Shadow and Deflection Angle of Rotating Black Holes in Perfect Fluid Dark Matter with a Cosmological Constant

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The presence of dark matter around a black hole remarkably affects its spacetime. We consider the effects of dark matter on the shadow of a new solution to the Einstein equations that describes a rotating black hole in the background of perfect dark matter fluid (PFDM), along with its extension to nonzero cosmological constant Λ. Working in Boyer-Lindquist coordinates, we consider the effects of the PFDM parameter α on the shadow cast by a black hole with respect to an observer at position \((r_o, \theta_o)\). By applying the Gauss-Bonnet theorem to the optical geometry we find that notable distortions from a Kerr black hole can occur. We describe their dependence on α and Λ.
I. INTRODUCTION

Astronomical predictions in recent years give us reason to expect that most galaxies contain supermassive black holes at their centre. In particular, astrophysical observations strongly suggest the presence of a black hole, Sgr A*, at the centre of our galaxy. An interferometric instrument, \textit{Gravity}, is actively working to obtain more precise observations of this supermassive black hole \cite{1}. Simultaneously, a network of dishes all around the Earth has been developed using the VLBI technique to secure the shape and shadow of SgrA*. Known as the Event Horizon Telescope (EHT) \cite{2}, this project is successfully collecting signals from radio sources, and it may be that we shall soon observe the first silhouette of a supermassive black hole. This data may also eventually provide us with a means for testing the general theory of relativity in the strong-field regime.

It is therefore necessary to advance our theoretical research of black hole silhouettes (or shadows) to best evaluate the soon-expected observational data. Synge was the first to propose the apparent shape of a spherically symmetric black hole \cite{3}. After that Luminet \cite{4} discussed the appearance of a Schwarzschild black hole surrounded by an accretion disk. The shadow of a Kerr black hole was first studied by Bardeen \cite{5}. Recent astrophysical advances have motivated many authors to invest in theoretical investigations of black hole shadows, including Kerr-Newman black holes \cite{6}, naked singularities with deformation parameters \cite{7}, Kerr-Nut spacetimes \cite{8} and more. The shadows of black holes in Chern-Simons modified gravity, Randall-Sundrum braneworlds, and Kaluza-Klein rotating black holes have been studied in \cite{9,10,11}. Some authors have also tried to test theories of gravity by using the observations obtained from shadow of Sgr A* \cite{12,13,14,15}. A short review on shadows of a black hole is recently done in \cite{16}.

The method used for computing shadows of (rotating) black holes is more or less the same in all cases. An observer is placed at a very large distance (effectively infinity) away from the black hole, and it is from the viewpoint of this observer that the shadow is determined; typically celestial coordinates are introduced. More generally, the bending of light in a given spacetime background is a result of the spacetime curvature due to the presence of a massive body, and the deflection for a given impact parameter is obtained by solving the geodesic equations. Another geometric method for computing the deflection of light \cite{29} involves integration over a domain outside the light ray using the Gauss-Bonnet theorem. This method really shows the global aspect of the lensing effect in terms of the topology of the spacetime. Subsequently this method was applied to study lensing in different black hole/wormhole geometries \cite{31,38}.

For asymptotically flat black holes these methods are fine, but in the presence of a cosmological
constant there is an additional subtlety in that the position of the observer needs to be fixed. While the effect of the cosmological constant on the deflection of light has been investigated by several authors, unfortunately there seems to be no general agreement on the final results [39–46]. The main difficulty relies on the fact that the main assumption according to which the source and observer are located at infinity is no longer valid in the case of non-asymptotically flat spacetimes.

The Standard model of cosmology suggests that our universe is compiled of 27% dark matter and 68% dark energy, while the rest is baryonic matter. Though dark matter has not been directly detected, observational evidence for its existence can be found in abundance. Examples include galactic rotation curves [20], the dynamics of galaxy clusters [21], and the measurements of cosmic microwave background anisotropies obtained through PLANCK [22].

It is therefore natural to ask how black hole solutions might depend on dark matter. Recently a generalization of the Kerr-(A)dS solution in the presence of dark matter (PFDM) was obtained [25]. This solution had a number of interesting features. The size of its ergosphere decreased with increasing $|\alpha|$, where $\alpha$ parametrizes the strength of the dark matter contribution to the metric. Stable orbits were shown to exist, and the dependence of the rotational velocity on $\alpha$ was determined. However no observational consequences of this solution were considered.

Motivated by the above we investigate here the deflection of light and the shadow of the rotating PFDM black hole [25]. We note that the shadow of black hole in the presence of quintessence [17, 18] and in a dark matter halo [19] has been previously considered. We intend here to use techniques recently employed [27] for computing the shadow of a rotating black hole with cosmological constant. We begin by fixing the location of the observer in Boyer-Lindquist coordinates $(r_0, \theta_0)$, the respective radial and polar angular coordinates of the observer. Instead of considering photon rays coming from past, we follow them from the location $(r_0, \theta_0)$ to the past. The behaviour of such light-like geodesics can be characterized into two categories: those that venture so close to the outer horizon $r = r_+$ of the black hole that they are absorbed by it due to the gravitational pull, and those that ultimately escape to their original source in the past. The boundary between these two regions encloses a dark region called the shadow.

Despite the continued debate over the effects of the cosmological constant, we shall follow recent work by Ishihara et al. [17, 50] that takes into consideration finite-distance corrections in two particular spacetimes: Schwarzschild-de Sitter spacetime and an exact solution in Weyl conformal gravity [17]. To address issues with non-asymptotically flat spacetimes we shall consider the effects of the PFDM parameter $\alpha$ and cosmological constant on the deflection angle assuming finite distance corrections.
II. BLACK HOLES IN PERFECT FLUID DARK MATTER BACKGROUND

Amongst the many dark matter models that have been suggested is the perfect fluid dark matter model, which was initially proposed by Kiselev [23], and entailed construction of a new class of spherically symmetric black hole metrics in the presence of PDFM. In the spherically symmetric case this class of black holes was distinguished by a new term in the metric function that grows logarithmically with distance from the black hole. Only recently has this class been generalized to include rotation [25], providing a PDFM version of the Kerr-(A)dS solution. The metric is given by

\[
\begin{align*}
 ds^2 &= -\frac{\Delta_r}{\Xi^2\Sigma} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Delta_\theta \sin^2 \theta}{\Xi^2\Sigma} (adt - (r^2 + a^2) d\phi)^2 \\
&+ \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2
\end{align*}
\]

where

\[
\begin{align*}
 \Delta_r &= r^2 - 2Mr + a^2 - \frac{\Lambda}{3} r^2 (r^2 + a^2) + \alpha r \log \left( \frac{r}{|\alpha|} \right), \\
\Delta_\theta &= 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \\
\Xi &= 1 + \frac{\Lambda}{3} a^2, \\
\Sigma &= r^2 + a^2 \cos^2 \theta
\end{align*}
\]

with the mass parameter of the black hole being \( M \). The parameter indicating the presence of perfect fluid dark matter is \( \alpha \). This solution reduces to a rotating black hole in a PFDM background when \( \Lambda = 0 \), and to the Kerr-(A)dS solution for \( \alpha = 0 \). The PDFM stress-energy tensor in the standard orthogonal basis of the Kerr-(A)dS metric can be written in diagonal form \([\rho, p_r, p_\theta, p_\phi]\), where

\[
\rho = -p_r = \frac{\alpha r}{8\pi \Sigma^2}, \quad p_\theta = p_\phi = \frac{\alpha r}{8\pi \Sigma^2} \left( r - \frac{\Sigma}{2r} \right)
\]

For \( \Lambda \neq 0 \), the solution can either be a Kerr-Anti-de Sitter (\( \Lambda < 0 \)) or Kerr-de Sitter (\( \Lambda > 0 \)) metric. The horizons of the black hole are the solutions of \( \Delta_r = 0 \) i.e.

\[
\frac{\Lambda}{3} r^4 + \left( \frac{\Lambda}{3} a^2 - 1 \right) r^2 + 2Mr - a^2 + \alpha r \log \left( \frac{r}{|\alpha|} \right) = 0.
\]

Usually we have inner and outer horizon for Kerr and Kerr-de Sitter black holes and along with a cosmological horizon for Anti-de Sitter black holes. Imposing the requirement that PDFM does not change the number of horizons as compared to its Kerr counterpart, the parameter \( \alpha \) is constrained such that [25]

\[
\alpha \in \begin{cases} (-7.18M, 0) \cup (0, 2M) & \text{if } \Lambda = 0, \\
(\alpha_{\text{min}}, 0) \cup (0, \alpha_{\text{max}}) & \text{if } \Lambda \neq 0
\end{cases}
\]
where $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$ respectively satisfy

\[
\alpha_{\text{min}} + \alpha_{\text{min}} \log \left( \frac{2M}{-\alpha_{\text{min}}} \right) = 2M + H(\Lambda)
\]

\[
\alpha_{\text{max}} + \alpha_{\text{max}} \log \left( \frac{2M}{\alpha_{\text{max}}} \right) = 2M + H(\Lambda)
\]

and

\[
H(\Lambda) = -\text{sgn}(\Lambda) \left( \frac{32}{3\Lambda} M^3 + \frac{2}{3} \Lambda a^2 \right)
\]

and we see if $a = 0$ that $H > 0$ for $\Lambda < 0$ and $H < 0$ for $\Lambda > 0$.

### III. PHOTON REGION

For the spacetime (2), geodesic motion is governed by the Hamilton Jacobi equation [26]:

\[
- \frac{\partial S}{\partial \tau} = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},
\]

where $\tau$ is an affine parameter, $x^\mu$ represents the four-vector $(t, r, \theta, \phi)$ and $S$ is Hamilton’s principal function, which can be made separable by introducing an ansatz such that

\[
S = \frac{1}{2} \delta \tau - E t + L \phi + S_r(r) + S_\theta(\theta),
\]

where energy $E$ and angular momentum $L$ are constants of motion related to the associated Killing vectors $\partial/\partial t$ and $\partial/\partial \phi$. For timelike geodesics $\delta = 1$ and for null geodesics $\delta = 0$. Thus by solving Eq. (9) the resulting equations describing the propagation of a particle are

\[
\Sigma \dot{t} = \Xi^2 \left( (r^2 + a^2)E - aL \right) \frac{\Delta_r}{\Delta_\theta},
\]

\[
\Sigma^2 \dot{r}^2 = \Xi^2 \left( (r^2 + a^2) E - aL \right)^2 - \Delta_r \Delta_\theta \delta = R(r),
\]

\[
\Sigma^2 \dot{\theta}^2 = - \frac{\Xi^2}{\sin^2 \theta} \left( aE \sin^2 \theta - L \right)^2 - a^2 \delta \cos^2 \theta + C \Delta_\theta = \Theta(\theta),
\]

\[
\Sigma \dot{\phi} = \frac{a \Xi^2 \left( (r^2 + a^2)E - aL \right)}{\Delta_r} - \frac{\Xi^2 (aE \sin^2 \theta - L)}{\Delta_\theta \sin^2 \theta}
\]

for both null and time-like geodesics. In the above equations, besides the two constants of motion $E$ and $L$, we also have the Carter constant $C$ [28]. As we are interested in black hole shadows, henceforth we consider only null geodesics, for which $\delta = 0$. To reduce the number of parameters we write $\xi = L/E$ and $\eta = C/E^2$, and rescale $R/E^2 \rightarrow R$ and $\Theta/E^2 \rightarrow \Theta$. Then Eq. (11) and (12) respectively yield

\[
R = \Xi^2 \left( (r^2 + a^2) - a \xi \right)^2 - \Delta_r \eta
\]
and

\[ \Theta = \eta \Delta_\theta - \frac{\Xi^2}{\sin^2 \theta} (a \sin^2 \theta - \xi)^2. \]  

(15)

The of the photon region is defined as the region of space where gravity is strong enough that photons are forced to travel in orbits. Circular photon orbits only exist in the equatorial plane for rotating black holes, and there are two such types, retrograde and prograde. To determine the photon region we require \( r = r_s \) such that

\[ R(r_s) = 0 \quad \text{and} \quad \left. \frac{dR(r)}{dr} \right|_{r=r_s} = 0, \]  

(16)

along with the condition that

\[ \Theta(\theta) \geq 0 \quad \text{for} \quad \theta \in [0, \pi] \]  

(17)

By solving (16) we obtain the value of \( \xi \) and \( \eta \) at \( r = r_s \) to be

\[ a \xi(r_s) = r_s^2 + a^2 - 4r_s \frac{\Delta_r(r_s)}{\Delta'_r(r_s)}, \]  

(18)

\[ \eta(r_s) = \frac{16r_s^2 \Xi^2 \Delta_r(r_s)}{(\Delta'_r(r_s))^2} \]  

(19)

and by inserting Eqs (18) and (19) in condition (17) we find the condition

\[ (4r \Delta_{r,s} - \Sigma \Delta'_{r,s})^2 \leq 16a^2 r^2 \Xi^2 \Delta_{r,s} \Delta_\theta \sin^2 \theta \]  

(20)

that describes the photon region. For \( \Lambda = \alpha = 0 \), eq. (20) yields in the equatorial plane the Kerr result \( r = 2m \left( 1 + \cos \left( \frac{2}{3} \cos^{-1} \left( \pm \frac{|a|}{m} \right) \right) \right) \).

Photon orbits can be stable or unstable. Their stability provides us with the boundary of the black hole shadow. The boundary is given by the condition

\[ \frac{d^2 R(r_s)}{dr^2} > 0 \]

which can be written as

\[ \frac{R''}{8E^2 \Xi^2} \Delta_r^2 > 2r \Delta_r \Delta'_r + r^2 \Delta_r^2 - 2r^2 \Delta_r \Delta''_r \]  

(21)

where we have restored the factor of \( E \).
IV. SHADOWS OF THE PFDM BLACK HOLE

As noted above, in the presence of a cosmological constant the position of the observer needs to be fixed, employing the technique recently introduced in [27]. So we fix the observer in Boyer-Lindquist coordinates \((r_0, \theta_0)\), where \(r_0\) is the radial coordinate and \(\theta_0\) is angular coordinate of observer. We also assume that the observer is in domain of outer communication i.e. \(\Delta_r > 0\) and we consider the trajectories of light rays sent from position \((r_0, \theta_0)\) to the past.

We now define orthonormal tetrads \((e_0, e_1, e_2, e_3)\) at the observer’s position \((r_0, \theta_0)\) such that
\[
e_0 = \frac{\Xi^2}{\sqrt{\Delta_r \Sigma}} \left( \left(r^2 + a^2\right) \partial_t + a \partial_\phi \right)|_{(r_0, \theta_0)},
\]
\[
e_1 = \frac{\Delta \theta}{\Sigma} \partial_\theta |_{(r_0, \theta_0)},
\]
\[
e_2 = -\frac{\Xi^2}{\sqrt{\Delta_r \Sigma} \sin \theta} \left( \partial_\phi + a \sin^2 \theta \partial_t \right)|_{(r_0, \theta_0)},
\]
\[
e_3 = -\frac{\Delta_r}{\Sigma} \partial_r |_{(r_0, \theta_0)},
\]
where \(e_0\) is observer’s four velocity, \(e_0 \pm e_3\) are tangent to the direction of principal null congruences and \(e_3\) is along the spatial direction pointing towards the centre of the black hole. Let the coordinates of the light ray are described as \(\lambda(s) = (r(s), \theta(s), \phi(s), t(s))\), then a vector tangent to \(\lambda(s)\) is given by
\[
\dot{\lambda} = \dot{r} \partial_r + \dot{\theta} \partial_\theta + \dot{\phi} \partial_\phi + i \partial_t.
\]
This tangent vector can also be described in terms of orthonormal tetrads and celestial coordinates \(\rho\) and \(\sigma\) as
\[
\dot{\lambda} = \beta \left( -e_0 + \sin \rho \cos \sigma e_1 + \sin \rho \sin \sigma e_2 + \cos \rho e_3 \right),
\]
where the scalar factor \(\beta\) is obtained from Eq. \((26)\) and \((27)\) such that
\[
\beta = g(\dot{\lambda}, e_0) = \frac{\Xi^2 a L - E(r^2 + a^2)}{\sqrt{\Delta_r \Sigma}} |_{(r_0, \theta_0)}.
\]
Our next aim is to define the celestial coordinates, \(\rho\) and \(\sigma\) in terms of parameters \(\xi\) and \(\eta\). To do so we compare the coefficients of \(\partial_r\) and \(\partial_\phi\) in Eq. \((26)\) and \((27)\) and thus we obtain
\[
\sin \rho = \sqrt{1 - \frac{i^2 \Sigma^2}{\Xi^4 \left((r^2 + a^2) E - a L\right)^2}} |_{(r_0, \theta_0)}.
\]
FIG. 1. Shadows cast by a rotating black hole in PFDM background for different values of $\alpha$; all quantities are in units of $M$. 
and

\[
\sin \sigma = \frac{\sqrt{\Delta_\theta \sin \theta}}{\sqrt{\Delta_r \sin \rho}} \left( \frac{\Sigma \Delta_r}{\Xi^2 ((r^2 + a^2) E - a L) \dot{\phi} - a} \right) \bigg|_{(r_0, \theta_0)}
\]  (30)

Using Eqs. (11) and (13), we can present the above two equations in terms of parameter \( \xi \) and \( \eta \) as

\[
\sin \rho = \pm \sqrt{\frac{\Xi^2 (\Xi^2 - 1) ((r^2 + a^2) - a \xi)^2 + \Delta_r \eta}{\Xi^2 (r^2 + a^2 - a \xi)}} \bigg|_{(r_0, \theta_0)}
\]  (31)

and

\[
\sin \sigma = \frac{\sqrt{\Delta_r \sin \theta}}{\Xi^2 \sqrt{\Delta_\theta \sin \rho}} \left[ \frac{a - \xi \csc^2}{a \xi - (r^2 + a^2)} \right] \bigg|_{(r_0, \theta_0)}
\]  (32)

The boundary of shadow of the black hole can be presented graphically by projecting a stereographic projection from the celestial sphere onto to a plane with the Cartesian coordinates

\[
x = -2 \tan \left( \frac{\rho}{2} \right) \sin(\sigma),
\]  (33)

\[
y = -2 \tan \left( \frac{\sigma}{2} \right) \cos(\sigma).
\]  (34)

Figure 1 allows us to distinguish the silhouette cast by a rotating black hole in presence of perfect fluid dark matter \( (\alpha \neq 0) \) from that of Kerr black hole \( (\alpha = 0) \). For \( \alpha < 0 \) we find that the shadow of the black hole gets larger and more circular as \( \alpha \) becomes increasingly negative. However for \( \alpha > 0 \) the effect on the shadow is no longer monontic. For small \( \alpha > 0 \) the shadow shrinks whilst maintaining its asymmetric shape. However once \( \alpha \gtrsim 0.8 \), the shadow begins to grow, becoming increasingly circular and shifting leftward relative to its \( \alpha = 0 \) Kerr counterpart.

Our study thus indicates that presence of perfect fluid dark matter can have considerable effects on a black hole silhouette. The rotational distortion of a Kerr black hole is diminished for sufficiently large \( |\alpha| \), even for large spin \( (a = 0.84) \). The next effect is that the PFDM ‘cancels out’ the rotational distortion of the shadow.

Figure 2 shows the effects of cosmological constant on the shadow for different values of parameter \( \alpha \). We see that for small \( |\Lambda| \) the shadow maintains its shape for a given \( \alpha \), increasing for the AdS case \( \Lambda < 0 \) and decreasing for the dS case \( \Lambda > 0 \).

V. DEFLECTION OF LIGHT

A. Deflection angle without a cosmological constant

In this section we proceed to study the deflection angle of light applying the Gauss Bonnet Theorem (GBT) over the optical geometry under the assumption that the distance from the source
FIG. 2. Variation in shadow of a rotating black hole in PFDM background w.r.t cosmological constant. All quantities are in units of $M$.

$(S)$ to the receiver $(R)$ is finite. In order to see more clearly the effect of the PFDM parameter and the cosmological constant on the deflection of light first consider $\Lambda = 0$.

Let $T$ be a two-dimensional orientable surface with boundaries $\partial T_a (a = 1, 2, \ldots, N)$, and let the jump angles between the curves be $\theta_a (a = 1, 2, \ldots, N)$. In terms of this construction the GBT
can be stated as follows \[47, 48\]
\[
\int_T \kappa dS + \sum_{a=1}^{N} \int_{\partial T_a} \kappa_g dl + \sum_{a=1}^{N} \theta_a = 2\pi, \tag{35}
\]
in which $\kappa$ is the Gaussian curvature of the surface $T$, $dS$ gives the surface area element, $\kappa_g$ is known as the geodesic curvature of $\partial T_a$, and finally $l$ is the line element along the boundary. It is convenient to find first the black hole optical metric by imposing the null condition $ds^2 = 0$, and then by solving the spacetime metric for $dt$, yielding the generic form
\[
dt = \pm \sqrt{\gamma_{ij} dx^i dx^j + \beta_i dx^i}, \tag{36}
\]
where $i, j$ run from 1 to 3. Furthermore $\gamma_{ij}$ and $\beta_i$ in our case are found as follows
\[
\gamma_{ij} dx^i dx^j = \left(\frac{r^4}{\Delta r (\Delta r - a^2 \sin^2 \theta)}\right) dx^2 + \left(\frac{r^4}{\Delta r - a^2 \sin^2 \theta}\right) d\theta^2,
\]
\[
+ \left(\frac{\sin^2 \theta \Delta r (r^2 + \cos^2 \theta a^2)}{\Delta r - a^2 \sin^2 \theta}\right) d\phi^2, \tag{37}
\]
\[
\beta_i dx^i = -\frac{\sin^2 \theta (a^2 + r^2 - \Delta r) a}{\Delta r - a^2 \sin^2 \theta} d\phi. \tag{38}
\]

Next, let $\pi - \Psi_R$ and $\Psi_S$ be the corresponding inner angles measured at the vertex $R$ and $S$, and consider a quadrilateral domain $\infty_R \square \infty_S$ embedded in a curved space as can be seen from figure 3. The quadrilateral $\infty_R \square \infty_S$ consists of the light ray, two outgoing radial lines from $R$ and from $S$ and a circular arc segment $C_r$ at a coordinate distance $r_c$ ($r_c \to \infty$) located at the lens $L$ (see \[47–49\] for more details). Moreover, let $\phi_R$ and $\phi_S$ be the longitudes of the $R$ and the $S$, then we can define the quantity $\phi_{RS} \equiv \phi_R - \phi_S$, which gives the coordinate separation angle between the $R$ and $S$. By construction it follows that one can find a general relation for the deflection angle given in terms of $\Psi_R$, $\Psi_S$ and $\phi_{RS}$, by the following compact form \[47–49\]
\[
\hat{\alpha} \equiv \Psi_R - \Psi_S + \phi_{RS}. \tag{39}
\]

That being said, basically one can find the deflection angle $\hat{\alpha}$ by just computing $\Psi_R$, $\Psi_S$ and $\phi_{RS}$ and applying the last relation. Note that this method will be used later on in the case of non-vanishing $\Lambda$. There is, however, another way to find $\hat{\alpha}$ given the Gaussian curvature $\kappa$ and geodesic curvature $\kappa_g$. To do this, one simply has to integrate the Gaussian curvature $\kappa$ over the quadrilateral $\infty_R \square \infty_S$ domain. We recall that for an asymptotically flat spacetime $\kappa_g \to 1/r_C$ and $dl \to r_c d\phi$ as $r_c \to \infty$, implying the relation $\int_{C_r} \kappa_g dl \to \phi_{RS}$. Taking into account this information the GBT can be rewritten as follows \[47–49\]
\[
\hat{\alpha} = -\int_{\infty_R \square \infty_S} \kappa dS + \int_{S}^{R} \kappa_g dl. \tag{40}
\]
In what follows we are going to use this particular form of the GBT to calculate the finite distance corrections on the deflection angle of light. Considering the deflection of light in the equatorial plane and applying the definition of the Gaussian curvature we find the following result for $K$ in leading order

$$K = \frac{R_{\theta \phi \phi}}{\det \gamma} = \frac{1}{\sqrt{\det \gamma}} \left[ \frac{\partial}{\partial \phi} \left( \sqrt{\det \gamma} \Gamma_{rr}^{\phi \phi} \right) - \frac{\partial}{\partial r} \left( \sqrt{\det \gamma} \Gamma_{r \phi}^{r \phi} \right) \right] = -\frac{2M}{r^3} - \frac{\alpha \left( 3 - 2 \log \left( \frac{r}{|\alpha|} \right) \right)}{2r^3} + \mathcal{O} \left( \frac{Ma^2M}{r^4}, \frac{a^2M}{r^5} \right)$$

(41)
in both $M$ and $\alpha$.

On the other hand, the geodesic curvature of the light ray for the stationary spacetime can be obtained by \[49, 50\]

$$\kappa_g = -\sqrt{\gamma_{\theta \theta}} \beta_{\phi, r}. \tag{42}$$

Using this last equation we obtain the following result

$$\kappa_g = -\frac{1}{\sqrt{r^8 \sin^2 \theta (r^2 + \cos^2 \theta a^2)^2}} \frac{\Delta_r - a^2 \sin^2 \theta}{r^4} \beta_{\phi, r} = -\frac{2aM}{r^3} + \mathcal{O} \left( \frac{a \alpha}{r^4} \left[ (2M + r) \log \left( \frac{r}{|\alpha|} \right) - (M + r) \right] \right)$$

(44)

for the leading order term.

We can proceed to find the deflection angle by evaluating first the integration of $K$ over the quadrilateral $R^\infty_S \cap S^\infty_R$ in terms of the following integral

$$-\int \int_{R^\infty_S \cap S^\infty_R} KdS = \int_{\phi_S}^{\phi_R} \int_{\infty}^{r(\phi)} \left( -\frac{2M}{r^3} - \frac{\alpha \left( 3 - 2 \log \left( \frac{r}{|\alpha|} \right) \right)}{2r^3} \right) \sqrt{\det \gamma} d\phi dr + \mathcal{O} \left( \frac{Ma^2M}{r^4}, \frac{a^2M}{r^5} \right)$$

$$= \int_{\phi_S}^{\phi_R} d\phi \int_0^{\sin \phi \frac{b}{2}} \left[ 2M + \frac{3\alpha}{2} - \alpha \log \left( \frac{1}{|\alpha|u} \right) \right] du$$

$$= \int_{\phi_S}^{\phi_R} \left( \frac{2M \sin \phi}{b} + \alpha \sin \phi \frac{b}{2b} \right) d\phi + \int_{\phi_S}^{\phi_R} \left( \frac{-\sin \phi \alpha}{b} \log \left( \frac{b}{|\alpha| \sin \phi} \right) \right) d\phi$$

$$= \frac{2M}{b} \left( \sqrt{1 - b^2u_R^2 + \sqrt{1 - b^2u_S^2}} \right) + \frac{\alpha}{2b} \left( \sqrt{1 - b^2u_R^2 + \sqrt{1 - b^2u_S^2}} \right)$$

$$- \frac{\alpha}{b} \left( \sqrt{1 - b^2u_R^2 + \sqrt{1 - b^2u_S^2}} \right) \left( 1 + \log \left( \frac{b}{|\alpha|} \right) - 2 \log 2 \right) + \mathcal{O} \left( \frac{Ma^2M}{b^2}, \frac{a^2M}{b^3} \right) \tag{45}$$

in which we have used the relations $\cos \phi_S = \sqrt{1 - u_R^2b^2} + \mathcal{O}(Mu_S, au_S, \alpha u_S)$ and $\cos \phi_R = -\sqrt{1 - u_R^2b^2} + \mathcal{O}(Mu_R, au_R, \alpha u_R)$. Note that in the above integral we have introduced a new
variable $u = r^{-1}$, which is related to the finite radial distance of the source (receiver) from the black hole as follows $u_{S,R} = r_{S,R}^{-1}$. The integral of $κ_g$ can be evaluated easily, yielding

$$\int_{S}^{R} κ_g dl = \int_{S}^{R} \left( -\frac{2Ma}{r^3} + O\left(\frac{\alpha M a}{r^4}, \frac{a M^2}{r^4}\right) \right) dl
= -\frac{2Ma}{b^2} \int_{\phi_{S}}^{\phi_{R}} (\cos \vartheta d\vartheta) + O\left(\frac{\alpha M a}{b^3}, \frac{a M^2}{b^3}\right)
= -\frac{2Ma}{b^2} (\sin \phi_{R} - \sin \phi_{S}) + O\left(\frac{\alpha M a}{b^3}, \frac{a M^2}{b^3}\right)
= \frac{2Ma}{b^2} \left( \sqrt{1 - b^2u_{R}^2} + \sqrt{1 - b^2u_{S}^2} \right) + O\left(\frac{\alpha M a}{b^3}, \frac{a M^2}{b^3}\right),  \quad (46)$$

where we have used the equation for the light ray orbit $r = b/\cos \vartheta + O(M, a, \alpha)$, with $l = b \tan \vartheta + O(M, a, \alpha)$ and $\sin \phi_{R,S} = \mp \sqrt{1 - u_{R,S}^2b^2} + O(Mu_{R,S}, au_{R,S}, \alpha u_{R,S})$. Finally putting together these results we obtain the deflection angle

$$\dot{\alpha} = -\int_{\cal R}^{\cal S} K dS + \int_{S}^{R} κ_g dl
= \frac{2M}{b} \left( \sqrt{1 - b^2u_{R}^2} + \sqrt{1 - b^2u_{S}^2} \right)
- \frac{\alpha}{b} \zeta + \frac{2Ma}{b^2} \left( \sqrt{1 - b^2u_{R}^2} + \sqrt{1 - b^2u_{S}^2} \right) + O\left(\frac{M\alpha}{b^2}, \frac{a M^2}{b^3}, \frac{\alpha M a}{b^4}\right), \quad (47)$$

where

$$\zeta = \left( \sqrt{1 - b^2u_{R}^2} + \sqrt{1 - b^2u_{S}^2} \right) \left( 1 + \log \left[ \frac{b}{|\alpha|} \right] \right) - 2 \log 2. \quad (48)$$

This result shows that the Kerr deflection angle is strongly affected by the PFDM parameter under the effect of finite distance corrections. In particular we see that the deflection angle is proportional to the PFDM parameter which belongs in the interval $\alpha \in (-7.18M, 0) \cup (0, 2M)$.  

---

**FIG. 3.** The figure shows the quadrilateral $\cal R \cap \cal S$ domain of integration.
Finally if we assume that the $S$ and $R$ are located at the null infinity, i.e., $u_S \to 0$ and $u_R \to 0$, yielding $\zeta = 2(1 + \log(b/|\alpha|)) - 2 \log 2$ resulting with the deflection angle

$$\hat{\alpha} \to \frac{4M}{b} - \frac{\alpha}{b} \left(1 - 2 \log 2 + 2 \log\left(\frac{b}{|\alpha|}\right)\right) \pm \frac{4Ma}{b^2}. \quad (49)$$

As a special case we can find the Kerr deflection angle by setting $\alpha = 0$. Note that the $\pm$ sign corresponds for the retrograde and prograde light ray, respectively.

FIG. 4. On the left side we plot the deflection angle $\hat{\alpha}$ as a function of the impact factor $b$ and the PFDM parameter $\alpha \in (0, 2M)$. On the right side we plot $\hat{\alpha}$ as a function of the impact factor $b$ and the PFDM parameter $\alpha \in (-7.18M, 0)$. We have chosen $M = 1$ and $a = 0.8$ in both plots.

B. Deflection angle with a cosmological constant

Let us proceed by considering a more general scenario, namely by including the cosmological constant. In addition we are going to study the orbit equation on the equatorial plane. The Lagrangian from the metric (2) is found to be

$$2 \mathcal{L} = -A(r) \dot{t}^2 - 2H(r) \dot{t} \dot{\phi} + B(r) \dot{r}^2 + D(r) \dot{\phi}^2 \quad (50)$$
where dot represents a derivation to the affine parameter $\lambda$. Furthermore we have introduced the following relations

\begin{align}
A(r) &= \left( \frac{\Delta_r - a^2}{\Xi^2 r^2} \right), \\
B(r) &= \frac{r^2}{\Delta_r}, \\
H(r) &= \frac{a r^2 + a^3 - \Delta_r a}{\Xi^2 r^2}, \\
D(r) &= \frac{r^4 + 2a^2 r^2 + a^4 - \Delta_r a^2}{\Xi^2 r^2}.
\end{align}

Due to the spacetime symmetries there are two Killing vectors associated with the metric (2), implying two constants of motion defined as follows

\begin{align}
-p_t &= \frac{\partial L}{\partial \dot{t}} = E = A(r)\dot{t} + H(r)\dot{\phi}, \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = L = -H\dot{t} + D(r)\dot{\phi}.
\end{align}

The light ray equation can be obtained by defining first the impact parameter as

\begin{equation}
b = \frac{L}{E} = \frac{D(r)\frac{d\phi}{dt} - H(r)}{H(r)\frac{d\phi}{dt} + A(r)},
\end{equation}

then by introducing a new variable as $u = 1/r$, with the following important relation

\begin{equation}
\frac{\dot{r}}{\dot{\phi}} = \frac{dr}{d\phi} = -\frac{1}{u^2} \frac{du}{d\phi}.
\end{equation}

Then the light ray equation is given by the differential equation

\begin{equation}
\left( \frac{du}{d\phi} \right)^2 = \frac{u^4 \left( A(u)D(u) + H^2(u) \right)}{B(u) \left( H(u) - A(u)b^2 \right)^2} \equiv F(u).
\end{equation}

From the GBT we recall the following relation for the deflection angle

\begin{equation}
\hat{\alpha} \equiv \Psi_R - \Psi_S + \phi_{RS} = \Psi_R - \Psi_S + \int_{u_R}^{u_0} \frac{du}{\sqrt{F(u)}} + \int_{u_S}^{u_0} \frac{du}{\sqrt{F(u)}},
\end{equation}

where the inverse of the closest approach can be approximated with the impact factor i.e., $u_0 = b$.

Using the unit tangent vector $e_i$ along the light ray orbit in $\mathcal{M}$ which satisfies the relation $\gamma_{ij}e^i e^j = 1$, it is possible obtain the angles at the $R$ and $S$ in terms of the following quantity

\begin{equation}
\sin \Psi = \frac{H(r) + A(r)b}{\sqrt{A(r)D(r) + H^2(r)}}.
\end{equation}
From Eq. (61) it is possible to determine the quantity $\Psi_R - \Psi_S$ which in our case is found to be

\[
\Psi_R - \Psi_S = \Psi_{R}^{Kerr} - \Psi_{S}^{Kerr} - \frac{b\Lambda}{6} \left( \frac{1}{u_R \sqrt{1 - b^2 u_R^2}} + \frac{1}{u_S \sqrt{1 - b^2 u_S^2}} \right)
+ \frac{b\Lambda M}{6} \left( \frac{2b^2 u_R^2 - 1}{(1 - b^2 u_R^2)^{3/2}} + \frac{2b^2 u_S^2 - 1}{(1 - b^2 u_S^2)^{3/2}} \right)
+ \frac{a\Lambda}{3} \left( \frac{1}{u_R \sqrt{1 - b^2 u_R^2}} + \frac{1}{u_S \sqrt{1 - b^2 u_S^2}} \right)
+ \frac{\alpha b}{2} \left( \frac{u_R^2 \log\left( \frac{1}{u_R |\alpha|} \right)}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S^2 \log\left( \frac{1}{u_S |\alpha|} \right)}{\sqrt{1 - b^2 u_S^2}} \right)
- \frac{\alpha \Lambda b}{12} \left( \frac{(2b^2 u_R^2 - 1)}{(1 - b^2 u_R^2)^{3/2}} + \frac{(2b^2 u_S^2 - 1)}{(1 - b^2 u_S^2)^{3/2}} \right) + \mathcal{O}(a\Lambda M, aM/b, \alpha a/b^2, M^2\Lambda, \alpha^2\Lambda),
\]

where

\[
\Psi_{R}^{Kerr} - \Psi_{S}^{Kerr} = (\arcsin(bu_R) + \arcsin(bu_S) - \pi)
- bM \left( \frac{u_R^2}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S^2}{\sqrt{1 - b^2 u_S^2}} \right)
+ 2aM \left( \frac{u_R^2}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S^2}{\sqrt{1 - b^2 u_S^2}} \right).
\]

Our next goal is to compute the quantity $\phi_{RS}$ in leading order terms by evaluating the integral
of the angular coordinate $\phi$ in terms of the following equation

$$
\phi_{RS} = \int_{S}^{R} d\phi
$$

$$
= \phi_{RS}^{Kerr} - \frac{\alpha}{2b} \left( \sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) - \frac{\alpha \Xi}{b}
$$

$$
- \frac{\alpha}{b} \left[ \frac{(2 - b^2 u_R^2) \log \left( \frac{1}{u_R |\alpha|} \right)}{2 \sqrt{1 - b^2 u_R^2}} + \frac{(2 - b^2 u_S^2) \log \left( \frac{1}{u_S |\alpha|} \right)}{2 \sqrt{1 - b^2 u_S^2}} \right]
$$

$$
+ \frac{\Lambda b^3}{6} \left( \frac{u_R}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S}{\sqrt{1 - b^2 u_S^2}} \right)
$$

$$
+ \frac{M \Lambda b}{6} \left[ \frac{2 - 3b^2 u_R^2}{(1 - b^2 u_R^2)^{3/2}} + \frac{2 - 3b^2 u_S^2}{(1 - b^2 u_S^2)^{3/2}} \right]
$$

$$
+ \frac{\alpha \Lambda}{3} \left( \frac{1 - 2b^2 u_R^2}{u_R \sqrt{1 - b^2 u_R^2}} + \frac{1 - 2b^2 u_S^2}{u_S \sqrt{1 - b^2 u_S^2}} \right)
$$

$$
+ \frac{\alpha \Lambda b}{12} \left( \frac{1}{\sqrt{1 - b^2 u_R^2}} + \frac{1}{\sqrt{1 - b^2 u_S^2}} \right) + \mathcal{O}(\alpha \Lambda M, \alpha M/b, \alpha a/b^2, M^2 \Lambda, \alpha^2 \Lambda),
$$

(64)

where we have introduced

$$
\phi_{RS}^{Kerr} = \pi - (\arcsin(b u_R) - \arcsin(b u_S))
$$

$$
+ \frac{2M}{b} \left[ \frac{1}{\sqrt{1 - b^2 u_R^2}} \left( \frac{1}{2} \frac{1}{b^2 u_R^2} \right) + \frac{1}{\sqrt{1 - b^2 u_S^2}} \left( \frac{1}{2} \frac{1}{b^2 u_S^2} \right) \right]
$$

$$
- \frac{2aM}{b^2} \left( \frac{1}{\sqrt{1 - b^2 u_R^2}} + \frac{1}{\sqrt{1 - b^2 u_S^2}} \right),
$$

(65)

and

$$
\Xi = 2 \log(b) - \log\left[ (1 + \sqrt{1 - b^2 u_R^2})(1 + \sqrt{1 - b^2 u_S^2}) \right].
$$

(66)

Going back to Eq. (60) and after some algebraic manipulations we obtain the following result
for the deflection angle
\[
\hat{\alpha} = \frac{2M}{b} \left( \sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) - \frac{\alpha}{2b} \left( \sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right)
\]
\[
- \frac{\alpha}{b} \left[ \Xi + \left( \sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) \log \left( \frac{1}{|\alpha|} \right) \right]
\]
\[
- \frac{\Lambda b}{6} \left[ \frac{1}{\sqrt{1 - b^2 u_R^2}} + \frac{1}{\sqrt{1 - b^2 u_S^2}} \right]
\]
\[
+ \frac{M \Lambda b}{6} \left[ \frac{1}{\sqrt{1 - b^2 u_R^2}} + \frac{1}{\sqrt{1 - b^2 u_S^2}} \right]
\]
\[
+ \frac{a \Lambda b}{12} \left[ \frac{2 - 3b^2 u_R^2}{(1 - b^2 u_R^2)^{3/2}} + \frac{2 - 3b^2 u_S^2}{(1 - b^2 u_S^2)^{3/2}} \right]
\]
\[
\pm \frac{2M a}{b^2} \left( \sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) + \mathcal{O}(aM, \alpha M/b, \alpha a/b^2, M^2 \Lambda, \alpha^2 \Lambda).
\] (67)

In the case of a vanishing cosmological constant we recover Eq. (47). Although there are terms that diverge in the case of a nonzero cosmological constant (in the limit \( bu_R \to 0 \) and \( bu_S \to 0 \)), from a physical point of view we know that an observed star or galaxy is located at a finite distance from us. In other words, the limit \( bu_R \to 0 \) and \( bu_S \to 0 \), is not allowed in this case but one can include only a certain finite distance which leads to further simplified relation
\[
\hat{\alpha} \sim \frac{4M}{b} - \frac{\alpha}{b} \left( 1 - 2 \log 2 + 2 \log \left( \frac{b}{|\alpha|} \right) \right) - \frac{\Lambda b}{6} \left( \frac{1}{u_R} + \frac{1}{u_S} \right) + \frac{M \Lambda b}{3} + \frac{\alpha \Lambda b}{3}
\]
\[
\pm \frac{4M a}{b^2} \pm \frac{2\alpha \Lambda}{3} \left( \frac{1}{u_R} + \frac{1}{u_S} \right).
\] (68)

Finally we see that, besides the effect of DFDM parameter \( \alpha \), there are additional corrections including a finite contribution term \( \sim \alpha \Lambda b/3 \), and a divergent term \( a\Lambda/(u_R^{-1} + u_S^{-1}) \). Here the limit \( bu_R \to 0 \) and \( bu_S \to 0 \) is not allowed – in other words, we can consider only finite distance corrections. In this sense, the last equation generalizes a previous result reported in Ref. [47].

VI. CONCLUSION

The existence of dark matter around black holes located at the centers of most of large galaxies plays an important role in many astrophysical phenomena. Motivated by this fact, in this paper we have studied the effects of perfect fluid dark matter and a cosmological constant on the shadow of a rotating black hole. Our work provides a possible tool for observation of dark matter via shadows, perhaps using the high resolution imaging of the Event Horizon Telescope.
We have shown that the different shadow shapes are found by varying the PFDM parameter, mass, spin parameter, and the cosmological constant. Through graphs we have demonstrated that size of shadow of our black hole decreases for $\alpha < 1$ but after that we see an increase in its size. In addition we have done a detailed analyses on the effect of those parameters on the deflection angle of light using a recent geometric method by means of the the Gauss-Bonnet theorem applied to the optical geometry. Due to the presence of the cosmological constant we have included finite distance correction on the deflection angle.

More specifically, in the case of vanishing $\Lambda$, we found $K$ and $\kappa_g$, then the deflection angle is simply found by integrating over the quadrilateral $\infty_R \Box \infty_S$ domain. Our results show that the PFDM parameter strongly affects the deflection of light under finite distance corrections. As a special case, in the limit $bu_R \to 0$ and $bu_S \to 0$ the standard Kerr deflection angle is modified. In the case of a non-vanishing $\Lambda$ we followed an alternative method by computing the quantity $\Psi_R - \Psi_S + \phi_{RS}$ which gives the deflection angle. Besides $\alpha$, here we found a finite contribution term $\sim \alpha \Lambda b/3$, and a divergent term $a\Lambda/(u_R^{-1} + u_S^{-1})$. We pointed out that in this case the limit $bu_R \to 0$ and $bu_S \to 0$ is not allowed. In this context, the presence of the divergent terms is not problematic due to the finite distance corrections. After all, by observations we can only observe a given star or a galaxy in finite distance from us.

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[1] Gravity Collaboration, A and A, 602, 94 (2017).
[2] www.eventhorizontelescope.org
[3] J. L. Synge, MNRAS, 131, 463 (1966).
[4] J. P. Luminet, A&A, 75, 228 (1979).
[5] J. M. Bardeen, In Black Holes (Les Astres Occlus), p. 215239 (1973)
[6] A. de Vries, Class. Quantum Grav., 17, 123 (2000)
[7] K. Hioki and K. I. Maeda, Phys. Rev. D 80, 024042 (2009).
[8] A. Abdujabbarov, F. Atamurotov, Y. Kucukakca, B. Ahmedov, and U. Camci, Astrophys. Space Sci. 344, 429 (2012).
[9] L. Amarilla, E. F. Eiroa, and G. Giribet, Phys. Rev. D 81, 124045 (2010).
[10] L. Amarilla and E. F. Eiroa, Phys. Rev. D 85, 064019 (2012).
[11] L. Amarilla and E. F. Eiroa, Phys. Rev. D 87, 044057 (2013).
[12] C. Bambi and K. Phys. Rev. D, 79, 043002 (2009).
[13] A. E. Broderick, T. Johannsen, A. Loeb and D. Psaltis, Astro. Phys. J. 784, 7 (2014).
[14] T. Johannsen, A. E. Broderick, P. M. Plewa, et al., Phys. Rev. Lett, 116, 031101 (2016).
[15] Y. Mizuno, Z. Younsi, C. M. Fromm, et al., Nature Astronomy 2, 585 (2018).
[16] P. V. P. Cunha and C. A. R. Herdeiro, Gen. Rel. and Grav., 50, 42 (2018).
[17] B. Pratap Singh, arXiv:1711.02898 (2017).
[18] A. Abdujabbarov, B. Toshmatov, Z. Stuchlik and B. Ahmedov, Inter. J. Mod. Phys. D 26, 1750051 (2017).
[19] X. Hou, Z. Xu, M. Zhou, J. Wang (2018). arXiv:1804.08110.
[20] V. C. Rubin, Jr. W. K. Ford and N. Thomard,Astro. Phys. J. 238, 471 (1980).
[21] F. Zwicky, Helvetica Physica Acta, 6, 110 (1933).
[22] Planck Collaboration: P. A. R. Ade, N. Aghanim, et al., A& A, 571, A16 (2014).
[23] V.V. Kiselev (2003) arXiv:gr-qc/0303031.
[24] M. H. Li and K. C. Yang, Phys. Rev. D 86, 123015 (2012).
[25] Z. Xu, X. Hou and J. Wang, Class. Quantum Grav. 35, 115003 (2018).
[26] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press (1998).
[27] A. Grenzebach, V. Perlick and C. Lämmerzahl, Phys. Rev. D 89, 124004 (2014).
[28] B. Carter, Commun. Math. Phys. 10, 280 (1968).
[29] G. W. Gibbons and M. C. Werner, Class. Quant. Grav. 25, 235009 (2008).
[30] M. C. Werner, Gen. Rel. Grav. 44, 3047-3057 (2012).
[31] K. Jusufi, Int. J. Geom. Meth. Mod. Phys. 14, no. 12, 1750179 (2017);
[32] K. Jusufi, Phys. Rev. D 98, 064017 (2018)
[33] K. Jusufi, Phys. Rev. D 98, 044016 (2018)
[34] K. Jusufi, F. Rahaman and A. Banerjee, Annals Phys. 389, 219 (2018)
[35] K. Jusufi, N. Sarkar, F. Rahaman, A. Banerjee and S. Hansraj, Eur. Phys. J. C (2018) 78: 349.
[36] K. Jusufi, M. C. Werner, A. Banerjee and A. Ovgun, Phys. Rev. D 95, no. 10, 104012 (2017)
[37] G. Crisnejo, E. Gallo, Phys. Rev. D 97, 124016 (2018)
[38] G. Crisnejo, E. Gallo, A. Rogers, [arXiv:1807.00724[gr-qc]]
[39] F. Kottler, Annalen. Phys. 361, 401 (1918).
[40] K. Lake, Phys. Rev. D 65, 087301 (2002).
[41] W. Rindler and M. Ishak, Phys. Rev. D 76, 043006 (2007); M. Ishak and W. Rindler, Gen. Relativ. Gravit. 42, 2247 (2010).
[42] M. Park, Phys. Rev. D 78, 023014 (2008).
[43] M. Sereno, Phys. Rev. Lett. 102, 021301 (2009).
[44] A. Bhadra, S. Biswas, and K. Sarkar, Phys. Rev. D 82, 063003 (2010).
[45] F. Simpson, J. A. Peacock, and A. F. Heavens, Mon. Not. R. Astron. Soc. 402, 2009 (2010).
[46] H. Arakida, and M. Kasai, Phys. Rev. D 85, 023006 (2012).
[47] A. Ishihara, Y. Suzuki, T. Ono, T. Kitamura, H. Asada, Phys. Rev. D 94, 084015 (2016)
[48] A. Ishihara, Y. Suzuki, T. Ono, H. Asada, Phys. Rev. D 95, 044017 (2017)
[49] T. Ono, A. Ishihara, H. Asada, Phys. Rev. D 96, 104037 (2017)
[50] T. Ono, A. Ishihara, H. Asada, Phys. Rev. D 98, 044047 (2018)