Inverse scattering transform for the Tzitzéica equation

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Abstract

The inverse scattering transform is extended to investigate the Tzitzéica equation. A set of sectionally analytic eigenfunctions and auxiliary eigenfunctions are introduced. We note that in this procedure, the auxiliary eigenfunctions play an important role. Besides, the symmetries of the analytic eigenfunctions and scattering data are discussed. The asymptotic behaviors of the Jost eigenfunctions are derived systematically. A Riemann-Hilbert problem is constructed to study the inverse scattering problem. Lastly, some novel exact solutions are obtained for reflectionless potentials.

\textbf{Keywords:} The Tzitzéica equation; Lax pair; The inverse spectral transform; Vanishing boundary condition; Explicit solutions.

1 Introduction

The inverse scattering transform method is an effective tool to study the integrable nonlinear equation with sufficient decay boundary condition or non-zero boundary condition. Besides, some novel and interesting properties for solutions can be given by

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the inverse scattering transform. There are so many works involving these problems. We here only refer to some of them \[1\]–\[10\]. In contrast to \(2 \times 2\) matrix linear spectral problems, the \(3 \times 3\) or higher order matrix problems are more difficult to be investigated by the inverse scattering transform method \[11\]–\[14\].

In the present paper, we will study the Tzitzéica equation by the inverse scattering transform

\[
u_{xt} = e^u - e^{-2u},
\]

where \(u = u(x, t)\) and has sufficient decay for all \(t\) as \(|x| \to \infty\). Here, the subscripted variables \(x\) and \(t\) in Eq. (1.1) denote the corresponding partial differentiation. We note that the Tzitzéica equation \(1.1\) can be rewritten as other forms \[15, 16\]

\[
(ln h)_{xt} = h - h^{-2},
\]

\[
u_{xx} - \nu_{tt} = e^u - e^{-2u},
\]

It can be directly calculated that Eq. (1.1) is equivalent to Eq. (1.2) with the transform \(\ln h = u\). The Tzitzéica equation (1.1) was initially proposed as the model describing special surfaces in differential geometry, where the ratio \(k/d^4\) is constant. The parameter \(k\) is the Gauss curvature of the surface and \(d\) is the distance from the origin to the tangent plane at a fixed point. In \[17–19\], the Tzitzéica equation can be reduced from the two-dimensional Toda lattice equation via the conjugate nets. Furthermore, the relations between soliton theory and affine differential geometry were presented in \[17\].

It is shown that Eq. (1.1) plays an important role in many fields, such as geometry \[20\], soliton theory \[19, 21, 23\] and gas dynamics \[24\]. The Tzitzéica equation was studied by many methods, such as the Darboux transformation \[25, 26\], Bäcklund transformation \[27\], the algebro-geometric approach \[28, 30\], the Hirota bilinear method \[31, 32\], the inverse scattering method \[10, 33\] and the dressing method \[31\]. In addition, there are some other results for the Tzitzéica equation, such as the discretization \[34\], ultradiscretization \[35\], symmetries \[36\], conservation laws \[21\] and others \[37–42\].

This paper is to obtain the explicit solution of the Tzitzéica equation through inverse scattering transforms. To this end, a Riemann-Hilbert (RH) problem with the corresponding six jump matrices and six jump conditions is constructed. We note that it is difficult to construct a set of analyticity eigenfunctions in every domains.
Thus, introducing the auxiliary eigenfunctions and the adjoint problem, we obtain the
piecewise meromorphic function. To avoid the overly cumbersome work, we consider
the symmetry conditions associated with the Tzitzéica equation. By using the Cauchy
projector, the RH problem is transformed to a system of matrix algebraic-integral
equations. Based on the algebraic-integral equations, some explicit solutions of Eq.(1.1)
for the case of reflectionless potentials are given.

The paper is organized as follows. In section 2, the direct problem of the Tzitzéica
equation Eq. (1.1) is considered. In section 3, the discrete spectrum is discussed in
detail. In section 4, the inverse problem is investigated. In the last section, based on
the RH problem, some exact solutions of Eq. (1.1) are constructed.

2 Spectral analysis

In this section, we will consider the direct scattering problem associated with Eq. (1.1).

2.1 Preliminaries: Lax pair, scattering matrix

The Tzitzéica equation (1.1) can be derived through the compatibility condition of the
following Lax pair

\[ \psi_{0x} = U_0(x, \lambda, u) \psi_0, \quad (2.1a) \]
\[ \psi_{0t} = V_0(t, \lambda, u) \psi_0, \quad (2.1b) \]

where

\[ U_0(x, \lambda, u) = \begin{pmatrix} 0 & 0 & \lambda e^u \\ e^{-u} & u & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V_0(t, \lambda, u) = \begin{pmatrix} u_t & e^{-u} & 0 \\ 0 & 0 & e^u \\ \lambda^{-1} & 0 & 0 \end{pmatrix}. \quad (2.2) \]

Since the trace of the matrices \( U_0 \) and \( V_0 \) are not zero, we make the following eigen-
function transformation

\[ \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} e^{-u} & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu^2 \end{pmatrix} \begin{pmatrix} \psi_{10} \\ \psi_{20} \\ \psi_{30} \end{pmatrix}, \quad (2.3) \]

and have the second Lax pair:

\[ \psi_x = U(x, \lambda, u) \psi, \quad (2.4a) \]
\[ \psi_t = V(t, \lambda, u)\psi, \quad (2.4b) \]

with

\[ U(x, \lambda, u) = \begin{pmatrix} -u_x & 0 & \nu \\ \nu & u_x & 0 \\ 0 & \nu & 0 \end{pmatrix}, \quad V(t, \lambda, u) = \begin{pmatrix} 0 & \frac{e^{-2u}}{\nu} & 0 \\ 0 & 0 & \frac{e^{nu}}{\nu} \end{pmatrix}, \quad (2.5) \]

\( \nu \) is the spectral parameter. Thus, the determinant of the fundamental matrix solution of (2.4) is \( x \) and \( t \) independent. In the following, we will discuss the spectral analysis from the spectral problem (2.4). Under the vanishing boundary condition \( u \to 0, x \to \pm \infty \), we have

\[ U \to U_{\pm} = \begin{pmatrix} 0 & 0 & \nu \\ \nu & 0 & 0 \\ 0 & \nu & 0 \end{pmatrix}, \quad x \to \pm \infty, \quad (2.6) \]

\[ V \to V_{\pm} = \begin{pmatrix} 0 & \frac{1}{\nu} & 0 \\ 0 & 0 & \frac{1}{\nu} \end{pmatrix}, \quad x \to \pm \infty. \]

For convenience, we introduce an invertible matrix \( A \), such that

\[ A^{-1}U_{\pm}A = \nu \sigma, \quad A^{-1}V_{\pm}A = \nu^{-1} \sigma^{-1}, \quad (2.7) \]

where

\[ \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \\ 1 & 1 & 1 \end{pmatrix}. \quad (2.8) \]

Here, \( \lambda = \nu^2, \alpha = e^{2\pi i} \). Now we introduce the matrix solutions \( \psi_{\pm}(x, t, \nu) \) of (2.4a) satisfying the following condition

\[ \psi_{\pm}(x, t, \nu) = Ae^{\nu x \sigma + \nu^{-1} \sigma^{-1} t} + o(1), \quad x \to \pm \infty. \quad (2.9) \]

Thus

\[ \text{det} \psi_{\pm}(x, t, \nu) = 3(\alpha - \alpha^2) = \gamma. \quad (2.10) \]

For the sake of analyticity, we introduce new Jost solutions \( \mu_{\pm}(x, t, \nu) \) defined by

\[ \psi_{\pm}(x, t, \nu) = \mu_{\pm}(x, t, \nu)e^{\nu x \sigma + \nu^{-1} \sigma^{-1} t}, \quad (2.11) \]
with their corresponding asymptotic behaviour

\[ \mu_\pm(x, t, \nu) \to A, \quad x \to \pm \infty. \quad (2.12) \]

In addition, they are linked by the scattering matrix \( s(\nu) = (s_{ij}) \), namely

\[ \psi_-(x, t, \nu) = \psi_+(x, t, \nu)s(\nu). \quad (2.13) \]

### 2.2 Analyticity of the Jost function

In order to study the analytic properties of the matrix Jost functions, we need to rewrite the spectral equation. Inserting the representations \( (2.11) \) into the Lax pair \( (2.4) \), we get

\[ (A^{-1}\mu_\pm)_x(x, t, \nu) = \nu[s, A^{-1}\mu_\pm(x, t, \nu)] + \Omega A^{-1}\mu_\pm(x, t, \nu), \quad (2.14a) \]

\[ (A^{-1}\mu_\pm)_t(x, t, \nu) = \nu^{-1}[\sigma^{-1}, A^{-1}\mu_\pm(x, t, \nu)] + \tilde{\Omega} A^{-1}\mu_\pm(x, t, \nu), \quad (2.14b) \]

where

\[ \Omega(x, t, \nu) = A^{-1}\Delta U(x, t, \nu)A, \quad \Delta U(x, t, \nu) = U(x, t, \nu) - U_\pm(x, t, \nu), \]

\[ \tilde{\Omega}(x, t, \nu) = A^{-1}\Delta V(x, t, \nu)A, \quad \Delta V(x, t, \nu) = V(x, t, \nu) - V_\pm(x, t, \nu), \]
that is

\[ \Omega(x, t, \nu) = \frac{1}{3}(\alpha - \alpha^2)u_x \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \]

\[ \tilde{\Omega}(x, t, \nu) = \frac{1}{3\nu} \begin{pmatrix} 2e^u + e^{-2u} - 3 & \alpha(e^{-2u} - e^u) & \alpha^2(e^{-2u} - e^u) \\ \alpha(e^{-2u} - e^u) & \alpha^2(e^{-2u} + 2e^u - 3) & e^{-2u} - e^u \\ \alpha^2(e^{-2u} - e^u) & e^{-2u} - e^u & \alpha(e^{-2u} + 2e^u - 3) \end{pmatrix}. \]

In the analysis that follows, we choose \( t = 0 \) and the expressions of \( \mu_{\pm}(x, 0, \nu) \) can be given from Eq. (2.14)

\[ \mu_{\pm}(x, 0, \nu) = A + \int_{-\infty}^{x} A e^{\nu\phi(x-\xi)} \Omega(\xi) A^{-1} \mu_{\pm}(\xi) d\xi, \quad (2.15) \]

**Theorem 1.** If \( u(x) \in L^1(-\infty, a) \) or \( L^1(a, +\infty) \) for any constant \( a \in \mathbb{R} \), the columns of \( \mu_{\pm}(x, 0, \nu) \) can be analytically extended into the corresponding regions of the complex \( \nu \)-plane:

\[
\begin{align*}
\mu_{+1}(x, 0, \nu) & : \nu \in D_3 \cup D_4, \quad \mu_{-1}(x, 0, \nu) : \nu \in D_1 \cup D_6, \\
\mu_{+2}(x, 0, \nu) & : \nu \in D_1 \cup D_2, \quad \mu_{-2}(x, 0, \nu) : \nu \in D_3 \cup D_5, \\
\mu_{+3}(x, 0, \nu) & : \nu \in D_5 \cup D_6, \quad \mu_{-3}(x, 0, \nu) : \nu \in D_2 \cup D_3,
\end{align*}
\]

where the subscript \( j \) refer to the columns and \( D_n(n = 1, 2, \ldots, 6) \) are defined by

\[
\begin{align*}
D_1 & = \left\{ \nu \in \mathbb{C} \mid R_2 < R_3 < R_1 \right\}, \quad D_2 = \left\{ \nu \in \mathbb{C} \mid R_2 < R_1 < R_3 \right\}, \\
D_3 & = \left\{ \nu \in \mathbb{C} \mid R_1 < R_2 < R_3 \right\}, \quad D_4 = \left\{ \nu \in \mathbb{C} \mid R_1 < R_3 < R_2 \right\}, \\
D_5 & = \left\{ \nu \in \mathbb{C} \mid R_3 < R_1 < R_2 \right\}, \quad D_6 = \left\{ \nu \in \mathbb{C} \mid R_3 < R_2 < R_1 \right\},
\end{align*}
\]

namely

\[ D_n = \left\{ \nu \in \mathbb{C} : \text{arg} \nu \in \left( \frac{(n-1)\pi}{3}, \frac{n\pi}{3} \right) \right\}, \quad n = 1, \ldots, 6. \]

Here \( l_i(\nu)(i = 1, 2, 3) \) are the diagonal entries of matrices \( \nu \sigma \).

**Proof** Setting \( J_{\pm}(x, 0, \nu) = A^{-1} \mu_{\pm}(x, 0, \nu), \) (2.15) can be rewritten in the column
form

\[ J_{±1}(x, 0, ν) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_{±∞}^{x} e^{νσ(x-ξ)}Ω(ξ)e^{-ν(x-ξ)} J_{±1}(ξ, t, ν) dξ, \quad (2.18a) \]

\[ J_{±2}(x, 0, ν) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \int_{±∞}^{x} e^{νσ(x-ξ)}Ω(ξ)e^{-αν(x-ξ)} J_{±2}(ξ, t, ν) dξ, \quad (2.18b) \]

\[ J_{±3}(x, 0, ν) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \int_{±∞}^{x} e^{νσ(x-ξ)}Ω(ξ)e^{-α^2ν(x-ξ)} J_{±3}(ξ, t, ν) dξ. \quad (2.18c) \]

We consider the integral equation of \( J_{±1}(x, 0, ν) \), that is

\[ J_{±1}(x, 0, ν) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_{+∞}^{x} κ_{±}(x, ξ, ν) J_{±1}(ξ, ν) dξ, \quad (2.19) \]

where \( κ_{±}(x, ξ, ν) = e^{ν(x-ξ)}σe^{-ν(x-ξ)}A^{-1} U A \). Then \( ∥κ_{±}(x, ξ, ν)∥ ≤ 6ρ|u_ξ|, ν ∈ D_3 \cup D_4 \),

where \( ρ = \max\{∥A∥, ∥A∥^{-1}\}, ∥ΔU∥ = 2|u_x| \).

Furthermore, we define the Neumann series as

\[ J_{±1}(x, 0, ν) = \sum_{j=0}^{∞} C^{(j)}(x, 0, ν), \quad ∥C^{(j)}(x, 0, ν)∥ ≤ \frac{μ(x)}{j!}, \]

where

\[ μ(x) = 6ρ \int_{+∞}^{x} |u_ξ|dξ, \quad C^{(0)}(x, 0, ν) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \]

\[ C^{(j+1)}(x, 0, ν) = \int_{+∞}^{x} e^{νσ(x-ξ)}Ω(ξ)e^{-ν(x-ξ)}C^{(j)}(ξ, t, ν) dξ. \]

This proves that \( J_{±1}(x, 0, ν) \) is analytic in domain \( ν ∈ D_3 \cup D_4 \). The others can be proved similarly. \( \square \)

Under the same hypotheses as in Theorem 1, the scattering coefficients can be analytically extended from \( Σ \) (the contour \( Σ \) is defined as the boundary of domain \( D \).
Figure 2: Domain $D^+$, $D^-$ and their boundary curve $\partial D$.

i.e $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \cup \Sigma_6$ and $\Sigma_j = D_j \cap D_{j+1}$, $D_{6+1} = D_1$) to the following regions:

$$s_{11} : \nu \in D_1 \cup D_6, \quad s_{22} : \nu \in D_4 \cup D_5, \quad s_{33} : \nu \in D_2 \cup D_3,$$

$$r_{11} : \nu \in D_3 \cup D_4, \quad r_{22} : \nu \in D_1 \cup D_2, \quad r_{33} : \nu \in D_5 \cup D_6,$$

where $s_{ij} = (s(\nu))_{ij}$ and $r_{ij} = (s^{-1}(\nu))_{ij}$. Next, we define six solutions $M_n(x, 0, \nu)$ of (2.14), which take the following form

$$(M_n)_{ij}(x, 0, \nu) = \delta_{ij} + \int_{\gamma_{ij}^n} \left( e^{\nu \sigma(x-x')} \Omega(x', 0, \nu) M_n(x', 0, \nu) dx' \right)_{ij},$$

where the contours $\gamma_{ij}^n$, $(n = 1, \ldots, 6, i, j = 1, 2, 3)$ are defined as

$$\gamma_{ij}^n = \begin{cases} \gamma_1, & \Re(l_i) > \Re(l_j), \\ \gamma_2, & \Re(l_i) < \Re(l_j), \end{cases}$$

and $\gamma_2: (-\infty, x)$, $\gamma_1: (x, +\infty)$.

For concretely

$$\gamma^1 = \begin{pmatrix} \gamma_2 & \gamma_1 & \gamma_1 \\ \gamma_2 & \gamma_2 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_2 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \gamma_2 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_2 \\ \gamma_1 & \gamma_1 & \gamma_2 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} \gamma_2 & \gamma_2 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_2 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} \gamma_2 & \gamma_2 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_1 \\ \gamma_1 & \gamma_2 & \gamma_2 \end{pmatrix},$$

$$\gamma^5 = \begin{pmatrix} \gamma_2 & \gamma_2 & \gamma_1 \\ \gamma_1 & \gamma_2 & \gamma_2 \\ \gamma_2 & \gamma_2 & \gamma_2 \end{pmatrix}, \quad \gamma^6 = \begin{pmatrix} \gamma_2 & \gamma_1 & \gamma_1 \\ \gamma_2 & \gamma_2 & \gamma_1 \\ \gamma_2 & \gamma_2 & \gamma_2 \end{pmatrix}.$$
We note that $\mu_{\pm}(x, t, \nu)$ have the same analytic properties as $\mu_{\pm}(x, 0, \nu)$, and $M_n(x, t, \nu)$ can be defined in the same way as in (2.21), where $e^{\nu \partial (x-x') \pm \nu^{-1} \partial t}$ should be replaced by $e^{\nu \partial (x-x') \pm \nu^{-1} \partial t}$. Supposing that
\[
M_n(x, t, \nu) = \mu_{+}(x, t, \nu)e^{\nu \partial x + \nu^{-1} \partial t}S_n(\nu), \quad \nu \in D_n, \quad (2.22)
\]
we have
\[
S_n(\nu) = s(\nu)T_n(\nu),
\]
\[
S_n(\nu) = \lim_{x \to \infty} e^{-\nu \partial x - \nu^{-1} \partial t}M_n(x, t, \nu), \quad (2.23)
\]
\[
T_n(\nu) = \lim_{x \to -\infty} e^{-\nu \partial x - \nu^{-1} \partial t}M_n(x, t, \nu),
\]
where $S_n$ can be given in terms of the entries of $s(\nu) = (s_{ij}(\nu))$ and $s^{-1}(\nu) = (r_{ij}(\nu))$ as follows
\[
S_1(\nu) = \begin{pmatrix}
               s_{11}(\nu) & 0 & 0 \\
               r_{21}(\nu) & r_{22}(\nu) \\
               r_{31}(\nu) & 0 & r_{33}(\nu)
             \end{pmatrix}, \quad S_2(\nu) = \begin{pmatrix}
               0 & 0 & s_{13}(\nu) \\
               r_{22}(\nu) & 1 & s_{23}(\nu) \\
               s_{33}(\nu) & r_{33}(\nu) & 0
             \end{pmatrix},
\]
\[
S_3(\nu) = \begin{pmatrix}
               0 & r_{12}(\nu) & s_{13}(\nu) \\
               -r_{11}(\nu) & r_{11}(\nu) & s_{23}(\nu) \\
               0 & 0 & s_{33}(\nu)
             \end{pmatrix}, \quad S_4(\nu) = \begin{pmatrix}
               0 & s_{12}(\nu) & 0 \\
               0 & 0 & s_{22}(\nu) \\
               0 & s_{33}(\nu) & 0
             \end{pmatrix},
\]
\[
S_5(\nu) = \begin{pmatrix}
               r_{23}(\nu) & s_{12}(\nu) & 0 \\
               0 & s_{22}(\nu) & 0 \\
               -r_{31}(\nu) & s_{32}(\nu) & 1
             \end{pmatrix}, \quad S_6(\nu) = \begin{pmatrix}
               s_{11}(\nu) & 0 & 0 \\
               s_{21}(\nu) & r_{22}(\nu) & 0 \\
               s_{31}(\nu) & r_{32}(\nu) & r_{33}(\nu)
             \end{pmatrix}.
\]

2.3 Adjoint problem and auxiliary eigenfunctions

It is remarked that, for the Tzitzéica equation (1.1), traditional inverse scattering transform may fail to construct a set of analytic eigenfunctions in any given domain. One efficient technique is to introduce the adjoint Lax pair [3, 11]
\[
\tilde{\psi}_x = \tilde{U}(x, \nu, u)\tilde{\psi}, \quad (2.24a)
\]
\[
\tilde{\psi}_t = \tilde{V}(x, \nu, u)\tilde{\psi}, \quad (2.24b)
\]
where
\[
\tilde{U}(x, \nu, u) = U^*(x, -\nu^*, u), \quad \tilde{V}(x, \nu, u) = V^*(x, -\nu^*, u),
\]
and the star denotes the complex conjugate.

We note that if $\phi(x,t,\nu)$ and $\varphi(x,t,\nu)$ are two arbitrary vector eigenfunctions of the adjoint problem (2.24), then

$$T_1 = \frac{T_1}{3\gamma\gamma^*} [\phi \times \varphi](x,t,\nu), \quad T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.25)$$

is an eigenfunction of the linear system (2.4). Here the symbol $\times$ denotes the usual cross product.

Suppose $\tilde{\psi}_\pm(x,t,\nu)$ defined similarly as in (2.9) are fundamental matrix solutions of the (2.24a), where $\tilde{A}$ should be $\tilde{A} = A^*$, then there exists an invertible $3 \times 3$ matrix $\tilde{s}(\nu)$ such that

$$\tilde{\psi}_-(x,t,\nu) = \tilde{\psi}_+(x,t,\nu)\tilde{s}(\nu).$$

Introducing the modified adjoint eigenfunction $\tilde{\mu}_\pm(x,t,\nu)$ defined by

$$\tilde{\psi}_\pm = \tilde{\mu}_\pm(x,t,\nu)e^{-\nu\sigma^{-1}x - \nu^{-1}\sigma t},$$

and proceeding the same analysis for $\mu_\pm(x,t,\nu)$, we know that different collum of $\tilde{\mu}_\pm(x,t,\nu)$ can be analyticity extended into the following domains

$$\tilde{\mu}_{+1}(x,t,\nu) : \nu \in D_1 \cup D_6, \quad \tilde{\mu}_{-1}(x,t,\nu) : \nu \in D_3 \cup D_4,$$

$$\tilde{\mu}_{+2}(x,t,\nu) : \nu \in D_2 \cup D_3, \quad \tilde{\mu}_{-2}(x,t,\nu) : \nu \in D_5 \cup D_6,$$

$$\tilde{\mu}_{+3}(x,t,\nu) : \nu \in D_4 \cup D_5, \quad \tilde{\mu}_{-3}(x,t,\nu) : \nu \in D_1 \cup D_2. \quad (2.26)$$

Then the associated scattering coefficients can be similarly extended in the following region

$$\tilde{s}_{11} : \nu \in D_3 \cup D_4, \quad \tilde{s}_{22} : \nu \in D_5 \cup D_6, \quad \tilde{s}_{33} : \nu \in D_1 \cup D_2, \quad (2.27a)$$

$$\tilde{r}_{11} : \nu \in D_1 \cup D_6, \quad \tilde{r}_{22} : \nu \in D_2 \cup D_3, \quad \tilde{r}_{33} : \nu \in D_4 \cup D_5. \quad (2.27b)$$

Using (2.25) and the solutions $\tilde{\psi}_\pm$ of the adjoint problem (2.24), we construct six novel solutions for the Lax pair (2.4) as

$$\chi_1(x,t,\nu) = \frac{T_1}{3\gamma\gamma^*}[\tilde{\psi}_{-3} \times \tilde{\psi}_{+1}](x,t,\nu), \quad (2.28a)$$
\[ \chi_2(x, t, \nu) = \frac{T_1}{3\gamma^*}[\tilde{\psi}_{+2} \times \tilde{\psi}_{-3}](x, t, \nu), \quad (2.28b) \]

\[ \chi_3(x, t, \nu) = \frac{T_1}{3\gamma^*}[\tilde{\psi}_{-1} \times \tilde{\psi}_{+2}](x, t, \nu), \quad (2.28c) \]

\[ \chi_4(x, t, \nu) = \frac{T_1}{3\gamma^*}[\tilde{\psi}_{+3} \times \tilde{\psi}_{-1}](x, t, \nu), \quad (2.28d) \]

\[ \chi_5(x, t, \nu) = \frac{T_1}{3\gamma^*}[\tilde{\psi}_{-2} \times \tilde{\psi}_{+3}](x, t, \nu), \quad (2.28e) \]

\[ \chi_6(x, t, \nu) = \frac{T_1}{3\gamma^*}[\tilde{\psi}_{+1} \times \tilde{\psi}_{-2}](x, t, \nu). \quad (2.28f) \]

Here, \( \chi_1(x, t, \nu), \ldots, \chi_6(x, t, \nu) \) are called the auxiliary eigenfunctions. Due to the Tzitzéica equation has six different domains of analyticity, we need to define six auxiliary eigenfunctions. Furthermore, \( \chi_n(x, t, \nu) \) is analytic in domain \( \nu \in D_n, n = 1, \ldots, 6. \)

In addition, for \( \nu \in \Sigma, \) the Jost functions \( \psi_{\pm} \) in (2.9) and the solutions \( \tilde{\psi}_{\pm} \) of the adjoint problem (2.24) satisfy the following relations

\[ \psi_{\pm 1}(\nu) = \frac{T_1}{3\gamma^*}[\psi_{\pm 2} \times \psi_{\pm 3}](x, t, \nu), \quad (2.29a) \]

\[ \psi_{\pm 2}(\nu) = \frac{T_1}{3\gamma^*}[\psi_{\pm 1} \times \psi_{\pm 2}](x, t, \nu), \quad (2.29b) \]

\[ \psi_{\pm 3}(\nu) = \frac{T_1}{3\gamma^*}[\psi_{\pm 3} \times \psi_{\pm 1}](x, t, \nu), \quad (2.29c) \]

\[ \tilde{\psi}_{\pm 1}(\nu) = \frac{T_1}{3\gamma}[\psi_{\pm 2} \times \psi_{\pm 3}](x, t, \nu), \quad (2.29d) \]

\[ \tilde{\psi}_{\pm 2}(\nu) = \frac{T_1}{3\gamma}[\psi_{\pm 1} \times \psi_{\pm 2}](x, t, \nu), \quad (2.29e) \]

\[ \tilde{\psi}_{\pm 3}(\nu) = \frac{T_1}{3\gamma}[\psi_{\pm 3} \times \psi_{\pm 1}](x, t, \nu). \quad (2.29f) \]

Based on the above facts, the scattering matrices \( s(\nu) \) and \( \tilde{s}(\nu) \) are related by

\[ \tilde{s}(\nu) = \Gamma \left( s^{-1}(\nu) \right)^T \Gamma, \quad \Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.30) \]
For all $\nu \in \Sigma$, the Jost functions and the auxiliary eigenfunctions obey the following equations
\begin{equation}
\psi_{+1}(\nu) = \frac{1}{r_{22}(\nu)}[\chi_2(\nu) + \psi_{+2}(\nu)r_{21}(\nu)] = \frac{1}{r_{33}(\nu)}[\chi_5(\nu) + \psi_{+3}(\nu)r_{31}(\nu)], \tag{2.31a}
\end{equation}
\begin{equation}
\psi_{+2}(\nu) = \frac{1}{r_{12}(\nu)}[\chi_3(\nu) + \psi_{+1}(\nu)r_{12}(\nu)] = \frac{1}{r_{33}(\nu)}[\chi_6(\nu) + \psi_{+3}(\nu)r_{32}(\nu)], \tag{2.31b}
\end{equation}
\begin{equation}
\psi_{+3}(\nu) = \frac{1}{r_{22}(\nu)}[\chi_1(\nu) + \psi_{+2}(\nu)r_{23}(\nu)] = \frac{1}{r_{11}(\nu)}[\chi_4(\nu) + \psi_{+1}(\nu)r_{13}(\nu)], \tag{2.31c}
\end{equation}
\begin{equation}
\psi_{-1}(\nu) = \frac{1}{s_{33}(\nu)}[\chi_2(\nu) + \psi_{-3}(\nu)s_{31}(\nu)] = \frac{1}{s_{22}(\nu)}[\chi_5(\nu) + \psi_{-2}(\nu)s_{21}(\nu)], \tag{2.31d}
\end{equation}
\begin{equation}
\psi_{-2}(\nu) = \frac{1}{s_{33}(\nu)}[\chi_3(\nu) + \psi_{-3}(\nu)s_{32}(\nu)] = \frac{1}{s_{11}(\nu)}[\chi_6(\nu) + \psi_{-2}(\nu)s_{12}(\nu)], \tag{2.31e}
\end{equation}
\begin{equation}
\psi_{-3}(\nu) = \frac{1}{s_{11}(\nu)}[\chi_1(\nu) + \psi_{-1}(\nu)s_{13}(\nu)] = \frac{1}{s_{22}(\nu)}[\chi_4(\nu) + \psi_{-2}(\nu)s_{23}(\nu)]. \tag{2.31f}
\end{equation}

### 2.4 symmetries

**First symmetry.** It is readily verified that if $\psi(x, t, \nu)$ is a fundamental matrix solution of the Lax pair (2.4), so is $T_1 \left( \psi_\dagger(x, t, -\nu^*) \right)^{-1}$, where $T_1$ is defined in (2.25).

For all $\nu \in \Sigma$, the Jost functions satisfy the symmetry
\begin{equation}
T_1 \left( \psi_\dagger_{\pm}(x, t, -\nu^*) \right)^{-1} \frac{\Gamma}{3} = \psi_{\pm}(x, t, \nu),
\end{equation}
where
\begin{equation}
\left( \psi_{\pm}^{-1}(x, t, \nu) \right)^T = \frac{1}{\det \psi_{\pm}(x, t, \nu)} \left( \psi_{\pm2} \times \psi_{\pm3}, \psi_{\pm3} \times \psi_{\pm1} \times \psi_{\pm2} \right)(x, t, \nu).
\end{equation}
This symmetry implies a condition between the scattering matrix and its inverse
\begin{equation}
s^{-1}(\nu) = \Gamma s_\dagger(-\nu^*) \Gamma^{-1}, \quad \nu \in \Sigma, \tag{2.32}
\end{equation}
or, in component form
\begin{equation}
r_{11}(\nu) = s_{11}^*(\nu^*), \quad r_{12}(\nu) = s_{31}^*(\nu^*), \quad r_{13}(\nu) = s_{21}^*(\nu^*), \tag{2.33a}
\end{equation}
\begin{equation}
r_{21}(\nu) = s_{13}^*(\nu^*), \quad r_{22}(\nu) = s_{33}^*(\nu^*), \quad r_{23}(\nu) = s_{23}^*(\nu^*). \tag{2.33b}
\end{equation}
\[ r_{31}(\nu) = s_{12}^*(-\nu^*), \quad r_{32}(\nu) = s_{32}^*(-\nu^*), \quad r_{33}(\nu) = s_{22}^*(-\nu^*). \quad (2.33c) \]

The Schwarz reflection principle of (2.32) gives to the following results

\[ r_{11}(\nu) = s_{11}^*(-\nu^*), \quad \nu \in D_3 \cup D_4, \quad (2.34a) \]
\[ r_{22}(\nu) = s_{33}^*(-\nu^*), \quad \nu \in D_1 \cup D_2, \quad (2.34b) \]
\[ r_{33}(\nu) = s_{22}^*(-\nu^*), \quad \nu \in D_5 \cup D_6. \quad (2.34c) \]

The Jost functions and the auxiliary eigenfunctions obey the following symmetry relations

\[ \psi_{+1}^*(-\nu^*) = \frac{T_1}{3\gamma r_{22}(\nu)}[\chi_6(\nu) \times \psi_{+3}(\nu)] = \frac{T_1}{3\gamma r_{23}(\nu)}[\psi_{+2}(\nu) \times \chi_{1}(\nu)], \quad (2.35a) \]
\[ \psi_{-1}^*(-\nu^*) = \frac{T_1}{3\gamma s_{22}(\nu)}[\chi_3(\nu) \times \psi_{-3}(\nu)] = \frac{T_1}{3\gamma s_{23}(\nu)}[\psi_{-2}(\nu) \times \chi_{4}(\nu)], \quad (2.35b) \]
\[ \psi_{+2}^*(-\nu^*) = \frac{T_1}{3\gamma r_{22}(\nu)}[\chi_2(\nu) \times \psi_{+2}(\nu)] = \frac{T_1}{3\gamma r_{11}(\nu)}[\psi_{+1}(\nu) \times \chi_{3}(\nu)], \quad (2.35c) \]
\[ \psi_{-2}^*(-\nu^*) = \frac{T_1}{3\gamma s_{22}(\nu)}[\chi_5(\nu) \times \psi_{-2}(\nu)] = \frac{T_1}{3\gamma s_{11}(\nu)}[\psi_{-1}(\nu) \times \chi_{6}(\nu)], \quad (2.35d) \]
\[ \psi_{+3}^*(-\nu^*) = \frac{T_1}{3\gamma r_{11}(\nu)}[\chi_4(\nu) \times \psi_{+1}(\nu)] = \frac{T_1}{3\gamma r_{33}(\nu)}[\psi_{+3}(\nu) \times \chi_{5}(\nu)], \quad (2.35e) \]
\[ \psi_{-3}^*(-\nu^*) = \frac{T_1}{3\gamma s_{11}(\nu)}[\chi_1(\nu) \times \psi_{-1}(\nu)] = \frac{T_1}{3\gamma s_{33}(\nu)}[\psi_{-3}(\nu) \times \chi_{2}(\nu)], \quad (2.35f) \]

Besides, the auxiliary eigenfunctions and the Jost functions satisfy another symmetry conditions

\[ \chi_{1}^*(x, t, -\nu^*) = \frac{T_1}{3\gamma}[\psi_{-3} \times \psi_{+1}](x, t, \nu), \quad \nu \in D_3, \quad (2.36a) \]
\[ \chi_{2}^*(x, t, -\nu^*) = \frac{T_1}{3\gamma}[\psi_{+2} \times \psi_{-3}](x, t, \nu), \quad \nu \in D_2, \quad (2.36b) \]
\[ \chi_{3}^*(x, t, -\nu^*) = \frac{T_1}{3\gamma}[\psi_{-1} \times \psi_{+2}](x, t, \nu), \quad \nu \in D_1, \quad (2.36c) \]
\[ \chi_{4}^*(x, t, -\nu^*) = \frac{T_1}{3\gamma}[\psi_{+3} \times \psi_{-1}](x, t, \nu), \quad \nu \in D_6, \quad (2.36d) \]
\[ \chi_{5}^*(x, t, -\nu^*) = \frac{T_1}{3\gamma}[\psi_{-2} \times \psi_{+3}](x, t, \nu), \quad \nu \in D_5, \quad (2.36e) \]
\[
\chi_6^*(x, t, -\nu^*) = \frac{T_1}{3\gamma}[\psi_+ \times \psi_-](x, t, \nu), \quad \nu \in D_4. \quad (2.36f)
\]

**Second symmetry.** We note that if \( \psi_\pm(x, t, \nu) \) is a solution of the Lax pair (2.4), so is \( \psi_\pm(x, t, \alpha^2 \nu) \).

For all \( \nu \in \Sigma \), the Jost eigenfunctions satisfy the symmetry

\[
\psi_\pm(\nu) = \alpha \sigma \psi_\pm(\alpha^2 \nu) \Delta, \quad \Delta = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}. \quad (2.37)
\]

In component form, (2.37) yields

\[
\psi_{\pm 1}(\nu) = \alpha \sigma \psi_{\pm 2}(\alpha^2 \nu), \quad \nu \in \mathbb{R}_1 \subseteq \mathbb{R}_j, \quad j = 2, 3, \quad (2.38a)
\]
\[
\psi_{\pm 2}(\nu) = \alpha \sigma \psi_{\pm 3}(\alpha^2 \nu), \quad \nu \in \mathbb{R}_2 \subseteq \mathbb{R}_j, \quad j = 1, 3, \quad (2.38b)
\]
\[
\psi_{\pm 3}(\nu) = \alpha \sigma \psi_{\pm 1}(\alpha^2 \nu), \quad \nu \in \mathbb{R}_3 \subseteq \mathbb{R}_j, \quad j = 1, 2. \quad (2.38c)
\]

The scattering matrix satisfies the symmetry

\[
s(\alpha^2 \nu) = \Delta s(\nu) \Delta^{-1}, \quad \nu \in \Sigma. \quad (2.39)
\]

Componentwise, we have

\[
s_{11}(\nu) = s_{22}(\alpha^2 \nu), \quad s_{12}(\nu) = s_{23}(\alpha^2 \nu), \quad s_{13}(\nu) = s_{21}(\alpha^2 \nu), \quad (2.40a)
\]
\[
s_{21}(\nu) = s_{32}(\alpha^2 \nu), \quad s_{22}(\nu) = s_{33}(\alpha^2 \nu), \quad s_{23}(\nu) = s_{31}(\alpha^2 \nu), \quad (2.40b)
\]
\[
s_{31}(\nu) = s_{12}(\alpha^2 \nu), \quad s_{32}(\nu) = s_{13}(\alpha^2 \nu), \quad s_{33}(\nu) = s_{11}(\alpha^2 \nu). \quad (2.40c)
\]

Beside, the analyticity properties of the scattering matrix imply that

\[
s_{11}(\nu) = s_{22}(\alpha^2 \nu), \quad \nu \in D_1 \cup D_6, \quad s_{22}(\nu) = s_{33}(\alpha^2 \nu), \quad \nu \in D_4 \cup D_5, \quad s_{33}(\nu) = s_{11}(\alpha^2 \nu), \quad \nu \in D_2 \cup D_3, \quad (2.41a)
\]
\[
r_{11}(\nu) = r_{22}(\alpha^2 \nu), \quad \nu \in D_3 \cup D_4, \quad r_{22}(\nu) = r_{33}(\alpha^2 \nu), \quad \nu \in D_1 \cup D_2, \quad r_{33}(\nu) = r_{11}(\alpha^2 \nu), \quad \nu \in D_5 \cup D_6. \quad (2.41b)
\]

Finally, taking (2.31), (2.37) and (2.39) into consideration, we find

\[
\chi_1(\nu) = \alpha \sigma \chi_5(\alpha^2 \nu) = \alpha^2 \sigma^2 \chi_3(\alpha \nu), \quad \nu \in D_1, \quad (2.42a)
\]
Componentwise, we have

\[ \chi_2(\nu) = \alpha \sigma \chi_6(\alpha^2 \nu) = \alpha^2 \sigma^2 \chi_4(\alpha \nu), \quad \nu \in D_2. \]  

(2.42b)

Third symmetry. It is the fact that \( \psi_{\pm}(x, t, \nu) \) and \( \psi_{\pm}^*(x, t, \nu^*) \) are both the solutions of the system (2.4).

For all \( \nu \in \Sigma \), the Jost eigenfunctions satisfy the symmetry

\[ \psi_{\pm}(x, t, \nu) = \psi_{\pm}^*(x, t, \nu^*) \Gamma, \]  

(2.43)

or equivalently

\[ \psi_{\pm \pm}(x, t, \nu) = \psi_{\pm \pm}^*(x, t, \nu^*), \quad \nu \in \Re l_1 \subseteq \Re l_j, \quad j = 2, 3, \]  

(2.44a)

\[ \psi_{\pm \pm}(x, t, \nu) = \psi_{\pm \pm}^*(x, t, \nu^*), \quad \nu \in \Re l_2 \subseteq \Re l_j, \quad j = 1, 3, \]  

(2.44b)

\[ \psi_{\pm \pm}(x, t, \nu) = \psi_{\pm \pm}^*(x, t, \nu^*), \quad \nu \in \Re l_3 \subseteq \Re l_j, \quad j = 1, 2. \]  

(2.44c)

In the same way, the scattering matrices admit the symmetry

\[ s^*(\nu^*) = \Gamma s(\nu) \Gamma. \]  

(2.45)

Componentwise, we have

\[ s_{11}^*(\nu^*) = s_{11}(\nu), \quad s_{12}^*(\nu^*) = s_{13}(\nu), \quad s_{13}^*(\nu^*) = s_{12}(\nu), \]  

(2.46a)

\[ s_{21}^*(\nu^*) = s_{31}(\nu), \quad s_{22}^*(\nu^*) = s_{33}(\nu), \quad s_{23}^*(\nu^*) = s_{32}(\nu), \]  

(2.46b)

\[ s_{31}^*(\nu^*) = s_{21}(\nu), \quad s_{32}^*(\nu^*) = s_{23}(\nu), \quad s_{33}^*(\nu^*) = s_{22}(\nu). \]  

(2.46c)

In addition, the analyticity properties of the scattering matrix give the following results

\[ s_{11}(\nu) = s_{11}^*(\nu^*), \quad \nu \in D_1 \cup D_6, \quad s_{22}(\nu) = s_{33}^*(\nu^*), \quad \nu \in D_4 \cup D_5, \]  

(2.47a)

\[ s_{33}(\nu) = s_{22}^*(\nu^*), \quad \nu \in D_2 \cup D_3, \]  

(2.47b)

\[ r_{11}(\nu) = r_{11}^*(\nu^*), \quad \nu \in D_3 \cup D_4, \quad r_{22}(\nu) = r_{33}^*(\nu^*), \quad \nu \in D_1 \cup D_2, \]  

(2.47b)

\[ r_{33}(\nu) = r_{22}^*(\nu^*), \quad \nu \in D_5 \cup D_6. \]  

(2.47b)

The auxiliary eigenfunctions satisfy the symmetries

\[ \chi_1(\nu) = \frac{\gamma}{\gamma^*} \chi_6^*(\nu^*), \quad \nu \in D_1. \]  

(2.48a)

\[ \chi_2(\nu) = \frac{\gamma}{\gamma^*} \chi_5^*(\nu^*), \quad \nu \in D_2; \]  

(2.48b)

\[ \chi_3(\nu) = \frac{\gamma}{\gamma^*} \chi_4^*(\nu^*), \quad \nu \in D_3. \]  

(2.48c)

The matrix spectral function \( s(\nu) \) has the following properties:

(i) \( s(\nu) = I + o(\frac{1}{\nu}), \quad s^{-1}(\nu) = I + o(\frac{1}{\nu}), \quad \nu \rightarrow \infty, \)

(ii) \( \det s(\nu) = 1, \quad \det s^{-1}(\nu) = 1. \)
3 Discrete spectrum

3.1 Discrete spectrum

In the following, we switch our attention to discuss the discrete spectrum for the Tzitzéica equation. To this end, we introduce the following six $3 \times 3$ matrices

$$G_1(x,t,\nu) = \left( \psi_{-1}(x,t,\nu), \psi_{+2}(x,t,\nu), \chi_1(x,t,\nu) \right), \quad \nu \in D_1, \quad (3.1a)$$

$$G_2(x,t,\nu) = \left( \chi_2(x,t,\nu), \psi_{+2}(x,t,\nu), \psi_{-3}(x,t,\nu) \right), \quad \nu \in D_2, \quad (3.1b)$$

$$G_3(x,t,\nu) = \left( \psi_{+1}(x,t,\nu), \chi_3(x,t,\nu), \psi_{-3}(x,t,\nu) \right), \quad \nu \in D_3, \quad (3.1c)$$

$$G_4(x,t,\nu) = \left( \psi_{+1}(x,t,\nu), \psi_{-2}(x,t,\nu), \chi_4(x,t,\nu) \right), \quad \nu \in D_4, \quad (3.1d)$$

$$G_5(x,t,\nu) = \left( \chi_5(x,t,\nu), \psi_{-2}(x,t,\nu), \psi_{+3}(x,t,\nu) \right), \quad \nu \in D_5, \quad (3.1e)$$

$$G_6(x,t,\nu) = \left( \psi_{-1}(x,t,\nu), \chi_6(x,t,\nu), \psi_{+3}(x,t,\nu) \right), \quad \nu \in D_6. \quad (3.1f)$$

We note that the above matrix function are analytic in their respective domain. From (2.31), we know that

$$\det \left( G_1(x,t,\nu) \right) = s_{11}(\nu)r_{22}(\nu)\gamma, \quad \nu \in D_1, \quad (3.2a)$$

$$\det \left( G_2(x,t,\nu) \right) = s_{33}(\nu)r_{22}(\nu)\gamma, \quad \nu \in D_2, \quad (3.2b)$$

$$\det \left( G_3(x,t,\nu) \right) = r_{11}(\nu)s_{33}(\nu)\gamma, \quad \nu \in D_3, \quad (3.2c)$$

$$\det \left( G_4(x,t,\nu) \right) = r_{11}(\nu)s_{22}(\nu)\gamma, \quad \nu \in D_4, \quad (3.2d)$$

$$\det \left( G_5(x,t,\nu) \right) = r_{33}(\nu)s_{22}(\nu)\gamma, \quad \nu \in D_5, \quad (3.2e)$$

$$\det \left( G_6(x,t,\nu) \right) = s_{11}(\nu)s_{33}(\nu)\gamma, \quad \nu \in D_6. \quad (3.2f)$$

Thus, we can easily observe that the columns of $G_1(x,t,\nu)$ become linearly dependent at the zeros of $s_{11}(\nu)$ and $r_{22}(\nu)$. 

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According to the symmetry conditions about the scattering matrices discussed in section 2.4, we can locate the distribution of the discrete spectrum. For convenience, we let \( \nu_0 \in D_2 \) be a discrete eigenvalue of the scattering problem. It is easy to see that \( \nu_0 \) can be divided into two types of eigenvalue.

1. The first type is that \( \nu_0 = -\nu_0^* \).
2. The second type is that \( \nu_0 \neq -\nu_0^* \).

When \( \nu_0 = -\nu_0^* \), the symmetry conditions about the scattering matrices imply that

\[
\begin{align*}
  s_{33}(\nu_0) = 0 &\iff s_{22}(\nu_0^*) = 0 &\iff s_{11}(\alpha \nu_0^*) = 0 \\
  &\iff s_{11}(\alpha^2 \nu_0) = 0 &\iff s_{22}(\alpha \nu_0) = 0 &\iff s_{33}(\alpha^2 \nu_0^*) = 0 \\
  &\iff r_{11}(\alpha^2 \nu_0^*) = 0 &\iff r_{11}(\alpha \nu_0) = 0 &\iff r_{22}(\nu_0) = 0 \\
  &\iff r_{22}(\alpha \nu_0^*) = 0 &\iff r_{33}(\nu_0^*) = 0 &\iff r_{33}(\alpha^2 \nu_0) = 0.
\end{align*}
\]

When \( \nu_0 \neq -\nu_0^* \), we can derive

\[
\begin{align*}
  s_{33}(\nu_0) = 0 &\iff s_{22}(\nu_0^*) = 0 &\iff s_{11}(\alpha \nu_0^*) = 0 \\
  &\iff s_{11}(\alpha^2 \nu_0) = 0 &\iff s_{22}(\alpha \nu_0) = 0 &\iff s_{33}(\alpha^2 \nu_0^*) = 0 \\
  &\iff r_{11}(\alpha^2 \nu_0^*) = 0 &\iff r_{11}(\alpha \nu_0) = 0 &\iff r_{22}(\nu_0) = 0 \\
  &\iff r_{22}(\alpha \nu_0^*) = 0 &\iff r_{33}(\nu_0^*) = 0 &\iff r_{33}(\alpha^2 \nu_0) = 0.
\end{align*}
\]

In addition, for \( \nu_0 = -\nu_0^*, \nu_0 \in D_2 \), Jost functions and the auxiliary eigenfunctions admit the following statements

1: \( \chi_2(\nu_0) = 0 \),
2: \( \chi_4(\alpha \nu_0) = 0 \),
3: \( \chi_6(\alpha^2 \nu_0) = 0 \),
4: There exists a constant \( b_2 \) such that \( \psi_{+2}(-\nu_0^*) = b_2 \psi_{-3}(-\nu_0^*) \),
5: There exists a constant \( b_4 \) such that \( \psi_{+3}(-\alpha^2 \nu_0^*) = b_4 \psi_{-1}(-\alpha^2 \nu_0^*) \),
6: There exists a constant \( b_6 \) such that \( \psi_{+1}(-\alpha \nu_0^*) = b_6 \psi_{-2}(-\alpha \nu_0^*) \).

Similarly, when \( \nu_0 \neq -\nu_0^*, \nu_0 \in D_2 \), the following statements are equivalent

1: \( \chi_1(\alpha \nu_0^*) = 0 \),
2: \( \chi_3(\alpha^2 \nu_0^*) = 0 \),
3: \( \chi_5(\nu_0^*) = 0 \),
4: There exists a constant \( b_1 \) such that \( \psi_{+1}(-\alpha^2 \nu_0) = b_1 \psi_{-3}(-\alpha^2 \nu_0) \),
There exists a constant $b_3$ such that $\psi_{+2}(-\alpha \nu_0) = b_3 \psi_{-1}(-\alpha \nu_0)$,

(6): There exists a constant $b_5$ such that $\psi_{+3}(-\nu_0) = b_5 \psi_{-2}(-\nu_j)$.

In what follows, we assume that the zeros of $s_{jj}(\nu)$ and $r_{jj}(\nu)(j = 1, 2, 3)$ are simple.

(i) If $\nu_0$ is an eigenvalue of the first type, we can directly get $\chi_1(\alpha \nu_0^*) = \chi_2(\nu_0) = 0$. As a result, we observe that

$$\psi_{+2}(\nu_0) = f_0 \psi_{-3}(\nu_0), \quad \psi_{+1}(\alpha^2 \nu_0^*) = \hat{f}_0 \psi_{-3}(\alpha^2 \nu_0^*), \quad (3.5a)$$

$$\psi_{+2}(\alpha \nu_0^*) = \hat{f}_0 \psi_{-1}(\alpha \nu_0^*), \quad \psi_{+3}(\alpha^2 \nu_0) = \hat{f}_0 \psi_{-1}(\alpha^2 \nu_0), \quad (3.5b)$$

$$\psi_{+3}(\nu_0^*) = \hat{f}_0 \psi_{-2}(\nu_0^*), \quad \psi_{+1}(\alpha \nu_0) = \hat{f}_0 \psi_{-2}(\alpha \nu_0), \quad (3.5c)$$

where $f_0, \hat{f}_0, \tilde{f}_0, \tilde{f}_0, \hat{f}_0$ and $\hat{f}_0$ are the associated proportionality constants.

(ii) Suppose $\nu_0$ is an eigenvalue of the second type. In this case, we find that

$$\chi_2(\nu_0) = \hat{c}_0 \psi_{-3}(\nu_0), \quad \psi_{+2}(-\nu_0^*) = \hat{d}_0 \chi_2(-\nu_0^*), \quad (3.6a)$$

$$\chi_1(\alpha \nu_0^*) = c_0 \psi_{-1}(\alpha \nu_0^*), \quad \psi_{+2}(-\nu_0) = d_0 \chi_1(-\nu_0), \quad (3.6b)$$

$$\chi_3(\alpha^2 \nu_0^*) = \hat{c}_0 \psi_{-3}(\alpha^2 \nu_0^*), \quad \psi_{+1}(-\alpha^2 \nu_0) = \hat{d}_0 \chi_3(-\alpha^2 \nu_0), \quad (3.6c)$$

$$\chi_4(\alpha \nu_0) = \hat{c}_0 \psi_{-2}(\alpha \nu_0), \quad \psi_{+1}(-\alpha \nu_0^*) = \hat{d}_0 \chi_4(-\alpha \nu_0^*), \quad (3.6d)$$

$$\chi_5(\nu_0^*) = \hat{c}_0 \psi_{-2}(\nu_0^*), \quad \psi_{+3}(-\nu_0) = \hat{d}_0 \chi_5(-\nu_0), \quad (3.6e)$$

$$\chi_6(\alpha^2 \nu_0) = \hat{c}_0 \psi_{-1}(\alpha^2 \nu_0), \quad \psi_{+3}(-\alpha^2 \nu_0^*) = \hat{d}_0 \chi_6(-\alpha^2 \nu_0^*), \quad (3.6f)$$

where, $d_0, \hat{d}_0, \tilde{d}_0, \hat{d}_0, \hat{d}_0, \hat{d}_0, \hat{d}_0, \hat{d}_0, \hat{d}_0, \hat{d}_0, \hat{d}_0, \hat{d}_0$ and $\hat{c}_0$ are the associated proportionality constants.

### 3.2 Symmetries of the norming constants

To obtain the residue conditions for the inverse problem, we rewrite the equations (3.5) and (3.6) in terms of the modified eigenfunctions. Setting $\{\zeta_j\}_{j=1}^{N_i}$ be the set of all eigenvalues of the first type, then from (2.11) and (3.5), we have

$$\mu_{+2}(\zeta_j) = f_j \mu_{-3}(\zeta_j) e^{(\alpha^2 - \alpha) \zeta_j + (\alpha - \alpha^2) \zeta_j t}, \quad (3.7a)$$

$$\mu_{+1}(\alpha^2 \zeta_j^*) = \hat{f}_j \mu_{-3}(\alpha^2 \zeta_j^*) e^{(\alpha - \alpha^2) \zeta_j^* + (\alpha^2 - \alpha) \zeta_j^* t}, \quad (3.7b)$$

$$\mu_{+2}(\alpha \zeta_j^*) = \hat{f}_j \mu_{-1}(\alpha \zeta_j^*) e^{(\alpha - \alpha^2) \zeta_j^* + (\alpha^2 - \alpha) \zeta_j^* t} \quad (3.7c)$$
In order to simplify the calculation, we introduce the modified auxiliary eigenfunctions

\[ \mu_3 (\alpha^2 \zeta_j) = \tilde{f}_j \mu_{-1} (\alpha^2 \zeta_j) e^{(\alpha^2 - \alpha) \zeta_j x + (\alpha - \alpha^2) \zeta_j t} \] (3.7d)

\[ \mu_3 (\zeta_j^*) = \tilde{f}_j \mu_{-2} (\zeta_j^*) e^{(\alpha - \alpha^2) \zeta_j^* x + (\alpha^2 - \alpha) \zeta_j^* t}, \] (3.7e)

\[ \mu_1 (\alpha \zeta_j) = \tilde{f}_j \mu_{-2} (\alpha \zeta_j) e^{(\alpha^2 - \alpha) \zeta_j x + (\alpha - \alpha^2) \zeta_j t}. \] (3.7f)

Let \( \{ \nu_j \}_{j=1}^{N_2} \) be the eigenvalues of the second type, we obtain that

\[ m_2 (\nu_j) = \tilde{c}_j e^{(\alpha - 1) \nu_j x + (\alpha - 1) \nu_j^* t} \mu_{-3} (\nu_j), \] (3.9a)

\[ m_2 (-\nu_j^*) = \tilde{d}_j e^{(\alpha - 1) \nu_j^* x + (\alpha - 1) \nu_j t} m_2 (-\nu_j^*), \] (3.9b)

\[ m_3 (\alpha^2 \nu_j^*) = \tilde{c}_j e^{(\alpha - 1) \nu_j^* x + (\alpha - 1) \nu_j^* t} \mu_{-3} (\alpha^2 \nu_j^*), \] (3.9c)

\[ m_4 (\alpha \nu_j) = \tilde{c}_j e^{(\alpha - 1) \nu_j x + (\alpha - 1) \nu_j t} \mu_{-2} (\alpha \nu_j), \] (3.9d)

\[ m_5 (\nu_j^*) = \tilde{c}_j e^{(\alpha - 1) \nu_j^* x + (\alpha - 1) \nu_j t} \mu_{-2} (\nu_j^*), \] (3.9e)

\[ m_6 (\alpha^2 \nu_j) = \tilde{c}_j e^{(\alpha - 1) \nu_j x + (\alpha - 1) \nu_j t} \mu_{-1} (\alpha^2 \nu_j), \] (3.9f)

where \( \bar{\nu}_j = 1/\nu_j \).

The norming constants in Eqs. (3.7) and (3.9) admit the following symmetry relations

\[ f_j = \tilde{f}_j = \tilde{f}_j, \quad \tilde{f}_j = \tilde{f}_j = \tilde{f}_j, \quad f_j = \tilde{f}_j = \tilde{f}_j, \] (3.10a)

\[ c_j = \tilde{c}_j = \tilde{c}_j, \quad \tilde{c}_j = \tilde{c}_j = \tilde{c}_j, \quad c_j^* = -\frac{\gamma^*}{\gamma} \tilde{c}_j, \] (3.10b)
where
\[ f_j = -f_j^* \frac{r_{22}(\nu_j)}{s_{33}(\nu_j)}, \quad d_j = -\frac{c^*_j}{s_{33}(-\nu_j^*)}, \quad \hat{d}_j = \frac{\gamma^*}{\gamma s_{33}(-\nu_j^*)} c_j. \] (3.10d)

Here the dot denotes the derivative with respect to the parameter \( \nu \).

### 3.3 Trace formula

In this section, we consider the associated trace formula. Assume that \( s_{33}(\nu) \) has the simple zeros \( \{ \zeta_j : \zeta_j = -\zeta_j^*, \zeta_j \in D_2 \} \) and \( \{ \nu_j : \nu_j \neq -\nu_j^*, \nu_j \in D_2 \} \) according to the symmetry of scattering data (3.3) and (3.4), we define

\[
\tilde{s}_{11}(\nu) = \prod_{j=1}^{N_1} \frac{(\nu + \alpha^2 \zeta_j)(\nu + \alpha \zeta_j^*)}{(\nu - \alpha^2 \zeta_j)(\nu - \alpha \zeta_j^*)} \prod_{j=1}^{N_2} \frac{(\nu + \alpha \nu_j^*)(\nu - \alpha \nu_j^*)(\nu - \nu_j)(\nu - \nu_j^*)}{(\nu - \alpha \nu_j)(\nu + \alpha \nu_j)(\nu - \nu_j)(\nu - \nu_j^*)} s_{11}(\nu),
\]

\[
\tilde{s}_{22}(\nu) = \prod_{j=1}^{N_1} \frac{(\nu + \alpha \zeta_j)(\nu - \zeta_j^*)}{(\nu - \alpha \zeta_j)(\nu + \zeta_j^*)} \prod_{j=1}^{N_2} \frac{(\nu + \alpha \nu_j)(\nu - \alpha \nu_j)(\nu + \nu_j)(\nu - \nu_j^*)}{(\nu - \alpha \nu_j)(\nu + \alpha \nu_j)(\nu + \nu_j)(\nu - \nu_j^*)} s_{22}(\nu).
\]

(3.11a)

(3.11b)

Here, \( \tilde{s}_{11}(\nu) \) and \( \tilde{s}_{22}(\nu) \) are analytic in \( \nu \in D_1 \cup D_6 \) and \( \nu \in D_4 \cup D_5 \), respectively, whereas they have no zeros. After a few calculation, we find that the analytic scattering coefficients of the scattering matrix take the following form,

\[
s_{11}(\nu) = \prod_{j=1}^{N_1} \frac{(\nu - \alpha^2 \zeta_j)(\nu + \alpha^2 \zeta_j^*)}{(\nu + \alpha^2 \zeta_j)(\nu - \alpha^2 \zeta_j^*)} \prod_{j=1}^{N_2} \frac{(\nu - \alpha \nu^*_j)(\nu - \nu_j^*)(\nu + \nu_j^*)(\nu - \nu_j)(\nu + \nu_j)}{(\nu + \alpha \nu^*_j)(\nu + \nu_j^*)(\nu - \nu_j)(\nu + \nu_j)}
\times \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \log(1 - s_{12}(\eta)s_{12}(-\eta) - s_{13}(\eta)s_{13}(-\eta)) d\eta \right),
\]

(3.12a)

\[
s_{22}(\nu) = \prod_{j=1}^{N_1} \frac{(\nu + \alpha \zeta_j)(\nu - \zeta_j^*)}{(\nu + \alpha \zeta_j)(\nu + \zeta_j^*)} \prod_{j=1}^{N_2} \frac{(\nu + \alpha \nu_j)(\nu + \alpha \nu_j^*)(\nu + \nu_j^*)(\nu + \nu_j)(\nu + \nu_j^*)}{(\nu - \alpha \nu_j)(\nu + \alpha \nu_j)(\nu + \nu_j)(\nu + \nu_j^*)}
\times \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \log(1 - s_{12}(\eta)s_{12}(-\eta) - s_{13}(\eta)s_{13}(-\eta)) d\eta \right).
\]

(3.12b)
4 Inverse problem

4.1 Riemann-Hilbert problem

In this section, the piecewise meromorphic function $M(x,t,\nu)$ in six regions for the Tzitzéica equation were presented. According to the analyticities of the Jost functions and the auxiliary eigenfunctions, we define six piecewise meromorphic functions as follows

\[ M_1(x,t,\nu) = \left( \frac{\mu_{-1} \mu_{+2} m_1}{r_{22} s_{11}}, \nu \in D_1, \right) \]

\[ M_2(x,t,\nu) = \left( \frac{m_2}{s_{33}}, \frac{\mu_{+2}}{r_{22}}, \mu_{-3}, \nu \in D_2, \right) \]

\[ M_3(x,t,\nu) = \left( \frac{\mu_{+1}}{r_{11}}, \frac{m_3}{s_{33}}, \mu_{-3}, \nu \in D_3, \right) \]

\[ M_4(x,t,\nu) = \left( \frac{\mu_{+1}}{r_{11}}, \frac{\mu_{-2}}{s_{22}}, \mu_{+3}, \nu \in D_4, \right) \]

\[ M_5(x,t,\nu) = \left( \frac{m_5}{s_{22}}, \mu_{-2}, \frac{\mu_{+3}}{r_{33}}, \nu \in D_5, \right) \]

\[ M_6(x,t,\nu) = \left( \frac{\mu_{-1}}{r_{11}}, \frac{m_6}{s_{11}}, \mu_{+3}, \nu \in D_6, \right) \]

(4.1)

It is readily verified that $M_n(x,t,\nu)$ satisfy the following jump conditions

\[ M_+(x,t,\nu) = M_-(x,t,\nu) J(x,t,\nu), \quad \nu \in \Sigma, \]

(4.2)

where the matrices $M_+(x,t,\nu), M_-(x,t,\nu)$ and $J(x,t,\nu)$ admit the following definitions

\[ M(x,t,\nu) = \begin{cases} 
M_+(x,t,\nu), & \nu \in D_1 \cup D_3 \cup D_5, \\
M_-(x,t,\nu), & \nu \in D_2 \cup D_4 \cup D_6. 
\end{cases} \]

(4.3)
Asymptotic behaviors

\[ \mu_\pm(x, t, \nu) = \begin{pmatrix} 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{\nu} \begin{pmatrix} \eta - \frac{2}{3} u_x & \alpha \eta - \frac{2}{3} \alpha u_x & \alpha^2 \eta - \frac{2}{3} \alpha^2 u_x \\ \eta + \frac{1}{3} u_x & \eta + \frac{1}{3} u_x & \eta + \frac{1}{3} u_x \\ \eta + \frac{1}{3} u_x & \alpha^2 \eta + \frac{1}{3} \alpha^2 u_x & \alpha \eta + \frac{1}{3} \alpha u_x \end{pmatrix} + O\left(\frac{1}{\nu^2}\right), \quad \nu \to \infty, \]  

where \( \eta = \frac{1}{3} \int_{-\infty}^{x} u_\xi^2 d\xi \), which implies that

\[ \mathbf{M}_\pm(x, t, \nu) \to A, \quad \nu \to \infty. \]
4.2 The case of no poles

In this case, matrix functions $M_{\pm}(x, t, \nu)$ are analytic in their respective domains. Applying the Cauchy operator $\frac{1}{2\pi i} \int_{\Sigma} \frac{M_n J_n}{\zeta - \nu} d\zeta$ to the equation (4.5), we have the following integral representation

$$M_{\pm}(x, t, \nu) = A + \frac{1}{2\pi i} \int_{\Sigma_2 \cup \Sigma_4 \cup \Sigma_6} \frac{M_n J_n}{\zeta - \nu} d\zeta + \frac{1}{2\pi i} \int_{\Sigma_1 \cup \Sigma_5} \frac{M_{n+1} J_n}{\zeta - \nu} d\zeta,$$

where

$$\zeta \in \Sigma_n, \quad J_n = e^{\zeta x+\zeta^{-1}s^{-1}t} e^{-\zeta x-\zeta^{-1}s^{-1}t}.$$

4.3 The case of poles

In this case, we assume that $s_{33}(\nu)$ has the simple zeros $\{\zeta_j : \zeta_j = -\zeta_j^*\}_{j=1}^{N_1}$ and $\{\nu_j : \nu_j \neq -\nu_j^*\}_{j=1}^{N_2}$ in $D_2$. The meromorphic matrices defined in (4.1) satisfy the following residue conditions

$$M_{1,\alpha\zeta_j^*}(x, t, \nu) = \bar{F}_j \left(0, \mu_{-1}(\alpha\zeta_j^*), 0\right), M_{2,\zeta_j^*}(x, t, \nu) = F_j \left(0, \mu_{-3}(\zeta_j), 0\right),$$

(4.9a)

$$M_{3,\alpha^2\zeta_j^*}(x, t, \nu) = \hat{F}_j \left(\mu_{-3}(\alpha^2\zeta_j^*), 0, 0\right), M_{4,\alpha\zeta_j^*}(x, t, \nu) = \hat{F}_j \left(\mu_{-2}(\alpha\zeta_j), 0, 0\right),$$

(4.9b)

$$M_{5,\zeta_j^*}(x, t, \nu) = \hat{F}_j \left(0, 0, \mu_{-1}(\alpha^2\zeta_j)\right), M_{6,\alpha^2\zeta_j}(x, t, \alpha^2\zeta_j) = \hat{F}_j \left(0, 0, \mu_{-3}(\alpha^2\zeta_j)\right),$$

(4.9c)

$$M_{1,-\alpha\nu_j}(x, t, \nu) = D_j \left(0, \frac{m_1}{s_{11}}(-\alpha\nu_j), 0\right), M_{1,\alpha\nu_j^*}(x, t, \nu) = C_j \left(0, 0, \mu_{-1}(\alpha\nu_j^*)\right),$$

(4.10a)

$$M_{2,-\nu_j^*}(x, t, \nu) = \hat{D}_j \left(0, \frac{m_2}{s_{33}}(-\nu_j^*), 0\right), M_{2,\nu_j}(x, t, \nu) = \hat{C}_j \left(\mu_{-3}(\nu_j), 0, 0\right),$$

(4.10b)

$$M_{3,-\alpha^2\nu_j}(x, t, \nu) = \hat{D}_j \left(\frac{m_3}{s_{33}}(-\alpha^2\nu_j^*), 0, 0\right), M_{3,\alpha^2\nu_j^*}(x, t, \nu) = \hat{C}_j \left(\mu_{-3}(\alpha^2\nu_j^*), 0, 0\right),$$

(4.10c)

$$M_{4,-\alpha\nu_j^*}(x, t, \nu) = \hat{D}_j \left(\frac{m_4}{s_{22}}(-\alpha\nu_j^*), 0, 0\right), M_{4,\alpha\nu_j}(x, t, \nu) = \hat{C}_j \left(0, 0, \mu_{-2}(\alpha\nu_j)\right),$$

(4.10d)

$$M_{5,-\nu_j}(x, t, \nu) = \hat{D}_j \left(0, 0, \frac{m_5}{s_{22}}(-\nu_j)\right), M_{5,\nu_j^*}(x, t, \nu) = \hat{C}_j \left(\mu_{-2}(\nu_j^*), 0, 0\right),$$

(4.10e)
\[ \mathbf{M}_{\alpha^2\nu_j}(x, t, \nu) = \hat{D}_j \left( 0, 0, \frac{m_0}{s_{11}}(-\alpha^2 \nu_j^*) \right), \quad \mathbf{M}_{\alpha \nu_j}(x, t, \alpha^2 \nu_j) = \hat{C}_j \left( 0, \mu_{-1}(\alpha^2 \nu_j), 0 \right), \]

where

\[ \hat{F}_j(x, t, \nu) = \frac{\tilde{f}_j}{r_{22}(\alpha \nu_j^*)} e^{(\alpha - \alpha^2) \zeta_j^* x + (\alpha - \alpha^2) \zeta_j^* t}, \]
\[ \hat{F}_j(x, t, \nu) = \frac{\tilde{f}_j}{r_{22}(\alpha \nu_j^*)} e^{(\alpha - \alpha^2) \zeta_j^* x + (\alpha - \alpha^2) \zeta_j^* t}, \]
\[ \hat{F}_j(x, t, \nu) = \frac{\tilde{f}_j}{r_{33}(\nu_j)} e^{(\alpha - \alpha^2) \zeta_j^* x + (\alpha - \alpha^2) \zeta_j^* t}, \]

\[ \hat{C}_j(x, t) = \frac{c_j}{s_{11}(\alpha \nu_j^*)} e^{(\alpha^2 - 1) \nu_j x + (\alpha^2 - 1) \nu_j^* t}, \]
\[ \hat{C}_j(x, t) = \frac{c_j}{s_{33}(\nu_j)} e^{(\alpha^2 - 1) \nu_j x + (\alpha^2 - 1) \nu_j^* t}, \]
\[ \hat{C}_j(x, t) = \frac{c_j}{s_{33}(\alpha \nu_j^*)} e^{(\alpha) \nu_j x + (\alpha) \nu_j^* t}, \]
\[ \hat{C}_j(x, t) = \frac{c_j}{s_{22}(\alpha \nu_j^*)} e^{(\alpha - 1) \nu_j x + (\alpha - 1) \nu_j^* t}, \]
\[ \hat{C}_j(x, t) = \frac{c_j}{s_{11}(\alpha \nu_j^*)} e^{(\alpha^2 - 1) \nu_j x + (\alpha^2 - 1) \nu_j^* t}. \]

It is remarked that

\[ \hat{F}_j(x, t, \nu) = F_j^*(x, t, \nu), \quad \hat{F}_j(x, t, \nu) = \alpha F_j^*(x, t, \nu), \quad \hat{F}_j(x, t, \nu) = \alpha F_j(x, t, \nu), \]
\[ \hat{F}_j(x, t, \nu) = \alpha^2 F_j^*(x, t, \nu), \]
\[ \hat{F}_j(x, t, \nu) = \alpha \hat{F}_j(x, t, \nu), \]
\[ \hat{D}_j(x, 0, \nu) = \alpha \hat{D}_j(x, t, \nu), \]
\[ \hat{D}_j(x, t, \nu) = \alpha \hat{D}_j(x, t, \nu), \]
\[ \hat{D}_j(x, t, \nu) = \alpha \hat{D}_j(x, t, \nu), \]
\[ \hat{D}_j(x, t, \nu) = \alpha \hat{D}_j(x, t, \nu). \]
and

\[ C_j(x,t,\nu) = \alpha \hat{C}_j(x,t,\nu) = \alpha^2 \check{C}_j(x,t,\nu), \hat{C}_j(x,t,\nu) = \alpha^2 \check{C}_j(x,t,\nu), \]

\[ \hat{D}_j(x,t,\nu) = (\check{C}_j(x,t,\nu))^*, \check{C}_j(x,t,\nu) = (\hat{D}_j(x,t,\nu))^*. \]

The key to solving the RH problem of (4.2) is to convert it into a mixed system of algebraic-integral equations. We similarly apply \( \frac{1}{2\pi i} \int_{\Sigma} \frac{f}{\zeta - \nu} d\zeta \) to both sides of the jump condition (4.3). Taking into account the symmetries properties of the involved functions, we can easily obtain

\[ m^+_1(x,t,\nu) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{j=1}^{N_1} \left\{ \frac{\hat{F}_{j\mu-3}(\alpha^2\zeta_j^*)}{\nu - \alpha^2\zeta_j} + \frac{\hat{F}_{j\mu-2}(\alpha\zeta_j)}{\nu - \alpha\zeta_j} \right\} + \sum_{j=1}^{N_2} \left\{ \frac{\check{C}_{j\mu-3}(\nu_j^*)}{\nu - \nu_j^*} + \frac{\check{C}_{j\mu-2}(\nu_j^*)}{\nu - \nu_j^*} \right\} \]

\[ + \frac{\hat{D}_j}{\nu + \alpha^2\nu_j} m_3 \frac{(-\alpha^2\nu_j^*)}{s_{33}} + \frac{\check{C}_{j\mu-2}(\nu_j^*)}{\nu - \nu_j^*} + \frac{\check{C}_{j\mu-3}(\alpha^2\nu_j^*)}{\nu - \alpha^2\nu_j} \]

\[ + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \check{J}_n)_1(\zeta)}{\zeta - z} d\zeta, \]

\[ m^+_2(x,t,\nu) = \begin{pmatrix} \alpha^2 \\ \alpha \\ 1 \end{pmatrix} + \sum_{j=1}^{N_1} \left\{ \frac{\hat{F}_{j\mu-1}(\alpha\zeta_j^*)}{\nu - \alpha\zeta_j^*} + \frac{F_{j\mu-3}(\zeta_j)}{\nu - \zeta_j} \right\} + \sum_{j=1}^{N_2} \left\{ \frac{\check{D}_j m_{211}(-\alpha\nu_j)}{\nu + \alpha\nu_j} \right\} \]

\[ + \frac{\hat{D}_j}{\nu + \nu_j^*} m_2 \frac{(-\nu_j^*)}{s_{33}} + \frac{\check{C}_{j\mu-3}(\alpha^2\nu_j^*)}{\nu - \alpha^2\nu_j} + \frac{\check{C}_{j\mu-1}(\alpha^2\nu_j)}{\nu - \alpha^2\nu_j} \]

\[ + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \check{J}_n)_2(\zeta)}{\zeta - z} d\zeta, \]

\[ m^-_2(x,t,\nu) = \begin{pmatrix} \alpha^2 \\ \alpha \\ 1 \end{pmatrix} + \sum_{j=1}^{N_1} \left\{ \frac{\hat{F}_{j\mu-1}(\alpha\zeta_j^*)}{\nu - \alpha\zeta_j^*} + \frac{F_{j\mu-3}(\zeta_j)}{\nu - \zeta_j} \right\} + \sum_{j=1}^{N_2} \left\{ \frac{\check{D}_j m_{211}(-\alpha\nu_j)}{\nu + \alpha\nu_j} \right\} \]

\[ + \frac{\hat{D}_j}{\nu + \nu_j^*} m_2 \frac{(-\nu_j^*)}{s_{33}} + \frac{\check{C}_{j\mu-3}(\alpha^2\nu_j^*)}{\nu - \alpha^2\nu_j} + \frac{\check{C}_{j\mu-1}(\alpha^2\nu_j)}{\nu - \alpha^2\nu_j} \]

\[ + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \check{J}_n)_2(\zeta)}{\zeta - z} d\zeta, \]

(4.13)
\[ m_3^\pm(x, t, \nu) = \left( \frac{\alpha}{\alpha^2} \right) + \sum_{j=1}^{N} \left\{ \frac{\dot{F}_j \mu_{-2}(\zeta_j^s)}{\nu - \zeta_j^s} + \frac{\tilde{F}_j \mu_{-1}(\alpha^2 \zeta_j^s)}{\nu - \alpha^2 \zeta_j^s} \right\} + \sum_{j=1}^{N_2} \left\{ \frac{C_j \mu_{-1}(\alpha \nu_j^s)}{\nu - \alpha \nu_j^s} \right\} \\
+ \frac{\dot{C}_j \mu_{-2}(\alpha \nu_j)}{\nu - \alpha \nu_j} + \frac{\tilde{D}_j \rho_{s22}(\nu_j)}{\nu + \nu_j s_{22}} + \frac{\tilde{D}_j \rho_{s11}(\nu_j)}{\nu + \alpha^2 \nu_j^s s_{11}} \right\} \\
+ \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \tilde{J}_n)_{33}(\zeta)}{\zeta - z} d\zeta. \tag{4.16} \]

\[ m_3^\pm(x, t, \nu) = \left( \frac{\alpha}{\alpha^2} \right) + \sum_{j=1}^{N} \left\{ \frac{\dot{F}_j \mu_{-2}(\zeta_j^s)}{\nu - \zeta_j^s} + \frac{\tilde{F}_j \mu_{-1}(\alpha^2 \zeta_j^s)}{\nu - \alpha^2 \zeta_j^s} \right\} + \sum_{j=1}^{N_2} \left\{ \frac{C_j \mu_{-1}(\alpha \nu_j^s)}{\nu - \alpha \nu_j^s} \right\} \\
+ \frac{\dot{C}_j \mu_{-2}(\alpha \nu_j)}{\nu - \alpha \nu_j} + \frac{\tilde{D}_j \rho_{s22}(\nu_j)}{\nu + \nu_j s_{22}} + \frac{\tilde{D}_j \rho_{s11}(\nu_j)}{\nu + \alpha^2 \nu_j^s s_{11}} \right\} \\
+ \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \tilde{J}_n)_{33}(\zeta)}{\zeta - z} d\zeta. \tag{4.17} \]

To finish the potential reconstruction about the scattering data, we consider the expansion of (4.13) at \( \nu \to \infty \), and compare it with (4.17). And the \( O(\nu^{-1}) \) terms imply the representation of the potential matrix about the eigenfunctions at certain eigenvalues, which can be reduced by taking \( \nu = -\alpha^2 \nu_j^s, \nu_j^s \) in (4.14), \( \nu = \alpha \zeta_j^s \) in (4.15), \( \nu = \alpha^2 \zeta_j^s \) in (4.16) and \( \nu = \nu_j, -\alpha \nu_j^s \) in (4.17). Then the potential can be reconstructed as

\[ U_x = \sum_{j=1}^{N_1} \left\{ \hat{F}_j \left( \mu_{-23} - \mu_{-13} \right)(\alpha^2 \zeta_j^s) + \hat{\tilde{F}}_j \left( \mu_{-22} - \mu_{-12} \right)(\alpha \zeta_j) \right\} \\
+ \sum_{j=1}^{N_2} \left\{ \hat{C}_j \left( \mu_{-23} - \mu_{-13} \right)(\nu_j) + \hat{\tilde{D}}_j \left( \frac{m_{23}}{s_{33}} - \frac{m_{13}}{s_{33}} \right)(-\alpha^2 \nu_j) \right\} \tag{4.18} \\
+ \hat{\tilde{D}}_j \left( \frac{m_{24}}{s_{22}} - \frac{m_{14}}{s_{22}} \right)(-\alpha \nu_j^s) \right\} \\
- \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \tilde{J}_n)_{21} - (M - \tilde{J}_n)_{11}}{\zeta} d\zeta. \]
5 Explicit solutions

In the reflectionless case, it is convenient to write (4.17) as fractional equation

$$U_x = \frac{\det(G' + b^*Y^T)}{\det G'} - \frac{\det(G + bY^T)}{\det G},$$

where

$$G = I - F, \quad G' = I - F', \quad b = (b_1, \ldots, b_{2N_1+4N_2})^T, \quad Y = (y_1, \ldots, y_{2N_1+4N_2})^T,$$

and $F_{ij}, b_j, y_j$ are given by (A.1)-(A.2), (A.4)-(A.9) in the appendix.

(1) I-explicit solution.
Assuming $N_1 = 1$ and $N_2 = 0$, and setting $\zeta_1 = ie^s, s \in \mathbb{R}, c_0, d_0 \in \mathbb{R}$, we then find the following explicit solution

$$u(x, t, \nu) = \partial^{-1} \left\{ -\frac{6e^s \sinh(c) \sin(d - \frac{\pi}{3})}{\sinh^2(c) + 2 \cos(d + \frac{2\pi}{3}) \cosh(c) + 2 \cos(2d - \frac{4\pi}{3})} \right\},$$

where

$$c = c_0 + \sqrt{3}(e^s x + e^{-s} t), \quad d = d_0.$$

(2) II-explicit solution.
Choosing $N_1 = 0$ and $N_2 = 1$, and supposing $\nu_1 = e^{\epsilon_1+im}, \epsilon_0, \epsilon_1, \eta_0, \eta_1 \in \mathbb{R}$, we get another explicit solution for the Eq. (1.1) as

$$u(x, t, \nu) = \partial^{-1} \left\{ 3\sqrt{3}e^{\epsilon_1} A \frac{A}{B} \right\},$$

where

$$A = -\omega \sin 2b + e^{2\xi - 3\tau} \tan \eta_1 \omega \sinh(2\tau + 2a)$$
$$- \frac{3}{8} \omega e^{2\xi - \tau + \epsilon_0} [\sin(s + \gamma) \cos b \sinh(\tau + a) - \sin b \cos(s + \gamma) \cosh(\tau + a)]$$
$$+ \frac{1}{2} e^{6\xi - 3\tau} [\sin b \cos s \cosh(3a + 3\tau) + \sin s \cos b \sinh(3a + 3\tau)]$$
$$- \frac{1}{2} e^{2\xi - \tau} [\sin s \cos b \sinh(a + \tau) + \sin 3b \cos s \cosh(a + \tau)],$$

$$B = \left[ \cos b - e^{2\xi - \tau} \cosh(a + \tau) \right]^2 \left\{ e^{4\xi - 2\tau} \cosh^2(a + \tau) + \cos^2 b - \frac{\cos^2(\eta_1 - \frac{\pi}{3})}{\cos \eta_1 \cos(\eta_1 + \frac{\pi}{3})} \right\}.$$
with

\[ a = \epsilon_0 + e^{\epsilon_1} \left( \frac{\sqrt{3}}{2} \sin \eta_1 - \frac{3}{2} \cos \eta_1 \right) x - e^{-\epsilon_1} \left( \frac{3}{2} \cos \eta_1 - \frac{\sqrt{3}}{2} \sin \eta_1 \right) t, \]
\[ b = \eta_0 + e^{\epsilon_1} \left( \frac{3}{2} \sin \eta_1 + \frac{\sqrt{3}}{2} \cos \eta_1 \right) x - e^{-\epsilon_1} \left( \frac{\sqrt{3}}{2} \cos \eta_1 + \frac{3}{2} \sin \eta_1 \right) t, \]
\[ s = \eta_1 + \frac{2}{3} \pi, \quad \tau = \frac{1}{2} \ln \left\{ \frac{\tan \eta_1}{\tan(\eta_1 + \frac{\pi}{3})} \right\}, \quad \xi = \frac{1}{2} \ln \left\{ \frac{\tan \eta_1}{\tan(\eta_1 - \frac{\pi}{3})} \right\}, \]
\[ \gamma = -\frac{2\pi i}{3} - i \frac{1}{2} \ln \left\{ \frac{e^{i(4m - \frac{2\pi}{3})} + 3e^{-4im} + 2e^{\frac{2\pi}{3}}}{2 + e^{-i(4m + \frac{2\pi}{3})} + 3e^{i(4m + \frac{2\pi}{3})}} \right\}, \]
\[ \omega = \frac{\cos^2(\eta_1 - \frac{\pi}{3})}{\cos \eta_1 \cos(\eta_1 + \frac{\pi}{3})}. \]

**Appendix**

**A Reflectionless potentials**

Through calculating, we can give the coefficients \( b_j(x, t) \) and \( y_j(x, t) \) as follows

\[ b_j(x, t, \nu) = \begin{cases} 
\alpha^2, & j = 1, \ldots, N_1, \\
\alpha, & j = N_1 + 1, \ldots, 2N_1, \\
\alpha, & j = 2N_1 + 1, \ldots, 2N_1 + N_2, \\
\alpha^2, & j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, \\
\alpha^2, & j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, \\
\alpha, & j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2, 
\end{cases} \tag{A.1} \]

\[ y_j(x, t, \nu) = \begin{cases} 
\hat{F}_j, & j = 1, \ldots, N_1, \\
\hat{F}_{j-N_1}, & j = N_1 + 1, \ldots, 2N_1, \\
\hat{C}_{j-2N_1}, & j = 2N_1 + 1, \ldots, 2N_1 + N_2, \\
\hat{C}_{j-2N_1-N_2}, & j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, \\
\hat{D}_{j-2N_1-2N_2}, & j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, \\
\hat{D}_{j-2N_1-3N_2}, & j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2. 
\end{cases} \tag{A.2} \]

For convenience, these definitions are given

\[ F^{(1)}_n(x, t, \nu) = F_n(x, t, \nu) \frac{\alpha^2}{\nu - \zeta_n}, \quad F^{(2)}_n(x, t, \nu) = \hat{F}_n(x, t, \nu) \frac{\alpha^2}{\nu - \alpha \zeta_n}. \tag{A.3a} \]
\[
F_n^{(3)}(x, t, \nu) = \tilde{F}_n(x, t, \nu) \frac{\alpha}{\nu - \alpha^2 \zeta_n}, \quad F_n^{(4)}(x, t, \nu) = \dot{F}_n(x, t, \nu) \frac{\alpha}{\nu - \zeta_n}; \quad (A.3b)
\]
\[
C_n^{(1)}(x, t, \nu) = \tilde{C}_n(x, t, \nu) \frac{\alpha^2}{\nu - \alpha^2 \nu_n}, \quad C_n^{(2)}(x, t, \nu) = \dot{C}_n(x, t, \nu) \frac{\alpha^2}{\nu - \alpha^2 \nu_n}; \quad (A.3c)
\]
\[
C_n^{(3)}(x, t, \nu) = \tilde{C}_n(x, t, \nu) \frac{\alpha}{\nu - \nu_n}, \quad C_n^{(4)}(x, t, \nu) = \dot{C}_n(x, t, \nu) \frac{\alpha}{\nu - \nu_n}; \quad (A.3d)
\]
\[
D_n^{(1)}(x, t, \nu) = D_n(x, t, \nu) \frac{\alpha^2}{\nu + \alpha \nu_n}, \quad D_n^{(2)}(x, t, \nu) = \dot{D}_n(x, t, \nu) \frac{\alpha^2}{\nu + \nu_n}; \quad (A.3e)
\]
\[
D_n^{(3)}(x, t, \nu) = \dot{D}_n(x, t, \nu) \frac{\alpha}{\nu + \nu_n}, \quad D_n^{(4)}(x, t, \nu) = \ddot{D}_n(x, t, \nu) \frac{\alpha}{\nu + \alpha \nu_n}. \quad (A.3f)
\]

Additionally, these terms \(F_{ij}\) can be derived from (A.3)

\[
F_{ij}(x, t, \nu) = F_j^{(1)}(\alpha \zeta_i) \quad (i, j = 1, \ldots, N_1), \quad (A.4a)
\]

For \(i = 1, \ldots, N_1, j = N_1 + 1, \ldots, 2N_1,

\[
F_{ij}(x, t, \nu) = F_{j-N_1}^{(2)}(\alpha \zeta_i). \quad (A.4b)
\]

For \(i = 1, \ldots, N_1, j = 2N_1 + 1, \ldots, 2N_1 + N_2,

\[
F_{ij}(x, t, \nu) = C_{j-2N_1}^{(1)}(\alpha \zeta_i). \quad (A.4c)
\]

For \(i = 1, \ldots, N_1, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2,

\[
F_{ij}(x, t, \nu) = C_{j-2N_1-N_2}^{(2)}(\alpha \zeta_i). \quad (A.4d)
\]

For \(i = 1, \ldots, N_1, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2,

\[
F_{ij}(x, t, \nu) = D_{j-2N_1-2N_2}^{(1)}(\alpha \zeta_i). \quad (A.4e)
\]

For \(i = 1, \ldots, N_1, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2,

\[
F_{ij}(x, t, \nu) = D_{j-2N_1-3N_2}^{(2)}(\alpha \zeta_i). \quad (A.4f)
\]

For \(i = N_1 + 1, \ldots, 2N_1, j = 1, \ldots, N_1,

\[
F_{ij}(x, t, \nu) = F_j^{(3)}(\alpha^2 \zeta_i^{*}; N_1). \quad (A.5a)
\]

For \(i = N_1 + 1, \ldots, 2N_1, j = N_1 + 1, \ldots, 2N_1,

\[
F_{ij}(x, t, \nu) = F_{j-N_1}^{(4)}(\alpha^2 \zeta_i^{*}). \quad (A.5b)
\]
For $i = N_1 + 1, \ldots, 2N_1, j = 2N_1 + N_1, \ldots, 2N_1 + N_2$,

$$F_{ij}(x, t, \nu) = C_{j-2N_1}^{(3)}(\alpha^2 \zeta_{i-N_1}^*).$$  \hfill (A.5c)

For $i = N_1 + 1, \ldots, 2N_1, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2$,

$$F_{ij}(x, t, \nu) = C_{j-2N_1-N_2}^{(4)}(\alpha^2 \zeta_{i-N_1}^*).$$  \hfill (A.5d)

For $i = N_1 + 1, \ldots, 2N_1, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$,

$$F_{ij}(x, t, \nu) = D_{j-2N_1-2N_2}^{(3)}(\alpha^2 \zeta_{i-N_1}^*).$$  \hfill (A.5e)

For $i = N_1 + 1, \ldots, 2N_1, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$,

$$F_{ij}(x, t, \nu) = D_{j-2N_1-3N_2}^{(4)}(\alpha^2 \zeta_{i-N_1}^*).$$  \hfill (A.5f)

For $i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 1, \ldots, N_1$,

$$F_{ij}(x, t, \nu) = F_j^{(3)}(\nu_{i-2N_1}).$$ \hfill (A.6a)

For $i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = N_1 + 1, \ldots, 2N_1$,

$$F_{ij}(x, t, \nu) = F_{j-N_1}^{(4)}(\nu_{i-2N_1}).$$ \hfill (A.6b)

For $i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 2N_1 + 1, \ldots, 2N_1 + N_2$,

$$F_{ij}(x, t, \nu) = C_{j-2N_1}^{(3)}(\nu_{i-2N_1}).$$ \hfill (A.6c)

For $i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2$,

$$F_{ij}(x, t, \nu) = C_{j-2N_1-N_2}^{(4)}(\nu_{i-2N_1}).$$ \hfill (A.6d)

For $i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$,

$$F_{ij}(x, t, \nu) = D_{j-2N_1-2N_2}^{(3)}(\nu_{i-2N_1}).$$ \hfill (A.6e)

For $i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$,

$$F_{ij}(x, t, \nu) = D_{j-2N_1-3N_2}^{(4)}(\nu_{i-2N_1}).$$ \hfill (A.6f)
For $i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 1, \ldots, N_1,$

$$F_{ij}(x, t, \nu) = F^{(1)}_{j} (\nu_{i-2N_1-2N_2}^*).$$ \hspace{1cm} (A.7a)

For $i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = N_1 + 1, \ldots, 2N_1,$

$$F_{ij}(x, t, \nu) = F^{(2)}_{j-N_1} (\nu_{i-2N_1-N_2}^*).$$ \hspace{1cm} (A.7b)

For $i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 2N_1 + 1, \ldots, 2N_1 + N_2,$

$$F_{ij}(x, t, \nu) = C^{(1)}_{j-2N_1} (\nu_{i-2N_1-N_2}^*).$$ \hspace{1cm} (A.7c)

For $i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2,$

$$F_{ij}(x, t, \nu) = C^{(2)}_{j-2N_1-N_2} (\nu_{i-2N_1-N_2}^*).$$ \hspace{1cm} (A.7d)

For $i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2,$

$$F_{ij}(x, t, \nu) = D^{(1)}_{j-2N_1-2N_2} (\nu_{i-2N_1-N_2}^*).$$ \hspace{1cm} (A.7e)

For $i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2,$

$$F_{ij}(x, t, \nu) = D^{(2)}_{j-2N_1-3N_2} (\nu_{i-2N_1-N_2}^*).$$ \hspace{1cm} (A.7f)

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = 1, \ldots, N_1,$

$$F_{ij}(x, t, \nu) = F^{(1)}_{j} (-\alpha^2 \nu_{i-2N_1-2N_2}).$$ \hspace{1cm} (A.8a)

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = N_1 + 1, \ldots, 2N_1,$

$$F_{ij}(x, t, \nu) = F^{(2)}_{j-N_1} (-\alpha^2 \nu_{i-2N_1-2N_2}).$$ \hspace{1cm} (A.8b)

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = 2N_1 + 1, \ldots, 2N_1 + N_2,$

$$F_{ij}(x, t, \nu) = C^{(1)}_{j-2N_1} (-\alpha^2 \nu_{i-2N_1-2N_2}).$$ \hspace{1cm} (A.8c)

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2,$

$$F_{ij}(x, t, \nu) = C^{(2)}_{j-2N_1-N_2} (-\alpha^2 \nu_{i-2N_1-2N_2}).$$ \hspace{1cm} (A.8d)
For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$, $j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$,

$$F_{ij}(x, t, \nu) = D_{j-2N_1-2N_2}^{(1)}(-\alpha^2 \nu_{i-2N_1-2N_2}).$$  \hfill (A.8e)

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$, $j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$,

$$F_{ij}(x, t, \nu) = D_{j-2N_1-3N_2}^{(2)}(-\alpha^2 \nu_{i-2N_1-2N_2}).$$  \hfill (A.8f)

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, $j = 1, \ldots, N_1$,

$$F_{ij}(x, t, \nu) = F^{(3)}_{ij}(-\alpha \nu^*_{i-2N_1-3N_2}).$$  \hfill (A.9a)

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, $j = N_1 + 1, \ldots, 2N_1$,

$$F_{ij}(x, t, \nu) = F^{(4)}_{ij}(-\alpha \nu^*_{i-2N_1-3N_2}).$$  \hfill (A.9b)

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, $j = 2N_1 + 1, \ldots, 2N_1 + N_2$,

$$F_{ij}(x, t, \nu) = C^{(3)}_{j-2N_1-N_2}(-\alpha \nu^*_{i-2N_1-3N_2}).$$  \hfill (A.9c)

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, $j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2$,

$$F_{ij}(x, t, \nu) = C^{(3)}_{j-2N_1-N_2}(-\alpha \nu^*_{i-2N_1-3N_2}).$$  \hfill (A.9d)

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, $j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$,

$$F_{ij}(x, t, \nu) = D^{(3)}_{j-2N_1-2N_2}(-\alpha \nu^*_{i-2N_1-3N_2}).$$  \hfill (A.9e)

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, $j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$,

$$F_{ij}(x, t, \nu) = D^{(4)}_{j-2N_1-3N_2}(-\alpha \nu^*_{i-2N_1-3N_2}).$$  \hfill (A.9f)

For $i, j = 1, \ldots, N_1$,

$$F'_{ij}(x, t, \nu) = \alpha^2 F^{(1)}_{ij}(\alpha \zeta_i).$$  \hfill (A.10a)

For $i = 1, \ldots, N_1$, $j = N_1 + 1, \ldots, 2N_1$,

$$F'_{ij}(x, t, \nu) = \alpha^2 F^{(2)}_{j-N_1}(\alpha \zeta_i).$$  \hfill (A.10b)

For $i = 1, \ldots, N_1$, $j = 2N_1 + 1, \ldots, 2N_1 + N_2$,

$$F'_{ij}(x, t, \nu) = \alpha^2 C^{(1)}_{j-2N_1}(\alpha \zeta_i).$$  \hfill (A.10c)
For $i = 1, \ldots, N_1, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2$,

$$F'_{ij}(x, t, \nu) = \alpha^2 C^{(2)}_{j-2N_1-N_2}(\alpha \zeta_i).$$  \hfill (A.10d)

For $i = 1, \ldots, N_1, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$,

$$F'_{ij}(x, t, \nu) = \alpha^2 D^{(4)}_{j-2N_1-2N_2}(\alpha \zeta_i).$$  \hfill (A.10e)

For $i = 1, \ldots, N_1, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$,

$$F'_{ij}(x, t, \nu) = \alpha^2 D^{(2)}_{j-2N_1-3N_2}(\alpha \zeta_i).$$  \hfill (A.10f)

For $i = N_1 + 1, \ldots, 2N_1, j = 1, \ldots, N_1$,

$$F'_{ij}(x, t, \nu) = \alpha F^{(3)}_j(\alpha^2 \zeta^*_i-N_1).$$  \hfill (A.11a)

For $i = N_1 + 1, \ldots, 2N_1, j = N_1 + 1, \ldots, 2N_1$,

$$F'_{ij}(x, t, \nu) = \alpha F^{(4)}_j(\alpha^2 \zeta^*_i-N_1).$$  \hfill (A.11b)

For $i = N_1 + 1, \ldots, 2N_1, j = 2N_1 + 1, \ldots, 2N_1 + N_2$,

$$F'_{ij}(x, t, \nu) = \alpha C^{(3)}_{j-2N_1}(\alpha^2 \zeta^*_i-N_1).$$  \hfill (A.11c)

For $i = N_1 + 1, \ldots, 2N_1, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2$,

$$F'_{ij}(x, t, \nu) = \alpha C^{(4)}_{j-2N_1-N_2}(\alpha^2 \zeta^*_i-N_1).$$  \hfill (A.11d)

For $i = N_1 + 1, \ldots, 2N_1, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$,

$$F'_{ij}(x, t, \nu) = \alpha D^{(3)}_{j-2N_1-2N_2}(\alpha^2 \zeta^*_i-N_1).$$  \hfill (A.11e)

For $i = N_1 + 1, \ldots, 2N_1, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$,

$$F'_{ij}(x, t, \nu) = \alpha D^{(4)}_{j-2N_1-3N_2}(\alpha^2 \zeta^*_i-N_1).$$  \hfill (A.11f)

For $i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 1, \ldots, N_1$,

$$F'_{ij}(x, t, \nu) = \alpha F^{(3)}_j(\nu_{i-2N_1}).$$  \hfill (A.12a)
For \( i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = N_1 + 1, \ldots, 2N_1, \)
\[
F'_{ij}(x,t,\nu) = \alpha F^{(4)}_{j-N_1}(\nu_{i-2N_1}). \tag{A.12b}
\]

For \( i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 2N_1 + 1, \ldots, 2N_1 + N_2, \)
\[
F'_{ij}(x,t,\nu) = \alpha C^{(3)}_{j-2N_1}(\nu_{i-2N_1}). \tag{A.12c}
\]

For \( i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, \)
\[
F'_{ij}(x,t,\nu) = \alpha C^{(4)}_{j-2N_1-N_2}(\nu_{i-2N_1}). \tag{A.12d}
\]

For \( i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, \)
\[
F'_{ij}(x,t,\nu) = \alpha D^{(3)}_{j-2N_1-2N_2}(\nu_{i-2N_1}). \tag{A.12e}
\]

For \( i = 2N_1 + 1, \ldots, 2N_1 + N_2, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2, \)
\[
F'_{ij}(x,t,\nu) = \alpha D^{(4)}_{j-2N_1-3N_2}(\nu_{i-2N_1}). \tag{A.12f}
\]

For \( i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 1, \ldots, N_1, \)
\[
F'_{ij}(x,t,\nu) = \alpha^2 F^{(1)}_{j}(\nu^*_{i-2N_1-N_2}). \tag{A.13a}
\]

For \( i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = N_1 + 1, \ldots, 2N_1, \)
\[
F'_{ij}(x,t,\nu) = \alpha^2 F^{(2)}_{j-N_1}(\nu^*_{i-2N_1-N_2}). \tag{A.13b}
\]

For \( i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 2N_1 + 1, \ldots, 2N_1 + N_2, \)
\[
F'_{ij}(x,t,\nu) = \alpha^2 C^{(1)}_{j-2N_1}(\nu^*_{i-2N_1-N_2}). \tag{A.13c}
\]

For \( i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, \)
\[
F'_{ij}(x,t,\nu) = \alpha^2 C^{(2)}_{j-2N_1-N_2}(\nu^*_{i-2N_1-N_2}). \tag{A.13d}
\]

For \( i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, \)
\[
F'_{ij}(x,t,\nu) = \alpha^2 D^{(1)}_{j-2N_1-2N_2}(\nu^*_{i-2N_1-N_2}). \tag{A.13e}
\]
For $i = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha^2 D_{j-2N_1-3N_2}^{(2)}(\nu^{*}_{i-2N_1-N_2}). \quad (A.13f) \]

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = 1, \ldots, N_1$, 
\[ F'_{ij}(x, t, \nu) = \alpha^2 F_j^{(1)}(-\alpha^2 \nu_{i-2N_1-N_2}). \quad (A.14a) \]

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = N_1 + 1, \ldots, 2N_1$, 
\[ F'_{ij}(x, t, \nu) = \alpha^2 F_{j-N_1}^{(2)}(-\alpha^2 \nu_{i-2N_1-N_2}). \quad (A.14b) \]

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = 2N_1 + 1, \ldots, 2N_1 + N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha^2 C_{j-2N_1-N_2}^{(1)}(-\alpha^2 \nu_{i-2N_1-N_2}). \quad (A.14c) \]

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha^2 C_{j-2N_1-N_2}^{(2)}(-\alpha^2 \nu_{i-2N_1-N_2}). \quad (A.14d) \]

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha^2 D_{j-2N_1-N_2}^{(1)}(-\alpha^2 \nu_{i-2N_1-N_2}). \quad (A.14e) \]

For $i = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha^2 D_{j-2N_1-3N_2}^{(2)}(-\alpha^2 \nu_{i-2N_1-N_2}). \quad (A.14f) \]

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2, j = 1, \ldots, N_1$, 
\[ F'_{ij}(x, t, \nu) = \alpha F_{j-N_1}^{(3)}(-\alpha \nu_{i-2N_1-3N_2}). \quad (A.15a) \]

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2, j = N_1 + 1, \ldots, 2N_1$, 
\[ F'_{ij}(x, t, \nu) = \alpha F_{j-N_1}^{(4)}(-\alpha \nu_{i-2N_1-3N_2}). \quad (A.15b) \]

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2, j = 2N_1 + 1, \ldots, 2N_1 + N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha C_{j-2N_1-N_2}^{(3)}(-\alpha \nu_{i-2N_1-3N_2}). \quad (A.15c) \]

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2, j = 2N_1 + N_2 + 1, \ldots, 2N_1 + 2N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha C_{j-2N_1-N_2}^{(4)}(-\alpha \nu_{i-2N_1-3N_2}). \quad (A.15d) \]

For $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2, j = 2N_1 + 2N_2 + 1, \ldots, 2N_1 + 3N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha D_{j-2N_1-3N_2}^{(3)}(-\alpha \nu_{i-2N_1-3N_2}). \quad (A.15e) \]

Finally, for $i = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2, j = 2N_1 + 3N_2 + 1, \ldots, 2N_1 + 4N_2$, 
\[ F'_{ij}(x, t, \nu) = \alpha D_{j-2N_1-3N_2}^{(4)}(-\alpha \nu_{i-2N_1-3N_2}). \quad (A.15f) \]
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