Abstract Error Groups Via Jones Unitary Braid Group Representations at $q=i$

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Abstract

In this paper, we classify a type of abstract groups by the central products of dihedral groups and quaternion groups. We recognize them as abstract error groups which are often not isomorphic to the Pauli groups in the literature. We show the corresponding nice error bases equivalent to the Pauli error bases modulo phase factors. The extension of these abstract groups by the symmetric group are finite images of the Jones unitary representations (or modulo a phase factor) of the braid group at $q = i$ or $r = 4$. We hope this work can finally lead to new families of quantum error correction codes via the representation theory of the braid group.

Key Words: Abstract Error Group, Nice Error Base, Unitary Braid Representation, Extraspecial Two-Groups

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1 Introduction

Quantum error correction codes (QECC) are devised to protect quantum information and computation from various kinds of noise. These codes [1, 2, 3, 4, 5] are binary stabilizer codes with abelian normal subgroups in the error model specified by the real or complex Pauli groups. To explore non-binary QECC allowing arbitrary normal subgroups later called Clifford codes, Knill [6, 7] introduces nice error bases as well as abstract error groups, which are refined by Klappenecker and Rötteler [8, 9] with the help of the Clifford theorem [10].

In this paper, we recognize the abstract groups exploited in [12, 13, 14] as a type of the abstract error groups, as is not done before to the author’s best knowledge. They are often not isomorphic to the Pauli groups, though they give rise to binary stabilized codes as the Pauli groups do. Furthermore, the extension of these abstract error groups by the symmetric group are isomorphic to finite images of the Jones unitary braid group representations (UBGR) at \( q = \exp(\frac{2\pi i}{r}) \), \( r = 4 \) (or modulo a phase factor), see [11, 12]. Moreover, see [13, 14], some of these UBGR act as unitary basis transformation matrices from the product basis to the Greenberger-Horne-Zeilinger (GHZ) states [15]. QECC involving UBGR, for example, QECC using GHZ states [16] are explored with the Shor nine-qubit code [1] as the simplest example.

![Diagram](image)

Figure 1: From the Jones UBGR at \( q = i, r = 4 \) to QECC using GHZ states

Figure 1 is a diagrammatic exposition on the network formed by the present paper and related references, and it also partly explains our motivations of writing the present paper. The plan of this paper is organized as follows. Section 2 is a preliminary on the dihedral group, quaternion group and extraspecial two-groups. Section 3 focuses on the classification of our abstract error groups, and compares them with the real or complex Pauli groups. Section 4 presents associated nice error bases equivalent to the Pauli error bases modulo phase factors, and then with them yields the Jones UBGR at \( q = i, r = 4 \) (or modulo a phase factor). Section 5 briefly remarks our further research.
2 Dihedral groups and quaternion groups

We sketch basic facts used in the following sections about the dihedral group $D$, quaternion group $Q$ and extraspecial two-groups $G$. Note that relevant notations and definitions are consistent with those in Gorenstein’s book [10].

The dihedral group $D$ with two generators $k$ and $h$ is a set defined as

$$D = \{k, h|k^2 = 1, h^2 = -1, hk = -kh\},$$

whereas two generators $k$ and $h$ of the quaternion group $Q$ satisfy very similar but distinct algebraic relations,

$$Q = \{k, h|k^2 = h^2 = -1, hk = -kh\}.$$  \hfill (2.2)

Both groups have the same order $|D| = |Q| = 8$ and the center $\mathbb{Z}_2 = \{\pm 1\}$, but they are not isomorphic to each other. The number of cyclic subgroups of order 4 in the dihedral group $D$ is 1, whereas this number in the quaternion group $Q$ is 3. The dihedral group $D$ is the symmetry group of a square. The quaternion group $Q$ has a presentation in terms of quaternion, i.e., $\{\pm 1, \pm i, \pm j, \pm k\}$ satisfying $i^2 = j^2 = k^2 = ijk = -1$.

A group $G$ of the form $G = H \circ K$ with the center $Z(G)$ is called the central product of its two subgroups $H$ and $K$ if and only if $hk = kh$ for all $h \in H$, $k \in K$ and $H \cap K \subseteq Z(G)$. The central product $D^2 \equiv D \circ D$ is isomorphic to the central product $Q^2 \equiv Q \circ Q$, and hence the central product of dihedral groups $D$ and quaternion groups $Q$ is either isomorphic to $QD^r-1$ or isomorphic to $D^r$, namely,

$$QD^r-1 \equiv Q \circ D \circ \cdots \circ D, \quad D^r \equiv D \circ \cdots \circ D,$$

which are not isomorphic to each other. Since $D^2 \equiv Q^2$, the central product of $D$ and $Q$ is also isomorphic to $DQ^{s-1}$ or $Q^s$, and the number of cyclic subgroups of order 4 in $DQ^{s-1}$ is denoted by $n$ as well as this number in $Q^s$ is by $m$, respectively given by

$$n = (2^{2s} + (-2)^s)/2, \quad m = (2^{2s} - (-2)^s)/2.$$  \hfill (2.4)

Extraspecial two-groups $G$ are the central product of quaternion groups $Q$ and dihedral groups $D$, and so there are two extraspecial two-groups, $DQ^{r-1}$ or $Q^r$ with the same order $2^{2r+1}$. They can be defined in the other way if the following hold for a group $G$: a) the center $Z(G)$ is the cyclic group $\mathbb{Z}_2 = \{\pm 1\}$; b) the quotient group $G/Z(G)$ is a nontrivial elementary abelian 2-group; c) the order is $2^{2r+1}$. With the terminology in [12], a central product consisting of a $\mathbb{Z}_2$ group, dihedral groups $D$ and quaternion groups $Q$ is called nearly extraspecial two-group with the center $\mathbb{Z}_2 \times \mathbb{Z}_2$, and a central product in terms of a $\mathbb{Z}_4$ group, dihedral groups $D$ and quaternion groups $Q$ is called almost extraspecial two-group with the center $\mathbb{Z}_4$.

3 Abstract error groups $E_n^r$

We identify a type of abstract groups $E_n^r$ as abstract error groups, classify them by central products of dihedral groups, quaternion groups, and $\mathbb{Z}_2$ or $\mathbb{Z}_4$, and then compare them with the Pauli groups accordingly.
3.1 Abstract groups $E_n^\nu$

A finite group $G$ is called an abstract error group \[9\] if it has an irreducible faithful unitary representation $\phi$ with the degree defined by $\deg \phi = |G : Z(G)|^{1/2}$, i.e., $Tr\phi(1) = |G/Z(G)|^{1/2}$. Let $d$ denote $Tr\phi(1)$. The set $\xi$ of unitary matrices, $\xi = \{\phi(g)\mid g \in G/Z(G)\}$ forms a nice error basis \[8\] in the $d$ dimensional Hilbert space. The quotient group $G/Z(G)$ is called the index group associated with the nice error basis $\xi$, with the order $|G/Z(G)| = d^2$.

The abstract group $E_n^\nu$ has been recently explored \[12\] \[13\] \[14\], and it is yielded by generators $e_1, \cdots, e_n$ satisfying:

\begin{align*}
(a) & \quad e_i^2 = \nu, \quad i = 1, \cdots, n, \\
(b) & \quad e_ie_{i+1} = -e_{i+1}e_i, \quad i = 1, \cdots, n-1, \\
(c) & \quad e_ie_j = e_je_i, \quad |i-j| \geq 2, \quad i, j = 1, \cdots, n
\end{align*} \hspace{1cm} (3.1)

with $\nu$ an element of the center $Z(E_n^\nu)$, either $\nu = 1$ or $\nu = -1$. The center $Z(E_n^\nu)$ is $Z_2 = \{\pm 1\}$ and the center $Z(E_{2k}^\nu)$ is $\{\pm 1, \pm e_1e_3 \cdots e_{2k+1}\}$ isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The order of $E_n^\nu$ is $|E_n^\nu| = 2^{n+1}$. The quotient group $E_n^\nu/Z_2$ is isomorphic to the elementary abelian two-group, and hence $E_{2k}^\nu$ is isomorphic to extraspecial two-groups.

The abstract groups $E_n^\nu$ can be easily verified to be abstract error groups, with the help of irreducible representation theory \[12\]. $E_{2k}^\nu$ has a $2^k$-dimensional faithful irreducible unitary representation $\rho$ with the degree $2^k = |E_{2k}^\nu/Z_2|^{1/2}$. $E_{2k+1}^\nu$ has two inequivalent $2^k$ dimensional faithful irreducible representations $\lambda_1$ and $\lambda_2$ with the same degree $2^k = |E_{2k+1}^\nu/Z(E_{2k+1}^\nu)|^{1/2}$.

The related nice error bases are respectively denoted by $\rho_{2k}$, $\lambda_{1,2k}$, $\lambda_{2,2k}$ (see Subsection 4.1).

3.2 The classification of $E_n^\nu$

We firstly classify the abstract error groups $E_n^{-1}$ and secondly classify $E_n^1$.

**Theorem 3.2.** Abstract error groups $E_n^{-1}$ are isomorphic to central products of quaternion groups $Q$, dihedral groups $D$, $\mathbb{Z}_2$ or $\mathbb{Z}_4$. They are classified into three cases:

\[\begin{align*}
E_{8j}^{-1} & \cong D_{4j}, \\
E_{8j+1}^{-1} & \cong Z_4 \circ D_{4j}, \\
E_{8j+2}^{-1} & \cong QD_{4j}, \\
E_{8j+3}^{-1} & \cong Z_2 \circ QD_{4j}, \\
E_{8j+4}^{-1} & \cong QD_{4j+1}, \\
E_{8j+5}^{-1} & \cong Z_4 \circ QD_{4j+1}, \\
E_{8j+6}^{-1} & \cong D_{4j+3}, \\
E_{8j+7}^{-1} & \cong Z_2 \circ D_{4j+3} 
\end{align*}\] \hspace{1cm} (3.3)

which are respectively isomorphic to extraspecial two-groups, almost extraspecial two-groups, and nearly extraspecial two-groups.
Proof. We classify $E_{2n}^{-1}$, and then use the classification of $E_{2n}^{-1}$ to classify $E_{2n+1}^{-1}$, The proof can be read in the three steps.

(1). Even cases. The abstract error group $E_{2n}^{-1}$ is verified as the central product of its two subgroups $E_{2n-2}^{-1}$ and $E_2^{-1}$, i.e., $E_{2n}^{-1} \cong E_{2n-2}^{-1} \circ E_2^{-1}$. The group $E_{2n-2}^{-1}$ is generated by $e_1, e_2, \ldots, e_{2n-2}$, and $E_2^{-1}$ denotes the subgroup $< e_1 e_3 \cdots e_{2n-1}, e_{2n} >$ generated by $e_1 e_3 \cdots e_{2n-1}$ and $e_{2n}$. The two generators of $E_2^{-1}$ satisfy the algebraic relations,

$$
e_{2n}^2 = -1, \quad (e_1 e_3 \cdots e_{2n-1})^2 = (-1)^n,
\quad (e_1 e_3 \cdots e_{2n-1}) e_{2n} = -e_{2n} (e_1 e_3 \cdots e_{2n-1}),$$

and hence we observe that $E_2^{-1}$ at $n = 2k$ is isomorphic to the dihedral group $D$ as well as $E_2^{-1}$ at $n = 2k + 1$ is isomorphic to the quaternion group $Q$.

The intersection set of $E_{2n-2}^{-1}$ and $E_2^{-1}$ is $Z_2 = \{ \pm 1 \}$ since they have different generators. Two generators of $< e_1 e_3 \cdots e_{2n-1}, e_{2n} >$ are commutative with generators of $E_{2n-2}^{-1}$ due to the defining relations of $E_{2n}^{-1}$. The number of group elements $g$ of $E_{2n}^{-1}$ having the form $g = g_1 g_2$ with $g_1 \in E_{2n-2}^{-1}, g_2 \in E_2^{-1}$, is counted in the following

$$2 \times 2^{2n-2} \times 2^2 = 2^{1+2n}$$

where the integers from the left to the right respectively denote the order of the center $Z_2$, the order of $E_{2n-2}^{-1} / Z(E_{2n-2}^{-1})$, the order of $E_2^{-1} / Z(E_2^{-1})$, and the order of $E_{2n}^{-1}$. Such a counting completes our proof that $E_{2n}^{-1} = E_{2n-2}^{-1} \circ E_2^{-1}$, namely,

$$E_{2n}^{-1} \cong E_{2n-2}^{-1} \circ D, \quad E_{4k+2}^{-1} \cong E_{4k}^{-1} \circ Q.$$

Solving the above recursive formula between $E_{2n}^{-1}$ and $E_{2n-2}^{-1}$ leads to $E_{4k}^{-1} \cong Q^k D^k$ and $E_{4k+2}^{-1} \cong Q^{k+1} D^k$. With the isomorphic relation $D^2 \cong Q^2$, we finally classify $E_{2n}^{-1}$ into four classes,

$$E_{8j}^{-1} \cong D^{4j}, \quad E_{8j+2}^{-1} \cong Q D^{4j}, \quad E_{8j+4}^{-1} \cong Q D^{4j+1}, \quad E_{8j+6}^{-1} \cong D^{4j+3}.$$

(2). Odd cases. Let us now study the classification of the abstract error group $E_{2n+1}^{-1}$. The group element $e_1 e_3 \cdots e_{2n+1}$ commutes with all elements of $E_{2n+1}^{-1}$, and so it is in the center $Z_4(E_{2n+1}^{-1})$. The subgroup $< e_1 e_3 \cdots e_{2n+1} >$ generated by $e_1 e_3 \cdots e_{2n+1}$ is either isomorphic to $Z_4$ at $n = 2k$ or isomorphic to $Z_2$ at $n = 2k + 1$, due to $(e_1 e_3 \cdots e_{2n+1})^2 = (-1)^{n+1}$. Hence $E_{2n+1}^{-1}$ is the central product of its two subgroups $E_{2n}^{-1}$ and $< e_1 e_3 \cdots e_{2n+1} >$, namely,

$$E_{4k+1}^{-1} \cong E_{4k}^{-1} \circ Z_4, \quad E_{4k+3}^{-1} \cong E_{4k+2}^{-1} \circ Z_2$$

which give rise to the classification of $E_{2n+1}^{-1}$ as follows

$$E_{8j+1}^{-1} \cong E_{8j}^{-1} \circ Z_4, \quad E_{8j+3}^{-1} \cong E_{8j+2}^{-1} \circ Z_2,
E_{8j+5}^{-1} \cong E_{8j+4}^{-1} \circ Z_4, \quad E_{8j+7}^{-1} \cong E_{8j+6}^{-1} \circ Z_2.$$
Note that $E_{8j+3}^{-1}$ and $E_{8j+7}^{-1}$ have the center $Z_2 \times Z_2$ in terms of the first $Z_2 = \{\pm 1\}$ and the second $Z_2 = \langle e_1 e_3 \cdots e_{8j+3} \rangle$ or $Z_2 = \langle e_1 e_3 \cdots e_{8j+7} \rangle$.

(3). Concluding the steps (1) and (2), we prove our theorem and classify the abstract groups $E_n^{-1}$ into three categories: $E_{8j+1}^{-1}$, $E_{8j+2}^{-1}$, $E_{8j+4}^{-1}$ and $E_{8j+6}^{-1}$ are extraspecial two-groups with the center $Z_2$; $E_{8j+1}^{-1}$ and $E_{8j+5}^{-1}$ are almost extraspecial two-groups with the center $Z_4$; $E_{8j+3}^{-1}$ and $E_{8j+7}^{-1}$ are nearly extraspecial two-groups with the center $Z_2 \times Z_2$. □

**Theorem 3.10.** Abstract error groups $E_n^1$ are classified into two categories. The ones at $n = 2k$ are isomorphic to extraspecial two-groups, i.e., $E_{2k}^1 \cong D^k$, and the others at $n = 2k + 1$ are isomorphic to nearly extraspecial two-groups with the center $Z_2 \times Z_2$, i.e., $E_{2k+1}^1 \cong Z_2 \circ D^k$.

**Proof.** We follow the methodology of the proof of Theorem 3.2. It is easy to verify $E_2^1 \cong D$ and $< e_1 e_3 \cdots e_{2k-1}, e_{2k} > \cong D$ which give rise to $E_{2k}^1 \cong D^k$. It is also explicit that $< e_1 e_3 \cdots e_{2k-1} > \cong Z_2$ leads to $E_{2k+1}^1 \cong Z_2 \circ D^k$ with the center $Z_2 \times Z_2$. □

**Remarks 3.11.** Based on the above two theorems, we are able to state that the abstract groups $E_n^{-1}$ are abstract error groups which are often not isomorphic to the Pauli groups, see the following subsection.

### 3.3 Comparisons of $E_n^\nu$ with the Pauli groups

A two-dimensional Hilbert space $\mathcal{H}_2 \cong \mathbb{C}^2$ over the complex field $\mathbb{C}$ is called a *qubit* in quantum information and computation. The symbol $I_2$ denotes the 2-dimensional identity operator or $2 \times 2$ identity matrix. The Pauli matrices $X, Y, Z$ have the conventional forms,

$$
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

(3.12)

respectively denoting the bit-flip, phase-flip, and bit-phase flip operations on a single qubit in quantum error correction theory. An $n$-fold tensor product in terms of $I_2$ and Pauli matrices $X, Y, Z$ has a simpler notation. For example, a 9-fold tensor product $Z_1 Z_2 = Z \otimes Z \otimes (I_2)\otimes^7$ where the notation $(I_2)\otimes^7$ denotes a 7-fold tensor product of $I_2$.

In the literature, the standard abstract error groups for the $k$-qubit Hilbert space $\mathcal{H}_2^{\otimes k}$ are the real Pauli group $\mathcal{P}_k$ or the complex Pauli group $\mathcal{P}_k'$, and the related Pauli error bases are also denoted by $\mathcal{P}_k$ or $\mathcal{P}_k'$. The Pauli group $\mathcal{P}_k$ with the center $Z(\mathcal{P}_k) = Z_2$ has the generators,

$$
\mathcal{P}_k : \quad X_1, X_2, \cdots, X_k; Z_1, Z_2, \cdots, Z_k
$$

(3.13)

which satisfy algebraic relations defining the Pauli group,

\[
\begin{align*}
X_i^2 &= Z_i^2 = 1, & X_i Z_i &= -Z_i X_i, & i = 1, \cdots, k \\
X_i X_j &= X_j X_i, & Z_i Z_j &= Z_j Z_i, & i, j = 1, \cdots, k \\
X_i Z_j &= Z_j X_i, & i \neq j \text{ and } i, j = 1, \cdots, k.
\end{align*}
\]

(3.14)
Each pair of $X_i, Z_i$ yields a dihedral group $D$ so that $\mathcal{P}_k$ is isomorphic to the central product of dihedral groups, i.e., $\mathcal{P}_k \cong D^k$. Hence the real Pauli group $\mathcal{P}_k$ is an extraspecial two-group. The complex Pauli group $\mathcal{P}_k'$ is isomorphic to an almost extraspecial two-group $Z_4 \circ D^k$ with the center $Z_4$ generated by the imaginary unit $i$, i.e., $Z_4 = \{ \pm i, \pm 1 \}$.

In the following, we compare abstract error groups $\mathbf{E}_{2k}^\nu$, $\mathbf{E}_{2k+1}^\nu$ with the Pauli groups:

1. The generators of the Pauli group $\mathcal{P}_k$ have the form in terms of generators of $\mathbf{E}_{2k}^{-1}$,

\[
Y_i = e_{2i}, \quad i = 1, \ldots, k; \\
Z_i = (-\sqrt{-1})^{i_1 e_3 \cdots e_{2i-1}}, \\
X_i = Z_i Y_i = (-\sqrt{-1})^{i_1 e_3 \cdots e_{2i-1} e_{2i}}; \quad (3.15)
\]

2. The real Pauli groups $\mathcal{P}_k$ are isomorphic to $\mathbf{E}_{2k}^1$;

3. The complex Pauli groups $\mathcal{P}_k'$ with the center $Z_4$ are not isomorphic to $\mathbf{E}_{2k+1}^1$ with the center $Z_2 \times Z_2$;

4. $\mathbf{E}_{8j}^{-1}, \mathbf{E}_{8j+6}^{-1}$ are respectively isomorphic to $\mathcal{P}_{4j}, \mathcal{P}_{4j+3}$;

5. $\mathbf{E}_{8j+2}^{-1}, \mathbf{E}_{8j+4}^{-1}$ are very interesting cases since they contain a quaternion group $Q$ which is not in the Pauli groups;

6. $\mathbf{E}_{8j+1}^{-1}$ are isomorphic to the complex Pauli groups $\mathcal{P}_{4j}'$;

7. $\mathbf{E}_{8j+7}^{-1}$ are isomorphic to $\mathbf{E}_{8j+7}^1$, and both have the same center $Z_2 \times Z_2$.

As an example, we compare $\mathcal{P}_2$ with $\mathbf{E}_4^{-1}$. With the help of the formula (2.3), we observe that the real Pauli group $\mathcal{P}_2 \cong D^2$ has 20 order-2 elements,

\[
\pm 1, \quad \pm X_1, \quad \pm X_2, \quad \pm Z_1, \quad \pm Z_2, \quad \pm X_1 X_2, \quad \pm X_1 Z_2, \quad \pm X_2 Z_1, \quad \pm Z_1 Z_2, \quad \pm X_1 X_2 Z_1 Z_2 \quad (3.16)
\]

and $12 = 2^4 - (-2)^2$ order-4 elements,

\[
\pm X_1 Z_1, \quad \pm X_2 Z_2, \quad \pm X_1 X_2 Z_1, \quad \pm X_1 Z_1 Z_2, \quad \pm X_1 X_2 Z_2, \quad \pm X_2 Z_1 Z_2. \quad (3.17)
\]

\[
\begin{array}{cccccc}
  \mathcal{P}_k & \mathbf{E}_{2k}^1 & \mathbf{E}_{2k}^{-1} & \mathcal{P}_k' & \mathbf{E}_{2k+1}^1 & \mathbf{E}_{2k+1}^{-1} \\
  k = 1 & D & D & Q & \mathbb{Z}_4 \circ D & \mathbb{Z}_2 \circ Q \\
  k = 2 & D^2 & D^2 & QD & \mathbb{Z}_4 \circ D^2 & \mathbb{Z}_2 \circ QD \\
  k = 3 & D^3 & D^3 & D^3 & \mathbb{Z}_4 \circ D^3 & \mathbb{Z}_2 \circ D^3 \\
  k = 4 & D^4 & D^4 & D^4 & \mathbb{Z}_4 \circ D^4 & \mathbb{Z}_2 \circ D^4 \\
\end{array}
\]

Table 1: Comparisons of $\mathbf{E}_{2k}^\nu$, $\mathbf{E}_{2k+1}^\nu$ with $\mathcal{P}_k, \mathcal{P}_k'$, $k = 1, 2, 3, 4$. 


whereas the abstract error group \( E^{-1}_{4} \cong DQ \) has 12 order-2 elements,
\[
\pm 1, \pm e_1 e_3, \pm e_1 e_4, \pm e_2 e_4, \pm e_1 e_2 e_4, \pm e_1 e_3 e_4,
\]  
and \( 20 = 2^4 + (-2)^2 \) order-4 elements,
\[
\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_1 e_2, \pm e_2 e_3 e_4, \pm e_1 e_2 e_3 e_4.
\]  

Table 1 lists more examples for comparisons of our abstract error groups \( E'_{2k}, E'_{2k+1} \) with the Pauli groups \( P_k, P'_{k}, k = 1, 2, 3, 4 \).

**Remark 3.20.** The index groups \( E'_{n}/Z_2 \) are elementary abelian groups and so associated QECC are still stabilizer codes according to the Clifford code theory \[8\], which is the same as the Pauli groups.

### 4 Nice error bases and Jones UBGR at \( q = i, r = 4 \)

We recognize unitary irreducible representations of abstract error groups \( E'_{n} \) as nice error bases, and set up a Jones UBGR at \( q = i, r = 4 \) (or modulo a phase factor) in terms of these bases.

#### 4.1 Nice error bases associated with \( E'_{n} \)

Given an abstract error group \( G \) and its faithful irreducible representation \( \phi \), the associated nice error basis \( \xi = \{ \phi(g) | g \in G \} \) satisfies the defining relations \[8\]:

a). \( \phi(1) \) is the \( d \otimes d \) identity matrix, \( d = Tr\phi(1) \);

b). \( Tr \phi(g) = 0 \), for all \( g \in G/Z(G) \) and \( g \neq 1 \);

c). \( \phi(g) \phi(h) = \omega(g, h) \phi(gh) \) for all \( g, h \in G/Z(G) \);

where the set of complex numbers \( \omega(g, h) \) forms a cyclic group, and hence the nice error basis \( \xi \) is a projective faithful irreducible representation of the index group \( G/Z(G) \).

Now we study nice error bases associated with the abstract error groups \( E'_{n} \). Interested readers are invited to see \[12\] for irreducible representations of \( E'_{n} \). 1) Even cases at \( \nu = -1 \). The \( 2^k \)-dimensional irreducible representation \( \rho \) of \( E_{-1}^{2k} \) has the form,

\[
\begin{align*}
\rho(e_1) &= \sqrt{-1} Z_1, \\
\vdots \\
\rho(e_{2i-1}) &= \sqrt{-1} Z_{i-1} Z_i, \quad i = 1, 2, \cdots, k, \\
\rho(e_{2i}) &= Y_i, \\
\vdots \\
\rho(e_{2k}) &= Y_k.
\end{align*}
\]  
(4.1)
2) Odd cases at $\nu = -1$. The two inequivalent $2^k$-dimensional irreducible representations of $E^{-1}_{2k+1}$ are respectively denoted by $\lambda_i$, $i = 1, 2$, in which $\lambda_i(e_1)$, $\lambda_i(e_2), \ldots$, $\lambda_i(e_{2k})$ have the same form as in the representation $\rho$, except that $\lambda_i(e_{2k+1})$ is specified to be $\lambda_i(e_{2k+1}) = \pm \sqrt{-1}Z_k$, the + sign for $\lambda_1$ and - sign for $\lambda_2$.

The irreducible representation $\rho$ of $E^{-1}_{2k}$ leads to the nice error bases $\rho_{2k}$ given by

$$\rho_{2k} = \{\rho(g) | g \in E^{-1}_{2k} / Z(E^{-1}_{2k})\}$$

which is found to satisfy the above defining relations of nice error bases. The nice error bases associated with irreducible representations $\lambda_1, \lambda_2$ of the abstract error group $E^{-1}_{2k+1}$ are respectively denoted by $\lambda_1, \lambda_2$. It is explicit that $\lambda_1, \lambda_2$ represent the same nice error bases as $\rho_{2k}$ in the $2^k$-dimensional vector space.

As all generators of $E^{-1}_{2k}$ are rescaled by the imaginary unit $i$, the resulted group is isomorphic to the abstract error group $E^{-1}_{0}$. Hence nice error bases $\rho'_{2k}$ at $n = 2k$ and $\lambda'_{1,2k}, i = 1, 2$, at $n = 2k + 1$ associated with $E^{-1}_{n}$ are respectively obtained by $\rho_{2k}, \lambda_{1,2k}$ and $\lambda_{2,2k}$ times the imaginary unit $i$.

**Remark 4.2.** Nice error bases $\rho_{2k}$ ([11]) are recast in terms of the Pauli matrices with relevant phase factors, though associated abstract error groups $E^{-1}_{2k}$ are often not isomorphic to the Pauli groups $P_k$. $\rho_{2k}$ ([11]) can be proved to be equivalent to the Pauli error bases $P_k$ ([13]) modulo phase factors, in view of the definition [17] for the equivalence of two unitary error bases.

### 4.2 Jones UBGR at $q = i, r = 4$ via nice error bases

Artin’s braid group $B_n$ on $n$ strands has a presentation in terms of generators $b_1, \ldots, b_{n-1}$ satisfying the commutation relation,

$$b_ib_j = b_jb_i, \quad |i - j| \geq 2$$

and the braid relations,

$$b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}, \quad 1 \leq i \leq n - 2.$$  \hfill (4.4)

The symmetric group $S_n$ includes all possible permutations of $n$ objects, and it is generated by transpositions $(i, i + 1)$ between $i$-th object and $(i + 1)$-th object, $1 \leq i \leq n - 1$. $B_n$ has a finite-index normal subgroup $PB_n$ generated by all conjugates of the squares of the generators of $B_n$, see [12]. $PB_n$ is called the pure braid group and can be understood as the kernel of the surjective homomorphism $B_n \to S_n$ given by $b_i \mapsto (i, i + 1)$. In other words we have an isomorphism $S_n \cong B_n/PB_n$.

**Theorem 4.5.** Let $\{T_1, \ldots, T_{n-1}\}$ be the images of the generators $\{e_1, \ldots, e_{n-1}\}$ of $E^{-1}_{n-1}$ under a representation $\phi_{n-1}$ of $E^{-1}_{n-1}$ such that:

(a) $T_i^2 = -Id$,

(b) $T_iT_j = T_jT_i$ if $|i - j| > 1$, 

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(c) \( T_i T_{i+1} = -T_{i+1} T_i \) for all \( 1 \leq i \leq (n - 2) \).

Then the set of matrices \( \{ \hat{R}_1, \ldots, \hat{R}_{n-1} \} \) defined by \( \hat{R}_i = \frac{1}{\sqrt{2}} (I + T_i) \) gives a representation of \( B_n \) by \( b_i \rightarrow \hat{R}_i \). If in addition the \( T_i \) are anti-Hermitian (i.e. \( T_i = -T_i^\dagger \)), the \( B_n \) representation is unitary.

**Proof.** This theorem is firstly used [13] and then proved as Lemma 3.8 in [14]. \( \square \)

In terms of the \( 2^k \)-dimensional nice error bases \( \rho_{2^k} \), namely, \( T_i = \rho(e_i) \), the finite image \( \hat{\rho}_{2^k} \) [12] of the irreducible UBGR has the form,

\[
\hat{\rho}_{2^k} : \quad \hat{R}_i = d_i, \\
\vdots \\
\hat{R}_{2i-1} = D_{i-1,i}, \quad i = 2, \ldots, k, \\
\hat{R}_{2i} = f_i, \\
\vdots \\
\hat{R}_{2k} = f_k,
\]

where \( d = e^{i \frac{\pi}{2} Z} \), \( f = e^{i \frac{\pi}{2} Y} \) and lower indices of \( d_1, f_i, f_k \) have the same meaning as that of \( Z_1, Z_i, Z_k \), and the \( k \)-fold tensor product \( D_{i-1,i} \) denotes a form

\[
D_{i-1,i} = (I_2)^{\otimes i-1} \otimes D \otimes (I_2)^{\otimes k-i-1}, \quad D = e^{i \frac{\pi}{4} Z \otimes Z}.
\]

Note that the finite images \( \hat{\lambda}_{1,2^k} \) or \( \hat{\lambda}_{2,2^k} \) are associated with nice error bases \( \lambda_{1,2^k} \) and \( \lambda_{2,2^k} \), and \( \hat{R}_{2k+1} = d_k \) for \( \hat{\lambda}_{1,2^k} \) or \( \hat{R}_{2k+1} = \bar{d}_k \) for \( \hat{\lambda}_{2,2^k} \) where \( \bar{ \cdot } \) denotes the complex conjugation.

The irreducible representation \((\pi_{2^k+1}, (C^2)^{\otimes 2^k+1})\) of the braid group \( B_{2^{k+1}} \) under the \( \hat{R} \) matrices (4.6) are defined by \( \pi_{2^k+1}(b_i) = \hat{R}_i \). The finite image of the pure braid group \( \mathcal{PB}_n \) under this representation is denoted by \( H_{2^{k+1}} = \pi_{2^k+1}(\mathcal{PB}_{2^{k+1}}) \) which is isomorphic to the abstract error group \( E_{2^k} \), namely, \( H_{2^{k+1}} \cong E_{2^k}^{-1} \). The finite image of the braid group \( B_{2^{k+1}} \) under this representation denoted by \( G_{2^{k+1}} = \pi_{2^k+1}(B_{2^{k+1}}) \) is the extension of \( E_{2^k}^{-1} \) by the symmetric group \( S_{2^{k+1}} \), namely, \( G_{2^{k+1}} / E_{2^k}^{-1} \cong S_{2^{k+1}} \). Furthermore, the Jones UBGR at \( q = i, r = 4 \) [11] is obtained by rescaling the \( \hat{R} \) matrices (4.6) in the way \( \hat{R}_i' = -e^{-i \frac{\pi}{4}} \hat{R}_i \). The images of the braid group \( B_{2^{k+1}} \) and the pure braid group \( \mathcal{PB}_{2^{k+1}} \) under the representation \( \pi_{2^k+1}(b_i) = \hat{R}_i' \) are respectively denoted by \( G'_{2^{k+1}} = \pi_{2^k+1}'(B_{2^{k+1}}) \), \( H'_{2^{k+1}} = \pi_{2^k+1}'(\mathcal{PB}_{2^{k+1}}) \)

It is easy to verify \( H'_{2^{k+1}} \cong E_{2^k} \) and \( G'_{2^{k+1}} / E_{2^k} \cong S_{2^{k+1}} \). Interested readers are invited to consult [11, 12] for details.

**Remark 4.7.** The Jones UBGR at \( q = i, r = 4 \) (or modulo a phase factor) can be regarded as unitary basis transformation matrices from the product states to GHZ states [15], see Ref. [13, 14] and the references therein. Furthermore, the author has already initiated the project of constructing QECC involving unitary braid representations, for examples, QECC using GHZ states [16].
5 Concluding remarks

In this paper, we recognize finite images of the Jones unitary pure braid group representations at $q = i, r = 4$ as a type of abstract error groups. We classify them by central products of dihedral groups and quaternion groups, and then realize that they are often not isomorphic to the Pauli groups in the literature. In our further research, we will continue to explore interesting QECC associated with unitary representations of the braid group and develop related fault-tolerant quantum computation, for example, QECC using GHZ states [16]. Furthermore, we will revisit topics using the Pauli error bases in quantum information and computation with our nice error bases [4], for example, quantum teleportation [18] and noiseless subsystems [19].

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