Parameterization of Retrofit Controllers

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Abstract—This study investigates a parameterization of all retrofit controllers. Retrofit control can achieve modular design of control systems, i.e., independent design of subcontrollers only with its corresponding subsystem model. In the framework, the network system to be controlled is regarded as an interconnected system composed of the subsystem of interest and an environment. Existing studies have shown all retrofit controllers can be characterized as a constrained Youla parameterization under the assumption that the subsystem of interest is stable. It has also been discovered that all of the retrofit controllers that belong to a tractable class have a distinctive internal structure under the additional assumption that abundant measurement is available. Due to the structure, the retrofit controller design problem under the assumptions can be reduced to a standard problem. The aim of this paper is to extend the results without the technical assumptions. It is found that the existing results can be generalized through the general Youla parameterization and the geometric control theory. The result indicates that retrofit controllers can readily be designed even in the general case.

Index Terms—Distributed design, large-scale systems, network systems, retrofit control, Youla parameterization

I. INTRODUCTION

Modern cyber technologies of sensing, communicating, and actuating physical systems enable us to handle highly complex and large-scale network systems [1], [2]. A major obstacle of designing a controller for such large-scale systems stems from restriction of information structure within the controller to be designed. Large efforts on designing structured controllers have been devoted to overcoming the challenge along the line of work on decentralized and distributed control [3], [4], [5]. The substantial research facilitates structured controller design in a computationally efficient way.

The subsequent problem considered in this paper is on distributed design of subcontrollers inside the whole structured controller. The existing methods are built on the premise that there exists a unique controller designer. However, in actual network systems, there are often multiple independent controller designers. For example, a power grid is typically governed by multiple companies each of whom is responsible for managing the corresponding part of the grid. Accordingly, each controller for frequency regulation is independently designed by each designer [6]. While integrated controller design by a unique designer is referred to as centralized design, independent design of subcontrollers by multiple designers is referred to as distributed design [7].

Although the notion of distributed design is indispensable for developing scalable controller design methods, there are several difficulties to be resolved. The primal difficulty of distributed design is that, from the perspective of a single controller designer only the model information of their corresponding subsystem is available while the model information of the other part, called environment here, is unknown. Moreover, even if the entire model information is provided at some time instant, the environment possibly varies depending on other controller designers’ action. Several studies on distributed design can be found, such as the deadbeat control approach [7], [8], an integral quadratic constraint approach [9], and system level synthesis [10], [11]. However, the approaches do not fully meet the requirements from both of the theoretical and practical perspectives.

Retrofit control [12], [13], [14] is a newly developed approach to resolve the issue of distributed design. In the retrofit control framework, the network system to be controlled is regarded as an interconnected system composed of the subsystem of interest and an unknown environment. A characterization and a parameterization of all retrofit controllers have been derived under the following technical assumptions:

- the subsystem of interest is stable and
- the interconnection signal or the internal state of the subsystem of interest is measurable

in the existing researches. In more detail, it has been shown that all retrofit controllers can be characterized through the Youla parameterization with a linear constraint on the Youla parameter. Furthermore, it has also been discovered that all of the retrofit controllers that belong to a particular tractable class have a distinctive internal structure composed of a so-called rectifier and an internal controller when abundant measurement is available as stated in the technical assumptions.

The objective of this paper is to generalize the results without the above technical assumptions. First, we consider an interconnected system in which the subsystem of interest is possibly unstable. By considering the Youla parameterization for not the subsystem of interest but the closed-loop system composed of the subsystem of interest and a possible environment, we can generalize the above existing result. It turns out that all retrofit controllers can be obtained through the above modified Youla parameterization with the linear constraint on the modified Youla parameter being the same as the constraint of the existing result. Further, we reveal that all of the retrofit controllers that belong to the above tractable class have a similar structure through an explicit description of the inverse system obtained with the geometric control theory. Owing to the fact, it is shown that the retrofit controller design problem can be reduced to a conventional stabilizing controller design problem even in the general situation.
This paper is organized as follows. Sec. [II] gives mathematical preliminaries. In Sec. [III] we review the exiting retrofit control framework and pose the problems treated in this paper. We provide a characterization of all retrofit controllers for network systems with possibly unstable subsystems in Sec. [IV]. In Sec. [V] we give a parameterization of all output-rectifying retrofit controllers, introduced below, under the general output feedback case. Sec. [VI] draws the conclusion.

II. MATHEMATICAL PRELIMINARIES

In this section, we provide mathematical preliminaries necessary for the discussion in this paper.

Notation: We denote the set of the real numbers by \( \mathbb{R} \), the \( n \times m \)-dimensional real Euclidean vector space by \( \mathbb{R}^n \), the set of the \( n \times m \) real matrices by \( \mathbb{R}^{n \times m} \), the identity matrix by \( I \), the transpose of a matrix \( M \) by \( M^T \), the matrix where matrices \( M_i \) for \( i = 1, \ldots, m \) are concatenated vertically by \( \col(M_i)_{i=1}^m \), the block-diagonal matrix whose diagonal blocks are composed of \( M_i \) for \( i = 1, \ldots, m \) by \( \diag(M_i)_{i=1}^m \), the image of a matrix \( M \) by \( \im M \), the kernel of a matrix \( M \) by \( \ker M \), the set of real \( n \times m \) transfer matrices by \( \mathbb{R}^{n \times m} \), the set of proper transfer matrices in \( \mathbb{R}^{n \times m} \) by \( \mathbb{R}^P^{n \times m} \) and the set of stable transfer matrices in \( \mathbb{R}^{n \times m} \) by \( \mathbb{R}H_\infty^{n \times m} \). When the dimensions of the spaces are clear from the context, we omit the superscript. A matrix \( M \in \mathbb{R}^{n \times m} \) is said to be injective and surjective when \( \ker M = \{0\} \) and \( \im M = \mathbb{R}^n \), respectively. We say that a controller \( K \) stabilizes a system \( G \) when the feedback system composed of \( G \) and \( K \) becomes internally stable, i.e., the transfer matrices from all inputs to all outputs belong to \( \mathbb{R}H_\infty \) [15 Chap. 5]. When a transfer matrix \( G \) has a realization \( C(sI - A)^{-1}B + D \), the shorthand is defined by

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} := G.
\]

Projection Matrix: A square matrix \( M \) is said to be a projection matrix along \( \ker M \) onto \( \im M \) when \( M^2 = M \). For a projection matrix \( M \), there exists a matrix \( P \) being injective such that \( P^T M = M \). Conversely, for a matrix \( P \) being injective and a left inverse \( P^T \), \( P P^T \) and \( I - PP^T \) become projection matrices. We employ the notations \( P \) and \( P^\dagger \) to represent the matrices such that \( P^\dagger = I - PP^T \) and \( P^T \) hold. For the matrices, \( \im P = \ker P^\dagger \) and \( \ker P = \im P^\dagger \).

Youla Parameterization: A collection of stable transfer matrices \( N_r, M_r, U_r, V_r, N_i, M_i, U_i, V_i \in \mathbb{R}H_\infty \) defines a doubly coprime factorization of \( G \in \mathbb{R}P \) if \( G = N_r M_i^{-1} N_i^{-1} \) and

\[
\begin{bmatrix} V_i & -U_i \\ -N_i & M_i \end{bmatrix} \begin{bmatrix} M_r & U_r \\ N_i & V_i \end{bmatrix} = I. \tag{1}
\]

If \( G \) is stabilizable and detectable, there always exists a doubly coprime factorization [15 Chap. 12]. Then the set of all controllers that stabilize \( G \), denoted by \( \mathcal{K} \), is given by

\[
\mathcal{K} = \{ K = (U_r + M_r Q (V_r + N_i Q)^{-1} : Q \in \mathbb{R}H_\infty \}. \tag{2}
\]

An important special case is given when \( G \) itself is stable. In this case, a doubly coprime factorization of \( G \) is given by \( N_r = G, M_r = I, U_r = 0, V_r = 1 \), and then the elements of \( \mathcal{K} \) are given by \( K = Q(I + G Q)^{-1} \). We refer to \( Q \in \mathbb{R}H_\infty \) as the Youla parameter of \( K \) for \( G \).

III. REVIEW OF EXISTING RETROFIT CONTROL

In this section, we first review the retrofit control based on the formulation in [13]. Further, we pose the problems treated in this paper.

A. Definition of Retrofit Controllers

We consider an interconnected system in Fig. [1] where

\[
\begin{bmatrix} w \\ y \end{bmatrix} = \begin{bmatrix} G_{wy} & G_{wu} \\ G_{gy} & G_{gy} \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}
\]

is referred to as a subsystem of interest for retrofit control, and \( v = G w \) is referred to as its environment. The interconnected system from \( u \) to \( y \) is given by

\[
G_{pre} := G_{ya} + G_{yv} G_{wv} (I - G_{wv} G_{yv})^{-1} G_{wu},
\]

which we refer to as a preexisting system. In the system representation, \( w, v \) denote the interconnection signals between \( G \) and \( G \) and \( y, u \) denote the measurement output and the control input. The interconnected system models a large-scale network system managed by multiple operators. From a single operator’s viewpoint, \( G \) and \( G \) represent its managed subsystem and the other subsystems with interconnection, respectively (for the detail of the modeling process, see [13]). We describe a state-space representation of the subsystem of interest [3] as

\[
\begin{align*}
\dot{x} &= Ax + Lv + Bu & y &= Cx
\end{align*}
\]

where \( x \) is the state of \( G \). It should be noted that exogenous input and evaluation output are not considered because this paper focuses just on stability analysis.

The purpose of retrofit control is to build a design method of the dynamical controller \( K \) generating the control input according to \( u = Ky \) only with the model information on the subsystem of interest \( G \). As a premise for controller design, let us suppose that the preexisting system \( G_{pre} \) without the controller \( K \) is assured to be internally stable, which
can be achieved by introducing a globally coordinating controller [13]. Under this assumption, in order to reflect the obscurity of the model information on $G$, we introduce the set of admissible environments as

$$
\Omega := \{\bar{G} : G_{\text{pre}} \text{ is internally stable}\}.
$$

The role of the controller $K$ is to improve a control performance without losing its internal stability. Following the discussion above, we reach the definition of retrofit controllers.

**Definition 1** The controller $K$ is said to be a retrofit controller if the resultant control system is internally stable for any environment $\bar{G} \in \Omega$.

The retrofit controller is a plug-and-play type controller and has an advantage that every system operator can independently introduce their own controllers.

**B. Characterization of Retrofit Controllers for Stable Subsystems**

A characterization of all retrofit controllers can be obtained under the assumption that the subsystem of interest $G$ is stable. The following assumption is made.

**Assumption 1** The subsystem of interest $G$ is stable, namely, $G \in \mathcal{RH}_\infty$.

The first existing result indicates that all retrofit controllers can be characterized as a constrained Youla parameterization under Assumption 1. The following proposition holds [13].

**Proposition 1** Let Assumption 1 hold. Consider the Youla parameterization of $K$ given by $K = (I + QG_{yu})^{-1}Q$ where $Q$ denotes the Youla parameter of $K$ for $G_{yu}$. Then $K$ is a retrofit controller if and only if

$$
G_{wu}QG_{yw} = 0
$$

and $Q \in \mathcal{RH}_\infty$.

The interpretation of the conditions is given as follows. We first transform the closed-loop system with $K$ depicted by Fig. 2(a) into Fig. 2(b). Then the transfer matrices of the subloops, depicted by boxes of broken lines in Fig. 2(b), are exactly the same as the Youla parameters for $G_{yu}$ and $G_{wu}$ from the assumption that $G$ is stable and $G_{\text{pre}}$ is internally stable. Because $\bar{Q}$ can be taken as an arbitrary element in $\mathcal{RH}_\infty$ from the definition of $\Omega$, the linear constraint (5) is necessary and sufficient for the internal stability.

Assumption 1 is essential to the claim. Owing to the assumption, the transfer matrices with respect to the loops are equal to the Youla parameters themselves. Without the assumption the same transformation cannot be applied for the stability analysis. Alternative conditions for unstable subsystems have not been discussed so far.

**C. Explicit Parameterization of Tractable Retrofit Controllers with Abundant Measurement**

For retrofit controller design, it suffices to find an appropriate Youla parameter $Q \in \mathcal{RH}_\infty$ that satisfies the constraint (5) under a desired performance criterion, but the constraint on $Q$ cannot directly be handled by a standard controller design technique. To surmount the difficulty, a tractable class of retrofit controllers is introduced [13]:

**Definition 2** The controller $K$ is said to be an output-rectifying retrofit controller if

$$
KG_{yw} = 0
$$

and $Q := (I - KG_{yu})^{-1}K \in \mathcal{RH}_\infty$.

The following proposition is obvious from the relationship between $Q$ and $K$.

**Proposition 2** Let Assumption 1 hold. Then a controller that satisfies the conditions in Definition 2 is a retrofit controller.

The class is tractable in the sense that all output-rectifying retrofit controllers can explicitly be parameterized with a free parameter under technical assumptions on abundance of measurement. Two cases are considered in the existing study:

**Assumption 2** The interconnection signal $v$ is measurable in addition to the measurement output $y$.

**Assumption 3** The measurement output $y$ is the state of $G$, namely, $x$ in (4).

An explicit parameterization of all output-rectifying retrofit controllers is given as follows [13].

**Proposition 3 (a)** Let Assumptions 1 and 2 hold. Then $K$ is an output-rectifying retrofit controller if and only if there exists a proper transfer matrix $\hat{K}$ such that $K = \hat{K}R$ and $\hat{Q} := (\hat{K} - KG_{yu})^{-1}\hat{R} \in \mathcal{RH}_\infty$ where $R := [I - G_{yu}]$.

**Proposition 3 (b)** Let Assumptions 1 and 3 hold. Then $K$ is an output-rectifying retrofit controller if and only if there exists a proper transfer matrix $\hat{K}$ such that $K = \hat{K}R[\hat{P}^T \hat{P}]^T$
and $\hat{Q} := (I - \hat{K}\hat{G}_{yu})^{-1}\hat{K} \in \mathcal{RH}_\infty$ where $R := [I - \hat{G}_{yu}]$ with $\hat{G}_{yu} := (sI - PAP^T)^{-1}PAP^T$ and $\hat{G}_{yu} := (sI - PA^{-1})PB$ with a certain matrix $P$ satisfying $PL = 0$.

The claims imply that, in both cases, all output-rectifying retrofit controllers have the structure illustrated by Fig. 3. The structure has two distinct characteristics. One is that a measurement signal $y$ is rectified through the rectifier to remove the effect of $v$. The other is that the free parameter $\hat{K}$ is characterized as a stabilizing controller for $\hat{G}_{yu}$, which is $G_{yu}$ itself or a reduced-order model of $G_{yu}$. Since $\hat{K}$ can be an arbitrary stabilizing controller for $G_{yu}$, the design problem of an output-rectifying retrofit controller is reduced to a standard controller design problem, which is readily handled by existing techniques. It should be emphasized that the claims rely not only on Assumptions 2 or 3 but also Assumption 1.

**D. Objective of this Paper**

The objective of this paper is to generalize the existing results without the technical assumptions. The research questions here to be addressed are as follows:

Q1: Does there exist a simple characterization of all retrofit controllers without Assumption 1?
Q2: What is the appropriate definition of output-rectifying retrofit controllers without Assumption 1?
Q3: Do all output-rectifying retrofit controllers have a structure similar to Fig. 3 without Assumptions 2 and 3?

Each of the questions corresponds to Propositions 1, 2, and 3 addressed above. The goal of this paper is to provide solid answers to the questions.

**IV. Characterization of Retrofit Controllers**

The aim of this section is to give an answer to Q1 by deriving a characterization of all retrofit controllers for possibly unstable subsystems. The idea as follows. Consider a doubly coprime factorization of $G_{wu}$ as (1) and denote the Youla parameter of $G_{wu}$ in (2) by $Q$. Then

$$\mathcal{G}(I - G_{wu})^{-1} = U_t M_t + M_t Q M_l$$

holds [10, Sec. 4.5], the right-hand side of which we denote by $\mathfrak{M}(Q)$. This relationship implies that the original block diagram in Fig. 2(a) can be transformed into that in Fig. 2(b) where the Youla parameter $Q$ explicitly appears. Moreover, the transfer matrix from $\hat{v}$ to $\hat{w}$ in Fig. 2(a), denoted by $\hat{G}_{wu}$, can be written by $G_{wu} = G_{wu} Q G_{wu}$ where

$$\hat{Q} := (I - \hat{K}\hat{G}_{yu})^{-1}\hat{K}$$

and $\hat{G}_{yu} := G_{yu} + G_{wu} U_t M_t G_{wu}$. Then we obtain the further transformed block diagram in Fig. 2(b), which expressly represents its loop transfer matrix. The block diagram of Fig. 2(b) suggests that the transfer matrix from $\hat{v}$ to $\hat{w}$ must be zero for retrofit controllers as in Proposition 1 because $\hat{Q}$ can be an arbitrary element in $\mathcal{RH}_\infty$.

In fact, this expectation is true. The following theorem, our first main result, provides a necessary and sufficient condition of retrofit controllers for possibly unstable subsystems.

**Theorem 1** Consider a doubly coprime factorization of $G_{wu}$. Then $\hat{K}$ is a retrofit controller if and only if

$$G_{wu} \hat{Q} G_{wu} = 0$$

and $\hat{Q} \in \mathcal{RH}_\infty$ where $\hat{Q}$ is defined in (7).

**Proof:** For stability analysis, we consider external inputs $(\delta_u, \delta_y, \delta_v, \delta_w)$ as perturbations from $(u, y, v, w)$ as in Fig. 5. The internal stability is equivalent to that the sixteen transfer matrices from $(\delta_u, \delta_y, \delta_v, \delta_w)$ to $(u, y, v, w)$ all belong to $\mathcal{RH}_\infty$. Note that the transfer matrices from $(u, y, \delta_v, \delta_w)$ to $(y, v, w)$ belong to $\mathcal{RH}_\infty$ since the system is internally stable when $K = 0$. The transfer matrices are given by

$$y = \{G_{yu} + G_{wu} \mathfrak{M}(Q) G_{wu}\} u + \delta_y + G_{yu} \{I + \mathfrak{M}(Q) G_{wu}\} \delta_y + G_{yu} \mathfrak{M}(Q) \delta_w,$$

$$v = \mathfrak{M}(Q) G_{wu} u + \mathfrak{M}(Q) \delta_w + \{I + \mathfrak{M}(Q) G_{wu}\} \delta_v,$$

$$w = \{I + G_{wu} \mathfrak{M}(Q)\} G_{wu} u + \{I + G_{wu} \mathfrak{M}(Q)\} G_{wu} \delta_v + \{I + G_{wu} \mathfrak{M}(Q)\} \delta_w,$$

for the derivation of which the relationship $(I - \mathcal{G}_{wu})^{-1} = I + \mathfrak{M}(Q) G_{wu}$ is used.

We here show the sufficiency. Assume that (8) holds and $\hat{Q} \in \mathcal{RH}_\infty$. For the internal stability, it suffices to show the stability of the transfer matrices in terms of $u$ because of
stability of the transfer matrices in (9). By simple algebra, we have
\[ u = \hat{Q}Xu + (I + \hat{Q}\tilde{G}_{gw})\delta_u + \hat{Q}\delta_y + \hat{Q}G_{gy}\{I + \tilde{M}(\tilde{Q})G_{wu}\} \delta_v + \hat{Q}G_{wv}\tilde{M}(\tilde{Q})\delta_w \] (10)

where
\[ X := G_{wu}M_1\tilde{M}M_1G_{wu}. \]

Because the transfer matrices in (10) except for \( \hat{Q} \) appear in (9), it suffices to show \((I - \hat{Q}X)^{-1} \in \mathcal{RH}_\infty\) to prove the stability of the transfer matrices in (10). Now we have
\[ (I - \hat{Q}X)^{-1} = I + \hat{Q}X \]
because \((\hat{Q}X)^2 = 0\) for any \( \hat{Q} \) from (9). Because \( G_{wu}M_1 \) and \( M_1G_{wu} \) belong to \( \mathcal{RH}_\infty \) [16 Theorem 4.5.1] and \( \tilde{Q} \in \mathcal{RH}_\infty \), it turns out that \( QX \in \mathcal{RH}_\infty \). Hence \( I + QX \in \mathcal{RH}_\infty \) and \((I - QX)^{-1} \in \mathcal{RH}_\infty \).

We next show the necessity. Assume that \( K \) is a retrofit controller, i.e., the system is internally stable for any \( G \in \mathcal{G} \). We focus on the transfer matrix from \( \delta_y \) to \( u \), which is given by \( T_{uy} := (I - \hat{Q}X)^{-1}\tilde{Q} \in \mathcal{RH}_\infty \) from (10). First, when choosing \( Q = 0 \) we have \( X = 0 \) and \( T_{uy} = Q \), which leads to \( Q \in \mathcal{RH}_\infty \). Next,
\[ (I - QX)^{-1} = I + T_{uy}X \in \mathcal{RH}_\infty \]
since \( X \in \mathcal{RH}_\infty \) as shown in the sufficiency part. From the Sylvester’s determinant identity [17 Fact 2.14.13] and the Nyquist stability criterion [15 Theorem 5.8],
\[ \det(I - \tilde{Q}(j\omega)Y(j\omega)) = \det(I - \tilde{Q}(j\omega)X(j\omega)) \neq 0, \quad \forall \omega \in \mathbb{R} \]
where \( Y := M_1G_{wu}\tilde{Q}G_{gy}M_1 \). If \( Y(j\omega_0) \neq 0 \) for some \( \omega_0 \in \mathbb{R} \), then we can construct \( \tilde{G} \in \mathcal{G} \) such that \( \det(I - \tilde{Q}Y)(j\omega_0) = 0 \) as in the proof of the small-gain theorem [15 Theorem 9.1], which leads to that \((I - QX)^{-1} \notin \mathcal{RH}_\infty \) for such \( \tilde{G} \). Thus \( Y = 0 \). This is equivalent to (8) because \( M_1 \) and \( M_1 \) are invertible in \( R \).

Theorem 1 provides a clear answer to Q1 posed in Sec. III-D. In (8), the variable \( Q \), the Youla parameter of \( K \) for \( G_{gy} \), is simply replaced with \( \tilde{Q} \), the Youla parameter of \( K \) for \( G_{gy} \) in (8). In conclusion, all retrofit controllers, even for unstable subsystems, can be characterized by the Youla parameter \( \tilde{Q} \in \mathcal{RH}_\infty \) with the linear constraint (8).

V. EXPLICIT PARAMETERIZATION OF OUTPUT-RECTIFYING RETROFIT CONTROLLERS

The aim of this section is to give answers to Q2 and Q3 in Sec. III-D based on the result in the previous section.

A. Output-Rectifying Retrofit Controllers for Unstable Subsystems

An answer to Q2 is given as a direct conclusion of Theorem 1. The following theorem holds.

**Theorem 2** If \( K \) satisfies the conditions in Definition 2, then \( K \) is a retrofit controller for unstable subsystems as well.

**Proof**: Suppose that \( KG_{gy} = 0 \) holds and \( Q \in \mathcal{RH}_\infty \). Then \( \tilde{Q}G_{gy} = (I - KG_{gy})^{-1}KG_{gy} = 0 \) regardless of the choice of coprime factorization of \( G_{wu} \). Moreover, \( \tilde{Q} = (I - K(G_{gy} + G_{gy}U\tilde{M}_1G_{wu}))^{-1}K = (I - KG_{gy})^{-1}K = Q \), which implies that \( Q \in \mathcal{RH}_\infty \) if \( Q \in \mathcal{RH}_\infty \). Thus \( K \) is a retrofit controller from Theorem 1.

Theorem 2 implies that the existing definition of output-rectifying retrofit controllers can still be employed even when \( G \) is unstable.

B. Explicit Parameterization of Output-Rectifying Retrofit Controllers

In this subsection, we give an answer to Q3. Our aim is to investigate an intrinsic structure of all solutions of (6) constrained by \( Q \in \mathcal{RH}_\infty \) without Assumptions 2 and 3. Since (6) is a linear equation in \( R \), the set of the solutions can possibly be characterized by a subspace in the vector space \( R \), which is obtained by extending \( R \). In fact, we can find the bases spanning the solution space through a left inverse of \( G_{gy} \). For establishing a left inverse of \( G_{gy} \), we introduce the notion of relative degree for multi-input and multi-output systems.

**Definition 3** Consider a strictly proper transfer matrix \( G \in R_{p \times m} \). Let \( (A, B, C) \) be a realization of \( G \) and \( c_i \) be the \( i \)th row of \( C \) for \( i = 1, \ldots, p \). Then \( G \) is said to have relative degree \((r_1, \ldots, r_p)\) if for \( i = 1, \ldots, p \)
\[ c_iA^kB = 0, \quad c_iA^{r_i-1}B \neq 0, \quad \forall k \leq r_i - 2 \]
and \( \text{col}(c_iA^{r_i-1}B)_{i=1}^p \) is injective.

Let the dimensions of \( y \) and \( v \) be \( p \) and \( m \), respectively. The following assumption is made.

**Assumption 4** The transfer matrix \( G_{gy} \) has relative degree \((r_1, \ldots, r_p)\) satisfying \( r_1 \leq r_2 \leq \cdots \leq r_p \). Moreover, \( r_m < r_{m+1} \) and \( \text{col}(c_iA^{r_i-1}L)_{i=1}^m \) is injective.

The assumption states that \( G_{gy} \) has a relative degree arranged in ascending order. It is also assumed that the \( m \) rows of \( G_{gy} \) that have the lowest relative degrees are uniquely determined. Although the latter assumption seems restrictive, but actually generality is not lost as long as the partial transfer matrix that have relative degrees lower than or equal to \( r_m \) is left invertible as shown in the following lemma.

**Lemma 1** Assume that \( G_{gy} \) has a relative degree \((r_1, \ldots, r_p)\) satisfying \( r_1 \leq \cdots \leq r_p \) and \( \text{col}(c_iA^{r_i-1}L)_{i=1}^m \) is injective. Then there exists a matrix \( T \in \mathbb{R}^{p \times p} \) such that \( TG_{gy} \) satisfies Assumption 4.
Proof: Let $i, j$ be the maximum nonnegative integers such that $r_{m-i} = r_{m+j} = r_m$. From the assumption, $\dim \ker \text{col}(c_k A^{r_k-1} L_k^{m-i-1}) = i + 1$. Thus there exists a matrix $U \in \mathbb{R}^{(i+1) \times m}$ such that $\ker U \cap \ker \text{col}(c_k A^{r_k-1} L_k^{m-i-1}) = \{0\}$. Let $\tilde{T} \in \mathbb{R}^{(i+j+1) \times (i+j+1)}$ be a coordinate transformation matrix such that

$$\tilde{T} \text{col}(c_k A^{r_k-1} L_k^{m+j}) = [U^T 0]^T.$$ 

Let $T = \text{diag}(I, \tilde{T}, I)$ and then $T$ satisfies the condition. □

Intuitively, $T$ is given by changing the coordinate of the outputs from $m_i$ to $m_j$ to satisfy that the transfer matrices with respect to the first $m$ outputs have the relative degree $r_m$ and the others have larger relative degrees.

We construct a left inverse of $G_{yy}$, namely, a possibly improper transfer matrix $G^\dagger_{yy}$, such that $G^\dagger_{yy} G_{yy} = I$. Because it suffices to consider $m$ independent outputs for constructing a left inverse, we consider the first $m$ elements of $y$. Defining the matrices

$$P := [0 \ I], \quad \bar{P} := [I \ 0]$$

such that

$$\text{col}(y_i)_{i=1}^m = \bar{P} y$$

where $y_i$ is the $i$th element of $y$, we treat $\bar{P} G_{yy}$ instead of $G_{yy}$ itself. Define $\xi$ as the derivatives of $\bar{P} y$ by

$$\xi = D \bar{P} y$$

with a differential operator $D := \text{diag}(D_i)_{i=1}^m$ where

$$D_i := \text{col} \left( d^i-1 \frac{d}{dt} \right)_{j=1}^{r_i}.$$ 

Then from (4) we have $\xi = S x$ with

$$S := \text{col} \left( \text{col}(c_i A^{r_i-1})_{j=1}^m \right)_{i=1}^m.$$ 

It can be shown that $S$ is surjective and there exists $\bar{S}$ such that $S$ and $\bar{S}$ complete the coordinates and $\bar{S} L = 0$ [18]. Considering the coordinate transformation $x \mapsto (z, \xi)$ with $z = \bar{S} x$, we have

$$\bar{P} G_{yy} := \begin{bmatrix} \frac{z}{\xi} & S A S^\dagger & S A S^\dagger & \cdots & S A S^\dagger \\ \bar{P} y & \bar{P} C S^\dagger \end{bmatrix} v$$

where $SL$ is injective. The state-space representation (12) is referred to as a normal form [18]. Let us suppose that the signals in (12) satisfy the equations. Since $SL$ is injective, $v$ is uniquely determined by

$$v = (SL)\dagger (-SA S^\dagger z - SA S^\dagger \xi + \hat{\xi}) = -(SL)\dagger SA S^\dagger z - (SL)\dagger SA S^\dagger \xi + (SL)\dagger \frac{d}{dt} \bar{P} y$$

from (11). Since $(\bar{P} G_{yy})^{-1} \bar{P} G_{yy} = I$ and $(SL)\dagger SA S^\dagger = 0$, we have a left inverse $G^\dagger_{yy} := (\bar{P} G_{yy})^{-1} \bar{P}$ by

$$G^\dagger_{yy} := \begin{bmatrix} \bar{S} A S^\dagger & \bar{S} A S^\dagger & \cdots & \bar{S} A S^\dagger \\ 0 & \bar{L} \bar{S} A S^\dagger & \cdots & \bar{L} \bar{S} A S^\dagger \\ -\bar{L} \bar{S} A S^\dagger & \bar{L} \bar{S} A S^\dagger & \cdots & \bar{L} \bar{S} A S^\dagger \\ \bar{L} \bar{S} A S^\dagger & \bar{L} \bar{S} A S^\dagger & \cdots & \bar{L} \bar{S} A S^\dagger \end{bmatrix} s D \xi \bar{P}$$

with $\hat{K} = K R PT \bar{P}^T$ and $\hat{Q} = (1 - \hat{K} \hat{G}_{yy})^{-1} K \hat{G}_{yy}$, which is given by the Laplace transformation of $D$.

We have the following theorem, our second main result, providing a parameterization of all output-rectifying retrofit controllers.

Theorem 3 Let Assumption 4 hold. The controller $K \in \mathcal{R}P$ is an output-rectifying retrofit controller if and only if there exists a proper transfer matrix $\hat{K} \in \mathcal{R}P$ such that

$$K = \hat{K} R PT \bar{P}^T \hat{P}^T$$

and $\hat{Q} := (I - \hat{K} \hat{G}_{yy})^{-1} K \hat{G}_{yy}$, which is given by the Laplace transformation of $D$.

Theorem 3 implies that all output-rectifying retrofit controllers have the structure of the controller $\hat{K}$ in Fig. 6 composed of a rectifier $R$ and an internal controller $\hat{K}$, which is the same as the existing ones shown in Fig. 5. The internal controller $\hat{K}$ belongs to the class where the transfer matrices $\hat{Q}$ and $\hat{Q} \hat{G}_{yy}$ are stable. Here $\hat{Q}$ and $\hat{Q} \hat{G}_{yy}$ are obtained through $\hat{G}_{yy}$, a reduced-order model of $G_{yy}$. Note that, although differentiators explicitly appear in the expressions of $\hat{G}_{yy}$ and $\hat{G}_{yy}$, the transfer matrices are proper and the dimensions of those internal states are invariant even when the differentiators are removed from the representations. In other words, those transfer matrices can still be interpreted as reduced-order models of $G_{yy}$ and $G_{yy}$ without the differentiators.
Before proving Theorem 3, we show that Theorem 3 is a generalization of the existing results by confirming that the claim coincides with Proposition 3(b) when y = x.

**Example 1** We consider G in (4) and suppose that C = I and L is injective without loss of generality. We employ the notation \(G_{xy}\) and \(G_{xx}\) instead of \(G_{vy}\) and \(G_{yy}\). To fulfill Assumption 4 let B be the coordinate transformation \(T := [L^T \ T^T]^T\) and consider TG_{xy}. Then TG_{xy} satisfies Assumption 4 and \(r_1 = \cdots = r_m = 1\). The states of the normal form with respect to the transformed system are given by \(\xi := L^1 x\) and \(z := T^T x\) and hence \(S = L^1, S = T^T\). Moreover, since \((r_1, \ldots, r_m)\) are all one, \(D\xi = I\) and \(Z_i\) disappears in (13) for any i. Thus, the transfer matrices in Theorem 3 for \(G_{xy}\) are given by

\[
\hat{G}_{xy} = \begin{bmatrix}
L^T A & L^T C \\
I & 0
\end{bmatrix},
\text{ and } \hat{G}_{xy} = \begin{bmatrix}
L^T A & L^T B \\
I & 0
\end{bmatrix}.
\]

Because \(T^T\) satisfies \(T^T L = 0\), we can take \(T^T = P\) and the result coincides with Proposition 3(b).

We give a proof of Theorem 3 in the rest of this subsection. In the claim of Theorem 3 two conditions are required: one is on the structure of \(K\) and the other is on the class of the parameter \(K\). We treat the conditions separately and show the following propositions corresponding to each condition.

**Proposition 4** Under Assumption 4 the relationship (6) holds if and only if there exists a proper transfer matrix \(K\) such that (14) holds. Moreover, \(\hat{G}_{yy}\) is proper.

**Proposition 5** Assume that there exists a proper transfer matrix \(\hat{K}\) such that (14) holds for a given \(\hat{K}\). Then \(Q \in \mathcal{RH}\) if and only if \(Q \in \mathcal{RH}\) and \(Q \hat{G}_{yy} \in \mathcal{RH}\). Moreover, \(\hat{G}_{yy}\) is proper.

We first prove Proposition 4. The following lemma holds.

**Lemma 2** Let \(G_{yy}^{-1}\) be a left inverse of \(G_{yy}\) such that \(G_{yy}G_{yy}^{-1}\) is proper. Then (6) holds for a proper transfer matrix \(K\) if and only if there exists a proper transfer matrix \(\hat{K}\) such that \(K = \hat{K}\Xi\) with \(\Xi := \Pi(I - G_{yy}G_{yy}^{-1})\) where \(\Pi\) is a matrix being surjective and satisfying \(\Pi G_{yy}(s)G_{yy}(s)|_{s=+\infty} = 0\).

**Proof:** Let the dimensions of the signals v, y, u be m, p, q, respectively. As a preparation, we give all solutions to (6) in \(\mathcal{R}^{q \times m}\). Because \(\mathcal{R}^{q \times m}\) is a vector space on the field of scalar transfer functions \(\mathcal{R}\) and \(\dim \text{im}(G_{yy}) = m\) since its left inverse exists, the fundamental theorem on homomorphisms leads to that all solutions in \(\mathcal{R}\) can be represented by \(K = \hat{K}\Xi\) with \(\Xi \in \mathcal{R}^{q \times (p-m)}\) if \(\Xi\) satisfies \(\Xi G_{yy} = 0\) and \(\text{im}\Xi = \mathcal{R}^{p-m}\). The former condition is obvious. The condition on \(\text{im}\Xi\) is also satisfied, because \(\Xi\) is right-invertible since the feedthrough term of \(\Xi\), \(\Pi\) itself, is right-invertible. Thus all solutions to (6) in \(\mathcal{R}^{q \times m}\) can be characterized by \(K = \hat{K}\Xi\).

We here show that the parameter \(\hat{K}\) in the characterization is proper if and only if \(K\) is proper. Because \(\Xi\) is proper from the assumption, \(K\) is proper if \(\hat{K}\) is proper. Conversely, because \(\Xi\) is right-invertible there exists a proper transfer matrix \(\Xi\) such that \(\Xi \Xi = I\), \(\hat{K} = \hat{K}\Xi\) and \(\hat{K}\) is proper if \(\hat{K}\) is proper. Therefore, the claim holds.

**Lemma 3** implies that all proper transfer matrices that satisfy (6) can be characterized with \(\Xi\), obtained through a left inverse of \(G_{yy}\) such that \(G_{yy}G_{yy}^{-1}\) is proper.

Based on Lemma 2 we can prove Proposition 4 as follows.

**Proof of Proposition 4** From Lemma 2 it suffices to show that the left inverse \(G_{yy}^{-1}\) in (13) satisfies that \(G_{yy}G_{yy}^{-1}\) is proper, \(PG_{yy}(s)G_{yy}(s)|_{s=+\infty} = 0\), and \(P(I - G_{yy}G_{yy}^{-1}) = R[P^T P^T]^T\).

First, note that \(G_{yy}G_{yy}^{-1} = P^T PG_{yy}G_{yy}^{-1} + P^T P\).

From Assumption 4 which guarantees \(r_i > r_m\) for \(i = m+1, \ldots, p\), \(P^T PG_{yy}G_{yy}^{-1}\) is strictly proper. The inverse of (13) satisfies \(G_{yy}^{-1} = (P^T)^{-1} P\) and hence we have \(P^T P\), which is proper. Therefore \(G_{yy}^{-1} = P^T PG_{yy}G_{yy}^{-1} + P^T P\) is proper and \(PG_{yy}(s)G_{yy}(s)|_{s=+\infty} = 0\).

Next, we show that \(P(I - G_{yy}G_{yy}^{-1}) = R[P^T P^T]^T\). From Assumption 4 \(PCS^1 = 0\). Thus we have

\[
P_G = PCS^1 \xi + PCS^1 z = PCS^1 z
\]

from (12). Now because \(\xi = S\xi S^1 z + S\xi S^1 z = S\xi S^1 z + S\xi S^1 D\), it turns out that

\[
PG_{yy} = \begin{bmatrix}
\hat{A} \\
C
\end{bmatrix} D_S G_{yy} = \hat{G}_{yy} PC_{yy}
\]

Then \(PG_{yy}G_{yy}^{-1} = G_{yy}^{-1} PG_{yy} G_{yy}^{-1} = G_{yy}^{-1} P\).

Thus \(P(I - G_{yy}G_{yy}^{-1}) = P - G_{yy}^{-1} P = R[P^T P^T]^T\). Finally, because \(P(I - G_{yy}G_{yy}^{-1})\) is proper, \(\hat{G}_{yy} = -P(I - G_{yy}G_{yy}^{-1})\) is proper.

We next prove Proposition 5.

**Proof of Proposition 5** We first show that

\[
G_{yy} = \Xi G_{yy} = \hat{G}_{yy}
\]

where \(\Xi := R[P^T P^T]^T\). From the definition of \(\Xi\), a state-space representation of \(\Xi G_{yy}\) is given by

\[
\Xi G_{yy} := \begin{cases}
\hat{z} = S\hat{A}\hat{S} z + S\hat{A}\hat{S} \xi + SBu \\
\hat{\xi} = S\hat{A}\hat{S} \xi + S\hat{A}\hat{S} z + SBu \\
\hat{y} = CS\hat{S} z + C\hat{S} \xi \\
\hat{\zeta} = S\hat{A}\hat{S} \zeta + S\hat{A}\hat{S} D\zeta \\
\hat{\eta} = -PC\hat{S} \zeta + Py
\end{cases}
\]

with the coordinate transformation \(z = Sx, \zeta = S\xi\). Since \(PCS^1 = 0\), we have \(\hat{y} = PCS^1 (z - \zeta)\). Let \(\phi := z - \zeta\) whose dynamics is given by

\[
\hat{\phi} = S\hat{A}\hat{S} \phi + S\hat{A}\hat{S} (\zeta - D\zeta) + SBu.
\]

Consider a state-space representation of \(\hat{P} G_{yy}\) as \(\hat{x} = Ax + Bu, \hat{y} = PC\hat{x}\). Since

\[
\frac{d\hat{y}}{d\hat{t}} = c_i A^j x + c_i A^{j-1} Bu + \cdots + c_i B \frac{d^{j-1} u}{d\hat{t}^{j-1}}
\]

for \(i = 1, \ldots, m\) and \(j = 0, \ldots, r_i - 1\), it turns out that

\[
D\hat{P}\hat{y} = Sx + \text{col} (Z_i \hat{D}_i)_{i=1}^m u
\]  

(16)
where
\[
Z_i := \begin{bmatrix} 0 & \ldots & 0 \\
c_i B & 0 & \ldots \\
\vdots & \ddots & \ddots \\
c_i A^{-2} B & \ldots & c_i B 0 \end{bmatrix}, \quad \hat{D}_i := \text{col} \left( \frac{d^{j-1}}{dt^{j-1}} L \right)_{j=1}^{r_i}.
\]

From (16), we have \( \phi = \overline{SA} S^\dagger \phi - \overline{SA} S^\dagger \text{col} \left( Z_i \hat{D}_i \right)^m \) + \( \overline{SB} u \). Therefore, we have
\[
\Xi G_{yu} = \begin{bmatrix} \overline{SA} S^\dagger \\ \overline{PC^\dagger} S \\ 0 \end{bmatrix} \begin{bmatrix} B - \overline{SA} S^\dagger \text{col} \left( Z_i \hat{D}_i \right)^m \\ 0 \end{bmatrix}
\]
which is equal to \( \hat{G}_{yu} \). Moreover, because \( \Xi \) and \( G_{yu} \) are proper, \( \hat{G}_{yu} = \Xi G_{yu} \) is proper.

Under \( \Xi G_{yu} = G_{yu} \), it turns out that \( Q = \hat{Q} P - \hat{Q} G_{yu} \) from \( K = K \Xi \). Thus, if \( Q \) and \( \hat{Q} G_{yu} \) belong to \( \mathcal{R} \mathcal{H}_\infty \), \( Q \) belongs to \( \mathcal{R} \mathcal{H}_\infty \). The converse also holds because \( Q = Q P \) and \( \hat{Q} G_{yu} = -Q P \).

The above discussion proves Theorem 5.

C. Retrofit Controller Design

For design of an output-rectifying retrofit controller, the structure is given in Theorem 5. Thus the design problem of \( K \) is reduced to the design problem of the internal controller \( \hat{K} \).

In Theorem 5 not only \( Q \in \mathcal{R} \mathcal{H}_\infty \) but also \( \hat{Q} G_{yu} \in \mathcal{R} \mathcal{H}_\infty \) are required for the class of \( K \). The additional condition is arisen from the absence of Assumption 1. The block diagram of the entire closed-loop system with an output-rectifying retrofit controller is illustrated in Fig. 6. To guarantee the internal stability, it is necessary to deal with the measurement noises \( \delta_{py} \) and \( \delta_{ty} \). Thus \( \hat{Q} \) and \( \hat{Q} G_{yu} \), the closed-loop transfer matrices from \( \delta_{py} \) and \( \delta_{ty} \) to \( u \), must belong to \( \mathcal{R} \mathcal{H}_\infty \). In contrast, because \( G_{yu} \) can be constructed to be stable when Assumption 1 holds 13, the second condition vanishes in Propositions 6(a) and (b).

The additional condition \( \hat{Q} G_{yu} \in \mathcal{R} \mathcal{H}_\infty \) seems to make the retrofit controller synthesis problem difficult because the constraint cannot be handled with a standard controller design method. We here provide a decision policy in terms of the internal controller \( \hat{K} \) without Assumption 1. The solution is given by simply designing a stabilizing controller for \( G_{yu} \) as shown by the following proposition.

**Proposition 6** Assume that \( \hat{K} \) stabilizes \( \hat{G}_{yu} \). Then both of \( \hat{Q} \) and \( \hat{Q} G_{yu} \) belong to \( \mathcal{R} \mathcal{H}_\infty \).

**Proof:** It is obvious that \( \hat{Q} \in \mathcal{R} \mathcal{H}_\infty \). We suppose \( K \) to be a static controller for simplicity, although the following proof can be readily extended to the case of dynamical controllers. Let \( L \) and \( B \) be matrices such that \((A, L, C) \) and \((A, B, C) \) are realizations of \( \hat{G}_{yu} \) and \( G_{yu} \), respectively. From the assumption, \( A + B \hat{K} C \) is Hurwitz. Because
\[
\hat{Q} G_{yu} = \begin{bmatrix} A & 0 \\ B \hat{K} C & A + B \hat{K} C \end{bmatrix} L \begin{bmatrix} K C \\ K C \end{bmatrix} 0
\]
we have \( \hat{Q} G_{yu} \in \mathcal{R} \mathcal{H}_\infty \). \( \square \)

**VI. CONCLUSION**

Existing works have shown that all retrofit controllers can be characterized by Youla parameterization with a linear constraint and also that all output-rectifying retrofit controllers can be parameterized with an internal stabilizing controller under several technical assumptions. In this paper, we have proven that retrofit controllers have the same property even without the technical assumptions, fully exploiting the general Youla parameterization and an explicit description of the inverse system obtained through the geometric control theory. The results provide a retrofit controller synthesis method, to which standard controller design methods can be applied.

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