Staggering behavior of the low lying excited states of even-even nuclei in a Sp(4,R) classification scheme

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Abstract

We implement a high order discrete derivative analysis of the low lying collective energies of even-even nuclei with respect to the total number of valence nucleon pairs $N$ in the framework of $F$- spin multiplets appearing in an algebraic $sp(4, R)$ classification scheme. We find that for the nuclei of any given $F$- multiplet the respective experimental energies exhibit a $\Delta N = 2$ staggering behavior and for the nuclei of two united neighboring $F$- multiplets well pronounced $\Delta N = 1$ staggering patterns are observed. Those effects have been reproduced successfully through a generalized $sp(4, R)$ model energy expression and explained in terms of the step-like changes in collective modes within the $F$- multiplets and the alternation of the $F$- spin projection in the united neighboring multiplets. On this basis we suggest that the observed $\Delta N = 2$ and $\Delta N = 1$ staggering effects carry detailed information about the respective systematic manifestation of both high order $\alpha$ particle-like quartetting of nucleons and proton (neutron) pairing interaction in nuclei.

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1 Introduction

The systematic behavior of the low-lying collective states of even-even nuclei has been extensively studied in different phenomenological and microscopic model approaches [1, 2, 3, 4]. In particular, some algebraic classification schemes provide a convenient way to systematize the general collective characteristics of these states in terms of the basic nuclear structure quantities, such as the nuclear mass number, the proton or neutron numbers, as well as the total number of neutron and proton pairs in nuclear valence shells [1, 5]. These quantities and their combinations can naturally be treated as the quantum numbers of the irreducible representations associated with the respective groups of symmetry. In this framework the low lying collective energy levels exhibit various kinds of smooth systematic dependence on the different quantum numbers, which can be evaluated empirically [1, 5].

From another point of view, however, the current studies of various fine effects in the structure of collective interactions and the respective energy spectra suppose the presence of quite a complicated behavior of nuclear collective characteristics. Thus, the application of the discrete approximation of a high order derivative of given nuclear characteristics as a function of particular physical quantity reveals various kinds of staggering effects which carry detailed information about the fine properties of nuclear interactions and the respective high order correlations in the collective dynamics of the system.

Typical examples are the $\Delta I = 1$ staggering effects (with $I$ being the angular momentum) in the collective $\gamma$- bands [7] and nuclear octupole bands [1], the $\Delta I = 1, 2$ and 4, staggering in superdeformed nuclear bands [10, 11, 12, 13] and the $\Delta I = 2$ staggering in the ground-state bands (gsb) of normally deformed nuclei [14].

Another example is the odd-even staggering in nuclear binding energies [15, 16, 17] or the behavior of the nuclear masses [13] as a function of the atomic number $A$, for which the influence of different pairing, $\alpha$—like quartetting and mean-field effects are considered to play a role.

In the above aspect, a natural question is whether the application of a high order discrete derivative analysis to the low lying collective states in wide range of nuclear chart would provide any staggering behavior of the respective energies as functions of the quantum numbers of an appropriate classification scheme. The answer of this question could open a new fine tool in the interpretation and the systematic study of nuclear collective interactions and
their symmetries.

The purpose of the present work is to examine the above problem with respect to the low lying collective states of even-even nuclei in the framework of an algebraic \( Sp(4, R) \) classification scheme \[19\]. The basic assumption of this scheme is that the structure of nuclear proton and neutron valence shells can be characterized in a unique way by the quantum numbers of the irreducible representations (irreps) of the algebra \( su(2) \) and the ladder representations of the algebra \( u(1, 1) \) which are subalgebras of the boson representation of \( Sp(4, R) \), considered as a classification group. As a result the even-even nuclei of each major (proton and neutron) shell can be classified into sets of \( F \)-spin multiplets \[3\], where any nucleus is uniquely identified by the total number of valence bosons \( N \) (pairs of protons and neutrons) and the third projection \( F_0 \) of the \( F \)-spin. It is known that in each \( F \)-spin multiplet the energies of the low lying yrast states exhibit a smooth and periodic behavior as functions of \( N \). On this basis, a unified theoretical description of gsb energies in even-even nuclei has been obtained \[3\].

In the present paper, we apply the high order discrete derivative analysis to the both experimental and theoretical \( 2^+ \) energies as functions of the quantum number \( N \) in the framework of the respective \( F \)-multiplets appearing in the \( Sp(4, R) \) scheme. The same analysis is also applied to the next higher \( 4^+ \) and \( 6^+ \) levels of the yrast bands of the classified nuclei and some typical results are presented. As it will be seen the experimental data exhibit considerable \( \Delta N = 2 \) staggering behavior in all considered \( F \)-multiplets (with \( N \) even or odd) and also very well pronounced \( \Delta N = 1 \) staggering effect in the cases when two neighboring \( F \)-multiplets are united into one single multiplet (with both even and odd \( N \)). In addition, we extend the study in some isotopic chains, where a tedious \( \Delta N = 1 \) staggering effect can be observed also but its general magnitude is quite small compared to the patterns observed in the \( F \)-spin multiplets.

We will see that these phenomena are reproduced theoretically by a generalized phenomenological energy expression derived in the framework of the classification scheme. On this basis we are able to provide an adequate physical interpretations of the observed staggering patterns in terms of the related collective model characteristics and the high order \( \alpha \)-like quartetting and pairing nucleon interactions as well.

In Sec. 2 we briefly outline the algebraic base of the \( Sp(4, R) \) classification scheme. The high order discrete derivative formalism and its meaning in the context of the present theory is given in Sec. 3. Experimental and theoretical
staggering patterns with $\Delta N = 2$ in the $F$ - multiplets and with $\Delta N = 1$ in the combined (neighboring) $F$ - multiplets and isotopic chains are presented in Sec. 4. A detailed analysis of their form and amplitudes in the different multiplets is also given. In Sec.5 the obtained results are analyzed in terms of the model characteristics of the classification scheme. Some comments and conclusions are given in Sec. 6.

2 Low-lying collective states in the $Sp(4, R)$ classification scheme

The nuclear $Sp(4, R)$ classification scheme is based on the use of two kinds of bosons, proton ($\pi$) and neutron ($\nu$) bosons, interpreted as pairs of the respective nucleons, confined in the valence orbits. The boson representation of the algebra $sp(4, R)$ is generated through the respective creation and annihilation operators as follows [5]:

$$\pi^\dagger \pi^\dagger, \nu^\dagger \nu^\dagger, \pi^\dagger \nu^\dagger, \pi \nu, \nu \nu, \pi \pi, N_\pi = \pi^\dagger \pi, N_\nu = \nu^\dagger \nu, F_+ = \pi^\dagger \nu, F_- = \nu^\dagger \pi.$$  \hfill (1)

The following subset of the above operators (1) :

$$F_+, \quad F_-, \quad F_0 = \frac{1}{2}(N_\pi - N_\nu)$$  \hfill (2)

generates a $su_F(2)$ algebra so that a $F-$spin quantum number is assigned to the considered boson system with a mathematical structure similar to that of the isospin.

The operator $N = N_\pi + N_\nu$ commutes with all the operators (2) acting in this way as a first order invariant of an $u(2) \supset su_F(2) \otimes u_N(1)$ [6]. The space $H$, in which the boson representation of $sp(4, R)$ acts is reducible and the operator $(-1)^N$ splits it into two irreducible spaces $H_+$ and $H_-$, corresponding to $N$ even or odd respectively. Acting on these spaces the operator of the total number of particles $N$ reduces them into a direct sum of a totally symmetric unitary irreps of $su_F(2)$, labeled by $N = 0, 2, 4, ...$ for $H_+$ and $N = 1, 2, 3, ...$ for $H_-$. The operator $F_0$ (2) does not differ essentially from the first order Casimir operator of the noncompact $u(1, 1) \subset sp(4, R)$. Hence it reduces the spaces $H_+$ and $H_-$ to the ladder series defined by its fixed
eigenvalues. The states from a given ladder, are defined by the eigenvalue of $F_0$ and are obtained by the subsequent action of the operator $\pi^\dagger \nu^\dagger$ on the lowest weight state. The operator $F_0$ reduces each $u(2)$—representation (with fixed value of $N$) to the representations of $u_\pi(1) \oplus u_\nu(1)$ labeled by $N_\pi$ and $N_\nu$ respectively. The same is obtained by reducing the $u(1,1)$ ladders with the operator $N$.

In terms of the well known and rather successful language of IBM-2, the algebraic operators (1) obtain the physical meaning of basic nuclear characteristics inherent for the $Sp(4, R)$ classification scheme. Thus the eigenvalues of the operators $N_\pi = \frac{1}{2}(N_p - N_p^{(1)})$ and $N_\nu = \frac{1}{2}(N_n - N_n^{(1)})$ are naturally interpreted as the numbers of valence proton and neutron pairs for a nucleus of a given shell. Here $N_p^{(1)}$ and $N_n^{(1)}$ are the proton and neutron numbers respectively of the double magic nucleus at the beginning of the shell, considered as the “vacuum state” for the symplectic representations. A basic point in the present classification scheme is that the major shells, which are defined by their bordering magic numbers $(N_p^{(1)}, N_n^{(1)} | N_p^{(2)}, N_n^{(2)})$ (with $N_p^{(2)} > N_p^{(1)}$ and $N_n^{(2)} > N_n^{(1)}$) are mapped on two $Sp(4, R)$- multiplets $(N_p^{(1)}, N_n^{(1)} | N_p^{(2)}, N_n^{(2)}) \pm$, with $N$ being even (+) or odd (−). In these terms the total number of valence bosons, $N$, and the third projection of the $F$-spin, $F_0$, are exactly the operators that reduce the $sp(4, R)$ spaces to a definite vector in one of the $u_\pi(1) \oplus u_\nu(1)$ subspaces, corresponding to a given nucleus with fixed $N$ and $F_0$. In this way the nuclei of given major shell are naturally classified into $F_0$ - multiplets (with fixed $F_0$ and $N$ taking all its possible values), unified in each of the symplectic spaces $H_+$ and $H_-$. The nuclei belonging to each of these multiplets differ by $\Delta N = 2$, an $\alpha$-like cluster (two protons and two neutrons) created by the operator $\pi^\dagger \nu^\dagger$. It is important to remark that another kind of $\Delta N = 2$ nuclear sequences can be defined in the symplectic spaces $H_+$ and $H_-$ by the action of the operators $\pi^\dagger \pi^\dagger$ (or $\nu^\dagger \nu^\dagger$) on the minimal weight states of each $F_0$ - multiplet. These sequences correspond to the even-even nuclei from a given isotonic (or isotopic chain), which differ by four protons (or neutrons) with $F_0$ differing by $\Delta F_0 = +1$ (or $-1$).

Furthermore, if the anticommutation relations of the boson creation and annihilation operators $\pi^\dagger, \pi$ and $\nu^\dagger, \nu$ are formally introduced, one can easily see that $sp(4, R)$ is the even part of a superalgebra, which has the single boson operators as odd generators. These operators practically relate the even and odd symplectic multiplets. On this basis any two neighboring
multiplets from $H_+$ and $H_-$, can be united into one single odd–even $F_0$-multiplet in which the neighboring nuclei are determined by the subsequent alternative action of the operators $\pi^\dagger (\Delta F_0 = +\frac{1}{2})$ and $\nu^\dagger (\Delta F_0 = -\frac{1}{2})$ and thus differ alternatively by $\Delta F_0 = \pm \frac{1}{2}$. In addition the nuclei belonging to such odd–even $F_0$-multiplets differ by $\Delta N = 1$. In the same way the isotonic and isotopic nuclear chains of even even nuclei with $\Delta N = 1$ can be obtained by the respective consecutive action of $\pi^\dagger (\Delta F_0 = +\frac{1}{2})$ and $\nu^\dagger (\Delta F_0 = -\frac{1}{2})$ without a change in the sign of $\Delta F_0$.

The above formalism has been applied for classifying the even-even nuclei into symplectic spaces mapped on the respective major nuclear shells [19]. Moreover, it provides a natural systematic of the yrast energy bands observed in these nuclei. It has been shown that the low-lying yrast energies exhibit a smooth and periodic behavior in each $F_0$-multiplet, which allows their unified description in terms of the nuclear collective characteristics inherent for the $Sp(4,R)$ classification scheme. On this basis the following generalized energy expression has been proposed [5]:

$$E_{yrast}(h_k, I, \omega) = \alpha \{h_k\} I(I + \omega).$$

(3)

The second term of Eq. (3) represents the generalized collective interaction characterized by the geometrical parameter $\omega$. The latter has the physical meaning of a measure for the interplay between the vibrational ($\omega > 20$) and the rotational ($\omega = 1$) collective modes (3) and reflects the respective changes in nuclear shape. Since in heavy nuclei the shape is changed from almost spherical at the beginning of given shell to axially deformed in the midshell region and back towards spherical at its closure, the parameter $\omega$ changes respectively from $\omega > 20$ to $\omega = 1$ and then again to $\omega > 1$. The values of $\omega$ have been determined [5] for each of the classified nuclei by using the experimental yrast energy ratios $\frac{E_{I+2}}{E_I}$, which are known to provide a reliable criteria [24] of nuclear collectivity. So, the assigned integer values of $\omega$ for each of the classified nuclei (fixed values of $N$ and $F_0$) determine its type of collectivity.

The first term in Eq. (3), $\alpha \{h_k\}$, is a dynamical (inertial) coefficient determined as a function of the six quantum numbers $h_k, k = 1, 2, ..., 6$, which are specific for each nucleus in a given shell. Generally, the variables $h_k$ depend on the number of protons $N_p$ and neutrons $N_n$ and the four magic numbers $N_p^1, N_n^1, N_p^2, N_n^2$ and can be expressed in terms of the classification quantum numbers $N$ and $F_0$. Thus for a fixed nuclear shell the inertial
parameter is given by the expression:

\[ \alpha \{ h_k \} = A_1 + A_2 N + A_3 F_0 + A_4 N^2 + A_5 F_0^2 + A_6 N F_0, \]  

(4)

where \( A_i, i = 1, 2, \ldots, 6 \) are phenomenological parameters determined overall for all nuclear shells [3].

In general, the geometrical parameter \( \omega \) has its specific values for each of the classified nuclei. The inertial parameter \( \alpha \{ h_k \} \) is a function only of \( N \) in each \( F_0 \)-multiplets, and of the two classification numbers in the combined odd-even and isotopic multiplets.

3 High order discrete derivative analysis of collective energies in the \( Sp(4, R) \) scheme

As it became clear from Sec. 2, the even-even nuclei of given \( F_0 \)-multiplet appearing in the considered \( Sp(4, R) \) classification scheme are uniquely determined by the subsequent even (or odd) values of the valence boson number \( N \). This allows us to apply the high order discrete derivative analysis to the low lying gsb energies \( E(N) \) for a given angular momentum \( I = 2, 4, 6 \) in any considered \( F_0 \)-multiplet, as a function of the quantum number \( N \), by analogy to the staggering analysis of rotational band energies applied in terms of the angular momentum \( I \) [8, 9, 25].

Since for any given \( F_0 \)-multiplet the values of the quantum number \( N \) of the neighboring members differ by \( \Delta N = 2 (\alpha-\text{particle}) \), the following quantity can be introduced:

\[
Stg(2N) = \frac{1}{16}(6\Delta_2 E(N) - 4\Delta_2 E(N - 2) - 4\Delta_2 E(N + 2) + \Delta_2 E(N + 4) + \Delta_2 E(N - 4)),
\]

(5)

where

\[
\Delta_2 E(N) = E(N + 2) - E(N).
\]

(6)

The point function \( Stg(2N) \) is proportional to the discrete approximation of the fourth derivative of the function \( \Delta_2 E(N) \), and obviously, to the fifth derivative of the energy \( E(N) \):

\[
Stg(2N) = \frac{1}{32}(10E(N + 2) + 5E(N - 2) + E(N + 6)
- [10E(N) + 5E(N + 4) + E(N - 4))].
\]

(7)
As such, it naturally obtains zero values for any polynomial form of $E(N)$ of power less or equal to four. Therefore, any non-zero values of the quantity $Stg(2N)$ will imply a higher order functional dependence. Moreover, any staggering (zigzagging) behavior of the $Stg(2N)$-values will suggest the presence of quite complicated non-polynomial dependence of the lowest collective energy and the respective nuclear interactions on the quantum number $N$.

On the above basis it is expected that the application of the quantity $Stg(2N)$ to the low-lying collective nuclear states within the $F_0$-multiplets of $Sp(4, R)$ could provide a detailed information about the possible influence of $\alpha$-particle-like high order quartetting interaction $(\pi^\dagger \nu^\dagger \pi \nu)$ of nucleons on the systematic behavior of collective excitation energies.

We should remark, that the application of the function $Stg(2N)$ for the energies of the nuclei in the isotopic and isotonic chains is restricted, since even in the larger shells there is not enough number of points (observed excitation energies). Actually this is due to the circumstance that the available data are separated in the two spaces $H_+$ and $H_-$. The above restriction is naturally released in the extended regions of exotic nuclei (such as with $N_n = N_p$ and close to the drip lines), where the number of newly obtained experimental data grows continuously. Now, we can extend the framework of the fine systematic analysis of low-lying collective energies by considering the unified odd-even $F_0$-multiplets. Within these multiplets the neighboring members differ by $\Delta N = 1$ (a proton or neutron pair) in a way that the increasing $N$ corresponds to a subsequent alternative adding of proton and neutron pairs to the valence shells. In this case the systematic behavior of the excitation energies can be characterized analogously to Eq. (5) by a discrete approximation of the fourth derivative of the energy difference for $\Delta N = 1$:

$$Stg(1N) = \frac{1}{16}(6\Delta_1E(N) - 4\Delta_1E(N - 1) - 4\Delta_1E(N + 1) + \Delta E_1(N + 2) + \Delta E_1(N - 2)),$$  \hspace{1cm} (8)

where

$$\Delta_1E(N) = E(N + 1) - E(N).$$  \hspace{1cm} (9)

The use of the same order in the discrete derivative approximations, (3) and (8), for the $F_0$-multiplets with $\Delta N = 2$ and the combined odd-even and isotopic multiplets with $\Delta N = 1$ respectively, allows one to treat the $\alpha$-particle quartetting and the different kinds of nucleon pairing correlations in
nuclei on the same footing, as well as to compare quantitatively their fine systematic influence on nuclear collectivity. Also, in this way the present classification scheme allows the standard analysis of the $\Delta N = 1$ staggering effect in isotopic chains of even even nuclei. In the isotonic chains this analysis is limited due to the Coulomb restriction on the possible proton numbers, but nevertheless it seems to be perspective in view of the current progress in the experimental investigations of proton rich nuclei.

4 $\Delta N = 2$ and $\Delta N = 1$ staggering patterns for the low lying collective energies. Experiment and theory

We have applied the functions $Stg(2N)$ (5) and $Stg(1N)$ (8) for the analysis of the experimental data [24, 27, 28] on the lowest collective energies of even-even nuclei classified in the respective symplectic multiplets. In all the considered cases these functions are oscillating in respect to the zero of their scale, with a changing amplitude, i.e. exhibit staggering patterns, which will be analyzed bellow.

The function (5) has been used in all possible cases, in which the $F_0$-multiplets of the considered $Sp(4,R)$ classification scheme [5], contain more than 8 nuclei. For all of them we observe $\Delta N = 2$ staggering behavior of the $2^+\!$ energy. Some typical staggering patterns including either odd or even $F_0$-multiplets are shown on Fig. 1.

We see, that for the different $F_0$-multiplets the general scale of the staggering amplitude depends essentially on the particular valence shells. Thus, for the two multiplets $F_0 = 3/2$ (Fig. 1(a)) and $F_0 = 0, 1, 2$ (Fig. 1(c)), belonging to the essentially different (light and heavy) valence shell configurations $(28,50|50,82)_-$ and $(50,82|82,126)_+$ respectively, the staggering amplitude generally differs by more than one order in magnitude. While for the lighter shells $(28,50|50,82)_-$ the function $Stg(2N)$ oscillates between $-0.8$ and $0.6$ MeV (Fig. 1(a)), for the heavier (rare earth) shells it drops in the interval $[-0.06,0.04]$ MeV (Fig. 1(c)). Some intermediate staggering magnitude in the interval $[-0.03,0.05]$ MeV is observed for the multiplet $F_0 = -4$ (Fig. 1(b)), which belongs to the ”intermediate” valence space $(50,50|82,82)_+$.

As a result, we find that the general $\Delta N = 2$ staggering scale tends to
be large for the light nuclear valence shells and essentially smaller, by more than one order in magnitude, for the heavy nuclear regions. However, we should remark that the more detailed comparison is restricted due to the fact that the "length" of the $F_0^-$ multiplets increases towards the heavier valence shells and, as it will be pointed out below, more complicated structure of the staggering patterns appears there.

So, in the region of heavy nuclei we find that for any particular $F_0^-$ multiplet the $\Delta N = 2$ staggering function strongly varies in dependence on the quantum number $N$. On Fig. 1(c) we show three well developed (long enough) patterns obtained for the multiplets $F_0 = 0, 1, 2$ of the rare earth valence shells $(50, 82|82, 126)_+$. We find that in the beginning of any multiplet ($N = 4 - 12$) as well as near its end ($N > 20$) a considerable staggering amplitudes of about 0.05 MeV are observed, while in the middle of the multiplet ($N = 15 - 20$), which corresponds to the effective middle of the valence shells, the staggering effect is essentially suppressed to practically zero amplitude. Also, we remark that in some cases (see the multiplet $F_0 = 2$
on Fig. 1(c) the transitions between the beginning, middle and the end of the multiplet are characterized by irregularities in the zigzagging curve with subsequent changes in the phase of the staggering pattern. This is a specific indication for some kind of phase transition in nuclear collectivity within the particular multiplet.

Further, in Fig. 1(d) we illustrate the $\Delta N = 2$ staggering patterns obtained for the experimental levels with angular momentum $I = 2, 4, 6$ within the long $F_0 = 1$ multiplet. The different curves are characterized with the same phases of the oscillation, while the amplitude increases with the increasing of the angular momentum, conserving the same dependence on the quantum number $N$.

Now, let us consider the fine structure of the odd–even $F_0$–multiplets obtained by uniting two neighboring $F_0$-multiplets from the spaces $H_+$ and $H_-$. The respective $2^+_1$- energy sequences are obtained by switching between the subsequent members of the two considered multiplets. The unified odd–even $F_0$–multiplets contain much larger number of nuclei than the single multiplets, which allows one to extend the analysis by applying the discrete function (8) to all the nuclear shells, including the first $(28, 28|50, 50)$ and the last (open) $(82, 126|126, \ldots)$ major shells. In all of them we found that the experimental energies exhibit rather well pronounced $\Delta N = 1$ staggering patterns. Some typical examples illustrating different combinations of odd–even multiplets from all the shells are shown in Fig. 2.

First of all, we see that the general scale of the $\Delta N = 1$ staggering is larger by almost one order of magnitude than the scale in the $\Delta N = 2$ staggering patterns of the respective separate $F_0$-multiplets. (For example, compare Fig. 2 (c) and Fig. 1(c).)

Further, we find that the trend of the decrease of the staggering amplitude with the increase of the shell dimension is observed with no exceptions in this case, like in the $\Delta N = 2$ case. This can be seen, for the odd-even multiplets with $F_0 = \{-5/2, -2\}$ from the shell $(28, 28|50, 50)$ on Fig. 2(a) and from the shell $(82, 126|126, \ldots)$ on Fig. 2(d), where the maximal staggering amplitude drops from $0.2 MeV$ in the first one, to $0.003 MeV$ in the last one. This decrease is gradual through the consecutive shells (Fig. 2(a) -(d)).

As in the $\Delta N = 2$ case, we observe that in the beginning and the end of any particular unified odd–even multiplets the staggering amplitude is larger, than in the middle of it. It is however important to remark that in the middle of the unified $F_0$–multiplets in the "lighter" shells $(28, 28|50, 50)_\pm$, $(28, 50|50, 82)_\pm$ and $(50, 50|82, 82)_\pm$ the decrease in the staggering magni-
Figure 2: The experimental values of $Stg(1N)$, (8), for the odd-even $F_0$ -
multiplets: (a) the odd-even multiplet $F_0 = \{-5/2, -2\}$ from (28, 28|50, 50); (b) the odd-even multiplet $F_0 = \{1, 3/2\}$ from (28, 50|50, 82); (c) the odd-
even multiplet $F_0 = \{1, 3/2\}$ from (50, 82|82, 126); (d) the odd-even multiplet $F_0 = \{-5/2, -2\}$ from (82, 126|126,...); (e) the odd-even multiplet $F_0 = \{-4, -7/2\}$ from (50, 50|82, 82) for $I = 2, 4, 6$.
tude is not very well expressed, as could be seen for the multiplets \( F_0 = \{-5/2; -2\} \) on Fig. 2(a), \( F_0 = \{1,3/2\} \) on Fig. 2(b) and \( F_0 = \{-4; -7/2\} \) on Fig. 2(e) respectively. In the middle of all the combined \( F_0 \)-multiplets in the rare-earth \((50,82|82,126)\) region the staggering amplitude is almost vanishing, as illustrated for the \( F_0 = \{1,3/2\} \) multiplet on Fig. 2(c).

The application of (8) to the energies of the next exited states \( I = 4,6 \) in the combined odd-even multiplets, shows as in the \( \Delta N = 2 \) staggering, that all the typical features of the energy behavior of the \( 2^+ \)-states, outlined above are even enhanced with the increase of the angular momentum. An example for one of the longest combined multiplets \( F_0 = \{-4; -7/2\} \), from the shell \((50,50|82,82)\) is given on Fig. 2(e).

The nuclear isotonic chains obtained in the scheme are too short, so the \( Stg(1N) \) function is investigated only in the isotopic chains of even-even nuclei, obtained by the consecutive action of the boson operators \( \nu^+ \) (pairs of neutrons). It does not exhibit a pronounced staggering effect. An exception is presented on Fig. 3, for the Os isotopes, but the maximal amplitude in this case is two orders smaller \((0.006 MeV)\) than the odd-even \( \Delta N = 1 \) staggering in the same shell (Fig. 2(c)).

Figure 3: The experimental values of \( Stg(1N) \), (3) as function of the neutron numbers \( N_n \), obtained for the Os isotopic multiplet from \((50,82|82,126)\).

Now, we turn to the analysis of above phenomena in terms of the collective model characteristics inherent for the applied \( Sp(4,R) \) classification of the theoretical energies (3) are obtained as functions of coefficients \( \alpha \{ h_k \} \) and through them on the classification numbers \( N, F_0 \) (4) and the relevant value of \( \omega \) for each nucleus. Thus, the theoretical energies \( E_{\text{yrast}}(h_k, I, \omega) \) successfully reproduce the respective experimental data (3) allowing us to apply our fine systematic analysis by situating them into the considered multiplets of the classification.
Figure 4: The experimental and theoretical values of the functions: (a) \( Stg(2N) \), for the multiplet \( F_0 = 0 \) from \((50, 82|82, 126)_+ \); (b) \( Stg(1N) \), for the odd-even multiplet \( F_0 = \{0, 1/2\} \) from \((50, 82|82, 126) \); (c) same as (b) for the odd-even multiplet \( F_0 = \{1, 1/2\} \) from \((50, 82|82, 126) \) for \( I = 2 \); (d) same as (c) for \( I = 4 \); (e) same as (c) for \( I = 6 \).
Hence, we apply the staggering filters (3) and (8) on the theoretically obtained by means of (3) energies of the low lying collective states of even–even nuclei in all the multiplets, in which experimental $\Delta N = 2$ and $\Delta N = 1$ staggering was observed. We found that the behavior of the (3) and (8) is reproduced rather well by the theoretical energies. This result is illustrated for $2^+_1$ energies from the multiplet $F_0 = 0$ of the shell $(50, 82|82, 126)_+$ and for the odd-even $F_0 = \{0, 1/2\}$ multiplet of the same shell on Fig. 4(a) and Fig. 4(b), respectively. We see that in both cases with $\Delta N = 2$ and $\Delta N = 1$ the specific systematic dependence of the staggering magnitude on the quantum number $N$ is reproduced. Moreover in the ends of the multiplets the larger amplitudes are reproduced together with the correct signs. The theoretical amplitude is a bit smaller than the experimental one and its irregularity is more clear in the region of the well deformed nuclei. As illustrated on Fig. 4(c), 4(d) and 4(e) the staggering behaviour of the energies of the states with $I = 2, 4, 6$ respectively in the combined multiplet $F_0 = \{1/2, 1\}$ from the shell $(50, 82|82, 126)$ is reproduced by the generalized formula (3), including the sign and typical features of the amplitude for the higher states. This means, that all the presented in this work analysis for the first exited $2^+_1$ states is appropriate for all the higher yrast states, which were included in the fitting procedure, when (3) was deduced in [5]. The slight discrepancies in the theoretical reproduction of the staggering amplitudes are due to the accuracy of the phenomenological description and the restriction to integer values of $\omega$.

5 Model analysis

A relevant model interpretation of the presented investigation, can be obtained on the basis of the geometrical factor $\omega$, non explicitly depending on $N$ and $F_0$, defining the nuclei in the multiplet. Its phenomenological values vary in the limits $\omega = 1 \div 30$ and specify the character of the collective motion. In decreasing order these values consequently characterize the collective excitations of nearly spherical, vibrational nuclei (large values of $\omega \approx 15 - 30$; $U(5)$—limit of IBM1); the transitions region (with fast decrease of $\omega$) including the $\gamma$—soft nuclei with $\omega \approx 5$; (O(6) - limit) and the region of well deformed, rotational nuclei ($\omega = 1$; $SU(3)$). In the lighter shells (Fig. 1(a), 1(b), 2(a), 2(e)) we have a transition only from spherical to $\gamma$—soft
nuclei and back to almost the same value of \( \omega \), which corresponds to the transition from the U(5) to the O(6) limit of the IBM1 \[29\] (only one side of the Casten’s triangle \[30\]). The typical model distribution of \( 2^+ \) energy levels between the different \( \omega \)-modes of collective motion is illustrated on Fig. 5 for the long \( F_0 = 0 \) multiplet of the shell \((50, 82|82, 126)\)\+. There the thin curves represent various \( \omega \)-fixed modes of the \( E_{\text{yrast}}(h_k, 2, \omega) \) as functions of the valence pairs number \( N \), while the thick curve connects its phenomenologically determined values at a specific \( \omega \). We see that in the beginning of the multiplet \((N = 4)\) the theoretical \( 2^+ \) energies start from the high \( \omega = 17 \) ”vibrational” curve, but with the increase of \( N \) they rapidly dump down (through \( \omega = 8, 6 \)) to the ”rotational iso-line” with \( \omega = 1 \) which ”receives” all levels in the middle region \( N = 10 - 22 \). Further, in the end of the multiplet (shell closure region \( N = 24 - 30 \)) several jumps up to the transitional \((\omega = 3, 6, 8)\) collective modes are observed.

Namely, the above rather large discrete jumps of energy as a function of \( N \) between the different \( \omega \)-lines cause the pronounced \( \Delta N = 2 \) staggering effect in the beginning and the end of the theoretically ”filled” \( F_0 \)-multiplet. The small (but non zero) staggering amplitudes appearing in the middle region can be considered as the effect of the five-point discrete derivative, \((7)\), which has a ”memory” of the preceding states and also ”propagates” the information about ”what happens” in the ends.

At the same time, the change in the geometric factor \( \omega \), through its dependence on \( N \), reflects phenomenologically the respective change in the intrinsic
nuclear structure and its influence on the character of collective motion. On this basis we can directly interpret the experimental $\Delta N = 2$ staggering patterns as the result of rapid change in nuclear collective properties with respect to different intrinsically determined modes, related in this case to the accumulation of $\alpha-$like clusters.

Now, we can directly extend the above considerations with respect to the $\Delta N = 1$ staggering patterns corresponding to changes in the intrinsic nuclear structure with alternating kinds of nucleon pairs. In the combined symplectic multiplets the analysis concerning the geometric factor $\omega$ and its contribution to the fine systematical behavior of $2^+_1$ energies is still valid for their members, but their more gradual change must be taken into account. It provides an analogous explanation of the large $\Delta N = 1$ staggering amplitudes in the ends of given odd-even multiplet. However, in this case there is another factor that appears to be decisive especially for the essentially larger staggering scale compared to the $\Delta N = 2$ case. In the single $F_0$ multiplet the difference between two neighboring energies is entirely determined by the quantum number $N$ and the geometric factor $\omega$ (all other model quantities and parameters are fixed). In the combined multiplets there is additional dependence on the quantum number $F_0$ (4). The alternative change of the $F_0$-values with $+\frac{1}{2}$ then $-\frac{1}{2}$ in the two neighboring $F_0$- multiplets provides respective change in the inertial factors $\alpha\{h_k\}$ (4) and thus a’priori produces a mutual energy oscillation in the respective energy sequences. Hence, even in the energy difference (4) some staggering is obtained, due to its dependence on the oscillating values of $F_0$ as seen on Fig. 6.

The latter results in magnifying the $\Delta N = 1$ staggering background. Thus, our theoretical analysis suggest that the form and the sensitivity of
\( \Delta N = 1 \) staggering effect to the nuclear shell structure are determined by discrete changes of the quantum number \( \omega \) in respect to \( N \), but its general magnitude is also determined by the fluctuations of the \( F_0 \)-values. The magnitude and the form of the staggering pattern, together with the respective set of phenomenological \( \omega \)-values provide both relevant qualitative and quantitative characteristics of nuclear collectivity in the framework of the \( Sp(4, R) \) classification scheme.

6 Conclusions

The results presented so far allow us to outline the following systematic behavior of the experimentally observed low lying collective energy levels in the framework of the considered \( Sp(4, R) \) model scheme. The energies of the levels classified in any single \( F_0 \)-multiplet of \( Sp(4, R) \) exhibit \( \Delta N = 2 \) staggering behavior, while the levels of more general odd-even -multiplets provide respective \( \Delta N = 1 \) staggering patterns of their fifth order discrete derivative. In both cases the general staggering magnitude is relatively large for the light nuclear valence shells and considerably smaller for the valence shells of heavy nuclei. In addition, both \( \Delta N = 2 \) and \( \Delta N = 1 \) staggering patterns of heavy nuclei are characterized by relatively large amplitudes in the ends and their suppression in the middle region of the respective single and combined odd-even \( F_0 \)-multiplets. On the other hand, we find that the \( \Delta N = 1 \) staggering effect is by an order of magnitude stronger and more regular in form than the \( \Delta N = 2 \) staggering. For the odd-even \( F_0 \)-multiplets the staggering is inherent even for the energy differences of the neighboring states \([9]\), with \( F_0 \) integer and half integer. The weakest staggering effect is observed in the isotopic nuclear chains, where there is no oscillation of the \( \Delta F_0 \) values.

The application of the \( Sp(4, R) \) classification scheme provides relevant physical interpretation of the experimentally observed \( \Delta N = 2 \) and \( \Delta N = 1 \) staggering effects in terms of the nucleon correlations in nuclear valence shells. The \( \Delta N = 2 \) effect indicates very fine systematic behavior of nuclear collectivity associated with the simultaneous adding of proton and neutron pairs (\( \alpha \)-like clusters) to valence shells, relevant to the effect of four nucleon interaction. In the \( \Delta N = 1 \) staggering effect the situation is different. Here the filling of valence shells is realized in two ways:
1. in the odd-even multiplets by the alternative adding of proton and neutron pairs, which is reflected by the alternative change of the $F_0$—values and thus strongly magnifies the staggering effect,

2. in the isotopic multiplets by adding of only neutron pairs, which gives a very small staggering.

In these two cases the observed effect is due to the pairing correlations, which are of lower order, but its magnitude strongly depends on the charge fluctuations in the consecutive adding of the nucleon pairs.

This analysis provides an indication of the different ways of influence of the different orders of valence nucleon correlations (pairing and quartetting) on the fine systematic characteristics of the nuclear collectivity.

The general formula for the yrast energies $E_{yrast}(h_k, I, \omega)$ obtained for the even-even nuclei, as classified in the symplectic multiplets reproduces rather accurately the experimental $\Delta N = 2$ and $\Delta N = 1$ staggering. The geometric parameter $\omega$, introduced in [5], not only unifies the description of the energies of the ground state bands of the spherical, transitional and well-deformed nuclei, but also is the main reason for obtaining the effects of the shape transitions reflected in both staggering effects. The amplitude of the staggering generally follows the same trend as $\omega$, so it could be considered as relevant characteristic of the type of collectivity of nuclear spectra and respectively the nuclear shape. In the region of the well deformed nuclei, when $\omega = 1$ the staggering effect is strongly suppressed, which is observed in the rare-earth and actinide shells and is the reason for the larger scale in the light ones.

The generalizations of the interaction coefficient $\alpha\{h_k\}$ (4), as functions of the classification quantum numbers, case is important for the reproduction of the staggering effects. In the $\Delta N = 2$ case $F_0$ is fixed, as well as in the isotopic multiplets with $\Delta N = 1$, while in the combined odd-even multiplets its values oscillate from a state to state. This reflects in the bigger values of the amplitudes and their regularity in the $Stg(1N)$ function in the combined multiplets.

The investigation in this work, reveals once again that the symplectic classification scheme is rather convenient for a generalization and reproduction of important physical properties of the even-even nuclei. It also has a predicting power, since in the consideration of the staggering $\Delta N = 1$ effects in the last shell, we used experimental data [28], which was not included in
the fitting procedure in [5]. As the ladder series of the boson representations are infinite dimensional, the symplectic multiplets can be extended with the newly obtained data for exotic nuclei and they can be considered as a new test for this kind of fine structure investigations. Moreover the suggested quantitative analysis could provide a rather fine estimation of the possible energy regions, where low lying collective excitations of exotic nuclei can be expected.

In conclusion, the above results show that the investigation of the different types of the staggering effects based on a convenient classification scheme is an useful tool to understand not only the collective properties of the lowest states of even–even nuclei, but also provides various quantitative considerations on the fine structure of the changes in nuclear collectivity.

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