BIANCHI COSMOLOGICAL MODELS IN THE MINIMUM QUADRATIC POINCARE GAUGE THEORY OF GRAVITY

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Abstract

Within the framework of the minimum quadratic Poincare gauge theory of gravity in the Riemann-Cartan spacetime the dynamics of homogeneous anisotropic Bianchi types I-IX spinning-fluid cosmological models is investigated. A basic equation set for these models is obtained and analyzed. In particular, exact solutions for the Bianchi type-I spinning-fluid and Bianchi type-V perfect-fluid models are found in analytic form.

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I. INTRODUCTION

As is well known in the modern theories of fundamental physical interactions the principle of local gauge symmetry takes the most important position. On the basis of this principle some unified gauge theories - the Weinberg-Salam theory, quantum chromodynamics, Grand Unified theories, etc. - that successfully describe electromagnetic, weak, and strong interactions of elementary particles, are constructed and carefully studied. Therefore, it seems natural and attractive to construct the theory of gravity as the gauge theory of spacetime symmetries and to study its consequences. During the last 35 years various versions of gauge theory of gravity have been suggested and investigated. These theories differ one from another by the choice of the gauge group (Lorentz, Poincare, de Sitter, metric-affine, etc.) and gravitational Lagrangian (linear, quadratic ...). However, the importance of the Poincare symmetry in particle physics lead one to consider the Poincare gauge theory (PGT) (see [1-5]) as the most suitable for description of the gravitational interaction.

In the PGT the gravity is described by two set of gauge fields: the tetrad fields \( h^i_\mu \), and the Lorentz connection coefficients \( A^{ik}_\nu \), the spacetime continuum is represented by a four-dimensional differentiable Riemann-Cartan manifold \( U_4 \).

The corresponding gauge field strength tensors are the torsion:

\[
S^i_{\mu\nu} = \partial_\nu h^i_\mu - h_k_{[\mu} A^{ik}_{\nu]},
\]

and the curvature:

\[
F^{ik}_{\mu\nu} = 2\partial_{[\mu} A^{ik}_{\nu]} + 2A^{ij}_{[\mu} A^k_{j|\nu]}.
\]

The sources of the gravitational field in this theory are the canonical energy-momentum tensor (EMT) \( t^\mu_i \), and the tetrad spin current (TSC) \( J_{ik}^\nu \).

Investigation of gravitating systems in the approximation of homogeneous isotropic spaces shows, that by including in the gravitational Lagrangian the terms quadratic in the curvature and torsion tensors the PGT allows to prevent the appearance of the gravitational singularity with infinite energy density [3, 5-8]. In connection with avoiding the cosmological
singularity problem, the study of more general homogeneous anisotropic models in the PGT is of certain interest.

In this article we shall study the dynamics of homogeneous anisotropic models within the framework of the so-called minimum quadratic (Poincare) gauge theory of gravity (MQGT) - the simple version of the PGT with the following gravitational Lagrangian:

$$L_g = h(-\chi + f_0 F + \alpha F^2), \quad F_{\mu\nu} = F^\lambda_{\mu\lambda\nu}, \quad F = F^\mu_{\mu}, \quad (1)$$

where $h = det h^i_\mu$, $F$ is the scalar curvature of the Riemann-Cartan spacetime $U_4$, $f_0 = (16\pi G)^{-1}$, (G is the Newton’s gravitational constant), $\chi$ is a cosmological term, and $\alpha < 0$ is a dimensionless coefficient (in the sequel we use all the notations of [9-12] unless otherwise stated). As was shown in [5, 9] the MQGT satisfies all restrictions following from the simultaneous consideration of the quantization problem (the theory without ghosts and tachyons) [13], Birkhoff’s theorem [14], and the problem of cancelling the metric singularity in the homogeneous isotropic cosmological models [6]. Satisfying the correspondence principle with General Relativity (GR) for the systems with rather small energy density, the given theory gives a chance to overcome some difficulties of GR, which appear on the description of gravitating systems under extremal conditions (extremely high energy density, spinning matter...).

The paper is organized as follows. In section 2, for spinning-fluid systems we obtain the gravitational equations of MQGT in the synchronous frame of references. In section 3, we derive the equations of motion for Bianchi models in the MQGT. In section 4 and 5 we find analytic solutions of the obtained basic equation set in the simple cases: Bianchi spinning-fluid type-I and Bianchi perfect-fluid type-V models, respectively. Some interesting properties of the found solutions are analyzed in section 6.
II. MQGT IN THE SYNCHRONOUS FRAME OF REFERENCE.

By independent variation of Lagrangian (1) with respect to gauge fields $h^i_{\mu}$, and $A^i_{ik\nu}$ we obtain two field equations of MQGT in the form:

$$2(f_0 + 2\alpha F)F_{\mu i} - (f_0 + \alpha F)Fh_{\mu i} + \chi h_{\mu i} = -t_{\mu i},$$

$$2\nabla_\nu[(f_0 + 2\alpha F)h_{[\mu}h_{\nu]i}] = -J_{ik\nu}.$$  (2)

It follows from (2) that $F = \frac{1}{2f_0}(t^\mu_{\mu} + 4\chi)$. Then, it is easily to solve eq. (3):

$$S^{\lambda}_{\mu\nu} = (1 - 4\beta\chi - \beta t^\sigma_{\sigma})^{-1}\left(-\frac{1}{2f_0}(J_{\mu\nu}^{\lambda} + \delta^\lambda_{[\mu}J_{\nu]\sigma^\sigma}) + \frac{\beta}{2}\delta^\lambda_{[\mu}\partial_{\nu]}t^\sigma_{\sigma}\right).$$  (4)

By virtue of (4), after some simple transformations (see Ref. [9]) we can represent (2) in the form of the Einstein’s equation with the effective EMT $T_{\mu\nu}^{\lambda\mu}$ constructed from EMT $t^\mu_{\nu}$ and TSC $J_{ik\nu}$ as:

$$G^{\lambda\mu}(\{\}) = -\frac{1}{2f_0}T_{\mu\nu}^{\lambda\mu},$$

(5)

where $G^{\lambda\mu}(\{\})$ is the ordinary Einstein’s tensor and

$$T_{\mu\nu}^{\lambda\mu} = (1 - 4\beta\chi - \beta t^\sigma_{\sigma})^{-1}\left[T^{\lambda\mu} - \frac{\beta}{4}g^{\lambda\mu}(t^\sigma_{\sigma} + 4\chi)^2 + \chi g^{\lambda\mu} + \frac{1}{2f_0}(1 - 4\beta\chi - \beta t^\sigma_{\sigma})^{-1}\left[-4J^{\lambda\sigma}_{\rho\sigma}(J^{\mu\rho}_{\nu\sigma} - J^{\mu\rho}_{\sigma\nu}J^{\lambda\sigma}_{\rho\sigma}) + \frac{1}{2}g^{\lambda\mu}(4J^{\nu\rho}_{\sigma\rho} + J^{\mu\rho}_{\sigma\rho}J^{\lambda\sigma}_{\rho\sigma})\right] + 2f_0\left[\tilde{\nabla}^\mu\partial^\lambda M - \frac{1}{2}\partial^\mu\partial^\lambda M - g^{\lambda\mu}(\tilde{\nabla}_{\sigma}\partial^\sigma M + \frac{1}{4}\partial^\mu M\partial^\sigma M)\right]\right].$$

(6)

where $M = \ln[1 - 4\beta\chi - \beta t^\sigma_{\sigma}]$, $\beta = -\frac{\alpha}{f_0} > 0$, $T^{\lambda\mu} = t^{\lambda\mu} + (\nabla_\nu + 2S_\nu)(J^{\nu\rho}_{\sigma\rho} + J^{\nu\rho}_{\sigma\rho}J^{\lambda\sigma}_{\rho\sigma})$, $S_\nu = S^\sigma_{\nu\sigma}$, $\nabla_\nu$ and $\tilde{\nabla}_\nu$ denote covariant derivatives calculated by means of the total connection $\Gamma^{\lambda\mu\nu}$ and the Christoffel symbols $\gamma^{\lambda\mu\nu}$ respectively.

As the source of gravitational fields we consider the perfect fluid with angular spin momentum (spinning fluid). In [15-18] a variational formalism for relativistic dynamics of spinning fluid was developed in the Riemann-Cartan spacetime. In the simplest spinning fluid model the canonical EMT $t^{\lambda\mu}$ and TSC $J^{\lambda\mu\nu}$ take the following form:

$$t^{\lambda\mu} = (\rho + p)u^{\lambda}\nu^{\mu} - pg^{\lambda\mu} + j^{\lambda\nu}_{\sigma}u^{\mu}\nu\frac{\partial}{\partial u^{\nu}}u^{\sigma},$$

(7)
\[ J_{\mu\nu}^\lambda = \frac{1}{2}j_{\mu\nu}u^\lambda, \quad (8) \]

where \( \rho \) is the invariant energy density, \( p \) is the pressure, \( u^\mu \) is the 4-velocity vector, \( j_{\mu
u} = j_{\nu\mu} \) is the spin density tensor. By using the relations: \( j_{\mu\nu}u^\nu = 0 \) from (7) we get \( t^\sigma_\sigma = \rho - 3p \). Then, the torsion tensor (4) can be rewritten as

\[ S^\lambda_{\mu\nu} = (1 - 4\beta\chi - \beta t^\sigma_\sigma)^{-1} \left( - \frac{1}{4f_0} j_{\mu\nu}u^\lambda + \frac{\beta}{2} \delta^\lambda_{[\mu} \partial_{\nu]} t^\sigma_\sigma \right). \quad (9) \]

In view of (7)-(9) and the equations of rotational motion (see [15,16]) the effective EMT \( T^\lambda_{\mu\nu} \) can be transformed to the form:

\[ T^\lambda_{\ef\mu\nu} = (1 - 4\beta\chi - \beta t^\sigma_\sigma)^{-1} \left\{ \left[ (\rho + p - \frac{j^2}{4f_0}(1 - 4\beta\chi - \beta t^\sigma_\sigma)^{-1})u^\lambda u^\mu - g^\lambda\mu \right] (p - \chi + \frac{\beta}{4} (t^\sigma_\sigma + 4\chi)^2 - \frac{j^2}{8f_0}(1 - 4\beta\chi - \beta t^\sigma_\sigma)^{-1}) + j^\nu(\mu u^\mu) \partial_\nu M - u^\nu u^\tau (\mu \tilde{\nabla}_\nu j^\lambda)^\tau - \tilde{\nabla}_\nu (j^\nu(\lambda u^\mu)) \right\} + 2f_0 \left[ \tilde{\nabla}_\nu \tilde{\nabla}^\lambda M - \frac{1}{2} \partial^\mu M \partial^{\lambda} M - g^\lambda\mu (\tilde{\nabla}_\nu \partial^\nu M + \frac{1}{4} \partial_\nu M \partial^{\nu} M) \right], \quad (10) \]

where \( j^2 = \frac{1}{2}j_{\mu\nu}j^{\mu\nu} \).

Following [19] we rewrite the MQGT gravitational equations in the synchronous frame of reference, in which the spacetime metric takes the form:

\[ ds^2 = dt^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta, \quad (11) \]

where \( t \) is a synchronous time, \( x^\alpha - \) spatial coordinates (first greek letters \( \alpha, \beta \ldots \) pass values 1, 2, 3). As in [19] let us introduce the notation:

\[ \mathcal{H}_{\alpha\beta} = \dot{\gamma}_{\alpha\beta} \quad (12) \]

(a dot denotes differentiation with respect to the synchronous time \( t \)). The quantities \( \mathcal{H}_{\alpha\beta} \) make up a 3-dimensional tensor, raising or lowering of one’s indices, and so the covariant differentiation is calculated by the 3-metric \( \gamma_{\alpha\beta} \). In terms of \( \gamma_{\alpha\beta} \) and \( \mathcal{H}_{\alpha\beta} \) the Einstein tensor’s components can be expressed as:

\[ G_0^0 = \frac{1}{8} \mathcal{H}_{\beta}^\alpha \mathcal{H}_{\alpha}^\beta - \frac{1}{8} \dot{\gamma}^\alpha \mathcal{H}_{\alpha}^\beta - \frac{1}{2} P^\alpha, \]

5
\[ G^0 = \frac{1}{2}(\nabla_\alpha \mathcal{H}^\beta_\beta - \nabla_\beta \mathcal{H}^\beta_\alpha), \]
\[ G^\beta_\alpha = \frac{1}{2} \dot{\mathcal{H}}^\beta_\alpha + \frac{1}{4} \gamma \mathcal{H}^\beta_\alpha + P^\beta_\alpha - \frac{1}{2} \delta^\beta_\alpha (\dot{\mathcal{H}}^\gamma_\delta + \frac{1}{4} \gamma \mathcal{H}^\gamma_\delta + \frac{1}{4} \mathcal{H}^\gamma_\delta \mathcal{H}^\delta_\gamma + P^\gamma_\delta), \]  

(13)

where \( \gamma = \text{det}(\gamma_{\alpha\beta}) \), \( P^\beta_\alpha \) is a 3-dimensional Ricci tensor constructed from \( \gamma_{\alpha\beta} \). Concerning the source of gravitational field we will consider only the case of rest matter, when \( u^0 = 1, u^a = 0 \), (hence we have \( j_{\mu 0} = 0 \)). Using these relationships we obtain the following expressions for the components of \( T^{\lambda\mu}_{\text{eff}} \) in the synchronous frame of reference:

\[ T^0_{0(\text{eff})} = (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1}[\rho + \chi - \frac{\beta}{4} (t^\sigma_\sigma + 4\chi)^2 - \frac{1}{8 f_0} i^2 (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1}] - 2 f_0 \left( \frac{1}{2} M^0_\alpha - \frac{3}{4} M^2 \right), \]

\[ T^0_{\alpha(\text{eff})} = -\frac{1}{2} (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} \nabla_\beta j^\beta_\alpha, \]

\[ T^{\beta}_{\alpha(\text{eff})} = -(1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} \left[ \left( p - \chi + \frac{\beta}{4} (t^\sigma_\sigma + 4\chi)^2 - \frac{1}{8 f_0} i^2 (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} \right) \delta^\alpha_\beta - \frac{1}{4} (\mathcal{H}^\beta_\gamma j^\gamma_\alpha + \mathcal{H}^\beta_\gamma j^\gamma_\alpha) \right] + 2 f_0 \left[ \frac{1}{2} \dot{M} \mathcal{H}^\beta_\alpha + (- \ddot{M} + \frac{1}{2} \dot{M} \mathcal{H}^\gamma_\gamma + \frac{1}{4} M^2) \delta^\beta_\alpha \right]. \]

Note that covariant differentiation \( \nabla \) of the quantity \( j^\beta_\alpha \) in the expression of \( T^0_{\alpha(\text{eff})} \) is calculated by the 3-metric \( \gamma_{\alpha\beta} \).

### III. Bianchi Models in the MQGT: A Basic Equation Set

Let us proceed to derive the equations of motion for homogeneous anisotropic (Bianchi) cosmological models. Homogeneity of the 3-space implies invariance of the spatial metric:

\[ \text{d}l^2 = \gamma_{\alpha\beta} \text{d}x^\alpha \text{d}x^\beta \]

at each fixed moment of time \( t \) with respect to the 3-dimensional group of motion \( G_3 \) (see [19-21]). So that local characters of homogeneous anisotropic models are determined fully by types of the corresponding motion’s group G, which was classified by Bianchi with respect to independent sets of the structure contants \( C^c_{ab} \). Here and later on, the first latin letters \( a, b, c... \) take the values 1, 2, 3 and number of the basis’s vectors.
Let \( X_a = e^\alpha_a \frac{\partial}{\partial x^\alpha} \), \( a = 1, 2, 3 \) be generators of \( G \), that satisfy the following commutation relations:

\[
[X_a, X_b] = C^c_{ab} X_c,
\]

where \( e^\alpha_a \) are the basis's vectors of a 3-dimensional space with the metric \( \gamma_{\alpha\beta} \). As we see from (15) the structure constants are antisymmetric in their lower indices: \( C^c_{ab} = -C^c_{ba} \), and satisfy the Jacobi identity:

\[
C^e_{ab} C^d_{cc} + C^e_{bc} C^d_{ea} + C^e_{ca} C^d_{eb} = 0.
\]

We also introduce basis's vectors \( e^\alpha_a \) that satisfy the relationships:

\[
e^\alpha_a e^\alpha_b = \delta^a_b; \quad e^\alpha_a e^\beta_a = \delta^\alpha_\beta.
\]

Expanding the spatial metric with respect to the basis \( (e^\alpha_a) \) we obtain the following time-dependent functions:

\[
\eta_{ab}(t) = \gamma_{\alpha\beta} e^\alpha_a e^\beta_b.
\]

The Bianchi classification of homogeneous spaces reduces to determining all nonequivalent sets of structure constants. In place of the three-index constants we introduce a set of two-index quantities, defined by the dual transformation: \( C^e_{ab} = \varepsilon_{abc} C^d_{dc} \), \( \varepsilon_{abc} = \varepsilon_{abc} \) is the unit antisymmetric symbol with \( \varepsilon_{123} = 1 \). Following [22], we decompose nonsymmetric \( C^{ab} \) into a symmetric tensor \( n^{ab} \) and a vector \( a_c \) as:

\[
C^{ab} = n^{ab} + \varepsilon^{abc} a_c.
\]

It is easy to diagonalize the symmetric tensor \( n^{ab} \); let \( n_1, n_2, n_3 \) be its eigenvalues. It then follows from the Jacobi identity that if \( a_a \) exits it is an eigenvector of \( n^{ab} \) corresponding to a zero eigenvalue:

\[
n^{ab} a_a = 0.
\]

Without loss of generality the vector \( a_a \) can be written as \( a_a = (a, 0, 0) \) with \( n_1 a = 0 \). Then, we get either \( n_1 \) or \( a \) is equal to zero. Bianchi models with \( a = 0 \) were defined by
Ellis and MacCallum as A-class models, also known as tensor models. Bianchi models for
which \( n_1 \) is set equal to zero are known as B-class models - vector models. By rescaling
the operators \( X_1, X_2, X_3 \) one can set the eigenvalues \( n_1, n_2, n_3 \) to be equal to \( 0, \pm 1 \). The
classification of these models is given in the tables A, and B, respectively (see [19-20]):

**Table A: Classification of A-class models \((a = 0)\)**

| Type | \( n_1 \) | \( n_2 \) | \( n_3 \) |
|------|------------|------------|------------|
| I    | 0          | 0          | 0          |
| II   | 1          | 0          | 0          |
| VII_0| 1          | 1          | 0          |
| VI_0 | 1          | -1         | 0          |
| IX   | 1          | 1          | 1          |
| VIII | 1          | 1          | -1         |

**Table B: Classification of B-class models \((n_1 = 0)\)**

| Type | \( a \) | \( n_2 \) | \( n_3 \) |
|------|---------|------------|------------|
| V    | 1       | 0          | 0          |
| IV   | 1       | 0          | 1          |
| III  | 1       | -1         | 1          |
| VI_h | \( a > 0 \) | -1         | 1          |
| VII_h| \( a > 0 \) | 1          | 1          |

By virtue of (12) and (15)-(17) the expansions of \( \mathcal{H}_a^\beta, \mathcal{P}_a^\beta, (\tilde{\nabla}_\alpha \mathcal{H}_\beta^\beta - \tilde{\nabla}_\beta \mathcal{H}_\alpha^\alpha), \) and \( \tilde{\nabla}_\beta j_{\alpha}^\beta \)
with respect to \( \{e^a_{\alpha}\} \) lead to the following expressions (see [19]):

\[
\mathcal{H}_b^a = \mathcal{H}_a^\beta e_{\alpha}^a e_{\beta}^b = \eta_{\alpha c} \tilde{\nabla}_c, \quad \mathcal{H}_a^a = \mathcal{H}_a^\alpha, \quad \text{(18)}
\]

\[
\mathcal{P}_a^b = \frac{1}{2\eta} [2\mathcal{C}^{bd} \mathcal{C}_{ad} + \mathcal{C}^{ab} \mathcal{C}_{ad} + \mathcal{C}^{bd} \mathcal{C}_{da} - \mathcal{C}^{d}_{d(a} \mathcal{C}_{b)c} \mathcal{C}_{c} + \delta_{a}^{b}(\mathcal{C}_{d}^{d})^2 - 2\mathcal{C}^{cd} \mathcal{C}_{cd}], \quad \text{(19)}
\]

\[
e_{\alpha}^a(\tilde{\nabla}_\alpha \mathcal{H}_\beta^\beta - \tilde{\nabla}_\beta \mathcal{H}_\alpha^\alpha) = \mathcal{H}_b^c (\mathcal{C}_{b}^{a} - \delta_{a}^{b} \mathcal{C}_{d}^{c}), \quad \text{(20)}
\]
The components of $G_{\lambda \mu}$, written in the basis \( \{ e^a \} \), which contain only the time-dependent functions, where the corresponding components of $G_{\lambda \mu}$ in the case when Walker models respectively.

Thus, we obtain the MQGT basic equation set describing the dynamics of Bianchi (types I-IX) spinning-fluid cosmological models, written in the basis \( \{ e^a \} \), in the following form:

\[
G^0 \hspace{0.5cm} = \hspace{-0.05cm} -\frac{1}{2f_0}T^0 \hspace{-0.05cm}, \hspace{0.5cm} G_a \hspace{0.5cm} = \hspace{-0.05cm} \frac{1}{2f_0}T_a \hspace{-0.05cm}, \hspace{0.5cm} G^b \hspace{0.5cm} = \hspace{-0.05cm} \frac{1}{2f_0}T^a \hspace{-0.05cm},
\]

where the corresponding components of $G_{\lambda \mu}$ and $T_{\lambda \mu}^{\text{eff}}$ are determined by the expressions (22)-(23), which contain only the time-dependent functions.

Note that these equations reduce to the corresponding equations of the Einstein-Cartan theory in the case when \( \beta = 0 \), and coincide with the ordinary Einstein’s equations of GR in the case when \( \beta = 0 \) and \( j_{\mu \nu} = 0 \).

In the following sections we will study Bianchi models of types I and V, which are the direct anisotropic generalization of the flat and the open isotropic Friedmann-Robertson-Walker models respectively.
We now study the dynamics of the simplest case of homogeneous anisotropic - Bianchi type-I - models, when all of the structure constants $C_{ab}^c = 0$ (see also [12]). Then, the metric of 3-space can be written in the diagonal form:

$$\eta_{ab} = \text{diag}(r_1^2(t), r_2^2(t), r_3^2(t)).$$  \hspace{1cm} (25)

In this case we note that under the presence of spin momentum $j_{ab} \neq 0$ the 3-space will be either isotropic or axially symmetric. In fact, for $a \neq b$ we find from (22) $G_a^a = 0$ and from (23)

$$T^{b}_{a(\text{eff})} = -\frac{1}{4}(1-4\beta\chi - \beta t_{\sigma})^{-1}(j^{b}_{c}H^{c}_a + j^{c}_{a}H^{b}_c) =$$

$$= -\frac{1}{4}(1-4\beta\chi - \beta t_{\sigma})^{-1}(H^{b}_a - H^{a}_b)j_{ab} \eta^{ab}$$

there is no sum on indexes $a, b$ here). It follows that

$$\left(\frac{\dot{r}_a}{r_a} - \frac{\dot{r}_b}{r_b}\right)j_{ab} = 0 \quad (a, b = 1, 2, 3; \; a \neq b).$$ \hspace{1cm} (26)

In the spinless matter case ($j_{ab} = 0$) relation (19) is automatically satisfied. If at least two of the three components of $j_{ab}$ are not equal to zero, then (26) implies $\frac{\dot{r}_1}{r_1} = \frac{\dot{r}_2}{r_2} = \frac{\dot{r}_3}{r_3}$, which leads to the isotropic 3-space: $r_1 = r_2 = r_3$. If only one of the three components of $j_{ab}$ is nonzero, for example $j_{12}$, it follows that the 3-space will be axially symmetric $r_1 = r_2 \neq r_3$.

By analogy with GR we introduce new variables: $r = (r_1 r_2 r_3)^{1/3}$, $h = (\dot{r}/r)$, $h_k = (\dot{r}_k/r_k)$ ($k = 1, 2, 3$). Let us consider the general case, when $t_\sigma^a = \rho - 3p \neq 0$. Then, it is easy to find

$$H^{b}_a = 2 \text{ diag}(h_1, h_2, h_3); \quad \eta = r^6; \quad P^{b}_a = 0.$$ \hspace{1cm} (27)

By using (27) we can rewrite (22) and (23) as

$$G_0^a = \frac{1}{2} \sum_{k=1}^{3} h_k^2 - \frac{9}{2} h^2.$$

\footnote{The special case of radiation $t_\sigma^a = \rho - 3p = 0$ was carefully studied in [11].}
\[ G^a_{\alpha} = \dot{h}_a - 3\dot{h} + 3hh_a - \frac{1}{2} \sum_{k=1}^{3} h_k^2 - \frac{9}{2} h^2, \]

\[ T_{0}^{0(eff)} = (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} \left[ \rho + \chi - \frac{\beta}{4} (t^\sigma_\sigma + 4\chi)^2 \right] - \frac{j^2}{8f_0} (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-2} - 6f_0 \left( \frac{1}{2} \ddot{M}h + \frac{1}{4} \dot{M}^2 \right), \]

\[ T_{2}^{a (eff)} = -(1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} \left[ p - \chi + \frac{\beta}{4} (t^\sigma_\sigma + 4\chi)^2 \right] + \frac{j^2}{8f_0} (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-2} - 2f_0 \left[ \ddot{M} + \dot{M}(3h - h_a) + \frac{1}{4} \dot{M}^2 \right]. \] (28)

After some manipulations from (24) and (28) we obtain (for the case of \( \chi = 0 \) see also [10])

\[ h_k = h + s_k r^{-2} (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1}; \quad \sum_{k=1}^{3} s_k = 0, \] (29)

\[ \left[ h - \frac{\beta}{2} (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} (t^\sigma_\sigma) \right]^2 = (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} \left[ \frac{1}{6f_0} \left( \rho + \chi - \frac{\beta}{4} (t^\sigma_\sigma + 4\chi)^2 \right) \right] + (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} \left[ \rho - p + 2\chi - \frac{\beta}{2} (t^\sigma_\sigma + 4\chi)^2 \right] + \beta \left( (t^\sigma_\sigma)^2 + 5h(t^\sigma_\sigma) \right), \] (30)

\[ \dot{h} + 3h^2 = (1 - 4\beta \chi - \beta t^\sigma_\sigma)^{-1} \left[ \frac{1}{4f_0} \left( \rho - p + 2\chi - \frac{\beta}{2} (t^\sigma_\sigma + 4\chi)^2 \right) + \beta \left( (t^\sigma_\sigma)^2 + 5h(t^\sigma_\sigma) \right) \right], \] (31)

where \( s_k (k = 1, 2, 3) \) are integration constants and \( P = \frac{1}{6} \sum_{k=1}^{3} s_k^2 \) is a measure of the anisotropy.

In the case considered, the spin conservation law (by using the equation of rotation of fluid in [15]) takes the following simple form:

\[ (j) + 3hj = 0. \] (32)

and "the energy-momentum conservation law" \( \tilde{\nabla}_\mu T_{\nu(\mu)}^{\lambda} = 0 \) yields

\[ \dot{\rho} + 3h(\rho + p) = 0. \] (33)

From these conservation laws we obtain

\[ r = exp \left( -\frac{1}{3} \int \frac{d\rho}{\rho + p(\rho)} \right); \quad j r^3 = J_0 = const. \] (34)
Assuming \( \rho = \rho(r) \), \( j = j(r) \) in accordance with (34), it is easy to find the solution of (30) in an analytic form:

\[
t - t_0 = \int_{r_0}^{r} \frac{\Phi_1(r)}{\Phi_2^{1/2}(r)} \, dr,
\]

where

\[
\Phi_1(r) = 3 \left[ 1 - 4\beta \chi - \beta t_\sigma^\sigma + \frac{3\beta}{2} (\rho + p) \left( 1 - 3 \frac{dp}{d\rho} \right) \right],
\]

\[
\Phi_2(r) = \frac{3}{2f_0} (1 - 4\beta \chi - \beta t_\sigma^\sigma) \left[ \rho + \chi - \frac{\beta}{4} (t_\sigma^\sigma + 4\chi)^2 \right] + \frac{A}{r^6},
\]

and \( A = 9 \left( \frac{f_2^2}{48f_0} \right) \). It follows from (29) that

\[
r_k = r \exp \left( s_k \int r^{-3} (1 - 4\beta \chi - \beta t_\sigma^\sigma)^{-1} \, dt \right).
\]

Thus, under the given equation of state \( p = p(\rho) \) the analytic solutions (34)-(36) adequately describe the dynamics of the Bianchi spinning-fluid models. Note that the solution (35) makes sense only if \( \Phi_2(r) \geq 0 \). For the case of vanishing cosmological term \( \chi = 0 \) the study of these solutions (for more details see Ref [23]) shows that we can construct Bianchi type-1 cosmological models, which are regular in the metric and in the torsion, by fulfilling the condition \( A < 0 \) (i.e. when the spin effect dominates over the effect of anisotropy). For the case of \( \chi \neq 0 \) and the linear equation of state:

\[
p = \frac{1 - \gamma}{3} \rho, \quad 0 < \gamma \leq 1,
\]

a complete qualitative analysis of properties of the obtained solutions is performed in [12] by using methods of qualitative theory of dynamic systems [24].

**V. BIANCHI TYPE-V PERFECT-FLUID MODELS: ANALYTIC SOLUTIONS.**

In this section we restrict ourselves to the spinless matter case of the perfect fluid, when \( j_{\mu\nu} = 0 \). As was shown in [25] in the general case of Bianchi models (except for type-I) the metrics of anisotropic models can not be diagonalized in the presence of spin momentum.
Taking \((x_1, x_2, x_3)\) as local coordinates we can write the interval of Bianchi type-V models in the diagonal form \([25, 26]\):

\[
ds^2 = dt^2 - r_1^2(t)dx_1^2 - e^{2\chi(t)}(r_2^2(t)dx_2^2 - r_3^2(t)dx_3^2).
\]

For the models considered we have the following structure constants: \(C_{21} = -C_{22} = C_{31} = -C_{33} = 1\). Then, it is easy to find the following relationships:

\[
\mathcal{H}_a^b = 2 \text{diag}(h_1, h_2, h_3); \quad \eta = r^6; \quad \mathcal{P}_a^b = -\frac{2}{r_1^4} \delta_a^b.
\] (38)

By using (38) and the restriction: \(j_{ab} = 0\) from (22) and (23) we find:

\[
G_0^0 = \frac{1}{2} \sum_{k=1}^3 h_k^2 - \frac{9}{2} h^2 + \frac{3}{r_1^2},
\]

\[
G_1^0 = 2h_1 - (h_2 + h_3),
\]

\[
G_2^a = \dot{h}_a - 3\dot{h} + 3hh_a - \frac{1}{2} \sum_{k=1}^3 h_k^2 - \frac{9}{2} h^2 + \frac{1}{r_1^2},
\]

\[
T_{0(\text{eff})}^0 = -(1 - 4\beta \chi - \beta t_0^\sigma)^{-1}[\rho - \chi - \frac{\beta}{4}(t_0^\sigma + 4\chi)^2] - 6f_0\left(\frac{1}{2} \dot{M}h + \frac{1}{4} \dot{M}^2\right),
\]

\[
T_{2(\text{eff})}^a = -(1 - 4\beta \chi - \beta t_0^\sigma)^{-1}[p - \chi + \frac{\beta}{4}(t_0^\sigma + 4\chi)^2] - 2f_0\left[\ddot{M} + \dot{M}(3\dot{h} - h_a) + \frac{1}{4} \ddot{M}^2\right].
\] (39)

It is clear that equation \(G_0^0 = -\frac{1}{2f_0} T_{0(\text{eff})}^0 = 0\) leads to the relationships:

\[
h = h_1 = \frac{1}{2} (h_2 + h_3).
\] (40)

It follows from (40) that \(r_1 = r\). By virtue of (40) it is easy to show that the equations \(G_2^a = -\frac{1}{2f_0} T_{a(\text{eff})}^a\) have the following integrals of motion:

\[
h_2 = h + \frac{s}{r^3}(1 - 4\beta \chi - \beta t_0^\sigma)^{-1}; \quad h_3 = h - \frac{s}{r^3}(1 - 4\beta \chi - \beta t_0^\sigma)^{-1},
\] (41)

and consequently

\[
r_2 = \exp\left(s \int r^{-3}(1 - 4\beta \chi - \beta t_0^\sigma)^{-1} dt\right); \quad r_3 = \exp\left(s \int r^{-3}(1 - 4\beta \chi - \beta t_0^\sigma) dt\right),
\] (42)

where \(s\) is an integration constant and has the sense of a measure of the anisotropy. By virtue of (40)-(41) the equation \(G_0^0 = -\frac{1}{2f_0} T_{0(\text{eff})}^0\) can be transformed to

\[
\left[h - \frac{\beta}{2}(1 - 4\beta \chi - \beta t_0^\sigma)^{-1}(t_0^\sigma)^2\right]^2 = (1 - 4\beta \chi - \beta t_0^\sigma)^{-1}\left[\frac{1}{6f_0}\left(\rho + \chi - \frac{\beta}{4}(t_0^\sigma + 4\chi)^2\right) +
\right.
\]

\[
\left.\left[\rho + \chi - \frac{\beta}{4}(t_0^\sigma + 4\chi)^2\right] + \frac{1}{2f_0} \dot{M}h + \frac{1}{4} \dot{M}^2\right].
\]

13
\[(1 - 4\beta\chi - \beta t_\sigma^\sigma)^{-1} \frac{s^2}{3r^6} + \frac{1}{r^2}, \quad (43)\]

The obtained equation is a direct generalization of the corresponding equation for open homogeneous isotropic models in MQGT (see [6]), that is the case, when the measure of the anisotropy \( s = 0 \). In the given case the "energy-momentum conservation law" \( \tilde{\nabla}_\mu T^{\lambda\mu}_{\text{eff}} = 0 \) has the same form as eq. (33). In a way similar to that of section 4 we can find solutions of eq. (43) in an analytic form:

\[
t - t_0 = \int_{r_0}^{r} \frac{\Phi_3(r)}{\Phi_4^{1/2}(r)} \, dr,
\]

where

\[
\Phi_3(r) = \frac{1}{r} \left[ 1 - 4\beta\chi - \beta t_\sigma^\sigma + \frac{3\beta}{2} (\rho + p) \left( 1 - \frac{3dp}{d\rho} \right) \right],
\]

\[
\Phi_4(r) = \frac{1}{6f_0} \left( 1 - 4\beta\chi - \beta t_\sigma^\sigma \right) \left[ \rho + \chi - \frac{\beta}{4} (t_\sigma^\sigma + 4\chi)^2 \right] + \frac{1}{r^2} \left( 1 - 4\beta\chi - \beta t_\sigma^\sigma \right)^2 + \frac{s^2}{3r^6},
\]

\[\text{VI. DISCUSSION}\]

The exact solutions (37) and (44), that were found in analytic form, describe dynamics of the spinning-fluid Bianchi type-1 and perfect-fluid Bianchi type-V respectively. The corresponding torsion functions have been written in (9). A brief analysis shows that the given solutions are, in general, "weakly singular" i.e have following properties:

- an energy density is finite at any time: \( \rho \leq \rho_{cr} < \infty \);
- a "volume" of "3-space" is positive \( V = r^3 = (r_1r_2r_3) \geq V_{cr} > 0 \) at any time;
- one or two of the three metric functions \( r_k \) may become zero at a finite moment of time when \( 1 - 4\beta\chi - \beta(\rho - 3p) = 0 \)
- torsion functions diverge \( S(t) = \infty \), when \( 1 - 4\beta\chi - \beta(\rho - 3p) = 0 \)

However, in some cases (depending on the values of the spin density, the anisotropy measure, the cosmological constant, the parameter \( \beta \), and the equation of state) we can find truly regular solutions with finite energy density, nonzero metric functions and finite
torsion functions. Using methods of qualitative theory of dynamical systems we can make a detailed analysis of properties of every possible solutions of the MQGT equations for the models considered.

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