A REMARK ON THE KOTTWITZ HOMOMORPHISM

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Abstract. We prove that for any split, almost simple, connected reductive group \( G \) over a \( p \)-adic field \( F \), the Kottwitz homomorphism \( \kappa : G(F) \to \Omega \) exhibits a homomorphic section \( \Omega \to G(F) \). We then extend this result to certain additional split connected reductive groups.

1. Introduction

Let \( G \) be a connected reductive group over a \( p \)-adic field \( F \). In [Kot97], Kottwitz defined a canonical homomorphism

\[
\kappa : G(F) \to X^*(Z(\hat{G}))^{Fr}.
\]

This homomorphism is surjective and, in the case that \( G \) is split, simplifies to a homomorphism

\[
\kappa : G(F) \to X^*(Z(\hat{G})) \cong X_*(T)/Q^\vee.
\]

In this note, we show that the map \( \kappa \) has a homomorphic section in the case that \( G \) is split and almost simple, as well as for certain additional split groups. More specifically, fix a fundamental alcove in the building of \( G \) corresponding to a maximal split torus \( T \), and let \( \Omega \) be the subgroup of the extended affine Weyl group \( W \) that stabilizes \( C \). We show that there is a homomorphic section of the canonical projection \( N_G(T) \to \Omega \), where \( N_G(T) \) is the normalizer of a maximal torus \( T \) in \( G \). If \( G \) is almost-simple, then this section can be described as follows: it is known (see Proposition 3.3) that \( \Omega \) may be identified with a collection of elements \( \{1, \epsilon_i \times w_i \} \subseteq W = X_*(T) \rtimes W_o \), where \( \epsilon_i \) are certain fundamental coweights and \( W_o \) is the finite Weyl group. By [Spr98, §9.3.3], there is a canonical map \( \mathcal{N}_o : W_o \to N_G(T) \) (denoted \( \phi \) in loc. cit.) that is compatible with the projection \( N_G(T) \to W_o \). We may then consider the map

\[
\iota : \Omega \to N_G(T)
\]

\[
\epsilon_i w_i \mapsto \epsilon_i (\varpi^{-1}) \mathcal{N}_o (w_i),
\]

where \( \varpi \) is a uniformizer in \( F \). The map \( \iota \) is a section of the projection \( N_G(T) \to \Omega \), and it turns out that \( \iota \) is a homomorphism in all cases except the adjoint group of type \( D_l \) where \( l \) is odd, and some cases in type \( A_l \) (see Theorem 3.5 and Remark 3.6). Nonetheless, we can still use \( \iota \) to construct a homomorphic section for all almost-simple \( p \)-adic groups (see Theorem 3.4). We then show that for certain split connected groups with connected center, the Kottwitz homomorphism exhibits a homomorphic section (see Proposition 4.3).

We would like to remark that if \( G \) is any split connected reductive group with simply connected derived group, then \( \kappa \) has a homomorphic section. This follows from the fact that \( \Omega \) is a free abelian group of finite rank, isomorphic to a free quotient of \( X_*(T) \). Then one constructs a section by taking a homomorphic section of \( X_*(T) \to \Omega \) and then composing that section with the map \( X_*(T) \to T, \lambda \mapsto \lambda(\varpi^{-1}) \). In particular, the image of this section lies in \( T \), not just \( N_G(T) \). The situation where \( \Omega \) is finite is much more subtle, which is what this paper is about.

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2. Preliminaries

Let $G$ be a split connected reductive group over a $p$-adic field $F$. Fix a pinning $(B,T,\{X_\alpha\})$ for $G$. This gives rise to a set of non-zero roots $\Phi$ of $G$ with respect to $T$, a set of positive roots $\Pi$ in $\Phi$, and a basis $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ of the set of positive roots, so that $l$ is the rank of $G$. We recall that for each $\alpha \in \Phi$, there exists an isomorphism $u_\alpha$ of $F$ onto a unique closed subgroup $U_\alpha$ of $G$ such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$, for $t \in T, x \in F$ [Spr98 §8.1.1].

Let $X^*(T), X_*(T)$ be the character, cocharacter lattices of $T$, respectively. Let $Q$ be the lattice generated by $\Phi$, and $P^\vee$ the coweight lattice. Namely, $P^\vee$ is the $Z$-dual of $Q$ relative to the standard pairing $(\cdot, \cdot) : X^*(T) \times X_*(T) \to Z$. We let $\Phi^\vee$ be the set of coroots, $Q^\vee$ the lattice generated by $\Phi^\vee$, and $P$ the weight lattice. Then $P^\vee$ is spanned by the $l$ fundamental coweights, which are denoted $\epsilon_1, \epsilon_2, ..., \epsilon_l$. We recall that the $\epsilon_i$ are defined by the relation $(\epsilon_i, \alpha_j) = \delta_{ij}$. If $\alpha$ is a root, we denote its associated coroot by $\alpha^\vee$.

We now let $W_\circ = N_G(T)/T$ be the Weyl group of $G$ relative to $T$. For each $\alpha \in \Phi$, we let $s_\alpha \in W$ be the simple reflection associated to $\alpha$. Then the $u_\alpha$ may be chosen such that for all $\alpha \in \Pi$, $u_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ lies in $N_G(T)$ and has image $s_\alpha$ in $W_\circ$ (see [Spr98 §8.1.4]). Relative to the pinning that we have chosen, there is a canonical, well-defined map $\mathcal{N}_\circ : W_\circ \to N_G(T)$ [Spr98 §9.3.3] (the map is denoted $\phi$ in loc.cit.), defined by $\mathcal{N}_\circ(w) = n_{\beta_1}n_{\beta_2} \cdots n_{\beta_m}$ for a reduced expression $w = s_{\beta_1}s_{\beta_2} \cdots s_{\beta_m}$.

2.1. The map $\mathcal{N}_\circ$. In this section, we recall a result about the map $\mathcal{N}_\circ$ from [Ros16].

Definition 2.1. For $u,v \in W_\circ$, we define
\[ \mathcal{F}(u,v) = \{ \alpha \in \Pi \mid v(\alpha) \in -\Pi, u(v(\alpha)) \in \Pi \}. \]

The following proposition describes the failure of $\mathcal{N}_\circ$ to be a homomorphism.

Proposition 2.2. [Ros16] Proposition 3.1.2] For $u,v \in W_\circ$,
\[ \mathcal{N}_\circ(u) \cdot \mathcal{N}_\circ(v) = \mathcal{N}_\circ(u \cdot v) \cdot \prod_{\alpha \in \mathcal{F}(u,v)} \alpha^\vee(-1). \]

Definition 2.3. For $w \in W_\circ, i \in \mathbb{N}$, we define
\[ \mathcal{F}_w(i) = \{ \alpha \in \Pi \mid w^i(\alpha) \in -\Pi, w^{i+1}(\alpha) \in \Pi \}. \]

Corollary 2.4. If $w \in W_\circ$ and $n \in \mathbb{N}$, then
\[ \mathcal{N}_\circ(w)^n = \mathcal{N}_\circ(w^n) \cdot \prod_{m=1}^{n-1} \prod_{\alpha \in \mathcal{F}_w(m)} \alpha^\vee(-1). \]

Proof. By Proposition 2.2, $\mathcal{N}_\circ(w)^2 = \mathcal{N}_\circ(w^2) \cdot \prod_{\alpha \in \mathcal{F}_w(1)} \alpha^\vee(-1)$. Multiplying by $\mathcal{N}_\circ(w)$ on the left and using Proposition 2.2 again, we get $\mathcal{N}_\circ(w)^3 = \mathcal{N}_\circ(w^3) \cdot \prod_{\alpha \in \mathcal{F}_w(2)} \alpha^\vee(-1) \cdot \prod_{\alpha \in \mathcal{F}_w(1)} \alpha^\vee(-1)$. Continuing in this way, the claim follows. \qed

3. Embedding $\Omega$ into $G$

Let $G$ be a split, almost-simple $p$-adic group. We set $W = N_G(T)/T_0$, where $T_0$ is the maximal bounded subgroup of $T$. The group $W$ is the extended affine Weyl group, and we note that we have a semidirect product decomposition $W = X_*(T) \rtimes W_\circ$. We also set $\Omega = W/W^\circ$, where $W^\circ = Q^\vee \rtimes W_\circ$ is the affine Weyl group. We therefore have a canonical projection $N_G(T) \to \Omega$. This projection is exactly the restriction of $\kappa$ to $N_G(T)$.

The group $\Omega$ can be identified with the subgroup of $W$ that stabilizes a fundamental alcove $C$. Moreover, it is known that $\Omega$ acts on the set $\{1 - \alpha_0, \alpha_1, \alpha_2, ..., \alpha_l\}$, where $\alpha_0$ is the highest root in $\Phi$. The action of $\Omega$ on this set can be found in [M65, p. 18-19]. We let $\Omega_{\text{ad}}$ be the analogous group for the adjoint group $G_{\text{ad}}$.

It is known that there exists in $W_\circ$ an element $w_\Delta$ such that $w_\Delta(\Delta) = -\Delta$. The element $w_\Delta$ is unique and satisfies $w_\Delta^2 = 1$. Moreover, if we denote the subset $\Delta - \{\alpha_i\}$ by $\Delta_i$, then the subgroup $W_i$ of $W_\circ$ generated
by $s_{\alpha_1}, ..., \hat{s}_{\alpha_i}, ..., s_{\alpha_i}$ ($\hat{s}_{\alpha_i}$ means that $s_{\alpha_i}$ is omitted) contains an element $w_\Delta$, such that $w_\Delta(\Delta_i) = -\Delta_i$ and $w_\Delta^2 = 1$.

We recall the following result from [IM65].

Proposition 3.1. [IM65 Proposition 1.18] The mapping from the set $\{0\} \cup \{\epsilon_i : (\alpha_0, \epsilon_i) = 1\}$ onto $\Omega_{\text{ad}}$ defined by $0 \mapsto 1, \epsilon_i \mapsto \epsilon_i w_\Delta, w_\Delta$ is bijective.

The notation $\rho_i$ (and sometimes $\rho$) is used in [IM65] to denote the element $\epsilon_i w_{\Omega_1}, w_\Delta$. We will adopt the same notation. We will also let $S_{\text{ad}}$ denote the set $\{0\} \cup \{\epsilon_i : (\alpha_0, \epsilon_i) = 1\}$. We note that every lattice between $Q^\vee$ and $P^\vee$ arises as $(Q^\vee, S)$, for some subset $S \subseteq S_{\text{ad}}$. If $S$ is such a subset, we will talk of the almost-simple $p$-adic group $G$ that is determined by the lattice $(Q^\vee, S)$. We note in particular that if $\Omega_G$ denotes the omega group for $G$, then one can see that $\Omega_G = W/W^0 \cong (1, \rho_i : \epsilon_i \in S)$.

We now assume that $G$ is not simply connected. For otherwise, $\Omega = 1$, so the claim that $\kappa : G \to \Omega$ has a homomorphic section is vacuous.

Let $\varpi$ be a uniformizer of $F$. There is a natural map $X_*(T) \to N_G(T)$ given by $\lambda \mapsto \lambda(\varpi^{-1})$ (see [IM67] p. 31). We also have the map $N_\varphi : W \to N_G(T)$. Coupling these maps together, we obtain a natural map

$$W \to N_G(T)$$

$$(\lambda, w) \mapsto \lambda(\varpi^{-1})N_\varphi(w)$$

for $\lambda \in X_*(T), w \in W$. Most of the time, we will write $\lambda w$ instead of $(\lambda, w)$. Proposition 3.1 gives us a set-theoretic embedding $\Omega \hookrightarrow W$. We can then consider the composite map $\Omega \hookrightarrow W \to N_G(T)$, which gives us a section of the canonical projection $N_G(T) \to \Omega$:

$$\iota : \Omega \to N_G(T)$$

$$\omega = \epsilon_i w_\Delta, w_\Delta \mapsto \epsilon_i(\varpi^{-1})N_\varphi(w_\Delta, w_\Delta)$$

That $\iota(\omega)$ is well-defined follows from the fact that $\epsilon_i \in X_*(T)$ by our definition of $G$ earlier. That $\iota$ is a section follows from Proposition 3.1. In particular, $\iota$ is injective. We will sometimes identify $\epsilon_i$ with $\epsilon_i(\varpi^{-1})$ for ease of notation.

We will show that $\iota$ is a homomorphic embedding for all types except $A_l$, and the specific case when $G$ is adjoint of type $D_l$ where $l$ is odd. Nonetheless, we will still produce a homomorphic embedding $\Omega \hookrightarrow N_G(T)$ which is a section of $N_G(T) \to \Omega$, in these two outlier cases.

Suppose $\omega = \epsilon_i w_\Delta, w_\Delta$ is a generator of $\Omega$, whose order is $r$. Propositions 3.2 and 3.3 will be dedicated to showing that $\iota(\omega)$ also has order $r$. Let $w_i = w_\Delta, w_\Delta$ for convenience of notation. We compute

$$\iota(\omega)^r = (\epsilon_i(\varpi^{-1})N_\varphi(w_i))^r = \epsilon_i(\varpi^{-1}) \cdot (N_\varphi(w_i)\epsilon_i(\varpi^{-1})N_\varphi(w_i)^{-1}) \cdot (N_\varphi(w_i)^2 \epsilon_i(\varpi^{-1})N_\varphi(w_i)^{-2})$$

$$\cdots \cdot (N_\varphi(w_i)^{r-1} \epsilon_i(\varpi^{-1})N_\varphi(w_i)^{1-r})N_\varphi(w_i)^r = (\epsilon_i + w_i(\epsilon_i) + w_i^2(\epsilon_i) + \cdots + w_i^{r-1}(\epsilon_i))(\varpi^{-1})N_\varphi(w_i)^r.$$ We will now show that $\epsilon_i + w_i(\epsilon_i) + w_i^2(\epsilon_i) + \cdots + w_i^{r-1}(\epsilon_i) = 0$ and $N_\varphi(w_i)^r = 1$.

Proposition 3.2. $\epsilon_i + w_i(\epsilon_i) + w_i^2(\epsilon_i) + \cdots + w_i^{r-1}(\epsilon_i) = 0$.

Proof. We compute $w_i^j(\epsilon_i)$ for $j = 1, 2, ..., r - 1$. The tables on pages 18-19 of [IM65] give the values of $i$ for each type, and the explicit action of $w_i$ on the set $\{-\alpha_0, \alpha_1, \alpha_2, ..., \alpha_l\}$. We also note that the order of $\omega$ equals the order of $w_i$ (see [IM65] p. 18). We begin with type $B_l$ and end with type $A_l$ (since, computationally, $A_l$ is the most intricate).

- In type $B_l$, we have that $r = 2$ and $i = 1$, so we wish to show that $\epsilon_1 + w_1(\epsilon_1) = 0$. Note that $w_1(\alpha_1) = -\alpha_0$ and $w_1$ fixes the other simple roots. To compute $w_1(\epsilon_1)$, we pair $w_1(\epsilon_1)$ with all of the simple roots. By Weyl-invariance of the inner product $(\cdot, \cdot)$ and the fact that $w_1^2 = 1$, we have

$$(w_1(\epsilon_1), \alpha_j) = \begin{cases} 
0 & \text{if } j \neq 1 \\
(\epsilon_1, -\alpha_0) & \text{if } j = 1
\end{cases}$$

But since $\alpha_0 = \alpha_1 + 2(\alpha_2 + ... + \alpha_l)$, we have that $(\epsilon_1, -\alpha_0) = 1$. Therefore, $w_1(\epsilon_1) = -\epsilon_1$, so that $\epsilon_1 + w_1(\epsilon_1) = 0$.

- In type $C_l$, we have that $i = l$, and the same argument as in type $B_l$ holds. Indeed, $w_l(\alpha_1) = -\alpha_0$, $w_l$ permutes the simple roots other than $\alpha_1$, and $\alpha_0 = 2(\alpha_1 + ... + \alpha_{l-1}) + \alpha_l$. Thus, $w_l(\epsilon_1, \alpha_1) = -1$, so $w_l(\epsilon_1) = -\epsilon_1$, and the result follows.
• We consider type $D_l$. We first consider the case that $l$ is odd and $G$ is adjoint. In this case $\Omega \cong \mathbb{Z}/4\mathbb{Z}$ and it is enough to consider $i = l$. The claim is that $\epsilon_i + w_i(\epsilon_i) + w_i^2(\epsilon_i) + w_i^3(\epsilon_i) = 0$. One can see from the table on [IM65] p. 19 that $w_l$ permutes $\alpha_2, \alpha_3, ..., \alpha_{l-2}$, and also acts by $-\alpha_0 \mapsto \alpha_1 \mapsto \alpha_1 \mapsto \alpha_{l-1} \mapsto -\alpha_0$. We therefore conclude that $(w_l(\epsilon_i), \omega_j) = (\epsilon_i, w_l^2(\alpha_j)) = 0$ if $j = 2, 3, ..., l - 2$. Moreover, since $w_l^2(\alpha_1) = \alpha_1, w_l^2(\alpha_{l-1}) = \alpha_1$, $w_l^2(\alpha_l) = -\alpha_0$, we conclude that $(w_l(\epsilon_i), \alpha_1) = 1, (w_l(\epsilon_i), \alpha_{l-1}) = 0$, and $(w_l(\epsilon_i), \alpha_l) = -1$. Therefore, $w_l(\epsilon_i) = \epsilon_l - \epsilon_l$. One can compute similarly that $w_l^2(\epsilon_i) = \epsilon_{l-1} - \epsilon_l$ and $w_l^3(\epsilon_i) = -\epsilon_{l-1}$. Therefore, $\epsilon_i + w_l(\epsilon_i) + w_l^2(\epsilon_i) + w_l^3(\epsilon_i) = 0$.

We now consider the case where $l$ is odd and $G$ is neither adjoint nor simply connected. We have that $\rho_l^2 = \rho_l$ generates $\Omega$. Thus, we need to show that $\epsilon_1 + w_1(\epsilon_1) = 0$. First, we note that $w_l$ fixes $\alpha_j$, for $j = 2, 3, ..., l - 2$, it exchanges $-\alpha_0$ and $\alpha_1$, and it exchanges $\alpha_{l-1}$ and $\alpha_l$. Since $w_l$ has order 2, we compute that

\[(w_1(\epsilon_1), \alpha_j) = (\epsilon_1, w_1(\alpha_j)) = \begin{cases} 0 & \text{if } j \neq 1, 6 \\ -1 & \text{if } j = 1 \\ 1 & \text{if } j = 6 \end{cases}\]

which implies that $w_1(\epsilon_1) = -\epsilon_1 + \epsilon_6$. Similarly one may compute that $w_l^2(\epsilon_1) = -\epsilon_6$. Therefore, $\epsilon_1 + w_1(\epsilon_1) + w_l^2(\epsilon_1) = 0$.

• Type $E_l$ is analogous to types $B_l$ and types $C_l$. Just note that in this case we have $i = 1$ and $w_1(\alpha_1) = -\alpha_0$, and from [IM65] p. 19 we see that the coefficient of $\alpha_1$ in $\alpha_6$ is 1.

• We finally consider type $A_l$. We may identify roots and co-roots, fundamental weights and fundamental coweights. Recall that we may take $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ to be $\alpha_i = \alpha_i^\vee = e_i - e_{i+1}$ for $1 \leq i \leq l$. The corresponding fundamental coweights are

$\epsilon_i = \epsilon_i^\vee = \{\frac{1}{l}{(l+1-i)}(e_1 + e_2 + ... + e_i) - i(e_{i+1} + e_{i+2} + ... + e_{l+1})\} \quad \text{if} \quad i \leq l$

We recall that the isogenies of type $A_l$ are in one to one correspondence with the subgroups of $\Omega_{ad} = \mathbb{Z}/(l+1)\mathbb{Z}$. The element $\rho_l$ generates $\Omega_{ad}$. Let $a, b \in \mathbb{N}$ such that $l + 1 = ab$. Let $\omega = \rho_l^b$, so that $\omega^b = 1$. Let $G$ be the group of type $A_l$ that is given by the subgroup $\langle \omega \rangle$ of $\Omega_{ad}$. In particular, its associated cocharacter lattice, which we denote by $X_*(A_l^\vee)$, is given by \langle $Q^\vee, e_a$ \rangle. For ease of notation, let $n = l + 1$. Then $\omega = e_a w_a$, where $w_a$ is the a-th power of the n-cycle (1 2 ... n). We need to show that $\epsilon_a + w_a(\epsilon_a) + ... + w_a^{b-1}(\epsilon_a) = 0$. A computation shows that this sum is

$\frac{1}{n}[(n-a)(e_1 + e_2 + ... + e_a) - a(e_{a+1} + ... + e_n)]$

$+ \frac{1}{n}[(n-a)(e_{a+1} + e_{a+2} + ... + e_{2a}) - a(e_{2a+1} + ... + e_n + e_1 + e_2 + ... + e_a)]$

$+ ... + \frac{1}{n}[(n-a)(e_{n-a+1} + e_{n-a+2} + ... + e_n) - a(e_1 + e_2 + ... + e_{n-a})],$

which equals zero.

\[\Box\]

Proposition 3.3. $N_{\sigma}(w_i)^r = 1$. 

Proof. We again assume that $G$ is not simply connected. We proceed on a type by type basis, beginning with $B_l$ and ending again with type $A_l$. To compute $\mathcal{N}_0(w_l)^2$, we use Corollary [2,4]. We remind the reader again that the order of $\epsilon_l w_l$ equals the order of $w_l$ (see [LM65, p. 18]).

- Suppose that $G$ is of type $B_l$ and adjoint. We recall that the roots may be identified with the functionals $\pm \epsilon_i (1 \leq i \leq l)$ and $\pm \epsilon_i \pm \epsilon_j (1 \leq i < j \leq l)$. The corresponding coroots may be identified (in the obvious way) with the functionals $\pm 2\epsilon_i, \pm \epsilon_i \pm \epsilon_j$. The fundamental coweights corresponding to the standard choice of simple roots are given by $\epsilon_i = \epsilon_1 + ... + \epsilon_i$, for $1 \leq i \leq l$.

We note that the cocharacter lattice of type $B_l$ adjoint is $\langle Q^+, \epsilon_1 \rangle$. The action of $w_1$ exchanges $-\epsilon_0$ and $\epsilon_1$. Therefore, $F_{w_1}(1)$ is the set of positive roots that contain $\epsilon_1$. In other words, $F_{w_1}(1) = \{e_1 + e_j : j = 2, 3, ..., l\} \cup \{e_1 - e_j : j = 2, 3, ..., l\} \cup \{e_1\}$. One may therefore compute that

$$\sum_{\alpha \in F_{w_1}(1)} \alpha^\vee = \sum_{j>1} (e_1 + e_j)^\vee + \sum_{j>1} (e_1 - e_j)^\vee + e_1^\vee = \sum_{j>1} (e_1 + e_j) + \sum_{j>1} (e_1 - e_j) + 2e_1,$$

where we have identified $e_1^\vee$ with $2e_1$ in the usual way. Writing $e_1 + e_j$ and $e_1 - e_j$ as sums of simple coroots, one may compute that

$$\sum_{j>1} (e_1 + e_j) + \sum_{j>1} (e_1 - e_j) + 2e_1 = 2l\alpha_1^\vee + 2l\alpha_2^\vee + ... + 2l\alpha_{l-1}^\vee + l\alpha_l^\vee.$$

Noting that $e_1 = \alpha_1^\vee + \alpha_2^\vee + ... + \alpha_{l-1}^\vee + \frac{1}{2}\alpha_l^\vee$, we have that $\sum_{\alpha \in F_{w_1}(1)} \alpha^\vee = 2l\epsilon_1$. Therefore, $\mathcal{N}_0(w_1)^2 = (\epsilon_1)(-1)^{2l} = 1$.

- We now turn to type $C_l$ adjoint. We recall that the roots may be identified with the functionals $\pm 2\epsilon_i (1 \leq i \leq l)$ and $\pm \epsilon_i \pm \epsilon_j (1 \leq i < j \leq l)$. The corresponding coroots may be identified (in the obvious way) with the functionals $\pm \epsilon_i, \pm \epsilon_i \pm \epsilon_j$. The fundamental coweights corresponding to the standard choice of simple roots are $\epsilon_i = \epsilon_1 + ... + \epsilon_i$, for $1 \leq i < l$, and $\epsilon_l = \frac{1}{2}(\epsilon_1 + e_2 + ... + \epsilon_l)$.

We note that the cocharacter lattice of type $C_l$ adjoint is $\langle Q^+, \epsilon_1 \rangle$. The action of $w_l$ exchanges $-\epsilon_0$ and $\epsilon_l$. Therefore, $F_{w_l}(1)$ is the set of all positive roots that contain $\epsilon_l$, so that $F_{w_l}(1) = \{e_i + e_j : 1 \leq i < j \leq l\} \cup \{2\epsilon_i : 1 \leq i \leq l\}$. One may therefore compute that

$$\sum_{\alpha \in F_{w_l}(1)} \alpha^\vee = \sum_{i<j} (e_i + e_j)^\vee + \sum_{i} (2e_i)^\vee = \sum_{i<j} (e_i + e_j) + \sum_{i} e_i.$$

Writing $e_i + e_j$ and $e_i$ as sums of simple coroots, we may compute that

$$\sum_{i<j} (e_i + e_j) + \sum_{i} e_i = l\alpha_1^\vee + 2l\alpha_2^\vee + 3l\alpha_3^\vee + ... + l^2\alpha_l^\vee.$$

Recalling that $\epsilon_l = \frac{1}{2}(\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + ... + l\alpha_l^\vee)$, we have that $\sum_{\alpha \in F_{w_l}(1)} \alpha^\vee = 2l\epsilon_l$. Therefore, $\mathcal{N}_0(w_l)^2 = (\epsilon_l)(-1)^{2l} = 1$.

- We now consider type $D_l$. We recall that the root system of type $D_l$ is realized as the set of all $\pm \epsilon_i \pm \epsilon_j$, with $i < j$. Since all roots $\alpha$ satisfy $|\alpha|^2 = 2$, we may identify roots and co-roots, fundamental weights and fundamental co-weights. Recall that we may take $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ to be

$$\alpha_i = \alpha_i^\vee = \begin{cases} e_i - e_{i+1} & \text{if } i \leq l - 1 \\ e_{l-1} + e_l & \text{if } i = l \end{cases}$$

The corresponding fundamental weights are

$$\epsilon_i = \epsilon_i^\vee = \begin{cases} e_1 + ... + e_i & \text{if } i < l - 1 \\ \frac{1}{2}(e_1 + ... + e_{l-1} - e_l) & \text{if } i = l - 1 \\ \frac{1}{2}(e_1 + ... + e_{l-1} + e_l) & \text{if } i = l \end{cases}$$

First we consider the case where $l$ is even. By [LM65, p. 19], $\Omega_{\text{ad}}$ is generated by the elements $\rho_1, \rho_{l-1}, \rho_l$, and the actions of their corresponding Weyl elements $w_1, w_{l-1}, w_l$ on the set
\{-\alpha_0, \alpha_1, \alpha_2, ..., \alpha_l\} are given by
\begin{align*}
w_1(-\alpha_0) &= \alpha_1, \quad w_1(\alpha_1) = -\alpha_0, \quad w_1(\alpha_i) = \alpha_i \quad (2 \leq i \leq l - 2) \\
w_1(\alpha_{l-1}) &= \alpha_l, \quad w_1(\alpha_l) = \alpha_{l-1}, \\
w_l(-\alpha_0) &= \alpha_l, \quad w_l(\alpha_l) = -\alpha_0, \quad w_l(\alpha_i) = \alpha_{l-i} \quad (1 \leq i \leq l - 1). \quad w_{l-1} = w_lw_1.
\end{align*}

Consider the isogeny given by the subgroup \langle \rho_l \rangle \text{ in } \Omega_{\text{ad}}. Its associated cocharacter lattice, which we denote by \(X_s(D_l^l)\), is given by \(X_s(D_l^l) = (Q^\vee, \epsilon_l)\). We must compute \(\mathcal{F}_{w_l}(1)\). As \(w_l\) exchanges \(-\alpha_0\) and \(\alpha_l\), \(\mathcal{F}_{w_l}(1)\) is the set of positive roots that contain \(\alpha_l\). By \cite{Bou02} Plate IV, the sum of all of the (co)roots in \(\mathcal{F}_{w_l}(1)\) is then equal to
\[(l-1)(\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + (l-2)\alpha_{l-2} + \frac{1}{2}(l-2)\alpha_{l-1} + \frac{1}{2}(l+1)\alpha_l)\].

But notice that \(\epsilon_l = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + (l-2)\alpha_{l-2} + \frac{1}{2}(l-2)\alpha_{l-1} + \frac{1}{2}(l+2)\alpha_l)\). Therefore, \(\mathcal{N}_\alpha(w_l)^2 = (\epsilon_l)(-1)^{(l-1)} = 1\).

We now consider the isogeny given by the subgroup \langle \rho_{l-1} \rangle \text{ of } \Omega_{\text{ad}}. Its associated cocharacter lattice, which we denote by \(X_s(D_l^{l-1})\), is given by \(X_s(D_l^{l-1}) = (Q^\vee, \epsilon_{l-1})\). Since \(w_l\) exchanges \(-\alpha_0\) and \(\alpha_l\), \(\mathcal{F}_{w_l}(1)\) is the set of all positive roots containing \(\alpha_l\). By \cite{Bou02} Plate IV, the sum of all of the (co)roots in \(\mathcal{F}_{w_l}(1)\) is
\[(l-1)(\alpha_1 + 2\alpha_2 + 2\alpha_3 + ... + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l)\].

But \(X_s(D_l^{l-1}) = (Q^\vee, \epsilon_{l-1})\), and \(\epsilon_{l-1} = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + (l-2)\alpha_{l-2} + \frac{1}{2}(l-2)\alpha_{l-1} + \frac{1}{2}(l+2)\alpha_l)\). Thus, \(\mathcal{N}_\alpha(w_{l-1})^2 = (\epsilon_{l-1})(-1)^{(l-1)} = 1\).

We now turn to \(D_l\) with \(l\) odd. First we consider the adjoint case, denoting the associated cocharacter lattice by \(X_s(D_l^{ad})\). To show that \(\mathcal{N}_\alpha(w_l)^4 = 1\), we need to compute the sum
\[\gamma := \sum_{\alpha \in \mathcal{F}_{w_l}(1)} \alpha + \sum_{\beta \in \mathcal{F}_{w_l}(2)} \beta + \sum_{\delta \in \mathcal{F}_{w_l}(3)} \delta.\]

Noting that
\[
\begin{align*}
\mathcal{F}_{w_l}(1) &= \{\alpha \in \Pi : \alpha \text{ contains } \alpha_{l-1} \text{ but doesn't contain } \alpha_1\} \\
\mathcal{F}_{w_l}(2) &= \{\alpha \in \Pi : \alpha \text{ contains } \alpha_1 \text{ but doesn't contain } \alpha_l\} \\
\mathcal{F}_{w_l}(3) &= \{\alpha \in \Pi : \alpha \text{ contains } \alpha_l\},
\end{align*}
\]
once computes that
\[\gamma = 2(l-2)(\alpha_1 + 2\alpha_2 + ... + (l-2)\alpha_{l-2}) + 2(\alpha_1 + \alpha_2 + ... + \alpha_{l-2})
\]
\[+ (l-2)(l-1)\alpha_{l-1} + \left(\frac{(l-3)(l-2)}{2} + \frac{(l-1)l}{2}\right)\alpha_l.\]

Modulo \(2X_s(D_l^{ad})\), \(\gamma\) is equivalent to \(\alpha_{l-1} + \alpha_l\). But \(\alpha_{l-1} + \alpha_l \equiv 2\epsilon_1 \text{ (mod } 2X_s(D_l^{ad}))\), so \(\mathcal{N}_\alpha(w_l)^4 = 1\) as needed.

We now consider the group \(G\), of type \(D_l\), with \(l\) odd, that is neither simply connected nor adjoint. We need to show that \(\mathcal{N}_\alpha(w_l^2)^2 = 1\). First, we recall that \(w_l^2\) fixes \(\alpha_i\), for \(i = 2, 3, ..., l-2\) and it exchanges \(-\alpha_0\) and \(\alpha_1\), and exchanges \(\alpha_{l-1}\) and \(\alpha_l\). We must therefore count the positive roots that contain \(\alpha_1\). But this has already been computed in the \(D_l\) cases with \(l\) even, and our results there imply that \(\mathcal{N}_\alpha(w_l^2)^2 = 1\), noting that the cocharacter lattice in the current case is given by \(\langle Q^\vee, \epsilon_1 \rangle\).
• We now turn to the group $G$ of type $E_6$ and adjoint. We follow here [Bou02 Plate V], which has different conventions than [IM65]. The Weyl element $w$ in question acts by $\alpha_1 \mapsto \alpha_6 \mapsto -\alpha_0$. Therefore, we need to compute the sum of all roots $\alpha$ that contain $\alpha_1$, together with all roots that contain $\alpha_0$ that also do not contain $\alpha_1$. One computes that this sum is
\[
\sum_{\alpha \in F_{w}(1)} \alpha^\vee + \sum_{\beta \in F_{w}(2)} \beta^\vee = 16\alpha_1 + 16\alpha_2 + 24\alpha_3 + 32\alpha_4 + 24\alpha_5 + 16\alpha_6 \in 2Q^\vee.
\]
Therefore, $\mathcal{N}_\omega(w)^3 = 1$.

• We now turn to $E_7$ adjoint. We need to show that $\mathcal{N}_\omega(w)^2 = 1$, where $w$ is the Weyl element in question. We follow here [Bou02 Plate VI], which has different conventions than [IM65]. Using the fact that $w$ exchanges $\alpha_7$ and $-\alpha_0$, one counts that the sum of all of the positive roots that contain $\alpha_7$ is
\[
\sum_{\alpha \in F_{w}(1)} \alpha^\vee = 18\alpha_1 + 27\alpha_2 + 36\alpha_3 + 54\alpha_4 + 45\alpha_5 + 36\alpha_6 + 27\alpha_7.
\]
But this sum is exactly equal to $18e_7$, so $\mathcal{N}_\omega(w)^2 = (e_7)(-1)^{18} = 1$.

• We finally consider type $A_l$. We re-adopt our conventions and notation from the proof of Proposition 5.24 in the case of type $A_l$. That is, we let $a, b \in \mathbb{N}$ such that $l + 1 = ab$. Let $\omega = \rho_1^a$, so that $\omega^b = 1$, and for ease of notation, let $n = l + 1$. Then $\omega = \epsilon_a w_a$, where $w_a$ is the $a$-th power of the $n$-cycle $(1 2 \cdots n)$.

A computation then shows that
\[
\sum_{m=1}^{b-1} \sum_{\alpha \in F_{w_a(m)}} \alpha^\vee = (n - a)[e_1 + e_2 + \ldots + e_a] + (n - 3a)[e_{a+1} + e_{a+2} + \ldots + e_{2a}]
\]
\[+ (n - 5a)[e_{2a+1} + \ldots + e_{3a}] + \ldots + (a - n)[e_{n-a+1} + e_{n-a+2} + \ldots + e_{n-1} + e_n].
\]
We denote this sum by $\gamma$. We recall that the cocharacter lattice of this isogeny is given by $X_*(A_l^n) = \langle Q^\vee, \epsilon_a \rangle$, where
\[
\epsilon_a = \frac{1}{n}[(n - a)(e_1 + e_2 + \ldots + e_a) - a(e_{a+1} + \ldots + e_n)].
\]
Suppose first that $n$ is odd, so that $a$ is also odd. Therefore, $n - a, n - 3a, n - 5a, \ldots, a - n$ are all even, so one can see that $\gamma \in 2Q^\vee$, which implies that $\mathcal{N}_\omega(w_a)^b = 1$. Suppose now that $n$ is even. Then
\[
\gamma - n\epsilon_a = (n - 2a)[e_{a+1} + e_{a+2} + \ldots + e_{2a}] + (n - 4a)[e_{2a+1} + \ldots + e_{3a}]
\]n\ldots + (2a - n)[e_{n-a+1} + e_{n-a+2} + \ldots + e_{n-1} + e_n].
\]
One can see that $\gamma - n\epsilon_a =: \eta \in 2Q^\vee$. Therefore, $\gamma = \eta + n\epsilon_a$ is twice a cocharacter, so $\mathcal{N}_\omega(w_a)^b = 1$.

\[\] 

Remark 3.4. The previous argument in the case of type $A_l$ depends on the group not being simply connected. Otherwise, it may not be that $\mathcal{N}_\omega(w_a)^b = 1$. Indeed, if $n$ is even, we relied on the fundamental coweight $\epsilon_a$ being contained in the cocharacter lattice in order to conclude that $\mathcal{N}_\omega(w_a)^b = 1$. If $n$ is odd, however, it was automatic that $\mathcal{N}_\omega(w_a)^b = 1$. Indeed, this does not conflict with a basic known example: if $G = SL(n)$ and $w$ is the long Weyl element, then
\[
\mathcal{N}_\omega(w)^n = \begin{cases} -1 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}
\]

Theorem 3.5. For $G$ a split, almost simple, $p$-adic group, there exists an embedding $\Omega \hookrightarrow N_G(T)$ that is also a section of the canonical map $N_G(T) \to \Omega$.

Proof. Suppose that $\Omega$ is cyclic of order $n$. We have shown that if $\omega = \epsilon_i w_{\Delta_{w}} \in \Omega$ is a generator, then $\iota(\omega)^n = 1$. But in fact $\iota(\omega)$ has order $n$. To see this, note that if $m < n$, then $\iota(\omega)^m = (\epsilon_i + w_i(\epsilon_i) + \ldots + w_i^{m-1}(\epsilon_i))(\varpi^{-1}) \cdot \mathcal{N}_\omega(w_i)^m$. But $\mathcal{N}_\omega(w_i)^m$ has a nontrivial projection to $W_0$ since $w_i$ has order $n$. Therefore, $\iota(\omega)^m$ has a nontrivial projection to $W_0$ as well, so in particular must be nontrivial. Since $\iota(\omega)$ has order $n$,
we may define a homomorphism $\Omega \to N_G(T)$ by sending $\omega^j$ to $i(\omega)^j$, and one may check that this map is in fact a section of the map $N_G(T) \to \Omega$.

It remains to consider the case where $G$ is adjoint of type $D_l$ with $l$ even, since its fundamental group is not cyclic. We denote the associated cocharacter lattice by $X_*(D_l^{ad})$. We show that $i$ is a homomorphism in this case. Recall that in Proposition 3.3 we showed that $i(\omega)^2 = 1$ for each $\omega \in \Omega$. We need to show that $i(\rho_1 \rho_l) = i(\rho_l) i(\rho_1)$, $i(\rho_1 \rho_{l-1}) = i(\rho_l) i(\rho_{l-1})$, and $i(\rho_{l-1} \rho_l) = i(\rho_{l-1}) i(\rho_l)$. We will carry out the case $i(\rho_1 \rho_l) = i(\rho_l) i(\rho_1)$, noting that the other cases are similar. First note that $i(\rho_1 \rho_l) = i(\rho_l) i(\rho_1) = e_l (\rho_1) N_\alpha(w_1) = e_l N_\alpha(w_1) = e_l N_\alpha(w_1) e_l N_\alpha(w_1)^{-1} N_\alpha(w_1) N_\alpha(w_1)$. One can compute that $e_l N_\alpha(w_1) e_l N_\alpha(w_1)^{-1} = e_l$, so it suffices to show that $N_\alpha(w_1) N_\alpha(w_1) = N_\alpha(w_{l-1})$. By Corollary 2.2 we need to show that $\prod_{\alpha \in \mathcal{F}(w_1, w_l)} \alpha^\gamma = 1$. One computes that

$$\mathcal{F}(w_1, w_l) = \{ \alpha \in \Pi : w_l(\alpha) \in -\Pi, w_1 w_l(\alpha) / \Pi \} = \{ \alpha \in \Pi : \alpha \text{ contains } \alpha_l \text{ and } \alpha \text{ does not contain } \alpha_{l-1} \}.$$  

This last set, by [Bou2, Plate IV], is the set $\{ e_i + e_l : 1 \leq i < l \}$. Adding these roots together gives $\gamma := \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + (l-2)\alpha_{l-2} + (l-1)\alpha_l$. But $X_*(D_l^{ad})$ contains $\alpha_{l-1}, \alpha_l$, and we see that $\gamma = (2-l)\epsilon_{l-1} + le_l$, which lives in $2X_*(D_l^{ad})$ since $l$ is even. The result follows. \hfill \Box

**Remark 3.6.**

1. It is not difficult to show that $i$ is a homomorphism in the case that $G$ is adjoint of type $E_6$. Since we cannot claim this for all types, we do not include the computation.

2. In the case that $G$ is adjoint of type $D_l$ where $l$ is odd, one can show that $i$ is not a homomorphism. In fact, one can show that $i(\rho_l)^2 = i(\rho_l^2)$, but it turns out that $i(\rho_l)^3 \neq i(\rho_l^3)$. This boils down to computing that the sum of all (co)roots in $\mathcal{F}(w_1(2))$ equals $le_1 - (e_1 + e_2 + \ldots + e_l)$, which when evaluated at $-1$ is nontrivial.

3. In the case that $G$ is type $A_l$, it turns out that $i$ is sometimes a homomorphism and sometimes not. For example, if $a = 1$ (in the notation of Proposition 3.3), then the group in consideration if $PGL_n$ (recall that in our notation, $n = l + 1 = ab$), and one can show that $i(\rho_l)^2 \neq i(\rho_l^2)$. On the other hand, if both $n$ and $a$ are even, then $i$ is a homomorphism.

## 4. Beyond split almost-simple groups

One may ask about generalizing Theorem 3.5 to more general connected reductive groups. The biggest obstacle to generalizing the result, using the methods in this paper, revolves around the fact that if $W_\circ(\Omega)$ denotes the projection of $\Omega$ onto the finite Weyl group, then $N_\alpha(W_\circ(\Omega)) : W_\circ(\Omega) \to N_G(T)$ is not necessarily a homomorphism. This problem occurred in some $A_l$ types, as well as adjoint $D_l$ with $l$ odd. But in these cases, we were able to skirt this issue by adjusting $i$ as in Theorem 3.5 using the fact that $\Omega$ is cyclic.

On the other hand, we are able to extend our result to certain additional split connected reductive groups. Note first that since $G_{ad}$ is a product of split, almost-simple groups, Theorem 3.5 gives a section $s_{G_{ad}}$ of $\kappa_{G_{ad}} : G_{ad}(F) \to \Omega_{G_{ad}}$.

**Definition 4.1.** Call a homomorphic section $s_{G}$ of $\kappa_{G}$ good if it is compatible with the one constructed for $G_{ad}$. In other words, the following diagram commutes:

$$\begin{array}{ccc}
G(F) & \longrightarrow & G_{ad}(F) \\
\kappa_{G} \downarrow & & \kappa_{G_{ad}} \downarrow \\
\Omega_G & \longrightarrow & \Omega_{ad}
\end{array}$$

**Remark 4.2.** Recall that when $G_{der} = G_{ac}$, there is an easy way to produce a homomorphic section with values in $T(F)$. However, this will not generally make the diagram commute, so it is not good.

**Proposition 4.3.** Let $G$ be a split connected reductive group over $F$. Let $C$ be an alcove in the apartment corresponding to a split maximal torus $T$, with associated extended affine Weyl group $W = X_*(T) \rtimes W_\circ$. Then:

1. If $Z = Z(G)$ is connected, then the induced map $G(F)/Z(\mathcal{O}_F) \to \Omega_G$ has a good homomorphic section (the analogue of the diagram above commutes).
(2) If $Z$ is connected and $\Omega_G \cong Z$ (e.g. $G = GSp(2m)$), then $\kappa_G$ has a good homomorphic section.

(3) If $Z$ is connected, $\Omega_G \cong \mathbb{Z}^n$, with $n > 1$, and $(|\Omega_{G_{\text{ad}}}|, q(q-1)) = 1$, where $q$ is the cardinality of the residue field, then $\kappa_G$ has a good homomorphic section.

**Proof.** We start with (1). It follows from Theorem 5.5 that $\kappa_{G_{\text{ad}}}$ has a homomorphic section $s_{G_{\text{ad}}}$, since $G_{\text{ad}}$ is known to be a product of almost-simple groups. Moreover, if $\kappa_Z$ denotes the Kottwitz homomorphism for $Z(F)$, then $\kappa_Z$ also has a homomorphic section, which we denote $s_Z$. As $H^1(F, Z) = 1$, we have a commutative diagram of exact sequences

$$
\begin{array}{cccccccc}
1 & \longrightarrow & Z(F) & \longrightarrow & G(F) & \longrightarrow & G_{\text{ad}}(F) & \longrightarrow & 1 \\
\kappa_Z & & \downarrow \kappa_G & & \downarrow \kappa_{G_{\text{ad}}} & & \downarrow \kappa_{G_{\text{ad}}} & & \\
1 & \longrightarrow & \Omega_Z & \longrightarrow & \Omega_G & \longrightarrow & \Omega_{G_{\text{ad}}} & \longrightarrow & 1
\end{array}
$$

We naturally have $Z(F)/Z(\mathcal{O}_F) \cong X_*(Z)$, therefore obtaining another diagram

$$
\begin{array}{cccccccc}
1 & \longrightarrow & Z(F)/Z(\mathcal{O}_F) & \longrightarrow & G(F)/Z(\mathcal{O}_F) & \longrightarrow & G_{\text{ad}}(F) & \longrightarrow & 1 \\
\kappa_Z & & \downarrow \kappa_G & & \downarrow \kappa_{G_{\text{ad}}} & & \downarrow \kappa_{G_{\text{ad}}} & & \\
1 & \longrightarrow & \Omega_Z & \longrightarrow & \Omega_G & \longrightarrow & \Omega_{G_{\text{ad}}} & \longrightarrow & 1
\end{array}
$$

where $\kappa_Z, \kappa_G$ are the induced maps. Let $\kappa_{G_{\text{ad}}}$ denote the map induced from $\kappa_{G_{\text{ad}}}$ on $s_{\text{ad}}(\Omega_{G_{\text{ad}}})$. Then we have a commutative diagram of exact sequences:

$$
\begin{array}{cccccccc}
1 & \longrightarrow & Z(F)/Z(\mathcal{O}_F) & \longrightarrow & G(F)/Z(\mathcal{O}_F) & \longrightarrow & G_{\text{ad}}(F) & \longrightarrow & 1 \\
\kappa_Z & & \downarrow \kappa_G & & \downarrow \kappa_{G_{\text{ad}}} & & \downarrow \kappa_{G_{\text{ad}}} & & \\
1 & \longrightarrow & \Omega_Z & \longrightarrow & \Omega_G & \longrightarrow & \Omega_{G_{\text{ad}}} & \longrightarrow & 1
\end{array}
$$

We have that $\kappa_Z, \kappa_{G_{\text{ad}}}$ are isomorphisms, so by the five lemma, $\kappa_G$ is an isomorphism, and thus the map $\kappa_G : G(F)/Z(\mathcal{O}_F) \to \Omega_G$ has a homomorphic section.

We now prove (2). Make an initial choice of a homomorphic section $s^0_Z$ of $\kappa_Z$. Given $\sigma \in \Omega_G$, let $s^0(\sigma)$ be any lift in $G(F)$ of $s^0_{G_{\text{ad}}} (pr(\sigma)) \in N_{G_{\text{ad}}}(T_{G_{\text{ad}}})(F)$; it automatically lies in $N_G(T)(F)$. It might happen that $s^0$ is not a section of $\kappa_G$. However, for all $\sigma \in \Omega_G$, we have $pr(\kappa_G(s^0(\sigma))) = \kappa_{G_{\text{ad}}}(pr(s^0(\sigma))) = \kappa_{G_{\text{ad}}}(s^0_{G_{\text{ad}}} (pr(\sigma))) = pr(\sigma)$. Thus, the difference between $\sigma$ and $\kappa_G(s^0(\sigma))$ belongs to $\Omega_Z$. Since $\kappa_Z$ is surjective, we may alter each $s^0(\sigma)$ by an element $z^0_\sigma \in Z(F)$ in such a way that $\sigma \mapsto s^0(\sigma)z^0_\sigma$ is a section of $\kappa_G$.

So we may assume $s^0$ is a set-theoretic section of $\kappa_G$, taking values in $N_G(T)(F)$. Because $s^0_{G_{\text{ad}}}$ is homomorphic, the map

$$(\sigma_1, \sigma_2) \mapsto s^0(\sigma_1)s^0(\sigma_2)s^0(\sigma_1\sigma_2)^{-1}$$

is a 2-cocycle of $\Omega_G$ with values in $Z(F)$, with $\Omega_G$ acting trivially on $Z(F)$. Therefore, we get an element of $H^2(\Omega_G, Z(F))$. This group parameterizes isomorphism classes of extensions of $\Omega_G$ by $Z(F)$ where the induced action of $\Omega_G$ on the normal subgroup $Z(F)$ is trivial (i.e. $Z(F)$ is central in the extension group). We claim that the extension corresponding to the 2-cocycle is the direct product $Z(F) \times \Omega_G$. This follows because $\Omega_G = Z$ and $H^2(Z, A) = 1$ for any abelian group $A$ with trivial $Z$-action.

The fact that the extension is trivial means that the 2-cocycle defining it is a 2-coboundary. This means that we may alter our initial choice of set-theoretic section $s^0$ to give a homomorphism $s : \Omega_G \to G(F)$, taking values again in $N_G(T)(F)$.

The problem now is that $s$ might not be a section of $\kappa_G$, which we take care of as before. By construction, $\sigma^{-1}\kappa_G(s(\sigma)) \in \Omega_Z$ for every $\sigma \in \Omega_G$. So we may define $z_\sigma := s^0_{Z}(\sigma(\kappa_G(s(\sigma)))^{-1}) \in Z(F)$, for $\sigma \in \Omega_G$. Note that $\sigma \mapsto z_\sigma$ is a homomorphism $\Omega_G \to Z(F)$. Now define

$s_G(\sigma) := z_\sigma s(\sigma)$.

Then $s_G$ is the desired homomorphic section of $\kappa_G$ in case (2).
In case (3), the same argument works, as long as we can prove that the 2-cocycle defined by \( s^0 \) is still a 2-coboundary. But when \( n > 1 \) it is no longer true that \( H^2(\mathbb{Z}, A) \) always vanishes for abelian groups \( A \) with trivial \( \mathbb{Z}^n \)-action. Nevertheless, we will show that the extension corresponding to the given 2-cocycle is still trivial. Write \( \hat{e}_i = s^0(e_i) \), where \( e_i \) corresponds to a standard basis vector in \( \Omega_G \cong \mathbb{Z}^n \). Then the extension is the exact sequence

\[
1 \rightarrow Z(F) \rightarrow Z(F)(\hat{e}_1, \ldots, \hat{e}_n) \xrightarrow{\kappa_G} \Omega_G \rightarrow 1.
\]

Write \( N := |\Omega_{G_{\text{ad}}}| \). As \( \text{pr}(\hat{e}_j) \in \text{im}(s_{G_{\text{ad}}}) \cong \Omega_{G_{\text{ad}}} \), we have \( \text{pr}(\hat{e}_j)^N = 1 \) and hence \( \hat{e}_j^N \in Z(F) \). Moreover, \( \hat{e}_i \hat{e}_j \hat{e}_i^{-1} \hat{e}_j^{-1} \in Z(F) \). We may write

\[
a \hat{e}_j = \hat{e}_i \hat{e}_j \hat{e}_i^{-1},
\]

for some \( a \in Z(F) \). Raising to the \( N \)-th power, we get

\[
a^N \hat{e}_j^N = \hat{e}_j^N,
\]

and hence \( a^N = 1 \). Therefore, \( a \in Z(O_F) \). Moreover, since \( N \) is coprime to the pro-order of the profinite group \( Z(O_F) \), we conclude that \( a = 1 \), and therefore the elements \( \hat{e}_i \) pairwise commute. Therefore, the extension is an abelian group. But then the extension is trivial, since \( \Omega_G \cong \mathbb{Z}^n \).

This concludes the proof of the proposition. But we make one additional comment. By construction, the map \( s_{G_{\text{ad}}} \) has image in \( Z(F) \) and so gives a homomorphic section \( s_Z \) of \( \kappa_Z \). This section might be different from the initial choice \( s^0_Z \). But now we have a commutative diagram

\[
\begin{array}{c}
1 \rightarrow Z(F) \rightarrow G(F) \rightarrow G_{\text{ad}}(F) \rightarrow 1 \\
\kappa_Z \downarrow s_Z & \kappa_G \downarrow s_G & \kappa_{G_{\text{ad}}} \downarrow s_{G_{\text{ad}}} \\
1 \rightarrow \Omega_Z \rightarrow \Omega_G \rightarrow \Omega_{G_{\text{ad}}} \rightarrow 1
\end{array}
\]

\[\square\]

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