Asynchronous Schemes for Stochastic and Misspecified Potential Games and Nonconvex Optimization

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Abstract

The distributed computation of equilibria and optima has seen growing interest in a broad collection of networked problems. We consider the computation of equilibria of convex stochastic Nash games characterized by a possibly nonconvex potential function. In fact, any stationary point of the potential function is a Nash equilibrium of the associated game. Consequently, there is an equivalence between asynchronous best-response (BR) schemes applied on Nash game and player-specific block-coordinate descent (BCD) schemes implemented on the potential function which is block-wise convex. Our focus is on two classes of stochastic Nash games: (P1): A potential stochastic Nash game, in which each player solves a parameterized stochastic convex program; and (P2): A misspecified generalization, where the player-specific stochastic program is complicated by a parametric misspecification with the unknown parameter being the solution to a stochastic convex optimization problem. In both settings, exact proximal BR solutions are generally unavailable in finite time since they necessitate solving parameterized stochastic programs. Consequently, we design two asynchronous inexact proximal BR schemes to solve problems (P1) and (P2), where in each iteration a single player is randomly chosen to compute an inexact proximal BR solution (via stochastic approximation) with rivals’ possibly outdated information while the other players’ equilibrium strategies are kept invariant. Yet, in the misspecified regime (P2), each player possesses an extra estimate of the misspecified parameter and updates its estimate by a projected stochastic gradient (SG) algorithm with an increasing batch of sampled gradients. By imposing suitable conditions on the inexactness sequences, we prove that the iterates produced by both schemes converge almost surely to a connected subset of the set of Nash equilibria. In effect, we provide what we believe is amongst the first randomized block-coordinate descent schemes for stochastic nonconvex (but block convex) optimization problems equipped with almost-sure convergence guarantees. We further show that the associated gap function converges to zero in mean. These statements can be extended to allow for accommodating weighted potential games and generalized potential games (characterized by coupled strategy sets). Finally, we present preliminary numerics based on applying the proposed schemes to congestion control and Nash-Cournot games.

1 Introduction

Nash games, rooted in the seminal work by [2], have seen wide applicability in a broad range of engineered systems, such as power grids, communication networks, transportation networks and sensor networks. In the N-player Nash game, each player maximizes a prescribed payoff over a player-specific strategy set, given the rivals’ strategies. Nash’s eponymous solution concept, Nash equilibrium (NE), requires that at an equilibrium, no

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player can improve its payoff by unilaterally deviating from its equilibrium strategy. Potential games represent an important subclass of Nash games, formally introduced in [3], that arise naturally in the modeling of many applications, ranging from congestion control [4], routing problem [5] in communication networks, network Cournot competition [6, 7, 8, 9], and a host of other control-theoretic problems [10, 11, 12]. We refer the interested reader to the survey [13] on potential game for additional references. An interesting extension is the class of near potential games that are close to a potential game, for which some learning dynamics (for instance, best-response, fictitious play, logit response) are discussed in [14]. Additional decomposition algorithms are proposed in [15] to solve generalized potential games in which the player-specific strategy set depends on the strategies selected by the other agents.

**Motivation.** While prior algorithmic efforts have considered deterministic regimes (cf. [5, 6, 7]), there have been attempts to contend with stochastic generalizations via stochastic gradient schemes (cf. [16, 17, 9]). Yet, gradient-based schemes require communication after every update, an often undesirable in certain applications in cellular networks. Further, in many regimes, the payoff functions are defined by parameters that are unavailable (such as parameters of inverse-demand functions, congestion metrics, etc.). To this end, we consider inexact best-response schemes applied to the stochastic Nash game (or equivalently inexact block coordinate-descent schemes applied to the stochastic potential function) that are characterized by communication after computing a BR step, generally leading to lower overall communication. In addition, we allow for resolving parametric misspecification in players’ payoffs by equipping each player with a simultaneous learning step.

**Problems of interest.** We consider two classes of $N$-player potential stochastic Nash games with players indexed by $i$ where $i \in \mathcal{N} \triangleq \{1, 2, \ldots, N\}$.

**P1:** Potential Stochastic Nash Games. Suppose the $i$th player’s strategy is denoted by $x_i$ with a strategy set $X_i \subseteq \mathbb{R}^{n_i}$, implying that a feasible strategy $x_i$ satisfies $x_i \in X_i$, and let $n \triangleq \sum_{i \in \mathcal{N}} n_i$. Additionally, suppose player $i$’s objective (or negative of the utility function) is denoted by $f_i(x_i, x_{-i})$, which depends on its own strategy $x_i$ and on the vector of rival strategies $x_{-i} \triangleq \{x_j\}_{j \neq i}$. Suppose $X$ and $X_{-i}$ are defined as $X \triangleq \prod_{i=1}^{N} X_i$ and $X_{-i} \triangleq \prod_{j \neq i=1}^{N} X_j$, respectively. Given rival strategies $x_{-i}$, the $i$th player is faced by the following parameterized stochastic optimization problem:

$$
\min_{x_i \in X_i} f_i(x_i, x_{-i}) \triangleq \mathbb{E}[\psi_i(x_i, x_{-i}; \xi(\omega))],
$$

where $\psi_i : X \times \mathbb{R}^d \to \mathbb{R}$ is a scalar-valued function and the expectation is taken with respect to the random vector $\xi : \Omega \to \mathbb{R}^d$ defined on the probability space $(\Omega, \mathcal{F}_x, \mathbb{P}_x)$. Our interest lies in a subclass of Nash games, qualified as potential, of which each member is associated with a potential function $P : X \to \mathbb{R}$ such that for any $i \in \mathcal{N}$ and for any $x_{-i} \in X_{-i}$ and for all $x_i, x'_i \in X_i$:

$$
P(x_i, x_{-i}) - P(x'_i, x_{-i}) = f_i(x_i, x_{-i}) - f_i(x'_i, x_{-i}).
$$

Then the Nash game, in which the $i$th player solves (1) given $x_{-i}$, is called a potential stochastic Nash game. We aim to compute an NE $x^* = \{x^*_i\}_{i=1}^{N}$ such that for any $i \in \mathcal{N}$, the following holds:

$$
f_i(x^*_i, x_{-i}^*) \leq f_i(x_i, x_{-i}^*), \quad \forall x_i \in X_i.
$$

In other words, a feasible strategy tuple $x^* \in X$ is an NE if no player can improve its payoff by unilaterally deviating from the chosen strategy $x^*_i$. 


(P2): Misspecified Potential Stochastic Nash Games. In many practical settings, players' payoffs may be parameterized by a vector unknown to the player but may be learnt by solving a learning problem constructed through the a priori aggregation of data. This notion of resolving parametric misspecification has been studied extensively in the field of dynamical systems by [18], [19], [20], amongst others, where firms compete in a Cournot duopoly without observing rival actions directly. Recent work has examined the development of coupled SA schemes for resolving misspecified stochastic optimization [21], and stochastic Nash games [9]. In the second part of the paper, we consider a generalization of the potential game (1) in which the $i$th player’s problem is represented as follows:

$$\min_{x_i \in X_i} f_i(x_i, x_{-i}, \theta^*) \triangleq E \left[ \psi_i(x_i, x_{-i}, \theta^*; \xi(\omega)) \right],$$

(4)

where $\theta^* \in \mathbb{R}^m$ is unknown to the players, $\xi : \Omega \rightarrow \mathbb{R}^d$ is defined on the probability space $(\Omega, F_x, P_x)$, and $\psi_i : X \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a scalar-valued function. For instance, in the context of Nash-Cournot games, $\theta^*$ may represent the slope and intercept of a linear inverse demand function [18] while in the context of routing games, it may capture the unknown parameters of the demand function. We consider the case where $\theta^*$ can be estimated through solving a stochastic programming:

$$\min_{\theta \in \Theta} g(\theta) := E \left[ g(\theta, \eta) \right],$$

(5)

where $\Theta \in \mathbb{R}^m$ is a closed and convex set, $\eta : \Lambda \rightarrow \mathbb{R}^p$ is defined on the probability space $(\Lambda, F_\theta, P_\theta)$, and $g : \Theta \times \mathbb{R}^p \rightarrow \mathbb{R}$ is a scalar-valued function. We assume that the players have access to a common set of data that allow for learning $\theta^*$, while updating their beliefs regarding the equilibrium strategies requires solving a subsequent stochastic program. In such an instance, the problem of interest is to compute the correctly specified Nash equilibrium, defined as follows, holds for $i \in N$:

$$f_i(x_i^*, x_{-i}^*, \theta^*) \leq f_i(x_i, x_{-i}^*, \theta^*), \quad \forall x_i \in X_i.$$

(6)

Stochastic Nash games. Both (P1) and (P2) belong to the class of static stochastic Nash games, as opposed to their dynamic variants, represent a class of Nash games in which the payoff functions are expectation-valued. Tractable sufficiency conditions for existence of equilibria to such games were first provided in [22] in regimes where player objectives were convex and could be nonsmooth. SA schemes for a subclass of convex stochastic Nash games were presented under Lipschitzian assumptions in [16] via an iterative regularization technique while in [17], Lipschitzian requirements were relaxed by utilizing a randomized smoothing approach. In both instances, a.s. convergence to an NE is guaranteed under suitable monotonicity assumptions on the variational map and the steplength sequences. More recently, [9] presented a set of distributed coupled SA schemes for resolving misspecified monotone stochastic Nash games. Though gradient-based schemes exist for solving stochastic Nash games are characterized by ease of implementation and lower complexity in terms of each player step, such schemes are characterized by four shortcomings: (i) These schemes are not necessarily rational in that they require taking gradient steps rather than BR steps; (ii) Players communicate after every gradient step, necessitating a high level of communication; (iii) Convergence theory is reliant on a relatively strong monotonicity assumption on the gradient map; and (iv) the schemes are synchronous. To address these shortcomings, we develop implementable asynchronous inexact proximal best-response schemes, a class of techniques that can cope with delays and have far lower communication overhead.
Best-response and coordinate-descent schemes. In BR schemes, each player selects a BR strategy, given current rival strategies (cf. [23, 24]). There have been efforts to extend BR schemes to engineered settings (cf. [25]), where a BR can be expressed in a closed form. Recently, [26] have proposed several variants of the BR schemes to solve the two-stage noncooperative games with risk-averse players. Proximal BR schemes appear to have first discussed in [27], where convergence guarantees were provided under suitable contractive properties on the proximal BR map. However, it is unnecessary to impose such a condition in potential games since the decrease of a single player’s objective function leads to a corresponding decrease in the value of the potential function. In fact, [15] propose several (regularized) Gauss-Seidel BR schemes for generalized potential games and show that a limit point of the generated sequence is an NE when every player’s problem is convex. Recall that stationary points of the potential function are NE of the original game when the player-specific problems are convex. Thus, block coordinate descent (BCD) methods, where the coordinates are partitioned into several blocks (each corresponding to a player in the associated Nash game) and at each iteration, a single block is chosen to update while the other blocks remain unchanged. This avenue may be employed for obtaining either stationary points of the potential function (if nonconvex) or global minimizers (if convex). Its original format dates back to [28], where blocks were updated cyclically. Convergence has been extensively studied for both convex and nonconvex regimes with either differentiable or nondifferentiable objectives (cf. [29, 30, 31]). Notice that the asynchronous BCD schemes, where at each epoch a single block is chosen, are, in essence, identical to asynchronous BR schemes. [32] considers a randomized BCD method that performs a gradient update on a randomly selected block and proves the global rate of convergence is $O(\frac{1}{k})$ for merely convex functions and linear for strongly convex (S.C.) function, respectively. Extensions to convex nonsmooth regimes have been studied extensively (cf. [33, 34]) while in nonconvex regimes, [34, 35, 36] examined convergence theory. [37] proposed an accelerated gradient method to solve nonconvex and possibly stochastic optimization problems, and analyzed the convergence properties. [38] designed a randomized stochastic projected gradient algorithm to solve a class of constrained stochastic composite optimization problems, and established the iteration complexity of the proposed algorithm. We summarize much of the prior work in Table 1, where we observe that there is no available a.s. convergence theory for potential stochastic Nash games (or nonconvex stochastic programs) via BR (or CD) schemes.

Contributions: In [40, 39], while rate statements and iteration complexity bounds are provided for inexact proximal BR schemes for stochastic Nash games under a contractive requirement on the proximal BR map, even asymptotic guarantees are unavailable without such an assumption. Motivated by this gap, we aim to design convergent implementable asynchronous BR schemes such that at each epoch, a single player updates its strategy while the other players keep their strategies invariant. In our settings, each player-specific subproblem involves solving a stochastic program whose exact solution is generally unavailable in finite time, necessitating inexact solutions. Accordingly, we propose two classes of asynchronous inexact proximal BR schemes to compute NE of problems (P1) and (P2), and make the following contributions: (i). In Section II, we propose an asynchronous inexact proximal BR scheme to solve (P1). In each iteration, a single agent is randomly chosen to inexactly solve a stochastic optimization problem, given (possibly outdated) rival strategies via an SA scheme. By imposing suitable conditions on (P1) and on the inexactness sequences, in a regime that allows for communication delays, we prove that the iterates converge a.s. to a connected subset of the set of Nash equilibria and that the gap function converges in mean to zero. Extensions are provided to generalized stochastic potential games (with coupled strategy sets) and weighted potential games. (ii). In Section III, we extend the regime to contend with the misspecified stochastic Nash game (P2) where every player updates its strategy and its belief regarding the misspecified param-
eter (via variable sample-size SA schemes), given rival strategies. Asymptotic guarantees analogous to Section II are provided and we additionally show that the belief regarding the misspecified parameter converges a.s. to its true counterpart. (iii) We provide some preliminary numerics on congestion control and Nash-Cournot games in Section IV, and conclude the paper in Section V.

Notations: When referring to a vector $x$, it is assumed to be a column vector while $x^T$ denotes its transpose. Generally, $\|x\|$ denotes the Euclidean vector norm, i.e., $\|x\| = \sqrt{x^Tx}$. For a nonempty closed convex set $X \subset \mathbb{R}^m$, we use $\Pi_X[x]$ to denote the Euclidean projection of a vector $x \in \mathbb{R}^m$ on $X$, i.e., $\Pi_X[x] = \min_{y \in X} \|x - y\|$. We write a.s. as the abbreviation for “almost surely”. We use $E[z]$ to denote the expectation of a random variable $z$. For a real number $x$, the floor function $\lfloor x \rfloor$ denotes the largest integer smaller than $x$. Denote by $I_N \in \mathbb{R}^{N \times N}$ the identity matrix and by $J_N \in \mathbb{R}^{N \times N}$ the matrix with each entry being 1. We use $\otimes$ to denote the Kronecker product and $[A]_{i,j}$ to denote the $(i, j)$-th entry of the matrix $A$.

2 Asynchronous Inexact Best Response Schemes

In Section 2.1, we propose an asynchronous inexact proximal BR scheme to compute an equilibrium of the stochastic potential game (P1). Then in Section 2.2 we introduce some basic assumptions, based on which, we proceed to prove the almost sure convergence and convergence in mean of the generated sequence to a Nash equilibrium in Section 2.3. Finally, in Section 2.4, we discuss some possible extensions including the generalized potential games allowing for coupled strategy sets, and weighted potential games.

2.1 Algorithm Design

In standard potential games, a natural approach is an asynchronous BR method where in each iteration, one player updates its strategy by solving a problem (I), given its rivals’ strategies, referred to as the best-response problem. However, as shown in [15], the convergence of such schemes might not hold even if each player’s BR problem is

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**Table 1:** A list of some recent research papers on Nash games and (non)convex optimization.
convex. Accordingly, in \cite{15}, the authors propose and prove convergence of the iterates produced by a regularized scheme in which each player’s objective is modified by adding a quadratic proximal term. Motivated by this scheme, we define the proximal stochastic BR problem as follows for $\mu > 0$:

$$
T_i(x) \triangleq \arg\min_{y_i \in X_i} \left[ \mathbb{E} \left[ \psi_i(y_i, x_{-i}; \xi(\omega)) \right] + \frac{\mu}{2} \| y_i - x_i \|^2 \right].
$$

(7)

Since $T_i(x)$, the minimizer of a stochastic optimization problem (7) defined on a general probability space, is generally unavailable in finite time, we utilize Monte-Carlo sampling schemes in obtaining inexact solutions \cite{41}. We now propose an asynchronous inexact proximal BR scheme (Alg. 1) to compute a Nash equilibrium of this game. The scheme is defined as follows: At major iteration $k \geq 0$, randomly pick a single $i_k \in \mathcal{N}$ with $\mathbb{P}(i_k = i) = p_i > 0$. If $i_k = i$, then player $i$ is chosen to initiate an update at major iteration $k$ and computes an inexact proximal BR solution to problem (7) with possibly outdated data $y^i_k \triangleq (x_{1,k-d_i(k)}, \ldots, x_{N,k-d_i(k)})$, where $d_i(k)$ are random nonnegative integers capturing communication delays from player $j$ to player $i$ at time $k$. Set $d_i(k) = 0$ without loss of generality.

**Algorithm 1** Asynchronous inexact proximal BR scheme

Let $k := 0$, $x_{i,0} \in X_i$ for $i \in \mathcal{N}$. Suppose $\{d_1(k), \ldots, d_N(k)\}_{k \geq 0}$ and $\{\varepsilon_{i,k}\}_{k \geq 1}$ denote nonnegative communication delay sequences and inexactness sequences for $i \in \mathcal{V}$. Additionally $0 < p_i < 1$ for $i \in \mathcal{N}$ such that $\sum_{i=1}^{N} p_i = 1$.

(S.1) Pick $i_k = i \in \mathcal{N}$ with probability $p_i$ and set $y^i_k = (x_{1,k-d_i(k)}, \ldots, x_{N,k-d_i(k)})$.

(S.2) If $i_k = i$, then player $i$ updates $x_{i,k+1} \in X_i$ as follows:

$$
x_{i,k+1} := T_i(y^i_k) + \varepsilon_{i,k+1}.
$$

(8)

Otherwise, $x_{j,k+1} := x_{j,k}$ if $j \notin i_k$.

(S.3) If $k > K$, stop; Else, $k := k + 1$ and return to (S.1).

\section{2.2 Assumptions and Preliminary Results}

For notational simplicity, let $\xi$ denote $\xi(\omega)$ throughout the paper. We begin by imposing assumptions on $X_i$, $f_i$, $\psi_i$, and on the second moments of $\psi_i$.

**Assumption 1** Let the following hold.

(a) The feasible set $X_i$ is closed, compact, and convex;
(b) $f_i(x_i, x_{-i})$ is convex and continuously differentiable in $x_i$ over an open set containing $X_i$ for every $x_{-i} \in X_{-i}$.

Further, there exists a Lipschitz constant $L$ such that for every $i \in \mathcal{N}$ the following holds:

$$
\| \nabla_{x_i} f_i(x) - \nabla_{x_i} f_i(x') \| \leq L \| x - x' \| \quad \forall x, x' \in X;
$$

(c) For all $x_{-i} \in X_{-i}$ and any $\omega \in \Omega$, $\psi_i(x_i, x_{-i}; \xi(\omega))$ is differentiable in $x_i$ over an open set containing $X_i$ such that $\nabla_{x_i} f_i(x_i, x_{-i}) = \mathbb{E}[\nabla_{x_i} \psi_i(x_i, x_{-i}; \xi)]$;

(d) For any $i \in \mathcal{N}$ and all $x \in X$, there exists a constant $M > 0$ such that $\mathbb{E}[\| \nabla_{x_i} \psi_i(x_i, x_{-i}; \xi) \|^2] \leq M^2$. 


Remark 1

(i) Let $f_i$ be the standard SA algorithm defined as follows:

\[ f_i(x) = \left\{ \begin{array}{ll}
0 & \text{if } i = i_k, \\
\infty & \text{otherwise},
\end{array} \right. \]

where $i_k$ denotes the index of the worker who received the update $y_k$. The following hold:

\[ \sum_{k=1}^{\infty} \mathbb{E} [ ||x_{i,k+1}^t||^2 | F_k] < \infty \text{ a.s., and } \sum_{k=1}^{\infty} \mathbb{E} [ ||x_{i,k+1}^t|| | F_k] < \infty \text{ a.s.,} \]

(c) and there exists a positive integer $\tau$ such that for any $i, j \in \mathcal{N}$ and any $k \geq 0$, $d_{ij}(k) \in \{0, \ldots, \tau\}$.

By Assumption 1, it is clear that $T_i(x)$ defined by (7) requires solving a strongly convex stochastic program. Thus, an approximation of the solution to the problem (7) with $x = y_k^i$, characterized by (8), can be computed via the standard SA algorithm defined as follows:

\[ z_{i,k}^{t+1} := \Pi_{X_i}\left[ z_{i,k}^t - \gamma_{i,t} \left[ \nabla x_i \psi_i(z_{i,k}^t, x_{i,k}^t, x_{i,k}^t) + \mu(z_{i,k}^t - x_{i,k}^t) \right] \right], \tag{9} \]

where $\gamma_{i,t} = \frac{1}{\mu(t+1)}$ and $z_{i,k}^t = x_{i,k}$. Let (9) be applied from $t = 1, \ldots, j_{i,k}$ for obtaining $x_{i,k+1}$ and set $x_{i,k+1} = z_{i,k}^{j_{i,k}}$. Define $\xi_{i,k}^{l,i} \triangleq (\xi_{i,k}^{l,1}, \ldots, \xi_{i,k}^{l,i_N})$, $\mathcal{F}_k \triangleq \sigma\{x_0, \ldots, x_k, \{d_{ij}(k)\}_{i \in \mathcal{N}}\}$, and $\mathcal{F}_k \triangleq \sigma\{\mathcal{F}_k, d_{ij}(k), i \in \mathcal{N}\}$. Then by Algorithm 1, it follows that $x_k$ is adapted to $\mathcal{F}_k$ and $y_k^i$ is adapted to $\mathcal{F}_k$. Hence $T_i(y_k^i)$ is adapted to $\mathcal{F}_k$ by Definition 7. Analogous to the proof of [39, Lemma 3], we obtain the following result for the SA scheme (9).

**Lemma 1**  Let Assumption 1 hold. Consider the asynchronous inexact proximal BR scheme given by Algorithm 1. Assume that the random variables $\{\xi_{i,k}^{l,i}\}_{1 \leq t \leq j_{i,k}}$ are iid, and that for any $i \in \mathcal{N}$ the random vector $\xi_{i,k}$ is independent of $\mathcal{F}_k$. Then we have the following for any $t \geq 1$:

\[ \mathbb{E} [ ||z_{i,k}^t - T_i(y_k^i)||^2 | F_k] \leq Q_i/(t + 1) \text{ a.s.,} \]

where $Q_i \triangleq \frac{2M^2}{\mu^2} + 2D_{X_i}^2$ with $D_{X_i} \triangleq \sup\{d(x_i, x_i') : x_i, x_i' \in X_i\}$.

**Remark 1** (i) Let $1_{[i_k = i]}$ denote the indicator function of the event $i_k = i$, defined as follows:

\[ 1_{[i_k = i]} \triangleq \begin{cases} 
1 & \text{if } i_k = i, \\
0 & \text{if } i_k \neq i.
\end{cases} \]
Define $\Gamma_{i,k}$ as follows: $\Gamma_{i,0} \triangleq 1$ and $\Gamma_{i,k} \triangleq 1 + \sum_{t=0}^{k-1} 1\{t=i\}$ for all $k \geq 1$. Then for every $\omega \in \Omega$, there exists a sufficiently large $\hat{k}(\omega)$ that is possibly contingent on the sample path $\omega$ such that for any $i \in \mathcal{N}$:

$$\Gamma_{i,k} \geq \frac{k p_i}{2} + 1 \quad \forall k \geq \hat{k}(\omega). \quad (10)$$

The proof can be found in Lemma 7 of [42].

(ii) Set $j_{i,k} \triangleq \left\lceil \Gamma_{i,k}^{2(1+\delta)} \right\rceil$ and $x_{i,k+1} = z_{j_{i,k}}$. Then by Lemma 1, we have that

$$\mathbb{E}\|x_{i,k+1} - T_i(x_k)\|^2 |\mathcal{F}_k| \leq \frac{Q_i}{j_{i,k} + 1} \leq \frac{Q_i}{\Gamma_{i,k}^{2(1+\delta)}} \triangleq \alpha_{i,k}^2.$$ 

Thus, $\left\lfloor \Gamma_{i,k}^{2(1+\delta)} \right\rfloor$ steps of (9) suffice for obtaining a solution to (8) with $\varepsilon_{i,k}$ satisfying $\mathbb{E}\|\varepsilon_{i,k+1}\|^2 |\mathcal{F}_k| \leq \alpha_{i,k}^2$ a.s. Then by the conditional Jensen’s inequality, $\mathbb{E}\|\varepsilon_{i,k+1}\| |\mathcal{F}_k| \leq \alpha_{i,k} a.s.$ By invoking (10), we obtain that for all $i \in \mathcal{N}$, $\sum_{k=1}^{\infty} \alpha_{i,k}^2 < \infty$ a.s., and $\sum_{k=1}^{\infty} \alpha_{i,k} < \infty$ a.s. Then Assumption 3(b) holds.

The following result following from equation (18) in [43] establishes an equivalence between Nash equilibria of the stochastic Nash game (1) and solutions to the corresponding variational inequality problem (11).

Lemma 2 Let Assumptions 1(a), 1(b), and 2 hold. Then $x^*$ is an NE of the potential game (1) if and only if $x^*$ is a solution to the following problem:

$$\nabla_x P(x^*)^T(y - x^*) \geq 0 \quad \forall y \in X. \quad (11)$$

Further, the set of Nash equilibria is nonempty and compact.

2.3 Convergence Analysis

We now establish the a.s. convergence and convergence in mean of the iterates produced by Algorithm 1. Parts of the proof are inspired by Theorem 4.1 in [36] and Theorem 4.3 in [15].

Theorem 1 (a.s. convergence to Nash equilibrium) Let $\{x_k\}$ be generated by Algorithm 1. Suppose Assumptions 1, 2 and 3 hold. We further assume that the parameter $\mu$ utilized in (7) satisfies $\mu > \frac{L}{2} + \sqrt{2L \tau}$. Then the following hold:

(a) (square summability): For any $i \in \mathcal{N}$, $\sum_{k=0}^{\infty} \|T_i(y_k) - x_k\|^2 < \infty$ a.s.

(b) (cluster point is an NE): For almost all $\omega \in \Omega$, every limit point of $x_k(\omega)$ is a Nash equilibrium.

(c) (a.s. convergence to a connected subset of the set $X^*$ of Nash equilibria): There exists a connected subset $X^*_c \subset X^*$ such that $d(x_k, X^*_c) \longrightarrow 0$ a.s.

Proof. By Assumption 1(b), we have the following bound:

$$f_i(T_i(y_k), x_{-i,k}) \leq f_i(x_k) + \nabla_x f_i(x_k)^T (T_i(y_k) - x_{i,k}) + \frac{L}{2} \|T_i(y_k) - x_{i,k}\|^2. \quad (12)$$
Since $T_i(y_k^i)$ is a global minimum of (7) and $x_{i,k} \in X_i$, by the optimality condition we have that

$$0 \leq (\nabla_{x_i} f_i(T_i(y_k^i), y_{i,k}^i) + \mu(T_i(y_k^i) - x_{i,k}))^T(x_{i,k} - T_i(y_k^i))$$

$$= -(T_i(y_k^i) - x_{i,k})^T \nabla_{x_i} f_i(T_i(y_k^i), y_{i,k}^i) - \mu\|T_i(y_k^i) - x_{i,k}\|^2$$

$$= -(T_i(y_k^i) - x_{i,k})^T \nabla_{x_i} f_i(x_{i,k}, y_{i,k}^i) - \mu\|T_i(y_k^i) - x_{i,k}\|^2$$

$$- (\nabla_{x_i} f_i(T_i(y_k^i), y_{i,k}^i) - \nabla_{x_i} f_i(x_{i,k}, y_{i,k}^i))^T(T_i(y_k^i) - x_{i,k})$$

$$\leq -\nabla_{x_i} f_i(y_k^i) (T_i(y_k^i) - x_{i,k}) - \mu\|T_i(y_k^i) - x_{i,k}\|^2,$$

where the last inequality follows by $(\nabla_{x_i} f_i(x_i, x_{-i}) - \nabla_{x_i} f_i(x_i', x_{-i}))^T(x_i - x_i') \geq 0 \forall x_i, x_i' \in X_i, \forall x_{-i} \in X_{-i}$ from Assumption [5]b. Adding terms (12) and (13), we have the following inequality:

$$f_i(T_i(y_k^i), x_{-i,k}) \leq f_i(x_k) + (\nabla_{x_i} f_i(x_k) - \nabla_{x_i} f_i(y_k^i))^T(T_i(y_k^i) - x_{i,k}) - \left(\mu - \frac{L}{2}\right)\|T_i(y_k^i) - x_{i,k}\|^2.$$

By Assumption [5]b, we obtain the following sequence of inequalities for any $C > 0$:

$$(\nabla_{x_i} f_i(x_k) - \nabla_{x_i} f_i(y_k^i))^T(T_i(y_k^i) - x_{i,k}) \leq \|\nabla_{x_i} f_i(x_k) - \nabla_{x_i} f_i(y_k^i)\|\|T_i(y_k^i) - x_{i,k}\|$$

$$\leq L\|x_k - y_k^i\|\|T_i(y_k^i) - x_{i,k}\| \leq L^2 \frac{C}{2}\|x_k - y_k^i\|^2 + C\|T_i(y_k^i) - x_{i,k}\|^2 \quad \text{(by ab} \leq a^2 + b^2)$$

$$= \frac{L^2}{2C} \sum_{j=1}^{N} \|x_{j,k} - x_{j,k-d_{ij}(k)}\|^2 + \frac{C}{2}\|T_i(y_k^i) - x_{i,k}\|^2$$

$$= \frac{L^2}{2C} \sum_{j=1}^{N} \sum_{h=k-d_{ij}(k)+1}^{k} (x_{j,h} - x_{j,h-1})^2 + \frac{C}{2}\|T_i(y_k^i) - x_{i,k}\|^2. \quad \text{(Term 1)}$$

Term 1 on the right of the expression may be further bounded as follows:

$$\text{Term 1} \leq \frac{L^2}{2C} \sum_{j=1}^{N} \sum_{h=k-d_{ij}(k)+1}^{k} \|x_{j,h} - x_{j,h-1}\|^2 \quad \text{(by Jensen’s inequality)}$$

$$\leq \frac{L^2\tau}{2C} \sum_{j=1}^{N} \sum_{h=k-\tau+1}^{h=k-\tau+1} \|x_{j,h} - x_{j,h-1}\|^2 \quad \text{(by } d_{ij}(k) \in \{0, 1, \ldots, \tau\})$$

$$= \frac{L^2\tau}{2C} \sum_{h=k-\tau+1}^{k} \|x_h - x_{h-1}\|^2$$

$$\leq \frac{L^2\tau}{2C} \sum_{h=k-\tau+1}^{k} (h-k+\tau)\|x_h - x_{h-1}\|^2 + \frac{L^2\tau^2}{2C}\|x_{k+1} - x_k\|^2$$

$$\leq \frac{L^2\tau}{2C} \sum_{h=k-\tau+1}^{k+1} (h-(k+1)+\tau)\|x_h - x_{h-1}\|^2 + \frac{L^2\tau^2}{2C}\|x_{k+1} - x_k\|^2 \quad \text{by } V_k$$

By substituting (16) into (15), and by invoking (14), we obtain the following:

$$f_i(T_i(y_k^i), x_{-i,k}) \leq f_i(x_k) + V_k - V_{k+1} - \left(\mu - \frac{L + C}{2}\right)\|T_i(y_k^i) - x_{i,k}\|^2 + \frac{L^2\tau^2}{2C}\|x_{k+1} - x_k\|^2. \quad \text{(17)}$$
By Assumptions 1(c), 1(d) and the Jensen’s inequality, the following holds for any $x \in X$:
\[
\|\nabla x_i f_i(x_i, x_{-i})\| = \|E[\nabla x_i \psi_i(x_i, x_{-i}; \xi)]\| \leq \|E[\|\nabla x_i \psi_i(x_i, x_{-i}; \xi)\|^2]\| \leq M.
\]
(18)

Then from Algorithm 1 by invoking Assumption 2 we may obtain the following bound:
\[
P(x_{k+1}) - P(x_k) = P(x_{i_k,k+1}, x_{-i,k}) - P(x_{i_k,k}, x_{-i,k}) = f_{i_k}(x_{i_k,k+1}, x_{-i,k}) - f_{i_k}(x_{i_k,k}, x_{-i,k})
\]
\[
= f_{i_k} \left( T_{i_k} (y_{i_k}^k), x_{-i,k} \right) - f_{i_k}(x_k) + f_{i_k}(x_{i_k,k+1}, x_{-i,k}) - f_{i_k} \left( T_{i_k} (y_{i_k}^k), x_{-i,k} \right)
\]
\[
\leq f_{i_k} \left( T_{i_k} (y_{i_k}^k), x_{-i,k} \right) - f_{i_k}(x_k) + M\|\varepsilon_{i_k,k+1}\| \quad \text{(by Cauchy–Schwarz inequality and (18)),}
\]
where $z_{i_k,k+1} = \beta_{i_k,k} x_{i_k,k+1} + (1 - \beta_{i,k,k}) T_{i_k} (y_{i_k}^k)$ for some $\beta_{i,k,k} \in (0, 1)$. By Algorithm 1, we have that
\[
\|x_{k+1} - x_k\|^2 = \|x_{i_k,k+1} - x_{i_k,k}\|^2 \leq 2\|T_{i_k} (y_{i_k}^k) - x_{i_k,k}\|^2 + 2\|\varepsilon_{i_k,k+1}\|^2.
\]
(20)

Therefore, by combining (17) and (19) we have the following inequality:
\[
P(x_{k+1}) + V_{k+1} \leq P(x_k) + V_k - \left( \mu - \frac{L + C}{2} - \frac{L^2 \tau^2}{C} \right) \|T_{i_k} (y_{i_k}^k) - x_{i_k,k}\|^2
\]
\[
+ M\|\varepsilon_{i_k,k+1}\| + \frac{L^2 \tau^2}{C} \|\varepsilon_{i_k,k+1}\|^2
\]
\[
\leq P(x_k) + V_k - C\|T_{i_k} (y_{i_k}^k) - x_{i_k,k}\|^2 + M \sum_{i=1}^{N} \|\varepsilon_{i,k+1}\| + \frac{L^2 \tau^2}{C} \sum_{i=1}^{N} \|\varepsilon_{i,k+1}\|^2.
\]
(21)

Therefore, by taking expectations conditioned on $\mathcal{F}_k$, we obtain the following:
\[
E \left[ P(x_{k+1}) + V_{k+1} | \mathcal{F}_k \right] \leq E \left[ P(x_k) + V_k | \mathcal{F}_k \right] - C E \left[ \|T_{i_k} (y_{i_k}^k) - x_{i_k,k}\|^2 | \mathcal{F}_k \right]
\]
\[
+ M \sum_{i=1}^{N} E \left[ \|\varepsilon_{i,k+1}\| | \mathcal{F}_k \right] + \frac{L^2 \tau^2}{C} \sum_{i=1}^{N} E \left[ \|\varepsilon_{i,k+1}\|^2 | \mathcal{F}_k \right].
\]
(22)

Since $T_i (y_i^k)$ $\forall i \in \mathcal{N}$ is adapted to $\mathcal{F}_k$ and $i_k$ is independent of $\mathcal{F}_k$, by [44] Corollary 7.1.2 and $P(i_k = i) = p_i$ we have the following equation:
\[
E \left[ \|T_{i_k} (y_{i_k}^k) - x_{i_k,k}\|^2 | \mathcal{F}_k \right] = E_{i_k} \left[ \|T_{i_k} (y_{i_k}^k) - x_{i_k,k}\|^2 \right] = \sum_{i=1}^{N} p_i \|T_i (y_i^k) - x_{i,k}\|^2.
\]
(23)

Since $x_k$ and $V_k$ are adapted to $\mathcal{F}_k$, by (22) and (23) we have the following:
\[
E \left[ P(x_{k+1}) + V_{k+1} | \mathcal{F}_k \right] \leq P(x_k) + V_k - C \sum_{i=1}^{N} p_i \|T_i (y_i^k) - x_{i,k}\|^2
\]
\[
+ M \sum_{i=1}^{N} E \left[ \|\varepsilon_{i,k+1}\| | \mathcal{F}_k \right] + \frac{L^2 \tau^2}{C} \sum_{i=1}^{N} E \left[ \|\varepsilon_{i,k+1}\|^2 | \mathcal{F}_k \right].
\]
(24)

\footnote{Let the random vectors $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ on $(\Omega, \mathcal{F}, P)$ be independent of one another and let $f$ be a Borel function on $\mathbb{R}^{m \times n}$ with $|E[f(X,Y)]| \leq \infty$. If for any $x \in \mathbb{R}^m$, $g(x) = \begin{cases} E[f(x,Y)] & \text{if } |E[f(x,Y)]| \leq \infty, \\ 0 & \text{otherwise} \end{cases}$, then $g$ is a Borel function with $g(X) = E[f(X,Y)|\sigma(X)]$.}
By setting $C = \sqrt{2L\tau}$ we derive $\frac{C^2}{2} + \frac{L^2\tau^2}{C^2} = \sqrt{2L\tau}$. Thus, by taking $\mu > \frac{C}{2} + \sqrt{2L\tau}$ we have that $\bar{C} > 0$.

(a) By Assumption 3(b), the terms in $\|\epsilon_{i,k+1}\|$ are summable. We may then invoke Robbins-Siegmund theorem [45 Theorem 1], allowing us to conclude that

$$\sum_{k=0}^{\infty} \sum_{i=1}^{N} p_i \|T_i(y^k_i) - x_{i,k}\|^2 < \infty \text{ a.s.}$$

(25)

Since $p_i \in (0, 1)$, we obtain (a).

(b) By result (a) we have the following for any $i \in \mathcal{N}$:

$$\lim_{k \to \infty} \|T_i(y^k_i) - x_{i,k}\| = 0, \quad \text{a.s.}$$

(26)

Let $\bar{x}(\omega)$ be a cluster point of sequence $\{x_k(\omega)\}$. Then there exists a subsequence $\mathcal{K}(\omega)$ such that

$$\lim_{k \to \infty, k \in \mathcal{K}(\omega)} x_k(\omega) = \bar{x}(\omega).$$

(27)

Then by (26) and (27), we have that

$$\lim_{k \to \infty, k \in \mathcal{K}(\omega)} T_i(y^k_i(\omega)) = \bar{x}_i(\omega) \quad \forall i \in \mathcal{N}.$$  

(28)

We intend to show that $\bar{x}(\omega)$ is a Nash equilibrium. Assume the converse holds. Then there exists an $i \in \mathcal{N}$ and a vector $\bar{y}_i \in X_i$ such that

$$f_i(\bar{y}_i, \bar{x}_{-i}(\omega)) < f_i(\bar{x}_i(\omega), \bar{x}_{-i}(\omega)).$$

By Assumption 1(b), the directional derivative of $f_i$ at point $(\bar{x}_i(\omega), \bar{x}_{-i}(\omega))$ along vector $q_i = \bar{y}_i - \bar{x}_i(\omega)$ exists, and we have

$$f'_i(\bar{x}_i(\omega), \bar{x}_{-i}(\omega); q_i) = (\bar{y}_i - \bar{x}_i(\omega))^T \nabla_{x_i} f_i(\bar{x}_i(\omega), \bar{x}_{-i}(\omega)) \quad \text{(by convexity)}$$

$$= \inf_{\lambda > 0} \frac{f_i(\bar{x}_i(\omega) + \lambda q_i, \bar{x}_{-i}(\omega)) - f_i(\bar{x}_i(\omega), \bar{x}_{-i}(\omega))}{\lambda} \quad \text{(by definition)}$$

$$\leq f_i(\bar{y}_i, \bar{x}_{-i}(\omega)) - f_i(\bar{x}_i(\omega), \bar{x}_{-i}(\omega)) \quad \text{(by setting } \lambda = 1)$$

$$= f_i(\bar{y}_i, \bar{x}_{-i}(\omega)) - f_i(\bar{x}_i(\omega), \bar{x}_{-i}(\omega)) < 0.$$

(29)

Recall that $T_i(y^k_i)$ is defined as a global minimum of a convex optimization problem (7). Since $\bar{y}_i \in X_i$, by the optimality condition for a constrained convex programming, we obtain that

$$\mu (\bar{y}_i - T_i(y^k_i))^T (T_i(y^k_i) - x_{i,k}) + (\bar{y}_i - T_i(y^k_i))^T \nabla_{x_i} f_i(T_i(y^k_i), x_{-i,k}) \geq 0.$$  

(30)

Since $\nabla_{x_i} f_i(\cdot)$ is continuous by Assumption 1 by taking limits to $k \to \infty, k \in \mathcal{K}(\omega)$, we obtain the following inequality from (26), (27) and (28):

$$(\bar{y}_i - \bar{x}_i(\omega))^T \nabla_{x_i} f_i(\bar{x}_i(\omega), \bar{x}_{-i}(\omega)) \geq 0,$$

which contradicts (29). Thus, the assumption that $\bar{x}(\omega)$ is not an NE does not hold, proving (b).

(c) By result (b) it is clear that $d(x_k, X^*) \to 0 \text{ a.s.}$ Assume the converse holds and that $X^*_c$ is disconnected with some positive probability. Then there exist at least two closed connected sets $X^*_{c_1}$ and $X^*_{c_2}$ such that $X^*_c = X^*_{c_1} \cup X^*_{c_2}$ with $d(X^*_{c_1}, X^*_{c_2}) > 0$. By hypothesis, the sequence $\{x_k\}$ cannot converge to either $X^*_{c_1}$ or $X^*_{c_2}$ a.s.,
and $x_k$ visits $X_1^*, X_2^*$ infinitely often. Define $\rho = \frac{1}{3}d(X_1^*, X_2^*)$. By $d(x_k, X_1^*) \xrightarrow{k \to \infty} 0$ we know there exists $k_0$ such that

$$x_k \in B(X_1^*, \rho) \cup B(X_2^*, \rho) \quad \forall k \geq k_0,$$

where $B(A, \rho)$ denotes the $\rho$-neighborhood of $A$. Define

$$n_0 \triangleq \inf\{k > k_0, d(x_k, X_1^*) < \rho\},$$

$$m_p \triangleq \inf\{k > n_p-1, d(x_k, X_2^*) < \rho\},$$

and $n_p \triangleq \inf\{k > m_p, d(x_k, X_2^*) < \rho\}$, $p \geq 1$.

Then $\{n_p\}$ and $\{m_p\}$ are infinite sequences by the converse of result (c). By (31) we have $x_{n_p} \in B(X_1^*, \rho)$ and $x_{n_p-1} \in B(X_2^*, \rho)$ for any $p \geq 1$. Then by $d(X_1^*, X_2^*) = 3\rho$, it follows that $\|x_{n_p} - x_{n_p-1}\| > \rho \forall p \geq 1$ with some positive probability. Then by taking expectations, we have the following:

$$\mathbb{E}[\|x_{n_p} - x_{n_p-1}\|] > 0. \quad (32)$$

By taking expectations on both sides of (24), we have the following:

$$\mathbb{E}[P(x_{k+1}) + V_{k+1}] \leq \mathbb{E}[P(x_k) + V_k] - \overline{C} \sum_{i=1}^{N} p_i \mathbb{E}[\|T_i(y_k^i) - x_{i,k}\|^2]$$

$$+ M \sum_{i=1}^{N} \mathbb{E}[\|\epsilon_{i,k+1}\|] + \frac{L^2\tau^2}{C} \sum_{i=1}^{N} \mathbb{E}[\|\epsilon_{i,k+1}\|^2]. \quad (33)$$

Since $\mathbb{E}[\|\epsilon_{i,k+1}\| | \mathcal{F}_k]$ and $\mathbb{E}[\|\epsilon_{i,k+1}\|^2 | \mathcal{F}_k]$ are nonnegative random variables for $k \geq 1$, by Assumption 3(b) we have the following:

$$\sum_{k=1}^{\infty} \mathbb{E}[\|\epsilon_{i,k+1}\|] = \mathbb{E}\left[\sum_{k=1}^{\infty} \mathbb{E}[\|\epsilon_{i,k+1}\| | \mathcal{F}_k]\right] < \infty, \quad \text{and}$$

$$\sum_{k=1}^{\infty} \mathbb{E}[\|\epsilon_{i,k+1}\|^2] = \mathbb{E}\left[\sum_{k=1}^{\infty} \mathbb{E}[\|\epsilon_{i,k+1}\|^2 | \mathcal{F}_k]\right] < \infty. \quad (34)$$

Then by (33), we have that

$$\overline{C} \sum_{k=0}^{\infty} \sum_{i=1}^{N} p_i \mathbb{E}[\|T_i(y_k^i) - x_{i,k}\|^2] \leq \mathbb{E}[P(x_0) + V_0] - \liminf_{k \to \infty} \mathbb{E}[P(x_k) + V_k]$$

$$+ M \sum_{i=1}^{N} \sum_{k=0}^{\infty} \mathbb{E}[\|\epsilon_{i,k+1}\|] + \frac{L^2\tau^2}{C} \sum_{k=0}^{\infty} \sum_{i=1}^{N} \mathbb{E}[\|\epsilon_{i,k+1}\|^2] < \infty,$$

where the second inequality holds by the boundedness of $P(x_k) + V_k$ since $P(\cdot)$ is continuous and $X$ is compact. Hence by Jensen’s inequality we have that

$$\lim_{k \to \infty} \mathbb{E}[\|T_i(y_k^i) - x_{i,k}\|] = 0 \quad \forall i \in \mathcal{N}. \quad (35)$$

Note that $\|x_{k+1} - x_k\| \leq \sum_{i=1}^{N} (\|\epsilon_{i,k+1}\| + \|T_i(y_k^i) - x_{i,k}\|).$ Then by (34) we obtain the following:

$$\lim_{k \to \infty} \mathbb{E}[\|x_{k+1} - x_k\|] = 0. \quad (36)$$
This contradicts \([32]\), and hence the converse does not hold. Consequently, result (c) follows by.

Theorem 1 shows that the estimates generated by Algorithm 1 converge a.s. to the set of Nash equilibria. If the set of Nash equilibria contains isolated points, then for almost all \(ω \in Ω\), \(x_k(ω)\) converges to an NE. Further, if the potential game \([1]\) admits a unique NE, then the iterates converge a.s. to the NE. In what follows, we discuss the convergence in mean of the iterates. Since the potential function is employed as a vehicle to analyze convergence of the iterates, a natural approach would have to been to employ the value of the potential function. However, the iterates may converge to stationary points which are not necessarily global minimizers and as a consequence, we need an appropriate metric. We note from Lemma 2 that a stationary point of \(\min_{x \in X} P(x)\) is given by a solution to the variational inequality problem \(VI(X, \nabla_x P)\) that requires an \(x \in X\) such that

\[
(y - x)^T \nabla_x P(x) \geq 0 \quad \forall y \in X.
\]

Suppose \(X^*\) denotes the set of solutions to \(VI(X, F)\). A merit function for ascertaining the departure from solvability of the VI is a gap function. It may be recalled from \([46]\) that a function \(G(x)\) is called a gap function if it satisfies two properties:

(i) \(G(\cdot)\) is sign restricted over the set \(X\);

(ii) \(G(x) = 0\) if and only if \(x\) solves \(VI(X, F)\).

We now consider a primal gap function \([46]\) Theorem 3.1] that has found a fair amount of applicability in the context of variational inequality problems.

**Definition 1** Let \(X \subseteq \mathbb{R}^n\) be a nonempty, closed, and convex set. Let \(F : X \rightarrow \mathbb{R}^n\) and let \(G : X \rightarrow \mathbb{R}^+\) be defined as follows:

\[
G(x) = \sup_{y \in X} F(x)^T (x - y) \quad \forall x \in X.
\]

The following result shows the mean convergence of the iterates in the sense that the limit of \(\mathbb{E}[G(x_k)]\) is zero. This is analogous to showing that expected sub-optimality of iterates tends to zero in the context of stochastic optimization problems.

**Theorem 2 (Convergence in mean)** Let \(\{x_k\}\) be generated by Algorithm 1 Suppose that Assumptions 2 and 3 hold, and, in addition, that the parameter \(μ\) utilized in 7 satisfies \(μ > \frac{L}{2} + √2Lτ\). Then we have that

\[
\lim_{k \rightarrow ∞} \mathbb{E}[G(x_k)] = 0.
\]

**Proof.** By Assumption 2 we have the following for any \(i \in N\) and any \(\bar{y}_i \in X_i:\)

\[
(x_{i,k} - \bar{y}_i)^T \nabla_{x_i} P(x_k) = (x_{i,k} - \bar{y}_i)^T \nabla_{x_i} f_i(x_k)
\]

\[
= (x_{i,k} - T_i(y_k^i))^T \nabla_{x_i} f_i(T_i(y_k^i), x_{-i,k}) + (T_i(y_k^i) - \bar{y}_i)^T \nabla_{x_i} f_i(T_i(y_k^i), x_{-i,k})
\]

\[
- (x_{i,k} - \bar{y}_i)^T (\nabla_{x_i} f_i(T_i(y_k^i), x_{-i,k}) - \nabla_{x_i} f_i(x_k)).
\]

Then we have the following sequence of inequalities for any \(i \in N\) and any \(\bar{y}_i \in X_i:\)

\[
(x_{i,k} - \bar{y}_i)^T \nabla_{x_i} P(x_k) \leq μ \left(\bar{y}_i - T_i(y_k^i)\right)^T \left(T_i(y_k^i) - x_{i,k}\right) + (x_{i,k} - T_i(y_k^i))^T \nabla_{x_i} f_i(T_i(y_k^i), x_{-i,k})
\]

\[
+ (\bar{y}_i - x_{i,k})^T (\nabla_{x_i} f_i(T_i(y_k^i), x_{-i,k}) - \nabla_{x_i} f_i(x_k)) \quad \text{(by (30))}
\]

\[
\leq μ \|\bar{y}_i - T_i(y_k^i)\| \|T_i(y_k^i) - x_{i,k}\| + \|x_{i,k} - T_i(y_k^i)\| \|\nabla_{x_i} f_i(T_i(y_k^i), x_{-i,k})\|
\]

\[
+ \|\bar{y}_i - x_{i,k}\| \|\nabla_{x_i} f_i(T_i(y_k^i), x_{-i,k}) - \nabla_{x_i} f_i(x_k)\| \quad \text{(by Cauchy–Schwarz inequality)}
\]

\[
\leq (μD_{X_i} + M + LD_{X_i}) \|T_i(y_k^i) - x_{i,k}\| \quad \text{(by Assumptions (a), (b) and (13)).}
\]
By summing these inequalities over $i$, we have that

$$
G(x_k) = \sup_{\bar{y} \in X} (x_k - \bar{y})^T \nabla P(x_k) = \sum_{i=1}^N \sup_{\bar{y}_i \in X_i} (x_{i,k} - \bar{y}_i)^T \nabla_{x_i} P(x_k)
$$

\begin{equation}
\leq \sum_{i=1}^N (\mu D_{X_i} + M + LD_{X_i}) \| T_i(y^i_k) - x_{i,k} \|.
\end{equation}

Then, by taking expectations on both sides of (37), we obtain that

$$
\mathbb{E}[G(x_k)] \leq \sum_{i=1}^N (\mu D_{X_i} + M + LD_{X_i}) \mathbb{E} \left[ \| T_i(y^i_k) - x_{i,k} \| \right] \implies \lim_{k \to \infty} \mathbb{E}[G(x_k)] \leq 0 \quad \text{(by 35)}.
$$

However, $G(x_k) \geq 0$ since $x_k \in X$, implying that $\lim_{k \to \infty} \mathbb{E}[G(x_k)] = 0$, giving us the required result.

We now define an alternative proximal-linear map as follows:

$$
T^L_i(x) = \arg\min_{y_i \in X_i} \left[ (y_i - x_i)^T \nabla_{x_i} f_i(x) + \frac{\mu}{2} \| y_i - x_i \|^2 \right].
$$

Corollary 1 (a.s., mean convergence under proximal-linear map) Let $\{x_k\}$ be generated by Algorithm 1 in which $T_i(y_i^k)$ is replaced by $T^L_i(y_i^k)$. Suppose Assumptions 1-2 and 3 hold, and $\mu > \frac{L}{2} + \sqrt{2}L\tau$. Then the results of Theorem 1 and Theorem 2 hold.

**Proof.** Since $T^L_i(y_i^k)$ is a global minimum of (38) and $x_{i,k} \in X_i$, by the optimality condition we have that

$$
0 \leq -\nabla^T_{x_i} f_i(y_i^k) (T_i(y_i^k) - x_{i,k}) - \mu \| T_i(y_i^k) - x_{i,k} \|^2
$$

which is indeed the last inequality in Equation (13). Then by inequality (39), similar to the proof of Theorem 1 and Theorem 2, we conclude the corollary.

### 2.4 Generalized Potential Nash games and Weighted Potential Games

We now consider the generalized Nash setting and the weighted potential game.

#### 2.4.1 Generalized potential Nash games

We now extend the separable constraint to the coupled constraint. Suppose there exists a nonempty closed set $C \subseteq \mathbb{R}^n$ such that each player $i$’s feasible set $X_i(x_{-i}) = \{ x_i \in X_i : (x_i, x_{-i}) \in C \}$ depends on the rivals’ strategies $x_{-i}$, where $X_i \subseteq \mathbb{R}^{n_i}$ are nonempty closed sets such that $\prod_{i=1}^N X_i \cap C$ is nonempty. We say that a point $x \in \mathbb{R}^n$ is feasible if $x_i \in X_i(x_{-i})$ for any $i \in \mathcal{N}$. The aim of player $i$ is to choose a strategy $x_i$ that solves the following stochastic program:

$$
\min_{x_i \in X_i(x_{-i})} f_i(x_i, x_{-i}) \triangleq \mathbb{E}[\psi_i(x_i, x_{-i}, \xi(\omega))].
$$

Assume that there exists a continuous potential function $P : C \cap \prod_{i=1}^N X_i \to \mathbb{R}$ such that for any $i \in \mathcal{N}$ and any $x_{-i}$ with $X_i(x_{-i})$ being nonempty we have the following equality:

$$
P(x_i, x_{-i}) - P(x'_i, x_{-i}) = f_i(x_i, x_{-i}) - f_i(x'_i, x_{-i}) \quad \forall x_i, x'_i \in X_i(x_{-i}).
$$

14
Proof. By multiplying both sides of (14) with Theorem 1 and Theorem 2 hold. used in (7) (1) game Corollary 3 (Weighted potential stochastic Nash games) Let Algorithm 1 be applied to the stochastic Nash game (40). Then computing the proximal BR map $T(x)$ of the problem (40) also involves solving a strongly convex stochastic program. Let Algorithm 1 without communication delays be applied to the generalized stochastic potential game with the starting point $x_0$ being feasible, where it is required that $x_{i,k+1} \in X_i(x_{-i,k})$. This requirement can be guaranteed by using the projected SG scheme to approximate the proximal BR solutions. Similar to Lemma 4.1 in [15], we can show that $x_{k+1}$ generated by Algorithm 1 is feasible. We then conclude the following result.

**Corollary 2 (Generalized potential stochastic Nash games)** Let Algorithm 1 be applied to the stochastic generalized Nash game (40) that satisfies (41), where $x_0$ is feasible, $y_k^i = x_k$ in (S.1), and in (S.2) it is required that $x_{i,k+1} \in X_i(x_{-i,k})$. By imposing suitable conditions, the results of Theorem 1 and Theorem 2 hold as well. The proof is similar to that of Theorem 1 and Theorem 2. Nevertheless, the case with communication delays might not carry over to generalized potential game since $y_k^i$ might be infeasible.

### 2.4.2 Weighted potential games

We now extend the exact potential game to a weighted potential game, in which there exist positive numbers $w_1, \ldots, w_N$ such that, for any $i \in N$ and any $x_{-i}$ the following equality holds:

$$P(x_i, x_{-i}) = P(x_i', x_{-i}) = w_i \left( f_i(x_i, x_{-i}) - f_i(x_i', x_{-i}) \right) \quad \forall x_i, x_i' \in X_i(x_{-i}).$$

**Proof.** By multiplying both sides of (14) with $w_i$ and rearranging the terms we obtain that

$$w_i \left( f_i \left( T_i(y_k^i), x_{-i,k} \right) - f_i(x_k) \right) \leq w_i \left( \nabla_x f_i(x_k) - \nabla_x f_i(y_k^i) \right)^T \left( T_i(y_k^i) - x_{i,k} \right) - w_i \left( \mu - \frac{L}{2} \right) \| T_i(y_k^i) - x_{i,k} \|^2$$

(43)

$$\leq w_{\max} \left\| \nabla_x f_i(x_k) - \nabla_x f_i(y_k^i) \right\| \| T_i(y_k^i) - x_{i,k} \| - w_{\min} \left( \mu - \frac{L}{2} \right) \| T_i(y_k^i) - x_{i,k} \|^2 \quad \text{(since } \mu - \frac{L}{2} > 0).$$

By substituting (15) and (16) into (43) we have the following bound for any $C > 0$:

$$w_i \left( f_i \left( T_i(y_k^i), x_{-i,k} \right) - f_i(x_k) \right) \leq w_{\max} (V_k - V_{k+1}) + \frac{L^2 \tau^2 w_{\max}}{2C} \| x_{k+1} - x_k \|^2$$

$$- \left( w_{\min} \left( \mu - \frac{L}{2} \right) - \frac{C w_{\max}}{2} \right) \| T_i(y_k^i) - x_{i,k} \|^2.$$
Then by (42), similar to (19) we have that
\[ P(x_{k+1}) - P(x_k) \leq w_{ik} \left( f_{ik} \left( T_{ik} (y^i_k), x_{-ik,k} \right) - f_{ik} (x_k) + M \| \varepsilon_{ik,k+1} \| \right), \]
which incorporating with (20) and (44) yields the following inequality:
\[ P(x_{k+1}) - P(x_k) \leq w_{\max}(V_k - V_{k+1}) + \frac{L^2 \tau^2 w_{\max}}{C} \| \varepsilon_{ik,k+1} \|^2 + M w_{ik} \| \varepsilon_{ik,k+1} \|
- \left( w_{\min} \left( \frac{\mu}{2} - \frac{C w_{\max}}{2} - \frac{L^2 \tau^2 w_{\max}}{C} \right) \right) \| T_{ik} (y^i_k) - x_{ik,k} \|^2. \quad (45) \]

By rearranging the terms, we obtain the following bound:
\[ P(x_{k+1}) + w_{\max} V_{k+1} \leq P(x_k) + w_{\max} V_k - C_w \| T_{ik} (y^i_k) - x_{ik,k} \|^2
+ \frac{L^2 \tau^2 w_{\max}}{C} \sum_{i=1}^N \| \varepsilon_{i,k+1} \|^2 + M \sum_{i=1}^N w_i \| \varepsilon_{i,k+1} \|. \]

Then by taking expectations conditioned on \( \mathcal{F}_k \), by (23) and by invoking that \( x_k \) and \( V_k \) are adapted to \( \mathcal{F}_k \), we obtain that
\[ \mathbb{E} \left[ P(x_{k+1}) + w_{\max} V_{k+1} \big| \mathcal{F}_k \right] \leq P(x_k) + w_{\max} V_k - \tilde{C}_w \sum_{i=1}^N \| T_{i}(y^i_k) - x_{i,k} \|^2
+ \frac{L^2 \tau^2 w_{\max}}{C} \sum_{i=1}^N \mathbb{E} \left[ \| \varepsilon_{i,k+1} \|^2 \mid \mathcal{F}_k \right] + M \sum_{i=1}^N w_i \mathbb{E} \left[ \| \varepsilon_{i,k+1} \| \mid \mathcal{F}_k \right]. \quad (46) \]

By setting \( C = \sqrt{2} L \tau \), we derive \( \frac{C w_{\max}}{2} + \frac{L^2 \tau^2 w_{\max}}{C} = \sqrt{2} L \tau w_{\max} \). Since \( \mu > \frac{L}{2} \sqrt{2} + \sqrt{2} L \tau w_{\max} \), by invoking \( \tilde{C}_w \) defined in (45), we have that \( \tilde{C}_w > 0 \). Then similar to the proof of Theorem 1 and Theorem 2 we obtain the results.

\[ \square \]

3 Misspecified Potential Stochastic Nash Games

In this section, we propose a framework that combines the asynchronous inexact proximal BR scheme with joint learning to resolve the misspecified stochastic Nash game (P2). Under suitable conditions, we prove the a.s. convergence and convergence in mean of the produced strategy vector to the set of Nash equilibria. In addition, we show that for every \( i \in \mathcal{N} \), player \( i \)'s belief regarding the misspecified parameter \( \theta_{i,k} \) converges to \( \theta^* \) in an a.s. sense as \( k \to \infty \).

3.1 Algorithm Design and Assumptions

We impose the following conditions on the misspecified problem.

**Assumption 4**  
(a) For every \( i \in \mathcal{N} \), \( X_i \) is a closed, compact, and convex set; \( f_i(x_i, x_{-i}; \theta) \) is convex and continuously differentiable in \( x_i \) over an open set containing \( X_i \) for every \( x_{-i} \in X_{-i} \) and every \( \theta \in \Theta \).
(b) For every \( i \in \mathcal{N} \), \( \nabla_x f_i(x; \theta^*) \) is Lipschitz continuous in \( x \) with Lipschitz constant \( L_x \), i.e.,
\[ \| \nabla_x f_i(x; \theta^*) - \nabla_x f_i(x'; \theta^*) \| \leq L_x \| x - x' \| \quad \forall x, x' \in X. \]
Further, there exists a constant $L_{\theta^*}$ such that for any $x \in X$ and every $i \in \mathcal{N}$:

$$\|\nabla_{x_i} f_i(x_i, x_{-i}; \theta) - \nabla_{x_i} f_i(x_i, x_{-i}; \theta^*)\| \leq L_{\theta^*}\|\theta - \theta^*\| \quad \forall \theta \in \Theta.$$ (46)

(c) The function $g(\theta)$ is strongly convex with convexity constant $\mu_g$ and is continuously differentiable in $\theta$ on an open set containing $\Theta$ with the gradient function being $L_{\theta^*}$-Lipschitz continuous.

(d) There exists a function $P(\cdot; \cdot) : X \times \Theta \to \mathbb{R}$ such that for any $i \in \mathcal{N}$ and every $x_{-i} \in X_{-i}$:

$$P(x_i, x_{-i}; \theta^*) - P(x'_i, x_{-i}; \theta^*) = f_i(x_i, x_{-i}; \theta^*) - f_i(x'_i, x_{-i}; \theta^*) \quad \forall x_i, x'_i \in X_i. \quad (47)$$

In the following, for notational simplicity, we define $P(x) \triangleq P(x; \theta^*)$ as the potential function of the problem (4).

**Assumption 5**  
(a) For any $i \in \mathcal{N}$, all $x_{-i} \in X_{-i}$, any $\theta \in \Theta$ and any $\xi \in \mathbb{R}^d$, $\psi_i(x_i, x_{-i}; \theta; \xi)$ is differentiable in $x_i$ over an open set containing $X_i$ such that $\nabla_{x_i} f_i(x_i, x_{-i}; \theta) = \mathbb{E}[\nabla_{x_i} \psi_i(x_i, x_{-i}; \theta; \xi)]$.  
(b) For any $i \in \mathcal{N}$ and any $x \in X$, there exists a constant $M_1 > 0$ such that $\mathbb{E}[\|\nabla_{x_i} \psi_i(x_i, x_{-i}; \theta; \xi)\|^2] \leq M_1^2$.  
(c) For any $\eta \in \mathbb{R}^p$, $g(\theta, \eta)$ is differentiable in $\theta$ over an open set containing $\Theta$ such that

$$\nabla g(\theta) = \mathbb{E}[\nabla g(\theta; \eta)].$$

If $T_i(x, \theta)$ is defined as follows:

$$T_i(x, \theta) \triangleq \arg\min_{y_i \in X_i} \left[ f_i(y_i, x_{-i}; \theta) + \frac{\mu}{2}\|y_i - x_i\|^2 \right], \quad \mu > 0, \quad (48)$$

then $T_i(x, \theta)$ is uniquely defined by invoking Assumption 4(a). Additionally, we may claim the Lipschitz continuity of $T_i(x, \cdot)$ based on the following Lemma.

**Lemma 3**  
There exists a constant $L_t$ such that for any $i \in \mathcal{N}$ and any $x \in X$:

$$\|T_i(x, \theta) - T_i(x, \theta^*)\| \leq L_t\|\theta - \theta^*\| \quad \forall \theta \in \Theta. \quad (49)$$

**Proof.** By the optimality condition, $T_i(x, \theta)$ is the solution to the following variational inequality problem:

Find $x_i^*$ such that $(y_i - x_i^*)^T \left( \nabla_{x_i} f_i(x_i^*, x_{-i}; \theta) + \mu(x_i^* - x_i) \right) \geq 0 \quad \forall y_i \in X_i.$

Note that by Assumption 4(a), $\nabla_{x_i} f_i(y_i, x_{-i}; \theta) + \mu(y_i - x_i)$ is strongly monotone in $y_i \in X_i$ for any $x \in X$ and any $\theta \in \Theta$. By triangle inequality, we have the following for any $y_i \in X_i$ and any $x, x' \in X$:

$$\|\nabla_{x_i} f_i(y_i, x_{-i}; \theta) + \mu(y_i - x_i) - \nabla_{x_i} f_i(y_i, x'_{-i}; \theta^*) - \mu(y_i - x'_i)\|$$

$$= \|\nabla_{x_i} f_i(y_i, x_{-i}; \theta) - \nabla_{x_i} f_i(y_i, x_{-i}; \theta^*) + \nabla_{x_i} f_i(y_i, x_{-i}; \theta^*) - \nabla_{x_i} f_i(y_i, x'_{-i}; \theta^*) + \mu(x'_i - x_i)\|$$

$$\leq \|\nabla_{x_i} f_i(y_i, x_{-i}; \theta) - \nabla_{x_i} f_i(y_i, x_{-i}; \theta^*)\| + \|\nabla_{x_i} f_i(y_i, x_{-i}; \theta^*) - \nabla_{x_i} f_i(y_i, x'_{-i}; \theta^*)\| + \mu\|x'_i - x_i\| \quad \text{(by Assumption 4(b)).}$$

Then for any $y_i \in X_i$ and any $i \in \mathcal{N}$, $\nabla_{x_i} f_i(y_i, x_{-i}; \theta) + \mu(y_i - x_i)$ is Lipschitz continuous in $x, \theta$ at $X \times \theta^*$. Therefore, by Lemma 2.4 in [47], we obtain the result. \qed

We propose an asynchronous inexact proximal BR scheme that is coupled with learning (Alg. 2) to compute a Nash equilibrium of the misspecified potential stochastic game. The scheme is defined as follows: At major
iteration \( k \geq 0 \), randomly pick \( i \in \mathcal{N} \) with \( \mathbb{P}(i_k = i) = p_i > 0 \). If \( i_k = i \), then player \( i \) is chosen to initiate an update at major iteration \( k \) and computes an inexact proximal BR solution to problem (48) with \( x = y_{i_k}^k, \theta = \theta_{i,k} \). Here \( y_{i_k}^k \triangleq (x_{1,k} - d_{i_1}(k), \ldots, x_{N,k} - d_{i_N}(k)) \) denotes the outdated data with \( d_{ij}(k) \) capturing communication delays from player \( j \) to player \( i \) at time \( k \), while \( \theta_{i,k} \), denoting the estimate of \( \theta^* \) given by player \( i \) at time \( k \), is learnt via the variable sample-size SA (51) with \( N_{i,k} \) sampled gradients.

**Algorithm 2** Asynchronous inexact proximal BR scheme with stochastic learning

Let \( k := 0, x_{i,0} \in X_i \) for \( i \in \mathcal{N} \). Suppose \( \{d_{i_1}(k), \ldots, d_{i_N}(k)\}_{k \geq 0} \) and \( \{\varepsilon_{i,k}\}_{k \geq 1} \) denote nonnegative communication delay sequences and inexactness sequences for \( i \in \mathcal{V} \). Additionally \( 0 < p_i < 1 \) for \( i \in \mathcal{N} \) such that \( \sum_{i=1}^N p_i = 1 \).

(1) Pick \( i_k = i \in \mathcal{N} \) with probability \( p_i \) and set \( y_{i_k}^k = (x_{1,k} - d_{i_1,k}, \ldots, x_{N,k} - d_{i_N,k}) \).

(2) If \( i_k = i \), then player \( i \) updates \( x_{i,k+1} \in X_i \) and \( \theta_{i,k+1} \in \Theta_i \) as follows:

\[
x_{i,k+1} := T_i(y_{i_k}^k, \theta_{i,k}) + \varepsilon_{i,k+1},
\]

\[
\theta_{i,k+1} := \Pi_{\Theta_i} \left[ \theta_{i,k} - \frac{\beta}{N_{i,k}} \sum_{p=1}^{N_{i,k}} \nabla g \left( \theta_{i,k}, \eta_{i,k}^p \right) \right],
\]

where \( \nabla g \left( \theta_{i,k}, \eta_{i,k}^p \right) \) denotes the sampled gradient, and \( \beta > 0 \) is the step size; Otherwise, \( x_{j,k+1} := x_{j,k}, \theta_{j,k+1} = \theta_{j,k} \) if \( j \neq i \).

(3) If \( k > K \), stop; Else, \( k := k + 1 \) and return to (1).

We then list the following conditions concerning the communication delays, observation noise of the gradient function \( \nabla g(\theta) \) as well as the inexactness sequence \( \{\varepsilon_{i,k}\} \) utilized in Algorithm 2.

**Assumption 6** Define \( d_i(k) \triangleq (d_{i_1}(k), \ldots, d_{i_N}(k)) \) and \( \mathcal{F}_k \triangleq \sigma \{x_0, \ldots, x_k, \theta_0, \ldots, \theta_k, \{d_i(k)\}_{i \in \mathcal{N}}\} \), where \( \theta_k = (\theta_{1,k}, \ldots, \theta_{N,k}) \). The following hold:

(a) \( \{i_k\} \) is an iid sequence, where \( i_k \) is independent of \( \mathcal{F}_k \) for all \( k \geq 1 \).

(b) For any \( i \in \mathcal{N} \), the noise term \( \{\varepsilon_{i,k}\} \) satisfies the following condition:

\[
\sum_{k=1}^{\infty} \mathbb{E} \left[ \|\varepsilon_{i,k+1}\|^2 | \mathcal{F}_k \right] < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \mathbb{E} \left[ \|\varepsilon_{i,k+1}\| | \mathcal{F}_k \right] < \infty \quad \text{a.s.}
\]

(c) There exists a positive integer \( \tau \) such that for any \( i, j \in \mathcal{N} \) and any \( k \geq 0 \), \( d_{ij}(k) \in \{0, \ldots, \tau\} \).

(d) Define \( e_{i,k}^p = \nabla g \left( \theta_{i,k}, \eta_{i,k}^p \right) - \nabla g \left( \theta_{i,k} \right) \). There exists a constant \( M_2 > 0 \) such that for any \( k \geq 0 \):

\[
\mathbb{E} \left[ \|e_{i,k}^p\|^2 | \mathcal{F}_k \right] \leq M_2^2 \quad \forall p = 1, \ldots, N_{i,k}.
\]

Analogous to the computation of the inexact best-response (8) in Algorithm 1, we still utilize SA to compute (50). By (48) it is seen that the computation of \( T_i(x, \theta) \) requires solving a strongly convex stochastic program. Thus, an inexact solution to the problem (48), characterized by (50), can also be computed via the SA algorithm defined as follows:

\[
z_{i,k+1}^t := \Pi_{X_i} \left[ z_{i,k}^t - \gamma_{i,t} \left[ \nabla x_i \psi_i(z_{i,k}^t, y_{i,k}^t; \theta_{i,k}; \xi_{i,k}^t) + \mu(z_{i,k}^t - x_{i,k}) \right] \right],
\]
where $\gamma_{i,t} = \frac{1}{\mu(t+1)}$ and $z^1_{i,k} = x_{i,k}$. Let algorithm (52) be employed from $t = 1, \ldots, j_{i,k}$ to obtain $x_{i,k+1}$ and set $x_{i,k+1} = z^j_{i,k}$. Define $\xi_{i,k} = (\xi^1_{i,k}, \ldots, \xi^j_{i,k})$, $\eta_{i,k} = (\eta^1_{i,k}, \ldots, \eta^j_{i,k})$, $F_i \triangleq \sigma\{x_0, \eta_{i,l}, \xi_{i,l}, d_{i,l}, 0 \leq l \leq k - 1\}$ and $F_k \triangleq \sigma\{F_i, d_i(k), i \in V\}$. Then by Algorithm 2, it follows that $x_k$ and $\theta_k$ are adapted to $F_k$, and hence $y^i_k$ is adapted to $F_k$. Thus, $T_1(y^i_k, \theta_{i,k})$ is adapted to $F_k$ by its definition (48). Then by Assumption 4 and 5 we obtain the same error bound as that of Lemma 1. Consequently, Assumption 6(b) is satisfied by setting $j_{i,k} = \left[\Gamma^2(1+\delta)\right]$, where $\delta > 0$ and $\Gamma_{i,k}$ is defined in Remark 1.

3.2 Convergence Analysis

Theorem 3 (a.s. convergence for inexact BR with learning) Let $\{x_k\}$ and $\{\theta_k\}$ be generated by Algorithm 2. Suppose Assumptions 2, 3, and 6 hold. We further assume that the parameter $\mu$ used in (48) satisfies $\mu > \frac{L_2}{2} + \sqrt{3L_2}\tau$, $\beta \in (0, 2\mu_g/L_2)$ and $N_{i,k} = \left[\Gamma^2(1+\delta)\right]$ for some $\delta > 0$. Then the following hold:

(a) For any $i \in N$, $\sum_{k=1}^{\infty} \|\theta_{i,k} - \theta^*\|^2 < \infty \ a.s.$, and $\sum_{k=1}^{\infty} \|\theta_{i,k} - \theta^*\| < \infty \ a.s.$

(b) For any $i \in N$, $\sum_{k=0}^{\infty} \|T_1(y^i_k; \theta^*) - x_{i,k}\|^2 < \infty \ a.s.$

(c) For almost all $\omega \in \Omega$, every limit point of $x_k(\omega)$ is a Nash equilibrium.

(d) There exists a connected subset $X^*_c \subset X^*$ such that $d(x_k, X^*_c) \xrightarrow{k \to \infty} 0 \ a.s.$

Proof. (a) For $i_k = i$, by $e^p_{i,k}$ defined in Assumption 6(d) we can rewrite (51) as follows:

$$\theta_{i,k+1} = \Pi_{\Theta} \left[\theta_{i,k} - \beta \nabla g(\theta_{i,k}) - \beta \bar{e}_{i,k}\right],$$

where $\bar{e}_{i,k} = \frac{1}{N_{i,k}} \sum_{p=1}^{N_{i,k}} e^p_{i,k}$. Note that $\theta^* = \Pi_{\Theta} [\theta^* - \beta \nabla g(\theta^*)]$ by the optimality condition for any $\beta > 0$. Then by the non-expansivity of the Euclidean projector, $\|\theta_{i,k+1} - \theta^*\|^2$ may be bounded as follows:

$$\|\theta_{i,k+1} - \theta^*\|^2 \leq \|\Pi_{\Theta} \left[\theta_{i,k} - \beta \nabla g(\theta_{i,k}) - \beta \bar{e}_{i,k}\right] - \Pi_{\Theta} \left[\theta^* - \beta \nabla g(\theta^*)\right]\|
\leq \|\theta_{i,k} - \theta^* - \beta (\nabla g(\theta_{i,k}) - \nabla g(\theta^*)) - \beta \bar{e}_{i,k}\|
\leq \|\theta_{i,k} - \theta^*\|^2 + \beta^2 \|\nabla g(\theta_{i,k}) - \nabla g(\theta^*)\|^2 + \beta^2 \|\bar{e}_{i,k}\|^2
2\beta (\theta_{i,k} - \theta^*)^T (\nabla g(\theta_{i,k}) - \nabla g(\theta^*)) - 2\beta \bar{e}_{i,k}^T (\theta_{i,k} - \theta^* - \beta (\nabla g(\theta_{i,k}) - \nabla g(\theta^*))).

Then by taking expectations conditioned on $F_k$, by Assumptions 4(c) and 5(c) we have the following:

$$\mathbb{E} \left[\|\theta_{i,k+1} - \theta^*\|^2 \|F_k\right] \leq \left(1 + \beta^2 \frac{L_2}{2} - 2\beta \mu_g\right) \|\theta_{i,k} - \theta^*\|^2 + \beta^2 \mathbb{E} \left[\|\bar{e}_{i,k}\|^2 \|F_k\right]
\leq q \|\theta_{i,k} - \theta^*\|^2 + \frac{\beta^2 M_2^2}{N_{i,k}} \ (\text{by Assumption 4(d)}).$$

If $\beta \in (0, 2\mu_g/L_2)$, it follows that $q \in (0, 1)$. While for $i_k \neq i$, we have $\mathbb{E} \left[\|\theta_{i,k+1} - \theta^*\| \|F_k\right] = \|\theta_{i,k} - \theta^*\|^2.$

Thus, by $\mathbb{P}(i_k = i) = p_i$ we obtain the following bound:

$$\mathbb{E} \left[\|\theta_{i,k+1} - \theta^*\|^2 \|F_k\right] \leq \left(1 - q p_i + q (1 - q)\right) \|\theta_{i,k} - \theta^*\|^2 + \frac{p_i \beta^2 M_2^2}{N_{i,k}}.$$
By invoking Remark 1, we have that \( \frac{1}{N_{i,k}} \leq \infty \) a.s. Then by Theorem 1 in [45], we get \( \sum_{k=1}^{\infty} \|\theta_{i,k} - \theta^*\|^2 \leq \infty \) a.s. By invoking the conditional Jensen’s inequality, \( \left( \mathbb{E} \left[ \|X\| F \right] \right)^2 \leq \mathbb{E} \left[ \|X\|^2 \right] F \) and by (54), we have that

\[
\mathbb{E} \left[ \|\theta_{i,k+1} - \theta^*\|^2 \right] \leq \sqrt{1 - q_i} \|\theta_{i,k} - \theta^*\| + \frac{\sqrt{p_i} \beta M_2}{\sqrt{N_{i,k}}} \leq \left( 1 - \sqrt{1 - q_i} \right) \|\theta_{i,k} - \theta^*\| + \frac{\sqrt{p_i} \beta M_2}{\gamma_{i,k}^1 + \delta_i}.
\]

Since \( q_i \in (0, 1) \), by Theorem 1 in [45] and by invoking Remark 1, we have that \( \sum_{k=1}^{\infty} \|\theta_{i,k} - \theta^*\| \leq \infty \) a.s.

(b) By Assumption 4(b), \( \nabla_{x_i} f_i(x_i, x_{-i}; \theta^*) \) is Lipschitz continuous in \( x \in X \) with Lipschitz constant \( L_x \). Then similar to [17] we obtain the following inequality for any \( C > 0 \):

\[
f_i \left( T_i(y_{k}^{i}, x_{-i}; \theta^*) \right) + V_{k+1} \leq f_i(x_k; \theta^*) + V_k + \frac{L_x^2 \tau^2}{2C} \|x_{k+1} - x_k\|^2
\]

(55)

where \( V_k \triangleq \frac{L_x^2 \tau^2}{2C} \sum_{h=k-\tau+1}^{k} (h-k+\tau) \|x_h - x_{h-1}\|^2 \). By Assumptions 5(a), 5(b), and Jensen’s inequality, the following holds for any \( x \in X \):

\[
\|\nabla_{x_i} f_i(x_i, x_{-i}; \theta)\| = \|\mathbb{E}[\nabla_{x_i} \psi_i(x_i, x_{-i}; \theta; \xi)\| \| \leq \sqrt{\mathbb{E}[\|\nabla_{x_i} \psi_i(x_i, x_{-i}; \theta; \xi)\|^2] \leq M_1.
\]

Then by the mean-value theorem and Cauchy-Schwarz inequality, we have that

\[
f_i \left( x_{i,k+1}, x_{-i,k}; \theta^* \right) - f_i \left( T_i(y_{k}^{i}, x_{-i,k}; \theta^*) \right) = \left( x_{i,k+1} - T_i(y_{k}^{i}, \theta^*) \right)^T \nabla_{x_i} f_i \left( x_{i,k+1}, x_{-i,k}; \theta^* \right) \leq M_1 \|x_{i,k+1} - T_i(y_{k}^{i}, \theta^*)\|.
\]

(56)

where \( z_{i,k+1} = \beta_{i,k} x_{i,k+1} + (1 - \beta_{i,k}) T_i(y_{k}^{i}, \theta^*) \) for some \( \beta_{i,k} \in (0, 1) \). By the triangle inequality, Lemma 3 and (50), we have the following:

\[
\|x_{i,k+1} - T_i(y_{k}^{i}, \theta^*)\| \leq \|T_i(y_{k}^{i}, \theta_{i,k}) - T_i(y_{k}^{i}, \theta^*)\| + \|\varepsilon_{i,k+1}\| \leq L_t \|\theta_{i,k} - \theta^*\| + \|\varepsilon_{i,k+1}\|.
\]

(57)

We then substitute (57) in (56) to obtain the following bound:

\[
f_i \left( x_{i,k+1}, x_{-i,k}; \theta^* \right) - f_i \left( T_i(y_{k}^{i}, \theta^*), x_{-i,k}; \theta^* \right) \leq M_1 L_t \|\theta_{i,k} - \theta^*\| + M_1 \|\varepsilon_{i,k+1}\|.
\]

(58)

Then by Algorithm 2 and Assumption 4(d), we may obtain the following bound:

\[
P(x_{k+1}) - P(x_k) = f_k \left( x_{i,k+1}, x_{-i,k}; \theta^* \right) - f_k \left( x_{i,k}, x_{-i,k}; \theta^* \right)
\]

\[
= f_k \left( x_{i,k+1}, x_{-i,k}; \theta^* \right) - f_k \left( T_i(y_{k}^{i}, \theta^*), x_{-i,k}; \theta^* \right) \leq \|T_i(y_{k}^{i}, \theta^*), x_{-i,k}; \theta^* \| - f_k \left( x_{i,k}; \theta^* \right)
\]

\[
\leq M_1 L_t \|\theta_{i,k} - \theta^*\| + M_1 \|\varepsilon_{i,k+1}\| + V_k - V_{k+1}
\]

(59)

\[
- \left( \mu - \frac{L_x + C}{2} \right) \|T_i(y_{k}^{i}, \theta^*), x_{i,k}); \theta^* \| - x_{i,k})^2 + \frac{L_x^2 \tau^2}{2C} \|x_{k+1} - x_k\|^2 \quad \text{by (55) and (58).}
\]

By the update rule in Algorithm 2 and (57), we have that

\[
\|x_{k+1} - x_k\| = \|x_{i,k+1} - x_{i,k}\| = \|x_{i,k+1} - T_i(y_{k}^{i}, \theta^*), x_{i,k}; \theta^*\| - x_{i,k},\theta^*\| \leq L_t \|\theta_{i,k} - \theta^*\| + \|\varepsilon_{i,k+1}\| + \|T_i(y_{k}^{i}, \theta^*), x_{i,k}; \theta^*\| - x_{i,k},\theta^*\|,
\]
which when combined with (59) and \( (a + b + c)^2 \leq 3(a^2 + b^2 + c^2) \) yields the following inequality:

\[
P(x_{k+1}) + V_{k+1} \leq P(x_k) + V_k + M_1 L_t \|\theta_{i,k} - \theta^*\| + M_1 \|\varepsilon_{i,k+1}\| + \frac{3L^2_t\tau^2}{2C} \|\varepsilon_{i,k+1}\|^2
\]

\[
- \left( \mu - \frac{L_x + C}{2} - \frac{3L^2_t\tau^2}{2C} \right) \|T_{ik}(y^{i_k}_k, \theta^*) - x_{i,k}\|^2 + \frac{3L^2_t\tau^2}{2C} \|\theta_{i,k} - \theta^*\|^2
\]

\[
\leq P(x_k) + V_k - \tilde{C} \|T_{ik}(y^{i_k}_k, \theta^*) - x_{i,k}\|^2 + M_1 L_t \|\theta_{i,k} - \theta^*\|
\]

\[
+ \frac{3L^2_t\tau^2}{2C} \|\theta_{i,k} - \theta^*\|^2 + M_1 \sum_{i=1}^N \|\varepsilon_{i,k+1}\| + \frac{3L^2_t\tau^2}{2C} \sum_{i=1}^N \|\varepsilon_{i,k+1}\|^2.
\]

Thus, by taking expectations conditioned on \( F_k \), we obtain

\[
\mathbb{E} \left[ P(x_{k+1}) + V_{k+1} \mid F_k \right] \leq \mathbb{E} \left[ P(x_k) + V_k \mid F_k \right] - \tilde{C} \mathbb{E} \left[ \|T_{ik}(y^{i_k}_k, \theta^*) - x_{i,k}\|^2 \mid F_k \right]
\]

\[
+ M_1 L_t \mathbb{E} \left[ \|\theta_{i,k} - \theta^*\| \mid F_k \right] + \frac{3L^2_t\tau^2}{2C} \mathbb{E} \left[ \|\theta_{i,k} - \theta^*\|^2 \mid F_k \right] + \sum_{i=1}^N \mathbb{E} \left[ \|\varepsilon_{i,k+1}\| \mid F_k \right] + \frac{3L^2_t\tau^2}{2C} \sum_{i=1}^N \mathbb{E} \left[ \|\varepsilon_{i,k+1}\|^2 \mid F_k \right].
\]  (60)

Since \( \theta_{i,k}, T_i(y^{i}_k; \theta^*) \) \( \forall i \in N \) are adapted to \( F_k \) and \( i_k \) is independent of \( F_k \), by Corollary 7.1.2 in [44] and \( \mathbb{P}(i_k = i) = p_i \) we have the following equalities:

\[
\mathbb{E} \left[ \|T_i(y^{i}_k, \theta^*) - x_{i,k}\|^2 \mid F_k \right] = \mathbb{E}_{i_k} \left[ \|T_i(y^{i}_k, \theta^*) - x_{i,k}\|^2 \right] = \sum_{i=1}^N p_i \|T_i(y^{i}_k, \theta^*) - x_{i,k}\|^2,
\]

\[
\mathbb{E} \left[ \|\theta_{i,k} - \theta^*\| \mid F_k \right] = \sum_{i=1}^N p_i \|\theta_{i,k} - \theta^*\|, \quad \text{and} \quad \mathbb{E} \left[ \|\theta_{i,k} - \theta^*\|^2 \mid F_k \right] = \sum_{i=1}^N p_i \|\theta_{i,k} - \theta^*\|^2.
\]  (61)

Since \( x_k \) and \( V_k \) are adapted to \( F_k \), by invoking (60) and (61) we obtain the following bound:

\[
\mathbb{E} \left[ P(x_{k+1}) + V_{k+1} \mid F_k \right] \leq P(x_k) + V_k - \tilde{C} \sum_{i=1}^N p_i \|T_i(y^{i}_k, \theta^*) - x_{i,k}\|^2
\]

\[
+ M_1 L_t \sum_{i=1}^N p_i \|\theta_{i,k} - \theta^*\| + \frac{3L^2_t\tau^2}{2C} \sum_{i=1}^N p_i \|\theta_{i,k} - \theta^*\|^2
\]

\[
+ M_1 \sum_{i=1}^N \mathbb{E} \left[ \|\varepsilon_{i,k+1}\| \mid F_k \right] + \frac{3L^2_t\tau^2}{2C} \sum_{i=1}^N \mathbb{E} \left[ \|\varepsilon_{i,k+1}\|^2 \mid F_k \right].
\]  (62)

By setting \( C = \frac{\sqrt{3L^2_t\tau^2}}{2} \) we derive \( \frac{C}{2} + \frac{3L^2_t\tau^2}{2C} = \sqrt{3L^2_t\tau^2} \). Thus, by taking \( \mu > \frac{L_x + C}{2} + \sqrt{3L^2_t\tau^2} \) it follows that \( \tilde{C} > 0 \). Therefore, by Theorem 1 in [45], Assumption 6(a) and by invoking result (a), we have that \( \sum_{i=1}^\infty \sum_{i=1}^N p_i \|T_i(y^{i}_k, \theta^*) - x_{i,k}\|^2 < \infty \) a.s. Then by \( p_i \in (0, 1) \) we obtain (b).

(c) The proof is similar to that of Theorem 1(b).

(d) Since \( \mathbb{E} \left[ \|\varepsilon_{i,k+1}\| \mid F_k \right], \|\theta_{i,k} - \theta^*\|, \text{and} \|T_i(y^{i}_k; \theta^*) - x_{i,k}\|^2 \) are nonnegative for \( k \geq 1 \), by Assumption
\( b \), results (a) and (b) we have the following for any \( i \in \mathbb{N} \):

\[
\sum_{k=1}^{\infty} \mathbb{E}[\|\varepsilon_{i,k+1}\|] = \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbb{E} \left[ \|\varepsilon_{i,k+1}\| \mathcal{F}_k \right] \right] < \infty, \]

\[
\sum_{k=1}^{\infty} \mathbb{E}[\|\theta_{i,k} - \theta^*\|] < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \mathbb{E} \left[ \|T_i(y^*_k; \theta^*) - x_{i,k}\|^2 \right] < \infty. \]

Thus, by the Jensen’s inequality we have the following for any \( i \in \mathbb{N} \):

\[
limit_{k \to \infty} \mathbb{E}[\|\varepsilon_{i,k+1}\|] = 0, \quad \lim_{k \to \infty} \mathbb{E}[\|\theta_{i,k} - \theta^*\|] = 0, \quad \text{and} \quad \lim_{k \to \infty} \mathbb{E} \left[ \|T_i(y^*_k; \theta^*) - x_{i,k}\| \right] = 0. \tag{63} \]

Then by the triangle inequality and (57), we obtain that

\[
\mathbb{E}[\|x_{i,k+1} - x_{i,k}\|] \leq \mathbb{E}[\|x_{i,k+1} - T_i(y^*_k, \theta^*)\|] + \mathbb{E}[\|x_{i,k} - T_i(y^*_k, \theta^*)\|] \\
\leq L_t \mathbb{E}[\|\theta_{i,k} - \theta^*\|] + \mathbb{E}[\|\varepsilon_{i,k+1}\|] + \mathbb{E}[\|x_{i,k} - T_i(y^*_k, \theta^*)\|] \to 0, \quad \text{as} \quad k \to \infty.
\]

This implies that \( \lim_{k \to \infty} \mathbb{E}[\|x_{i,k+1} - x_{i,k}\|] = 0 \). The result follows by proceeding as in Theorem 1(c). \( \square \)

Theorem 3 shows that the estimates of the equilibrium strategy and the misspecified parameter generated by Algorithm 2 converge a.s. to the set of Nash equilibria and to \( \theta^* \), respectively. Define the gap function \( G(x; \theta^*) \triangleq \sup_{y \in X} \nabla P(x; \theta^*)^T(x - y) \). The following result shows the convergence in mean of \( x_k \), characterized by the convergence of \( G(x_k) \) to zero in the mean sense.

**Theorem 4 (Convergence in mean of gap function.)** Let \( \{x_k\} \) and \( \{\theta_k\} \) be generated by Algorithm 2. Suppose Assumptions 4, 5, and 6 hold, and, in addition, that \( \mu > \frac{L^2}{2} + \sqrt{3}L_x \tau, \beta \in (0, 2\mu_g/L_g^2) \) and \( N_{i,k} = \left\lceil \frac{2^{1+\delta}}{\epsilon_{i,k}} \right\rceil \) for some \( \delta > 0 \). Then \( \lim_{k \to \infty} \mathbb{E} \left[ G(x_k; \theta^*) \right] = 0 \).

**Proof.** Similar to the proof of Theorem 2, we have the following bound:

\[
\mathbb{E} \left[ G(x_k; \theta^*) \right] \leq \sum_{i=1}^{N} (\mu D_{X_i} + M + L_x D_{X_i}) \mathbb{E} \left[ \|T_i(y^*_k; \theta^*) - x_{i,k}\| \right] \implies \lim_{k \to \infty} \mathbb{E} \left[ G(x_k; \theta^*) \right] \leq 0 \quad (\text{by} \ 63).\]

However, \( G(x_k; \theta^*) \geq 0 \) since \( x_k \in X \), implying that \( \lim_{k \to \infty} \mathbb{E} \left[ G(x_k; \theta^*) \right] = 0 \), giving us the required result. \( \square \)

**Remark 2**

(i) Algorithm 2 might be extended to generalized and weighted potential games with misspecified parameters since Algorithm 1 is applicable to generalized and weighted potential games as shown in Section 2.4.

(ii) Note that in Algorithm 2, the update of \( \theta_{i,k} \) given by (51) uses multiple sampled gradients instead of a single sampled gradient utilized in the standard SA scheme. This enables us to firstly show the summability of \( \|\theta_{i,k} - \theta^*\|^2 \) and \( \|\theta_{i,k} - \theta^*\| \), and then prove the summability of \( \|T_i(y^*_k; \theta^*) - x_{i,k}\|^2 \). Nevertheless, the standard projected SG method might be still applicable but with a different selection of Lyapunov function in the convergence analysis. Actually, the recent work [9] considered the monotone stochastic Nash games and presented a set of coupled SA schemes distributed across players in which the first scheme updates each player’s strategy via a projected SG step while the second scheme updates each player’s belief regarding its misspecified parameter using an independently specified learning problem. The authors showed that the generated sequences converge a.s. to the true equilibrium strategy and the true parameter, and that convergence in the equilibrium strategy achieved the optimal rate of convergence in a mean-squared sense with a quantifiable degradation in the rate constant.
4 Preliminary numerics

In this section, we empirically validate the performance of Algorithm 1 and Algorithm 2 on the problem of congestion control and misspecified stochastic Nash-Cournot games, respectively.

4.1 Congestion Control

We consider a congestion control problem on a connected network characterized by a set of nodes \( V = \{1, \cdots, V\} \) and a set of links \( L = \{1, \cdots, L\} \) connecting the nodes. There are \( N \) users in the network, where each player \( i \) aims at sending a flow rate \( x_i \in C_i = \{x_i \in \mathbb{R} : 0 \leq x_i \leq x_{i,\text{max}}\} \) from the source node \( s_i \) to the destination node \( d_i \) through a path \( L_i \) in the network. The upper bound \( x_{i,\text{max}} \) on user \( i \)'s flow rate might represent a player-specific physical limitation. The payoff function of player \( i \) takes as the difference of a player-specific pricing function and a utility function \( U_i \) associated to the flow \( x_i \) parameterized by uncertainty \( \xi_i, \zeta_i \):

\[
\psi_i(x_i, x_{-i}; \xi_i, \zeta_i) = \sum_{l \in L_i} P_l \left( \sum_{j \in L_j} x_j \right) - U_i(x_i, \xi_i, \zeta_i).
\]

The first term can be interpreted as the price that player \( i \) pays for the network resources with \( P_l \) depending on the aggregated flows on the link \( l \). Suppose that \( P_l, l \in L \) is convex and \( U_i, i \in \mathcal{N} \) is concave on \([0, x_{i,\text{max}}]\). Typical examples for the pricing and utility functions are given by the following:

\[
P_l = \frac{a_l}{b_l - \sum_{j \in L_j} x_j} \quad \text{and} \quad U_i = \xi_i \log(1 + x_i + \zeta_i),
\]

where \( \xi_i, \zeta_i \) are random variables. Suppose each link \( l \in L \) in the network has a positive capacity \( c_l \). Let us introduce a routing matrix \( A \in \mathbb{R}^{L \times N} \), where \([A]_{l,i} = 1\) if \( l \in L_i \), and \([A]_{l,i} = 0\), otherwise. The capacity constraints of all links can be expressed in the vector form as \( Ax \leq c \) with \( c = \text{col}\{c_l\}_{l=1}^L \).

For a fixed feasible \( x_{-i} \), we derive the bound of the user \( i \)'s flow rate \( x_i \) denoted by

\[
0 \leq X_i(x_{-i}) = \min_{l \in L_i} \{c_l - \sum_{j \neq i} A_{l,j} x_j\}.
\]

The \( i \)th user aims at solving the following problem:

\[
\min_{x_i \in C_i \cap X_i(x_{-i})} f_i(x_i, x_{-i}) = \mathbb{E}[\psi_i(x_i, x_{-i}; \xi_i, \zeta_i)].
\]

Thus, the resulting problem is a generalized potential game with the coupled constraint \( X = \{x \in \mathbb{R}^n : Ax \leq c\} \) and the potential function defined as follows:

\[
P(x) = \sum_{l \in L} \frac{a_l}{b_l - \sum_{j \in L_j} x_j} - \sum_{i \in \mathcal{N}} \mathbb{E}[U_i(x_i, \xi_i, \zeta_i)].
\]

Further, it is shown in Theorem 3.1 of [48] that the congestion control problem has a unique inner NE under appropriately chosen parameters.

We conducted numerical simulations for a network of \( V = 8 \) nodes and \( L = 12 \) links shown in Figure 1. The parameters \( a_l, b_l \) of the utility function \( P_l \) and the capacity constraint \( c_l \) associated to link \( l \in L \) are given in Figure 1 as well. There are \( N = 8 \) users sending flows through the network depicted in Figure 1. The link paths of user \( i \in \mathcal{N} \) as well as local parameters \( \xi_i, \zeta_i, x_{i,\text{max}}, p_l \) are given in Table 2 where \( U[\tau_1, \tau_2] \)
denotes the uniform distribution over the interval $[\tau_1, \tau_2]$. For any $k \geq 0, i, j \in \mathcal{N}$, the communication delays $d_{ij}(k)$ are independently generated from a uniform distribution on the set $\{0, 1, \ldots, \tau\}$ with $\tau = 4$. We carry out simulations for Algorithm 1, where the inexact solution (8) satisfying Assumption 3 are computed via the SA scheme (SA$_{i,k}$) with $j_{i,k} = \lceil \Gamma_{i,k} \rceil$ and $\mu = 1$. The estimates of each users equilibrium flow rates are shown in Figure 2, which demonstrates the almost sure convergence of the iterates generated by Algorithm 1. Figure 3 displays the trajectory of the mean gap function $\mathbb{E}[G(x_k)]$ calculated by averaging across 50 sample paths, which demonstrates convergence in mean of the estimates generated by Algorithm 1.

![Figure 1: A network with 8 nodes and 12 links.](image)

| Parameters | Links $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|------------|----------|---|---|---|---|---|---|---|---|---|----|----|----|
| $a_l$      |          | 5 | 4 | 3 | 5 | 4 | 3 | 4 | 3 | 4 | 4  | 4  | 3  |
| $b_l$      |          | 6 | 10| 8| 6 | 9 | 5 | 5 | 6 | 6  | 8  | 9  |    |
| $c_l$      |          | 5 | 8 | 6 | 5 | 8 | 4 | 5 | 4 | 4  | 5  | 7  | 8  |

Table 2: Link paths and local parameters of all users

![Figure 2: Flow rates of players (a single sample path)](image)

![Figure 3: Trajectory of $\mathbb{E}[G(x_k)]$](image)
4.2 Nash-Cournot Games with Misspecified Parameters

We apply Algorithm\textsuperscript{2} to the networked Nash-Cournot game \textsuperscript{[8,7]}. Suppose there are $N$ firms, regarded as the set of players $\mathcal{N} = \{1, \ldots, N\}$, competing over $L$ markets denoted by $\mathcal{L} = \{1, \ldots, L\}$. Firm $i \in \mathcal{N}$ sells its products $x_i = (x_{i,1}, \ldots, x_{i,n_i}) \in \mathbb{R}^{n_i}$ to each connected market with $n_i$ denoting the number of markets connected to firm $i$. We use matrix $A_i \in \mathbb{R}^{L \times n_i}$ to specify the participation of firm $i$ in the markets, where $[A_i]_{j,p} = 1$ if firm $i$ delivers its production $x_{i,p}$ to market $j$, and $[A_i]_{j,p} = 0$, otherwise. The production cost function of firm $i$ is given by $c_i(x_i; \xi_i) = (c_i + \xi_i)^T x_i$ for some positive parameter $c_i \in \mathbb{R}^{n_i}$ and random disturbance $\xi_i$ with mean zero. Denote by $A = [A_1, \ldots, A_N]$, by $Ax = \sum_{i=1}^{N} A_i x_i \in \mathbb{R}^L$ and by $S_j = [Ax]_j$ the aggregated products of all connected firms delivered to market $j$, where $[Ax]_j$ denotes the $j$-th entry of the vector $Ax$. Furthermore, the price of products sold in market $j \in \mathcal{L}$ is assumed to follow a linear function corrupted by noise:

$$p_j(S_j; \zeta_j) = a_j^* + \zeta_j - b_j^* S_j,$$

where $a_j^* > 0, b_j^* > 0$ are the pricing parameters, and the random disturbance $\zeta_j$ is zero-mean. Then firm $i \in \mathcal{N}$ has a stochastic payoff function defined as follows:

$$\psi_i(x; \theta^*; \xi_i, \zeta_i) = c_i(x_i; \xi_i) - \sum_{j \in \mathcal{L}} p_j(S_j; \zeta_j)[A_i]_{j,j} = (c_i + \xi_i)^T x_i - \left( a^* + \zeta - B^* A \right)^T A_i x_i,$$

where $a^* = \text{col}\{a_1^*, \ldots, a_N^*\}$, $\zeta = \text{col}\{\zeta_1, \ldots, \zeta_L\}$, $B^* = \text{diag}\{b_1^*, \ldots, b_L^*\}$, and $\theta^* = (a^*, B^*)$ is unknown to all companies. Suppose firm $i \in \mathcal{N}$ has finite production capacity $X_i = \{x_i \in \mathbb{R}^{n_i} : 0 \leq x_i \leq \text{cap}_i\}$. In this networked Cournot competition, firm $i$ minimizes $c_i^T x_i - (A_i x_i)^T a^* + (A_i x_i)^T B^* \sum_{i=1}^{N} A_i x_i$ over $X_i$. If $P(x)$ is defined as

$$P(x) \triangleq \sum_{i=1}^{N} c_i^T x_i - \left( \sum_{i=1}^{N} A_i x_i \right)^T a^* + \left( \text{col}\{A_i x_i\}_{i=1}^{N} \right)^T \chi \left( \text{col}\{A_i x_i\}_{i=1}^{N} \right),$$

where $\chi = \frac{1}{2}(I_N + J_N) \otimes B^*$. Then for any $i \in \mathcal{N}$ and for any $x_{-i} \in X_{-i}$, equation \textsuperscript{[4,7]} holds for all $x_i, x_i' \in X_i$. Thus, the Nash-Cournot game admits a potential function $P(x)$. By definition of $f_i(x_i, x_{-i}; \theta^*), \nabla x_i f_i(x_i, x_{-i}; \theta^*)$ depends on $a_j^*, b_j^*, j \in \mathcal{L}$ if firm $i$ sells its products to market $j$. Each firm $i$ can observe the historic data about the aggregated sales $S_j$ in market $j$ and the price of products $p_j = a_j^* + \zeta_j - b_j^* S_j$ if the market $j$ is connected to firm $i$. As such, firm $i$ is able to learn the pricing parameters $a_j^*, b_j^*$ of the connected market $j$ through solving the following problem:

$$\min_{a_j \geq 0, b_j \geq 0} \mathbb{E}[(a_j - b_j S_j - p_j)^2] \quad (64)$$

In the numerical investigation, there are $V = 13$ firms to sell their products to $L = 7$ markets with the network shown in Figure\textsuperscript{4}. Suppose each component of $\text{cap}_i$ and the cost pricing parameter $c_i$ of the firm $i \in \mathcal{N}$ satisfy uniform distributions specified by $U[5, 8]$ and $U[2, 4]$. The pricing parameters $a_j^*, b_j^*$ of market $j \in \mathcal{L}$ are drawn from uniform distributions $U[4, 6]$ and $U[0.2, 0.4]$, respectively. Suppose the random variables $\xi_i, i \in \mathcal{N}$ and $\zeta_j, j \in \mathcal{L}$ are drawn from uniform distributions $U[-c_i^*/8, c_i^*/8]$ and $U[-a_j^*/8, a_j^*/8]$, respectively. Suppose the historic aggregated sales $S_j$ in market $j \in \mathcal{L}$ satisfies the uniform distribution $U[0, 5]$. For any $k \geq 0, i, j \in \mathcal{N}$, the communication delays $d_{ij}(k)$ are independently generated from a uniform distribution on the set $\{0, 1, \ldots, \tau\}$ with $\tau = 4$. We carry out simulations for Algorithm\textsuperscript{2} where the inexact solution \textsuperscript{[50]} satisfying Assumption\textsuperscript{6} is
Figure 4: Networked Nash-Cournot: An edge from $C_i$ to $M_j$ implies firm $C_i$ sells its products to market $M_j$.

Figure 5: The estimates of $a^*$, $b^*$ and $x^*$ (a single sample path)

computed via the SA scheme (52) with $N_{i,k} = j_{i,k} = \lfloor \Gamma_{i,k}^3 \rfloor$, $p_i = 1/N \forall i \in \mathcal{N}$, $\beta = 0.1$ and $\mu = 5$. The scaled errors of learning schemes for the unknown parameters $a^*$, $b^*$ and the Nash equilibrium $x^*$ are provided in Figs. 5 where $a_k^i$ and $b_k^i$ denotes the estimates of $a^*$ and $b^*$ given by firm $i$ at time $k$. The figure demonstrates the almost sure convergence of Algorithm 2.

Comparison with the asynchronous SG method: Suppose there are no communication delays among the players, i.e., $\tau = 0$. Set $p_i = 1/N \forall i \in \mathcal{N}$, $\beta = 0.1$ and $\mu = 5$. Let $N_{i,k} = \lfloor \Gamma_{i,k}^3 \rfloor$ steps of the SA scheme (52) be taken at major iteration $k$ to obtain an inexact solution to (7), where $\Gamma_{i,k}$ is defined in Lemma 1. Set $j_{i,k} = \lfloor \Gamma_{i,k}^3 \rfloor$ in equation (50). We then carry out simulations for both Algorithm 2 and the asynchronous SG algorithm, which indeed is Algorithm 2 with equations (50) and (51) replaced by

$$x_{i,k+1} = \Pi_{X_i} [x_{i,k} - \gamma_{i,k} \nabla \psi_i(x_k, \theta_{i,k}; \xi_{i,k})],$$

$$\theta_{i,k+1} = \Pi_{\Theta} [\theta_{i,k} - \beta \gamma_{i,k} \nabla g (\theta_{i,k}, \eta_{i,k})],$$

where $\gamma_{i,k} = \frac{1}{\Gamma_{i,k}^3}$. We compare the two methods for the estimates of the equilibrium strategy $x^*$ in terms of (i) the total number of the gradients steps (iteration complexity), and (ii) the communication overhead for achieving the same accuracy. Let $K(\epsilon)$ denotes the smallest total number of SG steps the players have carried out to make $\mathbb{E} \left[ \frac{\|x_k - x^*\|}{1 + \|x^*\|} \right] < \epsilon$. The empirically observed relationship between $\epsilon$ and $K(\epsilon)$ for both methods are shown in Figure 6(a), where the empirical errors are calculated by averaging across 50 trajectories. From the figure, it is seen that the iteration complexity are of the same orders while the constant of SG is superior to that of Algorithm 2. Since SG requires the rivals’ newest information to carry out a single gradient step, the resulting communication overhead
is proportional to the total number of gradient steps. In contrast, Algorithm \(2\) carries out multiple gradient steps without requiring the most recent rivals’ information. Further, the communication overhead of the two methods are shown in Figure 6(b). From the results in Figure 6 upon termination, Algorithm \(2\) requires approximately about 10 times more gradient steps than the standard SG method while characterized by approximately 500 times less communication overhead.

5 Concluding Remarks

This paper develops an asynchronous inexact proximal BR scheme (combined with joint learning) to compute the Nash equilibrium of a stochastic potential Nash game (possibly corrupted by misspecification). When player-specific problems are convex, we show that the estimates generated by the proposed schemes converge a.s. to a connected subset of the NE set with uniformly bounded communication delays. Since the game is characterized by a possibly nonconvex potential function, the schemes can be viewed as randomized block coordinate descent schemes for a stochastic nonconvex optimization problem which is block-wise convex. We further show that the gap function converges to zero in mean for both schemes. Finally, we apply the developed methods to the congestion control problem and the Nash-Cournot game, and demonstrate the simulation results.

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