Fluctuations of gravitational waves in Eddington inspired Born-Infeld theory

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In this paper we review the EiBI gravity in the presence of a cosmological constant and its tensor perturbations analysis. We show the existence of gravitational waves in the past-time, seeing as a result the smooth transition between high-energy densities (where the EBI dynamics plays its role) and low-energy densities (GR). We obtain the fluctuation spectrum for the graviton in this theory, where for small values of $k$ the fluctuations are strongly suppressed and for large values of $k$ these fluctuations vanish during the De Sitter expansion.

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I. INTRODUCTION

One of the greatest jigsaws in the current physics research is to understand the nature of dark energy and dark matter [1–4]. Currently, dark energy is one of the main classes of models to describe the cosmic late-time acceleration, which has been confirmed by a large number of observations such as measurements SNIa [5], BAO [6], CMBR anisotropies [7], LSS [8] and WL [9]. Future projects and surveys [10] are underway to discover the underlying cause of this phenomena. Recently, the first multimessenger gravitational-wave (GW) observation of a binary neutron star made by LIGO-Virgo detector network set a way to infer cosmological parameters independently of the cosmic distance ladder [11], getting a better value for the Hubble constant – and by extension, a better understanding of dark energy – could be right on the horizon.

In the light of rich observed data, either we just know some properties of each component of the dark sector or one might have a new proposal of the gravitational theory without the need of these dark components instead. Some attempts has been done in order to achieve these issues, e.g in [12] was presented a class of bigravity with solutions that can be interpolate between matter and acceleration epochs. In [13, 14] was presented a non-conventional formulation in terms of the affine connection to independently of the cosmic distance ladder [11], getting a better value for the Hubble constant – and by extension, a better understanding of dark energy – could be right on the horizon.

In latest works, further considerations about the tensor perturbations in EiBI were made [17–20]. Moreover, the aim of this paper is to take a step forward in order to calculate the fluctuations of the EiBI tensor perturbations and compute the graviton mass at two limits: for low-energy densities (General Relativity -GR-) and high-energy densities (Eddington limit).

This paper is organised as follows: In Sec. II we will review the field equations for the EiBI theory. In Sec. III we summarise the limits in this theory. In Sec. IV we calculate the EW equation for the EiBI theory and it will be thoroughly discussed. Also, as a main goal of this paper, we compute the evolution of the graviton mass at both, high and low densities. In Sec. V we explore the fluctuation spectrum in this theory.

II. EIBI FIELD EQUATIONS

From (1) we can calculate the Einstein field equations by varying with respect to the metric $g_{\mu\nu}$ and the variation with respect to the connection fixes the affine connection to be $\Gamma^\rho_{\mu\nu}$:

$$\sqrt{\frac{q}{g}}(q^{-1})^\mu_{\nu} - \lambda g^\mu_{\nu} = -\kappa T^\mu_{\nu},$$

$$q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu},$$

where $\lambda = 1 + \kappa \Lambda$, and $\kappa$ is a constant with the inverse dimensions of $\Lambda$. Notice that these field equations are obtained from independent variation of the metric and $\Gamma$. The auxiliary tensor $q_{\mu\nu}$ is not the space-time metric and $\lambda$ can be related to the cosmological constant term from a GR point of view.

III. LIMITS IN THE EIBI THEORY

Focusing on the dynamics of homogeneous and isotropic metric in (2)-(3), we consider a line element with time and spatial components for each metric:

$$g^{00} = 1, \quad g^{ij} = a^{-2} \delta^{ij},$$

$$q^{00} = X^{-2}, \quad q^{ij} = (aY)^{-2} \delta^{ij},$$

where $X$ and $Y$ are constants.
Eqs. (2)-(3) can be solved analytically using (4) to derive the conventional Friedmann cosmology at late-times. The zero-component evolution equation with \(|q/g| = |XY^3|\) is:

\[
3\kappa \left( H + \frac{\dot{Y}}{Y} \right)^2 = X^2 \left( 1 - \frac{3}{2Y^2} \right) + \frac{1}{2}
\]  

(5)

where

\[
|X| = \frac{(1 + \kappa \rho_T)^2}{(1 + \kappa \rho_T)(1 - \kappa \rho_T)^{1/4}},
\]

(6)

\[
|Y| = [(1 + \kappa \rho_T)(1 - \kappa \rho_T)^{1/4}],
\]

(7)

with \(\rho_T = \rho + \Lambda\) and \(P_T = P - \Lambda\). Let us assume radiation domination as: \(\rho_T = \rho\) and \(P_T = P = \rho/3\), we find that \(X\) and \(Y\) at late times behaves as:

\[
|X| \approx 1 - \frac{5}{6} \kappa \rho + O(\kappa^2),
\]

(8)

\[
|Y| \approx a + \frac{a}{3} \kappa \rho + O(\kappa^2).
\]

(9)

If \(X = Y = 1\) the latter reduces to the low-energy densities limit (GR limit). Now, considering high energy densities (Eddington limit) \(\rho \rightarrow \rho_B\), where the subindex \(B\) indicates the existence of a minimum value for the scale factor \([14]\) then the approximation for the variables \(X\) and \(Y\) are:

\[
|X| = \frac{(1 - \bar{\rho})^2}{(1 + \bar{\rho}) (1 - \bar{\rho})^{1/4}},
\]

(10)

\[
|Y| = [(1 + \bar{\rho})(1 - \bar{\rho})^{1/4}],
\]

(11)

where we introduce \(\bar{\rho} = \kappa \rho\). We see a critical point at \(\bar{\rho} = \rho_B = 3\). Rewriting (5) we obtain:

\[
3H^2 = \frac{1}{\kappa} \left[ \bar{\rho} - 1 + \frac{1}{3\sqrt{3}} \sqrt{(\bar{\rho} + 1)(3 - \bar{\rho})^3} \right] \times \left[ \frac{(1 + \bar{\rho})(3 - \bar{\rho})^2}{(3 + \bar{\rho})^2} \right],
\]

(12)

where for \(\bar{\rho} \ll 1\) we have \(H^2 \approx \rho/3\). Eq.(12) has critical points for \(H(\rho_B) = 0\) in a maximum density \(\rho_B = 0, -1, 3\). Each critical point appear when \(Y^2 = 3X^2/(2X^2 + 1)\). Notice that each critical density has an analytical solution that corresponds to an expansion of the scale factor depending of the sign of \(\kappa\) (see Figure 1):

- When \(\bar{\rho} (\kappa \approx 1)\), \(X = Y = 1\), then we have a minimum scale factor at \(a = 0\) and the universe its stationary and has a minimum size \(a = a_B \approx 10^{-32}(\kappa)^{1/4}a_0\), where \(a_0\) is the scale factor today. This replace the usual Big Bang singularity of Einstein’s model by a cosmic bounce.

- When \(\bar{\rho} = 3(\kappa > 0)\), \(X = Y = 0\) and the solution is exponential-like \((a/a_B) - 1 \approx e^{-t/a}\), which corresponds to a loitering solution.

- When \(\bar{\rho} = 1 (\kappa < 0)\), \(X = (3\cdot4^3)^{1/4}/9\) and \(Y = (4^3)^{1/4}\), with solution \((a - a_B) \approx |t - t_B|^2\), which corresponds to a bouncing solution.

**FIG. 1.** The evolution of \(X\) (blue solid line) and \(Y\) (purple dashed line) from the low-energy to high-energy density limit. We observe two critical points for each limit at \(\rho \rightarrow \rho_B = 0\) and \(\rho \rightarrow \rho_B = 3/\kappa\), respectively. The yellow-dotted curve represents the evolution of \(Y^2 = \frac{3X^2}{2X^2 + 1}\) that gives \(H^2 = 0\).

Given that the solution for the radiation is \(\rho = \rho_0/a^4 = \rho_0/(a_B + \Delta a)^4\), we can expand the density around the small variation of \(a (\Delta a)\),

\[
\rho \propto \frac{\rho}{a_B^4} + 4\rho_0 \frac{\Delta a}{a_B} + O(\Delta a^2),
\]

(13)

\[
\rho \propto \rho_B + 4\rho_B \frac{\Delta a}{a_B} + O(\Delta a^2),
\]

(14)

where \(\rho_B = \rho_0 a_B^{-4}\) is the maximum density. As \(a = a_B + \Delta a\) then \((a/a_B) - 1 = \Delta a/a_B\),

\[
\rho \propto \rho_B + 4\rho_B \left( \frac{a}{a_B} - 1 \right) + O \left[ \frac{a^2}{a_B^2} \left( \frac{a}{a_B} - 1 \right)^2 \right],
\]

\[
a \propto 1 + (t - t_B) + O[(t - t_B)^2].
\]

(15)

At early times (15) shows a universe with a maximum density and constant scale factor.

**IV. EIBI TENSOR PERTURBATIONS**

We can consider a perturbed homogeneous and isotropic spacetime by choosing the two metrics to be of the form:

\[
g^{00} = 1, \quad g^{ij} = a^{-2}(\delta^{ij} - h^{ij}),
\]

\[
q^{00} = X^{-2}, \quad q^{ij} = (aY)^{-2}(\delta^{ij} - \gamma^{ij}),
\]

(16)

where \(h_{ij}\) and \(\gamma_{ij}\) are traceless and transverse, i.e \(\partial_i h^{ij} = \partial_i \gamma^{ij} = 0, h_{ii} = \gamma_{ii} = 0\), respectively. To construct the perturbed field equations we compute the quantities:

\[
(q^{-1})^{ij} = (aY)^{-2}(\delta^{ij} - \gamma^{ij}),
\]

(17)

\[
(g)^{ij} = a^2(\delta^{ij} - h^{ij}),
\]

(18)

\[
\delta T^{ij} = -P a^2 h^{ij},
\]

(19)
where we take $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}$ with $u^\mu = (1,0,0,0)$.

An interesting results derived from the field equations is that $\gamma_{ij} = h_{ij}$, i.e it was found in [16] that $\gamma_{ij}$ is completely locked to the behaviour of $h_{ij}$. After following this consideration we can write the evolution equation for $h_{ij}$ as

$$h_{ij}'' + \left( 4 \frac{(aY)'(aY)}{(aY)^2} - 2 \frac{a'}{a} \right) h_{ij}' + \left[ \frac{X''}{X} + 2 \frac{X'(aY)'}{(aY)} + (aY)'' \right] h_{ij} - \frac{4}{X^2} \frac{X'(aY)'}{(aY)} a' \left( \frac{a'}{a} - 2 \frac{(aY)'}{(aY)} \right) \frac{X}{X^2} - 2k^2 \frac{X'(aY)'}{(aY)} k^2 + \frac{X(aY)^3 + \alpha^3 \lambda}{\kappa \lambda a X^2(aY)^2} h_{ij} = 0. \tag{20}$$

where the prime denotes derivatives w.r.t the conformal time $\eta$. This graviton equation can be rewritten by using $[aY]$ and the component $R_{00}$ to obtain

$$\left[ \kappa \sqrt{-(\kappa \rho + 1)(\kappa \rho - 3)(\kappa \rho + 1)(\kappa^2 \rho^2 + 3)} \right] h_{ij}'' + \left[ \frac{2}{9} \sqrt{3} \kappa (\kappa \rho - 1)(\kappa \rho - 3) \sqrt{-(\kappa \rho - 3)^3} \sqrt{(9\kappa \rho + \sqrt{3}(\kappa \rho + 1)(\kappa \rho - 3)^2 - 9)(\kappa \rho + 1)} \right] h_{ij}' + \left( (\kappa^2 \rho^2 + 3) \sqrt{-(\kappa \rho + 1)(\kappa \rho - 3)^3} \left[ \frac{2}{3} a^2 \sqrt{-(\kappa \rho + 1)(\kappa \rho - 3)^3} + k^2 \kappa (\kappa \rho - 3) \right] \right) \frac{2 \sqrt{3}}{3} (\kappa \rho + 1)(\kappa \rho - 3)^3 h_{ij} = 0. \tag{21}$$

At low-energy densities, if we expand the r.h.s of (21) we obtain

$$\frac{1}{\sqrt{3}} \left( 2 \sqrt{\kappa} h_{ij}' + \sqrt{3} k^2 h_{ij} \right) + \frac{4}{3} \left( 3 \kappa^2 + k^2 \kappa \rho \right) h_{ij} \kappa + O(\kappa^2) \approx 0, \tag{22}$$

where at late times ($\kappa \ll 1$) and using $3H^2 = \rho$ we recover the Helmholtz equation.

At high-energy densities, (21) has a critical point $\rho_B = 3/\kappa$, therefore

$$h_{ij}'' = 0 \quad \Rightarrow \quad h_{ij} \approx h_0 \eta, \tag{23}$$

then $h_{ij}$ grows linearly at early times for

$$\lim_{\rho \to 3} \left[ \kappa \sqrt{-(\kappa \rho + 1)(\kappa \rho - 3)(\kappa \rho + 1)(\kappa^2 \rho^2 + 3)} \right] = 0. \tag{24}$$

Performing the numerical integration of (20)-(21) we see the predicted behaviour (Figure 2): From the evolution of the scale factor we notice the smooth transition between high-energy densities (where the EBI dynamics plays its role) and low-energy densities (GR) (Figure 3). From the solution $h_{ij}$, we notice the linear grow of the GW (23) and as time evolves we have the damped oscillations in the GR limit.

**A. Graviton mass**

Consider the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa (T_{\mu\nu} + T_{\mu\nu}^{\text{mass}}), \tag{25}$$

where usually the extra term $T_{\mu\nu}^{\text{mass}}$ depends on the graviton mass $m_g$ and the background metric

$$T_{\mu\nu}^{\text{mass}} = -m_g \left( (g_0^{-1})_{\mu\sigma} \left[ (g_0 - g_0)_{\sigma\rho} - \frac{1}{2} (g_0)_{\sigma\rho} (g_0)^{\alpha\beta} (g_0 - g_0)_{\alpha\beta} \right] (g_0 - g_0)_{\nu\lambda} \right), \tag{26}$$
where if $m_g \to 0$ we recover the usual Einstein field equations. If we consider the background metric with a small perturbation to obtain for this mass term:

$$T_{\text{mass}}^{\mu \nu} = -m_g \left\{ h_{\mu \nu} - \frac{1}{2} \left[ ((g_0)^{-1})^{\alpha \beta} h_{\alpha \beta} \right] (g_0)_{\mu \nu} \right\}.$$  \hspace{1cm} (27)

where, for $\delta T_{\mu \nu} = 0$, $\delta G_{\mu \nu} = -\kappa \delta T_{\mu \nu}^{\text{mass}}$, we can rewrite the perturbed equation as:

$$\delta h_{ij}'' + 2H \delta h_{ij}' + (k^2 + m_g^2) \delta h_{ij} = 0.$$ \hspace{1cm} (28)

This equation is similar to the equation for a free massive scalar field in a flat FRW background. Now, from the perturbation of $(3) \delta R_{ij} = \delta q_{ij} - \delta g_{ij}$ we obtain:

$$\delta R_{ij} + \frac{a^2}{\kappa} \left( \frac{XY^3}{\lambda} + 1 \right) \delta h_{ij} = 0.$$ \hspace{1cm} (29)

If we compare the latter with $(28)$ we obtain

$$m_g^2 = \frac{1}{\kappa} \left[ \frac{(1 + \kappa \rho_T)(1 - \kappa P_T)^3}{1 + \kappa \lambda} + 1 \right],$$ \hspace{1cm} (30)

which is the graviton mass that takes the following values:

- $\frac{XY^3}{\lambda} < -1$ gives a large tachyonic mass $m_g^2 < 0$ implying the unstable evolution of tensor perturbations,
- $\frac{XY^3}{\lambda} > -1$, the growth of the tensor perturbations is suppressed.

Eq. $(28)$ reduces to

$$h_{ij}'' + 2H h_{ij}' + k^2 h_{ij} = 0, \quad \text{for } p^2 \gg m_g^2,$$ \hspace{1cm} (31)

and

$$h_{ij}'' + 2H h_{ij}' + m_g^2 = 0, \quad p^2 \ll m_g^2, \quad \lambda \left( \frac{nk^2}{a^2} - 1 \right) \ll XY^3,$$ \hspace{1cm} (32)

where $p^2 \equiv a^{-2} k^2$ is the physical momentum.

At high-energy densities ($\kappa \rho \to 3$) and in radiation regime (Figure 4)

$$m_{g_{\text{rad}}}^2 \propto \frac{1}{\kappa} \left[ (1 + \bar{\rho}) \left( 1 - \frac{1}{3} \bar{\rho} \right) + 1 \right] = \frac{1}{\kappa}.$$ \hspace{1cm} (33)

When $m_g > 1$ there is a growth of the tensor perturbations, after the graviton crosses the critical point $\bar{\rho} = 3/\kappa$ at low-energy density $(0 < m_g < 1)$ the growth is suppressed.

V. FLUCTUATIONS IN GW EIBI

We rewrite the graviton equation (20) as:

$$h_{ij}'' + F h_{ij}' + (G + J k^2) h_{ij} = 0,$$ \hspace{1cm} (34)

where

$$F = 2 \left[ \frac{(aY)'}{(aY)} - \frac{a'}{a} \right],$$ \hspace{1cm} (35)

$$G = \frac{X''}{X} + 2 \left( \frac{X'}{X} \right) \left[ (aY)' \right] (aY) - \left( \frac{(aY)'}{(aY)} \right)^2 \right] + 2 \left( \frac{a'}{a} \right)^2,$$ \hspace{1cm} (36)

$$J = 1 - 2 \left( \frac{X'}{X} \right) \left[ (aY)' \right] (aY)' \right],$$ \hspace{1cm} (37)

$$\xi = \frac{X(aY)'^3 + a^3 \lambda}{\kappa \lambda a X^2 (aY)^2}.$$ \hspace{1cm} (38)
We define the fluctuation spectrum as the standard deviation and the variance is

\[ \omega_k := \sqrt{k^2 - \frac{z''}{z}}, \quad (41) \]

and at high-energy densities:

\[ \omega_k := \sqrt{G + Jk^2 - \frac{z''}{z}}, \quad (42) \]

If the expansion is rapid enough, \( \omega_k \) becomes imaginary.

Will use the following definition for the quantum fluctuations where \( g_B \) is an observable with mean value

\[ g_B = \langle \Omega | g_B | \Omega \rangle = \langle 0 | Aa + Ba^\dagger | 0 \rangle = 0, \quad (43) \]

and the variance is

\[ [\delta g_B]^2 \equiv \langle \Omega | g_B^2 | \Omega \rangle \approx a^{-2}k^3 |u_k|^2. \quad (44) \]

We define the fluctuation spectrum as the standard deviation as a function of \( k \)

\[ \delta g_k := a^{-1}k^{3/2}|u_k|. \quad (45) \]

### A. Solutions of Eq.(39)

- **Case 1.** Minkowski spacetime \( z = 1 \). Solving (39) and using (41)-(45), the solutions for the mode function and the fluctuation spectrum are

\[ u_k,\eta = (1/\sqrt{\pi})e^{ik}, \]

\[ \delta g_k = k^{3/2}/\sqrt{k} = k^{3/2}. \quad (46) \]

We observed that at small \( k \) (large scale), the fluctuations are strongly suppressed. Analogous to the numerical solutions, when the scale factor is constant we observed a fast growing of the graviton mode \( h_{ij} \).

- **Case 2.** De Sitter spacetime. We consider \( z = e^{\alpha t} = -(\alpha \eta)^{-1} \), with \( \eta(t) = \int z(t') dt' \). The frequency is

\[ \omega_{k,\eta} = k^2 - \frac{2}{\eta^2}. \quad (47) \]

The modes \( k \) oscillate if \( |\eta| \gg \sqrt{2}/k \) and the \( \omega \) is an imaginary quantity when \( |\eta| \ll \sqrt{2}/k \). The solutions are

\[ u_{k,\eta} = -\sqrt{\frac{2}{\pi k^3}} [(C_1 k\eta + C_2) \cos (k\eta) - (C_1 + C_2 k\eta) \sin (k\eta)], \quad (48) \]

and

\[ \delta g_{k,\eta} = -\alpha \eta \sqrt{\frac{2}{\pi}} [(C_1 k\eta + C_2) \cos (k\eta) - (C_1 + C_2 k\eta) \sin (k\eta)], \quad (49) \]

with \( C_1 \) and \( C_2 \) constants of integration. Notice that as \( t \to -\infty \) we have \( \eta \to -\infty \), but as \( t \to \infty \) we have \( \eta \to 0^+ \). At large \( k \) we have the usual fluctuation spectrum for Minkowski spacetime. As \( t \to \infty \), the fluctuations vanishes during the De Sitter expansion.

### B. Solutions for Eq.(40)

For (40) is not so simple to consider the same assumptions as the latter case since there is a dependence of \( X \) and \( aY \) in \( G \) and \( J \). Therefore, let us consider an expansion over \( 1/\eta \).

We can rewrite the expressions for \( X \) and \( Y \) as:

\[ |Y| = |(1 + \kappa \rho r)(1 - \kappa \rho r)^{1/4}|, \quad (50) \]

\[ |X| = \frac{1}{|Y|}, \quad (51) \]

where with \( \rho = r_0 - \frac{\lambda}{\kappa} \). The expressions for the total density and pressure are

\[ P_T = \frac{1}{\kappa}(1 - \lambda \pi_0 - \pi_0 \rho) = -\frac{1}{\kappa}\pi_0 r_0, \quad (52) \]

\[ \rho_T = r_0 - \frac{1}{\kappa}, \quad (53) \]

and now

\[ |X| = \frac{1}{\kappa r_0 (1 + \pi_0 r_0)^{1/4}}, \quad (54) \]

\[ |Y| = \kappa r_0 (1 + \pi_0 r_0)^{1/4}. \quad (55) \]

In the asymptotic past these expansions are reduced to \( p_1 = \pi_0 \) and \( p_2 = r_0 \). We rewrite (40) for the ER as:

\[ u'' + \left( -\frac{z''}{z} + \xi_0 + k^2 \right) u = 0, \quad (56) \]

where

\[ \xi_0 = \frac{(1 - \pi_0 r_0)(1 + \pi_0 r_0)^{1/2} - \lambda \kappa r_0}{\kappa r_0 [\kappa \lambda (1 - \pi_0 r_0)(1 + \pi_0 r_0)]^{1/4}} \quad (57) \]

For the Minkowski case, we obtain similar solutions as in (46) but with an extra constant term \( \xi_0 \) in the exponential. For the DeSitter case also we obtain oscillating solution, but the harmonic functions will be weighted by a \((\sqrt{\xi_0} + k)^{-1}\) term. When \( \pi_0, r_0 \ll 1 \) there is an increase on the amplitude of the fluctuation spectrum. In Figure 5 we show the power spectrum of the graviton equation \( (P_g \propto |\delta g_k|^2) \).

\[ ^1 \text{This last condition is only correct if the integral constant of } \eta \text{ vanishes.} \]
The EiBI theory has been a successful proposal for modify gravity theories, in which it is replaced the usual Big Bang singularity of Einstein’s model by a cosmic bounce. Also, it was observed that this proposal suffers from a tensor instability. In regards to this, here we have discussed the evolution of the EiBI-GW equation. Furthermore, we obtain the value for the graviton mass in EiBI gravity at high and low energy densities, where for $k \ll 1$ the fluctuations are strongly suppressed and for $k \gg 1$ these ones vanish during the De Sitter expansion.

Work still needs to be done before compare with current observations. Although within this paper, we review the importance of use the EiBI theory in future test of GW.

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