DUAL REPRESENTATIONS OF LAPLACE TRANSFORMS OF BROWNIAN EXCURSION AND GENERALIZED MEANDERS

WŁODZIMIERZ BRYC AND YIZAO WANG

ABSTRACT. The Laplace transform of the $d$-dimensional distribution of Brownian excursion is expressed as the Laplace transform of the $(d+1)$-dimensional distribution of an auxiliary Markov process, started from a $\sigma$-finite measure and with the roles of arguments and times interchanged. A similar identity holds for the Laplace transform of a generalized Brownian meander, which is expressed as the Laplace transform of the same auxiliary Markov process, with a different initial law.

1. INTRODUCTION

Consider the Brownian excursion standardized to have length 1 and conditioned to be positive. This is also the Brownian bridge conditioned to stay positive, or the 3-dimensional Bessel process conditioned to hit zero at $t = 1$, and its finite-dimensional distributions are given by formula (2.1) below. We let $B_{ex}^x = (B_{ex}^x_t)_{t \in [0,1]}$ denote this process throughout. Brownian excursion has been extensively investigated in the literature. See for example Bertoin and Pitman (1994) and Revuz and Yor (1999). It also appears in asymptotic analysis of various combinatorial problems, see for example Pitman (2006) and Janson (2007).

The purpose of this note is to introduce a “dual representation” that ties the Laplace transforms of finite-dimensional distributions of Brownian excursion and another Markov process, denoted by $(X_t)_{t \geq 0}$ throughout, with state space $[0, \infty)$ and transition probabilities

$$P(X_t \in dy \mid X_s = x) = p_{t-s}(x, y)dy, \quad 0 \leq s < t, x \geq 0$$

with

$$p_t(x, y) = \frac{2t\sqrt{y}}{\pi ((y-x)^2 + 2(x+y)t^2 + t^4)}, \quad t > 0, y \geq 0.$$  \hfill (1.1)

This is a positive self-similar Markov process that arises as the tangent process at the boundary of support of so-called $q$-Brownian motions and $q$-Ornstein–Uhlenbeck processes; see Bryc and Wang (2016, Proposition 2.2) and Wang (2017, Theorem 3.1). It can also be obtained from the construction in Biane (1998) applied to the $1/2$-stable free Lévy process by including appropriate drift. The derivation of the transition probability density function, following Biane’s approach, can be found in Bryc and Wang (2016, Section 3).

Our main result is the following identity which was needed in Bryc and Wang (2017), where we investigated, by essentially computing the Laplace transforms,
the fluctuations of asymmetric simple exclusion processes with open boundaries in the steady state. These processes are representative non-equilibrium models Derrida (2007); Derrida et al. (2004) that have attracted much attention recently in probability and mathematical physics.

Let $E_x\left[\cdot\right]$ denote the expectation with respect to the law of $(X_t)_{t\geq 0}$ starting at $X_0 = x > 0$.

**Theorem 1.1.** For $d \in \mathbb{N}$, let $s_0 = 0 < s_1 < s_2 < \cdots < s_d$ and $t_0 = 0 \leq t_1 < \cdots < t_d \leq 1 = t_{d+1}$. Then,

\[
E_x\left[\exp\left(-\sum_{k=1}^{d} (s_k - s_{k-1})B_{t_k}^ex\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} E_x\left[\exp\left(-\frac{1}{2} \sum_{k=0}^{d} (t_{k+1} - t_k)X_{s_k}\right)\right] \frac{1}{\sqrt{x}} dx.
\]

The left-hand side of (1.2) is the Laplace transform of the joint distribution of Brownian excursion, while the right-hand side is the Laplace transform of the process $(X_t)_{t\geq 0}$ with the arguments and time indices interchanged. On the right-hand side, the initial distribution of $(X_t)_{t\geq 0}$ is the stationary $\sigma$-finite measure $(x/(2\pi))^{1/2}1_{\{x>0\}}dx$.

We are aware of only a couple of results that connect Laplace transforms of stochastic processes by interchanging the argument and time parameters. One such result is the formula for the joint generating function of the finite Asymmetric Simple Exclusion Process in Bryc and Wesołowski (2017, Theorem 1). Another result of this type is the formula Bertoin and Yor (2001, Eq. (2)) for the univariate Mellin transform of a positive self-similar Markov process, see also Hirsch and Yor (2012).

As an immediate consequence of Theorem 1.1, however, a family of such dualities can be derived easily for generalized Brownian meanders defined in (3.1) below, including some of the generalized Bessel meanders considered in Mansuy and Yor (2008, Section 3.7). The dualities, see Theorem 3.1 and Corollary 3.3, have similar forms as in (1.2), but they differ in the choice of the $\sigma$-finite measure for the initial law on the right-hand side of (1.2).

Due to its importance, here we state the duality formula for Brownian meander $(B_t^{me})_{0 \leq t \leq 1}$, which is a special case $\delta = 1$ of Corollary 3.3 below. Recall that Brownian meander can be defined as the Brownian motion conditioned to stay positive over the time interval $[0, 1]$. See for example Bertoin and Pitman (1994); Pitman (1999, 2006). We also need this formula for our analysis in Bryc and Wang (2017) for asymmetric simple exclusion processes.

**Theorem 1.2.** For $d \in \mathbb{N}$, let $s_0 = 0 < s_1 < s_2 < \cdots < s_d$ and $t_0 = 0 \leq t_1 < t_2 < \cdots < t_d \leq 1 = t_{d+1}$. Then

\[
E_x\left[\exp\left(-\sum_{k=1}^{d} (s_k - s_{k-1})B_{1-t_k}^{me}\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} E_x\left[\exp\left(-\frac{1}{2} \sum_{k=0}^{d} (t_{k+1} - t_k)X_{s_k}\right)\right] \frac{1}{\sqrt{x}} dx.
\]
The paper is organized as follows. In Section 2 we recall some known facts about Brownian excursion and prove Theorem 1.1. In Section 3 we prove the dual representations for generalized Brownian meanders, with Brownian meander as a special case.

2. Proof of dual representation of Brownian excursion

We first recall some facts on Brownian excursion $\mathbb{E}^{ex}$. For our purposes it is convenient to define it as a Markov process that starts at $\mathbb{E}^{ex}_0 = 0$, ends at $\mathbb{E}^{ex}_1 = 0$, and has transition probabilities

$$P(\mathbb{E}^{ex}_t \in dy \mid \mathbb{E}^{ex}_s = x) = \begin{cases} \sqrt{8\pi} \ell_1(y) \ell_1^{-1}(y) & \text{if } s = 0 < t < 1, x = 0, y > 0 \\ g_{t-s}(x,y) \frac{\ell_1^{-1}(y)}{\ell_1^{-1}(x)} & \text{if } 0 < s < t < 1, x > 0, y > 0, \end{cases}$$

with

$$\ell_t(y) = \frac{1}{\sqrt{2\pi t}} y \exp\left(\frac{y^2}{2t}\right) 1_{\{y>0\}}$$

and

$$g_t(y_1,y_2) = \frac{1}{\sqrt{2\pi t}} \left[ \exp\left(-\frac{(y_1 - y_2)^2}{2t}\right) - \exp\left(-\frac{(y_1 + y_2)^2}{2t}\right) \right] \mathbf{1}_{\{y_1,y_2>0\}}.$$ 

Equivalently, the joint probability density function at time points $0 < t_1 < \cdots < t_d < 1$ is

$$f_{t_1,\ldots,t_d}(y_1,\ldots,y_d) = \sqrt{8\pi} \ell_{t_1}(y_1) \ell_{t_d}(y_d) \prod_{k=1}^{d-1} g_{t_{k+1} - t_k}(y_k, y_{k+1}).$$

See Itô and McKean (1965, page 76), Durrett et al. (1977), or Revuz and Yor (1999, page 464).

Proof of Theorem 1.1. The proof consists of rewriting the right-hand side of (1.2). We first assume $t_1 > 0$ and $t_d < 1$. With $t_0 = 0$, we have, using transition density functions of the process $X$ and letting $dx$ denote $dx_0 \cdots dx_d$,

$$\int_0^\infty \mathbb{E}_{x_0} \left[ \exp\left(-\frac{1}{2} \sum_{k=0}^d (t_{k+1} - t_k) X_{x_k}\right) \right] \sqrt{x_0} \, dx_0$$

$$= \int_{\mathbb{R}_{+}^{d+1}} dx_1 \cdots dx_d \exp\left(-\frac{1}{2} \sum_{k=0}^d (t_{k+1} - t_k)x_k\right) \prod_{k=1}^d p_{x_{k-1}}(x_{k-1},x_k)$$

$$= 2^{d+1} \int_{\mathbb{R}_{+}^{d+1}} dx_1 \cdots dx_d \exp\left(-\frac{1}{2} \sum_{k=0}^d (t_{k+1} - t_k)x_k^2\right) \prod_{k=1}^d x_k p_{x_{k-1}}(x_{k-1}^2,x_k^2),$$

where we apply in the last equality changes of variables from $x_k$ to $x_k^2$, $k = 0, \ldots, d$.

We derive a different expression of the products of transition density functions, thanks to the factorization. Namely, recalling (1.1), we write

$$p_s(x^2, y^2) = \frac{2sy}{\pi(s^2 + (y-x)^2)(s^2 + (y+x)^2)}$$

$$= \frac{1}{2\pi x} \left( \frac{s}{s^2 + (y-x)^2} - \frac{s}{s^2 + (y+x)^2} \right).$$
We now use the elementary Laplace transform
\[
\frac{s}{s^2 + a^2} = \int_0^\infty e^{-sy} \cos(ay) dy
\]
to get
\[
yp_s(x^2, y^2) = \frac{y}{2\pi x} \int_0^\infty e^{-sz} [\cos((y-x)z) - \cos((y+x)z)] dz
\]
\[
= \frac{1}{\pi x} \int_0^\infty e^{-sz} \sin(xz) \sin(yz) dz.
\]
Therefore, writing \(d\vec{y} = dy_1 \cdots dy_d\),
\[
x_0^2 \prod_{k=1}^d x_k p_{s_k-s_k-1}(x_k^2, x_k^2) = \frac{1}{\pi^d} \int_{\mathbb{R}^d_+} d\vec{y} \exp \left(- \sum_{k=1}^d (s_k - s_k-1) y_k \right)
\]
\[
\times x_0 \sin(x_0 y_1) \times x_d \sin(x_d y_d) \times \prod_{k=1}^{d-1} \sin(x_k y_{k+1}) \sin(x_k y_k).
\]
Then, noticing that the integrand is non-negative and interchanging the order of integrations, we rewrite the right-hand side of (2.2) as
\[
(2.3) \quad \frac{2^{d+1}}{\pi^d} \int_{\mathbb{R}^d_+} d\vec{y} \exp \left(- \sum_{k=1}^d (s_k - s_k-1) y_k \right)
\]
\[
\times \int_0^\infty x_0 e^{-t_1 x_0^2/2} \sin(x_0 y_1) dx_0 \times \int_0^\infty x_d e^{-(1-t_d)x_d^2/2} \sin(x_d y_d) dx_d
\]
\[
\times \prod_{k=1}^{d-1} \int_0^\infty e^{-(t_{k+1}-t_k)x_k^2/2} \sin(x_k y_{k+1}) \sin(x_k y_k) dx_k.
\]
By straightforward calculation we have
\[
\int_0^\infty x e^{-tx^2/2} \sin(xy) dx = \frac{y}{t} \int_0^\infty e^{-tx^2/2} \cos(xy) dx = \pi \ell_t(y).
\]
In the last step above we used the formula for the cosine transform
\[
\int_0^\infty e^{-tx^2/2} \cos(ax) dx = \sqrt{\frac{\pi}{2t^{1/2}}} e^{-a^2/(2t)},
\]
which follows from the characteristic function of a Gaussian distribution with mean zero and variance \(1/t\). We also have
\[
\int_0^\infty e^{-tx^2/2} \sin(xy_1) \sin(xy_2) dx
\]
\[
= \frac{1}{2} \int_0^\infty e^{-tx^2/2} \left[ \cos(x(y_1 - y_2)) - \cos(x(y_1 + y_2)) \right] dx
\]
\[
= \sqrt{\frac{\pi}{8t^{1/2}}} \left( e^{-(y_1-y_2)^2/(2t)} - e^{-(y_1+y_2)^2/(2t)} \right) = \frac{\pi}{2} \ell_t(y_1, y_2).
\]
Thus, recalling (2.1), expression (2.3) now becomes,

$$4\pi \int_{\mathbb{R}^d} d\bar{y} \exp \left( -\sum_{k=1}^{d} (s_k - s_{k-1})y_k \right) \ell_{t_1}(y_1)\ell_{t_2-d}(y_d) \prod_{k=1}^{d-1} g_{t_{k+1} - t_k}(y_k, y_{k+1})$$

$$= \sqrt{2\pi}E \left[ \exp \left( -\sum_{k=1}^{d} (s_k - s_{k-1})E_{t_k} \right) \right].$$

We have proved the theorem with $0 < t_1, t_d < 1$.

Now assume $t_d = 1$ and $t_1 > 0$. Recall that $E_{t_1} = 0$. If $d = 1$, then the left-hand side of (1.2) becomes 1, and the right-hand side becomes $(\sqrt{2\pi})^{-1} \int_0^\infty e^{-x/2} \sqrt{\pi} dx = 1$. If $d > 1$, we see that the desired identity is reduced to the same type of identity with fewer arguments, namely with $0 < s_1 < \cdots < s_{d-1}$ and $0 < t_1 < \cdots < t_{d-1} < 1 = t_d$. Such an identity has been proved in the first part of the proof.

It remains to prove the theorem for $t_1 = 0$. If $d = 1$, then the left-hand side of (1.2) is 1 and the right-hand side equals

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty E_x \left[ e^{-X_{s_1}/2} \right] \sqrt{\pi} dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x/2} \sqrt{\pi} dx = 1,$$

where we used the fact that $X$ is a Markov process with stationary distribution $\sqrt{\pi} \mathcal{N}(x, y)$ (as $\sqrt{\pi} p_s(x, y) = \sqrt{\pi} p_s(y, x)$ for $x, y \geq 0, s > 0$). For $d \geq 2$, consider $\tilde{s}_k := s_{k+1} - s_1$, $\tilde{t}_k := t_{k+1} - t_1$, $k = 0, \ldots, d - 1$ and $t_d := 1$. In this way, the left-hand side of (1.2) becomes $E[\exp(-\sum_{k=1}^{d-1}(\tilde{s}_k - \tilde{s}_{k-1})E_{\tilde{t}_k})]$, and the right-hand side becomes

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty E_x \left[ \exp \left( -\frac{1}{2} \sum_{k=0}^{d-1} (\tilde{t}_{k+1} - \tilde{t}_k)X_{\tilde{s}_k + s_1} \right) \right] \sqrt{\pi} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty E_x \left[ \exp \left( -\frac{1}{2} \sum_{k=0}^{d-1} (\tilde{t}_{k+1} - \tilde{t}_k)X_{\tilde{s}_k} \right) \right] \sqrt{\pi} dx.$$

Above in the last step we used the stationarity of $X$ again. Since necessarily $\tilde{t}_1 = t_2 > 0$, when $d \geq 2$ the desired identity (1.2) becomes an identity for Laplace transform of $(d - 1)$-dimensional distribution of Brownian excursion at time points $0 < t_1 < \cdots < t_{d-1} \leq 1$ that we have already proved before. This completes the proof. \hfill \square

3. APPLICATION TO GENERALIZED BROWNIAN MEANDERS

In this section, we let $\nu(du)$ be a probability measure on $[0, 1]$ such that $\nu(\{0\}) = 0$ and consider the generalized Brownian meander $(B_t^{(\nu)})_{0 \leq t \leq 1}$ defined by

$$B_t^{(\nu)} = \frac{1}{\sqrt{\nu}} B_{Vt}, \quad t \in [0, 1],$$

where $V$ is a random variable with law $\nu$ and independent from $E_{t_1}$. Explicit examples, including Brownian meander, will be discussed later in this section.

Noting that the function $v \mapsto e^{-x/v} u^{-3/2}$ is bounded on $(0, 1]$ for $x > 0$, we define

$$\varphi_{\nu}(x) = \frac{\sqrt{2\pi e x^2 / 2}}{\sqrt{2\pi}} \int_{(0, 1]} e^{-x/2v} u^{-3/2} \nu(du).$$
The key step is the following identity

\[ \text{For Theorem 3.1.} \]

\[ \text{with } (s_k, \ldots, s_1) \prec t \]

Thanks to the self-similarity of process \( X \),

(1) Once we establish (3.3), integrating with respect to \( \nu \) on both sides yields the desired result, by Fubini’s theorem.

Now we prove (3.3). We first rewrite (1.2), with \( t \) replaced by \( 1 - t \), as processes \((B^e_t)_{t \in [0,1]}\) and \((B^c_{1-t})_{t \in [0,1]}\) have the same law. So (1.2) becomes

\[ \exp \left( - \sum_{k=1}^d \frac{s_k - s_{k-1}}{\sqrt{v}} \right) \]

\[ = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp \left( - \frac{t_k - t_{k-1}}{\sqrt{v}} \right) dx. \]

This identity remains valid, if we replace the increasing sequence \((s_k)_{k=0,\ldots,d}\) with \((s_k/\sqrt{v})_{k=0,\ldots,d}\) and replace the increasing sequence \((t_k)_{k=1,\ldots,d+1}\) with \((\tilde{t}_k)_{k=1,\ldots,d+1}\) with \( \tilde{t}_k := 1 - v(1 - t_k) \) (formally we just replace \( s_k \) by \( s_k/\sqrt{v} \) and \( 1 - t_k \) by \( v(1 - t_k) \), but we also need to verify the same monotonicity of the replaced sequences, and that \( t_1 \geq 0, \tilde{t}_d \leq 1 \)). In this way we get

\[ \exp \left( - \sum_{k=1}^d \frac{s_k - s_{k-1}}{\sqrt{v}} \right) \]

\[ = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp \left( - \frac{v(t_{k+1} - t_k)X_{s_k} - \frac{1}{2}(1 - (1 - t_1))X_0}{\sqrt{v}} \right) dx. \]

Thanks to the self-similarity of process \( X \),

\[ ((X_{\lambda s})_{s \geq 0}, P_x) \overset{d}{=} (\lambda^2 (X_s)_{s \geq 0}, P_{x/\lambda^2}) \] for \( x > 0, \lambda > 0 \),

\[ \text{Theorem 3.1. For } d \in \mathbb{N}, \text{ let } s_0 = 0 < s_1 < s_2 < \cdots < s_d \text{ and } t_0 = 0 \leq t_1 < t_2 < \cdots < t_d = 1 = t_d+1. \text{ Then} \]

\[ \mathbb{E} \left[ \exp \left( - \sum_{k=1}^d (s_k - s_{k-1}) \mathbb{B}^{(v)}_{1-t_k} \right) \right] \]

\[ = \int_0^\infty \mathbb{E}_x \left[ \exp \left( - \frac{1}{2} \sum_{k=0}^d (t_{k+1} - t_k) X_{s_k} \right) \right] \varphi_\nu(x) dx. \]
where \( P_x(\cdot) = P(\cdot \mid X_0 = x) \) (see Bryc and Wang (2016)), we have
\[
\int_0^\infty E_x \left[ \exp \left( -\frac{1}{2} \sum_{k=1}^d v(t_{k+1} - t_k)X_{s_k} - \frac{1}{2} (1 - v(1 - t_1))X_0 \right) \right] \sqrt{v} \, dx
\]
\[
= \int_0^\infty E_{vx} \left[ \exp \left( -\frac{1}{2} \sum_{k=1}^d (t_{k+1} - t_k)X_{s_k} - \frac{1}{2} (1 - v(1 - t_1))X_0 \right) \right] \sqrt{x} \, dx
\]
\[
= \int_0^\infty E_{vx} \left[ \exp \left( -\frac{1}{2} \sum_{k=0}^d (t_{k+1} - t_k)X_{s_k} - \frac{1}{2} \left( \frac{1}{v} - 1 \right)X_0 \right) \right] \sqrt{x} \, dx
\]
\[
= \int_0^\infty E_x \left[ \exp \left( -\frac{1}{2} \sum_{k=0}^d (t_{k+1} - t_k)X_{s_k} \right) \right] e^{-x(1/v-1)/2} \frac{\sqrt{x}}{\sqrt{v/2}} \, dx.
\]
This yields (3.2), and completes the proof.

Next, we specialize Theorem 3.1 to the case of Beta distribution. The formula shall involve the confluent hypergeometric function
\[
\psi(\alpha, \beta, x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xv} u^{\alpha-1} (1 + u)^{\beta-1} du, \alpha > 0, \beta \in \mathbb{R}, x > 0.
\]
See for example Lebedev (1965, page 268).

**Corollary 3.2.** For \( d \in \mathbb{N}, \) let \( s_0 = 0 < s_1 < s_2 < \cdots < s_d \) and \( t_0 = 0 \leq t_1 < \cdots < t_d = 1 \). If \( v(du) = u^{\alpha-1}(1 - v)^{\beta-1} / B(\alpha, \beta) \) with \( \alpha, \beta > 0, \) then
\[
E \left[ \exp \left( -\sum_{k=1}^d (s_k - s_{k-1})\mathbb{E}^{(v)}_{s_{k-1}} \right) \right]
= \frac{\Gamma(\alpha + \beta)}{\sqrt{2\pi} \Gamma(\alpha)} \int_0^\infty E_x \left[ \exp \left( -\frac{1}{2} \sum_{k=0}^d (t_{k+1} - t_k)X_{s_k} \right) \right] \psi \left( \beta \frac{5}{2} - \alpha, \frac{x}{2} \right) \sqrt{x} \, dx.
\]

**Proof.** We apply Theorem 3.1 with
\[
\varphi_v(x) = \frac{1}{\sqrt{2\pi} B(\alpha, \beta)} \int_0^1 e^{-x(1/v-1)/2} u^{\alpha-5/2}(1 - v)^{\beta-1} \, dv
\]
\[
= \frac{1}{\sqrt{2\pi} B(\alpha, \beta)} \int_0^\infty e^{-ux/2}(1 + u)^{3/2-(\alpha+\beta)} u^{\beta-1} \, du
\]
\[
= \frac{\sqrt{\pi} \Gamma(\beta)}{\sqrt{2\pi} B(\alpha, \beta)} \psi \left( \beta \frac{5}{2} - \alpha, \frac{x}{2} \right),
\]
where in the second step we applied a change of variable \( u = 1/v - 1. \)

Our results are also related to generalized Bessel meanders that were introduced in Mansuy and Yor (2008, Sections 3.6 and 3.7). There are three equivalent ways to define a generalized Bessel meander \( \mathbb{E}^{(\delta, \delta')} \) for \( \delta, \delta' > 0. \) (This is the process under the law \( M^{\delta, \delta'} \) in Mansuy and Yor (2008), and strictly speaking the authors did not give it a name there but only mentioned ‘generalized meanders’ in the section titles.) Firstly, it can be defined as a randomized Bessel bridge of dimension \( \delta + \delta', \) using Beta(\( \delta/2, \delta'/2 \)) distribution. For this approach, see Mansuy and Yor (2008, Theorem 3.12). Secondly, it can be defined as \( ((R_t^2 + (R_t')^2)^{1/2})_{t \in [0,1]} \), where \( R \) is
a $\delta$-dimensional Bessel bridge (pinned down at $R_0 = 0$ and $R_1 = 0$), $R'$ is a $\delta'$-dimensional Bessel process starting from 0, and $R$ and $R'$ are independent. Thirdly, the law $M^{\delta,\delta'}$ of $\mathbb{B}^{(\delta,\delta')}$, viewed as a probability measure on $C([0,1])$, is absolutely continuous with respect to the probability measure $P_{\text{BES}(\delta+\delta')}$ on $C([0,1])$ induced by a $(\delta + \delta')$-dimensional Bessel process starting from 0; more precisely

$$M^{\delta,\delta'} = \frac{c_{\delta,\delta'}}{X_1^{\delta}} P_{\text{BES}(\delta+\delta')}, \quad \delta, \delta' > 0,$$

with

$$c_{\delta,\delta'} = M^{\delta,\delta'}(X_1^{\delta}) = \frac{2^{\delta/2} \Gamma(\delta + \delta')/2}{\Gamma(\delta' / 2)}.$$

Here, for $\omega$ from the canonical space $C([0,1])$, $X_1(\omega) = \omega_1$. For the second and third characterizations, see Mansuy and Yor (2008, Theorem 3.9); the corresponding formula (3.8) in Theorem 3.9 therein has a typo and is corrected here.

The generalized Bessel meanders $\mathbb{B}^{(\delta,\delta')}$ and the generalized Brownian meanders $\mathbb{B}^{(\nu)}$ with Beta distribution $\nu$ in Corollary 3.2 are in general different processes. However, since a Bessel bridge of dimension 3 is a Brownian excursion, generalized Bessel meanders (3.7) with

$$\delta \in (0, 3) \quad \text{and} \quad \delta' = 3 - \delta,$$

become a special case of the generalized Brownian meanders introduced in (3.1). This case covers a couple of examples investigated in the literature. In particular, it is known that $\mathbb{B}^{(1,2)}$ is the Brownian meander $\mathbb{B}^{\me}$ and in this case the relation (3.7) is originally due to Imhof (1984). The process $\mathbb{B}^{(2,1)}$, known as Brownian co-meander, has also been investigated before, and the relation (3.7) is due to Biane et al. (1987). See also Yen and Yor (2013, Theorem 7.4.1). Our Corollary 3.2 simplifies and takes the following form.

**Corollary 3.3.** In the notations of Corollary 3.2, we have for all $\delta \in (0, 3)$,

$$E \left[ \exp \left( - \sum_{k=1}^{d} (s_k - s_{k-1}) B^{(\delta,3-\delta)}_1 - t_k \right) \right] = \frac{1}{2^{\delta/2} \Gamma(\delta/2)} \int_0^\infty E_x \left[ \exp \left( - \frac{1}{2} \sum_{k=0}^{d} (t_{k+1} - t_k) X_{s_k} \right) \right] x^{\delta/2-1} dx.$$

**Proof.** From (3.6) with $\alpha = \delta/2$, $\beta = 3/2 - \alpha$ it follows that

$$\varphi_{\nu}(x) = \frac{\sqrt{x} \Gamma(\alpha + \beta)}{\sqrt{2\pi} \Gamma(\alpha) \Gamma(\beta)} \int_0^\infty e^{-ux/2} u^{\beta-1} du = \frac{2^{\beta} \Gamma(\alpha + \beta)}{\sqrt{2\pi} \Gamma(\alpha)} x^{1/2-\beta} = \frac{2^{\delta/2} \Gamma(\delta/2)}{\Gamma(\delta/2)}.$$

**Remark 3.4.** When $d = 1$, expression (3.10) becomes

$$E \left[ e^{-\mathbb{B}^{(3,3-\delta)}_t} \right] = \frac{1}{2^{\delta/2} \Gamma(\delta/2)} \int_0^\infty E_x \left[ e^{-tX_s} e^{-(1-t)x/2} \right] x^{\delta/2-1} dx,$$
This differs from Bertoin and Yor (2001, Eq. (2)), who developed a dual representation of Laplace transforms of univariate distributions in the form of univariate integrals
\[ E \left[ e^{-sU_t} \right] = E \left[ e^{-tR_s} \right], \quad s, t > 0. \]
But, clearly, there is some similarity in how the roles of arguments and times are interchanged.

**Remark 3.5.** It is natural to interpret Beta(0, 3/2) and Beta(3/2, 0) as degenerate laws. The limiting case of Beta(\(\delta/2, 3/2 - \delta/2\)) as \(\delta \uparrow 3\) is \(V = 1\), so with \(B^{(3,0)} = B^e\) Corollary 3.3 can be viewed as an extension of Theorem 1.1. On the other hand, as \(\delta \downarrow 0\), one can check that \(B^{(\delta,3-\delta)}\) converges to the 3-dimensional Bessel process. In this limit, the right-hand side of (3.5) becomes an undefined expression. The 3-dimensional Bessel process is also considered in Hirsch and Yor (2012, Theorem 6.1). Laplace transforms of squared Bessel processes have been investigated in Pitman and Yor (1982). However, we do not see immediate connection between our identities to the results there.

**Acknowledgement.** The authors thank Jim Pitman for insightful comments on several early versions of the paper, and a few key references including in particular Mansuy and Yor (2008), which helped us improve significantly the paper. The authors also thank Joseph Najnudel for several inspiring discussions. WB’s research was supported in part by the Charles Phelps Taft Research Center at the University of Cincinnati. YW’s research was partially supported by NSA grant H98230-16-1-0322 and Army Research Laboratory grant W911NF-17-1-0006.

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WŁODZIMIERZ BRYC, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF CINCINNATI, 2815 COMMONS WAY, CINCINNATI, OH, 45221-0025, USA.

E-mail address: wlodzimierz.bryc@uc.edu

YIZAO WANG, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF CINCINNATI, 2815 COMMONS WAY, CINCINNATI, OH, 45221-0025, USA.

E-mail address: yizao.wang@uc.edu