NUMBERS WHICH ARE ORDERS ONLY OF CYCLIC GROUPS

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Abstract. We call $n$ a cyclic number if every group of order $n$ is cyclic. It is implicit in work of Dickson, and explicit in work of Szele, that $n$ is cyclic precisely when $\gcd(n, \phi(n)) = 1$. With $C(x)$ denoting the count of cyclic $n \leq x$, Erdős proved that

$$C(x) \sim e^{-\gamma} x / \log \log \log x, \quad \text{as } x \to \infty.$$ 

We show that $C(x)$ has an asymptotic series expansion, in the sense of Poincaré, in descending powers of $\log \log \log x$, namely

$$e^{-\gamma} x \left(1 - \frac{\gamma}{\log \log \log x} + \frac{\gamma^2 + \frac{1}{6} \pi^2}{(\log \log \log x)^2} - \frac{\gamma^3 + \frac{1}{3} \gamma \pi^2 + \frac{2}{45} \zeta(3)}{(\log \log \log x)^3} + \ldots \right).$$

1. Introduction

Call the positive integer $n$ cyclic if the cyclic group of order $n$ is the unique group of order $n$. For instance, all primes are cyclic numbers. It is implicit in work of Dickson [Dic05], and explicit in work of Szele [Sze47], that $n$ is cyclic precisely when $\gcd(n, \phi(n)) = 1$, where $\phi(n)$ is Euler’s totient. (In fact, this criterion had been stated as “evident” already by Miller in 1899 [Mil99, p. 235].) If $C(x)$ denotes the count of cyclic numbers $n \leq x$, Erdős proved in [Erd48] that

$$(1) \quad C(x) \sim e^{-\gamma} x / \log \log \log x,$$

as $x \to \infty$, where $\gamma$ is the Euler–Mascheroni constant. Thus, the relative frequency of cyclic numbers decays to 0 but “with great dignity” (Shanks).

Several authors have investigated analogues of (1) for related counting functions from enumerative group theory. See, for example, [May79, MMS84, War85, Sri87, EMM87, EM88, NS98, Sr91, NP18]. Our purpose in this note is somewhat different; we aim to refine the formula (1). Begunts [Beg01], optimizing the method of [Erd48], showed that $C(x)$ is given by $e^{-\gamma} / \log \log \log x$ up to a multiplicative error of size $1 + O(\log \log \log \log x / \log \log \log x)$ (the same result appears as Exercise 2 on p. 390 of [MV07]). We improve this as follows.

Theorem 1.1. The function $C(x)$ admits an asymptotic series expansion, in the sense of Poincaré (see [JBS1] §1.5), in descending powers of $\log \log \log x$. Precisely: There is a sequence of real numbers $c_1, c_2, c_3, \ldots$ such that, for each fixed positive integer $N$ and all
Our proof of Theorem 1.1 yields the following explicit determination of the constants $c_k$. Write the Taylor series for the $\Gamma$-function, centered at 1, in the form $\Gamma(1 + z) = 1 + C_1 z + C_2 z^2 + \ldots$. Then the coefficients $c_1, c_2, \ldots$ are determined by the formal relation

$$1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots = \exp(0! C_1 z + 1! C_2 z^2 + 2! C_3 z^3 + \ldots).$$

For computations of the $C_k$ and $c_k$, it is useful to recall that

$$\Gamma(1 + z) = \exp \left( -\gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) z^k \right).$$

(This is one version of a well-known expansion for the digamma function; see, e.g., entries 5.7.3 and 5.7.4 in [OLBC10].) The first few $c_k$ are given by

$$c_1 = -\gamma, \quad c_2 = \gamma^2 + \frac{1}{2} \zeta(2) = \gamma^2 + \frac{\pi^2}{12}, \quad c_3 = -\left( \gamma^3 + \frac{1}{4} \gamma \pi^2 + \frac{2}{3} \zeta(3) \right).$$

Owing to (2), each $c_k$ belongs to the ring $\mathbb{Q}[\gamma, \zeta(2), \zeta(3), \ldots, \zeta(k)]$. From the fact that the coefficients of $\log \Gamma(1 + z)$ are alternating in sign, one deduces that both the $C_k$ and the $c_k$ are alternating as well. Moreover,

$$|c_k| \geq (k - 1)!|C_k| \geq (k - 1)!\zeta(k)/k \geq (k - 1)!/k$$

for each $k \geq 2$. It follows that the series $1 + c_1 / \log \log \log x + c_2 / (\log \log \log x)^2 + \ldots$ is purely an asymptotic series, in that it diverges for all values of $x$.

The proof of Theorem 1.1 has many ingredients in common with the related work cited above (see also [PP, Pol]). But we must be more careful about error terms than in earlier papers, and somewhat delicate bookkeeping is required to wind up with a clean result.

**Notation.** The letters $p$ and $q$ are reserved for primes. We use $K_0, K_1, K_2, \ldots$ for absolute positive constants. To save space, we write $\log_k$ for the $k$th iterate of the natural logarithm.

## 2. Lemmata

We will use Mertens’ theorem in the following form, which is a consequence of the prime number theorem with the classical $x \exp(-K_0 \sqrt{\log x})$ error estimate of de la Vallée Poussin.

**Lemma 2.1.** There is an absolute constant $c$ such that, for all $X \geq 3$,

$$\sum_{p \leq X} \frac{1}{p} = \log_2 X + c + O(\exp(-K_1 \sqrt{\log X})).$$

Moreover, for all $X \geq 3$,

$$\prod_{p \leq X} \left(1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log X} \left(1 + O(\exp(-K_2 \sqrt{\log X}))\right).$$

large $x$, 

$$C(x) = \frac{e^{-\gamma x}}{\log \log \log x} \left(1 + \frac{c_1}{\log \log \log x} + \frac{c_2}{(\log \log \log x)^2} + \cdots + \frac{c_N}{(\log \log \log x)^N}\right) + O_N \left(\frac{x}{(\log \log \log x)^{N+2}}\right).$$
The following sieve result is a special case of [HR74, Theorem 7.2].

**Lemma 2.2.** Suppose that \( X \geq Z \geq 3 \). Let \( P \) be a set of primes not exceeding \( Z \). The number of \( n \leq X \) coprime to all elements of \( P \) is

\[
X \prod_{p \in P} \left(1 - \frac{1}{p}\right) \left(1 + O\left(\exp\left(-\frac{1}{2 \log Z}\right)\right)\right).
\]

The final estimate of this section was proved independently by Pomerance (see Remark 1 of [Pom77]) and Norton (see the Lemma on p. 699 of [Nor76]).

**Lemma 2.3.** For every positive integer \( m \) and every \( X \geq 3 \),

\[
\sum_{\substack{p \leq X \\
p \equiv 1 \pmod{m}}} \frac{1}{p} = \frac{\log_2 X}{\phi(m)} + O\left(\frac{\log (2m)}{\phi(m)}\right).
\]

### 3. Proof of Theorem 1.1

#### 3.1. Outline.

We summarize the strategy of the proof, deferring the more intricate calculations to later sections. Put \( y = \frac{\log_2 x}{2 \log_3 x} \) and \( z = (\log_2 x) \cdot \exp(\sqrt{\log_3 x}) \).

Let us call the prime \( p \) a **standard divisor** of \( \gcd(n, \phi(n)) \) if there is a prime \( q \leq x^{1/\log_2 x} \) dividing \( n \) with \( q \equiv 1 \pmod{p} \). Clearly, each standard divisor of \( (n, \phi(n)) \) is a divisor of \( \gcd(n, \phi(n)) \).

Let \( S_0 \) be the set of \( n \leq x \) with no prime factor in \([2, y]\). For each positive integer \( k \), let \( S_k \) be the set of \( n \in S_0 \) having exactly \( k \) distinct prime factors from the interval \((y, z]\), all of which divide \( n \) to the first power only, and at least one of which is a standard divisor of \( \gcd(n, \phi(n)) \). We will estimate \( C(x) \) by

\[
(3) \quad \# \left( S_0 \setminus \bigcup_{1 \leq k \leq \log_3 x} S_k \right) = \#S_0 - \sum_{1 \leq k \leq \log_3 x} \#S_k.
\]

Suppose \( n \) is counted by \( C(x) \) but not by \( (3) \). Then \( n \) has a prime factor \( p \leq y \). Since \( n \) is counted by \( C(x) \), it must be that \( p \nmid \phi(n) \), so that \( n \) is not divisible by any \( q \equiv 1 \pmod{p} \). By Lemma 2.2, for a given \( p \) the number of those \( n \leq x \) is \( \ll x \prod_{q \leq x, q \equiv 1 \pmod{p}} (1 - 1/q) \leq x \exp(-\sum_{q \leq x, q \equiv 1 \pmod{p}} 1/q) \). And by Lemma 2.3

\[
\sum_{\substack{q \leq x \\
q \equiv 1 \pmod{p}}} \frac{1}{q} = \frac{1}{p-1} \log_2 x + O(1) \geq 2 \log_3 x + O(1).
\]

Thus, the number of \( n \) corresponding to a given \( p \) is \( \ll x \exp(-2 \log_3 x) = x/(\log_2 x)^2 \). Summing on \( p \leq y \), we deduce that the total number of \( n \) counted by \( C(x) \) but not \( (3) \) is \( O(x/\log_2 x) \).

Working from the opposite side, suppose that \( n \) is counted by \( (3) \) but not by \( C(x) \). Then at least one of the following holds:
(i) there is a prime $p > y$ for which $p^2 \mid n$,
(ii) there is a prime $p > z$ that divides $n$ and $\phi(n)$,
(iii) there is a prime $p$ in $(y, z]$ dividing $n$ and a prime $q \equiv 1 \pmod{p}$ dividing $n$ with $q > x^{1/\log\log x}$,
(iv) $n$ has more than $\log x$ different prime factors in $(y, z]$.

The number of $n \leq x$ for which (i) holds is $\ll x \sum_{p>z} 1/p^2 \ll x/y \log y \ll x/\log^2 x$. In order for (ii) to hold but (i) to fail, there must be a prime $q \equiv 1 \pmod{p}$ dividing $n$. Clearly, there are most $x/pq$ such $n$ corresponding to a given $p, q$. Thus, the number of $n$ that arise this way is

$$\ll x \sum_{p>z} \frac{1}{p} \sum_{q \leq x \atop q \equiv 1 \pmod{p}} \frac{1}{q} \ll x \sum_{p>z} \frac{\log_2 x + \log p}{p^2} \ll \frac{x \log_2 x}{z} = \frac{x}{\exp(\sqrt{\log x})}.$$ 

For similar reasons, the number of $n \leq x$ for which (iii) holds is

$$\sum_{y<p\leq z} \frac{1}{p} \sum_{x^{1/\log_2 x} < q \leq x \atop q \equiv 1 \pmod{p}} \frac{1}{q} \ll x \sum_{p>y} \frac{\log_3 x}{p^2} \ll \frac{x \log_3 x}{\log^2 x}.$$ 

To handle (iv), observe that $\sum_{y<p\leq z} 1/p \leq K_3/\sqrt{\log x} < 1/2$ for large values of $x$. Thus, the number of $n \leq x$ for which (iv) holds is (crudely) at most

$$x \sum_{k>\log_3 x} \left( \sum_{y<p\leq z} 1/p \right)^k \leq 2x(K_3/\sqrt{\log x})^\log x \leq x/\log^2 x.$$

Collecting estimates, we conclude that

$$C(x) = \# \left( S_0 \setminus \bigcup_{1 \leq k \leq \log_3 x} S_k \right) + O(x/\exp(\sqrt{\log x})).$$

Since the error term is $O_N(x/(\log_3 x)^{N+2})$ for any fixed $N$, for the sake of proving Theorem 1.1 we may replace $C(x)$ by $\#(S_0 \setminus \bigcup_{1 \leq k \leq \log_3 x} S_k)$.

In §3.2 we prove suitable estimates for the numbers $\#S_k$ and in §3.3 we tie everything together and complete the proof of Theorem 1.1.

3.2. Estimating $\#S_k$. The case $k = 0$ is easy to dispense with. By Lemmas 2.1 and 2.2

$$\#S_0 = e^{-\gamma} \frac{x}{\log y} + O(x/\exp(K_4\sqrt{\log_3 x})).$$

Now suppose that $1 \leq k \leq \log_3 x$. In order for the integer $n \leq x$ to be counted by $S_k$, it is necessary and sufficient than $n = p_1 \cdots p_k m$, where (a) $p_1, \ldots, p_k$ are distinct primes belonging to $(y, z]$, (b) the integer $m$ is free of prime factors in $[2, z]$, and (c) $m$ has a prime factor $q \leq x^{1/\log_2 x}$ with $q \equiv 1 \pmod{p_i}$ for some $i = 1, 2, \ldots, k$.

Fix distinct primes $p_1, \ldots, p_k \in (y, z]$. We will count the number of $n \in S_k$ for which $p_1, \ldots, p_k$ are the prime divisors of $n$ in $(y, z]$. To get at this, we count all $n = p_1 \cdots p_k m \leq x$
where condition (b) holds and then subtract the contribution from \( n \) for which (b) holds but (c) fails. By Lemma 2.2, this is approximately

\[
\frac{x}{p_1 \cdots p_k} \prod_{p \leq z} \left( 1 - \frac{1}{p} \right) \left( 1 - \prod_{z < q \leq x^{1/\log_2 x}} \left( 1 - \frac{1}{q} \right) \right). \tag{5}
\]

In fact, taking \( X = x/p_1 \cdots p_k \) (which exceeds \( x^{1/2} \)) and \( Z = x^{1/\log \log x} \) in Lemma 2.2, we see that the error in this approximation is (very crudely) bounded by \( O(x/(p_1 \cdots p_k \log x)) \).

Now we replace \( \prod_{p \leq z} (1 - 1/p) \) in (5) with \( e^{-\gamma}/\log z \). This introduces another error of size \( x/(p_1 \cdots p_k \exp(K_5 \sqrt{\log_3 x})) \).

It remains to estimate the product over \( q \) in (5). We have that

\[
\prod_{z < q \leq x^{1/\log_2 x}, \, q \equiv 1 \pmod{p_i}} \left( 1 - \frac{1}{q} \right) = \exp \left( - \sum_{z < q \leq x^{1/\log_2 x}, \, q \equiv 1 \pmod{p_i}} \frac{1}{q} + O \left( \sum_{q > z} \frac{1}{q^2} \right) \right)
\]

Continuing, we observe that

\[
\sum_{z < q \leq x^{1/\log_2 x}, \, q \equiv 1 \pmod{p_i}} \frac{1}{q} = \sum_{i=1}^{k} \sum_{z < q \leq x^{1/\log_2 x}, \, q \equiv 1 \pmod{p_i}} \frac{1}{q} + O \left( \sum_{1 \leq i < j \leq k} \sum_{z < q \leq x^{1/\log_2 x}, \, q \equiv 1 \pmod{p_i p_j}} \frac{1}{q} \right),
\]

and that the \( O \)-term here is

\[
\ll \sum_{1 \leq i < j \leq k} \frac{\log_2 x}{p_i p_j} \ll \left( \frac{k}{2} \right) \frac{(\log_3 x)^2}{\log_2 x} \ll (\log_3 x)^4 / \log_2 x.
\]

Moreover,

\[
\sum_{i=1}^{k} \sum_{z < q \leq x^{1/\log_2 x}, \, q \equiv 1 \pmod{p_i}} \frac{1}{q} = \sum_{i=1}^{k} \left( \frac{\log_2 x}{p_i - 1} + O \left( \frac{\log_3 x}{p_i} \right) \right)
\]

\[
= \sum_{i=1}^{k} \frac{\log_2 x}{p_i} + O \left( \frac{k (\log_3 x)^2}{\log_2 x} \right) = \sum_{i=1}^{k} \frac{\log_2 x}{p_i} + O \left( \frac{(\log_3 x)^3}{\log_2 x} \right).
\]

Therefore,

\[
\prod_{z < q \leq x^{1/\log_2 x}, \, q \equiv 1 \pmod{p_i}} \left( 1 - \frac{1}{q} \right) = \left( \prod_{i=1}^{k} \exp \left( - \frac{\log_2 x}{p_i} \right) \right) \left( 1 + O \left( \frac{(\log_3 x)^4}{\log_2 x} \right) \right)
\]

\[
= \prod_{i=1}^{k} \exp \left( - \frac{\log_2 x}{p_i} \right) + O \left( \frac{(\log_3 x)^4}{\log_2 x} \right).
\]
Now collect estimates. We find that the number of $n \in \mathcal{S}_k$ where $p_1, \ldots, p_k$ are the prime divisors of $n$ from $(y, z)$ is

\[
(6) \quad \frac{e^{-\gamma}}{\log z} \left( \frac{1}{p_1 \cdots p_k} - \prod_{i=1}^{k} \frac{\exp(-(\log_2 x)/p_i)}{p_i} \right) + O \left( \frac{x}{p_1 \cdots p_k \exp(K_5 \sqrt{\log_3 x})} \right).
\]

Finally, we sum (6) over all sets of distinct primes $p_1, \ldots, p_k \in (y, z]$. The $O$-terms contribute $O(x/\exp(K_5 \sqrt{\log_3 x}))$. Next we look at the contribution from the $1/p_1 \cdots p_k$ terms. On the one hand, the multinomial theorem immediately implies that

\[
\sum_{y<p_1<p_2<\cdots<p_k\leq z} \frac{1}{p_1 \cdots p_k} \leq \frac{1}{k!} \sigma_0^k, \quad \text{where} \quad \sigma_0 := \sum_{y<p \leq z} \frac{1}{p}.
\]

(We have $\sigma_0 \asymp 1/\sqrt{\log_3 x}$ for large $x$ by Mertens’ theorem.) On the other hand,

\[
\sum_{p_1, \ldots, p_k \in (y, z]} \frac{1}{p_1 \cdots p_k} = \sum_{y<p_{k-1} \leq z} \frac{1}{p_1 \cdots p_{k-1}} \sum_{p_k \not\in \{p_1, \ldots, p_{k-1}\}} \frac{1}{p_k} \geq \left( \sigma_0 - \frac{k-1}{y} \right) \sum_{y<p_{k-1} \leq z} \frac{1}{p_1 \cdots p_{k-1} \cdot p_k}.
\]

We can estimate the sum over $p_1, \ldots, p_{k-1}$ in a similar way. Iterating, we find that

\[
\sum_{p_1, \ldots, p_{k-1} \in (y, z]} \frac{1}{p_1 \cdots p_{k-1}} \geq \prod_{i=0}^{k-2} \left( \sigma_0 - \frac{i}{y} \right)^{k-1} \geq \left( \sigma_0 - \frac{2(\log_3 x)^2}{\log_2 x} \right)^k,
\]

so that

\[
\sum_{y<p_1<p_2<\cdots<p_k\leq z} \frac{1}{p_1 \cdots p_k} \geq \frac{1}{k!} \left( \sigma_0 - \frac{2(\log_3 x)^2}{\log_2 x} \right)^k.
\]

Combining the upper and lower bounds,

\[
\sum_{y<p_1<p_2<\cdots<p_k\leq z} \frac{1}{p_1 \cdots p_k} = \frac{1}{k!} \sigma_0^k \left( 1 + O \left( \frac{(\log_3 x)^3}{\log_2 x} \right) \right)^k = \frac{1}{k!} \sigma_0^k + O \left( \frac{1}{k!} \frac{(\log_3 x)^4}{\log_2 x} \right).
\]

The contribution from the terms of the form $\prod_{i=1}^{k} \exp(-(\log_2 x)/p_i)/p_i$ can be handled similarly. Put

\[
\sigma_1 := \sum_{y<p \leq z} \frac{\exp(-(\log_2 x)/p)}{p}.
\]

Clearly, $\sigma_1 \leq \sum_{y<p \leq z} 1/p \ll 1/\sqrt{\log_3 x}$. Since $\exp(-(\log_2 x)/p) \gg 1$ when $p \geq \log_2 x$, we also have that $\sigma_1 \leq \sum_{\log_2 x<p \leq z} 1/p \gg 1/\sqrt{\log_3 x}$. Now a computation completely parallel to the one shown above yields

\[
\sum_{y<p_1<p_2<\cdots<p_k\leq z} \prod_{i=1}^{k} \frac{\exp(-(\log_2 x)/p_i)}{p_i} = \frac{1}{k!} \sigma_1^k + O \left( \frac{1}{k!} \frac{(\log_3 x)^4}{\log_2 x} \right).
\]
Piecing everything together, we conclude that

\[ \#S_k = e^{-\gamma} \frac{x}{\log z} \left( \frac{\sigma_0^k}{k!} - \frac{\sigma_1^k}{k!} \right) + O \left( \frac{x}{\exp(K_6 \sqrt{\log_3 x})} \right). \]

### 3.3. Denouement.

Summing (7) over positive integers \( k \leq \log_3 x \), keeping in mind that \( \sigma_0, \sigma_1 \ll 1/\sqrt{\log_3 x} \), we find that

\[ \sum_{1 \leq k \leq \log_3 x} \#S_k = e^{-\gamma} \frac{x}{\log z} \left( \exp(\sigma_0) - \exp(\sigma_1) \right) + O \left( \frac{x}{\exp(K_6 \sqrt{\log_3 x})} \right). \]

By Mertens’ theorem, \( \exp(\sigma_0) = \frac{\log x}{\log y} \left( 1 + O(1/\exp(K_7 \sqrt{\log_3 x})) \right) \). So recalling (4),

\[ \#S_0 - \sum_{1 \leq k \leq \log_3 x} \#S_k = e^{-\gamma} \frac{x}{\log z} \exp(\sigma_1) + O(x/\exp(K_8 \sqrt{\log_3 x})). \]

By another application of the prime number theorem with the de la Vallée Poussin error term,

\[ \sigma_1 = \int_y^z \frac{\exp(-(\log_2 x)/t)}{t \log t} d\theta(t) = \int_y^z \frac{\exp(-(\log_2 x)/t)}{t \log t} dt + O(1/\exp(K_8 \sqrt{\log_3 x})), \]

and thus

\[ \#S_0 - \sum_{1 \leq k \leq \log_3 x} \#S_k = e^{-\gamma} \frac{x}{\log z} \exp \left( \int_y^z \frac{\exp(-(\log_2 x)/t)}{t \log t} dt \right) + O(x/\exp(K_9 \sqrt{\log_3 x})). \]

We proceed to analyze the integral appearing in this last estimate. Making the change of variables \( u = (\log_2 x)/t \),

\[ \int_y^z \frac{\exp(-(\log_2 x)/t)}{t \log t} dt = \frac{1}{\log_3 x} \int_{(\log_2 x)/z}^{\log_3 x} \frac{\exp(-u)}{u} \left( 1 - \frac{\log u}{\log_3 x} \right)^{-1} du. \]

Inside the domain of integration, \( \log u \ll \sqrt{\log_3 x} \), and so for each fixed positive integer \( M \),

\[ \left( 1 - \frac{\log u}{\log_3 x} \right)^{-1} = 1 + \left( \frac{\log u}{\log_3 x} \right) + \left( \frac{\log u}{\log_3 x} \right)^2 + \cdots + \left( \frac{\log u}{\log_3 x} \right)^M + O_M(\log_3 x)^{-M+1/2}. \]

Thus,

\[ \frac{1}{\log_3 x} \int_{(\log_2 x)/z}^{\log_3 x} \frac{\exp(-u)}{u} \left( 1 - \frac{\log u}{\log_3 x} \right)^{-1} du = \sum_{k=0}^{M} \frac{1}{(\log_3 x)^{k+1}} \int_{(\log_2 x)/z}^{\log_3 x} \frac{\exp(-u)}{u} \log^k u du + O \left( \frac{1}{(\log_3 x)^{M+3/2}} \int_{(\log_2 x)/z}^{\log_3 x} \frac{\exp(-u)}{u} du \right). \]
The $O$-term here is $\ll (\log_3 x)^{-\frac{1}{2}(M+3)} \int_{\log_2 x}^{\log_3 x} du/u \ll (\log_3 x)^{-1-\frac{1}{2}M}$. To handle the main term, we integrate by parts to find that

$$\int_{\log_2 x}^{\log_3 x} \frac{\exp(-u)}{u} \log^k u \, du = \exp(-u) \frac{\log^{k+1} u}{k+1} \bigg|_{u=\log_2 x}^{u=\log_3 x} + \frac{1}{k+1} \int_{\log_2 x}^{\log_3 x} \exp(-u) \log^{k+1} u \, du.$$ 

For each $0 \leq k \leq M$, and all large $x$,

$$\exp(-u) \frac{\log^{k+1} u}{k+1} \bigg|_{u=\log_2 x}^{u=\log_3 x} = -\frac{1}{k+1} \log \left( \frac{\log_2 x}{z} \right)^{k+1} + O_M(1/\exp(K_{10} \sqrt{\log_3 x})),$$

while

$$\frac{1}{k+1} \int_{\log_2 x}^{\log_3 x} \exp(-u) \log^{k+1} u \, du = \frac{1}{k+1} \int_0^\infty \exp(-u) \log^{k+1} u \, du + O_M(1/\exp(K_{11} \sqrt{\log_3 x})) = \frac{1}{k+1} \Gamma(k+1)(1) + O_M(1/\exp(K_{11} \sqrt{\log_3 x})) = k! C_{k+1} + O_M(1/\exp(K_{11} \sqrt{\log_3 x})).$$

Assembling our results,

$$\int_y^z \exp(-((\log_2 x)/t) \, dt = -\sum_{k=0}^{M} \frac{1}{k+1} \left( \frac{\log((\log_2 x)/z)}{\log_3 x} \right)^{k+1} + \sum_{k=0}^{M} \frac{k! C_{k+1}}{(\log_3 x)^{k+1}} + O_M((\log_3 x)^{-1/2} M)$$

$$= \log \left( 1 - \frac{\log((\log_2 x)/z)}{\log_3 x} \right) + \sum_{k=0}^{M} \frac{k! C_{k+1}}{(\log_3 x)^{k+1}} + O_M((\log_3 x)^{-1/2} M)$$

$$= \log \frac{\log z}{\log_3 x} + \sum_{k=0}^{M} \frac{k! C_{k+1}}{(\log_3 x)^{k+1}} + O_M((\log_3 x)^{-1/2} M).$$

We now choose $M = 2N$, where $N$ is as in Theorem 1.1. In the last displayed sum on $k$, the terms of the sum with $k \geq N$ may be absorbed into the error. Doing so and exponentiating,

$$\exp \left( \int_y^z \frac{\exp(-((\log_2 x)/t) \, dt} {t \log t} \right)$$

$$= \frac{\log z}{\log_3 x} \exp \left( \sum_{1 \leq k \leq N} \frac{(k-1)! C_k}{(\log_3 x)^k} \right) \left( 1 + O_N((\log_3 x)^{-1-N}) \right).$$
so that

\[
e^{-\gamma} \frac{x}{\log z} \exp \left( \int_y^z \exp\left( -\frac{(\log_2 x)/t}{t \log t} \right) dt \right)
\]

\[
e^{-\gamma} \frac{x}{\log_3 x} \exp \left( \sum_{1 \leq k \leq N} \frac{(k - 1)!C_k}{(\log_3 x)^k} \right) \left( 1 + O_N((\log_3 x)^{-1-N}) \right)
\]

\[
e^{-\gamma} \frac{x}{\log_3 x} \exp \left( \sum_{1 \leq k \leq N} \frac{(k - 1)!C_k}{(\log_3 x)^k} \right) + O_N(x(\log_3 x)^{-2-N}).
\]

This expression describes \(#(S_0 \setminus \bigcup_{1 \leq k \leq \log_3 x} S_k)\), by \([9]\), and so also describes \(C(x)\), by

the discussion in \([3.1]\) Theorem 1.1 follows, along with the description of the constants \(c_k\) appearing in the introduction.

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