Semisymmetric Graphs from Polytopes

Barry Monson∗
University of New Brunswick
Fredericton, New Brunswick, Canada E3B 5A3 (bmonson@unb.ca)

Tomaž Pisanski†
University of Ljubljana, FMF
Jadranska 19, Ljubljana 1111, Slovenia;
and Faculty of Education, University of Primorska,
Cankarjeva 5, Koper 6000, Slovenia (tomaz.pisanski@fmf.uni-lj.si)

Egon Schulte‡
Northeastern University
Boston MA 02115, USA (schulte@neu.edu)

Asia Ivić Weiss§
York University
Toronto, Ontario, Canada M3J 1P3 (weiss@yorku.ca)

Abstract
Every finite, self-dual, regular (or chiral) 4-polytope of type \{3, q, 3\} has a trivalent 3-transitive (or 2-transitive) medial layer graph. Here, by dropping self-duality, we obtain a construction for semisymmetric trivalent graphs (which are edge- but not vertex-transitive). In particular, the Gray graph arises as the medial layer graph of a certain universal locally toroidal regular 4-polytope.

Key Words: semisymmetric graphs, abstract regular and chiral polytopes.

AMS Subject Classification (1991): Primary: 05C25. Secondary: 51M20.

∗Supported by the NSERC of Canada, grant 4818
†Supported by Ministry of Higher Education, Science and Technology of Slovenia grants P1-0294,J1-6062,L1-7230.
‡Supported by NSA-grant H98230-05-1-0027
§Supported by the NSERC of Canada, grant 8857
1 Introduction

The theory of symmetric trivalent graphs and the theory of regular polytopes are each abundant sources of beautiful mathematical ideas. In [22], two of the authors established some general and unexpected connections between the two subjects, building upon a rich variety of examples appearing in the literature (see [3], [7], [11], [28] and [29]). Here we develop these connections a little further, with specific focus on semisymmetric graphs. In particular, we reexamine the Gray graph, described in [2, 3] and [18, 24], and here appearing as the medial layer graph of an abstract regular 4-polytope.

We begin with some basic ideas concerning symmetric graphs [1, ch. 18-19]. Although some of the following results generalize to graphs of higher valency, for brevity we shall assume outright that $G$ is a simple, finite, connected trivalent graph (so that each vertex has valency 3).

By a $t$-arc in $G$ we mean a list of vertices $[v] = [v_0, v_1, \ldots, v_t]$ such that $\{v_{i-1}, v_i\}$ is an edge for $1 \leq i \leq t$, but no $v_{i-1} = v_{i+1}$. Tutte has shown that there exists a maximal value of $t$ such that the automorphism group $\text{Aut}(G)$ is transitive on $t$-arcs. We say that $G$ is $t$-transitive if $\text{Aut}(G)$ is transitive on $t$-arcs, but not on $(t + 1)$-arcs in $G$, for some $t \geq 1$. Tutte also proved the remarkable result that $t \leq 5$ ([1, Th.18.6]). Any such arc-transitive graph is said to be symmetric.

Each fixed $t$-arc $[v]$ in a $t$-transitive graph $G$ has stabilizer sequence $$\text{Aut} G \supset B_t \supset B_{t-1} \supset \ldots \supset B_1 \supset B_0,$$ where the subgroup $B_j$ is the pointwise stabilizer of $\{v_0, \ldots, v_{t-j}\}$. Since $\text{Aut}(G)$ is transitive on $r$-arcs, for $r \leq t$, the subgroup $B_j$ is conjugate to that obtained from any other $t$-arc. In particular, $B_t$ is the vertex stabilizer, whereas $B_0$ is the pointwise stabilizer of the whole arc. In fact, $B_0 = \{e\}$ is trivial ([1 Prop. 18.1]), so that $\text{Aut}(G)$ acts sharply transitively on $t$-arcs.

Each $t$-arc $[v]$ has two successors, $t$-arcs of the form $[v^{(k)}] := [v_1, \ldots, v_t, y_k]$, where $v_{t-1}, y_1, y_2$ are the vertices adjacent to $v_t$. The shunt $\tau_k$ is the (unique) automorphism of $G$ such that $[v] \tau_k = [v^{(k)}]$. Also let $\alpha$ be the unique automorphism which reverses the basic $t$-arc $[v]$. Then $\alpha$ has period 2 and $\alpha \tau_1 \alpha$ equals either $\tau_1^{-1}$ or $\tau_2^{-1}$. We shall say that $G$ is of type $t^+$ or $t^-$, respectively. We can now assemble several beautiful results concerning $\text{Aut}(G)$ (see [1] ch. 18).

**Theorem 1** Suppose $G$ is a finite connected $t$-transitive trivalent graph, with $1 \leq t$, and suppose $G$ has $N$ vertices. Then

(a) For $0 \leq j \leq t - 1$ we have $|B_j| = 2^j$. Also, $|B_t| = 3 \cdot 2^{t-1}$ and $$|\text{Aut}(G)| = 3 \cdot N \cdot 2^{t-1}.$$ (b) The stabilizers $B_j$ are determined up to isomorphism by $t$:  

|Aut(G)| = 3 \cdot N \cdot 2^{t-1} .
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
$t$ & $B_1$ & $B_2$ & $B_3$ & $B_4$ \\
\hline
1 & $\mathbb{Z}_2$ & $\mathbb{Z}_4$ & $\mathbb{Z}_5$ & $\mathbb{Z}_6$ \\
\hline
2 & $\mathbb{Z}_2 \times \mathbb{Z}_2$ & $\mathbb{S}_3$ & $\mathbb{S}_4$ & $\mathbb{S}_5$ \\
\hline
3 & $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ & $\mathbb{D}_{12}$ & $\mathbb{D}_8$ & $\mathbb{S}_4$ \\
\hline
4 & $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ & $\mathbb{S}_4 \times \mathbb{Z}_2$ & $\mathbb{S}_4 \times \mathbb{Z}_2$ & \\
\hline
5 & $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ & $\mathbb{S}_4 \times \mathbb{Z}_2$ & $\mathbb{S}_4 \times \mathbb{Z}_2$ & \\
\hline
\end{tabular}
\end{center}

(c) $\mathcal{G}$ is one of 7 types: $1^-, 2^+, 3^-, 4^+, 4^- \text{ or } 5^+.$

(Here $\mathbb{Z}_k$ is the cyclic group of order $k$, $\mathbb{D}_{2k}$ is the dihedral group of order $2k$, $\mathbb{S}_k$ is the symmetric group of degree $k$.)

Useful lists of symmetric trivalent graphs appear in [22] and [23]. We refer to [22] for a description of several interesting examples.

We now briefly describe some key properties of abstract regular and chiral polytopes, referring again to [22] for a short discussion, and to [23, 25, 26] for details. An \textit{(abstract)} $n$-polytope $\mathcal{P}$ is a partially ordered set with a strictly monotone rank function having range $\{-1, 1, \ldots, n\}$. An element $F \in \mathcal{P}$ with $\text{rank}(F) = j$ is called a \textit{j-face}; typically $F_j$ will indicate a $j$-face; and $\mathcal{P}$ has a unique least face $F_{-1}$ and unique greatest face $F_n$. Each maximal chain or \textit{flag} in $\mathcal{P}$ must contain $n+2$ faces. Next, $\mathcal{P}$ must satisfy a \textit{homogeneity property}: whenever $F < G$ with $\text{rank}(F) = j - 1$ and $\text{rank}(G) = j + 1$, there are exactly two $j$-faces $H$ with $F < H < G$, just as happens for convex $n$-polytopes. It follows that for $0 \leq j \leq n-1$ and any flag $\Phi$, there exists a unique \textit{adjacent flag} $\Phi^j$, differing from $\Phi$ in just the rank $j$ face. With this notion of adjacency the flags of $\mathcal{P}$ form a \textit{flag graph} (not to be confused with the medial layer graphs appearing below). The final defining property of $\mathcal{P}$ is that it should be \textit{strongly flag-connected}. This means that the flag graph for each section is connected. Whenever $F \leq G$ are faces of ranks $j \leq k$ in $\mathcal{P}$, the section $G/F := \{H \in \mathcal{P} \mid F \leq H \leq G\}$ is thus in its own right a $(k-j-1)$-polytope.

Since our main concern is with 4-polytopes, we now tailor our discussion to that case. A (rank 4) polytope $\mathcal{P}$ is \textit{equivelar} of type $\{p_1, p_2, p_3\}$ if, for $j = 1, 2, 3$, whenever $F$ and $G$ are incident faces of $\mathcal{P}$ with $\text{rank}(F) = j - 2$ and $\text{rank}(G) = j + 1$, then the rank 2 section $G/F$ has the structure of a $p_j$-gon (independent of choice of $F < G$). Thus, each 2-face (polygon) of $\mathcal{P}$ is isomorphic to a $p_1$-gon, and there are $p_3$ of these arranged around each 1-face (edge) of $\mathcal{P}$; and in every 3-face (facet) of $\mathcal{P}$, each 0-face is surrounded by an alternating cycle of $p_2$ edges and $p_2$ polygons.

The automorphism group $\text{Aut}(\mathcal{P})$ consists of all order preserving bijections on $\mathcal{P}$. If $\mathcal{P}$ also admits a duality (order reversing bijection), then $\mathcal{P}$ is said to be \textit{self-dual}; clearly $\text{Aut}(\mathcal{P})$ then has index 2 in the group $D(\mathcal{P})$ of all automorphisms and dualities. (Note that $D(\mathcal{P}) = \text{Aut}(\mathcal{P})$ when $\mathcal{P}$ is not self-dual.) If $\mathcal{P}$ is self-dual and equivelar, then it has type $\{p_1, p_2, p_3\}$. 

\textbf{Definition 1} Let $\mathcal{P}$ be a 4-polytope. The associated medial layer graph $\mathcal{G}(\mathcal{P})$, or briefly $\mathcal{G}$, is the simple graph whose vertex set is comprised of all 1-faces and 2-faces in $\mathcal{P}$, two such taken to be adjacent when incident in $\mathcal{P}$.
Remarks: Any medial layer graph $\mathcal{G}$ is easily seen to be bipartite and connected. Note that the more desirable phrase 'medial graph' already has a somewhat different meaning in the literature on topological graph theory.

To further focus our investigations, we henceforth assume that $\mathcal{P}$ is equivelar of type $\{3, q, 3\}$, where the integer $q \geq 2$. Thus $\mathcal{G}$ is trivalent, with vertices of two types occurring alternately along cycles of length $2q$. We say that a $t$-arc in $\mathcal{G}$ is of type 1 (resp. type 2) if its initial vertex is a 1-face (resp. 2-face) of $\mathcal{P}$. The fact that certain polygonal sections of $\mathcal{P}$ are triangular immediately implies that the action of $D(\mathcal{P})$ on $\mathcal{G}$ is faithful, so that we may regard $D(\mathcal{P})$, or Aut($\mathcal{P}$), as a subgroup of Aut($\mathcal{G}$) (see [22, §2]).

In Figure 1 we show a fragment of a polytope $\mathcal{P}$ of type $\{3, 6, 3\}$. The vertices of $\mathcal{G}$ are here represented as black and white discs, and the edges of $\mathcal{G}$ are indicated by heavy lines.

![Figure 1: A fragment of a polytope of type $\{3, q, 3\}$, with $q = 6$.](image)

Since we shall soon assume that $\mathcal{P}$ is quite symmetric, it is useful now to fix a base flag $\Phi = \{F_{-1}, F_0, F_1, F_2, F_3, F_4\}$ in $\mathcal{P}$. Given this, it is convenient to define $v_1 := F_1, v_2 := F_2$, and in general let $v_0 = v_{2q}, v_1, v_2, \ldots, v_{2q-1} = v_{-1}$ denote alternate edges and polygons in the rank 2 section $F_3/F_0$ of $\mathcal{P}$. Thus each $v_j$ is adjacent in $\mathcal{G}$ to $v_{j \pm 1}$, taking subscripts mod $2q$. We also let $w_j$ be the third vertex adjacent to $v_j$ in $\mathcal{G}$, as indicated in Figure 1.

We turn now to two significant classes of highly symmetric polytopes. First we recall that $\mathcal{P}$ is regular when Aut($\mathcal{P}$) acts transitively on the flags of $\mathcal{P}$. Assuming still that $n = 4$, we observe that for $0 \leq j \leq 3$, there exists a (unique) automorphism $\rho_j$ mapping the base flag $\Phi$ to the adjacent flag $\Phi^j$. Then Aut($\mathcal{P}$) is generated by the involutions $\rho_0, \rho_1, \rho_2, \rho_3$, which satisfy at least the relations...
\[ \rho_0^3 = \rho_1^3 = \rho_2^3 = (\rho_0 \rho_2)^2 = (\rho_0 \rho_3)^2 = (\rho_1 \rho_3)^2 = \epsilon, \]
\[ (\rho_0 \rho_1)^{p_1} = (\rho_1 \rho_2)^{p_2} = (\rho_2 \rho_3)^{p_3} = \epsilon, \]

with \( 2 \leq p_1, p_2, p_3 \leq \infty \). Indeed, \( \mathcal{P} \) is equivelar of type \{\( p_1, p_2, p_3 \)\}. (As before we will actually have \( p_1 = p_3 = 3 \) and \( p_2 = q \) in our applications.)

Furthermore, an intersection condition on standard subgroups holds:
\[ \langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle \]

for all \( I, J \subseteq \{0, 1, 2, 3\} \). In short, \( \text{Aut}(\mathcal{P}) \) is a very particular quotient of a Coxeter group with string diagram.

Conversely, suppose that \( \Gamma = \langle \rho_0, \ldots, \rho_3 \rangle \) is a string C-group, namely any group generated by specified involutions satisfying (1) and (2). Then one may construct a regular 4-polytope \( \mathcal{P} = \mathcal{P}(\Gamma) \), of type \{\( p_1, p_2, p_3 \)\}, with \( \text{Aut}(\mathcal{P}) = \Gamma \). We refer to [22, Def. 2] or [23, Thms. 2E11 and 2E12] for details of the construction. Note also that \( \mathcal{P} \) is self-dual if and only if \( \text{Aut}(\mathcal{P}) \) admits an involutory group automorphism \( \delta \) such that \( \delta \rho_j \delta = \rho_{3-j} \) for \( j = 0, 1, 2, 3 \). Such a polytope \( \mathcal{P} \) admits a polarity (i.e. involutory duality) which reverses the basic flag \( \Phi \). Thus \( D(\mathcal{P}) \simeq \text{Aut}(\mathcal{P}) \rtimes \mathbb{Z}_2 \) (see [23, 2B17 and 2E12]).

The upshot of Theorem 2 in [22] is that \( \mathcal{G}(\mathcal{P}) \) is 3-transitive when \( \mathcal{P} \) is finite, regular and self-dual of type \{3, q, 3\}.

For any regular polytope \( \mathcal{P} \), the rotations \( \sigma_j := \rho_{j-1} \rho_j \) generate a subgroup \( \text{Aut}(\mathcal{P})^+ \) having index 1 or 2 in \( \text{Aut}(\mathcal{P}) \). In the latter case, \( \mathcal{P} \) is said to be directly regular, and certain properties of the \( \sigma_j \) lead, in a natural way, to a parallel theory of chiral polytopes (see [25, 26] for details).

A polytope \( \mathcal{P} \) of rank \( n \geq 3 \) is said to be chiral if it is not regular, but there do exist automorphisms \( \sigma_1, \ldots, \sigma_{n-1} \) such that \( \sigma_j \) fixes all faces in \( \Phi \setminus \{F_j-1, F_j\} \) and cyclically permutes consecutive \( j \)-faces of \( \mathcal{P} \) in the rank 2 section \( F_{j+1}/F_{j-2} \) of \( \mathcal{P} \). The automorphism group of \( \mathcal{P} \) now has two flag orbits, with adjacent flags always in different orbits. Again taking \( n = 4 \), it is even possible in the chiral case to choose automorphisms \( \sigma_1, \sigma_2, \sigma_3 \) which generate \( \text{Aut}(\mathcal{P}) \) and satisfy at least the relations
\[ \sigma_1^{p_1} = \sigma_2^{p_2} = \sigma_3^{p_3} = \epsilon, \]
\[ (\sigma_1 \sigma_2)^2 = (\sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 = \epsilon, \]

for some \( 2 \leq p_1, p_2, p_3 \leq \infty \). Once more \( \mathcal{P} \) is equivelar of type \{\( p_1, p_2, p_3 \)\}. Here too the specified generators satisfy a revised intersection condition:
\[ \langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{\epsilon\} = \langle \sigma_2 \rangle \cap \langle \sigma_3 \rangle, \]
\[ \langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle. \]

Conversely, if a group \( \Lambda = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \) satisfies (3) and (4), then there exists a chiral or directly regular 4-polytope \( \mathcal{P} = \mathcal{P}(\Lambda) \) of type \{\( p_1, p_2, p_3 \)\}. We refer to [22] or [25] Thm. 1] for the details of the construction. The directly regular
case occurs if and only if $\Lambda$ admits an involutory automorphism $\rho$ such that $(\sigma_1)\rho = \sigma_1^{-1}$, $(\sigma_2)\rho = \sigma_2^{-1}\sigma_2$ and $(\sigma_3)\rho = \sigma_3$.

A chiral polytope $\mathcal{P}$ can be self-dual in two subtly different ways (see [16] or [26, §3]). $\mathcal{P}$ is properly self-dual if it admits a polarity $\delta$ which reverses the base flag $\Phi$ and so preserves the two flag orbits. In $D(\mathcal{P})$ we then have $\delta^2 = \epsilon$ and $\delta\sigma_j\delta = \sigma_{4-j}$, for $j = 1, 2, 3$. In contrast, $\mathcal{P}$ is improperly self-dual if there exists a duality $\delta$ which exchanges the two flag orbits. In fact, we can choose $\delta$ so that $\delta^2 = \sigma_1\sigma_2\sigma_3$ (so $\delta$ has period 4); and $\delta^{-1}\sigma_1\delta = \sigma_3^{-1}$, $\delta^{-1}\sigma_2\delta = \sigma_1\sigma_2\sigma_1^{-1}$, $\delta^{-1}\sigma_3\delta = \sigma_1$.

In Theorem 5 of [22] we find that $\mathcal{G}$ is 2-transitive when $\mathcal{P}$ is finite, chiral and self-dual of type $\{3, q, 3\}$; more specifically, $\mathcal{G}$ is then of type $2^+$ (resp. $2^-$) if and only if $\mathcal{P}$ is properly (resp. improperly) self-dual.

In the above results, the self-duality of the polytope $\mathcal{P}$ serves as a natural guarantee that the medial layer graph $\mathcal{G}$ be vertex-transitive. Now ignoring duality, it is quite clear from the symmetry of $\mathcal{P}$ that $\text{Aut}(\mathcal{G})$ is transitive on the edges of $\mathcal{G}$, and separately, at least, is also transitive on $t$-arcs of types 1 or 2, for some $t \geq 2$. We thus ask whether $\text{Aut}(\mathcal{G})$ can be transitive on all such $t$-arcs, thereby making $\mathcal{G}$ symmetric, even when $\mathcal{P}$ is not self-dual. In fact, we shall see that this cannot happen, and so we make the following

**Definition 2** A finite regular graph $\mathcal{G}$ is semisymmetric if $\text{Aut}(\mathcal{G})$ acts transitively on the edges of $\mathcal{G}$ but not transitively on the vertices of $\mathcal{G}$.

**Remarks.** To be quite clear about terminology, we recall that a ‘regular’ graph has all vertices of some fixed degree $k$. Semisymmetric graphs are a little elusive and hence of considerable interest. The so-called Gray graph is the earliest known example of a trivalent semisymmetric graph; see [1], [2, 3], or [18, 24] for neat descriptions, and [8] for another interesting ‘small’ example. A census of such graphs, with at most 768 vertices, appears in [9].

It is easy to check that a connected, semisymmetric graph $\mathcal{G}$ is bipartite, say with vertices of types 1 and 2. In analogy to the symmetric case, we define $\mathcal{G}$ to be $(t_1, t_2)$-semitransitive if, for $j = 1, 2$, $\text{Aut}(\mathcal{G})$ is transitive on $t_j$-arcs emanating from vertices of type $j$ (but of course not transitive on longer such arcs). In brief, we say then that $\mathcal{G}$ is ss of type $(t_1, t_2)$. The theory of such graphs seems to be largely uncharted, although it was proved in [27] that each $t_j \leq 7$. A further generalization is the notion of a locally $s$-arc transitive graph; see [13] for a detailed survey, or [14, 15] for more specific investigations. We note that the ‘$s$-arc transitivity’ discussed in the papers just cited has a more general meaning than that employed here.

In the next section we develop some machinery for manufacturing semisymmetric trivalent graphs from non-self-dual regular or chiral 4-polytopes of type $\{3, q, 3\}$.
2 Vertex-transitive medial layer graphs

We begin by letting $\mathcal{P}$ be a regular polytope of type $\{3, q, 3\}$, with medial layer graph $\mathcal{G}$. Theorem 2 below characterizes the case in which $\text{Aut}(\mathcal{G})$ is vertex-transitive.

**Theorem 2** Suppose that $\mathcal{P}$ is a finite regular 4-polytope of type $\{3, q, 3\}$ with medial layer graph $\mathcal{G}$. Then if $\mathcal{G}$ is vertex-transitive, $\mathcal{G}$ must actually be 3-transitive and $\mathcal{P}$ must be self-dual.

**Proof.** Suppose that $\mathcal{G}$ is vertex-transitive. Then $\mathcal{G}$ must be transitive on 3-arcs. In fact, $\text{Aut}(\mathcal{G})$ is already known to be transitive on the 3-arcs of each type, and any element of $\text{Aut}(\mathcal{G})$ which maps a vertex $x$ of $\mathcal{G}$ to a vertex $s$ of different type must necessarily also map a 3-arc with initial vertex $x$ to a 3-arc of the other type, with initial vertex $s$.

Next we show that $\text{Aut}(\mathcal{G})$ is actually sharply transitive on 3-arcs, that is, $\mathcal{G}$ is 3-transitive. We need to exclude the possibility that $\mathcal{G}$ is $t$-transitive for $t = 4$ or 5. In the notation of the previous section, we now have $\text{Aut}(\mathcal{P}) = \langle \rho_0, \ldots, \rho_3 \rangle$. It is also useful to specify a few more vertices in Figure 1: let $x$ be the two other vertices adjacent to $w_1$, and likewise let $s := (v_4)\rho_0$ and $z$ be the two others adjacent to $w_2$.

The case $t = 4$ can be ruled out as in [22, Thm. 2], using the fact that the stabilizer $B_4$ of a vertex in a finite connected 4-transitive trivalent graph must be isomorphic to $S_4$. In fact, the element $\eta := \rho_0\rho_2\rho_3$ in $\text{Aut}(\mathcal{P})$ is an automorphism of $\mathcal{G}$ that stabilizes the vertex $v_1 = F_1$ of $\mathcal{G}$ and has order 6; and it permutes the vertices at distance 2 from $v_1$ in the 6-cycle $(x w_0 v_3 y v_{-1} w_2)$. However, $S_4$ does not contain an element of order 6.

The elimination of the case $t = 5$ is more elaborate. When $t = 5$, the stabilizer $B_5(v_1)$ of the vertex $v_1 = F_1$ of $\mathcal{G}$ in $\text{Aut}(\mathcal{G})$ must be isomorphic to $S_4 \times Z_2$. However, the stabilizer of $v_1$ in $\text{Aut}(\mathcal{P})$ is just the subgroup $\langle \rho_0, \rho_2, \rho_3 \rangle \cong S_4 \times Z_2$. We claim that $\rho_0$ is the central element of $B_5(v_1)$, determining the factor $Z_2$. In fact, viewing $S_4 \times Z_2$ as the symmetry group $[4, 3]$ of the 3-cube $\{4, 3\}$, we observe that its only subgroups of type $S_3$ are those that fix a vertex of the cube, and that the central inversion is the only non-trivial element in $[4, 3]$ that commutes with a subgroup of this kind. Hence $\rho_0$, which determines the factor $Z_2$ in $S_4 \times Z_2$, is the central element of $B_5(v_1)$.

Since the vertex-stabilizers in $\text{Aut}(\mathcal{G})$ are all conjugate, we similarly obtain that $\rho_1$, $\rho_2$ and $\rho_3$ are the central elements in the stabilizers of the vertices $w_2$, $w_1$ and $v_2 = F_2$, respectively, denoted by $B_5(w_2)$, $B_5(w_1)$ and $B_5(v_2)$ (see Figure 1). Now consider an element $\delta$ in $\text{Aut}(\mathcal{G})$ which maps the 3-arc $[w_1, v_1, v_2, w_2]$ to the reversed 3-arc $[w_2, v_2, v_1, w_1]$. Since the $\rho_j$'s are distinguished as central elements of their respective vertex-stabilizers, we must therefore have $\delta^{-1}\rho_j\delta = \rho_{3-j}$ for $j = 0, 1, 2, 5$. Suppose for a moment that $\delta$ is an involution. Then it follows that conjugation by $\delta$ in $\text{Aut}(\mathcal{G})$ induces an involutory group automorphism of $\text{Aut}(\mathcal{P})$, so that necessarily $\mathcal{P}$ is self-dual (see [23, 2E12]), contrary to our assumption that $t = 5$. (Recall from [22, Thm. 2] that the medial layer graph of $\mathcal{P}$ must be 3-transitive if $\mathcal{P}$ is self-dual.)
It remains to prove that we may take $\delta$ to be an involution. First observe that $\delta^2$ belongs to the pointwise stabilizer of the 3-arc $[w_1, v_1, v_2, w_2]$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Theorem 1(b)). In all cases, $\delta^4 = \epsilon$ and $x\delta^2 = x$ or $y$. If $x\delta^2 = x$, then $(x\delta)^2 = x\delta$, so that $\delta^2$ fixes the 5-arc $[x, w_1, v_1, v_2, w_2, x\delta]$. Then $\delta^2 = \epsilon$ as desired. Otherwise, $x\delta^2 = y$ and $y\delta^2 = x$. Consider the unique automorphism $\gamma$ of $\mathcal{G}$ which fixes the 4-arc $[x, w_1, v_1, v_2, w_2]$ pointwise but interchanges $s$ and $z$ (i.e. $x\delta$ and $y\delta$). Then $\gamma\delta$ reverses the 3-arc $[w_1, v_1, v_2, w_2]$, so we may replace $\delta$ by $\gamma\delta$. But

$$x(\gamma\delta)^2 = x\delta\gamma\delta = y\delta^2 = x,$$

so now $\gamma\delta$ is the desired involution.

It follows that $\mathcal{G}$ is 3-transitive. Now we apply the methods of [22, §4]. In particular, associated with $\mathcal{G}$ is a certain subgroup $\Gamma$ of $\text{Aut}(\mathcal{G})$ with a canonically defined set of four involutory generators (see [22, Def. 3]), and this subgroup $\Gamma$ is the automorphism group of a certain self-dual ranked partially ordered set (see [22, Thm. 3]). In the present context we can actually identify the generators of $\Gamma$ with the generators $\rho_j$ for $\text{Aut}(\mathcal{P})$ (and hence $\Gamma$ with $\text{Aut}(\mathcal{P})$), and then also the new partially ordered set with $\mathcal{P}$ itself. Thus $\mathcal{P}$ is self-dual. This completes the proof. $\square$

The situation for chiral polytopes is quite similar. We give fewer details in the proof, which relies more closely on ideas used in establishing [22, Thm. 5].

**Theorem 3** Suppose that $\mathcal{P}$ is a finite chiral 4-polytope of type $\{3, q, 3\}$ with medial layer graph $\mathcal{G}$. Then if $\mathcal{G}$ is vertex-transitive, $\mathcal{G}$ must actually be 2-transitive and $\mathcal{P}$ must be self-dual.

**Proof.** Let $\mathcal{G}$ be vertex-transitive. First observe that $\mathcal{G}$ is transitive on 2-arcs. In fact, $\text{Aut}(\mathcal{P})$ (and hence $\text{Aut}(\mathcal{G})$) is transitive on the 2-arcs of each type, and the vertex-transitivity allows us again to swap the two kinds of 2-arcs. It follows that $\mathcal{G}$ is $t$-transitive for $t = 2, 3, 4$ or 5. We must establish that $t = 2$. We now have $\text{Aut}(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$.

Suppose first that $t = 3$. We apply the methods of [22, §4] to prove that $\mathcal{P}$ must actually be regular, not chiral. In fact, because $\mathcal{G}$ is 3-transitive, we again have a subgroup $\Gamma$ of $\text{Aut}(\mathcal{G})$ with canonically defined generators $\rho_0, \rho_1, \rho_2, \rho_3$ (see [22, Def. 3]). Consulting [22, Lemma 1] and its proof we find that the products $\rho_0\rho_1, \rho_1\rho_2, \rho_2\rho_3$ can be identified with the generators $\sigma_1, \sigma_2, \sigma_3$ of $\Gamma(\mathcal{P})$ acting on $\mathcal{G}$. (It is crucial here that $\mathcal{G}$ be 3-transitive.) Moreover, the self-dual “regular” ranked poset (with a flag-transitive action) associated with $\Gamma$ as in [22, Thm. 3] is actually isomorphic to $\mathcal{P}$. In fact, this poset can be defined completely in terms of the generators $\rho_0\rho_1, \rho_1\rho_2, \rho_2\rho_3$ of the “rotation subgroup” $\Gamma^+$ of $\Gamma$ (see [25, p.510]), that is, in terms of the generators $\sigma_1, \sigma_2, \sigma_3$ of $\text{Aut}(\mathcal{P})$. However, the poset associated with $\sigma_1, \sigma_2, \sigma_3$ is just $\mathcal{P}$ itself. Hence $\mathcal{P}$ must be regular. It follows that we cannot have $t = 3$.

To rule out the cases $t = 4, 5$ we mimic part of the proof of [22, Thm. 5], which utilized certain universal relations satisfied by generators of $\text{Aut}(\mathcal{G})$, as
described in [6 §1]. In each case it is impossible to achieve \((\sigma_2\sigma_3)^2 = \epsilon\), given the other relations in [6]. (We note that Theorem 5 of [22] has almost the same hypotheses as here, except that \(\mathcal{P}\) is there assumed to be self-dual; but this self-duality is used only to guarantee that the medial layer graph \(\mathcal{G}\) be vertex-transitive.)

Thus we must have \(t = 2\). Now, as in the proof of [22 Thm. 6], the sharp transitivity of \(\text{Aut}(\mathcal{G})\) on 2-arcs enables a definition of a duality on \(\mathcal{P}\), whether \(\mathcal{G}\) is of type 2\(^+\) or 2\(^-\); see [22, eqns. (7) and (13)].

3 Graphs from polytopes of type \(\{3, q, 3\}\)

There is a wealth of finite trivalent semisymmetric graphs that are medial layer graphs of regular or chiral polytopes \(\mathcal{P}\) of type \(\{3, 6, 3\}\). Necessarily, by Theorems 2 and 3, \(\mathcal{P}\) must not be self-dual. However, before exploring such polytopes we must first review some key constructions.

For any pair \(s = (s, t)\) of integers satisfying \(s^2 + st + t^2 > 1\), the toroidal map \(\{3, 6\}_s\) has the structure of a finite 3-polytope (or polyhedron), usually chiral, but regular just when \(st(s-t) = 0\). Referring to [23 1D], we merely note here that \(\{3, 6\}_s\) is obtained from the regular triangular tessellation \(\{3, 6\}\) of the Euclidean plane by factoring out a suitable subgroup of the group of translation symmetries. Taking \(v = s^2 + st + t^2\), we find that \(\{3, 6\}_s\) has \(v\) vertices, \(3v\) edges, \(2v\) triangular facets and a rotation group \(\langle \sigma_1, \sigma_2 \rangle\) of order \(6v\). The toroidal map \(\{6, 3\}_t\) can be constructed similarly and is dual to \(\{3, 6\}_s\), both as a map on a compact surface and as an abstract polyhedron.

In any regular (or chiral) \(n\)-polytope \(\mathcal{P}\), all facets are isomorphic to a particular \((n-1)\)-polytope, say \(\mathcal{M}\); likewise each vertex-figure \(\mathcal{N}\) (maximal section over a vertex in \(\mathcal{P}\)) is isomorphic to one \((n-1)\)-polytope \(\mathcal{N}\). Conversely, given regular \((n-1)\)-polytopes \(\mathcal{M}, \mathcal{N}\), there may or may not exist a regular \(n\)-polytope \(\mathcal{P}\) with facets \(\mathcal{M}\) and vertex-figures \(\mathcal{N}\); but if one such polytope exists, then there is a universal polytope of this type, denoted

\[\{\mathcal{M}, \mathcal{N}\}\, ,\]

and from which all others are obtained by identifications [24 4A]. Somewhat more intricate results like this hold for chiral polytopes [25 26].

3.1 Medial layer graphs of finite universal polytopes.

Rephrasing the introductory remarks above, we observe that every (finite) regular polytope \(\mathcal{P}\) of type \(\{3, 6, 3\}\) has certain facets \(\{3, 6\}_s\) and vertex-figures \(\{6, 3\}_t\), with \(s = (s^k, 0^{2-k})\), \(t = (t^l, 0^{2-l})\); here, \(s \geq 2\) if \(k = 1\) and \(s \geq 1\) if \(k = 2\); likewise \(t \geq 2\) if \(l = 1\) and \(t \geq 1\) if \(l = 2\). In particular, \(\mathcal{P}\) is a quotient of the (generally infinite) universal regular 4-polytope

\[\mathcal{P}_{s, t} := \{\{3, 6\}_s, \{6, 3\}_t\}\, .\]
(See [23 Section 11E] for details. In some cases, the only available construction for \( P_{s,t} \) is via the corresponding string C-group, which in turn is naturally defined by a presentation encoding the local structure of the polytope. We can expect no simple expression for the order of the group.)

For certain small parameter values these universal polytopes are known to be finite; however, the finite polytopes \( P_{s,t} \) have not yet been completely enumerated. Clearly, if \( s \neq t \), then \( P \) cannot be self-dual and hence its medial layer graph is semisymmetric.

We list in Table 1 data for the medial layer graphs \( G_{s,t} \) of those universal polytopes \( P_{s,t} \) which are known to be finite; in the last column we use ‘ss-(\( t_1, t_2 \))’ or ‘3+’, respectively, to indicate that \( G_{s,t} \) is semisymmetric of type \( (t_1, t_2) \) or 3-transitive. (The type of the last semisymmetric graph with 40320 vertices seems to be beyond brute force calculation in GAP [12], for example.) Recall that \( N \) is the number of vertices.

| \( s \)   | \( t \)   | \( N \) | Transitivity type |
|----------|----------|--------|------------------|
| (1, 1)   | (1, 1)   | 18     | 3+               |
| (1, 1)   | (3, 0)   | 54     | ss-(4, 3)        |
| (2, 0)   | (2, 0)   | 40     | 3+               |
| (2, 0)   | (2, 2)   | 120    | ss-(3, 3)        |
| (3, 0)   | (3, 0)   | 486    | 3+               |
| (3, 0)   | (2, 2)   | 6912   | ss-(3, 3)        |
| (3, 0)   | (4, 0)   | 40320  | ss-(?, ?)        |

Table 1: The medial layer graphs of the known finite polytopes \( P_{s,t} \).

When \( s = t \), the universal polytope \( P_{s,t} \) of \( \S \) 3.1 is self-dual and generally has many self-dual quotients. For example, when the standard representations of the crystallographic Coxeter groups \([3,6,3]\) and \([3,\infty,3]\) are reduced modulo an odd prime \( p \), we obtain interesting self-dual (in one case, non-self-dual) regular polytopes of types \{3, 6, 3\} or \{3, p, 3\}, respectively, with automorphism groups isomorphic to finite reflection groups over the finite field \( \mathbb{Z}_p \) (see [20 (28),(31)]). The exception occurs for \([3,6,3]\) with \( p = 3 \), yielding the non-self-dual polytope \( P_{(1,1),(3,0)} \), whose medial layer graph is the Gray graph (see Section 3.4). All other polytopes obtained by this construction have finite trivalent symmetric graphs as medial layer graphs. In particular, when \( p > 3 \), the polytopes obtained from \([3,6,3]\) have facets \{3, 6\}(\( p,0 \)) and vertex-figures \{6, 3\}(\( p,0 \)) and hence are quotients of \( P_{(p,0),(p,0)} \).

### 3.2 Non-constructive methods.

Even if the universal polytope \( P_{s,t} \) of 3.1 is not finite, we often can still establish the existence of semisymmetric medial layer graphs through non-constructive methods by appealing to [23 Thm. 4C4]. Recall that a group \( \Gamma \) is residually...
finite if, for each finite subset of $\Gamma \setminus \{e\}$, there exists a homomorphism of $\Gamma$ onto a finite group such that no element of the subset is mapped to the identity element.

Suppose that $Q$ is an infinite regular 4-polytope with facets $\{3,6\}_s$ and vertex-figures $\{6,3\}_t$, whose group $\Gamma(Q)$ is residually finite. Then [23] Thm. 4C4, applied with $P_1 = \{3,6\}_s$ and $P_2 = \{6,3\}_t$, says that there are infinitely many finite regular 4-polytopes with facets $\{3,6\}_s$ and vertex-figures $\{6,3\}_t$, which are quotients of $Q$. When $s \neq t$, these polytopes yield trivalent semisymmetric graphs.

Such polytopes $Q$ are known to exist at least for certain parameter values, including $s = (s,s)$ and $t = (s,0)$ or $(3s,0)$, with $s \geq 2$ (but excluding the pair $s = (2,2)$ and $t = (2,0)$). In fact, inspection of the methods employed in the proof of [23] Thm. 11E5 reveals the existence of certain infinite regular 4-polytopes $Q$ with facets $\{3,6\}_s$ and vertex-figures $\{6,3\}_t$, whose group $\Gamma(Q)$ is a semi-direct product of an infinite, finitely generated, 4-dimensional complex linear group by a small group ($S_3$, in fact); then $\Gamma(Q)$ itself also is a complex linear group, in a space of dimension larger than 4 (see [23] pp. 415-416). By a theorem of Malcev [17], every finitely generated linear group is residually finite. Thus $\Gamma(Q)$ is residually finite.

In summary, we obtain the following

**Theorem 4** Let $s \geq 2$, and let $s := (s,s)$ and $t := (s,0)$ or $(3s,0)$, but excluding the pair $s = (2,2)$ and $t = (2,0)$. Then there are infinitely many finite trivalent semisymmetric graphs which are medial layer graphs of finite regular polytopes with facets $\{3,6\}_s$ and vertex-figures $\{6,3\}_t$.

As a final application of these methods, we mention a similar such theorem for symmetric graphs.

**Theorem 5** For each $q \geq 5$, there are infinitely many finite, trivalent symmetric (indeed 3-transitive) graphs which are medial layer graphs of finite self-dual regular polytopes of type $\{3,q,3\}$.

**Proof.** The Coxeter group $[3,q,3]$ is the automorphism group of the self-dual universal regular polytope $P := \{3,q,3\}$. In particular, $D(P) \cong [3,q,3] \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the polarity $\delta$ that fixes the base flag of $P$ ($\delta$ corresponds to the symmetry of the string Coxeter diagram). Hence $D(P)$ is residually finite, since $[3,q,3]$ is residually finite. Now adapt the proofs of [23] Thm. 4C4 and [23] Cor. 4C5, applying Malcev’s theorem to $D(P)$ in place of $\text{Aut}(P)$, and requiring that $\delta$ does not become trivial under the homomorphisms onto finite groups. Then the latter guarantees the self-duality of the resulting quotients of $P$; hence their medial layer graphs are symmetric. □

### 3.3 Polytopes and graphs from the Eisenstein integers.

Next we consider from [21] §6 a family of regular or chiral polytopes $Q^A_m$, again of type $\{3,6,3\}$. Here, the parameter $m$ is chosen from $D := \mathbb{Z}[\omega]$, the domain
of Eisenstein integers. (Recall that $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.) The construction begins with a certain group $H_m$ of $2 \times 2$ matrices over the residue class ring $\mathbb{D}_m := \mathbb{D}/(m)$; and then any subgroup $A$ of the unit group of $\mathbb{D}_m$, with $-1 \in A$, is said to be admissible. Without going into many details, we note simply that the rotation group $H_m^A = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ for $Q_m^A$ is obtained from the matrix group by factoring out the subgroup consisting of scalar multiples of the identity, with scalars from $A$. Thus $Q_m^A$ is finite when $m \neq 0$. On the other hand, $H_0^A$ is the infinite rotation group for $Q_0^\pm$, which is isomorphic to the regular honeycomb $\{3, 6, 3\}$ of hyperbolic space $\mathbb{H}^3$.

If the Eisenstein prime $1 - \omega$ does not divide $m$, the polytope will be self-dual. Interesting as it is, we leave this case behind (see [22]). Suppose therefore that

$$m = (1 - \omega)^e d,$$

where $e \geq 1$ and $d \in \mathbb{D} \setminus \{0\}$. To avoid degeneracies, we also assume that $d$ is a non-unit if $e = 1$. It follows from [21] Thm. 6.1 that $Q_m^A$ is a finite quotient of the universal polytope

$$\{ \{3, 6\}_{(c,b)}, \{6, 3\}_{(\pm 1, \pm \omega)} \},$$

where $m = c - b\omega$, for certain $b, c \in \mathbb{Z}$. Since the facets of $Q_m^A$ are clearly not dual to its vertex-figures, $Q_m^A$ itself cannot be self-dual. Furthermore, $Q_m^A$ is regular if $m \mid n$ and $A = \bar{A}$ (i.e. the scalar subgroup is invariant under complex conjugation); otherwise, $Q_m^A$ is chiral. Consequently, by Theorems 2 and 3 above, we obtain a trivalent, semisymmetric medial layer graph $G_m^A$ with

$$N = 2\left\lfloor \frac{(m\bar{m})^3}{12 \cdot |A|} \prod_{\pi|m} (1 - (\pi\bar{\pi})^{-2}) \right\rfloor$$

vertices. (The product here is over all non-associated prime divisors $\pi$ of $m$.) We can summarize our construction in the following

**Theorem 6** Suppose the Eisenstein integer $m$ satisfies $m\bar{m} = 3k$, for some rational integer $k > 1$; and let $A$ be any admissible group of scalars. Then $G_m^A$ is a finite, trivalent semisymmetric graph.

**Remarks.** When $m = 3 = (1 - \omega)^2(-\omega^2)$ and $A = \{\pm 1\}$, we get the dual of the universal polytope $\{\{3, 6\}_{(1,1)}, \{6, 3\}_{(3,0)}\}$ mentioned in §3.1 above. The medial layer graph $G_3^{\pm 1}$ is the Gray graph, which we examine more closely below. (For easier reading we omit the brackets from $\{\pm 1\}$.) Similarly, for $m = 2(1 - \omega)$ we find that $Q_2^{\pm 2\omega}$ is the dual of the universal polytope $\{\{3, 6\}_{(2,0)}, \{6, 3\}_{(2,2)}\}$ described in §3.1. The medial layer graph has 120 vertices. According to the census in [9], we have thus described the unique trivalent semisymmetric graphs with these orders.

We note that $Q_m^A$ itself is not usually the universal polytope for the specified toroidal facets and vertex-figures. Certainly we get a proper quotient of the universal cover when $|A| > 2$, which is possible when $m$ has distinct prime divisors. In any case, the scalar group $A$ has order $2^a$ and depends in an intricate way on the prime factorization of $m$ in $\mathbb{D}$; see [21] pp. 105-106].
3.4 The Gray graph.

The Gray graph $C$ is the smallest trivalent, semisymmetric graph (see [9]). Following [3], we define $C$ to be the (bipartite) incidence graph of cubelets and columns in a $3 \times 3 \times 3$ cube. Thus vertices of the first type are the 27 cubelets; and vertices of the second type are the $9 + 9 + 9$ columns of 3 cubelets parallel to edges of the cube. It is not hard to check that $|\text{Aut}(C)| = 1296$ [3, Thm. 1.1]. Recent work has concerned various interesting features of the graph ([18] and [19]); and here, of course, we construct it in a new way.

Before confirming that $G_{3}^{\pm 1}$ really is isomorphic to $C$, we develop a more concrete geometric description. First of all, using GAP it is easy to check that $|\text{Aut}(G_{3}^{\pm 1})| = 1296$. Somewhat unexpectedly this is 4 times the order of $\text{Aut}(Q_{3}^{\pm 1})$. This discrepancy hints that we might examine a related embedding of the honeycomb $\{3,6,3\}$ into a different hyperbolic honeycomb $\{3,3,6\}$.

![Figure 2: A tetrahedral tile in the hyperbolic honeycomb $\{3,3,6\}$.](image)
Let us pick an arbitrary tetrahedral tile $T$ of $\{3, 3, 6\}$ and denote its centre by $F_3$. In Figure 2 $F_0$ is a vertex of $T$ (and is an ideal point on the sphere at infinity); let $F_1$ be the centre of an edge of $T$ through $F_0$ and $F_2$ the centre of a triangle of $T$ with that edge. The points $F_i$ ($i = 0, 1, 2, 3$) are the vertices of a fundamental region for the hyperbolic Coxeter group $[3, 3, 6]$. Thus, taking $\rho_i$ to be the reflection in the face opposite $F_i$ in this fundamental region, we have $[3, 3, 6] = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$.

Now let $E_3$ be the vertex of the tetrahedron $T$ which does not belong to the triangle centred at $F_2$. Then $F_0, F_1, F_2$ and $E_3$ are vertices of a new fundamental region for the hyperbolic Coxeter group $[3, 6, 3]$, appearing here as the subgroup of $[3, 3, 6]$ generated by

$$\rho_0, \rho_1, \rho'_2 := \rho_2 \rho_3 \rho_2 \text{ and } \rho_3.$$

In fact, $[3, 6, 3]$ has index 4 in $[3, 3, 6]$ (see [23, 11G]).

Through the edge containing $F_1$ there are six triangles of $\{3, 3, 6\}$, but only three of them belong to the honeycomb $\{3, 6, 3\}$, as we indicate in Figure 3. Hence, the vertices of the medial layer graph of $\{3, 6, 3\}$ are comprised of just ‘half’ the edges and ‘one quarter’ of the triangles of $\{3, 3, 6\}$.

![Figure 3: The medial layer graph of $\{3, 6, 3\}$ inscribed in $\{3, 3, 6\}$.](image)

Now a typical 2-face of $\{3, 6, 3\}$ is the ideal triangle $\{3\}$ with vertex $F_0$, edge $F_1$ and center $F_2$ in the hyperbolic plane $p$ which serves as the mirror for the reflection $\rho_3$. But $p$ is perpendicular to the mirrors for reflections $\rho_0, \rho_1$ and $\rho := \rho_2 \rho_3 \rho_2 = \rho_2 (\rho_3 \rho_2)^2$ in $[3, 3, 6]$. Since the latter two mirrors are parallel at $F_0$, we see that $\langle \rho_0, \rho_1, \rho \rangle \approx [3, \infty]$. Hence the honeycomb $\{3, 3, 6\}$ cuts the plane $p$ into the triangles of the regular tessellation $\{3, \infty\}$ (see Figure 4).
At this point we recall from §3.3 that the polytopes $Q^A_m$ ultimately arise from a modular representation of the rotation subgroup in the reflection group $[3, 6, 3]$ for $\{3, 6, 3\}$. A parallel construction based on the group $[3, 3, 6]$ is described in [21, §4], thereby yielding a family of polytopes $P^A_m$ of type $\{\{3, 3\}, \{3, 6\}\}_{(b,c)}$. When $1 - \omega$ divides $m$, the rotation group $H^A_m$ for $Q^A_m$ still has index 4 in the rotation group $G^A_m$ for $P^A_m$. In particular, when $m = 3$ we obtain the universal polytope

$$P^{\pm 1}_3 = \{\{3, 3\}, \{3, 6\}\}_{(3,0)}$$

whose automorphism group has order 1296 [23 11B5]. (Note that the rotation group for $Q^{\pm 1}_3$ has order just 324.) To see that the larger group of order 1296 is the automorphism group for the Gray graph, we must consider how reduction modulo $m = 3$ affects the picture in $H^3$.

In identifying vertices of $\{3, 3, 6\}$ to obtain the universal polytope $P^{\pm 1}_3$, we note that $\rho_1 \rho = \rho_1 \rho_2 (\rho_3 \rho_2)^2$ has order 3, so that the tessellation $\{3, \infty\}$ in the plane $p$ collapses to a ‘medial tetrahedron’ $\{3, 3\}$. At the same time, the ‘inscribed’ honeycomb $\{3, 6, 3\}$ collapses to $Q^{\pm 1}_3$ and its medial layer graph to $G^{\pm 1}_3$.

Now working in $P^{\pm 1}_3$, we define a graph $M$ whose vertex set consists of all $27 = 1296/(2 \cdot 24)$ of the medial tetrahedra from $P^{\pm 1}_3$, together with all $27 = (27 \cdot 3)/3$ pairs of opposite edges from such tetrahedra. A vertex representing a tetrahedron is adjacent to a vertex representing a pair of edges whenever the tetrahedron contains the edges (see Figure 5). Evidently, $G^{\pm 1}_3$ is isomorphic to $M$ and so inherits all 1296 automorphisms of $P^{\pm 1}_3$. In other words,

$$\text{Aut}(G^{\pm 1}_3) \cong \text{Aut}(P^{\pm 1}_3).$$

(This group appears as $[112]^3 \rtimes Z_2$ in [23 11B5].)
Figure 5: The graph constructed from the ‘medial tetrahedra’.

Now it is a simple matter to identify $G_{3}^{\pm 1} \cong M$ with the Gray graph $C$: take the medial tetrahedra to be the columns in the $3 \times 3 \times 3$ cube, and pairs of opposite edges to be the cubelets themselves. But each edge of $\{3,6,3\}$ belongs to three facets $\{3,6\}$; so after reducing modulo $m = 3$, each of our 27 pairs of edges must lie on three distinct medial tetrahedra.

We note finally that it is not at all clear, for general moduli $m = (1 - \omega)^{e}d$, just how large $\text{Aut}(G_{m}^{\pm})$ is relative to its subgroup $\text{Aut}(Q_{m}^{A})$.

For example, when $m = 2(1 - \omega)$, $Q_{2(1-\omega)}^{\pm 1}$ is a regular polytope whose full reflection group of order 720 is isomorphic to $\text{Aut}(G_{2(1-\omega)}^{\pm 1})$. A similar isomorphism holds for the chiral polytope obtained when $m = (1 - \omega)(1 + 3\omega)$. But when $m = 3(1 - \omega)$, the reflection group $\text{Aut}(Q_{3(1-\omega)}^{\pm 1})$ once more has index 4 in the $\text{Aut}(G_{3(1-\omega)}^{\pm 1})$, whose order is 34992.

Based on this flimsy evidence, we conjecture that the index is always 4 whenever $m = (1 - \omega)^{e}$, for $e \geq 2$.

**Some history and words of thanks.** At this point we happily thank Izak Bouwer for several comments concerning the provenance of the Gray graph. In 1968, Izak gave the first published description [2]. A year later, in private correspondence with him, Dr. Marion C. Gray (1902–?) wrote that she had encountered the graph while investigating ‘completely symmetric networks’. (This happened about 1932, early in her career at Bell Labs.) In fact, with some uncertainty, Dr. Gray even attributed the graph to R. D. Carmichael. Perhaps the configurations described in [5] were an inspiration.

It is also a pleasure to thank the referees for several suggestions and for pointing out related material in [13] and [27].
References

[1] N. Biggs, Algebraic Graph Theory (2nd. Ed.), Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.

[2] I.Z. Bouwer, An edge but not vertex transitive cubic graph, Canad. Math. Bull. 11 (1968) 533–535.

[3] I.Z. Bouwer, On Edge but not Vertex Transitive Regular Graphs, J. Combinatorial Theory Ser. B 12 (1972) 32-40.

[4] I.Z. Bouwer, et al (eds.), The Foster Census: R.M. Foster’s Census of Connected Symmetric Trivalent Graphs, The Charles Babbage Research Centre, Winnipeg, 1988.

[5] R.D. Carmichael, Tactical configurations of rank two, Amer. Journal of Math. 53 (1931) 217-240.

[6] M. Conder and P. Lorimer, Automorphism groups of symmetric graphs of valency 3, J. Combin. Theory Ser. B 47 (1989) 60-72.

[7] M. Conder and P. Dobcsányi, Trivalent symmetric graphs on up to 768 vertices, J. Combin. Math. Combin. Comput. 40 (2002) 41-63.

[8] M. Conder, A. Malnič, D. Marušič, T. Pisanski and P. Potočnik, The edge-transitive but not vertex-transitive cubic graph on 112 vertices, J. Graph Theory 50 (2005) 25-42.

[9] M. Conder, A. Malnič, D. Marušič and P. Potočnik, A census of semisymmetric cubic graphs on up to 768 vertices, J. Algebraic Combin., 23 Number 3 (2006) 255-294.

[10] H.S.M. Coxeter, The edges and faces of a 4-dimensional polytope, Congr. Numer. 28 (1980) 309-334.

[11] H.S.M. Coxeter and A. Ivić Weiss, Twisted honeycombs \{3, 5, 3\}t and their groups, Geom. Dedicata 17 (1984), 169-179.

[12] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.3 (2002), [http://www.gap-system.org](http://www.gap-system.org).

[13] M. Giudici, C.H. Li and C.E. Praeger, Analysing finite locally s-arc transitive graphs, Trans. Amer. Math. Soc. 356 (2004) no. 1 291-317.

[14] M. Giudici, C.H. Li and C.E. Praeger, Some locally 3-arc transitive graphs constructed from triality, J. Algebra 285 (2005) 11-28.

[15] M. Giudici, C.H. Li and C.E. Praeger, A new family of locally 5-arc transitive graphs, European J. Combin., to appear.

[16] I. Hubard and A. Ivić Weiss, Self-duality of chiral polytopes, J. Combinatorial Theory Ser. A 111 (2005) 128-136.
[17] A.I. Malcev, On faithful representations of infinite groups of matrices (Russian), Mat. Sb. 8 (1940) 405-422. (English translation: Amer. Math. Soc. Transl. (2) 45 (1965) 1-18.)

[18] D. Marušič and T. Pisanski, The Gray graph revisited, J. Graph Theory 35 (2000) 1-7.

[19] D. Marušič, T. Pisanski and S. Wilson, The genus of the Gray graph is 7, European J. of Combin. 26 (2005) 377-385.

[20] B. Monson and E. Schulte, Reflection groups and polytopes over finite fields, II, Adv. in Appl. Math. (2006), in press.

[21] B. Monson and A. Ivić Weiss, Eisenstein integers and related C-groups, Geom. Dedicata 66 (1997) 99-117.

[22] B. Monson and A. Ivić Weiss, Medial layer graphs of equivelar 4-polytopes, European J. of Combin. (2005), in press.

[23] P. McMullen and E. Schulte, Abstract Regular Polytopes, Encyclopedia of Mathematics and its Applications 92, Cambridge University Press, Cambridge, 2002.

[24] T. Pisanski and M. Randić, Bridges between geometry and graph theory, Geometry at work, MAA Notes 53, Math. Assoc. America, Washington, DC, 2000, 174-194.

[25] E. Schulte and A. Ivić Weiss, Chiral polytopes, Applied Geometry and Discrete Mathematics (“The Victor Klee Festschrift”, P. Gritzmann and B. Sturmfels eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science 4, Amer. Math. Soc. and Assoc.Computing Machinery, 1991, 493-516.

[26] E. Schulte and A. Ivić Weiss, Chirality and projective linear groups, Discrete Math. 131 (1994) 221-261.

[27] R. Weiss, s-transitive graphs, Algebraic Methods in Graph Theory, Vol. I,II (Szeged, 1978), North-Holland, Amsterdam, 1981, 827-847.

[28] A. Ivić Weiss, On trivalent graphs embedded in twisted honeycombs, Combinatorics '81 (Rome, 1981), North-Holland Math. Stud. 78, North-Holland, Amsterdam, 1983, 781-787.

[29] A. Ivić Weiss, An infinite graph of girth 12, Trans. Amer. Math. Soc. 283 (1984) no. 2 575-588.