NEW DIGIT RESULTS AND SEVERAL PROBLEMS

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Abstract. We give some new relations for Newman digit sums respectively different modulos and put some problems. In particular, for the odd prime modulos we put an important conjecture.

1. Introduction

As in [7] put for $q > 1$

$$n = \sum_{k=0}^{v} a_k q^k, \quad 0 \leq a_k < q, \quad \sigma_q(n) = \sum_{k=0}^{v} a_k.$$  

Denote for $x \in \mathbb{N}, \quad l \in [0, m - 1]

$$S_{m,l,q}(x) = \sum_{0 \leq n < x; n \equiv l \, (\text{mod} \, m)} (-1)^{\sigma_n(n)}$$

In the case $q = 2$ we write $S_{m,l,2} = S_{m,l}, \quad \sigma_2(n) = \sigma(n)$. We call (2) a generalized Newman sum.

In [7] we gave a quite another proof of the Coquet’s estimates for $S_{3,0}(x)$ and a fast algorithm for its calculation. Professor J.-P. Allouche kindly informed me about a misprint in Coquet’s theorem: for odd $x$

$$\eta = \eta(x) = (-1)^{\sigma(3x-1)}$$

(but $(-1)^{\sigma(3x-3)}$ as in [2]; sf. [1], pp.98-99)

An important role in our proof belongs to the formula: for an even $n$

$$S_{3,0}(4n) = 3S_{3,0}(n).$$

The method of proof [3] in [7] allows to obtain several new relations for some Newman digit sums and to formulate a very important conjecture.
2. SOME NEW DIGIT RELATIONS

We use the following simple relations for $S_{m,l}(x)$, $x \in \mathbb{N}$.

If $m$ is odd then

$$S_{m,l}(2x) = \begin{cases} S_{2m,l}(2x) + S_{2m,l+m-1}(2x), & \text{if } l \text{ is even} \\ -S_{2m,l-1}(2x) + S_{2m,l+m}(2x), & \text{if } l \text{ is odd} \end{cases}$$

$$= \begin{cases} S_{m,l/2}(x) - S_{m,\frac{l+1}{2}}(x), & \text{if } l \text{ is even} \\ -S_{m,\frac{l-1}{2}}(x) + S_{m,\frac{l+1}{2}}(x), & \text{if } l \text{ is odd}. \end{cases} \quad (4)$$

If $m$ is even then

$$S_{m,l}(2x) = \begin{cases} S_{m,l/2}(x), & \text{if } l \text{ is even} \\ -S_{m,\frac{l-1}{2}}(x), & \text{if } l \text{ is odd}. \end{cases} \quad (5)$$

Note that (5) reduces the calculations to the case of an odd $m$. Hence, for an odd $m$ we should solve the system (4) to get an equation for e.g. $S_{m,0}(x)$ only.

The calculations by this method are rather long and sometimes complicated. Nevertheless, we obtained the following relations for $x, y \in \mathbb{N}$, the first of which was obtained in [7] (here as in [7] $S_{m,0}([y, y + z]) = S_{m,0}(y + z) - S_{m,0}(y)$):

$$(6) \quad S_{3,0}([8x, 8y]) = 3S_{3,0}([2x, 2y]),$$

$$(7) \quad S_{5,0}([32x, 32y]) = 5S_{5,0}([2x, 2y]),$$

$$(8) \quad S_{7,0}([128x, 128y]) = -7S_{7,0}([2x, 2y]),$$

$$(9) \quad S_{9,0}([512x, 512y]) = 3S_{9,0}([128x, 128y]) + 3S_{9,0}([8x, 8y]) - 9S_{9,0}([2x, 2y]).$$

Besides, by similar way we obtained the following relation for $S_{5,0,4}([x, y])$:

if $x$ is divisible by 32 then

$$(10) \quad S_{5,0,4}([256x, 256y]) = 10S_{5,0,4}([16x, 16y]) - 5S_{5,0,4}([x, y]).$$
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Using (6)-(10) as in [7] it could be proved that

\[(11)\quad |3S_{5,0}(n)| = O(n^{\frac{\log 5}{\log 16}}) = O(n^{0.58048})\]

\[(12)\quad |S_{7,0}(n)| = O(n^{0.46789...}),\]

\[(13)\quad |S_{9,0}(n)| = O(n^{0.79248...})(as\ for\ S_{3,0}(n)),\]

\[(14)\quad |S_{5,0,4}(n)| = O(n^{0.81092...}).\]

3. SOME CONJECTURES AND PROBLEMS

1) To find a method (probably, a variant of the method of generating functions) for an automatic obtaining of relations of type (6)-(10). To find a general digit equation of this type (at least, for the base 2).

2) According to (6)-(8) we have in particular that

\[(15)\quad S_{3,0}(2^3) = 3, \ S_{5,0}(2^5) = 5, \ S_{7,0}(2^7) = -7.\]

Denote

\[a_n = S_{n,0}(2^n).\]

By the further direct calculations for the prime values of \(n\) we obtained a very astonishing sequence:

\[a_3 = 3, \ a_5 = 5, \ a_7 = -7, \ a_{11} = 11, \ a_{13} = 13,\]

\[(16)\quad a_{17} = 697, \ a_{19} = 19, \ a_{23} = -23, \ a_{29} = 29, \ldots\]

It was very difficult for us to believe that \(a_{17} = 697!\)

It this connection recall a remarkable result of M.Drmota and M.Skalba [3]: the only primes \(p \leq 1000\) satisfying \(S_{p,0}(n) > 0\) (at least, for sufficiently large \(n\)) are 3, 5, 17, 43, 257, 683.

Therefore it is natural to conjecture that for primes different from 17, 43, 257, 683, ... we have
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(17) \[ S_{p,0}(2^p) = \pm p. \]

Note that, (17) satisfies also for 3 and 5 because of the numbers 2^3 and 2^5 are small.

Besides we conjecture that always \( p | S_{p,0}(2^p) \).

3) In the connection with the results (7), (8) it is interesting to find the sharp estimates in these cases similar to [2] and [7].

4) In our opinion, it is very interesting to find a generalization of (10) for \( S_{2k+1,0,2k}(x) \) and get the sharp estimates.

We conjecture that not only \( S_{2k+1,0,2k}(x) > 0 \), \( k \geq 1 \), but also the Newman-like phenomena becomes more and more strong with the enlargement of \( k \). Moreover, if

\[ S_{2k+1,0,2k}(x) = O(x^{\lambda_k}) \]

then we conjecture that \( \lim_{k \to \infty} \lambda_k = 1. \)

5) We conjecture that, if \( d | m \) then the characteristic polynomial which corresponds to the relation of considered type for \( S_{m,0,q}(x) \) is divisible by one for \( S_{d,0,q}(x) \).

6) We conjecture that if \( (m, 3) = 1 \) then for any \( k \in (1, \frac{m}{3}) \),

\[ |S_{m,0}(x)| = o(|S_{3k,0}(x)|). \]

Note that, if 6) is true then it could be proved our Conjecture 2 [5] which until now has only a heuristic justification [6].

Remark. Put

(18) \[ G_{m,0}^{(i)}(x) = \sum_{0 \leq n < x, n \equiv 0 \pmod{m}, \sigma(n) \equiv i \pmod{2}} 1, \quad i = 0, 1. \]

It is a special case of the Gelfond digit sum. It is evident that

\[ G_{m,0}^{(0)}(x) + G_{m,0}^{(1)}(x) = \sum 1 = [\frac{x}{m}] + 1, \]

\[ G_{m,0}^{(0)}(x) - G_{m,0}^{(1)}(x) = S_{m,0}(x). \]
By the Gelfond theorem

\begin{equation}
G_{m,0}^{(i)}(x) = \frac{x}{2m} + O\left(x \frac{\ln 3}{\ln 4}\right).
\end{equation}

Thus, estimates (7), (8) make more precise the remainder term in (19) in the cases of \( m = 5 \) and \( m = 7 \).

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