Introduction to Radio Astronomical Polarimetry
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ABSTRACT

These workshop notes present an introduction to the concepts and mathematical foundations of polarimetry. One of the main goals of this workshop is to develop an understanding of the relationships between a physical description of the signal path (e.g. gain, delay, rotation, coupling, etc.), the corresponding transformations of the electric field vector, and the equivalent transformations of the Stokes parameters. The adopted algebraic/geometric approach is either directly copied from or heavily inspired by the work of Britton (2000) and Hamaker (2000), and some justification is provided for preferring these over other approaches and parameterizations.

1. POLARIZATION

For a plane-propagating transverse electromagnetic wave, there exist two independent solutions to Maxwell’s equations, representing two orthogonal senses of polarization. Radio receiver systems must differentiate between these two senses in order to fully describe the vector state of the observed radiation. A dual-polarization receiver is therefore designed with two receptors, or probes, that are ideally sensitive to orthogonal polarizations. Define the transverse electric field vector,

\[ e(t) = \begin{pmatrix} e_0(t) \\ e_1(t) \end{pmatrix} \]  

(1)

where \( e_0(t) \) and \( e_1(t) \) are the complex-valued analytic signals associated with two real-valued time series, providing the instantaneous amplitudes and phases of the two orthogonal senses of polarization (see Appendix A for more details).

The polarization of electromagnetic radiation is described by the second-order statistics of \( e \), as represented by the complex-valued 2 \( \times \) 2 coherency matrix (Born & Wolf 1980)

\[ \rho = \langle e \otimes e^\dagger \rangle = \begin{pmatrix} \langle e_0 e_0^\dagger \rangle & \langle e_0 e_1^\dagger \rangle \\ \langle e_1 e_0^\dagger \rangle & \langle e_1 e_1^\dagger \rangle \end{pmatrix}. \]  

(2)

Here, the angular brackets denote an ensemble average, \( \otimes \) is the tensor product, and \( e^\dagger \) is the Hermitian transpose of \( e \).

The coherency matrix is self-adjoint, or Hermitian, i.e. \( \rho = \rho^\dagger \), and can be written as a linear combination of four Hermitian basis matrices,

\[ \rho = S_\mu \sigma_\mu / 2, \]  

(3)

where \( S_\mu \) are the four real-valued Stokes parameters, Einstein notation is used to imply a sum over repeated indexcs, \( 0 \leq \mu \leq 3 \), \( \sigma_0 \) is the 2 \( \times \) 2 identity matrix, and

\[ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \]  

(4)

are the Pauli matrices. The Pauli matrices are traceless (i.e. \( \text{Tr} [\sigma_i] = 0 \), where \( \text{Tr} \) is the matrix trace operator) and satisfy

\[ \sigma_i \sigma_j = \delta_{ij} \sigma_0 + i \epsilon_{ijk} \sigma_k, \]  

(5)

where \( \epsilon_{ijk} \) is the permutation symbol and summation over the index \( k \) is implied. Using these properties, it is easily shown that

\[ S_\mu = \text{Tr}(\sigma_\mu \rho). \]  

(6)

Equivalent expressions of the Stokes parameters are given by

\[ S_\mu = \langle e^\dagger \sigma_\mu e \rangle \]  

(7)

and

\[ S_\mu = \sigma_\mu : \rho \]  

(8)

where the \( : \) operator represents tensor double contraction, a tensor product followed by contraction over two pairs of indices. The double contraction of two matrices \( A \) and \( B \) yields a scalar quantity defined by

\[ A : B \equiv A_{\mu}^\nu B_{\nu}^\mu. \]  

(9)
Equation (3) expresses the coherency matrix as a linear combination of Hermitian basis matrices; Equation (8) represents the Stokes parameters as the projections of the coherency matrix onto the basis matrices. Because $\rho$ is Hermitian, the Stokes parameters are real-valued.

Any one of Equations (6) through (8) can be used to derive the following expressions for the Stokes parameters

\begin{align*}
S_0 &= \langle |e_0(t)|^2 \rangle + \langle |e_1(t)|^2 \rangle \\
S_1 &= \langle |e_0(t)|^2 \rangle - \langle |e_1(t)|^2 \rangle \\
S_2 &= 2 \text{Re} \langle [e_0^*(t)e_1(t)] \rangle \\
S_3 &= 2 \text{Im} \langle [e_0^*(t)e_1(t)] \rangle.
\end{align*}

It proves useful to organize the four real-valued Stokes parameters into scalar and vector components, $[S_0, S]$, where $S_0$ is the total intensity and $S = (S_1, S_2, S_3)$ is the polarization vector.

Given a Cartesian basis in which the radiation propagates in the direction of the $\hat{x}$ axis, and the electric field is measured by its projection along the $\hat{x}$ and $\hat{y}$ axes,

\[ e(t) = \begin{pmatrix} e_x(t) \\ e_y(t) \end{pmatrix} \quad (14) \]

and $S = (Q, U, V)$. Some receivers employ waveguide structures (for example, a quarter waveplate) to convert from linear to circular polarization. In a basis defined in terms of orthogonal senses of circular polarization (e.g. see van Straten et al. 2010, for more details),

\[ e(t) = \begin{pmatrix} e_l(t) \\ e_r(t) \end{pmatrix} \quad (15) \]

and $S = (V, Q, U)$. To foster an intuitive understanding of the mathematical definitions of the Stokes parameters, consider some special cases of polarization state.

1.1. 100% Polarized Radiation

Six special cases of fully polarized radiation are considered in the Cartesian basis; they are organized into the following three groups:

1. Stokes Q: $e_x = 0$ or $e_y = 0$
2. Stokes U: $e_x = e_y$ or $e_x = -e_y$
3. Stokes V: $e_x = -ie_y$ or $e_x = ie_y$

1.1.1. Stokes Q

If $e_y = 0$, then the radiation is 100% linearly polarized with the electric field vector oscillating only along the $x$-axis. In this case, the total intensity $S_0$ is equal to the variance of $e_x$, $S_1 = S_0$ and $S_2 = S_3 = 0$. This corresponds to positive Stokes $Q$.

If $e_x = 0$, then the radiation is 100% linearly polarized with electric field vector oscillating only along the $y$-axis. In this case, the total intensity $S_0$ is equal to the variance of $e_y$, $S_1 = -S_0$ and $S_2 = S_3 = 0$. This corresponds to negative Stokes $Q$.

1.1.2. Stokes U

If $e_x = e_y$, then the radiation is 100% linearly polarized with the electric field vector oscillating only along an axis that is rotated by 45 degrees with respect to the $x$-axis. In this case, the variances of $e_x$ and $e_y$ are equal, the total intensity $S_0$ is equal to twice the variance of $e_x$, $S_1 = 0$, $S_2 = S_0$, and $S_3 = 0$. This corresponds to positive Stokes $U$.

If $e_x = -e_y$, then the radiation is 100% linearly polarized with the electric field vector oscillating only along an axis that is rotated by -45 degrees with respect to the $x$-axis. In this case, the variances of $e_x$ and $e_y$ are equal, the total intensity $S_0$ is equal to twice the variance of $e_x$, $S_1 = 0$, $S_2 = -S_0$, and $S_3 = 0$. This corresponds to negative Stokes $U$.

1.1.3. Stokes V

If $e_x = -ie_y$, then the phase of $e_y$ leads that of $e_x$ by 90 degrees and the radiation is 100% circularly polarized with the electric field vector tracing a counter-clockwise circle in the $x$-$y$ plane; this is defined by the IEEE as left-hand circularly polarized (LCP). In this case, the variances of $e_x$ and $e_y$ are equal, the total intensity $S_0$ is equal to twice the variance of $e_x$, $S_1 = 0$, $S_2 = 0$, and $S_3 = S_0$. This corresponds to positive Stokes $V$, which is contrary to the IAU convention (see van Straten et al. 2010, for more details).

If $e_x = ie_y$, then the phase of $e_x$ leads that of $e_y$ by 90 degrees and the radiation is 100% circularly polarized with the electric field vector tracing a clockwise circle in the $x$-$y$ plane; this is defined by the IEEE as right-hand circularly polarized (RCP). In this case, the variances of $e_x$ and $e_y$ are equal, the total intensity $S_0$ is equal to twice the variance of $e_x$, $S_1 = 0$, $S_2 = 0$, and $S_3 = -S_0$. This corresponds to negative Stokes $V$.

1.2. Arbitrarily Polarized Radiation

It is easy to see in the case of Stokes Q that partially polarized radiation can be produced by an incoherent superposition of orthogonally polarized (and 100% polarized) modes. Incoherent means only that the modes are uncorrelated (i.e. $S_2 = S_3 = 0$) and not necessarily statistically independent. When only one mode is present, $S_1 = \pm S_0$ and the signal is 100% polarized;
when incoherent modes are present with equal power, $S_1 = 0$ and the signal is unpolarized.

In fact, any polarized state can be represented as an incoherent sum of orthogonally polarized states. This can be seen by expressing the coherency matrix as a similarity transformation known as its eigen decomposition,

$$\rho = \mathbf{R} \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \mathbf{R}^{-1}. \quad (16)$$

Here, $\mathbf{R} = (e_0 \ e_1)$ is a $2 \times 2$ matrix with columns equal to the eigenvectors of $\rho$, and $\lambda_m$ are the corresponding eigenvalues, given by $\lambda = (S_0 \pm |\mathbf{S}|)/2 = (1 \pm P)S_0/2$, where $P = |\mathbf{S}|/S_0$ is the degree of polarization ($0 \leq P \leq 1$). If the signal is completely polarized, then $\lambda_1 = 0$ and the degree of polarization $P = 1$. If the signal is unpolarized, then there is a single 2-fold degenerate eigenvalue, $\lambda = S_0/2$; the degree of polarization $P = 0$, and $\mathbf{R}$ is undefined; that is, an unpolarized signal is unpolarized in any basis.

If the eigenvectors are normalized such that $e_k^\dagger e_k = 1$, then equation (16) is equivalent to a congruence transformation by a unitary matrix $(\mathbf{R}^\dagger = \mathbf{R}^{-1})$. In the natural basis defined by $\mathbf{R}^\dagger$, the eigenvalues $\lambda_m$ are equal to the variances of two uncorrelated signals received by orthogonally polarized receptors described by the eigenvectors. The total intensity, $S_0 = \lambda_0 + \lambda_1$; the polarized intensity, $S_1 = |\mathbf{S}| = \lambda_0 - \lambda_1$; and $S_2 = S_3 = 0$. That is, $\mathbf{R}^\dagger$ rotates the basis such that the mean polarization vector points along $S_1$.

The Stokes parameters may be mapped onto a point in the Poincaré sphere by $\mathbf{p} = S/|S|$. In this three-dimensional space, 100% polarized radiation is represented by a point on the surface of the unity-radius Poincaré sphere, partially polarized radiation is represented by a point that lies a distance $|\mathbf{p}| < 1$ from the centre, and unpolarized radiation defines the origin.

Note that for each pair of special cases considered in Sections 1.1.1 through 1.1.3, the Stokes parameters associated with the two orthogonally polarized states have opposite signs. This is generally true and, in the following section, it is shown that orthogonally polarized states have anti-parallel Stokes polarization vectors that occupy antipodal points on the Poincaré sphere.

2. LINEAR TRANSFORMATIONS

In the narrow-band (or quasi-monochromatic) approximation of an electromagnetic wave (see Appendix B), the response of a single receptor is defined by the Jones vector, $\mathbf{r}$, such that the voltage induced in the receptor by the incident electric field is given by the scalar product, $v = \mathbf{r}^\dagger \mathbf{e}$.

The state of polarization to which a receptor maximally responds is completely described by the three components of its associated Stokes polarization vector, $S_k = \mathbf{r}^\dagger \sigma_k \mathbf{r}$. Therefore, it is possible to define a receptor using the spherical coordinates of $\mathbf{S}$ (Chandrasekhar 1960). For example, in the linear basis, the spherical coordinates include the gain, $g = |\mathbf{r}| = |\mathbf{S}|^{1/2}$, the orientation,

$$\theta = \frac{1}{2} \tan^{-1} \frac{S_2}{S_1}, \quad (17)$$

and the ellipticity,

$$\epsilon = \frac{1}{2} \sin^{-1} \frac{S_3}{|\mathbf{S}|}, \quad (18)$$

such that

$$\mathbf{r} = g \begin{pmatrix} \cos \theta \cos \epsilon + i \sin \theta \sin \epsilon \\ \sin \theta \cos \epsilon - i \cos \theta \sin \epsilon \end{pmatrix}. \quad (19)$$

Section 4 describes the rationale for adopting this parameterization.

A dual-receptor feed is represented by the Hermitian transpose of a Jones matrix with columns equal to the Jones vector of each receptor,

$$\begin{pmatrix} \mathbf{r}_0 \mathbf{r}_1 \end{pmatrix}^\dagger = \begin{pmatrix} r^*_{00} & r^*_{01} \\ r^*_{10} & r^*_{11} \end{pmatrix}. \quad (20)$$

The receptors in an ideal feed respond to orthogonal senses of polarization (ie. the scalar product, $\mathbf{r}_0^\dagger \mathbf{r}_1 = 0$) and have identical gains (ie. $\mathbf{r}_0^\dagger \mathbf{r}_0 = \mathbf{r}_1^\dagger \mathbf{r}_1$). In general, any linear transformation of the electric field vector may be represented by

$$\mathbf{e}'(t) = \mathbf{J} \mathbf{e}(t), \quad (21)$$

where $\mathbf{J}$ is a $2 \times 2$ complex-valued Jones matrix.

Substitution of $\mathbf{e}' = \mathbf{J} \mathbf{e}$ into the definition of the coherency matrix yields the congruence transformation,

$$\rho' = \mathbf{J} \rho \mathbf{J}^\dagger, \quad (22)$$

Using Equations (3) and (6), the congruence transformation of the coherency matrix can be expressed as an equivalent linear transformation of the associated Stokes parameters by a real-valued $4 \times 4$ Mueller matrix $\mathbf{M}$, as defined by

$$S'_\mu = M^\nu_\mu S^\nu, \quad (23)$$

where

$$M^\nu_\mu = \frac{1}{2} \text{Tr} [\sigma_\mu \mathbf{J} \sigma_\nu \mathbf{J}^\dagger] \quad (24)$$

or equivalently,

$$M^\nu_\mu = \frac{1}{2} \sigma_\mu \mathbf{J} : \sigma_\nu \mathbf{J}^\dagger. \quad (25)$$
Although there is an unique Mueller matrix for every Jones matrix, the converse is not true. Mueller matrices that do not have an equivalent Jones matrix are known as “impure” or “depolarizing”; e.g. see Section 6 and Appendix E of Hamaker et al. (1996). For more information about depolarizing Mueller matrices, see Lu & Chipman (1996). Other approaches to representing the polarization state and transformations of polarization state are compared and contrasted in Appendix D.

If \( J \) is non-singular, it can be decomposed into the product of a Hermitian matrix and a unitary matrix known as its polar decomposition,

\[
J = J \mathbf{B}_m (\beta) \mathbf{R}_n (\phi), \tag{26}
\]

where \( J = |J|^{1/2} \) and \( |J| \) is the determinant of \( J \); \( \mathbf{B}_m (\beta) \) is positive-definite Hermitian, i.e.

\[
[\mathbf{B}_m (\beta)]^\dagger = \mathbf{B}_m (\beta);
\]

and \( \mathbf{R}_n (\phi) \) is unitary, i.e.

\[
[\mathbf{R}_n (\phi)]^\dagger = [\mathbf{R}_n (\phi)]^{-1}.
\]

Under the congruence transformation of the coherency matrix, the Hermitian matrix

\[
\mathbf{B}_m (\beta) = \sigma_0 \cosh \beta + \hat{\mathbf{m}} \cdot \mathbf{\sigma} \sinh \beta \tag{27}
\]

effects a Lorentz boost of the Stokes four-vector along the \( \hat{\mathbf{m}} \) axis by a hyperbolic angle \( 2\beta \). In the above equation, \( \mathbf{\sigma} \) is a 3-vector whose components are the Pauli spin matrices. As the Lorentz transformation of a spacetime event mixes temporal and spatial dimensions, the polarimetric boost mixes total and polarized intensities, thereby altering the degree of polarization. In contrast, the unitary matrix

\[
\mathbf{R}_n (\phi) = \sigma_0 \cos \phi + i \hat{\mathbf{n}} \cdot \mathbf{\sigma} \sin \phi \tag{28}
\]

rotates the Stokes polarization vector about the \( \hat{\mathbf{n}} \) axis by an angle \( 2\phi \). As the orthogonal transformation of a vector in Euclidean space preserves its length, the polarimetric rotation leaves the degree of polarization unchanged.

Each axis-angle parameterization of \( \mathbf{B}_m (\beta) \) and \( \mathbf{R}_n (\phi) \) has three free parameters: a unit vector that defines the axis of symmetry of the transformation and an angle. Combined with the real and imaginary parts of the complex-valued \( J \), Equation (26) has eight degrees of freedom, as expected for a complex-valued \( 2 \times 2 \) matrix. However, the coherency matrix is insensitive to the absolute phase of \( J \) because \( J \) (on the left) is multiplied by \( J^* \) (on the right) in Equation (2). Therefore, only seven degrees of freedom matter in single-dish polarimetry and \( J \) can be replaced by the real-valued absolute gain \( G = |J| \).

Both \( \mathbf{B}_m (\beta) \) and \( \mathbf{R}_n (\phi) \) are unimodular (i.e. \( |\mathbf{B}_m (\beta)| = 1 \) and \( |\mathbf{R}_n (\phi)| = 1 \); therefore, because \( |AB| = |A||B| \), congruence transformation of the coherency matrix by either \( \mathbf{B}_m (\beta) \) or \( \mathbf{R}_n (\phi) \) preserves the determinant. The determinant of the coherency matrix is therefore an invariant of boost and rotation transformations (but not of scalar multiplication). Using Equation (3) it is easy to show that the Lorentz invariant of a Stokes four-vector is equal to four times the determinant of the coherency matrix; that is,

\[
S^2 \equiv S_0^2 - |S|^2 = 4|\rho| \tag{29}
\]

As with the spacetime null interval, no linear transformation of the electric field can alter the degree of polarization of a completely polarized source. Also, no non-singular transformation can convert partially polarized radiation into 100% polarized radiation.

3. PHYSICAL PROPERTIES

3.1. Hermitian Transformations (Lorentz Boosts)

A Hermitian Jones matrix corresponds to

- a Lorentz boost of the Stokes four-vector;
- the polconversion defined by Hamaker (2000); and
- an optical diattenuator.

Physically, boosts arise from the differential amplification and non-orthogonality of the feed receptors (Britton 2000, and references therein). For a pair of orthogonal receptors with different gains, \( g_0 \) and \( g_1 \), define the orthonormal receptors, \( \hat{\mathbf{r}}_0 = r_0 / g_0 \) and \( \hat{\mathbf{r}}_1 = r_1 / g_1 \), and substitute into equation (20) to yield

\[
J = (r_0 \quad r_1)^\dagger = G \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma^{-1} \end{pmatrix} (\hat{\mathbf{r}}_0 \quad \hat{\mathbf{r}}_1)^\dagger, \tag{30}
\]

where \( G = (g_0 g_1)^\dagger \) is the absolute gain and \( \Gamma = (g_0 / g_1)^\dagger \) parameterizes the differential gain matrix. Equation (30) is a polar decomposition and, by substituting \( \Gamma = \exp(\beta) \), the differential gain matrix may be expressed in the form of equation (27),

\[
\mathbf{B}_m (\beta) = \begin{pmatrix} e^\beta & 0 \\ 0 & e^{-\beta} \end{pmatrix} = \sigma_0 \cosh \beta + \sigma_1 \sinh \beta, \tag{31}
\]

where \( \hat{\mathbf{m}} = (1, 0, 0) \) and

\[
\beta = \frac{1}{2} \ln \frac{g_0}{g_1}. \tag{32}
\]

\(^1\) There is an error following equation (14) in Britton (2000), where it should read \( \beta = \ln(g_0 / g_1) / 2. \)
In the case of linearly polarized receptors, the \( \vec{m} \) axis lies in the Stokes \( Q-U \) plane; for circularly polarized receptors, \( \vec{m} \) corresponds to \( \pm \) Stokes \( V \). To first order, \( \beta = \gamma/2 \), where \( \gamma = g_0/g_1 - 1 \) is the differential gain ratio.

For a pair of non-orthogonal receptors, first consider the spherical coordinate system introduced in Equation (19). The orientations and ellipticities of orthogonal receptors satisfy \( \theta_0 - \theta_1 = \pm \pi/2 \) and \( \epsilon_0 = -\epsilon_1 \). If \( \delta_\theta \) and \( \delta_\epsilon \) parameterize the departure from orthogonality in each of these angles, such that \( \theta'_0 = \theta_0 + \delta_\theta \) and \( \epsilon'_0 = \epsilon_0 + \delta_\epsilon \), then

\[
|r_{0}^\dagger r_1| = g_0 g_1 (\sin \delta_\theta \cos (\delta_\epsilon + 2\epsilon_0) + i \cos \delta_\theta \sin \delta_\epsilon). \quad (33)
\]

However, this description is specific to linearly polarized receptors. In the circular basis, \( \epsilon_0 = \pi/4 \), and the non-orthogonality is completely described by \( \delta_\epsilon \); \( \delta_\theta \) becomes degenerate with the differential phase of the receptors. As such a degeneracy exists at the poles of any spherical coordinate system, a basis-independent parameterization of non-orthogonality is sought.

To this end, it proves useful to consider the geometric relationship between the Stokes polarization vector of each receptor. In particular, the scalar product,

\[
S_{0\cdot} S_{1} = 2|r_{0}^\dagger r_1|^2 - |r_0|^2 |r_1|^2, \quad (34)
\]

shows that orthogonally polarized receptors have anti-parallel Stokes polarization vectors \( (S_{0\cdot} S_{1} = -|S_0||S_1|) \). Furthermore, where \( \Theta \) is the angle between \( S_0 \) and \( S_1 \), the angle \( \delta = (\pi - \Theta)/2 \) parameterizes the magnitude of the receptor non-orthogonality, such that

\[
\sin \delta = \frac{|r_{0}^\dagger r_1|}{|r_0||r_1|}. \quad (35)
\]

It is much simpler to relate \( \delta \) to the boost transformation that results from non-orthogonal receptors. To determine the boost component of an arbitrary matrix, \( \mathbf{J} \), the polar decomposition (eq. [26]) is multiplied by its Hermitian transpose to yield

\[
|\mathbf{J}^\dagger| = |\det \mathbf{J}| \mathbf{B}_m(2\beta) = |\det \mathbf{J}| \mathbf{B}_{\vec{m}}(2\beta). \quad (36)
\]

For a pair of receptors with gain, \( g \), substitution of equation (20) into equation (36) yields

\[
|\mathbf{J}^\dagger| = \begin{pmatrix} g^2 & W \\ W^* & g^2 \end{pmatrix}, \quad (37)
\]

where \( W = r_{0}^\dagger r_1 \). Substitute \( W = g^2 e^{-i\Phi} \tanh 2\beta \), so that \( |\det \mathbf{J}| = |\det (\mathbf{J}^\dagger)|^\dagger = (g^4 - |W|^2)^\dagger = g^2 \text{sech} 2\beta \), and

\[
\mathbf{B}_{\vec{m}}(2\beta) = \frac{|\mathbf{J}^\dagger|}{|\det \mathbf{J}|} = \sigma_0 \cosh 2\beta + \vec{m} \cdot \sigma \sinh 2\beta, \quad (38)
\]

where \( \vec{m} = (0, \cos \Phi, \sin \Phi) \) and

\[
\beta = \frac{1}{2} \tanh^{-1} \frac{|r_{0}^\dagger r_1|}{g^2}. \quad (39)
\]

To first order, \( \beta \sim \delta/2 \) (see eq. [35]), which is consistent with the approximation in equation (19) of Britton (2000).

Due to the combined effects of differential gain and receptor non-orthogonality, the boost axis, \( \vec{m} \), can have an arbitrary orientation. For example, to first order in the polar coordinate system best-suited to the linear basis, \( \vec{m} \propto (\gamma, \delta_\theta, \delta_\epsilon) \). Furthermore, the instrumental boost can vary as a function of both time and frequency for a variety of reasons. For example, the parallactic rotation of the receiver feed during transit of the source changes the orientation of \( \vec{m} \) with respect to the equatorial coordinate system. Also, to keep the signal power within operating limits, some instruments employ active attenuators that introduce differential gain fluctuations on short timescales. Furthermore, the mismatched responses of the filters used in downconversion typically lead to variation of \( \gamma \) as a function of frequency.

3.2. Unitary Transformations (Euclidean Rotations)

A unitary Jones matrix corresponds to

- a Euclidean rotation of the Stokes polarization three-vector;
- the polrotation defined by Hamaker (2000); and
- an optical retarder.

Unitary transformations arise from things like rotations, differential phase delays, and symmetric cross-coupling of the receptors. For example,

- the observatory rotates with respect to a coordinate system that is fixed on the sky,
- birefringence in the magnetized interstellar medium causes a phase delay between LCP and RCP, observed as frequency-dependent Faraday rotation of the plane of linear polarization
- the signal paths through which the orthogonal polarizations propagate through the instrument have different path lengths

4. CRITERIA FOR MODEL SELECTION

In the published literature, many different approaches to modeling the instrumental response have been introduced and adopted. Choosing a suitable model can be guided by the following criteria, which are described in more detail in the following sub-sections. The selected model should be
1. complete, such that the parameter space spans all possible transformations;

2. self-consistent, such that fundamental properties of the system are preserved; and

3. numerically stable, at least in the vicinity of the anticipated solution.

4.1. Completeness

When considering only linear transformations of the electric field, as represented using Jones matrices, there are 7 degrees of freedom (dof) that must be modelled. If the absolute gain is treated as a scalar multiplier, then the matrix component of the transformation must be described by 6 parameters. In the language of the framework developed by Britton (2000), these 6 dof must describe the three boosts that mix total intensity and Stokes Q, U, and V (3 dof) and the three rotations about the Stokes Q, U, and V axes (3 dof).

Equation (22) of Heiles et al. (2001) describes 5 of these 6 dof:

1. the differential gain, \( \Delta G \) (Eq. 20) describes the mixing between Stokes I and Q;

2. the cross-coupling amplitude \( \epsilon \) and phase \( \phi \) (Eq. 18) describe the mixing of Stokes I with U;

3. and Stokes I with V;

4. the differential phase, \( \Psi \) (Eq. 20) describes the rotation about the Stokes Q axis; and

5. the coupling amplitude \( \alpha \) (Eq. 15) describes the rotation about the Stokes U axis.

The rotation about the Stokes V axis (which is missing in the above list) is equivalent to physically rotating the receiver about the line of sight. As noted in Section 3.4 of Heiles et al. (2001), this rotation is “impossible to measure without calibration sources whose position angles are accurately known.” The same conclusion is reached in Appendix B of van Straten (2004).

When modelling variations of the observed Stokes parameters as a function of parallactic angle, care must also be taken regarding the degenerate mixing between Stokes I and V, as described in Appendix B of van Straten (2004) and repeated in Appendix C of this paper. In Section 5 of Heiles et al. (2001), this degeneracy is eliminated by assuming that Stokes V of the calibrator source is zero. This assumption is typically invalid when the calibrator source is a pulsar; therefore, observations of other sources with known circular polarization must be incorporated to constrain the instrumental mixing between I and V (e.g. Liao et al. 2016).

The assumption that Stokes V is zero also leads to the \((\alpha, \psi)\) ambiguity described in section 5.1 of Heiles et al. (2001). Note that \( \alpha \) is defined in Equation (15) as the rotation of Q into V (about the Stokes U axis) and \( \psi \) is defined in Equation (20) as the rotation of U into V (about the Stokes Q axis). When comparing the solutions \((\alpha_1, \psi_1) = (0, \psi_0)\) and \((\alpha_2, \psi_2) = (90, \psi_0 + 180)\), it appears that only the signs of Q and U have been changed. However, the switch also changes the sign of Stokes V, which goes unnoticed because it is assumed to be zero.

4.2. Self-consistency

Many treatments begin with a description of linear transformations of the electric field, as represented by Jones matrices. In this case, the resulting Mueller matrix should be pure. However, to simplify the products of these Mueller matrices, some authors introduce a number of small-value approximations. These small-value approximations typically result in an impure Mueller matrix that is inconsistent with the assumed linear response to the electric field.

For example, after assuming that the differential gain \( \Delta G \) is small, Heiles et al. (2001) arrive at Equation 20,

\[
M_A = \begin{pmatrix}
1 & \Delta G/2 & 0 & 0 \\
\Delta G/2 & 1 & 0 & 0 \\
0 & 0 & \cos \psi & -\sin \psi \\
0 & 0 & \sin \psi & \cos \psi
\end{pmatrix}.
\]

To see that the above transformation is impure, consider a signal that is 100% polarized such that the polarization vector lies in the U–V plane, \( I' = U^2 + V^2 \), and \( Q = 0 \). After transformation by the above Mueller matrix, forming \( S' = M_A S \), the total intensity is unmodified (\( I' = I \)), the resulting Stokes \( Q' = \Delta G/2 \) and, because rotation about the Stokes Q axis by \( \Psi \) preserves length in the U–V plane, \( U'^2 + V'^2 = U^2 + V^2 = I' \).

Therefore, the degree of polarization of the transformed signal

\[
P' = \frac{\sqrt{Q'^2 + U'^2 + V'^2}}{I'} = \sqrt{1 + \frac{\Delta G^2}{4}} > 1.
\]

As no linear transformation of the electric field vector can produce over-polarization, \( M_A \) must be impure. This example demonstrates the pitfalls that can be encountered when parameterizing the response of an antenna using Mueller matrices and small-value approximations.

4.3. Numerical Stability

The parameterization of the instrumental response to polarized radiation introduced in Appendix II of
Conway & Kronberg (1969) forms the basis of the approach applied in several subsequent papers on radio polarization; e.g. Equation (A2) of Stinebring et al. (1984) and Equation (16) of Heiles et al. (2001). This parameterization is unstable in the vicinity of the ideal solution, which makes this model unsuitable for use with methods of parameter estimation such as the Levenberg-Marquardt algorithm for non-linear least-squares minimization.

For example, consider Equation (16) of Heiles et al. (2001)

$$e' = \begin{pmatrix} 1 & \epsilon_1 e^{i\phi_1} \\ \epsilon_2 e^{-i\phi_2} & 1 \end{pmatrix} e,$$

(42)

which describes the undesirable cross coupling between the two receptors of an imperfect feed. When attempting to jointly determine the model parameters $\phi_k$ and $\epsilon_k$, where $k \in \{1, 2\}$ (e.g. using a least-squares fit to experimental data) this parameterization can lead to instability because the phase angle that describes the cross coupling $\phi_k$ becomes undetermined as the amount of cross coupling $\epsilon_k$ approaches zero. More formally, the partial derivative of the model with respect to $\phi_k$ approaches zero as $\epsilon_k$ approaches zero, causing the curvature (or Hessian) matrix used in techniques like Levenberg-Marquardt to become ill-conditioned (i.e. inversion is prone to large numerical errors) or even singular (i.e. non-invertible). This problem persists in Equation (18) of Heiles et al. (2001), where $\phi$ is poorly constrained when $\epsilon$ is small. A similar instability appears in Eq. (10) of Heiles et al. (2001); however, it is assumed (in Section 3.3) that $\chi = \pi/2$.

In Equation (42), the numerical instability of the model arises in the vicinity of a nearly ideal response (i.e. in the neighbourhood of $\epsilon_k = 0$). In contrast, Equation (19) (cf. Eq. (15) of Britton 2000) becomes unstable only at the poles, where $\epsilon = \pm \pi/4$, $\cos \epsilon = \pm \sin \epsilon$ and the orientation $\theta$ becomes degenerate with either the absolute phase of the signal (which is lost during detection) or the differential phase (when considering a pair of receptors with opposite ellipticities). However, because this parameterization is chosen for nominally linearly polarized receptors, this region of instability lies well away from the region in which the anticipated solution lies.

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APPENDIX

A. THE ANALYTIC SIGNAL

The voltage signal from each receptor is a real-valued function of time, or process, that may be represented by its associated analytic signal. The analytic signal, also known as Gabor’s complex signal, is a complex-valued representation of a real-valued process that provides its instantaneous amplitude and phase. In order to define the analytic signal associated with a process, \( x(t) \), it is first necessary to define the Hilbert transform (Papoulis 1965),

\[
\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} \, d\tau.
\]

The discontinuity in \( \hat{x}(t) \) at \( t = \tau \) is avoided by taking the Cauchy principal value. The analytic signal associated with \( x(t) \) is then defined by

\[
z(t) = x(t) + i\hat{x}(t).
\]

As it is derived from the real-valued process, the analytic signal does not contain any additional information. However, the analytic signals associated with two orthogonal senses of polarization, \( e_0(t) \) and \( e_1(t) \), permit calculation of the coherency matrix. The analytic signal therefore proves to be a useful representation in radio polarimetric studies.

A.1. The Quadrature Filter

The Hilbert transformation of Equation (A1) may also be written as the convolution, \( \hat{x}(t) = h(t) * x(t) \), where

\[
h(t) = \frac{1}{\pi t},
\]

and the * symbol is used to represent the convolution operation,

\[
h(t) * x(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau.
\]

By the convolution theorem, this transformation is equivalent to \( \hat{X}(\nu) = H(\nu)X(\nu) \), where

\[
H(\nu) = \begin{cases} 
-i & \nu > 0 \\
i & \nu < 0 
\end{cases}
\]

is the Fourier transform of \( h(t) \), known as the quadrature filter (Papoulis 1965). Referring to the other commonly used functions and their Fourier transforms in Table 1, it is trivial to show that the Hilbert transform of \( \cos(\nu_0 t) \) is equal to \( \sin(\nu_0 t) \), enabling the quadrature filter to be understood as a 90° phase shifter. Using the quadrature filter, it can also be shown that the Fourier transform of the analytic signal, \( Z(\nu) \), is equal to zero for \( \nu \) less than zero:

\[
Z(\nu) = X(\nu) + i\hat{X}(\nu) = X(\nu) + iH(\nu)X(\nu) = \begin{cases} 
2X(\nu) & \nu > 0 \\
0 & \nu < 0 
\end{cases}
\]

Conversely, the analytic signal associated with \( x(t) \) may be produced by suppression of the negative frequencies in \( X(\nu) \).

B. NARROW-BAND APPROXIMATION TO POLARIMETRIC TRANSFORMATIONS

Regardless of feed design, the electric field component of the radiation impinging on the receiver feed horn induces a voltage in each of the receptors and these voltages are propagated through separate signal paths. Each voltage signal therefore experiences a different series of amplification, attenuation, mixing, and filtering before sampling or detection.
is performed. Whereas efforts are made to match the components of the observatory equipment, each will realistically have a unique frequency response to the input signal. Even a simple mismatch in signal path length will result in a relative phase difference between the two polarizations that varies linearly with frequency.

In fact, any physically realizable system will transform the radiation in a manner that depends on frequency. Where variations across the smallest bandwidth available may be considered negligible, post-detection calibration and correction techniques may be used to invert the transformation and recover the original polarimetric state. However, the transformation may vary significantly across the band, causing the polarization vector to combine destructively when integrated in frequency. This phenomenon is known as “bandwidth depolarization” of the signal, and results in irreversible decimation of the degree of polarization. It is therefore desirable to perform polarimetric corrections at sufficiently high spectral resolution.

Consider a linear system with impulse response, \( j(t) \). Presented with an input signal, \( e(t) \), the output of this system is given by the convolution, \( e'(t) = j(t) * e(t) \). In the two-dimensional case, each output signal is given by a linear combination of the input signals,

\[
\begin{align*}
e'_1(t) &= j_{11}(t) * e_1(t) + j_{12}(t) * e_2(t), \\
e'_2(t) &= j_{21}(t) * e_1(t) + j_{22}(t) * e_2(t).
\end{align*}
\]

By defining the analytic vector, \( e(t) \), with elements \( e_1(t) \) and \( e_2(t) \), and the \( 2 \times 2 \) impulse response matrix, \( j(t) \), with elements \( j_{mn}(t) \), we may express the propagation of a transverse electromagnetic wave by the matrix equation,

\[
e'(t) = j(t) * e(t).
\]

By the convolution theorem, Equation (B7) is equivalent to

\[
E'(\nu) = J(\nu)E(\nu),
\]

where \( J(\nu) \) is the frequency response matrix with elements \( J_{mn}(\nu) \), and \( E(\nu) \) is the vector spectrum. In the case of monochromatic light, or under the assumption that \( J(\nu) \) is constant over all frequencies, matrix convolution reduces to simple matrix multiplication in the time domain, as traditionally represented using the Jones matrix. However, because these conditions are not physically realizable, the Jones matrix finds its most meaningful interpretation in the frequency domain.

The average auto- and cross-power spectra are summarized by the average power spectrum matrix, defined by the vector direct product, \( \mathbf{P}(\nu) = \langle \mathbf{E}(\nu) \otimes \mathbf{E}^{\dagger}(\nu) \rangle \), where \( \mathbf{E}^{\dagger} \) is the Hermitian transpose of \( \mathbf{E} \) and the angular brackets denote time averaging. More explicitly:

\[
\mathbf{P}(\nu) = \begin{pmatrix}
\langle E_1(\nu)E_1^{\ast}(\nu) \rangle & \langle E_1(\nu)E_2^{\ast}(\nu) \rangle \\
\langle E_2(\nu)E_1^{\ast}(\nu) \rangle & \langle E_2(\nu)E_2^{\ast}(\nu) \rangle
\end{pmatrix}.
\]

| \( \pi(t) \) | \( X(\nu) \) |
|----------------|----------------|
| \( \cos(2\pi\nu_0 t) \) | \( \frac{1}{2}(\delta(\nu + \nu_0) + \delta(\nu - \nu_0)) \) |
| \( \sin(2\pi\nu_0 t) \) | \( \frac{i}{2}(\delta(\nu + \nu_0) - \delta(\nu - \nu_0)) \) |
| \( h(t) = (\pi t)^{-1} \) | \( H(\nu) = \begin{cases} -i & \nu > 0 \\ i & \nu < 0 \end{cases} \) |
| \( \pi(t) = \text{sinc}(\pi \Delta \nu t) \) | \( \Pi(\nu/\Delta \nu) = \begin{cases} 0 & |\nu/\Delta \nu| > 1/2 \\ 1/2 & |\nu/\Delta \nu| = 1/2 \\ 1 & |\nu/\Delta \nu| < 1/2 \end{cases} \) |

Table 1. Useful Fourier Transform pairs. The left column lists functions of time. In the right column, the corresponding Fourier transform is given as a function of oscillation frequency, \( \nu \). The filters, \( H(\nu) \) and \( \Pi(\nu) \), are known as the quadrature and rectangle functions, respectively.
Each component of the average power spectrum matrix, \( \overline{P}_{mn}(\nu) \), is the Fourier transform pair of the average correlation function, \( \overline{\rho}_{mn}(\tau) \) (Papoulis 1965). Therefore, \( \overline{P}(\nu) \) may be related to the commonly used coherency matrix,

\[
\rho = (E(t) \otimes E^\dagger(t)) = \overline{\rho}(0) = \frac{1}{2\pi} \int_{\nu_0-\Delta\nu}^{\nu_0+\Delta\nu} \overline{P}(\nu) d\nu,
\]

where \( \nu_0 \) is the centre frequency and \( 2\Delta\nu \) is the bandwidth of the observation. The average power spectrum matrix may therefore be interpreted as the coherency spectral density matrix and, in the narrow band limit \( \Delta\nu \to 0 \), \( \rho = \overline{P}(\nu_0)/2\pi \).

Using Equations B8 and B9 it is easily shown that a two-dimensional linear system transforms the average power spectrum as

\[
\overline{P}'(\nu) = J(\nu) \overline{P}(\nu) J^\dagger(\nu).
\]

This matrix equation is a congruence transformation, and forms the basis on which the frequency response of the system will be related to the input (source) and output (measured) coherency spectrum. For brevity in this paper, all symbolic values are assumed to be a function of frequency, \( \nu \).

C. DEGENERACY UNDER COMMUTATION REVISITED

Using 2 \times 2 Jones matrices and coherency matrices, Appendix B of van Straten (2004) proves that no unique solution to the polarization measurement equation can be derived from observations of only unknown sources at multiple parallactic angles. Here, the proof is repeated using Mueller matrices and Stokes parameters. Consider an observation of a pulsar that consists of Stokes parameters measured as a function of pulse phase,

\[
S'(\phi) = \begin{pmatrix}
I'(\phi) \\
Q'(\phi) \\
U'(\phi) \\
V'(\phi)
\end{pmatrix}
\]

The measured Stokes parameters are related to the unknown pulsar-intrinsic Stokes parameters \( S(\phi) \) by a Mueller matrix \( M \) that describes the unknown instrumental response. If the receiver also rotates with respect to the sky, this transformation can be written as

\[
S'(\phi) = MR(\Phi)S(\phi)
\]

where \( R(\Phi) \) represents a rotation about the line of sight by the parallactic angle \( \Phi \). Given \( M \) and \( S(\phi) \) that satisfy the above equation for all \( \Phi \) and \( \phi \), it is possible to define a family of solutions, \( M_u = MU^{-1} \) and \( S_u(\phi) = US(\phi) \), where \( U \) is any matrix that commutes freely with \( R(\Phi) \) for all values of \( \Phi \); i.e. \( R(\Phi)U = UR(\Phi) \). In this case

\[
\begin{align*}
S'(\phi) &= M_u R(\Phi)S_u(\phi) \\
S'(\phi) &= MU^{-1} R(\Phi)US(\phi) \\
S'(\phi) &= MU^{-1} UR(\Phi)S(\phi) \\
S'(\phi) &= MR(\Phi)S(\phi)
\end{align*}
\]

Mueller matrices that commute with \( R(\Phi) \) include transformations that

- rotate the Stokes polarization vector about the Stokes V axis (in the Q-U plane); and
- mix Stokes I and Stokes V, such as a Lorentz boost (Britton 2000) or polarizance transformation (Lu & Chipman 1996) along the Stokes V axis.

Owing to this degeneracy, there is no unique solution and other constraints or assumptions must be introduced to constrain the above two degrees of freedom.

D. ALTERNATIVE REPRESENTATIONS OF POLARIZATION STATE AND TRANSFORMATIONS

The mathematical representations of polarization state and transformations of polarization state adopted in this work have equivalent substance but slightly different form to the mathematical description of polarization and transformations presented by Hamaker et al. (1996). Table 2 summarizes the key differences in formal notation.
Table 2. Mapping between the representations of polarization state and transformations of polarization state adopted in this work and those adopted by Hamaker et al. (1996).

| Property                                      | This work                                      | Hamaker et al. (1996)                   |
|-----------------------------------------------|-----------------------------------------------|----------------------------------------|
| polarization state                           | coherency matrix, Equation (2)                | coherency vector, Equation (3)         |
| corresponding Stokes parameters              | via Pauli matrices, Equations (3) and (8)    | via coordinate transformation, Equation (8) |
| transformations                               | congruence transformation, Equation (22)     | matrix operation, Equation (6)         |
| corresponding Mueller matrix                 | via Pauli matrices, Equation (24)            | congruence transformation, Equation (7) |

In Section 3.2 of Hamaker et al. (1996), the four-dimensional coherency vector is transformed by a $4 \times 4$ matrix operator given by the Kronecker product of $2 \times 2$ Jones matrices. In Section 3.4 of Hamaker et al. (1996), the coherency vector is related to the Stokes parameters using a $4 \times 4$ coordinate transformation, and this same coordinate transformation is used to compute the Mueller matrix associated with a given $4 \times 4$ matrix operator (e.g. as in Eq. [18]). Hamaker et al. (1996) focus on synthesis imaging and therefore two different Jones matrices $J_A$ and $J_B$ enter into the Kronecker product in Equation (6) of their work; these are the Jones matrices for each member of the pair of antennas that are cross-correlated to form the coherency vector of visibilities. The notation adopted in this work can be easily adapted to represent cross-correlations between antennas. In this case, the cross-coherency matrix

$$\rho_c \equiv \langle e_a \otimes e_b^\dagger \rangle = \begin{pmatrix}
\langle e_{a,0} e_{b,0}^* \rangle & \langle e_{a,0} e_{b,1}^* \rangle \\
\langle e_{a,1} e_{b,0}^* \rangle & \langle e_{a,1} e_{b,1}^* \rangle
\end{pmatrix}$$

(D18)

is no longer Hermitian; the cross-Stokes parameters

$$S_{c,\mu} = \sigma_\mu : \rho_c$$

(D19)

are complex-valued; and the cross-congruence transformation

$$\rho' = J_a \rho_c J_b^\dagger,$$

(D20)

yields a complex-valued cross-Mueller matrix

$$M_{\nu} = \frac{1}{2} \sigma_\mu J_a : \sigma_\nu J_b^\dagger.$$

(D21)