Hamiltonian dynamics and Noether symmetries in Extended Gravity Cosmology

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Abstract We discuss the Hamiltonian dynamics for cosmologies coming from Extended Theories of Gravity. In particular, minisuperspace models are taken into account searching for Noether symmetries. The existence of conserved quantities gives selection rule to recover classical behaviors in cosmic evolution according to the so called Hartle criterion, that allows to select correlated regions in the configuration space of dynamical variables. We show that such a statement works for general classes of Extended Theories of Gravity and is conformally preserved. Furthermore, the presence of Noether symmetries allows a straightforward classification of singularities that represent the points where the symmetry is broken. Examples for nonminimally coupled and higher-order models are discussed.

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1 Introduction

Different points of view can be assumed in order to deal with Quantum Cosmology. It can be considered as the first step towards the construction of a complete theory of Quantum Gravity. Moreover, its goal is to find out the law of initial conditions from which our classical universe started its evolution. However, with respect to other theories of physics such as Electromagnetism, General Relativity (GR) or Quantum Mechanics, boundary conditions for the evolution of the system universe cannot be obviously set from outside. In standard theories, the approach is to search for some field equations (e.g. Maxwell’s or Einstein’s equations or Schrödinger’s equation) and then impose, from the outside, the laws of initial or boundary conditions (the Cauchy problem). In Cosmology, by definition, there is no rest of the universe so that boundary conditions must be fundamental laws of physics. In this sense, a part the fact that Quantum Cosmology is a viable outline to achieve Quantum Gravity, it can be considered as an autonomous branch of physics due to the problem of finding initial conditions [1].

However, not only the conceptual difficulties, but also mathematical ones make Quantum Cosmology difficult to handle. For example, the superspace of geometrodynamics [2] has infinite degrees of freedom so that it is not possible to fully integrate the Wheeler-De Witt (WDW) equation. Moreover, the Hilbert space of states describing universes is not available and then it is not clear how to interpret the solutions of WDW equation in the framework of probability theory [3].

Despite of these shortcomings, several results have been obtained and Quantum Cosmology has become a sort of paradigm in theoretical physics. For example the infinite-dimensional superspace can be restricted to suitable finite-dimensional configuration spaces, the so-called minisuperspaces. In this case, the above mathematical difficulties can be circumvented since the WDW equation reduces to a partial differential equation and, in principle, can be integrated. The initial value problem can be approached in some simplified ways as, for example, the so called no boundary condition by Harte and Hawking [4] and the tunneling from nothing by Vilenkin [5]. Both schemes give reasonable laws for initial conditions from which our clas-
sical universe could be started. However also other approaches are possible [6]. However, it is better to stress that Quantum Cosmology is not fully satisfactory in view of solving Quantum Gravity issues but is a useful working scheme despite of different interpretations of results.

For example, the Hartle criterion [7] is an interpretative scheme for the solutions of the WDW equation. Hartle proposed to look for peaks of the wave function of the universe: if it is strongly peaked, we have correlations among the geometrical and matter degrees of freedom; if it is not peaked, correlations are lost. In the first case, the emergence of classical trajectories (i.e. universes) is expected. The analogy to non-relativistic Quantum Mechanics is straightforward. If we have a wave function, solution of the Schrödinger equation in presence of a potential barrier, an oscillatory regime is possible on and outside the barrier; a decreasing exponential behavior is present under the barrier. The situation is analogous in Quantum Cosmology: now the potential barrier has to be replaced by the superpotential $U(h^{ij}, \phi)$, where $h^{ij}$ are the components of the 3-metric of geometrodynamics and $\phi$ is a generic scalar field describing the matter content. More precisely, the wave function of the universe can be written as

$$\Psi[h_{ij}(x), \phi(x)] \sim e^{-m_p^2 S},$$

where $m_p$ is the Planck mass and

$$S \equiv S_0 + m_p^2 S_1 + O(m_p^{-4}),$$

is the action which can be expanded. We have to note that there is no normalization factor due to the lack of a probability interpretative scheme. Inserting $S$ into the WDW equation (that we will derive below) and equating similar powers of $m_p$, one obtains the Hamilton-Jacobi equation for $S_0$. Similarly, one gets equations for $S_1, S_2, \ldots$, which can be solved considering results of previous orders. We need only $S_0$ to recover the semi-classical limit of Quantum Cosmology [8]. If $S_0$ is a real number, we get oscillating WKB modes and the Hartle criterion is recovered since $\Psi$ is peaked on a phase-space region defined by

$$\pi_{ij} = m_p \frac{\delta S_0}{\delta h^{ij}}, \quad \pi_\phi = m_p \frac{\delta S_0}{\delta \phi},$$

where $\pi_{ij}$ and $\pi_\phi$ are classical momenta conjugates to $h^{ij}$ and $\phi$. The semi-classical region of superspace, where $\Psi$ has an oscillating structure, is the Lorentz one otherwise it is Euclidean. In the latter case, we have $S = 1I$ and

$$\Psi \sim e^{-m_p^2 I},$$

where $I$ is the action for the Euclidean solutions of classical field equations (instantons). This scheme, at least at semi classical level, solves the problem of initial conditions. Given an action $S_0$, Eqs. (3) imply $n$ free parameters (one for each dimension of the configuration space $Q \equiv \{h^{ij}, \phi\}$) and then $n$ first integrals of motion. The general solution of the field equations implies $2n - 1$ parameters (one for any Hamilton equation plus the energy constraint). As a consequence, the wave function is peaked on a subset of the general solution and the boundary conditions on the wave function (e.g. Harte-Hawking or Vilenkin one) imply initial conditions for the classical solutions. The issue is now to search for a method capable of selecting these constants of motion. Alternatively, can the Hartle criterion (and the emergence of classical trajectories) be implemented by some general approach without arbitrarily choosing regions of the phase-space where momenta $\pi$ are constant? In this review we discuss this question considering the Hamiltonian formalism for Extended Gravity Cosmologies. We want to show that the existence of Noether symmetries implies a subset of the general solution of the WDW equation where the oscillating behaviors are selected. Viceversa, the Hartle criterion can be always related to a Noether symmetry and then to the classical trajectories. For classical trajectories, we mean solutions of the standard cosmological equations. In particular we restrict the discussion to minisuperspace models but it is clear that it could work for the complete field theory as soon as the method could be extended to the whole superspace. We remember that Extended Theories of Gravity (ETGs) have recently become a sort of paradigm in the study of gravitational interaction based on corrections and enlargements of the Einstein scheme [9][10][11]. The scheme consists, essentially, in adding higher-order curvature invariants and/or non-minimally coupled scalar fields into dynamics resulting from the effective action of Quantum Gravity [12]. All these models are not the complete theory of Quantum Gravity but are needed as approaches toward it. Furthermore, unification schemes as Superstrings, Supergravity or Grand Unified Theories, consider effective actions where non-minimal couplings to the geometry or higher-order terms come out. These contributions come from one-loop or higher-loop corrections in the high-curvature regimes [13][14]. In addition, these approaches have gained interest in cosmology due to the fact that they naturally exhibit inflationary behaviors [15][16][17][18][19][20][21][22][23].

Besides, ETGs are going to play an interesting role to describe also the today observed Universe. However, the energy and curvature regimes are very different with respect to the primordial epochs. In fact, the good quality data of last years have made it possible to build up a more realistic picture of the observed universe. Type Ia Supernovae (SNeIa) [24], anisotropies in the Cosmic Microwave Background Radiation (CMBR) [25], and matter power spectrum inferred from large galaxy surveys [26] represent evidences for a substantial revision of the Cosmological Standard Model. In particular, the concordance $\Lambda CDM$ model predicts that baryons contribute only for $\sim 4\%$ of the total matter-energy budget, while the cold dark matter (CDM) represents the bulk of the matter content ($\sim 25\%$) and the cosmological constant $\Lambda$ plays the role of the so called "dark energy" ($\sim 70\%$) [27]. Although being the best fit to a wide range of data [28], the $\Lambda CDM$ model is severely affected by strong theoretical shortcomings [29] that have motivated the search for alternative models [30]. Dark energy models mainly rely on the implicit assumption that standard GR is the correct theory of gravity. Nevertheless, its validity on large astrophysical and cosmological scales has never been tested [31], and it is therefore conceivable that both cosmic speed up and missing matter represent signals of a breakdown of gravitation law as we conceive it at small scales. This means that one could consider the possibility that the Hilbert- Einstein Lagrangian, linear in the Ricci scalar $R$, could be generalized in order to cure the "dark side" shortcomings. The simplest choice is a function $f(R)$, that can be encompassed in the ETGs being a "minimal" extension of GR. It has been widely shown that $f(R)$-gravity can easily match the dark energy and dark matter issues at cosmological and astrophysical scales [32][33][34][35][36][37][38][39][40][41][42][43][44][45][46][47][48][49].

\[1\] Up to now, there is no final evidence that dark energy and dark matter exist at fundamental quantum level [52].
In this review paper, we want to discuss the Hamiltonian dynamics for minisuperspace models coming from Extended Theories of Gravity. This problem is of fundamental interest for several reasons. First of all, it is the first step towards the Quantum Cosmology of such theories. Second, it allows to disentangle the further gravitational degrees of freedom defining their role into dynamics, using, in particular, conformal transformations. Finally, Hamiltonian formalism allows to select conserved quantities (Noether charges and currents) that assume a main role in Quantum Cosmology since can be connected to the emergence of classical universes. It is worth noticing that the Noether Symmetry Approach is that we are going to develop here, has been applied to different situations (see e.g. [77, 78, 80, 90, 102]) being a method extremely useful for such exact solutions.

The paper is organized as follows. In Sec. 2 we recall the Hamiltonian formalism approach to GR and the problem of quantization starting from the well-established results of the Arnowitt-Deser-Misner (ADM) formalism. In Sec. 3 we introduce the Minisuperspace Approach to Quantum Cosmology considering also its limits. Sect. 4 is devoted to the Noether Symmetry Approach and to its connection to Quantum Cosmology. The existence of Noether symmetries allows to select conserved momenta that acquire a straightforward relevance in view of the Hartle criterion. Extended Theories of Gravity are introduced in Sec. 5 while conformal transformations are considered in Sec. 6 In Sect. 7 we discuss examples of minisuperspace models coming from EFTGs. It is easy to show that the point-like Hamiltonian coming from the Legendre transformation of the starting minisuperspace Lagrangian can be easily worked out and transformed in the corresponding WDW equation as soon as Noether symmetries are identified. Exact solutions are derived for such minisuperspace models and a singularity classification is presented according to [31, 62]. Moreover, we analyze the conformal equivalence in view of Noether symmetries showing how they transform under conformal transformations (Sec. 5). Discussion and conclusions are drawn in Sect. 8.

2 The Hamiltonian formulation of General Relativity and the problem of quantization

Let us start with a summary of the so-called canonical formulation of GR according to the so called ADM formalism. In order to achieve the Hamiltonian formulation of GR, we have to consider a 3-surface on which the 3-metric is h_{ij}, with matter fields defined on it. The 3-manifold is embedded in a 4-manifold whose metric is g_{uv}. Following the ADM formalism, the embedding is described by the so-called (3+1) form of g_{uv}, that is

\[ ds^2 = g_{uv} dx^u dx^v = - \left( N^2 - N_i N^i \right) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j, \]

where N and N_i are the lapse and shift arbitrary functions and describe the way in which the coordinates on a 3-surface is related to the preceding and following 3-manifold. The action is, for the moment, and standard one of GR minimally coupled to matter, that is

\[ S = \frac{m_p^2}{16\pi} \int_M d^4x \sqrt{-g} \left( R - 2\Lambda \right) + 2 \int_{\partial M} d^3x \sqrt{-h} K + S_{(m)}, \]

where K is the trace of the extrinsic curvature K_{ij} at the boundary \partial M of the 4-manifold M, and is given by

\[ K_{ij} = \frac{1}{2N} \left[ -\frac{\partial h_{ij}}{\partial t} + 2D_i (N_j) \right]. \]

\[ D_i \] represents the covariant derivative on the 3-manifold. The action for the matter scalar field is

\[ S_{(m)} = -\frac{1}{2} \int d^3x \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right], \]

that in term of (3+1)-variables, is

\[ S = \frac{m_p^2}{16\pi} \int d^3x dt N\sqrt{-h} \left[ K_{ij} K^{ij} - K^2 + \frac{1}{4} \left( R - 2\Lambda \right) \right] + S_{(m)}. \]

The Hamiltonian form of the action is then

\[ S = \int d^3x dt \left[ h_{ij} \pi^{ij} + \phi \pi_\phi - N\mathcal{H} - N^i \mathcal{H}_i \right], \]

where \( \pi^{ij} \) and \( \pi_\phi \) are the momenta conjugate to \( h_{ij} \) and \( \phi \) respectively. The momenton constraint is

\[ \mathcal{H}_i = -2D_j \pi^{ij} + \mathcal{H}_i^{(m)} = 0, \]

while the proper Hamiltonian constraint is

\[ \mathcal{H} = \frac{16\pi}{m_p^2} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{m_p^2}{16\pi} \sqrt{-h} \left( R - 2\Lambda \right) + \mathcal{H}^{(m)} = 0, \]

where \( G_{ijkl} \) is the so-called De Witt metric explicitly given by

\[ G_{ijkl} = \frac{1}{2} \sqrt{h} \left( h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right). \]

These constraints correspond to the time-space and time-time components of the Einstein field equations respectively. The canonical quantization procedure is essentially based on them, as we will see below.

The so-called superspace is the framework where classical dynamics takes place: it is the space of all 3-metric and matter field configurations \( (h_{ij}(x), \phi(x)) \) defined on a 3-manifold. It is infinite dimensional, with a finite number of coordinates \( (h_{ij}(x), \phi(x)) \) at every point \( x \) of the 3-manifold. The De Witt metric and metric on the matter fields determine the metric on superspace. It has the important property that its signature is hyperbolic at every point \( x \) in the 3-space. The signature of the De Witt metric does not depend on the signature of standard space-time.
The quantum state of the system can be represented by a wave functional $\Psi[h_{ij}, \phi]$ in the canonical quantization approach. An important characteristic of this wave function is that it does not depend explicitly on the coordinate time $t$. This is because the 3-surfaces are compact, and thus their intrinsic geometry fixes almost uniquely their relative position in the 4-manifold.

Following the Dirac quantization, the wave function is assumed to be annihilated by the classical constraints after they have been "transformed" into operators, that is

$$\pi^{ij} \to -i \frac{\delta}{\delta h_{ij}}, \quad \pi_{\phi} \to -i \frac{\delta}{\delta \phi}.$$  \hspace{1cm} (14)

The equations for $\Psi$ are the momentum constraint

$$\mathcal{H}_0 \Psi = 2iD_j \frac{\delta \Psi}{\delta h_{ij}} + \mathcal{H}_h^{(m)} \Psi = 0,$$  \hspace{1cm} (15)

and the WDW equation

$$\mathcal{H} \Psi = \left[ -G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + \sqrt{h^{(3)}} R - 2\Lambda + \mathcal{H}^{(m)} \right] \Psi = 0.$$  \hspace{1cm} (16)

The momentum constraint implies that the wave function is the same for configurations $(h_{ij}(x), \Phi(x))$ that are related by coordinate transformations in the 3-surface. The momentum constraint $(15)$ is the quantum mechanical expression of the invariance of the theory under 3-dimensional diffeomorphisms. Similarly, the WDW eq. $(16)$ represents the reparameterization invariance of the theory. Such an equation is a second-order hyperbolic functional differential equation describing the dynamical evolution of the wave function in superspace (the Wave Function of the Universe).

Another approach to canonical quantization is to derive the wave function by path integrals. In this case, the wave function is an Euclidean functional integral over a class of 4-metrics and matter fields, weighted by $e^{-\mathcal{L}}$, where $\mathcal{L}$ is the Euclidean action of gravity plus matter fields, that is

$$\Psi[h_{ij}, \phi, \mathcal{L}] = \sum_{\mathcal{M}} \int \mathcal{D}g_{\mu\nu} \mathcal{D}\phi e^{-\mathcal{L}}.$$  \hspace{1cm} (17)

The sum is over a given class of manifolds $\mathcal{M}$ (where $\mathcal{B}$ is their boundary), and over a class of 4-metrics $g_{\mu\nu}$ and matter fields $\phi$ which induce the 3-metric $h_{ij}$ and matter field configuration $\phi$ on the 3-surface $B$. If the 4-manifold has topology $\mathbb{R} \times B$, the path integral assumes the form

$$\Psi[h_{ij}, \phi, \mathcal{L}] = \int \mathcal{D}N^\mu \int \mathcal{D}h_{ij} \mathcal{D}\phi \exp(-I[g_{\mu\nu}, \phi]).$$  \hspace{1cm} (18)

The delta-functional fixes the gauge condition $N^\mu = \chi^\mu$ and $\Delta_N$ is the Faddeev-Popov determinant. The 3-metric and matter field are integrated over a class of paths $(h_{ij}(x, \tau), \phi(x, \tau))$ matching the argument of wave function on the 3-surface $B$ where we can assume $\tau = 1$, that is

$$h_{ij}(x, 1) = \tilde{h}_{ij}(x), \quad \phi(x, 1) = \tilde{\phi}(x).$$  \hspace{1cm} (19)

The paths are fully specified assuming, at the initial point, $\tau = 0$. The WDW equation and momentum constraints, $(15)$, $(16)$ can be considered as a quantum invariance under 4-dimensional diffeomorphisms. The wave functions generated by the path integral $(15)$ have to satisfy the WDW equation and momentum constraints, providing that the path integral is invariantly constructed $(16)$. The solution of the WDW equation is generated by the path integral and strictly depends on how the initial and boundary conditions are chosen; thus the problem of boundary conditions is crucial in canonical quantization.

3 The Minisuperspace Approach to Quantum Cosmology

Since the superspace is infinite dimensional and very difficult to handle, minisuperspaces are restrictions of it where some symmetries are imposed a priori on the metric and the related matter fields. This approach allows to construct useful toy models that, as we said, are not the full Quantum Gravity but can give indications towards it. The simplest minisuperspace consists in restricting to homogeneous and isotropic metrics and matter fields. In general, a minisuperspace involves the 4-metric $\Psi$, a lapse function $N = N(t)$ assumed homogeneous, and the shift function $N^i = 0$ set to zero. One obtains

$$ds^2 = -N^2(t) dt^2 + h_{ij}(x, t) dx^i dx^j.$$  \hspace{1cm} (20)

Being the 3-metric $h_{ij}$ homogeneous, it is described by a finite number of functions of $t$, $q^\alpha(t)$, where $\alpha = 0, 1, 2 \ldots (n - 1)$. Any Bianchi-type cosmology is suitable for such an analysis $(12)$. The Hilbert-Einstein action $(10)$ can be recast as

$$S[h_{ij}, N, N^i] = \frac{m_p^2}{16\pi} \int dt \int d^3x N \sqrt{h} \left[ K_{ij} K^{ij} - K^2 + \delta^{(3)} R - 2\Lambda \right],$$  \hspace{1cm} (21)

and, in general, one gets

$$S[q^\alpha(t), N(t)] = \int_0^1 dt N \left[ \frac{1}{N^2} f_{\alpha\beta}(q) q^\alpha q^\beta - U(q) \right] \equiv \int \mathcal{D}q d\mathcal{L},$$  \hspace{1cm} (22)

where, $f_{\alpha\beta}(q)$ is the reduced De Witt metric, with signature, $(−,+,...)$. The $t$ integration can range from 0 to 1 by shifting $t$ and scaling the lapse function. Including matter also leads to an action of this form, and then the $q^\alpha$ functions include matter variables as well as 3-metric components. The $(−)$ part of the signature corresponds to a gravitational variable.

Eq. $(22)$ has the form of a relativistic point particle action where the particles moves on a n-dimensional curved space-time with a self-interaction potential. The variation with respect to $q^\alpha$, gives the equations of motion

$$\frac{1}{N} \frac{d}{dt} \left( \frac{q^\alpha}{N} \right) + \frac{1}{N^2} \Gamma^\alpha_{\beta\gamma} q^\beta q^\gamma + f^\alpha_{\beta\gamma} \frac{\partial U}{\partial q^\beta} = 0,$$  \hspace{1cm} (23)

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols derived from the metric $f_{\alpha\beta}$. Varying with respect to $N$, one gets

$$\frac{1}{2N^2} f_{\alpha\beta} q^\alpha q^\beta + U(q) = 0,$$  \hspace{1cm} (24)

that is a constraint equation.

Eqs $(23)$ and $(24)$ describe geodesic motion in minisuperspace with a forcing term. The general solution of $(23)$, $(24)$
and the canonical Hamiltonian is
\[ U = \sum_{\alpha} p_\alpha \dot{q}^\alpha - f_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta / N, \] (25)
and the canonical Hamiltonian is
\[ \mathcal{H}_c = p_\alpha \dot{q}^\alpha - \mathcal{L} = N \left[ \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q) \right] \equiv N \mathcal{H}, \] (26)
where \( f^{\alpha\beta}(q) \) is the inverse metric on minisuperspace. The Hamiltonian form of the action is
\[ S = \int_0^1 dt \left[ p_\alpha \dot{q}^\alpha - N \mathcal{H} \right]. \] (27)
This equation means that the lapse function \( N \) is a Lagrange multiplier and then the Hamiltonian constraint has to be
\[ \mathcal{H}(q^\alpha, p_\alpha) = \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q) = 0. \] (28)
This is the minisuperspace reduction equivalent to the Hamiltonian constraint of the full theory \[12\], integrated over the spatial hypersurfaces. Also the momentum constraint \[11\] is identically satisfied.

At this point, the canonical quantization procedure requires a time-independent wave function \( \Psi(q^\alpha) \) that has to be annihilated by the quantum operator corresponding to the classical constraint \[26\]. This fact gives rise to the WDW equation,
\[ \hat{\mathcal{H}}(q^\alpha, -i \frac{\partial}{\partial q^\alpha}) \Psi(q^\alpha) = 0. \] (29)
Since the metric \( f^{\alpha\beta} \) depends on \( q \) there is a factor ordering issue in \[29\]. This may be solved by requiring that the quantization procedure is covariant in minisuperspace, that is unchanged by field redefinitions of the 3-metric and matter fields, \( q^\alpha \rightarrow \tilde{q}^\alpha(q^\alpha) \). This fact restricts the possible operator orderings to
\[ \hat{\mathcal{H}} = -\frac{1}{2} \nabla^2 + \xi \mathcal{R} + U(q), \] (30)
where \( \nabla^2 \) and \( \mathcal{R} \) are the Laplacian and curvature of the minisuperspace metric \( f_{\alpha\beta} \) and \( \xi \) is an arbitrary constant.

The constant \( \xi \) is fixed as soon as the minisuperspace metric is defined by the form of the action up to a conformal factor. From a classical viewpoint, the constraint \[25\] can be multiplied by an arbitrary function of \( q, \Omega^{-2}(q) \), and the constraint is identical in form but has metric \( f_{\alpha\beta} = \Omega^2 f_{\alpha\beta} \) and potential \( \tilde{U} = \Omega^{-2} U \). The same is true in the actions \[22\] and \[27\] if one rescales the lapse function, \( N \rightarrow \tilde{N} = \Omega^{-2} N \). However, the quantum theory should also be insensitive to such rescaling. This is achieved if the metric dependent part of the operator \[29\] is conformally covariant, that is the constant \( \xi \) is the conformal coupling
\[ \xi = -\frac{n-2}{8(n-1)}, \] (31)
for \( n \geq 2 \), where \( n \) is the space dimension \[8, 50\].

The wave function of the universe can be obtained also by the path integral formalism. However, we do not consider such an approach any more here referring the interested reader to Ref.\[8\].

Before concluding this section, an important issue has to be addressed. It is how to interpret the probability measure in Quantum Cosmology. In fact, given a wave function \( \Psi(q^\alpha) \), defined in a minisuperspace, one needs to a probability measure. The question is to define a suitable probability measure. The WDW equation is a sort of Klein-Gordon equation and a current can be defined as
\[ J = i \left( \Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right). \] (32)
It is conserved and satisfies the relation
\[ \nabla \cdot J = 0, \] (33)
thanks to the structure of the WDW equation. As in the case of the Klein-Gordon equation (and, in general, of hyperbolic equations), the probability derived from such a conserved current can be affected by negative probabilities. Due to this shortcoming, the correct measure to use should be
\[ dP = |\Psi(q^\alpha)|^2 dV, \] (34)
where \( dV \) is a volume element of minisuperspace. Also this assumption can be problematic since one of the coordinates \( q^\alpha \) is "time", so that \[33\] is the analogue of interpreting \( |\Psi(x, t)|^2 \) in ordinary quantum mechanics as the probability of finding the particle in the space-time interval \( dx dt \). This means that a careful discussion on the meaning of time in Quantum Cosmology has to be pursued. For details and alternative proposals see \[34\].

### 4 The Noether Symmetry Approach

As we said before, minisuperspaces are restrictions of the superspace of geometrodynamics. They are finite-dimensional configuration spaces on which point-like Lagrangians can be defined. Cosmological models of physical interest can be defined on such minisuperspaces (e.g., Bianchi models). According to the above discussion, a crucial role is played by the conserved currents that allow to interpret the probability measure and then the physical quantities obtained in Quantum Cosmology. In this context, the search for general methods to achieve conserved quantities and symmetries become relevant. The so-called Noether Symmetry Approach \[36\], as we will show, can be extremely useful to this purpose.

Before taking into account specific models, let us remind some properties of the Lie derivative and the derivation of the Noether theorem \[52\]. Let \( L_X \) be the Lie derivative
\[ (L_X \omega)(\xi) = \frac{d}{dt} \omega(g^t, \xi), \] (35)
where \( \omega \) is a differential form of \( \mathbb{R}^n \) defined on the vector field \( \xi \), \( g^t \) is the differential of the phase flux \( \{ g \} \) given by the vector field \( X \) on a differential manifold \( \mathcal{M} \). Let \( p_t = p_{g^t} \) be the action of a one-parameter group able to act on functions, vectors and forms on the vector spaces \( C^\infty(\mathcal{M}), D(\mathcal{M}) \), and \( \Lambda(\mathcal{M}) \) constructed starting from \( \mathcal{M} \). If \( g_t \) takes the point \( m \in \mathcal{M} \)
$M$ in $g_t(m)$, then $\rho_t$ takes back on $m$ the vectors and the forms defined on $g_t(m)$; $\rho_t$ is a pull back $^{[55]}$. Then the properties
\[ \rho_{t+t} = \rho_t \rho_s, \] (36)
holds since
\[ g_{t+t} = g_t \circ g_s. \] (37)
On the functions $f, g \in C^\infty(M)$ we have
\[ \rho_t(fg) = (\rho_t f)(\rho_t g); \] (38)
on the vectors $X, Y \in D(M)$,
\[ \rho_t [X, Y] = [\rho_t X, \rho_t Y]; \] (39)
on the forms $\omega, \mu \in \Lambda(M)$
\[ \rho_t (\omega \wedge \mu) = (\rho_t \omega) \wedge (\rho_t \mu). \] (40)
$L_X$ is the infinitesimal generator of the one–parameter group $\rho_t$, and, being a derivative on the algebra $C^\infty(M)$, $D(M)$, and $\Lambda(M)$, the following properties have to hold
\[ aL_X(fg) = (L_X f)g + f(L_X g), \] (41)
\[ L_X[Y, Z] = [L_X Y, Z] + [Y, L_X Z], \] (42)
\[ L_X(\omega \wedge \mu) = (L_X \omega) \wedge \mu + \omega \wedge (L_X \mu), \] (43)
which are nothing else but the Leibniz rules for functions, vectors and differential forms, respectively. Moreover,
\[ L_X f = X f, \] (44)
\[ L_X Y = dX(Y) = [X, Y], \] (45)
\[ L_X d\omega = dL_X \omega, \] (46)
where $ad$ is the self–adjoint operator and $d$ is the external derivative by which a $p$–form becomes a $(p+1)$–form.
The discussion can be specified by considering a Lagrangian $L$ which is a function defined on the tangent space of configuration $TQ \equiv \{q_i, \dot{q}_i\}$. In this case, the vector field $X$ is
\[ X = \alpha^i(q) \frac{\partial}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial}{\partial \dot{q}^i}, \] (47)
where dot means derivative with respect to $t$, and
\[ L_X L = X L = \alpha^i(q) \frac{\partial L}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial L}{\partial \dot{q}^i}. \] (48)
The condition
\[ L_X L = 0, \] (49)
implies that the phase flux is conserved along $X$; this means that a constant of motion exists for $L$ and the Noether theorem holds. In fact, taking into account the Euler-Lagrange equations
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \] (50)
and defining the inner derivative
\[ i_X \theta_L = \langle \theta_L, X \rangle, \] (54)
we get, as above,
\[ i_X \theta_L = \Sigma_0, \] (55)
if condition $^{[19]}$ holds. This representation is useful to identify cyclic variables. Using a point transformation on vector field $^{[47]}$, it is possible to get
\[ \dot{X} = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \left[ \frac{d}{dt} (i_X dQ^k) \right] \frac{\partial}{\partial \dot{Q}^k}. \] (56)
If $X$ is a symmetry also $\dot{X}$ has this property, then it is always possible to choose a coordinate transformation so that
\[ i_X dQ^1 = 1, \quad i_X dQ^i = 0, \quad i \neq 1, \] (57)
and then
\[ \dot{X} = \frac{\partial}{\partial Q^1}, \quad \frac{\partial}{\partial \dot{Q}^1} = 0. \] (58)
It is evident that $Q^1$ is the cyclic coordinate and the dynamics can be reduced $^{[54]}$. However, the change of coordinates is not unique and a clever choice is always important. Furthermore, it is possible that more symmetries are found. In this case more cyclic variables exists. For example, if $X_1, X_2$ are the Noether vector fields and they commute, $[X_1, X_2] = 0$, we obtain two cyclic coordinates by solving the system
\[ i_{X_1} dQ^1 = 1, \quad i_{X_2} dQ^1 = 1, \] (59)
\[ i_{X_1} dQ^1 = 0, \quad i \neq 1; \quad i_{X_2} dQ^1 = 0, \quad i \neq 2. \]
If they do not commute, this procedure does not work since commutation relations are preserved by diffeomorphisms. In this case
\[ X_3 = [X_1, X_2], \] (60)
is again a symmetry since
\[ L_{X_3} L = L_{X_1} L_{X_2} L - L_{X_2} L_{X_1} L = 0. \] (61)
If $X_3$ is independent of $X_1, X_2$ we can go on until the vector fields close the Lie algebra $^{[56]}$. A reduction procedure by cyclic coordinates can be implemented in three steps: i) we choose a symmetry and obtain new coordinates as above. After this first reduction, we get a new Lagrangian $\tilde{L}$ with a cyclic coordinate; ii) we search for new symmetries in this new space and apply the reduction technique until it is possible; iii) the process stops if we select a pure kinetic Lagrangian where all coordinates are cyclic. This case is not very common and often it is not physically relevant. Going back to the point of view interesting in Quantum Cosmology, any symmetry selects a constant of motion. Alternatively, using the Cartan one–form
\[ \theta_L \equiv \frac{\partial L}{\partial \dot{q}^i} dq^i, \] (53)
and the non–degenerate transformation
\[ Q' = Q'(q), \quad \dot{Q}'(q) = \frac{\partial Q'}{\partial q} dq^i, \] (54)
is performed. However the Jacobian determinant $\mathcal{J} = \| \frac{\partial Q'}{\partial q} \| \dot{q}^i$ has to be non–zero.
constant conjugate momentum since, by the Euler-Lagrange equations
\[ \frac{\partial \mathcal{L}}{\partial q^i} = 0 \iff \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \Sigma_i, \]
Vice versa, the existence of a constant conjugate momentum means that a cyclic variable has to exist. In other words, a Noether symmetry exists.

Further remarks on the form of the Lagrangian \( \mathcal{L} \) are necessary at this point. We shall take into account time-independent, non-degenerate Lagrangians \( \mathcal{L} = \mathcal{L}(q^i, \dot{q}^i) \), i.e.
\[ \frac{\partial \mathcal{L}}{\partial t} = 0, \quad \det H_{ij} \equiv \det \left| \frac{\partial^2 \mathcal{L}}{\partial q^i \partial q^j} \right| \neq 0, \]
where \( H_{ij} \) is the Hessian. As in usual analytic mechanics, \( \mathcal{L} \) can be set in the form
\[ \mathcal{L} = T(q^i, \dot{q}^i) - V(q^i), \]
where \( T \) is a positive-defined quadratic form in the \( \dot{q}^i \) and \( V(q^i) \) is a potential term. The energy function associated with \( \mathcal{L} \) is
\[ E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L}(q^i, \dot{q}^i), \]
and by the Legendre transformations
\[ \mathcal{H} = \pi_j \dot{q}^j - \mathcal{L}(q^i, \dot{q}^i), \quad \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}, \]
we get the Hamiltonian function and the conjugate momenta. Considering again the symmetry, the condition \( \mathcal{L} \) and the vector field \( X \) in Eq. \( 17 \) give a homogeneous polynomial of second degree in the velocities plus an inhomogeneous term in the \( q^i \). Due to \( 19 \), such a polynomial has to be identically zero and then each coefficient must be independently zero. If \( n \) is the dimension of the configuration space (i.e. the dimension of the minisuperspace), we get \( 1 + n(n+1)/2 \) partial differential equations whose solutions assign the symmetry, as we shall see below. Such a symmetry is over-determined and, if a solution exists, it is expressed in terms of integration constants instead of boundary conditions. In the Hamiltonian formalism, we have
\[ [\Sigma_j, \mathcal{H}] = 0, \quad 1 \leq j \leq m, \]
as it must be for conserved momenta in quantum mechanics and the Hamiltonian has to satisfy the relations
\[ L_{\Gamma} \mathcal{H} = 0, \]
in order to obtain a Noether symmetry. The vector \( \Gamma \) is defined by \( 55 \)
\[ \Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{\dot{q}}^i \frac{\partial}{\partial \dot{q}^i}. \]
These considerations can be applied to the minisuperspace models of Quantum Cosmology and to the interpretation of the wave function of the universe. As discussed above, by a straightforward canonical quantization procedure, we have
\[ \pi_j \rightarrow \hat{\pi}_j = -i \partial_j, \quad \mathcal{H} \rightarrow \hat{\mathcal{H}}(q^i, -i \partial_{\dot{q}^i}). \]
It is well known that the Hamiltonian constraint gives the WDW equation, so that if \( |\Psi> \) is a state of the system (i.e. the wave function of the universe), dynamics is given by
\[ \mathcal{H} |\Psi> = 0, \]
where we write the WDW equation in an operatorial way. If a Noether symmetry exists, the reduction procedure outlined above can be applied and then, from \( 121 \) and \( 56 \), we get
\[ \pi_1 \equiv \frac{\partial \mathcal{L}}{\partial Q^1} = i L_{\Sigma_1}, \quad \pi_2 \equiv \frac{\partial \mathcal{L}}{\partial Q^2} = i L_{\Sigma_2}, \]
\[ \ldots \ldots \]
depending on the number of Noether symmetries. After quantization, we get
\[ -i \partial_l |\Psi> = \Sigma_l |\Psi> , \]
\[ -i \partial_2 |\Psi> = \Sigma_2 |\Psi> , \]
\[ \ldots \ldots \]
which are nothing else but translations along the \( Q^l \) axis singled out by corresponding symmetry. Eqs. \( 24 \) can be immediately integrated and, being \( \Sigma_1 \) real constants, we obtain oscillatory behaviors for \( |\Psi> \) in the directions of symmetries, i.e.
\[ |\Psi> = \sum_{l=1}^{m} e^{i \Sigma_l Q^l} |\chi(Q^l)> , \quad m \leq l \leq n, \]
where \( m \) is the number of symmetries, \( l \) are the directions where symmetries do not exist, \( n \) is the total dimension of minisuperspace. Vice versa, dynamics given by \( 24 \) can be reduced by \( 24 \) if and only if it is possible to define constant conjugate momenta as in \( 23 \), that is oscillatory behaviors of a subset of solutions \( |\Psi> \) exist only if Noether symmetry exists for dynamics.

The \( m \) symmetries give first integrals of motion and then the possibility to select classical trajectories. In one and two-dimensional minisuperspaces, the existence of a Noether symmetry allows the complete solution of the problem and to get the full semi-classical limit of Quantum Cosmology \( 54 \). In conclusion, we can state that in the semi-classical limit of quantum cosmology, the reduction procedure of dynamics, connected to the existence of Noether symmetries, allows to select a subset of the solution of WDW equation where oscillatory behaviors are found. This fact, in the framework of the Hartle interpretative criterion of the wave function of the universe, gives conserved momenta and trajectories which can be interpreted as classical cosmological solutions. Vice-versa, if a subset of the solution of WDW equation has an oscillatory behavior, due to Eq. \( 19 \), conserved momenta exist and Noether symmetries are present. In other words, Noether symmetries select classical universes.

In what follows, we will show that such a statement holds for general classes of minisuperspaces and allows to select exact classical solutions. In this sense, the presence of Noether symmetries is a selection criterion for classical universes. Before this, let us discuss the general problem of Extended Theories of Gravity and their conformal properties. As we will see, most of theories of gravity can be conformally related to the Einstein one plus a suitable number of scalar fields. In this sense, the above standard minisuperspace approach works for any theory of gravity.

5 Extending General Relativity

In Sect.1, we discussed several issues, coming from fundamental physics, astrophysics and cosmology, that lead to take into
account effective theories where the gravitational action has to be generalized with respect to the standard Hilbert-Einstein one. In Quantum Cosmology, the question of the effective action of gravity is crucial since, in general, we do not know the initial conditions from which our classical, observed universe emerged. This means that general criteria to study minisuperspace models coming from Extended Gravity are extremely relevant towards a full theory of Quantum Gravity.

In this section, without pretending to be complete, we outline the main features of higher-order and scalar-tensor gravity as examples of Extended Theories of Gravity. For a detailed discussion, see [10, 11].

We will consider two main features:

- first, the geometry can couple non-minimally to some scalar field;
- second, derivatives of the metric components of order higher than second may appear.

In the first case, we say that we have scalar-tensor gravity, and in the second case we have higher order theories. Combinations of non-minimally coupled and higher order terms can also emerge in effective Lagrangians, producing mixed higher order/scalar-tensor gravity [10, 11, 59]. A general class of higher-order scalar-tensor theories in four dimensions is given by the action

\[ S = \int d^4x \sqrt{-g} \left[ L(R, \Box R, \Box^2 R, \ldots, \Box^k R, \phi) + \frac{\epsilon}{2} \nabla_\mu \phi \nabla_\nu \phi + \mathcal{L}^{(m)} \right], \]  

(76)

where \( F \) is an unspecified function of curvature invariants and of a scalar field \( \phi \). The term \( \mathcal{L}^{(m)} \), as above, is the minimally coupled ordinary matter contribution, considered here as a perfect fluid; \( \epsilon \) is a constant which specifies the theory. Actually its values can be \( \epsilon = \pm 1 \), 0 fixing the nature and the dynamics of the scalar field which can be a standard scalar field, a phantom field or a field without dynamics (see [10, 11, 60, 61, 62] for details). In the metric approach, the field equations are obtained by varying with respect to \( g_{\mu\nu} \). We get

\[ G^{\mu\nu} = \frac{1}{2} \left[ \kappa T^{\mu\nu}(F - GR) + \frac{1}{2} \epsilon T^{\mu\nu} \right] + \frac{1}{2} \sum_{i=1}^{n} (g^{\mu\lambda} \frac{\partial F}{\partial \Box_i R}) g^{\lambda\sigma} \Box_i R \sigma + \frac{1}{2} \sum_{i=1}^{n} (g^{\mu\lambda} \frac{\partial F}{\partial \Box_i R}) g^{\lambda\sigma} (\Box_i R)_\sigma \]  

(77)

where \( G^{\mu\nu} \) is the above Einstein tensor and

\[ G \equiv \sum_{j=0}^{n} \Box_j \left( \frac{\partial F}{\partial \Box_j R} \right) . \]  

(78)

The differential Eqs. (77) are of order \((2k + 4)\). The stress-energy tensor is due to the kinetic part of the scalar field and to the ordinary matter:

\[ T_{\mu\nu} = T^{(m)}_{\mu\nu} + \frac{\epsilon}{2} \phi,_{\mu} \phi,_{\nu} - \frac{1}{2} \phi^2 \phi,_{\mu} \phi,_{\nu} \]  

(79)

The (possible) contribution of a potential \( V(\phi) \) is contained in the definition of \( F \). From now on, we shall indicate by a capital \( F \) a Lagrangian density containing also the contribution of a potential \( V(\phi) \) and by \( F(\phi), f(R) \), or \( f(R, \Box R) \) a function of such fields without potential.

By varying with respect to the scalar field \( \phi \), we obtain the Klein-Gordon equation

\[ \epsilon \Box \phi = - \frac{\partial F}{\partial \phi}. \]  

(80)

The simplest extension of GR is achieved assuming,

\[ F = f(R) \]  

(81)

in the action. The standard Hilbert-Einstein action is, of course, recovered for \( f(R) = R \). Varying with respect to \( g_{\alpha\beta} \), we get

\[ f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f'(R) + \frac{1}{2} \epsilon \Box \phi f'(R) \]  

(82)

and, after some manipulations,

\[ g_{\mu\nu} = \frac{1}{f'(R)} \left\{ \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R) + \frac{\epsilon}{2} \nabla_\mu \nabla_\nu \epsilon - \frac{1}{2} \Box f'(R) \right\} , \]  

(83)

where the gravitational contribution due to higher-order terms can be simply reinterpreted as a stress-energy tensor contribution. This means that additional and higher-order terms in the gravitational action act, in principle, as a stress-energy tensor, related to the form of \( f(R) \). Considering also the standard perfect-fluid matter contribution, we have

\[ G_{\alpha\beta} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\alpha\beta} \left[ f'(R) - R f'(R) \right] + \frac{\epsilon}{2} \Box f'(R) + \frac{\epsilon}{2} \nabla_\alpha \nabla_\beta f'(R) \right\} + \frac{\kappa T^{(m)}_{\alpha\beta}}{f'(R)} = \frac{T_{\alpha\beta}^{(\text{curv})}}{f'(R)} + \frac{\epsilon}{2} \Box f'(R) \]  

(84)

where \( T_{\alpha\beta}^{(\text{curv})} \) is an effective stress-energy tensor constructed by the extra curvature terms. In the case of GR, \( T_{\alpha\beta}^{(\text{curv})} \) identically vanishes while the standard, minimal coupling is recovered for the matter contribution. The peculiar behaviour of \( f(R) = R \) is due to the particular form of the Lagrangian itself which, even though it is a second order Lagrangian, can be non-covariantly rewritten as the sum of a first order Lagrangian plus a pure divergence term. The Hilbert-Einstein Lagrangian can be in fact recast as follows:

\[ L_{\text{HE}} = L_{\text{HE}} \sqrt{-g} = \left[ p_{\alpha\beta} (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}) + \nabla_\alpha p_{\alpha\beta} \nabla_\beta \right], \]  

(85)

where:

\[ p_{\alpha\beta} = \sqrt{-g} a_{\alpha\beta} = \frac{\partial L}{\partial R_{\alpha\beta}} . \]  

(86)
where a quantity constructed out with the variation of \( \Gamma \). Since \( u_{\alpha \beta} \) is not a tensor, the above expression is not covariant; however a standard procedure has been studied to recast covariance in the first order theories. This clearly shows that the field equations should consequently be second order and the Hilbert-Einstein Lagrangian is thus degenerate.

From the action \([64]\), it is possible to obtain another interesting case by choosing

\[
F = F(\phi) R - V(\phi), \quad \epsilon = -1.
\]

In this case, we get

\[
S = \int V(\phi) \left[ F(\phi) R - \frac{1}{2} g^{\mu \nu} \phi_{, \mu} \phi_{, \nu} - V(\phi) \right], \quad (88)
\]

\( V(\phi) \) and \( F(\phi) \) are generic functions describing respectively the potential and the coupling of a scalar field \( \phi \). The Brans-Dicke theory of gravity is a particular case of the action \([85]\) for \( V(\phi) = 0 \). The variation with respect to \( g_{\mu \nu} \) gives the second-order field equations

\[
F(\phi) G_{\mu \nu} = F(\phi) \left[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \right] = -\frac{1}{2} T_{\mu \nu}^{(\phi)} - g_{\mu \nu} \nabla_{\sigma} F(\phi) - F(\phi)_{, \mu \nu}, \quad (89)
\]

here \( \nabla_{\sigma} \) is the d’Alember operator with respect to the metric \( g \). The energy-momentum tensor relative to the scalar field is

\[
T_{\mu \nu}^{(\phi)} = \phi_{, \mu} \phi_{, \nu} - \frac{1}{2} g_{\mu \nu} \phi_{, \sigma} \phi_{, \sigma} + g_{\mu \nu} V(\phi). \quad (90)
\]

The variation with respect to \( \phi \) provides the Klein-Gordon equation, i.e. the field equation for the scalar field:

\[
\square \phi - R F_{\phi}(\phi) + V_{\phi}(\phi) = 0, \quad (91)
\]

where \( F_{\phi}(\phi) = \frac{dF(\phi)}{d\phi} \) and \( V_{\phi}(\phi) = \frac{dV(\phi)}{d\phi} \). This last equation is equivalent to the Bianchi contracted identity \([85]\).

### 6 Conformal Transformations

Conformal transformations are mathematical tools that are very useful in Extended Theories of Gravity in order to disentangle the further gravitational degrees of freedom coming from extended actions (see \([65, 66, 67]\) for reviews). In Quantum Cosmology, they are transformations configuration spaces in minisuperspace models. The idea is to perform a conformal rescaling of the spacetime metric \( g_{\mu \nu} \rightarrow \tilde{g}_{\mu \nu} \). Often a scalar field is present in the theory and the metric rescaling is accompanied by a (nonlinear) redefinition of this field \( \phi \rightarrow \tilde{\phi} \). New dynamical variables \( \{ \tilde{g}_{\mu \nu}, \tilde{\phi} \} \) are thus obtained. The scalar field redefinition serves the purpose of casting the kinetic energy density of this field in canonical form. The new set of variables \( \{ \tilde{g}_{\mu \nu}, \tilde{\phi} \} \) is called the Einstein conformal frame, while \( \{ g_{\mu \nu}, \phi \} \) constitute the Jordan frame. When a scalar degree of freedom \( \phi \) is present in the theory, as in scalar tensor or \( f(R) \) gravity, it generates the transformation to the Einstein frame in the sense that the rescaling is completely determined by a function of \( \phi \). In principle, infinitely many conformal frames could be introduced, giving rise to as many representations of the theory.

Let the pair \( \{ M, g_{\mu \nu} \} \) be a spacetime, with \( M \) a smooth manifold of dimension \( n \geq 2 \) and \( g_{\mu \nu} \) a Lorentzian or Riemannian metric on \( M \). The point-dependent rescaling of the metric tensor

\[
g_{\mu \nu} \rightarrow \tilde{g}_{\mu \nu} = \Omega^2 g_{\mu \nu}, \quad (92)
\]

where the conformal factor \( \Omega(x) \) is a nowhere vanishing, regular function, is called a Weyl or conformal transformation. Due to this metric rescaling, the lengths of spacelike and timelike vectors are changed, while null vectors and null intervals of the metric \( g_{\mu \nu} \) remain null in the rescaled metric \( \tilde{g}_{\mu \nu} \). The light cones are left unchanged by the transformation \([92]\) and the spacetimes \( \{ M, g_{\mu \nu} \} \) and \( \{ M, \tilde{g}_{\mu \nu} \} \) exhibit the same causal structure; the converse is also true \([83]\). A vector that is timelike, spacelike, or null with respect to the metric \( g_{\mu \nu} \) has the same character with respect to \( \tilde{g}_{\mu \nu} \), and vice versa.

In the ADM decomposition of the metric, using the lapse function \( N \) and the shift vector \( N^i \), the transformation properties of these quantities follow from Eq. \([93]\):

\[
\tilde{N} = \Omega N, \quad \tilde{N}^i = N^i, \quad \tilde{h}_{ij} = \Omega^2 h_{ij}. \quad (93)
\]

The ADM mass of an asymptotically flat spacetime \([74]\) does not change under the conformal transformation and scalar field redefinition \([73]\). The transformation properties of various geometrical quantities are useful \([74, 75]\). Some of them are:

\[
\tilde{g}^{\mu \nu} = \Omega^{-2} g^{\mu \nu}, \quad \tilde{g} = \Omega^{2n} g, \quad (94)
\]

for the inverse metric and the metric determinant,

\[
\Gamma^{\alpha \beta \gamma}_{\mu} \rightarrow \Gamma^{\alpha \beta \gamma}_{\mu} + \Omega^{-1} \left( \delta^{\alpha}_{\beta} \nabla_{\gamma} \Omega + \delta^{\alpha}_{\gamma} \nabla_{\beta} \Omega - g_{\beta \gamma} \nabla^{\alpha} \Omega \right), \quad (95)
\]

for the Christoffel symbols,

\[
\tilde{R}_{\alpha \beta \gamma}^{\delta} = R_{\alpha \beta \gamma}^{\delta} + 2 \Omega g^{\alpha \sigma} \nabla_{\beta} \nabla_{\gamma} \Omega \nabla_{\sigma} (\ln \Omega) + 2 \nabla_{\alpha} (\ln \Omega) g^{\alpha \beta} \nabla_{\gamma} \nabla_{\delta} \Omega + 2 \nabla_{\alpha} (\ln \Omega) \delta^{\alpha}_{\beta} \tilde{x}^{\gamma} \Omega \nabla_{\delta} (\ln \Omega) + 2 \nabla_{\alpha} (\ln \Omega) \delta^{\alpha}_{\delta} \tilde{x}^{\gamma} \Omega \nabla_{\beta} (\ln \Omega), \quad (96)
\]

for the Riemann tensor,

\[
\tilde{R}_{\alpha \beta} = R_{\alpha \beta} - (n - 2) \nabla_{\alpha} \nabla_{\beta} (\ln \Omega) + (n - 2) \nabla_{\alpha} \nabla_{\beta} (\ln \Omega) + \nabla_{\alpha} g^{\alpha \sigma} \nabla_{\sigma} \nabla_{\beta} (\ln \Omega) + \nabla_{\alpha} g^{\alpha \sigma} \nabla_{\sigma} \nabla_{\beta} (\ln \Omega), \quad (97)
\]

for the Ricci tensor, and

\[
\tilde{R} \equiv \tilde{g}^{\alpha \beta} \tilde{R}_{\alpha \beta} = \Omega^{-2} \left[ R - 2(n - 1) \square (\ln \Omega) + (n - 1)(n - 2) \frac{\square \nabla_{\alpha} \nabla_{\beta} (\ln \Omega)}{\Omega^2} \right], \quad (98)
\]

\footnote{See \([60, 70, 71]\) for the possibility of continuation beyond singular points of the conformal factor.}
for the Ricci scalar. In the case of \( n = 4 \) space-time dimensions, the transformation property of the Ricci scalar can be written as
\[
\tilde{R} = \Omega^{-2} \left[ R - \frac{\delta g_{\alpha\beta} \Omega}{\sqrt{\Omega}} \right] = \Omega^{-2} \left[ R - \frac{12 \sqrt{\Omega} (\nabla\Omega)}{\Omega} + 3 \frac{g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega}{\Omega^2} \right].
\] (99)
The Weyl tensor \( C_{\alpha\beta\gamma} \) with the last index contravariant is conformally invariant,
\[
\tilde{C}_{\alpha\beta\gamma} = C_{\alpha\beta\gamma},
\] (100)
but the same tensor with indices raised or lowered with respect to \( C_{\alpha\beta\gamma} \) is not. This property explains the name \textit{conformal tensor} used for \( C_{\alpha\beta\gamma} \). If the original metric \( g_{\alpha\beta} \) is Ricci-flat (i.e., \( R_{\alpha\beta} = 0 \)), the conformally transformed metric \( \tilde{g}_{\alpha\beta} \) is not Eq. (77). In the conformally transformed world the conformal factor \( \Omega \) plays the role of an effective form of matter and this fact has consequences for the physical interpretation of the theory. A vacuum metric in the Jordan frame is not such in the Einstein frame, and the interpretation of what is matter and what is gravity becomes frame-dependent [77]. However, if the Weyl tensor vanishes in one frame, it also vanishes in the conformally related frame. Conformally flat metrics are mapped into conformally flat metrics, a property used in cosmology when mapping FRW universes (which are conformally flat) into each other. In particular, de Sitter spaces with scale factor \( a(t) = a_0 \exp(\dot{H}t) \) and a constant scalar field as the material source are mapped into similar de Sitter spaces. Since, in general, tensorial quantities are not invariant under conformal transformations, neither are the tensorial equations describing geometry and physics. An equation involving a tensor field \( \psi \) is said to be \textit{conformally invariant} if there exists a number \( w \) (the \textit{conformal weight} of \( \psi \)) such that, if \( \psi \) is a solution of a tensor equation with the metric \( g_{\alpha\beta} \) and the associated geometrical quantities, \( \tilde{\psi} = \Omega^w \psi \) is a solution of the corresponding equation with the metric \( \tilde{g}_{\alpha\beta} \) and the associated geometrical quantities. In addition to geometric quantities, one needs to consider the behavior of common forms of matter under conformal transformations. It goes without saying that most forms of matter or fields are not conformally invariant: invariance under conformal transformations is a very special property. In general, the covariant conservation equation for a (symmetric) stress-energy tensor \( T_{\alpha\beta}^{(m)} \) representing ordinary matter,
\[
\nabla^\beta T_{\alpha\beta}^{(m)} = 0,
\] (101)
is not conformally invariant [75]. The conformally transformed \( \tilde{T}_{\alpha\beta}^{(m)} \) satisfies the equation
\[
\nabla^\beta \tilde{T}_{\alpha\beta}^{(m)} = -\tilde{T}^{\alpha\beta} \nabla_\beta (\ln \Omega).
\] (102)
Clearly, the conservation equation (101) is conformally invariant only for a matter component that has vanishing trace \( T^{(m)} \) of the energy-momentum tensor. This feature is associated with light-like behavior; examples are the electromagnetic field and a radiative fluid with equation of state \( P^{(m)} = \rho^{(m)}/3 \). Unless \( T^{(m)} = 0 \), Eq. (102) describes an exchange of energy and momentum between matter and the scalar field \( \Omega \), reflecting the fact that matter and the geometric factor \( \Omega \) are directly coupled in the Einstein frame description. Since the geodesic equation ruling the motion of free particles in GR can be derived from the conservation equation (101) (geodesic hypothesis), it follows that timelike geodesics of the original metric \( g_{\alpha\beta} \) are not geodesics of the rescaled metric \( \tilde{g}_{\alpha\beta} \) and \textit{vice-versa}. Particles in free fall in the world \( (M, g_{\alpha\beta}) \) are subject to a force proportional to the gradient \( \nabla\Omega \) in the rescaled world \( (M, \tilde{g}_{\alpha\beta}) \) (this is often identified as a fifth force acting on all massive particles and, therefore, it can be said that no massive test particles exist in the Einstein frame). The stress-energy tensor definition in terms of the matter action \( S^{(m)} = \int d^4x \sqrt{-g} \tilde{L}^{(m)} \),
\[
\tilde{T}_{\alpha\beta}^{(m)} = -\frac{2}{\sqrt{-g}} \frac{\delta \tilde{L}^{(m)}}{\delta g^{\alpha\beta}},
\] (103)
together with the rescaling \( \tilde{g}_{\alpha\beta} \) of the metric, yields
\[
\tilde{T}_{\alpha\beta}^{(m)} = \Omega^{-4} T_{\alpha\beta}^{(m)}, \quad \tilde{T}_{\alpha\beta}^{(m)} = \Omega^{-4} T_{\alpha\beta}^{(m)}, \quad \tilde{T}_{\alpha\beta}^{(m)} = \Omega^{-4} T_{\alpha\beta}^{(m)}.
\] (104)

The last equation makes it clear that the trace vanishes in the Einstein frame if and only if it vanishes in the Jordan frame.

In this context, it is relevant to discuss the behavior of the Klein-Gordon equation. In fact, the source-free equation \( \Box \phi = 0 \) in the absence of self-interactions is not conformally invariant. However, its generalization
\[
\Box \phi - \frac{n - 2}{4(n - 1)} R \phi = 0,
\] (105)
for \( n \geq 2 \) is conformally invariant as pointed out in the above discussion of minisuperspace quantization procedure [78, 79]. It is reasonable to allow for the possibility that the scalar \( \phi \) acquires a mass or other potential at high energies and, accordingly, in particle physics and in cosmology it is customary to introduce a potential energy density \( V(\phi) \) for the Klein-Gordon scalar. We have already discussed how a non-minimal coupling between \( \phi \) and the Ricci curvature arises. The introduction of non-minimal coupling with \( \xi \neq 0 \) makes the theory a scalar-tensor one.

The Klein-Gordon equation is conformally invariant in four space-time dimensions if \( \xi = 1/6 \) and \( V = 0 \) or \( V = \lambda \phi^4 \) [78, 80]. Even a constant potential \( V \), equivalent to a cosmological constant, corresponds to an effective mass for the scalar (not to be identified with a real mass) [81] which breaks conformal invariance [82].

Although unintuitive, it is not difficult to understand why a quartic potential preserves conformal invariance on the basis of dimensional considerations. Conformal invariance corresponds to the absence of a characteristic length (or mass) scale in the physics. In general, the potential \( V(\phi) \) contains dimensional parameters (such as the mass \( m \) in \( V = m^2 \phi^2 / 2 \)) but, when \( V = \lambda \phi^4 \), the dimension of \( V \) (a mass to the fourth power) is carried by \( \phi^4 \) and the self-coupling constant \( \lambda \) is dimensionless, i.e., there is no scale associated to \( V \) in this case. We will discuss these cases in the minisuperspace framework by considering the related Noether symmetries.

7 Extended Minisuperspace Models

In what follows, we shall give realizations of the above approach for minisuperspace cosmological models derived from
Extended Theories of Gravity. As we saw in previous sections, the existence of a Noether symmetry for a given minisuperspace is a sort of selection rule to recover classical behaviors in cosmic evolution. The so-called Hartle criterion to select correlated regions in the configuration space of dynamical variables is directly connected to the presence of a Noether symmetry and we will show that such a statement works for minisuperspace models coming from Extended Gravity.

The approach is connected to the search for Lagrange multipliers. In fact, imposing Lagrange multipliers allow to modify the dynamics and select the form of effective potentials. By integrating the multipliers, solutions can be achieved. In our case, such solutions are cosmological ones.

On the other hand, the Lagrange multipliers are constraints capable of reducing dynamics in scalar-tensor and higher-order theories. Technically they are anholonomic constraints being time-dependent. They give rise to field equations which describe dynamics of the further degrees of freedom coming from Extended Theories of Gravity. This fact is extremely relevant to deal with new degrees of freedom under the standard of effective scalar fields [27]. Below, we give minisuperspace examples and obtain exact cosmological solutions for non-minimally coupled and higher-order theories. In particular, we show that, by imposing Lagrange multipliers, a given minisuperspace model becomes canonical and Noether symmetries, if exist, can be found out. Finally, it is possible to show that Noether symmetries allow also to classify finite-time singularities. In other words, as soon as symmetries are broken, singularities emerge at finite.

### 7.1 Scalar-Tensor Gravity Cosmologies

Let us take into account a nonminimally coupled theory of gravity of the form

\[
S = \int d^4x \sqrt{-g} \left[ F(\phi)R + \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right],
\]

(106)

where, as said, \(F(\phi)\) and \(V(\phi)\) are respectively the coupling and the potential of a scalar field [26]. We are using physical units \(8\pi G = c = h = 1\), so that the standard Einstein coupling is recovered for \(F(\phi) = -1/2\).

Let us restrict to a FRW minisuperspace. The Lagrangian in (106) becomes point-like, that is

\[
\mathcal{L} = 6a^2 \dot{F} - 6a^2 \ddot{F} - 6kaF + a^3 \left( \frac{\dot{\phi}}{2} - V \right),
\]

(107)

in terms of the scale factor \(a\). The configuration space of such a Lagrangian is \(Q \equiv \{a, \phi\}\), i.e., a bidimensional minisuperspace. A Noether symmetry exists if [19] holds. In this case, it has to be

\[
X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \phi \frac{\partial}{\partial \alpha} + \phi \frac{\partial}{\partial \beta},
\]

(108)

where \(\alpha, \beta\) depend on \(a, \phi\). This vector field acts on the \(Q\) minisuperspace. The system of partial differential equation given by [19] is

\[
a F(\phi) \left[ \alpha + 2a \frac{\partial a}{\partial a} + aF'(\phi) \left( \beta + a \frac{\partial \beta}{\partial a} \right) = 0, \right.
\]

(109)

\[
3\alpha + 12F'(\phi) \left( \alpha \frac{\partial a}{\partial a} + 2a \frac{\partial \beta}{\partial a} \right) = 0,
\]

(110)

\[
a F''(\phi) + \left[ 2a + a \frac{\partial a}{\partial a} + \phi \frac{\partial \beta}{\partial a} \right] F'(\phi) +
\]

\[
+ 2 \frac{\partial a}{\partial a} F(\phi) + \frac{\phi \partial a}{\partial a} F(\phi) = 0,
\]

(111)

\[
[3aV(\phi) + a\beta V'(-\phi)]a^2 + 6k[aF(\phi) + a\beta F'(\phi)] = 0.
\]

(112)

Prime indicates the derivative with respect to \(\phi\). The number of equations is 4 as it has to be, being \(n = 2\) the \(Q\)-dimension. Several solutions exist for this system [13][31][34]. They determine also the form of the model since the system (109)-(112) gives \(a, \beta, F(\phi)\) and \(V(\phi)\). For example, if the spatial curvature is \(k = 0\), a solution is

\[
\alpha = \frac{2}{3} p(s) \beta \phi^{m + 1} \phi^{m(s) - 1}, \quad \beta = \beta(a) \phi^{m(s)},
\]

(113)

\[
F(\phi) = D(s) a^2, \quad V(\phi) = \lambda \phi^{2p(s)},
\]

(114)

where

\[
D(s) = \frac{(2s + 3)^2}{4(s + 1)(s + 2)},
\]

(115)

\[
p(s) = \frac{3(s + 1)}{2s + 3},
\]

(116)

\[
m(s) = \frac{2s^2 + 6s + 3}{2s + 3},
\]

(117)

and \(s, \lambda\) are free parameters. The change of variables [37] gives

\[
w = \sigma_0 a^3 \phi^{2p(s)}, \quad z = \frac{3}{\beta_0 \lambda(s)} a^{-1} \phi^{1 - m(s)},
\]

(118)

where \(\sigma_0\) is an integration constant and

\[
\lambda(s) = \frac{6s}{2s + 3}.
\]

(119)

Lagrangian (107) becomes, for \(k = 0\),

\[
\mathcal{L} = \gamma(s) \dot{w}^3 - \lambda w,
\]

(120)

where \(z\) is cyclic and

\[
\gamma(s) = \frac{2s + 3}{12\sigma_0 \lambda(s)(s + 1)(s + 2)}.
\]

(121)

The conjugate momenta are

\[
\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \gamma(s) \dot{w}^3 \dot{w}, \quad \pi_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = \gamma(s) \dot{w}^3 \dot{z},
\]

(122)

and the Hamiltonian is

\[
\mathcal{H} = \frac{\pi_z \pi_w}{\gamma(s) \dot{w}^3} + \lambda w.
\]

(123)
The Noether symmetry is given by

\[ \pi_s = \Sigma_0. \]  

(124)

Quantizing Eqs. (123), we get

\[ \pi \rightarrow -i\partial_\tau, \quad \pi_w \rightarrow -i\partial_w, \]  

(125)

and then the WDW equation

\[ \left[ (i\partial_\tau)(i\partial_w) + \tilde{\lambda}w^{1+s/3}\right]|\Psi> = 0, \]  

(126)

where \( \tilde{\lambda} = \gamma(s)\lambda \). The quantum version of constraint (124) is

\[ -i\partial_\tau|\Psi> = \Sigma_0|\Psi>, \]  

(127)

so that dynamics results reduced. A straightforward integration of Eqs. (126) and (127) gives

\[ |\Psi> = |\Omega(w) > |\chi(z) > e^{i\tilde{\lambda}w^{2+s/3}}, \]  

(128)

which is an oscillating wave function and the Hartle criterion is recovered. In the semi-classical limit, we have two first integrals of motion: \( \Sigma_0 \) (i.e. the equation for \( \pi_s \)) and \( E_L = 0, \) i.e. the Hamiltonian (123) which becomes the equation for \( \pi_w \). Classical trajectories in the configuration space \( Q \equiv \{w, z\} \) are immediately recovered

\[ w(t) = [k_1t + k_2]^{(s+3)/s}, \quad z(t) = [k_1t + k_2]^{(s+6)/(s+3)} + z_0, \]  

(129)

(130)

then, going back to \( Q \equiv \{a, \phi\} \), we get the classical cosmological behaviour

\[ a(t) = a_0(t - t_0)^{l(s)}, \]  

(131)

\[ \phi(t) = \phi_0(t - t_0)^{q(s)}, \]  

(132)

where

\[ l(s) = \frac{2s^2 + 9s + 6}{s(s + 3)}, \quad q(s) = \frac{2s + 3}{s}, \]  

(133)

which means that Hartle criterion selects classical universes. Depending on the value of \( s \), we get Friedman, power-law, or pole-like behaviors.

An important remark has to be done at this point. Noether symmetries are a useful tool also to classify singularities. For example, for \( s = -2 \pm \sqrt{2} \), Eq. (123) assumes the value \( l(s) = -1 \). In this case, we get the solution

\[ a(t) = \frac{a_0}{(t - t_0)}. \]  

(134)

This means that given values of the parameter \( s \) leads to future singularities of the scale factor \( a(t) \) where the symmetry is broken. In order to obtain a comprehensive classification of singularities, we need also to know the behavior of the density \( \rho \) and the pressure \( p \) related to the scalar field \( \phi \). In our scalar-tensor models, they are

\[ \rho = -\frac{1}{2D(s)\phi^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + 12D(s)H\dot{\phi}\phi\right], \]  

(135)

\[ p = -\frac{1}{2D(s)\phi^2}\left[\frac{1}{2}\dot{\phi}^2 - V(\phi) + 4D(s)\left(\ddot{\phi}^2 + 2H\dot{\phi} - \dot{\phi}^2\right)\right], \]  

(136)

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. These equations correspond to the minimally coupled analogue forms

\[ p = \frac{1}{2}\dot{\phi}^2 + V(\phi), \]  

(137)

\[ p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \]  

(138)

Clearly, also \( \rho \) and \( p \) are functions of \( s \) and then it is easy to achieve the classification of finite-time singularities according to the value of parameter \( s \) (see also [61]; that is

- Type I (“Big Rip”) : For \( t \rightarrow t_0, a \rightarrow \infty, \rho \rightarrow \infty \) and \( |p| \rightarrow \infty \). This also includes the case of \( \rho, p \) being finite at \( t_0 \).
- Type II (“sudden”) : For \( t \rightarrow t_0, a \rightarrow a_0, \rho \rightarrow \rho_0 \) and \( |p| \rightarrow \infty \).
- Type III : For \( t \rightarrow t_0, a \rightarrow a_0, \rho \rightarrow \infty \) and \( |p| \rightarrow \infty \).
- Type IV : For \( t \rightarrow t_0, a \rightarrow a_0, \rho \rightarrow 0, |p| \rightarrow 0 \) and higher derivatives of \( H \) diverge. This also includes the case when \( p \) (or) both of them tend to some finite values while higher derivatives of \( H \) diverge.

In conclusion, the presence of Noether symmetries allows a full classification of singularities which, from another point of view, correspond to the breaking points of symmetries.

Finally, if we take into account generic Bianchi models, the configuration space is \( Q \equiv \{a_1, a_2, a_3, \phi\} \) and more than one symmetry can exist as it is shown in [59]. The considerations on the oscillatory regime of the wave function of the universe and the recovering of classical behaviors are exactly the same.

### 7.2 Fourth-Order Gravity Cosmologies

Similar arguments work for higher-order gravity minisuperspaces. In particular, let us consider fourth-order gravity given by the action

\[ S = \int d^4x \sqrt{-g} f(R), \]  

(139)

where \( f(R) \) is a generic function of scalar curvature. If \( f(R) = R + 2\Lambda \), the standard second-order gravity is recovered. We are discarding matter contributions for the sake of simplicity. Reducing the action to a point-like, FRW one, we have to write

\[ S = \int dt L(a, \dot{a}; R, \dot{R}), \]  

(140)

where \( dt \) means derivative with respect to the cosmic time. The scale factor \( a \) and the Ricci scalar \( R \) are the canonical variables. This position could seem arbitrary since \( R \) depends on \( a, \dot{a}, a, \dot{a} \), but it is generally used in canonical quantization [55], [56].

The definition of \( R \) in terms of \( a, \dot{a}, a, \dot{a} \) introduces a constraint which eliminates second and higher order derivatives in action (140), and yields to a system of second order differential equations in \( \{a, R\} \). Action (140) can be written as

\[ S = 2\pi^2 \int dt \left\{ a^3 f(R) - \lambda \left[ R + 6 \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}, \]  

(141)

where the Lagrange multiplier \( \lambda \) is derived by varying with respect to \( R \). It is

\[ \lambda = a^3 f'(R). \]  

(142)
Here prime means derivative with respect to $R$. To recover the analogy with previous scalar-tensor models, let us introduce the auxiliary field

$$p \equiv f'(R),$$

so that the Lagrangian in (131) becomes

$$\mathcal{L} = 6a^2 \dot{p} + 6a^2 \ddot{\phi} - 6kap - a^4 W(p),$$

which is of the same form as (107), apart from the kinetic term. This is an Helmhotz-like Lagrangian [74] and $a, p$ are independent fields. The potential $W(p)$ is defined as

$$W(p) = h(p)p - r(p),$$

where

$$r(p) = \int f'(R)dR = \int p dR = f(R), \quad h(p) = R,$$

such that $h = (f')^{-1}$ is the inverse function of $f'$. The configuration space is now $Q \equiv \{a, p\}$ and $p$ has the same role of the above $\phi$. Condition (139) is now realized by the vector field

$$X = \alpha(a, p) \frac{\partial}{\partial a} + \beta(a, p) \frac{\partial}{\partial p} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{p}},$$

and explicitly gives the system

$$a \left[ \alpha + 2a \dot{\alpha} \right] p + a \left[ \beta + a \ddot{\beta} \right] = 0,$$

$$a^2 \dot{\alpha} = 0,$$

$$2a + a \ddot{\alpha} + 2 \dot{p} \ddot{\alpha} + a \dot{\beta} = 0,$$

$$6k[\alpha + \beta a] + a^2 [3aW + a \ddot{W}] = 0.$$  

The solution of this system, i.e. the existence of a Noether symmetry, gives $\alpha$, $\beta$ and $W(p)$. It is satisfied for

$$\alpha = \alpha(a), \quad \beta(a, p) = \beta_0 a^p,$$

where $s$ is a parameter and $\beta_0$ is an integration constant. In particular,

$$s = 0 \rightarrow \alpha(a) = -\frac{\beta_0}{3} a, \quad \beta(p) = \beta_0 p,$$

$$W(p) = W_0 p, \quad k = 0,$$

$$s = -2 \rightarrow \alpha(a) = -\frac{\beta_0}{a}, \quad \beta(a, p) = \beta_0 \frac{p}{a^2},$$

$$W(p) = W_1 p^3, \quad k,$$

where $W_0$ and $W_1$ are constants. Let us discuss separately the solutions (149) and (151).

7.2.1 The case $s = 0$

The induced change of variables $Q \equiv \{a, p\} \rightarrow \tilde{Q} \equiv \{w, z\}$ can be

$$w(a, p) = a^3 p, \quad z(p) = \ln p.$$  

Lagrangian (144) becomes

$$\tilde{\mathcal{L}}(w, \dot{w}, \dot{z}) = \dot{z} \dot{w} - 2w^2 + \frac{\dot{w}^2}{w} - 3W_0 w.$$  

and, obviously, $z$ is the cyclic variable. The conjugate momenta are

$$\pi_w \equiv \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{w}} = \dot{w} - 4\dot{z} = \Sigma_0,$$

$$\pi_w \equiv \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{w}} = \dot{z} + 2 \dot{\dot{w}}.$$  

and the Hamiltonian is

$$\mathcal{H}(w, \pi_w, \pi_z) = \pi_w \pi_z - \frac{\dot{w}^2}{w} + 2w\pi_w^2 + 6W_0 w.$$  

By canonical quantization, reduced dynamics is given by

$$\left[ \dot{\delta}_\tau^2 - W_0 \ddot{\delta}_\tau - W_0 \dot{\delta}_\tau \dot{\delta}_\tau + 6W_0 \dot{\delta}_\tau \right] |\Psi > = 0,$$

$$-i\delta \tau |\Psi > = \Sigma_0 |\Psi >.$$  

However, we have done simple factor ordering considerations in the WDW equation (160). Immediately, the wave function has an oscillatory factor, being

$$|\Psi > \sim e^{i\Sigma_0 \zeta} \chi(w) >.$$  

The function $|\chi >$ satisfies the Bessel differential equation

$$\left[ w^2 \ddot{\chi} + \left( \frac{\Sigma_0}{2} - \frac{\chi^2}{2} - 3W_0 w^2 \right) \right] \chi(w) = 0,$$

whose solutions are linear combinations of Bessel functions $Z_\nu(w)$

$$\chi(w) = w^{1/2 - i\Sigma_0/4} Z_\nu(\lambda w),$$  

where

$$\nu = \pm \frac{1}{4} \sqrt{4 - 9\Sigma_0^2 - i4\Sigma_0}, \quad \lambda = \pm \sqrt{\frac{W_0}{2}}.$$  

The oscillatory regime for this component depends on the reality of $\nu$ and $\lambda$. The wave function of the universe, from Noether symmetry (150), is then

$$\Psi(z, w) \sim e^{i\Sigma_0 z -(1/4) \ln w} w^{1/2} Z_\nu(\lambda w).$$  

For large $w$, the Bessel functions have an exponential behavior, so that the wave function (160) can be written as

$$\Psi \sim e^{i\Sigma_0 z -(\Sigma_0/4) \ln w \pm \lambda w}.$$  

Due to the oscillatory behavior of $\Psi$, Hartle’s criterion is immediately recovered. By identifying the exponential factor of (167) with $S_0$, we can recover the conserved momenta $\pi_s, \pi_w$ and select classical trajectories. Going back to the old variables, we get the cosmological solutions

$$a(t) = a_0 e^{(\lambda/6)t} \exp \left\{ -\frac{z_1}{3} e^{-2(\lambda/3)t} \right\},$$

$$p(t) = p_0 e^{(\lambda/6)t} \exp \left\{ z_1 e^{-2(\lambda/3)t} \right\},$$  

where $a_0, p_0$ and $z_1$ are integration constants. It is clear that $\lambda$ plays the role of a cosmological constant and inflationary behavior is asymptotically recovered.
7.2.2 The case $s = -2$

The new variables adapted to the foliation for the solution \[153\] are now
\[w(a, p) = ap, \quad z(a) = a^2.\] (170)
and Lagrangian \[144\] assumes the form
\[\mathcal{L}(a, \dot{a}, \dot{z}, \dot{z}) = 3\dot{w} - 6kw - W_1w^3,\] (171)
The conjugate momenta are
\[\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = 3\dot{w} = \Sigma_1,\] (172)
\[\pi_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = 3\dot{z}.\] (173)
The Hamiltonian is given by
\[\mathcal{H}(a, \pi_z, \pi_w) = \frac{1}{3} \pi_z \pi_w + 6kw + W_1w^3.\] (174)
Going over the same steps as above, the wave function of the universe is given by
\[\Psi(z, w) \sim e^{i\Sigma_1 z + 9kw^2 + (3W_1/4)w^4}},\] (175)
and the classical cosmological solutions are
\[a(t) = \pm \sqrt{h(t)}, \quad p(t) = \pm \frac{c_1 + (\Sigma_1/3) t}{\sqrt{h(t)}},\] (176)
where
\[h(t) = \left(\frac{W_1\Sigma_1^3}{36}\right) t^4 + \left(\frac{W_1w_1\Sigma_1^2}{6}\right) t^3 + \left(k\Sigma_1 + \frac{W_1w_1^2\Sigma_1}{2}\right) t^2 + w_1(6k + W_1w_1^2)t + z_2.\] (177)
\[w_1, \Sigma_1, \text{and } z_2 \text{ are integration constants.} \text{ Immediately we see that, for large } t\]
\[a(t) \sim t^2, \quad p(t) \sim \frac{1}{t},\] (178)
which is a power-law inflationary behavior. An extensive discussion of Noether symmetries in $f(R)$ gravity is in \[154\].

7.3 Higher than Fourth-Order Gravity Cosmologies

Minisuperspaces which are suitable for the above analysis can be found for higher than fourth-order theories of gravity as
\[\mathcal{S} = \int d^4x \sqrt{-g} f(R, \Box R).\] (179)
In this case, the configuration space is $Q \equiv \{a, R, \Box R\}$ considering $\Box R$ as an independent degree of freedom \[155\[156\[22\]. The FRW point–like Lagrangian is formally
\[\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R}, \Box R, (\Box R))\] (180)
and the constraints
\[R = -6\left[\left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right],\] (181)
\[\Box R = \dot{R} + \frac{3a}{a} \dot{R},\] (182)
holds. Using the above Lagrange multiplier approach, we get the Helmholtz point–like Lagrangian
\[\mathcal{L} = 6a\dot{a}^2p + 6a^2a \dot{p} - 6kap - a^3hq - a^3W(p, q),\] (183)
where
\[p \equiv \frac{\partial f}{\partial \dot{R}}, \quad q \equiv \frac{\partial f}{\partial \Box R},\] (184)
\[W(p, q) = h(p)p + g(q)q - f,\] (185)
and
\[h(p) = R, \quad g(q) = \Box R, \quad f = f(R, \Box R).\] (186)
Now the minisuperspace is 3-dimensional but, again, the Noether symmetries can be recovered. Cases of physical interest \[22\] are
\[f(R, \Box R) = f_0R + f_1R^2 + f_2R \Box R;\] (187)
\[f(R, \Box R) = f_0R + f_1\sqrt{R \Box R},\] (188)
discussed in details in \[155\]. Also here the existence of the symmetry selects the form of the model and allows to reduce the dynamics. Once it is identified, we can perform the change of variables induced by foliation using Eqs. \[155\], if a symmetry is present, is two symmetries are present. In both cases,
\[Q \equiv \{a, R, \Box R\} \rightarrow \tilde{Q} \equiv \{z, u, w\},\] (189)
where one or two variables are cyclic in Lagrangian \[155\]. Taking into account, for example, the case \[155\], we get
\[\tilde{\mathcal{L}} = 3[wu^2 - kw] - F_1 \left[3wu^2u + 3w^2\dot{u}^2 + \frac{w^3}{2a^2} - 3kww\right],\] (190)
where we assume $F_0 = -1/2$, the standard Einstein coupling, $z$ is the cyclic variable and
\[z = R, \quad u = \sqrt{\frac{\Box R}{R}, \quad w = a}.\] (191)
The conserved quantity is
\[\Sigma_0 = \frac{w^3 u}{2a^2}.\] (192)
Using the canonical procedure of quantization and deriving the WDW equation from \[155\], the wave function of the universe is
\[|\Psi| \sim e^{i\Sigma_0 t}|\chi(u)| > |\Theta(u)| > ,\] (193)
where $\chi(u)$ and $\Theta(u)$ are combinations of Bessel functions. The oscillatory subset of the solution is evident and the Hartle criterion is recovered. In the semi–classical limit, using the conserved momentum \[192\], we obtain the cosmological behavoirs
\[a(t) = a_0 t, \quad a(t) = a_0 t^{1/2}, \quad a(t) = a_0 e^{kt^2},\] (194)
depending on the choice of boundary conditions.

However, the above considerations of singularities at finite time holds also for higher-order theories of gravity. Also in these cases, they can be classified according to the presence of Noether symmetries.
8 Noether Symmetries and Conformal Transformations

To conclude the discussion, we want now to analyze the compatibility between the conformal transformation connecting Jordan frame and Einstein frame, and the presence of Noether symmetries. In other words, we want to analyze how minisuperspace models, derived by the Noether Symmetry Approach, behave under the action of conformal transformation.

Since the existence of a Noether symmetry implies the existence of a vector field \( X \) along which \( L_X \mathcal{L} = 0 \), this happens if the Lie derivative of the Lagrangian \( \mathcal{L} \) in the Einstein frame and in the Jordan frame. We want to investigate how this vector field acts on one of the above point-like Lagrangians under a conformal transformation. Let us consider the minisuperspace model (106).

Thus, a given lift-vector field of the form \( [\bar{X}] \)

\[
\bar{X}_n = \bar{\alpha} \frac{\partial}{\partial \bar{a}} + \bar{\beta} \frac{\partial}{\partial \bar{\phi}} + \bar{\alpha}' \frac{\partial}{\partial \bar{a}'},
\]

where \( \bar{\alpha} = \alpha(a, \phi), \bar{\beta} = \beta(a, \phi) \), corresponds, under conformal transformations, to the lift-vector field on the configuration space \( (\bar{a}, \bar{\phi}) \)

\[
\bar{X}_n = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \alpha' \frac{\partial}{\partial a'} + \beta' \frac{\partial}{\partial \phi'},
\]

where \( \alpha = \alpha(a, \phi), \beta = \beta(a, \phi) \) are connected to \( \alpha = \alpha(a, \phi), \beta = \beta(a, \phi) \) through a Jacobian matrix. Here the prime means the derivative with respect to the time \( \eta \). The Lie derivative of the Lagrangian \( \mathcal{L}_n \) along the vector field \( X_n \) corresponds then to the Lie derivative of \( \mathcal{L}_n \) along \( \bar{X}_n \).

\[
\mathcal{L}_{X_n} \mathcal{L}_n = L_{\bar{X}_n} \bar{\mathcal{L}}_n.
\]

Therefore, if \( X_n \) is a Noether vector field relative to \( \mathcal{L}_n \) one has

\[
L_{\bar{X}_n} \mathcal{L}_n = 0,
\]

and, from \( [\bar{X}] \), \( \bar{X}_n \) is a Noether vector field relative to \( \bar{\mathcal{L}}_n \). The choice of \( \eta \) as time-coordinate is in a formal point of view, but in order to analyse the phenomenology relative to a given model and to obtain then quantities comparable with the observational data, the appropriate choice of time-coordinate is the cosmic time \( \tau \). The problem with the cosmic time is that it is not preserved by the conformal transformation. Thus the conformal transformation we are considering does not take simply the form of a “coordinate transformation” on the phase space \( \{a, \phi\} \), therefore its compatibility with the presence of a Noether symmetry cannot be easily verified. Of course it must hold also under such a choice of time-coordinate.

We do not decide to verify such a compatibility directly. Rather, we analyze how the Lie derivative \( L_{\bar{X}_n} \mathcal{L}_n \) in the Jordan frame is transformed under the time transformation which connects \( \tau \) with \( \eta \).

The explicit expression of \( L_{\bar{X}_n} \mathcal{L}_n \) is given by

\[
L_{\bar{X}_n} \mathcal{L}_n = 6 \left[ 2F \frac{\partial F_{\bar{a}}}{\partial a} + \left( \beta + a \frac{\partial \beta}{\partial a} \right) F_{\bar{a}} \right] a^2 +
+ a \left[ \alpha + 6F \frac{\partial F_{\bar{a}}}{\partial \phi} + a \frac{\partial \beta}{\partial \phi} \right] \phi^2 +
+ 6 \left[ a \beta F_{\phi} \left( \alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \phi} \right) F_{\phi} +
+ 2F \frac{\partial a}{\partial \phi} + \left( \frac{a^2 \phi}{6} \right) a' \phi' \right.
\]

\[
- a^3 (4aV + a \beta V_{\phi}) - 6a (2F \alpha + F_{\phi} a \beta) \kappa,
\]

in which we have taken into account that

\[
\alpha' = \frac{\partial \alpha}{\partial a} \alpha' + \frac{\partial \alpha}{\partial \phi} \phi'; \quad \beta' = \frac{\partial \beta}{\partial a} \alpha' + \frac{\partial \beta}{\partial \phi} \phi'.
\]

Eq. (199) under the transformation becomes

\[
L_{\bar{X}_n} \mathcal{L}_n = 6 \left[ 2F \frac{\partial F_{\bar{a}}}{\partial a} + \left( \beta + a \frac{\partial \beta}{\partial a} \right) F_{\bar{a}} \right] a^2 \bar{a}^2 +
+ a \left[ \alpha + 6F \frac{\partial F_{\bar{a}}}{\partial \phi} + a \frac{\partial \beta}{\partial \phi} \right] a^2 \bar{\phi}^2 +
+ 6 \left[ a \beta F_{\phi} \left( \alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \phi} \right) F_{\phi} +
+ 2F \frac{\partial a}{\partial \phi} + \left( \frac{a^2 \beta}{6} \right) \bar{a}^2 \bar{\phi}^2 \right.
\]
in which we have taken into account that
\[ \dot{a} = \frac{\partial \alpha}{\partial a} + \frac{\partial \alpha}{\partial \phi} \beta; \quad \dot{\beta} = \frac{\partial \beta}{\partial a} + \frac{\partial \beta}{\partial \phi}. \] (205)

We remind that the dot means the derivative with respect to \( t \).

Comparing (202) with (204), we obtain that, the Lie derivative \( L_{X_\eta} L_{\eta} \) becomes
\[ L_{X_\eta} L_{\eta} = aL_{X_\eta} L_{\eta} - (L_{X_\eta} a) E_t, \] (206)
being \( L_{X_\eta} a = a \) where \( E_t \) is the energy function.

It can be seen that the same relation as (206) holds in the Einstein frame, that is
\[ L_{X_{\bar{\eta}}} \bar{L}_{\eta} = \bar{a} L_{X_{\bar{\eta}}} \bar{L}_{\eta} - (\bar{L}_{X_{\bar{\eta}}} \bar{a}) \bar{E}_t, \] (207)
with obvious meaning of \( X_{\bar{\eta}} \) and \( \bar{E}_t \). This implies that, if \( X_{\eta} \) is a Noether vector field relative to \( L_{\eta} \), that is, if (198) holds, the corresponding vector field \( X_{\bar{\eta}} \) is such that
\[ L_{X_{\bar{\eta}}} \bar{L}_{\eta} - \frac{x_{\bar{\eta}} a}{a} E_t = 0. \] (208)

It means also that, when the cosmic time is taken as time-coordinate, the conformal transformation preserves the expression given by the righthand side of (206) and not the Lie derivative along a given vector field \( X_\eta \). Relation (208) represents a more general way to express the presence of a first integral for the Lagrangian \( L_t \); associated to (208), we have the conserved quantity
\[ \frac{\partial L_t}{\partial a} \dot{a} + \frac{\partial L_t}{\partial \phi} \dot{\phi} - E_t \int \frac{L_t}{a} \frac{\dot{a}}{a} dt = \text{const}, \] (209)
which, of course, holds on the solutions of the Euler–Lagrange equations. The vector field \( X_t \) verifying (208) can thus be seen as a generalized Noether vector field and the conformal transformation [55] preserves this generalized symmetry. That is, if \( X_t \) is a Noether vector field, in the sense of (208), relative to \( L_t \) then \( X_{\bar{\eta}} \) is a Noether vector field relative to \( \bar{L}_t \) in the same sense, that is
\[ L_{\bar{X}_{\bar{\eta}}} \bar{L}_{\bar{\eta}} - \frac{L_{X_{\bar{\eta}}}}{a} \frac{\dot{a}}{a} \bar{E}_t = 0. \] (210)
In terms of conformal time, the first integral relative to (198) for the Lagrangian \( L_{\eta} \) is given by
\[ \frac{\partial L_{\eta}}{\partial a} a + \frac{\partial L_{\eta}}{\partial \phi} \beta = \text{const}. \] (211)
We see that the expression (211) corresponds to (209) under the transformation, except for a term in the energy function. In fact, Eq. (211) explicitly written, is
\[ (12F \alpha' + 6F_\phi a \phi') \alpha + (6F_\phi a \alpha' + a^2 \phi') \beta = \text{const}, \] (212)
while (209) is
\[ -E_t \int a \frac{\dot{a}}{a} dt + (12F a \dot{a} + 6F_\phi a^2 \dot{\phi}) \alpha + (6F_\phi a^2 \dot{a} + a^3 \phi') \beta = \text{const}. \] (213)
Taking into account that, the expressions of energy function, in the conformal time and in the cosmic time, are related by
\[ E_{\eta} = a E_t, \] (214)
we have that (212), becomes
\[ \frac{E_t}{a} \int a \dot{a} dt + (12F a \dot{a} + 6F_\phi a^2 \phi') \alpha + (6F_\phi a^2 \dot{a} + a^3 \phi') \beta = \text{const} \] (215)
which coincides with (212) except for the term in \( E_{\eta} \), but \( E_{\eta} = 0 \) corresponds to the first order Einstein equation, expressed in the time \( \eta \), therefore, there is equivalence between the two formulations [55]. As already said, some authors have formulated the existence of a Noether vector field imposing
\[ L_{X_t} L_t = 0, \] (216)
using the cosmic time as time-coordinate; condition (216), after the analysis we have done till now, turns out to be less general than (210). By the way, condition (216) has the interesting property that it implies the possibility to define some new coordinates on the configuration space \( \{a, \phi\} \), such that the Lagrangian has a cyclic coordinate [55, 56], reducing in this way the Euler-Lagrange equations. In fact, one can always define new coordinate, say \( \{z, w\} \), in the configuration space of the Lagrangian, such that the lift-vector field assumes the form \( X_t = \frac{\partial}{\partial z} \), so that one has \( L_{X_t} L = \frac{\partial L}{\partial z} \); in case Eq. (216) holds, one has that \( z \) is cyclic. In the generalized case we are considering, it is no longer possible to get this behavior, since \( L_{X_t} L \neq 0 \) and consequently \( z \) is no longer cyclic. In this case, one has to use the first integral (209) together with the relation on the energy function to reduce the Euler-Lagrange equations.

This problem corresponds, in the Einstein frame, to
\[ -\bar{E}_t \int \frac{L_{X_{\bar{\eta}}} \bar{a}}{a} dt + \frac{\partial L_{\bar{\eta}}}{\partial a} \bar{a} + \frac{\partial L_{\bar{\eta}}}{\partial \phi} \beta = \text{const}. \] (217)
Thus, finding the solutions of some cosmological model by the existence of a Noether symmetry (and therefore fixing the class of model compatible with it) in the Einstein frame, one gets, via the conformal transformation, the solutions to the class of models in the Jordan frame corresponding to the one given in the Einstein frame. This result is extremely important in minisuperspace Quantum Cosmology since allow to select all the equivalent initial (boundary) conditions.

Let us discuss a significant example. A solvable cosmological model in the Einstein frame is the model where the dynamical potential of the scalar field is constant and the spatial curvature is zero. The Lagrangian is given by
\[ \bar{L}_t = -3a \dot{a}^2 + \frac{1}{2} a^3 \dot{\phi}^2 - \dot{a}^2 \Lambda. \] (218)
We have that such a model in the Einstein frame corresponds, in the Jordan frame, to the class of models with (arbitrarily given) coupling \( F \) and potential \( V \) connected by the relation
\[ \frac{V}{4F^2} = \Lambda. \] (219)
We can thus fix the potential \( V \) and obtain from (218) the corresponding coupling. Once we have the solutions relative to the constant potential in the Einstein frame, we can obtain the solutions of all the non-minimally coupled models in the class defined by relation (219).

Let us consider the case
\[ V = \lambda \dot{\phi}^4, \quad \lambda > 0, \] (220)
which correspond to a “chaotic inflationary” potential. The corresponding coupling is quadratic in $\phi$, that is

$$F = k_0 \phi^2,$$

in which

$$k_0 = -\frac{1}{2} \sqrt{\frac{\lambda}{\bar{\lambda}}}$$

We get the conformal transformation through which we obtain the solutions in the Jordan frame, i.e.

$$a = \frac{\bar{a}}{\phi \sqrt{-2k_0}},$$

$$d\phi = \phi \sqrt{\frac{2k_0}{12k_0 - 1}} d\bar{\phi},$$

$$dt = \frac{d\bar{t}}{\phi \sqrt{-2k_0}}.$$

As we can see from these relations, it has to be $k_0 < 0$. Integrating the second of (224), we have $\phi$ in terms of $\bar{\phi}$

$$\phi = \alpha_0 e^{\sqrt{\frac{2k_0}{12k_0 - 1}} \bar{\phi}}.$$

We want now to consider the aspects connected with the point of view of the Noether symmetries. It is possible to show that, in the context of generalized Noether symmetries, the nonstandard coupled model with quartic potential and negative quadratic coupling admits a Noether symmetry, while such a result has not been found in the previous analysis of Noether symmetries (see [83, 86]). The system of equations for the Noether vector field obtained from (210) is given by

$$\frac{\partial \bar{\alpha}}{\partial \bar{a}} = 0,$$

$$\bar{\alpha} + \bar{\alpha} \frac{\partial \bar{\beta}}{\partial \bar{\phi}} = 0,$$

$$\frac{6}{\bar{\phi}} \frac{\partial \bar{\alpha}}{\partial \bar{\phi}} - \bar{\alpha} \frac{\partial \bar{\beta}}{\partial \bar{\phi}} = 0,$$

$$4\alpha \bar{V} + \bar{\alpha} \bar{\beta} \bar{V} = 0.$$

Substituting $\bar{V} = \bar{\lambda}$ in the fourth of (225), one gets $\bar{\alpha} = 0$; from the second one gets $\bar{\beta} = \text{const}$; the first and the third turn out to be identical verified. It is immediate to see that the Lagrangian (218) presents a Noether symmetry, since it does not depend on $\bar{\phi}$; being, in this particular case, $L_{\bar{\bar{\lambda}}} \bar{\alpha} = \bar{\alpha} = 0$. This result is compatible with the presence of a cyclic coordinate in the Lagrangian. Performing the conformal transformation given by (226) on the Noether vector field

$$\bar{\alpha} = 0, \quad \bar{\beta} = \bar{\beta}_0,$$

we have

$$\alpha = -\alpha \sqrt{\frac{2k_0}{12k_0 - 1}} \bar{\beta}_0, \quad \beta = \phi \sqrt{\frac{2k_0}{12k_0 - 1}} \bar{\beta}_0.$$

This means that (223) is a Noether vector field relative to the corresponding Lagrangian in the Jordan frame, with potential given by (220) and coupling given by (221). It is easy to verify that (225) holds.

In conclusion, the Noether Symmetry Approach is compatible with conformal transformations and allows to relate classes of conformally equivalent minisuperspace models

9 Discussion and Conclusions

The purpose of this paper has been to outline the canonical Hamiltonian approach to Quantum Cosmology taking into account minisuperspace models coming from Extended Theories of Gravity. After a quick summary of the Hamiltonian formulation of GR and the problem of canonical quantization in the ADM formalism, we discussed the minisuperspace approach to Quantum Cosmology. This one does not give a satisfactory solution to the full Quantum Gravity problem, however it is a useful scheme to set the problem of boundary conditions from which should emerge classical universes, that is cosmological dynamical models that could be reasonably observed with standard astrophysical tools. A main role in this approach is played by the identification of conserved quantities that give rise to peaked behaviors in the wave function of the universe. Such a function is the solution of the WDW equation, the corresponding of Schrödinger equation in Quantum Cosmology.

Peaked behaviors means correlations among variables and then the possibility to obtain classical universes according to the Hartle interpretative criterion. These conserved quantities can naturally be related to the Noether symmetries of the theory. The existence of symmetries depends, in several cases, by the identification of suitable Lagrange multipliers that allow to recast the point-like Lagrangian of the given minisuperspace model in a canonical form. In this sense, the Noether symmetries can be considered as "constraints" of the theory that allow to reduce the dynamics and recover classical solutions. The emergence of singularities at finite for such solutions means that symmetries are broken for certain values of the parameters.

Reversing the argument, if the wave function of the universe is related to the probability to get a classical cosmological solution, the existence of Noether symmetries tell us when the Hartle criterion works.

Some remarks are necessary at this point. First of all, we have to stress that the wave function is only related to the probability to get a certain behavior but it is not the probability amplitude since, till now, Quantum Cosmology is not a unitary theory. Furthermore, the Hartle criterion works in the context of an Everett-type interpretation of Quantum Cosmology [99, 100] which assumes the ideas that the universe branches into a large number of copies of itself whenever a measurement is made. This point of view is called Many Worlds interpretation of Quantum Cosmology. Such an interpretation is just one way of thinking and gives a formulation of Quantum Mechanics designed to deal with correlations internal to individual, isolated systems. The Hartle criterion gives an operative interpretation of such correlations. In particular, if the wave function is strongly peaked in some region of configuration space, we predict that we will observe the correlations which characterize that region. On the other hand, if the wave function is smooth in some region, we predict that correlations which characterize
that region are precluded to the observations. If the wave function is neither peaked nor smooth, no predictions are possible from observations. In other words, we can read the correlation of some region of minisuperspace as casual connection.

The analogy with standard Quantum Mechanics is straightforward. By considering the case in which the individual system consists of a large number of identical subsystems, one can derive from the above interpretation, the usual probabilistic interpretation of quantum mechanics for the subsystems. We have worked out this approach for minisuperspace models coming from Extended Theories of Gravity showing that identification of suitable Noether symmetries allows to completely solve the dynamical system. Furthermore, we have seen that if a Noether symmetry is present, it is preserved by the conformal transformation which connects Jordan and Einstein frames. In this sense, conformally equivalent classes of minisuperspaces can be selected.

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