On unrolled Hopf algebras

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ABSTRACT

We show that the definition of unrolled Hopf algebras can be naturally extended to
the Nichols algebra $B(V)$ of a Yetter–Drinfeld module $V$ on which a Lie algebra $\mathfrak{g}$ acts by
biderivations. As a special case, we find unrolled versions of the small quantum group.

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1. Introduction

1.1.

In the recent papers [9, 11], a so-called unrolled version of quantum sl(2) was
introduced, with applications to quantum topology; the definition was generalized
to simple finite-dimensional Lie algebras in [10]. In this paper, we propose a general-
ization of this notion and embed it into the appropriate conceptual context.

Recall that the unrolled quantum sl(2) is defined as the smash product of
$U_q(sl(2))$ by the universal enveloping algebra of the Lie algebra of dimension 1.
Our starting point is the observation in Lemma 2.6: given an action of the universal
enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ on a Hopf algebra $H$, the smash product is
a Hopf algebra, if and only if $\mathfrak{g}$ acts on $H$ by biderivations. We next observe that, if $V$
is a Yetter–Drinfeld module over a group $G$, then the Lie algebra $\mathfrak{b}0V := \text{End}_G^G(V)$

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of endomorphisms of the Yetter–Drinfeld module $V$ acts by biderivations on the Nichols algebra $B(V)$. Hence, we can form the Hopf algebra $(B(V) \# k G) \rtimes U(b \mathfrak{d} V)$ which we call the unrolled bosonization of $V$. If dim $V$ is finite, then its Gelfand–Kirillov dimension can be expressed in terms of the Gelfand–Kirillov dimension of $B(V)$ and the dimension of $b \mathfrak{d} V$.

The construction of unrolled bosonizations extends to a Lie subalgebra $g$ of $b \mathfrak{d} V$, pre- or post-Nichols algebras (in the place of $B(V)$), and to deformations thereof, provided that the action of the Lie algebra $g$ preserves the relevant defining relations. In particular, we define the unrolled version of the quantum double of a finite-dimensional Nichols algebra of diagonal type.

1.2. Preliminaries

Fix a field $k$ and let $H$ be a Hopf algebra over $k$. We use standard notation: $\Delta$, $\varepsilon$, $S$, $\overline{S}$ are respectively the comultiplication, the counit, the antipode (always assumed to be bijective) and the inverse of the antipode.

We denote by $\mathcal{YD}_H$ the category of Yetter–Drinfeld modules over $H$ as in [5]. For $V, W \in \mathcal{YD}_H$, we denote by $\text{Hom}_H^H(V, W)$, $\text{End}_H^H(V)$, $\text{Aut}_H^H(V)$ the spaces of morphisms, respectively endomorphisms, automorphisms in $\mathcal{YD}_H$. Let $R$ be a Hopf algebra in the braided monoidal category $\mathcal{YD}_H$, with comultiplication denoted by $r \mapsto r^{(1)} \otimes r^{(2)}$. Recall that the bosonization $R \# H$ is the Hopf algebra over $k$ with underlying vector space $R \otimes H$, smash product multiplication and smash coproduct comultiplication; i.e. for all $r, s \in R$, $a, b \in H$,

$$\begin{align*}
(r \# a)(s \# b) &= r(a_{(1)} \cdot s) \# a_{(2)} b, \\
\Delta(r \# a) &= r^{(1)} \# (r^{(2)})(-1) a_{(1)} \otimes (r^{(2)})_{(0)} \# a_{(2)}. 
\end{align*}$$

Here we write $r \# h$ for $r \otimes h$.

We also introduce the category $\mathcal{YD}_H^H = \mathcal{YD}_H \# _{\text{Hopf}} \mathcal{YD}$ of right–right Yetter–Drinfeld modules over $H$. Thus $M \in \mathcal{YD}_H^H$ means that $M$ is a right $H$-module and a right $H$-comodule (with coaction $\rho$), and satisfies the compatibility axiom

$$\rho(m \cdot h) = m_{(0)} \cdot h_{(2)} \otimes S(h_{(1)})m_{(1)}h_{(3)}, \quad m \in M, \quad h \in H. \quad (1.3)$$

The tensor category $\mathcal{YD}_H^H$ is braided, with braiding $c(m \otimes n) = n \cdot m_{(1)} \otimes m_{(0)}$, for all $m, n \in M, N \in \mathcal{YD}_H^H$. For right–right Yetter–Drinfeld modules $V, W \in \mathcal{YD}_H^H$, we use the notions $\text{Hom}_H^H(V, W)$, $\text{End}_H^H(V)$, $\text{Aut}_H^H(V)$ as before.

Let $T$ be a Hopf algebra in the braided monoidal category $\mathcal{YD}_H^H$ of right–right Yetter–Drinfeld modules, with comultiplication denoted by $t \mapsto t^{(1)} \otimes t^{(2)}$. In this case, the bosonization $H \# T$ is the Hopf algebra over $k$ with underlying vector space $H \otimes T$, smash product multiplication and smash coproduct comultiplication; i.e.

$$\begin{align*}
(a \# t)(b \# u) &= ab_{(1)} \# (t \cdot b_{(2)})u, \\
\Delta(a \# t) &= a_{(1)} \# ((t^{(1)})_{(0)} \otimes a_{(2)}(t^{(1)})_{(1)} \# t^{(2)},
\end{align*}$$

for all $t, u \in R$, $a, b \in H$. Here we write $h \# t$ for $h \otimes t$. 

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If $\Gamma$ is an abelian group, then we denote by $k^\chi_\Gamma$ the one-dimensional object in $k\Gamma\mathcal{YD}$ with coaction given by the group element $g \in \Gamma$ and action given by the character $\chi \in \hat{\Gamma}$. For a Yetter–Drinfeld module $V \in k^\Gamma\mathcal{YD}$, the corresponding isotypic component is denoted by $V^\chi_\Gamma$. A Yetter–Drinfeld module has a natural structure of a braided vector space. For a braided vector space $V$, denote by $B(V)$ its Nichols algebra and by $J = J(V)$ its ideal of defining relations, cf. [5]; so that $B(V) \simeq T(V)/J(V)$.

2. Unrolled Hopf Algebras

2.1.

Let $L$ be a Hopf algebra. Recall that a (left) $L$-module algebra is an algebra $A$ which is also an $L$-module with action $\cdot : L \otimes A \rightarrow A$ such that for all $\ell \in L$ and all $a, b \in A$ the compatibility conditions

$$\ell \cdot (ab) = (\ell(1) \cdot a)(\ell(2) \cdot b),$$

$$\ell \cdot 1 = \varepsilon(\ell)1$$

for product and unit hold. It is well-known that (2.1) and (2.2) mean that $A$ is an algebra in the monoidal category $L\mathcal{M}$ of left $L$-modules.

In this paper, we are interested in the case of a Hopf algebra $H$ that is also an $L$-module algebra, where $L$ is a Hopf algebra as well. In this case, we impose the following consistency conditions:

$$\Delta(\ell \cdot a) = \ell(1) \cdot a \otimes \ell(2) \cdot a,$$

$$\varepsilon(\ell \cdot a) = \varepsilon(\ell)\varepsilon(a),$$

$$\ell(1) \otimes \ell(2) \cdot a = \ell(2) \otimes \ell(1) \cdot a,$$

for all $\ell \in L$ and all $a, b \in H$. Then $H \times L := H \otimes L$ with the tensor product structure as a coalgebra and with the smash product $\ast$ for the algebra structure is a Hopf algebra; see [15, 1.2.10] (in this second paper a different notation is used). We shall say that $H$ is a $L$-module Hopf algebra.

Remark 2.1. The following perspective shows that it is natural to impose these consistency conditions. The category $L\mathcal{M}$ of left $L$-modules is monoidal, but not braided; thus $H$ cannot be interpreted as a Hopf algebra in $L\mathcal{M}$. Still, it can be interpreted in terms of monads. Recall that $A$ has the structure of an algebra in the monoidal category $L\mathcal{M}$ of left $L$-modules, if and only if the endofunctor $T : L\mathcal{M} \rightarrow L\mathcal{M}$, $T(X) = A \otimes X$ has the structure of a monad.

Also recall [8] that a bimonad structure on a monad $T$ on a monoidal category consists of a comonoidal structure on the functor $T$, i.e. a natural transformation

$$T_2 : T(X \otimes Y) = H \otimes (X \otimes Y) \rightarrow T(X) \otimes T(Y) = (H \otimes X) \otimes (H \otimes Y),$$

such that...
and a morphism $T_0 : T(1) \rightarrow 1$. They have to obey axioms generalizing coassociativity and counitality. If $H$ is a bialgebra in a braided monoidal category, the monad $T(-) = H \otimes -$ can be endowed via the coproduct $\Delta : H \rightarrow H \otimes H$ with the natural transformation

$$T_2(a \otimes x \otimes y) = (a_{(1)} \otimes x) \otimes (a_{(2)} \otimes y),$$

where we used Sweedler notation for $\Delta$. The morphism $T_0$ is induced from the counit $\varepsilon : H \rightarrow k$.

Now let $L$ be another Hopf algebra and $H$ be an $L$-module algebra. The fact that $T_2$ is a morphism in $L \mathcal{M}$ is then equivalent to the consistency conditions (2.3) and (2.5), while condition (2.4) amounts to the fact that $\varepsilon$ is a morphism in $L \mathcal{M}$. Thus $T(-) = H \otimes -$ is a bimonad on the monoidal category $L \mathcal{M}$, if and only if the requirements (2.3)–(2.5) hold. It is a Hopf monad, if and only if $H$ is a Hopf algebra. The Hopf monad in $\text{Vec}_k$ (i.e. Hopf algebra) $H \ltimes L$ corresponds to the forgetful functor as described in [8, Proposition 4.3].

**Remark 2.2.** Here is another way to interpret $H \ltimes L$, dual to [4, 1.1.5]. Let $H$ be a $L \mathcal{YD}$-algebra. Then $H$, endowed with the trivial coaction, is a Hopf algebra in $L \mathcal{YD}$ and $H \ltimes L \simeq H \# L$. Indeed, (2.5) is equivalent to the compatibility in $L \mathcal{YD}$.

### 2.2.

Now turn to the situation of two Hopf algebras $H$ and $U$, provided with a non-degenerate bilinear form $(|) : H \otimes U \rightarrow k$. We extend this bilinear form to a non-degenerate bilinear form $(|) : H \otimes H \otimes U \otimes U \rightarrow k$ by

$$(a \otimes \tilde{a} | u \otimes \tilde{u}) := (a | \tilde{u})(\tilde{a} | u), \quad \text{for } a, \tilde{a} \in H, \ u, \tilde{u} \in U. \quad (2.6)$$

We assume that the pairing $(|)$ is such that for every $a, \tilde{a} \in H$, $u, \tilde{u} \in U$, the following identities hold

$$(a \tilde{a} | u) = (a \otimes \tilde{a} | \Delta(u)) = (a | u_{(2)})(\tilde{a} | u_{(1)}), \quad (1 | u) = \varepsilon(u), \quad (2.7)$$

$$(a | u \tilde{u}) = (\Delta(a) | u \otimes \tilde{u}) = (a_{(2)} | u)(a_{(1)} | \tilde{u}), \quad (a | 1) = \varepsilon(a), \quad (2.8)$$

$$(S(a) | u) = (a | S(u)). \quad (2.9)$$

Such a pairing is called a Hopf pairing on $H$ and $U$.

**Lemma 2.3.** Assume that the two Hopf algebras $H$ and $U$ are $L$-modules and that there is a Hopf pairing on $H$ and $U$. Assume that the pairing is compatible with the $L$-action involving the antipode of $L$,

$$(\ell \cdot a | u) = (a | S(\ell) \cdot u), \quad a \in H, \ u \in U, \ \ell \in L. \quad (2.10)$$

Then the Hopf algebra $H$ is an $L$-module Hopf algebra, if and only if $U$ is so.
**Proof.** Let \( \ell \in L, u, v \in U \) and \( a \in H \). We compute

\[
(a | \ell \cdot (uv)) = (S(\ell) \cdot a | uv) = ((S(\ell) \cdot a)_{(2)} | u)((S(\ell) \cdot a)_{(1)} | v);
\]

\[
(a | (\ell_1 \cdot u)(\ell_2 \cdot v)) = (a_{(2)} | \ell_{(1)} \cdot u)(a_{(1)} | \ell_{(2)} \cdot v)
\]

\[= (S(\ell_{(1)}) \cdot a_{(2)} | u)(S(\ell_{(2)}) \cdot a_{(1)} | v)
\]

\[= (S(\ell_{(1)}) \cdot a_{(2)} | u)(S(\ell_{(2)}) \cdot a_{(1)} | v).
\]

Hence (2.3) holds for \( U \) if and only if \( (a | \ell \cdot (uv)) = (a | (\ell_{(1)} \cdot u)(\ell_{(2)} \cdot v)) \) for all \( \ell \in L, u, v \in U, a \in H \), and only if \( ((\ell \cdot a)_{(2)} | u)((\ell \cdot a)_{(1)} | v) = (\ell_{(2)} \cdot a_{(2)} | u)(\ell_{(1)} \cdot a_{(1)} | v) \) for all \( \ell \in L, u, v \in U, a \in H \), if and only if (2.3) holds for \( L \). Thus (2.4) holds for \( H \) if and only if (2.3) holds for \( U \).

Similarly (2.4) holds for \( U \) if and only if (2.4) holds for \( H \) and vice versa. Finally, (2.3) holds for \( H \) if and only if it holds for \( U \):

\[
\ell_{(1)} \otimes \ell_{(2)} \cdot u = \ell_{(2)} \otimes \ell_{(1)} \cdot u, \quad \forall u \Leftrightarrow S(\ell_{(1)})(a \otimes \ell_{(2)} \cdot u)
\]

\[= S(\ell_{(2)})(a \otimes \ell_{(1)} \cdot u), \quad \forall u, a \Leftrightarrow S(\ell_{(1)})(S(\ell_{(2)}) \cdot a | u),
\]

\[= S(\ell_{(2)})(S(\ell_{(1)}) \cdot a | u), \quad \forall u, a \Leftrightarrow S(\ell_{(2)})(S(\ell_{(1)}) \cdot a | u)
\]

\[= S(\ell_{(1)})(S(\ell_{(2)}) \cdot a | u), \quad \forall u, a \Leftrightarrow S(\ell_{(2)} \otimes S(\ell_{(1)}) \cdot a
\]

\[= S(\ell_{(1)} \otimes S(\ell_{(2)}) \cdot a, \quad \forall a.
\]

\( \square \)

2.3.

We next extend our construction to Hopf algebras in braided monoidal categories. To this end, let now \( K \) be a Hopf algebra, \( B \) a Hopf algebra in the braided category \( B^k \text{YD}^k \). Let \( L \) be another Hopf algebra as before, and assume that \( B \) is also an \( L \)-module algebra. We extend the action of the Hopf algebra \( L \) to the bosonization \( H := B^k \text{YD} \) by \( \ell \cdot (b \# k) := (\ell \cdot b) \# k \), for \( \ell \in L, b \in B \) and \( k \in K \):

Then straightforward verifications show that:

- The bosonization \( H \) is a \( L \)-module algebra \( \Leftrightarrow \) The actions of \( L \) and \( K \) on \( B \) commute.
- Equation (2.4) holds for \( H \Leftrightarrow \) Equation (2.4) holds for \( B \).
  From now on, we assume that this is the case.
- Equation (2.3) holds for \( H \Leftrightarrow \) Equation (2.3) holds for \( B \) and the action of \( L \) on \( B \) is a morphism of \( K \)-comodules for all \( \ell \in L \).
- Equation (2.5) holds for \( H \Leftrightarrow \) Equation (2.5) holds for \( B \).

In other words, the action of \( L \) on the bosonization \( H = B^k \text{YD} \) satisfies (2.4), (2.3) and (2.5), if and only if so does the action of \( L \) on \( B \), and the homothety \( \eta_{\ell} \) for \( \ell \in L \) is a morphism of Yetter–Drinfeld modules, \( \eta_{\ell} \in \text{End}_K^B \) for all \( \ell \in L \). This leads to the following.
Definition 2.4. An \( L \)-module braided Hopf algebra is a Hopf algebra \( B \) in the braided category \( \mathcal{K} \mathcal{YD} \) that is also a \( L \)-module algebra, that satisfies (2.4), (2.3) and (2.5), and such that the homothety \( \eta_\ell \in \text{End}_\mathcal{K} B \) for all \( \ell \in L \).

We have just seen: for an \( L \)-module braided Hopf algebra, the bosonization \( H := B \# K \) is an \( L \)-module Hopf algebra over \( k \) and we can form the Hopf algebra \( H \rtimes L = (B \# K) \rtimes L \).

As in Sec. 2.2, we consider the situation with non-degenerate pairings; this time internal to the braided monoidal category \( \mathcal{K} \mathcal{YD} \) instead of vect \( k \). Concretely, let \( E \) be another Hopf algebra in the category \( \mathcal{K} \mathcal{YD} \) provided with a non-degenerate bilinear form \( (\cdot | \cdot) : B \otimes E \to k \), and extend it by (2.6) to a pairing \( B \otimes B \otimes E \otimes E \to k \).

The fact that the pairing is internal to the category \( \mathcal{K} \mathcal{YD} \) means that the bilinear form \( (\cdot | \cdot) \) is a morphism in the monoidal category \( \mathcal{K} \mathcal{YD} \), where \( k \) is endowed with the structure of a trivial Yetter–Drinfeld module.

We assume that for every \( a, \tilde{a} \in B \), \( u, \tilde{u} \in E \), the conditions (2.7)–(2.9) of a Hopf pairing, relating coproduct, product, unit and counit of \( B \) and \( E \) hold.

Then we have in the braided category \( \mathcal{K} \mathcal{YD} \) exactly the same situation we considered in Lemma 2.3 in the braided category vect \( k \). The same calculations, this time in the category \( \mathcal{K} \mathcal{YD} \), yield the following.

Lemma 2.5. Assume that both \( B \) and \( E \) are \( L \)-modules and that condition (2.10) on the Hopf pairing \( (\cdot | \cdot) \) holds. Then \( B \) is a \( L \)-module braided Hopf algebra, if and only if \( E \) is so.

2.4.

Let \( \mathfrak{g} \) be a Lie algebra over the field \( k \). We specialize to \( L \)-module braided Hopf algebras where the Hopf algebra \( L = U(\mathfrak{g}) \) is the universal enveloping algebra of \( \mathfrak{g} \). Then the conditions (2.11) and (2.12) in the definition of an \( L \)-module Hopf algebra \( H \) just mean that \( \mathfrak{g} \) acts on \( H \) by \( k \)-derivations, while condition (2.10) is for free, due to the cocommutativity of \( U(\mathfrak{g}) \). Condition (2.13) amounts to the condition

\[
\Delta(x \cdot a) = x \cdot a_{(1)} \otimes a_{(2)} + a_{(1)} \otimes x \cdot a_{(2)}, \quad \varepsilon(x \cdot a) = 0, \quad (2.11)
\]

for all \( x \in \mathfrak{g} \) and \( a \in H \). In other words, condition (2.11) tells us that \( \mathfrak{g} \) acts on \( H \) by \( k \)-coderivations. We summarize all conditions by saying that \( \mathfrak{g} \) acts on \( H \) by \( \mathfrak{k} \)-biderivations: \( \mathfrak{g} \) acts by endomorphisms that are simultaneously \( k \)-derivations and \( k \)-coderivations. Thus we have the following.

Lemma 2.6. Let \( H \) be a Hopf algebra and let \( \mathfrak{g} \) be a Lie algebra acting on \( H \) by \( \mathfrak{k} \)-biderivations. Then \( H \) is a \( U(\mathfrak{g}) \)-module Hopf algebra and we can form the Hopf algebra \( H \rtimes U(\mathfrak{g}) \).
On unrolled Hopf algebras

The following remarks on biderivations are useful:

⋄ For any Hopf algebra $H$, the subspace $\text{Bider}_k(H) := \{ x \in \text{Der}_k(H) : x \text{ is a coderivation} \}$ is a Lie subalgebra of $\text{Der}_k(H)$.

⋄ If $x \in \text{Der}(H)$ and if $a, b \in H$ fulfill (2.11) for $x$, then so does their product $ab$. Hence it is enough to check the biderivation property (2.11) for a given derivation $x$ on a family of generators of $H$.

**Remark 2.7.** Let $H$ be a Hopf algebra and let $\mathfrak{g}$ be a Lie algebra acting on $H$ by $k$-coderivations. Let $H_0$ be the coradical, and $(H_n)_{n \geq 0}$ the coradical filtration, of $H$. If $H_0$ is $\mathfrak{g}$-stable, then $H_n$ is $\mathfrak{g}$-stable for all $n \geq 0$ by the defining condition (2.11). Hence $\mathfrak{g}$ acts on $\text{gr} H$ by $k$-coderivations.

Assume that $H_0$ is a Hopf subalgebra, that $\mathfrak{g}$ acts on $H$ by $k$-biderivations and that $H_0$ is $\mathfrak{g}$-stable. Then $\mathfrak{g}$ acts on the graded object $\text{gr} H$ by $k$-biderivations.

Notice that $\mathfrak{g}$ may act on $H$ by $k$-biderivations with $H_0$ not being $\mathfrak{g}$-stable. For instance, let $x \in H$ primitive. Then $D = \text{ad}_x$ is a $k$-biderivation. If there exists $g \in G(H)$ such that $gx = qxg$ with $q \in k^\times - \{1\}$, then $D(g) = (1 - q)xg / \in H_0$.

**2.5.** In this context, suppose that $H$ is pointed and set $G := G(H)$ the group of group-like elements of $H$. Let $\mathfrak{g}$ act on $H$ by derivations; assume that $\mathfrak{g}$ acts trivially on $kG$. Let $g, t \in G$ and $\mathcal{P}_{g, t}(H) := \{ a \in H : \Delta(a) = g \otimes a + a \otimes t \}$ the space of $(g, t)$ skew-primitive elements. Then the coderivation property (2.11) implies that $\mathcal{P}_{g, t}(H)$ is a $\mathfrak{g}$-submodule for all $g, t \in G$. Summarizing, we have the following.

**Lemma 2.8.** Let $\mathfrak{g}$ be a Lie algebra acting by derivations on a pointed Hopf algebra $H$, $G = G(H)$. Assume that:

- $\mathfrak{g}$ acts trivially on $kG$.
- $H$ is generated by group-like and skew-primitive elements.

Then the following are equivalent:

1. $\mathfrak{g}$ acts on $H$ by $k$-biderivations, i.e. (2.11) holds.
2. $\mathcal{P}_{g, t}(H)$ is a $\mathfrak{g}$-submodule for all $g, t \in G$.
3. $\mathcal{P}_{g, 1}(H)$ is a $\mathfrak{g}$-submodule for all $g \in G$.

**2.6.** Let $K$ be a Hopf algebra and $V \in K\mathcal{YD}$. It is well-known that every $d \in \text{Hom}_K(V, T(V))$ extends uniquely to a derivation $D \in \text{Der}(T(V))$ on the tensor algebra $T(V)$ by $D(1) = 0$ and

$$D_{|_{T^n(V)}} = \sum_{1 \leq j \leq n} \text{id}_{T^{j-1}(V)} \otimes d \otimes \text{id}_{T^{n-j}(V)},$$  

(2.12)
for $n > 0$. Thus every Lie algebra map $\mathfrak{g} \to \text{End}(V)$ extends to a Lie algebra map $\mathfrak{g} \to \text{Der}(T(V))$.

**Proposition 2.9.** Let $V \in \mathcal{C}_0$. Every morphism of Lie algebras $\mathfrak{g} \to \text{End}_{\mathcal{C}_0}(V)$ extends to an action of the universal enveloping algebra $U(\mathfrak{g})$ on $T(V)^\#K$ and to an action on $B(V)^\#K$, giving rise to the Hopf algebras $(T(V)^\#K) \rtimes U(\mathfrak{g})$ and $(B(V)^\#K) \rtimes U(\mathfrak{g})$.

**Proof.** As explained, the action of $\mathfrak{g}$ on $V$ extends uniquely to an action of $\mathfrak{g}$ on the tensor algebra $T(V)$ by derivations. Formula (2.12) and the assumptions imply that this action is by morphisms in the category $\mathcal{C}_0$. By definition, (2.3) holds in $V$, hence it holds in $T(V)$. By Sec. 2.3, the action extended to $T(V)^\#K$ satisfies the requirements in Sec. 2.1, hence we can form $(T(V)^\#K) \rtimes U(\mathfrak{g})$. Second, the action of $\mathfrak{g}$ on $T^\circ(V)$ commutes with that of the braid group $\mathcal{B}_n$; since the kernel of the projection $T^\circ(V) \to \mathcal{B}_n^\circ(V)$ is the kernel of the quantum symmetrizer, $\mathfrak{g}$ acts on the Nichols algebra $\mathcal{B}(V)$ with the desired requirements.

**Definition 2.10.** Let $K$ be a Hopf algebra, $V \in \mathcal{C}_0$ and $\mathfrak{g}$ a Lie subalgebra of $\text{End}_K(V)$. We call the Hopf algebra $(\mathcal{B}(V)^\#K) \rtimes U(\mathfrak{g})$ the unrolled bosonization of the Nichols algebra of $V$ by $\mathfrak{g}$.

One may define unrolled versions of bosonizations of pre-Nichols or post-Nichols algebras, see e.g. [13], or of deformations of Nichols algebras, provided that the ideals of defining relations are preserved by the action of $\mathfrak{g}$.

2.7. **Finite GK-dim**

Our main reference for this subsection is [14]. Let $A$ be an associative $k$-algebra. We say that a finite-dimensional subspace $V \subseteq A$ is GK-deterministic if

$$\text{GK-dim } A = \lim_{n \to \infty} \log_n \dim \sum_{0 \leq j \leq n} V^n.$$ 

**Lemma 2.11 ([2, Lemma 2.2]).** Let $K$ be a Hopf algebra, $R$ a Hopf algebra in $\mathcal{C}_0$, $A$ a $K$-module algebra and $B$ an $R$-module algebra in $\mathcal{C}_0$. Assume that the actions of $K$ on $A$, of $K$ on $B$, of $K$ on $R$, and of $R$ on $B$ are locally finite.

(a) $\text{GK-dim } A^\#K \leq \text{GK-dim } A + \text{GK-dim } K$. If either $K$ or $A$ has a GK-deterministic subspace, then $\text{GK-dim } A^\#K = \text{GK-dim } A + \text{GK-dim } K$.

(b) $\text{GK-dim } B^\#R \leq \text{GK-dim } B + \text{GK-dim } R$. If either $R$ or $B$ has a GK-deterministic subspace, then $\text{GK-dim } B^\#R = \text{GK-dim } B + \text{GK-dim } R$.

Clearly, a finite-dimensional Lie algebra $\mathfrak{g}$ is a GK-deterministic subspace of $U(\mathfrak{g})$. Thus we have the following.
Example 2.12. Let $H$ be a Hopf algebra and let $\mathfrak{g}$ be a Lie subalgebra of $\text{Bider}_k(H)$ such that $\text{GK-dim } H, \dim \mathfrak{g} < \infty$. If the action of $\mathfrak{g}$ on $H$ is locally finite, then

$$\text{GK-dim}(H \rtimes U(\mathfrak{g})) = \text{GK-dim } H + \dim \mathfrak{g} < \infty. \quad (2.13)$$

Here are some particular cases:

- If $H$ is a finite-dimensional Hopf algebra and $\mathfrak{g}$ is a Lie subalgebra of $\text{Bider}_k(H)$, then

$$\text{GK-dim}(H \rtimes U(\mathfrak{g})) = \dim \mathfrak{g} < \infty.$$  

- Let $K$ be a Hopf algebra, $V \in K^\mathcal{YD}$, $\mathfrak{g}$ a Lie subalgebra of $\text{bd}_V$, $B \in K^\mathcal{YD}$ a pre-Nichols algebra of $V$ and $E \in K^\mathcal{YD}$ a post-Nichols algebra of $V$. Assume that the action of $\mathfrak{g}$ descends to $B$ and $E$, $\text{GK-dim } K < \infty$, $\dim V < \infty$, $\text{GK-dim } B < \infty$, $\text{GK-dim } E < \infty$. Clearly, $\dim \mathfrak{g} < \infty$ and $\mathfrak{g}$ acts locally finitely on $B\#K$ and $E\#K$. If either $K$ or $B$, respectively $E$, have a GK-deterministic subspace, then

$$\text{GK-dim}((B\#K) \rtimes U(\mathfrak{g})) = \text{GK-dim } B + \text{GK-dim } K + \dim \mathfrak{g} < \infty,$$

$$\text{GK-dim}((E\#K) \rtimes U(\mathfrak{g})) = \text{GK-dim } E + \text{GK-dim } K + \dim \mathfrak{g} < \infty.$$

3. The Dual Construction

3.1. Let $J$ be a Hopf algebra. A $J$-comodule coalgebra is a coalgebra $C$ which is also a right $J$-comodule with coaction $\varrho : C \to C \otimes J$, $\varrho(c) = c_{[0]} \otimes c_{[1]}$, and counit $\varepsilon_C$ such that for all $c \in C$

$$(c_{(1)})_{[0]} \otimes (c_{(2)})_{[0]} \otimes (c_{(1)})_{[1]}(c_{(2)})_{[1]} = (c_{[0]})_{(1)} \otimes (c_{[0]})_{(2)} \otimes c_{[1]}, \quad (3.1)$$

$$\varepsilon_C(c_{[0]})c_{[1]} = \varepsilon_C(c). \quad (3.2)$$

Here (3.1) and (3.2) mean that $C$ is a coalgebra in the monoidal category $M^J$ of right $J$-comodules. Assume that $C = H$ is a Hopf algebra and a $J$-comodule coalgebra that satisfies

$$(ab)_{[0]} \otimes (ab)_{[1]} = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}, \quad (3.3)$$

$$\varrho(1) = 1 \otimes 1, \quad (3.4)$$

$$a_{[0]} \otimes ja_{[1]} = a_{[0]} \otimes a_{[1]}j, \quad (3.5)$$

$j \in J, a, b \in H$; (3.3) and (3.5) say that $H$ is a $J$-comodule algebra. Then $J \ltimes H := J \otimes H$ with the tensor product structure as an algebra and with the smash coproduct
Proof. Let \( H \) and \( U \) be Hopf algebras, provided with a non-degenerate Hopf pairing \((\ ) : H \otimes U \to k\).

Lemma 3.1. Assume that \( H \) and \( U \) are \( J \)-comodules and that the pairing is compatible with \( J \)-coaction involving the antipode of \( J \), i.e.

\[
(\alpha_{[0]} | u)\alpha_{[1]} = (\alpha | u_{[0]})S(u_{[1]}), \quad \alpha \in H, \ u \in U.
\]  

(3.6)

Then \( H \) is a \( J \)-comodule Hopf algebra if and only if \( U \) is so.

Proof. Let \( u, v \in U, a, b \in H \). We compute

\[
((ab)_{[0]} | u)(ab)_{[1]} = (ab | u_{[0]})S(u_{[1]}) = (a | (u_{[0]})(2))(b | (u_{[1]})(1))S(u_{[1]});
\]

\[
(a_{[0]}b_{[0]} | u)a_{[1]}b_{[1]} = (a_{[0]} | u_{[2]})(b_{[0]} | u_{[1]})a_{[1]}b_{[1]}
\]

\[
= (a | (u_{[2]})(0))(b | (u_{[1]})(0))S((u_{[2]})(1))S((u_{[1]})(1))
\]

\[
= (a | (u_{[2]})(0))(b | (u_{1})(0))S((u_{[1]})(1)(u_{[2]})(1)).
\]

Hence (3.6) holds for \( U \) if and only if (3.6) holds for \( H \) and vice versa. Similarly (3.2) holds for \( U \) if and only if (3.2) holds for \( H \) and vice versa. Finally, (3.5) holds for \( H \) if and only if it holds for \( U \):

\[
(a_{[0]} | u)ja_{[1]} = (a | u_{[0]})jS(u_{[1]}),
\]

\[
(a_{[0]} | u)ja_{[1]} = (a | u_{[0]})S(u_{[1]}). \quad \Box
\]

3.3.

Let now \( K \) be a Hopf algebra, \( B \) a Hopf algebra in \( YD_K \) and also a \( J \)-comodule coalgebra. Extend the coaction of \( J \) to \( H = K \# B \) by \( g(k \# b) = k \# b_{[0]} \otimes b_{[1]} \), \( b \in B \) and \( k \in K \). Then

- \( H \) is a \( J \)-comodule coalgebra \( \Leftrightarrow \) the coactions of \( J \) and \( K \) on \( B \) commute, i.e. for all \( b \in B \)

\[
(b_{[0]})(0) \otimes b_{[1]} \otimes (b_{[0]})(1) = (b_{[0]})(0) \otimes (b_{[0]})(1) \otimes b_{[1]} \in B \otimes K \otimes J.
\]  

(3.7)

- Equation (3.6) holds for \( H \) \( \Leftrightarrow \) (3.6) holds for \( B \). Assume this is the case.

- Equation (3.3) holds for \( H \) \( \Leftrightarrow \) (3.3) holds for \( B \) and the action of \( k \) on \( B \) is a morphism of \( J \)-comodules for all \( k \in K \).

- Equation (3.5) holds for \( H \) \( \Leftrightarrow \) (3.5) holds for \( B \).

\(^a\)In [4, p. 10] a left version is presented, with a different notation. The proof is equally straightforward.
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In other words, the coaction of $J$ on $H = K \# \mathcal{B}$ satisfies (3.4), (3.3) and (3.5), if and only if so does the coaction of $J$ on $\mathcal{B}$, and the coaction of $J$ on $\mathcal{B}$ commutes both with the action and the coaction of $K$. This can be phrased also as: the homothety $\eta_\ell$ for $\ell \in J^*$ is a morphism of Yetter–Drinfeld modules, i.e. $\eta_\ell \in \text{End}_K^K \mathcal{B}$.

**Definition 3.2.** A $J$-comodule braided Hopf algebra is a Hopf algebra $\mathcal{B}$ in the braided category $\mathcal{YD}_K^K$ that is also a $J$-comodule coalgebra, that satisfies (3.4), (3.3) and (3.5), and such that the coaction of $J$ on $\mathcal{B}$ commutes both with the action and the coaction of $K$. In such a case, the bosonization $H = K \# \mathcal{B}$ is a $J$-comodule Hopf algebra and we can form the Hopf algebra $J \ltimes H = J \ltimes (K \# \mathcal{B})$.

As in Sec. 3.2, we consider the situation with non-degenerate pairings; this time internal to the braided monoidal category $\mathcal{YD}_K^K$ instead of vect$^k$. Concretely, let $\mathcal{E}$ be a Hopf algebra in $\mathcal{YD}_K^K$ provided with a non-degenerate bilinear form $(\cdot) : \mathcal{B} \otimes \mathcal{E} \to k$, and extend it by (2.6) to a pairing $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{E} \otimes \mathcal{E} \to k$.

- The fact that the pairing is internal to the category $\mathcal{YD}_K^K$ means that the bilinear form $(\cdot)$ is a morphism in the monoidal category $\mathcal{YD}_K^K$, where $k$ is endowed with the structure of a trivial Yetter–Drinfeld module.
- We assume that for every $a, \tilde{a} \in \mathcal{B}$, $u, \tilde{u} \in \mathcal{E}$, the conditions (2.7)–(2.9) of a Hopf pairing, relating coproduct, product, unit and counit of $\mathcal{B}$ and $\mathcal{E}$ hold.

Then we have in the braided category $\mathcal{YD}_K^K$ exactly the same situation we considered in Lemma 3.1 in the braided category vect$^k$. The same calculations, this time in the category $\mathcal{YD}_K^K$, yield the following.

**Lemma 3.3.** Assume that both $\mathcal{B}$ and $\mathcal{E}$ are $J$-comodules and that (3.6) holds. Then $\mathcal{B}$ is a $J$-comodule braided Hopf algebra, if and only if $\mathcal{E}$ is so.

**3.4.**

Let $G$ be an affine algebraic group over $k$ and let $J = k[G]$ be the algebra of functions on $G = \text{Hom}_{\text{alg}}(J, k)$. Here we use the convention (2.6), i.e.

$$\langle \gamma \eta, j \rangle = \langle \gamma, j(2) \rangle \langle \eta, j(1) \rangle, \quad \gamma, \eta \in G.$$  

Thus, being a (right) $J$-comodule means being a rational (right) $G$-module: $m \cdot \gamma = m_{(0)} \eta m_{(1)}$; which of course is equivalent to being rational left $G$-module. So, in what follows we work with left rational modules. The conditions (3.1) and (3.2), respectively (3.3) and (3.4), in the definition of $J$-comodule Hopf algebra just say that $G$ acts on $H$ by coalgebra, respectively algebra, automorphisms, while (3.5) is automatic by the commutativity of $k[G]$. We summarize our findings.

**Proposition 3.4.** Let $H$ be a Hopf algebra and let $G$ be an affine algebraic group acting rationally on $H$ by Hopf algebra maps. Then $H$ is a $k[G]$-comodule Hopf algebra and we can form $k[G] \ltimes H$.  

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Remark 3.5. Since $J$ is commutative, $\text{GK-dim}(k[G] \rtimes H) = \dim G + \text{GK-dim } H$, see e.g. [14, 3.10].

3.5. Let $K$ be a Hopf algebra and $V \in \mathcal{YD}^K_K$, $\dim V < \infty$. Then $\text{Aut}^K_K(V)$ is an algebraic group, whose Lie algebra is $\text{End}^K_K(V)$. Every morphism of algebraic groups $G \to \text{Aut}^K_K(V)$ extends to an action of $G$ on $T(V)$ by Hopf algebra automorphisms in $\mathcal{YD}^K_K$; hence it descends to an action of $G$ on $B(V)$ by Hopf algebra automorphisms in $\mathcal{YD}^K_K$. It extends to an action of $G$ on $K \# B(V)$, trivially on $K$, giving rise to the Hopf algebra $k[G] \rtimes (K \# B(V))$. One may define analogous actions of these Hopf algebras from bosonizations of pre-Nichols or post-Nichols algebras, or of deformations of Nichols algebras, provided that the ideals of defining relations are preserved by the action of $G$.

4. Hopf Algebras Arising from Nichols Algebras of Diagonal Type

4.1. Let $\theta \in \mathbb{N}, I = I_\theta = \{1, 2, \ldots, \theta\}$. Denote by $(\alpha_i)_{i \in \mathbb{I}}$ the canonical basis of $\mathbb{Z}^\theta$.

Let $(V, c)$ be a braided vector space of diagonal type of dimension $\theta$; let $(x_i)_{i \in \mathbb{I}}$ be a basis of $V$. Since $(V, c)$ is assumed to be of diagonal type, there is a matrix $q = (q_{ij})_{i, j \in \mathbb{I}} \in (k^\times)^{\mathbb{I} \times \mathbb{I}}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $i, j \in \mathbb{I}$. Then the tensor algebra $T(V)$ and the Nichols algebra $B(V)$ are $\mathbb{Z}^\theta$-graded (as braided Hopf algebras), by $\text{deg } x_i = \alpha_i, i \in \mathbb{I}$.

Let $K$ be a Hopf algebra. To realize the braided vector space $(V, c)$ as a Yetter–Drinfeld module over $K$ we need some extra data.

A pair $(g, \chi) \in G(K) \times \text{Hom}_\text{alg}(K, k)$ is called a YD-pair if $\chi(a) g = \chi(a(2)) a(1) g S(a(3))$ for all $a \in K$. This implies $g \in Z(G(K))$.

Then $k^\times g := k$ with coaction given by $g$ and action given by $\chi$ is a simple object in $^K_K \mathcal{YD}$.

A principal realization of the braided vector space $(V, c)$ over the Hopf algebra $K$ is a family $((g_i, \chi_i))_{i \in \mathbb{I}}$ of YD-pairs such that

$$
\chi_j(g_i) = q_{ij}, \quad \text{for all } i, j \in \mathbb{I}.
$$

A principal realization allows us to see braided vector space as a Yetter–Drinfeld module, $V \in {}^K_K \mathcal{YD}$, by declaring $x_i \in V^g_{\chi_i}, i \in \mathbb{I}$. Let $d_{\theta}^g = \dim V^g = |\{i \in \mathbb{I} : (g_i, \chi_i) = (g, \chi)\}|$. Then

$$
\text{bd}_V = \text{End}^K_K(V) \simeq \bigoplus_{g \in \Gamma, \chi \in \overline{\Gamma}} gl(d_{\theta}^g, k).
$$
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Despite the notation, the Lie algebra \( b \mathfrak{d}_V \) depends on the way the braided vector space \( V \) is realized as a \( K \)-Yetter–Drinfeld module and not merely on the braided vector space \( V \) itself.

For \( h = (h_i)_{i \in I} \in k^I \) we denote by \( D_h \in \text{End}(V) \) the map defined by \( D_h(x_i) = h_i x_i, i \in I \). By abuse of notation, we denote by \( D_h \) the corresponding derivation of \( T(V) \# k \Gamma \) or \( B(V) \# k \Gamma \). Let

\[
 t_V = \{ D_h : h \in k^I \} \subseteq b \mathfrak{d}_V.
\]

The abelian Lie algebra \( t_V \) depends only on \((V,c)\). If \((g_i, \chi_i) = (g_j, \chi_j)\) implies \( i = j \), then \( b \mathfrak{d}_V = t_V \).

**Remark 4.1.** The action of the Lie algebra \( t_V \) preserves the \( \mathbb{Z}^0 \)-grading. Indeed, let \( h \in k^I \) and let \( \alpha \mapsto h_\alpha \) be the unique group homomorphism \( \mathbb{Z}^0 \to k \) such that \( h_\alpha = h_i, i \in I \). Then \( D_h \) acts by \( h_\beta \) in the homogeneous component \( T(V)_\beta \) for all \( \beta \in \mathbb{Z}^0 \). Hence every Hopf ideal \( I \) of \( T(V) \) generated by \( \mathbb{Z}^0 \)-homogeneous elements is stable under \( t_V \) and \( t_V \) acts by derivations and coderivations on \( T(V)/I \).

**Remark 4.2.** In fact, the \( \mathbb{Z}^0 \)-grading is tantamount to a comodule structure over the group algebra \( k\mathbb{Z}^0 \), which is the algebra of functions on the algebraic torus \( \mathbb{T}_V \); \( t_V \) is its Lie algebra, and the action of \( t_V \) is the derivation of the natural action of \( \mathbb{T}_V \).

### 4.2.

From now on, we assume that \( \text{char} k = 0 \). We keep the notation above and assume that \( \dim B(V) < \infty \). The classification of the finite-dimensional Nichols algebras of diagonal type was given in [12]. An efficient set of defining relations of \( B(V) \), i.e. generators of the ideal \( \mathcal{J}_q \), was provided in [9]. Besides \( B(V) \), there are two other Hopf algebras in \( k \mathcal{YD} \) that are expected to play a role in representation theory:

(a) ( [9] [7] ) The **distinguished pre-Nichols algebra** of \((V,c)\) is the quotient \( \tilde{B}(V) := T(V)/\mathcal{I}_q \) by a suitable ideal \( \mathcal{I}_q \). Thus, there are projections \( T(V) \twoheadrightarrow \tilde{B}(V) \twoheadrightarrow B(V) \).

(b) ( [13] ) The **Lusztig algebra** of \((V,c)\) is the graded dual \( \mathcal{L}(V) \) of \( \tilde{B}(V) \).

**Proposition 4.3.** Let \( K \) be a Hopf algebra provided with a principal realization of \((V,c)\) and let \( L = U(t_V) \). Then \( B(V) \) and \( L(V) \) are \( L \)-module braided Hopf algebras in \( k \mathcal{YD} \) and we can form the unrolled bosonizations \( (B(V) \# K) \times L \) and \( (\mathcal{L}(V) \# K) \times L \).

**Proof.** The claim for \( \tilde{B}(V) \) follows from Remark 4.1 and implies the one for \( \mathcal{L}(V) \) by Lemma 2.5.

**Example 4.4.** If \( \theta = 1 \) and \( q \) is a root of 1 of even order, then we recover the construction in [9] [11].
4.3.

Let \((V,c)\) be of diagonal type with \(\dim B(V) < \infty\). Fix a principal realization over the group algebra \(k\Gamma\), where \(\Gamma\) is abelian. Then each of the Hopf algebras \(B(V)\), \(\tilde{B}(V)\) and \(L(V)\) in \(\mathcal{YD}\) gives rise to Hopf algebras \(u(V)\), \(U(V)\), \(\tilde{U}(V)\), respectively; they are suitable Drinfeld doubles of the bosonizations \(B(V)\#k\Gamma\), \(\tilde{B}(V)\#k\Gamma\) and \(L(V)\#k\Gamma\). See [3] [7] [13]. If \(q\) is symmetric, then we may divide that Drinfeld double by a central Hopf subalgebra. If furthermore \(q\) is of Cartan type, then we recover the small and the De Concini–Procesi quantum group, respectively. Then we may define unrolled quantum groups

\[
\begin{align*}
\quad u(V) \rtimes U(t_V), & \quad U(v) \rtimes U(t_V), & \quad U(V) \rtimes U(t_V).
\end{align*}
\]

Indeed, the Lie algebra \(t_{V \oplus W}\) acts on \(T(V \oplus W)\#k\Gamma\), but if \(\zeta \in k^{2n}\), then \(D_\zeta\) preserves the relations of the quantum double if and only if \(\zeta\) belongs to the image of the map \(t_V \to t_{V \oplus W}, \xi \mapsto (\xi, -\xi)\).

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References

[1] N. Andruskiewitsch, I. Angiono, A. García Iglesias, A. Masuoka and C. Vay, Lifting via cocycle deformation, \textit{J. Pure Appl. Algebra} \textbf{218}(4) (2014) 684–703.

[2] N. Andruskiewitsch, I. Angiono and I. Heckenberger, On finite GK-dimensional Nichols algebras over abelian groups, preprint (2016), arXiv:1606.02521.

[3] N. Andruskiewitsch, I. Angiono and F. Rossi Bertone, The quantum divided power algebra of a finite-dimensional Nichols algebra of diagonal type, \textit{Math. Res. Lett.} \textbf{24} (2017) 619–643.

[4] N. Andruskiewitsch and S. Natale, Examples of self-dual Hopf algebras, \textit{J. Math. Sci. Univ. Tokyo} \textbf{6} (1999) 181–215.

[5] N. Andruskiewitsch and H.-J. Schneider, Pointed Hopf algebras, in \textit{New Directions in Hopf Algebras}, Mathematical Sciences Research Institute Series (Cambridge University Press, 2002), pp. 1–68.

[6] I. Angiono, On Nichols algebras of diagonal type, \textit{J. Reine Angew. Math.} \textbf{683} (2013) 189–251.

[7] I. Angiono, Distinguished pre-Nichols algebras, \textit{Transf. Groups} \textbf{21} (2016) 1–33.

[8] A. Bruguières, S. Lack and A. Virelizier, Hopf monads on monoidal categories, \textit{Adv. Math.} \textbf{227} (2011) 745–800.
On unrolled Hopf algebras

[9] F. Costantino, N. Geer and B. Patureau-Mirand, Some remarks on the unrolled quantum group of sl(2), *J. Pure Appl. Algebra* **219** (2015) 3238–3262.

[10] N. Geer and B. Patureau-Mirand, The trace on projective representations of quantum groups, *Lett. Math. Phys.* **108** (2018) 117–140.

[11] N. Geer, B. Patureau-Mirand and V. Turaev, Modified quantum dimensions and re-normalized link invariants, *Compos. Math.* **145** (2009) 196–212.

[12] I. Heckenberger, Classification of arithmetic root systems, *Adv. Math.* **220** (2009) 59–124.

[13] I. Heckenberger, Lusztig isomorphism for Drinfel’d doubles of bosonizations of Nichols algebras of diagonal type, *J. Algebra* **323** (2010) 2130–2182.

[14] G. Krause and T. Lenagan, *Growth of Algebras and Gelfand–Kirillov Dimension*, Rev. edn., Graduate Studies in Mathematics, Vol. 22 (American Mathematical Society, Providence, RI, 2000), x+212 pp.

[15] R. K. Molnar, Semi-direct products of Hopf algebras, *J. Algebra* **47** (1977) 29–51.