A NOTE ON BLOWUP LIMITS IN 3D RICCI FLOW

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Abstract. We prove that Perelman’s ancient ovals occur as blowup limit in 3d Ricci flow through singularities if and only if there is an accumulation of spherical singularities.

1. Introduction

In [18, Section 1.4], Perelman constructed ancient ovals for the Ricci flow. These are ancient 3-dimensional Ricci flows that for \( t \to 0 \) converge to a round point, but for \( t \to -\infty \) look like a long neck capped off by two Bryant solitons. Uniqueness of Perelman’s ancient ovals has been proved in recent important work by Brendle-Daskalopoulos-Sesum [8]. While 3-dimensional Ricci flow is now extremely well understood, one of the few remaining open problems is whether or not Perelman’s ancient ovals actually occur as blowup limit in 3d Ricci flow through singularities. In this short note, we prove that Perelman’s ancient ovals occur as blowup limit if and only if there is an accumulation of spherical singularities.

For the purpose of this note it is most convenient to describe 3-dimensional Ricci flow through singularities as metric flows, as introduced by Bamler [3]. A metric flow is given by a set \( X \), a time-function \( t: X \to \mathbb{R} \), complete separable metrics \( d_t \) on the time-slices \( X_t = t^{-1}(t) \), and probability measures \( \nu_{x,s} \in \mathcal{P}(X_s) \) such that the Kolmogorov consistency condition and a certain sharp gradient estimate for the heat flow hold (see Section 2 for details). We remark that by our recent work [9], any metric flow \( X \) arising as a noncollapsed limit of smooth Ricci flows or arising from a 3-dimensional singular Ricci flow from [16] is a weak solution of the Ricci flow in the sense of Haslhofer-Naber [14], namely for almost every \((p,t)\) the estimate

\[
|\nabla_p \mathbb{E}_{(p,t)}[F]| \leq \mathbb{E}_{(p,t)}[|\nabla^H F|]
\]

holds for all cylinder functions \( F \) on path-space. However, for our present purpose it is enough to assume that the Ricci flow equation \( L_0 g = -2\text{Ric}(g) \) holds on the regular part.

Next, we recall from [3] Section 6.8) that a tangent flow at some space-time point \( x \in X \) is an \( F \)-limit of parabolic rescalings by any sequence \( \lambda_i \to \infty \) of \( (X,(\nu_{x,i,s})_{s \leq t(x,i)}) \), where as above \( \nu_{x,i,s} \) denotes the conjugate heat kernel measures centered at \( x \). We say that \( X \) has a spherical singularity at \( x \), if some (and thus any) tangent flow at \( x \) is a round shrinking sphere. More generally than tangent flows, which are given as rescaling limits around a fixed center \( x \), one can consider rescalings around any sequence of space-time points \( x_i \to x \). A blowup limit at \( x \) is any \( F \)-limit of parabolic rescalings by any sequence \( \lambda_i \to \infty \) of \( (X,(\nu_{x,i,s})_{s \leq t(x,i)}) \). While tangent flows are always self-similarly shrinking as a consequence of Perelman’s monotonicity formula, the analysis of general blowup limits is much more involved. Finally, we say that there is an accumulation of spherical singularities at \( x \in X \), if the flow \( X \) has spherical singularities at a sequence of pairwise distinct space-time points \( x_i \to x \). Using the above notions, we can now state our main result:

Theorem 1.1. Perelman’s ancient ovals occur as blowup limit in a 3-dimensional Ricci flow through singularities if and only if there is an accumulation of spherical singularities.

In essence, we use a similar scheme of proof as in our prior joint work with Hershkovits [10], where we proved a corresponding result for the mean curvature flow. Namely, if there is a sequence of spherical singularities.
We recall from \cite[Definition 3.30 and Definition 4.25]{Bamler2014} that a metric flow \( X \) consists of a set \( \mathcal{X} \), a time-function \( t : \mathcal{X} \to \mathbb{R} \), complete separable metrics \( d_t \) on the time-slices \( \mathcal{X}_t = t^{-1}(t) \), and probability measures \( \nu_{x,t} \in \mathcal{P}(\mathcal{X}_t) \), called conjugate heat kernel measures, such that:

- \( \nu_{x,t(x)} = \delta_x \) for all \( x \in \mathcal{X} \), and for all \( t_1 \leq t_2 \leq t_3 \) in \( I \) and all \( x \in \mathcal{X}_t \), we have the Kolmogorov consistency condition \( \nu_{x,t_1} = \int_{\mathcal{X}_{t_2}} \nu_{x,t_2} \, dv_{x,t_2} \).
- For all \( s < t \) in \( I \), any \( T > 0 \), and any \( T^{-1/2} \)-Lipschitz function \( f_s : \mathcal{X}_s \to \mathbb{R} \), setting \( v_s = \Phi \circ f_s \), where \( \Phi : \mathbb{R} \to (0, 1) \) denotes the antiderivative of \((4\pi)^{-1/2}e^{-s^2/4}\), the function \( v_t : \mathcal{X}_t \to \mathbb{R} \), \( x \mapsto \int_{\mathcal{X}_s} v_s \, dv_{x,s} \) is of the form \( v_t = \Phi \circ f_t \) for some \((t-s+T)^{-1/2}\)-Lipschitz function \( f_t : \mathcal{X}_t \to \mathbb{R} \).

\[ \nu_{x,t(x)}(x) := \int_{\mathcal{X}} v_s \, dv_{x,s}, \quad \mu_s := \int_{\mathcal{X}_t} \nu_{x,t} \, d\mu_t(x) \]  

We recall from \cite[Definition 3.30 and Definition 4.25]{Bamler2014} that a metric flow \( \mathcal{X} \) is called \( H_n \)-concentrated, where \( H_n = (n-1)\pi^2/2 + 4 \), if for all \( s \leq t \) in \( I \) and all \( x_1, x_2 \in \mathcal{X}_t \) we have the variance bound

\[ \text{Var}(\nu_{x_1,s}, \nu_{x_2,s}) \leq d^2(x_1, x_2) + H_n(t-s), \]

and is called future-continuous at \( t_0 \) if for all conjugate heat flows \( (\mu_t)_{t \in I} \) with finite variance and \( t_0 \in I \), the function \( t \mapsto \int_{\mathcal{X}} \int_{\mathcal{X}} d_t \, d\mu_t \, d\mu_t \) is right continuous at \( t_0 \). Throughout this note, we will work with an \( H_3 \)-concentrated future-continuous metric flow \( \mathcal{X} \) that satisfy the partial regularity results from \cite{Bamler2014}, and such that \( \mathcal{L}_{\partial_t} g = -2\text{Ric}(g) \) and \( R(g) \geq -6 \) holds on the regular part.
Let $\mathcal{X}$ be a metric flow as above. Given any rescaling factors $\lambda_i \to \infty$ and space-time points $x_i \to x$, we consider the sequence of parabolically rescaled flows

$$\mathcal{X}^{x_i, \lambda_i} = \left( \mathcal{X}, \lambda_i^2 (t-t(x_i)), (d_{x_i}^{\lambda_i}(t-t(x_i)))_{t \in I}, (\nu_{x_i}^{\lambda_i}(s-t(x_i)))_{x \in \mathcal{X}, s \in I, s \leq t(x_i)} \right),$$

equipped with the parabolically rescaled adjoint heat kernel measures

$$\nu_{x_i}^{\lambda_i} = \nu_{x, t(x_i) + \lambda_i^{-2} s}.$$  

By Bamler’s compactness theory [3], the metric flow pair $(\mathcal{X}^{x_i, \lambda_i}, (\nu_{x_i}^{\lambda_i})_{s \leq 0})$ subsequentially converges to a limit $(\mathcal{X}^{\infty}, (\nu^{\infty})_{s \leq 0})$, called a blowup limit at $x$. A priori the limit is just a metric flow pair obtained in the sense of $\mathbb{F}$-convergence on compact time-intervals within some correspondence, as defined in [3] Section 6]. However, since we are working in dimension 3, by [4] Theorem 2.44] (see also Perelman [17, 18]) all blowup limits are $\kappa$-noncollapsed, have nonnegative curvature, and are smooth with bounded curvature on compact time intervals. Hence, by the local regularity theorem [4] Theorem 2.29] (see also Hein-Naber [15]) the convergence is actually locally smooth. Finally, recall that if $(\mathcal{X}^{x_i, \lambda_i}, (\nu_{x_i}^{\lambda_i})_{s \leq 0})$ converges to a blowup limit $(\mathcal{X}^{\infty}, (\nu^{\infty})_{s \leq 0})$, and if for some sequence of space-time points $\tilde{x}_i$ the sequence of probability measures $\tilde{\nu}^i = (\tilde{\nu}^{x_i, \lambda_i})_{s \leq 0}$ converges to the same limiting measure $\nu^{\infty}$, then by Bamler’s change of base-point theorem [3] Theorem 6.40] the sequence $(\mathcal{X}^{\tilde{x}_i, \lambda_i}, \tilde{\nu}^i)$ also converges to the same limit $(\mathcal{X}^{\infty}, (\nu^{\infty})_{s \leq 0})$.

3. The proofs

To prove Theorem 1.1 we proceed by establishing the following two propositions.

**Proposition 3.1.** If there is a sequence of pairwise distinct spherical singularities $x_i$ converging to a neck or quotient neck singularity at $x \in \mathcal{X}$, then Perelman’s ancient ovals occur as blowup limit at $x$.

Here, by neck or quotient neck singularity at $x$ we mean that some tangent flow at $x$ is either a round shrinking $S^2 \times \mathbb{R}$ or one of its $\mathbb{Z}_2$-quotients, respectively.

**Proof.** Fix a small enough constant $\varepsilon > 0$. Given $x \in \mathcal{X}$, we consider the rescaled and restricted flow

$$\mathcal{X}^x_\varepsilon := \left( \mathcal{X}^{x, 1/r_\alpha}, (\nu^{x, 1/r_\alpha})_{s \in (-\varepsilon^{-1}, 0]} \right)$$
on dyadic scales $r_\alpha = 2^\alpha$, where $\alpha \in \mathbb{Z}$. We say that $\mathcal{X}$ is $\varepsilon$-selfsimilar around $x$ at scale $r_\alpha$, if

$$d_\mathcal{F}(\mathcal{X}_\varepsilon^x, S) < \varepsilon$$

for some metric soliton $S$ that becomes extinct at time zero, where $d_\mathcal{F}$ denotes the $\mathcal{F}$-distance on the time interval $(-\varepsilon^{-1}, 0]$ between metric flow pairs [3] Definition 5.8]. Since we are working in dimension 3, as a consequence of [4] Theorem 2.44] and [18] the only metric solitons are flat $\mathbb{R}^3$, the round shrinking $S^3$, the round shrinking $\mathbb{R} \times S^2$, as well as finite quotients thereof (note also that a lower bound for the Nash entropy, which we always have near any neck singularity, gives an upper bound for the order of the quotient group). In particular, as a consequence of the local regularity theorem [4] Theorem 2.29] we actually have

$$d_{C^{1, \varepsilon/\varepsilon}}(\mathcal{X}_\varepsilon^x, S) < \varepsilon$$

with $\varepsilon(\varepsilon) \to 0$ as $\varepsilon \to 0$, where the $C^{[1/\varepsilon]}$-distance is modulo diffeomorphisms on $B_{1/\varepsilon} \times (-\varepsilon^{-1}, \varepsilon]$.

For any space-time point $x \in \mathcal{X}$ we denote by $S(x)$ the largest spherical scale, i.e. the supremum of $r_\alpha$ such that $\mathcal{X}$ is $\varepsilon$-close around $x$ at scale $r_\alpha$ to a round shrinking sphere, and by $Z(x)$ the infimum of $r_\alpha$ such that $\mathcal{X}$ is $\varepsilon$-close around $x$ at scale $r_\alpha$ to a round shrinking cylinder or to one of its $\mathbb{Z}_2$-quotients.

Now assume $x_i \in \mathcal{X}$ is a sequence of spherical singularities converging to a neck or quotient neck singularity $x \in \mathcal{X}$. Since the flow has a spherical singularity at $x_i$ it clearly holds that $S(x_i) > 0$. On the other hand, recall that $x_i \to x$ by definition means convergence in the natural topology [3] Definition 3.43] and by [3]...
Proposition 3.45] is equivalent to \( t(x_i) \rightarrow t(x) \) and \( \nu_{x_i,s} \rightarrow \nu_{x,s} \) for all \( s < t(x) \) in the \( W_1 \)-Wasserstein distance. Since the flow has a (quotient) neck singularity at \( x \), applying the change of base-point theorem [3, Theorem 6.40], as recalled in the previous section, we thus obtain a sequence of metric flows around \( x_i \) rescaled by \( \lambda_i \rightarrow \infty \) which still converges to a (quotient) neck. This implies

\[
\lim_{i \to \infty} Z(x_i) = 0.
\]

Next, recall that the pointed Nash entropy, which is monotone by [17], is defined as

\[
N_{x_i}(\tau) = \int_{X_i(t(x_i))} f_{x_i}(y, t(x_i) - \tau) \, d\nu_{x_i, t(x_i) - \tau}(y) = \frac{n}{2},
\]

where \( f_x \) is the (almost everywhere defined) potential function for the heat kernel measure, namely

\[
d\nu_{x, t(x) - \tau}(y) = (4\pi\tau)^{-n/2}e^{-f_x(y, t(x) - \tau)} \, d\mathrm{Vol}_{g_{t(x) - \tau}}(y).
\]

By Hamilton’s roundness estimate [13] we have \( S(x_i) \leq Z(x_i) \). On the other hand, we claim that

\[
Z(x_i) \leq CS(x_i).
\]

To see this, recall that by the almost rigidity case of the monotonicity formula from [4, Theorem 2.20] we can find a constant \( \delta > 0 \) such that if \( N_{x_i}(r_{\alpha_1}) - N_{x_i}(r_{\alpha_1 + N[\epsilon^{-1}]}) < \delta \) then \( X \) is \( \epsilon \)-close around \( x_i \) at scale \( r_{\alpha_1} \) to a metric soliton. Recall that since we are working in dimension 3 the only metric solitons are flat \( \mathbb{R}^3 \), the round shrinking \( S^3 \), the round shrinking \( \mathbb{R} \times S^2 \), as well as finite quotients thereof. Now, if \( r_{\alpha_1} = S(x_i) \) is the largest spherical shape, then by monotonicity there must be some \( \beta_i \in \{ \alpha_1 + 1, \ldots, \alpha_i + N[\epsilon^{-1}] \} \), where \( N < \infty \) is a positive integer independent of \( i \), such that \( X \) is \( \epsilon \)-close around \( x_i \) at scale \( r_{\beta_i} \) to a metric soliton. To see this, note since the flow has a (quotient) neck singularity at \( x_i \), by the semicontinuity of the Nash entropy from [4, Proposition 4.37], which is applicable thanks to our assumption that the scalar curvature is bounded below, we get a uniform entropy bound \( N_{x_i}(\tau_0) \geq -Y \) at some fixed scale \( \tau_0 > 0 \). Hence, if there was no \( \beta_i \) such that \( N_{x_i}(r_{\beta_i}) - N_{x_i}(r_{\beta_i + N[\epsilon^{-1}]}) < \delta \), then for \( i \) large enough by monotonicity we would obtain \( N_{x_i}(r_{\alpha_1 + N[\epsilon^{-1}]}) \leq N_{x_i}(r_{\alpha_i}) - N\delta \), which yields a contradiction provided we choose \( N > Y/\delta \). Now, if \( X \) was \( \epsilon \)-close around \( x_i \) at scale \( r_{\beta_i} \) to a round shrinking sphere or a round shrinking nontrivial spherical space form, then we would obtain a contradiction with the definition of \( S(x_i) \) or with Hamilton’s convergence theorem [13], respectively. Hence, \( X \) is \( \epsilon \)-close around \( x_i \) at scale \( r_{\beta_i} \) to a round shrinking \( \mathbb{R} \times S^2 \) or one of its \( \mathbb{Z}_2 \)-quotients. This shows that \( Z(x_i)/S(x_i) \leq 2N[\epsilon^{-1}] \), and thus proves [17].

Now, considering the rescaled flows \( (X_{x_i,1/S(x_i)}, (\nu_{x_i,1/S(x_i)})_{s \leq 0}) \) by Bamler’s compactness theory [3] we can pass to a subsequential limit \( (X^\infty, (\nu^\infty_{x_{imax},s})_{s \leq 0}) \). By [4, Theorem 2.44] (see also [11] [18]) any such 3-dimensional blowup limit \( X^\infty \) is \( \kappa \)-noncollapsed, has nonnegative curvature, and is smooth – with bounded curvature on compact time intervals – until it becomes extinct. Moreover, by construction \( X^\infty \) satisfies \( S(x^\infty) \geq 1 \) and \( Z(x^\infty) \leq C \). Hence, by the recent classification of compact \( \kappa \)-solutions from Brendle-Daskalopoulos-Sesum [8] we conclude that \( X^\infty \) must be an ancient oval. This proves the proposition.

**Proposition 3.2.** If there are only finitely many spherical singularities near \( x \in X \), then Perelman’s ancient ovals do not occur as blowup limit at \( x \).

**Proof.** Suppose towards a contradiction that there are \( x_i \rightarrow x \) and \( \lambda_i \rightarrow \infty \) such that \( (X^{x_i,\lambda_i}, (\nu^{x_i,\lambda_i}_{x_{imax,s}})_{s \leq 0}) \) has as \( F \)-limit an ancient oval, say \( (X^\infty, (\nu^\infty_{x_{imax,s}})_{s \leq 0}) \), that becomes extinct at time zero. Since the time \( -1 \) slices converge smoothly by the local regularity theorem [4, Theorem 2.29], and since the ancient ovals
are compact with positive curvature, it follows that the flow $X$ has a spherical singularity at some nearby space-time point $y_i \in X$ with $y_i \to x$. Now, using the change of base-point theorem [3, Theorem 6.40] we see that $(X^{y_i, \lambda_i}, (\nu^{y_i, \lambda_i}))_{i \leq 0}$ still $\mathcal{F}$-converges to an ancient oval. However, since there are only finitely many spherical singularities, we infer that $y_i = x$ for large $i$, and passing to a subsequential limit we obtain a tangent flow at $x$, which is selfsimilar. This is a contradiction, and thus proves the proposition. □

We can thus conclude that our main theorem holds true:

**Proof of Theorem 1.1.** Fix $x \in X$. Suppose first that there is an accumulation of spherical singularities at $x$. Note that $x$ must be a singular point by the local regularity theorem [4, Theorem 2.29]. Recall that since we are working in dimension 3, as a consequence of [4, Theorem 2.44] and [18] the only nontrivial metric solitons are the round shrinking $S^3$ and the round shrinking $\mathbb{R} \times S^2$, as well as finite quotients thereof. Note that spherical space forms cannot occur as tangent flow at $x$, since round singularities are isolated. Indeed, as a consequence of Hamilton’s classical theorem [13], at any spherical singularity $x$ the tangent flow is unique and becomes extinct in a unique singular point. Hence, if there was a sequence $x_i$ of singular points converging to $x$ in the natural topology, then for $i$ large enough we would have $x_i = x$. Thus, any tangent flow at $x$ must be a round shrinking cylinder or one of its $\mathbb{Z}_2$-quotients, i.e. the flow $X$ has a neck or quotient neck singularity at $x$. Hence, by Proposition 3.2, Perelman’s ancient ovals occur as blowup limit at $x$. Conversely, if there is no accumulation of spherical singularities at $x$, then, by Proposition 3.2, Perelman’s ancient ovals cannot occur as blowup limit at $x$. This proves the theorem. □

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