Robust energy transfer mechanism between non-resonant triads in nonlinear wave systems

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A new generic physical mechanism of energy transfer is found which is based on the existence of non-resonant interactions in nonlinear systems. The basic mechanism consists of a transfer of energy from a “source” triad to a “target” triad, the triads being connected via two common modes. For the mechanism to occur it is essential that the target triad be non-resonant. Since the mode to which the energy is transferred does not require initial energy, the mechanism is more robust than the modulational instability and can be seen as a generalisation of it. Moreover, the transfer efficiency can be optimised by tuning appropriately the size of the nonlinearity at an intermediate level. In other words, for a given initial condition the energy transfer can be made more efficient by a simple rescaling of the modes’ initial amplitudes by a common factor, the optimal value of which depends on a type of critical balance between the nonlinear frequency of oscillations and the linear frequency mismatch of non-resonant triads in the target triad. At nonlinearity levels satisfying this critical balance, the energy transfer efficiency can attain peak values of up to 90% while the efficiency observed in the highly-nonlinear region is only about 30%. The energy transfer mechanism is analyzed extensively in a simple model of ODEs and the effect of optimal enstrophy transfer is also observed in the Charney-Hasegawa-Mima PDE.

NONLINEAR WAVE SYSTEMS IN FINITE DOMAINS

Interactions of three waves, typically referred to as triad interactions, are important in many physical systems such as Rossby waves in the atmosphere, drift waves in magnetically-confined plasmas and Alfvén waves in astrophysical magnetohydrodynamic turbulence. It is often assumed in the literature that the most efficient energy transfers take place between three modes which are in exact resonance. Another usual assumption is that higher wave amplitudes give rise to more efficient energy transfers. It will be shown here that these assumptions do not hold in certain regimes, particularly in finite-sized systems. The main result of this paper is that a detailed balance between nonlinearity and linear frequency mismatch of non-resonant triads is required in order to maximise the efficiency of energy transfers.

Consider systems of waves interacting nonlinearly in a periodic spatial domain of dimension d. In the limit of infinitesimally small nonlinearity, the Fourier components $\mathbf{a}_k$ of the wave field (corresponding to wavevectors $\mathbf{k} \in \mathbb{Z}^d$) satisfy

$$\mathbf{a}_k \approx e^{-i\omega_k t},$$

where $\omega_k$ is the so-called dispersion relation. In the nonlinear case the wave system is better described in the interaction representation, which is the set of complex amplitudes $b_k = a_k e^{i\omega_k t}$. The evolution equations to leading order in nonlinearity, are

$$\partial_t b_k = \sum_{k_1, k_2 \in \mathbb{Z}^d} Z^b_{12} \delta^b_{12} b_{k_1} b_{k_2} e^{i(\omega_{k_1} - \omega_{k_2} + \omega_k + \omega_{k_1} - \omega_{k_2}) t},$$

where additional terms related to permutations have been omitted for brevity. Coefficients $Z^b_{12}$ are the so-called interaction coefficients, $\omega_j \equiv \omega_{k_j}$ and the Kronecker symbol $\delta^b_{12}$ defines the triad as the set of three wave vectors $k_1, k_2, k$ satisfying $k_1 + k_2 = k$. The advantage of using the interaction representation is that the variables $b_k$ evolve slowly compared to $a_k$. Note that system (1) is not necessarily Hamiltonian. Resonant triads are defined by $\omega_{k_1} - \omega_{k_2} = 0$. Non-resonant triads satisfy $\omega_{k_1} - \omega_1 - \omega_2 \neq 0$, and are physically more sensible than resonant ones. The mechanism presented here leads to efficient energy transfers towards non-resonant triads.

A ROBUST NONLINEAR INSTABILITY

In the limit of very small amplitudes, contributions from most of the terms in the sums appearing in the RHS of equation (1) tend to cancel individually because the exponential terms $e^{i(\omega_k + \omega_1 - \omega_2) t}$ rotate faster than the amplitudes $b_k$, so that their contribution averages to zero for intermediate time scales. In other words, the nonlinear frequency of the oscillations of $b_k$ is much lower than any of the linear frequency mismatches $\omega_k \pm \omega_1 \pm \omega_2$ appearing in the exponentials. An obvious exception to this rule occurs when some of the frequency mismatches $(\omega_k \pm \omega_1 \pm \omega_2)$ are equal to zero. This exception is interesting on its own but is irrelevant for the mechanism discussed here.

As amplitudes are increased from infinitesimally small to finite values, the nonlinear frequency of the oscillations of $b_k$ increases, typically proportional to the amplitudes. Thus, at some intermediate value of the amplitudes the nonlinear frequencies become comparable to the linear frequencies. Therefore, at least some of the terms in the sums in equation (1) will contain zero modes i.e. terms that remain constant in time for a while. These terms contribute to the evolution of the modes $b_k$ since they do not average to zero for intermediate time scales. Therefore they lead to sustained growth even from zero initial condition. This phenomenon can be understood as an instability that relies on non-resonant interactions and is triggered by tuning the amplitudes appropriately. It is quite easy to find the effect in numerical simulations, as will be seen in the next sections and it should be relatively easy to find in experiments provided the initial conditions can be controlled to some extent.

STRONG ENERGY TRANSFER MECHANISM

In order to understand the nonlinear instability, consider what might be termed “the atom” of this process of energy transfer. Here, the discussion is restricted to two triads connected by
two common modes i.e. there are only four evolving modes $b_1$, $b_2$, $b_3$ and $b_4$ with corresponding wavenumbers $k_1, k_2, k_3, k_4$ satisfying the 3-wave conditions $k_1 + k_2 = k_3$ and $k_2 + k_3 = k_4$. The evolution equations for this reduced system can be written as

\[
\begin{align*}
\dot{b}_1 &= S_1 b_2^* b_3 e^{i\delta, t}, \\
\dot{b}_2 &= S_2 b_1^* b_3 e^{i\delta, t} + T_1 b_2^* b_4 e^{i\delta, t}, \\
\dot{b}_3 &= S_3 b_1 b_2 e^{-i\delta, t} + T_2 b_2^* b_4 e^{i\delta, t}, \\
\dot{b}_4 &= T_3 b_2 b_3 e^{-i\delta, t},
\end{align*}
\]

(2)

where the frequency mismatches are defined as $\delta = \omega_1 + \omega_2 - \omega_3$ and $\delta_T = \omega_2 + \omega_3 - \omega_4$ for the ‘source’ and ‘target’ triads respectively. The six interaction coefficients $S_1$, $S_2$, $S_3$, $T_1$, $T_2$ and $T_3$ are assumed to be real and nonzero. The main aim of this paper is to understand how energy is transferred from the source triad $(b_1, b_2, b_3)$ to the target mode $b_4$, with initial conditions $(b_1(0), b_2(0), b_3(0)) \neq 0$ and $b_4(t=0) = 0$.

Only models where the evolution is bounded are considered here. This implies that for each individual triad there must be one active mode and two passive modes i.e. the three interaction coefficients $S_1$, $S_2$ and $S_3$ do not have the same sign and likewise for $T_1$, $T_2$ and $T_3$. The full dynamical system now has two functionally-independent quadratic invariants

\[
\begin{align*}
I_1 &= -S_1 T_3 |b_1|^2 + S_1 T_1 |b_2|^2 - S_1 T_1 |b_4|^2, \\
I_2 &= -S_1 T_3 |b_1|^2 + S_1 T_3 |b_2|^2 - S_1 T_2 |b_4|^2.
\end{align*}
\]

(3)

The physical condition of boundedness indicates the existence of at least one positive-definite quadratic invariant that involves the four square amplitudes $|b_j|^2$. Using the explicit form above for $I_1$ and $I_2$, the boundedness condition leads to nine allowed, non-equivalent combinations of signs for the interaction coefficients. Such classification will be discussed in a subsequent paper, since it is not too relevant for the analysis of the instability. In what follows, the boundedness condition is assumed valid.

Consider the physical situation whereby the initial energy is in the first triad only i.e. $b_4(0) = 0$. The question is whether $b_4$ remains small for subsequent times or not. Assuming that it does, system (2) can be approximated by

\[
\begin{align*}
\dot{b}_1 &= S_1 b_2^* b_3 e^{i\delta, t}, \\
\dot{b}_2 &= S_2 b_1^* b_3 e^{i\delta, t}, \\
\dot{b}_3 &= S_3 b_1 b_2 e^{-i\delta, t}, \\
\dot{b}_4 &= T_3 b_2 b_3 e^{-i\delta, t}.
\end{align*}
\]

(4)

At this level of approximation, the first three equations form a closed system that can be integrated analytically. They describe the so-called isolated triad. For simplicity of presentation the case $\delta_T = 0$ is discussed. There exist three conservation laws, the triad Hamiltonian $H \equiv \Im(b_1 b_2^* b_3^*)$ and the two Manley-Rowe invariants $I_1 = -S_1 |b_1|^2 + S_1 |b_3|^2$ and $I_2 = -S_3 |b_1|^2 + S_3 |b_2|^2$. In the amplitude-phase representation $b_j(t) = |b_j(t)| e^{i\varphi_j(t)}$ for $j = 1, 2, 3$, all real amplitudes $|b_1(t)|$, $|b_2(t)|$, $|b_3(t)|$ and the dynamical phase $\varphi(t) \equiv \varphi_1(t) + \varphi_2(t) - \varphi_3(t)$ are periodic functions with the same period $\frac{2\pi}{\delta}$, where $\Gamma = \Gamma_1 I_1, I_2, H$ is the so-called nonlinear frequency broadening. The phases are quasiperiodic, that is $\varphi_j(t) \equiv \varphi_{j\ell}(t) + \Omega_j t$ where $\varphi_{j\ell}(t)$ has period $\frac{2\pi}{\delta}$ and the constants $\Omega_j$ are precession frequencies of the individual phases, defined as $\Omega_j = -\frac{S_1^* H}{2\pi} \int_0^{2\pi} \frac{1}{|b_j(t)|^2} dt$, $j = 1, 2$, and $\Omega_3 = \Omega_1 + \Omega_2$.

The fourth equation in system (1) can immediately be integrated by quadratures to give

\[
\begin{align*}
b_4(t) &= T_3 \int_0^t |b_2(t')||b_3(t')| e^{i(\varphi_{22}(t') - \varphi_{23}(t')) - \delta_T t'} dt' \\
&= \int_0^t f_{\text{net}}(t') e^{i(\Omega_1 + \Omega_2 - \delta_T) t'} dt',
\end{align*}
\]

(5)

where $f_{\text{net}}(t) \equiv T_3 |b_2(t)||b_3(t)| e^{i(\varphi_{22}(t) + \varphi_{33}(t))} + \varphi_{23}(t)}$ is a periodic complex function with period $\frac{2\pi}{\delta_T}$. Thus, if the frequencies satisfy

\[
n\Gamma + \Omega_2 + \Omega_3 = \delta_T \quad \text{for some } n \in \mathbb{Z},
\]

(6)

then an instability occurs whereby the amplitude $b_4(t)$ grows as $\kappa n t$ without bound, where $\kappa n$ is the $n$-th Fourier coefficient of the function $f_{\text{net}}(t)$. As this coefficient decays as $n \to \infty$, it turns out that the instability has a practical application only for a few values of $n$.

Equation (6) is a balance between an amplitude-dependent nonlinear frequency ($n\Gamma + \Omega_2 + \Omega_3$) pertaining to the source triad and a constant frequency mismatch $\delta_T$ pertaining to the target triad. Since the LHS quantity is proportional to the amplitudes of the initial condition in the source triad, it follows that one can in principle “tune” the initial amplitudes in order to satisfy the balance provided $\delta_T$ is non-zero. It is worth emphasising that if $\delta_T = 0$, there is no way to trigger the instability by tuning the initial amplitude. Therefore this mechanism favours energy transfers towards non-resonant triads.

Quantitative evidence of the strong energy transfer mechanism is now presented. Returning to the full nonlinear ODE system (2), the parameters used are $\delta_T = 0$, $\delta_T = -\frac{\pi}{8}$, $S_1 = 1$, $S_2 = 9$, $S_3 = -8$, $T_1 = -\epsilon$, $T_2 = \frac{\epsilon}{4}$ and $T_3 = -\frac{\epsilon}{2}$. The constant $\epsilon$ is arbitrary and it serves as an interpolation between system (3) when $\epsilon = 0$ and the full system (2) when $\epsilon = 1$.

The existence of the energy transfer mechanism is robust with respect to the choice of initial conditions, however the actual values of amplitude at which resonances occur depend on this choice. With the generic initial amplitudes $b_1(0) = 0.007772 A$, $b_2(0) = 0.0385822 A$, $b_3(0) = -0.0358876 I$ and $b_4(0) = 0$, where $A$ is an arbitrary positive constant, the terms in equations (4) can be computed explicitly with the aid of the analytic or numerical solutions giving $\Gamma = 0.27267 A$, $\Omega_2 = -0.04935 A$ and $\Omega_3 = 0.12014 A$. Thus the balance equation (6) becomes

\[
A = \delta_T \frac{0.0708 + 0.27267 n}{0.0796 + 0.3068 n}, \quad n \in \mathbb{Z}.
\]

(7)

By inspection, positive values of $A$ are obtained only for negative $n$. As previously mentioned, if $|n|$ is too large, the strength of the instability coming from the Fourier coefficient of $f_{\text{net}}(t)$ will be negligible and in practice, only the cases $n = -1$ and $n = -2$ are relevant. These give the predicted values of $A$ for which resonance occurs to be $A_{-1} = 4.40$ and $A_{-2} = 1.87$. In order to check the strength of the actual energy transfers of the full ODEs, the relative contribution of the mode $b_4$ to the positive definite quadratic invariants $E$, defined in equation (5),

\[
E = \frac{72}{5} |b_1(t)|^2 + \frac{9}{5} |b_2(t)|^2 + \frac{8}{3} \epsilon |b_4(t)|^2
\]

\[
F = \frac{9}{5} |b_2(t)|^2 + \frac{81}{40} |b_3(t)|^2 + 2 \epsilon |b_4(t)|^2
\]

(8)

is shown in Fig. 1 (top). A similar contribution is made to invariant $F$ defined in equation (8). It shows that the strong
Above. From Fig. 2, it can be seen that as $\epsilon \rightarrow 0$, it is attained very close to the predicted efficiency. The effect predicted by the simple analysis is very close to what happens in the numerics of the ODE system. Fig. 1 (bottom) provides a different perspective. The figure shows the relative transfer of energy to $B_4$ as a function of $tA$. Additionally, a range of values of $\epsilon$ has been considered. It is important to note that each $\epsilon$ provides a different dynamical system with the coefficients $T_2, T_3 \propto \epsilon$ as noted above. From Fig. 2 it can be seen that as $\epsilon$ is deformed from 1 to lower values, the position of the peak becomes thinner and thinner and converges to the predicted value of $A_{-1}$.

Also notice that the position of a second resonance at $A_{-2}$ is visible but with much less intensity. This situation is reminiscent of the persistence theorem of resonant manifolds [4,5].

**NATURE OF THE INSTABILITY**

There is good evidence to support the existence of a persistent manifold for any value of $\epsilon$. This manifold is directly related to the effect of strong energy transfer. Looking again at Fig. 2, fix $\epsilon = 0.1$ and let $A$ vary from small to large values, or from left to right on the plot. Notice that the rise in efficiency is very smooth until $A$ is increased beyond $A_{\text{crit}}$, when there is a steep ridge at the right of the peak. At $A = A_{\text{crit}}$, the solution of the evolution equations (2) gets arbitrarily close to a periodic solution. This result is established numerically using a bisection method. Fig. 3 shows time evolutions of the relative transfer of invariant $E$ towards $B_4$ as a function of time for three values $A$, namely $A_{\text{crit}}$ and $A_{\text{crit}} \pm 10^{-5}A_{\text{crit}}$. It is evident that the periodic solution is unstable: for $A \lesssim A_{\text{crit}}$, the trajectories eventually escape from the periodic solution towards high values of $|b_4|$, leading to a strong transfer of invariant $E$. For $A \gtrsim A_{\text{crit}}$, instead, the trajectories escape towards small values of $|b_4|$. Further numerical studies confirm that the solution is periodic. The character of the periodic solution is explored by computing the Lyapunov exponents using the continuous QR method as implemented in [6]. It is important to stress that the dynamical system (2) has only six Lyapunov exponents, and each of these must add to zero because the system is volume-preserving. One of these

![FIG. 1. Top: Relative transfer of E as a function of tA. Bottom: Energy transfer efficiency to B4 as function of tA.](image)

![FIG. 2. Efficiency of the transfer of energy to B4 as a function of log10A and log10\epsilon. Dashed vertical lines correspond to the predicted resonances A_{-3}, A_{-2}, A_{-1}, from left to right. The horizontal line is the case \epsilon = 1 and provides the efficiency profile shown in figure 1 bottom. The marked cross is the global peak of efficiency (94%), located at A = 7.36, \epsilon = 0.97.](image)

![FIG. 3. Plot of B4's contribution to E for A = A_{\text{crit}} = 5.320174 (middle curve), A = A_{\text{crit}} + 10^{-5}A_{\text{crit}} (lower curve) and A = A_{\text{crit}} - 10^{-5}A_{\text{crit}} (upper curve).](image)
four exponents vanishes since it corresponds to the direction tangent to the periodic orbit. Three Lyapunov exponents remain whose sum is zero. As a general result, the biggest exponent in size has opposite sign to the remaining two.

In the numerical study, the biggest exponent in size is positive, and a consistent ratio $-1 : -2 : 3$ between the three exponents is found, for the full length of a simulation that takes about 30 periods of the periodic solution. So the picture is that the stable manifold of the periodic orbit is two-dimensional and the unstable manifold is one-dimensional. This supports the intuition from Fig. 3, that the escape from the periodic orbit along the unstable (repulsive) direction is strong compared to the stable (attractive) direction.

**NEW TURBULENCE PARADIGM**

A general picture of wave turbulence can be derived from the previous analyses. When all possible triads are taken into account, most are non-resonant and form clusters of connected triads. Thus, for a given initial condition energy can flow through clusters via a chain of transfers between adjacent triads. Each of these individual transfers can be understood as a particular case of Fig. 2 where the efficiency of the transfer from the source triad to the target triad depends on where the corresponding $A - \epsilon$ point lies on the plot. Recall that the resonant values of $A$ (leading to high efficiency) are determined by a type of critical balance between the nonlinear frequency of the source triad and the linear frequency mismatch of the target triad. In addition, the quantity $\epsilon$ is basically the ratio between the interaction coefficients of the source and target triads. There is a resonant structure for this quantity: the energy transfer efficiency depends non-monotonically on the value of $\epsilon$. Therefore, it may be hypothesised that a strongly turbulent system is one where the most active triads are those naturally selected according to this $A - \epsilon$ criterion.

Using this as a guide, a cluster of modes has been chosen for the initial condition of pseudospectral simulations (resolution $128^3$) of the Charney-Hasegawa-Mima (CHM) equation [6] [8], an evolution equation for the stream function $\psi$. Just as the ODE system has two independent invariants, so too does the CHM equation, understood to be energy and enstrophy. Using a brute-force search method [9], all triads with $\delta\gamma$ greater than a certain value were identified and the modes in these triads are initially excited according to $\psi = \Re(5A k^{-3} e^{-0.3k})^{1/2} e^{i\varphi_k}$ where $\varphi_k$ are the wave phases. Similar to the results from the ODE calculations, Fig. 4 illustrates how enstrophy transfer is maximized for an intermediate, finite range of $A$, showing that the mechanism applies to arbitrary clusters of triads and is robust with respect to the choice of initial conditions.

**CONCLUSIONS**

A robust energy transfer mechanism towards non-resonant triads has been established analytically and verified numerically for reduced ODE models as well as direct numerical simulations of a full PDE model. The implications of this mechanism are numerous and their impact in experiments in several physical systems is significant; for example, it could be used (i) for the understanding of turbulent cascades as a natural selection mechanism of triads (ii) as a possible mechanism responsible for rogue wave generation (triggered either by forcing or dissipation) (iii) as a new paradigm for a complete theory of wave turbulence (iv) industrial applications in terms of optimizing the size of a system with respect to efficiency and (v) in the case of four-wave interactions, the mechanism could lead to application in nonlinear optics and telecommunications. The mechanism exists even in the case when quadratic terms in equation (1) can be eliminated by mapping to normal forms, which occurs when no triad is resonant.

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