Efficient quantum gate teleportation in higher dimensions

Nadish de Silva

Centre for Quantum Information and Foundations, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, UK

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The Clifford hierarchy is a nested sequence of sets of quantum gates critical to achieving fault-tolerant quantum computation. Diagonal gates of the Clifford hierarchy and ‘nearly diagonal’ semi-Clifford gates are particularly important: they admit efficient gate teleportation protocols that implement these gates with fewer ancillary quantum resources such as magic states. Despite the practical importance of these sets of gates, many questions about their structure remain open; this is especially true in the higher-dimensional qudit setting. Our contribution is to leverage the discrete Stone–von Neumann theorem and the symplectic formalism of qudit stabilizer theory towards extending the results of Zeng et al (2008) and Beigi & Shor (2010) to higher dimensions in a uniform manner. We further give a simple algorithm for recursively enumerating all gates of the Clifford hierarchy, a simple algorithm for recognizing and diagonalizing semi-Clifford gates, and a concise proof of the classification of the diagonal Clifford hierarchy gates due to Cui et al. (2016) for the single-qudit case. We generalize the efficient gate teleportation protocols of semi-Clifford gates to the qudit setting and prove that every third-level gate of one qudit (of any prime dimension) and of two qutrits can be implemented efficiently. Numerical evidence gathered via the aforementioned algorithms supports the conjecture that higher-level gates can be implemented efficiently.

1. Introduction

Quantum computers hold great promise as tools for solving problems beyond the capabilities of existing
classical devices. However, their realization in practice requires surmounting the challenges posed by the need for fine control over quantum systems. The need to protect highly sensitive quantum data against errors induced by environmental noise motivates the theories of quantum error correction and fault-tolerant quantum computation. The former enables the encoding of data into larger systems to allow the detection and correction of reasonable errors; the latter enables performing quantum gates on encoded data using potentially faulty hardware.

Quantum error correction is complicated by the fact that the no-cloning theorem forbids the copying of quantum data and, thus, naive redundancy-based schemes. Most common quantum error correction schemes are based on stabilizer codes [1] that are constructed from Pauli gates (definition 2.1). The related Clifford gates play important roles as they are sufficient for encoding and decoding stabilizer codes and admit simple fault-tolerant implementations on encoded data. However, they do not form a sufficiently rich set of gates to perform an arbitrary quantum computation. To achieve quantum universality, one requires the ability to fault-tolerantly perform at least one non-Clifford gate. (Basic introductions can be found in §4.5 of [2] and Chapter 10 of [3].)

Gottesman & Chuang [4], building on Shor [5], introduced quantum gate teleportation: a variation on standard quantum teleportation that enables performing certain non-Clifford gates on an input state once given access to an appropriate magic state. Magic states are quantum resources that can be prepared in advance of a computation. Gottesman and Chuang also introduced the Clifford hierarchy: sets of gates that admit fault-tolerant gate teleportation protocols. For a fixed number \( n \) of qubits, the Clifford hierarchy (definition 2.5) forms a nested sequence \( C_k^n \) of sets of \( n \)-qubit gates; the positive integer \( k \) denotes the level of the Clifford hierarchy. The Pauli and Clifford gates form the first and second levels, respectively. Gates of the third level can be implemented with magic states of \( 2^n \) qubits. The standard example of a one-qubit third-level gate is the \( T = \sqrt{Z} \) gate, which can be performed using the magic state \( (1/\sqrt{2})(I \otimes T)(|00\rangle + |11\rangle) \). Gates of higher levels can be implemented via a recursive procedure.

With the goal of reducing the resource costs of fault-tolerant quantum computation, Zhou et al. [6] introduced the one-bit teleportation protocol that enables the implementation of certain Clifford hierarchy gates, e.g. diagonal Clifford hierarchy gates, using half the ancillary resources required in the original protocol. The class of Clifford hierarchy gates admitting these efficient teleportation protocols was expanded by Zeng et al. [7] to include the semi-Clifford gates (definition 2.6). That is, those which are, in a sense, diagonalizable by Clifford gates. For example, using this efficient protocol, the \( T \) gate can be performed using the single-qubit magic state \( |T\rangle = |0\rangle + e^{i\pi/4}|1\rangle \). In a typical computation, these savings are multiplied by the many times such a gate is required.

The importance of the families of Clifford and semi-Clifford gates motivated the study of their structure [7–11]. A question of particular interest is: For which pairs \((n, k)\) are all \( n \)-qubit \( k \)th-level gates semi-Clifford?

- In the case of one- or two-qubit gates, all gates of the Clifford hierarchy are semi-Clifford [7].
- In the case of three-qubit gates, all gates of the third level of the Clifford hierarchy are semi-Clifford [7].
- In the case of \( n > 2, k > 3 \), there exist \( n \)-qubit \( k \)th-level gates that are not semi-Clifford [7].
- In the case of \( n > 3, k = 3 \), there exist \( n \)-qubit third-level gates that are not semi-Clifford (Gottesman–Mochon).

In this work, we will extend the study of semi-Clifford gates and their efficient gate teleportation protocols to the higher-dimensional qudit setting by raising the question: For which triples \((d, n, k)\), where \( d \) is a prime dimension, are all \( n \)-qudit \( k \)th-level gates semi-Clifford? Cui et al. [9] recently characterized the diagonal gates of the \( n \)-qudit Clifford hierarchy; no prior work exists on qudit semi-Clifford gates. We further give an algorithm for constructing all gates of the Clifford hierarchy.
The magic state distillation procedure for generating magic states is more efficient for qudits than in the qubit case [12]. It is possible that practically realizable quantum computers will one day be based on qudits rather than qubits. In any case, the mathematical and algorithmic techniques introduced below, as well as conjectures we propose with supporting numerical evidence, should benefit the wider project of elucidating the complete structure of the Clifford hierarchy and the semi-Clifford gates.

The unifying theme of the results below is the application of the discrete Stone–von Neumann theorem towards studying the Clifford hierarchy. The Stone–von Neumann theorem [13,14] asserts the essential equivalence of all representations of the fundamental quantum commutation relations and was originally motivated by the problem of unifying the matrix and wave mechanics pictures of early quantum theory [15]. We thus generalize techniques implicitly employed by Beigi & Shor [8] in the case of qubit third-level gates. We show that \( k \)th-level gates correspond to certain tuples of \( k - 1 \)th-level gates and argue that this perspective is useful in understanding all levels of the Clifford hierarchy in higher dimensions.

(a) Summary of main results

— A streamlined proof of the discrete Stone–von Neumann theorem (lemma 3.8) and a compact expression for the unitary that carries the basic Pauli gates to prescribed, admissible targets: any unitary representation of the Heisenberg group (theorem 3.9).
— A simple algorithm for recursively enumerating all gates of the Clifford hierarchy (Algorithm 1).
— A simplified statement and elementary proof of Cui et al.’s classification of diagonal gates of the Clifford hierarchy (theorem 4.6) in the single-qudit case.
— A generalization of the efficient gate teleportation protocol of Zhou et al. for qubit semi-Clifford gates to the qudit case (§5a).
— A novel strengthening of a characterization of semi-Clifford gates (theorem 5.4) and an algorithm for recognizing and diagonalizing semi-Clifford gates (Algorithm 2).
— A proof that all third-level gates of one-qudit (of any prime dimension) or two-qutrits are semi-Clifford (theorem 5.5) and numerical evidence suggesting that this extends to other \((d, n, k)\).

(b) Notation

We shall denote the imaginary unit by \( i \) to distinguish it from our use of \( i \) as an indexing variable.

Suppose \( d \) is prime, \( n \geq 1 \), and \( \omega = e^{i2\pi/d} \). The set \( \{1, \ldots, n\} \) is denoted by \([n]\). For \( \hat{z} \in \mathbb{Z}_d^n \), the ket \(|\hat{z}\rangle = |z_1 \cdots z_n\rangle\).

Definition 1.1. For any \( n \geq 1 \) and function \( f : \mathbb{Z}_d^n \to \mathbb{C} \), the diagonal matrix \( D[f] \in M_{dn}(\mathbb{C}) \) is defined by

\[
D[f]|\hat{z}\rangle = f(z)|\hat{z}\rangle.
\]

Identity matrices are denoted by \( I \) with dimensions given by context. The group of \( d^n \times d^n \) unitary complex matrices is denoted by \( U(d^n) \). Given \( n \) unitaries \( U_1, \ldots, U_n \) and a vector \( \vec{p} \) of \( n \) integers, we denote by \( U^{\vec{p}} \) the product \( U_1^{p_1} \cdots U_n^{p_n} \).

2. Mathematical background

Here, we summarize the necessary mathematical background. We first define the groups of Pauli gates and Clifford gates in prime dimension. We then define the Clifford hierarchy gates and semi-Clifford gates. Finally, we introduce the canonical commutation relations obeyed by conjugate position and momentum operators in the continuous case and the basic Pauli gates in the discrete case. The historical context of the Stone–von Neumann theorem as providing unitary equivalences between different representations of the canonical commutation relations is given; in
the next section, we understand Clifford hierarchy gates as examples of the unitaries witnessing these equivalences.

(a) Qubit and qudit Pauli gates

The Pauli gates are the most basic quantum gates used in quantum computing. They are of fundamental importance to quantum error correction as stabilizer codes work by encoding data as common eigenvectors of commuting sets of Pauli gates. In the single qubit or qudit case, the $Z$ and $X$ gates are defined in the standard basis $\{|z\}\forall z\in\mathbb{Z}_d$ of $\mathbb{C}^d$ by:

$$Z|z\rangle = \omega^z|z\rangle \quad X|z\rangle = |z + 1 \pmod{d}\rangle.$$

For $n > 1$ and $i \in [n]$, define $Z_i \in U(d^n)$ to be a tensor product of $n - 1$ identity matrices of size $d \times d$ with $Z$ in the $i$th factor: $\mathbb{I} \otimes \cdots \otimes Z \otimes \cdots \otimes \mathbb{I}; X_i$ is defined similarly. The $Z_i, X_i$ are the basic Pauli gates and satisfy the Weyl commutation relations

$$Z_i X_i = \omega X_i Z_i,$$

and, for $i \neq j$, $[Z_i, X_j] = [Z_i, Z_j] = [X_i, X_j] = 0$. As we shall see in §2e, any set of gates exhibiting these commutation relations is a unitary rotation of the set of basic Pauli gates; further, the group of unitaries (up to phase) is in bijection with the sets of gates exhibiting these commutation relations.

**Definition 2.1.** The group of Pauli gates is the subgroup of $U(d^n)$ generated by the basic Pauli gates and denoted

$$C^n_1 = \{\omega^c Z^\hat{p} X^\hat{q} \in \mathbb{Z}_d, (\hat{p}, \hat{q}) \in \mathbb{Z}_d^n\}.$$

We will later require the two following simple lemmas. The first indicates how $X_i$ gates can be commuted past diagonal matrices and the second characterizes diagonal matrices as those commuting with all $Z_i$.

**Lemma 2.2.** For $n \geq 1$ and a function $f : \mathbb{Z}_d^n \to \mathbb{C}$,

$$X_i D[f] = D[T_i f] X_i,$$

where $T_i f(z) = f(z_1, \ldots, z_i - 1, \ldots, z_n)$ denotes a translation in the $i$th component of $f$.

**Proof.** Both sides map $|\hat{z}\rangle$ to $f(z_1, \ldots, z_i - 1, \ldots, z_n)$.

**Lemma 2.3.** For $n \geq 1$, a matrix $M \in M_{d^n}(\mathbb{C})$ is diagonal if and only if it commutes with $Z_i$ for all $i \in [n]$.

**Proof.** If $M$ is diagonal, it commutes with the diagonal $Z_i$. Conversely, if $M$ commutes with each $Z_i$, it commutes with each rank-1 projector onto a standard basis vector $|\hat{z}\rangle = d^{-n} \sum_{\hat{p} \in \mathbb{Z}_d^n} \omega^{-\hat{p} \cdot \hat{z}} Z^\hat{p}$ and is therefore diagonal.

(b) The Clifford group and the symplectic phase space formalism

The Pauli gates form the basis of the error-correcting codes necessary for practical quantum computation. The set of Clifford gates can be performed fault-tolerantly on data encoded using such stabilizer codes.

**Definition 2.4.** The Clifford gates are those unitaries that preserve the group of Pauli gates under conjugation:

$$C^n_2 = \{G \in U(d^n) | GC^n_1 G^* \subseteq C^n_1\}.$$

Being the normalizer of a subgroup of the unitaries, the set of Clifford gates $C^n_2$ is a group. When $d$ is an odd prime, the Pauli and Clifford groups admit a rich phase space formalism in terms of a discrete symplectic vector space [16]. The phase space is $\mathbb{Z}_d^{2n}$ and a typical phase point is usually denoted $(\hat{p}, \hat{q})$. The phase space is equipped with a symplectic bilinear product
\[ [\cdot, \cdot] : \mathbb{Z}_d^{2n} \times \mathbb{Z}_d^{2n} \to \mathbb{Z}_d \] defined by \([(\hat{p}, \hat{q}), (\hat{p}', \hat{q}')] = \sum_{i=1}^{n} p_i q_i' - p'_i q_i.\] This notation can be distinguished from its use as the commutator of operators from context. To each phase point, we associate the Pauli gate \(W(\hat{p}, \hat{q}) = \omega^{-1} \hat{p}^T \hat{q} Z^d X^d\) where \(\omega^{-1}\) denotes the multiplicative inverse of \(2\) in \(\mathbb{Z}_d\). They obey the multiplication law: \(W(\hat{p}_1, \hat{q}_1)W(\hat{p}_2, \hat{q}_2) = \omega^{-2 \langle \hat{p}_1, \hat{d}_1 \rangle \langle \hat{p}_2, \hat{d}_2 \rangle} W(\hat{p}_1 + \hat{p}_2, \hat{q}_1 + \hat{q}_2).\)

A set of \(m\) Pauli gates \(\{\omega^e Z^d X^d\}_{e \in [m]}\) is independent if no non-trivial product of them is a multiple of the identity. This is equivalent to the set \(\{(\hat{p}_i, \hat{q}_i)\}_{i \in [m]}\) being a linearly independent subset of \(\mathbb{Z}_d^{2n}\).

The Clifford gates, up to phase, are in correspondence with affine symplectic linearizations of the phase space. First, we define the group of Clifffords up to phase: \([C_2^n] = C_2^n/T\). The group \(Sp(n, \mathbb{Z}_d) \times \mathbb{Z}_d^{2n}\) of affine symplectic transformations of \(\mathbb{Z}_d^{2n}\) are pairings of \(2n \times 2n\) symplectic matrices over \(\mathbb{Z}_d\) and translations in \(\mathbb{Z}_d^{2n}\) with the composition law: \((S, v) \circ (T, w) = (ST, Sw + v)\). There is a (Weil or metaplectic) projective representation \(\rho : Sp(n, \mathbb{Z}_d) \times \mathbb{Z}_d^{2n} \to [C_2^n]\) that is an isomorphism between the groups of affine symplectic transformations and Clifffords up to phase. In terms of the explicit description of the representation \(\mu : Sp(n, \mathbb{Z}_d) \to [C_2^n]\), given by Neuhauser ([17], Theorem 4.1), \(\rho(S, v) = [W(v)\mu(S)]\).

(c) The Clifford hierarchy

While Cliffford gates can be implemented fault-tolerantly, they are not a sufficiently rich gate set to perform arbitrary quantum computations. Motivated by the need to implement non-Clifford gates fault-tolerantly, Gottesman-Chuang introduced the Clifford hierarchy.

**Definition 2.5.** The Clifford hierarchy is an inductively defined sequence of sets of gates. For \(k > 1\), the \(k\)th level of the Clifford hierarchy is the set:

\[ C_k^n = \{ G \in U(d^n) \mid GC_k^n G^* \subseteq C_{k-1}^n \}. \]

The levels of the Clifford hierarchy are nested: \(C_k^n \subseteq C_{k+1}^n\). While the first two levels form groups, higher levels do not. However, the sets \(\{G \in C_k^n \mid G\) is diagonal\} do form groups. The sets \(C_k^n\) are closed under left or right multiplication by Cliffford gates: for \(k > 1\), \(C_k^n C_k^n C_k^n = C_k^n [7]\).

Strict third-level gates \(G \in C_3^n \setminus C_2^n\) can be fault-tolerantly implemented via quantum gate teleportation to achieve universality when given access to an appropriate resource magic state of \(2n\) qudits. The problem of implementing a non-Clifford gate is thus reduced to the problem of preparing a magic state; a task which can be done in advance of a computation. Higher-level gates can be implemented via a recursive procedure requiring additional ancillary resources.

(d) Semi-Clifford gates and one-bit teleportation

Zhou et al. [6] introduced a simplified gate teleportation protocol, based on Bennett–Gottesman’s one-bit teleportation, capable of fault-tolerantly implementing certain qubit Cliffford hierarchy gates using half the ancillary resources required in the original Gottesman–Chuang protocol. This class of gates includes the diagonal Cliffford hierarchy gates. Zeng et al. [7] introduced the notion of semi-Clifford gates that are ‘nearly diagonal’ in the sense of being within Cliffford corrections of diagonal gates:

**Definition 2.6.** A gate \(G \in U(d^n)\) is semi-Clifford if \(G = C_1 DC_2\), where \(C_1, C_2 \in C_2^n\) and \(D\) is diagonal.

**Definition 2.7.** For \(k \geq 1\) the \(k\)th-level semi-Clifford gates are as follows:

\[ SC_k^n = \{ G \in C_k^n \mid G = C_1 DC_2, \text{ where } C_1, C_2 \in C_2^n, D\) is diagonal\}.

Zeng et al. gave a protocol expanding the class of gates that can be fault-tolerantly implemented via one-bit gate teleportation to include the semi-Clifford gates. We generalize this protocol to the qudit case in §5a.
(e) The Heisenberg group and the Stone–von Neumann theorem

The basic Pauli gates $Z_i, X_i$ are finite-dimensional analogues of translations in momentum and position. To understand this and to give historical context and mathematical motivation for the techniques used below, we here review the Heisenberg group, the canonical commutation relations and their exponentiated form due to Weyl in the infinite-dimensional case.

The Heisenberg group is a group of square matrices over a base field whose elements are given by $2n + 1$ parameters.

**Definition 2.8.** For any $n \geq 1$ and field $\mathbb{F}$, the $n$-Heisenberg group $\mathbb{H}_n(\mathbb{F})$ is

$$
\begin{pmatrix}
1 & p_1 & \cdots & p_n & c \\
1 & & \ddots & & q_1 \\
\vdots & & \ddots & \ddots & \ddots \\
1 & & & \ddots & q_n \\
1 & & & & 1
\end{pmatrix} \in M_{n+2}(\mathbb{F}) | c \in \mathbb{F}^n, \hat{p} \in \mathbb{F}^n, \hat{q} \in \mathbb{F}^n
$$

The group of Pauli gates $C_n^q$ arises from a representation of the Heisenberg group over the finite field $\mathbb{Z}_d$: the matrix with entries described by $(c, \hat{p}, \hat{q}) \in \mathbb{Z}_d \times \mathbb{Z}_d^4 \times \mathbb{Z}_d^4$ gives the Pauli gate $a^c \hat{p}^{\hat{q}} X^i$. When the Heisenberg group is constructed over the field $\mathbb{R}$, however, its standard representation arises from translations in momentum and position.

The canonical commutation relations are the mathematical branch point at which quantum theory diverges from classical theory. For a system of one continuous degree of freedom, with $P$ and $Q$ representing momentum and position:

$$[P, Q] = i.$$

Systems with multiple degrees of freedom are represented by tuples $\{(P_i, Q_i)\}_{i=1}^n$ satisfying $[P_i, Q_i] = i$ and, for $i \neq j$, $[P_i, Q_j] = [P_i, P_j] = [Q_i, Q_j] = 0$. As noted by Weyl, these equations have no solutions with $P$ or $Q$ a bounded operator. To sidestep this technical issue, he introduced his exponentiated form. Let $U(s) = e^{isP}$ and $V(t) = e^{itQ}$ be two groups of unitaries indexed by the parameters $s, t \in \mathbb{R}$. They obey the Weyl relations:

$$U(s)V(t) = e^{isP}V(t)U(s),$$

and are similarly generalized to multiple degrees of freedom: for $i \neq j$, $[U_i(s), V_j(t)] = [U_i(s), U_j(t)] = [V_i(s), V_j(t)] = 0$.

The canonical commutation relations are instantiated both by Heisenberg’s infinite-dimensional matrices and by Schrödinger’s wave operators on $L^2(\mathbb{R})$ (which act on functions $f \in L^2(\mathbb{R})$ by $X(f)(x) = xf(x)$ and $P(f)(x) = i(\partial/\partial x)f(x)$, respectively). The motivation of the Stone–von Neumann theorem was to assure the equivalence of the matrix mechanics picture and the wave mechanics picture of quantum theory. It asserts that all manifestations of the Weyl relations are unitarily equivalent. The following modern statement of the theorem is found in e.g. [18].

**Theorem 2.9 (Stone–von Neumann [13,14], 1930).** Suppose $\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n$ are self-adjoint operators that satisfy the Weyl relations and act irreducibly on a Hilbert space $\mathcal{H}$, i.e. the only closed subspaces of $\mathcal{H}$ invariant under all $e^{iA}$ and $e^{iB}$ are trivial. There exists a unitary map $G : \mathcal{H} \rightarrow L^2(\mathbb{R})$, unique up to a phase, such that

$$G e^{iA}G^{-1} = e^{iX} \quad \text{and} \quad G e^{iB}G^{-1} = e^{iP}.$$
3. The discrete Stone–von Neumann theorem

As shown in the previous subsection, the Pauli gates Z and X of one qubit or qudit of prime dimension \( d \) are the discrete analogues of exponentiated forms of \( P \) and \( Q \); that is, they correspond to conjugate variables with respect to a Fourier transformation. They satisfy the algebraic relations: \( Z^d = X^d = 1 \) and \( ZX = \omega XZ \). Clearly, if we conjugate \( Z \) and \( X \) by any unitary gate \( G \), the gates \( U = GZG^\ast \) and \( V = GXG^\ast \) satisfy the same relations.

**Definition 3.1.** An ordered pair of unitaries \((U, V) \in U(d^2) \times U(d^2)\) is a conjugate pair if

1. \( U^d = 1 \) and \( V^d = 1 \),
2. \( UV = \omega VU \).

Remarkably, if \((U, V)\) is any pair of unitaries satisfying these relations, there exists an essentially unique gate \( G \) such that \( U = GZG^\ast \) and \( V = GXG^\ast \). As we will see, this is a consequence of the discrete Stone–von Neumann theorem and leads to a bijective correspondence between gates \( G \) (up to phase) and conjugate pairs \((U, V)\). The remainder of this paper demonstrates the utility of studying gates of the Clifford hierarchy and semi-Clifford gates via their action under conjugation on basic Pauli gates.

(a) Single qudits

**Lemma 3.2.** Suppose \( U, V \in U(d) \) satisfy \( UV = \omega VU \). Then, \( U^pV^q \) are traceless for \((p, q) \in \mathbb{Z}_d^2\) with \((p, q) \neq (0, 0)\).

**Proof.** If \( q \neq 0 \), \( \text{Tr}(U^pV^q) = \text{Tr}(U^{p-1}UV^q) = \omega^p\text{Tr}(U^{p-1}V^qU) = \omega^p\text{Tr}(U^pV^q) \). Since, \( \omega^q \neq 1 \), this expression vanishes. A similar argument holds if \( p \neq 0 \). \( \blacksquare \)

**Lemma 3.3.** Suppose \((U, V) \in U(d) \times U(d)\) is a conjugate pair. Then, the matrices \( \{U^pV^q | (p, q) \in \mathbb{Z}_d^2\} \) are orthogonal in \( M_d(\mathbb{C}) \) with the Hilbert–Schmidt inner product \( \langle A, B \rangle_{HS} = \text{Tr}(A^\ast B) \) and hence form a basis of \( M_d(\mathbb{C}) \).

**Proof.** \( \langle U^pV^q, U^p'V^q' \rangle_{HS} = \text{Tr}(V^{-q}U^{-p}U^pV^q) \propto \text{Tr}(U^pV^q) \) as all terms can be commuted freely by multiplying by a power of \( \omega \). This vanishes unless \( p_1 \equiv p_2 \) and \( q_1 \equiv q_2 \) (mod \( d \)). Since \( U, V \) are unitary, their products are non-zero. An orthogonal set of non-zero matrices is linearly independent. \( \blacksquare \)

**Lemma 3.4** (discrete Stone–von Neumann theorem, single-qudit version). Suppose \((U, V)\) and \((\tilde{U}, \tilde{V}) \in U(d) \times U(d)\) are two conjugate pairs. There is a unitary \( G \), unique up to phase, such that \( \tilde{U} = GUG^\ast \) and \( \tilde{V} = GVG^\ast \).

**Proof.** We define \( \phi(U) = \tilde{U} \) and \( \phi(V) = \tilde{V} \) and prove that this extends to a unique \( * \)-automorphism of \( M_d(\mathbb{C}) \). The \( * \)-automorphisms of simple matrix algebras are in correspondence with unitaries up to phase as a consequence of the Skolem–Noether theorem.

We first define \( \phi(U^pV^q) = \tilde{U}^p\tilde{V}^q \). As the matrices \( U^pV^q \) and \( \tilde{U}^p\tilde{V}^q \) form two bases by lemma 3.3, this is a well-defined vector space automorphism via its unique linear extension to all of \( M_d(\mathbb{C}) \). It is easy to check that it respects \( * \) and matrix multiplication. Thus, \( \phi \) is an inner \( * \)-automorphism induced by a unitary \( G \).

**Theorem 3.5.** The unitary \( G \) that carries \((Z, X)\) to \((U, V)\) under conjugation is given by

\[
G(z) = Vz|u_0\rangle,
\]

where \( |u_0\rangle = G|0\rangle \) is an eigenvector of \( U \) with eigenvalue 1.

**Proof.** Apply \( G \) to both sides of the equation: \( |z\rangle = X^2|0\rangle \). \( \blacksquare \)
(b) Multiple qudits

The next definition captures the commutation relations of the basic Pauli gates of $n$ qubits or qudits. Conjugate pairs are in correspondence with single-qudit gates (up to phase); conjugate tuples are in correspondence with multi-qudit gates (up to phase).

Definition 3.6. A conjugate tuple $(\{U_1, V_2\}, \ldots, \{U_n, V_n\}) \in (\mathcal{U}(d^n) \times \mathcal{U}(d^n))^n$ is a set of $n$ conjugate pairs such that elements of distinct pairs commute.

Essentially the same proofs work to establish the multi-qudit generalizations of the above results.

Lemma 3.7. Suppose $(\{U_i, V_i\})_{i \in [n]}$ is a conjugate tuple. The matrices $\{\hat{U} \hat{V} | (p, q) \in \mathbb{Z}_d^{2n}\}$ are orthogonal in $M_{d^n}(\mathbb{C})$ with the Hilbert–Schmidt inner product and hence form a basis of $M_{d^n}(\mathbb{C})$.

Lemma 3.8 (discrete Stone–von Neumann theorem). Suppose $(\{U_i, V_i\})_{i \in [n]}$ and $(\{\tilde{U}_i, \tilde{V}_i\})_{i \in [n]}$ are two conjugate tuples. There is a unitary $G$, unique up to phase, such that, for all $i \in [n]$, $\tilde{U}_i = GU_iG^*$ and $\tilde{V}_i = GV_iG^*$.

Theorem 3.9. The unitary $G$ that carries $(\{Z_i, X_i\})_{i \in [n]}$ to $(\{U_i, V_i\})_{i \in [n]}$ under conjugation is given by $G(\hat{z}) = \hat{V}^2 |u_0\rangle$, where $\hat{z} \in \mathbb{Z}_d^n$ and $|u_0\rangle = G|0\rangle$ is a simultaneous eigenvector of the $U_1, \ldots, U_n$ with eigenvalue 1.

The simplest way to compute $|u_0\rangle$ is as the eigenvector with eigenvalue 1 of the rank-1 projector $d^{-n} \sum_{p \in \mathbb{Z}_d^n} \hat{U}^p$.

(c) An algorithm for enumerating the Clifford hierarchy gates

We have shown a bijective correspondence between gates $G$ (up to phase) and conjugate tuples. The following definition is used to characterize those conjugate tuples corresponding to gates of the Clifford hierarchy.

Definition 3.10. A conjugate tuple $(\{U_i, V_i\})_{i \in [n]}$ is $k$-closed if it generates a group of $k$th-level gates. Equivalently:

$$\{\hat{U} \hat{V} | (p, q) \in \mathbb{Z}_d^{2n}\} \subseteq C_k^n.$$

Theorem 3.11. Gates of the $k + 1$th level of the Clifford hierarchy, up to phase, are in bijective correspondence with $k$-closed conjugate tuples.

Proof. The correspondence is given by the map that sends $G \in [C_{k+1}^n]$ to the tuple $(\{U_i = GZ_iG^*, V_i = GX_iG^*\})_{i \in [n]}$.

Conjugation by $G$ preserves the order of all matrices and the commutation relations between them, so, as $(\{Z_i, X_i\})_{i \in [n]}$ is a conjugate tuple, so is $(\{U_i, V_i\})_{i \in [n]}$. For any $(p, q) \in \mathbb{Z}_d^{2n}$, the product $\hat{U}^p \hat{V}^q = G(Z^pX^q)G^* \in C_k^n$ as it is the conjugation of a Pauli gate by a $C_k^n$ gate.

Conversely, given a $k$-closed conjugate tuple $(\{U_i, V_i\})_{i \in [n]}$, we can apply lemma 3.8 to find a unitary $G$, unique up to phase, such that $U_i = GZ_iG^*$, $V_i = GX_iG^*$. The definition of $k$-closedness ensures that the conjugation of any Pauli gate by $G$ is in $C_k^n$ and thus, that, $G \in C_{k+1}^n$.

Since the Clifford gates form a group, the condition of 2-closedness is fulfilled by any conjugate tuple of Clifford gates. Therefore:

Theorem 3.12. Gates of the third level of the Clifford hierarchy, up to phase, are in bijective correspondence with conjugate tuples of Clifford gates.

The question of whether the assumption of $k$-closedness in theorem 3.11 for $k > 2$ (those $k$ for which $C_k^n$ are not groups) is actually a necessary one remains open. That is, while $C_k^n$ may not be a group, does it contain a copy of the Heisenberg group whenever it contains its generators? Numerical investigations suggest that the assumption of $k$-closedness is not necessary for $(d, n, k)$
with $n = 1$. We are therefore led to the conjecture that all conjugate tuples of $k$th-level gates are $k$-closed. Equivalently:

**Conjecture 3.13.** Gates of the $k + 1$th level of the Clifford hierarchy, up to phase, are in bijective correspondence with conjugate tuples of $k$th-level gates.

We can use theorem 3.11 to describe a simple algorithm for recursively enumerating all gates of the Clifford hierarchy that works for any prime dimension and number of qudits.

**Algorithm 1.** Recursively enumerate the $C^n_k$ gates (up to phase)

1. Generate $[C^n_1]$: the $d^{2n}$ Pauli gates without phase.
2. For $k = 2$ to $\infty$:
   a. Select those elements of $[C^n_{k-1}]$ with order $d$ for some choice of phase.
   b. From all pairs of these elements, select the conjugate pairs.
   c. From $n$-tuples of conjugate pairs, select the conjugate tuples.
   d. From the conjugate tuples, select the $k$-closed conjugate tuples (see Conjecture 3.13).
   e. From the $k$-closed conjugate tuples, generate the $[C^n_k]$ gates using Theorem 3.11.

In step 2-a, the elements of $[C^n_{k-1}]$ we are interested in are those for which raising any representative to the $d$th power gives a diagonal matrix with constant diagonal element. From this constant, we can easily extract the choice of phase to correct that representative to give one of order $d$. Note that the list of $k$-closed conjugate tuples generated by step 2-d is not complete. That is because the conjugate pairs selected in step 2-b are selected from an enumeration of $[C^n_{k-1}]$ that ignores phase. An ordered pair $(U, V)$ is a conjugate pair if and only if, for any $p, q \in \mathbb{Z}_d$, $(\omega^p U, \omega^q V)$ is a conjugate pair. Therefore, in executing step 2-e, every $k$-closed conjugate tuple found by step 2-d generates $d^{2n}$ gates of $[C^n_k]$ by introducing an arbitrary choice of discrete phase factors into the elements of the conjugate tuple.

We computed $[C^n_1]$ in the $n = 1$ case for small $d$ and $k$. In the $d = 3$ case, the sizes found were as follows: 9, 216, 1944, 72680, 69336. In the $d = 5$ case, the sizes found were 25, 3000, 7500, 435000, 2235000. In the $d = 7$ case, the sizes found were: 49, 16464, 806736, 6338640.

In order to check $k$-closure in step 2-d, one must implement a function to determine whether a gate is in $C^n_k$. This can be defined recursively by conjugating all Pauli gates and checking if they are in $C^n_{k-1}$.

Deeper understanding of the structure of the Clifford hierarchy can lead to efficiency gains in the practical execution of this algorithm. For example, establishing Conjecture 3.13 would eliminate the need for step 2-d. A better grip on the lifting of the projective Weil representation to the ordinary representation could aid in optimizing 2-b.

4. **Diagonal gates of the Clifford hierarchy**

In this section, we give a concise, elementary proof of Cui et al.’s characterization of diagonal gates of the qudit Clifford hierarchy in the single-qudit case. Thus, $d$ is hereafter restricted to denoting an odd, prime dimension. For convenience, in this section, we drop superscripts $n$ indicating the number of qudits. For an integer $m \geq 1$, denote the $d^m$th primitive root of unity:

$$\omega_m = e^{2\pi i / d^m}.$$  

As we shall prove below, diagonal gates of the Clifford hierarchy are the gates of form $D[\omega_m^\phi]$ where $\phi : \mathbb{Z}_d \subseteq \mathbb{Z}_{d^m} \to \mathbb{Z}_{d^m}$ is a polynomial with coefficients in $\mathbb{Z}_{d^m}$. The levels of the Clifford hierarchy to which such a gate belongs is determined by $m$, the degree of $\phi$ and the coefficients of $\phi$ that are divisible by $d$. 
(a) Preliminary definitions

**Definition 4.1.** For \(k \geq 1\), the group of diagonal \(k\)th-level gates (up to phase) is

\[
\mathcal{D}_k = \{D \in \mathcal{C}_k | D \text{ is diagonal and } D(0) = |0\rangle\}.
\]

The second condition ensures that \(\mathcal{D}_k\) does not contain two gates that are related by a phase factor.

Any integer \(k \geq 1\) can be uniquely expressed as

\[
k = (m_k - 1)(d - 1) + a_k
\]

with \(a_k \in \{1, \ldots, d - 1\}\). We will suppress subscripts; call \(m \geq 1\) the precision of \(k\) and \(a\) the degree of \(k\).

**Definition 4.2.** Denote by \(\mathcal{R}_k\) the set of rank-\(k\) polynomials:

\[
\mathcal{R}_k = \left\{ \phi : \mathbb{Z}_{dm} \to \mathbb{Z}_{dm} \mid \phi \text{ has degree at most } d - 1, \phi(0) = 0 \right\}.
\]

Note that \(\mathcal{R}_k\) is an additive subgroup of the group of polynomials over \(\mathbb{Z}_{dm}\) and thus \(\mathcal{R}_k \cong \mathbb{Z}_{dm} \times \mathbb{Z}_{(d-1)a}\). Each copy of \(\mathbb{Z}_{dm}\) tracks the coefficient for the terms of degree \(1, \ldots, a\) while each copy of \(\mathbb{Z}_{(d-1)a}\) tracks the coefficients of the terms of degree \(a + 1, \ldots, d - 1\) after having divided out by a factor of \(d\). We can see immediately that \(|\mathcal{R}_k| = d^k\).

We will also require the notion of a polynomial \(\xi\) being of rank \(k\) up to a constant, i.e. there exists \(C \in \mathbb{Z}_{dm}\) such that \(\xi + C \in \mathcal{R}_k\). If \(\phi \in \mathcal{R}_k\) and \(\psi : \mathbb{Z}_{dm} \to \mathbb{Z}_{dm}\) is a polynomial of degree at most \(d - 1\), then \(\phi + d\psi\) is rank \(k\) up to a constant with \(C = -d\psi(0)\).

We now define the sets of gates that we will prove are the diagonal \(k\)th-level gates.

**Definition 4.3.**

\[
\Delta_k = \{D[\omega_m^Φ] | φ \in \mathcal{R}_k\}.
\]

Recall that we defined the construction of diagonal gates \(D[f]\) for \(f : \mathbb{Z}_d \to \mathbb{C}\) whereas the polynomials \(φ\) in the definition above take elements of \(\mathbb{Z}_{dm}\) as their inputs. Thus, when interpreting \(φ\) acting on \(\mathbb{Z}_d\), we are implicitly precomposing with the identity inclusion \(1 : \mathbb{Z}_d \hookrightarrow \mathbb{Z}_{dm}\) where \(1(z) = z\).

To appreciate the significance of this seemingly trivial point, consider the action of the translation operator \(T : \mathbb{C}^{\mathbb{Z}_d} \to \mathbb{C}^{\mathbb{Z}_d}\) defined by \(Tf(z) = f(z - 1 \mod d)\). To apply \(T^q\) to a polynomial \(φ\), and thus to \(\omega_m^Φ\), we can simply substitute each instance of \(z\) with \((T^q)(z)\). If this expression is interpreted over \(\mathbb{Z}_d\), this would be \(z - q\). However, as we are considering polynomials over \(\mathbb{Z}_{dm}\), we require a correction term in our substitution:

\[
z \mapsto (z - q) + d \chi_q(z),
\]

where \(\chi_q\) is the characteristic function of \([0, q]\). We have thus established:

**Lemma 4.4.** Suppose \(φ : \mathbb{Z}_{dm} \to \mathbb{Z}_{dm}\) and \(q \in \mathbb{Z}_d\). Then, \(T^q(\omega_m^Φ)(z) = \omega_m^{φ((z-q)+d \chi_q(z))}\).

(b) A simple proof of Cui et al.’s characterization of \(\mathcal{D}_k\)

**Lemma 4.5 (Zeng et al., 2008).** Suppose \(D_{k+1} \in \mathcal{D}_{k+1}\). There exist \(D_k \in \mathcal{D}_k\) and \(θ \in [0, 2π)\) such that

\[
D_{k+1} XD_{k+1}^* = e^{iθ} D_k X.
\]

**Proof.** Let \(D_k = D_{k+1} XD_{k+1}^* X^*\). It is diagonal as it commutes with \(Z\); the phase acquired as a \(Z\) passes through \(X\) is cancelled by the one acquired as it passes through \(X^*\). Thus, \(D_k ZD_k^* = Z \in \mathcal{C}_{k-1}\). Further, \(D_k XD_k^* = (D_{k+1} XD_{k+1}^*) X (D_{k+1} XD_{k+1}^*)^* \in \mathcal{C}_{k-1}\). \(D_k\) can be corrected by a phase factor to ensure that \(D_k(0) = |0\rangle\). ◻
Theorem 4.6 (Cui et al., 2016). \( \mathcal{D}_k = \Delta_k \) for all \( k \in \mathbb{Z}^+ \).

**Proof.** We proceed by induction on \( k \). The \( k = 1 \) case is immediate. So, let us assume that \( \mathcal{D}_k = \Delta_k \) and prove that \( \mathcal{D}_{k+1} = \Delta_{k+1} \). This will require two steps. First, we will count the elements of \( \mathcal{D}_{k+1} \) and find that \( |\mathcal{D}_{k+1}| = |\Delta_{k+1}| \). Then we will show that \( \Delta_{k+1} \subseteq \mathcal{D}_{k+1} \). As \( |\mathcal{D}_{k+1}| \) and \( |\Delta_{k+1}| \) are finite sets, this will complete our proof.

**Step 1:** \( |\mathcal{D}_{k+1}| = |\Delta_{k+1}| \). Recall that \( \mathcal{D}_{k+1} \subseteq \mathcal{D}_k \) is determined by its conjugate pair and, by the preceding lemma,

\[
(\mathcal{D}_{k+1} \mathcal{Z}^* D_{k+1}, \mathcal{D}_{k+1} \mathcal{X} D_{k+1}^*) = (Z, e^{i\theta} D_k X).
\]

There are \( d^k \) possible choices for \( D_k \) and as we shall now show, for each one, \( d \) possible choices of \( \theta \) such that \( e^{i\theta} D_k X \) has order \( d \). Suppose \( D_k = D[\omega_m^\phi] \) where \( m \) is the precision of \( k \). By repeatedly applying lemma 2.2,

\[
(D_k X)^d = \left( \prod_{j=0}^{d-1} D[T^j(\omega_m^\phi)] \right) X^d = D[\omega_m \sum_{j \in \mathbb{Z}_d^*} \phi(j)].
\]

Thus, \( e^{i\theta} = e^{i\theta_0} = \omega_m^{-\bar{\phi}} \omega^\alpha \) with \( \bar{\phi} = (1/d) \sum_j \phi(j) \), the average value of \( \phi \) over \( \mathbb{Z}_d \), and for any \( \alpha \in \mathbb{Z}_d^* \).

Each choice of \( (D_k, \theta_0) \) yields a distinct conjugate pair with \( Z \) and hence an element of \( \mathcal{D}_{k+1} \). This will follow from theorem 3.11 once we establish that \( (Z, e^{i\theta_0} D_k X) \) is \( k \)-closed: i.e. \( \mathbb{Z}^p (e^{i\theta_0} D_k X)^d \in \mathcal{C}_k \) for all \( p, q \in \mathbb{Z}_d^* \). As

\[
\mathbb{Z}^p (e^{i\theta_0} D_k X)^d = e^{i\theta_0} Z^p \left( \prod_{j=0}^{d-1} D[T^j(\omega_m^\phi)] \right) X^q
\]

and given that \( \mathcal{C}_k \) is closed under multiplication by Pauli gates and phase factors, it is sufficient to show that \( \prod_{j=0}^{d-1} D[T^j(\omega_m^\phi)] \in \mathcal{C}_k \). Each factor is in the group \( \{ G \in \mathcal{C}_k | G \) is diagonal\} as \( D[T^j(\omega_m^\phi)] = X^j D[\omega_m^\phi] X^{-j} \in \mathcal{C}_k \) and our conclusion follows.

Thus, \( |\mathcal{D}_{k+1}| = d^k \cdot d = d^{k+1} = |\mathcal{R}_{k+1}| = |\Delta_{k+1}| \).

**Step 2:** \( \Delta_{k+1} \subseteq \mathcal{D}_{k+1} \). For this step, \( m, a \) denote the precision and degree of \( k+1 \), not that of \( k \).

Suppose \( D[\omega_m^\phi] \in \Delta_{k+1} \) with \( \phi \in \mathcal{R}_{k+1} \), i.e. \( \phi(z) = \sum_{j=1}^{d-1} \phi_j z^j \) with \( \phi_{a+1}, \ldots, \phi_{a+1} \equiv 0 \) (mod \( d \)). We will show that \( D[\omega_m^\phi] \in \mathcal{D}_{k+1} \). It is sufficient to show that \( D[\omega_m^\phi](X^q Z^p) D[\omega_m^{-\phi}] \in \mathcal{C}_k \) for \( p, q \in \mathbb{Z}_d^* \) as every Pauli gate is of the form \( X^q Z^p \) up to phase. As, by applying lemma 2.2,

\[
D[\omega_m^\phi](X^q Z^p) D[\omega_m^{-\phi}] = D[\omega_m^\phi] X^q D[\omega_m^{-\phi}] Z^p = D[\omega_m^\phi] D[T^q(\omega_m^{-\phi})](X^q Z^p),
\]

it is sufficient to prove that \( D[\omega_m^\phi \cdot T^q(\omega_m^{-\phi})] \in \mathcal{C}_k \) for any \( q \in \mathbb{Z}_d^* \).

By lemma 4.4, \( D[\omega_m^\phi] = D[\omega_m^\phi \cdot T^q(\omega_m^{-\phi})] \) with

\[
\xi(z) = \sum_{j=1}^{d-1} \phi_j (z^j - (z - q) + d \chi_q(z)^j) = \sum_{j=1}^{d-1} \phi_j \left[ z^j - \sum_{\beta=0}^{j-1} \binom{j}{\beta} (z - q)^{j-\beta} d^{\beta-1} \chi_q(z)^\beta \right].
\]

We now separate out the \( \beta = 1 \) and higher terms of the latter inner sum and divide by their common factor of \( d \). Noting that a polynomial with \( d \) prescribed values can be constructed with degree at most \( d - 1 \), let \( \psi : \mathbb{Z}_d^m \to \mathbb{Z}_d^m \) be a polynomial of degree at most \( d - 1 \) that, on inputs in \( \mathbb{Z}_d^m \), coincides with the resulting expression:

\[
\psi|_{\mathbb{Z}_d^m}(z) = \sum_{j=1}^{d-1} \phi_j \left[ \sum_{\beta=1}^{j-1} \binom{j}{\beta} (z - q)^{j-\beta} d^{\beta-1} \chi_q(z)^\beta \right].
\]

We define

\[
\xi'(z) = \sum_{j=1}^{d-1} \phi_j [z^j - (z - q)^j] - d \psi(z)
\]
and note that $\xi'|_{Z_d}(z) = \xi|_{Z_d}(z)$ and thus that $D[\omega_m^\xi] = D[\omega_m^\xi]$. It is therefore sufficient for us to prove that $\xi'$ is a rank-$k$ polynomial up to a constant. By the Remark following definition 4.2, we may add $d\psi$ to $\xi'$ whilst preserving its rank up to a constant. We may similarly drop all terms for $j > a$ as these $\phi_j \equiv 0 \bmod d$. The coefficient for the $z^a$ term in the resulting expression,

$$\sum_{j=1}^a \phi_j (z^j - (z - q)^j),$$

vanishes, and so what remains has rank $k$ up to a constant. This implies that there exists $C \in Z_{d^m}$ such that $\xi' + C \in R_k$ and, thus, $D[\omega_m^\xi \cdot T^i(\omega_m^{-\phi})] = D[\omega_m^{\xi'}] \in C_k$ as $\omega_m^{-C} D[\omega_m^{\xi'}] \in D_k$. ■

5. Semi-Clifford operators and efficient gate teleportation

In this section, we first generalize the efficient gate teleportation protocol of Zhou et al. for qubit semi-Clifford gates to the qudit case. We then give an algorithm for recognizing semi-Clifford gates that, importantly, also yields Clifford gates diagonalizing them. Finally, we prove that all third-level gates of one-qudit (of any prime dimension) or two-qutrits are semi-Clifford. The proof for the two-qutrit case likely extends to higher dimensions.

(a) Efficient qudit gate teleportation

We will now construct a circuit gadget that implements a semi-Clifford third-level gate using half the ancillary quantum resources as required in the original gate teleportation protocol due to Gottesman–Chuang. It is a generalization to the qudit case of the qubit circuit due to Zhou et al. [6].

We consider now the one-qudit case. Suppose $G \in SC^3$, i.e. $G = C_1 DC_2$ for $C_1, C_2 \in C_2, D \in D_3$. The qudit generalization of the Hadamard gate is defined by $H|z\rangle = (1/\sqrt{d}) \sum_j \omega_z^j |j\rangle$. Given access to a magic state $|M\rangle = D|+\rangle$, we can perform $G$ on an input state $|\psi\rangle$ with the following circuit:

![Circuit Diagram]

Crucially, the elements of the gadget are Clifford operations and so can be implemented fault-tolerantly. Preparing the magic state can be done fault-tolerantly and with greater efficiency than in the qubit case [12].

Three key differences manifest only in the qudit case. First, the need for the $H^2$ gate with the action $H^2|z\rangle = |z - z (\bmod d)\rangle$, which is simply the identity in the qubit case. Second, the qubit CNOT gate is generalized to the CX (alternatively, CSUM) gate with the action $CX|z_1\rangle|z_2\rangle = |z_1\rangle|z_1 + z_2 (\bmod d)\rangle$. Finally, the need for $X^*$ in the final gate, which is simply $X$ in the qubit case.

The validity of this circuit is most easily demonstrated by first considering the single-qudit $X$-teleportation circuit:

![Single-Qudit X-Teleportation Circuit]

Consider the action of this circuit on an input state $|0\rangle \otimes |\psi\rangle = \sum_j \psi_j |0\rangle|j\rangle$. It is mapped by $H \otimes H^2$ to $(1/\sqrt{d}) \sum_j \psi_j |i\rangle|j - i\rangle$, which is then mapped by CX to $(1/\sqrt{d}) \sum_j \psi_j |i\rangle|i - j\rangle$. A measurement outcome of $\tilde{j}$ on the second qudit collapses the state to $\sum_j \psi_j |j + \tilde{j}\rangle|\tilde{j}\rangle$. Applying the classically controlled correction $X^{-\tilde{j}}$ to the first qudit and discarding the second qudit yields $|\psi\rangle$.

For any gate $G$ commuting with the control qudit of CX, we can apply $G$ at the end of the circuit to yield an output of $G|\psi\rangle$ and commute the gate $G$ backwards in time until it is absorbed.
into the stage of preparing a magic state:

\[
\text{MAGIC STATE: } |M\rangle = G |+\rangle
\]

\[
\begin{array}{c}
\begin{array}{c}
|0\rangle \\
|\psi\rangle
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
H
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
G
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
GX^*G^*
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
G |\psi\rangle
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
H^2
\end{array}
\end{array}
\end{array}
\]

This construction is particularly useful for \( G \in D_3 \) as \( GX^*G^* \) is guaranteed to be Clifford and diagonal gates commute with the control of a controlled gate. From this, we can generalize to implement \( G \in SC_3 \) and obtain our first circuit: teleport the state \( C_2 |\psi\rangle \) and apply \( C_1 D \) at the end of the circuit.

These arguments are straightforwardly parallelized for \( n \)-qudit gates. Zhou et al. show, in the qubit case, how a recursive construction can implement higher-level semi-Clifford gates with savings on ancillary resources.

(b) All \( C^1_3 \) gates admit efficient gate teleportation

Before proving the main theorem of this subsection (theorem 5.5), we will give an alternative characterization of semi-Clifford gates in terms of their action on basic Pauli gates. This characterization is a mild strengthening of proposition 5 of [7] that enables more efficient computations and simpler analytic proofs.

The following definition characterizes these subsets of phase points that correspond to the images (up to phase) of \( Z_1, \ldots, Z_n \) under conjugation by a Clifford.

**Definition 5.1.** A Lagrangian semibasis of a symplectic vector space of dimension \( 2n \) is a linearly independent set of \( n \) vectors \( \{ v_1, \ldots, v_n \} \) satisfying \( [v_i, v_j] = 0 \) for all \( i, j \in [n] \).

**Lemma 5.2.** For any Lagrangian semibasis \( \{(p_1, q_1), \ldots, (p_n, q_n)\} \subseteq \mathbb{Z}^{2n}_d \), there is a Clifford \( C \in C^n_d \) such that \( CZ_iC^* = W(p_i, q_i) \) for all \( i \in [n] \).

**Proof.** A Lagrangian semibasis can be extended to a symplectic basis of \( \mathbb{Z}^{2n}_d \). The Clifford \( C \) arising from the symplectic transformation that maps the standard basis to this symplectic basis yields the desired action.

**Lemma 5.3.** For any set \( \{P_1, \ldots, P_n\} \) of \( n \) independent, commuting Pauli gates, there is a Clifford \( C \in C^n_d \) such that \( CZ_iC^* = P_i \) for all \( i \in [n] \).

**Proof.** There exists a Lagrangian semibasis \( \{(\hat{p}_i, \hat{q}_i)\} \subseteq \mathbb{Z}^n_d \) and \( x_i \in \mathbb{Z}_d \) such that each Pauli \( \hat{P}_i = \omega^{x_i} W(\hat{p}_i, \hat{q}_i) \). Find \( C' \in C^n_d \) such that \( C'Z_iC'^* = W(\hat{p}_i, \hat{q}_i) \) using the previous lemma. Then, \( C = C' X_1^{x_1} \cdots X_n^{x_n} \) yields the desired action.

**Theorem 5.4.** Suppose \( G \in C^n_k \) and denote by \( U_i = GZ_iG^* \), \( V_i = GX_iG^* \) elements of \( C^n_{k-1} \). \( G \) is semi-Clifford if and only if there exists a Lagrangian semibasis \( \{(\hat{p}_i, \hat{q}_i)\} \subseteq \mathbb{Z}^{2n}_d \) such that, for each \( i \in [n] \), \( U_i^\dagger V_i^\dagger \) is a Pauli gate.

**Proof.** For notational simplicity, \( C^n_d, D^n_d, SC^n_k \) shall be denoted by \( C_k, D_k, SC_k \).

\[
G \in SC_k \iff (1) \exists C_1, C_2 \in C_k, D \in D_k \text{ s.t. } G = C_1 D C_2
\]

\[
\iff (2) \exists C_1, C_2 \in C_k \text{ s.t. } C_1 G C_2 = D_k
\]

\[
\iff (3) \exists C_1, C_2 \in C_k \text{ s.t. } \forall i \in [n], C_1 G C_2 Z_i C_2^* G^* C_1^* = Z_i
\]

\[
\iff (4) \exists C_2 \in C_k \text{ s.t. } \{G C_2 Z_i C_2^* G^*\}_{i \in [n]} \text{ are } n \text{ independent, commuting Pauli gates}
\]

\[
\iff (5) \exists \text{ a Lagrangian semibasis } \{(\hat{p}_i, \hat{q}_i)\}_{i \in [n]} \subseteq \mathbb{Z}^{2n}_d \text{ s.t. } \forall i \in [n], U_i^\dagger V_i^\dagger \text{ is a Pauli gate.}
\]
The first two equivalences are straightforward and the third equivalence follows from lemma 2.3: the fact that a matrix is diagonal if and only if it commutes with $Z_i$ for all $i \in [n]$.

For the direction $\Rightarrow$, we note that as the $Z_i$ are independent and commuting Paulis, so are $C^*_1 Z_i C_1$. The converse follows by applying lemma 5.3 to construct $C^*_1$ and hence $C_1$.

For the direction $\Rightarrow$, the Lagrangian semibasis arises from the images $C_2 Z_i C^*_2 = \omega^W(\hat{p}_i, \hat{q}_i)$. Conversely, we again apply lemma 5.3 to construct $C^*_1$ and hence $C_1$.

In addition to providing a useful technical characterization, employed below, it can be used to algorithmically find the Cliffords that diagonalize a given semi-Clifford gate in a relatively efficient manner.

Algorithm 2. Recognize and diagonalize semi-Clifford gates

1. Check if the given gate $G$ is in $C^n_k$; terminate if it is not. Otherwise, store $U_i = GZ_i G^*$, $V_i = GX_i G^*$.
2. For each Lagrangian subspace $L$ of $Z^n_d$:
   a. Choose a Lagrangian semibasis $\{(\hat{p}_i, \hat{q}_i)\}_{i \in [n]}$ of $L$.
   b. For each $i \in [n]$
      i. Check if $U_i \hat{p}_i V_i \hat{q}_i$ is a Pauli gate; return to step 2 if it is not.
   c. If a Lagrangian semibasis satisfies the criterion of Theorem 5.4 store it and go to step 4.
3. If no Lagrangian semibasis satisfies the criterion of Theorem 5.4, terminate.
4. Construct $C_2$ as a Clifford satisfying $C_2 Z_i C^*_2 = \hat{Z} \hat{X}$ using Lemma 5.3.
5. Construct $C^*_1$ as a Clifford satisfying $C^*_1 Z_i C_1 = U \hat{p} \hat{q}$ using Lemma 5.3.
6. Return $C_1$ and $C_2$.

A simple way of generating an exhaustive list of the Lagrangian semibases needed for step 2-a, is to first construct the list of vectors in $Z^n_d$ with the leading non-zero component equal to 1 (i.e. discard non-zero scalar multiples) and to select from this list those subsets of size $n$ with pairwise vanishing symplectic product.

Theorem 5.5. Every third-level gate of one qudit (of any prime dimension) is semi-Clifford: $\text{SC}_1^3 = C_1^3$.

Proof. Suppose $G \in C_3^1$ is a one-qudit third-level gate and let $U = GZ G^*$, $V = GX G^*$ be its conjugate pair of Cliffords. Let $(S, v), (T, w) \in \text{Sp}(1, Z_d) \times Z_2^d$ be such that $\rho(S, v) = [U]$ and $\rho(T, w) = [V]$. As $UV = \omega^W U$, it follows that $[U][V] = [V][U]$ and thus that $S, T$ commute in $\text{Sp}(1, Z_d)$. We can thus define a group homomorphism:

$$\phi_G : Z_2^d \rightarrow \text{Sp}(1, Z_d) : (p, q) \mapsto S^p T^q.$$

The order of $\text{Sp}(1, Z_d)$ is $d(d^2 - 1)$ and, as this is not divisible by $|Z_2^d| = d^2$, $\phi_G$ cannot be injective. Therefore, there exists a non-zero vector $(p, q)$, i.e. a Lagrangian semibasis for $Z^n_d$ such that $S^p T^q$ is the identity from which we can conclude that $U \hat{p} \hat{q}$ is a Pauli gate. By the previous theorem, $G \in \text{SC}_3^1$.

Numerical evidence supports the conjecture that every $k$th-level gate of one qudit (of any prime dimension) is semi-Clifford. Indeed, we will suggest an even bolder conjecture (see Conjecture 5.10 below).
(c) All two-qutrit $C_3^2$ gates admit efficient gate teleportation

We now show that all third-level gates of two-qutrits are semi-Clifford. First, we establish general lemmas enabling reduction to the case of gates whose conjugate tuples consist of Cliffords of the form $DX^\alpha$ for $D \in D_3^2$, $\alpha \in Z_d^2$ up to phase.

Denote by $Q \subseteq Sp(n, Z_d)$ the abelian subgroup of symplectic matrices of the form $\left( \begin{array}{cc} I_n & 0 \\ 0 & I_n \end{array} \right)$ for $n \times n$ symmetric matrices $b$; $|Q| = d^n(n+1)/2$. Under the explicit projective representation of $Sp(n, Z_d)$ of [17, Theorem 4.1], the image of $Q$ is a subgroup of diagonal Cliffords; each one is of the form $D[\omega^\phi]$ for $\phi : Z_d^n \to Z_d$, the homogeneous quadratic polynomial $\hat{z} \mapsto -2^{-1}z^Tb\hat{z}$.

By the next lemma, any commuting set of order $d$ symplectic matrices over $Z_d$ can be ‘simultaneously diagonalized’.

Lemma 5.6. Suppose that $B \subseteq Sp(n, Z_d)$ is a set of commuting symplectic matrices such that $S^d = [I]$ for all $S \in B$. There exists $R \in Sp(n, Z_d)$ such that $R(B)R^{-1} \subseteq Q$.

Proof. The subgroup $\langle B \rangle$ generated by $B$ is either trivial (in which case, the lemma follows immediately) or it is a $d$-subgroup of $Sp(n, Z_d)$. Therefore, $\langle B \rangle \subseteq A \subseteq J$, where $J$ is a Sylow $d$-subgroup of $Sp(n, Z_d)$ and $A$ is a maximal abelian subgroup of $J$.

By theorem 2.5 of [20], the Sylow $d$-subgroups of $Sp(n, Z_d)$ contain a unique maximal abelian subgroup of order $d^n(n+1)/2$. Note that every non-identity element of $Q$ has order $d$ and so $Q$ is contained in a Sylow $d$-subgroup $K$ of $Sp(n, Z_d)$. As all Sylow $d$-subgroups are conjugate, there exists $R \in Sp(n, Z_d)$ such that $R(B)R^{-1} = K$.

As $Q$ has order $d^n(n+1)/2$, it is the maximal abelian subgroup of $K$. Conjugation by $R$ must carry the unique maximal abelian subgroup of $J$ to that of $K$ and so $R(B)R^{-1} \subseteq Q$ implies the lemma.

The next lemma shows that an arbitrary $n$-qudit third-level gate can be conjugated by a Clifford to construct an $n$-qudit third-level gate whose conjugate tuple is of special form: diagonal Cliffords multiplied by a Pauli. Thus, the former is semi-Clifford if the latter is.

Lemma 5.7. Suppose that $G \in C_3^d$ is a third-level gate with the corresponding conjugate tuple $\{(U_i, V_i)\}_{i \in [n]}$ of Clifford gates. Suppose further that $(S_1, v_1), (T_1, w_1) \in Sp(n, Z_d) \times Z_d^{2n}$ are such that $\rho(S_1, v_1) = [U_1]$ and $\rho(T_1, w_1) = [V_1]$ as elements of $[C_3^n]$. There exists a Clifford gate $C \in C_3^n$ such that $CGC^*$ has a conjugate tuple $\{([\omega^x D_1^x X_1^\alpha, \omega^y E_1^y X_1^\beta])\}_{i \in [n]}$ where $D_1, E_1 \in D_3^2; x_i, y_i \in Z_d; \alpha_i, \beta_i \in Z_d^n$.

Proof. Apply the preceding lemma to $B = \{S_1, T_1, \ldots, S_n, T_n\}$ and take $C = \rho(R, 0)$ (with any phase).

The situation where the conjugate tuples are diagonal Cliffords is easier to tackle as we can characterize them in a simpler combinatorial fashion. We can describe diagonal Cliffords and Paulis using parameters in $Z_d$ and characterize the condition of a set of such gates forming a conjugate tuple as these parameters satisfying certain polynomial equations.

Lemma 5.8. Suppose $U = \omega^x DX^\alpha$, $V = \omega^y EX^\beta$, for $D, E \in D_3^2; x, y \in Z_d$; and $\alpha, \beta \in Z_d^2$. Suppose further that $D = D[\omega^\phi]Z^\delta$ and $E = E[\omega^\psi]Z^\hat{\delta}$ where $\phi(\hat{z}) = d_1\zeta_1^2 + d_2\zeta_2^2 + d_3z_1z_2$; $\psi(\hat{z}) = e_1\zeta_1^2 + e_2\zeta_2^2 + e_3z_1z_2$; and $\hat{\alpha}, \hat{\beta} \in Z_d^2$.

Then, $UV = \omega^c VU$ for $c \in Z_d$ if and only if, modulo $d$:

\[
\begin{align*}
2d_1\beta_1 + d_3\beta_2 - 2c_1\alpha_1 - e_3\alpha_2 & = 0 \\
d_3\beta_1 + 2d_2\beta_2 - e_3\alpha_1 - 2e_2\alpha_2 & = 0 \\
\psi(\hat{\alpha}) + \hat{\alpha} \cdot \hat{\beta} - \phi(\hat{\beta}) \cdot \hat{\beta} & \equiv c.
\end{align*}
\]

Proof. Simplify the expression $UVU^*V^* = \omega^c I$ by commuting $X_1, X_2$ terms to the right by repeatedly applying lemma 2.2. The effect of translating the polynomials defining $U$ and $V$ is to multiply by a Pauli correction; the above equations ensure that this correction is simply the desired phase factor $\omega^c$. 

These lemmas can be applied towards computationally verifying that each two-qutrit third-level gate is semi-Clifford. Computations would be intractable without them. We then indicate how the proof might be analytically generalized to higher dimensions.

**Theorem 5.9.** Every third-level gate of two qutrits is semi-Clifford.

*Proof.* Let $G$ be a two-qutrit third-level gate. By lemma 5.7, we can assume without loss of generality that $G$ has a conjugate tuple of the form $((\omega^{x_1} D_1 X^{\hat{t}_1}, \omega^{y_1} E_1 X^{\hat{b}_1}), (\omega^{x_2} D_2 X^{\hat{t}_2}, \omega^{y_2} E_2 X^{\hat{b}_2}))$, where $D_i, E_i \in D_3^2$, $x_i, y_i \in \mathbb{Z}_d$, and $\hat{a}_i, \hat{b}_i \in \mathbb{Z}_d^2$. We can ignore the discrete phase factors (integer powers of $\omega$) as doing so results in another conjugate tuple whose corresponding gate is, by theorem 5.4, semi-Clifford if and only if the original one is.

Thus, $U = GZ_1 G^*$, $V = GX_1 G^*$, $S = GZ_2 G^*$, $T = GX_2 G^*$ are characterized by four septuples of elements of $\mathbb{Z}_d$ that satisfy the 18 equations of lemma 5.8 describing the commutation relations $UV = \omega VU$, $ST = \omega TS$, $US = SU$, $UT = TU$, $VS = SV$, $VT = TV$. One can exhaustively compute all such quadruples of septuples by first computing the conjugate pairs and then by finding the pairs of these which give conjugate tuples; we find there to be 4 199 049 such conjugate tuples.

One can then apply theorem 5.4 to verify that each conjugate tuple arises from a semi-Clifford gate. It is sufficient to verify that the kernel of the matrix

$$
\begin{pmatrix}
u_1 & s_1 & t_1 \\
u_2 & s_2 & t_2 \\
u_3 & s_3 & t_3
\end{pmatrix}
$$

contains a Lagrangian semibasis, where $u_i, v_i, s_i, t_i$ are the coefficients of the homogeneous quadratic polynomial of $U, V, S, T$ respectively. This is because for any Pauli $P_1$ and homogeneous quadratic $\phi$, there is a Pauli $P_2$ such that $P_1 D[\omega^\phi] = D[\omega^\phi] P_2$.

One path to generalizing this result to higher dimensions would be to analytically derive from the 18 equations of lemma 5.8 characterizing conjugate tuples the existence of a Lagrangian semibasis in the kernel of the above matrix. It thus seems very likely to be true that every third-level gate of two qudits (of any prime dimension) is semi-Clifford. In analogy with the result of Zeng et al. that all two-qubit gates of any level are semi-Clifford, one might even conjecture that every third-level gate of two qudits (of any prime dimension) is semi-Clifford.

**Conjecture 5.10.** Every $k$th-level gate of one or two qudits (of any prime dimension) is semi-Clifford.

### 6. Conclusions and open problems

Understanding the structure of the Clifford hierarchy and the semi-Clifford gates, i.e. those admitting efficient implementation via the one-bit gate teleportation protocol described above, in the qudit case is essential for bolstering the viability of qudit fault-tolerant quantum computation.

We have developed a perspective on studying the qudit Clifford hierarchy via the discrete Stone–von Neumann theorem. This focus on studying Clifford gates via their actions by conjugation on basic Pauli gates, first employed by Beigi & Shor in the qubit third-level case, is fruitfully extended to the widest possible generality.

Technically, this perspective enables a simple proof of Cui et al.’s classification of diagonal Clifford hierarchy gates (in the single-qudit case), which raises the question: Might it more easily admit generalization to a classification of all Clifford hierarchy gates? It further enables a novel characterization of semi-Clifford gates that serves as the basis for proving that all third-level gates of one-qudit and two-qutrits are semi-Clifford.

These technical developments lead to simple algorithms for recursively enumerating all members of the Clifford hierarchy that works for any $(d, n, k)$ and for recognizing and diagonalizing semi-Clifford gates.
We have employed these algorithms to find numerical evidence that supports a number of conjectures. Establishing these conjectures promises to aid efforts to fully classify the Clifford hierarchy.

**Conjecture 1.** Gates of the $k+1$th level of the Clifford hierarchy, up to phase, are in bijective correspondence with conjugate tuples of $k$th-level gates.

**Conjecture 2.** Every $k$th-level gate of one or two qudits (of any prime dimension) is semi-Clifford.

While these conjectures may not hold for all $(d,n,k)$, they do hold for some such triples and, so, may therefore be interpreted as questions: For which $(d,n,k)$ do they hold?

A future direction of research is to give a complete classification of the Clifford hierarchy. One might begin by classifying the third-level gates which, by theorem 3.12, correspond to conjugate tuples of Clifford gates. Using the explicit metaplectic representation of Neuhauser [17] could be useful in characterizing these tuples.

**Data accessibility.** Code used for numerical calculations is accessible at https://github.com/ndesilva/cliffordhierarchy

**Competing interests.** We declare we have no competing interests.

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