Renormalization group analysis of resonantly interacting anyons

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(Dated: August 2007)

We formulate a field theory for resonantly interacting anyons, that enables us to perform a perturbative calculation near the fermionic limit. We derive renormalization group equations for three-body and four-body couplings at one-loop order. In addition to two fixed points, we find a limit cycle behavior in the four-body coupling, which implies an infinite set of bound states in the four-anyon system.

PACS numbers: 05.30.Pr, 11.10.Hi, 21.45.+v

I. INTRODUCTION

Physics in two-dimensional space has intriguing features that cannot be found in three dimensions. One such example is anyons, quantum particles having fractional statistics [1, 2]. The statistics of anyons interpolates continuously between usual fermion and boson cases. Anyons are not only theoretically interesting but also can be realized as quasiparticles in the fractional quantum Hall effect [3]. Recently it is argued that anyonic quasiparticles may be created using rotating Bose-Einstein condensates [4] or cold atoms confined in an optical lattice [5]. Furthermore anyons are attracting renewed interest in the context of quantum computation [6].

In this article, we investigate nonrelativistic anyons interacting via a contact interaction. Motivated by recent experimental realization of fermions and bosons at infinite scattering length, we focus on the case where the two-body interaction is tuned to the resonance at threshold, i.e., two anyons form a bound state with zero binding energy. Because of its scale invariance, such a system should be described by nonrelativistic conformal field theories (CFT). A field-theoretical description of resonantly interacting anyons known so far allows for perturbative calculations only near the bosonic limit [7, 8] [23].

Here we propose an alternative description that becomes perturbative near the fermionic limit and study a few-body problem of resonantly interacting anyons. For more than two anyons, additional three-body and four-body contact interactions turn out to be necessary to ensure the renormalizability of the theory. In the renormalization group (RG) equations of three-body and four-body couplings, we will find two nontrivial fixed points and a limit cycle behavior of the four-body coupling. Our finding implies the existence of an infinite set of discrete bound states in the four-anyon system, where the ratio of two successive binding energies is given by

\[
\frac{E_{n+1}}{E_n} \to \exp \left( -\frac{\pi}{3|\alpha|} \right) \tag{1}
\]

near the fermionic limit \( |\alpha| \ll 1 \).

II. ANYONS IN QUANTUM MECHANICS

Before going on to the field-theoretical formulation of anyons with the contact interaction, it is instructive to review two-anyon problem in quantum mechanics. Anyons are characterized by a phase \( e^{i\pi(\alpha+1)} \) acquired by the wave function under the exchange of two anyons. Without loss of generality, the statistics parameter \( \alpha \) is assumed to be in the interval \([-1, 1]\). For the later convenience, we chose \( \alpha \) so that \( \alpha = 0 \) corresponds to fermions and \( |\alpha| = 1 \) corresponds to bosons.

In the center of mass frame, the two-body wave function can be expressed as \( \Psi_\alpha(r) = e^{i\alpha \varphi} \Phi(r, \varphi) \), where \( r \) and \( \varphi \) are relative distance and angle between two anyons. \( \Phi(r, \varphi) \) is a fermionic wave function which is antisymmetric under \( \varphi \to \varphi + \pi \). Since \( \Psi_\alpha(r) \) obeys a free Schrödinger equation, the wave function expanded in partial waves \( \Phi(r, \varphi) = \sum_l e^{il\varphi} R_l(r) \) satisfies (below \( \hbar = 1 \) and anyon mass \( m = 1 \))

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(l + \alpha)^2}{r^2} + E \right] R_l(r) = 0 \tag{2}
\]

for \( r \neq 0 \), where the odd integer \( l \) is an orbital angular momentum.

The contact interaction between anyons can be introduced by allowing a singular solution that behaves as \( R_l(r) \sim r^{-|l+\alpha|} \) at origin \( r \to 0 \), i.e., when the positions of two anyons coincide [9–11]. Because of the square integrability of the wave function, the contact interaction is available only for \( l = -1 \) when \( \alpha > 0 \) or for \( l = 1 \) when \( \alpha < 0 \). In this channel, the contact interaction is equivalent to imposing the following boundary condition on the wave function at origin:

\[
R_{\pm 1}(r) \to \frac{1}{r^{1-|\alpha|}} - \text{sgn}(a) \left( \frac{r}{a} \right)^{1-|\alpha|} + O(r^{1+|\alpha|}), \tag{3}
\]

where \( a \) is a real parameter analogous to the scattering length in three spatial dimensions. There are two scale-independent boundary conditions; \( a \to 0 \) and \( a \to \infty \). The former corresponds to the usual non-colliding limit. Since the binding energy is given by \[10]\]

\[
E_b = -\frac{4}{a^2} \left[ \frac{\Gamma(2-|\alpha|)}{\Gamma(|\alpha|)} \right]^{1-|\alpha|}, \tag{4}
\]
when $a$ is positive, the resonant limit corresponds to $a \to \infty$.

An interesting observation is that the normalization integral of the wave function (3) is logarithmically divergent at $r \to 0$ in the fermionic limit $|\alpha| \to 0$. Therefore, if the statistics is close to fermion, the two-body wave function is concentrated near the origin, and the anyon pair forms a pointlike dimer. The same situation has been observed in two-component fermions at infinite scattering length near four spatial dimensions [12]. This observation has led to a field theory composed of weakly coupled fermions and dimers [13, 14]. Similarly, as we shall see below, it is possible to formulate a field theory describing resonantly interacting anyons, which allows for a perturbative analysis near the fermionic limit.

### III. FIELD-THEORETICAL FORMULATION

The field-theoretical representation of anyons is provided by a nonrelativistic field $\psi$ minimally coupled to a Chern-Simons (CS) gauge field $(a_0, a)$ [15, 16]:

$$\mathcal{L}_\psi = \frac{1}{4\pi\alpha} \partial_t a \times a - \frac{1}{2\pi\alpha} a_0 \nabla \times a - \frac{1}{2\zeta} (\nabla \cdot a)^2$$

$$+ \psi^* (i\partial_t - a_0 + \mu) \psi - \frac{1}{2} |(\nabla - i a)| \psi|^2.$$  \hspace{1cm} (5)

In our definition of $\alpha$, $\psi$ is a fermionic field. $\mu$ is the anyon chemical potential and here we consider the system at zero density $\mu = 0$. We shall work in the Coulomb gauge $\zeta = 0$.

Motivated by the above argument, it is natural to add the following terms involving the dimer field $\phi$ to (5):

$$\mathcal{L}_\phi = \phi^* (i\partial_t - 2a_0 - \epsilon_0) \phi - \frac{1}{4} |(\nabla - 2ia)| \phi|^2$$

$$+ g \phi^* \psi (-i\partial_x \pm \partial_y) \psi + g \psi^* (-i\partial_x \pm \partial_y) \psi \phi.$$  \hspace{1cm} (6)

$\phi$ is a bosonic field with mass $2m$ and has the orbital angular momentum $l = \mp 1$. The upper sign corresponds to $\alpha > 0$ and the lower sign corresponds to $\alpha < 0$ throughout this article. $\phi$ also couples to the CS gauge field in a gauge invariant way. $g$ is a dimensionless coupling between the dimer and a pair of anyons in $p$-wave. $\epsilon_0$ is a cutoff-dependent bare “mass gap”, which is fine-tuned so that the $\phi$ propagator has a pole at $p_0 = p = 0$, i.e.,

$$\epsilon_0 = 2g^2 \int_0^\Lambda \frac{dk}{(2\pi)^2} + O(\alpha g^2).$$  \hspace{1cm} (7)

This condition is equivalent to the fine-tuning to the resonant limit (zero binding energy).

The other marginal operators one can add to the Lagrangian density are three-body and four-body contact interactions:

$$\mathcal{L}_\delta = v_3 \phi^* \psi^* \phi + v_4 \phi^* \phi \phi.$$  \hspace{1cm} (8)

Note that terms as $\nabla \times a \psi^* \psi$ and $\nabla \times a \phi^* \phi$ are also allowed, but they can be absorbed by the redefinition of $v_3$ and $v_4$ if we use the CS Gauss law

$$\nabla \times a = -2\pi\alpha (\psi^* \psi + 2\phi^* \phi).$$  \hspace{1cm} (9)

Thus, the most general renormalizable Lagrangian density composed of $\psi$ and $\phi$ becomes $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\phi + \mathcal{L}_\delta$. First, we concentrate on the two-body sector, i.e., a renormalization of the two-body coupling $g$.

### IV. TWO-BODY ANOMALY

The renormalization of the theory can be performed in the standard way. There is a one-loop self-energy diagram for $\phi$ that is logarithmically divergent [Fig. 1(a)]. Integrating out modes in the momentum shell $e^{-s}\Lambda < k < \Lambda$, we obtain

$$\Sigma(p) = -\frac{g^2}{\pi} \left( p_0 - \frac{p^2}{4} \right) \ln \frac{\Lambda}{e^{-s}\Lambda}.$$  \hspace{1cm} (10)

which corresponds to the wave-function renormalization of $\phi$: $Z_\phi = 1 - (g^2/\pi)s$. The vertices of $\phi$ with $a_0$ or $a$ are also renormalized in the same way. Thus, the anomalous dimension of $\phi$ is found to be

$$\gamma_\phi = \frac{1}{2} \frac{\partial \ln Z_\phi}{\partial s} = \frac{g^2}{2\pi}.$$  \hspace{1cm} (11)

The vertex $g \phi^* \psi (-i\partial_x \pm \partial_y) \psi$ is renormalized by the logarithmically divergent one-loop diagram in Fig. 1(b), which is evaluated as

$$i|\alpha|g \left( (p - q)_x \pm i(p - q)_y \right) \ln \frac{\Lambda}{e^{-s}\Lambda}.$$  \hspace{1cm} (12)

As a result, a RG equation that governs the running of $g$ is given by

$$\frac{dg}{ds} = -g \gamma_\phi + |\alpha|g = -\frac{g^3}{2\pi} + |\alpha|g.$$  \hspace{1cm} (13)

There is a nontrivial fixed point located at

$$g_*^2 = 2\pi|\alpha|.$$  \hspace{1cm} (14)
At this fixed point, the theory is a nonrelativistic CFT describing anyons at the two-body resonance. Since $g^* \sim \alpha \ll 1$ in the fermionic limit, one can perform a perturbative analysis in terms of $\alpha$.

In order to confirm the connection of physics described by this fixed point with resonantly interacting anyons, we compute two physical quantities at $g = g^*$ using perturbative expansions in terms of $\alpha$ and compare them with exact results in quantum mechanics. First, the amplitude of two-anyon scattering at the tree level (Fig. 2) is given by

$$A(p, \varphi) = \frac{4i\pi\alpha}{\sin \varphi} - 4g^*e^{+i\varphi}$$

(15)

where $p$ is the magnitude of the relative momentum and $\varphi$ is the scattering angle. The exact scattering amplitude obtained by solving the Schrödinger equation (2) with the boundary condition (3) at $a \to \infty$ is

$$f_{\text{exact}}(p, \varphi) = \frac{\sin \pi\alpha e^{+2i\varphi}}{\sqrt{\pi p} \sin \varphi} \quad \text{for} \quad \varphi \neq 0, \pi.$$ (16)

Thus, up to a kinematical factor, the perturbative result (15) reproduces the correct expanded form of the exact result to the leading order in $\alpha$.

The ground state energy of two anyons in a harmonic potential can be directly extracted from the scaling dimension of the operator $\phi$ [17]. At the fixed point, the scaling dimension of $\phi$ becomes

$$\Delta_\phi = 1 + \gamma_\phi = 1 + |\alpha| + O(\alpha^2),$$ (17)

from which the ground state energy is given by $E = \Delta_\phi \omega$ with $\omega$ being the oscillator frequency. This result coincides with the exact result obtained in quantum mechanics,

$$E_{\text{exact}} = (1 + |\alpha|) \omega,$$ (18)

including the energy of center of mass motion. The corresponding relative wave function is

$$\Phi(r, \varphi) = e^{+ir\varphi}r^{-1-|\alpha|}e^{-\omega r^2/4},$$ (19)

which satisfies the boundary condition (3) with $a \to \infty$.

These results establish that the fixed point (14) describes the physics of anyons with the two-body contact interaction tuned to the resonance.

V. RG EQUATIONS FOR $v_3$ AND $v_4$

Now we study the renormalization of the three-body and four-body couplings $v_3$ and $v_4$. First, we shall look at the one-loop diagram in Fig. 3. Integrating out modes in the momentum shell $e^{-s} \Lambda < k < \Lambda$, we obtain

$$i\frac{g^2}{2\pi} [(p + q)_i + i\epsilon^j(p - q)_j] \ln \frac{\Lambda}{e^{-s}\Lambda}.$$ (20)

The first term renormalizes the vertex $-ia \cdot \phi^* \nabla \phi/2$ and is consistent with the wave-function renormalization of $\phi$. The second term generates the following vertex in the Lagrangian density:

$$\delta L = \mp g^2 s \nabla \cdot a \phi^* \phi.$$ (21)

Using the CS Gauss law (9), the generated vertex $\delta L$ is converted into the renormalization of the couplings $v_3$ and $v_4$.

The other one-loop diagrams that contribute to the renormalization of $v_3$ and $v_4$ are depicted in Figs. 4 and 5, respectively. Taking into account the contribution of the wave-function renormalization of $\phi$, RG equations that govern the running of $v_3$ and $v_4$ are given by

$$\frac{dv_3}{ds} = \frac{20}{3}\alpha^2 - \frac{22}{3}|\alpha|v_3 + \frac{2}{3\pi}v_4^3,$$ (22)

$$\frac{dv_4}{ds} = -20\alpha^2 + 4|\alpha|v_3 - 4|\alpha|v_4 + \frac{2}{\pi}v_4^2.$$ (23)

Here $g = g^*$ was substituted. We note that the $\beta$ function of $v_3$ ($v_4$) is a quadratic function of $v_3$ ($v_4$) to all orders. Higher order corrections appear only in their coefficients and do not alter the following discussions as far as the perturbative expansion is valid $|\alpha| \ll 1$.

Equation (22) has two fixed points

$$v_3^* = \pi|\alpha| \quad \text{and} \quad v_4^* = 10\pi|\alpha|,$$ (24)

each of which corresponds to a scale-independent short range boundary condition on the three-body wave function. The stable fixed point ($v_3^* = \pi|\alpha|$) corresponds to
the regular wave function without the three-body resonance, while the unstable one \( (v_3^* = 10|\alpha|) \) corresponds to the three-body resonance at threshold. When \( v_3 \) is at the former fixed point \( v_3^* = \pi|\alpha| \), Eq. (23) also has stable and unstable fixed points at

\[
v_4^* = -2\pi|\alpha| \quad \text{and} \quad v_4^* = 4\pi|\alpha|.
\]

(25)

The unstable fixed point \( v_4^* = 4\pi|\alpha| \) corresponds to the four-body resonance at threshold. However, when \( v_3 \) is at the other fixed point corresponding to the three-body resonance, Eq. (23) does not have a fixed point. In this case, the solution of the coupled equations (22) and (23) is

\[
\begin{align*}
    v_3(s) &= 10\pi|\alpha|, \\
    v_4(s) &= \pi|\alpha| + 3\pi|\alpha| \tan(6\alpha|s + C|),
\end{align*}
\]

(26)

where \( C \) is a constant of integration. We thus find a RG limit cycle where \( v_3(s) \) is a constant and \( v_4(s) \) is a periodic function of \( s \) with a primitive period \( \pi/(6|\alpha|) \). The system tuned to this limit cycle has a discrete scaling symmetry with a scaling factor \( \exp[\pi/(6|\alpha|)] \) in momentum. It especially implies a geometric spectrum of energy eigenvalues [18] in the four-anyon system when two-body and three-body interactions are simultaneously tuned to the resonance. When the statistics is close to fermion \( |\alpha| \ll 1 \), the ratio of two successive energy eigenvalues is given by

\[
E_{n+1}/E_n \rightarrow \exp\left[-\frac{\pi}{3|\alpha| + O(\alpha^2)}\right].
\]

(27)

There are an infinite number of bound states with an accumulation point at zero energy. This resembles Efimov states in three-body systems at infinite scattering length in three spatial dimensions [19–21], whose experimental evidence has been recently reported using cold atoms in Ref. [22].

Near the bosonic limit, on the other hand, the Efimov states do not exist in a few-anyon system. This is because the two-body resonant interaction becomes just zero interaction in the bosonic limit. Therefore, there should be at least one critical value of \( \alpha \) where the four-anyon bound states disappear. Determination of such a critical \( \alpha \) is, however, beyond the scope of our perturbative approach.

The RG flow diagram obtained from Eqs. (22) and (23) is shown in Fig. 6. We see two fixed points located at

\[
(v_3^*, v_4^*) = (\pi|\alpha|, -2\pi|\alpha|) \quad \text{and} \quad (\pi|\alpha|, 4\pi|\alpha|),
\]

(28)

and the limit cycle given by the solution (26).

VI. SCALING DIMENSIONS OF OPERATORS

It is worthwhile to determine scaling dimensions of \( N \)-body operators at the fixed points because they are observable as energy eigenvalues in a harmonic potential [17]. At the one-loop level, anomalous dimensions of three-body and four-body composite operators \( \phi \psi \) and \( \phi \phi \) are, respectively, computed as

\[
\begin{align*}
\gamma_{\phi \psi} &= \frac{8|\alpha|}{3} - \frac{2v_3^*}{3\pi} + O(\alpha^2), \\
\gamma_{\phi \phi} &= -\frac{2v_4^*}{\pi} + O(\alpha^2).
\end{align*}
\]

(29)

From those, we obtain the scaling dimension of the lowest \((2n + 1)\)-body operator \( \phi^n \psi \) as

\[
\Delta_{\phi^n \psi} = n + 1 + n\gamma_{\phi} + n\gamma_{\phi \psi}
\]

(30)

and that of the lowest \((2n)\)-body operator \( \phi^n \) as

\[
\Delta_{\phi^n} = n + n\gamma_{\phi} + \frac{n(n - 1)}{2}\gamma_{\phi \phi}
\]

(31)

up to the order \( \alpha \). Then the ground state energy of \( N \) anyons in a harmonic potential is given by the scaling...
dimension of the $N$-body operator times $\omega$. The leading-order results can be easily understood by recalling that, in the fermionic limit $|\alpha| \rightarrow 0$, fermion pairs at the $p$-wave resonance form point-like bosons and they do not interact with each other or with extra fermions. Since these particles can occupy the same state, the ground state energy is simply given by the number of particles multiplied by the lowest single particle energy $\omega$ in a harmonic potential.

VII. MANY-BODY PHYSICS

An interesting extension of our work is to the many-body system of anyons. Here we discuss some qualitative features of such a system. When the two-body resonant interaction is present, we have choices to further introduce the three-body and four-body resonant interactions. If the three-body resonant interaction is introduced, the Efimov effect occurs in the four-anyon system (the limit cycle in Fig. 6) and accordingly the corresponding many-body system can not be stable. Instead, if the four-body resonant interaction is introduced (the upper fixed point in Fig. 6), the corresponding many-body system will be unstable because of the attractive interaction between bosonic dimers. Therefore the only stable many-body system seems to be the case where the two-body resonant interaction is present without the three-body and four-body resonant interactions (the lower fixed point in Fig. 6).

In this case, we define $\xi$ as a ratio of the ground state energy $E$ to that of “noninteracting” anyons $E_0$ at the same density and statistics; $\xi = E/E_0|_{n,\alpha}$. Since the system has no intrinsic dimensionful quantity, $\xi$ can depend only on the statistics parameter $\alpha$. Although it is a difficult many-body problem to determine $\xi$ in general $\alpha$, the problem becomes treatable near the fermionic limit. In the fermionic limit, the ground state energy $E$ vanishes because resonantly interacting anyons form non-interacting point-like bosons, while $E_0$ remains nonzero because of the Fermi energy. Therefore we have $\xi \rightarrow 0$ in the fermionic limit $\alpha \rightarrow 0$. Corrections to $\xi$ near the fermionic limit are computable using the perturbative framework developed in this article. It will be important in future works to further study many-body properties of resonantly interacting anyons, for example, whether they exhibit superfluidity, and their implications for real planar systems.

VIII. CONCLUSION

We have investigated anyons with a contact interaction by formulating a field theory that becomes perturbative near the fermionic limit. We identified a fixed point that describes anyons at the two-body resonance. In RG equations for three-body and four-body couplings, we found two nontrivial fixed points and computed scaling dimensions of $N$-body operators which are observable as a ground state energy of $N$ anyons in a harmonic potential.

Furthermore, we found a limit cycle behavior in the four-body coupling, which implies an infinite set of discrete bound states (Efimov states) in the four-anyon system when both the two-body and three-body interactions are tuned to the resonance. Such a system provides a unique opportunity to study Efimov-type physics within a perturbative framework. It should be possible to verify these results by solving a few-body Schrödinger equation for anyons with suitable short range boundary conditions and hopefully by experiments in the future. We believe this work opens up new universal physics in the system of anyons in two spatial dimensions.

Acknowledgments

The author thanks D. T. Son for useful discussions. This work was supported by JSPS Postdoctoral Program for Research Abroad.

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