RAMIFICATION THEORY FOR DEGREE $p$ EXTENSIONS OF ARBITRARY VALUATION RINGS IN MIXED CHARACTERISTIC $(0, p)$

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June 23, 2017

Abstract

We previously obtained a generalization and refinement of results about the ramification theory of Artin-Schreier extensions of discretely valued fields in characteristic $p$ with perfect residue fields to the case of fields with more general valuations and residue fields. As seen in [VT16], the “defect” case gives rise to many interesting complications. In this paper, we present analogous results for degree $p$ extensions of arbitrary valuation rings in mixed characteristic $(0, p)$ in a more general setting. More specifically, the only assumption here is that the base field $K$ is henselian. In particular, these results are true for defect extensions even if the rank of the valuation is greater than 1. A similar method also works in equal characteristic, generalizing the results of [VT16].

Contents

1 Introduction .............................................. 3
   1.1 Invariants of Ramification Theory .................... 3
   1.2 Main Results ........................................ 3
   1.3 Outline of the Contents .............................. 4

2 Preliminaries ............................................ 4
   2.1 Definitions .......................................... 4
   2.2 Classical Invariants ................................. 5

3 Swan Conductor, Best $h$ and Defect ...................... 6
   3.1 Complete Discrete Valuation Case .................... 6
   3.2 Best $h$ and Swan Conductor: Classical Case and General Case ........................................ 6
   3.3 Defect, $J_{\sigma}$ and Best $h$ ........................ 7

4 Proof of Theorem 1.3 .................................... 8
   4.1 Case $p = 2$ ......................................... 8
   4.2 Case $p > 2$ ......................................... 8

5 Filtered Union in the Defect Case ......................... 9
   5.1 Preparation for the Proof ............................ 9
   5.2 Proof of Theorem 5.1 ................................ 10

6 Refined Swan Conductor and Proof of Theorem 1.5 ....... 11
   6.1 Refined Swan Conductor $rsw$ in the Defectless Case ........................................ 12
   6.2 Proof of Theorem 1.5 ................................. 13
      6.2.1 Defectless Case ................................. 13
6.2.2 Preparation for the defect case ........................................ 14
6.2.3 Refined Swan Conductor and Proof of Theorem 1.5 in the defect case ............... 15

7 Results for the non-Kummer Case ........................................ 17
  7.1 Invariants for $L'|K'$ .................................................. 17
  7.2 Main Results for $L'|K'$ .................................................. 18

8 Generalizing the Results of [VT16] to Defect Extensions of Rank $> 1$ ................. 19
1 Introduction

Let $K$ be a henselian valued field of mixed characteristic $(0, p)$ with arbitrary valuation and $L|K$ a non-trivial Galois extension of degree $p$. We present a generalization and refinement of the classical ramification theory in this case. In [VT16], we considered Artin-Schreier extensions, when the defect is trivial or the valuation is of rank 1. The results we present in this paper are true without such assumptions. We also remark that similar methods can be used to improve the results of [VT16] and it is possible to remove the aforementioned assumptions.

First we consider Kummer extensions $L|K$, where $K$ contains a primitive $p^{th}$ root $\zeta$ of unity. The general case is then reduced to this case, by using tame extensions and Galois invariance.

1.1 Invariants of Ramification Theory

Let $K$ be a valued field of characteristic 0 with henselian valuation ring $A$, valuation $v$ and residue field $k$ of characteristic $p > 0$. We assume that $K$ contains a primitive $p^{th}$ root of 1, let us denote it by $\zeta_p = \zeta$. Let $L = K(\alpha)$ be the (non-trivial) Kummer extension defined by $\alpha^p = h$ for some $h \in K^\times$. For any $a \in K^\times$, $h$ and $h \alpha^p$ give rise to the same extension $L$. Let $B$ be the integral closure of $A$ in $L$. Since $A$ is henselian, it follows that $B$ is a valuation ring. Let $w$ be the unique valuation on $L$ that extends $v$ and let $l$ denote the residue field of $L$. We denote the value group of $K$ by $\Gamma_K := v(K^\times)$. The Galois group $\text{Gal}(L/K) = G$ is cyclic of order $p$, generated by $\sigma : \alpha \mapsto \zeta \alpha$. Let $\theta := \zeta - 1$

Let $\mathcal{A} = \{h \in K \mid$ the solutions of the equation $\alpha^p = h$ generate $L$ over $K\}$. Consider the ideals $\mathcal{J}_\sigma$ and $\mathcal{H}$ of $B$ and $A$ respectively, defined as below:

$$\mathcal{J}_\sigma = \left(\left\{ \frac{\sigma(b)}{b} - 1 \mid b \in L^\times \right\} \right) \subset B$$

(1.1)

$$\mathcal{H} = \left(\left\{ \frac{3^p}{h - 1} \mid h \in \mathcal{A} \right\} \right) \subset A$$

(1.2)

It is not apparent from the definition that $\mathcal{H}$ is indeed a subset of $A$, we prove that in Lemma 3.9. Our first result compares these two invariants via the norm map $N_{L|K} = N$, by considering the ideal $N_\sigma$ of $A$ generated by the elements of $N(\mathcal{J}_\sigma)$. We also consider the ideal $\mathcal{I}_\sigma = (\{\sigma(b) - b \mid b \in B\})$ of $B$. The ideals $\mathcal{I}_\sigma$ and $\mathcal{J}_\sigma$ play the roles of $i(\sigma)$ and $j(\sigma)$ (the Lefschetz numbers in the classical case, as explained in 2.2), respectively, in the generalization.

1.2 Main Results

We will prove the following results in sections 4 and 6, respectively. Then extend them to the non-Kummer case, in section 7.

Theorem 1.3. If $L|K$ is as in 1.1, we have the following equality of ideals of $A$:

$$\mathcal{H} = N_\sigma$$

(1.4)

Theorem 1.5. For $L|K$ as in 1.1, we consider the $A$-module $\omega_A^1$ of logarithmic differential 1-forms and the $B$-module $\omega_B^1|A$ of relative logarithmic differential 1-forms. Then

(i) There exists a unique homomorphism of $A$-modules $\text{rs} : \mathcal{H} / \mathcal{H}^2 \to \omega_A^1 / (\mathcal{I}_\sigma \cap A) \omega_A^1$ such that for all $h \in \mathcal{A}$,

$$\frac{3^p}{h - 1} \mapsto \frac{1}{h - 1} d\log h.$$

(ii) There is a $B$-module isomorphism $\varphi_\sigma : \omega_B^1|A / \mathcal{J}_\sigma \omega_B^1 \cong \mathcal{J}_\sigma / \mathcal{J}_\sigma^2$ such that for all $x \in L^\times$, $d\log x \mapsto \frac{\sigma(x)}{x} - 1$.

(iii) Furthermore, these maps induce the following commutative diagram:
The maps $\overline{\Delta N}, N$ are induced by the norm map $N$.

The map $\text{rsw}$ in (i) is a refined generalization of the refined Swan conductor of Kato for complete discrete valuation rings [KK89].

Remark 1.6. If $L|K$ is unramified ($e_L|K = 1, l|k$ separable of degree $p$), then we have $i(\sigma) = j(\sigma) = 0, I_\sigma = J_\sigma = B$ and $H = A$. Consequently, our main results are trivially true. From now on, we assume that $L|K$ is either wild ($e_L|K = p, l|k$ trivial), ferocious ($l|k$ purely inseparable of degree $p$) or with defect.

1.3 Outline of the Contents

We begin, in section 2, with a preliminary discussion of Kähler differentials, defect and classical invariants of ramification theory. Section 3 contains the description of Swan conductor in the defectless case and some results that connect defect with the ideal $J_\sigma$.

We prove Theorem 1.3 in section 4. In the next section, we use it to prove Theorem 5.1. This allows us, in the defect case, to express the ring $B$ as a filtered union of rings of the form $A[x|A$, where the elements $x \in L^\times$ are chosen in a particular way.

The generalized and refined definition of the refined Swan conductor $\text{rsw}$ is presented in section 6. First we define it in the defectless case and then extend the definition to defect extensions. We also prove Theorem 1.5, first for defectless extensions and then for defect extensions, using Theorem 5.1. Results that can be proved in a manner similar to the Artin-Schreier case are presented without proofs.

In the seventh section, we extend the main results to the non-Kummer case. The last section consists of some remarks about how the results of [VT16] can be generalized to Artin-Schreier defect extensions of higher rank valuations, in a similar fashion.

2 Preliminaries

2.1 Definitions

Definition 2.1. Differential 1-Forms

(i) Let $R$ be a commutative ring. The $R$-module $\Omega^1_R$ of differential 1-forms over $R$ is defined as follows: $\Omega^1_R$ is generated by

- The set $\{db \mid b \in R\}$ of generators.
- The relations are the usual rules of differentiation: For all $b, c \in R$,
  - (Additivity) $d(b + c) = db + dc$
  - (Leibniz rule) $d(bc) = cdb + bdc$

(ii) For a commutative ring $A$ and a commutative $A$-algebra $B$, the $B$-module $\Omega^1_{B|A}$ of relative differential 1-forms over $A$ is defined to be the cokernel of the map $B \otimes_A \Omega^1_A \to \Omega^1_B$.

Definition 2.2. Logarithmic Differential 1-Forms

(i) For a valuation ring $A$ with the field of fractions $K$, we define the $A$-module $\omega^1_A$ of logarithmic differential 1-forms as follows: $\omega^1_A$ is generated by

- The set $\{db \mid b \in A\} \cup \{d\log x \mid x \in K^\times\}$ of generators.
• The relations are the usual rules of differentiation and an additional rule: For all \( b, c \in A \) and for all \( x, y \in K^\times \),

(a) (Additivity) \( d(b + c) = db + dc \)

(b) (Leibniz rule) \( d(bc) = cdb + bdc \)

(c) (Log 1) \( d\log(xy) = d\log x + d\log y \)

(d) (Log 2) \( b \, d\log b = db \) for all \( 0 \neq b \in A \)

(ii) Let \( L|K \) be an extension of henselian valued fields, \( B \) the integral closure of \( A \) in \( L \) and hence, a valuation ring.

We define the \( B \)-module \( \omega^1_B|A \) of logarithmic relative differential 1-forms over \( A \) to be the cokernel of the map \( B \otimes_A \omega^1_A \to \omega^1_B \).

Definition 2.3. Defect

Let \( K \) be a henselian valued field of mixed characteristic \((0, p)\) and \( L|K \) a non-trivial Galois extension of degree \( p > 0 \). Let \( e_{L|K} := (w(L^\times) : w(K^\times)) \) denote the ramification index and \( f_{L|K} := [l : k] \) the inertia degree of \( L|K \). Then \( p = d_{L|K} e_{L|K} f_{L|K} \), where \( d_{L|K} \) is a positive integer, called the defect of the extension. Since \( p \) is a prime, \( d_{L|K} \) is either 1 or \( p \).

For a more general discussion on defect, see [FVK06].

2.2 Classical Invariants

Let \( K \) be a complete discrete valued field of residue characteristic \( p > 0 \) with normalized valuation \( v \), valuation ring \( A \) and perfect residue field \( k \). Consider \( L|K \), a finite Galois extension of \( K \). Let \( e_{L|K} \) be the ramification index of \( L|K \) and \( G = \text{Gal}(L|K) \). Let \( w \) be the valuation on \( L \) that extends \( v \), \( B \) the integral closure of \( A \) in \( L \) and \( l \) the residue field of \( L \). In this case, we have the following invariants of ramification theory:

• The Lefschetz number \( i(\sigma) \) and the logarithmic Lefschetz number \( j(\sigma) \) for \( \sigma \in G \setminus \{1\} \) are defined as

\[
  i(\sigma) = \min \{ v_L(\sigma(a) - a) \mid a \in B \} \\
  j(\sigma) = \min \left\{ v_L \left( \frac{\sigma(a)}{a} - 1 \right) \mid a \in L^\times \right\}
\]

Both the numbers are non-negative integers.

• For a finite dimensional representation \( \rho \) of \( G \) over a field of characteristic zero, the Artin conductor \( \text{Art}(\rho) \) and the Swan conductor \( \text{Sw}(\rho) \) are defined as

\[
  \text{Art}(\rho) = \frac{1}{e_{L|K}} \sum_{\sigma \in G \setminus \{1\}} i(\sigma)(\dim(\rho) - \text{Tr}(\rho(\sigma))) \\
  \text{Sw}(\rho) = \frac{1}{e_{L|K}} \sum_{\sigma \in G \setminus \{1\}} j(\sigma)(\dim(\rho) - \text{Tr}(\rho(\sigma)))
\]

Both these conductors are non-negative integers. This is a consequence of the Hasse-Arf Theorem (see [S] chapters 4, 6).

The invariants \( j(\sigma) \) and \( \text{Sw}(\rho) \) are the parts of \( i(\sigma) \) and \( \text{Art}(\rho) \), respectively, which handle the wild ramification. We wish to generalize these concepts to arbitrary valuation rings. Let us begin with the case of discrete valuation rings, possibly with imperfect residue fields.
3 Swan Conductor, Best $h$ and Defect

3.1 Complete Discrete Valuation Case

The following lemma classifies Kummer extensions of complete discrete valued fields.

Lemma 3.1. (See [OH87], [XZ14].)

Let $L/K$ be an extension of complete discrete valued fields, $\pi$ a prime element of $K$. We use the notation of 1.1, $L = K(\alpha)$ is given by $\alpha^p = h$. We can choose $h$ with either $v(h) = 1$ or $v(h) = 0$ such that $t = v(h - 1)$ is maximal.

Then we have the following cases:

(i) $h = 1 + c\pi^t; \pi \notin \{x^p - x \mid x \in k\}$.

(ii) $h = c\pi$.

(iii) $h = 1 + c\pi^t; 0 < t < e'p, (t, p) = 1$.

(iv) $h = u$.

(v) $h = 1 + u\pi^t; 0 < t < e'p, p | t$.

In the case (i), $L/K$ is unramified. In (ii) and (iii), it is wild and in the last two cases, it is ferocious. We compute $j(\sigma)$ in each case.

(i) $i(\sigma) = j(\sigma) = w \left( (\sigma - 1) \left( \frac{1}{\alpha - 1} \right) \right) = 0$.

(ii) $j(\sigma) = w \left( \frac{\alpha(\alpha)}{\alpha} - 1 \right) = w(\frac{z}{p}) = e'p$.

(iii) $j(\sigma) = w \left( \frac{\alpha(\alpha - 1)}{\alpha - 1} - 1 \right) = w(\frac{z}{p}) - w(\alpha - 1) = e'p - t$.

(iv) $j(\sigma) = w \left( \frac{\alpha(\alpha)}{\alpha} - 1 \right) = w(\frac{z}{p}) = e'$.

(v) $j(\sigma) = w \left( \frac{\alpha(\alpha - 1)}{\alpha - 1} - 1 \right) = w(\frac{z}{p}) - w(\alpha - 1) = e' - t/p$.

3.2 Best $h$ and Swan Conductor: Classical Case and General Case

Definition 3.2. Let $L/K$ be as in Lemma 3.1. We do not require $k$ to be perfect. We define the Swan conductor of this extension by

$$Sw(L/K) := \min_{h \in \mathfrak{A}} v \left( \frac{\alpha^p}{h - 1} \right)$$

This definition coincides with the classical definition of $Sw(L/K)$ when $k$ is perfect.

Any element $h$ of $\mathfrak{A}$ that achieves this minimum value is called best $h$.

It is well-defined up to multiplication by $a^p; a \in K^\times$.

Remark 3.4. Lemma 3.1 explicitly describes best $h$. $Sw(L/K)$ is 0 in (i), $e'p$ in (ii), (iv) and $e'p - t$ in (iii), (v).

We generalize the definition of best $h$ to arbitrary extensions as in 1.1.

Definition 3.5. Let $L/K$ be as in 1.1. An element $h$ of $\mathfrak{A}$ is called best if

$$v \left( \frac{\alpha^p}{h - 1} \right) = \inf_{g \in \mathfrak{A}} v \left( \frac{\alpha^p}{g - 1} \right)$$

If $h$ is best, $\mathcal{H}$ is the principal ideal generated by $\left( \frac{\alpha^p}{h - 1} \right)$ and plays the role of $Sw(L/K)$ in the generalization. We cannot, however, guarantee the existence of best $h$ in general.
3.3 Defect, \( \mathcal{J}_\sigma \) and Best \( h \)

**Lemma 3.7.** Let \( L/K \) be as in 1.1, except that we don’t require \( \zeta \in K \). Assume further that \( L/K \) is either wild or ferocious. Then

1. There exists \( \mu \in L^\times \) such that either \( w(L^\times)/w(K^\times) \) is of order \( p \) and generated by \( w(\mu) \) or \( l|k \) is purely inseparable of degree \( p \) and generated by the residue class \( \overline{\mu} \) of \( \mu \).

2. Let \( \mu \) be as above, \( x_i \in K \) for \( 0 \leq i \leq p - 1 \). Then \( \sum_{i=0}^{p-1} x_i \mu^i \in B \) if and only if \( x_i \mu^i \in B \) for all \( i \).

3. \( \text{dlog} \mu \) generates the \( B \)-module \( \omega_{1|A}^1 \).

**Proof.** See Lemma 1.11, Lemma 1.12, Lemma 1.13 of [VT16].

**Proposition 3.8.** Let \( L/K \) be as in 1.1, except that we don’t require \( \zeta \in K \). Then \( \mathcal{J}_\sigma \) is a principal ideal of \( B \) if and only if \( L/K \) is defectless.

**Proof.** See the proof of Proposition 3.10 of [VT16].

**Lemma 3.9.** For \( L/K \) as in 1.1, \( \mathcal{H} \) is an integral ideal of \( A \).

**Proof.** Let \( h \in \mathfrak{A}, a \in m_A \) such that \( h-1 = a \mathfrak{q}^p \) and set \( \beta = (\alpha - 1)/3 \). Recall that \( p \) divides \( \mathfrak{q}^p \) and \( p \mathfrak{q}^2 \) divides \( \mathfrak{q}^p + p \mathfrak{q} \). For some \( x, x' \in B \), we have \( 1 + a \mathfrak{q}^p = (1 + a \mathfrak{q})^p = 1 + p \mathfrak{q} + (a \mathfrak{q} + p \mathfrak{q})^2 x = 1 + a \mathfrak{q}^p - a \mathfrak{q} \mathfrak{q} + a \mathfrak{q}^2 \mathfrak{q}^2 x = 1 + (a \mathfrak{q}^p - a \mathfrak{q} \mathfrak{q} + a \mathfrak{q}^2 \mathfrak{q}^2 \mathfrak{q}^2 / \mathfrak{q}^2) x. \) Hence, \( a = p \mathfrak{q}^{(2-p)} x' + \mathfrak{q}^p - \mathfrak{q} \mathfrak{q} ). \) Since \( K \) is henselian, \( \beta \in K \) and this contradicts our assumption that \( L/K \) is non-trivial.

**Lemma 3.10.** Let \( L/K \) be as in 1.1. If \( L/K \) is defectless, then we can find best \( h \) satisfying exactly one of the following properties:

(i) \( h = 1 + e\mathfrak{q}; \mathfrak{q} \notin \{x^p - x \mid x \in k\} \).

(ii) \( h = ct; p \nmid v(t) \).

(iii) \( h = 1 + ct; 0 < v(t) < e'p, p \nmid v(t) \).

(iv) \( h = u \).

(v) \( h = 1 + ut; 0 < v(t) < e'p, p \mid v(t) \).

where \( t \in m_A, u, c \in A^\times, \mathfrak{q} \notin k^p \).

**Proof.** Follows from Proposition 3.8.

4 Proof of Theorem 1.3

Let \( L/K \) be as in 1.1. First we prove \( \mathcal{H} \subset \mathcal{N}_\sigma \).

Let \( h \in \mathfrak{A} \), we want to show that \( \left( \frac{h}{h-1} \right) A \subset \mathcal{N}_\sigma \). We observe that \( N(\alpha) = (-1)^{p-1} h \) and \( N(\alpha - 1) = \prod_{i=0}^{p-1} (\sigma^i \alpha - 1) = (-1)^{p-1} (\alpha^p - 1) = (-1)^{p-1} (h - 1) \).

If \( v(h) > 0, \left( \frac{h}{h-1} \right) A = (\mathfrak{q}^p) A \). Note that \( N \left( \frac{\sigma(\alpha)}{\alpha} - 1 \right) = N(\mathfrak{q}) = \mathfrak{q}^p \).

If \( v(h) = 0 \), consider \( N \left( \frac{\sigma(\alpha - 1)}{\alpha - 1} - 1 \right) = N \left( \frac{\sigma(\alpha)}{\alpha} \right) = \frac{\mathfrak{q}^p}{h-1}. \) Since \( h \) is a unit, \( \frac{\mathfrak{q}^p}{h-1} \) and \( \frac{\mathfrak{q}^p}{h} \) generate the same ideal of \( A \). Thus, it follows that \( \mathcal{H} \) is a subset of \( \mathcal{N}_\sigma \).

Next, we prove the reverse inclusion \( \mathcal{N}_\sigma \subset \mathcal{H} \). If \( L/K \) is defectless, this follows directly from Proposition 3.8 and Lemma 3.10. Proof in the defect case requires some work.
Let $L/K$ be a defect extension as in 1.1. The value group $\Gamma = \Gamma_K$ need not be an ordered subgroup of $\mathbb{R}$. Let $v$ denote the valuation on $L$ and also on $K$. Given any $b \in L \setminus K$, we want to show that $N\left(\frac{\alpha(b) - b}{b}\right) \in \mathcal{H}$. It is enough to consider the case when $b$ is a unit. For any $\alpha$ such that $\alpha^p = h$ generates $K(\alpha) = L$,

$$N\left(\frac{\alpha}{\alpha - 1}\right) = N(\alpha - 1) = \mathfrak{p}^p \in \mathcal{H}$$  \hspace{1cm} (4.1)$$

If $v(\sigma(b) - b) = v\left(\frac{\alpha(b) - b}{b}\right) \geq v(3)$, then $N\left(\frac{\alpha(b) - b}{b}\right) \in (\mathfrak{p}^p) \subset \mathcal{H}$. Thus, we may assume

$$v(\sigma(b) - b) < v(3)$$  \hspace{1cm} (4.2)$$

We divide the proof into two cases: $p = 2$ and $p > 2$.

### 4.1 Case $p = 2$

**Proof.** In this case, $\sigma^2 = id$, $\zeta = -1$ and $\mathfrak{p} = 2$. Let $b \in B^\times \setminus A$.

Since $v(\sigma(b) - b) < v(2) = v(2b)$, $v(\text{Tr}(b)) = v(\sigma(b) - b + 2b) = v(\sigma(b) - b)$. Define $x_b = x := \frac{\sigma(b) - b}{\text{Tr}(b)} \in B^\times$.

Clearly, $\mathfrak{s}(x) = -x$ and hence, $x^2 = -N(x) = h \in A^\times$. As $\sigma$ does not fix $x, L = K(x)$.

We have $x - 1 = \frac{2b}{\text{Tr}(b)} = \frac{\sigma(x-1)-1}{x-1} = \frac{\sigma(b)}{b} - 1$.

Therefore, $N\left(\frac{\sigma(b)}{b} - 1\right) = N\left(\frac{\sigma(x-1)}{x-1} - 1\right) = N\left(\frac{-2x}{x-1}\right) = \frac{b^2h}{h-1}$ is an element of $\mathcal{H}$. \hfill $\square$

### 4.2 Case $p > 2$

**Proof.** Consider the formal expression $\text{Tr}_{L|K} = \text{Tr} = \frac{\sigma^p - 1}{\sigma - 1} = \prod_{i=1}^{p-1} (\sigma - \zeta^i)$. Given $b \in B^\times \setminus A$, define

$$\gamma_b = \gamma := \left(\frac{b^{p-1}}{g^p(b)}\right) \text{ and } y_b = y := \left(\prod_{i=2}^{p-1} (\sigma - \zeta^i)\right)(\gamma), \text{ where } g(T) = \min_K(b).$$

Then $(\sigma - \zeta)(y) = \text{Tr}(\gamma) = 1$, i.e., $\sigma(y) = \zeta y + 1$.

Next define $x_b = x := 1 + \mathfrak{s}y$. Then $\sigma(x) = 1 + \mathfrak{s}(\zeta y + 1) = 1 + \mathfrak{s} + \zeta(\mathfrak{s}y) = \zeta x$.

Consequently, $x^p = N(x) = h \in K$ and $L = K(x)$. We compare $v\left(\frac{\sigma(b)}{b} - 1\right) = v(\sigma(b) - b) =: s$ and $v\left(\frac{v(\sigma(b) - b)}{s-1}\right) = v\left(\frac{v(\sigma(y) - 1)}{y-1}\right) = v\left(1 + \mathfrak{s}y\right) =: s'$.

For $1 \leq i \leq p - 1$, let $v\left(\left(\prod_{j=p-i}^{p-1} (\sigma - \zeta^j)\right)(\gamma)\right) = v\left(\left(\prod_{j=p-i+1}^{p-1} (\sigma - \zeta^j)\right)(\gamma)\right) + c_i; c_i \geq 0$. Then

(i) $v(\gamma) = -(p-1)s \Rightarrow \sum_{i=1}^{p-1} c_i = (p-1)s$.

(ii) $\sum_{i=1}^{p-2} c_i = v(y) - v(\gamma) = v(y) + (p-1)s$.

(iii) $c_{p-1} = v((\sigma - \zeta)(y)) - v(y) = -v(y)$.

Since $v(\mathfrak{p}) > s$, we have $v(\mathfrak{p}) > 0 \Rightarrow v(x) = v(1 + \mathfrak{s}y) = 0, -v(y) = s' = c_{p-1}$.

Consider $B_b := A[\{\sigma^i(b) | 1 \leq i \leq p - 1\}] \subset B$.

It is invariant under the action of $\sigma^i - \zeta^i$ for all $1 \leq i \leq p - 1, 0 \leq j \leq p - 1$.

For the ideal $J_{\sigma,b} := \{(\sigma(t) - t | t \in B_b)\}$ of $B_b$, the ideal $J_{\sigma,b}B$ of $B$ is finitely generated and therefore, principal. Observe that

$$\gamma = \frac{b^{p-1}N(g^p(b))/g^p(b)}{N(g^p(b))} \in \left(\frac{1}{N(g^p(b))}\right) B_b$$
Therefore, \( c_i \geq s = v(\sigma(b) - b) \) for all \( 1 \leq i \leq p - 2 \).
Consequently, we have \( s + v(y) = s - s' = s - c_{p - 1} \geq 0 \)
\[ \Rightarrow \left( \frac{\sigma(b)}{b} - 1 \right) \subset \left( \frac{\sigma(x - 1)}{x - 1} - 1 \right) \]
\[ \Rightarrow \left( N \left( \frac{\sigma(b)}{b} - 1 \right) \right) \subset \left( \frac{\sigma(x - 1)}{x - 1} - 1 \right) \]
\[ \Rightarrow \left( \frac{N}{N} \right) \subset \left( \frac{N}{\frac{\sigma(x - 1)}{x - 1} - 1} \right) = \left( \frac{\sigma h}{h - 1} \right) = \left( \frac{\sigma h}{h - 1} \right) \subset \mathcal{H} . \]
This concludes the proof.

\[ \square \]

Remark 4.3. In [VT16], we used an argument that required the rank of the valuation to be 1. The above argument, however, works for valuations of arbitrary rank.

Corollary 4.4. For \( L/K \) as in 1.1, the following statements are equivalent:

1. Best \( h \) exists.
2. \( \mathcal{H} \) is a principal ideal of \( A \).
3. \( \mathcal{J}_\sigma \) is a principal ideal of \( B \).
4. \( L/K \) is defectless.

5 Filtered Union in the Defect Case

Let \( L/K \) be a defect extension as in 1.1. We will write the ring \( B \) as a filtered union of rings \( A[x] \) and study the extensions \( K(x)/K \) for a better understanding of \( L/K \).

Let \( \mathfrak{H} := \{ h \in \mathfrak{H} \mid h \in A^\times, h - 1 \in \mathfrak{m}_A \} \). We note that in the defect case, \( \mathcal{H} = \left( \left\{ \frac{x^p}{h - 1} \mid h \in \mathcal{H} \right\} \right) \).

Theorem 5.1. Consider \( \mathcal{I} = \{ \alpha \in L \mid \alpha^p = h \in \mathcal{H} \} \). For each \( \alpha \in \mathcal{I} \), we can find \( \alpha' \in B^\times \cap \alpha K^\times \) such that \( B = \bigcup_{\alpha \in \mathcal{I}} A[\alpha'] \) is a filtered union, that is, the following are true:

(i) For any \( \alpha_1, \alpha_2 \in \mathcal{I} \), either \( A[\alpha_1'] \subset A[\alpha_2'] \) or \( A[\alpha_2'] \subset A[\alpha_1'] \).
(ii) Given any \( \beta \in B \), there exists \( \alpha \in \mathcal{I} \) such that \( \beta \in A[\alpha'] \).

Definition 5.2. For \( \alpha \in \mathcal{I} \), define \( \alpha' := \gamma_\alpha \left( \frac{\alpha - 1}{\delta} \right) \) where \( \gamma_\alpha = \gamma \in A \) such that \( \alpha' \in B^\times \). We will show that these \( \alpha' \)’s satisfy the conditions of Theorem 5.1. Note that the ring \( A[\alpha'] \) does not depend on the choice of \( \gamma \).

5.1 Preparation for the Proof

Lemma 5.3. If \( \alpha_1, \alpha_2 \in \mathcal{I} \) such that \( v(\alpha_1 - 1) \leq v(\alpha_2 - 1) \), then \( A[\alpha_1'] \subset A[\alpha_2'] \).

Proof. Since \( \alpha_1, \alpha_2 \in B^\times \) generate the same extension, (by Remark 5.4) we have \( \sigma \left( \frac{\alpha_1}{\alpha_2} \right) = \frac{\sigma h}{\alpha_2} =: u \in K \cap B^\times = A^\times \).

\[ \alpha_1' = \gamma_1 (\alpha_1 - 1) \]
\[ = \left( \frac{\gamma_1}{\gamma_2} \right) \gamma_2 \left( \frac{\alpha_2 u - 1}{\delta} \right) \]
\[ = \left( \frac{\gamma_1}{\gamma_2} \right) \gamma_2 \left( \frac{\alpha_2 u - u + u - 1}{\delta} \right) \]
\[ = \left( \frac{\gamma_1}{\gamma_2} \right) u \gamma_2 \left( \frac{\alpha_2 - 1}{\delta} \right) + \left( \frac{\gamma_1}{\gamma_2} \right) \gamma_2 \left( \frac{u - 1}{\delta} \right) \]
\[ = \left( \frac{\gamma_1}{\gamma_2} \right) u \alpha_2' + \gamma_1 \left( \frac{u - 1}{\delta} \right) \]

9
\[ v(\alpha_1 - 1) \leq v(\alpha_2 - 1) \Rightarrow v(\gamma_1) \geq v(\gamma_2) \text{ and } u \in A^x. \] Hence, \( \left( \frac{1}{u} \right) u\alpha'_2 \in A[\alpha'_2]. \)

Furthermore, \( v(u - 1) = v(\alpha_1 - \alpha_2) + v(\alpha_2) = v((\alpha_1 - 1) - (\alpha_2 - 1)) \geq v(\alpha_1 - 1). \]

Hence, \( v(\gamma_1 \left( \frac{u - 1}{u} \right)) \geq v(\alpha'_1) = 0. \) Since \( \gamma_1 \left( \frac{u - 1}{u} \right) \in K, \) we see that \( \alpha'_1 \in A[\alpha'_2]. \)

**Remark 5.4.** Let \( \sigma(\alpha_1) = \zeta \alpha_1 \) and \( \sigma(\alpha_2) = \zeta^i \alpha_2 \) for some \( 1 \leq i \leq p - 1. \) Consider the unique number \( j \) satisfying \( 1 \leq j \leq p - 1 \) and \( ij \equiv 1 \mod p. \) Clearly, \( \alpha_2 \) and \( \alpha'_2 \) give the same extension \( L|K \) and \( \sigma(\alpha'_2) = \zeta^j \alpha'_2 = \zeta \alpha'_2. \) Thus, if \( i \neq 1, \) we can replace \( \alpha_2 \) by \( \alpha'_2. \)

**Lemma 5.5.** Given any \( b \in B, \) there exists \( \alpha \in \mathcal{I} \) such that \( (\sigma(b) - b) \subset (\sigma(\alpha') - \alpha'). \)

**Proof.** It is enough to consider the case \( b \in B^x. \) Let \( \nu_0 = v(\sigma(b) - b). \) Since this is the defect case, \( \mathcal{I}_\sigma = \mathcal{J}_\sigma \) and hence, \( \mathcal{H} = \mathcal{N}_\sigma = \{ N(\sigma(x) - x) \mid x \in B^x \}. \) In particular, all the elements of valuation greater than or equal to \( p\nu_0 \) are in \( \mathcal{H}. \) Pick some \( \alpha \in \mathcal{I}, \) \( \alpha^p = h \) and let \( c \) be as follows.

\[ c := v(\sigma(\alpha') - \alpha'). \] \( \sigma = v(\sigma(\alpha') - \alpha') = v(\alpha \gamma_\alpha) = v \left( \frac{3}{\alpha - 1} \right) = \frac{1}{p} v \left( \frac{3^p}{h - 1} \right). \]

By definition of \( \mathcal{H}, \) it is possible to choose \( \alpha \) such that \( pc = v \left( \frac{3^p}{h - 1} \right) \leq p\nu_0. \) Hence, there exists \( \alpha \in \mathcal{I} \) such that \( c \leq \nu_0. \)

**Lemma 5.6.** For \( x, y \in L, \) we have \( (\sigma - 1)^n(xy) = \sum_{k=0}^n \binom{n}{k}(\sigma - 1)^{n-k}(x)(\sigma - 1)^{k}(\sigma^{n-k}(y)) \)

In particular, for \( n = 1, \) \( (\sigma - 1)(xy) = (\sigma - 1)(x)y + x(\sigma - 1)(y). \)

### 5.2 Proof of Theorem 5.1

As in the case of Artin-Schreier extensions (see section 5.4 of [VT16]), it is enough to prove the following result:

**Proposition 5.7.** Given any \( \beta \in B^x, \) there exists \( \alpha \in \mathcal{I} \) such that

\[ \left( \prod_{i=1}^{p-1} (\sigma - \zeta^i) \right) \left( \frac{1}{F'(\alpha')} A[\alpha', \beta] \right) = \frac{1}{F'(\alpha')} A[\alpha', \beta] \in B. \] Here, \( F \) denotes the minimal polynomial of \( \alpha' \) over \( K. \)

**Proof.** For any \( \alpha \in \mathcal{I} \) with \( \alpha^p = h, \) we have

(i) \( \sigma(\alpha') = \gamma \left( \frac{\alpha - 1}{3} \right) = \gamma \left( \frac{\alpha - \zeta - \zeta^{-1}}{3} \right) = \zeta \alpha' + \gamma \)

(ii) \( (\sigma - 1)(\alpha') = 3\alpha' + \gamma = \gamma \alpha. \) Note that \( v(\gamma) < v(\zeta) = v(\gamma) + v(\alpha) \).

(iii) \( F'(\alpha') = \prod_{i=1}^{p-1} (\alpha' - \sigma^i(\alpha')) = \prod_{i=1}^{p-1} \left( \frac{\alpha' - \sigma^i(\alpha)}{\frac{1}{F'(\alpha')}} \right) = p \left( \frac{2\alpha}{3} \right)^{p-1} = \frac{h^p}{\alpha}. \)

(iv) Since \( v(\beta) = v \left( \frac{3^p}{h - 1} \right), \) \( v(F'(\alpha')) = v(\gamma^{p-1}). \)

(v) For any \( 1 \leq i \leq p - 1, \) \( \sigma^i \left( \frac{1}{F'(\alpha')} \right) = \zeta \left( \frac{1}{F'(\alpha')} \right) \)

We want to show that for a “special” \( \alpha, \) for all \( 0 \leq m, j \leq p - 1, \)

\[ v \left( \left( \prod_{i=1}^{p-1} (\sigma - \zeta^i) \right) \left( \frac{1}{F'(\alpha')} A[\alpha', \beta] \right) \right) \geq 0 \] (5.8)

For any \( x \in L, 1 \leq i \leq p - 1, \)

\[ (\sigma - \zeta^i) \left( \frac{x}{F'(\alpha')} \right) = \frac{\sigma(x) \zeta^i}{F'(\alpha')} = x \zeta^i \left( \frac{1}{F'(\alpha')} \right) \left( \sigma - \zeta^{-1} \right)(x) \text{ by (v) above.} \]

Thus, (5.8) is equivalent to

\[ v \left( \zeta^{-1} \left( \prod_{i=1}^{p-1} (\sigma - \zeta^{-1}) \right) (\alpha'^m \beta^j) \right) = v \left( \left( \prod_{i=1}^{p-1} (\sigma - \zeta^{-1}) \right) (\alpha'^m \beta^j) \right) \geq v(F'(\alpha')) = (p - 1)v(\gamma) \] (5.9)
Since \( \prod_{i=1}^{p-1}(\sigma - \zeta^{i-1}) = \prod_{i=1}^{p-1}(\sigma - 1 - (\zeta^{i-1} - 1)) \) and \( v(\gamma) \leq v(\zeta^{i-1} - 1) \), it is enough to show
\[
v((\sigma - 1)^{p-1}(\sigma^{m} \beta^{j})) \geq v(F'(\alpha^{j})) = (p-1)v(\gamma)
\]  
(5.10)

The rest follows from Lemma 5.5, Lemma 5.6 and the following argument. This is taken directly from [VT16], it is worth noting that we did not use the rank 1 assumption in these steps and therefore, the argument is valid for higher rank valuations.

(Step 1) Construction of the special \( \alpha' \)

We begin with an \( \alpha_0 \) satisfying \( (\sigma(\beta) - \beta) \subset (\sigma(\alpha'_0) - \alpha'_0) \). Let \( (\sigma - 1)(\beta) = b_1\gamma_0 ; b_1 \in B \). Therefore, \( (\sigma - 1)^2(\beta) = (\sigma - 1)(b_1)\gamma_0 \). We don’t know much about the valuation of \( (\sigma - 1)(b_1) \), however. Let \( \alpha_1 \) be such that \( ((\sigma - 1)(b_1)) \subset ((\sigma - 1)(\alpha'_1)) \). Write \( (\sigma - 1)(b_1) = b_2\gamma_1 \). Now we can write \( (\sigma - 1)^2(\beta) = b_2\gamma_1\gamma_0 \). Using this process, we can find \( b_i \)'s and \( \alpha_i \)'s such that \( (\sigma - 1)^i(\beta) = b_i\gamma_{i-1}...\gamma_1\gamma_0 ; \) where \( b_i \in B \).

Let \( \gamma \) be the \( \gamma_j \) with smallest valuation involved in the expression for \( i = p - 1 \). Let \( \alpha \) denote the corresponding \( \alpha_j \). We will show that this \( \alpha \) satisfies the required property.

(Step 2) Proof for \( \beta \)

\( (\sigma(\beta) - \beta) \subset (\sigma(\alpha'_0) - \alpha'_0) \subset (\sigma(\alpha') - \alpha') = (\gamma) \), since \( v(\gamma) \leq v(\gamma_0) \). Due to the choice of \( \gamma \), we also have \( v((\sigma - 1)^t(\beta)) \geq tv(\gamma) \) for all \( 1 \leq t \leq p - 1 \). In particular, this is true for \( t = p - 1 \), proving the statement (5.10) for the case \( i = 0, j = 1 \).

(Step 3) Terms \( \alpha^{m} \beta^{j} \)

For the terms of the form \( \beta^{j} \), we use induction on \( j \) and Lemma 5.6. Valuation of each term in the expansion is at least \( (p-1)v(\gamma) \). In fact, by a similar argument, \( v((\sigma - 1)^k(\beta^{j})) \geq kv(\gamma) \) for all \( 1 \leq k \leq p - 1 \). For the general terms \( \alpha^{m} \beta^{j} \), first note that \( (\sigma - 1)^k(\alpha') = (\sigma - 1)^{k-1}(\gamma) = 0 \) for all \( k > 1 \). Therefore, (again using the identity), we have
\[
(\sigma - 1)^{p-1}(\alpha^{m} \beta^{j}) = \alpha^{m}(\sigma - 1)^{p-1}(\beta^{j}) + (p-1)(\sigma - 1)(\alpha^{m})(\sigma - 1)^{p-2}(\sigma(\beta^{j}))
\]
Once again, both these terms have valuation \( \geq (p-1)v(\gamma) \).

\( \square \)

6 Refined Swan Conductor and Proof of Theorem 1.5

**Definition 6.1.** Let \( K \) be as in 1.1. For any \( x \in K^{\times} \), we define elements \( \delta \log x \in \mathbb{H}_x \otimes \omega^1_A \) as described below. \( \mathbb{H}_x \) is the ideal of \( A \) given by \( \mathbb{H}_x := (x - 1)A \cap A \cap \left( \frac{1}{x-1} \right) A \) if \( x \neq 1 \) and \( \mathbb{H}_1 := (0) \).

- \( \delta \log 1 := 0 \).
- If \( 0 \neq x \in m_A, \delta \log x := 1 \otimes \text{dlog } x \).
- If \( 1 \neq x \in A^\times, \delta \log x := (x - 1) \otimes \frac{\text{dlog}(x-1)}{x} \).
- If \( x \in K \setminus A, \delta \log x := -\delta \log (1/x) \).

Furthermore, for any \( 0 \neq b \in A \), we define elements \( \delta b \in b\mathbb{H}_b \otimes \omega^1_A \) by \( \delta b := b\delta \log b = 0 \).

**Lemma 6.2.** For all \( b, c \in A \) and for all \( x, y \in K^{\times} \), we have
1. (Log 1) \( \delta \log(xy) = \delta \log x + \delta \log y \in (\mathbb{H}_x \cup \mathbb{H}_y \cup \mathbb{H}_{xy}) \otimes \omega^1_A \).
2. (Additivity) \( \delta(b + c) = \delta b + \delta c \in (b\mathbb{H}_b \cup c\mathbb{H}_c \cup (b+c)\mathbb{H}_{b+c}) \otimes \omega^1_A \).
3. (Leibniz rule) \( \delta(bc) = c\delta b + b\delta c \in (b\mathbb{H}_b \cup c\mathbb{H}_c \cup (bc)\mathbb{H}_{bc}) \otimes \omega^1_A \).
6.1 Refined Swan Conductor $\text{rsw}$ in the Defectless Case

We first define the refined Swan conductor for the defectless case (below) and then extend the definition to the defect case.

**Definition 6.3.** Let $L|K$ be as in 1.1 and defectless, given by best $h$. Consider the ideal $I$ of $A$ defined by

$$I := \{ x \in K \mid v(x) \geq \left( \frac{p-1}{p} \right) \log \left( \frac{3^p}{h-1} \right) \}$$

We note that this definition only depends on the valuation of $h - 1$ and hence, is independent of the choice of best $h$. The refined Swan conductor $\text{rsw}$ of this extension is defined to be the $A$-homomorphism

$$\delta \log h : \mathcal{H} \to \omega^1_A / \mathfrak{I} \omega^1_A$$

given by

$$r \mapsto \frac{r}{3^p} \delta \log h.$$

We will show in Lemma 6.5 that this definition is independent of the choice of best $h$.

**Remark 6.4.** We can also view $\text{rsw}$ as an element of $\left( \frac{1}{3^p} \right) \mathbb{H}_h \otimes \omega^1_A$. This definition is consistent with Kato’s definition in [KK89]. We note that $\mathbb{H}_h$ is independent of choice of best $h$.

**Lemma 6.5.** Let $L|K$ be defectless, given by best $h$. Then the refined Swan conductor of this extension, i.e., the $A$-homomorphism $\delta \log h$, is independent of the choice of $h$.

**Proof.** Let $h$ and $a^p h$ be best; $0, 1 \neq a \in A$. Then the difference between the two $A$-homomorphisms is given by $\delta \log a^p h - \delta \log h = \delta \log a^p = p \delta \log a$. For an element $r$ of $\mathcal{H} = \left( \frac{3^p}{h-1} \right) = \left( \frac{3^p}{a^p h-1} \right)$, we have

$$(p \delta \log a)(r) = \begin{cases} \frac{r}{3^p} p \otimes \log a; & 0 \neq a \in m_A \\ \frac{r}{3^p} p(a-1) \otimes \frac{\log(a-1)}{a}; & 1 \neq a \in A^\times \end{cases}$$

We wish to show that $(p \delta \log a)(r)$ belongs to $I \omega^1_A$. Considering the formulas above and observing that $\frac{r}{3^p} p$ has the same valuation as $\frac{r}{3^p} p(a-1)$ when $0 \neq a \in m_A$, it is enough to show that $\frac{r}{3^p} p(a-1) \in I$ for $0, 1 \neq a \in A$.

$$v(\frac{r}{3^p} p(a-1)) \geq v(\frac{1}{h-1} p(a-1)) = v(a-1) + v(p) - v(h-1) = v(a-1) + (p-1) v(3) - v(h-1)$$

Thus, it is enough to prove the inequality

$$v(a-1) + (p-1) v(3) - v(h-1) \geq \left( \frac{p-1}{p} \right) \log \left( \frac{3^p}{h-1} \right) = \frac{1}{p} v(h-1) + (p-1) v(3) - v(h-1)$$

That is,

$$v(a-1) \geq \frac{1}{p} v(h-1) \quad (6.6)$$

When $h-1 \in A^\times$, this follows simply from $v(a^p - 1) = pv(a-1) \geq 0 = v(h-1)$, since $a \in A$.

When $h-1 \in m_A$, $h \in A^\times$ and hence, Equation (6.6) follows from

$$v(h-1) = v(a^p h - 1) = v((a^p - 1) h + h - 1) \geq \min(v(a^p - 1), v(h-1))$$

where the first equality is a consequence of $h$ and $a^p h$ both being best.

**Corollary 6.7.** Let $L|K$ and $h$ be as above. Then the ideal $\mathfrak{I}'$ of $A$ generated by

$$\left\{ \frac{a-1}{h-1} \mid a \in A, \text{a^p h is best} \right\}$$

is contained in the ideal $I$. 

12
6.2 Proof of Theorem 1.5

Lemma 6.8. Let $L/K$ be as in 1.1. Then

(a) The norm map $N_{L|K} = N$ induces the surjective ring homomorphism $N : B \to A/(I_\sigma \cap A)$.

(b) The map $\varphi_\sigma : \omega _{|B/A}| / J_\sigma \omega _{|B/A} \to J_\sigma / J_\sigma ^2$ given by $\log x \mapsto \frac{\sigma (x)}{x} - 1$ is a surjective $B$-module homomorphism.

Proof. See Lemmas 1.6 and 6.1 of [VT16].

6.2.1 Defectless Case

Let $L/K$ be as in 1.1 and defectless, given by best $h$.

Lemma 6.9. When $L/K$ is defectless, $\mathcal{H} \subset I \subset I_\sigma \cap A$

Proof. Let $pw \left( \frac{1}{\alpha - 1} \right) = pv = w \left( \frac{\sqrt{p}}{p - 1} \right)$. By definition, $I = \{ x \in K \mid w(x) \geq (p - 1)\nu \}$ and hence, contains $\mathcal{H}$. Now we need to show that $I_\sigma$ contains all the elements $x$ of $L$ satisfying $w(x) \geq (p - 1)\nu$. Using the characterization in Lemma 3.10, we see that in Case (i), $\nu = 0$ and the result follows trivially. Thus, we may assume that $L/K$ is either wild or ferocious.

Without loss of generality, we may further assume $0 \leq v(h) < pv(3)$. We will divide the proof in two cases:

- Case $p > 2$: If $h \in m_A, \nu = w(3)$. Since $\alpha \in B, 3\alpha = (\sigma - 1)(\alpha) \in I_\sigma$. By our assumption on $h$, $w(3\alpha) < 2w(3) \leq (p - 1)w(3)$ and hence, $I \subset I_\sigma \cap A$.

  When $h \in A^\times$, we consider the element $b$ of $I_\sigma$ given by
  \[ b = (\sigma - 1) \left( \frac{3}{\alpha - 1} \right) = \left( \frac{-3^2}{(\alpha - 1)(\zeta \alpha - 1)} \right). \]

  Since $w(b) = 2\nu \leq (p - 1)\nu, I \subset I_\sigma \cap A$.

- Case $p = 2$: In this case, $(p - 1) = 1$ and $\frac{3}{\alpha - 1} = 2$. By Lemma 3.7, the ideal $I_\sigma$ of $B$ is generated by the elements $(\sigma - 1)(x\mu) = x(\sigma - 1)(\mu); x \in K, x\mu \in B$ where $\mu$ is either $\alpha$ or $\alpha - 1$.

  Since $(\sigma - 1)(\alpha - 1) = (\sigma - 1)(\alpha) = -2\alpha, I_\sigma$ is generated by $(2\alpha)K \cap B$.

  When $\alpha - 1 \in B^\times, w \left( \frac{2}{\alpha - 1} \right) = w(2) > w \left( \frac{2\alpha}{h} \right) \geq 0$. Since $\frac{2\alpha}{h} \in I_\sigma, I \subset I_\sigma \cap A$.

  If $\alpha - 1 \in m_B$ and $\mathcal{E}_{L/K} = I, I_\sigma = J_\sigma = \left( \frac{2}{\alpha - 1} \right) B$ and therefore, $I \subset I_\sigma \cap A$.

  The last remaining case is when $\alpha - 1 \in m_B$ and $\mathcal{E}_{L/K} = 2$. As in the preceding case, $I = J_\sigma \cap A$. Since $I_\sigma \neq J_\sigma$ in general, we use a different strategy.

  Let $w(2) = 2e', 0 < w(h - 1) = 2s < 4e', w(\alpha - 1) = s, \nu = 2e' - s$.

  If there exists an element $b \in A$ such that $s < w(b) < 2e'$, then $\frac{2\alpha}{h} \in I_\sigma$ and $w \left( \frac{2\alpha}{h} \right) < \nu$. Now suppose that there is no such element. In particular, $2s \geq 2e'$. Any element $x$ of $I$ must satisfy $w(x) > 2e' - s$. If there is an $x \in I$ such that $2e' > w(x) > 2e' - s$, then $2s > w \left( \frac{x(h - 1)}{2} \right) > s$. By the assumption above, we must have $w \left( \frac{x(h - 1)}{2} \right) \geq 2e'$ and hence, $w(x) \geq 2e' + 2e' - 2s = 2\nu$.

  Since
  \[ w \left( (\sigma - 1) \left( \frac{2}{\alpha - 1} \right) \right) = w \left( \frac{-4\alpha}{h - 1} \right) = 4e' - 2s = 2\nu, \]

  we have $I \subset I_\sigma$.

Proof of Theorem 1.5 in the defectless case. We will use the characterization in Lemma 3.10.
• Case (i): \( h = 1 + e\pi; \pi \notin \{ x^p - x \mid x \in K \} \). As mentioned earlier, \( L|K \) is unramified in this case and the result is trivially true.

• Case (ii): In this case, \( h \in m_A \) and \( p \nmid v(h) \). Consequently, the \( B\)-module \( \omega_B^1 \) is generated by \( d\log \alpha \) and the diagram is given by the following maps:

\[
\begin{array}{ccc}
\omega \log \alpha & \xrightarrow{\varphi_{\sigma}} & \omega_{\tilde{B}} \\
N \downarrow & & \downarrow N \\
N(b) \omega \log N(\alpha) & = N(b) \omega \log h & \xleftarrow{\text{rsw}} & N(b) \omega_B^p
\end{array}
\]

For \( b \in B \), \( b \omega \log(\alpha) \in \text{Ker}(\varphi_{\sigma}) \iff b_{\tilde{B}} \in (3^2) \). Hence, \( \varphi_{\sigma} \) is an isomorphism. Since \( \mathcal{H} \subseteq \mathfrak{I} \subseteq (\mathcal{I}_\sigma \cap A) \), the map \( \text{rsw} \) is well-defined, independent of the choice of \( h \).

• Cases (iii)-(v): In these cases, \( 1 \neq h \in A^\times, \alpha \in B^\times \) and the \( B\)-module \( \omega_B^1 \) is generated by \( \omega \log(\alpha - 1) \). The diagram is given by the following maps:

\[
\begin{array}{ccc}
b \omega \log(\alpha - 1) & \xrightarrow{\varphi_{\sigma}} & b \left( \frac{d\log(h)}{h} \right) \\
N \downarrow & & \downarrow N \\
N(b) \omega \log N(\alpha - 1) & \xleftarrow{\text{rsw}} & N(b) \left( \frac{d\log(h - 1)}{h - 1} \right)
\end{array}
\]

It is easy to verify that \( \varphi_{\sigma} \) is an isomorphism.

Since \( \mathcal{H} \subseteq \mathfrak{I} \subseteq (\mathcal{I}_\sigma \cap A) \), the map \( \text{rsw} \) is well-defined, independent of the choice of \( h \). By definition,

\[
\text{rsw} \left( N(b) \left( \frac{3^2 h}{h - 1} \right) \right) := N(b) \left( \frac{h}{h - 1} \right) \left( h - 1 \right) \left( \frac{d\log(h - 1)}{h} \right) = N(b) \log(h - 1) = N(b) \log(\alpha - 1).
\]

The rest follows.

\[ \square \]

6.2.2 Preparation for the defect case

Let \( L|K \) be a defect extension as in 1.1. Recall that \( B = \cup_{\alpha \in \mathcal{J}} A[\alpha'] \) is a filtered union, where \( \mathcal{J} = \{ \alpha \in L \mid \alpha^p = h \in A^\times, h - 1 \in m_A \} \) and for each \( \alpha \in \mathcal{J} \), we have \( \gamma_\alpha = \gamma \in A \) such that \( \alpha' := \gamma \frac{\alpha - 1}{\alpha - 1} \in B^\times \). Since there is defect, we consider \( \Omega_B^1 \) instead of \( \omega_B^1 \) and \( \omega_A^1 \), respectively. Fix some \( \alpha_0 \in \mathcal{J} \) as the starting point. Let \( v(\alpha_0 - 1) - v(\tilde{\alpha}) = -v(\gamma_0) = -\mu < 0 \). We may only consider the subset \( \mathcal{J}_0 := \{ \alpha \in \mathcal{J} \mid v(\alpha - 1) > v(\alpha_0 - 1) \} \) of \( \mathcal{J} \).

Lemma 6.10. Let \( \alpha \in \mathcal{J}_0, \alpha^p = h_\alpha, \alpha_0^p = h_0 \). Let \( F_0(X) \) and \( F_0(X) \) denote the minimal polynomials over \( K \) of \( \alpha' \) and \( \alpha_0' \), respectively. Consider \( c_\alpha := F'_\alpha(\alpha'), c_0 := F'_0(\alpha_0') \) and the ratio \( \alpha_0\gamma_\alpha/c_\alpha = a_\alpha \in A \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\Omega_A[\alpha_0'][A] & \xrightarrow{\equiv} & A[\alpha_0']/\langle c_0 \rangle \\
\downarrow{\rho_\alpha} & & \downarrow{\iota_{\alpha}} \\
\Omega_A[\alpha'][A] & \xrightarrow{\equiv} & A[\alpha]/\langle c_{\alpha} \rangle
\end{array}
\]

\[
\begin{array}{ccc}
(\frac{1}{a_{\alpha}}) A[\alpha_0']/\langle c_0 \rangle & \xrightarrow{\equiv} & (\frac{1}{a_{\alpha}}) A[\alpha']/\langle c_{\alpha} \rangle \\
\downarrow{\iota_{\alpha}} & & \downarrow{\iota_{\alpha}}
\end{array}
\]

Here, the isomorphisms are given by \( b_0 d\alpha_0' \mapsto b_0 \mapsto b_0/a_0 \) and \( b d\alpha' \mapsto b \mapsto b/a_\alpha \) for all \( b_0 \in A[\alpha_0'] \) and \( b \in A[\alpha'] \). The vertical maps are given by multiplication by \( a_\alpha \).
$\alpha_0' = \gamma_0(\alpha_0 - 1)/3 = \gamma a(\alpha u - 1)/3 = \gamma a(\alpha - 1)/3 + \gamma a(u - 1)/3 = \alpha a' + \gamma_0(u - 1)/3$ (6.11)

Since $v(\alpha_0 - 1) = v(\alpha u - u + u - 1) < v(\alpha - 1), v(u - 1) = v(\alpha_0 - 1)$. Therefore, $\lambda = \gamma_0(u - 1)/3 \in A^\times$ and we have $d\alpha_0' = ada' + \alpha' da + d\lambda = ada'$, in $\Omega_{A[\alpha']|A}$.

Due to the defect, we have

$\mathcal{I}_\lambda = \mathcal{J}_\lambda = \left\{ \frac{\sigma(b)}{b} - 1 \mid b \in B^\times \right\} B = \left( \left\{ \sigma(b) - b \mid b \in B^\times \right\} \right) B$

By Theorem 5.1 and $v(\sigma(\alpha') - \alpha') = v\left( \left( \frac{\sigma做不到b}{b} \right) (-\lambda) \right) = v(\gamma_0 b)$, we have

$\mathcal{I}_\lambda = \mathcal{J}_\lambda = \left\{ \left( \left\{ \sigma(\alpha') - \alpha' \mid \alpha \in \mathcal{J}_\lambda \right\} \right) B = \left( \left\{ \gamma_0 \mid \alpha \in \mathcal{J}_\lambda \right\} \right) B \right.$ (6.12)

Lemma 6.13. Consider the fractional ideals $\Theta$ and $\Theta'$ of $L$ given by $\Theta = \{ x \in L \mid x\gamma_0 \in \mathcal{J}_\lambda \}$ and $\Theta' = \{ x \in L \mid x\gamma_0 \in \mathcal{J}_\lambda \}$. Then we have:

(a) $\Omega_{B|A} \cong \Theta / \Theta'$

(b) $\Theta / \mathcal{J}_\lambda \Theta \cong \mathcal{J}_\lambda / \mathcal{J}_\lambda^2$

Proof. (a) Let $I$ be the fractional ideal of $L$ generated by the elements $(\frac{h}{\alpha})$. Let $I'$ be the fractional ideal of $L$ generated by the elements $(\frac{h}{\alpha})$. Under the isomorphisms described in the preceding discussion, we can identify each $\Omega_{A[\alpha] | A}$ with $(\frac{h}{\alpha}) \beta A[\alpha]/(\frac{h}{\alpha}) A[\alpha]$. Taking limit over $\alpha$’s, we can identify $\Omega_{B|A}$ with $I / I'$. Since $-v(h_\alpha) = v(\gamma_0) - v(h_\alpha) = v(\gamma_0) - \mu, I = \Theta$. Similarly, $v(c_\alpha) = (p - 1) v(h_\alpha)$ implies that $v(\frac{h_\alpha}{c_\alpha}) = pv(\gamma_0) - \mu$ and hence, $I' = \Theta'$.

(b) This follows from the fact that $\Theta \cong \mathcal{J}_\lambda$ as $B$-modules, via the map $x\gamma_0 \alpha_0 : x \mapsto x\gamma_0 \alpha_0$.

6.2.3 Refined Swan Conductor and Proof of Theorem 1.5 in the defect case

Let $L|K$ be a defect extension as in 1.1 for the rest of this section.

Definition 6.14. Consider the ideals $\mathcal{I}_\alpha$ of $A$ defined for each $\alpha \in \mathcal{J}_\lambda$ by

$\mathcal{I}_\alpha := \left\{ x \in K \mid v(x) \geq \frac{(p - 1)}{p} v\left( \frac{h_\alpha}{h_\alpha - 1} \right) \right\}$

and let

$\mathcal{I} := \cup_{\alpha \in \mathcal{J}_\lambda} \mathcal{I}_\alpha$

We note that the definition of $\mathcal{I}_\alpha$ only depends on the valuation of $h_\alpha - 1$.

The refined Swan conductor $\text{rsw}$ of the extension $L|K$ is defined to be the $A$-homomorphism $\text{rsw} : \mathcal{H} \to \omega_A / \omega_A$ given by $r \mapsto \frac{r}{3^p} \delta \log h_\alpha$, where $r \in \left( \frac{h_\alpha}{h_\alpha - 1} \right)$ for some $\alpha \in \mathcal{J}_\lambda$.

We will show, as before, that this definition does not depend on the choice of $h_\alpha$.

Lemma 6.15. (i) The map $\text{rsw}$ in this case, is well-defined.
(ii) For each $\alpha \in \mathcal{I}_0$, $\left(\frac{3^p}{h_{\alpha-1}}\right) \subset \mathbb{I}_\alpha \subset \left(\frac{1}{\alpha-1}\right) A[\alpha'] \cap A$

(iii) $\mathcal{H} \subset \mathbb{I} \subset \mathcal{I}_\sigma \cap A$

Proof. (i) Let $h_1 = h_{\alpha_1}, h_2 = h_{\alpha_2}$ for some $\alpha_1, \alpha_2 \in \mathcal{I}_0$ and $r \in \left(\frac{3^p}{h_{\beta_1}}\right) \cap \left(\frac{3^p}{h_{\beta_2}}\right)$. It is enough to focus on the case when $v(h_1 - 1) \neq v(h_2 - 1)$. We imitate the proof of Lemma 6.5. Let $h_2 = a^p h_1; a \in A^\times, a \neq 1$ and without loss of generality, assume that $v(h_1 - 1) < v(h_2 - 1)$. It is enough to show that $v(a^p - 1) \geq v(h_1 - 1)$. Since

$$v(h_2 - 1) = v(a^p h_1 - 1) = v((a^p - 1) h_1 + (h_1 - 1)) > v(h_1 - 1),$$

we must, in fact, have $v(a^p - 1) = v(h_1 - 1)$. Hence, rsw is well-defined in this case.

(ii) The first part $\left(\frac{3^p}{h_{\alpha-1}}\right) \subset \mathbb{I}_\alpha$ is easy to see. The next part follows from

$$v\left(\frac{3}{\alpha - 1}\right) = \left(\frac{1}{p}\right) v\left(\frac{3^p}{h_{\alpha-1}}\right) \leq \left(\frac{p - 1}{p}\right) v\left(\frac{3^p}{h_{\alpha-1}}\right).$$

(iii) This follows directly from (ii), since $\mathcal{I}_\sigma = \mathcal{I}_\varphi = \left\{\left(\frac{3^p}{h_{\alpha-1}}\right) | \alpha \in \mathcal{I}_0\right\}$.  \hfill $\Box$

Proof of Theorem 1.5 for the defect case. Let $L|K$ be a defect extension as in 1.1. Lemma 6.10 and Lemma 6.13 allow us to write $\Omega_{B|A}^1 = \lim_{\alpha \to \mathcal{I}_0} \Omega_{A[\alpha]|A}^1$ and it is enough to consider the diagram for each $\alpha \in \mathcal{I}_0$:

$$\Omega_{A[\alpha']|A}^1 \left(\frac{3^p}{\alpha-1}\right) A[\alpha'] \xrightarrow{\varphi_\sigma} \Omega_{A[\alpha']|A}^1 \left(\frac{3^p}{\alpha-1}\right) A[\alpha'] \xrightarrow{\delta N} \Omega_{A}^1 \xrightarrow{\text{rsw}} \Omega_{A}^1 \left(\frac{3^p h_{\alpha}}{h_{\alpha-1}}\right) A,$$

where the maps are given by

$$bd_{\alpha'} \rightarrow \varphi_\sigma \rightarrow \varphi_\sigma \rightarrow \delta N \rightarrow N(\alpha') \rightarrow N(\alpha) \left(\frac{3^p h_{\alpha}}{h_{\alpha-1}}\right) \rightarrow$$

We note that in $\omega_{B|A}^1, \text{dlog}(\alpha - 1) = \text{dlog}(\alpha' + \text{dlog} \left(\frac{3^p}{\gamma_0}\right)) = \text{dlog}(\alpha') = \frac{\text{dlog}(\alpha')}{\alpha'} - 1 = \frac{3^p}{\alpha-1}$. At each $\alpha$-level, we observe the following:

(i) The map $\varphi_\sigma : \Omega_{A[\alpha']|A}^1 \left(\frac{3^p}{\alpha-1}\right) A[\alpha'] \rightarrow \left(\frac{3^p}{\alpha-1}\right) A[\alpha']$ is same as the one obtained from Lemma 6.13. This can be proved as follows.

By Lemma 6.13, $\Omega_{A[\alpha']|A}^1 \left(\frac{3^p}{\alpha-1}\right) A[\alpha'] \cong \left(\frac{1}{\alpha_0}\right) \left(\frac{3^p}{\alpha-1}\right) A[\alpha'] \cong \left(\frac{3^p}{\alpha-1}\right) A[\alpha']$ under the composition $d\alpha' \rightarrow \frac{1}{\alpha_0} \rightarrow \gamma_0 \alpha_0 \frac{1}{\alpha_0} = \alpha' \gamma_0$.

On the other hand, $\varphi_\sigma(d\alpha') = \alpha' \left(\frac{3^p}{\alpha-1}\right) = \alpha' \gamma_0$.

(ii) The map rsw is well-defined. We just need to verify that for $h = h_{\alpha}, \text{rsw} \left(\frac{3^p h_{\alpha}}{h_{\alpha-1}}\right) = \text{dlog}(h - 1)$. By definition,

$$\text{rsw} \left(\frac{3^p h_{\alpha}}{h_{\alpha-1}}\right) := \frac{h}{(h - 1)} \left(\frac{\text{dlog}(h - 1)}{h}\right) = \text{dlog}(h - 1) = \text{dlog} N(\alpha - 1).$$
7 Results for the non-Kummer Case

In the $p = 2$ case, we always have $\zeta = -1 \in K$. For the rest of this section, we will assume $p > 2$.

**Notation 7.1.** Let $K'$ be a valued field of characteristic 0 with henselian valuation ring $A'$, valuation $v'$ and residue field $k'$ of characteristic $p > 0$. Consider a non-trivial Galois extension $L'|K'$ of degree $p$, with Galois group $G' := \text{Gal}(L'|K')$. Let $w'$, $B'$, $l'$ denote the valuation, valuation ring and the residue field of $L'$. We consider the fields $K := K'(\zeta)$, $L := L'(\zeta)$ and the Kummer extension $L|K$ described by $\alpha^p = h$ for some $h \in K$.

The Galois group $G := \text{Gal}(L|K)$ is cyclic of order $p$, generated by $\sigma : \alpha \mapsto \zeta \alpha$. Let $\Lambda_K := \text{Gal}(K|K')$ and $\Lambda_L := \text{Gal}(L|L')$, we will omit the subscripts when the meaning is clear. Note that the order of $\Lambda$ is coprime to $p$. We will use the notation of 1.1 for the extension $L|K$.

7.1 Invariants for $L'|K'$

First we define the corresponding invariants for the extension $L'|K'$ as follows.

\[ I' := \langle \{\sigma(b) - b \mid b \in B'\} \rangle \subset B' \] (7.2)

\[ J' := \left( \left\{ \frac{\sigma(b)}{b} - 1 \mid b \in L'^{\times} \right\} \right) \subset B' \] (7.3)

\[ N' := \left( N_{L'|K'}(J') \right) \subset A' \] (7.4)

\[ \mathcal{H}' := \langle \mathcal{H} \rangle^\Lambda \subset A' \] (7.5)

We prove the following lemma in order to prove Proposition 7.7 and further results.

**Lemma 7.6.** Let $L|L'$ be as above, $[L : L'] = m$, where $m$ is a positive integer coprime to $p$. Assume that $L|L'$ is either unramified or totally ramified. Then there exists an $L'$-basis $\{b_i\}_{1 \leq i \leq m}$ of $L$ that satisfies the following properties.

(B1) $\{b_i\}_{1 \leq i \leq m}$ is also a $K'$-basis of $K$.

(B2) $\{b_i\}_{1 \leq i \leq m} \subset A$

(B3) If $L|L'$ is totally ramified, the valuations $\{w(b_i)\}_{1 \leq i \leq m}$ are all distinct modulo the value group of $L'$. If $L|L'$ is unramified, the residue classes $\{b_i\}_{1 \leq i \leq m}$ form a basis of the residue extension $l|l'$.

(B4) For any $0 \neq x = \sum_{i=1}^{m} x_i b_i; x_i \in L'$, we have $w(x) = \min_i w(x_i b_i)$.

(B5) For any $0 \neq x = \sum_{i=1}^{m} x_i b_i; x_i \in L'$ as above, $x \in B$ $\iff$ $x_i b_i \in B$ for all $1 \leq i \leq m$.

**Proof.** (B1-3): If $L|L'$ is totally ramified, the ramification indices $e_{L|L'}$ and $e_{K'|K'}$ are both equal to $m$. We can choose $m$ elements $\{b_i\}_{1 \leq i \leq m}$ of $K$ that have distinct valuations modulo the value group of $K'$. Without loss of generality, we may assume that they have non-negative valuations.

If $L|L'$ is unramified, $[l : l'] = m = [k : k']$ and we can choose units $\{b_i\}_{1 \leq i \leq m}$ of $K$ satisfying the required conditions.

(B4): If $L|L'$ is totally ramified, $w(x_i b_i)$ are all distinct by (B3), and therefore, exactly one term achieves the minimum valuation.

If $L|L'$ is unramified, it is possible for more than one term to have the minimum valuation. However, $x$ cannot have a greater valuation. This can be proved as follows. Without loss of generality, let $w(x_1 b_1) = \min_i w(x_i b_i)$. If $w(x) > w(x_1 b_1) = w(x_1) = \sum_{i=2}^{m} \left( \frac{x_i}{x_1} \right) b_i \in m_L$. Since $\overline{b_i}$ are $l'$-linearly independent, this is not possible.

(B5): This follows from (B4).
Proposition 7.7. We have the following relations between the invariants for $L|K$ and the invariants for $L'|K'$

1. $\mathcal{J}_\sigma = \mathcal{J}'B$
2. $(\mathcal{J}_\sigma)^A = \mathcal{J}'$
3. $N_\sigma = N'\alpha$
4. $(N_\sigma)^A = N''$
5. $(\mathcal{I}_\sigma)^A = \mathcal{I}'$

Proof. Let $[L : L'] = m = e_0f_0$, where $m$ is a positive integer coprime to $p$, $e_0$ is the ramification of $L|L'$ and $f_0$ is the inertia degree of $L|L'$. It is enough to consider the two cases where $L|L'$ is either unramified or totally ramified. This can be seen by considering the two extensions $T|L'$ and $L|T$, where $T$ is the maximal unramified subextension of $L|L'$.

Let $\{b_i\}_{1 \leq i \leq m}$ be an $L'$-basis of $L$ as described in Lemma 7.6.

1. Let $x = \sum_{i=1}^m x_ib_i; x_i \in L'$ be an element of $L'$. Since $\sigma$ fixes each $b_i$ (by (B1)), we have

$$\frac{\sigma(x)}{x} - 1 = \sum_{i=1, x_i \neq 0}^m \left( \frac{\sigma(x_i b_i)}{x_i b_i} - 1 \right) \frac{x_ib_i}{x_i} = \sum_{i=1, x_i \neq 0}^m \left( \frac{\sigma(x_i)}{x_i} - 1 \right) \frac{x_ib_i}{x_i}$$

For each $i$ with $x_i \neq 0$, $\frac{\sigma(x_i)}{x_i} - 1 \in J'$ and $\frac{x_ib_i}{x_i} \in B$ (by (B4)). Thus, $J_\sigma \subset J'B$. The reverse direction is trivial.

2. It is clear that $(\mathcal{J}_\sigma)^A$ contains $\mathcal{J}'$. For the reverse direction, consider the action of $\text{Tr}_{L|L'}$ on $\mathcal{J}_\sigma$. For $x$ as above,

$$\text{Tr}_{L|L'} \left( \frac{\sigma(x)}{x} - 1 \right) = \sum_{i=1, x_i \neq 0}^m \left( \frac{\sigma(x_i)}{x_i} - 1 \right) \text{Tr}_{L|L'} \left( \frac{x_ib_i}{x_i} \right) \in \mathcal{J}'$$

On the other hand, $\text{Tr}_{L|L'}$ acts on $(\mathcal{J}_\sigma)^A$ as multiplication by $m$. The rest follows from $(m, p) = 1$.

3. By 1, for any $x \in \mathcal{J}_\sigma = J'B$, there exists $y \in J'$ such that $x \in (y)B$. Thus, $N_\sigma = (N_L|K'(J')) A = N''A$

4. This follows from 2 and 3.

5. For any $x \in B$, $\text{Tr}_{L|L'} ((\sigma - 1)(x)) = (\sigma - 1) (\text{Tr}_{L|L'}(x)) \in \mathcal{I}'. $ The rest of the proof is quite similar to the proof of 2.

7.2 Main Results for $L'|K'$

Observe that $L'|K'$ and $L|K$ have the same defect. We have the analogues of the main results as follows.

Theorem 7.8. $\mathcal{H}' = N'$.

Proof. This is a direct consequence of Theorem 1.3 and Proposition 7.7.

Theorem 7.9. By taking $\Lambda$-invariant parts of the commutative diagram

$$\omega^1_{\mathcal{I}|A} / \mathcal{J}_\sigma \omega^1_{\mathcal{I}|A} \xrightarrow{\phi_\sigma} \mathcal{J}_\sigma / \mathcal{J}'^2 \xrightarrow{N_L|K} \mathcal{H} / \mathcal{H}^2$$
we have the following commutative diagram for $L'|K'$:

\[
\begin{array}{cccc}
\omega_{B'|A'|}^{1}/\mathcal{J}'\omega_{B'|A'}^{1} & \xrightarrow{\phi'} & \mathcal{J}'/\mathcal{J}'^2 \\
\xrightarrow{\Delta_N} & & & \xrightarrow{N_{L'|K'}} \\
\omega_{A'|(\mathcal{J}' \cap A')}^{1}/\omega_{A'}^{1} & \xleftarrow{rsw'} & \mathcal{H}'/\mathcal{H}'^2
\end{array}
\]

The maps $\Delta_N, \Delta_N'$ are induced by the norm maps $N_{L|K}$ and $N_{L'|K'}$, while the map $rsw'$ is the restriction of the map $rsw$ to $\mathcal{H}'/\mathcal{H}'^2$.

\textbf{Proof.} Validity and properties of the map $rsw'$ follow from the commutativity of the first diagram and properties of the map $rsw$. The rest follows from Proposition 7.7. \hfill \Box

\section{8 Generalizing the Results of [VT16] to Defect Extensions of Rank $> 1$}

In [VT16], we proved the main results under the assumption that the Artin-Schreier extension $L|K$ is defectless or has valuation of rank 1. However, we observed the following.

- In the case $p = 2$, the results were true regardless of the rank of the valuation. This led us to believe that the results should be true for defect extensions of higher rank, even when $p > 2$.

- Many of the key lemmas, such as Proposition 3.8, were proved without using the condition on the rank.

- If we could prove the result $\mathcal{H} = \mathcal{N}_r$ independent of the rank, the rest would follow.

We can easily modify the proof of Theorem 1.3 presented in 4.2 to fit the Artin-Schreier case. Similarly, we can imitate the proof of Lemma 6.13 and thus, the main results of [VT16] can be generalized to the higher rank defect case.

\textbf{Acknowledgments:} I am very grateful to Professor Kazuya Kato (University of Chicago) for his invaluable advice, helpful feedback during the writing process, and his constant support during the project.

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