LARGE-TIME BEHAVIOR OF SOLUTION TO AN INFLOW PROBLEM ON THE HALF SPACE FOR A CLASS OF COMPRESSIBLE NON-NEWTONIAN FLUIDS

ZHENHUA GUO, WENCHAO DONG AND JINJING LIU *

School of Mathematics and CNS
Northwest University, Xi’an 710069, China

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Abstract. In this paper, we study the large time behaviors of boundary layer solution of the inflow problem on the half space for a class of isentropic compressible non-Newtonian fluids. We establish the existence and uniqueness of the boundary layer solution to the non-Newtonian fluids. Especially, it is shown that such a boundary layer solution have a maximal interval of existence. Then we prove that if the strength of the boundary layer solution and the initial perturbation are suitably small, the unique global solution in time to the non-Newtonian fluids exists and asymptotically tends toward the boundary layer solution. The proof is given by the elementary energy method.

1. Introduction. The one-dimensional isentropic compressible non-Newtonian fluids on half space in the Eulerian coordinates read

\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p(\rho))_x &= (\mu|u_x|^{q-2}u_x)_x, & x > 0, t > 0.
\end{aligned}
\]

where \(\rho > 0\) and \(u\) are density and velocity, the pressure \(p\) is assumed to be a smooth function of \(\rho\) satisfying

\[
p'(\rho) > 0 \quad \text{and} \quad \rho p''(\rho) + 2p'(\rho) > 0, \quad \text{for} \quad \rho > 0, \tag{1.2}
\]

\(\mu > 0\) and \(q > 2\) are assumed to be constant.

In many fields, such as chemistry, biological fluids like blood, geology and glaciology, a large number of problems may arise with non-Newtonian fluids. In the past decades, mathematical aspects for the non-Newtonian fluids have been extensively studied and significant progress has been made, see [2–9, 12–14, 21, 23–27] and the references cited therein. Especially, on the study of incompressible non-Newtonian fluids:

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}(\Gamma) + \nabla p &= \rho f, \\
\text{div} u &= 0,
\end{aligned}
\]

\[\text{(1.3)}\]

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* Corresponding author.
where \( f \) is an external force, \( \Gamma \) denotes the viscous stress tensor which depends on the rate of strain \( E_{ij}(\nabla u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \). Ladyzhenskaya [12] proposed a special model for \( \Gamma \):

\[
\Gamma_{ij} = (\mu_1 + \mu_2 |E(\nabla u)|^{q-2})E_{ij}(\nabla u),
\]

(1.4)

In the case \( \mu_1 = 0 \) and \( \mu_2 > 0 \), if \( q < 2 \), it is a pseudo-plastic fluid, and if \( q > 2 \), then it is a dilatant fluid [2]. Physically, the model captures the shear thinning fluid for the case of 1 < \( q < 2 \), and captures the shear thickening fluid for the case of \( q > 2 \). The global existence of sufficiently regular solutions to two-dimensional and three-dimensional equations of compressible non-Newtonian fluids have been established by Mamontov [14]. Yuan and his cooperators [25, 26] obtained the existence and uniqueness of local and global solutions for one dimensional initial boundary value problem. Fang and Guo [3] gave the blow-up criterion for the local strong solutions, constructed an analytical solutions to a class of compressible non-Newtonian fluids with free boundaries in [4], and considered the existence and uniqueness of global classical solution for a initial boundary problem [6]. For weak solutions to the non-Newtonian fluids, Zhikov and Pastukhova [27] obtained the existence of weak solutions of initial boundary value problem for multidimensional cases. Guo and Zhu [9] investigated the partial regularity of the suitable weak solutions. Feireisl, Liao and Málek [8] studied mathematical properties of unsteady for three dimensional compressible non-Newtonian fluids in bounded domains and shown the long-time and large-data existence result of weak solutions with strictly positive density. The existence of weak solutions to a one-dimensional full compressible non-Newtonian fluids has been investigated by Fang, Kong and Liu [7]. Recently, Fang and Guo [5], Shi, Wang and Zhang [21] discussed the stability of rarefaction waves for the isentropic and nonisentropic compressible non-Newtonian fluids, respectively.

In this paper, we study the large time behaviors of boundary layer solutions of the inflow problem to the system (1.1) with the initial data

\[
(\rho, u)(0, x) = (\rho_0, u_0)(x) \to (\rho_+, u_+), \quad \text{as} \quad x \to +\infty,
\]

(1.5)

and the condition on the boundary \( x = 0 \):

\[
\begin{aligned}
&u(t, 0) = u_- > 0, \\
&\rho(t, 0) = \rho_- > 0, \quad t > 0,
\end{aligned}
\]

(1.6)

where \( \rho_\pm \) and \( u_\pm \) are prescribed constants.

The inflow problem was proposed by Matsumura [16] for a one-dimensional isentropic compressible Navier-Stokes equations and the author classified all possible large time behaviors of the solutions in terms of the boundary values. Then Matsumura and Nishihara [17] gave the rigorous proofs of the stability theorems on both the boundary layer solution and a superposition of the boundary layer solution and the rarefaction wave. Huang, Matsumura and Shi [11] established the asymptotic stability on both the viscous shock wave and a superposition of the viscous shock wave and the boundary layer solution under some smallness conditions for the inflow problem to the isentropic Navier-Stokes equations. For the full Navier-Stokes equations, Huang, Li and Shi [10] proved the existence of the subsonic boundary layer solution and the stability of this boundary layer solution and its superposition with the 3-rarefaction wave under some smallness assumptions. Qin and Wang [19] showed the asymptotic stability of not only the single contact wave but also the superposition of the subsonic boundary layer solution, the contact wave, and the rarefaction wave to the inflow problem under some smallness conditions. Then
they [20] also proved the asymptotic stability of the superposition of the transonic boundary layer solution, the 1-rarefaction wave, the viscous 2-contact wave, and the 3-rarefaction wave to the inflow problem under some smallness conditions. We also refer to [18] for the asymptotic stability of stationary solution to the inflow problem.

The nonlinear waves can be identified in the non-Newtonian fluids, such as shock phenomena in the blood fluids [1]. However, since the nonlinear constitutive relation between viscous stress tensor and rate of strain in non-Newtonian fluids, the nonlinear waves may exhibit some properties which are different from those in the Newtonian fluids. Our main goal is to investigate the boundary layer solution of the non-Newtonian fluids (1.1) and to show that it is asymptotically stable under small initial perturbations.

In the present paper, we establish the existence and uniqueness of the boundary layer solution to the inflow problem of the non-Newtonian fluids (1.1). In contrast with the case of the Newtonian fluids [17], we identify a peculiar feature of boundary layer solution, i.e., it is shown that the boundary layer solution to the non-Newtonian fluids (1.1) has a maximal interval of existence (see Section 2 below for the details). This characteristic of boundary layer solution do not develop in the Newtonian fluids. The effect of the nonlinear constitutive relation between viscous stress tensor and rate of strain for the non-Newtonian fluids (1.1) gives rise to such phenomenon of boundary layer solution.

Then we prove that if the initial data are close to the boundary layer solution and the strength of boundary layer solution is suitably small, a unique solution to the inflow problem of the non-Newtonian fluids (1.1) exists globally in time and tends toward the boundary layer solution as the time goes to infinity. The main point in our analysis is based on the maximal interval of existence of the boundary layer solution to the non-Newtonian fluids (1.1), which allows us to use the elemental energy method to study the asymptotic stability of the boundary layer solution for any \( q > 2 \).

The rest of this paper is organized as follows. In Section 2, we study the existence of the boundary layer solution for the non-Newtonian fluids (1.1) and state our main theorems. Sections 3 reformulates the original problem to obtain a initial-boundary-value problem, and establishes the a priori estimates for proving the main theorems. In Section 4, we give the proof of the a priori estimates and complete the proof of the main theorems.

**Notations:** Throughout this paper, function space \( L^p(\Omega) \), \( 1 \leq p \leq +\infty \), is an usual Lebesgue space on \( \Omega \subset [0, +\infty) \) with its norm

\[
\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty, \quad \|u\|_{L^\infty(\Omega)} = \text{ess sup}_{\Omega} |u(x)|.
\]

\( W^{k,p}(\Omega) \) denotes the Sobolev space with its norm

\[
\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{j \leq k} \int_{\Omega} \left| \frac{d^j u(x)}{dx^j} \right|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty.
\]

If \( p = 2 \), we usually write \( H^k(\Omega) = W^{k,2}(\Omega) \), and \( H^1_0(\Omega) \) is a closure of \( C_0^\infty(\Omega) \) with respect to \( H^1 \)-norm. Particularly, we denote

\[
\| \cdot \|_{L^p} = \| \cdot \|_{L^p(\mathbb{R}_+)} \quad \text{and} \quad \| \cdot \|_{W^{k,p}} = \| \cdot \|_{W^{k,p}(\mathbb{R}_+)}. \]
$C$ denotes the generic positive constants which are independent of time $t$ unless otherwise stated.

2. Boundary layer solution and main results. In this section, we study the boundary layer solution of (1.1) and state our main results.

We transform the inflow problem (1.1), (1.5) and (1.6) to the moving boundary problem in the Lagrangian coordinate

$$
\begin{align*}
&v_t - u_x = 0, \quad x > s_0 t, \quad t > 0, \\
&u_t + p(v)_x = \mu \left( \frac{|u_x|^{q-2} u_x}{v} \right)_x, \\
&(v, u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+), \quad \text{as} \quad x \to +\infty, \\
&(v, u)|_{x=s_0 t} = (v_-, u_-), \quad t > 0,
\end{align*}
$$

(2.1)

where $v = \frac{1}{\rho}$ represents the specific volume, the boundary moves with the constant speed $s_0 = -u_- < 0$ and

$$
p'(v) < 0 < p''(v), \quad \text{for} \quad v > 0.
$$

(2.2)

The corresponding hyperbolic system without viscosity is

$$
\begin{align*}
&v_t - u_x = 0, \\
&u_t + p(v)_x = 0,
\end{align*}
$$

(2.3)

which has two eigenvalues

$$
\lambda_1 = -\sqrt{-p'(v)}, \quad \lambda_2(v) = \sqrt{-p'(v)}.
$$

(2.4)

We define the sound speed $c(v)$ by

$$
c(v) = v\sqrt{-p'(v)}.
$$

(2.5)

Let $\Xi = \{(v, u) \in \mathbb{R}_+ \times \mathbb{R}_+ \}$ be the phase plane of $(v, u)$ and abbreviate $(v, u)$ to $w$. For any fixed left state $(v_-, u_-) := w_- \in \Xi$, solving the Riemann problem of (2.3) on $\Xi$, we can get the 1-rarefaction wave

$$
R_1(w_-) : \begin{cases}
\lambda_1 = -\sqrt{-p'(v)}, \\
u - u_- = \int_{v_-}^{v} \sqrt{-p'(s)} ds, \quad v_- < v, \quad w \in \Xi,
\end{cases}
$$

(2.6)

and the 2-rarefaction wave

$$
R_2(w_-) : \begin{cases}
\lambda_2 = \sqrt{-p'(v)}, \\
u - u_- = -\int_{v_-}^{v} \sqrt{-p'(s)} ds, \quad v_- > v, \quad w \in \Xi.
\end{cases}
$$

(2.7)

On the other hand, from the Rankine-Hugoniot condition

$$
\begin{align*}
-s[v] - [u] = 0, \\
-s[u] + [p(v)] = 0,
\end{align*}
$$

(2.8)

where $[q] = q_r - q_l$ with $q_l = q(t, x(t) - 0)$ and $q_r = q(t, x(t) + 0)$, $s$ is the velocity of shock waves, and the Lax entropy condition

$$
s < \lambda_1(w_l), \quad \lambda_1(w_r) < s < \lambda_2(w_r)
$$

for 1-shock;

$$
\lambda_1(w_l) < s < \lambda_2(w_l), \quad \lambda_2(w_r) < s
$$

for 2-shock, we can obtain 1-shock curve

$$
S_1(w_-) : u - u_- = -\sqrt{(v - v_-)(p(v_-) - p(v))}, \quad v_- > v, \quad w \in \Xi,
$$

(2.9)
and 2-shock curve

\[ S_2(w_-): \quad u - u_- = -\sqrt{(v - v_-)(p(v_-) - p(v))}, \quad v_- < v, \quad w \in \Xi. \quad (2.10) \]

2.1. Existence and uniqueness of boundary layer solution. We divide the domain \( \Xi \) into three regions (Figure 1):

\[
\Xi_{\text{super}} = \{ w \mid |u| > c(v), \quad w \in \Xi \},
\]

\[
\Gamma_{\text{trans}} = \{ w \mid |u| = c(v), \quad w \in \Xi \},
\]

\[
\Xi_{\text{sub}} = \{ w \mid |u| < c(v), \quad w \in \Xi \}.
\]

Call them the supersonic, transonic and subsonic regions, respectively.

![Figure 1](image)

When \((v_-, u_-) \in \Xi_{\text{sub}}\), \( \lambda_1(v_-) < s_0 < 0 \), i.e., the first wave speed \( \lambda_1(v_-) \) is less than the boundary speed \( s_0 \), hence the existence of a traveling wave solution \((V, U)(x - s_0 t)\) with \((V, U)(0) = (v_-, u_-), \quad (V, U)(+\infty) = (v_+, u_+), \) is expected. Substituting this into (2.1) gives

\[
\begin{cases}
- s_0 V' - U' = 0, \\
- s_0 U' + p'(V) = \mu \left( \frac{U'}{V'} \right)^{q-2} \frac{U'}{V'} ,
\end{cases}
\]

(2.11)

where \( t = d/d\xi \) and \( \xi = x - s_0 t > 0 \). We call this solution \( W(\xi) = (U, V)(\xi) \) the boundary layer solution (BL-solution).

Seek the condition for the existence of the BL-solution and investigate its properties. Integrating (2.11) over \((0, +\infty)\) and \((\xi, +\infty)\) yields

\[
\begin{cases}
- s_0 (v_+ - v_-) - (u_+ - u_-) = 0, \\
- s_0 (u_+ - u_-) + p(v_+) - p(v_-) = -\mu \frac{U'(0)}{V} \frac{q-2}{v_-},
\end{cases}
\]

(2.12)

and

\[
\begin{cases}
- s_0 (v_+ - V) - (u_+ - U) = 0, \\
- s_0 (u_+ - U) + p(v_+) - p(V) = -\mu \frac{U'}{V} \frac{q-2}{V'},
\end{cases}
\]

(2.13)

respectively. From (2.12) and (2.13), we get that

\[ s_0 = \frac{U(\xi)}{V(\xi)} = -\frac{u_+}{v_+} = -\frac{u_-}{v_-}. \]

(2.14)
Thus we define BL-line through \( w_- \in \Xi_{\text{sub}} \) by

\[
\text{BL}(w_-) = \{ w \mid \frac{u}{v} = \frac{u_-}{v_-} = -s_0, \ w \in \Xi \}. \tag{2.15}
\]

The line \( \text{BL}(w_-) \) always intersects the transonic line \( \Gamma_{\text{trans}} \) at the point \( w_* = (v_*, u_*) \). See Figure 2.

By (2.13), we obtain the ordinary differential equation of \( V \)

\[
\begin{cases}
\mu \left| s_0 V' \right|^q s_0 V' = -s_0^2 (V - v_+) + p(v_+) - p(V), \\
V(0) = v_-, \quad V(+\infty) = v_+,
\end{cases} \tag{2.16}
\]

from which, we can obtain one of our main results.

**Theorem 2.1** (Existence and uniqueness of BL-solution). Suppose \( u_- > 0 \) and \( w_+ \in \text{BL}(w_-) \).

(i) If \( w_- \in \Xi_{\text{sub}} \cup \Gamma_{\text{trans}} \) and \( u_+ > u_* \), then there is no BL-solution.

(ii) If \( w_- \in \Xi_{\text{sub}} \) and \( u_+ = u_* \), then there exists a unique degenerate BL-solution of (2.11). In addition, when \( 2 < q < 3 \), we have

\[
V'(\xi) > 0, \quad \text{for} \quad \xi > 0, \tag{2.17}
\]

and

\[
|W(\xi) - w_+| = O(1)\xi^{\frac{q-1}{q}}, \quad \xi \to +\infty; \tag{2.18}
\]

when \( q = 3 \), there exists positive constants \( \delta_1 \) and \( \delta_2 \) such that

\[
V'(\xi) > 0, \quad \text{for} \quad \xi > 0, \tag{2.19}
\]

and

\[
|w_+ - w_-|e^{-\delta_1 \xi} \leq |W(\xi) - w_+| \leq |w_+ - w_-|e^{-\delta_2 \xi}, \quad \xi \to +\infty; \tag{2.20}
\]

when \( q > 3 \), there exists a positive constant \( \xi_+ \) such that

\[
V'(\xi) > 0, \quad \text{for} \quad 0 < \xi < \xi_+, \tag{2.21}
\]

and

\[
V'(\xi) \equiv 0, \quad V(\xi) \equiv v_+, \quad \text{for} \quad \xi \geq \xi_+. \tag{2.22}
\]
(iii) If $w_\rightarrow \in \Xi_{\text{sub}}$, then there exists a unique BL-solution of (2.11). Furthermore, there exists a positive constant $\xi_+$ such that
\[ V'(\xi) > 0, \quad \text{for} \quad 0 < \xi < \xi_+, \tag{2.23} \]
and
\[ V'(\xi) \equiv 0, \quad V(\xi) \equiv v_+, \quad \text{for} \quad \xi \geq \xi_. \tag{2.24} \]

(iv) If $w_\rightarrow \in \Xi_{\text{sub}} \cup \Gamma_{\text{trans}}$ and $0 < u_+ < u_-$, then there exists a unique BL-solution of (2.11). Moreover, there exists a positive constant $\xi_+$ such that
\[ V'(\xi) < 0, \quad \text{for} \quad 0 < \xi < \xi_+, \tag{2.25} \]
and
\[ V'(\xi) \equiv 0, \quad V(\xi) \equiv v_+, \quad \text{for} \quad \xi \geq \xi_. \tag{2.26} \]

Proof. For part (i), if $(V,U)$ exists, then
\[ 0 < u_\ast < U(\xi) < u_+ \quad \text{and} \quad 0 < v_\ast < V(\xi) < v_+, \tag{2.27} \]
which implies $(V,U) \in \Xi_{\text{super}}$. Thus, we have $U > c(V) = V\sqrt{-p'(V)}$. Since $s_0 = -\frac{V}{V'}$, we then obtain
\[ s_0^2 > \left(\frac{c(V)}{V}\right)^2 = -p'(V). \tag{2.28} \]

By (2.2), one can check that
\[ -s_0^2(V - v_+) + p(v_+) - p(V) > 0, \tag{2.29} \]
thus, $V' < 0$ in (2.16). Due to $V(\infty) = v_+$, one can deduce that $v_\ast > v_+$, which contradicts with the condition $v_\ast < v_+$. Therefore, in this case, there is no BL-solution of (2.11).

For part (ii), in this case, we have
\[ 0 < u_\rightarrow < U(\xi) < u_+ \quad \text{and} \quad 0 < v_\rightarrow < V(\xi) < v_+, \tag{2.30} \]
which implies $(V,U) \in \Xi_{\text{sub}}$. Thus,
\[ U < c(V) = V\sqrt{-p'(V)}, \tag{2.31} \]
and
\[ s_0^2 < \left(\frac{c(V)}{V}\right)^2 = -p'(V). \tag{2.32} \]

By (2.2), it holds that
\[ -s_0^2(V - v_+) + p(v_+) - p(V) < 0, \tag{2.33} \]
which implies $V' > 0$ in (2.16). Hence, from (2.16), we have
\[ \begin{cases} \mu\left(\frac{-s_0^2 V'}{V'}\right)^{q-1} = s_0^2(V - v_+) + p(V) - p(v_+), \\ V(0) = v_\rightarrow, \quad V(\infty) = v_+, \end{cases} \tag{2.34} \]
from which, we can get an initial value problem of the ordinary differential equation
\[ \begin{cases} V' = \frac{V}{s_0} \left(\mu^{-1}(s_0^2(V - v_+) + p(V) - p(v_+))\right)^{\frac{1}{q-1}} := F(V), \\ V(0) = v_\rightarrow. \end{cases} \tag{2.35} \]
This equation can be solved using a small ruse [22]. Since \( F(V) > 0 \) when \( v_- \leq V(\xi) < v_+ \), we can divide both sides of \( V' = F(V) \) by \( F(V) \) and integrate with respect to \( \xi \):

\[
\xi = \int_0^\xi \frac{V'(\xi)}{F(V(\xi))} \, d\xi = \int_{v_-}^{V(\xi)} \frac{dV}{F(V)} := G(V(\xi)), \quad \text{for } v_- \leq V(\xi) < v_+.
\]

Since \( G(V) \) is strictly monotone, it can be inverted and we obtain a unique solution in a neighborhood of \( \xi = 0 \):

\[
\begin{cases}
V(\xi) = G^{-1}(\xi), \\
V(0) = G^{-1}(0) = v_-.
\end{cases}
\]

Now let us investigate the maximal interval of existence of this solution. Define

\[
\xi_+ = \int_{v_-}^{v_+} \frac{dV}{F(V)}. \tag{2.36}
\]

Then \( V(\xi) \in C^1(0, \xi_+) \) and \( \lim_{\xi \to \xi_+} V(\xi) = v_+ \).

Since \( u_+ = u_\ast \), i.e. \( w_+ \in \Gamma_{trans} \), we have \( u_+ = c(v_+) \). Thus,

\[
\frac{1}{2} p'(\eta)(v_+ - V)^2 \leq s^2(V - v_+) + p(V) - p(v_+) \leq p''(\eta)(v_+ - V)^2, \quad V < \eta < v_+.
\]

Returning to (2.35), we have

\[
C_1 (v_+ - V)^{\frac{2}{q-1}} \leq F(V) \leq C_2 (v_+ - V)^{\frac{2}{q-1}}, \tag{2.37}
\]

where \( C_1, C_2 \) are some positive constants.

Next, according to the quantities of \( q \), we discuss the BL-solution for the following three cases.

1) When \( 2 < q < 3 \), using (2.36) and (2.37), we calculate that

\[
\xi_+ \geq C \int_{v_-}^{v_+} (v_+ - V)^{-\frac{2}{q-1}} \, dV = +\infty. \tag{2.38}
\]

Hence there exists a unique degenerate BL-solution of (2.11) for \( \xi > 0 \).

Moreover, we deduce from (2.35) and (2.37) that

\[
C_1 \leq \frac{V'}{(v_+ - V)^{\frac{2}{q-1}}} \leq C_2. \tag{2.39}
\]

Integrating (2.39) with respect to \( \xi \), we have

\[
C_1 \xi \leq \int_0^\xi \frac{V'}{(v_+ - V)^{\frac{2}{q-1}}} \, d\xi \leq C_2 \xi, \tag{2.40}
\]

which leads to

\[
C_1 \frac{3 - q}{q - 1} \xi \leq (v_+ - V)^{\frac{q-3}{q-1}} - (v_+ - v_-)^{\frac{q-3}{q-1}} \leq C_2 \frac{3 - q}{q - 1} \xi. \tag{2.41}
\]

Thus, we obtain

\[
C_3 \xi^{\frac{q-1}{q-3}} \leq v_+ - V \leq C_4 \xi^{\frac{q-1}{q-3}}, \quad \text{as } \xi \to +\infty, \tag{2.42}
\]

where \( C_3, C_4 \) are some positive constants. Then, by (2.14), we can get

\[
|W(\xi) - w_+| = O(1)\xi^{\frac{q-1}{q-3}}, \quad \xi \to +\infty. \tag{2.43}
\]
(2) When \( q = 3 \), in a similar way to (2.38) and (2.39), one can get that \( \xi_+ = +\infty \) and
\[
C_1 \leq \frac{V'}{(v_+ - V)} \leq C_2. \tag{2.44}
\]
Integrating (2.44) over \((0, \xi)\) leads to
\[
\ln(v_+ - v_-) - C_2\xi \leq \ln(v_+ - V) \leq \ln(v_+ - v_-) - C_1\xi. \tag{2.45}
\]
Thus,
\[
(v_+ - v_-)e^{-C_2\xi} \leq v_+ - V \leq (v_+ - v_-)e^{-C_1\xi}, \tag{2.46}
\]
Therefore, by (2.14), we obtain
\[
|w_+ - w_-|e^{-\delta_1\xi} \leq |W(\xi) - w_+| \leq |w_+ - w_-|e^{-\delta_2\xi}, \quad \text{as } \xi \to +\infty, \tag{2.47}
\]
where \( \delta_1, \delta_2 \) are some positive constants.

(3) When \( q > 3 \), similarly, using (2.36) and (2.37), we calculate that
\[
\xi_+ = \int_{v_-}^{v_+} \frac{dV}{F(V)} \leq C \int_{v_-}^{v_+} (v_+ - V)^{-\frac{q-1}{q-3}} dV
\]
\[
= C\frac{q-1}{q-3}(v_+ - v_-)^{\frac{q-3}{2}} < +\infty.
\]
Thus, \( V(\xi_+) = v_+ \) and \( \lim_{x \to \xi_+} V'(\xi) = 0 \). Therefore \( V(\xi) \equiv v_+ \), for any \( \xi > \xi_+ \), if not, then we can check that there exists \( \bar{\xi} > \xi_+ \) such that \( V'(\bar{\xi}) < 0 \), which contradicts with \( V' > 0 \).

For part (iii), similar fashion to the proof of part (ii) yields \( V' > 0 \). On the other hand, since
\[
\mathbf{s}_0^2(V - v_+) + p(V) - p(v_+) = (-\mathbf{s}_0^2 - p'(\bar{\eta}))(v_+ - V), \quad \bar{\eta} \in (V, v_+), \tag{2.48}
\]
we obtain
\[
C_1(v_+ - V)^{\frac{q-1}{q}} \leq F(V) \leq C_2(v_+ - V)^{\frac{q-3}{2}}. \tag{2.49}
\]
Due to \( q > 2 \), using a analogous way to the proof of case (3) of part (ii), we can get the desired result.

For part (iv), in a similar analysis as above, we can conclude that (iv) holds. \( \Box \)

**Remark 2.1.** From the analysis above, it is clear to see that if \( q = 2 \), i.e., the case of Newtonian fluids, we always have \( \xi_+ = +\infty \). However, in the non-Newtonian fluids (1.1), the nonlinear constitutive relation between viscous stress tensor and rate of strain leads to the maximal interval of existence of the BL-solutions.

2.2. **Stability of boundary layer solution.** For any \( w_- \in \Xi_{\text{sub}} \) and \( w_+ \in BL(w_-)(u_- < u_+ < u_+) \), we assume that the initial data in (2.1) satisfies
\[
w_0 - W \in H^1(\mathbf{R}_+), \quad (u_0 - U)_x \in L^q(\mathbf{R}_+), \tag{2.50}
\]
and the compatibility condition
\[
w_0(0) = w_- . \tag{2.51}
\]
Then, we can prove the other main result of this paper.

**Theorem 2.2** (Stability of BL-solution). For any \( w_- \in \Xi_{\text{sub}} \) and \( w_+ \in BL(w_-) \) \( (u_- < u_+ < u_+) \), suppose that the assumptions (2.50) and (2.51) hold. Then there exists a small positive constant \( \sigma_0 \) such that if \( |w_+ - w_-| \) is suitably small and
\[
\|w_0 - W\|_{H^1} + \|(u_0 - U)_x\|_{L^q} \leq \sigma_0, \tag{2.52}
\]

the inflow problem (2.1) has a unique time global solution w satisfying \((w - W) \in C(0, +\infty; H^1_0)\) and \((u - U) \in L^8(0, +\infty; L^0)\), and the asymptotic behavior
\[
\lim_{t \to +\infty} \sup_{x \geq s_0} | w(t, x) - W(t, x) | = 0.
\]

**Remark 2.2.** Our result and method can also be generalized to discuss the stability of BL-solutions to (2.1) for the cases (ii) and (iv) shown in Theorem 2.1 and similar results to Theorem 2.2 are also expected for the two cases.

3. **Reformulation of the original problem.** Now, we begin to prove Theorem 2.2. Putting the perturbation \((\phi, \psi)(\xi, t)\) by
\[
(v, u)(x, t) = (V, U)(\xi) + (\phi, \psi)(\xi, t), \quad \xi = x - s_0 t, \tag{3.1}
\]
so that we can reformulate the inflow problem (2.1) as
\[
\begin{cases}
\phi_t - s_0 \phi_x - \psi_x = 0, & \xi > 0, \ t > 0, \\
\psi_t - s_0 \psi_x + p(V + \phi) - p(V) \big|_{\xi = 0} = \mu \left( \frac{|u_t + \psi_x|^{q-2} u_t + \psi_x}{V + \phi} - \frac{|u_t|^{q-2} u_t}{V} \right), & \phi, \psi \big|_{t = 0} = (\phi_0, \psi_0)(\xi), \ \psi \big|_{t = 0} = (v_0 - V, u_0 - U)(\xi).
\end{cases}
\tag{3.2}
\]

Then we establish the existence of solution to (3.2). First of all, we define the solution space \(X(0, T)\) for any \(0 \leq T < +\infty\),
\[
X(0, T) = \left\{ (\phi, \psi) \in C(0, T; H^1_0(\mathbb{R}_+)) \mid \phi \in L^2(0, T; L^2(\mathbb{R}_+)), \psi \in L^2(0, T; H^1(\mathbb{R}_+))), \psi \in L^q(0, T; L^q(\mathbb{R}_+)), \sup_{[0, T]} \|(\phi, \psi)\|_{H^1} \leq M, \inf_{\mathbb{R}_+ \times [0, T]} (V + \phi)(\xi, t) \geq m \right\},
\]
where \(m, M\) are positive constants. Set
\[
N(T) = \sup_{0 \leq t \leq T} \|(\phi, \psi)\|_{H^1} + \sup_{0 \leq t \leq T} \|\psi\|_{L^q}, \quad \delta_1 = |w_+ - w_-| \tag{3.3}
\]
and denote
\[
(\phi_\xi(\xi, 0), \psi_\xi(\xi, 0)) = (\phi_{0, \xi}, \psi_{0, \xi})(\xi). \tag{3.4}
\]

The global smooth solution in \(X(0, T)\) is constructed by the combination of the local existence and the a priori estimate. The proof of the existence of local solution is standard, and we are mainly concerned about the following a priori estimate.

**Proposition 3.1** (A priori estimate). For any \(q > 2\), let \((\phi, \psi) \in X(0, T)\) be the solution of (3.2) for a positive \(T\). Then there exists a constant \(\sigma > 0\) such that, if \(\delta_1\) is suitably small and \(N(T) \leq \sigma\), then it holds, for \(t \in [0, T]\), that
\[
\begin{align*}
&\|(\phi, \psi)\|_{H^1}^2 + \|\psi\|_{L^q}^2 + \int_0^t \left( (|\phi|_{L^2}^2 + |\psi|_{L^q}^2) d\tau \right) \\
&+ \int_0^t \int_0^{\xi(t)} ((\phi^2 + \phi_{\xi}^2) V_\xi + \psi_{\xi}^2) d\xi d\tau \\
&+ \int_0^t \int_0^{\xi(t)} (\phi^2 + \phi_{\xi}^2) V_\xi + \psi_{\xi}^2 d\xi d\tau \\
&+ \int_0^t |\psi_{\xi}^2(0, \tau) + |\psi_\xi(0, \tau)|^q d\tau + \int_0^t \int_0^{\xi(t)} |U_\xi + \psi_{\xi}||^{q-2} \psi_{\xi} + \psi_{\xi}^2 + \psi_{\xi}^2 d\xi d\tau \\
&+ \int_0^t \int_{\xi(t)}^{+\infty} |\psi_{\xi}||^{q-2} \psi_{\xi}^2 d\xi d\tau + \int_0^t \int_0^{+\infty} |U_\xi + \psi_{\xi}||^{q-2} |\psi_{\xi}||^{q-2} \psi_{\xi}^2 d\xi d\tau \\
&\leq C \left\{ \|(\phi_0, \psi_0)\|_{L^2}^2 + \|\phi_{0, \xi}\|_{L^q}^2 \right\}.
\end{align*}
\tag{3.5}
\]
Once the Proposition 3.1 is obtained, we can show the following global existence theorem, which implies Theorem 2.2 by the definition (3.1).

**Theorem 3.2.** For any \( q > 2 \), there exists a small positive constant \( \sigma_0 \) such that if \(|w_+ - w_-|\) is suitably small and

\[
\|(\phi_0, \psi_0)\|_{H^1} + \|\psi_0\|_{L^q_x} \leq \sigma_0,
\]

then (3.2) has a unique global solution \((\phi, \psi)\) in \( X(0, +\infty) \) satisfying

\[
\lim_{t \to +\infty} \sup_{\xi \geq 0} \|((\phi, \psi)(\xi, t)\| = 0.
\]

4. **Proof of Proposition 3.1.** In what follows, the analysis is always carried out under the a priori assumption for sufficiently small \( \sigma \):

\[
N(T) \leq \sigma \ll 1.
\]

Throughout this paper, we denote \( \Omega_1 = (0, \xi_+) \) and \( \Omega_2 = (\xi_+, +\infty) \).

**Lemma 4.1.** For any \( \alpha > 0 \), \( \beta \geq 0 \) and \( y \in \mathbb{R} \) there exists a constant \( C \) depend only on \( \alpha, \beta \) such that

\[
||1 + y|^{\alpha} - 1| \leq C(|y|^{\alpha + \beta} + |y|).
\]

**Proof.** Since

\[
\lim_{y \to \pm \infty} \frac{|1 + y|^{\alpha} - 1}{|y|^{\alpha + \beta}} = \begin{cases} 0, & \beta > 0, \\ 1, & \beta = 0, \end{cases}
\]

\[
\lim_{y \to 0^+} \frac{|1 + y|^{\alpha} - 1}{|y|} = \alpha \quad \text{and} \quad \lim_{y \to 0^-} \frac{|1 + y|^{\alpha} - 1}{|y|} = -\alpha.
\]

Then according to the definition of limit, we complete the proof of this lemma.

Using a same argument as in the proof of Lemma 4.1, we can get the following two lemmas.

**Lemma 4.2.** For any \( \alpha > 0 \), \( \beta \geq 0 \), \( y \in \mathbb{R} \) and \( 0 \leq z \leq \delta \), there exists a constant \( C \) depend only on \( \alpha, \beta \) such that

\[
|z + y|^{\alpha} \leq C(\delta^{\alpha} + |y|^{\alpha + \beta}).
\]

**Lemma 4.3.** For any \( \alpha > 0 \), \( \beta \geq 0 \), \( y \in \mathbb{R} \) and \( \epsilon > 0 \), there exists a constant \( C_{\epsilon} \) depend only on \( \epsilon, \alpha, \beta \) such that

\[
|y|^{\alpha} \leq \epsilon + C_{\epsilon}|y|^{\alpha + \beta}.
\]

Next, we have the following basic energy estimate.

**Lemma 4.4.** Under the assumptions of Proposition 3.1, it holds that

\[
\|(\phi, \psi)\|_{L^2_x}^2 + \int_0^t \int_0^{\xi_+} \phi^2 V_x d\xi d\tau + \int_0^t \int_0^{\xi_+} |U_x + \psi|^{q-2} \psi^2 d\xi d\tau + \int_0^t \int_{\xi_+}^{+\infty} |\psi\xi|^{q} d\xi d\tau
\]

\[
\leq C\|(\phi_0, \psi_0)\|_{L^2_x}^2 + C\delta \int_0^t \int_0^{\xi_+} \psi_x^2 d\xi d\tau,
\]

where \( \delta = \delta_1^{\frac{1}{q-2}} \) and \( 0 < \kappa < \min\{q-2, 1\} \) is a constant.
Proof. It follows from (2.13) that
\[ V_\xi \leq C\delta \quad \text{and} \quad U_\xi \leq C\delta. \]  
(4.4)

Set \( \Phi(v, V) = p(V)\phi - \int_0^V \psi/\phi(y)dy \). Multiplying equations (3.2)_1 and (3.2)_2 by \( p(V) - p(V + \phi) \) and \( \psi \), respectively, then we sum them up to get

\[
\left\{ \frac{1}{2} \frac{\partial^2 V}{\partial t^2} + \Phi(v, V) \right\}_t + \left\{ -s_0 \Phi(v, V) - \frac{s_0}{2} \frac{\partial^2 V}{\partial t^2} + \left( p(V + \phi) - p(V) \right) \psi \right.
- \mu \left( \left| \frac{U_\xi + \psi_\xi}{V + \phi} \right|^{q-2} \frac{U_\xi + \psi_\xi}{V + \phi} - \frac{U_\xi}{V} \right) \psi_\xi
+ \left\{ \mu \left( \left| \frac{U_\xi + \psi_\xi}{V + \phi} \right|^{q-2} \frac{U_\xi + \psi_\xi}{V + \phi} - \frac{U_\xi}{V} \right) \psi_\xi \right.
- s_0 V_\xi \left( p(V + \phi) - p(V) - p'(V)\phi \right) \right\}_t = 0. \]  
(4.5)

Integrating (4.5) both sides over \([0, +\infty) \times [0, t] \), we have

\[
\int_0^t \int_0^{\xi_{-}} \left\{ \frac{1}{2} \frac{\partial^2 V}{\partial t^2} + \Phi(v, V) \right\} d\xi d\tau - s_0 \int_0^t \int_0^{\xi_{-}} V_\xi \left( p(V + \phi) - p(V) - p'(V)\phi \right) d\xi d\tau
+ \int_0^t \int_0^{\xi_{-}} \mu \left( \left| \frac{U_\xi + \psi_\xi}{V + \phi} \right|^{q-2} \frac{\psi_\xi}{V + \phi} \right) d\xi d\tau
= \int_0^t \int_0^{\xi_{-}} \left\{ \frac{1}{2} \frac{\partial^2 V}{\partial t^2} + \Phi(v_0, V) \right\} d\xi d\tau
+ \int_0^t \int_0^{\xi_{-}} \mu \left( \left| \frac{U_\xi}{V} \right|^{q-2} \frac{U_\xi}{V + \phi} - \left| \frac{U_\xi + \psi_\xi}{V + \phi} \right|^{q-2} \frac{U_\xi + \psi_\xi}{V + \phi} \right) \psi_\xi d\xi d\tau. \]  
(4.6)

According to the definition of \( \Phi(v, V) \) and using the the properties of the BL-solution shown in Theorem 2.1 (iii), we get

\[
\|(\phi, \psi)\|_{L^2}^2 + \int_0^t \int_0^{\xi_{-}} \phi^2 V_\xi d\xi d\tau + \int_0^t \int_0^{\xi_{-}} \left| U_\xi + \psi_\xi \right|^{q-2} \psi_\xi^2 d\xi d\tau + \int_0^t \int_0^{\xi_{-}} \left| V_\xi \right|^q d\xi d\tau
\leq C \|(\phi_0, \psi_0)\|_{L^2}^2 + C \int_0^t \int_0^{\xi_{-}} \left| U_\xi + \psi_\xi \right|^{q-2} - U_\xi^{q-2} \left| U_\xi \psi_\xi \right| d\xi d\tau. \]  
(4.7)

Next, we estimate the last term on the right hand side of (4.7). If \( q > 4 \), applying Lemma 4.1 with \( y = \frac{\psi_\xi}{\phi} \), \( \alpha = q - 2 \) and \( \beta = 0 \), we obtain

\[
\int_0^t \int_0^{\xi_{-}} \left| U_\xi + \psi_\xi \right|^{q-2} - U_\xi^{q-2} \left| U_\xi \psi_\xi \right| d\xi d\tau
\leq C \int_0^t \int_0^{\xi_{-}} \left| \left| \psi_\xi \right|^{q-1} U_\xi + \psi_\xi^2 U_\xi^{q-2} \right| d\xi d\tau
\leq C\delta^{q-2} \int_0^t \int_0^{\xi_{-}} \psi_\xi^2 d\xi d\tau + C\delta \int_0^t \int_0^{\xi_{-}} |\psi_\xi|^{q-1} d\xi d\tau. \]  
(4.8)

Since

\[
\delta \int_0^t \int_0^{\xi_{-}} |\psi_\xi|^{q-1} d\xi d\tau \leq C\delta \int_0^t \int_0^{\xi_{-}} \|\psi_\xi\|_{L^2(\Omega)} \|\psi_\xi\|_{L^q(\Omega)}^{q-2} d\tau
\leq C\delta \int_0^t \int_0^{\xi_{-}} \|\psi_\xi\|_{L^2(\Omega)} \|\psi_\xi\|_{L^q(\Omega)}^{q-2} \|\psi_\xi\|_{L^2(\Omega)} d\tau
\]
\[
\begin{align*}
\leq C\delta \int_0^t (\|\psi_\xi\|_{L^4(\Omega_1)}^{2(q-4)} + \|\psi_\xi\|_{W^{1,2}(\Omega_1)}^2 + \|\psi_\xi\|_{L^2(\Omega_1)}^2) d\tau \\
\leq C\delta \sigma q^{-4} \int_0^t t^{\xi+} |q_\xi|^q d\xi d\tau + C\delta \int_0^t \int_0^{\xi+} \psi_\xi^2 + \psi_\xi^2 d\xi d\tau, \quad (4.9)
\end{align*}
\]

then,
\[
\begin{align*}
&\int_0^t \int_0^{\xi+} \|U_\xi + \psi_\xi\|_{L^2(\Omega_1)}^{q-2} - U_\xi^{q-2} \|U_\xi \psi_\xi\|_{L^2(\Omega_1)} d\xi d\tau \\
&\leq C(\delta q^{-2} + \delta) \int_0^t \int_0^{\xi+} \psi_\xi^2 + \psi_\xi^2 d\xi d\tau + C\delta \sigma q^{-4} \int_0^t \int_0^{\xi+} |q_\xi|^q d\xi d\tau + C\delta \int_0^t \int_0^{\xi+} \psi_\xi^2 d\xi d\tau. \quad (4.10)
\end{align*}
\]

If \(2 < q \leq 4\), in order to investigate the lower bound of \(q\), we take a positive constant \(\kappa(0 < \kappa < 1)\) such that \(2 < 2 + \kappa < q \leq 4\). Applying Lemma 4.1 with \(y = \frac{\psi_\xi}{\xi}, \alpha = q - 2\) and \(\beta = 1 - \kappa > 0\), one has
\[
\|U_\xi + \psi_\xi\|_{L^2(\Omega_1)}^{q-2} - U_\xi^{q-2} \|U_\xi \psi_\xi\|_{L^2(\Omega_1)} \leq C \left( \frac{|\psi_\xi|^{q-1-\kappa}}{U_\xi^{1-\kappa}} + |\psi_\xi U_\xi^{q-3} - 1\right),
\]
thus,
\[
\begin{align*}
&\int_0^t \int_0^{\xi+} \|U_\xi + \psi_\xi\|_{L^2(\Omega_1)}^{q-2} - U_\xi^{q-2} \|U_\xi \psi_\xi\|_{L^2(\Omega_1)} d\xi d\tau \\
&\leq C \int_0^t \int_0^{\xi+} \left( |\psi_\xi|^{q-\kappa} U_\xi^\kappa + \psi_\xi^2 U_\xi^{q-2} \right) d\xi d\tau \\
&\leq C\delta q^{-2} \int_0^t \int_0^{\xi+} \psi_\xi^2 d\xi d\tau + C\delta \kappa \int_0^t \int_0^{\xi+} |\psi_\xi|^{q-\kappa} d\xi d\tau. \quad (4.11)
\end{align*}
\]

Since \(\|\psi_\xi\|_{L^\infty(\Omega_1)} \leq \frac{2^{\alpha-2}}{\alpha-2} \|\psi_\xi\|_{W^{1,2}(\Omega_1)}^{\alpha-2} \|\psi_\xi\|_{L^2(\Omega_1)}^{\alpha-2} \) and \(\sup_{0 \leq t \leq T} \|\psi_\xi\|_{L^2} \leq \sigma\), we can check that
\[
\begin{align*}
&\delta \kappa \int_0^t \int_0^{\xi+} |\psi_\xi|^{q-\kappa} d\xi d\tau = \delta \kappa \int_0^t \|\psi_\xi\|_{L^4(\Omega_1)}^{q-\kappa} d\tau \\
&\leq C\delta \kappa \int_0^t \|\psi_\xi\|_{L^2(\Omega_1)}^{\kappa} \|\psi_\xi\|_{L^\infty(\Omega_1)}^{q-2\kappa} d\tau \\
&\leq C\delta \kappa \int_0^t (\|\psi_\xi\|_{L^2(\Omega_1)}^2 + \|\psi_\xi\|_{L^\infty(\Omega_1)}^{2\kappa-2q}) d\tau \\
&\leq C\delta \kappa \int_0^t \|\psi_\xi\|_{L^2(\Omega_1)}^2 d\tau + C\delta \kappa \int_0^t \|\psi_\xi\|_{L^4(\Omega_1)}^{q-2\kappa} d\tau \\
&\quad + C\delta \kappa \int_0^t \|\psi_\xi\|_{L^2(\Omega_1)}^{2\kappa-2q+4\kappa-4q} d\tau \\
&\leq C\delta \kappa \int_0^t \psi_\xi^2 d\xi d\tau + C\delta \kappa \int_0^t \psi_\xi^2 d\xi d\tau + C\delta \sigma \frac{2^{\alpha-2}}{\alpha-2} \int_0^t \int_0^{\xi+} \psi_\xi^2 d\xi d\tau, \quad (4.12)
\end{align*}
\]
then,
\[
\int_0^t \int_0^{\xi_+} \left| U_x + \psi_x \right|^{q-2} - U_x^{q-1} \right| U_x \psi_x \right| d\xi d\tau \\
\leq C(\delta^{q-2} + \delta^\sigma + \delta^\sigma \sigma^{\frac{q-2-q\sigma}{q-\sigma}} \frac{\|a\|_{\infty}}{\sigma-q-\sigma}) \int_0^t \int_0^{\xi_+} \psi^2 d\xi d\tau + C\delta^\sigma \int_0^t \int_0^{\xi_+} \psi^2 d\xi d\tau.
\]
(4.13)

Note that there exists \( a, b > 0 \) such that
\[
\left| U_x + \psi_x \right|^{q-2} \geq a |\psi_x|^{q-2} + b,
\]
(4.14)

from which, we have
\[
\left| U_x + \psi_x \right|^{q-2} \psi^2 \geq \frac{1}{2} |U_x + \psi_x|^{q-2} \psi^2 + a \frac{1}{2} |\psi_x|^q + b \frac{1}{2} \psi^2.
\]
(4.15)

Consequently, for any given \( q > 2 \) and \( 0 < \kappa < \min\{q - 2, 1\} \), combining (4.10), (4.13), (4.15) with (4.7), we can take \( \sigma \) small enough and restrict \( \delta \) as
\[
C(\delta^{q-2} + \delta + \delta^\sigma) \leq \min\{a, b\},
\]
(4.16)
to obtain the desired basic energy estimate (4.3).

**Remark 4.1.** It can be seen from the analysis above that, thanks to the properties of the maximal interval of existence for the BL-solution, we can obtain the desired basic energy estimate (4.3) for any \( q > 2 \) by elemental energy method.

**Lemma 4.5.** Under the assumptions of Proposition 3.1, it holds that
\[
\| (\phi, \psi) \|_{L^2_t}^2 + \int_0^t \int_0^{\xi_+} \left| \phi^2 \right| \omega \theta + \int_0^t \int_0^{\xi_+} \phi^2 V_x d\xi d\tau \\
+ \int_0^t \int_0^{\xi_+} |U_x + \psi_x|^{q-2} \psi^2 \omega \theta d\xi d\tau + \int_0^t \int_0^{\xi_+} |\psi_x|^{q-2} \psi^2 \omega \theta d\xi d\tau \\
\leq C \left\| (\phi_0, \psi_0) \right\|_{L^2_t}^2 + (C_\epsilon + \delta^{q-1} + \delta + \delta^2) \int_0^t \int_0^{\xi_+} \psi^2 \omega \theta d\xi d\tau \\
+ \delta \int_0^t \int_0^{\xi_+} |U_x + \psi_x|^{q-2} \psi^2 \omega \theta d\xi d\tau + \epsilon \int_0^t \int_0^{\xi_+} \phi^2 \omega \theta d\xi d\tau + C_\epsilon \sigma^2 \int_0^t \int_0^{\xi_+} |\psi_x|^{q} d\xi d\tau \\
+ \delta^2 \int_0^t \int_0^{\xi_+} \psi^2 V_x d\xi d\tau + \delta^{-q-1} \int_0^t \int_0^{\xi_+} \phi^2 \omega \theta d\xi d\tau,
\]
(4.17)

where \( \epsilon \ll 1 \) is a small and fixed constant, \( C_\epsilon \) is a positive constant depend on \( \epsilon \).

**Proof.** Multiplying (3.2)_2 by \(-\psi \xi \), differentiating (3.2)_1 with respect to \( \xi \) and multiplying the resulting equation by \(-\psi(V) \phi \xi \), then summing them up, we have
\[
\left( \frac{1}{2} \psi^2 \right)_t + \left( \frac{p' \psi}{2} \phi^2 \right)_t + \left( -\psi_t \psi \xi \psi^2 + \frac{s_0}{2} \psi^2 \right)_\xi + \frac{s_0 p'(\psi)}{2} \phi^2 \xi \psi^2 \\
+ \mu (q - 1) \frac{|U_x + \psi_x|^{q-2}}{\nu_q - 1} \psi^2 \xi \\
= \left( p'(\psi) - p'(V) \right) V_x \psi \xi \xi + \mu (q - 1) \left\{ \left( \frac{U_x}{V} \right)^{q-2} \left( \frac{U_x \xi}{V} \right) \\
- \frac{U_x V_x}{V^2} \right\} \psi \xi \xi \\
+ \frac{U_x + \psi_x}{\nu_q - 1} [U_x + \psi_x]^{q-2}(U_x + \psi_x)(V_x + \phi \xi) \psi \xi \xi.
\]
(4.18)
Integrating (4.18) both sides over \([0, +\infty) \times [0, t]\) by parts yields

\[
\| (\phi, \psi) \|_{L^2}^2 + \int_0^t \phi_\xi^2(0, \tau) d\tau + \int_0^t \int_{\xi^+}^\xi |U_\xi + \psi| q-2 |\psi| d\xi d\tau + \int_0^t \int_{\xi^+}^\xi \big|\psi\big| q-2 |\psi| d\xi d\tau
\]

\[
\leq C \left\{ \left( \| \phi \|_{W^{1,2}(\Omega)} \right) \| \psi \|_{L^2(\Omega)} \right\} + \int_0^t \left( |p'(v) - p'(V)| V_\xi \psi \psi \xi \right) d\xi d\tau
\]

\[
+ \int_0^t \int_{\xi^+}^\xi \big|U_\xi + \psi\big| q-2 - U_\xi^{q-2} \left| \left( U_\xi \psi \xi \xi + |U_\xi V_\xi \psi \xi \xi \right) d\xi d\tau
\]

\[
+ \int_0^t \int_{\xi^+}^\xi \big|U_\xi + \psi\big| q-2 |V_\xi \psi \xi \xi \xi| d\xi d\tau + \int_0^t \int_{\xi^+}^\xi |U_\xi + \psi| q-2 |\psi| \psi \xi d\xi d\tau
\]

\[
+ \int_0^t \int_{\xi^+}^\xi |\psi| q-2 |\psi| \psi \xi d\xi d\tau + \int_0^t \int_{\xi^+}^\xi \phi_\xi^2 |\psi| d\xi d\tau
\]

\[
= C \left\{ \| (\phi, \psi) \|_{L^2}^2 + \sum_{i=1}^7 A_i \right\}.
\] (4.19)

For the estimate of \(A_1, A_2\), using Young’s inequality with \(\epsilon\), we have

\[
A_1 = \int_0^t \phi_\xi^2(0, \tau) d\tau = \frac{1}{2} \int_0^t \| \psi \|_{L^2(\Omega)}^2 d\tau \leq C \int_0^t \| \psi \|_{L^2(\Omega)}^2 d\tau
\]

\[
\leq C \int_0^t \| \psi \|_{W^{1,2}(\Omega)} \| \psi \|_{L^2(\Omega)} d\tau
\]

\[
\leq \epsilon \int_0^t \int_{\xi^+}^\xi \phi_\xi^2 d\xi d\tau + C \epsilon \int_0^t \int_{\xi^+}^\xi \phi_\xi^2 d\xi d\tau,
\] (4.20)

and

\[
A_2 = \int_0^t \int_{\xi^+}^\xi \big| (p'(v) - p'(V)) V_\xi \psi \psi \xi \xi \right| d\xi d\tau \leq C \delta^\frac{1}{2} \int_0^t \int_{\xi^+}^\xi \big| \phi \sqrt{V_\xi \psi \xi \xi \xi d\xi d\tau
\]

\[
\leq C \delta^\frac{1}{2} \int_0^t \int_{\xi^+}^\xi \phi_\xi^2 d\xi d\tau + C \delta^\frac{1}{2} \int_0^t \int_{\xi^+}^\xi \phi_\xi^2 d\xi d\tau.
\] (4.21)

For the estimate of \(A_3\), when \(2 < q < 4\), employing Lemma 4.1 with \(y = \frac{\psi_{\xi}}{\psi_{\xi}}, \alpha = q - 2\) and \(\beta = 1\) and Cauchy’s inequality, one has

\[
\int_0^t \int_{\xi^+}^\xi \big| U_\xi + \psi \big| q-2 - U_\xi^{q-2} \left| \left( U_\xi \psi \xi \xi + |U_\xi V_\xi \psi \xi \xi \right) d\xi d\tau
\]

\[
\leq \int_0^t \int_{\xi^+}^\xi \big| \psi \big| q-1 \big| U_\xi \psi \xi \xi \right| \left( U_\xi \psi \xi \xi + |U_\xi V_\xi \psi \xi \xi \right) d\xi d\tau
\]

\[
\leq C \delta^q \int_0^t \int_{\xi^+}^\xi \psi_{\xi}^2 d\xi d\tau + C \delta \int_0^t \int_{\xi^+}^\xi \psi_{\xi}^{q-1} |\psi_{\xi}| d\xi d\tau
\]

\[
\leq C \delta^q \int_0^t \int_{\xi^+}^\xi \psi_{\xi}^2 d\xi d\tau + C \delta \int_0^t \int_{\xi^+}^\xi \psi_{\xi}^{2q-2} d\xi d\tau.
\]

Since

\[
\| \psi_{\xi} \|_{L^{2q-2}(\Omega)} \| \psi_{\xi} \|_{L^{2q-2}(\Omega)} \leq C \| \psi_{\xi} \|_{W^{1,2}(\Omega)} \| \psi_{\xi} \|_{L^2(\Omega)} \leq C \left( \| \psi_{\xi} \|_{W^{1,2}(\Omega)} + \| \psi_{\xi} \|_{L^2(\Omega)} \right),
\]
it follows that

\[
A_3 = \int_0^t \int_0^{\xi +} \left( |U_\xi + \psi_\xi|^{q-2} - U_\xi^{q-2} \right) \left( |U_\xi \psi_\xi| + |U_\xi V_\xi \psi_\xi| \right) d\xi d\tau \\
\leq C(\delta^{q-1} + \delta \sigma^{\frac{q-3}{q-2}} + \delta) \int_0^t \int_0^{\xi +} \psi_\xi^2 d\xi d\tau + C(\delta^{q-1} + \delta) \int_0^t \int_0^{\xi +} \psi_\xi^2 d\xi d\tau.
\]

When \( q \geq 4 \), since

\[
\|\psi_\xi\|_{L^{q-2}_x(\Omega_t)}^2 \leq C\|\psi_\xi\|^2_{L^2_x(\Omega_t)} \|\psi_\xi\|_{L^{q-4}_x(\Omega_t)} \leq C\|\psi_\xi\|_{L^{q-4}_x(\Omega_t)} \|\psi_\xi\|_{W^{1,2}_x(\Omega_t)} \|\psi_\xi\|_{L^2_x(\Omega_t)}^{\frac{q-4}{2}}
\]

then we conclude from \( \sup_{0 \leq t \leq T} \|\psi_\xi\|_{L^4} \leq \sigma \) and Lemma 4.1 with \( y = \frac{\psi_\xi}{U_\xi}, \alpha = q - 2, \beta = 0 \) that

\[
A_3 \leq \int_0^t \int_0^{\xi +} \left( |\psi_\xi|^{q-2} + |\psi_\xi|U_\xi^{q-3} \right) \left( |U_\xi \psi_\xi| + |U_\xi V_\xi \psi_\xi| \right) d\xi d\tau \\
\leq C\delta^{q-1} \int_0^t \int_0^{\xi +} (\psi_\xi^2 + \psi_\xi^2) d\xi d\tau + C\delta^2 \int_0^t \int_0^{\xi +} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau \\
+ C\delta^2 \int_0^t \int_0^{\xi +} |\psi_\xi|^{q-2} d\xi d\tau.
\]

By Cauchy’s inequality, \( A_4 \) can be estimated as

\[
A_4 = \int_0^t \int_0^{\xi +} |U_\xi + \psi_\xi|^{q-2} |V_\xi \psi_\xi\psi_\xi| d\xi d\tau \\
\leq \frac{C \delta}{2} \int_0^t \int_0^{\xi +} \left( |U_\xi + \psi_\xi|^{q-2} \psi_\xi^2 + |U_\xi + \psi_\xi|^{q-2} \psi_\xi^2 \right) d\xi d\tau.
\]

For the estimate of \( A_5 \), utilizing Lemma 4.2 with \( y = \psi_\xi, z = U_\xi, \alpha = q - 2, \beta = 0 \) and Young’s inequality with \( \epsilon \), we deduce that

\[
\int_0^t \int_0^{\xi +} |U_\xi + \psi_\xi|^{q-1} |\phi_\xi \psi_\xi| d\xi d\tau \\
\leq C \int_0^t \int_0^{\xi +} (\delta^{q-1} + |\psi_\xi|^{q-1}) |\phi_\xi \psi_\xi| d\xi d\tau \\
\leq C\delta^{q-1} \int_0^t \int_0^{\xi +} (\phi_\xi^2 + \psi_\xi^2) d\xi d\tau + \epsilon \int_0^t \int_0^{\xi +} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau \\
+ C\epsilon \int_0^t \int_0^{\xi +} |\psi_\xi|^{q} \phi_\xi^2 d\xi d\tau.
\]
In addition, we have
\[
\int_{\xi}^{\xi+} |\psi_\xi|^q \phi_\xi^2 d\xi \leq ||\psi_\xi||^q_{L^\infty(\Omega_1)} ||\phi_\xi||^2_{L^2(\Omega_1)}
\]
\[
\leq C \sup_{0 \leq t \leq T} ||\phi_\xi||^2_{L^2} \left( ||\psi_\xi||^q_{W^{1,2}(\Omega_1)} ||\psi_\xi||^2_{L^2(\Omega_1)} \right)
\]
\[
\leq C \sigma^2 \left( ||\psi_\xi||^q_{W^{1,2}(\Omega_1)} + ||\psi_\xi||^2_{L^2(\Omega_1)} \right)
\]
\[
\leq C \sigma^2 \left( ||\psi_\xi||^q_{W^{1,2}(\Omega_1)} + \sigma \delta^{-q} ||\psi_\xi||^q_{L^2(\Omega_1)} \right).
\]
(4.26)
Thus,
\[
A_5 \leq C \delta^{-1} \int_0^t \int_0^{\xi+} \phi_\xi^2 d\xi d\tau + \left( C \delta^{-1} + C \sigma^2 \right) \int_0^t \int_0^{\xi+} \psi_\xi^2 d\xi d\tau
\]
\[
+ C \sigma^2 \int_0^t \int_0^{\xi+} \psi_\xi^2 d\xi d\tau + \epsilon \int_0^t \int_0^{\xi+} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau
\]
\[
+ C \sigma^2 \sigma \delta^{-q} \int_0^t \int_0^{\xi+} |\psi_\xi|^q d\xi d\tau.
\]
(4.27)
A direct computation yields that
\[
\int_{\xi+}^{+\infty} |\psi_\xi|^q \phi_\xi^2 d\xi \leq |||\psi_\xi||^2||^q_{L^\infty(\Omega_2)} ||\phi_\xi||^2_{L^2(\Omega_2)}
\]
\[
\leq C \sup_{0 \leq t \leq T} ||\phi_\xi||^2_{L^2} \left( ||\psi_\xi||^q_{W^{1,2}(\Omega_2)} ||\psi_\xi||^2_{L^2(\Omega_2)} \right)
\]
\[
\leq C \sigma^2 \left( |||\psi_\xi||^2||^q_{L^2(\Omega_2)} + ||\psi_\xi||^2_{L^2(\Omega_1)} \right).
\]
Thus, by Cauchy’s inequality with \( \epsilon \) and Lemma 4.3 with \( y = \psi_\xi, \alpha = 1, \beta = q - 1 \), \( A_6 \) and \( A_7 \) are estimated as
\[
A_6 = \int_0^t \int_{\xi+}^{+\infty} |\psi_\xi|^q \phi_\xi^2 \psi_\xi \psi_\xi d\xi d\tau
\]
\[
\leq \epsilon \int_0^t \int_{\xi+}^{+\infty} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau + C_{\epsilon} \int_0^t \int_{\xi+}^{+\infty} |\psi_\xi|^{q} \phi_\xi^2 d\xi d\tau
\]
\[
\leq (\epsilon + C_{\epsilon} \sigma^2) \int_0^t \int_{\xi+}^{+\infty} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau + C_{\epsilon} \sigma^2 \int_0^t \int_{\xi+}^{+\infty} |\psi_\xi|^q d\xi d\tau,
\]
(4.28)
and
\[
A_7 = \int_0^t \int_{\xi+}^{+\infty} \phi_\xi^2 |\psi_\xi| d\xi d\tau \leq \int_0^t \int_{\xi+}^{+\infty} \phi_\xi^2 (\epsilon + C_{\epsilon} |\psi_\xi|^q) d\xi d\tau
\]
\[
\leq \epsilon \int_0^t \int_{\xi+}^{+\infty} \phi_\xi^2 d\xi d\tau + C_{\epsilon} \sigma^2 \int_0^t \int_{\xi+}^{+\infty} (|\psi_\xi|^{q-2} \psi_\xi^2 + |\psi_\xi|^q) d\xi d\tau,
\]
(4.29)
respectively. Then, collecting the estimates of \( A_i, (i = 1, ..., 7) \), we have
\[
\sum_{i=1}^{7} A_i \leq (\epsilon + C \delta^2 + C \delta^{-1} + C \delta + C \delta^2 \sigma^{-q} + C_{\epsilon} \sigma^2) \int_0^t \int_{\xi+}^{+\xi_{\psi_\xi}} \psi_\xi^2 d\xi d\tau
\]
\[
(\epsilon + C \delta^2) \int_0^t \int_{\xi+}^{+\xi_{\psi_\xi}} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau + C \delta \int_0^t \int_{\xi+}^{+\xi_{\psi_\xi}} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau
\]
\[
+ C \delta \int_0^t \int_{\xi+}^{+\xi_{\psi_\xi}} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau.
\]
\begin{align*}
+ (C_* + C\delta^{q-1} + C\delta \sigma^{4q-8} + C\delta + C\delta^2 \sigma^{q-4} + C_* \sigma^2) \int_0^t \int_0^{\xi_+} \psi_+^2 \phi d\xi d\tau \\
+ C\delta \int_0^t \int_0^{\xi_+} \phi^2 V_\xi d\xi d\tau + C\delta \int_0^t \int_0^{\xi_+} |U_\xi + \psi_+|q-2 \psi_+^2 d\xi d\tau \\
+ C\delta^{q-1} \int_0^t \int_0^{\xi_+} \phi^2 \phi_\xi d\xi d\tau + C_* \sigma^2 \delta^{q-4} \int_0^t \int_0^{\xi_+} |\psi_+|^q \phi d\xi d\tau \\
+ (\epsilon + C_* \sigma^2) \int_0^t \int_0^{\xi_+} |\psi_+|q-2 \psi_+^2 d\xi d\tau + C_* \sigma^2 \int_0^t \int_0^{\xi_+} |\psi_+|^q \phi d\xi d\tau \\
+ \epsilon \int_0^t \int_0^{\xi_+} \phi^2 \phi_\xi d\xi d\tau + C_* \sigma^2 \int_0^t \int_0^{\xi_+} |\psi_+|q-2 \psi_+^2 d\xi d\tau.
\end{align*}

Therefore, for any \(q > 2\), substituting (4.30) into (4.19), by choosing \(\epsilon\), \(\sigma\) small enough and letting

\[
C(\delta^{\frac{1}{4}} + \delta^{q-1} + \delta^{\frac{3}{4}}) \leq \frac{b}{4}, \quad C\delta^2 \leq \frac{a}{4}, \quad C\delta \leq \frac{1}{4},
\]

we can get (4.17).

\[\square\]

**Lemma 4.6.** Under the assumptions of Proposition 3.1, it holds that

\[
\int_0^t \int_0^{\xi_+} \phi^2 \phi d\xi d\tau \\
\leq C \int_0^t \int_0^{\xi_+} \phi^2 + \phi^2 + \phi^2_0 + \phi_0^2 \phi_\xi d\xi d\tau + C\delta^{\frac{1}{2}} \int_0^t \int_0^{\xi_+} \phi^2 V_\xi d\xi d\tau \\
+ C(\delta + \delta^{\frac{1}{2}}) \int_0^t \int_0^{\xi_+} \phi^2 V_\xi d\xi d\tau + (C\delta^{\frac{1}{2}} + C\delta + C_* \sigma^2) \int_0^t \int_0^{\xi_+} |\psi_+|^q \phi d\xi d\tau \\
+ C\delta \int_0^t \int_0^{\xi_+} |U_\xi + \psi_+|q-2 \psi_+^2 d\xi d\tau
\]

(4.32)

where \(\epsilon \ll 1\) is a small and fixed constant, \(C_*\) is a positive constant depend on \(\epsilon\).

**Proof.** Multiplying (3.2)_1 and (3.2)_2 by \(\psi_\xi\) and \(-\phi_\xi\), respectively, then summing them up, we have

\[
-\phi_t (v) \phi_\xi^2 + (\phi_\xi \psi_\xi) - (\phi_\xi \psi_\xi)_t \\
= \psi_\xi^2 + (p''(v) - p''(V)) V_\xi \phi_\xi + \mu(q - 1) \left\{ \frac{(U_\xi)}{(U)} \right\}^{q-2} \left( \frac{U_\phi - U_\phi}{V_\phi} \right) \\
- \frac{|U_\xi + \psi_\xi|^{q-2} (U_\xi + \psi_\xi)}{V^q} \phi_\xi.
\]

(4.33)

Integrating the above equation both sides over \([0, +\infty) \times [0, t]\), one has

\[
\int_0^t \int_0^{\xi_+} \phi_\xi^2 d\xi d\tau
\]
When \( q \) and \( q \) we have
\[
B - B + 1 = B
\]
For the estimate of \( \beta \leq 1 \) and Cauchy’s inequality, we obtain
\[
\int_0^t \int_0^{\xi^+} \left| U_\xi + \psi_\xi \right|^{q-2} \left| V_\xi \psi_\phi \right| d\xi d\tau \leq C \delta^{q-1} \int_0^t \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau + C \delta \int_0^t \||\psi_\xi||^{2q-2}_{L^{2q-2}(\Omega)} d\tau
\]
\[
\int_0^t \int_0^{\xi^+} \left| \psi_\xi \right|^{2q-1} d\xi \leq \||\psi_\xi||^{q-1}_{L^q(\Omega)} \sup_{0 \leq t \leq T} \||\psi_\xi||^{q}_{L^q}
\]
\[
\leq C \sigma^q \left( \||\psi_\xi||^{q}_{W^{1,q}(\Omega)} + \||\psi_\xi||^{q}_{L^q(\Omega)} \right)
\]
we have
\[
B \leq C \int_0^t \int_0^{\xi^+} \left| U_\xi \right|^{q-1} + \left| \psi_\xi \right|^{q-1} \left| U_\xi \right|^{q-3} \left| U_\xi \phi_\xi \right| + \left| U_\xi \psi_\phi \right| d\xi d\tau
\]
\[
\leq C \delta^{q-1} \int_0^t \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau + C \delta \int_0^t \int_0^{\xi^+} \phi_\xi^2 d\xi d\tau
\]
\[
\leq C \delta^{q-1} + \delta \frac{4}{q-3} \right) \int_0^t \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau + C \delta \int_0^t \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau
\]
\[
\leq C \delta^{q-1} + \delta \frac{4}{q-3} \right) \int_0^t \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau + C \delta \int_0^t \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau
\]
\[
\leq C \delta^{q-1} + \delta \frac{4}{q-3} \right) \int_0^t \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau + C \delta \int_0^t \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau
\]
For the estimate of $B_2$, it follows from $\sup_{0 \leq t \leq T} \|\phi_t\|_{L^2} \leq \sigma$ and the Sobolev inequality that

$$
\int_0^t \int_0^{\xi^+} |\psi_{\xi}|^q \phi_{\xi}^2 d \xi d \tau = \int_0^t \|\psi_{\xi}\|_{L^\infty(\Omega_1)} \int_0^{\xi^+} \phi_{\xi}^2 d \xi d \tau \\
\leq C \sup_{0 \leq t \leq T} \|\phi_t\|_{L^2} \int_0^t \|\psi_{\xi}\|_{H^1(\Omega_1)} \|\psi_{\xi}\|_{L^2(\Omega_1)} d \tau \\
\leq C \sigma^2 \int_0^t \left( \|\psi_{\xi}\|^2 \|W_{1,2}(\Omega_1)\| \|\psi_{\xi}\|^2 \|L^2(\Omega_1)\| \right) d \tau \\
\leq C \sigma^2 \int_0^t \int_0^{\xi^+} \left( |\psi_{\xi}|^q + |\psi_{\xi}|^{q-2} \phi_{\xi}^2 \right) d \xi d \tau.
$$

Then by Cauchy’s inequality, (4.38) and Lemma 4.2 with $y = \psi_{\xi}$, $z = U_\xi$, $\alpha = q - 2$ and $\beta = 2$, one can get

$$
B_2 = \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} V_{\xi} \psi_{\xi} \phi_{\xi} |d \xi d \tau
\leq C \delta \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau + C \delta \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau
\leq C \delta \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau + C \delta \int_0^t \int_0^{\xi^+} (\delta^{q-2} + |\psi_{\xi}|^q) \phi_{\xi}^2 d \xi d \tau
\leq C \delta \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau + C \delta \sigma^2 \int_0^t \int_0^{\xi^+} |\psi_{\xi}|^q d \xi d \tau.
$$

Similarly,

$$
B_3 = \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-1} \phi_{\xi}^2 d \xi d \tau \leq C \int_0^t \int_0^{\xi^+} (\delta^{q-1} + |\psi_{\xi}|^q) \phi_{\xi}^2 d \xi d \tau
\leq C \delta^{q-1} \int_0^t \int_0^{\xi^+} \phi_{\xi}^2 d \xi d \tau + C \sigma^2 \int_0^t \int_0^{\xi^+} |\psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau
\leq C \delta^{q-1} \int_0^t \int_0^{\xi^+} \phi_{\xi}^2 d \xi d \tau + C \sigma^2 \int_0^t \int_0^{\xi^+} |\psi_{\xi}|^q d \xi d \tau,
$$

and

$$
B_4 = \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} |\psi_{\xi} \phi_{\xi}| d \xi d \tau
\leq \epsilon \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau + C \epsilon \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau
\leq \epsilon \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau + C \epsilon \int_0^t \int_0^{\xi^+} \delta^{q-2} + |\psi_{\xi}|^q \phi_{\xi}^2 d \xi d \tau
\leq \epsilon \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau + C \epsilon \delta^{q-2} \int_0^t \int_0^{\xi^+} \phi_{\xi}^2 d \xi d \tau
\leq \epsilon \int_0^t \int_0^{\xi^+} |U_\xi + \psi_{\xi}|^{q-2} \phi_{\xi}^2 d \xi d \tau + C \epsilon \delta^{q-2} \int_0^t \int_0^{\xi^+} \phi_{\xi}^2 d \xi d \tau.
$$
For $B_5$, note that
\[|\psi_t|^{q-2} \leq \epsilon + C_\epsilon |\psi_t|^q,\] (4.42)
where we have used Lemma 4.3 with $y = \psi_t$, $\alpha = q - 2$ and $\beta = 2$, and
\[|\psi_t|^{q-1} \leq \epsilon + C_\epsilon |\psi_t|^q,\] (4.43)
where we have used Lemma 4.3 with $y = \psi_t$, $\alpha = q - 1$ and $\beta = 1$. Thus, by Cauchy’s inequality with $\epsilon$, (4.38), (4.42) and (4.43), one can compute that
\[
B_5 = \int_0^t \int_{\xi^+}^{+\infty} (|\psi_t|^q |\psi_t \phi_t| + |\psi_t|^{q-1} \phi_t^2) d\xi d\tau \\
\leq \epsilon \int_0^t \int_{\xi^+}^{+\infty} |\psi_t|^q |\psi_t \phi_t^2| d\xi d\tau + \int_0^t \int_{\xi^+}^{+\infty} \left( C_\epsilon |\psi_t|^q + |\psi_t|^{q-1} \phi_t^2 \right) d\xi d\tau \\
\leq \epsilon \int_0^t \int_{\xi^+}^{+\infty} |\psi_t|^q |\psi_t \phi_t^2| d\xi d\tau + \int_0^t \int_{\xi^+}^{+\infty} \left( \epsilon + C_\epsilon |\psi_t|^q \right) \phi_t^2 d\xi d\tau \\
\leq \epsilon \int_0^t \int_{\xi^+}^{+\infty} |\psi_t|^q |\psi_t \phi_t^2| d\xi d\tau + \epsilon \int_0^t \int_{\xi^+}^{+\infty} \phi_t^2 d\xi d\tau \\
+ C_\sigma \int_0^t \int_{\xi^+}^{+\infty} |\psi_t|^q |\psi_t \phi_t^2| d\xi d\tau + C_\sigma \int_0^t \int_{\xi^+}^{+\infty} |\psi_t|^q d\xi d\tau.
\]

Therefore, we obtain
\[
\sum_{i=1}^5 B_i \leq (C_\delta^{q-1} + C_\delta + C_\delta^{1/2} + C_\epsilon \delta^{q-2}) \int_0^t \int_{\xi^+}^{\xi^+} \phi_t^2 d\xi d\tau + \epsilon \int_0^t \int_{\xi^+}^{+\infty} \phi_t^2 d\xi d\tau \\
+ C_\delta^{q-1} + C_\delta \left( \frac{\alpha - q}{\delta} + \delta + \frac{1}{2} \sigma q \right) \int_0^t \int_{\xi^+}^{\xi^+} \phi_t^2 d\xi d\tau \\
+ C_\delta + C_\sigma \int_0^t \int_{\xi^+}^{\xi^+} |U_\xi + \psi_t|^{q-2} \phi_t^2 d\xi d\tau \\
+ \left( C_\delta \sigma^2 + C_\sigma \sigma^2 \right) \int_0^t \int_{\xi^+}^{\xi^+} |\psi_t|^{q-2} \phi_t^2 d\xi d\tau \\
+ \epsilon \int_0^t \int_{\xi^+}^{\xi^+} |U_\xi + \psi_t|^{q-2} \phi_t^2 d\xi d\tau \\
+ (\epsilon + C_\sigma \sigma^2) \int_0^t \int_{\xi^+}^{+\infty} |\psi_t|^{q-2} \phi_t^2 d\xi d\tau + C_\sigma \sigma^2 \int_0^t \int_{\xi^+}^{+\infty} |\psi_t|^{q} d\xi d\tau.
\]

Consequently, for any $q > 2$, recalling (4.14), by taking $\epsilon$, $\sigma$ suitably small and letting
\[C_\delta^{q-1} + C_\delta + C_\delta^{1/2} + C_\epsilon \delta^{q-2} \leq \frac{1}{2},\] (4.45)
we complete the proof of this lemma. □
Lemma 4.7. Under the assumptions of Proposition 3.1, it holds that

\[
\int_0^{+\infty} |\psi_\xi|^q d\xi + \int_0^t |\psi_\xi(0, \tau)|^q d\tau + \int_0^t \int_0^{+\infty} |U_\xi + \psi_\xi|^{q-2} |\psi_\xi|^2 \psi_\xi \xi d\xi d\tau \\
\leq C \left\{ \int_0^{+\infty} |\psi_0\xi|^q d\xi + (1 + \delta^q) \int_0^t \int_0^{+\infty} \phi_\xi^2 d\xi d\tau + \delta^2 \int_0^t \int_0^{\xi^+} \phi^2 V_\xi d\xi d\tau \\
+ (\delta + \delta^q + \sigma^2) \int_0^t \int_0^{\xi^+} |\psi_\xi|^q d\xi d\tau + \sigma^2 \int_0^t \int_0^{+\infty} |\psi_\xi|^q d\xi d\tau \\
+ \sigma^{q+2} \int_0^t \int_0^{+\infty} |\psi_\xi|^{q-2} \psi_\xi^2 \xi d\xi d\tau \right\}.
\]

(4.46)

Proof. Multiplying (3.2) by \(|\psi_\xi|^{q-2} \psi_\xi\xi\) gives

\[
\frac{1}{q(q-1)} \left( |\psi_\xi|^q \right)_\xi - \frac{1}{q-1} \left( |\psi_\xi|^{q-2} \psi_\xi \xi \right) + \frac{n}{q} \left( |\psi_\xi|^q \right)_\xi \\
= \left( \left( p'(v)(V_\xi + \phi_\xi) - p'(V)V_\xi \right) |\psi_\xi|^{q-2} \psi_\xi \xi \right) \\
+ \mu \left( \left( \frac{U_\xi'}{V} |\psi_\xi|^{q-2} - \frac{U_\xi + \psi_\xi}{V + \phi_\xi} \right) \right) |\psi_\xi|^{q-2} \psi_\xi \xi.
\]

(4.47)

Then integrating (4.47) over \([0, +\infty) \times [0, t]\) implies

\[
\int_0^{+\infty} |\psi_\xi|^q d\xi + \int_0^t |\psi_\xi(0, \tau)|^q d\tau + \int_0^t \int_0^{+\infty} |U_\xi + \psi_\xi|^{q-2} |\psi_\xi|^2 \psi_\xi \xi d\xi d\tau \\
\leq C \left\{ \int_0^{+\infty} |\psi_0\xi|^q d\xi + \int_0^t \int_0^{+\infty} |p'(v)\phi_\xi| |\psi_\xi|^{q-2} \psi_\xi \xi d\xi d\tau \\
+ \int_0^t \int_0^{\xi^+} \left| (p'(v) - p'(V))V_\xi \xi |\psi_\xi|^{q-2} \psi_\xi \xi \right| d\xi d\tau \\
+ \int_0^t \int_0^{\xi^+} \left| U_\xi + \psi_\xi|^{q-2} - U_\xi |^{q-2} \right| \left( |U_\xi |^{q-2} \psi_\xi \xi + |U_\xi V_\xi| |\psi_\xi|^{q-2} \psi_\xi \xi \right) d\xi d\tau \\
+ \int_0^t \int_0^{\xi^+} |U_\xi + \psi_\xi|^{q-2} |V_\xi| |\psi_\xi|^{q-1} \psi_\xi \xi d\xi d\tau + \int_0^t \int_0^{\xi^+} |U_\xi + \psi_\xi|^{q-1} |\phi_\xi| |\psi_\xi|^{q-2} \psi_\xi \xi d\xi d\tau \\
+ \int_0^t \int_0^{+\infty} |\psi_\xi|^{q-1} |\phi_\xi| |\psi_\xi|^{q-2} \psi_\xi \xi d\xi d\tau \right\} \\
= C \left\{ \int_0^{+\infty} |\psi_0\xi|^q d\xi + \sum_{i=1}^6 D_i \right\}.
\]

(4.48)

For the estimate of \(D_1\) and \(D_2\), using Cauchy’s inequality with \(\epsilon\) yields

\[
D_1 = \int_0^t \int_0^{+\infty} \left| p'(v)\phi_\xi |\psi_\xi|^{q-2} \psi_\xi \xi \right| d\xi d\tau \\
\leq \epsilon \int_0^t \int_0^{+\infty} |\psi_\xi|^{2q-4} \psi_\xi^2 d\xi d\tau + C \int_0^t \int_0^{+\infty} \phi_\xi^2 d\xi d\tau,
\]

(4.49)

and

\[
D_2 = \int_0^t \int_0^{\xi^+} \left| (p'(v) - p'(V))V_\xi |\psi_\xi|^{q-2} \psi_\xi \xi \right| d\xi d\tau \\
\leq C \delta^{\frac{1}{2}} \int_0^t \int_0^{\xi^+} |\psi_\xi|^{2q-4} \psi_\xi^2 d\xi d\tau + C \delta^{\frac{1}{2}} \int_0^t \int_0^{\xi^+} \phi_\xi^2 V_\xi d\xi d\tau.
\]

(4.50)
For the estimate of $D_3$, according to Lemma 4.3 with $y = \psi_\xi, \epsilon = 1, \alpha = q - 2$ and $\beta = 2$, we have

\[
\int_0^{\xi^+} |\psi_\xi|^q d\xi \leq \int_0^{\xi^+} (1 + C|\psi_\xi|^q) |\psi_\xi|^q d\xi
\]

\[
\leq \int_0^{\xi^+} |\psi_\xi|^q d\xi + C \sup_{0 \leq t \leq T} \|\psi_\xi\|_{L^q} \|\psi_\xi\|_{L^2(\Omega_1)}^2
\]

\[
\leq \int_0^{\xi^+} |\psi_\xi|^q d\xi + C\sigma^q \|\psi_\xi\|_{L^2(\Omega_1)}^2 \|\psi_\xi\|_{L^2(\Omega_1)}^2
\]

\[
\leq \int_0^{\xi^+} |\psi_\xi|^q d\xi + C\sigma^q \int_0^{\xi^+} (|\psi_\xi|^{q-2} \psi_\xi^2 + |\psi_\xi|^q) d\xi,
\]

then, we can arrive at

\[
D_3 = \int_0^t \int_0^{\xi^+} \left| U_\xi + \psi_\xi \right|^{q-2} d\xi dr
\]

\[
\leq C \int_0^{\xi^+} \left( |\psi_\xi|^{q-1} U_\xi \right) \left( |U_\xi| \psi_\xi \right. |^{q-2} \psi_\xi | + |U_\xi V_\xi \psi_\xi |^{q-2} \psi_\xi | \left. \right) d\xi dr
\]

\[
\leq C\delta^{q-1} \int_0^t \int_0^{\xi^+} |\psi_\xi|^{q-1} |\psi_\xi| d\xi dt + C\delta \int_0^{\xi^+} \left( |\psi_\xi|^{q-1} |\psi_\xi|^{q-2} \psi_\xi | \right) d\xi dt
\]

\[
\leq C\delta^{q-1} \int_0^t \int_0^{\xi^+} \left( |\psi_\xi|^q + |\psi_\xi|^{q-2} \psi_\xi^2 \right) d\xi dt
\]

\[
+ C\delta \int_0^t \int_0^{\xi^+} |\psi_\xi|^{2q-4} \psi_\xi^2 d\xi d\tau + C\delta \int_0^t \int_0^{\xi^+} |\psi_\xi|^{2q-2} d\xi d\tau
\]

\[
\leq C(\delta^{q-1} + \delta + \delta \sigma^q) \int_0^t \int_0^{\xi^+} |\psi_\xi|^q d\xi d\tau + C(\delta^{q-1} + \delta \sigma^q) \int_0^t \int_0^{\xi^+} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau
\]

\[
+ C\delta \int_0^t \int_0^{\xi^+} |\psi_\xi|^{2q-4} \psi_\xi^2 d\xi d\tau.
\]

For the estimate of $D_4$, applying Lemma 4.2 with $y = \psi_\xi, z = U_\xi, \alpha = q - 2$ and $\beta = 2$, we have

\[
\int_0^t \int_0^\xi |U_\xi + \psi_\xi|^{q-2} |V_\xi| |\psi_\xi| |^{q-1} \psi_\xi | d\xi d\tau
\]

\[
\leq C\delta \int_0^t \int_0^\xi |U_\xi + \psi_\xi|^{q-2} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau + C\delta \int_0^t \int_0^{\xi^+} |U_\xi + \psi_\xi|^{q-2} |\psi_\xi|^q d\xi d\tau
\]

\[
\leq C\delta \int_0^t \int_0^\xi |U_\xi + \psi_\xi|^{q-2} |\psi_\xi|^{q-2} \psi_\xi^2 d\xi d\tau + C\delta \int_0^{\xi^+} (\delta^{q-2} + |\psi_\xi|^q) |\psi_\xi|^q d\xi d\tau.
\]
It follows from the Sobolev inequality and $\sup_{0 \leq t \leq T} \| \psi_\xi \|_{L^q} \leq \sigma$ that

$$
\int_0^t \int_0^{\xi+} |\psi_\xi|^q |\psi_\xi|^q d\xi d\tau = \int_0^t \| \psi_\xi \|_{L^q}^2 \int_0^{\xi+} |\psi_\xi|^q d\xi d\tau \\
\leq C \sup_{0 \leq t \leq T} \| \psi_\xi \|_{L^q}^2 \int_0^t \| \psi_\xi \|_{W^{1,2}(\Omega_t)}^2 \| \psi_\xi \|_{L^q(\Omega_t)}^2 d\tau \\
\leq C \sigma^q \int_0^t \left( \| \psi_\xi \|_{W^{1,2}(\Omega_t)}^2 + \| \psi_\xi \|_{L^q(\Omega_t)}^2 \right) d\tau \\
\leq C \sigma^q \int_0^t \int_0^{\xi+} (|\psi_\xi|^q + |\psi_\xi|^{q-2} |\psi_\xi|^2) d\xi d\tau.
$$

(4.54)

Then, substituting (4.54) into (4.53) yields

$$
D_4 \leq C \delta \int_0^t \int_0^{\xi+} |U_\xi + \psi_\xi |^{q-2} |\phi_\xi| |\psi_\xi|^{q-2} |\psi_\xi|^2 d\xi d\tau \\
+ C(\delta^{q-1} + \delta \sigma^q) \int_0^t \int_0^{\xi+} |\psi_\xi|^q d\xi d\tau + C \delta \sigma^q \int_0^t \int_0^{\xi+} |\psi_\xi|^{q-2} |\psi_\xi|^2 d\xi d\tau.
$$

(4.55)

For the estimate of $D_5$, employing Cauchy’s inequality with $\epsilon$ leads to

$$
\int_0^t \int_0^{\xi+} |U_\xi + \psi_\xi |^{q-2} |\phi_\xi| |\psi_\xi|^{q-2} |\psi_\xi| d\xi d\tau \\
\leq \epsilon \int_0^t \int_0^{\xi+} |U_\xi + \psi_\xi |^{q-2} |\psi_\xi|^{q-2} |\psi_\xi|^2 d\xi d\tau + C \epsilon \int_0^t \int_0^{\xi+} |U_\xi + \psi_\xi |^q |\phi_\xi|^2 d\xi d\tau.
$$

(4.56)

Then by the Sobolev inequality and $\sup_{0 \leq t \leq T} \| \psi_\xi \|_{L^q} \leq \sigma$, we have

$$
\| \psi_\xi \|_{L^q(\Omega_t)}^{4q-4} \leq C \left( \| \psi_\xi \|_{W^{1,2}(\Omega_t)}^{4q-4} + \| \psi_\xi \|_{L^q(\Omega_t)}^{2q-2} \right) \\
\leq C \left( \| \psi_\xi \|_{W^{1,2}(\Omega_t)}^{2q-2} + \| \psi_\xi \|_{L^q(\Omega_t)}^{q(q-1)} \right) \\
\leq C \left( \| \psi_\xi \|_{W^{1,2}(\Omega_t)}^{2q-2} + \sigma^{q-2q} \| \psi_\xi \|_{L^q(\Omega_t)}^{q(q-1)} \right).
$$

(4.57)

Thus,

$$
\int_0^t \int_0^{\xi+} |\psi_\xi|^{2q-2} \phi_\xi^2 d\xi d\tau \\
\leq \int_0^t \| \psi_\xi \|_{L^q(\Omega_t)}^{4q-4} \int_0^{\xi+} \phi_\xi^2 d\xi d\tau \\
\leq C \sup_{0 \leq t \leq T} \| \phi_\xi \|_{L^q(\Omega_t)}^2 \int_0^t \left( \| \psi_\xi \|_{W^{1,2}(\Omega_t)}^{2q-2} + \sigma^{q-2q} \| \psi_\xi \|_{L^q(\Omega_t)}^{q(q-1)} \right) d\tau \\
\leq C \sigma^2 \int_0^t \int_0^{\xi+} |\psi_\xi|^{q-2} |\psi_\xi|^2 d\xi d\tau + C \left( \sigma^2 + \sigma^2 \sigma^{q-2q} \right) \int_0^t \int_0^{\xi+} |\psi_\xi|^q d\xi d\tau.
$$

(4.58)
Hence, utilizing Lemma 4.2 with $y = \psi_\xi, z = U_\xi, \alpha = q, \beta = 0$, Lemma 4.3 with $y = \psi_\xi, \epsilon = 1, \alpha = q - 2, \beta = 2$ and (4.38), we obtain
\[
\int_0^t \int_0^{\xi+} |U_\xi + \psi_\xi|^q \phi_\xi^2 |\psi_\xi|^{q-2} d\xi d\tau \\
\leq C \int_0^t \int_0^{\xi+} (\delta^q + |\psi_\xi|^q) \phi_\xi^2 |\psi_\xi|^{q-2} d\xi d\tau \\
= C \delta^q \int_0^t \int_0^{\xi+} |\psi_\xi|^{q-2} \phi_\xi^2 d\xi d\tau + C \int_0^t \int_0^{\xi+} |\psi_\xi|^q \phi_\xi^2 d\xi d\tau \\
\leq C \delta^q \int_0^t \int_0^{\xi+} (1 + |\psi_\xi|^q) \phi_\xi^2 d\xi d\tau + C \int_0^t \int_0^{\xi+} |\psi_\xi|^{2q-2} \phi_\xi^2 d\xi d\tau \\
\leq C \delta^q \int_0^t \int_0^{\xi+} \phi_\xi^2 d\xi d\tau + C \delta^q \sigma^2 \int_0^t \int_0^{\xi+} (|\psi_\xi|^{q-2} \psi_{\xi \xi} + |\psi_\xi|^q) d\xi d\tau \\
+ C \sigma^2 \int_0^t \int_0^{\xi+} |\psi_\xi|^{q-2} \psi_{\xi \xi} d\xi d\tau + C (\sigma^2 + \sigma^2 \sigma^{-2q}) \int_0^t \int_0^{\xi+} |\psi_\xi|^q d\xi d\tau.
\]
(4.59)
Substituting (4.59) into (4.56), we get
\[
D_5 \leq \epsilon \int_0^t \int_0^{\xi+} |U_\xi + \psi_\xi|^q |\psi_\xi|^{q-2} \psi_{\xi \xi} d\xi d\tau + C \delta^q \int_0^t \int_0^{\xi+} \phi_\xi^2 d\xi d\tau \\
+ C_\epsilon (\delta^q \sigma^2 + \sigma^2) \int_0^t \int_0^{\xi+} |\psi_\xi|^{q-2} \psi_{\xi \xi} d\xi d\tau \\
+ C \epsilon (\delta^q \sigma^2 + \sigma^2 + \sigma^2 \sigma^{-2q}) \int_0^t \int_0^{\xi+} |\psi_\xi|^q d\xi d\tau.
\]
(4.60)
For the estimate of $D_6$, similar fashion to (4.51) gives
\[
\int_0^{+\infty} |\psi_\xi|^{2q-2} d\xi \leq \int_0^{+\infty} |\psi_\xi|^q d\xi + C \sigma^q \int_0^{+\infty} (|\psi_\xi|^{q-2} \psi_{\xi \xi} + |\psi_\xi|^q) d\xi.
\]
(4.61)
Moreover,
\[
\int_0^{+\infty} |\psi_\xi|^{2q-2} \phi_\xi^2 d\xi \leq \left( \sup_{0 \leq \xi \leq T} \|\phi_\xi\|_L^2 \right) \|\psi_\xi|^{2q-2}\|_{L^\infty(\Omega_2)} \\
\leq \sigma^2 \|\psi_\xi|^{q-1}\|_{L^\infty(\Omega_2)} \\
\leq C \sigma^2 \|\psi_\xi|^{q-1}\|_{W^{1,2}(\Omega_2)} + \|\psi_\xi|^{q-1}\|_{L^2(\Omega_2)} \\
\leq C \sigma^2 \|(|\psi_\xi|^{q-1})_{\xi\in \Omega_2}\|_{L^2(\Omega_2)} + C \sigma^2 \|\psi_\xi|^{q-1}\|_{L^2(\Omega_2)} \\
= C \sigma^2 \int_0^{+\infty} |\psi_\xi|^{2q-4} \psi_{\xi \xi}^2 d\xi + C \sigma^2 \int_0^{+\infty} |\psi_\xi|^{2q-2} d\xi.
\]
(4.62)
Thus, by (4.61) and (4.62), $D_6$ can be estimated by
\[
D_6 = \int_0^t \int_0^{+\infty} |\psi_\xi|^{q-1} \phi_\xi |\psi_\xi|^{q-2} \psi_{\xi \xi} d\xi d\tau \\
\leq \epsilon \int_0^t \int_0^{+\infty} |\psi_\xi|^{2q-4} \phi_\xi^2 d\xi d\tau + C \epsilon \int_0^t \int_0^{+\infty} |\psi_\xi|^{2q-2} \phi_\xi^2 d\xi d\tau \\
\leq (\epsilon + C \sigma^2) \int_0^t \int_0^{+\infty} |\psi_\xi|^{2q-4} \phi_\xi^2 d\xi d\tau + C \sigma^2 \int_0^t \int_0^{+\infty} |\psi_\xi|^{2q-2} d\xi d\tau.
\]
Collecting the estimates of $D_i$, $(i = 1, \ldots, 6)$, we have

$$
\sum_{i=1}^{6} D_i \leq (\epsilon + C\delta^{\frac{1}{2}} + C\delta) \int_{0}^{t} \int_{0}^{\infty} |\psi|^{2q-4} \psi^2 \xi \xi^* d\xi d\tau \\
+ (C\delta^{q-1} + C\delta^{q} + C_\epsilon \delta^{q} \sigma^2 + C_\epsilon \sigma) \int_{0}^{t} \int_{0}^{\infty} |\psi|^{q-2} \psi^2 \xi d\xi d\tau \\
+ (\epsilon + C\delta) \int_{0}^{t} \int_{0}^{\infty} |\psi|^{q-2} \psi^2 \xi^* d\xi d\tau \\
+ (C_\epsilon + C_\epsilon \delta^2) \int_{0}^{t} \int_{0}^{\infty} \phi^2 \xi d\xi d\tau \\
+ (C_\delta + C_\delta \delta^2) \int_{0}^{t} \int_{0}^{\infty} \phi^2 \xi^* d\xi d\tau \\
+ (C_\epsilon + C_\epsilon \sigma^2) \int_{0}^{t} \int_{0}^{\infty} |\psi|^{2q-4} \psi^2 \xi \xi^* d\xi d\tau \\
+ (C_\epsilon + C_\epsilon \sigma^{q+2}) \int_{0}^{t} \int_{0}^{\infty} |\psi|^{q} d\xi d\tau \\
+ C_\epsilon \sigma^{q+2} \int_{0}^{t} \int_{0}^{\infty} |\psi|^{q-2} \psi^2 \xi d\xi d\tau.
$$

(4.63)

Then Lemma 4.7 can be verified by choosing $\epsilon$, $\sigma$ small enough and taking

$$
C\delta + C\delta^{\frac{1}{2}} \leq \min \left\{ \frac{a}{4}, \frac{1}{4}\right\}, \quad C\delta^{q-1} \leq \frac{b}{4}.
$$

(4.65)

The proof of Proposition 3.1. Now, adding $\lambda_1 \cdot (4.17)$, $\lambda_2 \cdot (4.32)$ and $\lambda_3 \cdot (4.46)$ to (4.3) for three fixed number $\lambda_1, \lambda_2, \lambda_3 > 0$ and recalling (4.14), then by taking $\epsilon$, $\sigma$ suitably small and letting

$$
\begin{cases}
C\lambda_1 (\delta^{q-1} + \delta + \delta^2) \leq \frac{b}{4}, \\
(C\lambda_1 + C\lambda_2) \delta \leq \frac{1}{4}, \\
C\lambda_1 \delta^{q-1} + C\lambda_3 \delta^q \leq \frac{b}{2}, \\
C\lambda_2 (\delta^{\frac{1}{2}} + \delta) + C\lambda_3 (\delta + \delta^q) \leq \frac{b}{4}, \\
(C\lambda_1 + C\lambda_2 + C\lambda_3) \delta^{\frac{1}{2}} \leq \frac{1}{2}, \\
C\delta^q + C\lambda_2 (\delta^{\frac{1}{2}} + \delta) \leq \frac{b}{2}, \\
C\lambda_2 \delta \leq \frac{b}{2}.
\end{cases}
$$

(4.66)
we have
\[
\begin{align*}
&\|\langle \phi, \psi \rangle \|_{L^2}^2 + \int_0^T \int_0^{\xi^+} \phi^2 V_\xi d\xi d\tau + \frac{a}{2} \int_0^T \int_0^{\xi^+} |\psi_\xi|^q d\xi d\tau + \frac{b}{2} \int_0^T \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau \\
&+ \frac{1}{2} \int_0^T \int_0^{\xi^+} |U_\xi + \psi_\xi|^q - |\psi_\xi|^q d\xi d\tau + \int_0^T \int_0^{\xi^+} |\psi_\xi|^q d\xi d\tau \\
&+ \lambda_1 \left\{ \| \langle \phi_\xi, \psi_\xi \rangle \|_{L^2}^2 + \int_0^T \psi_\xi^2 (0, \tau) d\tau + \int_0^T \int_0^{\xi^+} \phi_\xi^2 V_\xi d\xi d\tau \right\} \\
&\leq C \| (\phi_0, \psi_0) \|_{L^2}^2 + C \lambda_1 \| (\phi_0, \psi_0) \|_{L^2}^2 + C \lambda_1 \int_0^T \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau \\
&+ C \lambda_2 \int_0^T \int_0^{\xi^+} \left( \psi^2 + \phi_\xi^2 + \psi_\xi^2 + \phi_\xi^2 \right) d\xi d\tau + C \lambda_2 \int_0^T \int_0^{\xi^+} |U_\xi + \psi_\xi|^q - |\psi_\xi|^q d\xi d\tau \\
&+ C \lambda_3 \int_0^T \int_0^{\xi^+} |\psi_0, \xi|^q d\xi + C \lambda_3 \int_0^T \int_0^{\xi^+} \psi_\xi^2 d\xi d\tau.
\end{align*}
\]

Consequently, choosing \( \lambda_1, \lambda_2, \lambda_3 \) such that
\[
C \lambda_1 = \frac{b}{4}, \quad C \lambda_2 = \min \left\{ \frac{1}{4}, \frac{\lambda_1}{2} \right\}, \quad C \lambda_3 = \frac{\lambda_2}{2}, \quad (4.67)
\]
we obtain the inequality (3.5). In conclusion, from (3.5), if we let \( \| (\phi_0, \psi_0) \|_{H^1} + \| \psi_0, \xi \|_{L^q} \leq \sigma_0 \) and \( C (\sigma_0^2 + \sigma_0^q) \leq \min \{ \sigma^2, \sigma^q \} = \sigma^q \), we can obtain
\[
\begin{align*}
&\| \langle \phi, \psi \rangle \|_{H^1}^2 + \| \psi_\xi \|_{L^q}^q + \int_0^T \left( \| \phi_\xi \|_{L^2}^2 + \| \psi_\xi \|_{L^q}^q \right) d\tau \\
&\leq C \left( \| (\phi_0, \psi_0) \|_{H^1}^2 + \| \psi_0, \xi \|_{L^q}^q \right) (\leq \sigma^q), \quad (4.68)
\end{align*}
\]
which proves Proposition 3.1.

**The proof of Theorem 3.2.** From global estimate (4.68) above, we derive
\[
\| \phi_\xi \|_{L^2} + \| \psi_\xi \|_{L^q} \rightarrow 0, \quad \text{as} \quad t \rightarrow +\infty.
\]

Consequently, for all \( \xi \geq 0 \),
\[
\phi^2(\xi, t) = 2 \int_0^\xi \phi(\xi, t) \phi_\xi(\xi, t) d\xi \\
\leq 2 \| \phi(t) \|_{L^2} \| \phi_\xi(t) \|_{L^2} \rightarrow 0, \quad \text{as} \quad t \rightarrow +\infty, \quad (4.69)
\]
and
\[
|\psi(\xi, t)|^q = q \int_0^\xi |\psi(\xi, t)|^{q-1}\psi(\xi, t)\psi_\xi(\xi, t) d\xi \\
\leq q\|\psi(t)\|_{L^\frac{q}{q-1}}^{q-1}\|\psi(\xi(t))\|_{L^q} \\
= q\|\psi(t)\|_{L^\frac{q}{q-1}}^{q-1}\|\psi(t)\|_{L^q} \\
\leq C\left(\|\psi(t)\|_{W^{1,2}}\|\psi(t)\|_{L^2}\right)^{q-1}\|\psi(t)\|_{L^q} \to 0, \quad t \to +\infty.
\]

Thus the proof of Theorem 3.2 is completed.

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*E-mail address*: zhguo@nwu.edu.cn

*E-mail address*: wcdong04@163.com

*E-mail address*: ljj1241210126.com