \]
Then \( \rho(g) \geq 2^{-k/p} \rho(\lim_{n} f_{k,n}) > 2^{k/p} \) for all \( k \in \mathbb{N} \). That is, \( \rho(g) = \infty \).

On the other hand, since \( \rho \) is a function \( p \)-norm, \( \rho^p(g_n) \leq \sum_{k=1}^{n} 2^{-k} \leq 1 \) for all \( n \in \mathbb{N} \). Therefore \( \rho \) does not have the weak Fatou property. \( \square \)

**Proposition 3.16** (cf. [6, Theorem 1.8]). Let \( \lambda \) and \( \rho \) be two function quasi-norms over the same \( \sigma \)-finite measure space. Suppose that \( \rho \) has the weak Fatou property. Then \( \rho \) dominates \( \lambda \) if and only if \( L_\rho^+ \subseteq L_\lambda^+ \).

**Proof.** The direct implication is obvious. Suppose now that \( \rho \) does not dominate \( \lambda \). Then there is a sequence \((f_n)_{n=1}^{\infty}\) in \( L_0^+(\mu) \) such that \( 4^n \rho(f_n) < \lambda(f_n) \) for all \( n \in \mathbb{N} \). Set
\[ f = \sum_{n=1}^{\infty} \frac{2^{-n}}{\rho(f_n)} f_n. \]
Using that \( \rho \) has the rough Fatou property (due to Proposition 3.15) and Proposition 3.6, we obtain that \( \rho(f) < \infty \). Since
\[ 2^{-n} \lambda(f_n) \leq \sup_n \frac{2^{-n} \lambda(f_n)}{\rho(f_n)} \leq \sup_n 2^n = \infty, \]
the space \( L_\rho^+ \) is not contained in \( L_\lambda^+ \). \( \square \)

**Definition 3.17.** Let \( 0 < p \leq 1 \) and let \( \rho \) be a function quasi-norm over a \( \sigma \)-finite measure space \((\Omega, \Sigma, \mu)\). We say that \( \rho \) has the Riesz-Fischer \( p \)-property if for every sequence \((f_n)_{n=1}^{\infty}\) in \( L_0^+(\mu) \) with \( \sum_{n=1}^{\infty} \rho^p(f_n) < \infty \) we have \( \rho(\sum_{n=1}^{\infty} f_n) < \infty \).

**Lemma 3.18** (cf. [4, Theorem]). Let \( \rho \) be a \( p \)-convex function quasi-norm with the weak Fatou property. Then \( \rho \) has the Riesz-Fischer \( p \)-property.

**Proof.** Let \((f_n)_{n=1}^{\infty}\) be a sequence in \( L_0^+(\mu) \) with \( A := \sum_{n=1}^{\infty} \rho^p(f_n) < \infty \). If \( C \) denotes the \( p \)-convexity constant of \( \rho \), then
\[ \rho \left( \sum_{n=1}^{m} f_n \right) \leq C^{1/p} \left( \sum_{n=1}^{m} \rho^p(f_n) \right)^{1/p} \leq C^{1/p} A^{1/p}, \quad m \in \mathbb{N}. \]
Hence \( \lim_{m} \rho(\sum_{n=1}^{m} f_n) < \infty \), and therefore \( \rho(\sum_{n=1}^{\infty} f_n) < \infty \) (since \( \rho \) has the weak Fatou property). That is, \( \rho \) has the Riesz-Fischer \( p \)-property. \( \square \)

**Proposition 3.19.** Let \( 0 < p \leq 1 \) and let \( \rho \) be a function quasi-norm over a \( \sigma \)-finite measure space \((\Omega, \Sigma, \mu)\). The following are equivalent.
(i) \( \rho \) has the Riesz-Fischer \( p \)-property.
(ii) There is a constant \( C \) such that \( \rho^p(\sum_{n=1}^{\infty} f_n) \leq C \sum_{n=1}^{\infty} \rho^p(f_n) \) for every sequence \( (f_n)_{n=1}^{\infty} \) in \( L^+_0(\mu) \).
(iii) \( L_p(\mathbb{X}) \) is a quasi-Banach space for any (resp. some) nonzero quasi-Banach space \( \mathbb{X} \).
(iv) \( L_p \) is a \( p \)-convex quasi-Banach space.

Moreover, the optimal constant in (ii) is the \( L_\kappa \) convexity constant of \( L_p \). In particular, \( L_p \) is a \( p \)-Banach space if and only if (ii) holds with \( C = 1 \).

**Proof.** Assume that (ii) does not hold. Then for every \( k \in \mathbb{N} \) there is a sequence \( (f_{k,n})_{n=1}^{\infty} \) in \( L^+_0(\mu) \) such that

\[
\rho^p \left( \sum_{n=1}^{\infty} f_{k,n} \right) \geq k \quad \text{and} \quad \sum_{n=1}^{\infty} \rho^p(f_{k,n}) \leq 2^{-k}.
\]

Then \( \sum_{(k,n) \in \mathbb{N}^2} \rho^p(f_{k,n}) \leq 1 \), and also

\[
\rho^p \left( \sum_{(k,n) \in \mathbb{N}^2} f_{k,n} \right) \geq \rho^p \left( \sum_{n=1}^{\infty} f_{k,n} \right) \geq k
\]

for all \( k \in \mathbb{N} \). That is, \( \rho(\sum_{(k,n) \in \mathbb{N}^2} f_{k,n}) = \infty \). So (i) does not hold. In other words, (i) implies (ii).

Now assume (ii). Let \( \mathbb{X} \) be a nonzero quasi-Banach space with modulus of concavity \( \kappa \). By Lemma 3.11 (i), we already know that \( L_\rho(\mathbb{X}) \) is a quasi-normed space. Therefore, in order to obtain (iii), it suffices to prove that the series \( \sum_{n=1}^{\infty} f_n \) converges in \( L_\rho(\mathbb{X}) \) for every sequence \( (f_n)_{n=1}^{\infty} \) in \( L_\rho(\mathbb{X}) \) such that

\[
\sum_{n=1}^{\infty} \kappa^n \rho^p(p_{f_n}) < \infty. \quad (3.2)
\]

Using (ii) and Lemma 3.4 (ii), we obtain that \( \sum_{n=1}^{\infty} \kappa^n \| f_n \| \) converges a.e. in \( \Omega \); say it converges in \( \Omega \setminus N \) where \( \mu(N) = 0 \). Set \( g_n := f_n \chi_{\Omega \setminus N} \). Obviously \( \| g_n \| \leq \| f_n \| \), so \( \rho(\| g_n \|) \leq \rho(\| f_n \|) \) for all \( n \in \mathbb{N} \). Then (3.2) is also true if we put \( g_n \) instead of \( f_n \).

For all \( M, N \in \mathbb{N} \) with \( M \geq N \), we have \( \| \sum_{n=N}^{M} g_n \| \leq \sum_{n=N}^{M} \kappa^n \| g_n \| \). Since \( \sum_{n=1}^{\infty} \kappa^n \| g_n(t) \| \) converges for all \( t \in \Omega \), \( \sum_{n=1}^{\infty} g_n(t) \) is a Cauchy sequence in \( \mathbb{X} \). Therefore \( \sum_{n=1}^{\infty} g_n(t) =: f(t) \) converges for all \( t \in \Omega \). Let us see that \( \sum_{n=1}^{\infty} f_n \) converges to \( f \) in \( L_\rho(\mathbb{X}) \).

Notice that if a sequence \( (x_n)_{n=1}^{\infty} \) converges to \( x \) in \( \mathbb{X} \), since \( \| x \| \leq \kappa \| x \| + \kappa \| x - x_n \| \), we have \( \| x \| \leq \kappa \liminf_n \| x_n \| \). Recall that if two
functions $u, v$ in $L_0^+(\mu)$ are equal a.e., then $\rho(u) = \rho(v)$. Hence
\[
\rho \left( \left\| f - \sum_{n=1}^m f_n \right\| \right) = \rho \left( \left\| f - \sum_{n=1}^m g_n \right\| \right) = \rho \left( \left\| \sum_{n=m+1}^\infty g_n \right\| \right)
\leq \rho \left( \kappa \liminf_{M \to \infty} \left\| \sum_{n=m+1}^M g_n \right\| \right)
\leq \kappa \rho \left( \sum_{n=m+1}^\infty \kappa^n \rho(\|g_n\|) \right)^{1/p} \to 0.
\]

Therefore, we have proved that (ii) implies (iii).

Suppose that $L_\rho(X)$ is a quasi-Banach space for some nonzero quasi-Banach space $X$. Take a nonzero vector $x$ in $X$. Since obviously $\mathbb{F}$ is isomorphic to $\{tx: t \in \mathbb{F}\}$, which is a closed subset of $X$, it follows that $L_\rho$ is a quasi-Banach space using Lemma 3.11 (iv). By the Aoki-Rólewicz theorem, $L_\rho$ is $p$-convex for some $0 < p \leq 1$. Hence (iii) implies (iv).

Finally, assume that (iv) holds. Let $(f_n)_{n=1}^\infty$ be a sequence in $L_0^+(\mu)$ such that $\sum_{n=1}^\infty \rho^p(f_n) < \infty$. Since $L_\rho$ is $p$-convex (with constant $C$), for all $M, N \in \mathbb{N}$ with $M \geq N$ we have
\[
\rho \left( \sum_{n=N}^M f_n \right) \leq C^{1/p} \left( \sum_{n=N}^M \rho^p(f_n) \right)^{1/p}.
\]

Therefore $(\sum_{n=1}^m f_n)_{m=1}^\infty$ is a Cauchy sequence in the quasi-Banach space $L_\rho(X)$, so it converges to a function $f$ in $L_\rho(X)$. By Proposition 3.13, there is a subsequence $(\sum_{n=1}^{m_j} f_n)_{j=1}^\infty$ that converges to $f$ a.e., say in $\Omega \setminus \mathcal{N}$ where $\mu(\mathcal{N}) = 0$. Since $(\sum_{n=1}^\infty f_n)_{m=1}^\infty$ is non-decreasing, it follows that it converges to $f$ in $\Omega \setminus \mathcal{N}$. That is, $\sum_{n=1}^\infty f_n = f$ a.e., and therefore $\rho(\sum_{n=1}^\infty f_n) = \rho(f) < \infty$. Hence (iv) implies (i).

\section*{3.2. Absolute continuity and domination.}

\begin{dfn}
Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. We say that $f \in L_\rho^+$ is absolutely continuous with respect to $\rho$ if
\[
\lim_n \rho(f_n) = \rho(\lim_n f_n)
\]
for every non-increasing sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$ with $f_1 \leq f$. If the above holds only in the case when $\lim_n f_n = 0$, we say that $f$ is dominating. We denote by $L_\rho^a$ (resp. $L_\rho^d$) the set consisting of all $f \in L_0(\mu)$ such that $|f|$ is absolutely continuous (resp. dominating). We say that $\rho$ is absolutely continuous (resp. dominating) if $L_\rho^a = L_\rho$.
\end{dfn}
(resp. $L^d_\rho = L_\rho$). If $\chi_E \in L^d_\rho$ (resp. $L^d_\rho$) for every $E \in \Sigma(\mu)$, we say that $\rho$ is \emph{locally absolutely continuous} (resp. locally dominating).

Notice that domination is preserved under equivalence of function quasi-norms, but absolute continuity is not. Proposition 3.21 below yields that if the function quasi-norm is continuous (in the sense that $\lim_n \|x_n\|_\rho = \|x\|_\rho$ whenever $(x_n)_{n=1}^\infty$ and $x$ in $L_\rho$ satisfy $\lim_n \|x_n - x\|_\rho = 0$), then both concepts are equivalent. Notice that any function $p$-norm, $0 < p \leq 1$, is continuous. So, the existence of non-continuous function quasi-norms is a ‘pathology’ which only occurs in the non-locally convex setting. We must point out that, since it is by no means clear whether absolutely continuous norms are continuous, the terminology could be somewhat confusing. Notwithstanding, we prefer to use terminology similar to that it is customary within framework of function norms.

**Proposition 3.21.** Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. Suppose that $f \in L^+_\rho$ is dominating. Then $\lim_n x_n = x$ in $L_\rho(X)$ for every quasi-Banach space $X$ and every sequence $(x_n)_{n=1}^\infty$ in $L_0(\mu, X)$ with $\lim_n x_n = x$ a.e. and $\|x_n\| \leq f$ a.e. for all $n \in \mathbb{N}$.

**Proof.** Let $N$ be a null set such that $\sup_n \|x_n(\omega)\| \leq f(\omega) < \infty$ and $\lim_n x_n(\omega) = x(\omega)$ for all $\omega \in \Omega \setminus N$. Then $\|x(\omega)\| \leq \kappa f(\omega)$ for all $\omega \in \Omega \setminus N$, where $\kappa$ is the modulus of concavity of the quasi-norm $\|\cdot\|$. Set
\[
 f_n = \sup_{j \geq n} \|x_j - x\| \chi_{\Omega \setminus N}, \quad n \in \mathbb{N}.
\]

The sequence $(f_n)_{n=1}^\infty$ in $L^+_0(\mu)$ decreases to 0, and $\sup_n f_n \leq \kappa (\kappa + 1) f$. Consequently, $\lim_n \rho(f_n) = 0$. Since $\|x_j - x\|_\rho \leq \rho(f_n)$ whenever $j \geq n$ we are done. \hfill \Box

**Proposition 3.22** (cf. [6, Proposition 3.6]). Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, and let $f$ be a function in $L^+_\rho$. Then, $f$ is dominating if and only if
\[
 \lim_n \rho(f \chi_{A_n}) = 0
\]
whenever the sequence $(A_n)_{n=1}^\infty$ in $\Sigma$ decreases to $\emptyset$.

**Proof.** The direct implication is obvious. Conversely, suppose that $\rho(f \chi_{A_n}) \to 0$ whenever $(A_n)_{n=1}^\infty$ decreases to $\emptyset$. Let $(f_n)_{n=1}^\infty$ be a non-increasing sequence of functions in $L^+_0(\mu)$ such that $f_1 \leq f$ and $f_n \to 0$. Let us prove that $\rho(f_n) \to 0$.

Let $\kappa$ be the modulus of concavity of $\rho$, and fix $\varepsilon > 0$.\hfill \Box
Assume first that \( \mu(\Omega) < \infty \). Then \( \rho(\chi_{\Omega}) < \infty \), and we can set \( s = \varepsilon/(2\kappa \rho(\chi_{\Omega})) \). For each \( n \in \mathbb{N} \), let \( B_n = \{ f_n < s \} \subseteq \Omega \). It is a non-decreasing sequence in \( \Sigma \) whose union is \( \Omega \). Since \( f_n \leq f\chi_{\Omega\setminus B_n} + s\chi_{B_n} \), we have

\[
\rho(f_n) \leq \kappa \rho(f\chi_{\Omega\setminus B_n}) + \kappa s \rho(\chi_{B_n}) \leq \kappa \rho(f\chi_{\Omega\setminus B_n}) + \varepsilon/2 < \varepsilon
\]

for \( n \) sufficiently large.

Now suppose that \( \mu(\Omega) = \infty \). Let \( (\Omega_m)_{m=1}^{\infty} \) be a non-decreasing sequence in \( \Sigma(\mu) \) whose union is \( \Omega \). Take \( m \) such that \( \kappa \rho(f\chi_{\Omega\setminus \Omega_m}) < \varepsilon/2 \). Since \( f_n \leq f_n\chi_{\Omega_m} + f\chi_{\Omega\setminus \Omega_m} \), using that \( \mu(\Omega_m) < \infty \) and the previous case, we have

\[
\rho(f_n) \leq \kappa \rho(f_n\chi_{\Omega_m}) + \kappa \rho(f\chi_{\Omega\setminus \Omega_m}) \leq \kappa \rho(f_n\chi_{\Omega_m}) + \varepsilon/2 < \varepsilon
\]

for \( n \) sufficiently large.

Given a function quasi-norm \( \rho \) and a set \( E \in \Sigma \) we define

\[
\Phi[E, \rho](t) = \sup \{ \rho(\chi_A) : A \in \Sigma, A \subseteq E, \mu(A) \leq t \},
\]

and we set \( \Phi[\rho] = \Phi[\Omega, \rho] \). Notice that the function \( \Phi[E, \rho] \) is non-negative and non-decreasing. In particular, there exists the limit of \( \Phi[E, \rho](t) \) when \( t \to 0^+ \).

**Corollary 3.23.** A function quasi-norm \( \rho \) is locally dominating if and only if \( \lim_{t \to 0^+} \Phi[E, \rho](t) = 0 \) for every \( E \in \Sigma(\mu) \).

**Proof.** If \( \lim_{t \to 0^+} \Phi[E, \rho](t) = 0 \) for every \( E \in \Sigma(\mu) \), using Proposition 3.22 we obtain that \( \rho \) is locally dominating. Now assume that \( s := \lim_{t \to 0^+} \Phi[E, \rho](t) > 0 \) for some \( E \in \Sigma(\mu) \). Then there is a sequence \( (A_n)_{n=1}^{\infty} \) of measurable subsets of \( E \) such that \( \mu(A_n) \leq 1/2^n \) and \( \rho(\chi_{A_n}) > s/2 \) for all \( n \in \mathbb{N} \). Set \( B_n = \bigcup_{k=n}^{\infty} A_k \). The sequence \( (B_n)_{n=1}^{\infty} \) decreases to a null set and \( \rho(\chi_{B_n}) \geq s/2 \) for all \( n \in \mathbb{N} \), so \( \chi_{B_1} \) is not dominating. Hence \( \rho \) is not locally dominating. \( \square \)

**Definition 3.24.** Let \( \rho \) be a function quasi-norm over a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \). We say say \( L \subseteq L_{\rho} \) is an order ideal with respect to \( \rho \) if it is an order ideal and it is closed in \( L_{\rho} \).

**Lemma 3.25** (cf. [6, Theorem 3.8]). Let \( \rho \) be a function quasi-norm over a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \). Then \( L^d_{\rho} \) is an order ideal with respect to \( \rho \).

**Proof.** It is straightforward that \( L^d_{\rho} \) is a subspace of \( L_0(\mu) \). If a function \( f \) belongs to \( L^d_{\rho} \), obviously \( f \) also belongs to \( L^d_{\rho} \). Let \( f \) and \( g \) be real-valued functions in \( L^d_{\rho} \). Set \( A = \{ \omega \in \Omega : |f(\omega)| < |g(\omega)| \} \). Let \( (h_n)_{n=1}^{\infty} \) be a sequence in \( L^+_0(\mu) \) decreasing to 0 with \( h_1 \leq \max\{|f|, |g|\} \). Since
\[ |f| \text{ and } |g| \text{ are dominating, } h_1 \chi_A \leq |g|, \text{ and } h_1 \chi_{\Omega \setminus A} \leq |f|, \text{ we obtain that } \lim_n \rho(h_n \chi_A) = 0 \text{ and } \lim_n \rho(h_n \chi_{\Omega \setminus A}) = 0. \text{ Hence } \lim_n \rho(h_n) = 0. \text{ Therefore } \max\{|f|, |g|\} \text{ is dominating. This implies that } \max\{f, g\} \text{ is also dominating, so } L^d_\rho \text{ is an order ideal.}

Now we prove that \( L^d_\rho \) is closed in \( L_\rho \). Let \((f_j)_{j=1}^\infty\) be a sequence in \( L^d_\rho \) that converges in \( L_\rho \) to a function \( f \). Let \((g_n)_{n=1}^\infty\) be a non-increasing sequence in \( L^+_0(\mu) \) with \( g_1 \leq |f| \) and \( \lim_n g_n = 0 \). Then \( g_n \leq \min\{g_n, |f_j|\} + |f - f_j| \) for each \( j \in \mathbb{N} \). Consequently, if \( \kappa \) is the modulus of concavity of \( \rho \), we have

\[ \rho(g_n) \leq \kappa \rho(\min\{g_n, |f_j|\}) + \kappa \rho(|f - f_j|). \]

Hence \( \lim_n \rho(g_n) = 0 \). So \(|f|\) is dominating, as we wanted to prove. \( \Box \)

**Definition 3.26.** Let \( \rho \) be a function quasi-norm over a \( \sigma \)-finite measure space \((\Omega, \Sigma, \mu)\). We denote by \( L^b_\rho \) the closure of \( S(\mu) \) in \( L_\rho \). We say that \( \rho \) is minimal if \( L^d_\rho = L_\rho \).

**Lemma 3.27 (cf. [6, Proposition 3.10]).** Let \( \rho \) be a function quasi-norm over a \( \sigma \)-finite measure space \((\Omega, \Sigma, \mu)\). Then \( L^b_\rho \) is an order ideal with respect to \( \rho \). Moreover \( L^{b,+}_\rho \) is the closure in \( L_\rho \) of

\[ \mathcal{C} = \{f \in L^+_0(\mu) : \|f\|_\infty < \infty, \quad \mu_f(0) < \infty\}. \]

**Proof.** It is obvious that \( L^b_\rho \) is an order ideal in \( L_0(\mu) \), and it is closed in \( L^b_\rho \) by definition. Hence \( L^b_\rho \) is an order ideal with respect to \( \rho \).

Let \( f \) be a function in \( \mathcal{C} \), and set \( E := \{0 < f < \infty\} \subseteq \Omega \). Since \( \mu(E) < \infty \), we have \( \rho(\chi_E) < \infty \). Fix \( \varepsilon > 0 \), and let \( 0 \leq g \leq f \) be a simple function such that \( \|f - g\|_\infty < \varepsilon / \rho(\chi_E) \). Then

\[ \rho(f - g) \leq \|f - g\|_\infty \rho(\chi_E) < \varepsilon. \]

This means that \( \mathcal{C} \) is contained in \( L^{b,+}_\rho \). Therefore, the closure of \( \mathcal{C} \) in \( L_\rho \) is also contained in \( L^{b,+}_\rho \). On the other hand, it is obvious that every non-negative simple function which is finite a.e. belongs to \( \mathcal{C} \). So the second part of the statement follows. \( \Box \)

**Proposition 3.28 (cf. [6, Theorem 3.11]).** For any function quasi-norm \( \rho \) over a \( \sigma \)-finite measure space \((\Omega, \Sigma, \mu)\) we have \( L^{d,+}_\rho \subseteq L^{b,+}_\rho \).

**Proof.** It is enough to prove that \( L^{d,+}_\rho \subseteq L^{b,+}_\rho \). Let \( f \) be a function in \( L^{d,+}_\rho \). Let \((A_n)_{n=1}^\infty\) be an increasing sequence in \( \Sigma(\mu) \) whose union is \( \{f > 0\} \subseteq \Omega \). Pick an increasing sequence \((f_j)_{j=1}^\infty\) of measurable positive simple functions with \( \lim_n f_n = f \). We have \( \lim_n \rho(f - f \chi_{A_n}) = 0 \) and \( \lim_j \rho(f \chi_{A_n} - f_j \chi_{A_n}) = 0 \) for each \( n \in \mathbb{N} \). Since \( f_j \chi_{A_n} \in L^{b,+}_\rho \) for all \( j, n \in \mathbb{N} \), we infer that \( f \in L^{b,+}_\rho \). \( \Box \)
Corollary 3.29. A function quasi-norm \( \rho \) is locally dominating if and only if \( L^d_\rho = L^b_\rho \). Moreover if \( \rho \) is dominating, then \( \rho \) is minimal.

Proof. It is a straightforward consequence of Proposition 3.28.

Since we could need to deal with non-continuous function quasi-norms, we give some results pointing to ensure that \( \lim_n \| x_n \|_\rho = \| x \|_\rho \) under the assumption that \( (x_n)_{n=1}^\infty \) converges to \( x \).

Lemma 3.30. Let \( \rho \) be a function quasi-norm over a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \) with the Fatou property, and let \( (f_n)_{n=1}^\infty \) be a sequence in \( L^0_\rho(\mu) \). Then

\[
\rho(\liminf_n f_n) \leq \liminf_n \rho(f_n).
\]

Proof. Just apply Fatou property to \( \inf_{k \geq n} f_k, n \in \mathbb{N} \).

Lemma 3.31. Let \( \rho \) be a function quasi-norm with the Fatou property and \( X \) be a Banach space. If \( x \in L_\rho(X) \) and \( (x_n)_{n=1}^\infty \subseteq L_\rho(X) \) satisfy \( \sup_n \| x_n \| \leq \| x \| \) and \( \lim_n \rho(\| x_n - x \|) = 0 \), then \( \lim_n \rho(\| x_n \|) = \rho(\| x \|) \).

Proof. Obviously, \( \limsup_n \rho(\| x_n \|) \leq \rho(\| x \|) \). Let us see now that \( \rho(\| x \|) \leq \liminf_n \rho(\| x_n \|) \). Let \( (y_n)_{n=1}^\infty \) be a subsequence of \( (x_n)_{n=1}^\infty \) such that \( \lim_n \rho(\| y_n \|) = \liminf_n \rho(\| x_n \|) \). Since \( \lim_n \rho(\| x - y_n \|) = 0 \), by Lemma 3.12 we have \( \lim_n \rho(\| x \| - \| y_n \|) = 0 \). Then Proposition 3.13 guarantees the existence of a subsequence \( (z_n)_{n=1}^\infty \) of \( (y_n)_{n=1}^\infty \) such that \( \lim_n \| z_n \| = \| x \| \). Using Lemma 3.30 we obtain

\[
\rho(\| x \|) = \rho(\liminf_n \| z_n \|) \leq \liminf_n \rho(\| z_n \|) = \lim_n \rho(\| y_n \|),
\]

as we wanted to prove.

Lemma 3.32. Let \( \rho \) be a function quasi-norm over a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \) with the Fatou property, and let \( X \) be a Banach space. If \( x \in L_0(\mu, X) \) and \( (x_n)_{n=1}^\infty \subseteq L_0(\mu, X) \) satisfy \( \lim_n x_n = x \) a.e., and \( \sup_n \| x_n \| \leq g \) for some \( g \in L^0_\rho(\mu) \), then \( \lim_n \rho(\| x_n \|) = \rho(\| x \|) \).

Proof. Note that since \( \lim_n x_n = x \) a.e. and \( X \) is a Banach space, we have \( \lim_n \| x_n \| = \| x \| \) a.e. Consider two particular cases. First, suppose that \( \| x_n \| \leq \| x \| \) for all \( n \in \mathbb{N} \). Obviously, \( \limsup_n \rho(\| x_n \|) \leq \rho(\| x \|) \). Then, by Lemma 3.30, \( \rho(\| x \|) \leq \liminf_n \rho(\| x_n \|) \). Second, suppose that \( \| x_n \| \geq \| x \| \) for all \( n \in \mathbb{N} \). Obviously, \( \liminf_n \rho(\| x_n \|) \geq \rho(\| x \|) \). Set \( g_n = \sup_{k \geq n} \| x_k \| \). Then \( g \geq g_1 \) and \( (g_n)_{n=1}^\infty \) is non-increasing with \( \lim_n g_n = \| x \| \) a.e. Using the absolute continuity of \( g \), we have \( \limsup_n \rho(\| x_n \|) \leq \lim \rho(g_n) = \rho(\| x \|) \). In the general case, set \( g_n = \min\{\| x_n \|, \| x \|\} \) and \( h_n = \max\{\| x_n \|, \| x \|\} \). Then both \( (\rho(g_n))_{n=1}^\infty \) and \( (\rho(h_n))_{n=1}^\infty \) converge to \( \rho(\| x \|) \). Since \( g_n \leq \| x_n \| \leq h_n \), the statement follows.
Proposition 3.33. Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ with the Fatou property, and let $X$ be a Banach space. If $x \in L_0(\mu, X)$ and $(x_n)_{n=1}^{\infty} \subseteq L_0(\mu, X)$ satisfy $\lim_n \|x - x_n\|_\rho = 0$, and $\sup_n \|x_n\| \leq g$ for some $g \in L_{\rho}^{\infty}$, then $\lim_n \rho(\|x_n\|) = \rho(\|x\|)$.

Proof. It suffices to prove that any subsequence of $(x_n)_{n=1}^{\infty}$ has a further subsequence $(y_n)_{n=1}^{\infty}$ with $\lim_n \|y_n\|_\rho = \|x\|_\rho$. But this follows combining Proposition 3.13 with Lemma 3.32. \qed

3.3. The role of lattice convexity and Minkowski-type inequalities. Function spaces built from function quasi-norms have a lattice structure. Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. Given $0 < p \leq \infty$, we say that $\rho$ is lattice $p$-convex (resp. concave) if $L_\rho$ is. Equivalently, $\rho$ is lattice $p$-convex (resp. concave) if and only if there is a constant $C$ such that $G \leq CH$ (resp. $H \leq CG$) for every $n \in \mathbb{N}$ and $(f_j)_{j=1}^{n}$ in $L_0^+(\mu)$, where

$$G = \rho((\sum_{j=1}^{n} f_j^p)^{1/p}), \quad H = (\sum_{j=1}^{n} \rho^p(f_j))^{1/p}.$$  

If the above holds for disjointly supported families, we say that $\rho$ satisfies an upper (resp. lower) $p$-estimate.

If $\rho$ is lattice $p$-convex, then it is $\overline{p}$-convex, where $\overline{p} = \min\{1, p\}$. The notions of 1-convexity and lattice 1-convexity are equivalent. This identification does not extend to $p < 1$ since there are function quasi-norms over $\mathbb{N}$ which are lattice $p$-convex for no $p > 0$ (see [19]). Kalton [18] characterized quasi-Banach lattices (in particular, function quasi-norms) that are $p$-convex for some $p$ as those that are $L$-convex. We say that a function quasi-norm is $L$-convex if there is $0 < \varepsilon < 1$ such that if $f$ and $(f_j)_{j=1}^{n}$ in $L_0^+(\mu)$ satisfy

$$\max_{1 \leq j \leq n} f_j \leq f \quad \text{and} \quad \frac{1}{n} \sum_{j=1}^{n} f_j \geq (1 - \varepsilon)f,$$

then $\max_{1 \leq j \leq n} \rho(f_j) \geq \varepsilon \rho(f)$.

Given $0 < r < \infty$, the $r$-convexified quasi-norm $\rho^{(r)}$ is defined by

$$\rho^{(r)}(f) = \rho^{1/r}(f^r).$$

It is straightforward to check that $\rho^{(r)}$ is a function quasi-norm. If $\rho$ has the Fatou (resp. weak Fatou) property, then $\rho^{(r)}$ does have. If $\rho$ is $p$-convex (resp. concave), then $\rho^{(r)}$ is $pr$-convex (resp. concave). We set

$$L_{\rho}^{(r)} = L_{\rho^{(r)}}.$$

A question implicit in Section 3.2 is whether any $p$-convex function quasi-norm with the weak Fatou property is equivalent to a function.
$p$-norm with the Fatou property. For function norms the answer to this question is positive, and its proof relies on using the associated gauge $\rho'$ given by

$$\rho'(f) = \sup \left\{ \int_\Omega fg \, d\mu : g \in L_0^+(\mu), \rho(g) \leq 1 \right\}.$$

In fact, we have the following.

**Lemma 3.34** (see [6, Theorem 2.2]). Let $\rho$ be a function quasi-norm fulfilling (F.7). Then $\rho'$ is a function norm with the Fatou property.

**Proof.** It is a routine checking. □

**Theorem 3.35** (cf. [6] and [35, Theorem 112.2]). Let $\rho$ be a function norm with the weak Fatou property. Suppose that $\rho$ satisfies (F.7). Then $\rho''$ is equivalent to $\rho$. Moreover, if $\rho$ has the Fatou property, then $\rho'' = \rho$.

In the non-locally convex setting, it is hopeless to try to obtain full information for $\rho$ from the associated function norm $\rho'$. Nonetheless, the following is a partial positive answer to the aforementioned question.

**Proposition 3.36.** Let $0 < p < \infty$ and let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. Suppose that $\rho$ is $p$-convex, has the weak Fatou property, and that for every $E \in \Sigma(\mu)$ there is a constant $C_E$ such that $\int_E f^p \, d\mu \leq C_E \rho(f)$ for all $f \in L_0^+(\mu)$. Then $\rho$ is equivalent to a function $p$-norm with the Fatou property. In fact, there is $G \subset L_0^+(\mu)$ such that $\rho$ is equivalent to the function quasi-norm $\lambda$ given by

$$\lambda(f) = \sup_{g \in G} \left( \int_\Omega f^p g \, d\mu \right)^{1/p}.$$

**Proof.** The function quasi-norm $\rho^{1/p}$ is $1$-convex and, then, equivalent to a function norm $\sigma$. The properties of $\rho$ yields that $\sigma$ satisfies (F.7) and has the weak Fatou property. By Theorem 3.35, $\sigma$ is equivalent to the function norm $\sigma''$. Consequently, $\rho$ is equivalent to the function quasi-norm $\sigma''$. □

**Definition 3.37.** Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, and let $(\Theta, T, \nu)$ be another $\sigma$-finite measure space. Given $f \in L_0^+(\mu \otimes \nu)$ and $g \in L_0^+(\nu \otimes \mu)$ we set

$$\rho[1, f] : \Theta \to [0, \infty], \quad \rho[1, f](\theta) = \rho(f(\cdot, \theta)); \quad \text{and}$$

$$\rho[2, g] : \Theta \to [0, \infty], \quad \rho[2, g](\theta) = \rho(g(\theta, \cdot)).$$
Proposition 3.38. Let \( \rho \) be a locally absolutely continuous function quasi-norm over a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \) with the Fatou property. Let \( (\Theta, \mathcal{T}, \nu) \) be another \( \sigma \)-finite measure space. Let \( f \in L^+_0(\mu \otimes \nu) \) and \( g \in L^+_0(\nu \otimes \mu) \). Then \( \rho[1, f] \) and \( \rho[2, g] \) are measurable functions.

Proof. It suffices to prove the result for \( f \). The Fatou property yields that if the result holds for a non-decreasing sequence \( (f_n)_{n=1}^{\infty} \), then it also holds for \( \lim_{n} f_n \). Consequently, we can suppose that \( \mu(\Omega) < \infty \) and that \( f \) is a measurable simple function. Given a measurable simple positive function \( f \) we denote by \( M_f \) the set consisting of all \( E \) in the product \( \sigma \)-algebra \( \Sigma \otimes \mathcal{T} \) such that the result holds for \( f + t\chi_E \) for every \( t \geq 0 \). The absolute continuity and the Fatou property yields that \( M_f \) is a monotone class for any measurable simple function \( f \). Therefore, if \( \mathcal{R} \) denotes the algebra consisting of all finite disjoint unions of measurable rectangles, the monotone class theorem yields that \( \mathcal{R} \subseteq M_f \) implies \( \Sigma \otimes \mathcal{T} \subseteq M_f \). Let \( \mathcal{C}_r \) denote the cone consisting of all positive functions measurable with respect to \( \mathcal{R} \). Given \( n \in \mathbb{N} \), let \( \mathcal{C}[n] \) be the cone consisting of all measurable non-negative functions which take at most \( n - 1 \) different positive values. It is straightforward to check that the result holds for all functions in \( \mathcal{C}_r = \mathcal{C}_r + \mathcal{C}[1] \). Suppose that the result holds for all functions in \( \mathcal{C}_r + \mathcal{C}[n] \). Then \( \mathcal{R} \subseteq M_f \) for all \( f \in \mathcal{C}_r + \mathcal{C}[n] \). Consequently, \( \Sigma \otimes \mathcal{T} \subseteq M_f \) for all \( f \in \mathcal{C}_r + \mathcal{C}[n] \). In other words, the result holds for all functions in \( \mathcal{C}_r + \mathcal{C}[n + 1] \). By induction, the result holds for every \( f \in \mathcal{C} := \bigcup_{n=1}^{\infty} \mathcal{C}_r + \mathcal{C}[n] \). Since \( \mathcal{C} \) is the cone consisting of all measurable simple non-negative functions, we are done. \( \square \)

Proposition 3.38 allows us to iteratively apply function quasi-noms to measurable functions defined on product spaces. A Minkowski-type inequality is an inequality that compares the gauges that appear when iterating in different ways.

Definition 3.39. Let \( \rho \) and \( \lambda \) be locally absolutely continuous function quasi-norms with the Fatou property over \( \sigma \)-finite measure spaces \( (\Omega, \Sigma, \mu) \) and \( (\Theta, \mathcal{T}, \nu) \) respectively. Given \( f \in L^+_0(\mu \otimes \nu) \) we set
\[
(\rho, \lambda)[1, 2](f) = \rho(\lambda[2, f]), \quad (\lambda, \rho)[2, 1](f) = \lambda(\rho[1, f]).
\]
We say that the pair \((\rho, \lambda)\) has the Minkowski’s integral inequality (MII for short) property if there is a constant \( C \) such that
\[
(\rho, \lambda)[1, 2](f) \leq C(\lambda, \rho)[2, 1](f)
\]
for all \( f \in L^+_0(\mu \otimes \nu) \).
The following result is obtained from the corresponding one for function norms [27]. We do not know whether a direct proof which circumvent using lattice convexity is possible.

**Theorem 3.40.** Let \( \rho \) and \( \lambda \) be locally absolutely continuous \( L \)-convex function quasi-norms with the Fatou property. Then \( (\rho, \lambda) \) has the MII property if and only if there is \( 0 < p \leq \infty \) such that \( \lambda \) is lattice \( p \)-convex and \( \rho \) is lattice \( p \)-concave.

**Proof.** Pick \( 0 < s < \infty \) such that \( \rho(s) \) and \( \lambda(s) \) are 1-convex. Since

\[
(\rho(s), \lambda(s))[1, 2](f) = ((\rho, \lambda)[1, 2](f^s))^{1/s},
\]

\( (\rho, \lambda) \) has the MII property if and only if \( (\rho(s), \lambda(s)) \) does have. It turn, by [27, Theorems 2.3 and 2.5], \( (\rho(s), \lambda(s)) \) has the MII property if and only if there is \( q \in (0, \infty] \) such that \( \lambda(s) \) is lattice \( q \)-convex and \( \rho(s) \) is lattice \( q \)-concave. This latter condition is equivalent to the existence of \( p \in (0, \infty] \) (related with \( q \) by \( q = sp \)) as desired. \( \Box \)

Given \( 0 < p < \infty \) and a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \), the Lebesgue space \( L_p(\mu) \) is absolutely continuous and lattice \( p \)-convex. Moreover, if \( \mu \) is infinite-dimensional, then \( L_p(\mu) \) is not lattice \( q \)-concave for any \( q < p \). Consequently, we have the following.

**Proposition 3.41.** Let \( 0 < p < \infty \) and \( \rho \) be a locally absolutely continuous \( L \)-convex function quasi-norm over an infinite-dimensional \( \sigma \)-finite measure space. Given another \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \) such that \( L_0(\mu) \) is infinite-dimensional, the pair \( (\rho, L_p(\mu)) \) has the MII property if and only if \( \rho \) is \( p \)-concave.

Another Köthe space of interest for us is the weak Lorentz space \( L_{1,\infty}(\mu) \) defined from the function quasi-norm

\[
f \mapsto \sup_{s>0} s \mu f(s) = \sup_{s>0} s \mu \{ \omega \in \Omega : f(\omega) \geq s \}, \quad f \in L_0^+(\mu).
\]

We will denote by \( \| \cdot \|_{1,\infty} \) the quasi-norm in \( L_{1,\infty}(\mu) \). We infer from the properties of the distribution function that \( L_{1,\infty}(\mu) \) is continuous, has the Fatou property, and it is locally dominating. Kalton [17] proved that then \( L_{1,\infty}([0, 1]) \) is lattice \( p \)-convex for any \( p < 1 \). We emphasize that the milestone paper [18] allows to achieve this convexity result regardless the \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \). In fact, given \( 0 < p < 1 \), the \( p^{-1/2} \)-convexified of \( L_{1,\infty}(\mu) \), namely the Lorentz space \( L_{p^{-1/2},\infty} \), is locally convex [15]. Therefore, by [18, Theorem 2.2], \( L_{p^{-1/2},\infty} \) is lattice \( p^{1/2} \)-convex. Consequently, \( L_{1,\infty}(\mu) \) is lattice \( p \)-convex. Since \( L_{1,\infty}(\mu) \) is not locally convex unless finite-dimensional [15], we have the following.
Theorem 3.42. Let $\rho$ be a locally absolutely continuous $L$-convex function quasi-norm, and let $(\Omega, \Sigma, \mu)$ an infinite-dimensional $\sigma$-finite measure space. Then $(\rho, L_{1,\infty}(\mu))$ has the MII property if and only if $\rho$ is $p$-concave for some $p < 1$.

3.4. Conditional expectation in quasi-Banach function spaces.
Given a sub-$\sigma$-algebra $\Sigma_0 \subseteq \Sigma$, we denote by $L^+_0(\mu, \Sigma_0)$ the set consisting of all non-negative $\Sigma_0$-measurable functions. Given $f \in L^+_0(\mu)$ there is a unique $g \in L^+_0(\mu, \Sigma_0)$ such that $\int_A f d\mu = \int_A g d\mu$ for all $A \in \Sigma_0$. We say that $g$ is the conditional expectation of $f$ with respect to $\Sigma_0$, and we denote $E(f, \Sigma_0) := g$.

Definition 3.43. Let $\rho$ be a function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. We say that $\rho$ is leveling if there is a constant $C$ such that $\rho(E(f, \Sigma_0)) \leq C \rho(f)$ for every finite sub-$\sigma$-algebra $\Sigma_0$ and every $f \in L^+_0(\mu)$.

This terminology follows that used in [11]. We remark that Ellis and Halperin imposed leveling function norms to satisfy the above definition with $C = 1$. Not imposing conditional expectations to be contractive turns the notion stable under equivalence.

Given a function quasi-norm $\rho$, a sub-$\sigma$-algebra $\Sigma_0$, and a quasi-Banach space $X$, we denote by $L\rho(\Sigma_0, X)$ the space consisting of all $\Sigma_0$-measurable functions in $L\rho(X)$. Note that, if $\rho|_{\Sigma_0}$ is the restriction of $\rho$ to $\Sigma_0$, then $L\rho(\Sigma_0, X) = L\rho|_{\Sigma_0}(X)$. For further reference, we write down an elementary result.

Lemma 3.44. Let $\rho$ be a leveling function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. Then there is a constant $C$ such that for any finite sub-$\sigma$-algebra $\Sigma_0$ there is positive projection $T: l\rho \to l\rho(\Sigma_0)$ such that $\|T\| \leq C$ and $\int_A f d\mu = \int_A T(f) d\mu$ whenever $f \geq 0$ or $\int_A |f| d\mu < \infty$.

Definition 3.45. If $\rho$, $\Sigma_0$ and $T$ are as in Lemma 3.44, we denote $E[\rho, \Sigma_0] := T$.

Lemma 3.46. Leveling function quasi-norms satisfy (F.7).

Proof. Let $\rho$ be a leveling function quasi-norm over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. Given $A \in \Sigma(\mu)$ with $\mu(A) > 0$, let $\Sigma_0$ be the smallest $\sigma$-algebra containing $A$. For all $f \in L^+_0(\mu)$ we have

$$\int_A f d\mu \leq \frac{\mu(A)}{\rho(\chi_A)} \rho(E(f, \Sigma_0)) \leq C \frac{\mu(A)}{\rho(\chi_A)} \rho(f).$$

It is known that, if $q \geq 1$, $L_q(\mu)$ has the conditional expectation property. Locally convex Lorentz and Orlicz spaces do have. More
generally, we have the following. Recall that a measure space is said to be resonant if either is non-atomic or it consists of equi-measurable atoms.

**Theorem 3.47.** Let \( \rho \) be a rearrangement invariant function norm over a resonant measure space. If \( \rho \) satisfies (F.7), then it is leveling.

**Proof.** By Calderón-Mitjagin Theorem (see [7, 24], and also [6, Theorem 2.2]), \( L_\rho \) is an interpolation space between \( L_1 \) and \( L_\infty \). Since both \( L_1 \) and \( L_\infty \) are leveling, the result follows by interpolation. \( \square \)

3.5. **Function quasi-norms over \( \mathbb{N} \).** Suppose that \( \rho \) is a function quasi-norm over \( \mathbb{N} \) endowed with the counting measure. In this particular case, \( \rho \) is locally dominating, and the space of integrable simple functions is the space \( c_{00} \) consisting of all eventually null sequences. Concerning the density of \( c_{00} \) in \( L_\rho \) we have the following.

**Proposition 3.48.** Let \( \rho \) be a function quasi-norm over \( \mathbb{N} \). Then \( \rho \) is not minimal if and only if \( \ell_\infty \) is a subspace of \( L_\rho \), in which case \( L_\rho \) has block basic sequence equivalent to the unit vector system of \( \ell_\infty \).

Before tackling the proof of Proposition 3.48 we give an auxiliary lemma that will be used a couple of times.

**Lemma 3.49.** Let \( \rho \) be a function quasi-norm over \( \mathbb{N} \) and let \( (a_n)_{n=1}^\infty \) be a sequence in \( L_\rho \). Then \( (a_n)_{n=1}^\infty \) does not belong to \( L_\rho \) if and only if there is an increasing sequence \( (m_k)_{k=1}^\infty \) of non-negative integers such that

\[
\inf_{k \in \mathbb{N}} \rho((|a_n|)_{n=1+m_{2k-1}}^{m_{2k}}) > 0.
\]

**Proof.** Use that \( (a_n)_{n=1}^\infty \in L_\rho \setminus L_\rho^b \) if and only if the series \( \sum_{n=1}^\infty a_n e_n \) does not converge. \( \square \)

**Proof of Proposition 3.48.** Assume that \( L_\rho^b \neq L_\rho \). By Lemma 3.49, there is \( (a_n)_{n=1}^\infty \) in \( [0, \infty)^\mathbb{N} \) such that, if

\[
x_k = \sum_{n=1+m_{2k-1}}^{m_{2k}} a_n e_n, \quad k \in \mathbb{N},
\]

then \( \inf_k \|x_k\|_\rho > 0 \) and \( \sup_m \|\sum_{k=1}^m x_k\| < \infty \). So, \( (x_k)_{k=1}^\infty \) is a block basic sequence as desired. \( \square \)

**Corollary 3.50.** Let \( \rho \) be a function quasi-norm over \( \mathbb{N} \). If \( \rho \) satisfies a lower \( p \)-estimate for some \( p < \infty \), then \( \rho \) is minimal and \( L \)-convex.

**Proof.** Our assumptions yields that \( \ell_\infty \) is not finitely represented in \( L_\rho \) by means of block basic sequences. Then, result follows from Proposition 3.48 and [18, Theorem 4.1]. \( \square \)
Notice that function quasi-norms over \( \mathbb{N} \) are closely related to unconditional bases. In fact, if \( \rho \) is a function quasi-norm over \( \mathbb{N} \), then the unit vector system \( (e_n)_{n=1}^{\infty} \) is an unconditional basis of \( L^b_{\rho} \). Reciprocally, if \( (x_n)_{n=1}^{\infty} \) is an unconditional basis of a quasi-Banach space \( X \), then the mapping

\[
\rho ((a_n)_{n=1}^{\infty}) = \sup \left\{ \left\| \sum_{n=1}^{\infty} b_n x_n \right\| : (b_n)_{n=1}^{\infty} \in c_{00}, \forall n \in \mathbb{N} \right\}
\]

defines a function quasi-norm over \( \mathbb{N} \), and the linear map given by \( x_n \mapsto e_n \) extends to an isomorphism from \( X \) onto \( L^b_{\rho} \).

4. The galb of a quasi-Banach space

In this section we deal with function quasi-norms associated with gbals of quasi-Banach spaces.

**Definition 4.1.** A function quasi-norm over \( \mathbb{N} \) is said to be symmetric (or rearrangement invariant) if \( \rho(f) = \rho(g) \) whenever \( g = (b_n)_{n=1}^{\infty} \) is a rearrangement of \( f = (a_n)_{n=1}^{\infty} \), i.e., there is a permutation \( \pi \) of \( \mathbb{N} \) such that \( b_n = a_{\pi(n)} \) for all \( n \in \mathbb{N} \).

The symmetry of \( \rho \) allows us to safely define \( \rho(f) \) for any countable family of non-negative scalars \( f = (a_j)_{j \in J} \). In the language of bases, if \( \rho \) is a symmetric function-quasi-norm, then the unit vector system is a 1-symmetric basis of \( L^b_{\rho} \).

**Definition 4.2.** Given a quasi-Banach space \( X \) and a sequence \( f = (a_n)_{n=1}^{\infty} \) in \( [0, \infty]^\mathbb{N} \), we define

\[
\lambda_{X}(f) = \sup \left\{ \left\| \sum_{n=1}^{N} a_n x_n \right\| : N \in \mathbb{N}, \|x_n\| \leq 1 \right\}
\]

if \( a_n < \infty \) for all \( n \in \mathbb{N} \), and \( \lambda_{X}(f) = \infty \) otherwise.

**Proposition 4.3.** Let \( X \) be a quasi-Banach space. Then \( \lambda_{X} \) is a symmetric function quasi-norm with modulus of concavity at most that of \( X \). Moreover,

(i) \( \lambda_{X} \) is locally absolutely continuous.
(ii) \( \lambda_{X} \) has the Fatou property.
(iii) If \( Y \) is a subspace of \( X \), then \( \lambda_{X} \) dominates \( \lambda_{Y} \).
(iv) If \( X \) and \( Y \) are isomorphic, then \( \lambda_{X} \) and \( \rho_{Y} \) are equivalent.
(v) \( (\lambda_{X}, \lambda_{X})[1, 2] \) dominates \( \lambda_{X} \) (regarded as a function quasi-norm over \( \mathbb{N}^2 \)).
(vi) If \( X \) is a \( p \)-Banach space, \( 0 < p \leq 1 \), then \( \lambda_{X} \) is a function \( p \)-norm.
(vii) If $\mathbb{X}$ a $p$-convex quasi-Banach lattice, $0 < p \leq 1$, then $\lambda_{\mathbb{X}}$ is lattice $p$-convex.

**Proof.** We will prove (vii), and we will leave the other assertions, which are reformulations of results from [33], as an exercise for the reader. Notice that $\ell_1$ is a $p$-convex lattice, that is, we have

$$\sum_{n=1}^{\infty} \left( \sum_{j=1}^{J} |a_{n,j}|^p \right)^{1/p} \leq \left( \sum_{j \in J} \left( \sum_{n=1}^{\infty} |a_{n,j}|^p \right)^{1/p} \right), \quad a_{n,j} \in \mathbb{F}.$$ 

Hence, the lattice defined by the quasi-norm

$$g = (x_n)_{n=1}^{\infty} \mapsto \left\| \sum_{n=1}^{\infty} x_n \right\|, \quad g \in \mathbb{X}^N,$$

is $p$-convex. Let $C$ denote its $p$-convexity constant. Let $f_j = (a_{j,n})_{n=1}^{\infty} \in [0, \infty)^N$, $1 \leq j \leq J$. Given $(x_n)_{n=1}^{N} \in B^N_{\mathbb{X}}$ we have

$$\left\| \sum_{n=1}^{N} \left( \sum_{j=1}^{J} a_{j,n}^p \right)^{1/p} x_n \right\| \leq \left\| \sum_{n=1}^{N} \left( \sum_{j=1}^{J} |a_{j,n}|^p x_n \right)^{1/p} \right\| \leq C \left( \sum_{j=1}^{J} \left\| \sum_{n=1}^{N} a_{j,n} |x_n|^p \right\|^{1/p} \right) \leq C \left( \sum_{j=1}^{J} \lambda_{\mathbb{X}}^p(f_j) \right)^{1/p}.$$ 

Consequently, $\lambda_{\mathbb{X}}(\sum_{j=1}^{J} |f_j|^p)^{1/p} \leq C(\sum_{j=1}^{J} \lambda_{\mathbb{X}}^p(f_j))^{1/p}$. □

**Definition 4.4.** Let $\mathbb{X}$ be a quasi-Banach space. We denote $\mathcal{G}(\mathbb{X}) = L_{\lambda_{\mathbb{X}}}$, and we say that $\mathcal{G}(\mathbb{X})$ is the galb of $\mathbb{X}$. The positive cone of $\mathcal{G}(\mathbb{X})$ will be denoted by $\mathcal{G}^+(\mathbb{X})$, and $\mathcal{G}_b(\mathbb{X})$ stands for the closure of $c_{00}$ in $\mathcal{G}(\mathbb{X})$.

Roughly speaking, it could be said that the galb of a space is a measure of its convexity. The notion of galb was introduced and developed by Turpin, within the more general setting of "espaces vectoriels à convergence", in a series of papers [31, 32] and a monograph [33]. In this section we restrict ourselves to galbs of locally bounded spaces and touch only a few aspects of the theory and summarize without proofs the properties that are more relevant to our work.

**Proposition 4.5** (see [33]). Let $\mathbb{X}$ be a quasi-Banach space. Then $\mathcal{G}(\mathbb{X}) \subset \ell_1$, and $\mathcal{G}(\mathbb{X}) = \ell_1$ if and only if $\mathbb{X}$ is locally convex.
Proposition 4.6 (see [33]). Let $X$ be a quasi-Banach space. Then $\mathcal{G}(\mathcal{F}(X)) = \mathcal{F}(X)$.

Proposition 4.7 (see [33]). Let $X$ be a quasi-Banach space and $0 < p \leq 1$. Then $X$ is $p$-convex if and only if $\ell_p \subseteq \mathcal{G}(X)$.

Proposition 4.8 (see [31]). Let $X$ be a quasi-Banach space. Then the mapping

$$B : \mathcal{G}(X) \times c_0(X) \to X, \quad ((a_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}) \mapsto \sum_{n=1}^{\infty} a_n x_n$$

is well-defined, and defines a bounded bilinear map.

It is natural to wonder whether the map $B$ defined as in Proposition 4.8 can be extended to a continuous bilinear map defined on $\mathcal{G}(X) \times \ell_\infty(X)$. In fact, the authors of [20], perhaps taking for granted that the answer to this question is positive, defined a sequence $(a_n)_{n=1}^{\infty}$ to be in the gaib of $X$ if $\sum_{n=1}^{\infty} a_n x_n$ converges for every bounded sequence $(x_n)_{n=1}^{\infty}$. If we come to think of it, we obtain the following.

Lemma 4.9. Let $X$ be a quasi-Banach space and let $f = (a_n)_{n=1}^{\infty} \in \mathbb{F}^N$. Then, $f \in \mathcal{G}_b(X)$ if and only if $\sum_{n=1}^{\infty} a_n x_n$ converges for every bounded sequence $(x_n)_{n=1}^{\infty}$ in $X$.

Proof. Let $G$ denote the set consisting of all sequences $f = (a_n)_{n=1}^{\infty} \in \mathbb{F}^N$ such that $\sum_{n=1}^{\infty} a_n x_n$ converges for every bounded sequence $(x_n)_{n=1}^{\infty}$ in $X$. It is routine to check that $G$ is a closed subspace of $\mathcal{G}(X)$ which contains $c_0$. Consequently, $\mathcal{G}_b(X) \subseteq G$. Assume that $f = (a_n)_{n=1}^{\infty} \in \mathcal{G}(X) \setminus \mathcal{G}_b(X)$. Then, by Lemma 3.49, there are $\delta > 0$ and an increasing sequence $(m_k)_{k=1}^{\infty}$ of non-negative integers such that $\rho(|a_n|_{n=1}^{m_k}) \cdot \rho(m_k_{n=1}^{m_{k-1}}) > \delta$ for all $k \in \mathbb{N}$. Consequently, there is $(x_n)_{n=1}^{\infty}$ in the unit ball of $\ell_\infty(X)$ such that

$$\left\| \sum_{n=1}^{m_{2k}} a_n x_n \right\| \geq \delta, \quad \text{for } k \in \mathbb{N}.$$

We infer that $\sum_{n=1}^{\infty} a_n x_n$ does not converge.

Corollary 4.10. Let $X$ be a quasi-Banach space. Then the mapping

$$B' : \mathcal{G}_b(X) \times \ell_\infty(X) \to X, \quad ((a_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}) \mapsto \sum_{n=1}^{\infty} a_n x_n$$

is well-defined, and defines a continuous bilinear map. Moreover, if $\mathcal{G}_b(X) \not\subseteq G \subseteq \mathcal{G}(X)$, then $B'$ can not be extended to a continuous bilinear map defined on $G \times \ell_\infty(X)$. 

Proof. It follows from Lemma 4.9 and, alike the proof of Proposition 4.8, the Open Mapping Theorem.

In light of Corollary 4.10, the following question arise.

Question 4.11. Is \( G(X) \) minimal for any quasi-Banach space \( X \)?

Corollary 3.50 alerts us of the connection between Question 4.11 and the existence of lower estimates for \( \lambda_X \). Lattice concavity also plays a key role when studying galbs of vector-valued spaces.

Definition 4.12. We say that a symmetric function quasi-norm \( \lambda \) over \( \mathbb{N} \) galbs a quasi-Banach space \( X \) if \( \lambda \) dominates \( \lambda_X \), i.e., \( L_\lambda \subseteq G(X) \). We say that \( \lambda \) galbs a function quasi-norm \( \rho \) if it galbs \( L_\rho \). If \( \lambda \) galbs itself, we say that \( \lambda \) is self-galbed.

Remark 4.13. Given \( 0 < p \leq 1 \), the function quasi-norm defining \( \ell_p \) is self-galbed. More generally, \( \lambda_X \) is self-galbed for any quasi-Banach space \( X \) (see Proposition 4.6).

Proposition 4.14. Let \( \rho \) and \( \lambda \) be locally absolutely continuous \( L \)-convex function quasi-norms with the Fatou property. Suppose that \( \lambda \) galbs a quasi-Banach space \( X \). If there is \( 0 < p < \infty \) such that \( \lambda \) is \( p \)-concave and \( \rho \) is \( p \)-convex, then \( \lambda \) galbs \( L_\rho(X) \).

Proof. By Theorem 3.40, the pair \( (\lambda, \rho) \) has the MII property for some constant \( C \). Since \( \lambda \) galbs \( X \), there is a constant \( K > 0 \) such that \( \lambda \) \( K \)-dominates \( \lambda_X \). Therefore, if \( (a_n)_{n=1}^\infty \) is a sequence in \( L_\lambda \), and \( f_1, \ldots, f_N \) belong the unit ball of \( L_\rho(X) \), we have

\[
\rho \left( \left\| \sum_{n=1}^N a_n f_n \right\| \right) \leq \rho \left( \lambda_X \left( (\|a_n\|f_n\|)_{n=1}^N \right) \right) \\
\leq K \rho \left( \lambda \left( (a_n \|f_n\|)_{n=1}^N \right) \right) \\
\leq CK \lambda \left( (\|f_n\|)_{n=1}^N \right) \\
\leq CK \lambda \left( (a_n)_{n=1}^N \right) \\
\leq CK \lambda((a_n)_{n=1}^\infty).
\]

Hence \( (a_n)_{n=1}^\infty \) belongs the galb of \( L_\rho(X) \).

Proposition 4.14 gives, in particular, that if \( \lambda \) is a 1-concave function quasi-norm which galbs \( X \), then it galbs \( L_1(\mu, X) \). As we plan to develop an integral for functions belonging to a suitable subspace of \( L_1(\mu, X) \), the following question arises.

Question 4.15. Is \( G(X) \) 1-concave for any quasi-Banach space \( X \)?
Note that a positive answer to Question 4.15 would yield a positive answer to Question 4.11. To properly understand Question 4.15, we must go over the state-of-the-art of the theory galbs.

We point out that all known examples suggest a positive answer to Question 4.15. Galbs of Lorentz spaces were explored through several papers [8, 9, 28–30] within the study of convolution operators, and all computed galbs occur to be Orlicz sequence spaces modeled after a concave Orlicz function. Also, Turpin [33] proved that the galb of any locally bounded Orlicz space is an Orlicz sequence space modeled after a concave Orlicz function. Recall that an Orlicz function \( \varphi \) is a non-null left-continuous non-decreasing function \( \varphi : [0, \infty) \rightarrow [0, \infty) \) such that \( \lim_{t \rightarrow 0^+} \varphi(t) = 0 \). Given an Orlicz function \( \varphi \), with the convention that \( \varphi(\infty) = \infty \), the gauge

\[
\rho \varphi (f) = \inf \left\{ t > 0 : \sum_{n=1}^{\infty} \varphi \left( \frac{a_n}{t} \right) \leq 1 \right\}, \quad f \in [0, \infty]^{\mathbb{N}}
\]

is a function quasi-norm if and only if

\[
\lim_{t \to 0^+} \sup_{u \in (0,1]} \frac{\varphi(tu)}{\varphi(u)} = 0. \tag{4.1}
\]

(see [33]), in which case \( \rho \varphi \) has the Fatou property. If (4.1) holds, the Orlicz sequence space \( \ell_{\varphi} \) is the Köthe space associated with \( \rho \varphi \).

**Proposition 4.16.** Let \( \varphi \) be a concave Orlicz function fulfilling (4.1). Then \( \rho \varphi \) is lattice 1-concave.

**Proof.** Let \( (f_j)_{j=1}^{J} \) be a finite family consisting of non-negative sequences. We will prove that

\[
H := \sum_{j=1}^{J} \rho \varphi (f_j) \leq G := \rho \varphi \left( \sum_{j=1}^{J} f_j \right).
\]

To that end, it suffices to prove that if \( G < \infty \) and \( 0 < t < H \), then, \( t < G \). Assume without loss of generality that \( \rho \varphi (f_j) > 0 \) for all \( j \).

Then, pick \( (t_j)_{j=1}^{J} \) such that \( \sum_{j=1}^{J} t_j = t \) and \( 0 < t_j < \rho (f_j) \). Then, if \( f_j = (a_{j,n})_{n=1}^{\infty} \), \( a_{j,n} < \infty \) for all \( n \in \mathbb{N} \), and

\[
\sum_{n=1}^{\infty} \varphi \left( \frac{a_{j,n}}{t_j} \right) > 1, \quad j = 1, \ldots, J.
\]

Consequently,

\[
\sum_{n=1}^{\infty} \varphi \left( \frac{\sum_{j=1}^{J} a_{j,n}}{t} \right) = \sum_{n=1}^{\infty} \varphi \left( \sum_{j=1}^{J} \frac{t_j a_{j,n}}{t_j} \right) \geq \sum_{n=1}^{\infty} \sum_{j=1}^{J} \frac{t_j}{t} \varphi \left( \frac{a_{j,n}}{t_j} \right) > 1.
\]
Therefore, \( t < G \).

The lattice convexity of spaces of galbs is also quite unknown. It is known that if the gauge \( \lambda_\varphi \) associated with an Orlicz function \( \varphi \) is function quasi-norm, so that \( \ell_\varphi \) is a quasi-Banach lattice, then there is \( p > 0 \) such that

\[
\sup_{0 < u, t \leq 1} \frac{\varphi(tu)}{u^p \varphi(t)} < \infty
\]  

(4.2)

(see [16, Proposition 4.2]). Moreover, if (4.2) holds for a given \( p \), then \( \ell_\varphi \) is a \( p \)-convex lattice. Therefore, \( \ell_\varphi \) is \( L \)-convex. The behavior of general spaces of galbs is unknown.

**Question 4.17.** Is \( \lambda_X \) an \( L \)-convex function quasi-norm for any quasi-Banach space \( X \)?

Note that Proposition 4.3 (vii) partially solves in the positive Question 4.17.

5. **Topological tensor products built by means of symmetric function quasi-norms over \( \mathbb{N} \)**

**Definition 5.1.** Let \( X \) and \( Y \) be quasi-Banach spaces and \( \lambda \) be a symmetric minimal function quasi-norm with the Fatou property. We define

\[
\| \cdot \|_{X \otimes \lambda Y} : X \otimes Y \to [0, \infty)
\]

by

\[
\| \tau \|_{X \otimes \lambda Y} = \inf \left\{ \lambda \left( \sum_{j=1}^n \| x_j \| \| y_j \| \right) : \tau = \sum_{j=1}^n x_j \otimes y_j \right\}.
\]

It is clear that \( \| \cdot \|_{X \otimes \lambda Y} \) is a semi-quasi-norm whose modulus of concavity is at most that of \( \lambda \), and that \( \| x \otimes y \|_{X \otimes \lambda Y} \leq \| x \| \| y \| \) for all \( x \in X \) and \( y \in Y \).

**Definition 5.2.** Let \( X \) and \( Y \) be quasi-Banach spaces and \( \lambda \) be a symmetric minimal function quasi-norm with the Fatou property. The quasi-Banach space built from \( \| \cdot \|_{X \otimes \lambda Y} \) will be called the *topological tensor product of \( X \) and \( Y \) by \( \lambda \)*, and will be denoted by \( X \otimes \lambda Y \). The canonical norm-one bilinear map from \( X \times Y \) to \( X \otimes \lambda Y \) given by \((x, y) \mapsto x \otimes y\) will be denoted by \( T_\lambda[X, Y] \).

**Proposition 5.3.** Let \( X, Y, U \) and \( V \) be quasi-Banach spaces, and let \( \lambda \) be a symmetric minimal function quasi-norm with the Fatou property.

(i) If \( \lambda \) is a function \( p \)-norm, \( 0 < p \leq 1 \), then \( X \otimes \lambda Y \) is a \( p \)-Banach space.
(ii) \( \mathcal{G}(L_\lambda) \subseteq \mathcal{G}(X \otimes_\lambda Y) \).

(iii) If \( \lambda \) galbs \( U \), there is a constant \( C \) such that for every bounded bilinear map \( B: X \times Y \to U \) there is a unique linear map \( B_\lambda: X \otimes_\lambda Y \to U \) such that \( B_\lambda \circ T_\lambda [X, Y] = B \) and \( \|B_\lambda\| \leq C\|B\| \).

(iv) If \( R: X \to U \) and \( S: Y \to V \) are bounded linear operators, then there is a unique bounded linear operator \( R \otimes_\lambda S: X \otimes_\lambda Y \to U \otimes_\lambda V \) such that \( (R \otimes_\lambda S) \circ T_\lambda [X, Y] = T_\lambda [U, V] \circ (R, S) \).

(v) If \( U \) is complemented in \( X \) through \( R \) and \( V \) is complemented in \( Y \) through \( S \), then \( U \otimes_\lambda V \) is complemented in \( X \otimes_\lambda Y \) through \( R \otimes_\lambda S \). Moreover, if \( U^c \) and \( V^c \) are such that \( X \simeq U \oplus U^c \) and \( Y \simeq V \oplus V^c \), then
\[
X \otimes_\lambda Y \simeq (U \otimes_\lambda V) \oplus (U \otimes_\lambda V^c) \oplus (U^c \otimes_\lambda V) \oplus (U^c \otimes_\lambda V^c).
\]

(vi) Let \( \rho \) be a symmetric minimal function quasi-norm with the Fatou property. If \( \rho \) dominates \( \lambda \), then there is a bounded linear map \( I: X \otimes_\rho Y \to X \otimes_\lambda Y \) such that \( I \circ T_\rho [X, Y] = T_\lambda [X, Y] \).

(vii) There is a constant \( C \) such that if \( (x_j)_{j=1}^\infty \) in \( X \) and \( (y_j)_{j=1}^\infty \) in \( Y \) are such that
\[
H = \lambda \left( (\|x_j\| \|y_j\|)_{j=1}^\infty \right) < \infty.
\]
then \( \sum_{j=1}^\infty x_j \otimes y_j \) converges in \( X \otimes_\lambda Y \) to a vector \( \tau \in X \otimes_\lambda Y \) with \( \|\tau\|_{X \otimes_\lambda Y} \leq CH \). Conversely, for all \( \tau \in X \otimes_\lambda Y \) and \( \varepsilon > 0 \) there are \( (x_n)_{n=1}^\infty \) in \( X \) and \( (y_n)_{n=1}^\infty \) in \( Y \) such that, if
\[
f := (\|x_j\| \|y_j\|)_{j=1}^\infty,
\]
then \( \lambda(f) \leq \varepsilon + C\|\tau\|_{X \otimes_\lambda Y} \) and \( \tau = \sum_{j=1}^\infty x_j \otimes y_j \). Moreover, if \( \lambda \) is a function \( p \)-norm, we can pick \( C = 1 \). And, if \( X_0 \) and \( Y_0 \) are dense subspaces of \( X \) and \( Y \) respectively, we can pick \( x_j \in X_0 \) and \( y_j \in Y_0 \) for all \( j \in \mathbb{N} \).

(viii) If \( \lambda \) galbs \( X \) and \( Y \) is finite dimensional, then \( X \otimes_\lambda Y \simeq X^n \), where \( n = \dim(\mathcal{Y}) \). To be precise, if \( (\mathbf{y}_j)_{j=1}^n \) is a basis of \( Y \), the map \( R: X^n \to X \otimes_\lambda Y \) given by \( (x_j)_{j=1}^n \mapsto \sum_{j=1}^n x_j \otimes \mathbf{y}_j \) is an isomorphism.

(ix) If \( \lambda \) galbs \( X \) and \( Y \) has the point separation property, then \( \|\cdot\|_{X \otimes_\lambda Y} \) is a quasi-norm on \( X \otimes Y \).

\[\textbf{Proof.}\] A simple computation yields (i).

Let \( f = (a_k)_{k=1}^\infty \in [0, \infty)^N \), and let \( (\tau_k)_{k=1}^m \) in \( X \otimes Y \) be such that \( \|\tau_k\|_{X \otimes Y} \leq 1 \). Then, given \( \varepsilon > 0 \), for each \( k = 1, \ldots, m \) there is an expansion
\[
\tau_k = \sum_{j=1}^{n_k} b_{k,j} x_{k,j} \otimes y_{k,j},
\]
with \( \max \{ \| x_{k,j} \|, \| y_{k,j} \| \} \leq 1 \) for all \((k,j) \in \mathcal{N} := \{(k,j) \in \mathbb{N}^2 : 1 \leq k \leq m, 1 \leq j \leq n_k \}\) and \( \lambda((b_{k,j})_{k=1}^m) \leq 1 + \varepsilon \). The expansion
\[
\tau := \sum_{k=1}^m a_k \tau_k = \sum_{(k,j) \in \mathcal{N}} a_k b_{k,j} x_{k,j} \otimes y_{k,j}
\]
gives
\[
\| \tau \|_{\mathcal{X} \otimes \mathcal{Y}} \leq \|(a_k b_{k,j})_{(k,j) \in \mathcal{N}} \|_{\lambda} \leq (1 + \varepsilon) \lambda L_\lambda(f).
\]
Consequently, \( \lambda_{\mathcal{X} \otimes \mathcal{Y}}(f) \leq \lambda L_\lambda(f) \), and we obtain (ii).

Let us prove (iii). Let \( C \) be such that \( \| \sum_{j=1}^n a_j u_j \| \leq C \lambda((a_j)_{j=1}^n) \) for all \((a_j)_{j=1}^n \) in \([0, \infty]^n\) and \((u_j)_{j=1}^n \) in \( B_\mathcal{U} \). Given a bounded bilinear map \( B : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{U} \), let \( B_0 : \mathcal{X} \otimes \mathcal{Y} \rightarrow \mathcal{U} \) be the linear map defined by \( B(x \otimes y) = B(x, y) \). Given \( \tau = \sum_{j=1}^n x_j \otimes y_j \in \mathcal{X} \otimes \mathcal{Y} \) we have
\[
\| B_0(\tau) \| \leq C \lambda((\| B(x_j, y_j) \|)_{j=1}^n) \leq C \| B \| \lambda((\| x_j \|, \| y_j \|)_{j=1}^n).
\]
Consequently, \( \| B_0(\tau) \| \leq C \| B \| \| \tau \|_{\mathcal{X} \otimes \mathcal{Y}} \). We infer that \( B_0 \) ‘extends’ to an operator as desired.

Now we prove (iv). Let \( \tau \in \mathcal{X} \otimes \mathcal{Y} \). The mere definitions of the semi-quasi-norms involved give
\[
\|(R \otimes \lambda) S\|_{\mathcal{U} \otimes \mathcal{Y}} \leq \inf \left\{ \lambda \left( (\| R(x_j) \|, \| S(y_j) \|)_{j=1}^n \right) \right\} : \tau = \sum_{j=1}^n x_j \otimes y_j \leq \| R \| \| S \| \| \tau \|_{\mathcal{X} \otimes \mathcal{Y}}.
\]

For statement (v), it suffices to consider the case when \( \mathcal{V} = \mathcal{Y} \) and \( S_u = \text{Id}_\mathcal{Y} \). Let \( I : \mathcal{U} \rightarrow \mathcal{X} \) and \( P : \mathcal{X} \rightarrow \mathcal{U} \) be such that \( P \circ I = \text{Id}_\mathcal{U} \). Then \( (P \otimes \lambda \text{Id}_\mathcal{Y}) \circ (I \otimes \lambda \text{Id}_\mathcal{Y}) = \text{Id}_{\mathcal{U} \otimes \mathcal{Y}} \). Let \( J : \mathcal{U}_c \rightarrow \mathcal{X} \) and \( Q : \mathcal{X} \rightarrow \mathcal{U}_c \) be such that \( Q \circ J = \text{Id}_{\mathcal{Y}_c} \) and \( J \circ Q + I \circ P = \text{Id}_\mathcal{X} \). Then
\[
(I \otimes \lambda \text{Id}_\mathcal{Y}) \circ (P \otimes \lambda \text{Id}_\mathcal{Y}) + (J \otimes \lambda \text{Id}_\mathcal{Y}) \circ (Q \otimes \lambda \text{Id}_\mathcal{Y}) = \text{Id}_{\mathcal{X} \otimes \mathcal{Y}}.
\]

Statement (vi) is immediate from definition.

Let us prove (vii). Assume without lost of generality that \( \lambda \) is function \( p \)-norm for some \( 0 < p \leq 1 \). If (5.1) holds, then \( \sum_{j=1}^\infty x_j \otimes y_j \) is a Cauchy series. Therefore, it converges to \( \tau \in \mathcal{X} \otimes \mathcal{Y} \). The continuity of the quasi-norm \( \| \cdot \|_{\mathcal{X} \otimes \mathcal{Y}} \) yields
\[
\| \tau \|_{\mathcal{X} \otimes \mathcal{Y}} = \lim_m \left\| \sum_{j=1}^m x_j \otimes y_j \right\|_{\mathcal{X} \otimes \mathcal{Y}} \leq H.
\]
Conversely, let \( \tau \in \mathcal{X} \otimes \mathcal{Y} \) and \( \varepsilon > 0 \). Assume that \( \mathcal{X}_0 \) and \( \mathcal{Y}_0 \) are dense subspaces of \( \mathcal{X} \) and \( \mathcal{Y} \) respectively. Pick \( (\tau_n)_{n=1}^\infty \) in \( \mathcal{X}_0 \otimes \mathcal{Y}_0 \) such
that \[ \lim_n \| \tau - \tau_n \|_{X \otimes \epsilon Y} = 0, \] and pick a sequence \((\varepsilon_n)_{n=1}^\infty\) of positive numbers with
\[ \varepsilon_1 > \| \tau \|_{X \otimes \epsilon Y} > \left( \sum_{n=1}^\infty \varepsilon_n^p \right)^{1/p} - \varepsilon. \]

Passing to a subsequence we can suppose that \(\| \tau_n - \tau_{n-1} \|_{X \otimes \epsilon Y} < \varepsilon_n\) for all \(n \in \mathbb{N}\), with the convention \(\tau_0 = \tau\). Therefore, for all \(n \in \mathbb{N}\), we can write
\[ \tau_n - \tau_{n-1} = \sum_{j=1}^{j_n} x_{j,n} \otimes y_{j,n}, \quad R_n := \lambda \left( \left( \| x_{j,n} \| \| y_{j,n} \| \right)_{(j,n) \in \mathcal{N}} \right)^{1/p} < \varepsilon_n. \]

Let \(\mathcal{N} = \{(j,n) \in \mathbb{N}^2 : 1 \leq j \leq j_n\}\). Then
\[ \lambda \left( \left( \| x_{j,n} \| \| y_{j,n} \| \right)_{(j,n) \in \mathcal{N}} \right) \leq \left( \sum_{n=1}^\infty R_n \right)^{1/p} \leq \varepsilon + \| \tau \|_{X \otimes \epsilon Y}. \]

Hence, we can safely define \(\tau' = \sum_{(j,n) \in \mathcal{N}} x_{j,n} \otimes y_{j,n}\), and we have
\[ \tau = \sum_{n=1}^{\infty} \sum_{j=1}^{j_n} x_{j,n} \otimes y_{j,n} = \sum_{n=1}^{\infty} (\tau_n - \tau_{n-1}) = \lim_n \tau_n = \tau. \]

Now we prove (viii). The mapping \(R\) is linear and bounded, and \(R(\mathbb{X}^n)\) spans \(\mathbb{X} \otimes \epsilon \mathbb{Y}\). Since \(\lambda\) galbs \(\mathbb{X}\), there is a bounded linear map \(S : \mathbb{X} \otimes \epsilon \mathbb{Y} \to \mathbb{X}^n\) such that \(S(x \otimes y_j) = xe_j\) for all \(x \in \mathbb{X}\) and \(j = 1, \ldots, n\). Taking into account that \(S \circ R = \text{Id}_{\mathbb{X}^n}\), we are done.

Finally, let \(\mathbb{V}\) be finite-dimensional subspace of \(\mathbb{Y}\). Since \(\mathbb{V}\) is complemented in \(\mathbb{Y}\), \(\mathbb{X} \otimes \epsilon \mathbb{V}\) is complemented in \(\mathbb{X} \otimes \epsilon \mathbb{Y}\) via the canonical map. Hence, it suffices to consider the case when \(\mathbb{Y}\) is finite dimensional. In this particular case, statement (ix) follows from (viii). \(\Box\)

6. **Topological Tensor Products as Spaces of Functions and Integrals for Spaces of Vector-Valued Functions**

Let us give another approach to the proof of Proposition 5.3 (ix). Given quasi-Banach spaces \(\mathbb{X}\) and \(\mathbb{Y}\), let \(B : \mathbb{X} \times \mathbb{Y} \to \ell_\infty(B_{\mathbb{Y}}^*, \mathbb{X})\) be defined by \(B(x, y)(y^*) = y^*(y)x\). Since \(B\) is linear and bounded, if \(\lambda\) galbs \(\mathbb{X}\), there is a bounded linear map \(B_\lambda : \mathbb{X} \otimes \epsilon \mathbb{Y} \to \ell_\infty(B_{\mathbb{Y}}^*, \mathbb{X})\) given by \(B_\lambda(x \otimes y)(y^*) = y^*(y)x\). If \(\mathbb{Y}\) has the point separation property, then \(B_\lambda\) is one-to-one on \(\mathbb{X} \otimes \mathbb{Y}\). Consequently, no vector in \(\mathbb{X} \otimes \mathbb{Y}\) is norm-zero.

Note the injectivity of \(B_\lambda\) on \(\mathbb{X} \otimes \mathbb{Y}\) does not imply the injectivity of \(B_\lambda\) on its closure \(\overline{\mathbb{X} \otimes \mathbb{Y}}\). That is, we can not, a priori, identify vectors in \(\mathbb{X} \otimes \mathbb{Y}\) with functions defined over \(B_{\mathbb{Y}}^*\). More generally, if \(\mathbb{Y}\) embeds in \(\mathbb{F}^\Omega\) for some set \(\Omega\), then \(\mathbb{X} \otimes \mathbb{Y}\) embeds into \(\mathbb{X}^\Omega\), and it is natural.
to wonder if the character of the members of $X \otimes Y$ is preserved when taking the completions, that is, if we can regard the vectors in $X \otimes \lambda Y$ as $X$-valued functions defined on $\Omega$. In this section, we address this question in the case when $Y$ is a Köthe space.

Given a quasi-Banach space $X$ and a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ we have a canonical linear map $J[X, \mu] : X \otimes L_0(\mu) \to L_0(\mu, X)$, $x \otimes f \mapsto xf$.

It is routine to check that $J[X, \mu]$ is one-to-one. Suppose that $\lambda$ is a symmetric function quasi-norm and $\rho$ is a function quasi-norm over $(\Omega, \Sigma, \mu)$ such that $\lambda$ is $p$-concave and $\rho$ is $p$-convex for some $0 < p < \infty$. Then $\lambda$ is minimal (see Corollary 3.50). So, we can safely define $X \otimes \lambda L_\rho$.

If, moreover, $\lambda$ galbs $X$, then $\lambda$ also galbs $L_\rho(X)$ (see Proposition 4.14). Hence, if $\rho$ has the weak Fatou property, there is a bounded linear canonical map $J[\rho, X, \lambda] : X \otimes \lambda L_\rho \to L_\rho(X)$, $x \otimes f \mapsto xf$.

Consider the range $L_\rho^\lambda(X) := J[\rho, X, \lambda](X \otimes \lambda L_\rho)$ of this operator endowed with the quotient topology. If $J[\rho, X, \lambda]$ is one-to-one, then $L_\rho^\lambda(X)$ is a space isometric to $X \otimes \lambda L_\rho$ which embeds continuously into $L_\rho(X)$. This is our motivation to studying the injectivity of $J[\rho, X, \lambda]$. Vogt [34] gave a positive answer to this question in the case when $\lambda$ is the function quasi-norm associated with $\ell_p$ for some $0 < p \leq 1$ and $\rho$ is the function quasi-norm associated with $L_q(\mu)$ for some $p \leq q \leq \infty$. A detailed analysis of the proof of [34, Satz 4] reveals that it depends heavily on the fact that $\lambda$ is both $p$-convex and $p$-concave and $\rho$ is both $q$-convex and $q$-concave. So, it is hopeless to try to extend this result using analogous ideas. In this paper, we use an approach based on conditional expectations.

Before going on, let us mention that if $\lambda$ is the function quasi-norm associated with $\ell_1$ (and $\rho$ and $X$ are 1-convex), then a routine computation yields that $J[\rho, X, \lambda]$ is an isometric embedding when restricted to $X \otimes S(\mu)$. We infer that $J[\rho, X, \lambda]$ is an isometric embedding and that $L_\rho^\lambda(X)$ consists of all strongly measurable functions in $L_\rho(X)$.

**Lemma 6.1.** Let $\lambda$ be a minimal symmetric function quasi-norm. For $i = 1, 2$, let $\rho_i$ be a function quasi-norm with the weak Fatou property over a $\sigma$-finite measure space $(\Omega_i, \Sigma_i, \mu_i)$, and let $X_i$ be a quasi-Banach space galbed by $\lambda$. Suppose that the bounded linear operators $S : X_1 \to X_2$, $T : L_{\rho_1} \to L_{\rho_2}$ and $R : L_{\rho_1}(X_1) \to L_{\rho_2}(X_2)$ satisfy

$$R(x f) = S(x) T(f), \quad x \in X_1, \ f \in L_{\rho_1}.$$
Then, $R$ restricts to a bounded linear map from $L^\lambda_{\rho_1}(X_1) \to L^\lambda_{\rho_2}(X_2)$.

Proof. Our assumptions yield a commutative diagram

$$
\begin{array}{ccc}
X_1 \otimes_\lambda L_{\rho_1} & \xrightarrow{S \otimes_\lambda T} & X_2 \otimes_\lambda L_{\rho_2} \\
J_{[\rho_1,X_1,\lambda]} \downarrow & & \downarrow J_{[\rho_2,X_2,\lambda]} \\
L_{\rho_1}(X_1) & \xrightarrow{R} & L_{\rho_2}(X_2).
\end{array}
$$

We infer that $R$ maps the range of the map $J_{[\rho_1,X_1,\lambda]}$ into the range of the map $J_{[\rho_2,X_2,\lambda]}$. That is, there is a linear map $R[\lambda]: L_{\rho_1}(X_1) \to L_{\rho_2}(X_2)$ such that the diagram

$$
\begin{array}{ccc}
X_1 \otimes_\lambda L_{\rho_1} & \xrightarrow{S \otimes_\lambda T} & X_2 \otimes_\lambda L_{\rho_2} \\
J_{[\rho_1,X_1,\lambda]} \downarrow & & \downarrow J_{[\rho_2,X_2,\lambda]} \\
L^\lambda_{\rho_1}(X_1) & \xrightarrow{R[\lambda]} & L^\lambda_{\rho_2}(X_2)
\end{array}
$$

commutes. Since both $L^\lambda_{\rho_1}(X_1)$ and $L^\lambda_{\rho_2}(X_2)$ are endowed with the quotient topology and $S \otimes_\lambda T$ is continuous, so is $R[\lambda]$. □

Let $\lambda$ be a 1-concave symmetric function quasi-norm that galbs a quasi-Banach space $X$. Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. If $\rho$ is the function quasi-norm defining $L_1(\mu)$, we denote $L^\lambda_{\rho}(X) = L^\lambda_{\rho}(\mu, X)$. Given $A \in \Sigma$, we set $L^\lambda_{\rho}(A, \mu, X) = L^\lambda_{\rho}(\mu|_A, X)$. The bounded linear operator

$$I[\mu]: L_1(\mu) \to \mathbb{F}, \quad f \mapsto \int_\Omega f \, d\mu$$

yields a bounded linear operator

$$I[\mu, X, \lambda]: X \otimes_\lambda L_1(\mu) \to X, \quad x \otimes f \mapsto x \int_\Omega f \, d\mu.$$

**Definition 6.2.** Suppose that a 1-concave symmetric function quasi-norm $\lambda$ galbs a quasi-Banach space $X$. We say that the pair $(\lambda, X)$ is amenable if $I[\mu, X, \lambda](\tau) = 0$ whenever $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure and $\tau \in X \otimes_\lambda L_1(\mu)$ satisfies $J[L_1(\mu), X, \lambda](\tau) = 0$.

In other words, $(\lambda, X)$ is amenable if and only if for every $\sigma$-finite measure $\mu$ there is an operator

$$I[\mu, X, \lambda]: L^\lambda_{\rho}(\mu, X) \to X.$$
such that the diagram

\[
\begin{array}{ccc}
X \otimes_{\lambda} L_1(\mu) & \xrightarrow{J[\mu, X, \lambda]} & L^\lambda_1(\mu, X) \\
& \downarrow J[\mu, X, \lambda] & \downarrow I[\mu, X, \lambda] \\
& X & \rightarrow \end{array}
\]

commutes. The bounded linear operator \(I[\mu, X, \lambda]\) satisfies

\[
I[\mu, X, \lambda](xf) = x \int_{\Omega} f d\mu, \quad x \in X, \quad f \in L^1(\mu).
\]

So, we must regard it as ‘integral’ for functions in \(L^\lambda_1(\mu, X)\). Loosely speaking, that \((\lambda, X)\) is amenable means that there is an integral for functions in \(L^\lambda_1(\mu, X)\).

**Definition 6.3.** Let \(X\) be a quasi-Banach space. We say that a net \((T_i)_{i \in I}\) in \(L(X)\) is a bounded approximation of the identity if \(\sup_i \|T_i\| < \infty\) and \(\lim_i T_i(x) = x\) for all \(x \in X\). We say that \(X\) has the BAP if it has a bounded approximation of the identity consisting of finite-rank operators.

Note that if a net \((T_i)_{i \in I}\) in \(L(X)\) is uniformly bounded then the set \(\{x \in X: \lim_i T_i(x) = x\}\) is closed. This yields the following elementary result.

**Lemma 6.4.** Let \(X\) be a quasi-Banach space. Let \((P_i)_{i \in I}\) be a net consisting of uniformly bounded projections with \(P_j \circ P_i = P_i\) if \(i \leq j\) and \(\cup_{i \in I} P_i(X)\) is dense in \(X\). Then \((P_i)_{i \in I}\) is a bounded approximation of the identity.

If \(\rho\) satisfies \((F.7)\), then for every \(A \in \Sigma(\mu)\) we have a bounded linear map

\[
S[A, \rho]: L_\rho \to L_1(A, \mu), \quad f \mapsto f|_A.
\]

**Theorem 6.5.** Let \(\lambda\) be a 1-concave symmetric function quasi-norm, let \(\rho\) be a leveling function quasi-norm with the weak Fatou property over a \(\sigma\)-finite measure space \((\Omega, \Sigma, \mu)\), and let \(X\) be a quasi-Banach space. Suppose that \((\lambda, X)\) is amenable. Then \(J[\rho, X, \lambda]\) is one-to-one.

**Proof.** Let \(A \in \Sigma(\mu)\). By Lemma 3.46, \(\rho\) satisfies \((F.7)\). Therefore, for each quasi-Banach space \(Y\) there is a bounded linear operator

\[
S[A, \rho, Y]: L_\rho(Y) \to L_1(A, \mu, Y), \quad f \mapsto f|_A.
\]
Set \( S[A, \rho, F] = S[A, \rho] \). It is routine to check that the diagram

\[
\begin{array}{ccc}
X \otimes \lambda L_\rho & \xrightarrow{\text{Id}_X \otimes \lambda S[A, \rho]} & X \otimes \lambda L_1(A, \mu) \\
J[\rho, X, \lambda] & & J[L_1(A, \mu), X, \lambda] \\
L_\mu(X) & \xrightarrow{S[A, \rho, X]} & L_1(A, \mu, X) \\
\end{array}
\]

commutes. Using that \((\lambda, X)\) is amenable we obtain the commutative diagram

\[
\begin{array}{ccc}
X \otimes \lambda L_\rho & \xrightarrow{\text{Id}_X \otimes \lambda S[A, \rho]} & X \otimes \lambda L_1(A, \mu) \\
J[\rho, X, \lambda] & & J[L_1(A, \mu), X, \lambda] \\
L_\mu(X) & \xrightarrow{S[A, \rho, X]} & L_1(A, \mu, X) \\
\end{array}
\] (6.1)

Suppose that \( \mu(\Omega) < \infty \). Let \( \Sigma_0 \) be a finite sub-\( \sigma \)-algebra. If \( \Sigma_0 \) is generated by the partition \((A_j)_{j=1}^n\) of \( \Omega \) consisting of nonzero measure sets, then

\[
\mathbb{E}(\rho, \Sigma_0) = \sum_{j=1}^n \frac{\chi_{A_j}}{\mu(A_j)} I[\mu|_{A_j}] \circ S[A_j, \rho].
\]

By Proposition 5.3 (viii), there is an isomorphism \( S: X^n \to X \otimes \lambda L_\rho(\Sigma_0) \) such that

\[
S((x_j)_{j=1}^n) = \sum_{j=1}^n x_j \otimes \frac{\chi_{A_j}}{\mu(A_j)}, \quad x_j \in X.
\]

Therefore,

\[
\text{Id}_X \otimes \lambda \mathbb{E}(\rho, \Sigma_0) = S \circ (I[\mu|_{A_j}, X, \lambda] \circ (\text{Id}_X \otimes S[A_j, \rho]))_{j=1}^n.
\]

Combining this identity with the commutative diagrams (6.1) associated with each set \( A_j \) yields a bounded linear map \( R: L_\rho(X) \to X \otimes \lambda L_\rho(\Sigma_0) \) such that the diagram

\[
\begin{array}{ccc}
X \otimes \lambda L_\rho & \xrightarrow{\text{Id}_X \otimes \lambda \mathbb{E}(\rho, \Sigma_0)} & X \otimes \lambda L_\rho(\Sigma_0) \\
J[\rho, X, \lambda] & & \\
L_\rho(X) & \xrightarrow{R} & X \otimes \lambda L_\rho(\Sigma_0) \\
\end{array}
\]

commutes. The operators \( \text{Id}_X \otimes \lambda \mathbb{E}(\rho, \Sigma_0) \) are uniformly bounded projections. Let \((\Sigma_i)_{i \in I}\) a non-decreasing net of finite \( \sigma \)-algebras whose union generates \( \Sigma \). By Lemma 6.4, \((\text{Id}_X \otimes \lambda \mathbb{E}(\rho, \Sigma_i))_{i \in I}\) is a bounded approximation of the identity. We infer that \( J[\rho, X, \lambda] \) is one-to-one, as wanted, in the particular case that \( \mu(\Omega) < \infty \).
In general, let $R[A, X]: L_\rho(X) \to L_\rho(A, X)$ be the canonical projection on a set $A \in \Sigma(\mu)$. Set $R[A] = R[A, F]$. Since $R[A, X]$ is bounded, applying Lemma 6.1 yields a bounded linear operator $R[A, X, \lambda]$ such that the diagram

$$
\begin{array}{ccc}
X \otimes_L L_\rho & \xrightarrow{\text{Id}_X \otimes R[A]} & X \otimes_L L_\rho(A) \\
\downarrow & & \downarrow \\
L_\lambda^X & \xrightarrow{R[A, X, \lambda]} & L_\lambda^A
\end{array}
$$

commutes. Let $(A_n)_{n=1}^\infty$ be a non-decreasing sequence in $\Sigma(\mu)$ whose union is $\Omega$. By Lemma 6.4, $(\text{Id}_X \otimes L_\rho(A_n))_{n=1}^\infty$ is a bounded approximation of the identity. Since $J[\rho|A_n, X, \lambda]$ is one-to-one (by the previous particular case), it follows that $J[\rho, X, \lambda]$ is one-to-one. \hfill $\square$

Notice that the applicability of Theorem 6.5 depends on the existence of amenable pairs. In the optimal situation, we would be able to choose $\lambda$ to be the smallest symmetric function quasi-norm which gals the quasi-Banach space $X$. Thus, the following question arises.

**Question 6.6.** Let $X$ be a quasi-Banach space. Is $(\lambda_X, X)$ amenable?

As long as there is no general answer to Question 6.6, we will focus on the spaces of gals that have appeared in the literature. We next prove that for all of them Question 6.6 has a positive answer.

**Theorem 6.7.** Let $\phi$ be a concave Orlicz function fulfilling (4.1). Suppose that $\lambda_\phi$ gals a quasi-Banach space $X$. Then $(\lambda_\phi, X)$ is amenable.

**Proof.** Assume that $\phi(1) = 1$. Assume by contradiction that there is a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, a positive sequence $\alpha = (a_j)_{j=1}^\infty$ in $\ell_\phi$, a sequence $(f_j)_{j=1}^\infty$ in the unit ball of $L_1(\mu)$, and a sequence $(x_j)_{j=1}^\infty$ in the unit ball of $X$ such that $\sum_{j=1}^\infty a_j x_j f_j = 0$ in $L_\phi(X)$ and

$$
x := \sum_{j=1}^\infty a_j x_j \int_\Omega f_j d\mu \neq 0.
$$

The following claim will be used a couple of times.

**Claim.** If $(\Omega_k)_{k=1}^\infty$ is a non-decreasing sequence in $\Sigma(\mu)$ such that $\Omega \setminus \cup_{k=1}^\infty \Omega_k$ is a null set, then $\sum_{j=1}^\infty a_j x_j \int_{\Omega_k} f_j d\mu \neq 0$ for some $k \in \mathbb{N}$.

**Proof of the claim.** Since $\lim_k \int_{\Omega_k} f_j d\mu = \int_\Omega f_j d\mu$ for all $j \in \mathbb{N}$ and $\lambda_\phi$ is dominating, we have

$$
\lim_k \left\| \left( a_j \int_{\Omega} f_j d\mu \right)_{j=1}^\infty - \left( a_j \int_{\Omega_k} f_j d\mu \right)_{j=1}^\infty \right\|_\phi = 0.
$$
Since $\ell_p$ embeds continuously in $G_b(X)$,
\[
\lim_k \left\| \sum_{j=1}^{\infty} a_j x_j \int_{\Omega} f_j \, d\mu - \sum_{j=1}^{\infty} a_j x_j \int_{\Omega_k} f_j \, d\mu \right\| = 0.
\]
This limit readily gives our claim.

The claim allows us to assume that $\mu(\Omega) < \infty$. By Proposition 5.3 (vii), we can assume that $f_j \in S(\mu)$ for all $j \in \mathbb{N}$. Also, we can assume without loss of generality that $\lambda_\varphi(\alpha) < 1$, so that $\sum_{j=1}^{\infty} \varphi(a_j) < 1$. Set
\[
F_m = \sum_{j=m+1}^{\infty} \varphi(a_j) |f_j|, \quad m \in \mathbb{N} \cup \{0\}.
\]
We have $\int_{\Omega} F_0 \, d\mu < \infty$. Therefore, $F_0 < \infty$ a.e. By Severini–Egorov theorem, $\lim_m F_m = 0$ quasi-uniformly. By Proposition 3.13, there is an increasing sequence $(J_n)_{n=1}^{\infty}$ such that, if
\[
G_n = \sum_{j=1}^{J_n} a_j x_j f_j, \quad n \in \mathbb{N},
\]
then $\lim_n G_n = 0$ a.e. Taking into account the claim, we can assume without loss of generality that $\lim_m F_m = 0$ uniformly and that $\lim_n G_n = 0$ pointwise.

Pick $0 < \varepsilon < 1$. There is $m_0 \in \mathbb{N}$ such that $\lambda_\varphi((a_j)_{m_0+1}^{\infty}) < \varepsilon$, i.e.,
\[
A := \sum_{j=m_0+1}^{\infty} \varphi \left( \frac{a_j}{\varepsilon} \right) < 1.
\]
Let $m \geq m_0$ be such that
\[
F_m(\omega) \leq \frac{\varepsilon(1 - A)}{\mu(\Omega)}, \quad \omega \in \Omega.
\]
Let $\Sigma_0$ be a finite $\sigma$-algebra such that $f_j$ is $\Sigma_0$-measurable for all $1 \leq j \leq m$. Let $(A_h)_{h=1}^{H}$ be a partition of $\Omega$ which generates $\Sigma_0$. Pick points $\omega_h \in A_h$ for each $1 \leq h \leq H$, and set
\[
g_j = f_j - \sum_{h=1}^{H} f_j(\omega_h) \chi_{A_h}, \quad j \in \mathbb{N}.
\]
Since $g_j = 0$ for all $1 \leq j \leq m$ we have
\[
x = \lim_n \sum_{j=1}^{J_n} a_j x_j \int_{\Omega} f_j \, d\mu - \sum_{h=1}^{H} \mu(A_h) \lim_n G_n(\omega_h)
\]
\[= \lim_{n} \sum_{j=1}^{J_n} a_j x_j \int_{\Omega} g_j \, d\mu = \lim_{n} \sum_{j=m+1}^{J_n} a_j x_j \int_{\Omega} g_j \, d\mu.\]

Notice that
\[
\left\| \sum_{j=m+1}^{J_n} a_j x_j \int_{\Omega} g_j \, d\mu \right\| \leq \lambda_{\mathcal{X}}((a_j b_j)_{j=m+1}^{\infty}),
\]
where \(b_j = |\int_{\Omega} g_j \, d\mu|\). Recall that if a sequence \((u_n)_{n=1}^{\infty}\) converges to \(x\) in \(\mathcal{X}\), then \(\|x\| \leq \kappa \liminf \|u_n\|\), where \(\kappa\) is the modulus of concavity of \(\mathcal{X}\). Therefore, since \(\lambda_{\varphi}\) galbs \(\mathcal{X}\), we have
\[
\|x\| \leq \kappa \lambda_{\mathcal{X}}((a_j b_j)_{j=m+1}^{\infty}) \leq \kappa C \lambda_{\varphi}((a_j b_j)_{j=m+1}^{\infty}) \leq \varepsilon.
\]
Using the concavity of \(\varphi\) and that \(\varepsilon < 1\), we have
\[
\sum_{j=m+1}^{\infty} \varphi \left( \frac{a_j b_j}{\varepsilon} \right) \leq \sum_{j=m+1}^{\infty} \max\{1, b_j\} \varphi \left( \frac{a_j}{\varepsilon} \right)
\]
\[
\leq \sum_{j=m+1}^{\infty} \left(1 + \sum_{h=1}^{H} |f_j(\omega_h)| \mu(A_h) \right) \varphi \left( \frac{a_j}{\varepsilon} \right)
\]
\[
\leq \sum_{j=m+1}^{\infty} \varphi \left( \frac{a_j}{\varepsilon} \right) + \sum_{h=1}^{H} \sum_{j=m+1}^{\infty} \frac{1}{\varepsilon} \mu(A_h) |f_j(\omega_h)| \varphi(a_j)
\]
\[
= \sum_{j=m+1}^{\infty} \varphi \left( \frac{a_j}{\varepsilon} \right) + \frac{1}{\varepsilon} \sum_{h=1}^{H} \mu(A_h) F_m(\omega_h)
\]
\[
\leq A + \frac{1}{\varepsilon} \sum_{h=1}^{H} \mu(A_h) \frac{\varepsilon(1 - A)}{\mu(\Omega)} = 1.
\]
Therefore \(\|x\| \leq \kappa C \varepsilon\). Letting \(\varepsilon\) tend to 0 we arise to absurdity. \(\square\)

Given a quasi-Banach space \(\mathcal{X}\), a \(\sigma\)-finite measure space \((\Omega, \Sigma, \mu)\), a symmetric function quasi-norm \(\lambda\) such that \((\lambda, \mathcal{X})\) is amenable, and a function \(f: \Omega \to \mathcal{X}\), we say that \(f\) is \(\lambda\)-integrable if \(f \in L_{\lambda}^1(\mu, \mathcal{X})\), and we write
\[
\int_{\Omega}^\lambda f \, d\mu = \mathcal{I}[\mu, \mathcal{X}, \lambda](f).
\]
A natural question is whether \(\int_{\Omega}^\lambda f \, d\mu = \mathcal{I}[\mu, \mathcal{X}, \lambda](f)\) really depends on \(\lambda\). That is, do we have \(\mathcal{I}[\mu, \mathcal{X}, \lambda_1](f) = \mathcal{I}[\mu, \mathcal{X}, \lambda_2](f)\) whenever \((\lambda_1, \mathcal{X})\) and \((\lambda_2, \mathcal{X})\) are amenable pairs? This question is equivalent to the following one. Given function quasi-norms \(\rho_1\) and \(\rho_2\) over the same
σ-finite measure space \((\Omega, \Sigma, \mu)\) we define a function quasi-norm \(\rho_1 \cap \rho_2\) by

\[
(\rho_1 \cap \rho_2)(f) = \inf \{ \rho_1(g) + \rho_2(h) : g, h \in L_0^+(\mu), \, f = g + h \},
\]

for each \(f \in L_0^+(\mu)\). It can be proved that if \(\rho_1\) and \(\rho_2\) are \(p\)-concave (resp. \(p\)-convex), \(0 < p < \infty\), then \(\rho_1 \cap \rho_2\) is \(p\)-concave (resp. \(p\)-convex).

**Question 6.8.** Let \(X\) be a quasi-Banach space, and let \(\lambda_1\) and \(\lambda_2\) be symmetric function quasi-norms such that \((\lambda_1, X)\) and \((\lambda_2, X)\) are amenable. Is \((\lambda_1 \cap \lambda_2, X)\) amenable?

Of course, a positive answer to Question 6.6 would yield a positive answer to Question 6.8.

### 7. The fundamental theorem of calculus

Let \(X\) be a quasi-Banach space and let \(\lambda\) be a symmetric function quasi-norm such that \((\lambda, X)\) is amenable. If \(d \in \mathbb{N}, A \subseteq \mathbb{R}^d\) is measurable, and \(\mu\) is the Lebesgue measure on \(A\), we set \(L_1^\lambda(A, X) = L_1^\lambda(\mu, X)\) and, for \(f \in L_1^\lambda(A, X)\),

\[
\int_A f(x) \, dx = \int_A^\lambda f \, d\mu.
\]

A function \(f : \mathbb{R}^d \rightarrow X\) is said to be locally \(\lambda\)-integrable if \(f|_A \in L_1^\lambda(A, X)\) for every bounded measurable \(A \subseteq \mathbb{R}^d\).

Given \(d \in \mathbb{N}\), we denote by \(Q\) the set consisting of all \(d\)-dimensional open cubes. If \(y \in \mathbb{R}^d\), the set \(Q[y]\) consisting of all \(Q \in Q\) such that \(y \in Q\) is a directed set when ordered by inverse inclusion. We denote by \(Q \in Q \rightarrow y\) the convergence with respect to that directed set.

The following improves [1, Theorem 5.2].

**Theorem 7.1.** Let \(X\) be a quasi-Banach space and \(\lambda\) be a symmetric function quasi-norm. Suppose that \(\lambda\) is \(p\)-concave for some \(0 < p < 1\) and that \((\lambda, X)\) is amenable. Then, for any locally \(\lambda\)-integrable function \(f : \mathbb{R}^d \rightarrow X\),

\[
\lim_{Q \in Q \rightarrow y} \frac{1}{|Q|} \int_Q^\lambda f(x) \, dx = f(y) \quad \text{a.e. } y \in \mathbb{R}^d.
\]

**Proof.** Set

\[
M[\lambda, \lambda](f)(y) = \sup_{Q \subseteq \mathbb{R}^d} \frac{1}{|Q|} \left\Vert \int_Q^\lambda f(x) \, dx \right\Vert, \quad f \in L_1^\lambda(\mathbb{R}^d, X), \; y \in \mathbb{R}^d.
\]

If \(\kappa\) is the modulus of concavity of \(X\), we have

\[
M[\lambda, \lambda](f + g) \leq \kappa M[\lambda, \lambda](f) + \kappa M[\lambda, \lambda](g), \quad f, g \in L_1^\lambda(\mathbb{R}^d, X).
\]

The result holds for functions in the set

\[
\mathcal{F} = \{ x \chi_Q : x \in X, \; Q \in Q \}.
\]
Since $\mathcal{F} = L_1^\lambda(\mathbb{R}^d, X)$, it suffices to prove that the maximal function $M[X, \lambda]$ is bounded from $L_1^\lambda(\mathbb{R}^d, X)$ into $L_{1,\infty}(\mathbb{R}^d)$. Let $M$ be the classical Hardy-Littlewood maximal function. Let $f = \sum_{j=1}^\infty x_j f_j$, where $(x_j)_{j=1}^\infty$ is in the unit ball of $X$ and $(f_j)_{j=1}^\infty$ in $L_1(\mathbb{R}^d)$ satisfies $\lambda((\|f_j\|_1)_{j=1}^\infty) < \infty$, be an expansion of $f \in L_1^\lambda(\mathbb{R}^d, X)$. We have

$$M[X, \lambda](f) \leq \lambda((M(f_j))_{j=1}^\infty).$$

By Theorem 3.42, the pair $(\lambda, L_{1,\infty}(\mathbb{R}^d))$ has the MII property. Since $M$ maps $L_1(\mathbb{R}^d)$ into $L_{1,\infty}(\mathbb{R}^d)$,

$$\|M[X, \lambda](f)\|_{1,\infty} \leq C_1 \lambda((\|M(f_j)\|_{1,\infty})_{j=1}^\infty) \leq C_1 C_2 \lambda((\|f_j\|_1)_{j=1}^\infty),$$

where the constants $C_1$ and $C_2$ do not depend on $f$. Consequently, there is constant $C$ such that $\|M[X, \lambda](f)\|_{1,\infty} \leq C\|f\|_{L_1^\lambda(\mathbb{R}^d, X)}$ for all $f \in L_1^\lambda(\mathbb{R}^d, X)$.

References

[1] F. Albiac and J. L. Ansorena, Integration in quasi-Banach spaces and the fundamental theorem of calculus, J. Funct. Anal. 264 (2013), no. 9, 2059–2076.
[2] F. Albiac and N. J. Kalton, Topics in Banach space theory, Second, Graduate Texts in Mathematics, vol. 233, Springer, [Cham], 2016. With a foreword by Gilles Godefroy.
[3] F. Albiac, J. L. Ansorena, M. Cúth, and M. Doucha, Lipschitz free $p$-spaces for $0 < p < 1$, Israel J. Math. 240 (2020), no. 1, 65–98.
[4] I. Amemiya, A generalization of Riesz-Fischer’s theorem, J. Math. Soc. Japan 5 (1953), 353–354.
[5] T. Aoki, Locally bounded linear topological spaces, Proc. Imp. Acad. Tokyo 18 (1942), 588–594.
[6] C. Bennett and R. Sharpley, Interpolation of operators, Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988.
[7] A.-P. Calderón, Spaces between $L^1$ and $L^\infty$ and the theorem of Marcinkiewicz, Studia Math. 26 (1966), 273–299.
[8] M. Carro, L. Colzani, and G. Sinnamon, From restricted type to strong type estimates on quasi-Banach rearrangement invariant spaces, Studia Math. 182 (2007), no. 1, 1–27.
[9] L. Colzani and P. Sjögren, Translation-invariant operators on Lorentz spaces $L(1,q)$ with $0 < q < 1$, Studia Math. 132 (1999), no. 2, 101–124.
[10] M. M. Day, The spaces $L^p$ with $0 < p < 1$, Bull. Amer. Math. Soc. 46 (1940), 816–823.
[11] H. W. Ellis and I. Halperin, Function spaces determined by a levelling length function, Canad. J. Math. 5 (1953), 576–592.
[12] F. L. Hernández, The $p$-convexity of Orlicz spaces, Collect. Math. 34 (1983), no. 3, 233–245.
[13] ———, On the galb of weighted Orlicz sequence spaces. I, Bull. Polish Acad. Sci. Math. 32 (1984), no. 3–4, 193–202.
[14] ______, On the gap of weighted Orlicz sequence spaces. II, Arch. Math. (Basel) 45 (1985), no. 2, 158–168.
[15] R. A. Hunt, On $L(p, q)$ spaces, Enseign. Math. (2) 12 (1966), 249–276.
[16] N. J. Kalton, The convexity type of quasi-banach spaces, Unpublished (1977).
[17] ______, Linear operators on $L_p$ for $0 < p < 1$, Trans. Amer. Math. Soc. 259 (1980), no. 2, 319–355.
[18] ______, Convexity conditions for nonlocally convex lattices, Glasgow Math. J. 25 (1984), no. 2, 141–152.
[19] ______, Banach envelopes of nonlocally convex spaces, Canad. J. Math. 38 (1986), no. 1, 65–86.
[20] N. J. Kalton, N. T. Peck, and J. W. Roberts, An $F$-space sampler, London Mathematical Society Lecture Note Series, vol. 89, Cambridge University Press, Cambridge, 1984.
[21] W. A. J. Luxemburg and A. C. Zaanen, Notes on Banach function spaces. II, Nederl. Akad. Wetensch. Proc. Ser. A 66 = Indag. Math. 25 (1963), 148–153.
[22] B. Maurey, Intégration dans les espaces $p$-normés, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 26 (1972), 911–931.
[23] S. Mazur and W. Orlicz, Sur les espaces métriques linéaires. I, Studia Math. 10 (1948), 184–208.
[24] B. S. Mitjagin, An interpolation theorem for modular spaces, Mat. Sb. (N.S.) 66 (108) (1965), 473–482.
[25] S. Rolewicz, On a certain class of linear metric spaces, Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 471–473, XL.
[26] S. Rolewicz, Metric linear spaces, Second, Mathematics and its Applications (East European Series), vol. 20, D. Reidel Publishing Co., Dordrecht; PWN—Polish Scientific Publishers, Warsaw, 1985.
[27] A. R. Schep, Minkowski’s integral inequality for function norms, Operator theory in function spaces and Banach lattices, 1995, pp. 299–308.
[28] P. Sjögren, Translation-invariant operators on weak $L^1$, J. Funct. Anal. 89 (1990), no. 2, 410–427.
[29] ______, Convolutors on Lorentz spaces $L^{1,q}$ with $1 < q < \infty$, Proc. London Math. Soc. (3) 64 (1992), no. 2, 397–417.
[30] E. M. Stein and N. J. Weiss, On the convergence of Poisson integrals, Trans. Amer. Math. Soc. 140 (1969), 35–54.
[31] P. Turpin, Espaces et intersections d’espaces $d’Orlicz$ non localement convexes, Studia Math. 46 (1973), 167–195.
[32] ______, Opérateurs linéaires entre espaces d’Orlicz non localement convexes, Studia Math. 46 (1973), 153–165.
[33] ______, Convexités dans les espaces vectoriels topologiques généraux, Dissertationes Math. (Rozprawy Mat.) 131 (1976), 221.
[34] D. Vogt, Integrationstheorie in $p$-normierten Räumen, Math. Ann. 173 (1967), 219–232.
[35] A. C. Zaanen, Riesz spaces. II, North-Holland Mathematical Library, vol. 30, North-Holland Publishing Co., Amsterdam, 1983.
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES, UNIVERSIDAD DE LA RIOJA, LOGROÑO, 26004 SPAIN

Email address: joseluis.ansorena@unirioja.es

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES, 00-656 WARSAWA, UL. ŚNIADECKICH 8, POLAND

Email address: gbello@impan.pl