No Signalling and Unknowable Bohmian Particle Positions

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ABSTRACT: We exhibit in a model with simple dynamics, specifically a particle in a square box or two particles in one dimensional boxes, that if an experimenter can prepare the initial wave function of a system, the maximal information about the positions of Bohmian particles that is compatible with “no signalling” is that they are distributed according to $|\psi(x)|^2$. In particular, the positions cannot be prepared independently from the wave function. Any sharper “actual” position of the particle must be inaccessible since it could be used to send signals instantaneously. This is a consequence of the non-local character of the Bohmian dynamical law.
1 Introduction

Bohmian Mechanics (for an introduction, see [1]) is often presented as an approach to quantum theory that has an “ontology”, meaning that it is about particles that are characterized by their positions and is in that sense realistic. The price to pay is that the theory has non-local equations of motions but is claimed to be observationally indistinguishable from the textbook version of quantum mechanics (which one could loosely call Copenhagen style lacking a better name irrespective if one believes in collapse, many worlds, decoherence or the like).

The Bohmian theory is based on the observation that the quantum mechanical current

\[ j = \bar{\psi} \nabla \psi - (\nabla \bar{\psi}) \psi \]  

(1.1)

is conserved for Schrödinger type Hamilton operators

\[ H = -\Delta + V(x). \]  

(1.2)

From that one defines a velocity field

\[ v(x) = j/|\psi|^2 = 2\Re(\nabla \psi / \psi) \]  

(1.3)

and postulates “particles” whose position \( q(t) \) follow the velocity field, i.e.

\[ \frac{d}{dt} q = v(q). \]  

(1.4)

If one starts with a statistical ensemble of such particles with probability density \( |\psi(x)|^2 \) at the initial time then at all times the evolved probability density will be given by \( |\psi(x,t)|^2 \). Note well that the wave function \( \psi \) evolves according to the time dependent Schrödinger equation and that thus there is no feedback from the particle positions \( q \) to the wave function.
In the Bohmian framework, it is emphasised that all measurements in the end can be traced back to position measurements (which could be the position of a pointer on a scale or the position of a black spot on a photo plate in a double slit experiment) so it is understood that the $q$ are the positions of the “real” or actual particles. They have objective (realistic, deterministic and in that sense classical) trajectories but they are quantum in the sense that their dynamical law (1.4) is very different from Newton’s second law. Still, they produce the predictions of standard quantum mechanics as their probability density traces at all times the probability density implied by the absolute value square of the wave function assuming it did at an initial time.

The existence of trajectories appears to be in direct conflict with standard lore of quantum mechanics as well as the existence of classical probability densities seems to clash with Bell-type inequalities. But this tension is usually relieved upon the realization that as long as one only considers observables that are functions of the positions only (and not of momenta) these all commute. And since one deals only with a commutative algebra of observables, all states can indeed be realized as probability densities on some classical (configuration) space.

This argument is only true as long as one considers only positions at one instant of time as in general the positions at different time fail to commute. This fact, in the context of the Bohm theory, was emphasised by [2, 3]. In those papers, positional observables that lead to (violations) of Bell type inequalities were constructed and here, we strongly build on these works.

In this note, we will analyse a particular simple example of the construction in [2] and spell out the consequences.

Our model, consisting of a particle in a box, has the advantage of being elementary solvable so one has full analytic control at all stages as it comes without devices like double slits, Stern-Gerlach devices or beam splitters that do not have explicitly known Hamiltonians and in which technicalities could be hidden or suspected to be hidden. All observables considered are diagonal in position representation and thus directly expressible in terms of Bohmian particle positions. The model’s understanding only requires the most elementary quantum mechanics.

2 The Model

Often, inequalities of Bell type that demonstrate that Quantum Mechanics cannot be a local, realistic theory (in the very general meaning of having a space of states that is a simplex so all states can uniquely decomposed into extremal states) are expressed in terms of entangled qubits which are then thought of as realized by spin or helicity degrees of freedom. For a discussion in the Bohmian context this can cause problems or confusion since their Hamiltonians are generally not of Schrödinger type (1.2) and the Bohmian trajectories do not directly apply.

Using Stern-Gerlach type experiments where an inhomogeneous magnetic field that couples to a spin degree of freedom one can translate spin states to positions. But those
have the problem that they cannot be easily accessible to analytic study since they lead to complicated dynamics.

Of course, the only structural property of a qubit is that it lives in a two dimensional Hilbert space and has a non-commutative algebra of operators is acting on it. This can also be realized as a two dimensional subspace of a positional Hilbert space and this is what we will do in this note following[2, 3].

Our system is simply a free particle in a two dimensional square box, i.e. the Hilbert space is

\[ H = L^2([-\pi/2,\pi/2]^2) \quad \text{and} \quad H = -\Delta \]  

(2.1)

with Dirichlet boundary conditions.

At times, it will be convenient to emphasise the bipartite nature of the two coordinates by the use of an equivalent tensor product language:

\[ H = L^2([-\pi/2,\pi/2]) \otimes L^2([-\pi/2,\pi/2]) \]  

(2.2)

and (very explicitly) a non-interacting time evolution given in terms of

\[ H = H_1 \otimes I + I \otimes H_2 \quad \text{with} \quad H_i = -\frac{\partial^2}{\partial x_i^2}, \]  

(2.3)

where \( I \) denotes the identity operator. Thus, instead of one particle in a two dimensional box, this set-up equivalently describes two independent particles in one dimensional boxes (i.e. intervals). In the following, we will employ both these one and two particle interpretations interchangeably. For the two particle interpretation, it can be beneficial to think of the two intervals as widely separated to visualize that anything performed on particle one must not have measurable consequences on particle two as otherwise no-signalling would be violated. But for the moment, we stick with the “one particle in a square box” point of view.

The particle is prepared to be in the state given by the wave function

\[ \psi(x_1, x_2) = \frac{\sqrt{2}}{\pi} (\cos(x_1) \sin(2x_2) - \sin(2x_2) \cos(x_1)). \]  

(2.4)

With the ground state \( \psi_1(x_i) = \sqrt{2/\pi} \cos(x_i) \) of energy 1 and first excited state \( \psi_2(x_i) = \sqrt{2/\pi} \sin(2x_i) \) of \( H_i \) of energy 4, our state can be written as the entangled state

\[ \psi = \frac{1}{\sqrt{2}} (\psi_1 \otimes \psi_2 - \psi_2 \otimes \psi_1). \]  

(2.5)

From this form, it is obvious that \( \psi \) is an eigenstate of \( H \) with energy 5 and thus stationary. This is also reflected by in the Bohmian theory where the velocity field vanishes

\[ v = 0 \]  

(2.6)

as the wave function is real and thus according to (1.3), the Bohmian particles do not move at all.

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We will consider two statements about the particle:

\[ A : \text{The particle is in the right half of the box} \]  
\[ B : \text{The particle is in the upper half of the box.} \]

As observables, they can be expressed by the multiplication operators by

\[ A = \text{sgn}(x_1) = \Sigma \otimes I \quad B = \text{sgn}(x_2) = I \otimes \Sigma, \]

where \( \Sigma \) is the operator that multiplies by the sign function in position space. As functions of different coordinates, they commute and are thus simultaneously observable. As can be observed from figure 1, by symmetry, both have vanishing expectation value but their outcome is anti-correlated. One could imagine to do this experiment by inserting a horizontal or vertical wall in the box and then detecting the particle in one of the two now separated halves.

We can apply a (Heisenberg picture) time evolution and obtain

\[ A_t = e^{itH} A e^{-itH} = (e^{itH_1} \Sigma e^{-itH_1}) \otimes I \]

and similarly for \( B_t \). As \( \psi \) is stationary and \( \psi_1 \) and \( \psi_2 \) are symmetric and anti-symmetric, respectively, one finds \( \langle \psi_i, \Sigma \psi_i \rangle = 0 \) and thus

\[ \langle \psi, A_t \psi \rangle = \langle \psi, B_t \psi \rangle = 0. \]

As the time evolution does not mix the two tensor factors, the two observables even commute for different times

\[ [A_s, B_t] = 0 \]

and thus can be observed without one disturbing the measurement of the other.

For the direct application of the Bohmian framework, it is essential that all these measurements are measurements of positions so that one can relate the actual observation directly to \( q \). To make use of the quantum nature of entanglement, however, we need a non-commutative set of observables which we have in observing \( A \) or \( B \) at different times as the Hamiltonian does not commute with \( A \) or \( B \) and thus

\[ [A_s, A_t] \neq 0 \neq [B_s, B_t] \quad \text{for } t - s \notin \pi \mathbb{Z}. \]
A short calculation yields for their correlation
\[
\langle \psi, A_s B_t \psi \rangle = -\cos \left[ (E_2 - E_1)(s-t) \right] |\langle \psi_1, \Sigma \psi_2 \rangle|^2,
\]
(2.14)
where \(\langle \psi_1, \Sigma \psi_2 \rangle = 8/3\pi\). We find that \(A\) and \(B\) are totally anti-correlated when measured at the same time but then oscillate to correlation and back.

If one is not happy with the Heisenberg picture of time dependent observables, one can re-express this time dependence in terms of the Copenhagen interpretation invoking collapse of the wave function after the first measurement. One would say that the state is stationary until the first measurement upon which (depending on the outcome), the wave function is projected to zero in one half of the box which results in a state that is no longer an eigenstate of the total Hamiltonian and thus oscillates which explains the outcome of the second measurement being an oscillating function of time.

This is the case even though the measurements of \(A_s\) and \(B_t\) act in different factors of the Hilbert space and due to their commutativity do not influence each other.

The spectral projectors for the two eigen-spaces of the operator \(A\) are
\[
P_{\pm} = \frac{1}{2}(I \pm \Sigma) \otimes I = \theta(\pm x) \otimes I,
\]
(2.15)
with the Heavyside step function denoted \(\theta\).

According to the collapse prescription, after a measurement of this observables yields the \(i\)th eigenvalue, the state collapses to
\[
\psi_c = \frac{P_i \psi}{\|P_i \psi\|}.
\]
(2.16)
As from there on, we are only interested only in particle (or coordinate) two, we trace over the Hilbert space factor of particle one and obtain a reduced density matrix
\[
\rho_r = \text{tr}_1 |\psi_c \rangle \langle \psi_c |.
\]
(2.17)
Without the measurement, this reduced density matrix was simply \(\frac{1}{2} I_2\) in the \(\psi_1, \psi_2\) basis, but as
\[
\langle \psi_1 | \theta | \psi_2 \rangle = \frac{4}{3\pi} \neq 0,
\]
(2.18)
after the measurement contains off-diagonal entries
\[
\rho_r = \begin{pmatrix}
\frac{1}{2} & \frac{4}{3\pi} \\
\frac{4}{3\pi} & \frac{1}{2}
\end{pmatrix}.
\]
(2.19)
This density matrix is diagonalized in the basis of eigenvectors \((\psi_1 \pm \psi_2)/\sqrt{2}\) with eigenvalues \(\frac{1}{2} \pm \frac{4}{3\pi} = 0.924413\) and \(0.0755868\).

Clearly, these eigenvectors are no longer eigenfunctions of the Hamiltonian of particle 2 and thus this state is no longer stationary. The probability density for the second particle to be found in a specific position is plotted in Figure 2 and one can see the oscillations that lead to the time dependent oscillation (2.14) of the particle two between left an right.
This is conditioned on the first particle being found in the left half of the interval. If it would have been found in the right half, the probability density would oscillate in the opposite direction. If the outcome of the measurement on particle one is not known, both distributions would have to be superimposed (mixed) yielding the stationary distribution just as without any measurement on particle one at all. So the distribution of particle two oscillates only conditioned on the outcome of the measurement of particle one.

3 The Bohmian perspective

After this analysis in the Copenhagen language we will perform the same in the Bohmian framework. In order for it to cover the oscillating correlation as a function of the different times of observation it cannot keep the particle velocity \( v \) zero at all times. So, also here does the measurement of the \( x_1 \) coordinate (“left” or “right”) influence the velocity of the Bohmian particles in the \( x_2 \)-direction: They have to oscillate up and down. Even thought there is no explicit collapse in the Bohmian framework, the measurement is again
measuring the position of particle one (resulting in a moving particle two). We don’t have to know about the detailed workings of the measurement process to conclude that the particle has to start moving as otherwise one would not measure the two-time correlation function eq. (2.14) that depends on the difference of the two times. If it were different there would be an immediate observational difference between the orthodox and the Bohmian approach.

The Bohmian ontology is only about the momentary positions of the particles so there is no immediate problem. It can be seen that this is in fact everything one can know about these particles. This is clearest in the realization of the model as two particles moving in two one-dimensional intervals. If one could observe their velocity, this would violate No Signalling: By observing the velocity of particle two one would know if a measurement has been performed on particle one. This is independent of knowing the outcome of the experiment. So it must be in principle impossible to measure the velocity, it is not just that so far nobody cared to measure it. The particles only have a position unlike for example the particles in classical Hamiltonian mechanics which have both a position and a momentum.

One could try to measure the velocity by measuring the position twice, separated by a short time interval. But then one could argue that the first measurement would necessarily disrupt the particle so much that the measured velocity is no longer the velocity of the particle before trying to measure it.

But there is more: It is usually assumed that we don’t have any more specific information about the initial position of the particles except that it is distributed according to $|\psi|^2$ (this is known as the “quantum equilibrium hypothesis” in the literature). Again, this is not about our voluntary ignorance. If we had any more specific knowledge about the particles’ initial position, in particular if there were a way to prepare it while still preparing the initial wave-function to be $\psi$, we could measure the position of particle two at time $t$ and thus (possibly probabilistically) determine if it moved since it had been prepared which would imply that a measurement at particle one had taken place, possibly at a large space-like distance.

The oscillating trajectories of the Bohmian particle two after particle one has been measured (assuming a similar collapse at least as an effective description of the measurement process) are shown in Fig. 3. Those are key to our claim that if any information about those particles beyond what is already contained in $|\psi(x)|^2$ would be accessible to observation, this information could be used to transmit information instantaneously from particle one to an observer of particle two (the impossibility of knowing the distribution of the particles has also been discussed in [4–6] based on general arguments).

To this end, let us assume the system of the two particles has been set up in the quantum state according to (2.4). Let us further assume that Alice’s information about the position of the Bohmian particle two is described by a probability density $\varrho(x)$. In particular, if Alice knew the position of that particle, $\varrho$ would be a $\delta$-function. But her knowledge could be more coarse grained and be described by a more general probability density. Just knowing that the quantum state is given by (2.4) is described by $\varrho(x)$ being
1. Bohmian trajectories of particle two after particle one has been measured in one half of its interval

2. equal to $|\psi(x)|^2$, so let us parametrize it as

$$\rho(x) = \lambda(x)|\psi(x)|^2$$

Equation (3.1)

3. Having any more specific information about the location of the Bohmian particle would correspond to a non-constant $\lambda$.

4. Now, Bob wants to transmit one bit of information to Alice. If that bit is “0” he does nothing but if it is “1” he observes if particle one is in the left or the right half (of course without telling Alice the outcome of his observation).

5. For Alice waits a short moment of time. If the bit was “0” then the Bohmian velocity was vanishing throughout the experiment. No Bohmian particle moves and she finds her particle two at a position described by the original probability density $\rho$.

6. If, however, Bob had transmitted a “1” and had thus done an observation on particle one, particle two starts moving with a velocity field (1.3) and the probability density changes due to a non-vanishing divergence of the current

$$\nabla \left( \lambda|\psi|^2 \nabla \psi / \psi \right) = (\nabla \lambda) \cdot |\psi|^2 \nu$$

Equation (3.2)

which is non-vanishing as long as $\lambda$ is not constant. Thus this change in position can be detected by observing the position of particle two which follows a different distribution than $\rho$.

7. Only in the case where $\lambda$ is spatially constant, Alice cannot detect (not even probabilistically) the difference between Bob sending a 0 and a 1. So we conclude that no-signalling implies that Alice’s knowledge about the Bohmian particles cannot be better than what is already given by the probability density indicated by the wave function. The positions of the particles, the additional ingredient of the Bohmian interpretation, thus must not be knowable (beyond what is already known in terms of $\psi$) if no-signalling holds.

4 Signalling and Semi-Classics

Non-local signalling has of course never been observed in the real world and if it were it would create immediate problems with causality at least as long as the world is believed to be realtivistic.
Furthermore, there is no experiment that can distinguish between the Bohmian interpretation of quantum theory and the “orthodox” version. This is because it is possible to take the Bohmian perspective and then simply ignore the particle positions $q$ to get back to the orthodox view.

From a structural stand point, this is possible because besides the equation of motion (1.4) for $q$, the particle positions do not appear in any other equation of motion, in particular they don’t appear in the Schrödinger equation that governs the time evolution of the wave function. There is no feed-back. This is in contrast to the situation for example in electrodynamics with charged particles: There, the motion of the particles feels the electromagnetic forces due to the field-strength but the motion of the sources also influences the electro-magnetic field.

If the Bohmian particle postions would source any other field, one could use an observation of that field as a proxy for the particle positions and their motions and build a signalling device based on the set-up described in this note.

There is an attempt to use Bohmian notions as a semi-classical approximation [7, 8]: The idea is to treat particles quantum mechanically but couple those to a classical field (for example electro-magnetic or gravitational) via $\partial_\mu F^{\mu\nu} = j_\nu$ with (in our notation)

\[ j^0(x, t) = \sum_k e\delta(x - q_k(t)), \quad j^i = \sum_k ev^i_k(t)\delta(x - q_k(t)). \]

In the situation of the previous section, particle two is at rest before the measurement of particle one is performed. It would only create an electrostatic Coulomb field. But as soon as the measurement of particle one takes place, particle two starts oscillating and not only creates a magnetic field but also an electro-magnetic wave. One could use a radio receiver close to particle two to detect that a measurement of particle one has been performed possibly very far away in the universe.

This shows that for the Bohmian theory to be non-signalling (and not to get in tension with causality), it is essential that no observable degree of freedom couples directly to the particle positions $q$, they have to remain invisible.

5 Conclusion

In order not to violate no-signalling, one must not be able to know more about the particle positions than their $|\psi|^2$-distribution, in particular, it must be impossible to prepare another (possibly purer) initial state (of knowledge about the position) of the particles while keeping the preparation of the wave function or just have more information about the initial state than this particular probability distribution. In classical physics, the fact that a state is given by a probability density that is not a single $\delta$-function peak expresses ignorance about the microscopic details.

In a Bohmian world, any further information would immediately lead to the possibility to send signals faster than the speed of light.

As a consequence, compared to “orthodox quantum mechanics”, Bohmian mechanics has further elements of reality (the particle positions) with a deterministic equation of
motion but without the experimenter’s influence on obtaining knowledge of or controlling the initial conditions further than what is given by the probability distribution already encoded in the wave function. In this sense, these particles are like the proverbial angels on the tip of a pin: They exist but interaction with them is very limited: If we observe their position we can have no information where they came from.

This should be contrasted to the Copenhagen interpretation: As we have explained, the collapse of the wave function is also global and instantaneous (and thus leads to oscillations of the wave function conditioned on the outcome of the first measurement). But the wave function is not directly observable. In this approach it is clear that causality is only to be imposed at the level of observables. And here it is clear that locality (and thus causality) hold simply by the observation that any observable of the form $X \otimes I$ commutes with any observable of the form $I \otimes Y$ and thus actions of Bob on his system cannot influence anything that Alice experiences in her system.

Acknowledgments

I would like to thank various unnamed members of the “Workgroup Mathematical Foundations of Physics” as well as Tim Maudlin and Ward Struyve for helpful discussions that helped to sharpen the points made in this paper.

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