Semisimple-direct-injective modules

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Abstract

The notion of simple-direct-injective modules which are a generalization of injective modules unifies $C_2$ and $C_3$-modules. In the present paper, we introduce the notion of the semisimple-direct-injective module which gives a unified viewpoint of $C_2$, $C_3$, SSP properties and simple-direct-injective modules. It is proved that a ring $R$ is Artinian serial with the Jacobson radical square zero if and only if every semisimple-direct-injective right $R$-module has the SSP and, for any family of simple injective right $R$-modules $\{S_i\}$, $\bigoplus_i S_i$ is injective. We also show that $R$ is a right Noetherian right $V$-ring if and only if every right $R$-module has a semisimple-direct-injective envelope if and only if every right $R$-module has a semisimple-direct-injective cover.

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1. Introduction

Throughout this article, unless otherwise stated, all rings have unity and all modules are unital. A right $R$-module $M$ is called

a $C_1$-module provided that every submodule of $M$ is essential in a direct summand of $M$;

a $C_2$-module (or direct-injective) provided that $A$ is a direct summand in $M$ whenever $A$ is a submodule of $M$ such that $A$ is isomorphic to a direct summand in $M$ and

a $C_3$-module if $A$ and $B$ are direct summands in $M$ and $A \cap B = 0$, then $A + B$ is a direct summand in $M$.

It is easy to see that each $C_2$-module is also a $C_3$-module. Conversely, for each module $M$, if $M \oplus M$ is a $C_3$-module, then $M$ is a $C_2$-module (see also [1, Corollary 2.6]). However, $C_3$ is a weaker property in general: if $R$ is any integral domain which is not a field, then

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R is C3, but not C2. Recently, the classes of Ci-modules (i = 1, 2, 3) are studied and generalizations of them are considered ([1, 5, 6, 11, 13, 15]).

We recall also that a module M has the summand sum property (SSP) if the sum of two direct summands is a direct summand of M ([10] and [16]). Clearly, modules having (SSP) are C3.

Recently, Camillo, Ibrahim, Yousif and Zhou [5] obtained that every simple submodule which is isomorphic to a direct summand is itself a direct summand if and only if the sum of any two simple direct summands with zero intersection is again a direct summand [5, Proposition 2.1]. Such modules are called simple-direct-injective (see also [11]). In the present paper, we introduce the concept of semisimple-direct-injective modules. A module is called semisimple-direct-injective if every semisimple submodule isomorphic to a summand is itself a summand, or equivalently if the sum of any two semisimple summands (with zero intersection) is again a summand (see Proposition 2.1). Theorem 3.4 in [5] addressed the question of when every simple-direct-injective module is C3, and they proved that every simple-direct-injective right R-module is C3 if and only if R is an Artinian serial ring with Jacobson radical square zero. In Theorem 2.10, we prove that R is an Artinian serial ring with Jacobson radical square zero if and only if every semisimple-direct-injective right R-module has the SSP and \( \oplus \{ S_i \} \) is injective for any family of simple injective modules \( \{ S_i \} \).

Enochs [7] introduced the notation of injective cover as the dual notation of the injective envelope, and proved that a ring R is right Noetherian if and only if every right R-module has an injective cover. In Section 3, we are concerned with semisimple-direct-injective envelopes and covers, namely sdi-envelopes and sdi-covers. In Theorem 3.4, it is shown that the classes of semisimple-direct-injective modules over a ring R provide for sdi-envelopes and sdi-covers only if R is a right Noetherian V-ring.

A ring is called a right V-ring if every simple right R-module is injective. In Section 4, we study some natural connections between V-rings and semisimple-direct-injective modules which are similar to simple-direct-injective modules. For instance, we obtain that a ring is right Noetherian and a right V-ring if and only if every right R-module is semisimple-direct-injective if and only if every direct sum of two semisimple-direct-injective modules is semisimple-direct-injective (Theorem 2.11).

Throughout this article, a submodule N of an R-module M is called essential in M, denoted by \( N \leq_e M \), if for any nonzero submodule L of M, \( L \cap M \neq 0 \). We write \( J(R) \) and \( \text{Soc}(R) \) for the Jacobson radical and the socle of R, respectively. We also write \( N \leq_d M \) and \( E(M) \) to indicate that N is a direct summand of M and the injective envelope of M, respectively. For a nonempty subset X of a ring R, the left annihilator of X in R is \( l(X) = \{ r \in R : rx = 0 \text{ for all } x \in X \} \). For any \( a \in R \), we write \( l(a) \) for \( l(\{ a \}) \). Right annihilators are defined similarly. General background material can be found in [3], [6], [12] and [17].

2. Semisimple-direct-injective modules

Proposition 2.1. The followings are equivalent for a right R-module M.

1. For any semisimple submodules A, B of M with \( A \cong B \leq_d M \), A is a summand of M.
2. For any semisimple summands A, B of M with \( A \cap B = 0 \), the sum \( A \oplus B \) is a summand of M.
3. For any semisimple summands A, B of M, \( A + B \leq_d M \).
4. If \( M = A_1 \oplus A_2 \) with \( A_1 \) semisimple and \( f : A_1 \to A_2 \) is a homomorphism, then \( \text{Im}(f) \leq_d A_2 \).

Proof. (1) \( \Rightarrow \) (2) Assume M = A \( \oplus A' \) and let \( \pi : A \oplus A' \to A' \) be the canonical projection. Then \( A \oplus B = A \oplus \pi(B) \) is a direct summand of M as \( \pi(B) \cong B \).
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\[ \text{Example 2.6. Let } Q := \bigoplus_{i=1}^{\infty} F_i \text{ with } F_i := \mathbb{Z}_2 \text{ and } R \text{ be the subring of } Q \text{ generated by } \bigoplus_{i=1}^{\infty} F_i \text{ and } 1_Q. \text{ Then } R \text{ is a commutative, non self-injective V-ring and } \text{Soc}(R) \text{ is essential in } R. \text{ We deduce that } R \text{ is not Noetherian. Thus one infers that there exists a simple-direct-injective module over } R \text{ which is not semisimple-direct-injective.} \]

(ii) Let \( V \) be an infinite-dimensional vector space over \( F \). Let \( Q := \text{End}_F(V) \), \( J := \{ x \in Q : \text{dim}_F(xV) < +\infty \} \) and \( R := F + J \). Then \( R \) is a right V-ring (see [9, Example 6.19]) and \( R \) is not right Noetherian. Similarly (i), there is a simple-direct-injective right \( R \)-module which is not semisimple-direct-injective.

\[ \text{Example 2.6. If } M \text{ is an indecomposable right } R\text{-module which is not simple, then } M \oplus E(M) \text{ is a semisimple-direct-injective module. Indeed, by [5, Lemma 3.3], } M \oplus E(M) \text{ has no simple summands.} \]
Example 2.7. Given a field $F$ and an isomorphism $F \to \mathcal{F} \subseteq F$ defined by $a \mapsto \pi$, let $R$ be the right $F$-space on basis $\{1, t\}$ with multiplication given by $t^2 = 0$ and $at = \pi a$ for all $a \in F$. Assume that $1 < \dim_{\mathcal{T}}(F) < \infty$. By Example 2.6, $R \oplus E(R)$ is a semisimple-direct-injective module which is not C3 (has not the SSP) by [5, Example 3.6].

Proposition 2.8. If $M = \bigoplus_{i \in I} E_i$ is a direct sum of indecomposable injective right $R$-modules $E_i$, then $M$ is a semisimple-direct-injective module.

Proof. Let $A$ be the sum of the simples $E_i$ and $B$ be the sum of the non-simple ones. If $S$ is isomorphic to a semisimple direct summand of $M$, then all simple summands of $S$ are clearly injective, so that $S \cap B = 0$. Since $(B \oplus S) \cap A$ is a direct summand of $A$, we get the former is a direct summand of $M$, whence $S$ is a direct summand of $M$. \hfill $\Box$

Corollary 2.9. Let $\{S_i\}_I$ be a family of simple injective modules and $\{E(S_j)\}_J$ be a family of injective envelopes of simple non-injective modules $S_j$. Then $M = (\bigoplus_{i \in I} S_i) \oplus (\bigoplus_{j \in J} E(S_j))$ is a semisimple-direct-injective module.

A module is uniserial if the lattice of its submodules is totally ordered under inclusion. A ring $R$ is called uniserial if $R_R$ is a uniserial module. A ring $R$ is called serial if both modules $R_R$ and $R_R$ are direct sums of uniserial modules.

Now we investigate when semisimple-direct-injective modules have the SSP.

Theorem 2.10. The followings are equivalent for a ring $R$:

$(1)$ $R$ is an Artinian serial ring with $J(R)^2 = 0$.

$(2)$ (a) Every semisimple-direct-injective right $R$-module is a C3-module.

(b) For any family of simple injective modules $\{S_i\}_I$, $\bigoplus \ S_i$ is injective.

$(3)$ (a) The right socle of $R$ is finitely generated.

(b) Every semisimple-direct-injective right $R$-module is quasi-injective.

Proof. $(1) \Rightarrow (3)$ For any module $M$ over an Artinian serial ring $R$ with $J(R)^2 = 0$, we have a decomposition $M = A \oplus M$, where $A$ is semisimple and $B$ is a sum of injective serial modules of length 2 by [6, 13.5]. So, it is obvious that semisimple-direct-injective right $R$-modules are precisely those with $A$ orthogonal to $B$. In this case, $B$ is injective and $A$ is injective relative to $B$. Thus, $M$ is quasi-injective.

$(3) \Rightarrow (2)$ As each quasi-injective module is a C3-module, one only needs to verify (b): If every semisimple-direct-injective right $R$-module is quasi-injective and every module having the zero socle is a semisimple-direct-injective module, then $R$ is right semi-Artinian (i.e., all nonzero modules have nonzero socle). So, $E(R_R) = E(T_1) \oplus E(T_2) \oplus \cdots \oplus E(T_n)$ where each $T_i$ is a minimal right ideal of $R$. Let $\{S_i\}_I$ be a family of simple right $R$-modules. Let $M := (\bigoplus_{i \in I} E(S_i)) \oplus (\bigoplus_{j = 1}^\infty E(T_j))$. By Lemma 2.8, $M$ is a semisimple-direct-injective module and so, by (3-b), $M$ is a quasi-injective module. Now one infers that $\bigoplus_{i} E(S_i)$ is $E(R_R)$-injective and hence it is injective.

$(2) \Rightarrow (1)$ We first prove $R$ is right Noetherian. Let $\{S_i\}_I$ be a family of simple right $R$-modules. We claim that $\bigoplus_{i} E(S_i)$ is an injective module. By [4, Theorem 1.3], one infers that there exists an infinite subset $J$ of $\mathbb{N}$ such that $\bigoplus J E(S_i)$ is injective. Write $\mathbb{N} = J_1 \cup J_2$ such that $S_i$ is injective if $i \in J_1$ and $S_j$ is not injective if $j \in J_2$. By the assumption, $\bigoplus J_1 E(S_i)$ is injective. Now we can assume that $|J_2|$ is infinite. Note that $M = (\bigoplus_{j \in J_2} E(S_j)) \oplus E(\bigoplus_{j \in J_2} E(S_j))$ has no simple summands. Hence $M$ is a semisimple-direct-injective module, and so it is a C3-module. So, $\bigoplus_{j \in J_2} E(S_j)$ is an injective module. Thus $R$ is right Noetherian. Now, by the same proof of $(1) \Rightarrow (3)$ of Theorem 3.4 in [5], one infers that $R$ is an Artinian serial ring with $J(R)^2 = 0$. \hfill $\Box$

The following observations give some connections between (right Noetherian) right $V$-rings and semisimple-direct-injective modules.
Theorem 2.11. The following conditions are equivalent for a ring $R$:

1. $R$ is a right Noetherian and right V-ring.
2. Every right $R$-module is semisimple-direct-injective.
3. Direct sum of two semisimple-direct-injective right $R$-modules is semisimple-direct-injective.

Proof. Recall that $R$ is a right Noetherian and right V-ring if and only if every semisimple module is injective.

$(1) \implies (2), (3)$ are obvious.

$(2) \implies (1)$ If $A$ is a semisimple right $R$-module, then, by the assumption, $M = A \oplus E(A)$ is a semisimple-direct-injective module. By Proposition 2.1, $A$ is a direct summand of $E(A)$ and hence $A$ is injective. Thus $R$ is a right Noetherian right V-ring.

$(3) \implies (1)$ is similar to $(2) \implies (1)$.

Corollary 2.12. $R$ is semisimple Artinian if and only if every semisimple-direct-injective right $R$-module is injective.

Proof. Assume that every semisimple-direct-injective right $R$-module is injective. We deduce that every semisimple right $R$-module is injective. So, $R$ is a right Noetherian right V-ring.

If $R$ is not right semi-Artinian, there exists a non-zero right $R$-module $M$ with $\text{Soc}(M) = 0$. Clearly, $M$ and its submodules are injective, a contradiction.

We recall Example 2.3 before the following corollary.

Corollary 2.13. Let $R$ be a right V-ring. Then $R$ is right Noetherian if and only if every simple-direct-injective right $R$-module is semisimple-direct-injective.

In [5, Theorem 4.4], authors give a new answer to Fisher’s question [8]: When are regular rings right V-rings? They proved that a regular ring $R$ is a right V-ring if and only if every cyclic right $R$-module is simple-direct-injective. Recall that a ring $R$ is called (von Neumann) regular if for every $a \in R$, there exists some $b \in R$ such that $a = aba$.

Theorem 2.14. Let $R$ be a regular ring. The following conditions are equivalent:

1. $R$ is a right V-ring.
2. Every cyclic right $R$-module is semisimple-direct-injective.
3. Every cyclic right $R$-module is simple-direct-injective.

Proof. This follows from [5, Theorem 4.4] and Example 2.3.

A right $R$-module $M$ is called strongly soc-injective if for any right $R$-module $N$ and any semisimple submodule $K$ of $N$, every $R$-homomorphism $f : K \to M$ extends to $N$ [2]. By [2, Proposition 16], a right $R$-module $M$ is strongly soc-injective if and only if $M = E \oplus T$, where $E$ is injective and $\text{Soc}(T) = 0$. It is easy to see that every strongly soc-injective module is semisimple-direct-injective.

Proposition 2.15. The following conditions are equivalent for a ring $R$:

1. $R$ is a right Noetherian right V-ring.
2. Every semisimple-direct-injective module is strongly soc-injective.

Proof. $(1) \implies (2)$. Let $M$ be a semisimple-direct-injective module. Assume that $\text{Soc}(M)$ is non-zero. Hence, $M$ has a decomposition $M = \text{Soc}(M) \oplus T$ such that $\text{Soc}(M)$ is injective and $\text{Soc}(T) = 0$. Thus, $M$ is a strongly soc-injective module.

$(2) \implies (1)$ Let $M$ be a semisimple module. Then, $M$ is a strongly soc-injective module, write $M = E \oplus T$, where $E$ is injective and $\text{Soc}(T) = 0$. Furthermore, we have $T = \text{Soc}(T)$ and so $M = E$ is injective.
Recall that a right $R$-module $M$ is called \textit{mininjective} if, for every simple right ideal $K$ of $R$, each $R$-homomorphism $f : K \to M$ extends to $g : R \to M$; that is, $f = m \cdot$ is multiplication by some $m \in M$ ([13]).

\textbf{Lemma 2.16} ([13, Theorem 2.36]). \textit{The following conditions are equivalent for a ring $R$:}

(1) Every right $R$-module is mininjective.

(2) Every cyclic right $R$-module is mininjective.

(3) $K^2 \neq 0$ for every simple right ideal $K$ of $R$.

(4) $\text{Soc}(R_R) \cap J(R) = 0$.

(5) $R$ is right mininjective and $\text{Soc}(R_R)$ is projective as a right $R$-module.

A ring $R$ is called right \textit{universally mininjective} if it satisfies the conditions in Lemma 2.16.

\textbf{Lemma 2.17.} \textit{The following conditions are equivalent for a ring $R$:}

(1) $R$ is right universally mininjective.

(2) $R$ is right semisimple-direct-injective and every minimal right ideal of $R$ is projective as a right $R$-module.

\textbf{Proof.} \ $(1) \Rightarrow (2)$. Assume that $R$ is right universally mininjective. Then, every minimal right ideal of $R$ is a direct summand of $R_R$ by Lemma 2.16. It follows that $R$ is a right simple-direct-injective ring, and so it is semisimple-direct-injective.

$(2) \Rightarrow (1)$. We show that $R$ is right mininjective. Indeed, let $K$ be a minimal right ideal of $R$. Then, $K$ is a projective module, and so $K$ is isomorphic to a direct summand of $R_R$. We have that $R$ is right semisimple-direct-injective and obtain that $K$ is a direct summand of $R_R$. We deduce that $R$ is right mininjective. Thus, $R$ is right universally mininjective by Lemma 2.16. \hfill \Box

\textbf{Theorem 2.18.} \textit{The following conditions are equivalent for a ring $R$:}

(1) $R$ is semisimple Artinian.

(2) $R$ satisfies the following conditions:

(a) $R$ is right semisimple-direct-injective with $\text{Soc}(R_R) \leq_e R_R$ and projective as a right $R$-module.

(b) Every ascending chain

$$r(a_1) \subseteq r(a_2a_1) \subseteq \cdots$$

terminates for every infinite sequence $a_1, a_2, \ldots$ of elements in $R$.

\textbf{Proof.} \ $(1) \Rightarrow (2)$ This is obvious.

$(2) \Rightarrow (1)$ By $(2-a)$, $R$ is a right universally mininjective ring and $\text{Soc}(R_R) \leq \text{Soc}(R_R)$ by Lemma 2.17. Hence $\text{Soc}(R_R)$ is essential in $R_R$. From [14, Theorem 2.2] we infer that $R$ is a right perfect ring. Furthermore, $\text{Soc}(R_R) \cap J(R) = 0$ and $\text{Soc}(R_R) \leq_e R_R$, which implies that $J(R) = 0$. Thus $R$ is a semisimple Artinian ring. \hfill \Box

We denote the nil radical $N(R)$ of $R$ by $N(R) = \sum \{ I \mid I$ is nil right ideal of $R \}$.

\textbf{Corollary 2.19.} \textit{If $N(R) = 0$, $\text{Soc}(R_R) \leq_e R_R$ and every ascending chain}

$$r(a_1) \subseteq r(a_2a_1) \subseteq \cdots$$

\textit{terminates for every infinite sequence $a_1, a_2, \ldots$ of elements in a ring $R$, then $R$ is a semisimple Artinian ring.}

\textbf{Proof.} Let $I$ be an arbitrary minimal right ideal of $R$. From the hypothesis $N(R) = 0$ it immediately follows that $I^2 \neq 0$. Therefore, $I$ is a direct summand of $R_R$. It follows that $R$ is right semisimple-direct-injective and every minimal right ideal of $R$ is projective as a right $R$-module. Thus $R$ is a semisimple Artinian ring. \hfill \Box
**Corollary 2.20** ([17, 4.3]). A right Artinian ring $R$ with $N(R) = 0$ is semisimple Artinian.

We finish this section with the study of the following question:

"Does there exist a right semisimple-direct-injective ring that is not left semisimple-direct-injective?"

Rings of formal triangular matrices also serve as a source of examples of rings with non-symmetrical properties. Below we give an example of a formal triangular matrices ring that answers positively the previous question.

Given the $R$-$S$-bimodule $M$ we denote

$$l(M) = \{r \in R \mid rM = 0\}, \quad r(M) = \{s \in S \mid Ms = 0\}$$

**Theorem 2.21.** The following conditions are equivalent for a formal triangular matrices ring $K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$

1. $K$ is a right semisimple-direct-injective ring;
2. (a) For any semisimple submodules $A, B$ of $l(M)$ with $A \cong B \leq_d R_R$, $A$ is a summand of $R_R$.
   
   (b) For any semisimple submodules $A, B$ of $S_S$ with $A \cong B \leq_d S_S$, $A$ is a summand of $S_S$ and $A \leq r(M)$.

**Proof.** (1) $\Rightarrow$ (2) (a) Let $A$ be a semisimple submodule of $R_R$, $A \cong B \leq_d R_R$ and $A, B \leq l(M)$. Then, there exists a submodule $B'$ of $R_R$ such that $R_R = B \oplus B'$. It follows that there is a decomposition $K_K = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} B' & M \\ 0 & S \end{pmatrix}$. We have that an $K$-isomorphism $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \cong \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ of $K$-modules and obtain that there exists a submodule $L$ of $K_K$ such that we have a decomposition $K_K = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \oplus L$. Let $A' = \{a \in R \mid \exists m \in M, \exists s \in S : \begin{pmatrix} a & m \\ 0 & s \end{pmatrix} \in L\}$. One can check that $R_R = A \oplus A'$.

(b) Let $A$ be a semisimple submodule of $S_S$, $A \cong B \leq_d S_S$. Using arguments similar to those in the proof of (a), we can show that $A \leq S_S$. Assume that $MA \neq 0$. Then, there exists a simple submodule $A_0$ of $A$ such that $MA_0 \neq 0$. One can check that there is an isomorphism $K$-modules $\begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix} \cong \begin{pmatrix} 0 & MA_0 \\ 0 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix} \leq_d K_K$, then we get a contradiction with the condition of (1). It follows that $MA = 0$ or $A \leq r(M)$.

(2) $\Rightarrow$ (1) Firstly, let $A$ be a simple submodule of $K_K$, $A \cong A' \leq_d K_K$. It follows, from the condition of (2), that either $A' = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} K = \begin{pmatrix} eR & 0 \\ 0 & 0 \end{pmatrix}$, or $A' = \begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix} K = \begin{pmatrix} 0 & 0 \\ 0 & e'S \end{pmatrix}$ for some $e^2 = e \in R$ and $e^2 = e' \in S$.

Assume that $A' = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} K$ and $f : A' \to A$ is an isomorphism of $K$-modules. Since $A' \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$, then $S = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$, where $A_0$ is a simple submodule of $R_R$.

Assume that $A' = \begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix} K$ and $f : A' \to A$ is an isomorphism of $K$-modules. Since $A' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$ then $f(\begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix}) = \begin{pmatrix} m & 0 \\ 0 & s \end{pmatrix}$ with $m \in M, s \in S$. We have

$f(\begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix}) = f(\begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix})$ and get $\begin{pmatrix} m & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} me' & 0 \\ 0 & se' \end{pmatrix}$.

Thus $A = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$, where $B$ is a simple submodule of $S_S$.\]
Now, we assume that \( A \) is a semisimple submodule of \( K_K \) and \( A \cong B \leq_d K_K \). It follows, from the above reasoning, that there are submodules \( C, C' \) of \( R_R \) and \( D, D' \) of \( S_S \), such that 
\[
A = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} C' & 0 \\ 0 & D' \end{pmatrix}. 
\]
Since \( A \cong B \), it is easy to verify that \( C_R \cong C'_R \) and \( D_R \cong D'_R \). We have that \( B \leq_d K_K \) and obtain that \( C' \leq_d R_R \) and \( D' \leq_d S_S \). Then, it follows, from the conditions of (2), that there are submodules \( E \leq R_R, F \leq S_S \) such that we have a decomposition \( C \oplus E = R_R, D \oplus F = S_S \). Thus, we have a decomposition \( K_K = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \oplus \begin{pmatrix} E & M \\ 0 & F \end{pmatrix} = A \oplus \begin{pmatrix} E & M \\ 0 & F \end{pmatrix} \).

\( \square \)

Example 2.22. Let \( Q := \prod_i F_i \) with \( F_i := \mathbb{Z}_2 \) and \( R \) be the subring of \( Q \) generated by \( \bigoplus_{i=1}^{\infty} F_i \) and \( 1_Q \). Consider the right action \( R \) on \( T_2(\mathbb{Z}_2) = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} \) which are defined by the relations
\[
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} (\alpha 1_Q + \beta) = \begin{pmatrix} a\alpha & b \\ 0 & c\alpha \end{pmatrix},
\]
where \( \alpha \in \mathbb{Z}_2, \beta \in \bigoplus_{i=1}^{\infty} F_i \). Then \( T_2(\mathbb{Z}_2) \) is \( T_2(\mathbb{Z}_2) \)-R-bimodule. Consider the formal triangular matrices ring \( K = \begin{pmatrix} T_2(\mathbb{Z}_2) & T_2(\mathbb{Z}_2)T_2(\mathbb{Z}_2)R \\ 0 & R \end{pmatrix} \). Since the ring \( T_2(\mathbb{Z}_2) \) is not left (and right) semisimple-direct-injective, it follows, from the left-sided analogue of Theorem 2.21, that the ring \( K \) is not left semisimple-direct-injective. Since \( l(T_2(\mathbb{Z}_2)T_2(\mathbb{Z}_2)R) = 0 \) and \( r(T_2(\mathbb{Z}_2)T_2(\mathbb{Z}_2)R) = \text{Soc}(R) \), then conditions (2)(a) and (2)(b) of Theorem 2.21 hold. Thus, the ring \( K \) is right semisimple-direct-injective.

3. Semisimple-direct-injective envelopes and covers

An \( R \)-homomorphism \( g : E \rightarrow M \) is called a semisimple-direct-injective cover (a C3-cover [1], respectively) for short an sdi-cover, of a right \( R \)-module \( M \) if \( E \) is a semisimple-direct-injective module (a C3 module, respectively) such that:

(i) Any diagram
\[
\begin{array}{ccc}
E & \xrightarrow{g} & M \\
\downarrow{\exists \alpha} & & \downarrow{g'} \\
E' & & 
\end{array}
\]

with \( E \) a semisimple-direct-injective module (a C3 module, respectively), can be commutatively completed.

(ii) If any endomorphism \( \alpha : E \rightarrow E \) satisfies \( g\alpha = g \), then \( \alpha \) must be an automorphism of \( E \).

Dually, the notion of the semisimple-direct-injective envelope can be defined.

Lemma 3.1. Assume that \( N \) is a non-injective semisimple module. Then the module \( M = N \oplus E(N) \) does not have an sdi-envelope and an sdi-cover.

Proof. Consider the inclusion map (note that, it is the semisimple-direct-injective envelope monomorphism)
\[
\iota : N \oplus E(N) \rightarrow E,
\]
where \( E \) is a semisimple-direct-injective module. Since the modules \( N \) and \( E(N) \) are semisimple-direct-injective, there exist \( f_1 : E \rightarrow N \) and \( f_2 : E \rightarrow E(N) \) such that \( f_1 \iota = \pi_1 \), where \( \pi_1 : M \rightarrow N \) and \( \pi_2 : M \rightarrow E(N) \) are the projections. Now there exists \( f : E \rightarrow \)
$N \oplus E(N)$ such that $\pi_i f = f_i$, which implies that $(\iota f)i = \iota$. Since $E$ is semisimple-direct-injective envelope of $M$, we have $\iota f$ is an isomorphism. It follows that $E \cong N \oplus E(N)$ is a semisimple-direct-injective module. Thus $N = E(N)$ is injective, a contradiction.

The rest is similar. \qedhere

**Lemma 3.2.** If $A$ is a C3-module and $A \oplus E(A)$ has a C3-cover, then $A$ is injective.

**Proof.** This is similar to Lemma 3.1. \qedhere

**Theorem 3.3.** The followings are equivalent for a ring $R$:

1. $R$ is an Artinian serial ring with $J(R)^2 = 0$.
2. Every simple-direct-injective right $R$-module has a C3-cover.
3. (a) Every semisimple-direct-injective right $R$-module has a C3-cover.
   (b) The module $\oplus S_i$ is injective for any family of simple injective modules $\{S_i\}_I$.

**Proof.** (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Consider the family $\{E_i\}_{i \in I}$ of injective right $R$-modules $E_i$, $i \in I$. By the assumption, $M = E \oplus (\oplus_{i \in I} E_i)$ with $E = E(\oplus_{i \in I} E_i)$ has a C3-cover, say $\alpha: C \to M$. Let $E_i := E$ and $\iota: E_i \to M$ be the inclusion maps for all $i \in I \cup \{i_0\}$. Since $E_i$ is injective (hence simple-direct-injective), there exists a linear map $\beta_i: E_i \to C$ such that $\alpha\beta_i = \iota_i$. Hence $id = \oplus \iota_i = \alpha(\oplus \beta_i)$ which implies that $M$ is a direct summand of $C$. So $M$ is a C3-module. By [5, Lemma 3.2], $\oplus_{i \in I} E_i$ is injective. Thus $R$ is right Noetherian.

We next prove that $R$ is right semi-Artinian. Without loss of generality, we can assume that $M$ is a non-zero indecomposable right $R$-module with $\text{Soc}(M) = 0$ (since $R$ is right Noetherian). Then $M$ is a C3-module. Since $\text{Soc}(M \oplus E(M)) = 0$, we get $M \oplus E(M)$ is a simple-direct-injective module. By the assumption, $M \oplus E(M)$ has a C3-cover. By Lemma 3.2, $M$ is injective. Hence $M$ is uniform and every submodule of $M$ is C3. Let $N$ be a non-zero arbitrary submodule of $M$. By the same argument, we have $N$ is injective. So, $N$ is a direct summand of $M$. This shows that $M$ is a semisimple module, a contradiction. Thus, every non-zero indecomposable right $R$-module has non-zero socle. It follows that $R$ is right semi-Artinian and hence $R$ is right Artinian.

By the same technique of [5, Theorem 3.4 (1) $\Rightarrow$ (3)], we can obtain that every right $R$-module is a direct sum of a semisimple module and a family of injective uniserial modules of length 2. Thus $R$ is an Artinian serial ring with $J(R)^2 = 0$.

(1) $\iff$ (3) This is similar to (1) $\iff$ (2). \qedhere

Now, we can prove that the classes of semisimple-direct-injective modules over a ring $R$ provide for sdi-envelopes and sdi-covers only if $R$ is a right Noetherian right V-ring:

**Theorem 3.4.** The following conditions are equivalent:

1. $R$ is a right Noetherian right V-ring.
2. Every right $R$-module has an sdi-cover.
3. Direct sums of semisimple-direct-injective modules have sdi-covers.
4. Every right $R$-module has an sdi-envelope.
5. Direct sums of semisimple-direct-injective modules has an sdi-envelope.

**Proof.** (1) $\Rightarrow$ (2), (3) are obvious.

(2) $\Rightarrow$ (1) For any semisimple direct right $R$-module $S$, then by the assumption, $M = S \oplus E(S)$ has an sdi-cover, say $\alpha: C \to M$. Let $\iota_1: S \to M$ and $\iota_2: E(S) \to M$ be the inclusion maps for all $i = 1, 2$. Note that $S$ and $E(S)$ are semisimple-direct-injective modules, and there are homomorphisms $\beta_1: S \to C, \beta_2: E(S) \to C$ such that $\alpha\beta_1 = \iota_1$. Clearly, $\text{id}_M = \iota_1 \oplus \iota_2 = \alpha(\iota_1 \oplus \iota_2)$. This shows that $M$ is isomorphic to a direct summand of $C$, which implies that $M$ is a semisimple-direct-injective module. Hence $S$ is injective.

(3) $\Rightarrow$ (1) is similar to (2) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1) Let $N$ be an arbitrary semisimple module. Assume that $\iota: M = N \oplus E(N) \to E$ is the sdi-envelope, where $E$ is a simple-direct-injective module. Since $N$ and $E(N)$...
are semisimple-direct-injective modules, there exist $f_1 : E \to N$, $f_2 : E \to E(N)$ such that $f_i \pi = \pi_i$, where $\pi_1 : M \to N_i$ and $\pi_2 : M \to E(N)$ are the projections. There exists $\phi : E \to M$ such that $\pi_i \phi = f_i$ for all $i = 1, 2$. It follows that $\phi \iota = id_M$, and so the monomorphism $\iota$ splits. Thus $N \oplus E(N)$ is isomorphic to a direct summand of $E$. It follows that $N \oplus E(N)$ is also a semisimple-direct-injective module. Hence $N$ is injective.

$(5) \Rightarrow (1)$ is similar to $(4) \Rightarrow (1)$. □

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