Abstract—In this paper, we obtain new sum capacity results for the Gaussian many-to-one and one-to-many interference channels. Simple Han-Kobayashi (HK) schemes, i.e., HK schemes with Gaussian signaling, no time-sharing, and no common-private power splitting, achieve sum capacity under the channel conditions for which the new results are obtained. First, by careful Fourier-Motzkin elimination, we obtain the HK achievable rate region for the $K$-user Gaussian many-to-one and one-to-many channels in simplified form, i.e., only in terms of the $K$ rates $R_1, R_2, \ldots, R_K$. We also obtain the achievable sum rate using Fourier-Motzkin elimination. Then, to obtain sum capacity results, we derive genie-aided upper bounds that match the achievable sum rate of simple HK schemes under certain channel conditions.

I. INTRODUCTION

The $K$-user Gaussian Interference channel (IC) has $K$ distinct transmit-receive pairs that interfere with each other. The capacity region or even the sum capacity are not known in general. The sum capacity of the Gaussian IC is known under some channel conditions [1]–[5]. In [1], the capacity region and sum capacity for the 2-user IC were determined under strong interference conditions. In [2]–[5], the sum capacity of the $K$-user Gaussian IC was obtained under noisy interference conditions. Under these conditions, Gaussian signaling and treating interference as noise at each receiver is sum rate optimal. In [3], the sum capacity of the 2-user Gaussian IC under mixed interference conditions was also obtained.

The many-to-one Gaussian IC and one-to-many Gaussian IC are special cases of the Gaussian IC where only one receiver experiences interference and only one transmitter causes interference, respectively. Even for this simpler topology, exact capacity results are hard to obtain. The one-to-many IC and many-to-one IC were studied in [5]–[10]. In [6], [7], approximate capacity and degrees of freedom results are obtained for the many-to-one and one-to-many ICs. The sum capacity under noisy interference conditions, where Gaussian signaling and treating interference as noise is sum rate optimal, is obtained for the many-to-one and one-to-many Gaussian ICs in [5], [8]. The same results can also be obtained as a special case of the result in [2]. Recently, for the many-to-one Gaussian IC, channel conditions under which Gaussian signalling and a combination of treating interference as noise and interference decoding is sum rate optimal were obtained in [9]. Other special cases of the Gaussian IC, namely the cyclic IC and cascade IC have been studied in [11], [12].

In this paper, we obtain new sum capacity results for Gaussian many-to-one and one-to-many ICs. First, by careful Fourier-Motzkin elimination, we obtain the Han-Kobayashi (HK) achievable rate region for the $K$-user Gaussian many-to-one and one-to-many channels in simplified form, i.e., only in terms of the $K$ rates $R_1, R_2, \ldots, R_K$. Then, we focus on simple HK schemes with Gaussian signaling, no timesharing, and no common-private power splitting. We derive genie-aided sum capacity upper bounds that match the achievable sum rates of simple HK schemes under some channel conditions. Thus, we obtain new sum capacity results for a larger subset of possible channel conditions than currently known in existing literature in [5], [8], [9]. In these simple HK schemes, either the interference from a particular transmitter is decoded fully or gets treated as noise. For the many-to-one case, we consider schemes where $k$ out of $K$-1 interfering signals are decoded at receiver 1. For the one-to-many case, we consider schemes where $k$ out of $K$-1 receivers decode the interfering signal.

The channel models (in standard form) for the Gaussian many-to-one and one-to-many ICs are shown in Fig. 1. As an illustration of the new results in this paper, the channel conditions under which sum capacity results are obtained for the 3-user many-to-one and one-to-many ICs are shown in Figs. 2 and 3. In the figures, the shaded regions represent the new regions where sum capacity is determined in this paper.
Fig. 1: Channel Models in standard form

(a) Many-to-one IC
(b) One-to-many IC

Fig. 2: Channel conditions where sum capacity is obtained for the 3-user many-to-one IC, $P_1 = P_2 = P_3 = 2$.

II. CHANNEL MODELS IN STANDARD FORM

The received signals in the Gaussian many-to-one IC in standard form are given by:

\[ y_1 = x_1 + \sum_{j=2}^{K} h_j x_j + z_1 \]  
\[ y_i = x_i + z_i, \quad i = 2, 3, \ldots, K, \]

where $x_i$ is transmitted from transmitter $i$, $z_i \sim \mathcal{N}(0, 1)$ for each $i$. The average power constraint at transmitter $i$ is $P_i$. Similarly, the received signals in the Gaussian one-to-many IC in standard form are given by:

\[ y_i = x_i + h_i x_K + z_i, \quad i = 1, 2, 3, \ldots, K - 1 \]
\[ y_K = x_K + z_K. \]

Fig. 3: Channel conditions where sum capacity is obtained for the 3-user one-to-many IC, $P_1 = P_2 = P_3 = 2$.

III. ACHIEVABLE RATE REGION FOR HAN-KOBAYASHI (HK) SCHEME IN SIMPLIFIED FORM

A. Many-to-one IC

Let $W_i$ be the message at transmitter $i$. For each $i = 2, 3, \ldots, K$, the message is split into two parts $W_i = \{W_{i0}, W_{i1}\}$, where $W_{i0}$ is common message that gets decoded at receiver $i$ and also at receiver 1, and $W_{i1}$ is the private message that gets decoded only at receiver $i$. The HK achievable rate region in simplified form in the Theorem below is stated for the discrete memoryless channel, and can be readily extended to the Gaussian many-to-one IC with average power constraints using standard approaches [13, 14].

Theorem 1. For the discrete memoryless $K$-user many-to-one IC, the HK achievable rate region is given by the set of all $(R_1, R_2, \ldots, R_K)$ that satisfy:

\[ R_1 + \sum_{j=2}^{K} R_j \leq \sum_{j \in \mathcal{N}} I(X_j; Y_j | Q, U_j) \]
\[ + I(U_N X_1; Y_1 | U_{F-N}, Q), \forall \mathcal{N} \subseteq \mathcal{F} \]  
\[ R_i \leq I(X_i; Y_i | Q), i \in [2 : K] \]

where $U_A = \{U_i, i \in A\}$, $\mathcal{F} = \{2, 3, \ldots, K\}$ and $(Q, U_2, U_3, \ldots, U_K, X_1, X_2, \ldots, X_K)$ is distributed as

\[ p(q, u_2, \ldots, u_K, x_1, \ldots, x_K) = p(q)p(x_1 | q)\prod_{i=2}^{K}(p(u_i | q)p(x_i | u_i, q). \]

Proof. See Appendix A. \qed
Corollary 1. The achievable sum rate $S$ for a discrete memoryless many-to-one IC satisfies:

$$S \leq \sum_{i \in \mathcal{N}} I(X_i; Y_i | Q) + \sum_{i \in \mathcal{F} \setminus \mathcal{N}} I(X_i; Y_i | Q)$$

$$+ I(U_N X_i; Y_i | U_{\mathcal{F} \setminus \mathcal{N}}, Q), \forall N \subseteq \mathcal{F}, \quad (8)$$

where $\mathcal{F} = \{2, 3, \ldots, K\}$.

Proof. See Appendix B for proof.

Simple HK schemes: Consider HK schemes with Gaussian signaling, no timesharing, and no commonprivate power splitting, i.e., $X_i \sim \mathcal{N}(0, P_i)$, $\forall 1 \leq i \leq K$, $Q$ is constant, and $U_i = X_i, i \in \mathcal{B}$ and $U_i = \phi, i \notin \mathcal{B}$ for a fixed $\mathcal{B} \subseteq \{2, 3, \ldots, K\}$. The set $\mathcal{B}$ denotes the indices of the set of transmit messages decoded at receiver 1. For simple HK schemes, we get the following sum rate result directly from Corollary 1.

Corollary 2. The achievable sum rate of a simple HK scheme over the Gaussian many-to-one IC satisfies:

$$S \leq \frac{1}{2} \sum_{i \in \mathcal{B}} \log(1 + P_i) + \frac{1}{2} \sum_{i \in \mathcal{M}} \log(1 + P_i)$$

$$+ \frac{1}{2} \log \left(1 + \frac{P_1 + \sum_{i \in \mathcal{B} \setminus \mathcal{M}} h_i^2 P_i}{1 + \sum_{i \in \mathcal{B}} h_i^2 P_i}\right), \forall \mathcal{M} \subseteq \mathcal{B} \quad (9)$$

for a fixed $\mathcal{B} \subseteq \{2, 3, \ldots, K\}$.

B. One-to-many IC

Let $\mathcal{I}$ denote the set of indices of the receivers at which interference is decoded, and $\mathcal{J}$ be the set of receivers at which interference is treated as noise, i.e., $\mathcal{J} = \{1, 2, \ldots, K-1\} \setminus \mathcal{I}$. Let $W_i$ be the message at transmitter $i$. The message $W_K$ gets split into two parts $W_K = \{W_{K0}, W_{K1}\}$, where $W_{K0}$ represents the common message that gets decoded at every receiver in $\mathcal{I}$ and $W_{K1}$ is the private message that gets decoded only at receiver $K$.

Theorem 2. For the discrete memoryless $K$-user one-to-many IC, the HK achievable rate region is given by the set of all $(R_1, R_2, \ldots, R_K)$ that satisfy

$$R_i \leq I(X_i; Y_i | Q), i \in \mathcal{J}$$

$$R_i \leq I(X_i; Y_i | Q, U), i \in \mathcal{I}$$

$$R_i + R_K \leq I(X_i, U; Y_i | Q) + I(X_K; Y_K | Q, U), i \in \mathcal{I}$$

$$R_K \leq I(X_K; Y_K | Q),$$

where $(Q, U, X_1, X_2, \ldots, X_K)$ is distributed as

$$p(q, u, x_1, x_2, \ldots, x_K) = p(q) \prod_{i=1}^{K-1} p(x_i | q) p(u | q) p(x_K | u, q).$$

Proof. See Appendix C.

Simple HK scheme: Let $X_i \sim \mathcal{N}(0, P_i)$, $\forall 1 \leq i \leq K$, $Q$ is constant, and $U = X_K$. From Theorem 2, we directly get the following result.

Corollary 3. The achievable rate region for the simple HK scheme over the Gaussian one-to-many IC is given by:

$$R_i \leq \frac{1}{2} \log(1 + \frac{P_i}{1 + h_i^2 P_K}), i \in \mathcal{J}, \quad (10)$$

$$R_i \leq \frac{1}{2} \log(1 + P_i), i \in \mathcal{I}, \quad (11)$$

$$R_i + R_K \leq \frac{1}{2} \log(1 + P_i + h_i^2 P_K), i \in \mathcal{I}, \quad (12)$$

$$R_K \leq \frac{1}{2} \log(1 + P_K). \quad (13)$$

Corollary 4. The achievable sum rate $S$ for the simple HK scheme over the Gaussian one-to-many IC when $\mathcal{J} = \phi$ satisfies

$$S \leq \sum_{j=1}^{K-1} \frac{1}{2} \log(1 + P_j), \quad (14)$$

$$S \leq \sum_{j=1}^{K-1} \frac{1}{2} \log(1 + P_j) + \frac{1}{2} \log(1 + P_i + h_i^2 P_K), \quad (15)$$

$\forall 1 \leq i \leq K - 1$.

Proof. See appendix D.

IV. SUM CAPACITY RESULTS

A. Gaussian many-to-one IC

Consider the simple HK scheme with $\mathcal{B} = \{2, 3, \ldots, k\}$, i.e., interference from transmitters 2 to $k$ are decoded at receiver 1. We choose successive indices 2 to $k$ only for notational convenience, and the results can be generalized to any set of $k - 1$ indices by just relabeling the transmitters. For this case, from (9), we have the following $2^{k-1}$ sum rate constraints:

$$S \leq \frac{1}{2} \sum_{i=k+1}^{K} \log(1 + P_i) + \frac{1}{2} \sum_{i \in \mathcal{M}} \log(1 + P_i)$$

$$+ \frac{1}{2} \log \left(1 + \frac{P_1 + \sum_{i \in \mathcal{B} \setminus \mathcal{M}} h_i^2 P_i}{1 + \sum_{i=k+1}^{K} h_i^2 P_i}\right), \forall \mathcal{M} \subseteq \mathcal{B}. \quad (16)$$
The least of these $2^{k-1}$ upper bounds will determine the maximum achievable sum rate for this simple HK scheme. We will now discuss two cases below where we can show that the simple HK scheme achieves sum capacity.

**Case 1 ($\mathcal{M}\overline{T}_{k_0}$):** Here we consider the case when the inequality corresponding to $\mathcal{M} = \mathcal{B}$ in (16) is the dominant inequality, i.e., its right hand side is the least.

**Theorem 3.** For the $K$-user Gaussian many-to-one IC satisfying the following channel conditions:

$$
\prod_{i \in B - \mathcal{N}} (1 + P_i) (1 + \sum_{j=k+1}^{K} h_i^2 P_j + P_i) \leq 1 + \sum_{i \notin \mathcal{N}} h_i^2 P_i + P_i, \forall \mathcal{N} \subset \mathcal{B}, \mathcal{N} \neq \mathcal{B},
$$

(17)

where $\mathcal{B} = \{2, 3, \ldots, k\}$, $k \in \{1, 2, \ldots, K\}$, the sum capacity is given by

$$
S = \frac{1}{2} \log \left( 1 + \frac{P_1}{1 + \sum_{j=k+1}^{K} h_i^2 P_j} \right) + \sum_{i=2}^{1} \frac{1}{2} \log(1 + P_i).
$$

(19)

**Proof.** The converse or upper bound has already been proved in [9, Thm. 7] under the condition (18) using the genie-aided channel in Fig. 4a. This sum rate can be achieved by the simple HK scheme if the inequality corresponding to the $\mathcal{M} = \mathcal{B}$ case is the dominant inequality in (16). This inequality is dominant if the conditions in (17) are satisfied.

**Remark 1.** The case of $k = 1$ is taken to be $\mathcal{B} = \emptyset$ resulting in condition (18) alone, thereby recovering the sum capacity result for treating all interference as noise in [5].

**Remark 2.** The achievability conditions in (17) are less stringent than the achievability conditions in [9] since joint decoding in the simple HK scheme is better than the successive interference cancellation decoding used in [9]. This can be noted in Fig. 2 where the region obtained using this theorem includes an additional shaded region for the case $\mathcal{M}\overline{T}_{k_0}$ compared to the result in [9].

**Case 2 ($\mathcal{M}\overline{T}_{k_1}$):** Here we consider the case when the inequality corresponding to $\mathcal{M} = \mathcal{B}\setminus\{k\} = \{2, 3, \ldots, k - 1\}$ in (16) is the dominant inequality.

**Theorem 4.** For the $K$-user Gaussian many-to-one IC satisfying the following channel conditions:

$$
\prod_{i \in B - \mathcal{N}} (1 + P_i) \left( 1 + P_i + \sum_{j=k+1}^{K} h_i^2 P_j + \sum_{j=k}^{K} h_i^2 P_j \right) \geq \prod_{i \in B - \mathcal{N}} (1 + P_i) (1 + P_i + \sum_{j=k}^{K} h_i^2 P_j)
$$

(20)

\forall \mathcal{N} \subset \mathcal{B}, \mathcal{N} \neq \{2, 3, \ldots, k - 1\}$ and $\mathcal{B} = \{2, 3, \ldots, k\}$

$$
\sum_{i=k+1}^{K} h_i^2 \leq 1 - \left( \frac{1 + \sum_{i=k+1}^{K} h_i^2 P_i}{h_k} \right)^2,
$$

(21)

the sum capacity is given by

$$
S = \sum_{i=2}^{K} \frac{1}{2} \log(1 + P_i) + \frac{1}{2} \log \left( 1 + \frac{P_1 + h_i^2 P_k}{1 + \sum_{i=k+1}^{K} h_i^2 P_i} \right).
$$

**Proof.** For achievability, we note that the sum capacity can be achieved by the simple HK scheme if the inequality corresponding to the $\mathcal{M} = \mathcal{B}\setminus\{k\}$ case is the dominant inequality in (16). This inequality is dominant if the conditions in (20) are satisfied.

For the converse or upper bound, we consider the genie-aided channel in Fig. 4b, where a genie provides the signal $s_i^n = \{x_i^n, x_i^n, \ldots, x_i^n\}$ to receiver 1 and the signal $s_k^n = \sum_{i=k}^{K} h_i x_i^n + \mathbf{w}^n$ to receiver $k$, where $\mathbf{w}^n$ is i.i.d. $\mathcal{N}(0, 1)$, and $\mathbf{w}$ and $z_k$ are jointly Gaussian with

$$
(z_k \ w) \sim \mathcal{N} \left( \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right), \text{ with } \rho = \frac{1 + \sum_{i=k+1}^{K} h_i^2 P_i}{h_k}.
$$

The sum capacity can now be bounded as follows.

$$
nS + nC \leq I(x_1^n; y_1^n | s_1^n) + \sum_{i=2, i \neq k}^{K} I(x_i^n; y_i^n) + I(x_k^n; y_k^n, s_k^n)
$$
\[ h(y^n_1|s^n_1) - h \left( \sum_{i=k}^{K} h_i x^n_i + z^n_i \right) \]
\[ + \sum_{i=2,i\neq k}^{K} \left( h(y^n_i) - h(z^n_i) \right) + h(s^n_k) \]
\[ + h(y^n_k|s^n_k) - h(y^n_1, s^n_k|x^n_k) \]
\[ \leq nh(y_{1G}|s_{1G}) + \sum_{i=2}^{K} \left( nh(y_{iG}) - nh(z_i) \right) \]
\[ + \sum_{i=k+1}^{K} \left( h(y^n_i) - h(z^n_i) \right) + nh(y_{kG}|s_{kG}) \]
\[ - h(s^n_k|z^n_k) - h(z^n_k) \]
\[ \leq nh(y_{1G}|s_{1G}) + \sum_{i=2}^{K} \left( nh(y_{iG}) - nh(z_i) \right) \]
\[ + nh(y_{kG} \sum_{i=k}^{K} h_i x_{iG} + w) - nh(z_k) \]
\[ - nh(\sum_{i=k+1}^{K} s_{kG} z_k) + \sum_{i=k+1}^{K} (nh(y_{iG}) - nh(z_i)) \]
\[ = nI(x_{1G}; y_{1G}|s_{1G}) + \sum_{i=2,i\neq k}^{K} nI(x_{iG}; y_{iG}) \]
\[ + nI(x_{kG}; y_{kG}, s_{kG}) \]
\[ \geq nI(x_{1G}; y_{1G}|s_{1G}) + \sum_{i=2,i\neq k}^{K} nI(x_{iG}; y_{iG}) \]
\[ + nI(x_{kG}; y_{kG}) \]
\[ = nI(x_{1G}, x_{kG}; y_{1G}|s_{1G}) + \sum_{i=2,i\neq k}^{K} nI(x_{iG}; y_{iG}), \]

where \( x_{iG} \) denotes a zero mean Gaussian random variable with variance \( P_i \), and \( s_{iG} \) and \( y_{iG} \) represent the Gaussian side information and output that result when all the inputs are Gaussian as described in [5]. (a) follows from the fact that Gaussian inputs maximize differential entropy, and \( h(\frac{1}{n} \sum_{i=k+1}^{K} h_i x^n_i + z^n_i) = h(\frac{1}{n} \sum_{i=k+1}^{K} h_i x^n_i + w^n) \), (b) follows from [9, Lemma 2] and from the fact that \( h(\frac{1}{n} \sum_{i=k+1}^{K} h_i x^n_i + w^n|z^n_k) \) can be written as \( h(\frac{1}{n} \sum_{i=k+1}^{K} x^n_i + w^n) \) where \( w_n \sim \mathcal{N}(0, 1 - \rho^2) \), if (21) holds, (c) follows from the fact that \( x_{kG} \rightarrow s_{kG} \rightarrow y_{kG} \) forms a Markov Chain according to [5, Lemma 8].

The results in Theorems 3 and 4 for the Gaussian \( K \)-user many-to-one IC are now listed in Table I for the 3-user case.

| Strategy | Channel conditions |
|----------|---------------------|
| MT2a | (i) \( h_2^4 \leq 1 + P_i + h_2^2 P_3 \), \( h_2^4 \leq 1 - \left( \frac{1 + h_2^2 P_3}{h_2^2} \right)^2 \), \( h_2^4 \geq 1 \)  
(ii) \( h_2^4 \leq 1 + P_i + h_2^2 P_3 \), \( h_2^4 \leq 1 - \left( \frac{1 + h_2^2 P_3}{h_2^2} \right)^2 \), \( h_2^4 \geq 1 \) |
| MT3a | \( h_3^2 \geq 1 + P_i + h_3^2 P_3, h_3^2 \leq 1 + P_i, h_3^2 \geq 1, 1 + P_i + h_3^2 P_3 \geq (1 + P_3)(1 + P_3) - (1 + P_3) \) |

TABLE I: Channel conditions under which sum capacity is achieved using simple HK schemes in Theorems 3 and 4 for the 3-user Gaussian many-to-one IC. Conditions for MT1 and MT2 are already given in [9]. These conditions are plotted in Fig. 2 for a given set of power constraints.

### B. Gaussian One-to-many IC

Consider the simple HK scheme where interference from transmitter \( K \) is decoded at \( k \) receivers. Without loss of generality, we can consider the set these \( k \) receivers to be \( I = \{1, 2, \ldots, k\} \) and \( J = \{k + 1, k + 2, \ldots, K - 1\} \) (other choices can be easily handled by relabeling the receivers). We denote this scheme to be \( OI_k \).

\[ 1 + P_i \leq |h_i|^2, 1 \leq i \leq k, \]
\[
\sum_{j=k+1}^{K-1} \frac{|h_j|^2 P_k + |h_j|^2}{|h_j|^2 P_k + 1} \leq 1, \quad (23)
\]

the sum capacity is given by
\[
S = \frac{1}{2} \sum_{i=1}^{k} \log(1 + P_l) + \frac{1}{2} \log(1 + P_K) + \\
\frac{1}{2} \sum_{j=k+1}^{K-1} \log \left(1 + \frac{P_j}{1 + |h_j|^2 P_K} \right). \quad (24)
\]

**Proof.** For achievability, consider the achievable rate region in Corollary 3 for the simple HK scheme \(OI_k\). Under (22), constraint (12) is redundant. From the remaining constraints (10), (11), and (13), we get the achievable sum rate to be equal to the sum capacity in the theorem statement.

For the converse, consider the genie-aided channel in Fig. 5a, where a genie provides \(x_k\) to receivers 1 to \(k\). The first \(k\) receivers can now achieve the point-to-point channel capacities without any interference. The genie-aided channel can be considered to be a combination of these \(k\) point-to-point channels and a Gaussian one-to-many IC with users \(k + 1\) to \(K\) of the original channel. The sum capacity of the \(k\) point-to-point channels corresponds to the first term in the right-hand side of (24). The sum capacity of the Gaussian one-to-many IC with users \(k + 1\) to \(K\) is upper bounded by the sum of the second and third terms in (24) under condition (23) [5, Thm. 5]. Thus, we have the required sum capacity result.

Now, we consider the special case where \(\mathcal{I} = \{1, 2, \ldots, K - 1\}\) and \(\mathcal{J} = \phi\), i.e., the interference gets decoded at all receivers. For this special case, we now have a sum capacity result for conditions not included in Theorem 5. We will denote this case \(OI_{K-1}\).

**Theorem 6.** For the \(K\)-user Gaussian one-to-many IC satisfying the following conditions:
\[
1 \leq h_i^2 \leq 1 + P_l \quad (25)
\]
\[
\frac{h_i^2}{1 + P_l} \leq \frac{h_j^2}{1 + P_l}, 1 \leq i \leq K - 1 \text{ and } i \neq l \quad (26)
\]
for any \(l \in \{1, 2, \ldots, K - 1\}\), the sum capacity is
\[
S = \frac{1}{2} \sum_{j=1, j \neq l}^{K-1} \log(1 + P_j) + \frac{1}{2} \log(1 + P_l + h_i^2 P_K). \quad (27)
\]

**Proof.** For achievability, consider the achievable sum rate in corollary 4. The sum capacity in (27) is the right-hand side of the inequality corresponding to \(i = l\) in

| Strategy | Channel conditions |
|----------|--------------------|
| \(OI_0\) | \(\sum_{j=1}^{K-1} h_j^2 P_k + h_j^2 \leq 1\) |
| \(OI_1\) | (i) \(h_1^2 \geq 1 + P_1\), \(h_2^2 \leq 1\) |
| \(OI_2\) | (i) \(h_1^2 \geq 1 + P_1\), \(h_2^2 \leq 1 + P_2\) |
| \(OI_{21}\) | (i) \(h_1^2 \leq 1 + P_1\), \(h_2^2 \geq 1 + P_2\) |

**TABLE II:** Channel conditions under which sum capacity is achieved using simple HK schemes in Theorems 5 and 6 for the 3-user Gaussian one-to-many IC. These conditions are plotted in Fig. 2 for a given set of power constraints.

Corollary 4. This inequality is the dominant inequality under conditions (25) and (26).

For the converse, consider the genie-aided channel (shown in Fig. 5b for \(l = 1\)), where a genie provides \(x_k\) to all receivers 1 to \(K - 1\) except receiver \(l\). The genie-aided channel is a combination of \(K - 2\) point-to-point channels and a Gaussian one-sided IC with users \(l\) and \(K\) of the original channel. The sum capacity of the \(K - 2\) point-to-point channels corresponds to the first term in (27). The sum capacity of the Gaussian one-sided IC with users \(l\) and \(K\) is upper bounded by the second term in (27) under condition (25) [15, Thm. 2]. Thus, we have the required result.

The results in Theorems 5 and 6 for the Gaussian \(K\)-user one-to-many IC are now listed in Table II for the 3-user case.

**V. CONCLUSIONS**

We derived new sum capacity results for the \(K\)-user Gaussian many-to-one and one-to-many ICs. For both the many-to-one and one-to-many ICs, two new classes of channel conditions under which sum capacity is achieved were determined (cases \(\mathcal{MI}_0, \mathcal{MI}_1, \mathcal{OI}_k, \mathcal{OI}_{K-1}\)). In all these cases, simple HK schemes with Gaussian signaling, no time-sharing and no common-private power splitting achieve sum capacity.

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The simplified rate region in (5) and (6) in terms of only the $R_i$'s can be obtained using Fourier-Motzkin elimination. The main steps of the Fourier-Motzkin elimination are provided below.

We eliminate the variables in the following sequence: $T_2, T_3, \ldots, T_K$. After eliminating $T_2, T_3, \ldots, T_k$, the set of inequalities is given by:

$$R_1 + \sum_{i \in \mathcal{N}} R_i + \sum_{i \in \mathcal{S}} T_i \leq \sum_{i \in \mathcal{N}} I(X_j; Y_j | Q, U_j) + I(U_N, U_{S}, X_1; Y_1 | U_{F-(S \cup \mathcal{N})}, Q), \forall \mathcal{N} \subseteq \{2, 3, \ldots, k\}, \mathcal{S} \subseteq \{k+1, \ldots, K\}.$$

For $k+1 \leq i \leq K$

$$R_i - T_i \leq I(X_i; Y_i | Q, U_i),$$
$$T_i \geq 0, T_i \leq R_i.$$

For $2 \leq i \leq K$

$$R_i \leq I(X_i; Y_i | Q).$$

This can be proved by induction.

Setting $k = K$, we get the required inequalities in (5) and (6) after elimination of $T_2, T_3, \ldots, T_K$.

B. Proof of Corollary 1

Corollary 1 is also proved using Fourier-Motzkin elimination starting from the result in Theorem 1. First, we substitute $R_1 = S - \sum_{i=2}^{K} R_i$. Then, we eliminate the variables in the following sequence: $R_2, R_3, \ldots, R_K$. After eliminating $R_2, R_3, \ldots, R_k$, the set of inequalities is given by:

$$S - \sum_{i \in \mathcal{B}} R_i \leq \sum_{i \in (S-\mathcal{B})} I(X_i; Y_i | Q, U_i) + \sum_{i \in \mathcal{N}} I(X_i; Y_i | Q, U_i) + \sum_{i \in \mathcal{M} \setminus \mathcal{N}} I(X_i; Y_i | Q) + I(U_{S-\mathcal{B}}, U_{\mathcal{N}}, X_1; Y_1 | U_{\mathcal{M} \setminus \mathcal{N}}, U_{\mathcal{B}}, Q), \forall \mathcal{B} \subseteq \mathcal{S} \text{ and } \mathcal{S} = \{k+1, \ldots, K\} \text{ and } \forall \mathcal{N} \subseteq \mathcal{M} \text{ and } \mathcal{M} = \{2, 3, \ldots, k\}, \text{ and for } k+1 \leq i \leq K$$

$$R_i \leq I(X_i; Y_i | Q).$$

This can be proved by induction.

Setting $k = K$, we get the required result in (8) after elimination of $R_2, R_3, \ldots, R_K$.

C. Proof of theorem 2

Let $S$ denote the rate of the private message $W_{K1}$ and $T$ denote the rate of the common message $W_{K0}$. Note that $R_K = S + T$. Using standard analysis of HK schemes, we get the following achievable rate region in terms of $\{R_i\}$ and $\{T\}$:

$$R_i \leq I(X_i; Y_i | Q), i \in \mathcal{J}.$$
\[ R_i \leq I(X_i; Y_i | U, Q), \ i \in \mathcal{I} \]
\[ R_i + T \leq I(X_i, U; Y_i | Q), \ i \in \mathcal{I} \]
\[ R_K - T \leq I(X_K; Y_K | U, Q) \]
\[ R_K \leq I(X_K; Y_K | Q) \]

Using Fourier-Motzkin elimination to eliminate \( T \), we get the rate region in Theorem 2.

D. Proof of corollary 4

Given \( J = \phi \), we get the following rate constraints:

\[ R_i \leq \frac{1}{2} \log(1 + P_i), \ 1 \leq i \leq K - 1 \]
\[ R_i + R_K \leq \frac{1}{2} \log(1 + P_i + h_i^2 P_K), \ 1 \leq i \leq K - 1 \]
\[ R_K \leq \frac{1}{2} \log(1 + P_K). \]

First, we substitute \( R_K = S - \sum_{i=1}^{K-1} R_i \). Then, we eliminate the variables in the following sequence: \( R_1, R_2, \ldots, R_{K-1} \). After eliminating \( R_1, R_2, \ldots, R_k \), the set of inequalities is given by:

\[ R_i \leq \frac{1}{2} \log(1 + P_i), \ k + 1 \leq i \leq K - 1 \]
\[ S - \sum_{j=k+1}^{K-1} R_j \leq \sum_{j=1}^{k} \frac{1}{2} \log(1 + P_j) + \frac{1}{2} \log(1 + P_K) \]
\[ S - \sum_{j=k+1}^{K-1} R_j \leq \sum_{j=1, j \neq i}^{k} \frac{1}{2} \log(1 + P_j) + \frac{1}{2} \log(1 + P_i + h_i^2 P_K), \ 1 \leq i \leq k \]
\[ S - \sum_{j=k+1, j \neq i}^{K-1} R_j \leq \sum_{j=1}^{k-1} \frac{1}{2} \log(1 + P_j) + \frac{1}{2} \log(1 + P_i + h_i^2 P_K), \ k + 1 \leq i \leq K - 1 \]

This can be proved by induction.

Setting \( k = K - 1 \), i.e., after elimination of \( R_1, R_2, \ldots, R_{K-1} \), we get the required inequalities in (14) and (15).