INDEPENDENT SETS IN ALMOST-REGULAR GRAPHS AND THE CAMERON-ERDŐS PROBLEM FOR NON-INARIANT LINEAR EQUATIONS

PETER HEGARTY

ABSTRACT. We propose a generalisation of the Cameron-Erdős conjecture for sum-free sets to arbitrary non-translation invariant linear equations over $\mathbb{Z}$ in three or more variables and, using well-known methods from graph theory, prove a weak form of the conjecture for a class of equations where the structure of the maximum-size sets avoiding solutions to the equation has been previously obtained.

1. INTRODUCTION

A set of integers $A$ is said to be sum-free if there are no solutions in $A$ to the equation $x + y = z$. In 1988, Cameron and Erdős [CE] proposed their famous conjecture that the number of sum-free subsets of $[1, n]$ is $O(2^{n/2})$, as $n \to \infty$. The number $n/2$ arises naturally, since $[n/2]$ is easily seen to be the maximum size of a sum-free subset of $[1, n]$. Indeed, the set of odd numbers in the interval, together with the interval $(n/2, n]$, provide two explicit examples of sum-free sets of maximum size. Moreover, it is not hard to show that, for any odd $n$, these are the only two examples, whereas for $n$ even there is one more, namely $[n/2, n]$. Since any subset of a sum-free set is still sum-free, the main point of the Cameron-Erdős conjecture is that it says in essence that there are few large sum-free sets.

As is well-known, this conjecture has been proven in recent years independently by Green [G] and Sapozhenko [Sa]. Somewhat less well-known is that many authors have worked on the following generalisation of the problem. Let $k > l$ be positive integers. A set $A$ of integers is said to be $(k, l)$-sum-free if there are no solutions in $A$ to the equation $x_1 + \cdots + x_k = y_1 + \cdots + y_l$. For fixed $k$ and $l$, there are two natural examples of a ‘large’ $(k, l)$-sum-free subset of $\{1, \ldots, n\}$, namely

$$A_1 = \left( \frac{l}{k} n, n \right),$$

and

$$A_2 = \{ m \in [1, n] : m \equiv r \pmod{\rho} \},$$

where $\rho$ is the smallest natural number not dividing $k - l$ and $r$ is any integer relatively prime to $\rho$. Note that $|A_1| \approx \left( \frac{k-1}{k} \right) n$ and $|A_2| \approx \frac{1}{\rho} n$. The simplest and most natural
generalisation of the Cameron-Erdős conjecture to this setting is then the assertion that the number of \((k, l)\)-sum-free subsets of \([1, n]\) is \(O(2^{\frac{k-l}{k-l}})\), where \(c_{k,l} = \max\{\frac{k-l}{k-l}, \frac{1}{\rho}\}\). There have been a considerable number of papers written on this problem, notably, in chronological order, [CTa], [CTh], [B], [Sc1], [Sc2] and [L]. The state-of-the-art is due to Lev [L], who established the conjecture (indeed, a more precise asymptotic) for all but finitely many pairs \((k, l)\).

The original problem has also been generalised in a quite different direction, namely where the interval \([1, n]\) is replaced by a finite abelian group (the notion of ‘sum-free’ being defined in the same way), in particular \(\mathbb{Z}_n\). We will not be concerned with this direction here, but the interested reader is referred to [GR].

Rather we take our inspiration from the previous generalisation. Let \(L\) denote an arbitrary linear equation over \(\mathbb{Z}\), which we may write as

\[
L : a_1x_1 + \cdots + a_kx_k = b, \quad a_1, \ldots, a_k, b \in \mathbb{Z}.
\]

A set \(A\) of integers is said to be \(L\)-free\(^3\) if there are no non-trivial solutions in \(A\) to (1.3). For an explanation of the meaning of ‘non-trivial’, see [R1] - trivial solutions only arise when the equation is translation invariant, i.e.: \(\sum a_i = b = 0\). Let \(M_L(n)\) denote the maximum size of an \(L\)-free subset of \([1, n]\) and \(S_L(n)\) the total number of \(L\)-free subsets. It is natural to ask if, for any \(L\), one has \(S_L(n) = O(2^{M_L(n)})\)? However, at this level of generality, the answer is easily seen to be negative. For suppose \(L\) is translation invariant. It follows from Szemerédi’s celebrated theorem [Sz] that \(M_L(n) = o(n)\), and from this it is easily deduced that \(S_L(n)/M_L(n) \rightarrow \infty\). Indeed, this observation was already made by Cameron and Erdős [CE]. For the specific examples of Sidon sets and sets avoiding arithmetic progressions of a fixed length (the latter referring to a system of linear equations rather than a single one), they asked if one could still recover a weaker result, namely is

\[
S_L(n) = O(2^{(1+o(1))M_L(n)}) \quad ?
\]

As far as we know, absolutely no progress has been made on this question, for any translation invariant \(L\). But this weaker conjecture is still interesting for historical reasons, since the first significant advances on the Cameron-Erdős conjecture for sum-free sets were made (independently) by Alon [A] and Calkin [C], who both proved (1.4) in that case. They employed considerably different arguments. Alon’s is based on an estimate for the number of independent sets in a regular graph, and works equally well in the group setting. Calkin makes use of Szemerédi’s theorem. Both approaches had a profound influence on future work. Sapozhenko’s proof of the sum-free conjecture crucially employs a version of Alon’s graph theoretical result (Green’s proof is different), which is both an improvement and an extension to graphs which are ‘nearly’ regular\(^4\): see Lemma 2.4 below. Calkin’s methods found greater purchase in the papers cited above extending the analysis to \((k, l)\)-sum-free sets.

Let us now state formally the conjecture we wish to propose in this paper:

\[^3\]In [H], the term ‘\(L\)-avoiding’ was used.
\[^4\]In a slightly different direction, Kahn [K1] [K2] proved an optimal result for bipartite graphs.
Conjecture 1.1. Let $\mathcal{L}$ be a linear equation over $\mathbb{Z}$ in at least three variables. For each $n \in \mathbb{N}$ let $M_{\mathcal{L}}(n)$ and $S_{\mathcal{L}}(n)$ denote, respectively, the maximum size of and the total number of $\mathcal{L}$-free subsets of $\{1, ..., n\}$. Then, as $n \to \infty$,
\begin{equation}
S_{\mathcal{L}}(n) = O_{\mathcal{L}}(2^{(1+o(1))M_{\mathcal{L}}(n)})
\end{equation}
and, if $\mathcal{L}$ is not translation invariant, then in fact
\begin{equation}
S_{\mathcal{L}}(n) = O_{\mathcal{L}}(2^{M_{\mathcal{L}}(n)}).
\end{equation}

We shall refer to (1.5) as the weak form of our conjecture and to (1.6) as the strong form (for non-invariant equations).

Remark 1.2. The restriction to three or more variables is necessary, since even the weak conjecture fails miserably in two variables. For example, consider the equation $\mathcal{L} : 2x = y$. It is known (see the Proposition in Section 1 of [H]) that a greedy choice mechanism yields the infinite $\mathcal{L}$-free set
\begin{equation}
A = \{u \cdot 4^i : u \text{ odd}, i \geq 0\}
\end{equation}
and that, for every $n$, $A \cap [1, n]$ is an $\mathcal{L}$-free subset of $\{1, ..., n\}$ of maximum size. Hence $M_{\mathcal{L}}(n)/n \to 2/3$. On the other hand, a set $B \subseteq \{1, ..., n\}$ is $\mathcal{L}$-free if and only if $B$ is an independent set in the graph $G = (V, E)$ where $V = \{1, ..., n\}$ and $\{x, y\} \in E$ if $2x = y$ or $2y = x$. This graph consists of $\lceil n/2 \rceil$ disjoint chains, one for each odd number in $V$. In a chain of length $l$, the number of independent sets is the Fibonacci number $F_{l+1}$ - here $F_0 = F_1 = 1$ and $F_t = F_{t-1} + F_{t-2}$ for all $t > 1$. To simplify the calculation of an estimate for $S_{\mathcal{L}}(n)$, suppose $n$ is a power of two, say $n = 2^r$. Then the graph $G$ contains one chain of length $r + 1$ and, for each $j = 1, ..., r$, $2^{t-1}$ chains of length $r - j$. It follows that
\begin{equation}
S_{\mathcal{L}}(2^r) = F_{r+2} \times \prod_{j=0}^{r-1} F_{r-j}.
\end{equation}

One easily checks that, for example, $F_t \geq \gamma^{t-1}$ for all $t$, where $\gamma = \frac{1+\sqrt{5}}{2}$. Thus $S_{\mathcal{L}}(2^r) \geq \gamma^{\sigma_r}$ where
\begin{equation}
\sigma_r = (r + 1) + \sum_{j=0}^{r-1} 2^j(r - 1 - j) = 2^r.
\end{equation}

In other words, if $n$ is a power of two, then $S_{\mathcal{L}}(n) \geq \gamma^n$. Since $\gamma > 2^{2/3}$, this means that (1.5) does not hold for $\mathcal{L}$.

It is quite easy to see that there is nothing special about this choice of $\mathcal{L}$ and that (1.5) fails to hold for any non-invariant equation in two variables.

In considering Conjecture 1.1 a very big problem seems to present itself, namely, for very few equations is the function $M_{\mathcal{L}}(n)$ even known. The structure of maximum $\mathcal{L}$-free sets is, for most $\mathcal{L}$, shrouded in mystery. This problem was introduced in very general terms by Ruzsa [R1] [R2], but remains intractable. In particular, Ruzsa
\footnote{For the remainder of this paper, the word ‘maximum’ is used synonymously with ‘maximum size’, and should not be confused with ‘maximal’, which denotes a weaker property having to do with location in a poset of sets ordered by inclusion.}
conjectured that maximum \( L \)-free sets should always ‘look like’ (1.1) or (1.2), but several examples are now known where this is not so. In this paper, we will consider a collection of equations in three variables where the structure of the maximal \( L \)-free sets has been obtained in great detail [BHKLS], [H], and is usually more complicated than (1.1) or (1.2). This background material is presented in Section 2 below. Our main result (Theorem 2.2) is that (1.5) holds for these equations. While this may seem like a very miniscule step on the way to establishing (1.5) or (1.6) in general, we believe our result motivates further study since we obtain it using the same graph theoretical tools developed by Alon, Sapozhenko et al. It thus suggests that these methods may have even wider applicability, at least to equations in three variables, to which they seem particularly suited. Indeed, as the number of variables is increased, other techniques may be more suitable, as in the papers cited above dealing with \((k, l)\)-sum-free sets. We shall return to this discussion in the last section.

The rest of the paper is organised as follows. In Section 2, the main result is introduced and some preliminary lemmas listed. In Section 3, we prove the main result in a special case which illustrates all the main ideas involved and is the most technically challenging. This section is the heart of the paper. In Sections 4 and 5, we prove Theorem 2.2 in all remaining cases, giving only the briefest outline in the latter section. Section 6 is devoted to summarising remarks and suggestions for future work.

2. Background material, statement of main result and preliminary lemmas

We will be concerned with the following three groups of equations:

- **GROUP 1**: \( x + y = cz, c > 2 \).
- **GROUP 2**: \( x + by = cz, b > 1, c > \frac{(b+1)b^2}{4} \).
- **GROUP 3**: \( b(x + y) = cz, b > 1, \text{GCD}(b, c) = 1 \).

Let us first recall what is known about the maximum \( L \)-free subsets of \( \{1, \ldots, n\} \), when \( L \) is one of the equations in Groups 1-3. To help with notation, we shall denote the equation \( ax + by = cz \) by \( L_{a,b,c} \), and drop the subscripts whenever it is clear to which equation we are referring. To begin with, it was proven in [CG] that \( M_{L_{1,1,3}}(n) = \lceil n/2 \rceil \) and that the odd numbers in \([1, n]\) form the unique maximum \( L \)-free set for all \( n \gg 0 \). In [BHKLS] it was shown that, for \( c > 3 \), \( M_{L_{1,1,c}}(n) = f(c) \cdot n + O(1) \), where

\[
 f(c) = \frac{(c - 2)(c^4 - 4)}{c(c^4 - 2c^2 - 4)}, \tag{2.1}
\]

and that every maximum \( L \)-free subset of \( \{1, \ldots, n\} \) has a bounded symmetric difference from the set

\[
 A_{1,1,c}(n) = \left( \frac{2}{c} u_c v_c n, u_c v_c n \right] \cup \left( \frac{2}{c} u_c n, u_c n \right] \cup \left( \frac{2}{c} n, n \right], \tag{2.2}
\]

where

\[
 u_c = \frac{2(c^2 - 2)}{c^4 - 2c^2 - 4}, \quad v_c = \frac{2}{c^2 - 2}. \tag{2.3}
\]
For equations in Group 2 it was proven in [Hi] that $M_{L_1,b,c}(n) = f(b, c) \cdot n + O(1)$, where

$$f(b, c) = \left(1 - \frac{b + 1}{c}\right) \left(1 + \frac{b + 1}{c^2 - b(b + 1)}\right),$$

and that every maximum $L$-free set has bounded symmetric difference from

$$A_{1,b,c}(n) = \left(\frac{b + 1}{c} w_{b,c} n, w_{b,c} n\right) \cup \left(\frac{b + 1}{c} n, n\right),$$

where

$$w_{b,c} = \frac{b + 1}{c^2 - b(b + 1)}.$$  \hfill (2.6)

For Group 3 equations, maximum free sets were also exhibited in [Hi], though not completely classified. There are three different cases:

(i) If $c > 2b$ then $M_{L_2,b,b,c}(n) = \left(1 - \frac{2}{c}\right) n + O(1)$ and a maximum $L$-free set is

$$A_{b,b,c}(n) = \left[\frac{2b}{c} n, n\right] \cup \left\{x \in [1, \frac{2b}{c} n] : x \not\equiv 0 \pmod{b}\right\}. \hfill (2.7)$$

(ii) If $2 \leq c < 2b$ then $M_{L_2,b,b,c}(n) = \left(1 - \frac{1}{b}\right) n + O(1)$ and a maximum $L$-free set is

$$A'_{b,b,c}(n) = \{x \in [1, n] : x \not\equiv 0 \pmod{b}\}. \hfill (2.8)$$

(iii) If $c = 1$ then $M_{L_3,b,b,1}(n) = \left(1 - \frac{1}{2b}\right) n$ and a maximum $L$-free set is

$$A''_{b,b,1}(n) = \left[\frac{n}{2b}, n\right]. \hfill (2.9)$$

**Remark 2.1.** To use terminology introduced in [Li], the sets in (2.2), (2.5) and (2.9) are non-archimedean, those in (2.8) and for $L_1,1,3$ are archimedean, while that in (2.7) is mixed. We will return to this point in Section 6.

Let us now state our main result.

**Theorem 2.2.** Let $L$ be any of the equations in groups 1-3 above. Then, as $n \to \infty$, $S_L(n) = O(2^{(1+o(1))M_L(n)})$.

Our proof will make repeated use of the following two lemmas from the literature. The first one is a very simple observation:

**Lemma 2.3.** For each $\epsilon > 0$ there exists $\delta > 0$ such that, for all sufficiently large integers $n$, the number of subsets of $\{1, \ldots, n\}$ of size at most $\delta n$ is at most $2^{\epsilon n}$.

**Proof.** See Lemma 2 of [C]. \hfill \square

The next result is the crucial one. Let $G = (V, E)$ be a simple, loopless graph. Following [Sa], we say that $G$ is a $(p, l, k, m, \alpha, \beta, \theta)$-graph if it meets the following conditions:

1. $|V| = p$.
2. Every vertex has degree at least $l$ and at most $m$.
3. The fraction of vertices with degree less than $k - \theta$ is at most $\alpha$, and the fraction whose degree exceeds $k + \theta$ is at most $\beta$.

The following result is part of Theorem 2 of [Sa]:
**Lemma 2.4.** Let $G$ be a $(p, l, k, m, \alpha, \beta, \theta)$-graph and let $I(G)$ denote the number of independent sets in $G$. Then for large enough $p$,

$$I(G) \leq 2^{\frac{2}{7} \left(1+\alpha(1/l)+\beta(m/k)+O(\theta/k+\sqrt{\log k/k})\right)}.$$  

(2.10)

3. **Proof of Theorem 2.2 for equations $\mathcal{L}_{1,c}$ with $c > 3$**

The proof of the theorem in this case will illustrate all the key ideas involved, as well as being the most technically challenging. We thus present it in detail. Our approach is similar to that followed in [BHKLS], but since we are not interested in the detailed structure of large $\mathcal{L}$-free sets the presentation will be modified to suit our present purposes. For the rest of this section, $c$ denotes a fixed integer greater than 3 and $\mathcal{L}$ denotes the equation $\mathcal{L}_{1,c}$.

We will seek to apply Lemma 2.4, and for that we will apply a strategy of ‘pairing off’ elements in an $\mathcal{L}$-free set $A$ with elements not in $A$. The latter is accomplished by means of the following simple fact:

**Lemma 3.1.** Let $A$ be an $\mathcal{L}$-free subset of $\{1, \ldots, n\}$ and suppose $x \in A \cap [1, \frac{2n}{c}]$. Then for each integer $y \in [cx - n, n]$, at most one of the numbers $y$ and $cx - y$ lies in $A$.

*Proof.* Obvious. See page 2 of [BHKLS].

To obtain the structure of maximum $\mathcal{L}$-free sets, the basic idea is to show that, if an $\mathcal{L}$-free set $A$ contains an element far from the three intervals identified in (2.2), then Lemma 3.1 can be applied to pair off a lot of numbers which keeps the size of $A$ small. For Theorem 2.2, one essentially needs to extend this idea to show that, if an $\mathcal{L}$-free set contains a lot of elements outside these ‘good’ intervals, then the resulting collection of pairings will, when taken altogether, keep the number of possibilities for $A$ small. This is where Lemma 2.4 comes in. Let’s now get to the details. The first step is

**Lemma 3.2.** Let $A$ be an $\mathcal{L}$-free subset of $\{1, \ldots, n\}$. The number of possibilities for $A \cap (n/c, n]$ is $2^{(1 - \frac{c}{n})n(1 + o(1))}$.

*Proof.* Let $\epsilon > 0$ be given. Let $\delta > 0$ be as in Lemma 2.3, and $n$ be much, much larger than is necessary for the conclusion of that lemma to hold (how much larger will become clear in due course). Let $A$ be an $\mathcal{L}$-free subset of $\{1, \ldots, n\}$. We distinguish two types of sets:

I. Those $A$ for which $|A \cap (\frac{n}{c}, \frac{2n}{c})| < \delta \left(\frac{n}{c}\right)$,

II. All other $A$.

If $A$ is of type I, then clearly the number of possibilities for $A \cap (\frac{n}{c}, n]$ is $O(2^{(1 - \frac{c}{n} + \frac{\delta}{n})n})$. So it remains to consider type II sets. Let $A$ be any such set. Divide the interval $(\frac{n}{c}, \frac{2n}{c}]$ into subintervals each of length $[\sqrt{n}]$ or $[\sqrt{n}]$. There must exist at least one such subinterval, say $\mathcal{I} = [s, t]$, such that $t < \frac{2n}{c} - \left(\frac{3}{2}\right)\frac{n}{c}$ and $|\mathcal{I} \cap A| \geq \frac{s}{2} \sqrt{\frac{n}{c}}$. First consider a fixed $t$. Note that there are certainly no more than $2^{\sqrt{n}}$ possibilities for $\mathcal{I} \cap A$, so we also consider a fixed choice of the latter. Let $G = (V, E)$ be the following graph:

$V = [cs - n, n]$ and $(x, y) \in E$ if there exists some $z \in \mathcal{I} \cap A$ such that $x + y = cz$. 


By Lemma 3.1, \( A \cap [c^s - n, n] \) must be an independent set in this graph. We now wish to apply Lemma 2.4. The graph \( G \) is a \(( p, l, k, m, \alpha, \beta, \theta)\)-graph with parameters

\[
p = 1 + n - (c^s - n) > \frac{\delta}{2} n, \tag{3.1}
\]

\[
l = 1, k = m = |\mathcal{F} \cap A| \geq \frac{\delta}{2} \sqrt{n}, \tag{3.2}
\]

\[
\beta = \theta = 0, \quad \alpha = \frac{c(t - s)}{p} = O_\epsilon(1/\sqrt{n}). \tag{3.3}
\]

To simplify notation, put \( c^s - n := \kappa n \), where \( \kappa = \kappa_t = 1 - \Theta_\epsilon(1) \), so that \( p = 1 + (1 - \kappa)n \). From Lemma 2.4, it follows that the number of possibilities for \( A \cap (\kappa n, n] \) is \( O(2^{\frac{(1-\epsilon)\kappa n}{1+O_\epsilon(1/\sqrt{n})}}) \). Note that this estimate is not affected if we now allow the set \( \mathcal{F} \cap A \) to vary, still for a fixed \( t \). Hence, for a fixed choice of \( t \), the number of possibilities for a type II set equals the previous number times the number of possibilities for \( A \cap (\kappa n, n] \). The latter factor is certainly no more than the number of possibilities for \( A \cap (\kappa n, n] \).

To summarise, we have shown that, for given \( \epsilon > 0 \) and all \( n \) sufficiently large (depending on \( \epsilon \)), if \( A \) is an \( \mathcal{L} \)-free subset of \( \{1, \ldots, n\} \) then the number \( \pi_n \) of possibilities for \( A \cap (\kappa n, n] \) satisfies a recursion

\[
\pi_n = O \left( 2^{(1-\frac{4}{c^2-2})n} + \sum_{1 < \kappa < \kappa^0} 2^{\frac{(1-\epsilon)\kappa n}{1+O_\epsilon(1/\sqrt{n})} \tau_{\kappa n}} \right), \tag{3.4}
\]

where \( \kappa^0 = 1 - \Theta_\epsilon(1) \) and the number of terms in the sum is \( O(n) = 2^{O(\log n)} \). Clearly, we can deduce Lemma 3.2 by iterating this recursion \( O_\epsilon(1) \) times for fixed \( \epsilon \) and \( n \), and then letting \( \epsilon \to 0 \) as \( n \to \infty \).

**Lemma 3.3.** Let \( A \) be an \( \mathcal{L} \)-free subset of \( \{1, \ldots, n\} \). The number of possibilities for \( A \cap \left( \frac{4n}{c^2-2}, n \right] \) is \( 2^{f_1(c)n(1+o(1))} \), where \( f_1(c) = (1 - \frac{4}{c^2-2})(1 + \frac{2}{c^2-2}) \).

**Proof.** Let \( \epsilon > 0 \) be given, \( \delta \) as in Lemma 2.3 and \( n \) sufficiently large. Let \( \tau_n \) denote the number of possibilities for \( A \cap \left( \frac{4n}{c^2-2}, n \right] \), where \( A \) is \( \mathcal{L} \)-free. This time, we consider three types of \( \mathcal{L} \)-free sets:

I. Those \( A \) for which \( |A \cap \left( \frac{n}{c^2-2}, \frac{2n}{c^2-2} \right]| < \delta \left( \frac{n}{c^2-2} \right) \) and \( |A \cap \left( \frac{2n}{c^2-2}, \frac{3n}{c^2-2} \right]| < \delta \left( \frac{n^2-2c-2}{c^2-2} \right) \),

II. Those \( A \) for which \( |A \cap \left( \frac{n}{c^2-2}, \frac{2n}{c^2-2} \right]| < \delta \left( \frac{n}{c^2-2} \right) \), but \( |A \cap \left( \frac{2n}{c^2-2}, \frac{3n}{c^2-2} \right]| \geq \delta \left( \frac{n^2-2c-2}{c^2-2} \right) \),

III. Those \( A \) for which \( |A \cap \left( \frac{n}{c^2-2}, \frac{2n}{c^2-2} \right]| \geq \delta \left( \frac{n}{c^2-2} \right) \).

First, regarding sets of type III, we can argue as in the proof of Lemma 3.2 and deduce that their number is

\[
O \left( \sum_{\frac{n}{c^2-2} < \kappa < \kappa^0} 2^{\frac{(1-\epsilon)\kappa n}{1+O_\epsilon(1/\sqrt{n})} \tau_{\kappa n}} \right), \tag{3.5}
\]
for some constant $\kappa^0 = 1 - \Theta_\epsilon(1)$.

Secondly, regarding sets $A$ of type I, the number of possibilities for $A \cap \left(\frac{2n}{c^2-2}, n\right]$ is by assumption at most $2^n[(1-\frac{\delta}{2})+O(\epsilon)]$. By Lemma 3.2, we also know that the number of possibilities for $A \cap \left(\frac{4n}{c^2-2}, \frac{2n}{c^2-2}\right]$ is $O\left(2^{(1-\frac{\delta}{2})n(1+O(\epsilon))}\right)$. Hence, the number of possibilities for $A \cap \left(\frac{4n}{c^2-2}, n\right]$, when $A$ is of type I, is clearly $O\left(2^{f_1(\epsilon)n(1+O(\epsilon))}\right)$.

To deal with type II sets will require a further application of Lemma 2.4. Let $A$ be any such set and divide the interval $\left(\frac{2n}{c^2-2}, \frac{n}{c}\right]$ into subintervals each of length $\lfloor \sqrt{n} \rfloor$ or $\lceil \sqrt{n} \rceil$. Let $\mathcal{I} = [s, t]$ be the rightmost subinterval such that $|\mathcal{I} \cap A| \geq \frac{\delta}{2}\sqrt{n}$. We must have $t \geq \frac{2n}{c^2-2} + \frac{\delta}{2} \left(\frac{c^2-2c-2n}{c^2-2}\right)$. Fix $t$ and set $s := \frac{2n}{c^2-2} + s_1$, $t := \frac{2n}{c^2-2} + t_1$ and $T := \min\{t, \frac{n}{c^2-2} + ct\}$. To simplify notation, assume that $n$ is a multiple of $c^2 - 2$.

For a fixed choice of $\mathcal{I} \cap A$, let $G = (X, Y, E)$ be the following bipartite graph: $X = [1, T]$, $Y = [ct - T, ct]$ and $(x, y) \in E$ if there is some $z \in \mathcal{I} \cap A$ such that $x + y = cz$. Notice that $Y \subseteq \left(\frac{2n}{c}, n\right]$, so in particular $X \cap Y = \emptyset$. By Lemma 3.1, $A \cap (X \cup Y)$ must be an independent set in this graph. The graph $G$ is easily seen to be a $(p, l, k, m, \alpha, \beta, \theta)$-graph with $p = 2T$ and the remaining parameters as in (3.2) and (3.3). From Lemma 2.4 it follows that the number of possibilities for $A \cap (X \cup Y)$ is $O\left(2^{T(1+O(\epsilon))}\right)$. Now we have

$$A = (A \cap (X \cup Y)) \cup (A \cap (T, t]) \cup \left(A \cap \left(t, \frac{2n}{c}\right]\right) \cup \left(A \cap \left(\frac{2n}{c}, n\right] \setminus Y\right).$$

(3.6)

We have just given an upper bound for the number of possibilities for the first intersection. Trivial bounds for the second and fourth are $O\left(2^{t-T}\right)$ and $O\left(2^{(1-\frac{\delta}{2})n-T}\right)$ respectively. Our assumptions also imply that there are no more than $2^{(\frac{2n}{c^2-2}-t)}$ possibilities for the third intersection. Since $t - T \leq \frac{n}{c^2-2}$, it follows by an easy computation that the number of possibilities for a type II set is $O\left(2^{f_1(\epsilon)n(1+O(\epsilon))}\right)$. All of this was for a fixed choice of $t$ and $\mathcal{I} \cap A$, but allowing these to vary will not affect our estimate.

To summarise, we have shown that, given $\epsilon > 0$, then for all sufficiently large $n$,

$$\tau_n = O\left(2^{f_1(\epsilon)n(1+O(\epsilon))} + \sum_{c^2-2 \leq \kappa < \kappa^0} 2^{n(1+O(\epsilon))}\right),$$

(3.7)

for some $\kappa^0 = 1 - \Theta_\epsilon(1)$. Iterating this recursion $O_\epsilon(1)$ times for fixed $\epsilon$ and then letting $\epsilon \to 0$ as $n \to \infty$ clearly leads to a proof of the lemma.

Before we start with the proof of the theorem proper, let us note one further fact, whose proof is a straightforward iteration of the argument used in the proof of Lemma 3.2:

**Lemma 3.4.** Given $\epsilon_1, \epsilon_2 > 0$ there exists $\delta > 0$ such that, for all sufficiently large $n$, we can distinguish two types of $\mathcal{L}$-free subsets $A$ of $\{1, \ldots, n\}$.

**Type 1:** Those $A$ for which there is some $y \in (\epsilon_1 n, n]$ such that $|A \cap \left(\frac{2n}{c}, \frac{n}{c}\right]| < \delta \left(\frac{n}{c}\right)$.

For each such $y$, there are $O\left(2^{n(1+O(\epsilon_2))}\right)$ possibilities for $A \cap \left[y, n\right]$. 


TYPE 2: All other $\mathcal{L}$-free sets $A$. In this case, there are $O(2^{(1-\epsilon)n(1+O(\epsilon))})$ other possibilities for $A \cap (\epsilon_1 n, \bar{n}]$.

We are now ready to prove Theorem 2.2 for the equation $\mathcal{L}_{1,1,c}$. Let $n$ be a large positive integer and let $\xi_n$ denote the number of $\mathcal{L}$-free subsets of $\{1, ..., n\}$. Let $\epsilon$ be a small positive real number (in the end we’ll send $\epsilon \to 0$), let $\delta$ be as in Lemma 2.3 and divide the $\mathcal{L}$-free sets into the same three types as in the proof of Lemma 3.3. By the same arguments as used in the proof of that lemma, one finds that the number of type III sets is $O(\sum_{0 < \kappa < \kappa_0} 2^{(1-\epsilon)n(1+O(\sqrt{\kappa}))}\xi_{\kappa n})$ for some $\kappa_0 = \kappa_0^0 < 1$, and the number of type II sets is $O(2^{(\delta-2)n(1+O(\delta))})$. We want to set up a recursion which will yield the theorem and thus it remains to bound the number of type I sets. Let $\xi_{I,n}$ denote the number of type I sets. We will show that, for any fixed constant $\phi$ satisfying

$$
\frac{4}{c^2(c^4 - 2c^2 - 2)} < \phi < \frac{8}{c^2(c^4 - 4)},
$$

we have

$$
\xi_{I,n} = O\left(2^{f(c)n(1+O(\epsilon))} + 2^{(1-\frac{1}{2})n(1+O(\epsilon))}\xi_{\frac{2n}{c^2}(\delta + \phi)n}\right).
$$

Notice that, since $\phi < \frac{8}{c^2(c^4 - 4)}$, one has

$$
f(c) \left(\frac{2}{c^2} + \phi\right) + \left(1 - \frac{2}{c}\right) < f(c).
$$

It is then clear that (3.9), together with the estimates for the numbers of sets of types II and III, suffice to set up a recursion which will yield the theorem.

So for the remainder of the proof, we fix a choice of $\phi$ satisfying (3.8), and our aim is to prove (3.9). Let $A$ be an $\mathcal{L}$-free set of type I, and we consider two cases:

CASE 1: There is at least one $t \in (n(\frac{2}{c^2} + \phi), \frac{2n}{c^2}]$ such that the interval $J = [t - \sqrt{n}, t]$ contains at least $\delta \sqrt{n}$ elements of $A$.

CASE 2: There is no such $t$.

In CASE 2 we can immediately deduce that

$$
\left|A \cap \left(n \left(\frac{2}{c^2} + \phi\right), \frac{2n}{c^2}\right]\right| \leq (1 + o(1))\delta n \left(\frac{2}{c} - \frac{2}{c^2} - \phi\right),
$$

and from there an upper bound on the number of possibilities for $A$ corresponding to the second term on the right hand side of (3.9).

So to CASE 1. Our aim is to show that the number of possibilities for $A$ in this case is $O(2^{f(c)n(1+O(\epsilon))})$. For each $t \in (n(\frac{2}{c^2} + \phi), \frac{2n}{c^2}]$, let $\mathcal{F}_t$ be the family of all type-I $\mathcal{L}$-free sets $A$ such that $t$ is the largest integer for which the property in CASE 1 holds. Since $n = 2^{o(n)}$ it suffices to show that for each individual $t$ one has

$$
|\mathcal{F}_t| = O(2^{f(c)n(1+O(\epsilon))}),
$$

for we can then sum over all $t$ and it will not affect our estimate.
So we consider a fixed $t$. Let $t = n \left( \frac{2}{c^2} + \phi_t \right)$ and $\mathcal{I} = \mathcal{I}_1 = [s, t]$. We also consider a fixed choice of $\mathcal{I} \cap A$, as this will not affect our estimate either. Our argument will now follow a familiar pattern. Put $s = \frac{2}{c^2} + s_1$ and $t = \frac{2}{c^2} + t_1$. Let $G = (X, Y, E)$ be the bipartite graph where $X = [1, ct_1], Y = \left[ \frac{2n}{c} \right] \cup \left[ \frac{2n}{c} + ct_1 \right]$ and $(x, y) \in E$ if $x + y = cz$ for some $z \in \mathcal{I} \cap A$. Note that $t_1 \leq \frac{4n}{c(c^2 - 2)}$, so $X$ and $Y$ are certainly disjoint. We have

$$A = (A \cap (X \cup Y)) \cup (A \cap (ct_1, t]) \cup \left( A \cap (t, \frac{2n}{c}] \right) \cup \left( A \cap (\frac{2n}{c}, n \setminus Y) \right).$$

As in the proof of Lemma 3.3, we can bound the number of possibilities for the first intersection by $O(2^{ct_1(1 + O(1/\sqrt{n})))}$. By definition of $t$, there are $O(2^{O(c)n})$ possibilities for the third intersection. Trivially, there are $O(2^{1 - \frac{2}{c^2}}n - ct_1)$ possibilities for the fourth intersection. Unlike in Lemma 3.3 however, we will need a non-trivial bound on the number of options for the second intersection. It is now that we use the left-hand inequality of (3.8). It is easily checked to imply that $ct_1 > \frac{2n}{c^2 - 2}$. Hence, since $t = \Theta(n)$, we can use Lemma 3.3 to conclude that there are $O(2^{f_1(c) - t(1 + O(1)))}$ possibilities for the second intersection in (3.13). Another easy computation shows that this is already enough to yield (3.12) if and only if $\phi_t \leq \frac{8}{c^2(c^2 - 2)}$, in other words, if and only if $t \leq (\frac{1}{c^2} - 4) n$. So it remains to prove (3.12) when $t > (\frac{1}{c^2} - 4) n$. Note that, in that case, $ct_1 > \frac{4t}{c^2 - 2}$.

At this point, we use Lemma 3.4. Since $t_1 = \Theta(t)$ it implies that either there are $O(2^{\frac{1}{2}(1 + O(c)))}$ possibilities for $A \cap (ct_1, t]$, which is even better than what Lemma 3.3 yields, or there must be some $y \in (ct_1, t]$ satisfying the requirements of Lemma 3.4. As usual, we may consider a fixed $y$, since summing over $y$ at the end will not affect our estimates. Trivially, we’ll still have only $O(2^{(1 - \frac{2}{c^2})n - O(c)))$ possibilities for $A \cap (ct_1, t]$, which remains better than what Lemma 3.3 gives, unless $y/c - ct_1 = \Theta(n)$ and positive, and $|A \cap (ct_1, \frac{y}{c})| > \delta (\frac{2}{c} - ct_1)$. Thus, in turn, we may assume that there is a largest integer $z \in (ct_1 + \sqrt{n}, \frac{y}{c}]$ such that $A$ contains at least $\delta \sqrt{n}$ elements from the interval $s_z = [z - \sqrt{n}, z]$. Again, we may consider a fixed $z$, since summing over different $z$ will not affect an estimate like (3.12). To be precise, let $\mathcal{F}_{z,y,t}$ be the family of $\mathcal{L}$-free sets we are now considering for fixed $z, y, t$ satisfying all our assumptions to date. It just remains to prove that

$$|\mathcal{F}_{z,y,t}| = O(2^{f(c)n(1 + O(c)))}).$$

To simplify notation later on, let $ct_1 := qt$, where $q > \frac{4}{c(c^2 - 2)}$. Regarding the intersection $A \cap (ct_1, t]$, for fixed $z, y$ we can still trivially bound the number of possibilities by $O(2^{(1 - \frac{2}{c^2})t + (z - qt) + O(c)t})$ and, together with the estimates for the other terms in (3.13), this is easily checked to already yield (3.14) unless $z > y(\frac{2}{c^2} + r)$, where

$$r = \frac{c^4 - c^3 - 4}{c^4 - 2c^2 - 4} \left( q - \frac{4}{c(c^2 - 2)} \right).$$

Note, in particular, that $r > 0$. So we can now assume furthermore that $z$ satisfies this condition. From this point onwards, we work with a different decomposition of a set.
We already know there are \( O(2^{(1-\frac{2}{3})(t-y)/(1+O(\epsilon))}) \) possibilities for the second intersection and \( O(2^{(1-\frac{2}{3})n(1+O(\epsilon))}) \) possibilities for the third. We deal with the first intersection by one final application of graph theory. Fix a choice of \( A \subseteq \mathcal{I}_z \cap A \) which, as usual, doesn’t affect anything. Put \( z = \frac{2y}{c^2} + z_1 \) and\( Z = \min \{z, \frac{4y}{c(\epsilon^2-2)} + cz_1\} \). Let \( G = (\mathcal{X}, \mathcal{Y}, \mathcal{E}) \) be the bipartite graph where \( \mathcal{X} = [1, Z], \mathcal{Y} = [cZ, cz] \) and \((x, y) \in \mathcal{E} \) if \( x + y = c\zeta \) for some \( \zeta \in \mathcal{I}_z \cap A \). Lemma 2.4 can be applied in the usual manner to conclude that there are \( O(2^{Z(1+O(1/\sqrt{n}))}) \) possibilities for \( A \cap (\mathcal{X} \cup \mathcal{Y}) \). We now consider a decomposition of \( A \cap [1, y] \) analogous to (3.6), namely

\[
A \cap [1, y] = (A \cap (\mathcal{X} \cup \mathcal{Y})) \cup (A \cap [Z, z]) \cup \left( A \cap [z, \frac{2y}{c}] \right) \cup \left( A \cap \left( \frac{2y}{c}, y \right) \right).
\]  

(3.17)

We have just given an upper bound for the number of possibilities for the first intersection. Trivial bounds for the second and fourth are \( O(2^Z) \) and \( O(2^{(1-\frac{2}{3})y-Z}) \) respectively, while our definition of \( z \) means there are \( O(2^{O(\epsilon)y}) \) possibilities for the third. Plugging these estimates into (3.17), then the resulting estimate for the number of options for \( A \cap [1, y] \) back into (3.16), and finally a straightforward computation shows that (3.14) holds provided the variable \( r \) satisfies

\[
r > \left(1 + o(1)\right) \frac{c^2(c-2)}{(c-1)(c^4 - 2c^2 - 4)} \left( q - \frac{4}{c(c^2 - 2)} \right).
\]  

(3.18)

This is consistent with (3.15) for all \( n \gg 0 \), hence the proof of Theorem 2.2 for the equation \( \mathcal{L}_{1,1,c} \) is complete.

4. Proof of Theorem 2.2 for equations \( \mathcal{L}_{1,b,c} \)

Let integers \( b, c \) be given with \( b > 1 \) and \( c > \frac{(b+1)b^2}{b-1} \). The argument that follows is similar to that in Section 3, in fact shorter but with some new technicalities. The latter arise because we don’t have as simple a pairing mechanism as in Lemma 3.1 since, if \( z \in A \), then there exists an integer solution to \( x + by = cz \) if and only if \( x \equiv cz \pmod{b} \).

We start with a lemma analogous to Lemma 3.2 :

Lemma 4.1. Let \( A \) be an \( \mathcal{L} \)-free subset of \( \{1, \ldots, n\} \). The number of possibilities for \( A \cap (n/c, n] \) is \( O(2^{(1-\frac{b+1}{c})n(1+o(1))}) \).

Proof. Let \( \epsilon > 0 \) be given. Let \( \delta > 0 \) be as in Lemma 2.3 and \( n \) be very large. Let \( A \) be an \( \mathcal{L} \)-free subset of \( \{1, \ldots, n\} \). We distinguish two types of sets :

I. Those \( A \) for which \( |A \cap (\frac{n}{c}, \frac{(b+1)n}{c})| < \delta(\frac{n}{c}) \),

II. All other \( A \).

If \( A \) is of type I, then clearly there are \( O(2^{(1-\frac{b+1}{c})n(1+O(\epsilon))}) \) possibilities for \( A \cap (\frac{n}{c}, \frac{(b+1)n}{c}) \). Now consider type II sets. Divide the interval \( (\frac{n}{c}, \frac{(b+1)n}{c}) \) into subintervals, each of length \( \lfloor \sqrt{n} \rfloor \) or \( \lceil \sqrt{n} \rceil \), such that \( [bn/c] \) is a right-hand endpoint of some such interval. At least one of the following two possibilities must arise :

\[ ... \]
(i) There is some such interval \( \mathcal{I} = [s, t] \) such that \( |\mathcal{I} \cap A| \geq \delta \sqrt{n} \) and \( \mathcal{I} \subseteq \left( \frac{n}{c}, \frac{bn}{c} \right] \).

(ii) There is some such interval \( \mathcal{I} = [s, t] \) such that \( \mathcal{I} \subseteq \left( \frac{bn}{c}, \frac{(b+1)n}{c} \right] \), \( t < \left( \frac{b+1}{c} \right) \frac{n}{\epsilon} \) and \( |\mathcal{I} \cap A| \geq \frac{\delta}{2} \sqrt{n} \).

First suppose (i) occurs. There is at least one residue class \( \chi \pmod{b} \) such that \( \mathcal{I} \cap A \) contains at least \( \frac{\delta}{6} \sqrt{n} \) elements congruent to \( \chi \pmod{b} \). Let \( \mathcal{I}_\chi \) denote the corresponding set of elements of \( \mathcal{I} \cap A \). For a fixed choice of the parameters \( t, \chi \) and \( \mathcal{I}_\chi \), let \( G = (X, Y, E) \) be the following bipartite graph \( : X = \{ x \in [1, n] : x \equiv \chi \pmod{b} \}, \ Y = \left[ \frac{cs-n}{b}, \frac{ct-1}{b} \right] \) and \( (x, y) \in E \) if there exists some \( z \in \mathcal{I}_\chi \) such that \( x + by = cz \). The graph \( G \) is a \((p, l, k, m, \alpha, \beta, \theta)\)-graph where

\[
p = \frac{2n}{b} + O_{\epsilon, b}(\sqrt{n}), \tag{4.1}
\]

\[
l = 0, \ k = m = |\mathcal{I}_\chi| \geq \frac{\delta}{6} \sqrt{n}, \tag{4.2}
\]

\[
\beta = \theta = 0, \ \alpha = O_{\epsilon, b}(1/\sqrt{n}). \tag{4.3}
\]

Now \( A \cap (X \cup Y) \) must correspond to an independent set in this graph so, by Lemma 2.4, the number of possibilities for this intersection, for a fixed \( b \), is \( O(2^{\frac{\pi}{2} + O_{\epsilon}(1/\sqrt{m})}) \).

We now use the trivial decomposition

\[
A = [A \cap (X \cup Y)] \sqcup [A \cap ([1, n] \setminus (X \cup Y))]. \tag{4.4}
\]

Note that \( X \) and \( Y \) are not disjoint, but that \( |X \cup Y| = (\frac{2}{b} - \frac{1}{b^2}) n + O(\sqrt{n}) \). Hence the number of possibilities for the second intersection in (4.4) is trivially \( O(2^{\frac{1-b}{2-b} n(1 + O(1/\sqrt{m}))}) \), and hence the total number of possibilities for \( A \) is \( O(2^{\frac{1-b}{2-b} n(1 + O(1/\sqrt{m}))}) \). Since \( c > \frac{(b+1)b^2}{b^2-1} \), this number is also \( O(2^{\frac{1-b}{2-b} n(1 + O(1/\sqrt{m}))}) \). Allowing \( t, \chi \) and \( \mathcal{I}_\chi \) to vary will not affect this estimate.

Now we turn to Case (ii). As before, choose a congruence class \( \chi \pmod{b} \) such that \( |\mathcal{I}_\chi| \geq \frac{\delta}{6} \sqrt{n} \). With the same three parameters as before fixed, let the real number \( \kappa \) be defined by setting \( cs := (b+1)n - (1-\kappa)n \). Note that \( 0 < \kappa < 1 \) and \( \kappa = 1 - \Theta_\delta(1) = 1 - \Theta_\epsilon(1) \). This time the bipartite graph \( G = (X, Y, E) \) we wish to consider is the following: \( X = \{ x \in [\kappa n, n] : x \equiv \chi \pmod{b} \}, \ Y = [\left( 1 - \frac{1-\Theta_\epsilon(1)}{b} \right) n, n] \) and \( (x, y) \in E \) if \( x + by = cz \) for some \( z \in \mathcal{I}_\chi \). By Lemma 2.4, there are \( O(2^{\frac{2\pi}{2} + O_{\epsilon}(1/\sqrt{m})}) \) possibilities for \( A \cap (X \cup Y) \). We decompose a generic \( A \) as

\[
A = (A \cap [1, \kappa n]) \sqcup (A \cap (X \cup Y)) \sqcup (A \cap (\kappa n, n]) \setminus (X \cup Y) \tag{4.5}
\]

and deduce by a similar argument as in Case (i) that the number of possibilities for \( A \cap (n/c, n] \) is now \( O(2^{\frac{1-b}{2-b} n(1 - \kappa)(1 + O_{\epsilon}(1/\sqrt{m}))}) \). Allowing \( \chi \) and \( \mathcal{I}_\chi \) to vary does not affect our estimate (note that \( \kappa \) depends only on \( t \)).

To summarise, what we have demonstrated is a recursion for \( \pi_n \) of the form

\[
\pi_n = O \left( 2^{\left( \frac{1-b}{2-b} n(1 + O(\epsilon)) \right)} + \sum_{0 < \kappa < 1 - \Theta_\epsilon(1)} 2^{\left( \frac{1-b}{2-b} n(1 - \kappa)(1 + O(\epsilon)) \right)} \pi_{\kappa n} \right), \tag{4.6}
\]
where the number of terms in the sum is $O(n) = 2^{o(n)}$. By the usual reasoning, this is enough to establish our lemma. \hfill $\square$

We do not need an analogue of Lemma 3.3 this time round (since the extremal $\mathcal{L}$-free sets consist of two intervals only), so can go straight ahead to the proof of the theorem proper. Let $n$ be large and $\xi_n$ denote the number of $\mathcal{L}$-free subsets of $\{1, \ldots, n\}$. Let $\epsilon > 0, \delta$ as in Lemma 2.3 and $n$ very large. Let $\phi$ be any fixed constant satisfying

$$\frac{b+1}{c^2-b} < \phi < \frac{b+1}{c^2-(b^2-1)},$$

and divide the $\mathcal{L}$-free sets into the following three categories:

I. Those sets $A$ for which $|A \cap (\frac{n}{c}, \frac{(b+1)n}{c}| > \delta (\frac{bn}{c})$.

II. Those $A$ for which $|A \cap (\phi n, \frac{(b+1)n}{c}| < c\delta n(\frac{b+1}{c} - \phi)$.

III. All other $A$.

By following the argument in the proof of Lemma 4.1 we can establish an upper bound for the number of sets of type I identical to the right-hand side of (4.6), just with $\pi$ replaced by $\xi$. Regarding sets of type II, one immediately has these are at most $O(2^{(1-\frac{b+1}{c})n(1+O(\epsilon))\xi_{\phi n}})$ in number. One easily checks that the right-hand inequality of (4.7) implies that

$$f(b, c)\phi + \left(1 - \frac{b+1}{c}\right) < f(b, c).$$

Thus, in setting up a recursion for $\xi_n$ which will yield the theorem, sets of types I and II do not cause any problems. It remains to consider type-III sets, so let $\xi_{III, n}$ denote their number. The definition of these sets clearly implies the existence of some largest $t \in (\phi n, \frac{n}{c})$ such that the interval $\mathcal{I} = [t-\sqrt{n}, t] = [s, t]$ contains at least $\delta\sqrt{n}$ elements of $A$. For each possible choice of such a $t$, let $\mathcal{F}_t$ denote the corresponding family of type-III sets. It suffices to show that

$$|\mathcal{F}_t| = O(2^{f(b,c)n(1+O(\epsilon))}).$$

So consider a fixed $t$. Pick out a congruence class $\chi \ (\text{mod} \ b)$ with the usual properties. For fixed $\chi$ and $\mathcal{I}_\chi$, put $s := \frac{(b+1)n}{c^2} + s_1, t := \frac{(b+1)n}{c^2} + t_1, T := \min\{t, ct\}$. Let $G = (X, Y, E)$ be the following bipartite graph: $X = \{x \in (\frac{(b+1)n}{c}, \frac{(b+1)n}{c}+T] : x \equiv \chi \ (\text{mod} \ b)\}, Y = [1, \frac{T}{b}]$ and $(x, y) \in E$ if $x + by = cz$ for some $z \in \mathcal{I}_\chi$. We decompose an $\mathcal{L}$-free set $A$ as

$$A = (A \cap (X \cup Y)) \cup \left(A \cap (\frac{T}{b}, t]\right) \cup \left(A \cap (t, \frac{(b+1)n}{c}]\right) \cup \left(A \cap ((\frac{(b+1)n}{c}, n] \setminus Y)\right).$$

(4.10)

In the usual manner, Lemma 2.4 implies a bound for the first intersection of the form $O(2^{f(b,c)(1+O(1/\sqrt{n}))})$. The definition of $t$ implies that the possibilities for the third intersection are negligible. We use a trivial bound for the fourth. For the second, it is now that we use the left-hand inequality of (4.7). This is easily checked to guarantee that $T/b > t/c$, and hence we can get a bound from Lemma 4.1. Putting these four bounds together and letting $\chi$ and $\mathcal{I}_\chi$ vary is then easily checked to yield (4.9).

This completes the proof of Theorem 2.2 for the equations $\mathcal{L}_{1,b,c}$.\hfill $\square$
5. Proof in the remaining cases

The reason for presenting our arguments in the previous two sections in detail was that those presented in [BHKLS] and [H] for establishing the sizes of the maximum $L$-free subsets of $\{1, ..., n\}$ needed to be modified to a form more suitable for proving Theorem 2.2. For the remaining equations, no such modifications are necessary and things proceed smoothly.

5.1. The equation $L_{1,1,3}$. That $\lceil n/2 \rceil$ is the maximum size of an $L$-free subset of $\{1, ..., n\}$ is easily established by induction on $n$. One divides the $L$-free sets into two types, those that intersect $(n/3, 2n/3]$ and those that don’t. For the latter type, the induction is set up immediately. For the former, one chooses a $z \in A \cap (n/3, n/2]$ and pairs off elements in the interval $[3x - n, n]$ using Lemma 3.1. Then the induction step can be applied.

To prove Theorem 2.2, one proceeds analogously. For given $\epsilon, \delta$ one divides the $L$-free sets into two types according as to whether $|A \cap (n/3, n/2]| > \delta(n)$ or not. One then proceeds as in Section 3 to obtain a recursive estimate for the number $\xi_n$ of $L$-free sets, but only one application of Lemma 2.4 will be necessary now. We omit further details.

5.2. The equations $L_{b,b,c}$. In a similar manner, the argument establishing the size of a maximum $L$-free subset of $\{1, ..., n\}$, as presented in [H], is already in such a form that it can be directly modified to prove Theorem 2.2. Thus, there is nothing new going on here either.

6. Concluding remarks

The main purpose of this paper has been simply to propose Conjecture 1.1, as this does not seem to have been done in the existing literature. We have proven the weak form of it for some new equations, using well-established methods. Obviously, one would like to know if the strong form holds for these equations and, indeed, if one can obtain a more precise asymptotic for the number $\xi_n$ of $L$-free sets. One would also like to generalise our results to the group setting. Perhaps this can be done with only technical modifications of existing results for sum-free sets.

It is nevertheless a more fundamental question as to whether even the weak form of Conjecture 1.1 holds for any $L$ whatsoever. I suspect that, if a counterexample exists, then one can find one already in three variables. The intractability of the problem seems to lie in the difficulty in computing $M_L(n)$ in the first place for most $L$. Indeed, what our paper seems to show is that, in situations where $M_L(n)$ can be computed by some kind of pairing-off procedure (which seems most natural in three variables), something like Lemma 2.4 gives (1.5) as a bonus almost ‘for free’. One can wonder if there might be a way to prove (1.5) without knowing $M_L(n)$.

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E-mail address: hegarty@chalmers.se

MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG, 41296 GOTHENBURG, SWEDEN