MORITA’S DUALITY FOR SPLIT REDUCTIVE GROUPS

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Abstract. In this paper, we extend the work in Morita’s Theory for the Symplectic Groups [?] to split reductive groups. We construct and study the holomorphic discrete series representations and the principal series representations of a split reductive group $G$ over a $p$-adic field $F$ as well as a duality between certain sub-representations of these two representations.

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Notations

Let $p$ be a prime, $F$ a finite extension of $\mathbb{Q}_p$, $\mathfrak{o}$ the ring of integers of $F$, $\varpi$ the uniformizer of $\mathfrak{o}$, $| |$ the normalized absolute value, and $F^{\text{alg}}$ an algebraic closure of $F$. Let $K$ be an extension of $F$ with an absolute value extending $| |$, and $\mathfrak{o}_K$ the valuation ring of $K$. We assume that $K$ is complete with respect to $| |$, and moreover, $K$ is spherically complete whenever topological properties of the $K$-vector spaces are under consideration.

1. Introduction

In a series of their papers, [4], [5] and [6], Morita and Murase started the work on the representation theory for $\text{SL}(2, F)$ with coefficient field $K$, especially the holomorphic discrete series representations and their duality relations with the principal series representations. The principal series representations, or more generally the induced representations, appeared in many literatures, notably in Féaux de Lacroix’s work [2] on the locally analytic representations. On the other hand, the holomorphic discrete series representations were not extensively studied. The holomorphic discrete series representations of $\text{SL}(n + 1, F)$ associated to a rational representation of $\text{GL}(n, F)$ were introduced by Schneider in [8], in order to understand the de Rham complex over Drinfel’d’s space as representations of $\text{SL}(n + 1, F)$. In another direction, the holomorphic discrete series representations of $\text{Sp}(2n, F)$ associated to a $K$-rational representation of $\text{GL}(n, F)$ were constructed in our recent work [7]. Furthermore, the algebraization and generalization of Morita’s duality were established in [7].

The purpose of this paper is to generalize Morita’s theory from $\text{Sp}(2n, F)$ to a split reductive group $G$. We are able to do the generalization owing to the entirely algebraic construction of Morita’s theory for $\text{Sp}(2n, F)$. Therefore we closely follow the main ideas presented in [7].

In the first paragraph, we recollect some notions on split reductive groups, and construct an $F$-regular function $f$ on $G$ that characterizes the parabolic big cell associated to a parabolic subgroup. $f$ will appear extensively in the study of the rigid symmetric space associated to $G$ and the holomorphic series representations. $f$ corresponds to the determinant function on $\text{M}(n, F)$ used in the definition of $p$-adic Siegel upper half-space in [7] (see Example 2.5 and 4.3).

The principal series representation $(C^\text{an}_{\sigma}(\mathfrak{h}, V), T_{\sigma})$ is another interpretation of the parabolic induction from a locally analytic $K$-representation $(\sigma, V)$ of the Levi
subgroup (cf. [5] and [7]). In the second paragraph, applying the general results of Féraux de Lacroix on the induced representations of $F$-Lie groups ([2]), we see that $(C_{\sigma'}^{an}(\mathcal{S}, V), T_{\sigma'})$ is a locally analytic representation of $G$ over a $K$-vector space of compact type.

The third paragraph is dedicated to the construction and study of the rigid analytic symmetric space $\Omega$ which is the foundation of the holomorphic discrete series representations. The examples are the $p$-adic upper half-plane for $SL(2, F)$ (cf. [4]), Drinfel’d’s space for $SL(n + 1, F)$ (cf. [8]) and the $p$-adic Siegel upper half-space for $Sp(2n, F)$ (cf. [7]). $\Omega$ has been studied by van der Put and Voskuil in [14]. However, we shall use another approach following [10] and [7] to construct the admissible affinoid covering, which enables us to obtain precise descriptions of the rigid analytic functions on $\Omega$. We prove that the space $\mathcal{O}_K(\Omega)$ of $K$-rigid analytic functions on $\Omega_K$ is a nuclear $K$-Fréchet space.

In the fourth paragraph, for a $K$-rational representation $(\sigma, V)$ of the Levi subgroup, we construct the holomorphic discrete series representation $(\mathcal{O}_{\sigma}(\Omega), \pi_{\sigma})$ defined over the nuclear $K$-Fréchet space of $V$-valued rigid analytic functions on $\Omega$. We prove that its dual representation is locally analytic.

Since the strong duality defines a contra-variant equivalence between the category of $K$-vector spaces of compact type and the category of nuclear $K$-Fréchet spaces (cf. [11]), it is natural to expect certain duality relations between the subquotient spaces of the principal series representations and those of the holomorphic discrete series representations. For $SL(2, F)$, Morita analytically constructed a duality of this kind via the residues (cf. [5]). There does not seem to be any direct way to generalize Morita’s duality. However, as we did in [7], we may algebraically interpret Morita’s duality in a weaker form and generalize it to any split reductive group $G$. These are done in the last paragraph. For a $K$-rational representation $(\sigma, V)$ of the Levi subgroup, we algebraically define a closed sub-representation $B_{\sigma'}(\mathcal{S}, V')$ of $C_{\sigma'}^{an}(\mathcal{S}, V')$ and a closed sub-representation $\mathcal{N}_{\sigma}(\Omega)$ of $\mathcal{O}_{\sigma}(\Omega)$, and construct a duality between them. As discussed in [7], our duality for $SL(2, F)$ is equal to Morita’s duality composed with Casselman’s intertwining operator (cf. [5]).

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2. A Lemma on the Split Reductive Groups

2.1. Split reductive groups. We adopt the notations in [3] Part II, Chapter 1.

Let \( G \) be a connected split reductive algebraic group over \( F \), \( T \) a split maximal torus of \( G \). We have the decomposition of Lie algebra \( \mathfrak{g} \) of \( G \) (over \( F \)) in the form

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,
\]

where \( \mathfrak{g}_0 \) is the Lie algebra of \( T \) and \( R \) is the root system of \( G \) with respect to \( T \).

Each \( \mathfrak{g}_\alpha \) is of rank 1 over \( F \), and we denote \( U_\alpha \simeq G_a(F) \) the root subgroup of \( G \) corresponding to \( \alpha \).

Let \( W / \sim \equiv N_G(T)/T \) be the Weyl group of \( R \). For \( w \in W \), we also denote \( w_a \) representing element in \( N_G(T) \).

Choose a positive system \( R^+ \) and denote \( S \) the corresponding set of simple roots. Let \( B^+ \) denote the corresponding Borel subgroup and \( B^- \) its opposite Borel subgroup, \( U^\pm = U(\pm R^+) \) the unipotent radical of \( B^\pm \).

Throughout this article we fix a subset \( I \) of \( S \), and denote \( R_I = R \cap \mathbb{Z}I \), \( W_I \) the Weyl group of \( R_I \), \( R^+_I \) the standard parabolic subgroup of \( G \) corresponding to \( R^+_I \), \( P^- \) its opposite parabolic subgroup, \( U^\pm = U(\pm R^+_I) \) the unipotent radical of \( P^\pm \), \( L \) the common Levi subgroup of \( P^+ \) and \( P^- \).

\( L \) is a split reductive group with split maximal torus \( T \), root system \( R_I \), positive system \( R^+_L = R_I \cap R^+ \) and Weyl group \( W_I \). Let \( U^\pm_L = U(\pm R^+_L) \).

We recall ([3] Part II.1.7) that for any closed and unipotent subset \( R' \) of \( R \) (that is, \( (\mathbb{N} \alpha + \mathbb{N} \beta) \cap R \subset R' \) for any \( \alpha, \beta \in R' \) and \( R' \cap (-R') = \emptyset \)), for instance \( \pm R^+, \pm R^+_I \) and \( \pm R^+_L \), the multiplication induces, for any ordering of \( R' \), an isomorphism of schemes over \( F \)

\[
\prod_{\alpha \in R'} U_\alpha \simeq U(R').
\]

2.2. Parabolic big cell. We have the Bruhat decomposition of \( G \) ([3] Part II.1.9)

\[
G = \bigcup_{w \in W} B^-wB^+ = \bigcup_{w \in W} U^-_GwTU^+_G.
\]

Let \( C \) denote the parabolic big cell

\[
C = \bigcup_{w \in W_I} U^-_GwTU^+_G = U^-(\bigcup_{w \in W_I} U^-_LwTU^+_L)U^+ = U^-LU^+ = U^-P^+ = P^-U^+.
\]

Then \( G \) is the disjoint union of \( C \) and \( U^-_Gu TU^+_G \) for all \( w \in W_I \).

Let \( r = |R^+_I| = \dim U^+ \). We consider the adjoint representation of \( G \) on \( \wedge' \mathfrak{g} \) over \( F \). From the decomposition (2.1) of \( \mathfrak{g} \) we obtain a direct sum decomposition
of $\wedge^r g$. Choosing $X_\alpha$ a nonzero element in $g_\alpha$ for each $\alpha \in R$ and a basis of $g_0$ we obtain a basis of $\wedge^r g$ with respect to this decomposition and containing $Y = \wedge_{\alpha \in R_i^+} X_\alpha$. For $g \in G$ we define $f(g)$ to be the coefficient of $Y$ in the expansion of $\text{Ad}(g)Y$ in the chosen basis of $\wedge^r g$. Then $f$ is a regular function on $G$ over $F$.

There is a partial order on $\mathbb{Z}S$: $\gamma < \delta$ iff $\delta - \gamma$ is a sum of positive roots. On the set of the unordered $r$-tuples $[\gamma_1, ..., \gamma_r]$ of elements in $\mathbb{Z}S$ which, if we consider the group action of the symmetric group $S_r$ on $(\mathbb{Z}S)^r$ via coordinate permutation, may be seen as the set of $S_r$-orbits in $(\mathbb{Z}S)^r$, we define $[\gamma_1, ..., \gamma_r] < [\delta_1, ..., \delta_r]$ iff there exists $s \in S_r$ such that $\gamma_j \leq \delta_{s(j)}$ for all $1 \leq j \leq r$ and $\gamma_j < \delta_{s(j)}$ for at least one $j$.

We adopt the convention that $g_\gamma = 0$ if $\gamma \in \mathbb{Z}S$ is nonzero and not a root. Then $[g_\beta, g_\alpha] \subseteq g_{\alpha + \beta}$, and therefore

$$\text{Ad}(u_\beta)X_\alpha \in X_\alpha + \sum_{i \geq 1} g_{\alpha + i\beta}, \quad u_\beta \in U_\beta.$$  

If $\alpha, \beta \in R^+_i$, then it is clear that either $\alpha + i\beta \in R^+_i$ or $g_{\alpha + i\beta} = 0$. The same statement holds for $\alpha \in R^+_i$ and $\beta \in R_I$, since the $\beta$-string through $\alpha$ lies in $R^+_i$. Therefore $U_{\beta}$ fixes $Y$ for any $\beta \in R^+_i \cup R_I = R^+ \cup (-R^+_i)$. (2.2) implies that

$$(2.3) \quad Y \text{ is invariant under } U_{\gamma}^+ \text{ and } U_{\gamma}^-.$$

If we let $\beta$ be negative roots, then (2.2) also implies that for $v^- \in U_{G}^-,$

$$(2.4) \quad \text{Ad}(v^-)X_\alpha \in X_\alpha + \sum_{\gamma < \alpha} g_{\gamma}.$$  

For $t \in T$,

$$(2.5) \quad \text{Ad}(t)Y = \prod_{\alpha \in R^+_i} \alpha(t)Y.$$  

For $w \in W$ there exists a constant $c_{w,\alpha} \in F^\times$ satisfying

$$(2.6) \quad \text{Ad}(w)X_\alpha = c_{w,\alpha}X_{w\alpha}.$$  

In view of (2.2), we see that $w$ preserves $R^+_i$ iff $w$ normalizes $U^+.$ Since $N_G(U^+) = P^+$ and $w \in P^+$ iff $w \in W_I$,  

$$(2.7) \quad w \text{ preserves } R^+_i \text{ iff } w \in W_I.$$  

Since $P^+ = \bigcup_{w \in W_I} U^-_wTU_G^+,$ if we write $p^+ = u^-wv^+$ ($v^+ \in U^+_G,$ $t \in T,$ $w \in W_I,$ $u^- \in U^-_G$), then it follows from (2.3), (2.5), (2.6) and (2.7) that

$$\text{Ad}(p^+)Y = \text{sign}(w) \prod_{\alpha \in R^+_i} c_{w,\alpha} \alpha(t) \cdot Y.$$
where sign(w) denotes the sign of the permutation w on $R_I^+$. Moreover, it follows from (2.4) that

$$\text{Ad}(v^{-wtv^+})Y \in \text{sign}(w) \prod_{\alpha \in R_I^+} c_{w,\alpha} \alpha(t) \cdot Y + \sum_{[\gamma] | [\alpha] \in R_I^+} g_{Y_{\gamma}}.$$ 

So

$$f(v^{-wtv^+}) = \text{sign}(w) \prod_{\alpha \in R_I^+} c_{w,\alpha} \alpha(t).$$ 

Similarly, for $w \notin W_I$,

$$\text{Ad}(v^{-wtv^+})Y \in \sum_{[\gamma] \leq [w \alpha] \in R_I^+} g_{Y_{\gamma}}.$$ 

It follows from the proof of (2.3) that $\alpha + \beta \in R_I^+$ or $g_{\alpha + \beta} = 0$ if $\alpha \in R_I^+$ and $\beta \in R^+$, so $\alpha \in R_I^+$ and $\delta \geq \alpha$ imply $\delta \in R_I^+$ or $g_{\delta} = 0$. Therefore if $\delta \in \{0\} \cup R - R_I^+$ and $\gamma \leq \delta$ then $\gamma \not\in R_I^+$, and hence (2.7) implies that $Y$ does not appear in the expression of $\text{Ad}(v^{-wtv^+})Y$, so $f(v^{-wtv^+}) = 0$.

We conclude with the following lemma.

**Lemma 2.1.** Let the notations be as above, then

1. For $p^+ \in P^+$,

$$\text{Ad}(p^+)Y = f(p^+)Y,$$

and hence $f$ is an $F$-rational character on $P^+$.

2. For $w \in W_I$,

$$f(v^{-wtv^+}) = \text{sign}(w) \prod_{\alpha \in R_I^+} c_{w,\alpha} \alpha(t), \quad v^\pm \in U_{G}^\pm, \ t \in T.$$

3. $f$ vanishes on $U_{G}^- w T U_{G}^+$ for $w \notin W_I$.

In particular,

4. $C$ is an open $F$-subscheme of $G$, and $F[C] = F[G]_f$.

5. $f$ is right invariant under $U_{G}^+$ and left invariant under $U_{G}^-$. 

**Example 2.2.** [cf. [3] Part II 1.9 and [13] §5 Theorem 7] If $I = \emptyset$, then $L = T$ and $U^\pm = U_{G}^\pm$. Lemma 2.1 implies that $f(u^{-tu^+}) = \prod_{\alpha \in R^+} \alpha(t)$ and $f(u^{-wtu^+}) = 0$ for all nontrivial $w, u^\pm \in U_{G}^\pm$ and $t \in T$. 
Example 2.3. Let $G = \text{SL}(n+1, F)$. Write $g = \begin{pmatrix} A & B \\ C & d \end{pmatrix}$ with $A \in \text{M}(n, F), B \in \text{M}(n, 1; F), C \in \text{M}(1, n; F)$ and $d \in F$. Let

$$U^+ = \left\{ \begin{pmatrix} I_n & 0 \\ C & 1 \end{pmatrix} \in G \right\},$$

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in G \right\}. $$

Some calculations show that

$$f(g) = d^{n+1}. $$

Example 2.4. More generally, we consider $G = \text{GL}(n, F)$. Let $(n_1, ..., n_s)$ be a partition of $n$. Write $g = (g_{ij})_{1 \leq i,j \leq s}$ with $g_{ij} \in \text{M}(n_i, n_j; F)$. Let $U^+$ be the subgroup consisting of the matrices $u = (u_{ij})$ such that $u_{ij} = 0$ for $j < i$ and $u_{ii} = I_{n_i}$, and $L$ the subgroup consisting of the matrices $l = (l_{ij})$ such that $l_{ij} = 0$ for $i \neq j$. $L \cong \prod_{1 \leq i \leq s} \text{GL}(n_i, F)$.

For $l \in L$,

$$f(l) = \prod_{1 \leq j < i \leq s} \det(l_{ij})^{n_j} \det(l_{jj})^{-n_i} = \prod_{1 \leq i \leq s} \det(l_{ii})^{\sum_{j<i} n_j - \sum_{j<i} n_k}. $$

The computation of the explicit formula for $f$ involves the process of block lower (or upper) triangularization, and it turns out to be complicated if $s \geq 3$. For $s = 2$, we have

$$f(g) = \det(g_{22})^{n_1 + n_2} \det(g)^{-n_2}. $$

The situation is the same if $G = \text{SL}(n, F)$. For instance, if $s = 2$, then

$$f(g) = \det(g_{22})^{n_1 + n_2}. $$

Example 2.5. We consider

$$G = \text{Sp}(2n, F) = \left\{ g \in \text{GL}(2n, F) : \left( \begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right) g \right\}. $$
Write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in \text{M}(n, F)$. Let

$$U^+ = \left\{ \begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix} : C \in \text{Sym}(n, F) \right\},$$

$$L = \left\{ \begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix} : D \in \text{GL}(n, F) \right\},$$

where $\text{Sym}(n, F)$ is the group of symmetric matrices of order $n$ over $F$. Some calculations show that

$$f(g) = \det(D)^{n+1}.$$

3. Ind$_p^G \sigma$ and the principal series $(C^{an}_\sigma(\mathbb{S}, V), T_\sigma)$

Induced representations, especially the parabolic inductions, are of extreme importance in Lie theory. For $p$-adic Lie groups, they were studied by Féaux de Lacroix in his work ([2]) on the locally analytic representations.

We first recall the notion of locally analytic representations over $K$ of an $F$-Lie group.

**Definition 3.1** (cf. [2] and [11] §3). A locally analytic representation $(\sigma, V)$ of an $F$-Lie group $G$ on a barreled locally convex Hausdorff $K$-vector space $V$ is a continuous representation such that the orbit maps are $V$-valued locally analytic functions on $G$. More precisely, for any $v \in V$ there exists a BH-space $W$ of $V$ (that is, a Banach space $W$ together with a continuous injection $W \hookrightarrow V$) such that $g \mapsto \sigma(g)v$ expands (in a neighborhood of the unit element) to a power series with $W$-coefficients.

Let $(\sigma, V)$ be a locally analytic representation of the Levi subgroup $L$. $\sigma$ extends to a representation of $P^-$ defined by $\sigma(ul) = \sigma(l) (l \in L, u \in U^-)$.

**Definition 3.2.** Let Ind$_p^G \sigma$ be the space of $V$-valued locally analytic functions $\phi$ on $G$ satisfying

$$\phi(p^-g) = \sigma(p^-)\phi(g), \quad \text{for all } g \in G, p^- \in P^-.$$

On Ind$_p^G \sigma$ we have a $G$-action defined by right translation.

Since the quotient space $P^- \backslash G$ is compact, we obtain from [2] 4.1.5 the following proposition.

**Proposition 3.3.** Ind$_p^G \sigma$ is a locally analytic representation of $G$. 
Next, we introduce the principal series representation which serves as another description of $\text{Ind}_G^P \sigma$.

Let $\mathcal{S}$ and $\overline{\mathcal{S}}$ denote the $G$-homogeneous spaces $U \setminus G$ and $P^- \setminus G$ respectively, and denote $\hat{g} := \text{pr}_G^G(g)$. Because $P^- \cong U^- \rtimes L$, there is a left $L$-action on $\mathcal{S}$, and $\overline{\mathcal{S}} = L \setminus \mathcal{S}$.

**Definition 3.4.** Let $C^\text{an}_\sigma(\mathcal{S}, V)$ be the space of $V$-valued locally analytic functions $\varphi$ on $\mathcal{S}$ satisfying

$$\varphi(l \hat{g}) = \sigma(l) \varphi(\hat{g}), \quad \text{for all } \hat{g} \in \mathcal{S} \text{ and } l \in L.$$

The principal series representation $(C^\text{an}_\sigma(\mathcal{S}, V), T_\sigma)$ of $G$ is defined via

$$(T_\sigma(g) \varphi)(\hat{g}') := \varphi(\hat{g}' \cdot g).$$

**Lemma 3.5.**

(1) The representation $\text{Ind}_G^P \sigma$ is (naturally) isomorphic to $(C^\text{an}_\sigma(\mathcal{S}, V), T_\sigma)$.

(2) $\text{Ind}_G^P \sigma$ is isomorphic to $C^\text{an}(\mathcal{S}, V)$.

(3) Let $\iota$ be a locally analytic section of $\text{pr}_G^G(\mathcal{S})$, then $\iota$ induces an isomorphism

$$\iota^* : C^\text{an}_\sigma(\mathcal{S}, V) \to C^\text{an}(\overline{\mathcal{S}}, V) \quad \varphi \mapsto \varphi \circ \iota.$$

**Proof.** (1) From a locally analytic section $\bar{\iota}$ of $\text{pr}_G^G$ we obtain an isomorphism ([2] 4.3.1)

$$\bar{\iota}^* : \text{Ind}_G^G \text{I} \cong C^\text{an}(\overline{\mathcal{S}}, V), \quad \phi \mapsto \phi \circ \bar{\iota}.$$ 

By restriction to the subspaces, $\iota^*$ induces an isomorphism, independent of $\iota$, between $\text{Ind}_G^G \sigma$ and $C^\text{an}_\sigma(\mathcal{S}, V)$. G-equivariance is evident.

(2) A locally analytic section $\bar{\iota}$ of $\text{pr}_G^G$ induces an isomorphism $\bar{\iota}^*$ from $\text{Ind}_G^G \sigma$ onto $C^\text{an}(\overline{\mathcal{S}}, V)$ (ibid.).

(3) Choose $\bar{\iota}$ and $\iota$ compatible with $\iota$, that is, $\bar{\iota} = \iota \circ \iota$, then the assertion follows from (1) and (2).

Q.E.D.

Compactness of $\overline{\mathcal{S}}$ implies that $C^\text{an}(\overline{\mathcal{S}}, V)$ is of compact type ([11] Lemma 2.1).

By [11] Proposition 1.2, Theorem 1.3 and [9] Proposition 16.10, we have the following corollary.

**Corollary 3.6.** Suppose that $K$ is spherically complete. Let $B$ be a closed subspace of $C^\text{an}_\sigma(\mathcal{S}, V)$, then both $B$ and $C^\text{an}_\sigma(\mathcal{S}, V)/B$ are of compact type. In particular, they are reflexive, bornological, and complete. Moreover, their strong duals $B^*_b$ and $(C^\text{an}_\sigma(\mathcal{S}, V)/B)^*_b$ are nuclear Fréchet spaces.
4. Rigid analytic symmetric space $\Omega$

The rigid analytic symmetric space $\Omega$ associated to $G$ (with respect to a parabolic $P^+$) was constructed by van der Put and Voskuil in [14]. Some examples are the $p$-adic upper half-plane, Drinfel’d’s space and the $p$-adic Siegel upper half-space, which are associated to $\text{SL}(2, F)$, $\text{SL}(n + 1, F)$ and $\text{Sp}(2n, F)$ respectively (cf. [4], [10] and [7]).

4.1. Definition of the symmetric space $\Omega$. Let $G$, $P^\pm$, $U^\pm$, $L$ and $C$ denote the $F$-rigid analytifications of $G$, $P^\pm$, $U^\pm$, $L$ and $C$ respectively. $f$ defines a rigid analytic function on $G$.

Since $f$ is left invariant under $U^-$ (Lemma 2.1 (5)), we may define

$$f(\hat{g}, u) := f(g \cdot u)$$

for $\hat{g} \in \hat{G}$ and $u \in U^-.$

Definition 4.1. Let

$$\Omega := \{ u \in U^- : g \cdot u \in C, \text{ for all } g \in G \}$$

$$= \{ u \in U^- : f(\hat{g}, u) \neq 0, \text{ for all } \hat{g} \in \hat{G} \}.$$

We call $\Omega$ the symmetric space associated to $G$ with respect to $P^+$.

Example 4.2. In the situation of Example 2.3, $U^- \cong \mathbb{A}^n_F$ and $(z_1, ..., z_n) \in \Omega$ is given by the inequalities

$$c_1z_1 + ... + c_nz_n + d \neq 0 \quad \text{for all nonzero } (c_1, ..., c_n, d) \in F^{n+1}.$$

Therefore $\Omega$ is Drinfel’d’s space.

Example 4.3. In the situation of Example 2.5, $U^- \cong \text{Sym}(n)$ and $Z \in \Omega$ is given by the inequalities

$$\det(CZ + D) \neq 0 \quad \text{for all } C^TD = D^TC, \text{ rank}(C D) = n.$$

Therefore $\Omega$ is the $p$-adic Siegel upper half-space.

We may also interpret $\Omega$ to be the complement of all the $G$-translations of $(G - C)/P^+ = G/P^+ - U^-$ in $G/P^+$. Therefore we have a left $G$-action on $\Omega$ (induced from the left $G$-action on $G/P^+$). We denote $g * u$ the action of $g \in G$ on $u \in \Omega$. We have $g * u = \text{pr}_{U^-}(g \cdot u).$
4.2. **Automorphy factor.** We define the *automorphy factor*

\[
j : G \times \Omega \rightarrow P^+ \quad \quad (g, u) \mapsto (g \cdot u)^{-1} \cdot g \cdot u.
\]

Then \(j(g, u) = \text{pr}_{P^+}^C (g \cdot u)\), and straightforward computations show

\[
(4.1) \quad j(g_1 g_2, u) = j(g_1, g_2 \cdot u) j(g_2, u).
\]

For any \(u \in U^-\), \(j(u, u) = 1_G\), and hence (4.1) implies \(j(u \cdot g, u) = j(g, u)\), so we may define \(j(\hat{g}, u) := j(g, u)\).

For \(l \in L\), since \(l \cdot u = l \cdot u \cdot l^{-1}\), we have \(j(l, u) = l\), and by (4.1)

\[
(4.2) \quad j(l \cdot \hat{g}, u) = l \cdot j(\hat{g}, u).
\]

Since \(f\) is left invariant under \(U^-\) (Lemma 2.1(5)), it follows that

\[
(4.3) \quad f(j(\hat{g}, u)) = f(\hat{g}, u).
\]

From Lemma 2.1 (1), (4.1) and (4.3), we see that

\[
(4.4) \quad f(\hat{g}_1 \hat{g}_2, u) = f(\hat{g}_1, g_2 \cdot u) f(\hat{g}_2, u).
\]

Moreover, Lemma 2.1 (1), (4.2) and (4.3) imply

\[
(4.5) \quad f(l \cdot \hat{g}, u) = f(l) f(\hat{g}, u).
\]

4.3. **\(F\)-rigid analytic structure on \(\Omega\).** van der Put and Voskuil defined an affinoid covering of \(\Omega\) using Bruhat-Tits Buildings ([14]). In this paper we choose another approach following the construction of affinoid covering of Drinfel’d’s space in [10] and that of \(p\)-adic Siegel upper half-space in [7]. We endow \(\Omega\) with a structure of \(F\)-rigid analytic variety and show that it is an admissible open subset of \(U^-\) and, in particular, an open rigid analytic subspace of \(U^-\) (and therefore \(G/P^+\)).

We realize \(G\) as a Zariski closed subgroup of \(\text{GL}(n, F)\) such that \(T\) consists of diagonal matrices and \(B^+\) consists of lower triangular matrices. Then \(f(g)\) extends to an \(F\)-regular function on \(\text{GL}(n, F)\) with respect to the coordinates \(g_{i,j}\) (\(1 \leq i, j \leq n\)), and \(\det(g)^r f(g)\) is a homogeneous \(F\)-polynomial in \(g_{i,j}\). We denote \(N\) the degree of \(\det(g)^r f(g)\) and let \(M\) be an integer such that all the coefficients have absolute values bounded by \(|\varpi|^{NM}\).

**Lemma 4.4.** \(\Omega\) is nonempty.
Proof. It suffices to prove that $G$-translations of $G - C$ do not cover $G$. For $g \in G$, $g \cdot (G - C)$ is the locus of $f(g^{-1} \cdot g) = 0$. With the embedding of $G$ into $GL(n, F)$ we view $f(g^{-1} \cdot g)$ as a rational function in $g_{ij}$ with $F[G]$-coefficients, and, for a given $g \in G$, $f(g^{-1} \cdot g)$ is a nonzero $F$-rational function in $g_{ij}$. It is not hard to see that there are choices of $g_{ij} \in F^{alg}$ with appropriate absolute values so that the non-vanishing monomials in $f(g^{-1} \cdot g)$ are of distinct absolute values in $|F^{alg}| \setminus |\sigma|^Q$ modulo $|F| \equiv |\sigma|^F$. Therefore there exists $g \in G$ such that $f(g^{-1} \cdot g)$ is nonzero for all $g \in G$, and consequently $g$ lies in the complement of all the $g \cdot (G - C)$. Q.E.D.

Let $G_o$ and $L_o$ denote the intersections of $G$ and $L$ with $GL(n, o)$ respectively, and denote $\bar{S}_o = \text{pr}^G_o(G_o)$.

We recall Iwasawa’s decomposition

$$G = B^{-1} G_o.$$  

Then $G = P^- \cdot G_o$ and $\bar{S} = L \cdot \bar{S}_o$, so (4.5) and Lemma 2.1 imply

$$\Omega = \{ u \in U^- : f(\hat{g}, u) \neq 0, \text{ for any } \hat{g} \in \bar{S}_o \}.$$  

For $u \in U^-$ an upper triangular matrix with diagonal entries 1, let

$$|u| := \max_{1 \leq i \leq j \leq n} |u_{ij}| = \max_{1 \leq i \leq j \leq n} \left\{ 1, |u_{ij}| \right\}.$$  

For any nonnegative integer $m$ and $\hat{g} \in \bar{S}_o$, we define

$$B(m; \hat{g}) := \{ u \in U^- : |f(\hat{g}, u)| < |u|^N |\sigma|^{N(M+m)} \}.$$  

**Lemma 4.5.** If $m$ is a nonnegative integer and $g_1, g_2 \in G_o$ such that $g_1 \equiv l \cdot g_2 \mod \sigma^{Nm+1}$ for some $l \in L_o$, then

$$B(m; \hat{g}_1) = B(m; \hat{g}_2).$$  

**Proof.** Since $f|_{L_o}$ is a continuous $F$-character (Lemma 2.1 (1)) and $L_o$ is compact, the image of $L_o$ under $f$ is contained in $\varphi^\infty$. Therefore (4.5) implies $|f(\hat{l}_{\hat{g}_2}, u)| = |f(\hat{g}_2, u)|$, and hence $B(m; \hat{g}_2) = B(m; \hat{l}_{\hat{g}_2})$. So we may assume $g_1 \equiv g_2 \mod \sigma^{Nm+1}$.

We choose $\lambda \in (F^{alg})^\times$ such that $|\lambda| = |u|$. Since $|\lambda^{-1}u_{ij}| \leq 1$, 

$$g_1 \cdot \lambda^{-1}u \equiv g_2 \cdot \lambda^{-1}u \mod \sigma^{Nm+1},$$

and the matrices on both sides have entries with absolute values $\leq 1$. Applying $\det' \cdot f$, we obtain

$$\lambda^{-N} \det(g_1)' f(g_1 \cdot u) \equiv \lambda^{-N} \det(g_2)' f(g_2 \cdot u) \mod \sigma^{NM+Nm+1},$$

where $\det' \cdot f$ is the determinant of the matrix obtained from $f$ by removing the last column and row.
and consequently

\[ |u|^N |f(\hat{g}_1, u)| < |\sigma|^{N(M+m)} \iff |u|^N |f(\hat{g}_2, u)| < |\sigma|^{N(M+m)}. \]

Therefore \( B(m; \hat{g}_1) = B(m; \hat{g}_2) \).

Q.E.D.

Let

\[ \Omega(m; \hat{g}) := U^- \setminus B(m; \hat{g}) \]

\[ \Omega(m) := \bigcap_{\hat{g} \in \hat{S}_0} \Omega(m; \hat{g}). \]

For a given \( u \in \Omega \), \( |f(\hat{g}, u)| \) has a positive lower bound on \( \hat{S}_0 \). Therefore

\[ \Omega = \bigcup_{m=0}^{\infty} \Omega(m). \]

Let \( \hat{S}^{(m)} \) be any finite subset of \( \hat{S}_0 \) including a set of representatives in \( \hat{S}_0 \) for \( \text{pr}^G_{\hat{S}}(L_0(G_0/G_0(Nm+1))) \), where \( G_0(Nm+1) \) denotes the congruence subgroup \( (I_n + \sigma^{Nm+1}M(n, o)) \cap G \). Then Lemma 4.5 implies that

\[ \Omega(m) = \bigcap_{\hat{g} \in \hat{S}^{(m)}} \Omega(m; \hat{g}). \]

Moreover, we may assume that \( \hat{S}^{(m)} \) contains \( \hat{I}_n \).

\[ \Omega(m; \hat{I}_n) = \left\{ u \in U^- : |\sigma^{M+m}u_{ij}| \leq 1 \right\} \]

is an admissible open affinoid subset of \( U^- \). \( \Omega(m) \) is the intersection of finitely many rational sub-domains of \( \Omega(m; \hat{I}_n) \):

\[ \left\{ u \in \Omega(m; \hat{I}_n) : \left| \frac{\sigma^{N(M+m)}u_{Nj}^N}{f(\hat{g}, u)} \right| \leq 1, 1 \leq i \leq j \leq n \right\}, \]

with \( \hat{g} \) ranging on \( \hat{S}^{(m)} \setminus \{\hat{I}_n\} \). Therefore \( \Omega(m) \) is an affinoid variety.

We conclude that \( (\Omega(m))_{m=0}^{\infty} \) constitutes an admissible affinoid covering of \( \Omega \) so that \( \Omega \) admits a rigid analytic variety structure (see [1] 9.3). According to [1] 9.1.2 Lemma 3 (compare [1] 9.1.4 Proposition 2), the following proposition implies that \( \Omega \) is an admissible open subset of \( U^- \).

Proposition 4.6. Any morphism from an affinoid variety to \( U^- \) with image in \( \Omega \) factors through some \( \Omega(m) \).
Proof. The argument is similar to the third proof of [10] §1 Proposition 1.

Let $X$ be an affinoid variety, $\Phi : X \rightarrow U$ a morphism from $X$ to $U$ with image in $\Omega$. For any $\hat{g} \in \mathcal{S}_0$,

$$x \mapsto \Phi(x)^N_{ij} \frac{f(\hat{g}, \Phi(x))}{f(\hat{g}, \Phi(x))}, \quad 1 \leq i \leq j \leq n,$$

are $F$-rigid analytic functions on $X$. By the maximum modulus principle ([1] §6.2 Proposition 4 (i)), there exists a positive integer $m_\hat{g}$ such that

$$\max_{1 \leq i \leq j \leq n} \max_{x \in X} \left| \Phi(x)^N_{ij} \frac{f(\hat{g}, \Phi(x))}{f(\hat{g}, \Phi(x))} \right| \leq |\sigma|^{-N(M+m_\hat{g})}.$$

In other words, $\Phi(X) \subset \Omega(m_\hat{g} ; \hat{g})$. In view of Lemma 4.5, $m_\hat{g}$ can be chosen locally constant. Therefore the compactness of $\mathcal{S}_0$ implies that there exists a positive integer $m$ such that $\Phi(X) \subset \Omega(m)$. Q.E.D.

Finally, we prove that the morphisms of $g$-translations from $\Omega(m)$ into $\Omega$ indeed factor through the same $\Omega(m')$ for all $g \in G_0$.

**Lemma 4.7.** For any nonnegative integer $m$, there exists a nonnegative integer $m'$ such that for all $g \in G_0$,

$$g * \Omega(m) \subset \Omega(m').$$

**Proof.** Let $u \in \Omega(m)$. Then

$$1 \leq |u| \leq |\sigma|^{-M+m}, \quad (4.6)$$

and

$$\frac{|u|^N}{|f(\hat{g}, \Phi(x))|} \leq |\sigma|^{-N(M+m)} \quad \text{for any } g \in G_0. \quad (4.7)$$

$$g * u = \text{pr}_{U^*}^C (g \cdot u), \quad \text{and since } \text{pr}_{U^*}^C \text{ is } F\text{-regular on } C, \text{Lemma 2.1 (4) implies that there exist positive integers } s \text{ and } t \text{ such that all the entries of}$$

$$\det(g \cdot u)^f f(g \cdot u)^s \cdot g * u = \det(g)^f f(g \cdot u)^s \cdot g * u$$

are $F$-polynomials with variables the entries of $g \cdot u$. Let $D$ be the height of $g \cdot u$. Let $L$ be an integer such that the absolute values of all the coefficients are bounded by $|\sigma|^L$. Since $g \in G_0$, the entries of $g \cdot u$ have absolute values $\leq |u|$ and $|\det(g)| = 1$, then

$$|g * u| \leq \frac{|\sigma|^L|u|^D}{|f(\hat{g}, \Phi(x))|^r}. \quad (4.8)$$
It follows from (4.4), (4.8), (4.6) and (4.7) that for any \( g_1 \in G_g \),

\[
\frac{|g * u|^N}{|f(\hat{g}_1, g * u)|} \leq \frac{|\sigma|^{N_L}|u|^{N_D}}{|f(\hat{g}_1, u)||f(\hat{g}, u)|^{N_s - 1}}
\]

\[
= \frac{|\sigma|^{N_L}|u|^{N_D - N_s}}{|f(\hat{g}_1, u)| |f(\hat{g}, u)|^{N_s - 1}} \cdot \frac{|u|^N}{|f(\hat{g}_1, g * u)|},
\]

\[
\leq |\sigma|^{N(L - \max(D, N_s)(M + m))}.
\]

Therefore \( g * \Omega(m) \subset \Omega(m') \) for any \( m' \geq -M - L + \max(D, N_s)(M + m) \). Q.E.D.

4.4. **Rigid analytic functions on \( \Omega \).** Let \( \mathcal{O}(\Omega(m)) \) denote the space of \( F \)-rigid analytic functions on \( \Omega(m) \). Then \( \mathcal{O}(\Omega(m)) \) is an \( F \)-affinoid algebra with the supremum norm.

Let \( \mathcal{O}(\Omega) \) be the \( F \)-algebra of \( F \)-rigid analytic functions on \( \Omega \), that is, the projective limit of \( \mathcal{O}(\Omega(m)) \),

\[
\mathcal{O}(\Omega) := \lim_{\longleftarrow m} \mathcal{O}(\Omega(m)).
\]

\( \mathcal{O}(\Omega) \) is endowed with the projective limit topology.

From the construction of \( \Omega(m) \) we see that the \( F \)-affinoid algebra \( \mathcal{O}(\Omega(m)) \) is equal to

\[
F \left( \sigma^{M+m} u_{ij}, \frac{\sigma^{N(M+m)} u_{ij}^N}{f(\hat{g}, u)} : 1 \leq i \leq j \leq n, \hat{g} \in \mathcal{S}(m) - \mathcal{I}_n \right).
\]

Therefore \( \psi \in \mathcal{O}(\Omega(m)) \) has an expansion in the following form which converges with respect to the supremum norm \( \| \cdot \|_{\mathcal{O}(\Omega(m))} \):

\[
\psi(u) = \sum_{(f_\ell) \in (\mathcal{I}_n)^{\mathcal{S}(m)}} P_{(f_\ell)}(u) \prod_{\hat{g} \in \mathcal{S}(m)} f(\hat{g}, u)^{-f_\ell},
\]

where \( P_{(f_\ell)}(u) \) are polynomials in the coordinates of \( u \) with coefficients in \( F \).

In view of (4.5), the assumption that \( \mathcal{S}(m) \) is contained in \( \mathcal{S}_0 \) is quite artificial, and it is more convenient and natural to choose \( \mathcal{S}(m) \) to be an arbitrary finite subset of \( \mathcal{S} \) whenever we consider the expansion of \( \psi \in \mathcal{O}(\Omega(m)) \).

\( \psi \in \mathcal{O}(\Omega) \) may be considered as an \( F^{\text{alg}} \)-valued function on \( \Omega \) such that, restricting on each \( \Omega(m) \), \( \psi \) has an expansion (4.10) which converges with respect to \( \| \cdot \|_{\mathcal{O}(\Omega(m))} \). In particular, \( f(\hat{g}, u)^{-1} \in \mathcal{O}(\Omega) \) for any \( \hat{g} \in \mathcal{S} \).

Since all the generators of \( \mathcal{O}(\Omega(m)) \) in (4.9) are \( F \)-rigid analytic functions on \( \Omega(m') \) for any \( m' \geq m \) and therefore on \( \Omega \), we obtain the following proposition.
**Proposition 4.8.**

1. \( \Omega \) is a Stein space, that is, the image of \( \mathcal{O}(\Omega(m) + 1) \) under the transition homomorphism in \( \mathcal{O}(\Omega) \) is dense for any nonnegative integer \( m \).
2. The image of \( \mathcal{O}(\Omega) \) under the transition homomorphism in \( \mathcal{O}(\Omega(m)) \) is dense.

Let \( \mathcal{O}_K(\Omega(m)) \) and \( \mathcal{O}_K(\Omega) \) denote \( \mathcal{O}(\Omega(m)) \otimes_F K \) and \( \mathcal{O}(\Omega) \otimes_F K \) respectively. If we let \( \Omega_K(m) \) and \( \Omega_K \) denote the extensions of the ground field \( K/F \) of \( \Omega(m) \) and \( \Omega \) respectively (see [1] §9.3.6), then \( \mathcal{O}_K(\Omega(m)) \) and \( \mathcal{O}_K(\Omega) \) are the spaces of \( K \)-rigid analytic functions on \( \Omega_K(m) \) and \( \Omega_K \) respectively.

**Proposition 4.9.** Let \( K \) be spherically complete. \( \mathcal{O}_K(\Omega) \) is a nuclear \( K \)-Fréchet space.

**Proof.** By [9] Proposition 19.9, it suffices to prove that all the \( \mathcal{O}_K(\Omega(m)) \) constitute a compact projective system.

Consider the \( \mathcal{O}_K(\Omega(m) - 1) \)-norms of the generators of \( \mathcal{O}_K(\Omega(m)) \) (see 4.9), then

\[
\sup_{u \in \Omega(m - 1)} \max \max_{g \in \text{alg}(m - 1), 1 \leq j \leq n} \left\{ |g^{M+m} u_j|, \frac{|g^{N(M+m)} u_j|}{f(g, u)} \right\} \leq |\sigma|.
\]

[12] Lemma 1.5 implies that the transition homomorphism from \( \mathcal{O}_K(\Omega(m)) \) to \( \mathcal{O}_K(\Omega(m - 1)) \) is compact.  

Q.E.D.

[11] Theorem 1.3 and Proposition 1.2 imply the following corollary.

**Corollary 4.10.** Suppose \( K \) is spherically complete. Let \( \mathcal{N} \) be a closed subspace of \( \mathcal{O}_K(\Omega) \), then \( \mathcal{N} \) and \( \mathcal{O}_K(\Omega)/\mathcal{N} \) are nuclear Fréchet spaces, and their strong duals \( \mathcal{N}^*_b \) and \( (\mathcal{O}_K(\Omega)/\mathcal{N})^*_b \) are of compact type.

5. **Holomorphic discrete series** \( \mathcal{O}_\sigma(\Omega), \pi_\sigma \)

Let \( \Omega \) (resp. \( \Omega(m) \)) denote \( \Omega_K(K) \) (resp. \( \Omega_K(m)(K) \)). Restricting to \( \Omega \) (resp. \( \Omega(m) \)), we view \( K \)-rigid analytic functions in \( \mathcal{O}_K(\Omega) \) (resp. \( \mathcal{O}_K(\Omega(m)) \)) as \( K \)-valued functions on \( \Omega \) (resp. \( \Omega(m) \)), and abbreviate \( \mathcal{O}_K(\Omega(m)) \) (resp. \( \mathcal{O}_K(\Omega) \)) to \( \mathcal{O}(\Omega(m)) \) (resp. \( \mathcal{O}(\Omega) \)).

Let \( (V, \sigma) \) be a \( d \)-dimensional \( K \)-rational representation of \( L \). \( \sigma \) extends to a representation of \( \mathbb{P}^+ \).

Let \( \mathcal{O}_\sigma(\Omega) := \mathcal{O}(\Omega) \otimes V \) and \( \mathcal{O}_\sigma(\Omega(m)) := \mathcal{O}(\Omega(m)) \otimes V \).

For any \( g \in G \) and \( \psi \in \mathcal{O}_\sigma(\Omega) \), let \( \pi_\sigma(g)\psi \) be the \( V \)-valued function on \( \Omega \) as follows

\[
(\pi_\sigma(g)\psi)(u) := \sigma(j(g^{-1}, u))^{-1}\psi(g^{-1} * u).
\]
Lemma 5.1. \( \pi_\sigma(g)\psi \in \mathcal{O}_\sigma(\Omega) \).

Proof. Since \( \sigma \) is \( K \)-rational, each coordinate of \( \sigma(j(g^{-1}, u))^{-1} \) is a product of a \( K \)-polynomial in the coordinates of \( j(g^{-1}, u) \) and a power of \( \det(j(g^{-1}, u))^{-1} = \det(g) \). Note that \( j(g^{-1}, u) = \text{pr}_C(g^{-1} \cdot u) \), and since \( \text{pr}_C \) is \( F \)-regular on \( C \), each coordinate of \( j(g^{-1}, u) \) is a product of an \( F \)-polynomial in the coordinates of \( g^{-1} \cdot u \) and powers of \( \det(g^{-1} \cdot u)^{-1} = \det(g) \) and \( f(g^{-1} \cdot u)^{-1} \). Therefore each coordinate of \( \sigma(j(g^{-1}, u))^{-1} \) has a finite expansion of the form (4.10), and hence belongs to \( \mathcal{O}(\Omega) \).

Similarly, the coordinates of \( \psi(g^{-1} \cdot u) \) also have expansions of the form (4.10). By Proposition 4.6, for any \( m \), \( g^{-1} \)-translation maps \( \Omega(m) \) into some \( \Omega(m') \), and hence the norm of each coordinate of \( \psi(g^{-1} \cdot u) \) on \( \Omega(m) \) is bounded by the norm of the corresponding coordinate of \( \psi \) on \( \Omega(m') \). Therefore \( \psi(g^{-1} \cdot u) \in \mathcal{O}_\sigma(\Omega) \).

We conclude that \( \pi_\sigma(g)\psi \in \mathcal{O}_\sigma(\Omega) \). Q.E.D.

It follows from the automorphy relation (4.1) that \( \pi_\sigma \) is an action of \( G \) on \( \mathcal{O}_\sigma(\Omega) \).

Definition 5.2. We call \( (\mathcal{O}_\sigma(\Omega), \pi_\sigma) \) the holomorphic (rigid analytic) discrete series representation of \( G \).

Lemma 5.3. Let \( m \) and \( m' \) be as in Lemma 4.7. Then there exists a constant \( c \) depending on \( \sigma \) and \( m \) such that

\[
\|\pi_\sigma(g)\psi\|_{\mathcal{O}_\sigma(\Omega(m))} \leq c\|\psi\|_{\mathcal{O}_\sigma(\Omega(m'))},
\]

for all \( g \in G_o \).

Proof. The proof is similar to the arguments in Lemma 5.1, but instead of Proposition 4.6 we apply Lemma 4.7.

Using the expressions for the coordinates of \( \sigma(j(g^{-1}, u))^{-1} \) in the first paragraph of the proof of Lemma 5.1, we see that their \( \mathcal{O}(\Omega(m)) \)-norms are uniformly bounded on \( G_o \), so there is a constant \( c > 0 \) such that

\[
\max_{g \in G_o} \max_{u \in \Omega(m)} \|\sigma(j(g^{-1}, u))^{-1}\|_{\text{End}(V)} \leq c.
\]
Consequently,

\[
\max_{g \in G_0} \|\pi_\sigma(g)\psi\|_{\mathcal{O}(\Omega(m))} \\
= \max_{g \in G_0} \max_{u \in \Omega(m)} \|((\pi_\sigma(g)\psi)(u))\|_V \\
\leq \max_{g \in G_0} \max_{u \in \Omega(m)} ||\sigma(j(g^{-1}, u))^{-1}||_{\text{End}(V)} \cdot \max_{g \in G_0} \max_{u \in \Omega(m)} ||\psi(g^{-1} \ast u)||_V \\
\leq c \max_{u \in \Omega(m')} ||\psi(u)||_V \\
= c||\psi||_{\mathcal{O}(\Omega(m'))}.
\]

Q.E.D.

It follows from Lemma 5.3 that, for each \(m\), the map

\[
G_0 \times \mathcal{O}_\sigma(\Omega) \rightarrow \mathcal{O}_\sigma(\Omega(m)) \\
(g, \psi) \mapsto (\pi_\sigma(g)\psi)|_{\Omega(m)}.
\]

is continuous. Since \(\mathcal{O}_\sigma(\Omega)\) is the projective limit of \(\mathcal{O}_\sigma(\Omega(m))\), we obtain the following corollary.

**Corollary 5.4.** \((\mathcal{O}_\sigma(\Omega), \pi_\sigma)\) is a continuous \(G\)-representation.

Moreover, we shall prove that the dual representation of \(\pi_\sigma\) is locally analytic. For this, we recall that a coordinate chart at \(1_G\) is obtained from the decomposition of the Bruhat big cell (see (2.2))

\[U^-U_1^-TU_1^+U^+ \simeq A_F^{\text{dim } g_0}.\]

**Lemma 5.5.** Let \(m\) and \(m'\) be as in Lemma 4.7. Let \(B\) be any parameterized (as in (5.1)) open neighborhood of \(1_G\) contained in \(G_0\). For any \(\psi \in \mathcal{O}_\sigma(\Omega(m'))\), the orbit map

\[
B \rightarrow \mathcal{O}_\sigma(\Omega(m)) \\
g \mapsto (\pi_\sigma(g)\psi)|_{\Omega(m)}
\]

is an \(\mathcal{O}_\sigma(\Omega(m))\)-valued analytic function, namely, it can be expanded as a convergent power series with variables the coordinate parameters of \(B\) and coefficients in the Banach space \(\mathcal{O}_\sigma(\Omega(m))\).

**Proof.** Once we have obtained a formal expansion of \(\pi_\sigma(g)\psi\) into a power series with variables the coordinate parameters of \(B\) and coefficients in \(\mathcal{O}_\sigma(\Omega(m))\), Lemma 5.3 would imply that the expansion is indeed convergent. In view of (5.1),
it suffices to consider \( \pi_\sigma(g)\psi(u) \) for \( g \) in \( U^-, U^-_L \) and \( T \) (note that \( U^+ \) and \( U^+_L \) are the conjugations of \( U^- \) and \( U^-_L \) by the long Weyl element).

Let \( u \in U^- \), then
\[
\pi_\sigma(u)\psi(u) = \psi(u^{-1} \cdot u).
\]

Let \( \mathcal{O}(\Omega(m))[[u]] \) denote the ring of formal power series \( \varphi(u) \) in the coordinates \( u_\alpha (\alpha \in R^-_I) \) with coefficients in \( \mathcal{O}(\Omega(m)) \), where \( \varphi(u) \) is expressed as
\[
\varphi(u) = \sum_{\alpha \in R^+_I} a_\alpha \cdot \varphi^\alpha, \quad a_\alpha \in \mathcal{O}(\Omega(m)), \quad \varphi^\alpha := \prod_{\alpha \in R^+_I} u_\alpha^{\alpha}.
\]

If the constant term \( a_0 \) is a unit in \( \mathcal{O}(\Omega(m)) \), then \( \varphi(u) \) is invertible in \( \mathcal{O}(\Omega(m))[[u]] \).

Note that, for \( \hat{g} \in \mathfrak{g} \), the constant term in the expansion of \( f(\hat{g}, u^{-1} \cdot u) \) is \( f(\hat{g}, u) \), and it is invertible in \( \mathcal{O}(\Omega(m)) \), so \( f(\hat{g}, u^{-1} \cdot u)^{-1} \) belongs to \( \mathcal{O}(\Omega(m))[[u]] \). Therefore, in view of the expansion form (4.10), the coordinates of \( \psi(u^{-1} \cdot u) \) expand into a formal power series in \( u_\alpha \) whose coefficients are series in \( \mathcal{O}(\Omega(m)) \), but it follows from Lemma 5.3 that the coefficients are indeed convergent series in \( \mathcal{O}(\Omega(m)) \) for \( u \in B \). So each coordinate of \( \psi(u^{-1} \cdot u) \) belongs to \( \mathcal{O}(\Omega(m))[[u]] \).

For \( l \in U^-_L \) or \( T \),
\[
\pi_\sigma(l)\psi(u) = \sigma(l)^{-1}\psi(l^{-1} \cdot u \cdot l).
\]
The arguments are similar. Q.E.D.

**Corollary 5.6.** Let \( U^+_0 = U^+ \cap G_b \), then the power series expansion of
\[
f(j(u^+, u))^{-1} = f(u^+ \cdot u)^{-1}, \quad u^+ \in U^+_0,
\]
on \( U^+_0 \) converges in \( \mathcal{O}(\Omega(m)) \).

**Proof.** Since \( f \) is an \( F \)-rational character on \( P^+ \) (Lemma 2.1 (1)), if we put \( \sigma = f \)
and \( \psi \equiv 1 \), then \( (\pi_j g^{-1})l(u) = f(j(g, u))^{-1} \). Therefore our assertion follows from Lemma 5.5. Q.E.D.

Now consider the dual representation \( \pi^*_{\sigma} \) of \( G \) on \( \mathcal{O}_{\sigma}(\Omega)^*_b \equiv \lim_{m \to \infty} \mathcal{O}(\Omega(m))^*_b \). The transition homomorphisms \( \mathcal{O}_{\sigma}(\Omega(m))^*_b \to \mathcal{O}_{\sigma}(\Omega(m'))^*_b \) are injective (see Proposition 4.8 (2)). Lemma 4.7 implies that, for any \( g \in G_b \), \( \pi^*_{\sigma}(g) \) maps \( \mathcal{O}(\Omega(m))^*_b \) into \( \mathcal{O}(\Omega(m'))^*_b \) via
\[
\langle \psi, \pi^*_{\sigma}(g)\mu \rangle = \langle (\pi_\sigma g^{-1})\psi \rangle_{\Omega(m)}, \quad \mu \in \mathcal{O}(\Omega(m))^*_b, \psi \in \mathcal{O}_{\sigma}(\Omega(m')).
\]
We deduce from Lemma 5.5 that, for any $\mu \in \mathcal{O}_\sigma(\Omega(m))^*$, the orbit map
\[
B^{-1} \rightarrow \mathcal{O}_\sigma(\Omega(m'))^*_b \\
g \mapsto \pi^*_\sigma(g)\mu
\]
is an $\mathcal{O}_\sigma(\Omega(m'))^*_b$-valued analytic function. Therefore we obtain the following corollary.

**Corollary 5.7.** $(\mathcal{O}_\sigma(\Omega)^*_b, \pi^*_\sigma)$ is locally analytic.

### 6. Duality

In the following, we assume that $K$ is spherically complete. Let $(V, \sigma)$ be a $d$-dimensional $K$-rational representation of L. We choose a basis $v_1, \cdots, v_d$ of $V$ and denote by $v_1^*, \cdots, v_d^*$ the corresponding dual basis of the dual space $V^*$. $(V^*, \sigma^*)$ denotes the dual representation of $(V, \sigma)$.

#### 6.1. Duality operator $I_\sigma$

For $u \in \Omega$ and $v^* \in V^*$, let $\varphi_{u,v^*}$ be the $V^*$-valued locally analytic function on $\hat{\mathcal{S}}$:

\[
\varphi_{u,v^*}(\hat{g}) := \sigma^*(j(\hat{g}, u))v^*. 
\]

In view of (4.2), $\varphi_{u,v^*}$ belongs to $C^an_{\sigma^*}(\hat{\mathcal{S}}, V^*)$. Let $B^0_{\sigma^*}(\hat{\mathcal{S}}, V^*)$ be the subspace of $C^an_{\sigma^*}(\hat{\mathcal{S}}, V^*)$ spanned by $\varphi_{u,v^*}$, $B_{\sigma^*}(\hat{\mathcal{S}}, V^*)$ the closure of $B^0_{\sigma^*}(\hat{\mathcal{S}}, V^*)$. From (4.1), we see that $B^0_{\sigma^*}(\hat{\mathcal{S}}, V^*)$ and therefore $B_{\sigma^*}(\hat{\mathcal{S}}, V^*)$ are $G$-invariant.

For any continuous linear functional $\xi \in B_{\sigma^*}(\hat{\mathcal{S}}, V^*)^*$, we define a $V$-valued function on $\Omega$:

\[
I_{\sigma}(\xi)(u) := \sum_{k=1}^d \langle \varphi_{u,v_k^*}, \xi \rangle v_k, \quad u \in \Omega. 
\]

$I_{\sigma}(\xi)$ is independent of the choice of the basis $\{v_k\}_{k=1}^d$. Evidently, $I_{\sigma}$ is injective.

**Lemma 6.1.** $I_{\sigma}$ is $G$-equivariant, that is,

\[
I_{\sigma}(T_{\sigma^*}^*(g)\xi) = \pi_{\sigma}(g)I_{\sigma}(\xi),
\]

for any $g \in G$. 

Proposition 6.2.

Proof. Step 1. We denote by $i$ the inclusion: $B_{\sigma^*}(\mathfrak{S}, V^*) \hookrightarrow C^\mathrm{an}_{\sigma^*}(\mathfrak{S}, V^*)$, $i^*$ its adjoint operator. Because of our assumption that $K$ is spherically complete, Hahn-Banach Theorem ([9] Corollary 9.4) implies that $i^*$ is surjective. Since $C^\mathrm{an}_{\sigma^*}(\mathfrak{S}, V^*)_b$ and $B_{\sigma^*}(\mathfrak{S}, V^*)_b$ are both Fréchet spaces (Corollary 3.6), the open mapping theorem ([9] Proposition 8.6) implies that $i^*$ is open. Therefore the continuity of $I_{\sigma} \circ i^*$ implies that of $I_{\sigma}$. Consequently, (1) and (2) are equivalent to:

(1') $I_{\sigma} \circ i^*(\xi) \in \mathcal{O}_{\sigma}(\Omega)$ for any $\xi \in C^\mathrm{an}_{\sigma^*}(\mathfrak{S}, V^*)$;
(2') $I_{\sigma} \circ i^*: (C^\mathrm{an}_{\sigma^*}(\mathfrak{S}, V^*)_b, \sigma^*) \rightarrow (\mathcal{O}_{\sigma}(\Omega), \pi_{\sigma})$ is a continuous homomorphism of $G$-representations.

Since $G$-equivariance is proved in Lemma 6.1, for (2') it remains to show the continuity of $I_{\sigma} \circ i^*$. For convenience, we still denote $I_{\sigma} \circ i^*$ by $I_{\sigma}$.

Step 2. Let $[\overline{U}_k]_k$ be a finite disjoint open covering of $\mathfrak{T}$ satisfying:

1. $U^+_0 \in [\overline{U}_k]_k$ (note that the open subscheme $P^-\setminus C$ of $\mathfrak{T}$ is identified with $U^+$);
2. each $\overline{U}_κ$ is (right) translated into $U^+_0$ by some $g_κ ∈ G$.

Let $U_κ$ be the preimage of $\overline{U}_κ$ under $pr^S_S$.

For $ξ ∈ C^an(\overline{S}, V^*)_b$, we write $I_σ(ξ)$ in integral:

$$I_σ(ξ)(u) = \sum_{k=1}^d \int_{\overline{S}} ϕ_{u;v_k^*} dξ \cdot v_k$$

$$= \sum_{k=1}^d \sum_k \int_{U_κ} \varphi_{u;v_k^*} dξ \cdot v_k$$

$$= \sum_k π_σ(g_κ)(\sum_{k=1}^d \int_{U_κ} \varphi_{u;v_k^*} d(T_σ^*(g_κ^{-1})ξ) \cdot v_k)$$

where $v_{k;g_κ} = σ(j(g_κ^{-1}, u))v_k$ is defined in the proof of Lemma 6.1. Therefore it suffices to consider

$$(6.1) \sum_{k=1}^d \int_{U} \varphi_{u;v_k^*} dξ' \cdot v_k,$$

where $U$ ranges on $\{U_κ·g_κ\}_κ$ and $ξ'$ is the image of $ξ$ under $C^an(\overline{S}, V^*)_b → C^an(U, V^*)_b$.

For the open subset $U = pr^H_H(U)$ of $U^+_0$, we have the isomorphism induced from a locally analytic section $ι$ of $pr^H_H(U)$ (see Lemma 3.5 (3)):

$$(6.2) C^an(U, V^*)_b ≃ C^an(\overline{U}, V^*)_b.$$

Then (6.1) is equal to

$$\overline{I}_{σ,\overline{U}}(\overline{ξ})(u) := \sum_{k=1}^d \int_{\overline{U}} (σ^*(j(u^+, u))v_k^*) dξ(u^+) \cdot v_k,$$

where $ξ$ is the image of $ξ'$ in $C^an(\overline{U}, V^*)_b$ via the isomorphism (6.2).

Therefore it suffices to prove that $\overline{I}_{σ,\overline{U}}(\overline{ξ})$ is rigid analytic on $Ω(m)$, and that the map

$$C^an(\overline{U}, V^*)_b \to O_σ(Ω(m))$$

$$ξ \mapsto \overline{I}_{σ,\overline{U}}(\overline{ξ})|_Ω(m)$$

is continuous for all $m$.

Step 3. Since $σ^*$ is $K$-rational, using the same arguments in the proof of Lemma 5.1 and applying Corollary 5.6, we obtain an expansion

$$σ^*(j(u^+, u))v_k^* = \sum_{ℓ=1}^d \left( \sum_{u^+ ∈ l^k} a_{-kℓ}(u) · (u^+)ℓ\right)v_κ^*,$$
with \( a_{\ell,k} \in \mathcal{O}^r(\Omega(m)) \) such that
\[
\lim_{|\ell| \to \infty} \| a_{\ell,k} \|_{\mathcal{O}^r(\Omega(m))} \cdot \| (u^+) \|_{C^0(U^2)} = 0,
\]
and moreover, there is a constant \( c' > 0 \), depending only on \( m, \sigma \) and \( \{ v_k \}_{k=1}^d \), such that
\[
\| a_{\ell,k} \|_{\mathcal{O}^r(\Omega(m))} \cdot \| (u^+) \|_{C^0(U^2)} \leq c'.
\]

Then
\[
I_{\sigma,\Omega}^{-1}(\xi)(u) = \sum_{k=1}^d \left( \sum_{\ell=1}^d \sum_{\xi} \int_{\mathbb{R}} (u^+)^{\ell} \cdot v_\xi^* d\xi (u^+) \cdot a_{\ell,k} (u) v_k.\right)
\]

We have
\[
\left| \int_{\mathbb{R}} (u^+)^{\ell} \cdot v_\xi^* d\xi (u^+) \right| \leq \| (u^+) \|_{C^0(U^2)} \cdot \| v_\xi^* \|_{V^*} \cdot \| \xi \|_{C^0(U^2)}.
\]

(6.3) and (6.6) imply that the expansion (6.5) of \( I_{\sigma,\Omega}^{-1}(\xi) \) converges in \( \mathcal{O}^r(\Omega(m)) \).

(6.4) and (6.6) imply
\[
\left\| I_{\sigma,\Omega}^{-1}(\xi) \right\|_{\mathcal{O}^r(\Omega(m))} \leq \max_{1 \leq k, \ell \leq d} c' \| v_\xi^* \|_{V^*} \| v_k \|_{V} \cdot \| \xi \|_{C^0(U^2)}.
\]

and therefore the continuity follows. Q.E.D.

6.2. Duality operator \( J_\sigma \). Let \( \mathcal{N}_r(\Omega) \) denote the image of \( I_\sigma \).

We consider \( J_\sigma \), the adjoint operator of \( I_\sigma \), which is an injective continuous linear operator from \( \mathcal{N}_r(\Omega) \) to \( (B_{r^c}(\mathfrak{g}, V^*))^* \) given by \( B_{r^c}(\mathfrak{g}, V^*) \) is reflexive according to Corollary 3.6).

For any \( \mu \in \mathcal{N}_r(\Omega)^* \) and \( \xi \in B_{r^c}(\mathfrak{g}, V^*)^* \), we have
\[
\langle J_\sigma(\mu), \xi \rangle = \langle I_\sigma(\xi), \mu \rangle.
\]

For \( \hat{g} \in \mathfrak{g} \) and \( v \in V \), we define the Dirac distribution \( \xi_{\hat{g},v} \in B_{r^c}(\mathfrak{g}, V^*)^* \) as follows:
\[
\langle \varphi, \xi_{\hat{g},v} \rangle = \langle v, \varphi(\hat{g}) \rangle_V, \quad \varphi \in B_{r^c}(\mathfrak{g}, V^*),
\]
and a \( V \)-valued rigid analytic function \( \psi_{\hat{g},v} \) on \( \Omega \):
\[
\psi_{\hat{g},v}(u) := \sigma(j(\hat{g}, u))^{-1} v.
\]

**Lemma 6.3.**
\[
I_\sigma(\xi_{\hat{g},v}) = \psi_{\hat{g},v}.
\]

**Proof.** Straightforward from the definitions. Q.E.D.

Then we obtain a formula for \( J_\sigma \).
**Proposition 6.4.** For any continuous linear functional \( \mu \in \mathcal{N}_G(\Omega)^* \), we have

\[
J_\sigma(\mu)(\hat{g}) = \sum_{k=1}^{d} \langle \psi_{\hat{g}, v_k}, \mu \rangle v_k^*.
\]

**Proof.** Straightforward from Lemma 6.3 and (6.7). Q.E.D.

### 6.3. Image of \( I_\sigma \)

Let \( \mathcal{N}_G^0(\Omega) \) denote the subspace of \( \mathcal{O}_G(\Omega) \) spanned by \( \psi_{\hat{g}, v} \) for all \( \hat{g} \in \mathcal{G} \) and \( v \in V \). Then it follows from (4.1) that \( \mathcal{N}_G^0(\Omega) \) is \( \mathcal{G} \)-invariant, and Lemma 6.3 implies \( \mathcal{N}_G^0(\Omega) \subset \mathcal{N}_G(\Omega) \).

From (6.8), we see that \( J_\sigma \) factors through \( \mathcal{N}_G^0(\Omega)^* \), and (6.8) defines an injective map from \( \mathcal{N}_G^0(\Omega)^*_b \) into \( \mathcal{B}_{\mathcal{G}}(\mathcal{G}, V^*) \). Since \( J_\sigma \) is injective and the natural map \( \mathcal{N}_G(\Omega)^*_b \to \mathcal{N}_G^0(\Omega)^*_b \) is surjective (Hahn-Banach Theorem), \( \mathcal{N}_G^0(\Omega)^*_b = \mathcal{N}_G(\Omega)^*_b \). Therefore Hahn-Banach Theorem implies the following lemma.

**Lemma 6.5.** \( \mathcal{N}_G^0(\Omega) \) is dense in \( \mathcal{N}_G(\Omega) \).

**Theorem 6.6.**

1. \( I_\sigma \) is an isomorphism from \( \mathcal{B}_{\mathcal{G}}(\mathcal{G}, V^*)_b \) to \( \mathcal{N}_G(\Omega) \).
2. \( \mathcal{N}_G(\Omega) \) is the closure of \( \mathcal{N}_G^0(\Omega) \) in \( \mathcal{N}_G(\Omega) \).

**Proof.** Let \( \iota \) be a locally analytic section of \( \text{pr}^\mathcal{G} \) and denote \( \mathcal{R} = \iota(\mathcal{G}) \).

1. Let \( \mathcal{N}_G^0(\Omega(m)) \) be the image of \( \mathcal{N}_G^0(\Omega) \) in \( \mathcal{O}_G(\Omega(m)) \).

   Since \( \psi_{\hat{g}, v_k} = \pi_{\mathcal{G}}(g^{-1}) v_k \), we see that the map

   \[
   \mathcal{G} \to \mathcal{O}_G(\Omega(m)),
   \]

   \[
   \hat{g} \mapsto \psi_{\hat{g}, v_k},
   \]

   is locally analytic (Lemma 5.5). Since \( \mathcal{R} \) is compact, \( \rho_m = \min \min_{1 \leq k \leq d} \| \psi_{\hat{g}, v_k} \|_{\mathcal{O}_G(\Omega(m))} \) is positive. Let \( \mathcal{L} \) denote the lattice \( \sum_{k=1}^{d} \sum_{g \in \mathcal{R}} \psi_{g, v_k} \) in \( \mathcal{N}_G^0(\Omega) \). Then, for each \( m \), the image of \( \mathcal{L} \) in \( \mathcal{N}_G^0(\Omega(m)) \) contains the ball of radius \( \rho_m \) centered at zero, and therefore the interior of \( \mathcal{L} \) is a nontrivial open lattice.

2. According to Lemma 3.5, \( \iota \) induces an isomorphism \( \iota^* \) between \( \mathcal{C}^m(\mathcal{G}, V^*) \) and \( \mathcal{C}^m(\mathcal{G}, V^*) \), and hence an isomorphism between \( \mathcal{B}_{\mathcal{G}}(\mathcal{G}, V^*) \) and its image which we denote to be \( \mathcal{B}(\mathcal{G}, V^*) \).

Let \( I \) be any (finite) disjoint open chart covering \( \{ \mathcal{U}_k \}_k \) of \( \mathcal{G} \). We recall that \( \mathcal{C}^m(\mathcal{G}, V^*) \) is defined to be the inductive limit, indexed with all the \( I \), of the \( K \)-Banach algebras \( \mathcal{F}_I(\mathcal{G}, V^*) = \prod_k \mathcal{O}(\mathcal{U}_k, V^*) \), where \( \mathcal{O}(\mathcal{U}_k, V^*) \) denotes the space of
$K$-analytic functions on $\mathbb{U}_K$ (cf. [2] 2.1.10 and [11] §2). The inductive limit structure is naturally induced onto $B(\mathbb{S}, V^*)$, say $B(\mathbb{S}, V^*) = \lim_{\lambda \in \Lambda} E_I(\mathbb{S}, V^*)$. Moreover, the strong dual space $B(\mathbb{S}, V^*)^*_b$ is the projective limit of $E_I(\mathbb{S}, V^*)^*_b$.

3. Consider

$$(i^{\omega-1})^* \circ I_{\sigma}^{-1}|_{\mathcal{H}_{\sigma}^0(\Omega)} : \mathcal{H}_{\sigma}^0(\Omega) \rightarrow B(\mathbb{S}, V^*)^*_b$$

$$\psi_{\kappa, \psi} \mapsto (i^{\omega-1})^*(\xi_{\kappa, \psi}).$$

Let $\hat{g} \in \mathcal{R}$.

$$\left\| (i^{\omega-1})^*(\xi_{\kappa, \psi}) \right\|_{E_I(\mathbb{S}, V^*)^*_b} = \max_{\varphi \in E_I(\mathbb{S}, V^*)} \frac{\langle \varphi, (i^{\omega-1})^*(\xi_{\kappa, \psi}) \rangle}{\|\varphi\|_{E_I(\mathbb{S}, V^*)}}$$

$$= \max_{\varphi \in E_I(\mathbb{S}, V^*)} \frac{\langle \varphi, \xi_{\kappa, \psi} \rangle}{\|\varphi\|_{E_I(\mathbb{S}, V^*)}}$$

$$= \max_{\varphi \in E_I(\mathbb{S}, V^*)} \max_{\hat{g} \in \mathcal{R}} \|\varphi(\hat{g})\|_{V^*} \leq \|v\|_{V^*}.$$

Therefore the image of $\mathcal{L}$ under $(i^{\omega-1})^* \circ I_{\sigma}^{-1}|_{\mathcal{H}_{\sigma}^0(\Omega)}$ in $B(\mathbb{S}, V^*)^*_b$ is bounded, since its image in $E_I(\mathbb{S}, V^*)^*_b$ are all norm-bounded by $\max_{1 \leq \lambda \leq d} \|v_\lambda\|_{V^*}$. Because $\mathcal{H}_{\sigma}^0(\Omega)$ is metrizable, it is bornological ([9] Proposition 6.14), and therefore $I_{\sigma}^{-1}|_{\mathcal{H}_{\sigma}^0(\Omega)}$ is continuous ([9] Proposition 6.13). Therefore $I_{\sigma}$ induces an isomorphism between $I_{\sigma}^{-1}(\mathcal{H}_{\sigma}^0(\Omega))$ and $\mathcal{H}_{\sigma}^0(\Omega)$, and consequently $I_{\sigma}$ induces an isomorphism between their completions, which, in view of Lemma 6.5, must be $B_{\sigma^{-1}}(\mathbb{S}, V^*)^*_b$ and $\mathcal{H}_{\sigma}^0(\Omega)$ respectively.

Q.E.D.

**Corollary 6.7.** $I_{\sigma}$ is an isomorphism of $G$-representations from $(\mathcal{H}_{\sigma}^0(\Omega)^*_b, \pi_{\sigma}^*)$ to $(B_{\sigma^{-1}}(\mathbb{S}, V^*), T_{\sigma^*})$.

### 7. Concluding remarks

In [7] §3 we briefly reviewed Morita’s theory of $\text{SL}(2, F)$ and discussed the relation between $I_{\sigma}$ and Morita’s duality and Casselman’s operator for

$$\sigma_s \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = z^s$$

with $s$ a positive integer (for $s$ non-positive, $I_{\sigma}$ is an isomorphism between two $(-s + 1)$-dimensional $G$-representations, which is of less interest).
To illustrate this connection, we consider the special case $s = 2$. $\mathcal{O}_{\sigma_2}(\Omega)$ is canonically isomorphic to the space $\Omega^1(\Omega)$ of holomorphic 1-forms on the upper half plane $\Omega \cong K - F = \mathbb{P}^1(K) - \mathbb{P}^1(F)$ via $\psi(u) \mapsto \psi(u)du$. It may be shown that $\mathcal{N}_{\sigma_2}(\Omega)$ corresponds to the subspace of $\Omega^1(\Omega)$ with zero residue at each point of $\mathbb{P}^1(F)$ (see [4] and [7]). On the other hand, $\mathcal{N}_{\sigma_2}(\Omega)$ is canonically isomorphic to the space $\Omega^1(\Omega)$ of holomorphic 1-forms on the upper half plane $\Omega \cong K - F = \mathbb{P}^1(K) - \mathbb{P}^1(F)$. $D_0$ has two closed $G$-invariant subspaces, the spaces $P_0$ and $P_{0\text{loc}}$ consisting of constants and locally constant functions on $\mathbb{P}^1(F)$ respectively. The classical Morita’s duality is established via residues. More precisely, for each $\psi \in \mathcal{O}_{\sigma_2}(\Omega)$, define a linear functional $M_2(\psi)$ of $D_0$ by

$$\langle \varphi, M_2(\psi) \rangle = \text{the sum of residues of the 1-form } \varphi(u)\psi(u) \, du \text{ on } \mathbb{P}^1(F).$$

Morita’s duality $M_2$ induces $G$-isomorphisms $\mathcal{O}_{\sigma_2}(\Omega) \cong (D_0/P_0)_{\mathcal{B}_0}$ and $\mathcal{N}_{\sigma_2}(\Omega) \cong (D_0/P_{0\text{loc}})_{\mathcal{B}_0}$. Moreover, Casselman’s intertwining operator $S_0 : \varphi \mapsto d\varphi$

induces a $G$-isomorphism between $D_0/P_{0\text{loc}}$ and the space $D_{-2}$ of locally analytic 1-forms on $\mathbb{P}^1(F)$, a space which is isomorphic to $C^\text{an}_{\sigma_{-2}}(\mathfrak{S})$ (see [5] and [7]). The connection between our duality operator $I_{\sigma_2}$ and Morita’s duality $M_2$ was found in [7] Theorem 3.6 as the following commutative diagram

$$\begin{array}{ccc}
\mathcal{N}_{\sigma_2}(\Omega) & \xrightarrow{M_2} & (D_0/P_{0\text{loc}})^*_{\mathcal{B}_0} \\
\downarrow{I_{\sigma_2}} & & \downarrow{S_0} \\
B_{\sigma_{-2}}(\mathfrak{S})^*_{\mathcal{B}_0} & = & C^\text{an}_{\sigma_{-2}}(\mathfrak{S})^*_{\mathcal{B}_0} \cong (D_{-2})^*_{\mathcal{B}_0}
\end{array}$$

A generalization of Morita’s duality seems quite hard in view of its analytic construction via residues. The first step towards this would be finding other closed sub-representations of $(C^\text{an}_\sigma(\mathfrak{S}, V), T_\sigma)$ and $(\mathcal{O}_\sigma(\Omega), \pi_\sigma)$. This work is done completely for $\text{SL}(2, F)$ in Morita and Murase’s [4], [5] and [6], where the complete classifications of the sub-quotient spaces of holomorphic discrete series and the principal series are conjectured and claimed (Morita attempted to prove this, but his proof contained a serious gap).

A further question is on the irreducibility. We conjecture that $(\mathcal{N}_\sigma(\Omega), \pi_\sigma)$ and $(B_{\sigma}(\mathfrak{S}, V^*), T_{\sigma^*})$ are topologically irreducible $G$-representations if $\sigma$ is irreducible. For $\text{SL}(2, F)$, this conjecture was claimed in [6] Theorem 1 (i) and a proof for $F = \mathbb{Q}_p$ was given by Schneider and Teitelbaum in [11].
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