Blaschke’s problem for timelike surfaces in pseudo-Riemannian space forms

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Abstract

We show that isothermic surfaces and S-Willmore surfaces are also the solutions to the corresponding Blaschke’s problem for both spacelike and timelike surfaces in pseudo-Riemannian space forms. For timelike surfaces both Willmore and isothermic, we obtain a description by minimal surfaces similar to the classical results of Thomsen.

Keywords: Blaschke’s problem; timelike S-Willmore surfaces; timelike isothermic surfaces; timelike minimal surfaces

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1 Introduction

Isothermic surfaces and Willmore surfaces are important objects in differential geometry. Especially, they are surface classes invariant under conformal transforms. Although seemed so distinct to each other, they may be introduced as the only non-trivial solutions to a problem in the category of conformal differential geometry[14], i.e. the Blaschke’ problem:

**Blaschke’s Problem:** Let $S$ be a sphere congruence with two envelopes

$$f, f' : M^2 \to S^3,$$

such that these envelopes induce the same conformal structure. Characterize such sphere congruences and envelop surfaces.

Blaschke posed this question and solved it in [2], namely

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**Theorem**  The non-trivial solution to the Blaschkes problem is either a pair of isothermic surfaces forming Darboux transform to each other, or a pair of dual Willmore surfaces with their common mean curvature spheres. (Here non-trivial means the two envelopes are not congruent up to Möbius transforms.)

Recently, Ma considered the arbitrary co-dimensional case and proved that the generalized Darboux pair of isothermic surfaces as well as S-Willmore surfaces in $S^n$ are the full non-trivial solution to the Blaschke’s problem in [14]. There have been some other kinds of generalization as to Blaschke’ problem after Ma’s work, see [7, 13].

On the other hand, there have been several kinds of research concerning the surface theory in pseudo-Riemannian space forms from different viewpoints, for example, see [8], [11], [18]. When dealing with the conformal geometry of such surfaces, it is natural to consider the Blaschke’s problem, which is just the main content of this paper. We obtain the similar results for both spacelike and timelike surfaces in pseudo-Riemannian space forms as [14]. For the spacelike case, the theorems and proofs are the same as Ma’s results, so we omit it and focus on the timelike case. The Blaschke’ problem for timelike surfaces can be stated as below:

**Blaschke’s Problem for timelike surfaces:** Let $S$ be a timelike 2-sphere congruence with two timelike envelopes $f, f' : M^2 \to Q^n_r$, such that these timelike envelopes induce the same conformal structure. Characterize such sphere congruences and envelop surfaces.

Here a timelike 2-sphere congruence in $Q^n_r$ means a map into the Grassmannian manifold $G_{2,2}(\mathbb{R}^{n+2}_{r+1}) := \{4\text{-dim (2,2)-type subspaces of } \mathbb{R}^{n+2}_{r+1}\}$. For the equivalence of them, we prefer to [12]. And for the notion of $Q^n_r$, see Section 2. Our main result is the following theorem:

**Theorem A** The non-trivial solution to the Blaschkes problem of timelike surfaces is either a pair of timelike isothermic surfaces forming Darboux transform to each other, or a pair of dual timelike S-Willmore surfaces with their common mean curvature spheres.

We also give a characteristic of timelike isothermic Willmore surfaces in $Q^3_1$ as follows, which is similar to the classical results of Thomsen [19]:

**Theorem B** Any timelike isothermic Willmore surface in $Q^3_1$ is conformally equivalent to a timelike minimal surface in some 3-dimensional Lorentzian space form $R^3_1$, $S^3_1$, or $H^3_1$. 
This paper is organized as follows. In Section 2, the pseudo-Riemannian conformal space $Q^n_r$, the general theory about timelike surfaces in $Q^n_r$ and the characterization of timelike Willmore surfaces and timelike isothermic surfaces are introduced. Then we prove Theorem A, B in Section 3 and Section 4 respectively.

2 Timelike surfaces in pseudo-Riemannian space forms

2.1 conformal geometry of $Q^n_r$

Let $\mathbb{R}^m_s$ be the space $\mathbb{R}^m$ equipped with the quadric form

$$\langle x, x \rangle = \sum_{i=1}^{m-s} x_i^2 - \sum_{i=m-s+1}^m x_i^2.$$  

We denote by $C^{m-1}_s$ the light cone of $\mathbb{R}^m$. The quadric

$$Q^n_r = \{ [x] \in \mathbb{R}P^{n+1} | x \in C^{n+1}_r \setminus \{0\} \}$$

is exactly the projectived light cone. The standard projection $\pi : C^n_s \setminus \{0\} \to Q^n_r$ is a fiber bundle with fiber $\mathbb{R} \setminus \{0\}$. It is easy to see that $Q^n_r$ is equipped with a $(n-r, r)$–type pseudo-Riemannian metric induced from projection $S^{n-r} \times S^r \to Q^n_r$. Here

$$S^{n-r} \times S^r = \{ x \in \mathbb{R}^{n+2}_r | \sum_{i=1}^{n-r+1} x_i^2 = \sum_{i=n-r+2}^{n+2} x_i^2 = 1 \} \subset C^{n+1}_r \setminus \{0\}$$

endowed with a $(n-r, r)$–type pseudo-Riemannian metric $g(S^{n-r}) \oplus (-g(S^r))$, where $g(S^{n-r})$ and $g(S^r)$ are standard metrics on $S^{n-r}$ and $S^r$. So there is a conformal structure of $(n-r, r)$–type pseudo-Riemannian metric $[h]$ on $Q^n_r$. By a theorem of Cahen and Kerbrat [4], we know that the conformal group of $(Q^n_r, [h])$ is exactly the orthogonal group $O(n-r+1, r+1)/\{\pm1\}$, which keeps the inner product of $\mathbb{R}^{n+2}_r$ invariant and acts on $Q^n_r$ by

$$T([x]) = [xT], \ T \in O(n-r+1, r+1).$$

For the three n-dimensional $(n-r, r)$–type pseudo-Riemannian space forms with constant sectional curvature $c = 0, +1, -1$, they are defined by

$$R^n_r, \ S^n_r := \{ x \in \mathbb{R}^{n+1}_r | \langle x, x \rangle = 1 \}, \ H^n_r := \{ x \in \mathbb{R}^{n+1}_r | \langle x, x \rangle = -1 \}.$$
Each of them could be embedded as a proper subset of $Q^n_r$:
\begin{align*}
\phi_0 : R^n_r &\to Q^n_r, \quad \phi_0(x) = \left[\left(\frac{-1+\langle x,x \rangle}{2}, x, \frac{1+\langle x,x \rangle}{2}\right)\right]; \\
\phi_+ : S^n_r &\to Q^n_r, \quad \phi_+(x) = [(x, 1)]; \\
\phi_- : H^n_r &\to Q^n_r, \quad \phi_-(x) = [(1, x)].
\end{align*}

It is easy to verify that these maps are conformal embeddings. Thus $Q^n_r$ is the proper space to study the conformal geometry of these $(n-r, r)-$type pseudo-Riemannian space forms.

### 2.2 Basic equations of Timelike Surfaces in $Q^n_r$

Let $y : M \to Q^n_r$ be a timelike surface. For any open subset $U \subset M$, we call $Y : U \to C^{n+1}_r$ a local lift of $y$ if $y = \pi \circ Y$. Two different local lifts differ by a scaling, so the metric induced from them are conformal to each other. Choose asymptotic coordinates $(u, v)$ on $U$, such that for some lift $Y$
\[ \langle Y_u, Y_u \rangle = \langle Y_v, Y_v \rangle = 0. \]

Such property inversely holds for any lift, showing that asymptotic coordinates are conformal invariant.

For such a surface there is a decomposition $M \times \mathbb{R}^6_2 = V \oplus V^\perp$, where
\[ V := \operatorname{Span}\{Y, Y_u, Y_v, Y_{uv}\} \]

is a 4-dimensional $(2,2)$-type subbundle independent of choice of $y$ and $(u, v)$. $V^\perp$ has a $(n-r-1, r-1)-$type metric, which might be identified with the normal bundle of $y$.

For a local asymptotic coordinate $(u, v)$, there is a local lift $Y$ such that $\langle Y_u, Y_v \rangle = \pm \frac{1}{2}$. We can adjust $v$ so that
\[ \langle Y_u, Y_v \rangle = \frac{1}{2}. \]

We call $Y$ a canonical lift with respect to $(u, v)$. So there is a unique $N \in \Gamma(V)$ satisfying
\[ \langle N, Y_u \rangle = \langle N, Y_v \rangle = \langle N, N \rangle = 0, \langle N, Y \rangle = -1. \]

Given frames as above, we note that $Y_{uu}$ and $Y_{uv}$ are orthogonal to $Y, Y_u$ and $Y_v$. So there must be two functions $s_1, s_2$ and two section $\kappa_1, \kappa_2 \in \Gamma(V^\perp)$ such that
\begin{align*}
\begin{cases}
Y_{uu} = -\frac{a_1}{2}Y + \kappa_1, \\
Y_{uv} = -\frac{a_2}{2}Y + \kappa_2.
\end{cases}
\end{align*}

These define four basic invariants $\kappa_1, \kappa_2$ and $s_1, s_2$ dependent on $(u, v)$. Similar to the case in Möbius geometry, $\kappa_i$ and $s_i$ are called the conformal Hopf differential and the Schwarzian derivative of $y$, respectively (compare [5],[15]).
Let $\psi \in \Gamma(V^\perp)$ denote a section of the normal bundle, and $D$ the normal connection, we can derive the structure equations as below:

\[
\begin{align*}
Y_{uu} &= -\frac{s_1}{2} Y + \kappa_1, \\
Y_{vv} &= -\frac{s_2}{2} Y + \kappa_2, \\
Y_{uv} &= -\langle \kappa_1, \kappa_2 \rangle Y + \frac{1}{2} N, \\
N_u &= -2\langle \kappa_1, \kappa_2 \rangle Y_u - s_1 Y_v + 2D_v\kappa_1, \\
N_v &= -s_2 Y_u - 2\langle \kappa_1, \kappa_2 \rangle Y_v + 2D_u\kappa_2, \\
\psi_u &= D_u\psi + 2\langle \psi, D_u\kappa_1 \rangle Y - 2\langle \psi, \kappa_1 \rangle Y_v, \\
\psi_v &= D_v\psi + 2\langle \psi, D_v\kappa_2 \rangle Y - 2\langle \psi, \kappa_2 \rangle Y_u.
\end{align*}
\]

The conformal Gauss equations, Codazzi equations, and Ricci equations as integrable conditions are:

\[
\begin{align*}
\frac{1}{2} s_{1v} &= 3\langle \kappa_1, D_u\kappa_2 \rangle + \langle D_u\kappa_1, \kappa_2 \rangle, \\
\frac{1}{2} s_{2u} &= \langle \kappa_1, D_v\kappa_2 \rangle + 3\langle D_v\kappa_1, \kappa_2 \rangle; \\
D_v D_u \kappa_1 + \frac{s_2}{2} \kappa_1 &= D_u D_v \kappa_2 + \frac{s_1}{2} \kappa_2; \\
R^D_{uv} := D_u D_v \psi - D_v D_u \psi &= 2\langle \psi, \kappa_1 \rangle \kappa_2 - 2\langle \psi, \kappa_2 \rangle \kappa_1.
\end{align*}
\]

### 2.3 Timelike Willmore surfaces

**Definition 2.1.** Let $y : M \to Q^n_r$ be an immersed timelike surface. The Willmore functional of $y$ is defined as:

\[W(y) := 2 \int_M \langle \kappa_1, \kappa_2 \rangle dudv.\]

$y$ is called a Willmore surface, if it is a critical surface of the Willmore functional with respect to any timelike variation of the map $y : M \to Q^n_r$.

It is direct to check that $W(y)$ is well-defined. Timelike Willmore surfaces can be characterized as follows, which is similar to the results of spacelike case [11, 12, 18, 20].

**Theorem 2.2.** For a timelike surface $y : M^2 \to Q^n_r$, the following three conditions are equivalent:

(i) $y$ is a timelike Willmore surface.

(ii) The conformal Gauss map

\[G : M \to G_{2,2}(\mathbb{R}^{n+2}_r), \ G(p) := V_p, \ \forall p \in M\]

of $y$ is harmonic.

(iii) The two Hopf differential $\kappa_1, \kappa_2$ satisfy the following Willmore equation:

\[D_v D_u \kappa_1 + \frac{s_2}{2} \kappa_1 = D_u D_v \kappa_2 + \frac{s_1}{2} \kappa_2 = 0.\]
For the proof, we prefer to [8, 18, 20]. We also note that the calculation of Euler-Lagrange equations of Willmore functional by Wang in [20] is valid for timelike submanifolds in Lorentzian space forms, and then leads to Theorem 2.2.

Now we define timelike S-Willmore surfaces as:

**Definition 2.3.** A timelike Willmore surface \( y : M \to Q^n \) is called an S-Willmore surface if it satisfies \( D_v \kappa_1 \parallel \kappa_1, D_u \kappa_2 \parallel \kappa_2 \), i.e., if there exist two functions \( \mu_1, \mu_2 \) such that

\[
D_v \kappa_1 + \mu_1 \kappa_1 = D_u \kappa_2 + \mu_2 \kappa_2 = 0. \tag{13}
\]

### 2.4 Timelike isothermic surfaces

**Definition 2.4.** Let \( y : M \to Q^n \) be a conformal timelike surface without umbilic points. It is called \((\pm)\)-isothermic if around each point of \( M \) there exists an asymptotic coordinate \((u, v)\) and canonical lift \( Y \) such that the Hopf differentials \( \kappa_1 = \pm \kappa_2 \). Such a coordinate \((u, v)\) is called an adapted coordinate.

\( \kappa_1 = \pm \kappa_2 \) together with the conformal Ricci equations in (11) shows that the normal bundle of \( y \) is flat. This is an important property of isothermic surfaces, which guarantees that all shape operators commute and the curvature lines could still be defined. Setting \( u = s + t, v = s - t \), the two fundamental forms of an isothermic surface, with respect to some parallel normal frame \( \{e_\alpha\} \), are of the form

\[
I = e^{2\rho}(ds^2 - dt^2), \quad II = \sum_\alpha (b_{\alpha 1}ds^2 - b_{\alpha 2}dt^2)e_\alpha, \tag{14}
\]

if \( y \) is \((+)\)-isothermic and

\[
I = e^{2\rho}(ds^2 - dt^2), \quad II = \sum_\alpha (b_{\alpha 1}(ds^2 - dt^2) - b_{\alpha 2}dsdt)e_\alpha \tag{15}
\]

if \( y \) is \((-)\)-isothermic. Note that \((\pm)\)-isothermic surfaces are called real and complex isothermic surface separately in [9]. And our notions here follow [11].

### 3 Proof of Theorem A

Denote \( y \) and \( \hat{y} \) the pair of surfaces in the Blaschke’s problem. Let \((u, v)\) be an asymptotic coordinate of \( y \) and \( Y \) the relevant canonical lift. Choose a lift \( \hat{Y} \) of \( \hat{y} \) such that \( \langle Y, \hat{Y} \rangle = -1 \). Then the sphere congruence tangent to \( Y \) and passing \( \hat{Y} \) is

\[
\text{Span}\{Y, Y_u, Y_v, \hat{Y}\}.
\]
By the conditions of Theorem A, we know that \((u, v)\) is also asymptotic coordinate of \(\hat{Y}\), and
\[
\text{Span}\{Y, Y_u, Y_v, \hat{Y}\} = \text{Span}\{\hat{Y}, \hat{Y}_u, \hat{Y}_v, Y\}.
\] (16)
Assume that
\[
\hat{Y} = N + 2aY_u + 2bY_v + (2ab + \frac{1}{2}(\xi, \xi))Y + \xi,
\] (17)
where \(\xi \in \Gamma(V^\perp)\). Differentiating shows
\[
\begin{align*}
\hat{Y}_u &= b\hat{Y} + \rho_1(Y_u + bY) + \theta_1(Y_v + aY) + \eta_1 + (\langle \xi, \eta_1 \rangle)Y, \\
\hat{Y}_v &= a\hat{Y} + \theta_2(Y_u + bY) + \rho_2(Y_v + aY) + \eta_2 + (\langle \xi, \eta_2 \rangle)Y.
\end{align*}
\] (18)
Here
\[
\begin{align*}
\rho_1 &= 2a_u - 2\langle \kappa_1, \kappa_2 \rangle + \frac{1}{2}(\xi, \xi), \quad \rho_2 = 2b_v - 2\langle \kappa_1, \kappa_2 \rangle + \frac{1}{2}(\xi, \xi); \\
\theta_1 &= 2b_u - 2b^2 - s_1 - 2\langle \xi, \kappa_1 \rangle, \quad \theta_2 = 2a_v - 2a^2 - s_2 - 2\langle \xi, \kappa_2 \rangle; \\
\eta_1 &= D_u\xi - b\xi + 2D_v\kappa_1 + 2a\kappa_1, \quad \eta_2 = D_v\xi - a\xi + 2D_u\kappa_2 + 2b\kappa_2.
\end{align*}
\] (19)
By (16), there must be \(\eta_1 = \eta_2 = 0\) and \(\rho_1 = \rho_2 = 0\) or \(\eta_1 = \eta_2 = 0\) and \(\theta_1 = \theta_2 = 0\). From \(\eta_1 = \eta_2 = 0\), we obtain
\[
D_v\kappa_1 = -\frac{1}{2}D_u\xi + \frac{b}{2}\xi - a\kappa_1, \quad D_u\kappa_2 = -\frac{1}{2}D_v\xi + \frac{a}{2}\xi - b\kappa_2.
\]
So
\[
D_uD_v\kappa_1 + \frac{s_2}{2}\kappa_1 = D_v\left(-\frac{1}{2}D_u\xi + \frac{b}{2}\xi - a\kappa_1\right) + \frac{s_2}{2}\kappa_1
\]
\[
= -\left(\frac{\theta_2}{2} + 2\langle \xi, \kappa_2 \rangle\right)\kappa_1 + \left(\frac{b_v}{2} - \frac{ab}{2}\right)\xi + \frac{a}{2}D_u\xi + \frac{b}{2}D_v\xi - \frac{1}{2}D_vD_u\xi.
\]
And
\[
D_uD_v\kappa_2 + \frac{s_1}{2}\kappa_2 = D_u\left(-\frac{1}{2}D_v\xi + \frac{a}{2}\xi - b\kappa_2\right) + \frac{s_1}{2}\kappa_2
\]
\[
= -\left(\frac{\theta_1}{2} + 2\langle \xi, \kappa_1 \rangle\right)\kappa_2 + \left(\frac{a_u}{2} - \frac{ab}{2}\right)\xi + \frac{b}{2}D_v\xi + \frac{a}{2}D_u\xi - \frac{1}{2}D_uD_v\xi.
\]
Plus the conformal Codazzi equation (10) and conformal Ricci equation (11), we get
\[
\frac{a_u}{2}\xi - \frac{\theta_1}{2}\kappa_2 = \frac{b_v}{2}\xi - \frac{\theta_2}{2}\kappa_1.
\] (20)
This equation works when concerning the isothermic case.
Besides this, by the conformal Gauss equation (9), we see that

\[
\theta_1 v = 2b_{uv} - 4bb_v - s_{1v} - 2\langle \xi, \kappa_1 \rangle_v
\]

\[
= (\rho_2 + 2\langle \kappa_1, \kappa_2 \rangle - \frac{1}{2} \langle \xi, \xi \rangle) u - 2b(\rho_2 + 2\langle \kappa_1, \kappa_2 \rangle - \frac{1}{2} \langle \xi, \xi \rangle)
\]

\[
- 6\langle \kappa_1, D_u \kappa_2 \rangle - 2\langle D_u \kappa_1, \kappa_2 \rangle - 2\langle D_v \xi, \kappa_1 \rangle - 2\langle \xi, D_v \kappa_1 \rangle
\]

\[
= \rho_2u - 2b\rho_2 + (-4D_u \kappa_2 - 4b\kappa_2 - 2D_v \xi, \kappa_1) - \langle \xi, 2D_v \kappa_1 - b\xi + D_u \xi \rangle
\]

\[
= \rho_2u - 2b\rho_2.
\]

i.e.

\[
\theta_1 v = \rho_2u - 2b\rho_2. \tag{21}
\]

Similarly we obtain

\[
\theta_2 u = \rho_1v - 2a\rho_1. \tag{22}
\]

Now let us prove Theorem A in the following three cases.

1. **The S–Willmore case:** \( \theta_1 = \theta_2 = 0, \xi = 0 \)

Since \( \xi = 0 \) and \( \eta_1 = \eta_2 = 0 \), (19) reduces to

\[
2D_v \kappa_1 + a\kappa_1 = 2D_u \kappa_2 + b\kappa_2 = 0. \tag{23}
\]

Together with

\[
\left\{ \begin{array}{c}
D_v D_v \kappa_1 + \frac{a}{2} \kappa_1 = \theta_1 \kappa_1 = 0, \\
D_u D_u \kappa_2 + \frac{a}{2} \kappa_2 = \theta_2 \kappa_2 = 0,
\end{array} \right. \tag{24}
\]

we see that \( Y \) is a timelike S-Willmore surface. To verify \( \hat{Y} \), direct calculation shows that

\[
\hat{Y}_{uv} = (\cdots) Y \mod \{ \hat{Y}, \hat{Y}_u, \hat{Y}_v \}, \quad \hat{\kappa}_1 = \rho_1 \kappa_1, \quad \hat{\kappa}_2 = \rho_2 \kappa_2.
\]

So \( \hat{y} \) is S-Willmore by Theorem 2.2 since \( \hat{Y} \) shares the same asymptotic coordinate and the same conformal Gauss map with \( Y \). So \( y \) and \( \hat{y} \) are a pair of dual S-Willmore surfaces.

2. **The isothermic case:** \( \rho_1 = \rho_2 = 0 \)

From the definition of \( \rho_1 \) and \( \rho_2 \), we see that \( a_u = b_v \). Substituting into (20) obtains

\[
\theta_1 \kappa_2 = \theta_2 \kappa_1. \tag{25}
\]

By use of (21) and (22), we have

\[
\theta_{1v} = \theta_{2u} = 0.
\]
So 
\[ \theta_1 = \theta_1(u), \ \theta_2 = \theta_2(v). \]

By choosing new asymptotic coordinate \((\tilde{u}, \tilde{v})\) we can derive

\[ \tilde{\kappa}_1 = \theta_2 \kappa_1 = \theta_1 \kappa_2 = \pm \kappa_2, \]  

(26)

where \(\pm\) corresponds to \((\pm)\)-isothermic surface. Notice that we must choose the \((\tilde{u}, \tilde{v})\) such that

\[ \left| \frac{\partial(\tilde{u}, \tilde{v})}{\partial(u,v)} \right| > 0 \]

to ensure that \(\langle Y_{\tilde{u}}, Y_{\tilde{v}} \rangle > 0\).

To show that \(\hat{Y}\) is also \((\pm)\)-isothermic surface as \(y\), we can suppose that \(\kappa_1 = \pm \kappa_2\). So \(\theta_1 = \pm \theta_2\) and \(\theta_{1v} = \theta_{2u} = 0\) show that \(\theta_1 = \pm \theta_2 = \theta = \text{const.}\) Then

\[ \hat{Y}_u = b\hat{Y} + \theta(Y_v + aY) \Rightarrow Y_v = -aY + \frac{1}{\theta}(\hat{Y}_u - b\hat{Y}), \]

\[ \hat{Y}_v = a\hat{Y} \pm \theta(Y_u + bY) \Rightarrow Y_u = -bY \pm \frac{1}{\theta}(\hat{Y}_v - a\hat{Y}). \]

So \(\hat{Y}\) also satisfies the conditions of case 2, which means \(\hat{Y}\) is also \((\pm)\)-isothermic as \(Y\). In fact, \(\hat{y}\) is the Darboux transform of \(\theta\)-parameter of \(y\) and vice versa.

3. The trivial case: \(\theta_1 = \theta_2 = 0, \ \xi \neq 0\)

In this case, from (21) and (22), we can see that

\[ \rho_{1v} - 2a\rho_1 = \rho_{2u} - 2b\rho_2 = 0. \]

Together with (20), we see that \(a_u = b_v\). So \(\rho_1 = \rho_2 = \rho \neq 0\). Consider the vector

\[ \frac{1}{\rho}\hat{Y} - Y, \]

we have

\[ \left( \frac{1}{\rho}\hat{Y} - Y \right)_u = -b\left( \frac{1}{\rho}\hat{Y} - Y \right), \quad \left( \frac{1}{\rho}\hat{Y} - Y \right)_v = -a\left( \frac{1}{\rho}\hat{Y} - Y \right). \]

This means that \(\frac{1}{\rho}\hat{Y} - Y\) is a fixed direction, showing that this is the trivial case.

4 Proof of Theorem B

Let \(y : M \rightarrow Q^3\) be a timelike \((+)-\)isothermic Willmore surface with the adapted asymptotic coordinate \((u,v)\) and canonical lift \(Y\). Then

\[ \kappa_1 = \kappa_2, \ \kappa_2 = D_u \kappa_1 + \frac{s_2}{2} \kappa_1 = D_u \kappa_2 + \frac{s_1}{2} \kappa_2 = 0. \]
Assume that 
\[ \kappa_1 = \kappa_2 = kE, \]
where \( E \) is a unit section of the conformal normal bundle. If \( k = 0 \) in a neighborhood, \( y \) is contained in some \( S_2^3 \) and then minimal in some \( S_3^3 \).

So we can suppose \( k \neq 0 \) in a open subset \( U \subset M \). Set
\[ \dot{Y} = N + 2aY_u + 2bY_v + (2ab)Y, \]
with
\[ a = -\frac{\kappa_v}{\kappa}, \quad b = -\frac{\kappa_u}{\kappa}. \]

From the calculation in Section 3, we can verify that \( \dot{Y} \) is just the dual Willmore surface of \( Y \) and
\[ \dot{Y}_u = a\dot{Y} + \rho(Y_u + bY), \quad \dot{Y}_v = b\dot{Y} + \rho(Y_v + aY), \]
where
\[ \rho = a_u - 2k^2 = b_v - 2k^2, \quad \rho_u = 2b\rho, \quad \rho_v = 2a\rho, \]
by use of the Willmore equations as above in Section 3.

Consider the vector field \( Y_0 = \dot{Y} - \rho Y \).

Differentiating it leads to
\[ Y_{0u} = \dot{Y}_u - \rho_u Y - \rho Y_u = b(\dot{Y} - \rho Y) = bY_0, \quad Y_{0v} = \dot{Y}_v - \rho_v Y - \rho Y_v = aY_0. \]

This means that \( Y_0 \) is a point in \( Q_3^3 \).

(i) If \( \langle Y_0, Y_0 \rangle = 0, \rho \equiv 0 \). So \([\dot{Y}]\) reduces to a point. By some conformal transform, we can set \( \dot{Y} = f_1(1, 0, 0, 0, 1) \) with some function \( f_1 \) and \( Y = e^{-\omega}\left( -\frac{1}{2}(x,x), \frac{1}{2}(x,x) \right) \) for some timelike surface \( x : U \to R^1_3 \) with \( \langle x_u, x_v \rangle = \frac{1}{2}e^{2\omega} \).

The structure equations of \( x \) is:
\[
\begin{align*}
    x_{uu} &= 2\omega_u x_u + \Omega n, \quad x_{vv} = 2\omega_v x_v + \Omega n, \quad x_{uv} = \frac{1}{2}e^{2\omega} Hn, \\
    n_u &= -Hx_u - 2\Omega e^{-2\omega} x_v, \quad n_v = -2\Omega e^{-2\omega} x_u - Hx_v.
\end{align*}
\]

So
\[ k = e^{-\omega}\Omega, \quad a = \omega_v - \frac{\Omega_v}{\Omega}, \quad b = \omega_u - \frac{\Omega_u}{\Omega}, \]
\[ N = e^{\omega}(1 + H\langle x, n \rangle, Hn, 1 + H\langle x, n \rangle) - 2\omega_v Y_u - 2\omega_u Y_v + 2\omega_v \omega_u Y, \]
\[ \dot{Y} = e^{\omega}(1 + H\langle x, n \rangle, Hn, 1 + H\langle x, n \rangle + (\cdots)Y_u + (\cdots)Y_v + (\cdots)Y). \]

Since \( \dot{Y} = f_1(1, 0, 0, 0, 1) \), the coefficient of \( n \) must be zero, i.e. \( H = 0 \), which means that \( x \) is a timelike minimal surface in \( R^1_3 \).
(ii) If $\langle Y_0, Y_0 \rangle < 0$, by some conformal transform, we can set $Y_0 = f_2(0, 0, 0, 0, 1)$ with some function $f_2$ and $Y = e^{-\omega}(x, 1)$ for some timelike surface $x : U \rightarrow S^3_1$ with $\langle x_u, x_v \rangle = \frac{1}{2} e^{2\omega}$. Similar to case (i), it is direct to show that $x$ is just a minimal surface in $S^3_1$.

(iii) If $\langle Y_0, Y_0 \rangle > 0$, similar treatments as above show that $Y_0 = f_3(1, 0, 0, 0, 0)$ with some function $f_3$ and $Y = e^{-\omega}(1, x)$ for some timelike minimal surface $x : U \rightarrow H^3_1$ with $\langle x_u, x_v \rangle = \frac{1}{2} e^{2\omega}$.

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