An application of the cubic spline on Shishkin mesh for the approximation of a function and its derivatives in the presence of a boundary layer

I A Blatov\textsuperscript{1}, A I Zadorin\textsuperscript{2}, E V Kitaeva\textsuperscript{3}

\textsuperscript{1} Volga Region State University of Telecommunications and Informatics, ul. L’va Tolstogo,23, 443010, Samara, Russia
\textsuperscript{2} Sobolev Institute of Mathematics, pr. Koptyuga,4, 630090, Novosibirsk, Russia
\textsuperscript{3} Korolev Samara State University, Moskovskoe sh., 34, 443086, Samara, Russia

E-mail: zadorin@ofim.oscsbras.ru, blatow@mail.ru

Abstract. The problem of the approximate calculation of the derivatives of functions having large gradients in the region of an exponential boundary layer is considered. The problem is that the application of the classical formulas of the numerical differentiation on the uniform grids to functions with large gradients leads to significant errors. It is proposed to apply a cubic spline interpolation on a Shishkin grid that condensed in the boundary layer. It is proposed to approximate the derivatives on the basis of the spline differentiation. The error of such approximation is estimated taking into account the uniformity of the estimate with respect to the small parameter.

1. Introduction

Various convection-diffusion processes with predominant convection are modeled on the basis of singularly perturbed problems. The solution of singular perturbed problem with a small parameter $\varepsilon$ at the highest derivative has large gradients in the boundary layer. The problem of the interpolation of functions with large gradients in the boundary layer is of interest. It is shown \cite{1} that the interpolation of functions with large gradients in the boundary layer with using of cubic spline on a uniform grid is inefficient. It is proved that the interpolation error grows without limit when the small parameter $\varepsilon$ tends to zero. In \cite{1} it is proved that in the case of Shishkin mesh \cite{2} the interpolation error grows if $\varepsilon$ decreases. The modification of the cubic spline on Shishkin mesh is made, as result the error of the cubic spline becomes uniform with respect to $\varepsilon$.

The problem of calculating the derivatives of functions with large gradients, given at the mesh nodes, is also relevant. The application of polynomial formulas of numerical differentiation to functions with large gradients in the boundary layer can lead to errors of the order of $O(1)$. Formulas of numerical differentiation which are exact on the boundary layer component were proposed and studied in \cite{3, 4, 5}. The error of such formulas is $\varepsilon$-uniform.

Splines are widely used to approximate functions and derivatives \cite{6}, \cite{7}. But it is of interest to investigate the accuracy of the spline interpolation in the case when the function has large gradients in the boundary layer region. Cubic and parabolic splines on the Shishkin mesh are...
investigated in [1, 8, 9]. An exponential spline for a function with large gradients was constructed and investigated in [10].

In this paper, we investigate the possibility of calculating the derivatives of functions with large gradients based on the application of a cubic spline on a Shishkin mesh that condensed in the boundary layer. With this approach, we calculate the derivative as a smooth function of its argument.

Introduce the notations. Let us Ω = \{x_n\} be the mesh of the interval [0, 1], where \( x_0 = 0 < x_1 < \cdots < x_N = 1 \). By \( S(\Omega, 3, 1) \) we denote the space of cubic splines of defect 1 on the mesh \( \Omega \). By \( C \) and \( C_j \) we denote positive constants independent of the parameter \( \varepsilon \) and the number of mesh nodes. We write \( f = O(g) \) if \( |f| \leq C|g| \). Suppose that \( C[a, b] \) is the space of functions continuous on \([a, b]\) with the norm \( \| \cdot \|_{C[a, b]} \).

2. Formulation of the problem
Let us a function \( u(x) \) be decomposed in the form of the sum of regular and singular components:

\[
    u(x) = q(x) + \Phi(x), \ x \in [0, 1],
\]

where

\[
|q^{(j)}(x)| \leq C_1, \ |\Phi^{(j)}(x)| \leq \frac{C_1}{\varepsilon^j} e^{-\alpha x / \varepsilon}, \ 0 \leq j \leq 4,
\]

and functions \( q(x), \Phi(x) \) are not explicitly specified, \( \alpha > 0, \varepsilon \in (0, 1] \). According to (2), the regular component \( q(x) \) has derivatives that are bounded up to the fourth order, and the singular component \( \Phi(x) \) has derivatives that are not \( \varepsilon \)-uniformly bounded. The component \( \Phi(x) \) is responsible for the large gradients of the function \( u(x) \) in the boundary layer. The solution of a singularly perturbed problem has the representation (1) [2].

Let us show that the classical difference formulas on the uniform grid are unacceptable for calculating the derivatives of functions with large gradients. Consider the formula

\[
    u'(x) \approx \frac{u_n - u_{n-1}}{h}, \ x_{n-1} \leq x \leq x_n.
\]

Let us \( u(x) = e^{-x / \varepsilon} \). Then \( \varepsilon ((u_1 - u_0) / h - u'(0)) = e^{-1} \) if \( \varepsilon = h \). So, the relative error of this formula doesn’t decrease if \( h \to 0 \) and \( \varepsilon = h \).

In order for the error to be uniform with respect to the parameter \( \varepsilon \), we will apply the Shishkin mesh [2]. This is all the more urgent, because the Shishkin mesh is widely used to construct difference schemes that converge uniformly in the parameter \( \varepsilon \). To restore the derivative as a smooth function of its argument we apply a cubic spline. We use the values of the function at the mesh nodes.

In accordance with [2], we define the steps of the mesh \( \Omega \):

\[
    h_n = h = \frac{2\sigma}{N}, \ n = 1, \ldots, \frac{N}{2}, \quad h_n = H = \frac{2(1 - \sigma)}{N}, \ n = \frac{N}{2} + 1, \ldots, N,
\]

where

\[
    \sigma = \min \left\{ \frac{1}{2}, \frac{4\varepsilon}{\alpha} \ln N \right\}.
\]

We assume that the function \( u(x) \) is given at the nodes of the mesh \( \Omega \), \( u_n = u(x_n), n = 0, 1, \ldots, N \).

We define the cubic spline \( S(x, u) \in SL(\Omega, 3, 1) \) for the function \( u(x) \) from the interpolation conditions

\[
    S(x_n, u) = u(x_n), \ 0 \leq n \leq N, \ S'(0, u) = u'(0), \ S'(1, u) = u'(1).
\]
In accordance with [1], for some constants $C, \beta$ that do not depend on $\varepsilon, N$ the following error estimate is valid

$$
\| u(x) - S(x, u) \|_{C[x_n, x_{n+1}]} \leq C \begin{cases} 
N^{-4} \ln^4 N, 0 \leq n \leq N/2 - 1, \\
\frac{1}{N^3} + \frac{1}{\varepsilon N^4} e^{-\beta(n-N/2)}, \frac{N}{2} \leq n \leq N - 1.
\end{cases}
$$

(6)

Our goal is to investigate the error of the numerical differentiation formulas based on a cubic spline $S(x, u)$:

$$
u'(x) \approx S'(x, u), \ u''(x) \approx S''(x, u), \ x \in [0, 1].$$

Derivatives $u'(0), u'(1)$ used in (5) can be found with a given accuracy through the values of the function $u(x)$ at the nodes of Shishkin mesh.

3. Error estimation

We estimate the error in approximation of the derivatives on the basis of spline interpolation. In the boundary layer, the first derivative of the order $O(\varepsilon^{-1})$ and the second derivative of the order $O(\varepsilon^{-2})$. The derivatives are not bounded uniformly in $\varepsilon$, which can be close to zero. Therefore, in the boundary layer we give estimates of the relative errors.

**Theorem 1** Suppose that the function $u(x)$ has the representation (1), (2). Then for some constant $C$ the following error estimates are valid:

$$
|u'(x) - S'(x, u)| \leq C \frac{\ln^3 N}{N^3}, \ x \leq \sigma,
$$

(7)

$$
u(x) - S'(x, u) \leq C \left[ \frac{1}{N^3} + \frac{1}{\varepsilon N^4} e^{-\alpha(x-\sigma)/\varepsilon} + \frac{1}{\varepsilon N^4} e^{-\beta(n-N/2)} \right], \ x > \sigma,
$$

(8)

$$
\varepsilon^2 |u''(x) - S''(x, u)| \leq C \frac{\ln^2 N}{N^2}, \ x \leq \sigma,
$$

(9)

$$
u''(x) - S''(x, u) \leq C \left[ \frac{1}{N^2} + \frac{1}{\varepsilon^2 N^4} e^{-\alpha(x-\sigma)/\varepsilon} + \frac{1}{\varepsilon N^3} e^{-\beta(n-N/2)} \right], \ x > \sigma,
$$

(10)

where $\alpha, \beta$ corresponds to (2), (6).

**Proof.** Fix an arbitrary grid interval $[x_n, x_{n+1}]$. We denote by $P_n(x)$ the Taylor polynomial of degree 3 of the function $u(x)$ with the center of the decomposition at the point $x_{n+3}$. Then

$$
u(x) = P_n(x) + \frac{1}{3!} \int_{x_{n+3}}^{x} (x - s)^3 u^{(4)}(s)ds.
$$

(11)

From (2), (11) we obtain

$$
\| u(x) - P_n(x) \|_{C[x_n, x_{n+1}]} \leq C_2 \frac{\ln^4 N}{N^4}, \ 0 \leq n \leq N/2 - 1.
$$

(12)

Let us $N/2 \leq n \leq N - 1$, then

$$
|u(x) - P_n(x)| = \left| \int_{x_{n+3}}^{x} (x - s)^3 u^{(4)}(s)ds \right| \leq C \int_{x}^{x_{n+3}} (s - x)^3 (1 + \varepsilon^{-4} e^{-\alpha s/\varepsilon})ds \leq CH^4 +
$$

$$
+ C e^{-\alpha s/\varepsilon} \int_{x}^{x_{n+3}} (s - x)^3 e^{-\alpha(x-\sigma)/\varepsilon}ds = CH^4 + C e^{-\alpha s/\varepsilon} \frac{1}{\varepsilon} \int_{x}^{x_{n+3}} \left( \frac{s - x}{\varepsilon} \right)^3 e^{-\alpha(x-\sigma)/\varepsilon}ds \leq
$$

(13)
Similarly, virtue of the equivalence of norms in the space of polynomials \[7\] for some constant \(C \) the difference \(u(x) - P_n(x)\) is a polynomial of degree three on the interval \([x_n, x_{n+1}]\). By virtue of the equivalence of norms in the space of polynomials \[7\] for some constant \(C_7\)

\[
S(x, u) - P_n(x) \mid_{[x_n, x_{n+1}]} \leq C_7 \|S(x, u) - P_n(x)\|_{C[x_n, x_{n+1}]}, \quad n \geq N/2.
\]  

The difference \(S(x, u) - P_n(x)\) is a polynomial of degree three on the interval \([x_n, x_{n+1}]\). By virtue of the equivalence of norms in the space of polynomials \[7\] for some constant \(C_7\)

\[
h_{n+1} \mid S'(x, u) - P'_n(x) \mid_{C[x_n, x_{n+1}]} \leq C_7 \|S'(x, u) - P'_n(x)\|_{C[x_n, x_{n+1}]},
\]

Using (20)–(22) we obtain

\[
\|S'(x, u) - P'_n(x)\|_{C[x_n, x_{n+1}]} \leq C_8 \frac{\ln^3 N}{\varepsilon N^3}, \quad 0 \leq n < N/2,
\]

\[
\|S''(x, u) - P''_n(x)\|_{C[x_n, x_{n+1}]} \leq C_8 \left[ \frac{1}{N^3} + \frac{1}{\varepsilon N^4} e^{-\beta(n-\frac{3}{2})} \right], \quad n \geq N/2.
\]

Similarly,

\[
\|S''(x, u) - P''_n(x)\|_{C[x_n, x_{n+1}]} \leq C_8 \frac{\ln^3 N}{\varepsilon N^3}, \quad 0 \leq n < N/2,
\]

\[
\|S''(x, u) - P''_n(x)\|_{C[x_n, x_{n+1}]} \leq C_8 \left[ \frac{1}{N^2} + \frac{1}{\varepsilon N^4} e^{-\beta(n-\frac{3}{2})} \right], \quad n \geq N/2.
\]

Using the estimates (12), (15), (16), (18), (19), (23)–(26), we obtain (7)–(10). The theorem is proved.
4. Modification of cubic spline

In accordance with (6), the error estimate of the spline \(S(x, u)\) is not \(\varepsilon\)-uniform. In [1] it is proved that this estimate is unimprovable. A modification of the spline \(S(x, u)\) is proposed, in which the error estimate becomes \(\varepsilon\)-uniform. The modified spline \(S_M(x, u)\) is constructed by replacing the interpolation node \(x_{N/2}\) by \((x_{N/2} + x_{N/2+1})/2\). In [1] the following estimate is proved:

\[
\| u(x) - S_M(x, u) \|_{C[0,1]} \leq CN^{-4}\ln^4 N.
\] (27)

Let us study the problem of calculating the derivatives of the function \(u(x)\) of the form (1) on the basis of the spline \(S_M(x, u)\).

By analogy with Theorem 1, the following estimates are obtained:

\[
\varepsilon|u'(x) - S'_M(x, u)| \leq C\frac{\ln^3 N}{N^3}, \quad x \leq \sigma;
\]

\[
|u'(x) - S'_M(x, u)| \leq C\left[\frac{1}{N^3} + \frac{1}{\varepsilon N^4}e^{-\alpha(x-\sigma)/\varepsilon}\right], \quad x > \sigma,
\]

\[
\varepsilon^2|u''(x) - S''_M(x, u)| \leq C\frac{\ln^2 N}{N^2}, \quad x \leq \sigma,
\]

\[
|u''(x) - S''_M(x, u)| \leq C\left[\frac{1}{N^2} + \frac{1}{\varepsilon^2 N^4}e^{-\alpha(x-\sigma)/\varepsilon}\right], \quad x > \sigma,
\]

where \(\sigma\) corresponds to (4).

On the basis of estimates (28)–(31), it is possible to identify the region outside the boundary layer in which the absolute error estimates are \(\varepsilon\)-uniform:

\[
\varepsilon|u'(x) - S'_M(x, u)| \leq C\frac{\ln^3 N}{N^3}, \quad x \leq \sigma; \quad |u'(x) - S'_M(x, u)| \leq \frac{C}{N^3}, \quad x \geq \sigma, \varepsilon N \geq 1;
\]

\[
\varepsilon|u'(x) - S'_M(x, u)| \leq C\frac{\ln^3 N}{N^3}, \quad x \leq \sigma^*, \varepsilon N \leq 1;
\]

\[
|u'(x) - S'_M(x, u)| \leq \frac{C}{N^3}, \quad x \geq \sigma^* = \sigma - \frac{\varepsilon}{\alpha} \ln(\varepsilon N), \varepsilon N \leq 1;
\]

\[
\varepsilon^2|u''(x) - S''_M(x, u)| \leq C\frac{\ln^2 N}{N^2}, \quad x \leq \sigma; \quad |u''(x) - S''_M(x, u)| \leq \frac{C}{N^2}, \quad x > \sigma, \varepsilon N \geq 1;
\]

\[
\varepsilon^2|u''(x) - S''_M(x, u)| \leq C\frac{\ln^2 N}{N^2}, \quad x \leq \sigma^{**}, \varepsilon N \leq 1;
\]

\[
|u''(x) - S''_M(x, u)| \leq \frac{C}{N^2}, \quad x > \sigma^{**} = \sigma - \frac{2\varepsilon}{\alpha} \ln(\varepsilon N), \varepsilon N \leq 1.
\]

5. Numerical results

We define the function of the form (1)

\[
u(x) = \cos \frac{\pi x}{2} + e^{-\frac{x}{2}}, \quad x \in [0, 1].
\] (32)

The results of the calculations are summarized in three tables. The tables show the maximum errors in computing the derivatives on the basis of splines, calculated at the nodes of the thickened mesh obtained from the given computational mesh by dividing each of its mesh intervals into 10 equal parts.
Table 1. Errors and calculated accuracy orders of cubic spline on the uniform grid in the calculation of the first derivative of the function (32)

| $\varepsilon$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ |
|---------------|-------|-------|-------|-------|-------|-------|
| $10^{-1}$     | $1.11 \cdot 10^{-4}$ | $1.38 \cdot 10^{-5}$ | $1.73 \cdot 10^{-6}$ | $2.16 \cdot 10^{-7}$ | $2.70 \cdot 10^{-8}$ | $3.38 \cdot 10^{-9}$ |
| $10^{-2}$     | $3.00$ | $3.00$ | $3.00$ | $3.00$ | $3.00$ | $3.00$ |

Table 2. Errors and calculated accuracy orders of cubic spline on Shishkin mesh in the calculation of the first derivative of the function (32)

| $\varepsilon$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ |
|---------------|-------|-------|-------|-------|-------|-------|
| $10^{-1}$     | $1.11 \cdot 10^{-4}$ | $1.38 \cdot 10^{-5}$ | $1.72 \cdot 10^{-6}$ | $2.16 \cdot 10^{-7}$ | $2.70 \cdot 10^{-8}$ | $3.38 \cdot 10^{-9}$ |
| $10^{-2}$     | $3.00$ | $3.00$ | $3.00$ | $3.00$ | $3.00$ | $3.00$ |

The Table 1 contains relative errors and orders of accuracy in computing the first derivative based on the cubic spline $S(x, u)$ on the uniform grid. Spline interpolation loses the convergence property for small values of the parameter $\varepsilon$.

The Table 2 contains relative errors and orders of accuracy in computing the first derivative based on the cubic spline $S(x, u)$ on Shishkin mesh (3). The results of the calculations are consistent with the error estimate (7).

The Table 3 contains relative errors and orders of accuracy in computing the second derivative based on the spline $S(x, u)$ on the mesh (3). The results are consistent with the estimate (9).

The errors in computing derivatives based on the modified spline $S_M(x, u)$ are close to the errors in the application of the cubic spline $S(x, u)$.
Table 3. Errors and calculated accuracy orders of cubic spline on Shishkin mesh in the calculation of the second derivative of the function (32)

| $\varepsilon$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | $2^7$ | $2^8$ |
|---------------|-------|-------|-------|-------|-------|-------|
| $10^{-1}$     | 2.00  | 2.00  | 2.00  | 2.00  | 2.00  | 2.00  |
| $10^{-2}$     | 2.76·10^{-2} | 7.54·10^{-3} | 1.96·10^{-3} | 5.00·10^{-4} | 1.26·10^{-4} | 1.99 |
| $10^{-3}$     | 1.73  | 1.88  | 1.94  | 1.97  | 1.97  | 1.99  |
| $10^{-4}$     | 1.07·10^{-1} | 4.94·10^{-2} | 1.97·10^{-2} | 7.11·10^{-3} | 2.40·10^{-3} | 1.57 |
| $10^{-5}$     | 0.85  | 1.12  | 1.33  | 1.47  | 1.47  | 1.57  |
| $10^{-6}$     | 1.07·10^{-1} | 4.94·10^{-2} | 1.97·10^{-2} | 7.11·10^{-3} | 2.40·10^{-3} | 1.57 |

6. Conclusion
The problem of calculating the derivatives of functions with large gradients in the exponential boundary layer is investigated on the basis of the spline interpolation. The cubic spline and its modification are considered. On the basis of differentiation of these splines an approximation of the first and second derivatives on the entire interval is obtained. The estimates of the error are uniform in the small parameter. In this case, the relative error is estimated in the boundary layer, and the absolute error is estimated outside the boundary layer.

Acknowledgments
The work was supported by the program of fundamental scientific researches of the SB RAS 1.1.3., project 0314-2016-0009.

References
[1] Blatov I A, Zadorin A I and Kitaeva E V 2017 Cubic spline interpolation of functions with high gradients in boundary layers Comput. Math. Math. Phys. 57 9-28
[2] Miller J J H, O’Riordan E and Shishkin G I 2012 Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions (Singapore: World Scientific)
[3] Zadorin A I and Zadorin N A 2010 Spline interpolation on a uniform grid for functions with a boundary-layer component Comput. Math. Math. Phys. 50 211-223
[4] Zadorin A I and Zadorin N A 2012 Interpolation formula for functions with a boundary layer component and its application to derivatives calculation Sib. Electron. Math. Rep. 9 445-455
[5] Zadorin A I 2016 Interpolation formulas for functions with large gradients in the boundary layer and their application Modeling and Analysis of Information Systems 23 377-384
[6] Ashberg J H, Nilson E N and Walsh J L 1967 The theory of splines and their applications (New York: Academic Press)
[7] De Boor C 1978 Practical guide to splines (New York: Springer-Verlag)
[8] Blatov I A, Zadorin A I and Kitaeva E V 2017 Parabolic spline interpolation for functions with large gradient in the boundary layer Sib. Mat. Zhurn. 58 578-590
[9] Blatov I A, Zadorin A I and Kitaeva E V 2017 On the uniform convergence of parabolic spline interpolation on the class of functions with large gradients in the boundary layer Numer. Anal. Appl. 10 108-119
[10] Blatov I A, Zadorin A I and Kitaeva E V 2018 On the uniform in the parameter convergence of the exponential spline- interpolation with the presence of a boundary layer Comput. Math. Math. Phys. 58 348-363