A new physical-space approach to decay for the wave equation with applications to black hole spacetimes

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October 26, 2009

Abstract

We present a new general method for proving global decay of energy through a suitable spacetime foliation, as well as pointwise decay, starting from an integrated local energy decay estimate. The method is quite robust, requiring only physical space techniques, and circumvents use of multipliers or commutators with weights growing in $t$. In particular, the method applies to a wide class of perturbations of Minkowski space as well as to Schwarzschild and Kerr black hole exteriors.

1 Introduction

The wave equation

$$\Box_g \phi = 0$$  \hspace{1cm} (1)

on general Lorentzian background metrics $g$ appears ubiquitously in mathematical physics, often in the context of the linearisation of a field theory governed by a system of non-linear hyperbolic p.d.e.’s. The most classical example occurs perhaps in the context of the Euler equations of fluid mechanics, in which case $g$ represents the so-called acoustical metric. Another important source for (1) arises from problems in general relativity, where $g$ represents the metric of spacetime, and (1) can be viewed as a poor-man’s linearisation of the Einstein equations. For such metrics $g$, the global causal structure is often much richer than that of Minkowski space. In this context, perhaps the most interesting examples are so-called “black hole” metrics. Briefly, these are metrics which—like Minkowski space—possess a natural asymptotic structure representing future null infinity $\mathcal{I}^+$, but where the past of $\mathcal{I}^+$ has a non-trivial complement in the spacetime.

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A fundamental problem for \((1)\) on suitable metrics \(g\) is to obtain “decay estimates”, that is to say, estimates giving upper bounds for the rates of decay for \(\phi\) toward the future, in terms of quantities computable initially on a Cauchy surface. This classical problem can be studied as an end in itself, but it is often an essential step in proving stability results for non-linear equations which give rise to \((1)\) as described above. From the latter point of view, there is an important trade off between how much decay one proves and how robust is the argument to prove it. The history of non-linear wave equations in the past 30 years, culminating in part in the monumental stability proofs (see for instance \([4]\)) based on robust estimates for equations of the form \((1)\), are a testament to how important it is to get this balance right.

The now standard approach to obtaining robust decay estimates proceeds by using weighted multiplier estimates (going back to Morawetz \([15]\)) and weighted commutators (going back to Klainerman \([10]\)), both originating in one form or another from the symmetries of Minkowski space. In Minkowski space, full use of this method (applying both weighted multipliers and weighted commutators) obtains (a) weighted energy decay, which can be expressed for instance as

\[
\int_{t=\tau} \left( \left( (t^2+r^2+1)\phi^2 + (t+r+1)^2((\partial_t + \partial_r)\phi)^2 
+ (|t-r|+1)^2((\partial_t - \partial_r)\phi)^2 + (t^2+r^2)|\nabla\phi|^2 \right) \right) \\
\leq C \int_{t=0} \left( \phi^2 + r^2((\partial_t\phi)^2 + (\partial_r\phi)^2 + |\nabla\phi|^2) \right),
\]

where \(\nabla\) denotes angular derivatives, (b) pointwise decay for \(\phi\) of the form

\[
|\phi| \leq C \sqrt{E}(|t-r|+1)^{-1/2}(|t+r|+1)^{-1}
\]

for \(t \geq 0\), where \(E\) represents a higher order weighted energy at \(t = 0\), and (c) pointwise estimates for derivatives of \(\phi\), with additional decay, for instance:

\[
|\partial_t \phi| \leq C \sqrt{E}(|t-r|+1)^{-3/2}(|t+r|+1)^{-1}.
\]

Obtaining (a) requires only use of inverted time translations \((t^2+r^2)\partial_t + 2t r \partial_r\) as a multiplier. Obtaining the full (b) and (c) requires both use of inverted time translations and commutation with the full algebra of commutators associated to \(\Box\).

Various applications to non-linear problems and other geometric settings have required variants of the vector field method where the use of certain vector field multipliers and/or commutators is avoided and less information is obtained concerning (a), (b), or (c) above. Klainerman and Sideris \([11]\) obtained decay results by commuting only with \(t\partial_t + r \partial_r\) and angular momentum operators, and this, used together with integrated local decay estimates in the style of Morawetz \([16]\), has proven useful in a number of problems concerning obstacles, multiple speeds, etc. \([4, 9, 14]\). A common feature of all these approaches, however, is the necessity of always applying at least some multipliers or commutators with weights growing in \(t\).
In the present paper, we give a related but different approach to the problem of proving suitable decay estimates for solutions to (1) which, as we shall see, has in principle a wider range of validity and may be even more useful for non-linear problems. In particular, the method here avoids both multipliers and commutators with weights in $t$. Our approach does not start from scratch, but begins with certain “more basic” estimates as ingredients, estimates which have already been proven (either classically or much more recently) in many cases of interest, by various methods. The method by which these ingredient estimates are obtained is in fact of no importance for the considerations here. As we shall see, our approach applies to the study of (1) on Minkowski space and sufficiently small perturbations thereof, but also to Schwarzschild and Kerr black hole spacetimes, and immediately yields essentially the decay consistent with non-linear stability. The common feature which these regimes share are

(i) An integrated local energy decay estimate

(ii) Good asymptotics towards null infinity allowing for a hierarchy of $r$-weighted energy identities

(iii) Uniform boundedness of energy

Estimates of the form (1), already referred to above, were first proven by Morawetz [16] for perturbations of Minkowski space, while for Schwarzschild, estimates of the form (1) are due to several authors (see Section 4.4.1 of [8]). For slowly rotating Kerr spacetimes $(\mathcal{M}, g_{\mathcal{M},a})$ with parameters $|a| \ll M$, such estimates have been shown by [8, 17] (see also [2]), whereas, for the general sub-extremal Kerr case $|a| < M$, this will appear in a forthcoming paper. In the black hole case, the “integrated decay” estimate (1), to be useful here, must be understood in the sense of an estimate which does not degenerate at the horizon $\mathcal{H}^+$, an estimate which is indeed possible in view of a multiplier construction intimately connected to the celebrated red-shift effect (see [5, 8]). In the Schwarzschild and Kerr case, the integrated decay estimate necessarily degenerates, however, near the analogue of the photon sphere, and this loss is related to the trapping phenomenon; a non-degenerate estimate can nevertheless be obtained after losing differentiability. Similar phenomena occur in the simpler setting of obstacle problems in Minkowski space, to which this method also applies.

We note that the hierarchy of $r$-weighted energy identities (ii) applied here corresponds to the use of a multiplier of the form $r^p(\partial_t + \partial_r)$ for $0 \leq p \leq 2$.

In principle, our approach can retrieve the “full decay statements” of the traditional vector field method as would follow from the combined use of [15] and [10], except that the energy decay statements will in general lose derivatives if the integrated local energy decay statement does so. In this short note, we shall here only give the first step of the approach. This begins by obtaining $\tau^{-2}$ decay of the energy flux through a suitable foliation $\Sigma_\tau$ (whose leaves approach null infinity), where $\tau$ is the natural parameter of the foliation. This is precisely the analogue of (2). (See the discussion of Section 7.) This then leads
immediately to pointwise results (in Section 5) which are the analogue of
\[ |r^{1/2} \phi | \leq Ct^{-1}, \quad |r\phi| \leq Ct^{-1/2}. \]

Again, no use is made of commutation with vector fields with weights in \( t \).

We outline in Section 8 how to extend to the method to achieve the analogue of (4). (4).

We consider the case of Minkowski space first in Section 3, followed by Schwarzschild in Section 4 to illustrate the structure of the argument in each of these settings. As shall be apparent, the nature of the argument is such that elements (i)–(iii), suitably interpreted, are sufficient and a very general theorem can in fact be formulated. This general theorem includes in particular the Kerr case. See Section 6. We leave the precise formulation of the most general assumptions for a follow-up paper.

2 Notation

Let \((M, g)\) be a Lorentzian manifold and \(\phi\) a solution to (1) on \(M\). We recall the energy-momentum tensor
\[ T_{\mu\nu}[\phi] = \phi,_{\mu}\phi,_{\nu} - \frac{1}{2} g_{\mu\nu} \phi,^{\alpha}\phi,_{\alpha}. \]

Given a vector field \(X\), we define the currents
\[ J_X^{\mu}[\phi] = T_{\mu\nu} [\phi] X^{\nu}, \quad K_X^{\mu}[\phi] = T^{\mu\nu} [\phi] \pi_X^{\nu\mu}, \]
where \(\pi_X^{\mu\nu} = \frac{1}{2} \mathcal{L}_X g_{\mu\nu}\). Recall that
\[ \nabla^{\mu} J_X^{\mu}[\phi] = K_X^{\mu}[\phi]. \]

If \(X\) is Killing, then \(K_X^{\mu}[\phi] = 0\), in which case \(J_X^{\mu}[\phi]\) is a conserved current.

Integrands without an explicit measure of integration are to be understood with respect to the induced volume form. For the case of null hypersurfaces \(\Sigma\), integrands of the form \(\int_{\Sigma} J_X^{\mu}[n^{\mu}_{\Sigma}]\) are to be understood with respect to a choice of volume form on \(\Sigma\) and corresponding normalisation for \(n^{\mu}\) such that the integral represents the correct term in the energy identity. Alternatively, one can interpret this simply as the integral of the 3-form \(\int_{\Sigma} * J_X\), where * denotes the Hodge star operator associated to the spacetime metric.

3 The Minkowski case

Let us start by considering the wave equation in Minkowski space
\[ \Box \phi = 0. \]

For convenience, we may assume for this discussion that \(\phi\) is smooth and compactly supported on a Cauchy hypersurface. This will ensure that all integrals
are a priori defined. By a density argument, our results will only require the finiteness of the quantity on the right hand side of the final estimate and the relation $\phi \to 0$ at spacelike infinity.

Let $(t, r, \theta, \phi)$ denote a choice of standard spherical polar coordinates on Minkowski space. We shall let $\nabla$ denote the induced covariant derivative on the spheres of constant $r$, and $\Delta$ the induced Laplacian. We will also define the null coordinates $u = t - r$, $v = t + r$. Finally, let us introduce the notation $T$ to denote the coordinate vector field $\partial_t$ with respect to $(t, r)$ coordinates.

For our argument it will be useful to consider the following foliation: Fix $R > 0$ and consider the hypersurfaces $\Sigma_\tau$ defined as the union of the spacelike $\{ t = \tau \} \cap \{ r \leq R \}$ and the null $\{ u = (\tau - R) \} \cap \{ v \geq (\tau + R) \}$.

The following estimate can be shown by the original method of Morawetz:\[\text{(ILED-Mink)}\]

The initials of the label stand for “integrated local energy decay”. Since $n^\alpha = T^\alpha$ in $r \leq R$, we could replace the first term on the right hand side more explicitly with the quantity $(\partial_t \phi)^2 + (\partial_r \phi)^2 + |\nabla \phi|^2$, but we prefer this more geometric formulation to compare with other situations. Our goal is to use (ILED-Mink) to derive decay for the energy flux through $\Sigma_\tau$, and finally, pointwise decay estimates.

We rewrite the wave equation in the form

$$-\partial_u \partial_v \psi + \Delta \psi = 0, \quad \psi := r\phi,$$

multiply the equation by $r^p \partial_v \psi$ and integrate by parts in the region $D^\tau_{\tau_1}$ bounded by the two null hypersurfaces $u_1 = (\tau_1 - R)$, $u_2 = (\tau_2 - R)$ and the timelike hypersurface $r = R$: 

![Diagram](image)
We obtain
\[
\int_{u=\tau_2-R}^{\tau_2-R,v \geq \tau_2+R} r^p (\partial_v \psi)^2 \sin \theta \, d\theta \, d\phi \, dv \\
+ \int_{D^2_{\tau_1}} r^{p-1} \left( p(\partial_v \psi)^2 + (2-p)|\nabla \psi|^2 \right) \sin \theta \, d\theta \, d\phi \, du \, dv + \int_{T^2_{\tau_1-R}} r^p |\nabla \psi|^2 \sin \theta \, d\theta \, d\phi \, du \\
= \int_{u=\tau_1-R}^{\tau_1-R,v \geq \tau_1+R} r^p (\partial_v \psi)^2 \sin \theta \, d\theta \, d\phi \, dv \\
+ \int_{\tau_1}^{\tau_2} r^p \left( |\nabla \psi|^2 - (\partial_v \psi)^2 \right) \sin \theta \, d\theta \, d\phi \, d\tau_{v=R}.
\tag{p-WE-Mink}
\]

The initials “WE” denote “weighted energy”. We have written explicitly the measure of integration to emphasize that the expected $r^2$ part of the volume form is now in fact included in the quadratic terms in $\psi$ (remember: $\psi = r\phi$). Thus, the factors $r^p$, $r^{p-1}$ are to be viewed as weights. We see that the left hand side of the above identity is positive definite for $p \leq 2$.

We note that identity (p-WE-Mink) can be reinterpreted in terms of the energy identity in $D^2_{\tau_1}$ satisfied by the currents $J^V_{[\phi]}$, $K^V_{[\phi]}$ corresponding to $V = r^p \partial_v$ (where $\partial_v$ is understood with respect to $(u,v)$ coordinates), suitably modified, however, by appropriate 0'th order terms.

Let us first apply (p-WE-Mink) with $p = 2$. Then, observing that the last term on the right hand side of (p-WE-Mink) can be controlled (after a bit of averaging in $R$) by the left hand side of (ILED-Mink), we obtain
\[
\int_{u=\tau_2-R,v \geq \tau_2+R} r^2 (\partial_v \psi)^2 \sin \theta \, d\theta \, d\phi \, dv + \int_{D^2_{\tau_1}} r (\partial_v \psi)^2 \sin \theta \, d\theta \, d\phi \, du \, dv \\
\leq \int_{u=\tau_1-R,v \geq \tau_1+R} r^2 (\partial_v \psi)^2 \sin \theta \, d\theta \, d\phi \, dv + C \int_{\Sigma_{\tau_1}} J^T_{\alpha}[\phi] n^\alpha_{\Sigma_{\tau_1}}. \tag{5}
\]

This implies that we can find a dyadic sequence of $\tau_n \to \infty$ with the property that
\[
\int_{u=\tau_n-R,v \geq \tau_n+R} r (\partial_v \psi)^2 \sin \theta \, d\theta \, d\phi \, dv \\
\leq C\tau_n^{-1} \left[ \int_{u=\tau_1-R,v \geq \tau_1+R} r^2 (\partial_v \psi)^2 \sin \theta \, d\theta \, d\phi \, dv + \int_{\Sigma_{\tau_1}} J^T_{\alpha}[\phi] n^\alpha_{\Sigma_{\tau_1}} \right].
\]
We now apply \([p\text{-WE-Mink}]\) with \(p = 1\) to the region \(D_{\Sigma_1}^n\), to obtain
\[
\int_{\tau_n{\tau_n-R,v\geq\tau_n-R}} r(\partial_v\psi)^2 \sin \theta \, d\theta \, d\phi \, dv + \int_{D_{\Sigma_1}^n} ((\partial_v\psi)^2 + |\nabla\psi|^2) \sin \theta \, d\theta \, d\phi \, dv \\
\leq C_{\tau_1}^{-1} \left[ \int_{\tau_n{\tau_n-R,v\geq\tau_n-R}} r(\partial_v\psi)^2 \sin \theta \, d\theta \, d\phi \, dv + \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha \right] \\
+ C \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha. \tag{6}
\]

Adding a multiple of the estimate \([\text{ILED-Mink}]\), we obtain
\[
\int_{\tau_n} \int_{\tau_n-R} (J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha + \phi^2) \\
+ \int_{\tau_n{\tau_n-R,v\geq\tau_n-R}} r(\partial_v\psi)^2 \sin \theta \, d\theta \, d\phi \, dv + \int_{D_{\Sigma_1}^n} ((\partial_v\psi)^2 + |\nabla\psi|^2) \sin \theta \, d\theta \, d\phi \, dv \\
\leq C_{\tau_1}^{-1} \left[ \int_{\tau_n{\tau_n-R,v\geq\tau_n-R}} r(\partial_v\psi)^2 \sin \theta \, d\theta \, d\phi \, dv + \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha \right] \\
+ C \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha. \tag{6}
\]

We now observe that
\[
\int_{\tau_n} (\partial_v(\phi))^2 \, dv = \int_{\tau_n} [r^2(\partial_v\phi)^2 + 2r\partial_v\phi \phi + \phi^2] \, dv = \int_{\tau_n} r^2(\partial_v\phi)^2 \, dv - r \phi^2_{|v=\tau_n}.
\]
Substituting this into (6) we obtain
\[
\int_{\tau_n} \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha \leq C_{\tau_1}^{-1} \left[ \int_{\tau_n{\tau_n-R,v\geq\tau_n-R}} r(\partial_v\psi)^2 \sin \theta \, d\theta \, d\phi \, dv + \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha \right] \\
+ C \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha. \tag{7}
\]

Finally, in view also of the energy bound
\[
\int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha \leq \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha, \tag{EB-Mink}
\]
which holds for all \(\tau \geq \tau'\), the estimate (7) easily implies that
\[
\int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha \leq C_{\tau_1}^{-2} \left[ \int_{\tau_n{\tau_n-R,v\geq\tau_n-R}} r(\partial_v\psi)^2 \sin \theta \, d\theta \, d\phi \, dv + \int_{\Sigma_1} J_{\alpha}^T[\phi] n_{\Sigma_1}^\alpha \right] \tag{8}
\]
for any \(\tau \geq \tau_1\).
Remark 1. Note that (8) essentially corresponds to (2) of the discussion of the vector field method. In view of the absence of weights in $t$ in the multiplier used for the proof, it is already immediate from the above that (8) follows similarly on sufficiently small non-stationary perturbations of Minkowski space. This estimate is in itself very useful for applications to global existence problems for quasilinear equations, and previously was not available without commutation with inverted time translations with their quadratic weights in $t$. See the discussion in Section 7.

4 The Schwarzschild case

We now consider the wave equation (11) on a Schwarzschild background $(M, g)$ with parameter $M$. See Section 2 of [8]. Let $\mathcal{D}$ denote the closure of (a connected component of) the domain of outer communications, let $\Sigma_\tau$ be a translation-invariant family of hypersurfaces connecting the event horizon $\mathcal{H}^+$ and null infinity $I^+$ as below:

For definiteness, one can choose the family $\Sigma_\tau$ explicitly as follows: Given a constant $R > 3M$, first define a Regge-Wheeler coordinate $r^* = r + 2M \log (r - 2M) - R$, the null coordinates $u = t - r^*$, $v = t + r^*$ and the coordinate $t^* = t + 2M \log (r - 2M)$. Then $\Sigma_\tau$ can be defined to coincide with $t^* = \tau$ for $r \leq R$, and to coincide with $u = \tau$ for $r \geq R$. Let us define $D^\tau_{\tau_1}$ analogously as before, i.e., with the present normalisation of the null coordinates: $D^\tau_{\tau_1} = \{ r \geq R \} \cap \{ \tau_1 \leq u \leq \tau_2 \}$.

It will be useful to fix a translation-invariant timelike vector field $N$ on $J^+(\Sigma_\tau)$ such that $N = \partial_t$ for $r \geq R$, say. (The coordinate vector field $\partial_{t^*}$ in $(r, t^*)$ coordinates will do for instance.) Let us note that (EB-Mink) is replaced by

$$\int_{\Sigma_\tau} J^N_\alpha [\phi] n^{\alpha}_{\Sigma_\tau} \leq C \int_{\Sigma_\tau} J^N_\alpha [\phi] n^{\alpha}_{\Sigma_\tau}.$$  (EB-Schw)

This non-degenerate energy boundedness statement was originally proven in [5] but can in fact be shown independently of decay results, as in [7] and Section 3 of [8].

The analogue of (ILED-Mink) is

$$\int_\tau^\infty \int_{r \leq R} (\chi J^N_\alpha [\phi] n^{\alpha}_{\Sigma_\tau} + (\partial_r \phi)^2 + \phi^2) \leq C \int_{\Sigma_\tau} J^N_\alpha [\phi] n^{\alpha}_{\Sigma_\tau},$$  (ILED1-Schw)
where $\chi$ is a weight which vanishes quadratically at $r = 3M$. This type of estimate, which does not degenerate on the horizon, was originally proven in [4].

For background, see Section 4.4.1 of [8]. Commuting with $T$, we may obtain for instance

$$
\int_r^\infty \int_{r \leq R} J^N_\alpha [\phi] n^{\Sigma_\alpha}_2 \lesssim C \int_{\Sigma_\tau} J^N_\alpha [T \phi] n^{\Sigma_\alpha}_2 + C \int_{\Sigma_\tau} J^N_\alpha [\phi] n^{\Sigma_\alpha}_2. \quad \text{(ILED2-Schw)}
$$

We will outline more explicitly further on in this section how the above are used in the Schwarzschild case. Let us immediately remark, however, that (ILED1-Schw) would be sufficient for the use of (ILED-Mink) to obtain the analogue of (5), whereas one would have to use (ILED2-Schw) to obtain (6), necessitating the appearance of a higher order quantity on the right hand side. This will mean that, in the Schwarzschild case, the final estimate loses differentiability, as expected.

In this latter use of (ILED2-Schw), it is essential that one has the non-degenerate quantity $J^N_\alpha [\phi]$ on the horizon.

Before adapting the argument of Section 3 to Schwarzschild, it remains to derive the analogue of (p-WE-Mink).

As before, we write the wave equation (1) as

$$
- \partial_u \partial_v \psi + (1 - \frac{2M}{r}) \Delta \psi - \frac{2M(1 - \frac{2M}{r})}{r^3} \psi = 0, \quad \psi := r \phi.
$$

We may rewrite this in the form

$$
- \frac{1}{1 - \frac{2M}{r}} \partial_u \partial_v \psi + \Delta \psi - \frac{2M}{r^3} \psi = 0.
$$

Observe that this implies that

$$
- \frac{1}{2} \partial_u \left( \frac{r^p}{1 - \frac{2M}{r}} |\partial_v \psi|^2 \right) + \frac{1}{2} \partial_v \left( \frac{r^p}{1 - \frac{2M}{r}} |\partial_u \psi|^2 \right)
$$

$$
- \frac{1}{2} \partial_v \left( \frac{2Mr^p}{r^3} \psi^2 \right) + \frac{1}{2} \partial_u \left( \frac{2Mr^p}{r^3} \psi^2 \right) \psi^2 + r^p \Delta \psi \partial_v \psi = 0,
$$

which generates the following additional terms in the analogue of (p-WE-Mink):

- a boundary term at null infinity of the right sign, proportional to

$$
2M \int_{\mathcal{T}_{\Sigma}^{\tau_2, \tau_1 - R}} \tau^p \psi^2 \sin \theta \ d\theta \ d\phi \ du,
$$

and the bulk terms

$$
- M \int_{D^2_{\tau_2}} \partial_v \left[ \psi^2 \sin \theta \ d\theta \ d\phi \ du \right] \left( 3 - p \right) M \int_{D^2_{\tau_1}} \psi^2 \sin \theta \ d\theta \ d\phi \ du \ dv
$$

$$
- \frac{1}{2} M \int_{D^2_{\tau_2}} \partial_u \left[ \psi^2 \sin \theta \ d\theta \ d\phi \ du \right] \left( 3 - p \right) M \int_{D^2_{\tau_1}} \psi^2 \sin \theta \ d\theta \ d\phi \ du \ dv.
$$

The precise loss of differentiability in (ILED2-Schw) is wasteful. Using a refined estimate (see [13]) would lead to an improved result here in the sense of loss of differentiability, but this would take us outside the realm of physical space methods.
Both expressions have the right sign as long as $0 < p \leq 3$ and $r$ is sufficiently large. Note also the new $(1 - 2M/r)$ weights in the analogue of \[\text{(p-WE-Mink)},\]
which of course will play no role for large $r$.

Suppressing the spherical integration from the notation, we thus obtain, for $R$ sufficiently large, the estimate

\[
\int_{u=\tau_2, v \geq \tau_2 + R} r^p (\partial_v \psi)^2 dv + \int_{D_{\tau_2}^r} r^{p-1} \left( p (\partial_v \psi)^2 + (2 - p) |\nabla \psi|^2 \right) dudv + \int_{T_{\tau_1}^r} r^p |\nabla \psi|^2 d\tau \leq C \int_{u=\tau_1 - R, v \geq \tau_1 + R} r^p (\partial_v \psi)^2 dv + \int_{\tau_1} r^p \left( |\nabla \psi|^2 + (\partial_v \psi)^2 \right) d\tau |_{r=R}.
\]

(Given \[\text{ILED1-Schw}}, \text{ILED2-Schw}}, \text{p-WE-Schw}}, \text{EB-Schw}}, one repeats

the Minkowski argument until before (9). At this point, one adds the estimate \[\text{ILED2-Schw}} to obtain

\[
\int_{\tau_{n-1}}^{\tau_n} \int_{r \leq R} \left( J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha + \phi^2 \right) + \int_{u=\tau_n - R, v \geq \tau_n + R} r (\partial_v \psi)^2 dv + \int_{D_{\tau_n}^r} r (|\nabla \psi|^2 + (\partial_v \psi)^2) dudv
\]

\[
\leq C \tau_n^{-1} \left[ \int_{u=\tau_1 - R, v \geq \tau_1 + R} r^2 (\partial_v \psi)^2 dv + \int_{\Sigma_{\tau_1}} J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha \right]
\]

\[
+C \int_{\Sigma_{\tau_{n-1}}} J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha + C \int_{\Sigma_{\tau_{n-1}}} \int_{\Sigma_{\tau_1}} J^N_{\alpha} [T \phi] n_{\Sigma_r}^\alpha.
\]

One obtains as before

\[
\int_{\tau_{n-1}}^{\tau_n} \int_{\Sigma_{\tau}} J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha \leq C \tau_n^{-1} \left[ \int_{u=\tau_1 - R, v \geq \tau_1 + R} r^2 (\partial_v \psi)^2 dv + \int_{\Sigma_{\tau_1}} J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha \right]
\]

\[
+C \int_{\Sigma_{\tau_{n-1}}} J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha + C \int_{\Sigma_{\tau_{n-1}}} \int_{\Sigma_{\tau_1}} J^N_{\alpha} [T \phi] n_{\Sigma_r}^\alpha
\]

\[
\leq C \tau_n^{-1} \int_{\Sigma_{\tau_1}} J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha + C \int_{\Sigma_{\tau_1}} J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha + C \int_{\Sigma_{\tau_1}} J^N_{\alpha} [T \phi] n_{\Sigma_r}^\alpha.
\]

Identifying a good dyadic sequence, applying the boundedness statement \[\text{EB-Schw}}, and using the above inequality again with $\phi$ replaced by $T \phi$, we obtain finally

\[
\int_{\Sigma_{\tau}} J^N_{\alpha} [\phi] n_{\Sigma_r}^\alpha \leq C \tau^{-2} \sum_{i=0}^2 \int_{\Sigma_{\tau_1}} r^2 J^N_{\alpha} [T^i \phi] n_{\Sigma_r}^\alpha.
\]

5 Pointwise estimates

Starting from the energy decay bounds \[\text{[S]}, \text{[10]}\] proven above, one can obtain pointwise decay estimates after additional commutations and applications of
weighted Sobolev inequalities. To obtain decay near null infinity, we shall require commutator vector fields with weights in \( r \), but, again in accordance with the philosophy of our method, no weights in \( t \) will be used.

In the Schwarzschild and Minkowski space, one approach for this is to use only commutation with the so-called angular momentum operators \( \Omega_i \), defined as a standard basis for the Lie algebra corresponding to rotations. See [5].

An alternative argument (see Section 5.3.8 of [8]) uses commutation with cut-off versions of \( \Omega_i \), supported only near infinity, together with additional commutations with \( T \) and the red-shift vector field \( N \) (or equivalently \( Y \)) and an appeal to elliptic estimates. This argument is more robust, and moreover, can also be used to obtain non-degenerate pointwise estimates of higher derivatives up to and including the horizon.

From this latter approach, we obtain finally:

\[
|r^{1/2} \phi| \leq C \sqrt{E \tau^{-1}}, \quad |r \phi| \leq C \sqrt{E \tau^{-1/2}},
\]

where

\[
E = \sum_{|\alpha| \leq 2} \sum_{\Gamma = T, N, \Omega_i} \int_{\Sigma_0} r^2 (J^N_\mu (\Gamma^\alpha \phi) + J^N_\mu (\Gamma^\alpha T \phi) + J^N_\mu (\Gamma^\alpha TT \phi)) n^\mu_{\Sigma_0}.
\]

We note that the second two terms of the integrand arise from the loss in (ILED2-Schw) and are thus unnecessary in the case of Minkowski space, where, in addition, of course, we take \( N = T \).

6 A general formulation and the Kerr case

It should be clear from the structure of the above argument that the properties (i), (ii), (iii), as captured by suitable estimates in the form of (ILED1-Schw)–(ILED2-Schw), (p-WE-Schw), (EB-Schw), are in fact sufficient for obtaining pointwise-in-time energy decay of the form (10). In particular, we note that the stationarity of the metric is not used directly, as long as one has the above estimates (i), (ii), (iii), and (in the case where there is trapping), as long as one can absorb the errors arising from commuting with \( T \). We shall give a general formulation in a follow-up paper. The results apply to a wide class of background metrics and can include obstacles with trapped rays, black holes, etc.

Of particular importance are the Kerr spacetimes \((M, g_{a,M})\). See Section 5.1 of [8] for a discussion. In the case \( |a| \ll M \), the estimate associated to (ii) follows by the results of [8, 17], whereas the estimate associated to (iii) follows from [7].

For the general case \( |a| < M \), the estimates associated to (i) and (ii) will follow by forthcoming results. On the other hand, a computation reveals that the analogue of (i) follows as in the Schwarzschild case. Thus, one obtains precisely (10) for Kerr, where \( T \) is the stationary Killing field, \( N \) is a globally timelike translation-invariant vector field, and \( \Omega_i \) are angular momentum operators on an ambient background Schwarzschild metric.
In the general formulation, to obtain pointwise estimates as in (11), the assumptions necessary for (10) have to be supplemented with assumptions regarding the existence of sufficiently many good commutators. Again, we leave the most general formulation to a follow-up paper. We remark here, however, that the necessary properties can be immediately explicitly verified for Kerr, for the entire range $|a| < M$, in view of the timelike nature of the span of $\partial_t$ and $\partial_\phi$ outside the horizon, and the positivity of the surface gravity ensuring the validity of the commutation Theorem 7.2 of [8]. Thus, as in Schwarzschild, one obtains the pointwise (11) for Kerr, where the vector fields are interpreted as in the previous paragraph.

7 Discussion

As mentioned before, the energy decay estimate (8) corresponds exactly to what one can obtain in Minkowski space by applying the multiplier $J^Z_\mu$ associated to the generator of inverted time translations

$$Z = u^2 \partial_u + v^2 \partial_v = (t^2 + r^2) \partial_t + 2tr \partial_r.$$  

This argument for obtaining decay estimates in Minkowski space goes back to Morawetz [15]. Analogues of the current $J^Z_\mu$ can be constructed in the Schwarzschild case [5 3] and slowly rotating Kerr case $|a| \ll M$ [8 2], and were used in conjunction with an integrated decay type inequality, to obtain decay of energy in $\Sigma_\tau$. Already, however, in the Kerr case, there is a loss in the power of decay which one can obtain by this method, which is in principle proportional to the parameter $a$. See Section 5.3.6 of [8]. The difficulty with these constructions arises from error terms in the region $r \leq R$ which inherit large weights in $t$. These problems could be even more profound in the setting of non-linear applications. This issue was in fact our motivation for the present work.

It is amusing to compare the role of null infinity $\mathcal{I}^+$ in the present argument with the role of the cosmological horizon in the Schwarzschild-de Sitter case studied in [6]. There, the analogue of the iteration process applied here could be continued indefinitely to yield decay bounds faster than any polynomial rate. The reason is that in the analogue of (p-WE-Schw), the weights of the boundary terms coincide with the weights of the bulk terms. This is precisely related to the red-shift. Indeed, this is reflected already in the Schwarzschild case at the event horizon by the fact that $J^N [\phi]$ appears without degeneration at $r = 2M$ on both sides of inequality (ILED1-Schw). A similar relation holds at the cosmological horizon of Schwarzschild-de Sitter. In contrast, in the present asymptotically flat case, the weights in (p-WE-Schw) are polynomial in $r$ and drop by 1 power when comparing bulk and boundary terms. This is why it is important to keep the whole $p$-hierarchy of estimates (p-WE-Schw) in mind for the method to be useful.

We mention finally a forthcoming Fourier-based method of Tataru and collaborators for obtaining pointwise estimates for stationary metrics starting again
from (i) and good asymptotic behaviour of the metric\textsuperscript{4}.

8 Refinements: the full decay of the vector field method

As explained in the introduction, the “full vector field method” yields faster pointwise decay of the form \textsuperscript{3}. One way to prove such a result starting from our \textsuperscript{10} would be to apply one commutation with the scaling vector field \(t \partial_t + r \partial_r\). (See work of Luk \textsuperscript{12} for the Schwarzschild case.) This would be contrary, however, to the philosophy of our current approach, which is to avoid any multipliers and commutators with weights in \(t\). It turns out that our approach can in fact be extended to yield more decay without such commutations, by instead commuting with \(\partial_v\) (again understood in a null \((u, v)\) coordinate system), and \(\Omega\), which allows applying the hierarchy of estimates \textsuperscript{p-WE-Mink} or \textsuperscript{p-WE-Schw} for larger weights in \(r\), i.e. for larger values of the parameter \(p\). This leads to additional decay for a higher-order energy flux through \(\Sigma_{\tau}\), which can then be transformed into the usual decay for \(\phi\) using also ideas of \textsuperscript{12}. This will be discussed in a forthcoming follow-up paper.

Acknowledgments

This work was conducted when I.R. was visiting Cambridge in February 2009. I.R. thanks the University of Cambridge for hospitality. M.D. is supported in part by a grant from the European Research Council. I.R. is supported in part by NSF grant DMS-0702270.

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