Well-posedness for a class of generalized variational-hemivariational inequalities involving set-valued operators

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Abstract
The aim of present work is to study some kinds of well-posedness for a class of generalized variational-hemivariational inequality problems involving set-valued operators. Some systematic approaches are presented to establish some equivalence theorems between several classes of well-posedness for the inequality problems and some corresponding metric characterizations, which generalize many known results. Finally, the well-posedness for a class of generalized mixed equilibrium problems is also considered.

Keywords: Generalized variational-hemivariational inequality; Set-valued operator; \( \alpha \)-well-posedness; Monotonicity

1 Introduction
Nowadays, well-posedness has been drawing great attention in the field of optimization problems and related problems such as variational inequalities, hemivariational inequalities, fixed point problems, equilibrium problems, and inclusion problems (see [1, 5, 9, 11, 17, 19, 21, 23, 33]). The classical concept of well-posedness for a global minimization problem was first introduced by Tikhonov [35], which required the existence and uniqueness of a solution to the global minimization problem and the convergence of every minimizing sequence toward the unique solution. Thereafter, the concept of well-posedness has been generalized to variational inequalities. The initial notion of well-posedness for variational inequality is due to Lucchetti and Patrone [28]. Fang [13, 14] generalized two kinds of well-posedness for a mixed variational inequality problem in a Banach space. For further results on the well-posedness of variational inequalities, we refer to [2, 4, 12–14, 16, 22, 27, 28] and the references therein.

As an important and useful generalization of variational inequality, hemivariational inequality, which was first studied by Panagiotopoulos [32], has a great development in recent years by several works [6, 29, 31]. Many authors are interested in generalizing the concept of well-posedness to hemivariational inequalities. In 1995, Goeleven and Mentagui [15] generalized the concept of the well-posedness to a hemivariational inequality and presented some basic results concerning the well-posed hemivariational inequality. Recently, using the concept of approximating sequence, Xiao et al. [37, 38] introduced a concept of well-posedness for a hemivariational inequality and a variational-hemivariational
Inequality. Ceng, Lur, and Wen [3] considered an extension of well-posedness for a minimization problem to a class of generalized variational-hemivariational inequalities with perturbations in reflexive Banach spaces. For more recent works on the well-posedness for variational-hemivariational inequalities, we refer to [3, 15, 18, 19, 26, 37, 38] and the references therein.

In the last years, many authors studied the existence results for some types of hemivariational inequalities involving set-valued operators [34, 36, 39]. In 2011, Zhang and He [39] studied a kind of hemivariational inequalities of Hartman–Stampacchia type by introducing the concept of stable quasimonotonicity. They supposed that the constraint set is a bounded (or unbounded), closed, and convex subset in a reflexive Banach space. The authors gave sufficient conditions for the existence and boundedness of solutions. In 2013, Tang and Huang [34] generalized the result of [39] by introducing the concept of stable \( \phi \)-quasimonotonicity and obtained some existence theorems when the constrained set is nonempty, bounded (or unbounded), closed, and convex in a reflexive Banach space. Hereafter, Wangkeeree and Preechasilp [36] generalized the results of [34] and [39] by introducing the concept of stable \( f \)-quasimonotonicity. Very recently, Liu and Zeng obtained some existence results for a class of hemivariational inequalities involving the stable \((g,f,\alpha)\)-quasimonotonicity [25], a result on the well-posedness for mixed quasivariational hemivariational inequalities [26], and some existence results for a class of quasimixed equilibrium problems involving the \((f,g,h)\)-quasimonotonicity [24].

Let \( K \) be a nonempty, closed, and convex subset of a real Banach space \( X \) with its dual \( X^* \), and let \( F : K \to P(X^*) \) be a set-valued operator, where \( P(X^*) \) is the set of all nonempty subsets of \( X^* \). Let \( T : K \to X^* \) be a perturbation, and let \( f \in X^* \) be a given element. Let

\[
\begin{align*}
\delta (f) :\ R \cup \{ \pm \infty \} \to \mathbb{R} & = \{ u \in K : g(u, v) \neq -\infty, \forall v \in K \} \neq \emptyset. \\
\end{align*}
\]

Let \( J : X \to R \) be a locally Lipschitz function, and let \( \phi^* (u, v) \) denote the generalized directional derivative in the sense of Clarke of a locally Lipschitz functional \( J : X \to R \) at \( u \) in the direction \( v \). In this paper, we discuss the following generalized variational-hemivariational inequality (GVHVI):

Find \( u \in K \) such that, for some \( u^* \in F(u) \),

\[
\langle u^* + Tu - f, v - u \rangle + g(u, v) + f^*(u; v - u) \geq 0, \quad \forall v \in K.
\]

Now, let us consider some particular cases of GVHVI.

(a) If \( T \equiv 0, f \equiv 0, \) and \( g \equiv 0 \), then GVHVI is reduced to the following form:

Find \( u \in K \) and \( u^* \in F(u) \) such that

\[
\langle u^*, v - u \rangle + f^*(u; v - u) \geq 0, \quad \forall v \in K.
\]

The existence of solutions to this inequality was recently studied by Zhang and He [39].

(b) If \( T \equiv 0 \) and \( f \equiv 0 \), and \( g(u, v) = \phi(v) - \phi(u) \) for all \( u, v \in K \), then GVHVI is reduced to the following form:

Find \( u \in K \) and \( u^* \in F(u) \) such that

\[
\langle u^*, v - u \rangle + \phi(v) - \phi(u) + f^*(u; v - u) \geq 0, \quad \forall v \in K.
\]

The existence of solutions to this inequality was studied by Tang and Huang [34].
(c) If \( T \equiv 0 \) and \( f \equiv 0 \), then GVHVI is reduced to the following form:
Find \( u \in K \) and \( u^* \in F(u) \) such that
\[
\langle u^*, v - u \rangle + g(u, v) + J(u; v - u) \geq 0, \quad \forall v \in K.
\]

The existence of solutions to this inequality was studied by Wangkeeree and Preechasilp [36].

Inspired by previous works, we study the well-posedness for GVHVI, which generalizes many known works. Under relatively weak conditions, we establish some equivalence results and some metric characterizations for the strong and weak \( \alpha \)-well-posed GVHVI in the generalized sense. In particular, we present equivalence results on weak \( \alpha \)-well-posedness for GVHVI, which were considered by few authors.

This paper is organized as follows. In Sect. 2, we recall some basic preliminaries of single-valued and set-valued mappings, metric concepts, Clarke’s generalized directional derivative, and some classes of well-posedness for GVHVI. In Sect. 3, we show some equivalence results for the well-posedness for GVHVI and some corresponding metric characterizations. Theorems 3.3, 3.5, and 3.6 are the main results in this section. In the last section, we also present the well-posedness for a class of generalized mixed equilibrium problems.

2 Preliminaries

Let \( R, R_+, \) and \( N \) be the sets of real numbers, nonnegative real numbers, and natural numbers, respectively. Let \( X \) be a real Banach space with norm \( \| \cdot \|_X \). Denote by \( X^* \) its dual space and by \( \langle \cdot, \cdot \rangle_X \) the duality pairing between \( X^* \) and \( X \). Let \( X_w \) be the Banach space \( X \) with weak topology.

**Definition 2.1** Let \( K \) be a nonempty subset of \( X \). A function \( f : K \to R \) is said to be

(i) convex on \( K \) if for all finite subsets \( \{u_1, \ldots, u_n\} \subset K \) and \( \{\lambda_1, \ldots, \lambda_n\} \subset R_+ \) such that \( \sum_{i=1}^n \lambda_i = 1 \) and \( \sum_{i=1}^n \lambda_i u_i \in K \), we have
\[
f \left( \sum_{i=1}^n \lambda_i u_i \right) \leq \sum_{i=1}^n \lambda_i f(u_i);
\]

(ii) (weakly) upper semicontinuous (u.s.c. for short) at \( u \) if for any sequence \( \{u_n\}_{n \geq 1} \subset K \) with \( u_n \to u \) \( u_n \to u \), we have
\[
\limsup_{n \to \infty} f(u_n) \leq f(u).
\]

(iii) (weakly) lower semicontinuous (l.s.c. for short) at \( u \), if for any sequence \( \{u_n\}_{n \geq 1} \subset K \) with \( u_n \to u \) \( u_n \to u \), we have
\[
\liminf_{n \to \infty} f(u_n) \geq f(u).
\]

The function \( f \) is said to be (weakly) u.s.c. (l.s.c.) on \( K \) if \( f \) is (weakly) u.s.c. (l.s.c.) at all \( u \in K \).
Definition 2.2 ([20]) Let $K$ be a nonempty subset of $X$. An operator $\beta : K \to X$ is said to be affine if for any $u_i \in K$ ($i = 1, 2, \ldots, n$) and $\lambda_i \in [0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\beta \left( \sum_{i=1}^{n} \lambda_i u_i \right) = \sum_{i=1}^{n} \lambda_i \beta(u_i).$$

Definition 2.3 A set-valued operator $F : K \to P(X^*)$ is said to be

(i) lower semicontinuous (l.s.c.) at $u_0$ if for any $u^*_0 \in F(u_0)$ and sequence $\{u_n\}_{n=1}^{\infty} \subset K$ with $u_n \to u_0$, there exists a sequence $u^*_n \in F(u_n)$ that converges to $u^*_0$.

(ii) lower hemicontinuous (l.h.c.) if the restriction of $F$ to every line segment of $K$ is lower semicontinuous with respect to the weak topology in $X^*$.

Definition 2.4 A set-valued operator $F : K \to P(X^*)$ is said to be monotone if for all $u, v \in K$,

$$\langle v^* - u^*, v - u \rangle \geq 0, \quad \forall u^* \in F(u), \forall v^* \in F(v).$$

Definition 2.5 Let $S$ be a nonempty subset of $X$. The measure $\mu$ of noncompactness for the set $S$ is defined by

$$\mu(S) := \inf \left\{ \epsilon > 0 : S = \bigcup_{i=1}^{n} S_i, \text{diam } |S_i| < \epsilon, i = 1, 2, \ldots, n \right\},$$

where \text{diam } |S_i|$ is the diameter of the set $S_i$.

Now, let us recall the definitions of the Clarke generalized directional derivative and generalized gradient for a locally Lipschitz function $\varphi : X \to R$ (see [6, 10]). The Clarke generalized directional derivative $\varphi^0(u; v)$ of $\varphi$ at the point $u \in X$ in the direction $v \in X$ is defined as

$$\varphi^0(u; v) := \limsup_{\lambda \to 0^+, \zeta \to u} \frac{\varphi(\zeta + \lambda v) - \varphi(\zeta)}{\lambda}.$$ 

The Clarke subdifferential or generalized gradient of $\varphi$ at $u \in X$, denoted by $\partial \varphi(u)$, is the subset of $X^*$ given by

$$\partial \varphi(u) := \left\{ u^* \in X^* : \varphi^0(u; v) \geq [u^*, v]_X, \forall v \in X \right\}.$$ 

Lemma 2.6 ([6], Proposition 2.1.1) Let $\varphi : X \to R$ be locally Lipschitz of rank $L_u > 0$ near $u$. Then

(i) $\varphi^0(u; v)$ is u.s.c. as a function of $(u, v)$ and, as a function of $v$ alone, is Lipschitz of rank $L_u$ near $u$ on $X$ and satisfies

$$\left| \varphi^0(u; v) \right| \leq L_u \|v\|_X;$$

(ii) the gradient $\partial \varphi(u)$ is a nonempty, convex, and weakly* compact subset of $X^*$ bounded by a Lipschitz constant $L_u$ near $x$;
We end this section with the notions of several classes of $\alpha$-approximating sequences and $\alpha$-well-posedness for GVHVI. Let $\alpha : X \to \mathbb{R}$ be a functional.

**Definition 2.7** A sequence $\{u_n\}$ in $K$ is an $\alpha$-approximating sequence for GVHVI if there exist $\{u^*_n\}$ in $X^*$ with $u^*_n \in F(u_n)$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that, for every $n \in \mathbb{N}$,

$$
\langle u^*_n + Tu_n - f, v - u_n \rangle + g(u_n, v) + J(u_n; v - u_n) \geq -\epsilon_n \alpha(v - u_n), \quad \forall v \in K.
$$

In particular, if $\alpha(\cdot) = \|\cdot\|_X$, then $\{u_n\}$ is said to be an approximating sequence for GVHVI.

**Definition 2.8** GVHVI is said to be strongly (respectively, weakly) $\alpha$-well-posed if it has a unique solution $u$ and every $\alpha$-approximating sequence $\{u_n\}$ strongly (respectively, weakly) converges to $u$. In particular, if $\alpha(\cdot) = \|\cdot\|_X$, then GVHVI is said to be strongly (respectively, weakly) well-posed.

**Definition 2.9** GVHVI is said to be strongly (respectively, weakly) $\alpha$-well-posed in the generalized sense if the solution set $\Gamma$ of GVHVI is nonempty and every $\alpha$-approximating sequence $\{u_n\}$ has a subsequence that strongly (respectively, weakly) converges to some point of $\Gamma$. In particular, if $\alpha(\cdot) = \|\cdot\|_X$, then GVHVI is said to be strongly (respectively, weakly) well-posed in the generalized sense.

**Remark 2.10** Strong $\alpha$-well-posedness (in the generalized sense) implies weak $\alpha$-well-posedness (in the generalized sense), but the converse is not true in general.

### 3 The characterizations of well-posedness for GVHVI

In this section, we establish metric characterizations and derive some conditions under which GVHVI is strongly (weakly) $\alpha$-well-posed.

For any $\epsilon > 0$, we define the following two sets:

$$
\Omega_\alpha(\epsilon) = \left\{ u \in K : \exists u^* \in F(u) \text{ such that } \langle u^* + Tu - f, v - u \rangle + g(u, v) + f^*(u; v - u) \geq -\epsilon \alpha(v - u), \forall v \in K \right\}
$$

and

$$
\Phi_\alpha(\epsilon) = \left\{ u \in K : \langle v^* + Tu - f, v - u \rangle + g(u, v) + f^*(u; v - u) \geq -\epsilon \alpha(v - u), \forall v \in K, \forall v^* \in F(v) \right\}.
$$

Denote by $\Gamma$ the set of solutions to GVHVI. It is clear that $\Gamma = \Omega_0(\epsilon)$.

**Lemma 3.1** Assume that:

(i) $K$ is a nonempty closed subset of a real Banach space $X$;
(ii) $T : K \to X^*_w$ is continuous;
(iii) $g : K \times K \to \mathbb{R}$ is u.s.c. with respect to the first variable;
(iv) $\alpha : X \to \mathbb{R}$, is such that $\liminf_{n \to \infty} \alpha(v_n) \leq \alpha(v)$ whenever $v_n \to v$.

Then, for every $\epsilon > 0$, the set $\Phi_\alpha(\epsilon)$ is closed in $X$.

Proof Let $\{u_n\} \subset \Phi_\alpha(\epsilon)$ be a sequence such that $u_n \to u$ in $X$. Then $u \in K$, and, for all $v \in K$ and $v^* \in F(v)$,

$$\{v^* + Tu_n - f(v - u_n) + g(u_n, v) + f^*(u_n, v - u_n) \geq -\epsilon \alpha(v - u_n).$$

By the assumptions and the properties of $f^*$ we have

$$\{v^* + Tu - f(v - u) + g(u, v) + f^*(u; v - u)
\geq \limsup_{n \to \infty} \left[\{v^* + Tu_n - f(v - u_n) + g(u_n, v) + f^*(u_n, v - u_n)\}\right]
\geq \limsup_{n \to \infty} -\epsilon \alpha(v - u_n)
= -\epsilon \liminf_{n \to \infty} \alpha(v - u_n)
\geq -\epsilon \alpha(v - u),$$

and hence

$$\{v^* + Tu - f(v - u) + g(u, v) + f^*(u; v - u) - \epsilon \alpha(v - u), \quad \forall v \in K, \forall v^* \in F(v),$$

which shows that $u \in \Phi_\alpha(\epsilon)$. \hfill \qed

Lemma 3.2 Assume that:

(i) $K$ is a nonempty convex subset of a real Banach space $X$;
(ii) $F : K \to P(X^*)$ is l.h.c. and monotone;
(iii) $g : K \times K \to \mathbb{R}$ is convex with respect to the second variable;
(iv) $\alpha : X \to \mathbb{R}$, is convex with $\alpha(tv) = t\alpha(v)$ for all $t \geq 0$ and $v \in X$.

Then $\Omega_\alpha(\epsilon) = \Phi_\alpha(\epsilon)$ for all $\epsilon > 0$.

Proof We first show that $\Omega_\alpha(\epsilon) \subset \Phi_\alpha(\epsilon)$. Indeed, take arbitrary $u \in \Omega_\alpha(\epsilon)$. Then there exists $u^* \in F(u)$ such that

$$\{u^* + Tu - f(v - u) + g(u, v) + f^*(u; v - u) \geq -\epsilon \alpha(v - u), \quad \forall v \in K.$$

According to the monotonicity of $F$, we obtain

$$\{v^* + Tu - f(v - u) + g(u, v) + f^*(u; v - u) \geq -\epsilon \alpha(v - u), \quad \forall v \in K, \forall v^* \in F(v),$$

which means that $u \in \Phi_\alpha(\epsilon)$. Therefore $\Omega_\alpha(\epsilon) \subset \Phi_\alpha(\epsilon)$.

Now we show that $\Phi_\alpha(\epsilon) \subset \Omega_\alpha(\epsilon)$. Indeed, take arbitrary $u \in \Phi_\alpha(\epsilon)$. Then

$$\{v^* + Tu - f(v - u) + g(u, v) + f^*(u; v - u) \geq -\epsilon \alpha(v - u), \quad \forall v \in K, \forall v^* \in F(v).$$
Since the set $K$ is convex, for any $v \in K$ and $\lambda \in [0, 1]$, taking $v_1 := \lambda v + (1 - \lambda)u \in K$ in this inequality, we have

$$\left\{ v_1^* + Tu - f, v_2 - u \right\} + g(u, v_1) + f^*(u; v_2 - u) \geq -\epsilon \alpha(v_2 - u), \quad \forall v_1^* \in F(v_2).$$

Then by (iii), (iv), and the properties of $J^*$ we obtain

$$\left\{ v_2^* + Tu - f, v - u \right\} + g(u, v) + f^*(u; v - u) \geq -\epsilon \alpha(v - u), \quad \forall v_2^* \in F(v). \tag{3.1}$$

Let $u^* \in F(u)$ be fixed, and let $v_2^* \in F(v)$ be such that $v_2^* \rightharpoonup u^*$ in $X^*$ (the existence of such a sequence is ensured by the fact that $F$ is l.h.c.). Taking the limit as $\lambda \to 0$ in (3.1), we obtain

$$\left\{ u^* + Tu - f, v - u \right\} + g(u, v) + f^*(u; v - u) = \lim_{\lambda \to 0} \left[ \left\{ v_2^* + Tu - f, v - u \right\} + g(u, v) + f^*(u; v - u) \right] \geq -\epsilon \alpha(v - u),$$

which implies that $u \in \Omega_\alpha(\epsilon)$. The proof is complete. \qed

The following result is a consequence of Lemmas 3.1 and 3.2.

**Theorem 3.3** Assume that:

(i) $K$ is a nonempty closed convex subset of a real Banach space $X$;

(ii) $F : K \to P(X^*)$ is l.h.c. and monotone;

(iii) $T : K \to X_0^*$ is continuous;

(iv) $g : K \times K \to R$ is u.s.c. with respect to the first variable and convex with respect to the second variable;

(v) $\alpha : X \to R_+$ is continuous and convex with $\alpha(tv) = t\alpha(v)$ for all $t \geq 0$ and $v \in X$.

Then $\Omega_\alpha(\epsilon) = \Phi_\alpha(\epsilon)$ is closed in $X$ for all $\epsilon > 0$. Moreover, $\Gamma = \Omega_\alpha(0) = \Phi_\alpha(0)$, that is, GVHVI is equivalent to the following problem:

Find $u \in K$ such that

$$\left\{ v^* + Tu - f, v - u \right\} + g(u, v) + f^*(u; v - u) \geq 0, \quad \forall v \in K, v^* \in F(v).$$

**Theorem 3.4** GVHVI is strongly $\alpha$-well-posed if and only if $\Gamma$ is nonempty and

$$\lim_{\epsilon \to 0} \text{diam}(\Omega_\alpha(\epsilon)) = 0.$$

**Proof** The proof is similar to that of Theorem 4.3 in [26] by the assumptions of $g$. \qed

**Theorem 3.5** Assume that all the assumptions of Theorem 3.3 are satisfied. Then GVHVI is strongly $\alpha$-well-posed if and only if

$$\Omega_\alpha(\epsilon) \neq \emptyset \quad \forall \epsilon \geq 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \text{diam}(\Omega_\alpha(\epsilon)) = 0. \tag{3.2}$$
By the assumptions we obtain that $t \to 0$ as $\epsilon \to 0$, which is a contradiction.

Suppose that $\text{GVHVI}$ is strongly $\alpha$-well-posed. Then $\text{GVHVI}$ has a unique solution $u \in K$, and thus $\Gamma \not= \emptyset$. Now, we prove that (3.2) holds. Clearly, $\Omega_\epsilon(\epsilon) \supset \Gamma \not= \emptyset$. For the second part of (3.2), arguing by contradiction, let us assume that $u \in K$ for $\text{GVHVI}$. Then, there exist $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ satisfying $u_n^{(1)} \in K$ and $u_n^{(2)} \in \Omega_\epsilon(\epsilon_n)$ approximating sequences of $\text{GVHVI}$, and thus

$$
\|u_n^{(1)} - u_n^{(2)}\| > \beta > 0.
$$

From (3.3) and (3.4) we have

$$
0 < \beta < \|u_n^{(1)} - u_n^{(2)}\| \leq \|u_n^{(1)} - u\| + \|u_n^{(2)} - u\| \to 0,
$$

which is a contradiction.

Conversely, assume that condition (3.2) holds. Let $\{u_n\}$ in $K$ be an $\alpha$-approximating sequence for $\text{GVHVI}$. Then, there exist $\{u_n^*\}$ in $X^*$ with $u_n^* \in F(u_n)$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that, for every $n \in N$,

$$
\{u_n^* + Tu_n, v - u_n\} \geq \epsilon_n \alpha(v - u_n), \quad \forall v \in K,
$$

that is, $u_n \in \Omega_\epsilon(\epsilon_n)$ for all $n \in N$. By condition (3.2) we deduce that the sequence $\{u_n\}$ is a Cauchy sequence, and so $\{u_n\}$ converges strongly to some point $u \in K$. Let us show that $u \in K$ is a solution for $\text{GVHVI}$. By the monotonicity of $F$ we obtain that, for every $n \in N$,

$$
\{v^* + Tu_n - f, v - u_n\} \geq \epsilon_n \alpha(v - u_n), \quad \forall v \in K, v^* \in F(v).
$$

By the assumptions we obtain that

$$
\{v^* + Tu - f, v - u\} \geq \epsilon_n \alpha(v - u_n)
$$

which implies that

$$
\{v^* + Tu - f, v - u\} + g(u, v) + f^*(u; v - u) \geq 0, \quad \forall v \in K, \forall v^* \in F(v).
$$
It follows from Theorem 3.3 that there exists \( u^* \in F(u) \) such that
\[
\left\{ u^* + Tu - f, v - u \right\} + g(u, v) + f^*(u; v - u) \geq 0, \quad \forall v \in K.
\]
Then \( u \in K \) is a solution of GVHVI.

Finally, we prove that the solution \( u \) is unique. If there exists another solution \( u' \in K \), then \( u, u_1 \in \Omega_\alpha(\epsilon) \) for all \( \epsilon > 0 \), and
\[
0 < \|u - u'\| \leq \text{diam}(\Omega_\alpha(\epsilon)) \to 0 \quad \text{as} \ \epsilon \to 0,
\]
which is a contradiction. This completes the proof. \( \square \)

**Theorem 3.6** Assume that:

(i) \( K \) is a nonempty closed convex subset of a real reflexive Banach space \( X \);
(ii) \( F : K \to P(X^*) \) is l.h.c. and monotone;
(iii) \( T : K \to X^* \) is compact;
(iv) \( g : K \times K \to R \) is weakly u.s.c. with respect to the first variable and convex with respect to the second variable;
(v) \( \limsup_{n \to \infty} J^*(u_n; v - u_n) \leq J^*(u; v - u) \) for all \( v \in X \) whenever \( u_n \to u \) as \( n \to \infty \);
(vi) \( \alpha : X \to R_+ \) is a continuous and convex functional with \( \alpha(tv) = t\alpha(v) \) for all \( t \geq 0 \) and \( v \in X \).

Then GVHVI is weakly \( \alpha \)-well-posed if and only if GVHVI has a unique solution and there exists \( \epsilon_0 > 0 \) such that \( \Omega_\alpha(\epsilon_0) \) is nonempty and bounded.

**Proof** The necessity is obvious. We now prove the sufficiency. Let \( \{u_n\} \) be an \( \alpha \)-approximating sequence for GVHVI. Then, there exist \( \{u_n^*\} \) in \( X^* \) with \( u_n^* \in F(u_n) \) and a nonnegative sequence \( \{\epsilon_n\} \) with \( \epsilon_n \to 0 \) as \( n \to \infty \) such that, for every \( n \in N \),
\[
\left\{ u_n^* + Tu_n - f, v - u_n \right\} + g(u_n, v) + f^*(u_n; v - u_n) \geq -\epsilon_n\alpha(v - u_n)
\]
for all \( v \in K \). We claim that the sequence \( \{u_n\} \) is bounded in \( X \). Indeed, since \( \Omega_\alpha(\epsilon_0) \) is bounded and \( \Omega_\alpha(\epsilon) \subset \Omega_\alpha(\epsilon_0) \) for all \( \epsilon \in (0, \epsilon_0) \), there exists \( n_0 \in N \) such that \( \epsilon_{n_0} \in (0, \epsilon_0) \) and \( u_n \in \Omega_\alpha(\epsilon_0) \) for all \( n \geq n_0 \), which shows that \( \{u_n\} \) is bounded in \( X \).

Since the Banach space \( X \) is reflexive, we can choose a subsequence of \( \{u_n\} \), denoted by \( \{u_{n_k}\} \), such that \( u_{n_k} \to \overline{u} \) as \( k \to \infty \) for some \( \overline{u} \in X \). Let us show that \( \overline{u} \in K \) is a solution for GVHVI. Obviously, \( \overline{u} \in K \). By the monotonicity of \( F \) we obtain that
\[
\left\{ v^* + Tu_n - f, v - u_n \right\} + g(u_n, v) + f^*(u_n; v - u_n) \\
\geq \left\{ u_{n_k}^* + Tu_{n_k} - f, v - u_{n_k} \right\} + g(u_{n_k}, v) + f^*(u_{n_k}; v - u_{n_k}) \\
\geq -\epsilon_{n_k}\alpha(v - u_{n_k}), \quad \forall v \in K, v^* \in F(v), \forall n \in N.
\]
By the assumptions, we obtain that
\[
\left\{ v^* + T\overline{u} - f, v - \overline{u} \right\} + g(\overline{u}, v) + f^*(\overline{u}; v - \overline{u}) \\
\geq \limsup_{n \to \infty} \left\{ v^* + Tu_n - f, v - u_n \right\} + g(u_n, v) + f^*(u_n; v - u_n) \\
\geq \limsup_{n \to \infty}\left[ v^* + Tu_n - f, v - u_n \right] + g(u_n, v) + f^*(u_n; v - u_n)
\]
\[
\geq \limsup_{n \to \infty} -\varepsilon_n \alpha(v - u_n)
= \limsup_{n \to \infty} \alpha(-\varepsilon_n (v - u_n))
= 0,
\]

which implies that

\[
(v^* + T\overline{n} - f, v - \overline{n}) + g(\overline{n}, v) + f^*(\overline{n}; v - \overline{n}) \geq 0, \quad \forall v \in K, \forall v^* \in F(v).
\]

It follows from Theorem 3.3 that there exists \( \overline{n}^* \in F(\overline{n}) \) such that

\[
(u^* + T\overline{n} - f, v - \overline{n}) + g(\overline{n}, v) + f^*(\overline{n}; v - \overline{n}) \geq 0, \quad \forall v \in K,
\]

Therefore \( \overline{n} \in K \) is a solution to problem GVHVI, and so we get that GVHVI is weakly \( \alpha \)-well-posed by the uniqueness of the solution to problem GVHVI. This completes the proof.

Remark 3.7 In the theorem, condition (v) can be found in [30], and the condition that there exists \( \varepsilon_0 > 0 \) such that \( \Omega_\alpha(\varepsilon_0) \) is nonempty and bounded can be replaced by the conditions that \( K \) is bounded or that there exists \( n_0 \in \mathbb{N} \) such that, for every \( u \in K \setminus B_{n_0} \), there exists \( v \in K \) with \( \|v\| < \|u\| \) such that

\[
\sup_{u^* \in F(u)} (u^* + Tu - f, v - u) + g(u, v) + f^*(u; v - u) \leq -\frac{1}{n_0}.
\]

See [34, 36, 39] for more detail.

Next, we give some equivalence results for the strong \( \alpha \)-posedness in the generalized sense.

Theorem 3.8 Assume that all the assumptions of Theorem 3.5 are satisfied. Then GVHVI is strongly \( \alpha \)-well-posed in the generalized sense if and only if \( \Gamma \) is nonempty compact and

\[
\lim_{\varepsilon \to 0} e(\Omega_\alpha(\varepsilon), \Gamma) = 0,
\]

where \( e(A, B) := \sup_{a \in A} d(a, B) \) with \( d(a, B) := \inf_{b \in B} \|a - b\| \).

Proof The proof is similar to that of Theorem 5.1 in [26] by the assumptions of \( g \).

Theorem 3.9 Assume that all the assumptions of Theorem 3.5 are satisfied. Then GVHVI is strongly \( \alpha \)-well-posed in the generalized sense if and only if

\[
\Omega_\alpha(\varepsilon) \neq \emptyset, \quad \forall \varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon \to 0} \mu(\Omega_\alpha(\varepsilon)) = 0.
\]

Proof The proof is similar to that of Theorem 3.2 in [3] by the assumptions of \( g \).
Theorem 3.10 Assume that all the assumptions of Theorem 3.6 are satisfied. Then $GVHVI$ is weakly $\alpha$-well-posed in the generalized sense if and only if there exists $\epsilon_0 > 0$ such that $\Omega_\alpha(\epsilon_0)$ is nonempty and bounded.

Proof The proof is similar to that of Theorem 3.6 by the assumptions of $g$. $\square$

4 Well-posedness for GMEP

In this section, we consider the following generalized mixed equilibrium problem (GMEP):

Find $u \in K$ such that, for some $u^* \in F(u)$,

$$[u^*, \eta(u, v)] + \langle Tu - f, v - u \rangle + g(u, v) + h(u, v) \geq 0, \quad \forall v \in K,$$

where $\eta : K \times K \to X$ is an operator. The existence of solutions to this problem when $T \equiv 0$ and $f \equiv 0$ can be found in [25].

To study GMEP, we introduce the concept of $\eta$-monotonicity (see [7, 8]).

Definition 4.1 Let $F : K \to P(X^*)$ be a set-valued operator. $F$ is said to be $\eta$-monotone if there exists a function $\eta : K \times K \to X$ such that, for all $u, v \in K$,

$$[v^* - u^*, \eta(u, v)] \geq 0, \quad \forall u^* \in F(u), \forall v^* \in F(v). \quad (4.1)$$

Remark 4.2 If $\eta(u, v) = v - u$ for all $u, v \in X$, then (4.1) becomes

$$[v^* - u^*, v - u] \geq 0, \quad \forall u^* \in F(u), \forall v^* \in F(v),$$

that is, $F$ is monotone.

For any $\epsilon > 0$, we define the following two sets:

$$\Omega_{\eta, \alpha}(\epsilon) = \{ u \in K : \exists u^* \in F(u) \text{ such that } [u^*, \eta(u, v)] + \langle Tu - f, v - u \rangle + g(u, v) + h(u, v) \geq -\epsilon \alpha(v - u), \forall v \in K \}$$

and

$$\Phi_{\eta, \alpha}(\epsilon) = \{ u \in K : [v^*, \eta(u, v)] + \langle Tu - f, v - u \rangle + g(u, v) + h(u, v) \geq -\epsilon \alpha(v - u), \forall v \in K, \forall v^* \in F(v) \}.$$
Then $\Omega_{\eta,\alpha}(\epsilon) = \Phi_{\eta,\alpha}(\epsilon)$ is closed in $X$ for all $\epsilon > 0$. Moreover, $\Gamma_{\eta} = \Omega_{\eta,0}(\epsilon) = \Phi_{\eta,0}(\epsilon)$, that is, GMEP is equivalent to the following problem:

Find $u \in K$ such that

$$\langle v^* + Tu - f, \eta(u, v) \rangle + g(u, v) + h(u, v) \geq 0, \quad \forall v \in K, v^* \in F(v).$$

**Theorem 4.4** Assume that all the assumptions of Theorem 3.5 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u, u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to \mathbb{R}$ be such that:

(i) $h(u, u) = 0$ for all $u \in X$,

(ii) for all $v \in K$, $h(\cdot, v)$ is u.s.c.,

(iii) for all $u \in K$, $h(u, \cdot)$ is convex.

Then GMEP is strongly $\alpha$-well-posed if and only if

$$\Omega_{\eta,\alpha}(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \quad \text{and} \quad \lim_{\epsilon \to 0} \mu(\Omega_{\eta,\alpha}(\epsilon)) = 0.$$

**Theorem 4.5** Assume that all the assumptions of Theorem 3.6 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u, u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to \mathbb{R}$ be such that:

(i) $h(u, u) = 0$ for all $u \in X$,

(ii) for all $v \in K$, $h(\cdot, v)$ is weakly u.s.c.,

(iii) for all $u \in K$, $h(u, \cdot)$ is convex.

Then GMEP is weakly $\alpha$-well-posed if and only if GMEP has a unique solution and there exists $\epsilon_0 > 0$ such that $\Omega_{\eta}(\epsilon_0)$ is nonempty and bounded.

**Theorem 4.6** Assume that all the assumptions of Theorem 3.5 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u, u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to \mathbb{R}$ is such that:

(i) $h(u, u) = 0$ for all $u \in X$,

(ii) for all $v \in K$, $h(\cdot, v)$ is u.s.c.,

(iii) for all $u \in K$, $h(u, \cdot)$ is convex.

Then GMEP is strongly $\alpha$-well-posed in the generalized sense if and only if

$$\Omega_{\eta,\alpha}(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \quad \text{and} \quad \lim_{\epsilon \to 0} \mu(\Omega_{\eta,\alpha}(\epsilon)) = 0.$$

**Theorem 4.7** Assume that all the assumptions of Theorem 3.6 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u, u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to \mathbb{R}$ be such that:

(i) $h(u, u) = 0$ for all $u \in X$,

(ii) for all $v \in K$, $h(\cdot, v)$ is weakly u.s.c.,

(iii) for all $u \in K$, $h(u, \cdot)$ is convex.

Then GMEP is weakly $\alpha$-well-posed in the generalized sense if and only if there exists $\epsilon_0 > 0$ such that $\Omega_{\eta}(\epsilon_0)$ is nonempty and bounded.

5 Conclusion

In this paper, inspired by the previous works, we study the well-posedness for GVHVI. Under relatively weak conditions for the data $F$, $T$, $g$, $J$ (see Theorems 3.3 and 3.6), we
provide some equivalence results for the strong and weak $\alpha$-well-posed GVHVI in the
generalized sense. Our results generalize and improve many known results and can be
applied to many other problems.

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