A duality theorem for ergodic actions of compact quantum groups on $C^*$–algebras

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Abstract

The spectral functor of an ergodic action of a compact quantum group $G$ on a unital $C^*$–algebra is quasitensor, in the sense that the tensor product of two spectral subspaces is isometrically contained in the spectral subspace of the tensor product representation, and the inclusion maps satisfy natural properties. We show that any quasitensor $*$–functor from $\text{Rep}(G)$ to the category of Hilbert spaces is the spectral functor of an ergodic action of $G$ on a unital $C^*$–algebra.

As an application, we associate an ergodic $G$–action on a unital $C^*$–algebra to an inclusion of $\text{Rep}(G)$ into an abstract tensor $C^*$–category $\mathcal{T}$.

If the inclusion arises from a quantum subgroup $K$ of $G$, the associated $G$–system is just the quotient space $K \backslash G$. If $G$ is a group and $\mathcal{T}$ has permutation symmetry, the associated $G$–system is commutative, and therefore isomorphic to the classical quotient space by a subgroup of $G$.

If a tensor $C^*$–category has a Hecke symmetry making an object $\rho$ of dimension $d$ and $\mu$–determinant 1 then there is an ergodic action of $S_\mu U(d)$ on a unital $C^*$–algebra having the $(\iota, \rho)$ as its spectral subspaces. The special case of $S_\mu U(2)$ is discussed.

1 Introduction

A theorem in [6] asserts that any abstract tensor $C^*$–category with conjugates and permutation symmetry is the representation category of a unique compact group, thus generalizing the classical Tannaka–Krein duality theorem, where one starts from a subcategory of the category of Hilbert spaces (see, e.g., [14]).

The content of this paper fits into the program of generalizing the abstract duality theorem of [6] to tensor $C^*$–categories without permutation symmetry.
1 INTRODUCTION

Our interest in this problem is motivated by the fact that tensor $C^*$-categories with conjugates, but also with a unitary symmetry of the braid group, arise from low dimensional QFT [10].

In [25], [26] and [27] Woronowicz introduced compact quantum groups, and generalized the classical representation theory of compact groups. He proved a Tannaka–Krein duality theorem, asserting that the representation categories of compact quantum groups are precisely the tensor $^*$-subcategories of categories of Hilbert spaces where every object has a conjugate. This theorem allowed him to construct the quantum deformations $S_\mu U(d)$ of the classical $SU(d)$ groups by real a parameter $\mu$.

Therefore if a tensor $C^*$-category $\mathcal{T}$ with conjugates can be embedded into a category of Hilbert spaces, then $\mathcal{T}$ is necessarily the representation category of a compact quantum group.

In [17] the first named author characterized the representation category of $S_\mu U(d)$ among braided tensor $C^*$-categories $\mathcal{T}$ with conjugates. It follows that if an object $\rho$ has a symmetry of the Hecke algebra type $H_\infty(\mu^2)$ making $\rho$ of dimension $d$ and with $\mu$-determinant one, then there is a faithful tensor $^*$-functor $\text{Rep}(S_\mu U(d)) \to \mathcal{T}$.

The notion of subgroup of a compact quantum group $G$ was given by Podles in [20], who computed all the subgroups of the quantum $SU(2)$ and $SO(3)$ groups. In the same paper the author introduced quantum quotient spaces, and he showed that these spaces have an action of $G$ which splits into the direct sum of irreducibles, with multiplicity bounded above by the Hilbert space dimension.

Later Wang proved in [23] that compact quantum group actions on quotient spaces are ergodic, as in the classical case, and he found an example of an ergodic action on a commutative $C^*$-algebra which is not a quotient action.

It turns out that a compact quantum subgroup $K$ of $G$ gives rise to an inclusion of tensor $C^*$-categories $\text{Rep}(G) \to \text{Rep}(K)$, and that, by Tannaka–Krein duality, every tensor $^*$-inclusion of $\text{Rep}(G)$ into a subcategory of the category of Hilbert spaces, taking a representation $u$ to its Hilbert space and acting trivially on the arrows, is of this form.

In the case where a tensor $^*$-inclusion $\rho : \text{Rep}(G) \to \mathcal{T}$ into an abstract tensor $C^*$-category is given, it is natural to look for a tensor $^*$-functor of $\mathcal{T}$ into the category of Hilbert spaces, acting as the embedding functor on $\text{Rep}(G)$. This amounts to asking whether $\rho$ arises as an inclusion associated with a quantum subgroup of $G$.

The aim of this paper is twofold. Assume that we have a tensor $^*$-inclusion $\rho : \text{Rep}(G) \to \mathcal{T}$. The first result is the construction of a unital $C^*$-algebra $\mathcal{B}$ associated with the inclusion, and an ergodic action of $G$ on $\mathcal{B}$ whose spectral subspaces are the spaces $\langle \iota, \rho_u \rangle$, where $u$ varies in the set of unitary irreducible representations of $G$. The relevance of this construction to abstract duality for compact quantum groups will be discussed elsewhere [7]. We exhibit two interesting particular cases of this construction.

If $\mathcal{T} = \text{Rep}(K)$, with $K$ a compact quantum subgroup of $G$, the ergodic system thus obtained is just the quotient space $K \backslash G$ (Theorem 11.1). If instead $G$ is a group and $\mathcal{T}$ has a permutation symmetry, then $\mathcal{B}$ turns out to be
1 INTRODUCTION

commutative, and can therefore be identified with the $C^*$–algebra of continuous functions over a quotient of $G$ by a point stabilizer subgroup (Theorem 10.2).

We apply our construction to the case where an abstract tensor $C^*$–category $T$ has an object of dimension $d$ and $\mu$–determinant one and we find ergodic $C^*$–actions of $S_\mu U(d)$ (Theorem 10.3). We also discuss the particular case where $d = 2$ (Cor. 10.5).

Our second aim is to characterize, among all ergodic actions, those which are isomorphic to the quotient spaces by some quantum subgroup. By a well known theorem by Høegh–Krohn, Landstad and Størmer [11], an irreducible representation of a compact group $G$ appears in the spectrum of an ergodic action of $G$ on a unital $C^*$–algebra with multiplicity bounded above by its dimension.

As a generalization to compact quantum groups, Boca shows that the multiplicity of a unitary irreducible representation of the quantum group in the action is, instead, bounded above by its quantum dimension [3].

In [2], Bichon, De Rijdt and Vaes construct examples of ergodic actions of $S_\mu U(2)$ in which the multiplicity of the fundamental representation can be any integer $n$ with $2 \leq n \leq \mu + \frac{1}{\mu}$. Therefore these actions are not on quotient spaces of $S_\mu U(2)$. They also give a simpler proof of Boca’s result by introducing a new invariant, the quantum multiplicity of a spectral representation, which they show to be bounded below by the multiplicity and above by the quantum dimension.

The main tool for constructing their examples is a duality theorem, proved in that paper, between ergodic quantum actions for which the quantum multiplicity equals the quantum dimension, and certain maps associating to any intertwining operator between tensor products of irreducible representations, a linear map between the tensor products of the corresponding associated Hilbert spaces, respecting composition, tensor products and the $\ast$–involution.

In our generalization to the case where the quantum multiplicity is not maximal, our main tool is the spectral functor associated with a generic ergodic action (Sect. 7). This is the covariant $\ast$–functor that associates to any unitary, finite dimensional representation $u$ of $G$, the dual space $L_u$ of the space all multipllets in $B$ transforming like $u$ under the action $\eta$. The space $L_u$, by ergodicity, is known to become a Hilbert space in a natural way.

We stress that the functor $L$ satisfies two crucial properties: the first one is that $L_{u \otimes v}$ naturally contains a copy of $L_u \otimes L_v$, in such a way that the copy of $L_u \otimes L_v \otimes L_z$ contained in both $L_{u \otimes v \otimes z}$ and $L_u \otimes L_v \otimes L_z$ is the same. The second property is that the projection from $L_{u \otimes v \otimes z}$ onto $L_u \otimes L_v \otimes L_z$ actually takes $L_u \otimes L_v \otimes L_z$ onto $L_u \otimes L_v \otimes L_z$ (Theorem 7.3).

We call any functor $\mathcal{F}$ from a generic tensor $C^*$–category $T$ to the category of Hilbert spaces satisfying the above properties, quasitensor (Sect. 3). We show that quasitensor functors, like the tensor ones, have the property that if $\mathcal{F}(\rho)$ is a conjugate of $\rho$ then the Hilbert space $\mathcal{F}(\mathcal{F}(\rho))$ must be a conjugate of $\mathcal{F}(\rho)$, although this conjugate must be found in the image category of $\mathcal{F}$ enriched with the projection maps from the spaces $\mathcal{F}(\rho \otimes \sigma)$ onto $\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma)$ (Theorem 3.7).
This result easily shows that $\mathcal{F}(\rho)$ is automatically finite dimensional and endowed with an *intrinsic dimension*, in the sense of [15], bounded below by the Hilbert space dimension of $\mathcal{F}(\rho)$ and above by the intrinsic dimension of $\rho$ (Cor. 3.8).

This result applied to the spectral functor $\mathcal{L}$ allows us to identify the quantum multiplicity of a spectral irreducible representation of an ergodic action of $\mathcal{L}$, with the intrinsic dimension of $\mathcal{L}_u$, and to recover the multiplicity bound theorems of [3] and [2]. The maximal quantum multiplicity case corresponds to the case where for any irreducible $u$ with conjugate $\pi$, $\mathcal{L}_\pi$ is already a conjugate of $\mathcal{L}_u$ in the image of $\mathcal{L}$ (see Cor. 7.5 for a precise statement).

The spectral functor determines uniquely the $^{\ast}$–algebra structure of the dense $^{\ast}$–subalgebra of spectral elements. Our main result is a duality theorem for ergodic $C^*$–actions of compact quantum groups, showing that any quasitensor $^{\ast}$–functor $\mathcal{F}$ from the representation category of a compact quantum group $G$ to the category of Hilbert spaces is the spectral functor of an ergodic $G$–action over a unital $C^*$–algebra (Theorem 9.1).

Isomorphisms between two constructed ergodic systems correspond bijectively to unitary natural transformations between the corresponding functors splitting as tensor products over a tensor product subspace (Prop. 9.4).

Our main application of the Duality Theorem 9.1 is to inclusions of tensor $C^*$–categories. In fact, quasitensor $^{\ast}$–functors defined on the representation category of a compact quantum group $G$ arise very naturally from tensor $^{\ast}$–functors $\rho : \text{Rep}(G) \rightarrow \mathcal{F}$: just take the map associating with a unitary $G$–representation $u$ the Hilbert space $(\iota, \rho_u)$, and with an intertwiner $T \in (u, v)$ between two representations, the map acting on $(\iota, \rho_u)$ as left composition with $\rho(T)$ (Example 3.5). Therefore for any tensor $^{\ast}$–functor $\rho : \text{Rep}(G) \rightarrow \mathcal{F}$, our duality theorem provides us with an ergodic $G$–action over a unital $C^*$–algebra, having the spaces $(\iota, \rho_u)$ as its spectral subspaces (Theorem 10.1).

Among quasitensor functors, those arising from quantum quotient spaces share the property of being subobjects of the embedding functor $H$ associating to any representation $u$ its Hilbert space $H_u$. In fact, for these functors, one can find a natural unitary transformation identifying the spectral subspace $\mathcal{L}_u$ with the subspace $K_u$ of $H_u$ of all vectors invariant under the restriction of $u$ to the subgroup (Theorem 7.7). For maximal compact quantum groups, we characterize algebraically the spaces of invariant vectors $K_u$ for a unique maximal subgroup $K$ (Theorem 5.5).

From this we derive our second main result. In order that a maximal ergodic action $(\mathcal{B}, \eta)$ be isomorphic to a compact quantum quotient space, it is necessary and sufficient that there exists, for any representation $u$ of $G$, an isometry from the spectral subspace $\mathcal{L}_u$ onto some subspace $K_u$ of the representation Hilbert space, satisfying certain coherence properties with the tensor products (Theorem 11.3).
2 Preliminaries

2.1 Compact quantum groups and their representations

In this paper $G = (A, \Delta)$ will always denote a compact quantum group in the sense of [27]: a unital $C^*$-algebra $A$ with a unital $^*$-homomorphism (the coproduct) $\Delta : A \to A \otimes A$ such that

a) $\Delta \otimes I \circ \Delta = I \otimes \Delta \circ \Delta$, with $I$ the identity map on $A$,

b) the sets $\{b \otimes I \Delta(c), b, c \in A\}$ $\{I \otimes b \Delta(c), b, c \in A\}$ both span dense subspaces of $A \otimes A$.

Let $H$ be a finite dimensional Hilbert space, and form the free right $A$–module $H \otimes A$. The natural $A$–valued inner product makes it into a right Hilbert $A$–module.

A unitary representation of $G$ with Hilbert space $H_u$ can be defined as a linear map $u : H_u \to H_u \otimes A$, with $H_u$ a finite dimensional Hilbert space, such that

$$(u(\psi), u(\phi)) = (\psi, \phi)I, \quad \psi, \phi \in H_u,$$

$$u \otimes I \circ u = I \otimes \Delta \circ u,$$

$$u(H_u)I \otimes A \text{ is total in } H_u \otimes A.$$  

If $u$ is a unitary representation, the matrix coefficients of $u$ are the elements of $A$:

$$u_{\phi,\psi} := \ell_{\phi}^* \circ u(\psi), \quad \psi, \phi \in H_u,$$

with $\ell_{\phi} : A \to H \otimes A$ the operator of tensoring on the left by $\phi$. Let $u : H_u \to H_u \otimes A$ be any linear map, and let $(\phi_i)$ be an orthonormal basis of $H_u$. Consider the matrix $(u_{ij})$ with entries in $A$, where $u_{ij} := \ell_{\phi_i}^* u(\phi_j)$. Then $u$ is a unitary representation if and only if the matrix $(u_{ij})$ is unitary and $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

The linear span $A_\infty$ of all the matrix coefficients is known to be a unital dense $^*$–subalgebra of $A$, and a Hopf $^*$–algebra $G_\infty = (A_\infty, \Delta)$ with the restricted coproduct [27].

The category $\text{Rep}(G)$ of unitary representations of $G$ with arrows the spaces $(u, v)$ of linear maps $T : H_u \to H_v$ intertwining $u$ and $v$: $T \otimes I \circ u = v \circ T$ is a tensor $C^*$–category with conjugates [26]. Recall that the tensor product $u \otimes v$ of two representations $u$ and $v$ and the conjugate representation $\overline{u}$ are those representations with Hilbert spaces $H_u \otimes H_v$ and $H_{\overline{u}}$ and coefficients,

$$(u \otimes v)_{\phi \otimes \phi', \psi \otimes \psi'} := u_{\phi,\psi} v_{\phi',\psi'}, \quad \phi, \psi \in H_u, \phi', \psi' \in H_v,$$

$$\overline{u}_{\phi,\psi} := (u_{\psi,\phi})^*, \quad \phi, \psi \in H_u,$$

respectively, where $j : H_u \to H_{\overline{u}}$ is an antilinear invertible intertwiner.

2.2 Spectra of Hopf $^*$–algebra actions
Let $\mathcal{C}$ be a unital $^*$–algebra and $G = (\mathcal{A}, \Delta)$ a compact quantum group. Consider the dense Hopf $^*$–subalgebra $G_\infty = (\mathcal{A}_\infty, \Delta)$ of $G$ and a unital action $\eta : \mathcal{C} \to \mathcal{C} \odot \mathcal{A}_\infty$ of it on $\mathcal{C}$ (where $\odot$ denotes the algebraic tensor product): $\eta$ is a unital $^*$–homomorphism such that $\eta \otimes \iota \circ \eta = \iota \otimes \Delta \circ \eta$.

We define the spectrum of $\eta$, denoted $sp(\eta)$, to be the set of all unitary $G$–representations $u : H_u \to H_u \otimes \mathcal{A}$ for which there is a faithful linear map $T : H_u \to \mathcal{C}$ intertwining $u$ with $\eta$:

$$\eta \circ T = T \otimes \iota \circ u.$$ 

In other words, representing $u$ as a matrix $u = (u_{ij})$ with respect to some orthonormal basis of $H_u$, $u \in sp(\eta)$ if and only if there exists a multiplet $(c_1, \ldots, c_d)$, with $d$ the dimension of $u$, constituted by linearly independent elements of $\mathcal{C}$, such that

$$\eta(c_i) = \sum_j c_j \otimes u_{ji}.$$ 

For compact quantum groups this notion was introduced by Podles in [20], as a generalization of the classical notion for an action of a compact group on a $C^*$–algebra [9]. We show some simple properties of the spectrum.

**2.1 Proposition**

a) If $u \in sp(\eta)$ and if $z$ is a unitary representation of $G$ such that $(z, u)$ contains an isometry, then $z \in sp(\eta)$,

b) if $u \in sp(\eta)$ and $\overline{u}$ is a unitary representation equivalent to the complex conjugate $u^*$ then $\overline{u} \in sp(\eta)$. Here $u^*$ denotes the representation, in general not unitary, whose matrix elements are the adjoints of those of $u$.

**Proof** a) If $S \in (z, u)$ is an isometry and $T : H_u \to \mathcal{C}$ is a faithful intertwining map then $T \circ S : H_z \to \mathcal{C}$ is a faithful intertwining map:

$$\eta \circ (TS) = (\eta \circ T)S = (T \otimes \iota) \circ uS = (TS) \otimes \iota z.$$ 

b) If $(c_1, \ldots, c_d)$ is a linearly independent multiplet transforming like the unitary representation $u$, then $(c_1^*, \ldots, c_d^*)$ is a linearly independent multiplet transforming like the complex conjugate representation $u^*$, which, in general, is just an invertible representation, but equivalent to a unitary representation [25]. Let $\mu = (\mu_{ij}) \in (\overline{\pi}, u_\ast)$ be an invertible intertwiner with a unitary representation $\overline{\pi}$. Set $f_j := \sum_p \mu_{pj} c^*_p$. The multiplet $(f_1, \ldots, f_d)$ is linearly independent since $\mu$ is invertible, and transforms like $\overline{\pi}$:

$$\eta(f_j) = \sum_{p,r} \mu_{pj} c^*_r \otimes u^*_{rp} =$$

$$\sum_r c^*_r \otimes (u_\ast \mu)_{r,j} = \sum_r c^*_r \otimes (\mu \overline{\pi})_{r,j} =$$
\[ \sum_{r,s} \mu_{rs} \xi_r \otimes \pi_{s,j} = \sum_s f_s \otimes \pi_{s,j}. \]

We denote the linear span of all spectral multiplets by \( \mathcal{C}_{sp} \). Part a) of the previous proposition tells us that \( \mathcal{C}_{sp} \) is generated, as a linear space, by those nonzero multiplets transforming according to unitary irreducible \( G \)-representations (such multiplets are automatically linearly independent by irreducibility).

2.2 Proposition

a) If \( T : H_u \rightarrow \mathcal{C} \) is any linear map satisfying \( \eta \circ T = T \otimes \iota \circ u \) then the image of \( T \) lies in \( \mathcal{C}_{sp} \).

b) \( \mathcal{C}_{sp} \) is a unital \( \ast \)-subalgebra of \( \mathcal{C} \) invariant under the \( G_\infty \)-action: \( \eta(\mathcal{C}_{sp}) \subset \mathcal{C}_{sp} \otimes A_\infty \).

Proof a) We can assume \( T \neq 0 \). Let us write \( u \) as a direct sum of unitary irreducible subrepresentations \( u_1, \ldots, u_p \). Let \( H_i \) be the subspace corresponding to \( u_i \), for \( i = 1, \ldots, p \). Since each \( u_i \) is irreducible, \( T \) is either faithful or zero on \( H_i \). We can assume that there is a \( q \leq p \), \( q \geq 1 \) such that the restriction \( T_i \) of \( T \) to \( H_i \) is faithful for \( i = 1, \ldots, q \) and \( T = 0 \) on \( H_i \) for \( i > q \). Thus any element \( T\psi \) in the image of \( T \) can be written as a sum \( \sum_{k \leq q} T_k \psi_k \) with \( T_k \psi_k \in \mathcal{C}_{sp} \) now.

b) The trivial representation is clearly in the spectrum with spectral subspace \( C I \), so \( \mathcal{C}_{sp} \) contains the identity. If \( T : H_u \rightarrow \mathcal{C} \) and \( S : H_v \rightarrow \mathcal{C} \) are faithful maps intertwining \( u \) and \( v \) respectively with \( \eta \), then the map

\[ H_u \otimes H_v \rightarrow \mathcal{C}, \quad \psi \otimes \phi \rightarrow T(\psi)S(\phi), \quad \psi \in H_u, \phi \in H_v \]

intertwines the tensor product \( u \otimes v \) with \( \eta \). By a) \( T(\psi)S(\phi) \) lies in \( \mathcal{C}_{sp} \), so \( \mathcal{C}_{sp} \) is an algebra. Furthermore the proof of part b) of the previous proposition shows that \( \mathcal{C}_{sp} \) is a \( \ast \)-subalgebra.

Consider an action \( \eta : \mathcal{B} \rightarrow \mathcal{B} \otimes A \) of \( G = (A, \Delta) \) on a unital \( C^* \)-algebra \( \mathcal{B} \) (with \( \otimes \) the minimal tensor product): a unital \( \ast \)-homomorphism such that \( \eta \otimes \iota \circ \eta = \iota \otimes \Delta \circ \eta \).

In the \( C^* \)-algebraic case, one can similarly define, the spectrum of the action, \( sp(\eta) \), and the spectral \( \ast \)-subalgebra \( \mathcal{B}_{sp} \), a unital \( \ast \)-subalgebra invariant under the action of \( G_\infty \). In the case where \( \mathcal{B} = A \) and \( \eta = \Delta \), \( A_{sp} = A_\infty \).

There is an alternative way of introducing the notion of spectrum. Let \( u \) and \( v \) be corepresentations of a Hopf \( C^* \)-algebra on Hilbert spaces \( H \) and \( K \) respectively. Then we may define a coaction \( \eta \) on the space \( (H, K) \) of linear mappings from \( H \) to \( K \) by setting

\[ \eta(b) := vb \otimes 1u^*, \quad b \in (H, K). \]

The above formulae may be used to define a coaction on \( (H, K) \otimes \mathcal{C} \) for a \( C^* \)-algebra \( \mathcal{C} \) with the variables in \( \mathcal{C} \) being spectators. If we are now given in
addition a $C^*$–algebra $\mathcal{C}$ carrying a coaction $\eta$ of $\mathcal{A}$, then we can let it act on $(H, K) \otimes \mathcal{C}$ with the variables in $(H, K)$ being spectators. Combining it with the coaction of $\mathcal{A}$ on $(H, K)$ defined by $u$ and $v$ as above we get a coaction $\beta$ on $(H, K) \otimes \mathcal{C}$. Since $\mathcal{A}$ is not, in general, commutative, this coaction must be spelled out in detail. We have

$$\beta(a) := \hat{v}\eta(a)\hat{u}^*, \quad a \in (H, K) \otimes \mathcal{C},$$

where $\hat{v}$ denotes $v$ with $1_\mathcal{C}$ inserted between the tensor product factors in $v$. Choosing orthonormal bases $\psi_p, \chi_r$ in $H$ and $K$ and expressing $u$, $v$ and $a$ in terms of these bases, the coaction may be written

$$\beta(a) = \sum_{p,q,r,s} \chi_r \psi_p^* \otimes (1_\mathcal{C} \otimes v_{rs} \eta(a_{sq}) 1_\mathcal{C} \otimes u_{pq}^*).$$

Using this or the previous formula, it is easy to check that $\beta$ is a coaction.

The set of fixed points will be denoted $(u \otimes \eta, v \otimes \eta)$. Restricting ourselves to finite-dimensional unitary corepresentations and allowing $\mathcal{C}$ to be just a $\ast$–algebra we define the spectral category $\text{Sp}(\mathcal{C}, \eta)$ by letting $(u \otimes \eta, v \otimes \eta)$ be the set of arrows from the object $u \otimes \eta$ to the object $v \otimes \eta$ with the obvious law of composition. After choosing orthonormal bases, and setting

$$a = \sum_{s,t} \psi_t \varphi_s^* \otimes a_{ts},$$

$a \in (u \otimes \eta, v \otimes \eta)$ reads in coordinates

$$\eta(a_{nm}) = \sum_{t,s} I_\mathcal{C} \otimes v_{nt}^* a_{ts} \otimes I_\mathcal{A} \otimes u_{sm}.$$ 

There is also a tensor product of a restricted nature in $\text{Sp}(\mathcal{C}, \eta)$. If $a \in (u \otimes \eta, mg \otimes \eta)$, $a = \sum_{s,t} \varphi_s \psi_t^* \otimes a_{st}$, and $b \in (v \otimes \eta, m_\eta \otimes \eta)$, $b = \sum_{p,q} \lambda_p \mu_q^* \otimes b_{pq}$, where $m$ and $n$ are positive integers, then

$$a \top b := \sum_{p,q,s,t} \varphi_s \lambda_p \psi_t^* \mu_q^* \otimes a_{st} b_{pq}$$

is in $(u \top v \otimes \eta, mn_\eta \otimes \eta)$. In fact, $\eta(a_{st}) = \sum_f a_{sf} \otimes u_{ft}$ and $\eta(b_{pq}) = \sum_g b_{pg} \otimes v_{gq}$ so that

$$\eta(a_{st} b_{pq}) = \sum_{f,g} a_{sf} b_{pg} \otimes u_{ft} v_{gq},$$

as required. Notice that if $a$ and $b$ are unitaries then so is $a \top b$.

One can check that if $S \in (u', u)$, $T \in (v, v')$ and $X \in (u \otimes \eta, v \otimes \eta)$ then $T \otimes 1X S \otimes 1 \in (u' \otimes \eta, v' \otimes \eta)$.

An element of $\mathcal{C}_\eta := (u \otimes \eta, \iota \otimes \eta)$, where $\iota$ denotes the trivial corepresentation, will be called a $u$–multiplet and may be thought of as a multiplet transforming according to the corepresentation $u$. More precisely, we have

$$\eta(c_j) = \sum_j c_j \otimes u_{ji}.$$
Thus these multiplets coincide with those introduced above when defining the spectrum. In fact, if we define the coordinates of $T : H_u \to \mathcal{C}$ with respect to an orthonormal basis by $T \phi_i := T_i$ and the coordinates of $S \in (u \otimes \eta, \iota \otimes \eta)$ by $S := \sum_i \phi^* \otimes S_i$ then setting $S_i := T_i$ defines a canonical isomorphism from the set of intertwining maps to $(u \otimes \eta, \iota \otimes \eta)$.

If $\eta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{A}$ and $\beta : \mathcal{D} \to \mathcal{D} \otimes \mathcal{A}$ are two coactions of $\mathcal{A}$ and $k : \mathcal{C} \to \mathcal{D}$ is a morphism commuting with the coactions, i.e. $k \otimes I_A \circ \eta = \beta \circ k$ then there is an induced functor $k^* : \text{Sp} \eta \to \text{Sp} \beta$, $k^*(u \otimes \eta) := (u \otimes \beta)$ and $k^*(C) = 1_{(H_u, H_v)} \otimes k(C)$ for $C \in (u \otimes \eta, v \otimes \eta)$. Note that $k^*$ being a $*$–functor of $C^*$–categories will map unitaries to unitaries and isometries to isometries.

2.3 Multiplicities of spectral representations

We call an action $\eta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ of $G = (\mathcal{A}, \Delta)$ on a unital $C^*$–algebra $\mathcal{B}$ nondegenerate if $\eta(\mathcal{B}) I \otimes \mathcal{A}$ is dense in $\mathcal{B} \otimes \mathcal{A}$. In [20] the following result is proven.

2.3 Theorem [20] Let $\eta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ be a nondegenerate action of a compact quantum group $G$ on $\mathcal{B}$. Then for any irreducible $u \in \text{sp}(\eta)$ there is a subspace $W_u \subset \mathcal{B}$ containing any spectral multiplet tranforming like $u$, such that

a) $W_u$ splits $W_u = \oplus_{i \in I_u} W_u^i$ into an algebraic direct sum of subspaces $W_u^i$, each of them corresponding to $u$.

b) If $\hat{\eta}$ is a complete set of inequivalent irreducibles in $\text{sp}(\eta)$ then the linear span of $\{W_u, u \in \hat{\eta}\}$, which coincides with the spectral $*$–subalgebra $\mathcal{B}_{sp}$, is dense in $\mathcal{B}$.

The cardinality of the set $I_u$ is called the multiplicity of the irreducible $u$ in $\eta$, and it will be denoted $\text{mult}(u)$.

2.4 Quantum subgroups and quotient spaces

A compact quantum subgroup $K = (\mathcal{A}', \Delta')$ of $G$, as introduced in [20], is a compact quantum group for which there exists a surjective $*$–homomorphism $\pi : \mathcal{A} \to \mathcal{A}'$ such that $\pi \otimes \pi \circ \Delta = \Delta' \circ \pi$.

A closed bi–ideal $\mathcal{I}$ of $\mathcal{A}$ is a norm closed two–sided ideal of $\mathcal{A}$ such that $\Delta(\mathcal{I}) \subset \mathcal{A} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{A}$.

There is a surjective correspondence from quantum subgroups of $G$ and closed bi–ideals of $\mathcal{A}$, which associates to a subgroup defined by the surjection $\pi$, the kernel of $\pi$. Subgroups defined by surjections with the same kernel are isomorphic as Hopf $C^*$–algebras [18].

The subgroup $K$ acts (on the left) on the $C^*$–algebra $\mathcal{A}$ via $\delta := \pi \otimes \iota \circ \Delta : \mathcal{A} \to \mathcal{A}' \otimes \mathcal{A}$. 
The fixed point algebra
\[ A^\delta := \{ T \in A : \delta(T) = I \otimes T \} \]
is defined to be the quantum quotient space of right cosets. This algebra has a
natural right action of \( G \):
\[ \eta_K := \Delta \mid_{A^\delta} : A^\delta \to A^\delta \otimes A, \]
known to be nondegenerate and ergodic [23], see also [18]. We set:
\[ K \setminus G := (A^\delta, \eta_K). \]
One can similarly consider the action of \( K \) on the right on \( A \):
\[ \rho := \iota \otimes \pi \circ \Delta : A \to A \otimes A', \]
with fixed point algebra
\[ A^\rho := \{ T \in A : \delta(T) = I \otimes T \}, \]
called the quantum quotient space of left cosets. This algebra carries a left
action of \( G \):
\[ \eta^K := \Delta \mid_{A^\rho} : A^\rho \to A \otimes A^\rho, \]
and we set
\[ G/K := (A^\rho, \eta^K). \]
As in the group case, unitary representations of \( G \) can be restricted to unitary
representations of \( K \) on the same Hilbert space:
\[ u \mid_K := \iota \otimes \pi \circ u : H_u \to H_u \otimes A'. \]

2.4 Proposition [18] If \( T \in (u, v) \) then \( T \in (u \mid_K, v \mid_K) \) as well. So the
map \( \text{Rep}(G) \to \text{Rep}(K) \), taking \( u \to u \mid_K \), and acting trivially on the arrows
defines a faithful tensor \(*\)–functor. The smallest full tensor \(*\)–subcategory of
\( \text{Rep}(K) \) with subobjects and direct sums containing the \( u \mid_K \), for \( u \in \text{Rep}(G) \),
is \( \text{Rep}(K) \).

We shall consider the subspace \( K_u \) of \( H_u \) of \( K \)–invariant vectors for the
restricted representation \( u \mid_K \): this is the set of all \( k \in H_u \) for which \( u \mid_K (k) = k \otimes I \). The dimension of \( K_u \) is the multiplicity of the trivial representation \( \iota_K \)
of \( K \) in \( u \mid_K \).

2.5 Proposition If \( K \) is a compact quantum subgroup of \( G \) defined by the
surjection \( \pi \) then for any representation \( u \) of \( G \) and vectors \( \psi \in H_u \), \( k \in K_u \),
the elements \( u_{\psi,k} - (\psi, k)I \) and \( u_{k,\psi} - (k, \psi)I \) belong to the closed bi–ideal \( \ker \).

Proof Notice then that for any \( \psi \in H_u \) and \( k \in K_u \)
\[ (u \mid_K)_{\psi,k} = \pi(u_{\psi,k}) = (\psi, k)I, \]
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so \( u_{\psi,k} - (\psi,k)I \in \ker \pi \). On the other hand, picking an orthonormal basis \((\psi_i)\) of \( H_u \), the condition for \( k \) to be a fixed vector for \( u \restriction_K \) can be written

\[
\sum_i \pi(u_{ji})(\psi_i, k) = (\psi_j, k)I,
\]

where \( u_{ji} := u_{\psi_i, \psi_j} \). Since the matrix \((u_{ji})\) is unitary, this condition can be rewritten in terms of \( u^* \) and reads

\[
\sum_j \pi(u^*_{rj})(\psi_j, k) = (\psi_r, k)I,
\]

or, taking the adjoint,

\[
\sum_j \pi(u_{rj})(k, \psi_j) = (k, \psi_r)I.
\]

Thus we also have, for \( k \in K_u, \psi \in H_u, u \in \text{Rep}(G) \),

\[
(u \restriction_K)_{k,\psi} = \pi(u_{k,\psi}) = (k, \psi)I.
\]

As in the group case, we can construct elements of the coset spaces using invariant vectors for the subgroup: for any representation \( u \) of \( G \), if we pick vectors \( k \in K_u, \psi \in H_u \), the coefficient \( u_{k,\psi} \) lies in \( \mathcal{A}_\delta \) and \( u_{\psi,k} \) lies in \( \mathcal{A}_\rho \). We only show the former: if \((\psi_j)\) is an orthonormal basis of \( H_u \):

\[
\delta(u_{k,\psi_j}) = \pi \otimes \iota \circ \Delta(u_{k,\psi_j}) = \\
\sum_i \pi(u_{k,\psi_i}) \otimes u_{ij} = \sum_i I \otimes (k, \psi_i)u_{ij} = I \otimes u_{k,\psi_j}.
\]

Fix a complete set \( \hat{G} \) of unitary irreducible representations of \( G \), and set

\[
\hat{G}_K := \{ u \in \hat{G} : K_u \neq 0 \}.
\]

2.6 Proposition The linear space generated by the matrix coefficients \( \{u_{u,\psi}\} \), (resp. \( \{u_{\psi,k}\} \)) as \( u \in \hat{G}_K, k \in K_u, \psi \in H_u \), vary, coincides with \( \mathcal{A}_\delta \) (resp. \( \mathcal{A}^\rho \)).

Proof Let \( V \) denote the linear space defined in the statement. We show, for completeness, that \( V \) is dense. Consider the conditional expectation \( E : \mathcal{A} \to \mathcal{A}_\delta \) onto the fixed point algebra, obtained by averaging the \( K \)-action: \( E(T) := h' \otimes \iota_A \circ \delta(T) \), with \( h' \) the Haar measure of \( K \). Since \( \mathcal{A} \) is generated, as a Banach space, by the coefficients of its irreducible unitary representations \( u_{\psi',\psi}, \psi, \psi' \in H_u \), the set \( \{ E(u_{\psi',\psi}) : u \in \hat{G}, \psi, \psi' \in H_u \} \), is total in \( \mathcal{A}_\delta \). A computation shows that \( E(u_{\psi',\psi}) = \sum_k h' \circ \pi(u_{\psi',\psi_k})u_{\psi_k,\psi} \), with \( \{\psi_k\} \) any orthonormal
basis of $H_u$. By [25], $h'\pi(u_{\psi',\phi}) = h'(u_{\mid K})_{\psi',\phi} = 0$ for all $\psi', \phi \in H_u$, unless $u_{\mid K}$ contains the identity representation. Assume then that this is the case. Replace, if necessary, that orthonormal basis with another one such that $u_{\mid K}$, when represented as a matrix with entries in $A'$, becomes diagonal of the form \(\text{diag}(I, \ldots, I, v)\), with $v$ a unitary representation of $K$ which does not contain the trivial representation, so that $h'\pi(u_{\psi',\phi}) = 0$ unless $i \leq \dim(K_u)$ and $k = i$. In that case, $h' \circ \pi(u_{\psi',\phi}) = 1$. Therefore $E(u_{\psi',\psi}) = u_{\psi',\psi}$ if $\psi' \in K_u$, and $E(u_{\psi',\psi}) = 0$ if $\psi' \notin K_u$. We have thus shown that $V$ is norm dense in $A^\delta$. One can easily check that for any $u \in \hat{G}_K$, $\phi \in K_u$, the map $\psi \in H_u \to u_{\phi,\psi} \in A^\delta$ intertwines $u$ with $\eta_K$, so $V$ is contained in $A^{sp}_{A^\delta}$. Podles shows in [20] that the subspace $W_u$ described in Theorem 2.3 coincides with the linear span of \(\{u_{\phi,\psi}\}\), $\phi \in K_u$, $\psi \in H_u$, so by Theorem 2.3, b) $V = A^{sp}_{A^\delta}$.

As a consequence, the above subspace is a unital $^{*}$-algebra endowed with the restricted action of the Hopf $^{*}$-algebra $G_{\infty} := (A_{\infty}, \Delta)$, still denoted by $\eta_K : A^\delta_{sp} \to A^\delta_{sp} \otimes A_{\infty}$.

2.5 Stabilizer and kernel

In this subsection we examine situations that should give rise to a compact quantum subgroup.

Let $\alpha$ be an action of a compact quantum group $G := (A, \Delta)$ on a $C^{*}$-algebra $\mathcal{B}$ and $\varphi$ a state of $\mathcal{B}$. We look for an appropriate notion of the stabilizer of $\varphi$ under the action. Consider

$$S_{\varphi} := \{\pi \in \text{Rep}A : \varphi \otimes \pi(\alpha(B)) = \varphi(B) \otimes 1_{\pi}, \ B \in \mathcal{B}\}.$$ 

Note that if $G$ is a compact group, the irreducible representations of $A$ are equivalent to characters and labelled by the elements of $G$. Thus the irreducible part of $S_{\varphi}$ corresponds exactly to the stabilizer of $\varphi$ under the action. We now investigate the stability properties of $S_{\varphi}$. If $W \in (\pi', \pi)$ is an isometry and $\pi \in S_{\varphi}$ then

$$\varphi \otimes \pi'(\alpha(B)) = W^* \varphi \otimes \pi(\alpha(B))W = W^* \varphi(B) \otimes 1_{\pi}W = \varphi(B) \otimes 1_{\pi'},$$

so that $\pi' \in S_{\varphi}$. Similarly, if $W_i \in (\pi_i, \pi)$ are isometries with $\sum_i W_iW_i^* = 1_{\pi}$ and $\pi_i \in S_{\varphi}$ for all $i$, then

$$\varphi \otimes \pi(\alpha(B)) = \sum_i W_i \varphi \otimes \pi_i(\alpha(B))W_i^* = \sum_i \varphi(B) \otimes W_iW_i^* = \varphi(B)1_{\pi},$$

so that $\pi \in S_{\varphi}$ and $S_{\varphi}$ is closed under subobjects and direct sums. If $\pi, \pi' \in S_{\varphi}$ then

$$\varphi \otimes (\pi \otimes \pi')(\alpha(B)) = \varphi \otimes \pi \otimes \pi'(\iota \otimes \Delta \circ \alpha(B)) = \varphi \otimes \pi \otimes \pi'(\alpha \otimes \iota \circ \alpha)(B).$$

For any $B \in \mathcal{B}$ and $A \in \mathcal{A}$, we have

$$\varphi \otimes \pi \otimes \pi'(\alpha \otimes \iota)(B \otimes A) = \varphi(B) \otimes 1_{\pi} \otimes \pi'(A) = \varphi \otimes \iota \otimes \pi'(B \otimes 1_{\pi} \otimes A).$$
Thus for any \( C \in \mathcal{B} \otimes \mathcal{A} \), we shall have \( \varphi \otimes \pi \otimes \pi' \circ \alpha \otimes \iota(C) = \varphi \otimes \iota \otimes \pi'(\hat{C}) \), where \( \hat{C} \) denotes \( C \) with a \( 1_\pi \) inserted in the middle. Taking \( \hat{C} = \alpha(B) \), we get

\[
\varphi \otimes (\pi \ast \pi')(\alpha(B)) = \varphi \otimes \iota \otimes \pi'(\hat{\alpha(B)}) = \varphi(B) \otimes 1_{\pi \ast \pi'},
\]

so that \( \pi \ast \pi' \in S_{\varphi} \).

Before pursuing this line of reasoning, we ask when a surjective morphism \( \pi : \mathcal{A} \to \pi(\mathcal{A}) \) defines a quantum subgroup of \( \mathcal{A} \). We must be able to equip \( \pi(\mathcal{A}) \) with a comultiplication \( \Delta' \) such that \( \pi \otimes \pi \Delta = \Delta' \pi \). This equation will have a solution if and only if \( \pi(A) = 0 \) implies \( \pi \otimes \pi \Delta(A) = 0 \). Any solution is unique since \( \pi \) is surjective and for the same reason \( \Delta' \) will be a comultiplication because \( \Delta \) is a comultiplication. Finally, \( \pi(\mathcal{A}) \otimes I \Delta'(\pi(\mathcal{A})) \) and \( I \otimes \pi(\mathcal{A}) \Delta'(\pi(\mathcal{A})) \) are dense in \( \pi(\mathcal{A}) \otimes \pi(\mathcal{A}) \) since \( \mathcal{A} \otimes I \Delta(\mathcal{A}) \) and \( I \otimes \mathcal{A} \Delta(\mathcal{A}) \) are dense in \( \mathcal{A} \otimes \mathcal{A} \). Thus we have a simple necessary and sufficient condition for a surjective morphism \( \pi : \mathcal{A} \to \pi\mathcal{A} \) to define a quantum subgroup which may be written

\[
\ker \pi \ast \pi \supset \ker \pi.
\]

We now pick one representative \( \pi_i \) from each equivalence class of finite-dimensional irreducibles in \( S_{\varphi} \) and form the direct sum, say, \( \pi = \oplus_{i \in I} \pi_i \). To show that \( \pi \) defines a quantum subgroup we must show that \( \pi_i(A) = 0 \) for \( i \in I \) implies \( \pi_j \ast \pi_k(A) = 0 \). But this is the case since, by the results above, \( \pi_j \ast \pi_k \) is a finite direct sum of the \( \pi_i, i \in I \).

As an example of an infinite dimensional representation belonging to a stabilizer, consider the ergodic action \( \eta_K \) of \( G = (A, \Delta) \) on the quantum quotient space \( A^\delta \) by a quantum subgroup \( K \), and let \( e \) be an everywhere defined counit of \( \mathcal{A} \), which we restrict to a state \( \varphi \) of \( A^\delta \). Let \( \pi : \mathcal{A} \to \mathcal{A}' \) be the surjection defining the subgroup, regarded as a representation of \( \mathcal{A} \). We compute \( \varphi \otimes \pi(\eta_K(B)) \) on the elements \( B = u_{k,\psi} \), with \( k \in K_u, \psi \in H_u \):

\[
\varphi \otimes \pi(\eta_K(u_{k,\psi})) = \varphi \otimes \pi(\sum_j u_{k,\psi_j} \otimes u_{\psi_j,\psi}) =
\]

\[
\sum_j (k, \psi_j) \otimes \pi(u_{\psi_j,\psi}) = I \otimes \pi(u_{k,\psi}) = I \otimes (k, \psi) 1_{\pi}
\]

by prop. 2.5, and this expression equals in turn \( \varphi(u_{k,\psi}) \otimes 1_{\pi} \). Since the elements \( u_{k,\psi} \) span a dense subspace of \( A^\delta \), we can conclude that \( \pi \in S_{\varphi} \).

Similarly, we can look for the kernel of the action of a compact quantum group. Letting \( \alpha : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A} \) be the action of a compact quantum group on a \( C^* \)-algebra \( \mathcal{B} \), let

\[
K_\alpha := \{ \pi \in \text{Rep} \mathcal{A} : 1_{\mathcal{B}} \otimes \pi \alpha(B) = B \otimes 1_{\pi}, B \in \mathcal{B} \}.
\]

Arguing as above, we may verify that \( K_\pi \) is closed under subobjects, direct sums but there seems to be no reason for it to be closed under tensor products.

Having failed to define the kernel of an action \( \alpha \), we can at least define an action to be faithful when the irreducible part of \( K_\alpha \) reduces to the trivial representation \( \iota \).
We may similarly talk of the kernel of a representation \( u : H \to H \otimes A \) of a quantum group by defining
\[
M_u := \{ \pi \in \text{Rep} A : 1_H \otimes \pi u(\psi) = \psi \otimes 1_\pi, \; \psi \in H \}.
\]
Arguing as above, we may verify that \( M_u \) is closed under subobjects, direct sums and tensor products and proceed to define the kernel of \( u \) as the quantum subgroup associated with \( M_u \). \( u \) is faithful when the irreducible part of \( M_u \) reduces to the trivial representation \( \iota \).

## 3 Quasitensor functors and a finiteness theorem

### 3.1 Definition

Let \( \mathcal{T} \) and \( \mathcal{R} \) be strict tensor \( C^* \)-categories [6]. We shall always assume that the tensor units are irreducible: \((\iota,\iota) = \mathbb{C}\). A (covariant) \( \ast \)-functor \( \mathcal{F} : \mathcal{T} \to \mathcal{R} \) will be called quasitensor if
\[
\mathcal{F}(\iota) = \iota, \tag{3.1}
\]
if for objects \( \rho, \sigma \in \mathcal{T} \) there is an isometry
\[
S_{\rho,\sigma} : \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \to \mathcal{F}(\rho \otimes \sigma) \tag{3.2}
\]
such that
\[
S_{\rho,\iota} = S_{\iota,\rho} = 1_{\mathcal{F}(\rho)}, \tag{3.3}
\]
\[
S_{\rho \otimes \sigma,\tau} \circ S_{\rho,\sigma} \otimes 1_{\mathcal{F}(\tau)} = S_{\rho,\sigma \otimes \tau} \circ 1_{\mathcal{F}(\rho)} \otimes S_{\sigma,\tau} =: S_{\rho,\sigma,\tau}, \tag{3.4}
\]
\[
E_{\rho,\sigma \otimes \tau} \circ E_{\rho,\sigma} \leq E_{\rho,\sigma,\tau}, \tag{3.5}
\]
with \( E_{\rho,\sigma} \in (\mathcal{F}(\rho \otimes \sigma), (\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma))) \) the range projection of \( S_{\rho,\sigma} \) and \( E_{\rho,\sigma \otimes \tau} \in (\mathcal{F}(\rho \otimes \sigma \otimes \tau), (\mathcal{F}(\rho \otimes \sigma) \otimes \mathcal{F}(\tau))) \) the range projection of \( S_{\rho,\sigma,\tau} \), and if
\[
\mathcal{F}(S \otimes T) \circ S_{\rho,\sigma} = S_{\rho',\sigma'} \circ \mathcal{F}(S) \otimes \mathcal{F}(T), \tag{3.6}
\]
for any other pair of objects \( \rho', \sigma' \) and arrows \( S \in (\rho, \rho') \), \( T \in (\sigma, \sigma') \).

In this paper we shall only deal with the case where \( \mathcal{R} \) is the tensor \( C^* \)-category \( \mathcal{H} \) with objects Hilbert spaces and arrows from a Hilbert space \( H \) to a Hilbert space \( H' \), the set \((H,H')\) of all bounded linear mappings from \( H \) to \( H' \). We shall assume that \( \mathcal{H} \) is strictly tensor, namely that the tensor product between the objects has been realized in a strictly associative way.

In order to economize on brackets, we evaluate tensor products of arrows before composition. Moreover in the sequel, for the sake of simplicity, we shall occasionally identify, with a less precise but lighter notation, \( \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \) with a subspace of \( \mathcal{F}(\rho \otimes \sigma) \), the image of \( S_{\rho,\sigma} \). Equations (3.2)–(3.6) shall then be written:
\[
\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \subset \mathcal{F}(\rho \otimes \sigma), \tag{3.7}
\]
\[
E_{\rho,\iota} = E_{\iota,\rho} = 1_{\mathcal{F}(\rho)}, \tag{3.8}
\]
\[ E_{\rho,\sigma} \otimes 1_{\mathcal{F}(\tau)} \circ E_{\rho \otimes \sigma,\tau} = 1_{\mathcal{F}(\rho)} \otimes E_{\sigma,\tau} \circ E_{\rho,\sigma \otimes \tau} =: E_{\rho,\sigma,\tau}. \]  
\[ E_{\rho \otimes \sigma,\tau}(\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma \otimes \tau)) \subset \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \otimes \mathcal{F}(\tau), \]  
\[ \mathcal{F}(S \otimes T) |_{\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma)} = \mathcal{F}(S) \otimes \mathcal{F}(T). \]  
Equation (3.10) combined with (3.9) requires that the projection onto \( \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \otimes \mathcal{F}(\tau) \) actually takes the subspace \( \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \otimes \mathcal{F}(\tau) \) onto \( \mathcal{F}(\rho) \otimes \mathcal{F}(\sigma) \otimes \mathcal{F}(\tau) \). Therefore we necessarily have
\[ E_{\rho,\sigma,\tau} = E_{\rho \otimes \sigma,\tau} \circ E_{\rho,\sigma \otimes \tau} = E_{\rho,\sigma \otimes \tau} \circ E_{\rho \otimes \sigma,\tau}. \]  

Notice that any tensor *–functor from \( \mathcal{F} \) to \( \mathcal{H} \) is quasitensor.

3.2 Example Assume that \( \mathcal{F} = \text{Rep}(G) \), the representation category of a compact quantum group \( G \). Then the embedding functor \( H : \text{Rep}(G) \to \mathcal{H} \) associating to each representation \( u \) its Hilbert space \( H_u \) and acting trivially on the arrows, is tensor, and therefore quasitensor.

Quasitensor *–functors arise naturally in abstract tensor \( C^* \)-categories.

3.3 Proposition Let \( \mathcal{F} \) be a tensor \( C^* \)-category with \( (\iota, \iota) = \mathbb{C}1_\iota \). For any object \( \rho \) of \( \mathcal{F} \), consider the Hilbert space \( \hat{\rho} := (\iota, \rho) \), with inner product
\[(\phi, \phi')_\iota := \phi^* \circ \phi', \quad \phi, \phi' \in (\iota, \rho).\]  
For \( T \in (\rho, \sigma) \) define a bounded linear map \( \hat{T} : \hat{\rho} \to \hat{\sigma} \) by \( \hat{T}(\phi) = T \circ \phi \). This is a quasitensor *–functor.

Proof It is easy to check that, for \( T \in (\rho, \sigma), S \in (\sigma, \tau), \) \( \sqrt{S} \circ \hat{T} = \hat{S} \circ T \) and that \( \hat{T}^* = T^* \). This shows that we have a *–functor such that \( i = (\iota, \iota) = \mathbb{C} \) and (3.1) is satisfied. For \( \phi \in \hat{\rho} = (\iota, \rho), \psi \in \hat{\sigma} = (\iota, \sigma) \), the map \( \phi \otimes \psi \to \phi \otimes \psi = \phi \otimes 1_\iota \circ \psi \in \rho \otimes \sigma = (\iota, \rho \otimes \sigma) \) defines an isometric map from \( \hat{\rho} \otimes \hat{\sigma} \) to \( \hat{\rho} \otimes \hat{\sigma} \). The copy of \( \hat{\rho} \otimes \hat{\sigma} \otimes \hat{\tau} \) sitting inside \( \hat{\rho} \otimes \hat{\sigma} \otimes \hat{\tau} \) is the subspace generated by elements \( \phi \otimes 1_{\sigma \otimes \tau} \circ \psi \otimes 1_\iota \circ \eta \), for \( \phi \in \hat{\rho}, \psi \in \hat{\sigma}, \eta \in \hat{\tau} \), and it coincides with the copy of the same Hilbert space sitting inside \( \hat{\rho} \otimes \hat{\sigma} \otimes \hat{\tau} \), hence (3.9) holds. We now check (3.10). First notice that if \( \phi_i \) is an orthonormal basis of \( \hat{\sigma} \) then every finite sum \( \sum_{\text{finite}} 1_{\rho} \otimes (\phi_i \circ \phi_i^*) \) is an element of \( (\rho \otimes \sigma, \rho \otimes \sigma) \), defining, by composition, a projection map from \( \rho \otimes \sigma \) onto a subspace of \( \hat{\rho} \otimes \hat{\sigma} \). The strong limit of this net converges, in the strong topology defined by \( \rho \otimes \sigma \), to the projection map \( E_{\rho,\sigma} \). This shows that the projection map of \( E_{\rho \otimes \sigma,\tau} \) is the strong limit of \( \sum_{i} 1_{\rho \otimes \sigma} \otimes \psi_i \circ \psi_i^* \), with \( \psi_i \in \hat{\tau} \) an orthonormal basis. Thus \( E_{\rho \otimes \sigma,\tau} \) takes \( \hat{\rho} \otimes \hat{\sigma} \otimes \hat{\tau} \) into \( \hat{\rho} \otimes \hat{\sigma} \otimes \hat{\tau} \), and the proof of (3.10) is complete.

The following proposition is of help in constructing more examples.

3.4 Proposition Let \( S, \mathcal{F} \) be tensor \( C^* \)-categories. If \( \mathcal{G} : S \to \mathcal{F} \) is a tensor *–functor and \( \mathcal{F} : \mathcal{F} \to \mathcal{H} \) is quasitensor then \( \mathcal{F} \circ \mathcal{G} \) is quasitensor.
We shall call

\[ \alpha(3.7) \text{Theorem} \]

to showing the following theorem.

3.7 Theorem Denote, with abuse of notation, by

\[ \alpha(3.7) \] * tensor

\[ \alpha(3.7) \] of the conjugate equations for

\[ \alpha(3.7) \] tensor

\[ \alpha(3.7) \]–functor Rep(3.3).

We now obtain a 'concrete' quasitensor

\[ \alpha(3.7) \] * G

Furthermore if

\[ \alpha(3.7) \] E

In Sec. 7 we shall exhibit examples of quasitensor

\[ \alpha(3.7) \]–functors associated to

\[ \alpha(3.7) \] and get

\[ \alpha(3.7) \]–algebra

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Hence (3.7) and (3.11) hold. Set

\[ \alpha(3.7) \] E^F_{\alpha,\beta} := E\hat{F}_{\alpha,\beta}. \]

This is the orthogonal projection from \( E(\rho \otimes \sigma) \) to \( E(\rho) \otimes E(\sigma) \), and it is easy to check that it satisfies properties (3.8), (3.9) and (3.10).

Remark One can similarly show that if \( \mathcal{F} \) is tensor and \( \mathcal{G} \) is quasitensor then \( \mathcal{F} \circ \mathcal{G} \) is quasitensor.

3.5 Example Let \( G \) be a compact quantum group, \( \mathcal{I} \) be a tensor \( C^* \)-category and \( \rho : \text{Rep}(G) \rightarrow \mathcal{I} \) is a tensor \( * \)-functor. If we compose \( \rho \) with the quasitensor \( * \)-functor \( \mathcal{I} 

\[ \alpha(3.6) \]–functor

\[ \alpha(3.6) \]–functor Rep(3.3) described in Prop. 2.4. We now obtain a 'concrete' quasitensor \( * \)-functor, that we shall denote, with abuse of notation, by \( K : \text{Rep}(G) \rightarrow \mathcal{H} \), where now \( K_u := (\iota, \rho_u) \) and defined on arrows by

\[ \alpha(3.6) \]–functor

\[ \alpha(3.6) \]–functor

Remark One can similarly show that if \( \mathcal{F} \) is tensor and \( \mathcal{G} \) is quasitensor then \( \mathcal{F} \circ \mathcal{G} \) is quasitensor.

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\[ \alpha(3.6) \]–functor

\[ \alpha(3.6) \]–functor

In Sec. 7 we shall exhibit examples of quasitensor \( * \)-functors associated to ergodic actions of compact quantum groups. The rest of this section is devoted to showing the following theorem.

3.6 Example Apply the construction described in the previous example to the tensor \( C^* \)-category \( \mathcal{I} = \text{Rep}(K) \), where \( K \) is a compact quantum subgroup of \( G \), with \( \rho \) given by the canonical tensor \( * \)-functor \( \text{Rep}(G) \rightarrow \text{Rep}(K) \) described in Prop. 2.4. We now obtain a 'concrete' quasitensor \( * \)-functor, that we shall denote, with abuse of notation, by \( K : \text{Rep}(G) \rightarrow \mathcal{H} \), where now \( K_u := (\iota, \rho_u) \) and defined on arrows by

\[ \alpha(3.6) \]–functor

\[ \alpha(3.6) \]–functor Rep(3.3) described in Prop. 2.4. We now obtain a 'concrete' quasitensor \( * \)-functor, that we shall denote, with abuse of notation, by \( K : \text{Rep}(G) \rightarrow \mathcal{H} \), where now \( K_u := (\iota, \rho_u) \) and defined on arrows by

\[ \alpha(3.6) \]–functor

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In Sec. 7 we shall exhibit examples of quasitensor \( * \)-functors associated to ergodic actions of compact quantum groups. The rest of this section is devoted to showing the following theorem.

3.7 Theorem Let \( \mathcal{I} \) be a strict tensor \( C^* \)-category and \( \mathcal{F} : \mathcal{I} \rightarrow \mathcal{H} \) a quasitensor \( * \)-functor. If \( \rho \) has a conjugate \( \mathcal{I} \) in \( \mathcal{I} \) then \( \mathcal{F}(\rho) \) is finite dimensional and

\[ \alpha(3.7) \text{dim} \mathcal{F}(\rho) = \text{dim} \mathcal{F}(\mathcal{I}) \]

Furthermore if \( \mathcal{F}(\rho) \neq 0 \) and if \( R \in (\iota, \rho \otimes \rho) \) and \( \mathcal{R} \in (\iota, \rho \otimes \rho) \) is a solution of the conjugate equations for \( \rho \) in \( \mathcal{I} \) then \( \hat{R} := S_{\rho,\rho} \circ \mathcal{F}(\mathcal{R}) \in \mathcal{F}(\mathcal{I}) \otimes \mathcal{F}(\rho) \) and \( \mathcal{R} := S_{\rho,\rho}\circ \mathcal{F}(\mathcal{R}) \in \mathcal{F}(\rho) \otimes \mathcal{F}(\rho) \) is a solution of the conjugate equations for \( \mathcal{F}(\rho) \) in \( \mathcal{H} \).

Proof For the sake of simplicity, we shall use the simplified notation (3.7)–(3.11). This amounts to replacing \( S_{\rho,\rho} \) by the identity and \( S_{\rho,\rho} \) by \( E_{\rho,\rho} \). Apply the functor \( \mathcal{F} \) to the relation

\[ \alpha(3.7) \]

\[ \alpha(3.7) \]

and get

\[ \alpha(3.7) \]

and

\[ \alpha(3.7) \]

Finally, we have

\[ \alpha(3.7) \]
Using successively (3.11), (3.8), (3.9), (3.10), we get
\[ F(1_\rho \otimes R)(\psi) = \psi \otimes F(R), \quad \psi \in F(\rho), \]
\[ F(R \otimes 1_\rho)(\phi) = F(R) \otimes \phi, \quad \phi \in F(\rho). \]
So
\[ (\phi, \psi) = (\phi, F(R \otimes 1_\rho)^* \circ F(1_\rho \otimes R)\psi) = (F(R) \otimes \phi, \psi \otimes F(R)) = (F(R) \otimes \phi, E_{\rho \otimes \overline{\pi}, \rho}(\psi \otimes F(R)) = (E_{\rho, \overline{\pi}} \circ F(R) \otimes \phi, \psi \otimes E_{\overline{\pi}, \rho} \circ F(R)), \]
so
\[ \overline{R}^* \otimes 1_{\overline{\tau}(\rho)} \circ 1_{\overline{\tau}(\rho)} \otimes \hat{R} = 1_{\overline{\tau}(\rho)}. \]
At this point we can start over with \( \overline{\tau} \) and obtain the relation
\[ \hat{R}^* \otimes 1_{\overline{\tau}(\overline{\tau})} \circ 1_{\overline{\tau}(\overline{\tau})} \otimes \hat{R} = 1_{\overline{\tau}(\overline{\tau})}, \]
which implies that \( F(\overline{\tau}) \) is finite dimensional as well with the same dimension as \( F(\rho) \).

Recall that in a tensor \( C^* \)–category with conjugates, the infimum of all the
\[ d_{R, \overline{R}}(\rho) := \| R \| \| \overline{R} \| \]
is the **intrinsic dimension** of \( \rho \), denoted \( d(\rho) \) [13]. If \( \rho \) is irreducible (in the sense that \( (\rho, \rho) = \mathbb{C} \) the spaces \( (\iota, \tau \otimes \rho) \) and \( (\iota, \rho \otimes \overline{\tau}) \) are one dimensional, so any solution of the conjugate equations is of the form \( \lambda R, \mu \overline{R} \) with \( \mu \lambda = 1 \).
Therefore in this case for any solution \( (R, \overline{R}) \) of the conjugate equations,
\[ d(\rho) = \| R \| \| \overline{R} \|. \]

### 3.8 Corollary
If \( \mathcal{F} : \mathcal{T} \to \mathcal{H} \) is a quasitensor \( * \)–functor and if \( \rho \) is an object of \( \mathcal{T} \) with a conjugate defined by \( R \) and \( \overline{R} \) then
\[ \dim(\mathcal{F}(\rho)) \leq d_{R, \overline{R}}(\mathcal{F}(\rho)) \leq d_{R, \overline{R}}(\rho). \]
Furthermore \( d_{R, \overline{R}}(\mathcal{F}(\rho)) = d_{R, \overline{R}}(\rho) \) if and only if \( \mathcal{F}(R) \in \text{Image} S_{\tau, \rho} \) and \( \mathcal{F}(R) \in \text{Image} S_{\rho, \overline{\tau}}. \)

**Proof** Note that
\[ \| \mathcal{F}(R) \|^2 = \| \mathcal{F}(R)^* \mathcal{F}(R) \| = \| \mathcal{F}(R^* R) \| = \| R \|^2, \]
so \( \| \hat{R} \| \leq \| R \| \) and, similarly, \( \| \hat{R} \| \leq \| R \| \). Thus the last inequality follows. This also shows that \( d_{R, \hat{R}}(\mathcal{F}(\rho)) = d_{R, R}(\rho) \) if and only if \( \| \hat{R} \| = \| \mathcal{F}(R) \| \) and \( \| R \| = \| \mathcal{F}(R) \| \), and the last statement follows.

Let \( J : \mathcal{F}(\rho) \to \mathcal{F}(\hat{R}) \) be the antilinear invertible associated to \( \hat{R}, \hat{R} \) by \( J \psi = r^*_\psi \circ \hat{R} \). Then

\[
d^2_{R, \hat{R}}(\mathcal{F}(\rho)) = \text{Trace}(J J^*) \text{Trace}((J J^*)^{-1}).
\]

The first inequality is now a consequence of the elementary fact that for any positive invertible matrix \( Q \in M_n \),

\[
n^2 \leq \text{Trace}(Q) \text{Trace}(Q^{-1}).
\]

In Sect. 7 we shall relate this result to the work of [2].

Let \( \mathcal{F} : S \to \mathcal{F} \) be a quasitensor functor and \( R, \hat{R} \) be a solution of the conjugate equations for an object \( \rho \) of \( S \). Then, as we have seen, \( \hat{R} := E_{\rho, \hat{\rho}} \circ \mathcal{F}(R) \) and \( \hat{R} := E_{\rho, \hat{\rho}} \circ \mathcal{F}(\hat{R}) \) is a solution of the conjugate equations for \( \mathcal{F}(\rho) \). We show that this construction has certain functoriality properties. Given \( T \in (\rho, \rho') \) define \( T^* \) by

\[
T^* \otimes 1_\rho \circ R_\rho := 1_{\rho'} \otimes T^* \circ R_{\rho'}.\]

Then

\[
\mathcal{F}(T^* \otimes 1_\rho \circ R_\rho) = \mathcal{F}(1_{\rho'} \otimes T^*) \circ \mathcal{F}(R_{\rho'}),
\]

so

\[
E_{\rho, \rho'} \circ \mathcal{F}(T^* \otimes 1_\rho \circ R_\rho) = E_{\rho, \rho'} \circ \mathcal{F}(1_{\rho'} \otimes T^*) \circ \mathcal{F}(R_{\rho'}),
\]

and then

\[
\mathcal{F}(T^*) \otimes 1_{\mathcal{F}(\rho)} \circ \hat{R}_\rho = 1_{\mathcal{F}(\rho')} \otimes \mathcal{F}(T^*) \circ \hat{R}_{\rho'}.
\]

It follows that

\[
\mathcal{F}(T^*) \otimes 1_{\mathcal{F}(\rho)} \circ \hat{R}_\rho = \mathcal{F}(T^*) \otimes 1_{\mathcal{F}(\rho)} \circ \hat{R}_\rho,
\]

from which we get

\[
\mathcal{F}(T^*) = \mathcal{F}(T^*).
\]

Now suppose that \( R, \hat{R} \) is a standard solution of the conjugate equations. Then \( R = \sum_i \hat{W}_i \otimes W_i \circ R_i \), where \( R_i \in (\iota, \hat{\rho}_i, \rho_i) \) and \( \hat{R}_i \in (\iota, \rho_i, \hat{\rho}_i) \) are normalized solutions of the conjugate equations for the irreducible \( \rho_i \). Thus \( \mathcal{F}(R) = \sum_i \mathcal{F}(\hat{W}_i \otimes W_i) \circ \mathcal{F}(R_i) \) giving \( \hat{R} = E_{\rho, \hat{\rho}} \circ \mathcal{F}(R) = \sum_i \mathcal{F}(\hat{W}_i) \otimes \mathcal{F}(W_i) \circ \hat{R}_i \). We similarly get

\[
\hat{R} = \sum_i \mathcal{F}(W_i) \otimes \mathcal{F}(\hat{W}_i) \circ \hat{R}_i.
\]

Consequently, if the \( \mathcal{F}(\rho_i) \) are irreducible, \( \hat{R}, \hat{R} \) is a standard solution of the conjugate equations for \( \mathcal{F}(\rho) \).
For future use we note that $S_{\rho, \sigma}$ continues to be a natural transformation when antilinear intertwiners are allowed, as follows from the next result.

3.9 Lemma If $\mathcal{F} : \mathcal{F} \rightarrow \mathcal{H}$ is a quasitensor functor then for the antilinear operators $J$ associated with solutions $\hat{R}, \hat{R}$ of the conjugate equations we have

$$J_{\rho \otimes \sigma} S_{\rho, \sigma} = S_{\sigma, \rho} J_{\rho} \circ \theta_{\rho, \sigma},$$

provided $R_{\rho \otimes \sigma} := 1_{\hat{\sigma}} \otimes R_{\rho} \otimes 1_{\sigma} \circ R_{\rho'}$.

Remark For a discussion of antilinear arrows see [19], where it is pointed out that $J_{\rho} \otimes J_{\rho} \circ \theta_{\rho, \sigma}$ is a natural tensor product to use for antilinear arrows.

Proof. Identifying $\mathcal{F}(\rho) \otimes \mathcal{F}(\sigma)$ with a subspace of $\mathcal{F}(\rho \otimes \sigma)$, we have to show that

$$J_{\rho \otimes \sigma} \psi \otimes \phi = J_{\sigma} \phi \otimes J_{\rho} \psi, \quad \psi \in \mathcal{F}(\rho), \quad \phi \in \mathcal{F}(\sigma).$$

If $\chi \in \mathcal{F}(\rho \otimes \sigma)$ then $r_{\psi, \phi}^* \circ \hat{R}_{\rho \otimes \sigma} = J_{\rho \otimes \sigma} \chi$. Hence

$$J_{\rho \otimes \sigma} \psi \otimes \phi = r_{\psi, \phi}^* \circ \hat{R}_{\rho \otimes \sigma} =$$

$$r_{\psi, \phi}^* \circ E_{\sigma \otimes \rho, \rho \otimes \sigma} \mathcal{F}(1_{\rho} \otimes R_{\rho} \otimes 1_{\sigma} \circ R_{\sigma}) =$$

$$r_{\psi, \phi}^* \circ 1_{\mathcal{F}(\sigma \otimes \rho)} \circ E_{\sigma, \rho \otimes \sigma} \mathcal{F}(1_{\rho} \otimes R_{\rho} \otimes 1_{\sigma} \circ R_{\sigma}).$$

On the other hand by (3.9),

$$1_{\mathcal{F}(\sigma \otimes \rho)} \circ E_{\rho, \sigma} \circ E_{\sigma \otimes \rho, \rho \otimes \sigma} = E_{\sigma \otimes \rho, \rho \otimes \sigma} \circ 1_{\mathcal{F}(\sigma \otimes \rho)},$$

hence the last term above equals

$$r_{\psi, \phi}^* \circ E_{\sigma \otimes \rho, \rho \otimes \sigma} \circ 1_{\mathcal{F}(\sigma)} \circ E_{\sigma \otimes \rho, \rho \otimes \sigma} \circ \mathcal{F}(1_{\rho} \otimes R_{\rho} \otimes 1_{\sigma} \circ R_{\sigma}).$$

(3.13)

Now by (3.11),

$$\mathcal{F}(1_{\rho} \otimes R_{\rho}^* \otimes 1_{\sigma}) \circ E_{\sigma \otimes \rho, \rho \otimes \sigma} = \mathcal{F}(1_{\rho} \otimes R_{\rho}^*) \circ 1_{\mathcal{F}(\sigma)} \circ E_{\sigma \otimes \rho, \rho \otimes \sigma} =$$

$$E_{\sigma, \sigma} \circ \mathcal{F}(1_{\rho} \otimes R_{\rho}^*) \circ 1_{\mathcal{F}(\sigma)} \circ E_{\sigma, \sigma},$$

hence taking the adjoint,

$$E_{\sigma \otimes \rho, \rho \otimes \sigma} \circ \mathcal{F}(1_{\rho} \otimes R_{\rho} \otimes 1_{\sigma}) = \mathcal{F}(1_{\rho} \otimes R_{\rho}) \otimes 1_{\mathcal{F}(\sigma)} \circ E_{\sigma, \sigma} =$$

$$1_{\mathcal{F}(\sigma)} \otimes \mathcal{F}(R_{\rho}) \otimes 1_{\mathcal{F}(\sigma)} \circ E_{\sigma, \sigma},$$

and (3.13) becomes

$$r_{\psi, \phi}^* \circ E_{\sigma \otimes \rho, \rho \otimes \sigma} \circ 1_{\mathcal{F}(\sigma)} \circ \mathcal{F}(R_{\rho}) \otimes 1_{\mathcal{F}(\sigma)} \circ E_{\sigma, \sigma} \circ \mathcal{F}(R_{\rho}).$$

(3.14)

Now

$$E_{\sigma \otimes \rho, \rho \otimes \sigma} \circ E_{\sigma, \rho \otimes \rho} = 1_{\mathcal{F}(\sigma)} \otimes E_{\sigma, \rho \otimes \rho} \circ E_{\sigma, \rho \otimes \rho},$$

and

$$E_{\sigma, \rho \otimes \sigma} \circ 1_{\mathcal{F}(\sigma)} \circ \mathcal{F}(R_{\rho}) = 1_{\mathcal{F}(\sigma)} \otimes \mathcal{F}(R_{\rho})$$

so (3.14) equals

$$r_{\psi, \phi}^* \circ 1_{\mathcal{F}(\sigma)} \circ \hat{R}_{\rho} \otimes 1_{\mathcal{F}(\sigma)} \circ \hat{R}_{\rho} = J_{\rho} \phi \otimes J_{\rho} \psi$$

as required.
4 Concrete quasitensor functors

Let $G$ be a compact quantum group. In this section we construct quasitensor $^*$-functors $\text{Rep}(G) \to \mathcal{H}$ which associate a subspace $K_u$ of the representation Hilbert space $H_u$ with the representation $u$, and generalize the invariant vectors functor associated with a compact quantum subgroup $K$ of a compact quantum group $G$ described in Example 3.6.

For each unitary representation $u$ of $G$, we suppose assigned a subspace $K_u$ of the representation Hilbert space $H_u$, and we look for sufficient conditions on the projection maps $E_u : H_u \to K_u$.

4.1 Lemma For $u \in \text{Rep}(G)$, let $E_u : H_u \to H_u$ be an orthogonal projection such that

$$E_u = 1_C,$$

$$TE_u = E_u T, \quad T \in (u,v),$$

$$E_u \otimes E_v = I \otimes E_v \circ E_u \otimes v.$$

Then the $^*$-functor:

$$K_u := E_u H_u,$$

$$K_T := T \left. \right|_{K_u} \in (K_u,K_v)$$

is quasitensor.

Proof Property (3.1) follows from (4.1). Multiplying both sides of (4.3) on the right by $E_u \otimes v$ and taking the adjoint gives

$$E_u \otimes E_v \circ E_u \otimes v = E_u \otimes E_v = E_u \otimes v,$$

so the subspace $K_u \otimes K_v$ of $H_u \otimes H_v$ is contained in $K_u \otimes v$, and this shows (3.7). We show (3.11). For $S \in (u,u')$, $T \in (v,v')$:

$$K_S \otimes T \left. \right|_{K_u \otimes K_v} = ((S \otimes T) \left. \right|_{K_u \otimes v} \right) \left. \right|_{K_u \otimes K_v} =$$

$$(S \otimes T) \left. \right|_{K_u \otimes K_v} = K_S \otimes K_T.$$

In this case the projection $E_{u,v} : K_u \otimes v \to K_u \otimes K_v$ is given by the restriction of $E_u \otimes E_v$ to $K_u \otimes v$, so (3.8) follows easily. Since the copy of $K_u \otimes K_v \otimes K_z$ sitting inside $K_u \otimes K_v \otimes K_z$ and $K_u \otimes v \otimes K_z$ is, in both cases, the subspace $K_u \otimes v \otimes K_z$ of $H_u \otimes H_v \otimes H_z$, (3.9) is satisfied. It remains to check (3.10). If $H$ and $H'$ are Hilbert spaces, we consider the operators of tensoring on the right of $H$ by vectors $\phi \in H'$:

$$r_\phi : \psi \in H \to \psi \otimes \phi \in H \otimes H'.$$

Now (4.3) implies that $E_u \otimes E_v \circ E_u \otimes v = I \otimes E_v \circ E_u \otimes v$. Thus if $\psi \in K_u \otimes v$ and $\phi \in K_v$ then $r_\psi(\phi) \in K_u$. If then $\psi' \in K_z$ then $\psi' (r_\phi \psi) \in K_z \otimes K_u$.

Now choose $\phi = \phi_k$, an orthonormal basis of $K_v$, apply the operator $r_{\phi_k}$, use the fact that on $K_z \otimes v$, $\sum_k r_{\phi_k} r_{\phi_k}^*$ converges strongly to the projection map $E_{z \otimes u,v}$ onto $K_z \otimes v$, and obtain the desired relation.
The property that the isometric inclusion map $K_u \otimes K_v \subset K_{u \otimes v}$ be simply the identity map is well described by the notion of quasitensor natural transformation.

4.2 Definition Let $\mathcal{F}, \mathcal{G}$ be quasitensor $\ast$–functors from a tensor $C^\ast$–category $\mathcal{T}$ to the tensor $C^\ast$–category of Hilbert spaces $\mathcal{H}$. Let $S^\mathcal{F}_{\rho,\sigma}$ and $S^\mathcal{G}_{\rho,\sigma}$ be the defining set of isometries for $\mathcal{F}$ and $\mathcal{G}$ respectively. A natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ will be called quasitensor if for objects $\rho, \sigma \in \mathcal{T}$,

$$\eta : \mathcal{F}(\iota) = \mathbb{C} \to \mathcal{G}(\iota) = \mathbb{C}$$

is the identity map

$$\eta_{\rho \otimes \sigma} \circ S^\mathcal{F}_{\rho,\sigma} = S^\mathcal{G}_{\rho,\sigma} \circ \eta_\rho \otimes \eta_\sigma.$$

4.3 Proposition Let $K : \text{Rep}(G) \to \mathcal{H}$ be the quasitensor $\ast$–functor obtained from projections $(E_u)$ satisfying properties (4.1)–(4.3). Then the inclusion map $W_u : K_u \to H_u$ defines a quasitensor natural transformation from the functor $K$ to the embedding functor $H : \text{Rep}(G) \to \mathcal{H}$.

We next show that the invariant vectors functors associated with quantum subgroups, fit into this description. Let $K$ be a compact quantum subgroup of $G$ with Haar measure $h'$, and, for an invertible representation $u$ of $G$, let $E^K_u : H_u \to H_u$ be the idempotent onto the subspace of $u \mid_K$–fixed vectors obtained averaging over the $K$–action: $E^K_u(\psi) = \iota \otimes h' \circ u \mid_K (\psi)$. If $u$ is unitary, $E^K_u$ is a selfadjoint projection. Indeed,

$$(E^K_u(\psi_1), \psi_2) = h'(\pi(u_{j_1})) = h'(\pi(u_{j_2}')) =$$

$$h'(\kappa'(\pi(u_{j_1})) = h'(\pi(u_{j_2})) = (\psi_1, E^K_u(\psi_2)),$$

since the Haar measure $h'$ of $K$ is left invariant by the coinverse $\kappa'$ [25].

4.4. Lemma If $u$ and $v$ are invertible and finite dimensional $G$–representations, and $K$ is a compact quantum subgroup of $G$ then for any $T \in (u \mid_K, v \mid_K)$, thus in particular for any $T \in (u, v)$,

$$T \circ E^K_u = E^K_v \circ T.$$  

Proof For $\psi \in H_u$,

$$(T \circ E^K_u(\psi) = T \iota \otimes h' \circ u \mid_K (\psi) =$$

$$\iota \otimes h'(T \otimes Iu \mid_K (\psi)) = \iota \otimes h'(v \mid_K (T(\psi))) = E^K_v \circ T(\psi).$$

We thus obtain another proof of the fact that the invariant vectors functor is quasitensor.
4.5 Theorem If $K$ is a compact quantum subgroup of $G$, the projections $u \rightarrow E^K_u$ satisfy properties (4.1)–(4.3). Therefore the associated invariant vectors functor is a quasitensor $^*$–functor and the inclusion map $K_u \rightarrow H_u$ is a quasitensor natural transformation from the functor $K$ to the embedding functor $H$.

Proof We check properties (4.1)–(4.3) on the projections $E^K_u$. Clearly (4.1) is verified, and (4.2) follows from the previous lemma. Since the tensor product of $K$–invariant vectors is a $K$–invariant vector for the tensor product representation, we have: $E^K_u \otimes E^K_v \leq E^K_{u \otimes v}$, and (4.3) becomes equivalent to

$$E^K_u \otimes E^K_v E^K_{u \otimes v} = I \otimes E^K_v \circ E^K_{u \otimes v},$$

or, in other words, to the fact that if $\psi \in K_v$, $\eta \in K_{u \otimes v}$ then $r^*_\psi(\eta) \in K_u$. In order to verify this property, we use the fact that the restriction to $K$ of a tensor product representation is the tensor product of the restrictions, so

$$\eta \in K_{u \otimes v} = (t, (u \otimes v) | K) = (t, u | K) \otimes v | K).$$

Using $\psi \in K_v = (t, v | K)$, too, gives,

$$r^*_\psi(\eta) = 1_{u|K} \otimes r^*_\psi \circ \eta \in (t, u | K) = K_u,$$

where the operations are to be understood in the tensor $C^*$–category $\text{Rep}(K)$.

Remark Notice that a similar argument shows that the projections $E^K_u$ also satisfy

$$E^K_u \otimes E^K_v = E^K_{u \otimes v} + E^K_v \circ E^K_{u \otimes v}. \quad (4.4)$$

Remark Notice that if $G$ is a maximal quantum group (i.e. obtained from its smooth part $A_{\infty}$ by completing with respect to the maximal $C^*$–seminorm) and $K$ is the trivial subgroup (corresponding to the counit $\varepsilon : A \rightarrow \mathbb{C}$ of $G$) then the associated functor $K$ coincides with the embedding functor $H : \text{Rep} G \rightarrow \mathcal{M}$, which is tensor. At the other extreme, if $K = G$, $K_u = (t, u)$, and this is not a tensor functor, as $K_u = K_{\pi} = 0$, for example, if $u$ is irreducible, but $K_{\pi \otimes u} = (t, \pi \otimes u) \neq 0$.

We next construct certain maps which will be useful later on when describing multiplicities of spectral representations in quantum quotient spaces. Given a unitary representation $u : H_u \rightarrow H_u \otimes A$, we set

$$u^K := E^K_u \otimes I \circ u : H_u \rightarrow K_u \otimes A.$$

For any pair of vectors $\phi, \psi \in H_u$, consider the coefficients of $u^K$:

$$u^K_{\phi, \psi} := t^*_\phi \circ u^K(\psi) = t^*_\phi E^K_\phi \circ u(\psi) = u_{E^K_\phi, \psi},$$

which belong to $A^K_{sp}$. So the range of $u^K$ is actually contained in $K_u \otimes A^K_{sp}$:

$$u^K : H_u \rightarrow K_u \otimes A^K_{sp}.$$

The map $u \rightarrow u^K$ satisfies the following properties.

4.6 Proposition Let $u$ and $v$ be unitary representations of $G$. 
a) For any intertwiner \( T \in (u, v) \),
\[
K_T \otimes I \circ u^K = v^K \circ T,
\]

b) For \( \phi \in K_u, \phi' \in K_v, \psi \in H_u, \psi' \in H_v, \)
\[
(u \otimes v)^K_{\phi \otimes \phi', \psi \otimes \psi'} = u^K_{\phi, \psi} v^K_{\phi', \psi'}.
\]

**Proof**
a) \[
v^K \circ T = E^K_v \otimes I \circ v \circ T = E^K_v \otimes I \circ T \otimes I \circ u = T \otimes I \circ E^K_u \otimes I \circ u = K_T \otimes I u^K.
\]

Property b) follows from the fact that \( K_u \otimes K_v \subset K_{u \otimes v} \), so
\[
(u \otimes v)^K_{\phi \otimes \phi', \psi \otimes \psi'} = u \otimes v^K_{\phi \otimes \phi', \psi \otimes \psi'} = u^K_{\phi, \psi} v^K_{\phi', \psi'}.
\]

5 Characterizing the invariant vectors functor

For each \( u \in \text{Rep}(G) \), let \( E_u \) be an orthogonal projection on the representation Hilbert space \( H_u \) satisfying properties (4.1)–(4.3). In terms of the Hilbert spaces \( K_u = E_u H_u \), these conditions can be written
\[
K_\iota = H_\iota = \mathbb{C}, \quad \text{(5.1)}
\]
\[
TK_u \subset K_v, \quad T \in (u, v), \quad \text{(5.2)}
\]
\[
r^*_k K_{u \otimes v} \subset K_u, \quad k \in K_v, \quad \text{(5.3)}
\]
\[
K_u \otimes K_v \subset K_{u \otimes v}. \quad \text{(5.4)}
\]

As a consequence of these conditions, one gets
\[
(\iota, \iota \otimes v) \subset K_{u \otimes v}. \quad \text{(5.5)}
\]

In fact, pick \( T \in (\iota, u \otimes v) \). Properties (5.1) and (5.2) show that for any \( \lambda \in K_\iota = \mathbb{C}, T\lambda \in K_{u \otimes u}, \) so \( T = T1 \in K_{u \otimes u} \). In particular, thanks to (5.3),
\[
r^*_k T \in K_u, \quad T \in (\iota, u \otimes v), \quad k \in K_v. \quad \text{(5.6)}
\]

Now let \( \overline{u} \) be a conjugate of \( u \) defined by \( R \in (\iota, \overline{u} \otimes u), \overline{R} \in (\iota, u \otimes \overline{u}) \), and let \( j : H_u \rightarrow H_{\overline{u}} \) be the associated antilinear invertible intertwiner defined by:
\[
R = \sum_i j \phi_i \otimes \phi_i.
\]
5 CHARACTERIZING THE INVARIANT VECTORS Functor

\[
\mathcal{R} = \sum_j j^{-1}\psi_j \otimes \psi_j,
\]

with \((\phi_i)\) and \((\psi_j)\) orthonormal bases of \(H_u\) and \(H_\pi\) respectively, that we choose to complete the orthonormal bases of the corresponding subspaces \(K_u\) and \(K_\pi\). Property (5.6) applied to \(R\) and \(\mathcal{R}\) shows that

\[
jK_u = K_\pi.
\]  

(5.7)

When \(G\) is a group, conditions (5.1), (5.2), (5.4) and (5.7) are known to characterize a subgroup of \(G\) with the property that each \(K_u\) is the invariant subspace of \(u\) in restriction to the subgroup \([21]\).

We next show that the antilinear invertible operator \(j^*\) defined by a solution of the conjugate equations may be regarded as an antilinear intertwiner from \(\bar{u}\) to \(u\). The intertwining property of \(R\), when expressed in coordinate form, reads

\[
\sum_{i,p} u_{ji} \bar{u}_{jp} \mathcal{R}_{(ip)} = \mathcal{R}_{(jq)}.
\]

Expressed in terms of \(j\) and using the unitarity of \(u\), we get

\[
\sum_p \bar{u}_{qp}(\psi_p, j^{-1}\varphi_k) = \sum_j u^*_{kj}(\psi_q, j^{-1}\varphi_j).
\]

Now, if \(A \in \mathcal{A}\),

\[
(\psi_q \otimes A, j^{-1}\varphi) = \sum_{i,j} (\psi_q, j^{-1}\varphi_j) A^* u^*_{ji}(\varphi, \varphi_i) = \sum_p A^* \bar{u}_{qp}(\psi_p, j^{-1}\varphi)
\]

\[
= \sum_p ((\psi_q \otimes A, \bar{u}\psi_p)(\psi_p, j^{-1}\varphi) = (\psi_q \otimes A, \bar{u} j^{-1}\varphi).
\]

Thus \(j^{-1}\varphi = \bar{u} j^{-1}\varphi\) and therefore \(j^* \otimes^* u = \bar{u} j^*\), which is the form of the intertwining relation for an antilinear operator. Note that this intertwining relation could alternatively be deduced as above by starting with \(R\) rather than \(\mathcal{R}\).

Now, in view of what happens for compact groups, the question naturally arises of whether, given a compact quantum group \(G\) and subspaces \(K_u \subset H_u\) for each finite-dimensional unitary representation \(u\) of \(G\), conditions (5.1), (5.2), (5.4) and (5.7) are still sufficient for the existence of a unique compact quantum subgroup of \(G\) whose the subspaces of invariant vectors are the \(K_u\).

Uniqueness does not hold in general for compact quantum groups. In fact, one can have a proper subgroup \(K\) of a compact quantum group \(G\) with \(K_u = (\iota, u)\) for any representation \(u\) of \(G\): consider a group \(G\) with a nonfaithful Haar measure \(h\) and form the reduced group \(G_{\text{red}} = (A_{\text{red}}, \Delta_{\text{red}})\) obtained completing the dense Hopf \(*\)-subalgebra in the norm defined by the GNS representation \(\pi_h\). Since \(\pi_h: A \rightarrow A_{\text{red}} \) is a surjection intertwining the corresponding coproducts, \(G_{\text{red}}\) becomes a subgroup of \(G\). Since \(h\) is not faithful, \(\pi_h\) has a nontrivial kernel.
In this sense, $G_{\text{red}}$ is a proper subgroup of $G$. Furthermore $G_{\text{red}}$ has a faithful Haar measure, while $G$ has not, so $G$ and $G_{\text{red}}$ are not isomorphic as Hopf $C^*$-algebras. However, $G$ and $G_{\text{red}}$ have the same representation categories, and therefore the same spaces of invariant vectors.

We could have also considered the maximal compact quantum group, $G_{\text{max}} = (A_{\text{max}}, \Delta_{\text{max}})$ obtained completing the dense Hopf $^*$-subalgebra with respect to the maximal $C^*$-seminorm. There is again a surjection $\pi : A_{\text{max}} \to A$ intertwining the coproducts, so $G$ is a subgroup of $G_{\text{max}}$, and we still have

$$\text{Rep}(G_{\text{max}}) = \text{Rep}(G) = \text{Rep}(G_{\text{red}}),$$

so $G_{\text{red}}$ is also a proper subgroup of $G_{\text{max}}$ with the same spaces of invariant vectors as $G_{\text{max}}$. This example suggests that one possible way to get uniqueness is to restrict attention to maximal compact quantum groups.

As far as existence is concerned, we start by showing the following result.

**5.1 Theorem** Let $G = (A, \Delta)$ be a compact quantum group, and, for each $u \in \text{Rep}(G)$, let $K_u$ be a subspace of the representation Hilbert space $H_u$ satisfying conditions (5.1), (5.2), (5.4) and (5.7). Then there exists a compact quantum subgroup $K$ of $G$ such that $K_u \subset (t_K, u | K)$ for $u \in \text{Rep}(G)$ and such that the linear span of $\{u_{k,\phi}, k \in K_u, \phi \in H_u, u \in \text{Rep}(G)\}$ is a unital $^*$-subalgebra of the quantum quotient space $A^\delta$.

**Proof** Consider a complete set $A$ of irreducible representations $u$ of $G$ such that $K_u \neq 0$. Let $M$ denote the linear span generated by the set

$$\{x_{\phi, k}^u := u_{\phi, k} - (\phi, k)I, u \in A, k \in K_u, \phi \in H_u\},$$

in the Hopf $C^*$-algebra $A$. It is easy to check that

$$\Delta(x_{\phi, k}^u) = \sum_i u_{\phi_i, \phi} \otimes x_{\phi_i, k}^u + x_{\phi, k}^u \otimes I,$$

with $(\phi_i)$ an orthonormal basis of $H_u$. Therefore $\Delta(M) \subset A_{\infty} \otimes M + M \otimes C(I).$ Let $J$ be the closed two–sided ideal of $A$ generated by $M$. Then $\Delta(J) \subset A \otimes J + J \otimes A$, so $J$ is a closed bi–ideal. Consider the associated compact quantum subgroup $K = (A/J, \Delta')$ of $G$ with coproduct $\Delta'(q(a)) = q \otimes q \circ \Delta(a)$, where $q : A \to A/J$ is the canonical surjection. We show that $K_u \subset (t_K, u | K)$. For $k \in K_u$,

$$u | K (k) = \iota \otimes q \circ u(k) = \iota \otimes q(\sum_i \phi_i \otimes u_{\phi, k}) = \sum_i \phi_i \otimes (u_{\phi, k}) = \sum_i \phi_i \otimes (\phi_i, k)I = k \otimes I.$$

We are left to show that the linear span $V$ of all the $u_{k,\phi}$ is a unital $^*$-subalgebra of $A^\delta$. Since $K_u \subset (t, u | K)$, $V$ is contained in $A^\delta$, and $I \in V$, as $K_i = \mathbb{C}$. Therefore it suffices to show that $V$ is a $^*$-subalgebra of $A$. On the other hand
the $^*$–algebra structure of $A$ recalled at the end of subsection 2.1 and properties (5.4) and (5.7) show that $V$ is a $^*$–subalgebra. 

**Remark** Under conditions (5.1), (5.2), (5.4) and (5.7) alone, one can not, in general, identify the subspace $K_u$ with the space of all the invariant vectors $(\iota_K, u \upharpoonright_K)$ for some quantum subgroup $K$ of $G$. In fact there is an example, due to Wang [23] of an ergodic action $\delta$ on a commutative $C^*$–algebra $C$ which is not a quotient action. Now the commutativity of $C$ allows a faithful embedding of $(C_{sp}, \delta)$ into a quantum quotient space by a quantum subgroup and a construction of subspaces $K_u$ satisfying the above equations (see Theorem 11.4). Such an embedding, though, can not extend to an isomorphism of the whole of $C$ and therefore the $K_u$ can not be the spaces of all the invariant vectors.

We shall be able, though, to give a positive answer if we replace (5.4) by a stronger, still necessary, condition, motivated by the following argument.

In the group case, the representation category of $G$ contains, among its intertwiners, the *permutation symmetry* operators $\theta_{u,v} \in (u \otimes v, v \otimes u)$ permute the order of factors in the tensor product. Consequently, for $u, v, z \in \text{Rep}(G)$,

$$1_u \otimes \phi \otimes 1_z \circ k \in K_{u \otimes v \otimes z}, \quad k \in K_{u \otimes z}, \phi \in K_v. \quad (5.8)$$

In fact, by (5.4), $\phi \otimes k \in K_{v \otimes u \otimes z}$, so $1_u \otimes \phi \otimes 1_z \circ k = (\partial_{v,u} \otimes 1_z)\phi \otimes k$ and this is an element of $K_{u \otimes v \otimes z}$ thanks to (5.2).

On the other hand, (5.8) is still a necessary condition for the $K_u$ to be the invariant subspaces of the restriction of $u$ to a quantum subgroup, as one can easily show using the same argument as the one used, in Theorem 4.5, to show the necessity of condition (4.3). Therefore, in the quantum group case, it seems natural to replace (5.4) by the stronger condition (5.8).

Assume then that we have a functor $K$ associating to any representation $u \in \text{Rep}(G)$ a subspace $K_u \subset H_u$ satisfying (5.1), (5.2), (5.7) and (5.8). We consider, for $u, v \in \text{Rep}(G)$, the subspace of $(H_u, H_v)$ defined by

$$<H_u, H_v> := \{R^* \otimes 1_v \circ 1_u \otimes \phi, \phi \in K_{\overline{\pi} \otimes v}\},$$

where $R \in (\iota, u \otimes \overline{\pi})$ is an intertwiner arising from a solution of the conjugate equations for $u$ in $\text{Rep}(G)$.

**5.2 Proposition** The space $<H_u, H_v>$ is independent of the choice of $\overline{\pi}$ and $\overline{R}$.

**Proof** If $R' \in (\iota, u \otimes \overline{\pi})$ arises from another solution to the conjugate equations then there exists a unitary $U \in (\overline{\pi}, \overline{\pi})$ such that $R' = 1_u \otimes U \circ \overline{R}$ [15]. Since $U \otimes 1_v \in (\overline{\pi} \otimes \pi, \overline{\pi} \otimes v)$, $U \otimes 1_v K_{\overline{\pi} \otimes v} = K_{\overline{\pi} \otimes v}$ by (5.2). Therefore any $\phi' \in K_{\overline{\pi} \otimes v}$ is of the form $U \otimes 1_v \phi$ with $\phi \in K_{\overline{\pi} \otimes v}$. This implies $R'^* \otimes 1_v \circ 1_u \otimes \phi' = \overline{R^*} \otimes 1_v \circ 1_u \otimes \phi$.

**5.3 Proposition** For $u, v \in \text{Rep}(G)$, $(u, v) \subset <H_u, H_v>$.

**Proof** Fix a pair $R \in (\iota, \overline{\pi} \otimes u) \overline{R} \in (\iota, u \otimes \overline{\pi})$ solving the conjugate equations. Pick $T \in (u, v)$ and set $\phi_T := 1_v \otimes T \circ R$, So $\phi_T \in (\iota, \overline{\pi} \otimes v) \subset K_{\overline{\pi} \otimes v}$. Since

$$\overline{R^*} \otimes 1_v \circ 1_u \otimes \phi_T = \overline{R^*} \otimes 1_v \circ 1_u \otimes 1_{u \otimes \overline{\pi}} \otimes T \circ 1_u \otimes R =$$
we can conclude that $T \in < H_u, H_v >$.

**5.4 Proposition** The Hilbert spaces $H_u$ and the subspaces $< H_u, H_v >$, as $u$ and $v$ vary in $\text{Rep}(G)$, form, respectively, the objects and the arrows of a tensor $\ast$–subcategory of the category of Hilbert spaces. This category contains the image of $\text{Rep}(G)$ under the embedding functor $H$, therefore it has conjugates.

**Proof** Since $(u, v) \subset < H_u, H_v >$, $1_u \in < H_u, H_u >$ for $u \in \text{Rep}(G)$. We show that if $T \in < H_u, H_v >$ and $S \in < H_v, H_u >$ then $S \circ T \in < H_u, H_v >$. In fact, writing $T = \overline{R}_u \otimes 1_v \circ 1_u \otimes \phi$ and $S = \overline{R}_v \otimes 1_z \otimes 1_v \otimes \psi$ then

$$
S \circ T = \overline{R}_u \otimes 1_z \circ 1_v \otimes \psi \circ \overline{R}_u \otimes 1_v \circ 1_u \otimes \phi =
$$

$$
\overline{R}_v \otimes 1_z \circ \overline{R}_u \otimes 1_v \otimes \psi \circ 1_u \otimes \phi =
$$

$$
\overline{R}_u \otimes 1_z \circ 1_u \otimes \psi \circ 1_v \otimes \phi =
$$

$$
\overline{R}_u \otimes 1_z \circ 1_u \otimes (1_v \otimes \overline{R}_v \otimes 1_z (\phi \otimes \psi)) \in < H_u, H_v >
$$

as $\phi \otimes \psi \in K_{1_v \otimes 1_z}$ and $(1_v \otimes \overline{R}_v \otimes 1_z)(\phi \otimes \psi) \in K_{1_v \otimes 1_u \otimes 1_z}$ by (5.2).

We next show that $< H_u, H_v > \ast = < H_v, H_u >$. Pick $\phi \in K_{1_v \otimes 1_u \otimes 1_v}$ and set $T = \overline{R}_u \otimes 1_v \circ 1_u \otimes \phi$. Then $T^\ast = 1_u \otimes \phi^* \circ \overline{R}_u \otimes 1_v$. We use the solution $R_{1_v \otimes 1_u} := 1_v \otimes \overline{R}_u \otimes 1_u \otimes R_u \in (u, \pi \otimes u \otimes \overline{\pi} \otimes v)$ and $R_{1_v \otimes 1_u} := 1_v \otimes \overline{R}_v \otimes 1_u \circ R_u \in (u, \pi \otimes v \otimes \overline{\pi} \otimes u)$ of the conjugate equations for $\pi \otimes v$. Thanks to (5.9), $\psi := 1_v \otimes \phi^* \circ \overline{R}_v \otimes v = j \phi \in \pi_{1_v \otimes 1_u \otimes v}$. We have:

$$
\overline{R}_v \otimes 1_u \circ 1_v \otimes \psi = \overline{R}_v \otimes 1_u \circ 1_v \otimes \phi^* \circ 1_v \otimes \overline{R}_u \otimes 1_v \circ 1_u \otimes R_v =
$$

$$
1_u \otimes \phi^* \circ \overline{R}_u \otimes 1_v \circ \overline{R}_v \otimes 1_v \circ 1_u \otimes R_v = 1_u \otimes \phi^* \circ \overline{R}_u \otimes 1_v = T^\ast.
$$

Therefore $T^\ast \in < H_v, H_u >$.

We are left to show that if $S = \overline{R}_u \otimes 1_v \circ 1_u \otimes \phi \in < H_u, H_v >$, $T = \overline{R}_u \otimes 1_v \circ 1_u \otimes \psi \in < H_u, H_v >$ then $S \circ T \in < H_u \otimes v, H_v \otimes u >$. Consider the following solution to the conjugate equations for $u \otimes v$: $R_{u \otimes v} = 1_v \otimes R_u \circ 1_v \otimes R_v$, $\overline{R}_{u \otimes v} = 1_u \otimes \overline{R}_u \circ 1_v \otimes \overline{R}_u$. Since $\eta := 1_v \otimes \phi \circ 1_v \otimes \psi \in K_{1_v \otimes 1_v \otimes v} \otimes v^\prime}$ by (5.8) then $\overline{R}_{u \otimes v} \circ 1_u \otimes \eta \in < H_u \otimes v, H_v \otimes u^\prime >$. On the other hand,

$$
\overline{R}_u \otimes 1_u \otimes \eta =
$$

$$
\overline{R}_u \otimes 1_u \otimes \phi \circ 1_v \otimes \psi =
$$

$$
\overline{R}_u \otimes 1_u \otimes \phi \circ 1_v \otimes 1_u \otimes \psi =
$$

$$
S \otimes 1_v \circ 1_u \otimes T = S \otimes T.
$$

A compact quantum subgroup of $G$ will be called maximal if it is maximal as a compact quantum group (recall that this means that the norm on the
dense Hopf $^*$–subalgebra coincides with the maximal $C^*$–seminorm). The above results lead to the following theorem.

\section{Characterizing the Invariant Vectors Functor}

\begin{theorem}
Consider the tensor $C^*$–category of finite dimensional unitary representations of a maximal compact quantum group $G$. Suppose for each representation $u$ of $G$ on a Hilbert space $H_u$ there is given a subspace $K_u \subset H_u$ satisfying properties (5.1), (5.2), (5.7) and (5.8). Then the $K_u$ are the subspaces of invariant vectors for a unique maximal compact quantum subgroup of $G$.
\end{theorem}

\begin{proof}
Consider the category $\mathcal{J}$ obtained completing the spaces $< H_u, H_v >$, for $u, v \in \text{Rep}(G)$, with respect to subobjects and direct sums. This is still a tensor $^*$–category of Hilbert spaces with conjugates, now with subobjects and direct sums. Thanks to Woronowicz’s Tannaka–Krein Theorem, we can construct a Hopf $^*$–algebra $(\mathcal{C}, \Delta')$. Since $\text{Rep}(G) \subset \mathcal{J}$, this Hopf $^*$–algebra is a model, in the sense of $[26]$, for the Hopf $^*$–algebra constructed from $\text{Rep}(G)$, which, in turn, is isomorphic to $(A_{\infty}, \Delta)$. Therefore we can find a $^*$–epimorphism $\pi : A_{\infty} \to \mathcal{C}$ such that $\Delta' \circ \pi = \pi \otimes \pi \circ \Delta$. Set $\delta := \ker(\pi)$. This is obviously a $^*$–ideal, but also an algebraic coideal, as if $a \in \delta$ then $b := \pi(\Delta(a))$ must belong to the kernel of $\pi \otimes \mathcal{C}$. Since $\ker(\pi \otimes \mathcal{C}) = \delta \otimes \mathcal{C} = \text{Image}(\delta \otimes \pi)$, we can find $c \in \delta \otimes A$ such that $b = \delta \otimes \pi(c)$. Set $d := \Delta(a) - c$, which is easily checked to lie in $\ker(\delta \otimes \pi)$, that in turn, equals $\delta \otimes \mathcal{J}$. Then $\Delta(a) = c + d \in \delta \otimes A \oplus \delta \otimes \mathcal{J}$. Consider the completion $\mathcal{A}'$ of $\mathcal{C}$ in the maximal $C^*$–seminorm, so $\Delta'$ extends to the completion. We thus obtain a compact quantum group $(\mathcal{A}', \Delta')$ and a $^*$–homomorphism $\tilde{\pi} : A_{\infty} \to \mathcal{A}'$ intertwining $\Delta$ with $\Delta'$ and with the same kernel as $\pi$, because the inclusion $\mathcal{C} \subset \mathcal{A}'$ is faithful. Therefore we can extend $\tilde{\pi}$ to a $^*$–homomorphism $\eta$ from $\mathcal{A}$ to $\mathcal{A}'$ intertwining the coproducts. This $^*$–homomorphism is a surjection, as its range contains $\mathcal{C}$, which is dense in $\mathcal{A}'$. Thus $(\mathcal{A}', \Delta')$ is a maximal compact quantum subgroup of $G$, that we denote, with abuse of notation, by $K$. Since $\eta(A_{\infty}) = \delta$, and for any quantum subgroup one always has $\eta(A_{\infty}) = A'_{\infty}$, we deduce that $\mathcal{C} = A'_{\infty}$. Therefore the representation category of the subgroup $K$ coincides with the category we started out with: $\text{Rep}(K) = \mathcal{J}$. In particular for any representation $u$ of $G$, $(u K, u \mid K) = < H_u, H_u >= K_u$. If $K_1 = (A_1, \Delta_1)$ is another maximal quantum subgroup of $G$ with spaces of invariant vectors given by the $K_u$, then by Frobenius reciprocity, for any pair of representations $u$ and $v$ of $G$, $(u \mid K_1, v \mid K_1)$ is canonically linearly isomorphic to

\[(<u,K_1 \otimes v \mid K_1> = (u,K_1 \otimes v \mid K_1) = K_{\pi \otimes v}v).
\]

But $(u | K_1, v | K_1)$ is also linearly isomorphic, according to the same isomorphism, to $K_{\pi \otimes v}$, so $(u | K_1, v | K_1) = (u | K, v | K)$. Hence $\text{Rep}(K_1) = \text{Rep}(K)$. It follows that there is a $^*$–isomorphism $\eta : A_{1\infty} \to A'_{\infty}$ intertwining the corresponding coproducts, which extends to a $^*$–isomorphism of the completions.

We now give two constructions of subspaces $K_u \subset H_u$ satisfying our equations making no mention of a compact quantum group. Let $\mathcal{I}$ be a tensor $C^*$–subcategory with conjugates of a tensor category of Hilbert spaces. The objects of $\mathcal{I}$ will be denoted by $u, v, \ldots$ as in the category of finite dimensional representations of a compact quantum group whilst the arrows will be written...
for example as $T \in (u,v)$. We suppose that $T$ comes equipped with a permutation symmetry $\varepsilon$. We now define

$$K_u = \{ \varphi \in H_u : \varepsilon(u,v)\varphi \otimes 1_v = 1_v \otimes \varphi \text{ for all objects } v \}. $$

Note that the defining condition can be replaced by $\varepsilon(v,u)1_v \otimes \varphi = \varphi \otimes 1_v$ for all objects $v$. But more is true: we have $\varepsilon(u,v)\varphi \otimes \psi = \psi \otimes \varphi$ provided either $\varphi \in K_u$ and $\psi \in H_v$ or $\varphi \in H_u$ and $\psi \in K_v$.

We check the validity of our equations (5.1), (5.2), (5.7), (5.4) for this definition of $K$. (5.1) is true by definition. If $T \in (u,v)$ and $\varphi \in K_u$ then

$$\varepsilon(v,w)T\varphi \otimes 1_w = 1_w \otimes T\varepsilon(u,w)\varphi \otimes 1_w = 1_w \otimes T\varphi. $$

Thus $(u,v)K_u \subset K_v$, which is (5.2). If $\varphi \in K_u$ and $\psi \in K_v$ then

$$\varepsilon(u \otimes v,w)\varphi \otimes \psi \otimes 1_w = \varepsilon(u,w) \otimes 1_v \otimes 1_v \otimes \varepsilon(v,w)\varphi \otimes \psi \otimes 1_w = 1_w \otimes \varphi \otimes \psi. $$

Thus $K_u \otimes K_v \subset K_{u \otimes v}$, which is (5.4). Let $R \in (\iota, \bar{u} \otimes u)$ and $\overline{R} \in (\iota, u \otimes \bar{u})$ be a solution of the conjugate equations and define the corresponding antilinear operator $j$ by: $j\varphi := r_\varphi^* \circ R = 1_{\bar{u}} \otimes \varphi^* R$. If $\varphi \in K_u$ then

$$j\varphi \otimes 1_v = 1_{\bar{u}} \otimes \varphi^* \otimes 1_v \circ R \otimes 1_v =
1_{\bar{u}} \otimes \varphi^* \otimes 1_v \circ \varepsilon(v, \bar{u} \otimes u) \circ 1_v \circ R =
1_{\bar{u}} \otimes \varphi^* \otimes 1_v \circ 1_u \otimes \varepsilon(v, u) \circ \varepsilon(v, \bar{u}) \circ 1_u \otimes 1_v \circ R =
1_{\bar{u}} \otimes 1_v \otimes \varphi^* \circ \varepsilon(v, \bar{u}) \circ 1_u \otimes 1_v \circ R.$$

Acting on the left by $\varepsilon(\bar{u}, v)$, we get

$$\varepsilon(\bar{u}, v)j\varphi \otimes 1_v = 1_v \otimes 1_{\bar{u}} \otimes \varphi^* \circ \varepsilon(\bar{u}, v) \otimes 1_u \otimes \varepsilon(v, \bar{u}) \otimes 1_u \otimes 1_v \circ R =
1_v \otimes j\varphi.$$

This proves (5.7).

In fact there is a further property, (5.8), valid here.

$$1_u \otimes \phi \otimes 1_z \circ k \in K_{u \otimes v \otimes z}, \quad k \in K_{u \otimes v}, \quad \phi \in K_v.$$  

In fact,

$$\varepsilon(u \otimes v \otimes z, w)(1_u \otimes \phi \otimes 1_z \circ k) \otimes 1_w =
\varepsilon(u \otimes v, w) \otimes 1_z \circ 1_{u \otimes v} \otimes \varepsilon(z, w) \circ 1_u \otimes \phi \otimes 1_z \otimes 1_w \circ k \otimes 1_w =
\varepsilon(u \otimes v, w) \otimes 1_z \circ 1_u \otimes \phi \otimes 1_w \otimes 1_z \circ 1_u \otimes \varepsilon(z, w) \otimes k \otimes 1_w =
1_w \otimes 1_u \otimes \phi \otimes 1_z \circ \varepsilon(u, w) \otimes 1_z \circ 1_u \otimes \varepsilon(z, w) \otimes k \otimes 1_w =
1_w \otimes 1_u \otimes \phi \otimes 1_z \circ \varepsilon(u \otimes v, w)k \otimes 1_w = 1_w \otimes (1_u \otimes \phi \otimes 1_z \circ k),$$

as required.

A second example of defining subspaces $K_u \subset H_u$ is the following.
Consider an inclusion $A \subseteq \mathcal{F}$ of $C^*$–algebras and suppose that $\mathcal{F}$ is a tensor $C^*$–category of endomorphisms with conjugates of $A$, where each object $\rho$ of the category is induced by a Hilbert space $H_\rho$ of support $I \in \mathcal{F}$. We suppose that $A' \cap \mathcal{F} = \mathbb{C}$ then this Hilbert space is unique. We thus have an embedding of $\mathcal{F}$ in a category of Hilbert spaces in $\mathcal{F}$.

Now suppose that $\mathcal{B}$ is a $C^*$–algebra with $A \subseteq \mathcal{B} \subseteq \mathcal{F}$ and set $K_\rho := H_\rho \cap \mathcal{B}$. The $K_\rho$ are Hilbert spaces in $\mathcal{B}$. In this case (5.1) and (5.2) are obvious. Pick a solution $R, \bar{R}$ of the conjugate equations of $\rho$. Note that $j_\rho(\psi) := \bar{R}(\psi)^* R \in H_\rho$, where $\psi \in H_\rho$. The inverse $j_\rho^{-1}$ of $j_\rho$ is defined by $j_\rho^{-1} := (\rho(\psi)^* \bar{R}) \bar{R}$.

It is easily checked that these operators are inverses of one another. If $\psi \in K_\rho$ then $j_\rho \psi \in \rho(\psi)^* \bar{R} \in \mathcal{B}$ and (5.7) follows. If $\phi \in K_\sigma$ and $\chi \in K_\rho \otimes \sigma$, then $\rho(\phi^*) \chi F = \rho(\sigma) \rho(F) \chi = \rho(F) \chi$. Thus $\rho(\phi^*) \chi \in H_\rho \cap \mathcal{B} = K_\rho$, verifying (5.3).

Note that we have a correspondence $K$ from $\mathcal{F}$ to the category of Hilbert spaces, where $K_T := T \upharpoonright K_\rho$ for $T \in (\rho, \sigma)$. It is not clear whether $K$ satisfies (5.8). The problem is that we do not know whether $\rho(\phi) \in \mathcal{B}$. On the other hand we do have

$$\psi^* K_\rho \subseteq K_\sigma,$$

since if $\chi \in K_\rho \sigma$ then $\psi^* \chi$ induces $\sigma$ and is thus in $H_\sigma$. But since $\psi$ and $\chi$ are in $\mathcal{B}$ their product is actually in $K_\sigma$.

### 6 A Galois correspondence

We set up a correspondence between functors $K : u \mapsto K_u$ satisfying (5.1), (5.2), (5.7) and (5.8) above and closed bi-ideals $\mathcal{J}$. Given $K$ we define $K_u^\perp$ to be the closed ideal generated by the $(u_{ij} - \delta_{ij})$ for all $i$ and $j \leq m_u$ and all corepresentations $u$, where $m_u$ is the Hilbert space dimension of $H_u$, and we have chosen an orthonormal basis of $H_u$ whose first $m_u$ vectors are an orthonormal basis of $K_u$. Obviously, if $K_1 \subseteq K_2$, then $K_1^\perp \subseteq K_2^\perp$. Given $\mathcal{J}$, we define the functor $\mathcal{J}_u^\perp$ by $\varphi \in \mathcal{J}_u^\perp$ if and only if $(u - \iota(u)) \varphi \in H_u \otimes \mathcal{J}$. Again, if $\mathcal{J}_1 \subseteq \mathcal{J}_2$, then $\mathcal{J}_{1,u}^\perp \subseteq \mathcal{J}_{2,u}^\perp$. Given $K$ and $u$, if $j \leq m_u$ then

$$(u - \iota(u)) \varphi_j = \sum_i \varphi_i \otimes (u_{ij} - \delta_{ij}) \in H_u \otimes K_u^\perp.$$ 

Thus $K_u \subseteq K_u^\perp$. Given $\mathcal{J}$, then if $J \in \mathcal{J}_u^\perp$, $J$ is in the closed ideal generated by the $(u_{ij} - \delta_{ij})$ for all $i, j \leq m_u$ and all $u$. But if $j \leq m_u$ then $u \varphi_j = \sum_i \varphi_i \otimes (u_{ij} - \delta_{ij}) \in H_u \otimes \mathcal{J}$, so $(u_{ij} - \delta_{ij}) \in \mathcal{J}$ and $\mathcal{J}_u^\perp \subseteq \mathcal{J}$. From these relations, we can derive the usual closure relations: $K_u^\perp = K_u^{\perp \perp}$ and $\mathcal{J}_u^\perp = \mathcal{J}_u^{\perp \perp}$.

Up to this point we have made no use of relations (5.1), (5.2), (5.7) and (5.8) but we now want to show that that $K_u = K_u^{\perp \perp}$ and, by Theorem 5.5, it follows that the $K_u$ are the spaces of invariant vectors for a unique maximal compact quantum subgroup $G'$ of $G$, where $G' = (A/\mathcal{J}, \Delta')$. We let $\pi : A \to A/\mathcal{J}$ denote the canonical surjection. Now, by definition, $\varphi \in \mathcal{J}_u^\perp$ if $(u - \iota(u)) \varphi \in \mathcal{J}_u^\perp$. 
But then $1 \otimes \pi u(\varphi) = \varphi \otimes I'$ so $\varphi \in K_u$. Conversely, if $\varphi \in K_u$, then $1 \otimes \pi u(\varphi) = 1 \otimes \pi(\varphi)$. But $\ker(1 \otimes \pi) = H_u \otimes \mathcal{J}$ so $\varphi \in \mathcal{J}^u$. Thus $K_u = \mathcal{J}^u$ and $K_u^{\perp \perp} = \mathcal{J}^{u \perp \perp} = \mathcal{J}^u = K_u$ as required.

7 Functors arising from ergodic actions

Consider a nondegenerate action $\eta : B \to B \otimes A$ of a compact quantum group $G = (A, \Delta)$ on a unital $C^*$-algebra $B$. If $\eta$ is ergodic ($B^n = CI$), much can be said about the spectrum of the action.

Boca showed that any irreducible representation in the spectrum of $\eta$ has a finite multiplicity $\text{mult}(u)$ bounded above by the quantum dimension $\text{q-dim}(u)$ of $u$ \cite{[3]}.

In \cite{[2]} it was shown that $\text{mult}(u)$ can be bigger than the Hilbert space dimension of $u$. We shall outline, with a different notation, the main tools used in \cite{[2]}.

For any unitary (not necessarily irreducible) representation $u$ of $G$ of finite dimension, consider the vector space $L_u$ of linear maps $T : H_u \to B$ intertwining $u$ with $\eta$. For $S, T \in L_u$, $\sum T(e_i)S(e_i)^*$ is independent of the choice of orthonormal basis $(e_i)$ of $H_u$, and defines an element of the fixed point algebra, which, though, reduces to the complex numbers. Thus $L_u$ is a Hilbert space with inner product:

$$(S, T) := \sum T(e_i)S(e_i)^*.$$  

Note that if we regard $S$ and $T$ as elements of $(u \otimes \eta, \iota \otimes \eta)$, then $(S, T)I = T \circ S^\ast$.

7.1 Proposition The Hilbert space $L_u$ is non-trivial if and only if $u$ contains a subrepresentation $v \in \text{sp}(\eta)$. In particular, if $u$ is irreducible then $u \in \text{sp}(\eta)$ if and only if $L_u \neq 0$.

For any intertwiner $A \in (u, v)$ let $L_A : L_v \to L_u$ be the linear map defined by:

$L_A(T) = T \circ A.$

It is easy to check that $L$ is a contravariant $^\ast$-functor.

We obtain a covariant functor passing to the dual space. Consider, for any $u \in \text{sp}(\eta)$ another Hilbert space, $\overline{L}_u$, which, as a vector space, is the complex conjugate of $L_u$, with inner product:

$$(\overline{S}, \overline{T}) := \sum S(e_i)T(e_i)^* = (T, S)$$  

where $(e_i)$ is an orthonormal basis of $H_u$. Identify the complex conjugate of $L_u$ with the dual of $L_u$, and set, for $A \in (u, v)$,

$\overline{L}_A : \phi \in \overline{L}_u \to \phi \circ L_A \in \overline{L}_v.$

This is now a covariant $^\ast$-functor. The functors $L$ and $\overline{L}$ are related by:

$\overline{L}_A(T) = L_A(\overline{T}), \quad A \in (v, u), T \in L_v.$
7.2 Definition The functor $\mathcal{L}$ will be called the spectral functor associated with the ergodic action $\eta: \mathcal{B} \to \mathcal{B} \otimes A$, and $\mathcal{L}_u$ the spectral subspace corresponding to the representation $u$.

7.3 Theorem The $^*$--functor $\mathcal{L}: \text{Rep}(G) \to \mathcal{H}$ is quasitensor. Therefore for any $u \in \text{Rep}(G)$, $\mathcal{L}_u$ is finite dimensional and $\dim(\mathcal{L}_u) = \dim(\mathcal{L}_\pi)$.

Proof The space $L_u$ is, by definition, the fixed point algebra, which reduces to the complex numbers, so $L_u = \mathbb{C} = \mathcal{T}_u$, and (3.1) follows. If $S \in L_u$, $T \in L_v$, then the map $S \otimes T: \psi \otimes \phi \in H_u \otimes H_v \to S(\psi)T(\phi)$ intertwines $u \otimes v$ with $\eta$, so $S \otimes T \in L_u \otimes L_v$. If $S' \in L_u$, $T' \in L_v$, is another pair, and $(e_i)$ and $(f_j)$ are orthonormal bases of $H_u$ and $H_v$ respectively, then

$$(S' \otimes T', S \otimes T)I = \sum_{i,j} S(e_i)T(f_j)(S'(e_i)T'(f_j))^* = (S', S)(T', T)I,$$

so the linear span of all $S \otimes T$ is just a copy of $L_u \otimes L_v$ sitting inside $L_u \otimes L_v$.

Moreover, if $A \in (u', u)$, $B \in (v', v)$, $S \in L_u$, $T \in L_v$, $e \in H_{u'}$, $f \in H_{v'}$

$$L_{A \otimes B}(S \otimes T)(e \otimes f) = (S \otimes T) \circ (A \otimes B)(e \otimes f) =$$

$$S \otimes T(Ae \otimes Bf) = S(Ae)T(Bf) =$$

$$L_A(S)(e)L_B(T)(f) = L_A(S) \otimes L_B(T)(e \otimes f).$$

Hence

$$L_{A \otimes B} \upharpoonright_{L_u \otimes L_v} = L_A \otimes L_B.$$ 

Next we define an inclusion

$$S_{u,v}: \mathcal{L}_u \otimes \mathcal{L}_v \to \mathcal{L}_{u \otimes v}$$

by taking $S \otimes T$ to $S \otimes T$, for $S \in L_u$, $T \in L_v$ and hence $S \otimes T \in L_u \otimes L_v$. As for $L$, one can check that this inclusion is an isometry from the tensor product Hilbert space $\mathcal{L}_u \otimes \mathcal{L}_v$ to the Hilbert space $\mathcal{L}_{u \otimes v}$, and (3.2) follows. The relation between $L$ and $\mathcal{L}$ on arrows shows that

$$\mathcal{L}_{A \otimes B} \upharpoonright_{\mathcal{L}_{u'} \otimes \mathcal{L}_{v'}} = \mathcal{L}_A \otimes \mathcal{L}_B,$$

and this shows (3.6). Relation (3.3) is obvious for $L$ and $\mathcal{L}$ as well. Equation (3.4) follows from the fact that if $R \in L_u$, $S \in L_v$, $T \in L_z$ then $(R \otimes S) \otimes T = R \otimes (S \otimes T)$. It remains to show property (3.5). We define, for any $S \in L_v$, $T \in L_{u \otimes v}$, a linear map $\tilde{S}(T): H_u \to \mathcal{B}$ by

$$\tilde{S}(T)(\psi) = \sum_{k} T(\psi \otimes f_k)S(f_k)^*,$$

which is independent of the choice of the orthonormal basis $(f_k)$ of $H_v$. We check that $\tilde{S}(T) \in L_u$. In fact, on an orthonormal basis $(\psi_i)$ of $H_u$,

$$\eta \circ \tilde{S}(T)(\psi_i) = \eta(\sum_{k} T(\psi_i \otimes f_k)S(f_k)^*) =$$
\[
\sum_k \eta(T(\psi_i \otimes f_k))\eta(S(f_k))^* = \sum_{r,s,k,p} (T(\psi_r \otimes f_s) \otimes u_{ri}v_{sk})S(f_p) \otimes v_{pk})^* = \\
\sum_{r,s,k,p} T(\psi_r \otimes f_s)S(f_p)^* \otimes u_{ri}v_{sk}v_{pk} = \sum_{r,s} T(\psi_r \otimes f_s)S(f_s)^* \otimes u_{ri} = \\
\sum_r \hat{S}(T)(\psi_r) \otimes u_{ri}.
\]

We have thus defined a linear map:
\[
\hat{S} : L_u \otimes v \to L_u.
\] (7.1)

Conjugating \( \hat{S} \) gives a linear map
\[
\hat{S} : T \in L_u \otimes v \to \overline{S(T)} \in L_u.
\]

Consider the operator
\[
r(S) : L_u \to L_u \otimes L_v
\]
which tensors on the right by \( \overline{S} \) and the previously defined isometric inclusion map
\[
S_{u,v} : L_u \otimes L_v \to L_{u \otimes v}.
\]

We claim that
\[
\hat{S}^* = S_{u,v} \circ r(S).
\]

In fact, for \( T \in L_{u \otimes v}, T' \in L_u, \)
\[
(T', \hat{S}^* T) = (\overline{S(T')} , T) = \\
(\overline{S(T')} , T) = \sum_{i,j} T'(\psi_i \otimes f_j)S(f_j)^* T(\psi_i)^* = \\
(\overline{T'} , T \otimes \overline{S}) = (\overline{T'} , S_{u,v} T \otimes \overline{S}) = \\
(\overline{T'} , S_{u,v} \circ r(S) T),
\]
and the claim is proved. Taking the adjoint,
\[
\hat{S} = r(\overline{S})^* \circ S^*_{u,v},
\]
which implies
\[
S^*_{u,v} = \sum_p r(\overline{S_p}) \hat{S}_p,
\]
with \( \overline{S_p} \) an orthonormal basis of \( L_v \). Replace \( u \) with a tensor product representation \( u \otimes z \), and compute, for \( T \in L_u, T' \in L_z \otimes v \)
\[
S^*_{u \otimes z,v} \circ S_{u,z \otimes v}(\overline{T} \otimes T') = \sum_p r(\overline{S_p}) \circ \hat{S}_p(\overline{T} \otimes \overline{T'}) = 
\]
7 FUNCTIONARISING FROM ERGODIC ACTIONS

Now for \( \psi \in H_u, \phi \in H_z \),

\[
\tilde{S}_p(T \otimes T')(\psi \otimes \phi) = \sum_k (T \otimes T')(\psi \otimes \phi \otimes f_k)S_p(f_k)^* = T(\psi) \sum_k T'(\phi \otimes f_k)S_p(f_k)^* = T \otimes \tilde{S}_p(T')(\psi \otimes \phi)
\]

so

\[
E_{u \otimes z,v} \circ S_{u,z \otimes v}(T \otimes T') = \sum_p T \otimes \tilde{S}_p(T') \otimes S_p
\]

and therefore

\[
E_{u \otimes z,v}(\text{Image}S_{u,z \otimes v}) \subset \text{Image}S_{u,z,v}
\]

and the proof is complete.

Let \( \overline{u} \) be a conjugate of \( u \) defined by \( R \in (\iota, \overline{u} \otimes u) \) and \( \overline{R} \in (\iota, u \otimes \overline{u}) \) solving the conjugate equations. We shall identify the solution \( \hat{R}, \hat{\overline{R}} \) of the conjugate equations for \( \overline{T}_u \) constructed in Theorem 3.7 and we shall relate it to the notion of quantum multiplicity introduced in [2].

We can write \( R = \sum_i j e_i \otimes e_i \) and \( \overline{R} = \sum_k j^{-1} f_k \otimes f_k \), with \( j : H_u \to H_{\overline{u}} \) an antilinear invertible map and \( (e_i) \) and \( (f_k) \) orthonormal bases of \( H_u \) and \( H_{\overline{u}} \), respectively.

For \( T \in L_u \), the multiplet \( T(e_i)^* \) transforms like the complex conjugate invertible representation \( u^* \), so the linear map

\[
\hat{T} := \overline{\psi} \in \overline{H}_u \to T(\psi)^* \in B, \quad \psi \in H_u
\]

intertwines \( u^* \) with \( \eta \).

Consider the linear map

\[
Q : H_{u^*} = \overline{H}_u \to H_{\overline{u}}
\]

obtained composing the (antiunitary) complex conjugation \( \overline{H}_u \to H_u \) with \( j^{-1} \). Regard \( \overline{H}_u \) as the Hilbert space for \( u^* \). The condition that \( R \) is an intertwiner can be written: \( Q \in (u^*, \overline{u}) \).

Thus \( \hat{T} \circ Q^{-1} : H_{\overline{u}} \to B \) intertwines \( \overline{u} \) with \( \eta \). We have therefore defined an antilinear invertible map

\[
J : L_u \to L_{\overline{u}},
\]

\[
J(T)(\psi) = \hat{T} \circ Q^{-1}(\psi) = T(j^*(\psi))^*,
\]

with inverse \( J^{-1} : L_{\overline{u}} \to L_u \),

\[
J^{-1}(S)(\phi) = S(j^{*-1}(\phi))^*, \quad \phi \in H_u.
\]
7 FUNCTORS ARISING FROM ERGODIC ACTIONS

If we conjugate $J$ with the antilinear invertible map $T \in L_u \to \overline{T} \in \overline{L}_u$, we obtain an antilinear invertible

$$\mathcal{J} : \mathcal{L}_u \to \mathcal{L}_{\overline{\pi}}.$$ 

If $u$ is irreducible, the quantum multiplicity of $u$ is defined in [2] by

$$q\text{-mult}(u)^2 \triangleq \text{Trace}(\mathcal{J} \mathcal{J}^*) \text{Trace}((\mathcal{J} \mathcal{J}^*)^{-1}).$$

7.4 Theorem If $u$ is irreducible, one has

$$\hat{R} = \sum \mathcal{J} T_i \otimes T_i,$$

$$\overline{R} = \sum \mathcal{J}^{-1} S_j \otimes S_j,$$

where $(T_i)$ and $(S_j)$ are orthonormal bases of $\mathcal{L}_u$ and $\mathcal{L}_{\overline{\pi}}$, respectively. Therefore

$$q\text{-mult}(u) = d_{R,\overline{\pi}}(\mathcal{L}_u).$$

Proof We need to show that for $\overline{T} \in \overline{L}_u$, $r(\overline{T})^* \circ \hat{R} = \mathcal{J} \overline{T}$. Recall that $\hat{R} = S_{\overline{\pi},u}^* T_R$, so, with the same notation as in the previous theorem,

$$r(\overline{T})^* \circ \hat{R} = r(\overline{T})^* \circ S_{\overline{\pi},u}^* \circ \overline{T}_R = \hat{T} \circ \overline{T}_R = \hat{T}(\overline{T}_R).$$

In the last equation we have identified $\overline{T}_R \in (\mathcal{L}_u, \mathcal{L}_{\overline{\pi} \otimes u})$ with the element $\mathcal{T}_R(1)$, with $1 \in L_i$ the intertwiner $1 \in H_s \to I \in \mathcal{B}$. Now

$$\mathcal{T}_R(1) = \overline{L_{R^*}(1)} = \overline{1 \circ R^*},$$

so

$$\hat{T}(\overline{T}_R) = \overline{T(1 \circ R^*)}.$$ 

But $1 \circ R^* \in \mathcal{L}_{\overline{\pi} \otimes u}$ takes $\xi \in H_{\overline{\pi} \otimes H_u}$ to $(R, \xi)I$, hence $\overline{T(1 \circ R^*)} \in \mathcal{L}_{\overline{\pi}}$ is the map taking $\psi \in H_{\overline{\pi}}$ to

$$\sum_i (1 \circ R^*)(\psi \otimes e_i)T(e_i)^* = \sum_i (R_i \psi \otimes e_i)T(e_i)^* =$$

$$\sum_i (je_i, \psi)T(e_i)^* = \sum_i (j^* \psi, e_i)T(e_i)^* = T(j^* \psi)^* = J(T)(\psi).$$

So $\hat{T}(1 \circ R^*) = J(T)$, and the proof is complete.

As a consequence, we obtain the following result, proved in [2], in turn extending Boca’s result.
7.5 Corollary Let η be an ergodic nondegenerate action of a compact quantum group G on a unital C*-algebra B. Then for any irreducible representation u of G,

\[ \text{mult}(u) \leq q \cdot \text{mult}(u) \leq q \cdot \text{dim}(u). \]

Furthermore \( q \cdot \text{mult}(u) = q \cdot \text{dim}(u) \) if and only if \( T_R \in \text{Image}S_{\pi,u} \) and \( T_R \in \text{Image}S_{u,\pi} \) for some (and hence any) solution \((R, \overline{R})\) to the conjugate equations for u.

**Proof** We are stating, thanks to the previous theorem, that

\[ \text{dim}(L_u) \leq d_{R,\overline{R}}(L_u) \leq d_{R,\overline{R}}(u), \]

and this follows from Corollary 3.8 applied to the functor \( L \).

In the last part of this section we relate the functor \( T \) associated to the \( G \)-action on a quantum quotient space with the functor \( K \) of invariant vectors.

7.6 Proposition Let \( K \) be a compact quantum subgroup of a compact quantum group \( G \). A unitary \( G \)-representation \( u \) contains a subrepresentation of \( \text{sp}(\eta_K) \) if and only if \( K_u \neq 0 \). In particular, if \( u \) is irreducible, then \( u \in \text{sp}(\eta_K) \) if and only if \( K_u \neq 0 \). In this case, \( \text{mult}(u) = \text{dim}(K_u) \).

**Proof** Proposition 2.6 shows that any unitary irreducible \( G \)-representation \( v \) for which \( K_v \neq 0 \) lies in \( \text{sp}(\eta_K) \). If \( u \) is any representation such that \( K_u \neq 0 \) then \( u \) contains an irreducible subrepresentation \( v \) such that \( K_v \neq 0 \). So \( v \in \text{sp}(\eta_K) \). Conversely, if \( u \) contains a spectral subrepresentation \( u' \) and if \( z \) is an irreducible component of \( u' \), then by Prop 2.1, \( z \in \text{sp}(\eta_K) \). By Prop. 2.6 \( K_z \neq 0 \), so \( K_u \neq 0 \).

For any \( \psi \in K_u \), we have a map \( T_\psi : H_u \to A^d_{\psi} \) defined by

\[ T_\psi(\phi) = \ell_\psi^* u(\phi), \]

which actually lies in \( L_u \), as on an orthonormal basis of \( H_u \):

\[ \eta_K \circ T_\psi(e_i) = \Delta(\ell_\psi^* u(e_i)) = \sum_k \ell_\psi^* u(e_k) \otimes \ell_{e_k}^* u(e_i) = T_\psi \otimes \iota \circ u(e_i). \]

7.7 Theorem Let \( K \) be a compact quantum subgroup of \( G \). The map

\[ U_u : \psi \in K_u \to T_\psi \in L_u \]

is a quasitensor natural transformation from the functor \( u \to K_u \) to the functor \( u \to L_u \) such that \( U_u \) is a unitary whenever \( K_u \neq 0 \).

**Proof** We first show naturality, namely that for \( A \in (u, v) \),

\[ L_A \circ U_u = U_v \circ K_A. \]
For \( \psi \in K_u \) the right hand side computed on \( \psi \) equals \( \overline{T_{A\psi}} \), and the left hand side equals
\[
L_A(T_\psi) = \overline{L_A(T_\psi)} = T_\psi \circ A^*.
\]

Now for \( \phi \in H_v \),
\[
T_\psi \circ A^*(\phi) = \ell^*_\psi u(A^*\phi) = \ell^*_\psi A^* \otimes \iota v(\phi) = \ell^*_A v(\phi) = T_{A\psi}(\phi),
\]
so
\[
\overline{T_\psi \circ A^*} = \overline{T_{A\psi}}.
\]

We show that \( U_u \) is isometric:
\[
\langle (T_\psi, T_\phi) I = \sum_k T_\psi(e_k) T_\phi(e_k)^* = 
\sum_k \ell^*_\psi u(e_k)((\ell_\phi)^* u(e_k))^* = (\psi, \phi) I
\]
by the unitarity of \( u \). Now, if \( u \) is irreducible, then \( U_u \) is known to be surjective.

More generally, if \( v_i \) are irreducible subrepresentations of \( u \) and \( S_i \in (v_i, u) \) are isometries such that \( \sum_i S_i S^*_i = I \) then by naturality \( U_u K_{S_i} = \overline{T_{S_i}} \circ U_{v_i} \), which shows that \( U_u U^*_u = I \). Finally, restricting \( U_u \otimes v \) to \( K_u \otimes K_v \) gives, for \( \psi \in K_u \), \( \psi' \in K_v \),
\[
U_{u \otimes v}(\psi \otimes \psi') = \overline{T_\psi \otimes T_{\psi'}}.
\]

On the other hand for \( \phi \in H_u \), \( \phi' \in H_v \),
\[
T_{\psi \otimes \psi'}(\phi \otimes \phi') = \ell^*_\psi \otimes \ell^*_{\psi'} (u \otimes v)(\phi \otimes \phi') = 
\ell^*_\psi u(\phi) \ell^*_{\psi'} v(\phi') = (T_\psi \otimes T_{\psi'})(\phi \otimes \phi')
\]
so
\[
\overline{U_{u \otimes v}(\psi \otimes \psi')} = \overline{T_\psi \otimes T_{\psi'}} = \\
\overline{T_\psi} \otimes \overline{T_{\psi'}} = U_u(\psi) \otimes U_v(\psi').
\]

\section{8 Multiplicity maps in ergodic actions}

Let \( \eta : B \to B \otimes A \) be a nondegenerate ergodic action of a compact quantum group \( G = (A, \Delta) \) on a unital \( C^* \)-algebra \( B \). Consider the Hilbert space \( \overline{T_u} \) associated with the unitary representation \( u \), and an orthonormal basis \( \{T_k\} \) in \( \overline{T_u} \). Let
\[
c_u : H_u \to \overline{T_u} \otimes B
\]
denote the linear map defined by
\[
c_u(\phi) = \sum_k \overline{T_k} \otimes T_k(\phi).
\]
This map does not depend on the choice of the orthonormal basis of \( L_u \). We shall call the map \( c_u \) the \textit{multiplicity map} of \( u \) in \( \eta \).

\section*{8.1 Proposition}
The map \( c_u \) is nonzero if and only if \( L_u \) is nonzero, i.e. if and only if \( u \) contains a spectral subrepresentation.

\textbf{Proof} We need to show that \( c_u = 0 \) implies \( L_u = 0 \). Indeed, if \( L_u \) were \( \neq 0 \) then for any orthonormal basis \( (T_k) \) of \( L_u \), \( T_k \neq 0 \). Since \( (T_k) \) is a linear basis of \( L_u \), we must have \( L_u = 0 \). A contradiction.

We can also represent \( c_u \) as the rectangular matrix, still denoted by \( c_u \), whose \( k \)-th row is \( \left( T_k(e_1), \ldots, T_k(e_d) \right) \), with \( (e_i) \) an orthonormal basis of \( H_u \), \( d = \dim(u) \), \( k = 1, \ldots, \dim(L_u) \).

\section*{8.2 Proposition}
Let \( u \) contain a subrepresentation in the spectrum of \( \eta \). Then the matrix \( c_u \) satisfies the following properties.

\begin{itemize}
  \item[a)] \( c_u c_u^* = I \),
  \item[b)] each row of \( c_u \) transforms like \( u \),
  \item[c)] for any multiplet \( c \) transforming like \( u \), \( cP_u = c \), with \( P_u \) the domain projection of \( u \).
\end{itemize}

Conversely, if \( c'_u \in M_{p, \dim(u)}(\mathcal{B}) \) satisfies a)--c) then \( p = \dim(L_u) \) and \( c'_u \) is of the form of \( c_u \).

\textbf{Proof} Property b) is obviously satisfied, a) and c) are a consequence of the fact that \( (T_k) \) is an orthonormal basis of \( L_u \). We show uniqueness. If \( c'_u \) is as in the statement then the rows of \( c'_u \) lie in \( L_u \) by b). Furthermore these rows must be an orthonormal basis of \( L_u \) by a) and c). In particular, \( p = \dim(L_u) \).

We next investigate the relationship between \( u \to c_u \) and the functor \( u \to \overline{L_u} \). Define general coefficients of \( c_u \): for \( \phi \in \overline{L_u} \), \( \psi \in \overline{H_u} \), set

\[ c^u_{\phi, \psi} := \ell^*_\phi c_u(\psi) \in \mathcal{B}. \]

\section*{8.3 Proposition}
The map \( c_u : H_u \to \overline{L_u} \otimes \mathcal{B} \) satisfies the following properties.

\begin{itemize}
  \item[a)] \( \overline{L_A} \otimes I ; c_u = c_u \circ A \), \quad \( A \in (u, v) \),
  \item[b)] for \( \phi \in \overline{L_u} \), \( \phi' \in \overline{L_v} \), \( \psi \in H_u \), \( \psi' \in H_v \),
  \hspace{1cm} c^{u \otimes v}_{\phi \otimes \phi', \psi \otimes \psi'} = c^u_{\phi, \psi} c^{v}_{\phi', \psi'}. \]
\end{itemize}

Let \( K \) be a compact quantum subgroup of \( G \) and \( \eta_K : A^\delta \to A^\delta \otimes A \) the ergodic action defining the quantum quotient space \( K \setminus G \). We can then associate to any representation \( u \) of \( G \) the multiplicity map \( c^K_u \).

In the last part of Sect. 4 we have also defined, for any representation \( u \), the map \( u^K = E^K_u \otimes I ; u : H_u \to K_u \otimes A^\delta \). Now consider the natural unitary
transformation \( U_u : K_u \to \overline{L}_u \) between the functors \( u \to K_u \) and \( u \to \overline{L}_u \) defined in the previous section.

**8.4 Proposition** If \( U : K \to \overline{L} \) is the unitary natural transformation defined in Theorem 7.7, then for any representation \( u \)

\[ U_u \otimes \iota \circ u^K = c_u^K. \]

**Proof** If \((\psi_k)\) is an orthonormal basis of \( K_u \), we can write \( u^K = \sum_k \psi_k \otimes T_{\psi_k} \).

The rest of the proof is now clear.

**8.5 Corollary** If \((\psi_k)\) and \((\epsilon_j)\) are orthonormal bases of \( K_u \) and \( H_u \) respectively, the matrix \((u^K_{\psi_k,\epsilon_j}) = (\epsilon_j^* u(\psi_k))\) satisfies properties a)–c) of Prop. 8.2.

We next show the linear independence of the coefficients of the \( c_u \)'s for inequivalent irreducibles. The proof is obtained by generalizing a result of Woronowicz [25] stating the linear independence of matrix coefficients of inequivalent irreducible representations of a compact matrix pseudogroup.

**8.6 Proposition** Let \( \eta : \mathcal{C} \to \mathcal{C} \odot \mathcal{A}_\infty \) be an action of the Hopf *-algebra \( G_\infty \) on a unital *-algebra \( \mathcal{C} \), and let \( S \) be a set of unitary, irreducible, pairwise inequivalent representations of \( G \) in the spectrum of \( \eta \). For each \( u \in S \), let \( c_u = (c_{ij}^u) \in \mathcal{M}_{p_u \dim(u)}(\mathcal{C}) \) satisfy a) and b) of Prop. 8.2. It follows that the set of matrix coefficients \( \{c_{ij}^u, i = 1, \ldots, p_u, j = 1, \ldots, \dim(u), u \in S\} \) is linearly independent in \( \mathcal{C} \).

**Proof** Let \( F \) be a finite subset of \( S \). Consider the linear subspace of \( \oplus_{u \in F} \mathcal{M}_{p_u \dim(u)} \),

\[ M := \{ \oplus_{u \in F} c^u_{\rho} := \oplus_{u \in F}(\rho(c_{ij}^u)), \rho \in \mathcal{C}', \} \]

with \( \mathcal{C}' \) the dual of \( \mathcal{C} \) as a vector space and also the subspace of \( \oplus_{u \in F} \mathcal{M}_{\dim(u)} \),

\[ B := \{ \oplus_{u \in F} \sigma = \oplus_{u \in F}(\sigma(u_{pq})), \sigma \in \mathcal{A}_\infty' \}. \]

Notice that \( MB \subseteq M \), since for \( \rho \in \mathcal{C}', \sigma \in \mathcal{A}_\infty' \), \( u \in S \), \( c^u_{\rho} u_{\sigma} = c^u_{\rho \sigma} \), where \( \rho \odot \sigma := \rho \otimes \sigma \odot \eta \). On the other hand, by Lemma 4.8 in [25], \( B = \oplus_{u \in F} \mathcal{M}_{d_u} \).

Thus, for each fixed \( u^0 \in F \), choosing \( \sigma \) such that \( u_{\sigma} = 0 \) for \( u \neq u^0 \) and \( u^0_{\rho} \) is a matrix unit \( \theta^0_{h,k} \), shows that for every \( \rho \in \mathcal{C}' \), every \( u^0 \in F \) and every \( h, k = 1, \ldots, \dim(u^0) \) there exists \( \rho' \in \mathcal{C}' \) such that \( c_{ij}^u \rho' \) is zero if \( u \neq u^0 \) and \( c_{ij}^u \rho' \) is the matrix with identically zero columns except for the \( k \)-th one, which coincides with the \( h \)-th column of \( c_{ij}^u \). Assume now that we have a vanishing finite linear combination in \( \mathcal{C} \): \( \sum_{\eta \in u} \lambda_{ij}^u c_{ij}^\eta = 0 \). Then for all \( \rho' \in \mathcal{C}' \), \( \sum \lambda_{ij}^u \rho'(c_{ij}^u) = 0 \). Choose \( F \) finite and large enough so that this sum runs over \( F \). By the previous arguments, for every \( \rho \in \mathcal{C}' \), every \( u^0 \in S \) and every choice of \( h, k \), \( \sum_{\eta} \lambda_{ij}^u \rho'(c_{ij}^\eta) = 0 \). It follows that

\[ \sum_i \lambda_{i,k}^u c_{i,k}^u = 0, \quad h, k = 1, \ldots, d_u. \]
Multiplying on the right by $c_{u_0}^{*p,h}$, summing up over $h$ and using $c_{u_0}c_{u_0}^{*} = I$ gives $\lambda_{p,k}^{u_0} = 0$ for all $u_0 \in S$, $p = 1, \ldots, p_{u_0}$, $k = 1, \ldots, \dim(u_0)$.

8.7. Theorem Let $\eta : B \to B \otimes A$ be a nondegenerate, ergodic $G$–action of a compact quantum group on a unital $C^*$–algebra $B$. Let $\tilde{\eta}$ be a complete set of unitary irreducible elements of $sp(\eta)$. Associate with any $u \in \tilde{\eta}$ a corresponding multiplicity map $c_u$. Then the set of all matrix coefficients

$$\left\{ \ell_{T_i}^u c_u(e_j), u \in \tilde{\eta}, \text{ o.n.b. of } T_u, (e_j) \text{ o.n.b. of } H_u \right\}$$

is a linear basis for the dense spectral $*$–subalgebra $B_{sp}$.

9 A duality theorem for ergodic $C^*$–actions

We have seen in Sect. 7 that an ergodic nondegenerate action $\eta$ of a compact quantum group $G$ on a unital $C^*$–algebra has an associated quasitensor $*$–functor $L : \text{Rep}(G) \to \mathcal{H}$ to the category of, necessarily finite dimensional, Hilbert spaces. In this section we shall conversely construct from a quasitensor $*$–functor $F$ an ergodic nondegenerate action having $F$ as its spectral functor.

9.1 Theorem Let $G = (A, \Delta)$ be a compact quantum group and $F : \text{Rep}(G) \to \mathcal{H}$ be a quasitensor $*$–functor. Then there exists a unital $C^*$–algebra $B_F$ with an ergodic nondegenerate $G$–action $\eta_F : B_F \to B_F \otimes A$ and a quasitensor natural unitary transformation from the associated spectral functor $L$ to $F$.

The proof of the above theorem is inspired by the proof of the Tannaka–Krein duality theorem given by Woronowicz in [26]. A similar result is obtained in [2] for unitary fibre functors.

Proof We start by considering a complete set $\tilde{\mathcal{F}}$ of inequivalent, unitary, irreducible representations $u$ of $G$ such that $F(u) \neq 0$, and form the algebraic direct sum

$$\mathcal{E}_F = \bigoplus_{u \in \tilde{\mathcal{F}}} F(u) \otimes H_u.$$

We shall make $\mathcal{E}_F$ into a unital $*$–algebra. Consider, for each $u \in \tilde{\mathcal{F}}$, orthonormal bases $(T_k)$ and $(e_i)$ of $F(u)$ and $H_u$ respectively, form the orthonormal basis $(T_k \otimes e_i)$ of $F(u) \otimes H_u$. The linear map

$$c_u : \phi \in H_u \to \sum_k T_k \otimes (T_k \otimes \phi) \in F(u) \otimes \mathcal{E}_F$$

is independent of the choice of orthonormal basis. Extend the definition of $c_u$ to all objects of $\text{Rep}(G)$: first set $c_u = 0$ if $u$ is irreducible but $F(u) = 0$. Then choose isometries $S_i \in (u_i, u)$, with $u_i \in \tilde{\mathcal{F}}$, such that $\sum_i S_i S_i^* = I$, and set

$$c_u := \sum_i F(S_i) \otimes I \circ c_{u_i} S_i^*.$$
If \( S'_j \in (v_j, u) \) is another set of orthogonal isometries with ranges adding up to the identity, \( S'_i S'_j \in (v_i, u_i) \) are always scalars, by irreducibility, and nonzero only if \( v_j = u_i \), so
\[
\mathcal{F}(S'_i S'_j) \otimes I_{c_{uv}} = c_u S'_i S'_j.
\]
Multiplying on the left by \( \mathcal{F}(S'_i) \), on the right by \( S''_j \), summing over \( i \) and \( j \) and using the fact that \( \mathcal{F} \) is a \( * \)-functor, shows that \( c_u \) is independent of the choice of the \( S'_i \)'s, and one now has:
\[
\mathcal{F}(A) \otimes I \circ c_u = c_v \circ A, \quad A \in (u, v), u, v \in \text{Rep}(G).
\]
For \( u, v \in \mathcal{H}, T \otimes \phi \in \mathcal{F}(u) \otimes H_u, T^\dagger \otimes \phi' \in \mathcal{F}(v) \otimes H_v \), set
\[
(T \otimes \phi)(T^\dagger \otimes \phi') := \ell_T^* \circ c_{u \otimes v}(\phi \otimes \phi').
\]
In this way \( \mathcal{C}_\mathcal{F} \) becomes an associative algebra with identity \( I = \mathbb{1} \in \mathcal{F}(u) \otimes H_u \).

We next define the \( * \)-involution on \( \mathcal{C}_\mathcal{F} \) with the help of the conjugate representation. This representation is defined, up to unitary equivalence, by intertwiners \( R \in (i, \mathfrak{T} \otimes u) \) and \( \overline{R} \in (i, u \otimes \mathfrak{T}) \) satisfying the conjugate equations. If \( u \) is irreducible, the spaces \( (i, \mathfrak{T} \otimes u) \) and \( (i, u \otimes \mathfrak{T}) \) are one dimensional. For \( u \in \mathcal{F} \) consider orthonormal bases \((T'_i, e'_i)\) of \( \mathcal{F}(\mathfrak{T}) \) and \( H_\mathfrak{T} \) respectively, and set, for \( T \in \mathcal{F}(u), \phi \in H_u \),
\[
(\ell_T^* c_u(\phi))^* := \sum_{i,p} (\mathcal{F}(R), S_\mathfrak{T}, u T'_i \otimes T) \ell_{T'}^* c_\mathfrak{T}(e'_p \phi \otimes e'_p, \overline{R}).
\]

Any other solution to the conjugate equations is of the form \((\lambda R, \mu \overline{R})\) with \( \lambda, \mu \in \mathbb{C} \) and \( \overline{\lambda} = 1 \), so the above definition is independent of the choice of \((R, \overline{R})\). Recall that the pair \((R, \overline{R})\) defines a solution \((\hat{R}, \overline{R})\) of the conjugate equations for \( \mathcal{F}(u) \) as in Theorem 3.7. So writing
\[
R = \sum_i je_i \otimes e_i = \sum_k f_k \otimes j^* f_k,
\]
\[
\overline{R} = \sum_k j^{-1} f_k \otimes f_k = \sum_i e_i \otimes j^{-1} e_i,
\]
with \((e_i)\) and \((f_k)\) o.n.b. of \( H_u \) and \( H_\mathfrak{T} \) respectively, and, again, with an analogous meaning of notation,
\[
\hat{R} = \sum_i J T_i \otimes T_i = \sum_k T'_k \otimes J^* T'_k,
\]
\[
\overline{\hat{R}} = \sum_k J^{-1} T'_k \otimes T'_k = \sum_i T_i \otimes J^{-1} T_i,
\]
we obtain for the adoint the equivalent form
\[
(\ell_T^* c_u(\phi))^* = \ell_{T'}^* c_\mathfrak{T}(j^{-1} \phi),
\]
which is obviously independent of the choice of the orthonormal bases. Replacing $u$ by $\overline{u}$, and therefore $R$ by $\overline{R}$ and $\hat{R}$ by $\overline{\hat{R}}$, and $j$ and $J$ in turn by their inverses, shows that

\[(\ell^*_T c_u(\phi))^* = \ell^*_T c_u(\phi).\]

We have thus defined an antilinear involutive map $\overline{} : \mathcal{C}_T \to \mathcal{C}_T$.

We next show that for $u, v \in \mathcal{F}$, $\phi \in H_u, T \in \mathcal{F}(u), \phi' \in H_v, T' \in \mathcal{F}(v)$,

\[
(\overline{T} \otimes \phi)(\overline{T'} \otimes \phi')^* = (\overline{T'} \otimes \phi')^*(\overline{T} \otimes \phi)^*.
\]

Choose irreducible representations $(u_r)$ and isometries $S_r \in \mathcal{F}(u_r, u \otimes v)$ with pairwise orthogonal ranges, adding up to the identity. Then with an obvious meaning of notation, the previous equation becomes

\[
\sum_r \ell^*_r J_r F(S_r) = \ell^*_r J_r T \otimes T' c_u(S_r)(j_r^{-1} R^*_r \otimes \phi) = \ell^*_r J_r T' \otimes T c_v(S_r')(j'_{r'} R^*_{r'} \otimes \phi').
\]

Now recall from [15] that in any tensor $C^*$–category with conjugates $\mathcal{T}$ there is an antilinear map

\[
(\rho, \sigma) \to (\overline{\rho}, \overline{\sigma})
\]

given by

\[
S \to \overline{S} := 1_\sigma \otimes R_{\overline{\rho}}^* \circ 1_\sigma \otimes S^* \otimes 1_\tau \circ R_{\overline{\tau}} \otimes 1_\sigma.
\]

Applying this to categories of Hilbert spaces, with conjugates defined by antilinear invertible maps $j_\rho, j_\sigma$, related in the usual way to $R_\rho$ and $R_\sigma$, one finds that

\[
\overline{S} = j_\sigma \circ S \circ j^{-1}_\rho.
\]

We claim, see Lemma 9.2, that $\mathcal{F}(\overline{S}) = \overline{\mathcal{F}(S)}$, where the latter is defined using the antilinear invertible maps $J_\rho, J_\sigma$. Also, if $\overline{\tau}$ is a conjugate of $\tau$ defined by $R_\tau$ and $\overline{R}_\tau$, then $\overline{\tau} \otimes \overline{\sigma}$ is a conjugate of $\sigma \otimes \tau$ defined by

\[
R_{\sigma \otimes \tau} = 1_\tau \otimes R_\sigma \otimes 1_\tau \circ R_\tau
\]

\[
\overline{R}_{\sigma \otimes \tau} = 1_\sigma \otimes R_{\overline{\tau}} \otimes 1_\sigma \circ R_{\overline{\sigma}}.
\]

In conclusion, we obtain an antilinear map

\[
S \in (\rho, \sigma \otimes \tau) \to \overline{S} \in (\overline{\rho}, \overline{\sigma} \otimes \overline{\tau}).
\]

Thus for an intertwiner $S$ from the Hilbert space $\rho$ to the Hilbert space $\sigma \otimes \tau$, $\overline{S}$ is the operator from $\overline{\rho}$ to $\overline{\sigma} \otimes \overline{\tau}$ given by

\[
\overline{S} = j_\tau \otimes j_\sigma \circ \overline{\vartheta}_{\sigma, \tau} \circ S \circ j^{-1}_\rho,
\]

with $\overline{\vartheta}_{\sigma, \tau} : \sigma \otimes \tau \to \tau \otimes \sigma$ the flip operator. Therefore the left hand side of the equation we want to establish equals

\[
\sum_r \ell^*_r J_r F(S_r) c_{\overline{S}_r}(j_r^{-1} R^*_r \otimes \phi) = \sum_r \ell^*_r J_r F(S_r)(\overline{S}_r(j_r^{-1} R^*_r \otimes \phi)).
\]
hilates the coefficients of all the irreducible representations\[\eta\]

We show that

\[\text{We introduce a state on } C\text{ obtained averaging over the group action. Since the Haar measure of } \omega, we see that }\]

\[\text{It is easy to check that }\]

\[\eta\] thanks to Prop. 3.9. So far we have obtained a unital \(\ast\)-algebra \(C_F\). Consider the linear map

\[\eta_F : C_F \to C_F \otimes A_\infty, \]

\[\eta_F(T \otimes \phi) = T \otimes u(\phi), \quad T \in \mathcal{F}(u), \phi \in H_u, u \in \hat{F}. \]

It is easy to check that \(\eta_F\) is multiplicative and that

\[\eta_F \otimes \iota \circ \eta_F = \iota \otimes \Delta \circ \eta_F. \]

We show that \(\eta_F\) preserves the involutions. Recall that the relation \(T \in (\iota, u \otimes \overline{u})\) is equivalent to the fact that the linear operator \(Q : \psi \in H_u \to j^{-1}\psi \in H_u\) lies in \((u, \overline{u})\), so

\[\eta_F((\ell_T^{u}(e_i)))^* = \eta_F((\ell_T^{u}(Q\overline{u}))) = \]

\[JT \otimes \overline{Q}\overline{u} = \sum_k JT \otimes j^{-1} \epsilon_k \otimes u_{ki} = \]

\[\eta_F((\ell_T^{u}(e_i)))^*. \]

We introduce a state on \(C_F\). Consider the linear functional \(h\) on \(C_F\) defined by

\[h(T \otimes \phi) = 0 \quad T \in \mathcal{F}(u), \phi \in H_u, u \in \hat{F}, u \neq \iota, \]

\[h(I) = 1. \]

We claim that \(h\) is a positive faithful state on \(C_F\), so \(C_F\) has a \(C^*\)-norm.

Consider the conditional expectation \(E\) onto the fixed point \(*\)-subalgebra obtained averaging over the group action. Since the Haar measure of \(G\) annihilates the coefficients of all the irreducible representations \(u\), except for \(u = \iota\), we see that \(E = h\), and we can conclude that \(\eta_F\) is an ergodic algebraic action.

It is now evident that, for \(u \in \hat{F}\), the maps, for \(T \in \mathcal{F}(u)\),

\[\gamma_T : \psi \in H_u \to T \otimes \psi \in C_F\]

are elements of the Hilbert space \(L_u\) associated with the algebraic ergodic space, and that elements of this form span \(L_u\). Therefore we obtain surjective linear maps, for \(u \in \hat{F}\),

\[V_u : T \in \mathcal{F}(u) \to \gamma_T \in T_u. \]
We show that $V_u$ is an isometry, and therefore a unitary. For $S, T \in \mathcal{F}(u)$, and an orthonormal basis $(\psi_i)$ of $H_u$, 

$$
(V_u(T), V_u(S)) I = \sum_i \gamma_T(\psi_i) \gamma_S(\psi_i)^* = 
\sum_i (T \otimes \psi_i) (S \otimes \psi_i)^* = \sum_i (T \otimes \psi_i) (JS \otimes j^*-1 \psi_i).
$$

Consider pairwise inequivalent irreducible representations $u_1, \ldots, u_N$ of $G$ and, for $\alpha = 1, \ldots, N$, isometries $s_{\alpha,j} \in (u_\alpha, u \otimes \mathcal{F})$, with ranges adding up to the identity. Then the last term above becomes 

$$
\sum_i \sum_\alpha \sum_j \mathcal{F}(s_{\alpha,j})^*(T \otimes JS) \otimes s_{\alpha,j}^* R_u = \sum_\alpha \sum_j \mathcal{F}(s_{\alpha,j})^*(T \otimes JS) \otimes s_{\alpha,j}^* R_u.
$$

Now $s_{\alpha,j}^* R_u \in (\iota, u_\alpha)$, hence $s_{\alpha,j}^* R_u = 0$ unless $u_\alpha = \iota$. In this case $N = 1$ and $s_{\alpha,j}$ can be chosen to coincide with $R_u \| R_u \|^{-1}$. We conclude that the last term above equals 

$$
\| R_u \|^{-2} \mathcal{F}(R_u)^*(T \otimes JS) \otimes R_u R_u = R_u T \otimes JS \otimes 1 = (S, T) 1 \otimes 1 = (T, S) I.
$$

We extend $V$ to a quasitensor natural transformation from $\mathcal{F}$ to $\mathcal{F}$. First set $V_u = 0$ if $u$ is irreducible but $\mathcal{F}(u) = 0$. Then for any representation $u \in \operatorname{Rep}(G)$, consider pairwise inequivalent irreducible representations $u_\alpha \in \hat{\mathcal{F}}$ and isometries $s_{\alpha,j} \in (u_\alpha, u)$ decomposing $u$. Define a unitary map $V_u : \mathcal{F}(u) \to \mathcal{L}_u$ by 

$$
V_u = \sum_\alpha \sum_j L_{s_\alpha,j} V_{u_\alpha} \mathcal{F}(s_{\alpha,j})^*.
$$

It is easy to check that $V_u$ does not depend on the choice of the isometries, it is a natural transformation and that one now has, as a consequence of the definition of multiplication in $\mathcal{C}_\mathcal{F}$, that for $u, v \in \mathcal{F}$, 

$$
V_{u \otimes v} |_{\mathcal{F}(u) \otimes \mathcal{F}(v)} = V_u \otimes V_v.
$$

From this relation one concludes, with routine computations, that $V$ is a quasitensor unitary natural transformation between $\mathcal{L}$ and $\mathcal{F}$. Recalling the definition of the maps $c_u$ at the beginning of the proof, one also has that, for $\psi \in H_u$, $u \in \mathcal{F}$, 

$$
V_u \otimes I c_u(\psi) = \sum_k \gamma T_k \otimes \gamma T_k(\psi),
$$

and $c_u$ is the multiplicity map associated with the functor $\mathcal{L}$ in the representation $u$. Therefore we can now say that $\mathcal{C}_\mathcal{F}$ is linearly spanned by entries.
of coisometries in matrix algebras over $\mathcal{C}_\sigma$, the maps $c_u$, so the maximal $C^*$-seminorm on $\mathcal{C}_\sigma$ is finite. Completing $\mathcal{C}_\sigma$ in the maximal $C^*$-seminorm yields a unital $C^*$-algebra $\mathcal{B}_\sigma$, a nondegenerate ergodic $G$-action

$$\eta_\sigma : \mathcal{B}_\sigma \to \mathcal{B}_\sigma \otimes \mathcal{A},$$

with spectral functor $\mathcal{T}$, and a natural unitary transformation from $\mathcal{T}$ to $\mathcal{F}$.

We now claim the following.

**Lemma 9.2** Let $S \in (u, v)$ and if $j_u : H_u \to H_\pi$, $j_v : H_v \to H_\pi$ are antilinear invertible maps defining conjugates of $u$ and $v$ respectively, with corresponding antilinear invertible maps $J_u : \mathcal{F}(u) \to \mathcal{F}(\pi)$, $J_v : \mathcal{F}(v) \to \mathcal{F}(\pi)$, in the sense of Theorem 3.7, then

$$\mathcal{F}(j_u S j_v^{-1}) = J_v \mathcal{F}(S) J_u^{-1}.$$

**Proof** Let $R_u$, $\overline{R}_u$ be the solution to the conjugate equations corresponding to $j_u$, and, similarly, $R_v$, $\overline{R}_v$ the solution corresponding to $j_v$. We need to show that

$$\mathcal{F}(1_\pi \otimes \overline{R}_u \circ 1_\pi \otimes S^* \otimes 1_\pi \circ R_u \otimes 1_\pi) = 1_{\mathcal{F}(\pi)} \otimes \overline{R}_u^* \circ 1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(S)^* \otimes 1_{\mathcal{F}(\pi)} \circ \overline{R}_v \otimes 1_{\mathcal{F}(\pi)}.$$

We shall use the simplified notation (3.7)–(3.11), replacing $S_{u,v}$ by the identity, $S_{u,v}^*$ by $E_{u,v}$, $R$ with $E_{u,u} \circ R$ and $\overline{R}$ by $E_{u,\pi} \circ \overline{R}$. Properties (3.11) and (3.8) show that, regarding $\mathcal{F}(R_v)$ as an element of $\mathcal{F}(\pi \otimes v)$,

$$\mathcal{F}(R_v \otimes 1_\pi) = \mathcal{F}(R_v) \otimes 1_{\mathcal{F}(\pi)}.$$

We next show that

$$\mathcal{F}(1_\pi \otimes \overline{R}_u) = 1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(\overline{R}_u)^* \circ E_{\pi,u \otimes \pi}.$$

In fact, if $\xi \in \mathcal{F}(\pi \otimes u \otimes \pi)$ and $\eta \in \mathcal{F}(\pi)$,

$$(\eta, \mathcal{F}(1_\pi \otimes \overline{R}_u) \xi) = (\eta, \mathcal{F}(1_\pi \otimes \overline{R}_u)^* \xi) =$$

$$(\mathcal{F}(1_\pi \otimes \overline{R}_u) \eta, \xi) = (\eta \otimes \mathcal{F}(\overline{R}_u), \xi) =$$

$$(\eta \otimes \mathcal{F}(\overline{R}_u), E_{\pi,u \otimes \pi} \xi) = (\eta, 1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(\overline{R}_u)^* \circ E_{\pi,u \otimes \pi} \xi).$$

Pick a vector $\zeta \in \mathcal{F}(\pi)$. Then

$$\mathcal{F}(1_\pi \otimes \overline{R}_u \circ 1_\pi \otimes S^* \otimes 1_\pi \circ R_v \otimes 1_\pi) \zeta =$$

$$1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(\overline{R}_u) \circ E_{\pi,u \otimes \pi}(\mathcal{F}(1_\pi \otimes S)^*(\mathcal{F}(R_v)) \otimes \zeta)$$

(9.1)

But thanks to (3.12),

$$E_{\pi,u \otimes \pi}(\mathcal{F}(1_\pi \otimes S)^*(\mathcal{F}(R_v)) \otimes \zeta) = E_{\pi,u \otimes \pi}(\mathcal{F}(1_\pi \otimes S)^*(\mathcal{F}(R_v)) \otimes \zeta) =$$
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\[ E_{\pi,u}(\mathcal{F}(1_\pi \otimes S)^*\mathcal{F}(R_v)) \otimes \zeta = \]
\[ E_{\pi,u} \circ \mathcal{F}(1_\pi \otimes S)^*(\mathcal{F}(R_v)) \otimes \zeta = 1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(S)^*(E_{\pi,v} \circ \mathcal{F}(R_v)) \otimes \zeta = \]
\[ (1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(S)^*(R_v)) \otimes \zeta = 1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(S)^* \otimes 1_{\mathcal{F}(\pi)} \circ R_v \circ 1_{\mathcal{F}(\pi)}(\zeta) = \]
\[ 1_{\mathcal{F}(\pi)} \otimes E_{u,\pi} \circ 1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(S)^* \otimes 1_{\mathcal{F}(\pi)} \circ \widehat{R}_v \circ 1_{\mathcal{F}(\pi)}(\zeta). \]

Substituting back in (9.1) gives
\[ 1_{\mathcal{F}(\pi)} \otimes \mathcal{F}(S)^* \otimes 1_{\mathcal{F}(\pi)} \circ \widehat{R}_v \circ 1_{\mathcal{F}(\pi)}(\zeta). \]

9.3 Lemma The linear functional \( h \) is a faithful state on the \( ^* \)-algebra \( \mathcal{C}_\pi \).

Proof For \( u, v \in \mathcal{F} \), pick \( T \in \mathcal{F}(u) \), \( S \in \mathcal{F}(v) \), \( \phi \in H_u \), \( \psi \in H_v \). Choose isometries \( s_\alpha \in (u_\alpha, \pi \otimes v) \), with \( u_\alpha \) irreducible, with orthogonal ranges and summing up to the identity. Then
\[ (T \otimes \phi)^*(S \otimes \psi) = \ell^\prime_{J_u T \otimes S} \circ c_{\pi \otimes v} (j_u^{-1} \phi \otimes \psi) = \]
\[ \sum_\alpha \ell^\prime_{J_u T \otimes S} \circ \mathcal{F}(s_\alpha) \otimes c_{\alpha} (s_\alpha j_u^{-1} \phi \otimes \psi). \]

Let \( A \) be the subset of all \( \alpha \) for which \( u_\alpha = \iota \). Then for \( \alpha \in A \), \( (\iota, \pi \otimes v) \) is always zero unless \( u = v \). If this is the case, there is, up to a phase, just one \( s_\alpha \), which is of the form \( \lambda_u R_u \), with \( \lambda_u = \|R_u\|^{-1} \), because \( u \) is irreducible and the space \( (\iota, \pi \otimes u) \) is one-dimensional. Thus
\[ h((T \otimes \phi)^*(S \otimes \psi)) = \delta_{u,v} |\lambda_u|^2 (\mathcal{F}(R_u)^* J_u T \otimes S)^* R_u j_u^{-1} \phi \otimes \psi) = \]
\[ \delta_{u,v} |\lambda_u|^2 (R_u^* J_u T \otimes S)^*(R_u^* j_u^{-1} \phi \otimes \psi)) = \delta_{u,v} |\lambda_u|^2 (J_T, J_S)(\phi, \psi). \]

If \( a \in \mathcal{C}_\pi \) is written in the form \( a = \sum_{u \in F} \mu^u_{ij} T^u_i \otimes \phi^u_j \) with \( F \) a finite set, \( T^u_i \in \mathcal{F}(u) \), \( \phi^u_j \in H_u \) orthonormal bases, the previous computation gives
\[ h(a^*a) = \sum_{u \in F} \sum_{i,j} \text{Trace}(j^u_{ij} j^u_{ij})^{-1} \mu^u_{ij} \mu^u_{ij} (T^u_i T^u_j) \geq 0. \]

If \( h(a^*a) = 0 \) then we can conclude that for all \( j, \sum_{t} \mu^u_{ij} T^u_i = 0 \), hence \( \mu^u_{ij} = 0 \) for all \( i, j, u \), so \( a = 0 \).

One can induce \( ^* \)-isomorphisms between two induced \( C^* \)-systems \((\mathcal{B}_\pi, \eta_\pi)\) and \((\mathcal{B}_\mathcal{G}, \eta_\mathcal{G})\) using natural transformations.

9.4 Proposition Let \( \mathcal{F}, \mathcal{G} : \text{Rep}(G) \to \mathcal{H} \) be two quasitensor \( ^* \)-functors and let \( U : \mathcal{F} \to \mathcal{G} \) be a unitary quasitensor natural transformation. Then there exists a unique \( ^* \)-isomorphism \( \alpha_u : \mathcal{B}_\mathcal{F} \to \mathcal{B}_\mathcal{G} \) intertwining the corresponding \( G \)-actions, such that
\[ \alpha_U(T \otimes \phi) = \overline{U_u} T \otimes \phi, \quad T \in \mathcal{F}(u), \phi \in H_u, u \in \hat{\mathcal{F}}. \]
Proof It is easy to check that that formula defines a linear multiplicative map 
\( \alpha_u \) commuting with the actions. It is also easy to check that, for \( T \in \mathcal{F}(u) \),
\[
U_{\pi }T^* \circ E_{\pi }^T \circ E_{\pi }^G \circ U_{\pi } \circ U_{\pi } \circ U_{\pi } \circ U_{\pi } = 
\]
Thus
\[
U_{\pi }J^G_u(T) = U_{\pi }T^* \circ E_{\pi }^T \circ E_{\pi }^G \circ \mathcal{F}(R_u) = 
\]
\[
r_{U_{\pi }T}^* \circ E_{\pi }^T \circ U_{\pi } \circ U_{\pi } \circ U_{\pi } \circ U_{\pi } \circ U_{\pi } \circ U_{\pi } = 
\]
which shows that \( \alpha_u \) is \( \ast \)-invariant. Thus \( \alpha_u \) extends to the completions in the
maximal \( C^* \)-seminorms.

10 Applications to abstract duality theory

As a first application of Theorem 9.1, we consider a tensor \( C^* \)-category \( T \) and
a faithful tensor \( \ast \)-functor \( \rho : \text{Rep}(G) \to T \) from the representation category of
a compact quantum group \( G \) to \( T \). Then, as shown in Example 3.5, one has a
quasitensor \( \ast \)-functor \( F_{\rho} : \text{Rep}(G) \to \mathcal{F} \) which associates to any representation
of \( G \) the Hilbert space \( (\iota, \rho_u) \). Therefore we can apply Theorem 9.1 to
and obtain an ergodic \( G \)-space canonically associated with the inclusion \( \rho \). We
summarize this in the following theorem.

10.1 Theorem Let \( \rho : \text{Rep}(G) \to T \) be a tensor \( \ast \)-functor from the representa-
tion category of a compact quantum group \( G \) to an abstract tensor \( C^* \)-category \( T \). Then there is a canonically associated ergodic nondegenerate action of \( G \) on
a unital \( C^* \)-algebra \( B \) whose spectral functor can be identified with \( F_{\rho} \).

The above \( G \)-space plays a central role in abstract duality theory for compact
quantum groups. This matter will be developed elsewhere. We notice
some consequences of the previous corollary.

The notion of permutation symmetry for an abstract tensor \( C^* \)-category has been introduced in \[6\]. If \( G \) is a compact group, \( \text{Rep}(G) \) has permutation symmetry defined by the intertwiners \( \vartheta_{u,v} \in (u \otimes v, v \otimes u) \) exchanging the order
of factors in the tensor product. The previous theorem allows us to recover the
following result.

10.2 Theorem If \( G \) is a compact group, \( \mathcal{F} \) is a tensor \( C^* \)-category with per-
mutation symmetry \( \varepsilon \) and \( \rho : \text{Rep}(G) \to \mathcal{F} \) is a tensor \( \ast \)-functor such that
\( \rho(\vartheta_{u,v}) = \varepsilon(\rho_u, \rho_v) \) then there exists a compact subgroup \( K \) of \( G \), unique up
to conjugation, and an isomorphism of the \( G \)-ergodic system associated with \( \rho \) and the ergodic \( C^* \)-system induced by the homogeneous space \( K \backslash G \) over \( G \).

Proof We show that the \( C^* \)-algebra associated with the quasitensor \( \ast \)-functor
\( u \to (\iota, \rho_u) \) is commutative. This will suffice, as it is well known that any
ergodic action of a compact group on a commutative $C^*$–algebra arises from the transitive $G$–action on a quotient space $K\backslash G$ by a point stabilizer subgroup, unique up to conjugation. We need to show that if $u$ and $v$ are irreducible then for $k \in (t, \rho_u)$, $k' \in (t, \rho_v)$, $\psi \in H_u$, $\psi' \in H_v$ then $\mathcal{K} \otimes \psi$ and $\mathcal{K} \otimes \psi'$ commute, as these elements span a dense $*$–subalgebra. Choose inequivalent irreducible representations $(u_\alpha)$ of $G$ and isometries $s_{\alpha,j} \in (u_\alpha, u \otimes v)$ such that $\sum_\alpha \sum_j s_{\alpha,j}^* s_{\alpha,j} = 1_{u \otimes v}$. Thus
\[
(\mathcal{K} \otimes \psi)(\mathcal{K} \otimes \psi') = \sum_\alpha \sum_j \rho(s_{\alpha,j}^*)(k \otimes 1_{\rho_v} \circ k') \otimes s_{\alpha,j}^*(\psi \otimes \psi'). \tag{10.1}
\]
Now
\[
\vartheta_{v,u} \psi' \otimes \psi = \psi \otimes \psi'
\]
and
\[
\varepsilon(\rho_v, \rho_u)(k' \otimes 1_{\rho_u} \circ k) = \varepsilon(\rho_v, \rho_u)(k' \otimes k) = k \otimes k' = k \otimes 1_{\rho_v} \circ k'.
\]
Thus the right hand side of (10.1) can be written
\[
\sum_\alpha \sum_j \rho(s_{\alpha,j}^*)(\varepsilon(\rho_v, \rho_u)k' \otimes 1_{\rho_u} \circ k) \otimes s_{\alpha,j}^*(\vartheta_{v,u} \otimes \vartheta_{v,u})(\psi' \otimes \psi). \tag{10.2}
\]
On the other hand $\varepsilon(\rho_u, \rho_v) \circ \rho(s_{\alpha,j}) = \rho(\vartheta_{u,v}) \circ \rho(s_{\alpha,j})$, and this is the map that takes an element $\xi \in (t, \rho_u)$ to the element
\[
\rho(\vartheta_{u,v}) \circ \rho(s_{\alpha,j}) \circ \xi = \rho(\vartheta_{u,v} \circ s_{\alpha,j}) \circ \xi \in (t, \rho_v \circ \rho_u).
\]
Thus
\[
\varepsilon(\rho_u, \rho_v) \circ \rho(s_{\alpha,j}) = \rho(\vartheta_{u,v} \circ s_{\alpha,j}).
\]
Set $t_{\alpha,j} := \vartheta_{u,v} \circ s_{\alpha,j} \in (u_\alpha, v \otimes u)$. Then (10.2) equals
\[
\sum_\alpha \sum_j \rho(t_{\alpha,j}^*)(k' \otimes 1_{\rho_u} \circ k) \otimes t_{\alpha,j}^*(\psi' \otimes \psi)
\]
Since the isometries $t_{\alpha,j} := \vartheta_{u,v} \circ s_{\alpha,j} \in (u_\alpha, v \otimes u)$ give an orthogonal decomposition of $v \otimes u$ into irreducibles, the last term above equals $(\mathcal{K} \otimes \psi')(\mathcal{K} \otimes \psi)$, and the proof is complete.

Recall that a $q$–Hecke symmetry for an object $\rho$ in a tensor $C^*$–category $\mathcal{I}$ is given by representations
\[
\varepsilon_n : H_n(q) \rightarrow (\rho \otimes_n) \rho \otimes_n
\]
of the Hecke algebras $H_n(q)$ such that
\[
\varepsilon_{n+1}(b) = \varepsilon_n(b) \otimes 1_\rho, \quad b \in H_n(q) \subset H_{n+1}(q),
\]
\( \varepsilon_{n+1}(\sigma(b)) = 1_{\rho} \otimes \varepsilon_n(b), \quad b \in H_n(q), \)

where \( \sigma : H_n(q) \to H_{n+1}(q) \) is the homomorphism taking each generator \( g_i \) of \( H_n(q) \) to \( g_{i+1} \).

Also recall that an object \( \rho \) of \( T \) is called \( \mu \)-special of dimension \( d \) if there is a \( \mu^2 \)-Hecke symmetry for \( \rho \) and an intertwiner \( R \in (\iota, \rho \otimes d) \) for some \( d \geq 2 \), satisfying

\[
R^* R = (d-1)! q (\varepsilon)^{d-1} 1_{\rho}, \quad (10.3)
\]

\[
R^* \otimes 1_{\rho \otimes d} R = (\varepsilon)^{d-1} 1_{\rho}, \quad (10.4)
\]

\[
RR^* = \varepsilon_d(A_d), \quad (10.5)
\]

\[
\varepsilon(g_1 \ldots g_d) R \otimes 1_{\rho} = -(\mu)^{d-1} 1_{\rho} \otimes R, \quad (10.6)
\]

where \( q := \mu^2 \), see [17] for notation.

If \( T \) admits a special object of dimension \( d \) for some \( \mu > 0 \), Theorem 6.2 in [17] then assures then the existence of a tensor \( ^* \)-functor \( \text{Rep}(S_{\mu}U(d)) \to T \) taking the fundamental representation \( u \) of \( S_{\mu}U(d) \) to \( \rho \) and the canonical intertwiner \( S \in (\iota, u \otimes d) \) to \( R \), where \( u \) is the fundamental representation of \( (S_{\mu}U(d)). \) We are thus in a position to apply Theorem 10.1.

10.3 Theorem Let \( \rho \) be a \( \mu \)-special object of a tensor \( C^* \)-category \( T \) with dimension \( d \geq 2 \) and parameter \( \mu > 0 \). Then there exists an ergodic non-degenerate action of \( S_{\mu}U(d) \) on a unital \( C^* \)-algebra \( B \) whose spectral functor can be identified on the objects with \( u \otimes \tau \rightarrow (\iota, \rho^\tau) \), with \( u \) the fundamental representation of \( S_{\mu}U(d) \).

In the particular case where \( d = 2 \) the notion of a \( \mu \)-special object of dimension 2 simplifies considerably.

10.4 Proposition If an object \( \rho \) of a tensor \( C^* \)-category \( T \) admits an intertwiner \( R \in (\iota, \rho \otimes 2) \) satisfying relations

\[
R^* R = (1 + q)1_{\iota}, \quad (10.7)
\]

\[
R^* \otimes 1_{\rho} R = -\mu 1_{\rho}, \quad (10.8)
\]

with \( \mu > 0 \) and \( q = \mu^2 \), then \( \rho \) can be made uniquely into a \( \mu \)-special object of dimension 2 with intertwiner \( R \).

Proof If \( \varepsilon \) is any \( \mu^2 \)-Hecke symmetry for \( \rho \) making \( \rho \) into a \( \mu \)-special object of dimension 2 with intertwiner \( R \), then (10.5) shows that \( RR^* = \varepsilon_2(A_2) \). Since \( A_2 = 1 + g_1, \varepsilon_2(g_1) = RR^* - 1_{\rho \otimes 2} \). Therefore for all \( n, i = 1, \ldots, n - 1 \),

\[
\varepsilon_n(g_i) = \varepsilon_n(\sigma^{i-1}(g_1)) = 1_{\rho \otimes i} \otimes \varepsilon_2(g_1) = 1_{\rho \otimes i} \otimes (RR^* - 1_{\rho \otimes 2})
\]
and the symmetry is uniquely determined. Let us then show the existence of a
symmetry using that formula. The orthogonal projection \( e = \frac{1}{1+q} RR^* \in (\rho^2, \rho^2) \)
satisfies the Temperley–Lieb relations

\[
e \otimes 1_\rho \circ 1_\rho \otimes e \circ e \otimes 1_\rho = (q + \frac{1}{q} + 2)^{-1} e \otimes 1_\rho,
\]

\[
1_\rho \otimes e \circ e \otimes 1_\rho \circ 1_\rho \otimes e = (q + \frac{1}{q} + 2)^{-1} 1_\rho \otimes e
\]

thanks to (10.8). Since the Temperley–Lieb algebra \( TL_n((q + \frac{1}{q} + 2)^{-1}) \) is the
quotient of the Hecke algebra \( H_n(q) \) by the ideal generated by \( A_n \) (see [5]),
there does exist a Hecke symmetry for \( \rho \) such that

\[
\varepsilon_2(g_1) = (q + 1)e - 1_\rho \otimes 2 = RR^* - 1_\rho \otimes 2.
\]

Since \( A_n = 1 + g_1 \), \( \varepsilon_2(A_n) = (q + 1)e = R \circ R^* \), so (10.5) follows.

We show (10.6) for \( d = 2 \):

\[
\varepsilon_3(g_2) \circ R \otimes 1_\rho = (1_\rho \otimes (RR^* - 1_\rho \otimes 2)) \circ R \otimes 1_\rho = -\mu 1_\rho \otimes R - R \otimes 1_\rho,
\]

hence

\[
\varepsilon_3(g_1g_2) \circ R \otimes 1_\rho = ((RR^* - 1_\rho \otimes 2) \otimes 1_\rho) \circ (-\mu 1_\rho \otimes R - R \otimes 1_\rho) =
\]

\[
\mu^2 R \otimes 1_\rho - (q + 1) R \otimes 1_\rho + \mu 1_\rho \otimes R + R \otimes 1_\rho = \mu 1_\rho \otimes R.
\]

Remark There is a canonical isomorphism from \( \text{Rep}(S_\mu U(2)) \) to \( \text{Rep}(A_o(F)) \)

Relations (10.7) and (10.8) can be implemented in Hilbert spaces with di-
mension \( \geq 2 \) in the following way. Let \( j \) be any antilinear invertible map on a
finite dimensional Hilbert space \( H \), and set \( R = \sum_i j e_i \otimes e_i \in H \otimes 2 \). Then, for
this \( R \), (10.7) and (10.8) become,

\[
\text{Trace}(j^*j) = 1 + q,
\]

\[
j^2 = -\mu.
\]

Consider the involutive antunitary map \( c \) of \( H \) acting trivially on the orthonor-
mal basis \( e_i \), and set \( F = jc \) and \( \overline{F} = cFc \). Then the above conditions can be
equivalently written

\[
\text{Trace}(F^*F) = 1 + q,
\]

\[
\overline{F}F = -\mu.
\]

The maximal compact quantum group with representation category generated
by \( R \) is the universal quantum group \( A_o(F) \) defined in [22]. Therefore the
fundamental representation of \( A_o(F) \) is a \( \mu \)-special object of dimension 2 in
\( \text{Rep}(A_o(F)) \). Theorem 6.2 in [17] then shows that there is a unique isomorphism
of tensor \( C^* \)-categories \( \text{Rep}(S_\mu U(2)) \to \text{Rep}(A_o(F)) \) taking the fundamental
representation $u$ of $S_uU(2)$ to the fundamental representation of $A_o(F)$ and the quantum determinant $S = \psi_1 \otimes \psi_2 - \mu \psi_2 \otimes \psi_1 \in (\iota, u^{\otimes 2})$ to $R = \sum j e_i \otimes e_i$.

For related results, see Cor. 5.4 in [2], where the authors find similar necessary and sufficient conditions for the existence of a monoidal equivalence between a generic pair of universal compact quantum groups, and the result of Banica [1], where it is shown that the fusion rules of $A_o(F)$ are the same as those of $SU(2)$.

10.5 Corollary Let $\rho$ be an object of a tensor $\text{C}^\ast$–category $\mathcal{T}$ with an intertwiner $R \in (\iota, \rho^{\otimes 2})$ satisfying conditions (10.7) and (10.8). Then the unique tensor $\ast$–functor $\text{Rep}(S_uU(2)) \to \mathcal{T}$ taking the fundamental representation $u$ to $\rho$ and the quantum determinant $S = \psi_1 \otimes \psi_2 - \mu \psi_2 \otimes \psi_1 \in (\iota, u^{\otimes 2})$ to $R$ gives rise to an ergodic nondegenerate action of $S_uU(2)$ on a unital $\text{C}^\ast$–algebra $B$ with spectral subspaces $\mathcal{T}_{u^{\otimes r}} = (\iota, \rho^r)$.

11 Actions embeddable into quantum quotient spaces

As a second application of the duality theorem 9.1, consider the invariant vectors functor $K$ associated with a compact quantum subgroup $K$ of $G$. We know that this is just a copy of the spectral functor of the quotient space $K\backslash G$.

11.1 Theorem Let $K$ be a compact quantum subgroup of a maximal compact quantum group $G$. Then the nondegenerate ergodic $G$–system associated with the invariant vectors functor $K$ is isomorphic to the quotient $G$–space $K\backslash G$.

Proof It is clear from the construction of the dense Hopf $\ast$–algebra $\mathcal{E}_K$ in Proposition 2.6 that $\mathcal{E}_K$ is $\ast$–isomorphic to the dense spectral subalgebra of $A^\delta$, in such a way that the constructed $G$–action corresponds to the right $G$–action on the right coset space. Therefore we are left to show that the maximal $\text{C}^\ast$–seminorm $\| \cdot \|_1$ on $\mathcal{E}_K = A^\delta_{sp}$ coincides with the restriction of the maximal $\text{C}^\ast$–seminorm $\| \cdot \|_2$ on $A_{\infty}$. Any Hilbert space representation of $A_{\infty}$ restricts to a Hilbert space representation on $A^\delta_{sp}$, so $\| \cdot \|_2 \leq \| \cdot \|_1$. Conversely, if $\pi$ is a Hilbert space representation for $A^\delta_{sp}$, we can induce it up to a Hilbert space representation $\tilde{\pi}$ of $A_{\infty}$ via the conditional expectation $m : A_{\infty} \to A^\delta_{sp}$ obtained averaging over the action $\delta$ of the subgroup. There is an isometry $V$ from the Hilbert space of $\pi$ to the Hilbert space of $\tilde{\pi}$ such that $V^*\tilde{\pi}(a)V = \pi(m(a))$, for $a \in A_{\infty}$. In particular, if $a \in A^\delta_{sp}$, $\pi(a) = V^*\tilde{\pi}(a)V$, so $\|\pi(a)\| \leq \|\tilde{\pi}(a)\| \leq \|a\|_2$, and $\|a\|_1 \leq \|a\|_2$.

11.2 Definition An ergodic $G$–action $\eta : B \to B \otimes A$ will be called maximal if $G$ is a maximal compact quantum group and if $B$ is obtained completing the dense spectral $\ast$–subagebra with respect to the maximal $\text{C}^\ast$–seminorm.

Combining the previous result with the abstract characterization of the invariant vectors functor $K$ given in Theorem 5.5 and with Prop. 9.4, gives the
following characterization of maximal ergodic systems isomorphic to quotient spaces.

11.3 Theorem Let \((B, \eta)\) be a maximal, nondegenerate, ergodic \(G\)–action. If there exists, for each unitary representation \(u\) of \(G\), a subspace \(K_u \subset H_u\) satisfying properties (5.1), (5.2), (5.7), (5.8) and a quasitensor natural unitary transformation from the spectral functor \(\mathcal{T}\) associated with \((B, \eta)\) to the functor \(K\), then there exists a unique maximal compact quantum subgroup \(K\) of \(G\) such that \((B, \eta) \simeq K\backslash G\).

The last application concerns a functor \(K\) satisfying conditions (5.1)–(5.4), which are weaker than the conditions describing the invariant vectors functor.

11.4 Theorem Let \(G = (A, \Delta)\) be a compact quantum group, and \(\zeta : \mathcal{C} \to \mathcal{C} \otimes A\) a nondegenerate ergodic \(G\)–action on a unital \(C^*\)–algebra \(\mathcal{C}\) with associated spectral functor \(L\). Then the following properties are equivalent:

a) \(\mathcal{C}_{sp}\) has a \(*\)–character,

b) there is a subfunctor \(K\) of the embedding functor \(H\) satisfying properties (5.1)–(5.4) and a quasitensor unitary natural transformation from \(L\) to \(K\),

c) there is a compact quantum subgroup \(K\) of \(G\) and a faithful \(*\)–homomorphism \(\phi : \mathcal{C}_{sp} \to A^\delta\) intertwining \(\zeta\) with the \(G\)–action on the compact quantum quotient space \(K\backslash G\).

Proof We first show that a) implies b). Let \(\chi\) be a \(*\)–character of \(\mathcal{C}_{sp}\), and define, for \(u \in \text{Rep}(G)\), the map \(\eta_u : \mathcal{T}_u \to H_u\) by

\[
\eta_u(T) = \sum_i \chi(T(e_i)^*)e_i,
\]

with \(e_i\) an orthonormal basis of \(H_u\). This map is an isometry, as

\[
(\eta_u(T), \eta_u(T')) = \sum_i \chi(T(e_i)^*)\chi(T'(e_i)^*) = \chi\left(\sum_i T(e_i)T'(e_i)^*\right) = (T, T').
\]

Actually \(\eta\) is a quasitensor natural transformation from \(\mathcal{T}\) to \(H\), as for \(A \in \{(u, v)\}, \mathcal{T} \in L_u\),

\[
\eta_u(T_A(T)) = \eta_u(T_A(T)) \sum_j \chi((TA^*(f_j))^*)f_j = \sum_{i,j} \chi(T(e_i)^*)(f_j, Ae_i)f_j = \sum_i \chi(T(e_i)^*)Ae_i = H_A(\eta_u(T)).
\]

and, for \(T \in L_u, T' \in L_v,\)

\[
\eta_u \otimes v(T \otimes T') = \eta(u \otimes v(T \otimes T')) = \sum_{i,j} \chi(T \otimes T'(e_i \otimes f_j)^*)e_i \otimes f_j = \sum_{i,j} \chi((T(e_i)T'(f_j))^*)e_i \otimes f_j =
\]
\[
\sum_i \chi(T(e_i^*))e_i \otimes \sum_j \chi(T'(f_j^*))f_j = \eta_u(T) \otimes \eta_v(T').
\]

It follows that the functor \(K_u := \eta_u(L_u), K_A := A \upharpoonright K_u\), for \(A \in (u, v)\) and \(u, v \in \text{Rep}(G)\) is a quasitensor \(*\)-subfunctor of \(H\). Therefore \(K\) satisfies properties (5.1), (5.2) and (5.4). We are left to show that (5.3) holds as well. Consider, for \(S \in L_u\), the map \(\hat{S} : L_u \otimes v \to \overline{L_u}\) defined in (7.1). A straightforward computation shows that
\[
\eta_u \circ \hat{S} = r^\ast_{\eta_L(S)} \circ \eta_u \otimes v.
\]
Therefore
\[
r^\ast_{K_u}, K_u \otimes v \subset \eta_u(\overline{L_u}) = K_u,
\]
and this is relation (5.3). We next show that b) implies c). Thanks to Lemma 4.1, \(K\) is a quasitensor subfunctor of \(H\). By Prop. 9.4 there is a faithful \(*\)-homomorphism \(\phi : \mathcal{C}^G \to \mathcal{C}^K\) intertwining the \(G\)-actions. Now, \(\mathcal{C}^K\), regarded as the \(*\)-algebraic \(G\)-system defined by \(K\), is \(*\)-isomorphic to the \(*\)-algebraic \(G\)-system defined by the linear span of the coefficients \(u_k, \varphi\), with \(k \in K_u, \varphi \in H_u, u \in \text{Rep}(G)\), with the restricted \(G\)-action, thanks to the \(*\)-algebraic structure of \(A\) recalled at the end of subsection 2.1. This is in turn a \(*\)-subsystem of some quantum quotient space \(K \backslash G\), by Theorem 5.1. On the other hand the system \((\mathcal{C}^\delta, \eta^\delta_{\mathcal{C}})\) is in turn \(*\)-isomorphic to \((\mathcal{C}^\delta, \zeta)\), and the proof is now complete. We are left to show that c) implies a). The range of \(\phi\) must be contained in the spectral \(*\)-subalgebra of \(A^\delta\) because of the intertwining relation between the \(G\)-actions. Since \(A^\delta_{sp}\) is contained in the dense \(*\)-subalgebra of \(A\) generated by the matrix coefficients, \(u_{\varphi, \psi}\), we can define a \(*\)-character \(\chi\) on \(A_{sp}\) simply by composing \(\phi\) with the counit \(e\) of \(G\).

**Remark** Under the assumptions of Theorem 11.4, if one has an everywhere defined \(*\)-character on \(C\) and if the action \(\zeta\) and the Haar measure of \(G\) are faithful, one can construct a faithful embedding of the whole system \((C, \zeta)\) into a compact quantum quotient space \(K \backslash G\), as shown in Theorem 7.4 in [13].

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