Intuitionistic Fuzzy Topological BH-Algebras

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Abstract: The intuitionistic fuzzification of a BH-algebra is considered and related results are investigated. The notion of equivalence relations on the family of all intuitionistic fuzzy BH-algebras of a BH-algebra is introduced, and then some properties are discussed. The concept of intuitionistic fuzzy topological BH-algebras is introduced, and some related results are obtained.

Keywords: intuitionistic fuzzy BH-algebra, intuitionistic fuzzy topological BH-algebra.

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1 Introduction

Y. Imai and K. Iseki [8] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [6, 7] Q.P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. In 1996, Jun, Roh and Kim introduced the notion of BH-algebra, which is a generalization of BCH-algebras [9]. In 2001, Q. Zhang, E.H. Roh and Y.B. Jun studied the fuzzy theory in BH-algebras [14]. C.H. Park introduced the notion of an interval-valued fuzzy BH-algebra in a BH-algebra and investigate related properties [10]. The concept of a fuzzy set, which was introduced in [13], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. D. H. Foster [5] combined the structure of a
fuzzy topological spaces with that of a fuzzy group, introduced by A. Rosenfeld [11], to formulate the elements of a theory of fuzzy topological groups. After the introduction of fuzzy sets by L. A. Zadeh [13], several researchers were conducted on the generalizations of the notion of fuzzy sets. The idea of intuitionistic fuzzy set was first published by K. T. Atanassov [1], as a generalization of the notion of fuzzy sets. In this paper, using the Atanassov’s idea, we establish the notion of intuitionistic fuzzy BH-algebras, equivalence relations on the family of all intuitionistic fuzzy BH-algebras, and intuitionistic fuzzy topological BH-algebras which are generalization of the notion of fuzzy topological BH-algebras. We investigate several properties, and show that the BH-homomorphic image and preimage of an intuitionistic fuzzy topological BH-algebra is an intuitionistic fuzzy topological BH-algebra.

2 Preliminaries

Definition 2.1 ([9]). Let X be a set with a binary operation $\ast$ and a constant 0. Then $(X, \ast, 0)$ is called a BH-algebra if it satisfies the following axioms:

1. $x \ast x = 0$,

2. $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$,

3. $x \ast 0 = x$

for all $x, y \in X$.

Definition 2.2. A non-empty set S of a BH-algebra X is called a BH-subalgebra of X if $x \ast y \in S$ for any $x, y \in X$.

Definition 2.3. A mapping $\theta : X \rightarrow Y$ of BH-algebras is called a BH-homomorphism if $\theta(x \ast y) = \theta(x) \ast \theta(y)$ for all $x, y \in X$.

Definition 2.4 ([5]). A fuzzy topology on a set X is a family $\tau$ of fuzzy sets in X which satisfies the following conditions:

1. for all $c \in [0, 1], k_c \in \tau$, where $k_c$ has a constant membership function,

2. if $A, B \in \tau$, then $A \cap B \in \tau$,

3. if $A_j \in \tau$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \tau$.

The pair $(X, \tau)$ is called a fuzzy topological space and members of $\tau$ are called open fuzzy sets.
Definition 2.5. An intuitionistic fuzzy set (IFS for short) \( D \) in \( X \) is an object having the form
\[
D = \{ \langle x, \mu_D(x), \nu_D(x) \rangle | x \in X \}
\]
where the functions \( \mu_D : X \to [0,1] \) and \( \nu_D : X \to [0,1] \) denote the degree of membership (namely \( \mu_D(x) \)) and the degree of nonmembership (namely \( \nu_D(x) \)) of each element \( x \in X \) to the set \( D \), respectively, and \( 0 \leq \mu_D(x) + \nu_D(x) \leq 1 \) for each \( x \in X \).

For the sake of simplicity, we shall use the notation \( D = \langle x, \mu_D, \nu_D \rangle \) instead of \( D = \{ \langle x, \mu_D(x), \nu_D(x) \rangle | x \in X \} \).

Let \( f \) be a mapping from a set \( X \) to a set \( Y \). If
\[
B = \{ \langle y, \mu_B(y), \nu_B(y) \rangle | y \in Y \}
\]
is an IFS in \( Y \), then the preimage of \( B \) under \( f \) denoted by \( f^{-1}(B) \), is the IFS in \( X \) defined by
\[
f^{-1}(B) = \{ \langle x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x) \rangle | x \in X \},
\]
and if
\[
D = \{ \langle x, \mu_D(x), \nu_D(x) \rangle | x \in X \}
\]
is an IFS in \( X \), then the image of \( D \) under \( f \), denoted by \( f(D) \), is the IFS in \( Y \) defined by
\[
f(D) = \{ \langle y, f_{sup}\mu_D(y), f_{inf}\nu_D(y) \rangle | y \in Y \},
\]
where
\[
f_{sup}(\mu_D)(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu_D(x), & \text{if } f^{-1}(y) \neq \phi \\
0, & \text{otherwise}
\end{cases}
\]
and
\[
f_{inf}(\nu_D)(y) = \begin{cases} 
\inf_{x \in f^{-1}(y)} \nu_D(x), & \text{if } f^{-1}(y) \neq \phi \\
1, & \text{otherwise}
\end{cases}
\]
for each \( y \in Y \).

3 Intuitionistic fuzzy BH-algebras

Definition 3.1. Let \( X \) be a BH-algebra. An IFS
\[
D = \langle x, \mu_D, \nu_D \rangle
\]
in \( X \) is called an intuitionistic fuzzy BH-algebra if it satisfies:
\[
\mu_D(x \ast y) \geq \min\{\mu_D(x), \mu_D(y)\}
\]
and
\[
\nu_D(x \ast y) \leq \max\{\nu_D(x), \nu_D(y)\} \text{ for all } x, y \in X.
\]
Example 3.2. Consider a BH-algebra \( X = \{0, a, b, c\} \) with the following Cayley table:

\[
\begin{array}{cccc}
\ast & 0 & a & b & c \\
0 & 0 & c & 0 & b \\
a & a & 0 & 0 & 0 \\
b & b & b & 0 & c \\
c & c & c & b & 0 \\
\end{array}
\]

Let

\[
D = \langle x, \mu_D(x), \nu_D(x) \rangle
\]

be an IFS in \( X \) by \( \mu_D(0) = 0.7, \mu_D(a) = 0.2, \mu_D(b) = 0.5, \mu_D(c) = 0.4 \) and \( \nu_D(0) = 0.2, \nu_D(a) = 0.8, \nu_D(b) = 0.3, \nu_D(c) = 0.5 \). Then

\[
\begin{align*}
D = \langle x, \mu_D, \nu_D \rangle
\end{align*}
\]

is an intuitionistic fuzzy BH-algebra.

Proposition 3.3. If an IFS

\[
D = \langle x, \mu_D, \nu_D \rangle
\]

in \( X \) is an intuitionistic fuzzy BH-algebra of \( X \), then

\[
\mu_D(0) \geq \mu_D(x)
\]

and

\[
\nu_D(0) \leq \nu_D(x) \text{ for all } x \in X.
\]

Proof. Let \( x \in X \). Then

\[
\mu_D(0) = \mu_D(x \ast x) \geq \min\{\mu_D(x), \mu_D(x)\} = \mu_D(x)
\]

and

\[
\nu_D(0) = \nu_D(x \ast x) \leq \max\{\nu_D(x), \nu_D(x)\} = \nu_D(x).
\]

Theorem 3.4. If \( \{D_i \mid i \in \land\} \) is an arbitrary family of intuitionistic fuzzy BH-algebras of \( X \), then \( \cap D_i \) is an intuitionistic fuzzy BH-algebra of \( X \), where

\[
\cap D_i = \{(x, \land D_i(x), \lor D_i(x)) \mid x \in X\}.
\]

Proof. Let \( x, y \in X \). Then

\[
\land D_i(x \ast y) \geq \land(\min\{\mu_{D_i}(x), \mu_{D_i}(y)\}) = \min\{\land D_i(x), \land D_i(y)\}
\]

and

\[
\lor D_i(x \ast y) \leq \lor(\max\{\nu_{D_i}(x), \nu_{D_i}(y)\}) = \max\{\lor D_i(x), \lor D_i(y)\}.
\]

Hence,

\[
\cap D_i = \langle x, \land D_i, \lor D_i \rangle
\]

is an intuitionistic fuzzy BH-algebra of \( X \).
Theorem 3.5. If an IFS \( D = \langle x, \mu_D, \nu_D \rangle \) in \( X \) is an intuitionistic fuzzy BH-algebra of \( X \), then so is \( D \), where \( D = \{ \langle x, \mu_D(x), 1 - \mu_D(x) \rangle | x \in X \} \).

Proof. It is sufficient to show that \( \overline{D} \) satisfies the second condition in Definition 3.1.

Let \( x, y \in X \). Then
\[
\mu_D(x * y) = 1 - \mu_D(x * y) \\
\leq 1 - \min \{ \mu_D(x), \mu_D(y) \} \\
= \max \{ 1 - \mu_D(x), 1 - \mu_D(y) \} \\
= \max \{ \mu_D(x), \mu_D(y) \}.
\]
Hence \( D \) is an intuitionistic fuzzy BH-algebra of \( X \). \( \square \)

Theorem 3.6. If an IFS \( D = \langle x, \mu_D, \nu_D \rangle \) in \( X \) is an intuitionistic fuzzy BH-algebra of \( X \), then the sets
\[
X_\mu := \{ x \in X | \mu_D(x) = \mu_D(0) \}
\]
and
\[
X_\nu := \{ x \in X | \nu_D(x) = \nu_D(0) \}
\]
are BH-subalgebras of \( X \).

Proof. Let \( x, y \in X_\mu \). Then \( \mu_D(x) = \mu_D(0) = \mu_D(y) \), and so \( \mu_D(x * y) = \min \{ \mu_D(x), \mu_D(y) \} = \mu_D(0) \). By using Proposition 3.3, we know that \( \mu_D(x * y) = \mu_D(0) \) or equivalently \( x * y \in X_\mu \). Now let \( x, y \in X_\nu \). Then \( \nu_D(x * y) \leq \max \{ \nu_D(x), \nu_D(y) \} = \nu_D(0) \), and by applying Proposition 3.3 we conclude that \( \nu_D(x * y) = \nu_D(0) \) and hence \( x * y \in X_\nu \). \( \square \)

Definition 3.7. Let \( D = \langle x, \mu_D, \nu_D \rangle \) be an IFS in \( X \) and let \( t \in [0, 1] \). Then the set \( U(\mu_D, t) := \{ x \in X | \mu_D(x) \geq t \} \) (resp. \( L(\nu_D, t) := \{ x \in X | \nu_D(x) \leq t \} \)) is called a \( \mu \)-level t-cut (resp. \( \nu \)-level t-cut) of \( D \).

Theorem 3.8. If an IFS \( D = \langle x, \mu_D, \nu_D \rangle \) in \( X \) is an intuitionistic fuzzy BH-algebra of \( X \), then the \( \mu \)-level t-cut and \( \nu \)-level t-cut of \( D \) are BH-subalgebras of \( X \) for every \( t \in [0, 1] \) such that \( t \in \text{Im}(\mu_D) \cap \text{Im}(\nu_D) \), which are called a \( \mu \)-level BH-subalgebra and a \( \nu \)-level BH-subalgebra respectively.

Proof. Let \( x, y \in U(\mu_D, t) \). Then \( \mu_D(x) \geq t \) and \( \mu_D(y) \geq t \). It follows that \( \mu_D(x * y) \geq \min \{ \mu_D(x), \mu_D(y) \} \geq t \) so that \( x * y \in U(\mu_D, t) \). Hence, \( U(\mu_D, t) \) is a BH-subalgebra of \( X \). Now let \( x, y \in L(\nu_D, t) \). Then \( \nu_D(x * y) \leq \max \{ \nu_D(x), \nu_D(y) \} \leq t \) and so \( x * y \in L(\nu_D, t) \). Therefore, \( L(\nu_D, t) \) is a BH-subalgebra of \( X \). \( \square \)

Theorem 3.9. Let \( D = \langle x, \mu_D, \nu_D \rangle \) be an IFS in \( X \) such that the sets \( U(\mu_D, t) \) and \( L(\nu_D, t) \) are BH-subalgebras of \( X \). Then \( D = \langle x, \mu_D, \nu_D \rangle \) is an intuitionistic fuzzy BH-algebra of \( X \).
Proof. Assume that there exist \( x_0, y_0 \in X \) such that \( \mu_D(x_0 \ast y_0) < \min\{\mu_D(x_0), \mu_D(y_0)\} \).

Let
\[
t_0 := 1/2(\mu_D(x_0 \ast y_0) + \min\{\mu_D(x_0), \mu_D(y_0)\}).
\]
Then
\[
\mu_D(x_0 \ast y_0) < t_0 < \min\{\mu_D(x_0), \mu_D(y_0)\}
\]
and so \( x_0 \ast y_0 \notin U(\mu_D, t_0) \), but \( x_0, y_0 \in U(\mu_D, t_0) \). This is a contradiction, and therefore
\[
\mu_D(x \ast y) \geq \min\{\mu_D(x), \mu_D(y)\} \quad \text{for all } x, y \in X.
\]

Now suppose that \( \nu_D(x_0 \ast y_0) > \max\{\nu_D(x_0), \nu_D(y_0)\} \) for some \( x_0, y_0 \in X \). Taking
\[
S_0 := 1/2(\nu_D(x_0 \ast y_0) + \max\{\nu_D(x_0), \nu_D(y_0)\}),
\]
then
\[
\max\{\nu_D(x_0), \nu_D(y_0)\} < S_0 < \nu_D(x_0 \ast y_0).
\]
It follows that \( x_0, y_0 \in L(\nu_D, S_0) \) and \( x_0 \ast y_0 \notin L(\nu_D, S_0) \), a contradiction. Hence
\[
\nu_D(x \ast y) \leq \max\{\nu_D(x), \nu_D(y)\} \quad \text{for all } x, y \in X.
\]

This completes the proof. \( \square \)

**Theorem 3.10.** Any BH-subalgebra of \( X \) can be realized as both a \( \mu \)-level BH-subalgebra and a \( \nu \)-level BH-subalgebra of some intuitionistic fuzzy BH-algebra of \( X \).

Proof. Let \( S \) be a BH-subalgebra of \( X \) and let \( \mu_D \) and \( \nu_D \) be fuzzy sets in \( X \) defined by

\[
\mu_D(x) := \begin{cases} 
  t, & \text{if } x \in S \\
  0, & \text{otherwise}
\end{cases}
\]

and

\[
\nu_D(x) := \begin{cases} 
  s, & \text{if } x \in S \\
  1, & \text{otherwise}
\end{cases}
\]

for all \( x \in X \) where \( t \) and \( s \) are fixed numbers in \((0, 1)\) such that \( t + s < 1 \).

Let \( x, y \in X \). If \( x, y \in S \), then \( x \ast y \in S \). Hence \( \mu_D(x \ast y) = \min\{\mu_D(x), \mu_D(y)\} \) and \( \nu_D(x \ast y) = \max\{\nu_D(x), \nu_D(y)\} \). If at least one of \( x \) and \( y \) does not belong to \( S \), then at least one of \( \mu_D(x) \) and \( \mu_D(y) \) is equal to 0, and at least one of \( \nu_D(x) \) and \( \nu_D(y) \) is equal to 1. It follows that
\[
\mu_D(x \ast y) \geq 0 = \min\{\mu_D(x), \mu_D(y)\}, \nu_D(x \ast y) \leq 1 = \max\{\nu_D(x), \nu_D(y)\}.
\]

Hence \( D = \langle x, \mu_D, \nu_D \rangle \) is an intuitionistic fuzzy BH-algebra of \( X \). Obviously,
\[
U(\mu_D, t) = S = L(\nu_D, s).
\]

This completes the proof. \( \square \)
**Theorem 3.11.** Let $\alpha$ be a BH-homomorphism of a BH-algebra $X$ into a BH-algebra $Y$ and $B$ an intuitionistic fuzzy BH-algebra of $Y$. Then $\alpha^{-1}(B)$ is an intuitionistic fuzzy BH-algebra of $X$.

**Proof.** For any $x, y \in X$, we have

$$
\mu_{\alpha^{-1}}(B)(x * y) = \mu_B(\alpha(x * y)) = \mu_B(\alpha(x) * \alpha(y)) \\
\geq \min\{\mu_B(\alpha(x)), \mu_B(\alpha(y))\} = \min\{\mu_{\alpha^{-1}}(B)(x), \mu_{\alpha^{-1}}(B)(y)\}
$$

and

$$
\nu_{\alpha^{-1}}(B)(x * y) = \nu_B(\alpha(x * y)) = \nu_B(\alpha(x) * \alpha(y)) \\
\leq \max\{\nu_B(\alpha(x)), \nu_B(\alpha(y))\} = \max\{\nu_{\alpha^{-1}}(B)(x), \nu_{\alpha^{-1}}(B)(y)\}.
$$

Hence $\alpha^{-1}(B)$ is an intuitionistic fuzzy BH-algebra in $X$. 

**Theorem 3.12.** Let $\alpha$ be a BH-homomorphism of a BH-algebra $X$ onto a BH-algebra $Y$. If $D = \langle x, \mu_D, \nu_D \rangle$ is an intuitionistic fuzzy BH-algebra of $X$, then $\alpha(D) = \langle y, \alpha_{\sup}\mu_D, \alpha_{\inf}\nu_D \rangle$ is an intuitionistic fuzzy BH-algebra of $Y$.

**Proof.** Let $D = \langle x, \mu_D, \nu_D \rangle$ be an intuitionistic fuzzy topological BH-algebra in $X$ and let $y_1, y_2 \in Y$. Noticing that

$$
\{x_1 * x_2 | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \subseteq \{x \in X | x \in \alpha^{-1}(y_1 * y_2)\},
$$

we have

$$
\alpha_{\sup}(\mu_D)(y_1 * y_2) = \sup\{\mu_D(x) | x \in \alpha^{-1}(y_1 * y_2)\} \\
\geq \sup\{\mu_D(x_1 * x_2) | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \\
\geq \sup\{\min\{\mu_D(x_1), \mu_D(x_2)\} | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \\
= \min\{\sup\{\mu_D(x_1) | x_1 \in \alpha^{-1}(y_1)\}, \sup\{\mu_D(x_2) | x_2 \in \alpha^{-1}(y_2)\}\} \\
= \min\{\alpha_{\sup}(\mu_D)(y_1), \alpha_{\sup}(\mu_D)(y_2)\}
$$

and

$$
\alpha_{\inf}(\nu_D)(y_1 * y_2) = \inf\{\nu_D(x) | x \in \alpha^{-1}(y_1 * y_2)\} \\
\leq \inf\{\nu_D(x_1 * x_2) | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \\
\leq \inf\{\max\{\nu_D(x_1), \nu_D(x_2)\} | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \\
= \max\{\inf\{\nu_D(x_1) | x_1 \in \alpha^{-1}(y_1)\}, \inf\{\nu_D(x_2) | x_2 \in \alpha^{-1}(y_2)\}\} \\
= \max\{\alpha_{\inf}(\nu_D)(y_1), \alpha_{\inf}(\nu_D)(y_2)\}
$$
Hence $\alpha(D) = (y, \alpha_{\text{sup}}(\mu_D), \alpha_{\text{inf}}(\nu_D))$ is an intuitionistic fuzzy BH-algebra in $Y$. Let $\Omega(X)$ denote the family of all intuitionistic fuzzy BH-algebras of $X$ and let $t \in [0, 1]$. Define binary relations $\tilde{\mu}$ and $\tilde{\nu}$ on $\Omega(X)$ as follows:

$$A \tilde{\mu} B \iff U(\mu_A, t) = U(\mu_B, t)$$

and

$$A \tilde{\nu} B \iff L(\nu_A, t) = L(\nu_B, t),$$

respectively, for $A = \langle x, \mu_A, \nu_A \rangle$ and $B = \langle x, \mu_B, \nu_B \rangle \in \Omega(X)$. Then clearly $\tilde{\mu}$ and $\tilde{\nu}$ are equivalence relations on $\Omega(X)$. For any $A = \langle x, \mu_A, \nu_A \rangle \in \Omega(X)$, let $[A]_{\mu}$ (resp. $[A]_{\nu}$) denote the equivalence class of $A = \langle x, \mu_A, \nu_A \rangle$ modulo $\tilde{\mu}$ (resp. $\tilde{\nu}$), and denote by $\Omega(X)\mid_{\tilde{\mu}}$ (resp. $\Omega(X)\mid_{\tilde{\nu}}$) the collection of all equivalence classes of $A$ modulo $\tilde{\mu}$ (resp. $\tilde{\nu}$). $\Omega(X)\mid_{\tilde{\mu}} := \{[A]_{\mu} \mid A = \langle x, \mu_A, \nu_A \rangle \in \Omega(X)\}$ (resp. $\Omega(X)\mid_{\tilde{\nu}} := \{[A]_{\nu} \mid A = \langle x, \mu_A, \nu_A \rangle \in \Omega(X)\}$). Now, let $S(X)$ denote the family of all BH-subalgebras of $X$ and let $t \in [0, 1]$. Define maps $\alpha_t$ and $\beta_t$ from $\Omega(X)$ to $S(X) \cup \{\phi\}$ by $\alpha_t(A) = U(\mu_A, t)$ and $\beta_t(A) = L(\nu_A, t)$, respectively, for all $A = \langle x, \mu_A, \nu_A \rangle \in \Omega(X)$. Then $\alpha_t$ and $\beta_t$ are clearly well-defined.

**Theorem 3.13.** For any $t \in (0, 1)$ the maps $\alpha_t$ and $\beta_t$ are surjective from $\Omega(X)$ to $S(X) \cup \{\phi\}$.

**Proof.** Let $t \in (0, 1)$. Note that $0 \tilde{=} \langle x, 0, 1 \rangle$ is in $\Omega(X)$, where 0 and 1 are fuzzy sets in $X$ defined by $\mu(x) = 0$ and $\mu(x) = 1$ for all $x \in X$. Obviously $\alpha_0(\mu) = U(0, t) = \phi = L(1, t) = \beta_t(0)$. Let $G(\neq \phi) \in S(X)$. Then $\Omega(X)\mid_{\tilde{\mu}} = \{G\}$. For $G = \langle x, \chi_G, \chi_G \rangle \in \Omega(X)$, we have $\alpha_t(G) = U(\chi_G, t) = G$ and $\beta_t(G) = L(\chi_G, t) = G$. Hence $\alpha_t$ and $\beta_t$ are surjective.

**Theorem 3.14.** The quotient sets $\Omega(X)\mid_{\tilde{\mu}}$ and $\Omega(X)\mid_{\tilde{\nu}}$ are equipotent to $S(X) \cup \{\phi\}$ for every $t \in (0, 1)$.

**Proof.** For $t \in (0, 1)$ let $\alpha_t^\ast$ (resp. $\beta_t^\ast$) be a map from $\Omega(X)\mid_{\tilde{\mu}}$ (resp. $\Omega(X)\mid_{\tilde{\nu}}$) to $S(X) \cup \{\phi\}$ defined by $\alpha_t^\ast([A]_{\mu}) = \alpha_t(A)$ (resp. $\beta_t^\ast([A]_{\nu}) = \beta_t(A)$) for all $A = \langle x, \mu_A, \nu_A \rangle \in \Omega(X)$.

If $U(\mu_A, t) = U(\mu_B, t)$ and $L(\nu_A, t) = L(\nu_B, t)$ for $A = \langle x, \mu_A, \nu_A \rangle$ and $B = \langle x, \mu_B, \nu_B \rangle \in \Omega(X)$, then $A \tilde{\mu} B$ and $A \tilde{\nu} B$; hence $[A]_{\mu} = [B]_{\mu}$ and $[A]_{\nu} = [B]_{\nu}$. Therefore, the maps $\alpha_t^\ast$ and $\beta_t^\ast$ are injective. Now let $G(\neq \phi) \in S(X)$. For $G = \langle x, \chi_G, \chi_G \rangle \in \Omega(X)$, we have $\alpha_t^\ast([G]_{\mu}) = \alpha_t(G) = U(\chi_G, t) = G$ and $\beta_t^\ast([G]_{\nu}) = \beta_t(G) = L(\chi_G, t) = G$. Finally, for $0 \tilde{=} \langle x, 0, 1 \rangle \in \Omega(X)$ we get

$$\alpha_t^\ast([0]_{\mu}) = \alpha_t(0) = U(0, t) = \phi$$

and $\beta_t^\ast([0]_{\nu}) = \beta_t(0) = L(1, t) = \phi$. This show that $\alpha_t^\ast$ and $\beta_t^\ast$ are surjective, and we are done. For any $t \in [0, 1]$, we define another relation $\mathcal{R}^t$ on $\Omega(X)$ as follows:

$$A, B \in \mathcal{R}^t \iff U(\mu_A, t) \cap L(\nu_A, t) = U(\mu_B, t) \cap L(\nu_B, t).$$

For any $A = \langle x, \mu_A, \nu_A \rangle, B = \langle x, \mu_B, \nu_B \rangle \in \Omega(X)$. Then the relation $\mathcal{R}^t$ is also an equivalence relation on $\Omega(X)$.

**Theorem 3.15.** For any $t \in (0, 1)$, the map $\phi_t : \Omega(X) \to S(X) \cup \{\phi\}$ defined by $\phi_t(A) = \alpha_t(A) \cap \beta_t(A)$ for each $A = \langle x, \mu_A, \nu_A \rangle \in \Omega(X)$ is surjective.

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Proof. Let $t \in (0, 1)$. For $0^\ast = \langle x, 0, 1 \rangle \in \Omega(X)$, we get

$$\phi_t(0) = \alpha_t(0) \cap \beta_t(0) = \cup(0, t) \cap L(1, t) = \phi.$$ 

For any $H \in \Omega(X)$, there exists $H = \langle x, \chi_H, \chi_H \rangle \in \Omega(X)$ such that

$$\phi_t(H) = \alpha_t(H) \cap \beta_t(H) = \cup(\chi_H, t) \cap L(\chi_H, t) = H.$$ 

This completes the proof.

Theorem 3.16. For any $t \in (0, 1)$, the quotient set $\Omega(X) | \mathcal{R}^t$ is equipotent to $S(X) \cup \{\phi\}$.

Proof. Let $t \in (0, 1)$ and let $\phi_t^* : \Omega(X) | \mathcal{R}^t \to S(X) \cup \{\phi\}$ be a map defined by $\phi_t^*([A]_{\mathcal{R}^t}) = \phi_t(A)$ for all $[A]_{\mathcal{R}^t} \in \Omega(X) | \mathcal{R}^t$. Assume that $\phi_t^*([A]_{\mathcal{R}^t}) = \phi_t^*([B]_{\mathcal{R}^t})$ for any $[A]_{\mathcal{R}^t}, [B]_{\mathcal{R}^t} \in \Omega(X) | \mathcal{R}^t$. Then

$$\alpha_t(A) \cap \beta_t(A) = \alpha_t(B) \cap \beta_t(B).$$ 

i.e., $U(\mu_A, t) \cap L(\nu_A, t) = U(\mu_B, t) \cap L(\nu_B, t)$. Hence $(A, B) \in \mathcal{R}^t$, and so $[A]_{\mathcal{R}^t} = [B]_{\mathcal{R}^t}$. Therefore $\phi_t^*$ is injective. Now for

$$0^\ast = \langle x, 0, 1 \rangle \in \Omega(X)$$

we have

$$\phi_t^*([0]_{\mathcal{R}^t}) = \phi_t(0^\ast) = \alpha_t(0^\ast) \cap \beta_t(0^\ast) = U(0, t) \cap L(1, t) = \phi.$$ 

For

$$H^\ast = \langle x, \chi_H, \chi_H \rangle \in \Omega(X)$$

we get

$$\phi_t^*([H]_{\mathcal{R}^t}) = \phi_t(H^\ast) = \alpha_t(H^\ast) \cap \beta_t(H^\ast) = \cup(\chi_H, t) \cap L(\chi_H, t) = H.$$ 

Thus $\phi_t^*$ is surjective. This completes the proof.

4 Intuitionistic fuzzy topological BH-algebras

In [4], Coker generalized the concept of fuzzy topological space, first initiated by Chang [3], to the case of intuitionistic fuzzy sets as follows.

Definition 4.1 ([4]). An intuitionistic fuzzy topology (IFT) on a non-empty set $X$ is a family $\Phi$ of IFSs in $X$ satisfying the following axioms:

(T1) $0^\ast, 1^\ast \in \Phi$,

(T2) $G_1 \cap G_2 \in \Phi$ for any $G_1, G_2 \in \Phi$,

(T3) $\cup_{i \in J} G_i \in \Phi$ for any family $\{G_i : i \in J\} \subseteq \Phi$.

In this case the pair $(X, \Phi)$ is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in $\Phi$ is called an intuitionistic fuzzy open set (IFOS for short) in $X$.

Definition 4.2. Let $(X, \Phi)$ and $(Y, \Psi)$ be two IFTSs. A mapping $f : X \to Y$ is said to be intuitionistic fuzzy continuous if the preimage of each IFS in $\Psi$ is an IFS in $\Phi$; and $f$ is said to be intuitionistic fuzzy open if the image of each IFS in $\Phi$ is an IFS in $\Psi$. 

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Definition 4.3. Let $D$ be an IFS in an IFTS $(X, \Psi)$. Then the induced intuitionistic fuzzy topology (IIFT for short) on $D$ is the family of IFSs in $D$ which are the intersection with $D$ of IFSs in $X$. The IIFT is denoted by $\Psi_D$, and the pair $(D, \Psi_D)$ is called an intuitionistic fuzzy subspace of $(X, \Psi)$.

Definition 4.4. Let $(D, \Phi_D)$ and $(B, \Psi_B)$ be intuitionistic fuzzy subspaces of IFTSs $(X, \Phi)$ and $(Y, \Psi)$, respectively, and let $f : X \rightarrow Y$ be a mapping. Then $f$ is a mapping of $D$ into $B$ if $f(D) \subset B$. Furthermore, $f$ is said to be relatively intuitionistic fuzzy continuous if for each IFS $V_B \in \Psi_B$, the intersection $f^{-1}(V_B) \cap D$ is an IFS in $\Phi_D$; and $f$ is said to be relatively intuitionistic fuzzy open if for each IFS $U_D \in \Phi_D$, the image $f(U_D)$ is an IFS in $\Psi_B$.

Proposition 4.5. Let $(D, \Phi_D)$ and $(B, \Psi_B)$ be intuitionistic fuzzy subspaces of IFTSs $(X, \Phi)$ and $(Y, \Psi)$ respectively, and let $f$ be an intuitionistic fuzzy continuous mapping of $X$ into $Y$ such that $f(D) \subset B$. Then $f$ is relatively intuitionistic fuzzy continuous mapping of $D$ into $B$.

Proof. Let $V_B$ be an IFS in $\Psi_B$. Then there exists $V \in \Psi$ such that $V_B = V \cap B$. Since $f$ is intuitionistic fuzzy continuous, it follows that $f^{-1}(V)$ is an IFS in $\Phi$. Hence

$$f^{-1}(V_B) \cap D = f^{-1}(V \cap B) \cap D = f^{-1}(V) \cap f^{-1}(B) \cap D = f^{-1}(V) \cap D$$

is an IFS in $\Phi_D$. This completes the proof.

For any BH-algebra $X$ and any element $a \in X$ we use $a_r$ to denote the selfmap of $X$ defined by $a_r(x) = x * a$ for all $x \in X$. \hfill \Box

Definition 4.6. Let $X$ be BH-algebra, $\Phi$ an IFT on $X$ and $D$ an intuitionistic fuzzy BH-algebra with IIFT $\Phi_D$. Then $D$ is called an intuitionistic fuzzy topological BH-algebra if for each $a \in X$ the mapping

$$a_r : (D, \Phi_D) \rightarrow (D, \Phi_D), x \mapsto x * a,$$

is relatively intuitionistic fuzzy continuous.

Theorem 4.7. Given BH-algebras $X$ and $Y$, and a BH-homomorphism $\alpha : X \rightarrow Y$, let $\Phi$ and $\Psi$ be the IFTs on $X$ and $Y$, respectively such that $\Phi = \alpha^{-1}(\Psi)$. If $B$ is an intuitionistic fuzzy topological BH-algebra in $Y$, then $\alpha^{-1}(B)$ is an intuitionistic fuzzy topological BH-algebra in $X$.

Proof. Let $a \in X$ and let $U$ be an IFS in $\Phi_{\alpha^{-1}}(B)$. Since $\alpha$ is an intuitionistic fuzzy continuous mapping of $(X, \Phi)$ into $(Y, \Psi)$, it follows from Proposition 4.5 that $\alpha$ is a relatively intuitionistic fuzzy continuous mapping of $(\alpha^{-1}(B), \Phi_{\alpha^{-1}}(B))$ into $(B, \Psi_B)$. Note that there exists an IFS $V$ in $\Psi_B$ such that $\alpha^{-1}(V) = U$. Then

$$\mu_{\alpha^{-1}}(U)(x) = \mu_U(a_r(x)) = \mu_U(x * a) = \mu_{\alpha^{-1}}(V)(x * a)$$

and

$$\nu_{\alpha^{-1}}(U)(x) = \nu_U(a_r(x)) = \nu_U(x * a) = \nu_{\alpha^{-1}}(V)(x * a)$$

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Since $B$ is an intuitionistic fuzzy topological BH-algebra in $Y$, the mapping
\[ b_r : (B, \Psi_B) \rightarrow (B, \Psi_B), \ y \mapsto y \ast b \]
is relatively intuitionistic fuzzy continuous for every $b \in Y$. Hence,
\[
\mu_{a_r^{-1}}(U)(x) = \mu_{\nu}(\alpha(x) \ast \alpha(a)) = \mu_{\nu}(\alpha(a)r(\alpha(x))) \\
= \mu_{\alpha(a)r(V)}^{-1}(\alpha(x)) = \mu_{\alpha(a)r^{-1}(V)}^{-1}(\alpha(x))
\]
and
\[
\nu_{a_r^{-1}}(U)(x) = \nu_{\nu}(\alpha(x) \ast \alpha(a)) = \nu_{\nu}(\alpha(a)r(\alpha(x))) \\
= \nu_{\alpha(a)r(V)}^{-1}(\alpha(x)) = \nu_{\alpha(a)r^{-1}(V)}^{-1}(\alpha(x))
\]
Therefore
\[
a_r^{-1}(U) = \alpha^{-1}(\alpha(a)r^{-1}(V)),
\]
and so
\[
a_r^{-1}(U) \cap \alpha^{-1}(B) = \alpha^{-1}(\alpha(a)r^{-1}(V)) \cap \alpha^{-1}(B)
\]
is an IFS in $\Phi_{a^{-1}}(B)$.

This completes the proof. \( \square \)

**Theorem 4.8.** Given BH-algebras $X$ and $Y$, and a BH-isomorphism $\alpha$ of $X$ to $Y$, let $\Phi$ and $\Psi$ be the IFTs on $X$ and $Y$ respectively such that $\alpha(\Phi) = \Psi$. If $D$ is an intuitionistic fuzzy topological BH-algebra in $X$, then $\alpha(D)$ is an intuitionistic fuzzy topological BH-algebra in $Y$.

*Proof.* It is sufficient to show that the mapping
\[ b_r : (\alpha(D), \Psi_{\alpha(D)}) \rightarrow (\alpha(D), \Psi_{\alpha(D)}), \ y \mapsto y \ast b \]
is relatively intuitionistic fuzzy continuous for each $b \in Y$. Let $U_D$ be an IFS in $\Psi_D$. Then there exists an IFS $U$ in $\Phi$ such that $U_D = U \cap D$. Since $\alpha$ is one-one, it follows that
\[
\alpha(U_D) = \alpha(U \cap D) = \alpha(U) \cap \alpha(D)
\]
which is an IFS in $\Psi_{\alpha(D)}$. This shows that $\alpha$ is relatively intuitionistic fuzzy open.

Let $V_{\alpha}(D)$ be an IFS in $\Psi_{\alpha}(D)$. The surjectivity of $\alpha$ implies that for each $b \in Y$ there exists $a \in X$ such that $b = \alpha(a)$. Hence
\[
\mu^{-1}(\alpha(br(V_{\alpha}(D)))(x)) = \mu_{\alpha^{-1}(\alpha(a)r^{-1}(V_{\alpha}(D)))(x)} \\
= \mu_{\alpha(a)r^{-1}(V_{\alpha}(D))}(\alpha(x)) \\
= \mu_{V_{\alpha}(D)}(\alpha(a)r(\alpha(x))) \\
= \mu_{V_{\alpha}(D)}(\alpha(x) \ast \alpha(a)) \\
= \mu_{V_{\alpha}(D)}(\alpha(x \ast a)) \\
= \mu_{\alpha^{-1}(V_{\alpha}(D))}(x \ast a) \\
= \mu_{\alpha^{-1}(V_{\alpha}(D))}(a_r(x)) \\
= \mu_{a_r^{-1}(V_{\alpha}(D))}(x)
\]
\[ \nu^{-1}\alpha(\operatorname{br}(V\alpha(D)))(x) = \nu_{\alpha^{-1}(\alpha(a)r^{-1}(V\alpha(D))})(x) \\
= \nu_{\alpha(a)r^{-1}(V\alpha(D))}(\alpha(x)) \\
= \nu_{V\alpha(D)}(\alpha(a)r(\alpha(x))) \\
= \nu_{V\alpha(D)}(\alpha(x)*\alpha(a)) \\
= \nu_{V\alpha(D)}(\alpha(x*a)) \\
= \nu_{\alpha^{-1}(V\alpha(D))}(x*a) \\
= \nu_{\alpha^{-1}(V\alpha(D))}(a_{r}(x)) \\
= \nu_{\alpha^{-1}(V\alpha(D))}(x) \]

Therefore
\[ \alpha^{-1}(b^{-1}_{r}(V\alpha(D))) = a^{-1}_{r}(\alpha^{-1}(V\alpha(D)))). \]

By hypothesis, the mapping
\[ a_{r} : (D, \Phi_{D}) \rightarrow (D, \Phi_{D}), x \mapsto x*a \]
is relatively intuitionistic fuzzy continuous and \( \alpha \) is a relatively intuitionistic fuzzy continuous map:
\[ (D, \Phi_{D}) \rightarrow (\alpha(D), \Psi_{\alpha(D)}). \]

Thus
\[ \alpha^{-1}(b^{-1}_{r}(V\alpha(D))) \cap D = a^{-1}_{r}(\alpha^{-1}(V\alpha(D)) \cap D \]
is an IFS in \( \Phi_{D} \). Since \( \alpha \) is relatively intuitionistic fuzzy open,
\[ \alpha(\alpha^{-1}(b^{-1}_{r}(V\alpha(D))) \cap D) = b^{-1}_{r}(V\alpha(D)) \cap \alpha(D). \]
is an IFS in \( \Psi_{\alpha(D)} \). This completes the proof. \( \square \)

References

[1] Atanassov, K. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 35, 87–96.

[2] Atanassov, K.T. (2015) One new algebraic object, Int.J.Bioautomation, 20 (S1), S75–S81.

[3] Chang, C. (1968). Fuzzy topological spaces. J. Math. Anal. Appl., 24, 182–190.

[4] Coker, D. (1997). An introduction to intuitionistic fuzzy topological spaces. Fuzzy Sets and Systems, 88, 81–89.

[5] Foster, D. H. (1979). Fuzzy topological groups. J. Math. Anal. Appl., 67, 549–564

[6] Hu, Q. P. & Li, X. (1983). On BCH-algebras. Math. Seminar Notes, 11, 313–320.

[7] Hu, Q. P. & Li, X. (1985). On Proper BCH-algebras. Math. Japan., 30, 659–661.
[8] Imai, Y & Iseki, K. (1966). On axiom systems of propositional calculi XIV. *Proc. Japan Academy*, 42, 19–22

[9] Jun, Y. B., Roh, E. H. & Kim, H. S. (1998) On BH-algebras, *Sci. Mathematicae* 1, 347-354.

[10] Park, C. H. (2006). Interval-valued fuzzy ideal in BH-algebras. *Advance in Fuzzy Set and System*, 1, 231–240.

[11] Rosenfeld, A. (1971). Fuzzy groups. *J. Math. Anal. Appl.*, 35, 512–517.

[12] Senapati, T., Bhowmik, M., & Pal, M. (2013) Intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals in BCK/BCI algebras, *Notes on Intuitionistic Fuzzy sets*, 19(1), 32–47.

[13] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8(3), 338–353.

[14] Zhang, Q., Roh, E. H., & Jun, Y. B. (2001). On Fuzzy BH-algebras. *J. Huanggang. Normal Univ.*, 21, 14–19.