On the Noisy Feedback Capacity of Gaussian Broadcast Channels

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Abstract—It is well known that, in general, feedback may enlarge the capacity region of Gaussian broadcast channels. This has been demonstrated even when the feedback is noisy (or partial-but-perfect) and only from one of the receivers. The only case known where feedback has been shown not to enlarge the capacity region is when the channel is physically degraded. In this paper, we show that for a class of two-user Gaussian broadcast channels (not necessarily physically degraded), passively feeding back the stronger user’s signal over a link corrupted by Gaussian noise does not enlarge the capacity region if the variance of feedback noise is above a certain threshold.

I. INTRODUCTION

It is well known that feedback does not increase the capacity of a memoryless point-to-point channel, a result which goes back to C. E. Shannon [1]. However, feedback has a positive impact in simplifying coding schemes and boosting error exponents [2]. With the discovery of capacity regions for several multiuser models in the ‘70s and ‘80s, it was of interest to find the impact of feedback in these models. For the discrete memoryless broadcast channel (BC), El Gamal [3] showed that feedback does not enlarge the capacity region when the channel is physically degraded, and Dueck [4] demonstrated a broadcast channel for which points outside its capacity region can be attained using feedback.

For a two-user scalar Gaussian broadcast channel (GBC), El Gamal [5] showed that the capacity region is unchanged by the presence of noiseless feedback, if one of the receivers is physically degraded with respect to the other. However, Ozarow and Leung [6] showed the surprising fact that the GBC capacity region is enlarged by feedback for a class of positively correlated noise processes at the receivers. The technique of [6] used full causal feedback from both the receivers. Recently it was shown that even perfect causal feedback from one of the receivers can enlarge the GBC capacity region [7]. In other related works, it was shown that the capacity enlargement can occur for the discrete memoryless broadcast channel even when the feedback is noisy [8], [9], or rate-limited [10], [11]. The GBC case was also considered in [9] and [11]. For the Gaussian multiple-access channel (MAC), it is known that noisy feedback, even to only one of the transmitters, always enlarges the capacity region [12]. Moreover, a duality has been shown between linear coding schemes for MAC and BC in the presence of noiseless feedback [13]. Thus, in general, there is an optimism about the availability of feedback enlarging the capacity region of broadcast channels when the channel is not physically degraded.

Against this backdrop, the purpose of this paper is to show that the anticipated capacity enlargement may not exist for all feedback models over a GBC. We show that, for a class of two-user scalar GBCs, when the stronger receiver’s signal (i.e., the signal of the receiver with the smaller noise variance) is passively fed back to the transmitter over a noisy link corrupted by independent Gaussian noise, any capacity enlargement is impossible if the feedback noise variance is above a certain threshold. Our class of channels is, in fact, the same as that studied by Ozarow and Leung [6] where they showed that the capacity region is enlarged by perfect feedback from both receivers. This class includes independent noises at the two users as well as a range of positive correlations.

We also study a related class of vector GBCs with partial-but-perfect feedback. Specifically, consider a vector GBC with a strong receiver employing two receive antennas and the other receiver and encoder only having single antennas. In the absence of feedback, the optimal scheme is superposition coding, along with maximal ratio combining at the strong receiver. Now, assume that the weak user has a higher noise variance with respect to the first antenna of the strong receiver. Then, we will show that even perfectly feeding back the second antenna output from the strong receiver does not enlarge the capacity region.

The organization of this paper is as follows. In the next section, we will describe the broadcast channel and the feedback model. This will be followed by our main result in Section III, which shows that feedback from the stronger user does not enlarge the capacity region if the feedback noise variance is above a certain threshold for a class of Gaussian noise models. A Gaussian BC with two antennas at the strong receiver and perfect feedback from one of the antennas will be considered in Section IV. Section V concludes the paper.

II. SYSTEM MODEL

Consider a memoryless two-user scalar Gaussian broadcast channel, where receiver 1 is the stronger receiver (i.e., its noise variance is not larger than the noise variance of the other user). Assume a noisy feedback link from the strong receiver to the transmitter, as shown in Figure 1.
In this model, $X$ represents the transmitted signal. The additive (forward) channel noises $Z_1, Z_2$ are zero-mean jointly Gaussian with variances $\sigma_1^2, \sigma_2^2$, respectively, and correlation coefficient $\rho$. As mentioned above, we will take $\sigma_1^2 \leq \sigma_2^2$. The additive noise on the feedback channel $Z_{fb}$ is assumed to be independent of $(Z_1, Z_2)$ and zero-mean Gaussian with variance $\sigma^2_1$. In this setup, we aim to send two independent messages, say $W_1$ and $W_2$, to the respective receivers. The transmitted symbol at the $i^{th}$ instant can be a function of the messages and the causal but noisy feedback of the stronger user’s signal, i.e.,

$$X_i = g_i(W_1, W_2, Y_i^{i-1} + Z_{fb}^{i-1}),$$  

where the symbol $U^{i-1} \triangleq (U_1, \cdots, U_{i-1})$. We will convey a pair of messages in $n$ uses of the channel, and the alphabet sizes of $W_1, W_2$ are $2^{nR_1}$ and $2^{nR_2}$, respectively. Consider the average probability of error over a uniform choice of the messages, which we denote by $P_e(n)$. We are interested in the capacity region $C_{\text{sbc-noisy-fb}}$ of our model under an average transmit power constraint of $P$.

**Definition 1.** The capacity region $C_{\text{sbc-noisy-fb}}$ is the closure of the set of all rate-pairs $(R_1, R_2)$ such that there is a sequence of encoder-decoders with $P_e(n) \to 0$ as $n \to \infty$.

Let $C_{\text{sbc-fb}}$ denote the capacity region without feedback. Clearly,

$$C_{\text{sbc-fb}} \subseteq C_{\text{sbc-noisy-fb}}.$$

The region $C_{\text{sbc-fb}}$ is a well known quantity, which can be achieved by superposition coding [14].

**III. Feedback Noise Variance Thresholds**

Our main result establishes a threshold for the feedback noise variance beyond which the feedback from stronger user does not enlarge the capacity region for a class of forward noise correlations (including independent and physically degraded cases).

**Theorem 1.** For $0 \leq \rho \leq \sqrt{\frac{\sigma_1^2}{\sigma_2^2}} \leq 1$, $C_{\text{sbc-noisy-fb}} = C_{\text{sbc-fb}}$

Notice that, when $\rho = \sqrt{\frac{\sigma_1^2}{\sigma_2^2}}$, the broadcast channel is physically degraded and the feedback noise variance threshold in (2) is 0. This is already implied by El Gamal’s result [5]. At the other extreme, when the forward channel noises are independent ($\rho = 0$), the result is as shown in Fig. 2. However, note that the theorem does not give a threshold for all positively correlated $Z_1, Z_2$. For example, when $Z_2 = 2Z_1$, we have $1 = \rho \geq \sqrt{\frac{\sigma_1^2}{\sigma_2^2}} = \frac{1}{2}$, and Theorem 1 does not apply. More specifically, the theorem above gives a threshold whenever we can write $Z_1, Z_2$ as (3)-(4), where $Z, Z_1, Z_2$ are independent. This is also the class of channels studied by Ozarow and Leung [6].

**Proof:** We start by noting that when $\sigma_1^2 \leq \sigma_2^2$ and $\rho \leq \sqrt{\frac{\sigma_1^2}{\sigma_2^2}}$, without loss of generality, we may write $Z_1, Z_2$ in the form

$$Z_1 = Z + \tilde{Z}_1,$$

$$Z_2 = Z + \tilde{Z}_2,$$

where $Z, \tilde{Z}_1, \tilde{Z}_2$ are independent zero-mean Gaussian random variables with variances $\varsigma^2, \varsigma_1^2, \varsigma_2^2$, respectively, given by

$$\varsigma^2 = \rho\sqrt{\frac{\sigma_1^2}{\sigma_2^2}},$$

$$\varsigma_1^2 = \sigma_1^2 - \rho\sqrt{\frac{\sigma_1^2}{\sigma_2^2}}$$

$$\varsigma_2^2 = \sigma_2^2 - \rho\sqrt{\frac{\sigma_1^2}{\sigma_2^2}}.$$

The key idea of the proof is to give receiver 1 access to the feedback noise (step (a) below). By Fano’s inequality,
where the last inequality follows (via concavity of log) from the power constraint and the memorylessness of the channel. We can relate the first term of (5) and the second term above through the entropy power inequality (EPI).

Lemma 2.

\[ 2^{\frac{n}{2}h(Y_2^n | W_2)} \geq 2^{\frac{n}{2} \sum_{i=1}^{n} h(Y_{i_1}|W_2,Y_{i_2}^{-1})} + 2\pi e(\sigma_2^2 - \sigma_1^2). \]  

The proof, which is along the lines of El Gamal [5, Lemma 1] and Blachman [15], is given in the appendix.

To finish the proof, we note that

\[ \frac{n}{2} \log 2\pi e(P + \sigma_2^2) \geq h(Y_2^n) \geq h(Y_2^n | W_2) \geq h(Z_2^n), \]

where the last inequality follows from (5) and (7). Substituting in (6), we get

\[ R_2 \leq \frac{1}{2} \log \left( 1 + \frac{(1 - \theta)P}{\theta P + \sigma_2^2} \right). \]

Furthermore, (5) and (7) will imply,

\[ R_1 \leq \frac{1}{2} \log \left( 1 + \frac{\theta P}{\sigma_1^2} \right). \]

Thus, we have shown that \((R_1, R_2) \in C_{\text{abc}} \ominus_{\text{fb}} \). and the proof is complete.

Theorem 1 can also be extended to more generalized feedback settings with additive Gaussian noise where the feedback noise may not be independent of the forward channel noises. This extension will be considered in a longer version of this paper.

IV. VECTOR BROADCAST CHANNEL WITH FEEDBACK

It may appear that the additive Gaussian noise in the feedback plays a critical role in the negative result that we presented in the last section. In this section we study a vector GBC where the signal from one of the antennas will be fed back perfectly, but this still does not enlarge the capacity region in certain settings as we will describe below.

Let us consider the two-user memoryless broadcast channel shown in Figure 3 below, where user 1 makes two independent observations \((Y_{i_1}, Y_{i_2})\) of each transmitted symbol. User 2 observes a scalar output \(Y_2\). For simplicity, we will assume that the noises on different antennas are independent. Specifically, let \(Z = (Z_2, Z_{i_1}, Z_{i_2})\) be a zero mean Gaussian random vector with a diagonal covariance matrix

\[ K_z = \begin{bmatrix} \sigma_2^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{bmatrix}. \]

Let us assume that the output symbols \(Y_{i_2}\) are perfectly fed back to the transmitter ( causally). Notice that \(Y_{i_1}\) as well as \(Y_2\)
are not fed back. Let $C_{\text{partial-fb}}$ and $C_{\text{no-fb}}$ denote the feedback and no-feedback capacity regions.

We study the case where $\sigma_{11}^2 \leq \sigma_{12}^2$, i.e., the second receiver has a larger noise variance than the noise variance at the antenna of the first receiver that is not fed back. Under this assumption, we will show that perfect causal feedback of antenna of the first receiver that is not fed back. Under this assumption, we will show that perfect causal feedback of $Y_{12}$ does not enlarge the capacity region. Notice that the 'no enlargement' result holds good irrespective of the link quality of $Y_{12}$. This is surprising, since we are feeding back a major portion of the output to receiver 1, particularly so when $\sigma_{12}^2 < \sigma_{11}^2$. We now state our main theorem.

**Theorem 3.** $C_{\text{partial-fb}} = C_{\text{no-fb}}$ if $\sigma_{11}^2 \leq \sigma_{12}^2$.

The proof is along the same lines as that of Theorem 1 presented in the last section.

**Proof:** Recall that the boundary of the no-feedback capacity region $C_{\text{no-fb}}$ is given by

$$R_1 \leq \frac{1}{2} \log \left( 1 + \theta P \left[ \frac{1}{\sigma_{11}^2} + \frac{1}{\sigma_{12}^2} \right] \right)$$

$$R_2 \leq \frac{1}{2} \log \left( 1 + \frac{(1-\theta)P}{\theta P + \sigma_{12}^2} \right),$$

for $\theta \in [0, 1]$. This is just the capacity region of the scalar broadcast channel resulting from receiver 1 pre-processing $Y_{11}$ and $Y_{12}$ by passing it through a maximal ratio combiner.

By Fano’s inequality,

$$nR_1 \approx I(W_1; Y^n_{11}) = h(Y^n_{11}) - h(Y^n_{11}|W_2)$$

$$\leq \frac{n}{2} \log 2\pi e(P + \sigma_{12}^2) - h(Y^n_{11}|W_2).$$

We know that there exists some $\theta \in [0, 1]$ such that

$$h(Y^n_{11}|W_2) = \frac{n}{2} \log 2\pi e(\theta P + \sigma_{12}^2).$$

Thus,

$$R_2 \leq \frac{1}{2} \log \left( 1 + \frac{(1-\theta)P}{\theta P + \sigma_{12}^2} \right),$$

Let us now consider the rates for receiver 1. We will write $(Y_{11}, Y_{12})$ as $Y_1$, and let $Y_1^*$ denote the $2 \times n$ matrix of received values at receiver 1. Again, by Fano’s inequality

$$nR_1 \approx I(W_1; Y_1^*|W_2)$$

$$= \sum_{i=1}^{n} I(W_1; Y_{1,i}|W_2, Y_1^{i-1}).$$

Let $\frac{1}{\theta P} = \frac{1}{\sigma_{11}^2} + \frac{1}{\sigma_{12}^2}$ and consider the following invertible transformation of $Y_1$:

$$\begin{bmatrix} Y_e \\ Y_{e,i}^{*} \end{bmatrix} = \sigma_e^2 \begin{bmatrix} \frac{1}{\sigma_{11}^2} & \frac{1}{\sigma_{12}^2} \\ -1 & 1 \end{bmatrix} Y = \begin{bmatrix} X + Z_e \\ Z_e^* \end{bmatrix},$$

where $Z_e = \sigma_e^2 \left( \frac{Z_{11}^e + Z_{12}^e}{\sigma_{11}^2 + \sigma_{12}^2} \right)$ is independent of $Z_{11}^e$. Note that $Z_e \sim \mathcal{N}(0, \sigma_e^2)$ is an independent identically distributed (i.i.d.) process with $\mathbb{E}[Z_e Z_{12}^*] = \frac{\sigma_e^2 \sigma_{12}^2}{\sigma_{11}^2 + \sigma_{12}^2}$. Since the $Z_e^*$ process is i.i.d. and independent of the $Z_e$ process, we can write,

$$\sum_{i=1}^{n} I(W_1; Y_{1,i}|W_2, Y_1^{i-1})$$

$$= \sum_{i=1}^{n} h(Y_{e,i}|W_2, Y_1^{i-1}) - h(Z_e,i)$$

$$\leq \sum_{i=1}^{n} h(Y_{e,i}|W_2, Y_1^{i-1}) - h(Z_e,i)$$

$$= \sum_{i=1}^{n} h(X + Z_e,i|W_2, Y_1^{i-1}) - h(Z_e,i),$$

where in step (a) we removed $Y_1^{i-1}$ from the conditioning to obtain the inequality. Since $Y_2$ has a larger noise variance than $Y_{11}$ (i.e., $\sigma_{12}^2 \geq \sigma_{11}^2$) and both of them are not fed back to the transmitter, we may write (also see footnote 1)

$$nR_1 \leq \sum_{i=1}^{n} h(X + Z_e,i|Y_1^{i-1}, W_2) - h(Z_e,i).$$

Consider an i.i.d. process $Z_d \sim \mathcal{N}(0, \sigma_d^2 - \sigma_e^2)$, independent of all other processes mentioned before. Using El Gamal’s version of EPI [5, Lemma 1],

$$2^\frac{1}{n} \sum_{i=1}^{n} h(Y_{e,i}|Y_1^{i-1}, W_2) \leq 2^\frac{1}{n} h(Y_{e,i}|W_2) - 2\pi e(\sigma_d^2 - \sigma_e^2)$$

$$= 2^\frac{1}{n} h(Y_{e,i}|W_2) - 2\pi e(\sigma_d^2 - \sigma_e^2)$$

$$= 2\pi e(\sigma_d^2 - \sigma_e^2) - 2\pi e(\sigma_e^2 - \sigma_e^2)$$

$$= 2\pi e(\sigma_d^2 + \sigma_e^2).$$

Collecting all these together, we have

$$R_1 \leq \frac{1}{2} \log \left( 1 + \frac{(1-\theta)P}{\theta P + \sigma_{12}^2} \right).$$

This completes our converse, and we have shown whenever $\sigma_{11}^2 \leq \sigma_{12}^2$, perfect causal feedback from the antenna $Y_{12}$ does not change the capacity region of our broadcast channel.

While we have assumed independent noise processes at the antennas, this can be extended to the case where the noises may be correlated. In fact, Theorem 1 can be obtained as a corollary of such a generalization.
V. Conclusion

We presented a class of two-user scalar Gaussian broadcast channels with passive noisy feedback from one of the receivers for which feedback does not enlarge the capacity region. Our result is in the form of a threshold on the feedback noise variance above which feedback cannot achieve points outside the no-feedback capacity region. We also saw a class of two-user vector Gaussian broadcast channels where perfect feedback of some components of the received signal from a user does not lead to an enlarged capacity region.

Our study raises the question whether the threshold in Theorem 1 is tight, i.e., is there a coding scheme which can achieve points outside the no-feedback capacity region whenever the feedback noise variance is below our threshold (and also in all cases where Theorem 1 does not apply).

It must be emphasized that we only considered passive feedback. With active feedback schemes no such thresholds (and also in all cases where Theorem 1 does not apply).

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APPENDIX

Proof of Lemma 2:

We prove this by induction on \( n \). For \( n = 1 \), the inequality follows from entropy power inequality [2, pg. 22] since we may write

\[
\begin{align*}
    h(Y_{2,1}|W_2) &= h(Y_{1,1} + Z'_{2,1}|W_2), \\
    \text{where } Z'_{2,1} &\text{ is zero-mean Gaussian with } \Var(Z'_{2,1}) = \sigma_2^2 - \sigma_1^2
\end{align*}
\]

and independent of everything else. This simply follows from the fact that \((Z_{1,1}, Z_{2,1})\) is independent of \((W_2, X_1)\), and \(\Var(Z_{1,1}) = \sigma_1^2 \leq \sigma_2^2 = \Var(Z_{2,1})\).

Suppose (7) is true for \( n = m - 1 \). We may write

\[
\begin{align*}
    h(Y_{2m}|(W_2, Y_{2m-1}^{m-1})) &= h(Y_{1m} + Z'_{2m}|(W_2, Y_{2m-1}^{m-1})), \\
    \text{where } Z'_{2m} &\text{ is zero-mean Gaussian with } \Var(Z'_{2m}) = \sigma_2^2 - \sigma_1^2
\end{align*}
\]

and independent of everything else. This follows from the fact that \((Z_{1m}, Z_{2m})\) is independent of \((W_2, Y_{2m-1}^{m-1}, X_m)\) and \(\Var(Z_{1m}) \leq \Var(Z_{2m})\). By (conditional) entropy power inequality [2, pg. 22],

\[
2^{2h(Y_{2m}|W_2, Y_{2m-1}^{m-1})} \geq 2^{2h(Y_{1m}|W_2, Y_{2m-1}^{m-1})} + 2^{2h(Z'_{2m})}.
\]

i.e.,

\[
2h(Y_{2m}|W_2, Y_{2m-1}^{m-1}) \geq \log \left(2^{2h(Y_{1m}|W_2, Y_{2m-1}^{m-1})} + 2\pi e(\sigma_2^2 - \sigma_1^2)\right).
\]

So,

\[
\begin{align*}
    2^{h(Y_{2m}^m|W_2)} &= m - 1 \frac{2}{m} h(Y_{2m}^m|W_2) + 2 \frac{m}{m} h(Y_{2m}|W_2, Y_{2m}^{m-1}) \\
    &\geq m - 1 \log \left(2^{\frac{2}{m} \sum_{i=1}^{m-1} h(Y_{1i}|W_2, Y_{2i}^{i-1})} + 2\pi e(\sigma_2^2 - \sigma_1^2)\right) \\
    &\quad + \frac{1}{m} \log \left(2^{2h(Y_{1m}|W_2, Y_{2m}^{m-1})} + 2\pi e(\sigma_2^2 - \sigma_1^2)\right) \\
    &\geq \log \left(2 \frac{m}{m} \sum_{i=1}^{m} h(Y_{1i}|W_2, Y_{2i}^{i-1}) + 2\pi e(\sigma_2^2 - \sigma_1^2)\right),
\end{align*}
\]

where (a) follows from the induction hypothesis and the EPI above, and (b) follows from convexity of \(\log(2^u + v)\) in \(u\) for \(v \geq 0\).