Co-degree threshold for rainbow perfect matchings in uniform hypergraphs

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Abstract

Let $k$ and $n$ be two integers, with $k \geq 3$, $n \equiv 0 \pmod{k}$, and $n$ sufficiently large. We determine the $(k-1)$-degree threshold for the existence of a rainbow perfect matchings in $n$-vertex $k$-uniform hypergraph. This implies the result of Rödl, Ruciński, and Szemerédi on the $(k-1)$-degree threshold for the existence of perfect matchings in $n$-vertex $k$-uniform hypergraphs. In our proof, we identify the extremal configurations of closeness, and consider whether or not the hypergraph is close to the extremal configuration. In addition, we also develop a novel absorbing device and generalize the absorbing lemma of Rödl, Ruciński, and Szemerédi.

1 Introduction

A hypergraph $H$ consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. Let $H_1$ and $H_2$ be two hypergraphs. If $V(H_1) \subseteq V(H_2)$ and $E(H_1) \subseteq E(H_2)$, then $H_1$ is said to be a subgraph of $H_2$ and we denote this by $H_1 \subseteq H_2$. Let $k$ be a positive integer and write $[k] = \{1, \ldots, k\}$. For a set $S$, let $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. A hypergraph $H$ is $k$-uniform if $E(H) \subseteq \binom{V(H)}{k}$, and a $k$-uniform hypergraph is also called a

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Given a partition \( H \) of \( k \)-graph, let \( T \subseteq V(H) \), let \( H - T \) denote the subgraph of \( H \) with vertex set \( V(H) \setminus T \) and edge set \( \{ e \in E(H) : e \subseteq V(H) \setminus T \} \).

Let \( H \) be a hypergraph and \( S \subseteq V(H) \). The neighborhood of \( S \) in \( H \) is \( N_H(S) = \{ T \subseteq V(H) \setminus S : T \cup S \subseteq E(H) \} \) and the degree of \( S \) in \( H \) is \( d_H(S) = |N_H(S)| \). For any positive integer \( l \), \( \delta_l(H) := \min\{d_H(S) : S \in \binom{V(H)}{l}\} \) is the minimum \( l \)-degree of \( H \). Note that \( \delta_1(H) \) is called the minimum vertex degree of \( H \). If \( H \) is a \( k \)-graph then \( \delta_{k-1}(H) \) is known as the minimum co-degree of \( H \). For a subset \( M \subseteq E(H) \), we let \( V(M) = \bigcup_{e \in M} e \).

A matching in a hypergraph \( H \) is a subset of \( E(H) \) consisting of pairwise disjoint edges, which is perfect if \( V(M) = V(H) \). While a maximum matching in a graph can be found in polynomial time [5], it is NP-hard to find even for 3-graphs [12]. Much effort has been devoted to finding good sufficient conditions for the existence of a large matching in uniform hypergraphs, including Dirac type conditions. A celebrated result in this area has been devoted to finding good sufficient conditions for the existence of a large matching in \( k \)-graphs. For integers \( k, n \), with \( n \geq k \geq 3 \) and \( n \equiv 0 \mod k \), let

\[
t(n, k) := \begin{cases} n/2 + 2 - k, & \text{if } k/2 \text{ is even and } n/k \text{ is odd,} \\ n/2 + 3/2 - k, & \text{if } k \text{ is odd and } (n - 1)/2 \text{ is odd,} \\ n/2 + 1/2 - k, & \text{if } k \text{ is odd and } (n - 1)/2 \text{ is even,} \\ n/2 + 1 - k, & \text{otherwise.} \end{cases}
\]

Rödl, Ruciński, and Szemerédi [28] proved the following result.

**Theorem 1.1 (Rödl, Ruciński, and Szemerédi 2009)** Let \( k, n \) be integers, with \( k \geq 3 \), \( n \equiv 0 \mod k \), and \( n \) sufficiently large. Let \( H \) be a \( k \)-graph on \( n \) vertices such that \( \delta_{k-1}(H) > t(n, k) \). Then \( H \) has a perfect matching.

Codegree condition \( \delta_{k-1}(H) > t(n, k) \) is best possible because of the following \( k \)-graphs \( H(n, k) \) on vertex set \( [n] \) from [20] (for odd \( k \)) and [28] (for even \( k \)): When \( k \) is odd, \( [n] \) has a partition \( A, B \) such that \( |A| \) is the unique odd integer from the set \( \{ n/2, n/2 - 1, n/2 - k \} \) and \( E(H(n, k)) = \{ e \in \binom{[n]}{k} : |e \cap A| \equiv 0 \mod 2 \} \). When \( k \) is even, \( V(H(n, k)) \) has a partition \( A, B \) such that

\[
|A| := \begin{cases} n/2 - 1, & \text{if } n/k \text{ is even,} \\ n/2 - 1, & \text{if } n/k \text{ is odd and } n/2 \text{ is odd,} \\ n/2, & \text{if } n/k \text{ is odd and } n/2 \text{ is even,} \end{cases}
\]

and \( E(H(n, k)) = \{ e \in \binom{[n]}{k} : |e \cap A| \equiv 1 \mod 2 \} \). Note that the sets \( A, B \) are called partition classes of \( H(n, k) \).

Let \( \mathcal{F} = \{ F_1, \ldots, F_t \} \) be a family of hypergraphs; a set of pairwise disjoint edges, one from each \( F_i \), is called a rainbow matching for \( \mathcal{F} \). In this case, we also say that \( \mathcal{F} \) admits a rainbow matching. There has been effort to extend results on matchings in hypergraphs to rainbow matchings, see for instance, [2][6][7][10][16][19][22][24][26]. The main result in this paper is a rainbow version of Theorem 1.1.
Theorem 1.2 Let \( k, n \) be integers with \( k \geq 3 \), \( n \equiv 0 \pmod{k} \), and \( n \) sufficiently large. Let \( \{F_1, \ldots, F_{n/k}\} \) be a family of \( k \)-graphs on the common vertex set \([n]\), such that \( \delta_{k-1}(F_i) > t(n,k) \) for \( i \in [n/k] \). Then \( \{F_1, \ldots, F_{n/k}\} \) admits a rainbow perfect matching.

It is easy to see that we derive Theorem 1.1 from Theorem 1.2 by setting \( F_1 = \ldots = F_{n/k} = H \). Moreover, if \( F_i = H(n,k) \) for \( i \in [n/k] \) then \( \{F_1, \ldots, F_{n/k}\} \) admits no rainbow perfect matching. So the co-degree bound in Theorem 1.2 is best possible. We point out that Theorem 1.2 for \( k = 2 \) is a result of Joos and Kim [13] and Akiyama and Frankl [1].

For \( n \equiv 0 \pmod{k} \), let \( \{F_1, \ldots, F_{n/k}\} \) be a family of \( k \)-graphs on the same vertex set \([n]\). Let \( X = \{x_1, \ldots, x_{n/k}\} \) be disjoint from \([n]\). We consider the hypergraph \( F(n,k) \) with vertex set \( X \cup [n] \) and edge set \( \bigcup_{i=1}^{n/k} \{\{x_i\} \cup e : e \in E(F_i)\} \). We denote this hypergraph by \( H(n,k) \) when \( F_i = H(n,k) \) for \( i \in [n/k] \) with same partition classes \( A, B \) of \([n]\), and refer to \( \mathcal{H}(n,k) \) as extremal configuration. It is easy to see the following is true.

Observation. \( \delta_{k-1}(F_i) > t(n,k) \) for \( i \in [n/k] \) implies that \( d_{F(n,k)}(S) > t(n,k) \) for any \( S \in \binom{V(F(n,k))}{k} \) with \( |S \cap X| = 1 \). \( \{F_1, \ldots, F_{n/k}\} \) admits a rainbow matching if, and only if, \( F(n,k) \) has a perfect matching.

So we will show that \( F(n,k) \) has a perfect matching. Indeed, we consider a more general class of hypergraphs. Let \( Q, V \) be two disjoint sets. A \((k + 1)\)-graph \( H \) with vertex set \( Q \cup V \) is said to be \((1,k)\)-partite with partition classes \( Q, V \) if, for each edge \( e \in E(H) \), \(|e \cap Q| = 1 \) and \(|e \cap V| = k \). A \((1,k)\)-partite \((k + 1)\)-graph \( H \) with partition classes \( Q, V \) is balanced if \( |V| = k|Q| \). We say that a subset \( S \subseteq V(H) \) is balanced if \( |S \cap V| = k|S \cap Q| \).

Theorem 1.3 Let \( k, n \) be integers with \( k \geq 3 \), \( n \equiv 0 \pmod{k} \), and \( n \) sufficiently large. Let \( F \) be a balanced \((1,k)\)-partite \((k + 1)\)-graph with partition classes \( X, [n] \), such that for any \( S \in \binom{V(F)}{k} \) with \( |S \cap X| = 1 \), \( d_F(S) > t(n,k) \). Then \( F \) admits a perfect matching.

To prove Theorem 1.3 we consider whether or not \( F \) is “close” to the extremal configuration \( \mathcal{H}(n,k) \). In Section 2, we describe several properties of the extremal configurations. In Section 3, we prove Theorem 1.3 for the case when \( F \) is close to the extremal configuration. In Section 4, we study absorbing devices for perfect matchings, and in Section 5, we study an absorbing device for near perfect matchings. We deal with the case when \( F \) is not close to the extremal configuration in Section 6 and offer some concluding remarks in Section 7.

2 Properties of Extremal configurations

We will often use the following \((1,k)\)-partite \((k + 1)\)-graphs as intermediate configuration to compare \((1,k)\)-partite \((k + 1)\)-graphs with \( \mathcal{H}(n,k) \). Suppose \( W, U \) form a partition of \([n]\) such that \(|W| = (1/2 \pm o(1))n \) and \(|U| = (1/2 \pm o(1))n \). For \( i \in \{0, 1\} \), let \( H^i_{n,k}(W, U) \) denote the \( k \)-graph with vertex set \([n]\) and edge set \( \{S \in \binom{[n]}{k} : |S \cap W| = i \pmod{2} \} \). When \(|W| = \lfloor n/2 \rfloor \), we denote \( H^i_{n,k}(W, U) \) by \( H^i_{n,k} \).

We need the following definition to quantify the difference between \( F(n,k) \) and \( \mathcal{H}(n,k) \). Let \( \varepsilon > 0 \) be a real number. Given two \( k \)-graphs \( H_1, H_2 \) with \( V(H_1) = V(H_2) \), we say that
$H_2$ is strongly $\varepsilon$-close to $H_1$ if $|E(H_1) \setminus E(H_2)| \leq \varepsilon|V(H_1)|^k$. We say that $H_2$ is weakly $\varepsilon$-close to $H_1$ if $\varepsilon(H_1, H_2) \leq \varepsilon|V(H_1)|^k$, where $\varepsilon(H_1, H_2)$ be the minimum of $|E(H_1) \setminus E(H_2)|$ taken over all isomorphic copies $H_2'$ of $H_2$ with $V(H_2') = V(H_2)$. It is easy to see that the following is true.

**Lemma 2.1** Let $\varepsilon > 0$ be a real number. Let $k, n$ be integers with $k \geq 3$, $n \equiv 0 \pmod{k}$ and $n$ is sufficiently large. Let $W, U$ be a partition of $[n]$ with $|W| = (1/2 \pm o(1))n$ and $|U| = (1/2 \pm o(1))n$. Then the following statements hold.

(i) If $k$ is odd then $H_{n,k}^i(U, W)$ and $H_{n,k}^j(W, U)$, $i, j \in \{0, 1\}$, are weakly $\varepsilon$-close to each other.

(ii) If $k$ is even then $H_{n,k}^i(U, W) = H_{n,k}^j(W, U)$ for $i \in \{0, 1\}$.

Let $k, n$ be integers with $k \geq 3$ and $n \equiv 0 \pmod{k}$. Let $m_0, m_1$ be integers between 0 and $n/k$ (inclusive). For convenience, define $H_{n,k}^{m_0,m_1}(W, U)$ as the $(1, k)$-partite $(k+1)$-graph $\mathcal{H}$ with partition classes $X$ and $[n]$ and a partition $W, U$ of $[n]$, such that $|X| = n/k$ and $\{|x \in X : N_\mathcal{H}(x) = H_{n,k}^i(W, U)| = m_i \text{ for } i \in \{0, 1\}\}$. For $i \in \{0, 1\}$, if $m_i = m$ and $m_0 + m_1 = n/k$ then we denote $H_{n,k}^{m_0,m_1}(W, U)$ by $H_{n,k}^i(W, U; m)$.

In the remainder of this section, we study $(1, k)$-partite $(k+1)$-graphs that $\mathcal{F}$ are close to some $\mathcal{H}(n, k)$, and consider those vertices in $\mathcal{F}$ that is contained in lots of edges of $\mathcal{H}(n, k)$. So we introduce the following concept. Let $k, n$ be integers with $k \geq 3$, $n \equiv 0 \pmod{k}$, and $n$ sufficiently large. Let $\mathcal{F}$ and $\mathcal{H}$ be $(1, k)$-partite $(k+1)$-graphs with partition classes $X, [n]$. A vertex $v$ of $\mathcal{F}$ is said to be $\alpha$-good with respect to $\mathcal{H}$ if $|N_{\mathcal{H}}(v) \setminus N_{\mathcal{F}}(v)| < \alpha|V(\mathcal{F})|^k$. Otherwise, $v$ is said to be $\alpha$-bad with respect to $\mathcal{H}$.

The following lemma shows that the number of bad vertices in $\mathcal{F}$ is small if $\mathcal{F}$ is close to $H_{n,k}^i(W, U; m)$ for some $i \in \{0, 1\}$.

**Lemma 2.2** Let $k, n$ be integers with $k \geq 3$ and $n \equiv 0 \pmod{k}$, and let $\varepsilon$ be a constant such that $0 < 1/n \ll \varepsilon \ll 1/k$. Let $\mathcal{F}$ be a $(1, k)$-partite $(k+1)$-graph with partition classes $X$ and $[n]$ where $|X| = n/k$. Let $0 \leq m \leq n/k$ be an integer. If $\mathcal{F}$ is strongly $\varepsilon$-close to some $H_{n,k}^{m}(W, U; m)$, where $|W| = n/2 \pm o(n)$ and $|U| = n/2 \pm o(n)$, then the number of $\varepsilon^{2/3}$-bad vertices in $\mathcal{F}$ with respect to $H_{n,k}^i(W, U; m)$ is at most $(1 + 1/k)(k + 1)^{3/2}n$.

**Proof.** Let $N$ be the set of $\varepsilon^{2/3}$-bad vertices in $\mathcal{F}$ with respect to $H_{n,k}^i(W, U; m)$. If $|N| > ((1 + 1/k)(k + 1)^{3/2}n \text{ then} \quad |E(H_{n,k}^i(W, U; m)) \setminus E(\mathcal{F})| > \frac{1}{k+1} |N|^{\varepsilon^{2/3}}|V(\mathcal{F})|^k \geq \varepsilon \left(1 + \frac{n}{k}\right)^{k+1}.

This contradicts the assumption that $\mathcal{F}$ is strongly $\varepsilon$-close to $H_{n,k}^i(W, U; m)$.\hfill \blacksquare

The next lemma says that we can find an edge in $\mathcal{F}$ which serves as “parity breaker”. This is the only place in the proof of Theorem 1.3 where we require $\delta_{k-1}(N_{\mathcal{F}}(x)) > t(n, k)$ for all $x \in X$.\hfill 4
Lemma 2.3 Let $k, n$ be integers with $k \geq 3$, $n \equiv 0 \pmod{k}$, and $n \geq 2k$. Let $F$ be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$, such that $\delta_{k-1}(N_F(x)) > t(n,k)$ for all $x \in X$. Then for any proper subset $X'$ of $X$ and any partition $W, U$ of $[n]$ with $\min\{|W|, |U|\} \geq k$, there exists $e_0$ with $e_0 \in E(F)$ or $e_0 = \emptyset$, such that, for $i \in \{0,1\}$,

(i) if $i = 0$ then $|W \setminus e_0| \equiv |X' \setminus e_0| \pmod{2}$;

(ii) if $i = 1$ then $|W \setminus e_0| \equiv |X \setminus (X' \cup e_0)| \pmod{2}$;

(iii) if $k - i$ is even then $|U \setminus e_0| \equiv |X' \setminus e_0| \pmod{2}$;

(iv) if $k - i$ is odd then $|U \setminus e_0| \equiv |X \setminus (X' \cup e_0)| \pmod{2}$.

Proof. Let $q := |X'|$ and fix $x \in X \setminus X'$. Then $\delta_{k-1}(N_F(x)) \geq t(n,k) + 1$. In all cases below, we may assume the assertion of this lemma does not hold for $e_0 = \emptyset$.

Case 1: $i = 0$ and $k - i = k$ is even.

Then (i) and (iii) are relevant and not both true for $e_0 = \emptyset$. Hence, $|W| \not\equiv q \pmod{2}$ or $|U| \not\equiv q \pmod{2}$. Therefore, since $n \equiv 0 \pmod{2}$ (as $n \equiv 0 \pmod{k}$ and $k$ is even), $|W| \not\equiv q \pmod{2}$, $|U| \not\equiv q \pmod{2}$ and $|W| \equiv |U| \pmod{2}$.

If there exists $e_0 \in E(F)$ with $x \in e_0$ such that $|e_0 \cap W| = 1$ or $|e_0 \cap U| = 1$, then both (i) and (iii) holds with this $e_0$. So we may assume that, for every $e \in E(F)$ with $x \in e$, $|e \cap W| \not\equiv 1$ and $|e \cap U| \not\equiv 1$.

Then for any $(k-1)$-set $S \subseteq U$, $N_{N_F(x)}(S) \subseteq U \setminus S$. Thus, $|U \setminus S| \geq |N_{N_F(x)}(S)| \geq t(n,k) + 1 \geq n/2 + 2 - k$. Thus $|U| \geq n/2 + 1$.

Similarly, we derive $|W| \geq n/2 + 1$. This leads to a contradiction as $n = |U| + |W|$.

Case 2: $i = 0$ and $k - i = k$ is odd.

Then (i) and (iv) are relevant and not both true for $e_0 = \emptyset$. So $|W| \not\equiv q \pmod{2}$ or $|U| \not\equiv n/k - q \pmod{2}$. Note that $n/k - q \equiv n - q \pmod{2}$. Thus since $|U| + |W| = n$, we have $|U| \not\equiv n - q \pmod{2}$ and $|W| \not\equiv q \pmod{2}$.

Subcase 2.1: $n$ is even; so $n/k$ is even.

If there exists $e_0 \in F$ such that $x \in e_0$ and $|e_0 \cap W| = 1$ or $|e_0 \cap W| = k - 2$ then (i) and (iv) holds for this $e_0$. So assume that for any $e \in E(F)$ with $x \in e$, $|e \cap W| \not\equiv 1$ and $|e \cap W| \not\equiv k - 2$.

Then for any $(k-1)$-subset $S \subseteq U$, $N_{N_F(x)}(S) \subseteq U \setminus S$. Thus, $|U \setminus S| \geq |N_{N_F(x)}(S)| \geq t(n,k) + 1 \geq n/2 + 2 - k$; so $|U| \geq n/2 + 1$.

Similarly, for any $(k-1)$-subset $T \subseteq [n]$ with $|T \cap W| = k - 2$, $N_{N_F(x)}(T) \subseteq W \setminus T$. Thus, $|W \setminus T| \geq |N_{N_F(x)}(T)| \geq t(n,k) + 1 \geq n/2 + 2 - k$; so $|W| \geq n/2$. However, this is a contradiction as $n = |W| + |U| \geq n + 1$.

Subcase 2.2. $n$ is odd; so $n/k$ is odd.

Suppose $q \equiv 0 \pmod{2}$. If there exists $e_0 \in E(F)$ such that $x \in e_0$ and $|e_0 \cap W| = 1$ or $|e_0 \cap W| = k - 2$ then (i) and (iv) hold for this $e_0$. So assume that for any $e \in E(F)$ with $x \in e$, $|e \cap W| \not\equiv 1$ and $|e \cap W| \not\equiv k - 2$. Then, for any $(k-1)$-subset $S \subseteq U$, $N_{N_F(x)}(S) \subseteq U \setminus S$ which implies $|U \setminus S| \geq |N_{N_F(x)}(S)| \geq t(n,k) + 1 \geq n/2 + 3/2 - k$; so $|U| \geq (n + 1)/2$.

Similarly, for any $(k-1)$-subset $T \subseteq [n]$ with $|T \cap W| = k - 2$, $N_{N_F(x)}(T) \subseteq W \setminus T$ and, hence, $|W \setminus T| \geq |N_{N_F(x)}(T)| \geq t(n,k) + 1 \geq n/2 + 3/2 - k$; so $|W| \geq (n - 1)/2$ and the
equality holds only when \((n - 1)/2\) is even. Thus since \(n = |W| + |U|\), \(|U| = (n + 1)/2\) and \(|W| = (n - 1)/2 \equiv 0 \pmod{2}\), which leads to a contradiction since \(|W| \not\equiv q \pmod{2}\) and \(q \equiv 0 \pmod{2}\).

Now assume \(q \equiv 1 \pmod{2}\). Then we have \(X' \neq \emptyset\), and fix \(x' \in X'\). If there exists an edge \(e_0 \in E(F)\) such that \(x' \in e_0\) and \(|e_0 \cap W| = 2\) or \(|e_0 \cap W| = k - 1\), then (i) and (iv) hold with this \(e_0\). So we may assume that for any \(e \in E(F)\) with \(x' \in e, \{e \cap W\} \neq 2\) and \(|e \cap W| \neq k - 1\). Then for any \((k - 1)\)-subset \(T \subseteq W, N_{N_{x'}}(T) \subseteq W \setminus T\) and, hence, \(|W \setminus T| \geq |N_{N_{x'}}(T)| \geq t(n, k) + 1 \geq n/2 + 3/2 - k\); so \(|W| \geq (n + 1)/2\). Similarly, for any \((k - 1)\)-subset \(S \subseteq U\) with \(|S \cap U| = k - 2, N_{N_{x'}}(S) \subseteq U \setminus S\) and, hence, \(|U \setminus S| \geq |N_{N_{x'}}(S)| \geq t(n, k) + 1 \geq n/2 + 3/2 - k\); so \(|U| \geq (n - 1)/2\) and the equality holds only when \((n - 1)/2\) is even. Thus since \(n = |W| + |U|, |U| = (n - 1)/2 \equiv 0 \pmod{2}\) and \(|W| = (n + 1)/2\), which leads to a contradiction since \(|U| \not\equiv n - q \pmod{2}\) and \(n - q \equiv 0 \pmod{2}\).

**Case 3:** \(i = 1\) and \(k - i = k - 1\) is odd.

So (ii) and (iv) are relevant and not both true for \(e_0 = \emptyset\). Thus, \(|W| \not\equiv n/k - q \pmod{2}\) or \(|U| \not\equiv n/k - q \pmod{2}\). Note that \(n\) is even as \(k\) is even. Since \(n = |W| + |U|, |W| \not\equiv n/k - q \pmod{2}\), \(|U| \not\equiv n/k - q \pmod{2}\) and \(|W| \equiv |U| \pmod{2}\).

Suppose \(q \equiv 1 \pmod{2}\). Then \(X' \neq \emptyset, \) and fix \(x' \in X'\). If there exists \(e_0 \in E(F)\) such that \(x' \in e_0\), and \(|e_0 \cap W| = 1\) or \(|e_0 \cap W| = k - 1\), then (ii) and (iv) holds for this \(e_0\), in this case, \(X \setminus (X' \cup e_0) = X \setminus X'\). So assume that, for any \(e \in E(F)\) with \(x' \in e, \{e \cap W\} \neq 1\) and \(|e \cap W| \neq k - 1\). Hence, for any \((k - 1)\)-subset \(S \subseteq [n]\) with \(S \subseteq W, N_{N_{x'}}(S) \subseteq W \setminus S\) and, hence, \(|W \setminus S| \geq |N_{N_{x'}}(S)| \geq t(n, k) + 1 \geq n/2 + 2 - k\); so \(|W| \geq n/2 + 1\). Similarly, \(|U| \geq n/2 + 1\). This is a contradiction since \(n = |W| + |U| \geq n + 2\).

So \(q \equiv 0 \pmod{2}\). If there exists \(e_0 \in E(F)\) with \(x \in e_0\) such that \(|e_0 \cap W| = 2\) or \(|e_0 \cap U| = 2\) then (ii) and (iv) holds with this \(e_0\). Hence, we may assume that, for any \(e \in E(F)\) with \(x \in e, \{e \cap W\} \neq 2\) and \(|e \cap U| \neq 2\).

Hence, for any \((k - 1)\)-subset \(S \subseteq [n]\) with \(|W \cap S| = k - 2, N_{N_{x'}}(S) \subseteq W \setminus S\) and, hence, \(|W \setminus S| \geq |N_{N_{x'}}(S)| \geq t(n, k) + 1 \geq n/2 + 2 - k\). So \(|W| \geq n/2\) and equality holds only when \(n/k\) is even or when \(n/k\) is odd and \(k/2\) is odd.

Similarly, \(|U| \geq n/2\) and equality holds only when \(n/k\) is even or when \(n/k\) is odd and \(k/2\) is odd. Since \(n = |W| + |U|, |W| = |U| = n/2\). If \(n/k\) is even then \(|W| = n/2\) is even (as \(k\) is even); however, \(|W|\) is odd since \(|W| \not\equiv n/k - q \pmod{2}\), a contradiction. If \(n/k\) is odd then \(k/2\) is odd and, hence, \(|W| = n/2\) is odd. However, \(|W|\) is even since \(|W| \not\equiv n/k - q \pmod{2}\), a contradiction.

**Case 4:** \(i = 1\) and \(k - i = k - 1\) is even.

Then (ii) and (iii) are relevant, and not both true for \(e_0 = \emptyset\). Hence, \(|W| \not\equiv n/k - q \pmod{2}\) or \(|U| \not\equiv q \pmod{2}\). Note that \(n = |W| + |U|\) and \(n/k - q \equiv n - q \pmod{2}\). So \(|W| \not\equiv n - q \pmod{2}\) and \(|U| \not\equiv q \pmod{2}\).

**Subcase 4.1:** \(n\) is even.

If there exists \(e_0 \in E(F)\) with \(x \in e_0\) such that \(|e_0 \cap U| = 1\) or \(|e_0 \cap U| = k - 2\), then (ii) and (iii) holds for this \(e_0\). So assume that for any \(e \in E(F)\) with \(x \in e, \{e \cap U\} \neq 1\) and \(|e \cap U| \neq k - 2\).

Then, for any \((k - 1)\)-subset \(S \subseteq W, N_{N_{x'}}(S) \subseteq W \setminus S\) and, hence, \(|W \setminus S| \geq |N_{N_{x'}}(S)| \geq t(n, k) + 1 \geq n/2 + 2 - k\); so \(|W| \geq n/2 + 1\). Similarly, for any \((k - 1)\)-subset
Let $E$ be a $k$-graph. For $j \in \{0,1\}, v \in V(H)$, and $S \subseteq V(H)$, we define

$$d^{j}_{H,S}(v) := |\{e \in E(H) : v \in e \text{ and } |e \cap S| \equiv j \pmod{2}\}|.$$

We begin with a lemma that allows us to find a matching in the hypergraphs in question covering any small fixed set of vertices. For convenience, we set the following parameters for a given integer $k$ for the remainder of this section: $\eta = 1/(4k!)$, $c = 1/(8(k + 1)!)$, $\varepsilon = 1/(80k^{5}k^{-5}(k!)^{3/2})$, and $\gamma = \frac{2^{3/2}k^{3}}{e^{2}k^{2}}$.

**Lemma 3.1** Let $k, n$ be integers with $k \geq 3$, $n \equiv 0 \pmod{2}$, and $n \gg 1/\varepsilon$. Let $\mathcal{F}$ be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X,[n]$, and assume $\delta_{k-1}(\mathcal{F}(x)) > t(n,k)$ for all $x \in X$. Let $W,U$ be a partition of $[n]$ such that $|\min\{W,|U|\}| \geq k$, and let $X_{0},X_{1}$ be a partition of $X$ such that $d^{j}_{\mathcal{F},W}(x) \geq \eta n^{k}$ for $j \in \{0,1\}$ and $x \in X_{j}$. Suppose there exists $i \in \{0,1\}$ such that $d^{i}_{\mathcal{F},X_{i-1}\setminus U}(v) \geq \eta n^{k}$ for all $v \in [n]$.

Then for any $e_{0}$ satisfying the conclusion of Lemma 2.4 and for any $N \subseteq X \cup [n]$ with $|N| \leq 2cn$, there exists a matching $M$ in $\mathcal{F}$ such that $N \subseteq V(M)$, $e_{0} \in M$ when $e_{0} \neq \emptyset$, $|V(M)| \leq (k + 1)2cn$, and $|W \setminus V(M)| \equiv |X_{1} \setminus V(M)| \pmod{2}$.

**Proof.** Let $N_{0} = N \cap X_{0}$ and $N_{1} = N \cap X_{1}$. Let $e_{0}$ be the edge satisfying the conclusion of Lemma 2.4. If $N \setminus e_{0} = \emptyset$ then let $M := \emptyset$ when $e_{0} = \emptyset$, and $M = \{e_{0}\}$ when $e_{0} \neq \emptyset$. So
assume $N \setminus e_0 \neq \emptyset$. Divide $N \setminus e_0$ into three pairwise disjoint sets: $N_0 \setminus e_0 = \{v_1, \ldots, v_r\}$, $N_1 \setminus e_0 = \{v_{r+1}, \ldots, v_s\}$, and $(N \setminus e_0) \cap [n] = \{v_{s+1}, \ldots, v_t\}$. We find the desired matching $M$ by covering the vertices in $N \setminus e_0$ greedily.

Suppose we have found the matching $M_u := \{e_0, e_1, \ldots, e_u\}$ for some $u \geq 0$ such that for $1 \leq j \leq u$, $\{v_j\} = e_j \cap N$ and for $1 \leq i \leq \min\{u, r\}$, $|e_j \cap W| \equiv 0 \pmod{2}$, for $r + 1 \leq j \leq \min\{u, s\}$, $|e_j \cap W| \equiv 1 \pmod{2}$, and for $s + 1 \leq j \leq \min\{u, t\}$, $|e_j \cap W| \equiv i \pmod{2}$. Now suppose a vertex from $\cup_{i=0}^u e_i \cup N$ is less than $\eta n$.

By assumption, $d^0_F W(v) \geq \eta n v$ for $v \in N_0$, $d^1_F W(v) \geq \eta n$ for $v \in N_1$ and $d^2_F X_{i-1} W(v) \geq \eta n$ for $v \in (N \setminus e_0) \cap [n]$. Thus there exists an edge $e_{u+1}$ in $F - \cup_{i=0}^u e_i$ such that $\{v_{u+1}\} = e_{u+1} \cap N$, $|e_{u+1} \cap W| \equiv 0 \pmod{2}$ when $u + 1 \leq r$, $|e_{u+1} \cap W| \equiv 1 \pmod{2}$ when $r < u + 1 \leq s$, and $|e_{u+1} \cap W| \equiv i \pmod{2}$ and $e_j \cap X \subseteq X_i$ when $s + 1 < u + 1 \leq t$. Continuing this process for at most $|N \setminus e_0| \leq 2$ steps, we obtain the desired matching $M$.

Let $H(W, U; r)$ denote the balanced $(1, k)$-partite $(k+1)$-graph with partition classes $X, [n]$ and a partition $W, U$ of $[n]$ such that for every $x \in X$, $N_{H(W, U; r)}(x) = \{e \in {n \choose 2} : |e \cap W| = r\}$. Now we show that if all the vertices of a $(1, k)$-partite $(k+1)$-graph $F$ are good with respect to $H(W, U; r)$, then there exists a perfect matching in $F$ consisting of edges intersecting $W$ exactly $r$ times.

**Lemma 3.2** Let $k, n, r$ be integers such that $k \geq 3$, $0 \leq r \leq k$, $n \equiv 0 \pmod{k}$, and $1/n \ll 1/k$. Let $F$ be a balanced $(1, k)$-partite $(k+1)$-graph with partition classes $X, [n]$, and let $W, U$ be a partition of $[n]$ with $|W| = rn/k$. Suppose all vertices in $F$ are $\gamma$-good with respect to $H(W, U; r)$. Then there exists a perfect matching $M$ in $F$ such that $|e \cap W| = r$ for all $e \in M$.

**Proof.** By symmetry between $W$ and $U$, we may assume $r \geq k/2$. Let $M$ be a maximum matching in $F$ such that, for every $e \in M$, $|e \cap W| = r$. Let $W_0 := W \setminus V(M)$ and $U_0 := U \setminus V(M)$. Since $r \geq k/2$, $|W_0| \geq |U_0|$.

Suppose $|M| < \frac{n}{4k}$. Then $|W_0| \geq \frac{n}{4k}$. By maximality of $M$, for $v \in W_0$, we have

$$|N_{H(W, U; r)}(v) \setminus N_F(v)| \geq |X \setminus V(M)| \left(\frac{|W_0|}{r - 1}\right) \left(\frac{|U_0|}{r - 1}\right).$$

Thus, if $r = k$ then

$$|N_{H(W, U; r)}(v) \setminus N_F(v)| \geq \frac{3n}{4k} \left(\frac{|W_0|}{k - 1}\right) \geq \gamma(n + n/k)^k.$$

Now suppose $r \leq k - 1$. Then

$$|U_0| = \frac{k - r}{r} |W_0| \geq \frac{k - r}{r} (n/4) \geq \frac{n}{4(k - 1)}.$$
So we have
\[ |N_{H(W,U;r)}(v) \setminus N_{\mathcal{F}}(v)| \geq \frac{3n}{4k}(|W_0| - r + 2)^{r-1}(|U_0| - k + r + 1)^{k-r}/k! \]
\[ \geq \gamma(n + n/k)^k \]
contradicting the fact that \( v \) is \( \gamma \)-good with \( \mathcal{H}(W,U;r) \).

Now, suppose for a contradiction that \( M \) is not a perfect matching. There exist \( x_{k+1} \in X \setminus V(M), \) distinct \( v_{k+1,1}, \ldots, v_{k+1,r} \in W_0, \) and distinct \( v_{k+1,r+1}, \ldots, v_{k+1,k} \in U_0. \)

Let \( \{e_1, e_2, \ldots, e_k\} \in \binom{M}{k} \) and write \( e_i := \{x_i, v_{i,1}, v_{i,1}, \ldots, v_{i,k}\} \), such that, for \( i \in [k], \) \( x_i \in X, \) \( v_{i,j} \in W \) for \( j \in [r], \) and \( v_{i,j} \in U \) for \( j \in [k] \setminus [r] \). For \( j \in [k+1], \) let \( f_j := \{x_j, v_{j+1,1}, v_{j+1,2}, \ldots, v_{j+1+k,k}\} \), with the addition in the subscripts modulo \( k+1 \) (except we write \( k+1 \) for \( 0 \)). Note that \( f_1, \ldots, f_{k+1} \) are pairwise disjoint and \( |f_j \cap W| = r \) for \( j \in [k+1]. \)

If \( f_j \in E(\mathcal{F}) \) for all \( j \in [k+1], \) then \( M' := (M \cup \{f_1, \ldots, f_{k+1}\}) \setminus \{e_1, \ldots, e_k\} \) is matching in \( \mathcal{F}, \) contradicting the maximality of \( |M| \). Hence, \( f_j \not\in E(\mathcal{F}) \) for some \( j \in [k+1]. \) Note that there are \( \binom{|M|}{k} \) choices of \( \{e_1, \ldots, e_k\} \subseteq M. \) Thus we have
\[
|\{e \in E(H(W,U;r)) \setminus E(\mathcal{F}) : e \cap \{v_{k+1,i} : i \in [k]\} = 1\}| \\
\geq \binom{|M|}{k} \\
\geq \left(\frac{n}{4k} - k + 1\right)^k / k! \\
\geq \frac{1}{k!(5k)^k} n^k \quad \text{(since } n \geq 20k^2) \\
> \gamma(k+1)(n + n/k)^k
\]
This implies that there exists \( x \in \{x_{k+1}, v_{k+1,1}, \ldots, v_{k+1,k}\} \) such that
\[ |N_{H(W,U;r)}(x) \setminus N_{\mathcal{F}}(x)| > \gamma(n + n/k)^k. \]
That is, \( x \) is not \( \gamma \)-good with respect to \( H(W,U;r), \) a contradiction.

After obtaining the matching \( M \) in \( \mathcal{F} \) from Lemma 3.2, we need to find a perfect matching in \( \mathcal{F} - V(M) \) to conclude the proof of Theorem 1.3 for the case when all vertices of \( \mathcal{F} - V(M) \) are good. The following lemma serves this purpose.

**Lemma 3.3** Let \( k, n \) be integers such that \( k \geq 3, n \equiv 0 \pmod{k}, \) and \( n \gg 1/\varepsilon. \) Let \( \mathcal{F} \) be a balanced \((1,k)\)-partite \((k+1)\)-graph with partition classes \( X, [n]. \) Let \( W', U' \) be a partition of \([n]\) with \( \min\{|U'|,|W'|\} \geq 1.1n/k + k, \) and let \( i \in \{0,1\} \) such that
\[
(i) \text{ if } i = 0 \text{ then } |W'| \equiv 0 \pmod{2}, \\
(ii) \text{ if } i = 1 \text{ then } |W'| \equiv n/k \pmod{2}, \\
(iii) \text{ if } k - i \equiv 0 \pmod{2} \text{ then } |U'| \equiv 0 \pmod{2}, \) and
\[ (iv) \text{ if } k - i \equiv 1 \pmod{2} \text{ then } |U'| \equiv n/k \pmod{2}. \]
If all vertices of $F$ are $\gamma$-good with respect to $\mathcal{H}_{n,k}^0(W',U')$, then there is a perfect matching in $F$.

**Proof.** First, suppose there exists an edge $e_1 \in E(F)$ such that $|W' \setminus e_1| = r_1x + r_2y$ and $|U' \setminus e_1| = (k - r_1)x + (k - r_2)y$, where $x, y, r_1, r_2$ are integers satisfying $x, y > 20k^2$ and $0 \leq r_1, r_2 \leq k$. We partition $W' \setminus e_1$ to $W_1, W_2$ such that $|W_1| = r_1x$ and $|W_2| = r_2y$, partition $U' \setminus e_1$ to $U_1, U_2$ such that $|U_1| = (k - r_1)x$ and $|U_2| = (k - r_2)y$, and partition $X \setminus e_1$ to $X_1, X_2$ such that $|X_1| = x$ and $|X_2| = y$. By assumption, for $i \in \{0, 1\}$, $0 \leq r_i \leq k$.

Hence, by Lemma 3.2, $\mathcal{H}_{n_k}^0(W',U')$ has a perfect matching, say $M_i$, consisting of edges containing exactly $r_i$ vertices from $W_i$. Now $M_1 \cup M_2$ (when $e_1 = 0$) or $M_1 \cup M_2 \cup \{e_1\}$ gives the desired matching.

Therefore, it suffices to prove the existence of such $e_1$. This is done in the following four cases.

**Case 1.** $i = 0$ and $k - i = k$ is even.

Then (i) and (iii) hold. So $|W'|$ and $|U'|$ are both even. Then, since all vertices of $F$ are $\gamma$-good with respect to $\mathcal{H}_{n,k}^0(W',U')$, there is an edge $e_1 \in F$ such that $|e_1 \cap W'| \equiv |W'| \pmod{k}$. Thus, $|W' \setminus e_1| \equiv 0 \pmod{k}$.

Since $n \equiv 0 \pmod{k}$, we have $|e_1 \cap U'| = k - |e_1 \cap W'| \equiv k - |W'| \pmod{k}$; hence, $|e_1 \cap U'| \equiv |U'| \pmod{k}$. So $|U' \setminus e_1| \equiv 0 \pmod{k}$.

Thus, we may take $r_1 = k$ and $r_2 = 0$. Note that $x = |W' \setminus e_1|/k > n/k^2 > 20k^2$ and $y = |U' \setminus e_1|/k > n/k^2 > 20k^2$.

**Case 2.** $i = 0$ and $k - i = k$ is odd.

Then (i) and (iv) hold. So $|W'|$ is even and $|U'| \equiv n/k \pmod{2}$. Since all vertices of $F$ are $\gamma$-good with respect to $\mathcal{H}_{n,k}^0(W',U')$, there is an edge $e_1 \in F$ such that $|e_1 \cap W'| \equiv |W'| \pmod{k - 1}$.

Write $|W' \setminus e_1| = (k - 1)x$ for some integer $x$. Then $|U' \setminus e_1| - x = n - x - |W'| - |e_1 \cap U'| = n - x - |W' \setminus e_1| - |W' \cap e_1| + |e_1 \cap U'| = n - k(x + 1) \equiv 0 \pmod{k}$. So we take $r_1 = k - 1$ and $r_2 = 0$.

Note that $x = |W' \setminus e_1|/(k - 1) > n/k^2 > 20k^2$ and

$$y = (|U' \setminus e_1| - x)/k = (|U' \setminus e_1| - |W' \setminus e_1|/(k - 1))/k = (|U' \setminus e_1| - (n - k - |U' \setminus e_1|)/(k - 1))/k = (|U' \setminus e_1| - n/k + 1)/(k - 1) \geq 0.1n/k^2 > 20k^2.$$

**Case 3.** $i = 1$ and $k - i = k - 1$ is odd.

Then (ii) and (iv) hold. So $|W'| \equiv n/k \pmod{2}$ and $|U'| \equiv n/k \pmod{2}$. Without loss generality, suppose that $|W'| \geq |U'|$. Since all vertices of $F$ are $\gamma$-good with respect to $\mathcal{H}_{n,k}^0(W',U')$, there is an edge $e_1 \in E(F)$ such that $|W' \setminus e_1| \equiv n/k - 1 \pmod{k - 2}$. So $|e_1 \cap W'| \equiv |W'| - n/k + 1 \pmod{k - 2}$.

Let $|W' \setminus e_1| = (k - 1)x + y$ and $x + y = n/k - 1$; then $x + (k - 1)y = n - k - |W' \setminus e_1| = |U' \setminus e_1|$. Moreover, $x = (|W' \setminus e_1| - n/k + 1)/(k - 2)$ and $y = (|U' \setminus e_1| - n/k + 1)/(k - 2)$. So we may set $r_1 = k - 1$ and $r_2 = 1$.  

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Note that $x = (|W' \setminus e_1| - n/k + 1)/(k - 2) > 0.1n/k^2 > 20k^2$ and $y = (|W' \setminus e_1| - n/k + 1)/(k - 2) > 0.1n/k^2 > 20k^2$.

**Case 4.** $i = 1$ and $k - i = k - 1$ is even.

Then (ii) and (iii) hold. So $|W'| \equiv n/k \pmod{2}$ and $|U'|$ is even. Since all vertices in $F$ are $\gamma$-good with respect to $\mathcal{H}_{n,k}^1(W', U')$, there is an edge $e_1 \in E(F)$ such that $|e_1 \cap U'| \equiv |U'| \pmod{k - 1}$.

Write $|U' \setminus e_1| = (k - 1)y$ for some integer $y$. Then

$$|W' \setminus e_1| - y = n - y - |U'| - |e_1 \cap W'|$$
$$= n - y - |U' \setminus e_1| - (|e_1 \cap U'| + |e_1 \cap W'|)$$
$$= n - y - (k - 1)y - k$$
$$= n - k(y + 1) \equiv 0 \pmod{k}.
$$

So let $r_1 = k$ and $r_2 = 1$.

Note that

$$x = (|W' \setminus e_1| - y)/k$$
$$= (|W' \setminus e_1| - |U' \setminus e_1|/(k - 1))/k$$
$$= (|W' \setminus e_1| - (n - k - |W' \setminus e_1|)/(k - 1))/k$$
$$= (|W' \setminus e_1| - n/k + 1)/(k - 1)$$
$$\geq 0.1n/k^2$$
$$> 20k^2.$$

Moreover $y = |U' \setminus e_1|/(k - 1) > n/k^2 > 20k^2$. □

We are ready to prove the main result in this section.

**Lemma 3.4** Let $k, n$ be integers with $k \geq 3$, $n \equiv 0 \pmod{k}$, and $n \gg 1/\epsilon$. Let $F$ be a balanced $(1, k)$-partite $(k + 1)$-graph with partition classes $X, [n]$, such that $\delta_{k-1}(N_{\mathcal{F}}(x)) > t(n, k)$ for all $x \in X$. Suppose $[n]$ has a partition $W, U$ with $|W| = n/2 \pm o(n)$ and $|U| = n/2 \pm o(n)$ such that $F$ is strongly $\epsilon$-close to $\mathcal{H}_{n,k}^1(W, U; m)$ for some $m \in [n/k]$. Then $F$ admits a perfect matching.

**Proof.** Since $F$ is strongly $\epsilon$-close to $\mathcal{H}_{n,k}^0(W, U; m)$ for some $m \in [n/k]$, the number of $\epsilon^{2/3}$-bad vertices in $F$ is at most $(k + 1)\epsilon^{1/3}(n/k + n) \epsilon^{2/3}$ (see Lemma 2.2). Let $B$ denote the set of $\epsilon^{2/3}$-bad vertices in $F$. Write $X_2 = X \cap B$ and $N = [n] \cap B$. For $i \in \{0, 1\}$,

$$X_i = \{x \in X \setminus X_2 \mid N_{\mathcal{H}_{n,k}^i(W, U; m)}(x) \cong H_{n,k}^i(W, U)\}.$$

Write $m_i := |X_i|$ for $i \in \{0, 1, 2\}$. For $i \in \{0, 1\}$ and $x \in X_i$, $x$ is $\epsilon^{2/3}$-good; hence,

$$|N_{\mathcal{H}_{n,k}^0(W, U; m)}(x) \setminus N_{\mathcal{F}}(x)| \leq \epsilon^{2/3}(n + n/k)^k \leq (1 + 1/k)^k \epsilon^{2/3}n^k.$$

So $N_{\mathcal{F}}(x)$ is strongly $(1 + 1/k)^k \epsilon^{2/3}$-close to $H_{n,k}^i(W, U)$.
Recall the definition of constants \( \eta = 1/(4k!) \), \( c = 1/(8(k+1)!) \), \( \varepsilon = 1/(80^k k^k (k+1)!) \)\(^{3/2} \) and \( \gamma = \frac{\varepsilon^{2/3} k}{c^2 k^k} \). Let \( F_i := (F - X_2) - X_{-i} \) for \( i \in \{0, 1\} \). Define a partition \( W_0, U_0 \) of \([n]\) as follows: If \( |X_0| \geq |X_1| \) then let

\[
W_0 = (W \setminus \{v \in W \cap N : d_{F_0,W}(v) \leq \eta m^k\}) \cup \{v \in U \cap N : d_{F_0,W}(v) > \eta m^k\}
\]

and

\[
U_0 = (U \setminus \{v \in U \cap N : d_{F_0,W}(v) > \eta m^k\}) \cup \{v \in W \cap N : d_{F_0,W}(v) \leq \eta m^k\}.
\]

If \( |X_0| < |X_1| \) then let

\[
W_0 = (W \setminus \{v \in W \cap N : d_{F_1,W}(v) \leq \eta m^k\}) \cup \{v \in U \cap N : d_{F_1,W}(v) > \eta m^k\}
\]

and

\[
U_0 = (U \setminus \{v \in U \cap N : d_{F_1,W}(v) > \eta m^k\}) \cup \{v \in W \cap N : d_{F_1,W}(v) \leq \eta m^k\}.
\]

Let \( X_{21} := \{x \in X_2 : d_{F,X_0}(x) \geq \eta m^k\} \) and \( X_{22} = X_2 \setminus X_{21} \). We apply Lemma 2.23 to \( F \) with \( X' = X_1 \cup X_{22} \) and \( i = 0 \) (if \( |X_0| \geq |X_1| \)), or \( X' = X_0 \cup X_{21} \) and \( i = 1 \) (when \( |X_0| < |X_1| \)). So there exists \( e_0 \) such that \( e_0 = \emptyset \) or \( e_0 \in E(F) \) satisfying (i)(ii)(iii)(iv) of Lemma 2.23.

Define

\[
N' := \begin{cases} 
B \cup X_1, & \text{if } |X_1| \leq cn \\
B \cup X_0, & \text{if } |X_0| \leq cn \\
B, & \text{otherwise.}
\end{cases}
\]

Then \( |N'| \leq 2cn \). Next, we apply Lemma 3.1 to \( F \) with \( W_0 \) and \( U_0 \) as partition of \([n]\) and with \( N' \) as the \( N \) in Lemma 3.1. Note that if \( |X_0| \geq |X_1| \), we set \( i = 0 \) and have \( d_{F,X_0,W_0}(v) \geq \eta m^k \) for all \( v \in [n] \), and that if \( |X_0| < |X_1| \), we set \( i = 1 \) and have \( d_{F,X_0,W_0}(v) \geq \eta m^k \) for all \( v \in [n] \). Hence, there is a matching \( M_1 \) in \( F \) with \( e_0 \in M_1 \) (when \( e_0 \neq \emptyset \)), and \( |V(M_1)| \leq (k+1)(k+1) \) such that

(i) \( N' \subseteq V(M_1) \);

(ii) \( |W_0 \setminus V(M_1)| \equiv |X_1 \setminus V(M_1)| \mod 2 \).

Let \( W_1 := W_0 \setminus V(M_1) \) and \( U_1 := U_0 \setminus V(M_1) \). Let \( X'_0 := X_0 \setminus V(M_1) \) and \( X'_1 = X_1 \setminus V(M_1) \). Note that \( |X'_i| > cn/2 \) or \( X'_i = \emptyset \) for \( i \in \{0, 1\} \). If \( |X_0| < cn \), then \( X'_0 = \emptyset \) and \( X'_1 = \emptyset \) and let \( W_{11} := W_1, U_{11} := U_1 \) and \( W_{10} = U_{10} = \emptyset \); if \( |X_1| < cn \), then \( X'_1 = \emptyset \) and let \( W_{10} := W_1, U_{10} := U_1 \) and \( W_{11} = U_{11} = \emptyset \). Otherwise, let \( W_{10} \cup U_{11} \) be a partition of \( W_1 \) and \( U_{10} \cup U_{11} \) be a partition of \( U_1 \) such that for \( i \in \{0, 1\} \),

(iii) \( |W_{1i}| \equiv 0 \mod 2, |W_{1i}| \equiv |X'_i| \mod 2 \), \( |W_{1i} \cup U_{1i}| = k|X'_i| \), and if \( X'_i \neq \emptyset \) then \( \min\{|W_{1i}|, |U_{1i}|\} \geq 1.1|X'_i| + k \).

Note that the existence of these partitions is guaranteed by (ii).

Let \( F'_i := F[X'_i \cup W_{1i} \cup U_{1i}] \) for \( i \in \{0, 1\} \). Since every vertex in \( V(F) \setminus B \) is \( \varepsilon^{2/3} \)-good with respect to \( \mathcal{H}^0_{n,k}(W, U; m) \), every vertex of \( F'_i \) is \( \gamma \)-good with respect to \( \mathcal{H}^0_{k,X'_i,k}(W_{1i}, U_{1i}) \).
For, otherwise, without loss of generality, suppose \( v \in V(F'_i) \) is \( \gamma \)-bad with respect to \( \mathcal{H}_{k|X'|,k}(W_1, U_1) \). Then
\[
|N_{\mathcal{H}_{n,k}(W,
\cup U);m}(v) \setminus N\mathcal{F}(v)| \geq |N_{\mathcal{H}_{k|X'|,k}(W_1, U_1)}(v) \setminus N\mathcal{F}_i(v)|
\geq \gamma((k + 1)|X'|)^k
\geq \varepsilon^{2/3} \frac{(k + 1)n^k}{k^k} \left( \frac{n}{2} \right)^k
= \varepsilon^{2/3} (n + n/k)^k,
\]
contradicting that \( v \in \mathcal{F} \) is \( \varepsilon^{2/3} \)-good on \( \mathcal{H}_{n,k}(W, U; m) \).

By Lemma 4.1 (Rodl, Ruciński, and Szemerédi [28]), \( F'_i \) contains a perfect matching \( M_{2i} \) for \( i = 0, 1 \) (and let \( M_{2i} = \emptyset \) if \( F'_i \) is empty). So \( M_1 \cup M_{20} \cup M_{21} \) is a perfect matching in \( \mathcal{F} \).

\section{Absorbing devices for perfect matchings}

We need the following lemma from [28]. For subsets \( N_1, \ldots, N_k \) of the vertex sets of a \( k \)-graph \( H \), let \( E_H(N_1, \ldots, N_k) := \{(v_1, \ldots, v_k) : v_i \in N_i \text{ for } i \in [k] \text{ and } \{v_1, \ldots, v_k\} \in E(H)\} \). Let \( e_H(N_1, \ldots, N_k) := |E_H(N_1, \ldots, N_k)| \). Given a \( k \)-graph \( H \), let \( \overline{\mathcal{P}} \) denote the \( k \)-graph with vertex set \( V(H) \) and edge set
\[
E(\overline{\mathcal{P}}) = \left\{ e \in \binom{V(H)}{k} : e \notin E(H) \right\}.
\]
The following lemma is Claim 5.1 in [28].

\textbf{Lemma 4.1 (Rodl, Ruciński, and Szemerédi [28])} Let \( n, k \) be two integers such that \( n \gg k \geq 3 \) and \( n \equiv 0 \pmod{k} \). Let \( H \) be a \( k \)-graph on \( n \) vertices. If \( \delta_{k-1}(H) \geq (1/2 - 1/\log n)n \) and \( H \) is not weakly \( \varepsilon \)-close to \( H(n,k) \) or \( \overline{H(n,k)} \), then at least one of the following holds.

\begin{itemize}
  \item[(i)] For all \( N_1, \ldots, N_k \subseteq V(H) \) with \( |N_i| \geq (1/2 - 1/\log n)n \), we have \( e_H(N_1, \ldots, N_k) \geq n^k/\log^3 n \).
  \item[(ii)] \(|\{(v_1, \ldots, v_{k-1}) \in \binom{V(H)}{k-1} : d_H(\{v_1, \ldots, v_{k-1}\}) > (1/2 + 2/\log n)n\}| \geq n^{k-1}/\log n \).
\end{itemize}

Next we define two types of absorbing devices for a given balanced set \( S \) of \( k + 1 \) vertices. Both are \((k + 1)\)-matchings. The vertices of each devices together with \( S \) induce a \((k + 1)\)-graph with a perfect matching.

\textbf{Absorbing device I:} Let \( \mathcal{H} \) be a \((1,k)\)-partite \((k + 1)\)-graph with partition classes \( X, V \). Given a balanced set \( S = \{x_0, v_1, \ldots, v_k\} \) with \( x_0 \in X \) and \( v_i \in V \) for \( i \in [k] \), a \((k + 1)\)-matching \( \{e_1, \ldots, e_k, g\} \) in \( \mathcal{H} \) is said to be \( S \)-\textit{absorbing} if \( \mathcal{H} \) has a \((k + 2)\)-matching \( \{e'_1, \ldots, e'_k, f, g\} \) such that

\begin{itemize}
  \item[(i)] \( e'_i \cap e_j = \emptyset \) for all \( i \neq j \),
\end{itemize}
for \( i \) of (pairwise disjoint) sets \( B_n \). So, there are at least \( E \) Note that the inclusion of the edge \( g \) is only for later convenience.

**Absorbing device II:** Let \( \mathcal{H} \) be a \((1, k)\)-partite \((k + 1)\)-graph with partition classes \( X, V \). Given a balanced set \( S = \{x_0, v_1, \ldots, v_k\} \) with \( x_0 \in X \) and \( v_i \in V \) for \( i \in [k] \), a \((k + 1)\)-matching \( \{e_1, \ldots, e_{k+1}\} \) in \( \mathcal{H} \) is said to be \( S \)-absorbing if \( \mathcal{H} \) has a \((k + 2)\)-matching \( \{e_1', \ldots, e'_{k+1}, f\} \) such that

(i) \( e_i \cap e_j = \emptyset \) for \( i, j \in [k] \), where \( i \neq j \),

(ii) \( e_i' \setminus e_i = \{v_i\} \) and \( |e_i \setminus e_i'| = 1 \) for \( i \in [k] \),

(iii) \( e'_{k+1} \cap e_k = e_k \setminus e'_{k+1}, |e_k \setminus e'_{k+1}| = 1 \);

(iv) \( f = \{x_0\} \cup \bigcup_{i \in [k+1]\setminus k}(e_i \setminus e_i') \).

Next, we show that if (i) or (ii) of Lemma 4.1 holds for \( \mathcal{F}(n, k) \), then for each balanced set \( S \) there are many \( S \)-absorbing devices in \( \mathcal{F}(n, k) \).

**Lemma 4.2** Let \( \mathcal{F} \) be a \((1, k)\)-partite \((k + 1)\)-graph with partition classes \( X, [n] \) such that \( \delta_{k-1}(N_{\mathcal{F}}(x)) \geq (1/2 - 1/\log n)n \) for each \( x \in X \). Let \( R := \{x \in X : N_{\mathcal{F}}(x) \text{ is not weakly } x \text{-close to } H(n, k) \text{ or } H(n, k)\} \). Let \( S = \{x_0, v_1, \ldots, v_k\} \) be a balanced \((k + 1)\)-set such that \( x_0 \in R \).

(i) If (i) of Lemma 4.1 holds for \( N_{\mathcal{F}}(x_0) \), then the number of \( S \)-absorbing devices I in \( \mathcal{F} \) is \( \Omega(n^{(k+1)^2}/\log^3 n) \).

(ii) If \(|R| = o(n)\) and \(|\{x \in R \setminus \{x_0\} : \text{ (ii) of Lemma 4.1 holds for } N_{\mathcal{F}}(x)\}| \geq n/\log n\), then the number of \( S \)-absorbing devices II in \( \mathcal{F} \) is \( \Omega(n^{(k+1)^2}/\log^4 n) \).

**Proof.** First, suppose (i) of Lemma 4.1 holds for \( N_{\mathcal{F}}(x_0) \). Since \( \delta_{k-1}(N_{\mathcal{F}}(x_0)) \geq (1/2 - 1/\log n)n \), there are \( \Omega(n^k) \) sets \( B_i \), for each \( i \in [k] \), such that \( B_i \cup \{v_i\} \in \mathcal{F} \). Consequently, there are \( \Omega(n^{k^2}) \) choices of (pairwise disjoint) such sets \( B_1, \ldots, B_k \). Let \( e_i' = B_i \cup \{v_i\} \) for \( i \in [k] \).

Since \( |N_{\mathcal{F}}(B_i)| \geq (1/2 - 1/\log n)n \) and (i) of Lemma 4.1 holds for \( N_{\mathcal{F}}(x_0) \), we have

\[
e_{\mathcal{F}}(x_0, N_{\mathcal{F}}(B_1), \ldots, N_{\mathcal{F}}(B_k)) \geq n^k/\log^3 n.
\]

So, there are at least \( n^k/\log^3 n \) choices of edges \( f = \{x_0, u_1, \ldots, u_k\} \) such that \( e_i := B_i \cup \{u_i\} \in E(\mathcal{F}) \) for \( i \in [k] \). Moreover, there are \( \Omega(n^{k+1}) \) choices \( g \) such that \( g \in E((\mathcal{F} - S) - \bigcup_{i=1}^{k} e_i) \). Hence, altogether there are

\[
(n^k/\log^3 n)\Omega(n^{k^2})\Omega(n^{k+1}) = \Omega(n^{(k+1)^2}/\log^3 n)
\]

choices of \( S \)-absorbing \((k + 1)\)-matchings \( \{e_1', \ldots, e_k', f, g\} \).

Next we show (ii). As in the argument for (i), since \(|R| = o(n)\), there are \( \Omega(n^{k^2}) \) choices of (pairwise disjoint) sets \( B_1, \ldots, B_k \) such that \( R \cap (\bigcup_{i=1}^{k} B_i) = \emptyset \) and \( e_i' = B_i \cup \{v_i\} \in E(\mathcal{F}) \) for \( i \in [k] \).
For $i \in [k - 1]$, we choose $u_i \in N_F(B_i)$, each in at least $(1/2 - 1/\log n)n$ ways, and let $e_i = B_i \cup \{u_i\}$. Let $y \in R \setminus \{x_0\}$ such that (ii) of Lemma \[4.3\] holds for $N_F(y)$. By assumption, there are at least $n/\log n$ such $y$. We select a $(k - 1)$-element sequence of vertices, say $T$, such that $d_{N_F(y)}(T) \geq (1/2 + 2/\log n)n$ and $T$ is disjoint from $S \cup B_k \cup \bigcup_{i=1}^{k-1} e_i$. Let $B_{k+1} = T \cup \{y\}$. Then we may pick $u_k, u_{k+1}$ such that $B_k \cup \{u_k\}, B_{k+1} \cup \{u_k\}, B_{k+1} \cup \{u_{k+1}\}, \{u_0, u_1, \ldots, u_{k-1}, u_{k+1}\} \in E(F)$. Note that $d_F(B_{k+1}) \geq (1/2 + 2/\log n)n$. We have $|N_F(B_{k+1}) \cap N_F(B_k)| \geq n/\log n$ and $|N_F(B_{k+1}) \cap N_F(x_0, u_1, \ldots, u_{k-1})| \geq n/\log n$. Thus there are at least $n/\log n$ choices for each of $x_k, x_{k+1}$. By Lemma \[4.3\] (ii), there are at least $n^{k-1}/\log n$ choices for $T$. Summarizing, we have chosen $B_1, \ldots, B_k, B_{k+1}, u_1, \ldots, u_{k-1}, T$ forming an $S$-absorbing $(k + 1)$-matching, in

$$\Omega(n^{k^2})(n^{k-1}/\log n)n^{k-1}(n/\log n)^3) = \Omega(n^{(k+1)^2}/\log^4 n)$$

ways.

**Absorbing device III:** Let $H$ be a $(1, k)$-partite $(k + 1)$-graph with partition classes $X, V$. Given a balanced set $S = \{x_0, v_1, \ldots, v_k\}$ with $x_0 \in X$ and $v_i \in V$ for $i \in [k]$, a $(k + 1)$-matching $\{e_1, \ldots, e_{k+1}\}$ in $H$ is said to be $S$-absorbing if $H$ has a $(k + 2)$-matching $\{e'_1, \ldots, e'_{k+1}, f\}$ such that

(i) $e_i \cap e'_j = \emptyset$ for $i, j \in [k]$, where $i \neq j$,

(ii) $e'_i \setminus e_i = \{v_i\}$ and $|e_i \setminus e'_i| = 1$ for $i \in [k]$,

(iii) $e'_{k+1} \setminus e_{k+1} = \{x_0\}$, $|e_{k+1} \setminus e'_{k+1}| = 1$,

(iv) $f = \bigcup_{i \in [k+1]} (e_i \setminus e'_i)$.

In the following lemma, we show that any $(1, k)$-partite $(k + 1)$-graph is close to a $(k + 1)$-graph with same partition classes, or contains an absorbing matching.

**Lemma 4.3** Let $0 < \varepsilon \ll \beta \ll 1$. Let $H$ be a balanced $(1, k)$-partite $(k + 1)$-graph with partition classes $X, [n]$. Let $R := \{x \in X : N_H(x)\text{ is not weakly }\varepsilon\text{-close to }H_{0, k, n}\text{ or }H_{1, k, n}\}$. Let $x_0 \in X \setminus R$ such that $N_H(x_0)$ is strongly $\varepsilon$-close to $H_{0, k, n}^i(W, U)$ for some $i \in \{0, 1\}$ and for some partition $(W, U)$ of $[n]$ with $|W| = (1/2 \pm o(1))n = |U|$. Suppose $|R| \leq n/\log n$ and $\delta_{k-1}(N_H(x)) \geq (1/2 - 1/\log n)n$ for every $x \in X$. If $H$ is not strongly $\beta$-close to $H_{0, k, n}^i(W, U; m)$ for any integer $0 \leq m \leq n/k$, then for any balanced $(k + 1)$-set $S$ with $x_0 \in S$, there are at least $\Omega(n^{(k+1)^2}/\log^4 n)$ $S$-absorbing devices I or III.

**Proof.** First, let $\alpha$ be a number with $\varepsilon \ll \alpha \ll \beta$. We prove

Claim 1. Let $x \in X$ and let $W_x, U_x$ be a partition of $[n]$ such that $|W_x| = (1/2 \pm o(1))n$ and $|U_x| = (1/2 \pm o(1))n$ and $N_H(x)$ is strongly $\varepsilon$-close to $H_{n, k}^j(W_x, U_x)$ for some $j \in \{0, 1\}$. For any $N_1, \ldots, N_k \subseteq [n]$ with $|N_i| \geq (1/2 - 1/\log n)n$ for $i \in [k]$, if $e_{N_H(x)}(N_1, N_2, \ldots, N_k) \leq n^k/\log n$ then, for $j \in [k]$, $|U_x \setminus N_j| \leq \alpha n$ or $|W_x \setminus N_j| \leq \alpha n$.

Otherwise, suppose, without loss of generality, $|U_x \setminus N_1| \geq \alpha n$ and $|W_x \setminus N_1| \geq \alpha n$. Let $D$ denote the set of $\sqrt{\varepsilon}$-bad vertices on $H_{n, k}^j(W_x, U_x)$; then $|D| \leq k\sqrt{\varepsilon}n$. Then, since $|N_1| \geq (1/2 - 1/\log n)n$, $|N_1 \setminus (U_x \cup D)| \geq \alpha n/2$ and $|N_1 \setminus (W_x \cup D)| \geq \alpha n/2$. Note for any $z \in N_1 \setminus (W_x \cup D)$ and $y \in N_1 \setminus (U_x \cup D)$, since $\varepsilon \ll \alpha$ and both $z$ and $y$ are $\sqrt{\varepsilon}$-good with
Thus, we have such that and 1
contradicting the assumption of Claim 1 and completing its proof. Hence,

\[ e_{N_H(x)}(z, N_2, \ldots, N_k) \geq \frac{n^{k-1}}{2k+1}! \geq 1 \]

contradicting the assumption of Claim 1 and completing its proof.

Let \( A = \{ x \in X \setminus R : |N_H(x) \cap N_H(x_0)| \geq \alpha n^k \} \) and let \( B = X \setminus (A \cup R) \). Let \( x \in B \).

Then

\[ |N_H(x) \cap E(H_{n,k}^i(W, U))| \leq |N_H(x) \cap N_H(x_0)| + |E(H_{n,k}^i(W, U)) \setminus N_H(x)| \leq \alpha n^k + \varepsilon n^k. \]

Hence, for any \( x \in B \), we have

\[
|E(H_{n,k}^i(W, U)) \setminus E(N_H(x))| = e(H_{n,k}^i(W, U)) - (|N_H(x)| - |N_H(x) \cap E(H_{n,k}^i(W, U)))
\leq e(H_{n,k}^i) - (e(H_{n,k}^i) - \alpha n^k - \varepsilon n^k) + |N_H(x) \cap E(H_{n,k}^i(W, U))|
\leq \varepsilon n^k + \alpha n^k + \varepsilon n^k + \alpha n^k + \varepsilon n^k \leq 3\alpha n^k.
\]

Thus, we have

**Claim 2.** For any \( x \in B \), \( N_H(x) \) is strongly \((3\alpha)\)-close to \( H_{n,k}^{i-1}(W, U) \), which is strongly \( 3\alpha \)-close to \( H_{n,k}^{i-1}(W, U) \).

**Claim 3.** \( |B| \leq (1 - \beta)n/k \); so \( |A| \geq \beta n/k - n/\log n \).

For, otherwise, \( |B| > (1 - \beta)n/k \). Then \( |A| + |R| < \beta n/k \) and

\[
|E(H_{n,k}^{1-i}(W, U; |B|)) \setminus E(\mathcal{H})| = \sum_{x \in B} |N_{H_{n,k}^{1-i}(W, U; |B|)}(x) \setminus N_H(x)| + (|A| + |R|) \binom{n}{k}
< \frac{n}{k} 4\alpha n^k + \beta n/k \binom{n}{k} \leq \beta(n + n/k)^{k+1},
\]
a contradiction since \( \mathcal{H} \) is not strongly \( \beta \)-close to \( H_{n,k}^{1-i}(W, U; |B|) \).

**Claim 4.** Let \( \mathcal{H}' = \mathcal{H}[A \cup [n]] \). Then, for any \( N_1, \ldots, N_k \subseteq [n] \) with \( |N_i| \geq (1/2 - 1/\log n)n \), either \( e_{\mathcal{H}'}(\{x_0\}, N_1, \ldots, N_k) \geq n^k/\log n \) or \( e_{\mathcal{H}'}(A, N_1, \ldots, N_k) \geq n^{k+1}/\log^3 n \).

Suppose on the contrary that there exist \( N_1, \ldots, N_k \subseteq [n] \) with \( |N_i| \geq (1/2 - 1/\log n)n \), such that

\[ e_{\mathcal{H}'}(\{x_0\}, N_1, \ldots, N_k) < n^k/\log n \] (1)

and

\[ e_{\mathcal{H}'}(A, N_1, \ldots, N_k) < n^{k+1}/\log^3 n. \] (2)
Since $N_H(x_0)$ is strongly $\varepsilon$-close to $H^{i_k}_{n,k}(W,U)$, the number of $\sqrt{\varepsilon}$-bad vertices in $N_H(x)$ with respect to $H^{i_k}_{n,k}(W,U)$ is at most $k\sqrt{\varepsilon}n$.

By Claim 1 and (1), $|U \setminus N_i| \leq an$ or $|W \setminus N_i| \leq an$ for each $i \in [k]$. Let $A_1 := \{x \in A : e_{N_H(x)}(N_1, \ldots, N_k) \geq n^k/\log n\}$ and $A_2 = A \setminus A_1$. By (2), $|A_1| \leq n/\log^2 n$.

Let $y \in A_2$. So $y \notin R$ and, thus, $N_H(y)$ is weakly $\varepsilon$-close to $H^0_{n,k}$ or $H^1_{n,k}$. Let $W_y, U_y$ be a partition of $[n]$ such that $N_H(y)$ is strongly $\varepsilon$-close to $H^j_{n,k}(W_y, U_y)$ for some $j \in \{0,1\}$. By Claim 1, we have for all $i \in [k]$, $|U_y \setminus N_i| \leq an$ or $|W_y \setminus N_i| \leq an$. Hence

$$|U_y \setminus U| \leq |U_y \setminus N_i| + |N_i \setminus U| \leq 2an$$ (3)

or

$$|U_y \setminus W| \leq |U_y \setminus N_i| + |N_i \setminus W| \leq 2an$$ (4)

We claim that $N_H(y)$ is strongly $(5\alpha)$-close to $H^j_{n,k}(W,U)$ if inequality (3) holds.

$$|E(H^j_{n,k}(W,U)) \setminus N_H(y)| \leq |E(H^j_{n,k}(W_y, U_y)) \setminus N_H(y)| + |E(H^j_{n,k}(W,U)) \setminus E(H^j_{n,k}(W_y, U_y))|$$

$$\leq \varepsilon n^k + 4\alpha n^k \leq 5\alpha n^k;$$

If inequality (3) holds then

$$|E(H^j_{n,k}(U, W)) \setminus N_H(y)| \leq |E(H^j_{n,k}(W_y, U_y)) \setminus N_H(y)| + |E(H^j_{n,k}(U, W)) \setminus E(H^j_{n,k}(W_y, U_y))|$$

$$\leq \varepsilon n^k + 4\alpha n^k \leq 5\alpha n^k.$$ Thus $N_H(y)$ is strongly $(5\alpha)$-close to $H^j_{n,k}(U, W)$.

Hence by Claim 2, for all $x \in X \setminus (A_1 \cup R)$, $N_H(x)$ is strongly $(5\alpha)$-close to $H^j_{n,k}(W,U)$ for some $j \in \{0,1\}$. For $j \in \{0,1\}$, let $X_j = \{x \in X : N_H(x)$ is strongly $(5\alpha)$-close to $H^j_{n,k}(W,U)\}$. Since $|A_1| \leq n/\log^2 n$, and $|R| \leq n/\log n$, we have $|X \setminus (X_0 \cup X_1)| = |A_1 \cup R| \leq 2n/\log n$. Hence

$$|E(H^0_{n,k}(W,U; |X_0|)) \setminus E(H)| \leq \sum_{j=0}^{1} \sum_{x \in X_j} |N_{H^0_{n,k}(W,U; |X_0|)}(x) \setminus N_H(x)| + (|A_1| + |R|)n^k$$

$$\leq \frac{n}{k}5\alpha n^k + 2n^{k+1}/\log n \leq \beta(n + n/k)^{k+1}.$$ So $H$ is strongly $\beta$-close $H^0_{n,k}(W,U; |X_0|)$, a contradiction. This concludes the proof of Claim 4.

Let $S = \{x_0, v_1, \ldots, v_k\}$ with $v_1, \ldots, v_k \in [n]$ all distinct. For each $i \in [k]$, there are $\Omega(n^k)$ sets $B_i$ such that $e'_i = B_i \cup \{v_i\} \in E(H)$, and there are $\Omega(n^{k^2})$ choices of (pairwise disjoint) such sets $B_1, \ldots, B_k$. For $i \in [k]$, we have $N_{H'}(B_i) \geq (1/2 - 1/\log n)n$ by assumption. So there are $\Omega(n^{k^2})$ choices of (disjoint) sets $B_1, \ldots, B_k$ such that by Claim 4, one of the following two inequalities holds:

$$e_{H'}(\{x_0\}, N_{H'}(B_1), \ldots, N_{H'}(B_k)) \geq n^k/\log n$$ (5)

$$e_{H'}(A, N_{H'}(B_1), \ldots, N_{H'}(B_k)) \geq n^{k+1}/\log^3 n.$$ (6)
First, assume (iii) holds. Then there at least \( n^k / \log n \) choices \( u_1, \ldots, u_k \) such that \( e_i = B_i \cup \{ u_i \} \in E(\mathcal{H}') \) for \( i \in [k] \) and \( \{ x_0, u_1, \ldots, u_k \} \in E(\mathcal{H}') \). Moreover, there are \( \Omega(n^{k+1}) \) choices of \( g \in E(\mathcal{H}') \) such that \( g \) is disjoint from \( \cup_{i=1}^k e_i \). Thus the number of \( S \)-absorbing devices I is at least
\[
\Omega(n^{k^2}) \Omega(n^k / \log n) \Omega(n^{k+1}) = \Omega(n^{(k+1)^2} / \log n).
\]

Now assume (i) holds. Then there are \( n^{k+1} / \log^3 n \) choices \( \{ y, u_1, \ldots, u_k \} \) such that \( y \in A \) and, for \( i \in [k] \), \( B_i \cup \{ u_i \} \in E(\mathcal{H}') \). By definition of \( A \), there are \( \Omega(n^k) \) choices \( B_{k+1} \) such that \( B_{k+1} \in N_{\mathcal{H}}(y) \cap N_{\mathcal{H}}(x_0) \). Let \( e_{k+1} = B_{k+1} \cup \{ y \} \). So there are at least \( \Omega(n^{k+1} / \log^3 n) \Omega(n^k \Omega(n^{k^2}) = \Omega(n^{(k+1)^2} / \log^3 n) \) different choices of \( \{ e_1, \ldots, e_{k+1} \} \) such that \( \{ e_1, \ldots, e_{k+1} \} \) is an \( S \)-absorbing device II.

To prove another absorbing lemma, we need to use Chernoff bounds, see [3].

**Lemma 4.4** Suppose \( X_1, \ldots, X_n \) are independent random variables taking values in \( \{0, 1\} \). Let \( X \) denote their sum and \( \mu = \mathbb{E}[X] \) denote the expected value of \( X \). Then for any \( 0 < \delta \leq 1 \),
\[
\mathbb{P}[X \leq (1-\delta)\mu] < e^{-\frac{\delta^2 \mu}{2}}.
\]

**Lemma 4.5** Let \( k \geq 3 \) be an integer and let \( 0 < \varepsilon < \beta \ll 1 \). There exists \( n_2 > 0 \) such that the following holds for any integer \( n > n_2 \). Let \( \mathcal{H} \) be a balanced \((1,k)\)-partite \((k+1)\)-graph with partition classes \( X, [n] \) such that and \( \delta_{k-1}(N_{\mathcal{H}}(x)) > t(n,k) \) for all \( x \in X \). Let \( R := \{ x \in X : N_{\mathcal{H}}(x) \) is not \( \varepsilon \)-close to \( H_{n,k}^0 \) or \( H_{n,k}^1 \} \), and let \( x_0 \in X \). Suppose one of the following three conditions holds for every \( S \in \binom{V(\mathcal{H})}{k+1} \) with \( S \cap X = \{ x_0 \} \):

(i) (i) of Lemma 4.1 holds for \( N_{\mathcal{H}}(x_0) \), and \( \mathcal{H} \) has \( \Omega(n^{(k+1)^2} / \log^4 n) \) \( S \)-absorbing devices I.

(ii) \( |\{ x \in R \setminus \{ x_0 \} : (ii) \) of Lemma 4.1 holds for \( N_{\mathcal{H}}(x) \} | \geq n / \log n \), and \( \mathcal{H} \) has \( \Omega(n^{(k+1)^2} / \log^4 n) \) \( S \)-absorbing devices II.

(iii) \( |R| \leq n / \log n \), \( x_0 \notin R \), and \( N_{\mathcal{H}}(x_0) \) is strongly \( \varepsilon \)-close to \( H_{n,k}^0(W,U) \) or \( H_{n,k}^1(W,U) \) for some partition \( (W,U) \) of \([n]\) with \(|W| = (1/2 + o(1))n = |U| \), \( \mathcal{H} \) is not strongly \( \beta \)-close to \( H_{n,k}^0(W,U;m) \) for any \( m \in [n/k] \), and \( \mathcal{H} \) has \( \Omega(n^{(k+1)^2} / \log^4 n) \) \( S \)-absorbing devices I or III.

Then there exists a matching \( M' \) in \( \mathcal{H} \) such that \( |M'| = O(\log^6 n) \) and, for each \( S \in \binom{V(\mathcal{H})}{k+1} \) with \( S \cap X = \{ x_0 \} \), \( M' \) contains an \( S \)-absorbing \((k+1)\)-matching.

**Proof.** For each balanced \((k+1)\)-set \( S \subseteq V(\mathcal{H}) \) with \( x_0 \in S \), let \( \Gamma(S) \) be the collection of \( S \)-absorbing \((k+1)\)-matchings. Then by Lemmas 4.2 and 4.3 \( |\Gamma(S)| = \Omega(n^{(k+1)^2} / \log^4 n) \). So we may choose constant \( \alpha := \alpha(k) > 0 \) such that
\[
|\Gamma(S)| \geq \alpha(k^2 + k)! \left( \frac{n^k}{k^k} \right) \left( \binom{n}{k(k+1)} / ((k!)^{k+1} \log^4 n) \right).
\]
Let $\mathcal{M}$ be the family obtained by choosing a sequence of balanced $(k+1)$-sets $(S_1, \ldots, S_{k+1})$ independently with probability
\[
p = \frac{(k!)^{k+1} \log^6 n}{(n/k)^{k+1} (k!)(k^2+k)!}.
\]

Note that $p < 1$ as we can choose $n_2$ large enough. Then
\[
E(\mathcal{M}) = p \binom{n/k}{k+1} \binom{n}{k(k+1)} (k^2 + k)!/(k!)^{k+1} = O(\log^6 n),
\]
and, for $(k+1)$-set $S \subseteq V(\mathcal{H})$ with $\{x_0\} = S \cap X$,
\[
E(|\mathcal{M} \cap \Gamma(S)|) \geq p\alpha(k^2 + k)! \binom{n/k}{k+1} \binom{n}{k(k+1)} /((k!)^{k+1} \log^4 n) = \alpha \log^2 n.
\]

By Lemma 4.4 and by choosing $n_2$ large enough, we have, for $n > n_2$ and for each
$S \in \binom{V(H)}{k+1}$ with $S \cap X = \{x_0\}$,
\[
\mathbb{P}[|\mathcal{M}| > 2\alpha \log^6 n] = \mathbb{P}[|\mathcal{M}| > 2E(|\mathcal{M}|)] \leq e^{-E(|\mathcal{M}|)/3} = e^{-(\log^6 n)/3}.
\]
So with probability at least $1 - o(1)$
\[
|\mathcal{M}| \leq 2\alpha \log^6 n. \tag{7}
\]

Again by Lemma 4.4 and by choosing $n_2$ large enough, we have, for $n > n_2$ and for ,
\[
\mathbb{P}[|\mathcal{M} \cap \Gamma(S)| \leq (\alpha \log^2 n)/2] \leq \mathbb{P}[|\mathcal{M} \cap \Gamma(S)| \leq E(|\mathcal{M} \cap \Gamma(S)|)/2]
\]
\[
\leq e^{-E(|\mathcal{M} \cap \Gamma(S)|)/8} \leq e^{-(\alpha \log^2 n)/8}.
\]

So by union bound and by choosing $n_2$ large, we have for $n > n_2$,
\[
\mathbb{P}[\exists S \in \binom{V(H)}{k+1} \text{ with } S \cap X = \{x_0\} : |\mathcal{M} \cap \Gamma(S)| \leq (\alpha \log^2 n)/2]
\]
\[
\leq \binom{n}{k} e^{-(\alpha \log^2 n)/8}
\]
\[
= 2n^{k-(\alpha \log n)/8} < 1/10.
\]

Thus, with probability at least $9/10$, for all $S \in \binom{V(H)}{k+1}$ with $S \cap X = \{x_0\}$, we have
\[
|\mathcal{M} \cap \Gamma(S)| \geq (\alpha \log^2 n)/2 > 1. \tag{8}
\]

Furthermore, the expected number of pairs of sequences $(S_1, \ldots, S_{k+1}), (T_1, \ldots, T_{k+1}) \in \mathcal{M}$ satisfying $(\bigcup_{i \in [k]} S_i) \cap (\bigcup_{i \in [k]} T_i) \neq \emptyset$ is at most
\[
\binom{(k^2 + k)!}{(k!)^k} \binom{n}{k} \binom{n}{k+1} \binom{n}{k^2 + k}
\]
\[
\binom{k+1}{1} \binom{n}{k} \binom{n}{k^2 + k} + \binom{k^2 + k}{1} \binom{n}{k} \binom{n}{k^2 + k - 1} \bigg) p^2 \leq 1/2.
\]
This completes the proof.

Hence, with positive probability, \( \mathcal{M} \) satisfies (7), (8), and (9). So we may assume that \( \mathcal{M} \) satisfies (7), (8), and (9). Let \( M \) be the union of \( \mathcal{M} \cap \Gamma(S) \) for all \( S \in \binom{V(H)}{k+1} \) with \( S \cap X = \{x_0\} \). Then \( M \) is the desired matching.}

\section{Absorbing devices for near perfect matchings}

Let \( H \) be a \((1,k)\)-partite \((k+1)\)-graph with partition classes \( Q, [n] \). For a set \( S \in \binom{V(H)}{k+2} \) with \( |S \cap Q| = 1 \), an edge \( e \in E(H) \) is said to be \emph{S-absorbing} if \( H[e \cup S] \) has a matching of size 2.

\begin{lemma}
Let \( k \geq 3 \), \( 0 < c < 1 \), and \( n \) be a sufficiently large integer, and let \( H \) be a \((1,k)\)-partite \((k+1)\)-graph with partition classes \( Q, [n] \) such that \( |Q| \leq (n-1)/k \). If \( \delta_{k-1}(N_H(v)) \geq cn \) for all \( v \in Q \), then, for any \( S \in \binom{V(H)}{k+2} \) with \( |S \cap Q| = 1 \), \( H \) has at least \( c^4 n^{k+1}/2 \) \( S \)-absorbing edges.
\end{lemma}

\begin{proof}
Let \( S = \{v\} \cup B \), where \( v \in Q \) and \( B = \{b_1, \ldots, b_k, b_{k+1}\} \in \binom{[n]}{k+1} \). Let \( B' = B \setminus \{b_k, b_{k+1}\} \). Since \( \delta_{k-1}(N_H(v)) \geq cn \), we have at least \( cn-2 \) choices for \( q \in [n] \setminus B \) such that \( B' \cup \{v, q\} \in E(H) \). For a \((k-2)\)-set \( A \) of \( V(H) \setminus (S \cup \{q\}) \) with \( |A \cap Q| = 1 \), since \( d_H(A \cup \{b_k, b_{k+1}\}) \geq cn \), there are at least \( cn-2k \) choices \( q' \in [n] \setminus (S \cup A \cup \{q\}) \) such that \( \{q', q'' \} \cup \{b_k, b_{k+1}\} \cup A \in E(H) \). Since \( d_H(A \cup \{q, q'\}) \geq cn \), there are at least \( cn-2k \) choices \( q'' \) such that \( A \cup \{q, q', q''\} \cup A \in E(H) \). Clearly, any such \( \{q, q', q''\} \cup A \) is \( S \)-absorbing.

Note that there are \((|Q|-1)\left(\begin{array}{c}n-k-2 \\k\end{array}\right)\) different choices for set \( A \). Hence the number of \( S \)-absorbing edges is at least

\[ (|Q|-1)\left(\begin{array}{c}n-k-2 \\k\end{array}\right) (cn-2)(cn-2k)(cn-2k) \geq c^4 n^{k+1}/2. \]

This completes the proof.

\end{proof}

Analogous to the absorbing results in [23,28], we prove the existence of small absorbing matching for \((1,k)\)-partite \((k+1)\)-graphs with \( |Q| \geq n/k - k^2 \).

\begin{lemma}
For any constant \( c > 0 \), there exists an integer \( n_0 > 0 \) with the following holds for every integer \( n \geq n_0 \): Let \( H \) be a \((1,k)\)-partite \((k+1)\)-graph with partition classes \( Q, [n] \) such that \( n/k - k^2 \leq |Q| \leq (n-1)/k \) and \( \delta(N_H(v)) \geq cn \) for all \( v \in S \). Then there is a matching \( M \) in \( H \) such that \( |M| \leq (32(k+3)/cd) \log n \) and, for each \( S \in \binom{V(H)}{k+2} \) with \( |S \cap Q| = 1 \), \( M \) contains at least \( 4(k+3) \log n \) \( S \)-absorbing edges.
\end{lemma}

\begin{proof}
Let \( C = 32(k+3)/cd \). Let \( M \) be the family obtained by choosing each edge independently with probability \( p = (C/2)n^{-(k+1)} \log n \). Thus \( E(|M'|) = |E(H)|p \leq n^{k+1}p = (C/2) \log n \).

\end{proof}
The number of intersecting pairs of edges in $E(H)$ is at most $|Q|\binom{n}{2} + |Q|^2 n\binom{n-1}{k-1} \leq n^{2k+1}$; so the expected number of intersecting pairs of edges in $M'$ is at most

$$n^{2k+1}p^2 \leq C^2 \log^2 n/(4n) = o(1).$$

By Markov’s inequality, with probability strictly larger than $1/3$, $M$ is a matching of size at most $C \log n$.

For a set $S \in \binom{V(H)}{k+1}$ with $|S \cap Q| = 1$, let $X_S$ denote the number of $S$-absorbing edges in $M$. Then by Lemma 5.1, we have

$$\mathbb{E}[X_S] \geq pc^4 n^{k+1}/2 = 8(k+3) \log n.$$

By Lemma 4.4

$$\mathbb{P}[X_S \leq \mathbb{E}[X_S]/2] \leq \exp(-\mathbb{E}[X_S]/8) = \exp(-(k+2) \log n) = n^{-(k+2)}.$$ 

Note that there are at most $|Q|\binom{n}{k} < n^k/k$ sets $S \in \binom{V(H)}{k+1}$ with $|S \cap Q| = 1$. It follows from union bound that, with probability strictly larger than $1/4$, $X_S \geq \mathbb{E}[X_S]/2 \geq 4(k+3) \log n$ for all $S \in \binom{V(H)}{k+1}$ with $|S \cap Q| = 1$. Thus, the desired $M$ exists.

6 Hypergraphs not close to extremal configurations

In this section, we prove Theorem 1.2 for hypergraphs that are not close to extremal configurations. For this, we need a result on almost perfect matchings in $(1,k)$-partite $(k+1)$-graphs.

**Lemma 6.1** Let $k, n$ be positive integers with $k \geq 3$. Let $H$ be a $(1,k)$-partite $(k+1)$-graph with vertex partition classes $Q, [n]$, where $k + 1 \leq |Q| \leq n/k$. If $\delta_{k-1}(N_H(v)) > n/k$ for every $v \in Q$, then $H$ has a matching covering all but at most $k-1$ vertices of $Q$.

**Proof.** Let $M$ be a maximum matching in $H$. We may assume $|Q \setminus V(M)| \geq k$; for, otherwise, $M$ gives the desired matching. Note

$$|Q \setminus V(M)| = n - k|M| \geq k|Q| - k|M| \geq k|Q \setminus V(M)| \geq k^2.$$ 

So there exist $k$ pair-disjoint $k$-sets in $[n]\setminus V(M)$, say $S_1, \ldots, S_k$ such that $|S_i \cap Q| = 1$ for $i \in [k]$. By maximality of $M$, $N_H(S_i) \subseteq V(M)$ for $i \in [k]$.

We claim that $\sum_{i=1}^k |N_H(S_i) \cap e| \leq k$ for all $e \in M$. For otherwise, there exist $e \in M$ and distinct $u, v \in e$ such that $u \in N_H(S_i) \cap e$ and $v \in N_H(S_j) \cap e$. Now $(M \setminus \{e\}) \cup \{S_i \cup \{u\}, S_j \cup \{v\}\}$ is a matching in $H$, contradicting the maximality of $M$.

Since $N_H(S_i) \subseteq V(M)$,

$$\sum_{i=1}^k |N_H(S_i)| = \sum_{e \in M} \sum_{i=1}^k |N_H(S_i) \cap e| \leq k|M| < n.$$ 

However, since $\delta_{k-1}(N_H(v)) > n/k$ for all $v \in Q$, we have $\sum_{i=1}^k |N_H(S_i)| \geq k\delta_{k-1}(N_H(v)) > n$, a contradiction. ■
Lemma 6.2 Let \( k, n \) be integers with \( k \geq 3 \). Let \( H \) be a \((1, k)\)-partite \((k + 1)\)-graph with partition classes \( Q, [n] \), where \( k + 1 \leq |Q| \leq (n - 1)/k \). If \( \delta_{k-1}(N_H(v)) \geq n/k + 32k^5(k + 3) \log n \) for every \( v \in Q \), then \( H \) has a matching covering \( Q \).

Proof. Let \( c = 1/k \). Since \( \delta_{k-1}(N_H(v)) \geq n/k + 32k^5(k + 3) \log n \) for every \( v \in Q \), by Lemma 5.2, \( H \) has a matching \( M \) of size at most \( 32k^4(k + 3) \log n \) such that for any set \( S \in \binom{V(H)}{k+2} \) with \( |S \cap Q| = 1 \), the number of \( S \)-absorbing edges in \( M \) is at least \( k + 1 \). Let \( H' = H - V(M) \), with partition classes \( Q \setminus V(M), [n] \setminus V(M) \). For every \( v \in Q \setminus V(M) \), we have

\[
\delta_{k-1}(N_{H'}(v)) \geq \delta_{k-1}(N_H(v)) - k|M| > n/k.
\]

Thus by Lemma 6.1, \( H' \) has a matching \( M' \) covering all but at most \( k \) vertices of \( Q \setminus V(M) \).

Let \( M_0 := M \) and \( M_0' = M' \). If \( M_0 \cup M_0' \) covers \( Q \), then we are done. Otherwise, there exists \( S_1 \in \binom{V(H) \setminus V(M_0 \cup M_0')}{k+2} \) with \( |S_1 \cap Q| = 1 \). Recall \( M_0 \) contains an \( S_1 \)-absorbing edge, say \( e_0 \); so \( H[S_1 \cup e_0] \) contains a matching \( X_0 \) of size 2. Then \( M_1' := M_0' \cup X_0 \) is a matching in \( H \). Let \( M_1 := M_0 \setminus \{e_0\} \). If \( M_1 \cup M_1' \) covers \( Q \), then we are done. Otherwise, since \( |Q \setminus V(M_1 \cup M_1')| \leq k \) and \( M \) has at least \( k + 1 \) \( S \)-absorbing matching for each \( S \in \binom{V(H)^s}{k+2} \) with \( |S \cap Q| = 1 \), we may repeat the above procedure. Thus, we obtain a maximal sequence of pairs of matchings \( M_0, M_0', M_1, M_1', \ldots, M_t, M_t' \), a sequence of \((k + 2)\)-sets \( S_1, \ldots, S_t \) with \( |S_i \cap Q| = 1 \) for \( i \in [t] \), and a sequence of matchings \( X_0, X_1, \ldots, X_t \). Now \( M_t \cup M_t' \) is a matching of \( H \) covering \( Q \).

Lemma 6.3 Let \( \varepsilon > 0 \) be a constant and \( k, n \) be integers with \( k \geq 3 \) and \( n \equiv 0 \) (mod \( k \)) such that \( 0 < 1/n \ll \varepsilon \ll 1/k \). Let \( \mathcal{F} \) be a balanced \((1, k)\)-partite \((k + 1)\)-graph with partition classes \( X \) and \([n]\). Let \( R := \{x \in X : N_F(x) \text{ is not weakly } \varepsilon\text{-close to } H_{n,k}^0 \text{ or } H_{n,k}^1\} \). Suppose

1. \( |R| > t(n, k) \) for all \( x \in X \);
2. \( |R| > n/\log n \), or \( |R| \leq n/\log n \) and \( \mathcal{F} \) is not strongly \( \varepsilon \)-close to \( H_{n,k}^0(W, U; m) \) for any \( m \in [n/k] \) and for any partition \( W, U \) of \([n]\) with \( |W| = n/2 \pm o(n) \) and \( |U| = n/2 \pm o(n) \).

Then \( \mathcal{F} \) admits a perfect matching.

Proof. Note that the conclusion of Lemma 6.3 holds. We define \( x_0 \in X \) as follows:

(i) If \( |R| > n/\log n \) then choose \( x_0 \in R \) such that, whenever possible, (i) of Lemma 6.3 holds for \( N_F(x_0) \).

(ii) If \( |R| \leq n/\log n \) then let \( x_0 \in X \setminus R \) and let \( W, U \) be a partition of \([n]\) with \( |W| = n/2 \pm o(n) = |U| \) such that \( N_F(x_0) \) is strongly \( \varepsilon \)-close to \( H_{n,k}^0(W, U) \) or \( H_{n,k}^1(W, U) \).

By Lemma 6.3, there exists a matching \( M \) in \( \mathcal{F} \) with \( |M| = O(\log^6 n) \) such that for any balanced \((k + 1)\)-set \( S \subseteq V(\mathcal{F}) \) containing \( x_0 \), \( \mathcal{F}[S \cup V(M)] \) has a perfect matching.

Let \( \mathcal{F}' = \mathcal{F} - (V(M) \cup \{x_0\}) \). Then \( k + 1 \leq |V(\mathcal{F}') \cap X| = |V(\mathcal{F}) \cap [n]|/k - 1 \). Moreover, for every \( v \in X \cap V(\mathcal{F}') \), we have

\[
\delta_{k-1}(N_{\mathcal{F}'}(v)) \geq \delta_{k-1}(N_{\mathcal{F}}(v)) - |V(M)| \geq n/2 - k - k|M| > n/2 - k - k \log^6 n > n/k + \log^2 n.
\]
Thus by Lemma 6.2, $F'$ has a matching $M'$ covering $X \setminus (V(M) \cup \{x_0\})$. Now it is easy to see that $S := V(F) - V(M \cup M')$ is a balanced $(k+1)$-set with $\{x_0\} = S \cap X$. Note that $F[S \cup V(M)]$ has a perfect matching $M''$. Therefore, $M' \cup M''$ is a perfect matching of $F$.

7 Concluding remarks

First, we point out that Theorem 1.3 follows immediately from Lemmas 3.4 and 6.3, which in turn implies Theorem 1.2.

Thus, we proved a rainbow version of the result of Rödl, Ruciński, and Szemerédi [28] that determines the co-degree threshold function for the existence of a perfect matching in a $k$-graph.

There are many results on various Dirac type conditions for the existence of a matching of certain size. One can ask questions about whether similar rainbow versions hold for those results. Our method of converting the rainbow matching problem to a matching problem for a special class of hypergraphs provides a way for establishing the rainbow versions by using existing tools for matching problems.

We list some results that their rainbow version may be studied using our approach. Rödl, Ruciński, and Szemerédi [28] proved that, for $n \neq 0 \mod k$, the minimum co-degree threshold that ensures a matching $M$ in a $k$-graph $H$ with $|V(M)| \geq |V(H)| - k$ is between $\lfloor n/k \rfloor$ and $n/k + O(\log n)$, and conjectured that this threshold function is $\lfloor n/k \rfloor$. This conjecture was proved recently by Han [8]. Treglown and Zhao [29, 30] determined the minimum $l$-degree threshold for perfect matchings in $k$-graphs for $k/2 \leq l \leq k - 1$.

Bollobás, Daykin, and Erdős [4] considered the minimum vertex degree for the appearance of matchings of certain size. They proved that for integer $k \geq 2$, if $H$ is a $k$-graph of order $n \geq 2k^3(m-1)$ and $\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1}$, then $H$ has a matching of size at least $m$. The bound on $n$ is improved to $n \geq 3k^2m$ by Huang and Zhao [11] recently. For 3-graphs, Kühn, Osthus, and Treglown [21] and, independently, Khan [14] determined the minimum vertex degree threshold for perfect matchings in 3-graphs, which improves an earlier result by Hán, Person, and Schacht [7]. The minimum vertex degree threshold for perfect matchings in 4-graphs is obtained by Khan in [15].

References

[1] J. Akiyama and P. Frankl, On the size of graphs with complete-factors, J. Graph Theory, 9 (1985), 197–201.

[2] R. Aharoni and D. Howard, A rainbow $k$-partite version of the Erdős-Ko-Rado theorem, Comb. Probab. Comput., 26 (2017), 321–337.

[3] N. Alon and J. Spencer, The Probabilistic Method, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley Sons, Hoboken, NJ, 2000, third edition, 2008.

[4] B. Bollobás, D.E. Daykin and P. Erdős, Sets of independent edges of a hypergraphs, Quart. J. Math. Oxford Ser., 27, (1976), 25–32.
[5] J. Edmonds, Path, trees and flowers, *Canadian J. Math*, 17, 449–467.

[6] P. Frankl and A. Kupavskii, Simple juntas for shifted families, *Discrete Analysis*, (2020), 14507.

[7] J. Gao, H. Lu, J. Ma and X. Yu, On the rainbow matching conjecture for 3-uniform hypergraphs, *arXiv:2011.14363*.

[8] J. Han, Near perfect matching in $k$-uniform hypergraph, *Combin., Probab. and Comput.*, 24 (2015), 723–732.

[9] H. Hán, Y. Person and M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, *SIAM J. Discret. Math.*, 23, (2009), 732–748.

[10] H. Huang, P. Loh and B. Sudakov, The size of a hypergraph and its matching number, *Combin. Probab. Comput.* 21 (2012), 442–450.

[11] H. Huang and Y. Zhao, Degree versions of the Erdős-Ko-Rado theorem and Erdős hypergraph matching conjecture, *J. Combin. Theory, Ser. A*, 150 (2017), 233–247.

[12] R. Karp, Reducibility among combinatorial problems, *Complexity of Computer Computations* (R. E. Miller; J. W. Thatcher; J. D. Bohlinger (eds.)), New York; Plenum. pp. 85–103.

[13] F. Joos and J. Kim, On a rainbow version of Dirac’s theorem, *Bull. London Math. Soc.*, 52 (2020) 498–504.

[14] I. Khan, Perfect matchings in 3-uniform hypergraphs with large vertex degree, *SIAM J. Discrete Math.*, 27(2013), 1021–1039.

[15] I. Khan, Perfect Matchings in 4-uniform Hypergraphs, *J. Combin. Theory, Ser. A*, 116(2016), 333—366.

[16] P. Keevash, N. Lifshitz, E. Long and D. Minzer, Hypercontractivity for global functions and sharp thresholds, *arXiv:1906.05568*.

[17] P. Keevash, N. Lifshitz, E. Long and D. Minzer, Global hypercontractivity and its applications, *arXiv:2103.04604*.

[18] N. Keller and N. Lifshitz, The Junta Method for Hypergraphs and Chvátal’s Simplex Conjecture, *arXiv:1707.02643*.

[19] S. Kiselev and A. Kupavskii, Rainbow matchings in $k$-partite hypergraphs, *Bulletin of the London Math. Society*, 53 (2021), N2, 360—369.

[20] D. Kühn and D. Osthus, Matchings in hypergraphs of large minimum degree, *J. Graph Theory*, 51 (2006), 269–280.

[21] D. Kühn, D. Osthus and A. Treglown, Matchings in 3-uniform hypergraphs, *J. Combin. Theory, Ser. B*, 103 (2013), 291-305.
[22] A. Kupavskii, Rainbow version of the Erdős matching conjecture via concentration, arXiv:2104.08083.

[23] H. Lu, Y. Wang and X. Yu, Almost perfect matchings in $k$-partite $k$-graphs, SIAM J. Discrete Math., 32 (2018), 522–533.

[24] H. Lu, Y. Wang and X. Yu, A better bound on the size of rainbow matchings, arXiv:2004.12561v3.

[25] H. Lu, Y. Wang and X. Yu, Rainbow perfect matchings for 4-uniform hypergraphs, arXiv:2105.08608.

[26] H. Lu, X. Yu and X. Yuan, Rainbow matchings for 3-uniform hypergraphs, J. Combin. Theory Ser. A, 183 (2021), No. 105489.

[27] V. Rödl, A. Ruciński and E. Szemerédi, Perfect matchings in uniform hypergraphs with large minimum degree, European J. Combin., 27 (2006), 1333–1349.

[28] V. Rödl, A. Ruciński, and E. Szemerédi, Perfect matchings in large uniform hypergraphs with large minimum collective degree, J. Combin. Theory Ser. A, 116 (2009), 616–636.

[29] A. Treglown and Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs, J. Combin. Theory Ser. A, 119 (2012), 1500–1522.

[30] A. Treglown and Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II, J. Combin. Theory Ser. A, 120 (2013), 1463–1482.