Geometric Description of Schrödinger Equation in Finsler and Funk Geometry

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For a system of $n$ non-relativistic spinless bosons, we show by using a set of suitable matching conditions that the quantum equations in the pilot-wave limit can be translated into a geometric language for a Finslerian manifold. We further link these equations to Euclidean timelike relative Funk geometry and show that the two different metrics in both of these geometric frameworks lead to the same coupling.

PACS numbers:

I. INTRODUCTION

Geometric theories, in particular the link between these theories and quantum mechanics is a very active field of research. Spacetimes equipped with Lorentzian metric have been serving as geometrical framework to discuss the dynamics of the particles for over hundred years. However, [1, 2] suggest that modifying gravity at times could be helpful for a more comprehensive depiction of nature. Several modifications of GR have been discussed in [3–5]. In particular, Finsler geometry is a natural generalization of the Lorentzian metric geometry. There are several evidences supporting this idea because this extension of GR maintains a simultaneous geometrization of gravity, observer and causality, e.g., see [6] and references therein.

There are also several generalizations of the Finsler geometry, i.e., Euclidean timelike relative Funk geometry, hyperbolic timelike Funk geometry, timelike relative spherical Funk geometry etc. These theories of timelike spaces were introduced in [7]. It was shown in [8] that these timelike spaces are merely timelike Finsler spaces. This approach entails that the metric of such a space is determined by timelike norm on each tangent space in such a way that the distance between two points is infimum of length of path joining them and the length is defined between events using a timelike distance function.

On the other hand, quantum mechanics is one of the most profound and successful (algebraic and linear) mathematical models which helps describe the physical reality around us. However, the quantum phenomenon is obscure and unpredictable because of their probabilistic nature. Classical mechanics however is predictable, geometric, and nonlinear. Geometric in the sense that phenomenon described by classical mechanics is usually defined over a symplectic manifold. The quest to unify these two seemingly different approaches to reality is thus natural and is open for a long time and there has been a lot of effort in this direction.

In order to reformulate quantum mechanics in geometrical terms, one needs to associate physical reality to objects and to define the background space. The well-known Copenhagen interpretation suggests that one can associate physical reality with a particle only after the measurement, before the measurement it is meaningless to talk about physical properties of a particle due to the wavefunction collapse upon measurement.

Although different interpretations provide different groundwork to study quantum mechanics, a particularly nice approach is the pilot-wave theory, which provides an extensive framework in order to perceive the theory conceptually and completely. The idea of merging quantum and geometry to provide a unified framework came into play few years after the development of pilot-wave theory and is still an ongoing issue. In order to associate physical reality to a set of particles, one needs to associate a pilot wave to each particle that guides the specific particle along its trajectory according to the guiding equation $m\frac{d\gamma}{dt} = \nabla_i S$.

A quantum phenomenon could be perfectly choreographed with a well chosen hidden variable approach and could subsequently be reformulated in geometrical way, refs. [27–32] represent such efforts for a geometrical rewriting of quantum laws. A slightly different approach linking quantum mechanics with topological properties, has also been worked on extensively, and a few of these could be found in refs. [33–35]. The geometrisation of the Klein-Gordon equation for a conformally curved spacetime was done in [36], and in [37] it was generalized for the nonrelativistic case, i.e., the Schrödinger equation. A Finslerian version of the Klein-Gordon equation was presented in [38] and in another relevant work two particle entanglement was written geometrically for Finsler spacetimes [39]. Details about Finsler space can be found in [34, 35] and references therein.

The purpose of this paper is to translate the quantum equations in the pilot-wave limit to a geometric language of both Finsler and timelike relative Funk spacetimes in the context of modified gravity (by adding extra dimensions in the configuration space).
This paper is organized as follows: Sec.II, describes the physical model and establishes the quantum equations in the pilot-wave limit. In sec.III, we present the geometric interpretation of quantum equations over Finsler space-time. In sec.IV, we present the geometric translation of these equations over timelike relative Funk space. Sec.V is the summary and conclusion.

II. THE MODEL

We will study a system of $n$-spinless bosons in the non-relativistic limit. Further, we assume that these bosons are free of any external potential in such a way that each particle can freely move along any of the three spatial dimensions. The wave equation for spin-0 bosons in the non-relativistic limit reduces to the Schrödinger equation of motion [30].

The Schrödinger equation for multi-particle system could be written as

$$
\sum_{a=1}^{n} \frac{\hbar}{2M_a} \frac{\partial}{\partial \vec{x}} \partial_{ab} + i\partial_0 \right) \psi(x_1, x_2, ..., x_n, t_1) = 0 \quad (1)
$$

where $\partial_0 = \frac{\partial}{\partial t_1}$. In the above equation $M_a$ represents the $n$-particle mass. The indices $b$ and $a$ are used to label the 3-D flat space and the particle (such that it specifies which one out of the $n$-particle is affected) respectively.

We are interested in pilot-wave theory that interprets not only the dynamics of the particles in a system but also accounts for their configuration. To switch from standard quantum interpretation of Eq.1 into the pilot-wave limit, the wave function is factored into amplitude and phase [21, 28, 31], as follows

$$
\psi = Re^{iS/\hbar} \quad (2)
$$

We are concerned with a system of free bosons in classical frame of reference, all sharing a universal time coordinate $t_1$. The quantum phase then reads

$$
S(t_a, \vec{x}_a) = -M t_1 + \tilde{S}(t_1, \vec{x}_a) \quad (3)
$$

The same is true for Hamilton’s principle function $S_H$, and the amplitude $R$ is given by

$$
R(t_a, \vec{x}_a) = R(t_1, \vec{x}_a) \quad (4)
$$

One can then rewrite the wavefunction in Eq.2 as

$$
\psi(t_1, \vec{x}_a) = R(t_1, \vec{x}_a)e^{i(\tilde{S}(t_1, \vec{x}_a) - Mt_1)/\hbar}. \quad (5)
$$

A. First Equation

In pilot-wave limit, spin-0 bosons evolve in space and time according to

$$
\left( \sum_{a=1}^{n} \frac{\hbar}{2M_a} \frac{\partial}{\partial \vec{x}} \partial_{ab} + i\partial_0 \right) R(t_1, \vec{x}_a)e^{i(\tilde{S}(t_1, \vec{x}_a) - Mt_1)/\hbar} = 0 \quad (6)
$$

Here $\partial R/\partial t_1 = 0$ as for large $t_1$ the amplitude on the average is zero. Using Taylor expansion in Eq.6 and ignoring higher order terms, we can get

$$
Q = \sum_{a=1}^{n} \frac{(\partial_{ab}R)(\partial_{ab}\tilde{S})}{2M_a} + \tilde{S} - M, \quad (7)
$$

Eq.7 aligns nicely with the classical Hamilton-Jacobi equation and represents the motion of bosons as a moving pilot-wave. The term $Q$ here represents the quantum potential $Q(x_1, x_2, ..., x_n, t_1)$ and $R(x_1, x_2, ..., x_n, t_1)$ is the amplitude of the associated pilot wave that guides the particle along its trajectory. This is the first non-relativistic quantum equation in pilot-wave interpretation.

B. Second Equation

For a multi-particle system conservation of probability in position space is found to be

$$
\partial_0 (\psi^* \psi) - \sum_{a=1}^{n} \frac{\partial}{\partial \vec{x}_a} \left( \frac{i\hbar}{2M_a} (\psi^* \frac{\partial}{\partial \vec{x}_a} \psi) \right) = 0 \quad (8)
$$

The conserved current with the definition in Eq.5 yields the following equation

$$
\partial_0 (R^2) + \sum_{a=1}^{n} \partial_{ab} \left( \frac{R^2}{M_a} (\partial_{ab} \tilde{S}) \right) = 0 \quad (9)
$$

Eq.9 is the second dBB equation.

C. Third Equation

Guided by the following equation

$$
p_a^b = M_a \frac{dx_a}{ds} = \partial_{ab} \tilde{S} \quad (9)
$$

these particles always have well defined trajectories in terms of their initial positions. One can specify the particle trajectory once the particle location at a particular instant of time is determined. This is the idea behind this framework, it treats that particular position as a hidden variable that gives insight into the physical reality of the system. This is third dBB equation.

D. Fourth Equation

The trajectory equation of motion for a system of $n$ non-relativistic bosonic particles is given by (see appendix B)
\[
\frac{d^2 x^A}{ds^2} = \sum_{c=1}^{n} \frac{(\partial^B \tilde{S})(\partial^c \partial_B \tilde{S})}{M_a^2} \tag{10}
\]

Here, \(ds = dt_1\) is the common time dimension for all bosons. Eq. (10) could serve as Newton’s acceleration law in the pilot-wave limit. This equation is a clear evidence of the essential feature (non-locality) of the pilot-wave interpretation. The term left of the Eq. (10) simply represents the acceleration of a specific particle which gets affected by a tidal force \(\sum_{c=1}^{n} \frac{(\partial^B \tilde{S})(\partial^c \partial_B \tilde{S})}{M_a^2}\) created by all bosons in the system. That is to say, the entire system is responsible for a particular position and hence for trajectory followed by each particle. This is analogous to the classical picture but different in the sense that as the particles get far apart (in classical theory) the strength of non-local influence decreases. However in hidden variable approach the non-local influence might be strengthened at large distance.

Eqs. (7)-(10) is a set of four non-relativistic equations for the case of multi-particle system in the pilot-wave interpretation. The parameters \(R\) (the amplitude of pilot wave), the quantum phase \(\tilde{S}\), and quantum potential \(Q\) in these equations depend on \(1 + 3n\) coordinates, \(3n\) of space and 1 of time.

In single index notation, the coordinates can be specified as

\[ [t_1, x^A] = [t_1, (x_1^1, x_1^2, x_1^3, x_2^1, x_2^2, x_2^3, ..., x_n^1, x_n^2, x_n^3)] \]

such that \(\partial_\alpha \rightarrow \partial^A\) and \(\partial_{ab} \rightarrow \partial_A\) with

\[ Q = \frac{\hbar^2}{2M_a} \frac{\partial^A \partial_A R(t_1, x_a)}{R(t_1, x_a)} \]

The set of four equations, obtained by manipulating non-relativistic wave equation in the context of hidden variable theory, can now be written as

\[ Q = \frac{(\partial^A \tilde{S})(\partial_A \tilde{S})}{2M_a} + \dot{\tilde{S}} - M, \tag{11} \]

\[ \partial_A \left( \frac{R^2 (\partial^A \tilde{S})}{M_a} \right) + \partial_b (R^2) = 0, \tag{12} \]

\[ p^A = M_a \frac{dx^A}{ds} = \partial^A \tilde{S}, \tag{13} \]

\[ \frac{d^2 x^A}{ds^2} = \frac{(\partial^B \tilde{S})(\partial^A \partial_B \tilde{S})}{M_a^2}. \tag{14} \]

### III. FINSLER GEOMETRY

Finsler geometry provides a geometric background for field theories and a framework to discuss the gravitational dynamics. Here we investigate the dynamics and geometry of spinless bosons over a non-metric general length measure Finsler spacetime. Finsler spacetime is actually a natural generalization of the Lorentzian metric theory of gravity. It is possible to extend GR to Finsler geometry if the spacetime is equipped with a general length measure \(\xi(\xi) = \hbar\) i.e.,

\[ g_{\Xi \Sigma} = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^\Xi \partial y^\Sigma} \tag{15} \]

where \(F\) is the separation between two events on the worldline. The Einstein-Hilbert action including matter field interaction is given by

\[ S[g, \sigma_i] = \int_M d^4x \sqrt{|g|} \left( \hat{R} + k \hat{L}_M [g, \sigma_i] \right) \]

Replacing

\[ \sqrt{|g|} = \sqrt{|\hat{g}|} \sqrt{|h|} \tag{16} \]

\[ \hat{R} = \hat{R}_{\Xi \Sigma} \tag{17} \]

The Einstein-Hilbert action could then be rewritten for Finsler space in terms of a homogenous Finsler function \(F\) (length between two events on worldline) over a tangent bundle \(TM\). Also we consider the circle \(S^1_p\) to be fibered over each point of the \(2nD\)-space manifold \(M\) in the tangent space \(T_pM\).

\[ S^1_p = \left\{ y \in T_pM \mid \sqrt{g_{\hat{p}}(y, y)} = 1 \right\} \]

The Einstein-Hilbert action could then be written as an action on the circle bundle \(\sum\) which is subset of tangent bundle \(TM\) (obtained by union over all tangent spaces \(T_pM\) i.e \(TM = \bigcup_{p \in M} T_pM\)) as

\[ S^1_p \subset T_pM \]

Introducing the notion

\[ q^\sigma = (\dot{x}^\beta, \theta^\alpha), \quad \dot{x}^\alpha = x_1^\alpha, x_2^\alpha, ..., x_n^\alpha, \quad \alpha = 1, 2, ..., n \]

The curvature of such space is specified by \(1 + 3n\) dimensional equation as

\[ P_s \left( \hat{R}_{\Xi \Sigma} + k \hat{L}_M \right) = \hat{R}_{\Xi \Sigma} + k \hat{L}_M \tag{18} \]

In this equation \(P_s\) is the operator to preserve symmetry between different particles \(x^\alpha_c\) and \(x^\alpha_a\), \(\hat{R}_{\Xi \Sigma}\) is Ricci tensor, \(\hat{L}_M\) is matter lagrangian and \(k\) is coupling constant.
representing the extent of interaction between particles and field.

The metric \( \hat{g} \) is transformed conformally, splitting it into conformal function \( \sigma(q,t_1) \) and a flat part \( \eta \), as

\[
\hat{g}^\Sigma = \sigma^{\frac{4}{n-1}} \eta^S A
\]  

This conformal mapping preserves the local angles and does not change the physics. The inverse of the metric is given by

\[
\hat{g}^\Sigma = \sigma^{\frac{4}{n-1}} \eta^S A
\]  

The lower Greek and lower Roman index are identified as \( \partial = \partial_A \) so that the adjoint derivatives are different in each notation i.e.,

\[
\hat{\sigma} = \hat{g}^{\Sigma A} \hat{\sigma}_A = \sigma^{\frac{4}{n-1}} \eta^A L \partial_L
\]

\[
\hat{\sigma} = \sigma^{\frac{4}{n-1}} \partial^A
\]

\[
\hat{\sigma} = \sigma^{\frac{4}{n-1}} \partial_A
\]

The resulting Einstein-Hilbert action in Finsler space for circle bundle becomes

\[
S[F,\sigma] = \int dt_1 \int d^2 x d^n \theta \sqrt{\hat{g}^h} (\hat{R}^\Sigma + k \hat{L}_M[g,\sigma])
\]  

where \( F \) is the Finsler function. Note that in Finsler spacetime, the matter-field coupling (given by \( k \)) is different from Einstein’s constant.

Using \( \hat{R}^\Sigma = \hat{g}^\Sigma \hat{R} \) , applying conformal transformation and choosing \( R = 0 \) for an atlas on this manifold, gives.

\[
S[F,\sigma] = \int dt_1 \int d^2 x d^n \theta \sqrt{\hat{g}^h} \left[ \sigma^{3n-5/3n-1} \frac{12n}{1-3n} \partial^A \sigma \partial_A \sigma + k \sigma^2 L_M \right]
\]  

The matter Lagrangian is given by

\[
L_M = \frac{2(\hat{\sigma}^2 \hat{S}_H)(\hat{\sigma}^2 \hat{S}_H)}{2 \hat{M}_G} + \frac{\partial \hat{S}_H}{\partial t_1} - \hat{M}
\]  

The equation of motion with this Lagrangian then gives

\[
\sigma^{3n-5/(3n-1)} \frac{12n}{1-3n} \frac{\partial^A \partial_A \sigma}{k \sigma} = \frac{2(\hat{\sigma}^2 \hat{S}_H)(\hat{\sigma}^2 \hat{S}_H)}{2 \hat{M}_G} + \frac{\partial \hat{S}_H}{\partial t_1} - \hat{M}
\]  

With the following matching conditions

\[
k = \sigma^{3n-5/(3n-1)} \frac{12n}{1-3n} \frac{2 \hat{M}_G}{h^2}
\]

\[
\sigma(q,t_1) = R(\bar{x}_a,t_1)
\]

\[
\hat{S}_H(q,t_1) = \hat{S}(\bar{x}_a,t_1)
\]

\[
M_a = \hat{M}_G
\]

It is clear that Eq.(23) is identical to Eq.(11), where \( q \) in these equations represents generalized coordinates \( q^i = (\bar{x}^\beta, \theta^a) \).

A. Geometric dual to second Equation

Using covariant conservation of stress-energy tensor gives (see appendix C)

\[
\frac{(\hat{\sigma}^2 \hat{S}_H)(\hat{\sigma}^2 \hat{S}_H)}{\hat{M}_G} + \frac{\partial \hat{S}_H}{\partial t_1} = 0
\]  

The Levi-Civita connection here is given by

\[
\hat{\gamma}^\Sigma = \frac{1}{2} \hat{g}^{\Sigma A} \left( \hat{\sigma}^2 \hat{g}^{\Sigma A} + \hat{\sigma}^2 \hat{g}^{\Sigma A} - \hat{\sigma}^2 \hat{g}^{\Sigma A} \right)
\]  

Using this connection in Eq.(24), we get (for details see appendix C)

\[
\left( \frac{\partial^2 \hat{S}_H}{\partial t_1} + \frac{\partial \sigma^2}{\partial \sigma} \right) = 0
\]

which is dual to Eq.(12). Furthermore, it is important to stress that the role of \( \sigma(q,t_1) \) is two-fold; firstly, it describes the probable location of particle at any particular instant of time, secondly, it represents the conformal function of the theory and accounts for the matter-field interaction.

B. Geometric dual to third Equation:

Guiding equation is given by

\[
\hat{\rho}^\Sigma = \hat{\rho}^\Sigma \hat{S}_H
\]  

which is identical to Eq.(13)

C. Geometric dual to trajectory Equation of motion

The geometric dual to the equation of motion is obtained as (see appendix D)

\[
\frac{d^2 \hat{x}^\Sigma}{ds^2} = \left( \frac{\hat{g}^\Sigma \hat{S}_H}{\hat{M}_G} \right)
\]  

where, \( \hat{s} = t_1 \) represents the single absolute time coordinate. With the given matching conditions from Eq.(28) is identical to Eq.(14). It is to be noted that
Although the Finsler geometry developed in this section follows from GR, the theory is strictly non-local. The non-locality is encoded by

1. Eq. (28): This equation is just parallel to the Newton’s equation of motion and clearly reflects that all the particles taken together completely describe each particle in the system.

2. Hamilton-Jacobi equation (28): This equation includes quantum potential which induces a force that acts on all the particles in orbit. It is worth noting that, we considered a system of free bosons but the theory induces a potential $Q$ that interlocks the position of particles such that it impossible to insulate one particle from the rest.

3. Extra Dimensions: We developed the model such that each particle resides in its own frame of reference with two spatial and one polar dimension and all (particles) sharing a universal time coordinate $t_1$. The symmetrization postulate requires all the reference frames to be identical such that changes made in any one of the reference frame demands a simultaneous change in the other frames too and hence induces non-locality in the system.

The dual to the quantum equations Eqs. (11)-(14) is merely geometrical rewriting of the quantum equations Eq. (11)-(14). We defined a set of matching conditions connect the quantum phase $\tilde{S}$ with those in the Finsler geometry. These matching conditions to connect the quantum equations Eqs. (11)-(14).

One must note that, Eq. (23) and Eqs. (26)-(28) are local. The non-locality is encoded by

$\frac{\sigma^{3n-5/3n-1}}{(3n-1)^{3n-5/3n-1}}\frac{12n}{1-3n} \frac{2\hat{M}_G}{h^2}$ (29)

This coupling depends upon the conformal function $\sigma^{3n-5/3n-1}$. The coupling constant in the particle action is given by

$k = \frac{4\pi G}{c^4} \frac{1}{y^{-3}y^{3n}}$ (30)

Comparing Eq. (29) and Eq. (30), we are lead to

$k = \frac{4\pi G}{c^4} \frac{1}{y^{-3}y^{3n}} \frac{1-3n}{12n} \frac{h^2}{2\hat{M}_G}$

So Eq. (29) becomes

$k = \frac{4\pi G}{c^4} \frac{1}{y^{-3}y^{3n}} \frac{1-3n}{12n} \frac{h^2}{2\hat{M}_G}$ (31)

One can deduce from Eq. (31) that $k < 0$, which implies that the bosons are weakly coupled to the field.

IV. APPLICATION TO FUNK GEOMETRY

The Finsler distance $F$ and the time-like relative Funk distance $F'$ are related as

$F'^2(x, y) = \log F^2(x, y)$ (32)

where $F'^2$ in Funk geometry is the timelike separation between two events. $F^2$ in the above equation given by

$F^2(x, y) = \frac{d(x, b(x, y))}{d(y, b(x, y))}$ (33)

where $d$ represents the non-reversible, non-symmetric Euclidean distance. So, the time-like relative Funk metric in terms of the genuine Finsler metric in Eq. (15) could be written as

$\hat{g}^{\Sigma} = F^{-2} \hat{g}^{\Sigma}$ (34)

The conformal transformation in Funk geometry is given by

$\hat{g}^{\Sigma} = F^{-2} \sigma^{\frac{4}{3n-1}} \eta^{SA}$ (35)

and the inverse Funk metric is transformed conformally as

$\hat{g}^{\Sigma} = F^2 \sigma^{\frac{4}{3n-1}} \eta^{SA}$ (36)
The particle action in this Euclidean time-like relative Funk geometry is given by

\[ S[F', \sigma_i] = \int dt_1 \int d^2n x d^n \theta \sqrt{g'}|h| \times \left[ \sigma^{-3n-3/3n-1} \frac{12n}{1-3n} \partial^A \sigma \partial_A \sigma + k_\sigma \frac{\pi^4}{\pi} L_M \right] \]

The matter Lagrangian \( L \) is

\[ F \]

The equation of motion with this Lagrangian then gives

\[ \sigma^{3n-9/(3n-1)} g^\Sigma \frac{12n}{1-3n} \partial^A \partial_A \sigma = \frac{2(\hat{\partial}^\Sigma \hat{S}_H)(\hat{\partial}_2 \hat{S}_H)}{2M_G} + \frac{\partial \hat{S}_H}{\partial t_1} - \dot{\hat{M}} \]

With the following matching conditions

\[ k = F^{-2} \sigma^{3n-5/(3n-1)} \frac{12n}{1-3n} \frac{2\hat{M}_G}{h^2} \]

\[ \sigma(q, t_1) = R(\tilde{x}_a, t_1) \]

\[ \hat{S}_H(q, t_1) = \hat{S}(\tilde{x}_a, t_1) \]

\[ M_a = \hat{M}_G \]

Eq.(38) is identical to Eq.(41)

A. Geometric dual to second equation

Again using the conservation of stress-energy tensor together with the Levi-Civita connection (as in the previous section) and following the same procedure as mentioned in appendix C, one obtains

\[ \left( \frac{\partial_A}{\hat{M}_G} \left[ \frac{\sigma^2 (\partial^A \hat{S}_H)}{\hat{M}_G} \right] \right) + \frac{\partial}{\partial t_1} (\sigma^2) = 0 \]

It easy to see that, with the defined matching conditions for time-like relative Funk geometry, Eq.(39) is identical to Eq.(12)

B. Geometric Dual to third equation

The particles are guided along their trajectory by

\[ \hat{p}^\Sigma = \hat{\partial}^\Sigma \hat{S}_H \]

which, with the defined matching conditions is identical to Eq.(13)

C. Geometric dual to trajectory equation of motion

The geometric dual to the equation of motion in time-like relative Funk geometry is obtained by following the procedure discussed in appendix D, as

\[ \frac{d^2 \hat{x}^\Sigma}{ds^2} = \frac{(\hat{\partial}^\Sigma \hat{S}_H)(\hat{\partial}_2 \hat{S}_H)}{M_G^2} \]

It is important to note that

- In Finsler geometry: Eq.(28), Hamilton-Jacobi equation Eq.(23) and extra dimensions are responsible for the non-local nature. In time-like relative Funk geometry however, there is an additional parameter encoding non-locality in the theory. Since \( F^2 \), the time-like separation is when the event is occurring inside the light cone and this as a result can effect another event. Or we can say there exist causality between the events.

- Like the Finsler geometry, it is not possible to restore gravity from the time-like relative Funk geometry because we introduced extra dimensions and an absolute single time coordinate \( t_1 \) shared by all the bosons.

- We specified a set of matching conditions to show that quantum equations Eqs.(11)-(14) have duality in time-like relative Funk geometry. These matching conditions connect the quantum phase \( \tilde{S} \) with the Hamilton principle function \( \tilde{S}_H \), the amplitude of pilot wave \( R \) with the conformal function of the metric \( \sigma \) and the mass \( M_a \) with the mass \( M_G \). The coupling constant defined in the time-like relative Funk geometry is

\[ k = F^{-2} \sigma^{3n-5/(3n-1)} \frac{12n}{1-3n} \frac{2\hat{M}_G}{h^2} \]

This coupling depends upon the conformal function \( \sigma^{3n-5/(3n-1)} \). The coupling constant in the action is given by

\[ k = \frac{4\pi G}{c^4} \frac{1}{y^2} \]

Comparing Eq.(42) and Eq.(43) gives

\[ \sigma^{3n-5} = \frac{4\pi G}{c^4} \frac{1}{y^2} \frac{F^2}{y^2} \frac{1-3n}{12n} \frac{h^2}{2M_G} \]

So Eq.(42) becomes

\[ k = \frac{4\pi G}{c^4} \frac{1}{y^2} \]

(44)
Note from Eq. (44) that in a time-like relative Funk geometry also \( k < 0 \) showing weak matter-field coupling. One can see that in this specific framework and with this choice of matching conditions, the two geometries lead to the same result. The only difference is that, in time-like relative Funk geometry the metric is a time-like distance function. We could picture this as if time-like relative Funk space a Finsler space equipped with time-like distance function. This is similar to as observed in \([8]\) in the framework of convex geometry with time-like distance function. This is similar to as observed in \([8]\) in the framework of convex geometry with time-like distance function. This is similar to as observed in \([8]\) in the framework of convex geometry with time-like distance function. This is similar to as observed in \([8]\) in the framework of convex geometry with time-like distance function.

\[
\sum_{n=1}^{\infty} \frac{\hbar}{2M_a} \frac{\partial^k_a \partial_{ab} R(t_1, \bar{x}_a)}{R(t_1, \bar{x}_a)} e^{i(\bar{S}(t_1, \bar{x}_a)-M t_1)/\hbar} = 0 \quad (45)
\]

From this equation, we get

\[
\sum_{a=1}^{n} \frac{\hbar}{2M_a} \frac{\partial^k_a \partial_{a b} R(t_1, \bar{x}_a)}{R(t_1, \bar{x}_a)} e^{i(\bar{S}(t_1, \bar{x}_a)-M t_1)/\hbar} + \frac{iR(t_1, \bar{x}_a)}{\hbar} e^{i(\bar{S}(t_1, \bar{x}_a)-M t_1)/\hbar} \partial_{\partial t_1} \left( \bar{S} - M t_1 \right) = 0 \quad (46)
\]

For large \( t_1, \) \( \frac{\partial R}{\partial t_1} \) on the average is zero. We used the Taylor expansion in the above equation and picked up the real part of the resulting equation to get

\[
\sum_{a=1}^{n} \frac{\hbar^2}{2M_a} \frac{\partial^k_a \partial_{a b} R(t_1, \bar{x}_a)}{R(t_1, \bar{x}_a)} = \sum_{a=1}^{n} \frac{(\partial^2_a \bar{S})(\partial_{a b} \bar{S})}{2M_a} + \bar{S} - M \quad (47)
\]

where, the dot represents the derivative with respect to \( t_1 \). Using the definition \( Q = \sum_{a=1}^{n} \frac{\hbar^2}{2M_a} \frac{\partial^k_a \partial_{a b} R(t_1, \bar{x}_a)}{R(t_1, \bar{x}_a)} \), we can write

\[
Q = \sum_{a=1}^{n} \frac{(\partial^2_a \bar{S})(\partial_{a b} \bar{S})}{2M_a} + \bar{S} - M \quad (48)
\]

### VI. APPENDIX

#### A. First Equation

From \((6)\)

\[
\left( \sum_{a=1}^{n} \frac{\hbar}{2M_a} \frac{\partial^k_a \partial_{ab} R(t_1, \bar{x}_a)}{R(t_1, \bar{x}_a)} e^{i(\bar{S}(t_1, \bar{x}_a)-M t_1)/\hbar} \right) = 0 \quad (45)
\]

#### B. Fourth Equation

To find the equation of motion, we used the guiding equation \( \frac{d^2 x^b}{ds^2} = \frac{\partial^k_a \bar{S}}{M_a} \). This leads to

\[
\frac{d^2 x^b}{ds^2} = \frac{d}{ds} \left( \frac{\partial^2_a \bar{S}}{M_a} \right) \quad (49)
\]

\[
\Rightarrow \frac{d^2 x^b}{ds^2} = \sum_{c=1}^{n} \partial_{cd} \left( \frac{\partial^2_c \bar{S}}{M_a} \right) \quad (50)
\]

#### C. Geometric dual to second Equation

The stress-energy tensor is given by

\[
T_{\Sigma \Xi} = -2 \frac{\delta L_M}{\delta \Sigma_{\Xi}} + \bar{g}^{\Sigma \Xi} L_M \quad (51)
\]
The above equation upon substituting matter Lagrangian gives

\[ \dot{T}^{\Sigma \Xi} = -2 \frac{\delta}{\delta \Theta} \left( g^{\Sigma \Xi} \frac{(\partial^\Xi \tilde{S}_H)(\partial^\Xi \tilde{S}_H)}{M_G} + \frac{\partial \tilde{S}_H}{\partial t_1} - \dot{M} \right) \]

\[ + \tilde{g}^{\Sigma \Xi} \frac{(\partial^\Xi \tilde{S}_H)(\partial^\Xi \tilde{S}_H)}{M_G} + \frac{\partial \tilde{S}_H}{\partial t_1} - \dot{\tilde{g}}^{\Sigma \Xi} \dot{M} \] (52)

Differentiating the term inside the bracket with respect to metric, we get

\[ \dot{T}^{\Sigma \Xi} = -2 \frac{(\partial^\Xi \tilde{S}_H)(\partial^\Xi \tilde{S}_H)}{M_G} + \frac{\partial \tilde{S}_H}{\partial t_1} - \dot{M} \]

\[ \tilde{g}^{\Sigma \Xi} \frac{(\partial^\Xi \tilde{S}_H)(\partial^\Xi \tilde{S}_H)}{M_G} \] (53)

Using covariant conservation of stress-energy tensor \( \nabla_\Sigma \dot{T}^{\Sigma \Xi} = 0 \) gives

\[ \nabla_\Sigma \left[ -2 \frac{(\partial^\Xi \tilde{S}_H)(\partial^\Xi \tilde{S}_H)}{M_G} + \frac{\partial \tilde{S}_H}{\partial t_1} - \dot{M} \right] = 0 \] (54)

Applying the covariant derivative and assuming the metric to be covariantly conserved i.e. \( \nabla_\Sigma \tilde{g}^{\Sigma \Xi} = 0 \), this simplifies to

\[ -2 \frac{(\partial^\Xi \tilde{S}_H)\nabla_\Sigma (\partial^\Xi \tilde{S}_H)}{M_G} - \frac{(\partial^\Xi \tilde{S}_H)\nabla_\Sigma (\partial^\Xi \tilde{S}_H)}{M_G} \]

\[ + \frac{\tilde{g}^{\Sigma \Xi} (\partial^\Xi \tilde{S}_H)(\partial^\Xi \tilde{S}_H)}{M_G} + \tilde{g}^{\Sigma \Xi} \nabla_\Sigma \frac{\partial \tilde{S}_H}{\partial t_1} = 0 \] (55)

From above equation it follows that

\[ \frac{(\partial^\Xi \tilde{S}_H)\nabla_\Sigma (\partial^\Xi \tilde{S}_H)}{M_G} = 0 \] (56)

\[ \frac{(\partial^\Xi \tilde{S}_H)\nabla_\Sigma (\partial^\Xi \tilde{S}_H)}{M_G} = 0 \] (57)

\[ \frac{(\partial^\Xi \tilde{S}_H)\nabla_\Sigma (\partial^\Xi \tilde{S}_H)}{M_G} + \nabla_\Sigma \frac{\partial \tilde{S}_H}{\partial t_1} = 0 \] (58)

Using the Levi-Civita connection \( \{24\}, \{25\} \) reads

For first term (a part \( \nabla_\Sigma (\partial^\Xi \tilde{S}_H) \) of it)

\[ \nabla_\Sigma (\partial^\Xi \tilde{S}_H) = \partial_\Sigma (\partial^\Xi \tilde{S}_H) + \frac{1}{2} \tilde{g}^{\Delta \Xi} \left( \partial_\Xi \tilde{g}_{\Delta \Xi} + \partial_\Xi \tilde{g}_{\Xi \Delta} - \partial_\Delta \tilde{g}_{\Xi \Xi} \right) (\partial^\Xi \tilde{S}_H) = 0 \] (59)

Using conformal transformation and doing some mathematics gives

\[ \sigma^{-2-6n} \partial_A (\sigma^2 \partial^A \tilde{S}_H) = 0 \] (60)

For the second term in \( \{55\} \)

\[ \tilde{g} \frac{\partial \tilde{S}_H}{\partial t_1} = \frac{\partial \tilde{S}_H}{\partial t_1} \] (61)

Using conformal transformation, we obtain

\[ \nabla_\Sigma \frac{\partial \tilde{S}_H}{\partial t_1} = \frac{\partial}{\partial t_1} (\partial_A \tilde{S}_H) \] (62)

From \( \{55\} \) with first \( \{60\} \) and second term \( \{62\} \)

\[ \sigma^{-2-6n} (\partial_\Sigma \tilde{S}_H) \partial_A \left[ \sigma^2 (\partial^A \tilde{S}_H) \right] = \frac{\partial}{\partial t_1} (\partial_A \tilde{S}_H) = 0 \] (63)

After some manipulations (In doing so we limit the temporal and spatial derivatives up to quadratic terms in \( \sigma \) only), we obtain

\[ \left( \frac{\partial_A \left[ \sigma^2 (\partial^A \tilde{S}_H) \right]}{M_G} + \frac{\partial}{\partial t_1} (\sigma^2) \right) = 0 \] (64)

D. Geometric dual to the trajectory Equation of motion

The total derivative is

\[ \frac{d}{ds} = \frac{d}{ds} \partial_\Xi \] (65)

Applying this relation to momenta, we get

\[ \frac{d^2 \partial_\Xi}{ds^2} = \partial_\Xi \frac{d^2 \partial_\Xi}{ds^2} \frac{\partial^\Xi \tilde{S}_H}{M_G} \] (66)

Using \( \{27\} \), this leads to

\[ \frac{d^2 \partial_\Xi}{ds^2} = \frac{\partial^\Xi \tilde{S}_H (\partial_\Xi (\partial^\Xi \tilde{S}_H))}{M_G^2} \] (67)
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