Classifying all Mutually Unbiased Bases in \textbf{Rel}

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Abstract

Finding all the mutually unbiased bases in various dimensions is a problem of fundamental interest in quantum information theory and pure mathematics. The general problem formulated in finite-dimensional Hilbert spaces is open. In the categorical approach to quantum mechanics one can find examples of categories which behave “like” the category of finite-dimensional Hilbert spaces in various ways but are subtly different. One such category is the category of sets and relations, \textbf{Rel}. One can formulate the concept of mutually unbiased bases here as well. In this note we classify all the mutually unbiased bases in this category by relating it to a standard question in combinatorics.

1 Introduction

In the early 1960s Julian Schwinger \cite{Schwinger1959, Schwinger1960a, Schwinger1960c, Schwinger1960b} initiated a new approach to the foundations of quantum mechanics by basing the subject on the algebra of measurements. A mature presentation of this approach appears in the recent book \cite{Schwinger2003} published posthumously. Schwinger identified a fundamental concept: mutually unbiased bases which lay at the heart of the algebra of measurements and the geometry of quantum states.

A basis in the state space of a quantum system defines a measurement \cite{Peres1995}. Two bases are said to be \textit{mutually unbiased} if each vector of the first basis has the same inner product with every vector of the second.

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Thus two mutually unbiased bases (MUBs) define *complementary* observables. Finding maximum sets of MUBs in a vector space is a challenging problem in geometry and remains open. It is known that there are at most \( d + 1 \) MUBs in spaces of dimension \( d \). It is possible to achieve this upper bound when \( d \) is a prime power; see for example \cite{WB04}. However, for other dimensions little is known. In particular for dimension 6 it is known how to construct 3 MUBs and numerical evidence suggests that there are no more, but no proof is known.

The recent categorical approach to the foundations of quantum mechanics initiated by Abramsky and Coecke \cite{AC04} and pursued by Selinger \cite{Sel07} and others \cite{BCV08, CPP08, CES09} gives a number of “alternative universes” in which theories like – but not identical to – orthodox Hilbert-space-based quantum mechanics can be explored. In particular the category of sets and binary relations \( \text{Rel} \) has many of the features required for quantum mechanics, but is also clearly different. In this note we classify all the possible MUBs in this category. In particular, we show that there are only 3 when the underlying set has \( 6^2 = 36 \) elements. This is suggestive of the situation with finite-dimensional Hilbert spaces where only 3 mutually unbiased bases are known for 6 dimensions. However, the analogy is not perfect. The category \( \text{Rel} \) behaves like vector spaces with the scalars being \( \{0, 1\} \). Given a set \( S \) of \( n \) elements we can regard the elements of \( S \) as basis “vectors” and subsets of \( S \) can be regarded as formal linear combinations with coefficients 0 or 1. The set with 36 elements can be thought of as analogous to the space of \( 6 \times 6 \) matrices, which is not exactly the same as the case of 6 dimensions.

The proofs are based on combinatorial structures called Latin squares. The proof hints at connections with group representation theory that may help resolve the open questions in the category of finite-dimensional Hilbert spaces.

In the program of categorical quantum mechanics there have been some recent papers that set the stage for the present work. First, Coecke, Paquette, Pavlovic and Vicary \cite{BCV08, CPP08} have developed a theory of measurements in the abstract categorical framework. They have defined a *classical structure* as a space equipped with a notion of copying and deleting satisfying some basic algebraic laws. They have showed that in the category \( \text{FDHilb} \) of finite dimensional complex Hilbert spaces and linear maps this amounts to choosing a basis in the space. Later Coecke and Duncan \cite{CD09} developed a theory of interacting observables and described a diagrammatic presentation of the algebra of such pairs of observables. Such pairs of observables correspond exactly to MUBs.
In a recent paper Pavlovic [Pav09] classified all the classical structures in
the category Rel. He showed that all such structures come from direct sums
of finite abelian groups; a more precise statement appears below. Roughly
speaking, one has to partition a set and then provide an abelian group
structure on each block of the partition. In our classification of MUBs it
turns out that the partition is crucial but the abelian group structure chosen
is not.

2 Background

We review some of the basic combinatorial background. We refer to the pa-
pers of Coecke et al. for the background on categorical quantum mechanics,
but we will review the definition of a classical structure.

Definition 2.1. A $d \times d$ Latin square is a $d \times d$ array filled with $d$
symbols such that each row and column contains exactly 1 copy of each symbol.

Definition 2.2. A partition of a set $X$ is a set of disjoint sets $\pi$
such that

\[
\bigcup_{S \in \pi} S = X.
\]

Definition 2.3. In a dagger symmetric monoidal category, $f : A \to B$ is
unitary if $f \circ f^\dagger = \text{id}_B$, $f^\dagger \circ f = \text{id}_A$.

It is easy to see that in Rel, a relation is unitary if and only if it is a
bijective function.

Definition 2.4. A classical structure in a dagger symmetric monoidal cat-
egory $(C, \otimes, I)$ is a triple $(X, \delta, \varepsilon)$ where $X \in C_0$ is an object in $C$, $\delta : X \to
X \otimes X$ is a morphism called the copying operation and $\varepsilon : X \to I$ is called
the deletion and for which $(X, \delta^\dagger, \varepsilon^\dagger, \delta, \varepsilon)$ forms a special Frobenius algebra.

Definition 2.5. A point $p : I \to X$ is called classical for the classical
structure $(X, \delta, \varepsilon)$ if $\delta \circ p = p \otimes p$.

Definition 2.6. For any point $p : I \to X$ and classical structure $(X, \delta, \varepsilon)$,
define $\Lambda(p) = \delta^\dagger \circ (p \otimes \text{id}_X) \circ \lambda_X^{-1} : X \to X$. $p$ is called unbiased for the
classical structure $(X, \delta, \varepsilon)$ if $\Lambda(p)$ is unitary.

In Rel, what this means is that a set $U$ is unbiased for the classical
structure $(X, \delta, \varepsilon)$ if and only if the relation $R$ defined by

\[
x R y \iff \exists z \in U \text{ such that } x \sim (y, z)
\]
is a bijection.
3 Classical Structures in Rel

In [Pav09], Pavlovic showed that

**Theorem 3.1.** Every classical structure \((X, \delta, \epsilon)\) in \(\text{FRel}\) on a set \(X\) comes from choosing

- a partition of the set \(X\)
- an abelian group operation \(\cdot_S\) on every set \(S\) in the partition

where \(\delta\) is defined as follows:
\[x \sim (y, z)\] if \(y, z\) are in the same set \(S\) in the partition and \(x = y \cdot_S z\). Here \(\epsilon\) is the set of all the group identities.

Every partition and set of group operations uniquely determine a classical structure.

The following theorem tells us what the classical and unbiased points of a classical structure are.

**Theorem 3.2.** Suppose \((X, \delta, \epsilon)\) is a classical structure with partition \(\pi\). Then the classical points of \(X\) are exactly the sets of \(\pi\), and the unbiased points are the sets obtained by taking one element from each set of \(\pi\).

Note that the classical and unbiased points do not depend on the group structures chosen on the sets in the partition, only on the partition.

**Proof.** First we will prove that the classical points are the sets in \(\pi\).

Suppose we have a classical point \(P\). Choose \(S \in \pi\) such that \(|P \cap S| \neq \emptyset\).

Let \(x \in P \cap S\). For \(y, z \in S\), \(x \sim (y, z)\) if and only if \(x = y \cdot_S z\).

So it follows that \(x \sim (y, y^{-1}x) \forall y \in S\). Since \(\delta \circ P = P \otimes P\), this implies \(S \subseteq P\).

Now, suppose \(\exists y \in P \setminus S\) (that is, \(P \neq S\)). Then \((x, y) \in P \times P = \delta \circ P\), so \(\exists z \in P\) such that \(z \sim (x, y)\). But this is impossible, since \(x\) and \(y\) are in different sets in the partition. So \(S = P\) and we are done.

Now we need to show that the unbiased points are the sets obtained by taking one element from each set of \(\pi\). Recall that a point \(P\) is unbiased if and only the relation \(R\) defined by
\[x R y \iff \exists z \in P\) such that \(x \sim (y, z)\)

is a bijection.
Suppose $S \in \pi$, $x \in S$. Then the set of things that $x$ is related to by $R$ is exactly $\{x \cdot u^{-1} : u \in S \cap P\}$, so has cardinality $|S \cap P|$. So $R$ is a bijection if and only if $|S \cap P| = 1$ for all $S \in \pi$.

**Remark 3.1.** The maps $\Lambda(p)$ with composition form a group isomorphic to the direct product of the groups chosen on the sets $S$ in the partition.

The following corollary will be useful in our analysis of complementary classical structures.

**Corollary 3.1.** If $S_1, S_2 \subseteq X$ are unbiased points for $(X, \delta, \epsilon)$, then $|S_1| = |S_2|$.

**Proof.** Every unbiased point $S$ is constructed by taking one element from each set in the partition, so $|S|$ is always the number of sets in the partition.

4 Complementary classical structures

In this section we give a complete characterisation of complementary classical structures in $\text{Rel}$ in terms of their partitions, and reduce the problem of finding complementary classical structures to the well-studied combinatorial problem of finding mutually orthogonal Latin squares (MOLS).

**Definition 4.1.** Two classical structures $(X, \delta, \epsilon)$ and $(X', \delta', \epsilon')$ are called complementary if each classical point of $X$ is an unbiased point of $X'$ and vice versa.

**Definition 4.2.** A partition is uniform if all of its parts have the same size.

**Definition 4.3.** Two partitions $\pi_1$ and $\pi_2$ are complementary if for every $S \in \pi_1, T \in \pi_2$, $|S \cap T| = 1$.

Consider two classical structures on $\{1,2\}$ with partitions $\{\{1\}, \{2\}\}, \{\{1,2\}\}$. We will write these partitions as

\[
\begin{array}{c}
1 \\
2
\end{array},
\begin{array}{c}
1 \\
2
\end{array}.
\]

respectively. The first has classical points $\{1\}, \{2\}$, and unbiased point $\{1,2\}$, and the second has classical point $\{1,2\}$, and unbiased points $\{1\}, \{2\}$. So these two observables are complementary.
Now consider a classical structure \((X, \delta, \epsilon)\) with partition

\[
\begin{array}{ccc}
1 & 3 \\
2 & & \\
\end{array}
\]

This has classical points \(\{1,3\}, \{2\}\) and unbiased points \(\{1,2\}, \{3,2\}\). We know by Corollary 3.1 that \(\{1,3\}\) and \(\{2\}\) cannot be unbiased points for the same classical structure since they have different cardinality, so there is no classical structure complementary to \(X\)! This does not happen in \(\text{FDHilb}\), where every observable in a space of dimension greater than 1 has a complementary observable. The same argument shows that any classical structure that does not have a uniform partition has no complementary classical structure.

Theorem 3.2 says that two classical structures are complementary if and only if their partitions are complementary.

Now, suppose we have a classical structure \(X\) with a uniform partition

\[
\pi = \begin{array}{cccc}
\pi_{11} & \pi_{12} & \cdots & \pi_{1k} \\
\pi_{21} & \pi_{22} & \cdots & \pi_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{\ell 1} & \pi_{\ell 2} & \cdots & \pi_{\ell k}
\end{array}
\]

Then the transpose partition

\[
\tau = \begin{array}{cccc}
\pi_{11} & \pi_{21} & \cdots & \pi_{\ell 1} \\
\pi_{12} & \pi_{22} & \cdots & \pi_{\ell 2} \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{1k} & \pi_{2k} & \cdots & \pi_{\ell k}
\end{array}
\]

is complementary to \(\pi\), so if we take a classical structure \(X'\) that has this partition, then \(X'\) is complementary to \(X\).

So from the previous discussion, we have the following theorem:

**Theorem 4.1.** A classical structure has a complementary classical structure if and only its partition is uniform. Further, two classical structures with partitions \(T_1\) and \(T_2\) are complementary if and only if the partitions \(T_1\) and \(T_2\) are complementary.

We also have the following corollary:

**Corollary 4.1.** If a classical structure’s partition is not square, then it cannot be part of a set of 3 or more mutually complementary classical structures (MCCS).
So to have more than 2 mutually complementary classical structures on \( n \) elements, we need \( n = d^2 \) for some \( d \) and also for the classical structures’ partitions to be square.

Now, let us consider square \( d \times d \) partitions.

**Example 4.1.** If \( d = 2 \), then we can construct 3 mutually complementary classical structures on 4 elements from the following 3 partitions:

\[
\begin{array}{ccc}
1 & 2 & \\
3 & 4 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & \\
2 & 4 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 4 & \\
2 & 3 & \\
\end{array}
\]

**Example 4.2.** Similarly, if \( d = 3 \), we can construct 4 mutually complementary classical structures on 9 elements from the following 4 partitions:

\[
\begin{array}{ccc}
1 & 2 & 3 & \\
4 & 5 & 6 & \\
7 & 8 & 9 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 4 & 7 & \\
2 & 5 & 8 & \\
3 & 6 & 9 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 6 & 8 & \\
2 & 4 & 9 & \\
3 & 5 & 7 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 5 & 9 & \\
2 & 6 & 7 & \\
3 & 4 & 8 & \\
\end{array}
\]

**Lemma 4.1.** From any set of \( k \) mutually complementary partitions on a set of \( d^2 \) elements we can construct \( k - 2 \) mutually orthogonal Latin squares.

**Proof.** Suppose we have partitions \( \pi_1, \pi_2, \ldots, \pi_k \). Then we can define a table

\[
T = \begin{array}{cccc}
T_{11} & T_{12} & \ldots & T_{1d} \\
T_{21} & T_{22} & \ldots & T_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
T_{d1} & T_{d2} & \ldots & T_{dd} \\
\end{array}
\]

such that the rows of \( T \) are the parts of \( \pi_1 \) and the columns of \( T \) are the parts of \( \pi_2 \).

Now, for any partition \( \pi \) of \( X \) complementary to both \( \pi_1 \) and \( \pi_2 \), we can define a Latin square \( L_\pi \) by assigning a symbol \( \alpha_S \) to each part \( S \) of \( \pi \), and letting

\[
L_{\pi_{ij}} = \alpha_S \text{ if and only if } T_{ij} \in S
\]

For any such partitions \( \sigma \) and \( \tau \), \( L_\sigma \) and \( L_\tau \) are orthogonal Latin squares if and only if \( \sigma \) and \( \tau \) are complementary partitions.

So from \( \pi_1, \pi_2, \ldots, \pi_k \) we get the \( k - 2 \) orthogonal Latin squares \( L_{\pi_3}, L_{\pi_4}, \ldots, L_{\pi_k} \). \( \square \)

**Example 4.3.** We can obtain the partitions in Example 4.2 from the table
and the orthogonal Latin squares

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}
\quad \quad
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{array}
\quad \quad
\begin{array}{ccc}
1 & 3 & 2 \\
2 & 1 & 3 \\
3 & 2 & 1 \\
\end{array}
\]

We can also prove the converse of this:

**Lemma 4.2.** From any set of \( k \) mutually orthogonal \( d \times d \) Latin squares, we can construct \( k + 2 \) mutually complementary partitions of a set \( X \) with \( d^2 \) elements.

**Proof.** Suppose we have \( k d \times d \) Latin squares \( L^1, \ldots, L^k \). Put the elements of \( X \) into a \( d \times d \) array \( T \).

Let \( \sigma \) and \( \tau \) be the partitions of \( X \) obtained from the rows and columns of \( X \) respectively.

For each Latin square \( L^i \), define the partition

\[
\pi_i = \{ \{ T_{jk} \in A : L^i_{jk} = \alpha \} : \alpha \text{ a symbol of } L^i \}
\]

Then the partitions \( \sigma, \tau, \pi_1, \ldots, \pi_k \) form a set of \( k + 2 \) mutually complementary partitions.

Since two classical structures \( X, X' \) are complementary if and only if their partitions are complementary, we have

**Theorem 4.2.** There are \( k \) classical structures on a set with \( d^2 \) elements if and only if there exist \( k - 2 d \times d \) mutually orthogonal Latin squares.

The equivalence of sets of mutually complementary partitions and MOLS can be found in [CD06].

Wocjan and Beth showed that if there are \( k d \times d \) Latin squares then there are \( k + 2 \) MUBS in \( \mathbb{C}^{d^2} \) [WB04], so we have the following corollary relating \( MCCS \) in \( \textsf{FRel} \) and \( \textsf{FDHilb} \).

**Corollary 4.2.** If there are \( k \) MCCS on \( d \) elements in \( \textsf{FRel} \), then there are \( k \) MUBS in \( \mathbb{C}^{d^2} \).
In particular, we know that there are \( d - 1 \) orthogonal \( d \times d \) Latin squares if \( d \) is a prime power, and that there are no pairs of orthogonal Latin squares on 6 elements [CD06], so we have the following corollary (the number of MUBS on 6 elements in \( \text{FDHilb} \) is unknown):

**Corollary 4.3.** There are at most \( d + 1 \) MCCS on \( d^2 \) elements. If \( d \) is a prime power, then there are \( d + 1 \) MCCS on \( d^2 \) elements. For \( d = 6 \), there are exactly 3 MCCS on \( d^2 \) elements.

## 5 Conclusions

The category \( \text{Rel} \) is a “toy” version of quantum mechanics where one can explore ideas in a simpler setting. In our proofs the structures that appear are partitions of sets and Latin squares. It suggests that for Hilbert spaces the relevant structures might be Young tableaux though, at present, we have no idea how to pursue this thought. Another intriguing link with algebra comes from that fact that Latin squares are the multiplication tables of loops (non-associative analogues of groups).

The proofs depend on specific properties of \( \text{Rel} \); this suggests that the abstract diagrammatic algebra, while useful for general results, will not help with tackling the problem of classifying MUBs in \( \text{FDHilb} \).

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