Gravitation and cosmology in a brane-universe

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Abstract
Recent theoretical developments have generated a strong interest in the “brane-world” picture, which assumes that ordinary matter is trapped in a three-dimensional sub-manifold, usually called brane, embedded in a higher dimensional space. The purpose of this review is to introduce some basic results concerning gravity in these models and then to present various aspects of the cosmology in a brane-universe.

1 Introduction
The idea that our world may contain hidden extra dimensions is rather old since one can trace this idea, in the modern context of general relativity, back to the beginning of the twentieth century, with the pioneering works of Kaluza and Klein, trying to reinterpret electromagnetism as a geometrical effect from a fifth dimension. The idea of extra-dimensions was revived more recently with the advent of string theory as the most promising avenue for reconciling gravity and quantum field theory. In order to get a consistent theory at the quantum level, ten spacetime dimensions are needed in superstring theories (eleven in M-theory), which means that six dimensions must be somehow hidden to four-dimensional observers such as us.

The simplest way to hide extra dimensions is to assume that they are flat, compact with a radius sufficiently small to be unobservable. Consider for example the case of one extra-dimension described by the coordinate $y$ and compactified via the identification

$$y \rightarrow y + 2\pi R,$$

$R$ being the “radius” of the extra-dimension. Any matter field, for example a scalar field, depends on both the ordinary spacetime coordinates $x^\mu$ and the extra-coordinate $y$. It can be Fourier expanded along the extra-dimension so that

$$\phi(x^\mu, y) = \sum_{p=-\infty}^{\infty} e^{ipy/R} \phi_p(x^\mu).$$

(1)

The corresponding Fourier modes $\phi_p$ are designated as Kaluza-Klein modes, and each of them can be seen as a four-dimensional scalar field satisfying the four-dimensional Klein-Gordon equation with the effective squared mass

$$M_p^2 = m^2 + \frac{p^2 R^2}{\kappa^2}.$$  

(2)

A simple way to identify an extra-dimension would thus be to detect the characteristic spectrum of the Kaluza-Klein modes. To do this one needs enough energy to excite at least the first Kaluza-Klein modes and the non-observation of Kaluza-Klein modes can be interpreted as meaning that the size of the extra-dimension is smaller than the inverse of the energy scale probed by the experiment. Present constraints from colliders thus imply

$$R \lesssim 1 \text{ (TeV)}^{-1}.$$  

Let us now turn to gravity. The natural way to extend Einstein gravity to higher dimensions is to start from the Einstein-Hilbert action defined in a generalized spacetime with, say, $n$ extra-dimensions:

$$S_{\text{grav}} = \int d^4x d^n y \frac{R}{2\kappa^2}.$$  

(3)
where $R$ is the scalar curvature in the $(4 + n)$-dimensional spacetime. Variation of the action including the matter part leads to the generalized Einstein equations which read

$$G_{AB} \equiv R_{AB} - \frac{1}{2} R g_{AB} = \kappa^2 T_{AB}. \quad (4)$$

This equation has exactly the same form as the familiar one, with the difference that all tensors are now $(4 + n)$-dimensional tensors.

In the static weak field regime (in a flat $(4 + n)$-dimensional spacetime) Einstein equations imply as usual the Poisson equation. However, the solution of Poisson’s equation depends on the number of space dimensions, and the general form of the Newtonian potential is

$$\phi_N (r) \propto \frac{G_{(4+n)}}{r^{n+1}}, \quad (5)$$

where the generalized Newton’s constant $G_{(4+n)}$ is proportional to the gravitational coupling $\kappa^2$ introduced above in the Einstein equations (4).

This means that, a priori, the presence of extra-dimensions implies that gravity is modified. The simplest way to recover the familiar gravity law is once more to compactify the extra-dimensions. The resulting gravity is

- the $(4 + n)$ dimensional gravity on scales smaller than the radius $R$ of compactification (assumed here to be the same in all extra-dimensions for simplicity)
- the usual 4-dimensional gravity on scales much larger than $R$, with the Newton’s constant given by
  $$G_{(4)} \sim \frac{G_{(4+n)}}{R^n}, \quad (6)$$
  or, equivalently, expressed in terms of the Planck mass,
  $$M_{(4)}^2 \sim \frac{M_{(4+n)}^2 R^n}{R^n}. \quad (7)$$

This result can be understood is the following way. In a compactified space, the gravitational field induced by a point mass $m$ can be computed by unwrapping the extra-space and by summing the contributions of all the images of the true mass $m$. At small distances with respect to $R$, the influence of the image masses can be ignored and one gets $(4 + n)$ dimensional gravity. By contrast, on scales much larger than $R$, all image masses contribute to the gravitational field and they can be assimilated to a continuous massive “line” with constant “linear” mass density. Applying then Gauss’ law to a cylinder surrounding the massive line yields the usual gravitational force with the above gravitational coupling.

As a consequence, like in particle physics, an upper constraint on the compactification radius can be deduced from the absence of any observed deviation from ordinary Newton’s law. The present experimental constraints yield (see e.g. [1])

$$R \lesssim 0.2 \text{ mm}. \quad (8)$$

The latest developments in the models with extra-dimensions come from the realization that the observational constraint on the size of extra-dimensions from gravity experiments is much weaker than that from accelerator experiments. This suggests the idea to decouple the extra-dimensions “felt” by ordinary particles from the extra-dimensions “felt” by gravity. Concretely, this can be realized by invoking a mechanism that confines fields of the particle physics Standard Model to a subspace with three spatial dimensions, called three-brane, within a higher dimensional space where gravity lives.

The purpose of this review, far from being exhaustive, is to present the basic results concerning gravity and cosmology in the brane scenarios where the self-gravity of the brane is taken into account, and to illustrate some more advanced aspects like cosmological perturbations or brane collisions. Complementary information can be found in two recent reviews, one by R. Maartens [2] and the other by V. Rubakov [3], the latter more focused on the particle physics aspects.
2 Braneworld models

The braneworld scenarios have attracted much attention only recently but one can find in the literature a few precursor works \[4\] considering our four-dimensional universe as a subspace of a larger spacetime. A wider interest in braneworlds has developed when this idea has emerged in the context of superstring theories and M-theory. An important step is the Horava-Witten supergravity \[5\], which is supposed to describe the effective low energy theory in the strong coupling limit of heterotic $E_8 \times E_8$ superstrings. This theory is based on an eleven-dimensional spacetime with an orbifold $S_1/Z_2$ eleventh dimension, with 11-dimensional supergravity in the bulk and a gauge group $E_8$ on each of the 10-dimensional boundaries corresponding to the worldsheets of 9-branes located at the two fixed points along the eleventh dimension. After compactification of six dimensions, one finds a five-dimensional spacetime with two four-dimensional boundaries where matter is supposed to be localized.

At the phenomenological level, Arkani-Hamed, Dimopoulos and Dvali (ADD) \[6\] then suggested the more radical step to assume that the fundamental Planck mass is of the order of the TeV so as to solve the famous hierarchy problem in particle physics, i.e. why there are so many orders of magnitude between the electroweak scale and the Planck scale. In these models the observed Planck scale seems huge simply because the volume of the extra dimensions is large, according to the formula \[7\]. The ADD proposal has had a tremendous impact, especially in the particle physics community.

In the general relativity and cosmology communities, more interest was suscitated by the later proposal of Randall and Sundrum \[7, 8\]. By contrast with the ADD models, their specificities are the following:

- there is only one extra-dimension
- the bulk spacetime is not flat but curved: it is a portion of Anti-de Sitter (AdS) with a negative cosmological constant

\[ \Lambda = \frac{6}{\ell^2} = -6\mu^2 \tag{9} \]

where $\ell$ has the dimension of length, and $\mu$ the dimension of mass.
- there is a tension, $\sigma$, in the brane(s), which is (are) supposed to be $Z_2$-symmetric like in the Horava-Witten model.

The metric can then be written in the form

\[ ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{10} \]

where $\eta_{\mu\nu}$ is the usual Minkowski metric and the warping factor is given by

\[ a(y) = e^{-|y|/\ell}. \tag{11} \]

There are in fact two models due to Randall and Sundrum, with essentially the same framework but which differ in the rôle assigned to the positive tension brane:

- RS1: two branes are bounding the AdS portion, one positive tension brane at $y = 0$ and one negative tension brane, the latter corresponding to our accessible world \[7\]. This model suggests a solution to the hierarchy problem, by showing that TeV energy scales on the hidden brane (the TeV brane) correspond to $M_P$ energy scales in our brane (the Planck brane), due to the exponential warping factor.
- RS2: in this model \[8\], our world corresponds to the positive tension brane which is located at $y = 0$. The negative tension brane is now facultative. The goal of this model is not to solve the hierarchy problem, but to show that an infinite extra dimension can lead to usual four-dimensional gravity, as explained below.

The tension of the branes is not a free parameter for the above setup to be valid. They must be fine-tuned with the AdS cosmological constant so as to satisfy the condition

\[ \frac{\kappa^2}{6}\sigma = \frac{1}{\ell}. \tag{12} \]
From now on, I will consider only the second model RS2 and show how ordinary gravity is approximately recovered. In order to study weak gravity, the usual method is to consider tensor-like linear perturbations of the metric. Denoting these fluctuations $h_{AB}$ so that

$$g_{AB} = \bar{g}_{AB} + h_{AB},$$

and imposing the gauge requirements $h_{Ay} = 0$, $h_{\nu \lambda \nu} = 0$ and $h_{\mu \mu} = 0$, the linearized Einstein equations yield the following wave equation for the gravitons

$$\left[ a^{-2}\Box^{(4)} + \partial_y^2 - \frac{4}{\ell^2} + \frac{4}{\ell}\delta(y) \right] h_{\mu\nu} = 0,$$

where the delta function takes into account the presence of the brane. This equation is separable and the general solution can be written as the superposition of solutions of the form

$$h_{\mu\nu}(x^\lambda, y) = u_m(y)e^{ik_\lambda x^\lambda} \epsilon_{\mu\nu},$$

where $k_\lambda k^\lambda = -m^2$ can be interpreted as the four-dimensional mass and $u_m(y)$ satisfies an ordinary differential equation, which is deduced from (14) by replacing the d’Alembertian operator with $m^2$. Following Randall and Sundrum, it is convenient to rewrite this equation in a Schrödinger-like form,

$$-\frac{d^2 \psi_m}{dz^2} + V(z) \psi_m = m^2 \psi_m,$$

where $\psi_m = a^{-1/2}u_m$ and $z = sgn(y)\ell(\exp(|y|/\ell) - 1)$. The potential $V(z)$ is “volcano”–shaped, its expression being given by

$$V(z) = \frac{15}{4(|z| + \ell)^2} - \frac{3}{\ell}\delta(z).$$

The delta function part of the potential leads to the existence of a zero mode, with the functional dependence

$$u_m \propto a^2,$$

which is (exponentially) localized near the brane and which reproduces the usual four-dimensional graviton, the four-dimensional gravitational coupling being given by

$$8\pi G_{(4)} = \kappa^2/\ell.$$  

In addition to this zero mode, there is a whole continuum of massive graviton modes, which induce some corrections to the usual gravitational law, although significant only on scales of the order of the AdS lengthscale $\ell$ and below. The resulting gravitational potential is of the form

$$V(r) = \frac{G_{(4)}}{r} \left( 1 + \alpha \frac{\ell^2}{r^2} \right).$$

The coefficient $\alpha = 1$, given initially by RS, was corrected to $\alpha = 2/3$ by a more detailed analysis [9], which considered explicitly the coupling of graviton to matter in the brane.

Finally, let us mention that a rewriting of the Einstein equations, in the Randall-Sundrum type models, leads to effective four-dimensional Einstein equations, which can be written in the form [10]

$$(^{(4)}G)_{\mu\nu} = 8\pi G_{(4)}\tau_{\mu\nu} + \kappa^4 \Pi_{\mu\nu} - E_{\mu\nu},$$

where $\tau_{\mu\nu}$ is the brane energy-momentum tensor (not including the tension $\sigma$), $\Pi_{\mu\nu}$ is quadratic in the brane energy momentum tensor,

$$\Pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\sigma} \tau^{\sigma}_\nu + \frac{1}{12} \tau_{\nu\mu} + \frac{1}{8} g_{\mu\nu} \left( \tau_{\rho\sigma} \tau^{\rho\sigma} - \frac{1}{3} \tau^2 \right),$$

and $E_{\mu\nu}$ is the projection of the five-dimensional Weyl tensor

$$E_{\mu\nu} = (^{(5)}C^A_{BCD} n_A n^C g^B_{\mu} g^D_{\nu}),$$
A being the unit vector normal to the brane. Although very nice, one must be aware that the above equation is only a rewriting of the five-dimensional Einstein's equations using the junction conditions, and in practical problems, the full system remains to be solved. In particular one must be careful with the interpretation of the gravitational coupling to brane matter, even in the linear case, because $E_{\mu \nu}$ will in general depend on the matter $\tau_{\mu \nu}$.

In a two-brane model, in contrast with the single brane model, what is obtained is Brans-Dicke type gravity with the radion, i.e. the interbrane separation playing the rôle of the Brans-Dicke scalar field. The model RS1 leads to a Brans-Dicke gravity which is incompatible with observations: this model can be saved only by introducing a more complicated setting, for example a scalar field in the bulk with a potential and couplings to the branes, which provides an stabilization mechanism for the radion.

3 Homogeneous cosmology in a brane-universe

Let us now turn to cosmology. The main motivation for exploring cosmology in models with extra-dimensions is that the potentially new effects could arise significantly only at very high energies, i.e. in the very early universe, and leave some relic imprints which could be tested today via cosmological observations. Before discussing the potentially rich but very difficult question of cosmological perturbations in the next section, one must first describe homogeneous cosmology, following [12] and [13] (see also [14]).

Let us thus consider a five-dimensional spacetime with three-dimensional isotropy and homogeneity, which contains a three-brane representing our universe. It is convenient, but not necessary, to work in a Gaussian normal coordinate system based on our brane-universe. Due to the spacetime symmetries, the metric is then of the form

$$ds^2 = -n(t,y)^2 dt^2 + a(t,y)^2 \delta_{ij} dx^i dx^j + dy^2,$$

(23)

where we have assumed that our brane-universe is spatially flat (but this can be generalized very easily to hyperbolic or elliptic spaces). In these coordinates, our brane-universe is always located at $y = 0$.

The energy-momentum tensor can be decomposed into a bulk energy-momentum tensor and a brane energy-momentum tensor, the latter being of the form

$$T_B^A = S_B^A \delta(y) = \{\rho_b, p_b, p_b, p_b, 0\} \delta(y),$$

(24)

where the delta function expresses the confinement of matter in the brane. $\rho_b$ and $P_b$ are respectively the total energy density and pressure in the brane and depend only on time. For simplicity, we neglect the bulk energy-momentum tensor but allow for the presence of a cosmological constant in the bulk, $\Lambda$, so that the five-dimensional Einstein equations read

$$G_{AB} + \Lambda g_{AB} = \kappa^2 T_{AB},$$

(25)

Because of the distributional nature of the energy-momentum tensor, one way to solve the Einstein equations is to solve them first in the bulk and then apply the junction conditions [15] for the metric at $y = 0$. According to the junction conditions, the metric must be continuous and the jump of the extrinsic curvature tensor $K_{AB}$ (related to the derivatives of the metric with respect to $y$) depends on the distributional energy-momentum tensor,

$$[K_B^A - K^A_B] = \kappa^2 S_B^A,$$

(26)

where the brackets here denote the jump at the brane, i.e. $[Q] = Q_{\{y=0^+\}} - Q_{\{y=0^-\}}$, and the extrinsic curvature tensor is defined by

$$K_{AB} = h^C_A \nabla_C n_B,$$

(27)

$n^A$ being the unit vector normal to the brane. As before, one can add the extra assumption that the brane is mirror symmetric so that the jump in the extrinsic curvature is twice its value on one side (see [15] for references where this is not assumed). Substituting the ansatz metric (23) in (26), one ends up with the two junction conditions:

$$\left(\frac{n'}{n}\right)_{0^+} = \frac{\kappa^2}{6} (3p_b + 2\rho_b), \quad \left(\frac{a'}{a}\right)_{0^+} = -\frac{\kappa^2}{6} \rho_b.$$

(28)
Going back to the bulk Einstein equations (their explicit form can be found in e.g. [13]), one can solve the \((t-y)\) component to get
\[
\dot{a}(t,y) = \alpha(t) n(t,y),
\]
and the integration of the component \((t-t)\) with respect to \(y\) and of the component \((y-y)\) with respect to time, yields the first integral
\[
(aa')^2 - \alpha^2 a^2 + \frac{\Lambda}{6} a^4 + C = 0,
\]
(30)
where \(C\) is an integration constant. When one evaluates this first integral at \(y = 0\), i.e. in our brane-universe, substituting the junction conditions given above in (28), one ends up with the following equation
\[
H_0^2 \equiv \frac{\dot{a}_0^2}{a_0^2} = \kappa^4 \rho_b^2 + \frac{\Lambda}{6} + \frac{C}{a^4}.
\]
(31)
where the subscript ‘0’ means evaluation at \(y = 0\). This equation is analogous to the (first) Friedmann equation, since it relates the Hubble parameter to the energy density, but it is different from the usual Friedmann equation \([H^2 = (8\pi G/3)\rho]\). Its most remarkable feature is that the energy density of the brane enters quadratically on the right hand side in contrast with the standard four-dimensional Friedmann equation where the energy density enters linearly. As for the energy conservation equation it is unchanged in this five-dimensional setup and still reads
\[
\dot{\rho}_b + 3H(\rho_b + p_b) = 0.
\]
(32)

In the simplest case where \(\Lambda = 0\) and \(C = 0\), one can easily solve the above cosmological equations for a perfect fluid with an equation of state \(p_b = w\rho_b\) and \(w\) constant. One finds that the evolution of the scale factor is given by
\[
a_0(t) \propto t^{\frac{1}{3(1+w)}}.
\]
(33)
In the most interesting cases for cosmology, radiation and pressureless matter, one finds respectively \(a \sim t^{1/4}\) (instead of the usual \(a \sim t^{1/2}\)) and \(a \sim t^{1/3}\) (instead of \(a \sim t^{2/3}\)). Such behaviour is problematic because it cannot be reconciled with nucleosynthesis. Indeed, the nucleosynthesis scenario depends crucially on both the microphysical reaction rates and the expansion rate of the universe. And changing in a drastic way the evolution of the scale factor between nucleosynthesis and now modifies dramatically the predictions for the light element abundances.

The above Friedmann law with the \(\rho_b^2\) term, but without the bulk cosmological constant (and without the \(C\) term) was first derived in [12], just before Randall and Sundrum proposed their models. In fact, the unusual Friedmann law can be related to a gravity which should be five-dimensional rather than four-dimensional. With the obtention in RS2 of a five-dimensional model yielding a four-dimensional gravity, one could expect that the corresponding cosmology should be compatible with the usual cosmology [17]. In fact, it is clear from (31) that a Minkowski brane with a tension, as in the RS2 model, requires the presence of a negative cosmological constant in the bulk to compensate the squared tension term and get \(H = 0\).

If one wants to go beyond a Minkowski geometry and consider non trivial cosmology in the brane, one must then assume that the total energy density in the brane, \(\rho_b\), consists of two parts,
\[
\rho_b = \sigma + \rho,
\]
(34)
the tension \(\sigma\), constant in time, and the usual cosmological energy density \(\rho\). Substituting this decomposition into (31), one obtains
\[
H^2 = \left(\frac{\kappa^4}{36} \sigma^2 - \mu^2\right) + \frac{\kappa^4}{18} \sigma \rho + \frac{\kappa^4}{36} \rho^2 + \frac{C}{a^4}.
\]
(35)
If one fine-tunes the brane tension and the bulk cosmological constant as in [12], the first term on the right hand side vanishes. The second term then becomes the dominant term if \(\rho\) is small enough and one thus recovers the usual Friedmann equation at low energy, with the identification
\[
8\pi G = \frac{\kappa^4}{6} \sigma,
\]
(36)
which is exactly the relation obtained in RS2 by combining (12) and (18).

The third term on the right hand side, quadratic in the energy density, provides a high-energy correction to the Friedmann equation which becomes significant when the value of the energy density approaches the value of the tension $\sigma$ and dominates at higher energy densities. In the very high energy regime, $\rho \gg \sigma$, one thus recovers the unconventional behaviour analysed before since the bulk cosmological constant becomes negligible. It is in fact not difficult to obtain explicit solutions for the scale factor, which interpolate between the low energy regime and the high energy regime.

Finally, the last term on the right hand side behaves like radiation and arises from the integration constant $C$. This constant $C$ is quite analogous to the Schwarzschild mass and it is related to the bulk Weyl tensor, which vanishes when $C = 0$. In a cosmological context, this term is constrained to be small enough at the time of nucleosynthesis in order to satisfy the constraints on the number of extra light degrees of freedom. In the matter era, this term then redshifts quickly and would be in principle negligible today.

In the present section, we have so far considered only the metric in the brane. The metric outside the brane can be also determined explicitly \cite{13}. In the special case $C = 0$, the metric has a much simpler form and its components are given by

$$a(t, y) = a_0(t) (\cosh \mu y - \eta \sinh \mu |y|) \quad (37)$$

$$n(t, y) = \cosh \mu y - \tilde{\eta} \sinh \mu |y| \quad (38)$$

where

$$\eta = 1 + \frac{\rho}{\sigma}, \quad \tilde{\eta} = \eta + \frac{\dot{\eta}}{H_0} \quad (39)$$

and we have chosen the time $t$ corresponding to the cosmic time in the brane. In the RS2 limit, $\rho = 0$, i.e. $\rho_b = \sigma$, which implies $\eta = \tilde{\eta} = 1$ and one recovers $a(t, y) = a_0 \exp(-\mu |y|)$.

In summary, we have obtained a cosmological model, based on a braneworld scenario, which appears to be compatible with current observations at low enough energies. Let us now quantify the constraints on the parameters of the model in order to ensure this compatibility with observations. As mentioned above an essential constraint comes from nucleosynthesis: the evolution of the universe since nucleosynthesis must be approximately the same as in usual cosmology. This is the case if the energy scale associated with the tension is higher than the nucleosynthesis energy scale, i.e.

$$M_c \equiv \sigma^{1/4} \gtrsim 1 \text{ MeV.} \quad (40)$$

Combining this with \cite{36} this implies for the fundamental mass scale (defined by $\kappa^2 = M^{-3}$)

$$M \gtrsim 10^4 \text{ GeV.} \quad (41)$$

There is however another constraint, which is not of cosmological nature: the requirement to recover ordinary gravity down to scales of the submillimeter order. This implies

$$\ell \lesssim 10^{-1} \text{ mm,} \quad (42)$$

which yields the constraint

$$M \gtrsim 10^8 \text{ GeV.} \quad (43)$$

Therefore the most stringent constraint comes, not from cosmology, but from gravity experiments in this particular model. So far, we have thus been able to build a model, which reproduces all qualitative and quantitative features of ordinary cosmology in the domains that have been tested by observations. The obvious next question is whether this will still hold for a more realistic cosmology that includes perturbations from homogeneity, and more interestingly, whether brane cosmology is capable of providing predictions that deviate from usual cosmology and which might tested in the future. This is still an open question today.
4 Brane cosmological perturbations

Endowed with a viable homogeneous scenario, one would like to explore the much richer domain of cosmological perturbations and investigate whether brane cosmology leads to new effects that could be tested in the forthcoming cosmological observations, in particular of the anisotropies of the Cosmic Microwave Background.

Brane cosmological perturbations is a difficult subject and although there are now many published works on this question (see e.g. [18, 19, 20, 21, 22, 23, 24, 25, 26]), no observational signature has yet been predicted. Below I will summarize some results concerning two different aspects of perturbations. The first aspect deals with the evolution of scalar type perturbations on the brane, the second aspect with the production of gravitational waves from quantum fluctuations during a de Sitter phase in the brane.

Let us first discuss scalar type cosmological perturbations in brane cosmology. Choosing a GN coordinate system the "scalarly" perturbed metric can be written

\[ ds^2 = -n^2(1 + 2A)dt^2 + 2n^2 \partial_i B dt dx^i + a^2 \left[ (1 + 2C) \delta_{ij} + 2 \partial_i \partial_j E \right] dx^i dx^j + dy^2, \]

where the perturbations turn out to coincide exactly with the standard scalar cosmological perturbations since we are in the GN gauge. One can find other gauge choices in the literature. Using the compact notation \( h_\alpha = \{A, B, C, E\} (\alpha = 1, \ldots, 4) \), the linearized Einstein equations

\[ \delta G_{AB} + \Lambda \delta g_{AB} = \kappa^2 \delta T_{AB} \]

yield, in the bulk, expressions of the form

\[ \delta G^{(5)}_{\alpha\beta} = \delta G^{(5)}_{\alpha\beta \rho} + [h_\alpha, h'_\beta, h''_{\rho}] = -\Lambda \delta g_{\alpha\beta} \]

where the brackets represent linear combinations of the perturbations and their derivatives. Brane matter enters only in the junction conditions, which at the linear level relate the first derivatives (with respect to \( y \)) of the metric perturbations \( h'_\alpha \) to brane matter perturbations \( \delta \rho, \delta P, \nu, \pi \). One can then substitute these relations back into the perturbed Einstein equations. \( \delta G_{05}^{(5)} = 0 \) then yields the usual perturbed energy conservation equation, whereas \( \delta G_{i5}^{(5)} = 0 \) yields the perturbed Euler equation. The other equations yield equations of motion for the perturbations where one recognizes the usual equations of motion in ordinary cosmology, but with two types of corrections:

- modification of the homogeneous background coefficients due to the additional terms in the Friedmann equation. These corrections are negligible in the low energy regime \( \rho \ll \sigma \). For long wavelength (larger than the Hubble scale) perturbations, one can thus obtain a transfer coefficient, \( T = 5/6 \), characterizing the high/low energy transition, i.e.

\[ \Phi_{\rho \ll \sigma} = \frac{5}{6} \Phi_{\rho \gg \sigma}. \]

- presence of source terms in the equations. These terms come from the bulk perturbations and cannot be determined solely from the evolution inside the brane. To determine them, one must solve the full problem in the bulk (which also means to specify some initial conditions in the bulk). From the four-dimensional point of view, these terms from the fifth dimension appear like external source terms and their impact is formally similar to that of “active seeds”, which have been studied in the context of topological defects.
Let us turn now to another facet of the brane cosmological perturbations: their origin. In standard cosmology, the main mechanism for producing the cosmological perturbations is inflation. One can thus try to generalize this mechanism to the context of brane cosmology. Brane inflation generated by a scalar field confined to the brane has been investigated in [27]. The spectrum of gravitational waves generated in such a scenario is however more subtle to compute because gravitational waves have an extension in the fifth dimension. It has been first computed in [28] and confirmed (and extended) in a different approach [29].

To compute the production of gravitational waves, one can approximate slow-roll brane inflation by a succession of de Sitter phases. The metric for a de Sitter brane (see also [30]) corresponds to a particular case of (38) with \( \eta = \tilde{\eta} \) and can be written as

\[
a(t, y) = a_0(t) \mathcal{A}(y), \quad n = \mathcal{A}(y),
\]

with

\[
\mathcal{A}(y) = \cosh \mu y - \left( 1 + \frac{\rho}{\sigma} \right) \sinh \mu |y|.
\]

The gravitational waves appear in a perturbed metric of the form

\[
ds^2 = -n^2 dt^2 + a^2 \left[ \delta_{ij} + E_{TT}^{ij} \right] dx^i dx^j + dy^2,
\]

where the ‘TT’ stands for transverse traceless. Decomposing \( E_{TT}^{ij} \) in Fourier modes of the form

\[
E_{ij} = E e^{i \vec{k}.\vec{x}} e_{ij},
\]

one gets a wave equation, which reads

\[
\ddot{E} + 3H_0 \dot{E} + \frac{k^2}{a_0^2} E = \mathcal{A}^2 E'' + 4 \mathcal{A} \dot{\mathcal{A}} E'.
\]

This equation is separable, and one can look for solutions \( E = \varphi_m(t) \mathcal{E}_m(y) \), where the time-dependent part must satisfy

\[
\ddot{\varphi}_m + 3H_0 \dot{\varphi}_m + \left[ m^2 + \frac{k^2}{a_0^2} \right] \varphi_m = 0,
\]

and the \( y \)-dependent part satisfies

\[
\mathcal{E}_m'' + 4 \frac{\mathcal{A}'}{\mathcal{A}} \mathcal{E}_m' + \frac{m^2}{\mathcal{A}^2} \mathcal{E}_m = 0.
\]

Like in the Minkowski case, the latter equation can be reformulated as a Schroedinger type equation,

\[
\frac{d^2 \Psi_m}{dz^2} - V(z) \Psi_m = -m^2 \Psi_m,
\]

after introducing the conformal coordinate \( z = \int dy/\mathcal{A}(y) \) and defining \( \Psi_m \equiv \mathcal{A}^{3/2} \mathcal{E}_m \). The potential is given by

\[
V(z) = \frac{15H_0^2}{4 \sinh^2(H_0z)} + \frac{9}{4} H_0^2 - 3\mu \left[ 1 + \frac{\rho}{\sigma} \right] \delta(z - z_b).
\]

The non-zero value of the Hubble parameter signals the presence of a gap between the zero mode and the continuum of Kaluza-Klein modes, as noticed earlier by [31].

The zero mode corresponds simply to

\[
\mathcal{E}_0 = C_1 \equiv \sqrt{\mu} F(H_0/\mu),
\]

where, imposing the normalization \( 2 \int_{z_b}^{\infty} |\Psi_0'| dz = 1 \), the constant \( C_1 \) has been expressed in terms of \( H_0 \) via the function

\[
F(x) = \left\{ \sqrt{1 + x^2} - x^2 \ln \left[ \frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right] \right\}^{-1/2}.
\]
Asymptotically, $F \simeq 1$ at low energies, i.e. $H_0 \ll \mu$, and $F \simeq \sqrt{3H_0/(2\mu)}$ at high energies, i.e. $H_0 \gg \mu$.

One can then evaluate the vacuum quantum fluctuations of the zero mode by using the standard canonical quantization. To do this explicitly, one writes the five-dimensional action for gravity at second order in the perturbations. Keeping only the zero mode and integrating over the fifth dimension, one obtains

$$S_g = \frac{1}{8\kappa_5^2} \sum_{\tau, \omega} \int d\eta d^3k k_o^2 \left[ \left( \frac{d\varphi_o}{d\eta} \right)^2 + k^2 \varphi_o^2 \right],$$

(64)

This has the standard form for a massless graviton in four-dimensional cosmology, apart from the overall factor $1/8\kappa_5^2$ instead of $1/8\kappa_4^2$. It follows that quantum fluctuations in each polarization, $\varphi_o$, have an amplitude of $2\kappa(H_o/2\pi)$ on super-horizon scales. Quantum fluctuations on the brane at $y = 0$, where $E_o = C_1\varphi_0$, thus have the typical amplitude

$$\frac{1}{2\kappa_4} \delta E_{\text{brane}} = \left( \frac{H_0}{2\pi} \right) F(H_0/\mu)$$

(65)

At low energies, $F = 1$ and one recovers exactly the usual four-dimensional result but at higher energies the multiplicative factor $F$ provides an enhancement of the gravitational wave spectrum amplitude with respect to the four-dimensional result. However, comparing this with the amplitude for the scalar spectrum obtained in [27], one finds that, at high energies ($\rho \gg \sigma$), the tensor over scalar ratio is in fact suppressed with respect to the four-dimensional ratio. An open question is how the gravitational waves will evolve during the subsequent cosmological phases, the radiation and matter eras.

5 Collisions of branes

The last topic I would like to discuss is of interest for scenarios with several branes which are allowed to collide. So far, I have focused on cosmology for a single brane embedded in a five-dimensional AdS spacetime. For homogeneous cosmology, if one assumes the bulk to be empty, then the derivation summarized above proves that the bulk and the other branes it might contain can affect our brane-universe only via the constant $C$ of the Weyl radiation term, where $C$ is a constant in time. This property can be understood, more formally, as a generalized Birkhoff theorem [32].

Another brane can however have a dramatic influence when it collides with the initial brane. This possibility, which could provide a new interpretation of the Big-Bang in our brane-universe, has raised some interest recently, in particular the ekpyrotic scenario [33] based on the five-dimensional reduction of Horava-Witten model [34], but other, simpler, models have also been proposed [35].

With K. Maeda and D. Wands [36], I have recently given a general analysis of the collision of $n$-branes in a $(2 + n)$-dimensional empty spacetime with $n$-dimensional isotropy and homogeneity, i.e. branes separated by patches of Sch-AdS spacetimes (allowing for different Schwarzschild-type mass and cosmological constant in each region), with the metric

$$ds^2 = -f(R)dT^2 + \frac{dR^2}{f(R)} + R^2 d\Omega_n^2,$$

(66)

where the ‘orthogonal’ metric $d\Omega_n^2$ does not depend on either $T$ or $R$. The well-known case of a Schwarzschild-(anti)-de Sitter spacetime corresponds to $f(R) = k - (C/R^{n-1} - (R/R)^2$.

Contrarily to the coordinates of the previous section, a brane is no longer at rest in this coordinate system. It can be described by its trajectory $(T(\tau), R(\tau))$, where $\tau$ is the proper time. An alternative way [37] to obtain the generalized Friedmann equation [32] is simply to write the junction conditions at the location of the moving brane, with the $Z_2$ symmetry assumption, by noting that the brane coordinate $R$ can be reinterpreted as the scale factor of the induced metric, as is clear from the metric [36]. It can also been shown explicitly that the expression for the metric [34] given above in GN coordinates can indeed be deduced from [36] by an appropriate change of coordinates [38].

To analyse the collision, it is very convenient to introduce an angle $\alpha$, which characterizes the motion of the brane with respect to the coordinate system [33], defined by

$$\alpha = \sinh^{-1}(\epsilon R/\sqrt{f}),$$

(67)
where $\epsilon = +1$ if $R$ decreases from “left” to “right”, $\epsilon = -1$ otherwise. Considering a collision involving a total number of $N$ branes, both ingoing and outgoing, thus separated by $N$ spacetime regions one can label alternately branes and regions by integers, starting from the leftmost ingoing brane and going anticlockwise around the point of collision (see Figure). The branes will thus be denoted by odd integers, $2k - 1$ ($1 \leq k \leq N$), and the regions by even integers, $2k$ ($1 \leq k \leq N$). Let us introduce, as before, the angle $\alpha_{2k-1|2k}$ which characterizes the motion of the brane $B_{2k-1}$ with respect to the region $R_{2k}$, and which is defined by

$$\sinh \alpha_{2k-1|2k} = \frac{\epsilon_{2k} R_{2k-1}}{\sqrt{f_{2k}}}.$$

Conversely, the motion of the region $R_{2k}$ with respect to the brane by the Lorentz angle $\alpha_{2k|2k-1} = -\alpha_{2k-1|2k}$. It can be shown that the junction conditions for the branes can be written in the form

$$\tilde{\rho}_{2k-1} \equiv \pm \frac{k^2}{n} \rho_{2k-1} R = \epsilon_{2k} \sqrt{f_{2k}} \exp (\pm \alpha_{2k-1|2k}) - \epsilon_{2k-2} \sqrt{f_{2k-2}} \exp (\mp \alpha_{2k-2|2k-1}),$$

with the plus sign for ingoing branes ($1 \leq k \leq N$), the minus sign for outgoing branes ($N+1 \leq k \leq N$). An outgoing positive energy density brane thus has the same sign as an ingoing negative energy density brane.

The advantage of this formalism becomes obvious when one writes the geometrical consistency relation that expresses the matching of all branes and spacetime regions around the collision point. In terms of the angles defined above, it reads simply

$$\sum_{i=1}^{2N} \alpha_i i + 1 = 0.$$
Moreover, introducing the generalized angles
\[ \alpha_{ij'} = \sum_{i=j}^{j-1} \alpha_{ij+1}, \]
if \( j < j' \), and \( \alpha_{j'j} = -\alpha_{jj'} \), the sum rule for angles (70) combined with the junction conditions (69) leads to the laws of energy conservation and momentum conservation. The energy conservation law reads
\[ \sum_{k=1}^{N} \tilde{\rho}_{2k-1} \gamma_{2k-1} = 0, \]
where \( \gamma_{jj'} \equiv \cosh \alpha_{jj'} \) corresponds to the Lorentz factor between the brane/region \( j \) and the brane/region \( j' \) and can be obtained, if \( j \) and \( j' \) are not adjacent, by combining all intermediary Lorentz factors (this is simply using the velocity addition rule of special relativity), or the relative angle formula (71). The index \( j \) corresponds to the reference frame with respect to which the conservation rule is written. Similarly, the momentum conservation law in the \( j \)-th reference frame can be expressed in the form
\[ \sum_{k=1}^{N} \tilde{\rho}_{2k-1} \gamma_{2k-1} \beta_{2k-1} = 0, \]
with \( \gamma_{jj'} \beta_{jj'} \equiv \sinh \alpha_{jj'} \). One thus obtains, just from geometrical considerations, conservation laws relating the brane energies densities and velocities before and after the collision point. Our results apply to any collision of branes in vacuum, with the appropriate symmetries of homogeneity and isotropy. An interesting development would be to extend the analysis to branes with small perturbations and investigate whether one can find scenarios which can produce quasi-scale invariant adiabatic spectra, as seems required by current observations.

Acknowledgements
I would like to thank the organizers of the 11th Workshop on General Relativity and Gravitation (Waseda University, Tokyo, Japan) for a very interesting meeting. I would also like to acknowledge the financial support of the Yamada Science Foundation for my visit to Japan.

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