Universal conductivity and central charges

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We discuss a class of critical models in $d \geq 2+1$ dimensions whose electrical conductivity and charge susceptibility are fixed by the central charge in a universal manner. We comment on possible bounds on conductivity, as suggested by holographic duality.

I. INTRODUCTION AND SUMMARY

It is not uncommon to find physical systems which are described by interacting conformal field theories (CFTs). A simple example is the liquid-gas critical point whose static correlations are described by the Ising CFT in $d = 3$. Recently, CFTs which are formulated in space-time (rather than just space) have received attention, partly due to their appearance in quantum critical phenomena \cite{1, 2}. Such CFTs are relativistic theories, even though their speed of “light” $v$ is not necessarily equal to $3 \times 10^8 \text{m/s}$. As a result, charge transport in these systems at non-zero temperature obeys simple scaling laws.

At very short distances, the effects of temperature are irrelevant, and the natural physical questions involve the leading short-distance singularities of the correlation functions. On the other hand, at long distances the effects of temperature become important, and the natural questions are related to thermodynamics and transport phenomena. In CFTs, however, short and long distances are related by a scaling symmetry, and one may anticipate a universal relation between the long-distance transport coefficients and the parameters which describe the short-distance singularities. Unfortunately, this expectation seems to be quantitatively true only in $1+1$ dimensions. The subject of this note is precisely the class of models where such universal relations between short- and long-distance transport parameters extend naturally to any dimension $d > 1+1$.

At zero temperature, the (Euclidean) vacuum correlation functions of the energy-momentum tensor $T_{\mu\nu}$ and a $U(1)$ conserved current $J_\mu$ in a CFT are fixed to be

\begin{equation}
\langle J_\mu(x)J_\nu(0) \rangle = \frac{k}{x^{2(d-1)}} \frac{1}{\omega_{d-1}^2} I_{\mu\nu},
\end{equation}

\begin{equation}
\langle T_{\mu\nu}(x)T_{\alpha\beta}(0) \rangle = \frac{c}{x^{2d}} \frac{1}{\omega_{d-1}^2} \left( I_{\mu\alpha} I_{\nu\beta} + I_{\mu\beta} I_{\nu\alpha} - \frac{2}{d} \delta_{\mu\nu} \delta_{\alpha\beta} \right),
\end{equation}

where $I_{\mu\nu} \equiv \delta_{\mu\nu} - 2x_\mu x_\nu/x^2$, and $k$ and $c$ are central charges, which are dimensionless constants. [We use units in which $\hbar = v = 1$ where $v$ is the speed of “light” in the CFT.] The factors of $\omega_{d-1} \equiv 2\pi^{d/2}/\Gamma(d/2)$ are inserted for notational convenience. At non-zero temperature $T$, the equilibrium state is characterized by pressure $P$, as well as by the charge susceptibility $\chi = \langle Q^2 \rangle/(VT)$, where $Q$ is the conserved charge associated with the current $J_\mu$, and we take the thermodynamic limit $V \to \infty$. The susceptibility can be evaluated by introducing a small chemical potential $\mu$, so
that $\chi(T) = \partial \rho / \partial \mu |_{\mu=0}$, where $\rho(T, \mu) = \langle Q \rangle / V$ is the charge density. In a CFT, temperature remains the only scale which dictates

$$P(T) = c'T^d, \quad \chi(T) = k'T^{d-2},$$

(3)

where $c'$ and $k'$ are dimensionless constants. Physically, $c$ and $c'$ provide a measure of the total number of degrees of freedom in the system, while $k$ and $k'$ measure the number of charged degrees of freedom. In two dimensions, $c$ is uniquely related to $c'$ [3, 4], while $k$ is uniquely related to $k'$:

$$c' = \frac{\pi}{6} c, \quad k' = \frac{1}{2\pi} k,$$

(4)

which means that thermodynamics is uniquely fixed by the central charges. The reason is that in two dimensional CFTs, the vacuum state is related to the thermal state by a symmetry transformation. We review this argument in the next section. In $d > 2$, the conformal symmetry group is not large enough to enforce a relation similar to Eq. (4), and therefore thermodynamics is not determined by the central charges. However, there does exist a large class of CFTs in $d > 2$, whose pressure is determined by the central charge $c$, resembling the two-dimensional case [5]. The crucial property of these models is that they admit a dual description in terms of classical gravity on a $(d+1)$ dimensional anti-de Sitter space (AdS). We will show that these CFTs also have the property that their susceptibility is determined by the central charge $k$, as for the two-dimensional case. Namely, we find the following relations:

$$\frac{c'}{c} = \frac{1}{4\pi^{d/2}} \left( \frac{4\pi}{d} \right)^d \frac{\Gamma(d/2)^3}{\Gamma(d)} \frac{(d-1)}{d(d+1)}; \quad \frac{k'}{k} = \frac{1}{2\pi^{d/2}} \left( \frac{4\pi}{d} \right)^{d-2} \frac{\Gamma(d/2)^3}{\Gamma(d)}.$$

(5)

Even though the ratios (5) are derived for integer $d \geq 3$, they can be analytically continued to any real positive $d$. In particular, in $d = 2$ they reproduce the universal relations (4).

The CFTs which admit a dual gravitational description have many more universal properties beyond the above relation between thermodynamics and the central charges. A surprising feature of these CFTs (and of their relevant deformations) is that momentum transport in these models is completely determined by thermodynamics. In particular, their viscosity is given by $\eta = s/4\pi$ in any dimension [6, 7], where $s = (\partial P / \partial T)$ is the entropy density. This is surprising because transport coefficients are typically determined by the mean-free path even in CFTs [8], and are not fixed by thermodynamics. We will show that charge transport in these models is also completely determined by thermodynamics. Namely, we find that the dc electrical conductivity $\sigma$ obeys a similar relation,

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad \frac{\sigma}{\chi} = \frac{1}{4\pi T} \frac{d}{d-2}.$$  

(6)

Again, even though the ratio $\sigma/\chi$ was derived for integer $d \geq 3$, it can be analytically continued to any real positive $d > 2$. We will see that the singularity at $d = 2$ is precisely what one expects in 1+1 dimensional CFTs.

The ratio of viscosity to entropy density was conjectured to be a universal lower bound, saturated by models with a dual gravity description [6]. Motivated by the viscosity bound conjecture, we discuss similar bounds on conductivity in relativistic CFTs which are saturated by models with gravity duals.
II. NO HYDRODYNAMICS IN 1+1 DIMENSIONS

In this section, we review the argument [9] that 1+1 dimensional CFTs have no hydrodynamic regime, and derive the relation (4) between the susceptibility and the central charge $k$.

In two dimensions, correlation functions at zero temperature and finite temperature can be related to each other [10]. This is because the transformation which maps a plane to a cylinder is a conformal transformation, and therefore is a symmetry transformation in a CFT. The finite temperature state is obtained by the exponential map $z = e^{2\pi i T w}$, where $z = x^0 + i x^1$ represents a point on the plane, $w = \tau + iy$ represents a point on the cylinder, and $\tau$ is Euclidean time which is periodic with period $1/T$.

For a scalar operator of dimension $\Delta$, a conformal transformation $x \rightarrow \bar{x}$ restricts the two-point correlation function as follows

$$\langle \phi(x_a) \phi(x_b) \rangle = D(x_a)^{-\Delta/d} D(x_b)^{-\Delta/d} \langle \phi(x_a) \phi(x_b) \rangle,$$  \hspace{1cm} (7)

where $D(x) = \left| \det(\partial x'/\partial x) \right|$ is the Jacobian of the coordinate transformation. For the above exponential map in $d=2$, we have $D(x) = 1/(2\pi T |x|)^2$, while the zero-temperature correlator is simply a power-law, $\langle \phi(x_a) \phi(x_b) \rangle = C_\phi/|x_a - x_b|^{2\Delta}$. From the transformation relation (7) we find the finite-temperature correlator of the scalar field,

$$\langle \phi(\tau, y) \phi(0) \rangle = C_\phi \left[ \frac{(\pi T)^2}{\sin[\pi T (\tau + iy)] \sin[\pi T (\tau - iy)]} \right]^\Delta.$$  \hspace{1cm} (8)

This expression is periodic under $\tau \rightarrow \tau + 1/T$ (as it should be), and reduces to the standard power-law result in the limit $T \rightarrow 0$. For models with a dual gravitational description, this form of the correlator was reproduced from small perturbations of the BTZ black hole in [11]. A similar argument can be applied to the density-density correlator on the plane in Eq. (1), which can be written as

$$C_{00}(x^0, x^1) = -\frac{k}{8\pi^2} \left\{ \frac{1}{(x^0 + ix^1)^2} + \frac{1}{(x^0 - ix^1)^2} \right\}. $$  \hspace{1cm} (9)

At finite temperature, we find

$$C_{\tau\tau}(\tau, y) = -\frac{k}{8\pi^2} \left\{ \frac{\pi T}{\sin[\pi T (\tau + iy)]} \right\}^2 + \left\{ \frac{\pi T}{\sin[\pi T (\tau - iy)]} \right\}^2.$$  \hspace{1cm} (10)

Again, this expression is periodic under $\tau \rightarrow \tau + 1/T$, and reduces to Eq. (9) when $T \rightarrow 0$. The charge susceptibility follows from the thermal density-density correlator,

$$\chi = -\frac{1}{T} \int d^{d-1}y \ C_{\tau\tau}(\tau, y) ,$$  \hspace{1cm} (11)

where the extra minus sign is due to the Euclidean signature. In $d=2$ dimensions, we use the explicit expression (10), and find $\chi = k/2\pi$, as stated earlier in Eq. (4).

The imaginary-time result (10) can be Fourier transformed,

$$C_{\tau\tau}(\omega_n, q) = \int_0^{1/T} d\tau \int dy \ C_{\tau\tau}(\tau, y) e^{-i\omega_n \tau - i q y},$$  \hspace{1cm} (12)
where \( \omega_n = 2\pi n T \) is the Matsubara frequency. When performing the Euclidean time integration, the domain can be extended to include the whole real axis, and one picks up contributions from an infinite sequence of poles in the complex \( \tau \) plane. For the density-density correlator one finds a simple expression

\[
C_{\tau\tau}(\omega_n, q) = \frac{k}{2\pi} \frac{q^2}{\omega_n^2 + q^2}.
\]

(13)

Analytic continuation to real frequency \( \omega \) produces the retarded correlator \( C^\text{ret}_{\tau\tau}(\omega, q) \) which only has light-cone singularities, but shows no hydrodynamic modes (as one would find in higher dimensions). Formal application of the Kubo formula now gives

\[
\sigma(\omega) = \text{Im} \frac{\omega}{q^2} C^\text{ret}_{\tau\tau}(\omega, q) = \frac{k}{2} \delta(\omega).
\]

The singularity in the dc limit \( \omega \rightarrow 0 \) is precisely what we have in the general result (6) when \( d=2 \).

### III. CHARGE SUSCEPTIBILITY

We will focus on quantum field theories which admit a dual description in terms of classical gravity in Anti-de Sitter (AdS) space within the AdS/CFT correspondence [12]. For such field theories, a large-volume thermal state in a \( d \)-dimensional CFT is described by a \( (d+1) \)-dimensional black hole in AdS. The black hole solution follows from the Einstein-Maxwell action,

\[
S = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left[ R + \frac{d(d-1)}{L^2} \right] - \frac{1}{4g_{d+1}^2} \int d^{d+1}x \sqrt{-g} F^2,
\]

(14)

where \( L^2 \) sets the value of the cosmological constant, and \( g_{d+1}^2 \) is the \( (d+1) \)-dimensional gauge coupling constant, which has dimension of \( \text{length}^{d-3} \). An equilibrium state at finite temperature and density is described by the Reissner-Nordstrom black hole in AdS. The thermodynamics of these black holes has been studied extensively, see for example [13]. The metric in the thermodynamic limit is given by

\[
ds^2 = \frac{r^2}{L^2} \left( -V(r) dt^2 + dx^2 \right) + \frac{L^2}{r^2} \frac{dr^2}{V(r)},
\]

(15)

where \( V(r) = 1 - m/r^d + m_q^2/r^{2d-2} \), and the boundary is at \( r \rightarrow \infty \). The parameter \( m \) determines the mass of the black hole, and \( m_q \) determines its charge. The background gauge field is \( A_t = \mu - C/r^{d-2} \), where the constant \( C \) is related to the charge density of the CFT. The chemical potential \( \mu \) is fixed by the condition that \( A_t \) vanishes at the horizon \( r = r_0 \), i.e.

\[
\mu = \frac{C}{r_0^{d-2}}.
\]

(16)

The charge density \( \rho \) is defined by the variation of the action with respect to the boundary value of the bulk gauge field \( A_t^{(b)} = A_t(r \rightarrow \infty) \),

\[
\rho = \frac{\delta S}{\delta A_t^{(b)}} = \frac{(d-2)C}{g_{d+1}^2 L^{d-1}}.
\]

(17)
To find the susceptibility, we need the relation between $\rho$ and $\mu$ to linear order in $\mu$. This means that in (16) it suffices to express $r_0$ in terms of temperature when $\mu \to 0$, and one finds $T = r_0 d/(4\pi L^2)$. From the definition of the chemical potential (16) we find $\rho(T, \mu) = \chi(T) \mu$, where the susceptibility$^1$ is

$$\chi = \frac{(d-2)L^{d-3}}{g^2_{d+1}} \left(\frac{4\pi}{d}\right)^{d-2} T^{d-2}. \quad (18)$$

The value of the central charge $k$ can be found from the results of Freedman et al. [17]:

$$k = \frac{L^{d-3}}{g^2_{d+1}} \frac{\Gamma(d)(d-2)}{2\pi^{d/2}\Gamma(d/2)} \omega_{d-1}^2. \quad (19)$$

Comparing with the susceptibility in (18), we find our result for $k'/k$ in Eq. (5).

**IV. ELECTRICAL CONDUCTIVITY**

The methods of the AdS/CFT correspondence also allow us to compute the electrical conductivity of CFTs with a dual gravity description. Since these models typically do not have dynamical $U(1)$ gauge fields, we first need to say what we mean by the conductivity. We imagine gauging a global $U(1)$ symmetry of the theory with a small coupling $e$, and work to leading order in $e$. The electrical conductivity is then defined with respect to this $U(1)$ gauge field. To leading order in $e$, the effects of the gauge field can be ignored, and the electromagnetic response can be determined from the original theory [18]. This essentially amounts to sending $J_\mu \to e J_\mu$, and a factor of $e^2$ will appear in both the conductivity and the susceptibility. The conductivity is determined from the real-time current-current correlation function in thermal equilibrium,

$$\sigma(\omega) \delta_{ij} = e^2 \text{Im} \frac{1}{\omega} C_{ij}^{ret}(\omega, q=0). \quad (20)$$

Here $C_{ij}^{ret}(\omega, q)$ is the retarded correlation function of the global $U(1)$ symmetry currents. The dc conductivity is $\sigma(\omega=0)$.

To evaluate $C_{ij}^{ret}(\omega, q)$, we use the standard AdS/CFT recipe of [11, 14], and consider Maxwell fields propagating on the $(d+1)$ dimensional background,

$$ds^2 = \frac{L^2}{z^2} \left(-f(z)dt^2 + dx^2 + \frac{dz^2}{f(z)}\right), \quad (21)$$

where $f(z) = 1 - (z/z_0)^d$, and the temperature of the CFT is $T = d/(4\pi z_0)$. This metric is obtained from (15) at $m_q=0$, changing the radial coordinate to $z = L^2/r$. The bulk action for the Maxwell field is given by (14). Translation invariance allows us to take the bulk gauge field proportional

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$^1$ The value of the susceptibility can also be deduced from the hydrodynamic current-current correlators. Namely, one has for the retarded charge density-charge density correlation function: $C_{ii}^{ret}(\omega, q) = \chi D q^2 / (i \omega - D q^2)$, where $D$ is the charge diffusion constant. Comparing this with the hydrodynamic correlators found in [14] for $d=4$, and in [15] for $d=3, 6$, one finds the susceptibility in $d = 3, 4, 6$ in agreement with the general expression (18).
to $e^{-i\omega t+i\mathbf{q}\cdot\mathbf{x}}$, and $\mathbf{q}=0$ is sufficient to find the conductivity using the Kubo formula (20). The component $A_i$ satisfies the equation

$$u^{d-3} \left[ \frac{f(u)}{u^{d-3}} A_i'(u) \right]' + \frac{w^2}{f(u)} A_i(u) = 0,$$

(22)

where $u = z/z_0$, and $w = \omega z_0$. The computation of the retarded correlation function requires the choice of an outgoing boundary condition at the horizon, i.e. $A_i(u) = (1-u)^{-iw/d} a(u)$, where $a(u)$ is regular at $u=1$. To find the dc conductivity, we solve the equation for $a(u)$ as a power expansion in frequency, $a(u) = a_0 + iwa_0 h(u) + \mathcal{O}(w^2)$. For arbitrary dimension $d$, the solution for $h(u)$ can be expressed in terms of Gauss’ hypergeometric function, and the integration constants are fixed by requiring that $h(u)$ vanishes at the horizon. The current-current retarded correlation function is evaluated from the on-shell boundary action,

$$S = \frac{(L/z_0)^{d-3}}{2g_{d+1}^2} \int \frac{d\omega}{2\pi} \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{A_i'(\omega, u) A_i(-\omega, u)}{z_0 u^{d-3}},$$

(23)

with the implicit limit $u \to 0$. The near-boundary expansion for $h(u)$ has the form $h(u) = h(0) + \ln(1-u)/d + u^{d-2}/(d-2) + \mathcal{O}(u^{2d-2})$ which allows us to read off $C_{ij}^{\text{ret}}(\omega, \mathbf{q}=0)$ to leading order in $\omega$. The Kubo formula (20) then gives the conductivity,

$$\sigma = \frac{e^2}{g_{d+1}^2} \left( \frac{L}{z_0} \right)^{d-3}.$$

(24)

On the other hand, the susceptibility (18) can be written as $\chi = (e^2/g_{d+1}^2) \left( L/z_0 \right)^{d-3}(d-2)/z_0$ and we arrive at the simple result (6) for the conductivity to susceptibility ratio. For systems in which charge transport proceeds by diffusion, conductivity is related to the diffusion constant $D$ by the Einstein relation $\sigma = \chi D$. Therefore, our result can be interpreted as a remarkably simple diffusion constant in $d$ spacetime dimensions,

$$D = \frac{1}{4\pi T} \frac{d}{d-2}.$$

(25)

One readily verifies that it agrees with the known results in $d=4$ [14], and $d=3,6$ [15]. The electrical conductivity takes a particularly simple form in 2+1 dimensional CFTs. In this case the equation (22) can easily be solved for all $\omega$, and one finds a frequency-independent optical conductivity [9],

$$\sigma(\omega) = \frac{e^2}{g_4^2}.$$

It is a peculiar feature of these models that because of strong quantum fluctuations the optical conductivity in 2+1 dimensions is frequency-independent, and shows no crossover regime at $\hbar \omega \sim k_BT$. It would be very exciting to find two-dimensional materials which have this property.

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2 We were informed by A. Starinets that he has independently obtained Eq. (25) [16].
V. A CONDUCTIVITY BOUND?

We have shown that in all CFTs with a classical gravity dual, the ratio of electrical conductivity to the static charge susceptibility is given by a very simple form (6). Given that in these models the thermodynamics is fixed by the central charges, and transport coefficients are fixed by thermodynamics, it follows that transport coefficients are uniquely fixed by the central charges, \( \eta \sim c T^{d-1} \) and \( \sigma \sim k T^{d-3} \). Therefore, the ratio of viscosity to conductivity is proportional to the ratio of the central charges,

\[
\frac{\eta}{\sigma} \sim \frac{c}{k}.
\]

If \( c \) and \( k \) indeed provide a suitable measure of the number of degrees of freedom in the system, it is not unreasonable to assume that the right-hand side of (26) is bounded from below because the number of charged degrees of freedom must be smaller than the total number of degrees of freedom. As a result, one could imagine that the conductivity and viscosity obey a bound of the kind \( \sigma \leq \lambda d \eta c^2 \), with some order one constant \( \lambda d \). However, one should keep in mind that the definition of \( \sigma \) (or \( \eta \)) involves an arbitrary choice of normalization for the corresponding current. [E.g., if the electromagnetic \( U(1) \) is chosen as a subgroup of a larger global symmetry group \( G \), this translates to an arbitrary choice of normalization for the generators of \( G \).] Therefore, any universal bound on conductivity will more naturally involve a quantity which is independent of the normalization, such as \( \sigma/\chi \).

The universal relation for \( \sigma/\chi \) in (6) looks similar to the universal relation for \( \eta/s \): both ratios become large at weak coupling due to a large mean-free path, and saturate at strong coupling in CFTs with a dual gravity description. Alternatively, one may wonder if there could be a lower bound for conductivity, similar to the conjectured lower bound \[6\] for viscosity,

\[
\frac{\sigma}{\chi} \geq \frac{\hbar v^2}{4\pi T} \frac{d}{d-2},
\]

where we have now restored \( \hbar \) and the speed of “light” \( v \). There is an important difference between the two ratios in Eq. (6): while \( \eta/s = \hbar/4\pi \) only contains \( \hbar \) (suppressing the Boltzmann constant), the corresponding ratio for the conductivity also contains the speed of “light”. The non-relativistic limit corresponds to \( v \to \infty \), and therefore the bound (27) cannot hold in non-relativistic systems. Within the AdS/CFT correspondence, this expectation is confirmed by several explicit computations \[21, 22\] where \( \sigma/\chi \) falls below the bound (27) once conformal symmetry is broken. In practice, the speed of “light” \( v \) is different in different materials, and is not bounded from below.

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3 There are known counter-examples to the decrease of \( c \) and \( k \) along renormalization group trajectories in supersymmetric theories \[19\], which suggests that \( c \) and \( k \) do not always unambiguously measure the number of degrees of freedom in the theory. However, in these examples \( k \) corresponds to the \( R \)-current (which is related to the energy-momentum tensor by supersymmetry), and therefore \( c \) and \( k \) are proportional to each other.

4 Alternatively, the ratio of viscosity to conductivity can be expressed as the ratio of the \( d+1 \) dimensional gauge coupling constant to the \( d+1 \) dimensional Newton’s constant, \( e^2 \eta/\sigma T^2 = (\pi/d^2)(L^2 g_{d+1}^2)/(G_N) \). From the dual gravitational point of view, such an inequality between \( c \) and \( k \) is related to a version of the “weak gravity conjecture” of Ref. \[20\] in AdS space. We thank John McGreevy for pointing out Ref. \[20\] to us.
Once the speed of light is fixed, it is not unreasonable to guess that Eq. (27) does represent a lower bound on conductivity in relativistic CFTs such as the Wilson-Fisher fixed point in the $O(N)$ model.

VI. DISCUSSION

We have argued that in a large class of CFTs in $d > 2$, there are universal relations between the thermodynamic and transport properties, and the central charges which dictate the short distance behaviour of current-current correlators. One way of defining this class of theories is that they possess dual descriptions within AdS at the level of classical gravity and Maxwell electrodynamics. For example, this universality determines the shear viscosity $\eta$ and “electrical” conductivity $\sigma$ in terms of the corresponding central charges and naturally leads to a conjectured bound on conductivity in physical systems, given in Eq. (27), in analogy with the well-studied viscosity bound conjecture.

It is natural to ask about the regime of validity, or alternatively the constraints on the CFTs which may enter such universality classes. Indeed, the analysis we have performed using the AdS/CFT duality required the validity of a classical gravity approximation, and thus some kind of a large-$N$ limit. On the gravity side of the duality, universality follows from the uniqueness of the lowest dimension operator which determines the dynamics of the metric and/or the gauge field dual to the current in question, i.e. the uniqueness of the Einstein-Hilbert and Maxwell actions respectively. From this point of view, once we move to finite $N$, it appears that a large number of higher derivative corrections will also be required, thus limiting the possibility for universal behaviour. Nonetheless, it would be interesting if additional symmetries on the bulk side could constrain the possible classes that might arise. Some hint in this direction is provided by the black hole solutions in string theory beyond the leading classical Einstein term [23].

With these issues in mind, it is clearly useful to have a concrete example with which to contrast the general holographic results. We will consider the 3-dimensional $O(N)$ model at large $N$ with fields $\phi^\alpha$, $\alpha = 1, \ldots, N$ subject to a constraint $\phi^\alpha \phi^\alpha = 1$. In this system, the ratio of $c'/c$ was computed in the large-$N$ limit by Sachdev [24], with the result,

$$\left( \frac{c'}{c} \right)_{O(N)} = \frac{8 \zeta(3)}{15 \pi} \approx 0.2041,$$

which differs by only a few percent from the holographic answer, $c'/c = \pi^3/162 \approx 0.1914$ [5]. A similar comparison is possible for the ratio $k'/k$, where a natural variant of the vector current studied above lies in the adjoint, $J^\mu_{\alpha\beta} = (\phi^\alpha \partial_\mu \phi^\beta - \phi^\beta \partial_\mu \phi^\alpha)$, and we can write

$$\langle J^\alpha_{\mu\beta}(x)J^\gamma_{\nu\delta}(0) \rangle = \frac{k}{x^4} \left( \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \right) \frac{\delta^{\alpha\gamma} \delta^{\beta\delta} - \delta^{\alpha\delta} \delta^{\beta\gamma}}{(4\pi)^2}. \tag{29}$$

A straightforward calculation of the central charge at large $N$ leads to $k = 2$ [25],\(^5\) while the charge

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\(^5\) Alternatively, one can think of the central charge $k$ as the dynamical conductivity in the regime $\omega \gg T$. For high frequencies, we can use the zero-temperature correlation functions (1), and obtain $\sigma(\omega \to \infty) = e^2 k/32$. 

susceptibility in this case was computed by Chubukov et al. [26]. Using these results, we find at large $N$,

$$\left(\frac{k'}{k}\right)_{O(N)} = \frac{\sqrt{5}}{2\pi} \ln \left(\frac{\sqrt{5} + 1}{2}\right) \approx 0.1713,$$

which differs by about 24% from the result $k'/k = \pi/24 \approx 0.1309$ for models with gravitational duals in AdS. On the other hand, the conductivity in the $O(N)$ model is large, $\sigma/\chi$ is $O(N)$ [2], reflecting the fact that the model becomes weakly coupled at large $N$. Therefore, the comparison with the $O(N)$ model in $d = 3$ provides an example of a situation where two systems have very similar static thermodynamic properties, but vastly different transport properties.

As another aspect of the constraints defining these universality classes, it is possible to consider even more restrictive models. Namely, for theories with four supercharges, e.g. $\mathcal{N} = 1$ supersymmetry in $d = 4$, there is a global $U(1)$ $R$-symmetry whose current lies in the same supermultiplet as the energy momentum tensor. If we use the $R$-current to determine $k$, it follows that $c$ and $k$ are not independent, and specifically that the ratio $c/k$ depends only on the dimension $d$. Therefore, for these systems the viscosity and the $R$-current conductivity are related by a simple dimension-dependent constant as given in Eq. (26). As an interesting corollary, in such models the thermodynamic properties, the viscosity, and the conductivity are fully determined by a single number, the central charge $c$.

In this paper we focused on a universality class of models in $d > 2$ spacetime dimensions whose thermodynamic properties, transport coefficients, and central charges were all related to each other. We would like to conclude by pointing out that there is more than one such universality class, and that universal relations such as Eq. (5) are not restricted to models which admit a dual gravitational description. Rather, they can arise as a natural consequence of a large-$N$ limit. As examples, we have in mind pairs of “parent” and “daughter” theories where the daughter is obtained by projection onto a sector invariant under a global discrete symmetry, such as those studied in [27, 28]. As a simple example, consider a parent $U(kN)$ gauge theory with matter fields in the adjoint representation, and project out by a global $\mathbb{Z}_k$ symmetry to form a daughter theory with $U(N)^k$ gauge group, and matter fields in the bi-fundamental representation. It was shown in [28, 29] that the parent and daughter theories are completely equivalent in the $\mathbb{Z}_k$-invariant sector at large $N$, provided that the $\mathbb{Z}_k$ symmetry is not spontaneously broken. In this and similar examples, since the energy-momentum tensor does not carry any global charges it follows that the correlation functions of $T_{\mu \nu}$ in the parent and daughter theories are proportional to each other. For example, viscosity is proportional to the two-point correlation function of $T_{\mu \nu}$, and thus all daughters have the same $\eta/s$ ratio. For CFTs, all daughters would also have the same $c'/c$ ratio. One can say that the universality class consists of all daughter theories, which (even though they may have different global symmetries and contain matter fields in different representations) share the same universal ratios, similar to Eqs. (5) and (6). In this simple example, the existence of such a universality class requires a large-$N$ limit in the field theory, but assumes nothing about strong coupling, supersymmetry, conformal symmetry, or a dual gravity description. The implications of such large-$N$ equivalences for universal transport properties deserve investigation, and we plan to
return to them in the future.

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