Quasi-exact solvability of Dirac equation with Lorentz scalar potential

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We consider exact/quasi-exact solvability of Dirac equation with a Lorentz scalar potential based on factorizability of the equation. Exactly solvable and \( sl(2) \)-based quasi-exactly solvable potentials are discussed separately in Cartesian coordinates for a pure Lorentz potential depending only on one spatial dimension, and in spherical coordinates in the presence of a Dirac monopole.

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I. INTRODUCTION

Exact solutions in quantum mechanics are hard to come by. This is especially so in the case of relativistic wave equations. In particular, for the Dirac equation only a few exactly solvable electromagnetic field configurations such as homogeneous magnetic fields \([1]\), homogeneous electrostatic fields \([2]\), constant parallel magnetic fields \([3]\) etc. are known.

While exact solvability is desirable, in practice it is not always possible to determine the whole spectrum. Recently, in non-relativistic quantum mechanics a new class of potentials which are intermediate to exactly solvable ones and non-solvable ones have been found. These are called quasi-exactly solvable (QES) problems for which it is possible to determine algebraically a part of the spectrum but not the whole spectrum \([4, 5, 6, 7, 8]\). Usually a QES problem admits a certain underlying Lie algebraic symmetry which is responsible for the quasi-exact solutions. Previous studies in QES problems deal mainly with the non-relativistic equations, and only rather recently did discussions of the quasi-exactly solvability of the Dirac equations appear. Particularly, it had been shown that the Dirac equation with Coulomb interaction supplemented by a linear radial Lorentz scalar potential \([9]\), and planar Dirac equation with Coulomb and homogeneous magnetic fields \([10]\) are QES systems. The Pauli equation and the Dirac equation coupled minimally to a stationary vector potential were also shown to be QES \([11]\). More recently, added to the list are the Dirac-Pauli equation coupled non-minimally to external electric fields \([12]\), and Dirac oscillator with Coulomb interaction supplemented by a linear radial Lorentz scalar potential \([13]\).

Interest in the Dirac equation with Lorentz scalar potential was mainly motivated by the MIT bag model of quark confinement \([14, 15]\). Unfortunately, QES and exactly solvable Lorentz scalar potentials in such Dirac equation are rather scanty in the literature. The system considered in \([3]\) provides the first example of a QES system. Motivated by this work, in \([16]\) a reparametrization of the radial Dirac equation with a scalar and a Lorentz scalar potential was used to show the existence of infinitely many potentials admitting two QES states. However, explicit construction of the QES potentials was not given, as the reparametrized equations are difficult to handle in practice. Later, a subclass of QES potentials admitting doublets and multiplets were constructed by restricting the forms of the wave functions \([17]\).

In \([11, 12]\) we have developed a procedure to construct exactly-solvable potentials and \( sl(2) \)-based QES potentials admitting any finite number of QES states. The method relies mainly on the factorizability, or equivalently, on the supersymmetric structure of the Hamiltonian of the system. In the present paper we would like to extend the procedure to the Dirac equation with Lorentz scalar potential. We will determine possible forms of the potential allowing factorization of the Dirac equation. Once factorization is achieved, the forms of the exactly solvable and QES potentials can be easily determined.

The organization of the paper is as follows. In Sect. II we outline the procedure presented in \([11, 12]\). Sect. III discusses the exact and quasi-exact solvability of the Dirac equation in one spatial dimension. Sect. IV is devoted to the three and four dimensional Dirac equations in Cartesian coordinates, with the potential depending only one spatial dimension. In Sect. V the Dirac equation with a Lorentz scalar and a magnetic monopole potential is considered. Sect. VI concludes the paper.

II. OUTLINE OF THE PROCEDURE

In this section we shall outline the general procedure we adopted in \([11, 12]\) for determining exact/quasi-exact solvability of the Dirac and the Pauli equation. This procedure makes full use of the close connection between quasi-exactly solvable systems and supersymmetry, or factorizability of the system.
Suppose for a Dirac system one can reduce the corresponding multi-component equations to a set of one-variable equations possessing one-dimensional supersymmetry after separating the variables in a suitable coordinate system. Typically the set of equations takes the form

\[
\begin{align*}
\left( \frac{d}{dr} + W(r) \right) \psi_- &= \mathcal{E}^+ \psi_+ , \\
\left( - \frac{d}{dr} + W(r) \right) \psi_+ &= \mathcal{E}^- \psi_- ,
\end{align*}
\]

(1)

where \( r \) is the basic variable, e.g. the radial coordinate, and \( \psi_{\pm} \) are, say, the two components of the radial part of the Dirac wave function. The superpotential \( W \) is related to the external field configuration. \( \mathcal{E}^{\pm} \) may involve the energy, the mass of the particle, and perhaps some conserved quantum numbers. We can rewrite this set of equations as

\[
\begin{align*}
A^- A^+ \psi_- &= \epsilon \psi_- , \\
A^+ A^- \psi_+ &= \epsilon \psi_+ ,
\end{align*}
\]

(2, 3)

with

\[
A^\pm \equiv \pm \frac{d}{dr} + W , \quad \epsilon \equiv \mathcal{E}^+ \mathcal{E}^- .
\]

(4)

Explicitly, the above equations read

\[
\left( - \frac{d^2}{dr^2} + W^2 \mp W' \right) \psi_\pm = \epsilon \psi_\pm .
\]

(5)

Here and below the prime indicates differentiation with respect to the basic variable. Eq. (5) clearly exhibits the supersymmetric structure of the system. The operators acting on \( \psi_{\pm} \) in Eq. (5) are said to be factorizable, i.e. as products of \( A^- \) and \( A^+ \). The ground state, with \( \epsilon = 0 \), is given by one of the following two sets of equations:

\[
\begin{align*}
A^+ \psi_{(0)}^-(r) &= 0 , \quad \psi_{(0)}^-(r) = 0 ; \\
A^- \psi_{(0)}^+(r) &= 0 , \quad \psi_{(0)}^+(r) = 0 ,
\end{align*}
\]

(6, 7)

depending on which solution is normalizable.

One can determine the forms of the external field that admit exact solutions of the problem by comparing the forms of the superpotential \( W \) with those listed in Table 4.1 of [18].

Similarly, from Turbiner’s classification of the \( sl(2) \) QES systems [5], one can determine the forms of \( W \), and hence the forms of external fields admitting QES solutions based on \( sl(2) \) algebra. The main ideas of the procedure are outlined below. We shall concentrate only on solution of the upper component \( \psi_- \), which is assumed to have a normalizable zero energy state.

Eq. (5) shows that \( \psi_- \) satisfies the Schrödinger equation \( H_- \psi_- = \epsilon \psi_- \), with

\[
H_- = A^- A^+ = - \frac{d^2}{dr^2} + V(r) ,
\]

(8)

with

\[
V(r) = W(r)^2 - W'(r) .
\]

(9)

We shall look for \( V(r) \) such that the system is QES. According to the theory of QES models, one first makes an “imaginary gauge transformation” on the function \( \psi_- \)

\[
\psi_-(r) = \phi(r) e^{-g(r)} ,
\]

(10)

where \( g(r) \) is called the gauge function. The function \( \phi(r) \) satisfies

\[
- \frac{d^2 \phi(r)}{dr^2} + 2g \frac{d \phi(r)}{dr} + [V(r) + g'' - g'^2] \phi(r) = \epsilon \phi(r) .
\]

(11)
For physical systems which we are interested in, the phase factor \( \exp(-g(r)) \) is responsible for the asymptotic behaviors of the wave function so as to ensure normalizability. The function \( \phi(r) \) satisfies a Schrödinger equation with a gauge transformed Hamiltonian

\[
H_G = -\frac{d^2}{dr^2} + 2W_0(r)\frac{d}{dr} + \left[V(r) + W_0' - W_0^2\right],
\]

where \( W_0(r) = g'(r) \).

Now if \( V(r) \) is such that the quantal system is QES, that means the gauge transformed Hamiltonian \( H_G \) can be written as a quadratic combination of the generators \( J^a \) of some Lie algebra with a finite dimensional representation. Within this finite dimensional Hilbert space the Hamiltonian \( H_G \) can be diagonalized, and therefore a finite number of eigenstates are solvable. For one-dimensional QES systems the most general Lie algebra is \( sl(2) \). Hence if Eq. (12) is QES then it can be expressed as

\[
H_G = \sum C_{ab}J^a J^b + \sum C_a J^a + \text{constant},
\]

where \( C_{ab}, C_a \) are constant coefficients, and the \( J^a \) are the generators of the Lie algebra \( sl(2) \) given by

\[
\begin{align*}
J^+ &= z^2 \frac{d}{dz} - Nz, \\
J^0 &= z \frac{d}{dz} - \frac{N}{2}, \quad N = 0, 1, 2 \ldots \\
J^- &= \frac{d}{dz}.
\end{align*}
\]

Here the variables \( r \) and \( z \) are related by \( z = h(r) \), where \( h(\cdot) \) is some (explicit or implicit) function. The value \( j = N/2 \) is called the weight of the differential representation of \( sl(2) \) algebra, and \( N \) is the degree of the eigenfunctions \( \phi \), which are polynomials in a \((N+1)\)-dimensional Hilbert space with the basis \( \{1, z, z^2, \ldots, z^N\} \):

\[
\phi = (z - z_1)(z - z_2) \cdots (z - z_N).
\]

The requirement in Eq. (13) fixes \( V(r) \) and \( W_0(r) \), and \( H_G \) will have an algebraic sector with \( N+1 \) eigenvalues and eigenfunctions. For definiteness, we shall denote the potential \( V \) admitting \( N+1 \) QES states by \( V_N \). From Eqs. (10) and (15), any one of the \( N+1 \) functions \( \psi_- \) in this sector has the general form

\[
\psi_- = (z - z_1)(z - z_2) \cdots (z - z_N) \exp \left(- \int^z W_0(r)dr\right),
\]

where \( z_i \) \((i = 1, 2, \ldots, N)\) are \( N \) parameters that can be determined by plugging Eq. (16) into Eq. (11). The algebraic equations so obtained are called the Bethe ansatz equations corresponding to the QES problem.

Now comes a crucial observation in our procedure: one can rewrite Eq. (16) as

\[
\psi_- = \exp \left(- \int^z W_N(r, \{z_i\})dr\right),
\]

with

\[
W_N(r, \{z_i\}) = W_0(r) - \sum_{i=1}^N \frac{h'(r)}{h(r) - z_i}.
\]

There are \( N+1 \) possible functions \( W_N(r, \{z_i\}) \) for the \( N+1 \) sets of eigenfunctions \( \phi \). Inserting Eq. (17) into \( H_-\psi_- = \epsilon\psi_- \), one sees that \( W_N \) satisfies the Ricatti equation

\[
W_N^2 - W_N' = V_N - \epsilon_N,
\]

where \( \epsilon_N \) is the energy parameter corresponding to the eigenfunction \( \psi_- \) given in Eq. (16) for a particular set of \( N \) parameters \( \{z_i\} \).

From Eqs. (8), (10) and (15) it is clear how one should proceed to determine the external fields so that the Dirac equation becomes QES based on \( sl(2) \): one needs only to determine the superpotentials \( W(r) \) according to Eq. (11) from the QES potentials \( V(r) \) classified in [3]. This is easily done by observing that the superpotential \( W_0 \) corresponding to \( N = 0 \) is related to the gauge function \( g(r) \) associated with a particular class of QES potential \( V(r) \).
by \( g'(r) = W_0(r) \). This superpotential gives the field configuration that allows the weight zero \( (j = N = 0) \) state, i.e. the ground state, to be known in that class. The more interesting task is to obtain higher weight states (i.e. \( j > 0 \)), which will include excited states. For weight \( j \ (N = 2j) \) states, this is achieved by forming the superpotential \( W_N(r, \{ z_i \}) \) according to Eq. (18). Of the \( N + 1 \) possible sets of solutions of the Bethe ansatz equations, the set of roots \( \{ z_1, z_2, \ldots, z_N \} \) to be used in Eq. (18) is chosen to be the set for which the energy parameter of the corresponding state is the lowest.

In the following three sections, we will employ this procedure to classify and construct QES Lorentz scalar potentials.

### III. TWO-DIMENSIONAL DIRAC EQUATION IN CARTESIAN COORDINATES

Let us consider a \((1 + 1)\)-dimensional Dirac Hamiltonian of the kind

\[
H = \alpha p + \beta (m + V_s(x)) \quad (20)
\]

where \( m \) is the mass of the fermion, \( p = -id/dx \), \( \alpha \) and \( \beta \) are the Dirac matrices, and \( V_s \) is the Lorentz scalar potential. Such model is of interest in the theory of nuclear shell model [15, 19], and as a model of the self-compatible field of a quark system [20]. Furthermore, if there exists zero energy solution, then the theory possesses spectral asymmetry and fractional fermion number [15, 21].

It has long been known that the system is supersymmetric [22, 23]. This can be easily shown as follows. Let the Dirac matrices be represented in terms of the Pauli matrices as

\[
\alpha_x = \sigma_2 \quad \text{and} \quad \beta = \sigma_1 \quad (21)
\]

In this representation the Dirac equation \( H\psi = E\psi \) for the 2-component wave function

\[
\psi(x) = \begin{pmatrix} \psi_-(x) \\ \psi_+(x) \end{pmatrix} \quad (22)
\]

takes the form:

\[
\begin{pmatrix} \frac{d}{dx} + U(x) & \psi_- \\ -\frac{d}{dx} + U(x) & \psi_+ \end{pmatrix} = E \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \quad (23)
\]

Here \( U(x) = V_s(x) + m \) and \( \epsilon = E^2 \). Eq. (23) is now in the factorized form, with \( U(x) \) playing the role of the superpotential. As such, this system can be dealt with according to the general procedure outlined in the last section.

#### A. Exactly solvable cases

The exactly solvable cases have been classified in Table 4.1 of [18]. For the present system, there are six types of exactly solvable field configurations, namely, (i) shifted oscillator-like; (ii) Morse-potential; (iii) Rosen-Morse II (hyperbolic); (iv) Scarf II (hyperbolic); (v) Rosen-Morse I (trigonometric); and (vi) Scarf I (trigonometric).

We note here that the case of linear (shifted oscillator) potential has been discussed in [24], and the Morse case in [25].

#### B. Quasi-exactly solvable cases

In this case, seven classes of QES potential \( U(x) \) can be constructed. These classes correspond to Class I to Class VI, and Class X in Turbiner’s classification [3]. We shall illustrate the construction of Class I potentials below. Potentials in the other classes can be constructed accordingly.

The QES potential belonging to Class I has the form

\[
V_N(x) = a^2 e^{-2\alpha x} - a [\alpha (2N + 1) + 2b] e^{-\alpha x} + a (2b - \alpha) e^{\alpha x} + c^2 e^{2\alpha x} + b^2 - 2ac \quad (24)
\]

Without loss of generality, we assume \( \alpha, a, c > 0 \) for definiteness. The corresponding gauge function \( g(x) \) is given by

\[
g(x) = \frac{a}{\alpha} e^{-\alpha x} + \frac{c}{\alpha} e^{\alpha x} + bx \quad (25)
\]
One should always keep in mind that the parameters selected must ensure convergence of the function \( \exp(-g(x)) \) in order to guarantee normalizability of the wave function. The potential \( V(x) \) that gives the ground state, with energy parameter \( \epsilon \equiv E^2 = 0 \), is generated by

\[
V_0 = U_0^2 - U_0',
\]

with

\[
U_0(x) = g'(x) = -ae^{-\alpha x} + ce^{\alpha x} + b.
\]

To obtain the potentials \( U_N(x) \) which admit solvable states with higher weights \( j \), we must first derive the Bethe ansatz equations. To this end, let us perform the change of variable \( z = h(x) = \exp(-\alpha x) \). Eq. (11) then becomes

\[
\left\{-\alpha z^2 \frac{d^2}{dz^2} + \left[2az^2 - (2b + \alpha)z - 2c\right] \frac{d}{dz} + \left[-2aNz - \frac{\epsilon}{\alpha}\right]\right\} \phi(z) = 0.
\]

For \( N > 0 \), there are \( N + 1 \) solutions which include excited states. Assuming \( \phi(z) = \prod_{i=1}^{N} (z - z_i) \) in Eq. (28), one obtains the Bethe ansatz equations which determine the roots \( z_i \)'s

\[
2az_i^2 - (2b + \alpha)z_i - 2c - 2\alpha \sum_{i \neq i} \frac{z_i^2}{z_i - z_l} = 0, \quad i = 1, \ldots, N,
\]

and the equation which gives the energy parameter in terms of the roots \( z_i \)'s

\[
\epsilon = 2ac \sum_{i=1}^{N} \frac{1}{z_i}.
\]

Each set of \( \{z_i\} \) determines a QES energy \( E \) with the corresponding polynomial \( \phi \).

As an example, let us construct \( U_1(x) \) which admits two solutions. This corresponds to the case with \( N = 1 \) and \( \phi(z) = z - z_1 \). According to Eq. (29), the root \( z_1 \) satisfies

\[
2az_1^2 - (2b + \alpha)z_1 - 2c = 0,
\]

which gives two solutions

\[
z_1^+ = \frac{(2b + \alpha) \pm \sqrt{(2b + \alpha)^2 + 16ac}}{4a}.
\]

The corresponding energy parameters are

\[
\epsilon^\pm = 2ac \frac{1}{z_1^\pm} = -\frac{\alpha}{2} \left(2b + \alpha \mp \sqrt{(2b + \alpha)^2 + 16ac}\right).
\]

For the parameters assumed here, the solution with root \( z_1^- = -|z_1^-| < 0 \) gives the ground state, while that with root \( z_1^+ > 0 \) gives the first excited state. The superpotential is constructed according to Eq. (18) as

\[
U_1(x) = U_0 - \frac{h'(x)}{h(x) - z_1} = -ae^{-\alpha x} + ce^{\alpha x} + \frac{\alpha}{1 + |z_1^-|e^{\alpha x}} + b.
\]

The ground state and the excited state have energy \( E^2 = \epsilon^- - \epsilon^- = 0 \) and \( E^2 = \epsilon^+ - \epsilon^- = \alpha \sqrt{(2b + \alpha)^2 + 16ac} \), respectively.

Potential \( U_N(x) \) admitting \( N + 1 \) solvable states can be constructed accordingly.
IV. THREE- AND FOUR-DIMENSIONAL DIRAC EQUATIONS IN CARTESIAN COORDINATES

A. Three-dimensional Dirac equation

A $(2+1)$-dimensional Dirac Hamiltonian has the form

$$H = \alpha \cdot p + \beta (m + V_s (x)),$$

where $p = (p_x, p_y) = -i (d/dx, d/dy)$, and $\alpha$, $\beta$ are Dirac matrices. The potential $V_s (x)$ is assumed to depend only on $x$.

The Dirac equation can also be cast into factorized, or supersymmetric form. We represent the Dirac matrices in terms of the Pauli matrices $\sigma$ as follows:

$$\alpha_x = \sigma_x, \quad \alpha_y = \sigma_y, \quad \beta = \sigma_z.$$  \hfill (36)

As $V (x)$ depends on $x$ only, the wave function can be written as

$$\psi = e^{ik_y y} \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix},$$  \hfill (37)

where $k_y$ is a real constant, and $f_{\pm}$ are real functions of $x$. The Dirac equation becomes

$$\begin{pmatrix} U(x) & p_x - i k_y \\ p_x + i k_y & -U(x) \end{pmatrix} \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix} = E \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix}. \hfill (38)$$

Here $U(x)$ is again defined as $U(x) = V_s (x) + m$. To cast Eq. (38) into supersymmetric form, we transform the wave function by a unitary matrix $T$:

$$\begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix} \rightarrow T^\dagger \begin{pmatrix} i \psi_-(x) \\ \psi_+(x) \end{pmatrix}, \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \hfill (39)$$

Then the Hamiltonian transforms as

$$H \rightarrow T^\dagger HT = \begin{pmatrix} -i k_y & i (-\frac{d}{dx} + U(x)) \\ -i (-\frac{d}{dx} + U(x)) & k_y \end{pmatrix}, \hfill (40)$$

and the Dirac equation becomes

$$\begin{pmatrix} \frac{d}{dx} + U(x) \\ -\frac{d}{dx} + U(x) \end{pmatrix} \psi_- = (E + k_y) \psi_+, \hfill (41)$$

$$\begin{pmatrix} \frac{d}{dx} + U(x) \\ -\frac{d}{dx} + U(x) \end{pmatrix} \psi_+ = (E - k_y) \psi_- , \hfill (42)$$

which is exactly in the same form as Eq. (1). Hence the exact/quasi-exact solvability of Eq. (42) can be discussed as in the one-dimensional case.

B. Four-dimensional Dirac equation

Let us now consider exact/quasi-exact solvability of four-dimensional Dirac equation with a Lorentz scalar potential dependent only on one spatial variable, say $z$. Such system is useful in describing $z$-dependent valence- and conduction-band edge of semiconductors near certain points in the Brillouin zone [26].

The Hamiltonian has the form

$$H = \alpha \cdot p + \beta (m + V_s (z))$$

$$= \begin{pmatrix} 0 & \alpha \cdot p - i U (z) \\ \alpha \cdot p + i U (z) & 0 \end{pmatrix}, \hfill (43)$$

where $p = (p_x, p_y, p_z) = -i (d/dx, d/dy, d/dz)$. Here $U (z) = V_s (z) + m$ as before, and we have chosen the Dirac matrices in the supersymmetric representation [26]

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \hfill (44)$$
To see that the Hamiltonian is factorizable, it is more convenient to consider its square $H^2$:

$$H^2 = \begin{pmatrix} H^- & 0 \\ 0 & H^+ \end{pmatrix},$$

(45)

where

$$H^\mp = -\nabla^2 + U^2(z) \pm U'(z)\sigma_z.$$

(46)

Let the wave function be in the form

$$\psi = e^{ik_x x + ik_y y} \left( \psi_-(z) \chi_- + \psi_+(z) \chi_+ \right).$$

(47)

Here $k_x$ and $k_y$ are real constants, $\psi_{\pm}(z)$ are real functions of $z$, and $\chi_{\pm}$ are two-component eigen-spinors of $\sigma_z$: $\sigma_z \chi_{\pm} = \pm \chi_{\pm}$. Then the eigenvalues problem of $H^2 \psi = E^2 \psi$ reads

$$\left( -\frac{d^2}{dz^2} + U^2 + U' \right) \psi_{\mp} = \epsilon \psi_{\mp},$$

(48)

with $\epsilon = E^2 - k_x^2 - k_y^2$. Now it is obvious that Eq. (48) is in the supersymmetric form Eq. (1), and construction of QES potentials $U(z)$ proceeds as in the previous cases.

V. DIRAC EQUATION WITH A LORENTZ SCALAR AND A MAGNETIC MONOPOLE POTENTIAL

Dirac equation with a spherical Lorentz scalar potential was originally motivated by the MIT bag model intended for describing quark confinement [14, 15]. Now we would like to determine the forms of spherical Lorentz scalar potentials which admit exact/quasi-exact solutions.

Generally, in the presence of an electromagnetic field $(V, A)$ and a Lorentz scalar potential $V_s$, the Dirac equation is given by

$$H = \alpha \cdot (p - ieA) + \beta U + eV,$$

(49)

where $e$ is the charge of the fermion and $U \equiv m + V_s$. Quasi-exact solvability of the system when $V = V_s = 0$ has been demonstrated in connection with the Pauli equation [11]. In the absence of the vector potential $A$ and $V$, the system has been shown in the last two sections to be QES in one spatial dimension. In other coordinates, it is rather difficult, if not impossible, to find QES potentials. In this section, we shall consider the system in spherical coordinates.

When $A = 0$, the Dirac equation in spherical coordinates can be separated into a two-component radial equation

$$\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} eV - E - U \frac{d}{dr} - \frac{\kappa}{r} \right) \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = 0.$$

(50)

The $\kappa$ in the centrifugal term is related to the total angular momentum quantum number $j = 1/2, 3/2, \ldots$ as $\kappa = \pm(j + 1/2)$. It follows that $\kappa$ is nonzero integer: $\kappa = \pm 1, \pm 2, \ldots$. As mentioned in the Introduction, QES potentials with both $V$ and $U$ are difficult to find. The first example was presented in [6], in which $V$ is taken to be Coulomb-like and $V_s$ is linear in $r$. Existence of infinitely many QES $V$ and $V_s$ with two solutions based on some reparametrization scheme was shown in [16], and explicit construction of a subclass of such doublet (and multiplet) QES potentials were presented in [17].

We want to determine possible forms of exactly solvable and QES potentials based on factorizability of the system. Unfortunately, the Dirac equation Eq. (50) is difficult to factorize, owing to the presence of the non-vanishing centrifugal term $\kappa/r$ [23]. Nevertheless, it has been shown in [28] that $\kappa$ could be zero in the presence of a Dirac magnetic monopole described by

$$A_r = A_\theta = 0, \quad A_\phi = g \frac{1 - \cos \theta}{r \sin \theta}.$$

(51)

Here $g$ is the magnetic charge of the Dirac monopole. In [28] it is shown that

$$j = \begin{cases} \frac{1}{2}, \frac{3}{2}, \ldots \\ |q| - \frac{1}{2}, |q| + \frac{1}{2}, |q| - \frac{3}{2}, \ldots \end{cases}$$

if $q = 0$;

$$|q| - \frac{1}{2}, |q| + \frac{1}{2}, |q| - \frac{3}{2}, \ldots \quad \text{if } q \neq 0,$$

(52)
where \( q \equiv e g = 0, \pm 1/2, 1, \ldots \) according to Dirac’s quantization condition, and

\[
\kappa = \pm \sqrt{\left(j + \frac{1}{2}\right)^2 - q^2}.
\]

(53)

Hence, \( \kappa = 0 \) when \( j = |q| - 1/2 \), which is possible only when the magnetic charge \( g \neq 0 \). In this case, the Dirac equation with a scalar and a Lorentz scalar potential in the presence of the Dirac monopole field is also reducible to the two-component radial equation (50), with \( \kappa = 0 \) (see [28] for technical details).

We now show that Eq. (50) is factorizable when \( V = 0 \) in the sector \( j = |q| - 1/2 (\kappa = 0) \). In fact, if the wave function is transformed to

\[
\begin{pmatrix} f \\ g \end{pmatrix} \to \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \equiv T^\dagger \begin{pmatrix} f \\ g \end{pmatrix}, \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

(54)

then the Dirac equation (50) changes to the following factorized form:

\[
\begin{align*}
\left( \frac{d}{dr} + U(r) \right) \psi_- &= E \psi_+, \\
\left( -\frac{d}{dr} + U(r) \right) \psi_+ &= E \psi_-.
\end{align*}
\]

(55)

Here \( U(r) \) plays the role of the superpotential \( W(r) \), and the energy parameter is \( \epsilon = E^2 \). The form of \( U(r) \) admitting exact/quasi-exact solutions can then be constructed according to the prescribed procedure.

### A. Exactly solvable cases

From the Table 4.1 of [18], it is found that there are four types of exactly solvable potentials \( U(r) \) for this system, namely, (i) 3D oscillator-like; (ii) Coulomb-like; (iii) Eckart potential; and (iv) generalized Pöschl-Teller.

### B. Quasi-exactly solvable cases

Three types of QES potentials \( U(r) \) can be constructed, namely, Class VII, VIII, and IX in [2]. We outline the construction of Class VII below.

The general potential in Class VII has the form

\[
V_N(r) = a^2r^6 + 2abr^4 + [b^2 - a(4N + 2\gamma + 3)]r^2 + \gamma(\gamma - 1)r^{-2} - b(2\gamma + 1),
\]

(56)

where \( a, b \) and \( \gamma \) are constants. The gauge function is

\[
g(r) = a4^\frac{1}{4}r^4 + b2^\frac{1}{2}r^2 - \gamma \ln r.
\]

(57)

We must have \( a, \gamma > 0 \) to ensure normalizability of the wave function. The potential

\[
U_0(r) = g'(r) = ar^3 + br - \frac{\gamma}{r}
\]

(58)

admits a QES ground state with energy \( E = 0 \), and the corresponding wave function with components \( \psi_- \propto \exp(-g_0(r)) \) and \( \psi_+ = 0 \).

To determine potentials admitting QES potentials \( V_N \) with higher weights, we need to obtain the Bethe ansatz equations for \( \phi \). Letting \( z = h(r) = r^\frac{1}{2} \), Eq. (51) becomes

\[
\left[ -4z \frac{d^2}{dz^2} + (4az^2 + 4bz - 2(2\gamma + 1)) \frac{d}{dz} -(4aNz + \epsilon) \right] \phi(z) = 0.
\]

(59)

For \( N = 0 \), the value of the \( \epsilon \) is \( \epsilon = 0 \). For higher \( N > 0 \) and \( \phi(r) = \prod_{i=1}^{N}(z - z_i) \), the potential \( U_N(r) \) is obtained from Eqs. (15):

\[
U_N(r) = U_0(r) - \sum_{i=1}^{N} \frac{h'(r)}{h(r) - z_i}.
\]

(60)
For the present case, the roots $z_i$’s are found from the Bethe ansatz equations
\[
2az_i^2 + 2bz_i - (2\gamma + 1) - \sum_{i \neq l} \frac{z_i}{z_i - z_l} = 0, \quad i = 1, \ldots, N, \tag{61}
\]
and $\epsilon$ in terms of the roots $z_i$’s is
\[
\epsilon = 2(2\gamma + 1) \sum_{i=1}^{N} \frac{1}{z_i}. \tag{62}
\]

For $N = 1$ the roots $z_1$ are
\[
z_1^\pm = \frac{-b \pm \sqrt{b^2 + 2a(2\gamma + 1)}}{2a}, \tag{63}
\]
and the values of $\epsilon$ are
\[
\epsilon^\pm = 2 \left( b \pm \sqrt{b^2 + 2a(2\gamma + 1)} \right). \tag{64}
\]

For $a > 0$, the root $z_1^- = -|z_1^+| < 0$ gives the ground state. With this root, one gets the potential
\[
U_1(r) = ar^3 + br - \frac{2r}{r^2 + |z_1^-|} - \frac{\gamma}{r}. \tag{65}
\]

QES potentials for higher degree $N$ can be constructed in the same manner.

VI. SUMMARY

The exact/quasi-exact solvability of the Dirac equation with a Lorentz scalar potential is examined in this paper. Possible forms of the potentials allowing factorization of the Dirac equation are identified in two, three, and four dimensions. Based on such factorizability, we have classified all exactly solvable potentials, and QES potentials based on $sl(2)$ Lie-algebra according to Turbiner’s calibration.

In this work and in [11, 12], we have classified exactly solvable and QES potentials in the Pauli equation and the Dirac equation coupled minimally to a stationary vector potential, the Dirac-Pauli equation (equivalent to generalized Dirac oscillator) coupled non-minimally to an external electric fields, and Dirac equation with a Lorentz scalar potential. Nonetheless, we should mention that we have by no means exhausted all possibilities of exact/quasi-exact potentials. Our classification and construction are based on the factorizability, or supersymmetric structure, of the system, with $sl(2)$ as the underlying symmetry. There are QES systems which are not factorizable, such as those considered in [3, 12, 16, 17], and systems which are not related to the Lie-algebra $sl(2)$, such as that discussed in [10]. It is an interesting and challenging task to develop new methods to classify and construct QES potentials in multi-component wave equations with any Lie-algebraic structures.

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