EXCESS DIMENSIONS FOR BRILL-NOETHER SCHEMES OF STABLE VECTOR BUNDLES

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Abstract. We extend a result by Fulton-Harris-Lazarsfeld in Brill-Noether theory of line bundles and, as well, a result by Aprodu-Sernesi in theory of Secant Loci, to the Brill-Noether locus of stable bundles inside the moduli space of higher rank stable vector bundles on a smooth projective algebraic curve. We give some consequences of this extended result.

1. Introduction

Let $C$ be a smooth projective algebraic curve of genus $g$. For integers $r$ and $d$ with $0 \leq 2r \leq d$, the Brill-Noether scheme of line bundles on $C$, denoted commonly in literature by $W^r_d$, parameterizes line bundles of degree $d$ with the space of global sections of dimension at least $r + 1$. These schemes have now been extensively studied in literature. On a general curve $C$, "The theorem of Brill-Noether theory" computes the dimensions of these schemes to be the expected dimension which is a quantity in terms of $d$, $r$ and $g$; (see [12]). The expected dimension, $d - (r + 1)(g - d + r)$, may fail to attain when $C$ is special in the sense of moduli.

The basic Martens' theorem bounds the dimension of $W^r_d$ in terms of $d$ and $r$, on arbitrary smooth projective curves. Various variations of this basic result have been obtained in the literature, see [22], [8], [9] and [3]. Another result concerning dimensions of Brill-Noether schemes of line bundles, which holds on arbitrary curves, is the excess dimension inequality. William Fulton, Joe Harris and Rob Lazarsfeld proved in [11] that the difference of dimensions of $W^r_d$ and $W^{r+1}_d$, (of $W^r_d$ and $W^{r+1}_d$), are controllable in terms of $r$, (in terms of $r$, $d$ and $g$); respectively. A significant consequence of their result is that if $\dim W^r_d \geq r+1$, then $W^{r-1}_d$ would be non-empty.

The schemes of secant loci of appropriate line bundles are generalizations of $W^r_d$, somehow. Marian Aprodu and Edoardo Sernesi have extended the excess dimension inequalities to the schemes of secant loci of very ample line bundles. Michael Kemeny used their result in studying the syzygies of curves of arbitrary gonalities, see [17]. As well, in [5], the author uses Aprodu-Sernesi' excess dimension result to obtain a Mumford type theorem for schemes of secant loci.

Since some bundles appeared in constructing $W^r_d$'s turn to be ample, no non-emptiness prerequisite is needed in Fulton-Harris-Lazarsfeld framework. But, this positivity property fails to be hold in Aprodu-Sernesi’ general setting. So, they enter weaker hypothesis, some

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non-emptiness assumptions, to study various degeneracy locus of morphisms of bundles, set in a diagram as 2.1 with \( t = 1 \). Consequently (F-H-L)’s result would be a special case of Aprodu-Sernesi’ result. In a diagram as 2.1 with \( t = 1 \), Aprodu and Sernesi compare dimensions of \( D_k(\sigma), D_{k+1}(\sigma) \) and \( D_k(\pi\sigma) \), where \( D_k(\sigma) \) is defined as

\[
D_k(\sigma) := \{ x \in X \mid \text{rk}(\sigma_x) \leq k \}.
\]

We study Aprodu-Sernesi’ setting in more general situation. We compare the degeneracy locus of morphisms involved in a similar diagram, when \( \text{rk}F_1 = n + t \), for \( t \in \mathbb{N} \). This is also related to a problem imposed by Fulton-Harris-Lazarsfeld in [11, Remark p.321]. A basic issue in our study is that the locus \( D_k(\pi\sigma) \) may fail to be included in \( D_{k+1}(\sigma) \). Instead we compare the dimensions of \( D_{k+1}(\sigma) \) and \( D_k(\pi\sigma) \cap D_{k+1}(\sigma) \), which is a slightly more general setting.

In section 3, we study the excess-dimension problem for Brill-Noether locus of stable vector bundles of rank \( r \), degree \( d \) and with the space of global sections at least of dimension \( k \), \( B_{k,r,d} \), inside the moduli space of stable vector bundles of rank \( r \geq 2 \). In order to do so, we need a divisor \( X \subset B_{k+1,r,d+1}^1 \). We construct such a divisor by using elementary transformations of bundles in \( B_{k,r,d}^1 \). Then, we apply theorem 2.4, by which we obtain our main result

**Theorem 1.1.** Assume that \( C \) is a smooth curve. If \( B_{k,r,d-1}^1 \) is nonempty, then

(a) \( \dim B_{k,r,d-1}^1 \geq \dim B_{k,r,d}^k - k \),

(b) \( \dim B_{k+1,r,d+1}^k \geq \dim B_{k,r,d}^k - (r(g-1) - d + k) \),

(c) \( \dim B_{k,r,d}^k \geq \dim B_{k,r,d}^k - (r(g-1) - d + 2k) \).

We use theorem 1.1 together with a comparision of dimension of tangent spaces to relate the smoothness, irreducibility and reducedness of \( B_{k,r,d}^k \) to those of \( B_{k,r,d+1}^k \) as

**Theorem 1.2.** Let \( E \in B_{k,r,d}^k \) and \( F \) be an elementary transformation of \( E \) by a general point \( p \in C \). Assume moreover that \( B_{k,r,d}^k \) is of expected dimension. Then, for \( 1 \leq d \leq r(g-1) \) and \( 1 \leq k \leq \frac{d}{2} + 1 \) we have

(a) The scheme \( B_{k,r,d}^k \) is smooth at \( E \) if and only if \( B_{k,r,d+1}^{k+1} \) is smooth at \( F \).
(b) Assume that \( B_{k,r,d}^k \) is irreducible, then \( B_{k,r,d+1}^{k+1} \) would be irreducible as well. If furthermore, \( B_{k,r,d}^k \) is reduced, then \( B_{k,r,d+1}^{k+1} \) would be reduced too.

**Notation.** Throughout, \( C \) denotes a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field \( \mathbb{K} \) of characteristic zero. For a sheaf \( F \) on \( C \), we abbreviate \( H^i(C, F) \) and \( h^i(C, F) \) to \( H^i(F) \) and \( h^i(F) \), respectively. As is conventionally, if \( \sigma : E \to F \) is a morphism of locally free sheaves \( E, F \) of ranks \( \alpha, \beta \) on an algebraic variety \( X \), respectively, then for \( p \in X \) the \( \alpha \times \beta \) matrix \( \sigma |_p \) acts on an \( \alpha \)-vector \( \nu \in E |_p \) by \( \nu \cdot \sigma |_p \).
2. Aprodi-Sernesi' general setting

Let $X$ be an integral algebraic variety and consider a diagram of vector bundles as

$$
\begin{array}{ccc}
0 & \longrightarrow & H \\
\sigma & \longrightarrow & F \\
\downarrow \pi & & \downarrow \pi \sigma \\
E & \longrightarrow & F_1
\end{array}
$$

where the top row is exact and $E, F_1, F$ are of ranks $m, n + t, n$, respectively, with $t \in \mathbb{N}$. Then, clearly $D_k(\sigma) \subseteq D_k(\pi \circ \sigma) \subseteq D_{k+1}(\sigma)$. If

$$\mu : M(m, n + t) \times M(n + t, n) \to M(m, n)$$

be the multiplication map, then we have $M_k(m, n + t) \times M(n + t, n) \subseteq \mu^{-1}(M_k(m, n)) \subseteq M_{k+1}(m, n + t) \times M(n + t, n)$. The diagram (2.1) induces a map $f : X \to M(m, n + t) \times M(n + t, n)$, and for $1 \leq k < \min\{m, n\}$ one has the identifications

$$D_k(\sigma) = f^{-1}(M_k(m, n + t) \times M(n + t, n)) \quad \text{and} \quad D_k(\pi \sigma) = f^{-1}(\mu^{-1}(M_k(m, n))).$$

Lemma 2.1. The schemes

$$[\mu^{-1}(M_k(m, n)) \cap (M_{k+1}(m, n + t) \times M(n + t, n))] \setminus (M(m, n + t) \times M_{n-1}(n + t, n))$$

and

$$[\mu^{-1}(M_k(m, n)) \setminus (M(m, n + t) \times M_{n-1}(n + t, n))]$$

are of dimensions $(m + n)(n + t) - (m - k - 1)(n + t - k - 1) - (n - k)$ and $(m + n)(n + t) - (m - k)(n + t - k - 1)$, respectively.

Proof. As for the first locus, set

$$\mathcal{A} := [\mu^{-1}(M_k(m, n)) \cap (M_{k+1}(m, n + t) \times M(n + t, n))] \setminus (M(m, n + t) \times M_{n-1}(n + t, n)),$$

and let $\pi_2 : \mathcal{A} \to M_{n-1}^c(n + t, n)$ be the restriction of the second projection map, where we set $M_{n-1}^c(n + t, n) := M(n + t, n) \setminus M_{n-1}(n + t, n)$. Observe that in computing the dimension of the fibers, one can replace a matrix $B \in M_{n-1}^c(n + t, n)$ by a matrix as

$$
\begin{pmatrix}
I_{n \times n} \\
0_{(t-n) \times n}
\end{pmatrix}.
$$

The map $\pi_2$ projects on $M_{n-1}^c(n + t, n)$ and a general element $(A, B) \in \mathcal{A}$ belonging to the fiber of $\pi_2$ on $B$ is depended on $\dim M_{k+1}(m, n + t) - (n - k)$ number of parameters, as required.

The dimension of $[\mu^{-1}(M_k(m, n)) \setminus (M(m, n + t) \times M_{n-1}(n + t, n))]$ can be computed by a similar argument as in proof of \cite Prop. 2.1, to which we refer.

\[\square\]

Theorem 2.2. Let $E, F_1, F, m, n + t, n$ and $t \in \mathbb{N}$ be as in the diagram (2.1).

(a) Assume that no irreducible component of $D_{k+1}(\sigma)$ contains $D_k(\sigma)$. If $D_k(\pi \sigma) \cap D_{k+1}(\sigma)$ is nonempty, then

$$\dim D_k(\pi \sigma) \cap D_{k+1}(\sigma) \geq \dim D_{k+1}(\sigma) - (n - k).$$
(b) If no irreducible component of $D_k(\pi \sigma)$ is contained in $D_{k-1}(\pi \sigma)$ and if $D_k(\sigma)$ is nonempty, then
\[
\dim D_k(\sigma) \geq \dim D_k(\pi \sigma) - (m - k),
\]

(c) Assume that no irreducible component of $D_k(\pi \sigma) \cap D_{k+1}(\sigma)$ is contained in $D_k(\sigma)$ and no irreducible component of $D_k(\pi \sigma)$ is contained in $D_{k-1}(\pi \sigma)$. If $D_k(\sigma)$ is nonempty, then
\[
\dim D_k(\sigma) \geq \dim D_{k+1}(\sigma) - (m + n - 2k).
\]

Proof. (a) Recall that, under the assumption, we have
\[
\dim D_{k+1}(\sigma) = \dim(D_{k+1}(\sigma) \setminus D_k(\sigma)).
\]
We use Lemma 2.1 to apply theorem [14, Thm. 17.24] to the morphism
\[
f : Z = (D_{k+1}(\sigma) \setminus D_k(\sigma)) \longrightarrow X = (M_{k+1}(m, n + t) \setminus M_k(m, n + t)) \times M(n + t, n)
\]
\[
\supseteq Y = [\mu^{-1}(M_k(m, n)) \cap M_{k+1}(m, n + t) \times M(n + t, n)] \setminus (M_k(m, n + t) \times M(n + t, n)),
\]
by which we get the result.

(b) As in (a), the assumption implies that $\dim D_k(\pi \sigma) = \dim(D_k(\pi \sigma) \setminus D_{k-1}(\pi \sigma))$. Now we apply Lemma 2.1 and [14, Thm. 17.24] to the morphism $f : Z \to X$ with
\[
Z := D_k(\pi \sigma) \setminus D_{k-1}(\pi \sigma), \quad X := \mu^{-1}(M_k(m, n)) \setminus \mu^{-1}(M_{k-1}(m, n))
\]
\[
Y := M_k(m, n + t) \times M(n + t, n).
\]

(c) This is a trivial consequence of parts (a) and (b). \qed

Since with settings of diagram 2.1 one has $D_k(\pi \sigma) \subset D_{k+t}(\sigma)$, we compare their dimensions:

**Theorem 2.3.** Assume $X$ is smooth and admits an ample line bundle. Let $E, F_1, F, m, n + t, n$ and $t \in \mathbb{N}$ be as in the diagram 2.7. Assume moreover that no irreducible component of $D_{k+t-1}(\pi \sigma)$ is contained in $D_{k+t}(\sigma)$. If $D_k(\sigma)$ is nonempty, then
\[
\dim D_k(\pi \sigma) \geq \dim D_{k+t}(\sigma) - t(n - k),
\]
\[
\dim D_k(\sigma) \geq \dim D_k(\pi \sigma) - t(m - k),
\]
\[
\dim D_k(\sigma) \geq \dim D_{k+t}(\sigma) - t(m + n - 2k).
\]

Proof. We prove 2.1 If $t = 1$, then the assertion is Theorem [11, Th. 3.1]. If $t \geq 2$, then the existence of an ample line bundle together with smoothness of $X$, implies that a Harder-Narasimhan filtration exists for $E$. Particularly, it has a sub line bundle $L$ with the torsion free quotient sheaf $\frac{E}{L}$. The torsion freeness of $\frac{E}{L}$ and that of $\frac{F}{H}$ imply that the quotient sheaf $\frac{E}{L}$ is torsion free, so both of them would be vector bundles.
We consider two diagrams as

\[(A): \begin{array}{c}
0 \rightarrow L \rightarrow F_1 \rightarrow 0, \\
\sigma \downarrow \pi_0 \sigma \downarrow \pi_0 \\
E \rightarrow E
\end{array}
\]

\[(B): \begin{array}{c}
0 \rightarrow L \rightarrow F_1 \rightarrow 0, \\
\pi_0 \sigma \downarrow \pi_0 \sigma \downarrow \pi_0 \\
E \rightarrow E
\end{array}
\]

The rest of the proof can be shrunk into the following inequalities:

\[(2.8) \quad \dim D_k(\pi \sigma) \geq \dim D_{k+t-1}(\pi_0 \sigma) - (t - 1)(n - k),
\]

\[(2.9) \quad \dim D_{k+t-1}(\pi_0 \sigma) \geq \dim D_{k+t}(\sigma) - (n - k).
\]

We prove \[2.8\] Using diagram (A) we obtain

\[(2.10) \quad D_{k+t-2}(\pi_0 \sigma) \subset D_{k+t-1}(\sigma).
\]

If an irreducible component of \(D_{k+t-2}(\pi \sigma)\) was contained in \(D_{k+t-2}(\pi_0 \sigma)\), then by \[2.10\] an irreducible component of \(D_{k+t-1}(\pi \sigma)\) would be contained in \(D_{k+t-1}(\sigma)\), which is absurd by assumption. Furthermore, by \(D_k(\sigma) \subset D_k(\pi_0 \sigma)\), the non-emptiness of \(D_k(\sigma)\) implies that of \(D_k(\pi_0 \sigma)\). Therefore, the induction hypothesis can be applied to diagram \[2.7\] (B), by which we obtain \[2.8\].

We prove \[2.9\] If in the diagram (A), an irreducible component of \(D_{k+t-1}(\pi_0 \sigma)\) was contained in \(D_{k+t-1}(\sigma)\), then from diagram (B), an irreducible component of \(D_{k+t-1}(\pi \sigma)\) would be contained in \(D_{k+t-1}(\sigma)\), which is absurd by assumption. Observe moreover that \(D_{k+t-1}(\sigma)\) is non-empty by non-emptiness of \(D_k(\sigma)\). This, together with theorem \[11\] Th. 3.1] applied to diagram \[2.7\] (A) gives \[2.9\].

Similar argument as in \[2.4\] works for \[2.5\] and \[2.6\] is a combination of \[2.4\] and \[2.6\] \(\square\)

We also need a dual version of \[11\] Th. 3.1, as

**Theorem 2.4.** Let \(E, F_1\) and \(F\) are of ranks \(m, n + 1\) and \(n\), respectively, which are set in a diagram

\[(2.11) \quad \begin{array}{c}
0 \rightarrow F_1 \rightarrow F \rightarrow L \rightarrow 0, \\
\sigma \downarrow \pi \sigma \downarrow \pi \sigma \downarrow \pi \sigma \\
E \rightarrow E
\end{array}
\]

Assume moreover that no irreducible component of \(D_k(\sigma_i)\) is contained in \(D_k(\sigma)\).

(i) If \(D_k(\sigma_i)\) is nonempty, then

\[(2.12) \quad \dim D_k(\sigma_i) \geq \dim D_{k+1}(\sigma) - (n - k),
\]

(ii) If \(D_k(\sigma)\) is nonempty, then

\[(2.13) \quad \dim D_k(\sigma) \geq \dim D_k(\sigma_i) - (m - k),
\]

\[(2.14) \quad \dim D_k(\sigma) \geq \dim D_{k+1}(\sigma) - (m + n - 2k).
\]
Proof. For a morphism $\gamma : E \to F$ of vector bundles on an integral algebraic scheme $X$, one has $D_k(\gamma) = D_k(\gamma^*)$, where $\gamma^* : F^* \to E^*$ is the dual homomorphism of $\gamma$. The theorem follows, because diagram (2.11) is a dual version of the diagram considered in proposition [11] Th. 3.1]. □

Remark 2.5. The result of [11] can be interpreted as a special case of comparing the dimension of degeneracy locus of morphisms, set in some diagram as 2.1 when $\text{rk} \mathbb{H} = 1$ and $E^* \otimes \mathbb{H}$ is ample. Aprodu and Sernesi generalized and studied this problem with a weaker assumption, the non-emptiness of some degeneracy locus, when the kernel is a line bundle. Fulton-Harris-Lazarsfeld propose to compare the degeneracy locus of morphism of vector bundles not necessarily with ample $E^* \otimes \mathbb{H}$; see [11] Remark page 321], where they left it unanswered. While the paper [11] is related to this problem in the case $\text{rk} \mathbb{H} = 1$, Theorems 2.2 and 2.3 push these studies forward to the case when $\mathbb{H}$ is a vector bundle of arbitrary rank.

3. Applications to Brill-Noether locus of stable vector bundles

3.0.1. A divisorial component in $B_{r,d+1}^1$. Let $U(r,d)$ denote the moduli space of stable vector bundles of rank $r$ and degree $d$. Then, $U(r,d)$ admits an etale finite cover on which there exists a Poincare bundle. Precisely, there exists an etale finite cover $\pi : \mathcal{U} \to U(r,d)$ and a bundle $\mathcal{E}_d \to \mathcal{U}$ such that for each $E \in \mathcal{U}$ one has $\mathcal{E}_d(E) \times C \cong \pi(E)$. Furthermore, for any $\lambda \in \mathbb{N}$, $U(r,d) \cong U(r,d + \lambda r)$ by an isomorphism $\theta$, where $\theta$ is given from $U(r,d + \lambda r)$ to $U(r,d)$ by $E \mapsto E \otimes \mathcal{O}(-D)$ with $D := p_1 + \cdots + p_\lambda$, for some fixed $p_i \in C$.

For an integer $\delta$ with $\delta := d + \lambda r > 2r(g - 1)$, consider the projections

$$U(r, \delta) \times C \overset{\pi_1}{\longrightarrow} C$$

and set $E := (\pi_2)_*(\mathcal{E}_\delta)$, $F := (\pi_2)_*(\mathcal{E}_\delta \otimes \pi_1^*\mathcal{O}_D(D))$. The bundles $E$ and $F$ would be of ranks $\delta - r(g - 1)$ and $\lambda r$, respectively. See [13].

The locus of stable vector bundles $E$, of rank $r$ and degree $d$ with the space of global sections of dimension at least $k$, $B_{r,d}^k$, is, up to the isomorphism $\theta$, the $(\delta - r(g - 1) - k)$-th degeneracy locus of the evaluating morphism of bundles

$$\gamma : E \to F.$$

For $k \geq 2$, one may restrict the bundles $E$ and $F$ to $B_{r,\delta}^1$, and as well the morphism $\gamma$ to a morphism of bundles on $B_{r,\delta}^1$. To ease the notations, we denote by the same letters the restrictions of $E$, $F$ and $\gamma$ on $B_{r,\delta}^1$. Then, $B_{r,d}^k$ is the $(\delta - r(g - 1) - k)$-th degeneracy locus of the restricted $\gamma$.

For a general point $p \in C$, consider the subset

$$B_{r,\delta+1,p}^1 := \{F \in B_{r,\delta+1}^1 \mid F \text{ is a general extension as } 0 \to E \to F \to \mathcal{O}_p \to 0, E \in B_{r,\delta}^1\}.$$
Theorem 3.1. The set $B_{r,\delta+1,p}^1$ is a closed subscheme of $B_{r,\delta+1}^1$ of codimension 1.

Proof. Claim: If $E \in U(r, \delta)$ and $p \in C$ is general, then a general extension $F_p$ as $0 \to E \to F_p \to \mathbb{C}_p \to 0$, is stable.

Proof of Claim: Dually, we prove that a general subbundle $F$ of $E$, set in an exact sequence as $0 \to F \to E \to \mathbb{C}_p \to 0$, is stable. Assume that $H$ is a proper subbundle of $F$ and let $K \subset E$ be the subbundle generated by $H$ in $E$. Then we have a diagram as

\begin{equation}
\begin{array}{cccccc}
0 & \to & F & \to & E & \to \mathbb{C}_p & \to 0, \\
0 & \to & H & \to & K & \to 0
\end{array}
\end{equation}

Since the dual elementary transformation is generic, the non-zero map $E_p \to \mathbb{C}_p$ induces a non-zero map $K_p \to \mathbb{C}_p$. This map gives rise to a map of sheaves as $K \to \mathbb{C}_p$, so making the above diagram commutative, in which the lower row is replaced by a row as $0 \to H \to K \to \mathbb{C}_p \to 0$. We therefore get, by stability of $E$, that

$$\mu(H) = \mu(K) - \frac{1}{r_H} < \mu(E) - \frac{1}{r_E} = \mu(F) + \frac{1}{r_F} - \frac{1}{r_H} < \mu(F),$$

as required.

The Claim implies that $B_{r,\delta+1,p}^1$ is a non-empty subset of $B_{r,\delta+1}^1$. Restrict the Poincare bundle $E_\delta$ to $B_{r,\delta}^1 \times C$ and denote its restriction by the same letter. Consider the sheaf

\begin{equation}
\mathbb{V}_\delta := R^1(\pi_1)_*(\mathcal{E}^i(E, (\pi_{2,\delta})^*\mathcal{O}_p)),
\end{equation}

where $\mathcal{E}^i(F,G)$ denotes the $i-th$ "ext" sheaves of $F$ and $G$. The sheaf $\mathbb{V}_\delta$ is a vector bundle on $B_{1,\delta}$ of rank $r$, by [2, pp. 166-167]; and the projective bundle

\begin{equation}
\alpha_\delta : \mathbb{P}(\mathbb{V}_\delta) \to B_{r,\delta}^1
\end{equation}

is the family of $\text{Ext}^1(E, \mathcal{O}_p)$'s as $E$ varies in $B_{r,\delta}^1$. Then, one has

$$B_{r,\delta+1,p}^1 = \text{Im}(\nu_\delta),$$

where the morphism $\nu_\delta : \mathbb{P}(\mathbb{V}_\delta) \to B_{r,\delta+1}^1$ assigns the stable bundle $F$ to each extension $0 \to E \to F \to \mathcal{O}_p \to 0$. Since $\mathbb{P}(\mathbb{V}_\delta)$ is complete, so $\text{Im}(\nu_\delta)$ is a closed sub scheme of $B_{r,\delta+1}^1$.

In order to prove that $B_{r,\delta+1,p}^1$ is of codimension 1 in $B_{r,\delta+1}^1$, observe that the fibers of both morphisms $\alpha_\delta$ and $\nu_\delta$ are $r$-dimensional. Indeed, for an extension $0 \to E \to F \to \mathcal{O}_p \to 0$, the fiber of $\nu_\delta$ on this extension corresponds to elements of $\text{Ext}^1(F^*, \mathcal{O}_p)$, which is trivially $r$-dimensional. Therefore

$$\dim B_{r,\delta+1,p}^1 = \dim B_{r,\delta}^1 = \dim B_{r,\delta+1}^1 - 1,$$

by [23, Theorem II.3.1], as required. \qed
Remark 3.2.  (i) There is no doubt that the Claim in proof of theorem 3.1 has to be known in the literature, but I have been unable to locate a reference, so I included an argument for its proof. A similar situation, but with different nature, has been appeared in [7, Lemma 1, Lemma 2].

(ii) Various enumerative and geometric properties, e.g. cohomology classes, very ampleness, etc., of $B_{r,\delta+1,p}^1$ might be of some independent interest to study. This might be the objective of an upcoming study.

3.0.2. Excess dimension for Brill-Noether locus’. In this subsection, we apply the generalized machinery of section 2 to Brill-Noether locus of stable vector bundles. In order to do, fixing a general $p \in C$, we deduce that there exists a diagram as in diagram (2.11) on the divisorial locus $B_{r,\delta+1,p}^1$. Then we compare degeneracy loci of the morphisms therein.

Recall that $(\pi_2)_*\mathcal{E}_{\delta+1}$ is a bundle on $U(r, \delta + 1)$ of rank $\delta - r(g - 1) + 1$, and we have an exact sequence of sheaves

$$(3.7) \quad 0 \to (\nu_3)_*(\alpha_\delta)^*(\mathcal{E}_\delta) \xrightarrow{i} ((\pi_2)_*\mathcal{E}_{\delta+1})_{|B_{r,\delta+1,p}^1} \to \mathbb{L} \to 0,$$

on $B_{r,\delta+1,p}^1$. Note then the sheaf $\mathbb{L}$ is torsion free. Indeed, the stalk of $\mathbb{L}$ at $F \in B_{r,\delta+1,p}^1$ coincides on the quotient space $H^0(F)$, which is non-zero. Therefore, $\mathbb{L}$ is a line bundle and (3.7) is a sequence of vector bundles. For a general $p \in C$, observing $p$ as an effective divisor of degree 1 on $C$ and with a diagram as (3.1) we have $\mathbb{L} = (\pi_2)_*(\pi_1^*p)$.

We now consider a diagram as

$$(3.8) \quad 0 \quad (\nu_3)_*(\alpha_\delta)^*((\pi_2)_*\mathcal{E}_\delta) \quad \xrightarrow{i} \quad ((\pi_2)_*\mathcal{E}_{\delta+1})_{|B_{r,\delta+1,p}^1} \quad \xrightarrow{\gamma_{\delta+1}} \quad \mathbb{L} \quad \xrightarrow{0} \quad 0,$$

where $\gamma_{\delta+1}$ is the composition

$$(\nu_3)_*(\alpha_\delta)^*((\pi_2)_*\mathcal{E}_\delta) \xrightarrow{\gamma_{\delta+1}} (\pi_2)_*((\pi_2)_*\mathcal{E}_\delta \otimes \pi^*(\mathcal{O}_D(D))) \xrightarrow{\alpha} (\pi_2)_*(\mathcal{E}_{\delta+1} \otimes \pi^*(\mathcal{O}_D(D)))_{|B_{r,\delta+1,p}^1}.$$ 

We have $\dim D_{\delta-r(g-1)-k}(\gamma_{\delta+1}) = \dim B_{r,d}^{k-1}$ and $\dim D_{\delta-r(g-1)-k}(\gamma_{\delta+1})_{|B_{r,\delta+1,p}^1} = \dim B_{r,d}^k$. Now we apply Theorem 2.4 to this situation, by which we obtain:

**Theorem 3.3.** If $B_{r,d}^k$ is nonempty, then

(a) $\dim B_{r,d}^k \geq \dim B_{r,d+1}^{k-1} - (r(g - 1) - d + k)$,

(b) $\dim B_{r,d}^k \geq \dim B_{r,d+1}^{k-1} - k$,

(c) $\dim B_{r,d+1}^k \geq \dim B_{r,d+1}^{k-1} - (r(g - 1) - d + 2k)$.

In particular, the scheme $B_{r,d+1}^k$ is of expected dimension if and only if $B_{r,d}^k$ to be of expected dimension.

**Proof.** (a) Taking the inequality 2.13 of theorem 2.4 into account, it suffices to prove that no irreducible component of $B_{r,d}^k$ is contained in $B_{r,d+1}^{k+1}$, which is immediate by [9, Remark 2.3] or by [20, Prop. 1.6].
(b) This is a direct consequence of part (a) together with taking residuals and applying the Riemann-Roch theorem.

(c) This is a combination of (a) and (b).

\[ \Box \]

**Example 3.4.** There are cases in which the inequalities in theorem 3.3 turn to be sharp, at least for \( r = 2 \). On general curves of genus \( g \geq 1 \) and for \( d \) with \( 3 \leq d \leq 2g - 1 \), M. Teixidor i Bigas computes the dimension of \( B_{2,d}^2 \) to be \( 2d - 3 \) and \( B_{2,d}^3 \) to be either empty or of dimension \( 3d - 2g - 6 \). See [24] and [23] Theorem 2]. So, the equality holds for such \( B_{2,d}^2, B_{2,d+1}^3, B_{2,d}^3, \) and \( B_{2,d+1}^3 \), in the allowable range.

Also inequality can hold in theorem 3.3 strictly, at least for \( r = 2 \). Let \( C \) be a general curve of genus 6. Then, the scheme \( B_{2,10}^3 \) is non-empty, because the locus

\[ B_{2,K}^3 := \{ E \in B_{2,10}^3 \mid \det(E) = K \}, \]

is non-empty. Furthermore, \( B_{2,10}^3 \) would have superabundant components of dimension at least 6, provided that \( B_{2,K}^3 \) is strictly contained in \( B_{2,10}^3 \). Assuming this to be hold, we would have \( \dim B_{2,10}^3 \geq 6 \). Once again, using [24] and [23] Theorem 2], we obtain \( \dim B_{2,9}^3 = 9 \). So 3.3(c) holds strictly. Furthermore \( \dim B_{2,11}^2 = \dim B_{2,9}^3 = 9 \), by residuation. So 3.3(b) holds strictly.

It remains only to prove that not any vector bundle \( E \in B_{2,10}^3 \) belongs to \( B_{2,K}^3 \), to which it suffices to represent a bundle \( E \in B_{2,10}^3 \) with \( \det(E) \neq K \). In order to see this, take line bundles \( L_1, L_2 \in W_5^1 \setminus W_5^2 \) with \( L_1 + L_2 \neq K \) and consider the extensions as

\[ 0 \to L_1 \to E \to L_2 \to 0. \]

Such an extension, belonging to \( B_{2,10}^3 \setminus B_{2,K}^3 \), exists according to [18] Proposition 3.1] and [19] Lemma 2.18].

**Remark 3.5.** The importance of theorem 3.3 goes back to the fact that, even on general curves, the Brill-Noether schemes of higher rank stable bundles may fail to have the expected dimension. So the dimension of these schemes seems to be far from being controllable by numerical parameters \( r, d, k \) and \( g \). Theorem 3.3 is an assertion about the dimension of these spaces, without imposing any circumstances on the curve \( C \).

3.0.3. A byproduct: As a consequence, we relate the smoothness, reducedness and irreducibility of higher rank Brill-Noether schemes with large degrees to those of higher rank Brill-Noether schemes with low degree. In order to do this, we first make a comparison among the dimension of the tangent spaces of these schemes. An analogue of this argument has been done in Theorem 4.1 Th. 3.1] for schemes of secant loci. See also [16] for a more recent reference on the structure of the tangent spaces of twisted Brill-Noether schemes, which are vast generalizations of Brill-Noether schemes.

**Lemma 3.6.** Let \( p \in C \) be a general point, \( E \in B_{r,d}^k \setminus B_{r,d+1}^{k+1} \) and \( F \) be an elementary transformation of \( E \) by \( p \). Then, for \( 1 \leq d \leq r(g - 1) \) and \( 1 \leq k \leq \frac{d}{2} + 1 \) we have

\[ \dim T_E(B_{r,d}^k) \geq \dim T_F(B_{r,d+1}^k) - k. \]
proof. Recall that

\[ T_E(B_{r,d}^k) = \{ \mu_E : H^0(E) \otimes H^0(K \otimes E^*) \rightarrow H^0(K \otimes E \otimes E^*) \} \perp, \]

where for a sub-vector space \( W \leq V \), by \( W^\perp \) we mean the annihilator vector space of \( W \) inside the dual of \( V \) and \( \mu_E \) denotes the Petri map of \( E \).

Notice that, under the hypothesis, the bundle \( E \) is special and so lemma [21 2.5] can be applied to obtain \( H^0(E) = H^0(F) \). Therefore \( F \in B_{r,d+1}^k \setminus B_{r,d+1}^{k+1} \). There exists now a diagram as

\[
\begin{array}{ccc}
H^0(E) \otimes H^0(K \otimes E^*) & \overset{\mu_E}{\longrightarrow} & H^0(K \otimes E \otimes E^*) \\
\downarrow i \quad & & \quad \downarrow \alpha \\
H^0(F) \otimes H^0(K \otimes F^*) & \overset{\mu_F}{\longrightarrow} & H^0(K \otimes F \otimes F^*) \\
& \quad \quad \beta \\
& H^0(K \otimes F \otimes E^*),
\end{array}
\]

where the morphisms \( i \), \( \alpha \) and \( \beta \) are all injective. Therefore, we will have \( \dim \ker(\mu_E) \geq \dim \ker(\mu_F) \). Once again lemma [21 2.5] together with the Riemann-Roch theorem implies that \( h^0(K \otimes F^*) = h^0(K \otimes E^*) - 1 \). Therefore,

\[ \dim \ker(\mu_E) \leq \dim \ker(\mu_F) + k, \]

implying \( \dim \ker(\mu_E) \leq \dim \ker(\mu_F) + k \), as required. \( \square \)

**Theorem 3.7.** Let \( E \in B_{r,d}^k \) and \( F \) be an elementary transformation of \( E \) by a general point \( p \in C \). Assume moreover that \( B_{r,d}^k \) is of expected dimension. Then, for \( 1 \leq d \leq r(g - 1) \) and \( 1 \leq k \leq \frac{d}{g} + 1 \) we have

(a) The scheme \( B_{r,d}^k \) is smooth at \( E \) if and only if \( B_{r,d+1}^k \) is smooth at \( F \).

(b) Assume that \( B_{r,d}^k \) is irreducible, then \( B_{r,d+1}^k \) would be irreducible as well. If furthermore, \( B_{r,d}^k \) is reduced, then \( B_{r,d+1}^k \) would be reduced too.

**Proof.** (a) Since \( B_{r,d}^k \) is of expected dimension so is the scheme \( B_{r,d+1}^k \) and vice versa, by theorem [3.3]. The smoothness of \( B_{r,d}^k \) at \( E \in B_{r,d}^k \) implies that \( \dim B_{r,d}^k = \dim T_E B_{r,d} \). This, by Lemma [3.6] together with the fact that \( B_{r,d+1}^k \) is of expected dimension, gives rise to the assertion.

(b) Claim I: If \( A \subseteq B_{r,d+1}^k \) is an irreducible component and if \( A \cap B_{r,d+1}^k \) is non-empty, then any irreducible component \( B \) of \( A \cap B_{r,d+1}^k \) satisfies \( \dim B \geq \dim A - k \).

**Proof of Claim I:** If indeed, \( \dim B < \dim A - k = \dim B_{r,d}^k \), then a similar argument as in theorem [3.1] implies that for general \( p \in C \), denoting by \( B_p \) the locus of general extensions of elements of \( B \) by \( O_p \), the locus \( B_p \) is irreducible and is of dimension \( < \dim B_{r,d+1}^k - k \). Then, one can extend this locus to a maximal chain \( B_p \subset W_1 \subset \cdots \subset W_t \) with \( \dim W_t \leq \dim B_{r,d+1}^k - 1 \), which is in contradiction with the expected dimensionality of \( B_{r,d+1}^k \).

Claim II: If \( p \in C \) is general, then any irreducible component \( A \) of \( B_{r,d+1}^k \) contains \( B_{r,d+1}^k \).

**Proof of Claim II:** Recall that for a general \( E \in A \) and a general subbundle \( H \) of \( E \) represented by an extension \( 0 \rightarrow H \rightarrow E \rightarrow O_p \rightarrow 0 \), the bundle \( H \) is stable. Indeed, the
Claim in proof of theorem 3.1 applied to $K \otimes E^*$ and $O_p$, implies that $K \otimes H^*$ is stable, so $H$ is stable. Therefore, for a general $E \in A$, there exists $H \in B_{r,d}^k(x \geq k-1)$, such that $F$ is the elementary transformation of $H$ by $O_p$. Once again using lemma [21] 2.5, we conclude that $x \geq k$. This establishes claim II.

If $A$ and $B$ are irreducible components of $B_{r,d+1}^k$, then $B_{r,d+1}^k \subset A \cap B$ and so for a general element $E \in B_{r,d}^k$, a general extension of $E$ by $p$, denoted by $F$, belongs to $\text{Sing}(B_{r,d+1}^k)$. Therefore

$$\dim T_F(B_{r,d+1}^k) > r^2(g-1) + 1 - k(r(g-1) - d + k - 1).$$

Since we also have $F \in B_{r,d+1}^k \setminus B_{r,d+1}^{k+1}$, lemma 3.6 gives a contradiction.

If, furthermore $B_{r,d}^k$ be reduced, then Claim I and Claim II together with a word by word repetition in the argument of proof of [10] Th. 4(ii)] proves that $B_{r,d+1}^k$ is reduced. $\square$

Remark 3.8. (i) Claims I, II and the argument for their proof are analogues of [10] Proposition 1 and Theorem 2, under some additional hypothesis, i.e. the scheme $B_{r,d}^k$ to be of expected dimension and irreducible. We don’t know if the irreducibility and expected dimensionality property are actually redundant for higher rank B-N locus.’

(ii) If $B_{r,d}^k$ is equi-dimensional, then according to the proof of theorem 3.7(b) the dimension of any irreducible component $A \subseteq B_{r,d+1}^k$ can’t exceed $\dim B + k$, where $B$ is an irreducible component of $B_{r,d+1}^k$. This specifically holds for $B_{r,d}^1$ for $1 \leq d \leq r(g-1)$ by [23] Theorem II.3.1]. When $B_{r,d}^k$ fails to be equi-dimensional, an inequality as $\dim B \geq \dim B_p - k$ is expected to hold, where $p \in C$ is general and $B_p$ is a component of $B_{r,d+1}^k$ whose general element is a general elementary transformation of a general element $E \in B$ by $C_p$. A complete answer to this problem, without equi-dimensional property for $B_{r,d}^k$, is unknown to me.

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