KILLING SPINORS AND SUPERSYMMETRY ON AdS

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Abstract

In this paper we construct several supersymmetric theories on AdS$_5$ background. We discuss the proper definition of the Killing equation for the symplectic Majorana spinors required in AdS$_5$ supersymmetric theories. We find that the symplectic Killing spinor equation involves a matrix $M$ in the USp($2N$) indices whose role was not recognized previously. Using the correct Killing spinors we explicitly confirm that the particle masses in the constructed theories agree with the predictions of the AdS/CFT correspondence. Finally, we establish correct $O(d-1,2)$ isometry transformations required to keep the Lagrangian invariant on AdS$_d$.

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1 Introduction

Recent work on the AdS/CFT correspondence [1, 2, 3] has brought renewed interest in the subject of supersymmetric field theory in anti-de Sitter space, particularly for AdS$_5$. We have found that several basic questions are not clearly discussed in the literature, and it is our aim to clarify them in the present paper. These questions include the proper Killing equation for the symplectic Majorana spinor required in AdS$_5$ SUSY, and the Lagrangian and transformation rules for the SU($N$) gauge multiplet, the conformal scalar multiplet, and the massive scalar multiplet.

It was a surprise to us that the symplectic Killing spinor equation involves a matrix $M$ in the USp(2$N$) indices whose role was not recognized previously. It turns out that $M$ also enters the transformation rules and the Lagrangian of the basic supermultiplets. In this paper we will develop a full description of these basic supermultiplets on AdS$_5$ using the properly defined symplectic Killing spinors.

In the body of the paper, we will work with a metric of $(+, -, \cdots, -)$ signature unless stated otherwise:

$$ds^2 = e^{2ar} \eta_{\alpha\beta} dx^\alpha dx^\beta - dr^2.$$  \hspace{1cm} (1.1)

With this choice of metric, the Ricci curvature is $R_{\mu\nu} = (d-1)a^2 g_{\mu\nu}$. We give a summary of results for the $(-, +, \cdots, +)$ signature in Appendix A. It is our hope that the results in this paper will prove useful for developing further understanding of physics on AdS$_5$.

2 Killing spinors on AdS

It is known that in $d = 5, 6, 7 \mod 8$ regular Majorana fermions cannot be defined [4]. Instead, in these dimensions we can define symplectic Majorana fermions which are spinors satisfying the following condition [4, 4, 4]:

$$\chi^i = C(\chi^i)^T$$  \hspace{1cm} (2.1)

where $C$ is the charge conjugation matrix and

$$\chi^i \equiv \chi^i \gamma_0.$$  \hspace{1cm} (2.2)

In general, $i = 1, 2, \ldots, 2n$ and the indices are raised and lowered with a symplectic metric $\Omega_{ij}$ which obeys

$$\Omega^T = -\Omega, \hspace{0.5cm} \Omega \Omega^* = -I.$$  \hspace{1cm} (2.3)

In this paper, we will only be interested in a pair of symplectic Majorana spinors, so $i = 1, 2$ and

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.4)
is the symplectic metric which will be used throughout this paper. To simplify our computations, we will only use objects with all the symplectic indices lowered by inserting the symplectic metric explicitly, for example, $\chi^i = \chi_j e^{ji} = -\epsilon_{ij} \chi^j$. Note that in our convention,

$$\epsilon^{ij} = -\epsilon_{ji} = \epsilon_{ij}, \quad (2.5)$$

Because of definition (2.2), we will treat $\chi^i$ as an object with its symplectic index down, but will at times employ $\chi_i$ notation to save space.

For the rest of this paper we will work in $d = 5$ unless explicitly stated otherwise. Because now the Clifford algebra contains $\gamma_5$, the Fierz transformations become simpler:

$$\begin{pmatrix} s \\ v \\ t \end{pmatrix} (4, 2; 3, 1) = -\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ 5 & -3 & 1 \\ 10 & 2 & -2 \end{bmatrix} \begin{pmatrix} s \\ v \\ t \end{pmatrix} (4, 1; 3, 2) \quad (2.6)$$

where

$$s(a, b; c, d) = \bar{\psi}_a \psi_b \bar{\psi}_c \psi_d$$
$$v(a, b; c, d) = \bar{\psi}_a \gamma_\mu \psi_b \bar{\psi}_c \gamma_\mu \psi_d$$
$$t(a, b; c, d) = -\frac{1}{2} \bar{\psi}_a \gamma_\mu \psi_b \bar{\psi}_c \gamma_\mu \psi_d. \quad (2.7)$$

Another useful identity to keep in mind in 5 dimensions is the symplectic Majorana flip formula [5]

$$\chi_i \gamma_{\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_{n-1}} \gamma_{\mu_n} \psi^j = \bar{\psi}_j \gamma_{\mu_n} \gamma_{\mu_{n-1}} \cdots \gamma_{\mu_2} \gamma_{\mu_1} \chi^i \quad (2.8)$$

which written in our notation becomes

$$\chi_i \gamma_0 \gamma_{\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_{n-1}} \gamma_{\mu_n} \psi^j = -\epsilon_{il} \epsilon_{jk} \psi_k \gamma_0 \gamma_{\mu_n} \gamma_{\mu_{n-1}} \cdots \gamma_{\mu_2} \gamma_{\mu_1} \chi^l. \quad (2.9)$$

This formula comes about because the charge conjugation matrix, $C$, is such that

$$C \gamma_\mu C^{-1} = \gamma_\mu^T, \quad (2.10)$$

which is different from 4 dimensions where there is a minus sign on the right hand side of the equation. Because of that minus sign, a Majorana Killing spinor equation in 4 dimensions can be defined in a straightforward manner [7]:

$$D_\mu \epsilon = i \frac{a}{2} \gamma_\mu \epsilon. \quad (2.11)$$

Note that because this equation satisfies the Ricci identity (see eq. (2.14) below), it can be interpreted as a Killing equation in arbitrary dimension for a complex unconstrained spinor, which was studied previously [8]. An extension of the above definition to 5 dimensions fails because the left hand side satisfies the symplectic Majorana condition (2.1) while the right hand side of the above equation does not, due to eq. (2.10). This
led us to consider a generalized form of the Killing equation for the symplectic Majorana Killing spinors:

\[ D_\mu \epsilon_i = i M_{ij} \frac{a}{2} \gamma_\mu \epsilon_j, \quad (2.12) \]

where \( M_{ij} \) is an unknown \( 2 \times 2 \) matrix, and \( D_\mu \psi = (\partial_\mu + \frac{1}{2} \omega_{\mu ab} \sigma^{ab}) \psi \). It is important to note that this form of the symplectic Killing spinor equation and the subsequent supersymmetry transformations stemming from it are compatible with AdS\(_5\) supergravity transformation rules [1], although Killing spinors were not discussed there. We obtain the properties of the matrix \( M_{ij} \) by applying the Ricci identity and the symplectic Majorana condition to eq. (2.12). Using eqs. (2.1) and (2.10) yields a condition on \( M_{ij} \), which written in the matrix form becomes

\[ M = \epsilon M^* \epsilon, \quad (2.13) \]

where \( \epsilon \) is the symplectic metric. On the other hand, Ricci identity yields

\[ [D_\mu, D_\nu] \epsilon_i \equiv \frac{1}{2} R_{\mu\nu ab} \sigma^{ab} \epsilon_i = a^2 \sigma_{\mu\nu} \epsilon_i = a^2 \sigma_{\mu\nu} (M^2)_{ij} \epsilon_j, \quad (2.14) \]

that is

\[ M^2 = 1. \quad (2.15) \]

Putting equations (2.13) and (2.15) together, we can easily obtain the most general form of the matrix \( M \):

\[ M = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} - \cos \theta \end{pmatrix} = \vec{x} \cdot \vec{\sigma} \quad (2.16) \]

where \( \theta \) and \( \phi \) are angles taking values between 0 and 2\( \pi \), and

\[ \vec{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \]

and \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) is a vector of Pauli matrices. Hence, matrix \( M \) can be interpreted as an element of the Lie algebra of USp(2). Furthermore, it is easily seen that the Killing spinor equation (2.12) is USp(2)-covariant: take \( M \) to be an allowed matrix appearing in eq. (2.12), then USp(2) rotate the Killing spinors, \( \epsilon'_i = U_{ij} \epsilon_j \), and write new Killing spinor equation for the rotated spinors – we obtain the same form of Killing spinor equation but with a different matrix \( M' = U^\dagger MU \) which satisfies both conditions (2.13) and (2.15), hence giving us a valid Killing spinor equation.

Complex Killing spinors on \( d \)-dimensional anti-de Sitter spacetime AdS\(_d\) for arbitrary \( d \) have been constructed before [8, 9]. They are solutions of eq. (2.11) and take the form (see Appendix A for conversion between signatures)

\[ \epsilon = e^{i \frac{a r}{2}} e^{\gamma_\tau} \left( 1 + \frac{i}{2} a x^\alpha \gamma_\alpha (1 - i \gamma_\tau) \right) \epsilon_0. \quad (2.17) \]
Solution of equation (2.12) can easily be obtained from the solution (2.17) by substituting $M_{ij} \gamma_{\mu}$ for every $\gamma_{\mu}$ in eq. (2.17):

$$\epsilon_i = \left( e^{\frac{i}{2} \alpha \gamma_{\alpha} M \gamma_{\rho}} \right)_{ij} \left( \delta_{jk} + \frac{i}{2} \alpha x^{\alpha} \gamma_{\alpha} (M_{jk} - i \delta_{jk} \gamma_{\rho}) \right) \xi_k \quad (2.18)$$

where $\xi_j$ is a pair of constant symplectic Majorana spinors.

Now, for each matrix $M$ above, let us construct a Dirac spinor as a linear combination of the two symplectic Majorana spinors. Assume the most general relation between two symplectic Majorana spinors and a Dirac spinor:

$$\psi = A \epsilon_1 + B \epsilon_2 \quad (2.19)$$

with the unknown coefficients $A$ and $B$. To be consistent, eq. (2.19) should produce the right equation for the Dirac Killing spinors, eq. (2.11), when combined with equation for the symplectic Majorana Killing spinors, eq. (2.12). Using eq. (2.16) we find that this condition is satisfied by the following normalized Dirac spinor

$$\psi = e^{i \frac{\phi}{2}} \cos \frac{\theta}{2} \epsilon_1 + e^{-i \frac{\phi}{2}} \sin \frac{\theta}{2} \epsilon_2, \quad (2.20)$$

where we could also choose $-$ instead of $+$ between the two terms. This expression will be useful later in the paper.

Finally, it’s worth noting a general form of the matrix $M$ for more than 2 spinors. In the case of $2n$ spinors, $M$ is a $2n \times 2n$ matrix which takes a block form

$$M = \begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix} \quad (2.21)$$

where $A$ and $B$ are $n \times n$ complex matrices which satisfy the following equations:

$$AB = BA^*$$

$$A^2 + BB^* = I. \quad (2.22)$$

It is easy to see that for $n = 1$, above equations yield precisely the matrix $M$ given in equation (2.16).

3 The on-shell USp(2) supersymmetric U(1) Yang-Mills theory on AdS$_5$

Let us start with the massless USp(2) supersymmetric Yang-Mills theory in flat 4+1 spacetime. SU(2) version of this theory has been developed by Zizzi [10]. U(1) theory is easily obtained from SU(2) theory:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} D_\mu \phi D^\mu \phi + \frac{i}{2} \bar{\chi} D \chi_i$$

(3.1)
and invariant under the following supersymmetry transformations:

\begin{align}
\delta A_\mu &= i\eta^i \gamma_\mu \chi_i \\
\delta \phi &= i\eta^i \chi_i \\
\delta \chi_i &= (\sigma_{i\mu} F^{\mu\nu} - \bar{D}\phi)\eta_i 
\end{align}

(3.2)

where \( \mu, \nu = 0, \ldots, 4 \) and \( i = 1, 2 \). To describe the same theory on AdS_5 not only do we need to have additional terms in the supersymmetry transformations but we will have nonzero mass terms for both the scalar \( \phi \) and the spinors \( \chi_i \) for the case of massless gauge potential. In fact, compactification of \( \mathcal{N} = 2 \) supergravity on S^5 [11] or AdS/CFT correspondence [3, 12] let us determine these masses:

\[ m^2(A_\mu) = 0, \quad |m(\psi)| = \frac{1}{2}, \quad m^2(\phi) = -4. \]

(3.3)

Hence, the U(1) Yang-Mills theory on AdS_5 should be the flat U(1) theory (3.1) plus the above mass terms:

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} D_\mu D^\mu \phi + \frac{i}{2} \bar{\chi} \mathcal{D} \chi_i - a_{ij} \chi_i \chi_j - \frac{1}{2} a^2 m^2 \phi^2. \]

(3.4)

Using the proper Killing spinor equation (2.12) for the symplectic Majorana spinors, we find that theory (3.4) is invariant under the supersymmetry transformations

\begin{align}
\delta A_\mu &= i\eta^i \gamma_\mu \chi_i \\
\delta \phi &= i\eta^i \chi_i \\
\delta \chi_i &= (\sigma_{i\mu} F^{\mu\nu} - \bar{D}\phi)\eta_i - 2ia\phi M_{ij} \eta_j 
\end{align}

(3.5)

where \( M_{ij} \) is the matrix given by eq. (2.16). Furthermore, supersymmetry determines the values of the masses in eq. (3.4):

\[ \mu = -\frac{1}{4} M, \quad m^2 = -4. \]

(3.6)

It is easy to show that given a definition of a properly normalized Dirac spinor as in eq. (2.20),

\[ \frac{i}{2} \bar{\chi} \mathcal{D} \chi_i = i\bar{\psi} \mathcal{D} \psi \]

(3.7)

and

\[ \frac{1}{2} M_{ij} \chi_i \chi_j = \bar{\psi} \psi. \]

(3.8)

Hence, this theory contains a Dirac spinor of mass equal to \( \frac{1}{2} \) and one real scalar of mass equal to \( -4 \). These masses agree completely with our previous predictions given in equation (3.3).
To complete the description of this theory we need to write down the supersymmetry algebra. Using eq. (3.5), we find that
\[ [\delta_1, \delta_2] \phi = 2i \eta^i \gamma^\mu \eta_{2i} D_\mu \phi \] (3.9)
and similarly a usual expression for \([\delta_1, \delta_2] A_\mu\) (up to equations of motion and gauge transformations) because just like in the scalar case above all the terms proportional to \(a\) cancel. However, supersymmetry algebra for the spinors is more interesting:
\[ [\delta_1, \delta_2] \chi_i = 2i D_\mu \chi_i \eta^j_1 \gamma^\mu \eta_{2j} + 3a M_{ij} \chi_j \eta^k_1 \eta_{2k} + \frac{a}{2} \gamma^{\mu \nu} \chi_i M_{kj} \eta^k_1 \gamma_{\mu \nu} \eta_{2j}, \] (3.10)
where we used spinor equations of motion
\[ D \chi_i = i a M_{ij} \chi_j \] (3.11)
and the following useful identities
\[ M_{nl} \delta_{ij} - M_{il} \delta_{nj} + M_{nj} \delta_{il} - M_{nj} \delta_{it} = 0 \]
\[ \epsilon_{nj} (M \epsilon)_{ik} + \delta_{kn} M_{ij} = M_{in} \delta_{jk} \]
\[ \epsilon_{nj} \epsilon_{ik} + \delta_{kn} \delta_{ij} = \delta_{in} \delta_{jk}. \] (3.12)
To explain the terms appearing in this algebra, first consider only the fermionic part of the Lagrangian
\[ \mathcal{L}_F = \frac{i}{2} \slashed{D} \chi_i + \frac{1}{4} a M_{ij} \chi_j. \] (3.13)
From the properties of the matrix \(M\) (eqs. (2.13) and (2.17)), it follows that there is an additional U(1) symmetry in the theory:
\[ \delta \chi_i = i M_{ij} \chi_j. \] (3.14)
This extra symmetry manifests itself in the supersymmetry algebra, as we see from the second term in eq. (3.10). Furthermore, supersymmetry algebra involves a term proportional to
\[ \gamma^{\mu \nu} \chi_i M_{kj} \eta^k_1 \gamma_{\mu \nu} \eta_{2j} \] (3.15)
which at first glance appears unusual. However, there is a clear and dimension independent explanation of this term. The proof of the following arguments is presented in Appendix B. Below, we chose to work in 4 dimensions because we do not wish to involve the symplectic indices. In 5 dimensions, the following discussion is slightly more involved but follows the same outline as the proof in Appendix B. To facilitate the explanation, let us look at the AdS\(_4\) supersymmetric theory [7]. If we compute supersymmetry algebra of the fermions using transformations (3.1) of that paper, we obtain (with our conventions)
\[ [\delta_1, \delta_2] L \psi = i D^\mu L \psi \gamma_1 \gamma_\mu \epsilon_2 + \frac{a}{4} \gamma^{\mu \nu} L \psi \gamma_1 \gamma_{\mu \nu} \epsilon_2. \] (3.16)
This algebra contains an “extra” term of the same form as eq. (3.15). The explanation of this “extra” term in the algebra lies in the fact that the naive isometry transformation
\[ \delta \psi = K^\mu D_\mu \psi, \] (3.17)
where \( K^\mu = i \epsilon_1 \gamma^\mu \epsilon_2 \) is an O(3,2) Killing vector, is actually not a symmetry of the kinetic term in curved space. On a curved manifold, we need to add more terms to this variation because \( D_\mu K_\nu \) no longer equals to 0. In particular, in AdS we need to add precisely the “extra term” in eq. (3.16) in order to recover a symmetry of the Lagrangian. Using the fact that
\[ D_\mu K_\nu = a \epsilon_1 \gamma^\mu \epsilon_2 \] (3.18)
we expect that in AdS\( _4 \) the full O(3,2) isometry requires the following transformation rule:
\[ \delta \psi = i D^\mu \psi \epsilon_1 \gamma^\mu \epsilon_2 + a \sigma^{\mu \nu} \psi \epsilon_1 \sigma_{\mu \nu} \epsilon_2 = K^\mu D_\mu \psi + \frac{1}{4} D^\mu K^\nu \gamma_{\mu \nu} \psi \] (3.19)
which can be verified to be a symmetry of the Lagrangian (see Appendix B). Hence, we indeed expect a term like (3.15) in the supersymmetry algebra of our theory on AdS\( _5 \).

Let us finally note that extending the above results to an SU(\( N \)) gauge theory is quite trivial. Assuming that all the matter fields are in the adjoint representation of the gauge group SU(\( N \)), the Lagrangian is
\[ L = -\frac{1}{4} F^{\mu \nu a} F_{\mu \nu}^a + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a + \frac{i}{2} \chi_i D^\alpha \chi^i + \frac{1}{4} a M_{ij} \chi^a \chi^j + 2 a^2 \phi^a \phi^a - \frac{i}{2} g f_{abc} \chi^a \phi^b \chi^c \] (3.20)
where \( a, b, c = 1, 2, \ldots, N^2 - 1 \) and the covariant derivatives are now defined as
\[ D_\mu \psi_i^a = \partial_\mu \psi_i^a + \frac{1}{2} \omega_\mu^{\rho \sigma} \sigma^{\nu \rho \sigma} \psi_i^a + g f_{abc} A^b_\mu \psi_i^c \]
\[ D_\mu \phi^a = \partial_\mu \phi^a + g f_{abc} A^a_\mu \phi^c \] (3.21)
\[ F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu. \]
With these definitions, the theory (3.20) is invariant under exactly the same transformations as before, keeping in mind the definitions above:
\[ \delta A^a_\mu = i T^a \gamma_\mu \chi^i \]
\[ \delta \phi^a = i T^a \chi^i \]
\[ \delta \chi^i_a = (\sigma_{\mu \rho} F^{\mu \rho a} - \partial \phi^a) \eta_i - 2 i a \phi^a M_{ij} \eta_j. \] (3.22)
Therefore, all the results, including the field masses and the supersymmetry algebra, remain exactly the same in the case of SU(\( N \)) gauge theory.
4 The on-shell USp(2) supersymmetric conformal scalar theory on AdS$_5$

This theory describes 2 massless symplectic Majorana fermions and 4 massive real scalar fields with the same mass for all 4 scalars. A generalization of this theory has been developed in flat 5-dimensional spacetime [13]. As in the previous section, we use the flat theory to develop this same theory on AdS$_5$:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^I \partial^\mu \phi^I + \frac{i}{2} \lambda_i \bar{\psi} \gamma^\lambda \lambda_i - \frac{1}{2} a^2 m^2 \phi^I \phi^I$$

where $\mu = 0, \ldots, 4$, $I = 1, \ldots, 4$, and $i = 1, 2$. The theory (4.1) is invariant under the supersymmetry

$$\delta \phi^I = (\sigma^I \epsilon)_ij \bar{\psi} \lambda_j$$
$$\delta \lambda_i = i(\epsilon \sigma^I)_j ij \phi^I \epsilon_j + a^2 \left(\sigma^I \epsilon^T M \epsilon_M \epsilon_j \phi^I\right)$$

where $\sigma^I = (\tilde{\sigma}, i1)$, $M$ is the matrix found in eq. (2.16), $\epsilon$ is the symplectic metric in the matrix form, and $\epsilon_i$ is a symplectic Majorana Killing spinor. Supersymmetry also determines the value of $m$ in this theory:

$$m^2 = -\frac{15}{4}.$$  

Note that this formula agrees with the mass formula for the conformally coupled scalar field in 5 dimensions [6, 14].

Now, we are ready to compute the supersymmetry algebra for this theory. After a short computation, we obtain

$$[\delta_1, \delta_2] \phi^I = 2i \tilde{e}_1 \gamma^\mu \epsilon_2 \partial_\mu \phi^I + a^2 \left(\sigma^I \sigma^J \epsilon^T M - M \sigma^J \sigma^I\right)_{ij} \epsilon_{1j} \phi^J.$$  

(4.4)

It appears (and can be confirmed by an explicit computation) that there exists an “extra” symmetry in this theory

$$\delta \phi^I = \tilde{\epsilon}_2 \left(\sigma^I \sigma^J \epsilon^T M - M \sigma^J \sigma^I\right)_{ij} \epsilon_{1j} \phi^J.$$  

(4.5)

In fact, this transformation represents rotation of the scalar fields $\delta \phi^I = i \tilde{\epsilon}_2 \epsilon_{1i} T^{IJ} \phi^J$ where $T$ is a $4 \times 4$ matrix

$$T = \begin{pmatrix}
0 & x_3 & -x_2 & -x_1 \\
-x_3 & 0 & x_1 & -x_2 \\
x_2 & -x_1 & 0 & x_3 \\
x_1 & x_2 & x_3 & 0
\end{pmatrix}$$

(4.6)

and $\tilde{x}$ is defined in eq. (2.10). Hence, for each fixed matrix $M$, this is a particular representation of the SO(2) subgroup of the obvious SO(4) symmetry of the scalar Lagrangian.
The spinor algebra, on the other hand, presents nothing new, although the previous extra term associated with the O(4,2) isometry does. In order to calculate this algebra, we need to know a few useful identities given below:

\[
(\epsilon \sigma^I)_{ji}(\sigma^I)_m - \sigma^I_{mi}\sigma^I_{jn} = -4\delta_{jm}\delta_{in}
\]

\[
(\epsilon \sigma^I)_{ji}(\sigma^I)_m + \sigma^I_{mi}\sigma^I_{jn} = 0
\]

\[
(\epsilon \sigma^I)_{ji}(M\sigma^I)_m + \sigma^I_{mi}(M\sigma^I)_{jn} = 0
\]

\[
(\epsilon M\sigma^I)_{ij}(\sigma^I)_m - \sigma^I_{mi}(M\sigma^I)_{jn} = -4\delta_{jm}M_{in}
\]

\[
(\epsilon M\sigma^I)_{ij}(\sigma^I)_m + (M\sigma^I)_{im}\sigma^I_{jn} = 0
\]

\[
(\epsilon M\sigma^I)_{ij}(\sigma^I)_m - (M\sigma^I)_{im}\sigma^I_{jn} = 0
\]

\[
(\epsilon M\sigma^I)_{ij}(\sigma^I)_m + (M\sigma^I T)_{im}\sigma^I_{jn} = 0
\]

\[
(\epsilon M\sigma^I)_{ij}(\sigma^I)_m - (M\sigma^I T)_{im}\sigma^I_{jn} = 0
\]

Then, using these identities we arrive at the following result

\[
[\delta_1,\delta_2]\lambda_i = 2iD_\mu\lambda_i\bar{\chi}_1^\mu\epsilon_{2j} + \frac{a}{2}\gamma^{\mu\nu}\lambda_i\lambda_j + \frac{a}{2}M_{ij}\phi^I\phi^J
\]

which is remarkably similar to the supersymmetric algebra for the spinors in AdS\(_4\) as given by eq. (3.16).

### 5 The on-shell USp(2) supersymmetric massive scalar theory on AdS\(_5\)

Similarly to the conformal scalar theory presented in the previous section, this theory describes 2 massive symplectic Majorana fermions and 4 massive real scalar fields. The theory is described by an action similar to that of conformal scalar theory given by eq. (4.1):

\[
\mathcal{L} = \frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^I + \frac{i}{2}\bar{\psi}\lambda_i - \frac{1}{2}a_\mu M_{ij}\bar{\lambda}^i\lambda^j - \frac{1}{2}a^2m^2_{IJ}\phi^I\phi^J
\]

The new feature here is a symmetric real 4\(\times\)4 matrix \(m^2_{IJ}\) which is not assumed to be diagonal \textit{apriori}. Also, note that we have already introduced the correct form of the spinor mass term according to the prescription in eq. (3.8) so that this theory contains a Dirac spinor of mass \(\mu\). The theory (5.1) can be shown to be invariant under the following supersymmetry transformation rules:

\[
\delta\phi^I = (\sigma^I)_i\phi^I
\]

\[
\delta\lambda_i = i(\epsilon\sigma^I)_{ij}\phi^I\epsilon_j + a\frac{3}{2}(\sigma^IT\epsilon M)_{ij}\epsilon_j\phi^I + a_\mu(M\epsilon\sigma^I)_{ij}\epsilon_j\phi^I
\]

provided that the scalar mass matrix, \(m^2_{IJ}\), takes a very specific form, which will be determined by the supersymmetry. Obtaining \(m^2_{IJ}\) is nontrivial, so we provide the necessary calculations below.
Using the supersymmetry transformation rules (5.2) to vary the action, we find that all the terms proportional to 1 and $a$ cancel but terms proportional to $a^2$ yield the following matrix equation for each $I$:

$$\left( \mu^2 - \frac{15}{4} \right) \sigma' \epsilon + \mu M \sigma' \epsilon M = m_{IJ} \sigma^J \epsilon.$$  (5.3)

One way to solve this equation is to expand everything in $\{\sigma^I\}$ basis and then set the coefficients of each $\sigma^I$ matrix to 0. Noting that $M \sigma^4 M = \sigma^4$, for $I = 1, 2, 3$ we write

$$M \sigma^I M = c_{IJ} \sigma^J$$  (5.4)

where $c$ is a $3 \times 3$ matrix easily found from eq. (2.16):

$$
\begin{pmatrix}
2x_1^2 - 1 & 2x_1x_2 & 2x_1x_3 \\
2x_1x_2 & 2x_2^2 - 1 & 2x_2x_3 \\
2x_1x_3 & 2x_2x_3 & 2x_3^2 - 1
\end{pmatrix}
$$  (5.5)

with $\vec{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$.

We note that for $I = 2$ eq. (5.3) decouples from the rest of the equations giving us

$$m_{12}^2 = m_{23}^2 = m_{24}^2 = 0, m_{22}^2 = \mu^2 + \mu - \frac{15}{4}. \quad (5.6)$$

The other 3 equations remain coupled:

$$-(\mu^2 - \frac{15}{4} - m_{11}^2) \sigma^3 = \mu c_{31} \sigma^1 + m_{13}^2 \sigma^1 - m_{14}^2 \sigma^2$$
$$-(\mu^2 - \frac{15}{4} - m_{33}^2) \sigma^1 = \mu c_{13} \sigma^1 + m_{13}^2 \sigma^3 + m_{34}^2 \sigma^2$$
$$-(\mu^2 - \frac{15}{4} - m_{44}^2) \sigma^2 = \mu c_{24} \sigma^1 + m_{34}^2 \sigma^3 - m_{14}^2 \sigma^3$$  (5.7)

where we used eq. (5.4) above. A few simple manipulations yield the values of the diagonal elements of the scalar mass matrix

$$m_{11}^2 = \mu^2 - \frac{15}{4} + \mu c_{33}$$
$$m_{33}^2 = \mu^2 - \frac{15}{4} + \mu c_{11}$$
$$m_{44}^2 = \mu^2 - \frac{15}{4} + \mu c_{22}. \quad (5.8)$$

Plugging these values back into eq. (5.7) we find all the other elements of the scalar mass matrix:

$$m_{13}^2 = -\mu c_{13}, \quad m_{14}^2 = \mu c_{23}, \quad m_{34}^2 = -\mu c_{12}. \quad (5.9)$$

Finally, putting eqs. (5.8) and (5.9) together we find the scalar mass matrix

$$m^2 = \begin{pmatrix}
\mu^2 - \frac{15}{4} + \mu c_{33} & 0 & -\mu c_{13} & \mu c_{23} \\
0 & \mu^2 + \mu - \frac{15}{4} & 0 & 0 \\
-\mu c_{13} & 0 & \mu^2 - \frac{15}{4} + \mu c_{11} & -\mu c_{12} \\
\mu c_{23} & 0 & -\mu c_{12} & \mu^2 - \frac{15}{4} + \mu c_{22}
\end{pmatrix} \quad (5.10)$$
where the rows and columns can be rearranged to give a block diagonal form. To find the physical values of the masses we need to diagonalize \( m^2 \) and read the physical masses off of the diagonal. This procedure yields

\[
m^2 = \mu^2 + \mu - \frac{15}{4}, \quad \mu^2 + \mu - \frac{15}{4}, \quad \mu^2 - \mu - \frac{15}{4}, \quad \mu^2 - \mu - \frac{15}{4}.
\]  

(5.11)

This answer is remarkable because these are precisely the masses we expect from the AdS/CFT correspondence [3, 12]: from the AdS/CFT correspondence we know that this theory should contain a complex scalar with conformal dimension \( \Delta \), a complex spinor with conformal dimension \( \Delta + \frac{1}{2} \), and another complex scalar with conformal dimension \( \Delta + 1 \), which in 5 dimensions gives the spinor mass

\[
\mu = \Delta + 1 - 2 = \Delta - \frac{3}{2}
\]

(5.12)

and the two scalar masses

\[
m^2 = \Delta(\Delta - 4), \\
m^2 = (\Delta + 1)(\Delta - 3)
\]

(5.13)

which in turn implies that the complex scalars in this theory should have their masses equal to

\[
m^2 = \left( \mu + \frac{3}{2} \right) \left( \mu - \frac{5}{2} \right) = \mu^2 - \mu - \frac{15}{4}, \\
m^2 = \left( \mu + \frac{5}{2} \right) \left( \mu - \frac{3}{2} \right) = \mu^2 + \mu - \frac{15}{4}.
\]

(5.14)

Hence, eq. (5.10) is the correct scalar mass matrix as the equations (5.11) and (5.14) are in exact agreement.

To complete the study of this theory we calculate the supersymmetry algebra for the spinors and scalars under the transformation rules (5.2). For the scalars, we obtain

\[
[\delta_1, \delta_2] \phi^I = 2i \epsilon_i^2 \gamma^\mu \epsilon_2 i \partial_\mu \phi^I \\
+ a \epsilon_i^2 \left( \sigma^I \sigma^J M - M \sigma^J \sigma^I \right)_{ij} \epsilon_1 j \phi^J \\
+ a \epsilon_i^2 \left( \sigma^I M^* \sigma^J - \sigma^J M^* \sigma^I \right)_{ij} \epsilon_1 j \phi^J
\]

(5.15)

which is almost the same as the supersymmetry algebra for the “conformal scalar” theory, eq. (4.4), with addition of a term proportional to the spinor mass, \( \mu \). Because the scalar mass term (5.10) breaks SO(4) symmetry down to SO(2) \( \times \) SO(2) (this fact becomes obvious once the scalar fields are rotated so that the scalar mass term becomes diagonal – see discussion below), the two non-derivative terms in the algebra must encode at least some part of this symmetry of the Lagrangian. To see this more clearly, let us assume that \( \phi^1, \phi^2 \) and \( \phi^3, \phi^4 \) have physical masses \( \mu^2 + \mu - \frac{15}{4} \) and \( \mu^2 - \mu - \frac{15}{4} \) respectively, so that each pair transforms under separate symmetries. First, note that even with this symmetry breaking, the transformation

\[
\delta \phi^I = i \epsilon_i^2 \left( \sigma^I M^* \sigma^J - \sigma^J M^* \sigma^I \right)_{ij} \epsilon_1 j \phi^J
\]

(5.16)
is by itself a symmetry of the kinetic part of the scalar Lagrangian because before the transformation (4.5) was a symmetry of the kinetic part as well. To establish this fact is nontrivial, but it all boils down to showing that

\[ \left[ \sigma^I M^* \sigma^J, M \right] - \left[ \sigma^J M^* \sigma^I, M \right] = 0 \quad (5.17) \]

for all values of \( I, J \). We can again rewrite the above transformation in a more compact form, \( \delta \phi^I = i \epsilon_2 \epsilon_1 \phi^J \phi^I \) where \( Q \) is a \( 4 \times 4 \) matrix

\[
Q = \begin{pmatrix}
0 & x_3 & -x_2 & -x_1 \\
x_3 & 0 & -x_1 & x_2 \\
x_2 & x_1 & 0 & -x_3 \\
x_1 & -x_2 & x_3 & 0
\end{pmatrix}.
\]

(5.18)

Now, we only need to understand how the transformation (5.15) acts on the mass term. To see this more clearly, let us redefine

\[
\phi^I = O^{IJ} \phi^J
\]

(5.19)

with the matrix \( O \) defined as follows:

\[
O = \begin{pmatrix}
x_3 & 0 & \sqrt{1-x_3^2} & 0 \\
0 & 1 & 0 & 0 \\
x_1 & 0 & \frac{x_1}{\sqrt{1-x_3^2}} & \frac{x_1}{\sqrt{1-x_3^2}} \\
x_2 & 0 & \frac{x_2}{\sqrt{1-x_3^2}} & \frac{x_2}{\sqrt{1-x_3^2}}
\end{pmatrix}.
\]

(5.20)

With this field redefinition, the scalar mass matrix becomes diagonal in precisely the way discussed above and the transformation (5.15) becomes

\[
\delta \phi^I = -i \epsilon_2 \gamma^\mu \epsilon_1 \partial_\mu \phi^I + i a \epsilon_2 \epsilon_1 \left[ 3(O^T TO)^{IJ} + 2 \phi (O^T QO)^{IJ} \right] \phi^J
\]

\[
\delta \phi^I = -i \epsilon_2 \gamma^\mu \epsilon_1 \partial_\mu \phi^I + i a \epsilon_2 \epsilon_1 \left[ 3(O^T TO)^{IJ} + 2 \phi (O^T QO)^{IJ} \right] \phi^J
\]

(5.21)

where \( T \) and \( Q \) are defined in eqs. (4.6) and (5.18) respectively. Using eq. (5.20) we can explicitly compute

\[
O^T TO = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
O^T QO = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

(5.22)

Hence the transformation (5.21) of the new fields encodes a particular SO(2) symmetry of the larger SO(2)×SO(2) symmetry of the scalar Lagrangian.
The spinor supersymmetry algebra is quite similar to that of the previously derived conformal scalar theory, eq. (4.8):

\[ [\delta_1, \delta_2] \lambda_i = 2i D_\mu \lambda_i \bar{\epsilon}_1^{\mu} \epsilon_{2j} - 2a\mu M_{ik} \lambda_k \bar{\epsilon}_1^{i} \epsilon_{2j} + \frac{a}{2} \gamma^{\mu\nu} \lambda_i M_{kj} \bar{\epsilon}_1^{k} \gamma_{\mu\nu} \epsilon_{2j} \] (5.23)

with a new term proportional to the spinor mass \( \mu \). As before, the third term with \( \gamma_{\mu\nu} \) in this algebra is exactly the extra symmetry term required to recover the \( O(4,2) \) symmetry of the Lagrangian, and the new term proportional to \( \mu \) is reminiscent of the \( U(1) \) symmetry (3.14) in the Yang-Mills theory described in Section 3.

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Appendix

A Summary of results in \((-, +, \cdots, +)\) signature

In this section we will summarize some of the formulas given in body of the text for the \((-, +, \cdots, +)\) signature. We do so because most of the recent literature on AdS uses this signature almost exclusively. Note that in this signature, the curvature takes the usual form, \( R_{\mu\nu} = -(d-1)a^2 g_{\mu\nu} \).

Most of the formulas can be converted to the \((-, +, \cdots, +)\) signature simply by changing \( \gamma_\mu \to -i\gamma_\mu \) for every \( \gamma_\mu \) in the formula. Hence, a complex unconstrained Killing spinor equation (2.11) becomes:

\[ D_\mu \epsilon = \frac{a}{2} \gamma_\mu \epsilon, \] (A.1)

its solution (2.17) becomes:

\[ \epsilon = e^{\frac{1}{2}ax^\alpha \gamma_\alpha} \left( 1 + \frac{1}{2}ax^\alpha \gamma_\alpha (1 - \gamma_\alpha) \right) \epsilon_0, \] (A.2)

symplectic Majorana Killing spinor equation (2.12) becomes:

\[ D_\mu \epsilon_i = M_{ij} \frac{a}{2} \gamma_\mu \epsilon_j, \] (A.3)
and its solution (2.18) becomes:

\[ \epsilon_i = \left( e^{\frac{1}{2} \alpha r M \gamma_i} \right)_{ij} \left( \delta_{jk} + \frac{1}{2} a x^\alpha \gamma_\alpha (M_{jk} - \delta_{jk} \gamma_r) \right) \xi_k. \] (A.4)

Other formulas, such as supersymmetry transformation rules, have to be checked carefully when changing signatures so that the properties of the fields, e.g. real or symplectic Majorana, are satisfied by the transformations. When this is done carefully, we find that the Yang-Mills theory transformation rules (3.5) become:

\[ \delta A_\mu = -i \eta^I \gamma_\mu \chi_i \]
\[ \delta \phi = \mathcal{M} \chi_i \]
\[ \delta \chi_i = (-\sigma_{\mu\nu} F^{\mu\nu} + i \mathcal{D}/\phi) \eta_i + 2ia \phi M_{ij} \eta_j, \] (A.5)

the conformal scalar theory transformation rules (4.2) become:

\[ \delta \phi^I = -i (\sigma^I \epsilon)_{ij} \epsilon \lambda_j \]
\[ \delta \lambda_i = (\epsilon \sigma^I)_{ij} \phi^j \epsilon_j - a^2 (\sigma^I T)_{ij} \epsilon_j \phi^I, \] (A.6)

and the massive scalar theory transformation rules (5.2) become:

\[ \delta \phi^I = -i (\sigma^I \epsilon)_{ij} \epsilon \lambda_j \]
\[ \delta \lambda_i = (\epsilon \sigma^I)_{ij} \phi^j \epsilon_j - a^2 (\sigma^I T)_{ij} \epsilon_j \phi^I + i a \mu (M \epsilon \sigma^I)_{ij} \epsilon_j \phi^I. \] (A.7)

### B Isometry transformations for spinors on AdS

In this section we will attempt to prove that the action of a free, massless spinor on AdS is invariant under

\[ \delta \psi = K^\mu D_\mu \psi + \frac{1}{4} D^\mu K^{\nu \mu \nu} \psi \] (B.1)

where \( K^\mu \) is an \( O(d - 1, 2) \) Killing vector. This proof remains true in any spacetime dimension and for any spinor whose action is given by the usual Lagrangian

\[ \mathcal{L} = i \bar{\psi} \mathcal{D} \psi. \] (B.2)

Although \( O(d - 1, 2) \) Killing vectors and their properties on AdS can be studied independently of the Killing spinors, we will use the definition of Killing vectors through Killing spinors (true only in some dimensions) as a shortcut to establish the following property of the Killing vectors:

\[ D_\mu D_\nu K_\rho = D_\mu D_\nu (\bar{\tau}_1 \gamma_\mu \phi_2) = a D_\mu (\bar{\tau}_1 \gamma_{\nu \mu} \phi_2) = a^2 (g_{\mu \rho} K_\nu - g_{\mu \nu} K_\rho) \] (B.3)

where in the intermediate steps we used the Killing spinor equation (2.11) and the properties of the Clifford algebra

\[ \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu \nu}. \] (B.4)
Note that although the intermediate steps in eq. (B.3) involve Killing spinors, the final result is expressed only in terms of the Killing vectors. Hence, this is a general property of the $O(d - 1, 2)$ Killing vectors. Similarly, we can establish another Killing vector property:

$$D_\mu K_\nu = -D_\nu K_\mu. \quad (B.5)$$

Now, let us vary the free action by

$$\delta_1 \psi = K^\mu D_\mu \psi \quad (B.6)$$

which yields:

$$\delta_1 \left( \overline{\psi} D \psi \right) = -\overline{\psi} K^\mu D_\mu \psi + \overline{\psi} K^\mu D_\mu \psi + \overline{\psi} \gamma_\mu D_\mu \psi D^K \mu
$$

$$= \overline{\psi} \gamma^\nu [D_\nu, D_\mu] \psi K^\mu + \overline{\psi} \gamma_\nu D_\mu \psi D^K \mu \quad (B.7)$$

$$= a^2 d - 1 \overline{\psi} \gamma_\mu \psi K^K + \overline{\psi} \gamma_\mu D_\mu \psi D^K \mu,$$

where to go from line 2 to line 3 we used the Ricci identity, eq. (2.14). Thus, it is clear that the above transformation is not a symmetry of the Lagrangian. Now, let us vary the action by

$$\delta_2 \psi = D^K \gamma^K \mu \psi \quad (B.8)$$

which gives

$$\delta_2 \left( \overline{\psi} D \psi \right) = -D^K \gamma^K \mu \psi D_\mu \psi + D^K \gamma^K \mu \psi D_\mu \psi + D^K \gamma^K \mu \psi K^K \mu
$$

$$= 2 D^K \gamma^K \mu \psi (\gamma^K \mu \gamma^K \mu - \gamma^K \mu \gamma^K \mu) D_\mu \psi + 2a^2 (d - 1) \overline{\psi} \gamma_\mu \psi K^K \mu
$$

$$= -4 \overline{\psi} \gamma_\mu D_\mu \psi D^K \mu - 2a^2 (d - 1) \overline{\psi} \gamma_\mu \psi K^K \mu \quad (B.9)$$

where to go from line 1 to line 2 we used the properties of the Clifford algebra and of the Killing vectors, eq. (B.3). Therefore, it is now clear that the action of a free, massless spinor on AdS is invariant under

$$\delta \psi = K^K \gamma^K \mu \psi \quad \text{under} \quad (\delta_1 + \delta_2) \psi = K^K \mu D^K \mu \psi + \frac{1}{4} D^K \mu K^K \mu \psi. \quad (B.10)$$

Let us finally note that for $d = 5$, the preceding proof can be applied verbatim to the case of the symplectic Majorana spinors if we note that on AdS$_5$

$$K^K = \bar{\epsilon}_{2i} \gamma^K \epsilon_{2i}$$

$$D^K \gamma^K \mu = a M_{ij} \bar{\epsilon}_{2i} \gamma^K \epsilon_{2j} = -D^K \gamma^K \mu$$

$$D^K \gamma^K \mu = a^2 (g^K \mu \gamma^K \mu - g^K \mu \gamma^K \mu) \gamma^K \mu. \quad (B.11)$$

Note that although the intermediate steps in eq. (B.3) involve Killing spinors, the final result is expressed only in terms of the Killing vectors. Hence, this is a general property of the $O(d - 1, 2)$ Killing vectors. Similarly, we can establish another Killing vector property:

$$D_\mu K_\nu = -D_\nu K_\mu. \quad (B.5)$$

Now, let us vary the free action by

$$\delta_1 \psi = K^\mu D_\mu \psi \quad (B.6)$$

which yields:

$$\delta_1 \left( \overline{\psi} D \psi \right) = -\overline{\psi} K^\mu D_\mu \psi + \overline{\psi} K^\mu D_\mu \psi + \overline{\psi} \gamma_\mu D_\mu \psi D^K \mu
$$

$$= \overline{\psi} \gamma^\nu [D_\nu, D_\mu] \psi K^K \mu + \overline{\psi} \gamma_\nu D_\mu \psi D^K \mu \quad (B.7)$$

$$= a^2 d - 1 \overline{\psi} \gamma_\mu \psi K^K \mu + \overline{\psi} \gamma_\mu D_\mu \psi D^K \mu,$$

where to go from line 2 to line 3 we used the Ricci identity, eq. (2.14). Thus, it is clear that the above transformation is not a symmetry of the Lagrangian. Now, let us vary the action by

$$\delta_2 \psi = D^K \gamma^K \mu \psi \quad (B.8)$$

which gives

$$\delta_2 \left( \overline{\psi} D \psi \right) = -D^K \gamma^K \mu \psi D_\mu \psi + D^K \gamma^K \mu \psi D_\mu \psi + D^K \gamma^K \mu \psi K^K \mu
$$

$$= 2 D^K \gamma^K \mu \psi (\gamma^K \mu \gamma^K \mu - \gamma^K \mu \gamma^K \mu) D_\mu \psi + 2a^2 (d - 1) \overline{\psi} \gamma_\mu \psi K^K \mu
$$

$$= -4 \overline{\psi} \gamma_\mu D_\mu \psi D^K \mu - 2a^2 (d - 1) \overline{\psi} \gamma_\mu \psi K^K \mu \quad (B.9)$$

where to go from line 1 to line 2 we used the properties of the Clifford algebra and of the Killing vectors, eq. (B.3). Therefore, it is now clear that the action of a free, massless spinor on AdS is invariant under

$$\delta \psi = K^K \gamma^K \mu \psi \quad \text{under} \quad (\delta_1 + \delta_2) \psi = K^K \mu D^K \mu \psi + \frac{1}{4} D^K \mu K^K \mu \psi. \quad (B.10)$$

Let us finally note that for $d = 5$, the preceding proof can be applied verbatim to the case of the symplectic Majorana spinors if we note that on AdS$_5$

$$K^K = \bar{\epsilon}_{2i} \gamma^K \epsilon_{2i}$$

$$D^K \gamma^K \mu = a M_{ij} \bar{\epsilon}_{2i} \gamma^K \epsilon_{2j} = -D^K \gamma^K \mu$$

$$D^K \gamma^K \mu = a^2 (g^K \mu \gamma^K \mu - g^K \mu \gamma^K \mu) \gamma^K \mu. \quad (B.11)$$
C Results in \( d \neq 5 \)

In this section, we give some results for dimensions other than 5. These results hold for those dimensions where symplectic Majorana spinors can be defined and the charge conjugation matrix can be chosen to satisfy eq. (2.10). We know [4] that both conditions can be satisfied in \( d = 5, 6 \mod 8 \). Also, it is conceivable that similar approach has to be taken for \( d = 8, 9 \mod 8 \) as the only way to define Majorana spinors there is to take a symmetric charge conjugation matrix that satisfies eq. (2.10). However, it is important to realize that the transformation rules given below do not describe supersymmetry in dimensions above 5, but instead describe some accidental symmetry of the free non-interacting Lagrangian. We give the transformation rules and particle masses consistent with these transformations for the theories discussed in the paper. Note that the spinor algebra following from these transformations will change because Fierz identities take different from in different dimensions.

**Yang-Mills theory** is invariant under

\[
\begin{align*}
\delta A_\mu &= i\sigma^I\gamma_\mu \chi_I \\
\delta \phi &= i\sigma^I\chi_I \\
\delta \chi_i &= (\sigma_{\mu\nu} F^{\mu\nu} - \partial \phi) \eta_i - ia(d - 3) \phi M_{ij} \eta_j
\end{align*}
\]  
(C.1)

with mass parameters

\[
\mu = -\frac{d - 4}{4} M, \quad m^2 = -2(d - 3).
\]  
(C.2)

**Conformal scalar theory** is invariant under

\[
\begin{align*}
\delta \phi^I &= (\sigma^I \epsilon)_{ij} \bar{\epsilon}^j \lambda_j \\
\delta \lambda_i &= i(\epsilon \sigma^I)_{ji} \bar{\phi}^I \epsilon_j + a \frac{d-2}{2} (\sigma^I T \epsilon M)_{ij} \epsilon_j \phi^I
\end{align*}
\]  
(C.3)

with mass of the scalars given by

\[
m^2 = -\frac{d(d - 2)}{4},
\]  
(C.4)

which is exactly the mass of the conformally coupled scalar in dimension \( d \) [7, 14].

**Massive scalar theory** is invariant under

\[
\begin{align*}
\delta \phi^I &= (\sigma^I \epsilon)_{ij} \bar{\epsilon}^j \lambda_j \\
\delta \lambda_i &= i(\epsilon \sigma^I)_{ji} \bar{\phi}^I \epsilon_j + a \frac{d-2}{2} (\sigma^I T \epsilon M)_{ij} \epsilon_j \phi^I + a \mu (M \epsilon \sigma^I \epsilon)_{ij} \epsilon_j \phi^I
\end{align*}
\]  
(C.5)

with the scalar masses given by

\[
m^2 = \mu^2 + \mu - \frac{d(d - 2)}{4}, \quad \mu^2 - \mu - \frac{d(d - 2)}{4}
\]  
(C.6)

each with multiplicity 2.
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