Holographic transports and stability in anisotropic linear axion model

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January 16, 2015

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Abstract

We study thermoelectric conductivities and shear viscosities in a holographically anisotropic model. Momentum relaxation is realized through perturbing the linear axion field. AC conductivity exhibits a coherent/incoherent metal transition. The longitudinal shear viscosity for prolate anisotropy violates the bound conjectured by Kovtun-Son-Starinets. We also find that thermodynamic and dynamical instabilities are not always equivalent, which provides a counter example of the Gubser-Mitra conjecture.
1 Introduction

One of the advantages of holography is that it provides a non-perturbative method for calculating transport coefficients of strongly coupled systems. Transport coefficients of anisotropic plasma are of interests because the quark-gluon-plasma created in RHIC and LHC is actually anisotropic and non-equilibrium during the period of time $\tau_{\text{out}}$ after the collision. The neutral anisotropic black brane solution was constructed from type IIB supergravity by Mateos and Trancanelli [1,2]. Interestingly, the shear viscosity longitudinal to the direction of anisotropy violates the viscosity bound [3], which is up to now the first example of such violation in Einstein gravity [4].

In [5,6], the R-charged version of anisotropic black brane solution was derived via non-linear Kaluza-Klein reduction of type IIB supergravity to five dimension. The non-linear Kaluza-Klein reduction of type IIB supergravity to five dimension, leads to the presence of an Abelian field in the action. The introduction of the $U(1)$ gauge field breaks the SO(6) symmetry and thus leads to the excitations of the Kaluza-Klein modes. In addition, we also considered the analytic continuation in which the anisotropy parameter is taken to be imaginary, resulting in an oblate anisotropy.

In this paper, we will calculate the optical conductivity (longitudinal to the anisotropy direction), DC thermoelectric conductivities and shear viscosities in this anisotropic system. An accurate realization of thermoelectric conductivities in real condensed matter systems requires to include mechanisms for momentum dissipation. Recently, it was suggested to introduce momentum relaxation in holography by exploiting spatially dependent sources for
scalar operators or using the massive gravity [7, 24]. The reduction action for the anisotropic black brane used in this paper, is the Einstein-Maxwell-Dilaton-Axion theory. We will show that momentum relaxation can be realized in our model by explicitly breaking the translational invariance (for works on spontaneously symmetry breaking see [25]). In addition, we will prove that for oblate anisotropy, the viscosity bound is not violated. Remarkably, the resistivity shows its linear temperature behavior, signaling the presence of “strange metals”.

On the condensed matter theory side, it is still lack of satisfying explanation of the linear temperature dependence of resistivity at sufficiently high temperature in materials such as organic conductors, heavy fermions, fullerenes, vanadium dioxide and pnitides. The linear temperature dependence of resistivity at high temperature, a signature of a breakdown of the Boltzman theory, is expected when the quasiparticle mean free path \( l \) becomes shorter than the lattice parameter \( \tilde{a} \). It is the self-consistency of the Boltzmann theory which requires that the charge carrier transport should satisfy the Mott-Ioffe-Regel (MIR) limit \( l \sim \tilde{a} \) or \( \kappa_F l \sim 1 \) [26, 27]. The violation of the Mott-Ioffe-Regel limit in the so called “bad metals” has led to an assertion that the standard theory of Fermi liquids cannot be used to describe such strongly correlated systems both below and above the MIR limit. Recent studies on strange metals from holography can be found for examples in [28–32].

Another purpose of this paper is to study the relations between Gubser-Mitra (GM) conjecture and the “wall of stability” [7, 8]. The GM conjecture states that gravitational backgrounds with a translationally invariant horizon yield an unstable tachyonic mode precisely whenever the specific heat of the black brane geometry becomes negative [33]. In other words, dynamical instabilities is correlated with the local thermodynamic instability of background spacetimes. A lot of evidences have been known for this conjecture [34–39].

On the other hand, it was found that for the shear modes of the hydrodynamics in massive gravity [8], the dispersion relation at zero momentum limit is given by

\[
\omega = -i\tau_{rel}^{-1} + \cdots,
\]

where \( \tau_{rel} \) denotes the momentum relaxation timescale and the ellipsis represents the momentum dependent terms. The linear fluctuations are purely decaying modes when \( \tau_{rel}^{-1} > 0 \), indicating that the black brane is stable under such dynamical perturbations. This is what we mean “wall of stability”. The “wall of stability” requires the momentum relaxation timescale to be \( \tau_{rel} \geq 0 \), otherwise the dual field will absorb momentum rather than dissipating it. It maybe interesting to relate the question of the wall of stability with the GM conjecture and ask the following question: does the regime with \( \tau_{rel} \geq 0 \) exactly correspond to thermodynamic stability regime of the black brane? We will prove that for isotropic and homogenous black brane solution given in massive gravity [7, 8] and Einstein-Maxwell-linear scalar theory [9], the dynamically unstable regime partially overlaps with the local thermodynamical unstable region, but not fully satisfying the GM conjecture. However, for the anisotropic black brane investigated here, the physics in \( \tau_{rel} \geq 0 \) does not necessarily request the local thermodynamic stability of the black brane.

It is well-known that the AdS black holes have rich phase structure. For neutral black
holes with spherical topology in asymptotically AdS spacetime, there is the so-called Hawking-Page phase transition due to a competing effect between the scale set by the volume of the spacetime and the scale determined by the temperature. For Reissner-Nordström-AdS (RN-AdS) black holes with \( S^{d-1} \) horizon topology, the phase diagram is analogous to the phase structure of van der Waals’ liquid-gas system \([40–45]\). For event horizons with topology \( R^{d-1} \), the black brane phase structure is usually considered as dominated by the black brane phase without any thermodynamic instabilities.

However, in \([5,6]\) we found that a planar black brane does not necessary to be thermodynamically stable by demonstrating that an anisotropic black brane has a branch of solution with negative specific heat. One may naturally connect the thermodynamic instability uncovered in this anisotropic but translationally symmetric system with the GM conjecture \([33]\). Therefore, in this paper, we will first compute the DC and optical conductivities with momentum relaxation and then check the GM conjecture.

The organization of the contents is as follows. In section 2, we briefly review on the anisotropic black brane solution and its thermodynamic properties. In section 3, we calculate the DC and AC conductivities. In section 4, we calculate the transverse and longitudinal shear viscosities for the prolate and oblate black brane solutions. We discuss relations between the Gubser-Mitra conjecture and wall of stability, and extend our discussion to the massive gravity and Einstein-Maxwell-linear scalar theories in section 5. We present our conclusions in the last section.

### 2 R-charged anisotropic black brane solution

The five dimensional axion-dilaton-Maxwell-gravity bulk action reduced from type IIB supergravity is written as \([5,6]\)

\[
S = \frac{1}{2\kappa^2} \left[ \int d^5x \sqrt{-g} \left( R + 12 - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{2\phi} (\partial \chi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) - 2 \int d^4x \sqrt{-\gamma} K \right],
\]

(2.1)

where we have set the AdS radius \( L = 1 \), and \( \kappa^2 = 8\pi G = \frac{\pi^2}{N_c} \). The counter term takes the form

\[
S_{ct} = \frac{1}{\kappa^2} \int d^4x \sqrt{\gamma} \left( 3 - \frac{1}{8} e^{2\phi} \partial_i \chi \partial^i \chi \right) - \log v \int d^4x \sqrt{\gamma} A,
\]

(2.2)

where \( A \) is the conformal anomaly in the axion-dilaton-gravity system, \( v \) is the Ferrerma-Graham coordinate, and \( \gamma \) is the induced metric on a \( v = v_0 \) surface.

The background solutions with anisotropy along the \( z \)-direction for the equations of motion are

\[
ds^2 = e^{-\frac{1}{2} \phi} r^2 \left( - F B dt^2 + dx^2 + dy^2 + H dz^2 \right) + \frac{e^{-\frac{1}{2} \phi} dr^2}{r^2 F}
\]

(2.3)

\[ A = A_t(r) dt, \quad \phi = \phi(r), \quad \chi = az
\]

(2.4)
The metric functions $\phi, F, B$ and $H = e^{-\phi}$ are functions of the radial coordinate $r$ only. The electric potential is given by $A_t(r) = \int_{r_H}^r dr Q\sqrt{B}e^{\frac{2\phi}{r^3}}/r^3$, where $Q$ is an integral constant related to the charge. A dimensionless charge can be introduced by defining $q \equiv \frac{Q}{2\sqrt{3}r_H}$ and the physical range of $q^2$ is $0 \leq q^2 < 2$. The horizon locates at $r = r_H$ with $F(r_H) = 0$ and the boundary is at $r \to \infty$ where $F = B = H = 1$. The asymptotic $AdS_5$ boundary condition requires the boundary condition $\phi(\infty) = 0$. We note that the above ansatz is invariant under the scaling $t \to \lambda t, x_i \to \lambda x_i, r \to \lambda^{-1}r$ and $a \to \lambda^{-1}a$. The Hawking temperature is given by

$$T = \frac{r_H^2 F'(r_H) \sqrt{B_H}}{4\pi} = \sqrt{\frac{r_H}{2}} \left[ \frac{r_H e^{-\frac{2\phi}{r^3}}}{16\pi} \left( 16 + \frac{a^2 e^{7\phi}}{r_H^2} \right) - \frac{e^{2\phi} q^2 r_H^2}{2\pi} \right],$$

through the Euclidean method. The numerical and semi-analytic black brane solution was given in \[5, 6\]. For example, we can plot the numerical solutions (2.3) in Fig.1 for prolate and oblate anisotropy, respectively.

![Figure 1: (Color online.) The metric functions for $a = 64.06$, $Q = 9.76$ (left), which corresponds to the prolate anisotropy and $a = 1.2i$, $Q = 1/10$ (right), with $u_H = 1$, which depicts the oblate anisotropy.](image)

The Ward identity obeys

$$\nabla^i \langle T_{ij} \rangle = \langle O_\phi \rangle \nabla_j \phi^{(0)} + \langle O_\chi \rangle \nabla_j \chi^{(0)} + F_{ij}^{(0)} \langle J_i \rangle. \quad (2.6)$$

If we consider the background fields only, the translational symmetry is unbroken because $\nabla^i \langle T_{ij} \rangle = 0$, with the facts that $\nabla_j \phi^{(0)} = 0, \langle O_\chi \rangle = 0$ and $F_{ij}^{(0)} = 0$. However, by considering the fluctuations around the background as we will do in later section, we can prove that the $\langle O_\chi \rangle$ is finite. This gives rise to the momentum dissipation and thus the translational invariance is broken.

Actually, the anisotropic black brane has very special thermodynamic properties, as pointed out in \[5, 6\]. Considered the prolate anisotropy with $a^2 > 0$, the black brane suffers
thermodynamic instabilities. This can be easily seen from (2.5) in the small horizon radii limit \( r_H \ll 1 \), that is to say

\[
T \sim \frac{\sqrt{\mathcal{B}_H a^2 e^{3\phi_H}}}{16\pi r_H},
\]

(2.7)

which in turn results in negative specific heat since \( \partial T/\partial r_H < 0 \). On the other hand, for the larger horizon radii with \( r_H \gg 1 \), we have

\[
T \sim \frac{\sqrt{\mathcal{B}_H a^2 e^{-\phi_H/2}}}{16\pi} \left( 16 - \frac{g^2 e^{2\phi_H}}{2\pi} \right) r_H.
\]

(2.8)

Thus in the prolate case, for a fixed temperature there are two branches of allowed black brane solutions: a branch with larger horizon radii and one with smaller. The smaller branch solution is unstable with negative specific heat. This situation is very similar to the case of Schwarzschild-AdS black holes with a spherical horizon. As to the oblate anisotropy case, the black brane is qualitatively the same as the planar Schwarzschild-AdS black brane. This means that, there is only one stable branch of black brane solution and the thermodynamics is dominated by this phase for all temperature without any negative specific heat. In general, the larger black brane branch will appear and will match to the small black brane solution at some temperature \( T_{\text{min}} \), below which there is only the thermal gas solution. At some higher temperature a first order phase transition (from thermal gas) to the large black brane branch will take place. So the system will be a doped Mott-like insulator up to a critical temperature and then there will be a first order phase transition to a conducting phase \([46,48]\).

In the following, we will prove that for the conductivities along the anisotropic direction, the oblate anisotropy with \( a^2 < 0 \) is forbidden because thermal conductivity is negative there, although it is thermodynamically more stable than the prolate case. Moreover, the oblate anisotropy corresponds to negative momentum relaxation time, resulting in another dynamical instabilities of the black brane.

3 DC electric conductivity with momentum relaxation

In \([5,6]\), we found that a prolate black brane could be thermodynamically unstable. In particular, for the prolate case \( a^2 > 0 \), the phase structure of the black brane is similar to that of Schwarzschild-AdS black hole with spherical horizon. In what follows, we will compute the DC conductivity with momentum dissipation in this prolate background.

In the \( r = 1/u \)-coordinates, the metric can be recast as

\[
ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)dx^2 + g_{yy}(r)dx^2 + g_{zz}(r)dz^2,
\]

(3.1)

where \( g_{xx} = g_{yy} \neq g_{zz}(r) \).

It was argued in several papers that axions having a spatially dependent source leads to the fact that momentum is dissipated at the linearized level \([9,49,50]\). For the scalar types of
metric perturbations, the independent variables are $h_{tt}$, $h_{tz}$, $h_{xx} = h_{yy}$, $h_{zz}$, $\delta \phi$, $\delta \chi$ together with the $t-$ and $z-$components of the gauge field $A_\mu$. In the zero momentum limit, it is easy to check that $h_{tz}, A_z$ and $\delta \chi$ decouple from other variables. Therefore, to compute the conductivity with momentum dissipation, we only need to consider linearized fluctuations of the form

$$
\delta g_{t(0)} = h_{tz}(t, r), \quad \delta A_z(t, r) = a_z(t, r), \quad \delta \chi = a^{-1} \bar{\chi}(t, r), \tag{3.2}
$$

and all the other metric and gauge perturbations vanished. Here we choose the gauge $h_{rz} = 0$ and the electromagnetic perturbation along the $z$ direction at zero momentum. We shall work in the Fourier decomposition

$$h_{tz}(t, r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} h_{tz}(\omega, r), \tag{3.3}$$

$$a_z(t, r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} a_z(\omega, r), \tag{3.4}$$

$$\bar{\chi}(t, r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \bar{\chi}(\omega, r). \tag{3.5}$$

The linearized equations of motion corresponding to the $(r, z)$ component of Einstein’s equation, the Maxwell equation and the dilaton equation are given by

$$0 = A'_z a_z + h'_t + \frac{g'_{zz}}{g_{zz}} h_t - \frac{g_{tt} e^{2\phi} \chi'}{i\omega}, \tag{3.6}$$

$$0 = a'' + \left( \frac{g'_{tt}}{2g_{tt}} - \frac{g'_{zz}}{2g_{zz}} + \frac{g'_{xx}}{g_{xx}} - \frac{g'_{rr}}{2g_{rr}} \right) a'_z + \frac{A'_t}{g_{tt}} (h'_t - \frac{g'_{zz}}{g_{zz}} h_t) + \frac{\omega^2 g''_{rr}}{g_{tt}} a_z, \tag{3.7}$$

$$0 = \bar{\chi}'' + \left( \frac{g'_{rr}}{2g_{rr}} + \frac{g'_{tt}}{2g_{tt}} - \frac{g'_{xx}}{g_{xx}} + \frac{g'_{zz}}{2g_{zz}} - 2\phi' \right) \bar{\chi}' + \frac{\omega^2 g_{rr}}{g_{zz} g_{tt}} \bar{\chi} - \frac{i\omega a^2 g_{rr}}{g_{zz} g_{tt}} h_t. \tag{3.8}$$

$$0 = h''_{tz} - \left( \frac{g'_{tt}}{2g_{tt}} + \frac{g'_{rr}}{2g_{rr}} + \frac{g'_{xx}}{2g_{xx}} - \frac{g'_{zz}}{g_{zz}} \right) h'_t + \left( \frac{g''_{rr} g'_{zz}}{2g_{rr} g_{zz}} + \frac{g'_{tt} g'_{zz}}{2g_{tt} g_{zz}} - \frac{g''_{zz}}{g_{zz}} + \frac{g''_{zz}}{2g_{zz}} \right) h_t
- \frac{g'_{xx} g'_{zz}}{g_{xx} g_{zz}} h_t
- \frac{g''_{zz} a^2}{g_{zz}} h_t
+ A'_t a'_z - i\omega g_{rr} e^{2\phi} \bar{\chi}, \tag{3.9}$$

where the prime denotes derivative with respect to $r$. For sake of convenience, we can eliminate $h_t$ by taking a radial derivative of (3.8) and substituting the expression for $h_t'$ in (3.7) and (3.8). We can introduce a new variable

$$\bar{\chi} = \omega^{-1} g_{xx} g_{zz} g_{tt} g_{rr} e^{2\phi} \chi', \tag{3.10}$$

and recast the equation of motion as

$$0 = g^{-1}_{xx} g^{-1}_{zz} \left( g_{xx} g_{zz} \frac{g_{tt}}{g_{rr}} \frac{1}{2} \sqrt{\frac{g_{tt}}{g_{rr}}} a'_z \right)' + \omega^2 \sqrt{\frac{g_{tt}}{g_{rr}}} a_z - \frac{A'_t Q}{g_{xx} g_{zz}} a_z - \frac{i Q \sqrt{g_{rr} g_{tt}}}{g_{xx} g_{zz}} \bar{\chi}, \tag{3.11}$$

and
0 = e^{2\phi}g_{xx}g_{zz}^{\frac{1}{2}}(e^{-2\phi}g_{xx}^{-1}g_{zz}^{-\frac{1}{2}} \sqrt{\frac{g_{tt}}{g_{rr}}} \tilde{\chi})' + \omega^2 \sqrt{\frac{g_{tt}}{g_{rr}}} \tilde{\chi} + a^2 e^{2\phi} \left(\frac{ig_{xx}A_i'}{\sqrt{g_{zz}}} a_z - \frac{\sqrt{g_{rr}} g_{tt}}{g_{zz}} \tilde{\chi}\right). \tag{3.12}

Following \cite{9,11}, we can rewrite the fluctuation equations (3.11) and (3.12) in the form

\[
\begin{pmatrix}
L_1 & 0 \\
0 & L_2
\end{pmatrix}
\begin{pmatrix}
a_z \\
\tilde{\chi}
\end{pmatrix} + \omega^2 \sqrt{\frac{g_{tt}}{g_{rr}}} \begin{pmatrix}
a_z \\
\tilde{\chi}
\end{pmatrix} = \mathcal{M} \begin{pmatrix}
a_z \\
\tilde{\chi}
\end{pmatrix}, \tag{3.13}
\]

where \(L_1\) and \(L_2\) are linear differential operators and \(\mathcal{M}\) is the mass matrix,

\[
\mathcal{M} = \begin{pmatrix}
Q^2 \frac{g_{rr} g_{tt} / (g_{xx} g_{zz})}{i a^2 g_{xx}^2} & i Q \sqrt{g_{rr} g_{tt} / (g_{xx} g_{zz})} \\
-ia^2 Q e^{2\phi} \sqrt{g_{rr} g_{tt} / g_{zz}} & a^2 e^{2\phi} \sqrt{g_{rr} g_{tt} / g_{zz}}
\end{pmatrix}. \tag{3.14}
\]

Clearly, there exists a massless mode, because \(\det \mathcal{M} = 0\). Let us introduce the following linear combinations

\[
\begin{align*}
\lambda_1 & = b^{-1}(r) \left( e^{2\phi} a_z + \frac{Q}{ia^2 g_{xx}^2} \tilde{\chi} \right), \tag{3.15} \\
\lambda_2 & = b^{-1}(r) \left( \frac{Q^2}{a^2 g_{xx}^2} a_z - \frac{Q}{ia^2 g_{xx}^2} \tilde{\chi} \right), \tag{3.16}
\end{align*}
\]

where

\[
b(r) = e^{2\phi} + \frac{Q^2}{a^2 g_{xx}^2}. \tag{3.17}
\]

We then obtain the master equation for the massless mode \(\lambda_1\)

\[
\left(e^{-2\phi} g_{xx} \sqrt{\frac{g_{tt}}{g_{rr} g_{zz}}} b(r) \lambda_1 - \sqrt{\frac{g_{tt}}{g_{rr} g_{zz}}} c(r) \lambda_2 \right)' + \omega^2 b(r) e^{-2\phi} g_{xx} \sqrt{\frac{g_{tt}}{g_{rr} g_{zz}}} \lambda_1 = 0, \tag{3.18}
\]

where \(c(r) = (g_{xx} e^{2\phi})' / (e^{2\phi} g_{xx})\). We can easily find that the following quantity is radially conserved at zero frequency

\[
\Pi = e^{-2\phi} g_{xx} \sqrt{\frac{g_{tt}}{g_{rr} g_{zz}}} b(r) \lambda_1' - \sqrt{\frac{g_{tt}}{g_{rr} g_{zz}}} c(r) \lambda_2. \tag{3.19}
\]

The DC membrane conductivity to each radial slice is defined as

\[
\sigma_{DC}(r) = \lim_{\omega \to 0} \frac{\Pi}{\omega \lambda_1} \bigg|_r. \tag{3.20}
\]

It was proved in \cite{9,11} that \(\sigma_{DC}(r)\) does not evolve radially (i.e., \(\sigma_{DC}(\infty) = \sigma_{DC}(r_H)\)). So it can be evaluated at the horizon. In order to evaluate (3.20), we note that the ingoing boundary conditions for the fields are given by

\[
\begin{align*}
a_z & = (r - r_H)^{-i \omega / (F'(r_H) \sqrt{B(r_H)})} [a_z^H + \mathcal{O}(r - r_H)], \tag{3.21} \\
\tilde{\chi} & = (r - r_H)^{-i \omega / (F'(r_H) \sqrt{B(r_H)})} [\tilde{\chi}_z^H + \mathcal{O}(r - r_H)]. \tag{3.22}
\end{align*}
\]

\[8\]
Substituting the metric functions (2.3) and (3.21) into (3.20), we finally obtain
\[ \sigma_{DC} = r_H e^{\phi(r_H)} \left( 1 + 12r_H^2 e^{-\phi(r_H)} \frac{Q^2}{a^2} \right), \]  
(3.23)
where \( q = \frac{Q}{2\sqrt{3}r_H} \). The first term in the round brackets is the conductivity due to the pair production at the horizon. The DC conductivity can also be recast as
\[ \sigma_{DC} = (g_{xx}g_{yy}g_{zz})^{1/2} \left|_{r=r_H} \right. + \frac{q^2}{a^2 e^{2\phi}(g_{xx}g_{yy}g_{zz})^{1/2}} \left|_{r=r_H} \right. 
= r_H e^{\phi(r_H)} + \frac{Q^2}{a^2 r_H^3} e^{-3\phi(r_H)/4}. \]  
(3.24)
This result agrees with [19, 49]. We will provide an alternative calculation on the DC conductivity later as a consistent check. The DC conductivity can be related to a scattering time \( \tau \) by [21]
\[ \sigma_{DC} = (g_{xx}g_{yy}g_{zz})^{1/2} \left|_{r=r_H} \right. + \frac{Q^2}{\epsilon + P_z} \tau_{rel}. \]  
(3.25)
The scattering rate is given by
\[ \Gamma = \tau_{rel}^{-1} = \frac{sa^2}{4\pi(\epsilon + P_z)}. \]  
(3.26)
For the prolate solution, \( \tau_{rel} > 0 \). For the oblate solution, \( \tau_{rel} < 0 \) in which the wall of stability violated. To the linear order, the Ward identity (2.6) reduces to
\[ \partial_t \langle T_{zz} \rangle = -a \delta \langle O_\chi \rangle + \rho \partial_t a_z. \]  
(3.27)
We have proved that the linearized contribution to the vev \( \langle O \chi \rangle \) is finite, although at zeroth order \( \langle O \chi \rangle \) is vanishing. This gives rise to the momentum dissipation and thus the translational invariance is broken.

- **DC conductivity with prolate anisotropy** We would like to comment on the temperature dependence of the DC electric conductivity. It was proven in [1] that the IR geometry of the prolate black brane is asymptotic Lifshitz at zero temperature and thus the DC electric conductivity obtained here behaves quiet different from those with near horizon geometry \( AdS_2 \times R^3 [9] \). There are two branches in high temperature phase: small and large black hole radius. As can be seen in figure 2 (middle), the conductivity goes up as the temperature increases, which corresponds to the larger horizon radii branch solution of the black brane. The system undergoes a first order Hawking-Page-like confinement/deconfinement phase transition as the temperature varies. It has been proved that the smaller horizon radius branch of solution has a negative specific heat so it is not physically realized. If small horizon radius \( r_H \) were stable, we could identify it as the dual to doped “bad metals” since its resistivity is linear in temperature and DC resistivity violates the MIR limit as shown in figure 2 (right). It suggests that we need to stabilize the small radius branch by twisting our present model, which suggests a direction to future model building.

In [46–48], the authors argued that the confinement phase is dual to the Mott-insulators. The strange metal behaviors can be obtained by doping a Mott insulator. In our set-up, the linear axion field’s parameter \( a \) might play the role of dopants.

- **DC conductivity with oblate anisotropy** The linear temperature dependence of the resistivity can also be observed for the oblate black brane solution provided the pair production term \( (g_{xx}g_{yy}g_{zz})^{1/2} \big|_{r=r_H} \) is large enough. It is interesting to observe that the DC conductivity is finite as \( T \to 0 \), corresponding to metallic behaviour as shown in figure 3.

![Figure 3: (Color online.) (Left) Oblate DC conductivity as a function of the temperature, where we set \( q = 0.0001 \) and \( a = 0.8i \). (Right) the resistivity shows violation of the Mott-Ioffe-Regel limit for the oblate anisotropy. The oblate black brane solution is everywhere thermodynamically stable.](image)

The trouble with this case is that the for \( a^2 < 0 \), the relaxation time \( \tau_{rel} \) becomes negative, hinting the black brane is unstable under such metric fluctuations. It also signals tachyon...
condensations to form a new ground state. But the nature of the new ground state is not at all
clear to us at this moment. It was proven in \cite{5} that the oblate black brane solution is ther-
modynamically stable at any temperature. We wish to understand why a thermodynamically
stable background has unstable quasinormal mode in the future.

\subsection{3.1 Optical conductivity}

In this section, we try to solve Eqs. (3.6-3.9) numerically. In order to calculate AC thermo-
electric conductivities numerically, we need evaluate the on-shell action that gives finite and
quadratic function of the boundary values. In general for the variation of the action, we have

\[ \delta S = \int \partial_{\mu} \left( \frac{\partial L}{\partial \partial_{\mu} \phi} \delta \phi \right) dr + \int E.O.M. \delta \phi dr. \] \hspace{1cm} (3.28)

We obtain the on-shell action in the momentum space,

\[ S = \lim_{r \to \infty} \frac{V_3}{2} \int \frac{d\omega}{2\pi} \sqrt{-g} \left[ g_{zz} h'_{tz} h_{tz} - \frac{a'_z a_z}{g_{rr} g_{zz}} - \frac{e^{2\phi} \tilde{\chi}' \tilde{\chi}}{g_{rr}} + \frac{A'_t a_z}{g_{rr} g_{tt}} h_{tz} - \mathcal{E} h_{tz} h_{tz} \right], \] \hspace{1cm} (3.29)

where $\mathcal{E}$ denotes the energy density. The Green function is defined via

\[ G_{ij} = \frac{\delta^2 S}{\delta \phi^i \delta \phi^j}. \]

Near the boundary ($r \to \infty$), the asymptotic solutions go as

\[ h_{tz} = r^2 h_{tz}^{(0)} + h_{tz}^{(2)} + \frac{1}{r^2} h_{tz}^{(3)} + \frac{1}{r^4} h_{tz}^{(4)} + \ldots \] \hspace{1cm} (3.30)

\[ a_z = a_z^{(0)} + \frac{1}{r^2} a_z^{(2)} + \ldots \] \hspace{1cm} (3.31)

\[ \tilde{\chi} = \chi^{(0)} + \frac{1}{r^2} \chi^{(2)} + \frac{1}{r^4} \chi^{(3)} + \ldots \] \hspace{1cm} (3.32)

Note that $a_z^{(0)}$ is the source of the electric current $J_z$ and $h_{tz}^{(0)}$ is dual to energy-momentum
tensor $T_{tz}$. In order to compute the AC electric conductivity numerically, we need impose
the ingoing boundary condition at the horizon and adopt the numerical method developed
in \cite{18} and \cite{50}.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$a/\mu$ & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\
\hline
$K/\mu^2$ & 0.751 & 0.762 & 0.743 & 0.710 & 0.698 \\
\hline
$\tau \mu$ & 536.715 & 134.337 & 64.016 & 39.909 & 27.456 \\
\hline
\end{tabular}
\end{center}

Table 1: Drude parameters for different $a/\mu$ at $T/\mu = 0.2658$.

The optical conductivity is given by

\[ \sigma(\omega) = -\frac{i G_{Jz Jz}}{\omega}. \] \hspace{1cm} (3.33)
The numerical computation shows that the optical conductivity takes the form of the Drude conductivity as

$$
\sigma(\omega) = \sigma_Q + \frac{K\tau}{1 - i\omega\tau},
$$

(3.34)

with $\sigma_Q = r_H e^{\phi(r_H)}$, constant $K$ and relaxation time $\tau$. Figure 4 shows how the optical conductivity changes as the anisotropic parameter $a$ (i.e. the dissipation strength) changes. Since the anisotropic parameter $a$ is non-vanishing, the $1/\omega$ pole in the imaginary part disappears. As shown in (1) when $a$ becomes bigger, the maximum value of the peak in real part decreases, which is in good agreement with the DC conductivity (3.23). We also emphasize that for the case $a/\mu = 0.5$, the numerical data deviates from the standard Drude model, implying that there might be a coherent/incoherent transition [50]. We expect the AC thermal and thermoelectric conductivities with momentum relaxation also show a Drude peak as given in [50] and we postpone such computation to a future study.

### 3.2 DC Thermo-electric conductivities

In this section, we provide an alternative calculating on the DC electric conductivity, thermoelectric conductivities (Seebeck coefficients), and the thermal conductivity by using the method developed by Donos and Gauntlett [49]. In this set-up, we consider a slightly different form of the black hole fluctuations

$$
\delta A_z = -Et + a_z(r), \quad h_{tz} = h_{t_z}(r), \quad h_{rz} = h_{r_z}(r), \quad \chi = a z + \delta\chi_1(r)
$$

(3.35)

where the temporal component of the four-potential $a_\mu$ corresponds to a constant electric field $E$ along $z$—direction. The equations of motion for these linearized fluctuations are given by

$$
h''_{tz} - \left( \frac{g'_{tt}}{2g_{tt}} + \frac{g'_{rr}}{2g_{rr}} + \frac{g'_{zz}}{2g_{zz}} - \frac{g'_x}{g_{xx}} \right) h'_{tz} + \left( \frac{g''_{rr}g'_{zz}}{2g_{rr}g_{zz}} + \frac{g'_{tt}g'_z}{2g_{tt}g_{zz}} - \frac{g''_{zz}}{2g_{zz}} \right) h_{tz} + \left( \frac{g'_{rr}g'_{zz}}{2g_{rr}g_{zz}} + \frac{g_{tt}g'_z}{2g_{tt}g_{zz}} - \frac{g''_{zz}}{2g_{zz}} \right).
$$

(3.36)
\[-\frac{g'_{xx}g'_{zz}}{g_{xx}g_{zz}} - \frac{g_{rr}e^{2\phi}a^2}{g_{zz}})h_{tz} + A'_t a'_z = 0, \quad (3.36)\]
\[a''_z + \left(\frac{g'_{tt}}{2g_{tt}} - \frac{g'_{rr}}{2g_{rr}} - \frac{g'_{zz}}{2g_{zz}} + \frac{g'_{xx}}{g_{xx}}\right)a'_z + \frac{A'_t}{g_{tt}} h'_{tz} - \frac{g'_{zz}A'_t}{g_{zz}g_{tt}} h_{tz} = 0, \quad (3.37)\]
\[2A'_t E g_{rr} - 2g_{rr}e^{2\phi}a \delta \chi_1' + \left(\frac{g'_{rr}g'_{zz}}{g_{rr}g_{zz}} - \frac{g'_{tt}g'_{zz}}{2g_{tt}g_{zz}} - \frac{g''_{zz}}{2g_{zz}} + 2g''_{xx} - \frac{g'_{zz}g'_{xx}}{g_{xx}g_{xx}}\right)h_{rz} = 0. \quad (3.38)\]

Note that the derivative of the scalar potential is given by
\[A'_t = -\frac{Q}{\left(\frac{g_{rr}g_{tt}}{g_{xx}}\right)^{1/2}}. \quad \text{The equation} \quad (3.38) \quad \text{can be solved easily}
\[h_{rz} = \frac{EQ \sqrt{g_{rr}}}{a^2 e^{2\phi} g_{xx} \sqrt{g_{tt}g_{zz}}} + \frac{\delta \chi_1'}{a}. \quad (3.39)\]

In order to solve the equations of motion for \(a_z\) and \(h_{tz}\), we need impose proper boundary conditions for the fluctuation fields at the event horizon \(r = r_H\) and at the conformal boundary \(r \rightarrow \infty\). We first assume that \(\delta \chi'\) is analytic at the event horizon and falls off fast at infinity. Regularity at the event horizon can be obtained by switching to Eddington-Finklestein coordinates
\[v = t - \frac{1}{4\pi} \ln(r - r_H). \quad (3.40)\]
In this coordinate, the gauge field is determined by the regularity as
\[a_z = -\frac{E}{4\pi T} \ln(r - r_H) + O(r - r_H). \quad (3.41)\]
From equation (3.36), we know that the regularity at the event horizon requires
\[h_{tz} = \left.\frac{EQ \sqrt{g_{zz}}}{a^2 g_{xx}}\right|_{r=r_H} + O(r - r_H). \quad (3.42)\]
Near the boundary \((r \rightarrow \infty)\), we have the fall-off of \(a_z \sim J^z r^{-2}\), where \(J^z\) denotes the charge density current. As to \(h_{tz}\), from equation (3.36), we can see that there are two independent solutions, one of which behaves as \(\sim c_1 r^3\) and the other as \(\sim r^{-2}\). We require that there are no sources associated to thermal gradients, and thus the coefficient \(c_1\) should be vanishing. We also demand \(\delta \chi'\) falls off fast enough so that it has no contribution to the boundary value of \(h_{rz}\).

We now turn to the computation of the DC conductivity. We can see that the conserved charge density current is indicated by the non-zero Maxwell equation (3.37), in which we can define the current as
\[J^z \equiv \sqrt{-g}f^{rz}, \quad f^{rz} = \frac{a'_z h_{tz}}{g_{rr}g_{zz}g_{tt}} + \frac{A'_t}{g_{rr}g_{zz}}. \quad (3.43)\]

13
We emphasize that the charge density current derived from the Maxwell equation is not identical to the radially conserved quantity given in (3.19). Since the current is radially conserved, $J^z$ can be evaluated both at the horizon and at the boundary. The DC electric conductivity along the $z$--direction is expressed as $\sigma_{DC} = J^z/E$. By using (3.41) and (3.42), we finally obtain the DC electric conductivity

$$\sigma_{DC} = \frac{g_{xx}}{\sqrt{g_{zz}}} \bigg|_{r=r_H} + \frac{Q^2 g_{tt}}{a^2 e^{2\phi} g_{xx} \sqrt{g_{zz}}} \bigg|_{r=r_H} = r_H e^{\phi(r_H)/4} + \frac{Q^2}{a^2 r_H^3} e^{-3\phi(r_H)/4}. \quad (3.44)$$

This result exactly agrees with (3.24) obtained in the previous section.

The conserved heat current $Q$ is defined through introducing a two-form associated with the Killing vector field $K = \partial_t$ and it is assumed as the following form [49].

$$Q = 2\sqrt{-g} \nabla^r K^z + A_t J^z = \sqrt{g_{tt}} g_{rr} g_{zz} g_{xx} \left( - g^{tt} h_{tz} \partial_t g_{tt} + \partial_r h_{tz} \right) + A_t J^z. \quad (3.45)$$

Notice that the quantity $Q$ is also radially conserved. We can evaluate $Q$ at the event horizon

$$Q = \sqrt{g_{xx}} h_{tx} \partial_t g_{tt} \bigg|_{r=r_H}. \quad (3.46)$$

The Seebeck coefficient can be obtained at the event horizon $r = r_H$ by using the expression

$$\alpha = s_{DC} = \frac{1}{T} \frac{Q}{E} = \frac{4\pi Q}{a^2 e^{2\phi}} \bigg|_{r=r_H}. \quad (3.47)$$

In order to calculate the thermal conductivity, we need consider perturbations with a source for the heat current. A consistent choice of the linearized fluctuations takes the form

$$\delta A_z = -E t + \zeta A_z t + a_z(r), \quad (3.48)$$

$$h_{tz} = -\zeta t \sqrt{g_{tt}} / \sqrt{g_{rr}} + h_{tz}(r), \quad (3.49)$$

$$h_{rz} = h_{rz}(r), \quad (3.50)$$

$$\chi = a z + \delta \chi_1(r), \quad (3.51)$$

where $\zeta$ is a constant. According to the holographic dictionary, the coefficient $\zeta$ corresponds to the thermal gradient $-\nabla_z T/T$ [51,53]. The choosing of the fluctuations ensure that all the time dependent terms drops out of the conserved current $J^z$ and $Q$. The equations of motion for $h_{tz}$ and $a_z$ remain the same as given in (3.36) and (3.37). We can solve the equation for $h_{rz}$ and obtain

$$h_{rz} = -\frac{Q(\zeta A_t - E) \sqrt{g_{rr}}}{a^2 e^{2\phi} \sqrt{g_{tt} g_{xz} g_{zz}}} - \frac{g_{zz} (g_{zz}^{-1} \zeta \sqrt{g_{tt} / g_{rr}})'}{a^2 e^{2\phi} \sqrt{g_{tt} / g_{rr}}} + \frac{\delta \chi_1'}{a}, \quad (3.52)$$

\[14\]
where \( \delta \chi_1 \) can be a constant at the event horizon. The regularity at the event horizon requires the gauge field take the form \( a_z = -\frac{E}{4\pi T} \ln(r - r_H) + \mathcal{O}(r - r_H) \). It was suggested in [49] that the horizon regularity condition for \( h_{tz} \) can be obtained by switching to the Kruskal coordinates

\[
h_{tz} = g_{zz} \sqrt{g_{tt} g_{rr}} |_{r=r_H} - \frac{\zeta \sqrt{g_{tt}}}{4\pi T \sqrt{g_{rr}}} \ln(r - r_H) + \mathcal{O}(r - r_H).
\]

Again, we impose the boundary conditions at the infinity: \( a_z \sim J_z r^{-2} \) and \( h_{ tz} \sim r^{-2} \). We emphasize that under the choice of the fluctuations given in (3.48), the form of the conserved current does not change. The conserved currents evaluated at the event horizon are given by

\[
J^z = \left[ E \left( \frac{g_{xx}}{\sqrt{g_{zz}}} + \frac{Q^2}{a^2 e^{2\phi} g_{xx} \sqrt{g_{zz}}} \right) + \frac{\zeta Q g'_{tt}}{a^2 e^{2\phi} \sqrt{g_{tt} g_{rt}}} \right] |_{r=r_H},
\]

\[
Q = \left[ E \frac{Q g'_{tt}}{a^2 e^{2\phi} \sqrt{g_{tt} g_{rt}}} + \zeta g'_{tt} g_{xx} \sqrt{g_{zz}} \frac{Q}{a^2 e^{2\phi} g_{rr} g_{tt}} \right] |_{r=r_H}.
\]

We finally obtain the DC thermoelectric conductivities in the \( z \) direction

\[
\sigma_{DC} = \frac{\partial}{\partial E} J^z = \left( \frac{g_{xx}}{\sqrt{g_{zz}}} + \frac{Q^2 g_{tt}}{a^2 e^{2\phi} g_{xx} \sqrt{g_{zz}}} \right) |_{r=r_H},
\]

\[
\bar{\alpha} = \frac{1}{T} \frac{\partial}{\partial E} Q = \frac{4\pi Q}{a^2 e^{2\phi}} |_{r=r_H},
\]

\[
\alpha = \frac{1}{T} \frac{\partial}{\partial \zeta} J^z = \frac{4\pi Q}{a^2 e^{2\phi}} |_{r=r_H},
\]

\[
\bar{\kappa} = \frac{1}{T} \frac{\partial}{\partial \zeta} Q = \frac{4\pi s T}{a^2 e^{2\phi}} |_{r=r_H}.
\]

It is clear that if the anisotropic parameter \( a^2 < 0 \), the \( \bar{\alpha}, \alpha \) and \( \bar{\kappa} \) will take negative value and thus become nonphysical because negative thermal conductivity means that thermal current can flow from lower temperature objects to higher temperature objects spontaneously.

### 4 Shear viscosities and viscosity bound

For the anisotropic fluid considered here, the viscosity tensor \( \eta_{ijkl} \) yields two shear viscosities out of five independent components [54]. In the \( u \)-coordinate, we work with the \( h_{uu} = 0 \) and \( A_u = 0 \) gauges and consider linearized fluctuations of the form: \( h_{xy} = e^{-i\omega t + ik_z z} h_{xy}(u) \) for the transverse shear viscosity, and \( h_{xz} = e^{-i\omega t + ik_y y} h_{xz}(u) \) for the longitudinal shear viscosity.

For the transverse tensor mode \( h_{xy} \), the equation of motion is given by

\[
0 = h_{yy}^{xx''} - \frac{3}{u} h_{yy}^{x' } + \frac{1}{2 \bar{H}} h_{yy}^{x' x} + \frac{\mathcal{F}'}{\mathcal{F}} h_{yy}^{x' x} - \frac{3}{4} \phi' h_{yy}^{x' x} + \frac{\mathcal{B}'}{2 \bar{B}} h_{yy}^{x' x} - \frac{k_2^2 h_{yy}^{x' x}}{\mathcal{F} \bar{H}} + \frac{\omega^2 h_{yy}^{x' x}}{\mathcal{F}^2 \bar{B}}.
\]

(4.1)
We introduce the following notation \[^{[55]}\]
\[
\mathcal{N}^{\mu\nu} = \frac{1}{2\kappa^2} g_{xx} \sqrt{-g} g^{\mu x} g^{\nu y}.
\] (4.2)

The equation of motion for \(h_{xy}\) can be written as
\[
\partial_u (\mathcal{N}^{uy} \partial_u h_y^x) - k_y^2 \mathcal{N}^{zy} h_y^x = \omega^2 \mathcal{N}^{ty} h_y^x = 0.
\] (4.3)

The Green function is
\[
G_{xy, xy} = \mathcal{N}^{uy} \frac{\partial_u h_y^x}{h_y^x}.
\] (4.4)

The shear viscosity is defined as
\[
\eta_{xy, xy} = -\frac{G_{xy, xy}}{i\omega}.
\] (4.5)

The flow equation for the transverse viscosity \(\eta\)
\[
\partial_u \eta_{xy, xy} = i\omega \left( \frac{\eta^2}{N_{uy}} + \mathcal{N}^{ty} \right) + \frac{i}{\omega} \mathcal{N}^{zy} k_y^2.
\] (4.6)

The transverse shear viscosity is easily obtained by demanding the horizon regularity,
\[
\eta_{xy, xy} = \left( -\mathcal{N}^{ty} \mathcal{N}^{uy} \right)^{\frac{1}{2}} \bigg|_{u = u_H} = \frac{e^{-\frac{5\phi}{4}}}{2\kappa^2 u_H^3} = \frac{s}{4\pi}.
\] (4.7)

For both prolate and oblate anisotropy, the transverse shear viscosities is exactly \(\frac{s}{4\pi}\) and thus the viscosity bound is satisfied. However, for the longitudinal tensor mode, we have the equation of motion for \(h_{zx}\):
\[
0 = h_{zx}'' - \frac{3}{u} h_{zx}' + \frac{1}{2} \mathcal{H}' h_{zx}' + \frac{\mathcal{F}'}{\mathcal{F}} h_{zx}' - \frac{3}{4} \phi' h_{zx}' + \frac{\mathcal{B}'}{2\mathcal{B}} h_{zx}' - \frac{k_y^2 h_y^x}{\mathcal{F}^2} + \omega^2 h_y^x.
\] (4.8)

We can recast it as
\[
\partial_u \eta_{xz, xz} = i\omega \left( \frac{\eta^2}{N_{uz}} + \mathcal{N}^{uz} \right) + \frac{i}{\omega} \mathcal{N}^{yz} k_z^2.
\] (4.9)

The horizon regularity requires
\[
\eta_{xz, xz} = \frac{s}{4\pi \mathcal{H}(u_H)}.
\] (4.10)

For prolate black brane solutions with \(\mathcal{H}(u_H) > 1\), the shear viscosity to entropy density ratio \(\frac{\eta_{xz, xz}}{s} = \frac{1}{4\pi \mathcal{H}(u_H)} < \frac{1}{4\pi}\) violates the KSS bound \([4, 55, 56]\). However, for the oblate black brane solution with \(\mathcal{H}(u_H) < 1\), the shear viscosity to entropy density ratio \(\frac{\eta_{xz, xz}}{s} = \frac{1}{4\pi \mathcal{H}(u_H)} > \frac{1}{4\pi}\) satisfies the KSS bound.

Comparing the shear viscosity obtained here with that of Einstein-Gauss-Bonnet gravity \([57-65]\), we can find that the anisotropic parameter \(a^2\) plays the same role as the Gauss-Bonnet (GB) coupling constant. When the (GB) coupling constant takes positive value, the viscosity bound is violated, but no violation for the negative-valued GB coupling.
5 Gubser-Mitra conjecture and “Wall of stability”

The Gubser-Mitra (GM) conjecture claims that gravitational backgrounds with a translationally invariant horizon develop a dynamical instability precisely whenever the specific heat of the black brane geometry becomes negative \[33\]. The GM conjecture was later refined as: it works provided that there is a unique background with a spatially uniform horizon and specified conserved charges \[34\]. A holographic realization of the GM conjecture was given in \[35\] by demonstrating that a tachyonic mode of the GM instability is dual to sound wave in the gauge theory.

On the other hand, the wall of stability is referred to the regime $\tau_{rel} \geq 0$. This means that the metric fluctuations are unstable when $\tau_{rel} < 0$, because it absorbs momentum, rather than dissipating it. Thus the fluctuations will grow exponentially in time. The wall of stability in fact impose some constraints on the anisotropic parameter from the dynamical side.

It is our purpose in this section to consider the GM conjecture by comparing the dynamical and thermodynamic instabilities in our anisotropic system. For completeness of our study, we also extend our discussions to the massive gravity theory \[7\] and the Einstein-Maxwell-linear scalar theory \[9\].

5.1 Dynamical and thermodynamic instabilities in the anisotropic background

In section 3, we proved that the relaxation time $\tau$ is proportional to $a^2$. That is to say, $\tau_{rel}$ is positive for the prolate anisotropy, but negative for the oblate anisotropy. As we already knew that the prolate solution has a thermodynamic instability at smaller horizon radius and the oblate solution suffers no thermodynamic instability. This means that the dynamical instability uncovered in this anisotropic background is not correlated with the thermodynamic instability. Therefore, our results do not obey the GM conjecture.

It would be interesting to examine the GM conjecture by considering the sound modes in our anisotropic media: whenever the specific heat of the prolate black brane is negative, the speed of sound in such a system should be imaginary. The speed of sound determined from the equation of state is given by

$$v_s^2 = \frac{\partial P}{\partial \mathcal{E}}.$$  \hspace{1cm} (5.1)

The thermodynamic potential in the grand canonical ensemble is found to be $G = -P_z = E - T s - \rho \mu - a \Phi \ [5,6]$. The entropy density can be written as

$$s = -\left( \frac{\partial G}{\partial T} \right)_{\mu, \Phi} = \left( \frac{\partial P_z}{\partial T} \right)_{\mu, \Phi}.$$ \hspace{1cm} (5.2)

The specific heat can be defined as

$$c_{\mu, \Phi} = \left( \frac{\partial \mathcal{E}}{\partial T} \right)_{\mu, \Phi}.$$ \hspace{1cm} (5.3)
The speed of sound then can be expressed as
\[ v_s^2 = \frac{s}{c_{\mu,\phi}}. \] (5.4)

This implies that for the case \( c_{\mu,\phi} < 0 \), the speed of sound is purely imaginary since the entropy density is always positive.

The transport coefficients computed in the previous sections are mainly working on the shear and tensor modes in which we did not find dynamical unstable modes correlated with a negative specific heat. But this cannot preclude that there is such a dynamical instability in sound modes. Considering the complexity and difficulty in the computation of the sound modes, we defer the study of the sound modes and GM conjecture to a future publication.

### 5.2 Instabilities of black brane in massive gravity model

The application of massive gravity in holography with broken diffeomorphism invariance in the bulk introduces a mass term for the graviton in such a way that one has momentum relaxation in the boundary dual field theory. The action of the four-dimensional massive gravity model is given by \([7, 11, 21]\)

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa_4^2} \left( R + \frac{6}{L^2} + \beta \left( [K]^2 - [\mathcal{K}]^2 \right) \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] + \frac{1}{2\kappa_4^2} \int_{z=Z_U} d^3x \sqrt{-g_b} 2K ,
\] (5.5)

where \( \beta \) is an arbitrary parameter having the dimension of a mass squared and \( (K^2)_\mu = g^{\mu\rho} f_{\rho\nu}, f_{\mu\nu} = \text{diag}(0, 0, 1, 1) \). The black brane solution in this massive gravity is given by

\[
ds^2 = \frac{L^2}{u^2} \left[ -f(u)dt^2 + dx^2 + dy^2 + \frac{1}{f(u)} du^2 \right],
\] (5.6)

\[ A_t = \mu (1 - \frac{u}{u_H}), \] (5.7)

\[
f(u) = \frac{\gamma^2 \mu^2 u^4}{2L^2 u_H^2} - \frac{\gamma^2 \mu^2 u^3}{2L^2 u_H} - \frac{u^3}{u_H^3} - \frac{\beta u^3}{u_H} + \beta u^2 + 1.
\] (5.8)

The black hole temperature is written as

\[ T = \frac{3}{4\pi u_H} - \frac{\gamma^2 \mu^2 u_H}{8\pi L^2} + \frac{\beta u_H}{4\pi}.
\] (5.9)

It is easy to check that for the case \( \beta < 0 \), the local stability condition \( \partial T / \partial u_H < 0 \) is always satisfied. This is to say, as the horizon radius \( r_H = 1/u_H \) increases, the black hole temperature goes up. However, for the case \( \beta > 0 \), there is a branch of black brane solution.
having $\partial T/\partial u_\eta > 0$. That is what we mean, the instability of the black brane because the heat capacity could become negative. The heat capacity is computed in the usual way

$$
c_p = \frac{\partial E}{\partial T} = \left( \frac{\partial E}{\partial u_\eta} \right)_\rho = \left( -\frac{3L^2}{u_\eta^4 \kappa_4^2} - \frac{\beta L^2}{u_\eta^2 \kappa_4^2} - \frac{\mu^2}{2u_\eta^2} \right) \frac{1}{\frac{\beta}{4\pi} - \frac{\kappa_4^2 \mu^2}{8L^2 \pi} - \frac{3}{4\pi u_\eta^4}}. \tag{5.10}
$$

Note that if $\partial T/\partial u_\eta > 0$, the heat capacity becomes negative since $\partial E/\partial u_\eta < 0$ for all range of the parameters (see also [42–45]). We find that for $\beta > \frac{6L^2 + \kappa_4^2 \mu^2 u_\eta^4}{2L^2 u_\eta^4} =: \beta_c$, the black brane is thermodynamically unstable.

It is interesting to note that the thermodynamic instability uncovered here is related to the dynamical instability of the dual fluid\(^1\). The momentum dissipation rate determined in terms of the graviton mass and the equilibrium thermodynamical quantities is given by [8]

$$
\tau^{-1}_{\text{rel}} = -\frac{s\beta}{2\pi (E + P)}.
\tag{5.11}
$$

The case $\beta > 0$ corresponding to $\tau_{\text{rel}} < 0$, means that the fluid turns to gaining momentum. The amplitude of the shear diffusion mode thus will grow exponentially in time and leads to instability of the system. Therefore, the wall of stability impose more tight constraints on the parameter $\beta$ than the black brane thermodynamics and the regime of the dynamical instability does not coincide with the regime of the thermodynamic instability completely. This implies that our result provide the counter example to the GM conjecture: the dynamical instability occurs even when the black brane is thermodynamically stable as we saw above in the window $0 < \beta < \beta_c$.

### 5.3 Instability of black brane in Einstein-Maxwell-linear scalars theory

In the holographic model consisting of Einstein-Maxwell theory with linear scalar fields, momentum relaxation can be realized through spatially dependent sources for operators dual to neutral scalars. The five-dimensional action can simply be written as

$$
S = \int_M \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{2} \sum_I (\partial \chi_I)^2 - \frac{1}{4} F^2 \right] d^{4+1}x - 2 \int_{\partial M} \sqrt{-\gamma} K d^4x. \tag{5.12}
$$

where $\Lambda = -12/(2l^2)$ and $\chi_I$ denotes an axion field. The resulting black brane are homogeneous and isotropic

$$
ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 dx^i dx^i, \quad A_t = \mu (1 - \frac{r_H^2}{r^2}), \quad \chi_I = \beta_I x^i, \tag{5.13}
$$

$$
f(r) = r^2 - \frac{\beta^2}{4} - \frac{m_0}{r^2} + \frac{\mu^2 r_H^4}{3r^4}, \quad m_0 = r_H^4 (1 + \frac{\mu^2}{3r_H^2} - \frac{\beta^2}{4r_H^4}). \tag{5.14}
$$

\(^1\)We would like to thank Richard Davison for figuring out this point.
The temperature of the black brane is given by

\[ T = \frac{1}{4\pi} \left( 4r_B - \frac{\beta^2}{2r_B} - \frac{2\mu^2}{3r_B} \right) \]  

(5.15)

One may notice that through an analytical continuation \( \beta \rightarrow i\beta \), the black brane temperature here becomes exactly that of the massive gravity. We stress that the sign of the \( \beta^2 \) is arbitrary.

The specific heat can be read as

\[ c_\rho = 8\pi r_B^3 \left( 4 - \frac{\beta^2}{2r_B^2} + \frac{2\mu^2}{3r_B^2} \right) \frac{1}{4 + \frac{\beta^2}{2r_B^2} + \frac{2\mu^2}{3r_B^2}} \]  

(5.16)

It is easy to find that for any given temperature \( T > 0 \) and positive \( \beta^2 > 0 \), the heat capacity is positive. If the constant \( \beta \) is analytically continued to an imaginary value, the black brane solution will generate an unstable branch with \( c_\rho < 0 \). Also the momentum dissipation rate is given by \( \tau_{rel}^{-1} = \frac{s\beta^2}{2\pi(E+P)} \), with a sign difference with that of massive gravity. Therefore, considering the parameter \( \beta^2 \) in the range \( \beta^2 \in (-\infty, \infty) \), we can conclude that the regimes of thermodynamic and dynamical instabilities does not equal to each other.

## 6 Conclusions

In summary, we have investigated various aspects of the dynamics of the linear perturbations, in particular the effect of the relaxation of momentum upon various observables, in the spatially anisotropic \( N = 4 \) super-Yang-Mills theory dual to the action (2.1). We computed the DC thermoelectric conductivities analytically. The optical conductivity was obtained through numerical computation. We also computed the shear viscosities and checked the viscosity bound for the prolate anisotropy and oblate anisotropy, respectively. Finally, we examined the relations between the GM conjecture and the wall of stability by comparing conditions for dynamical instabilities with conditions for thermodynamic instabilities in massive gravity and Einstein-Maxwell-linear scalar theory.

The optical conductivity matches the Drude model. In \[66\], it was found that the probe fermions in this anisotropic background are a non-Fermi liquid type without well defined quasi-particle states. Nevertheless our results in this paper shows that the dual system shows Drude type behaviour in AC conductivity although there is no coherent quasi-particle states. In small black hole radius branch, which is unstable, the DC electric conductivity shows the strange metal behavior with a linear temperature resistivity. It is a future project to deform the present model to stabilise this branch, which is very motivating phenomenologically.

The ratio of the shear viscosity to entropy density violated the viscosity bound for the prolate black brane solution. But for the oblate anisotropy, the viscosity bound is well satisfied. Moreover, for the momentum relaxation case, the oblate black brane solution is useless in calculating the conductivities: It may give nonphysical, negative thermoelectric conductivity. In all cases considered in this paper, the GM conjecture was not strictly satisfied.
In the future, it might be interesting to consider the holographic transports of anisotropic black branes with higher derivative gravity terms by adding chemical potential to the model constructed in [67] and investigate the diffusive bound as in [68].

Acknowledgements

We thank A. Buchel, R. G. Cai, R. Davison, B. Gouteraux, B. S. Kim, J. X. Lu, N. Iqbal, S. F. Wu and especially K.-Y. Kim for useful discussions. XHG was partly supported by NSFC, China (No.11375110). YL was partly supported by NSFC, China (No.11275208). SJS was supported by the NRF, Korea (NRF-2013R1A2A2A05004846). YL also acknowledges the support from Jiangxi young scientists (JingGang Star) program and 555 talent project of Jiangxi Province.

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