Mixing-time and large-decoherence in continuous-time quantum walks on one-dimension regular networks

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Abstract

In this paper, we study mixing and large decoherence in continuous-time quantum walks on one dimensional regular networks, which are constructed by connecting each node to its $2l$ nearest neighbors ($l$ on either side). In our investigation, the nodes of network are represented by a set of identical tunnel-coupled quantum dots in which decoherence is induced by continuous monitoring of each quantum dot with nearby point contact detector. To formulate the decoherent CTQWs, we use Gurvitz model and then calculate probability distribution and the bounds of instantaneous and average mixing times. We show that adding links to cycle network, in appearance of large decoherence, decreases the mixing times. Also, the mixing times are inversely proportional to squared parameter $l$, but are linearly proportional to the decoherence rate. Moreover, our analytical solutions justify the mentioned results for cycle in [18] and for long-range interacting cycle in [22].

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1 Introduction

The classical random walk is one of the most basic concepts of probability. When formulated with a finite number of states, it consists of randomly choosing paths between vertices on a graph, and many randomized algorithms for computing are such walks. Random walks have a clear appeal to both physicists and computer scientists: they are a powerful tool for both describing physical phenomena and writing algorithms. Not surprisingly then, the quantum mechanical analog of random walk (quantum walk) can be developed from both physical and computational directions as well. Quantum walks were formulated in studies involving the dynamics of quantum diffusion [1], but the analysis of quantum walks for use in quantum algorithms was first done by Farhi and Gutmann [2]. Quantum walks as noted earlier, are primarily of two types. Depending on the way the evolution operator is defined, they can be either discrete-time quantum walks (DTQWs) [3] or continuous-time quantum walks (CTQWs) [2]. In the CTQW, one can directly define the walk on the position space, whereas in the DTQW, it is necessary to introduce a quantum coin operation to define the direction in which the particle has to move. Unlike the classical case the DTQW, because of the coin space, does not reduce to the CTQW when we let the time step between repeated applications of the unitary operator tend to zero. In recent years, the DTQW has been studied on random environments [4], on quotient graphs [5], in phase space [6] and for single and entangled particles [7]. Also, the CTQW has been considered on star graph [8, 9], direct product of cayley graphs [10], quotient graphs [11], odd graphs [12], trees [13, 14] and ultrametric spaces [15]. All of these cited articles have focused on the coherent quantum walks, i.e. we have a closed system without any interaction with any environment. But in any realistic quantum system, e.g. quantum computer, interaction with the environment cannot be entirely eliminated. Investigations on the decoherent QW were given in [16, 17, 18, 19, 20]. Also, the effect of decoherence on the mixing time of CTQW has been provided on cycle [18], long-range interacting cycle [21, 22] and one-dimension (1D) regular network [23]. Long-range interacting cycle is constructed by connecting all the two nodes of distance $m$ on the cycle graph [24] while 1D regular network is a cycle graph in which every node is connected to its 2l nearest neighbors ($l$ on either side) [25]. Figs. 1(a) and (b) show sketches of LRIC with $N = 8$ and $m = 3$ and 1D regular network with $N = 8$ and $l = 3$, respectively. It is clear that 1D regular network can be regarded as the extensions of the long-range interacting cycle network in which the interacting parameters are $m = 1, 2, \cdots, l$. 

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Figure 1: Examples of LRIC with $N = 8$, $m = 3$(left) and 1D regular network with $N = 8$ and $l = 3$(right).

In [23], authors proved that the mixing time of CTQW on 1D regular network, for the small rates of decoherence, is inversely proportional to rate of decoherence, but is independent of parameter $l(l \geq 2)$(due to losing of quantum interference). In this paper, we provide CTQWs on 1D regular network in the appearance of large decoherence. We show that the bounds of instantaneous and average mixing times are proportional to decoherence parameter but, unlike the small decoherence case, are inversely proportional to the sum of squared interacting parameters($m$). Moreover, our analytical results for $l = 1$ and for any $m \neq 1$ are in agreement with the results mentioned in [18] and [22], respectively.

Our paper is structured as follows: In Sec. 2, we provide a brief summary of the main concepts and of the formulae concerning CTQWs and give the exact solutions to the transition probabilities on 1D regular network. Sec. 3 present the decoherent CTQWs on 1D regular network. We assume that the decoherence rate $\Gamma$ is large and calculate the probability distribution in Sec. 4. In Sec. 5, we obtain the lower and upper bounds of instantaneous and average mixing times. Conclusions and discussions are given in the last part, Sec. 6.

2 CTQWs on 1D regular network

Every network can be considered as a graph whose nodes are connected in a similar manner. From an algebraic point of view, to each graph there is a corresponding adjacency matrix $A = (A_{ij})$, which is a discrete version of the Laplace operator. The non-diagonal elements $A_{ij}$ equal 1 if nodes $i$ and $j$ are connected by a bond and 0 otherwise. The connectivity of node $i$ can be calculated as a sum of matrix elements $z_i = \sum_j A_{ij}$. The Laplacian operator is then defined as $L = Z - A$, where $Z$ is the diagonal matrix given by $Z_{ik} = z_i\delta_{ik}$. It is worth underlining that, being symmetric and non-negative definite, $L$ can generate both a probability conserving Markov process and unitary process. Thus, the Laplacian operator can work both as a classical transfer operator and as a tight-binding Hamiltonian of a quantum transport process [26] [27] [28].
The continuous-time random walks (CTRWs) are described by the following Master equation [29]:

\[ \frac{d}{dt} p_{k,j}(t) = \sum_{l=1}^{N} T_{kl} p_{l,j}(t), \]  

being \( p_{k,j}(t) \) the conditional probability that the walker is on node \( k \) at time \( t \) when it started from node \( j \). If the walk is symmetric with a site-independent transmission rate \( \gamma \), then the transfer matrix \( T \) is simply related to the Laplacian operator through \( T = -\gamma L \) (in the following we set \( \gamma = 1 \)).

The quantum-mechanical extension of the CTRW is called continuous-time quantum walk (CTQW). The CTQWs are obtained by identifying the Hamiltonian of the system with the classical transfer matrix, \( H = -T \) [27, 30]. The states \( |j\rangle \), representing the walker localized at the node \( j \), span the whole accessible Hilbert space and also provide an orthonormal basis set. In these basis the Schrödinger equation is

\[ i \frac{d}{dt} |k\rangle = H |k\rangle, \]

where we set \( m = 1 \) and \( \hbar = 1 \). The time evolution of state \( |j\rangle \) starting at time 0 is given by \( |j, t\rangle = U(t)|j\rangle \), where \( U(t) = \exp[-iHt] \) is the quantum-mechanical time evolution operator. Therefore, the behaviour of the walker can be described by the transition amplitude \( \alpha_{k,j}(t) \) from state \( |j\rangle \) to state \( |k\rangle \), which is

\[ \alpha_{k,j}(t) = \langle k|e^{-iHt}|j\rangle. \]

From Eq. (2), the \( \alpha_{k,j}(t) \) obeys the following Schrödinger equation:

\[ \frac{d}{dt} \alpha_{k,j}(t) = -i \sum_{l=1}^{N} H_{kl} \alpha_{l,j}(t). \]

Note that the squared magnitude of transition amplitude provides the quantum mechanical transition probability \( \pi_{k,j}(t) = |\alpha_{k,j}(t)|^2 \).

To get the exact solution of Eqs. (1) and (4), all the eigenvalues and eigenvectors of the transfer operator and Hamiltonian are required. We denote the \( n \)th eigenvalue and eigenvector of \( A \) by \( E_n \) and \( |q_n\rangle \), respectively. Now the classical probability is given by

\[ p_{k,j}(t) = \sum_{n=1}^{N} e^{-tE_n} \langle k|q_n\rangle \langle q_n|j\rangle, \]
and the quantum probability can write as

$$\pi_{k,j}(t) = \sum_{n,l=1}^{N} e^{-it(E_n - E_l)} \langle k | q_n \rangle \langle q_n | j \rangle \langle k | q_l \rangle \langle q_l | j \rangle,$$

(6)

In the following, we focus on 1D regular networks and study CTQWs on them. 1D regular network is constructed by connecting every node to its 2m nearest neighbors (m on either side), see Fig. 1(b). This network provide a good model to study various coupled dynamical systems, including biological oscillators [31], Josephson junction arrays [32], synchronization [33], small-world networks [34] and many other self-organizing systems. The Hamiltonian of the system is given by [23]

$$H_{ij} = \begin{cases} 
-2l, & \text{if } i = j; \\
1, & \text{if } i = j \pm m, m \in [1, l]; \\
0, & \text{Otherwise.} 
\end{cases}$$

(7)

This Hamiltonian acting on the state $|j\rangle$ can be written as

$$H|j\rangle = -(2l + 1)|j\rangle + \sum_{m=-l}^{l} |j + m\rangle.$$

(8)

which is the discrete version of the Hamiltonian for a free particle moving on a lattice. It is well known in solid state physics that the solutions of the Schrödinger equation for a particle moving freely in a regular potential are Bloch functions [35, 36]. We denote the Bloch states by $|\Phi_n\rangle$ and then the time independent Schrodinger equation can be written as

$$H|\Phi_n\rangle = E_n|\Phi_n\rangle.$$

(9)

The Bloch state $|\Phi_n\rangle$ can be expressed as a linear combination of the states $|j\rangle$ localized at nodes $j$,

$$|\Phi_n\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-i\theta_n j}|j\rangle.$$

(10)

Because of periodic boundary conditions, we have $\Phi_n(N) = \Phi_n(0)$, where $\Phi_n(x) = \langle x | \Phi_n \rangle$. This restricts the $\theta_n$-values to $\theta_n = \frac{2\pi n}{N}$, where $n = 0, 1, \ldots, N - 1$. We set Eq. (8) and (10) into (9) and obtain the eigenvalues of system as [23]

$$E_n = -2l + 2 \sum_{j=1}^{l} \cos(j\theta_n).$$

(11)
Figure 2: Fig. 2(a) shows point contact detector $j$ monitoring the electron in dot $j + 1$ and Fig. 2(b) shows point contact detector $j$ when electron is placed in dot $j$. Source and drain reservoirs are kept at zero temperature with chemical potentials $F_{l,j}$ and $F_{r,j}$, respectively. $E_j$ is the on-site node energy. $\Omega_{lr,j}$ and $\Omega'_{lr,j}$ are transmission probability of the detector for empty dot and for the occupied dot.

Thus Eq. (5) and (6) can be written as

\[ p_{k,j}(t) = \frac{1}{N} \sum_n e^{-tE_n} e^{-i(k-j)^2\pi N}, \] (12)

\[ \pi_{k,j}(t) = \frac{1}{N^2} \sum_{n,l} e^{it(E_n-E_l)} e^{-i(k-j)(n-l)^2\pi N}. \] (13)

### 3 The Decoherent CTQWs on 1D regular network

In this section, we take into account the effects of decoherence in quantum walks on 1D regular network. For this aim, we use the realistic physical model of the network in which decoherence induced by continuous monitoring of each network node with nearby point contact(PC) detector. In this model, our assumptions are as follows: each node is represented by a quantum dot which is continuously monitored by an individual point contact(PC), the walks are performed by an electron initially placed in one of the quantum dots, identical PCs are placed far enough from QDs so that the tunneling between them is negligible, Coulomb interaction between electrons in QD and PC is taken into account and all electrons are spin-less fermions [37]. Such a set up is shown schematically in Fig. 2, where detector $j$ is represented by a barrier, connected with two reservoirs at the potentials $F_{l,j}$ and $F_{r,j}$. The transmission probability of the barrier varies from $\Omega_{lr,j}$...
to $\Omega'_{lr,j}$, depending on whether or not the quantum dot is occupied by an electron.

In the following, we want to write Hamiltonian for the entire system. We consider simple continuous-time quantum walks are defined over an undirected graph with $N$ nodes in which each node is labelled by an integer $i \in [0, N-1]$. These walks can be well described by the following Hamiltonian [38, 39]:

$$H_s = \sum_{ij} \Delta_{ij}(t) (\hat{c}_i^\dagger \hat{c}_j + \hat{c}_i \hat{c}_j^\dagger) - \sum_j E_j(t) (\hat{c}_j^\dagger \hat{c}_j),$$

(14)

Here, each node $j$ corresponds to the quantum state $|j\rangle = \hat{c}_j^\dagger |0\rangle$ in which $|j\rangle$ denotes the state where the particle is located at node $j$. The two terms correspond to a hopping term with amplitudes $\Delta_{ij}(t)$ between nodes, and on-site node energies $E_j(t)$, both of which can depend on time. For the sake of simplicity, we drop all the on-site energies and assume that hopping amplitudes $\Delta_{ij}(t)$ between connected sites to be constant. Also, we renormalize the time, so that it becomes dimensionless [37]. Thus, Eq. (14) reduces to

$$H_s = \frac{1}{4} \sum_{j=0}^{N-1} \sum_z (\hat{c}_{j+z}^\dagger \hat{c}_j + \hat{c}_j \hat{c}_{j+z}).$$

(15)

Now, we study the electron transport process in the point contact $j$. The point contact is considered as a barrier, separated two reservoirs (the source and drain). All the levels in the source and drain are initially filled up to the Fermi energies. We call it as the vacuum state $|0\rangle$. Thus, the Hamiltonian of $j$-th point contact can be written as

$$H_{pc,j} = \sum_l E_{l,j} \hat{a}_{l,j}^\dagger \hat{a}_{l,j} + \sum_r E_{r,j} \hat{a}_{r,j}^\dagger \hat{a}_{r,j} + \sum_{l,r} \Omega_{lr,j} (\hat{a}_{l,j}^\dagger \hat{a}_{r,j} + \hat{a}_{r,j}^\dagger \hat{a}_{l,j}),$$

(16)

where $\hat{a}_{l,j}^\dagger (\hat{a}_{l,j})$ and $\hat{a}_{r,j}^\dagger (\hat{a}_{r,j})$ are the creation (annihilation) operators in the left and right reservoirs respectively, and $\Omega_{lr,j}$ is the hopping amplitude between the states $E_{l,j}$ and $E_{r,j}$ in the right and left reservoirs.

The interaction between the detector and the measured system is described by $H_{\text{int}}$. The presence of an electron in the left dot results in an effective
increase of the point contact barrier ($\Omega_{lr,j} = \Omega_{lr,j} + \delta\Omega_{lr,j}$). Therefore, the interaction Hamiltonian can be written as

$$H_{int} = \sum_{l,r} \delta\Omega_{lr,j} c_j^{\dagger} c_j (a_{l,j} a_{r,j} + a_{r,j}^{\dagger} a_{l,j}) .$$  \hspace{1cm} (17)

For simplicity, we assume that the hoping amplitudes are weakly dependent on states $E_{l,j}$ and $E_{r,j}$, so that $\Omega_{lr,j} = \Omega$, $\delta\Omega_{lr,j} = \delta\Omega$ and $F_{l,j}(F_{r,j}) = F_l(F_r)$. Gurvitz in [40] showed the appearance of decoherence leads to the collapse of the density matrix into the statistical mixture in the course of the measurement processes, thus the evolution of the reduced density matrix traced over all states of source and drain electrons is given by Bloch-type rate equations. The time dependent non-unitary evolution of reduced density matrix $\rho(t) = |\Phi(t)\rangle\langle\Phi(t)|$ in the Gurvitz model is given by [23]

$$\frac{d}{dt}\rho_{j,k}(t) = -\frac{i}{4}\left[ \sum_{m=-l}^{l} (\rho_{j+m,k} - \rho_{j,k+m}) \right] - \Gamma(1 - \delta_{j,k})\rho_{j,k}$$

$$= -\frac{i}{4}\left[ \sum_{m=1}^{l} (\rho_{j+m,k} - \rho_{j,k+m} + \rho_{j-m,k} - \rho_{j,k-m}) \right] - \Gamma(1 - \delta_{j,k})\rho_{j,k} .$$  \hspace{1cm} (18)

4 Large Decoherence

In [23], authors studied the effect of small decoherence ($\Gamma N \ll 1$) in CTQWs on 1D regular networks. They showed that the instantaneous mixing time upper bound and the average time mixing lower bound are independent of parameter $l(l \geq 2)$, but are proportional to the inverse of decoherence rate. Here, we want to study CTQWs on these networks in appearance of the large rates of decoherence. For this purpose, we use Gurvitz model and focus on the elements of matrix $\rho(t)$. Based on the initial conditions, the non-zero elements appear only along the major diagonal. Firstly, we rewrite Eq. (18) for the elements of major diagonal and also minor diagonals whose distances of the major diagonal are lesser than $l$. By dropping terms that are smaller than $1/\Gamma$, we have
\[
\begin{align*}
\rho_{j,j}'(t) &= -\frac{i}{4}\left[\sum_{m=1}^{l} (\rho_{j+m,j} - \rho_{j,j+m} + \rho_{j-m,j} - \rho_{j,j-m})\right], \\
\rho_{j,j+1}'(t) &= -\frac{i}{4}\left[\rho_{j+1,j+1} - \rho_{j,j}\right] - \Gamma \rho_{j,j+1}, \\
\rho_{j,j+2}'(t) &= -\frac{i}{4}\left[\rho_{j+2,j+2} - \rho_{j,j}\right] - \Gamma \rho_{j,j+2}, \\
&\vdots \\
\rho_{j,j+l}'(t) &= -\frac{i}{4}\left[\rho_{j+l,j+l} - \rho_{j,j}\right] - \Gamma \rho_{j,j+l}, \\
\end{align*}
\tag{19}
\]

For the simplicity sake, we use the following definitions
\[
\begin{align*}
a_j &= \rho_{j,j}, \\
d_j &= \rho_{j,j+1} - \rho_{j+1,j}, \\
f_j &= \rho_{j,j+2} - \rho_{j+2,j}, \\
&\vdots \\
q_j &= \rho_{j,j+l} - \rho_{j+l,j}. \\
\end{align*}
\tag{20}
\]

Then the above difference equation system can be written as
\[
\begin{align*}
a_j' &= -\frac{i}{4}\left[-d_j + d_{j-1} - f_j + f_{j-2} + \cdots - q_j + q_{j-l}\right], \\
d_j' &= -\frac{i}{2}\left[a_{j+1} - a_j\right] - \Gamma d_j, \\
f_j' &= -\frac{i}{2}\left[a_{j+2} - a_j\right] - \Gamma f_j, \\
&\vdots \\
q_j' &= -\frac{i}{2}\left[a_{j+l} - a_j\right] - \Gamma q_j, \\
\end{align*}
\tag{21}
\]

Differentiation of the above equation gives
\[
\begin{align*}
a_j'' &= -\frac{i}{4}\left[-d_j' + d_{j-1}' - f_j' + f_{j-2}' + \cdots - q_j' + q_{j-l}'\right], \\
d_j'' &= -\frac{i}{2}\left[a_{j+1}' - a_j'\right] - \Gamma d_j', \\
f_j'' &= -\frac{i}{2}\left[a_{j+2}' - a_j'\right] - \Gamma f_j', \\
&\vdots \\
q_j'' &= -\frac{i}{2}\left[a_{j+l}' - a_j'\right] - \Gamma q_j', \\
\end{align*}
\tag{22}
\]

We can guess the following solutions for the above equations
a_j = \sum_{k=0}^{N-1} A_k e^{\frac{2\pi jk}{N}} e^{-\gamma_k t}, \quad d_j = \sum_{k=0}^{N-1} D_k e^{\frac{2\pi jk}{N}} e^{-\gamma_k t},

f_j = \sum_{k=0}^{N-1} F_k e^{\frac{2\pi jk}{N}} e^{-\gamma_k t}, \quad \cdots, \quad q_j = \sum_{k=0}^{N-1} Q_k e^{\frac{2\pi jk}{N}} e^{-\gamma_k t},

(23)

in which $\gamma_k, A_k, D_k, \cdots$ and $Q_k$ are the unknown quantities. We set these solutions into Eqs. (22) and get

\[
\begin{align*}
\gamma_k A_k + i\left[ D_k (1 - e^{-\frac{2\pi ik}{N}}) + F_k (1 - e^{-\frac{4\pi ik}{N}}) + \cdots + Q_k (1 - e^{-\frac{2\pi kl}{N}}) \right] &= 0 \\
A_k [\frac{i}{2} (-e^{-\frac{2\pi ik}{N}} + 1)] + D_k (\gamma_k - \Gamma) &= 0, \\
A_k [\frac{i}{2} (-e^{-\frac{4\pi ik}{N}} + 1)] + F_k (\gamma_k - \Gamma) &= 0, \\
\vdots \\
A_k [\frac{i}{2} (-e^{-\frac{2\pi ik}{N}} + 1)] + Q_k (\gamma_k - \Gamma) &= 0,
\end{align*}
\]

(24)

It is evident that there are nontrivial solutions when the determinant of the coefficients matrix is zero.

\[(\gamma_k - \Gamma)^{(l-1)}[\gamma_k (\gamma_k - \Gamma) + \frac{1}{2} (\sin^2 (\frac{\pi k}{N}) + \sin^2 (\frac{2\pi k}{N}) + \cdots + \sin^2 (\frac{l\pi k}{N}))] = 0\]

(25)

Therefore, there are four values for $\gamma_k$

\[
\gamma_k = \begin{cases}
\gamma_{k,0} = 0, \\
\gamma_{k,1} = \Gamma, \\
\gamma_{k,2} = \Gamma - \frac{1}{2T} \sum_{m=1}^{l} \sin^2 (\frac{\pi km}{N}), \\
\gamma_{k,3} = \frac{1}{2T} \sum_{m=1}^{l} \sin^2 (\frac{\pi km}{N}),
\end{cases}
\]

(26)

The general solutions of Eqs. (22) are
and replacing the constant coefficients into the others, we get
decoherence, this distribution reduces to

$$\text{does not converge to any constant value.}$$

Since CTQWs are symmetric under time-inversion, the above distribution

Note that the probability distribution

By the initial condition $a_j(0) = \delta_{j,0}$ and $d_j = 0$, for $j = 0, \cdots, N - 1$, and replacing the constant coefficients into the others, we get

$$\begin{align*}
A_{k,1} & \simeq 0, \quad A_{k,2} \simeq \frac{1}{2\Gamma} \sum_{m=1}^{l} \sin^2 \left( \frac{\pi km}{N} \right), \quad A_{k,3} = 1, \\
D_{k,1} & \simeq 0, \quad D_{k,2} \simeq \frac{i}{\Gamma} \sin \left( \frac{\pi k}{N} \right) \exp \left( \frac{i\pi k}{N} \right), \quad D_{k,3} \simeq \frac{-i}{\Gamma} \sin \left( \frac{\pi k}{N} \right) \exp \left( \frac{-i\pi k}{N} \right), \\
F_{k,1} & \simeq 0, \quad F_{k,2} \simeq \frac{i}{\Gamma} \sin \left( \frac{2\pi k}{N} \right) \exp \left( \frac{2i\pi k}{N} \right), \quad F_{k,3} \simeq \frac{-i}{\Gamma} \sin \left( \frac{2\pi k}{N} \right) \exp \left( \frac{-2i\pi k}{N} \right), \\
\vdots \\
Q_{k,1} & \simeq 0, \quad Q_{k,2} \simeq \frac{i}{\Gamma} \sin \left( \frac{3\pi k}{N} \right) \exp \left( \frac{3i\pi k}{N} \right), \quad Q_{k,3} \simeq \frac{-i}{\Gamma} \sin \left( \frac{3\pi k}{N} \right) \exp \left( \frac{-3i\pi k}{N} \right).
\end{align*}$$

Note that the probability distribution $P(t)$ of the quantum walk is specified by the diagonal elements of $\rho(t)$, that is $P_j(t) = a_j(t)$. For large rates of decoherence, this distribution reduces to

$$a_j(t) = \frac{1}{N} \sum_{k=0}^{N-1} \exp \left[ -t \sum_{m=1}^{l} \sin^2 \left( \frac{\pi km}{N} \right) \right] \omega^{jk}. \quad (29)$$

Since CTQWs are symmetric under time-inversion, the above distribution
does not converge to any constant value.
5 Mixing time

The rate of convergence to probability distribution of CTQWs can expressed in terms of many quantities, but the one used commonly is mixing time. There are two different definitions of mixing time in the literature: the instantaneous mixing time and the average mixing time. To define the notion of the mixing time, we use the total variation distance between distributions $d_1$ and $d_2$ as $\|d_1 - d_2\|_{tv} = \sum_j |d_1(j) - d_2(j)|$.

(a) **Instantaneous mixing time**

The instantaneous mixing time is defined as the first time instant at which the probability distribution of the walker position is sufficiently close to uniform distribution $[38]$, i.e.

$$T_{ins} = \min\{T : \sum_{j=0}^{N-1} \|P_j(t) - \frac{1}{N}\|_{tv} \leq \epsilon\},$$

(30)

Adding a summation over $j$ to Eq. (29) gives us the total variation distance required to obtain the mixing time:

$$\sum_{j=0}^{N-1} |a_j(t) - \frac{1}{N}| = \sum_{j=0}^{N-1} \left| \frac{1}{N} \sum_{k=0}^{N-1} \exp\left(-\frac{t}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right)\omega^{jk} - \frac{1}{N} \right|$$

$$= \sum_{j=0}^{N-1} \left| \frac{1}{N} \sum_{k=1}^{N-1} \exp\left(-\frac{t}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right) \cos\left(\frac{2\pi jk}{N}\right) \right|$$

(31)

**Lower bound**

To find the lower bound, we apply the following way:

Firstly, we use the term $j = 0$ and then use the terms $k = 1, N - 1$

$$\sum_{j=0}^{N-1} |a_j(t) - \frac{1}{N}| \geq |a_0(t) - \frac{1}{N}| = \frac{1}{N} \left| \sum_{k=1}^{N-1} \exp\left(-\frac{t}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right) \right|$$

$$\geq \frac{2}{N} e^{-\frac{t}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi m}{N}\right)}.$$

(32)
Based on the instantaneous mixing time definition (Eq. (30)):

\[
\frac{2}{N} e^{-\frac{T_{\text{ins,lower}}}{2\Gamma}} \sum_{m=1}^{l} \sin^2\left(\frac{\pi m}{N}\right) = \epsilon.
\] (33)

Thus, the lower bound of mixing time has the form

\[
T_{\text{ins,lower}} = \frac{2\Gamma}{l} \sum_{m=1}^{l} \sin^2\left(\frac{\pi m}{N}\right) \ln\left(\frac{2}{N\epsilon}\right).
\] (34)

For \( N \gg 1 \)

\[
T_{\text{ins,lower}} \simeq \frac{2\Gamma N^2}{\pi^2} \ln\left(\frac{2}{N\epsilon}\right).
\] (35)

Note that for \( l = 1 \), we have

\[
T_{\text{ins,lower}} = \frac{2\Gamma N^2}{\pi^2} \ln\left(\frac{2}{N\epsilon}\right)
\] (36)

which is coincide with [18]’s result for cycle.

Also, it is evident that for any \( m \neq 1 \), relation(36) gives us the mentioned result in [22] for LRICs.

**Upper bound**

To obtain the upper bound of instantaneous mixing time, we use the following way:

\[
\sum_{j=0}^{N-1} |a_j(t) - \frac{1}{N}| = \sum_{j=0}^{N-1} \left| \frac{1}{N} \sum_{k=1}^{N-1} \exp\left[-\frac{t}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right] \cos\left(\frac{2\pi jk}{N}\right) \right| \leq \sum_{j=0}^{N-1} \sum_{k=1}^{N-1} \exp\left[-\frac{t}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right]
\] (37)
Because of \( \sin(x) > \frac{2x}{\pi} \) for \( 0 < x < \frac{\pi}{2} \) \([11]\), we have

\[
\sum_{j=0}^{N-1} |a_j(t) - \frac{1}{N}| \leq \frac{2}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{[N/2]} \exp\left[\frac{-t}{\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right]
\]

\[
\leq \frac{2}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{[N/2]} \exp\left[\frac{-t}{\Gamma} \sum_{m=1}^{l} \frac{2k^2m^2}{N^2}\right]
\]

\[
\leq \frac{2}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{\infty} \exp\left[\frac{-t}{\Gamma} \sum_{m=1}^{l} \frac{2km^2}{N^2}\right]
\]

that in the third inequality, we used of the relation \( k^2 \geq k \) for \( k \geq 1 \). After some algebra, we obtain

\[
\sum_{j=0}^{N-1} |a_j(t) - \frac{1}{N}| \leq \frac{2}{\exp\left[\frac{l}{\Gamma} \sum_{m=1}^{l} \frac{2m^2}{N^2}\right] - 1}.
\]

According to the instantaneous mixing time definition (Eq. (30)):

\[
\frac{2}{\exp\left[\frac{T_{\text{ins,upper}}}{\Gamma} \sum_{m=1}^{l} \frac{2m^2}{N^2}\right] - 1} = \epsilon.
\]

Therefore, the upper bound of instantaneous mixing time is

\[
T_{\text{ins,upper}} = \frac{\Gamma N^2}{2} \ln\left(\frac{2 + \epsilon}{\epsilon}\right).
\]

For \( l = 1 \) (cycle), we obtain

\[
T_{\text{ins,upper}} = \frac{\Gamma N^2}{2} \ln\left(\frac{2 + \epsilon}{\epsilon}\right)
\]
and for any $m \neq 1$ (long-range interacting cycle), we have

$$T_{\text{ins,upper}} = \frac{\Gamma N^2}{2(1 + m^2)} \ln\left(\frac{2 + \epsilon}{\epsilon}\right) \quad (43)$$

which are the same results mentioned in [18, 22].

(b) **Average mixing time**

The average mixing time is based on the time-averaged probability distribution, that is

$$\bar{P}(j, T) = \frac{1}{T} \int_0^T P(j, t) dt. \quad (44)$$

We can define the average mixing time as the number of time steps required for the average distribution to be $\epsilon$-close to the limiting distribution [32], i.e.

$$T_{\text{ave}} = \min\{T : \|\bar{P}(j, t) - \frac{1}{N}\|_{tv} \leq \epsilon\}. \quad (45)$$

Firstly, we want to calculate $\bar{P}(j, T)$ by Eq. (29)

$$\bar{P}(j, T) = \frac{1}{T} \int_0^T \frac{1}{N} \sum_{k=0}^{N-1} \exp\left[-\frac{t}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right] \omega^{jk} dt$$

$$= \frac{2\Gamma}{TN} \sum_{k=0}^{N-1} 1 - \exp\left[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right] \omega^{jk} \quad (46)$$

The total variation distance between the uniform distribution and the time-average distribution of the decoherent quantum walk is given by
\[
\sum_{j=0}^{N-1} |\tilde{P}(j, T) - \frac{1}{N}| = \sum_{j=0}^{N-1} \left| \frac{2\Gamma}{TN} \sum_{k=0}^{N-1} \frac{1 - \exp[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2(\frac{\pi km}{N})]}{\sum_{m=1}^{l} \sin^2(\frac{\pi km}{N})} \right| - \frac{1}{N} \]

\[
= \sum_{j=0}^{N-1} \frac{2\Gamma}{TN} \sum_{k=1}^{N-1} \frac{1 - \exp[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2(\frac{\pi km}{N})]}{\sum_{m=1}^{l} \sin^2(\frac{\pi km}{N})} \]

\[
\geq \frac{4\Gamma}{TN} \frac{1 - \exp[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2(\frac{\pi m}{N})]}{\sum_{m=1}^{l} \sin^2(\frac{\pi m}{N})}
\]

\[
where \text{the first inequality uses the term } j = 0 \text{ only and the second inequality uses the terms } k = 1, N - 1.
\]

Assuming \(T/\Gamma \gg 1\) (which is consistent with our other assumptions, \(N \gg 1\) and \(\Gamma N \gg 1\), it requires \(T \gg N\) gives

\[
\sum_{j=0}^{N-1} |\tilde{P}(j, T) - \frac{1}{N}| \geq \frac{4\Gamma}{TN} \frac{1 - \exp[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2(\frac{\pi m}{N})]}{\sum_{m=1}^{l} \sin^2(\frac{\pi m}{N})}
\]

\[
\text{Lower bound}
\]

The lower bound of average mixing time can be derived as follows:

\[
\sum_{j=0}^{N-1} |\tilde{P}(j, T) - \frac{1}{N}| \geq \frac{4\Gamma}{TN} \frac{1 - \exp[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2(\frac{\pi m}{N})]}{\sum_{m=1}^{l} \sin^2(\frac{\pi m}{N})}
\]

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Therefore, the lower bound of average mixing time is

$$T_{\text{ave,lower}} = \frac{4\Gamma N}{e\pi^2 \sum_{n=1}^{N} n^2}. \quad (50)$$

**Upper bound**

Now, the upper bound of average mixing time can be obtained as follows

$$\sum_{j=0}^{N-1} |\tilde{P}(j,t) - \frac{1}{N}| = \sum_{j=0}^{N-1} \left| \frac{2\Gamma}{TN} \sum_{k=1}^{N} \frac{1 - \exp\left[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right] \omega^{jk}}{\sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)} \right|$$

$$= \sum_{j=0}^{N-1} \left| \frac{2\Gamma}{TN} \sum_{k=1}^{N} \frac{1 - \exp\left[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right] \cos\left(\frac{2\pi jk}{N}\right)}{\sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)} \right|$$

$$\leq \sum_{j=0}^{N-1} \left| \frac{2\Gamma}{TN} \sum_{k=1}^{N} \frac{1 - \exp\left[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right]}{\sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)} \right|$$

$$\leq 4\Gamma \frac{[N/2]}{TN} \sum_{k=1}^{[N/2]} \frac{1 - \exp\left[-\frac{T}{2\Gamma} \sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)\right]}{\sum_{m=1}^{l} \sin^2\left(\frac{\pi km}{N}\right)}$$

$$\leq \frac{\Gamma}{T} \sum_{k=1}^{[N/2]} \frac{1 - \exp\left[-\frac{2T}{\Gamma} \sum_{m=1}^{l} \frac{k^2 m^2}{N^2}\right]}{\sum_{m=1}^{l} \frac{k^2 m^2}{N^2}}$$
We assume that $\Gamma/T \gg 1$ and obtain

$$\sum_{j=0}^{N-1} |\tilde{P}(j, t) - \frac{1}{N}| \leq \frac{\Gamma}{T} \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{N^2}{k^2 \sum_{m=1}^{l} m^2}$$

$$\leq \frac{\Gamma \zeta(2)}{T \sum_{m=1}^{l} m^2} \leq \frac{\Gamma \zeta(2)}{6T \sum_{m=1}^{l} m^2}$$

(52)

where $\zeta$ is Riemann zeta function [43].

Thus, the upper bound of average mixing time is

$$T_{ave, upper} = \frac{\Gamma N^2 \pi^2}{6 \epsilon \sum_{m=1}^{l} m^2}.$$  

(53)

Therefore

$$\frac{2 \Gamma N^2}{\pi^2 \sum_{m=1}^{l} m^2 \ln\left(\frac{2}{N \epsilon}\right)} < T_{mix, ins} < \frac{\Gamma N^2}{2 \sum_{m=1}^{l} m^2 \ln\left(\frac{2 + \epsilon}{\epsilon}\right)}$$

$$\frac{4 \Gamma N}{\epsilon \pi^2 \sum_{m=1}^{l} m^2} < T_{mix, ave} < \frac{\Gamma N^2 \pi^2}{6 \epsilon \sum_{m=1}^{l} m^2}$$

(54)

Comparing of the both mixing times shows that the instantaneous mixing happens earlier than the time-average mixing. The above relations show that the bounds of instantaneous and average mixing times in the decoherent CTQWs on cycle decreases with adding the newly edges. Moreover, the bounds of mixing time have linear dependence on decoherence rate, while one observes inverse dependence on the sum of squared interacting parameters.

### 6 Conclusion

We considered the continuous-time quantum walks on one-dimension regular network under large decoherence $\Gamma \gg 1$. For this, we used an analytical model developed by Gurvitz [40] and calculated the probability distribution.
analytically. Then we calculated the lower and upper bounds of instantaneous and average mixing times as

\[
\frac{2\Gamma N^2}{\sum_{m=1}^{l} \pi^2 m^2} \ln\left(\frac{2}{N\epsilon}\right) < T_{mix,ins} < \frac{\Gamma N^2}{2 \sum_{m=1}^{l} m^2} \ln\left(\frac{2 + \epsilon}{\epsilon}\right).
\]

\[
\frac{4\Gamma N}{\epsilon \pi^2 \sum_{m=1}^{l} m^2} < T_{mix,ave} < \frac{\Gamma N^2 \pi^2}{6 \epsilon \sum_{m=1}^{l} m^2}.
\]

Thus instantaneous mixing time is shorter than average one. The both mixing times are linearly proportional to the decoherence rate and are inversely proportional to the sum of squared interacting parameters\((m)\). We found that adding shortcuts to cycle network decreases the mixing times. Moreover, our analytical results for \(l = 1\) and any \(m \neq 1\) are agreement with the mentioned results in [18, 22].

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