AUTOMORPHIC FORMS
AND CUBIC TWISTS OF ELLIPTIC CURVES

Daniel Lieman

0. Introduction

One of the most classical problems in number theory is that of determining whether a given rational integer is the sum of two cubes of rational numbers. This “sum of two cubes” problem has been attacked from a variety of both classical and modern viewpoints; it would be nearly impossible to list here all of the various approaches taken and results obtained. An extensive compilation of the older history of the problem is given in Dickson [D].

This paper is an exposition of one of the connections between the curve

\[ E_D : x^3 + y^3 = D \]

and metaplectic forms. The approach we shall consider has the potential of providing collective information “on average” about the curves \( E_D \). It will not provide any information about \( E_D \) for a particular \( D \); for these types of results, the best modern information seems to be from Elkies, and from Rodriguez Villegas and Zagier. Here, we will describe the ingredients necessary to obtain two theorems concerning the curves \( E_D \).

**Theorem 1.** There are infinitely many cube–free \( D \) such that \( E_D \) has no rational solutions.

**Theorem 2.** Fix a prime \( p \neq 3 \), and any congruence class \( c \) modulo \( p \). Then there are infinitely many cube–free \( D \) congruent to \( c \) modulo \( p \) such that \( E_D \) has no rational solutions.

**Remark.** Theorem 1 is in fact weaker than a classical result of Sylvester, while Theorem 2 is only slightly stronger than Sylvester’s results combined with the Dirichlet theorem on primes in arithmetic progressions. Our main results here are Theorem 3, relating the L–series to \( E_D \) to a certain metaplectic form, and the machinery of Theorem 4 and the subsequent discussion, which allow one to obtain analytic information about these L–series.

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There are many connections between $E_D$ and automorphic forms. The curves $E_D$ are elliptic curves, and there is the famous conjecture that they are related to a form of weight 2. In addition, work of Waldspurger and the Shimura correspondence give the existence of a form $f$ of weight $3/2$ which is related to the values of the L-series of $E_D$. Indeed, Nekovar [N] has explicitly identified the corresponding form. (It is perhaps interesting to note that this form is a product of Dedekind $\eta$ functions, and that the formula for the value of the L-series of $E_D$ at the center of the critical strip given by Rodriguez Villegas and Zagier also depends on $\eta$ functions. It is not clear whether this is more than coincidence.)

The connection between the L-series of $E_D$ and metaplectic forms is simultaneously quite explicit and not well-understood. That is, we will see that the L-series arise as the Whittaker-Fourier coefficients of a certain metaplectic Eisenstein series on the cubic cover of $GL(3)$, and perhaps we will even make sense of why one might expect this to happen; we will not, however, be able to satisfactorily explain this occurrence. There is no known analog of Waldspurger’s result in this case.

It is clear that this is not an isolated occurrence of an L-series in the coefficients of a metaplectic form. Recent works of Bump and Hoffstein [BH], Bump, Friedberg and Hoffstein [BHF], [BHF2] and Goldfeld, Hoffstein, and Patterson [GHP] have discovered metaplectic Eisenstein series with Whittaker-Fourier coefficients which are (essentially) the Hecke L-series of the cubic residue symbol, the L-series of quadratic twists of elliptic curves with complex multiplication, and the L-series of quadratic twists of cuspidal newforms for the group $\Gamma_0(M)$. For all the cases mentioned above, one is able to apply various analytic methods to the appropriate metaplectic form and obtain interesting number-theoretic results.

This paper will survey the construction of the form with coefficients which are the L-series of cubic twists of elliptic curves with complex multiplication, the extraction of information about the curves from this form, and how much this process may be generalized using a similar construction to obtain L-series of higher order twists, or Hecke L-series of higher order residue symbols. It is a summary of the results of [L],[L2], and [BL], in a manner less terse than that usually found in research articles.

We should mention that we often ignore the question of attribution, especially for fundamental work, in what follows. We take for granted several highly insightful observations and constructions which are crucial to the general theory of metaplectic forms and the success in carrying out investigations within this framework. For a more detailed and far-reaching survey of the theory of metaplectic forms, consult the beautiful and quite readable article of Hoffstein [H].

1. Cubic twists of elliptic curves

We are interested in the family of curves

$$x^3 + y^3 = D.$$  

(1.1)

As mentioned above, these are elliptic curves (with Weierstrass form $y^2 = x^3 - 432D^2$), and the rich machinery of elliptic curves may be brought to bear. To each curve there is associated an L-series $L(E_D, s)$, and the following are well-known.

**Theorem (Mordell–Weil).** The set of rational solutions to $E_D$, together with a “point at infinity,” form a finitely generated abelian group $E_D(\mathbb{Q})$, under a certain geometric group law.
Theorem (Coates–Wiles). If \( L(E_D, 1) \neq 0 \), then \( E_D \) has only finitely many rational points.

**Proposition.** For \( |D| \geq 3 \), the torsion subgroup \( E_D(\mathbb{Q})_{\text{tors}} \subseteq E_D(\mathbb{Q}) \) is trivial.

The Mordell–Weil Theorem asserts that
\[
E_D(\mathbb{Q}) \cong \mathbb{Z}^{r_D} \oplus E_D(\mathbb{Q})_{\text{tors}}.
\]
Combined with the proposition, this shows that for all but a few \( D \) (in particular, \( D = 0, \pm 1, \pm 2 \)), one has
\[
E_D(\mathbb{Q}) \cong \mathbb{Z}^{r_D}.
\]

Now applying the Theorem of Coates–Wiles to (1.2), we obtain the statement
\[
L(E_D, 1) \neq 0 \implies r_D = 0.
\]

It is the behavior of \( r_D \) that we want to study, and we shall do so using the relationship (1.3).

We begin by setting some notation. Let \( \mathbb{K} \) denote, for the rest of the paper, \( \mathbb{Q}(\sqrt{-3}) \), and let \( \mathcal{O} \) denote the ring of integers in \( \mathbb{K} \). If \( a, b \in \mathcal{O}, b \equiv 1 \pmod{3} \), we write \( \left( \frac{a}{b} \right)_3 \) for the cubic residue symbol of \( a \mod b \). The following proposition follows from the exposition in Ireland and Rosen [IR], together with the Weierstrass form of \( E_D \), mentioned above.

**Proposition.** The \( L \)-series of the elliptic curve \( E_D \) is given by
\[
L(E_D, s) = \sum_{\substack{a \in \mathcal{O} \\ a \equiv 1 \pmod{3}}} \left( \frac{D}{a} \right)_3 \frac{|a|}{a} \mathcal{N}(a)^{-s+\frac{1}{2}}.
\]

**Remark.** Each of the curves \( E_D \) is a cubic twist of the curve \( E_1 \), and so the proposition says that the \( L \)-series is a natural object, that is, that the \( L \)-series of “\( E_1 \) twisted by \( D \)” is just the “\( L \)-series of \( E_1 \)” twisted by the cubic residue symbol \( \left( \frac{D}{3} \right)_3 \). On the other hand, one can also interpret the Proposition as saying that the \( L \)-series of \( E_D \) is just the Hecke \( L \)-series of the cubic residue symbol \( \left( \frac{D}{3} \right)_3 \) twisted by the grossencharacter \( \left| \frac{1}{\cdot} \right| \). This latter viewpoint is the basis of our arguments below.

**Remark.** The family of curves we are studying is a particularly interesting one, not merely for its long history, but also because of recent results. Zagier and Kramarz [ZK] have computed the analytic rank of \( E_D \) for \( D \) cube–free and less than 70,000 and have found that the distribution of curves with analytic rank \( \geq 2 \) is quite frequent (\( \sim 25\% \)). More impressively, they found that, writing \( X^* \) for the number of cube–free integers less than \( X \), the average
\[
\frac{1}{X^*} \cdot \# \{ D < X \mid D \text{ cube–free, with analytic rank of } E_D \geq 2 \}
\]
seemed to be independent of \( X \) (\( X \) ranging up to 70,000). This rather uniform distribution of curves of high analytic rank is somewhat surprising, and it is an
interesting question to see to what extent, if at all, this is reflected in the average values of the L–series of these curves.

**Remark.** The average value of the L–series of $E_D$ as $D$ varies is the subject of two different conjectures. Zagier [Z-K] has pointed out that the series

$$L_{av}(s) = \sum_n \frac{b_n}{n^s}$$

formed by setting $b_n$ to be the average of the $n^{th}$ coefficient of the L–series of $E_D$ as $D$ varies has an analytic continuation with a finite value at $s=1$, and that this makes it plausible that the numbers $L(E_D,1)$ as $D$ varies have a well–defined average value. Goldfeld and Viola [GV], on the other hand, have given (based on heuristic arguments) a very general conjecture, the specialization of which to this problem is

$$\sum_{\begin{smallmatrix} p < X \\ p \text{ prime} \end{smallmatrix}} L(E_p,1) L(E_{p^2},1) \sim c \cdot X \text{ as } X \to \infty.$$ 

(Goldfeld and Viola also explicitly calculate the expected value of the constant $c$). We will discuss average values further in section 5.

### 2. Overview of the Approach

We begin by recalling how analytic information about the behavior of a Dirichlet series can yield information about the behavior of the coefficients. By a Dirichlet series, we mean a function of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

satisfying

1. $f(s)$ converges in a half-plane (i.e. for $re(s)$ sufficiently large)
2. $f(s)$ has a meromorphic continuation to the entire complex plane.

The prototype to think of, of course, is the Riemann $\zeta$–function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

which satisfies the functional equation

$$\zeta^*(s) = \Gamma \left( \frac{s}{2} \right) \pi^{-\frac{s}{2}} \zeta(s) = \zeta^*(1 - s).$$

There are two ways a Dirichlet series, such as the $\zeta$–function, can yield information about its coefficients.

**The easy method (Obvious).** Suppose $f(s)$ is a Dirichlet series, as above, and that $f(s)$ has a pole at $s = s_0$. Then infinitely many of the coefficients $a_n$ are non-zero.
The hard method (Ikehara’s Tauberian Theorem). Suppose $f(s)$ is a Dirichlet series which converges for $\Re(s) > 1$, and has a pole of order 1 at $s = 1$ with residue $\alpha$. Further suppose that $f(s)$ is analytic on the line $\Re(s) = 1$ except for the pole at $s = 1$, and that each $a_n \geq 0$. Then

$$\sum_{n < X} a_n \sim \alpha X \quad \text{as } X \to \infty.$$  

Remark. Tauberian theorems are of course much more general than that given above. We use the simplest specialization to illustrate the general shape of such a theorem, and because the $\zeta$–function satisfies the hypothesis, allowing the reader to verify the truth of the method in this one case. For this paper, we will use only “the easy method.” We will discuss expected applications of Ikehara’s Tauberian theorem to our problem, but without results.

We are now able to outline, in more detail, the contents of this paper. Our first topic will be the metaplectic cover of $GL(r)$. We will define automorphic functions on this group (the metaplectic group), and see that built into the automorphy property is the $n^{th}$ order reciprocity symbol, thus making it plausible that the predicted L–series should occur in the coefficients of these forms. We will then construct a particular form on the cubic cover of $GL(3)$, with the property that its Whittaker–Fourier coefficients are the L–series of the elliptic curves, in the form given in (1.4). We will next use the Rankin–Selberg method to construct a Dirichlet series (more or less) of the form

$$\sum_{D=1}^{\infty} \frac{L(E_D,1)}{D^w}$$  

and to obtain analytic information about the behavior of this Dirichlet series in the variable $w$. Finally, we will discuss generalizations of this work: what is known, and what is conjectured for L–series of higher order twists, and higher order residue symbols.

3. Automorphic forms on the metaplectic group

There are several ways to describe functions on the metaplectic group. We take here the most concrete, though perhaps not the most direct; we will describe a multiplier system, and show that functions transforming with respect to this multiplier system are in fact functions on the double cover of $GL(2)$. We will then discuss how this construction generalizes to functions on the $n$–cover of $GL(r)$. Our construction will clearly embed the quadratic reciprocity law into the forms we will construct; it is this key point which explains, in part, why one might expect the Hecke L–series of quadratic residue symbols (or quadratic gauss sums, for that matter) to appear in the Fourier coefficients of a form constructed in this manner. A basic reference for this section is Hoffstein [H].

We begin by defining forms on the double cover of $GL(2)$. The first step is to construct the Kubota homomorphism. Let $\lambda = 1 + i$, with $i = \sqrt{-1}$, as usual, and write $\left(\frac{a}{b}\right)_2$ for the quadratic residue symbol. We let $\Gamma(\lambda^3)$ denote the principal congruence subgroup modulo $\lambda^3$, that is

$$\Gamma(\lambda^3) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Q}(i)) \mid a \equiv d \equiv 1 (\lambda^3), b \equiv c \equiv 0 (\lambda^3) \right\}.$$


We define a homomorphism $\kappa : \Gamma(\lambda^3) \to \{\pm 1\}$ by

$$\kappa\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \begin{cases} \left(\frac{a}{b}\right)_2 & \text{if } c \neq 0, \\ 1 & \text{if } c = 0. \end{cases}$$

That this is indeed a homomorphism follows from quadratic reciprocity.

The standard action of $GL(2, \mathbb{R})$ on the upper half plane may be generalized in the following way. Let $Z \subset GL(2, \mathbb{R})$ and $K \subset GL(2, \mathbb{R})$ denote the groups of scalar and orthogonal matrices, respectively; each coset $GL(2, \mathbb{R})/Z \cdot K$ has a unique representative of the form $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, where $x, y \in \mathbb{R}$, and $y > 0$ (this decomposition is called the Iwasawa decomposition; each such coset representative is called the Iwasawa coordinate). Further, if we let $GL(2, \mathbb{R})$ act on the upper half plane in the usual way (i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}$), then for $z = x + iy$, and $\gamma \in GL(2, \mathbb{R})$, we have $\gamma z = x' + iy'$, where $\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix}$ is the Iwasawa coordinate of $\gamma \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$. That is, the action of $GL(2, \mathbb{R})$ on the upper half plane is the same as the action of $GL(2, \mathbb{R})$ on $GL(2, \mathbb{R})/Z \cdot K$ by left multiplication.

This latter action is easy to generalize to larger fields, or to higher rank groups. For now, we consider the first generalization. Let $Z$ and $K$ now denote, respectively, the subgroups of scalar and unitary matrices within $GL(2, \mathbb{C})$. We may consider complex–valued functions on $GL(2, \mathbb{C})/Z \cdot K$ which satisfy

\begin{equation}
(*) \quad f(\gamma \tau) = \kappa(\gamma)f(\tau) \quad \text{for all } \gamma \in \Gamma(\lambda^3), \tau \in GL(2, \mathbb{C})/Z \cdot K.
\end{equation}

We now explain the correspondence between functions satisfying (*) and functions on the double cover of $GL(2, \mathbb{C})$. We form the double cover

$$\tilde{G} = \{(g, \epsilon) \mid g \in GL(2, \mathbb{C}), \epsilon = \pm 1\}$$

with componentwise multiplication $(g, \epsilon)(g', \epsilon') = (gg', \epsilon \epsilon')$. We also fix the subgroups

$$\tilde{\Gamma} = \{(g, \kappa(g)) \mid g \in \Gamma(\lambda^3)\}, \quad \tilde{Z} = \{(z, 1) \mid z \in Z\}, \quad \tilde{K} = \{(k, 1) \mid k \in K\}.$$

Consider now a function $\tilde{f}$ on $\tilde{G}/\tilde{Z} \cdot \tilde{K}$ satisfying: (1) $\tilde{f}$ is invariant under the action of $\tilde{\Gamma}$, and (2) $\tilde{f}\left((g, \epsilon)\right) = \epsilon \tilde{f}\left((g, 1)\right)$. Finally, define a function $f$ on $GL(2, \mathbb{C})/Z \cdot K$ by $f(g) = \tilde{f}\left((g, 1)\right)$. Then we have, for $\gamma \in \Gamma(\lambda^3)$,

$$f(\gamma g) = \tilde{f}\left((\gamma g, 1)\right) = \tilde{f}\left((\gamma, \kappa(\gamma)^2)\right) = \tilde{f}\left((\gamma, \kappa(\gamma)) (g, \kappa(\gamma))\right) = \tilde{f}\left((g, \kappa(\gamma))\right) = \kappa(\gamma) \tilde{f}\left((g, 1)\right) = \kappa(\gamma) f(g).$$

Thus functions on the double cover of $GL(2, \mathbb{C})$ which satisfy the conditions stated above correspond to functions on $GL(2, \mathbb{C})/Z \cdot K$ satisfying condition (*). (Functions on the double cover of $GL(2, \mathbb{C})$ satisfying the first condition (regardless of whether they satisfy the automorphy (second) condition) are called genuine.)
Automorphic forms are functions which satisfy (*) and which also satisfy a particular differential equation. This latter condition is a technical one, and we do not dwell on it here.

We now wish to show how to use a similar construction to find functions on the $n$-cover of $GL(r, \mathbb{C})$. It is clear how we would define functions on the $n$-cover of $GL(2, \mathbb{C})$; we would use the same construction as above, except that $\kappa$ would be derived from the $r^{th}$ order residue symbol. For higher rank groups, it becomes difficult to describe the homomorphism $\kappa$ necessary to define forms on covering group. Nonetheless, the $n$-cover of $GL(r)$ is defined, and metaplectic forms (automorphic forms on this covering group) do exist; the concrete framework we have chosen for our discussion is the wrong setting to discuss general metaplectic forms (see [KP] for a discussion of metaplectic forms on the covers of $GL(r)$).

One case in which $\kappa$ has been explicitly written down, for higher rank groups, is the case of the cubic cover of $GL(3, \mathbb{C})$. Because this is important for our applications, we review the construction here (details may be found in [BH]). As before, we write $Z$ and $K$ for the scalar matrices and unitary matrices of $GL(3, \mathbb{C})$, respectively; we will wish to define functions on the space $GL(3, \mathbb{C})/Z \cdot K$ which satisfy

$$f(\gamma \tau) = \kappa(\gamma) f(\tau)$$

for $\gamma$ in an appropriate discrete subgroup $\Gamma$, as before, and $\kappa : \Gamma \to \mu_3$ an appropriate homomorphism which takes values in the set of cube roots of unity. Such functions, as we have seen, correspond to functions on the cubic cover of $GL(3, \mathbb{C})$. Each coset of $GL(3, \mathbb{C})/Z \cdot K$ has a unique Iwasawa representative, as before, of the form

$$\begin{pmatrix} y_1 & y_2 & 1 \\ y_1 & 1 & \\ 1 & & \\ \end{pmatrix} \begin{pmatrix} 1 & x_2 & x_3 \\ 1 & x_1 & \\ 1 & & \\ \end{pmatrix},$$

where the $y_i$ are positive real numbers, and the $x_i$ are complex numbers. We may thus write down our desired functions concretely as functions of two real and three complex variables; this is precisely what we will do in the next section.

Turning now to the definition of $\kappa$, we recall our earlier notation of $\mathbb{K}$ and $\mathcal{O}$ for $\mathbb{Q}(\sqrt{-3})$ and its ring of integers, respectively. We write $\Gamma$ for the subgroup of all matrices in $SL(3, \mathcal{O})$ congruent to the identity modulo 3. We need a way of parametrizing matrices in $\Gamma$; to identify such a system, we first note that that $GL(3, \mathbb{C})$ possesses an involution

$$\iota : g \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \\ \end{pmatrix} \cdot \iota g^{-1} \cdot \begin{pmatrix} 1 & 1 \\ 1 & \\ \\ \end{pmatrix}.$$

The bottom rows of the matrices $g$ and $\iota(g)$ depend only on the orbit of $g$ in

$$GL(3, \mathbb{C})_\infty \setminus GL(3, \mathbb{C}),$$

where $GL(3, \mathbb{C})_\infty$ denotes the upper triangular unipotent matrices. We denote the bottom row of $g$ by the coordinates $A_1$, $B_1$, $C_1$, and the bottom row of $\iota(g)$ by $A_2$, $B_2$, $C_2$. Then we have a bijection between matrices $\gamma \in \Gamma$ and invariants satisfying

$$(A, B, C) = (A_1, B_1, C_1) = 1.$$
\[ A_1 C_2 + B_1 B_2 + C_1 A_2 = 0, \]
\[ A_1 \equiv A_2 \equiv B_1 \equiv B_2 \equiv 0 \ (3), \ C_1 \equiv C_2 \equiv 1 \ (3). \]

We may now define the appropriate Kubota homomorphism \( \kappa : \Gamma \to \mu_3 \). Given \( A_1, B_1, C_1, A_2, B_2, C_2 \) as above, we factor (with \( r_1 \equiv r_2 \equiv 1 \ (3), \ (C'_1, C'_2) = 1 \))
\[ B_1 = r_1 B'_1 \quad B_2 = r_2 B'_2 \]
\[ C_1 = r_1 r_2 C'_1 \quad C_2 = r_1 r_2 C'_2. \]

The function
\[ \kappa(g) = \left( \frac{B'_1}{C'_1} \right) \left( \frac{B'_2}{C'_2} \right) \left( \frac{C'_1}{C'_2} \right)^{-1} \left( \frac{A_1}{r_1} \right) \left( \frac{A_2}{r_2} \right) \]
is independent of the choice of factorization and is the desired homomorphism (cf. [BFH3, Theorem 2]).

Metaplectic forms on the cubic cover of \( GL(3, \mathbb{C}) \) are just complex–valued functions on \( GL(3, \mathbb{C})/Z \cdot K \) which satisfy
\[(**) \quad f(\gamma \tau) = \kappa(\gamma) f(\tau) \text{ for all } \gamma \in \Gamma, \tau \in GL(3, \mathbb{C})/Z \cdot K \]
and which also satisfy a particular differential equation (not the same equation as for forms on the double cover of \( GL(2, \mathbb{C}) \), but there is a global viewpoint which relates the two equations). For the rest of this paper, when we write “satisfying condition (**)”, (or condition (*)) we mean satisfying condition (**) (or condition (*)) and also satisfying that specified differential equation.

4. A certain metaplectic form

In a fundamental paper, Bump and Hoffstein [BH] constructed an Eisenstein series on the cubic cover of \( GL(3, \mathbb{C}) \) which had the property that its Whittaker–Fourier coefficients contained the Hecke L–series of the cubic residue symbol. As we noted in section 1, these L–series are nearly the L–series of the elliptic curves we wish to study; indeed, once twisted by a grossencharacter, they become exactly the L–series we wish to study. Our goal in this section, then, is to review the theory of metaplectic forms on \( GL(3, \mathbb{C}) \) and the construction of Bump and Hoffstein, and then to modify this construction to obtain a form from which we will be able to obtain information about the L–series of the curves \( E_D \).

We begin by setting notation, which will be constant throughout the rest of the paper. Let \( Z \) denote the scalar, and \( K \) the unitary matrices in \( GL(3, \mathbb{C}) \), as usual. We now write \( \Gamma^2 \) for the copy of \( SL(2, \mathbb{O}) \) embedded in the upper left corner of \( SL(3, \mathbb{O}) \), and \( \Gamma^2_\infty \) for the intersection of \( \Gamma^2 \) with the group of unipotent upper triangular matrices. We continue to let \( \Gamma \) denote the discrete group defined in the previous section, and write \( \Gamma_P \) for those elements of \( \Gamma \) which have a bottom row of \( (0 \ 0 \ 1) \) (this is the maximal parabolic subgroup of \( \Gamma \)). If \( \phi \) is a form satisfying (**), then we have a Fourier expansion
\[ \phi(\tau) = \sum_n \phi_{0,n}(\tau) + \sum_{\gamma \in \Gamma^2_\infty \setminus \Gamma} \phi_{m,n}(\gamma \tau). \]
The multiplicity-one theorem of Shalika asserts that
\[ \phi_{m,n} \left( \begin{pmatrix} y_1 y_2 & y_1 \\ y_1 & y_1 \end{pmatrix} \right) = \frac{a_{m,n}}{N(mn)} W(my_1, ny_2) \]
where \( W \) is the appropriate “Whittaker function” (more on Whittaker functions below). What Bump and Hoffstein did was to construct a metaplectic Eisenstein series \( \phi \) which had the property that, for \( mn \neq 0 \),
\[ (***) \quad a_{m,n} = N_{m,n} \times \sum_{a \in \mathcal{O}} \left( \frac{m^2 n}{a} \right)_3 N(a)^{-s + \frac{1}{2}} \]
where \( N_{m,n} \) is a finite sum, which increases in complexity with the divisibility of \( m \) and \( n \) by prime powers. (We will see below that since \( \phi \) is an Eisenstein series, it includes a (surpressed, in this notation) parameter \( s \) which is the parameter appearing in the right hand side of (***)�.

**Remark.** In order to obtain the particular \( L \)-series in (***), one must make a different choice of cubic symbol than the one actually made in [BH]. We will fix such a choice throughout. In particular, our symbol is conjugate to theirs.

In words, what Bump and Hoffstein did was construct the Eisenstein series on \( GL(3, \mathbb{C}) \) by inducing up from the cubic theta function on \( GL(2, \mathbb{C}) \). (cf. the last section of this paper. Their paper also contained several other fundamental results, which we will not cover here. The cubic theta function is a function on the cubic cover of \( GL(2, \mathbb{C}) \); see [Pa] for details). More precisely, they studied the form
\[ \phi(\tau) = \sum_{\gamma \in \Gamma_P \setminus \Gamma} \kappa(\gamma) I_s(\gamma \tau) \]
where
\[ I_s(\tau) = (y_1^2 y_2)^2 \theta_3 \left( \begin{pmatrix} y_2 & x_2 \\ 0 & 1 \end{pmatrix} \right) \]
(Here, \( y_i \) and \( x_i \) are the usual Iwasawa coordinates of \( \tau \) mentioned in the last section; \( \theta_3 \) is the \( GL(2, \mathbb{C}) \) cubic theta function. Note that \( I_s(\tau) \) is invariant under \( \Gamma_P \) (since \( (y_1^2 y_2) \) is).) We wish to modify the Bump-Hoffstein form in two ways. First, we wish to introduce the grossencharacter into the Whittaker-Fourier coefficients to construct a form which has the \( L \)-series of the desired elliptic curves in its Whittaker-Fourier coefficients; second, we will (in the construction) sum over a larger group, thus increasing the size of the discrete group \( \Gamma \) under which our form will be invariant.

**Remark.** This last modification will make the machinery of our next section much simpler than it would be otherwise. In general, one would like to make \( \Gamma \) as large as possible, but there are in fact limitations on the size of \( \Gamma \); these are imposed by the requirement that a Kubota homomorphism exist.

We begin by introducing the grossencharacter (see [L] for details of this process). The first step is to define a representation \( \hat{\rho} \) on the group
\[ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \mathbb{K}(3, \mathbb{C}) \quad | \quad \xi = \frac{R}{|R|}, \text{ where } R \in \mathcal{O}, R \equiv 1, 2 \pmod{3} \]
such that
\[ \tilde{\rho} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \pm \frac{R}{|R|} \]
(the sign is positive precisely when \( R \equiv 1 \pmod{3} \)). Next, we induce \( \tilde{\rho} \) up to the full group \( \mathbb{K}(3, \mathbb{C}) \) and let \( \rho \) be any (fixed now, once and for all) finite dimensional subrepresentation of this induced representation.

By abuse of notation, we will write \( \rho \) applied to a scalar matrix \( z \) times a unitary matrix \( k \). In these instances, what we mean is to rewrite the product \( z \cdot k \) as the product of a positive real scalar matrix times a unitary matrix (by multiplying \( z \) by the appropriate complex number of absolute value 1, and \( k \) by the conjugate of this number), and then to apply \( \rho \) to \( k \).

From Frobenius reciprocity, \( \rho \) restricted to (4.1) contains at least one copy of the original representation \( \tilde{\rho} \). Further, since \( \mathbb{K}(3, \mathbb{C}) \) is compact, we can define a functional \( T \) on \( \rho \) such that
\[ T(\rho(\tau)) = \tilde{\rho}(\tau) \]
for \( \tau \in (4.1) \), and
\[ T(\rho(\tau \cdot g)) = T(\rho(\tau))T(\rho(g)) \]
for \( \tau \in (4.1) \).

We may now finally define our “I-function” in terms of the \( I'_{w}(\tau) \) introduced by Bump and Hoffstein. For \( g \in GL(3, \mathbb{C}) \), fix an Iwasawa decomposition \( g = \tau \cdot z \cdot k \), where \( z \in \mathbb{Z}(3, \mathbb{C}) \), \( k \in \mathbb{K}(3, \mathbb{C}) \), and define
\[ I^\phi_{w}(g) = I'_w(\tau)T(\rho(z \cdot k)) \]
That this is well-defined follows from properties of \( \rho \). By abuse of notation, we will often refer to \( T(\rho(\gamma)) \) as \( \rho(\gamma) \). We also note that
\[ I^\phi_{w}(\gamma g) = I^\phi_{w}(g) \]
for \( \gamma \in \Gamma_P \).

We are now ready to define our main object of study in this paper. Let \( \Gamma^\mathbb{Z} \) denote those elements of \( SL(3, \mathbb{O}_{\sqrt{-3}}) \) which are congruent to an element of \( SL(3, \mathbb{Z}) \) mod 3. In addition, let \( \Gamma^\mathbb{Z}_P \) be those elements of \( \Gamma^\mathbb{Z} \) which are of the form
\[ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \]
We may now define the Eisenstein series
\[ \phi(\tau, w) = \sum_{\gamma \in \Gamma^\mathbb{Z}_P \backslash \Gamma^\mathbb{Z}} \kappa(\gamma)I^\phi_{w}(\gamma \tau) \]
and the normalized Eisenstein series
\[ \phi^*(\tau, w) = (2\pi)^{1-3w} \times \zeta \left( 3w - \frac{4}{3} \right) \zeta \left( 3w - \frac{2}{3} \right) \Gamma \left( \frac{3w}{2} - \frac{2}{3} \right) \Gamma \left( \frac{3w}{2} - \frac{1}{3} \right) \sum_{\gamma \in \Gamma^\mathbb{Z}_P \backslash \Gamma^\mathbb{Z}} \kappa(\gamma)I^\phi_{w}(\gamma \tau) \]
where \( \kappa(\gamma) \) the Kubota symbol defined and studied by [BFH] (recall that our cubic residue symbol is conjugate to theirs).

For the rest of this section, we fix the notation \( \phi^*(\tau, w) \) for this one particular Eisenstein series.

Recall the definition of the \( m \)-th Fourier coefficient of \( \phi^*(\tau, w) \).
\[ (4.2) \]
\[ \phi^*(\tau, w)_{m,n} = \int_{\mathbb{C}/3} \int_{\mathbb{C}/3} \int_{\mathbb{C}/3} \phi_w^* \left( \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ \xi_1 & 1 & \end{pmatrix} \right) e(-m\xi_1 - n\xi_2) \, d\xi_1 \, d\xi_2 \, d\xi_3. \]

(Note that the measures in (4.2) are complex Haar measures). This definition is actually quite natural, as these Fourier coefficients will appear in the evaluation of certain Rankin–Selberg convolutions. The following theorem shows the relationship of the Whittaker–Fourier coefficients of \( \phi^*(\tau, w)_{m,n} \) to the L–series of the elliptic curves (0.1).

**Theorem 3.** Suppose \( mn \neq 0 \). The \( m, n \)th Fourier coefficient \( \phi^*(\tau, w)_{m,n} \) is equal to
\[ \sum_{\alpha \geq 0} \sum_{\delta \leq \alpha} 3^{[\frac{\alpha-2\delta+1}{3}]} \left( \frac{3}{m'} \right)^{\alpha-2\delta} (3^\alpha)^{-1-3w} \]
\[ \cdot \sum_{b_2 \text{ mod } 3^{\alpha-\delta}} \left( \frac{3^{\alpha-\delta}}{b_2} \right) e \left( \frac{n \cdot b_2}{3^{\alpha-\delta}} \right) \sum_{C_2 \text{ mod } 3^\alpha} \left( \frac{3^\delta}{C_2} \right) e \left( \frac{m \cdot b_2^{-1} \cdot C_2}{3^\delta} \right) \]
\[ + 3 \sum_{\alpha \geq 0} 3^{[\frac{\alpha+1}{3}]} \left( \frac{3}{m'} \right)^{\alpha} (3^\alpha)^{-1-3w} \sum_{b_2 \text{ mod } 3^\alpha} \left( \frac{3^\alpha}{b_2} \right) e \left( \frac{nb_2}{3^\alpha} \right) \]
\[ + 9; \]
\[ W(w, y_1, y_2) \]

\[ (2\pi)^{1-3w} \zeta(3w - \frac{4}{3}) \zeta(3w - \frac{2}{3}) \Gamma \left( \frac{3w - 2}{3} \right) \Gamma \left( \frac{3w - 1}{3} \right) \]
\[ \cdot (y_1 y_2)^w \int_{\mathbb{C}} \int_{\mathbb{C}} \rho \left( \begin{pmatrix} -\xi_2 Q^{-\frac{1}{2}} & y_2 Q^{-\frac{1}{2}} & 0 \\ -y_2 & -\xi_2 P^{-\frac{1}{2}} Q^{-\frac{1}{2}} & y_1 \xi_2 P^{-\frac{1}{2}} \xi_3 P^{-\frac{1}{2}} \\ y_1 y_2 & y_1 \xi_2 P^{-\frac{1}{2}} & \xi_3 P^{-\frac{1}{2}} \end{pmatrix} \right) \]
\[ \times P^{1-\frac{3w}{2}} Q^{-1} K_\frac{1}{2} \left( 4\pi y_2 P^{\frac{1}{2}} Q^{-1} \right) e \left( -\xi_2 Q^{-1} \right) e(-\xi_2) \, d\xi_2 \, d\xi_3; \]

and \( A(w)_{m,n} \) is the product of the L-series
\[ \prod_{p|\ell \cdot \lambda^6 mn = 1, \ell \equiv 1 (3)} \left( 1 - \left( \frac{mn^2}{p} \right) \frac{|p|}{p} Np^{1-3w} \right)^{-1} \left( 1 - \left[ \frac{|p|}{p} \right] Np^{3-2w} \right) \]

with a factor
\[ \frac{1}{2} 3^{\frac{5}{2}} |mn^2|^{w-1} \]
\[ \sum_{A' \equiv 1 (3); 3|A'} \frac{|A'|}{A} |A'|^{1-3w} \sum_{A'=a'd';d'|3^\lambda ma'} \tau \left( \frac{ma'}{d'} \right) \]
\[ \times \sum_{b_1 \text{ mod } a'} \left( \frac{b_1}{a'} \right) \left( \frac{b_1}{a'} \right) \sum_{C_1 \text{ mod } d'} \left( \frac{C_1}{d'} \right) e \left( \frac{m b_1^{-1} C_1}{d'} \right). \]
5. The Rankin–Selberg Method

A Rankin–Selberg convolution, in a general sense, is a machine for constructing a Dirichlet series with coefficients which are built out of the Whittaker–Fourier coefficients of automorphic forms, and for providing analytic information about the behavior of that Dirichlet series. This information can then be translated, as we mentioned in section 1, into information about the collective behavior of the coefficients of the original forms.

The type of convolution we will discuss below is an example of a convolution discovered independently by Asai [A] and Patterson [Pa]. It has one particularly nice feature: recall that the Whittaker–Fourier coefficients of the form constructed in the previous section were indexed by integers in a quadratic field. The Asai–Patterson convolution produces a “sieved” Dirichlet series, made up of those coefficients with rational integral indices. For our application, this is precisely what we desire.

Recall that a $GL(3, \mathbb{C})$ metaplectic form automorphic with respect to $GL(3, \mathbb{O})$ has a Fourier expansion

$$
\phi(\tau) = \sum_{n \in \mathcal{O}} \phi_{0,n}(\tau) + \sum_{\gamma \in \Gamma_\infty^2(\mathbb{C}) \setminus \Gamma^2(\mathbb{C})} \sum_{m,n \in \mathcal{O}, m \neq 0} \phi_{m,n}(\gamma \tau).
$$

and that we have

$$
\phi_{m,n}(\tau) = \text{arithmetic part} \times W_{m,n}(y_1, y_2)e(mx_1)e(nx_2)
$$

where $e(\cdot)$ is an exponential function and $W_{m,n}(y_1, y_2)$ is Whittaker function. If $mn \neq 0$, then $W_{m,n}(y_1, y_2)$ is of rapid (at least exponential) decay in both $y_1$ and $y_2$ as $y_1 \to \infty$ and $y_2 \to \infty$. When $m$ or $n$ is zero, then $W_{m,n}(y_1, y_2)$ is of polynomial growth/decay in $y_1$ or $y_2$, respectively. Note also that if $mn \neq 0$, then $W_{m,n}(y_1, y_2) = W(my_1, ny_2)$; that is, the non-degenerate Whittaker functions are all the same.

If we define the functions $I_s(\tau) = (y_1y_2)^s$ and

$$
\zeta^*(s) = \zeta(s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),
$$

then the maximal parabolic Eisenstein series

$$
E(\tau, s) = \zeta^*(3s) \sum_{\gamma \in SL(3,\mathbb{Z})P \setminus SL(3,\mathbb{Z})} I_s(\gamma \tau)
$$

(again, $SL(3,\mathbb{Z})P$ denotes those elements of $SL(3,\mathbb{Z})$ with bottom row $(0 0 1)$) converges for $re(s) > 1$, has a meromorphic continuation to the entire complex plane with simple poles at $s = 0$ and $s = 1$, and has a functional equation as $s \mapsto 1 - s$.

When $\phi$ is a $GL(3, \mathbb{C})$ cusp form ($\phi_{m,n} = 0$ whenever $mn = 0$), then the convolution

$$
R(s, \phi) = \int_{SL(3,\mathbb{Z}) \setminus (GL(3,\mathbb{R})/Z \cdot K)} \phi(\tau)E(\tau, s)d^h\tau
$$

(where $d^h\tau$ is the invariant Haar measure

$$
d^h\tau = \frac{dx_1dx_2dx_3dy_1dy_2}{(my_1ny_2)^3})
$$
is convergent (since the non-degenerate Whittaker functions are of rapid decay) and inherits the analytic properties (in $s$) of $E(\tau, s)$. On the other hand, it is straightforward to verify that by unfolding the Eisenstein series, and then unfolding the Fourier expansion of $\phi$, we obtain

\[ R(s, \phi) = \sum_{m,n \in \mathbb{Z}, \; mn \neq 0} \frac{a_{m,n}}{|m^2n|^s} \int_0^\infty \int_0^\infty W(y_1, y_2)y_1^{2s-3}y_2^{s-3} dy_1 dy_2. \]

We know that $R(s, \phi)$ has a pole at $s = 1$, and so if we show that the double Mellin transform of the Whittaker function converges at $s = 1$, then we have established that the Dirichlet series

\[ \sum_{m,n \in \mathbb{Z}, \; mn \neq 0} \frac{a_{m,n}}{|m^2n|^s} \]

has a pole at $s = 1$, and we may apply the “easy method” of Section 2 to translate this information into information about the coefficients $a_{m,n}$. We may also use the functional equation of $R(s, \phi)$ to apply Tauberian type theorems to this Dirichlet series.

When $\phi$ is not a cusp form, the convolution above does not converge. Nonetheless, it is possible to define a Rankin–Selberg convolution which evaluates to the desired Dirichlet series, and to obtain (albeit with much more work than in the cuspidal case) the functional equation and meromorphic continuation of this convolution. For $GL(2, \mathbb{R})$, this extension of the theory of Rankin–Selberg convolutions to non–cuspidal forms was accomplished by Zagier [Z]. The $GL(3, \mathbb{C})$ convolution was extended to non–cuspidal forms in [L2], whose results we summarize below.

Let $F_u$ denote the sum of the degenerate terms of the Fourier expansion of $\phi$ (expanded with respect to the upper parabolic subgroup), namely

\[ F_u = \sum_{n \in O} \phi_{0,n}(\tau) + \sum_{\gamma \in \Gamma_2(\mathbb{C}) \setminus \Gamma_2(\mathbb{C})} \sum_{m \in O, \; m \neq 0} \phi_{m,0}(\gamma \tau). \]

We now may define the Rankin-Selberg convolution $R(\phi, \omega)$ by

\[ R(\phi, \omega) = \zeta^*(3s) \int_{SL(3,\mathbb{R}) \setminus (GL(3,\mathbb{R})/Z \cdot K)} [\phi(\tau) - F_u(\tau)] (y_1^2 y_2)^s d^h \tau. \]

Note that this is analogous to unfolding the Eisenstein series in the cuspidal convolution, and that it is still not difficult to evaluate this convolution directly, and obtain the desired Dirichlet series. The meromorphic continuation and functional equation are given by the following Theorem. (In the Theorem, we use the standard notation of $\tilde{\phi}(\tau) = \phi(\tau)$.)

**Theorem 4.** The Rankin-Selberg convolution converges for $Re(\omega)$ sufficiently large, has a meromorphic continuation to all $\omega$, and satisfies the functional equation

\[ R(\phi, \omega) = R(\tilde{\phi}, 1 - \omega). \]

If the 0,0$^{th}$ Fourier coefficient of $\phi$ is the polynomial

\[ \phi_{0,0} = \sum_{i=1}^k y_1^{\alpha_i} y_2^{\beta_i}, \]

then

\[ 1 = \sum_{m,n \in \mathbb{Z}, \; mn \neq 0} \frac{a_{m,n}}{|m^2n|^s} \int_0^\infty \int_0^\infty W(y_1, y_2)y_1^{2s-3}y_2^{s-3} dy_1 dy_2. \]
then $R(\phi, w)$ may have poles only at the locations $0, \frac{1}{3}, \frac{2}{3}, 1, \alpha_i - 1, 2 - \beta_i, 1 - \frac{\alpha_i}{2}$ and $\frac{\beta_i}{2}$.

Further, if the Fourier expansion of $\phi$ is given by (3.2) where for $mn \neq 0$ we have

$$\phi_{m,n}(\tau) = \frac{a_{m,n}}{|mn|^2}W(y_1, y_2)$$

then we may explicitly obtain

$$R(\phi, w) = \sum_{m,n \in \mathbb{Z}, mn \neq 0} \frac{a_{m,n}}{|m^2n|^w} \int_0^\infty \int_0^\infty W(y_1, y_2)y_1^{2w-3}y_2^{w-3} dy_1 dy_2.$$
convolution is thus constructed from a smaller group (recall that we need to “un-
fold” this Eisenstein series using the automorphy of the form in the convolution),
and has a more complicated functional equation. This is precisely why we summed
over a larger group (in constructing $\phi^*$) than Bump and Hoffstein did: so that we
could use the maximal parabolic Eisenstein series constructed from the full group
$SL(3, \mathbb{Z})$, and not from a congruence subgroup.

The order of the pole of the Dirichlet series made up from the $L(E_D, 1)$ is quite
important. We have seen that it has at most a double pole. The question of whether
it actually has a double pole, or merely a single pole, depends on whether the double
Mellin transform of the twisted Whittaker function (4.3) vanishes. If it does, the
Dirichlet series has a single pole (it is direct to verify that the single pole does
indeed have a non-zero residue). If not, the Dirichlet series has a double pole (the
Lemma above states that this double transform converges, but says nothing about
its actual value). This problem is quite hard: indeed, one has almost no information
about the representation $\rho$ with which to work. It might be possible to answer this
question directly (in the language above), but there is another viewpoint which
might provide a more insightful answer – that of representation theory, cf. the next
Section.

6. Future directions

There are two main paths to generalize the work above. One is to try to sharpen
the analytic machinery in order to obtain better average value results, or average
value results for $L(E_D, 1)$ where $D$ ranges over smaller sets. The other is to try
to find Eisenstein series which have other interesting $L$–series in their Whittaker–
Fourier coefficients; indeed, a nice long–term goal might be to try to find the cor-
respondence between $L$–series and the associated metaplectic forms.

We start by considering this latter problem. If we think of a metaplectic form
as a form on $GL(r, \mathbb{C})$ which satisfies a certain transformation property (such as
(*) or (**)), then we always have a Fourier expansion (due to the multiplicity–one
formula of Shalika, which asserts that the space of Whittaker functions is one–
dimensional for $GL(r, \mathbb{C})$). It is not always possible, however, to compute explicitly
the arithmetic part of the Whittaker–Fourier coefficients, as it was in section 4. The
reason for this is that if one defines metaplectic forms directly as automorphic forms
on the metaplectic group, in general there is no uniqueness of Whittaker functions.
So to have any hope of finding metaplectic forms with interesting $L$–series in their
Fourier coefficients, we would have have to prove the forms in question had unique
Whittaker models. Conversely, if one could find forms with unique Whittaker
models, it would be interesting to compute their Fourier coefficients.

Bump and Hoffstein [H] have conjectured that the Hecke $L$–series of the $n^{th}$ or-
der residue symbol should appear in the Fourier coefficients of the Eisenstein series
obtained by inducing the theta function on the $n$–cover of $GL(n-1, \mathbb{C})$ up to the
$n$–cover of $GL(n, \mathbb{C})$. (The case $n = 3$ is considered in their paper [BH].) Bump
and Lieman [BL] have shown that the corresponding local representation (the rep-
resentation obtained by inducing the exceptional representation on the $n$–cover of
$GL(n-1, \mathbb{F})$ up to the $n$–cover of $GL(n, \mathbb{F})$, where $\mathbb{F}$ is a local field) does indeed
have unique Whittaker models. This result provides evidence that the conjecture
of Bump and Hoffstein is provable (indeed, this conjecture is the target of an ongo-
ing investigation of Bump and Lieman). (Further, the viewpoint of this approach
would be quite new.)
(representation theory) is the correct one to determine whether the twisted Whittaker function, discussed at the end of Section 5, does indeed vanish.) Once this conjecture is proved, one could obtain the L-series of biquadratic twists of elliptic curves as Whittaker-Fourier coefficients of a metaplectic form on the 4-cover of \( GL(4, \mathbb{C}) \) by introducing a *groszencharacter* as above.

The other main avenue for future research is to develop improved analytic machinery. Farmer, Kumanduri and Lieman are working to obtain the functional equation for the twisted \( GL(3, \mathbb{C}) \) Asai-Patterson convolution discussed at the end of Section 5; among other applications, this would allow one to obtain average values of of \( L(E_D, 1) \) where \( D \) ranges over rational integers in a fixed conjugacy class modulo any prime \( p \neq 3 \). Farmer, Hoffstein and Lieman are also working on obtaining the functional equation for the L-series of \( GL(3, \mathbb{C}) \) forms twisted by a character. This will allow the computation of the average values of the L-series of \( E_D \) where \( D \) ranges not over rational integers, but over all cube-free integers in \( \mathcal{O} \).
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