STABLE COHOMOLOGY OF ALTERNATING GROUPS

FEDOR BOGOMOLOV\textsuperscript{1} AND CHRISTIAN BÖHNING\textsuperscript{2}

ABSTRACT. In this article we determine the stable cohomology groups $H^i_s(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z})$ of the alternating groups $\mathfrak{A}_n$ for all integers $n$ and $i$, and all primes $p$.

1. Introduction and preliminaries

Let $G$ be a finite group, $V$ a finite-dimensional generically free complex representation of $G$, and let $V^L \subset V$ be the nonempty open subset of $V$ on which the $G$-action is free. There is then a natural homotopy class of maps from the classifying space $B G$ to $V^L/G$ which, for each nonempty $G$-invariant Zariski open subset $U \subset V^L$ gives maps $H^i(G, \mathbb{Z}/p\mathbb{Z}) \to H^i(U/G, \mathbb{Z}/p\mathbb{Z})$. It turns out that the kernel $K_{G, V}$ of $H^i(G, \mathbb{Z}/p\mathbb{Z}) \to \varprojlim \oplus_{U} H^i(U/G, \mathbb{Z}/p\mathbb{Z})$ is independent of $V$, and the stable cohomology $H^i_s(G, \mathbb{Z}/p\mathbb{Z})$ is defined to be the quotient $H^i(G, \mathbb{Z}/p\mathbb{Z})/K_{G, V}$. Algebraically, $H^i_s(G, \mathbb{Z}/p\mathbb{Z})$ can be identified with the image of $H^i(G, \mathbb{Z}/p\mathbb{Z})$ in $H^i(\text{Gal}(K), \mathbb{Z}/p\mathbb{Z})$.

More accessible computationally and stable birational invariants of the function field $K$ are the unramified cohomology groups $H^i_{nr}(G, \mathbb{Z}/p\mathbb{Z})$ defined as follows: geometrically, the unramified cohomology classes $a$ inside $H^i_s(G, \mathbb{Z}/p\mathbb{Z})$ are those for which, given any divisorial valuation $\nu_D$ of $K$, there exists a normal model $X = X_D$ of $K$ on which $\nu_D$ has a center, an isomorphism $i : U_X \to U_{V^L/G}$ between nonempty open subsets $U_X$ of $X$ and $U_{V^L/G}$ of $V^L/G$, and a representative $a'$ of $a$ in $H^i(U_{V^L/G}, \mathbb{Z}/p\mathbb{Z})$ such that there is a class $b \in H^i(X, \mathbb{Z}/p\mathbb{Z})$ whose image in $H^i(U_X, \mathbb{Z}/p\mathbb{Z})$ coincides with $i^*(a')$. More algebraically, if $\mathcal{O}_\nu \subset K$ is the valuation ring of $\nu$, $\kappa_\nu = \mathcal{O}_\nu / \mathfrak{m}_\nu$ its residue field, $S = \text{Spec}(\mathcal{O}_\nu)$ with open subset the generic point $U = \text{Spec}(K) \subset S$

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and complement the closed point $Z = \text{Spec}(\kappa_\nu) \subset S$, one can write down the long exact sequence of étale cohomology with supports

$$\cdots \to H^i(S, \mathbb{Z}/p) \to H^i(U, \mathbb{Z}/p) \to H^{i+1}_Z(S, \mathbb{Z}/p) \to H^{i+1}(S, \mathbb{Z}/p) \to \cdots$$

where $H^i(U, \mathbb{Z}/p) \simeq H^i(\text{Gal}(K), \mathbb{Z}/p)$ and there is the cohomological purity isomorphism

$$H^i_Z(S, \mathbb{Z}/p) \simeq H^{i-2}(Z, \mathbb{Z}/p)$$

whence the preceding sequence becomes the Gysin sequence

$$\cdots \to H^i_{\text{ét}}(S, \mathbb{Z}/p) \xrightarrow{\partial_\nu} H^i(\text{Gal}(K), \mathbb{Z}/p) \xrightarrow{r_\nu} H^{i+1}_Z(S, \mathbb{Z}/p) \to H^{i+1}(S, \mathbb{Z}/p) \to \cdots$$

A class in $H^i(\text{Gal}(K), \mathbb{Z}/p)$ is clearly unramified according to the geometric definition if and only if it is in the image of all maps $r_\nu$ for $\nu$ running over the divisorial valuations of $K$, i.e. equivalently if it is in the kernel of all maps $\partial_\nu$, the residue maps. The preceding sequence has as topological analogue the Borel-Moore long exact sequence. The residue map

$$\partial_\nu : H^i(\text{Gal}(K), \mathbb{Z}/p) \to H^{i-1}(\text{Gal}(\kappa_\nu), \mathbb{Z}/p)$$

agrees -up to a sign- with the following map defined entirely within the framework of Galois cohomology (see e.g. [GMS], Chap. II of Serre’s part, §6 and §7): extend $\nu$ in some way to a valuation $\nu^*$ on $\bar{K}$ which is possible by Chevalley’s theorem; all such extensions are conjugate under $\Gamma = \text{Gal}(K)$, and $\nu^*$ defines subgroups $\Gamma_Z \subset \Gamma$ (the decomposition group, Zerlegungsgruppe) and $\Gamma_T \subset \Gamma$ (the inertia subgroup, Trägheitsgruppe) by the conditions that $\sigma \in \Gamma$ is in $\Gamma_Z$ if $\sigma \cdot \nu^*$ and $\nu^*$ are equivalent valuations, i.e. have the same valuation ring, and $\Gamma_T$ consists of those $\sigma$ such that $\sigma \cdot x - x \in M_{\nu^*}$ for all $x$ in the valuation ring of $\nu^*$ whose maximal ideal we denoted by $M_{\nu^*}$. The decomposition group can be identified with the Galois group $\text{Gal}(\bar{K}_\nu/K_\nu)$ of the completion of $K$ with respect to $\nu$. The residue map $\partial_\nu$ then factors over the restriction to the decomposition group

$$\partial_\nu : H^i(\text{Gal}(K), \mathbb{Z}/p) \to H^i(\text{Gal}(K_\nu), \mathbb{Z}/p) \xrightarrow{r_\nu} H^{i-1}(\text{Gal}(\kappa_\nu), \mathbb{Z}/p)$$

where the second arrow has the following description in the local situation: the Galois group $\Gamma_{K_\nu} = \text{Gal}(K_\nu)$ sits in the exact sequence

$$1 \to I \to \Gamma_{K_\nu} \to \Gamma_{\kappa_\nu} = \text{Gal}(\kappa_\nu) \to 1$$

where the surjection is given by the fact that $\nu$ extends uniquely to $\bar{K}_\nu$ and the residue field of the extension is an algebraic closure of $\kappa_\nu$. The kernel is the inertia subgroup which we denote by $I$ in this
context, and it is topologically cyclic, \( I \simeq \hat{\mathbb{Z}} \), corresponding to taking roots of the uniformizing parameter, and the preceding sequence splits, \( \Gamma_{K_u} \simeq \hat{\mathbb{Z}} \oplus \text{Gal}(\kappa_u) \). As \( \hat{\mathbb{Z}} \) has cohomological dimension 1, one gets \( H^i(\Gamma_{K_u}, \mathbb{Z}/p) \simeq H^i(\Gamma_{\kappa_u}, \mathbb{Z}/p\mathbb{Z}) \oplus H^{i-1}(\Gamma_{\kappa_u}, \mathbb{Z}/p\mathbb{Z}) \) and a projection, which is independent of the splitting, \( H^i(\text{Gal}(K_u), \mathbb{Z}/p) \to H^{i-1}(\text{Gal}(\kappa_u), \mathbb{Z}/p) \), defining the second arrow in the sequence of maps yielding \( \partial \nu \). More precisely, the Hochschild-Serre spectral sequence of the group extension of \( \Gamma_{\kappa_u} \) by \( I \)

\[
H^p(\Gamma_{\kappa_u}, H^q(I, \mathbb{Z}/p)) \implies H^p(\Gamma_{K_u}, \mathbb{Z}/p)
\]

reduces to a long exact sequence as \( H^i(I, \mathbb{Z}/p) = 0 \) for \( i \geq 2 \), \( H^0(I, \mathbb{Z}/p) = \mathbb{Z}/p \), \( H^1(I, \mathbb{Z}/p) = \text{Hom}(I, \mathbb{Z}/p) = \mathbb{Z}/p \), which reads

\[
\cdots \to H^i(\Gamma_{\kappa_u}, \mathbb{Z}/p) \to H^i(\Gamma_{\kappa_u}, \mathbb{Z}/p) \to H^{i-1}(\Gamma_{\kappa_u}, \text{Hom}(\hat{\mathbb{Z}}, \mathbb{Z}/p)) \to \to H^{i+1}(\Gamma_{\kappa_u}, \mathbb{Z}/p) \to H^{i+1}(\Gamma_{K_u}, \mathbb{Z}/p) \to \cdots
\]

and the fact that the extension splits implies that this long exact sequence breaks into short exact sequences

\[
0 \to H^i(\Gamma_{\kappa_u}, \mathbb{Z}/p) \to H^i(\Gamma_{K_u}, \mathbb{Z}/p) \xrightarrow{r} H^{i-1}(\Gamma_{\kappa_u}, \mathbb{Z}/p)
\]

where \( r \) can also be explicitly described in terms of cocycles, see [GMS], p.16.

In [B-P] the following theorem was proven (loc. cit, Theorem 5.1):

**Theorem 1.1.** There is a natural isomorphism \( H_\ast^s(\mathfrak{A}_{2n+1}, \mathbb{Z}/2\mathbb{Z}) \simeq H_\ast^s(\mathfrak{A}_{2n}, \mathbb{Z}/2\mathbb{Z}) \), and as a \( \mathbb{Z}/2\mathbb{Z} \)-vector space

\[
H_\ast^s(\mathfrak{A}_{2n}, \mathbb{Z}/2\mathbb{Z}) = \bigoplus_{0 \leq i \leq n} \mathbb{Z}/2\mathbb{Z} \cdot w_{2i} \oplus \bigoplus_{0 < i \leq n} \mathbb{Z}/2\mathbb{Z} \cdot u_1 \wedge w_{2i}
\]

where \( w_j \) are the (images in stable cohomology of) the Stiefel-Whitney classes in \( H^j(\mathfrak{A}_{2n}, \mathbb{Z}/2\mathbb{Z}) \) obtained from the cohomology ring of the real orthogonal group \( O(2n) \) via the inclusions \( \mathfrak{A}_{2n} \subset \mathfrak{S}_{2n} \subset \mathfrak{O}(2n) \). The class \( u_1 \) is a one-dimensional cohomology class which can be described as follows:

Putting \( N = 2n \), the group \( \mathfrak{S}_N \) acts generically freely on the complement \( \mathbb{C}^{N-1} - H \) of the braid hyperplane arrangement \( H \) in the standard permutation representation \( \mathbb{C}^{N-1} \), and \( (\mathbb{C}^{N-1} - H)/\mathfrak{S}_N \simeq \mathbb{C}^{N-1} - \Delta \), the complement of the discriminant. Taking a nonramified double covering \( \mathbb{C}^{N-1} - \Delta \) of \( \mathbb{C}^{N-1} - \Delta \) corresponding to the inclusion \( \mathfrak{A}_N \subset \mathfrak{S}_N \), one gets a description of \( u_1 \) as the generator of \( H^1(\mathbb{C}^{N-1} - \Delta) \) given by the root of the discriminant.
In our Theorem 3.6 we determine $H_*(\mathcal{A}_N, \mathbb{Z}/p\mathbb{Z})$ completely for odd primes $p$.

We base our approach to the computation of the stable cohomology of alternating groups on the following lemmas.

**Lemma 1.2.** Suppose a group is a product $G \times A$ of finite groups $G$ and $A$ with $A$ abelian. Then there is the Künneth decomposition

$$H_*^s(G \times A, \mathbb{Z}/p) \cong \bigoplus_{i+j=n} H_*^s(G, \mathbb{Z}/p) \otimes H_*^j(A, \mathbb{Z}/p).$$

**Proof.** It is known (Bogo93, Lemma 7.1) that if we choose a free presentation $\pi : \mathbb{Z}^n \twoheadrightarrow A$ of $A$, then the kernel of the stabilization map coincides with the kernel of $\pi^*$. In other words, if one realizes $\mathbb{Z}^n$ as the fundamental group of some algebraic torus $T$, with cover $T' \rightarrow T \simeq T'/A$ corresponding to $A$, and realizes $T'$ as a maximal torus in some $\text{GL}(W)$, then stabilization is achieved by considering the image of the cohomology of $A$ in the cohomology of $T \simeq T'/A \subset W^L/A$. This is so because one can find a product of circles $(S^1)^m$ in the complement of any divisor $D$ in $T \simeq (\mathbb{C}^*)^m$, the inclusion being a homotopy equivalence, so the cohomology of $T$ is already stable. The product of circles can be found by induction on the dimension $m$ of $T$; if $m = 1$, one chooses a circle in the complex plane $\mathbb{C}$ not passing through the finite number of points which $D$ consists of. If $m > 1$, one views $T$ as a subset of $W$, which is stratified into torus orbits of lower dimension. Each of these is isomorphic to an algebraic tori. Choose a codimension 1 torus orbit $T_1$ adjacent to $T$. The closure $\bar{D}$ inside $W$ of a divisor $D \subset T$ meets $T_1$ in a proper algebraic subset, and by the induction hypothesis there is a real submanifold $M \simeq (S^1)^{m-1}$ in the complement of $D \cap T_1$. If $x_1, \ldots, x_m$ are coordinates in $W$ such that $T = \{x_i \neq 0 \forall i\}$, $T_1 = \{x_1 = 0 \land x_j \neq 0 \forall j \neq 1\}$, then $W = W' \oplus \mathbb{C}$ where $W' = \{x_1 = 0\}$. If we choose a small circle $S_\varepsilon \subset \mathbb{C}$, then $M \times S_\varepsilon \subset W$ will be in a small neighbourhood of $M$ hence will not intersect $D$.

This argument can be made relative: note first that there is always a natural surjection $H_*^s(G, \mathbb{Z}/p) \otimes H_*^s(A, \mathbb{Z}/p) \twoheadrightarrow H_*^s(G \times A, \mathbb{Z}/p)$ as the Zariski topology on a product is finer than the product topology, and to show it is an isomorphism, it suffices to note the following: suppose $T \simeq (\mathbb{C}^*)^m \simeq T'/A$ is as before, and $V$ is a generically free $G$-representation, $V^L$ the open part where the action is free. Then if $D \subset (V^L/G) \times T$ is any divisor, there is always a divisor $D' \subset V^L/G$
and a relatively compact subset \( U^L/G \subset V^L/G - D' \) with a (trivial) iterated circle fibration \( U^L/G \times (S^1)^m \subset ((V^L/G) \times T) - D \) such that \( U^L/G \times (S^1)^m \) and \((V^L/G) \times T) - D \) are homotopy equivalent.

Indeed, viewing \( V^L/G \times T \subset V^L/G \times W \), the latter being a vector bundle, we have a zero section \( V^L/G \subset V^L/G \times W \). Moreover, \( \tilde{D} \cap V^L/G \), where \( \tilde{D} \) is the closure of \( D \) in \( V^L/G \times W \), will be contained in some divisor \( D' \). Shrinking \( V^L/G - D' \) slightly, we can find a relatively compact open subset \( U^L/G \subset V^L/G - D' \) homotopy equivalent to \( V^L/G - D' \) and with the claimed circle fibration.

We say that the stable cohomology \( H^*_s(G, \mathbb{Z}/p) \) is detected by abelian subgroups if the map induced by the restriction to abelian subgroups

\[
H^*_s(G, \mathbb{Z}/p) \longrightarrow \prod_A H^*_s(A, \mathbb{Z}/p)
\]

is injective (where \( A \) ranges over all abelian subgroups of \( G \)). We will then also use the following principle which follows from Lemma 1.2.

**Lemma 1.3.** Suppose \( G_1 \) and \( G_2 \) are finite groups such that at least one of \( H^*_s(G_1, \mathbb{Z}/p) \) or \( H^*_s(G_2, \mathbb{Z}/p) \) is detected by abelian subgroups. Then one has a K"{u}nneth formula in stable cohomology

\[
H^*_s(G_1 \times G_2, \mathbb{Z}/p) \simeq H^*_s(G_1, \mathbb{Z}/p) \otimes H^*_s(G_2, \mathbb{Z}/p).
\]

**Proof.** There is always the natural surjection

\[
H^*_s(G_1, \mathbb{Z}/p) \otimes H^*_s(G_2, \mathbb{Z}/p) \twoheadrightarrow H^*_s(G_1 \times G_2, \mathbb{Z}/p).
\]

Without loss of generality, we can assume that abelian subgroups \( A_i \) are a detecting family for the stable cohomology of \( G_1 \):

\[
H^*_s(G_1, \mathbb{Z}/p) \twoheadrightarrow \prod_{i \in I} H^*_s(A_i, \mathbb{Z}/p).
\]

Now by Lemma 1.2 there is an injection

\[
H^*_s(G_1, \mathbb{Z}/p) \otimes H^*_s(G_2, \mathbb{Z}/p) \hookrightarrow \prod_i H^*_s(A_i \times G_2, \mathbb{Z}/p).
\]

But \( i = (\prod \text{res}_{A_i \times G_2}) \circ p \) where

\[
\prod \text{res}_{A_i \times G_2} : H^*_s(G_1 \times G_2, \mathbb{Z}/p) \rightarrow \prod_i H^*_s(A_i \times G_2, \mathbb{Z}/p)
\]

is the product of restriction maps. Hence \( p \) is also injective.

**Lemma 1.4.** Let \( G \) be a finite group such that \( H^i_{nr}(G, \mathbb{Z}/p) = 0 \) for all \( i > 0 \). Then every stable class \( a \in H^*_s(G, \mathbb{Z}/p) \) is nontrivial on the centralizer \( C(g) \) of some element \( g \in G \), i.e. the restriction \( \text{res} : H^*_s(G, \mathbb{Z}/p) \rightarrow H^*_s(C(g), \mathbb{Z}/p) \) is nonzero.
Proof. With the notation established above we have the maps of groups

\[ I \subset \text{Gal}(K_\nu) \subset \text{Gal}(K) \to G, \]

and the image of the inertia subgroup \( I \) in \( G \) is cyclic, generated by \( g \) say, and the image of the decomposition group \( \text{Gal}(K_\nu) \) in \( G \) belongs to the centralizer \( C(g) \). As the residue map \( \partial_\nu \) factors over \( \text{Gal}(K_\nu) \), we obtain the assertion. \( \square \)

Recall the exact sequence

\[ 1 \to I \to \Gamma_{K_\nu} \to \Gamma_{K_\nu} \to 1 \]

where \( I \) is the inertia subgroup of the decomposition group \( \Gamma_{K_\nu} \) associated to the valuation \( \nu \) of \( K = \mathbb{C}(V)^G \). The following Lemma allows one to increase the usefulness of Lemma 1.4 in inductive arguments further.

**Lemma 1.5.** Let \( G \) be a finite group and let \( a \in H^n_\ast(G, \mathbb{Z}/p) \) be a stable class. For \( \nu \) a divisorial valuation of \( K \), the image of the topologically cyclic inertia subgroup \( I \) in \( G \) is cyclic, generated by \( h \) say. There is a natural class \( d_\nu(a) \in H^{n-1}_\ast(Z_G(h), \mathbb{Z}/p) \) such that the residue \( \partial_\nu(a) \in H^{n-1}(\Gamma_{K_\nu}, \mathbb{Z}/p) \) is the pull-back of \( d_\nu(a) \) to \( \Gamma_{K_\nu} \) via the maps \( \Gamma_{K_\nu} \subset \Gamma_{K_\nu} \cong I \oplus \Gamma_{K_\nu} \to Z(h) \).

**Proof.** Consider the commutative diagram

\[
\begin{array}{cccccc}
1 & \to & I & \to & \Gamma_{K_\nu} & \to & \Gamma_{K_\nu} & \to & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \to & \langle h \rangle & \to & Z(h) & \to & Z(h)/\langle h \rangle & \to & 1 \\
\downarrow & & f & \downarrow g & & & & & \\
1 & \to & \langle h \rangle & \to & \langle h \rangle \times Z(h) & \to & Z(h) & \to & 1 \\
\end{array}
\]

where the arrow \( a \) is the identity, \( b \) is the couple \((i_{\langle h \rangle}, \text{id}_{Z(h)})\) where \( i_{\langle h \rangle} : \langle h \rangle \to Z(h) \) is the inclusion, and \( c \) is the projection. Here \( f \) and \( g \) are defined as follows: the extension defining \( \Gamma_{K_\nu} \) splits, \( \Gamma_{K_\nu} \cong \hat{\mathbb{Z}} \oplus \Gamma_{K_\nu} \), so the map \( \Gamma_{K_\nu} \to Z(h)/\langle h \rangle \) lifts to a map \( g_1 : \Gamma_{K_\nu} \to Z(h) \). The map \( g \) is simply \((1, g_1)\). The map \( f \) is \((f_1, 1)\) where \( f_1 : I \to \langle h \rangle \) is the natural map. As \( \hat{\mathbb{Z}} \) has cohomological dimension one, \( H^i(\Gamma_{K_\nu}, \mathbb{Z}/p) \cong H^{i-1}(\Gamma_{K_\nu}, \mathbb{Z}/p) \oplus H^i(\Gamma_{K_\nu}, \mathbb{Z}/p) \) and the residue map \( \partial_\nu \) was defined by the restriction of \( a \to \Gamma_{K_\nu} \) and projecting to \( H^{i-1}(\Gamma_{K_\nu}, \mathbb{Z}/p) \). By the commutativity of the diagram, we may thus define a class \( d_\nu(a) \) with the requested properties as follows: we restrict \( a \in H^n_\ast(G, \mathbb{Z}/p) \) to \( Z(h) \), and then take the pull-back
1.2. This defines $\text{H}^n_\text{d}(\text{H}) \times \text{Z}/(h), \text{Z}/p)$ and project this unto the component $\text{H}^n_{\text{d}}(\text{H})$ in the K"unneth decomposition, using Lemma 1.2. This defines $d_n(a) \in \text{H}^n_{\text{d}}(\text{H})$, $\text{Z}/p).$

Define a subgroup $H$ of $G$ recursively to be an iterated centralizer if it is the centralizer of an element in $G$ or a centralizer of an element inside another iterated centralizer.

**Corollary 1.6.** Assume that $G$ is such that each iterated centralizer has trivial unramified cohomology. Then any element $a \in \text{H}^n_\text{d}(G, \text{Z}/p)$ is nontrivial on some abelian $p$-subgroup.

**Proof.** We use induction on the cohomological degree, hence assume that every element in $\text{H}^i_\text{d}(H, \text{Z}/p)$, for all $i < n$, and for all iterated centralizers $H$ in $G$, is nontrivial on some abelian subgroup. By assumption, we get from Lemma 1.5 that $d_n(a) \in \text{H}^n_{\text{d}}(\text{H})$ is nontrivial for some $h$. Hence $d_n(a)$ is nontrivial on some abelian $p$-subgroup $A$ of $\text{Z}(h)$. By the construction of $d_n(a)$ in Lemma 1.5 we have that $a$ will then be nontrivial when restricted to $\text{H}^n_{\text{d}}(\langle h, A \rangle, \text{Z}/p)$ where $\langle h, A \rangle$ is the abelian subgroup of $G$ generated by $h$ and $A$. □

The Steenrod power operations $S^j, P^j$ (see [Steen], [A-M] II.2) are natural transformations

$$S^j : \text{H}^j(X, \text{Z}/2) \to \text{H}^{j+i}(X, \text{Z}/2),$$

$$P^j : \text{H}^j(X, \text{Z}/p) \to \text{H}^{j+2i(p-1)}(X, \text{Z}/p), \text{p an odd prime},$$

on the category of CW-complexes with continuous maps $f : X \to Y$. By functoriality, applied to the map $B \to (V^G) - D$, where $D$ is some divisor, $S^j, P^j$ induce operations on $\text{H}^n_{\text{d}}(G, \text{Z}/p)$.

For later use, we recall here the structure theorem for the cohomology of wreath products due to Steenrod [Steen], Section VII, see also [Mann78], Theorem 3.1 and [A-M], IV. 4, Theorem 4.1. We suppress the $\text{Z}/p\text{Z}$-coefficients in cohomology groups now, i.e. write $H^*(X, \text{Z}/p\text{Z})$.

**Theorem 1.7.** Let $H$ be a group, and let $H \ltimes \text{Z}/p = (H)^p \rtimes \text{Z}/p$ be the wreath product where $\text{Z}/p\text{Z}$ acts by cyclically permuting the copies of $H$.

Let $\text{id} \times \Delta^p : \text{Z}/p\text{Z} \times H \to \text{Z}/p\text{Z} \times (H)^p$ be the inclusion $(\text{id} \times \Delta^p)(z, a) = (z; (a, \ldots, a))$ (p-times $a$) and denote by $t : H^*(H^p) \to H^*(H \ltimes \text{Z}/p)$ the transfer. Then the sequence

$$H^*(H^p) \xrightarrow{t} H^*(H \ltimes \text{Z}/p) \xrightarrow{(\text{id} \times \Delta^p)^*} H^*(\text{Z}/p\text{Z} \times H)$$
is exact.

Moreover, for any $u \in H^j(H)$ there is a class $P(u) \in H^{jp}(H \wr \mathbb{Z}/p)$ (constructed by Steenrod) such that

(i) If $j : H^p \to \mathbb{Z}/p\mathbb{Z} \times (H)^p$ is the natural inclusion, then $j^*(P(u)) = u^{\otimes p}$.

(ii) In the Künneth decomposition of $(\text{id} \times \Delta^p)^*(P(u))$ in $H^*(\mathbb{Z}/p\mathbb{Z} \times H)$ we have

$$(\text{id} \times \Delta^p)^*(P(u)) = \Sigma w_k \otimes D_k(u)$$

where $w_k$ is a generator of $H^k(\mathbb{Z}/p\mathbb{Z})$ and $D_k : H^q(H) \to H^{q-k}(H)$ are homomorphisms which satisfy

(iii)

$$\beta D_{2k}(u) = D_{2k-1}(u), \beta D_{2k-1}(u) = 0, \beta D_0(u) = 0$$

where $\beta$ is the Bockstein homomorphism, i.e. connecting homomorphism in the long exact sequence coming from the short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$ 

The maps $D_k$ are used originally by Steenrod to define the Steenrod powers $P^i$, hence the Hopf algebras (Steenrod algebras) $A(p)$.

More precisely,

$$P^i(u) = (-1)^{i+m(j+1)/2} (m!)^j D_{(p-1)(j-2)I}(u) \in H^{j+2(p-1)i}(H, \mathbb{Z}/p\mathbb{Z}) .$$

In this setting, any $c \in H^*(H \wr \mathbb{Z}/p)$ can be written as

$$c = t(c_1) + c_2 \cdot P(c_3)$$

with $c_1 \in H^*(H^p)$, $c_2 \in H^*(\mathbb{Z}/p\mathbb{Z})$ and $c_3 \in H^*(H)$.

Here we view $H^*(H \wr \mathbb{Z}/p)$ as a module over $H^*(\mathbb{Z}/p\mathbb{Z})$ via the cohomology pull-back induced by the surjection $H \wr \mathbb{Z}/p \to \mathbb{Z}/p\mathbb{Z}$. The Steenrod operations $P^i$ have the following formal properties:

1. $P^0 = \text{id}$.
2. If $\dim(x) = 2n$, then $P^n(x) = x^p$.
3. If $2i > \dim(x)$, then $P^i(x) = 0$.
4. (Cartan formula) $P^i(x \cup y) = \sum_{j=0}^i P^j(x) \cup P^{i-j}(y)$.

A consequence of the Bloch-Kato conjecture (now a theorem by the work of Voevodsky, Rost and many others) is

**Lemma 1.8.** The Steenrod cohomology operations $Sq^i$, $P^i$ are zero in stable cohomology $H^n_s(G, \mathbb{Z}/p\mathbb{Z})$. 

Proof. Any stable class \( a \in H^*_s(G, \mathbb{Z}/p) \subset H^*(\text{Gal}(K), \mathbb{Z}/p) \) arises as the pull-back from some torus \( T \simeq (\mathbb{C}^*)^m \), more precisely, \( a \) has a representative \( a' \) in the cohomology of \( H^*((V^L/G) - D, \mathbb{Z}/p) \) and there is a regular map \( f : (V^L/G) - D \to T \) together with a class \( \hat{a} \in H^*(T, \mathbb{Z}/p) \) with \( f^*(\hat{a}) = a' \). This follows from the fact, which is a consequence of the Bloch-Kato conjecture, that \( H^*(\text{Gal}_{\text{ab}}(K), \mathbb{Z}/p) \to H^*(\text{Gal}(K), \mathbb{Z}/p) \) is surjective where \( \text{Gal}_{\text{ab}}(K) \) is the abelianized Galois group \( \text{Gal}(K)/[\text{Gal}(K), \text{Gal}(K)] \).

Since the Steenrod power operations are trivial in the cohomology algebra of the torus \( T \) (which is an exterior algebra on one-dimensional generators), the assertion of the Lemma will then follow from the functoriality of the cohomology operations.

It remains to explain in some more detail how from the surjection \( H^*(\text{Gal}_{\text{ab}}(K), \mathbb{Z}/p) \to H^*(\text{Gal}(K), \mathbb{Z}/p) \) we get the map \( f : (V^L/G) - D \to T \). There is a finite abelian quotient \( \text{Gal}(K) \to A \) such that \( a \) is induced from a class \( a'' \) in the cohomology of \( A \). The group \( A \) corresponds to an unramified abelian covering \( \tilde{X} \) of some nonempty open affine subvariety \( X \subset V^L/G \). The coordinate ring \( \mathbb{C}[\tilde{X}] \) contains an arbitrary finite dimensional representation of \( A \) as the regular functions on an \( A \)-orbit in \( \tilde{X} \) are precisely the regular representation \( \mathbb{C}[A] \) and this is also a subrepresentation (not only a quotient) because \( A \) is reductive. In particular, embedding \( A \) in a torus of diagonal matrices in some \( GL_m(\mathbb{C}) \), one obtains a dominant regular map from an open subset \( X' \) of \( X = \tilde{X}/A \), hence some \( (V^L/G) - D \) to a torus \( T = (\mathbb{C}^m)^L/A \simeq (\mathbb{C}^*)^m \) for which it holds by construction that the image of the class \( a'' \) in the cohomology of \( T \) induces a representative \( a' \in H^*((V^L/G) - D, \mathbb{Z}/p) \) of \( a \). \( \square \)

Thus the techniques used in the present article are mainly topological in flavour; for the connection to motivic cohomology and further developments the reader may consult [Kahn-Su00], [Kahn11], [Ngu1], [Ngu2], [TY11].

2. Detection by elementary abelian \( p \)-subgroups

In this section we want to prove the following Theorem.

**Theorem 2.1.** Let \( p \) as always be an odd prime and \( \mathfrak{A}_N \) the alternating group on \( N \) letters. Then \( H^*_s(\mathfrak{A}_N, \mathbb{Z}/p) \) is detected by elementary abelian \( p \)-subgroups.

We denote by \( G_n = \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z} \) \((n \text{ factors})\) the iterated wreath product of \( n \) cyclic \( p \)-groups. This is the \( p \)-Sylow of \( \mathfrak{A}_p^e \). If \( N \) is
arbitrary (not necessarily a power of \( p \)) expand it in base \( p \):

\[
N = a_0 + a_1 p + \cdots + a_m p^m
\]

with \( 0 \leq a_j < p \), \( a_m \neq 0 \), and note that this gives rise to a natural inclusion

\[
i_{a_1, \ldots, a_m} : \mathcal{A}_{a_1, \ldots, a_m} := \prod_{1}^{a_1} \mathcal{A}_p \times \prod_{1}^{a_2} \mathcal{A}_{p^2} \times \cdots \times \prod_{1}^{a_m} \mathcal{A}_{p^m} \hookrightarrow \mathcal{A}_N
\]

and that a \( p \)-Sylow subgroup in \( \mathcal{A}_N \) is given by the product of \( p \)-Sylow subgroups in the factors in \( \mathcal{A}_{a_1, \ldots, a_m} \). Hence it follows from Lemma 1.3 that it is enough to prove Theorem 2.1 for \( N = p^n \).

First we will prove the weaker

**Theorem 2.2.** The stable cohomology \( H^*_s(\mathcal{A}_N, \mathbb{Z}/p) \) is detected by abelian \( p \)-subgroups.

The proof of Theorem 2.2 will follow from the structures of centralizers in complete monomial groups.

**Definition 2.3.** Let \( H \) be a group. The **complete monomial group** of degree \( m \) on \( H \) is the group \( \Sigma_m(H) := H \wr \mathfrak{S}_m = (H)^m \rtimes \mathfrak{S}_m \), where \( \mathfrak{S}_m \) is the symmetric group on \( m \) letters.

A **monomial cycle** in \( \Sigma_m(H) \) is an element of the form \((h_1, \ldots, h_m); \sigma)\) where \( \sigma \in \mathfrak{S}_m \) is a cycle. The **determinant class** of a monomial cycle is the conjugacy class in \( H \) of the product \( h_1 \cdot \ldots \cdot h_m \). The **length** of a monomial cycle is the length of the underlying cycle \( \sigma \).

We have to recall some results from [Ore] on the structure of conjugacy classes and centralizers in groups \( \Sigma_m(H) \):

1. Two monomial cycles in \( \Sigma_m(H) \) are conjugate if and only if they have the same length and determinant class ([Ore], Theorem 6).
2. Any element \( x \in \Sigma_m(H) \) can be written uniquely as a product of commuting monomial cycles (the underlying cycles in \( \mathfrak{S}_m \) have no common variables) ([Ore], Theorem 3).
3. From (1) and (2) follows the description of conjugacy classes in \( \Sigma_m(H) \): two elements \( x', x \in \Sigma_m(H) \) are conjugate if the monomial cycles in their decompositions in (2) can be matched in such a way that corresponding cycles have the same length and determinant class.
4. Let \( c = ((h, \ldots, 1, 1); \sigma) \) be a monomial cycle of length \( m \) in \( \Sigma_m(H) \). Then its centralizer in \( \Sigma_m(H) \) is an extension

\[
1 \rightarrow Z_H(h) \rightarrow Z_{\Sigma_m(H)}(c) \rightarrow \mathbb{Z}/m \rightarrow 0
\]
In other words, the centralizer $Z_{\Sigma_m(H)}(c)$ is generated by $c$ and the group $Z_H(h)$ embedded diagonally into $H^m \subset \Sigma_m(H)$.

(5) ([Ore], Theorem 8): Let $x = x_1 \cdot \ldots \cdot x_l$, $x_i = y_1^{(i)} \cdot \ldots \cdot y_{k_i}^{(i)}$ be the decomposition of an element $x$ in $\Sigma_m(H)$ into disjoint monomial cycles $y_j^{(i)}$ as in (2), where we group the cycles of equal length and determinant class together: for fixed $i$, all $y_j^{(i)}$ have determinant class $h_i$ and length $n_i$. Then the centralizer of $x$ in $\Sigma_m(H)$ has a description as

$$Z_{\Sigma_m(H)}(x) = \prod_{i=1}^l \Sigma_{k_i}(Z_{\Sigma_{n_i}(H)}(y_1^{(i)}))$$

where as in (4) the centralizer of $y_1^{(i)}$ (or any $y_j^{(i)}$) in the group $\Sigma_{n_i}(H)$ is an extension

$$1 \to Z_H(h_i) \to Z_{\Sigma_{n_i}(H)}(y_1^{(i)}) \to \mathbb{Z}/n_i\mathbb{Z} \to 0.$$ 

From the last fact (5) we get immediately

**Lemma 2.4.** Let $\Sigma_m(A)$ be a complete monomial group with $A$ abelian. Then the centralizer of any element $x$ in $\Sigma_m(A)$ is a product of groups of the same type

$$Z_{\Sigma_m(A)}(x) = \prod_{h=1}^M \Sigma_{k_h}(A_h)$$

where the $A_h$ are abelian.

*Proof.* It suffices to remark that any central extension of a cyclic group by an abelian group is again abelian. \(\square\)

To prove Theorem 2.2 it suffices, by the technique of Lemma 1.5 exposed above, to show the following.

**Lemma 2.5.** Let $p$ be an odd prime and let $x$ be an element of order a power of $p$ in a group $A \wr \mathfrak{A}_m = A^m \rtimes \mathfrak{A}_m$ where $A$ is an abelian $p$-group. Then there is a group

$$Z' = \prod_{h=1}^M A_h \wr \mathfrak{A}_{k_h}$$

where all the $A_h$ are abelian $p$-groups, and with the property that $Z'$ is contained in the centralizer $Z_{A \wr \mathfrak{A}_m}(x)$ and contains a $p$-Sylow of $Z_{A \wr \mathfrak{A}_m}(x)$. 
Proof. We consider the group \( A \wr \mathfrak{A}_m \) as a subgroup of the complete monomial group \( \Sigma_m(A) \). By Lemma 2.4 it suffices to determine the intersection of \( Z_{\Sigma_m(A)}(x) = \prod_{h=1}^{M} \Sigma_{k_h}(A_h) \) and \( A \wr \mathfrak{A}_m \). As \( p \) is odd, it is clear that the intersection contains

\[
Z' = \prod_{h=1}^{M} A_h \wr \mathfrak{A}_{k_h}
\]

and that the complete centralizer \( Z_{A \wr \mathfrak{A}_m}(x) \) is an extension

\[
1 \rightarrow Z' \rightarrow Z_{A \wr \mathfrak{A}_m}(x) \rightarrow (\mathbb{Z}/2)^r \rightarrow 0
\]

where \((\mathbb{Z}/2)^r\) is an elementary abelian \( 2 \)-group which can be identified with the kernel of the sign

\[
\prod_{h=1}^{M} \mathfrak{G}_{k_h} \subset \mathfrak{G}_{\Sigma_{k_h}} \rightarrow \{ \pm 1 \}
\]

modulo the subgroup \( \prod_{h=1}^{M} \mathfrak{A}_{k_h} \). The statement follows as \( p \) is odd. \( \square \)

Thus we obtain

Proof. (of Theorem 2.2) We will prove more generally that \( H^s_*(G, \mathbb{Z}/p) \) is detected by abelian \( p \)-subgroups where \( G \) is any group which is a product of groups \( A \wr \mathfrak{A}_m \) with \( A \) an abelian \( p \)-group. We have that \( H^s_*(G, \mathbb{Z}/p) \) is detected by \( H^s_*(\text{Syl}_p(G), \mathbb{Z}/p) \), and the higher unramified cohomology of \( \text{Syl}_p(G) \) is trivial. This follows immediately from \( [B-P] \), Lemma 2.4, namely, if one forms a wreath product of groups, each of which has stably rational generically free linear quotients, then the wreath product inherits this property.

Hence every element \( a \) in \( H^s_*(G, \mathbb{Z}/p) \) will, in the notation of Lemma 1.5, give a nontrivial \( d_v(a) \in H^{n-1}_s(Z_G(h), \mathbb{Z}/p) \) for some element \( h \in G \) of \( p \)-power order. Thus it will be enough to show that \( H^{n-1}_s(Z_G(h), \mathbb{Z}/p) \) is detected by abelian \( p \)-subgroups. But \( H^{n-1}_s(Z_G(h), \mathbb{Z}/p) \) is detected by \( H^{n-1}_s(\text{Syl}_p(Z_G(h)), \mathbb{Z}/p) \) and, by Lemma 2.5, \( \text{Syl}_p(Z_G(h)) \) is contained in a group which in turn is contained in \( Z_G(h) \) and is again a product of groups of type \( A \wr \mathfrak{A}_m \). Hence we can conclude by induction on the cohomological degree \( n \). \( \square \)

Now we prove Theorem 2.1. It will follow immediately from

Proposition 2.6. Let \( N = p^n \), and suppose that \( A \) is an abelian \( p \)-subgroup of \( \mathfrak{A}_N \). Thus one can write

\[
A = \prod_{i=1}^{k} ((\mathbb{Z}/(p^i)))^{r_i}, \quad l_i, \ r_i \in \mathbb{N}.
\]


If $A$ is not reduced to a single cyclic group $\mathbb{Z}/(p^I)$, then $A$ is contained in a product of alternating groups $\prod_{j=1}^h \mathfrak{A}_t \subset \mathfrak{A}_N$ with $t_h < N$ for all $h$.

Once we have this Proposition, the proof of Theorem 2.1 is an induction: it suffices to prove it for $N = p^n$, and we may assume that detection by elementary abelian subgroups holds for the stable cohomology of all $\mathfrak{A}_j$ with $j < N$. Now clearly, $H^1_s(\mathfrak{A}_N, \mathbb{Z}/p)$ is detected by elementary abelian $p$-subgroups, for $H^1_s(\mathfrak{A}_N, \mathbb{Z}/p) = 0$ unless $p = 3$ and $N = 3$ so that $\mathfrak{A}_N = \mathbb{Z}/3$. So we have to show that any stable class $a \in H^i_s(\mathfrak{A}_N, \mathbb{Z}/p)$ for $i \geq 2$ is nontrivial on an elementary abelian $p$-subgroup. By Theorem 2.2 $a$ is nontrivial on an abelian $p$-subgroup $A$ with $\text{rk}(A/pA) \geq 2$ (as the stable cohomology of $A$ is an exterior algebra on $\text{rk}(A/pA)$ generators). Such an $A$ is contained in a product of smaller alternating groups by Proposition 2.6. Thus the proof is complete by induction.

Proof. (of Proposition 2.6)
Denote by $X_N = \{1, \ldots, N\}$ the set of letters on which the ambient $S_N \supset A$ acts. Let

$$X_N = \prod_{\alpha} X_{N, \alpha}$$

be the decomposition of $X_N$ into $A$-orbits. Let $X_{N,00} =: X$ be a fixed orbit. This orbit is isomorphic to a quotient $\tilde{A}$ of $A$, hence a group of the same form

$$\tilde{A} = \prod_{i=1}^k (\mathbb{Z}/(p^I))^n_{i}$$

and the action of $A$ on this orbit is via the regular representation of $\tilde{A}$ on itself. In other words, $A$ embeds into a subgroup $\prod_{\alpha} \tilde{A}_{\alpha}$ of $\mathfrak{A}_N$ where each $\tilde{A}_{\alpha}$ is embedded into a subgroup $\mathfrak{A}_{\text{ord}(A_{\alpha})}$ via the regular representation.

In summary, it suffices to prove the statement of Proposition 2.6 for the case that the group $A$ in its statement is embedded into the ambient $\mathfrak{A}_N$ via the regular representation. We can write $A = A' \times \mathbb{Z}/(p^k)$ with $\text{rk}(A'/pA') < \text{rk}(A/pA)$. Moreover, by definition of the regular representation, the composition of arrows

$$A = A' \times \mathbb{Z}/(p^k) \hookrightarrow \mathfrak{A}_{|A'|} \times \mathbb{Z}/(p^k) \hookrightarrow \mathfrak{A}_{|A'|} \wr \mathbb{Z}/(p^k)$$

gives the regular representation of $A$ where the first arrow $\hookrightarrow$ from the left is induced by the regular representation of $A'$, the second such arrow embeds $\mathfrak{A}_{|A'|} \times \mathbb{Z}/(p^k)$ into the wreath product $\mathfrak{A}_{|A'|} \wr \mathbb{Z}/(p^k)$ by
sending \((a; \sigma)\) to \((a, a, \ldots, a; \sigma)\) as usual, and the last arrow embeds the wreath product \(\mathfrak{A}_{|A'|} \wr \mathbb{Z}/(p^k)\) into \(\mathfrak{A}_{|A'|p^k}\) by partitioning the set of \(|A'| \cdot p^k\) objects which \(\mathfrak{A}_{|A'|p^k}\) permutes into \(p^k\) disjoint groups of \(|A'|\) objects, and letting \(\mathbb{Z}/(p^k)\) act by cyclically rotating these groups, and letting \((\mathfrak{A}_{|A'|})p^k\) act via permutations within these groups. It follows that

\[ A \subset \mathfrak{A}_{|A'|} \times \mathfrak{A}_{p^k} \]

where now \(\mathfrak{A}_{p^k}\) is embedded into \(\mathfrak{A}_{|A'|}\) as arbitrary alternating (not only cyclic) permutations of the \(p^k\) groups of items. Note that elements of the two subgroups \(\mathfrak{A}_{p^k}\) and \(\mathfrak{A}_{|A'|}\) of the group \(\mathfrak{A}_{|A'|}\) commute, and the two subgroups intersect trivially, so that we do have a direct product. Moreover, if \(A\) is not reduced to a single cyclic group, we have that \(A'\) is not the trivial group, and \(p^k < |A|\). □

3. Stable cohomology of alternating groups

Let \(\mathfrak{A}_n\) be, as in the previous section, the alternating group on \(n\) letters, and let \(p\) be an odd prime (the case \(p = 2\) has been treated in [B-P]). We assume first \(n = p^m\) for simplicity.

We have to know the way elementary abelian \(p\)-subgroups sit inside \(\mathfrak{A}_n\) for the following. We summarize everything in the following Lemma which is proven by arguments analogous to those already used in the proof of Proposition 2.6.

**Lemma 3.1.** Suppose \(n = p^m\) and denote by \(I_m := \{\underline{i} = (i_1, \ldots, i_m) \in \mathbb{N}^m\}\) the set of all nonnegative integer sequences \(\underline{i}\) with

\[ p^m = i_1p + i_2p^2 + \cdots + i_mp^m = \sum_{j=1}^{m} i_j p^j. \]

Then there is a natural bijection between \(I_m\) and the set of conjugacy classes of maximal elementary abelian \(p\)-subgroups in \(\mathfrak{S}_{p^m}\). The subgroup \(T(i_1, \ldots, i_m)\) corresponding to \(\underline{i}\) can be described as follows: partition the set of integers \(X = \{1, \ldots, n\}\) into segments of \(p\) power lengths according to \(\underline{i}\):

\[ X = \bigcup_{j=1}^{m} \bigcup_{s=1}^{i_j} X^j_s \]

where \(X^j_s\) is a set with \(p^j\) elements,

\[ X^j_s = \{i_1p + \cdots + i_{j-1}p^{j-1} + (s-1)p^j, \ldots, i_1p + \cdots + i_{j-1}p^{j-1} + sp^j\} \]
for definiteness. The subset $X^i$ corresponds to a subgroup $\mathcal{G}_{p^i} = (\mathcal{G}_{p^i})^{X^i} \subset \mathcal{G}_{p^m}$ fixing all elements in $X$ outside $X^i$. Inside $(\mathcal{G}_{p^i})^{X^i}$ there is a copy of $(\mathbb{Z}/p\mathbb{Z})^i$, which we denote by $((\mathbb{Z}/p\mathbb{Z})^i)^X$, embedded via the regular representation, i.e. we identify the elements in $X^i$ with the elements of $((\mathbb{Z}/p\mathbb{Z})^i)^X$ and the permutation action is then given by left multiplication.

We denote $T(0, \ldots, p^{m-k}, \ldots, 0)$ (a single nonzero entry $p^{m-k}$ in the $k$-th place) by $T_{k,m}$.

Hence every maximal elementary abelian $p$-subgroup in $\mathcal{A}_n$ is conjugate -in $\mathcal{S}_n$ or $\mathcal{A}_n$, it is the same thing- to one contained in $\mathcal{A}_{p^{n-1}} \times \cdots \times \mathcal{A}_{p^{n-1}}$ ($p$ factors) or conjugate to $T_{m,m}$.

The proof is immediate if one notices that under the action of some elementary abelian $p$-subgroup $A$ the set $X$ breaks up into $A$ orbits of cardinality a $p$ power, and the action of $A$ restricted to an orbit embeds $A$ into the permutation group of the elements of the orbit in such a way that the image is conjugate to the image of the regular representation. The result is in [A-M] VI. 1, Thm. 1.3, but also [Mui], Chapter II, §2, where it is ascribed to Dixon. For the statement that the conjugacy classes of maximal elementary abelian $p$-subgroups in $\mathcal{A}_n$ are the same as in $\mathcal{S}_n$ one can appeal to the following Lemma which we will also use in other instances below (it is e.g. in [Mann85], p. 269).

**Lemma 3.2.** For $n = p^m$ the Weyl groups $W_{\mathcal{S}_n}(T_{m,m}) = N_{\mathcal{S}_n}(T_{m,m})/T_{m,m}$ resp. $W_{\mathcal{A}_n}(T_{m,m})$ of $T_{m,m} \cong (\mathbb{Z}/p\mathbb{Z})^m$ inside $\mathcal{S}_n$ resp. $\mathcal{A}_n$ are

$$W_{\mathcal{S}_n}(T_{m,m}) = \text{GL}_m(\mathbb{F}_p), \quad W_{\mathcal{A}_n}(T_{m,m}) = \text{GL}_m^+(\mathbb{F}_p)$$

where $\text{GL}_m^+(\mathbb{F}_p)$ is the kernel of the map $\text{GL}_m(\mathbb{F}_p) \to \mathbb{Z}/2\mathbb{Z}$ given by the determinant raised to the power $(p - 1)/2$.

In fact it is true that the Weyl group of any group $H$ in the embedding $H \hookrightarrow \mathcal{S}_n$ given by the regular representation is the group of outer automorphisms of $H$, which become all inner in $\mathcal{S}_{|H|}$. Both statements of the Lemma follow from this remark as $\text{Aut}((\mathbb{Z}/p\mathbb{Z})^m) = \text{GL}(m, \mathbb{F}_p)$.

Likewise, Lemma 3.2 implies that in the normalizer of any maximal elementary abelian $p$-subgroup in $\mathcal{S}_n$ there are elements which do not lie in $\mathcal{A}_n$. Hence conjugacy classes of these in the two groups coincide.

We will also use in an essential way the Cárdenas-Kuhn Theorem to calculate the stable cohomology of $\mathcal{A}_n$, so we recall the precise statement (see [A-M] III.5 for the proof).
Theorem 3.3. Let \( E \subseteq S \subseteq G \) be a closed system of finite groups, where the closedness means that every subgroup of \( S \) which is conjugate to \( E \) in \( G \) is already conjugate to \( E \) in \( S \). Let \( W_G(E) = N_G(E)/E \) resp. \( W_S(E) = N_S(E)/E \) be the Weyl groups of \( E \) in \( G \) resp. \( S \), and suppose that \( E \) is \( p \)-elementary and that \( W_S(E) \) contains a \( p \)-Sylow of \( W_G(E) \). Then the image of the restriction map

\[
\text{res}_E^G : H^*(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z})
\]

is equal to

\[
\text{im} \left( \text{res}_E^S : H^*(S, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z}) \right) \cap H^*(E, \mathbb{Z}/p\mathbb{Z})^{W_G(E)}.
\]

We will mostly use this in the form of the following

Corollary 3.4. Let \( S \) be a \( p \)-Sylow of a finite group \( G \), and let \( E \) be an elementary abelian \( p \)-subgroup of \( S \). Suppose that any subgroup of \( S \) conjugate to \( E \) in \( G \) is conjugate to \( E \) in \( S \). Then we have

\[
\text{im} \left( \text{res}_E^G : H^*(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z}) \right) = \text{im} \left( \text{res}_E^S : H^*(S, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(E, \mathbb{Z}/p\mathbb{Z}) \right) \cap H^*(E, \mathbb{Z}/p\mathbb{Z})^{W_G(E)}.
\]

Proof. It suffices to remark that \([G : S] \equiv [N_G(E) : N_S(E)] \neq 0 \text{ (mod } p)\). \(\square\)

Now if \( n \) is arbitrary (not necessarily a power of \( p \)), to understand \( H^*_s(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z}) \), expand \( n \) in base \( p \):

\[
n = a_0 + a_1p + \cdots + a_mp^m
\]

with \( 0 \leq a_j < p \), \( a_m \neq 0 \), and note that this gives rise to a natural inclusion

\[
i_{a_1, \ldots, a_m} : \mathfrak{A}_{a_1, \ldots, a_m} := \prod_{1}^{a_1} \mathfrak{A}_p \times \prod_{1}^{a_2} \mathfrak{A}_{p^2} \times \cdots \times \prod_{1}^{a_m} \mathfrak{A}_{p^m} \hookrightarrow \mathfrak{A}_n
\]

and that a \( p \)-Sylow subgroup in \( \mathfrak{A}_n \) is given by the product of \( p \)-Sylow subgroups in the factors in \( \mathfrak{A}_{a_1, \ldots, a_m} \). In the notation of Lemma 3.3.1, the group \( \mathfrak{A}_{a_1, \ldots, a_m} \) contains an elementary abelian \( p \)-subgroup

\[
E := \prod_{1}^{a_1} T_{1,1} \times \prod_{1}^{a_2} T_{1,2} \times \cdots \times \prod_{1}^{a_m} T_{1,m} \simeq (\mathbb{Z}/p\mathbb{Z})^{\frac{n-a_k}{p^k}}.
\]

Proposition 3.5. The group \( E \) detects the stable cohomology of \( \mathfrak{A}_n \), i.e.

\[
H^*_s(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*_s(E, \mathbb{Z}/p\mathbb{Z})
\]

is injective.
Proof. It will be sufficient to prove this for $n = p^m$ as a Künneth theorem holds in stable cohomology for groups whose stable cohomology is detected by abelian subgroups, cf. Lemma 1.3. Now \( \mathfrak{A}_{p^m} \) contains the wreath product

\[
\mathfrak{A}_{p^m} \wr \mathbb{Z}/p\mathbb{Z}
\]

which detects the stable cohomology of \( \mathfrak{A}_{p^m} \) as it contains a \( p \)-Sylow. Using induction, it will be sufficient to prove that

\[
H_s^*(\mathfrak{A}_{p^m}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_s^*(\mathfrak{A}_{p^{m-1}} \times \cdots \times \mathfrak{A}_{p^{m-1}})
\]

is injective for \( m > 1 \). By Lemma 3.1 and because \( H_s^*(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z}) \) is detected by elementary abelian \( p \)-subgroups, it will be sufficient to show that all positive-dimensional classes in \( H^*(T_{m,m}, \mathbb{Z}/p\mathbb{Z}) \) coming as restrictions from \( H^*(\mathfrak{A}_{p^m}, \mathbb{Z}/p\mathbb{Z}) \) are unstable. This follows from the calculation in [Mann85], Theorem 1.9, and the fact that the Bocksteins are zero in stable cohomology. \( \square \)

**Theorem 3.6.** Let \( p \) be an odd prime as before. Then \( H_s^*(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z}) = 0 \) in positive degrees unless \( p = 3 \). For \( p = 3 \) one has for \( k \in \mathbb{N} \)

\[
H_s^*(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \simeq H_s^*(\mathfrak{A}_{3k+1}, \mathbb{Z}/3\mathbb{Z}), \quad H_s^d(\mathfrak{A}_{3k+2}, \mathbb{Z}/3\mathbb{Z}) = 0, \quad d > 0,
\]

and

\[
H_s^d(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) \neq 0 \text{ for } d > 0 \iff d = k,
\]

where \( H_s^*(E, \mathbb{Z}/3\mathbb{Z}) = H_s^*((\mathbb{Z}/3\mathbb{Z})^k, \mathbb{Z}/3\mathbb{Z}) \) is an exterior algebra on one-dimensional generators \( e_1, \ldots, e_k \).

Basically, we would like to use the Cardéñas-Kuhn Theorem with the elementary abelian subgroup \( E \), and \( S = \text{Syl}_p(\mathfrak{A}_n), \quad G = \mathfrak{A}_n \), but it will be more transparent to break it up into several steps.

**Lemma 3.7.** For \( p \neq 3 \) an odd prime we have in positive degrees \( H_s^*(\mathfrak{A}_n, \mathbb{Z}/p\mathbb{Z}) = 0 \).

Proof. The Weyl group \( W_{\mathfrak{A}_n}(E) \) contains two obvious subgroups: (1) the group \( \mathfrak{A}_N \) permuting the \( N := (n - a_0)/p \) copies of \( \mathbb{Z}/p\mathbb{Z} \) in \( E \), (2) a product \( \prod N(\mathbb{Z}/p\mathbb{Z})^* \) where \( (\mathbb{Z}/p\mathbb{Z})^* \) is the subgroup of the group of units in \( \mathbb{Z}/p\mathbb{Z} \) given as the kernel of \( a \mapsto a^{(p-1)/2} \). The stable cohomology of \( E \) is an exterior algebra over \( \mathbb{Z}/p\mathbb{Z} \) on \( N \) generators \( e_1, \ldots, e_N \). The \( \mathfrak{A}_N \)-invariants are concentrated in degrees 0, 1, \( (N - 1) \), \( N \), one-dimensional in each case and generated by

\[
1, \quad e_1 + \cdots + e_N, \quad f_1 \wedge \cdots \wedge f_{N-1}, \quad e_1 \wedge \cdots \wedge e_N,
\]
where \( f_1, \ldots, f_{N-1} \) is a basis of the \( \mathfrak{A}_N \)-invariant complement to \( e_1 + \cdots + e_N \) in \( H^1(E, \mathbb{Z}/p\mathbb{Z}) \). All of these are not invariant under the scalings in \( \prod_{i=1}^{N}(\mathbb{Z}/p\mathbb{Z})^{*+} \) unless \( p = 3 \) when \( (\mathbb{Z}/p\mathbb{Z})^{*+} \) is reduced to \( \{1\} \).

**Lemma 3.8.** One has

1. \( H^d_s(\mathfrak{A}_{3k+2}, \mathbb{Z}/3\mathbb{Z}) = 0, \ d > 0 \).
2. There is a natural embedding

\[
H^s_*(\mathfrak{A}_{3k+1}, \mathbb{Z}/3\mathbb{Z}) \hookrightarrow H^s_*(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) .
\]

**Proof.** This is already contained in [B-P], Lemmas 4.1 and 4.2. For completeness, let us repeat the argument: the restriction \( \text{res} \mathfrak{A}_{3k+2} \rightarrow \mathfrak{A}_{3k+1} \) factors through the restriction map induced from the embedding \( Syl_3(\mathfrak{A}_{3k+2}) \hookrightarrow \mathfrak{A}_{3k+2} \); but \( H^s_*(\mathfrak{A}_{3k}, \mathbb{Z}/3\mathbb{Z}) = 0 \) in positive degrees as the stable cohomology of \( \mathfrak{A}_{3k} \) is detected by its elementary abelian 2-subgroup generated by a maximal set of commuting transpositions. This proves (1), and (2) follows from the fact that the 3-Sylows in \( \mathfrak{A}_{3k} \) and \( \mathfrak{A}_{3k+1} \) are the same.

**Lemma 3.9.** Let \( n = 3k \) or \( n = 3k + 1 \). Then the Weyl group \( N_{\mathfrak{A}_n}(E) \) of \( E \simeq (\mathbb{Z}/3\mathbb{Z})^k \) in \( \mathfrak{A}_n \) sits in an extension

\[
1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^{k-1} \rightarrow W_{\mathfrak{A}_n}(E) \rightarrow \mathfrak{S}_n \rightarrow 1
\]

where \( \mathfrak{S}_n \) acts by permuting the copies of \( \mathbb{Z}/3\mathbb{Z} \) in \( E \simeq (\mathbb{Z}/3\mathbb{Z})^k \), and the group \( (\mathbb{Z}/2\mathbb{Z})^{k-1} \) acts by sending an even number of the generators \( g_i \) in the \( i \)th copy of \( \mathbb{Z}/3\mathbb{Z} \) to their inverses \( g_i^{-1} \). In stable cohomology \( H^s_*(E, \mathbb{Z}/3\mathbb{Z}) = E(e_1, \ldots, e_k) \) (exterior algebra), the action of the group \( W_{\mathfrak{A}_n}(E) \) is generated by the (signed) transpositions sending \( e_i, e_j \) to \( e_j, -e_i \), and transformations corresponding to elements in \( (\mathbb{Z}/2\mathbb{Z})^{k-1} \) acting via sign changes \( e_i \mapsto -e_i \) on an even number of the \( e_i \).

**Proof.** Any element of the normalizer \( N_{\mathfrak{A}_n}(E) \) induces a well-defined permutation of the copies of \( \mathbb{Z}/3\mathbb{Z} \) in \( E \). This gives a map to \( \mathfrak{S}_n \) which is onto: note that conjugating the 3-cycle \((123)\) by \( \tau = (23) \) exchanges the two nontrivial elements \( g, g^{-1} \) in \( \mathbb{Z}/3\mathbb{Z} \). We can also transpose two copies of \( \mathbb{Z}/3\mathbb{Z} \) in \( E \) by conjugating an element in \( \mathfrak{A}_n \). Now suppose \( n \in N_{\mathfrak{A}_n}(E) \) induces the trivial element in \( \mathfrak{S}_n \), so fixes all the copies of \( \mathbb{Z}/3\mathbb{Z} \) in \( E \) (though not necessarily elementwise). Then the only possible nontrivial automorphism of each copy of \( \mathbb{Z}/3\mathbb{Z} \) is exchanging \( g \) and \( g^{-1} \) as before. To conclude the proof, it suffices to note that if \( n \) induces the identity in \( \text{Aut}((\mathbb{Z}/3\mathbb{Z})^k) \), then \( n \in (\mathbb{Z}/3\mathbb{Z})^k \).

We can now turn to the
Proof. (of Theorem 3.6) The remaining assertion not covered by Lemma 3.7 and Lemma 3.8 are that
\[ H^d_s(\mathfrak{A}_3k, \mathbb{Z}/3\mathbb{Z}) \neq 0 \text{ for } d > 0 \iff d = k, \] and
\[ H^k_s(\mathfrak{A}_3k, \mathbb{Z}/3\mathbb{Z}) \simeq \langle \det_k \rangle \text{ where } \text{res}_{E}^{\mathfrak{A}_3k}(\det_k) = e_1 \wedge \cdots \wedge e_k. \]
and that
\[ H^*_{s}(\mathfrak{A}_3n+1, \mathbb{Z}/3\mathbb{Z}) \to H^*_{s}(\mathfrak{A}_3n, \mathbb{Z}/3\mathbb{Z}) \]
is surjective (it is injective by (2) of Lemma 3.8).

We prove first the assertions in the displayed formula 1 above, and 2 will follow easily (we just have to check that the determinant class comes from \( H^*_{s}(\mathfrak{A}_3n+1, \mathbb{Z}/3\mathbb{Z}) \)). We apply the Cardéñas-Kuhn Theorem 3.3 with \( S = \text{Syl}_3(\mathfrak{A}_3k) \) containing \( E \) and \( G = \mathfrak{A}_3k \). Then
- The fact that \( E \simeq (\mathbb{Z}/3\mathbb{Z})^k \subset \text{Syl}_3(\mathfrak{A}_3k) \subset \mathfrak{A}_3k \) is a closed system has been checked in [Mui], Prop. 2.2: in fact, he checks that if \( A \) is any maximal elementary abelian \( p \)-subgroup of a symmetric group \( S_n \), then any subgroup of a \( p \)-Sylow \( \text{Syl}_p(S_n) \) containing \( A \) which is conjugate to \( A \) in \( \overline{S_n} \) is conjugate to \( A \) in \( \text{Syl}_p(S_n) \). This implies clearly the statement for the alternating groups we need.
- By the Cardéñas-Kuhn Theorem or rather its Corollary 3.4, we get that the image of the cohomology of \( \mathfrak{A}_3k \) in the cohomology of \( E \) is
\[
\text{im} \left( \text{res}_{E}^{\text{Syl}_3(\mathfrak{A}_3k)} : H^*(\text{Syl}_3(\mathfrak{A}_3k, \mathbb{Z}/3\mathbb{Z}) \to H^*(E, \mathbb{Z}/p\mathbb{Z}) \right) \cap H^*(E, \mathbb{Z}/3\mathbb{Z})^{W_{\mathfrak{A}_3k}(E)}.
\]
- By Theorem 1.7 and induction
\[ \text{res}_{E}^{\mathfrak{A}_3k} : H^*(A_{3k}, \mathbb{Z}/3\mathbb{Z}) \to H^*(E, \mathbb{Z}/3\mathbb{Z})^{W_{\mathfrak{A}_3k}(E)} \]
is surjective (compare also the argument in [Mui], Prop. 3.9 and Lemma 3.11).

Thus
\[ \text{res}_{E}^{\mathfrak{A}_3k} : H^*_s(A_{3k}, \mathbb{Z}/3\mathbb{Z}) \simeq H^*_s(E, \mathbb{Z}/3\mathbb{Z})^{W_{\mathfrak{A}_3k}(E)} \]
and by the description of the action of \( W_{\mathfrak{A}_3k} \) on the stable cohomology of \( E \), we find that only \( e_1 \wedge \cdots \wedge e_k \) remains spanning the positive dimensional invariants.

Finally, to prove the surjectivity of the arrow in the displayed formula 2 above, consider the inclusions \( E \subset \mathfrak{A}_3n \subset \mathfrak{A}_3n+1 \). Then \( \text{Syl}_3(\mathfrak{A}_3k) = \text{Syl}_3(\mathfrak{A}_3k+1) \) and, in exact analogy to the argument above, by [Mui],
Prop. 2.2, \( E \subset \text{Syl}_3(\mathfrak{A}_{3^{n+1}}) \subset \mathfrak{A}_{3n+1} \) is a closed system, so that by Cardénas-Kuhn the image of the cohomology of \( \mathfrak{A}_{3n+1} \) in the cohomology of \( E \) coincides with the image of the cohomology of \( \mathfrak{A}_{3n} \) in \( E \) (because also \( W_{\mathfrak{A}_{3n}}(E) \simeq W_{\mathfrak{A}_{3n+1}}(E) \)). □

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F. BOGOMOLOV, COURANT INSTITUTE OF MATHEMATICAL SCIENCES, 251 MERCER ST., NEW YORK, NY 10012, U.S.A., and

LABORATORY OF ALGEBRAIC GEOMETRY, GU-HSE, 7 VAVILOVA STR., MOSCOW, RUSSIA, 117312

E-mail address: bogomolo@courant.nyu.edu

CHRISTIAN BÖHNING, FACHBEREICH MATHEMATIK DER UNIVERSITÄT HAMBURG, BUNDESSTRASSE 55, 20146 HAMBURG, GERMANY

E-mail address: christian.boehning@math.uni-hamburg.de