Composite leptons and quarks constructed as triply occupied quasiparticles in quaternionic quantum mechanics

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We propose a set of rules for constructing composite leptons and quarks as triply occupied quasiparticles, in the quaternionic quantum mechanics of a pair of Harari-Shupe preons $T$ and $V$. The composites fall into two classes, those with totally antisymmetric internal wave functions, and those with internal wave functions of mixed symmetry. The mixed symmetry states consist of precisely the three spin 1/2 lepton families used in the standard model (48 particle states, not counting the doubling arising from antiparticles), plus one doublet of spin 3/2 quarks (24 particle states). The antisymmetric states consist of a set of spin 3/2 leptonic states with charges as in a standard model family (16 particle states), and a spin 1/2 leptonic fractionally charged doublet (4 particle states). We sketch ideas for deriving our rules from a fundamental quaternionic preonic field theory.

Although the repetitive family structure of the standard model leptons and quarks is strongly suggestive of further substructure, it has proved difficult to formulate economical preon model candidates. A useful step in this direction was taken in 1979 by Harari [1] and Shupe [2], who proposed a set of rules for generating the states in a single lepton-quark family (including the color tripling of the quarks) as triples constructed from two fundamental preons $T$ and $V$, with charges 1/3 and 0 respectively. In order to generate the color states this way, the Harari-Shupe rules require that $TTV$, $TVT$, and $VTT$ be counted as distinct states, which is not possible within standard quantum mechanics, but which I suggested [3] might be possible within quaternionic quantum mechanics, where the multiplication of Hilbert space scalars is noncommutative. I later noted [4] that if one includes spin states in the enumeration, there is the possibility of generating family structure as well as color structure dynamically; however, in the naive formulation of Ref. [4] (which proceeds by permutation of character strings, as in Refs. [1,2]) one gets only two spin 1/2 families together with a spin 3/2 family, and there seemed to be no plausible mechanism to generate the additional spin 1/2 states needed for a third family.

Despite the lack of progress in further developing the Harari-Shupe proposal, in the intervening years there has been considerable progress in the analysis of the structure of quaternionic quantum mechanics (see, e.g. [5,6]), which is summarized and considerably expanded on in my book [7]. In this Letter, I use the methods developed in Ref. [7] for the second quantization of many-body systems in quaternionic quantum mechanics, and in particular the construction given there of quasiparticles with quaternionic wave functions, to take a new look at the issue of preon models. We shall see that when the structure of quaternionic quasiparticle wave functions is taken into account by a plausible set of rules, the proposal of Refs. [1-4] to generate color and family structure from two fundamental preons can in fact be realized. We begin by stating the rules and showing how they are used to enumerate three quasiparticle composite states; we then turn to a discussion of how the rules might be derived from a fundamental relativistic preonic theory.

Let $p_n(\vec{r})$ and $p_n^\dagger(\vec{r})$ be ordinary fermionic annihilation and creation operators for the preonic states,

$$\{p_n\} = \{t_u, t_d, v_u, v_d\},$$

\begin{align}
\{p_m(\vec{r}), p_n(\vec{r}')\} &= 0, \\
\{p_m^\dagger(\vec{r}), p_n^\dagger(\vec{r}')\} &= 0, \\
\{p_m(\vec{r}), p_n^\dagger(\vec{r}')\} &= \delta_{mn}\delta^3(\vec{r} - \vec{r}'),
\end{align}
where \( t \) and \( v \) carry charges \( 1/3 \) and \( 0 \) respectively and the arrows indicate the spin. It is shown in Ref. [7] that these creation and annihilation operators can be treated as real quantities with respect to an appropriately defined quaternion algebra (a left-acting or operator quaternion algebra) \( 1, E_1, E_2, E_3 \), which obeys

\[
E_A E_B = -\delta_{AB} + \sum_C \epsilon_{ABC} E_C, \tag{2a}
\]

and which can be reexpressed in an obvious vector notation (with real number vectors \( \vec{a}, \vec{b} \)) as

\[
\vec{E} \cdot \vec{a} \vec{E} \cdot \vec{b} = -\vec{a} \cdot \vec{b} + \vec{a} \times \vec{b} \cdot \vec{E}. \tag{2b}
\]

According to Ref. [7], and as discussed below, quasiparticle annihilation operators are formed as superpositions of ordinary annihilation operators, with quaternion-valued wave functions as coefficients. As our first rule, we assume

**Rule 1.** The wave function appearing in the quasiparticle operators is assumed to be quaternion imaginary, and to have nonvanishing and linearly independent components along the three quaternion units \( \vec{E} \). The preon binding forces, and thus the ground state wave function, are assumed also to be flavor (i.e., \( t, v \)) and spin (i.e., \( \uparrow, \downarrow \)) independent.

Hence the annihilation operator \( P_n(\vec{R}) \) for a quasiparticle located at \( \vec{R} \) has the form

\[
P_n(\vec{R}) = \int d^3 r \, \vec{a}(\vec{r}) \cdot \vec{E} \, p_n(\vec{r} + \vec{R}), \tag{3a}
\]

which can be written in abbreviated form as

\[
P_n = \sum_{1} \vec{a}(1) \cdot \vec{E} \, p_n(1). \tag{3b}
\]

In Table 1 we explicitly write out the quasiparticle operators for the two spin states of the two preonic flavor states, in both the full notation of Eq. (3a) and the abbreviated notation of Eq. (3b). We shall assume that the wave function \( \vec{a}(\vec{r}) \) has support only for \( |\vec{r}| \) of order the preonic length scale, which we assume to be much smaller than length scales characterizing standard model physics, and that the wave function is unit normalized,

\[
\int d^3 r \, |\vec{a}(\vec{r})|^2 = 1. \tag{3c}
\]

The quasiparticle operator defined by Eqs. (3a,b) has a number of interesting properties. First of all, a simple calculation shows that any quasiparticle state can at most be triply occupied, since the fourth power of a quasiparticle operator vanishes [7] (this property, first noted in a different context by Govorkov [8], holds even when the internal wave function \( \vec{a} \) has a real part):

\[
P_n(\vec{R})^4 = 0. \tag{4a}
\]

Also, the quasiparticles satisfy parastatistics-like commutation relations [8] (here the assumption of a quaternion imaginary internal wave function is needed):

\[
[P_\ell, [P_m, P_n]] = 0, \text{ any } \ell, m, n,
\]

\[
[P_\ell^\dagger, [P_m^\dagger, P_n^\dagger]] = -2\delta_{\ell m} \sum_{1,2} \vec{a}(1) \cdot \vec{E} \, \vec{a}(1) \cdot \vec{a}(2) p_n^\dagger(2). \tag{4b}
\]

[If one were to replace Eq. (3c) by the stronger condition

\[
\int d^3 r \, a_A(\vec{r}) a_B(\vec{r}) = \delta_{AB}/3, \tag{4c}
\]
then the second line of Eq. (4b) would become the parastatistics commutator

$$[P^\dagger, [P_m, P_n^\dagger]] = (2/3)\delta_{tm}P^\dagger_n.$$ (4d)

However, we do not assume Eqs. (4c,d) in what follows.] Eqs. (4a,b) suggest that states of three quasiparticles will play a special role, and we identify them as composite leptons and quarks, which are formed and classified according to the following further rules:

**Rule 2.** Composite fermion states are identified with the quaternion real components of the independent products of three quasiparticle operators drawn from Table 1. In other words, a product $C = P_tP_mP_n$ which is already quaternion real is identified as a composite fermion operator, while if $C$ has the quaternion imaginary form $C = C_\ast \cdot \vec{E}$, then the real components $C_1, C_2, C_3$ are each identified as an independent composite fermion operator.

**Rule 3.** When the number of composite states is tripled as a result of the inequivalence of different orderings of $t$ and $v$, the composites are identified as colored quark states, corresponding to the fact that the underlying preonic forces can cause transitions between $t$ and $v$. When the number of composite states is tripled as a result of the inequivalence of different orderings of the spin labels $\uparrow$ and $\downarrow$, the composites are identified as states in different families, corresponding to the fact that the underlying preonic forces are assumed spin independent, and so do not cause transitions between $\uparrow$ and $\downarrow$.

The enumeration of the independent products of three quasiparticle operators proceeds much as the enumeration of states in the quark model [9,10], and is expedited by the use of a number of simple identities which follow from the defining equations given above. First of all, from Eq. (2b) we immediately get

$$P_tP_m = \sum^{1,2}_l [-\vec{a}(1) \cdot \vec{a}(2) + \vec{a}(1) \times \vec{a}(2) \cdot \vec{E}] p^\ell(1) p_m(2),$$ (5a)

allowing us to easily evaluate the successive products of quasiparticle operators. The classification of products according to their reality properties under quaternion conjugation (indicated by a bar $\ast$) is facilitated by noting that for any $\ell, m, n$ we have

$$P^\dagger \ast P_m P_n = P_n P_m P^\dagger,$$ (5b)

which follows from the facts that (i) the quaternion conjugate of a product of factors is the product of the conjugates of the factors in reverse order, (ii) the conjugate of an imaginary quaternion is minus itself, and (iii) reverse ordering the product of three fermionic annihilation operators just reverses the sign of the product. Finally, setting $n = \ell$ in the first line of Eq. (4b) gives the repeatedly used identity, again valid for any $\ell, m,$

$$P_tP_tP_m + P_mP_tP_t = 2P_tP_mP_t.$$ (5c)

The most complicated enumeration is for the charge 2/3, spin and helicity 1/2 states, which can have the three charge structures $TTT + VTT$, $TVT$, and $TTV - VTT$, combined with the two spin structures $\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2$ $\uparrow\uparrow\uparrow - \downarrow\uparrow\uparrow$, giving six possibilities in all. One readily finds, using Eq. (5b), that $TTV + VTT$ and $TVT$ combined with $\uparrow\uparrow\uparrow - \downarrow\uparrow\uparrow$ are imaginary, as is $TTV - VTT$ combined with $\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2$ $\uparrow\uparrow\uparrow$, and these correspond to the 3 families of charge 2/3 quarks shown in Table 3. The remaining three combinations are real, but only one is linearly independent after using Eqs. (4b) and (5c), and corresponds to the spin 1/2 lepton in Table 2. For the charge 1, spin and helicity 1/2 states, there is only one charge structure $TTT$, which when combined with the spin structure $\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow$ gives one imaginary composite, corresponding to the 3 families of charge 1 leptons in Table 3; the contribution of the spin structure $\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2$ $\uparrow\uparrow\uparrow$ vanishes in this case by Eq. (5c). The computations in the spin 3/2 cases, which involve totally symmetric spin structures, are similar.

The results of the calculation are summarized in Tables 2 and 3. Table 2 lists all states with a totally antisymmetric internal wave function proportional to $\vec{a}(1) \times \vec{a}(2) \cdot \vec{E}$. Table 3 lists all states with a mixed symmetry internal wave function proportional to $\vec{a}(1) \cdot \vec{a}(2)a_A(3)$ or to the two structures obtained from this one by permuting the labels 1,2,3 for fixed $A$, with $A$ an index which also takes the values 1,2,3, each value of $A$ being counted (by Rule 2) as a distinct state. Although there is no dynamics for mass generation in the model, experience with the quark model suggests that states with similar internal wave functions should
have roughly similar masses, at least when viewed on a preonic energy scale, and that the states with mixed symmetry internal wave functions should be lighter than those which are totally antisymmetric. Thus, it is encouraging that the spin 1/2 content of the mixed symmetry states corresponds, when interpreted by Rule 3, with precisely the content of the fermions used in the standard model. One striking feature of the spin 1/2 wave functions in Table 3 is that all three leptons, and the first two sets of quarks, have spin structure $\uparrow\downarrow\downarrow - \downarrow\uparrow\uparrow$, while the third set of quarks has spin wave function $\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2\uparrow\downarrow\uparrow$. Hence a strong mass operator dependence on spin state would result in a large mass splitting between the third set of quarks and the remaining quarks and leptons, roughly corresponding to what is experimentally observed for the quarks of the third family (particularly if the measure of the zeroth order quark mass of a family is taken as a geometric or arithmetic mean of the masses of the two charge states in that family.)

If the spin 3/2 mixed symmetry quarks are nearby, they may be observable at LEP and the Tevatron, or at the LHC. In all likelihood, they should not be unstable against magnetic dipole electromagnetic decay into spin 1/2 quarks, and hence should not be seen directly (through new types of mesons), but rather should only appear indirectly through enhanced production of standard model quarks (and mesons) in certain channels. If preliminary hints of an excess branching ratio into $b$ quarks at LEP [11] survive the accumulation of better statistics, then production of the spin 3/2 quarks of Table 3 could be considered as a possible explanation. However, since the spin 3/2 quarks cannot cancel chiral anomalies among themselves, they must have vector rather than chiral electroweak currents. This permits them to have mass terms even in the absence of electroweak spontaneous symmetry breaking, and so they could in principle have masses much larger than those of the standard model spin 1/2 fermions, which have chiral electroweak currents and must get their masses through the electroweak Higgs mechanism. Whether the states of Table 2 will be visible in the near future is hard to assess without a detailed dynamics; they could in principle lie at the preonic mass scale, which in turn may well be as high as the GUT scale, in which case they may never be directly observable in accelerator experiments.

We note that when any two of the composite annihilation operators of Tables 2 and 3 are anticommutated, one gets zero as expected for canonical fermions. However, when a composite annihilation operator is anticommutated with the adjoint of another, one gets both a c-number term and operator terms. The operator terms contribute significantly to vacuum expectations of products of composite operators only when there is an appreciable probability for composite particles to approach within the preonic distance scale, in which case their substructure becomes visible; at energies far below the preonic scale the operator terms in the anticommutators can be neglected. For general composites $C_t, C_m$ from Tables 2 and 3, the c-number parts of the anticommutators have the form

$$\{C_t(\vec{R}), C_m(\vec{R}^\dagger)\} = \delta^3(\vec{R} - \vec{R}^\dagger)c_{tm},$$

with $c_{tm}$ a real, symmetric matrix which is not in general in diagonal form. To construct the independent degrees of freedom, one must form linear combinations of the composites within each spin and charge sector of the Tables using the real, orthogonal matrix which diagonalizes the matrix $c_{tm}$ for that sector, and then do an appropriate renormalization to get a unit anticommutator. The considerations of this paragraph all have analogs in the ordinary quark model.

Let us now turn to a brief discussion of how one could try to justify the rules given above from a more fundamental preonic theory. In a recent paper [12] I proposed a generalized quantum dynamics which can apply to quaternionic as well as standard complex quantum field theories, and which leads to equations of motion that are invariant under operator valued gauge transformations. I noted there (see also Chaps. 12 and 13 of Ref. [7]) that the minimal fermionic model with maximal (two-sided) operator gauge invariance necessarily has two fermion fields, basically because in order to get the right hermiticity properties without breaking one of the gauge groups, one must use the two dimensional real matrix representation of the imaginary unit (given in standard Pauli matrix form as $-i\sigma_2$) in constructing the fermionic Lagrangian. Thus the minimal fermionic model contains the two fields $t$ and $v$ which are needed to make lepton quark composites. In addition, the model of Ref. [12] contains only vector couplings to its gauge gluons, and has an (anti)symmetrical structure in the two fundamental fermions. The vector-like structure means that the forces binding the preons are spin-independent, and the fermionic symmetry can plausible lead to binding forces which are flavor-symmetric, as assumed in Rule 1. Finally, the model of Ref. [12] has a global chiral invariance (although it cannot be decoupled into noninteracting chiral components of the fields), and so it is
possible for the lowest lying composites formed from the preons to have zero mass on the preonic mass scale, as is needed in a physically realistic preon model. For these reasons, it is natural to conjecture [7] that the minimal fermionic model of Ref. [12] gives the underlying relativistic preon dynamics.

Although there are many open questions in the generalized dynamics discussed in Ref. [12], let us suppose that there is a regime in which this dynamics is represented by a unitary operator dynamics in Hilbert space. We can than invoke a general result proved in Ref. [7] (see Ref. [13] for an earlier version), which asserts that for a general quaternionic operator Hamiltonian dynamics, except in the strictly zero energy sector the $S$-matrix in quaternionic Hilbert space is complex. This means that the asymptotic state dynamics will be represented by an effective complex quantum field theory acting on the asymptotic particle states. To justify Rule 2, one would then have to show that the asymptotic particles correspond to three quasiparticle composites, with an effective anti-self-adjoint time development operator of the form

$$\hat{H} = tr_E[\partial C^\dagger / \partial t C - C^\dagger \partial C / \partial t],$$

with $tr_E$ denoting a trace over the operator quaternion algebra spanned by $\vec{E}$. Given the role of traces in the dynamics formulated in Ref. [12], this form for the asymptotic dynamics is plausible.

Let us next consider the binding of preons into composites. Since the model of Ref. [12] has a QCD-like gauge structure [based on a quaternionic extension of an $SU(2) \times SU(2)$ gauge theory], it is not unreasonable to suppose that the formation of composite bound states can be treated much as in QCD. In QCD, although the light mesons are actually highly relativistic bound states, one finds that the classification of the low-lying hadronic states can be successfully carried out in the non-relativistic quark model [9], in which quark binding is treated in the shell model approximation. In the shell model, which is based on the Hartree or self-consistent field approximation, one assumes that each particle moves independently in a potential centered on the center of mass of the overall system. Taken over to our quaternionic preon model, the shell model anti-self-adjoint Hamiltonian $\hat{H}_3$ for the binding of three preons, with spin and flavor independent forces and with the center of mass chosen as the origin, takes the form

$$\hat{H}_3 = \sum_{n=1}^{3} \hat{H}(\vec{r}_n),$$

$$\hat{H}(\vec{r}) = \{( -I/2M)(\vec{\nabla}_{\vec{r}} - I\vec{A}(\vec{r})) \}^2 + \hat{U}(\vec{r}),$$

(8a)

with $M$ the preon effective mass and with $\vec{A}$ and $\hat{U}$ respectively a real vector potential and a quaternion imaginary scalar potential which are both functions of the preon coordinate $\vec{r}$. Since Eq. (8a) is the sum of identical one body Hamiltonians for the three particles, it can be rewritten in Fock space as [7]

$$\hat{H}_F = \int d^3r \, p^\dagger(\vec{r})\hat{H}(\vec{r})p(\vec{r}),$$

(8b)

with $\hat{H}(\vec{r})$ the one body Hamiltonian defined in Eq. (8a). Now let $a_\kappa(\vec{r})$ be a complete orthonormal set of one particle energy eigenstates of the one body Hamiltonian, obeying

$$\hat{H}(\vec{r})a_\kappa(\vec{r}) = a_\kappa(\vec{r})IE_\kappa.$$  

(8c)

Then defining the quasiparticle operator $p_\kappa$ and its adjoint by

$$p_\kappa = \int d^3r \, a_\kappa(\vec{r})p(\vec{r}), \quad p^\dagger_\kappa = \int d^3r \, p^\dagger(\vec{r})a_\kappa(\vec{r}),$$

(9a)

some simple algebra [7] (which parallels the corresponding derivation in standard complex quantum mechanics) shows that $\hat{H}_F$ can be rewritten as

$$\hat{H}_F = \sum_\kappa p^\dagger_\kappa IE_\kappa p_\kappa.$$  

(9b)
As shown in Ref. [7], the energy eigenstates in the one particle sector are exactly created by the quasiparticle operators $p^\dagger_\kappa$, but since the latter do not obey canonical commutators because of the noncommutativity of quaternionic wave functions, they do not behave as creation operators for independent quasiparticles in sectors with more than one particle.

Nevertheless, let us assume that it is reasonable to approximate the creation operator for the ground state in the three particle sector as a product of three ground state quasiparticle creation operators. We then get the product recipe for creating composites given in Rule 2. In general the ground state wave function $a_0(\vec{r})$ is not quaternion imaginary, but we now observe that if we rewrite it in symplectic component form,

$$a_0(\vec{r}) = a_0(\vec{r})_\alpha + Ja_0(\vec{r})_\beta,$$  \hspace{1cm} (10a)

with the $\alpha, \beta$ components in the complex subalgebra spanned by 1 and $I$, we can always find a complex phase $\zeta(\vec{r})$ which makes $\zeta(\vec{r})a_0(\vec{r})$ quaternion imaginary. (Simply take $\zeta$ as $I$ times the complex conjugate of the phase of the $\alpha$ symplectic component.) But the effect of this multiplication is to just induce a gauge transformation on the vector potential $\vec{A}(\vec{r})$, and a corresponding quaternion automorphism transformation of the scalar potential $\vec{U}$. Hence we can always pick a gauge for the potentials in the shell model Hamiltonian which makes the ground state wave function (but not simultaneously the wave functions of higher excited states) quaternion imaginary. Working in this gauge we get the first part of Rule 1, with the identification

$$a_0(\vec{r}) = -\vec{a}(\vec{r}) \cdot \vec{E},$$  \hspace{1cm} (10b)

for the wave function $\vec{a}(\vec{r})$ introduced above. Although it may seem objectionable to have to assume a specific gauge to formulate the model, this feature was also present in the original form of the BCS theory of superconductivity, and this analogy suggests that as in the case of superconductivity, gauge invariance should be restored by the proper inclusion of collective effects.

Finally, we note that since the three quasiparticle approximation to the three body ground state wave function is not exact, and since the shell model itself represents an approximation, there are residual forces which act on the composites. These residual forces can, in principle, give rise to the gauge fields of the standard model which act on the quarks and leptons. We have argued in Ref. [7] that because the left algebra structure of quaternionic Hilbert space can give rise to multi-quaternion algebras, it is possible to build up larger effective gauge groups than the underlying $SU(2) \times SU(2)$ preonic gauge group. However, since the underlying gauge group is vector-like, it seems reasonable to suppose that any larger gauge groups kinematically generated from it will still not couple spin $\uparrow$ to $\downarrow$ components, which is the basis for the identification of spin-associated tripling with family structure in Rule 3. In this picture the chiral structure of the weak interactions is not fundamental, but must arise from spontaneous symmetry breaking at a scale lying at or below the preonic energy scale.

Although the above sketch of how one might attempt to justify Rules 1-3 in a more fundamental theory is only schematic, and leaves much to be done, I believe that it is consistent with both our current experimental knowledge and with the known properties of complex and quaternionic quantum mechanics.

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Table 1

The four basic quaternionic quasiparticle annihilation operators used to construct the annihilation operators for three quasiparticle composite states centered at $\vec{R}$. The annihilation operators $t_{\uparrow,\downarrow}$, $v_{\uparrow,\downarrow}$ are conventional fermion annihilation operators, and are real numbers with respect to the quaternion algebra $E_A E_B = -\delta_{AB} + \sum_C \varepsilon_{ABC} E_C$, $A, B, C = 1, 2, 3$. The spin and flavor independent coefficient $\vec{a}(\vec{r}) \cdot \vec{E} = \sum_A a_A(\vec{r}) E_A$ is a quaternion imaginary internal ground state wave function. The notation of the abbreviated forms is used in Tables 2 and 3.

Full form:

$$
T_{\uparrow}(\vec{R}) = \int d^3 r \vec{a}(\vec{r}) \cdot \vec{E} t_{\uparrow}(\vec{r} + \vec{R}) \\
T_{\downarrow}(\vec{R}) = \int d^3 r \vec{a}(\vec{r}) \cdot \vec{E} t_{\downarrow}(\vec{r} + \vec{R}) \\
V_{\uparrow}(\vec{R}) = \int d^3 r \vec{a}(\vec{r}) \cdot \vec{E} v_{\uparrow}(\vec{r} + \vec{R}) \\
V_{\downarrow}(\vec{R}) = \int d^3 r \vec{a}(\vec{r}) \cdot \vec{E} v_{\downarrow}(\vec{r} + \vec{R})
$$

Abbreviated form:

$$
T_{\uparrow,\downarrow} = \sum_1 \vec{a}(1) \cdot \vec{E} t_{\uparrow,\downarrow}(1) \\
V_{\uparrow,\downarrow} = \sum_1 \vec{a}(1) \cdot \vec{E} v_{\uparrow,\downarrow}(1)
$$
Table 2

Three quasiparticle composites with a totally antisymmetric internal wave function structure, classified by spin, charge, and interaction type. For each charge 1 and charge 2/3 state in the table, there is a corresponding charge 0 and charge 1/3 state obtained by the interchange $T \leftrightarrow V$, $t \leftrightarrow v$. Similarly, negative helicity states are obtained from positive helicity ones by the interchange $\uparrow \leftrightarrow \downarrow$.

(1) Spin 3/2, charge 1, lepton

helicity 3/2

\[ T\uparrow T\uparrow T\uparrow = - \sum_{1,2,3} \vec{a}(1) \times \vec{a}(2) \cdot \vec{a}(3) t\uparrow(1)t\uparrow(2)t\uparrow(3) \]

helicity 1/2

\[ T\uparrow T\downarrow T\uparrow = - \sum_{1,2,3} \vec{a}(1) \times \vec{a}(2) \cdot \vec{a}(3) t\uparrow(1)t\downarrow(2)t\uparrow(3) \]

(2) Spin 3/2, charge 2/3, lepton

helicity 3/2

\[ T\uparrow V\uparrow T\uparrow = - \sum_{1,2,3} \vec{a}(1) \times \vec{a}(2) \cdot \vec{a}(3) t\uparrow(1)v\uparrow(2)t\uparrow(3) \]

helicity 1/2

\[ T\uparrow V\uparrow T\downarrow + T\uparrow V\downarrow T\uparrow + T\downarrow V\uparrow T\uparrow \]

\[ = - \sum_{1,2,3} \vec{a}(1) \times \vec{a}(2) \cdot \vec{a}(3) [t\uparrow(1)v\uparrow(2)t\downarrow(3) + t\uparrow(1)v\downarrow(2)t\uparrow(3) + t\downarrow(1)v\uparrow(2)t\uparrow(3)] \]

(3) Spin 1/2, charge 2/3, lepton

helicity 1/2

\[ T\uparrow V\uparrow T\downarrow + T\downarrow V\uparrow T\uparrow - 2T\uparrow V\downarrow T\uparrow \]

\[ = -2 \sum_{1,2,3} \vec{a}(1) \times \vec{a}(2) \cdot \vec{a}(3) [t\uparrow(1)v\uparrow(2)t\downarrow(3) - t\uparrow(1)v\downarrow(2)t\uparrow(3)] \]
Table 3
Three quasiparticle composites with a mixed symmetry internal wave function, classified by spin, charge, and interaction type. The index $A$ takes the values 1, 2, 3, resulting in three copies of each state. This tripling is interpreted as corresponding to three colors when it arises from $t, v$ reorderings, and as corresponding to three families when it arises from spin $\uparrow, \downarrow$ reorderings. For each charge 1 and charge 2/3 state in the table, there is a corresponding charge 0 and charge 1/3 state obtained by the interchange $T \leftrightarrow V$, $t \leftrightarrow v$. Similarly, negative helicity states are obtained from positive helicity ones by the interchange $\uparrow \leftrightarrow \downarrow$

(1) Spin 3/2, charge 2/3, quark

helicity 3/2

$$T_\uparrow T_\uparrow V_\uparrow - V_\uparrow T_\uparrow T_\uparrow = 2 \sum_{1,2,3,A} [\bar{a}(1) \cdot \bar{a}(3)a_A(2) - \bar{a}(2) \cdot \bar{a}(3)a_A(1)] t_\uparrow(1)t_\uparrow(2)v_\uparrow(3) E_A$$

helicity 1/2

$$T_\uparrow T_\uparrow T_\uparrow + T_\uparrow T_\uparrow V_\uparrow + T_\uparrow T_\uparrow T_\downarrow - (V_\uparrow T_\uparrow T_\uparrow + V_\uparrow T_\uparrow T_\downarrow + V_\downarrow T_\uparrow)$$

$$= 2 \sum_{1,2,3,A} [\bar{a}(1) \cdot \bar{a}(3)a_A(2) - \bar{a}(2) \cdot \bar{a}(3)a_A(1)] [t_\uparrow(1)t_\uparrow(2)v_\uparrow(3) + t_\downarrow(1)t_\downarrow(2)v_\uparrow(3) + t_\uparrow(1)t_\downarrow(2)v_\downarrow(3)] E_A$$

(2) Spin 1/2, charge 1, lepton

helicity 1/2

$$TTT \times (\uparrow\downarrow - \downarrow\uparrow) = T_\uparrow T_\uparrow T_\downarrow - T_\downarrow T_\uparrow T_\uparrow$$

$$= 2 \sum_{1,2,3,A} [\bar{a}(1) \cdot \bar{a}(3)a_A(2) - \bar{a}(2) \cdot \bar{a}(3)a_A(1)] t_\uparrow(1)t_\uparrow(2)t_\downarrow(3) E_A$$

(3) Spin 1/2, charge 2/3, quark

helicity 1/2

$$(TTV + VTT) \times (\uparrow\downarrow - \downarrow\uparrow)$$

$$= T_\uparrow T_\uparrow V_\downarrow + V_\uparrow T_\uparrow T_\downarrow - T_\downarrow T_\uparrow V_\uparrow - V_\downarrow T_\uparrow$$

$$= 2 \sum_{1,2,3,A} \{-\bar{a}(1) \cdot \bar{a}(2)a_A(3) t_\uparrow(1)t_\downarrow(2)v_\uparrow(3)$$

$$+ [\bar{a}(1) \cdot \bar{a}(3)a_A(2) - \bar{a}(2) \cdot \bar{a}(3)a_A(1)] [t_\uparrow(1)t_\uparrow(2)v_\downarrow(3) - t_\downarrow(1)t_\downarrow(2)v_\uparrow(3)] E_A$$

Spin 1/2, charge 2/3, quark

helicity 1/2

$$TVT \times (\uparrow\downarrow - \downarrow\uparrow)$$

$$= T_\uparrow V_\uparrow T_\downarrow - T_\downarrow V_\uparrow T_\uparrow$$

$$= 2 \sum_{1,2,3,A} [\bar{a}(2) \cdot \bar{a}(3)a_A(1) + \bar{a}(1) \cdot \bar{a}(3)a_A(2) - \bar{a}(2) \cdot \bar{a}(3)a_A(3)] t_\uparrow(1)t_\downarrow(2)v_\uparrow(3) E_A$$

Spin 1/2, charge 2/3, quark

helicity 1/2

$$(TTV - VTT) \times (\uparrow\downarrow + \downarrow\uparrow - 2\downarrow\downarrow)$$

$$= T_\uparrow T_\uparrow V_\uparrow - V_\uparrow T_\uparrow T_\downarrow + T_\downarrow T_\uparrow V_\uparrow - V_\downarrow T_\uparrow T_\uparrow - 2T_\uparrow T_\downarrow V_\uparrow + 2V_\uparrow T_\downarrow T_\uparrow$$

$$= 2 \sum_{1,2,3,A} \{3\bar{a}(1) \cdot \bar{a}(2)a_A(3) t_\uparrow(1)t_\downarrow(2)v_\uparrow(3)$$

$$+ [\bar{a}(1) \cdot \bar{a}(3)a_A(2) - \bar{a}(2) \cdot \bar{a}(3)a_A(1)] [t_\uparrow(1)t_\uparrow(2)v_\downarrow(3) - t_\downarrow(1)t_\downarrow(2)v_\uparrow(3)] E_A$$

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