Exponential decay for semilinear wave equations with viscoelastic damping and delay feedback

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Received: 18 September 2020 / Accepted: 2 June 2021 / Published online: 17 June 2021 © The Author(s), under exclusive licence to Springer-Verlag London Ltd., part of Springer Nature 2021

Abstract

In this paper we study a class of semilinear wave-type equations with viscoelastic damping and delay feedback with time variable coefficient. By combining semigroup arguments, careful energy estimates and an iterative approach we are able to prove, under suitable assumptions, a well-posedness result and an exponential decay estimate for solutions corresponding to small initial data. This extends and concludes the analysis initiated in Nicaise and Pignotti (J Evol Equ 15:107–129, 2015) and then developed in Komornik and Pignotti (Math Nachr, to appear, 2018), Nicaise and Pignotti (Evol Equ 18:947–971, 2018).

Keywords Evolution equations · Delay feedbacks · Stabilization · Wave equation

1 Introduction

Let $H$ be a Hilbert space and let $A$ be a positive self-adjoint operator with dense domain $D(A)$ in $H$ and compact inverse in $H$. Let us consider the system:

$$
egin{align*}
&u_{tt}(t) + Au(t) - \int_0^{+\infty} \mu(s)Au(t-s)ds + k(t)BB^*u_t(t-\tau) \\
&= \nabla \psi(u(t)), \quad t \in (0, +\infty), \\
&u(t) = u_0(t), \quad t \in (-\infty, 0], \\
&u_t(0) = u_1, \\
&B^*u_t(t) = g(t), \quad t \in (-\tau, 0),
\end{align*}
$$

(1.1)
where $\tau > 0$ represents the time delay, $B$ is a bounded linear operator of $H$ into itself, $B^*$ denotes its adjoint, and $(u_0(\cdot), u_1, g(\cdot))$ are the initial data taken in suitable spaces. Moreover, the delay damping coefficient $k: [0, +\infty) \to \mathbb{R}$ is a function in $L^1_{\text{loc}}((0, +\infty))$ such that

$$
\int_{t-\tau}^t |k(s)| \, ds < C^*, \quad \forall t \in (0, +\infty),
$$

for a suitable constant $C^*$, and the memory kernel $\mu: [0, +\infty) \to [0, +\infty)$ satisfies the following assumptions:

(i) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$;

(ii) $\mu(0) = \mu_0 > 0$;

(iii) $\int_0^{+\infty} \mu(t) \, dt = \tilde{\mu} < 1$;

(iv) $\mu'(t) \leq -\delta \mu(t)$, for some $\delta > 0$.

Furthermore, $\psi: D(A^{\frac{1}{2}}) \to \mathbb{R}$ is a functional having Gâteaux derivative $D\psi(u)$ at every $u \in D(A^{\frac{1}{2}})$. Moreover, in the spirit of [4], we assume the following hypotheses:

(H1) For every $u \in D(A^{\frac{1}{2}})$, there exists a constant $c(u) > 0$ such that

$$
|D\psi(u)(v)| \leq c(u)||v||_H \quad \forall v \in D(A^{\frac{1}{2}}).
$$

Then, $\psi$ can be extended to the whole $H$ and we denote by $\nabla\psi(u)$ the unique vector representing $D\psi(u)$ in the Riesz isomorphism, i.e.,

$$
\langle \nabla\psi(u), v \rangle_H = D\psi(u)(v), \quad \forall v \in H;
$$

(H2) for all $r > 0$ there exists a constant $L(r) > 0$ such that

$$
||\nabla\psi(u) - \nabla\psi(v)||_H \leq L(r)||A^{\frac{1}{2}}(u - v)||_H,
$$

for all $u, v \in D(A^{\frac{1}{2}})$ satisfying $||A^{\frac{1}{2}}u||_H \leq r$ and $||A^{\frac{1}{2}}v||_H \leq r$.

(H3) $\psi(0) = 0$, $\nabla\psi(0) = 0$ and there exists a strictly increasing continuous function $h$ such that

$$
||\nabla\psi(u)||_H \leq h(||A^{\frac{1}{2}}u||_H)||A^{\frac{1}{2}}u||_H,
$$

for all $u \in D(A^{\frac{1}{2}})$. We are interested in studying well-posedness and stability results, for small initial data, for the above model. Our results extend the ones of [18,19] where abstract evolution equations are analyzed and, in the specific case of memory damping, exponential decay is obtained essentially only in the linear case. Indeed, in the nonlinear setting, an extra standard frictional damping, not delayed, was needed in order to obtain existence and uniqueness of global solutions with exponentially decaying energy for suitably small initial data. Moreover, in [18,19] the delay damping coefficient $k(t)$ is assumed to be constant and the results there obtained require a smallness assumption on $\|k\|_{\infty}$. 

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The analysis of [18,19] has been extended in [15] by considering a time-variable delay damping coefficient \( k(t) \) as in the present paper. However, also in [15] an extra frictional not delayed damping was needed, in the case of wave-type equation with memory damping, when a locally Lipschitz continuous nonlinear term is included into the equation.

Then, here, we focus on wave-type equations with viscoelastic damping, delay feedback and source term, obtaining well-posedness and stability results for small initial data without adding extra frictional not delayed damping. So, we here improve and conclude the analysis developed in [15,18,19] for the class of models at hand. Other models with viscoelastic damping and time delay are studied in recent literature. The first result is due to [14], in the linear setting. In that paper a standard frictional damping, not delayed, is included into the model in order to compensate the destabilizing effect of the delay feedback. Actually, at least in the linear case, the viscoelastic damping alone can counter the destabilizing delay effect, under suitable assumptions, without needing other dampings. This has been shown, e.g., in [5,11,13,25]. The case of viscoelastic wave equation with intermittent delay feedback has been studied in [22], while [16] deals with a model for plate equation with memory, source term, delay feedback and standard not delayed frictional damping. Undelayed models for wave-type equations with memory damping have been instead previously studied in [3,4]; see also [2] for an undelayed Timoshenko model.

More extended is the literature in case of a frictional/structural damping, instead of a viscoelastic term, which compensates the destabilizing effect of a time delay and, for specific models, mainly in the linear setting, several stability results have been quite recently obtained under appropriate assumptions (see, e.g., [1,6,7,9,17,20,23,24]).

As in Dafermos [10], we define the function

\[
\eta_t(t) := u(t) - u(t - s), \quad s, t \in (0, +\infty),
\]

so that we can rewrite (1.1) in the following way:

\[
\begin{align*}
        u_{tt}(t) + (1 - \tilde{\mu}) Au(t) &+ \int_{0}^{+\infty} \mu(s) A \eta_t(s) \, ds + k(t) BB^* u_t(t - \tau) \\
&= \nabla \psi(u(t)), \quad t \in (0, +\infty), \\
\eta_t(t) &=-\eta_t(s) + u_t(t), \quad t, s \in (0, +\infty), \\
u(0) &= u_0(0), \\
u_t(0) &= u_1, \\
B^* u_t(t) &= g(t), \quad t \in (-\tau, 0), \\
\eta_0(s) &= \eta_0(0) - u_0(-s) \quad s \in (0, +\infty).
\end{align*}
\] (1.5)

Let us define, as in [15], the energy of model (1.1) as

\[
E(t) := E(u(t)) = \frac{1}{2} ||u_t(t)||^2_H + \frac{1 - \tilde{\mu}}{2} ||A^{\frac{1}{2}} u(t)||^2_H - \psi(u) \\
+ \frac{1}{2} \int_{0}^{+\infty} \mu(s) ||A^{\frac{1}{2}} \eta_t(s)||^2_H \, ds + \frac{1}{2} \int_{t-\tau}^{t} |k(s + \tau)| \cdot ||B^* u_t(s)||^2_H \, ds.
\]

(1.6)
Note that, apart from the last term, this is the natural energy for nonlinear wave-type equation with memory (cf., e.g., [4]). The additional term
\[
\frac{1}{2} \int_{t-\tau}^{t} |k(s + \tau)| \cdot \|B^* u_t(s)\|_{H}^2 \, ds
\]
has been first introduced in [17], and it is crucial in order to deal with the delay feedback.

We will show that, thanks to assumption (1.3), for solutions to system (1.1) corresponding to sufficiently small initial data, the energy is positive for any \( t \geq 0 \). Moreover, we will prove that an exponential decay estimate holds.

First of all, we will reformulate (1.5) (see (2.13)) as an abstract first-order equation. It is well known (see, e.g., [12]) that the operator \( A \) in problem’s formulation (2.13), corresponding to the linear undelayed part of the model, generates an exponentially stable semigroup \( \{S(t)\}_{t \geq 0} \), namely there exist two constants \( M, \omega > 0 \) such that
\[
||S(t)||_{\mathcal{L}(H)} \leq Me^{-\omega t}. \tag{1.7}
\]
Let us denote
\[
\|B\|_{\mathcal{L}(H)} = \|B^*\|_{\mathcal{L}(H)} = b. \tag{1.8}
\]
Our result will be obtained under an assumption on the coefficient \( k(t) \) of the delay feedback. More precisely, we assume (cf. [15]) that there exist two constants \( \omega' \in [0, \omega) \) and \( \gamma \in \mathbb{R} \) such that
\[
b^2 Me^{\omega't} \int_{0}^{t} |k(s + \tau)| \, ds \leq \gamma + \omega't, \quad \text{for all } t \geq 0. \tag{1.9}
\]
Note that (1.9) includes, as particular cases, \( k \) integrable or \( k \) in \( L^\infty \) with \( \|k\|_{\infty} \) sufficiently small.

**Theorem 1.1** Let us consider (1.5) and assume (1.9). Then, there exists \( \rho_0 > 0 \) such that if
\[
(1 - \tilde{\mu}) ||A^{\frac{1}{2}} u_0||_{H}^2 + ||u_1||_{H}^2 + \int_{0}^{+\infty} \mu(s) ||A^{\frac{1}{2}} \eta_0||_{H}^2 \, ds
\]
\[
+ \int_{0}^{\tau} |k(s)| \cdot ||B g(s - \tau)||_{H}^2 \, ds < \rho_0, \tag{1.10}
\]
then the solution \( u \) is globally defined and it satisfies
\[
E(t) \leq \bar{K} e^{-\beta t}, \tag{1.11}
\]
where \( \bar{K} \) is a constant depending only on the initial data and \( \beta > 0 \).

The paper is organized as follows. In Sect. 2 we give some preliminaries, writing system (1.1) in an abstract way. In Sect. 3 we prove the exponential decay of the energy associated with (1.1). Finally, in Sect. 4 some examples are illustrated.

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2 Preliminaries

Let $L^2_\mu((0, +\infty); D(A^{\frac{1}{2}}))$ be the Hilbert space of the $D(A^{\frac{1}{2}})$–valued functions in $(0, +\infty)$ endowed with the scalar product

$$\langle \varphi, \psi \rangle_{L^2_\mu((0, +\infty); D(A^{\frac{1}{2}}))} = \int_0^{\infty} \mu(s) \langle A^{\frac{1}{2}} \varphi, A^{\frac{1}{2}} \psi \rangle_H ds$$

and denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = D(A^{\frac{1}{2}}) \times H \times L^2_\mu((0, +\infty); D(A^{\frac{1}{2}})),$$

equipped with the inner product

$$\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \rangle_{\mathcal{H}} := (1 - \tilde{\mu}) \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} \tilde{u} \rangle_H + \langle v, \tilde{v} \rangle_H + \int_0^{\infty} \mu(s) \langle A^{\frac{1}{2}} w, A^{\frac{1}{2}} \tilde{w} \rangle_H ds.$$

(2.12)

Setting $U = (u, u_t, \eta_t)$ we can restate (1.1) in the abstract form

$$U'(t) = AU(t) - k(t)BU(t - \tau) + F(U(t)),$$

$$BU(t - \tau) = \tilde{g}(t) \text{ for } t \in [0, \tau],$$

$$U(0) = U_0,$$

(2.13)

where the operator $A$ is defined by

$$A \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -1 - \tilde{\mu} A + \int_0^{+\infty} \mu(s) A w(s) ds \\ -w_x + v \end{pmatrix}$$

with domain

$$D(A) = \{(u, v, w) \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \times L^2_\mu((0, +\infty); D(A^{\frac{1}{2}})) : (1 - \tilde{\mu})u + \int_0^{+\infty} \mu(s) w(s) ds \in D(A), \ w_x \in L^2_\mu((0, +\infty); D(A^{\frac{1}{2}}))\},$$

(2.14)

in the Hilbert space $\mathcal{H}$, and the operator $B : \mathcal{H} \to \mathcal{H}$ is defined by

$$B \begin{pmatrix} u \\ v \\ w \end{pmatrix} := \begin{pmatrix} 0 \\ BB^* v \\ 0 \end{pmatrix}.$$

Note that, by (1.8), it results $\|B\|_{L(\mathcal{H})} = b^2$. Moreover, $\tilde{g}(t) = (0, Bg(t - \tau), 0)$, $U_0 = (u_0(0), u_1, \eta_0)$ and $F(U) := (0, \nabla \psi(u), 0)^T$. From (H2) and (H3) we deduce that the function $F$ satisfies:
\[ F(0) = 0; \]
\[ (F2) \text{ for each } r > 0 \text{ there exists a constant } L(r) > 0 \text{ such that} \]
\[ \|F(U) - F(V)\|_\mathcal{H} \leq L(r)\|U - V\|_\mathcal{H} \quad (2.15) \]
whenever \( \|U\|_\mathcal{H} \leq r \) and \( \|V\|_\mathcal{H} \leq r \).

### 3 Stability result

In this section we want to prove Theorem 1.1, namely the exponential stability of system (1.1) for small initial data.

First, we give a stability result for abstract model (2.13) under a suitable well-posedness assumption. Then, we will show that system (1.1) satisfies this well-posedness assumption. Thus, the exponential stability estimate for solutions to (1.1) corresponding to small initial data can be deduced from the general result.

Our abstract result is the following.

**Theorem 3.1** Assume (1.9). Moreover, suppose that

\( (W) \) there exist \( \rho > 0, C_\rho > 0, \) with \( L(C_\rho) < \frac{\alpha - \alpha'}{M} \) such that if \( U_0 \in \mathcal{H} \) and if \( \tilde{g} \in C([0, \tau]; \mathcal{H}) \) satisfy

\[ \|U_0\|_\mathcal{H}^2 + \int_0^\tau |k(s)| \cdot \|\tilde{g}(s)\|_\mathcal{H}^2 \, ds < \rho^2, \quad (3.16) \]
then system (2.13) has a unique solution \( U \in C([0, +\infty); \mathcal{H}) \) satisfying

\[ \|U(t)\|_\mathcal{H} \leq C_\rho \text{ for all } t > 0. \]

Then, for every solution \( U \) of (2.13), with initial datum \( U_0 \) satisfying (3.16),

\[ \|U(t)\|_\mathcal{H} \leq \tilde{M} \left( \|U_0\|_\mathcal{H} + \int_0^\tau e^{\alpha s} |k(s)| \cdot \|\tilde{g}(s)\|_\mathcal{H} \, ds \right) e^{-(\alpha - \alpha' - ML(C_\rho))t}, \quad t \geq 0, \quad (3.17) \]
with \( \tilde{M} = Me^\gamma. \)

**Proof** By Duhamel’s formula, using (2.15), (1.7) and assumption (W), we have

\[ \|U(t)\|_\mathcal{H} \leq Me^{-\alpha t} \|U_0\|_\mathcal{H} + Me^{-\alpha t} \int_0^t e^{\alpha s} |k(s)| \cdot \|BU(s - \tau)\|_\mathcal{H} \, ds \]
\[ + ML(C_\rho)e^{-\alpha t} \int_0^t e^{\alpha s} \|U(s)\|_\mathcal{H} \, ds \]
\[ \leq Me^{-\alpha t} \|U_0\|_\mathcal{H} + Me^{-\alpha t} \int_0^t e^{\alpha s} |k(s)| \cdot \|BU(s - \tau)\|_\mathcal{H} \, ds \]
\[ + Me^{-\alpha t} \int_0^t e^{\alpha s} |k(s)| \cdot \|BU(s - \tau)\|_\mathcal{H} \, ds + ML(C_\rho)e^{-\alpha t} \int_0^t e^{\alpha s} \|U(s)\|_\mathcal{H} \, ds. \]
Hence, we obtain
\[ e^{ot} ||U(t)||_{\mathcal{H}} \leq M ||U_0||_{\mathcal{H}} + M \int_0^t e^{os} |k(s)| \cdot ||BU(s - \tau)||_{\mathcal{H}} ds \]
\[ + \int_0^t \left( Me^{os} |k(s + \tau)| \cdot ||B||_{\mathcal{L}(H)} + ML(C_\rho) \right) e^{os} ||U(s)||_{\mathcal{H}} ds, \]
and then
\[ e^{ot} ||U(t)||_{\mathcal{H}} \leq M ||U_0||_{\mathcal{H}} + M \int_0^t e^{os} |k(s)| \cdot ||\bar{g}(s)||_{\mathcal{H}} ds \]
\[ + \int_0^t \left( Mb^2 e^{os} |k(s + \tau)| + ML(C_\rho) \right) e^{os} ||U(s)||_{\mathcal{H}} ds. \]

Therefore, using Gronwall’s inequality,
\[ e^{ot} ||U(t)||_{\mathcal{H}} \leq M \left( ||U_0||_{\mathcal{H}} + \int_0^t e^{os} |k(s)| \cdot ||\bar{g}(s)||_{\mathcal{H}} ds \right) e^{Mb^2 e^{ot} \int_0^t |k(s + \tau)| ds + ML(C_\rho) t} \]
and so, from (1.9),
\[ ||U(t)||_{\mathcal{H}} \leq M e^\gamma \left( ||U_0||_{\mathcal{H}} + \int_0^t e^{os} |k(s)| \cdot ||\bar{g}(s)||_{\mathcal{H}} ds \right) e^{-\left(\omega - \omega' - ML(C_\rho)\right)t}. \]

This gives (3.17) with $\bar{M}$ as in the statement. \hfill \Box

In order to prove the stability result we need then to show that the well-posedness assumption (W) of Theorem 3.1 is satisfied for problem (1.1). For this, let us state the following lemma.

**Lemma 3.2** Let $u : [0, T) \to \mathbb{R}$ be a solution of (1.1). Assume that
\[ E(t) \geq \frac{1}{4} ||u_t(t)||_{\mathcal{H}}^2 \] (3.18)
for all $t \in [0, T)$. Then,
\[ E(t) \leq \bar{C}(t) E(0), \] (3.19)
for all $t \in [0, T)$, where
\[ \bar{C}(t) = e^{2b^2 \int_0^t (|k(s)| + |k(s + \tau)|) ds}. \] (3.20)
Proof Differentiating $E(t)$, we obtain

$$\frac{dE(t)}{dt} = \langle u_t, u_{tt} \rangle_H + (1 - \tilde{\mu})(A^{1/2}u, A^{1/2}u_t)_H - \langle \nabla \psi(u), u_t \rangle_H$$

$$+ \frac{1}{2} |k(t + \tau)| \cdot ||B^*u_t(t)||_H^2$$

$$- \frac{1}{2} |k(t)| \cdot ||B^*u_t(t - \tau)||_H^2 + \int_0^{+\infty} \mu(s)\langle A^{1/2}\eta'(s), A^{1/2}\eta_l(s) \rangle_H ds.$$

Then, from (1.1),

$$\frac{dE(t)}{dt} = -\int_0^{+\infty} \mu(s)\langle u_t(t), A\eta'(s) \rangle_H ds - k(t)\langle u_t, B B^*u_t(t - \tau) \rangle_H$$

$$+ \frac{1}{2} |k(t + \tau)| \cdot ||B^*u_t(t)||^2$$

$$- \frac{1}{2} |k(t)| \cdot ||B^*u_t(t - \tau)||^2 + \int_0^{+\infty} \mu(s)\langle A\eta'(s), \eta_l(s) \rangle_H ds.$$

Using the second equation of (1.5), we have that

$$\frac{dE(t)}{dt} = -k(t)\langle u_t, B B^*u_t(t - \tau) \rangle_H$$

$$+ \frac{1}{2} |k(t + \tau)| \cdot ||B^*u_t(t)||^2 - \frac{1}{2} |k(t)| \cdot ||B^*u_t(t - \tau)||^2_H$$

$$- \int_0^{+\infty} \mu(s)\langle A\eta'(s), \eta_l(s) \rangle_H ds.$$

Now, we claim that

$$\int_0^{+\infty} \mu(s)\langle \eta'_l, A\eta'(s) \rangle_H ds \geq 0.$$

Indeed, integrating by parts and recalling assumption (iv) on $\mu(\cdot)$, we deduce

$$\int_0^{+\infty} \mu(s)\langle \eta'_l, A\eta'(s) \rangle_H ds = -\frac{1}{2} \int_0^{+\infty} \mu'(s)||A^{1/2}\eta'(s)||_H^2 ds \geq 0.$$

Therefore, we have that

$$\frac{dE(t)}{dt} \leq -k(t)\langle B^*u_t, B^*u_t(t - \tau) \rangle_H$$

$$+ \frac{1}{2} |k(t + \tau)| \cdot ||B^*u_t(t)||_H^2 - \frac{1}{2} |k(t)| \cdot ||B^*u_t(t - \tau)||_H^2.$$
and so, using Cauchy–Schwarz inequality,
\[ \frac{dE(t)}{dt} \leq \frac{1}{2} |k(t)| \cdot ||B^*u_t(t)||^2_H + \frac{1}{2} |k(t)| \cdot ||B^*u_t(t - \tau)||^2_H + \frac{1}{2} |k(t + \tau)| \cdot ||B^*u_t(t)||^2_H - \frac{1}{2} |k(t)| \cdot ||B^*u_t(t - \tau)||^2_H \]
\[ \leq \frac{1}{2} (|k(t)| + |k(t + \tau)|)||B^*u_t(t)||^2_H. \]

Then, we deduce
\[ \frac{dE(t)}{dt} \leq \frac{1}{2} (|k(t)| + |k(t + \tau)|) b^2 ||u_t||^2_H \]
\[ = 2b^2 (|k(t)| + |k(t + \tau)|) \frac{1}{4} ||u_t||^2_H \]
\[ \leq 2b^2 (|k(t)| + |k(t + \tau)|) E(t), \]
where in the last inequality we have used (3.18). Hence, Gronwall’s inequality concludes the proof.

Before proving the well-posedness assumption (W) for solutions to (2.13), we need the following two lemmas.

**Lemma 3.3** Let us consider system (2.13) with initial data \( U_0 \in \mathcal{H} \) and \( \tilde{g} \in C([0, \tau]; \mathcal{H}) \). Then, there exists a unique local solution \( U(\cdot) \) defined on a time interval \( [0, \delta) \), with \( \delta \leq \tau \).

**Proof** Since \( t \in [0, \tau] \), we can rewrite abstract system (2.13) as an undelayed problem:
\[ U'(t) = AU(t) - k(t)\tilde{g}(t) + F(U(t)), \quad t \in (0, \tau), \]
\[ U(0) = U_0. \]
Then, we can apply the classical theory of nonlinear semigroups (see, e.g., [21]) obtaining the existence of a unique solution on a set \([0, \delta)\), with \( \delta \leq \tau \). \( \square \)

**Lemma 3.4** Let \( U(t) = (u(t), u_t(t), \eta_t) \) be a solution to (2.13) defined on the interval \([0, \delta)\), with \( \delta \leq \tau \). Let \( h \) be the strictly increasing function appearing in (1.3). Then,
1. if \( h(||A^{1/2}u_0(0)||_H) < \frac{1 - \bar{\mu}}{2} \), then \( E(0) > 0 \);
2. if \( h(||A^{1/2}u_0(0)||_H) < \frac{1 - \bar{\mu}}{2} \) and \( h \left( \frac{2}{(1 - \bar{\mu})^2} \bar{C}^{1/2}(\tau) E^{1/2}(0) \right) < \frac{1 - \bar{\mu}}{2} \), with \( \bar{C}(\tau) \)
defined in (3.20), then
\[ E(t) > \frac{1}{4} ||u_t||^2_H + \frac{1 - \bar{\mu}}{4} ||A^{1/2}u||^2_H \]
\[ + \frac{1}{4} \int_{t-\tau}^t |k(s + \tau)| \cdot ||B^*u_t(s)||^2_H ds + \frac{1}{4} \int_0^{+\infty} \mu(s) ||A^{1/2}\eta'(s)||^2_H ds, \]
(3.21)
for all $t \in [0, \delta)$. In particular,

$$E(t) > \frac{1}{4} \| U(t) \|^2_H, \quad \text{for all } t \in [0, \delta).$$

(3.22)

**Proof** We first deduce by assumption (H3) on $\psi$ that

$$|\psi(u)| \leq \int_0^1 |\langle \nabla \psi(su), u \rangle| ds \leq \frac{1}{2} h(\| A^{\frac{1}{2}} u \|_H) \| A^{\frac{1}{2}} u \|^2_H.$$  \hspace{1cm} (3.23)

Hence, under the assumption $h(\| A^{\frac{1}{2}} u_0(0) \|_H) < \frac{1-\tilde{\mu}}{2}$, we have that

$$E(0) = \frac{1}{2} \| u_1 \|^2_H + \frac{1 - \tilde{\mu}}{2} \| A^{\frac{1}{2}} u_0(0) \|^2_H - \psi(u_0(0))$$

$$+ \frac{1}{2} \int_{-\tau}^0 |k(s + \tau)| \cdot \| B^* u(\tau) \|^2_H ds$$

$$+ \frac{1}{2} \int_0^{+\infty} \mu(s) \| A^{\frac{1}{2}} \eta_0(s) \|^2_H ds$$

$$\geq \frac{1}{2} \| u_1 \|^2_H + \frac{1 - \tilde{\mu}}{4} \| A^{\frac{1}{2}} u_0(0) \|^2_H + \frac{1}{2} \int_{-\tau}^0 |k(s + \tau)| \cdot \| B^* u(\tau) \|^2_H ds$$

$$+ \frac{1}{2} \int_0^{+\infty} \mu(s) \| A^{\frac{1}{2}} \eta_0(s) \|^2_H ds > 0,$$

obtaining 1.

In order to prove the second statement, we argue by contradiction. Let us denote

$$r := \sup \{ s \in [0, \delta) : (3.21) \text{ holds } \forall t \in [0, s) \}.$$  \hspace{1cm} (3.24)

We suppose by contradiction that $r < \delta$. Then, by continuity, we have

$$E(r) = \frac{1}{4} \| u(r) \|^2_H + \frac{1 - \tilde{\mu}}{4} \| A^{\frac{1}{2}} u(r) \|^2_H + \frac{1}{4} \int_{r-\tau}^r |k(s + \tau)| \cdot \| B^* u(\tau) \|^2_H ds$$

$$+ \frac{1}{4} \int_0^{+\infty} \mu(s) \| A^{\frac{1}{2}} \eta(\tau) \|^2_H ds.$$  \hspace{1cm} (3.24)

Now, since from (3.24)

$$\frac{1 - \tilde{\mu}}{4} \| A^{\frac{1}{2}} u(r) \|^2_H \leq E(r),$$

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we can infer, by using Lemma 3.2, that
\[
\begin{align*}
    h(||A^{\frac{1}{2}} u(r)||_H) & \leq h \left( \frac{2}{(1 - \tilde{\mu})^2} E^{\frac{1}{2}}(r) \right) \\
    & < h \left( \frac{2}{(1 - \tilde{\mu})^2} \tilde{C}^{\frac{1}{2}}(r)E^{\frac{1}{2}}(0) \right) < \frac{1 - \tilde{\mu}}{2}.
\end{align*}
\]
(3.25)

Hence, we have that
\[
\begin{align*}
    E(r) = \frac{1}{2} ||u_t(r)||_H^2 + \frac{1}{2} ||A^{\frac{1}{2}} u(r)||_H^2 - \psi(u(r)) \\
    & + \frac{1}{2} \int_{r-\tau}^r |k(s + \tau)| \cdot ||B^* u_t(s)||_H^2 ds \\
    & + \frac{1}{2} \int_0^{+\infty} \mu(s) ||A^{\frac{1}{2}} \eta^r(s)||_H^2 ds \\
    > \frac{1}{4} ||u_t(r)||_H^2 + \frac{1 - \tilde{\mu}}{4} ||A^{\frac{1}{2}} u(r)||_H^2 + \frac{1}{4} \int_{r-\tau}^r |k(s + \tau)| \cdot ||B^* u_t(s)||_H^2 ds \\
    & + \frac{1}{4} \int_0^{+\infty} \mu(s) ||A^{\frac{1}{2}} \eta^r(s)||_H^2 ds,
\end{align*}
\]
where in the last estimate we used (3.23) and (3.25). This contradicts the maximality of \( r \). Hence, \( r = \delta \) and this concludes the proof.

\[\square\]

**Theorem 3.5** If hypothesis (1.9) is satisfied, then problem (2.13), with initial data \( U_0 \in \mathcal{H} \) and \( \tilde{g} \in C([0, \tau]; \mathcal{H}) \), satisfies the well-posedness assumption (W). Hence, for solutions of (2.13) corresponding to sufficiently small initial data exponential decay estimate (3.17) holds.

**Proof** Let us fix \( N \in \mathbb{N} \) such that
\[
2M^2 \left( 1 + e^{2\omega \tau} C^* \right) e^{2\gamma} e^{-(\omega - \omega')(N-1)\tau} < \frac{1}{1 + e^{a\tau} b^2 C^*},
\]
(3.26)
where \( C^* \) is the constant defined in (1.2), \( M, \omega \) satisfy (1.7), and \( \gamma \) and \( \omega' \) are the constants appearing in (1.9). Then, let \( \rho \) be a positive constant such that
\[
\rho \leq \frac{(1 - \tilde{\mu})^{\frac{1}{2}}}{2 \tilde{C}^{\frac{1}{2}}(N\tau)} \left( \frac{1 - \tilde{\mu}}{2} \right),
\]
(3.27)
with \( \tilde{C} \) defined in (3.20). Now, let us assume that the initial data \((u_0(0), u_1, \eta_0)\) and \(B^* u_t(s), s \in [-\tau, 0]\), satisfy the smallness assumption
\[
(1 - \tilde{\mu}) ||A^{\frac{1}{2}} u_0(0)||_H^2 + ||u_1||_H^2 + \int_0^0 |k(s + \tau)||B^* u_t(s)||_H^2 ds \\
+ \int_0^{+\infty} \mu(s) ||A^{\frac{1}{2}} \eta_0(s)||_H^2 ds < \rho^2,
\]
(3.28)
with \( \rho \) as in (3.27). Note that (3.28) is equivalent to
\[
\|U_0\|_H^2 + \int_0^\tau |k(s)| \cdot \|\tilde{g}(s)\|_H^2 ds < \rho^2.
\] (3.29)

From Lemma 3.3 we know that there exists a local solution defined on a time interval \([0, \delta)\), with \( \delta \leq \tau \). From (3.28) and (3.27) we have that
\[
h(||A^{1/2}u_0(0)||_H) < h \left( \frac{\rho}{(1 - \tilde{\mu})^{1/2}} \right) \leq h \left( \frac{1}{2\tilde{C}^{1/4}(N\tau)} h^{-1} \left( \frac{1 - \tilde{\mu}}{2} \right) \right) < \frac{1 - \tilde{\mu}}{2},
\] (3.30)

where we have used the fact that \( \tilde{C}(N\tau) > 1 \). This implies, from Lemma 3.4,
\[
E(0) > 0.
\]
Furthermore, from (3.23) and (3.30) we get
\[
E(t) \geq \frac{1}{4} ||u_t(t)||_H^2 + \frac{1 - \tilde{\mu}}{4} ||A^{1/2}u(t)||_H^2 + \frac{1}{2} \int_{-\tau}^0 |k(s + \tau)| \cdot ||B^*u_t(s)||_H^2 ds + \frac{1}{2} \int_{-\tau}^0 \mu(s) ||A^{1/2}\eta_0(s)||_H^2 ds < \rho^2,
\]
where the last estimate follows from (3.29). This gives, recalling (3.27),
\[
h \left( \frac{2}{(1 - \tilde{\mu})^{1/2}} \tilde{C}^{1/4}(N\tau) E^{1/2}(0) \right) < h \left( \frac{2}{(1 - \tilde{\mu})^{1/2}} \tilde{C}^{1/4}(N\tau) \rho \right)
\]
\[
\leq h \left( h^{-1} \left( \frac{1 - \tilde{\mu}}{2} \right) \right) = \frac{1 - \tilde{\mu}}{2}.
\] (3.31)

Since \( \tilde{C}(N\tau) \geq \tilde{C}(\tau) \), then
\[
h \left( \frac{2}{(1 - \tilde{\mu})^{1/2}} \tilde{C}^{1/4}(\tau) E^{1/2}(0) \right) \leq h \left( \frac{2}{(1 - \tilde{\mu})^{1/2}} \tilde{C}^{1/4}(N\tau) E^{1/2}(0) \right) < \frac{1 - \tilde{\mu}}{2}.
\] (3.32)

So, we can apply Lemma 3.4 and we obtain
\[
E(t) > \frac{1}{4} ||u_t(t)||_H^2 + \frac{1 - \tilde{\mu}}{4} ||A^{1/2}u(t)||_H^2
\]
\[
+ \frac{1}{4} \int_{-\tau}^0 |k(s + \tau)| \cdot ||B^*u_t(s)||_H^2 ds + \frac{1}{4} \int_0^{+\infty} \mu(s) ||A^{1/2}\eta^t(s)||_H^2 ds,
\]
for all \( t \in [0, \delta) \). In particular we have that
\[
E(t) > \frac{1}{4} ||u_t(t)||_H^2, \quad \text{for } t \in [0, \delta).
\]

Therefore, we can apply Lemma 3.2, obtaining
\[
E(t) \leq \tilde{C}(\tau) E(0) < \tilde{C}(\tau) \rho^2.
\]
for any \( t \in [0, \delta] \). Since

\[
0 < \frac{1}{4} ||u(t)||_H^2 + \frac{1 - \bar{\mu}}{4} ||A^{\frac{1}{2}} u(t)||_H^2 \\
+ \frac{1}{4} \int_{t-\tau}^{t} |k(s + \tau)| \cdot ||B^* u_t(s)||_H^2 ds + \frac{1}{4} \int_{0}^{+\infty} \mu(s)||A^{\frac{1}{2}} \eta'(s)||_H^2 ds \\
\leq E(t) \leq \bar{C}(\tau) E(0),
\]

(3.33)

for all \( t \in [0, \delta] \), then we can extend the solution to the entire interval \([0, \tau]\).

Now, observe that from (3.33) and (3.32) we have

\[
h(||A^{\frac{1}{2}} u(\tau)||_H) \leq h \left( \frac{2}{(1 - \bar{\mu})^{\frac{1}{2}}} \bar{C}^{\frac{1}{2}}(\tau) E^{\frac{1}{2}}(0) \right) < \frac{1 - \bar{\mu}}{2}.
\]

(3.34)

By continuity, (3.34) implies that there exists \( \delta' \in (0, \tau] \) such that

\[
h(||A^{\frac{1}{2}} u(t)||_H) < \frac{1 - \bar{\mu}}{2}, \quad \forall t \in [\tau, \tau + \delta').
\]

From this, arguing as before, we deduce

\[
E(t) > \frac{1}{4} ||u(t)||_H^2 + \frac{1 - \bar{\mu}}{4} ||A^{\frac{1}{2}} u(t)||_H^2 + \frac{1}{4} \int_{t-\tau}^{t} |k(s + \tau)| \cdot ||B^* u_t(s)||_H^2 ds \\
+ \frac{1}{4} \int_{0}^{+\infty} \mu(s)||A^{\frac{1}{2}} \eta'(s)||_H^2 ds,
\]

for any \( t \in [\tau, \tau + \delta') \). In particular, also in such an interval we have \( E(t) > \frac{1}{4} ||u(t)||_H^2 \). Hence, we can apply again Lemma 3.2 on the time interval \([0, \tau + \delta')\) obtaining from (3.19), since \( \delta' \leq \tau \), the estimate

\[
0 < E(t) \leq \bar{C}(2\tau) E(0).
\]

As before, we can then extend the solution the whole interval \([0, 2\tau]\). At time \( t = 2\tau \) we have that

\[
h(||A^{\frac{1}{2}} u(2\tau)||_H) \leq h \left( \frac{2}{(1 - \bar{\mu})^{\frac{1}{2}}} \bar{C}^{\frac{1}{2}}(2\tau) E^{\frac{1}{2}}(0) \right) < \frac{1 - \bar{\mu}}{2},
\]

where we have used (3.32). Moreover, if \( 3 \leq N \),

\[
h \left( \frac{2}{(1 - \bar{\mu})^{\frac{1}{2}}} \bar{C}^{\frac{1}{2}}(3\tau) E^{\frac{1}{2}}(0) \right) \leq h \left( \frac{2}{(1 - \bar{\mu})^{\frac{1}{2}}} \bar{C}^{\frac{1}{2}}(N\tau) E^{\frac{1}{2}}(0) \right) < \frac{1 - \bar{\mu}}{2}.
\]

Thus, one can repeat again the same argument. By iteration, we then find a unique solution to problem (2.13) on the interval \([0, N\tau]\), where \( N \) is the natural number.
fixed at the beginning of the proof. Moreover, definition (3.27) of $\rho$ ensures that

$$h(\|A^{1/2}u(N\tau)\|_{\mathcal{H}}) \leq h \left( \frac{2}{1 - \tilde{\mu}} \tilde{C}_{1/2}(N\tau) E_{1/2}(0) \right) < \frac{1 - \tilde{\mu}}{2},$$

where we have used (3.31). Note that, by construction, (3.21) and (3.22) are satisfied in the whole $[0, N\tau)$. Then, from (3.22),

$$\|U(t)\|_{\mathcal{H}}^2 \leq 4E(t) \leq 4\tilde{C}(N\tau)E(0) < 4\tilde{C}(N\tau)\rho^2$$

and so

$$\|U(t)\|_{\mathcal{H}} \leq 2\tilde{C}_{1/2}(N\tau)\rho, \quad \forall \, t \in [0, N\tau].$$

Thus, we have proved that, under assumption (3.29) on the initial data, there exists a unique solution $U(\cdot)$ to problem (2.13) defined on the time interval $[0, N\tau]$. Moreover,

$$\|U(t)\|_{\mathcal{H}} \leq C_{\rho} := 2\tilde{C}_{1/2}(N\tau)\rho.$$

So far we have fixed $\rho$ satisfying (3.27); now, eventually choosing a smaller $\rho$, we assume that $\rho$ satisfies the additional assumption

$$L(C_{\rho}) = L(2\tilde{C}_{1/2}(N\tau)\rho) < \frac{\omega - \omega'}{2M}.$$ 

Then, the well-posedness assumption (W) of Theorem 3.1 is satisfied on $[0, N\tau]$. Therefore, we obtain that $U$ satisfies exponential decay estimate (3.17) and then

$$\|U(t)\|_{\mathcal{H}} \leq M \left( \|U(0)\|_{\mathcal{H}} + \int_0^\tau |k(s)| e^{\omega s} \|\hat{g}(s)\|_{\mathcal{H}} ds \right) e^{\gamma} e^{-\frac{\omega - \omega'}{2} t}, \quad \forall \, t \in [0, N\tau]. \quad (3.35)$$

In particular,

$$\|U(N\tau)\|_{\mathcal{H}} \leq M \left( \|U(0)\|_{\mathcal{H}} + \int_0^\tau e^{\omega s} |k(s)| \cdot \|\hat{g}(s)\|_{\mathcal{H}} ds \right) e^{\gamma} e^{-\frac{\omega - \omega'}{2} N\tau}. \quad (3.36)$$

Now observe that, from Cauchy–Schwarz inequality,

$$\int_0^\tau e^{\omega s} |k(s)| \cdot \|\hat{g}(s)\|_{\mathcal{H}} ds \leq e^{\omega \tau} \int_0^\tau |k(s)| ds \left( \int_0^\tau |\hat{g}(s)| ds \right)^2 \leq e^{\omega \tau} \left( \int_0^\tau |k(s)| ds \right) \left( \int_0^\tau \|\hat{g}(s)\|_{\mathcal{H}}^2 ds \right)^{1/2}.$$
Hence,
\[ \| U(t) \|_{\mathcal{H}} \leq M \rho \left( 1 + e^{\omega t} C^* \right)^{1/2} e^{\gamma e^{-\frac{\omega-\omega'}{2} t}}, \ \forall t \in [0, N\tau], \]
and then
\[ \| U(t) \|_{\mathcal{H}}^2 \leq 2M^2 \rho^2 \left( 1 + e^{2\omega t} C^* \right) e^{2\gamma e^{-\omega \tau} (N-1)\tau}, \ \forall t \in [0, N\tau], \quad (3.37) \]
where \( C^* \) is the constant defined in (1.2). From (3.37), with \( t = N\tau \), we deduce
\[ \| U(N\tau) \|_{\mathcal{H}}^2 + \int_{N\tau}^{(N+1)\tau} e^{\omega(s-N\tau)} |k(s)| \cdot \| B^* u_t (s - \tau) \|_{\mathcal{H}}^2 \mathrm{d}s \]
\[ \leq 2M^2 \rho^2 \left( 1 + e^{2\omega t} C^* \right) e^{2\gamma e^{-\omega \tau} (N-1)\tau} + e^{\omega t} b^2 \int_{N\tau}^{(N+1)\tau} |k(s)| \cdot \| U(s - \tau) \|_{\mathcal{H}}^2 \mathrm{d}s. \quad (3.38) \]

Now, observe that, for \( s \in [N\tau, (N+1)\tau] \), it results \( s - \tau \in [(N-1)\tau, N\tau] \); then, from (3.37) we deduce
\[ \| U(s - \tau) \|_{\mathcal{H}}^2 \leq 2M^2 \rho^2 \left( 1 + e^{2\omega t} C^* \right) e^{2\gamma e^{-\omega \tau} (N-1)\tau}, \ \forall s \in [N\tau, (N+1)\tau]. \]

This last estimate, used in (3.38), gives
\[ \| U(N\tau) \|_{\mathcal{H}}^2 + \int_{N\tau}^{(N+1)\tau} e^{\omega(s-N\tau)} |k(s)| \cdot \| B^* u_t (s - \tau) \|_{\mathcal{H}}^2 \mathrm{d}s \]
\[ \leq 2M^2 \rho^2 \left( 1 + e^{2\omega t} C^* \right) e^{2\gamma e^{-\omega \tau} (N-1)\tau} \left( 1 + e^{\omega t} b^2 C^* \right). \quad (3.39) \]

From (3.39) and (2.26), we then deduce
\[ \| U(N\tau) \|_{\mathcal{H}}^2 + \int_{N\tau}^{(N+1)\tau} |k(s)| \cdot \| B^* u_t (s - \tau) \|_{\mathcal{H}}^2 \mathrm{d}s < \rho^2. \]

Thus (cf. with (2.29)), one can argue as before on the interval \([N\tau, 2N\tau] \) obtaining a solution on \([0, 2N\tau] \). Iterating this procedure we get a global solution satisfying
\[ \| U(t) \|_{\mathcal{H}} < C^\rho. \]

Therefore, we have showed that problem (2.13) satisfies the well-posedness assumption (W) of Theorem 3.1.

We have then proved that, for suitably small data, solutions to problem (2.13) are globally defined and their energies satisfy an exponential decay estimate. Therefore, we are ready to prove Theorem 1.1.
Proof of Theorem 1.1  Let $\rho_0 = \rho^2$, where $\rho$ is constant satisfying (3.27) with $N$ as in (3.26). From (3.23) there exists a constant $C > 0$ such that

$$E(t) \leq C\|U(t)\|_{\mathcal{H}}^2 + \frac{1}{2} \int_{t-\tau}^{t} |k(s + \tau)| \cdot \|B^* u_t(s)\|_{\mathcal{H}}^2 ds$$

for any $t \geq 0$. Now, by a direct application of Theorems 3.1 and 3.5 we have that there exist $K$, $\gamma > 0$ such that

$$\|U(t)\|_{\mathcal{H}}^2 \leq Ke^{-\gamma t}, \quad (3.40)$$

for all $t \geq 0$, if the initial data satisfy (1.10). Moreover, observe that there exists a constant $C > 0$ such that

$$\frac{1}{2} \int_{t-\tau}^{t} |k(s + \tau)| \cdot \|B^* u_t(s)\|_{\mathcal{H}}^2 ds \leq CC^* e^{-\gamma(t-\tau)}, \quad (3.41)$$

where we used the fact that

$$\frac{1}{2} \|u_t(s)\|_{\mathcal{H}}^2 \leq \|U(s)\|_{\mathcal{H}}^2 \leq Ke^{-\gamma s}, \quad \forall s \geq 0,$$

and assumption (1.2). Then, from (3.40) and (3.41) we obtain (1.11). \qed

4 Examples

4.1 The wave equation with memory and source term

Let $\Omega$ be a nonempty bounded set in $\mathbb{R}^n$, with boundary $\Gamma$ of class $C^2$. Moreover, let $\mathcal{O} \subset \Omega$ be a nonempty open subset of $\Omega$. We consider the following wave equation:

$$u_{tt}(x, t) - \Delta u(x, t) + \int_{0}^{+\infty} \mu(s) \Delta u(x, t - s) ds + k(t) \chi_{\mathcal{O}} u_t(x, t - \tau)$$

$$= |u(x, t)|^{\sigma} u(x, t), \quad \text{in } \Omega \times (0, +\infty),$$

$$u(x, t) = 0, \quad \text{in } \Gamma \times (0, +\infty),$$

$$u(x, t) = u_0(x, t), \quad \text{in } \Omega \times (-\infty, 0],$$

$$u_t(x, 0) = u_1(x), \quad \text{in } \Omega,$$

$$u_t(x, t) = g(x, t), \quad \text{in } \Omega \times (-\tau, 0],$$

where $\tau > 0$ is the time delay, $\mu : (0, +\infty) \rightarrow (0, +\infty)$ is a locally absolutely continuous memory kernel, which satisfies the assumptions (i)–(iv), $\sigma > 0$ and the damping coefficient $k(\cdot)$ is a function in $L^1_{loc}([0, +\infty))$ satisfying (1.2). Then, system (4.42) falls in form (1.1) with $A = -\Delta$ and $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. 

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Defining $\eta'_s$ as in (1.4), we can rewrite system (4.42) in the following way:

$$
\begin{align*}
u_{tt}(x, t) - (1 - \tilde{\mu})\Delta u(x, t) - \int_0^{+\infty} \mu(s)\Delta \eta'_s(x, s)ds + k(t)\chi_\Omega u_t(x, t - \tau)
= |u(x, t)|^q u(x, t), & \text{ in } \Omega \times (0, +\infty), \\
n\eta'_s(x, t) &= u_t(x, t), & \text{ in } \Omega \times (0, +\infty) \\
u(x, t) &= 0, & \text{ in } \Gamma \times (0, +\infty), \\
\eta'_s(x, s) &= 0, & \text{ for } t \geq 0, \\
u(0, x) &= u_0(x) := u_0(x, 0), & \text{ in } \Omega, \\
u_t(x, 0) &= u_1(x) := \frac{\partial u_0}{\partial t}(x, t)|_{t=0}, & \text{ in } \Omega, \\
\eta^0_0(x, s) &= \eta_0(x, s) := u_0(x, 0) - u_0(x, -s), & \text{ in } \Omega \times (0, +\infty), \\
u_t(x, t) &= g(x, t), & \text{ in } \Omega \times (-\tau, 0).
\end{align*}
$$

(4.43)

As before, we introduce the Hilbert space $L^2_{\mu}((0, +\infty); H^1_0(\Omega))$ endowed with the inner product

$$
(\phi, \psi)_{L^2_{\mu}((0, +\infty); H^1_0(\Omega))} := \int_\Omega \left( \int_0^{+\infty} \mu(s)\nabla \phi(x, s)\nabla \psi(x, s)ds \right) dx,
$$

and consider the Hilbert space

$$\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \times L^2_{\mu}((0, +\infty); H^1_0(\Omega)),
$$

equipped with the inner product

$$
\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}} := (1 - \tilde{\mu}) \int_\Omega \nabla u \nabla \tilde{u} dx + \int_\Omega v \tilde{v} dx + \int_0^{+\infty} \mu(s)\nabla w \nabla \tilde{w} ds dx.
$$

Setting $U = (u, u_t, \eta'_s)$, we can rewrite (4.45) in form (2.13), where

$$
\mathcal{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} (1 - \tilde{\mu})\Delta u + \int_0^{+\infty} \mu(s)\Delta w(s)ds \\ -w_x + v \end{pmatrix},
$$

with domain

$$D(A) = \{ (u, v, w) \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2_{\mu}((0, +\infty); H^1_0(\Omega)) : \}

$$

$$
(1 - \tilde{\mu})u + \int_0^{+\infty} \mu(s)w(s)ds \in H^2(\Omega) \cap H^1_0(\Omega), \quad w_x \in L^2_{\mu}((0, +\infty); H^1_0(\Omega)),
$$

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\( \mathcal{B}(u, v, \eta)^T := (0, \chi_{\Omega} v, 0)^T \) and \( F(U(t)) = (0, |u(t)|^\sigma u(t), 0)^T \). For any \( u \in H_0^1(\Omega) \) consider the functional

\[
\psi(u) := \frac{1}{\sigma + 2} \int_{\Omega} |u(x)|^{\sigma+2} dx.
\]

By Sobolev’s embedding theorem, we know that if \( 0 < \sigma < \frac{4}{n-2} \), then \( \psi \) is well-defined, and Gâteaux differentiable at any point \( u \in H_0^1(\Omega) \), with Gâteaux derivative given by

\[
D\psi(u)(v) = \int_{\Omega} |u(x)|^\sigma u(x)v(x)dx, \quad n
\]

for any \( v \in H_0^1(\Omega) \). Moreover, as in [4], if \( 0 < \sigma \leq \frac{2}{n-2} \), then \( \psi \) satisfies the assumptions (H1), (H2), (H3). Define the energy as follows:

\[
E(t) := \frac{1}{2} \int_{\Omega} |u_t(x, t)|^2 dx + \frac{1 - \tilde{\mu}}{2} \int_{\Omega} |\nabla u(x, t)|^2 dx - \psi(u(x, t)) \\
+ \frac{1}{2} \int_{t-\tau}^{t} \int_{\Omega} |k(s + \tau)\cdot u_t(x, s)|^2 dx ds \\
+ \frac{1}{2} \int_{0}^{+\infty} \mu(s) \int_{\Omega} |\nabla \eta'(x, s)|^2 dx ds.
\]

Theorem 3.1 applies to this model giving well-posedness and exponential decay of the energy for suitably small initial data, provided that condition (1.9) on the time delay holds for every \( t \geq 0 \).

### 4.2 The plate system with memory and source term

Let \( \Omega \) be a nonempty bounded set in \( \mathbb{R}^n \), with boundary \( \Gamma \) of class \( C^2 \). Let us denote \( v(x) \) the outward unit normal vector at any point \( x \in \Gamma \). Moreover, let \( \mathcal{O} \subset \Omega \) be a nonempty open subset of \( \Omega \). We consider the following viscoelastic plate system:

\[
\begin{align*}
&u_{tt}(x, t) + \Delta^2 u(x, t) - \int_{0}^{+\infty} \mu(s)\Delta^2 u(x, t-s)ds + k(t)\chi_{\mathcal{O}} u_t(x, t-\tau) \\
&= |u(x, t)|^\sigma u(x, t), \quad \text{in} \ \Omega \times (0, +\infty), \\
&u(x, t) = \frac{\partial u}{\partial v}(x, t) = 0, \quad \text{in} \ \Gamma \times (0, +\infty), \\
&u(x, t) = u_0(x, t) \quad \text{in} \ \Omega \times (-\infty, 0], \\
&u_t(x, 0) = u_1(x), \quad \text{in} \ \Omega, \\
&u_t(x, t) = g(x, t), \quad \text{in} \ \Omega \times (-\tau, 0],
\end{align*}
\]

where \( \tau > 0 \) is the time delay, \( \mu : (0, +\infty) \to (0, +\infty) \) is a locally absolutely continuous memory kernel, which satisfies the assumptions (i)-(iv), \( \sigma > 0 \) and the damping coefficient \( k(\cdot) \) is a function in \( L^1_{loc}([0, +\infty)) \) satisfying (1.2). This system
again falls in (1.1) for $A = \Delta^2$ with domain $D(A) = H^4(\Omega) \cap H^1_0(\Omega)$. Defining $\eta^t_s$ as in (1.4), we can rewrite system (4.44) in the following way:

$$
\begin{align*}
&u_{tt}(x, t) + (1 - \tilde{\mu}) \Delta^2 u(x, t) + \int_0^{+\infty} \mu(s) \Delta^2 \eta^t_s(x, s) ds + k(t) \chi_{\Omega} u_t(x, t - \tau) \\
&= |u(x, t)|^\sigma u(x, t), \quad \text{in} \quad \Omega \times (0, +\infty), \\
&\eta^t_s(x, s) = -\eta^t_s(x, s) + u_t(x, t), \quad \text{in} \quad \Omega \times (0, +\infty) \times (0, +\infty), \\
&u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = 0, \quad \text{in} \quad \Gamma \times (0, +\infty), \\
&\eta^t_s(x, s) = 0, \quad \text{in} \quad \Gamma \times (0, +\infty), \quad \text{for} \ t \geq 0, \\
&u(x, 0) = u_0(x, 0), \quad \text{in} \quad \Omega, \\
&u_t(x, 0) = u_1(x) := \frac{\partial u_0}{\partial t}(x, t) \bigg|_{t=0}, \quad \text{in} \quad \Omega, \\
&\eta^0_s(x, s) = \eta_0(x, s) := u_0(x, 0) - u_0(x, -s), \quad \text{in} \quad \Omega \times (0, +\infty), \\
&u_t(x, t) = g(x, t), \quad \text{in} \quad \Omega \times (-\tau, 0).
\end{align*}
$$

(4.45)

Then, arguing analogously to the previous example, one can recast (4.45) in form (2.13). Moreover, for $(n - 4)\sigma \leq 4$ (cf., e.g., [16]) the nonlinear source satisfies the required assumptions.

Theorem 3.1 applies then to this model giving well-posedness and exponential decay of the energy for suitably small initial data, provided that condition (1.9) on the time delay holds for every $t \geq 0$.

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