Classification of Elliptic Line Scrolls

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Abstract: The classification of projective elliptic line scrolls with the description of their singular loci is given. In particular we recover Atiyah Theorem by using classical methods.

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Introduction: Through this paper, a geometrically ruled surface, or simply a ruled surface, will be a $\mathbb{P}^1$-bundle over a smooth curve $X$ of genus $g$. It will be denoted by $\pi : S = \mathbb{P}(\mathcal{E}_0) \to X$ and we will follow the notation and terminology of R. Hartshorne’s book [3], V, Section 2. We will suppose that $\mathcal{E}_0$ is a normalized sheaf and $X_0$ is the section of minimum self-intersection that corresponds to the surjection $\mathcal{E}_0 \to \mathcal{O}_X(e) \to 0$, $\wedge^2 \mathcal{E} \cong \mathcal{O}_X(e)$. Consider the following question: which are the linear equivalence classes $D \sim mx_0 + bf$, $b \in \text{Pic}(X)$, that correspond to very ample divisors? When $g = 1$ and $m = 1$, a characterization is known ([3], V, Ex. 2.12), but the classification of elliptic scrolls obtained by Corrado Segre in [5] does not follow directly from this. A scroll is the image of a ruled surface $\pi : S = \mathbb{P}(\mathcal{E}_0) \to X$ by a unisecant complete linear system.

In this paper, I apply the results of the previous work [2] to obtain the classification of elliptic (line) scrolls in $\mathbb{P}^N$, $N \geq 3$, including the singular ones and degenerations. Theorems 1.7 and 2.13 provide the description of all scrolls in the decomposable and indecomposable cases respectively.

To obtain Theorem 2.13 we use that all indecomposable ruled surface is obtained from a decomposable one by applying a finite number of elementary transformations, see (2, 3.9). We give a proof of Atiyah Theorem (the unique indecomposable elliptic scrolls have $e = 0$ or $e = -1$) that seems geometrically interesting to us.

In Theorem 2.13, we prove that the elementary transformation at a point of an elliptic decomposable ruled surface is either decomposable or indecomposable with $e = 0, -1$. Proposition 2.3 says that an indecomposable elliptic ruled
surface with $e = 0$ is obtained from a decomposable one or from an indecomposable one with $e = -1$. Finally, in Proposition 2.6 we see that the elementary transform at a point of an indecomposable elliptic ruled surface with $e = -1$ is either a decomposable one or an indecomposable one with $e = 0$. This result uses Proposition 2.3, where we give a parameterization of the indecomposable ruled surface with $e = -1$ that makes it isomorphic to the symmetric product $S^2X$. Moreover, this result provides explicitly how an elliptic ruled surface is obtained from $X \times P^1$ by applying elementary transformations. Compare with [4], II, §2.

We give tables that show the classification of elliptic line scrolls in $P^N$, $N \geq 3$, and contain the results of C. Segre in [5].

Finally, in §3 we study the classes of 2-secant divisors in an elliptic ruled surface. We characterize those that are base-point-free and very ample, and those that contain irreducible elements. These results make exhaustive our study of elliptic scrolls.

In a forthcoming paper we will study the classification of elliptic scrolls of higher rank.

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1 Decomposable elliptic ruled surfaces.

We are going to study elliptic geometrically ruled surfaces; that is, ruled surfaces over a smooth curve $X$ of genus 1.

We begin by working with decomposable elliptic ruled surfaces, and we apply the results seen in ([2], section 3). Then, by using elementary transformations, we will study the indecomposable ones.

Let us first review some properties of divisors on an elliptic curve. We will apply them to study elliptic ruled surfaces.

**Lemma 1.1** Let $X$ be an elliptic curve. Given a point $P_0 \in X$, there exists an one-to-one correspondence between points $P$ of $X$ and divisors $b$ of $Pic_0(X)$. Under this correspondence $b \sim P - P_0$.

**Lemma 1.2** Let $X$ be a smooth elliptic curve and let $b$ be a divisor on $X$. Then:

1. $b$ is base-point-free if and only if $b \sim 0$ or $deg(b) \geq 2$. 

2. 


2. \( b \) is very ample if and only if \( \text{deg}(b) \geq 3 \).

**Proposition 1.3** Let \( S \) be a decomposable elliptic ruled surface and let \( |H| = |X_0 + bf| \) be a complete unisecant linear system. Then:

1. If \( e \sim 0 \), then \( |H| \) is base-point-free if and only if \( \text{deg}(b) \geq 2 \) or \( b \sim 0 \).
2. If \( e \not\sim 0 \), then \( |H| \) is base-point-free if and only if \( \text{deg}(b) \geq e + 2 \) or \( b \sim -e \) and \( e \geq 2 \).

**Proof:**

1. Let us suppose \( e \sim 0 \). By Proposition 2.3 in [2], \( |H| \) is base-point-free if and only if \( b \) is base-point-free, because \( b + e \sim b \) in this case. Applying Lemma 1.2 the conclusion follows.

2. Let us suppose \( e \not\sim 0 \). \( |H| \) is base-point-free if and only if \( b \) and \( b + e \) are base-point-free. By applying Lemma 1.2 we see that there are the following possibilities:

   (a) \( b \sim 0 \) and \( b + e \sim 0 \), but then \( e \sim 0 \), which contradicts our assumption.

   (b) \( b \sim 0 \) and \( \text{deg}(b + e) \geq 2 \), but then \( \text{deg}(e) \geq 2 \) and by ([2], V, 2.12), \( S \) is indecomposable, which is false by hypothesis.

   (c) \( \text{deg}(b) \geq 2 \) and \( b + e \sim 0 \), then \( b \sim -e \) and necessarily \( e = \text{deg}(-e) \geq 2 \).

   (d) \( \text{deg}(b) \geq 2 \) and \( \text{deg}(b + e) \geq 2 \), but since \( \text{deg}(b) \leq 0 \), it is enough that \( \text{deg}(b) \geq 2 + e \).

**Proposition 1.4** Let \( S \) be a decomposable elliptic ruled surface and let \( |H| = |X_0 + bf| \) be a complete unisecant linear system on \( S \). Then \( |H| \) is very ample if and only if \( \text{deg}(b) \geq 3 + e \).

**Proof:**

By Theorem 2.8 in [2], \( |H| \) is very ample if and only if \( b \) and \( b + e \) are very ample. Applying Lemma 1.2, we see that this condition holds when \( \text{deg}(b) \geq 3 \) and \( \text{deg}(b + e) \geq 3 \). Since \( e \geq 0 \), it is sufficient that \( \text{deg}(b) \geq 3 + e \).

**Proposition 1.5** Let \( S \) be a decomposable elliptic ruled surface. The unisecant linear systems with generic element irreducible are:
Lemma 1.6

1. If \( \mathbf{c} \sim 0 \), they are \(|X_0|\) and \(|X_0 + \mathbf{b}f|\) with \(\text{deg}(\mathbf{b}) \geq 2\).

2. If \( \mathbf{c} \not\sim 0 \), they are \(|X_0|\), \(|X_0 - \mathbf{c}f|\) and \(|X_0 + \mathbf{b}f|\) with \(\text{deg}(\mathbf{b}) \geq 1 + \mathbf{e}\).

Proof:

We will apply Theorem 2.5 in [2]. The linear system \(|X_0 + \mathbf{b}f|\) have irreducible elements if and only if either \(\mathbf{b} \sim 0\), or \(\mathbf{b} \sim -\mathbf{c}\), or \(\mathbf{b}\) and \(\mathbf{b} + \mathbf{c}\) are effective without common base points. Thus we have two cases:

1. If \(\mathbf{c} \sim 0\), then \(\mathbf{b} + \mathbf{c} \sim \mathbf{b}\). So \(|X_0 + \mathbf{b}f|\) has irreducible elements if \(\mathbf{b} \sim 0\) or \(\mathbf{b}\) is base-point-free. According to Lemma 1.2, we know that this happens when \(\mathbf{b} \sim 0\) or \(\text{deg}(\mathbf{b}) \geq 2\).

2. If \(\mathbf{c} \not\sim 0\), we have irreducible elements in \(|X_0|\), \(|X_0 - \mathbf{c}f|\) and \(|X_0 + \mathbf{b}f|\) when \(\mathbf{b}\) and \(\mathbf{b} + \mathbf{c}\) are effective without common base points.

\(\mathbf{b} + \mathbf{c}\) is effective if \(\text{deg}(\mathbf{b} + \mathbf{c}) \geq 0\), this is, if \(\text{deg}(\mathbf{b}) \geq \mathbf{e}\). Moreover, if \(\text{deg}(\mathbf{b}) = \mathbf{e}\), then \(\mathbf{b} \sim -\mathbf{c}\) and the system is \(|X_0 - \mathbf{c}f|\). Thus, if \(\text{deg}(\mathbf{b}) \geq 1 + \mathbf{e}\), then \(\mathbf{b}\) and \(\mathbf{b} + \mathbf{c}\) are effective and they can only have base points when \(\mathbf{e} = 0\). In this case, since \(\mathbf{e} \not\sim 0\) and \(\text{deg}(\mathbf{b}) = 1\), \(\mathbf{b}\) and \(\mathbf{b} + \mathbf{c}\) correspond to different points of \(X\). So they have not common base points.

Lemma 1.6 Let \(S\) be a decomposable elliptic ruled surface, let \(|H| = |X_0 + \mathbf{b}f|\) be a complete base-point-free linear system on \(S\) such that \(\mathbf{b} \not\sim 0\), and let \(\phi: S \rightarrow \mathbf{P}^N\) be the regular map defined by \(|H|\). Let \(D\) be an irreducible unisecant curve on \(S\) that is not linearly equivalent to \(X_0\) and \(X_1\), with \(D \sim X_0 + \mathbf{a}f\). Then:

1. If \(\mathbf{b} \not\sim -\mathbf{c}\), \(\phi(D)\) is linearly normal if and only if \(\text{deg}(\mathbf{a}) \leq \text{deg}(\mathbf{b})\) and \(\mathbf{a} \not\sim \mathbf{b}\).

2. If \(\mathbf{b} \sim -\mathbf{c}\), \(\phi(D)\) is linearly normal if and only if \(\text{deg}(\mathbf{a}) = 1 + \mathbf{e}\).

Proof:

Let us consider the trace of the complete linear system \(|H|\) on the curve \(D\):

\[
0 \rightarrow H^0(\mathcal{O}_S((\mathbf{b} - \mathbf{a})f)) \rightarrow H^0(\mathcal{O}_S(X_0 + \mathbf{b}f)) \xrightarrow{\alpha} H^0(\mathcal{O}_D(X_0 + \mathbf{b}f)) = H^0(\mathcal{O}_X(\mathbf{a} + \mathbf{b} + \mathbf{c})) \rightarrow 0
\]

The curve \(\phi(D)\) is linearly normal when \(|H|\) traces the complete linear system \(|\mathbf{a} + \mathbf{b} + \mathbf{c}|\) on \(D\); equivalently, when \(\alpha\) is a surjection \(h^0(\mathcal{O}_S(X_0 + \mathbf{b}f)) = h^0(\mathcal{O}_X(\mathbf{a} + \mathbf{b} + \mathbf{c})) + h^0(\mathcal{O}_X(\mathbf{b} - \mathbf{a}))\).

We know that \(h^0(\mathcal{O}_S(X_0 + \mathbf{b}f)) = h^0(\mathcal{O}_X(\mathbf{b})) + h^0(\mathcal{O}_X(\mathbf{b} + \mathbf{c}))\).
1. Let us first suppose that \( b \not\sim -e \).

\( D \) is not linearly equivalent to \( X_0 \) or \( X_1 \). So by Proposition 1.3, \( \deg(a + e) \geq 1 \). Moreover, \(|H|\) is base-point-free and, by hypothesis, \( b \not\sim 0 \) and \( b \not\sim -e \); by applying Proposition 1.3 we have that \( \deg(b) \geq 2 + e \) and then \( \deg(a + e + b) \geq 1 \) and the divisor \( a + e + b \) is nonspecial. By Riemann-Roch, \( h^0(O_X(a + b + e)) = \deg(a) + \deg(b) + \deg(e) \) and \( h^0(O_X(b - a)) = \deg(b) - \deg(a) + h^1(O_X(b - a)) \).

Since \( \deg(b) \geq 2 + e \), \( b \) and \( b + e \) are nonspecial on \( X \) and by Riemann-Roch, \( h^0(O_S(X_0 + b f)) = 2\deg(b) - e \). Therefore, \( h^0(O_S(X_0 + b f)) - (h^0(O_X(a + b + e)) + h^0(O_X(b - a))) = -h^1(O_X(b - a)) \), and the map \( \alpha \) is a surjection when \( b - a \) is nonspecial. That happens when \( b \not\sim a \) and \( \deg(a) \leq \deg(b) \).

2. Let us suppose that \( b \sim -e \).

Since \( |X_0 + b f| \) is base-point-free, \( h^0(O_S(X_0 + b f)) = h^0(O_X(b)) + h^0(O_X(b + e)) = e + 1 \).

Moreover, \( D \) is not linearly equivalent to \( X_0 \) or \( X_1 \). So by Proposition 1.3 we have that \( \deg(a + e) \geq 1 \) (equivalently, \( \deg(a) \geq 1 + e \)).

Thus, \( h^0(O_X(a + b + e)) = h^0(O_X(a)) = \deg(a) \) and \( h^0(O_X(b - a)) = h^0(O_X(e - a)) = 0 \); from this, the equality

\[ h^0(O_X(b - a)) = h^0(O_X(a + b + e) + h^0(O_X(b - a)) \]

holds if and only if \( \deg(a) = 1 + e \).

Finally, a direct application of the above results allows us to describe decomposable elliptic scrolls in \( \mathbb{P}^N \).

**Theorem 1.7** Let \( S \) be a decomposable elliptic ruled surface. Let \(|H| = |X_0 + b f|\) be a complete base-point-free linear system on \( S \), with \( b = \deg(b) \). Let \( \phi : S \to \mathbb{P}^N \) be the regular map defined by \( |H| \) on \( S \) and let \( R = \phi(S) \) be the image scroll. Then \( R \) is one of the following models:

1. When \( \phi \) is not birational:

   (a) If \( e \sim 0 \) and \( b \sim 0 \), then \( R \) is a line parameterizing the curves of \( |X_0| \). In fact, \( S \cong X \times \mathbb{P}^1 \) and \( \phi \) is the second projection.

   (b) If \( e \sim 0 \) and \( b = 2 \), then \( R \) is a smooth quadric in \( \mathbb{P}^3 \) and \( \phi \) is a 2:1 morphism from \( S \) onto \( R \).

   (c) If \( e = 2 \) and \( b \sim -e \), then \( R \) is a plane and \( \phi \) is a 2:1 morphism.
2. When $\phi$ is birational:

(a) If $e > 2$ and $b \sim -e$, then $R$ is a cone in $\mathbb{P}^e$ over a smooth linearly normal elliptic curve of degree $e$.

The singular locus of $R$ is the vertex of the cone.

$R$ has a family of linearly normal elliptic curves of degree $e$ that correspond to the hyperplane sections.

$R$ has unisecant elliptic curves of degree $d > e$ in the linear systems $|X_0 + af|$ with $\deg(a) = d$. These curves are smooth and linearly normal if and only if $d = 1 + e$. If $d > 1 + e$ then the curves have a singular point of multiplicity $d - e$ at the vertex of the cone.

(b) If $e = 0$, $e \not\sim 0$ and $b = 2$, then $R$ is an elliptic scroll of degree $4$ in $\mathbb{P}^3$.

It is generated by a $2 : 2$ correspondence between two disjoint lines.

The singular locus of $R$ are disjoint lines which generate it.

$R$ has unisecant elliptic curves of degree $d \geq 3$ in the linear systems $|X_0 + af|$ with $\deg(a) = d - 2$. These curves are linearly normal if and only if $d \leq 4$ and $a \not\sim b$ (if $a \sim b$ they are the hyperplane sections).

(c) If $e > 0$ and $b = e + 2$, then $R$ is an elliptic scroll of degree $e + 4$ in $\mathbb{P}^{e+3}$. It is generated by a $1 : 2$ correspondence between a line and a smooth linearly normal elliptic curve of degree $e + 2$, laying in disjoint spaces.

The singular locus of $R$ is the directrix line.

$R$ has a unique directrix curve of minimum degree $2$ and unisecant elliptic curves of degree $d \geq 3 + e$ in the linear systems $|X_0 + af|$ with $\deg(a) = d - 2$. These curves are linearly normal if and only if $d \leq 4 + e$ and $a \not\sim b$ (if $a \sim b$ they are the hyperplane sections).

(d) If $b \geq 3 + e$, then $R$ is a smooth elliptic scroll of degree $2b - e$ in $\mathbb{P}^{2b-e-1}$. It is generated by a $1 : 1$ correspondence between two smooth linearly normal elliptic curves of degrees $b$ and $b + e$, laying in disjoint spaces.

If $e \sim 0$, $R$ has a one-dimensional family of disjoint directrix curves of minimum degree $b$. Moreover, $R$ has unisecant elliptic curves of degree $d \geq b + 2$ in the linear systems $|X_0 + af|$ with $\deg(a) = d - b$. These curves are linearly normal if and only if $d \leq 2b$ and $a \not\sim b$ (if $a \sim b$ they are the hyperplane sections).

If $e \not\sim 0$, $R$ has a directrix curve of minimum degree $b - e$ and unisecant linearly normal elliptic curves of degree $b$ in the linear system $|X_0 - ef|$. Therefore, if $e > 0$ the minimum degree curve is unique, but if $e = 0$ and $e \not\sim 0$, there are two curves of minimum degree. Moreover, $R$ has unisecant elliptic curves of degree $d \geq b + 1$ in the linear systems $|X_0 + af|$ with $\deg(a) = d + e - b$. These curves are
linearly normal if and only if $d \leq 2b - e$ and $a \not\sim b$ (if $a \sim b$ they are the hyperplane sections).

Proof:

Note that, when $\phi$ is birational, the description of the scroll $R$ follows immediately from Theorem 2.9 in [2]. The families of irreducible unisecant curves are described in Proposition 1.3 and we know when they are linearly normal by Lemma 1.6. If $|H|$ is very ample, then $\phi$ is an isomorphism. Let us study other cases.

By Propositions 1.3 and 1.4, we know when $|H|$ is base-point-free but not very ample:

1. If $e \sim 0$ and $b \sim 0$, then $|H| = |X_0|$. The linear system has dimension one, so $R$ is a line parameterizing curves of $|X_0|$. Each curve of this linear system is isomorphic to $X$ and it is applied onto a point of $R$. From this, the surface $S$ is isomorphic to $X \times \mathbb{P}^1$ and $\phi$ is the second projection.

2. If $e \sim 0$ and $b = 2$, then, by Theorem 2.9 in [2], $\phi(X_0)$ and $\phi(X_1)$ are two disjoint lines given by the $2:1$ morphism defined by $|b|$ from $X$ onto $\mathbb{P}^1$. Moreover, $P$ and $Q$ apply on the same point of $\mathbb{P}^1$ when $b \sim P + Q$, this is, when $Q$ is a common base point of $b - P$ and $b + e - P$. Hence, generators $Qf$ and $Pf$ apply onto the same line on $R$ if $P + Q \sim b$. Consequently, a unique (double) line passes through each point of $\phi(X_0)$ and $\phi(X_1)$. $R$ is generated by a $1:1$ correspondence between two disjoint lines in $\mathbb{P}^3$, so $R$ is the smooth quadric of $\mathbb{P}^3$ and $\phi$ is a $2:1$ morphism.

3. If $e = 2$ and $b \sim -e$, then $R$ have degree $2$ in $\mathbb{P}^3$. Since $b + e \sim 0$ and $b = 2$, $\phi(X_0)$ is a point and $\phi(X_1)$ is a double line. The scroll is generated by lines meeting the point and the double line. Therefore, $R$ is a plane and $\phi$ is a $2:1$ map.

4. If $e > 2$ and $b \sim -e$, since $b + e \sim 0$, $\phi(X_0)$ is a point which meets all generators. Then $R$ is a cone with vertex $\phi(X_0)$ over the linearly normal elliptic curve $\phi(X_1)$.

By Proposition 1.3 we know that $S$ have unisecant irreducible curves in the linear systems $|X_0|$ (the vertex of the cone), $|X_1|$ (the hyperplane sections) and $|X_0 + af|$ with $\deg(a) \geq 1 + e$. These curves have degree $d = (X_0 + af).(X_0 - ef) = d$. If $\deg(a) \geq 1 + e$, the curves meets $X_0$ at $X_0.(X_0 + af) = \deg(a) - e \geq 2$ points. So they have a singular point of multiplicity $d - e$ at the vertex $\phi(X_0)$.

5. If $e \not\sim 0$ and $b = e + 2$, then $b - P$ and $b + e - P$ have not common base points for any $P \in X$. So $\phi$ is birational. By applying Theorem 2.9 in [2], our claims follow.
2 Indecomposable elliptic ruled surfaces.

Our goal is to find models of indecomposable elliptic ruled surfaces. According to Corollary 3.9 in [2], they are obtained from decomposable ones by applying a finite number of elementary transformations.

In the following theorem we apply Theorem 3.10 in [2], to study how a decomposable ruled surfaced is modified by an elementary transformation.

Theorem 2.1 Let \( \pi : S \rightarrow X \) be a decomposable elliptic ruled surface. Let \( x \in Pf \) be a point of \( S \). Let \( S \) be the elementary transform of \( S' \) at \( x \) corresponding to a normalized sheaf \( E' \) with divisor \( e' = -\deg(e') \). Let \( Y_0 \) be the minimum self-intersection curve of \( S' \). Then we have the following cases:

1. When \( e \geq 2 \):
   (a) If \( x \in X_0 \), then \( S' \) is decomposable, \( e' \sim e - P \) and \( Y_0 = X_0' \).
   (b) If \( x \notin X_0 \), then \( S' \) is decomposable, \( e' \sim e + P \) and \( Y_0 = X_0' \).

2. When \( e = 1 \):
   (a) If \( x \in X_0 \), then \( S' \) is decomposable, \( e' \sim e - P \) and \( Y_0 = X_0' \).
   (b) If \( x \notin X_0 \) and \( -e \sim P \), then \( S' \) is decomposable, \( e' \sim e + P \) and \( Y_0 = X_0' \).
   (c) If \( x \notin X_0 \) and \( x \in X_1 \) and \( e \sim -P \), then \( S' \) is decomposable, \( e' \sim 0 \) and \( Y_0 = X_0' \).
   (d) If \( x \notin X_0 \) and \( x \notin X_1 \) and \( e \sim -P \), then \( S' \) is indecomposable, \( e' \sim 0 \) and \( Y_0 = X_0' \).

3. When \( e = 0 \) and \( e \sim 0 \):
   (a) If \( x \in X_0 \), then \( S' \) is decomposable, \( e' \sim e - P \) and \( Y_0 = X_0' \).
   (b) If \( x \in X_1 \), then \( S' \) is decomposable, \( e' \sim -e - P \) and \( Y_0 = X_1' \).
   (c) If \( x \notin X_0 \), then \( x \notin X_1 \), \( S' \) is indecomposable, \( e' \sim e + P \) and \( Y_0 = X_0' \).

4. If \( e \sim 0 \), then \( S' \) is decomposable, \( e' \sim -P \) and \( Y_0 = X_0' \).

Proof:

It is sufficient to apply Theorem 3.10 in [2], directly:
1. When $e \geq 2$:
   If $x \in X_0$ we apply the point one of Theorem 3.10 in [2].
   If $x \notin X_0$, since $e \geq 2$, $h^0(\mathcal{O}_X(-e)) > 0$ and $-e$ has not base points. By point 3 of Theorem 3.10 in [2], $S'$ is decomposable.

2. If $e = 1$, then $h^0(\mathcal{O}_X(-e)) = 1 > 0$ and $-e$ has a unique base point. Applying the four points of Theorem 3.10 in [2], we deduce four assertions.

3. If $e = 0$ and $e \not\sim 0$, then $h^0(\mathcal{O}_X(-e)) = 0$ and any point of $X$ is a base point of $-e$. Using points 1, 2 and 3 of Theorem 3.10 in [2], the conclusion follows.

4. If $e \sim 0$, then $h^0(\mathcal{O}_X(-e)) = 1$ and $-e$ has not base points. By points 1 and 3 of Theorem 3.10 in [2], the ruled surface $S'$ is always decomposable, with $e' \sim -P$ and $Y_0 = X'_0$. 

We have obtained two models of indecomposable elliptic ruled surfaces by applying an elementary transformation to a decomposable one.

If we repeat this construction with these models, there could appear new indecomposable surfaces. However, we will see that there are no other models of indecomposable elliptic ruled surfaces.

In order to get how these ruled surfaces are modified by elementary transformations, we begin by investigating their families of unisecant curves.

**Proposition 2.2** Let $\pi : S \to X$ be an indecomposable elliptic ruled surface with invariant $e = 0$ and $e \sim 0$. The complete linear system $|H| = |X_0 + bf|$ has irreducible elements if and only if $b \sim 0$ or $\deg(b) \geq 1$. Moreover, $X_0$ is the unique minimum self-intersection curve.

**Proof:**

Let us see that $X_0$ is the unique minimum self-intersection curve. Because $e = 0$, if there existed a curve $C$ with $C^2 = 0$, then $C$ would not meet $X_0$. So the surface would have two disjoint unisecant curves and it would be decomposable.

Let $D \sim X_0 + bf$ be an irreducible element different from $X_0$. It satisfies $\pi_*(X_0 \cap D) \sim b \sim b$. Then, $b$ must be effective and $\deg(b) \geq 1$.

Conversely, suppose $\deg(b) \geq 1$:

1. If $\deg(b) \geq 2$, then $b$ is nonspecial. Moreover, $b$ and $b + e$ have no common base points. So, by Corollary 1.8 in [2], the linear system has irreducible elements.
2. If $\deg(b) = 1$, then,
\[ h^0(O_S(X_0 + bf)) = h^0(O_X(b)) + h^0(O_X(b + \epsilon)) = 2 \]
because $b$ is nonspecial. Reducible elements of $|X_0 + bf|$ will contain
generators and they will left a residual unisecant curve. As $\deg(b) = 1$
this curve must have self-intersection 0, and so it is $X_0$. From this,
reducible elements of $|H|$ are in the linear subsystem $|H - X_0| = |bf|$.
Since $h^0(O_S(bf)) = 1 < h^0(O_S(X_0 + bf))$, the generic element of $|H|$ is
irreducible.

**Proposition 2.3** Let $\pi : S \rightarrow X$ be an indecomposable elliptic ruled surface
with invariant $e = -1$ and $\epsilon \sim P$. The complete linear system $|H| = |X_0 + bf|
has irreducible elements if and only if $\deg(b) \geq 0$. Moreover, there is an
unidimensional family of curves of minimum self-intersection parameterized by
the base elliptic curve $X$. Two of these curves pass through any point of each
generator $Qf$, except through four points $\{x_i, 1 \leq i \leq 4\}$. Through these points
it passes a unique curve $D_i$, with $D_i \sim X_0 + (R_i - P)f$. The points $R_i$ are the
ramification points of the morphism $|Q + P| : X \rightarrow P^1$; that is, points satisfying
$2R_i \sim Q + P$.

**Proof:** Any unisecant curve $D \sim X_0 + bf$ has self-intersection greater than or
equal to $X_0^2$, so necessarily $\deg(b) \geq 0$.

If $\deg(b) \geq 1$, then $b$ and $b + \epsilon$ are effective divisors, $b$ is nonspecial and
$b + \epsilon$ is base-point-free. By applying Corollary 1.8 in [8], we see that the generic
element of $|X_0 + bf|$ is irreducible.

In Lemma 1.1 we saw that, given a point $P \in X$, a divisor $b$ of degree 0 can
be written as $b \sim Q - P$. In this way, points $Q$ of $X$ parameterize divisors of
degree 0 on $X$.

Let $b \sim Q - P$ be a divisor of degree 0 with $Q \neq P$. Then $h^0(b) = 0$ and $b$
is nonspecial. According to Remark 1.2 in [8], we see that
\[ h^0(O_S(X_0 + bf)) = h^0(O_X(b)) + h^0(O_X(b + \epsilon)) = 1. \]
Because $\deg(b - R) < 0$, then $h^0(O_S(X_0 + (b - R)f)) = 0$ for any point $R \in P$.
Therefore, by Proposition 1.7 in [8], $|X_0 + (Q - P)|$ contains a unique irreducible
curve that will be denoted by $D_Q$.

Summarizing, we have a family of curves $D_Q$ with self-intersection 1 param-
eterized by $X$. If $Q \neq P$, then we know that $dim(|D_Q|) = 0$. Let us see that there
is a unique curve in the linear system $|X_0| = |D_P|$. Considering the trace
of $|X_0|$ on $|D_Q|$ with $P \neq Q$, we find:
\[ 0 \rightarrow H^0(O_S((P - Q)f)) \rightarrow H^0(O_S(X_0)) \xrightarrow{\alpha} H^0(O_{D_Q}(Q)) \rightarrow H^1(O_S((P - Q)f)) \]
Since $P - Q \not\sim 0$, then $h^1(O_S((P - Q)f)) = 0$ and 
\[ h^0(O_S(X_0)) = h^0(O_X(P - Q)) + h^0(O_X(Q)) = 1 \]

We now study how curves of this family intersect. It is clear that $D_Q.D_R = 1$. Moreover, $\pi_*(D_Q \cap D_R) \sim Q + R - P$; consequently, two curves $D_Q$ and $D_R$ meet at a point on the generator $Tf$, with $T$ satisfying $T + P \sim Q + R$.

In this way, given a generator $Tf$ we define the map $\sigma : X \longrightarrow P^1$ by assigning $D_Q \cap Tf$ to each point $Q$ of $X$.

Let us fix a curve $D_Q$. Other curve $D_R$ meets $D_Q$ on $Tf$ when $Q + R \sim T + P$; that is, when $R \sim T + P - Q$. Hence, there is a unique curve meeting $D_Q$ on $Tf$. This curve coincides with $D_Q$ when $2Q \sim T + P$.

We have seen that the morphism $\sigma$ is not constant and it applies two points $Q, R$ satisfying $Q + R \sim T + P$ onto a point of $P^1$. The morphism $\sigma$ is defined by the linear system $|T + P|$ on $X$. It is a 2:1 morphism from an elliptic curve onto a line. By Hurwitz Theorem ([3], IV.2.4), $\sigma$ has exactly four ramifications $\{R_i, 1 \leq i \leq 4\}$ satisfying $2R_i \sim P + Q$. It follows that a unique curve of the family passes through them.

Remark 2.4 The above proposition allows to obtain a parameterization of the indecomposable elliptic ruled surface $P(E_0)$ with $e = -1$ and $e \sim P$ in the following way.

Let $S^2X$ be the divisors of degree 2 in $X$. We define the map:
\[
S^2X \xrightarrow{\tau} P(E_0) \quad Q + R \quad \longrightarrow \quad D_Q \cap D_R
\]

The map is well defined except at most at points of the diagonal of $S^2X$. But we have seen that the curves $D_Q, D_R$ meet at a point in the generator $Tf$ such that $Q + R \sim T + P$. Thus, equivalently, the map can be defined:
\[
S^2X \xrightarrow{\tau} P(E_0) \quad Q + R \quad \longrightarrow \quad D_Q \cap Tf/Q + R \sim P + T
\]

Therefore, the map is well defined in whole $S^2X$ and it is clearly an isomorphism.

The image of the diagonal of $S^2X$ by the map $\tau$ is the focal curve $C_f$ in $P(E_0)$. It meets each generator $Tf$ precisely at the four points corresponding to the ramifications of the morphism $\varphi_{|T + P|} : X \longrightarrow P^1$. By the above proposition we know that, through these points, it passes a unique curve of the family of minimum self-intersection curves.
Let us now see what happens when we apply an elementary transformation to models of indecomposable elliptic ruled surfaces.

**Proposition 2.5** Let $\pi : S \rightarrow X$ be an indecomposable elliptic ruled surface with invariant $e = 0$ and $c \sim 0$. Let $x$ be a point of $X$ with $\pi(x) = P$. Let $S'$ be the elementary transformation of $S$ at $x$ corresponding to a normalized sheaf $E'_0$ with invariant $e'$, $e' = -\deg(e')$. Let $Y_0$ be the minimum self-intersection curve of $S'$. Then:

1. If $x \in X_0$, then $S'$ is decomposable with $e' \sim -P$ and $Y_0 = X'_0$.
2. If $x \notin X_0$, then $S'$ is indecomposable with $e' \sim P$ and $Y_0 = X'_0$.

**Proof:**

1. If $x \in X_0$, then, by applying Theorem 3.8 in [3], we obtain that $e' \sim e - P \sim -P$ and $Y_0 = X'_0$. Moreover, by Proposition 2.2, we know that there exists an irreducible curve $D \sim X_0 + Pf$. But $\pi_*(D \cap X_0) \sim P$, so $D$ meets $X_0$ at $X_0 \cap Pf = x$. Since $x \in X_0$ and $x \in D$, according to elementary transformation properties we know that $D'X'_0 = DX_0 - 1 = 0$. From this, $S'$ has two disjoint unisecant curves so it is decomposable.

2. Let us suppose $x \notin X_0$. We know $X_0$ is the unique curve of self-intersection 0. Any other unisecant curve $D$ of $S$ is in a linear system $|X_0 + bf|$ with $\deg(b) \geq 1$, so $D^2 \geq 2$. By applying elementary transformation properties, we conclude $X'_0 = X_0^2 + 1 = 1$. For any other curve $D$, we know that $D'^2 > D^2 - 1 \geq 1$. Therefore, $X'_0$ is the minimum self-intersection curve of $S'$ and $e' \sim e + P$. Finally, since $\deg(e') > 0$, the ruled surface $S'$ is indecomposable.

**Proposition 2.6** Let $\pi : S \rightarrow X$ be an indecomposable elliptic ruled surface with invariant $e = -1$ and $c \sim P$. Let $x$ a point of $S$ with $\pi(x) = Q$. Let $S'$ be the elementary transform of $S$ at $x$ corresponding to a normalized sheaf $E'_0$ with invariant $e'$, with $e' = -\deg(e')$ and let $Y_0$ be the minimum self-intersection curve of $S'$. Then:

1. If $x \in D_R$ with $D_R \sim X_0 + (R - P)f$ and $2R \sim P + Q$, then $S'$ is indecomposable, $e' \sim 0$ and $Y_0 = D'_R$.
2. If $x \in D_R \cap D_T$ with $R \neq T$ and $R + T \sim P + Q$, then $S'$ is decomposable, $e' \sim 2R - P - Q$ and $Y_0 = D'_R$. 

12
Proof:

By Proposition 2.3 we know that two curves $D_Q \sim X_0 + (R - P)f$ and $D_T \sim X_0 + (T - P)f$ satisfying $R + T \sim P + Q$ pass through a generic point of the generator $Qf$; if $T = R$, then it passes a unique curve. Hence, we have two cases:

1. If it passes a unique curve $D_R$ through $x$, then $D_R' \sim D_R^2 = D_R^2 - 1 = 0$. Any other curve $D_T$ of minimum self-intersection does not pass through $x$ so $D_T^2 = D_T^2 + 1 = 2$. Other curve $D$ of $S$ is in a linear system $|X_0 + b f|$ with $\deg(b) \geq 1$, so $D^2 \geq 3$ and $D^2 \geq D^2 - 1 \geq 1$. From this we deduce that the minimum self-intersect curve of $S'$ is $D_R'$ and this meets any other curve: $D_R' \cdot D_T' = (D_R^2 + D^2)/2 \geq 1$. It follows that $S'$ is indecomposable and $e' \sim D_R^2 \sim (X_0 + (R - P)f)^2 + Q \sim 2R - P + Q \sim 0$.

2. If $x \in D_R \cap D_T$, then, applying elementary transformation properties, we have $D_R'^2 = D_T'^2 = 0$ and $D_R' \cdot D_T' = D_R \cdot D_T - 1 = 0$. Hence, the ruled surface $S'$ is decomposable, $D_R'$ is one of the minimum self-intersection curves of $S'$ and $e' \sim D_R'^2 \sim 2R - P + Q \sim 0$.

In the above proposition we have seen that any indecomposable elliptic ruled surface with invariant $e = 0$ or $e = -1$ is obtained by applying an elementary transformation to a decomposable one.

Moreover, if we apply an elementary transformation to any of two models, we do not obtain new models of indecomposable surfaces.

Thus, we have actually proved the following theorem:

**Theorem 2.7** There only exist two models of indecomposable elliptic ruled surfaces. They have invariants $e = 0$ and $e = -1$.

**Remark 2.8** The indecomposable ruled surface $S$ with invariant $e = -1$ and $e \sim P$ is obtained by projecting a decomposable one from a generic point on a generator. If we change the generator, $e$ is modified too.

However, both models are isomorphic. We have a family of minimum self-intersection curves on $S$ parameterized by $X$. If $e \sim P$, then $O_{X_0}(X_0) \sim O_X(P)$. Taking other curve $D_R \sim X_0 + (R - P)f$, we have $O_{D_R}(D_R) \cong O_X(2R - P)$. If we take $R$ satisfying $2R \sim P + Q$, then $O_{D_R}(D_R) \cong O_X(Q)$. So, by considering $D_Q$ as a minimum self-intersection curve, we obtain $e \sim Q$.

Note that changing minimum self-intersection curve corresponds to modifying the normalized model of $S$. If we have the normalized sheaf $E_0$ with $e \sim P$, we have...
when we consider $D_R$ as minimum self-intersection curve, we are taking a new normalized sheaf $\mathcal{E}_0' = \mathcal{E}_0 \otimes \mathcal{O}_X(R - P)$.

**Remark 2.9** Nagata Theorem asserts that every geometrically ruled surface $\pi : \mathbb{P}(\mathcal{E}_0) \to X$ is obtained from $X \times \mathbb{P}^1$ by applying a finite number of elementary transformations.

We can study this fact in the elliptic ruled surfaces. Let us remember that $X \times \mathbb{P}^1$ corresponds to the ruled surface $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X)$. By applying Theorem 2.1 we can indicate with detail which is the minimum number of elementary transformations that we need to obtain each model of elliptic ruled surface:

1. The decomposable elliptic ruled surface with $e = 0$ and $\mathfrak{e} \neq 0$ is obtained by applying 2 elementary transformations to $X \times \mathbb{P}^1$ in 2 generic points.
2. The decomposable elliptic ruled surface with $e = 1$ is obtained by applying 1 elementary transformation to $X \times \mathbb{P}^1$ in a generic point.
3. The decomposable elliptic ruled surface with $e > 1$ is obtained by applying $e$ elementary transformations to $X \times \mathbb{P}^1$ in $e$ points laying in the same directrix curve $X \times \{x_0\}$.
4. The indecomposable elliptic ruled surface with $e = 0$ and $\mathfrak{e} \neq 0$ is obtained by applying 2 elementary transformations to $X \times \mathbb{P}^1$ in 2 infinitely near points: the second elementary transformation is applied in a generic point of the exceptional divisor corresponding to the first one.
5. The indecomposable elliptic ruled surface with $e = -1$ is obtained by applying 3 elementary transformations to $X \times \mathbb{P}^1$ in 3 generic points.

Let us now study base-point-free linear systems on indecomposable elliptic ruled surfaces. In this way, we will describe indecomposable elliptic scrolls.

**Proposition 2.10** Let $S$ be an indecomposable elliptic ruled surface and let $|H| = [X_0 + b f]$ be a complete linear system on $S$. $|H|$ is base-point-free if and only if $\deg(b) \geq 2 + e$.

**Proof:**

There are two cases, with $e = 0$ or $e = -1$. In both of them, if $\deg(b) \geq 2$, then $b$ is nonspecial and $b$ and $b + e$ are base-point-free. By applying Corollary 1.8 in ([2]), we deduce that the linear system $|H|$ is base-point-free. If $\deg(b) \leq 1$ we treat each case independently:
1. Let us suppose \( e = 0 \).

If \( \deg(b) \leq 0 \), then, since \( h^0(\mathcal{O}_S(X_0 + bf)) \leq h^0(\mathcal{O}_X(b)) + h^0(\mathcal{O}_X(b + \epsilon)) \) and \( h^0(\mathcal{O}_S(X_0)) = 1 \), we have \( h^0(\mathcal{O}_S(H)) \leq 1 \) and the linear system has base points.

If \( \deg(b) = 1 \), then \( b + \epsilon \sim b \) has a base point and, according to Proposition 1.5 in \( [2] \), we see that linear system \( |H| \) has a base point.

2. Let us suppose \( e = -1 \).

If \( \deg(b) \leq 0 \), then we saw that the linear system \( |X_0 + bf| \) has at most a unique curve so it has base points.

If \( \deg(b) = 1 \), then we know that \( b \) is nonspecial, so \( h^0(\mathcal{O}_S(X_0 + Pf)) = h^0(\mathcal{O}_X(P)) + h^0(\mathcal{O}_X(P + \epsilon)) \) is very ample if and only if \( \deg(b) \geq 3 + e \).

\[
\text{Proposition 2.11} \quad \text{Let } S \text{ be an indecomposable elliptic ruled surface and let } |H| = |X_0 + bf| \text{ a complete linear system on } S. \text{ } |H| \text{ is very ample if and only if } \deg(b) \geq 3 + e.
\]

\[
\text{Proof:}
\]

According to Theorem 1.10 in \( [2] \), we know that the linear system \( |H| \) is very ample if and only if \( |H| \) and \( |H - Pf| \) are base-point-free for all \( P \in X \). By the above theorem, this happens when \( \deg(b) \geq e + 2 \) and \( \deg(b - P) \geq e + 2 \); that is, when \( \deg(b) \geq e + 3 \).

\[
\text{Lemma 2.12} \quad \text{Let } S \text{ be an indecomposable elliptic ruled surface and let } |H| = |X_0 + bf| \text{ be a complete base-point-free linear system on } S. \text{ } |H| \text{ defines a regular map } \phi : S \to \mathbb{P}^N. \text{ Let } D \text{ be an unisectant curve on } S, \text{ with } D \sim X_0 + af. \text{ The curve } \phi(D) \text{ is linearly normal if and only if } \deg(a) \leq \deg(b) \text{ and } a \neq b.
\]

\[
\text{Proof:}
\]

Let us consider the trace of \( |H| \) on the curve \( D \):

\[
0 \to H^0(\mathcal{O}_S((b - a)f)) \to H^0(\mathcal{O}_S(X_0 + bf)) \to H^0(\mathcal{O}_D(X_0 + bf)) = H^0(\mathcal{O}_X(a + b + \epsilon))
\]

The curve \( \phi(D) \) is linearly normal when \( |H| \) traces on \( D \) the complete linear system \( a+b+c \); that is, when \( \alpha \) is a surjection. This happens when \( h^0(\mathcal{O}_S(X_0 + bf)) = h^0(\mathcal{O}_X(a + b + \epsilon)) + h^0(\mathcal{O}_X(b - a)). \)

Since \( |H| \) is base-point-free, by Theorem 2.10, \( \deg(b) \geq 2 + e \geq 1 \). Then \( b \) is nonspecial and we have \( h^0(\mathcal{O}_S(X_0 + bf)) = h^0(\mathcal{O}_X(b)) + h^0(\mathcal{O}_X(b + \epsilon)). \)
Moreover, according to Propositions 2.2 and 2.3, we know that \( \deg(a) \geq 0 \). From this, \( \deg(a + e + b) \geq 2 \) and divisor \( a + e + b \) is nonspecial. By Riemann-Roch, \( h^0(O_X(a + b + e)) = \deg(a) + \deg(b) + \deg(e) \) and \( h^0(O_X(b - a)) = \deg(b) - \deg(a) + h^1(O_X(b - a)) \).

As \( \deg(b) \geq 2 + e \) then \( b \) and \( b + e \) are nonspecial on \( X \) and \( h^0(O_S(X_0 + b f)) = 2 \deg(b) - e \). Thus
\[
h^0(O_S(X_0 + b f)) - (h^0(O_X(a + b + e)) + h^0(O_X(b - a))) = -h^1(O_X(b - a)),
\]
and \( \alpha \) is a surjection when \( b - a \) is nonspecial; equivalently, when \( b \not\sim a \) and \( \deg(a) \leq \deg(b) \).

**Theorem 2.13** Let \( S \) be an indecomposable elliptic ruled surface. Let \( |H| = |X_0 + b f| \) be a complete base-point-free linear system on \( S \). Denote \( b = \deg(b) \). Let \( \phi : S \rightarrow \mathbb{P}^N \) be the regular map defined by \( |H| \) on \( S \) and let \( R = \phi(S) \) be the image scroll. Then \( R \) is one of the following models:

1. **When \( \phi \) is not birational:**
   (a) If \( e = -1 \) and \( b = 1 \), then \( R \) is a plane and \( \phi \) is a 3 : 1 map from \( S \) onto \( R \).

2. **When \( \phi \) is birational:**
   (a) If \( e = 0 \) and \( b = 2 \), then \( R \) is an elliptic scroll of degree 4 in \( \mathbb{P}^3 \). It is generated by a 1 : 2 correspondence with an united point between a line and a linearly normal elliptic curve meeting at a point.
   The singular locus of \( R \) is the double line.
   \( R \) has unisecant elliptic curves of degree \( d \geq 3 \) in the linear systems \( |X_0 + a f| \) with \( \deg(a) = d - 2 \). These curves are linearly normal if and only if \( d \leq 4 \) and \( a \not\sim b \) (if \( a \sim b \) they are the hyperplane sections).
   (b) If \( e = -1 \) and \( b \geq 2 \), then \( R \) is a smooth elliptic scroll of degree \( 2b + 1 \) in \( \mathbb{P}^{2b} \). It is generated by a 1 : 1 correspondence with an united point between two linearly normal elliptic curves of degree \( b + 1 \) meeting at a point.
   \( R \) has unisecant elliptic curves of degree \( d \geq b + 1 \) in the linear systems \( |X_0 + a f| \) with \( \deg(a) = d - b + 1 \). These curves are linearly normal if and only if \( d \leq 2b + 1 \) and \( a \not\sim b \) (if \( a \sim b \) they are the hyperplane sections).
   (c) If \( e = 0 \) and \( b \geq 3 \), then \( R \) is a smooth elliptic scroll of degree \( 2b \) in \( \mathbb{P}^{2b - 1} \). It is generated by a 1 : 1 correspondence with an united
point between two linearly normal elliptic curves of degrees $b$ and $b+1$ meeting at a point.

There is a unique unisectant curve of minimum degree $b$. Moreover, $R$ has unisectant elliptic curves of degree $d \geq b+1$ in the linear systems $|X_0 + af|$ with $\deg(a) = d - 1$. These curves are linearly normal if and only if $d \leq 2b$ and $a \not\sim b$ (if $a \sim b$ they are the hyperplane sections).

Proof:

We begin by studying what happens when $|H|$ is base-point-free but not very ample. By Proposition 1.3, $|H|$ is base-point-free when $\deg(b) \geq e + 2$.

Propositions 2.2 and 2.3 determine us the families of irreducible curves and by Lemma 2.12 we know when they are linearly normal.

Note that the degree of $R$ is $(X_0 + bf)^2 = 2b - e$. $R$ lies in $\mathbb{P}^N$ with $N = h^0(O_S(X_0 + bf))$. Since $b \geq e + 2 \geq 1$, $b$ is nonspecial, $h^0(O_S(X_0 + bf)) = h^0(O_X(b)) + h^0(O_X(b + c))$ and $N = 2b - e - 1$.

In order to find the singular locus of the scroll, we will use Theorem 1.10 in [2], $\phi$ is not an isomorphism at the base points of the linear systems $|H - Pf|$.

1. If $e = -1$ and $b = 1$ then $|H| = |X_0 + Pf|$. For any point $Q \in X$, $|X_0 + (P - Q)f|$ has a unique curve $D_Q$. Their points are base points of $|X_0 + (P - Q)f|$. Moreover, by Proposition 2.3, the curves $D_Q$ fill the surface, so all points of $S$ are base points for some system $|H - Qf|$ and $\phi$ is not birational. In fact, since $h^0(O_S(X_0 + Pf)) = 3$, $R$ is a plane. Finally, $(X_0 + Pf)^2 = 3$ so the morphism $\phi$ is a $3 : 1$ map from $S$ onto a plane.

2. Let us suppose $e = 0$ and $b = 2$. $R$ is a scroll of degree 4 in $\mathbb{P}^3$. In order to find the singular locus of $R$, we study the base points of the systems $|H - Pf|$.

Since $b = 2$, $b - P$ is nonspecial, so

$$h^0(O_S(H - Pf)) = h^0(O_X(b - P)) + h^0(O_X(b + c - P)) = 2.$$

Given $Q \in X$ with $P + Q \not\sim b$, $b - P - Q$ is nonspecial and $h^0(O_S(H - (P + Q)f)) = 0$. By applying Proposition 1.5 in [2], we see that $|H - Pf|$ have not base points on $Qf$.

If $Q \in X$, but $P + Q \sim b$, then $|H - Pf| = |X_0 + Qf|$. By Proposition 1.5 in [2], the linear system $|H - Pf|$ has a base point at $X_0 \cap Qf$. Since $h^0(O_S(X_0 + Qf)) = 1$ this base point is unique on $Qf$. Moving $P$ on $X$, we see that $b - P \sim Q$ becomes any point of $X$. So all points of $X_0$ are base points for some system $|X_0 + (b - P)f|$.  

17
It follows that \( \phi \) is not an isomorphism at points of \( X_0 \). The singular locus of \( R \) is \( \phi(X_0) \), which is given by the complete linear system \( |b| \) on \( X \). Since \( b = 2 \), \( \phi(X_0) \) is a double line.

Finally, by applying Proposition 2.2 and Lemma 2.12, we know that there is a linearly normal elliptic curve of degree 3 meeting \( X_0 \) at one point. Then \( R \) is generated by a 1 : 2 correspondence with a united point between line \( \phi(X_0) \) and the elliptic curve.

3. If \( e = -1 \) and \( b \geq 2 \), the linear system is very ample. Then, \( R \) is a nonsingular elliptic scroll of degree \( 2b + 1 \) in \( \mathbb{P}^{2b} \).

By applying Proposition 2.3 and Lemma 2.12, we see that there are two linearly normal elliptic curves of degree \( b + 1 \) meeting at one point. The scroll is generated by a 1 : 1 correspondence with a united point between these curves. (Actually, we know that there is an unidimensional family of these curves parameterized by \( X \)).

4. If \( e = 0 \) and \( b \geq 3 \), the linear system is very ample. Then, \( R \) is a nonsingular elliptic scroll of degree \( 2b \) in \( \mathbb{P}^{2b-1} \).

By applying Proposition 2.2 and Lemma 2.12, we see that there are two linearly normal elliptic curves of degrees \( b \) and \( b + 1 \) meeting at one point. The scroll is generated by a 1 : 1 correspondence with a united point between these curves.

Now, we present some tables where all the elliptic scrolls of \( \mathbb{P}^N \) are described. We explain with detail their projective generation and singular loci. We indicate the degree of the divisor \( b \) providing the linear system of hyperplane sections \( |H| = |X_0 + bf| \).

### TABLE 1. ELLIPTIC SCROLLS IN \( \mathbb{P}^3 \)

| \( e = -\partial(e) \) | \( \partial(b) \) | Irreducible elements. | Projective generation. | Sing. |
|----------------------|---------------|----------------------|----------------------|------|
| \( e = 0 \)          | \( c \sim 0 \) | 2                    | \( |X_0| \)            | \( C_2^2 \) |
|                      |               | \( |X_0 + af| \)       | \( deg(a) \geq 1 \)   |      |
|                      |               | \( C_2^2 \)           | 1 united point        |      |
|                      |               |                      | Degree 4.             |      |
| \( e = 0 \)          | \( c \sim P - Q \) | 2                    | \( |X_0| \)            | \( C_2^2 \) |
|                      |               | \( |X_1| \)            | \( C_2^2 \)           |      |
|                      |               | \( |X_0 + af| \)       | \( deg(a) \geq 1 \)   |      |
|                      |               | \( C_1^2 \)           | Degree 4.             |      |
|                      |               |                      |                      |      |
| \( e = 3 \)          | \( b \sim -c \) | 3                    | \( |X_0| \)            | \( V \) |
|                      |               | \( |X_1| \)            | Cone over \( C_1^3 \) |      |
|                      |               | \( |X_0 + af| \)       | and vertex \( V = X_0 \). |      |
|                      |               | \( deg(a) \geq 4 \)   | Degree 3.             |      |
|                      |               |                      | Speciality 1.         |      |
### TABLE 2. ELLIPTIC SCROLLS IN $\mathbb{P}^N$ ($N$ odd, $N \geq 5$).

| $e = -\partial(c)$ | $\partial(b)$ | Irreducible elements. | Projective generation. | Sing. |
|-------------------|---------------|-----------------------|------------------------|------|
| $e = 0$ $c \sim 0$ | $\frac{N+1}{2}$ | $|X_0|$, $|X_0 + af|$ | $deg(a) \geq 1$ | $C_1^{\frac{N+1}{2}}$ $\to$ $C_1^{\frac{N+1}{2}+1}$ 1 united point Degree $N+1$. |
| $e = 0$ $c \sim 0$ | $\frac{N+1}{2}$ | $|X_0|$, $|X_0 + af|$ | $deg(a) \geq 2$ | $C_1^{\frac{N+1}{2}}$ $\to$ $C_1^{\frac{N+1}{2}+1}$ Degree $N+1$. |
| $e \sim P - Q$ | $\frac{N+1}{2}$ | $|X_0|$, $|X_1|$, $|X_0 + af|$ | $deg(a) \geq 1$ | $C_1^{\frac{N+1}{2}}$ $\to$ $C_1^{\frac{N+1}{2}+1}$ Degree $N+1$. |
| $2 \leq e < N-3$ | $\frac{N+1+e}{2}$ | $|X_0|$, $|X_1|$, $|X_0 + af|$ | $deg(a) \geq e+1$ | $C_1^{\frac{N+1+e}{2}}$ $\to$ $C_1^{\frac{N+1+e}{2}+1}$ Degree $N+1$. |
| $e = N-3$ $c \sim 0$ | $\frac{N+1}{2}$ | $|X_0|$, $|X_1|$, $|X_0 + af|$ | $deg(a) \geq N-2$ | $C_1^{\frac{N+1}{2}}$ $\to$ $C_1^{\frac{N+1}{2}+1}$ Degree $N+1$. |
| $e = N$ $b \sim -c$ | $\frac{N}{2}$ | $|X_0|$, $|X_1|$, $|X_0 + af|$ | $deg(a) \geq N+1$ | Cone over $C_1^{\frac{N}{2}}$ and vertex $V(=X_0)$. Degree $N$. Speciality 1. |

### TABLE 3. ELLIPTIC SCROLLS IN $\mathbb{P}^N$ ($N$ even, $N \geq 4$).

| $e = -\partial(c)$ | $\partial(b)$ | Irreducible elements. | Projective generation. | Sing. |
|-------------------|---------------|-----------------------|------------------------|------|
| $e = -1$ $c \sim P$ | $\frac{N}{2}$ | $|X_0 + a|$ | $deg(a) \geq 0$ | $C_1^{\frac{N}{2}+1}$ $\to$ $C_1^{\frac{N}{2}+1}$ 1 united point Degree $N+1$. |
| $1 \leq e < N-3$ | $\frac{N+1+e}{2}$ | $|X_0|$, $|X_1|$, $|X_0 + af|$ | $deg(a) \geq e+1$ | $C_1^{\frac{N+1+e}{2}}$ $\to$ $C_1^{\frac{N+1+e}{2}+1}$ Degree $N+1$. |
| $e = N-3$ $c \sim 0$ | $\frac{N+1}{2}$ | $|X_0|$, $|X_1|$, $|X_0 + af|$ | $deg(a) \geq N-2$ | $C_1^{\frac{N+1}{2}}$ $\to$ $C_1^{\frac{N+1}{2}+1}$ Degree $N+1$. |
| $e = N$ $b \sim -c$ | $\frac{N}{2}$ | $|X_0|$, $|X_1|$, $|X_0 + af|$ | $deg(a) \geq N+1$ | Cone over $C_1^{\frac{N}{2}}$ and vertex $V(=X_0)$. Degree $N$. Speciality 1. |
3 2-secant linear systems on an elliptic ruled surface.

We study the families of 2-secant curves on an elliptic ruled surface. In order to get this, we work with the linear systems $|2X_0 + bf|$. We investigate when are they base-point-free and very ample. Then, we apply Bertini Theorems to determine when is the generic element irreducible. We begin by working with the decomposable elliptic ruled surfaces.

**Proposition 3.1** Let $S$ be a decomposable elliptic ruled surface and let $|H| = |2X_0 + bf|$ be a complete 2-secant linear system. Then, $|H|$ is base-point-free if and only if:

1. $b \sim -2e$ or $\deg(b) \geq 2e + 2$, when $e > 0$.
2. $b \sim 0$ or $\deg(b) \geq 2$, when $e = 0$ and $e \sim 0$.
3. $b \sim 0$ and $2e \sim 0$ or $\deg(b) \geq 2$, when $e = 0$ and $e \not\sim 0$.

**Proof:** We use Proposition 2.11 in [2]. The linear system $|H|$ is base-point-free when $b$ and $b + 2e$ are base-point-free:

1. If $e > 0$, then $b + 2e$ is base-point-free when $b \sim -2e$ or $b \geq 2e + 2$. In both cases, $b$ is base-point-free too, because $e > 0$.
2. Let us suppose $e = 0$ and $e \sim 0$. Then, it is sufficient that $b$ is base-point-free, that is, $b \sim 0$ or $b \geq 2$.
3. Let us suppose $e = 0$ and $e \not\sim 0$. Then, if $2e \sim 0$ we are in the above situation. When $2e \not\sim 0$, if $b \sim 0$, then $b + 2e \not\sim 0$ and it has base points; and conversely. Thus $b$ and $b + 2e$ are base-point-free when $b \geq 2$.

**Proposition 3.2** Let $S$ be a decomposable elliptic ruled surface and let $|H| = |2X_0 + bf|$ be a complete 2-secant linear system. Then, $|H|$ is very ample if and only if $\deg(b) \geq 2e + 3$.

**Proof:**

By Theorem 2.13 in [2], we know that the linear system is very ample if and only if $b$ and $b + 2e$ are very ample, that is, if $\deg(b) \geq 3$ and $\deg(b) \geq 2e + 3$. Since $e \geq 0$, it is sufficient that $b \geq 2e + 3$. 

20
Proposition 3.3 Let $S$ be a decomposable elliptic ruled surface and let $|H| = |2X_0 + bf|$ be a complete 2-secant linear system. Then the generic element of $|H|$ is irreducible if only if it satisfies one of the following conditions:

1. $\deg(b) \geq 2e + 2$.
2. $\deg(b) = 2e + 1$ and $e \neq 0$.
3. $b \sim -2e$ and $e > 0$.
   $b \sim -2e$, $e = 0$, $e \neq 0$ and $2e \sim 0$.

Moreover, if the generic element is irreducible then it is smooth too and its genus is $\deg(b + e) + 1$.

Proof:

We can estimate the genus by using the formula

$$g(C + D) = g(C) + g(D) + CD - 1$$

by using (3, V, ex. 1.3.) and by taking $C = X_0$ and $D = X_0 + bf$.

If $D \sim 2X_0 + bf$ is irreducible, the divisors $\pi_*(D \cap X_0) \sim b + 2e$ and $\pi_*(D \cap X_1) \sim b$ must be effective, so necessarily $\deg(b) \geq 2e$.

Let $\varphi_H : S \to \mathbb{P}^N$ be the rational map defined by the linear system $|H|$:

1. If $\deg(b) \geq 2e + 3$, the linear system $|H|$ is very ample. So the generic element is irreducible and smooth.
2. If $\deg(b) = 2e + 2$, the linear system $|H|$ is base-point-free. In particular, $b$, $b + e$ and $b + 2e$ are base-point-free. By Proposition 2.12 in [4], the map $\varphi_H$ applies the generators onto nonsingular conics. Moreover, the image of $X_0$ is given by the complete linear system $|b + 2e|$. Since it has degree 2, it is a double line. Then $\dim(\text{Im}(\varphi_H)) = 2$ and by Bertini Theorem, we deduce that the generic element of the linear system is irreducible and smooth.

3. If $\deg(b) = 2e + 1$ and $e > 0$, by Proposition 2.11 in [4], we see that the linear system $|H|$ has a unique base point at $X_0 \cap Pf$, where $b + 2e \sim P$. A generic generator $Qf$ is applied onto a nonsingular conic. The image of $X_1$ is given by the complete linear system $|b|$ on $X$. Since $\deg(b) \geq 3$, $\varphi_H(X_1)$ is a nonsingular elliptic curve. Thus $\dim(\text{Im}(\varphi_H)) = 2$ and by Bertini Theorem, the generic element of $|H|$ is irreducible and it has at most a singular point at $X_0 \cap Pf$. But $HX_0 = 1$, so the generic element of $H$ meets $X_0$ at a unique point and it is smooth at the points of $X_0$. 

21
4. If \( \text{deg}(b) = 2e + 1 \), \( e = 0 \) and \( e \neq 0 \), the linear system has base points at \( X_0 \cap P_0 f \) and \( X_1 \cap P_1 f \), where \( b + 2e \sim P_0 \) and \( b \sim P_1 \). Anyway, the generic generator is applied onto a nonsingular conic. Moreover, \( b + e \sim P \) and \( P \neq P_i \) because \( e \neq 0 \). Therefore, \( P f \) is applied onto a double line (see the proof of ([2], 2.11)). Thus \( \dim(\text{Im}(\varphi|_H)) = 2 \) and by Bertini Theorem the generic element of \( |H| \) is irreducible and it has at most singular points at \( X_i \cap P_i f \). Since \( HX_i = 1 \), the generic element is smooth at the points of \( X_0 \) and \( X_1 \).

5. If \( \text{deg}(b) = 2e + 1 \) and \( e \sim 0 \), then \( h^0(\mathcal{O}_{S}(2X_0 + bf)) = 3 = h^0(\mathcal{O}_{S}(2X_0)) \), so \( bf \) is a fixed component of the linear system and the generic element is reducible.

6. If \( \text{deg}(b) = 2e \), then necessarily \( b \sim -2e \), because \( b + 2e \) must be an effective divisor.

If \( e > 0 \), the linear system is base-point-free. The generators are applied onto nonsingular conics. The curve \( X_1 \) is applied onto an elliptic curve of degree \( b \geq 2 \), so \( \dim(\text{Im}(\varphi|_H)) = 2 \) and by Bertini Theorem, the generic element of the system is irreducible and smooth.

If \( e = 0 \) and \( e \neq 0 \), since \( b \) must be an effective divisor, \( -2e \sim 0 \). In this case the linear system is base-point-free. By Bertini Theorem the generic element is smooth. A smooth element in \( |H| \) does not contain generators. If it is reducible, it has two disjoint unisecant curves. They must be \( X_0 + X_1 \). Since \( h^0(\mathcal{O}_{S}(2X_0 - 2ef)) = 2 \), the reducible elements don’t fill the linear system. Thus the generic element is irreducible and smooth.

If \( e = 0 \) and \( e \sim 0 \), then \( h^0(\mathcal{O}_{S}(2X_0)) = 3 \). Since \( h^0(\mathcal{O}_{S}(X_0)) = 2 \), \( |H| = \{D + D'/D, D' \sim X_0\} \), so the generic element is reducible. In fact, \( \text{Im}(\varphi|_H) \) is a conic whose hyperplane sections parameterize the curves of \( |2X_0| \).

Now, let us study the 2-secant linear systems in the indecomposable ruled surfaces. First, we will generalize some results about \( m \)-secant divisors on decomposable ruled surfaces that appear in ([2], 2).

**Lemma 3.4** Let \( \pi : S \rightarrow X \) be a geometrically ruled surface and let \( \nu : S' \rightarrow S \) be the elementary transformation at the point \( x \in S \), \( x \in Pf \). Let \( C \) be a \( m \)-secant and \( a \) a divisor on \( X \). Then:

1. \( |\nu^*(C) + af| \cong |C + (a + mP)f - mx| \).
2. \( h^0(\mathcal{O}_{S'}(\nu^*(C) + af)) = h^0(\mathcal{O}_{S}(C + (a + mP)f - mx)) \).
Proof:

Let $|C + bf|$ be a $m$-secant complete linear system in $S$. Let $D$ be a curve of the linear system. Then $\nu^\ast(D) \sim \nu^\ast(C) + bf$ and $D' + \mu_x(D)Pf \sim \nu^\ast(C) + bf$. From this, $D' \sim \nu^\ast(C) + (b - \mu_x(D)P)f$ and the elements of the linear system $|\nu^\ast(C) + af|$ come from the elements of the linear systems $|C + (a + kP)f - kx|$: \[h^0(\mathcal{O}_S(\nu^\ast(C) + af)) = \max\{h^0(\mathcal{O}_S(C + (a + kP)f - kx))/k = 0, \ldots, m\}\] Since $|C + (a + kP)f - kx| \subset |C + (a + (k + 1)P)f - (k + 1)x|$, the conclusion follows.

Lemma 3.5 Let $S$ be a ruled surface and let $|H| = |mX_0 + bf|$ be a $m$-secant linear system on $S$. Then:

\[h^0(\mathcal{O}_S(X_0 + bf)) \leq \sum_{k=0}^{m} h^0(\mathcal{O}_X(b + k\mathfrak{e}))\]

Moreover, if $b, \ldots, b + (m - 1)\mathfrak{e}$ are nonspecial divisors then the equality holds and $h^1(\mathcal{O}_S(mX_0 + bf)) = h^1(\mathcal{O}_X(b + m\mathfrak{e}))$.

Proof:

The proof is by induction on $m$.

If $m = 1$ we consider the exact sequence:

\[0 \rightarrow \mathcal{O}_S(bf) \rightarrow \mathcal{O}_S(X_0 + bf) \rightarrow \mathcal{O}_{X_0}(X_0 + bf) \rightarrow 0\]

By applying cohomology we have:

\[
\begin{align*}
0 & \rightarrow H^0(\mathcal{O}_X(bf)) \rightarrow H^0(\mathcal{O}_S(X_0 + bf)) \rightarrow H^0(\mathcal{O}_X(b + \mathfrak{e})) \rightarrow \\
& \rightarrow H^1(\mathcal{O}_X(bf)) \rightarrow H^1(\mathcal{O}_S(X_0 + bf)) \rightarrow H^1(\mathcal{O}_X(b + \mathfrak{e})) \rightarrow \\
& \rightarrow 0
\end{align*}
\]

Then $h^0(\mathcal{O}_S(X_0 + bf)) \leq h^0(\mathcal{O}_X(b)) + h^0(\mathcal{O}_X(b + \mathfrak{e}))$. Moreover, we see that if $b$ is nonspecial the equality holds and $h^1(\mathcal{O}_S(X_0 + bf)) = h^1(\mathcal{O}_X(b + \mathfrak{e}))$.

Let us suppose that the formula holds for $m - 1$. Let $|H| = |mX_0 + bf|$. Consider the exact sequence:

\[0 \rightarrow \mathcal{O}_S(H - X_0) \rightarrow \mathcal{O}_S(H) \rightarrow \mathcal{O}_{X_0}(H) \rightarrow 0\]

By applying cohomology, we get the long exact sequence:

\[
\begin{align*}
0 & \rightarrow H^0(\mathcal{O}_S(H - X_0)) \rightarrow H^0(\mathcal{O}_S(H)) \rightarrow H^0(\mathcal{O}_X(b + m\mathfrak{e})) \rightarrow \\
& \rightarrow H^1(\mathcal{O}_S(H - X_0)) \rightarrow H^1(\mathcal{O}_S(H)) \rightarrow H^1(\mathcal{O}_X(b + m\mathfrak{e})) \rightarrow \\
& \rightarrow 0
\end{align*}
\]
where $H - X_0 \sim (m-1)X_0 + bf$. Then $h^0(\mathcal{O}_S(mX_0 + bf)) \leq h^0(\mathcal{O}_S((m-1)X_0 + bf)) + h^0(\mathcal{O}_X(b + mc))$. We see that if $b, \ldots, b + (m-2)c$ are nonspecial, then by induction hypothesis $h^1(\mathcal{O}_S((m-1)X_0 + bf)) = h^1(\mathcal{O}_X(b + (m-1)c))$. Moreover, if $b + (m-1)c$ is nonspecial too, then the equality holds and $h^1(\mathcal{O}_S(mX_0 + bf)) = h^1(\mathcal{O}_X(b + mc))$.

**Lemma 3.6** Let $S$ be a ruled surface and let $|H| = |mX_0 + bf|$ be a complete $m$-secant linear system. If

$$h^0(\mathcal{O}_S(mX_0 + (b-P)f)) = h^0(\mathcal{O}_S(mX_0 + bf)) - (m+1)$$

then the linear system is base-point-free on the generator $Pf$. Moreover, this is applied on a linearly normal smooth rational curve of degree $m$ by the rational map $\varphi_{|mX_0 + bf|} : S \rightarrow \mathbb{P}^N$.

**Proof:**

Let us consider the trace of the linear system $|H|$ on $Pf$:

$$0 \rightarrow H^0(\mathcal{O}_S(mX_0 + (b-P)f)) \rightarrow H^0(\mathcal{O}_S(mX_0 + bf)) \xrightarrow{\alpha} H^0(\mathcal{O}_{\mathbb{P}^1}(m))$$

If $h^0(\mathcal{O}_S(mX_0 + (b-P)f)) = h^0(\mathcal{O}_S(mX_0 + bf)) - (m+1)$, then the map $\alpha$ is a surjection and $|H|$ traces the complete linear system of divisors of degree $1$ in $\mathbb{P}^1$. Thus $|H|$ is base-point-free on the generator $Pf$ and it is applied onto a linearly normal rational curve of degree $m$ by the rational map $\varphi_{|mX_0 + bf|}$.

**Lemma 3.7** Let $S$ be a ruled surface and let $|H| = |mX_0 + bf|$ be a complete $m$-secant linear system:

1. If $P$ is a base point of $b + mc$, then $H$ has a base point at $Pf \cap X_0$.
2. If $|H|$ is very ample then $|b + mc|$ is very ample.

**Proof:**

Let us consider the trace of the linear system $|H|$ on the curve $X_0$:

$$H^0(\mathcal{O}_S(H - X_0)) \rightarrow H^0(\mathcal{O}_S(H)) \rightarrow H^0(\mathcal{O}_{X_0}(H)) \cong H^0(\mathcal{O}_X(b + mc))$$

1. If $P$ is base point of $b + mc$, then all divisors of $|H|$ meet $X_0$ at $X_0 \cap Pf$, so this is a base point of $|H|$.
2. If $|H|$ is very ample, then the rational map $\varphi_{|H|}$ is an isomorphism. In particular, the restriction $\varphi_{|H|}|_{X_0}$ is an isomorphism, so $b + mc$ must be very ample.
**Proposition 3.8** Let $S$ be a ruled surface and let $|H| = |mX_0 + bf|$ be a complete $m$-secant linear system. Let $\varphi_{|H|} : S \to \mathbb{P}^N$ be the rational map defined by $|H|$. If $\varphi_{|H|}$ is an isomorphism on the generators and the linear systems $|mX_0 + (b - P)f|$ are base-point-free then the linear system $|H|$ is very ample.

**Proof:**

Since $\varphi_{|H|}$ is an isomorphism on the generators, the linear system $|H|$ is base-point-free. Let us see that $|H|$ separates points and tangent vectors.

Let $x, y \in S$, $x \in Pf$, $y \in Qf$:

1. If $Q = P$, then, since the restriction of the rational map $\varphi_{|H|}$ to the generators defines an isomorphism, the linear system separates points on the same generator.

2. If $Q \neq P$, since the linear system $|mX_0 + (b - P)f|$ is base-point-free, there exists a divisor $D \sim mX_0 + (b - P)f$ such that $y \notin D$. Thus $D + Pf \sim H$, $x \in D + Pf$, but $y \notin D + Pf$.

Let $x \in S$, $x \in Pf$, $t \in T_X(S)$:

1. If $t \in T_x(Pf)$, then there is a divisor which meets $Pf$ at $x$ transversally, because the restriction of the rational map $\varphi_{|H|}$ to the generators is an isomorphism.

2. If $t \notin T_x(Pf)$, since the linear system $|mX_0 + (b - P)f|$ is base-point-free, there exists a divisor $D \sim mX_0 + (b - P)f$ such that $x \notin D$. Then, $D + Pf \sim H$, $x \in D + Pf$, but $t \notin T_x(Pf) = T_x(D + Pf)$.

Now, we restrict our attention to the study of 2-secant linear systems on the indecomposable elliptic ruled surface with $e = 0$. Let us remember that it is obtained by applying a elementary transformation $\nu^* : S \to S_0$ at the point $x \in Pf$, $x \notin X_0 \cup X_1$ to the surface $S_0 = \mathbb{P}(O_X \oplus O_X(-P))$.

**Proposition 3.9** Let $S$ be the indecomposable elliptic ruled surface with $e = 0$. Let $|H| = |2X_0 + bf|$ be an $m$-secant linear system on $S$.

1. If $\text{deg}(b) \geq 1$, then $h^0(O_S(2X_0 + bf)) = 3\text{deg}(b)$.

2. If $\text{deg}(b) = 0$ and $b \neq 0$ or $\text{deg}(b) \leq 0$, then $h^0(O_S(2X_0 + bf)) = 0$.

3. If $b \sim 0$, then $h^0(O_S(2X_0 + bf)) = h^0(O_S(2X_0)) = 1$.

**Proof:**
1. If $\deg(b) \geq 1$, then $b$ and $b + \mathbf{e}$ are nonspecial. We can apply Lemma 3.3 and we obtain that $h^0(O_S(2X_0 + bf)) = 3h^0(O_X(b)) = 3\deg(b)$.

2. If $\deg(b) = 0$ and $b \neq 0$, $b$ is nonspecial and by Lemma 3.3, $h^0(O_S(2X_0 + bf)) = 0$. If $\deg(b) < 0$, then the inequality $h^0(O_S(2X_0 + bf)) \leq 3h^0(O_X(b))$ holds, where $3h^0(O_X(b)) = 0$.

3. If $b \sim 0$, since $S$ is the elementary transformation of $S_0$ and by Lemma 3.4, $h^0(O_S(2X_0)) = h^0(O_{S_0}(2X_0 + 2Pf - 2x))$. As we see at the section 2) the linear system $|2X_0 + 2Pf|_{S_0}$ defines the linear subsystem of divisors of degree 2 of $P^1$ generated by the homogeneous polynomials $x_0^2$ and $x_1^2$ on $Pf$. From this, the elements of $|2X_0 + 2Pf|_{S_0}$ have at most double points in $X_0$ or $X_1$, except when they contain the generator $Pf$. Thus, since $x \notin X_0 \cup X_1$, $|2X_0 + 2Pf - 2x|_{S_0} = |2X_0 + Pf - x|_{S_0}$. But $x$ is not a base-point of any linear system, because it does not lie on the curves $X_0$ and $X_1$. Consequently:

\[
h^0(O_S(2X_0)) = h^0(O_{S_0}(2X_0 + 2Pf - 2x)) = h^0(O_{S_0}(2X_0 + Pf - x)) = h^0(O_{S_0}(2X_0 + Pf)) - 1 = 1
\]

Proposition 3.10 Let $S$ be the indecomposable elliptic ruled surface with $e = 0$. Let $|H| = |2X_0 + bf|$ be an $m$-secant linear system on $S$. Then:

1. The linear system $|H|$ is base-point-free if and only if $\deg(b) \geq 2$.

2. If $\deg(b) = 1$, then the linear system $|H|$ has a unique base point at $bf \cap X_0$.

Proof:

1. By Lemma 3.3, if the linear system $|H|$ is base-point-free, necessarily $b + 2\mathbf{e}$ is base-point-free, that is, $\deg(b) \geq 2$ or $b \sim 0$. But, if $b \sim 0$, then $h^0(O_X(2X_0)) = 1$ and the linear system has base points. Conversely, if $\deg(b) \geq 2$, by Lemma 3.3 we know that $h^0(O_S(H - Pf)) = h^0(O_S(H)) - 3$. Applying Lemma 3.6 we deduce that the linear system is base-point-free and the regular map defined by $|H|$ apply the generators onto nonsingular conics.

2. Let us suppose $\deg(b) = 1$ and let $b \sim P_0$. Then $h^0(O_X(H)) = 3$ and $h^0(O_X(H - Pf)) = 0$, except when $P_0 = P$. In this case $h^0(O_X(H - Pf)) = 1$.

26
Thus, the linear system $|H|$ has at most base points in the generator $P_0f$. Moreover, since $h^0(O_X(H - P_0f)) = h^0(O_X(H)) - 2$, the complete linear system $|H|$ traces a 1-codimension linear subsystem of the divisors of degree 2 on $P_0f$. Therefore the linear system $|H|$ can have at most a base-point in $P_0f$. Since $b + 2e \sim b$ has a base point $P_0$ and applying Lemma 3.7, the conclusion follows.

**Proposition 3.11** Let $S$ be the indecomposable elliptic ruled surface with $e = 0$. The $m$-secant linear system $|H| = |2X_0 + bf|$ is very ample if and only if $\deg(b) \geq 3$.

**Proof:** By Lemma 3.7, if the linear system is very ample, then $b + 2e$ is very ample, so $\deg(b) \geq 3$.

Conversely, if $\deg(b) \geq 3$, then $h^0(O_S(H - Pf)) = h^0(O_S(H)) - 3$ for any generator $Pf$ and $|H - Pf|$ is base-point-free. From Lemmas 3.6 and 3.8 we deduce that $|H|$ is very ample.

**Proposition 3.12** Let $S$ be the indecomposable elliptic ruled surface with $e = 0$. The generic element of the 2-secant linear system $|H| = |2X_0 + bf|$ is irreducible if and only if $\deg(b) \geq 1$. Moreover, if the generic element is irreducible then it is smooth and it has genus $\deg(b) + 1$.

**Proof:** We can estimate the genus by using the formula $g(C + D) = g(C) + g(D) + C.D - 1$ (§VI, ex. 1.3.) taking $C = X_0$ and $D = X_0 + bf$.

If $\deg(b) \geq 3$ the linear system $|H|$ is very ample, so the generic element is irreducible and smooth.

If $\deg(b) = 2$ the linear system $|H|$ is base-point-free. Since $h^0(O_S(H - Pf)) = h^0(O_S(H)) - 3$ for any $P \in X$, $|H|$ defines a regular map which applies the generators onto smooth conics. Let us see which is the image of $X_0$ by this map:

$0 \to H^0(O_S(H - X_0)) \to H^0(O_S(H)) \xrightarrow{\alpha} H^0(O_{X_0}(X_0 + bf)) \cong H^0(O_X(b))$}

Because $\deg(b) = 2$ and $h^0(O_S(2X_0 + bf)) - h^0(O_S(X_0 + bf)) = 2$, the linear system $|H|$ traces a complete linear system of degree 2 on $X$. Thus $X_0$ is applied onto a double line. $\text{Dim}(\text{Im}(\varphi|_H)) = 2$ and by the Bertini Theorem, the generic element is irreducible and smooth.

If $\deg(b) = 1$, then the linear system $|H|$ has a unique base point at $X_0 \cap Pf$, with $b \sim P$. The generators (except $Pf$) are applied onto nonsingular conics.
The image of $Pf$ is the projection of a smooth conic from a point, that is, a line. Thus, $\dim(\text{Im}(\varphi_H)) = 2$ and by Bertini Theorem, the generic element is irreducible and it has at most a singular point at $X_0 \cap Pf$. But $HX_0 = 1$, so the curves of $|H|$ are smooth at points of $X_0$.

If $\deg(b) \leq 0$, then $h^0(\mathcal{O}_S(H)) \leq 1$, so the linear system has base points. ■

Let us study the 2-secant linear systems in the indecomposable elliptic ruled surface with $e = -1$.

Let us remember that this surface is obtained by applying a elementary transformation $\nu : S \rightarrow S_0$ at a point $x \in Qf$, $x \notin X_0 \cup X_1$ to the decomposable elliptic ruled surface $S_0 = \mathbf{P}(\mathcal{O}_X \oplus \mathcal{O}_X(e_0))$ with $\deg(e_0) = 0$ and $e_0 \neq 0$. Thus $\mathfrak{c} \sim e_0 + Q$ and we will take $\mathfrak{c} \sim P_0$.

Moreover, we know that $S$ has an one-dimensional family of curves of minimum self-intersection parameterized by $X$. Given $Q \in X$ we have the curve $D_Q = X_0 + (Q - P_0)f$.

**Proposition 3.13** Let $S$ be the indecomposable elliptic ruled surface with $e = -1$ and $\mathfrak{c} \sim P_0$. Let $|H| = |2X_0 + bf|$ a 2-secant linear system on $S$.

1. If $\deg(b) \geq 0$, then $h^0(\mathcal{O}_S(2X_0 + bf)) = 3\deg(b) + 3$.
2. If $\deg(b) = -1$, we have that:
   
   (a) if $-2b \neq 2P_0$ or $-b \sim P_0$ then $h^0(\mathcal{O}_S(2X_0 + bf)) = 3\deg(b) + 3 = 0$.
   
   (b) if $-2b \sim 2P_0$ and $-b \neq P_0$ then $h^0(\mathcal{O}_S(2X_0 + bf)) = 1$.
3. If $\deg(b) \leq -2$, then $h^0(\mathcal{O}_S(2X_0 + bf)) = 0$.

**Proof:**

1. If $\deg(b) > 0$ and $b \neq 0$, then $b$ and $b + \mathfrak{c}$ are nonspecial divisors; by Lemma 3.3 we obtain that $h^0(\mathcal{O}_S(2X_0 + bf)) = 3\deg(b) + 3$.

If $b \sim 0$, then $|2X_0 + bf| = |2D_Q + 2(P_0 - Q)|$ for any $Q$ with $2(P_0 - Q) \neq 0$. Taking $X_0 = D_Q$, we can apply Lemma 3.3 again and we obtain $h^0(\mathcal{O}_S(2X_0 + bf)) = 3$.

2. Let us suppose $\deg(b) = -1$ and let $b \sim -P$. By Lemma 3.4 we know that $h^0(\mathcal{O}_S(2X_0 - Pf)) = h^0(\mathcal{O}_{S_0}(2X_0 + (2Q - P)f - 2x))$ where $S_0$ is the decomposable ruled surface with $\mathfrak{c} = P_0 - Q$.

If $P \neq P_0$, we take $Q = P$. Then $h^0(\mathcal{O}_{S_0}(2X_0 + Qf)) = 3$. Moreover, we know that $|2X_0 + Qf|_{S_0}$ traces the linear subsystem of divisors of degree 2 generated by the polynomials $\{\lambda_0x_0^2, \lambda_1x_0x_1, \lambda_2x_1^2\}$ on $Qf$, where
\(\lambda_i = h^0(\mathcal{O}_X(Q + i\mathfrak{c}_0)) - h^0(\mathcal{O}_X(i\mathfrak{c}_0)).\) In particular, \(\lambda_0 = 0\) and since \(x \notin X_0 \cup X_1\), there is not curves passing through \(x\) with multiplicity 2 in \([2X_0 + Qf]_{S_0}\), except the reducible elements that contain the generator \(Qf\). From this, \([2X_0 + Qf - 2x]_{S_0} = [2X_0 - x]_{S_0}\). If \(2\mathfrak{c}_0 \neq 0\), that is, if \(2P_0 \neq 2Q\), then \(h^0(\mathcal{O}_{S_0}(2X_0)) = 1\) and \(h^0(\mathcal{O}_{S_0}(2X_0 - x)) = 0\). If \(2\mathfrak{c} \sim 0\), that is, if \(2P_0 \sim 2Q\), then \(h^0(\mathcal{O}_{S_0}(2X_0)) = 2\) and because the linear system \([2X_0]_{S_0}\) is base-point-free, \(h^0(\mathcal{O}_{S_0}(2X_0 - x)) = 1\).

If \(P = P_0\), we take \(Q \neq P\). Arguing as in the above case, we see that \([2X_0 + (2Q - P)f - 2x]_{S_0} = [2X_0 + (Q - P) - x]_{S_0} = [X_0 + X_1 - x]_{S_0};\) but \(h^0(\mathcal{O}_{S_0}(X_0 + X_1)) = 1\) and since \(x \notin X_0 \cap X_1\), we deduce that \(h^0(\mathcal{O}_{S_0}(X_0 + X_1 - x)) = 0\).

3. If \(\deg(b) \leq 2\), then \([2X_0 + bf] \subset [2X_0 - Qf]\) where \(Q\) verifies \(2Q \neq 2P_0\).

   By the above discussion, \(h^0(\mathcal{O}_S(2X_0 + bf)) \leq h^0(\mathcal{O}_S(2X_0 - Qf)) = 0\). \(\blacksquare\)

**Proposition 3.14** Let \(S\) be the indecomposable elliptic ruled surface with \(e = -1\). Let \(|H| = [2X_0 + bf]\) be a 2-secant linear system on \(S\). The linear system \(|H|\) is base-point-free if and only if \(\deg(b) \geq 0\).

**Proof:**

If \(\deg(b) > 0\), then \(h^0(\mathcal{O}_S(2X_0 + bf)) - h^0(\mathcal{O}_S(2X_0 + (b - P)f)) = 3\) for any \(P \in X\); by Lemma 3.6 the linear system is base-point-free.

If \(\deg(b) = 0\), let us suppose that the linear system \([2X_0 + bf]\) has a base point at \(x \in Pf\). The family of curves \(\{D_Q\}\) fills the surface, so there exists a curve \(D_Q \sim X_0 + (Q - P)f\) passing through \(x\). Let us consider the trace of the linear system \(|H|\) on \(D_Q\):

\[
0 \rightarrow h^0(\mathcal{O}_S(X_0 + (P_0 - Q)f)) \rightarrow h^0(\mathcal{O}_S(2X_0 + bf)) \xrightarrow{\alpha} h^0(\mathcal{O}_{D_Q}(2X_0 + bf)) \cong h^0(\mathcal{O}_X(b + 2Q))
\]

We have that \(\dim(\text{Im}(\alpha)) = h^0(\mathcal{O}_S(2X_0 + bf)) - h^0(\mathcal{O}_S(X_0 + (b + P_0 - Q)f)) = 2 = h^0(\mathcal{O}_X(b + 2Q))\). Since \([b + 2Q]\) is base-point-free, the system can not have base points on \(D_Q\), so we get a contradiction.

If \(\deg(b) < 0\), then \(h^0(\mathcal{O}_S(2X_0 + bf)) \leq 1\) and the linear system has base points. \(\square\)

**Proposition 3.15** Let \(S\) be the indecomposable elliptic ruled surface with \(e = -1\). The 2-secant linear system \(|H| = [2X_0 + bf]\) is very ample if and only if \(\deg(b) \geq 1\).

**Proof:**
By Lemma 3.7 we know that if $|H|$ is very ample then $b + 2e$ must be very ample, that is, $\deg(b) \geq 1$.

Conversely, if $\deg(b) \geq 1$, then $h^0(O_S(|H - Pf|)) = h^0(O_S(|H|)) - 3$ for any generator $Pf$; moreover, $|H - Pf|$ is base-point-free. From Lemmas 3.6 and 3.8 we deduce that $|H|$ is very ample.

Proposition 3.16 Let $S$ be the indecomposable ruled surface with $e = -1$ and $c \sim P_0$. The generic element of the 2-secant linear system $|H| = |2X_0 + bf|$ is irreducible if and only if $\deg(b) \geq 0$ or $b \sim -Q$ with $2Q \sim P_0$ and $Q \not\sim P_0$ (that is, $-b$ is one of the three ramification points different from $P_0$ of the map defined by the divisor $2P_0$ on $X$). Moreover, if the generic element of $|H|$ is irreducible, then it is smooth and it has genus $2\deg(b) + 2$.

Proof:

The genus follows from the formula $g(C + D) = g(C) + g(D) + CD - 1$ (3.15, V, ex. 1.3.) taking $C = X_0$ and $D = X_0 + bf$.

If $\deg(b) \geq 1$ the linear system $|H|$ is very ample, so the generic element is irreducible and smooth.

If $\deg(b) = 0$ the linear system $|H|$ is base-point-free. It defines a regular map $\varphi$ which applies the generators onto smooth conics. Let us see which is the image of the curve $X_0$:

$0 \rightarrow H^0(O_S(H - X_0)) \rightarrow H^0(O_S(H)) \xrightarrow{\alpha} H^0(O_{X_0}(H)) \cong H^0(O_{X}(b + 2P_0))$

Since $\deg(b) + 2P_0 = 2$ and $h^0(O_S(2X_0 + bf)) - h^0(O_S(X_0 + bf)) = 2$, the linear system $|H|$ traces a complete linear system of degree 2 in $X_0$. Thus, $X_0$ is mapped onto a double line. $\dim(\text{Im}(\varphi|_H)) = 2$ and by Bertini Theorem the generic element of $|H|$ is irreducible and smooth.

If $\deg(b) \leq -1$, then $h^0(O_S(2X_0 + bf)) = 0$ except when $b \sim -Q$ with $2Q \sim P_0$ and $Q \not\sim P_0$. In this case $h^0(O_S(2X_0 + bf)) = 1$, that is, the linear system has a unique curve. If this curve were reducible, it would contain generators or an unisecant curve $D \sim X_0 + af$, with $\deg(a) \geq 0$. But $h^0(O_S(2X_0 + bf - Pf)) = h^0(O_S(2X_0 + bf - D)) = 0$. So the unique curve of the linear system $|H|$ is irreducible. Since $D_QH = 1$, $H$ meets each $D_Q$ at a unique point and it can not have singular points.
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