Algebras with finitely many conjugacy classes of left ideals
versus algebras of finite representation type

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Abstract

Let $A$ be a finite dimensional algebra over an algebraically closed field with the radical nilpotent of index 2. It is shown that $A$ has finitely many conjugacy classes of left ideals if and only if $A$ is of finite representation type provided that all simple $A$-modules have dimension at least 6.

1 Introduction

Let $A$ be a finite dimensional unital algebra over a field $K$ and let $U(A)$ denote the group of units of $A$. Following [13], we denote by $C(A)$ the semigroup of conjugacy classes of left ideals of $A$, with respect to the natural operation $[L_1][L_2] = [L_1L_2]$ for $L_1, L_2 \in L(A)$. Here $L(A)$ stands for the set of left ideals of $A$ and $[L]$ is the conjugacy class of $L \in L(A)$ in $A$. A study of $C(A)$ is in part motivated by a general program of searching for semigroup invariants of associative algebras [9]. The semigroup $C(A)$ is also related to the subspace semigroup of an associative algebra, studied in [11, 12], which is an analogue of the semigroup of closed subsets in an algebraic monoid. In the context of ring theory, various related actions of $U(A)$ have been considered on a ring $A$, see [8], and also [5, 6, 7], leading to certain finiteness conditions for $A$.

Finite dimensional algebras seem to be of a particular interest from the point of view of finiteness of $C(A)$. The class of algebras with $C(A)$ finite includes in particular every algebra of finite representation type, see [13], Theorem 6. Recall that these are algebras with finitely many isomorphism classes of finite dimensional indecomposable left modules. On the other hand, if $K$ is algebraically closed then the fact that $C(M_n(A))$ is finite for every $n \geq 1$ implies that $A$ is of finite representation type, [13], Theorem 7.

The aim of this paper is to explore this connection in more detail. We do this for a natural class of algebras, namely, the class of algebras $A$ over an algebraically closed field $K$ with the radical $J(A)$ nilpotent of index 2. A study of this class of algebras is motivated on one hand by the role it plays in representation theory of arbitrary algebras, see [15], and on the other hand by the fact that within this class the semigroup $C(A)$ determines the algebra $A$ up to isomorphism, see [9], Theorem 1.2. In particular, our work is motivated by the following problem.

**Problem.** Let $A$ be a finite dimensional algebra over an algebraically closed field. Assume that $J(A)^2 = 0$ and the lattice $I(A)$ of ideals of $A$ is distributive (we then simply say that $A$ is distributive). Find necessary and sufficient conditions for $A$ in order that $C(A)$ is a finite semigroup.

Some partial results in this direction can be found in [10]. The approach adopted there, based on matrix problems arising from certain abstract combinatorial structures, called skeletons, will be also present in this paper. However, as our results are related to numerous classical facts from representation theory of algebras, we will also reformulate the finiteness problem of $C(A)$ in the language of quivers and their representations.
Assume that \( K \) is an algebraically closed field and \( A \) is a finite dimensional \( K \)-algebra. Fix a maximal subset \( \{e_1, e_2, \ldots, e_k\} \) of a complete set of orthogonal primitive idempotents of \( A \) such that \( Ae_i \neq Ae_j \) as left \( A \)-modules, for any \( i \neq j \). Then the directed graph \( \Gamma(A) = (\Gamma(A)_0, \Gamma(A)_1) \) is called the (ordinary) quiver of \( A \) if the set of vertices \( \Gamma(A)_0 = \{1, 2, \ldots, k\} \) and given \( a, b \in \Gamma(A)_0 \), the arrows \( \alpha: a \to b \) that constitute the set \( \Gamma(A)_1 \) are in a bijective correspondence with the vectors in a basis of the \( K \)-vector space \( e_a(J(A)/J(A)^2)e_b \). With \( \Gamma(A) \) one associates the separated quiver \( \Gamma^s(A) = (\Gamma^s(A)_0, \Gamma^s(A)_1) \), where \( \Gamma^s(A)_0 = \Gamma(A)_0 \times \{0, 1\} \) and \( \Gamma^s(A)_1 = \{(i, 0), (j, 1)\} \) \( (i, j) \in \Gamma^s(A)_1 \).

Let us recall some known facts about \( C(A) \). The first one in an obvious finiteness condition connected with the lattice of two sided ideals \( I(A) \) of \( A \). It is known that \( I(A) \) is distributive if and only if \( J(A) = 0 \) and the lattice of ideals of \( A \) is distributive. Then \( C(A) \) is finite if and only if \( I(A) \) has no cycles (as an unoriented graph) and \( \dim(eJ(A)) \leq 3 \) for every primitive idempotent \( e \) of \( A \).

Clearly, the matrix algebra \( A = M_n(K[x]/(x^2)) \) is of finite representation type for every \( n \geq 1 \), while it does not satisfy the restriction on the dimension of \( eJ(A) \) if \( n > 1 \). It is also easy to construct examples of algebras \( A \) of infinite type such that \( C(A) = \emptyset \), see [9], Example 4.7. This can be accomplished via the following classical result.

**Theorem (Gabriel, [15], 11.8).** Let \( A \) be a finite dimensional algebra over an algebraically closed field \( K \). Assume that \( J(A)^2 = 0 \) and the lattice of ideals of \( A \) is distributive. Then \( C(A) \) is finite if and only if the separated quiver \( \Gamma^s(A) \) has no cycles (as an unoriented graph) and \( \dim(eJ(A)) \leq 3 \) for every primitive idempotent \( e \) of \( A \).

In contrast to Theorem 1.1 if the algebra is not basic, not only the structure of \( \Gamma^s(A) \) but also the sizes of the simple blocks of \( A/J(A) \) will play a key role. Our main result reads as follows.

**Theorem 1.2.** Let \( A \) be a finite dimensional algebra over an algebraically closed field \( K \). Assume that \( A/J(A) \cong M_{r_1}(K) \oplus M_{r_2}(K) \oplus \cdots \oplus M_{r_k}(K) \), with \( r_i \geq 6 \) for every \( i \), and \( J(A)^2 = 0 \). Then \( C(A) \) is finite if and only if \( A \) is of finite representation type.

Two consequences will follow. The first one complements Theorem 7 in [13], mentioned above.

**Corollary 1.3.** Let \( A \) be a finite dimensional algebra over an algebraically closed field and let \( J(A)^2 = 0 \). Then \( A \) is of finite representation type if and only if the semigroup \( C(M_n(A)) \) is finite.

The second corollary of Theorem 1.2 comes from the well known classification result of Gabriel concerning hereditary algebras.

**Theorem (Gabriel, [4]).** Assume that \( A \) is a hereditary algebra over an algebraically closed field \( K \). Then \( A \) is of finite representation type if and only if the underlying graph of its (ordinary) quiver is a disjoint sum of Dynkin graphs \( A_n, D_n, E_6, E_7, E_8 \).

Our Theorem 1.2 leads to the following consequence.

**Corollary 1.4.** Let \( A \) be a hereditary algebra over an algebraically closed field. Assume that the dimensions of all simple \( A \)-modules are greater than or equal to 6. If \( C(A) \) is finite then \( A \) is of finite representation type.

We start in Section 2 with stating a general problem of finiteness of the set of orbits under the action of linear groups on certain sets of block matrices. This is a modification of the standard matrix problem considered in the context of representation theory of pairs of partially ordered sets, see [10], §16.1. Then we show in Corollary 2.3 that the finiteness problem of the set of conjugacy classes of left ideals of \( A \) contained in \( J(A) \) can be reformulated in this language. Next, we show that the number of orbits of the matrix problem that we obtained is equal to the number of isomorphism classes of representations of a quiver that is dual to the separated quiver of \( \Gamma(A) \). The proof of the main theorem, relying also on a standard geometric argument, is then derived in Section 3. In Section 4 we explain some aspects of the proof, including the role played by the number 6, and we state some problems.
2 Skeletons, contours and representations

In this section we assume that $A$ is a finite dimensional distributive algebra over an algebraically closed field $K$, and that $J(A)^2 = 0$, if not specified otherwise. First, we establish some notation. Assume that

$$A/J(A) \cong M_{r_1}(K) \oplus M_{r_2}(K) \oplus \cdots \oplus M_{r_k}(K),$$  \hspace{1cm} (2.1)

where $r_1, k$ are positive integers. According to the classical Wedderburn-Malcev theorem we have the following decomposition of $A$ into the direct sum of linear subspaces:

$$A = \mathcal{A}' \oplus J(A) = A_1 \oplus A_2 \oplus A_3 \oplus \cdots \oplus A_k \oplus J(A),$$

where $A_i \cong M_{r_i}(K)$. Let $J_{ij} = f_i J(A) f_j$ where $f_i$ is the unity in $A_i$.

Since $J(A)^2 = 0$, it follows that $J_{ij}$ are minimal submodules of the $A_i - A_j$-bimodule $f_i A f_j$, provided they are nonzero. In this case $J_{ij}$ are also two-sided ideals of $A$. They can be, therefore, identified with the linear spaces of rectangular matrices $M_{r_i \times r_j}(K)$. In the matrix notation the following decomposition follows.

$$A = \mathcal{A}' \oplus J(A) = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} \oplus \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{1k} \\ J_{21} & J_{22} & \cdots & J_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ J_{k1} & J_{k2} & \cdots & J_{kk} \end{bmatrix},$$ \hspace{1cm} (2.2)

Hence, for $n := r_1 + r_2 + \cdots + r_k$, the algebra $A$ can be identified with a subalgebra of $M_n(K[x]/(x^2))$, where $x^2 = 0$, and $J_{ij} = f_i J(A) f_j$, provided that it is nonzero, can be identified with the respective sets of rectangular matrices of sizes $r_i \times r_j$ with entries in the two-sided ideal of the ring $K[x]/(x^2)$, generated by the coset of $x \in K[x]$.

Let $e_1, e_2, \ldots, e_n$ be the consecutive diagonal idempotents of rank one in $A$. Put $r_i = n_{i+1} - n_i$, where $0 = n_1 < n_2 < \cdots < n_{k+1} = n$. Then the sets

$$E_i = \{e_{n_{i+1}}, e_{n_{i+2}}, \ldots, e_{n_{i+1}}\},$$ \hspace{1cm} (2.3)

defined for $i = 1, 2, \ldots, k$, satisfy the condition: $A e_p A = A e_q A$ if and only if $e_p, e_q$ belong to the same $E_i$. Moreover, $f_i := e_{n_{i+1}} \cdots + e_{n_{i+1}}$ is a two-sided unity element of the algebra $A_i$, for $i = 1, 2, \ldots, k$, and the elements of $E_i$ are diagonal idempotents of rank one in $A_i$.

Consider the following decomposition of the linear space $K^n$ into the direct sum $K^{r_1} \oplus \cdots \oplus K^{r_k}$. For $1 \leq i \leq k$ let

$$J_i := \{j_1, j_2, \ldots, j_{r_i}\}$$ \hspace{1cm} (2.4)

be the sets of indices such that $J_{ij} \neq 0 \iff j \in J_i$. We also define

$$V_i := \{v \in K^n \mid \pi_j(v) = 0, j \notin J_i\},$$

where $\pi_j : K^n \to K^{r_j}$ is the natural projection. Put $a_i := \sum_{j \in J_i} v_j$. Then $V_i$ is isomorphic to $K^{a_i}$ as a linear space.

**Proposition 2.1.** The linear spaces $\{e A \subseteq K^n \mid e \in E_i\}$ are pairwise isomorphic, and they are isomorphic to $V_i$. Moreover, there exists a bijection between the following sets:

- the set $L(A)$ of left ideals of $A$ contained in $J(A)$,
- the set of $k$-tuples of linear subspaces $(W_1, \ldots, W_k)$, where $W_i \subseteq V_i$, for $1 \leq i \leq k$.

**Proof.** If $e_p, e_q$ are both orthogonal idempotents of rank one, contained in the same $E_i$, and if $e_{ts}$ are matrix units in $A_t$, for $n_t + 1 \leq t, s \leq n_t + 1$, then $e_p = e_{pp}$ and $e_q = e_{qq}$, where $e_{pq} e_{qq} J(A) = e_{qq} J(A)$ and $e_{pp} e_{pp} J(A) = e_{pp} J(A)$. Hence $e A \subseteq V_i$, for all $e \in E_i$.

Let $L \in L(A)$. Then, according to (2.2) we have

$$L = AL = (A' + J(A))L = A'L = A_1 L \oplus A_2 L \oplus \cdots \oplus A_k L.$$

Since $J(A)^2 = 0$ and $A_i A_j = 0$, for $1 \leq i, j \leq k$, it is clear that: $A_i A_j L = A_i A L = A_i L = A_i L$. Thus, every left ideal $L$ of $A$ that is contained in $J(A)$ can be expressed as a direct sum of left ideals of $A$ that are contained in $A_i J(A)$, respectively, for $1 \leq i \leq k$. Therefore, it is enough to prove that
there exists a 1-1 correspondence between left ideals of $A$ contained in $A, J(A)$ and linear subspaces of $V_i$.

Let $L \subseteq A, J(A)$ be a left ideal of $A$. Then

$$L = f_1L = \bigoplus_{e \in E_i} eL,$$

where $E_i$ is a set of primitive orthogonal idempotents of rank one in $A_i$, according to our earlier notation. Moreover, if $e_p, e_q$ are idempotents contained in the same $E_i$, then the subspaces $e_pL, e_qL$, treated as linear subspaces of $K^n x$, are equal to $W_x$, for some subspace $W_x \subseteq V_i$.

Conversely, take a linear subspace $V \subseteq V_i$. Consider the set of matrices of sizes $r_i \times n$, whose every row is (as a vector) contained in $V x$. It is clear, that this set forms a left ideal of $A$, contained in $A, J(A)$. Thus we obtain a 1-1 correspondence, as claimed.

We have therefore proved, that to every left ideal of $A$ that is contained in the radical $J(A)$, we can associate exactly one $k$-tuple of linear subspaces of certain type. Next we will show, that the problem of finiteness for the semigroup $C(A)$ can be reformulated in the language of a certain matrix problem that corresponds to the action of linear groups on $k$-tuples of subspaces associated to left ideals of $A$ via the above correspondence. To state this result, first we introduce the abstract notions of skeletons and contours, see [10].

**Definition 2.2.** Let $(s_1, s_2, \ldots, s_t)$ and $(r_1, r_2, \ldots, r_s)$ be sequences of nonnegative integers. Consider a subset $I \subseteq \{1, 2, \ldots, t\} \times \{1, 2, \ldots, s\}$ and a map $f : I \to \mathbb{N}^2$ defined by $f(i, j) = (s_i, r_j)$. The set of pairs:

$$S = \{(i, j), f(i, j) \mid (i, j) \in I\}$$

(or simply, the pair $(I, f)$) will be called a skeleton.

Any element $(i, j) \in I$ we shall call a block of the skeleton $S$ of sizes $s_i \times r_j$. The integer numbers $s_i, r_j$ will be called the height and the width of $(i, j)$, respectively. For certain reasons, that will become clear in a moment, we allow $s_i, r_j$ to be zero as well. By the $i$-th row of a skeleton $S$ we mean, for a fixed $i$, the set $\{(i, j) \in I\}$. A column of a skeleton is defined in a similar way. It is therefore clear that blocks of the same row have equal heights, and blocks of the same column have equal widths.

In [10] a convenient graphical notation was adopted in order to give a better intuition of these notions. It treats each skeleton $S = (I, f)$ as a diagram consisting of rectangular blocks arranged in rows and columns, in accordance with the description of $I$. These blocks will be of sizes determined by $f$. Consider an important example that illustrates this convention.

**Example 2.3.** Let the skeleton $S = (I, f)$ consist of the following eight blocks of sizes $2 \times 1, 2 \times 3, 4 \times 3, 4 \times 5, 6 \times 5, 6 \times 3, 6 \times 4, 2 \times 4:

\[
\begin{array}{cccc}
| & | & | & |
\end{array}
\]

This skeleton has four rows and five columns, and $I = \{(1, 5), (2, 3), (2, 4), (2, 5), (3, 2), (3, 3), (4, 1), (4, 2)\}$. If the row (or column) of blocks is of zero height (or width), we do not include it in the picture.

**Definition 2.4.** Let $S = (I, f)$ be a skeleton and let $K$ be any field. Assume that $f(i, j) = (s_i, r_j)$, for $(i, j) \in I$. Consider the $K$-subspace $M_S \subseteq M_{(s_1 + s_2 + \ldots + s_t) \times (r_1 + r_2 + \ldots + r_s)}(K)$ that consists of the following block matrices:

\[
\begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1s} \\
    a_{21} & a_{22} & \ldots & a_{2s} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{t1} & a_{t2} & \ldots & a_{ts}
\end{bmatrix},
\]

where $a_{ij}$ has $s_i$ rows and $r_j$ columns, and where $a_{ij} = 0$, provided that $(i, j) \notin I$. The linear space $M_S$ will be called the contour space of the skeleton $S$, and its elements – the contours of $S$. 

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Note that some matrix blocks \(a_{ij}\) defined above may have zero numbers of rows or columns, if the sizes of the corresponding blocks of \(S\) are zero. The following notions are defined by an analogy to the skeleton case: a block of a contour, the width and the height of a block, and also the row and the column of a contour.

**Definition 2.5.** Consider a skeleton \(S = (I, f)\). Assume that \(f(i, j) = (s_i, r_j)\), for \((i, j) \in I\). We define linear groups \(\mathcal{S} := \text{Gl}_{s_i}(K) \times \text{Gl}_{r_j}(K) \times \ldots \times \text{Gl}_{s_k}(K)\), and \(\mathcal{G} := \text{Gl}_{r_j}(K) \times \text{Gl}_{r_j}(K) \times \ldots \times \text{Gl}_{r_j}(K)\). If \(s_i\) (or \(r_j\)) is equal to zero, we treat \(\text{Gl}_{s_i}(K)\) (or \(\text{Gl}_{r_j}(K)\)) as the trivial linear group.

Consider the left action of the group \(\mathcal{S} \times \mathcal{G}\) on the space \(\mathcal{M}_S\) (where \(\mathcal{G}\) stands for a group antiisomorphic to \(\mathcal{S}\)) defined in the following way: for \(h = (h_1, h_2, \ldots, h_k) \in \mathcal{S}\) and \(g = (g_1, g_2, \ldots, g_k) \in \mathcal{G}\), and an element \(A = (a_{ij}) \in \mathcal{M}_S\), the result of the action of \((h, g) \in \mathcal{S} \times \mathcal{G}\) on \(A\) is denoted by \(h \cdot A \cdot g\), where:

\[
h \cdot A \cdot g := \begin{bmatrix}
h_{1a_{11}g_1} & h_{1a_{12}g_1} & \ldots & h_{1a_{1s_1}g_1} \\
h_{2a_{21}g_1} & h_{2a_{22}g_1} & \ldots & h_{2a_{2s_2}g_1} \\
\vdots & \vdots & \ddots & \vdots \\
h_{ia_{i1}g_1} & h_{ia_{i2}g_1} & \ldots & h_{ia_{is_i}g_1}
\end{bmatrix}.
\]

The orbits of \(\mathcal{M}_S\) with respect to the action \((2.5)\) shall be called \(\mathcal{S} - \mathcal{G}\)-orbits on \(\mathcal{M}_S\).

Observe that if the \(i\)-th row (or column) of the skeleton is of height (or width) zero, then all blocks in the \(i\)-th row (or column) of any contour in \(\mathcal{M}_S\) is a zero matrix. So every matrix of the form \(h_{ia_{ij}g_1}\) (or \(h_{ia_{ij}g_1}\)) can be formally defined as a zero matrix of sizes \(0 \times s_j\) (or \(s_i \times 0\)). Thus the action \((2.5)\) is well defined. It may seem cumbersome at first to assume that some blocks of skeletons may have a zero size and to be forced to add the above-mentioned formal assumptions to the definition. However, from the standpoint of our aim, that is – to define a skeleton for an algebra \(A\) that would reflect the „block structure of \(J(A)“\), this will greatly simplify further notation.

**Definition 2.6.** Let \(A\) be an algebra such that \(A/J(A)\) has \(k\) simple blocks of sizes \(r_i \times r_i\), as in \((2.1)\). For \(i \in \{1, \ldots, k\}\) put

\[
a_i = \begin{cases} 
\sum_{j \in J_i} r_j, & \text{when } f_iJ(A) \neq 0, \\
0, & \text{when } f_iJ(A) = 0.
\end{cases}
\]

where \(J_i\) stands for the sets of indices defined in \((2.4)\). Consider the set \(I_A = \{1, \ldots, k\}\) and the map \(f_A : I_A \to \mathbb{N}^2\) defined by \(f(i, j) = (a_i, r_j)\). The skeleton \((I_A, f_A)\), which we shall denote by \(S_A\), will be called the skeleton of the algebra \(A\). The contour space of \(S_A\) is denoted by \(\mathcal{M}_A\) and will be called the contour space of \(A\). The elements of this space we will call the contours of \(A\).

We are ready to state the matrix problem that will be investigated throughout this paper.

**Proposition 2.7.** The following sets are in a bijection:

- the set \(C_J(A)\) of conjugacy classes of nilpotent left ideals of the algebra \(A\),
- the set of \(\mathcal{S} - \mathcal{G}\)-orbits on the contour space \(\mathcal{M}_A\) of \(A\).

**Proof.** Observe that for the algebra \(A\) the groups \(\mathcal{S}\) and \(\mathcal{G}\) are of the following form:

\[
\mathcal{S} = \text{Gl}_{s_1}(K) \times \ldots \times \text{Gl}_{s_k}(K), \quad \mathcal{G} = \text{Gl}_{r_1}(K) \times \ldots \times \text{Gl}_{r_k}(K).
\]

According to Proposition \((2.7)\) the set \(L_J(A)\) of nilpotent left ideals of \(A\) is in a 1-1 correspondence with the set of \(k\)-tuples of linear subspaces \((W_1, W_2, \ldots, W_k)\), where \(W_i \subseteq V_i\). Consider the map \(f : \mathcal{M}_A \to L_J(A)\) defined as follows. Assume that for \(A \in \mathcal{M}_A\), the \(i\)-th row of the contour \(A\) is of the form \(A(i) = [a_{i1} a_{i2} \ldots a_{ik}]\), where \(a_{ij} \in M_{a_{ij}}(K)\). Let \(W_i\) be a subspace of \(V_i\) spanned by the vectors belonging to the consecutive rows of \(A(i)\), for \(1 \leq i \leq k\). Let \(L\) be an element of \(L_J(A)\) that corresponds to the \(k\)-tuple \((W_1, W_2, \ldots, W_k)\). Put \(f(A) := L\). Then \(f\) is clearly surjective, since to every \(k\)-tuple \((W_1, W_2, \ldots, W_k)\) we can associate exactly one contour \(A \in \mathcal{M}_A\) and, according to Proposition \((2.1)\) to every element of \(L_J(A)\) we can associate certain \(k\)-tuple \((W_1, W_2, \ldots, W_k)\). It is also clear that if \(A, B \in \mathcal{M}_A\), then \(f(A) = f(B)\) if and only if \(A, B\) belong to the same orbit of the left action of \(\mathcal{S}\) on \(\mathcal{M}_A\). Call this orbit \(\mathcal{S}_A\). Thus, if \(\overline{f} : \{\mathcal{S}_A \mid A \in \mathcal{M}_A\} \to L_J(A)\) is the map defined on the set of \(\mathcal{S}\)-orbits \(\mathcal{M}_A\) by the condition \(\overline{f}(\mathcal{S}_A) = f(A)\), then \(\overline{f}\) is a well defined bijection.

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Next, assume that the nilpotent left ideals \( L_1 \) and \( L_2 \) belong to the same conjugacy class in \( A \). Thus, there exists \( u \in U(A) \) such that \( L_1 u = L_2 \). Since \( U(A) = U(A') + J(A) \), then (2.2) allows us to assume that
\[
U(A) = \left[ \begin{array}{ccc}
\text{Gl}_{r_1}(K) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \text{Gl}_{r_k}(K)
\end{array} \right] \oplus J(A) \simeq \mathcal{G} \oplus J(A).
\]

Let \( \mathcal{A}_1, \mathcal{A}_2 \) be such contours that \( \mathcal{H}_{\mathcal{A}_i} = \mathcal{H}^{-1}(L_i) \), for \( i = 1, 2 \). Then
\[
[L_1] = [L_2] \in C(A) \iff \text{there exists } g \in \mathcal{G} \text{ such that } \mathcal{H}_{\mathcal{A}_1} g = \mathcal{H}_{\mathcal{A}_2}.
\]

For a fixed \( \mathcal{A} \in \mathcal{M}_A \), let \( \mathcal{H}_A \mathcal{G} \) stand for the set of contours of the form \( \{ h \cdot A \cdot g \mid h \in \mathcal{H}, g \in \mathcal{G} \} \). This set is actually the \( \mathcal{H} - \mathcal{G} \)-orbit of \( \mathcal{A} \) under (2.5). Thus, if we define a map \( \mathcal{A} \rightarrow \mathcal{H}_A \mathcal{G} \), the argument above implies that \( \mathcal{A} \) is a well defined bijection.

From Theorem 1.1 in [9] an important corollary follows.

**Corollary 2.8.** The semigroup \( C(A) \) of an algebra \( A \) is finite if and only if the number of \( \mathcal{H} - \mathcal{G} \)-orbits on the contour space \( \mathcal{M}_A \) of \( A \) is finite.

From this point on, we will concentrate on reformulating the problem of finiteness of the number of \( \mathcal{H} - \mathcal{G} \)-orbits on the contour space of an algebra \( A \) to the problem of counting isomorphism classes of representations of a certain quiver. This is done in a straightforward way as we can associate the contour space of \( A \) with the so-called representation space of the quiver obtained from the separated quiver \( \Gamma^s(A) \) of \( A \) by the inversion of all arrows. We need some notation first.

Recall that in our setting, the separated quiver of \( A \) consists of \( 2k \) vertices \( \{ 1, 2, \ldots, k \} \times \{ 0, 1 \} \), and there exists an arrow \( (i, e') \rightarrow (j, e'') \) in \( \Gamma^s(A) \) if \( e'' = 0, e'' = 1 \) and \( e'_j A e'_j \neq 0 \), where \( e'_i, e''_j \) are any primitive idempotents taken from the sets \( E_i, E_j \), defined in (2.3). In this case there are precisely \( d_{ij} \) arrows \( (i, 0) \rightarrow (j, 1) \), where \( d_{ij} : = \dim K e'_i A e'_j \). We recall from [15], Corollary 2.4c, that if the lattice \( I(A) \) is distributive, then \( \dim_K e(A) f \leq 1 \), for any primitive idempotents \( e, f \in A \). Therefore, the number of arrows between any two vertices in the quiver of an algebra is always equal to either 0 or 1.

Also, recall that for a quiver \( Q = (Q_0, Q_1) \) with the set of vertices \( Q_0 \) and the set of arrows \( Q_1 \) one may consider a K-linear representation, or more briefly, a representation \( \mathcal{S} = (M, \phi) \in Q_0, \phi \in Q_1 \) of \( Q \), where \( M \) is a K-linear space and \( \phi : M \rightarrow M \) is a linear map, for \( \alpha = (a, b) \).

Assume that \( Q_0 \) has \( n \) elements \( \{ 1, 2, \ldots, n \} \) and \( d = (d_1, \ldots, d_n) \in \mathbb{N}^n \). By \( \text{rep}_d(Q) \) we denote the set of representations of \( Q \) with \( M_i = K^{d_i} \), for all \( i \in Q_0 \). Consider the following linear space.

\[
\mathcal{A}(d) = \prod_{(i,j) \in Q_1} M_{d_i \times d_j}(K).
\]

This object is known as the representation space corresponding to the dimension vector \( d \). It admits a natural structure of an affine space. One can define an action of the affine algebraic group \( G(d) = \prod_{i \in Q_0} \text{Gl}_{d_i}(K) \) on \( \mathcal{A}(d) \) via the conjugation formula:
\[
(g \cdot x)_\alpha = g_j \cdot x_{(i,j)} \cdot g_i^{-1},
\]
where \( g = (g_i) \in G(d) \), whereas \( x_\alpha \) and \( (g \cdot x)_\alpha \) are the matrices standing on the \( \alpha \)-th coordinates in \( \mathcal{A}(d) \) and \( g \cdot \mathcal{A}(d) \), respectively, where \( \alpha = (i, j) \in Q_1 \). It is well known that two representations \( M \) and \( N \in \text{rep}_d(Q) \) are isomorphic if and only if the (natural) representatives of \( M \) and \( N \) in \( \mathcal{A}(d) \) belong to the same \( G(d) \)-orbit, (see [15], XX.2).

We may now come back to the algebraic problem for \( \mathcal{H} - \mathcal{G} \) orbits.

**Proposition 2.9.** For an algebra \( A \), consider an element \( d \in \mathbb{N}^{2k} \) of the form:
\[
d = (a_1, \ldots, a_k; r_1, \ldots, r_k),
\]
where \( a_i \) and \( r_j \) are as in (2.6). Let \( \Gamma^s(A) \) be the quiver obtained from the separated quiver \( \Gamma^s(A) \) by inverting all its arrows. Let \( \mathcal{A}(d) \) stand for the algebraic variety corresponding to the set \( \text{rep}_d(\Gamma^s(A)) \) of representations of \( \Gamma^s(A) \) with the dimension vector \( d \). There is a 1-1 correspondence between the set of \( \mathcal{H} - \mathcal{G} \) orbits on the contour space \( \mathcal{M}_A \) of \( A \) and the set of orbits of the action (2.9) of \( G(d) \) on \( \mathcal{A}(d) \).
Proof. It is clear, that if $S_A = (I_A, f_A)$ is the skeleton of $A$, then the set of vertices $\{(i,0), (j,1) | 1 \leq i, j \leq k \}$ of the separated quiver of $\Gamma^s(A)$ corresponds to the set of rows and columns of the skeleton $S_A$ (see Definition 2.6). Namely, the block $(i,j)$ in the $i$-th row and the $j$-th column of $S_A$ corresponds to the arrow in $\Gamma^s(A)$ going from $(i,0)$ to $(j,1)$. Therefore, we can clearly see that the variety $\mathbb{A}(d)$ that corresponds to the representation space $\text{rep}_d(\Gamma^s(A))$, is in fact of the form

$$\mathbb{A}(d) = \prod_{(i,j) \in I_A'} \mathbb{M}_d, \text{ where } I_A'$$

$$\text{is the set of such blocks of } S_A \text{ that have both height and width nonzero. This variety, treated as a vector space, is clearly isomorphic to the contour space } \mathbb{M}_A \text{ of } A. \text{ Moreover, the group } G(d) \text{ that acts on } \mathbb{A}(d) \text{ via (2.9) is isomorphic to the group } \mathbb{S} \times \mathbb{S}, \text{ where } \mathbb{S}_1, \mathbb{S}_2 \text{ are as in (2.7).}$$

To be precise – it is $\mathbb{S} \times \mathbb{S}$ with the trivial factors excluded. The group actions (2.8) and (2.9) of $\mathbb{S}_1 \times \mathbb{S}_2^0$ and $G(d)$ on $\mathbb{A}(d)$ and $\mathbb{M}_A$ are, in essence, identical. The assertion follows. \hfill \Box

As a result, from Corollary 2.8 we obtain the desired characterisation of algebras $A$ such that $C(A)$ is finite in the language of representation theory of $A$.

**Corollary 2.10.** Let $A$ be a radical square zero finite dimensional distributive algebra over an algebraically closed field. Let $d$ be the dimension vector defined in (2.10). The following conditions are equivalent:

1. the semigroup $C(A)$ is finite,
2. the number of isomorphism classes of representations in the set $\text{rep}_d(\Gamma^s(A))$ is finite.

## 3 Proof of the main theorem

In this section we prove the main result of this paper. Recall, that by the Tits quadratic form $q_Q$ of a quiver $Q = (Q_0, Q_1)$ with $Q_0 = \{1, 2, \ldots, n\}$ we mean the following integral form (see [1], VII.4):

$$q_Q(d) = \sum_{i \in Q_0} d_i^2 - \sum_{(i,j) \in Q_1} d_i d_j, \text{ where } d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$$

(3.1)

Recall the action (2.9), for the dimension vector $d$. Then the first summand of the sum (3.1) is clearly the dimension of the group $G(d)$ (as an algebraic variety) and the second one is the dimension of the algebraic variety $\mathbb{A}(d)$ associated to the representation space $\text{rep}_d(Q)$. By a classical argument of Tits it is known that if $q_Q(d)$ is equal to 0, for some $d$, then the number of isomorphism classes in $\text{rep}_d(Q)$ is infinite. Indeed, this follows from the fact, that the set $F = \{(a \cdot id_{d_1}, \ldots, a \cdot id_{d_n}, a^{-1} \cdot id_{d_1}, \ldots, a^{-1} \cdot id_{d_n}) | a \in K^* \}$, where $K^*$ stands for the multiplicative group of $K$, acts trivially on the variety $\mathbb{A}(d)$. Thus if $q_Q(d) = 0$, then the dimension of the group $G(d)/F$ acting on the variety $\mathbb{A}(d)$ is smaller than the dimension of the variety itself. Thus the number of elements in $\text{rep}_d(Q)$ is infinite, see [15], 8.8. Such vectors $d$ that $q_Q(d) = 0$ constitute the, so-called, radical of the integral form $q_Q$.

Let $\preceq$ be the natural coordinate-wise order on the set $\mathbb{Z}^n$, namely $(p_1, \ldots, p_n) \preceq (q_1, \ldots, q_n)$ if and only if $p_i \leq q_i$, for all $1 \leq i \leq n$. Recall that $x \in \mathbb{Z}^n$ is called positive if $x \neq 0$ and $x_i \geq 0$ for all $i$. We will simply say that $x > 0$. The following lemma is crucial.

**Lemma 3.1.** Consider an algebra $A$ and let $d$ be the dimension vector for the skeleton $S_A$ of $A$. If the semigroup $C(A)$ is finite then no positive vector $d' \preceq d$ is in the radical of $Q = \Gamma^s(A)$.

**Proof.** Assume that $C(A)$ is finite. Then $\text{rep}_d(Q)$ is finite, according to Corollary 2.10. Suppose, to the contrary, there was a positive $d' \preceq d$ such that $q_Q(d') = 0$. From the argument of Tits we know that there would be infinitely many isomorphism classes of representations in $\text{rep}_d(Q)$. Consider the dimension vector $d - d'$. Its coordinates are all nonnegative and we can consider the representation space $\text{rep}_{d-d'}(Q)$. Then the (obviously defined) direct sum $\text{rep}_d(Q) \oplus \text{rep}_{d-d'}(Q)$ is naturally contained in $\text{rep}_d(Q)$, see [1], III.1. Therefore $\text{rep}_d(Q)$ would have infinitely many isomorphism classes and we arrive to a contradiction. The assertion follows. \hfill \Box

The proof of our main result follows naturally.
Proof of Theorem 1.2. One implication is clear. If $\Gamma^*(A)$ is a disjoint union of Dynkin graphs, then by Gabriel’s theorem $A$ is of finite representation type and from Theorem 6 in [13] we know that $C(A)$ is finite.

Assume now that the semigroup $C(A)$ is finite. Let $d$ be the dimension vector of the skeleton $S_A$ of the algebra $A$. From Corollary 2.11 we know that the number of isomorphism classes of representations of $\Gamma^*(A)$ for the dimension vector $d$ is finite. We will show that the separated graph $\Gamma^*(A)$ of $A$ is a disjoint union of Dynkin graphs.

Without losing generality we may assume that $Q = \Gamma^*(A)$ is connected. Assume, to the contrary, that $Q$ is not Dynkin. Then it is known that $Q$, as an unoriented graph, must contain one of the following Euclidean graphs (see [1], Lemma VII.2.1):

\[
\begin{align*}
\tilde{A}_n & \quad n \geq 2 \\
\tilde{D}_n & \quad n \geq 4 \\
\tilde{E}_6 & \\
\tilde{E}_7 & \\
\tilde{E}_8 &
\end{align*}
\]

Therefore, also the quiver $\tilde{Q} = \Gamma^*(A)$ contains, as an unoriented graph, an Euclidean graph $E$. Let $d'$ be the projection of the dimension vector $d$ to the coordinates that correspond to the vertices of $E$. By the assumption on $A$ we know that the coordinates $r_i$, appearing in the decomposition of $A/J(A)$, are not smaller than 6. From the definition of the skeleton of $A$ it follows that all of the coordinates of $d'$ are nonzero and are not smaller than 6.

It is known that the radicals of the quadratic forms of the quivers with the underlying graphs being Euclidean are of form $Zq$, where $q$ is in of the following forms, see [1], Lemma VII.4.2:

\[
\begin{array}{cccc}
1 & 1 & \ldots & 1 \\
1 & 2 & \ldots & 2 \\
1 & 2 & 3 & 2 & 1 \\
1 & 2 & 3 & 4 & 3 & 2 & 1 \\
2 & 4 & 6 & 5 & 4 & 3 & 2 & 1
\end{array}
\]

where each of the generators listed above is presented with accordance to the structure of the respective graph: $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

Let $r'$ be a generator of the radical for the Euclidean graph $E$ contained in the graph of $\Gamma^*(A)$. Since all coordinates of $d'$ are greater than or equal to 6, then $r' \leq d'$. This easily implies that there exists a radical vector $r$ of the quiver $\Gamma^*(A)$ such that $r \leq d$. From Lemma 3.1 it follows that $C(A)$ is infinite and we arrive to the contradiction with the supposition that $\Gamma^*(A)$ was not Dynkin. This concludes the proof. □

Finally, we derive Corollary 1.3 which extends Theorem 7 in [13].

Proof of Corollary 1.3. If $A$ is of finite representation type then so is $M_6(A)$ and $C(M_6(A))$ is finite by Theorem 7 in [13]. On the other hand, if $C(M_6(A))$ is finite then the algebra $M_6(A)$ is distributive by Theorem 6 in [13]. Moreover, it is clear that $J(M_6(A))^2 = M_6(J(A))^2 = 0$ and that $M_6(A)$ satisfies the hypotheses of Theorem 1.2. Therefore, $M_6(A)$ is of finite representation type by Theorem 1.2. It follows that also $A$ is of finite representation type, as desired. □

A natural question arises whether the assertion of Theorem 1.2 can be extended to a wider class of algebras, and in particular whether the hypothesis $J(A)^2 = 0$ can be dropped. From the point of view of our proof, the main problem here is to find a translation of the conjugacy problem for left ideals of $A$ contained in $J(A)$ in the language of an appropriate matrix problem. While in general it is not clear how to accomplish this, it is worth mentioning that a complete extension is possible at least in the case of hereditary algebras.

Proof of Corollary 1.4. Since $C(A/J(A)^2)$ is finite, from Theorem 1.2 it follows that $A/J(A)^2$ is of finite type. Let $B$ be the basic subalgebra of $A$. Then $B/J(B)^2$ is the basic subalgebra of $A/J(A)^2$. Hence, $B/J(B)^2$ is of finite type. Notice that $B$ is hereditary, see Exercise 8.27 in [19]. Then $B$ is
isomorphic to the path algebra of the quiver $\Gamma(B/J(B)^2)$, by Theorem VII.1.7 in [1]. By Gabriel’s theorem, the unoriented quiver of $B/J(B)^2$ is a disjoint union of Dynkin graphs. Since the quiver of $B$ coincides with the quiver of $B/J(B)^2$, it follows also, from the same result, that $B$ is of finite type. Hence, $A$ is of finite type as well.

4 Remarks and questions

We conclude with two remarks giving some insight into possible generalisations of our Corollary [1] to the nonhereditary case. They also put more light on the presence of number 6 in our main result (though, the role played by this hypothesis is already clear from the proof of Theorem 1.2).

A direct proof of Theorem [1.2] by means of Corollary [2.8] is also possible. Namely, one can show that if a skeleton $S_1$ is contained (in a natural way, see [10] for a precise definition) in a skeleton $S_2$, then the number of orbits of $S_1$ is finite if the number of orbits of $S_2$ is finite. As an example, consider the skeleton introduced in Example [2.3]. While it is not a skeleton of any algebra in the sense of Definition 2.6, we can easily show that the skeleton of any algebra in the sense of Definition 2.6, we can easily show that the skeleton corresponding to the isomorphism classes of representations of the quiver whose underlying graph is Euclidean of type $E_8$ with the dimension vector equal to the generator of the radical of its quadratic form. One could easily construct a matrix problem for each radical vector of each Euclidean graph in the language of skeletons and contours. These would serve as „test skeletons“ for any matrix problems arising from the class of algebras with $J(A)^2 = 0$. Consequently, our result could also be obtained via appropriate results on skeletons and contours, without reference to representation theory methods, as it was done for some related problems in [10]. However, the importance of the number 6 in the main theorem may be better understood within the framework of representation theory language. We discuss this further below.

As said before, in general it is not clear how to relate the conjugacy classes of left ideals to certain matrix problems, as we did with the radical square zero algebras. Nevertheless, there is a way of constructing matrix problems for any finite dimensional algebra that is a direct generalisation of the one we have considered. For the dimension vector $d$, one may consider the representation space $rep_d(Q, I)$ of a quiver $Q$ bound by relations generated by an admissible ideal $I$, that belongs to $J(KQ)^2$, see [1], III. The generalisation of the geometric context of representations of quivers to the bound quiver case is quite natural, see [18], XX.2. Following our approach in Section 2 one could then define „bound skeletons and contours“ for any finite dimensional algebra $A$ and introduce a matrix problem similar to the one of [2.5].

In [2], Bongartz introduced a generalisation of the quadratic form of a quiver, by defining the Tits quadratic form $\hat{q}_A : Z^n \rightarrow Z$ of the basic algebra $A$ with an acyclic quiver. It was shown that if the algebra $A$ has finite representation type then $\hat{q}_A$ is weakly positive, that is: $\hat{q}_A(x) > 0$, for all $x \neq 0$. The reverse implication is false in general, but it remains valid for many important classes of algebras of small global dimension: tilted algebras, double tilted algebras, quasitilted algebras, coil enlargements of concealed algebras, generalised multicoil algebras and others (see [18], XX.2).

How is the number 6 related with these generalisations? A classical theorem of Ovsienko [14] states that if a weakly positive integral quadratic form $q(x)$ has a root, that is such $x$ that $q(x) = 1$, then $x_i \leq 6$, for all $i$. Of course, the quadratic forms of Dynkin graphs are weakly positive. And the radical vectors of quadratic forms of Euclidean graphs are exactly the roots of Dynkin graphs with an additional coordinate corresponding to the one dimensional vector space in the vertex that extends the Dynkin graphs to the Euclidean one.

Bongartz proved, that the following conditions are equivalent for the so-called simply connected algebras e.g. the algebras with the trivial fundamental group (for details, see [18], XX.2): (i) $A$ is representation finite, (ii) the Tits form $\hat{q}_A$ of $A$ is weakly positive, (iii) $A$ does not admit a convex subalgebra $C$ which is critical. The latter is such an algebra $A$ that every proper convex subalgebra of $A$ is of finite representation type (see [3], for details). There exists a classification of critical algebras by means of the so-called Bongartz-Happel-Vossieck list, see [17], XIV. Each algebra $A$ on that list is defined as a path algebra of a bound quiver and a radical vector of the Tits quadratic form of $A$ is included. The coordinates of each of these radical vectors are less than or equal to 6. This suggests, that perhaps for the class of simply connected algebras the list of all possible „test bound skeletons“ can be obtained similarly to the radical square zero case. These arguments motivate us to conclude this paper with the following conjecture.

Conjecture 4.1. Assume $A$ is a finite dimensional distributive algebra such that the basic subalgebra
$A^b$ of $A$ is simply connected. Suppose that all simple $A$-modules have dimension at least 6. The following conditions are equivalent:

(1) the algebra $A$ is of finite representation type,

(2) the semigroup $C(A)$ is finite.

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