The coefficient smoothing method application to the problem of gas pipeline glaciation

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Abstract. In northern seas the gas temperature in the pipeline may be lower than the water-ice phase transition temperature, so the glaciation process must be considered. The coefficient smoothing method for the problem of glaciation of the cylinder immersed in seawater is considered. The model of glaciation process is presented as a problem for the linear heat equation in domain with unknown moving boundary. The method of solution is based on the transition to the Dirichlet problem for the nonlinear two-dimensional heat equation in domain with fixed boundaries. The splitting method is applied to the solution of the problem for two-dimensional heat equation. Two approximations for the Dirac delta function are proposed. Calculation of uniform glaciation process was carried out with the using of nonlinear implicit finite-difference schemes. Time moments of the glaciation for different spatial layers are obtained. Results of calculations are compared with the results obtained by the front-tracking method.

1. Introduction
In northern seas during the process of gas transportation through the pipelines, gas temperature may be lower than the water-ice transition temperature [1, 2], so the pipeline glaciation becomes possible, when the gas flow cools down below a certain temperature. The problem of the heat transfer in the pipe and surrounding media is considered [3, 4, 5]. In [3] a model for the steady state heat transfer between the gas and pipeline surroundings is considered. The proposed model does not take into account time dependent heat accumulation. In [4] the model of unsteady heat transfer is proposed as an initial boundary value problem for the one-dimensional heat equation without phase changes. In [5] these models are compared to the cases of onshore and offshore pipelines. The results of the onshore buried pipeline demonstrate that the steady heat transfer model over predicts the amplitude of temperature changes. For the case of offshore pipeline some results were obtained only for the case of unsteady model. But it must be noted, that for the correct modelling of the pipeline glaciation, phase transitions, such as seawater-ice transition, must be taken into account. So in computational modelling of this process initial boundary value problem with moving unknown boundary and the Stefan condition must be considered [1].

There are many works dedicated to the numerical solution of Stefan problem, based on two main approaches. First is the front-tracking method (FTM), when the position of the phase boundary is continuously tracked. In this method, finite-difference or finite-element schemes with deformed or adaptive grids are employed and the explicit tracking of phase front is provided. There are many various realizations of this approach. In [6] the method with variable time step, when the phase boundary move from one space node to another is realized. Murray and Ladis in
provide an approach based on space grid with variable step, but with fixed number of space intervals between fixed and moving boundary. Crank in [8] discussed various finite-difference methods with variable space and time steps and a change of space variable to fix the moving boundary. Ermolaeva and Kurbatova in [1, 2] proposed FTM to the gas pipeline glaciation problem. As it is mentioned in [9], these methods are effective only in the one-dimensional case, an extension to the two- and three-dimensional problems leads to various difficulties.

Another approach to numerical solution is based on the fixed domain formulation. An example of this approach is a coefficient smoothing method (CSM) proposed in [10, 11, 12], which transform problem for the linear heat equation with Stefan condition on moving boundary to the problem for nonlinear heat equation with fixed boundary and smoothed nonlinear coefficients. In this formulation the Stefan condition on the phase boundary is automatically satisfied and the phase boundary is defined by value of temperature equal to the temperature of phase transition.

Presented paper devoted to the application of CSM to the problem of gas pipeline glaciation. In section 2 the problem is considered. In section 3 the method is described. In section 4 the splitting method is discussed. In section 5 the results of the solution of the uniform glaciation problem are presented and discussed. Some concluding remarks are made in section 6.

2. Mathematical model of gas pipeline glaciation
Mathematical model of pipeline glaciation is based on the following assumptions [1, 2]: glaciation process only in cross-section of the pipeline is considered and normal component of heat flux vector from the seawater to the front line of glaciation is considered as constant.

The two-dimensional heat equation for the temperature distribution in an ice layer in pipeline cross-section is written in polar coordinates as:

\[
S \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r k \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left( k \frac{\partial T}{\partial \varphi} \right),
\]

where \( T = T(t, r, \varphi) \) is the temperature, \( t \) is the time, \( r \) is a radial coordinate, \( \varphi \) is a polar angle, \( S = cp \), \( c \) is a specific heat (per volume unit), \( \rho \) is the density of sea ice, \( k \) is the coefficient of thermal conductivity. Eq. (1) is considered on time interval \( (0, T_{end}) \) and in space domain \( \Omega = \{(r, \varphi)| \varphi \in [0, 2\pi], r \in (R, \hat{R}(t, \varphi))\} \), where \( R \) is a radius of the pipeline and the function \( \hat{R}(t, \varphi) \) characterize the phase change boundary \( G = \{(r, \varphi)| \varphi \in [0, 2\pi], r = \hat{R}(t, \varphi)\} \) (fig. 1).

Boundary and initial conditions for eq. (1) are written as:

\[
T(t, R, \varphi) = T_0(t), \quad T(t, \hat{R}, \varphi) = T_\ast, \quad \hat{R}(0, \varphi) = R,
\]

where \( T_\ast \) is the phase transition temperature, \( T_0(t) \) is a known temperature of gas in pipeline estimated after gas flow calculations [2]:

\[
T_0(t) = \frac{m_1}{m_2 + t} + m_3,
\]

where \( m_1 = 15120 \), \( m_2 = 5040 \), \( m_3 = 268 \).

On the boundary of phase transition \( G \) the following Stefan condition [8, 10] is stated:

\[
\left. k \frac{\partial T}{\partial n} \right|_{G^+} = q_2 = Q \rho \frac{d}{dt} F(t, \hat{R}, \varphi),
\]

where \( Q \) is the latent heat of fusion of ice, \( q_2 \) is a constant normal component of heat flux from seawater to the boundary \( G \), \( dF(t, \hat{R}, \varphi)/dt \) is a normal component of the velocity of the boundary \( G \). In addition to conditions (2),(3) the periodic condition for \( T \) is stated:

\[
T(t, r, \varphi + 2\pi) = T(t, r, \varphi).
\]

As it can be seen, presented problem (1)–(4) is a problem for the linear eq. (1) in domain \( \Omega \) with unknown moving boundary \( G \), defined by conditions (2),(3).
3. The coefficient smoothing method

According to the approach, presented by A. A. Samarskii [10] condition (3) is implicitly included in eq. (1) and nonlinear equation is considered in domain with known steady boundaries.

Let us consider new domain $\Psi = \{(r, \varphi) | \varphi \in [0, 2\pi), r \in (R, R_1)\}$, where constant $R_1$ characterize radial fictitious boundary situated in seawater. It is assumed that in $\Psi$ the phase change boundary $G$ is situated and two phases — ice and seawater are considered. The phases are defined by constant densities $\rho_1$ and $\rho_2$, specific heat coefficients $c_1$ and $c_2$ and coefficients of thermal conductivity $k_1$ and $k_2$ respectively (fig. 2).

According to [10], the new function is introduced: $\tilde{S}(T) = S + \rho Q \tilde{\delta}(T - T^*), \ |T - T^*| < \Delta$, where $\Delta > 0$ characterize interval $(T_s - \Delta, T_s + \Delta)$, and $\tilde{\delta} \neq 0$. According to this approach, $S$ and $k$ are replaced by following nonlinear functions:

$$
\tilde{S}(T) = \begin{cases} 
S_1, & T < T_s - \Delta, \\
\frac{S_1 + S_2}{2} + \rho Q \delta(T - T_s, \Delta), & |T - T_s| < \Delta, \\
S_2, & T > T_s + \Delta,
\end{cases}
$$

$$
\tilde{k}(T) = \begin{cases} 
\frac{k_1 + k_2}{2}, & |T - T_s| < \Delta, \\
k_1, & T < T_s - \Delta, \\
k_2, & T > T_s + \Delta.
\end{cases}
$$

where $S_i = c_i \rho_i, \ i = 1, 2$.

In this way eq. (1) is replaced by following nonlinear equation:

$$
\tilde{S}(T) \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \tilde{k}(T) \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left( \tilde{k}(T) \frac{\partial T}{\partial \varphi} \right),
$$

in domain $\Psi$, with the following Dirichlet boundary conditions:

$$
T(t, R, \varphi) = T_0(t), \quad T(t, R_1, \varphi) = T_w,
$$

where $T_w$ is a temperature of seawater.

The initial condition for eq. (5) in $\Psi$ is stated as:

$$
T(0, r, \varphi) = f(r, \varphi).
$$

The phase change boundary $G$, situated in $\Psi$ is defined in the process of calculation by following condition:

$$
T(t, \tilde{R}(t, \varphi), \varphi) = T_s.
$$
4. The splitting method
According to [13], the differential operator in right part of eq. (5) is represented as the sum of two operators, defined by formulas:

\[ A_1(\ldots) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \tilde{k}(\ldots) \frac{\partial(\ldots)}{\partial r} \right), \quad A_2(\ldots) = \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left( \tilde{k}(\ldots) \frac{\partial(\ldots)}{\partial \varphi} \right). \]

Let us construct a computational grid consist of nodes \((t_n,r_k,\varphi_p)\), where \(r_m = mh_r, \varphi_p = ph_\varphi, t_n = n\tau, m = 0, \ldots, M, p = 0, \ldots, P, n = 0, \ldots, N\), where \(M, P, N\) are positive integers, \(h_r, h_\varphi, \tau\) are the steps on the spatial coordinates and time.

During the numerical solution on time interval \([t_n, t_{n+1}]\) two one-dimensional problems are considered. The first problem is considered for the function \(T_1 = T_1(t, r, \varphi)\), where \(\varphi\) is considered as a parameter. The problem for \(T_1\) is written as:

\[ \tilde{S}(T_1) \frac{\partial T_1}{\partial t} = A_1(T_1), \quad T_1(t_n, r, \varphi) = T_0(t), \quad T_1(t, \varphi, R) = T_0(t), \quad T_1(t, R_1, \varphi) = T_w. \tag{8} \]

The second problem is considered with operator \(A_2\) and its solution is noted as \(T_2 = T_2(t, r, \varphi)\), where \(r\) considered as a parameter. The problem is stated as:

\[ \tilde{S}(T_2) \frac{\partial T_2}{\partial t} = A_2(T_2), \quad T_2(t_n, r, \varphi) = T_1(t_{n+1}, r, \varphi), \quad T_2(t, r, 0) = T_2(t, r, 2\pi). \tag{9} \]

As it can be seen, the problem (8) at fixed \(\varphi\) and problem (9) at fixed \(r\) are considered as one-dimensional. These problems are solved on interval \((t_n, t_{n+1})\) sequentially. On the first step problem (8) is solved and function \(T_1\) is obtained at moment \(t_{n+1}\). This function is considered as initial condition for problem (9). After the solution of (9) the value of \(T\) at moment \(t_{n+1}\) is obtained as: \(T(t_{n+1}, r, \varphi) \approx T_2(t_{n+1}, r, \varphi)\).

Heat equation from problem (8) is discretized by implicit scheme, proposed in [14]:

\[ \tilde{S}(T_{1,m}^n) \frac{T_{1,m}^{n+1} - T_{1,m}^n}{\tau} = \frac{1}{h_r m} \left( a_{1,m+1} \frac{T_{1,m+1}^{n+1} - T_{1,m}^{n+1}}{h_r} - a_{1,m} \frac{T_{1,m}^{n+1} - T_{1,m-1}^{n+1}}{h_r} \right), \tag{10} \]

where \(T_{1,m}^n \approx T_1(t_n, r_m, \varphi), m = 2, \ldots, M - 1, \) values of \(\varphi\) are considered on grid constructed in \([0, 2\pi]\) with step \(h_\varphi, a_{1,m} = 0.5(r_m \tilde{k}(T_{1,m}^n) + r_{m-1} \tilde{k}(T_{1,m-1}^n))\). The scheme is realized with boundary conditions obtained from (8): \(T_{1,1}^{n+1} = T_0(t_{n+1}), \quad T_{1,M}^{n+1} = T_w\) by solution of system of linear algebraic equations by the tridiagonal matrix algorithm.

Heat equation from problem (9) is approximated by the same scheme:

\[ r_m^2 \tilde{S}(T_{2,p}^n) \frac{T_{2,p}^{n+1} - T_{2,p}^n}{\tau} = \frac{1}{\varphi} \left( a_{2,p+1} \frac{T_{2,p+1}^{n+1} - T_{2,p}^{n+1}}{h_\varphi} - a_{2,p} \frac{T_{2,p}^{n+1} - T_{2,p-1}^{n+1}}{h_\varphi} \right), \tag{11} \]

where \(T_{2,p}^n \approx T_2(t_n, r_p, \varphi), p = 2, \ldots, P - 1,\) values of \(r\) are considered on previously defined grid, \(a_{2,p} = 0.5(\tilde{k}(T_{2,p}^n) + \tilde{k}(T_{2,p-1}^n))\). Scheme (11) is solved with periodic conditions: \(T_{2,1}^{n+1} = T_{2,P}^{n+1}\). System of linear equations (11) for \(T_{2,p}^{n+1}\) is solved by circular tridiagonal matrix algorithm.

Delta function is approximated on \((T_s - \Delta, T_s + \Delta)\) by constant and quadratic functions:

\[ \tilde{\delta}(T - T_s, \Delta) = \begin{cases} \frac{1}{2\Delta}, & |T - T_s| < \Delta, \\ 0, & |T - T_s| > \Delta, \end{cases} \tag{12} \]

\[ \tilde{\delta}(T - T_s, \Delta) = \begin{cases} \frac{3}{4\Delta^2} (\Delta^2 - (T - T_s)^2), & |T - T_s| < \Delta, \\ 0, & |T - T_s| > \Delta, \end{cases} \tag{13} \]

the value of \(\Delta\) was equal to 0.01 during all calculations.
Table 1. The time of glaciation of spatial layers \([R, R + l]\) obtained by FTM (min.) [2].

| Layer \((l)\) | 1 mm | 1 cm | 2 cm | 3 cm | 4 cm | 5 cm | 6 cm |
|-------------|------|------|------|------|------|------|------|
| Time        | 8.6  | 113.5| 298.9| 571.0| 944.6| 1435.9| 2063.6|

Table 2. The time of glaciation of spatial layers of length \(R + l\) obtained by the CSM at various \(R_2\) and delta-functions approximations \((12)–(13)\) (min.).

| \(R_2\) | 1 cm | 1.25 cm | 1.5 cm |
|---------|------|---------|--------|
| Layer \((l)\) | \((12)\) | \((13)\) | \((12)\) | \((13)\) | \((12)\) | \((13)\) |
| 1 mm    | 8.7450| 8.8110  | \textbf{8.6460} | 8.7120 | 8.7780 | 8.8330 |
| 1 cm    | 114.96| 115.61  | \textbf{113.55} | 114.09 | 112.82 | 113.35 |
| 2 cm    | 302.54| 304.16  | \textbf{299.03} | 300.19 | 295.92 | 296.88 |
| 3 cm    | 578.25| 582.66  | \textbf{567.38} | 569.92 | 558.61 | 560.43 |
| 4 cm    | \textbf{953.77} | 962.15 | 931.09 | 936.60 | 912.25 | 915.90 |
| 5 cm    | \textbf{1438.3} | 1454.4 | 1400.4 | 1411.6 | 1367.4 | 1375.1 |
| 6 cm    | \textbf{2061.2} | 2088.2 | 1978.0 | 1998.4 | 1939.5 | 1954.5 |

Figure 3. The relative errors of the time of glaciation of layers \([R, R + l]\). 1 and 2 — case of \(R_2 = 1.5\) cm, 3, 4 — \(R_2 = 1.25\) cm, 5, 6 — \(R_2 = 1\) cm. 1, 3, 5 corresponds to quadratic approximation of Dirac delta function, 2, 4, 6 — to constant approximation.

5. Solution of the problem of uniform glaciation

The problem of uniform glaciation is solved in domain \(\Psi\) with the initial condition, which is independent on \(\varphi\): \(T(0, r, \varphi) = f(r)\), where \(f(r)\) is considered as a solution of the symmetrical stationary problem for the heat equation in \(\Psi\), when \(T = T(r)\):

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0, \quad T(R) = T_0(0), \quad T(R_1) = T_w. \tag{14}
\]

Solution of (14) is written as: \(T(r) = A \ln(r) + B\), where constants \(A\) and \(B\) are defined from boundary conditions. The value of \(R_1\) is defined as \(R_1 = R + l + R_2\), where interval \([R, R + l]\),
$l > 0$ characterize the thickness of ice layer (e.g., 1 cm) and value of $R_2$ is varied. Calculations are realized with the following values of the parameters of ice-water system, presented in SI system [2]: $R = 0.67$, $c_1 = 2100$, $\rho_1 = 928$, $Q = 335000$, $k_1 = 2.3$, $c_2 = 4200$, $\rho_2 = 1025$, $k_2 = 0.56$, $T_\ast = 271$.

In table 1 results of the solution of one-dimensional problem for uniform glaciation obtained by FTM in [2] are presented.

Calculations by CSM are performed on various time and spatial grids. In order to compare with the results presented in table 1, the time interval of 2200 minutes is considered. At table 2 the results of the calculation on spatial grid with $500 \times 500$ nodes and time grid with $2 \cdot 10^5$ nodes are presented. The cases of various values of $R_2$ and approximations of delta function (12) and (13) are considered. The results closed to results from table 1 are presented by bold font. For the comparison of the results obtained by two methods for every ice layer we compute the values of following ratio: $\varepsilon = |t_{ft} - t_{cs}|/t_{ft}$, where $t_{ft}$ is a glaciation time obtained by FTM, $t_{cs}$ is a time obtained by CSM. Subroutine with realization of the algorithm of FTM is tested on different Stefan problems with analytical solutions and demonstrate a high accuracy, so the values of $\varepsilon$ may be considered as relative errors for the results obtained by CSM. In the fig. 3 the plots of these relative errors for different values of $R_2$ and approximations (12) and (13) are presented. As it can be seen, the using of quadratic approximation (13) is better only for the case of $l = 3$ cm and $R_2 = 1.25$ cm. In other cases values approximately equal to the presented in table 1 are obtained in the case of (12) and $R_2 = 1$ and 1.25 cm respectively.

6. Conclusion

Presented paper is devoted to the numerical solution of the glaciation problem for the gas pipeline. CSM proposed by A. A. Samarskii is considered. The method is based on the transition to the Dirichlet problem for the nonlinear two-dimensional heat equation in domain with fixed boundaries. The splitting method is applied to the solution of the problem for two-dimensional heat equation. Two approximations of the Dirac delta function are proposed. Calculation of uniform glaciation process was carried out with the using of nonlinear implicit finite-difference schemes. Time moments of the glaciation for different spatial layers are obtained. Results of calculation are compared with the results obtained by the FTM.

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