CATEGORICAL DONALDSON-THOMAS THEORY FOR LOCAL SURFACES: $\mathbb{Z}/2$-PERIODIC VERSION

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Abstract. We prove two kinds of $\mathbb{Z}/2$-periodic Koszul duality equivalences for triangulated categories of matrix factorizations associated with $(-1)$-shifted cotangents over quasi-smooth affine derived schemes. We use this result to define $\mathbb{Z}/2$-periodic version of Donaldson-Thomas categories for local surfaces, whose $\mathbb{C}^*$-equivariant version was introduced and developed in the author’s previous paper. We compare $\mathbb{Z}/2$-periodic DT category with the $\mathbb{C}^*$-equivariant one, and deduce wall-crossing equivalences of $\mathbb{Z}/2$-periodic DT categories from those of $\mathbb{C}^*$-equivariant DT categories.

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1. Introduction

1.1. Background. This paper is a complement of the author’s previous paper [Todb], where we introduced and developed $\mathbb{C}^*$-equivariant categorical Donaldson-Thomas theory for local surfaces, i.e. Calabi-Yau 3-folds obtained as total spaces of canonical line bundles on surfaces. It was based on Koszul duality equivalence, formulated as follows. Let $Y$ be a smooth affine scheme over $\mathbb{C}$ and $V \to Y$ a vector bundle with a section $s$. Then the derived zero locus of $s$ is a quasi-smooth affine derived scheme $U := (s = 0) \hookrightarrow Y$.

On the other hand, the above data determines a function on the total space of $V^\vee \to Y$

$$w: V^\vee \to \mathbb{C}, \ w(x, v) = \langle s(x), v \rangle$$

for $x \in Y$ and $v \in V^\vee|_x$. The above function is of $\mathbb{C}^*$-weight two with respect to the fiberwise weight two $\mathbb{C}^*$-action on $V^\vee$. The Koszul duality equivalence proved in [S13, Shi12, Hir17, Todb] is an equivalence of triangulated categories

$$(1.1) \quad \Psi: \text{MF}_{\text{coh}}^\mathbb{C}^*(V^\vee, w) \xrightarrow{\sim} D^b_{\text{coh}}(\mathfrak{U}).$$

Here the left hand side is the triangulated category of $\mathbb{C}^*$-equivariant matrix factorizations of $w$ (cf. [Orl09]), and the right hand side is the derived category of dg-modules over $\mathfrak{U}$ with bounded coherent cohomologies. A key observation is that, under the equivalence (1.1), the supports of matrix factorizations correspond to singular supports of coherent sheaves introduced in [AG15].

The $\mathbb{C}^*$-equivariant DT category is then defined in [Todb] as a certain singular support quotient of the derived category of coherent sheaves on a quasi-smooth derived stack, which through the equivalence (1.1) is regarded as a gluing of categories of $\mathbb{C}^*$-equivariant matrix factorizations.
On the other hand, triangulated categories of matrix factorizations without \(\mathbb{C}^*\)-actions are in general \(\mathbb{Z}/2\)-periodic. As there is no \(\mathbb{C}^*\)-action on a general CY 3-fold, it is more suitable to define \(\mathbb{Z}/2\)-periodic version of DT categories rather than \(\mathbb{C}^*\)-equivariant ones. An obstruction toward the \(\mathbb{Z}/2\)-periodic version is a failure of the equivalence (1.1), i.e. the \(\mathbb{Z}/2\)-periodic triangulated categories of matrix factorizations \(\text{MF}^{Z/2}_{\text{coh}}(V^\vee, w)\) is not necessary equivalent to the \(\mathbb{Z}/2\)-periodic derived categories of coherent sheaves on \(\mathcal{U}\).

In this paper, we observe that the above issue is caused by a subtlety on a correct definition of \(\mathbb{Z}/2\)-periodic derived category for \(\mathcal{U}\). We show that there is a \(\mathbb{Z}/2\)-periodic version of the equivalence (1.1) by using either Positselski’s \(\mathbb{Z}/2\)-periodic derived categories of the second kind [Pos11], or singular support quotients of usual \(\mathbb{Z}\)-graded derived categories of coherent sheaves. We will use the latter \(\mathbb{Z}/2\)-periodic equivalence to define \(\mathbb{Z}/2\)-periodic DT categories. We will show that it is recovered from the \(\mathbb{C}^*\)-equivariant one up to idempotent completion, and use this result to deduce some wall-crossing equivalence of \(\mathbb{Z}/2\)-periodic DT categories from the results of \(\mathbb{C}^*\)-equivariant DT categories.

### 1.2. \(\mathbb{Z}/2\)-periodic Koszul duality

In this paper, we show the \(\mathbb{Z}/2\)-periodic version of the equivalence (1.1). It is stated as follows:

**Theorem 1.1.** (Theorem 3.17) There exist equivalences

\[
\text{MF}^{Z/2}_{\text{coh}}(V^\vee, w) \xrightarrow{\sim} \text{Ind} D^b_{\text{coh}}(\mathcal{U}_c) / \text{Ind} \mathcal{C}_{\text{Crit}(w) \times \{0\}} \xrightarrow{\sim} D^{\text{co}}(\mathcal{O}_U \mathcal{m}od^{Z/2}_2),
\]

which restricts to the commutative diagram

\[
\begin{array}{ccc}
\text{MF}^{Z/2}_{\text{coh}}(V^\vee, w) & \xrightarrow{\sim} & \text{MF}^{Z/2}_{\text{coh}}(V^\vee, w) \\
\psi_* & \sim & \psi_{Z/2} \\
D^b_{\text{coh}}(\mathcal{U}_c) / \mathcal{C}_{\text{Crit}(w) \times \{0\}} & \xleftarrow{\sim} & D^{\text{abs}}(\mathcal{O}_U \mathcal{m}od^{Z/2}_2).
\end{array}
\]

Here (\(\sim\)) indicates the idempotent completion and the horizontal arrows are fully-faithful with dense images.

Here we explain the notation of the above theorem. The derived scheme \(\mathcal{U}_c\) is defined by \(\mathcal{U}_c := \mathcal{U} \times \text{Spec} \mathbb{C}[\epsilon] \text{ with deg}(\epsilon) = -1\). Its \((-1)\)-shifted cotangent has underlying classical scheme given by \(\text{Crit}(w) \times \mathbb{A}^1\). The subcategory

\[\mathcal{C}_{\text{Crit}(w) \times \{0\}} \subset D^b_{\text{coh}}(\mathcal{U}_c)\]

consists of objects whose singular supports are contained in \(\text{Crit}(w) \times \{0\} \subset \text{Crit}(w) \times \mathbb{A}^1\). The triangulated categories \(D^{\text{co}}(\mathcal{O}_U \mathcal{m}od^{Z/2}_2)\), \(D^{\text{abs}}(\mathcal{O}_U \mathcal{m}od^{Z/2}_2)\) are derived categories of \(\mathbb{Z}/2\)-periodic dg \(\mathcal{O}_U\)-modules of the second kind [Pos11], which we will review in Subsection 2.1. These are not necessary equivalent to the usual \(\mathbb{Z}/2\)-periodic derived categories, e.g. an object with trivial cohomologies may be non-zero in these categories. The former category \(D^{\text{co}}(\mathcal{O}_U \mathcal{m}od^{Z/2}_2)\) is regraded as a \(\mathbb{Z}/2\)-periodic analogue of ind-coherent sheaves [Gai13] (see Remark 2.3), though it is not an ind-completion of the usual \(\mathbb{Z}/2\)-periodic derived category of coherent sheaves.

### 1.3. \(\mathbb{Z}/2\)-periodic categorical DT theory

We use the left vertical arrow in (1.2) to define \(\mathbb{Z}/2\)-periodic DT categories. Let \(\mathfrak{M}\) be a quasi-smooth and QCA derived stack (see Subsection 4.1) with classical truncation \(\mathcal{M} = t_0(\mathfrak{M})\), and \(\Omega_{\mathfrak{M}}[-1]\) its \((-1)\)-shifted cotangent derived stack. For an open substack

\[\mathcal{N}^\text{ss} \subset \mathcal{N} := t_0(\Omega_{\mathfrak{M}}[-1])\]

with complement \(\mathcal{Z}\), we set

\[\mathcal{Z} := \mathbb{C}^* (\mathcal{Z} \times \{1\}) \cup (\mathcal{N} \times \{0\}) \subset \mathcal{N} \times \mathbb{A}^1.\]
Here $\mathbb{C}^*$ acts on the fibers of $N \to M$ and $\mathbb{A}^1$ by weight two. Then $Z_\epsilon$ is a conical closed substack in the $(-1)$-shifted cotangent over $\mathfrak{M}_\epsilon := \mathfrak{M} \times \text{Spec} \mathbb{C}[\epsilon]$ for $\deg(\epsilon) = -1$. We define the $\mathbb{Z}/2$-periodic DT category $\mathcal{D}^{\mathbb{Z}/2}(N^{ss})$ by

$$\mathcal{D}^{\mathbb{Z}/2}(N^{ss}) := D^b_{\text{coh}}(\mathfrak{M}_\epsilon)/\mathcal{C}_{Z_\epsilon}.$$  

By the left vertical equivalence in (1.2), the above category may be regarded as a gluing of $\mathbb{Z}/2$-periodic categories of matrix factorizations. Its dg-enhancement $\mathcal{D}^{\mathbb{Z}/2}(N^{ss})_{\text{dg}}$ is similarly defined by taking the Drinfeld quotient instead of Verdier quotient. We show that, up to idempotent completion, the $\mathbb{Z}/2$-periodic DT category is recovered from the $\mathbb{C}^*$-equivariant DT category $\mathcal{D}^{\mathbb{C}^*}(N^{ss})$ defined in $[Todb]$.

**Theorem 1.2.** (Theorem 4.9) Suppose that $Z \subset N$ is $\mathbb{C}^*$-invariant and $\text{Ind} \mathcal{C}_Z \subset \text{Ind} D^b_{\text{coh}}(\mathfrak{M})$ is compactly generated. Then there is an equivalence

$$\mathcal{D}^{\mathbb{Z}/2}(N^{ss})_{\text{dg}} \simeq \mathcal{R} \text{Hom}(\mathbb{C}[u^{\pm 1}], \text{Ind} \mathcal{D}^{\mathbb{C}^*}(N^{ss})_{\text{dg}})^{op}.$$  

Here $\deg(u) = 2$, $\mathbb{C}[u^{\pm 1}]$ is regarded as a dg-category with one object, and $\mathcal{R} \text{Hom}(-, -)$ is an inner Hom of dg-categories $[Toe07]$.

### 1.4 Wall-crossing equivalences of $\mathbb{Z}/2$-periodic DT categories for local surfaces.

In $[Todb]$, we proved several wall-crossing equivalences or fully-faithful functors of $\mathbb{C}^*$-equivariant DT categories for local surfaces. We can apply these results to show wall-crossing equivalences or fully-faithful functors of $\mathbb{Z}/2$-periodic DT categories using Theorem 1.2. We give one of such examples: wall-crossing equivalence of DT categories for one dimensional stable sheaves.

Let $S$ be a smooth projective surface and

$$\pi: X = \text{Tot}_S(\omega_S) \to S$$

the associated local surface. For a stability condition $\sigma$ and a numerical class $v$, we denote by $M^\sigma_X(v)$ the moduli space of $\sigma$-semistable compactly supported coherent sheaves on $X$. Under the assumption that there is no strictly $\sigma$-semistable sheaves, we define the $\mathbb{Z}/2$-periodic DT category

$$\mathcal{D}^{\mathbb{Z}/2}(M^\sigma_X(v))$$

for $M^\sigma_X(v)$, following the construction of the basic model [13] (see Definition 5.1).

Let $v$ be a primitive numerical class of a one dimensional sheaf, and suppose that $\sigma$ lies in a chamber of the space of stability conditions. We have the following wall-crossing diagram

$$M^\sigma_X(v) \quad M^\sigma_X(v)$$

$$\quad M^\sigma_X(v)$$

which is a d-critical flop in the sense of [Toda], i.e. it is a d-critical analogue of flop in birational geometry. As an analogy of D/K equivalence conjecture $[BO]$, $[Kaw02]$, we conjecture that the DT categories are equivalent under the above wall-crossing diagram. In $[Todb]$, we proved the wall-crossing equivalence for $\mathbb{C}^*$-equivariant DT categories under some assumption of preservation of stability under push-forward to the surface. By directly applying Theorem 1.2 we have the following:

**Theorem 1.3.** Suppose that any $\sigma$-semistable sheaf on $X$ with numerical class $v$ push-forwards to a $\sigma$-semistable sheaf on $S$. Then we have an equivalence

$$\mathcal{D}^{\mathbb{Z}/2}(M^\sigma_X(v)) \sim \mathcal{D}^{\mathbb{Z}/2}(M^\sigma_X(v)).$$

The examples where the assumption of Theorem 1.3 is satisfied are discussed in $[Todb]$ Section 5.4.4), e.g. it is satisfied when the curve class of $v$ is reduced. In a similar way, we can also prove $\mathbb{Z}/2$-periodic version of wall-crossing equivalences or fully-faithful functors of DT categories for MNOP/PT moduli spaces, proved in $[Todb]$ for $\mathbb{C}^*$-equivariant case (see Remark 5.5).
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1.6. Notation and convention. In this paper, all the schemes or (derived) stacks are locally of finite presentation over \( \mathcal{C} \). For a scheme or derived stack \( Y \) and a quasi-coherent sheaf \( \mathcal{F} \) on it, we denote by \( S_{\mathcal{O}_Y}(\mathcal{F}) \) its symmetric product \( \oplus_{i \geq 0} \text{Sym}^i_{\mathcal{O}_Y}(\mathcal{F}) \). We omit the subscript \( \mathcal{O}_Y \) if it is clear from the context. For a derived stack \( \mathcal{M} \), we always denote by \( t_0(\mathcal{M}) \) the underived stack given by the truncation. For an algebraic group \( G \) which acts on \( Y \), we denote by \([Y/G]\) the associated quotient stack.

In this paper, all the dg-categories or triangulated categories are defined over \( \mathcal{C} \). The category of dg-categories is denoted by \( \text{dgCat} \). Its objects consist of dg-categories over \( \mathcal{C} \) with morphisms given by dg-functors. By Tabuada [Tab05], there is a cofibrantly generated model category structure on \( \text{dgCat} \). Its localization by weak equivalences is denoted by \( \text{Ho}(\text{dgCat}) \). An equivalence between \( \text{dgCat} \), whose localization by weak equivalences is denoted by \( \text{Ho}(\text{dgCat}) \). For a triangulated category \( D \) and its triangulated subcategory \( D' \subset D \), we denote by \( D/D' \) its Verdier quotient. In the case that \( D \) is a dg-category and \( D' \subset D \) is a dg-subcategory, its Drinfeld dg-quotient [DG13] is also denoted by \( D/D' \). We denote by \( D^{\text{op}} \subset D \) the subcategory of compact objects. A subcategory \( D' \subset D \) is called dense if any object in \( D \) is a direct summand of an object in \( D' \).

For a dg-category \( D \), we denote by \( \text{Ind}(D) \) its dg-categorical ind-completion of \( D \) (denoted as \( \widehat{D} \) in [Toë07] Section 7], also see [Lur09] Section 5.3.5 in the context of \( \infty \)-category). For dg-categories \( D_1, D_2 \), we denote by \( D_1 \otimes D_2 \) the dg-category whose set of objects is \( \text{Obj}(D_1) \times \text{Obj}(D_2) \) and \( \text{Hom}^*_{D_1 \otimes D_2}((E_1, E_2), (F_1, F_2)) = \text{Hom}^*_{D_1}(E_1, F_1) \otimes \text{Hom}^*_{D_2}(E_2, F_2) \).

By [Toë07] Corollary 6.4, for two dg-categories \( D_1, D_2 \), there exists an inner Hom of dg-category \( R\text{Hom}(D_1, D_2) \in \text{Ho}(\text{dgCat}) \) satisfying that (see [Toë07] Corollary 7.6) \( R\text{Hom}(D_1, \text{Ind}(D_2)) \simeq \text{Ind}(D_1^{\text{op}} \otimes D_2) \).

When we discuss limits, ind-completions, inner Hom for triangulated categories, we (implicitly or explicitly) take these functors on dg-enhancements or \( \infty \)-categorical enhancements (which will be obviously given in the context) and then take their homotopy categories.

2. Some background

In this section, we review several background used in this paper: derived categories of first and second kind in [Pos11], dg or triangulated categories of factorizations [Ori12] [EP15] [PV11], \( \mathbb{C}^{\ast} \)-equivariant Koszul duality [Lsi13] [Shi12] [Hir17] [Hod18] and singular supports of (ind) coherent sheaves [AG15].

2.1. Review of derived categories. Let \( R \) be a commutative differential graded algebra over \( \mathbb{C} \) with non-positive degrees. Let \( \Gamma \) be either \( \mathbb{Z} \) or \( \mathbb{Z}/2 \), and we regard \( R \) as a \( \Gamma \)-graded dg-algebra. We recall some basic terminology of derived categories of \( \Gamma \)-graded dg-modules over \( R \), following [Pos11].

We denote by \( \text{R-mod}^\Gamma \) the dg-category of \( \Gamma \)-graded dg-modules over \( R \), and \( \text{Ho}(\text{R-mod}^\Gamma) \) its homotopy category. We have the full dg or triangulated subcategories

\[ \text{Acy}_{\text{dg}} \subset \text{R-mod}^\Gamma, \text{Acy} \subset \text{Ho}(\text{R-mod}^\Gamma) \]

consisting of acyclic objects, i.e. \( \mathcal{H}^i(M) = 0 \) for all \( i \in \Gamma \). The dg or triangulated derived categories of dg \( R \)-modules are defined by the quotient categories

\[ D(\text{R-mod}^\Gamma)_{\text{dg}} := \text{R-mod}^\Gamma/\text{Acy}_{\text{dg}}, D(\text{R-mod}^\Gamma) := \text{Ho}(\text{R-mod}^\Gamma)/\text{Acy}. \]

Here we take the Drinfeld quotient [Drî04] for the quotient of dg-categories, and take the Verdier quotient for the quotient of triangulated categories. The former dg-category is a dg-enhancement
of the latter triangulated category, i.e. the homotopy category of \(D(R\text{-mod}^\Gamma)_{\text{dg}}\) is equivalent to \(D(R\text{-mod}^\Gamma)\). We also have the full dg or triangulated subcategories
\[
D_{\text{fg}}(R\text{-mod}^\Gamma)_{\text{dg}} \subset D(R\text{-mod}^\Gamma)_{\text{dg}}, \quad D_{\text{fg}}(R\text{-mod}^\Gamma) \subset D(R\text{-mod}^\Gamma)
\]
consisting of objects \(M\) such that \(\oplus_{i \in \Gamma} \mathcal{H}^i(M)\) is finitely generated over the commutative algebra \(\mathcal{H}^0(R)\). The above dg or triangulated categories for \(\Gamma = \mathbb{Z}\) are equivalent to (quasi) coherent sheaves on the affine derived scheme \(\text{Spec} \ R\), so they are ‘usual’ derived categories.

The above composition is fully-faithful and the second functor is also fully-faithful since the second functor in (2.4) factors through the functors
\[
D_{\text{fg}}(R\text{-mod}^\Gamma) \rightarrow \text{Ho}(R\text{-mod}^\Gamma) \rightarrow D^\text{co}(R\text{-mod}^\Gamma)
\]
which is fully-faithful by [Pos11, Section 11, Theorem 1]. Also since any coacyclic object is acyclic, there is a natural functor
\[
D^\text{co}(R\text{-mod}^\Gamma) \rightarrow D(R\text{-mod}^\Gamma).
\]
In the \(\mathbb{Z}\)-graded case, we have the following lemma which essentially follows from [Pos11, Section 3.4, Theorem 1].

**Lemma 2.1.** Suppose that \(\Gamma = \mathbb{Z}\), \(R^i = 0\) for \(i \ll 0\) and each \(\mathcal{H}^i(R)\) is finitely generated over \(\mathcal{H}^0(R)\). Then the functor (2.2) restricts to the equivalence
\[
D_{\text{fg}}^\text{abs}(R\text{-mod}^\Gamma) \cong D_{\text{fg}}(R\text{-mod}^\Gamma).
\]

**Proof.** Let \(\text{Ho}^+(R\text{-mod}^\mathbb{Z})\) be the homotopy category of \(\mathbb{Z}\)-graded dg \(R\)-modules \(M\) such that \(M^i = 0\) for \(i \ll 0\). We set
\[
\text{Acy}^+ := \text{Acy} \cap \text{Ho}^+(R\text{-mod}^\mathbb{Z}), \quad \text{Acy}^{\text{co}+,+} := \text{Acy}^{\text{co}+} \cap \text{Ho}^+(R\text{-mod}^\mathbb{Z}).
\]
Then we have natural fully-faithful functors by [Pos11, Section 3.4, Theorem 1 (b), (c)]
\[
\text{Ho}^+(R\text{-mod}^\mathbb{Z})/\text{Acy}^+ \rightarrow D(R\text{-mod}^\mathbb{Z}), \quad \text{Ho}^+(R\text{-mod}^\mathbb{Z})/\text{Acy}^{\text{co}+,+} \rightarrow D^\text{co}(R\text{-mod}^\mathbb{Z}).
\]
By the assumption on \(R\), the functor (2.1) factors through the functors
\[
D_{\text{fg}}^\text{abs}(R\text{-mod}) \rightarrow \text{Ho}^+(R\text{-mod}^\mathbb{Z})/\text{Acy}^{\text{co}+,+} \rightarrow D^\text{co}(R\text{-mod}^\mathbb{Z}).
\]
The above composition is fully-faithful and the second functor is also fully-faithful since the second functor in (2.3) is fully-faithful. Therefore the first functor in (2.4) is also fully-faithful. On the
other hand, by \cite[Section 3.4, Theorem 1 (a)]{Pos11} we have $\text{Acy}^+ = \text{Acy}^{\text{co}+}$. Therefore combined with the first functor in (2.3), we obtain the fully-faithful functors
\[ D^\text{abs}(R\text{-mod}_{\text{fg}}^\text{Z}) \hookrightarrow \text{Ho}^+(R\text{-mod}^\text{Z})/\text{Acy}^{\text{co}+,+} = \text{Ho}^+(R\text{-mod}^\text{Z})/\text{Acy}^+ \to D(R\text{-mod}^\text{Z}). \]

Any object in the image of the above composition has bounded finitely generated cohomologies by the assumption on $R$, so we obtain the fully-faithful functor
\[ (2.5) \quad D^\text{abs}(R\text{-mod}_{\text{fg}}^\text{Z}) \to D\text{tg}(R\text{-mod}^\text{Z}). \]

It remains to show that the above functor is essentially surjective. Note that any object in the target is given by a finite successive extensions of its cohomologies. Since each cohomology is finitely generated over $\mathcal{H}^0(R)$, it comes from the objects in the source of (2.5). Since $D\text{tg}$ is fully-faithful, we conclude that it is also essentially surjective. \qed

**Remark 2.2.** As we will see in Example 2.7, the result of Lemma 2.1 does not hold in the $\mathbb{Z}/2$-graded case.

**Remark 2.3.** By \cite[Section 3.11, Theorem 1]{Pos11}, the coderived category $D^c(R\text{-mod}^\text{F})$ is compactly generated by $D^\text{abs}(R\text{-mod}^\text{F})$. In the situation of Lemma 2.1, it follows that $D^c(R\text{-mod}^\text{Z})$ is equivalent to the category of ind-coherent sheaves on Spec $R$ studied in \cite{Gai13} (see \cite[Section H.3]{AG15}), and the functor (2.2) is a natural functor from ind-coherent sheaves to quasi-coherent sheaves.

### 2.2. Review of factorization categories.

Here we review the theory of factorizations associated with super-potentials. The basic references are \cite{Orl12, EP15, PV11}.

Let $\mathcal{X}$ be a noetherian smooth algebraic stack over $\mathbb{C}$, $\mathcal{L} \to \mathcal{X}$ a line bundle and $w \in \Gamma(\mathcal{X}, \mathcal{L} \otimes 2)$ a global section. A (quasi) coherent factorization of $w$ consists of
\[ (\mathcal{P}, d_P), \quad d_P: \mathcal{P} \to \mathcal{P} \otimes \mathcal{L} \]
where $\mathcal{P}$ is a (quasi) coherent sheaf on $\mathcal{X}$ and $d_P$ is a morphism of (quasi) coherent sheaves satisfying $d_P^2 = w$. The category of factorizations of $w$ naturally forms a dg-category denoted by $\text{MF}_*(\mathcal{X}, w)_{\text{dg}}$ for $\star \in \{\text{qcoh}, \text{coh}\}$, whose homotopy category $\text{HMF}_*(\mathcal{X}, w)$ is a triangulated category. Let
\[ \text{Acy}^\text{abs}_* \subset \text{HMF}_*(\mathcal{X}, w) \]
be the minimal thick triangulated subcategory which contains totalizations of short exact sequences of (quasi) coherent factorizations of $w$. An object in $\text{Acy}^\text{abs}_*$ is called absolutely acyclic. The triangulated category of factorizations of $w$ is defined by the Verdier quotient
\[ (2.6) \quad \text{MF}_*(\mathcal{X}, w) := \text{HMF}_*(\mathcal{X}, w)/\text{Acy}^\text{abs}_*, \quad \star \in \{\text{qcoh}, \text{coh}\}. \]

It admits a natural dg-enhancement by taking the Drinfeld quotient
\[ (2.7) \quad \text{MF}_*(\mathcal{X}, w)_{\text{dg}} := \text{MF}_*(\mathcal{X}, w)_{\text{dg}}/\text{Acy}^\text{abs}_*\text{dg}, \quad \star \in \{\text{qcoh}, \text{coh}\} \]
where $\text{Acy}^\text{abs}_*\text{dg}$ is the full dg-subcategory of $\text{MF}_*(\mathcal{X}, w)_{\text{dg}}$ consisting of absolutely acyclic objects in the homotopy category.

Let $j: \mathcal{U} \subset \mathcal{X}$ be an open substack with complement $\mathcal{Z} \subset \mathcal{X}$. We define
\[ (2.8) \quad \text{MF}_*(\mathcal{X}, w)|_{\mathcal{U}} := \text{Ker}(\text{MF}_*(\mathcal{X}, w) \xrightarrow{j^*} \text{MF}_*(\mathcal{U}, w|_{\mathcal{U}})). \]
Then we have the equivalence (cf. \cite[Theorem 1.10]{EP15})
\[ (2.9) \quad \text{MF}_*(\mathcal{X}, w)/\text{MF}_*(\mathcal{X}, w)_{\mathcal{Z}} \xrightarrow{\sim} \text{MF}_*(\mathcal{U}, w|_{\mathcal{U}}). \]

Let $\text{Crit}(w) \subset \mathcal{X}$ be the critical locus. We also have the equivalence (cf. \cite[Corollary 5.3]{PV11})
\[ (2.10) \quad \text{MF}_*(\mathcal{X}, w)_{\text{Crit}(w)} \xrightarrow{\sim} \text{MF}_*(\mathcal{X}, w). \]
We will use the following special construction of factorizations. Let $V \rightarrow X$ be a vector bundle with sections $s \in \Gamma(X, V \otimes L)$, $t \in \Gamma(X, V^\vee \otimes L)$ satisfying $w = \langle s, t \rangle$. Then we have the following factorization of $w$ (called Koszul factorization)

$$ (2.10) \quad \mathcal{K}_{s,t} := \left( \bigwedge^i V^\vee, d_k \right), \quad d_k : \bigwedge^i V^\vee \xrightarrow{\sim} \left( \bigwedge^{i-1} V^\vee \oplus \bigwedge^{i+1} V^\vee \right) \otimes L. $$

If $s$ is a regular section so that $Z := \{ s = 0 \} \subset X$ has codimension rank $V$, then we have an isomorphism in $\text{MF}_{\text{coh}}(X, w)$ (see Subsection 1.6), which is equivalent to $\text{Ind}_{\text{coh}}(U)$.

Its dg or triangulated derived categories of coherent sheaves are defined by

$$ (2.11) \quad \mathcal{K}_{s,t} \cong (O_Z, d_{O_Z} = 0). $$

We will use the following two special versions of factorization categories. Let $\mathcal{Y}$ be a smooth stack and set $X = \mathcal{Y} \times B\mu_2$, and $L$ to be the line bundle on $X$ induced by the weight one $\mu_2$-character. Then $\mathcal{L}^{\otimes 2} \cong O_X$, so any regular function $w : \mathcal{Y} \rightarrow \mathbb{C}$ determines a global section $w \in \Gamma(X, \mathcal{L}^{\otimes 2})$. We set

$$ \text{MF}^2(\mathcal{Y}, w) := \text{MF}_{\ast}(\mathcal{Y} \times B\mu_2, w), \, \ast \in \{ \text{qcoh}, \text{coh} \}. $$

The above triangulated category is $\mathbb{Z}/2$-periodic, i.e. $[2] \cong \text{id}$. When $\mathcal{Y}$ is an affine scheme, the above triangulated category is equivalent to Orlov’s triangulated category of matrix factorizations $\text{Ind}_{\text{coh}}(U)$.

Let $G^\ast$ acts on a smooth stack $\mathcal{Y}$ and set $X = [\mathcal{Y}/G^\ast]$, and $L$ to be the line bundle on $X$ induced by the weight one $G^\ast$-character. For a global section $w \in \Gamma(X, \mathcal{L}^{\otimes 2})$, we set

$$ \text{MF}^G(\mathcal{Y}, w) := \text{MF}_{\ast}([\mathcal{Y}/G^\ast], w), \, \ast \in \{ \text{qcoh}, \text{coh} \}. $$

For example, we will use the above construction for $\mathcal{Y} = [Y/G]$ for a noetherian scheme $Y$ with an action of an algebraic group $G$ and an action of $G^\ast$ which commutes with the $G$-action, and $w : Y \rightarrow \mathbb{C}$ is a $G$-invariant function with $G^\ast$-weight two.

For a closed substack $Z \subset \mathcal{Y}$, we define the subcategories

$$ \text{MF}^2(\mathcal{Y}, w)_Z \subset \text{MF}^G(\mathcal{Y}, w), \quad \text{MF}^G(\mathcal{Y}, w)_Z \subset \text{MF}^G(\mathcal{Y}, w) $$

in the similar way as (2.7). Here we assume that $Z$ is $G^\ast$-invariant in the latter case. The dg-categories $\text{MF}^G(\mathcal{Y}, w)_{dg}$, $\text{MF}^G(\mathcal{Y}, w)_Z$ are also defined in the similar way as (2.6).

2.3. C*-equivariant Koszul duality. Let $Y$ be a smooth affine $\mathbb{C}$-scheme and $V \rightarrow Y$ a vector bundle on it. Given a section $s : Y \rightarrow V$ of $V$, its derived zero locus $\mathcal{U}$ is given by

$$ (2.12) \quad \mathcal{U} = \text{Spec} \mathcal{R}(V \rightarrow Y, s) $$

where $\mathcal{R}(V \rightarrow Y, s)$ is the Koszul complex

$$ \mathcal{R}(V \rightarrow Y, s) := \left( \cdots \rightarrow \bigwedge^2 V^\vee \xrightarrow{s} V^\vee \xrightarrow{\delta} O_Y \rightarrow 0 \right). $$

Its dg or triangulated derived categories of coherent sheaves are defined by

$$ D^b_{\text{coh}}(\mathcal{U})_{dg} := D^b_{\text{fr}}(O_{\mathcal{U}}\text{-mod}^2)_{dg}, \quad D^b_{\text{coh}}(\mathcal{U}) := D^b_{\text{fr}}(O_{\mathcal{U}}\text{-mod}^2). $$

The ind-completion $\text{Ind} D^b_{\text{coh}}(\mathcal{U})$ of $D^b_{\text{coh}}(\mathcal{U})$ is defined to be the homotopy category of $\text{Ind} D^b_{\text{coh}}(\mathcal{U})_{dg}$ (see Subsection 1.6), which is equivalent to $D^b_{\text{coh}}(O_{\mathcal{U}}\text{-mod}^2)$ (see Remark 2.3).

Let $V^\vee \rightarrow Y$ be the total space of the dual vector bundle of $V$. There is an associated function on $V^\vee$, given by

$$ (2.13) \quad w : V^\vee \rightarrow \mathbb{C}, \quad w(x, v) = \langle s(x), v \rangle, \quad x \in Y, \quad v \in V^\vee|_x. $$

It is well-known that the critical locus of the above function is the classical truncation of the $(-1)$-shifted cotangent over $\mathcal{U}$ (see [JTT17]),

$$ t_0(\Omega_{\mathcal{U}}[-1]) = \text{Crit}(w) \subset V^\vee. $$
Let $\mathbb{C}^*$ acts on the fibers of $V^\vee \to Y$ by weight two, so that $w$ is of weight two. We have the following Koszul duality equivalence which relates derived category coherent sheaves on $\Omega$ with the triangulated category of $\mathbb{C}^*$-equivariant factorizations of $w$ (see [Tod14 Theorem 2.3.3, Lemma 2.3.10]):

**Theorem 2.4.** (cf. [Is13, Shi12, Hir17, Tod14]) There is an equivalence of triangulated categories

$$\Psi : \mathcal{MF}_{\text{coh}}^{C^*}(V^\vee, w) \simto D^b_{\text{coh}}(\Omega),$$

which extends to the equivalence

$$\Psi : \mathcal{MF}_{\text{coh}}^{C^*}(V^\vee, w) \simto \text{Ind} D^b_{\text{coh}}(\Omega).$$

The equivalence (2.14) is constructed in the following way. Let $\mathcal{K}_s$ be the following $\mathbb{C}^*$-equivariant factorization of $w$

$$\mathcal{K}_s := (\mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_\Delta, d_{K_s}).$$

Here the $\mathbb{C}^*$-action is given by the grading

$$\mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_\Delta = \mathcal{S}_{\mathcal{O}_Y}(V[-2]) \otimes_{\mathcal{O}_Y} \mathcal{S}_{\mathcal{O}_Y}(V^\vee[1]),$$

and the weight one map $d_{K_s}$ is given by

$$d_{K_s} = 1 \otimes d_{\mathcal{O}_\Delta} + \eta : \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_\Delta \to \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_\Delta(1),$$

where $\eta \in V \otimes \mathcal{O}_Y, V^\vee \subset \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_\Delta$ corresponds to $\text{id} \in \text{Hom}(V, V)$, and (1) indicates the shift of $\mathbb{C}^*$-weight by one. Then one can check that there is a quasi-isomorphism of dg-algebras

$$\mathcal{O}_\Delta \simto \text{Hom}_{\text{MF}_{\text{coh}}^{C^*}(V^\vee, w)_{sa}}(\mathcal{K}_s, \mathcal{K}_s),$$

and the equivalence $\Psi$ is given by (see the proof of [Tod14 Theorem 2.3.3])

$$\Psi(-) = \text{RHom}(\mathcal{K}_s, -), \quad \mathcal{MF}_{\text{coh}}^{C^*}(V^\vee, w) \to D^b_{\text{coh}}(\Omega).$$

Its quasi-inverse is given by

$$\Phi(-) = \mathcal{K}_s \otimes_{\mathcal{O}_\Delta} (-), \quad D^b_{\text{coh}}(\Omega) \to \mathcal{MF}_{\text{coh}}^{C^*}(V^\vee, w).$$

2.4. **Singular supports of (ind) coherent sheaves.** The theory of singular supports of coherent sheaves on $\Omega$ is developed in [AG15] following the earlier work [BIK08]. Here we recall its definition. Let $\text{HH}^*(\Omega)$ be the Hochschild cohomology

$$\text{HH}^*(\Omega) := \text{Hom}_{\mathcal{O}_\Delta \times \mathcal{O}_\Delta}(\Delta_\ast \mathcal{O}_\Delta, \Delta_\ast \mathcal{O}_\Delta).$$

Here $\Delta : \Omega \to \Omega \times \Omega$ is the diagonal. Then it is shown in [AG15 Section 4] that there exists a canonical map $\mathcal{H}^1(\mathcal{T}_\Omega) \to \text{HH}^2(\Omega)$, so the map of graded rings

$$\mathcal{O}_{\text{Crit}(w)} = S(\mathcal{H}^1(\mathcal{T}_\Omega)) \to \text{HH}^2(\Omega) \to \text{Nat}_{D^b_{\text{coh}}(\Omega)}(\text{id, id}[2\ast]).$$

Here $\text{Nat}_{D^b_{\text{coh}}(\Omega)}(\text{id, id}[2\ast])$ is the group of natural transformations from $\text{id}$ to $\text{id}[2\ast]$ on $D^b_{\text{coh}}(\Omega)$, and the right arrow is defined by taking Fourier-Mukai transforms associated with morphisms $\Delta_\ast \mathcal{O}_\Delta \to \Delta_\ast \mathcal{O}_\Delta[2\ast]$. The above maps induce the map for each $\mathcal{F} \in D^b_{\text{coh}}(\Omega)$,

$$\mathcal{O}_{\text{Crit}(w)} \to \text{Hom}^{2\ast}(\mathcal{F}, \mathcal{F}).$$

The above map defines the $\mathbb{C}^*$-equivariant $\mathcal{O}_{\text{Crit}(w)}$-module structure on $\text{Hom}^{2\ast}(\mathcal{F}, \mathcal{F})$, which is finitely generated by [AG15 Theorem 4.1.8]. Below a closed subset $Z \subset \text{Crit}(w)$ is called conical if it is closed under the fiberwise $\mathbb{C}^*$-action on $\text{Crit}(w)$. For $\mathcal{F} \in D^b_{\text{coh}}(\Omega)$, its singular support is the conical closed subset

$$\text{Supp}^{2\ast}(\mathcal{F}) \subset \text{Crit}(w)$$

defined to be the support of $\text{Hom}^{2\ast}(\mathcal{F}, \mathcal{F})$ as $\mathcal{O}_{\text{Crit}(w)}$-module.

For a conical closed subset $Z \subset \text{Crit}(w)$, let

$$C_Z \subset D^b_{\text{coh}}(\Omega), \quad \text{Ind} C_Z \subset \text{Ind} D^b_{\text{coh}}(\Omega)$$
be the triangulated subcategory consisting of objects whose singular supports are contained in \( Z \), and its ind-completion respectively. Their dg-enhancements

\[
C_{Z, dg} \subset D^b_{coh}(\mathcal{U})_{dg}, \quad \text{Ind} C_{Z, dg} \subset \text{Ind} D^b_{coh}(\mathcal{U})_{dg}
\]

are also defined to be consisting of objects with singular supports contained in \( Z \).

**Proposition 2.5.** ([Coh Prop 2.3.9]) The equivalence \( \Psi \) in Theorem 2.4 restricts to the equivalences

\[
\Psi: MF^C_{coh}(V^\vee, w)_Z \sim C_Z, \quad \Psi: MF^C_{qcoh}(V^\vee, w)_Z \sim \text{Ind} C_Z.
\]

In particular by (2.18), the equivalences in Theorem 2.4 descend to the equivalences

\[
\Psi: MF^C_{coh}(V^\vee \setminus Z, w) \sim D^b_{coh}(\mathcal{U})/C_Z, \quad \Psi: MF^C_{qcoh}(V^\vee \setminus Z, w) \sim \text{Ind} D^b_{coh}(\mathcal{U})/\text{Ind} C_Z.
\]

**3. \( \mathbb{Z}/2 \)-periodic Koszul duality**

In this section, we observe that a \( \mathbb{Z}/2 \)-periodic analogue of Theorem 2.4 does not hold for the usual \( \mathbb{Z}/2 \)-periodic derived categories, and need to use co (absolute) derived categories. We also use singular supports quotients to give another version of \( \mathbb{Z}/2 \)-periodic analogue of Theorem 2.4.

**3.1. A failure of \( \mathbb{Z}/2 \)-periodic Koszul duality.** We will give a \( \mathbb{Z}/2 \)-periodic version of the Koszul duality equivalence in Theorem 2.4. Namely we would like to replace the \( \mathbb{C}^* \)-equivariant factorization category \( MF^C_{coh}(V^\vee, w) \) with the \( \mathbb{Z}/2 \)-periodic factorization category \( MF^{\mathbb{Z}/2}_{coh}(V^\vee, w) \). A naive guess is to replace \( D^b_{coh}(\mathcal{U}) \) with the \( \mathbb{Z}/2 \)-periodic derived category

\[
D^{\mathbb{Z}/2}_{coh}(\mathcal{U}) := D_{dg}(\mathcal{O}_{\mathcal{U}}\text{-mod}^{\mathbb{Z}/2}).
\]

Let \( K^{\mathbb{Z}/2}_s \) be the factorization \( K_s \) of \( w \) regarded as a \( \mathbb{Z}/2 \)-graded factorization, i.e.

\[
K^{\mathbb{Z}/2}_s = K^{\text{even}}_s \oplus K^{\text{odd}}_s, \quad K^{\text{even}}_s = O_{V^\vee} \otimes O_V \wedge^{\text{even}} V^\vee, \quad K^{\text{odd}}_s = O_{V^\vee} \otimes O_V \wedge^{\text{odd}} V^\vee,
\]

with differential given by (2.10). As an analogy of (2.17), we can define the functor

\[
\Psi^{\mathbb{Z}/2}: MF^{\mathbb{Z}/2}_{coh}(V^\vee, w) \rightarrow D^{\mathbb{Z}/2}_{coh}(\mathcal{U}), \quad (\cdot) \mapsto \text{RHom}(K^{\mathbb{Z}/2}_s, \cdot).
\]

We see that the above functor does not necessarily give an equivalence.

**Example 3.1.** Let \( Y = \text{Spec } \mathbb{C}, V = \mathbb{A}^1, s = 0 \) so that \( \mathcal{U} = \mathcal{O}_s := \text{Spec } \mathbb{C}[e] \) with \( \text{deg}(e) = -1 \).

In this case, the Koszul factorization is \( K_s = \mathbb{C}[e, t] \) where \( \text{deg}(t) = 2 \) with differential given by the multiplication \( ct \). The functor (3.1) is

\[
\Psi^{\mathbb{Z}/2}: MF^{\mathbb{Z}/2}_{coh}(\mathbb{A}^1, 0) \rightarrow D^{\mathbb{Z}/2}_{coh}(\mathcal{O}_s), \quad (\cdot) \mapsto \text{RHom}(K^{\mathbb{Z}/2}_s, \cdot).
\]

Then for \( y \in \mathbb{A}^1 \) we have

\[
\Psi^{\mathbb{Z}/2}(\mathcal{O}_y) = (\mathbb{C} \oplus \mathbb{C} e^y, d = ye) \cong \begin{cases} (\mathbb{C}[e])[1], & y = 0, \\ 0, & y \neq 0. \end{cases}
\]

In particular \( \Psi^{\mathbb{Z}/2} \) is not fully-faithful.

Geometrically this issue occurs since the factorization \( K^{\mathbb{Z}/2}_s \) is supported at the zero section of \( V^\vee \rightarrow Y \) so that the functor (3.2) annihilates objects whose supports are away from the zero section. We note that the above issue does not happen for \( \mathbb{C}^* \)-equivariant case (as indicated by Theorem 2.4), since the support of any \( \mathbb{C}^* \)-invariant coherent sheaf of \( V^\vee \) intersects with the zero section (so the objects \( \mathcal{O}_y \) for \( y \neq 0 \) are not allowed in the \( \mathbb{C}^* \)-equivariant case).

**Example 3.2.** In Example 3.1, the functor (3.2) also sends \( \mathcal{O}_{\mathbb{A}^1} \) to \( \mathcal{O}_0 := \mathbb{C}[e]/(e) \). We have the isomorphisms of \( \mathbb{Z}/2 \)-graded vector spaces

\[
\text{Hom}_{MF^{\mathbb{Z}/2}_{coh}(\mathbb{A}^1, 0)}^{\mathbb{Z}/2}(\mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1}) = \mathbb{C}[t], \quad \text{Hom}_{D^{\mathbb{Z}/2}_{coh}(\mathcal{O}_0)}^{\mathbb{Z}/2}(\mathcal{O}_0, \mathcal{O}_0) = \mathbb{C}[t]
\]
where \( t \) is of even degree. The right hand side is computed by taking the projective resolution 
\[
\cdots \to C[e] \to C[e] \to O_0 \to 0.
\]

As the above resolution exhibits \( O_0 \) as a colimit of \(( \cdots \to C[e] \to C[e] )\), the Hom space in the right hand side of \((3.4)\) is computed by the limit of \( C \to C \to \cdots \) in the category of \( \mathbb{Z}/2 \)-graded vector spaces, that is \( C[t] \). The above computation also implies that \( \Psi^{Z/2} \) is not fully-faithful.

In the \( \mathbb{C}^* \)-equivariant case, we compute \( \text{Hom}^*_{D^b_{coh}(\bullet, \mathcal{O})}(O_0, O_0) \) as a limit of \( C \to C \to \cdots \) in the category \( \mathbb{Z} \)-graded vector spaces, which is \( C[t] \) where \( t \) is of degree two. Therefore we have
\[
\text{Hom}^*_{\mathbb{M}^*_{coh}(\bullet, 0)}(O_{\mathbb{A}^1}, O_{\mathbb{A}^1}) = \text{Hom}^*_{D^b_{coh}(\bullet)}(O_0, O_0) = C[t].
\]

### 3.2. \( \mathbb{Z}/2 \)-periodic Koszul duality via coderived categories

A failure of the functor \((3.1)\) to be an equivalence is caused by the incorrect definition of the target category. Indeed in the \( \mathbb{Z}/2 \)-periodic case, an analogue of Lemma \(2.1\) does not hold, so we need to distinguish two kinds of derived categories. Technically, a \( \mathbb{Z}/2 \)-periodic analogue of the functor \((2.1)\) is not well-defined from \( D_{coh}(\mathcal{U}) \), but well-defined from coderived category or absolute derived category. As we see below, the latter derived categories provide a correct formulation of \( \mathbb{Z}/2 \)-periodic Koszul duality.

**Proposition 3.3.** In the setting of Theorem \(2.4\) there is an equivalence
\[
(3.5) \quad \Psi^{Z/2} : \text{MF}_{qcoh}^{Z/2}(V^\vee, w) \to D^{co}(O_\mathcal{U}\text{-mod}^{Z/2}).
\]

**Proof.** Let \( \text{MF}_{qcoh}^{Z/2}(V^\vee, w)_{dg} \) be the dg-category of \( \mathbb{Z}/2 \)-periodic quasi-coherent factorizations (see the notation in Subsection \(2.2\)). We have the dg functor
\[
\text{Hom}^*_{\mathbb{K}^{Z/2}_s(\bullet, 0)} : \text{MF}_{qcoh}^{Z/2}(V^\vee, w)_{dg} \to O_\mathcal{U}\text{-mod}^{Z/2}.
\]

Here the \( O_\mathcal{U}\)-module structure on \( \text{Hom}^*_{\mathbb{K}^{Z/2}_s(\bullet, 0)} \) is induced by the \( O_\mathcal{U}\)-module structure on \( \mathbb{K}^{Z/2}_s = O_{V^\vee} \otimes_{O_{V^\vee}} O_\mathcal{U} \), i.e. the multiplication by the right factor. Since \( \mathbb{K}^{Z/2}_s \) is projective as \( O_{V^\vee}\)-module, the above functor preserves absolutely acyclic objects in the homotopy categories. Therefore it induces the functor
\[
\Psi^{Z/2} : \text{MF}_{qcoh}^{Z/2}(V^\vee, w) \to D^{co}(O_\mathcal{U}\text{-mod}^{Z/2}).
\]

Similarly we have the dg functor
\[
\mathbb{K}^{Z/2}_s \otimes_{O_\mathcal{U}} (-) : O_\mathcal{U}\text{-mod}^{Z/2} \to \text{MF}_{qcoh}^{Z/2}(V^\vee, w)_{dg}.
\]

Since \( \mathbb{K}^{Z/2}_s \) is projective as a \( \mathbb{Z}/2 \)-graded \( O_\mathcal{U}\)-module, and taking tensor product preserves direct sums, the above functor preserves coacyclic objects. As \( V^\vee \) is smooth, coacyclic objects in the homotopy category of \( \text{MF}_{qcoh}^{Z/2}(V^\vee, w)_{dg} \) coincide with absolutely acyclic objects by \([\text{Pos1}]\) Section 3.6, Theorem. Therefore the above functor induces the functor
\[
\Phi^{Z/2} : D^{co}(O_\mathcal{U}\text{-mod}^{Z/2}) \to \text{MF}_{qcoh}^{Z/2}(V^\vee, w).
\]

It is enough to show that \( \Psi^{Z/2} \circ \Phi^{Z/2} \cong \text{id} \) and \( \Phi^{Z/2} \circ \Psi^{Z/2} \cong \text{id} \).

We first show the isomorphism \( \Phi^{Z/2} \circ \Psi^{Z/2} \cong \text{id} \). By the definition of \( \Phi^{Z/2} \) and \( \Psi^{Z/2} \), we have
\[
\Phi^{Z/2} \circ \Psi^{Z/2}(-) = \mathbb{K}^{Z/2}_s \otimes_{O_\mathcal{U}} \mathbb{K}^{Z/2}_s \otimes_{O_{V^\vee}} (-).
\]

Here \( \mathbb{K}^{Z/2}_{s^2V} = O_{\mathcal{U}}^\vee \otimes_{O_{V^\vee}} O_{V^\vee} \) is the \( O_{V^\vee}\)-dual of \( \mathbb{K}^{Z/2}_s \) which is a factorization of \(-w\), where \( O_{\mathcal{U}}^\vee \) is the \( O_{\mathcal{U}} \)-dual of \( O_\mathcal{U} \);
\[
O_{\mathcal{U}}^\vee = S_{O_{V^\vee}}(V|1]) = \det(V)[-\text{rank}(V)] \otimes_{O_{V^\vee}} O_\mathcal{U}.
\]

We have
\[
(3.6) \quad \mathbb{K}^{Z/2}_s \otimes_{O_\mathcal{U}} \mathbb{K}^{Z/2}_{s^2V} = (O_{V^\vee} \otimes_{O_{V^\vee}} O_{V^\vee} \otimes_{O_{V^\vee}} O_{V^\vee}, 1 \otimes d_{O_{Y_2}} \otimes 1 + \eta_1 \otimes 1_{O_{V^\vee}} - 1_{O_{V^\vee}} \otimes \eta_2).
\]
Here \( \eta_1 \in V \otimes \mathcal{O}_V, V^\vee \) and \( \eta_2 \in V^\vee \otimes \mathcal{O}_V, V \) correspond to \( \tilde{t} \in \text{Hom}(V, V) \), which act on \( \mathcal{O}_V^\vee \otimes \mathcal{O}_V \mathcal{O}_U^\vee \) and \( \mathcal{O}_U^\vee \otimes \mathcal{O}_V, \mathcal{O}_V^\vee \) respectively. The above object \((3.6)\) is a factorization of the function

\[
-p_1^*w + p_2^*w : V^\vee \times Y V^\vee \to \mathbb{C},
\]

giving a Fourier-Mukai kernel of \( \Phi^{Z/2} \circ \Psi^{Z/2} \). Here \( p_1 : V^\vee \times Y V^\vee \to V^\vee \) is the projection onto the corresponding factor. The function \(-p_1^*w + p_2^*w\) is explicitly written as

\[
(3.7) \quad (-p_1^*w + p_2^*w)(x, v_1, v_2) = \langle s(x), v_2 - v_1 \rangle
\]

for \((x, v_1, v_2) \in V^\vee \times Y V^\vee \) with \( x \in Y, v_i \in V(y) \). Let \( \alpha, \beta \) be maps

\[
\alpha : V^\vee \times Y V^\vee \to V^\vee \times Y V^\vee \times Y V^\vee, \quad (x, v_1, v_2) \mapsto (x, v_1, v_2 - v_1),
\]

\[
\beta : V^\vee \times Y V^\vee \to V^\vee \times Y V^\vee \times Y V, \quad (x, v_1, v_2) \mapsto (x, v_1, v_2, s(x)).
\]

They are sections of the vector bundles

\[
V^\vee \times Y V^\vee \times Y V^\vee \to V^\vee \times Y V^\vee, \quad V^\vee \times Y V^\vee \times Y V \to V^\vee \times Y V^\vee
\]

over \( V^\vee \times Y V^\vee \), given by projections onto the left two factors respectively. By \((2.11)\), we have

\[
-p_1^*w + p_2^*w = \langle \alpha, \beta \rangle.
\]

By unraveling the right hand side of \((3.6)\), we see that \((3.6)\) is the Koszul factorization associated with \((\alpha, \beta)\) (see \((2.10)\)). Since \( \alpha \) is a regular section whose zero locus is the diagonal \( \Delta \subset V^\vee \times Y V^\vee \), by \((2.11)\) we have an isomorphism in \( \text{MF}_{\text{coh}}(V^\vee \times Y V^\vee, -p_1^*w + p_2^*w)\)

\[
\mathcal{K}_s^{Z/2} \otimes \mathcal{O}_U \mathcal{K}_s^{Z/2} \cong (\mathcal{O}_\Delta, d_{\mathcal{O}_\Delta} = 0).
\]

The above isomorphism implies that \( \Phi^{Z/2} \circ \Phi^{Z/2} \cong \text{id} \).

We next show the isomorphism \( \Psi^{Z/2} \circ \Phi^{Z/2} \cong \text{id} \). We have

\[
\Psi^{Z/2} \circ \Phi^{Z/2}(-) = \mathcal{K}_s^{Z/2} \otimes \mathcal{O}_V \mathcal{K}_s^{Z/2} \otimes \mathcal{O}_U (-).
\]

Here \( \mathcal{K}_s^{Z/2} \otimes \mathcal{O}_V \mathcal{K}_s^{Z/2} \) is

\[
(3.8) \quad \mathcal{K}_s^{Z/2} \otimes \mathcal{O}_V \mathcal{K}_s^{Z/2} = (\mathcal{O}_U^\vee \otimes \mathcal{O}_V \mathcal{O}_V^\vee \otimes \mathcal{O}_V \mathcal{O}_U^\vee, d_{\mathcal{O}_U} + d_{\mathcal{O}_U} - \eta_2 \otimes 1_{\mathcal{O}_U} + 1_{\mathcal{O}_U} \otimes \eta_1).
\]

The above object is a \( \mathbb{Z}/2 \)-graded \( \mathcal{O}_U \times Y U \)-module, which naturally lifts to a \( \mathbb{Z} \)-graded one \( \mathcal{K}_s^V \otimes \mathcal{O}_V \mathcal{K}_s^V \). On the other hand, the automorphism of the vector bundle \( V \times Y V \) on \( Y \) by \((x, y) \mapsto (x + y, x - y) / 2 \) induces the equivalence of derived schemes

\[
(3.9) \quad U \times Y U \cong U^\vee \times Y U.
\]

Here \( U^\vee = \text{Spec} \mathcal{O}_Y (V^\vee [1]) \) with zero differential on \( \mathcal{O}_Y (V^\vee [1]) \). Under the above equivalence, the object \( \Delta, \mathcal{O}_U \) corresponds to \( \mathcal{O}_Y \boxtimes \mathcal{O}_U \). We have the Koszul resolution of \( \mathcal{O}_Y \) as \( \mathbb{Z} \)-graded \( \mathcal{O}_Y \mathcal{O}_U \)-modules

\[
\cdots \to 2 (V^\vee [1]) \otimes \mathcal{O}_Y \mathcal{O}_U \to V^\vee [1] \otimes \mathcal{O}_V \mathcal{O}_U \to \mathcal{O}_Y \mathcal{O}_U \to \mathcal{O}_Y \to 0.
\]

By taking the \( \mathcal{O}_Y \)-dual of the above resolution, we obtain the following resolution

\[
0 \to \mathcal{O}_Y \to \mathcal{O}_U^\vee \to \mathcal{O}_U^\vee \otimes \mathcal{O}_V \mathcal{O}_U \to \mathcal{O}_U^\vee \otimes \mathcal{O}_V \mathcal{O}_U \to \mathcal{O}_V \mathcal{O}_U \to \mathcal{O}_Y \to 0.
\]

The totalization of the above sequence is a \( \mathbb{Z} \)-graded \( \mathcal{O}_U \)-module over \( \mathcal{O}_U \) which is acyclic and bounded below, hence it is coacyclic by \( \text{Pos}^{\mathcal{I}1} \) Section 3.4, Theorem 1 (a)]. By applying \( \boxtimes \mathcal{O}_U \) and pull-back by the equivalence \((3.3)\), we obtain the complex of \( \mathbb{Z} \)-graded \( \mathcal{O}_U \times Y U \)-modules

\[
0 \to \Delta, \mathcal{O}_U \to \mathcal{O}_U^\vee \otimes \mathcal{O}_V \mathcal{O}_U \to \mathcal{O}_U^\vee \otimes \mathcal{O}_V \mathcal{O}_U \to \mathcal{O}_U^\vee \otimes \mathcal{O}_V \mathcal{O}_U \to \mathcal{O}_V \mathcal{O}_U \to \cdots
\]

whose totalization is coacyclic. By unraveling the differentials and comparing with \((3.6)\), we see that the above complex is \( \Delta, \mathcal{O}_U \to \mathcal{K}_s^V \otimes \mathcal{O}_V \mathcal{K}_s^V \). It follows that we have the isomorphism

\[
\Delta, \mathcal{O}_U \cong \mathcal{K}_s^V \otimes \mathcal{O}_V \mathcal{K}_s^V.
\]
in $D^{\text{co}}(\mathcal{O}_{\mathfrak{U} \times \mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2})$. Therefore the object (3.8) is also isomorphic to $\Delta_* \mathcal{O}_{\mathfrak{U}}$ in $D^{\text{co}}(\mathcal{O}_{\mathfrak{U} \times \mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2})$. Since $\mathcal{O}_{\mathfrak{U}}$ is Gorenstein, by EP15 Section 1.7, Proposition an object in $D^{\text{co}}(\mathcal{O}_{\mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2})$ is represented by a $\mathbb{Z}/2$-graded dg $\mathcal{O}_{\mathfrak{U}}$-module $M$ which is flat over $\mathcal{O}_{\mathfrak{U}}$. For such an object $M$ and a coacyclic object $\mathcal{P} \in \text{Ho}(\mathcal{O}_{\mathfrak{U} \times \mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2})$, the object $M \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{P}$ is also coacyclic. Therefore the isomorphism $\Psi^{\mathbb{Z}/2} \circ \Phi^{\mathbb{Z}/2}(M) \cong M$ holds.

**Corollary 3.4.** There is an equivalence

$$\Psi^{\mathbb{Z}/2} : MF_{\text{coh}}^{\mathbb{Z}/2}(V^\vee, w) \to D^{\text{abs}}(\mathcal{O}_{\mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2}).$$

Here $(-)$ indicates idempotent completion.

**Proof.** The equivalence (3.5) restricts to the equivalence between the subcategories of compact objects. Since $MF_{\text{coh}}^{\mathbb{Z}/2}(V^\vee, w)$ is compactly generated by $MF_{\text{coh}}^{\mathbb{Z}/2}(V^\vee, w)$ (see BFK14 Proposition 3.15), and $D^{\text{co}}(\mathcal{O}_{\mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2})$ is compactly generated by compact objects $D^{\text{abs}}(\mathcal{O}_{\mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2})$ (see Pos11 Section 3.11, Theorem 1), we have equivalences

$$MF_{\text{coh}}^{\mathbb{Z}/2}(V^\vee, w) \cong MF_{\text{coh}}^{\mathbb{Z}/2}(V^\vee, w)^{\text{cp}}, \ D^{\text{abs}}(\mathcal{O}_{\mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2}) \cong D^{\text{co}}(\mathcal{O}_{\mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}/2})^{\text{cp}}.$$ 

Therefore the corollary follows. □

**Example 3.5.** In the setting of Example 3.4, let us consider the functor

$$\Psi^{\mathbb{Z}/2} : MF_{\text{coh}}^{\mathbb{Z}/2}(A^1, 0) \to D^{\text{co}}(\mathbb{C}[e]\text{-}\text{mod}^{\mathbb{Z}/2}).$$

For $y \in k^1$, we have

$$\Psi^{\mathbb{Z}/2}(\mathcal{O}_y) = (\mathbb{C} \oplus \mathbb{C}e^y, d = ye).$$

A difference from Example 3.4 is that we no longer have the second isomorphism in (3.3) so that $\Psi^{\mathbb{Z}/2}(\mathcal{O}_y) \neq 0$ in the coderived category.

**Remark 3.6.** The proof of Proposition 3.3 is also applied to the $\mathbb{Z}$-graded case, which shows the equivalence

$$\Psi : MF_{\text{coh}}^{\mathbb{Z}}(V^\vee, w) \to D^{\text{co}}(\mathcal{O}_{\mathfrak{U}}\text{-}\text{mod}^{\mathbb{Z}}).$$

As the right hand side is equivalent to $\text{Ind} D_{\text{coh}}(\mathfrak{U})$ (see Remark 2.3), we recover the equivalence (2.15) in Theorem 2.4.

### 3.3. $\mathbb{Z}/2$-periodic Koszul duality via singular supports

As we mentioned in Remark 2.3, the coderived category in the $\mathbb{Z}$-graded case is equivalent to the category of ind-coherent sheaves, whose general theory is well-established Gal13, ACL15 in derived algebraic geometry. On the other hand, the $\mathbb{Z}/2$-periodic coderived category is not yet well developed in the context of derived algebraic geometry, e.g., its singular support theory and higher categorical treatment are missing. In this subsection, we give another kind of $\mathbb{Z}/2$-periodic Koszul duality described in the framework of usual $\mathbb{Z}$-graded derived category of coherent sheaves and ind-coherent sheaves. The Koszul duality in this subsection will be used to give a global model for $\mathbb{Z}/2$-periodic DT category. We first prepare some lemmas.

**Lemma 3.7.** Let $\mathbb{C}^*$ acts on a $\mathbb{C}$-scheme $A$ which restricts to the trivial action on $\mu_2 \subset \mathbb{C}^*$. The inclusion $A \to A \times \mathbb{C}^*$, $x \mapsto (x, 1)$ induces the isomorphism of stacks

$$\nu : [A/\mu_2] \cong [(A \times \mathbb{C}^*)/\mathbb{C}^*].$$

Here $\mu_2$ acts on $A$ trivially, and $\mathbb{C}^*$ acts on $A \times \mathbb{C}^*$ by $t(x, u) = (t(x), t^2u)$. 

Proof. By the assumption that $\mathbb{C}^*$-action on $A$ restricts to the trivial $\mu_2$-action, the map $\mathbb{C}^* \times A \to A$, $(x, u) \mapsto u^{-1/2} \cdot x$ is well-defined and $\mathbb{C}^*$-invariant. By taking the quotient by $\mathbb{C}^*$, we obtain the morphism of stacks $[(A \times \mathbb{C}^*)/\mathbb{C}^*] \to A$. We also have the projection $[(A \times \mathbb{C}^*)/\mathbb{C}^*] \to [\mathbb{C}^*/\mathbb{C}^*] = B\mu_2$, so we obtain the morphism

$$[(A \times \mathbb{C}^*)/\mathbb{C}^*] \to A \times B\mu_2 = [A/\mu_2].$$

It is straightforward to check that the above morphism gives an inverse of (3.10).

Lemma 3.8. In the setting of Lemma 3.7, suppose that $A$ is smooth. Let $w : A \to \mathbb{C}$ be a regular function of $\mathbb{C}^*$-weight two, and $w_\epsilon : A \times \mathbb{C}^* \to C$ be defined by $w_\epsilon(x, t) = w(x)$. Then we have equivalences

$$\iota^* : \text{MF}^\mathbb{C}_\mathbb{C}(A \times \mathbb{C}^*, w_\epsilon) \sim \text{MF}^\mathbb{Z}/2(A, w), \quad \epsilon \in \{\mathbb{qcoh}, \mathbb{coh}\}.$$

Proof. The function $w_\epsilon$ is of $\mathbb{C}^*$-weight two, so it is a global section of the line bundle $\mathcal{O}(2)$ on $[(A \times \mathbb{C}^*)/\mathbb{C}^*]$ induced by the $\mathbb{C}^*$-character of weight two. By the isomorphism $\iota$ in Lemma 3.7, it is pulled back to the global section $w$ of the trivial line bundle on $[A/\mu_2]$. Therefore the lemma follows from Lemma 3.7.

We return to the setting of Theorem 2.4. We define the following affine derived scheme

$$\Omega_\epsilon := \Omega \times \text{Spec} \mathbb{C}[\epsilon],$$

where $\deg(\epsilon) = -1$ with zero differential. Note that $\Omega_\epsilon$ is the derived zero locus of $s_\epsilon := (s, 0) : Y \to V \times \mathbb{A}^1$ for the vector bundle $V \times \mathbb{A}^1 \to Y$. Therefore we have

$$t_0(\Omega_{\Omega_\epsilon}[-1]) = \text{Crit}(w_\epsilon) = \text{Crit}(w) \times \mathbb{A}^1.$$

Here $w_\epsilon : V^\vee \times \mathbb{A}^1 \to \mathbb{C}$ is given by $(x, v, t) \mapsto w(x, v) = \langle s(x), v \rangle$ as in Lemma 3.8. The following is another $\mathbb{Z}/2$-periodic version of Koszul duality equivalence:

Proposition 3.9. There is an equivalence of triangulated categories

$$\Psi_\epsilon : \text{MF}^\mathbb{C}_\mathbb{C}(V^\vee, w) \sim D^b_{\mathbb{coh}}(\Omega_\epsilon)/\mathbb{C}_{\text{Crit}(w) \times \{0\}}.$$

Proof. Let $\mathbb{C}^*$ acts on $V^\vee \times \mathbb{A}^1$ by weight two on fibers of $V^\vee \times \mathbb{A}^1 \to Y$. By Lemma 3.8, the left hand side of (3.11) is equivalent to $\text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{C}^*, w_\epsilon)$. We have the equivalence by (2.8)

$$\text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{A}^1, w_\epsilon)/\text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{A}^1, w_\epsilon)_{V^\vee \times \{0\}} \sim \text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{C}^*, w_\epsilon).$$

By Theorem 2.4, we have the equivalence

$$\Psi : \text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{A}^1, w_\epsilon) \sim D^b_{\mathbb{coh}}(\Omega_\epsilon).$$

By (2.10), the above equivalence restricts to the equivalence

$$\text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{A}^1)_{V^\vee \times \{0\}} \sim \text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{A}^1)_{\text{Crit}(w) \times \{0\}} \sim C_{\text{Crit}(w) \times \{0\}}.$$

Here the first equivalence follows from $\text{Crit}(w_\epsilon) \cap (V^\vee \times \{0\}) = \text{Crit}(w) \times \{0\}$ and the equivalence (2.9). By taking the Verdier quotients, we obtain the equivalence

$$\text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{A}^1, w_\epsilon)/\text{MF}^\mathbb{C}_\mathbb{C}(V^\vee \times \mathbb{A}^1, w_\epsilon)_{V^\vee \times \{0\}} \sim D^b_{\mathbb{coh}}(\Omega_\epsilon)/\mathbb{C}_{\text{Crit}(w) \times \{0\}}.$$

By combining the above equivalences, we obtain the equivalence (3.11).

Let $\Phi_\epsilon$ be a quasi-inverse of $\Psi_\epsilon$

$$\Phi_\epsilon : D^b_{\mathbb{coh}}(\Omega_\epsilon)/\mathbb{C}_{\text{Crit}(w) \times \{0\}} \sim \text{MF}^\mathbb{Z}/2(V^\vee, w).$$

It is described as follows:
Lemma 3.10. The equivalence \( \Psi_{\epsilon} \) is given by
\[
(3.14) \quad \Phi_{\epsilon}(M, d_M) = (\mathcal{O}_V \otimes \mathcal{O}_V M^{\mathbb{Z}/2}, 1 \otimes (d_M + \epsilon) + \eta).
\]

Here \( (M, d_M) \) is a \( \mathbb{Z} \)-graded dg \( \mathcal{O}_U \)-module, \( M^{\mathbb{Z}/2} = M^{\text{even}} \oplus M^{\text{odd}} \), and \( \eta \in V \otimes V^* \) corresponds to \( id \in \text{Hom}(V, V) \) as in (2.10).

Proof. By the construction of \( \Psi_{\epsilon} \), its quasi-inverse \( \Phi_{\epsilon} \) is a descendant of the composition
\[
D^b_{\text{coh}}(U_{\epsilon}) \cong \text{MF}^{\mathbb{C}^*}_{\text{coh}}(V^* \times \mathbb{A}^1, w_{\epsilon}) \rightarrow \text{MF}^{\mathbb{C}^*}_{\text{coh}}(V^* \times \mathbb{C}^*, w_{\epsilon}) \cong \text{MF}^{\mathbb{Z}/2}_{\text{coh}}(V^*, w).
\]

Here the first equivalence is given by Theorem 2.4, the second functor is given by the restriction to the open subset \( V^* \times \mathbb{C}^* \subset V^* \times \mathbb{A}^1 \), and the last equivalence is given in Lemma 3.8. The above composition is given by
\[
M \mapsto (|K_{\epsilon} \otimes \mathbb{C} \{\epsilon, t^{\pm 1}\}| \otimes \mathcal{O}_U, M)|_{t=1} = \mathcal{O}_V \otimes \mathcal{O}_V M^{\mathbb{Z}/2}.
\]

Here \( t \) has degree two and \( \mathbb{C} \{\epsilon, t^{\pm 1}\} \) has differential \( t \epsilon \). Under the above isomorphism, the differential on \( \mathcal{O}_V \otimes \mathcal{O}_V M^{\mathbb{Z}/2} \) is given by \( 1 \otimes (d_M + \epsilon) + \eta \). Therefore the lemma holds. \( \Box \)

Example 3.11. In the setting of Example 3.1, we have \( \mathcal{U} = \text{Spec} \mathbb{C}[\epsilon_1], U_{\epsilon} = \text{Spec} \mathbb{C}[\epsilon_1, \epsilon_2] \) where \( \text{deg}(\epsilon_1) = -1 \), and \( \text{Crit}(w_{\epsilon}) = \mathbb{A}^2 \). The equivalence in Proposition 3.10 is an equivalence
\[
\Psi_{\epsilon}: \text{MF}^{\mathbb{Z}/2}_{\text{coh}}(\mathbb{A}^1, 0) \cong D^b_{\text{coh}}(\mathcal{U}_{\epsilon})/\mathcal{C}_{\text{Crit}} \times \{0\}.
\]

For \( y \in \mathbb{A}^1 \), let \( \mathcal{E}_y \) be defined by
\[
\mathcal{E}_y := \mathbb{C}[\epsilon_1, \epsilon_2]/(\epsilon_2 + y \epsilon_1) \in D^b_{\text{coh}}(\mathcal{U}_{\epsilon}).
\]

Then from (3.14), we have the isomorphism in \( \text{MF}^{\mathbb{Z}/2}_{\text{coh}}(\mathbb{A}^1, 0) \)
\[
\Phi_{\epsilon}(\mathcal{E}_y) = (\mathbb{C}[\epsilon, t], d_{\mathbb{C}[\epsilon, t]} = (t - y)\epsilon) \cong (\mathcal{O}_y, d_{\mathcal{O}_y} = 0)[1].
\]

Therefore \( \Psi_{\epsilon}(\mathcal{O}_y) = \mathcal{E}_y[1] \). The object \( \mathcal{E}_y \) has singular support \( \mathbb{A}^1(y, 1) \subset \mathbb{A}^2 \), so in particular it is non-zero in \( D^b_{\text{coh}}(\mathcal{U}_{\epsilon})/\mathcal{C}_{\text{Crit}} \times \{0\} \).

We can also compare with the \( \mathbb{C}^* \)-equivariant Koszul duality. Let \( i: \mathcal{U} \rightarrow \mathcal{U}_{\epsilon} \) be the morphism induced by \( \text{Spec} \mathbb{C} \rightarrow \text{Spec} \mathbb{C}[\epsilon] \). We have the following lemma:

Lemma 3.12. The following diagram is commutative
\[
\begin{array}{ccc}
D^b_{\text{coh}}(\mathcal{U}) & \xrightarrow{i_*} & D^b_{\text{coh}}(\mathcal{U}_{\epsilon})/\mathcal{C}_{\text{Crit}}(w_{\epsilon}) \times \{0\} \\
\downarrow{\Phi_{\epsilon}} & \Phi_{\epsilon} & \Phi_{\epsilon} \\
\text{MF}^{\mathbb{C}^*}_{\text{coh}}(V^*, w) & \rightarrow & \text{MF}^{\mathbb{Z}/2}_{\text{coh}}(V^*, w).
\end{array}
\]

Here \( \Phi_{\epsilon} \) is the functor forgetting the \( \mathbb{C}^* \)-equivariant structure.

Proof. For \( M \in D^b_{\text{coh}}(\mathcal{U}) \), the element \( \epsilon \in \mathbb{C}[\epsilon] \) acts trivially on \( i_* M \). Therefore the commutativity of (3.14) is obvious from (3.13). \( \Box \)

As mentioned in [P˘ ad, Proposition 2.1], the results of Proposition 3.10 and Lemma 3.12 imply that we have the isomorphism of \( K \)-groups of triangulated categories of \( \mathbb{C}^* \)-equivariant factorizations and \( \mathbb{Z}/2 \)-periodic ones. Although we will not use the corollary below, we include it here for an independent interest:

Corollary 3.13. The forgetting functor induces the isomorphism of \( K \)-groups
\[
\text{forg}: K(\text{MF}^{\mathbb{C}^*}_{\text{coh}}(V^*, w)) \cong K(\text{MF}^{\mathbb{Z}/2}_{\text{coh}}(V^*, w)).
\]
Proof. By Lemma 3.12 it is enough to show that the morphism induced by the left vertical arrow in (3.16)
\[ i_*: K(D^b_{\text{coh}}(\mathcal{U})) \rightarrow K(D^b_{\text{coh}}(\mathcal{U})/C_{\text{Crit}(W) \times \{0\}}) \]
is an isomorphism. Since \( D^b_{\text{coh}}(\mathcal{U}) \) and \( C_{\text{Crit}(w) \times \{0\}} \) are idempotent complete, we have the exact sequence (see the argument of [PS, Lemma 1.10])
\[ (3.16) \quad K(C_{\text{Crit}(W) \times \{0\}}) \rightarrow K(D^b_{\text{coh}}(\mathcal{U})) \rightarrow K(D^b_{\text{coh}}(\mathcal{U})/C_{\text{Crit}(W) \times \{0\}}) \rightarrow 0. \]
Since \( \mathcal{U} \) and \( \mathcal{U}_c \) have isomorphic classical truncation \( \mathcal{U} \), we have the isomorphisms
\[ K(\text{Coh}(\mathcal{U})) \cong K(D^b_{\text{coh}}(\mathcal{U})) \cong K(D^b_{\text{coh}}(\mathcal{U})/C_{\text{Crit}(W) \times \{0\}}) \]
where the last isomorphism is induced by \( i_* \). Therefore it is enough to show that the first morphism in (3.16) is a zero map. The subcategory \( C_{\text{Crit}(w) \times \{0\}} \) is split generated by \( F \otimes C[e] \) for \( F \in D^b_{\text{coh}}(\mathcal{U}) \), so in particular any object \( M \in C_{\text{Crit}(w) \times \{0\}} \) satisfies \( i^* M \in D^b_{\text{coh}}(\mathcal{U}) \). Then from the triangle
\[ i_* i^* M[1] \rightarrow M \rightarrow i_* i^* M \]
in \( D^b_{\text{coh}}(\mathcal{U}_c) \), we conclude \( [M] = 0 \) in \( K(D^b_{\text{coh}}(\mathcal{U}_c)) \). \( \square \)

By taking the ind-completion of the equivalence in Proposition 3.9 we have the following corollary:

**Corollary 3.14.** The equivalence (3.11) extends to the equivalence
\[ \Psi_\varepsilon: \text{MF}^{Z/2}_{\text{coh}}(V^\vee, w) \rightarrow \text{Ind} D^b_{\text{coh}}(\mathcal{U}_c)/\text{Ind} C_{\text{Crit}(w) \times \{0\}}. \]

**Proof.** Since \( \text{MF}^{Z/2}_{\text{coh}}(V^\vee, w) \) is compactly generated by compact objects \( \text{MF}^{Z/2}_{\text{coh}}(V^\vee, w) \) (see [BFK13, Proposition 3.15]), we have the equivalence
\[ \text{Ind} \text{MF}^{Z/2}_{\text{coh}}(V^\vee, w) \cong \text{MF}^{Z/2}_{\text{coh}}(V^\vee, w). \]
We also have the natural equivalence (see [Todb, Proposition 3.2.7])
\[ \text{Ind} D^b_{\text{coh}}(\mathcal{U}_c)/\text{Ind} C_{\text{Crit}(w) \times \{0\}} \rightarrow \text{Ind} D^b_{\text{coh}}(\mathcal{U}_c)/\text{Ind} C_{\text{Crit}(w) \times \{0\}}. \]
The corollary now follows by taking the ind-completion of the equivalence (3.11). \( \square \)

Let \( Z \subset \text{Crit}(w) \) be a closed subset. We define \( Z_\varepsilon \subset \text{Crit}(w) \times \mathbb{A}^1 \) to be
\[ Z_\varepsilon := C^*(Z \times \{1\}) \cup (\text{Crit}(w) \times \{0\}). \]
Note that \( Z_\varepsilon \) is a conical closed subset in \( \text{Crit}(w) \times \mathbb{A}^1 \).

**Proposition 3.15.** The equivalence in Proposition 3.3 descends to the equivalence
\[ (3.17) \quad \Psi_\varepsilon: \text{MF}^{Z/2}_{\text{coh}}(V^\vee \setminus Z, w) \rightarrow D^b_{\text{coh}}(\mathcal{U}_c)/C_{Z_\varepsilon}. \]

**Proof.** The closed substack \([Z/\mu_2] \subset [V^\vee/\mu_2]\) corresponds to \( C^*(Z \times \{1\}) \subset [(V^\vee \times C^*)/C^*] \) under the isomorphism in Lemma 3.7. It follows that the equivalence in Lemma 3.8 restricts to the equivalence
\[ \text{MF}^{C^*}_{\text{coh}}(V^\vee \times C^*, w_z)_{C^*(Z \times \{1\})} \cong \text{MF}^{Z/2}_{\text{coh}}(V^\vee, w)_{Z}. \]
The equivalence (3.12) restricts to the equivalence
\[ \text{MF}^{C^*}_{\text{coh}}(V^\vee \times \mathbb{A}^1, w_z)_{Z_\varepsilon}/\text{MF}^{C^*}_{\text{coh}}(V^\vee \times \mathbb{A}^1, w_z)_{\text{Crit}(w) \times \{0\}} \cong \text{MF}^{C^*}_{\text{coh}}(V^\vee \times C^*)_{C^*(Z \times \{1\})}. \]
By (2.19), the equivalence (3.13) restricts to the equivalence
\[ \text{MF}^{C^*}_{\text{coh}}(V^\vee \times \mathbb{A}^1, w_z)_{Z_\varepsilon}/\text{MF}^{C^*}_{\text{coh}}(V^\vee \times \mathbb{A}^1, w_z)_{\text{Crit}(w) \times \{0\}} \rightarrow C_{Z_\varepsilon}/C_{\text{Crit}(w) \times \{0\}}. \]
By combining the above equivalences, the equivalence in Proposition 3.9 restricts to the equivalence
\[ \Psi_\varepsilon: \text{MF}^{Z/2}_{\text{coh}}(V^\vee, w)_{Z} \rightarrow C_{Z_\varepsilon}/C_{\text{Crit}(w) \times \{0\}}. \]
By taking the Verdier quotients and using \[28\], the equivalence in Proposition \[3.9\] descends to the equivalence \[3.17\].

\[3.4. \textbf{Comparison of two Koszul dualities.} \]

We compare the equivalences in Proposition \[3.3\] and Corollary \[3.14\]. Let \( \Upsilon \) be the functor

\[
\Upsilon : \Ho(\mathcal{O}_U\text{-mod}^Z) \to \Ho(\mathcal{O}_U\text{-mod}^{Z/2}), \quad (M, d_M) \mapsto (M^{Z/2}, d_M + \epsilon).
\]

Here \( M^{Z/2} = M^{\text{even}} \oplus M^{\text{odd}} \) is a \( Z/2 \)-graded \( \mathcal{O}_U \)-module forgetting the \( \mathbb{C}[\epsilon] \)-module structure, with differential \( d_M + \epsilon \). Since the above functor preserves coacyclic objects, it induces the functor

\[
(3.18) \quad \Upsilon : D^{\text{co}}(\mathcal{O}_U\text{-mod}^Z) \to D^{\text{co}}(\mathcal{O}_U\text{-mod}^{Z/2}).
\]

Note that the source of the above functor is equivalent to \( \text{Ind} D^b_{\text{coh}}(\mathcal{U}_\epsilon) \) by Remark \[28\].

\[\textbf{Lemma 3.16.} \quad \text{The functor} \,(3.18)\, \text{descends to the functor}
\]

\[
(3.19) \quad \Upsilon : \text{Ind} D^b_{\text{coh}}(\mathcal{U}_\epsilon)/\text{Ind} C_{\text{Crit}(w) \times \{0\}} \to D^{\text{co}}(\mathcal{O}_U\text{-mod}^{Z/2}).
\]

\[\text{Proof.} \quad \text{Note that} \, C_{\{0\}} \subset D^b_{\text{coh}}(\text{Spec} \, \mathbb{C}[\epsilon]) \text{coincides with the subcategory of perfect complexes on} \text{Spec} \, \mathbb{C}[\epsilon] \,(\text{see} \, \text{AG15 Theorem 4.2.6}), \text{which is generated by} \, \mathbb{C}[\epsilon]. \text{Since} \, \text{Ind} D^b_{\text{coh}}(\mathcal{U}_\epsilon) \text{is generated by exterior products (see} \, \text{AG15 Lemma 4.6.4}), \text{it is enough to show that the functor} \,(3.18)\, \text{sends objects} \, M \otimes \mathbb{C}[\epsilon] \text{for} \, M \in D^b_{\text{coh}}(\mathcal{U}_\epsilon) \text{to zero. By the definition of} \, \Upsilon, \text{we have}
\]

\[
\Upsilon(M \otimes \mathbb{C}[\epsilon]) = \text{Tot}(M^{Z/2} \to M^{Z/2}) \cong 0.
\]

Therefore the lemma holds. \[\square\]

As a summary of the results of this section, we have the following:

\[\textbf{Theorem 3.17.} \quad \text{We have the commutative diagram of equivalences}
\]

\[
(3.20) \quad \begin{array}{ccc}
\text{MF}_{\text{qcoh}}^{Z/2}(V^\vee, w) & \xrightarrow{\sim} & \text{MF}_{\text{qcoh}}^{Z/2}(V^\vee, w) \\
\Psi \downarrow & & \sim \downarrow \Psi^{Z/2} \\
\text{Ind} D^b_{\text{coh}}(\mathcal{U}_\epsilon)/\text{Ind} C_{\text{Crit}(w) \times \{0\}} & \xrightarrow{\Upsilon} & D^{\text{co}}(\mathcal{O}_U\text{-mod}^{Z/2}),
\end{array}
\]

which restricts to the commutative diagram

\[
(3.21) \quad \begin{array}{ccc}
\text{MF}_{\text{coh}}^{Z/2}(V^\vee, w) & \xrightarrow{\sim} & \text{MF}_{\text{coh}}^{Z/2}(V^\vee, w) \\
\Psi \downarrow & & \sim \downarrow \Psi^{Z/2} \\
D^b_{\text{coh}}(\mathcal{U}_\epsilon)/C_{\text{Crit}(w)\times\{0\}} & \xrightarrow{\Upsilon} & D^{\text{coh}}(\mathcal{O}_U\text{-mod}_{\text{ig}}^{Z/2}).
\end{array}
\]

Here the horizontal arrows are fully-faithful with dense images.

\[\text{Proof.} \quad \text{The commutativity of the diagram} \,(3.20)\, \text{follows from Lemma 3.10 together with the isomorphism}
\]

\[
(\text{O}_{V^\vee} \otimes_{\mathcal{O}_Y} M^{Z/2}, 1 \otimes (d_M + \epsilon) + \eta) \cong K_{s}^{Z/2} \otimes_{\mathcal{O}_U} (M^{Z/2}, d_M + \epsilon)
\]

for \((M, d_M) \in D^b_{\text{coh}}(\mathcal{U}_\epsilon)\). Then the functor \,(3.19)\, is an equivalence since the vertical arrows in the diagram \,(3.20)\, are equivalences by Proposition \[3.3\] and Corollary \[3.14\]. The commutative diagram \,(3.21)\, follows from Corollary \[3.14\] and Proposition \[3.9\]. \[\square\]

\[\textbf{4.} \, \text{Z}/2\text{-periodic DT categories for} \,(−1)\text{-shifted cotangents}
\]

In this section, we give a basic model of \( \mathbb{Z}/2 \)-periodic DT categories for \((−1)\)-shifted cotangents over quasi-smooth derived stacks, based on the \( \mathbb{Z}/2 \)-periodic Koszul duality in the previous section. We then compare it with the \( \mathbb{C}^\ast \)-equivariant DT category studied in \[Tod1\].
4.1. **Quasi-smooth derived stacks.** Below, we denote by \( \mathcal{M} \) a derived Artin stack over \( \mathbb{C} \). This means that \( \mathcal{M} \) is a contravariant \( \infty \)-functor from the \( \infty \)-category of affine derived schemes over \( \mathbb{C} \) to the \( \infty \)-category of simplicial sets

\[
\mathcal{M} : dAff^{op} \to SSets
\]
satisfying some conditions (see [Toë14] Section 3.2 for details). Here \( dAff^{op} \) is defined to be the \( \infty \)-category of commutative simplicial \( \mathbb{C} \)-algebras, which is equivalent to the \( \infty \)-category of commutative differential graded \( \mathbb{C} \)-algebras with non-positive degrees. The classical truncation of \( \mathcal{M} \) is denoted by

\[
\mathcal{M} : = t_0(\mathcal{M}) : Aff^{op} \hookrightarrow dAff^{op} \to SSets
\]
where the first arrow is a natural functor from the category of affine schemes to affine derived schemes.

For an affine derived scheme \( \mathcal{U} = \text{Spec } R \) for a cdga \( R \), we set \( D_qcoh(\mathcal{U})_{dg} : = D(R\text{-mod}^1)_{dg} \). The dg-category of quasi-coherent sheaves on \( \mathcal{M} \) is defined to be the limit in the \( \infty \)-category of dg-categories (see [Toë14])

\[
D_{qcoh}(\mathcal{M})_{dg} : = \lim_{\mathcal{U} \to \mathcal{M}} D_qcoh(\mathcal{U})_{dg}.
\]
Here the limit is taken for the \( \infty \)-category of smooth morphisms \( \alpha : \mathcal{U} \to \mathcal{M} \) for affine derived schemes \( \mathcal{U} \) with 1-morphisms given by smooth morphisms \( \mathcal{U} \to \mathcal{U}' \) commuting with maps to \( \mathcal{M} \). The homotopy category of \( D_{qcoh}(\mathcal{M})_{dg} \) is denoted by \( D_{qcoh}(\mathcal{M}) \), which is a triangulated category.

We have the dg and triangulated subcategories

\[
D^b_{coh}(\mathcal{M})_{dg} \subset D_{qcoh}(\mathcal{M})_{dg}, \quad D^b_{coh}(\mathcal{M}) \subset D_{qcoh}(\mathcal{M})
\]
consisting of objects which have bounded coherent cohomologies.

A morphism of derived stacks \( f : \mathcal{M} \to \mathcal{N} \) is called **quasi-smooth** if \( Lf \) is perfect such that for any point \( x \to \mathcal{M} \) the restriction \( Lf|_x \) is of cohomological amplitude \([-1, 1]\). Here \( Lf \) is the \( f \)-relative cotangent complex. A derived stack \( \mathcal{M} \) over \( \mathbb{C} \) is called **quasi-smooth** if \( \mathcal{M} \to \text{Spec } \mathbb{C} \) is quasi-smooth.

By [BBBJ15] Theorem 2.8, the quasi-smoothness of \( \mathcal{M} \) is equivalent to that \( \mathcal{M} \) is a 1-stack, and any point of \( \mathcal{M} \) lies in the image of a 0-representable smooth morphism \( \alpha : \mathcal{U} \to \mathcal{M} \), where \( \mathcal{U} \) is an affine derived scheme obtained as a derived zero locus as in (2.12). In this case, we have

\[
D^b_{coh}(\mathcal{M})_{dg} = \lim_{\mathcal{U} \subset \mathcal{M}} D^b_{coh}(\mathcal{U})_{dg}
\]
where the limit is taken for the \( \infty \)-category \( \mathcal{I} \) of smooth morphisms \( \alpha : \mathcal{U} \to \mathcal{M} \) where \( \mathcal{U} \) is equivalent to an affine derived scheme of the form (2.12). In this paper when we write \( \lim_{\mathcal{U} \subset \mathcal{M}}(\cdot) \) for a quasi-smooth \( \mathcal{M} \), the limit is always taken for the \( \infty \)-category \( \mathcal{I} \) as above.

Following [DG13] Definition 1.1.8, a derived stack \( \mathcal{M} \) is called **QCA** (**quasi-compact and with affine automorphism groups**) if the following conditions hold:

(i) \( \mathcal{M} \) is quasi-compact;
(ii) The automorphism groups of its geometric points are affine;
(iii) The classical inertia stack \( I_M : = \Delta \times_{\mathcal{M} \times \mathcal{M}} \Delta \) is of finite presentation over \( \mathcal{M} \).

Let \( \mathcal{M} \) be a quasi-smooth derived stack. We denote by \( \text{Ind } D^b_{coh}(\mathcal{M})_{dg} \) the dg-category of its ind-coherent sheaves (see [Gai13])

\[
\text{Ind } D^b_{coh}(\mathcal{M})_{dg} : = \lim_{\mathcal{U} \subset \mathcal{M}} \text{Ind } D^b_{coh}(\mathcal{U})_{dg},
\]
where the limit is taken for the \( \infty \)-category \( \mathcal{I} \) as above. The QCA condition will be useful since we have the following theorem:

**Theorem 4.1.** ([DG13] Theorem 3.3.5) If \( \mathcal{M} \) is QCA, then \( \text{Ind } D^b_{coh}(\mathcal{M})_{dg} \) is compactly generated with \( (\text{Ind } D^b_{coh}(\mathcal{M})_{dg})^{cp} = D^b_{coh}(\mathcal{M})_{dg} \). In particular, we have

\[
\text{Ind } D^b_{coh}(\mathcal{M})_{dg} = \text{Ind } (D^b_{coh}(\mathcal{M})_{dg}).
\]
Let $\mathcal{M}$ be a quasi-smooth derived stack. We denote by $\Omega_{\mathcal{M}}[-1]$ the $(-1)$-shifted cotangent derived stack of $\mathcal{M}$

$$p : \Omega_{\mathcal{M}}[-1] := \text{Spec}_{\mathcal{M}} \mathcal{S}(T_{\mathcal{M}}[1]) \to \mathcal{M}.$$ 

Here $T_{\mathcal{M}} \in D_{\text{coh}}^b(\mathcal{M})$ is the tangent complex of $\mathcal{M}$, which is dual to the cotangent complex $L_{\mathcal{M}}$ of $\mathcal{M}$. The derived stack $\Omega_{\mathcal{M}}[-1]$ admits a natural $(-1)$-shifted symplectic structure [PTVV13, Cal19], which induces the d-critical structure [Joy15] on its classical truncation $M$

$$p_0 : \mathcal{N} := t_0(\Omega_{\mathcal{M}}[-1]) \to \mathcal{M}.$$ 

Let $\mathcal{M}_1, \mathcal{M}_2$ be quasi-smooth derived stacks with truncations $\mathcal{M}_i = t_0(\mathcal{M}_i)$. Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism. Then the morphism $f^* L_{\mathcal{M}_2} \to L_{\mathcal{M}_1}$ induces the diagram

$$t_0(\Omega_{\mathcal{M}_1}[-1]) \xrightarrow{f^\diamond} t_0(\Omega_{\mathcal{M}_2}[-1] \times_{\mathcal{M}_2} \mathcal{M}_1) \xrightarrow{f^*} t_0(\Omega_{\mathcal{M}_2}[-1]).$$

The morphism $f$ is quasi-smooth if and only if $f^\diamond$ is a closed immersion, $f$ is smooth if and only if $f^\diamond$ is an isomorphism (see [Todb, Lemma 3.1.2]).

Let us take a conical closed substack

$$\mathcal{Z} \subset \mathcal{N} = t_0(\Omega_{\mathcal{M}}[-1]).$$

Here $\mathcal{Z}$ is called conical if it is invariant under the fiberwise $\mathbb{C}^*$-action on $\mathcal{N} \to \mathcal{M}$. Let $\alpha : \mathcal{U} \to \mathcal{M}$ be a smooth morphism such that $\mathcal{M} = \text{Spec} \mathcal{S}(\mathcal{U})$.

We have the associated conical closed subscheme

$$\alpha^* \mathcal{Z} := \alpha^\diamond (\alpha^\bullet)^{-1}(\mathcal{Z}) \subset t_0(\Omega_{\mathcal{U}}[-1]) = \text{Crit}(w).$$

Here $w$ is given as in (2.13). As in [AG15], we define

$$C_{\mathcal{Z}, \text{dg}} := \lim_{\mathcal{U} \to \mathcal{M}} C_{\mathcal{Z}, \text{dg}} \subset D_{\text{coh}}^b(\mathcal{M})_{\text{dg}}, \text{ Ind } C_{\mathcal{Z}, \text{dg}} := \lim_{\mathcal{U} \to \mathcal{M}} \text{ Ind } C_{\mathcal{Z}, \text{dg}} \subset \text{ Ind } D_{\text{coh}}^b(\mathcal{M})_{\text{dg}},$$

whose homotopy categories are denoted as $C_Z$, Ind $C_Z$ respectively.

4.2. Definition of $\mathbb{Z}/2$-periodic DT categories. For a quasi-smooth and QCA derived stack $\mathcal{M}$, let us take an open substack $\mathcal{N}^\text{ss}$ and its complement $\mathcal{Z}$,

$$\mathcal{N}^\text{ss} \subset \mathcal{N}, \mathcal{Z} := \mathcal{N} \setminus \mathcal{N}^\text{ss}.$$ 

In the case that $\mathcal{N}^\text{ss}$ is $\mathbb{C}^*$-invariant so that $\mathcal{Z}$ is a conical closed substack, the $\mathbb{C}^*$-equivariant dg or triangulated DT categories were defined in [Todb] as the Drinfeld or Verdier quotient

$$D^T(\mathcal{N}^\text{ss})_{\text{dg}} := D_{\text{coh}}^b(\mathcal{M})_{\text{dg}} / C_Z, \text{dg}, \quad D^T(\mathcal{N}^\text{ss}) := D_{\text{coh}}^b(\mathcal{M}) / C_Z.$$ 

The idea of the above definition was based on the Koszul duality equivalence in Theorem 2.4, which gives an interpretation of the above category as a gluing dg-categories of $\mathbb{C}^*$-equivariant factorizations.

We give a definition of $\mathbb{Z}/2$-periodic DT categories based on Proposition 3.15 instead of Theorem 2.4. For a quasi-smooth derived stack $\mathcal{M}$, we set

$$\mathcal{M}_e := \mathcal{M} \times \text{Spec} \mathbb{C}[\varepsilon]$$

where $\deg(\varepsilon) = -1$. Then we have

$$t_0(\Omega_{\mathcal{M}_e}[-1]) = t_0(\Omega_{\mathcal{M}}[-1]) \times t_0(\Omega_{\text{Spec} \mathbb{C}[\varepsilon]}[-1]) = \mathcal{N} \times \mathbb{A}^1.$$ 

We define

$$\mathcal{Z}_e := \mathbb{C}^*(\mathcal{Z} \times \{1\}) \cup (\mathcal{N} \times \{0\}) \subset \mathcal{N} \times \mathbb{A}^1.$$
Here $\mathbb{C}^*$ acts on fibers of $\mathcal{N} \times \mathbb{A}^1 \to \mathcal{M}$ by weight two. Note that $\mathcal{Z}_e$ is a conical closed substack, though $\mathcal{Z}$ may not be conical. Note that if $\mathcal{Z}$ is conical, then $\mathcal{Z}_e = (\mathcal{Z} \times \mathbb{A}^1) \cap (\mathcal{N} \times \{0\})$.

**Definition 4.2.** The $\mathbb{Z}/2$-periodic dg or triangulated DT categories $\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss})_{dg}$, $\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss})$ are defined by Drinfeld or Verdier quotients

\[
\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss})_{dg} := D^b_{\text{coh}}(\mathcal{M}_e)_{dg}/\mathcal{C}_Z,_{dg}, \quad \mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss}) := D^b_{\text{coh}}(\mathcal{M}_e)/\mathcal{C}_Z.
\]

If $\mathcal{M}$ is an affine derived scheme of the form (2.12), then $\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss})$ is equivalent to the derived category of $\mathbb{Z}/2$-periodic factorizations by Proposition 3.15. In general, the triangulated category $\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss})$ is $\mathbb{Z}/2$-periodic by the following lemma:

**Lemma 4.3.** The triangulated category $\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss})$ is $\mathbb{Z}/2$-periodic.

**Proof.** Let $i: \mathcal{M}_e \hookrightarrow \mathcal{M}$ be the closed immersion induced by the closed immersion Spec $\mathbb{C}[e] \hookrightarrow$ Spec $\mathbb{C}$. Then for any $\mathcal{E} \in D^b_{\text{coh}}(\mathcal{M}_e)$ there is a distinguished triangle

\[
\mathcal{E}[1] \to i^*i_*\mathcal{E} \to \mathcal{E} \to \mathcal{E}[2].
\]

The morphism $i$ is nothing but the projection $\mathcal{M}_e \to \mathcal{M}$, and $i^*i_*\mathcal{E} = i_*\mathcal{E} \boxtimes \mathbb{C}[e]$. Since $\mathbb{C}[e] \in \mathcal{C}(\mathcal{0})$ in $D^b_{\text{coh}}(\text{Spec } \mathbb{C}[e])$, we have $i^*i_*\mathcal{E} \in \mathcal{C}(\mathcal{N}^{ss})$ by [AG15] Lemma 4.6.4. Therefore the morphism $\mathcal{E} \to \mathcal{E}[2]$ in the sequence (4.1) is an isomorphism in $\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss})$.

Let $\mathcal{W} \subset \mathcal{M}$ be a closed substack, and $\mathcal{M}_o \subset \mathcal{M}$ be the open substack whose classical truncation is $\mathcal{M} \setminus \mathcal{W}$. We have the following closed substack

\[
\mathcal{Z}_o := \mathcal{Z} \setminus p_0^{-1}(\mathcal{W}) \subset \mathcal{N}_o := t_0(\Omega_{\mathcal{M}_o}[-1]).
\]

Note that $\mathcal{N}_o = \mathcal{N} \setminus p_0^{-1}(\mathcal{W})$, and we have the following open immersion

\[
\mathcal{N}_o^{ss} := \mathcal{N}_o \setminus \mathcal{Z}_o \hookrightarrow \mathcal{N} \setminus \mathcal{Z} = \mathcal{N}^{ss}.
\]

The following lemma is an analogue of [Todb] Lemma 3.2.9):

**Lemma 4.4.** Suppose that $p_0^{-1}(\mathcal{W}) \subset \mathcal{Z}$, so that the open immersion (4.2) is an isomorphism. Then the restriction functor gives an equivalence

\[
\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss}) \cong \mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}_o^{ss}).
\]

**Proof.** Since $p_0^{-1}(\mathcal{W}) \times \mathbb{A}^1 \subset \mathcal{Z}_e$ and $(\mathcal{Z}_o)_e = \mathcal{Z}_e \setminus (p_0^{-1}(\mathcal{W}) \times \mathbb{A}^1)$, we can directly apply [Todb] Lemma 3.2.9] to obtain the lemma.

**4.3. Comparison with $\mathbb{C}^*$-equivariant DT categories.** For a conical closed substack $\mathcal{Z} \subset t_0(\Omega_{\mathcal{M}_o}[-1])$ with $\mathcal{N}^{ss} = \mathcal{N} \setminus \mathcal{Z}$, we have both of $\mathbb{C}^*$-equivariant DT category $\mathcal{D}T^{\mathbb{C}^*}(\mathcal{N}^{ss})$ and $\mathbb{Z}/2$-periodic one $\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^{ss})$. In this subsection, we compare these categories. We first prove some lemmas.

**Lemma 4.5.** Let $\mathcal{M}$ be a quasi-smooth and QCA derived stack and $\mathcal{Z} \subset t_0(\Omega_{\mathcal{M}_o}[-1])$ be a conical closed substack. Then $(\text{Ind } \mathcal{C}_Z)^{\text{fp}} = \mathcal{C}_Z$.

**Proof.** The lemma is a proved in the last part of [Todb Theorem 7.2.2]. Since we have $D^b_{\text{coh}}(\mathcal{M}) = (\text{Ind } D^b_{\text{co}h}(\mathcal{M}))^{\text{fp}}$ by [DG13] Theorem 3.3.5], we have

\[
\mathcal{C}_Z \subset (\text{Ind } D^b_{\text{co}h}(\mathcal{M}))^{\text{fp}} \cap \text{Ind } \mathcal{C}_Z \subseteq (\text{Ind } \mathcal{C}_Z)^{\text{fp}}.
\]

As for the converse direction $(\text{Ind } \mathcal{C}_Z)^{\text{fp}} \subset \mathcal{C}_Z$, we take an object $\mathcal{E} \in (\text{Ind } \mathcal{C}_Z)^{\text{fp}}$ and a smooth morphism $\alpha: \mathcal{U} \to \mathcal{M}$ where $\mathcal{U}$ is an affine derived scheme of the form (2.12). Since the pull-back $\alpha^*: \text{Ind } \mathcal{C}_Z \to \text{Ind } \mathcal{C}_{\alpha^*Z}$ admits a continuous right adjoint $\alpha^{\text{ind}}_*: \text{Ind } \mathcal{C}_{\alpha^*Z} \to \text{Ind } \mathcal{C}_Z$, we have $\alpha^{\text{ind}}_*\mathcal{E} \in (\text{Ind } \mathcal{C}_{\alpha^*Z})^{\text{fp}} = \mathcal{C}_{\alpha^*Z}$. Here the last identity follows from [AG15 Corollary 9.2.7, Corollary 9.2.8]. Therefore we have $\mathcal{E} \in \mathcal{C}_Z$.

\[\square\]
Lemma 4.6. Let $\mathcal{M}_i$ for $i = 1, 2$ be quasi-smooth and QCA derived stacks, and $Z_i \subset t_0(\Omega_{\mathcal{M}_i}[-1])$ be conical closed substacks. Suppose that $\text{Ind}C_{Z_i}$ are compactly generated. Then the subcategories

$$\text{Ind}C_{p_i|Z_i \cap p_i^\star Z_2}, \text{Ind}C_{p_i|Z_1 \cup p_i^\star Z_2} \subset \text{Ind}D^b_{\text{coh}}(\mathcal{M}_1 \times \mathcal{M}_2)$$

are compactly generated. Here $p_i : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i$ is the projection.

Proof. We first show that $\text{Ind}C_{Z_i} \cap p_i^\star Z_2$ is compactly generated. By the assumption and Lemma 4.5, $\text{Ind}C_{Z_i}$ are compactly generated with compact objects $C_{Z_i}$. Therefore the argument of Corollary 4.20 shows that

$$\text{Ind}C_{\alpha^\star} Z, dg \ominus C_{Z_2, dg} \simeq \lim_{U_1 \to \mathcal{M}_1} \lim_{U_2 \to \mathcal{M}_2} \text{Ind}(C_{\alpha^\star} Z, dg \ominus C_{\alpha^\star} Z, dg).$$

By [AG13, Lemma 4.6.4], we have the equivalence

$$\text{Ind}(C_{\alpha^\star} Z, dg \ominus C_{Z_2, dg}) \xrightarrow{\sim} \text{Ind}(C_{p_i|Z_i \cap p_i^\star Z_2, dg}).$$

It follows that we have the equivalence

$$\text{Ind}C_{\alpha^\star} Z, dg \ominus C_{Z_2, dg} \simeq \text{Ind}(C_{\alpha^\star} Z, dg \ominus C_{Z_2, dg}).$$

So $\text{Ind}C_{\alpha^\star} Z, dg \ominus C_{Z_2, dg}$ is generated by compact objects $C_{Z_1} \oplus C_{Z_2}$. Then the compact generation for $\text{Ind}C_{p_i|Z_i \cup p_i^\star Z_2}$ holds since it is generated by $\text{Ind}C_{p_i|Z_1}$ and $\text{Ind}C_{p_i^\star Z_2}$ and the latter categories are compactly generated by the above argument.

Theorem 4.9. In the setting of Lemma 4.5, the subcategory $\text{Ind}C_{Z_\epsilon} \subset \text{Ind}D^b_{\text{coh}}(\mathcal{M}_\epsilon)$ is compactly generated.

Proof. Since $Z_\epsilon = (Z \times \mathbb{A}^1) \cup (N \times \{0\})$ and $\text{Ind}C_{\{0\}} \subset \text{Ind}D^b_{\text{coh}}(\bullet, \epsilon)$ is compactly generated by $O_0 = \mathbb{C}[\epsilon]/(\epsilon)$, the lemma follows from Lemma 4.6.

Lemma 4.8. In the setting of Lemma 4.7, we have an equivalence

$$\text{Ind} DT^{Z/2}(N_{as})_{dg} \xrightarrow{\sim} \lim_{U \to Z_{\epsilon}} \text{Ind}(D^b_{\text{coh}}(U_{\epsilon})_{dg} / C_{\alpha^\star} Z, dg).$$

Here $\alpha = \alpha \times id : U_{\epsilon} \to \mathcal{M}_\epsilon$.

Proof. By Lemma 4.7 and Lemma 4.8, the category $\text{Ind}C_{Z_\epsilon}$ is compactly generated with compact objects $C_{Z_\epsilon}$. Then the lemma follows from [Toda, Proposition 3.2.7].

Let $\mathbb{C}[u^\pm 1]$ be the dg algebra with $dg(u) = 2$ and zero differential. We regard it as a dg category with one object. The following result implies that one can recover $\mathbb{Z}/2$-periodic DT category from the $C^*$-equivariant one up to idempotent completion.

Theorem 4.9. Let $\mathcal{M}$ be a quasi-smooth and QCA derived stack and $Z \subset t_0(\Omega_{\mathcal{M}}[-1])$ be a conical closed substack. Suppose that $\text{Ind}C_{Z}$ is compactly generated. Then there is an equivalence

$$(4.3) \quad \text{Ind} DT^{Z/2}(N_{as})_{dg} \simeq R\text{Hom}(\mathbb{C}[u^\pm 1], \text{Ind} DT^{C^*}(N_{as})_{dg}),$$

which restricts to the equivalence

$$(4.4) \quad DT^{Z/2}(N_{as})_{dg} \simeq R\text{Hom}(\mathbb{C}[u^\pm 1], \text{Ind} DT^{C^*}(N_{as})_{dg} \circ p).$$

Proof. Similarly to Lemma 4.8, we have the equivalence (see [Toda, Proposition 3.2.7])

$$\text{Ind} DT^{C^*}(N_{as})_{dg} \xrightarrow{\sim} \lim_{U \to Z_{\epsilon}} \text{Ind}(D^b_{\text{coh}}(U_{\epsilon})_{dg} / C_{\alpha^\star} Z, dg).$$

Therefore the right hand side of (4.3) is

$$\lim_{U \to Z_{\epsilon}} R\text{Hom}(\mathbb{C}[u^\pm 1], \text{Ind}(D^b_{\text{coh}}(U_{\epsilon})_{dg} / C_{\alpha^\star} Z, dg)) \simeq \lim_{U \to Z_{\epsilon}} \text{Ind}(D^b_{\text{coh}}(U_{\epsilon})_{dg} / C_{\alpha^\star} Z, dg) \otimes \mathbb{C}[u^\pm 1]).$$

By Theorem 2.4, there is an equivalence

$$\text{MF}_{\text{coh}}(C^*, 0)_{dg} \simeq D^b_{\text{coh}}(\bullet_{\epsilon})_{dg} / C_{\{0\}, dg}$$
where the $\mathbb{C}^*$-action on $\mathbb{C}^*$ in the left hand side is of weight two. The above functor sends $(\mathcal{O}_C, d\mathcal{O}_C, = 0)$ to $\mathcal{O}_0 = \mathbb{C}[\epsilon]/(\epsilon)$, so we have the quasi-isomorphism of dg-algebra

$$
\mathbb{C}[u^{\pm 1}] \xrightarrow{\sim} \text{Hom}^{\ast}_{\mathcal{D}^b_{\text{coh}}(\bullet, \mathcal{O}_0; \mathcal{O}_0)}(\mathcal{O}_0, \mathcal{O}_0).
$$

Therefore there exists a natural functor

$$(4.5) \quad (D^b_{\text{coh}}(\mathcal{U})_{\text{dg}}/\mathcal{C}_\alpha^* Z_{\text{dg}}) \otimes \mathbb{C}[u^{\pm 1}] \to D^b_{\text{coh}}(\mathcal{U})_{\text{dg}}/\mathcal{C}_\alpha^* Z_{\text{dg}}$$

sending $\mathcal{E}$ to $\mathcal{E} \boxtimes \mathcal{O}_0$. Under the equivalence $(2.20)$, the above functor corresponds to the functor

$$
\text{MF}^\ast_{\text{coh}}(V^\vee \setminus \alpha^* Z, w)_{\text{dg}} \otimes \mathbb{C}[u^{\pm 1}] \to \text{MF}^\ast_{\text{coh}}((V^\vee \setminus \alpha^* Z) \times_{\mathcal{C}^*} \mathbb{C}^*, w)_{\text{dg}}
$$

sending $\mathcal{P}$ to $\mathcal{P} \boxtimes \mathcal{O}_C$. The above functor is fully-faithful by [BFK14] Lemma 3.52, so the functor $(4.5)$ is also fully-faithful. By taking the ind-completion, we obtain the fully-faithful functor

$$(4.6) \quad \text{Ind}((D^b_{\text{coh}}(\mathcal{U})_{\text{dg}}/\mathcal{C}_\alpha^* Z_{\text{dg}}) \otimes \mathbb{C}[u^{\pm 1}]) \to \text{Ind}(D^b_{\text{coh}}(\mathcal{U})_{\text{dg}}/\mathcal{C}_\alpha^* Z_{\text{dg}}).$$

Since $\text{Ind}D^b_{\text{coh}}(\mathcal{U})$ is generated by exterior products by [GR17] Proposition 6.3.4, and $D^b_{\text{coh}}(\bullet)$ is generated by $\mathcal{O}_0$, the functor $(4.6)$ is essentially surjective, hence it is an equivalence. By taking the limit of the equivalence $(4.6)$ for all $\alpha: \mathcal{U} \to \mathcal{M}$ and using Lemma 4.8 we obtain the equivalence $\mathbf{(4.3)}$.

Any object in the subcategory

$$
\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^\text{ss})_{\text{dg}} \subset \text{Ind} \mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^\text{ss})_{\text{dg}}
$$

is compact by [Todb] Proposition 3.2.7, Lemma 7.1.2, which generates $\text{Ind} \mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^\text{ss})_{\text{dg}}$ by Theorem 4.1. Therefore we have

$$
\mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^\text{ss})_{\text{dg}} = \left(\text{Ind} \mathcal{D}T^{\mathbb{Z}/2}(\mathcal{N}^\text{ss})_{\text{dg}}\right)^{\text{cp}}.
$$

By combining with the equivalence $(4.3)$, we obtain the equivalence $\mathbf{(4.4)}$. \hfill $\square$

5. $\mathbb{Z}/2$-periodic DT categories for local surfaces

In this section, we define $\mathbb{Z}/2$-periodic DT categories for local surfaces following basic model in Definition 4.2. We then use Theorem 4.1 and results in [Todb] to prove some wall-crossing equivalences for $\mathbb{Z}/2$-periodic DT categories for one dimensional stable sheaves on local surfaces. by push-forward.

5.1. Derived moduli stacks of coherent sheaves on surfaces. Let $S$ be a smooth projective surface over $\mathbb{C}$. We consider the derived Artin stack

$$
\text{Perf}_S: \text{dAff}^{op} \to \text{SSets}
$$

which sends an affine derived scheme $T$ to the $\infty$-groupoid of perfect complexes on $T \times S$, constructed in [TV07]. We have the open substack

$$
\mathcal{M}_S \subset \text{Perf}_S
$$

corresponding to perfect complexes on $S$ quasi-isomorphic to coherent sheaves on $S$. Since any object in $\text{Coh}(S)$ is perfect as $S$ is smooth, the derived Artin stack $\mathcal{M}_S$ is the derived moduli stack of objects in $\text{Coh}(S)$. It is well-known that $\mathcal{M}_S$ is a quasi-smooth derived Artin stack (see [Todb] Section 3.4).

Let $\mathcal{N}(S)$ be the numerical Grothendieck group of $S$

$$
\mathcal{N}(S) := K(S)/\equiv
$$

where $F_1, F_2 \in K(S)$ satisfies $F_1 \equiv F_2$ if $\text{ch}(F_1) = \text{ch}(F_2)$. Then $\mathcal{N}(S)$ is a finitely generated free abelian group. We have the decompositions into open and closed substacks

$$
\mathcal{M}_S = \coprod_{v \in \mathcal{N}(S)} \mathcal{M}_S(v), \quad \mathcal{M}_S = \bigcup_{v \in \mathcal{N}(S)} \mathcal{M}_S(v)
$$

where each component corresponds to sheaves $F$ with $[F] = v$. 

Note that the automorphism group of a sheaf $F$ on $S$ contains a one dimensional torus $\mathbb{C}^* \subset \text{Aut}(F)$ given by the scalar multiplication. Therefore the inertia stack $I_{\mathcal{M}_S}$ of $\mathcal{M}_S$ admits an embedding $(\mathbb{C}^*)_\mathcal{M}_S \subset I_{\mathcal{M}_S}$. As in [Todb Subsection 3.2.4], we have the $\mathbb{C}^*$-rigidification

$$\mathcal{M}_S(v) \to \mathcal{M}_S^{\mathbb{C}^*-\text{rig}}(v), \quad (5.1)$$

which rigidifies the above $\mathbb{C}^*$-automorphisms, i.e., the automorphism group of $F$ in $\mathcal{M}_S^{\mathbb{C}^*-\text{rig}}(v)$ is $\text{Aut}(F)/\mathbb{C}^*$.

5.2. Moduli stacks of compactly supported sheaves on local surfaces. For a smooth projective surface $S$, we consider its total space of the canonical line bundle:

$$X = \text{Tot}_S(\omega_S) \xrightarrow{\pi} S.$$ 

Here $\pi$ is the projection. We denote by $\text{Coh}_{\text{cpt}}(X) \subset \text{Coh}(X)$ the subcategory of compactly supported coherent sheaves on $X$. We consider the classical Artin stack

$$\mathcal{M}_X : \text{Aff}^{\text{op}} \to \text{Groupoid}$$

whose $T$-valued points for $T \in \text{Aff}$ form the groupoid of $T$-flat families of objects in $\text{Coh}_{\text{cpt}}(X)$. We have the decomposition into open and closed substacks

$$\mathcal{M}_X = \coprod_{v \in N(S)} \mathcal{M}_X(v)$$

where $\mathcal{M}_X(v)$ corresponds to compactly supported sheaves $F$ on $X$ with $[\pi_* F] = v$. By pushing forward to $S$, we have the natural morphism

$$\pi_* : \mathcal{M}_X(v) \to \mathcal{M}_S(v), \quad F \mapsto \pi_* F. \quad (5.2)$$

Moreover we have an isomorphism of stacks over $\mathcal{M}_S(v)$ (see [Todb, Lemma 3.4.1])

$$\eta : \mathcal{M}_X(v) \xrightarrow{\cong} t_0(\Omega_{\mathcal{M}_S(v)}[-1]). \quad (5.3)$$

Similarly to (5.1), (5.3), we also have the $\mathbb{C}^*$-rigidification and the isomorphism

$$\mathcal{M}_X(v) \to \mathcal{M}_X(v)^{\mathbb{C}^*-\text{rig}}, \quad \mathcal{M}_X(v)^{\mathbb{C}^*-\text{rig}} \xrightarrow{\cong} t_0(\Omega_{\mathcal{M}_S(v)}^{\mathbb{C}^*-\text{rig}}[-1]). \quad (5.4)$$

5.3. Definition of $\mathbb{Z}/2$-periodic DT categories for local surfaces. Let $A(S)_C$ be the ample cone

$$A(S)_C := \{ B + iH : H \text{ is ample} \} \subset \text{NS}(S)_C.$$ 

Then any element $\sigma = B + iH \in A(S)_C$ determines the $\sigma$-stability on $\text{Coh}_{\text{cpt}}(X)$ in a standard way (see [Todb, Definition 3.4.3]). We have the open substacks

$$\mathcal{M}_{X, \sigma}^{\text{st}}(v) \subset \mathcal{M}_X^\sigma(v) \subset \mathcal{M}_X(v)$$

corresponding to $\sigma$-stable sheaves, $\sigma$-semistable sheaves, respectively. By the GIT construction of the moduli stack $\mathcal{M}_X^\sigma(v)$, it admits a good moduli space (cf. [Alp13, Section 1.2])

$$\mathcal{M}_X^\sigma(v) \to M_X^\sigma(v) \quad (5.5)$$

where $M_X^\sigma(v)$ is a quasi-projective scheme whose closed points correspond to $\sigma$-polystable objects. When we have the equality $\mathcal{M}_X^{\sigma, \text{st}}(v) = \mathcal{M}_X^\sigma(v)$, then the morphism (5.5) is a $\mathbb{C}^*$-gerbe so that

$$M_X^\sigma(v) = M_X^\sigma(v)^{\mathbb{C}^*-\text{rig}}$$

holds. Note that it admits a $\mathbb{C}^*$-action induced by the fiberwise weight two $\mathbb{C}^*$-action on $\pi : X \to S$.

The moduli stack of $\sigma$-semistable sheaves $\mathcal{M}_X^\sigma(v)$ is of finite type, while $\mathcal{M}_S(v)$ is not quasi-compact in general, so in particular it is not QC. So we take a quasi-compact open derived substack $\mathcal{M}_S(v)_{qc} \subset \mathcal{M}_S(v)$ satisfying the condition

$$\pi_* (\mathcal{M}_X^\sigma(v)) \subset t_0(\mathcal{M}_S(v)_{qc}). \quad (5.6)$$
Here \( \pi_* \) is the morphism \([5,2]\). Note that \( \mathcal{M}_S(v)_{qc} \) is QCA. By the isomorphism \( \eta \) in \([5,3]\), we have the conical closed substack
\[
Z^{\sigma-us} := t_0(\Omega_{\mathcal{M}_S(v)_{qc}}[-1]) \setminus \eta(\mathcal{M}_X^\sigma(v)) \subset t_0(\Omega_{\mathcal{M}_S(v)_{qc}}[-1]).
\]
By taking the \( \mathbb{C}^* \)-rigidification and using the isomorphism \([5,4]\), we also have the conical closed substack
\[
(Z^{\sigma-us})^{\mathbb{C}^*\text{-rig}} \subset t_0(\Omega_{\mathcal{M}_S(v)_{qc}}^{\mathbb{C}^*\text{-rig}}[-1]).
\]
We apply Definition \([4,2]\) for \( \mathcal{M} = \mathcal{M}_S(v)_{qc}^{\mathbb{C}^*\text{-rig}} \) and \( \mathcal{Z} = (Z^{\sigma-us})^{\mathbb{C}^*\text{-rig}} \) to give the following definition:

**Definition 5.1.** Suppose that \( \mathcal{M}^\sigma_{X-st}(v) = \mathcal{M}^\sigma_X(v) \) holds. The \( \mathbb{Z}/2 \)-periodic dg or triangulated DT categories for \( \mathcal{M}^\sigma_X(v) \)
\[
(5.7)
\]
are defined by setting \( \mathcal{M} = \mathcal{M}_S(v)_{qc}^{\mathbb{C}^*\text{-rig}} \) and \( \mathcal{Z} = (Z^{\sigma-us})^{\mathbb{C}^*\text{-rig}} \) in Definition \([4,2]\).

**Remark 5.2.** By Lemma \([4,4]\), the categories \((5.7)\) are independent of a choice of \( \mathcal{M}_S(v)_{qc} \) satisfying the condition \([5,0]\) up to equivalence.

### 5.4. Categorical wall-crossing for moduli spaces of one dimensional sheaves

We denote by
\[
\text{Coh}_{\leq 1}(X) \subset \text{Coh}_{\text{cpt}}(X)
\]
the abelian subcategory consisting of sheaves \( F \) with \( \dim \text{Supp}(F) \leq 1 \). We also denote by \( N_{\leq 1}(S) \) the subgroup of \( N(S) \) spanned by sheaves \( F \in \text{Coh}_{\leq 1}(S) \). Note that we have an isomorphism
\[
N_{\leq 1}(S) \xrightarrow{\sim} \text{NS}(S) \oplus \mathbb{Z}, \quad F \mapsto (l(F), \chi(F))
\]
where \( l(F) \) is the fundamental one cycle of \( F \). Below we identify an element \( v \in N_{\leq 1}(S) \) with \((\beta, n) \in \text{NS}(S) \oplus \mathbb{Z}\) by the above isomorphism. We write \([F] = (\beta, n)\) for \( F \in \text{Coh}_{\leq 1}(S) \) by the above isomorphism. For \( \sigma = B + iH \in A(S)_C \), we set
\[
\mu_\sigma(F) := \frac{n - B \cdot \beta}{H \cdot \beta} \in \mathbb{Q} \cup \{\infty\}.
\]
Then \( F \) is \( \sigma \)-\((\text{semi})\)stable if and only for any subsheaf \( 0 \neq F' \subseteq F \) we have the inequality \( \mu_\sigma(F') < (\leq) \mu_\sigma(F) \).

We fix a primitive element \( v = (\beta, n) \in N_{\leq 1}(S) \) such that \( \beta > 0 \). Here we write \( \beta > 0 \) if \( \beta = [C] \) for a non-zero effective divisor \( C \) on \( S \). For each decomposition
\[
v = v_1 + v_2, \quad v_i = (\beta_i, n_i), \quad \beta_i > 0
\]
we define
\[
W_{v_1, v_2} := \{ \sigma \in A(S)_C : \mu_\sigma(v_1) = \mu_\sigma(v_2) \} = \{ B + iH \in A(S)_C : (n_1\beta_2 - n_2\beta_1) \cdot H = B\beta_1 \cdot H\beta_2 - B\beta_2 \cdot H\beta_1 \}.
\]
Since \( v \) is primitive, \( W_{v_1, v_2} \subseteq A(S)_C \) and \( W_{v_1, v_2} \) is a real codimension one hypersurface in \( A(S)_C \). For a fixed \( v \), the set of hypersurfaces \( W_{v_1, v_2} \) are called **walls**. It is easy to see that the walls are locally finite. Also each connected component
\[
C \subset A(S)_C \setminus \bigcup_{v_1 + v_2 = v} W_{v_1, v_2}
\]
is called a **chamber**. From the construction of walls, the moduli stacks \( \mathcal{M}_S^\sigma(v) \), \( \mathcal{M}_X^\sigma(v) \) are constant if \( \sigma \) is contained in a chamber, but may change when \( \sigma \) crosses a wall. Moreover if \( \sigma \) lies in a chamber, they consist of \( \sigma \)-stable sheaves.
Let us take \( \sigma \in A(S) \) which lies on a wall and take \( \sigma_{\pm} \in A(S) \) which lie on its adjacent chambers. Then by [Toda, Theorem 8.3], the wall-crossing diagram

\[
\begin{array}{ccc}
M^v_X(\sigma^+) & \longrightarrow & M^v_X(\sigma^-) \\
\downarrow & & \downarrow \\
M^v_X(v) & \longrightarrow & M^v_X(v)
\end{array}
\]

is a d-critical flop, which is a d-critical analogue of usual flops in birational geometry. Therefore following the discussion in [Toda, Subsection 1.1.2], we propose the following conjecture:

**Conjecture 5.3.** There exists an equivalence of \( \mathbb{Z}/2 \)-periodic DT categories:

\[
DT^{\mathbb{Z}/2}(M^\sigma_X(v)) \simeq DT^{\mathbb{Z}/2}(M^\sigma_X(v)).
\]

In particular \( DT^{\mathbb{Z}/2}(M^\sigma_X(v)) \) is independent of a choice of generic \( \sigma \) up to equivalence.

The result of Theorem 4.9 reduces the above conjecture (up to idempotent completion) to the case of \( \mathbb{C}^* \)-equivariant version, which was proved in [Toda] in several cases. Indeed we have the following:

**Theorem 5.4.** In the setting of Conjecture 5.3, suppose that the following condition holds

\[
\mathcal{M}^v_X(v) \subset \pi^{-1}(\mathcal{M}^v_S(v)).
\]

Then we have an equivalence

\[
(5.8)
\]

**Proof.** It is proved in [Toda, Theorem 1.4.4] that there is an equivalence

\[
DT^{\mathbb{C}^*}(M^\sigma_X(v)) \simeq DT^{\mathbb{C}^*}(M^\sigma_X(v)).
\]

The above equivalence is given by windows, i.e. there exists a triangulated subcategory \( W \subset D_{coh}(\mathcal{M}^v_S(v)) \) such that the composition functors

\[
W \to D_{coh}(\mathcal{M}^v_S(v)) \to DT^{\mathbb{C}^*}(\mathcal{M}^v_S^+(v))
\]

are equivalences. The right functors are just quotient functors, so they lift to dg functors

\[
(5.9)
\]

Let \( W_{dg} \subset D_{coh}(\mathcal{M}^v_S(v))_{dg} \) be the full dg-subcategory whose objects are isomorphic to objects in \( W \) in the homotopy category. By composing \( W_{dg} \subset D_{coh}(\mathcal{M}^v_S(v))_{dg} \) with dg-functors (5.9), we have dg-functors

\[
DT^{\mathbb{C}^*}(\mathcal{M}^v_S^+(v))_{dg} \leftarrow W_{dg} \to DT^{\mathbb{C}^*}(\mathcal{M}^v_S^-(v))_{dg}
\]

which induce equivalences on homotopy categories, i.e. equivalences of dg-categories in the convention of Subsection 1.6. By taking ind-completions, applying the inner homomorphism of dg-categories \( Rhom(\mathcal{C}, \mathcal{D}) \) and taking the subcategories of compact objects, we obtain an equivalence by Theorem 4.9

\[
(5.8)
\]

Therefore we obtain the equivalence (5.8).

**Remark 5.5.** In [Toda], we proved several other wall-crossing equivalences or fully-faithful functors between \( \mathbb{C}^* \)-equivariant DT categories under d-critical flops or d-critical flips. By the same argument of Theorem 5.4 we can also prove \( \mathbb{Z}/2 \)-periodic version of these equivalences or fully-faithful functors.

For example in [Toda, Theorem 5.5.5], we proved the existence of a fully-faithful functor under MNOP/PT wall-crossing for reduced curve classes. Namely let \( I_n(X, \beta) \) and \( P_n(X, \beta) \) be MNOP
and PT moduli spaces on the local surface X respectively (see \cite{Todb} Section 4.2). Similarly to Definition \ref{def:Z/2-per} we can define $\mathbb{Z}/2$-periodic MNOP and PT categories

$$\mathcal{DT}^{\mathbb{Z}/2}(I_n(X, \beta)), \mathcal{DT}^{\mathbb{Z}/2}(P_n(X, \beta))$$

following Definition \ref{def:Z/2-per}. Since $I_n(X, \beta) \to P_n(X, \beta)$ is a d-critical flip, we expect the existence of a fully-faithful functor

\begin{equation}
\mathcal{DT}^{\mathbb{Z}/2}(P_n(X, \beta)) \hookrightarrow \mathcal{DT}^{\mathbb{Z}/2}(I_n(X, \beta)).
\end{equation}

The argument of Theorem \ref{thm:Z/2-per} also proves the existence of a fully-faithful functor \eqref{eq:DT-per} when $\beta$ is a reduced curve class, up to idempotent completions.

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