Are Unitarizable Groups Amenable?

by

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Abstract

We give a new formulation of some of our recent results on the following problem: if all uniformly bounded representations on a discrete group $G$ are similar to unitary ones, is the group amenable? In §5, we give a new proof of Haagerup’s theorem that, on non-commutative free groups, there are Herz-Schur multipliers that are not coefficients of uniformly bounded representations. We actually prove a refinement of this result involving a generalization of the class of Herz-Schur multipliers, namely the class $M_d(G)$ which is formed of all the functions $f : G \to \mathbb{C}$ such that there are bounded functions $\xi_i : G \to B(H_i, H_{i-1})$ ($H_i$ Hilbert) with $H_0 = \mathbb{C}, H_d = \mathbb{C}$ such that

$$f(t_1 t_2 \cdots t_d) = \xi_1(t_1) \xi_2(t_2) \cdots \xi_d(t_d). \quad \forall t_i \in G$$

We prove that if $G$ is a non-commutative free group, for any $d \geq 1$, we have

$$M_d(G) \neq M_{d+1}(G),$$

and hence there are elements of $M_d(G)$ which are not coefficients of uniformly bounded representations. In the case $d = 2$, Haagerup’s theorem implies that $M_2(G) \neq M_3(G)$.

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0 Introduction

The starting point for this presentation is the following result proved in the particular case $G = \mathbb{Z}$ by Sz.-Nagy (1947).

**Theorem 0.1 (Day, Dixmier 1950).** Let $G$ be a locally compact group. If $G$ is amenable, then every uniformly bounded (u.b. in short) representation $\pi : G \to B(H)$ ($H$ Hilbert) is unitarizable. More precisely, if we define

$$|\pi| = \sup\{\|\pi(t)\|_{B(H)} \mid t \in G\},$$

then, if $|\pi| < \infty$, there exists $S : H \to H$ invertible with $\|S\| \|S^{-1}\| \leq |\pi|^2$ such that $t \to S^{-1}\pi(t)S$ is a unitary representation.

**Note.** We say that $\pi : G \to B(H)$ ($H$ Hilbert) is unitarizable if there exists $S : H \to H$ invertible such that $t \to S^{-1}\pi(t)S$ is a unitary representation. We will mostly restrict to discrete groups, but otherwise all representations $\pi : G \to B(H)$ are implicitly assumed continuous on $G$ with respect to the strong operator topology on $B(H)$.

**Definition 0.2.** We will say that a locally compact group $G$ is unitarizable if every uniformly bounded (u.b. in short) representation $\pi : G \to B(H)$ is unitarizable.

In his 1950 paper, Dixmier [19] asked two questions which can be rephrased as follows:

**Q1:** Is every $G$ unitarizable?

**Q2:** If not, is it true that conversely unitarizable $\Rightarrow$ amenable?

In 1955, Ehrenpreis and Mautner answered Q1; they showed that $G = \text{SL}_2(\mathbb{R})$ is not unitarizable. Their work was clarified and amplified in 1960 by Kunze-Stein [35]. See Remark 0.7 below for more recent work in this direction. Here of course $G = \text{SL}_2(\mathbb{R})$ is viewed as a Lie group, but a fortiori the discrete group $\text{SL}_2$ underlying $\text{SL}_2(\mathbb{R})$ fails to be unitarizable, and since every group is a quotient of a free group and “unitarizable” obviously passes to quotients, it follows (implicitly) that there is a non-unitarizable free group, from which it is easy to deduce (since unitarizable passes to subgroups, see Proposition 0.5 below) that $\mathbb{F}_2$ the free group with 2 generators is not unitarizable. In the 80’s, many authors, notably Mantero–Zappa [16–17], Pytlik-Szwarc [70], Bożejko–Fendler [9], Bożejko [8], ..., and also Miotkowski [41], Szwarc [73–76], Wysoczański [81] (for free products of groups), produced explicit constructions of u.b. non-unitarizable representations on $\mathbb{F}_2$ or on $\mathbb{F}_\infty$ (free group with countably infinitely many generators), see [45] for a synthesis between the Italian approach and the Polish one. See also Valette’s papers [78–79] for the viewpoint of groups acting on trees, (combining Pimsner [68] and [70]) and [82] for recent work on Coxeter groups. This was partly motivated by the potential applications in Harmonic Analysis of the resulting explicit formulae (see e.g. [24] and [10]). For instance, if we denote by $|t|$ the length of an element...
in $\mathbb{F}_\infty$ (or in $\mathbb{F}_2$), they constructed an analytic family $(\pi_z)_{z \in \mathbb{D}}$ indexed by the unit disc in $\mathbb{C}$ of u.b. representations such that, denoting by $\delta_e$ the unit basis vector of $\ell_2(G)$ at $e$ (unit element), we have

$$\forall \ t \in G \quad z^{|t|} = \langle \pi_z(t)\delta_e, \delta_e \rangle.$$ 

Thus the function $\varphi_z : t \rightarrow z^{|t|}$ is a coefficient of a u.b. representation on $G = \mathbb{F}_\infty$. However, it can be shown that for $z \notin \mathbb{R}$, the function $\varphi_z$ is not the coefficient of any unitary representation, whence $\pi_z(\cdot)$ cannot be unitarizable. A similar analysis can be made for the so-called spherical functions.

Since unitarizable passes to subgroups (by “induction of representations”, see Proposition 0.5 below) this implies

**Corollary 0.3.** Any discrete group $G$ containing $\mathbb{F}_2$ as a subgroup is not unitarizable.

**Remark 0.4.** Thus if there is a discrete group $G$ which is unitarizable but not amenable, this is a non-amenable group not containing $\mathbb{F}_2$. The existence of such groups remained a fundamental open problem for many years until Olshanskii [51]–[52] established it in 1980, using the solution by Adian–Novikov (see [1]) of the famous Burnside problem, and also Grigorchuk’s cogrowth criterion ([25]). Later, Adian (see [2]) showed that the Burnside group $B(m,n)$ (defined as the universal group with $m$ generators such that every group element $x$ satisfies the relation $x^n = e$) are all non-amenable when $m \geq 2$ and odd $n \geq 665$. Obviously, since every element is periodic, such groups cannot contain any free infinite subgroup. We should also mention Gromov’s examples ([26, §5.5]) of infinite discrete groups with Kazhdan’s property T (hence “very much” non-amenable) and still without any free subgroup. In any case, it is natural to wonder whether the infinite Burnside groups are counterexamples to the above Q2, whence the following.

**Question.** Are the Burnside groups $B(m,n)$ unitarizable?

In the next statement, we list the main stability properties of unitarizable groups.

**Proposition 0.5.** Let $G$ be a discrete group and let $\Gamma$ be a subgroup.

(i) If $G$ is unitarizable, then $\Gamma$ also is.

(ii) If $\Gamma$ is normal, then $G$ is unitarizable only if both $\Gamma$ and $G/\Gamma$ are unitarizable.

**Proof.** (i) Consider a u.b. representation $\pi : \Gamma \rightarrow B(H)$. By Mackey’s induction, we have an “induced” representation $\hat{\pi} : G \rightarrow B(\hat{H})$ with $\hat{H} \supset H$ that is still u.b. (with the same bound) and hence is unitarizable. Moreover, for any $t$ in $\Gamma$, $\hat{\pi}(t)$ leaves $H$ invariant, and $\hat{\pi}(t)|_H = \pi(t)$. Hence, the original representation $\pi$ must also be unitarizable. (See [60, p. 43] for full details).

(ii) Let $q : G \rightarrow G/\Gamma$ be the quotient map and let $\pi : G/\Gamma \rightarrow B(H)$ be any representation. Then, trivially, $\pi$ is u.b. (resp. unitarizable) iff the same is true for $\pi \circ q$. Hence $G$ unitarizable implies $G/\Gamma$ unitarizable.

**Remark 0.6.** In (ii) above, we could not prove that conversely if $\Gamma$ and $G/\Gamma$ are unitarizable then $G$ is (although the analogous fact for ideals in an operator algebra is true, see [66, Exercise 27.1]). In particular, we could not verify that the product of two unitarizable groups is unitarizable, however, it is known, and even for semi-direct products, if one of the groups is amenable, see [50], see also [66] for related questions. Of course this should be true if unitarizable is the same as amenable. Similarly, it is not clear that a directed increasing union of a family $(G_i)_{i \in I}$ of unitarizable groups is unitarizable. Actually, we doubt that this is true in general. However, it is true (and easy to
check) if the family \((G_1)_{i \in I}\) is “uniformly” unitarizable, in the following sense: there is a function \(F: \mathbb{R}^+ \to \mathbb{R}^+\) such that, for any \(i \in I\) and any u.b. representation \(\pi: G_1 \to B(H)\), there is an invertible operator \(S: H \to H\) with \(\|S\| \|S^{-1}\| \leq F(|\pi|)\) such that \(t \to S^{-1} \pi(t) S\) is a unitary representation.

**Remark 0.7.** There is an extensive literature continuing Kunze and Stein’s work first on \(SL(2, \mathbb{R})\) \[35\] and later on \(SL(n, \mathbb{C})\) \[30\]–\[38\], and devoted (among other things) to the construction of non-unitarizable uniformly bounded (continuous) representations on more general Lie groups. We should mention P. Sally \[72\]–\[73\] for \(SL_2\) over local fields (see also \[46\]) and the universal covering group of \(SL(2, \mathbb{R})\), Lipsman \[39\]–\[40\] for the Lorentz groups \(SO_+(n, 1)\) and for \(SL(2, \mathbb{C})\). See the next remark for a synthesis of the current state of knowledge. We refer the reader to Cowling’s papers \([13\), \[14\) for more recent work and a much more comprehensive treatment of uniformly bounded representations on continuous groups. See also Lohoué’s paper \[43\]. All in all, it seems there is a consensus among specialists that discrete groups should be where to look primarily for a counterexample (i.e. unitarizable but not amenable), if it exists. The next remark hopefully should explain why.

**Remark 0.8.** (Communicated by Michael Cowling). For an almost connected locally compact group \(G\) (that is, \(G/G_e\) is compact, where \(G_e\) is the connected component of the identity \(e\)), unitarizability implies amenability. The first step of the argument for this is based on structure theory. The group \(G\) has a compact normal subgroup \(N\) such that \(G/N\) is a finite extension of a connected Lie group (see \[43\] p. 175). Suppose that \(G\) is unitarizable. Then a fortiori \(G/N\) is unitarizable. If we can show that \(G/N\) is amenable, then \(G\) will be amenable, and we are done. So we may suppose that \(G\) is a finite extension of a connected Lie group. A similar argument reduces to the case where \(G\) is a connected Lie group, and a third reduction (factoring out the maximal connected normal amenable subgroup) leads to the case where \(G\) is semisimple and non-compact. It now suffices to show that a non-compact connected semisimple Lie group \(G\) (which is certainly non-amenable) is not unitarizable.

So let \(G\) be a non-compact connected semisimple Lie group. We consider the representations \(\pi_\lambda\) of \(G\) unitarily induced from the characters \(man \mapsto \exp(i\lambda \log a)\) of a minimal parabolic subgroup \(MAN\). When \(\lambda\) is real, \(\pi_\lambda\) is unitary, and, according to B. Kostant \[34\], \(\pi_\lambda\) is unitarizable only if there is an element \(w\) of the Weyl group \((g, a)\) such that \(w\lambda = \bar{\lambda}\). Take a simple root \(\alpha\). If \(z\) is a complex number and there exists \(w\) in the Weyl group such that \(w (z\alpha) = (z\alpha)^{-1} = \bar{z}\alpha\), then \(z\) is either purely real and \(w\alpha = \alpha\) or \(z\) is purely imaginary and \(w\alpha = -\alpha\). Thus if \(z\) is neither real nor imaginary, then \(\pi_{z\alpha}\) is not unitarizable. However, if the imaginary part of \(z\) is small enough, then \(\pi_{z\alpha}\) is uniformly bounded. Indeed, using the induction in stages construction (see \[14\], and also \[3\]), we can make the representation uniformly bounded at the first stage, which involves a real rank one group only (see \[13\] for the construction of the relevant Hilbert space) and then induce unitarily thereafter to obtain a uniformly bounded representation.

The contents of this paper are as follows. In §1, we describe our contribution on the above problem Q2, namely Theorem \[14\] which says that if we assume unitarizability with a specific quantitative bound then amenability follows. We explain the main ideas of the proof in \[22\] there we introduce our main objects of study in this paper namely the spaces \(M_d(G)\). The latter are closely related on one hand to the space of “multipliers of the Fourier algebra,” (which in our notation corresponds to \(d = 2\)) and on the other hand to the space \(UB(G)\) of coefficients of uniformly bounded representations on \(G\), that we compare with the space \(B(G)\) of coefficients of unitary representations on \(G\). We have, for all \(d \geq 2\)

\[B(G) \subset UB(G) \subset M_d(G) \subset M_2(G).\]
Our methods lead naturally to a new invariant of \( G \), namely the smallest \( d \) such that \( M_d(G) = B(G) \), that we denote by \( d_1(G) \) (we set \( d_1(G) = \infty \) if there is no such \( d \)). We have \( d_1(G) = 1 \) iff \( G \) is finite and \( d_1(G) = 2 \) iff \( G \) is infinite and amenable (see Theorem \ref{thm:amenable}). Moreover, we have \( d_1(G) = \infty \) when \( G \) is any non-Abelian free group. Unfortunately, we cannot produce any group with \( 2 < d_1(G) < \infty \), and indeed such an example would provide a negative answer to the above Q2. While the main part of the paper is partially expository, \S5 contains a new result. We prove there that if \( G = \mathbb{F}_\infty \) (free group with countably infinitely many generators) then \( M_d(G) \neq M_{d+1}(G) \) for all \( d \geq 2 \). As a corollary we obtain a completely different proof of Haagerup’s unpublished result that \( M_2(G) \neq UB(G) \).

Let \( H \) be a Hilbert space. Actually, although it is less elementary, it is more natural to work with the \( B(H) \)-valued (or say \( B(\ell_2) \)-valued) analogue of the spaces \( M_d(G) \). The space \( M_d(G) \) corresponds to \( \text{dim}(H) = 1 \) using \( \mathbb{C} \simeq B(\mathbb{C}) \). In the \( B(\ell_2) \)-valued case, the analogue of \( d_1(G) \) is denoted simply by \( d(G) \). We have obviously \( d_1(G) \leq d(G) \) (but we do not have examples where this inequality is strict). Our results for \( d(G) \) run parallel to those for \( d_1(G) \).

Although our methods (especially in the \( B(\ell_2) \)-valued case) are inspired by the techniques of “operator space theory” and “completely bounded maps” (see e.g. \cite{22}, \cite{54} or \cite{67}), we have strived to make our presentation accessible to a reader unfamiliar with those techniques. This explains in particular why we present the scalar valued (i.e. \( \text{dim}(H) = 1 \) case first.

We recall merely that a linear map \( u: A \to B(H) \) defined on a \( C^* \)-algebra \( A \) is called completely bounded (c.b. in short) if the maps \( u_n : M_n(A) \to M_n(B(H)) \) defined by

\[
u_n([a_{ij}]) = [u(a_{ij})] \quad \forall[a_{ij}] \in M_n(A)
\]

are bounded uniformly over \( n \) when \( M_n(A) \) and \( M_n(B(H)) \) are each equipped with their unique \( C^* \)-norm, i.e. the norm in the space of bounded operators acting on \( H \oplus \cdots \oplus H \) (\( n \)-times).

We also recall that, for any locally compact group \( G \), the \( C^* \)-algebra of \( G \) (sometimes called “full” or “maximal” to distinguish it from the “reduced” case) is defined as the completion of the space \( L_1(G) \) for the norm defined by

\[
\|f\| = \sup \left\| \int_G f(t)\pi(t)dt \right\| \quad \forall f \in L_1(G),
\]

where the supremum runs over all (continuous) unitary representations \( \pi \) on \( G \).

In particular, the following result essentially due to Haagerup (\cite{28}) provides a useful (although somewhat abstract) characterization of unitarizable group representations.

**Theorem 0.9.** Let \( G \) be a locally compact group and let \( C^*(G) \) denote the (full) \( C^* \)-algebra of \( G \). Let \( \pi: G \to B(H) \) be a uniformly bounded (continuous) representation. The following are equivalent:

(i) \( \pi \) is unitarizable.

(ii) The mapping \( \tilde{\pi} : f \to \int f(t)\pi(t)dt \) from \( L_1(G) \) to \( B(H) \) extends to a completely bounded map from \( C^*(G) \) to \( B(H) \).

More generally, for an arbitrary bounded continuous function \( \varphi : G \to B(H) \), the following are equivalent:

(i)' There is a unitary representation \( \sigma : G \to B(H_\sigma) \) and operators \( \xi, \eta : H \to H_\sigma \) such that

\[
\varphi(t) = \xi^*\sigma(t)\eta \quad \forall t \in G.
\]
(ii)' The mapping \( \tilde{\varphi} : f \to \int f(t) \varphi(t) dt \) extends to a completely bounded map from \( C^*(G) \) to \( B(H) \).

Proof. If \( \pi \) is unitarizable, say we have \( \pi(\cdot) = \xi \sigma(\cdot) \xi^{-1} \) with \( \sigma \) unitary, then, by definition of \( C^*(G) \), \( \sigma \) extends to a \( C^* \)-algebra representation \( \tilde{\sigma} \) from \( C^*(G) \) to \( B(H) \). Then, if we set \( \hat{\pi}(\cdot) = \xi \tilde{\sigma}(\cdot) \xi^{-1} \), \( \hat{\pi} \) extends \( \pi \) and satisfies (ii). Thus (i) \( \Rightarrow \) (ii). Conversely, if we have a completely bounded extension \( \hat{\pi} : C^*(G) \to B(H) \), then by [23] Th. 1.10], there is \( \xi \) invertible on \( H \) such that \( \xi^{-1} \hat{\pi}(\cdot) \xi \) is a \( \ast \)-homomorphism (in other words a \( C^* \)-algebra morphism) and in particular \( t \to \xi^{-1} \hat{\pi}(t) \xi \) is a unitary representation, hence \( \pi \) is unitarizable. The proof of the equivalence of (i)' and (ii)' is analogous. That (ii)' \( \Rightarrow \) (i)' follows from the fundamental factorization of c.b. maps (see e.g. [60] Chapt. 3], [63] p. 23], or [22]). The converse is obvious. \( \square \)

In particular, this tells us that unitarizability is a countably determined property:

**Corollary 0.10.** Let \( \pi : G \to B(H) \) be a uniformly bounded representation on a discrete group \( G \). Then \( \pi \) is unitarizable iff its restriction to any countable subgroup \( \Gamma \subset G \) is unitarizable.

Proof. If \( \pi \) is not unitarizable, then, by Theorem 0.9 there is a sequence \( a^n \) with \( a^n \in M_n(C^*(G)) \) and \( \| a^n \| \leq 1 \), \( a^n_{ij} \in \ell_1(G) \) and such that \( \| \hat{\pi}(a^n_{ij}) \|_{M_n(B(H))} \to \infty \) when \( n \to \infty \). Since each entry \( a^n_{ij} \) is countably supported, there is a countable subgroup \( \Gamma \subset G \) such that all the entries \( \{a^n_{ij} | n \geq 1, 1 \leq i, j \leq n \} \) are supported on \( \Gamma \). This implies (by Theorem 0.9 again) that \( \pi|_\Gamma \) is not unitarizable. This proves the “if” part. The converse is trivial. \( \square \)

**Corollary 0.11.** If all the countable subgroups of a discrete group \( G \) are unitarizable, then \( G \) is unitarizable.

Proof. This is an immediate consequence of the preceding corollary. \( \square \)

Remark. Let \( G \) be a locally compact group and let \( \Gamma \subset G \) be a closed subgroup, we will say that \( \Gamma \) is \( \sigma \)-compactly generated if there is a countable union of compact subsets of \( G \) that generates \( \Gamma \) as a closed subgroup. Then the preceding argument suitably modified shows that, in the setting of Theorem 1.9 if \( \pi|_\Gamma \) is unitarizable for any \( \sigma \)-compactly generated closed subgroup \( \Gamma \subset G \), then \( \pi \) is unitarizable.

1 **Coefficients of uniformly bounded representations**

It will be useful to introduce the space \( B(G) \) of “coefficients of unitary representations” on a (discrete) group \( G \) defined classically as follows.

We denote by \( B(G) \) the space of all functions \( f : G \to \mathbb{C} \) for which there are a unitary representation \( \pi : G \to B(H) \) and vectors \( \xi, \eta \in H \) such that

\[
(1.1) \quad \forall t \in G \quad f(t) = (\pi(t) \xi, \eta),
\]

This space can be equipped with the norm

\[
\| f \|_{B(G)} = \inf \{ \| \xi \| \| \eta \| \}
\]

where the infimum runs over all possible \( \pi, \xi, \eta \) as above. As is well known, \( B(G) \) is a Banach algebra for the pointwise product. Moreover, \( B(G) \) can be identified with the dual of the “full” \( C^* \)-algebra of \( G \), denoted by \( C^*(G) \).

More generally, let \( c \geq 1 \) and let \( G \) be a semi-group with unit. In that case, we may replace “representations’ by unital semi-group homomorphisms. Indeed, note that (since a unitary operator
is nothing but an invertible contraction with contractive inverse) a unitary representation on a group $G$ is nothing but a unital semi-group homomorphism $\pi : G \to B(H)$ such that $\sup \{ \| \pi(t) \| : t \in G \} = 1$. For any semi-group homomorphism $\pi : G \to B(H)$, we again denote

$$|\pi| = \sup \{ \| \pi(t) \| : t \in G \}.$$ 

In the sequel, unless specified otherwise, $G$ will denote a semi-group with unit.

We denote by $B_c(G)$ the space of all functions $f : G \to \mathbb{C}$ for which there is a unital semi-group homomorphism $\pi : G \to B(H)$ with $|\pi| \leq c$ together with vectors $\xi, \eta$ in $H$ such that (1.1) holds. Moreover, we denote

$$\| f \|_{B_c(G)} = \inf \{ \| \xi \| \| \eta \| : f(\cdot) = \langle \pi(\cdot)\xi, \eta \rangle \text{ with } |\pi| \leq c \}.$$ 

Note that when $c = 1$ and $G$ is a group, $B_1(G) = B(G)$ with the same norm, since $|\pi| = 1$ iff $\pi$ is a unitary representation. For convenience of notation, we set $B(G) = B_1(G)$ also for semi-groups. In the group case, $B_c(G)$ appears as the space of coefficients of u.b. representations with bound $\leq c$.

Theorem 1.1 ([61]). The following properties of a discrete group $G$ are equivalent.

(i) $G$ is amenable.

(ii) $\exists K \exists \alpha < 3$ such that for every u.b. representation $\pi : G \to B(H)$ $\exists S : H \to H$ invertible with $\| S \| \| S^{-1} \| \leq K \| \pi \|^\alpha$ such that $S^{-1}(\pi(\cdot)S$ is a unitary representation.

(iii) $\exists K \exists \alpha < 3$ such that for any $c > 1 B_c(G) \subset B(G)$ and we have

$$\forall f \in B_c(G) \quad \| f \|_{B(G)} \leq K c^\alpha \| f \|_{B_c(G)}.$$ 

(iii)’ Same as (iii) with $\alpha = 2$ and $K = 1$.

(iii)’ Same as (iii) with $\alpha = 2$ and $K = 1$. 

The following result partially answers Dixmier’s question Q2.
where the infimum runs over all possible ways to write $f$ (2.1) (see however § important for the proofs of all the results below, but we prefer to skip this in the present exposition so-called Haagerup tensor product (see [12]). The connection is explained in detail in [61], and is motivated by the work of Christensen-Sinclair on “completely bounded multilinear maps” and the

Here of course we use the identification $B\otimes_m$  in the group case, but we could also take $M_2(G)$ implies $M_2(G) = B(G)$ does not really use the discreteness of the group, whence the result in full generality.

This observation, concerning the extension to general locally compact groups, was also made independently by Nico Spronk [74].

To explain the proof of Theorem 1.1, we will need some additional notation.

First part of the proof of Theorem 1.1. That (i) ⇒ (ii)' is the Dixmier-Day result mentioned at the beginning. The implications (ii)' ⇒ (ii) and (iii)' ⇒ (iii) are trivial. Moreover, (ii) ⇒ (iii) and (ii)' ⇒ (iii)' are easy to check: indeed consider $f(\cdot) = (\pi(\cdot)\xi, \eta)$ with $|\pi| \leq c$. Then, if $S$ is such that $\hat{\pi} = S\pi(\cdot)S^{-1}$ is a unitary representation, we have $f(\cdot) = (\pi(\cdot)\xi, \eta) = (\hat{\pi}(\cdot) S\xi, (S^{-1})^*\eta)$, hence the coefficients of $\pi$ are coefficients of $\hat{\pi}$ and

\[
\|f\|_{B(G)} \leq \|S\xi\| \|S^{-1}\| \|\eta\| \leq \|S\| \|S^{-1}\| \|\xi\| \|\eta\|,
\]

whence

\[
\|f\|_{B(G)} \leq \|S\| \|S^{-1}\| \|f\|_{B(G)}.
\]

Now if (ii) holds we can find $S$ as above with $\|S\| \|S^{-1}\| \leq Kc^\alpha$, thus (ii) ⇒ (iii), and similarly (ii)' ⇒ (iii)' . Thus it only remains to prove that (iii) ⇒ (i). \qed

2 The spaces of multipliers $M_d(G)$

To explain the proof of Theorem 1.1 we will need some additional notation.

Notation. Let $d \geq 1$ be an integer. Let $G$ be a a semigroup with unit. We are mainly interested in the group case, but we could also take $G = \mathbb{N}$.

Let $M_d(G)$ be the space of all functions $f: \ G \to \mathbb{C}$ such that there are bounded functions $\xi_i: \ G \to B(H_i, H_{i-1})$ ($H_i$ Hilbert) with $H_0 = \mathbb{C}$, $H_d = \mathbb{C}$ such that

(2.1) \quad \forall \ t_i \in G \quad f(t_1t_2 \ldots t_d) = \xi_1(t_1)\xi_2(t_2)\ldots\xi_d(t_d).

Here of course we use the identification $B(H_0, H_d) = B(\mathbb{C}/\mathbb{C}) \simeq \mathbb{C}$. We define

\[
\|f\|_{M_d(G)} = \inf\{\sup_{t_1 \in G} \|\xi_1(t_1)\| \ldots \sup_{t_d \in G} \|\xi_d(t_d)\|\}
\]

where the infimum runs over all possible ways to write $f$ as in 2.1.

The definition of the spaces $M_d(G)$ and of the more general spaces $M_d(G; H)$ appearing below is motivated by the work of Christensen-Sinclair on “completely bounded multilinear maps” and the so-called Haagerup tensor product (see [12]). The connection is explained in detail in [61], and is important for the proofs of all the results below, but we prefer to skip this in the present exposition (see however § below).
When \( d = 2 \), and \( G \) is a group, the space \( M_2(G) \) is the classical space of “Herz-Schur multipliers” on \( G \). This space also coincides (see [9] or [60, p. 110]) with the space of all c.b. “Fourier multipliers” on the reduced \( C^* \)-algebra \( C^*_r(G) \). The question whether the space \( M_2(G) \) coincides with the space of coefficients of u.b. representations (namely \( \bigcup_{c>1} B_c(G) \)) remained open for a while but Haagerup showed that it is not the case. More precisely, he showed that if \( G = F_\infty \), we have

\[
\forall \ c > 1 \quad B_c(G) \not\subset M_2(G).
\]

We give a different proof of a more precise statement in [27] below.

For \( d > 2 \), in the group case, the spaces \( M_d(G) \) are not so naturally interpreted in terms of “Fourier” multipliers. In particular, in spite of the strong analogy with the multilinear multipliers introduced in [21] (those are complex valued functions on \( G^d \)), there does not seem to be any significant connection.

In the case \( G = \mathbb{N} \), the space \( M_3(G) \) is characterized in [65] as the space of “completely shift bounded” Fourier multipliers on the Hardy space \( H_1 \), but this interpretation is restricted to \( d = 3 \) and uses the commutativity.

**Remark 2.1.** Note the following easily checked inclusions, valid when \( G \) is a group or a semigroup with unit:

\[
B(G) = B_1(G) \subset UB(G) = \bigcup_{c>1} B_c(G) \subset M_d(G) \subset M_{d-1}(G) \subset \cdots
\]

\[
\cdots \subset M_2(G) \subset M_1(G) = \ell_\infty(G),
\]

and we have clearly

\[
\forall m \leq d \quad \|f\|_{M_m(G)} \leq \|f\|_{M_d(G)}.
\]

Moreover, we have

\[
\forall \ f \in B_c(G) \quad \|f\|_{M_d(G)} \leq c^d \|f\|_{B_c(G)}.
\]

Indeed, if \( f(\cdot) = \langle \pi(\cdot)\xi, \eta \rangle \) with \( |\pi| \leq c \), then we can write

\[
f(t_1t_2 \ldots t_d) = \langle \pi(t_1) \ldots \pi(t_d)\xi, \eta \rangle = \xi_1(t_1)\xi_2(t_2) \ldots \xi_d(t_d)
\]

where \( \xi_1(t_1) \in B(H_\pi, \mathbb{C}) \), \( \xi_d(t_d) \in B(\mathbb{C}, H_\pi) \) and \( \xi_i(t_i) \in B(H_\pi, H_\pi) \) \((1 < i < d)\) are defined by \( \xi_1(t_1)h = \langle \pi(t_1)h, \eta \rangle \) \((h \in H_\pi)\) \( \xi_d(t_d)\lambda = \lambda\pi(t_d)\xi \) \((\lambda \in \mathbb{C})\) and \( \xi_i(t_i) = \pi(t_i) \) \((1 < i < d)\). Therefore, we have

\[
\|f\|_{M_d(G)} \leq \sup \|\xi_1\| \sup \|\xi_2\| \ldots \sup \|\xi_d\|
\]

\[
\leq |\pi|^d \|\xi\| \|\eta\| \leq c^d \|\xi\| \|\eta\|
\]

whence the announced inequality (2.3).

**Remark 2.2.** It is easy to see (using tensor products) that \( M_d(G) \) is a unital Banach algebra for the pointwise product of functions on \( G \): for any \( f, g \) in \( M_d(G) \) we have

\[
\|fg\|_{M_d(G)} \leq \|f\|_{M_d(G)}\|g\|_{M_d(G)}.
\]

The function identically equal to 1 on \( G \) is the unit and has norm 1.
Remark. Let \( h : \Gamma \to G \) be a unital homomorphism between two groups (or two semi-groups with unit). Then for any \( f \) in \( M_d(G) \) the composition \( f \circ h \) is in \( M_d(\Gamma) \) with \( \|f \circ h\|_{M_d(\Gamma)} \leq \|f\|_{M_d(G)} \). The proof is obvious.

In particular, if \( \Gamma \subset G \) is a subgroup we have \( \|f|\|_{M_d(\Gamma)} \leq \|f\|_{M_d(G)} \). Moreover, if \( \Gamma \) is a normal subgroup in a group \( G \) and if \( q : G \to G/\Gamma \) is the quotient map, then \( \|f\|_{M_d(G/\Gamma)} = \|f \circ q\|_{M_d(G)} \).

(Indeed, the equality can be proved easily using an arbitrary pointwise lifting \( \rho : G/\Gamma \to G \).) Let \( \Gamma \subset G \) be again an arbitrary subgroup of a group \( G \). Given a function \( f : \Gamma \to \mathbb{C} \), we let \( \tilde{f} : G \to \mathbb{C} \) be the extension of \( f \) vanishing outside \( \Gamma \). Then, it is rather easy to see that \( \|\tilde{f}\|_{M_d(G)} = \|f\|_{M_d(\Gamma)} \) (but the analogue of this for \( d > 2 \) seems unclear).

It is well known that \( 1 \Gamma \) is in the unit ball of \( B(G) \) (hence a fortiori of \( M_d(G) \)) hence by Remark 2.2 we have for any \( f \) in \( M_d(G) \):

\[ \|f \cdot 1\Gamma\|_{M_d(G)} \leq \|f\|_{M_d(G)}. \]

The proof of (iii) \( \Rightarrow \) (i) in Theorem 1.1 uses the following criterion for amenability due to Marek Bożejko [7].

**Theorem 2.3.** Let \( G \) be a discrete group. Then \( G \) is amenable iff \( B(G) = M_2(G) \).

**Remark.** We do not know whether \( B(G) = M_3(G) \Rightarrow G \) amenable.

**Sketch of proof of Theorem 2.3** The only if part is quite easy. Let us sketch the proof of the “if” part. Assume \( B(G) = M_2(G) \). Then there is a constant \( K \) such that, for any \( f \) in the space \( \mathbb{C}[G] \) of all finitely supported functions \( f : G \to \mathbb{C} \), we have

\[ \|f\|_{B(G)} \leq K \|f\|_{M_2(G)}. \]

Let \( \varepsilon : G \to \{ -1, 1 \} \) be a “random choice of signs” indexed by \( G \), and let \( \mathbb{E} \) denote the expectation with respect to the corresponding probability. We will estimate the average of the norms of the pointwise product \( \varepsilon f \). More precisely we claim that there are numerical constants \( C' \) and \( C'' \) (independent of \( f \)) such that

\[ \left( \sum_{t \in G} |f(t)|^2 \right)^{1/2} \leq C' \mathbb{E} \|\varepsilon f\|_{B(G)} \]

\[ \mathbb{E} \|\varepsilon f\|_{M_2(G)} \leq C'' \left\| \sum |f(t)|^2 \lambda(t) \right\|_{L^1(G)}. \]

Using this it is easy to conclude: indeed we have

\[ \sum |f(t)|^2 \leq (C'KC'')^2 \left\| \sum |f(t)|^2 \lambda(t) \right\|_{L^1(G)} \]

and by the well known Kesten-Hulanicki criterion (cf. e.g. [60], Th. 2.4), this implies that \( G \) is amenable.

We now return to the above claims. The inequality \( 2.4 \) can be seen as a consequence of the fact (due to N. Tomczak-Jaegermann [77]) that \( B(G) \) is of cotype 2 (a Banach space \( B \) is called of cotype 2 if there is a constant \( C \) such that for any finite sequence \( (x_i) \) in \( B \), the following inequality holds:

\[ \sum \|x_i\|^2 1/2 \leq C \text{Average}_{\pm} \|\sum \pm x_i\|). \]

As for \( 2.5 \), it is proved in [7] using an idea due to Varopoulos [80]. However, more recently the following result was proved in [59]: Consider all possible ways to have the following decomposition

\[ f(t_1t_2) = \alpha(t_1, t_2) + \beta(t_1, t_2) \quad \forall t_1, t_2 \in G \]
and let

\[
|||f||| = \inf \left\{ \sup_{t_1} \left( \sum_{t_2} |\alpha(t_1, t_2)|^2 \right)^{1/2} + \sup_{t_2} \left( \sum_{t_1} |\beta(t_1, t_2)|^2 \right)^{1/2} \right\}
\]

where the infimum runs over all possible decompositions as in (2.6).

Then (cf. [59]) there is a numerical constant \(\delta > 0\) such that

(2.7)
\[
\delta |||f||| \leq \mathbb{E} \|\varepsilon f\|_{M_2(G)} \leq |||f||| \quad \forall f \in \mathbb{C}[G].
\]

Note that the right-hand side of (2.7) is an immediate consequence of the following inequality

(2.8)
\[
\|f\|_{M_2(G)} \leq |||f|||,
\]

and the latter is easy: we simply write

\[
f(t_1 t_2) = \langle \xi_1(t_1), \xi_2(t_2) \rangle + \langle \eta_1(t_1), \eta_2(t_2) \rangle
\]

where

\[
\xi_1(t_1) = \sum_{t_2} \alpha(t_1, t_2) \delta_{t_2}, \quad \xi_2(t_2) = \delta_{t_2}
\]

and

\[
\eta_2(t_2) = \sum_{t_1} \beta(t_1, t_2) \delta_{t_1}, \quad \eta_1(t_1) = \delta_{t_1},
\]

and (2.8) follows.

Moreover, we also have

(2.9)
\[
|||f||| \leq \left\| \sum |f(t)|^2 \lambda(t) \right\|_{C^*_\lambda}^{1/2},
\]

therefore (2.7) follows from (2.9) and (2.10) with \(C'' = 1\). The inequality (2.7) follows from the following observation: \(\|\sum |f(t)|^2 \lambda(t)\|_{C^*_\lambda} \leq 1\) iff there is a decomposition of the form

(2.10)
\[
|f(t_1 t_2)| = |a(t_1, t_2)|^{1/2} \cdot |b(t_1, t_2)|^{1/2} \quad \forall t_1, t_2 \in G
\]

for kernels \(a, b\) on \(G \times G\) such that

(2.11)
\[
\sup_{t_1} \sum_{t_2} |a(t_1, t_2)|^2 \leq 1 \quad \text{and} \quad \sup_{t_2} \sum_{t_1} |b(t_1, t_2)|^2 \leq 1.
\]

Then (2.10) and (2.11) imply that \(|||f||| \leq 1\), and hence (2.7) follows by homogeneity. Indeed, by a compactness argument, these assertions are immediate consequences of the following Lemma.

This Lemma gives a converse to Schur’s classical criterion for boundedness on \(\ell_2\) of matrices with positive entries (we include the proof for lack of a suitable reference). See [64] for more information on this.
Lemma 2.4. Let \( n \geq 1 \). Let \( \{f_{ij} \mid 1 \leq i, j \leq n\} \) be complex scalars such that the matrix \( |f_{ij}|^2 \) has norm \( \leq 1 \) as an operator on the Euclidean space \( \ell_2^n \). Then there are \((a_{ij})\) and \((b_{ij})\) with

\[
\sup_i \sum_j |a_{ij}|^2 \leq 1 \quad \text{and} \quad \sup_j \sum_i |b_{ij}|^2 \leq 1
\]

such that \( |f_{ij}| = |a_{ij}|^{1/2} |b_{ij}|^{1/2} \). Therefore, there are \((\alpha_{ij})\) and \((\beta_{ij})\) with

\[
\sup_i (\sum_j |\alpha_{ij}|^2)^{1/2} + \sup_j (\sum_i |\beta_{ij}|^2)^{1/2} \leq 1
\]

such that \( f_{ij} = \alpha_{ij} + \beta_{ij} \).

Proof. By perturbation and compactness arguments, we can assume that \( |f_{ij}| > 0 \) for all \( i, j \). Let \( T = [|f_{ij}|^2] \). We may assume \( \|T\| = 1 \). Let \( \xi = (\xi_i) \) be a Perron–Frobenius vector for \( T^*T \) so that \( \xi_i > 0 \) for all \( i \) and \( T^*T \xi = \xi \). Let \( \eta = T \xi \), so that \( T^* \eta = \xi \). If we then set \( |a_{ij}|^2 = |f_{ij}|^2 \xi_j \xi_i^{-1} \) and \( |b_{ij}|^2 = |f_{ij}|^2 \xi_j \xi_i^{-1} \eta_i \) we obtain the first assertion. By the arithmetic-geometric mean inequality we have \( |f_{ij}| \leq g_{ij} \) with \( g_{ij} = 2^{-1}(|a_{ij}| + |b_{ij}|) \). If we then set

\[
\alpha_{ij} = 2^{-1} |a_{ij}| g_{ij}^{-1} \quad \text{and} \quad \beta_{ij} = 2^{-1} |b_{ij}| g_{ij}^{-1},
\]

we obtain the second assertion. \( \square \)

Remark. The proof of Theorem 2.3 sketched above shows that \( G \) is amenable if \( \exists K \forall f \in \mathbb{C}[G] \)

\[
\|f\|_{B(G)} \leq K \|f\|_{M_2(G)}.
\]

Remark. Note that (2.7) and (2.8) show that

\[
\sup_{\varepsilon} \|\varepsilon f\|_{M_2(G)} \leq \delta^{-1} \|\varepsilon f\|_{M_2(G)}.
\]

The proof of the implication (iii) \( \Rightarrow \) (i) in Theorem 1.1 rests on the following.

Key Lemma 2.5 (Implicit in [61]). Let \( f \in B(G) \). Fix \( d \geq 1 \). Then, for any \( c \geq 2 \), we have

\[
\|f\|_{B_c(G)} \leq 2\|f\|_{M_d(G)} + 2e^{-(d+1)}\|f\|_{B(G)}.
\]

More generally, for any \( 1 \leq \theta < c \), we have for any \( f \) in \( B_\theta(G) \) and any \( d \geq 1 \)

\[
\|f\|_{B_c(G)} \leq \left( \sum_{m=0}^{d} (\theta/c)^m \right) \cdot \|f\|_{M_m(G)} + \left( \sum_{m>d} (\theta/c)^m \right) \cdot \|f\|_{B_\theta(G)}.
\]

Remark. The proof of the key lemma uses ideas from two remarkable papers due to Peller [54] and Blecher and Paulsen [5].

Proof of (iii) \( \Rightarrow \) (i) in Theorem 1.1. Assume (iii). Then using the key lemma with \( d = 2 \) (and \( \theta = 1 \)), we have for all \( f \) in \( B(G) \) and all \( c \geq 2 \)

\[
\|f\|_{B(G)} \leq K' c^\alpha \|f\|_{B_c(G)} \leq 2K' c^\alpha \|f\|_{M_2(G)} + 2K' c^\alpha \|f\|_{B(G)}.
\]
But we can choose $c = c(K, \alpha)$ large enough so that $2Ke^{\alpha - 3} = 1/2$ (say) and then we obtain

$$
\left(1 - \frac{1}{2}\right) \|f\|_{B(G)} \leq 2Ke^\alpha \|f\|_{M_2(G)}
$$

so that we conclude

$$
\|f\|_{B(G)} \leq 4Kc(K, \alpha)^\alpha \|f\|_{M_2(G)},
$$

hence, by Theorem 2.3, $G$ is amenable.

The proof of the key lemma is based on the following result (of independent interest) which is “almost” a characterization of $B_c(G)$.

**Theorem 2.6.** Fix a number $c \geq 1$. Consider $f \in \bigcap_{m \geq 1} M_m(G)$ such that $\sum_{m} c^{-m}\|f\|_{M_m(G)} < \infty$. Then $f \in B_c(G)$ and moreover

$$
\|f\|_{B_c(G)} \leq |f(e)| + \sum_{m \geq 1} c^{-m}\|f\|_{M_m(G)}.
$$

Conversely, for all $f$ in $B_c(G)$, we have

$$
\sup_{m \geq 1} c^{-m}\|f\|_{M_m(G)} \leq \|f\|_{B_c(G)}.
$$

**Note.** (2.13) is easy and has been proved already (see (2.2)). The main point is (2.12).

**Proof.** This is essentially [61, Theorem 1.12] and the remark following it. For the convenience of the reader, we give some more details.

In [61] the natural predual of $B_c(G)$ is considered and denoted by $\tilde{A}_c$. By [61, Th. 1.7], any $x$ in the open unit ball of $\tilde{A}_c$ can be written as

$$
x = \sum_{m=0}^\infty c^{-m}x_m
$$

where each $x_m$ is an element of $C^*(G)$ which is the image, under the natural product map of an element $X_m$ in the unit ball of $\ell_1(G) \otimes_h \cdots \otimes_h \ell_1(G)$ ($m$ times). This implies by duality, for all $m > 0$

$$
|\langle f, x_m \rangle| \leq \|f\|_{M_m(G)}.
$$

In the particular case $m = 0$, $x_0$ is a multiple of the unit $\delta_e$ by a scalar of modulus $\leq 1$. Whence

$$
|\langle f, x \rangle| \leq |f(e)| + \sum_{m=1}^\infty c^{-m}\|f\|_{M_m(G)}
$$

and a fortiori, by (2.2) and (2.3)

$$
\leq |f(e)| + \sum_{m=1}^d c^{-m}\|f\|_{M_d(G)} + \sum_{m=d+1}^\infty c^{-m}\|f\|_{B(G)}.
$$

\[\square\]
Corollary 2.7. Let $UB(G) = \bigcup_{c>1} B_c(G)$ be the space of coefficients of u.b. representations on $G$. Then $f \in UB(G)$ iff $\sup_{m \geq 1} \|f\|_{M_m(G)}^{1/m} < \infty$. More precisely, let $c(f)$ denote the infimum of the numbers $c \geq 1$ for which $f \in B_c(G)$. Then, we have

$$c(f) = \limsup_{m \to \infty} \|f\|_{M_m(G)}^{1/m}.\]

Proof of Key Lemma 2.5. This is an easy consequence of (2.12), (2.13) and the obvious inequalities

$$|f(e)| \leq \|f\|_{\ell_\infty(G)} \leq \|f\|_{M_2(G)} \leq \cdots \leq \|f\|_{M_d(G)} \leq \cdots \leq \|f\|_{B(G)}. \quad \Box$$

In the case $c = 1$, Theorem 2.6 seems to degenerate but actually the following “limiting case” can be established, as a rather simple dualization of a result in [5].

Proposition 2.8. Consider a function $f : G \to \mathbb{C}$. Then $f \in B(G)$ iff $f \in \bigcap_{m \geq 1} M_m(G)$ with $\sup_{m} \|f\|_{M_m(G)} < \infty$. Moreover we have

$$\|f\|_{B(G)} = \sup_{m \geq 1} \|f\|_{M_m(G)}.\]

Remarks. The same argument shows the following. Given a real number $\alpha \geq 0$, we say that $G$ satisfies the condition $(C_\alpha)$ if there is $K \geq 0$ such that for any $f$ in $B(G)$ we have

$$\forall \ c > 1 \quad \|f\|_{B(G)} \leq Kc^\alpha \|f\|_{B_c(G)}.$$

Then the preceding argument shows that if $d \leq \alpha < d + 1$, $(C_\alpha)$ implies that $B(G) = M_d(G)$ (with equivalent norms).

We are thus led to define the following quantities:

$$\alpha_1(G) = \inf\{\alpha \geq 0 \mid G \text{ satisfies } (C_\alpha)\}$$

$$d_1(G) = \inf\{d \in \mathbb{N} \mid M_d(G) = B(G)\}.$$  

With this notation, the preceding argument shows that $d_1(G) \leq \alpha_1(G)$. A priori $\alpha_1(G)$ is a real number, but (although we have no direct argument for this) it turns out that it is an integer:

Theorem 2.9. Assume $B(G) = \bigcup_{c>1} B_c(G)$. Then $\alpha_1(G) < \infty$ and moreover

$$\alpha_1(G) = d_1(G).$$

In particular, we have

$$M_d(G) = M_{d+1}(G) \quad \forall d \geq \alpha_1(G).$$

Actually, for the last assertion to hold, it suffices to have much less:

Theorem 2.10 ([62]). Let $G$ be a semigroup with unit. Assume that there are $1 \leq \theta < c$ such that $B_\theta(G) = B_c(G)$. Then there is an integer $D$ such that $B_\theta(G) = M_D(G)$, and in particular, we have

$$M_d(G) = M_{d+1}(G) \quad \forall d \geq D.$$
Remark. Let $G$ be a locally compact group. Let $G_d$ be $G$ equipped with the discrete topology. In [29], Haagerup proves that if a function $\phi: G \to \mathbb{C}$ belongs to $M_2(G_d)$ and is continuous, then it belongs to $M_2(G)$ (with the same norm). We do not know if the analogous statement is valid for $M_3(G)$ or $M_d(G)$ when $d \geq 3$.

Remark 2.11. Let $I_1, \ldots, I_d$ be arbitrary sets. We will denote by $M_d(I_1, \ldots, I_d)$ the space of all functions $f: I_1 \times \cdots \times I_d \to \mathbb{C}$ for which there are bounded functions $f_i$

$$f_i: I_i \to B(H_i, H_{i-1}) \quad (\text{here } H_i \text{ are Hilbert spaces with } H_d = H_0 = \mathbb{C})$$

such that

$$\forall b_i \in I_i \quad f(b_1, \ldots, b_d) = f_1(b_1) \cdots f_d(b_d).$$

We equip this space with the norm

$$\|f\| = \inf \left\{ \prod_{i=1}^{d} \sup_{b_i \in I_i} \|f_i(b_i)\| \right\}$$

where the infimum runs over all possible such factorizations.

In particular, if $I_1 = I_2 = \cdots = I_d = G$, then for any function $\varphi$ in $M_d(G)$, we have

$$\|\varphi\|_{M_d(G)} = \|\Phi\|_{M_d(G, \ldots, G)}$$

where $\Phi$ is defined by

$$\Phi(t_1, \ldots, t_d) = \varphi(t_1 t_2 \ldots t_d).$$

By a well known trick, one can check that $f \to \|f\|_{M_d(I_1, \ldots, I_d)}$ is subadditive (and hence is a norm) on $M_d(I_1, \ldots, I_d)$, i.e. that we have for all $f, g$ in $M_d(I_1, \ldots, I_d)$:

$$\|f + g\|_{M_d(I_1, \ldots, I_d)} \leq \|f\|_{M_d(I_1, \ldots, I_d)} + \|g\|_{M_d(I_1, \ldots, I_d)} \quad (2.14)$$

Let us quickly sketch this: Let $f, g$ be in the open unit ball of $M_d(I_1, \ldots, I_d)$. Then by homogeneity we can assume

$$f(b_1, \ldots, b_d) = f_1(b_1) \cdots f_d(b_d) \quad \text{and} \quad g(b_1, \ldots, b_d) = g_1(b_1) \cdots g_d(b_d) \quad (2.15)$$

with $\sup \|f_j(b)\| < 1$ and $\sup \|g_j(b)\| < 1$ for all $j$. Then we can write for any $0 \leq \alpha \leq 1$

$$(\alpha f + (1 - \alpha)g)(b_1, \ldots, b_d) = F_1(b_1) \cdots F_d(b_d)$$

where

$$F_1(b_1) = \left[ \begin{array}{c} \alpha^{1/2} f_1(b_1) \\ (1 - \alpha)^{1/2} g_1(b_1) \end{array} \right] \quad (\text{row matrix with operator entries})$$

and

$$F_j(b_j) = \begin{bmatrix} f_j(b_j) & 0 \\ 0 & g_j(b_j) \end{bmatrix} \quad 2 \leq j \leq d - 1$$

and

$$F_d(b_d) = \begin{bmatrix} \alpha^{1/2} f_d(b_d) \\ (1 - \alpha)^{1/2} g_d(b_d) \end{bmatrix} \quad (\text{column matrix with operator entries})$$

Then it is easy to check that $\sup_b \|F_j(b)\| < 1$ for all $1 \leq j \leq d$ and hence we obtain

$$\|\alpha f + (1 - \alpha)g\|_{M_d(I_1, \ldots, I_d)} < 1.$$

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Moreover, $M_d(I_1, \ldots, I_d)$ is a unital Banach algebra for the pointwise product, i.e. for any $f, g$ in $M_d(I_1, \ldots, I_d)$ we have

$$
\|fg\|_{M_d(I_1, \ldots, I_d)} \leq \|f\|_{M_d(I_1, \ldots, I_d)} \|g\|_{M_d(I_1, \ldots, I_d)}.
$$

Indeed, if we assume (2.15) then we have

$$(fg)(b_1, \ldots, b_d) = (f_1(b_1) \otimes g_1(b_1)) \cdots (f_d(b_d) \otimes g_d(b_d)),$$

and (2.16) follows easily from this. Obviously, the function identically equal to 1 on $I_1 \times \ldots \times I_d$ is a unit for this algebra and it has norm 1 in $M_d(I_1, \ldots, I_d)$.

**Example 2.12.** To illustrate the preceding concepts, we recover here the following result from [70]: Let $G = F_{\infty}$ and let $W(1) \subset G$ be the subset of all the words of length 1. Then the indicator function of $W(1)$ is in $UB(G)$. Indeed, we claim that for any bounded function $\varphi$ with support in $W(1)$ we have

$$
\forall d \geq 1 \ \ \ |\varphi|_{M_d(G)} \leq 2^d |\varphi|_{\ell_\infty(G)}.
$$

Thus (by Corollary 2.7) we have $\varphi \in B_c(G)$ for $c > 2$ (actually, this is known for all $c > 1$).

However it can be shown that for any such function we have

$$
\left(\sum_{t \in G} |\varphi(t)|^2\right)^{1/2} \leq 2|\varphi|_{B(G)}.
$$

Thus for instance the indicator function of $W(1)$ is in $B_c(G)$ for all $c > 2$ but not in $B(G)$ (in particular this shows that $G$ is not unitarizable). Note however that since $1\chi_{W(1)}$ belongs to $M_d(G)$ for all $d \geq 1$, this does not distinguish the various classes $M_d(G)$ or $B_c(G)$, but this task is completed in §5.

**Proof of (2.18).** First it suffices to prove this for a finitely supported $\varphi$, with support in $W(1)$. Then (2.18) is an immediate consequence of an inequality proved first by Leinert [42], and generalized by Haagerup [30, Lemma 1.4]: Any $\psi$ finitely supported, with support in $W(1)$ satisfies $\|\sum \lambda(t)\psi(t)\| \leq 2(\sum |\psi(t)|^2)^{1/2}$. The inequality (2.18) can be deduced from this by duality, using the fact that, if $\varphi$ is finitely supported, then $|\varphi|_{B(G)} = \sup |\langle \varphi, \psi \rangle|$ where the sup runs over all $\psi$ finitely supported on $G$ such that $\|\sum \lambda(t)\psi(t)\| \leq 1$.

**Proof of (2.17).** Let $\varphi$ be a function with support in $W(1)$. Consider the set

$$
\Omega = \{(t_1, \ldots, t_d) \in G^d \mid t_1t_2\ldots t_d \in W(1)\}.
$$

Clearly, when $t_1t_2\ldots t_d$ has length one, it reduces to a single letter (i.e. a generator or its inverse). Clearly this letter must “come” from either $t_1, t_2, \ldots$ or $t_d$. Thus we have

$$
\Omega = \Omega_1 \cup \ldots \cup \Omega_d
$$

where $\Omega_j$ is the set of $(t_1, \ldots, t_d)$ in $\Omega$ such that the single “letter” left after reduction comes from $t_j$. Hence we have

$$
1_{\Omega} = \sum_j 1_{\Omega_j} \prod_{i<j} [1 - 1_{\Omega_i}].
$$
For any $\theta$ in $G$, we introduce the operator $\xi(\theta) \in B(\ell_2(G))$ defined as follows: Assume $\theta = a_1a_2 \ldots a_k$ (reduced word where $a_q \in W(1)$ for all $q$), with $k \geq 1$, then we set $a_0 = a_{k+1} = e$ and

$$\xi(\theta) = \sum_{q=1}^{k} \varphi(q)e_{a_1 \ldots a_{q-1},(a_{q+1} \ldots a_k)^{-1}}$$

where, as usual, $e_{s,t}$ denotes the operator defined by $e_{s,t}(\delta_t) = \delta_s$ and $e_{s,t}(\delta_x) = 0$ whenever $x \neq t$. Moreover, if $\theta = e$ (empty word, corresponding to $k = 0$), we set $\xi(\theta) = 0$. Then it is a simple verification that

$$\langle \lambda(t_1) \ldots \lambda(t_{j-1})\xi(t_j)\lambda(t_{j+1}) \ldots \lambda(t_d)1, \delta_e \rangle = \varphi(t_1t_2 \ldots t_d)1_{\{t_1 \ldots t_d \in \Omega_j\}}.$$ 

A moment of reflection shows that $\|\xi(\theta)\| = \sup_{\theta \in G} \|\varphi(a_q)\|$ hence $\sup_{\theta \in G} \|\xi(\theta)\| \leq \|\varphi\|_{\ell_\infty(G)}$. This shows with the notation introduced in Remark 2.11 that if we set

$$\Phi_j(t_1, \ldots, t_d) = \varphi(t_1t_2 \ldots t_d)1_{\{t_1, t_2, \ldots, t_d \in \Omega_j\}}$$

we have

$$\|\Phi_j\|_{M_d(G, \ldots, G)} \leq \sup_{t \in G} \|\xi(t)\| \leq \|\varphi\|_{\ell_\infty(G)},$$

and hence with $\varphi = 1$ identically, we find $\|1_{\Omega_j}\|_{M_d(G, \ldots, G)} \leq 1$, and

$$\|1 - 1_{\Omega_j}\|_{M_d(G, \ldots, G)} \leq 2$$

hence by Remark 2.11, we have

$$\|\varphi\|_{M_d(G, \ldots, G)} = \|\Phi\|_{M_d(G, \ldots, G)} = \left\| \sum_j \Phi_j \prod_{i < j} [1 - 1_{\Omega_i}] \right\|_{M_d(G, \ldots, G)} \leq 2^d \|\varphi\|_{\ell_\infty(G)}$$

which completes the proof of (2.17). \[\square\]

**Example 2.13.** Let $G$ be a free group.

(i) Let $\psi_d: G^d \to \{0, 1\}$ be the indicator function of the set formed by all the $d$-tuples $(t_1, \ldots, t_d)$ of reduced words such that $t_i \neq e$ for all $i$ and the product $t_1t_2 \ldots t_d$ allows no reduction. Then

$$\|\psi_d\|_{M_d(G, \ldots, G)} \leq 5^{d-1}$$

(ii) A fortiori, for any subsets $I_1 \subset G, \ldots, I_d \subset G$, we have

$$\|\psi_d|_{I_1 \times \ldots \times I_d}\|_{M_d(I_1, \ldots, I_d)} \leq 5^{d-1}.$$ 

**Proof.** Fix $1 \leq j \leq d - 1$. Let $A_j$ be the subset of $G^d$ formed of all $(t_1, \ldots, t_d)$ in $G^d$ such that $t_j \neq e$, $t_{j+1} \neq e$ and such that $t_jt_{j+1}$ does reduce, i.e. $|t_jt_{j+1}| < |t_j| + |t_{j+1}|$. Also let $B_j = \{t \in G^d \mid t_j \neq e, t_{j+1} \neq e\}$. We will use the fact that

$$\psi_d = \prod_{j=1}^{d-1} 1_{B_j} - 1_{A_j}.$$
Observe that for all $t = (t_j) \in G^d$

$$1_{B_j}(t) = (1 - 1_{t_j = e})(1 - 1_{t_{j+1} = e})$$

and using $1_{t_j = e} = \langle \delta_{t_j}, \delta_e \rangle$, it is easy to deduce from this with (2.61) and (2.64) that

$$\|1_{B_j}\|_{M_d(G, \ldots, G)} \leq 4.$$

Now, for any $x$ in $G$ with $x \neq e$ let us denote by $F(x)$ and $L(x)$ respectively the first and last letter of $x$ (i.e. $F(x)$ and $L(x)$ are equal to a generator or the inverse of one). Then it is easy to check that for any $t = (t_j)$ in $G^d$ we have

$$1_{A_j}(t) = (\alpha(t_j), \beta(t_{j+1}))$$

where $\alpha(t) = \delta_{L(t)}$ and $\beta(t) = \delta_{F(t)^{-1}}$ if both $|t| > 0$ and $|s| > 0$ and $\alpha(e) = \beta(e) = 0$. This implies immediately that

$$\|1_{A_j}\|_{M_d(G, \ldots, G)} \leq 1,$$

hence $\|1_{B_j} - 1_{A_j}\|_{M_d(G, \ldots, G)} \leq 5$, and since, by (2.10), $M_d(G, \ldots, G)$ is a Banach algebra, by (2.20) we obtain

$$\|\psi_d\|_{M_d(G, \ldots, G)} \leq 5^{d-1}. \quad \square$$

3 The predual $X_d(G)$ of $M_d(G)$

The definition of the spaces $B_c(G)$ and $M_d(G)$ shows that they are dual spaces. There is a natural duality between these spaces and the group algebra $\mathbb{C}[G]$ which we view as the convolution algebra of finitely supported functions on $G$. Indeed, for any function $f : G \to \mathbb{C}$ and any $g$ in $\mathbb{C}[G]$, we set

$$< g, f > = \sum_{t \in G} g(t)f(t).$$

Then we define the spaces $X_d(G)$ and $\tilde{A}_c$ respectively as the completion of $\mathbb{C}[G]$ for the respective norms

$$\|g\|_{X_d(G)} = \sup\{ |< g, f > | \mid f \in M_d(G), \|f\|_{M_d(G)} \leq 1 \}$$

and

$$\|g\|_{\tilde{A}_c} = \sup\{ |< g, f > | \mid f \in M_d(G), \|f\|_{B_c(G)} \leq 1 \}.$$ 

Obviously, we can also write

$$\|g\|_{\tilde{A}_c} = \sup\{ \| \sum_{t \in G} g(t)\pi(t) \| \mid \pi : G \to B(H), |\pi| \leq c \}.$$

This last formula shows that $\tilde{A}_c$ is naturally equipped with a Banach algebra structure under convolution: we have $\|g \ast g_2\|_{\tilde{A}_c} \leq \|g\|_{\tilde{A}_c}\|g_2\|_{\tilde{A}_c}$. However, the analog for the spaces $X_d(G)$ fails in general. This was the basic idea used by Haagerup [27] to prove that $M_2(\mathbb{F}_\infty) \neq UB(\mathbb{F}_\infty)$. Indeed, Haagerup used spherical functions to show that $X_2(\mathbb{F}_\infty)$ is not a Banach algebra under convolution (see Remark [22] below), which implies by the preceding remarks that $X_2(\mathbb{F}_\infty) \neq \tilde{A}_c$ for any $c$, hence $M_2(\mathbb{F}_\infty) \neq B_c(\mathbb{F}_\infty)$ for any $c$, from which $M_2(\mathbb{F}_\infty) \neq UB(\mathbb{F}_\infty)$ follows easily by Baire’s classical theorem.

Note that, in sharp contrast, for $G = \mathbb{N}$, it is known that $X_2(G)$ is a Banach algebra (due to G. Bennett), but not an operator algebra (see [35] for details).
Although $X_d(G)$ is not in general a Banach algebra under convolution, it satisfies the following property: if $g_1 \in X_d(G)$ and $g_2 \in X_k(G)$, then $g_1 \ast g_2 \in X_{d+k}(G)$ and

\[ \| g_1 \ast g_2 \|_{X_{d+k}(G)} \leq \| g_1 \|_{X_d(G)} \| g_2 \|_{X_k(G)}. \]

Therefore, Haagerup’s result in [27] implies that $X_2(F_{\infty}) \neq X_4(F_{\infty})$ (equivalently $M_2(F_{\infty}) \neq M_4(F_{\infty})$), since otherwise $X_2(F_{\infty})$ would be a Banach algebra under convolution.

To verify (3.1), we will need an alternate description of the space $X_d(G)$, which uses the Haagerup tensor product and the known results on multilinear cb maps (cf. [12, 55]). These results show that $X_d(G)$ may be identified with a quotient (modulo the kernel of the natural product map) of the Haagerup tensor product $\ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)$ of $d$ copies of $\ell_1(G)$ equipped with its “maximal operator space structure”. More explicitly, one can prove that the space $X_d(G)$ coincides with the space of all functions $g : G \to \mathbb{C}$ for which there is an element $\hat{g} = \sum_{t \in G^d} \hat{g}(t_1, \ldots, t_d) \delta_{t_1} \otimes \ldots \otimes \delta_{t_d}$ in $\ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)$ such that

\[ \forall t \in G \quad g(t) = \sum_{t_1 \ldots t_d = t} \hat{g}(t_1, \ldots, t_d) \]

and moreover we have

\[ \| g \|_{X_d(G)} = \inf \{ \| \hat{g} \|_{\ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)} \}. \]

In addition the norm of an element $\hat{g}$ in the space $\ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)$ can also be explicit as follows:

\[ \| \hat{g} \|_{\ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)} = \sup \left\{ \left\| \sum_{t \in G^d} \hat{g}(t_1, \ldots, t_d) x_{t_1} \ldots x_{t_d} \right\| \right\} \]

where the supremum runs over all families $(x_{t_i})_{t_i \in G}$ in the unit ball of $B(\ell_2)$. Actually (by e.g. [67, prop. 6.6]), the supremum remains the same if we restrict it to the case when the $d$ families actually coincide with a single family $(x_{t_i})_{t_i \in G}$ in the unit ball of $B(\ell_2)$.

Clearly, if $\hat{g}_1 \in \ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)$ ($d$ times) and $\hat{g}_2 \in \ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)$ ($k$ times) we have $\hat{g}_1 \otimes \hat{g}_2 \in \ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)$ ($d + k$ times) and $\| \hat{g}_1 \otimes \hat{g}_2 \| \leq \| \hat{g}_1 \|_\ell \| \hat{g}_2 \|$. From this, (3.1) follows easily using (3.2).

**Remark.** Assume that $M_d(G) = M_{2d}(G)$. Then passing to the preduals, $X_d(G) = X_{2d}(G)$ with equivalent norms. By (3.1) with $k = d$, this implies that $X_d(G)$ is (up to isomorphism) a Banach algebra under convolution. Moreover, since the product in $X_d(G)$ is “induced” by the Haagerup tensor product, Blecher’s characterization of operator algebras (see [1] which extends [3]) shows that $X_d(G)$ must be (unitaly) isomorphic to a (unital) operator algebra. Combined with Theorem 2.10 this implies

**Theorem 3.1.** In the situation of Theorem 2.10 the following assertions are equivalent:

(i) There is a $\theta \geq 1$ such that $B_\theta(G) = B_c(G)$ for all $c > \theta$.

(ii) There are $\theta \geq 1$ and an integer $d$ such that $B_\theta(G) = M_d(G)$.

(iii) There is an integer $d$ such that $M_d(G) = M_{2d}(G)$.

(iv) There is an integer $d$ such that $X_d(G)$ is (up to isomorphism) a unital operator algebra under convolution.
Proof. By Theorem\, 2.10 (i) implies (ii). By (2.2) and (2.3), (ii) implies (iii). The preceding remark shows that (iii) implies (iv). Finally, assume (iv). Then there is a unital operator algebra $A \subset B(H)$ and a unital isomorphism $u: X_d(G) \to A$. Let $\theta = \|u\|$ and $K = \|u^{-1}\|$. Clearly $u$ restricted to the group elements defines a unital homomorphism $\pi$ with $|\pi| \leq \theta$. By the very definition of $\|g\|_\theta$, this implies $\|u(g)\| \leq \|g\|_\theta$ for all finitely supported $g$, hence $\|g\|_{X_d(G)} \leq K\|u(g)\| \leq K\|g\|_{\theta}$. Conversely, we trivially have (see (2.3)) $\|g\|_{\theta} \leq \theta^d\|g\|_{X_d(G)}$. Thus we obtain $X_d(G) = X_{\theta}$, hence by duality $M_d(G) = B_{\theta}(G)$, which (recalling the basic inclusions (2.2) and (2.3)) implies (i). 

Remark 3.2. Haagerup’s proof in [27] that $M_2(\mathbb{F}_\infty) \neq UB(\mathbb{F}_\infty)$ can be outlined as follows. Let $G = \mathbb{F}_n$ with $2 \leq n < \infty$. Assume $M_2(G) = UB(G)$.

Step 1: By Baire’s theorem, there exists $c > 1$ such that $M_2(G) = B_c(G)$ with equivalent norms.

Step 2: This implies that $X_2(G)$ is a Banach algebra under convolution (because the predual of $B_c(G)$ is clearly an operator algebra, see Theorem 3.1 above). Hence, there is $C > 0$ such that for all $f, g$ finitely supported we have $\|f * g\|_{X_2(G)} \leq C\|f\|_{X_2(G)}\|g\|_{X_2(G)}$.

Step 3: By an averaging argument, the radial projection $f \to f_R$ defined by

$$f_R(t) = \sum_{s: \ |s| = |t|} f(s) \cdot \text{card}\{s \ | \ |s| = |t|\}^{-1}$$

is bounded on $M_2(G)$ so that $\|f_R\|_{M_2(G)} \leq \|f\|_{M_2(G)}$ for any $f$ in $M_2(G)$.

Step 4: Let $\varphi_z$ be the spherical function on $G$ equal to $z$ on words of length 1 (cf. e.g. [24]). This means that $\varphi_z(t) = \varphi(|t|)$ where $\varphi$ is determined inductively by: $\varphi(0) = 1, \varphi(1) = z$ and

$$\varphi(k + 1) = \frac{2n}{2n - 1} \varphi(1)\varphi(k) - \frac{1}{2n - 1} \varphi(k - 1)$$

for all $k \geq 2$. The spherical property of $\varphi_z$ implies that for any finitely supported radial function $f$ we have $\varphi_z * f = \langle \varphi_z, f \rangle \varphi_z$, and hence if $g$ is another finitely supported radial function, we have

$$\langle \varphi_z, f * g \rangle = \langle \varphi_z, f \rangle \langle \varphi_z, g \rangle.$$

Moreover, if $|z| < 1$ then $\varphi_z \in M_2(G)$ (actually $\varphi_z \in UB(G)$, see [44]). Thus, in short, although Step 5 below says it is unbounded, $\|\varphi_z\|_{M_2(G)}$ is finite whenever $|z| < 1$. Therefore, if $|z| < 1$, $f \to \langle \varphi_z, f \rangle$ defines a continuous multiplicative unital functional on the Banach subalgebra which is the closure of the set of finitely supported radial functions in $X_2(G)$. Clearly, this implies that $\langle \varphi_z, f \rangle$ is in the spectrum of $f$, hence its modulus is majorized by the spectral radius of $f$ in the latter Banach algebra, and this is $\leq C\|f\|_{X_2(G)}$ by Step 2. Thus we obtain for $f$ radial $|\langle \varphi_z, f \rangle| \leq C\|f\|_{X_2(G)}$. Now, for $f$ finitely supported but not necessarily radial, we have

$$|\langle \varphi_z, f \rangle| = |\langle \varphi_z, f_R \rangle| \leq C\|f_R\|_{X_2(G)}$$

hence by Step 3

$$\text{sup}_{|z| < 1} \|\varphi_z\|_{M_2(G)} = \infty.$$

This implies $\|\varphi_z\|_{M_2(G)} \leq C$. But this contradicts the next and final step proved in [27]:

Step 5: $\text{sup}_{|z| < 1} \|\varphi_z\|_{M_2(G)} = \infty$.

It is not clear to us how to extend this argument to $M_d$ in place of $M_2$. The analogue of Step 3 is not clear to us (but seems likely to be true). Moreover, note that the analogue of Step 2 would require assuming $X_d(G) = X_{2d}(G)$, so it would seem that the argument would lead, at best, to $M_d(G) \neq M_{2d}(G)$. 

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4 The $B(H)$-valued case

Up to now, we have mainly concentrated on properties of spaces of coefficients or of analogous spaces of complex valued functions on $G$. We now turn to the more general $B(H)$-valued case which is entirely similar to the preceding treatment (corresponding to dim$(H) = 1$). More generally, for any u.b. representation $\pi: G \to B(H)$ let us define

$$\text{Sim}(\pi) = \inf\{\|S\| \|S^{-1}\| \mid S^{-1}\pi(\cdot)S \text{ is a unitary representation}\},$$

and let

$$\alpha(G) = \inf\{\alpha \geq 0 \mid \exists K \forall \pi: G \to B(H) \text{ u.b. } \text{Sim}(\pi) \leq K|\pi|^\alpha\}.$$

Then again the same phenomenon arises:

**Theorem 4.1.** Let $G$ be a discrete group. If $G$ is unitarizable then $\alpha(G) < \infty$. Moreover $\alpha(G) \in \mathbb{N}$.

We will now explain what replaces $d_1(G)$ in this case.

First, we need to generalize the space $B(G)$, from complex values to operator values. Let $H$ be a Hilbert space and let $G$ be a semi-group with unit. We denote by $B_c(G; H)$ the space of all functions $f: G \to B(H)$ for which there are a u.b. unitary representation $\pi: G \to B(H_\pi)$ with $|\pi| \leq c$ and operators $\xi: H_\pi \to H$ and $\eta: H \to H_\pi$ such that

$$\forall \ t \in G \quad f(t) = \xi\pi(t)\eta.$$

We define

$$\|f\|_{B_c(G; H)} = \inf\{\|\xi\| \|\eta\|\}$$

where the infimum runs over all possible such representations.

Here again, in the group case we will denote $B_1(G; H)$ simply by $B(G; H)$ to emphasize that $|\pi| \leq 1$ means that $\pi$ is a unitary representation.

Similarly, we denote by $M_d(G; H)$ the space of functions $f: G \to B(H)$ for which there are bounded functions $\xi_i: G \to B(H_i, H_i^{-1})$, $1 \leq i \leq d$, with $H_0 = H_d = H$, $H_i$ Hilbert such that

$$\forall (t_1, \ldots, t_d) \in G^d \quad f(t_1 t_2 \ldots t_d) = \xi_1(t_1)\xi_2(t_2)\ldots\xi_d(t_d).$$

We equip this space with the norm

$$\|f\|_{M_d(G; H)} = \inf\{\sup_{t_1 \in G} \|\xi_1(t_1)\|\ldots\sup_{t_d \in G} \|\xi_d(t_d)\|\}.$$

Note that there is also an obvious $B(H)$-valued generalization of the spaces $M_d(I_1, \ldots, I_d)$ introduced in Remark 2.11 above. Let us denote it by $M_d(I_1, \ldots, I_d; H)$. Then, as before, for any $\varphi: G \to B(H)$, let $\Phi: G^d \to B(H)$ be defined by $\Phi(t_1, \ldots, t_d) = \phi(t_1 t_2 \ldots t_d)$. Then we have

$$\|\varphi\|_{M_d(G; H)} = \|\Phi\|_{M_d(G, \ldots, G; H)}.$$

Clearly, when dim$(H) = 1$, we recover the previous spaces $B(G)$, $B_c(G)$ and $M_d(G)$. The following extensions of the previous results can be proved:

**Theorem 4.2.** Consider a function $f: G \to B(H)$. Then $f \in B_1(G; H)$ iff $f \in \bigcap_{m \geq 1} M_m(G; H)$ and $\sup_m \|f\|_{M_m(G; H)} < \infty$. Moreover we have

$$\|f\|_{B_1(G; H)} = \sup_{m \geq 1} \|f\|_{M_m(G; H)}.$$
On the other hand, \( f \in \bigcup_{c>1} B_c(G; H) \) \( \text{iff} \ f \in \bigcap_m M_m(G; H) \) and \( \sup_{m \geq 1} \|f\|^{1/m}_{M_m(G; H)} < \infty \). In addition
\[
\limsup_{m \to \infty} \|f\|^{1/m}_{M_m(G; H)} = \inf\{c \mid f \in B_c(G; H)\}.
\]

**Notation.** Let us denote by \( \hat{K} \) the completion of the algebraic tensor product and is such that for any \( (K_1 \otimes K_2) \to x_1 \otimes x_2 \) the bilinear mapping from \( (K_1 \otimes K_2) \to K(H) \otimes (E_1 \otimes E_2) \) which is defined on rank one tensors by
\[
(k_1 \otimes e_1) \otimes (k_2 \otimes e_2) = (k_1 k_2) \otimes (e_1 \otimes e_2).
\]

The Haagerup tensor product \( E_1 \otimes^h E_2 \) can be characterized as the unique operator space which is a completion of the algebraic tensor product and is such that for any \( x \in K(H) \otimes [E_1 \otimes_h E_2] \) we have
\[
\|x\|_{\text{min}} = \inf\{\|x_1\|_{\text{min}} \|x_2\|_{\text{min}}\}
\]
where the infimum runs over all factorization of the form
\[
x = x_1 \otimes x_2
\]
with \( x_1 \in K(H) \otimes E_1 \) and \( x_2 \in K(H) \otimes E_2 \).

By definition of the Haagerup tensor product, \( (x_1, x_2) \to x_1 \otimes x_2 \) extends to a contractive bilinear mapping from \( (K(H) \otimes_{\text{min}} E_1) \times (K(H) \otimes_{\text{min}} E_2) \) to \( K(H) \otimes_{\text{min}} [E_1 \otimes_h E_2] \). We will still denote by \( (x_1, x_2) \to x_1 \otimes x_2 \) this extension, and similarly for \( d \)-fold tensor products.

The space \( X_d(G; H) \) is defined as the space of all functions \( g : G \to K(H) \) for which there is an element \( \hat{g} = \sum_{G^d} \hat{g}(t_1, \ldots, t_d) \otimes \delta_{t_1} \otimes \ldots \otimes \delta_{t_d} \) in \( K(H) \otimes_{\text{min}} [\ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)] \) such that
\[
\forall t \in G \quad g(t) = \sum_{t_1 \ldots t_d = t} \hat{g}(t_1, \ldots, t_d)
\]
and moreover we have
\[
\|g\|_{X_d(G; H)} = \inf\{\|\hat{g}\|_{K(H) \otimes_{\text{min}} [\ell_1(G) \otimes_h \ldots \otimes_h \ell_1(G)]}\}.
\]
In addition the norm of an element $\hat{g}$ in the space $K(H) \otimes_{\min} [\ell_1(G) \otimes_h \cdots \otimes_h \ell_1(G)]$ can also be explicited as follows:

$$\|\hat{g}\|_{K(H) \otimes_{\min} [\ell_1(G) \otimes_h \cdots \otimes_h \ell_1(G)]} = \sup \left\{ \| \sum_{G^d} \hat{g}(t_1, \ldots, t_d) \otimes x_{t_1}^1 \cdots x_{t_d}^d \|_{B(H \otimes \ell_2)} \right\}$$

where the supremum runs over all families $(x_t^i)_{t} \in G$, $(x_t^i)_{t} \in G$ in the unit ball of $B(\ell_2)$. Here again (by e.g. [67, prop. 6.6]) the supremum is the same if we restrict the supremum to the case when the $d$ families are all equal to a single family $(x_t)_{t} \in G$ in the unit ball of $B(\ell_2)$.

By definition of the Haagerup tensor product, we have also

$$\|\hat{g}\|_{K(H) \otimes_{\min} [\ell_1(G) \otimes_h \cdots \otimes_h \ell_1(G)]} = \inf \left\{ \|g_1\|_{K(H) \otimes_{\min} \ell_1(G)} \cdots \|g_d\|_{K(H) \otimes_{\min} \ell_1(G)} \right\}$$

where the infimum runs over all factorizations of $\hat{g}$ of the form

$$(4.3) \quad \hat{g} = g_1 \otimes g_2 \otimes \cdots \otimes g_d,$$

with $g_1, g_2, \cdots, g_d \in K(H) \otimes_{\min} \ell_1(G)$. Equivalently, (4.3) means that for all $(t_i)$ in $G^d$ we have

$$\hat{g}(t_1, \ldots, t_d) = g_1(t_1)g_2(t_2)\cdots g_d(t_d),$$

where the product is in $K(H)$.

From this, we deduce that

$$\|g\|_{X_d(G; H)} = \inf \left\{ \|g_1\|_{K(H) \otimes_{\min} \ell_1(G)} \cdots \|g_d\|_{K(H) \otimes_{\min} \ell_1(G)} \right\},$$

where the infimum runs over all factorizations of $g$ (as a generalized $d$-fold convolution) of the form

$$g(t) = \sum_{t_1 \cdots t_d = t} g_1(t_1)g_2(t_2)\cdots g_d(t_d) \quad (g_i \in K(H) \otimes_{\min} \ell_1(G)).$$

In particular, (4.4) implies that for any integers $d, k$, we have

$$\|g\|_{X_{d+k}(G; H)} = \inf \left\{ \|x\|_{\min} \|y\|_{\min} \right\},$$

where the infimum runs over all pairs $x \in X_d(G; H)$, $y \in X_k(G; H)$ such that $g(t) = \sum_{t_1t_2 = t} x(t_1)y(t_2)$ ($K(H)$-valued convolution).

The next statement is the $B(H)$-valued analogue of Theorem 3.1.

**Theorem 4.4 ([62]).** Let $G$ be a semigroup with unit. Let $H = \ell_2$. The following assertions are equivalent:

(i) There is a $\theta \geq 1$ such that $B_\theta(G; H) = B_c(G; H)$ for all $c > \theta$.

(i') There is a $\theta \geq 1$ such that $B_\theta(G; H) = B_c(G; H)$ for some $c > \theta$.

(ii) There are $\theta \geq 1$ and an integer $d$ such that $B_\theta(G; H) = M_d(G; H)$.

(iii) There is an integer $d$ such that $M_d(G; H) = M_{d+1}(G; H)$.

(iv) There is an integer $d$ such that $X_d(G)$ is (up to complete isomorphism) a unital operator algebra under convolution.
Proof. In \[61\] the above Key Lemma \[25\] is actually proved in the \(B(H)\)-valued case. Therefore all the preceding statements numbered between 2.5 and 2.9 remain valid in the \(B(H)\)-valued case. Thus exactly the same reasoning as for Theorem \[5.1\] yields the equivalence of (i), (i') and (ii). Clearly (ii) implies (iii).

Assume (iii). Then, passing to the preduals (here we mean the preduals of \(M_d(G)\) and \(M_{d+1}(G)\) in the operator space sense), we find \(X_d(G; H) = X_{d+1}(G; H)\). This proves that (iii) implies (iv).

Thus exactly the same reasoning as for Theorem 3.1 yields the equivalence of (i), (i') and (ii). Clearly (ii) implies (iii).

Assume (iii). Then, passing to the preduals (here we mean the preduals of \(M_d(G)\) and \(M_{d+1}(G)\) in the operator space sense), we find \(X_d(G; H) = X_{d+1}(G; H)\). This implies \(X_{d+1}(G; H) = X_{d+2}(G; H)\). Indeed, by \[4.5\] for any \(g\) in the open unit ball of \(X_{d+1}(G; H)\), we can find \(x\) in the unit ball of \(X_d(G; H)\) and \(y\) in the unit ball of \(X_1(G; H)\) such that

\[
g(t) = \sum_{t_1, t_2 = t} x(t_1) y(t_2).
\]

Now since \(X_d(G; H) = X_{d+1}(G; H)\), \(x \in X_{d+1}(G; H)\) hence by \[1.5\] \(g\) must be in \(X_{d+2}(G; H)\).

Now from \(X_{d+1}(G; H) = X_{d+2}(G; H)\) we deduce \(X_{d+2}(G; H) = X_{d+3}(G; H)\), and so on, so that we must have \(X_d(G; H) = X_{2d}(G; H)\), or equivalently \(X_d(G) = X_{2d}(G)\) completely isomorphically.

Note that, by \[4.4\] or \[4.5\], the convolution product defines a completely contractive linear map \(p\) from \(X_d(G) \otimes H \to X_{2d}(G)\), hence since \(X_d(G) = X_{2d}(G)\) completely isomorphically, \(p\) is c.b. from \(X_d(G) \otimes H \to X_d(G)\), which implies by Blecher’s result in \[4\] that \(X_d(G)\) is completely isomorphic to an operator algebra. This proves that (iii) implies (iv).

Finally, assume (iv). Then, there are a unital subalgebra \(A \subset B(H)\) and a unital homomorphism \(u: X_d(G) \to A\) which is also a complete isomorphism. Let \(\theta = \|u\|_{cb}\) and \(C = \|u^{-1}\|_{cb}\). Let \(\pi(t) = u(\delta_t)\) \((t \in \theta)\). Then \(\pi\) is a u.b. representation of \(G\) with \(|\pi| \leq \theta\). By the maximality of \(A\), for any \(x \in \mathbb{C}[G]\), we must have

\[
\|u(x)\| \leq \|x\|_{\tilde{A}}.
\]

hence \(\|x\|_{X_d(G)} \leq C \|x\|_{\tilde{A}}\). By duality, this implies that for all \(\varphi\) in \(M_d(G)\) we have

\[
\|\varphi\|_{B_d(G)} \leq C \|\varphi\|_{M_d(G)}.
\]

Moreover, the same arguments with coefficients in \(B(H)\) yield the c.b. version of this, so that we obtain, for all \(\varphi\) in \(M_d(G; H)\)

\[
\|\varphi\|_{B_d(G; H)} \leq C \|\varphi\|_{M_d(G; H)}.
\]

Thus we obtain (ii) and hence also (i), establishing \((iv) \Rightarrow (i)\). \(\square\)

Remark. The preceding argument shows that (iii) and (iv) are equivalent for the same \(d\).

5 A case study: The free groups

We wish to prove here the following.

Theorem 5.1. For any \(d \geq 2\),

\[
M_d(\mathbb{F}_\infty) \neq M_{d-1}(\mathbb{F}_\infty).
\]

More precisely let \(\{g_1, g_2, \ldots\}\) be the free generators of \(\mathbb{F}_\infty\), and for any \(n\) let \(W_{d,n}\) be the subset of \(\mathbb{F}_\infty\) formed of all the words \(w\) (of length \(d\)) of the form \(w = g_{i_1} g_{i_2} \ldots g_{i_d}\) with \(1 \leq i_j \leq n\) for any \(1 \leq j \leq d\). Then, for any \(n\), there is a function \(f_{d,n}\): \(\mathbb{F}_\infty \to \mathbb{C}\) supported on \(W_{d,n}\) and unimodular on \(W_{d,n}\) such that

\[
n^{-\frac{1}{d}} \leq \|f_{d,n}\|_{M_d(\mathbb{F}_\infty)} \quad \text{and} \quad \|f\|_{M_{d-1}(\mathbb{F}_\infty)} \leq C(d) n^{-\frac{1}{d}},
\]

where \(C(d)\) is a constant depending only on \(d\).
Let
\[ UB(G) = \bigcup_{c > 1} B_c(G). \]
Since we have obviously inclusions \( UB(G) \subset M_d(G) \subset M_{d-1}(G) \) for any group \( G \), this implies

**Corollary 5.2.** For any \( d \geq 1 \),
\[ M_d(\mathbb{F}_\infty) \neq UB(\mathbb{F}_\infty). \]

For \( d = 2 \) this is the main result of [27]. Note however that Theorem 5.1 yields a function \( f \) supported in the words of length 3 that is in \( M_2(G) \) but not in \( M_3(G) \) and hence not in \( UB(G) \). It is easy to see that 3 is minimal here, i.e. any function supported in the words of length 2 that is in \( M_2(G) \) must be in \( UB(G) \) (see Proposition 5.8 below).

Let \( I_1, \ldots, I_d \) be arbitrary sets. Recall the notation from Remark 2.11. We denote by \( M_d(I_1, \ldots, I_d) \) the space of all functions \( f : I_1 \times \cdots \times I_d \to \mathbb{C} \) for which there are bounded functions \( f_i \)
\[ f_i : I_i \to B(H_i, H_{i-1}) \quad \text{(here} \ H_i \text{ are Hilbert spaces with} \ H_d = H_0 = \mathbb{C} \text{) such that} \]
\[ f(b_1, \ldots, b_d) = f_1(b_1) \cdots f_d(b_d) \quad \forall b_i \in I_i. \]
We equip this space with the norm
\[ \| f \| = \inf \left\{ \prod_{i=1}^d \sup_{b \in I_i} \| f_i(b) \| \right\} \]
where the infimum runs over all possible such factorizations.

Let \( J_i \subset I_i \) be arbitrary subsets. Note that we obviously have
\[ \| f \|_{M_d(I_1, \ldots, I_d)} \leq \| f \|_{M_d(I_1, \ldots, I_d)}. \]
Moreover, for any function \( g : J_1 \times \cdots \times J_d \to \mathbb{C} \) let \( \tilde{g} : I_1 \times \cdots \times I_d \to \mathbb{C} \) be the extension of \( \tilde{g} \) equal to zero outside \( J_1 \times \cdots \times J_d \). Then it is easy to check that
\[ \| \tilde{g} \|_{M_d(I_1, \ldots, I_d)} = \| g \|_{M_d(j_1, \ldots, j_d)}. \]  
(5.1)

We will relate these spaces to \( M_d(\mathbb{F}_\infty) \) via the following observation. Given a function \( \varphi : \mathbb{F}_\infty \to \mathbb{C} \) supported by \( W_{d,n} \), we can define \( f : [1, \ldots, n]^d \to \mathbb{C} \) by
\[ \forall i_j \in [1, \ldots, n] \quad f(i_1, i_2, \ldots, i_d) = \varphi(g_1, g_2, \ldots, g_d). \]
We have then obviously if \( I = [1, \ldots, n] \)
\[ \| f \|_{M_d(I, \ldots, I)} \leq \| \varphi \|_{M_d(\mathbb{F}_\infty)}. \]
(5.2)
The main idea for the proof of Theorem 5.1 is to compare \( \| \varphi \|_{M_d(\mathbb{F}_\infty)} \) with certain norms of \( f \) of the form \( M_{d-1}(I_1, \ldots, I_{d-1}) \) when \( f \) is viewed as depending on less than \( d \) variables, by blocking together certain variables, so that \( I_1 = I^{p_1}, I_2 = I^{p_2}, \ldots \) with \( p_1 + p_2 + \cdots + p_{d-1} = d \).

**Remark.** With the notation used in operator space theory, the space \( M_d(I_1, \ldots, I_d) \) can be identified with the dual of the Haagerup tensor product \( \ell_1(I_1) \otimes_h \cdots \otimes_h \ell_1(I_d) \), where the spaces \( \ell_1(I_i) \) are equipped (as usual) with their maximal operator space structure in the sense of e.g. [22] or [67].
Consider now a partition $\pi = (\alpha_1, \ldots, \alpha_k)$ of $[1, \ldots, d]$ into disjoint intervals ("blocks") with $k < d$, so that at least one $\alpha_i$ has $|\alpha_i| > 1$. Let $I(\alpha_i) = \prod_{q \in \alpha_i} I_q$.

We have a natural mapping from $M_d(I_1, \ldots, I_d)$ to $M_k(I(\alpha_1), \ldots, I(\alpha_k))$ associated to the canonical identification

$$I_1 \times \cdots \times I_d = I(\alpha_1) \times \cdots \times I(\alpha_k).$$

It is easy to check that this mapping is contractive. For simplicity of notation, we denote $M(\pi) = M_k(I(\alpha_1), \ldots, I(\alpha_k))$.

Moreover, it is useful to observe that if $\pi'$ is another partition of $[1, \ldots, d]$ that is finer than $\pi$ (i.e. such that every block of $\pi$ is a union of certain blocks of $\pi'$), then we have $M(\pi') \subseteq M(\pi)$ and for any $f: [1, \ldots, n]^d \to \mathbb{C}$

$$\|f\|_{M(\pi)} \leq \|f\|_{M(\pi')}.$$  \hfill (5.3)

Note however that since the set of all partitions is only partially ordered (and not totally ordered), the intersection $\bigcap_{\pi} M(\pi)$ over all partitions with $k$ blocks does not reduce to one of the $M(\pi)$. We equip this intersection $\bigcap_{\pi} M(\pi)$ with its natural norm, namely:

$$\|f\| = \max_{\pi} \|f\|_{M(\pi)},$$

where the maximum runs over all $\pi$ with at most $d - 1$ blocks.

The main point in the proof of Theorem 5.1 is the following.

**Lemma 5.3.** Assuming $I_1, \ldots, I_d$ are infinite sets, then for any $d > 1$ the natural mapping

$$\Phi: M_d(I_1, \ldots, I_d) \longrightarrow \bigcap_{\pi} M(\pi)$$

is not an isomorphism.

To prove this, we will use two more lemmas.

**Lemma 5.4.** For any $f$ in $M_d(I_1, \ldots, I_d)$ and any fixed $j$ in $[1, 2, \ldots, d]$ we have

$$\|f\|_{M_d(I_1, \ldots, I_d)} \leq \sup_{I_1 \times \cdots \times I_d} |f| \cdot \left( \prod_{m \neq j} |I_m| \right)^{1/2}.$$

Proof. Let us write

$$f(i_1, \ldots, i_d) = f(a, i_j, b).$$

Then for each fixed $j$, the matrix $(f(a, i_j, b))_{a,b}$ defines an operator $\xi(i_j)$ from $H = \ell_2(I_{j+1}) \otimes \cdots \otimes \ell_2(I_d)$ to $K = \ell_2(I_1) \otimes \cdots \otimes \ell_2(I_{j-1})$ and its norm can be majorized as follows (observe that an $n \times m$ matrix $(a_{ij})$ has norm bounded by $\sup_{ij} |a_{ij}| \sqrt{n} \sqrt{m}$)

$$\|\xi(i_j)\| \leq \sup_{ij} |f| \cdot \left( \prod_{m \neq j} |I_m| \right)^{1/2}.$$
Moreover we have

\[ f(i_1, \ldots, i_d) = a_1(i_1) \cdots a_j(i_{j-1}) \xi(i_j)b_{j+1}(i_{j+1}) \cdots b_d(i_d) \]

with \( a_m : K \to K \) and \( b_m : H \to H \) defined by \( a_m(i_m) = 1 \otimes \cdots \otimes 1 \otimes e_{i_m} \otimes 1 \cdots \otimes 1 \) and \( b_m(i_m) = 1 \otimes \cdots \otimes e_{i_m} \otimes 1 \cdots \otimes 1 \), where the middle term is in the place of index \( m \).

\[ \text{Lemma 5.5 (Marius Junge). Assume } |I_1| = \cdots = |I_d| = n. \text{ Then the natural identity map from } \ell_\infty(n^d) \text{ to } M_d(I_1, \ldots, I_d) \text{ has norm } n^{-\frac{d-1}{2}}. \text{ Equivalently, we have} \]

\[ \sup \left\{ \left\| \sum z_{i_1i_2\ldots i_d} e_{i_1} \otimes \cdots \otimes e_{i_d} \right\|_{M_d(I_1, \ldots, I_d)} \mid |z_{i_1i_2\ldots i_d}| \leq 1 \right\} = n^{\frac{d-1}{2}}. \]

\[ \text{Proof. I am grateful to Marius Junge for kindly providing this lemma in answer to a question of mine for } d = 3. \text{ Let } C \text{ be the left side of (5.4). The fact that } C \leq n^{\frac{d-1}{2}} \text{ follows from Lemma 5.4. The main point is the converse. To prove this, consider the function} \]

\[ \psi(i_1, \ldots, i_d) = a_{i_1i_2}a_{i_2i_3} \cdots a_{i_{d-1}i_d} \]

where \( (a_{ij}) \) is an \( n \times n \) unitary matrix with \( |a_{ij}| = n^{-1/2} \). Let \( \xi_i(x) = \langle x, e_i \rangle \). Then we can write

\[ \psi(i_1, \ldots, i_d) = \langle ae_{i_1i_2}ae_{i_2i_3} \cdots e_{i_{d-1}i_d}a e_{i_1}, e_1 \rangle = \xi_{i_1}(ae_{i_1i_2}ae_{i_2i_3} \cdots e_{i_{d-1}i_d}a e_{i_1}) \]

where \( a \) appears \( d - 1 \) times, from which it follows that

\[ \| \psi \|_{M_d(I_1, \ldots, I_d)^*} \leq n. \]

Indeed, if \( \| f \|_{M_d(I_1, \ldots, I_d)} < 1 \). We may assume \( f(b_1, \ldots, b_d) = f_1(b_1) \cdots f_d(b_d) \quad \forall b_i \in I_i \) with \( \| f_i(b_i) \| < 1 \) \( (b_i \in I_i) \) for all \( i \). Then we have

\[ \sum \psi(i_1, \ldots, i_d) f(i_1, \ldots, i_d) = \left[ \sum \xi_{i_1} \otimes f_1(i_1) \right] [a \otimes I] \left[ \sum e_{i_2i_3} \otimes f_2(i_2) \right] \cdots \]

hence

\[ \left| \sum \psi(i_1, \ldots, i_d) f(i_1, \ldots, i_d) \right| \leq \left\| \sum \xi_{i_1} \otimes f_1(i_1) \right\| a_d^{-1} \left\| \sum f_d(i_d) \otimes e_{i_d} \right\| \]

\[ \leq \sqrt{n} \| a \|^{-d-1} \sqrt{n} \leq n, \]

which establishes (5.5). Therefore we must have

\[ \sum |\psi(i_1, \ldots, i_d)| \leq Cn, \]

whence

\[ n^d(n^{-1/2})^{d-1} \leq Cn \]

which yields \( C \geq n^{\frac{d-1}{2}} \) as announced.

Note: The preceding proof uses implicitly ideas from operator space theory namely the identity \( M_d(I_1, \ldots, I_d) = \ell_\infty^n \otimes_h \cdots \otimes_h \ell_\infty^n \) \( (d \text{ times}) \), for which we refer to e.g. 22 or 67.
Remark. Let
\[ \varepsilon(p, q) = \exp\{ipq/n\}, \]
and let \( a_{pq} = \varepsilon(p, q)n^{-1/2} \). Thus, the \( n \times n \) unitary matrix \( a = (a_{pq}) \) represents the Fourier transform on the group \( \mathbb{Z}/n\mathbb{Z} \). Let
\[ F_{d,n}(i_1, \ldots, i_d) = \varepsilon(i_1, i_2)\varepsilon(i_2, i_3)\ldots\varepsilon(i_{d-1}, i_d). \]

Then the preceding proof yields
\[ \|F_{d,n}\|_{M_d(I_1, \ldots, I_d)} = n^{d-1}. \]

Proof of Lemma 5.3. By (5.1), it suffices to show that if \( |I_1| = \cdots = |I_d| = n \) then \( \|\Phi^{-1}\| \geq \sqrt{n} \) for all \( n \). Thus we now assume \( |I_1| = \cdots = |I_d| = n \) throughout this proof. By Lemma 5.4 for any \( \pi \) we have \( \|\Phi^{-1}: \ell_\infty(n^d) \to M(\pi)\| \leq n^{d-2} \). (Indeed, we can choose \( \alpha_j \) with \( |\alpha_j| \geq 2 \), hence \( |I(\alpha_j)| \geq n^2 \).) It follows that
\[ \left\| \Phi^{-1}: \ell_\infty(n^d) \to \bigcap_\pi M(\pi) \right\| \leq n^{d-2}. \]

Note that for the mapping underlying \( \Phi^{-1} \) we have
\[ \|\ell_\infty(n^d) \to M_d(I_1, \ldots, I_d)\| \leq \|\ell_\infty(n^d) \to \bigcap_\pi M(\pi)\| \times \|\bigcap_\pi M(\pi) \to M_d(I_1, \ldots, I_d)\| \]

Thus the above estimate together with Lemma 5.5 implies
\[ \sqrt{n} \leq \|\Phi^{-1}: \bigcap_\pi M(\pi) \to M_d(I_1, \ldots, I_d)\|. \quad \square \]

Let \( G = \mathbb{F}_n \) with \( 2 \leq n \leq \infty \) and let \( I \) denote the set of generators of \( G \). Let \( W_d \) be the set of all elements of \( G \) which are a product of exactly \( d \) generators. Let \( F: I^d \to \mathbb{C} \) be a function and let \( f: G \to \mathbb{C} \) be the function defined on \( W_d \) by
\[ f(i_1i_2\ldots i_d) = F(i_1, i_2, \ldots, i_d) \]
and equal to zero outside \( W_d \).

Lemma 5.6. With the above notation, we have
\[ \|F\|_{M_d(I_1, \ldots, I)} \leq \|f\|_{M_d(\mathbb{F}_n)} \quad \text{and} \quad \|f\|_{M_{d-1}(\mathbb{F}_n)} \leq C(d) \sup_\pi \|F\|_{M(\pi)} \]
where the supremum runs over all (nontrivial) partitions of \([1, \ldots, d]\) into \( K \) disjoint intervals (= blocks), with \( K \leq d-1 \) and where \( C(d) \) is a constant depending only on \( d \).

Proof. The inequality \( \|F\|_{M_d(I_1, \ldots, I)} \leq \|f\|_{M_d(\mathbb{F}_n)} \) is essentially obvious by going back to the definitions, so we will now concentrate on the converse direction.
Consider \( t_1, \ldots, t_{d-1} \) in \( G = \mathbb{F}_n \) such that their product \( t_1t_2\ldots t_{d-1} \) belongs to \( W_d \), i.e. \( t_1t_2\ldots t_{d-1} \) can be written as a reduced word of the form \( g_i, g_{i+1} \ldots g_{i_d} \) where \( \{g_i : i \in I\} \) denotes the free generators of \( \mathbb{F}_n \). Since the letters \( g_i, \ldots, g_{i_d} \) remain after successive reductions in the product \( t_1t_2\ldots t_{d-1} \) it is easy to check that each \( t_i \) contributes a block of \( p_i \) letters in \( x \) with \( \sum_{i=1}^{d-1} p_i = d \) (we
allow \( p_i = 0 \). This means that when \( p_i > 0 \), \( t_i \) can be written as a reduced word \( x_i a_i y_i^{-1} \) with \( |a_i| = p_i \) and when \( p_i = 0 \) we set \( a_i = e \), so that

\[
(5.7) \quad t_1 t_2 \ldots t_{d-1} = a_1 a_2 \ldots a_{d-1}.
\]

Thus to each \( t = (t_1, \ldots, t_{d-1}) \) as above we can associate \( p(t) = (p_1, \ldots, p_{d-1}) \). Actually, we have a problem here: this \( p(t) \) is unambiguously defined when \( d = 3 \) (hence we only have to consider products of two elements). But when \( d > 3 \) (and thus \( d - 1 > 2 \)) there might be several reductions of \( t_1 t_2 \ldots t_{d-1} \) leading to the same element of \( W_d \), thus there might be several possibilities for \( p(t) \). For instance, when \( d = 1 \), denoting the generators by \( a,b,c \), the product \( abcd = (ab)(b^{-1}a^{-1}c)(c^{-1}abcd) \) (we mean here \( t_1 = ab, t_2 = b^{-1}a^{-1}c, t_3 = c^{-1}abcd \) allows \( p(t) = (0,0,4) \) but also \( p(t) = (2,0,2) \) or \( p(t) = (1,0,3) \). We prefer to ignore this difficulty for the moment while still treating the general case, so let us assume \( d = 3 \) so that \( p(t) \) is always well defined. Moreover, if we delete the indices for which \( p_i = 0 \) (and \( a_i = e \), we obtain a partition \( \pi(t) \) into \( k \) blocks \( (\alpha_1, \ldots, \alpha_k) \) with \( k \leq d - 1 \). Then we can rewrite \((5.7)\) as

\[
(5.8) \quad t_1 t_2 \ldots t_{d-1} = b_1 b_2 \ldots b_k
\]

with \( b_m \in W_{|\alpha_m|} \). Here we implicitly mean that the non-reduced product \( t_1 t_2 \ldots t_d \) can be viewed (just by adding parenthesis) as a product

\[
c_1 b_1 c_2 b_2 c_3 \ldots b_k c_{k+1}
\]

where each of the intermediate products \( c_1, \ldots, c_{k+1} \) reduces to \( e \). Moreover, the \( k \)-tuple \((b_1, \ldots, b_k)\) determines a \( k \)-tuple \((\tilde{b}_1, \ldots, \tilde{b}_k)\) with \( \tilde{b}_1 \in I_m^1, \tilde{b}_2 \in I_m^2, \ldots, \tilde{b}_k \in I_m^k \). Now fix \( \varepsilon > 0 \). For any partition \( \pi = (\alpha_1, \alpha_2, \ldots, \alpha_k) \), we can “factorize” \( F \) as follows:

\[
F(j_1, \ldots, j_k) = \eta_1^\pi(j_1) \ldots \eta_k^\pi(j_k) \quad (j_m \in I_m^\pi)
\]

where \( \eta_m^\pi \) are \( B(H_m, H_{m-1}) \)-valued functions \( (H_k = H_0 = \mathbb{C}) \) such that

\[
(5.10) \quad \prod_m \sup_m \| \eta_m^\pi(j_m) \| \leq \| F \|_{M(\pi)}(1 + \varepsilon).
\]

Let \( \hat{f} \) be the function defined on \( G^{d-1} \) by \( \hat{f}(t_1, \ldots, t_{d-1}) = f(t_1 t_2 \ldots t_{d-1}) \). Consider now the disjoint decomposition \( \hat{f} = \sum_p \hat{f}_p \) where \( \hat{f}_p = \hat{f} \cdot 1_{\{t: \ p(t) = p\}} \) where the first sum runs over all choices of \( p = (p_1, \ldots, p_{d-1}) \) with \( p_i \geq 0 \) and \( \sum p_i = d \).

We claim that \( \| \hat{f}_p \|_{M_{d-1}(G \ldots G)} \leq \| F \|_{M(\pi)} \) where \( \pi \) is the partition associated to \( p = (p_1, \ldots, p_{d-1}) \) after removal of the empty blocks.

To prove this claim, we will produce a factorization formula for \( \hat{f}_p(t_1, t_2, \ldots, t_{d-1}) \), namely we will show

\[
\hat{f}_p(t_1, t_2, \ldots, t_{d-1}) = \langle \xi_1^p(t_1) \ldots \xi_{d-1}^p(t_{d-1}) \delta_e, \delta_\pi \rangle.
\]

To define \( \xi_i^p(t_i) \) we must distinguish whether \( p_i = 0 \) or not.

If \( p_i = 0 \) we set \( \xi_i^p(\theta) = \xi_i^p(\theta) \otimes 1 \) (here \( \lambda(\theta) \) denotes as usual left translation by \( \theta \) on \( \ell_2(G) \)). On the other hand, if \( p_i > 0 \) so that \( i \) corresponds to a block \( \alpha_m \) of \( \pi \) with \( |\alpha_m| = p_i \), we write

\[
\xi_i^p(\theta) = \sum e_{x,y} \otimes \eta^\pi_m(\tilde{a})
\]

where the sum runs over all ways to decompose \( \theta \) as \( x \cdot a \cdot y^{-1} \) as a reduced product, with \( a \) a product of generators such that \( |a| = p_i \), and where \( \tilde{a} \) denotes the element of \( I_m^\pi \) corresponding to
a (x and y\(^{-1}\) being initial and final segments in the reduced word \(\theta\); we allow here \(x = e\) or \(y = e\)). In case \(\theta\) does not admit any such decomposition (i.e., \(\theta\) does not admit any subword in \(W_{p_i}\)), we set \(\xi^p_i(\theta) = 0\).

Note that we have \(\|\xi^p_i(\theta)\| \leq \sup_a \|\eta^p_m(\hat{a})\|\). Indeed, when \(\theta\) is fixed, in the various ways to write \(\theta = x \cdot a \cdot y^{-1}\) as a reduced product as above, all the \(x\)’s appearing will be distinct since they have different length, and similarly all the \(y\)’s will be distinct, so the various operators \(e_{x,y} \otimes \eta^p_m(\hat{a})\) have both orthogonal ranges and orthogonal domains, so that the norm of their sum is majorized by the maximum norm of each term.

A (tedious but straightforward) verification shows that if \(t_1 t_2 \ldots t_{d-1} \in W_d\) with \(p(t) = p\), and if \(t_1 t_2 \ldots t_{d-1} = b_1 b_2 \ldots b_k\) as described in \((5.8)\), then we have using \((5.9)\)

\[
(\xi^p_i(t_1) \ldots \xi^p_{d-1}(t_{d-1}) \delta_e, \delta_e) = \eta^p_1(b_1) \ldots \eta^p_k(b_k)
= F(b_1, \ldots, b_k)
\]

whence by \((5.6)\) and \((5.8)\)

\[
f(t_1 \ldots t_{d-1}).
\]

Moreover, if \(p(t) \neq p\) the left side of \((5.8)\) vanishes. Indeed, if that left side is non-zero, then we must have

\[
t_1 \cdots t_{d-1} = [x_1 a_1 y_1^{-1}] [x_2 a_2 y_2^{-1}] \cdots [x_{d-1} a_{d-1} y_{d-1}^{-1}] = a_1 a_2 \cdots a_{d-1}
\]

with \(a_i \in W_{p_i}\) if \(p_i > 0\), and \(a_i = e\) otherwise, \(a_i\) being a subword of \(t_i\), in such a way that the product of all the terms figuring in between two successive \(a_i\)’s with \(p_i > 0\) reduces to \(e\), as well as the product of all the terms preceding the first \(a_i\) with \(p_i > 0\), and that of all the terms after the last \(a_i\) with \(p_i > 0\). This implies \(p(t) = p\) and deleting the \(a_i\)’s equal to \(e\) we obtain \(t_1 \cdots t_{d-1} = b_1 \cdots b_k\) and \((5.8)\) is then easy to check.

Thus we have the announced factorization of \(\tilde{f}_p\); the latter implies

\[
\|\tilde{f}_p\|_{M_{d-1}(G, \ldots, G)} \leq \prod \sup \|\eta^p_m\| \leq \|F\|_{M(\pi)} (1 + \varepsilon).
\]

Using \(\tilde{f} = \sum \tilde{f}_p\), this yields \(\|f\|_{M_{d-1}(G, \ldots, G)} = \|\tilde{f}\|_{M_{d-1}(G, \ldots, G)} \leq C_d \|F\|_{M(\pi)} (1 + \varepsilon)\) (here \(C_d\) is the number of possible \(p\)’s) thus completing the proof of the lemma, at least in the case \(d = 3\). Since there are only four possibilities for \(p\) (namely \((3,0)\), \((0,3)\), \((1,2)\), and \((2,1)\)) we obtain \(C_d \leq 4\).

Now in the general case, the problem is that, for each \(t = (t_1, \ldots, t_{d-1}) \in G^{d-1}\) such that \(t_1 \cdots t_{d-1} \in W_d\), there is a multiplicity of possible \(p(t)\)’s (or of possible associated partitions \(\pi(t)\)): each such \(t\) admits \(N(t)\) possible distinct \(p(t)\)’s. However, we of course have a bound for this: \(1 \leq N(t) \leq N_d\) where the upper bound \(N_d\) depends only on \(d\). If \(t_1 \cdots t_{d-1} \notin W_d\), we set \(N(t) = 0\). Then, we think of \(p(t)\) as a multivalued function and we define

\[
\tilde{f}_p = \tilde{f} \cdot 1_{\{t: \ p \in p(t)\}}.
\]

Then the preceding shows again that

\[
\|\sum \tilde{f}_p\|_{M_{d-1}(G, \ldots, G)} \leq C_d \|F\|_{M(\pi)} (1 + \varepsilon),
\]

but, since the sum is no longer disjoint, we have

\[
\forall t \in G^{d-1} \sum \tilde{f}_p(t) = N(t) \tilde{f}(t).
\]

Consider now the special case when \(F\) is identically equal to 1. Note that \(\|F\|_{M(\pi)} \leq 1\) and \(N(t) \tilde{f}(t) = N(t)\) in this case. Thus, the preceding identity and \((5.12)\) shows that the function
$N: G^{d-1} \to \mathbb{R}$ is in $M_{d-1}(G, \ldots, G)$ with norm $\leq C_d$. To conclude, we will multiply by a function equal to $1/N$ on the support of $N$ and we will bound its norm in $M_{d-1}(G, \ldots, G)$ by a constant $C_d'$. Since $M_{d-1}(G, \ldots, G)$ is a Banach algebra for the pointwise product, this will yield the desired result. (Alternately, we could use a disjointification trick, as above for (2.19).

Let $P$ be a polynomial such that $P(k) = \frac{1}{k}$ for all $k = 1, 2, \ldots, N_d$. To fix ideas, we let $P$ be determined by Lagrange interpolation. Since $M_{d-1}(G, \ldots, G)$ is a Banach algebra, $P(N) \in M_{d-1}(G, \ldots, G)$ and since $P$ depends only on $d$, we have $\|P(N)\|_{M_{d-1}(G, \ldots, G)} \leq C_d'$ for some $C_d'$ depending only on $d$. Then we can write

$$P(N) \cdot \sum_p \tilde{f}_p = P(N) \cdot N \tilde{f} = \tilde{f}$$

hence we conclude

$$\|f\|_{M_{d-1}(G)} = \|\tilde{f}\|_{M_{d-1}(G, \ldots, G)} \leq \|P(N)\|_{M_{d-1}(G, \ldots, G)} \|\sum_p \tilde{f}_p\|_{M_{d-1}(G, \ldots, G)} \leq C_d'\|f\|_{M(\pi)} (1 + \varepsilon).$$

Proof of Theorem 5.1. Assume $M_d(\mathbb{F}_\infty) = M_{d-1}(\mathbb{F}_\infty)$. Then there must exist a constant $C'$ such that for all $f$ in $M_{d-1}(\mathbb{F}_\infty)$ we have

$$\|f\|_{M_d(\mathbb{F}_\infty)} \leq C'\|f\|_{M_{d-1}(\mathbb{F}_\infty)}.$$

Then by Lemma 5.6 we find that $\Phi^{-1}: \bigcap M(\pi) \to M_d(I, \ldots, I)$ is bounded, which contradicts Lemma 5.3 for any $d > 1$. Now, let $F_{d,n}$ be as in the remark following Lemma 5.5 and let $f_{d,n}$ be defined by

$$f_{d,n}(g_{i_1} g_{i_2} \ldots g_{i_d}) = F_{d,n}(i_1, \ldots, i_d)$$

and $f_{d,n}(t) = 0$ if $t \notin W_{d,n}$. Then by the latter remark and by Lemma 5.6 we have

$$n^{d-1} \leq \|f_{d,n}\|_{M_d(\mathbb{F}_\infty)},$$

and also by Lemma 5.6 and Lemma 5.4

$$\|f_{d,n}\|_{M_d(\mathbb{F}_\infty)} \leq C(4\sup_{\pi}\|F_{d,n}\|_{M(\pi)} \leq C(4)n^{d-1}.$$  

This complete the proof.

Note that in the special case $d = 3$, we obtain a very explicit example: Namely the function $f_{3,n}$ supported on $W_{3,n}$ and defined there for $1 \leq p, q, r \leq n$ by

$$f_{3,n}(g_p g_q g_r) = \exp(i(p + r)q/n),$$

satisfies

$$n \leq \|f_{3,n}\|_{M_3(G)} \quad \text{but} \quad \|f_{3,n}\|_{M_2(G)} \leq 4n^{1/2}.$$

Remark 5.7. The proof of Lemma 5.6 can be modified to show that $\|f\|_{M_d(\mathbb{F}_\infty)} \leq C'(d)\|F\|_{M_d(I, \ldots, I)}$ for some constant $C'(d)$ depending only on $d$. In particular, we have $\|1_{W_{d,n}}\|_{M_d(\mathbb{F}_\infty)} \leq C'(d)$ and hence $\|1_{W_{d,n}}\|_{M_d(\mathbb{F}_\infty)} \leq C'(d)$.

Note however:
Proposition 5.8. Any function \( f : \mathbb{F}_\infty \to \mathbb{C} \), supported on \( W_d \), that is in \( M_d(\mathbb{F}_\infty) \) must necessarily be in \( UB(\mathbb{F}_\infty) \).

**Proof.** Indeed, \( f|_{W_d} : W_d \to \mathbb{C} \) admits an extension to a function \( \hat{f} : \mathbb{F}_\infty \to \mathbb{C} \) that is in \( C^*(\mathbb{F}_\infty)^* = B(\mathbb{F}_\infty) \). This follows from [67, Corollary 8.13]. Now, since, by Remark 5.7, \( W_d \) belongs to \( UB(\mathbb{F}_\infty) \), the pointwise product \( f = \hat{f} \cdot 1_{W_d} \) also belongs to \( UB(\mathbb{F}_\infty) \). \( \square \)

This shows in particular that a function in \( M_2(\mathbb{F}_\infty) \setminus M_3(\mathbb{F}_\infty) \) cannot be supported on \( W_2 \). Thus the above example \( f_{3,n} \) supported on \( W_3 \) appears somewhat “minimal”.

Let \( G \) be any free group. Recall that we denote by \( \mathcal{W}(d) \) the set of all words of length \( d \) in the generators and their inverses. Note that the inclusion \( W_d \subset \mathcal{W}(d) \) is strict. We chose to concentrate on \( W_d \) (rather than on \( \mathcal{W}(d) \)) because then the ideas are a bit simpler and Lemma 5.6 is somewhat prettier in that case: Indeed, that lemma identifies the spaces \( \{ f \in M_d(G) \mid \text{supp}(f) \subset W_d \} \) and \( \{ f \in M_{d-1}(G) \mid \text{supp}(f) \subset W_d \} \) with two distinct spaces of functions on \( G^d \), thus reducing, in some sense, a problem in harmonic analysis to one in functional analysis. However, most of our results hold with suitable modification for functions with support in \( \mathcal{W}(d) \). We will merely describe them with mere indication of proof.

Fix \( d \geq 1 \) and let \( k \leq d \). Let \( f : G \to \mathbb{C} \) be a function supported on \( \mathcal{W}(d) \). Let \( \pi \) be a partition of \( [1, \ldots, d] \) in \( k \) disjoint consecutive blocks (intervals) \( \alpha_1, \ldots, \alpha_k \) so that \( |\alpha_1| + \cdots + |\alpha_k| = d \) and \( |\alpha_i| \geq 1 \) for all \( i = 1, \ldots, k \). We will denote by \( k(\pi) \) the number of blocks, i.e. we set \( k(\pi) = k \). We define \( f_\pi : \mathcal{W}(|\alpha_1|) \times \cdots \times \mathcal{W}(|\alpha_k|) \to \mathbb{C} \) by

\[
f_\pi(x_1, x_2, \ldots, x_k) = f(x_1 x_2 \ldots x_k).
\]

Note that \( f_\pi(x_1, \ldots, x_k) = 0 \) if the product \( x_1 x_2 \ldots x_k \) is not a reduced word, since then it has length \( < d \). For any function \( F : \mathcal{W}(|\alpha_1|) \times \cdots \times \mathcal{W}(|\alpha_k|) \to \mathbb{C} \), we denote again

\[
(5.14) \quad \|F\|_{M(\pi)} = \|F\|_{M([\alpha_1], \ldots, [\alpha_k])}.
\]

The preceding proofs (mainly Lemma 5.6) then yield

**Theorem 5.9.** With the preceding notation, we have for any function \( f \) with support in \( \mathcal{W}(d) \) and for any integer \( K \geq 1 \)

\[
\sup_{k(\pi) \leq K} \|f_\pi\|_{M(\pi)} \leq \|f\|_{M_K(G)} \leq C(d, K) \sup_{k(\pi) \leq K} \|f_\pi\|_{M(\pi)}
\]

where \( C(d, K) \) is a constant depending only on \( d \) and \( K \).

**Remark.** Let \( \pi_0 \) be the partition of \( [1, \ldots, d] \) into singletons, so that \( k(\pi_0) = d \). When \( K \geq d \), we have for any \( f \) as in \( (5.14) \)

\[
(5.15) \quad \sup_{k(\pi) \leq K} \|f_\pi\|_{M(\pi)} = \|f_{\pi_0}\|_{M(\pi_0)}.
\]

Indeed, we have \( k(\pi_0) \leq K \) and moreover it is easy to see using [53] that if a partition \( \pi \) is less fine than another one \( \pi' \) (i.e. every block in \( \pi \) is the union of certain blocks of \( \pi' \)) we have

\[
\|f_\pi\|_{M(\pi)} \leq \|f_{\pi'}\|_{M(\pi')}.
\]

Since any \( \pi \) is less fine than \( \pi_0 \), (5.15) follows immediately.
Thus, for all $K \geq d$, the norms $\|f\|_{M_K(G)}$ are equivalent on functions $f$ with support in $W(d)$. Indeed, by (5.14) they are equivalent to $\|f_{\pi_0}\|_{M(\pi)}$. In sharp contrast when $K < d$, in particular when $K = d - 1$, they are no longer equivalent.

**Corollary 5.10.** Let $G$ be any group containing a (non-Abelian) free subgroup. Then, for any $\theta > 1$ and any $c > \theta$ we have $B_c(G) \neq B_\theta(G)$, consequently there is a representation $\pi : G \to B(H)$ with $|\pi| \leq c$ that is not similar to any representation $\pi'$ with $|\pi'| \leq \theta$.

**Proof.** Assume by contradiction that the conclusion fails. Then, a fortiori it must also fail for $G = F_\infty$, by an “induction” argument as in Proposition 0.5. Hence, by Theorem 2.10 we obtain $M_d(F_\infty) = M_{d+1}(F_\infty)$ for some $d$, contradicting Theorem 5.1.

**Remark.** In the case of $G = \text{SL}_2(\mathbb{R})$, Michael Cowling showed me a very concrete proof (that he attributed to Haagerup) of the conclusion of Corollary 5.10. That proof uses the estimates of Kunze and Stein from [35], the Bruhat decomposition of $\text{SL}_2(\mathbb{R})$ and the amenability of the subgroup of triangular matrices in $\text{SL}_2(\mathbb{R})$.

**Remark.** In [65], we study the same question as in this section but for $G = N$. We answer a related question of Peller concerning power bounded operators (=uniformly bounded representations of $G = N$), by showing

$$M_2(N) \neq M_3(N).$$

On the other hand, the main result of [33] implies that if $H = \ell_2$ we have for any $d$

$$M_d(N; H) \neq M_{d+1}(N; H).$$

However, the same question for $H = C$ remains open when $G = N$.

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