Universal graded characters and limit of Lusztig $q$-analogues

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Abstract

Let $G$ be a symplectic or orthogonal complex Lie group with Lie algebra $\mathfrak{g}$. As a $G$-module, the decomposition of the symmetric algebra $S(\mathfrak{g})$ into its irreducible components can be explicitly obtained by using identities due to Littlewood. We show that the multiplicities appearing in the decomposition of the $k$-th graded component of $S(\mathfrak{g})$ do not depend on the rank $n$ of $\mathfrak{g}$ providing $n$ is sufficiently large. Thanks to a classical result by Kostant, we establish a similar result for the $k$-th graded component of the space $H(\mathfrak{g})$ of $G$-harmonic polynomials. These stabilization properties are equivalent to the existence of a limit in infinitely many variables for the graded characters associated to $S(\mathfrak{g})$ and $H(\mathfrak{g})$. The limits so obtained are formal series with coefficients in the ring of universal characters introduced by Koike and Terada.

From Hesselink expression of the graded character of harmonics, the coefficient of degree $k$ in the Lusztig $q$-analogue $K_{\lambda,\mu}(q)$ associated to the fixed partition $\lambda$ thus stabilizes for $n$ sufficiently large. By using Morris-type recurrence formulas, we prove that this is also true for the polynomials $K_{\lambda,\mu}(q)$ where $\mu$ is a nonempty fixed partition. This can be reformulated in terms of a stability property for the dimension of the components of the Brylinski-Kostant filtration. We also associate to each pair of partitions $(\lambda,\mu)$ formal series $K_{\lambda,\mu}^{\text{se}}(q)$ and $K_{\lambda,\mu}^{\text{sp}}(q)$, which can be regarded as natural limit of the Lusztig $q$-analogues. One gives a duality property for these limits and obtains simple expressions when $\lambda$ is a row or a column partition.

1 Introduction

The multiplicity $K_{\lambda,\mu}$ of the weight $\mu$ in the irreducible finite-dimensional representation $V^{\theta}(\lambda)$ of the simple Lie group $G$ with Lie algebra $\mathfrak{g}$ can be written in terms of the ordinary Kostant partition function $\mathcal{P}$ defined by the equality

$$\prod_{\alpha \text{ positive root}} \frac{1}{1 - e^\alpha} = \sum_{\beta} \mathcal{P}(\beta)e^\beta$$

where $\beta$ runs on the set of nonnegative integral combinations of positive roots of $\mathfrak{g}$. Thus $\mathcal{P}(\beta)$ is the number of ways the weight $\beta$ can be expressed as a sum of positive roots. Then, one derives from the Weyl character formula

$$K_{\lambda,\mu}^{\theta} = \sum_{w \in W^\theta} (-1)^{t(w)}\mathcal{P}(w(\lambda + \rho) - (\mu + \rho))$$

(1)

where $W^\theta$ is the Weyl group of $\mathfrak{g}$. 
The Lusztig \( q \)-analogue of weight multiplicity \( K_{\lambda,\mu}^q(q) \) is obtained by substituting the ordinary Kostant partition function \( P \) by its \( q \)-anologue \( P_q \) in (1). Namely \( P_q \) is defined by the equality

\[
\prod_{\alpha \text{ positive root}} \frac{1}{(1 - qe^\alpha)} = \sum_{\beta} P_q(\beta)e^\beta
\]

and we have

\[
K_{\lambda,\mu}^q(q) = \sum_{w \in W^g} (-1)^{\ell(w)}P_q(w(\lambda + \rho) - (\mu + \rho)).
\]

As shown by Lusztig [14], \( K_{\lambda,\mu}^q(q) \) is a polynomial in \( q \) with nonnegative integer coefficients. Many interpretations of the Lusztig \( q \)-analogues exist. In particular, they can be obtained from the Brylinski-Kostant filtration of weight spaces [1]. The polynomials \( K_{\lambda,\theta}^q(q) \) appear in the graded character of the harmonic polynomials associated to \( g [4] \). We also recover the Lusztig \( q \)-analogues as the coefficients of the expansion of the Hall-Littlewood polynomials on the basis of Weyl characters (see [17]). This notably permits to prove that there are affine Kazhdan-Lusztig polynomials. In [11] Lascoux and Schützenberger have obtained a combinatorial expression for \( K_{\lambda,\theta}^q(q) \) in terms of the charge statistic on the semistandard tableaux of shape \( \lambda \) and evaluation \( \mu \). By using the combinatorics of crystal graphs introduced by Kashiwara and Nakashima [5], we have also established similar formulas [12], [13] for the Lusztig \( q \)-analogues associated to the symplectic and orthogonal Lie algebras when \((\lambda, \mu)\) satisfies restrictive constraints.

Consider \( \lambda, \mu \) two partitions of length at most \( m \). These partitions can be regarded as dominant weights for \( g = gl_n, so_{2n+1}, sp_{2n} \) or \( so_{2n} \) when \( n \geq m \). Then, \( K_{\lambda,\mu}^{gl_n}(q) \) does not depend on the rank \( n \) considered. Such a property does not hold for the Lusztig \( q \)-analogues \( K_{\lambda,\mu}^g(q) \) when \( g = so_{2n+1}, sp_{2n} \) or \( so_{2n} \) which depend in general on the rank of the Lie algebra considered. Write

\[
K_{\lambda,\mu}^g(q) = \sum_{k \geq 0} K_{\lambda,\mu}^{g,k}q^k.
\]

We first establish in this paper that for \( g = so_{2n+1}, sp_{2n} \) or \( so_{2n} \), the coefficient \( K_{\lambda,\mu}^{g,k} \) stabilizes when \( n \) tends to the infinity. More precisely, \( K_{\lambda,\mu}^{g,k} \) does not depend on the rank \( n \) of \( g \) providing \( n \geq 2k + a \) where \( a \) is the number of nonzero parts of \( \mu \) (Theorem [4.3.1]). By Brylinski’s interpretation of the coefficients \( K_{\lambda,\mu}^{g,k} [1] \), one then obtains that the dimension of the \( k \)-th component of the Brylinski-Kostant filtration associated to the finite-dimensional irreducible representations of \( g = so_{2n+1}, sp_{2n} \) or \( so_{2n} \) stabilizes for \( n \) sufficiently large (Theorem [4.4.2]). Observe that this stabilization is immediate for \( g = gl_n \) since the polynomials \( K_{\lambda,\theta}^{gl_n}(q) \) does not depend on \( n \). For \( g = so_{2n+1}, sp_{2n} \) or \( so_{2n} \) the Brylinski-Kostant filtration depends in general on the rank considered and it seems difficult to obtain the dimension of its components by direct computations.

Our methods is as follows. We obtain the explicit decomposition of the symmetric algebra \( S(g) \) considered as a \( G \)-module into its irreducible components by using identities due to Littlewood. This permits to show that the multiplicities appearing in the decomposition of the \( k \)-th graded component of \( S(g) \) do not depend on the rank \( n \) of \( g \) providing \( n \) is sufficiently large. Thanks to a classical result by Kostant, we establish a similar result for the \( k \)-th graded component of the space \( H(g) \) of \( G \)-harmonic polynomials. These stabilization properties is equivalent to the existence of a limit in infinitely many variables for the graded characters associated to \( S(g) \) and \( H(g) \). The limits so obtained are formal series with coefficients in the ring of universal characters introduced.
by Koike and Terada. From Hesslink expression \[^4\] of the graded character of \(H(\mathfrak{g})\), one then derives that \(K^{g,k}_{\lambda,\emptyset}\) stabilizes for \(n\) sufficiently large. By using Morris-type recurrence formulas for the Lusztig \(q\)-analogues \[^3\], we prove that this is also true for the coefficients \(K^{g,k}_{\lambda,\mu}\) where \(\mu\) is a nonempty fixed partition. We also observe that these formulas permit to give an explicit lower bound for the degree of the \(q\)-analogues \(K^{g,k}_{\lambda,\mu}(q)\) such that \(K^{g,k}_{\lambda,\mu}(q) \neq 0\). We establish that the limits of the coefficients \(K^{so,2n+1,k}_{\lambda,\mu}\) and \(K^{so,2n+1,k}_{\lambda,\mu}\) are the same. Write \(K^{sp,k}_{\lambda,\mu}\) and \(K^{so,k}_{\lambda,\mu}\) respectively for the limits of the coefficients \(K^{so,2n+1,k}_{\lambda,\mu}\) and \(K^{sp,2n,k}_{\lambda,\mu}\) when \(n\) tends to the infinity.

The stabilization property of the coefficients \(K^{g,k}_{\lambda,\mu}\) suggests then to introduce the formal series

\[
K^{so}_{\lambda,\mu}(q) = \sum_{k \geq 0} K^{so,k}_{\lambda,\mu} q^k \quad \text{and} \quad K^{sp}_{\lambda,\mu}(q) = \sum_{k \geq 0} K^{sp,k}_{\lambda,\mu} q^k.
\]

These series belong \(\mathbb{N}[[q]]\) and can be regarded as natural limits of the polynomials \(K^{g}_{\lambda,\mu}(q)\). We establish a duality result between the formal series \(K^{so}_{\lambda,\emptyset}(q)\) and \(K^{sp}_{\lambda,\emptyset}(q)\) (Theorem 5.3.1). Namely, we have

\[
K^{so}_{\lambda,\emptyset}(q) = K^{sp}_{\lambda',\emptyset}(q)
\]

where \(\lambda'\) is the conjugate partition of \(\lambda\). Note that (2) do not hold in general if we replace the formal series \(K^{so}_{\lambda,\mu}(q)\) and \(K^{sp}_{\lambda,\mu}(q)\) by the polynomials \(K^{g,k}_{\lambda,\mu}(q)\). We also give recurrence formulas \[^3\], \[^4\] for the series \(K^{so}_{\lambda,\mu}(q)\) and \(K^{sp}_{\lambda,\mu}(q)\). Thanks to these recurrence formulas, one derives simple expressions for the formal series \(K^{X}_{\lambda,\emptyset}(q)\) when \(\lambda\) is a row or a column partition (Proposition 5.4.1). Note that we have not find so simple formulas for the Lusztig \(q\)-analogues \(K^{g}_{\lambda,\mu}(q)\) even in the cases when \(\lambda\) is a column or a row partition. Moreover the duality (2) is false in general for the polynomials \(K^{g}_{\lambda,\mu}(q)\). This suggests that the study of the series \(K^{so}_{\lambda,\mu}(q)\) and \(K^{sp}_{\lambda,\mu}(q)\) which is initiated in this paper, could be easier than that of the Lusztig \(q\)-analogues.

The paper is organized as follows. In Section 2 we recall the necessary background on symplectic and orthogonal Lie algebras, universal characters, and Lusztig \(q\)-analogues which is needed in the sequel. In Section 3 we introduce universal graded characters as limit in infinitely many variables for the graded characters associated to \(S(\mathfrak{g})\) and \(H(\mathfrak{g})\). We obtain the stabilization property of the coefficients \(K^{g,k}_{\lambda,\mu}\) in Section 4 and reformulate this result in terms of the Brylinski-Kostant filtration. In Section 5, we introduce the formal series \(K^{so}_{\lambda,\mu}(q)\) and \(K^{sp}_{\lambda,\mu}(q)\), establish recurrence formulas which permit to compute them by induction, prove the duality (2) and give explicit formulas for \(K^{so}_{\lambda,\mu}(q)\) and \(K^{sp}_{\lambda,\mu}(q)\) when \(\lambda\) is a row or a column partition. Finally in Section 6 we have added a few considerations on the possibility to define Hall-Littlewood polynomials from the formal series \(K^{so}_{\lambda,\mu}(q)\) and \(K^{sp}_{\lambda,\mu}(q)\).

2 Background

2.1 Convention for the root systems of types B, C and D

In the sequel \(G\) is one of the complex Lie groups \(Sp_{2n}, SO_{2n+1}\) or \(SO_{2n}\) and \(\mathfrak{g}\) its Lie algebra. We follow the convention of \[^7\] to realize \(G\) as a subgroup of \(GL_N\) and \(\mathfrak{g}\) as a subalgebra of \(\mathfrak{gl}_N\) where

\[
N = \begin{cases} 
2n & \text{when } G = Sp_{2n} \\
2n+1 & \text{when } G = SO_{2n+1} \\
2n & \text{when } G = SO_{2n}
\end{cases}
\]
With this convention the maximal torus $T$ of $G$ and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ coincide respectively with the subgroup and the subalgebra of diagonal matrices of $G$ and $\mathfrak{g}$. Similarly the Borel subgroup $B$ of $G$ and the Borel subalgebra $\mathfrak{b}_+$ of $\mathfrak{g}$ coincide respectively with the subgroup and subalgebra of upper triangular matrices of $G$ and $\mathfrak{g}$. This gives the triangular decomposition $\mathfrak{g} = \mathfrak{b}_+ \oplus \mathfrak{h} \oplus \mathfrak{b}_-$ for the Lie algebra $\mathfrak{g}$. Let $e_i, h_i, f_i, i \in \{1, \ldots, n\}$ be a set of Chevalley generators such that $e_i \in \mathfrak{b}_+, h_i \in \mathfrak{h}$ and $f_i \in \mathfrak{b}_-$ for any $i$.

Let $d_N$ be the linear subspace of $\mathfrak{gl}_N$ consisting of the diagonal matrices. For any $i \in \{1, \ldots, n\}$, write $\varepsilon_i$ for the linear map $\varepsilon_i : d_N \to \mathbb{C}$ such that $\varepsilon_i(D) = \delta_i$ for any diagonal matrix $D$ whose $(i, i)$-coefficient is $\delta_i$. Then $(\varepsilon_1, \ldots, \varepsilon_n)$ is an orthonormal basis of the Euclidean space $\mathfrak{h}^*_\mathbb{R}$ (the real part of $\mathfrak{h}^*$). We denote by $<\cdot, \cdot>$ the usual scalar product on $\mathfrak{h}^*_\mathbb{R}$.

Let $R$ be the root system associated to $G$. We can take for the simple roots of $\mathfrak{g}$

$$
\begin{align*}
\Sigma^+ &= \{\alpha_n = \varepsilon_n \text{ and } \alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1 \text{ for the root system } B_n\} \\
\Sigma^+ &= \{\alpha_n = 2\varepsilon_n \text{ and } \alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1 \text{ for the root system } C_n\} \\
\Sigma^+ &= \{\alpha_n = \varepsilon_n + \varepsilon_{n-1} \text{ and } \alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1 \text{ for the root system } D_n\}
\end{align*}
$$

(3)

Then the sets of positive roots are

$$
\begin{align*}
R^+ &= \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } B_n \\
R^+ &= \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } C_n \\
R^+ &= \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } D_n
\end{align*}
$$

We denote by $R$ the set of roots of $G$. For any $\alpha \in R$, let $\alpha^\vee = \frac{\alpha}{<\alpha, \alpha>}$ be the coroot corresponding to $\alpha$. The Weyl group of the Lie group $G$ is the subgroup of the permutation group of the set $\{\overline{n}, \overline{2}, \overline{1}, 1, 2, \ldots, n\}$ generated by the permutations

$$
\begin{align*}
\begin{cases}
s_i = (i, i + 1)(i, i + 1), & i = 1, \ldots, n-1 \text{ and } s_n = (n, \overline{n}) \text{ for the root systems } B_n \text{ and } C_n \\
s_i = (i, i + 1)(i, i + 1), & i = 1, \ldots, n-1 \text{ and } s'_n = (n, \overline{n-1})(n-1, \overline{n}) \text{ for the root system } D_n
\end{cases}
\end{align*}
$$

where for $a \neq b$ $(a, b)$ is the simple transposition which switches $a$ and $b$. We identify the subgroup of $W^g$ generated by $s_i = (i, i + 1)(i, i + 1), i = 1, \ldots, n-1$ with the symmetric group $S_n$. We denote by $\ell$ the length function corresponding to the above set of generators. The action of $w \in W^g$ on $\beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{h}^*_\mathbb{R}$ is defined by

$$
w \cdot (\beta_1, \ldots, \beta_n) = (\beta_1^{w^{-1}}, \ldots, \beta_n^{w^{-1}})
$$

where $\beta_i^w = \beta_{w(i)}$ if $w(i) \in \{1, \ldots, n\}$ and $\beta_i^w = -\beta_{w(i)}$ otherwise. We denote by $\rho$ the half sum of the positive roots of $R^+$. The dot action of $W^g$ on $\beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{h}^*_\mathbb{R}$ is defined by

$$
w \circ \beta = w \cdot (\beta + \rho) - \rho.
$$

(4)

Write $P$ and $P^+$ for the weight lattice and the cone of dominant weights of $G$. As usual we consider the order on $P$ defined by $\beta \leq \gamma$ if and only if $\beta - \gamma \in Q^+$.

For any positive integer $m$, denote by $P_m$ the set of partitions with at most $m$ nonzero parts. Let $P_m(k), k \in \mathbb{N}$ be the subset of $P_m$ containing the partitions $\lambda$ such that $|\lambda| = \lambda_1 + \cdots + \lambda_m = k$.

Set $P = \cup_{m \in \mathbb{N}} P_m$ and $P_m[k] = \cup_{a \leq k} P_m(a)$.

Each partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n$ will be identified with the dominant weight $\sum_{i=1}^n \lambda_i \varepsilon_i$. Then the irreducible finite-dimensional polynomial representations of $G$ are parametrized by the partitions
of \( P \). For any \( \lambda \in P \), denote by \( V^\theta(\lambda) \) the irreducible finite-dimensional representation of \( G \) of highest weight \( \lambda \). The representation \( V^\theta(1) \) associated to the partition \( \lambda = (1) \) is called the vector representation of \( G \). For any weight \( \beta \in P \) and any partition \( \lambda \in P \), we write \( V^\theta(\lambda)_{\beta} \) for the weight space associated to \( \beta \) in \( V^\theta(\lambda) \).

We denote by \( Q \) the root lattice of \( g \) and write \( Q^+ \) for the elements of \( Q \) which are linear combination of positive roots with nonnegative coefficients.

The exponents \( \{m_1, ..., m_n\} \) of the root system \( R \) verifies \( m_i = 2i - 1 \), \( i = 1, ..., n \) when \( R \) is of type \( B_n \) or \( C_n \) and

\[
m_i = 2i - 1, \quad i \in \{1, ..., n - 1\} \quad \text{and} \quad m_n = n - 1
\]

when \( R \) is of type \( D_n \).

**Remarks:**

(i) : The integer \( n - 1 \) appears twice in the exponents of a root system of type \( D_n \) when \( n \) is even.

(ii) : The exponents \( m_i, i = 1, ..., n - 1 \) are the same for the three root systems of type \( B_n, C_n \) or \( D_n \).

As customary, we identify \( P \) the lattice of weights of \( G \) with a sublattice of \((\frac{1}{2} \mathbb{Z})^n\). For any \( \beta = (\beta_1, ..., \beta_n) \in P \), we set \( |\beta| = \beta_1 + \cdots + \beta_n \). We use for a basis of the group algebra \( \mathbb{Z}[\mathbb{Z}^n] \), the formal exponentials \( (e^\beta)_{\beta \in \mathbb{Z}^n} \) satisfying the relations \( e^{\beta_1}e^{\beta_2} = e^{\beta_1+\beta_2} \). We furthermore introduce \( n \) independent indeterminates \( x_1, ..., x_n \) in order to identify \( \mathbb{Z}[\mathbb{Z}^n] \) with the ring of polynomials \( \mathbb{Z}[x_1, ..., x_n, x_1^{-1}, ..., x_n^{-1}] \) by setting \( e^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} = x^\beta \) for any \( \beta = (\beta_1, ..., \beta_n) \in \mathbb{Z}^n \).

Write \( s^{\text{gl}_n}_\lambda \) for the Weyl character (Schur function) of \( V^{\text{gl}_n}(\lambda) \) the finite-dimensional \( \text{gl}_n \)-module of highest weight \( \lambda \). The character ring of \( GL_n \) is \( \Lambda_n = \mathbb{Z}[x_1, ..., x_n]^{\text{sym}} \) the ring of symmetric functions in \( n \) variables.

For any \( \lambda \in P \), we denote by \( s^{\text{gl}_n}_\lambda \) the Weyl character of \( V^{\text{gl}_n}(\lambda) \). Let \( R^{\text{gl}} \) be the character ring of \( G \). Then

\[
R^{\text{gl}} = \mathbb{Z}[x_1, ..., x_n, x_1^{-1}, ..., x_n^{-1}]^{\text{W}^{\text{gl}}}
\]

is the \( \mathbb{Z} \)-algebra with basis \( \{ s^{\text{gl}_n}_\lambda \mid \lambda \in P \} \).

In the sequel we will suppose \( n \geq 2 \) when \( g = \text{sp}_{2n} \) or \( \text{so}_{2n+1} \) and \( n \geq 4 \) when \( g = \text{so}_{2n} \).

For each Lie algebra \( g = \text{so}_N \) or \( \text{sp}_N \) and any partition \( \nu \in P \), we denote by \( V^{\text{gl}_N}(\nu) \downarrow^{\text{gl}_N}_g \) the restriction of \( V^{\text{gl}_N}(\nu) \) to \( g \). Set

\[
V^{\text{gl}_N}(\nu) \downarrow^{\text{gl}_N}_g = \bigoplus_{\lambda \in P} V^{\text{so}_N}(\lambda) \otimes b^{\text{so}_N}_{\nu, \lambda},
\]

\[
V^{\text{so}_2n}(\nu) \downarrow^{\text{so}_2n}_g = \bigoplus_{\lambda \in P} V^{\text{so}_2n}(\lambda) \otimes b^{\text{so}_2n}_{\nu, \lambda}.
\]

This define in particular the branching coefficients \( b^{\text{so}_N}_{\nu, \lambda} \) and \( b^{\text{so}_2n}_{\nu, \lambda} \). The restriction map \( r^{\text{gl}} \) is defined by setting

\[
r^{\text{gl}} : \begin{cases} \mathbb{Z}[x_1, ..., x_N]^{\text{sym}} \to R^{\text{gl}} \\ s^{\text{gl}_N}_{\nu} \mapsto \text{char}(V^{\text{gl}_N}(\nu) \downarrow^{\text{gl}_N}_g) \end{cases}.
\]

We have then

\[
r^{\text{gl}}(s^{\text{gl}_N}_{\nu}) = \begin{cases} s^{\text{gl}_N}(x_1, ..., x_n, x_n^{-1}, ..., x_1^{-1}) \quad \text{when} \ N = 2n \\ s^{\text{gl}_N}(x_1, ..., x_n, 0, x_n^{-1}, ..., x_1^{-1}) \quad \text{when} \ N = 2n + 1 
\end{cases}.
\]
Let $\mathcal{P}_n^{(2)}$ and $\mathcal{P}_n^{(1,1)}$ be the subsets of $\mathcal{P}_n$ containing the partitions with even length rows and the partitions with even length columns, respectively. When $\nu \in \mathcal{P}_n$ we have the following formulas for the branching coefficients $b_{\nu,\lambda}^{s0_N}$ and $b_{\nu,\lambda}^{p2n}$:

**Proposition 2.1.1** (see [6] appendix p 295)

Consider $\nu \in \mathcal{P}_n$. Then:

1. $b_{\nu,\lambda}^{s0_{2n+1}} = b_{\nu,\lambda}^{s0_{2n}} = \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\lambda,\gamma}^{\nu}$
2. $b_{\nu,\lambda}^{p2n} = \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c_{\lambda,\gamma}^{\nu}$

where $c_{\gamma,\lambda}^{\nu}$ is the usual Littlewood-Richardson coefficient corresponding to the partitions $\gamma, \lambda$ and $\nu$.

Note that the equality $b_{\nu,\lambda}^{s0_{2n+1}} = b_{\nu,\lambda}^{s0_{2n}}$ becomes false in general when $\nu \notin \mathcal{P}_n$.

As suggested by Proposition 2.1.1, the manipulation of the Weyl characters is simplified by working with infinitely many variables. In [6], Koike and Terada have introduced a universal character ring for the classical Lie groups. This ring can be regarded as the ring $\Lambda = \mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}}$ of symmetric functions in countably many variables. It is equipped with three natural $\mathbb{Z}$-bases indexed by partitions, namely

\[ B^{sp} = \{ s_{\lambda}^{sp} \mid \lambda \in \mathcal{P} \}, \ B^{so} = \{ s_{\lambda}^{so} \mid \lambda \in \mathcal{P} \} \text{ and } B^{gl} = \{ s_{\lambda}^{gl} \mid \lambda \in \mathcal{P} \}. \]  

(7)

We have in particular the decompositions:

\[ s_{\nu}^{gl} = \sum_{\lambda \in \mathcal{P}} \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\lambda,\gamma}^{\nu} s_{\lambda}^{so} \text{ and } s_{\nu}^{gl} = \sum_{\lambda \in \mathcal{P}} \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c_{\lambda,\gamma}^{\nu} s_{\lambda}^{sp}. \]  

(8)

We denote by $\varphi$ the linear involution defined on $\Lambda$ by

\[ \varphi(s_{\lambda}^{so}) = s_{\lambda}^{sp}. \]  

(9)

For any positive integer $n$, denote by $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}}$ the ring of symmetric functions in $n$ variables. Write

\[ \pi_n : \mathbb{Z}[x_1, \ldots, x_n, \ldots]^{\text{sym}} \rightarrow \mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}} \]

for the ring homomorphism obtained by specializing each variable $x_i, i > n$ at 0. Then $\pi_n(s_{\lambda}^{gl}) = s_{\lambda}^{gl}$. Let $\pi^{sp_{2n}}$ and $\pi^{s0_{N}}$ be the specialization homomorphisms defined by setting $\pi^{sp_{2n}} = r^{sp_{2n}} \circ \pi_{2n}$ and $\pi^{s0_{N}} = r^{s0_{N}} \circ \pi_{N}$. For any partition $\lambda \in \mathcal{P}_n$ one has

\[ s_{\lambda}^{sp_{2n}} = \pi^{sp_{2n}}(s_{\lambda}^{sp}) \text{ and } s_{\lambda}^{s0_{N}} = \pi^{s0_{N}}(s_{\lambda}^{so}). \]

When $\lambda \notin \mathcal{P}_N$, we have $\pi^{sp_{2n}}(s_{\lambda}^{sp}) = 0$ and $\pi^{s0_{N}}(s_{\lambda}^{so}) = 0$. The situation is more complicated when $\lambda \in \mathcal{P}_N$ but $\lambda \notin \mathcal{P}_n$, that is if $d(\lambda)$ the number of parts of $\lambda$ verifies $n < d(\lambda) \leq N$. In this case one shows by using determinantal identities for the Weyl characters (see [6] Proposition 2.4.1) that

\[ \pi^{sp_{2n}}(s_{\lambda}^{sp}) = \pm s_{\eta_C}^{sp_{2n}}, \ \pi^{s0_{2n+1}}(s_{\lambda}^{so}) = \pm s_{\eta_B}^{s0_{2n+1}} \text{ and } \pi^{s0_{2n}}(s_{\lambda}^{so}) = \pm s_{\eta_D}^{s0_{2n}} \]

where the signs $\pm$ and the partitions $\eta_B, \eta_C, \eta_D$ are determined by simple combinatorial procedures.

Note that we have in this case $|\eta_X| \leq |\lambda|$ for $X = B, C, D$.

We shall also need the following proposition (see [6] Corollary 2.5.3).
Proposition 2.1.2  Consider a Lie algebra $g$ of type $X_n \in \{B_n, C_n, D_n\}$. Let $\lambda \in \mathcal{P}_r$ and $\mu \in \mathcal{P}_s$. Suppose $n \geq r + s$ and set

$$V^g(\lambda) \otimes V^g(\mu) = \bigoplus_{\nu \in \mathcal{P}_n} V^g(\nu) \otimes d^\nu_{\lambda, \mu}.$$  

Then the coefficients $d^\nu_{\lambda, \mu}$ do not depend on the rank $n$ of $g$ neither of its type $B, C$ or $D$.

Remark: The previous proposition follows from the decompositions

$$s^{sp}_\lambda \times s^{sp}_\mu = \sum_{\nu \in \mathcal{P}} d^\nu_{\lambda, \mu} s^{sp}_\nu$$

and

$$s^{so}_\lambda \times s^{so}_\mu = \sum_{\nu \in \mathcal{P}} d^\nu_{\lambda, \mu} s^{so}_\nu$$

for any $\lambda, \mu \in \mathcal{P}$, in the ring $\Lambda$.

2.2 Lusztig $q$-analogues

The $q$-analogue $\mathcal{P}_q$ of the Kostant partition function associated to the root system $R$ of the Lie algebra $g$ is defined by the equality

$$\prod_{\alpha \in R^+} \frac{1}{1 - q e^\alpha} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q(\beta) e^\beta.$$  

Note that $\mathcal{P}_q^{B_n}(\beta) = 0$ if $\beta \notin Q^+$. Given $\lambda$ and $\mu$ two partitions of $\mathcal{P}_n$, the Lusztig $q$-analogues of weight multiplicity is the polynomial

$$K^g_{\lambda, \mu}(q) = \sum_{w \in W^g} (-1)^{\ell(w)} \mathcal{P}_q(w \circ \lambda - \mu).$$

It follows from the Weyl character formula that $K^g_{\lambda, \mu}(1)$ is equal to the dimension of $V^g(\lambda)$.

Theorem 2.2.1  (Lusztig [14])

For any partitions $\lambda, \mu \in \mathcal{P}_n$, the polynomial $K^g_{\lambda, \mu}(q)$ has nonnegative integer coefficients.

We write

$$K^g_{\lambda, \mu}(q) = \sum_{k \geq 0} K^{g,k}_{\lambda, \mu} q^k. \quad (10)$$

Then

$$K^{g,k}_{\lambda, \mu}(q) = \sum_{w \in W^g} (-1)^{\ell(w)} \mathcal{P}^k(w \circ \lambda - \mu) \quad (11)$$

where for any $\beta \in \mathbb{Z}^n$, $\mathcal{P}^k(\beta)$ is the number of ways of decomposing $\beta$ as a sum of $k$ positive roots.

Remark: One verifies easily that $K^g_{\lambda, \mu}(q) \neq 0$ only if $\lambda \geq \mu$. Moreover, when $|\mu| = |\lambda|$, one has $K^g_{\lambda, \mu}(q) = K^{g}_\lambda(\mu)$ where $K^{g}_\lambda(\mu)$ is the Kostka polynomial associated to $(\lambda, \mu)$, i.e. the Lusztig $q$-analogue associated to the partitions $\lambda, \mu$ for the root system $A_{n-1}$.

We also introduce the Hall-Littlewood polynomials $Q^g_\mu$, $\mu \in \mathcal{P}_n$ defined by

$$Q^g_\mu = \sum_{\lambda \in \mathcal{P}_n} K^g_{\lambda, \mu}(q) s^g_\lambda.$$
2.3 The symmetric algebra $S(g)$

Considered as a $G$-module, $g$ is irreducible and we have

$$\begin{cases}
\mathfrak{so}_{2n+1} \simeq V^g(1,1) \text{ and } \dim(\mathfrak{so}_{2n+1}) = n(2n + 1) \\
\mathfrak{sp}_{2n} \simeq V^g(2) \text{ and } \dim(\mathfrak{sp}_{2n}) = n(2n + 1) \\
\mathfrak{so}_{2n} \simeq V^g(1,1) \text{ and } \dim(\mathfrak{so}_{2n}) = n(2n - 1)
\end{cases} \tag{12}$$

Let $S(g)$ be the symmetric algebra associated to $g$ and set

$$S(g) = \bigoplus_{k \geq 0} S^k(g)$$

where $S^k(g)$ is the $k$-th symmetric power of $g$. By Proposition 2.1.1 and (12), we have

$$g \simeq V^{\mathfrak{gl}_N(1,1)} \downarrow^{g}_{\mathfrak{so}_N} \text{ for } g = \mathfrak{so}_N \text{ and } g \simeq V^{\mathfrak{gl}_{2n}(2)} \downarrow^{g}_{\mathfrak{sp}_{2n}} \text{ for } g = \mathfrak{sp}_{2n}.$$ 

This implies the following isomorphisms

$$S^k(g) \simeq S^k(V^{\mathfrak{gl}_N(1,1)}) \downarrow^{g}_{\mathfrak{so}_N} \text{ for } g = \mathfrak{so}_N \text{ and } S^k(g) \simeq S^k(V^{\mathfrak{gl}_{2n}(2)}) \downarrow^{g}_{\mathfrak{sp}_{2n}} \text{ for } g = \mathfrak{sp}_{2n}. \tag{13}$$

for any nonnegative integer $k$.

**Example 2.3.1** By using the Weyl dimension formula (see [3] page 303), one easily obtain the decompositions

$$S^2(V^{\mathfrak{gl}_N(1,1)}) \simeq V^{\mathfrak{gl}_N(1,1,1)} \oplus V^{\mathfrak{gl}_N(2,2)}$$

and

$$S^2(V^{\mathfrak{gl}_{2n}(2)}) \simeq V^{\mathfrak{gl}_{2n}(4)} \oplus V^{\mathfrak{gl}_{2n}(2,2)}.$$ 

Hence by (13) and Proposition 2.1.1 this gives

$$S^2(g) \simeq V^g(1,1,1) \oplus V^g(2,2) \oplus V^g(2,0) \oplus V^g(0) \text{ for } g = \mathfrak{so}_N$$

and

$$S^2(\mathfrak{sp}_{2n}) \simeq V^{\mathfrak{sp}_{2n}(4)} \oplus V^{\mathfrak{sp}_{2n}(2,2)} \oplus V^{\mathfrak{sp}_{2n}(1,1)} \oplus V^{\mathfrak{sp}_{2n}(0)}.$$ 

**Remark:** By the previous formulas, the multiplicities appearing in the decomposition of the square symmetric power of the Lie algebra $g$ of type $X_n \in \{B_n, C_n, D_n\}$ do not depend on its rank providing $n \geq 2$. We give in Proposition 3.1.1 the general explicit decomposition of $S^k(g)$ into its irreducible components.

3 Graded characters

3.1 Graded character of the symmetric algebra

Let $V$ be a $G$ or $GL_n$-module. For any nonnegative integer $k$, write $S^k(V)$ for the $k$-th symmetric power of $V$ and set $S(V) = \bigoplus_{k \geq 0} S^k(V)$. Then $S^k(V)$ and $S(V)$ are also $G$-modules. The graded character of $S(V)$ is defined by

$$\text{char}_q(S(V)) = \sum_{k \geq 0} \text{char}(S^k(V))q^k.$$
Denote by $W(V)$ the collection of weight of the module $V$ counted with their multiplicities. Then we have

$$
\text{char}_q(S(V)) = \prod_{\beta \in W(V)} \frac{1}{1 - qe^{\beta}}.
$$

The weights of the Lie algebra $\mathfrak{g}$ of rank $n$ considered as a $G$-module are such that

$$W(\mathfrak{g}) = \{\alpha \in R, 0, \ldots, 0\ \text{n times}\}.$$

Thus the graded character $\text{char}_q(S(\mathfrak{g}))$ of $S(\mathfrak{g})$ verifies

$$\text{char}_q(S(\mathfrak{g})) = \frac{1}{(1-q)^n} \prod_{\alpha \in R} \frac{1}{1 - qx^{\alpha}}.$$  (14)

**Proposition 3.1.1** For any nonnegative integer $k$, we have

$$
\begin{align*}
\text{char}_q(S^k(\mathfrak{so}_N)) &= \sum_{\lambda \in \mathcal{P}} \sum_{\nu \in \mathcal{P}_{N}^{(1,1)}(2k)} b_{\nu,\lambda}^{\mathfrak{so}_N} s_{\lambda}^{\mathfrak{so}_N}, \\
\text{char}_q(S^k(\mathfrak{sp}_{2n})) &= \sum_{\lambda \in \mathcal{P}} \sum_{\nu \in \mathcal{P}_{2n}^{(2)}(2k)} b_{\nu,\lambda}^{\mathfrak{sp}_{2n}} s_{\lambda}^{\mathfrak{sp}_{2n}}
\end{align*}
$$

where $b_{\nu,\lambda}^{\mathfrak{so}_N}$ and $b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}$ are the branching coefficients defined in [6].

**Proof.** Suppose first $\mathfrak{g} = \mathfrak{sp}_{2n}$. Recall the classical identity

$$
\prod_{1 \leq i < j \leq 2n} \frac{1}{1 - q_{x_i,x_j}} = \sum_{\nu \in \mathcal{P}_{2n}^{(2)}} s_{\nu}^{\mathfrak{gl}_{2n}}
$$

due to Littlewood. It immediately implies the decomposition

$$
\prod_{1 \leq i < j \leq 2n} \frac{1}{1 - q_{x_i,x_j}} = \sum_{\nu \in \mathcal{P}_{2n}^{(2)}} q^{\nu} s_{\nu}^{\mathfrak{gl}_{2n}} = \sum_{k \geq 0} \sum_{\nu \in \mathcal{P}_{2n}^{(2)}(2k)} s_{\nu}^{\mathfrak{gl}_{2n}} q^k.
$$

By applying the restriction map $r_{\mathfrak{sp}_{2n}}$, this gives

$$
\frac{1}{(1-q)^n} \prod_{1 \leq i < j \leq n} \frac{1}{1 - q^2} \sum_{1 \leq r \leq s \leq n} \frac{1}{1 - q_{x_r,x_s}} = \sum_{\nu \in \mathcal{P}_{2n}^{(2)}(2k)} s_{\nu}^{\mathfrak{gl}_{2n}} (x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) q^k.
$$

From (5), this can be rewritten on the form

$$
\text{char}_q(S(\mathfrak{sp}_{2n})) = \frac{1}{(1-q)^n} \prod_{\alpha \in R} \frac{1}{1 - qx^{\alpha}} = \sum_{k \geq 0} \sum_{\lambda \in \mathcal{P}} \sum_{\nu \in \mathcal{P}_{2n}^{(2)}(2k)} b_{\nu,\lambda}^{\mathfrak{so}_N} s_{\lambda}^{\mathfrak{sp}_{2n}} q^k.
$$
which gives the desired identity by considering the coefficient in $q^k$.
When $g = \mathfrak{so}_{2n+1}$ or $g = \mathfrak{so}_{2n}$, one uses the identity
\[
\prod_{1 \leq i < j \leq 2n} \frac{1}{1 - q^{x_i x_j}} = \sum_{\nu \in \mathcal{P}_{2n}^{(1,1)}} q^{\nu_+} s_{\nu}^{\mathfrak{so}_{2n}} = \sum_{k \geq 0} \sum_{\nu \in \mathcal{P}_{2n}^{(1,1)}(2k)} s_{\nu}^{\mathfrak{so}_{2n}} q^k
\]
and our result follows by similar arguments. ■

In the sequel, we set
\[
m_{k,\lambda}^{\mathfrak{so}_N} = \sum_{\nu \in \mathcal{P}^{(1,1)}(2k)} b_{\nu,\lambda}^{\mathfrak{so}_N} \quad \text{and} \quad m_{k,\lambda}^{\mathfrak{sp}_{2n}} = \sum_{\nu \in \mathcal{P}^{(2)}_{2n}} b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}.
\]
Thus we have
\[
\text{char}_q(S^k(\mathfrak{so}_N))) = \sum_{\nu \in \mathcal{P}_{n}} m_{k,\lambda}^{\mathfrak{so}_N} s_{\lambda}^{\mathfrak{so}_N} \quad \text{and} \quad \text{char}_q(S^k(\mathfrak{sp}_{2n})) = \sum_{\nu \in \mathcal{P}_{n}} m_{k,\lambda}^{\mathfrak{sp}_{2n}} s_{\lambda}^{\mathfrak{sp}_{2n}}.
\]

### 3.2 Universal graded characters $\text{char}_q(\mathfrak{sp})$ and $\text{char}_q(\mathfrak{so})$

**Proposition 3.2.1** Consider a nonnegative integer $k$ and a partition $\lambda \in \mathcal{P}_m$. Suppose $n \geq 2k$. Then we have the identities
\[
m_{k,\lambda}^{\mathfrak{so}_{2n+1}} = m_{k,\lambda}^{\mathfrak{so}_{2n}} = \sum_{\nu \in \mathcal{P}^{(1,1)}(2k)} \sum_{\gamma \in \mathcal{P}^{(1)}_{2k}} c_{\nu,\gamma}^{\lambda} \quad \text{and} \quad m_{k,\lambda}^{\mathfrak{sp}_{2n}} = \sum_{\nu \in \mathcal{P}^{(2)}_{2n}} \sum_{\gamma \in \mathcal{P}^{(1)}_{2k}} c_{\nu,\gamma}^{\lambda}.
\]
In particular, the multiplicities $m_{k,\lambda}^{\mathfrak{so}_{2n}}, m_{k,\lambda}^{\mathfrak{so}_{2n+1}}$ and $m_{k,\lambda}^{\mathfrak{sp}_{2n}}$ do not depend on $n$.

**Proof.** For any $\nu \in \mathcal{P}_{N}(2k)$ we have $\nu \in \mathcal{P}_{N}(n)$ since $n \geq 2k$. We can thus deduce from Propositions 2.1.1 and 3.1.1 the decompositions
\[
m_{k,\lambda}^{\mathfrak{so}_N} = \sum_{\nu \in \mathcal{P}^{(1,1)}(2k)} \sum_{\gamma \in \mathcal{P}^{(1)}_{2k}} c_{\nu,\gamma}^{\lambda} \quad \text{and} \quad m_{k,\lambda}^{\mathfrak{sp}_{2n}} = \sum_{\nu \in \mathcal{P}^{(2)}_{2n}} \sum_{\gamma \in \mathcal{P}^{(1)}_{2k}} c_{\nu,\gamma}^{\lambda}.
\]
Since $c_{\nu,\gamma}^{\lambda} = 0$ when $|\lambda| + |\gamma| \neq 2k$, $m_{k,\lambda}^{\mathfrak{so}_N}$ and $m_{k,\lambda}^{\mathfrak{sp}_{2n}}$ can be rewritten as in (15) and thus, do not depend on $n$. ■

We set
\[
m_{k,\lambda}^{\mathfrak{so}_N} = \lim_{n \to \infty} m_{k,\lambda}^{\mathfrak{so}_{2n+1}} = \sum_{\nu \in \mathcal{P}^{(1,1)}(2k)} \sum_{\gamma \in \mathcal{P}^{(1)}_{2k}} c_{\nu,\gamma}^{\lambda} \quad \text{and} \quad m_{k,\lambda}^{\mathfrak{sp}_{2n}} = \lim_{n \to \infty} m_{k,\lambda}^{\mathfrak{sp}_{2n}} = \sum_{\nu \in \mathcal{P}^{(2)}_{2n}} \sum_{\gamma \in \mathcal{P}^{(1)}_{2k}} c_{\nu,\gamma}^{\lambda}.
\]

**Lemma 3.2.2** For any nonnegative integer $k$ and any partition $\lambda$, we have
1. $m_{k,\lambda}^{\mathfrak{sp}} = m_{k,\lambda}^{\mathfrak{so}}$,
2. $m_{k,\lambda}^{\mathfrak{sp}} = m_{k,\lambda}^{\mathfrak{so}} = 0$ if $|\lambda| > 2k$

where $m_{k,\lambda}^{\mathfrak{sp}}$ and $m_{k,\lambda}^{\mathfrak{so}}$ are the multiplicities defined in (16).
Proposition 3.2.3

By (14) and (17), we have then defined by setting

\[ \text{char}_q(S) = \prod_{1 \leq i < j} \frac{1}{1-qx_i x_j} \]

Note that \( \text{char}_q(S) \) and \( \text{char}_q(S') \) belong to the ring \( \mathbb{Z}[[q]] \) of formal series with coefficients in \( \Lambda \).

For any \( F = \sum_{k \geq 0} c_k q^k \) in \( \Lambda[[q]] \), the specialization homomorphisms \( \pi^{sp_{2n}}, \pi^{so_{2n+1}} \) and \( \pi^{so_{2n}} \) are then defined by setting

\[ \pi^{sp}(F) = \sum_{k \geq 0} \pi_q^u(c_k) q^k. \]

Similarly, the linear involution \( \varphi \) (see (9)) is defined on \( \Lambda[[q]] \) by

\[ \varphi(F) = \sum_{k \geq 0} \varphi(c_k) q^k. \]

Observe that \( \{ q^k s_{\lambda}^{gl} \mid k \in \mathbb{N}, \lambda \in \mathcal{P} \}, \{ q^k s_{\lambda}^{so} \mid k \in \mathbb{N}, \lambda \in \mathcal{P} \} \) and \( \{ q^k s_{\lambda}^{sp} \mid k \in \mathbb{N}, \lambda \in \mathcal{P} \} \) are \( \mathbb{Z} \)-bases of \( \Lambda[[q]] \).

Proposition 3.2.3 We have the decompositions

\[ \text{char}_q(S) = \sum_{k \geq 0} \sum_{\lambda \in \mathcal{P}} m_{k,\lambda}^{sp} s_{\lambda}^{sp} q^k \]

and

\[ \text{char}_q(S') = \sum_{k \geq 0} \sum_{\lambda \in \mathcal{P}} m_{k,\lambda}^{sp} s_{\lambda}^{sp} q^k. \]

Proof. One can write

\[ \text{char}_q(S) = \prod_{1 \leq i < j} \frac{1}{1-qx_i x_j} = \sum_{k \geq 0} \sum_{\nu \in \mathcal{P}(1,1)[2k]} s_{\nu}^{gl} \ q^k = \sum_{k \geq 0} \sum_{\lambda \in \mathcal{P}} m_{k,\lambda}^{so} s_{\lambda}^{so} q^k \]

\[ \text{char}_q(S') = \prod_{1 \leq i < j} \frac{1}{1-qx_i x_j} = \sum_{k \geq 0} \sum_{\nu \in \mathcal{P}(2)[2k]} s_{\nu}^{gl} \ q^k = \sum_{k \geq 0} \sum_{\lambda \in \mathcal{P}} m_{k,\lambda}^{sp} s_{\lambda}^{sp} q^k \]

where the rightmost equalities follow from (9). By using 1 of Lemma 3.2.2, one derives the following corollary:

Corollary 3.2.4 We have \( \varphi(\text{char}_q(S)) = \text{char}_q(S) \).

Remark: By 2 of Lemma 3.2.2, one has

\[ \text{char}_q(S^k) = \sum_{\lambda \in \mathcal{P}[2k]} m_{k,\lambda}^{sp} s_{\lambda}^{sp} \]

and

\[ \text{char}_q(S^k) = \sum_{\lambda \in \mathcal{P}[2k]} m_{k,\lambda}^{sp} s_{\lambda}^{sp} \]

where \( \mathcal{P}[2k] \) is the set of partitions \( \lambda \) such that \( |\lambda| \leq 2k \).
3.3 Universal graded character for harmonic polynomials

Let \( g \) be a Lie algebra of type \( X_n \in \{ B_n, C_n, D_n \} \). Since the symmetric algebra \( S(g) \) can be regarded as a \( G \)-module, one can consider

\[
S(g)^G = \{ x \in S(g) \mid g \cdot x = x \text{ for any } g \in G \}
\]

the ring of the \( G \)-invariants in \( S(g) \). By a classical theorem of Kostant \cite{6} we have

\[
S(g) = H(g) \otimes S(g)^G
\]

(23)

where \( H(g) \) is the ring of \( G \)-harmonic polynomials. The ring \( S(g)^G \) is generated by algebraically independent homogeneous polynomials of degrees \( d_i = m_i + 1 \) and the graded character of \( S(g)^G \) considered as a \( G \)-module verifies

\[
\text{char}_q(S(g)^G) = \prod_{i=1}^{n} \frac{1}{1 - q^{d_i}}.
\]

(24)

By (23) the graded character of \( H(g) \) can be written

\[
\text{char}_q(H(g)) = \frac{\text{char}_q(S(g))}{\text{char}_q(S(g)^G)} = \prod_{i=1}^{n} (1 - q^{d_i}) \text{char}_q(S(g)) = \sum_{k \geq 0} \text{char}(H^k(g))q^k.
\]

We define the universal graded characters \( \text{char}_q(H(\mathfrak{sp})) \) and \( \text{char}_q(H(\mathfrak{so})) \) by setting

\[
\text{char}_q(H(\mathfrak{so})) = \prod_{i \geq 1} (1 - q^{2i}) \text{char}_q(S(\mathfrak{so})) = \prod_{i \geq 1} (1 - q^{2i}) \prod_{1 \leq i < j} \frac{1}{1 - qx_ix_j}
\]

(25)

\[
\text{char}_q(H(\mathfrak{sp})) = \prod_{i \geq 1} (1 - q^{2i}) \text{char}_q(S(\mathfrak{sp})) = \prod_{i \geq 1} (1 - q^{2i}) \prod_{1 \leq i < j} \frac{1}{1 - qx_ix_j}
\]

(26)

The universal characters \( \text{char}_q(S(\mathfrak{sp})) \) and \( \text{char}_q(S(\mathfrak{so})) \) belong to \( \Lambda[[q]] \). Moreover \( \prod_{i \geq 1} (1 - q^{2i}) \in \Lambda[[q]] \) since it is a formal series in \( q \) with integer coefficients. Hence \( \text{char}_q(H(\mathfrak{sp})) \) and \( \text{char}_q(H(\mathfrak{so})) \) also belong to \( \Lambda[[q]] \). We set

\[
\text{char}_q(H(\mathfrak{sp})) = \sum_{k \geq 0} \sum_{\lambda \in \mathcal{P}} K_{\lambda,0}^{\mathfrak{sp},k} s_{\lambda}^{\mathfrak{sp}} q^k \quad \text{and}
\]

\[
\text{char}_q(H(\mathfrak{so})) = \sum_{k \geq 0} \sum_{\lambda \in \mathcal{P}} K_{\lambda,0}^{\mathfrak{so},k} s_{\lambda}^{\mathfrak{so}} q^k.
\]

Lemma 3.3.1 For any nonnegative integer \( k \), we have

\[
\text{char}_q(H^k(\mathfrak{so})) = \sum_{\lambda \in \mathcal{P}[2k]} K_{\lambda,0}^{\mathfrak{so},k} s_{\lambda}^{\mathfrak{so}} \quad \text{and} \quad \text{char}_q(H^k(\mathfrak{sp})) = \sum_{\lambda \in \mathcal{P}[2k]} K_{\lambda,0}^{\mathfrak{sp},k} s_{\lambda}^{\mathfrak{sp}}.
\]

(26)

Moreover, for any nonnegative integer \( n \geq 2k \)

\[
\pi_{\mathfrak{so}N}^{\mathfrak{so}}(\text{char}_q(H^k(\mathfrak{so}))) = \text{char}_q(H^k(\mathfrak{so}_N)) \quad \text{and} \quad \pi_{\mathfrak{sp}2n}^{\mathfrak{sp}}(\text{char}_q(H^k(\mathfrak{sp}))) = \text{char}_q(H^k(\mathfrak{sp}_{2n})).
\]

(27)
Proof. By definition of the coefficients \( K_{\lambda,\emptyset}^{g_0,k} \), we have

\[
\text{char}_q(H^k(g_0)) = \sum_{\lambda \in \mathcal{P}} K_{\lambda,\emptyset}^{g_0,k} s_{\lambda}^{g_0}. \tag{28}
\]

Write \([k/2]\) for the quotient of the Euclidean division of \( k \) by 2. Then by (25), \( K_{\lambda,\emptyset}^{g_0,k} \) is the coefficient of in \( q^k s_{\lambda}^{g_0} \) appearing in the expansion of

\[
\psi_k = \prod_{i=1}^{[k/2]} (1 - q^{2i}) \sum_{a=0}^{k} q^a \text{char}_q(S^a(g_0)) \tag{29}
\]
on the basis \( \{q^k s_{\lambda}^{g_0} \mid k \in \mathbb{N}, \lambda \in \mathcal{P}\} \). Indeed, for any \( i > [k/2] \), we have \( i > k \). By (22),

\[
\text{char}_q(S^a(g_0)) = \sum_{\lambda \in \mathcal{P}[2a]} m_{a,\lambda}^{g_0} s_{\lambda}^{g_0}
\]

for any nonnegative integer \( a \). Since the integers \( a \) appearing in (29) as lower than \( k \), we have by 2 of Lemma 3.2.2 \( m_{a,\lambda}^{g_0} = 0 \) for any partition \( \lambda \) such that \( |\lambda| > 2k \). Hence, the coefficient of \( \psi_k \) on \( q^k s_{\lambda}^{g_0} \) is equal to 0 when \( |\lambda| > 2k \). This permits to consider only the partitions of \( \mathcal{P}[2k] \) and yields (26). The proof is the same for \( \text{char}_q(H^k(sp_2)) \).

Remark: Since (27) only holds for \( n \geq 2k \), we have

\[
\pi^{g_0N}(\text{char}_q(H(g_0))) \neq \text{char}_q(H(g_0)) \quad \text{and} \quad \pi^{sp_{2n}}(\text{char}_q(H(sp_{2n}))) \neq \text{char}_q(H(sp_{2n}))
\]
i.e. there does not exist identities analogous to (19) for the \( \text{char}_q(H(g_0)) \) and \( \text{char}_q(H(sp_{2n})) \).

4 Stabilization of the coefficients \( K_{\lambda,\mu}^{g,k} \)

4.1 Stabilization of the coefficients \( K_{\lambda,\emptyset}^{g,k} \)

The theorem below shows that the coefficients of the expansion of \( \text{char}_q(H(g)) \) on the basis of Weyl characters are the Lusztig \( q \)-analogues associated to the zero weight (i.e. \( \mu = \emptyset \)).
Theorem 4.1.1 (Hesselink [4]) We have

\[ \text{char}_q(H(\mathfrak{g})) = \sum_{\lambda \in \mathcal{P}_n} K^g_{\lambda,0}(q)s_{\lambda}^g = \sum_{k \geq 0} \sum_{\lambda \in \mathcal{P}_n} K^{g,k}_{\lambda,0}q^k s_{\lambda}^g. \]

The multiplicity of \( V^\theta(\lambda) \) in the decomposition of \( H^k(\mathfrak{g}) \) in its irreducible components is equal to \( K^{g,k}_{\lambda,0} \).

Now fix a nonnegative integer \( k \) and choose a rank \( n \geq 2k \). The partitions \( \lambda \) appearing in (26) verify \(|\lambda| \leq 2k\). Hence they belong to \( \mathcal{P}_n \) for \( n \geq 2k \). Since \( \pi^{so_N}(s^\lambda_{\mathfrak{so}}) = s^\lambda_{\mathfrak{so}} \) and \( \pi^{sp_{2n}}(s^\lambda_{\mathfrak{sp}}) = s^\lambda_{\mathfrak{sp}} \) for any \( \lambda \in \mathcal{P}_n \) this gives

\[ \pi^{so_N}(\text{char}_q(H^k(\mathfrak{so}_N))) = \sum_{\lambda \in \mathcal{P}_n} K^{so,k}_{\lambda,0} s_{\lambda}^{so_N} \quad \text{and} \quad \pi^{sp_{2n}}(\text{char}_q(H^k(\mathfrak{sp}_{2n}))) = \sum_{\lambda \in \mathcal{P}_n} K^{sp,k}_{\lambda,0} s_{\lambda}^{sp_{2n}}. \]

By using (27), one obtains

\[ \text{char}_q(H^k(\mathfrak{so}_N)) = \sum_{\lambda \in \mathcal{P}_n} K^{so,k}_{\lambda,0} s_{\lambda}^{so_N} \quad \text{and} \quad \text{char}_q(H^k(\mathfrak{sp}_{2n})) = \sum_{\lambda \in \mathcal{P}_n} K^{sp,k}_{\lambda,0} s_{\lambda}^{sp_{2n}}. \quad (30) \]

We can now state the stabilization result for the coefficients \( K^{g,k}_{\lambda,0} \).

**Proposition 4.1.2** Let \( m, k \) be nonnegative integers. Consider \( \lambda \in \mathcal{P}_m \) and \( \mathfrak{g} \) a Lie algebra of type \( X_n \in \{B_n, C_n, D_n\} \). Suppose \( n \geq 2k \), then

\[ K^{sp_{2n},k}_{\lambda,0} = K^{sp,k}_{\lambda,0} \quad \text{and} \quad K^{so_{2n+1},k}_{\lambda,0} = K^{so,k}_{\lambda,0} = K^{so_{2n},k}_{\lambda,0}. \]

In particular the coefficients \( K^{sp_{2n},k}_{\lambda,0}, K^{so_{2n+1},k}_{\lambda,0} \) and \( K^{so_{2n},k}_{\lambda,0} \) do not depend on the rank \( n \).

**Proof.** By Theorem 4.1.1 we have

\[ \text{char}_q(H^k(\mathfrak{so}_N)) = \sum_{\lambda \in \mathcal{P}_n} K^{so_N,k}_{\lambda,0} s_{\lambda}^{so_N} \quad \text{and} \quad \text{char}_q(H^k(\mathfrak{sp}_{2n})) = \sum_{\lambda \in \mathcal{P}_n} K^{sp_{2n},k}_{\lambda,0} s_{\lambda}^{sp_{2n}}. \]

The Proposition then comes by identifying the coefficients appearing in these decompositions with those appearing in (30). \( \blacksquare \)

Thus the coefficients \( K^{g,k}_{\lambda,0} \) depends only on \( \lambda, k \) and the type \( X = C \) or \( X \in \{B, D\} \) of the Lie algebra considered when \( \text{rank}(\mathfrak{g}) \geq 2k \). Proposition 4.1.2 can also be rewritten on the form

\[ \lim_{n \to \infty} K^{sp_{2n},k}_{\lambda,0} = K^{sp,k}_{\lambda,0} \quad \text{and} \quad \lim_{n \to \infty} K^{so_N,k}_{\lambda,0} = K^{so,k}_{\lambda,0}. \]
4.2 Recurrence formulas for the Lusztig $q$-analogues

We now recall recurrence formulas established in [12] and [13] which permit to express the Lusztig $q$-analogues associated to the root system of type $X_n \in \{B_n, C_n, D_n\}$ in terms of those associated to the root system of type $X_{n-1}$. They can be regarded as generalizations of the Morris recurrence formula which holds for the Lusztig $q$-analogues of type $A$.

Suppose $g$ is of rank $n$ and consider $\nu \in P_m$ with $m \leq n$. For any nonnegative integer $l$, the decomposition of the $g$-module $V^\nu(\nu) \otimes V^\theta(l)$ into its irreducible components can be written

$$V^\nu(\nu) \otimes V^\theta(l) \simeq \bigoplus_{\lambda \in P_n} V^\theta(\lambda) \otimes p^{\nu,\lambda}_{\gamma,l}.$$  

This decomposition can be regarded as the analogue of the Pieri rule in type $X_n$.

**Remark:** The multiplicities $p^{\nu,\lambda}_{\gamma,l}$ are not necessarily equal to 0 or 1 as in the original Pieri rule (i.e. for $\mathfrak{gl}_n$). Moreover we can have $p^{\nu,\lambda}_{\gamma,l} \neq 0$ when $|\lambda| < |\gamma| + l$. Nevertheless, when $|\lambda| = |\gamma| + l$, one shows that the Pieri rule for type $X_n$ coincide with the original one. It means that $p^{\nu,\lambda}_{\gamma,l} = 1$ if $\lambda$ is obtained by adding an horizontal strip of length $l$ to $\gamma$ (i.e. the $l$ boxes of $\lambda/\gamma$ belong to distinct columns) and $p^{\nu,\lambda}_{\gamma,l} = 0$ otherwise.

The following Lemma is a consequence of Proposition 2.1.2.

**Lemma 4.2.1** Consider $\gamma \in P_m$ and $l \in \mathbb{N}$. Suppose that $g$ is a Lie algebra of type $X_n \in \{B_n, C_n, D_n\}$ with $n > m$. Then the multiplicities $p^{\nu,\lambda}_{\gamma,l}$ are independent of the rank $n$ and the type $X \in \{B, C, D\}$ of $g$.

We set

$$p^{\nu,\lambda}_{\nu,l} = \lim_{n \to \infty} p^{\nu,\lambda}_{\nu,l}.$$  

Let $\mu \in P_m$. Write $p$ maximal in $\{1, ..., m\}$ such that $\nu_p - p - \mu_1 + 1 \geq 0$. For any $s \in \{1, ..., p\}$ let $\gamma(s)$ be the partition of length $m - 1$ such that

$$\gamma(s) = \begin{cases} (\nu_2, ..., \nu_m) & \text{if } s = 1 \\ (\nu_1 + 1, \nu_2 + 1, ..., \nu_{s-1} + 1, \nu_{s+1}, ..., \nu_m) & \text{if } s \geq 2 \end{cases}.$$  

Finally set

$$R_s = \nu_s - s - \mu_1 + 1.$$  

**Theorem 4.2.2** [13]

With the above notation, we have for any partitions $\lambda, \mu \in P_m$:

(i) : $K^{so_{2n+1}}(q) = \sum_{s=1}^{p} (-1)^{s-1} \times q^{R_s} \times \sum_{r+2a=R_s} \sum_{\lambda \in P_{m-1}} p^{so_{2n-1},\lambda}_{\gamma(s),r} K^{so_{2n-1}}_{\lambda,\mu^{\lambda}}(q)$

(ii) : $K^{sp_{2n}}(q) = \sum_{s=1}^{p} (-1)^{s-1} \times \sum_{r+2a=R_s} q^{r+a} \times \sum_{\lambda \in P_{m-1}} p^{sp_{2n-2},\lambda}_{\gamma(s),r} K^{sp_{2n-2}}_{\lambda,\mu^{\lambda}}(q)$

(iii) : $K^{so_{2n}}(q) = \sum_{s=1}^{p} (-1)^{s-1} \times q^{R_s} \times \sum_{r+2a=R_s} \sum_{\lambda \in P_{m-1}} p^{so_{2n-2},\lambda}_{\gamma(s),r} K^{so_{2n-2}}_{\lambda,\mu^{\lambda}}(q)$

where $a \in \mathbb{N}$ and $\mu^{\lambda} = (\mu_2, ..., \mu_m)$.  

15
By using similar arguments to those given in Example 4 page 243 of [15], one shows that the polynomials \( K^q_{\lambda,\mu}(q) \) are monic of degree
\[
\begin{aligned}
\sum_{i=1}^{n} (n-i+1)(\lambda_i - \mu_i) &\quad \text{for } g = \mathfrak{so}_{2n+1} \\
\sum_{i=1}^{n} (n-i+1/2)(\lambda_i - \mu_i) &\quad \text{for } g = \mathfrak{sp}_{2n} \\
\sum_{i=1}^{n} (n-i)(\lambda_i - \mu_i) &\quad \text{for } g = \mathfrak{so}_{2n}
\end{aligned}
\]

Thank to the recurrence formulas of Theorem 4.2.2, one can derive a lower bound for the lowest degree appearing in \( K^q_{\nu,\mu}(q) \) when \( K^q_{\nu,\mu}(q) \neq 0 \).

**Proposition 4.2.3** For any partitions \( \nu, \mu \in \mathcal{P}_n \), \( K^q_{\nu,\mu}(q) = 0 \) or has degree at least \( \frac{|\nu| - |\mu|}{2} \).

**Proof.** We give the proof for \( g = \mathfrak{sp}_{2n} \), the arguments are essentially the same for \( g = \mathfrak{so}_{2n+1} \) and \( g = \mathfrak{so}_{2n} \). We proceed by induction on \( n \). For \( n = 1 \), one has \( K^q_{\nu,\mu}(q) = 0 \) or \( K^q_{\nu,\mu}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \).

Now suppose \( K^q_{\lambda,\mu}(q) \) has degree at least \( \frac{|\lambda| - |\mu|}{2} \) or is equal to 0. Consider \( \nu, \mu \in \mathcal{P}_n \) such that \( K^q_{\nu,\mu}(q) \neq 0 \). We apply recurrence formula (ii) of Theorem 4.2.2. Since \( K^q_{\nu,\mu}(q) \neq 0 \), there exist integers \( s \) and \( r \leq R_s \) such that \( K^q_{\lambda,\mu}(q) \neq 0 \) with \( p_{\gamma(s),r} \neq 0 \). One then have \( |\lambda| \geq |\gamma(s)| - r \).

The definition \( (\ref{31}) \) of \( \gamma(s) \) gives \( |\gamma(s)| = |\nu| - \nu_s + s - 1 \). By the induction hypothesis, the polynomial \( q^{r+s}K^q_{\lambda,\mu}(q) \) has degree at least
\[
d = r + a + \frac{|\lambda| - |\mu| + \mu_1}{2} \geq \frac{1}{2} r + a + \frac{|\nu| - \nu_s + s - 1 - |\mu| + \mu_1}{2}.
\]

One other hand, we have \( \frac{1}{2} r + a = \frac{1}{2} R_s \) for \( r + 2a = R_s \). Recall that \( R_s = \nu_s - s - \mu_1 + 1 \). This finally gives
\[
d \geq \frac{\nu_s - s - \mu_1 + 1}{2} + \frac{|\nu| - \nu_s + s - 1 - |\mu| + \mu_1}{2} \geq \frac{|\nu| - |\mu|}{2}.
\]

**Remark:** By the previous proposition, the coefficients \( K^q_{\nu,\mu} \) are all equal to zero when \( k < \frac{|\nu| - |\mu|}{2} \).

This implies in particular that we have the decomposition
\[
Q^q_{\mu} = \sum_{\nu \in \mathcal{P}_n} K^q_{\nu,\mu}(q) s^q_{\nu} = \sum_{k \geq 0} \sum_{\nu \in \mathcal{P}_n[2k+|\mu|]} K^q_{\nu,\mu} s^q_{\nu} q^k \tag{33}
\]
for the Hall-Littlewood functions \( Q^q_{\mu} \).

Since the integer \( p \) and the partitions \( \gamma(s) \) defined above do not depend on the rank of \( g \), we obtain from Lemma 4.2.4 and Theorem 4.2.2:

**Corollary 4.2.4** (of Theorem 4.2.2)
For any partitions \( \lambda, \mu \in \mathcal{P}_m \) and any integer \( n \geq m \)

(i) \( K^q_{\lambda,\mu}(q) = \sum_{s=1}^{p} (-1)^{s-1} \times q^{R_s} \times \sum_{r+2a=R_s, \lambda \in \mathcal{P}_{m-1}} p_\lambda^{\gamma(s),r} K^q_{\lambda,\mu}(q) \)

(ii) \( K^q_{\nu,\mu}(q) = \sum_{s=1}^{p} (-1)^{s-1} \times q^{r+a} \sum_{r+2a=R_s} \sum_{\lambda \in \mathcal{P}_{m-1}} p_\lambda^{\gamma(s),r} K^q_{\nu,\mu}(q) \)

(iii) \( K^q_{\nu,\mu}(q) = \sum_{s=1}^{p} (-1)^{s-1} \times q^{R_s} \times \sum_{r+2a=R_s} \sum_{\lambda \in \mathcal{P}_{m-1}} p_\lambda^{\gamma(s),r} K^q_{\nu,\mu}(q) \)
where \( a \in \mathbb{N} \) and \( \mu^b = (\mu_2, \ldots, \mu_m) \).

The Lemma below will be useful to derive the recurrence formulas of paragraph 5.2.

**Lemma 4.2.5** The partitions \( \lambda \) appearing in the right hand sides of the previous formulas for which there exists a pair \((\gamma(s), r)\) such that \( p_{\gamma(s), r}^\lambda \neq 0 \) must verify one of the following assertions:

1. \( \lambda = \nu \) and then \( \mu = \emptyset \), \( s = 1 \), \( r = R_1 = \nu_1 \), \( \gamma_1 = \nu^b \),
2. \( |\lambda| < |\nu| \),
3. \( |\lambda| = |
u| \) with \( \lambda \neq \nu \) and then \( \mu = \emptyset \) and \( |\lambda'| < |\nu'| \).

In particular \( |\lambda| = |\nu| \) only if \( \mu = \emptyset \).

**Proof.** Consider \( \lambda \) and \((\gamma(s), r)\) such that \( p_{\gamma(s), r}^\lambda \neq 0 \). We must have \( |\lambda| \leq r + |\gamma(s)| = R_s - (R_s - r) + |\gamma(s)| \). By definition of \( R_s \) [32] and \( \gamma(s) \) [31], we obtain \( |\lambda| \leq |\nu| - \mu_1 - (R_s - r) \). Thus \( |\lambda| < |\nu| \) when \( \mu \neq \emptyset \) and \( |\lambda| = |\nu| \) only if \( \mu = \emptyset \) and \( r = R_s \). This permits to restrict ourselves to the case when \( \mu = \emptyset \), \( |\lambda| = |\nu| \) and \( r = R_s \).

Suppose first \( \lambda = \nu \). Then we must have \( s = 1 \). Otherwise \( \gamma(s)_1 = \nu_1 + 1 \) and we would have \( \lambda_1 > \nu_1 \). This gives \( \mu = \emptyset \), \( s = 1 \), \( r = R_1 = \nu_1 \) and \( \gamma_1 = \nu^b \) as desired.

Now suppose \( |\lambda| = |\nu| \) with \( \lambda \neq \nu \). Observe that \( |\lambda'| < |\nu'| \) if and only if \( \lambda_1 > \nu_1 \). When \( s > 1 \), we have \( \gamma(s)_1 = \nu_1 + 1 > \nu_1 \), thus \( \lambda_1 > \nu_1 \) (see Remark before Lemma 4.2.1). When \( s = 1 \), we have \( \gamma(1) = (\nu_2, \ldots, \nu_n) = \nu^b \) and \( r = \nu_1 \). Since \( |\lambda| = |\nu| \) and \( p_{\nu', \nu}^\lambda \neq 0 \), \( \lambda \) is obtained by adding a horizontal strip of length \( \nu_1 \) on \( \nu^b \). This implies \( \lambda_1 > \nu_1 \) because the number of columns in \( \nu^b \) is equal to \( \nu_2 \leq \nu_1 \) and we have assumed \( \lambda \neq \nu \).

### 4.3 Stabilization of the coefficients \( K_{\lambda, \mu}^{g,k} \)

**Theorem 4.3.1** Consider \( m \) a nonnegative integer and \( \nu, \mu \) two partitions such that \( \nu \in \mathcal{P}_m \) and \( \mu \in \mathcal{P}_a \). Let \( g \) be a Lie algebra of type \( X_n \in \{B_n, C_n, D_n\} \) and \( k \) a nonnegative integer. Then for any \( n \geq 2k + a \), the coefficients \( K_{\nu, \mu}^{g,k} \) do not depend on the rank \( n \) of \( g \). Under these hypothesis, we have moreover \( K_{\nu, \mu}^{g,2^k+1} = K_{\nu, \mu}^{g,2^k} \).

**Proof.** Suppose first \( g = so_{2n+1} \). We proceed by induction on \( a \). Note that we can suppose \( a \leq m \), otherwise \( K_{\nu, \mu}^{so_{2n+1},k} = 0 \) for any rank \( n \). If \( a = 0 \), then \( \mu = \emptyset \) and the theorem follows directly from Proposition 4.1.2. Suppose now our theorem true for any partition \( \mu^b \) of length \( a - 1 \) with \( 1 \leq a < m \) and consider \( \mu \) a partition of length \( a \). We then apply the recurrence formulas of Corollary 4.2.4. It follows from [32] that the integers \( R_s \) appearing in these formulas do not depend on the rank \( n \) considered. This is also true for the multiplicities \( p_{\gamma(s),r}^\lambda \).

By our induction hypothesis, for any \( p \in \mathbb{N} \), the coefficients \( K_{\lambda, \mu}^{so_{2n-1},p} \) are independent of \( n \). Indeed \( \mu^b \in \mathcal{P}_{a-1} \) and \( so_{2n-1} \) has rank \( n - 1 \geq 2k + a - 1 \). The recurrence formulas of Corollary 4.2.4 imply that each coefficient \( K_{\nu, \mu}^{so_{2n+1},k} \) can be expressed in terms of the coefficients \( K_{\lambda, \mu}^{so_{2n-1},p} \), the integers \( R_s \) and \( p_{\gamma(s),r}^\lambda \). Moreover, this decomposition is independent of \( n \). Hence \( K_{\nu, \mu}^{so_{2n+1},k} \) does not depend on \( n \). By using similar arguments, we prove that \( K_{\nu, \mu}^{g,k} \) do not depend on \( n \) when \( g = sp_{2n} \) or \( so_{2n} \).
The equality $K_{\nu, \mu}^{\mathfrak{s}_02n+1, k} = K_{\nu, \mu}^{\mathfrak{s}_02n, k}$ is yet obtained by induction on $a$. It is true for $a = 0$ by Proposition 4.1.2 and the induction follows from the fact that the recurrence formulas of Corollary 4.2.4 are the same for $\mathfrak{s}_02n+1$ and $\mathfrak{s}_02n$.

**Remark:** The arguments used in the previous proof imply that it is possible to decompose any Lusztig $q$-analogue $K_{\nu, \mu}(q)$ such that $\mu \neq \emptyset$ in terms of the Lusztig $q$-analogues $K_{\lambda, \emptyset}(q)$. Moreover this decomposition is independent on the rank $n$ providing this rank is sufficiently large. In this case the decomposition is the same for $K_{\nu, \mu}^{\mathfrak{s}_02n+1}(q)$ and $K_{\nu, \mu}^{\mathfrak{s}_02n}(q)$. Nevertheless, these two polynomials do not coincide since $K_{\nu, \emptyset}^{\mathfrak{s}_02n+1}(q) \neq K_{\nu, \emptyset}^{\mathfrak{s}_02n}(q)$ in general. The previous Theorem also establishes the equality $K_{\nu, \mu}^{\mathfrak{s}_02n+1, k}(q) = K_{\nu, \mu}^{\mathfrak{s}_02n, k}(q)$ for any $k \leq \frac{n-a}{2}$ where $a$ is the number of nonzero parts in $\mu$.

By Theorem 4.3.1 it makes sense to set

$$K_{\nu, \mu}^{\mathfrak{s}_0} = \lim_{n \to \infty} K_{\nu, \mu}^{\mathfrak{s}_02n+1, k} = \lim_{n \to \infty} K_{\nu, \mu}^{\mathfrak{s}_02n, k}$$

and

$$K_{\nu, \mu}^{\mathfrak{s}_p} = \lim_{n \to \infty} K_{\nu, \mu}^{\mathfrak{s}_p2n+1, k} = \lim_{n \to \infty} K_{\nu, \mu}^{\mathfrak{s}_p2n, k}.$$  

(34)

### 4.4 A reformulation in terms of the Brylinski-Kostant filtration

Recall that the Lusztig $q$-analogue $K_{\nu, \mu}^{\mathfrak{g}}(q)$ can also be characterized from the Brylinski-Kostant filtration on the weight space $V^{\mathfrak{g}}(\lambda, \mu)$ \cite{1}. Take $e = e_1 + \cdots + e_n \in \mathfrak{u}_+$ for a principal nilpotent in $\mathfrak{g}$ compatible with $\mathfrak{h}$. The $e$-filtration of $V^{\mathfrak{g}}(\lambda, \mu)$ is the finite filtration $J_e(V^{\mathfrak{g}}(\lambda, \mu))$ such that

$$\{0\} \subset J^0_e(V^{\mathfrak{g}}(\lambda, \mu)) \subset J^1_e(V^{\mathfrak{g}}(\lambda, \mu)) \subset \cdots$$

where for any nonnegative integer $k$,

$$J^k_e(V^{\mathfrak{g}}(\lambda, \mu)) = \{v \in V^{\mathfrak{g}}(\lambda, \mu) \mid e^{k+1}(v) = 0\}.$$  

For completeness we also set $J^{-1}_e(V^{\mathfrak{g}}(\lambda, \mu)) = \{0\}$. The following theorem is a consequence of the main result of \cite{1}.

**Theorem 4.4.1 (Brylinski)** Consider $m$ a nonnegative integer and $\lambda, \mu \in \mathcal{P}_m$. Let $\mathfrak{g}$ be a Lie algebra of type $X_n \in \{B_n, C_n D_n\}$. Then

$$K_{\lambda, \mu}^{\mathfrak{g}}(q) = \sum_{k \geq 0} \dim(J^k_e(V^{\mathfrak{g}}(\lambda, \mu))/J^{k-1}_e(V^{\mathfrak{g}}(\lambda, \mu)))q^k.$$  

(35)

By using Theorem 4.3.1 the dimension of the space $J^k_e(V^{\mathfrak{g}}(\lambda, \mu))$ does not depend on the rank $n$ of $\mathfrak{g}$ providing $n$ is sufficiently large. More precisely, we have:

**Theorem 4.4.2** Consider $\lambda \in \mathcal{P}_m$, $\mu \in \mathcal{P}_n$ and $k \in \mathbb{N}$. Let $\mathfrak{g}$ be a Lie algebra of type $X_n \in \{B_n, C_n D_n\}$ with $n \geq 2k + a$. Then $\dim(J^k_e(V^{\mathfrak{g}}(\lambda, \mu)))$ is independent of the rank $n$ of $\mathfrak{g}$. Moreover we have in this case

$$\dim(J^k_e(V^{\mathfrak{s}_02n+1}(\lambda, \mu))) = \dim(J^k_e(V^{\mathfrak{s}_02n}(\lambda, \mu))).$$

**Proof.** We deduce from Theorem 4.3.1 and (35) that the coefficients

$$K_{\lambda, \mu}^{X_n,k} = \dim(J^k_e(V^{\mathfrak{g}}(\lambda, \mu))) - \dim(J^{k-1}_e(V^{\mathfrak{g}}(\lambda, \mu)))$$
does not depend on \( n \) providing \( n \geq 2k + a \). Under this hypothesis, one can write

\[
\dim(J^k_e(V^{\mathfrak{so}}(\lambda)_\mu)) = \sum_{a=0}^{k} K^\mathfrak{so}_{a,a} \quad \text{and} \quad \dim(J^k_e(V^{\mathfrak{sp}}(\lambda)_\mu)) = \sum_{a=0}^{k} K^\mathfrak{sp}_{a,a}.
\]

Hence \( \dim(J^k_e(V^{\mathfrak{so}}(\lambda)_\mu)) \) and \( \dim(J^k_e(V^{\mathfrak{sp}}(\lambda)_\mu)) \) are independent of \( n \). Moreover, we have

\[
\dim(J^k_e(V^{\mathfrak{so}_{2n+1}}(\lambda)_\mu)) = \dim(J^k_e(V^{\mathfrak{sp}_{2n}}(\lambda)_\mu)).
\]

**Remark:** In general, the spaces \( J^k_e(V^{\mathfrak{g}}(\lambda)_\mu) \) depend on the rank \( n \geq 2k + a \) considered although their dimension does not. An interesting problem could consist in the obtention of explicit bases for the weight spaces \( J^k_e(V^{\mathfrak{g}}(\lambda)_\mu) \).

5 Limit of Lusztig \( q \)-analogues

5.1 The formal series \( K^\mathfrak{so}_{\lambda,\mu}(q) \) and \( K^\mathfrak{sp}_{\lambda,\mu}(q) \)

The results of Theorem 4.3.1 suggest to introduce the formal series \( K^\mathfrak{so}_{\lambda,\mu}(q) \) and \( K^\mathfrak{sp}_{\lambda,\mu}(q) \) defined by

\[
K^\mathfrak{so}_{\lambda,\mu}(q) = \sum_{r \geq 0} K^\mathfrak{so}_{\lambda,\mu}^{r,k} q^k \in \mathbb{N}[[q]] \quad \text{and} \quad K^\mathfrak{sp}_{\lambda,\mu}(q) = \sum_{r \geq 0} K^\mathfrak{sp}_{\lambda,\mu}^{r,k} q^k \in \mathbb{N}[[q]]
\]

where the coefficients \( K^\mathfrak{so}_{\lambda,\mu}^{r,k} \) and \( K^\mathfrak{sp}_{\lambda,\mu}^{r,k} \) are those defined in (33). Then, \( K^\mathfrak{so}_{\lambda,\mu}(q) \) and \( K^\mathfrak{sp}_{\lambda,\mu}(q) \) can be regarded as the limits of the Lusztig \( q \)-analogues \( K^\mathfrak{so}_{\lambda,\mu}(q) \) and \( K^\mathfrak{sp}_{\lambda,\mu}(q) \) when the rang \( n \) of \( \mathfrak{g} \) tends to the infinity.

We have moreover

\[
\text{char}_q(H(\mathfrak{so})) = \sum_{\lambda \in \mathcal{P}} K^\mathfrak{sp}_{\lambda,\emptyset}(q) \mathfrak{s}^{\mathfrak{sp}}_{\lambda} \quad \text{and} \quad \text{char}_q(H(\mathfrak{so})) = \sum_{\lambda \in \mathcal{P}} K^\mathfrak{so}_{\lambda,\emptyset}(q) \mathfrak{s}^{\mathfrak{so}}_{\lambda}.
\]

(36)

Remarks:

(i) : When \( \mathfrak{g} = \mathfrak{sp}_{2n} \) or \( \mathfrak{so}_{2n} \), \( K^\mathfrak{sp}_{\lambda,\mu}(q) = 0 \) for any partitions \( \lambda, \mu \) such that \( |\lambda| - |\mu| \) is odd. Thus for such partitions we have also \( K^\mathfrak{sp}_{\lambda,\mu}(q) = K^\mathfrak{so}_{\lambda,\mu}(q) = 0 \).

(ii) : Observe that we may have \( K^{\mathfrak{so}_{2n+1}}_{\lambda,\mu}(q) \neq 0 \) even if \( |\lambda| - |\mu| \) is odd. In this case we have thus

\[
\lim_{n \to \infty} K^{\mathfrak{so}_{2n+1}}_{\lambda,\mu}(q) = 0
\]

for any nonnegative integer \( k \). Take as an example \( \lambda = (1) \) and \( \mu = \emptyset \). Then \( K^{\mathfrak{so}_{2n+1}}_{(1),\emptyset}(q) = q^{n-1} \) for any rank \( n \geq 2 \). Thus (37) is verified for any fixed degree \( k \).
5.2 Recurrence formulas for the series $K_{\lambda,\mu}^{\text{so}}(q)$ and $K_{\lambda,\mu}^{\text{sp}}(q)$

By taking the limit when $n$ tends to the infinity in the formulas of Corollary 4.2.4 (which do not depend on $n$), we obtain the identities

$$
K_{\lambda,\mu}^{\text{so}}(q) = \sum_{s=1}^{p} (-1)^{s-1} \times q^{R_s} \times \sum_{r+a=R_s} \sum_{\lambda \in \mathcal{P}} p_{\gamma(s),r}^{\lambda} K_{\lambda,\mu}^{\text{so}}(q)
$$

$$
K_{\nu,\mu}^{\text{sp}}(q) = \sum_{s=1}^{p} (-1)^{s-1} \times \sum_{r+a=R_s} q^{r+a} \sum_{\lambda \in \mathcal{P}} p_{\gamma(s),r}^{\lambda} K_{\lambda,\mu}^{\text{sp}}(q)
$$

where $a \in \mathbb{N}$ and $\mu^b = (\mu_2, \ldots, \mu_m)$. These identities yield recurrence formulas for the limit of $q$-analogues.

To see it, suppose first $\mu \neq \emptyset$. By Lemma 4.2.5, the formal series $K_{\lambda,\mu}^{\text{so}}(q)$ and $K_{\lambda,\mu}^{\text{sp}}(q)$ appearing in the right hand sides of (38) are such that $|\lambda| < |\nu|$. Thus formulas (38) permit to express the series $K_{\lambda,\mu}^{\text{so}}(q)$ and $K_{\lambda,\mu}^{\text{sp}}(q)$ respectively in terms of the series $K_{\lambda,\mu}^{\text{so}}(q)$ and $K_{\lambda,\mu}^{\text{sp}}(q)$ with $|\lambda| < |\nu|$. Now suppose $\mu = \emptyset$. Then by Lemma 4.2.5, $K_{\lambda,\emptyset}^{\text{so}}(q)$ and $K_{\lambda,\emptyset}^{\text{sp}}(q)$ also appear in the right hand side of (38) when $\gamma(s) = \gamma(1) = \nu$, $R_s = R_1 = \nu$. We can write

$$
K_{\lambda,\emptyset}^{\text{so}}(q) = \frac{1}{1-q^{\nu_1}} \left( q^{\nu_1} \sum_{r+2a=R_1} \sum_{\lambda \in \mathcal{P}} p_{\gamma(s),r}^{\lambda} K_{\lambda,\emptyset}^{\text{so}}(q) + \sum_{s=2}^{p-1} (-1)^{s-1} \times q^{R_s} \sum_{r+2a=R_s} \sum_{\lambda \in \mathcal{P}} p_{\gamma(s),r}^{\lambda} K_{\lambda,\emptyset}^{\text{so}}(q) \right)
$$

and

$$
K_{\lambda,\emptyset}^{\text{sp}}(q) = \frac{1}{1-q^{\nu_1}} \left( \sum_{r+2a=R_1} q^{r+a} \sum_{\lambda \in \mathcal{P}} p_{\gamma(s),r}^{\lambda} K_{\lambda,\emptyset}^{\text{sp}}(q) + \sum_{s=2}^{p} (-1)^{s-1} \times \sum_{r+2a=R_s} q^{r+a} \sum_{\lambda \in \mathcal{P}} p_{\gamma(s),r}^{\lambda} K_{\lambda,\emptyset}^{\text{sp}}(q) \right)
$$

where the series $K_{\lambda,\emptyset}^{\text{so}}(q)$ and $K_{\lambda,\emptyset}^{\text{sp}}(q)$ appearing in the right hand sides are such that $|\lambda| < |\nu|$ or, $|\lambda| = |\nu|$ and $|\lambda^b| < |\mu^b|$. Thus formulas (39) permit to express the series $K_{\lambda,\mu}^{\text{so}}(q)$ and $K_{\lambda,\mu}^{\text{sp}}(q)$ respectively in terms of the series $K_{\lambda,\emptyset}^{\text{so}}(q)$ and $K_{\lambda,\emptyset}^{\text{sp}}(q)$ with $|\lambda| < |\nu|$ or, $|\lambda| = |\nu|$ and $|\lambda^b| < |\mu^b|$. Observe that $|\lambda| + |\lambda^b| < |\nu| + |\nu^b|$. Hence one can compute the series $K_{\nu,\emptyset}^{\text{so}}(q)$ and $K_{\nu,\emptyset}^{\text{sp}}(q)$ by induction on $|\nu| + |\nu^b|$ starting from the obvious identity $K_{\emptyset,\emptyset}^{\text{so}}(q) = K_{\emptyset,\emptyset}^{\text{sp}}(q) = 1$.

It thus follows from the previous arguments that the series $K_{\nu,\mu}^{\text{so}}(q)$ and $K_{\nu,\mu}^{\text{sp}}(q)$ with $\mu \neq \emptyset$ can be computed by induction on $|\nu|$ from the series $K_{\nu,\emptyset}^{\text{so}}(q)$ and $K_{\nu,\emptyset}^{\text{sp}}(q)$. The series $K_{\nu,\emptyset}^{\text{so}}(q)$ and $K_{\nu,\emptyset}^{\text{sp}}(q)$ being obtained by induction on $|\nu| + |\nu^b|$ from $K_{\emptyset,\emptyset}^{\text{so}}(q) = K_{\emptyset,\emptyset}^{\text{sp}}(q) = 1$. We give in Proposition 5.3.1 explicit formulas for $K_{\nu,\emptyset}^{\text{so}}(q)$ and $K_{\nu,\emptyset}^{\text{sp}}(q)$ when $\nu$ is a row or a column partition.

5.3 A duality between the series $K_{\lambda,\emptyset}^{\text{so}}(q)$ and $K_{\lambda,\emptyset}^{\text{sp}}(q)$.

**Proposition 5.3.1** For any partition $\lambda$ we have the duality

$$
K_{\lambda,\emptyset}^{\text{so}}(q) = K_{\lambda',\emptyset}^{\text{sp}}(q)
$$
between the limits of the orthogonal and symplectic Lusztig $q$-analogues corresponding to the weight 0.

**Proof.** We have

$$\text{char}_q(H(\mathfrak{so})) = \prod_{i \geq 1} (1 - q^{2i}) \text{char}_q(S(\mathfrak{so})) \quad \text{and} \quad \text{char}_q(H(\mathfrak{sp})) = \prod_{i \geq 1} (1 - q^{2i}) \text{char}_q(S(\mathfrak{sp})).$$

Moreover by Corollary 3.2.1, $\varphi(\text{char}_q(S^k(\mathfrak{so}))) = \text{char}_q(S^k(\mathfrak{sp}))$ for any nonnegative integer $k$. This implies the equality

$$\varphi(\text{char}_q(H^k(\mathfrak{so}))) = \text{char}_q(H^k(\mathfrak{sp})) \quad \text{for any } k \in \mathbb{N}.$$  \hspace{1cm} (40)

Recall that

$$\text{char}_q(H^k(\mathfrak{so})) = \sum_{\lambda \in \mathcal{P}} K_{\lambda,0}^{\mathfrak{so},k} s_\lambda^\mathfrak{so} \quad \text{and} \quad \text{char}_q(H^k(\mathfrak{sp})) = \sum_{\lambda \in \mathcal{P}} K_{\lambda,0}^{\mathfrak{sp},k} s_\lambda^\mathfrak{sp}.$$

By using (40), this gives

$$\text{char}_q(H^k(\mathfrak{sp})) = \sum_{\lambda \in \mathcal{P}} K_{\lambda,0}^{\mathfrak{sp},k} s_\lambda^\mathfrak{sp} = \sum_{\lambda \in \mathcal{P}} K_{\lambda,0}^{\mathfrak{so},k} s_\lambda^\mathfrak{sp}.$$

Since the map

$$l : \mathcal{P} \to \mathcal{P} \quad \lambda \mapsto \lambda'$$

is bijective, we must have $K_{\lambda,0}^{\mathfrak{sp},k}(q) = K_{\lambda',0}^{\mathfrak{so},k}(q)$ for any nonnegative integer $k$ which proves the proposition. \hspace{1cm} \blacksquare

**Remark:** The duality of the previous theorem does not hold for the Lusztig $q$-analogues, that is $K_{\lambda,0}^{\mathfrak{so},2n}(q) = K_{\lambda',0}^{\mathfrak{so},2n+1}(q)$ in general. Nevertheless we have $K_{\lambda,0}^{\mathfrak{sp},2n,k}(q) = K_{\lambda',0}^{\mathfrak{so},2n,k}(q)$ when $k \leq \frac{n}{2}$ according to Proposition 4.1.2.

### 5.4 Some explicit formulas

We give below some explicit formulas for the series $K_{\nu,\mu}^{\mathfrak{so}}(q)$ and $K_{\nu,\mu}^{\mathfrak{sp}}(q)$ when $\nu$ is a column or a row partition. Note that we have not found such simple formulas for the Lusztig $q$-analogues $K_{\nu,\mu}^g(q)$ even in the case when $\nu$ is a row or a column.

**Proposition 5.4.1** Consider $l$ a nonnegative integer. Recall that $(2l)$ and $(1^{2l})$ are the row and column partitions of length and height $2l$, respectively. We have

$$K_{(2l),0}^{\mathfrak{sp}}(q) = K_{(1^{2l}),0}^{\mathfrak{so}}(q) = \frac{q^l}{\prod_{i=1}^{l}(1 - q^{2i})}$$

$$K_{(2l),0}^{\mathfrak{so}}(q) = K_{(1^{2l}),0}^{\mathfrak{sp}}(q) = \frac{q^{2l}}{\prod_{i=1}^{l}(1 - q^{2i})}.$$

**Proof.** We only give the proof for the first equality of the proposition. The proof for the second is similar.
We use the recurrence formula (39). We have then \( p = 1, R_1 = 2l \) and \( \gamma(1) = \emptyset \). Thus \( p^\lambda_{\gamma(1), r} \neq 0 \) only when \( \lambda = r \) and in this case \( p^\lambda_{\gamma(1), r} = 1 \). This yields for any \( l \geq 1 \)

\[
K_{(2l), \emptyset}^{sp}(q) = \frac{1}{1 - q^{2l}} \sum_{r+2a=2l} q^{r+a} K_{(r), \emptyset}^{sp}(q) = \frac{q^l}{1 - q^{2l}} \sum_{b=0}^{l-1} q^b K_{(2b), \emptyset}^{sp}(q) \tag{42}
\]

where the last equality is obtained by setting \( r = 2b \). By an immediate induction starting from \( K_{\emptyset, \emptyset}^{sp}(q) = 1 \), one derives the desired formula

\[
K_{(2l), \emptyset}^{sp}(q) = \frac{q^l}{\prod_{i=1}^{l} (1 - q^{2i})}
\]

by using the identity

\[
\sum_{b=0}^{l-1} q^b K_{(2b), \emptyset}^{sp}(q) = \frac{q^{2l}}{\prod_{i=1}^{l} (1 - q^{2i})} = \frac{1}{\prod_{i=1}^{l-1} (1 - q^{2i})}. \tag{43}
\]

We deduce then \( K_{(2l), \emptyset}^{sp}(q) = K_{(1^{2l}), \emptyset}^{sp}(q) \) from Theorem 5.3.1. □

Corollary 5.4.2 Consider \( m \) a nonnegative integer and \( \mu \) a partition with \( d \) nonzero parts. Then

1. \( K_{(m), \mu}^{sp}(q) \neq 0 \) and \( K_{(m), \mu}^{so}(q) \neq 0 \) only if \( m - |\mu| \in 2\mathbb{N} \). In this case

\[
K_{(m), \mu}^{sp}(q) = q^{h(\mu)} K_{(2l), \emptyset}^{sp}(q) = \frac{q^{h(\mu)+l}}{\prod_{i=1}^{l} (1 - q^{2i})} \quad \text{and} \quad K_{(m), \mu}^{so}(q) = q^{h(\mu)} K_{(2l), \emptyset}^{so}(q) = \frac{q^{h(\mu)+2l}}{\prod_{i=1}^{l} (1 - q^{2i})}
\]

where \( h(\mu) = \sum_{1 \leq i \leq d} (i - 1) \mu_i \) and \( l = \frac{m - |\mu|}{2} \).

2. \( K_{(1^m), \mu}^{so}(q) \neq 0 \) and \( K_{(1^m), \mu}^{sp}(q) \neq 0 \) only if \( \mu = (1^p) \) with \( m - p \in 2\mathbb{N} \) and in this case

\[
K_{(1^m), (1^p)}^{sp}(q) = K_{(1^{2l}), \emptyset}^{sp}(q) = \frac{q^{2l}}{\prod_{i=1}^{l} (1 - q^{2i})} \quad \text{and} \quad K_{(1^m), (1^p)}^{so}(q) = K_{(1^{2l}), \emptyset}^{so}(q) = \frac{q^{l}}{\prod_{i=1}^{l} (1 - q^{2i})}
\]

where \( l = \frac{m - |\mu|}{2} \).

Proof. 1 : We proceed by induction on \( d \) the number of nonzero parts of \( \mu \). If \( d = 0 \), the result follows from Proposition 5.4.1 Suppose \( b > 0 \) and apply the recurrence formula (38). We have \( p = 1, R_1 = m - \mu_1 \) and \( \gamma(1) = \emptyset \). This gives

\[
K_{(m), \mu}^{sp}(q) = \sum_{r+2a=m-\mu_1} q^{r+a} K_{(r), \mu}^{sp}(q).
\]

Since \( K_{(r), \mu}^{sp}(q) = 0 \) when \( r < |\mu| \), we can suppose \( r \geq |\mu| \) in the previous sum. This gives by using the induction hypothesis

\[
K_{(m), \mu}^{sp}(q) = q^{h(\mu)} \sum_{r+2a=m-\mu_1} q^{r+a} K_{(r-|\mu|), \emptyset}^{sp}(q).
\]
We must have \( r - |\mu^b| < 2N \), thus we can set \( b = \frac{r - |\mu^b|}{2} \). One then obtains

\[
K^{sp}_{(m),\mu}(q) = q^{h(\mu)} \sum_{b=0}^{l} q^{(l+b)} K^{sp}_{2b,0}(q) = q^{h(\mu)} \sum_{b=0}^{l} q^{l+b} K^{sp}_{2b,0}(q)
\]

where the last equality follows from the equalities \( l = \frac{m-|\mu^b|}{2} \) and \( h(\mu) = h(\mu^b) + |\mu| - \mu_1 \). By using (43), this gives

\[
K^{sp}_{(m),\mu}(q) = q^{h(\mu)} \frac{q^l}{\prod_{i=1}^{l} (1 - q^{2i})} = q^{h(\mu)} K^{sp}_{(2l),\emptyset}(q).
\]

The proof is similar for \( K^{so}_{(m),\mu}(q) \).

2 : By applying (43), we obtain this time \( p = 1, R_1 = 0 \) and \( \gamma(1) = \emptyset \). Hence

\[
K^{sp}_{(1m),(1p)}(q) = K^{sp}_{(1m-1),(1p-1)}(q).
\]

By an immediate induction, this gives \( K^{sp}_{(1m),(1p)}(q) = K^{sp}_{(1m-p),\emptyset}(q) = K^{sp}_{(1n),\emptyset}(q) \) and our formula follows from Proposition 5.4.1. □

6 Hall-Littlewood polynomials in infinitely many variables

6.1 The ring of graded universal characters

We now consider the ring \( \Delta \) generated over \( \mathbb{Z}[[q]] \) by the formal characters \( s^{{gl}}_{\lambda}, \lambda \in \mathcal{P} \) with multiplication defined by

\[
s^{{gl}}_{\lambda} \cdot s^{{gl}}_{\mu} = \sum_{\nu \in \mathcal{P}} c^{\nu}_{\lambda,\mu} s^{{gl}}_{\nu}
\]

and for any pair \( F = \sum_{\lambda \in \mathcal{P}} C_{\lambda} s^{{gl}}_{\lambda}, G = \sum_{\mu \in \mathcal{P}} C_{\mu} s^{{gl}}_{\mu} \)

\[
F \cdot G = \sum_{\nu \in \mathcal{P}} \sum_{\lambda,\mu \in \mathcal{P}} C_{\lambda} C_{\mu} c^{\nu}_{\lambda,\mu} s^{{gl}}_{\nu}.
\]

Observe that \( F \cdot G \) is well defined since we have \( c^{\nu}_{\lambda,\mu} = 0 \) if \( |\nu| \neq |\lambda| + |\mu| \) and thus \( \sum_{\lambda,\mu \in \mathcal{P}} C_{\lambda} C_{\mu} c^{\nu}_{\lambda,\mu} \) is finite. Then \( \mathcal{B}^{{gl}} = \{ s^{{gl}}_{\lambda} \mid \lambda \in \mathcal{P} \} \), is a \( \mathbb{Z}[[q]] \)-basis of \( \Delta \). We then defined the \( \mathbb{Z}[[q]] \)-bases \( \mathcal{B}^{{so}} = \{ s^{{so}}_{\lambda} \mid \lambda \in \mathcal{P} \} \) and \( \mathcal{B}^{{sp}} = \{ s^{{sp}}_{\lambda} \mid \lambda \in \mathcal{P} \} \) so that (43) is verified. We then write \( < \cdot, \cdot >^{{so}} \) and \( < \cdot, \cdot >^{{sp}} \) respectively for the inner scalar products which make the bases \( \mathcal{B}^{{so}} \) and \( \mathcal{B}^{{sp}} \) orthonormal.

6.2 Hall-Littlewood polynomials in infinitely many variables

For any partition \( \mu \), we set

\[
Q^{so}_{\mu} = \sum_{\lambda \in \mathcal{P}} K^{so}_{\lambda,\mu}(q)s^{so}_{\lambda} \quad \text{and} \quad Q^{sp}_{\mu} = \sum_{\lambda \in \mathcal{P}} K^{sp}_{\lambda,\mu}(q)s^{sp}_{\lambda}.
\]

Since \( K^{so}_{\lambda,\mu}(q) = K^{sp}_{\lambda,\mu}(q) = 0 \) when \( \lambda < \mu \) for the usual order on partitions, the transition matrices of the families \( \{ Q^{so}_{\lambda} \mid \lambda \in \mathcal{P} \} \) and \( \{ Q^{sp}_{\lambda} \mid \lambda \in \mathcal{P} \} \) respectively on the bases \( \mathcal{B}^{{so}} \) and \( \mathcal{B}^{{sp}} \) are upper-unitriangular. Hence \( \{ Q^{so}_{\lambda} \mid \lambda \in \mathcal{P} \} \) and \( \{ Q^{sp}_{\lambda} \mid \lambda \in \mathcal{P} \} \) are bases of \( \Delta \).
We then define the basis \( \{ P^s_{\lambda} | \lambda \in \mathcal{P} \} \) (resp. \( \{ P^{sp}_{\lambda} | \lambda \in \mathcal{P} \} \)) as the dual basis of \( \{ Q^s_{\lambda} | \lambda \in \mathcal{P} \} \) (resp. \( \{ Q^{sp}_{\lambda} | \lambda \in \mathcal{P} \} \)) with respect to \( \langle \cdot, \cdot \rangle >^s \) (resp. \( \langle \cdot, \cdot \rangle >^{sp} \)). Then \( P^s_{\lambda} \) and \( P^{sp}_{\lambda} \) can be regarded as Hall-Littlewood polynomials in infinitely many variables. One has the identities

\[
\begin{align*}
\lambda &= \sum_{\lambda \in \mathcal{P}} K^s_{\lambda, \mu}(q) P^s_{\mu} \\
\lambda &= \sum_{\lambda \in \mathcal{P}} K^{sp}_{\lambda, \mu}(q) P^{sp}_{\mu}.
\end{align*}
\]

**Remark:**

(i) : One cannot obtain Hall-Littelwood polynomials \( Q_{\lambda} \) in infinitely many variables by considering the limit when \( n \) tends to the infinity in (10) and (11). Indeed, the number of ways of decomposing a weight \( \beta \) as a sum of \( k \) positive roots (where \( k \) is a fixed nonnegative integer) may strictly increase with \( n \). As an example for \( k = 2 \) we have \( 2\varepsilon_1 = (\varepsilon_1 - \varepsilon_i) + (\varepsilon_1 + \varepsilon_i) \) for any \( i \in \{2, \ldots, n\} \).

(ii) : Recall that the Hall-Littlewood polynomial \( P^g_{\mu} \) (see [17]) associated to the partition \( \mu \) is defined by

\[
\begin{align*}
P^g_{\mu} &= \frac{1}{W^g_{\mu}(q)} \sum_{w \in W^g} w \left( e^\mu \prod_{\alpha \in R^+} \frac{1 - qe^{-\alpha} - e^{-\alpha}}{1 - e^{-\alpha}} \right)
\end{align*}
\]

where \( W^g_{\mu}(q) = \sum_{w \in W_{\mu}} q^{\ell(w)} \) with \( W^g_{\mu} \) the stabilizer of \( \mu \) in \( W^g \). Since the number of elements of \( W^g \) of fixed length strictly increase with the rank of \( g \), the polynomials \( W^g_{\mu}(q) \) have no limit in \( \mathbb{Z}[[q]] \). This implies that is not possible to define Hall-Littlewood polynomials in infinitely many variables by taking the limit when \( n \) tends to the infinity in (44).

(iii) : Similarly to (33), we have the decompositions

\[
\begin{align*}
Q^s_{\lambda} &= \sum_{\nu \in \mathcal{P}} K^s_{\nu, \mu}(q) s^s_{\nu} = \sum_{k \geq 0} \sum_{\nu \in \mathcal{P}[2k+|\mu|]} K^s_{\nu, \mu} s^s_{\nu} q^k \quad \text{and} \\
Q^{sp}_{\lambda} &= \sum_{\nu \in \mathcal{P}} K^{sp}_{\nu, \mu}(q) s^{sp}_{\nu} = \sum_{k \geq 0} \sum_{\nu \in \mathcal{P}[2k+|\mu|]} K^{sp}_{\nu, \mu} s^{sp}_{\nu} q^k.
\end{align*}
\]

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