Singularities, initial and boundary problems of the Tolman-Bondi model

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Abstract

Boundary problem for Tolman-Bondi model is formulated. One-to-one correspondence between singularities hypersurfaces and initial conditions of the Tolman-Bondi model is constructed.

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1 Introduction

Two theorems about asymptotic behavior of the Tolman-Bondi model have been proved by Olson and Silk (1979) with the assumption of no bang-time variation:

\[ t_0(r) = \text{constant}, \]  

where \( r \) is a co-moving coordinate. A general Tolman-Bondi model has been completely determined by the two basic functions: \( t_0(r) \) and full specific energy \( E_0(r) \). The formal sense of the functions \( t_0(r) \) and \( E_0(r) \) is the extra conditions of the Tolman-Bondi model. In the paper (Olson and Silk) the equation (1) determines the function \( t_0(r) \).

In the paper (Gromov 1997) the Tolman-Bondi model is formulated as the Cauchy problem for the equation of motion where initial conditions of the model are initial density \( \rho_0(r) \) and energy \( E_0(r) \) profiles.

The present paper is dedicated to generalisation of Olson and Silk’s idea (use the bang-time function as extra condition) in the frame of the approach (Gromov 1997). It is shown that the Tolman-Bondi model admits two different formal statements: initial and boundary problems. On this ground one-to-one correspondence between two singularities of density and two initial conditions of the Tolman-Bondi model is constructed.

2 Cauchy problem for the Tolman-Bondi model

In this section the Tolman-Bondi model (using \( c = 1 \) and \( G = 1 \)) is represented in the same way as in the paper (Gromov 1997). The co-moving coordinate ‘invariant mass’ \( M_i \) is used here. Using the \( M_i \) coordinate the Tolman-Bondi model is represented as the Cauchy problem for the equation of motion:

\[ 2 \dot{R}(M_i, t)R(M_i, t) + \dot{R}^2(M_i, t) + 1 - f^2(M_i) = 0, \]

\[ \dot{R} = \frac{\partial}{\partial \tau}, \]  

with initial conditions:

\[ R(M_i, 0) = R_0(M_i) = \left[ \frac{3}{4 \pi} \int_0^{M_i} \frac{f(x)}{\rho(x, 0)} \, dx \right]^{1/3}, \]
\[ \dot{R}(M_i, 0) = \dot{R}_0(M_i), \]  
\[ (4) \]

where \( R(M_i, t) \) is the Euler radial coordinate of the particle; \( \rho_0(M_i, t) \) is the density; \( \dot{R}(M_i, t) \) is the velocity; \( \rho_0(M_i, t) \) and \( \dot{R}(M_i, t) \) are given functions.

The function \( f(M_i) \) is defined by the equation

\[
\frac{1}{2} \left( \frac{3}{4 \pi} \right)^{1/3} \left( \int_0^{M_i} f(x) \frac{dx}{\rho_0(x)} \right)^{1/3} \left[ \dot{R}_0^2(M_i) - f^2(M_i) + 1 \right] = \int_0^{M_i} f(x) \, dx. \tag{5}
\]

The general solution of the equation (2) in the frame of approach (Gromov 1997) is:

\[
\pm t + t_0(M_i) = \int_0^{R(M_i, t)} \frac{d\tilde{R}}{\sqrt{f^2(M_i) - 1 + \frac{2}{\tilde{R}} \int_0^{M_i} f(\tilde{M}_i) d\tilde{M}_i}} \tag{6}
\]

where

\[
t_0(M_i) = \int_0^{R_0(M_i)} \frac{d\tilde{R}}{\sqrt{f^2(M_i) - 1 + \frac{2}{\tilde{R}} \int_0^{M_i} f(\tilde{M}_i) d\tilde{M}_i}}. \tag{7}
\]

Function \( t_0(M_i) \) is the bang-time function which has been used by Olson and Silk (1979).

3 Boundary problem for the Tolman-Bondi model

In this section the Tolman-Bondi model is studied as the boundary problem for the equation of motion.

The interval of the Tolman-Bondi space-time is:

\[
ds^2(M_i, t) = dt^2 - \frac{R^2(M_i, t)}{f^2(M_i)} \, dr^2 - R^2(M_i, t) \, d\Omega^2, \tag{8}
\]
where
\[ \dot{\theta} = \frac{\partial}{\partial M_i}, \quad d\Omega = d\theta^2 + \sin^2\theta d\phi, \] (9)
so that for metric coefficients we obtain:
\[ g_{11}(M_i, t) = \left( \frac{R'(r, t)}{f(r)} \right)^2, \] (10)
\[ g_{22}(M_i, t) = g_{33}(M_i, t) = R^2(M_i, t). \] (11)
The density is given by the equation
\[ 4\pi \rho(M_i, t) = \frac{f(M_i)}{R^2(r, t)} \frac{\partial R(M_i, t)}{\partial M_i}. \] (12)
Formula (12) becomes identity on the set of initial conditions (3). Two singularities of density
\[ \frac{\partial R(M_i, t_2)}{\partial M_i} = 0. \] (13)
and
\[ R(M_i, t_1) = 0 \] (14)
corresponds to two hypersurfaces in the space-time where metric coefficients are equal to zero:
\[ g_{11}(M_i, t) = 0, \] (15)
\[ g_{22}(M_i, t) = g_{33}(M_i, t) = 0 \] (16)
and metric is singular:
\[ g = 0. \] (17)
Let us name these hypersurfaces 'singular hypersurfaces'. Following Olson and Silk (1979) let us name also function the \( t_2(M_i) \) from (13) 'second bang-time function'. Equation (14) gives one of the boundary conditions for the equation of motion. To transform equation (13) to the form-like (14), let
us use the general solution of the Tolmn-Bondi model (6). First we rewrite the solution (6) in the form

\[ \pm t + t_0(M_i) \sqrt{\int_0^{M_i} f(\tilde{M}_i)d\tilde{M}_i} = \int_0^{R(M_i,t)} \frac{d\tilde{R}}{\sqrt{\frac{f^2(M_i) - 1}{\int_0^{M_i} f(\tilde{M}_i)d\tilde{M}_i} + \frac{2}{\tilde{R}}}} \] (18)

This equation defines the function \( R(M_i,t) \) implicitly. After differentiation \( \frac{\partial}{\partial M_i} \) and using equality (13) we obtain the following equation for \( R(M_i,t) \):

\[ \pm \frac{t_2 R(M_i,t_2)}{2 \sqrt{\int_0^{M_i} f d\tilde{M}_i}} + \frac{R_0'(M_i,t_2)}{\sqrt{\frac{f^2(M_i) - 1}{\int_0^{M_i} f(\tilde{M}_i)d\tilde{M}_i} + \frac{2}{R_0}}} = \frac{R'(M_i,t_2)}{\sqrt{\frac{f^2(M_i) - 1}{\int_0^{M_i} f(\tilde{M}_i)d\tilde{M}_i} + \frac{2}{R(M_i,t_2)}}} = 0 \] (19)

at time \( t = t_2 \), where \( t_2 \) is time when equation (13) is satisfied.

Let us study the flat TB model: \( f(M_i) = 1 \). Equation (14) corresponds to

\[ t_1(M_i) = t_0(M_i) \] (20)

and (13) corresponds to

\[ t_2(M_i) = \sqrt{2 M_i R_0(M_i) R_0'(M_i)}, \] (21)

that coincides with the results of paper (Gromov 1997).

\( R(M_i,t_1) = 0 \) and \( R(M_i,t_2) \) from (19) have now the sense of boundary conditions for the equation of motion.
4 Results and discussion

The TB model is reduced to the ordinary differential equation of motion (Gromov 1997) with extra conditions (initial or boundary) and completely defined by two functions of the set

\[ t_1(M_i), t_2(M_i), \rho_0(M_i), f(M_i), R_0(M_i), \dot{R}_0(M_i), \]

but not every two of them can be used simultaneously. Every two functions from the set

\[ \rho_0(M_i), f(M_i), R_0(M_i), \dot{R}_0(M_i) \]

produce Cauchy problem, but two bang-time functions

\[ t_1(M_i), t_2(M_i) \]

produce boundary problem for the equation of motion.

In accordance with the theorem about a correspondence between initial and boundary problems (Korn G.A. and Korn T.M, 1968) we can transform boundary problem into initial one, or initial problem into boundary one. This is supported by the formulas (19) and (7): if \( t_1(M_i) \) and \( t_2(M_i) \) are specified, we can calculate \( \rho_0(M_i) \) and \( f(M_i) \) and leave initial from boundary problem to the initial one. So, it is not possible to neglect one of two singularities of the TB model because it breaks the formal statement of the problem. We see also that two bang-time functions define two singular hypersurfaces of the TB model. Formulas (19) and (7) produce one-to-one correspondence between two set of functions (23) and (24). So, from the theorem it follows the implication:

One-to-one correspondence between singularities hypersurfaces and initial conditions of the Tolman-Bondi model exists.

The Tolman-Bondi model has spherical symmetry. This leads to the equality

\[ g_{22}(M_i, t) = g_{33}(M_i, t), \]

which corresponds to singularity \( R(M_i, t_1) = 0 \). One of the possible generalisations of the Tolman-Bondi model is small perturbation of the spherical symmetry, that provides the break down of the conditions (23):

\[ g_{22}(M_i, t) \neq g_{33}(M_i, t), \]
so one singularity \( R(M_i, t_1) = 0 \) will be split by the two different singularities like this:

\[
g_{22}(M_i, t) = 0, \quad (27)
\]

and

\[
g_{33}(M_i, t) = 0. \quad (28)
\]

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REFERENCES

Bondi H. (1947) MNRAS, 107, 410.
Gromov A. (1997) gr-qc/9612038.
Olson D. and Silk J. (1979) AphJ, 233, 395.
Tolman R.C. (1934) Proc.Nat.Acad.Sci (Wash), 20, 169.
Korn G. and Korn T. (1968) Mathematical Handbook.