Relaxing lockdown measures in epidemic outbreaks using selective socio-economic containment with uncertainty

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Abstract

After an initial phase characterized by the introduction of timely and drastic containment measures aimed at stopping the epidemic contagion from SARS-CoV2, many governments are preparing to relax such measures in the face of a severe economic crisis caused by lockdowns. Assessing the impact of such openings in relation to the risk of a resumption of the spread of the disease is an extremely difficult problem due to the many unknowns concerning the actual number of people infected, the actual reproduction number and infection fatality rate of the disease. In this work, starting from a compartmental model with a social structure, we derive models with multiple feedback controls depending on the social activities that allow to assess the impact of a selective relaxation of the containment measures in the presence of uncertain data. Specific contact patterns in the home, work, school and other locations for all countries considered have been used. Results from different scenarios in some of the major countries where the epidemic is ongoing, including Germany, France, Italy, Spain, the United Kingdom and the United States, are presented and discussed.

1 Introduction

"Phase two" is the key word after the most critical moment of the coronavirus emergency. The end of the pandemic will not immediately correspond to the disappearance of SARS-CoV2. This is why an intermediate phase is being carefully considered, with some activities that can be resumed immediately, regulating the reintegration of workers, for example through indicators measuring the impact of work activities on potential infections, increasing prevention measures, or through so-called immunity passports. Several question marks over convalescence times and the real extent of the contagion also raise fears of a second wave. It is essential to build scenarios that will help us understand how the situation might evolve in the future.

While in some countries, like Italy and France, the debate is open (and also to controversy), between those who would like to restart as soon as possible and those who, on the other hand, as a precautionary measure, would like to postpone the lockdown due to COVID-19, other countries, like Germany and Sweden, are preparing to restart (and in some cases the reduction of activity has never been total). The overall objective of this second phase is to limit the major damage to a country’s economy caused by the severe lockdown measures, but to avoid restarting the epidemic peak.

Among the many controversial aspects are, for example, the reopening of schools, sport activities and other social activities at different levels, which, while having less economic impact, have a very high social cost. Indeed, it is clear that it is difficult for the population to sustain an excessively long period of lockdown. It is therefore of primary importance to analyze the possibility of relaxing the control measures put in place by many countries in order to make them more sustainable on the socio-economic front, keeping the reproductive rate of the epidemic under control and without incurring health risks [23, 16, 19].

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The problem is clearly very challenging, traditional epidemiological models based on the assumption of homogeneous population mixing are inadequate, since the whole social and economic structure of the country is involved [29, 26, 11, 25, 21, 27]. On the other hand, interventions involving the whole population allow to use mathematical descriptions in analogy with classical statistical physics drawing on the statistical characteristics of a very large system of interacting individuals [1, 2, 3, 8, 18].

A further problem that cannot be ignored is the uncertainty present in the official data provided by the different countries in relation to the number of infected people. The heterogeneity of the procedures used to carry out the disease positivity tests, the delays in recording and reporting the results, and the large percentage of asymptomatic patients (in varying percentages depending on the studies and the countries but estimated by WHO at an average of around 80% of cases) make the construction of predictive scenarios affected by high uncertainty [28, 33, 44]. As a consequence, the actual number of infected and recovered people is typically underestimated, causing fatal delays in the implementation of public health policies facing the propagation of epidemic fronts.

In this research, we try to make a contribution to these problems starting from a description of the spread of the epidemic based on a compartmental model with social structure in the presence of uncertain data. This model allows not only to take into account the specific nature of the different activities involved through appropriate interaction functions derived from experimental interaction matrices [6, 35, 37, 24] but also to systematically include the uncertainty present in the data [9, 10, 13, 28, 33, 40].

The latter property is achieved by increasing the dimensionality of the problem adding the possible sources of uncertainty from the very beginning of the modelling. Hence, we extrapolate statistics by looking at the so-called quantities of interest, i.e. statistical quantities that can be obtained from the solution and that give some global information with respect to the input parameters. Several techniques can be adopted for the approximation of the quantities of interest. Here, following [4] we adopt stochastic Galerkin methods that allow to reduce the problem to a set of deterministic equations for the numerical evaluation of the solution in presence of uncertainties [43, 36, 15].

The main assumption made in this study is that the control measures adopted by the different countries cannot be described by the standard compartmental model but must necessarily be seen as external actions carried out by policy makers in order to reduce the epidemic peak. Most current research in this direction has focused on control procedures aimed at optimizing the use of vaccinations and medical treatments [5, 7, 12, 14, 30] and only recently the problem has been tackled from the perspective of non-pharmaceutical interventions [4, 31, 34, 22, 20]. For this purpose we derive new models based on multiple feedback controls that act selectively on each specific contact function and therefore social activity. Based on the data in [37] this allows to analyze the impact of containment measures in a differentiated way on family, work, school, and other activities.

In our line of approach, the classical epidemiological parameters that define the rate of reproduction of the infectious disease are therefore estimated only in the regime prior to lockdown and define an estimate of the reproductive rate in the absence of control. At this stage the estimation mainly serves to calibrate the model parameters and its variability will then be considered in the intrinsic uncertainty of these values. On the contrary, the control action is estimated in the first lockdown phase using the data available to date. Phase two, therefore, on the modelling front is characterized by a third temporal region following the first lockdown period, in which social characteristics become essential to quantify the impact of possible decisions of the various governments.

This makes it possible to carry out a systematic analysis for different countries and to observe the different behaviour of the control action in line with the dynamics observed and the measures taken by different governments. However, a realistic comparison between countries is an extremely difficult problem that would require a complex phase of renormalization of the data according to the different recording and acquisition methods used. In an attempt to provide comparative results altered as little as possible by assumptions that cannot be justified, we decided to adopt the same criteria for each country and therefore the scenarios presented, although based on realistic values,
do not aspire to have the value of a real quantitative prediction.

We present different simulation scenarios for various countries where the epidemic wave is underway, including Germany, France, Italy, Spain, the United Kingdom and the United States showing the effect of relaxing the lockdown measures in a selective way on the various social activities. The simulations suggest that premature lifting of these interventions will likely lead to transmissibility exceeding one again, resulting in a second wave of infection. On the other hand, a progressive loosening strategy in subsequent phases, as advocated by some governments, shows that, if properly implemented, may be capable to keep the epidemic under control by restarting various productive activities.

2 Methods

The starting model in our discussion is a SIR-type compartmental model with a social structure and uncertain inputs. The presence of a social structure is in fact essential in deriving appropriate sustainable control techniques from the population for a protracted period, as in the case of the recent COVID-19 epidemic. In addition we include the effects on the dynamics of uncertain data, such as the initial conditions on the number of infected people or the interaction and recovery rates.

2.1 A socially structured compartmental model with uncertainty

The heterogeneity of the social structure, which impacts the diffusion of the infective disease, is characterized by \( a \in \Lambda \subset \mathbb{R}_+ \) representing the age of the individual [25, 26]. We assume that the rapid spread of the disease and the low mortality rate allows to ignore changes in the social structure, such as the aging process, births and deaths. Furthermore, we introduce the random vector \( z = (z_1, \ldots, z_d) \in \mathbb{R}^d \), whose components are assumed to be independent real valued random variables taking into account various possible sources of uncertainty in the model. We assume to know the probability density \( p(z) : \mathbb{R}^d \rightarrow \mathbb{R}_+^d \) characterizing the distribution of \( z \).

We denote by \( s(z, a, t), i(z, a, t) \) and \( r(z, a, t) \) the densities at time \( t \geq 0 \) of susceptible, infectious and recovered individuals, respectively in relation to their age \( a \) and the source of uncertainty \( z \). The density of individuals of a given age \( a \) and the total population number \( N \) are deterministic conserved quantities in time, i.e.

\[
s(z, a, t) + i(z, a, t) + r(z, a, t) = f(a), \quad \int_{\Lambda} f(a) da = N.
\]

Hence, the quantities

\[
S(z, t) = \int_{\Lambda} s(z, a, t) da, \quad I(z, t) = \int_{\Lambda} i(z, a, t) da, \quad R(z, t) = \int_{\Lambda} r(z, a, t) da,
\]

(1)

denote the uncertain fractions of the population that are susceptible, infectious and recovered respectively.

In a situation where changes in the social features act on a slower scale with respect to the spread of the disease, the socially structured compartmental model with uncertainties follows the dynamics

\[
\frac{d}{dt} s(z, a, t) = -s(z, a, t) \sum_{j \in A} \int_{\Lambda} \beta_j(z, a, a_a) \frac{i(z, a_a, t)}{N} da_a,
\]

\[
\frac{d}{dt} i(z, a, t) = s(z, a, t) \sum_{j \in A} \int_{\Lambda} \beta_j(z, a, a_a) \frac{i(z, a_a, t)}{N} da_a - \gamma(z, a) i(z, a, t),
\]

(2)

\[
\frac{d}{dt} r(z, a, t) = \gamma(z, a) i(z, a, t)
\]
with initial condition \( i(z, a, 0) = i_0(z, a) \), \( s(z, a, 0) = s_0(z, a) \) and \( r(z, a, 0) = r_0(z, a) \). In (2) the functions \( \beta_j(z, a, a_*) \geq 0 \) represent transmission rates among individuals with different ages related to a specific activity characterized by the set \( A \), such as home, work, school, etc., and \( \gamma(z, a) \geq 0 \) is the recovery rate which may be age dependent. In the following we assume the quantities \( \beta_j(z, a, a_*) \) proportional to the contact rates in the various activities.

In the following, we introduce the usual normalization scaling

\[
\frac{s(z, a, t)}{N} \to s(z, a, t), \quad \frac{i(z, a, t)}{N} \to i(z, a, t), \quad \frac{r(z, a, t)}{N} \to r(z, a, t), \quad \int_A f(z, a) da = 1,
\]

and observe that the quantities \( S(t) \), \( I(t) \) and \( R(t) \) satisfy the SIR dynamic

\[
\frac{d}{dt} S(z, t) = -\sum_{j \in A} \int_A s(z, a, t) \int_A \beta_j(z, a, a_*) i(z, a_*, t) da_* da
\]

\[
\frac{d}{dt} I(z, t) = \sum_{j \in A} \int_A s(z, a, t) \int_A \beta_j(z, a, a_*) i(z, a_*, t) da_* da - \int_A \gamma(z, a) i(z, a, t) da,
\]

where the fraction of recovered is obtained from \( R(z, t) = 1 - S(z, t) - I(z, t) \). We refer to [26, 25, 21, 27] for analytical results concerning model (2) and (3) in a deterministic setting.

### 2.2 Multiple control of structured compartmental model

In order to characterize the action of a policy maker introducing a control over the system based on selective containment measures in relation to a specific social activity we consider the following optimal control setting

\[
\min_{u \in \mathcal{U}} J(u) := \int_0^T \mathcal{R}[I(\cdot, t)] dt + \frac{1}{2} \sum_{j \in A} \int_0^T \int_{\Lambda \times \Lambda} \nu_j(a, t) |u_j(a, a_*, t)|^2 d\text{da_*, dt},
\]

where \( u = (u_1, \ldots, u_J) \) is a vector of controls acting locally on the interaction between individuals of ages \( a \) and \( a_* \), the function \( \nu_j(a, t) > 0 \) is a selective penalization term and \( \mathcal{R}[I(\cdot, t)] \) is a suitable linear operator taking into account the presence of the uncertainties \( z \).

Examples of such operator that are of interest in epidemic modelling are the expectation with respect to uncertainties

\[
\mathcal{R}[I(\cdot, t)] = \mathbb{E}[I(\cdot, t)] = \int_{\mathbb{R}^d_z} I(z, t) p(z) dz
\]

or relying on deterministic data which underestimate the number of infected

\[
\mathcal{R}[I(\cdot, t)] = I(z_0, t),
\]

where \( z_0 \) is a given value such that \( I(z_0, t) \leq I(z, t) \), \( \forall z \in \mathbb{R}^d_z \) and \( t > 0 \).

In (4) the set \( \mathcal{U} \subseteq \mathbb{R}^L \) is the space of admissible controls \( u_j, j \in J \) defined as

\[
\mathcal{U} = \left\{ u \in \mathbb{R}^L | 0 \leq I(u_j)(a, t) \leq \min\{M, \min_z I(\beta_j)(z, a, t)\}, \forall (a, t) \in \Lambda \times [0, T], M > 0 \right\},
\]

where

\[
I(u_j)(a, t) = \frac{1}{I(z, t)} \int_\Lambda u_j(a, a_*, t) i(z, a_*, t) da_* \quad \text{and} \quad I(\beta_j)(z, a, t) = \frac{1}{I(z, t)} \int_\Lambda \beta_j(z, a, a_*) i(z, a_*, t) da_*.
\]
Note that, here we are considering less restrictive conditions on the space of admissible controls than in [4]. The above minimization is subject to the following dynamics

\[
\frac{d}{dt} s(z,a,t) = - s(z,a,t) \sum_{j \in A} \int_{A} (\beta_j(z,a,a_*) - u_j(a,a_*,t)) i(z,a_*,t) \, da_* \\
\frac{d}{dt} i(z,a,t) = s(z,a,t) \sum_{j \in A} \int_{A} (\beta_j(z,a,a_*) - u_j(a,a_*,t)) i(z,a_*,t) \, da_* \\
- \gamma(z,a) i(z,a,t) \\
\frac{d}{dt} r(z,a,t) = \gamma(z,a) i(z,a,t)
\]

with initial condition \( i(z,a,0) = i_0(z,a), \) \( s(z,a,0) = s_0(z,a) \) and \( r(z,a,0) = r_0(z,a)\).

Solving the above optimization problem, however, is generally quite complicated and computationally demanding when there are uncertainties as it involves solving simultaneously the forward problem (4)-(7) and the backward problem derived from the optimality conditions [4]. Furthermore, the assumption that the policy maker follows an optimal strategy over a long time horizon seems rather unrealistic in the case of a rapidly spreading disease such as the COVID-19 epidemic.

### 2.3 Feedback controlled compartmental models with uncertainty

In this section we consider short time horizon strategies which permits to derive suitable feedback controlled models. These strategies are suboptimal with respect the original problem (4)-(7) but they have proved to be very successful in several social modeling problems [1, 2, 3, 18]. To this aim, we consider a short time horizon of length \( h > 0 \) and formulate a time discretize optimal control problem through the functional \( J_h(u) \) restricted to the interval \([t, t + h]\), as follows

\[
\min_{u \in U} J_h(u) = \mathcal{R}[\{I(\cdot, t + h) + \frac{1}{2} \sum_{j \in A} \int_{A} \nu_j(a,t) u_j(a,a_*,t) \, da_*\}] \\
\text{subject to} \\
s(z,a,t + h) = s(z,a,t) - h s(z,a,t) \sum_{j \in A} \int_{A} (\beta_j(z,a,a_*) - u_j(a,a_*,t)) i(z,a_*,t) \, da_* \\
i(z,a,t + h) = i(z,a,t) + h s(z,a,t) \sum_{j \in A} \int_{A} (\beta_j(z,a,a_*) - u_j(a,a_*,t)) i(z,a_*,t) \, da_* \\
- h \gamma(z,a) i(z,a,t), \\
r(z,a,t + h) = r(z,a,t) + h \gamma(z,a) i(z,a,t).
\]

Recalling that the macroscopic information on the infected is

\[
I(z,t + h) = I(z,t) + h \sum_{j \in A} \int_{A} \left[ s(z,a,t) \int_{A} (\beta_j(z,a,a_*) - u_j(a,a_*,t)) i(z,a_*,t) \, da_* \right.
- \left. \gamma(z,a) i(z,a,t) \right] \, da,
\]

we can derive the minimizer of \( J_h \) computing \( \nabla_u J_h(u) \equiv 0 \) or equivalently

\[
\frac{\partial J_h(u)}{\partial u_j} = 0, \quad j \in A.
\]

Using (8) we can compute

\[
\mathcal{R} \left[ \frac{\partial I(\cdot, t + h)}{\partial u_j} \right] = \nu_j(a,t) u_j(a,a_*,t),
\]
The above assumption on \( R \) notation for \( h \) can apply the instantaneous strategies (10) into the discrete system (9) and pass to the limit for \( h \to 0 \). The resulting feedback controlled model reads

\[
\frac{d}{dt}s(z,a,t) = -s(z,a,t) \sum_{j \in A} \int_{\Lambda} \left( \beta_j(z,a,a_s) - \frac{R[s(\cdot,a,t)\iota(\cdot,a_s,t)]}{\kappa_j(a,a_s,t)} \right) i(z,a_s,t) \, da_s
\]

\[
\frac{d}{dt}i(z,a,t) = s(z,a,t) \sum_{j \in A} \int_{\Lambda} \left( \beta_j(z,a,a_s) - \frac{R[s(\cdot,a,t)\iota(\cdot,a_s,t)]}{\kappa_j(a,a_s,t)} \right) i(z,a_s,t) \, da_s - \gamma(z,a)i(z,a,t)
\]

\[
\frac{d}{dt}r(z,a,t) = \gamma(z,a)i(z,a,t).
\]

Finally, we provide sufficient conditions for the admissibility of the feedback control in terms of the penalization term. In fact, the dynamic must preserve the monotonicity of the susceptible population number \( s(z,a,t) \) for each age class and for each single control action over time.

By imposing the non-negativity of the total reproduction rate in (11) we get

\[
\mathcal{I}(\beta_j)(z,a,t) \geq \frac{1}{\kappa_j(a,t)} \mathcal{I}(R[s(\cdot,a,t)\iota(\cdot,t)])
\]

and assuming \( \beta_j(z,a,a_s) \geq \delta > 0, \forall (a,a_s) \in \Lambda \times \Lambda, z \in \mathbb{R}^d \) we have

\[
\kappa_j(a,t) \geq \mathcal{I}(R[s(\cdot,a,t)\iota(\cdot,t)]) / \delta, \quad j \in A.
\]

These inequalities have to be satisfied \( \forall a \in \Lambda \) and for every \( t \geq 0 \). In the case of a time independent penalization term \( \kappa_j = \kappa_j(a) \) we have the following admissibility conditions

\[
\kappa_j(a) \geq \mathcal{I}(R[s_0(\cdot,a)\tilde{I}(\cdot)]) / \delta,
\]

where \( R[\cdot] \) is defined by (5) or (6) and \( \tilde{I}(z) \) is the maximum reached by the total density of infectious.

### 3 Results

In this section we present the results of several simulation of the constrained compartmental model with uncertain data. Details of the stochastic Galerkin method used to deal efficiently with uncertain data may be found in [4, 36]. The data concerning the actual number of infected, recovered and deaths in the various country have been taken from the Johns Hopkins University Github repository [17] ad for the specific case of Italy from the Github repository of the Italian Civil Protection Department [38]. The social interaction functions \( \beta_j \) have been reconstructed from the dataset of age and location specific contact matrices related to home, work, school and other activities in [37]. Finally, the demographic characteristics of the population for the various country have been taken from the United Nations World Populations Prospects\(^1\). Other sources of data which have been used include the Coronavirus disease (COVID-2019) situation reports of the WHO\(^2\) and the Statistic and Research Coronavirus Pandemic (COVID-19) from OWD\(^3\).

\(^1\)https://population.un.org/wpp/
\(^2\)https://www.who.int/emergencies/diseases/novel-coronavirus-2019/
\(^3\)https://ourworldindata.org/coronavirus
3.1 Model calibration

Estimating epidemiological parameters is a very difficult problem that can be addressed with different approaches [9, 13, 40]. In the case of COVID-19 due to the limited number of data and their great heterogeneity is an even bigger problem that can easily lead to wrong results. Here, we restrict ourselves to identifying the deterministic parameters of the model through a suitable fitting procedure, considering the possible uncertainties due to such estimation as part of the subsequent uncertainty quantification process.

More precisely, we have adopted the following two-level approach in estimating the parameters. In the phase preceding the lockdown we estimated the epidemic parameters, and hence the model reproduction number \( R_0 \), in an uncontrolled regime. This estimate was then kept in the subsequent lockdown phase where we estimated as a function of time the value of the control penalty parameter. Both these two calibration steps were analyzed under the assumption of homogeneous mixing.

Therefore, we solved two separate constrained optimization problems. First we estimated \( \beta > 0 \) and \( \gamma > 0 \) by solving in the uncontrolled time interval \( t \in [t_0, t_u] \) a least square problem based on minimizing the relative \( L^2 \) norm of the difference between the reported number of infected \( I(t) \) and recovered \( R(t) \), and the theoretical evolution of the unconstrained model \( I(t) \) and \( R(t) \). As a function of the penalization introduced on the curve of the infected or recovered we have a better adaptation of the model to the respective curves thus obtaining a range for the reproduction number \( R_0 \). We observed that, in general, fitting the curve of infectious yields a larger estimated reproduction number compared to fitting the curve of recovered. On the other hand, the lack of reliable informations concerning the recovered in early stages of the disease suggests to adapt the model mainly to the curve of infectious and to introduce the uncertainty in the reproductive number using this estimated value as an upper bound of the reproduction number. Due to the heterogeneity of the data between the different countries, in order to have comparable results with reproduction numbers \( R_0 = \beta/\gamma \in (4.5, 9.5) \) we constrained the value of \( \beta \in (0, 0.6) \) and the value of \( \gamma \in (0.04, 0.06) \). In Table (1) we report the values obtained by averaging the optimization results obtained with a penalization factor of 0.01 and \( 10^{-6} \), respectively, over the recovered.

Next, we estimate the penalization \( \kappa = \kappa(t) > 0 \) in time by solving in the controlled time interval \( t \in [t_u, t_c] \) for a sequence of time steps \( t_i \) of size \( h \) the corresponding least square problems in \( [t_i + k_i h, t_i + k_i h] \), \( k_i, k_r \geq 1 \) integers, and where for the evolution we consider the values \( \beta_c \) and \( \gamma_c \) estimated in the first optimization step using the curve of infectious. The second fitting procedure has been activate up to last available data with daily time stepping \( (h = 1) \) and a window of seven days \( (k_t = 3, k_r = 4) \) for regularization along one week of available data. For consistency we performed the same optimization process used to estimate \( \beta \) and \( \gamma \), namely using two different penalization factors and then averaging the results. These optimization problems have been solved testing different optimization methods in combination with adaptive solvers for the system of ODEs. The results reported have been obtained using the Matlab functions \( \text{fmincon} \) in combination with \( \text{ode45} \).

The corresponding time dependent values for the controls as well as results of the model fitting with the actual trends of infectious are reported in Figure 1. The trends have been computed using a weighted least square fitting with the model function \( k(t) = a e^{bt} (1 - e^{ct}) \). For some countries, like France, Spain and Italy after an initial adjustment phase the penalty term converges towards a peak and has just started to decrease. This is consistent with a situation in which data concerning the number of reported infectious needs a certain period of time before being affected by the lockdown policy and can also be considered as an indicator of an unstable situation where reducing control may lead to a potential restart of the infectious curve. The penalty terms for the US and the UK clearly indicates that the pandemic is still in its growing phase and the situation is far from a controlled equilibrium state. The only exception is represented by Germany where the dynamic corresponds to a significative decrease in the penalization term as a result of a timely implementation of social distancing measures.
Figure 1: Model behavior with fitting parameters and actual trends in the number of reported infectious using the estimated control penalization terms after lockdown over time in the various countries.
| Country | Mar 5-Mar 22 | Mar 5-Mar 17 | Feb 24-Mar 9 | Mar 2-Mar 14 | Mar 8-Mar 23 | Mar 7-Mar 19 |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\beta$ | 0.3134      | 0.3164      | 0.3101      | 0.3686      | 0.2698      | 0.3716      |
| $\gamma$| 0.0483      | 0.0483      | 0.0494      | 0.0400      | 0.0482      | 0.0481      |
| $R_0$   | 6.487       | 6.5525      | 6.2710      | 9.2150      | 5.6010      | 7.7155      |

Table 1: Model fitting parameters in estimating attack values for the COVID-19 outbreak before lockdown in various countries.

Figure 2: Estimated disagreement in the total number of cases based on an IFR of 1.3% in the range 0.9%-2.0%. The uncertainty is measured as the estimated values divided by the reported cases. The country specific values are given on the top of each red bar, the average value of $c = 8.56$ is reported as a dashed green line.

3.2 Estimating actual infection trends with uncertain data

Next we focus on the influence of uncertain quantities on the controlled system with homogeneous mixing. According to recent results on the diffusion of COVID-19 in many countries the number of infected, and therefore recovered, is largely underestimated on the official reports, see e.g. [28, 33]. One possible way to understand this is based on a renormalization process of the reported data based on the estimated infection fatality rate (IFR) of Covid-19. Although estimating the true IFR is generally hazardous while an epidemic is underway, some studies have estimated an overall IFR around 1.3% with an age dependent credible interval [41, 39]. In the sequel we consider a range spanning between 0.9% - 2.0%. On the contrary the current fatality rate (CFR) may vary strongly from country to country accordingly to the differences in the number of people tested, demographics, health care system. One way to have in insight in the uncertainty of data is to use the estimated IFR ranges as normalization factors for the current data reported of total cases $I_{tot}$. This is done computing an estimated number of total confirmed cases as $\hat{I}_{tot} = 100 \times D_r / \text{IFR}$, where $D_r$ is the total number of confirmed deaths. The results of the variations $\hat{I}_{tot}/I_{tot}$ for the various countries are summarized in Figure 2 and are directly proportional to the CFR of the country. We are aware that the estimate obtained is certainly coarse, nevertheless it allows to get an idea of the disagreement between the data observed and expected in the various countries and therefore to be able to define a common scenario between the various countries.

In order to have an insight on global impact of uncertain parameters we consider a two-dimensional uncertainty $z = (z_1, z_2)$ with independent components such that

$$I(z,0) = I_0(1 + \mu z_1), \quad R(z,0) = R_0(1 + \mu z_1)$$

(14)
and
\[ \beta(z) = \beta_e + \alpha_\beta z_2, \quad \gamma(z) = \gamma_e + \alpha_\gamma z_2 \] (15)

where \( z_1, z_2 \) are chosen distributed as symmetric Beta functions in \([0, 1]\), \( i_0 \) and \( r_0 \) are the initial number of reported cases and recovered, and \( \beta_0, \gamma_e \) are the fitted values given in Table 1. In the following we will consider \( \mu = 2(c - 1) \) common for all countries such that \( \mathbb{E}[I(z, 0)] = cI(0) \), \( \mathbb{E}[R(z, 0)] = cR(0) \) where \( c = 8.56 \), the average value from Figure 2.

From a computational viewpoint we adopted the method developed in [4] based on a stochastic Galerkin approach. The feedback controlled model has been computed using an estimation of the total number of susceptible and infected reported, namely we have the control term

\[ u(t) = -\frac{1}{k(t)} S_\gamma(t) I_r(t), \] (16)

where \( S_\gamma(t) \) and \( I_r(t) \) are the model solution obtained from the reported data, and thus \( I_r(t) \) represents a lower bound for the uncertain solution \( I(z, t) \).

In Figure 3 we report the results concerning the evolution of estimated current infectious cases from the beginning of the pandemic in the reference countries using \( z_1 \sim B(10, 10) \) and \( \alpha_\beta = \alpha_\gamma = 0 \). In the inset figures the evolution of total cases is reported. The expected number of infectious is plotted with blue continuous line. Furthermore, to highlight the country-dependent underestimation of cases we report with dash-dotted lines both the expected evolutions, where the uncertain parameter \( c > 0 \) varies from country to country accordingly to the numbers on the top of the red bars in Figure 2.

In Figure 4 we report the evolution of reproduction number \( R_0 \) for the considered countries under the uncertainties in (15) obtained with \( \alpha_\beta = -3, \alpha_\gamma = 5 \) and \( z_2 \sim B(2, 2) \). It has been reported, in fact, that deterministic methods based on compartmental models overestimate the effective reproduction number [32]. The reproduction number is estimated from

\[ R_0(z_2, t) = \frac{\beta(z) - u(t)\chi(t > \bar{t})}{\gamma(z)} \]

being the control \( u(t) \) defined in (16) and \( \bar{t} \) is the country-dependent lockdown time. The estimated reproduction number relative to data is reported with x-marked symbols and represents an upper bound for \( R_0(z_2, t) \). The first day that the 50% confidence interval and the expected value fall below 1 is highlighted with a shaded green region. We can observe how the model estimates that for most countries in the first days of April the reproduction number \( R_0 \) has fallen below the threshold of 1. On the other hand, in the UK and the US the same condition was reached between the end of April and the beginning of May. In realistic terms these dates should be considered as overestimates as they are essentially based on observations without taking into account the delay in the data reported.

3.3 Relaxing control on the various social activities

We analyze the effects of the inclusion of age dependence and social interactions in the above scenario. The number of contacts per person generally shows considerable variability depending on age, occupation, country, in relation to the social habits of the population. However, some universal features can be extracted, which emerge as a function of specific social activities.

More precisely, we consider the social interaction functions corresponding to the contact matrices in [37] for the various countries. As a result we have four interaction functions characterized by \( \mathcal{A} = \{F, E, P, O\} \), where we identify family and home contacts with \( \beta_F \), education and school contacts with \( \beta_E \), professional and work contacts with \( \beta_P \), and other contacts with \( \beta_O \). These functions have been reconstructed over the age interval \( \Lambda = [0, a_{\text{max}}] \), \( a_{\text{max}} = 100 \) using linear interpolation. We report in Figure 5, as an example, the total social interaction functions for the various countries. The functions share a similar structure but with different scalings accordingly to the country specific features identified in [37].
Figure 3: Evolution of current and total cases for each country with uncertain initial data as in (14) based on the average uncertainty between countries. The 95% and 50% confidence levels are represented as shaded and darker shaded areas respectively. The dash-dotted lines denote the expected trends with a country dependent uncertainty from Figure 2.
Figure 4: Evolution of estimated reproduction number $R_0$ and its confidence bands for uncertain data in as in (15). The 95% and 50% confidence levels are represented as shaded and darker shaded areas respectively. The green zones denote the interval between the first day the 50% confidence band and the expected value fall below 1.

Figure 5: The total contact interaction function $\beta = \beta_F + \beta_E + \beta_P + \beta_O$ taking into account the contact rates of people with different ages. Family and home contacts are characterized by $\beta_F$, education and school contacts by $\beta_E$, professional and work contacts by $\beta_P$, and other contacts by $\beta_O$. 
In order to match the age-structured model with the homogeneous mixing model the social functions were normalized using the previously estimated parameters $\beta_c$ and $\gamma_c$ in accordance with

$$\beta_c = \frac{1}{a_{\text{max}}} \sum_{j \in A} \int_{\Lambda} \beta_j(a, a_e) \, da \, da_s,$$

$$\gamma_c = \frac{1}{a_{\text{max}}} \int_{\Lambda} \gamma(a) \, da.$$  \hspace{1cm} (17)

We considered a uniform recovery rate, together with an age-related recovery rate [42] as a decreasing function of the age in the form

$$\gamma(a) = \gamma_c + Ce^{-ra},$$  \hspace{1cm} (18)

with $r = 5$ and $C \in \mathbb{R}$ such that (17) holds. Clearly, this choice involves a certain degree of arbitrariness since there are not yet sufficient studies on the subject, nevertheless, as we will see in the simulations, it is able to reproduce more realistic scenarios in terms of age distribution of the infected without significantly altering the behaviour relative to the total number of infected.

In a similar spirit, to match the single control applied in the extrapolation of the penalization term $\kappa(t)$ to age dependent penalization factors $\kappa_j(a, t)$ we redistribute their values as

$$\kappa_j(a, t)^{-1} = \frac{w_j(t) \int_{\Lambda} \beta_j(a, a_e) \, da_s}{\sum_{j \in A} w_j(t) \int_{\Lambda} \beta_j(a, a_e) \, da_s} \kappa(t)^{-1}, \quad j \in A$$  \hspace{1cm} (19)

where $w_j(t) \geq 0$, are weight factors denoting the relative amount of control on a specific activity. In the lockdown period accordingly to other studies [37] we assume $w_E = 1.5$, $w_H = 0.2$, $w_P = 0.5$, $w_O = 0.6$, namely the largest effort of the control is due to the school closure which as a consequence implies more interactions at home. Work and other activities are equally impacted by the lockdown. In particular, these initial lockdown choices make it possible to have a good correspondence between the infectivity curves expected in the age dependent case and in the homogeneous mixing case. Therefore, these values have been set homogeneously for each country and correspond to the situation in the first lockdown period. We will discuss possible changes to these choices following a relaxation of the lockdown in the different scenarios presented below.

We divided the computation time frame into two zones and used different models in each zone, in accordance with the policy adopted by the various countries. The first time interval defines the period without any form of containment, the second the lockdown period. In the first zone we adopted the uncontrolled model with homogeneous mixing for the estimation of epidemiological parameters. Hence, in the second zone we compute the evolution of the feedback controlled age dependent model (11) with matching (on average) interaction and recovery rates (17) and with the estimated control penalization $\kappa(t)$. The initial values for the age distributions of susceptible have been taken from the specific demographic distribution of each country. More difficult is to get the same informations for the infected, since reported data are rather heterogeneous for the various country and the initial number of individuals is very small (we selected a time frame where the reported number of infectious is larger than 200). Therefore, we tested the available data against a uniform distribution. As there were no particular differences in the results, we decided to adopt a uniform initial distribution of the infected for all countries. In Figure 6 we report the age distribution of infected computed for each country at the end of the lockdown period using an age dependent recovery and a constant recovery. The differences in the resulting age distributions are evident. In subsequent simulations, to avoid an unrealistic peak of infection among young people, we decided to adopt an age-dependent recovery [42].

### 3.3.1 Scenario 1: Relaxing lockdown measures at different times

In the first scenario we analyze the effects on each country of the same relaxation of the lockdown measures at two different times. The first date is country specific accordingly to current available informations, the second is June 1st for all countries. For all countries we assumed a reduction of individual controls on the different activities by 20% on family activities, 35% on work activities and 30% on other activities without changing the control over the school. The behaviors of the
curves of infected people together with the relative 95% confidence bands are reported in Figure 7. The results show well the substantial differences between the different countries, with a situation in the UK and US that seems clearly premature to relax lockdown measures. On the contrary, Germany and, to some extent Spain, are in the most favorable situation to ease the lockdown without risking a new start of the infection. In all cases, however, it is clear that a further increase in the number of infected people should be expected.

3.3.2 Scenario 2: Impact of school and work activities

In order to highlight the differences in the behaviour of the infection according to the choices related to specific activities, such as school and work, we have considered the effects of a specific lockdown relaxation in these directions. Precisely for each country we have identified a range for such loosening which gives an indication of the maximum allowed opening of the activities before a strong departure of the infection.

It was assumed to relax the lockdown of the school with a mild resumption of family, work and other activities interactions by 5% for each 10% release of the school. The results are reported in Figure 8. Next, we perform a similar relaxation process oriented towards productive activities with a reduction of control on such activities at various percentages. Here we assumed no impact on school activities and a mild impact on family and other activities with a loosening at 5% for each 10% release of the work. The results are given in Figure 9. In both cases, the results show that the selected countries can be divided into 3 groups, Germany and Spain in a stable downward phase of the epidemic curve, France and Italy in a still transitional phase with greater risks in reopening, and the UK and US in full growth phase of the epidemic curve in which any relaxation of lockdown measures leads to a strong restart of the epidemic.

3.3.3 Scenario 3: Restarting activities while keeping the curve under control

One of the major problems in the application of very strong containment strategies, is the difficulty in maintaining them over a long period, both for the economic impact and for the impact on the population from a social point of view. The results of the previous scenarios have shown that it may be possible for some countries, like Germany, Spain, France, and Italy, to relax the lockdown measures albeit with some risk of an increase in the contagion curve. On the contrary, in all scenarios considered, the situation in the UK and the US suggests that any loosening of containment measures should be postponed. In the latter scenario, we consider a strategy based
Figure 7: Scenario 1: Effect on releasing containment measures in various countries at two different times. In all countries after lockdown we assumed a reduction of individual controls on the different activities by 20% on family activities, 35% on work activities and 30% on other activities by keeping the lockdown over the school.
Figure 8: **Scenario 2 - school:** Effect on releasing containment measures for school activities in various countries at two different times. Family, work and other activities are relaxed by 5% for each 10% release of the school activity.

Figure 9: **Scenario 2 - work:** Effect on releasing containment measures for productive activities in various countries at two different times. School is kept in lockdown. Family, and other activities are relaxed by 5% for each 10% release of the productive activity.
Figure 10: **Scenario 3:** Relaxing lockdown measures in a progressive way in two subsequent phases while keeping the epidemic peak under control. In the second phase only productive activities are restarted and partially home interactions and other activities. In a third phase school activities are also partially reopened (see Table 2).

on a two-stage opening of the blocking measures with a progressive approach. This possibility is analysed for the four countries where it might be possible to partially reopen production activities without restarting the contagion curve. For each country we have selected a progressive lockdown relaxation focused mainly on the opening of productive activities in the second phase and with a partial reprise of school activities in the third phase. The reduction of the controls are now country specific and the values are reported in Table 2. In Figure 10 we plot the resulting behavior for the expected number of current infectious. The simulations show that for all these countries, it is possible to relax the containment measures in a progressive way by keeping the infection curve under control. However, timing and intensity of the relaxation choices play a fundamental rule in the process.

### 4 Discussion

The approach of a second phase of the pandemic into a new normality is full of uncertainties from the point of view of social and economic planning. It is clear to the people that this period cannot be marked by an "all free", by a return to the old normality. For some time to come it will be necessary to respect rules of conduct and hygiene standards to which we have not been accustomed. There are many issues to be addressed, in particular how gradually to reopen the various social and economic activities without creating a new wave of infected and therefore deaths.

In order to analyze possible future scenarios, it is essential to have models capable of describing the impact of the epidemic according to the specific social characteristics of the country and the containment actions implemented. In this work, aware of the complexity of the problem, we have tried to provide a suitable modeling context to describe possible future scenarios in this direction.
More precisely, with the aid of a SIR model with specific feedback controls on social interactions capable to describe the selective action of a government in opening certain activities such as home, work, school and other activities, we can simulate their future impact with respect to the current epidemic trend. In particular, in an effort to take into account the high uncertainty in the data, the model has been formalized in the presence of uncertain input parameters that allow to explore hypothetical scenarios with appropriate confidence bands.

The simulation parameters have been obtained using data coming from several countries with “comparable situations” in terms of epidemic progress, such as Italy, France, Germany, Spain, the United Kingdom and the United States. The model is capable to describe accurately the reported data thanks to the introduction of the time dependent control action and therefore to provide potential useful indications thanks to the dependence of interactions between the population from the social context.

The results, in accordance with the observations, show situations with different levels of sensitivity to a hypothetical reopening of certain activities. The scenarios presented in order to be able to compare the various realities are largely hypothetical situations but they highlight very well the impact of the different social activities and how some countries such as the United Kingdom and the United States are still in an epidemic situation that suggests maintaining the actual lockdown measures before moving to a second phase. On the contrary, the simulations show how Germany before the other countries and secondly Spain, France and Italy, can aim at a gradual reopening of social and economic activities, keeping the epidemic curve under control, provided that they are resumed in a progressive way and within an appropriate time frame.

### Conclusions

The use of a SIR model with social structure modified through appropriate feedback controls allows to obtain simulations in agreement with the current epidemic scenarios in different countries, including Germany, France, Italy, Spain, the United Kingdom and the United States. The inclusion of uncertainty about the actual value of the number of infected people makes it possible to analyze the effects of the potential reopening of productive and social activities at different times. A multi-modelling approach aligned with the current epidemiological and demographic data, which includes experimental social interaction matrices for the different countries, permits to contextualize possible future scenarios. Further studies are being conducted on geographical dependence through spatial variables. This would make it possible to characterize control measures on a local rather than global basis.

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