Our response to the response hep-th/0608109 by Drummond

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We have carefully examined all the points raised by Drummond in his response hep-th/0608109 to our paper hep-th/0608109. We concede that Drummond is correct in claiming the non-existence of \( R^{-4} \) and \( R^{-5} \)-order effective string actions in the parity conserving sector, though only insofar as equivalence of field theories is considered at the classical level; the situation is unclear when quantum equivalence is taken into consideration. We still maintain the existence of such terms in the parity violating sector. Nevertheless we point out that all this has no consequence for our original proof of the nonexistence of order-\( R^{-3} \) terms. Apart from this we refute Drummond’s claims about our alleged use of field redefinitions as well as his criticism of our dropping \( R^{-4} \) terms in our analysis. We reject his contention that our work is merely a partial reconstruction of his original results and that our work contains technical and conceptual errors.

We do acknowledge the importance of the absence of terms pointed out by Drummond.

I. PRELIMINARY COMMENTS

There are several components to Drummond’s response to our work. We appreciate the very professional nature of his response as well as the fact that he has given full details of all his derivations making it easy for us to be focussed in our ‘response to the response’. We now give our replies, but before doing so point out a few typos in our original paper.

A. Typographical errors in our original paper

Eqn(17) of [2] should read:

\[
\frac{2}{a^2} \partial_+ Y^\mu = -4 \frac{\beta}{R^3} \partial_+^2 \partial_+^2 Y^\mu
-4 \frac{\beta}{R^3} \left\{ \partial_+^2 \{ \partial_+^2 Y^\mu(e_+ \cdot \partial_+ Y + e_- \cdot \partial_+ Y) \} + \partial_+^2 \{ \partial_+^2 Y^\mu(e_+ \cdot \partial_+ Y + e_- \cdot \partial_+ Y) \} - e_+^\mu \partial_-(\partial_+^2 \partial_+ Y - e_+^\mu \partial_+(\partial_+^2 Y - \partial_+^2 Y). \right. \quad (1)
\]

Eqn(18) of [2] should read:

\[
\frac{2}{a^2} \partial_+ Y_1^\mu = 4 \frac{\beta}{R^3} \left( e_+^\mu \partial_+ Y_0 \cdot \partial_+^2 Y_0 + e_-^\mu \partial_+^2 Y_0 \cdot \partial_+ Y_0 - \partial_+^2 Y_0 e_+ \cdot \partial_+^2 Y_0 - \partial_+ Y_0 e_- \partial_+^2 Y_0 \right). \quad (2)
\]

B. Fluctuation Field \( Y \) and \( X \)-uniformity

With no loss of generality one can make a functional shift and set \( X^\mu = X^{cl}_1 + Y^\mu \) where \( X^{cl}_1 = R(e_+^\mu \tau^+ + e_-^\mu \tau^-) \) which is a solution of the equation of motion (EOM) of the free part of the action. Since all additional terms in the effective action involve higher derivatives, \( X^{cl}_1 \) continues to be a solution of the full EOM, and \( Y^\mu \) continues to have the interpretation of a fluctuation around a classical background.

While carrying out field redefinitions of the \( Y \)-field, care should be taken not to upset the \((0,0)\) nature of the \( X \)-field. As this property is not explicit in the \( Y \)-formulation, it is very easy to upset.

Since the fluctuation field \( Y \) derives from the \( X \)-field, one could consider the following principle of “\( X \)-uniformity”: All expressions involving \( Y^\mu \) and its derivatives, \( e_+^\mu, R \) must be such they are derivable from expressions involving \( X^{cl}_1 \) and its derivatives only.

Many field redefinitions of the \( Y \)-field will not be permissible if this principle is adopted. In fact the field redefinitions found by Drummond involving \( R^{-2} \) and \( R^{-3} \) terms of the Polchinski-Strominger (PS) action are of a type that do not satisfy this principle.

If this principle of \( X \)-uniformity is applied, the \((0,0)\) property of \( Y^\mu \) is never upset. In fact, without this principle, it seems unclear how to define precisely the \((0,0)\) property for the fluctuation field. Still, this does not necessarily mean that such a definition is impossible in case \( X \)-uniformity is not imposed.

Further motivation for the imposition of \( X \)-uniformity is as follows. When it is broken, the fluctuation field \( Y \) and derived spectrum thereof no longer have a direct interpretation in terms of excitations about some background, since of course it is not possible in such a case to write down the background in question. Without such an interpretation, the physical significance or meaning of the fluctuations may be difficult to appreciate.

On the other hand, if one is given an effective theory describing fluctuations, there is a priori no clear reason why one should not define the field differently, and indeed one could set out with such an effective theory without even having seen the original \( X \)-field.

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II. EQUIVALENCE OF FIELD THEORIES

As many of the issues raised in this discussion centre around the issue of equivalence of field theories under a change of variables (what is called field redefinition) we review here known facts and establish some new points in the context of effective string theories.

As far as effective string theories are concerned two distinct possibilities exist as to their interpretation as quantum theories. One is to treat them as a tool to calculate the so-called tree diagrams only. In the early days of effective chiral symmetric field theories (of pions, for example) this was the attitude taken; in this case, equivalence of field theories would only mean equivalence as in classical field theories, as explained below in section II A.

Now, it is well known in chiral perturbation theory that such a limited approach would have missed many essential features like universal logarithmic behaviour and would have made a discussion of analytic properties of scattering amplitudes beyond perview. The more modern approach is to make the calculation of even loop amplitudes meaningful and the lack of renormalisability is handled through a larger set of arbitrary parameters. If we interpret the effective string theories this way, all features of a full quantum field theory are to be considered.

A. Classical field theories

Equivalence in classical field theories is easily proved; see for example [4] for an explicit demonstration for the case of point transformations, namely, cases where the redefinition does not involve time derivatives. At a classical level this can be extended also to more general redefinitions. This classical equivalence means that for such things as the currents and energy-momentum tensor, either using the lagrangian formalism on the original lagrangian and then transforming the fields to a new set gives the same results as first transforming the fields and then applying the formalism. The point is that the change of variables and the lagrangian formalism commute, and a field redefinition can thus be done at will to simplify the procedure without further thought.

When the redefinition in question is an infinitesimal one it is possible, classically, to eliminate terms in the action that are proportional to the leading order (in an appropriate sense) EOM. Equivalently, it can be stated that two classical actions differing from each other by EOM terms in this sense are physically equivalent. Therefore the PS prescription [5] of dropping all EOM terms as irrelevant is certainly justified at a classical level. The important question is to what extent this procedure is also valid quantum-mechanically. We are able to make some progress on this issue at least within the path integral formulation. We argue, somewhat formally, that up to order $R^{-3}$ classical equivalence also implies quantum equivalence, but beyond that things are uncertain. This means that our original proof of the absence of order-$R^{-3}$ corrections to the spectrum is still valid and complete. It also means that the claim of absence of order-$R^{-4}$ and $-R^{-5}$ corrections in the action made by Drummond in [6] is certainly true classically but its validity quantum-mechanically is not certain.

B. Canonical formulation of quantum field theories

Proofs of equivalence in the canonical, or operator, formalism of quantum field theories are not completely straightforward, the main obstacle being the non-commutativity of various operator expressions; this has been very clearly analysed in [4]. Although in that paper they do manage to prove the quantum equivalence for point transformations, such a proof for field redefinitions that go beyond point transformations does not seem to exist. In the context of effective string theories the field redefinitions considered are certainly not of the point-transformation type.

We shall therefore analyse this issue within the path integral formalism. There is of course an intimate connection (and usually assumed equivalence) between the operator and path integral formulations of quantum field theories. Though there are very important structural and conceptual differences, the final results for physically relevant issues are expected to be the same. This equivalence between canonical and path integral formulations is subtle and delicate as shown long ago by Lee and Yang [6].

C. Path integral formulation

In the path integral formulation, given a particular field definition, there are three important issues: the action; the transformation laws leaving the action invariant; the invariance of the measure under said transformation laws. For non-linear transformation laws the transformation of the measure can of course be quite involved.

In the case of the PS transformation law, it is easy to see that the naive measure $DX$ is indeed not invariant. We calculate now the change in the naive measure under the PS transformation law. Consider the relation between the untransformed field $Y^\mu$ and the (infinitesimally) transformed field $Y'^\mu$;
\[ Y'^\mu = Y^\mu + R e^\mu + e^- \partial_- Y^\mu + \frac{\beta}{R} a^2 \epsilon^\mu_+ \partial_+^2 \epsilon + \frac{\beta}{R^2} a^2 \partial_+^2 \epsilon [\partial_+ Y^\mu + 2 \epsilon^\mu_+ (e_- \partial_+ Y + e_+ \partial_- Y)] \\
+ \frac{2 \beta}{R^3} a^2 \partial_+^2 \epsilon [2 (e_+ \partial_- Y + e_- \partial_+ Y) + \partial_+ Y \cdot \partial_- Y] + \cdots. \]  

Denoting the naive measure by \(DY\), its change under the above transformation is \(DY' = J DY\), where \(J\) is the determinant of \(J^\mu_\nu (\xi, \xi') = \frac{\delta Y'^\nu}{\delta Y^\mu (\xi)}\) given by

\[ J^\mu_\nu (\xi, \xi') = \delta^\mu_\nu \delta^2 (\xi - \xi') + \delta^\mu_\nu \epsilon^\mu_\nu \partial_- \delta^2 (\xi - \xi') + \frac{2 \beta}{R^2} a^2 \partial_+^2 \epsilon (\epsilon^\mu_\nu e^\nu \partial_- \delta^2 + \epsilon^\nu \epsilon^\mu_\nu \partial_+ \delta^2) \\
+ \frac{2 \beta}{R^3} a^2 \partial_+^2 \epsilon \partial_+ Y^\mu (e_+ \nu \partial_- \delta^2 + \epsilon^\nu \partial_+ \delta^2) + \cdots. \]

The appearance of \(\delta^2 (\xi - \xi')\) and its derivatives makes the evaluation of \(J\) very delicate and a proper regularisation scheme, like for example the \(\zeta\)-function regularisation, is needed to do this carefully. There are two possible fates for these singular terms.

One possibility is that the singular terms from the measure could cancel against singular non-covariant expressions arising on a careful use of Feynman rules in the apparently covariant path integral. This is what was shown in [6]. The same situation shows up in perturbative quantum gravity also in [3].

The other possibility is that there are singular terms in the measure which do not vanish or cancel in the above fashion, leaving non-trivial factors to be taken into account. The only relevant changes are contained in the \(Y\)-dependent part of \(J\) and these come from those parts of the transformation law having at least quadratic terms in the fluctuation field \(Y^\mu\).

We conclude that at least formally, by which we mean that of course a careful regularised definition of the path integral in principle should be taken into account, one can say that the variation of the measure under the PS transformation law is order-\(R^{-3}\) and hence does not affect our original work where only variations that are of order \(R^{-2}\) are relevant. Nevertheless, the important point evident from the above expression is that when any analysis is carried out for order-\(R^{-4}\) terms and beyond, this could become an important issue and must be handled with care.

D. Jacobian for field redefinitions

When a generic field redefinition (change of variables in the path integral) is made, the new action is obtained from the old by simple substitution. The new transformation laws are what are induced by the redefinition. Unless the field redefinition is invertible it is not possible to work out the induced transformation laws. For infinitesimal changes in the fields, this can always be done.

Of course, this is not the whole picture; the change in the measure for path integration consequent to the change in variables should also be taken into account. Simple counting arguments of the type developed by us in [2] and improved upon by Drummond in [1] show that the most general field redefinitions have the structure

\[ \Delta Y^\mu = c_1 (R)[Y] + c_2 (R)[YY] + c_3 (R)[YYY] + \cdots \]

where \([Y], [YY], \ldots\) symbolically denote terms that are linear, quadratic, etc. in the \(Y\)-field. Further it can be shown that the coefficients \(c_n (R)\) can at most be of order \(R^{-(n+1)}\). The dominant field redefinition in fact has the form

\[ Y'^\mu = Y^\mu + \frac{\alpha}{R^2} \partial_+ Y^\mu + \frac{\beta}{R^3} \partial_- Y^\mu (e_+ \partial_+ Y + e_- \partial_- Y) + \cdots \]

The Jacobian for this transformation can again be formally computed as before. Again we encounter singular expressions and careful regularisation is required to make further progress. The non-trivial part of the Jacobian, by which we mean the \(Y\)-dependent terms, can only come from terms that are \textit{cubic} in the \(Y\)-field in the field redefinition and those are of order \(R^{-4}\). Once again these are irrelevant for the proof presented in [2].

We conclude this section with two points. Firstly, as far as the proof of the absence of \(R^{-3}\) corrections to the Nambu-Goto spectrum is concerned, quantum equivalence is the same as classical equivalence.

Secondly, if a careful evaluation of these Jacobians leads to local terms, one may be able to use Drummond’s results for absence of terms to order \(R^{-5}\) to show that classical equivalence implies quantum equivalence up to this higher order. A rough argument in support of this could be constructed by making field redefinitions and performing partial integration as done by Drummond to eliminate all terms up to order \(R^{-6}\). An inspection of
the field redefinitions carried out by Drummond in [1] reveals that at least formally the resultant jacobian also does not get any contributions to order $R^{-5}$. Nevertheless, this is only a sketch and we can have confidence in it only after it is carefully established. Until then, quantum equivalence to orders $R^{-4}$ and beyond should be taken as unproven.

### III. ORDER-$R^{-4}, R^{-5}$ TERMS IN THE EFFECTIVE STRING ACTION

In our proof of the absence of $R^{-3}$ corrections to the spectrum it was necessary to prove the absence of any extra order $R^{-3}$ terms in the action for effective strings beyond those already there by carrying out the expansion of the Polchinski-Strominger action to this order. For this we developed a systematic way of analysing the order at which various action terms could arise. In this analysis, in which we considered both parity-conserving as well as parity-violating terms (for the sake of completeness), we proved the result, crucial for analysis, that there are no order $R^{-3}$ terms. This enabled us to consistently expand the PS action [2] to order $R^{-3}$, and their stress tensor to order $R^{-2}$.

Though we did not state it explicitly in [2], this also implies that it is enough to expand the transformation law given by PS to include order-$R^{-2}$ terms. Incidentally, the transformation law is closed (satisfying the Virasoro algebra) to order $R^{-3}$; only at order $R^{-4}$ do the PS transformations fail to close.

Our analysis showed the potential existence of order $R^{-4}, R^{-5}$ terms. In [3] Drummond had claimed, without any systematic analysis, that the next corrections to the PS action would be order $R^{-6}$ and he listed four such terms. We had criticised this in [2].

Now we have thoroughly re-examined this question and agree with Drummond that our potential $R^{-5}$ and $R^{-4}$ terms in the parity conserving sector can all be reduced to order $R^{-6}$ terms modulo irrelevant terms (which according to us can simply be dropped provided the transformation laws are modified to keep all the relevant terms invariant).

#### A. Parity violating sector

In the parity violating sector, our claims of the existence of $R^{-4}, R^{-5}$ terms remain valid. We deal with this issue in some detail now. In the $D = 3$ case we had shown that

$$L^{-3} \epsilon_{\mu_1 \mu_2 \mu_3} \partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial_+ X^{\mu_3} \partial_+ X \cdot \delta^3 X$$

is potentially of order $R^{-3}$. This had been omitted in [3]. However around the classical background $X_\mu = e_\mu R + \tilde{e}_\mu R^{-1} X^\mu$, this becomes

$$- 8 R^{-3} \epsilon_{\mu_1 \mu_2 \mu_3} e_+^{\mu_1} e_+^{\mu_2} \partial_+ X^{\mu_3} e_+ \cdot \delta_3 Y$$

which we had shown can be reduced by partial integration to

$$8 R^{-3} \epsilon_{\mu_1 \mu_2 \mu_3} e_+^{\mu_1} e_+^{\mu_2} \partial_+ X^{\mu_3} e_+ \cdot \delta_3 Y$$

and hence rendered irrelevant. We consequently dropped them (for a discussion of Drummond’s criticism of this and our defense, see sections [IV] and [V]). As we were only interested in showing the non-existence of relevant $R^{-3}$ terms in [2], this was sufficient. In view of the present discussion we now analyse the $R^{-4}$ and $R^{-5}$ terms also. For that we give an improved treatment; by partial integration we can recast (7) as

$$L^{-3} \epsilon_{\mu_1 \mu_2 \mu_3} \partial_- X^{\mu_2} \partial_- (\partial_+ X^{\mu_1} \partial_+ X^{\mu_3} \partial_+ X \cdot \delta^2 X$$

$$L^{-3} \epsilon_{\mu_1 \mu_2 \mu_3} \partial_+ X^{\mu_1} \partial_+ X^{\mu_2} \partial_+ X^{\mu_3} \partial_+ X \cdot \delta^2 X$$

$$3L^{-4} \epsilon_{\mu_1 \mu_2 \mu_3} \partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial_+ X^{\mu_3} (\partial_-^2 X \cdot \partial_+ X)^2.$$ (10)

The first line produces irrelevant terms of order $R^{-3}$ and higher. We will discuss later how these and other irrelevant terms are treated while responding to criticisms on the use of field redefinitions. Both the remaining lines are of order $R^{-4}$ and higher. It should be noted that this has been done without any redefinition of fluctuation field and thus automatically obeys our principle of $X$-uniformity introduced in section [II].

If we do make a redefinition of the $Y$ field, we may further recast the $R^{-4}$ terms as

$$\frac{8}{R^2} \epsilon_{\mu_1 \mu_2 \mu_3} e_+^{\mu_1} e_+^{\mu_2} Y^{\mu_3} e_+ \cdot \partial_+ Y \{ \partial_+^2 Y^{\mu_2} - 6 e_+^{\mu_2} e_+ \cdot \partial_+^2 Y \}$$

(11)

These can again be reduced to irrelevant terms by partial integration, and therefore $R^{-4}$ terms may also be eliminated, at the expense of $X$-uniformity. Nevertheless, the $R^{-5}$ terms remain and we see no way to get rid of them, at least as of now.

In the case of $D = 4$ parity violating case

$$L^{-2} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial_+ X^{\mu_3} \partial_+ X^{\mu_4}$$

(12)

the order $R^{-2}$ term $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} e_+^{\mu_1} e_+^{\mu_2} \partial_+ Y^{\mu_3} \partial_+^2 Y^{\mu_4}$, which we had said could be reduced to an irrelevant term in [2], can actually be eliminated by partial integration as it reduces to

$$\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} e_+^{\mu_1} e_+^{\mu_2} \partial_+ Y^{\mu_3} \partial_+ Y^{\mu_4}$$

(13)

due to the complete antisymmetry of the $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$. This makes no difference for the way we have handled the irrelevant terms. For Drummond this would make a difference as what was thought to be an irrelevant term, necessitating a field redefinition and the attendant induced transformation law, is actually shown to vanish.

For the rest we cannot go beyond what we already presented in [2]. Thus in the parity violating sector there are indeed $R^{-4}, R^{-5}$ order terms.

We return later to the issue of how the irrelevant terms at order $R^{-3}$ should be handled in the $D = 4$ context also.

B. Irrelevance of the non-existence of $R^{-4}, R^{-5}$ terms to our original proof

Though we concede that we were in error in claiming that Drummond had missed these terms, and we also concede that the absence of these terms is very important for the issue of even higher order corrections to the spectrum, we point out that it is irrelevant for the main result of our paper which was to show that there are no order $R^{-3}$ corrections to the spectrum, as compared to Nambu-Goto (NG) theory.

Of course Drummond also has this result in \cite{3}, but the important point is he did not prove the absence of $R^{-3}$ terms. In that paper he merely asserted, though correctly, that there were no further action terms up to order $R^{-6}$. Considering the importance of the result in question, a systematic proof is absolutely necessary. In retrospect, this is how we should have worded our criticism of \cite{3}.

In his response \cite{1}, Drummond does prove systematically all his earlier claims; but to do so he employs the very methods of systematic analysis that we had developed in \cite{2}. Of course, he has added some improvements here by analysing in terms of pairs of contracted $X$ fields, and by his observation that for odd $N$ the condition $2M + 2 \geq 3N$ can be improved to $2M + 2 \geq 3N + 1$.

It is also worth pointing out that though his demonstration of the absence of $R^{-4}, R^{-5}$ order terms is certainly important, it is not of much use until the PS transformation laws that leave this combination of relevant and irrelevant terms invariant (modulo issues of quantum equivalence discussed earlier).

IV. OUR ALLEGED USE OF FIELD REDEFINITIONS

Drummond has argued that, notwithstanding our statements in \cite{2}, we have actually used field redefinitions of the same type used by him. Our reply to this follows (we further substantiate comments made here in section VI).

Drummond cites our treatment of the $D = 3$ parity violating case (and possibly our $D = 4$ parity violating case also) to make this point. If he insists on taking the parity violating cases seriously, then his claim that the next corrections occur only at $R^{-6}$ is already false, as elaborated above.

Even taking the parity violating cases seriously, what we have done does not amount to a field redefinition but only to a choice of field definition. This is an important distinction which we now explain fully.

A. Choice of field definition

The PS procedure can be stated algorithmically as follows: Firstly, write down all possible $(1,1)$ terms (in the sense used by PS); Secondly, discard all terms proportional to the leading order constraints and their derivatives; Finally, use integration by parts to relate equivalent terms.

At this point one will have terms with and without ‘mixed derivatives’, terms sporting mixed derivatives being what we have called irrelevant. The PS prescription then is to discard all irrelevant terms and find transformation laws that leave the relevant terms in the action invariant.

This is still ambiguous as not all the relevant terms are independent and some can be related to others through integration by parts and additional irrelevant terms. The unambiguous method is to first express all the relevant terms in terms of a particular choice of a minimal set (this choice in itself being arbitrary) and additional irrelevant terms. Next, one chooses a subset of the irrelevant terms and simply drops all the rest. Finally, one finds transformation laws that leave this combination of relevant and irrelevant terms invariant (modulo issues of quantum equivalence discussed earlier).

Now a particular choice of mixed derivative terms amounts to a choice of field definition. A different choice of the irrelevant terms amounts to yet another choice of field parametrisation. As long as the conditions for equivalence under field redefinitions hold, one can go from one parametrisation to another with the help of a field redefinition. The transformation laws in the new definition can be worked out as induced by the field redefinition (this is somewhat more than mere substitution).

Let us illustrate this with an example. Take the partial set of terms at order $R^{-2}$ to be

$$
\alpha \frac{\partial^2 X \cdot \partial^2 X}{L} + \frac{\beta \partial^2 X \cdot \partial_- X \partial^2 X \cdot \partial_+ X}{L^2} \\
\delta \frac{\partial_+ X \cdot \partial_- X}{L} + \eta \frac{\partial_- X \cdot \partial_+ X \partial_+ X \cdot \partial_- X}{L^2}
$$

In \cite{3} it is claimed that the two relevant actions here are equivalent modulo total derivatives. This is not true and this has some bearing on the issues discussed; using the identity

$$
\frac{\partial^2 X \cdot \partial^2 X}{L} = \frac{\partial^2 X \cdot \partial_- X \partial^2 X \cdot \partial_+ X}{L^2} \\
\frac{\partial_+ X \cdot \partial_- X}{L} - \frac{\partial_- X \cdot \partial_+ X \partial_+ X \cdot \partial_- X}{L^2} \\
+ \partial_- \left( \frac{\partial^2 X \cdot \partial_- X}{L} \right) - \partial_+ \left( \frac{\partial_+ X \cdot \partial_- X}{L} \right)
$$
\[ \text{can be rewritten as} \]

\[ \beta_{\gamma} \frac{\partial^2 X \cdot \partial_+ X}{L^2} + \delta_{\gamma} \frac{\partial_+ X \cdot \partial_+ X}{L} + \eta_{\gamma} \frac{\partial_+ X \cdot \partial_+ X \cdot \partial_+ X}{L^2} + \frac{\rho}{L} \]

\[ \text{with } \beta' = \beta + \alpha, \delta' = \delta + \alpha \text{ and } \eta' = \eta - \alpha. \]

Now the choice \( \delta' = 0, \eta' = 0 \) yields one field definition, say, \( X' \) with the effective action

\[ \beta \frac{\partial^2 X' \cdot \partial_+ X}{L^2} \]

while the choice \( \beta' = \alpha, \delta' = \alpha \) and \( \eta' = -\alpha \) gives another field definition, say, \( X^* \) with the effective action

\[ \frac{\partial^2 X^*}{L} \]

Thus specific choices for coefficients of the mixed derivative terms merely pick out specific parametrisations and have nothing to do with redefinitions. Having chosen a particular parametrisation, one has to work out the transformation laws leaving the action invariant, the consequent stress tensors, and so on.

In this particular example, though the two forms of relevant terms are related by a non-trivial field redefinition, the transformation laws for the two cases are identical. In fact there are families of field redefinitions that do not induce any additional terms in the transformation laws; for examples see section [IVC]. Of course, in generic cases field redefinitions induce additional terms in the transformation laws.

Care has to be taken to carry through this procedure consistently to all orders. This has to do with the fact that with field redefinitions of order \( \epsilon \) there is a residual error of order \( \epsilon^2 \) which should be included in the analysis of the most general effective action to be carried out to the next higher order.

**B. Other remarks regarding field redefinitions**

In [2], we had remarked that “subteleties regarding the field redefinition in eqns (2.17-2.19) and how this leads to a non-trivial energy momentum tensor are ignored.” To this Drummond’s response [1] has been: "Their claim that the field redefinition obscures the derivation of the energy momentum tensor in [3] is spurious - one simply rewrites expression (2.16) of [3] in terms of \( \tilde{Y} \)." Furthermore, Drummond claims that questions of changes in the path integral measure are unnecessary for the field redefinitions made by him in [3].

As for issues of measures and Jacobians, we have tried to address these in section [IVD] above. It appears that indeed these may not be issues of consequence to the orders required in proving the absence of corrections to the spectra in [3] and [2], but it is clear that these are matters that could not be merely asserted.

Turning now to the above rejection of our claim as spurious, there are some subtle issues here which we now discuss. We feel that Drummond has not quite appreciated our point about “... how this leads to a nontrivial energy momentum tensor ...”. It may be that our wording was opaque. It is of course true that substitution of \( \tilde{Y} \) in the PS energy momentum tensor yields his eqn.(2.20), but the subtlety we were referring to had to do with the fact that his redefinition eqn.(2.19) reduced the PS action modulo the terms of order higher than \( R^{-3} \) to the particular free action

\[ \mathcal{L}^{(0)} = \frac{1}{4\pi a^2} \partial_+ \tilde{Y} \cdot \partial_+ \tilde{Y} \]

and looking at this action, the casual reader would then have expected the standard stress tensor

\[ -\frac{1}{2a^2} \partial_+ \tilde{Y} \cdot \partial_+ \tilde{Y}. \]

Instead he has a different and non-trivial energy momentum tensor. What we meant was that he has ignored the issue of why eqn.(2.20) is not the correct choice and why the non-trivial energy momentum tensor he obtains by substituting his field redefinition is the correct choice. Clearly the issue we had in mind is not so spurious. We in fact resolve this issue in section [VI].

It is indeed not enough, as Drummond says, for the theory to remain conformal with critical central charge and non-trivial energy momentum tensor. Under field redefinitions central charge certainly cannot change, and neither can the conformal nature of a theory; this is somewhat beside the point, as the issue is really whether or not there are nontrivial effects on the spectrum.

An additional claim made by Drummond is that our procedure of iteratively solving field equations is exactly equivalent to his procedure involving field redefinition. Drummond does make some astute and interesting remarks about the similarities between the two methods, but we do not agree with his assertion that they are equivalent for the following reasons.

In our approach, based on the iterative solution to the EOM, we have no need to rework either the action or the transformation laws. In Drummond’s approach, both of these have to be done, though he did not compute the induced transformation law (see section [IVC]): he needs to compute the action to know about the two-point function relevant for the OPE calculation. He also needs to compute the transformation law to know whether the energy-momentum tensor obtained by substitution is the canonical one or not.

In our method, as we explain in section [IVD], we make use of no field redefinition. In consequence, there are no issues of quantum equivalence. In this respect, differences between the two procedures would be more pronounced at higher order. Indeed, iterative solutions to field equations are possible (in principle) to any order, but as Drummond himself admits, his trick really works only at order \( R^{-3} \).
In general solving the full EOM iteratively involves non-locality. It is a fortuitous circumstance here (perhaps because of 2-d) that it is quasi-local. Field redefinitions are typically local (not necessarily point transformations) by contrast, with nonlocal field redefinitions being a largely uninvestigated and difficult subject.

All one could have concluded had one designed a field redefinition based on the iterative solution to the full equations of motion is that $\bar{Y}$ would be solution of the EOM of eqn. (19), but of course that does not guarantee that the action is given by eqn. (19). There are many actions (including the quadratic part of the terms in PS) whose EOM is satisfied by such a redefined field. Significantly, they generically have different two-point functions and consequently different OPEs. It also does not guarantee that the transformation law is the standard one associated with free actions. One would have had to recompute both these, though Drummond did not do the latter. In his case, while the induced action turns out to be free, the induced transformation law turns out to be non-trivial. Our method did not require the consideration of these issues.

C. Further points on field redefinitions

All field redefinitions are ambiguous up to terms of the type

$$\Delta X^\mu = N^\mu, \quad N \cdot E = 0 \quad (21)$$

where $E^\mu$ is the EOM.

Not all field redefinitions induce changes in transformation laws. Some examples are:

$$\Delta_1 X^\mu = \frac{\partial_{+} X^\mu}{L}$$
$$\Delta_2 X^\mu = \frac{\partial_{+} X^\mu \partial_{+} X \cdot \partial_{-} X + \partial_{-} X^\mu \partial_{+} X \cdot \partial_{+} X}{L^2}$$
$$\Delta_3 X^\mu = \frac{\partial_{+} X^\mu \partial_{+} X \cdot \partial_{-} X - \partial_{-} X^\mu \partial_{+} X \cdot \partial_{+} X}{L^2}$$

It should be noted that $\Delta_3 X^\mu$ is of the form of an ambiguity. It is so even when $L^{-2}$ is replaced by any power of $L$, but the transformation law is unaffected only for $L^{-2}$. Only the choice $L^{-2}$ maintains the $(0,0)$ character of $X^\mu$.

V. OPE OF STRESS TENSOR AND VIRASORO ALGEBRA TO HIGHER ORDER

Drummond criticises by saying “It should be pointed out that DM do not bother to verify the operator product of the energy-momentum tensor and hence that the expression for the Virasoro generators really satisfies the Virasoro algebra to the next order.” In fact, slightly more can be said: although we did not mention it, we had noticed that the variation of the entire PS action, eqn (1) of 2, without truncating to any order, under the entire PS variation, eqn(2) of 2, again without any truncations, has only $\beta L$ terms and these are of order $R^{-4}$. Furthermore, the untruncated PS transformation has the closure property of Virasoro algebra also to order $R^{-4}$ as is very easily verified. The terms that spoil closure are again $\beta L$ terms. Taken together these guarantee that there are no issues with either the OPE of stress tensors or of the validity of the Virasoro algebra generated by their moments to the order to which we extended the PS results. The fact that our higher order energy-momentum tensor is conserved is another consistency check.

VI. OUR DROPPING ORDER $R^{-4}$ TERMS

Drummond has raised objections to our dropping order $R^{-4}$ terms in investigating the possible $R^{-3}$ corrections to the spectrum. His main argument here is that in his field definition he has shown that the PS action is of order $R^{-3}$ and that neglecting these $R^{-4}$ terms can only be done after taking due account of the changes in the transformation law induced by the field redenfition that allowed reducing the PS action to $O(R^{-4})$. Consequently, he claims, our treatment of the parity violating terms where we simply drop the $R^{-2}, R^{-3}$ terms as being reducible to mixed derivative terms is incorrect.

Again, as stated earlier, what we have done is drop all the irrelevant terms, and find the symmetry variations for what is retained. This means that, to the order we were originally interested in, the only relevant action terms are the PS term and the free action; the PS transformation law leaves them invariant, actually to order $R^{-3}$. Noting this, nothing more need be done.

As we have explained, our treatment merely amounts to a specific choice of field parametrisation and not to a redefinition. We feel that the distinction we made between these in section [IV.A] is an important one. In contrast to this straightforward procedure, Drummond suggests that we should have first determined the transformation laws that would leave invariant the free and PS terms along with the irrelevant terms in the parity violation action, then carried out the field redenfition that would remove the irrelevant terms, and finally computed the modifications to the transformation laws. We now show through explicit calculation of what he is proposing that these extra ingredients are totally unnecessary and our treatment of these and other similar terms in 2 is perfectly legitimate.

For this purpose consider a generic irrelevant term

$$\mathcal{L}_{irr} = F^\mu \partial_{+} X_\mu \quad (22)$$

where $F^\mu$ can be any general expression constructed out of derivatives of $X^\mu$. In particular it can also contain additional mixed derivative terms. Now we wish to modify the transformation law so that it leaves $\mathcal{L}_{irr}$ invariant. That this can always be done follows from the fact that the variation of the EOM is proportional to the
EOM (in a functional sense) which is just another way of saying the EOM is covariant under symmetry variations. Now we evaluate the variation of $\mathcal{L}_{\text{irr}}$ under

$$\delta^{(0)} X^\mu = \epsilon \partial_- X^\mu,$$

yielding

$$\delta^{(0)} \mathcal{L}_{\text{irr}} = \delta^{(0)} \mathcal{F}^\mu \partial_+ X_\mu + \mathcal{F}^\mu \delta^{(0)} (\partial_+ X_\mu) = \delta^{(0)} \mathcal{F}^\mu \partial_+ X_\mu - \epsilon \partial_- \mathcal{F}^\mu (\partial_+ X_\mu),$$

where we have dropped total derivative terms as usual. Therefore the modification to the transformation law under which $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{irr}}$ will be invariant is

$$\delta' X^\mu = 2\pi a^2 (\delta^{(0)} \mathcal{F}^\mu - \epsilon \partial_- \mathcal{F}^\mu).$$

On the other hand the field redefinition needed to remove the irrelevant term is

$$\Delta X^\mu = -2\pi a^2 \mathcal{F}^\mu.$$ (26)

This induces the additional terms

$$\tilde{\delta} X^\mu = \delta^{(0)} \Delta X^\mu - \epsilon \partial_- (\Delta X^\mu) = -2\pi a^2 \delta^{(0)} \mathcal{F}^\mu - \epsilon \partial_- \mathcal{F}^\mu$$

and this contribution exactly cancels the modification demanded by the irrelevant term.

Generalisation of these arguments to include an arbitrary number of relevant terms in the action is straightforward as long as one keeps track of the order of terms carefully.

We conclude that what Drummond claims to be the correct procedure is completely equivalent to simply dropping all irrelevant terms and working with the transformation laws that leave only the relevant terms invariant, which is exactly what we did in [2]. Thereafter dropping the irrelevant parity violating terms one was left with just the free action and PS terms as relevant and as the PS transformation laws expanded to the next higher order left them invariant, there was nothing wrong with our methodology.

In support of his argument Drummond says that if he had followed a similar procedure he would have dropped the entire PS action which, according to him, is not permissible. In the following section we instead take this suggestion seriously.

### VII. DROPPING THE PS ACTION

In the previous section we responded to some criticisms made by Drummond about our dropping of $\mathcal{O}(R^{-4})$ terms. He suggested that our methods could lead to our discarding of the whole PS action. In particular his argument seems to be that the PS action has left its trace in terms of a non-trivial energy momentum tensor and a non-trivial transformation law (though he did not work this out) remembering the parameters (i.e. $\beta$) of the irrelevant terms.

Following this argument, according to him our irrelevant terms also could in principle have left their non-trivial traces, which are simply not there in our way of handling things. Consider, though, that we have explicitly and quite generally proved in the previous section that the field redefinitions exactly compensate any modifications to the transformation laws brought forth by the irrelevant terms: How can these two apparently contradicting situations be reconciled?

Before we go on we wish to categorically state that the $R^{-4}$ order terms are totally irrelevant for our analysis in [2]. This is also true for Drummond’s analysis in [3]. The simple reason for this is that the variation of such terms under the PS transformation law can *at most* be of order $R^{-3}$ and consequently their contribution to the Virasoro generators can also be at most $R^{-3}$. The imprints in the transformation law that Drummond is possibly referring to are all from $R^{-2}$ or $R^{-3}$ terms.

Now, let us go on and examine the case of PS action more carefully. If we had simply dropped the irrelevant terms from the PS action we would have ended up only with (modulo $R^{-4}$ and higher order terms)

$$\frac{1}{2\pi a^2} \int \partial_+ Y \cdot \partial_- Y$$ (28)

Now according to our prescription we should just work with this relevant action and the transformation laws leaving it invariant. Somewhat surprisingly, it turns out that not just the standard

$$\delta^{(0)} Y^\mu = \epsilon \partial_- Y^\mu,$$

but *at least* the entire two-parameter family

$$\tilde{\delta}_{\epsilon} Y^\mu = \epsilon \partial_- Y^\mu + \epsilon e^\mu_+ R + \epsilon e^\mu_- R^{-1} \partial_+ \epsilon^\mu_+ - 2\epsilon \partial_+ e^\mu_+ e_+ \cdot \partial_- Y$$ (30)

leaves the above action invariant (exactly). Also, they constitute the conformal group (for fixed $R$ and $\beta'$) as can be verified by

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_{12}}; \quad \epsilon_{12} = \epsilon_1 \partial_+ \epsilon_2 - \epsilon_2 \partial_+ \epsilon_1$$ (31)

It should be emphasised that at this stage the parameter $\beta'$ has nothing to do with $\beta$.

The resulting canonical energy-momentum tensor is indeed the same with $\beta$ replaced by $\beta'$ as eqn(19) of [2] or eqn(2.20) of [3] (after correcting a sign error). The central charge depends on the free parameter $\beta'$ and can be adjusted for consistency in all dimensions.

On the other hand, if we had included the irrelevant terms of Drummond, the transformation law that would have left the above combination of relevant and irrelevant terms invariant would have been modified to

$$\delta^{\text{tot}} = \tilde{\delta}_{\epsilon} Y^\mu + \frac{\beta a^2}{R} \partial_+ \epsilon^\mu_+ e_+ \cdot \partial_- Y$$ (32)
If now one had done a field redefinition to remove the irrelevant terms there would be additional terms induced in the above equation. As shown above their effect would be to cancel the $\beta$-dependent terms and just leave eqn. (30).

If Drummond had computed the transformation law to which the PS-transformation law would have been modified by the field redefinition (which he actually did not, for all his admonitions to us the importance of working out the induced terms in transformation laws for field redefinitions) he would have found them to be just eqn. (32) with $\beta'$ replaced by $\beta$, modulo some harmless terms arising out of his not having paid attention to the many ambiguities in field redefinitions.

The question now is: How did $\beta'$ get to be replaced by $\beta$ which is a parameter belonging entirely to the irrelevant terms? The resolution is the following. According to the above construction what would leave the combination of relevant and irrelevant terms of this example would be eqn. (22), but this is clearly not the PS-transformation law expanded to suitable orders! Nevertheless in Drummond’s scheme of things he would have taken this to be PS-transformation suitably expanded. This is consistent with eqn. (22) if and only if $\beta'$ had been equated to $\beta$.

It seems then that, contrary to folklore, a free bosonic string theory with an adjustable central charge can be made consistent provided the conformal transformation law is chosen appropriately. This can be viewed as an alternative approach to the spectrum of free strings that appears consistent in all dimensions. How far such an approach to string theory can be extended further is currently under investigation [9].

**VIII. CONCLUSION**

We have examined a variety of issues bearing on our earlier work proving the absence of $R^{-3}$ corrections to the Nambu-Goto spectrum in effective string theories in light of Drummond’s criticisms of our paper. Here we summarise the main points.

We agree that as far as classical equivalence of field theories is concerned, terms alleged to have existed at order $R^{-4}, R^{-5}$ in effective string theories, can really be transformed away as irrelevant terms by integration by parts and field redefinitions in the parity conserving sector. In the parity violating sector no such conclusions can be drawn even classically. Quantum-mechanically, one has to exercise greater care in the transformation of measure (or equivalent technique in the canonical formalism) and at this point it is not at all clear whether order-$R^{-4}, R^{-5}$ terms play a rôle. In his original work Drummond only asserted the absence of these terms. While he did prove them systematically in his response to our work, these systematic proofs made use of the very techniques we developed in our original paper. The existence or otherwise of these terms is irrelevant for our original proof of the absence of $R^{-3}$ corrections to NG-spectrum in effective string theories. Drummond also had this result but he only asserted and did not prove the crucial absence of additional terms in the action at $R^{-3}$ order (apart from what are already contained in the PS action). We stress that what we really showed in [2] was that given the PS action and transformation laws there are no $R^{-3}$ corrections to the NG-spectrum staying entirely within the PS parametrisation.

We have carefully analysed Drummond’s remarks about field redefinition and show them to be incorrect through general proofs as well as explicit constructions.

In conclusion we wish to state that at least as far as we are concerned this debate with Drummond has enhanced our understanding of a variety of important issues connected with effective string theories.

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