Recent developments in the type IIB matrix model

Jun Nishimura\textsuperscript{1,2,*}

\textsuperscript{1} Theory Center, High Energy Accelerator Research Organization (KEK),
1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan
\textsuperscript{2} Department of Particle and Nuclear Physics, Graduate University for Advanced Studies (SOKENDAI),
1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan

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We review recent developments in the type IIB matrix model, which was conjectured to be a nonperturbative formulation of superstring theory. In the first part we review the recent results for the Euclidean model, which suggest that SO(10) symmetry is spontaneously broken. In the second part we review the recent results for the Lorentzian model. In particular, we discuss Monte Carlo results, which suggest that (3+1)-dimensional expanding universe emerges dynamically. We also discuss some results suggesting the emergence of exponential expansion and the power-law expansion at later times. The behaviors at much later times are studied by the classical equation of motion. We discuss a solution representing 3d expanding space, which suggests a possible solution to the cosmological constant problem.

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1 Introduction

Particle physics and cosmology have both entered a difficult and yet interesting era. We have successful phenomenological models supported by experiments and observations, but we still lack fundamental understanding of these models from a more microscopic point of view. On the particle physics side, we have the Standard Model, which has been established, in particular, by the recent discovery of the Higgs particle at LHC. But we do not have clear understanding for the origin of the Higgs particle, the matter fermions (the number of their generations, in particular) and the gauge group. On the cosmology side, we have the standard scenario of Inflation and the Big Bang, which have been strongly supported by the WMAP and PLANCK data for the cosmic microwave background, the theory for structure formation of galaxies, nucleosynthesis and so on. But we do not have clear understanding for the origin of the scalar field “inflaton”, its potential and its initial condition. Furthermore, we have serious naturalness problems in both particle physics and cosmology, which are presumably related to the lack of fundamental understanding of the phenomenological models. On the particle physics side, the hierarchy between the electroweak scale and the Planck scale is unnatural, whereas on the cosmology side, the accelerated expansion observed today suggests an unnaturally small but finite cosmological constant. It is widely believed that these problems can be solved by a fundamental theory that describes quantum gravity, and superstring theory has been studied as a promising candidate for such a theory.

The most fundamental issue in superstring theory concerns the reason why the dimensionality of our space-time appears to be four instead of ten as required from consistency of the theory. A conventional approach towards this issue is to compactify some of the dimensions leading to infinitely many consistent vacua, which are perturbatively stable. Each of them has different space-time dimensionality, different gauge symmetry and different matter contents. One can then hope to find a vacuum which explains what we observe now. The discovery of D-brane led to many interesting new ideas such as intersecting D-branes, D-brane Inflation and so on, which enriched both string phenomenology and string cosmology. However, an unavoidable feature of this conventional approach is that one obtains too many models and

* E-mail: jnishi@post.kek.jp
hence it is extremely hard to make predictions. In this regard we should not forget that all these perspectives are obtained from mostly perturbative studies of superstring theory, including at most the nonperturbative effects represented by the existence of D-branes. Therefore, a totally new perspective might appear if one studies superstring theory in a completely nonperturbative framework analogous to lattice gauge theory in the case of QCD. Let us recall that nonperturbative aspects of QCD such as confinement of quarks as well as the hadron mass spectrum can never be understood from perturbation theory.

The type IIB matrix model was proposed as a nonperturbative formulation of superstring theory [1]. It is a theory that is expected to define superstring theory beyond perturbative expansion just as lattice gauge theory does so in the case of QCD. The connection to perturbative formulations of superstring theory can be seen manifestly by considering type IIB superstring theory in ten dimensions in the worldsheet formalism [11] or in the light-cone string field formalism [2]. The model can be regarded as a natural extension [3] of the “one-matrix model”, which is established as a nonperturbative formulation of non-critical strings [4,5], where string worldsheets appear as Feynman diagrams in the matrix model and the large-N limit can be taken in such a way that diagrams with all different genera contribute. Despite its manifest connection to type IIB superstring theory within perturbation theory, the type IIB matrix model is expected to provide the unique theory underlying the web of dualities among various types of superstring theory. For this to be true, other types of superstring theory should be represented as perturbative vacua of the type IIB matrix model.

The type IIB matrix model is given by the action $S = S_0 + S_1$, where

$$S_0 = -\frac{1}{4g^2} \text{Tr} [A_\mu, A_\nu][A^\mu, A^\nu], \quad (1)$$

$$S_1 = -\frac{1}{2g^2} \text{Tr} (\Psi_\alpha (\partial \Gamma^\mu)_{\alpha\beta} [A_\mu, \Psi_\beta]). \quad (2)$$

Here $A_\mu (\mu = 1, \ldots, 10)$ are traceless $N \times N$ Hermitian matrices, whereas $\Psi_\alpha (\alpha = 1, \ldots, 16)$ are traceless $N \times N$ matrices with Grassmannian entries. The model has an SO(9, 1) symmetry, under which $A_\mu$ and $\Psi_\alpha$ transform as a vector and a Majorana-Weyl spinor, respectively. The Lorentz indices $\mu$ and $\nu$ in (1) and (2) are raised and lowered by the metric $\eta = \text{diag}(-1, 1, \ldots, 1)$.

Until quite recently, the type IIB matrix model was studied after making a “Wick rotation”, which amounts to replacing the Hermitian matrix $A_0$ by $A_0 = i A_{10}$, and treating the Hermitian matrix $A_{10}$ on equal footing as the matrices $A_i (i = 1, \ldots, 9)$ in the spatial directions. The Euclidean model obtained in this way has manifest SO(10) symmetry, and it is well defined as Monte Carlo studies with small matrices demonstrate [7]. In fact the partition function was proven to be finite for arbitrary matrix size [8]. In ref. [3], perturbative expansion around the diagonal configurations $A_\mu = \text{diag}(x_{1\mu}, \ldots, x_{N\mu})$ was studied and the low-energy effective theory for the diagonal elements was discussed. In particular, it was speculated that configurations with the $N$ points $\{\vec{x}_i; i = 1, \ldots, N\}$ distributed on a four-dimensional hypersurface in ten-dimensional Euclidean space may be favored due to some nontrivial interactions in the low-energy effective theory. If that really happens, it implies that the SO(10) symmetry is spontaneously broken down to SO(4) and that four-dimensional space-time is generated dynamically.

The recent developments in the type IIB matrix model we would like to discuss are the following. First in the Euclidean model, it was found that SSB indeed occurs, but the SO(10) symmetry is broken down to SO(3) [9]. The extent of space-time in the extended direction and that in the shrunken direction are calculated, and their ratio is found to be around five in the large-$N$ limit. Interpretation of these results is unclear, though, since the Wick rotation is not justifiable unlike in ordinary quantum field theory.

On the other hand, it was found that the Lorentzian model can be made well-defined by introducing infrared cutoffs and removing them in the large-$N$ limit [10]. Real-time evolution can be extracted from matrix configurations that dominate the path integral. It was shown that expanding three-dimensional space appears after a critical time [10], and the possibility of observing Inflation and the Big Bang has been discussed [11,12]. The behavior at much later times has been studied by solving the classical equation of motion [13], and a natural solution to the cosmological constant problem has been suggested [14]. Also the realization of the Standard Model in the type IIB matrix model has been discussed [15–17] by applying...
Fig. 1 a) The free energy for the SO($d$) symmetric vacuum is plotted against $d$. The horizontal line represents the value $f = \log 8 - \frac{1}{4} = 1.32944 \ldots$, corresponding to the prediction in ref. [7]. b) The extent of space-time $R^2$ and $r^2$ in the extended and shrunken directions, respectively, are plotted against $d$.

The idea of intersecting branes. The main message we would like to convey is that the Lorentzian version of type IIB matrix model seems to be indeed the correct nonperturbative formulation of superstring theory, which describes our Universe.

The rest of this article is organized as follows. In section 2 we discuss the recent results obtained in the Euclidean type IIB matrix model. In section 3 we discuss the recent results obtained in the Lorentzian type IIB matrix model. Section 4 is devoted to a summary and future prospects.

## 2 Euclidean type IIB matrix model

In this section we discuss the recent results obtained in the Euclidean type IIB matrix model.

In Fig. 1 we show the most recent results [9] based on the Gaussian expansion method [18] in the large-$N$ limit. On the left we plot the free energy for the SO($d$) symmetric vacuum. We find that $d = 3$ gives the minimum free energy, which implies that SO(10) symmetry is broken down spontaneously to SO(3). On the right we plot the extent of space-time in the extended directions (filled circles) and that in the shrunken directions (open circles). We find that the former increases as $d$ decreases, whereas the latter is almost independent of $d$.

The Gaussian expansion method has been applied also to a 6d version of the type IIB matrix model, which can be obtained by dimensionally reducing 6d super Yang-Mills (SYM) theory to a point [19], and it was found that the SO(6) symmetry is spontaneously broken down to SO(3). The mechanism of SSB is demonstrated by Monte Carlo studies in this case [20].

In order to probe the SSB of SO(6) rotational symmetry, we studied the “moment of inertia” tensor

$$ T_{\mu\nu} = \frac{1}{N} \text{Tr} \left( A_\mu A_\nu \right) $$

and its real positive eigenvalues $\lambda_n$ ($n = 1, \ldots, 6$) ordered as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_6$. The vacuum expectation values (VEVs) $\langle \lambda_n \rangle$, taken after the ordering for each configuration, play the role of order parameters. If they turn out to be unequal in the large-$N$ limit, it implies SSB of SO(6). It has been speculated that the phase of the fermion determinant induces the SSB [21, 22]. However, the effect of the phase is difficult to implement in Monte Carlo calculation due to the so-called sign problem.

In Fig. 2a we show the results for $\langle \lambda_n \rangle_0$ obtained in a model, which is obtained by simply omitting the phase of the fermion determinant [20]. We find that all the eigenvalues converge to the same value ($\ell^2 \sim 0.627$) at large $N$. In ref. [20] the effect of the phase has been studied by an analysis based on the factorization method [22, 23]. In Figs. 2b we show the results obtained for the SO(3) symmetric vacuum.
Fig. 2  a) The eigenvalues \( \langle \lambda_n \rangle_0 \) for the phase-quenched model are plotted against \( 1/N \). The solid line represents the value \( \ell^2 = 0.627 \), which is predicted by the Gaussian expansion method [19]. The other lines represent the fits to the behavior \( \langle \lambda_n \rangle_0 = \ell^2 + O(1/N) \). b) The Monte Carlo results obtained for the SO(3) symmetric vacuum. The intersection \( x \sim 0.35 \) gives the extent of space-time in the shrunken directions, which turns out to be in good agreement with the prediction \( 4 \) by the Gaussian expansion method.

See the original paper [20] for the details. From the intersecting point, we can obtain the extent of space-time in the shrunken directions. We find that it is given by \( x \sim 0.35 \), which should be compared with the value obtained by the Gaussian expansion method [19]

\[
x = \frac{r^2}{\ell^2} \sim \frac{0.223}{0.627} \sim 0.355 ,
\]

(4)

taking account of the chosen normalization. Thus we find that the two completely different methods give consistent results, which supports the validity of both calculations.

After all, we consider that the problem was in the Euclideanization. In quantum field theory, it can be fully justified as analytic continuation. That’s why we can use lattice gauge theory. On the other hand, it is subtle in gravitating theory, although it might be OK at the classical level. For instance, in quantum gravity based on the dynamical triangulation approach, it was found that problems with Euclidean gravity can be overcome in Lorentzian gravity [24]. As another example, we quote Coleman’s worm hole scenario for the cosmological constant problem. It was found recently [25] that a physical interpretation is possible only by considering the Lorentzian version instead of the original Euclidean version. Moreover, Euclidean theory is useless for studying the real time dynamics such as the expanding Universe. All these considerations led us to consider the Lorentzian version of the type IIB matrix model.

3 Lorentzian type IIB matrix model

In this section we discuss recent results obtained in the Lorentzian type IIB matrix model.

3.1 Definition of Lorentzian type IIB matrix model

We define the partition function of the Lorentzian model by [10]

\[
Z = \int dA \, d\Psi \, e^{iS} = \int dA \, e^{iS_A + \text{Pf} M(A)} ,
\]

(5)

where the Pfaffian \( \text{Pf} M(A) \) appears from integrating out the fermionic matrices \( \Psi_\alpha \). Note that in the Euclidean model, the Pfaffian is complex in general, and its phase plays a crucial role in the SSB of SO(10) symmetry as we have seen in the previous section. On the other hand, the Pfaffian in the Lorentzian model is real. Therefore, the mechanism of SSB that was identified in the Euclidean model is absent in the Lorentzian model.
In the definition (5), we have replaced the “Boltzmann weight” $e^{-S}$ in the Euclidean model by $e^{iS}$. This is theoretically motivated from the connection to the worldsheet theory [1]. The partition function (5) can also be obtained formally from pure $\mathcal{N} = 1$ SYM theory in $(9 + 1)$ dimensions by dimensional reduction. Note, however, that the expression (5) is ill-defined and requires appropriate regularization in order to make any sense out of it. It turns out that the integration over $A_\mu$ is divergent, and we need to introduce two constraints

$$\frac{1}{N} \text{Tr}(A_0)^2 \leq \kappa \frac{1}{N} \text{Tr}(A_i)^2,$$

$$\frac{1}{N} \text{Tr}(A_i)^2 \leq L^2.$$

This is in striking contrast to the Euclidean model, in which the partition function is shown to be finite without any regularization [7,8].

Note that $e^{iS_0}$ in the partition function (5) is a phase factor just as in the path-integral formulation of quantum field theories in Minkowski space. However, we can circumvent the sign problem by integrating out the scale factor of $A_\mu$, which essentially replaces the phase $e^{iS_0}$ by the constraint $S_0 \approx 0$. (Such a constraint is analogous to the one that appeared in the model inspired by space-time uncertainty principle [26].) Without loss of generality, we set $L = 1$ in (7), and thus we arrive at the model [10]

$$Z = \int \! dA \delta \left( \frac{1}{N} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) \right) \text{Pf}M(A) \delta \left( \frac{1}{N} \text{Tr}(A_i)^2 - 1 \right) \theta \left( \kappa - \frac{1}{N} \text{Tr}(A_0)^2 \right),$$

where $\theta(x)$ is the Heaviside step function. Since the Pfaffian $\text{Pf}M(A)$ is real in the present Lorentzian case, the model (8) can be studied by Monte Carlo simulation without the sign problem. Note that this is usually not the case for quantum field theories in Minkowski space.

### 3.2 Expanding 3d out of 9d

In ref. [10] we performed Monte Carlo simulation of the model [8]. In order to extract the “time evolution”, we diagonalize $A_0$, and define the eigenvectors $|t_a\rangle$ corresponding to the eigenvalues $t_a$ of $A_0$ ($a = 1, \ldots, N$) with the specific order $t_1 < \ldots < t_N$. The spatial matrix in this basis $\langle t_a|A_i|t_b\rangle$ is not diagonal, but it turns out that the off-diagonal elements decrease rapidly as one goes away from a diagonal element. This motivates us to define $n \times n$ matrices $\tilde{A}_{ij}^{(ab)}(t) \equiv \langle t_{\nu+a}|A_i|t_{\nu+b}\rangle$ with $1 \leq a, b \leq n$ and $t = \frac{1}{n} \sum_{\nu=1}^{n} t_{\nu+a}$ for $\nu = 0, \ldots, (N - n)$. These matrices represent the state of the Universe at fixed time $t$. (This point of view can be justified in the large-$N$ limit, in which more and more eigenvalues of $A_0$ appear around some value $t$ within a fixed interval $\delta t$.) The block size $n$ should be large enough to include non-negligible off-diagonal elements.

Let us study the spontaneous breaking of the SO(9) symmetry. As an order parameter, we define the $9 \times 9$ (positive definite) real symmetric tensor [10]

$$T_{ij}(t) = \frac{1}{n} \text{tr} \left\{ \tilde{A}_i(t) \tilde{A}_j(t) \right\},$$

which is an analog of (3) in the Euclidean model. The trace “tr" used here is taken over the $n \times n$ matrices. The 9 eigenvalues of $T_{ij}(t)$ are plotted against $t$ in Fig. 3. We find that 3 largest eigenvalues of $T_{ij}(t)$ start to grow at the critical time $t_c$, which suggests that the SO(9) symmetry is spontaneously broken down to SO(3) after $t_c$.

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1 Strictly speaking, the Pfaffian can flip its sign, but we find that the configurations with positive Pfaffian dominate as $N$ is increased. Hence, we just take the absolute value of the Pfaffian in actual simulation.
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3.3 Exponential/power-law expansion

In Fig. 3, we plot the extent of space

\[ R(t)^2 \equiv \frac{1}{n} \text{tr} \tilde{A}_i(t)^2 \]  

presented in ref. [11]. (Note that \( R(t)^2 \) is given by the sum of 9 eigenvalues of \( T_{ij}(t) \) defined in (9).) We normalize dimensionful quantities by \( R(t_c) \), where \( t_c \) is the “critical time” at which the spatial SO(9) symmetry is spontaneously broken down to SO(3). In fact the obtained \( R(t) \) can be nicely fitted with \( y = f(x) = a + (1 - a) \exp(bx) \), where we have imposed \( f(0) = 1 \), which follows from the chosen normalization. This implies that three spatial directions actually start to expand exponentially, which may be interpreted as the beginning of Inflation.

In order to confirm the exponential behavior for a longer time period, we need to increase the matrix size further, which makes the simulation too time-consuming. In ref. [12] we considered, instead, a simplified model that describes the behavior at early times. To motivate the model, let us decompose the fermionic action (2) into two terms as

\[ S_f \propto \text{Tr} \left( \Psi_\alpha \left( C T^0 \right)_{\alpha\beta} \left[ A_0, \Psi_\beta \right] \right) + \text{Tr} \left( \Psi_\alpha \left( C T^i \right)_{\alpha\beta} \left[ A_i, \Psi_\beta \right] \right). \]  

Due to the expanding behavior of the universe, the elements of the spatial matrices \( A_i \) become very large at late times. At early times, on the other hand, it is expected that the first term in (11) is more important, so we simply omit the second term in (11) as a simplification. Integrating out the fermionic matrices, we obtain the Pfaffian, which is now given by

\[ \text{PfM} (A) = \Delta^{2(d-1)}, \]  

where \( \Delta \equiv \prod_{i>j} (\alpha_i - \alpha_j) \) is the van der Monde determinant and we have written down the general results for dimensionally reduced SYM models with \( d \) spatial dimensions (\( d = 9 \) in the case of type IIB matrix model). The Pfaffian (12) obtained here causes a repulsive force between all the pairs of eigenvalues of \( A_0 \), which cancels the attractive force arising from the fluctuation of the bosonic matrices at the one-loop level. Due to this cancellation, the eigenvalues of \( A_0 \) can extend to infinity, which necessitates the cutoff (6) in the temporal direction.

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2 Strictly speaking, there are zero modes corresponding to \( \Psi_\alpha \) satisfying \([A_0, \Psi_\alpha] = 0\), which we simply neglect.
observed for some period after the critical time, but it changes into a linear behavior that refs. [27, 28] for other works on the classical solutions in the type IIB matrix model.)

two cutoffs that had to be introduced in order to make the model well-defined. Since the inequalities (6) and solution with an expanding behavior that can naturally solve the cosmological constant problem [14]. (See scribes our universe because we have a well-defined partition function. In particular, we find a classical increases with the cosmic expansion [13]. There are actually many classical solutions, which is reminiscent of the fact that superstring theory possesses infinitely many vacua that are perturbatively stable. However, unlike in perturbative superstring theory, we have the possibility to pick up the unique solution that de-

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value of E-folding, which is determined

The simplified model for early times with the Pfaffian replaced by [12] can be simulated with much less efforts. In ref. [12] we studied the $d = 5$ case, in which the rotational SO(5) symmetry is broken down to SO(3) at some critical time $t_c$ analogously to the $d = 9$ model. In Fig. 4a we plot the extent of space $R^2(t) / R^2(t_c)$ for the simplified model for early times in the $d = 5$ case with various $\kappa$ and $N$. The solid line is a fit to the exponential behavior $y = a + (1 - a) \exp(bx)$ with $a = 0.83(1)$ and $b = 2.3(1)$. b) The extent of space $R^2(t) / R(t_c)$ is plotted against $(t - t_c) / R(t_c)$ for the $d = 5$ quenched model with various $N$, which is considered as a simplified model for late times. The dashed line represents a fit $y = a + (1 - a) \exp(bx)$ to the early time behavior ($a = 0.870(3), 2.21(3)$), whereas the solid line represents a fit $y = cx + d$ to the late time behavior ($c = 17.0(1), d = -23.3(3)$).

The simplified model for early times with the Pfaffian replaced by [12] can be simulated with much less efforts. In ref. [12] we studied the $d = 5$ model, in which the rotational SO(5) symmetry is broken down to SO(3) at some critical time $t_c$ analogously to the $d = 9$ model. In Fig. 4a we plot the extent of space (10) as a function of $t$ for various $N$ and $\kappa$. This confirms the exponentially expanding behavior in the simplified model, which suggests that the first term of the fermionic action (11) is indeed important for the space to expand exponentially.

At late times, the second term in the fermionic action (11) becomes more important, and it is expected that the repulsive force represented by (12) is no more effective. In order to mimic such a situation, we considered a quenched model obtained by omitting the fermionic matrices completely (11). In this model, since the eigenvalues of $A_0$ attract each other, we do not need to introduce the cutoff (6) in the temporal direction. The extent of the eigenvalue distribution increases with $N$, however, and one can take both the continuum and infinite-volume limits. The breaking of SO(5) symmetry for the simplified model for early times in the Lorentzian type IIB matrix model. The extent of space $R^2(t) / R^2(t_c)$ is plotted against $(t - t_c) / R(t_c)$ for the $d = 5$ quenched model. The exponential behavior is observed for some period after the critical time, but it changes into a linear behavior $R^2(t) - t$ meaning that $R(t) \sim t^{1/2}$, which agrees with the expanding behavior of the Friedmann-Robertson-Walker (FRW) universe in the radiation dominated era. It would be interesting to confirm the transition from the exponential behavior to the power-law behavior directly in the original model (9). In particular, this will tell us the value of E-folding, which is determined dynamically in the Lorentzian type IIB matrix model.

3.4 Time-evolution at much later times

While the behaviors at much later times are difficult to study by direct Monte Carlo methods, the classical equation of motion is expected to become more and more valid at later times since the value of the action increases with the cosmic expansion [13]. There are actually many classical solutions, which is reminiscent of the fact that superstring theory possesses infinitely many vacua that are perturbatively stable. However, unlike in perturbative superstring theory, we have the possibility to pick up the unique solution that describes our universe because we have a well-defined partition function. In particular, we find a classical solution with an expanding behavior that can naturally solve the cosmological constant problem [14]. (See refs. [27, 28] for other works on the classical solutions in the type IIB matrix model.)

When we search for classical solutions in the Lorentzian model, it is important to take account of the two cutoffs that had to be introduced in order to make the model well-defined. Since the inequalities (6) and
First, define a sequence of commutation relations 

\[ T_{ij} = \frac{\lambda}{2} (A_i^2 - \kappa L^2) - \frac{\bar{\lambda}}{2} (A_i^2 - L^2) \]  

where \( \lambda \) and \( \bar{\lambda} \) are the Lagrange multipliers. Differentiating (13) with respect to \( A_0 \) and \( A_i \), we obtain 

\[ -[A_0, [A_0, A_i]] + [A_i, [A_j, A_i]] - \lambda A_i = 0 , \]  

\[ [A_j, [A_j, A_0]] - \bar{\lambda} A_0 = 0 , \]  

respectively.

A general prescription to solve the equations of motion (14) and (15) is given as follows [14]. Let us first define a sequence of commutation relations 

\[ [A_i, A_j] = iC_{ij} , \quad [A_i, C_{jk}] = iD_{ijk} , \quad [A_0, A_1] = iE_i , \]  

\[ [A_0, E_i] = iF_i , \quad [A_i, E_j] = iG_{ij} , \ldots , \]  

where \( 1 \leq i, j, k \leq 9 \) and the symbols on the right-hand side represent Hermitian operators newly defined. Then we determine the relationship among \( A_0, A_i, C_{ij}, D_{ijk}, E_i, F_i, G_{ij}, \ldots \) so that the equations of motion (14) and (15) and the Jacobi identities are satisfied. We obtain a Lie algebra in this way. Considering that all the operators are Hermitian, each unitary representation of the Lie algebra gives a classical solution.

As an example of SO(4) symmetric solution, we consider [14] 

\[ A_0 = bT_0 \otimes 1_k , \quad A_i = abT_1 \otimes M_i \quad (i = 1, 2, 3, 4) , \]  

where \( T_0 \) and \( T_1 \) are the two generators of the SU(1, 1) algebra 

\[ [T_0, T_1] = iT_2 , \quad [T_2, T_0] = iT_1 , \quad [T_1, T_2] = -iT_0 . \]  

\[ M_i \] are \( k \times k \) diagonal matrices defined by 

\[ M_i = \text{diag}(n_1^{(1)}, n_2^{(2)}, \ldots, n_k^{(k)}) , \quad |n_I| = 1 \quad (I = 1, \ldots, k) , \]  

and \( 1_k \) is the \( k \times k \) unit matrix. This is a solution of (14) and (15) for \( \lambda = -b^2 \) and \( \bar{\lambda} = -\alpha^2 b^2 \), and it represents \( (3 + 1) \)-dimensional space-time with \( R \times S^3 \) geometry.

Let us discuss the cosmological implications of this solution. As an irreducible unitary representation of the SU(1, 1) algebra, we consider the primary unitary series representation, in which the matrix elements of the generators are given as 

\[ (T_0)_{mn} = n \delta_{mn} , \]  

\[ (T_1)_{mn} = -\frac{i}{2} (n - i \rho) \delta_{m,n+1} + \frac{i}{2} (n + i \rho) \delta_{m,n-1} , \]  

\[ (T_2)_{mn} = -\frac{1}{2} (n - i \rho) \delta_{m,n+1} - \frac{1}{2} (n + i \rho) \delta_{m,n-1} , \]  

where \( m, n \in \mathbb{Z} \). In this case, \( A_1 \) has a tri-diagonal structure. Therefore, we extract \( 3 \times 3 \) submatrices \( \tilde{A}_0(n) \) and \( \tilde{A}_1(n) \). Then we find that the extent of space \( R(n) \) at a discrete time \( n \) becomes 

\[ R(n) = \sqrt{\frac{\alpha^2 b^2}{3} \left( n^2 + \rho^2 + \frac{1}{4} \right)} . \]
Let us take the continuum limit. We define the continuum time by \( t = nb \) and take the \( b \to 0 \) limit. We also take the \( \rho \to \infty \) limit at the same time so that \( t_0 \equiv \rho b \) is kept fixed. Then \( R(t) \) is given by

\[
R(t) = \sqrt{\frac{\alpha^2}{3} (t^2 + t_0^2)}.
\]

Here we naively identify \( R(t) \) with the scale factor of the FRW universe. Then, we obtain the Hubble parameter \( H \) and the parameter \( w \) as

\[
H \equiv \frac{\dot{R}}{R} = \frac{\alpha}{\sqrt{3} R^2} \sqrt{R^2 - \frac{\alpha^2 t_0^2}{3}}, \quad w \equiv -\frac{2 R}{3} \frac{d \ln H}{dR} - 1 = -\frac{2t_0^2}{3t^2} - \frac{1}{3},
\]

which are plotted against \( t \) in Figs. 5a and b, respectively. We find that \( w \) converges to \(-\frac{1}{3}\) as \( t \to \infty \), which corresponds to the expansion of universe with a constant velocity.

If we identify \( t_0 \) with the present time, the present value of \( w \) is \(-1\). This value of \( w \) corresponds to the cosmological constant, which explains the present accelerating expansion of the universe. Moreover, the corresponding cosmological constant becomes of the order of \((1/t_0)^4\), which suggests a possible solution to the cosmological constant problem. As we mentioned above, \( w \) increases with time and approaches \(-\frac{1}{3}\). This means that the cosmological constant actually vanishes in the future.

### 4 Summary and future prospects

We reviewed the recent developments in the type IIB matrix model, which was proposed as a nonperturbative formulation of superstring theory in 1996. While the Euclidean model has been shown to have interesting dynamical properties, their physical interpretation is yet to be clarified since the meaning of the Wick rotation is not obvious. On the other hand, the Lorentzian model remained untouched until recently because of its instability, but recent Monte Carlo studies revealed its surprising properties. First of all, a well-defined theory can be obtained by introducing cutoffs and removing them in the large-\( N \) limit. The notion of “time evolution” emerges dynamically. This is due to the nontrivial dynamical property of the model that the spatial matrices \( A_i \) have a band-diagonal structure when we diagonalize the temporal matrix \( A_0 \). The extracted time evolution shows that, after some “critical time”, the space undergoes the SSB of SO(9) symmetry and only three directions start to expand exponentially. The observed exponential expansion suggests the possibility that the Inflation is naturally realized in this model. (Note that we do not introduce a scalar field by hand, nor do we have to impose any particular initial condition.) We also observed the power-law \( t^{1/2} \) expansion in a simplified model for later times, which is reminiscent of the cosmic expansion of the FRW universe in the radiation dominated era. The behaviors at much later times
are expected to be captured by the classical equations of motion. We have discussed a solution, which suggests a natural solution to the cosmological constant problem.

It would be very important to observe directly the transition from the exponential expansion to the power-law expansion by Monte Carlo simulation. We speculate that the transition to commutative spacetime (as opposed to noncommutative one that is realized generically in the matrix model) occurs at the same time. It would also be interesting to calculate the density fluctuation to be compared with the cosmic microwave background. Another direction would be to read off the effective quantum field theory below the Planck scale from fluctuations around a classical solution that dominates at later times \[29\]. Along that direction, we should be able to see whether the Standard Model appears at low energy \[15–17\].

To conclude, we would like to list fundamental questions in particle physics and cosmology: the mechanism of Inflation, the initial condition problem, the cosmological constant problem, the hierarchy problem, dark matter, dark energy, baryogenesis, the origin of the Higgs field, the number of generations, etc.. It is conceivable that all these problems can be understood in a unified manner by a nonperturbative formulation of superstring theory. The recent developments reviewed above seem to suggest that the type IIB matrix model indeed have the potential for such a formulation.

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