What is the Equivalence theorem really?

Palash B. Pal

Center for Particle Physics, University of Texas, Austin, TX 78712, USA

Abstract

The precise statement of the equivalence theorem, between the longitudinally polarized states of a massive gauge boson and the corresponding goldstone mode, is discussed when the amplitude in question depends on masses of other particles (e.g., the Higgs boson) in the theory.

1 Introduction

In spontaneously broken gauge theories, some gauge bosons acquire mass. As a consequence, a longitudinal degree of polarization appears for each of them, which is accompanied by the disappearance of a spin-0 boson from the spectrum of the unbroken Lagrangian. Thus, in a vague sense, this spin-0 goldstone mode is “eaten up” by the gauge boson and it manifests itself as the longitudinal polarization state in the broken phase. One therefore vaguely expects that the amplitude of any process involving the longitudinal vector boson is equivalent to the amplitude of the same process calculated with the goldstone mode replaced in the outer lines for the longitudinal vector boson lines. This statement is formalized into the Equivalence theorem. There exist various precise statements of this theorem. The purpose of this article is to discuss them and suggest one which covers all known cases and follows from simple physical arguments.

Although the equivalence theorem should be valid for gauge models in general, we use the standard electroweak model here for the sake of notational simplicity. Here, $W^\pm$ and $Z$ are the massive gauge bosons, and let us denote by $w^\pm$ and $z$ the Higgs-Goldstone modes eaten up by them. We can write for any amplitude

$$A(W_\parallel^\pm, Z_\parallel, \cdots) = A(w^\pm, z, \cdots) + A', \quad (1)$$

where the subscript on the gauge bosons imply their longitudinal polarization states, and the ellipsis denote any other particle in the model, which include the Higgs boson and the fermions. If this equation is taken to be the definition of the quantity $A'$, equivalence theorem is in effect a statement about its value. Most simply and commonly, one encounters the following statement about $A'$:

$$A' = O(M_W/E), \quad (2)$$

where $E$ is the energy of the gauge boson. This means that, at energies much larger than $M_W$, one can replace the amplitude involving $W_\parallel^\pm$ and $Z_\parallel$ by the amplitude involving $w^\pm$ and $z$.

This statement is sufficient when the gauge boson is the only massive particle in the amplitude, but is incomplete and useless if the relevant amplitude depends on the mass of the Higgs boson and the fermions. One needs to know the behaviour of $A'$ as a function of these masses in order to make use of the theorem. For example, one can ask whether at a given value of $E$, the quantity $A'$ can be neglected for any value of the Higgs mass $M_H$, or does $M_H$ has to be in some special range? Ad hoc answers to such questions have been given by various authors. In what follows, we give examples to show that all such prescriptions can be understood in one simple statement of the equivalence theorem.

First, let us explain why one needs $E \gg M_W$ in order that $A'$ can be neglected. For any value of $M_W \neq 0$, the longitudinal components of the $W^\pm$-bosons are physical states. The “Higgs-Goldstone” modes $w^\pm$ are unphysical. On the other hand, for $M_W = 0$, the situation reverses since the $w^\pm$ are the physical states but the $W_\parallel^\pm$ are not,
as discussed below. The equivalence theorem then merely states that all observables are continuous in the limit \( M_W \to 0 \). In other words, in that limit, the amplitudes for any process with \( W_{\parallel}^\pm \) bosons is the same (apart from a phase maybe) with the amplitudes for the corresponding processes where the \( W_{\parallel}^\pm \) are replaced by \( w^\pm \). The limit \( M_W \to 0 \), of course, is realized physically if one deals with energies for which \( M_W/E \to 0 \).

Now, the \( W \)-mass is given by

\[
M_W = \frac{1}{2} g v ,
\]

where \( g \) is the SU(2)\(_L\) coupling constant and \( v \) is the vacuum expectation value (VEV) of the Higgs field. Thus, the mathematical limit of vanishing \( W \)-mass can be obtained in two ways:

\[
M_W \to 0 \quad \text{if} \quad \left\{ \begin{array}{l}
egthinspace \text{either } g \to 0 \\
egthinspace \text{or } v \to 0. \end{array} \right.
\]

Bagger and Schmidt \([5]\) clearly emphasized that the equivalence theorem has different interpretation in these two limits. For \( g = 0 \), we have only a global symmetry, so the Higgs mechanism should not work and the \( w^\pm \) and \( z \) should be physical states. On the other hand, for \( v = 0 \), we have a gauge theory whose symmetry is not broken, and this is why the \( w^\pm \) and \( z \) appear as physical states.

We will show that each of these cases yields a statement of the equivalence theorem, and in the presence of Higgs and fermion masses the two statements are different in the sense that they imply the equality of different parts of the amplitudes \( A(W_{\parallel}^\pm, Z_{\parallel}, \cdots) \) and \( A(w^\pm, z, \cdots) \). Moreover, whereas different ad-hoc prescriptions have been suggested to modify the statement of equivalence theorem as given in Eqs. \((1)\) and \((2)\), we will show that all these statements can be derived from the two limits given in Eqs. \((3)\) and \((4)\).

## 2 Examples of various processes

In this section, we consider the amplitudes of various \( 2 \to 2 \) scattering processes involving longitudinal gauge bosons and of the corresponding processes involving the Higgs-Goldstone modes. In particular, we discuss amplitudes which depend, in addition to the gauge boson masses, on the Higgs boson mass only, and show in what sense the equivalence theorem is vindicated in these calculations.

### 2.1 \( W_{\parallel}^+W_{\parallel}^- \to HH \)

Consider, for example, the process \( W_{\parallel}^+W_{\parallel}^- \to HH \). Calculation of the amplitude exists in the literature \([3]\) \([4]\). In terms of the usual Mandelstam variables \( s, t \) and \( u \), one obtains

\[
A(W_{\parallel}^+W_{\parallel}^- \to HH) = \frac{g^2}{4(1-4M_W^2/s)} \left\{ \frac{M_H^2}{M_W^2} \left[ 1 + \frac{3M_H^2}{s - M_H^2} + M_H^2 \left( \frac{1}{t - M_W^2} + \frac{1}{u - M_W^2} \right) \right] \\
+ 2 \left[ 1 - \frac{9M_H^2}{s - M_H^2} + 4 \frac{M_H^2}{s} \left( 1 + \frac{3M_H^2}{s - M_H^2} \right) \right] \\
+ 2 \left[ s - 2M_H^2 - 4M_W^2 + 8M_W^2/s \right] \left( \frac{1}{t - M_W^2} + \frac{1}{u - M_W^2} \right) \right\} ,
\]

whereas \([4]\)

\[
A(w^+w^- \to HH) = \frac{g^2}{4} \left\{ \frac{M_H^2}{M_W^2} \left[ 1 + \frac{3M_H^2}{s - M_H^2} + M_H^2 \left( \frac{1}{t - M_W^2} + \frac{1}{u - M_W^2} \right) \right] \\
+ 2 \left[ 1 + (s - M_H^2) \left( \frac{1}{t - M_W^2} + \frac{1}{u - M_W^2} \right) \right] \right\} ,
\]
As the authors of Ref. 8 observe, the terms enhanced by $M_H^2/M_W^2$ agree in the two amplitudes for the limit $M_W^2/s \to 0$. The remaining terms agree only when the extra condition $M_H^2/s \to 0$ is imposed.

It is easy to understand this in terms of the two limits mentioned in Eq. (4). Remember that the Higgs mass is $M_H$, so that in this limit, because they vanish for $M_H/\sqrt{s} \to 0$, we need not consider only the $g$-independent terms. However, Eq. 8 tells us that in this limit, $M_H$ also goes to zero, so equivalence theorem concerns only the terms with $M_H/\sqrt{s} \to 0$ in addition to $M_W/\sqrt{s} \to 0$.

2.2 $W^+ W^+$ scattering.

These amplitudes have been discussed in Ref. 10. In this case, one obtains

$$A(w^+ w^+ \to w^+ w^+) = \frac{g^2}{4} \left\{ \frac{M_H^2}{M_W^2} \left( \frac{t}{t-M_H^2} + \frac{u}{u-M_H^2} \right) + 4 \sin^2 \theta_W \left( \frac{s-u}{s} + \frac{s-t}{t} \right) + \frac{\cos^2 2\theta_W}{\cos^2 \theta_W} \left( \frac{s-u}{t} + \frac{s-t}{u} \right) \right\},$$

(11)

whereas the amplitude for the scattering using the longitudinal gauge bosons has the following extra terms for $M_W^2 \ll s$,

$$A' = -g^2 M_H^2 \left\{ \frac{4}{s} + \left( 1 + \frac{M_W^2}{M_H^2} + \frac{2 M_H^2}{s} \right) \left( \frac{1}{t-M_H^2} + \frac{1}{u-M_H^2} \right) \right\}.$$

(12)

Once again, notice that the leading terms in Eq. (11) are proportional to $g^2 M_H^2/M_W^2$, which means that they do not go to zero as $g \to 0$ because of the mass relation in Eq. (10). The terms in $A'$, on the other hand, all go to zero in this limit, so that equivalence theorem is verified. Note that the terms in Eq. (11) involving $\theta_W$ are irrelevant for this limit, because they vanish for $g \to 0$.

On the other hand, if we take $v \to 0$, it implies $M_H \to 0$ alongwith $M_W \to 0$, as mentioned earlier. In this case, equivalence theorem implies equality of terms having higher powers of the gauge coupling constant as well. As we see from the expression of $A'$ in Eq. (12), this is indeed true since $A'$ vanishes altogether for $M_H \to 0$.

In passing, we also want to note (11) using Eqs. (11) and (12), that $A'/A$ goes to zero also for $M_H^2 \to \infty$, so that $A'$ can be neglected in this case as well. The implication of this will be discussed in Sec. 3.

2.3 $W^+ W^- \to Z Z$.

The amplitude for $W^+ W^- \to Z Z$ has been calculated in Refs. 11, 7. Putting $M_W, M_Z = 0$ as is required to verify the equivalence theorem, we obtain from these calculations

$$A(W^+ W^- \to Z Z) = \frac{g^2}{4 t u (s-M_H^2)} \times \left\{ \left( \frac{M_H^2}{M_W^2} + 6 \right) s^3 + \left( -4 + 6 \frac{M_Z^2}{M_W^2} \right) M_H^2 s^2 \cos^2 \theta \left( \left( \frac{M_H^2}{M_W^2} + 2 \right) s^3 - \left( 8 + 6 \frac{M_Z^2}{M_W^2} \right) M_H^2 s^2 \right) \right\},$$

(13)

where $\theta$ is the scattering angle in the center-of-mass frame. The amplitude calculated using the Higgs-Goldstone modes in the outer lines, evaluated for $M_W = M_Z = 0$, is given by 8, 12

$$A(w^+ w^- \to z z) = \frac{g^2}{4 t u (s-M_H^2)} \times \left\{ \left( \frac{M_Z^2}{M_W^2} + 6 \right) s^3 + \left( -6 + 4 \frac{M_Z^2}{M_W^2} \right) M_H^2 s^2 \cos^2 \theta \left( \left( \frac{M_Z^2}{M_W^2} + 2 \right) s^3 - \left( 6 + 4 \frac{M_Z^2}{M_W^2} \right) M_H^2 s^2 \right) \right\}. \tag{14}$$
Thus

\[ A' = \frac{g^2}{2} \left( 1 + \frac{M_Z^2}{M_W^2} \right) \frac{M_H^2}{s - M_H^2}. \]  

(15)

Once again, it is easy to see that \( A' \) vanishes either when \( g = 0 \), in which case only the terms involving \( M_H^2/M_W^2 \) are important in the amplitudes of Eqs. (13) and (14); or when \( v = 0 \), which would imply \( M_H = 0 \) in addition to \( M_W = M_Z = 0 \). Once again, note that for \( M_H \to \infty \), \( A' \) is negligible in comparison with the amplitude calculated in either way. This is the subject of Sec. 3.

3 The case of \( M_H^2 \to \infty \).

In the calculations of some amplitudes which depend on \( M_H \), the authors have noticed that the difference between \( A(W_{+}^{+}Z_{||}, \cdots) \) and \( A(w^{+}, z, \cdots) \) becomes negligibly small even when \( M_H^2/s \to \infty \). This property has also been assumed to be true in demonstrating the equivalence theorem in some cases. In this section, we show which class of amplitudes show this property and what is the relation of this result with the equivalence theorem. Our conclusion is that this property is not, in general, a consequence of the equivalence theorem, but rather follows from an application of the decoupling theorem.

In a technical sense, the decoupling theorem is not expected to work for \( M_H \to \infty \) since the Higgs boson mass comes from physics at the electroweak scale. In other words, taking \( M_H \to \infty \) implies, via Eqs. (10), \( \lambda \to \infty \), which implies that the Higgs field has very large self-coupling. This invalidates perturbative calculations on which the above analysis is based.

However, one can blindly take a Feynman diagram, calculate its amplitude perturbatively, consider it as a mathematical expression and ask what happens to it if we take the mathematical limit of \( \lambda \to \infty \). If this limit is zero, we can say that the particular diagram decouples in the limit of \( M_H \to \infty \), although we emphasize that the physical significance of this statement is not clear. Now, if we deal with a process where all diagrams which depend on \( M_H \) decouple in the limit \( M_H \to \infty \), then taking the limit we will obtain an expression which depends only on the masses of the gauge bosons. The equivalence theorem should work with this expression when we take the limit of vanishing gauge boson masses.

We can see examples of this statement in the processes described in Sec. 2. For \( W_{||}^{+}W_{||}^{+} \) scattering as well as \( W_{||}^{+}W_{||}^{+} \to Z_{||}Z_{||} \), there are diagrams with a 4-point interaction or with \( W \)-exchange in the \( t \) and \( u \)-channels which are independent of \( \lambda \). The only \( \lambda \) dependence comes from the diagram with \( s \)-channel Higgs exchange, which goes like \( 1/M_H^2 \) for large Higgs mass, which means \( 1/\lambda \). Thus, this diagram decouples for large Higgs mass. The equivalence theorem should work for the other diagrams in this limit, whose amplitudes depend only on the gauge boson masses. Of course, this is easily seen from the expression for \( A' \) in Eq. (12) as well as from Eq. (15), and this was even noted by the authors who calculated these processes.

On the contrary, if we consider the process \( W_{||}^{+}W_{||}^{+} \to HH \), the \( s \)-channel Higgs exchange diagram now has an extra power of \( \lambda \) coming from the cubic vertex of Higgs bosons. This diagram is a constant for \( \lambda \to \infty \), i.e., it does not decouple. In this case, one should not expect the equivalence theorem to hold for large Higgs mass, and in fact a scrutiny of Eqs. (7) and (8) shows that it doesn’t.

4 Zeroes of the amplitude: the case of \( \gamma \gamma \to W_{||}^{+}W_{||}^{-} \)

In this section, we give an example where the equivalence theorem may appear to be invalidated due to some cancellations in the amplitude. Consider the scattering process \( \gamma(k)\gamma(k') \to W_{||}^{+}(p)W_{||}^{-}(p') \). In the center of energy frame, we can write the various momenta as

\[
\begin{align*}
    k^\mu &= E(1, 0, 0, 1), &  k'^\mu &= E(1, 0, 0, -1), \\
    p^\mu &= E(1, \beta \sin \theta, 0, \beta \cos \theta), &  p'^\mu &= E(1, -\beta \sin \theta, 0, -\beta \cos \theta),
\end{align*}
\]  

(16)

where \( \beta \) is the magnitude of the 3-velocity of the \( W \)-bosons. The polarization vectors are taken to be

\[
\begin{align*}
    \varepsilon^\mu &= (0, \sin \varphi, e^{i\delta} \cos \varphi, 0), &  \varepsilon'^\mu &= (0, \sin \varphi', e^{i\delta'} \cos \varphi', 0), \\
    \epsilon^\mu &= \frac{E}{M_W}(\beta, \sin \theta, 0, \cos \theta), &  \epsilon'^\mu &= \frac{E}{M_W}(\beta, -\sin \theta, 0, -\cos \theta).
\end{align*}
\]  

(17)
Note that $\varepsilon^\mu$ and $\varepsilon'^\mu$ represent the most general choice of the polarization vectors subject to the electromagnetic gauge invariance. Previous calculations of this amplitude have used some very special polarization states\(^1\). As we will show, our general choice increases our understanding of the equivalence theorem.

The tree-level diagrams for the process include a 4-point interaction, and $W$-exchange graphs in the $t$ and the $u$ channels. A simple calculation yields for the amplitude of the process

$$A(\gamma \gamma \rightarrow W^+ W^-) = -\frac{2e^2}{1 - \beta^2 \cos^2 \theta} \times \left[ \left( \sin \varphi \sin \varphi' + e^{i(\delta + \delta')} \cos \varphi \cos \varphi' \right) \left\{ -1 + (2 - \beta^2) \cos^2 \theta \right\} + 2(2 - \beta^2) \sin \varphi \sin \varphi' \sin^2 \theta \right].$$

(18)

On the other hand, the amplitude using the Higgs-Goldstone modes are obtained to be

$$A(\gamma \gamma \rightarrow W^+ W^-) = \frac{2e^2}{1 - \beta^2 \cos^2 \theta} \times \left[ \left( \sin \varphi \sin \varphi' + e^{i(\delta + \delta')} \cos \varphi \cos \varphi' \right) \left\{ -\frac{1}{2}(1 + \beta^2) + \beta^2 \cos^2 \theta \right\} + 2\beta^2 \sin \varphi \sin \varphi' \sin^2 \theta \right].$$

(19)

Notice that the two amplitudes are indeed equal in the limit $M_W \rightarrow 0$, i.e., $\beta \rightarrow 1$ for general values of the polarization vectors for the photons, and thus equivalence theorem is verified.

There is, however, one special case. This is when the two photons have the same circular polarization. This corresponds to the choice of $\delta = \delta' = \pi/2$, $\varphi = -\varphi' = \pi/4$. Notice that in this case, the exact amplitude becomes

$$A(\gamma \gamma \rightarrow W^+_\|W^-_\|) = 2e^2 \times \left[ \frac{1 - \beta^2}{1 - \beta^2 \cos^2 \theta} \right].$$

(20)

The amplitude of Eq. (19) is only half as large. Thus, looking at these amplitudes, one may want to conclude that the equivalence theorem does not hold in this case\(^1\).

One can argue about this statement. Certainly it is true that Eqs. (5) and (6) are valid still, because $A'$ also contains the factor $1 - \beta^2$, and therefore vanishes when $M_W = 0$, i.e., $\beta = 1$. So, in this sense, the equivalence theorem is still valid. What is different from the general case is that, here not only $A'$ vanishes in the limit, but so do the amplitudes themselves. Thus, in some sense here the equivalence theorem has no content, in that one cannot simply calculate the amplitude of $\gamma \gamma \rightarrow w^+ w^-$ and use it for the leading term of the amplitude of $\gamma \gamma \rightarrow W^+_\|W^-_\|$. Such a situation arises even for the most general photon polarization vector if one wants to calculate the process $\gamma Z \rightarrow W^+_\|W^-_\|$, which vanishes for $M_W = M_Z = 0$, and therefore the equivalence theorem is of no useful consequence.\(^1\)

There is another important point regarding the expression of Eq. (20). Suppose someone is interested in calculating the forward scattering amplitude of the process only using circularly polarized photons. In this case, one would put $\theta = 0$ from the very beginning. Looking back at Eq. (20), we now notice that the term in the square bracket is unity, so that the amplitude will turn out to be $2e^2$. On the other hand, using the Higgs-Goldstone modes, one would obtain an amplitude which is half as large. For this special case, one might then conclude that the equivalence theorem is wrong.

The point to emphasize is that, equivalence theorem is not expected to work for forward or backward scattering, i.e., when the scattering angle is 0 or $\pi$. We said earlier that the equivalence theorem concerns the limit $M_W \rightarrow 0$. Physically, this means that $M_W^2$ has to be much smaller than all the kinematical variables of the problem, viz., the invariant products of different external momenta (for 2 $\rightarrow$ 2 scattering problems, these are equivalent to the Mandelstam variables). In this problem, for example, using Eq. (14), one finds

$$\frac{M_W^2}{p \cdot k} = \frac{1 - \beta^2}{1 - \beta \cos \theta}.$$  

(21)

For the case of forward scattering, $\theta = 0$, this ratio in fact has a limiting value of 2 when $\beta \rightarrow 1$. Thus, the limit $M_W \rightarrow 0$ is certainly not realized here. Similarly, for backward scattering, $\theta = \pi$, it is easily seen that one faces the same problem with the ratio $M_W^2/p \cdot k'$. The lesson learned is this: the equivalence theorem may not work for some special choice of external momenta. One should be careful to check that the choice does not imply the smallness of any kinematical variable. For 2 $\rightarrow$ 2 scattering, this means that one needs to ensure $M_W^2 \ll s, |t|, |u|$.

\(^1\) I am grateful to D. A. Dicus for checking this statement by explicit calculation.
5 Summing up

We have thus shown that the equivalence theorem can always be understood in terms of the limits $g \to 0$ or $v \to 0$, as mentioned in Eqs. (5) and (6). These prescriptions, which can be derived from simple physical arguments mentioned in the Introduction, remove the arbitrariness of the various ad-hoc statements appearing in the literature about the validity of equivalence theorem in amplitudes dependent on the Higgs boson mass and fermion masses. Moreover, we have emphasized that these two limiting procedures really have different physical and mathematical content. Physically, given the energy, whether or not one can use the limit $v \to 0$ depends on what the Higgs mass really is. Thus, unless the Higgs mass is known, we cannot use this version of the equivalence theorem, i.e., Eq. (6), for a given energy. Mathematically, the two limits produce different limiting amplitudes, and therefore the part of the total amplitude equated with the corresponding amplitude using the Higgs-Goldstone modes is different for the two limiting cases.

There may be processes whose amplitude vanishes altogether for $M_W = M_Z = 0$, In Sec. [4] we have shown that for such processes, although the equivalence theorem holds, its content is trivial. It merely asserts that the amplitude with the corresponding Higgs-Goldstone modes also vanish for $M_W = M_Z = 0$, but does not say anything about the non-vanishing terms which occur for the non-vanishing values of the gauge boson masses.

Throughout the paper, we have used examples where the Higgs mass is the only mass other than the gauge boson masses. But from our discussion, the generalization for amplitudes involving fermion masses $m_f$ is obvious. For the limit $g \to 0$, the equivalence theorem dictates that the terms involving $g^2 m_f^2 / M_W^2$ (which are really independent of $g$) should be equal irrespective of the magnitude of $m_f$. On the other hand, for $v \to 0$, one can see the equivalence in other terms as well, but only for $m_f = 0$, since the fermion mass is also proportional to $v$. Examples of such equivalence can be seen in the calculation of $W^+W^-$ pair production from a lepton-antilepton pair [8].

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