APPROSSIMATION OF A STOCHASTIC TWO-PHASE FLOW MODEL BY A SPLITTING-UP METHOD

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Abstract. In this paper, we consider a stochastic Allen-Cahn Navier-Stokes system in a bounded domain of \( \mathbb{R}^d \), \( d = 2, 3 \). The system models the evolution of an incompressible isothermal mixture of binary fluids under the influence of stochastic external forces. We prove the existence of a global weak martingale solution. The proof is based on splitting-up method as well as some compactness method.

1. Introduction. In this paper, we consider a model of incompressible isothermal two-phase flow subjected to a random force in a bounded domain \( D \subset \mathbb{R}^d \) (\( d = 2, 3 \)) with \( C^2 \) boundary \( \partial D \). Let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}) \) be a stochastic basis and \( U \) be a separable Hilbert space containing an orthonormal basis \( \{e_j, j = 1, 2, \ldots\} \). Also, let \( \{\beta_j(t), t \in [0,T], j = 1, 2, \ldots\} \) be a given sequence of mutually independent standard real \( \mathcal{F}_t \)-Wiener processes and

\[
W(t) = \sum_{j=1}^{+\infty} \beta_j(t)e_j,
\]

(1.1)

be a \( U \)-valued cylindrical Wiener process defined on a stochastic basis \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}) \). Since \( U \) is arbitrary, the sum (1.1) does not generally converge in \( U \). For this reason, we will occasionally make use of a larger auxiliary space \( U_0 \supseteq U \) which is defined according to

\[
U_0 = \left\{ v = \sum_{k \geq 0} \alpha_k e_k : |v|_{U_0}^2 < \infty \right\}, \quad |v|_{U_0}^2 = \sum_{k \geq 1} \frac{\alpha_k^2}{k^2}.
\]

(1.2)

Note that the embedding of \( U \subset U_0 \) is Hilbert-Schmidt. Moreover, using standard martingale arguments with the fact that each \( \beta_j \) is almost surely continuous (see [7]), we have that, for almost every \( \omega \in \Omega \), \( W(\omega) \in C([0,T];U_0) \). Then we look
at a system of two-phase flow model, which is obtained by coupling the stochastic Navier-Stokes system for the velocity of the fluids with the Allen-Cahn equation for the phase parameter as follows:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p - K \mu \nabla \phi &= g_1(t, u) + g_2(t, u) \frac{\partial W}{\partial t}, \\
\text{div}(u) &= 0, \\
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi + \mu &= 0, \\
\mu &= -\varepsilon \Delta \phi + \alpha f(\phi),
\end{align*}
\]

in \((0,T) \times D\), subject to the boundary and initial conditions

\[
\begin{align*}
\partial_\eta \phi &= 0, & \text{on } \partial D \times (0,T), \\
u &= 0 & \text{on } \partial D \times (0,T), \\
(u, \phi)(0) &= (u_0, \phi_0) & \text{in } D,
\end{align*}
\]

where \(\eta\) is the outward normal to \(\partial D\).

In the model (1.3)-(1.4), the unknown functions are the velocity \(u = (u_1, u_2, \ldots, u_d)\) of the fluid, its pressure \(p\) and the order (phase) parameter \(\phi\). The external volume force \(g_1(t, u)\) is given. The term \(g_2(t, u) \frac{\partial W}{\partial t}\) represents random external forces depending eventually on \(u\) and defined by

\[
g_2(t, u) \frac{\partial W}{\partial t} = \frac{d}{dt} \left( \int_0^t g_2(s, u(s))dW_s \right).
\]

Also in the model (1.3)-(1.4), the quantity \(\mu\) is the chemical potential of the binary mixture which is given by the variational derivative of the following free energy functional

\[
F(\phi) = \int_D \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) dx,
\]

where, \(F(r) = \int_0^r f(\zeta)d\zeta\) is the suitable double-well potential. The quantities \(\nu\) and \(K\) are positive constants that correspond to the kinematic viscosity of the fluid and capillarity (stress) coefficient, respectively. Here, \(\varepsilon\) and \(\alpha\) are two positive parameters describing the interactions between the two phases. In particular, \(\varepsilon\) is related to the thickness of the interface separating the two fluids and it is reasonable to assume that \(\varepsilon \leq \alpha\). A typical example of potential \(F\) is of logarithmic type. However, this potential is very often replaced by a polynomial approximation of the type \(F(r) = \gamma_1 r^4 - \gamma_2 r^2\), with \(\gamma_1\) and \(\gamma_2\) are positive constants. Note that, \((1.3)_1\) can be replaced by

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u, \nabla)u + \nabla \tilde{p} + K \text{div}(\nabla \phi \otimes \nabla \phi) &= g_1(t, u) + g_2(t, u) \frac{\partial W}{\partial t}, \\
\text{where } \tilde{p} &= p - K \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right), \\
K \mu \nabla \phi &= \nabla \left( K \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) - K \text{div}(\nabla \phi \otimes \nabla \phi) \right).
\end{align*}
\]

The stress tensor \(\text{div}(\nabla \phi \otimes \nabla \phi)\) is considered as the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

The deterministic two-phase flow model has been investigated by several authors both theoretically and numerically (see for instance, [11, 19, 27, 28, 29]). In [27], the third author of this paper prove the existence and uniqueness of the weak and strong solution of Allen-Cahn Navier-Stokes (ACNS) model in two dimensional bounded
domain when the external force contains some delays. He also discusses the asymptotic behavior of the weak solutions and the stability of the stationary solutions. Also in [28] he has studied the three dimensional globally modified ACNS model and prove the existence and final fractal dimension of a pullback attractor in the space $V$ under appropriate properties on the time depending forcing term. In [29] the third author of this paper with his collaborators proposes a numerical scheme based on the implicit Euler scheme for the time discretization of the ACNS model in n-dimensional compact Riemannian manifolds both in dimension two and three. They proved that the discrete attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero. In [11], X. Feng et al. propose and prove the convergence of a numerical scheme based on the semi-implicit Euler scheme and finite element method for the fully discretisation of the ACNS model in three dimensional bounded case. 

The stochastic ACNS model (1.3)-(1.4) are often used as a complementary model to the deterministic one to better understand the role of small perturbation fluctuations present in two phases of the fluid flow. The existence and pathwise uniqueness of global probabilistic strong solution in two-dimensional case has been shown by the third author of this paper in [26]. The proof is based on the Galerkin approximation, the stopping time techniques and the principle of weak convergence in functional analysis. Moreover, he showed that the Galerkin solution converges in mean square to the probabilistic strong solution. In [25], the third author of the present paper, have studied the stochastic 3D globally modified ACNS system. He proved the existence of the unique strong solution (in the PDE and probabilistic sense) and used a limiting argument to derive the existence of a global weak martingale solution for the stochastic ACNS system in three-dimensional case. The method of the proof uses again a Galerkin approximation and some kind of local monotonicity of the non linear terms of the model. The stochastic two-phase separation model and stochastic two-phase flow have been studied in [1, 16]. In [1] the authors have proved a local existence and uniqueness result for the stochastic Cahn-Hilliard/Allen-Cahn model with a multiplicative space-time white noise via the techniques from semigroup theory. In [16], the authors consider a stochastic perturbation of the phase field alpha-Navier-Stokes model with vesicle-fluid interaction and prove the existence and uniqueness of solution in classical spaces of $L^2$ functions with estimates of non-linear terms and bending energy. 

In this paper, we use the splitting-up method to investigate the existence of a global weak martingale solution for the stochastic ACNS system in both two and three dimensional bounded domain. The numerical scheme consists of two systems. The first system is the deterministic ACNS system. The second is composed of a linear random PDE and the deterministic Allen-Cahn equation. The proof of our main result follows closely the approach used in [24] where the author uses the splitting-up method and some compactness results to establish the existence of a weak martingale solution of nonlinear stochastic hyperbolic equations. Let us note that the coupling between the stochastic Navier-Stokes equations and the Allen-Cahn system introduces in the coupled model a highly nonlinear term that makes the analysis more involved. The splitting-up method has been used to investigate issues of existence and numerical simulation for the several stochastic partial differential equations (see for instance, [2, 3, 4, 9, 15, 17, 18, 20, 24]). 

The rest of this paper is organized as follows. In Section 2, we introduce the mathematical setting of the model (1.3)-(1.4). The main result is stated in Section
and in Section 4, we introduce a splitting-up scheme for the problem (1.3)-(1.4), we derive a priori estimates for the approximating solutions and prove the crucial result of tightness of approximating solutions. In the very same section, we apply Prokhorov’s and Skorokhod’s compactness results and prove the main result.

2. Mathematical setting. In this section, we recall from [14, 26] a weak formulation of (1.3). Hereafter, as in [14], we will assume that the domain $D$ is bounded with a smooth boundary $\Gamma = \partial D$ (e.g. of class $C^2$) and that $f \in C^1(\mathbb{R})$ and satisfies

\[
\begin{cases}
\lim_{|r| \to +\infty} f'(r) > 0, \\
|f(r)| \leq c_f (1 + |r|^{m+1}), \quad \forall r \in \mathbb{R},
\end{cases}
\]

(2.1)

where $c_f$ is some positive constant, $m \in [2, +\infty)$ for $d = 2$ and $m = 2$ for $d = 3$ is fixed. We note that the classical cubic Allen-Cahn potential $f(r) = r^3 - r$, for all $r \in \mathbb{R}$ satisfies assumption (2.1).

Let us now recall from [14, 26] the functional set up of the model (1.3). If $X$ is a real Hilbert space with inner product $\langle . , . \rangle_X$, we will denote the induced norm by $|.|_X$ while $X^*$ will indicate its dual. We denote by $\langle . , . \rangle$ and $|.|_{L^2}$ respectively, the scalar product and associated norm in $(L^2(D))^d$, $d = 2, 3$ and by $(\nabla u, \nabla v)$ the scalar product in $(L^2(D))^d$ of the gradients of $u$ and $v$. We set

\[ V_1 = \{ u \in (C_0^\infty(D))^d : \nabla u = 0 \text{ in } D \}. \]

We denote by $H_1$ and $V_1$ the closure of $V_1$ in $(L^2(D))^d$ and $(H^1_0(D))^d$ respectively. $H_1$ is a Hilbert space equipped with the inner product of $(L^2(D))^d$, and $V_1$ is a Hilbert subspace of $(H^1_0(D))^d$ equipped with the gradient norm.

\[ \| u \|^2 = (\nabla u, \nabla u). \]

(2.2)

We denote by $A_0$ the Stokes operator with domain $D(A_0) = V_1 \cap (H^2(D))^d$, defined by $A_0 u = -\mathcal{P} \Delta u$, $u \in D(A_0)$, where $\mathcal{P}$ is the Leray operator i.e. the projection operator from $(L^2(D))^d$ into $H_1$. Then, $A_0$ is self-adjoint positive unbounded operator in $H_1$ which is associated with the scalar product defined above. Furthermore, $A_0^{-1}$ is a compact linear operator on $H_1$ and assuming that $\partial D$ is Lipschitz, $|A_0|_{L^2}$ defines in $D(A_0)$, a norm which is equivalent to the $(H^2(D))^d$-norm.

Note that from (2.1), we can find $\gamma > 0$ such that

\[ \lim_{|r| \to +\infty} f'(r) > 2\gamma > 0. \]

(2.3)

We define the linear positive unbounded operator $A_\gamma$ on $L^2(D)$ by

\[ A_\gamma = -\Delta \phi + \gamma \phi, \quad \forall \phi \in D(A_\gamma), \]

(2.4)

where

\[ D(A_\gamma) = \{ \phi \in H^2(D) : \frac{\partial \phi}{\partial \eta} = 0 \text{ on } \partial D \}. \]

(2.5)

Note that $A_\gamma^{-1}$ is a compact linear operator on $L^2(D)$ and $|A_\gamma|_{L^2}$ is a norm on $D(A_\gamma)$ which is equivalent to the $H^2(D)$-norm. Hereafter, we will set

\[ \| \psi \|^2 = (A_\gamma \psi, \psi) = \| \psi \|^2 + \gamma |\psi|_{L^2}^2, \quad \forall \psi \in H^1(D). \]

(2.6)

Now, we define the Hilbert spaces

\[ \mathcal{H} = H_1 \times H^1(D) \quad \text{and} \quad \mathcal{U} = V_1 \times D(A_\gamma), \]

(2.7)
endowed with the scalar products whose associated norms are respectively
\[(u, \phi)^2 = |u|_2^2 + \varepsilon \|\phi\|_\gamma^2 \quad \text{and} \quad \|(u, \phi)\|_H^2 = \|u\|_2^2 + |A_\gamma \phi|_{L^2}^2. \tag{2.8}\]

Hereafter, we set
\[H_2 = L^2(D), \quad V_2 = H^1(D), \quad f_r(y) = f(r) - \alpha^{-1} \varepsilon \gamma r \tag{2.9}\]
and observe that \(f_r\) also satisfies (2.3) with \(\gamma\) in the place of \(2\gamma\) since \(\varepsilon \leq \alpha\). Also, its primitive \(F_r(y) = \int_0^y f_r(s)ds\) is bounded from below.

In addition, we need to introduce the bilinear operator \(B_0, B_1\) (and their associated trilinear form \(b_0, b_1\)) as well as the coupling mapping \(R_0\), which are defined from \(D(A_0) \times D(A_0)\) into \(H_1\), \(D(A_0) \times D(A_\gamma)\) into \(L^2(D)\) and \(L^2(D) \times D(A_\gamma)\) into \(H_1\) respectively. More precisely, we set
\[
\langle B_0(u, v) \rangle = \int_D [u \cdot \nabla v] \cdot wdx = b_0(u, v, w), \ \forall u, v, w \in D(A_0),
\]
\[
\langle B_1(u, \phi, \psi) \rangle = \int_D [u \cdot \nabla \psi] \cdot \phi dx = b_1(u, \phi, \psi), \ \forall u \in D(A_0), \ \phi, \psi \in D(A_\gamma),
\]
\[
\langle R_0(\mu, \phi) \rangle = \int_D \mu \nabla \phi \cdot wdx, \ \forall w \in D(A_0), \ \phi \in D(A_\gamma), \ \mu \in L^2(D).
\]

Note that
\[
R_0(\mu, \phi) = P(\mu \nabla \phi). \tag{2.10}\]

We recall that, using the integration by part, a suitable generalized Hölder inequality and the Ladyzhenskaya inequality, we derive that \(b_0, b_1\) satisfy the following properties
\[
b_0(u, v, w) = -b_0(u, w, v) \quad \forall u, v, w \in V_1, \tag{2.11}\]
\[
b_0(u, v, v) = 0 \quad \forall u, v \in V_1, \tag{2.12}\]
\[
b_1(u, \phi, \psi) = -b_1(u, \psi, \phi) \quad \forall u \in V_1, \phi, \psi \in D(A_\gamma), \tag{2.13}\]
\[
b_1(u, \phi, \phi) = 0 \quad \forall u \in V_1, \phi \in D(A_\gamma), \tag{2.14}\]
\[
|b_0(u, v, w)| \leq |u|_{L^2} |v|_{H^1} |w|_{H^1} \leq c \left\{ \begin{array}{ll}
|u|_{L^2}^{1/2} \|u\|_{H^1} \|w\|_{H^1} & \text{for } d = 2, \\
|u|_{L^2}^{1/4} \|u\|_{H^1} \|w\|_{H^1}^{3/4} & \text{for } d = 3,
\end{array} \right. \tag{2.15}
\]

for all \(u, v, w \in V_1\),
\[
|b_1(u, \phi, \psi)| \leq |u|_{L^4} |\nabla \phi|_{L^4} |\psi|_{L^2}
\]
\[
\leq c \left\{ \begin{array}{ll}
|u|_{L^2}^{1/2} \|u\|_{H^1} \|\nabla \phi\|_{L^2}^{1/2} \|\nabla \phi\|_{H^1} \|\psi\|_{L^2}, & \text{for } d = 2, \\
|u|_{L^2}^{1/4} \|u\|_{H^1} \|\nabla \phi\|_{L^2}^{1/4} \|\nabla \phi\|_{H^1}^{3/4} \|\psi\|_{L^2}, & \text{for } d = 3,
\end{array} \right. \tag{2.16}
\]
\[
\leq c \left\{ \begin{array}{ll}
|u|_{L^2}^{1/2} \|u\|_{H^1} \|\phi\|_{L^2}^{1/2} \|A_\gamma \phi\|_{L^2} \|\psi\|_{L^2}, & \text{for } d = 2, \\
|u|_{L^2}^{1/4} \|u\|_{H^1} \|\phi\|_{L^2}^{1/4} \|A_\gamma \phi\|_{L^2}^{3/4} \|\psi\|_{L^2}, & \text{for } d = 3,
\end{array} \right. \tag{2.17}
\]

for all \(u, \phi, \psi \in V_1\).
for all $u \in V_1$, $\phi, \psi \in D(A_\gamma)$.

We then infer that $B_0$, $B_1$, and $R_0$ satisfy the following estimates.

$$
|B_0(u, v)|_{V_1^*} \leq c \left\{ \begin{array}{ll}
\|u\|_{L_2^2}^{1/2} \|\phi\|_{H^1} \|\phi\|_{H^1}^{1/2} \|\psi\|_{L_2^2}^{1/2}, & \text{for } d = 2, \\
\|u\|_{L_2^2}^{1/4} \|\phi\|_{H^1} \|\phi\|_{H^1}^{1/4} \|\psi\|_{L_2^2}^{1/4}, & \text{for } d = 3,
\end{array} \right.
$$

for all $u, v, w \in V_1$.

$$
|R_0(A_1 \phi, \rho)|_{V_1^*} \leq c \left\{ \begin{array}{ll}
\|\rho\|_{H^1}^{1/2} |A_\gamma \rho|_{L_2^2}^{1/2}, & \text{for } d = 2, \\
\|\rho\|_{H^1}^{1/4} |A_\gamma \rho|_{L_2^2}^{1/4}, & \text{for } d = 3,
\end{array} \right.
$$

for all $\phi, \rho \in D(A_\gamma)$.

Now, following the idea introduced in [26], we recall some notations and useful results related to the nonlinear coupling operator $R_0$. Then, we introduce the trilinear form $b_2$ defined by

$$
b_2(\phi, \psi, u) = -\sum_{i,j=1}^{d=2,3} \int_D \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \frac{\partial u}{\partial x_j} \, dx \quad \forall \phi, \psi \in W^{1,4}(D), \ u \in V_1.
$$

We can easily derive from (2.21) that for any $\phi, \psi \in D(A_\gamma)$, $u \in V_1$,

$$
|b_2(\phi, \psi, u)| \leq |\nabla \phi|_{L_1^4} |\nabla \psi|_{L_1^4} |\nabla u|_{L_1^2}
\leq c \left\{ \begin{array}{ll}
|\phi|_{H^1}^{1/2} |A_\gamma \phi|_{L_2^2}^{1/2} |\psi|_{H^1}^{1/2}, & \text{in } d = 2, \\
|\phi|_{H^1}^{1/4} |A_\gamma \phi|_{L_2^2}^{3/4} |\psi|_{H^1}^{1/4}, & \text{in } d = 3.
\end{array} \right.
$$

This prove that for any $u \in V_1$, the real valued mapping $b_2(\cdot, \cdot, u)$ defined on $D(A_\gamma) \times D(A_\gamma)$ is continuous. Consequently, there exists a bilinear operator $R_1$ defined on $D(A_\gamma) \times D(A_\gamma)$ with values in $V_1^*$ such that

$$
\langle R_1(\phi, \psi), u \rangle = b_2(\phi, \psi, u), \quad \forall \phi, \psi \in D(A_\gamma), \ u \in V_1,
$$

and from (2.22) we deduce that

$$
|R_1(\phi, \psi)|_{V_1^*} \leq c \left\{ \begin{array}{ll}
|\phi|_{H^1}^{1/2} |A_\gamma \phi|_{L_2^2} |\psi|_{H^1}^{1/2} |A_\gamma \psi|_{L_2^2}^{1/2}, & \text{in } d = 2, \\
|\phi|_{H^1}^{1/4} |A_\gamma \phi|_{L_2^2}^{3/4} |\psi|_{H^1}^{1/4} |A_\gamma \psi|_{L_2^2}^{3/4}, & \text{in } d = 3.
\end{array} \right.
$$

We recall from [26, Remark 2.2] that for $d = 2, 3$,

$$
R_0(A_\gamma \phi, \phi) = R_1(\phi, \phi), \quad \forall \phi \in D(A_\gamma).
$$

However, we infer from (2.24) that

$$
\|R_0(A_\gamma \phi, \phi)\|_{V_1^*} = \|R_1(\phi, \phi)\|_{V_1^*} \leq c \left\{ \begin{array}{ll}
|\phi|_{H^1} |A_\gamma \phi|_{L_2^2}, & \text{in } d = 2, \\
|\phi|_{H^1}^{1/2} |A_\gamma \phi|_{L_2^2}^{3/2}, & \text{in } d = 3.
\end{array} \right.
$$
Let us consider the following energy functional $E : \mathcal{H} \to \mathbb{R}_+$ defined by
\[
E(w, \psi) = |(w, \psi)|^2_{\mathcal{H}} + 2\alpha(F_0(\psi), 1) + c_1,
\]
where $c_1$ is sufficiently large so that $2\alpha(F_0(\psi), 1) + c_1 \geq 0$ for all $\psi \in H^1(D)$. Note that such a constant exists, since $F_0$ is bounded below.

For any Hilbert space $H$ with scalar product $(.,.)_H$, we will denote by $L^2(U; H)$ the separable Hilbert space of Hilbert-Schmidt operators from $U$ into $H$, and by $(.,.)_{L^2(U; H)}$ and $\|\cdot\|_{L^2(U; H)}$ the scalar product and its associated norm in $L^2(U; H)$.

For a separable Banach space $X$, $p \in [1, +\infty]$ and $T > 0$ we denote by $M^2_\mathcal{F}_t(0, T; X)$ the space of all processes $\psi \in L^p(\Omega \times (0, T), d\mathbb{P} \otimes dt; X)$ that are $\mathcal{F}_t$-progressively measurable. We will also denote by $L^p(\Omega; C([0, T]; X))$, for $1 \leq p < +\infty$, the space of all continuous and $\mathcal{F}_t$-progressively measurable $X$-valued processes $\{\psi_t; 0 \leq t \leq T\}$, satisfying
\[
E \left[ \sup_{t \in [0, T]} \|\psi_t\|_X^p \right] < +\infty.
\]

For any process $\psi \in L^p(\Omega \times (0, T), d\mathbb{P} \otimes dt; L^2(U; H))$ which is $\mathcal{F}_t$-progressively measurable, the stochastic integral of $\psi$ with respect to the Wiener process $W_t$, $t \in [0, T]$ is denoted by $\int_0^t \psi(s)dW_s$, $0 \leq t \leq T$ and is defined as the unique continuous $H$-valued $\mathcal{F}_t$-martingale, such that for all $h \in H$, we have
\[
\left( \int_0^t \psi(s)dW_s, h \right)_H = \sum_{j=1}^{+\infty} \int_0^t (\psi(s)e_j, h)_H d\beta_j(s), 0 \leq t \leq T,
\]
where the integral with respect to $d\beta_j(s)$ is understood in the sense of Itô, and the series converges component by component in $L^2(\Omega; (C([0, T]; \mathbb{R}))$. For some properties of this stochastic integral, the reader is referred to classical textbook such as [7].

For $r, p \geq 1$, we denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}; L^r(0, T; X))$ the space of functions $u = u(t, x, \omega)$ with values in $X$ defined on $[0, T] \times \Omega$ and such that
1) $u$ is measurable with respect to $(t, \omega)$ and for $t$, $u(t, \cdot)$ is $\mathcal{F}_t$-measurable.
2) $u \in X$ for almost $(t, \omega)$ and $|u|_{L^p(\Omega, \mathcal{F}, \mathbb{P}; L^r(0, T; X))} = \left[ \mathbb{E}\left( \int_0^T \|u(t)\|_X^r \, dt \right) \right]^{1/r} < \infty$, where $\mathbb{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$.

The space $L^p(\Omega, \mathcal{F}, \mathbb{P}; L^r(0, T; X))$ so defined is a Banach space.

When $r = +\infty$, the norm in $L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; X))$ is given by
\[
|u|_{L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; X))} = \left[ \mathbb{E} \sup_{0 \leq t \leq T} \|u(t)\|_X^p \right]^{1/p}.
\]

Now, we impose the following set of conditions on the nonlinear terms $g_1(.)$ and $g_2(.)$. However, hereafter, we assume that
\begin{enumerate}
\item[(H1)] $g_1 : (0, T) \times H_1 \to H_1$ is non random and such that for all $t \in [0, T]$, $g_1(t, \cdot)$ is continuous and sublinear mapping from $H_1$ into $H_1$. i.e., there exists a constant $K_1 > 0$, such that $dt$-a.e. in $(0, T)$,
\[
|g_1(t, v)|_{L^2} \leq K_1(1 + |v|_{L^2}).
\]
\item[(H2)] $g_2 : (0, T) \times H_1 \to L^2(U; H_1)$ is non random and such that for all $t \in [0, T]$, $g_2(t, \cdot)$ is a continuous and with linear growth from $H_1$ into $L^2(U; H_1)$. i.e., there exists a constant $K_2 > 0$, such that $dt$-a.e. in $(0, T)$,
\[
|g_2(t, v)|_{L^2(U; H_1)} \leq K_2(1 + |v|_{L^2}).
\]
\end{enumerate}
3. **Statement of the main result.** Now, we introduce the concept of the weak martingale solution to Problem (1.3)-(1.4).

**Definition 3.1.** A weak martingale solution of problem (1.3)-(1.4) is a system

\[ (\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_{t \in [0,T]}, \mathbb{P}', u, \phi, W) \]

such that

- \((\Omega', \mathcal{F}', \mathbb{P}')\) is a probability space, \(\{\mathcal{F}'_t\}_{t \in [0,T]}\) is a filtration,
- \(W(t)\) is an \(U\)-valued cylindrical Wiener process,
- The processes \(u\) and \(\phi\) are progressively \(\mathcal{F}_t\)-measurable and for all \(p \geq 2\),
  \[(u, \phi) \in L^p(\Omega', \mathcal{F}', \mathbb{P}'; L^\infty(0, T; \mathcal{H})) \cap L^p(\Omega', \mathcal{F}', \mathbb{P}'; L^2(0, T; \mathcal{U})), \quad (3.1)\]
- for almost all \((\omega, t) \in \Omega \times [0, T]\), the equation
  \[ u(t) + \nu \int_0^t A_0 u(s)ds + \int_0^t B_0(u, u)ds - K \int_0^t R_0(\varepsilon A_\gamma \phi, \phi)ds \]
  \[ = u_0 + \int_0^t g_1(s, u(s))ds + \int_0^t g_2(s, u(s))dW_s, \quad \text{in } V_1^*, \quad (3.2) \]
  \[ \phi(t) + \int_0^t \mu(s)ds + \int_0^t B_1(u, \phi)ds = \phi_0, \quad \text{in } V_2^*, \quad (3.3) \]
  \[ \mu = \varepsilon A_\gamma \phi + \alpha f_\gamma(\phi), \quad \text{in } V_2^*, \quad (3.4) \]

is satisfied. This means that for all \((v, \psi) \in V_1 \times V_2\), the integral identity

\[ (u(t), v) + \int_0^t (\nu A_0 u(s) + B_0(u, u) - KR_0(\varepsilon A_\gamma \phi, \phi), v)ds \]
\[ = (u_0, v) + \int_0^t (g_1(s, u(s)), v)ds + \left( \int_0^t g_2(s, u(s))dW_s, v \right), \quad (3.5) \]
\[ (\phi(t), \psi) + \int_0^t (\mu(s), \psi)ds + \int_0^t (B_1(u, \phi), \psi)ds = (\phi_0, \psi), \quad (3.6) \]
\[ (\mu, \psi) = \varepsilon (A_\gamma \phi, \psi) + \alpha (f_\gamma(\phi), \psi), \quad (3.7) \]

holds.

The main result of this paper is given in the following theorem.

**Theorem 3.2.** We assume that the hypotheses \((H1)-(H2)\) are satisfied and \(d = 2, 3\).

Let \(u_0 \in H_1\) and \(\phi_0 \in V_2\) such that

\[ \mathcal{E}(u_0, \phi_0) < \infty, \quad (3.8) \]

Then there exists a weak martingale solution of problem (1.3)-(1.4) in the sense of Definition 3.1.

4. **Proof of the main result.** The proof of the main result is done by using a numerical scheme based on the splitting-up method. This proof will be divided into three steps. In the first step, we present and analyze a numerical scheme based on the time discretisation. In the second step we establish some a priori estimates for the solution of the scheme in suitable functions spaces. In the third step, we prove the tightness for the approximating solutions. In the last step, we proceed with the passage to the limit in the equation and the conclusion of the proof of Theorem 3.2.
4.1. Splitting-Up algorithm. Let $N$ be a non-negative integer and $k = \frac{T}{N+1}$. We divide the interval $[0, T]$ into subintervals $[jk, (j+1)k)$, where $j = 0, 1, ..., N$. We consider the positive numbers $\rho, \eta \in (0, 1)$ and define the stochastic process $(u_N(t), \phi_N(t))$ and $(\bar{u}_N(t), \bar{\phi}_N(t))$ depending on $N$ by the relation

\[
\begin{align*}
\frac{d}{dt}u_N + \rho \nu A_0 u_N + B_0(u_N, u_N) - KR_0(\varepsilon A, \phi_N, \phi_N) &= 0, \quad \text{in } V_1^*, \ t \in [jk, (j+1)k],
\frac{d}{dt}\phi_N + \eta \mu N + B_1(u_N, \phi_N) &= 0, \quad \text{in } V_2^*,
\mu_N = \varepsilon A_\gamma \phi_N + \alpha f_\gamma(\phi_N), \quad \text{in } V_2^*,
(u_N, \phi_N)(jk) = (u_N^j, \phi_N^j),
\end{align*}
\]

and

\[
\begin{align*}
\bar{u}_N(t) + (1-\rho) \nu \int_{jk}^t A_0 \bar{u}_N ds = u_N((j+1)k-0) + \int_{jk}^t g_1(s, u_N) ds + \int_{jk}^t g_2(s, u_N(s)) dW(s), \quad \text{in } V_1^*, \ t \in [jk, (j+1)k],
\bar{\phi}_N(t) + (1-\eta) \int_{jk}^t \bar{\mu}_N(s) ds = \phi_N((j+1)k-0), \quad \text{in } V_2^*, \ t \in [jk, (j+1)k],
\bar{\mu}_N = \varepsilon A_\gamma \bar{\phi}_N + \alpha f_\gamma(\bar{\phi}_N), \quad \text{in } V_2^*,
\end{align*}
\]

with

\[
\begin{align*}
(u_N^{j+1}, \phi_N^{j+1}) &= (\bar{u}_N((j+1)k-0), \bar{\phi}_N((j+1)k-0)),
(u_N^j, \phi_N^j) &= (u_N((j+1)k), \phi_N((j+1)k)),
(u_N^0, \phi_N^0) &= (u_0, \phi_0),
\end{align*}
\]

where $v(r - 0)$ stands for the limit of $v$ from the left at $r$.

Note that the first problem (4.1) is deterministic and is known to be solvable (see [14]). The solutions $u_N$ and $\phi_N$ satisfy

\[(u_N, \phi_N) \in L^2(jk, (j+1)k; U) \cap L^\infty(jk, (j+1)k; H).\]

For the second equation (4.2), we note that (4.2)_1 is linear and admits a weak solution in the strong probabilistic sense as in [21, Chapters 2 and 3] where a more general equation was considered. The solution satisfies

\[\bar{u}_N \in L^2(\Omega, F, P; L^\infty(jk, (j+1)k; H_1) \cap L^2(\Omega, F, P; L^2(jk, (j+1)k; V_1)).\]

The equation (4.2)_2 is a deterministic Allen-Cahn equation and can be solvable using the Galerkin method as in [8] where the authors have proved the existence of weak solutions for the deterministic Cahn-Hilliard equation. A solution satisfies

\[\bar{\phi}_N \in L^2(jk, (j+1)k; V_2) \cap L^\infty(jk, (j+1)k; H_2).\]

We set for completeness, $(u_N(T), \phi_N(T)) = (u_N^{N+1}, \phi_N^{N+1})$ and note that $(u_N(\cdot), \phi_N(\cdot))$ is discontinuous at points $k, ..., (N+1)k$ and has left limits.

The following lemma gives us an equivalent form of the scheme (4.1)-(4.3).
Lemma 4.1. The stochastic processes \((u_N(t), \phi_N(t))\) and \((\bar{u}_N(t), \bar{\phi}_N(t))\) solve (4.1) - (4.2) if and only if they satisfy the following relations.

\[
\begin{align*}
\begin{cases}
    u_N(t) + \rho \nu \int_0^t A_0 u_N ds + (1 - \rho) \nu \int_0^{k[t/k]} A_0 \bar{u}_N ds + \int_0^t B_0(u_N, u_N) ds \\
    - \kappa \int_0^t R_0(\varepsilon A, \phi_N, \phi_N) ds = u_0 + \int_0^{k[t/k]} g_1(s, u_N(s)) ds \\
    + \int_0^t g_2(s, u_N(s)) dW(s) & \quad \text{in } V_1^*, \\
    \phi_N(t) + \eta \int_0^t \mu_N(s) ds + (1 - \eta) \int_0^{k[t/k]} \bar{\mu}_N(s) ds \\
    + \int_0^t B_1(u_N, \phi_N) ds = \phi_0 & \quad \text{in } V_2^*, \\
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
    \bar{u}_N(t) + \rho \nu \int_0^{k[t/k]+k} A_0 u_N ds + (1 - \rho) \nu \int_0^t A_0 \bar{u}_N ds \\
    + \int_0^{k[t/k]+k} (B_0(u_N, u_N) - \kappa R_0(\varepsilon A, \phi_N, \phi_N)) ds \\
    = u_0 + \int_0^t g_1(s, u_N(s)) ds + \int_0^t g_2(s, u_N(s)) dW(s) & \quad \text{in } V_1^*, \\
    \bar{\phi}_N(t) + \eta \int_0^{k[t/k]+k} \mu_N(s) ds + (1 - \eta) \int_0^t \bar{\mu}_N(s) ds \\
    + \int_0^{k[t/k]+k} B_1(u_N, \phi_N) ds = \phi_0 & \quad \text{in } V_2^*, \\
\end{cases}
\end{align*}
\]

for all \(t \in [0, T]\) with \([q]\) denoting the integer part of \(q\).

4.2. A priori estimates of the scheme. In this subsection, we will prove the suitable estimates of the solution of the scheme.

Lemma 4.2. We assume that the hypotheses (H1)-(H2) are satisfied. Let \(u_0 \in H_1\) and \(\phi_0 \in V_2\) such that \(E(u_0, \phi_0) < \infty\). Then the following estimates hold.

\[
\begin{align*}
\sup_{0 \leq t \leq T} E[\mathcal{E}(u_N, \phi_N(t))] & \leq C, \quad (4.6) \\
\sup_{0 \leq t \leq T} E[\mathcal{E}(\bar{u}_N, \bar{\phi}_N(t))] & \leq C, \quad (4.7) \\
E \int_0^T \left( \|\bar{u}_N\|^2 + |\bar{\mu}_N|^2 \right) ds + E \int_0^T \left( \|u_N\|^2 + |\mu_N|^2 \right) ds & \leq C, \quad (4.8) \\
\sup_{0 \leq t \leq T} E[\mathcal{E}^p(u_N, \phi_N(t))] & \leq C, \quad \forall p \geq 2, \quad (4.9) \\
\sup_{0 \leq t \leq T} E[\mathcal{E}^p(\bar{u}_N, \bar{\phi}_N(t))] & \leq C, \quad \forall p \geq 2. \quad (4.10)
\end{align*}
\]
Proof. Multiplying \((4.1)\) by \(K^{-1}u_N\) and integrating the result, we derive that, for all \(t \in [jk, (j+1)k)\),
\[
K^{-1}|u_N|^2_{L^2} + 2\nu K^{-1}\int_{jk}^t \|u_N\|^2 ds - 2\int_{jk}^t \langle R_0(\varepsilon A, \phi_N, \phi_N), u_N \rangle ds = K^{-1}\|u_N^1\|^2_{L^2}.
\] (4.11)

Multiplying also \((4.1)\) by \(\mu_N\), integrating the result and summing with \((4.11)\), we obtain
\[
E(u_N, \phi_N)(t) + 2\rho \nu K^{-1} \int_{jk}^t \|u_N\|^2 ds + 2\eta \int_{jk}^t \|\mu_N(s)\|^2_{L^2} ds = E(u_N^1, \phi_N^1),
\] (4.12)
which implies that
\[
E(u_N, \phi_N)((j+1)k - 0) + 2\rho \nu K^{-1} \int_{jk}^{(j+1)k} \|u_N\|^2 ds + 2\eta \int_{jk}^{(j+1)k} \|\mu_N\|^2_{L^2} ds
= E(u_N^1, \phi_N^1).
\] (4.13)

Now, we are going to estimate \(E(u_N^j, \phi_N^j)\). So, applying the Itô formula to \((4.2)\) over the interval \([jk, (j+1)k)\), we infer that for all \(t \in [jk, (j+1)k)\),
\[
\|\bar{u}_N\|^2_{L^2} + 2(1 - \rho)\nu \int_{jk}^t \|\bar{u}_N\|^2 ds
= \|u_N(jk)\|^2_{L^2} + 2 \int_{jk}^t (g_1(s, u_N), \bar{u}_N) ds + \int_{jk}^t |g_2(s, u_N)|^2_{L^2(U; H_1)} ds
+ \sum_{i=1}^{+\infty} \int_{jk}^t (g_2(s, u_N)e_i, \bar{u}_N) d\beta_i(s).
\] (4.14)

From \((4.2)\), we also deduce that \(\bar{\phi}_N\) satisfies
\[
\begin{aligned}
\frac{d}{dt} \phi_N(t) + (1 - \eta)\bar{\mu}_N &= 0, \quad \bar{\mu}_N = \varepsilon A, \bar{\phi}_N + \alpha f_\gamma(\bar{\phi}_N), \quad \text{on } [jk, (j+1)k),
\phi_N(jk) &= y_N((j+1)k - 0).
\end{aligned}
\] (4.15)

However, taking the inner product in \((4.15)\) with \(\bar{\mu}_N = \varepsilon A, \bar{\phi}_N + \alpha f_\gamma(\bar{\phi}_N)\), integrating the result, and adding with \((4.14)\), we obtain
\[
E(\bar{u}_N, \bar{\phi}_N)(t) + 2(1 - \rho)\nu \int_{jk}^t \|\bar{u}_N\|^2 ds + 2(1 - \eta) \int_{jk}^t \|\bar{\mu}_N\|^2_{L^2} ds
= E(\bar{u}_N, \bar{\phi}_N)(jk) + 2 \int_{jk}^t (g_1(s, u_N), \bar{u}_N) ds + \int_{jk}^t |g_2(s, u_N)|^2_{L^2(U; H_1)} ds
+ 2 \sum_{i=1}^{+\infty} \int_{jk}^t (g_2(s, u_N)e_i, \bar{u}_N) d\beta_i(s).
\] (4.16)

This implies that for all \(t \in [jk, (j+1)k)\),
\[
\mathbb{E}E(\bar{u}_N, \bar{\phi}_N)(t) + 2(1 - \rho)\nu \mathbb{E} \int_{jk}^t \|\bar{u}_N\|^2 ds + 2(1 - \eta)\mathbb{E} \int_{jk}^t \|\bar{\mu}_N\|^2_{L^2} ds
\leq \mathbb{E}E(\bar{u}_N, \bar{\phi}_N)(jk) + 2\mathbb{E} \int_{jk}^t (g_1(s, u_N), \bar{u}_N) ds + \mathbb{E} \int_{jk}^t |g_2(s, u_N)|^2_{L^2(U; H_1)} ds.
\] (4.17)
Using Hölder’s and Young’s inequalities, the condition (2.29) on $g_1$ and (4.12), we note that
\[
2\mathbb{E} \int_{j_k}^{t} (g_1(s, u_N), \bar{u}_N) ds \leq \mathbb{E} \int_{j_k}^{t} K^{-1} |\bar{u}_N|^{2}_{L^2} ds + C_K \mathbb{E} \int_{j_k}^{t} |g_1(s, u_N)|^{2}_{L^2} ds \\
\leq \mathbb{E} \int_{j_k}^{t} \mathcal{E}(\bar{u}_N, \bar{\phi}_N) ds + C_K (t-j_k) (1 + \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j)).
\] (4.18)

Similarly, in view of the condition (2.30) on $g_2$ and the equality (4.12), we get
\[
\mathbb{E} \int_{j_k}^{t} |g_2(s, u_N)|^{2}_{L^2(U;H_1)} ds \\
\leq K_2^2 \mathbb{E} \int_{j_k}^{t} (1 + |u_N|^{2}_{L^2}) ds \leq C_K (t-j_k) (1 + \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j)).
\] (4.19)

From (4.13), we infer that
\[
\mathbb{E} \left( \mathcal{E}(\bar{u}_N, \bar{\phi}_N)(j_k) \right) = \mathbb{E} \left( \mathcal{E}(u_N, \phi_N)((j + 1)k - 0) \right) \leq \mathbb{E} \left( \mathcal{E}(u_N^j, \phi_N^j) \right).
\] (4.20)

Combining (4.18)-(4.19) we derive from (4.17) that for all $t \in [j_k, (j + 1)k),$
\[
\mathbb{E} \mathcal{E}(\bar{u}_N, \bar{\phi}_N)(t) \leq \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j) + C_K k \left( 1 + \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j) \right) + \mathbb{E} \int_{j_k}^{t} \mathcal{E}(\bar{u}_N, \bar{\phi}_N) ds \\
\leq (1 + C_K k) \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j) + C_K k + \mathbb{E} \int_{j_k}^{t} \mathcal{E}(\bar{u}_N, \bar{\phi}_N) ds.
\] (4.21)

Applying the Gronwall inequality, we obtain that for all $t \in [j_k, (j + 1)k),$
\[
\mathbb{E} \mathcal{E}(\bar{u}_N, \bar{\phi}_N)(t) \leq \left[ (1 + C_K k) \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j) + C_K k \right] e^{(t-j_k)} \\
\leq \left[ (1 + C_K k) \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j) + C_K k \right] e^{k} \\
\leq (1 + C_K k) \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j) + C_K k, \quad \text{since } e^{k} \leq e^{T},
\] (4.22)

which yields
\[
\mathbb{E} \left( \mathcal{E}(u_N^j, \phi_N^j) \right) \leq (1 + Ck)^j ((\mathcal{E}(u_0, \phi_0) + 1)) \\
\leq \left( 1 + C \frac{T}{N+1} \right)^{N+1} (\mathcal{E}(u_0, \phi_0) + 1),
\] (4.23)

for any $N = 1, 2, ...$ and $j = 0, 1, ..., N$. Hence, we obtain
\[
\mathbb{E} \left( \mathcal{E}(u_N^j, \phi_N^j) \right) \leq C, \quad \forall N = 1, ..., \forall j = 0, 2, ..., N.
\] (4.24)

Using (4.24) we infer from (4.12) the inequality (4.6).

From (4.22) we derive that for all $j = 0, 1, ..., N$
\[
\sup_{j_k \leq t \leq (j + 1)k} \mathbb{E} \mathcal{E}(\bar{u}_N, \bar{\phi}_N)(t) \leq (1 + C_K) \mathbb{E} \mathcal{E}(u_N^j, \phi_N^j) + C_K.
\] (4.25)

However, using (4.24), the estimate (4.7) follows from (4.25).

Now we prove (4.8). Then, from (4.16) we note that
\[
\mathcal{E}(\bar{u}_N, \bar{\phi}_N)((j+1)k-0) + 2(1-\rho) \int_{j_k}^{(j+1)k} \|\bar{u}_N\|^2 ds + 2(1-\eta) \int_{j_k}^{(j+1)k} |\bar{u}_N|^2 ds
\]
Now, applying the Itô formula to the real process
\[ t \]
Using (4.7), the estimate (4.8) follows from (4.29).
\[ t \]
This implies that for all \( N \),
\[ \sum_{i=1}^{+\infty} \int_{j_k}^{(j+1)k} (g_2(s, u_N) e_i, \bar{u}_N) d\beta_i(s). \]
\[ (4.26) \]
Adding (4.13) and (4.26) we obtain that for all \( j = 0, 1, \ldots, N \),
\[ \mathcal{E}(u_N^{j+1}, \phi_N^{j+1}) + 2(1 - \rho) \nu K^{-1} \int_{j_k}^{(j+1)k} \| \bar{u}_N \|^2 ds + 2(1 - \eta) \int_{j_k}^{(j+1)k} |\bar{\mu}_N|_{L^2}^2 ds \]
\[ + 2 \rho \nu \int_{j_k}^{(j+1)k} \| u_N \|^2 ds + 2\eta \int_{j_k}^{(j+1)k} |\mu_N|_{L^2}^2 ds \]
\[ = \mathcal{E}(u_N^j, \phi_N^j) + 2 \int_{j_k}^{(j+1)k} (g_1(s, u_N), \bar{u}_N) ds \]
\[ + \int_{j_k}^{(j+1)k} |g_2(s, u_N)|_{L^2(U; H_1)}^2 ds + 2 \sum_{i=1}^{+\infty} \int_{j_k}^{(j+1)k} (g_2(s, u_N) e_i, \bar{u}_N) d\beta_i(s). \]
\[ (4.27) \]
Passing to the expectation and summing up the result from 0 to \( N \), we derive that
\[ 2(1 - \rho) \nu K^{-1} \mathbb{E} \int_0^T \| \bar{u}_N \|^2 ds + 2(1 - \eta) \mathbb{E} \int_0^T |\bar{\mu}_N|_{L^2}^2 ds \]
\[ + 2 \rho \nu \mathbb{E} \int_0^T \| u_N \|^2 ds + 2\eta \mathbb{E} \int_0^T |\mu_N|_{L^2}^2 ds \]
\[ \leq \mathcal{E}(u_0, \phi_0) + 2 \mathbb{E} \int_0^T (g_1(s, u_N), \bar{u}_N) ds + \mathbb{E} \int_0^T |g_2(s, u_N)|_{L^2(U; H_1)}^2 ds. \]
\[ (4.28) \]
Using (2.29) and (2.30) that
\[ \mathbb{E} \int_0^T \| \bar{u}_N \|^2 ds + \mathbb{E} \int_0^T |\bar{\mu}_N|_{L^2}^2 ds + \mathbb{E} \int_0^T \| u_N \|^2 ds + \mathbb{E} \int_0^T |\mu_N|_{L^2}^2 ds \]
\[ \leq C \mathcal{E}(u_0, \phi_0) + C(K_1, K_2) \int_0^T \mathbb{E}(1 + |u_N|_{L^2}^2) ds + \int_0^T \mathbb{E} |\bar{u}_N|_{L^2}^2 ds. \]
\[ (4.29) \]
Using (4.7), the estimate (4.8) follows from (4.29).

Let us remark that from (4.12) we infer that for all \( p \geq 2 \), for all \( j = 0, 1, \ldots, N \) and for all \( t \in [j_k, (j + 1)k) \),
\[ \mathbb{E} \mathcal{E}^p(u_N, \phi_N)(t) \leq \mathbb{E} \left[ \mathcal{E}^p(u_N^j, \phi_N^j) \right]. \]
\[ (4.30) \]
Now, applying the Itô formula to the real process \( \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) \), we derive that
\[ d\mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) \]
\[ = p \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(t) d\mathcal{E}(\bar{u}_N, \bar{\phi}_N)(t) + \frac{p(p - 1)}{2} \mathcal{E}^{p-2}(\bar{u}_N, \bar{\phi}_N)(t) \left( d\mathcal{E}(\bar{u}_N, \bar{\phi}_N)(t) \right)^2. \]
This implies that for all \( t \in [j_k, (j + 1)k), \ j = 0, 1, \ldots, N \),
\[ \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) + 2p(1 - \rho) \nu \int_{j_k}^{t} \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s) \| \bar{u}_N \|^2 ds \]
\[ + 2(1 - \eta) \int_{j_k}^{t} \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s) |\bar{\mu}_N|_{L^2}^2 ds \]
Let us estimate the integrals on the right-hand side of (4.31). However, using Young’s inequality and applying the same estimation than in (4.18), we note that

\begin{align*}
= & \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(jk) + 2p \int_{jk}^t \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s)(g_1(s, u_N), \bar{u}_N) \, ds \\
& + p \int_{jk}^t \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s)|g_2(s, u_N)|^2_{L^2(U; H_1)} \, ds \\
& + 2p \sum_{i=1}^{+\infty} \int_{jk}^t \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s)(g_2(s, u_N)e_i, \bar{u}_N) \, d\beta_i(s) \\
& + 2p(p-1) \int_{jk}^t \mathcal{E}^{p-2}(\bar{u}_N, \bar{\phi}_N)(s) \sum_{i=1}^{+\infty} (g_2(s, u_N)e_i, \bar{u}_N)^2 \, ds.
\end{align*}

So, we deduce that for all \( t \in [jk, (j+1)k) \), \( j = 0, \ldots, N \),

\begin{align*}
\mathbb{E}\mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) \leq & \mathbb{E}\mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(jk) + 2p \int_{jk}^t \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s)(g_1(s, u_N), \bar{u}_N) \, ds \\
& + p \int_{jk}^t \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s)|g_2(s, u_N)|^2_{L^2(U; H_1)} \, ds \\
& + 2p \sum_{i=1}^{+\infty} \int_{jk}^t \mathcal{E}^{p-2}(\bar{u}_N, \bar{\phi}_N)(s) \sum_{i=1}^{+\infty} (g_2(s, u_N)e_i, \bar{u}_N)^2 \, ds \tag{4.31}
\end{align*}

Let us estimate the integrals on the right-hand side of (4.31). However, using Young’s inequality and applying the same estimation than in (4.18), we note that

\begin{align*}
2p \int_{jk}^t \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s)(g_1(s, u_N), \bar{u}_N) \, ds \\
\leq & \frac{1}{2} \mathbb{E}\int_{jk}^t \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(s) \, ds + \frac{1}{2} \mathbb{E}\int_{jk}^t \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(s) \, ds + C \mathbb{E}\int_{jk}^t |g_1(s, u_N)|^{2p}_{L^2} \, ds \\
\leq & \mathbb{E}\int_{jk}^t \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(s) \, ds + C(t - jk) \left( 1 + \mathbb{E}\mathcal{E}^p(u_N^j, \phi_N^j) \right). \tag{4.32}
\end{align*}

By Young’s inequality and (4.19), we derive that

\begin{align*}
2p(p-1) & \mathbb{E}\int_{jk}^t \mathcal{E}^{p-2}(\bar{u}_N, \bar{\phi}_N)(s) \sum_{j=1}^{+\infty} (g_2(s, u_N)e_j, \bar{u}_N)^2 \, ds \\
\leq & C_{K,p} \mathbb{E}\int_{jk}^t \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s)|g_2(s, u_N)|^2_{L^2(U; H_1)} \, ds \\
\leq & \mathbb{E}\int_{jk}^t \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(s) \, ds + C(t - jk) \left( 1 + \mathbb{E}\mathcal{E}^p(u_N^j, \phi_N^j) \right). \tag{4.33}
\end{align*}

Also, applying Young’s inequality we can easily see that

\begin{align*}
p & \mathbb{E}\int_{jk}^t \mathcal{E}^{p-1}(\bar{u}_N, \bar{\phi}_N)(s)|g_2(s, u_N)|^2_{L^2(U; H_1)} \, ds \\
\leq & \mathbb{E}\int_{jk}^t \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(s) \, ds + C_{K,p} \mathbb{E}\int_{jk}^t |g_2(s, u_N)|^{2p}_{L^2(U; H_1)} \, ds \\
\leq & \mathbb{E}\int_{jk}^t \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(s) \, ds + C(t - jk) \left( 1 + \mathbb{E}\mathcal{E}^p(u_N^j, \phi_N^j) \right). \tag{4.34}
\end{align*}
Combining (4.32), (4.33), (4.34) and using (4.20) with the fact that \((t-jk)\leq k\), we infer from (4.31) that

\[
\mathbb{E}\mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) \leq (1 + Ck) \mathbb{E}\mathcal{E}^p(u^j_N, \phi^j_N) + Ck + \int_{jk}^t \mathbb{E}\mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(s)ds. \tag{4.35}
\]

Then, applying the deterministic Gronwall lemma, we arrive at

\[
\mathbb{E}\mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) \leq C_K \left[ (1 + Ck)\mathbb{E}\left(\mathcal{E}^p(u^j_N, \phi^j_N)\right) + Ck \right], \tag{4.36}
\]

which implies that

\[
\mathbb{E}\mathcal{E}^p(u^j_{N+1}, \phi^j_{N+1}) \leq C_K \left[ (1 + Ck)\mathbb{E}\left(\mathcal{E}^p(u^j_N, \phi^j_N)\right) + Ck \right]. \tag{4.37}
\]

Using the same arguments as above, we get for all \(j = 0, 1, ..., N\),

\[
\mathbb{E}\left(\mathcal{E}^p(u^j_N, \phi^j_N)\right) \leq C. \tag{4.38}
\]

Using (4.38), we infer from (4.30) that for all \(j = 0, 1, ..., N\) and for all \(t \in [jk, (j+1)k]\),

\[
\mathbb{E}\mathcal{E}^p(u_N, \phi_N)(t) \leq \mathbb{E}\left[\mathcal{E}^p(u^j_N, \phi^j_N)\right] \leq C. \tag{4.39}
\]

Passing to the supremum on (4.39), we obtain (4.9).

Using (4.38) and (4.36) we can derive that (4.10) holds and then ends the proof of Lemma 4.2.

**Lemma 4.3.** Under the conditions of Lemma (4.2), the following estimates hold

\[
\mathbb{E}\sup_{0 \leq t \leq T} \mathcal{E}(u_N, \phi_N)(t) \leq C, \tag{4.40}
\]

\[
\mathbb{E}\sup_{0 \leq t \leq T} \mathcal{E}^p(u_N, \phi_N)(t) \leq C, \quad \forall p \geq 2, \tag{4.41}
\]

\[
\mathbb{E}\sup_{0 \leq t \leq T} \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) \leq C, \quad \forall p \geq 1, \tag{4.42}
\]

\[
\mathbb{E}\left(\int_0^T \left(\|u_N\|^2 + |\mu_N|_{L^2}^2\right) ds\right)^p + \mathbb{E}\left(\int_0^T \left(\|\bar{u}_N\|^2 + |\bar{\mu}_N|_{L^2}^2\right) ds\right)^p \leq C, \quad \forall p \geq 2, \tag{4.43}
\]

\[
\mathbb{E}\left(\int_0^T \|u_N\|_{L^2}^2 ds\right)^p \leq C, \quad \forall p \geq 2, \tag{4.44}
\]

\[
\mathbb{E}\left(\int_0^T \|\bar{u}_N\|_{L^2}^2 ds\right)^p \leq C, \quad \forall p \geq 2. \tag{4.45}
\]

**Proof.** From (4.12), we obtain

\[
\sup_{t \in [0, T]} \mathcal{E}(u_N, \phi_N)(t) \leq \max_{j=0, 1, ..., N} \mathcal{E}(u^j_N, \phi^j_N). \tag{4.46}
\]

Now, we take arbitrary \(n \in \{0, 1, ..., N\}\). Then, adding up the relation (4.27) from \(0\) to \(j \leq n\), we arrive at

\[
\mathcal{E}(u^{n+1}_N, \phi^{n+1}_N) \leq \mathcal{E}(u_0, \phi_0) + 2 \int_0^{(n+1)k} (g_1(s, u_N), \bar{u}_N) ds \tag{4.47}
\]
Taking the maximum over $n$, and the expectation, we then deduce that
\[
E \max_{j=0,1,\ldots,N} \mathcal{E}(u^j_N, \phi^j_N)
\leq \mathcal{E}(u_0, \phi_0) + C_k \int_0^T (1 + \mathbb{E}\mathcal{E}(u_N, \phi_N)(s)) ds + C_k \int_0^T \mathbb{E}\mathcal{E}(\bar{u}_N, \bar{\phi}_N)(s) ds + 2E \left( \sup_{t \in [0,T]} \sum_{i=1}^{+\infty} \int_0^t (g_2(s, u_N)e_i, \bar{u}_N)d\beta_i(s) \right)^p.
\] (4.48)

Using Burkholder-Gundy’s inequality, the following holds
\[
2E \left( \sup_{t \in [0,T]} \sum_{i=1}^{+\infty} \int_0^t (g_2(s, u_N)e_i, \bar{u}_N)d\beta_i(s) \right)^p
\leq \left( \int_0^T |\bar{u}_N|^2_{L_2} |g_2(s, u_N)|^2_{L^2(U;H_1)} ds \right)^{1/2}.
\]
\[
\leq C \left( \int_0^T \mathbb{E}E^2(\bar{u}_N, \bar{\phi}_N)ds + CT + C \int_0^T \mathbb{E}\mathcal{E}^2(u_N, \phi_N)(s) ds \right)^{1/2}.
\] (4.49)

Using (4.49), (4.9) and (4.10), we then infer from (4.48) that
\[
E \max_{j=0,1,\ldots,N} \mathcal{E}(u^j_N, \phi^j_N) \leq C,
\] (4.50)
and from (4.46), we arrive at the estimate (4.40).

From (4.12), we derive that for all $p \geq 2$, 
\[
\sup_{t \in [0,T]} \mathcal{E}^p(u_N, \phi_N)(t) \leq \max_{j=0,1,\ldots,N} \mathcal{E}^p(u^j_N, \phi^j_N).
\] (4.51)

From (4.47) we can also deduce that
\[
\max_{j=0,1,\ldots,N} \mathcal{E}(u^j_N, \phi^j_N) \leq \mathcal{E}(u_0, \phi_0) + 2 \int_0^T (g_1(s, u_N), \bar{u}_N) ds + \int_0^T |g_2(s, u_N)|^2_{L^2(U;H_1)} ds + 2 \sup_{t \in [0,T]} \left( \sum_{j=1}^{+\infty} \int_0^t |g_2(s, u_N)e_j, \bar{u}_N|d\beta_i(s) \right)^p.
\] (4.52)

Squaring both side on this inequality by power $p$ and taking the expectation, we get
\[
E \max_{j=0,1,\ldots,N} \left[ \mathcal{E}(u^j_N, \phi^j_N) \right]^p \leq \mathcal{E}^p(u_0, \phi_0) + 2^pE \left( \int_0^T (g_1(s, u_N), \bar{u}_N) ds \right)^p
\]
\[
+ \mathbb{E} \left( \int_0^T |g_2(s, u_N)|^2_{L^2(U;H_1)} ds \right)^p + 2^pE \left( \sup_{t \in [0,T]} \left( \sum_{i=1}^{+\infty} \int_0^t |g_2(s, u_N)e_i, \bar{u}_N|d\beta_i(s) \right)^p \right).
\] (4.53)
We note that
\[ E \left( \int_0^T (g_1(s, u_N), \bar{u}_N) ds \right)^p \]
\[ \leq T^{p-1} E \left( \int_0^T |g_1(s, u_N)|_{L^2}^p |u_N|_{L^2}^p ds \right) \]
\[ \leq C_p T + C_p \int_0^T E \mathcal{E}^p(u_N, \phi_N) ds + C_p \int_0^T E \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N) ds, \quad (4.54) \]
\[ E \left( \int_0^T |g_2(s, u_N)|_{L^2(U;H^1)}^2 ds \right)^p \leq C_p E \int_0^T |g_2(s, u_N)|_{L^2(U;H^1)}^2 ds \]
\[ \leq C_p T + C_p \int_0^T E \mathcal{E}^p(u_N, \phi_N) ds + C_p \int_0^T E \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N) ds. \quad (4.55) \]

Using the martingale inequality, we get
\[ 2p E \left( \sup_{t \in [0,T]} \left| \sum_{i=1}^{+\infty} \int_0^t (g_2(s, u_N) e_i, \bar{u}_N) d\beta_i(s) \right|^p \right) \]
\[ \leq C E \left( \int_0^T |\bar{u}_N|_{L^2}^2 |g_2(s, u_N)|_{L^2(U;H^1)}^2 ds \right)^{p/2} \]
\[ \leq C_p T + C_p \int_0^T E \mathcal{E}^p(u_N, \phi_N) ds + C_p \int_0^T E \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N) ds. \quad (4.56) \]

Using (4.9) and (4.10) we infer from (4.53) that
\[ E \max_{j=0,1,\ldots,N} \left[ \mathcal{E}(u_N^j, \phi_N^j) \right]^p < C. \quad (4.57) \]

Since (4.57) holds, we then infer form (4.51) the estimate (4.41).

For the proof of (4.42), we start with (4.16) to derive that for all \( p \geq 1, \)
\[ \sup_{0 \leq t \leq T} E \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) \]
\[ \leq \max_{j=0,\ldots,N} \left[ \mathcal{E}(u_N^j, \phi_N^j) \right]^p + c \int_0^T |g_1(s, u_N)|_{L^2}^p |u_N|^p ds \]
\[ + c \int_0^T |g_2(s, u_N)|_{L^2(U;H^1)}^2 ds + 2 \max_{j=0,\ldots,N} \sup_{j_k \leq t \leq (j+1)k} \left| \int_{j_k}^{j_{k+1}} (g_2(s, u_N), \bar{u}_N) dW_s \right|^p. \quad (4.58) \]

Note that
\[ \max_{j=0,\ldots,N} \sup_{j_k \leq t \leq (j+1)k} \left| \int_{j_k}^{j_{k+1}} (g_2(s, u_N), \bar{u}_N) dW_s \right|^p \]
\[ \leq c \sup_{0 \leq t \leq T} \left| \int_0^t (g_2(s, u_N), \bar{u}_N) dW_s \right|^p + c \max_{j=0,\ldots,N} \left| \int_0^{j_k} (g_2(s, u_N), \bar{u}_N) dW_s \right|^p. \quad (4.59) \]

Using this last estimate, we obtain from (4.58) that
\[ E \sup_{0 \leq t \leq T} E \mathcal{E}^p(\bar{u}_N, \bar{\phi}_N)(t) \]
\[
\begin{align*}
&\leq \mathbb{E} \max_{j=0,1,\ldots,N} \left[ \mathcal{E}(u_N^j, \phi_N^j) \right]^p + c \int_0^T \mathbb{E}|g_1(s, u_N)|_{L_2}^p |\bar{u}_N|^p ds \\
&+ c \int_0^T \mathbb{E}|g_2(s, u_N)|_{L_2(U; H_1)}^p ds + c\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t (g_2(s, u_N), \bar{u}_N)dW_s \right|^p. 
\end{align*}
\]  

(4.60)

Using (4.54), (4.55) and (4.56), we derive that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \mathcal{E}^p(\bar{u}_N, \tilde{\phi}_N)(t) \leq C,
\]  

(4.61)

and then obtain (4.42).

From (4.27), we can infer that

\[
2\rho\nu\mathcal{K}^{-1} \int_0^T \|u_N\|^2 ds + 2\eta \int_0^T |\mu_N|_{L_2}^2 ds + 2(1-\rho)\nu \int_0^T |\bar{u}_N|^2 ds + 2(1-\eta) \int_0^T |\bar{\mu}_N|_{L_2}^2 ds 
\leq \mathcal{E}(u_0, \phi_0) + 2 \int_0^T (g_1(s, u_N), \bar{u}_N) ds + \int_0^T |g_2(s, u_N)|_{L_2(U; H_1)}^2 ds 
+ 2 \sup_{t \in [0, T]} \sum_{i=1}^{\infty} \int_0^t (g_2(s, u_N) e_i, \bar{u}_N) d\beta_i(s).
\]  

(4.62)

Squaring both of this inequality by \(p\), taking the expectation and using again (4.54), (4.55) and (4.56), we obtain

\[
\mathbb{E} \left( 2\rho\nu\mathcal{K}^{-1} \int_0^T \|u_N\|^2 ds + 2\eta \int_0^T |\mu_N|_{L_2}^2 ds \right)^p 
+ \mathbb{E} \left( 2(1-\rho)\nu \int_0^T |\bar{u}_N|^2 ds + 2(1-\eta) \int_0^T |\bar{\mu}_N|_{L_2}^2 ds \right)^p 
\leq C + C\mathbb{E} \int_0^T \mathcal{E}^p(u_N, \phi_N) ds + C\mathbb{E} \int_0^T \mathcal{E}^p(\bar{u}_N, \tilde{\phi}_N) ds \leq C.
\]  

(4.63)

Using (4.63), we deduce (4.43).

To prove (4.44), owing to (4.43) it is sufficient to prove that \(\mathbb{E}(\int_0^T |A_\gamma \phi_N|_{L_2}^2 ds)^p < C\). Since \(\mu_N = \varepsilon A_\gamma \phi_N + \alpha f_\gamma(\phi_N)\), we infer using (2.1) that

\[
|A_\gamma \phi_N|_{L_2}^2 = |\mu_N|_{L_2}^2 |A_\gamma \phi_N|_{L_2}^2 + \alpha |f_\gamma(\phi_N), A_\gamma \phi_N| 
\leq |\mu_N|_{L_2}^2 |A_\gamma \phi_N|_{L_2}^2 + \alpha |f_\gamma(\phi_N)|_{L_2}^2 |A_\gamma \phi_N|_{L_2}^2 
\leq \frac{\varepsilon}{4} |\mu_N|_{L_2}^2 + \frac{\varepsilon}{4} |A_\gamma \phi_N|_{L_2}^2 + \frac{\alpha^2}{\varepsilon} |f_\gamma(\phi_N)|_{L_2}^2 + \frac{\alpha^2}{\varepsilon} |A_\gamma \phi_N|_{L_2}^2 
\leq \frac{\varepsilon}{4} |\mu_N|_{L_2}^2 + \frac{\varepsilon}{4} |A_\gamma \phi_N|_{L_2}^2 + \frac{\alpha^2}{\varepsilon} C_f |D| + \frac{\alpha^2}{\varepsilon} C_f |\phi_N|_{L^{2m+2}} |A_\gamma \phi_N|_{L_2}^2.
\]

Using the Sobolev injection \(H^1(D) \hookrightarrow L^p(D)\) (\(p \geq 2\) with \(d = 2\) and \(2 \leq p \leq 6\) with \(d = 3\), we derive that

\[
\frac{\varepsilon}{2} |A_\gamma \phi_N|_{L_2}^2 \leq \frac{\varepsilon}{4} |\mu_N|_{L_2}^2 + \frac{\alpha^2}{\varepsilon} C_f |D| + \frac{\alpha^2}{\varepsilon} C_f |\phi_N|_{L_2}^{2m+2},
\]  

(4.64)

which implies that for \(p \geq 2\),

\[
\mathbb{E} \left( \int_0^T |A_\gamma \phi_N|_{L_2}^2 ds \right)^p \leq C\mathbb{E} \left( \int_0^T |\mu_N|_{L_2}^2 ds \right)^p + C\mathbb{E} \sup_{0 \leq t \leq T} \|\phi_N\|_{L^{2m+2}}^p
\]

(4.65)
\[ \leq C + C\mathbb{E} \sup_{0 \leq t \leq T} \mathcal{E}^{p(m+1)}(u_N, \phi_N). \] (4.65)

Using (4.41) and (4.43) we derive from (4.65) that \( \mathbb{E} \left( \int_0^T |A_y \phi_N|^2_{L^2_2} ds \right)^p \leq C \), which ends the proof of (4.44).

For the proof of (4.45), we note that since \( \bar{\mu}_N = \varepsilon A_y \bar{\phi}_N + \alpha f_y(\bar{\phi}_N) \), we derive easily that
\[
\mathbb{E} \left( \int_0^T |A_y \bar{\phi}_N|^2_{L^2_2} ds \right)^p \leq C \mathbb{E} \left( \int_0^T |\bar{\mu}_N|^2_{L^2_2} ds \right)^p + C \mathbb{E} \sup_{0 \leq t \leq T} \| \bar{\phi}_N \|^2_{\gamma}^{2(m+1)p}, \tag{4.66}
\]
which implies (4.45) and ends the proof of Lemma 4.3.

The another important estimates are given by the following lemma.

**Lemma 4.4.** *In the conditions of Lemma (4.2), there exists a constant* \( C > 0 \) *such that for any* \( N = 1, 2, \ldots \),
\[
\mathbb{E} \| u_N(t) - \bar{u}_N(t) \|^4_{V^*_1} \leq C \left( \frac{T}{N+1} \right)^{1/3}, \quad \forall t \in [0, T], \tag{4.67}
\]
\[
\mathbb{E} \| \phi_N(t) - \bar{\phi}_N(t) \|^2_{V^*_2} \leq \frac{CT}{N+1}, \quad \forall t \in [0, T], \tag{4.68}
\]
*for* \( d = 2 \) *and* \( d = 3 \).

**Proof.** Using Lemma 4.1 we can infer that \( u_N \) and \( \bar{u}_N \) satisfy
\[
u = - \rho \nu \int_t^{k[t/k]+k} A_0 u_N ds - \int_t^{k[t/k]+k} (B_0(u_N, u_N) - KR_0(\varepsilon A_y \phi_N, \phi_N)) ds
\]
\[- (1 - \rho) \nu \int_t^{k[t/k]} A_0 \bar{u}_N(s) ds + \int_t^{k[t/k]} g_1(s, u_N(s)) ds + \int_t^{k[t/k]} g_2(s, u_N(s)) dW(s).\]

Using Lemma 4.2, we note that
\[
\mathbb{E} \left\| \int_t^{k[t/k]+k} A_0 u_N ds \right\|^4_{V^*_1} \leq C_T k^{1/3} \left( \mathbb{E} \int_0^T \| u_N \|^2 ds \right)^{2/3} \leq C_T k^{1/3}, \tag{4.70}
\]
\[
\mathbb{E} \left\| \int_t^{k[t/k]} A_0 \bar{u}_N ds \right\|^4_{V^*_1} \leq C_T k^{1/3} \left( \mathbb{E} \int_0^T \| \bar{u}_N \|^2 ds \right)^{2/3} \leq C_T k^{1/3}. \tag{4.71}
\]

By Hölder’s inequality we arrive at
\[
\mathbb{E} \left[ \left\| \int_t^{k[t/k]+k} (B_0(u_N, u_N) - KR_0(\varepsilon A_y \phi_N, \phi_N)) ds \right\|_{V^*_1}^{4/3} \right] \leq C k^{1/3} \left( \mathbb{E} \int_0^T \| B_0(u_N, u_N) \|_{V^*_1}^{4/3} ds + C_k \mathbb{E} \int_0^T \| R_0(\varepsilon A_y \phi_N, \phi_N) \|_{V^*_1}^{4/3} ds \right). \tag{4.72}
\]

From (2.18) and Hölder’s inequality, we derive that for \( d = 2 \),
\[
\mathbb{E} \int_0^T \| B_0(u_N, u_N) \|_{V^*_1}^{4/3} ds.
\]

Using the Hölder inequality, we derive that
\[
\leq C \mathbb{E} \int_0^T |u_N|^{4/3}_L^2 \|u_N\|^{4/3}_L^2 \, ds \leq C \mathbb{E} \sup_{0 \leq s \leq T} |u_N|^2_{L^2} \int_0^T \|u_N\|^2_2 \, ds
\]

\[
\leq C \left( \mathbb{E} \sup_{0 \leq s \leq T} |u_N|^2_{L^2} \right)^{2/3} \left[ \mathbb{E} \left( \int_0^T \|u_N\|^2_2 \, ds \right)^{2/3} \right] \leq C. \tag{4.73}
\]

For \( d = 3 \),
\[
\mathbb{E} \int_0^T \|B_0(u_N, u_N)\|^{4/3}_{V_i^*} \, ds \tag{4.74}
\]
\[
\leq C \mathbb{E} \int_0^T \|\phi_N\|^{2/3}_\gamma A_\gamma \phi_N^2_{L^2} \, ds \leq C_T \mathbb{E} \sup_{0 \leq s \leq T} \|\phi_N\|^{2/3}_{\gamma_\gamma} \int_0^T |A_\gamma \phi_N|^2_{L^2} \, ds \tag{4.75}
\]
\[
\leq C_T \left( \mathbb{E} \sup_{0 \leq s \leq T} \|\phi_N\|^{2}_{\gamma_\gamma} \right)^{1/3} \left[ \mathbb{E} \left( \int_0^T |A_\gamma \phi_N|^2_{L^2} \, ds \right)^{2/3} \right] \leq C.
\]

However, using the Hölder inequality, we note that:

For the case where \( d = 2 \), from (2.19), we get
\[
\mathbb{E} \int_0^T \|R_0(\varepsilon A_\gamma \phi_N, \phi_N)\|^{4/3}_{V_i^*} \, ds
\]
\[
\leq C \mathbb{E} \int_0^T \|\phi_N\|^{2/3}_\gamma A_\gamma \phi_N^2_{L^2} \, ds \leq C_T \mathbb{E} \sup_{0 \leq s \leq T} \|\phi_N\|^{2/3}_{\gamma_\gamma} \int_0^T |A_\gamma \phi_N|^2_{L^2} \, ds \tag{4.76}
\]
\[
\leq C_T \left( \mathbb{E} \sup_{0 \leq s \leq T} \|\phi_N\|^{2}_{\gamma_\gamma} \right)^{1/3} \left[ \mathbb{E} \left( \int_0^T |A_\gamma \phi_N|^2_{L^2} \, ds \right)^{2/3} \right] \leq C.
\]

In the case where \( d = 3 \), using (2.26) we obtain
\[
\mathbb{E} \int_0^T \|R_0(\varepsilon A_\gamma \phi_N, \phi_N)\|^{4/3}_{V_i^*} \, ds
\]
\[
= \varepsilon^{4/3} \mathbb{E} \int_0^T \|R_1(\phi_N, \phi_N)\|^{4/3}_{V_i^*} \, ds \leq C \mathbb{E} \int_0^T \|\phi_N\|^{2/3}_\gamma A_\gamma \phi_N^2_{L^2} \, ds \tag{4.77}
\]
\[
\leq C_T \left( \mathbb{E} \sup_{0 \leq s \leq T} \|\phi_N\|^{2}_{\gamma_\gamma} \right)^{1/3} \left[ \mathbb{E} \left( \int_0^T |A_\gamma \phi_N|^2_{L^2} \, ds \right)^{2/3} \right] \leq C.
\]

Using the Hölder inequality, we derive that
\[
\mathbb{E} \left\| \int_{k[t/k]}^t g_1(s, u_N(s)) \, ds \right\|^{4/3}_{V_i^*} \tag{4.77}
\]
\[
\leq \mathbb{E} \left\| \int_{k[t/k]}^t g_1(s, u_N(s)) \, ds \right\|^{4/3}_{L^2} \leq (t - k[t/k])^{1/3} \mathbb{E} \int_0^T |g_1(s, u_N(s))|^4_{L^2} \, ds
\]
By the Burkholder-Davis-Gundy inequality, we get that

\[
\begin{align*}
& \leq k^{1/3} \left( \mathbb{E} \int_0^T |g_1(s, u_N(s))|^2_{L^2} \, ds \right)^{2/3} T^{1/3} \leq C_T k^{1/3} \left( 1 + \mathbb{E} \sup_{0 \leq s \leq T} |u_N(s)|_{L^2}^2 \right)^{2/3} \\
& \leq C_T k^{1/3}.
\end{align*}
\]

(4.78)

Using (4.70)-(4.79), we infer from (4.69) that

\[
\mathbb{E} \left\| u_N(t) - \bar{u}_N(t) \right\|_{V^*_1}^{4/3} \leq C k^{1/3} = C \left( \frac{T}{N+1} \right)^{1/3},
\]

which proves (4.67).

For the proof of (4.68), we note that using Lemma 4.1 we can also infer that \( \phi_N \) and \( \bar{\phi}_N \) satisfy

\[
\begin{align*}
\phi_N(t) - \bar{\phi}_N(t) &= - \eta \int_t^{k[t/k]+k} \mu_N \, ds - \int_t^{k[t/k]+k} B_1(u_N, \phi_N) \, ds - (1 - \eta) \int_t^{k[t/k]} \bar{\mu}_N(s) \, ds.
\end{align*}
\]

(4.80)

Owing to the embedding \( L^2(D) \hookrightarrow V^*_2 \), we remark that

\[
\begin{align*}
\mathbb{E} \left\| \int_t^{k[t/k]+k} \mu_N \, ds \right\|_{V^*_2}^2 &\leq \mathbb{E} \left( \int_t^{k[t/k]+k} \| \mu_N \|_{V^*_2} \, ds \right)^2 \leq C k \mathbb{E} \int_0^T \| \mu_N \|_{L^2}^2 \, ds \leq C k, \\
\mathbb{E} \left\| \int_t^{k[t/k]} \bar{\mu}_N \, ds \right\|_{V^*_2}^2 &\leq C k \mathbb{E} \int_0^T \| \bar{\mu}_N \|_{L^2}^2 \, ds \leq C k, \\
\mathbb{E} \left\| \int_t^{k[t/k]+k} B_1(u_N, \phi_N) \, ds \right\|_{V^*_2}^2 &\leq C k \mathbb{E} \int_0^T \| B_1(u_N, \phi_N) \|_{V^*_2}^2 \, ds.
\end{align*}
\]

(4.81)

(4.82)

(4.83)

For \( d = 2 \), we have

\[
\begin{align*}
\mathbb{E} \int_0^T \| B_1(u_N, \phi_N) \|_{V^*_2}^2 \, ds &\leq C \mathbb{E} \int_0^T |u_N|_{L^2} \| \phi_N \| \, ds \leq C \mathbb{E} \sup_{t \in [0,T]} \| \phi_N(t) \|_{L^2}^2 \int_0^T \| u_N(t) \|_{L^2}^2 \, dt \\
&\leq C \left[ \mathbb{E} \sup_{t \in [0,T]} \| \phi_N(t) \|_{L^2}^4 + \mathbb{E} \left( \int_0^T \| u_N(t) \|_{L^2}^2 \, dt \right)^2 \right] \leq C.
\end{align*}
\]

(4.84)
For $d = 3$,
\[
\mathbb{E} \int_0^T \| B_1(u_N, \phi_N) \|_{V^{2}}^2 \ ds \tag{4.85}
\]
\[
\leq C \mathbb{E} \int_0^T |u_N|_{L^1}^{1/2} |u_N|_{L^2}^{3/2} |\phi_N|_{L^2}^{1/2} |\phi_N|_{L^2}^{3/2} \ ds \leq C \mathbb{E} \sup_{t \in [0, T]} \| \phi_N(t) \|_{\gamma}^2 \int_0^T \| u_N(t) \|_2^2 \ dt
\]
\[
\leq C \left[ \mathbb{E} \sup_{t \in [0, T]} \| \phi_N(t) \|_{\gamma}^4 + \mathbb{E} \left( \int_0^T \| u_N(t) \|_2^2 \ dt \right)^2 \right] \leq C.
\]
Using (4.81)-(4.85), we infer from (4.80) the estimation (4.68) and ends the proof of Lemma 4.4.

The important estimates for the study of the tightness property of the approximating solutions are given by the following lemma.

**Lemma 4.5.** In the conditions of Lemma (4.2), there exists a constant $C > 0$ such that for $N = 1, 2, \ldots$,
\[
\mathbb{E} \left( \left\| u_N(t) - \int_0^{k[t/k]} g_2(s, u_N(s)) \, dW_s \right\|_{W^{1, \frac{4}{3}}(0, T; V_\gamma^*)}^{4/3} \right) \leq C, \tag{4.86}
\]
\[
\mathbb{E} \left( \left\| \int_0^{k[t/k]} g_2(s, u_N(s)) \, dW_s \right\|_{W^{1, \frac{4}{3}}(0, T; H_1)}^{4/3} \right) \leq C, \tag{4.87}
\]
\[
\mathbb{E} \left( \left\| u_N(t) - \int_0^t g_2(s, u_N(s)) \, dW_s \right\|_{W^{1, \frac{4}{3}}(0, T; V_\gamma^*)}^{4/3} \right) \leq C, \tag{4.88}
\]
\[
\mathbb{E} \left( \left\| \int_0^t g_2(s, u_N(s)) \, dW_s \right\|_{W^{1, \frac{4}{3}}(0, T; V_\gamma^*)}^{4/3} \right) \leq C, \tag{4.89}
\]
\[
\mathbb{E} \left( \| \phi_N \|_{W^{1, 2}(0, T; V_2^*)}^2 \right) \leq C, \tag{4.90}
\]
\[
\mathbb{E} \left( \| \phi_N \|_{W^{1, 2}(0, T; V_2^*)}^2 \right) \leq C, \tag{4.91}
\]
for $d = 2$ and $d = 3$.

**Proof.** For the prove of (4.86), we note that by (4.4) of Lemma 4.1, we can write
\[
u(t) - \int_0^{k[t/k]} g_2(s, u_N(s)) \, dW(s) \tag{4.92}
\]
\[
u(t) = u_0 - \nu \int_0^t A_0 u_N \, ds - (1 - \nu) \nu \int_0^{k[t/k]} A_0 \bar{u}_N \, ds
\]
\[
- \int_0^t B_0(u_N, u_N) \, ds + K \int_0^t R_0(\varepsilon A_\gamma \phi_N, \phi_N) \, ds + u_0 + \int_0^{k[t/k]} g_1(s, u_N(s)) \, ds,
\]
as equality in $V_1^*$.

By the inequalities (4.73)-(4.76) we note that
\[
\mathbb{E} \left( \left\| u_N(t) - \int_0^{k[t/k]} g_2(s, u_N(s)) \, dW_s \right\|_{W^{1, \frac{4}{3}}(0, T; V_\gamma^*)}^{4/3} \right)
\]
\(\leq c |u_0|^{4/3}_L + cE \int_0^T \|A_0 u_N\|^{4/3}_V ds + cE \int_0^T \|A_0 u_N\|^{4/3}_V ds + cE \int_0^T \|B_0(u_N, u_N)\|^{4/3}_N ds + cE \int_0^T \|g_1(s, u_N(s))\|^{4/3}_V ds.\)

\(\leq C + c \left( E \int_0^T \|u_N\|^2_V ds \right) + \left( E \int_0^T \|\bar{u}_N\|^2_V ds \right) + \left( E \int_0^T |g_1(s, u_N(s))|^2_{L^2} ds \right)\)

\(\leq C + \left( 1 + E \sup_{0 \leq s \leq T} |u_N(s)|^2_{L^2} \right)^{2/3} \leq C,\)

and the estimate (4.86) follows.

By the Burkholder-Davis-Gundy inequality, we obtain

\[ E \left( \left| \int_0^T g_2(s, u_N(s)) dW_s \right|^{4/3}_{W^{1,2}(0,T;H_1)} \right) \]

\[ \leq E \int_0^T |g_2(s, u_N(s))|^{4/3}_{L^2(U;H_1)} \leq \left( 1 + E \sup_{0 \leq s \leq T} |u_N(s)|^2_{L^2} \right)^{2/3} \leq C,\]

which gives the estimate (4.87).

The proof of (4.88) and (4.89) is very similar to proof of (4.86)-(4.87) and is omitted.

Now, we prove (4.90). So, one has

\[ E \|\phi_N\|^2_{W^{1,2}(0,T;V_2^*)} \]

\[ = E \int_0^T \|\phi_N\|^2_{V_2^*} ds + E \int_0^T \|d\phi_N\|^2_{V_2^*} ds \leq CE \int_0^T \|\phi_N\|^2_{V_2^*} ds + E \int_0^T \|d\phi_N\|^2_{V_2^*} ds.\]

We note that by (4.4)_2 of Lemma 4.1, we can write for almost \(t \in [0, T],\)

\[ \frac{d}{dt} \phi_N = -\eta \mu_N - B(u_N, \phi_N) \text{ in } V_2^*.\]

From (4.96) we arrive at

\[ \left\| \frac{d}{dt} \phi_N \right\|^2_{V_2^*} \leq 2\eta^2 \|\mu_N\|^2_{V_2^*} + 2 \|B(u_N, \phi_N)\|^2_{V_2^*} \]

\[ \leq 2c\eta^2 \|\mu_N\|^2_{L^2} + 2 \|B(u_N, \phi_N)\|^2_{V_2^*}.\]

Using (4.84) and (4.85) we infer from (4.97) that

\[ E \int_0^T \left\| \frac{d}{dt} \phi_N \right\|^2_{V_2^*} ds \leq 2c\eta^2 E \int_0^T \|\mu_N\|^2_{L^2} ds + 2E \int_0^T \|B(u_N, \phi_N)\|^2_{V_2^*} ds \leq C.\]

Using (4.98), we derive (4.90) from (4.95). For the proof of the last estimate of Lemma 4.4, we note that by (4.5)_2 of Lemma 4.1, we can write for almost \(t \in [0, T],\)

\[ \frac{d}{dt} \bar{\phi}_N = -\eta \bar{\mu}_N \text{ in } V_2^*.\]

However, we infer that

\[ E \left\| \bar{\phi}_N \right\|^2_{W^{1,2}(0,T;V_2^*)} ds \leq CE \int_0^T \left\| \bar{\phi}_N \right\|^2_{V_2^*} ds + c\eta^2 E \int_0^T \|\bar{\mu}_N\|^2_{L^2} ds \leq C,\]
and the estimate (4.91) follows. This ends the proof of Lemma 4.1.

The following compactness results play a crucial role in the proof of the tightness of the probability measures generated by the approximating sequences. The proof is given and commented in [12].

**Lemma 4.6.** Assume that \( X_0 \subset X \subset X_1 \) are Banach spaces, \( X_0 \) and \( X_1 \) being reflexive. Assume that the embedding \( X_0 \hookrightarrow X \) is compact, \( q \in (1, \infty) \), and \( \alpha \in (0, 1) \). Then the embedding

\[
L^q(0, T; X_0) \cap W^{\alpha, q}(0, T; X_1) \hookrightarrow L^q(0, T; X)
\]

is compact.

**4.3. Tightness property for the approximating solutions.** We start this subsection by state without proof the following result in [22, Chapter II, Theorem 3.2] which will be useful for the proof of the compactness property of the approximation solutions.

**Lemma 4.7.** Let \( X \) be a separable metric space with the property that there exists a complete separable metric space \( X_0 \) such that \( X \) is contained in \( X_0 \) as a topological subset and \( X \) is a Borel subset of \( X_0 \). Then every measure \( \mu \) on \( X \) is tight. In particular, if \( X \) itself is a complete separable metric space, every measure on \( X \) is tight.

Now define the constant sequence of \( U \)-valued cylindrical Wiener process by \( W_N = W \) for all \( N \geq 1 \) and consider the spaces \( \mathcal{X} = \mathcal{X}_u \times \mathcal{X}_u \times \mathcal{X}_\phi \times \mathcal{X}_\phi \times \mathcal{X}_W \) equipped with the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathcal{X}) \) where

\[
\mathcal{X}_u = L^{4/3}(0, T; H_1), \quad \mathcal{X}_\phi = L^2(0, T, H_2), \quad \mathcal{X}_W = C(0, T; U_0).
\]

For each \( N \geq 1 \), let \( \Phi_N : \Omega \rightarrow \mathcal{X} \) be the map defined by

\[
\Phi_N(\omega) = (u_N(\omega, \cdot), \bar{u}_N(\omega, \cdot), \phi_N(\omega, \cdot), \bar{\phi}_N(\omega, \cdot), W_N(\omega, \cdot)).
\]

Also, for each \( N \geq 1 \), we introduce a measure \( \Pi_N \) on \( (\mathcal{X}, \mathcal{B}(\mathcal{X})) \) given by

\[
\Pi_N(A) = \mathbb{P}\{(\Phi_N)^{-1}(A)\} \quad \text{for all} \quad A \in \mathcal{B}(\mathcal{X}).
\]

The following result gives us the tightness property of the measure \( \Pi_N \).

**Proposition 1.** For \( d = 2 \) and \( d = 3 \), the family of measures \( \{\Pi_N\}_{N \geq 1} \) is tight uniformly in \( (\mathcal{X}, \mathcal{B}(\mathcal{X})) \).

**Proof.** For any \( \varepsilon > 0 \), we should find the compact subsets \( H_\varepsilon \subset \mathcal{X}_W, \ K^1_\varepsilon, K^2_\varepsilon \subset \mathcal{X}_u \) and \( K^3_\varepsilon, K^4_\varepsilon \subset \mathcal{X}_\phi \) such that

\[
\mathbb{P}\{\omega : W_N(\omega, \cdot) \notin H_\varepsilon\} \leq \varepsilon/5,
\]

and

\[
\mathbb{P}\{\omega : (u_N(\omega, \cdot), \bar{u}_N(\omega, \cdot), \phi_N(\omega, \cdot), \bar{\phi}_N(\omega, \cdot)) \notin K^1_\varepsilon \times K^2_\varepsilon \times K^3_\varepsilon \times K^4_\varepsilon\} \leq 4\varepsilon/5.
\]

Since endowed with the uniform convergence \( \mathcal{X}_W \) is a Polish space, then it follows from [5, Theorem 6.8] that the space of probability measure on \( \mathcal{X}_W \) endowed with the Prokhorov’s metric is a separable and complete metric space. By construction, the family of probability law of \( \{W_N, N \geq 1\} \) is reduced to one element (owing to the fact that \( W_N = W \) for all \( N \geq 1 \) in the space of probability measure on \( \mathcal{X}_W \). Therefore, invoking Lemma 4.7 we easily deduce that the family law of \( \{W_N, N \geq 1\} \) is tight on \( \mathcal{X}_W \). Therefore, there exists a compact subset \( H_\varepsilon \) of \( \mathcal{X}_W \) such that (4.104) holds.
Next, let $M^1 > 0$, $M^2 > 0$, $M^3 > 0$ and $M^4 > 0$ (to be fixed later) and any \( \alpha \in (0, 1) \). We set for \( i = 1, 2, \) and \( j = 3, 4 \),

\[
K^i_\varepsilon = \left\{ v \in L^{4/3}(0, T; V_1) \cap W^{\alpha, 3/4}(0, T; V_1^*) : \right. \\
\left. \| \psi \|_{L^{4/3}(0, T; V_1)}^{4/3} + \| \psi \|_{W^{\alpha, 3/4}(0, T; V_1^*)} \leq (M^i_\varepsilon)^{4/3} \right\}
\]

\[
K^j_\varepsilon = \left\{ v \in L^2(0, T; V_2) \cap W^{\alpha, 2}(0, T; V_2^*) : \| \psi \|^2_{L^2(0, T; V_2)} + \| \psi \|^3_{W^{\alpha, 2}(0, T; V_2^*)} \leq (M^j_\varepsilon)^2 \right\}
\]

By Lemma 4.6, \( K^1_\varepsilon, i = 1, 2 \) are compact sets of \( L^{4/3}(0, T; H_1) \) and also \( K^j_\varepsilon, j = 3, 4 \) are compact sets of \( L^2(0, T; H_2) \).

Therefore, by the Markov inequality, the embedding \( W^{1, p}(0, T; V_2^*) \hookrightarrow W^{\alpha, p}(0, T; V_2^*) \) and \( L^2(0, T; V_1) \hookrightarrow L^{4/3}(0, T; V_1) \), we obtain

\[
P\left\{ \omega : u_N(\omega, \cdot), \bar{u}_N(\omega, \cdot), \varphi_N(\omega, \cdot), \tilde{\varphi}_N(\omega, \cdot) \notin K^1_\varepsilon \times K^2_\varepsilon \times K^3_\varepsilon \times K^4_\varepsilon \right\}
\]

\[
\leq \mathbb{P} \left\{ \omega : \| u_N \|_{L^{4/3}(0, T; V_1)} > \left( \frac{M^1_\varepsilon}{2} \right)^{4/3} \right\} + \mathbb{P} \left\{ \omega : \| \bar{u}_N \|_{L^{4/3}(0, T; V_1)} > \left( \frac{M^2_\varepsilon}{2} \right)^{4/3} \right\} + \mathbb{P} \left\{ \omega : \| \varphi_N \|_{L^2(0, T; V_2)} > \left( \frac{M^3_\varepsilon}{2} \right)^{2} \right\} + \mathbb{P} \left\{ \omega : \| \tilde{\varphi}_N \|_{L^2(0, T; V_2)} > \left( \frac{M^4_\varepsilon}{2} \right)^{2} \right\}
\]

\[
\leq \frac{2c}{(M^1_\varepsilon)^{4/3}} \left( \mathbb{E} \| u_N \|^2_{L^2(0, T; V_1)} \right)^{2/3} + \frac{2c}{(M^2_\varepsilon)^{4/3}} \left( \mathbb{E} \| \bar{u}_N \|^2_{L^2(0, T; V_1)} \right)^{2/3} + \frac{2c}{(M^3_\varepsilon)^{2}} \left( \mathbb{E} \| \varphi_N \|^2_{L^2(0, T; V_2)} \right) + \frac{2c}{(M^4_\varepsilon)^{2}} \left( \mathbb{E} \| \tilde{\varphi}_N \|^2_{L^2(0, T; V_2)} \right)
\]

Then, choosing \( M^i_\varepsilon = (5C_i/\varepsilon)^{3/4}, \) \( i = 1, 2 \) and \( M^i_\varepsilon = (5C_i/\varepsilon)^{1/2}, \) \( i = 3, 4 \), we get (4.105). From (4.104) and (4.105) we infer that

\[
\Pi_N(\varepsilon, H_\varepsilon \times K^1_\varepsilon \times K^2_\varepsilon \times K^3_\varepsilon \times K^4_\varepsilon) \geq 1 - \varepsilon.
\]

This ends the proof of Proposition 1. \( \square \)

4.4. **Application of Prokhorov’s and Skorokhod’s theorems.** Since the tightness of sequence \( \{ \Pi_N \} \) is proven, then applying the Prokhorov’s theorem (see [23]), we infer the existence of a probability measures \( \Pi \) such that, up to a subsequence,

\[
\Pi_N \rightharpoonup \Pi \text{ in } \mathcal{X}.
\]

Thus by making use of the Skorokhod embedding theorem (see [23]), we obtain, a new probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \), a sequence of random vector \( (\bar{W}_N, \bar{u}_N, \tilde{u}_N, \tilde{\varphi}_N, \tilde{\varphi}_N) \)
having the same law as \((W_N, u_N, \tilde{u}_N, \tilde{\phi}_N, \tilde{\phi}_N)\) and a random vector \((W, u, \bar{u}, \phi, \bar{\phi})\) having the probability law \(\mathbb{P}\) such that
\[
(W_N, \tilde{u}_N, \tilde{\phi}_N, \tilde{\phi}_N) \longrightarrow (W, u, \bar{u}, \phi, \bar{\phi}) \text{ in } \mathcal{X} \mathbb{P}\text{-a.s..} \tag{4.108}
\]

**Proposition 2.** The stochastic process \(\{W(t): t \in [0, T]\}\) obtained in (4.108) is a \(U\)-valued cylindrical Wiener process on \((\Omega', \mathcal{F}', \mathbb{P}')\). Furthermore, if \(0 \leq s < t \leq T\) then the increments \(W(t) - W(s)\) are independent of the \(\sigma\)-algebra generated by \((u(r), \bar{u}(r), \phi(r), \bar{\phi}(r), W(r))\) for \(r \in [0, s)\).

**Proof.** It is similar to [6, Lemma 5.2].

Now, let \(\mathcal{N}\) be the set of null sets of \(\mathcal{F}\) and for any \(t \geq 0\) and \(N \in \mathbb{N}\), let
\[
\mathcal{F}'_t = \sigma \left( \left( W_N(s), \tilde{u}_N(s), \tilde{\phi}_N(s), \tilde{\phi}_N(s) ; s \leq t \right) \cup \mathcal{N} \right), \tag{4.109}
\]}

be the completion of the natural filtration generated by \((W_N, \tilde{u}_N, \tilde{\phi}_N, \tilde{\phi}_N)\) and \((u, \bar{u}, \phi, \bar{\phi}, W)\) respectively. Also, owing to the fact that the law of \(W_N\) is equal to those of \(W, \tilde{u}_N, \tilde{\phi}_N, \tilde{\phi}_N\), we can easily see that \(W_N\) is a \(U\)-valued Wiener process adapted to the filtration \(\{\mathcal{F}'_t\}_{t \in [0, T]}\). Moreover, from the proof of Proposition 2, we see that \(W\) is a \(U\)-valued Wiener process adapted to the filtration \(\{\mathcal{F}'_t\}_{t \in [0, T]}\).

**Proposition 3.** The new random variable \((W_N, \tilde{u}_N, \tilde{\phi}_N, \tilde{\phi}_N, \tilde{\phi}_N)\) satisfies the initial scheme (4.1)-(4.2) on the new filtered probability space \((\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_{t \in [0, T]}, \mathbb{P}')\).

**Proof.** It is very similar to the method used in [24, Section 5, P. 956].

### 4.5. Passage to the limit

In this section we derive a priori estimates for the sequences \(\bar{u}_N, \tilde{\phi}_N, \tilde{\phi}_N\) and \(\phi_N\) obtained from the application of Prokhorov’s and Skorokhod’s compactness results. We have shown that they satisfy the initial scheme (4.1)-(4.2). Therefore they satisfy the same a priori estimates as \(u_N, \tilde{u}_N, \tilde{\phi}_N\) and \(\phi_N\) with \(\mathcal{E}'\) in the place of \(\mathcal{E}\). Namely,

\[
\mathbb{E}' \int_0^T \left( \|\tilde{u}_N\|_2^2 + \|\tilde{\mu}_N\|_{L_2}^2 \right) ds + \mathbb{E}' \int_0^T \left( \|\tilde{u}_N\|_2^2 + \|\tilde{\mu}_N\|_{L_2}^2 \right) ds \leq C, \tag{4.111}
\]

\[
\mathbb{E}' \sup_{0 \leq t \leq T} \mathcal{E}(\tilde{u}_N, \tilde{\phi}_N)(t) \leq C, \quad \mathbb{E}' \sup_{0 \leq t \leq T} \mathcal{E}^p(\tilde{u}_N, \tilde{\phi}_N)(t) \leq C, \quad \forall p \geq 2, \tag{4.112}
\]

\[
\mathbb{E}' \sup_{0 \leq t \leq T} \mathcal{E}(\tilde{\phi}_N, \tilde{\phi}_N)(t) \leq C, \quad \mathbb{E}' \sup_{0 \leq t \leq T} \mathcal{E}^p(\tilde{\phi}_N, \tilde{\phi}_N)(t) \leq C, \quad \forall p \geq 2, \tag{4.113}
\]

\[
\mathbb{E}' \left( \int_0^T (\|\tilde{u}_N\|_2^2 + \|\tilde{\mu}_N\|_{L_2}^2) ds \right)^p \leq C, \quad \mathbb{E}' \left( \int_0^T \left( \|\tilde{u}_N, \tilde{\phi}_N\|_u^2 ds \right)^p \right) \leq C, \quad \forall p \geq 2, \tag{4.114}
\]

\[
\mathbb{E}' \left( \int_0^T (\|\tilde{u}_N\|_2^2 + \|\tilde{\mu}_N\|_{L_2}^2) ds \right) \leq C, \quad \mathbb{E}' \left( \int_0^T \left( \|\tilde{u}_N, \tilde{\phi}_N\|_u^2 ds \right) \right) \leq C, \quad \forall p \geq 2. \tag{4.115}
\]

For almost \(t \in [0, T]\),

\[
\begin{cases}
\mathbb{E}' \|\tilde{u}_N(t) - \bar{u}_N(t)\|^4_{V^*_T} \leq C \left( \frac{T}{N + 1} \right)^{1/3}, \\
\mathbb{E}' \left\| \tilde{\phi}_N(t) - \bar{\phi}_N(t) \right\|^2_{V_2} \leq \frac{CT}{N + 1},
\end{cases} \tag{4.116}
\]

in \(d = 2\) and \(d = 3\).
From (4.116), we derive that for almost $t \in [0, T]$,
\[
\begin{align*}
\mathbb{E}' \|\tilde{u}_N(t) - \tilde{u}_N(t)\|_{V_d^2}^{4/3} & \to 0 \quad \text{as} \quad N \to \infty, \quad \text{for} \quad d = 2, 3, \\
\mathbb{E}' \|\tilde{\phi}_N(t) - \tilde{\phi}_N(t)\|_{V_d^2} & \to 0 \quad \text{as} \quad N \to \infty, \quad \text{for} \quad d = 2, 3,
\end{align*}
\]

(4.117)

where we infer using (4.108) that
\[
u = \bar{u}, \quad \phi = \bar{\phi}, \quad dt \otimes \mathbb{P}'\text{-a.e.}
\]
(4.118)

Applying the Banach-Alaoglu theorem, we derive that for $p \geq 2$, by extracting a new subsequence,
\[
\begin{align*}
(u_N, \phi_N) & \sim (u, \phi) \quad \text{in} \quad L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^\infty(0, T; \mathcal{H})), \\
(u_N, \phi_N) & \rightharpoonup (u, \phi) \quad \text{in} \quad L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0, T; \mathcal{U})), \\
\tilde{\mu}_N & \rightharpoonup \mu \quad \text{in} \quad L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0, T; \mathcal{H})), \\
\tilde{\bar{u}}_N & \rightharpoonup \bar{u} \quad \text{in} \quad L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0, T; V_1)), \\
\tilde{\bar{\phi}}_N & \rightharpoonup \bar{\phi} \quad \text{in} \quad L^p(\tilde{\Omega}', \tilde{\mathcal{F}}, \tilde{\mathbb{P}}'; L^2(0, T; D(A_*))), \\
\tilde{\bar{\mu}}_N & \rightharpoonup \bar{\mu} \quad \text{in} \quad L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0, T; \mathcal{H})).
\end{align*}
\]

(4.119)

Using (4.112), we infer that
\[
\begin{align*}
\mathbb{E}' \left( \int_0^T |\tilde{u}_N(s)|_{L^2}^{4/3} \, ds \right)^3 & \leq T^3 \mathbb{E}' \sup_{0 \leq s \leq T} |\tilde{u}_N(s)|_{L^2}^4 \leq C, \quad \text{for} \quad d = 2, 3, \\
\mathbb{E}' \left( \int_0^T \|\tilde{\phi}_N(s)\|_{\gamma}^2 \, ds \right)^2 & \leq T^2 \mathbb{E}' \sup_{0 \leq s \leq T} \|\tilde{\phi}_N(s)\|_{\gamma}^4 \leq C, \quad \text{for} \quad d = 2, 3.
\end{align*}
\]

(4.120) and (4.121)

Therefore, using the Vitali convergence theorem (see [13]), we infer from (4.108), (4.120) and (4.121) that
\[
\begin{align*}
\tilde{u}_N & \rightharpoonup u \quad \text{in} \quad L^{4/3}(\tilde{\Omega}', \tilde{\mathcal{F}}', \tilde{\mathbb{P}}'; L^{4/3}(0, T; \mathcal{H})), \quad \text{for} \quad d = 2, 3, \\
\tilde{\phi}_N & \rightharpoonup \phi \quad \text{in} \quad L^2(\tilde{\Omega}', \tilde{\mathcal{F}}', \tilde{\mathbb{P}}'; L^2(0, T; V_2)), \quad \text{for} \quad d = 2, 3,
\end{align*}
\]

(4.122) and (4.123)

and thus extracting a new subsequence still denoted by $(\tilde{u}_N, \tilde{\phi}_N)$ to save the notation, we can also assert that
\[
(\tilde{u}_N, \tilde{\phi}_N) \rightharpoonup (u, \phi) \quad \text{in} \quad \mathcal{H},
\]

(4.124)

for almost $w, t$ with respect to the measure $d\bar{\mathbb{P}}' \otimes dt$.

Since $\tilde{\mu}_N = \varepsilon A_* \tilde{\phi}_N + \alpha f_\gamma(\tilde{\phi}_N)$ in $V_2^*$ for almost $w, t$ with respect to the measure $d\bar{\mathbb{P}}' \otimes dt$, we then infer via the continuity of $f_\gamma$ that
\[
\bar{\mu} = \varepsilon A_* \tilde{\phi} + \alpha f_\gamma(\tilde{\phi}) \quad \text{in} \quad V_2^* \quad d\bar{\mathbb{P}}' \otimes dt \text{-a.e.}
\]

(4.125)

Also, since $\tilde{\mu}_N = \varepsilon A_* \tilde{\phi}_N + \alpha f_\gamma(\tilde{\phi}_N)$ in $V_2^*$ for almost $w, t$ with respect to the measure $d\bar{\mathbb{P}}' \otimes dt$, we then infer via the continuity of $f_\gamma$ (since $f$ is continuous) and the equality (4.118) that
\[
\mu = \varepsilon A_* \phi + \alpha f_\gamma(\phi) \quad \text{in} \quad V_2^* \\
= \varepsilon A_* \bar{\phi} + \alpha f_\gamma(\bar{\phi}), \quad \text{in} \quad V_2^* \\
= \bar{\mu}, \quad \text{in} \quad V_2^*.
\]

(4.126)
This implies that
\[ \mu = \bar{\mu}, \quad dt \otimes \mathbb{P} \text{-a.e.} \] (4.127)

**Lemma 4.8.** For all \( v \in V_1 \) and \( \psi \in V_2 \), the following convergence hold.

\[
\begin{align*}
\lim_{N \to +\infty} \int_0^t \langle A_0 \tilde{u}_N(s), v \rangle ds &= \int_0^t \langle A_0 u(s), v \rangle ds \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} \int_0^t \langle A_0 \tilde{u}_N(s), v \rangle ds &= \int_0^t \langle A_0 u(s), v \rangle ds \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} \int_0^t (\tilde{\mu}_N(s), \psi) ds &= \int_0^t (\mu(s), \psi) ds \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} \int_0^t (\tilde{\tilde{\mu}}_N(s), \psi) ds &= \int_0^t (\mu(s), \psi) ds \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} \int_0^t \langle B_0(\tilde{u}_N, \tilde{\phi}_N), v \rangle ds &= \int_0^t \langle B_0(u, \phi), v \rangle ds \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} \int_0^t \langle B_1(\tilde{u}_N, \tilde{\phi}_N), \psi \rangle ds &= \int_0^t \langle B_1(u, \phi), \psi \rangle ds \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} (\tilde{\mu}_N, \psi) &= (\mu, \psi) \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} (A_\gamma \phi_N, \psi) &= (A_\gamma \phi, \psi) \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} (f_\gamma(\tilde{\phi}_N), \psi) &= (f_\gamma(\phi), \psi) \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} \int_0^t (g_1(s, \tilde{u}_N(s)), v) ds &= \int_0^t (g_1(s, u(s)), v) ds \quad \text{in } L^1(\Omega' \times (0, T)), \\
\lim_{N \to +\infty} \left( \int_0^t g_2(s, \tilde{u}_N(s)) d\tilde{W}_N(s), v \right) &= \left( \int_0^t g_2(s, u(s)) dW_s, v \right) \quad \text{in } L^1(\Omega' \times (0, T)).
\end{align*}
\]

**Proof.** From the weak convergence (4.119), and the continuity property of the operator \( A_0 \), we derive that

\[
\lim_{N \to +\infty} \int_0^t \langle A_0 \tilde{u}_N - A_0 u, v \rangle ds = 0 \quad dt \otimes \mathbb{P} \text{-a.e.}
\]

Moreover, we note that

\[
\mathbb{E}' \int_0^T \left| \int_0^t \langle A_0 \tilde{u}_N, v \rangle ds \right|^4 dt \\
\leq C \|v\|^4 \mathbb{E} \int_0^T \left| \int_0^T \|\tilde{u}_N\| ds \right|^4 dt \leq C_T \|v\|^4 \mathbb{E}' \left( \int_0^T \|\tilde{u}_N\|^2 ds \right)^2 \leq C.
\]

Hence, using the Vitali’s convergence Theorem (see [13]), we infer that

\[
\lim_{N \to +\infty} \mathbb{E}' \int_0^T \left| \int_0^t \langle A_0 \tilde{u}_N - A_0 u, v \rangle ds \right| = 0.
\]
This proves (4.128). Using the same method, we also obtain (4.129), (4.130) and (4.131).

For the proof of (4.132), we note that

\[
\begin{align*}
\mathbb{E} \int_0^T \int_0^T & \langle B_0(\tilde{u}_N, \tilde{u}_N) - B_0(u, v) \rangle ds \ dt \\
\leq & \mathbb{E} \int_0^T \int_0^T \langle B_0(\tilde{u}_N - \tilde{u}, \tilde{u}_N), v \rangle ds \ dt + \mathbb{E} \int_0^T \int_0^T \langle B_0(\tilde{u}, \tilde{u}_N - u), v \rangle ds \ dt.
\end{align*}
\]

For \( d = 2 \), we obtain

\[
\begin{align*}
\mathbb{E}' \int_0^T \int_0^T & \langle B_0(\tilde{u}_N - \tilde{u}, \tilde{u}_N), v \rangle ds \ dt \\
\leq & \mathbb{E}' \int_0^T \int_0^T \| B_0(\tilde{u}_N - \tilde{u}, \tilde{u}_N), v \| ds dt \leq C_v \mathbb{E}' \int_0^T \| B_0(\tilde{u}_N - u, \tilde{u}_N) \|_{V_1'} ds \\
\leq & C_v \mathbb{E}' \int_0^T \| \tilde{u}_N - u \|^{1/2} \| \tilde{u}_N - u \|^{1/2} \| \tilde{u}_N \|^{1/2} \| \tilde{u}_N \|^{1/2} ds \\
\leq & C_v \left( \mathbb{E}' \int_0^T \| \tilde{u}_N - \tilde{u} \|_{L^2} \right)^{1/2} \left( \mathbb{E}' \int_0^T \| \tilde{u}_N - \tilde{u} \| L^2 \| \tilde{u}_N \| ds \right)^{1/2} \left( \mathbb{E}' \int_0^T \| \tilde{u}_N \|_{L^2}^2 \| \tilde{u}_N \|^2 ds \right)^{1/4} \left( \mathbb{E}' \int_0^T \| \tilde{u}_N \|^2 ds \right)^{1/8} \left( \mathbb{E}' \int_0^T \| \tilde{u}_N \|^2 ds \right)^{3/8} \\
\leq & C_v \left( \mathbb{E}' \int_0^T \| \tilde{u}_N - u \|_{L^2}^{4/3} ds \right)^{3/8} \left( \mathbb{E}' \sup_{0 \leq s \leq T} \| \tilde{u}_N \|_{L^2}^{4/3} \right)^{1/8} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|^2 ds \right)^2 \right)^{1/8} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|^2 ds \right)^2 \right)^{3/8} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|^2 ds \right)^2 \right)^{1/8} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|^2 ds \right)^2 \right)^{3/8}.
\end{align*}
\]

In (4.142), we have used the weak convergence (4.119) and the strong convergence (4.122).1

For \( d = 3 \), we note form (2.18) that

\[
\begin{align*}
\mathbb{E}' & \int_0^T \int_0^T \left( B_0(\tilde{u}_N - \tilde{u}, \tilde{u}_N), v \right) ds dt \\
\leq & T \| v \| \mathbb{E}' \int_0^T \| \tilde{u}_N - u \|_{L^2}^{1/4} \| \tilde{u}_N - u \|^{3/4} \| \tilde{u}_N \|_{L^2}^{1/4} \| \tilde{u}_N \|^{3/4} \| \tilde{u}_N \|^{1/4} \| \tilde{u}_N \|_{L^2} \| \tilde{u}_N \| ds \\
\leq & T \| v \| \left( \mathbb{E}' \int_0^T \| \tilde{u}_N - \tilde{u} \|_{L^2} \right)^{1/4} \left( \mathbb{E}' \int_0^T \| \tilde{u}_N - \tilde{u} \|_{L^2} \| \tilde{u}_N \|_{L^2} \| \tilde{u}_N \| ds \right)^{3/4} \left( \mathbb{E}' \int_0^T \| \tilde{u}_N \|_{L^2}^2 \| \tilde{u}_N \|^2 ds \right)^{3/8} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|^2 ds \right)^2 \right)^{1/8} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|^2 ds \right)^2 \right)^{3/8} \\
\leq & C_v \left( \mathbb{E}' \int_0^T \| \tilde{u}_N - u \|_{L^2}^{4/3} ds \right)^{1/16} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N - u \|_{L^2}^2 ds \right)^{1/4} \right) \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|_{L^2}^2 \| \tilde{u}_N \|^2 ds \right)^{3/4} \right)^{1/8} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|^2 ds \right)^2 \right)^{1/8} \left( \mathbb{E}' \left( \int_0^T \| \tilde{u}_N \|^2 ds \right)^2 \right)^{3/8}.
\end{align*}
\]
\[ C_v \left( E' \int_0^T |\tilde{u}_N - u|^{4/3}_{L^2} \, ds \right)^{\frac{3}{4}}. \]

In (4.143) we have used the weak convergence (4.119)\(_2\) and the strong convergence (4.122)\(_2\).

We also derive that for \( d = 2 \),
\[ E' \int_0^T \left| \int_0^t (B_0(\tilde{u}, \tilde{u}_N - u), v) \, ds \right| \, dt \]
\[ \leq TE' \int_0^T |b_0(u, \tilde{u}_N - u, v)| \, ds = TE' \int_0^T |b_0(u, v, \tilde{u}_N - u)| \, ds \]
\[ \leq CTE' \int_0^T |u|^{\frac{2}{3}}_L \|u\|^\frac{1}{3} \|\tilde{u}_N - u\|^{\frac{2}{3}}_L \|\tilde{u}_N - u\|^\frac{3}{2} \, ds \]
\[ \leq C_T \|v\| \left( E' \int_0^T |\tilde{u}_N - u|^{4/3}_{L^3} \, ds \right)^{\frac{3}{4}} \left( E' \int_0^T \|\tilde{u}_N - u\|^2 \, ds \right)^{\frac{1}{4}} \left( E' \int_0^T |u|^{\frac{2}{3}}_L \|u\|^\frac{1}{3} \, ds \right)^{\frac{1}{4}} \]
\[ \leq C_T \|v\| \left( E' \int_0^T |\tilde{u}_N - u|^{4/3}_{L^3} \, ds \right)^{\frac{3}{4}}. \]

For \( d = 3 \), we obtain
\[ E' \int_0^T \left| \int_0^t (B_0(\tilde{u}, \tilde{u}_N - u), v) \, ds \right| \, dt \]
\[ \leq TE' \int_0^T |b_0(u, v, \tilde{u}_N - u)| \, ds \leq CTE' \int_0^T |u|^{\frac{4}{3}}_L \|u\|^\frac{2}{3} \|\tilde{u}_N - u\|^{\frac{4}{3}}_L \|\tilde{u}_N - u\|^\frac{3}{2} \, ds \]
\[ \leq C_v \left( E' \int_0^T |\tilde{u}_N - u|^{4/3}_{L^3} \, ds \right)^{\frac{3}{4}} \left( E' \int_0^T \|\tilde{u}_N - u\|^2 \, ds \right)^{\frac{1}{4}} \left( E' \int_0^T |u|^{\frac{6}{3}}_L \|\tilde{u}_N\|^2 \, ds \right)^{\frac{1}{4}} \]
\[ \leq C_v \left( E' \int_0^T |\tilde{u}_N - u|^{4/3}_{L^3} \, ds \right)^{\frac{3}{4}}. \]

Using (4.142)-(4.145) and the strong convergence (4.122), we pass to the limit in (4.140) to infer (4.132).

Now we prove (4.133). However, owing to (2.25) we remark that
\[ E' \int_0^T \left| \int_0^t (R_0(\varepsilon A, \phi, \phi_N), v) \, ds \right| \, dt \]
\[ = \varepsilon E' \int_0^T \left| \int_0^t (R_1(\phi_N, \phi_N), \phi) \, ds + (\phi_N - \phi) \right| \, ds \]
\[ \leq \varepsilon E' \int_0^T (R_1(\phi_N - \phi, \phi) \, ds \right) dt + \varepsilon E' \int_0^T (R_1(\phi_N, \phi_N - \phi), v) \, ds \right) dt. \]

For \( d = 2 \), from (2.24) we arrive at,
\[ E' \int_0^T \left| \int_0^t (R_1(\phi_N, \phi_N), v) \, ds \right| \, dt \]
\[ \leq TE' \int_0^T \|R_1(\phi_N, \phi_N - \phi)\|_{V_1^s} \|v\| \, ds \]
\[ \leq T \| v \| \mathbb{E}' \int_0^T \left( \frac{\bar{\gamma} N}{\gamma} \right)^{1/2} A_{\gamma} \tilde{\phi}_N \|^{1/2} \tilde{\phi}_N - \phi \|^{1/2} \bigg\| A_{\gamma} \tilde{\phi}_N - A_{\gamma} N \phi \|^{1/2} ds \right) \quad (4.148) \]

\[ \leq C \left( \mathbb{E}' \int_0^T \left| A_{\gamma} \tilde{\phi}_N \right|^2 ds \right)^{1/2} \left( \mathbb{E}' \int_0^T \left| A_{\gamma} (\tilde{\phi}_N - \phi) \right|^2 ds \right)^{1/4} \left( \mathbb{E}' \int_0^T \left\| \tilde{\phi}_N - \phi \right\|^2 _\gamma ds \right)^{1/4} \]

\[ \leq C \left( \mathbb{E}' \int_0^T \left\| \tilde{\phi}_N - \phi \right\|^2 _\gamma ds \right)^{1/4} . \]

We note that in (4.147) we have used the weak convergence (4.119) and the strong convergence (4.123).

For \( d = 3 \), owing to (2.19), we obtain

\[ \mathbb{E}' \int_0^T \left| \int_0^T \langle R_1(\tilde{\phi}_N, \tilde{\phi}_N - \phi), v \rangle ds \right| dt \]

\[ \leq T \mathbb{E}' \int_0^T \left\| R_1(\tilde{\phi}_N, \tilde{\phi}_N - \phi) \right\|_{V'_1} \| v \| ds \]

\[ \leq cT \| v \| \left( \mathbb{E}' \int_0^T \left| A_{\gamma} \tilde{\phi}_N \right|^2 _{L^2} ds \right)^{1/2} \left( \mathbb{E}' \int_0^T \left| \tilde{\phi}_N - \phi \right|^2 _\gamma ds \right)^{1/4} \left( \mathbb{E}' \int_0^T \| A_{\gamma} \tilde{\phi}_N - A_{\gamma} N \phi \|^{1/2} _{L^2} ds \right) \]

\[ \leq cT \| v \| \left( \mathbb{E}' \int_0^T \left| A_{\gamma} \tilde{\phi}_N \right|^2 _{L^2} ds \right)^{1/2} \left( \mathbb{E}' \int_0^T \left| A_{\gamma} (\tilde{\phi}_N - \phi) \right|^2 _{L^2} ds \right)^{1/4} \left( \mathbb{E}' \int_0^T \left\| \tilde{\phi}_N - \phi \right\|^2 _\gamma ds \right)^{1/8} . \]

In the other hand, for \( d = 2 \), owing to (2.24),

\[ \mathbb{E}' \int_0^T \left| \int_0^T \langle R_1(\tilde{\phi}_N - \phi, \phi), v \rangle ds \right| dt \]

\[ \leq T \mathbb{E}' \int_0^T \left\| R_1(\tilde{\phi}_N - \phi, \phi) \right\|_{V'_1} \| v \| ds \]

\[ \leq cT \| v \| \left( \mathbb{E}' \int_0^T \left| \tilde{\phi}_N - \phi \right|^2 _\gamma ds \right)^{1/2} \left( \mathbb{E}' \int_0^T \left| A_{\gamma} \tilde{\phi}_N - A_{\gamma} \phi \right|^2 _{L^2} ds \right)^{1/2} \left( \mathbb{E}' \int_0^T \| A_{\gamma} \phi \|^{1/2} _{L^2} ds \right) \]

\[ \leq C \left( \mathbb{E}' \int_0^T \left| A_{\gamma} \tilde{\phi}_N - A_{\gamma} \phi \right|^2 _{L^2} ds \right)^{1/2} \left( \mathbb{E}' \int_0^T \left\| \tilde{\phi}_N - \phi \right\|^2 _\gamma ds \right)^{1/4} \left( \mathbb{E}' \int_0^T \left| A_{\gamma} \phi \right|^2 _{L^2} ds \right)^{1/2} \]

\[ \leq C \left( \mathbb{E}' \int_0^T \left\| \tilde{\phi}_N - \phi \right\|^2 _\gamma ds \right)^{1/4} . \]

For \( d = 3 \), since (2.24) holds, we obtain

\[ \mathbb{E}' \int_0^T \left| \int_0^T \langle R_1(\tilde{\phi}_N - \phi, \phi), v \rangle ds \right| dt \]

\[ \leq T \mathbb{E}' \int_0^T \left\| R_1(\tilde{\phi}_N - \phi, \phi) \right\|_{V'_1} \| v \| ds \]

\[ \leq C \left( \mathbb{E}' \int_0^T \left\| R_1(\tilde{\phi}_N - \phi, \phi) \right\|_{V'_1} \| v \| ds \right)^{1/4} . \]
≤ T \| \psi \| \mathbb{E}' \int_0^T \left| (B_1(\tilde{u}_N, \tilde{\phi}_N) - B_1(u, \phi), \psi) \right| ds \right| dt 

\leq \mathbb{E}' \int_0^T \left| \int_0^T (\tilde{B}(\tilde{u}_N - u, \tilde{\phi}_N), \psi) ds \right| dt + \mathbb{E}' \int_0^T \left| \int_0^T (\tilde{B}(\tilde{u}, \tilde{\phi}_N - \phi), \psi) ds \right| dt. 

For d = 2, from (2.18), we infer that

\mathbb{E}' \int_0^T \left| \int_0^T (B_1(\tilde{u}_N - u, \tilde{\phi}_N), \psi) ds \right| dt \leq T \| \psi \| \mathbb{E}' \int_0^T \left\| B_1(\tilde{u}_N - u, \tilde{\phi}_N) \right\|_{V_2'} ds 

\leq c_\psi \mathbb{E}' \int_0^T \left| \int_0^T (B_1(\tilde{u}_N - u, \tilde{\phi}_N), \psi) ds \right| dt 

\leq c_\psi \left( \mathbb{E}' \int_0^T |\tilde{u}_N - u|_{L_2}^2 \right)^{1/2} \left( \mathbb{E}' \int_0^T \left\| \tilde{\phi}_N \right\|_{\gamma}^2 ds \right)^{1/2} 

\leq c_\psi \left( \mathbb{E}' \int_0^T |\tilde{u}_N - u|_{L_2}^2 \right)^{3/8} \left( \mathbb{E}' \int_0^T \left\| \tilde{\phi}_N \right\|_{\gamma}^{4} ds \right)^{1/4} 

\leq c_\psi \left( \mathbb{E}' \int_0^T |\tilde{u}_N - u|_{L_2}^2 \right)^{3/8} \left( \mathbb{E}' \sup_{0 \leq s \leq T} \left\| \tilde{\phi}_N \right\|_{\gamma}^{4} \right)^{1/4} 

\leq c_\psi \left( \mathbb{E}' \int_0^T |\tilde{u}_N - u|_{L_2}^2 \right)^{3/8}. 

For d = 3, owing to (2.18), we derive that

\mathbb{E}' \int_0^T \left| \int_0^T (B_1(\tilde{u}_N - u, \tilde{\phi}_N), \psi) ds \right| dt \leq T \| \psi \| \mathbb{E}' \int_0^T \left\| B_1(\tilde{u}_N - u, \tilde{\phi}_N) \right\|_{V_2'} ds 

\leq c_\psi \mathbb{E}' \int_0^T \left| \int_0^T (B_1(\tilde{u}_N - u, \tilde{\phi}_N), \psi) ds \right| dt 

\leq c_\psi \left( \mathbb{E}' \int_0^T |\tilde{u}_N - u|_{L_2}^2 \right)^{3/16} \left( \mathbb{E}' \int_0^T \left\| \tilde{\phi}_N \right\|_{\gamma}^{16/13} ds \right)^{13/16} 

Using (4.147)-(4.151) and the strong convergence (4.123), we pass to the limit in (4.146) to infer (4.133). For the proof of (4.134), we remark that

\mathbb{E}' \int_0^T \left| \int_0^T (B_1(\tilde{u}_N - u, \tilde{\phi}_N), \psi) ds \right| dt
Also, for $d = 2$, since (2.18) holds, we deduce that

$$E' \int_0^T \left| \int_0^t \langle B_1(u, \tilde{\phi}_N - \phi), \psi \rangle ds \right| dt$$

$$\leq c_\psi E' \int_0^t \left| \int_0^T \left| B_1(u, \tilde{\phi}_N - \phi) \right|_{L^2} \right|_{V_2^1} ds \leq c_\psi E' \int_0^T \left| u \right|_{L^2} \left\| \tilde{\phi}_N - \phi \right\|_{L^2} ds$$

$$\leq c_\psi E' \int_0^t \left| u \right|_{L^2} \left\| \tilde{\phi}_N - \phi \right\|_{L^2} ds \leq c_\psi \left( E' \int_0^T \left| u \right|^2 ds \right)^{1/2} \left( E' \int_0^T \left\| \tilde{\phi}_N - \phi \right\|^2_{L^2} ds \right)^{1/2}.$$

For $d = 3$, form (2.18), we get

$$E' \int_0^T \left| \int_0^t \langle B_1(u, \tilde{\phi}_N - \phi), \psi \rangle ds \right| dt$$

$$\leq c_\psi E' \int_0^t \left| \int_0^T \left| B_1(u, \tilde{\phi}_N - \phi) \right|_{L^2} \right|_{V_2^1} ds \leq c_\psi E' \int_0^T \left| u \right|_{L^2} \left\| \tilde{\phi}_N - \phi \right\|_{L^2}^{1/4} dN \left( \tilde{\phi}_N - \phi \right)^{3/4} ds$$

$$\leq c_\psi E' \int_0^t \left| u \right|_{L^2} \left\| \tilde{\phi}_N - \phi \right\|_{L^2}^{1/4} dN \left( \tilde{\phi}_N - \phi \right)^{3/4} ds \leq c_\psi \left( E' \int_0^T \left| u \right|^2 ds \right)^{1/2} \left( E' \int_0^T \left\| \tilde{\phi}_N - \phi \right\|^2_{L^2} ds \right)^{1/2}.$$

Using (4.153)-(4.156) and the strong convergence (4.122)-(4.123), we pass to the limit in (4.152) to infer (4.134).

We also note that

$$E' \int_0^T \left| \langle A_1 \tilde{\phi}_N - A_1 \phi, \psi \rangle \right| ds \leq \left\| \psi \right\|_{L^\infty} E' \int_0^T \left| \tilde{\phi}_N - \phi \right| ds \leq \left\| \psi \right\|_{L^\infty} \left( E' \int_0^T \left\| \tilde{\phi}_N - \phi \right\|^2_{L^2} ds \right)^{1/2}.$$ 

This ends the proof of (4.136) with the help of the strong convergence (4.123).

For the proof of (4.137), we first assume that $\psi \in D(A_\gamma)$. However, owing to the fact that

$$D(A_\gamma) \hookrightarrow H^2(D) \hookrightarrow L^\infty(D),$$

we obtain

$$E' \int_0^T \left| \left( f_\gamma(\tilde{\phi}_N) - f_\gamma(\phi), \psi \right) \right| dt$$

$$\leq C_T \left\| \psi \right\|_{L^\infty} E' \int_0^T \int_D \left| f_\gamma(\tilde{\phi}_N) - f_\gamma(\phi) \right| dx ds$$

$$\leq C_T \left\| \psi \right\|_{D(A_\gamma)} E' \int_0^T \int_D \left| f_\gamma(\tilde{\phi}_N) - f_\gamma(\phi) \right| dx ds.$$ 

Since the strong convergence (4.123) holds, we derive that up to a subsequence

$$\tilde{\phi}_N \rightarrow \phi \quad \text{a.e. in } \Omega' \times (0, T) \times D.$$
Owing to the fact that $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f_\gamma : \mathbb{R} \to \mathbb{R}$ is also continuous. Hence, we obtain from (4.158) that

$$f_\gamma(\tilde{\phi}_N) \rightarrow f_\gamma(\phi) \quad \text{a.e. in} \quad \Omega' \times (0, T) \times D.$$ 

Using the assumption (2.1), the Sobolev embedding, Young inequality, we arrive at

$$\int_{\Omega \times (0, T) \times D} \left| f_\gamma(\tilde{\phi}_N) \right|^2 dx ds dP' = \mathbb{E}' \int_0^T \left| f_\gamma(\tilde{\phi}_N) \right|^2_{L^2} ds \leq C_f(1 + \mathbb{E}' \sup_{0 \leq s \leq T} \left\| \tilde{\phi}_N \right\|_{\gamma}^{2m+2}) \leq C.$$  

Thus, $\{f_\gamma(\tilde{\phi}_N)\}_{N \geq 1}$ is uniformly integrable over $\Omega \times (0, T) \times D$. Therefore, by Vitali convergence theorem (see [13]), we derive that

$$\lim_{N \to \infty} \mathbb{E}' \int_0^T \int_D \left| f_\gamma(\tilde{\phi}_N) - f_\gamma(\phi) \right| dx ds = 0.$$ 

Therefore, passing to the limit in (4.157) we obtain (4.137) since the injection of $D(A_{\gamma})$ in $V_2$ is dense.

For the proof of (4.138), we note that $\mathbb{P}'$-almost surely,

$$\int_0^t (g_1(s, \tilde{u}_N), v) ds \to \int_0^t (g_1(s, u), v) ds.$$  

Here, we have used the fact that the strong convergence implies the weak convergence. Also, we note that

$$\mathbb{E}' \int_0^T \left| \int_0^t (g_1(s, \tilde{u}_N), v) ds \right|^2 dt \leq 2T K_1^2 \mathbb{E}' \sup_{0 \leq s \leq T} (1 + \left| \tilde{u}_N(s) \right|_{L^2}^2) \leq C.$$  

Thus by (4.161), we observe that $\{\int_0^t (g_1(s, \tilde{u}_N), v) ds\}_{N \geq 1}$ is uniformly integrable in $L^1(\Omega' \times (0, T), \mathcal{F}', \mathbb{P}'; \mathbb{R})$. Hence thanks to the Vitali convergence theorem (see [13]), we conclude (4.138).

Lastly, following the same argument as in [3, 20, 24] we derive (4.139) and end the proof of Lemma 4.8.

Now, using Lemma 4.8, we can pass to the limit in (4.4) with the new random process $(\hat{W}_N, \tilde{u}_N, \tilde{\phi}_N, \hat{\phi}_N, \tilde{\phi}_N)$ and the new filtered probability space $(\Omega', \mathcal{F}', \mathcal{F}_t'_{t \in [0, T]}, \mathbb{P}')$ to derive that $(\Omega', \mathcal{F}', \mathcal{F}_t'_{t \in [0, T]}, \mathbb{P}, V, u, \phi, W)$ is a weak martingale solution of (1.3), (1.4) in the sense of the Definition 3.1. We then end the proof of Theorem 3.2.

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