LINK COMPLEXES OF SUBSPACE ARRANGEMENTS

AXEL HULTMAN

Abstract. Given a simplicial hyperplane arrangement \( H \) and a subspace arrangement \( A \) embedded in \( H \), we define a simplicial complex \( \Delta_{A,H} \) as the subdivision of the link of \( A \) induced by \( H \). In particular, this generalizes Steingrímsson’s coloring complex of a graph.

We do the following:

1. When \( A \) is a hyperplane arrangement, \( \Delta_{A,H} \) is shown to be shellable. As a special case, we answer affirmatively a question of Steingrímsson on coloring complexes.

2. For \( H \) being a Coxeter arrangement of type \( A \) or \( B \) we obtain a close connection between the Hilbert series of the Stanley-Reisner ring of \( \Delta_{A,H} \) and the characteristic polynomial of \( A \). This extends results of Steingrímsson and provides an interpretation of chromatic polynomials of hypergraphs and signed graphs in terms of Hilbert polynomials.

1. Introduction

In [10], Steingrímsson introduced the coloring complex \( \Delta_G \). This is a simplicial complex associated with a graph \( G \). The Hilbert polynomial of its Stanley-Reisner ring \( k[\Delta_G] \) is closely related to the chromatic polynomial \( P_G(x) \) in a way that is made precise in Section 5.

Answering a question of Steingrímsson, Jonsson [7] proved that \( \Delta_G \) is a Cohen-Macaulay complex by showing that it is constructible. In particular, \( \Delta_G \) being Cohen-Macaulay imposes restrictions on the Hilbert polynomial of \( k[\Delta_G] \), hence on \( P_G(x) \).

Since \( \Delta_G \) is a Cohen-Macaulay complex, a natural question, asked already in [10], is whether it is shellable — a stronger property than constructibility.

In [10], \( \Delta_G \) was defined in a combinatorially very explicit way. Another way to view \( \Delta_G \) is, however, as a simplicial decomposition of the link (i.e. intersection with the unit sphere) of the graphical hyperplane arrangement associated with \( G \). In this guise, \( \Delta_G \) appeared in work of Herzog, Reiner and Welker [6]. Adopting this point of view, one may define a similar complex \( \Delta_{A,H} \) for any subspace arrangement \( A \), as long as it has an embedding in a simplicial hyperplane arrangement \( H \).

This paper has two goals. The first is addressed in Section 4 where we show that \( \Delta_{A,H} \) is shellable whenever \( A \) consists of hyperplanes. In particular, this proves that the coloring complexes are shellable.

The chromatic polynomial of \( G \) is essentially the characteristic polynomial of the corresponding graphical hyperplane arrangement. Bearing this in mind, one may hope to extend the aforementioned connection between the Hilbert polynomial of \( k[\Delta_G] \) and \( P_G(x) \) to more general complexes \( \Delta_{A,H} \). Achieved in Section 5, our
second goal is to carry out this extension whenever \( H \) is a Coxeter arrangement of type \( A \) or \( B \). When \( A \) consists of hyperplanes and \( H \) is of type \( A \), Steingrímsson’s result is recovered.

We define the complexes \( \Delta_{A,H} \) in Section 3 after reviewing some necessary background in the next section.

2. Preliminaries

2.1. Subspace arrangements and characteristic polynomials. By the term subspace arrangement we mean a finite collection \( A = \{A_1, \ldots, A_t\} \) of linear subspaces, none of which contains another, of some ambient vector space. In our case, the ambient space will always be \( \mathbb{R}^n \) for some \( n \). To \( A \) we associate the intersection lattice \( L_A \) which consists of all intersections of subspaces in \( A \) ordered by reverse inclusion. (We emphasize the fact that \( A \) contains no strictly affine subspaces; in particular this implies that \( L_A \) is indeed a lattice.)

An important invariant of the arrangement \( A \) is its characteristic polynomial \( \chi(A; x) = \sum_{Y \in L_A} \mu(\hat{0}, Y)x^{\dim(Y)} \),

where \( \mu \) is the Möbius function of \( L_A \) which we think of as a function \( L_A \times L_A \to \mathbb{Z} \) with \( S \nleq T \Rightarrow \mu_A(S,T) = 0 \) (and similarly for \( A \setminus A \)).

Given a subspace \( A \in A \), we define two new arrangements, namely the deletion \( A \setminus A = A \setminus \{A\} \) and the restriction \( A/A = \max\{A \cap B | B \in A \setminus A\} \),

where \( \max\mathcal{S} \) denotes the collection of inclusion-maximal members of a set family \( \mathcal{S} \). Another way to think of \( A/A \) is as the set of elements covering \( A \) in \( L_A \). In this way, we may extend the definition of \( A/A \) to arbitrary \( A \in L_A \). We consider \( A \setminus A \) to be an arrangement in \( \mathbb{R}^n \), whereas \( A/A \) is an arrangement in \( A \).

When \( A \) is a hyperplane arrangement, the next result is standard. We expect the general case to be known, too, although we have been unable to find it in the literature.

**Theorem 2.1** (Deletion-Restriction). For a subspace arrangement \( A \) and any subspace \( A \in A \), we have

\[
\chi(A; x) = \chi(A \setminus A; x) - \chi(A/A; x).
\]

**Proof.** Choose \( Y \in L_A \). We claim that

\[
\mu_A(\hat{0}, Y) = \begin{cases} 
\mu_{A \setminus A}(\hat{0}, Y) - \mu_A(A, Y) & \text{if } Y \in L_{A \setminus A}, \\
-\mu_A(A, Y) & \text{otherwise,}
\end{cases}
\]

where \( \mu_A \) denotes the Möbius function of \( L_A \) which we think of as a function \( L_A \times L_A \to \mathbb{Z} \) with \( S \nleq T \Rightarrow \mu_A(S,T) = 0 \) (and similarly for \( A \setminus A \)).

The claim is true if \( Y = \hat{0} = \mathbb{R}^n \), so assume it has been verified for all \( Z < Y \) in \( L_A \). If \( Y \in L_A \setminus A \) we obtain

\[
\mu_A(\hat{0}, Y) = \sum_{\hat{0} \leq Z < Y} \mu_A(\hat{0}, Z) - \sum_{\hat{0} \leq Z < Y} \mu_{A \setminus A}(\hat{0}, Z) + \sum_{A \leq Z < Y} \mu_A(A, Z) = \mu_{A \setminus A}(\hat{0}, Y) - \mu_A(A, Y),
\]
as desired. If, on the other hand, \( Y \not\in L_{A\setminus A} \), then there is a unique largest element in \( L_{A\setminus A} \) which is below \( Y \) in \( L_A \), namely the join of all atoms (weakly) below \( Y \) except \( A \); call this element \( W \). If \( W = \hat{0} \), then \( Y = A \) and we are done. Otherwise, 

\[
\mu_A(\hat{0}, Y) = - \sum_{\hat{0} \leq Z < Y} \mu_A(\hat{0}, Z) = - \sum_{\hat{0} \leq Z < W} \mu_{A\setminus A}(\hat{0}, Z) + \sum_{A \leq Z < Y} \mu_A(A, Z) 
\]

\[
= \sum_{A \leq Z < Y} \mu_A(A, Z) = -\mu_A(A, Y),
\]

establishing the claim.

We conclude that 

\[
\chi(A; x) = \sum_{Y \in L_{A\setminus A}} \mu_{A\setminus A}(\hat{0}, Y)x^{\dim(Y)} - \sum_{Y \geq A} \mu_A(A, Y)x^{\dim(Y)}.
\]

Not every \( Y \) in the last sum belongs to \( L_{A\setminus A} \) in general; the latter is join-generated by the elements covering \( A \) in \( L_A \). However, it follows from Rota’s Crosscut theorem \cite{8} that for every \( Y \geq A \) in \( L_A \),

\[
\mu_A(A, Y) = \begin{cases} 
\mu_{A\setminus A}(A, Y) & \text{if } Y \in L_{A\setminus A}, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus,

\[
\sum_{Y \geq A} \mu_A(A, Y)x^{\dim(Y)} = \chi(A\setminus A; x),
\]

and the theorem follows. \( \square \)

Two (families of) hyperplane arrangements are of particular importance to us. The first is the \textit{braid arrangement} \( S_n \). This is an arrangement whose ambient space is \( \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0 \} \cong \mathbb{R}^{n-1} \). The \( \binom{n}{2} \) hyperplanes in \( S_n \) are given by the equations \( x_i = x_j \) for all \( 1 \leq i < j \leq n \).

The braid arrangement is the set of reflecting hyperplanes of a Weyl group of type \( A \). Considering type \( B \) instead, we find our second important family of arrangements. Explicitly, \( B_n \) is the arrangement of the \( n^2 \) hyperplanes in \( \mathbb{R}^n \) that are given by the equations \( x_i = \tau x_j \) for all \( 1 \leq i < j \leq n \), \( \tau \in \{-1, 1\} \), and \( x_i = 0 \) for all \( i \in \{1, \ldots, n\} \).

2.2. \textbf{Stanley-Reisner rings and \textit{h}-polynomials.} Let \( \Delta \) be a simplicial complex on the vertex set \( [n] \). Regarding the vertices as variables, we want to consider the ring of polynomials that live on \( \Delta \). To this end, for a field \( k \), we define the \textit{Stanley-Reisner ideal} \( I_\Delta \subseteq k[x_1, \ldots, x_n] \) by 

\[
I_\Delta = \langle \{x_{i_1} \cdots x_{i_t} \mid \{i_1, \ldots, i_t\} \not\subseteq \Delta \} \rangle.
\]

The quotient ring 

\[
k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta
\]

is the \textit{Stanley-Reisner ring} of \( \Delta \), which is a graded algebra with the standard grading by degree. When speaking of algebraic properties, such as Cohen-Macaulayness, of \( \Delta \) we have the corresponding properties of \( k[\Delta] \) in mind.

Given a simplicial complex \( \Delta \) of dimension \( d - 1 \), its \textit{h}-polynomial is 

\[
h(\Delta; x) = \sum_{i=0}^{d} f_{i-1}(x - 1)^{d-i},
\]
where \( f_i \) is the number of \( i \)-dimensional simplices in \( \Delta \) (including \( f_{-1} = 1 \) if \( \Delta \) is nonempty). One important feature of the \( h \)-polynomial is that it carries all information needed to compute the Hilbert series of \( k[\Delta] \). Specifically,

\[
\Hilb(k[\Delta]; x) = \frac{\overline{h}(\Delta; x)}{(1 - x)^d},
\]

where \( \overline{h} \) denotes the reverse \( h \)-polynomial:

\[
\overline{h}(\Delta; x) = x^d \overline{h}(\Delta; \frac{1}{x}).
\]

2.3. Shellable complexes. Suppose \( \Delta \) is a pure simplicial complex, meaning that all facets (maximal simplices) have the same dimension \( d - 1 \). A shelling order for \( \Delta \) is a total ordering \( F_1, \ldots, F_t \) of the facets of \( \Delta \) such that \( F_j \cap (\bigcup_{i<j} F_i) \) is pure of dimension \( d - 2 \) for all \( j = 2, \ldots, t \). We say that \( \Delta \) is shellable if a shelling order for \( \Delta \) exists.

One good reason to care about shellability is that it implies Cohen-Macaulayness.

3. The objects of study

Suppose \( \mathcal{H} \) is a hyperplane arrangement in \( \mathbb{R}^n \) such that \( \cap \mathcal{H} = \{0\} \). Then, \( \mathcal{H} \) determines a regular cell decomposition \( \Delta_{\mathcal{H}} \) of the unit sphere \( S^{n-1} \). In short, each point \( p \) on \( S^{n-1} \) has an associated sign vector in \( \{0, -, +\}^{|\mathcal{H}|} \) recording for each hyperplane \( h \in \mathcal{H} \) whether \( p \) is on, or on the negative, or on the positive side of \( h \) (for some choice of orientations of the hyperplanes). A cell in \( \Delta_{\mathcal{H}} \) consists of the set of points with a common sign vector. The face poset of \( \Delta_{\mathcal{H}} \) is the big face lattice of the corresponding oriented matroid, see [2].

If \( \Delta_{\mathcal{H}} \) is a simplicial complex, then \( \mathcal{H} \) is called simplicial. A prime example of a simplicial hyperplane arrangement is the collection of reflecting hyperplanes of a finite Coxeter group. In this case, \( \Delta_{\mathcal{H}} \) coincides with the Coxeter complex.

From now on, let \( \mathcal{H} \) be a simplicial hyperplane arrangement.

Consider an antichain \( A \) in \( L_{\mathcal{H}} \). We say that the subspace arrangement \( A \) is embedded in \( \mathcal{H} \). Observe that \( \cup A \cap S^{n-1} \), which is known as the link of \( A \), has the structure of a simplicial subcomplex of \( \Delta_{\mathcal{H}} \). This subcomplex is the principal object of study in this paper. We denote it \( \Delta_{A, \mathcal{H}} \).

**Example 3.1.** A graph \( G = ([n], E) \) determines a graphical hyperplane arrangement \( \tilde{G} \) in the \((n - 1)\)-dimensional subspace of \( \mathbb{R}^n \) given by the equation \( x_1 + \cdots + x_n = 0 \). There is one hyperplane in \( \tilde{G} \) for each edge in \( E \); the hyperplane corresponding to the edge \( \{i, j\} \) has the equation \( x_i = x_j \).

The arrangement \( \tilde{K}_n \) corresponding to the complete graph is nothing but the braid arrangement \( S_n \), which is simplicial. Any graph \( G \) thus determines a simplicial complex \( \Delta_{\tilde{G}, S_n} \). It coincides with Steinberg’s coloring complex of \( G \) which was denoted \( \Delta_G \) in the Introduction. The complex \( \Delta_{\tilde{G}, S_n} \) also appeared under the name \( \Delta_{m,J} \) in [6].

We remark that the homotopy type of the link of \( A \), hence of \( \Delta_{A, \mathcal{H}} \), can be computed in terms of the order complexes of lower intervals in \( L_A \) by a formula of Ziegler and Živaljević [13]. When \( A \) consists of hyperplanes we may simply note that \( \Delta_{A, \mathcal{H}} \) is homotopy equivalent to the \((n - 1)\)-sphere with one point removed for each connected region in the complement \( \mathbb{R}^n \setminus \cup A \). Denoting by \( R(A) \) the
number of such regions, $\Delta_{A,\mathcal{H}}$ is thus homotopy equivalent to a wedge of $R(A) - 1$ spheres of dimension $n - 2$ in this case. For the arrangements $\tilde{G}$ of Example 3.1 it is not difficult to see that $R(\tilde{G})$ equals the number $AO(G)$ of acyclic orientations of $G$. Thus, $\Delta_{\tilde{G}, S_n}$ has the homotopy type of a wedge of $AO(G) - 1$ $(n - 3)$-spheres (6, 7). In particular, the reduced Euler characteristic of $\Delta_{\tilde{G}, S_n}$ is $\pm AO(G) - 1$ (10, Theorem 17).

4. Shellability in the hyperplane case

Our goal in this section is to show that $\Delta_{A,\mathcal{H}}$ is shellable whenever $A$ consists of hyperplanes. Applied to the complexes $\Delta_{\tilde{G}, S_n}$ of Example 3.1 this answers affirmatively a question of Steingrímsson 10 which was restated in 7. The key tool is a particular class of shellings of $\Delta_{A,\mathcal{H}}$ determined by the poset of regions of $\mathcal{H}$ which we now define.

The complement $\mathbb{R}^n \setminus \cup \mathcal{H}$ is cut into disjoint open regions by $\mathcal{H}$. Restricting to the unit sphere, their closures are the facets of $\Delta_{\mathcal{H}}$. Let $F = F(\mathcal{H})$ be the set of such facets. For $R, R' \in F$, say that $h \in \mathcal{H}$ separates $R$ and $R'$ if their respective interiors are on different sides of $h$.

Choose a base region $B \in F$ arbitrarily. We have a distance function $\ell : F \to \mathbb{N}$ which maps a region $R$ to the number of hyperplanes in $\mathcal{H}$ which separate $R$ and $B$. Now, for two regions $R, R' \in F$, write $R \triangleleft R'$ if $R$ and $R'$ are separated by exactly one hyperplane in $\mathcal{H}$ and $\ell(R) = \ell(R') - 1$. The poset of regions $P_{\mathcal{H}}$ is the partial order on $F$ whose covering relation is $\triangleleft$. It was first studied by Edelman 6.

From the point of view of this paper, the most important property of $P_{\mathcal{H}}$ is the following.

**Theorem 4.1** (Theorem 4.3.3 in 2). Any linear extension of $P_{\mathcal{H}}$ is a shelling order for $\Delta_{\mathcal{H}}$.

We are now ready to state and prove the main result of this section.

**Theorem 4.2.** If $A$ consists of hyperplanes, then $\Delta_{A,\mathcal{H}}$ is shellable.

*Proof.* We proceed by induction over $|A|$. When $A = \{A\}$, we may apply Theorem 4.1 since $\Delta_{A,\mathcal{H}} = \Delta_{\mathcal{H}/A}$ in this case.

Now suppose $|A| \geq 2$ and that we have a shelling order for $\Delta_{A \setminus A,\mathcal{H}}$ for some $A \in A$. We will append the remaining facets to this order.

The remaining facets are the facets of $\Delta_{\{A\},\mathcal{H}} = \Delta_{\mathcal{H}/A}$. They are divided into equivalence classes in the following way: $F$ and $G$ belong to the same class iff their interiors belong to the same connected component of $\mathbb{R}^n \setminus \cup (A \setminus A)$ (or, equivalently, to the same connected component of $A \setminus \cup (A \setminus A)$). Observe that if $F$ and $G$ belong to different classes, then $F \cap G \in \Delta_{A \setminus A,\mathcal{H}}$. Thus, it is enough to show that the facets in any equivalence class can be appended to the shelling order for $\Delta_{A \setminus A,\mathcal{H}}$.

Without loss of generality, consider the class which contains the maximal element in $P_{\mathcal{H}/A}$, i.e. the region opposite to the base region. Call this class $C$. If $F \in C$ and $G \not\in C$ for $F, G \in P_{\mathcal{H}/A}$, then some hyperplane in $A/A \subseteq \mathcal{H}/A$ separates $F$ from $G$, and $G$ is on the positive side of this hyperplane. Thus, $F \not\triangleleft G$. This shows that $C$ is an order filter in $P_{\mathcal{H}/A}$. According to Theorem 4.2, $\Delta_{\mathcal{H}/A}$ has a shelling order which ends with the facets in $C$. Now observe that $(\cup C) \cap (\cup (P_{\mathcal{H}/A} \setminus C)) = (\cup C) \cap \Delta_{A \setminus A,\mathcal{H}}$. 
The facets in $C$ may therefore be appended in this order to the shelling order for $\Delta_{A \setminus A, H}$. 

5. The $h$-polynomial of $\Delta_{A, H}$

For brevity we write $h(A, H; x)$ meaning $h(\Delta_{A, H}; x)$ and similarly for $\overline{h}$. The following result of Steingrímsson serves as a motivating example for this section:

**Theorem 5.1** (Theorem 13 in [10]). Recall the complex $\Delta_{\hat{G}, S_n}$ defined in Example 3.1. We have

$$\frac{x \overline{h}(\hat{G}, S_n; x)}{(1 - x)^n} = \sum_{m \geq 0} (m^n - P_G(m)) x^m,$$

where $P_G$ is the chromatic polynomial of $G$.

This theorem is interesting because of the connection between the left hand side and the Hilbert series of the Stanley-Reisner ring $k[\Delta_{\hat{G}, S_n}]$. In [3], Brenti began a systematic study of which polynomials arise as Hilbert polynomials of standard graded algebras. A question left open in [3], and later answered affirmatively by Almkvist [1], was whether chromatic polynomials of graphs have this property. Theorem 5.1 implies something similar, namely that $(m + 1)^n - P_G(m + 1)$ is the Hilbert polynomial (in $m$) of a standard graded algebra; for details, see Corollary 5.7 below.

It is well-known that $P_G(x) = x \chi(\hat{G}; x)$; one way to prove it is to compare Theorem 2.1 with the standard deletion-contraction recurrence for $P_G$. The identity suggests the possibility of extending Theorem 5.1 to other complexes $\Delta_{A, H}$. This turns out to be possible at least if $H \in \{S_n, B_n\}$ and is the topic of this section.

Given a subspace $T$ of $\mathbb{R}^n$, let $d(T)$ denote its dimension. For a subspace arrangement $\mathcal{T}$, we also write

$$d(\mathcal{T}) = \max_{T \in \mathcal{T}} d(T).$$

**Lemma 5.2.** Let $A \in A$. Then,

$$h(A, H; x) = (x - 1)^{d(A) - d(A \setminus A)} h(A \setminus A, H; x) + (x - 1)^{d(A) - d(A)} h(\{A\}, H; x) - (x - 1)^{d(A) - d(A/A)} h(A/A, H/A; x).$$

**Proof.** Each simplex in $\Delta_{A, H}$ belongs to $\Delta_{A \setminus A, H}$ or to $\Delta_{\{A\}, H}$ or to both. Also, observe that $\Delta_{A \setminus A, H} \cap \Delta_{\{A\}, H} = \Delta_{A/A, H/A}$. Denoting by $f_i(S, \mathcal{T})$ the number of $i$-dimensional simplices in $\Delta_{S, \mathcal{T}}$, we thus obtain for all $i$

$$f_i(A, H) = f_i(A \setminus A, H) + f_i(\{A\}, H) - f_i(A/A, H/A).$$

The lemma now follows from the fact that $\dim(\Delta_{S, \mathcal{T}}) = d(S) - 1$. 

We may use Lemma 5.2 to recursively compute $h(A, H; x)$. As it turns out, this recursion is particularly useful when $H \in \{S_n, B_n\}$. The reason is given by the following two lemmata.

**Lemma 5.3.** We have

$$\frac{x \overline{h}(\Delta_{S_n}; x)}{(1 - x)^{n+1}} = \sum_{m \geq 0} m^n x^m.$$
and
\[ \frac{h(\Delta_{B_n}; x)}{(1 - x)^{n+1}} = \sum_{m \geq 0} (2m + 1)^n x^m. \]

**Proof.** The complexes \( \Delta_{S_n} \) and \( \Delta_{B_n} \) coincide with the Coxeter complexes of types \( A_{n-1} \) and \( B_n \), respectively. For the \( h \)-polynomials this implies that \( xh(\Delta_{S_n}; x) = A_n(x) \) and \( h(\Delta_{B_n}; x) = B_n(x) \), where \( A_n \) is the \( n \)th Eulerian polynomial and \( B_n \) is the \( n \)th \( B \)-Eulerian polynomial, see [4]. The assertions are well-known properties of these polynomials [4, Theorem 3.4.ii]. \( \square \)

**Lemma 5.4.**

(i) For any subspace \( A \in L_{S_n} \), we have
\[ xh(\{A\}, S_n; x) (1 - x)^{d(A) + 2} = \sum_{m \geq 0} m^{d(A) + 1} x^m. \]

(ii) For any subspace \( A \in L_{B_n} \), we have
\[ \frac{h(\{A\}, B_n; x)}{(1 - x)^{d(A) + 1}} = \sum_{m \geq 0} (2m + 1)^{d(A)} x^m. \]

**Proof.** A key property of \( S_n \) (\( B_n \)), which is readily checked, is that its restriction to any subspace in the intersection lattice is again a type \( A \) (\( B \)) hyperplane arrangement. Thus, \( \Delta_{\{A\}, S_n} = \Delta_{S_n/A} \cong \Delta_{S_{d(A) + 1}} \) (\( \Delta_{\{A\}, B_n} = \Delta_{B_n/A} \cong \Delta_{B_{d(A)}} \)). The assertions now follow from Lemma 5.3. \( \square \)

The leading term of \( \chi(A; x) \) is always \( x^n \), where \( n \) is the dimension of the ambient space. It is convenient to introduce the tail \( T(A; x) = x^n - \chi(A; x) \).

When \( A \) consists of hyperplanes, the following result coincides with Theorem 5.1.

**Theorem 5.5.** Suppose \( A \) is a subspace arrangement embedded in \( S_n \). Then,
\[ \frac{xh(\{A\}, S_n; x)}{(1 - x)^{d(A) + 2}} = \sum_{m \geq 0} mT(A; m)x^m. \]

**Proof.** We proceed by induction over \( |A| \), noting that \( |A \setminus A| < |A| \) and \( |A/A| < |A| \) for every \( A \in A \). If \( |A| = 1 \), we have \( \chi(A; m) = m^{n-1} - m^{d(A)} \), so that \( T(A; m) = m^{d(A)} \), and the theorem follows from part (i) of Lemma 5.3. \( \square \)
Theorem 5.6. Theorem 5.5 is easily adjusted to a proof of the next result. □

Now suppose $|A| \geq 2$ and pick a subspace $A \in \mathcal{A}$. Using Lemma 5.4 and the induction hypothesis, we obtain

$$\frac{x^{d(A)+1} h(A, S_n; \frac{1}{x})}{(1 - x)^{d(A)+2}} = \left(\frac{1 - x}{x}\right)^{d(A) - d(A^A)} \frac{x^{d(A)+1} h(A \setminus A, S_n; \frac{1}{x})}{(1 - x)^{d(A)+2}} + \left(\frac{1 - x}{x}\right)^{d(A) - d(A)} \frac{x^{d(A)+1} h(A, S_n_1; \frac{1}{x})}{(1 - x)^{d(A)+2}}$$

$$- \left(\frac{1 - x}{x}\right)^{d(A) - d(A^A)} \frac{x^{d(A)+1} h(A^A A, S_n / A; \frac{1}{x})}{(1 - x)^{d(A)+2}}$$

$$= \sum_{m \geq 0} m(m^{n-1} - \chi(A \setminus A; m)) x^m$$

$$+ \sum_{m \geq 0} m(m^{n-1} - (m^{n-1} - m^{d(A)})) x^m$$

$$- \sum_{m \geq 0} m(m^{d(A)} - \chi(A^A; m)) x^m$$

$$= \sum_{m \geq 0} m(m^{n-1} - \chi(A; m)) x^m,$$

where the last equality follows from Deletion-Restriction.

For completeness, we should also check the uninteresting case $|A| = 0$ which is not covered by the above arguments. Here, $\overline{h}()_n, 1; x = 0$ and $T()_n, 1; x = 0$, and the assertion holds. □

Corollary 5.7. Suppose $A$ is a subspace arrangement embedded in $\mathcal{B}_n$. Then,

$$\overline{h}(A, B_n; x) \overline{h}(A, B_n; x) = \sum_{m \geq 0} T(A; 2m + 1) x^m.$$

For subspace arrangements covered by Theorem 5.5 or Theorem 5.6, we may now draw the promised algebraic conclusions. To this end, for a simplicial complex $\Gamma$ and a subcomplex $\Gamma' \subseteq \Gamma$, let $\mathcal{J}_{\Gamma', \Gamma}$ be the ideal in the Stanley-Reisner ring $k[\Gamma]$ generated by the (equivalence classes of) monomials corresponding to simplices in $\Gamma$ that do not belong to $\Gamma'$.

Theorem 5.6. Suppose $A$ is a subspace arrangement embedded in $\mathcal{B}_n$. Then,

$$\overline{h}(A, B_n; x) = \sum_{m \geq 0} T(A; 2m + 1) x^m.$$
where the second equality follows from Theorem 5.6. This proves (i).

For (ii), we use that
\[ k[\Gamma'] \cong k[\Gamma]/J_{\Gamma', \Gamma}. \]

For the Hilbert series, this implies
\[ \text{Hilb}(k[\Gamma']; x) = \text{Hilb}(k[\Gamma]; x) - \text{Hilb}(J_{\Gamma', \Gamma}; x). \]

From part (i) and the fact that
\[ \text{Hilb}(k[\Gamma]) = \frac{h(\Delta_{B_n}; x)}{(1 - x)^{n+1}} = \frac{1}{x} \sum_{m \geq 0} m^n x^m, \]
we conclude
\[ \text{Hilb}(J_{\Gamma', \Gamma}; x) = \frac{1}{x} \sum_{m \geq 0} m^n x^m - \frac{1}{x} \sum_{m \geq 0} mT(A; m)x^m = \frac{1}{x} \sum_{m \geq 0} m\chi(A; m)x^m. \]

\[ \square \]

The situation for \( B_n \) is analogous, although we use cones instead of double cones. This is a manifestation of the fact that \( B_n \) and \( S_n \) differ by one in dimension.

**Corollary 5.8.** Suppose \( A \) is a subspace arrangement embedded in \( B_n \). Let \( \Gamma \) denote the cone over \( \Delta_{B_n} \), and write \( \Gamma' \) for the cone over \( \Delta_{A, B_n} \) with the same cone point. Then, the following holds:

(i) The Hilbert polynomial of \( k[\Gamma'] \) is \( F(k[\Gamma']; m) = T(A; 2m + 1) \).

(ii) The Hilbert polynomial of \( J_{\Gamma', \Gamma} \) is \( F(J_{\Gamma', \Gamma}; m) = \chi(A; 2m + 1) \).

**Proof.** Proceeding as in the proof of Corollary 5.7 using Theorem 5.6 instead of Theorem 5.5 we prove (i) by observing
\[ \text{Hilb}(k[\Gamma']; x) = \frac{h(A, B_n; x)}{(1 - x)^{d(A)+1}} = \sum_{m \geq 0} T(A; 2m + 1)x^m. \]

For (ii), note that
\[ \text{Hilb}(k[\Gamma]; x) = \frac{h(\Delta_{B_n}; x)}{(1 - x)^{n+1}} = \sum_{m \geq 0} (2m + 1)^n x^m. \]

Thus,
\[ \text{Hilb}(J_{\Gamma', \Gamma}; x) = \sum_{m \geq 0} (2m + 1)^n x^m - \sum_{m \geq 0} T(A; 2m + 1)x^m = \sum_{m \geq 0} \chi(A; 2m + 1)x^m. \]

\[ \square \]

Any hypergraph (without inclusions among edges) \( G \) on \( n \) vertices corresponds to a subspace arrangement \( \hat{G} \) embeddable in \( S_n \). The construction is virtually the same as in Example 3.1 with the hyperedge \( \{i_1, \ldots, i_t\} \) is associated the subspace given by \( x_{i_1} = \cdots = x_{i_t} \). As for ordinary graphs (the hyperplane case), we have \( x\chi(\hat{G}; x) = P_G(x) \), cf. [9, Theorem 3.4]. In this way, Corollary 5.7 allows us to interpret chromatic polynomials of hypergraphs in terms of Hilbert polynomials. For ordinary graphs, this is the content of Steingrímsson’s [10 Corollary 10].
Corollary 5.8, too, has an impact on chromatic polynomials. Any signed graph (in the sense of Zaslavsky \[11\]) \( G \) on \( n \) vertices corresponds to a hyperplane arrangement \( \hat{G} \subseteq B_n \), and vice versa. A signed graph \( G \) has a chromatic polynomial \( P_G(x) \), and \( P_G(x) = \chi(\hat{G}; x) \) \[12\].

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DEPARTMENT OF MATHEMATICS, KTH, SE-100 44 STOCKHOLM, SWEDEN

E-mail address: axel@math.kth.se