$R$–charge thermodynamical spectral sum rule
in $\mathcal{N} = 4$ Yang-Mills theory

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A sum rule related to the $R$–current correlator at vanishing three-momentum is derived in the $\mathcal{N} = 4$ supersymmetric Yang-Mills field theory at infinite 't Hooft coupling. For reference it is compared to the one in the free field theory, i.e. at the one loop perturbative approximation.

I. INTRODUCTION

Relativistic heavy ion physics is confronted with phenomena which are interpreted in terms of either a weakly interacting plasma (pQGP) or more recently due to experimental results produced at RHIC by the strongly-coupled quark-gluon plasma (sQGP) [1]. Transport properties are part of the main interest, especially the hydrodynamic coefficients [2], in order to find the proper interpretation.

In this note the current-current correlator at finite temperature [3, 4, 5] is considered. In analogy to the “hydrodynamic” sum rule derived in [6] a certain sum rule for the spectral density in a hot prototype gauge theory of sQGP is derived, i.e. in the $\mathcal{N} = 4$ supersymmetric Yang-Mills field theory.

II. DISPERSION RELATION

In the framework of linear response theory one considers the retarded correlator [7]

$$\chi(t) = i\Theta(t) < [O(t),O(0)] > ,$$

and its Fourier transform

$$\chi(\omega) = \int_{-\infty}^{+\infty} dt \ e^{i\omega t} \chi(t) .$$

The spectral density is defined by

$$\rho(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \ e^{i\omega t} < [O(t),O(0)] > ,$$

respectively by

$$< [O(t),O(0)] > = \int d\omega \ e^{-i\omega t} \rho(\omega) .$$

Inserting (4) into (2) one obtains

$$\chi(\omega + i\epsilon) = \chi^R(\omega) = \int_{-\infty}^{+\infty} d\omega' \frac{\rho(\omega')}{\omega' - \omega - i\epsilon} = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega' \frac{Im\chi(\omega')}{\omega' - \omega - i\epsilon} .$$
This basic dispersion relation is assumed to hold at finite temperature $T$ as well as at $T = 0$. Following the derivation given in [6] consider first

$$\chi^R(\omega, T) - \chi^R(\omega, T = 0) = \int_{-\infty}^{+\infty} d\omega' \frac{\rho(\omega', T) - \rho(\omega', T = 0)}{\omega' - \omega - i\epsilon},$$  

(6)

and subtract finally the piece at $\omega \to i\infty$ - the integral (6) is expected to vanish in this limit - to obtain [6, 10]

$$\chi^R(\omega, T) - \chi^R(\omega, T = 0) - \left[ \chi^R(\omega \to i\infty, T) - \chi^R(\omega \to i\infty, T = 0) \right]$$

$$= \int_{-\infty}^{+\infty} d\omega' \frac{\delta \rho(\omega')}{\omega' - \omega - i\epsilon},$$  

(7)

with

$$\delta \rho(\omega) = \rho(\omega, T) - \rho(\omega, T = 0).$$  

(8)

Expressed in terms of the retarded Green’s function $G_R(\omega) = -\chi^R(\omega)$ (7) can be written as

$$\delta G_R(\omega) - \delta G^\infty_R = -\int_{-\infty}^{+\infty} d\omega' \frac{\delta \rho(\omega')}{\omega' - \omega - i\epsilon},$$  

(9)

$\delta G_R(\omega) = G_R(\omega, T) - G_R(\omega, T = 0)$ and $\delta G^\infty_R = G_R(\omega = i\infty, T) - G_R(\omega = i\infty, T = 0)$.

Finally, using $\delta \rho(-\omega) = -\delta \rho(\omega)$ the result derived in [6]

$$\delta G_R(\omega) - \delta G^\infty_R = -2 \int_{0}^{+\infty} d\omega' \frac{\delta \rho(\omega')}{\omega' - \omega - i\epsilon},$$  

(10)

is obtained.

In the following the static limit $\omega = 0$ is considered

$$\delta G_R(\omega = 0) - \delta G^\infty_R = -2 \int_{0}^{+\infty} d\omega' \frac{\delta \rho(\omega')}{\omega'},$$  

(11)

i.e. a so-called thermodynamic sum rule [8].

III. EUCLIDEAN CORRELATOR

Let us start from the Euclidean correlator [3]

$$G(\tau) = \int_{0}^{\infty} d\omega \rho(\omega) \frac{\cosh[\omega(\tau - \beta/2)]}{\sinh(\omega\beta/2)},$$  

(12)

and take the integral ($\beta = 1/T$)

$$\int_{0}^{\beta} d\tau G(\tau) = \int_{0}^{+\infty} d\omega \frac{\rho(\omega)}{\omega}. $$

(13)

Together with the definition of the reconstructed correlator [9]

$$G^{\text{rec}}(\tau) = \int_{0}^{\infty} d\omega \rho(\omega; T = 0) \frac{\cosh[\omega(\tau - \beta/2)]}{\sinh(\omega\beta/2)},$$  

(14)
i.e. taking the spectral density at zero temperature, one finds equivalently to (11) but in terms of the Euclidean correlators \[10\],

\[
\frac{2}{\omega} \int_0^\infty d\omega \delta \rho(\omega) = \int_0^\beta d\tau \left[ G(\tau) - G^{\text{rec}}(\tau) \right].
\]

(15)

IV. R-CURRENT CORRELATOR

A. Strong coupling

In \[4\] the R-current correlator (at vanishing three-momentum) in the \(N = 4\) SYM theory is derived in the limit of infinite 't Hooft coupling, \(\lambda \rightarrow \infty\). The corresponding thermal spectral function reads

\[
\rho^{\text{strong}}(\omega, T) = \frac{N_c^2 \omega^2}{32\pi^2} \frac{\sinh(\omega \beta/2)}{\cosh(\omega \beta/2) - \cos(\omega \beta/2)},
\]

(16)

and

\[
\rho^{\text{strong}}(\omega, T = 0) = \frac{N_c^2 \omega^2}{32\pi^2}.
\]

(17)

From these expressions the integral (11) is obtained to vanish numerically, i.e. the sum rule in the strong coupling limit becomes,

\[
2 \int_0^{+\infty} d\omega \frac{\delta \rho^{\text{strong}}(\omega)}{\omega} = 0.
\]

(18)

This may be seen as follows: \(\delta \rho^{\text{strong}}(\omega)\) is oscillating around \(\delta \rho^{\text{strong}}(\omega) = 0\) - as can be seen in Fig. 3b of \[4\]. For large \(\omega\) it reads

\[
\delta \rho^{\text{strong}}(\omega) \approx \frac{N_c^2 \omega^2}{16\pi^2} e^{-\omega \beta/2} \cos(\omega \beta/2),
\]

(19)

such that

\[
\int_0^\infty dx \ x e^{-x} \cos x = \frac{\Gamma(2)}{2} \cos(2 \arctan 1) = 0.
\]

(20)

To make sure that for the sum rule (18) \(\delta G_R(\omega = 0) - \delta G_R^{\infty} = 0\), required by (11), let us use

\[
G_R(\omega) = \Pi(\omega) = \Pi^L(\omega) = \Pi^T(\omega),
\]

(21)

for vanishing three-momentum and given in \[4\] by

\[
\Pi(\omega) = \frac{N_c^2 T^2}{8} \left\{ i \frac{\omega}{2\pi T} + \frac{\omega^2}{(2\pi T)^2} \left[ \psi \left( \frac{1 - i}{4\pi T} \right) + \psi \left( \frac{1 + i}{4\pi T} \right) \right] \right\},
\]

(22)

with \(\psi\) the logarithmic derivative of the gamma function. This retarded correlator has a quasinormal spectrum \[11\] with poles located at \[4\]

\[
\omega = 2\pi T(\pm n - in), \quad n = 1, 2, \ldots.
\]

(23)
From (21) follows
\[ \delta G_R(\omega = 0) = 0 . \] (24)

Using
\[ \psi(z) \rightarrow \ln z - \frac{1}{2z} - \frac{1}{12z^2} + O\left(\frac{1}{z^4}\right), \quad z \rightarrow \infty , \] (25)

it follows
\[ \Pi(\omega = i\alpha \rightarrow -\frac{N^2}{16\pi^2} \alpha^2 \ln \alpha , \] (26)
in the limit \( \alpha \rightarrow \infty \), such that
\[ \delta G_R^\infty = 0 . \] (27)

This shows the validity of (18).

B. Weak coupling: free theory

As in [3] the spectral density of the free theory, for \( \lambda = 0 \), denoted by \( \rho_{J J}^L(\omega) \) is used as a reference for comparison. In Appendix A of [3] the one loop contribution for the \( \mathcal{N} = 4 \) theory is derived. Using this result the integral,
\[ 2 \int_0^{+\infty} d\omega \, \frac{\delta \rho_{\text{weak}}(\omega)}{\omega} \equiv 2 \int_0^{+\infty} d\omega \, \frac{\delta \rho_{J J}^L(\omega)}{\omega} \]
\[ = 2 \int_0^{+\infty} d\omega \left\{ \frac{N^2 T^2}{12} \delta(\omega) + \frac{N^2}{48\pi^2} \omega \left[ n_B(\omega/2T) - 2n_F(\omega/2T) \right] \right\} \]
\[ = \frac{N^2 T^2}{6} , \] (28)
is non-vanishing due to the contributions of the fermion and scalar quasiparticles.

V. CONCLUSION

Indeed comparing (18) with (28) there is a significant difference between the strong and weak coupling result for the sum rule of the current-current correlator, maybe even more significant than the difference between the Euclidean correlators \( G_{J J}(\tau) \) for the interacting and free theory as shown in Fig. 3(b) in [3], which is at most \( \sim 20\% \).

It is worth to keep in mind that these differences are coming from the qualitative different analytic structures of the retarded Green’s function \( G_R(\omega) \) as a function of \( \omega \) in the strong and weak coupling limit, respectively [12]. For strong coupling \( G_R(\omega) \) (22) contains only poles, i.e. the infinite sequence of quasinormal modes (23) obtained from perturbations about the \( AdS_5 \) black brane space-time [11, 13]. In the perturbative approximation of \( G_R(\omega) \) quasiparticles and branch cuts are present [7, 14].

Are QCD lattice calculations [15, 16] able to show a similar significant difference between strong and weak coupling behaviour by considering the integrals (15)?
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