Rocking behaviour of freestanding objects

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Abstract. The safeguard of museum exhibits like vases and statues, laboratory equipments, storage tanks, or even tall buildings and other structural elements subjected to earthquake and in general to time-dependent forces has been in the last years a topic focus. The classical mechanical problem of the motions of rigid objects simply supported on a base plane, developed in the first year of sixties, thanks to the seminal paper by Housner on the inverted pendulum, can be now take advance due to the development of calculation tools. In some cases the quality of motion can be determined varying the material parameters involved. This is a strong tool to reduce vulnerability, especially when rocking motion should be avoided and sliding motion is welcome, as in the case of artistic heritage. This paper focuses the attention on this last problem, common to a large class of both non-structural and structural elements that can lose their functionality because of earthquake motions. The results of a numerical modelling of sliding and rocking motion in presence of both different excitations and mechanical parameters are presented and compared with experimental data performed by the authors. The results developed are in good agreement with the laboratory tests and this assures the reliability of both the analytical procedure and the determination of the parameters involved. A model for a safeguard proposal is provided.

1. Introduction

The seminal paper by Housner [1] is the base of the modern studies on the rocking response of a rigid block supported on a base undergoing horizontal motion. The study was devoted to the understanding of the behaviour of tall, slender structures subjected to ground motion [2, 3]. Only in the following decades the model has been used to represent the behaviour of particular rigid bodies, i.e. the art objects in the museums in seismic areas [4, 5]. The seismic safety of artifacts is a research field of great interest, being part of research and policy in the more general field of Cultural Heritage, largely funded in recent European programmes [6, 7]. The motion of rigid bodies under base excitation includes the behaviour of piece of equipments, hospital devices, statues, storage tanks, or even that of tall buildings [8]. Despite its familiarity and apparent simplicity, the problem in case of earthquake excitations poses extreme difficulties when exact solutions are sought [9]. Six basic conditions [10] have been distinguished: rest, slide, rock, slide-rock, free flight and impact. Shenton [11] has shown that the motion of a rigid object simply supported on an uniformly accelerating rigid

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plane depends not only on the object shape and the base acceleration, but also on the friction coefficient. The start of rocking motion requires increasing friction with ground acceleration. It has been shown [12] that the range of sliding-rock is larger in the case of harmonic excitation. A rigid structure placed on a shaking base may enter into rocking motion that occasionally results in overturning [13, 14], so that appropriate models are needed to assess the museum building [15, 16]. In the case of art objects, where the limitation of displacements and avoiding of overturning is the issue, the development of appropriate exhibition systems is needed [17]. In the first studies performed in Japan [18], the computer simulation showed that the horizontal velocity as well as the acceleration must be taken into account as criteria for overturning. The simplest of the mathematical models that has received notable attention in the past has been the planar rocking of rigid rectangular blocks under harmonic base motions [19] although the influence of the ground motion properties has been analyzed since the first years of 1980’s [20, 21]. In the harmonic problem, the non-linearity arises not only in the displacements but also in the dissipation of energy due to impacts. Recent works [22] have tried to unify the piecewise formulation by Housner, with the same hypothesis of large friction coefficient and using impulsive forces to model impact. The results are in agreement with the classical formalism, but are unsuitable when rocking motion is to be avoided and sliding motion is welcome, like in the case of artifacts [23]. Another parameter to take into account is the object geometry, so that influence of slenderness has been largely investigated [24, 25]. A limited number of studies take into account sliding motion, especially in the case of slender objects, since in large part of papers the friction coefficient between the block and the base is assumed sufficiently large to prevent sliding. In the paper by Kounadis [26] the combination of rocking and sliding for the simple block case is reported. In the slender block problem closed form solutions are derived. The problem becomes more complex when the behaviour of two stacked rigid bodies is examined: the highly nonlinear formulation needs some simplifying assumptions [27, 28]. In the case of two superimposed blocks examined by Spanos [29] a large friction coefficient is considered to prevent sliding.

In this paper the dynamic behaviour of two stacked rigid bodies is examined. The two rigid bodies can represent respectively the slender art object and a further flat rigid body inserted between the moving base and the rigid block. The results can be useful in the determination of the optimal friction coefficient to be created in the surface inserted between the art object and the moving floor. The analysis show that the insertion of a rigid surface produces a relevant decrease of oscillations amplitude and in some cases avoids the overturning. In case of constant aspect ratio the value of the friction coefficients (i.e. static and dynamic) can influence the entities of the displacements only, while the interposition of the flat element totally changes the shape of motion.

2. Rocking and sliding motion of a single block

The rocking motion of one rigid block is examined in this paragraph. First, the motion of a symmetric rigid block with aspect ratio \( B/H \), simply supported on a moving plane with acceleration \( \ddot{x}_{b}(t) \) as in Figure 1 is examined. The static friction coefficient \( \mu_{s} \) takes into account the amount of force needed to start the sliding motion, while the force necessary to keep the object sliding is proportional to the kinetic friction coefficient \( \mu_{k} \), with \( \mu_{s} > \mu_{k} \). The block can rotate alternatively around the two base corners \( O \) and \( O' \) with rotation angle \( \theta \), clockwise positive. Impact, when the angle of rotation reverses [30] is the only dissipative event.

According the classical inverted pendulum theory [1], the velocity field after a perfect plastic and centered impact is related to the pre-impact field by means of a reduction coefficient \( e \). In particular \( e = \sqrt{r} \) being \( r \) the restitution coefficient in the Housner sense. The coefficient \( e \) can be assumed constant during the motion, so that the angular velocity \( \dot{\theta}(t) \) before it (Figure 1):

\[
\dot{\theta}(t) = e \dot{\theta}^{*}(t).
\] (1)
In these hypotheses the conservation of angular momentum about point $O'$ just before the impact and right after the impact is:

$$I_o - 2m R b \sin \alpha \dot{\theta}^- (t) = I_o \dot{\theta}^+ (t).$$

where $I_o$ is the moment of inertia (defined with respect to $O$ or $O'$), $R$ is the distance of the center of mass $G$ from the corners $O$ or $O'$ and $\alpha$ is the angle between $R$ and the vertical edge of the block (see Figure 1). The combination of (1) and (2), gives, for a rectangular block:

$$e = 1 - \frac{3}{2} \sin^2 \alpha \quad \text{with} \quad 0 < e < 1.$$  

Energy dissipation is involved for $e < 1$. Rocking motion is present when the static friction with the base plane prevents sliding. Adopting the notation by Shenton [11], let $f_x$ and $f_y$ be the horizontal and vertical reactions at the tip $O'$ of the block, at all times rocking motion holds true if:

$$|f_s| \leq \mu_f f_y.$$  

In other words, starting from an equilibrium configuration of the system and given the condition (4), the angular momentum of inertia forces is greater than that due to gravity force. The rocking motion, according to the D’Alembert principle, is governed by the following set of differential algebraic equations (DAEs):

$$I_o \dot{\theta}^- (t) + mg R \sin(-\alpha - \theta^- (t)) = -m \ddot{x}_s (t) R \cos(-\alpha - \theta^- (t)) \quad , \quad \theta^- (t) < 0$$

$$I_o \dot{\theta}^+ (t) + mg R \sin(\alpha - \theta^- (t)) = -m \ddot{x}_s (t) R \cos(\alpha - \theta^- (t)) \quad \text{,} \quad \theta^- (t) > 0$$

$$\dot{\theta}^- (t) = r \dot{\theta}^+ (t) \quad \text{,} \quad \theta^- (t) = 0$$

where $\ddot{x}_s (t)$ is the horizontal base acceleration and $L_o = I_o$ is the polar inertia moment with respect to the two points $O$ and $O'$. The rocking motion starts when $|\ddot{x}_s (t)| > g b/h$, being $g$ the gravity acceleration. The first two ordinary nonlinear differential equations are relative to the rotation motion around $O$ and $O'$ and the third algebraic equation relates the two angular velocities in $O$ and $O'$ and holds true at the impact instant only. The angle $\alpha = \arctan b/h$ takes into account the slenderness of the block. The system (5) can assume the following form:

$$\frac{I_o}{mR} \dot{\theta}^- (t) + g \text{sgn}(\theta^- (t)) \sin(\alpha - \text{sgn}(\theta^- (t)) \theta^- (t)) = -\ddot{x}_s (t) \cos(\alpha - \text{sgn}(\theta^- (t)) \theta^- (t)) \quad \theta^- (t) \neq 0$$

$$\dot{\theta}^- (t) = r \dot{\theta}^+ (t) \quad \text{,} \quad \theta^- (t) = 0. \quad \text{(6)}$$
where sgn(\cdot) is the signum function. The numerical solution of the DAEs (6) may be put more conveniently in terms of a key point displacement, considering two reference systems with origin in the two rotation points \( O \) and \( O' \), namely \( \mathcal{R}_1 = \{O, x, y\} \) for \( \theta(t) > 0 \) and \( \mathcal{R}_2 = \{O', x', y'\} \) for \( \theta(t) < 0 \). Let \( \theta(t) \) be the rotation function, the position of the point \( P \) at time \( t \) in the two frame systems above described is related to the position vector at the starting time:

\[
\mathbf{r}^{(i)}_p = \begin{bmatrix} x^{(i)}_p \\ y^{(i)}_p \end{bmatrix}, \quad \theta(t) < 0 ; \quad \mathbf{r}^{(2)}_p = \begin{bmatrix} x^{(2)}_p \\ y^{(2)}_p \end{bmatrix}, \quad \theta(t) > 0
\]

(7)

so that the actual position of the point \( P \) is given by the rotation matrix \( R \circ \theta(t) \) applied on \( \mathbf{r}^{(i)}_p \) and \( \mathbf{r}^{(2)}_p \):

\[
\begin{align*}
OP(t) &= R \circ \theta(t) \mathbf{r}^{(i)}_p, \quad \theta(t) > 0 \\
O'P(t) &= R \circ \theta(t) \mathbf{r}^{(2)}_p, \quad \theta(t) < 0
\end{align*}
\]

(8)

where the rotation matrix \( R \in SO(2) \), being \( SO(2) \) the orthogonal group of matrices with unit determinant, is:

\[
R \circ (\cdot) = \begin{bmatrix} \cos(\cdot) & \sin(\cdot) \\ -\sin(\cdot) & \cos(\cdot) \end{bmatrix}.
\]

(9)

from (8) the acceleration is derived as:

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} OP(t) &= \frac{\partial^2}{\partial t^2} [R \circ \theta(t)] \mathbf{r}^{(i)}_p, \quad \theta(t) > 0 \\
\frac{\partial^2}{\partial t^2} O'P(t) &= \frac{\partial^2}{\partial t^2} [R \circ \theta(t)] \mathbf{r}^{(2)}_p, \quad \theta(t) < 0
\end{align*}
\]

(10)

after some manipulations the (10) can be rewritten as follows:

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} OP &= \left[ \hat{\theta}(t) \hat{R} \circ \theta(t) - \hat{\theta}^2(t) R \circ \theta(t) \right] \mathbf{r}^{(i)}_p, \quad \theta(t) > 0 \\
\frac{\partial^2}{\partial t^2} O'P &= \left[ \hat{\theta}(t) \hat{R} \circ \theta(t) - \hat{\theta}^2(t) R \circ \theta(t) \right] \mathbf{r}^{(2)}_p, \quad \theta(t) < 0
\end{align*}
\]

(11)

where the first derivative of the rotation matrix \( \dot{R} \) belongs to the orthogonal group of matrices with \( \det(R) = 1 \):

\[
\dot{R} \circ (\cdot) = \begin{bmatrix} -\sin(\cdot) & \cos(\cdot) \\ -\cos(\cdot) & -\sin(\cdot) \end{bmatrix} \in SO(2).
\]

The horizontal component of relative acceleration can be deduced by (11):

\[
\ddot{x}(t) = \begin{cases} 
\frac{\partial^2}{\partial t^2} OP \cdot i, & \theta(t) > 0 \\
\frac{\partial^2}{\partial t^2} O'P \cdot i, & \theta(t) < 0
\end{cases}
\]

with \( i \) unit vector of \( x \) axis. The horizontal acceleration \( \ddot{x}(t) \) can be put in the explicit form:
\[ \ddot{x}(t) = \begin{cases} -[x_1 \cos(\theta(t)) + y_1 \sin(\theta(t))] \dot{\theta}(t) + [-x_1 \sin(\theta(t)) + y_1 \cos(\theta(t))] \ddot{\theta}(t), & \theta(t) > 0 \\ -[x_2 \cos(\theta(t)) + y_2 \sin(\theta(t))] \dot{\theta}(t) + [-x_2 \sin(\theta(t)) + y_2 \cos(\theta(t))] \ddot{\theta}(t), & \theta(t) < 0. \end{cases} \]  

(12)

The absolute acceleration:

\[ \ddot{x}_g(t) = \ddot{x}_x(t) + \ddot{x}(t). \]  

(13)

is the sum of the base acceleration and the block one.

The configuration of the block in case of sliding motion can be characterized by the translation of a generic point of the block with respect to the base. The friction force is function of the vertical forces applied on the block and is opposite to the motion. Starting from an equilibrium configuration, sliding motion begins when the maximum horizontal force due to the static friction coefficient is attained:

\[ m|\ddot{x}_x(t)| > m \mu_s (\ddot{y}_g(t) + g). \]  

(14)

The differential equation governing the sliding motion problem is:

\[ m(\ddot{x}_g(t) + \ddot{x}(t)) = -\text{sgn}(\ddot{x}(t)) \mu_s m(\ddot{y}_g(t) + g). \]  

(15)

The numerical procedure has been developed in the general case of nonzero vertical acceleration of the base whose equations are deducted by (14) and (15) with \( \ddot{y}_g(t) = 0 \):

\[ \text{Figure 2. The single sliding block} \]

The differential equation of sliding (15) is integrated until the relative velocity \( \dot{x}(t) \) is nonzero from the instant in which the friction contact force is exceeded by the inertial forces related to (13).

\[ m|\ddot{x}_x(t)| > m g \mu_s, \]

\[ m(\ddot{x}_g(t) + \ddot{x}(t)) = -\text{sgn}(\ddot{x}(t)) m g \mu_s. \]  

(16)

When the velocity becomes null the block is in relative equilibrium with the base (rest) until the external force attains a value able to reanimate the sliding motion.

3. Motion of two superimposed blocks

The statues can present damage due to different loading sources [31]. Among them, rocking motion could cause overturning in a freestanding object subjected to seismic base excitation, so that in case of artifacts the simple sliding motion is a desirable situation. Nevertheless relative displacements should be evaluated to avoid collisions [32, 33]. The results of an analysis about the motion of two stacked rigid blocks of different aspect ratio are here presented. A top slender block placed on a flat one, this last freestanding on a moving base are analyzed. The geometry of the lower flat block induces the only sliding motion, while rocking is the only possible motion for the stacked slender block 2, due to
the high friction coefficient with the base. The differential equations governing the problem have been derived and included in the numerical procedure developed with Mathematica®. The geometrical characteristics of the problem are reported in Figure 3. The system has two degrees of freedom, namely the rotation \( \theta \) (clockwise positive) of block 2 and the centroid position \( x_{G_1} \) of block 1. Assuming \( m_1 \) and \( m_2 \) masses of the two blocks whose center of masses are \( G_1 \) and \( G_2 \), \( M = m_1 + m_2 \) is the total mass of the system and \((x_g(t), y_g(t))\) are the components of the base motion. Let \( r_{O,G_1} \) and \( r_{O,G_2} \) be position vectors of \( G_2 \) relative to \( O' \) and \( O \) in the initial. Their components in the two Cartesian reference systems \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) pictured in Figure 3 are given by:

\[
\begin{align*}
r_{O,G_2} &= \begin{bmatrix} R \sin \theta \\ R \cos \theta \end{bmatrix}, \\
r_{O,G_2} &= \begin{bmatrix} R \sin \theta \\ -R \cos \theta \end{bmatrix}.
\end{align*}
\]

Figure 3. Double blocks at rest position (left) and with upper block in rocking mode (right)

The hypothesis of sliding motion for block 1 lead to null vertical component of its relative motion with respect to the base. The position of the center of mass \( G_1 \) is:

\[
x_{G_1}(t) = \begin{bmatrix} x_g(t) + x_{G_1}(t) \\ y_g(t) \end{bmatrix}
\]

and that of \( G_2 \) is:

\[
\begin{align*}
x_{G_2}'(t) &= x_{G_2}(t) + R \circ \theta(t) r_{O,G_1}, & \theta(t) < 0 \\
x_{G_2}(t) &= x_{G_2}(t) + R \circ \theta(t) r_{O,G_2}, & \theta(t) > 0
\end{align*}
\]

so that the actual position \( P(t) \) of the keypoint \( P \) is:

\[
\begin{align*}
O'P(t) &= R \circ \theta(t) r_p^{(1)}, & \theta(t) < 0 \\
OP(t) &= R \circ \theta(t) r_p^{(2)}, & \theta(t) > 0
\end{align*}
\]

where \( R \circ \theta(t) \) is the rotation matrix (9) that takes into account the rocking motion of block 2 and the position vectors \( r_p^{(1)} \) and \( r_p^{(2)} \) are represented in Figure 5.

The total kinetic energy of the system is:
\[ T(t) = T_1(t) + T_2(t) \]

with \( T_1(t) \) and \( T_2(t) \) the kinetic energies of block 1 and 2:

\[
T_1(t) = \frac{1}{2} m_1 \mathbf{\dot{x}}_{G_1} \cdot \mathbf{\dot{x}}_{G_1}, \quad T_2(t) = \frac{1}{2} [J_{G_2} \mathbf{\dot{\theta}}^2(t) + m_2 \mathbf{\dot{x}}_{G_2}(t) \cdot \mathbf{\dot{x}}_{G_2}(t)] \quad \theta(t) < 0
\]

\[
T_1(t) = \frac{1}{2} m_1 \mathbf{\dot{x}}_{G_1} \cdot \mathbf{\dot{x}}_{G_1}, \quad T_2(t) = \frac{1}{2} [J_{G_2} \mathbf{\dot{\theta}}^2(t) + m_2 \mathbf{\dot{x}}_{G_2}(t) \cdot \mathbf{\dot{x}}_{G_2}(t)] \quad \theta(t) > 0
\]  

being \( J_{G_2} \) the centroid moment of inertia of block 2. The potential energy of the two blocks is given by:

\[
V(t) = V_1(t) + V_2(t)
\]

with \( V_1(t) \) and \( V_2(t) \) the potential energies of block 1 and 2:

\[
V_1(t) = m_1 g x_{G_1} \cdot \mathbf{j}, \quad V_2(t) = m_2 g x_{G_2}' \cdot \mathbf{j} \quad \theta(t) < 0
\]

\[
V_1(t) = m_1 g x_{G_1} \cdot \mathbf{j}, \quad V_2(t) = m_2 g x_{G_2} \cdot \mathbf{j} \quad \theta(t) > 0
\]  

being \( \mathbf{j} \) the unit vector of the y axis. The friction force at the base of block 1 during the sliding movement is given by:

\[
F_{x,\text{friction}} = -\mu g (g + \dot{y}_g(t)) \ddot{x}_g(t) \text{sgn}(\dot{x}_g(t))
\]

so that the Lagrangian formulation of the problem states:

\[
L(t) = T(t) - V(t)
\]

with the two Lagrangian parameters:

\[
q_1(t) = x_{G_1}(t) \quad q_2(t) = \dot{\theta}(t).
\]

The motion is governed by two differential equations derived by the Euler-Lagrange relation:

\[
\frac{\partial^2 L(t)}{\partial t \partial q_k} - \frac{\partial L(t)}{\partial q_k} = Q_k(t) \quad k = 1, 2
\]

and \( Q_k(t) \) is the generalized non conservative force dual to \( q_k(t) \). The system assumes two different expressions according to the sign of \( \dot{\theta}(t) \). In view of equations (18)-(22), the DAEs can be expressed as:

\[
\begin{cases}
J_{G_2} \ddot{\theta}(t) - m_2 R \cos[\alpha - |\theta|](\ddot{x}_g(t) + \ddot{x}_{G_2}(t)) + m_2 R g \text{sgn}(\dot{\theta}(t)) \sin[\alpha - |\theta|] = 0, \quad \theta(t) \neq 0 \\
M(\ddot{x}_g(t) + \ddot{x}_{G_2}(t)) + \text{sgn}(\dot{\theta}(t))[-m_2 R \sin[\alpha - |\theta|] \dot{\theta}^2(t) - \cos[\alpha - |\theta|] \dot{\theta}(t)] + M \mu_g \dot{\theta}(t) = 0, \quad \theta(t) \neq 0
\end{cases}
\]

The motion problem (23) is composed by two ordinary nonlinear differential equations and a single algebraic one, that involves the pre-and-post-impact angular velocity of the top block during the rocking motion. It is worth noting that uncoupling of the two differential equations is not possible. The
DAEs (23) govern the motions of the two blocks, while the rocking of the top block when the flat block is at rest with respect to the ground is governed by the following relation:

\[ J_g \ddot{\theta}(t) - m_g R \cos(\alpha - \theta) \dot{x}_g(t) + m_g R g \text{sgn}(\dot{\theta}(t)) \sin(\alpha - \theta) = 0. \]  

(24)
derived from (23) with the condition \( q(t) = x_g(t) = 0 \). Equation (23), analogous of (16) with the condition \( \theta(t) = 0 \), describes the sliding motion of the two blocks:

\[ M(\ddot{x}_g(t) + \dot{x}_g(t)) = -\text{sgn}(\dot{x}_g(t)) M \mu_k g. \]  

(25)
The numerical procedure implemented in Mathematica © takes into account the only sliding motion for the lower block (i.e. a rigid flat pedestal), so that its mass is the only necessary mechanical parameter. The top slender block (i.e. an anthropomorphic statue) can undergo rocking motion and the procedure involves the aspect ratio \( B/H \). Given the geometrical conditions, at the starting point the two blocks are at rest with respect to the moving base, until (23) or (24) is activated according to the mechanical parameters involved.

![Figure 4. Possible motions considered](image)

The analysis of the dynamic system begins with the evaluation of the type of motion related to the base acceleration. Three possible patterns of motion have been examined (Figure 7):

a) rocking of the top block with base block at rest with respect to the ground
b) sliding motion of the two blocks as one rigid body
c) combined motion patterns: rocking of the top block and sliding of the lower one.

The conditions for the activation of the motion a) are given by:

\[
\begin{cases}
M \ddot{x}_g(t) < \text{sgn}(\dot{x}_g(t)) M \mu_k g \\
\dot{x}_g(t) h > g b.
\end{cases}
\]  

(26)
The time \( t_a \) is the instant at which:

\[ \dot{x}_g(t_a) h = g b. \]  

(27)
It corresponds to the change of motion to range b), whose conditions for the activation are given by:

\[
\begin{cases}
M \ddot{x}_g(t) > \text{sgn}(\dot{x}_g(t)) M \mu_k g \\
\dot{x}_g(t) h < g b.
\end{cases}
\]  

(28)
The balance of friction force with inertia forces corresponds to the time \( t_b \) in which:

\[ M \ddot{x}_g(t_b) = \text{sgn}(\dot{x}_g(t_b)) M \mu_k g. \]  

(29)
The conditions for the activation of the motion c) are given by:

\[
\begin{align*}
M \ddot{x}_g(t) &> \text{sgn}(\dot{x}_g(t))M \mu_k g \\
\dot{x}_g(t) &> gb.
\end{align*}
\]  

(30)

In general, the motion a) lasts until the ground acceleration does not allow the overcoming of the static friction force at base interface. The transition of motion from a) to c) occurs when:

\[
M \ddot{x}_g(t) - m_2 R \text{sgn}(\dot{\theta}(t)) \sin(\alpha - |\dot{\theta}(t)|) \ddot{\theta}(t) - \cos(\alpha - |\dot{\theta}(t)|) \dot{\theta}(t) \geq \text{sgn}(\dot{\theta}(t))M \mu_l g
\]

(31)

Equation (28) corresponds to the (23) substituting the kinematic friction coefficient with the static one and \(x_g(0) = \dot{x}_g(0) = 0\). The inertial forces due to the ground acceleration \(\ddot{x}_g(t)\) and those due to the rocking of block 2 are involved in (29) with a term containing \(\ddot{\theta}(t)\) and another containing \(\dot{\theta}(t)\) respectively projections on the \(x\)-axis of the centripetal and tangential interactions through the contact point \(O\) or \(O'\). The sliding motion of block 1 stops when \(\dot{q}_i(t) = \dot{x}_g(t) = 0\). A frequent circumstance involves the activation of the rocking of block 2: the differential equation of rocking (24) is integrated until conditions for sliding (29) are attained, so that the system (23) is integrated until the velocity of block 1 with respect to the ground is null. From this point on the program loop can restart and the procedure is able to choose the right motion according the dynamic conditions involved. In the examined cases equation (24) is activated before than the static friction force between the flat block and the ground is exceeded. Only in this last case the flat block initiates sliding [34].

4. Examples

Some results of the analysis have been reported in the following pictures. Where possible the vertical scale is homogeneous to allow comparisons.

| Table 1. Motion parameters common to all the analyses |
|-----------------------------------------------------|
| Data | \(\mu_i\) | \(\mu_k\) | \(B\) [m] | \(m_i\) [kg] | \(r\) [Hz] | \(f\) [ms\(^{-2}\)] | \(\omega\) [s\(^{-1}\)] | \(\ddot{x}(t)\) |
|------|------|------|------|------|------|------|------|------|
|      | 0.40 | 0.30 | 1    | 1    | 0.8  | 1.2  | 6.75 | 7.53 |

| Table 2. Variable motion parameters for the analyses |
|-----------------------------------------------------|
| Data | \(m_2\) [kg] | \(H\) [m] | n. of cycles | Data | \(m_2\) [kg] | \(H\) [m] | n. of cycles |
|------|-------|------|-------------|------|-------|------|-------------|
| Fig. 8 | 1     | 2    | 30          | Fig. 10 | 2     | 4    | 24          |
| Fig. 8 | 1     | 3    | 30          | Fig. 10 | 2     | 4    | 24          |
| Fig. 9 | 1     | 4    | 30          | Fig. 11 | 4     | 4    | 24          |
| Fig. 9 | 1     | 5    | 30          | Fig. 11 | 6     | 4    | 24          |
In Figures 5-6 the influence of the variation of the motion of the upper block mass has been analyzed for a fixed aspect ratio of the block and compared with that of the single block without pedestal. This last one is the same in the four cases examined, being independent on the mass.

**Figure 5.** Behaviour of the slender block for $m_2 / m_1 = 2$ (top) and $m_2 / m_1 = 3$ (bottom)

**Figure 6.** Behaviour of the slender for $m_2 / m_1 = 4$ (top) and $m_2 / m_1 = 6$ (bottom)
In Fig. 5 the flat pedestal has the only effect to retard the overturning of the upper block, while in the cases of increasing mass (see Fig. 6) the overturning is avoided interposing the pedestal. In the two cases related to low slenderness, the interposition of the flat pedestal does not change significantly the response of the upper block.

Figure 7. Behaviour of the slender block for $H / B = 2$ (top) and $H / B = 3$ (bottom)

Figure 8. Behaviour of the slender block for $H / B = 4$ (top) and $H / B = 5$ (bottom)
A different behaviour can be observed in the case of very slender blocks (Fig. 11), where the overturning of the upper block is avoided in presence of the flat rigid block at the base. In both cases the oscillations of the upper block begin before the sliding of the flat one and last during all the observation time. On the contrary, the sliding motion of the flat block presents a time delay at the beginning and several rest intervals during the observation time.

5. Conclusions
In seismic areas the sliding motion can be a desirable situation in case of fragile objects for which rocking due to earthquake motions can be cause of damage. This paper examines the real situation of a marble statue placed on a flat rigid base freestanding on a moving floor. The harmonic excitation has been considered for the analysis of the problem. The motion patterns examined in this paper involve the only sliding motion for the lower flat block and rocking for the stacked slender block, together with sliding of the whole complex of rigid bodies. The geometry of the blocks and the range of friction coefficients adopted allow for the accounting of these motions. The system of differential equations governing the problem has been derived and included in a numerical procedure on purpose developed. It has been shown that the presence of a rigid surface delays and in some cases avoids the overturning of a slender rigid artifact, especially in the case of increasing slenderness and increasing mass of the upper block. Future developments and applications to real cases of statues and hospital devices are envisaged as results of this study.

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