$D = 5$ M-theory radion supermultiplet dynamics

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Abstract

We show how the bosonic sector of the radion supermultiplet plus $d = 4, \mathcal{N} = 1$ supergravity emerge from a consistent braneworld Kaluza-Klein reduction of $D = 5$ M-theory. The radion and its associated pseudoscalar form an $SL(2,\mathbb{R})/U(1)$ nonlinear sigma model. This braneworld system admits its own brane solution in the form of a 2-supercharge supersymmetric string. Requiring this to be free of singularities leads to an $SL(2,\mathbb{Z})$ identification of the sigma model target space. The resulting radion mode has a minimum length; we suggest that this could be used to avoid the occurrence of singularities in brane-brane collisions. We discuss possible supersymmetric potentials for the radion supermultiplet and their relation to cosmological models such as the cyclic universe or hybrid inflation.

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1 Introduction

Heterotic string theory is one of the most promising string theories for phenomenological applications. In particular, models resembling the Standard Model can be obtained as low energy solutions of this theory. It is therefore not surprising that cosmological models inspired by heterotic string theory have recently been tried out. The proper setting for such theories lies within heterotic M–theory: Hořava and Witten showed how heterotic string theory arises from M–theory by compactifying 11–dimensional supergravity on the orbifold $S_1/\mathbb{Z}_2$, and by including boundary theories with a gauge group $E_8$ on each of the two orbifold fixed points in order to cancel gravitational anomalies [1]. For fields slowly varying across the orbifold this construction has as its low energy limit heterotic string theory with gauge group $E_8 \times E_8$ in ten dimensions. One can then compactify this on a Calabi–Yau manifold in order to retrieve 4–dimensional physics. However it turns out that for phenomenological reasons the size of the orbifold has to be about an order of magnitude larger than the Calabi–Yau size [2]. Hence going up in energy from an initial 4–dimensional point of view, the world would first look 5–dimensional and then 11–dimensional. Moreover the vacuum of the 5–dimensional theory is not flat space, but has been shown to consist of two parallel and static 3–branes located at the orbifold fixed points [3]. One may then try to identify one of the 3–branes with our visible universe. Cosmological scenarios, such as the cyclic universe of Steinhardt and Turok [5], for example, (or its predecessor, the ekpyrotic universe) [6], or the models of Brax and Davis [7], obtain an expanding universe on the brane worldvolume as a consequence of relative motion of the branes. Relative motion of the initially static branes can be achieved via the inclusion of a conjectured non–perturbative potential for the radion field determining the distance between the branes.

In this article, we will attempt to clarify certain mathematical issues concerning this setup: first of all we will identify the pseudoscalar partner of the radion in an $\mathcal{N} = 1$ chiral supermultiplet. We then show that it is consistent to truncate the original 5–dimensional theory down to the 4–dimensional worldvolume of the brane while keeping $d = 4$ gravity and the radion supermultiplet scalars, i.e. the 5–dimensional equations of motion reduce to 4–dimensional ones that are independent of the orbifold direction. The radion supermultiplet scalars form an $SL(2,\mathbb{R})/U(1)$ nonlinear sigma model. On the 4–dimensional worldvolume, we will construct a solitonic string solution supported by this sigma model, in which the pseudoscalar may be viewed as a 0–form gauge potential. This string has finite energy only if we one makes identifications in the target space under a discrete $SL(2,\mathbb{Z})$ subgroup of $SL(2,\mathbb{R})$, so that the reduced target space becomes $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)$ [11, 12, 13]. In fact, $SL(2,\mathbb{Z})$ has been conjectured to be preserved as a local symmetry in this sense of the full quantum string theory [15]. This has interesting consequences, especially if we look at the string solution from a 5–dimensional point of view. Indeed, in 5 dimensions the string arises as the intersection of a membrane with the boundary 3–branes, as we will show. And the $SL(2,\mathbb{Z})$
identification of the target space implies that the distance between the boundary branes has a minimum value, a result of potential significance for cosmological models relying on the collision of boundary branes.

In order to clarify the rôle of the pseudoscalar further, we will look at the conditions under which it itself can be truncated out. In fact, the pseudoscalar may be ignored entirely in a consistent way in the absence of a potential. However the question of whether one can truncate it must be reviewed when one includes a potential, since such a potential generically leads to interactions between the scalar and pseudoscalar. After these considerations we will show that the potential proposed for the cyclic universe cannot be embedded in heterotic M–theory, even if one neglects the $SL(2,\mathbb{Z})$ symmetry. It can, however, be approximated to some extend. We will give an example of such a supersymmetric approximation and will give the completion of it to a two–field potential after reinstating the pseudoscalar.

Finally we will turn our attention to the corrections that arise from integrating out the massive modes that occur in Calabi–Yau compactifications, which can couple to the massless sector. These corrections, in the presence of a superpotential, turn out to affect not only the kinetic terms of the radion and pseudoscalar, but also the potential of the theory. The importance of these corrections will be studied for known non-perturbative and flat potentials as well as a supersymmetric approximation to the cyclic universe potential.

# 2 A Consistent Truncation to Gravity and Scalars

The bosonic sector of the bulk action for 5–dimensional heterotic M–theory, including gravity and the universal hypermultiplet is given by [3]

$$ S_5 = \frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ R - \frac{3}{2} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{\sqrt{2}} \epsilon^{\alpha\beta\gamma\delta\epsilon} A_\alpha F_{\beta\gamma} F_{\delta\epsilon} - \frac{1}{2} V^{-2} \partial_\alpha V \partial^\alpha V - 2V^{-1} \partial_\alpha \xi \partial^\alpha \xi + \frac{1}{24} V^2 G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} - \frac{\sqrt{2}}{24} \epsilon^{\alpha\beta\gamma\delta\epsilon} G_{\alpha\beta\gamma\delta}(i(\xi \partial_\epsilon \bar{\xi} - \bar{\xi} \partial_\epsilon \xi) - 2\alpha A_\epsilon) - \frac{1}{3} V^{-2} \alpha^2 \right] . $$

(2.1)

The static 3–brane solution for this system is given by [3]

$$ ds_5^2 = H(y) dx^\mu dx^\nu \eta_{\mu\nu} + H^4(y) dy^2 $$

(2.2)

$$ V = H^3(y) , $$

(2.3)

with the codimension one harmonic function

$$ H(y) = \frac{\sqrt{2}}{3} \alpha |y| + c_0 , $$

(2.4)

and all other fields set to zero. In order to give a $D = 5$ Hořava–Witten geometry, the harmonic function is taken to have a second kink at $y = \pi \rho$. The solution then represents a pair of parallel 3–branes supported by the scalar $V$ at coordinate positions $y = 0, \pi \rho$. Note that the distance between
branes is given by \( d = \int_0^{\pi \rho} H^2 dy \), and is thus static but arbitrary since the coordinate position \( \pi \rho \) may be chosen at will.

We wish to extend the above solution to one in which the branes can move relative to each other, \textit{i.e.} in which the size of the orbifold can vary. Introducing the radion \( b(x^\mu) \), a function of the worldvolume coordinates only, we state the following ansatz:

\[
\begin{align*}
ds^2_5 &= e^{-b(y)} g_{\mu \nu} dx^\mu dx^\nu + e^b H^4(y) dy^2 + \frac{1}{4} \epsilon_{\mu \nu \rho \lambda} \partial_\mu \chi \partial_\nu \chi ,
\end{align*}
\]

This metric is a solution of the theory if \( V = e^b H^3 \).

Now the interbrane distance

\[
d = \int_0^{\pi \rho} e^b H^2 dy
\]

is a function of the worldvolume coordinates and is dynamical. Note that inclusion of the modulus \( b \) makes the parameter \( \rho \) redundant, as we may eliminate it by a rescaling of \( y \). So we shall set \( \rho = 1 \) hereafter. The branes will collide as \( b \to -\infty \), and in this limit \( V \to 0 \). Since \( V \), from the 11–dimensional point of view, represents the volume of the compactified Calabi–Yau space, we see that at the moment of collision a total of \( 1 + 6 = 7 \) dimensions disappear. Cosmological theories in which the branes collide, \textit{e.g.} the ekpyrotic universe, need to take into consideration that the Calabi–Yau dimensions also disappear momentarily. The way in which string theory handles this singularity is not clear at present, and some recent studies seem to indicate that such singularities may persist even in the full theory [8].

The domain wall solution preserves 4–dimensional \( \mathcal{N} = 1 \) supersymmetry on the brane worldvolume. Scalars therefore belong to chiral supermultiplets, and the radion must be paired with a pseudoscalar to form a complex scalar. Let us denote the pseudoscalar by \( \chi(x^\mu) \). It can also be obtained from a consistent truncation of the \( D = 5 \) theory, given by setting

\[
\begin{align*}
\mathcal{F}_{\mu y} &= H(y)^2 \partial_\mu \chi \\
G_{y \mu \nu \rho} &= e^{-b(y)} \epsilon_{\mu \nu \rho \lambda} \partial_\lambda \chi ,
\end{align*}
\]

where \( \epsilon_{\mu \nu \rho \lambda} \) is the 4-dimensional Levi-Civita tensor, and all other unrelated index structures of \( \mathcal{F} \) and \( G \) as well as the complex scalar \( \xi \) are set to zero. Taken together with \( d = 4 \) gravity and the radion mode \( b \), the 5–dimensional equations of motion reduce to a 4–dimensional system that can be summarized by the following effective action:

\[
S_4 = \frac{1}{2 \kappa^2} \int_{M_4} \sqrt{-g} \left( R^{(4)} - \frac{1}{2} \partial_\rho b \partial^\rho b - 4e^{-b} \partial_\rho \chi \partial^\rho \chi \right) .
\]
Note that, as a consequence of the consistency of this reduction, (2.10) may also be obtained directly by substituting the reduction ansatz (2.5,2.8,2.9) into (2.1), but the proper verification of this consistency is made using the \( D = 5 \) equations of motion.\(^1\)

Since the theory is supersymmetric it should be possible to cast the kinetic terms in Kähler form. For this purpose let us define a complex scalar

\[
\phi \equiv e^{\frac{b}{2}} + i\sqrt{2}\chi .
\] (2.11)

Then the Kähler potential can be identified to be

\[
K = -4 \ln (\phi + \bar{\phi}) .
\] (2.12)

From this, or more directly from the 4-dimensional effective action, it can be seen by inspection that the sigma model formed by the two scalars \( b \) and \( \chi \) has a \( SL(2,\mathbb{R}) \) symmetry. Also, in the notation of Ref. [3], our consistent truncation turns out to be the combination \( \phi = S = T \).

3 The Solitonic String solution

Our 4-dimensional bosonic theory consists of gravity and two scalars. The exponential coupling of the second scalar to the first indicates that this theory should have a brane solution. The scalar \( \chi \) acts as a 0-form gauge field, giving rise to a one-form field strength. In 4 dimensions, this is dual to a three-form field strength, which couples to a string solution. Explicitly, we have

\[
\begin{align*}
\eta_{\mu\nu}dx^{\mu}dx^{\nu} + h^4(r)dx^{m}dx^{m} & \quad \mu = 0, 1 \quad m = 2, 3 \\
e^{b} & = h^2(r) \\
\sqrt{2}\chi_m & = \epsilon_{mn}h(r),n
\end{align*}
\] (3.1)

with the harmonic function \( h(r) = 1 + ln(r) \) depending on the radial transverse distance \( r = \sqrt{x^m x^m} \). The solitonic string can be compared to a cosmic string: the spacetime is conical, with a singularity at the location of the string \( (r = 0) \). However this solitonic string possesses non-vanishing energy–momentum throughout space, unlike the cosmic string. Note that in this solution the metric is also degenerate at \( r = e^{-1} \), where there is a curvature singularity and infinite energy density. We shall shortly see how this is to be avoided; the above behavior will however remain valid at large \( r \).

In order to find a better behaved solution, let us introduce the complex field

\[
\tau \equiv \tau_1 + i\tau_2 \equiv \sqrt{2}\chi + ie^{\frac{b}{2}} = i\bar{\phi} .
\] (3.4)

\(^1\)The possibility of consistently reducing from a surrounding bulk supergravity theory to a braneworld supergravity multiplet was found in Refs [9], but such consistent reductions do not in general allow for the retention of matter multiplets. Hence the consistency of the \( d = 4 \) supergravity/radion supermultiplet system was not \textit{a priori} expected.
The 4–dimensional action can then be rewritten as
\[
S_4 = \frac{1}{2\kappa^2} \int_{M_4} \sqrt{-g} (R^{(4)} - 2 (\nabla \tau_1)^2 + (\nabla \tau_2)^2) \frac{\tau_2^2}{\tau_2^2} .
\tag{3.5}
\]
Now assume that \( \tau \) here depends only on \( x^2 \) and \( x^3 \), since we are looking for a string solution. Writing \( \partial = \frac{\partial}{\partial z} \) and \( z = x^2 + ix^3 \), the equation of motion for \( \tau \) is given by
\[
\partial \bar{\partial} \tau + 2 \frac{\partial \tau \partial \bar{\tau}}{\tau - \bar{\tau}} = 0 \, .
\tag{3.6}
\]
It is solved by any holomorphic \( \tau = \tau(z) \). Our metric ansatz reads
\[
d{s_4}^2 = -dt^2 + dx^2 + E^4(z, \bar{z}) dzd\bar{z} \, ,
\tag{3.7}
\]
with the harmonic function given by
\[
E(z, \bar{z}) = \frac{1}{2} \left[ h(z) + \bar{h}(\bar{z}) \right] \tag{3.8}
\]
where \( h(z) \) is a holomorphic function. The Einstein equations are solved for
\[
\tau(z) = ih(z) \, .
\tag{3.9}
\]
It turns out [11, 12, 13] that in order to have finite energy solutions one must make sigma model target-space identifications under an \( SL(2, \mathbb{Z}) \) subgroup of the \( SL(2, \mathbb{R}) \) symmetry of (3.5). \( \tau \) then takes its values in the upper half complex plane modulo \( SL(2, \mathbb{Z}) \) transformations, \( i.e. \) the moduli space becomes \( SL(2, \mathbb{Z}) \textbackslash SL(2, \mathbb{R})/U(1) \) (note that this makes \( \chi \) periodic). Physically inequivalent values of \( \tau \) lie in the fundamental domain \( F \) of the modular group, defined by \(-\frac{1}{2} \leq \tau_1 < \frac{1}{2} \) and \( |\tau| \geq 1 \). There exists a map, called the \( j \)–function, which maps \( F \) in a one–to–one and holomorphic fashion onto the complex plane [11]. Hence we can specify a holomorphic function and pull it back to \( F \) using \( j \), thus obtaining a solution for \( \tau \) which respects the \( SL(2, \mathbb{Z}) \) symmetry. A single string solution is given by
\[
j(\tau(z)) = z \, .
\tag{3.10}
\]
Multiple string solutions are obtained by taking \( j(\tau) \) to be a polynomial in \( z \). Asymptotically we have \( \tau \sim \frac{i\pi}{2} \ln(z) \), so that \( h(z) \sim \ln(z) \) and \( E(z, \bar{z}) \sim \ln(|z|) \) for large \(|z|\). This means that we recover a cylindrical symmetry, and actually have a conical spacetime at locations far away from the string.

Let us also check the supersymmetry conditions that this solution satisfies. These are derived from the requirement that one be able to set the fermionic fields to zero consistently with the surviving supersymmetry. The transformations of the spinor partner of \( \tau \) and of the gravitino lead to the following requirements:
\[
\delta \lambda = 0 \Rightarrow \gamma^a \partial_a \tau \bar{\epsilon} = 0 \tag{3.11}
\]
\[
\delta \psi_\mu = 0 \Rightarrow \partial_\mu \epsilon + \frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_b \epsilon - \frac{\partial_\mu (\tau + \bar{\tau})}{\tau - \bar{\tau}} \epsilon = 0 \, ,
\tag{3.12}
\]

where $\omega_{\mu}^{ab}$ is the spin connection and $\epsilon$ is a $d = 4$ Majorana spinor parameter. Now using $\gamma_z = \frac{1}{2}(\gamma_2 - i\gamma_3)$ and $\gamma_{\bar{z}} = \frac{1}{2}(\gamma_2 + i\gamma_3)$ we obtain

$$
\frac{1}{4} \omega_{\mu}^{ab} \gamma_a \gamma_b = \frac{\partial_\mu (\tau + \bar{\tau})}{\tau - \bar{\tau}} i\gamma_{23},
$$

(3.13)

with $\gamma_{23} = \frac{1}{2}(\gamma_2\gamma_3 - \gamma_3\gamma_2)$. Imposing the condition

$$
\gamma_{23} \epsilon = -i \epsilon
$$

(3.14)

then solves equation (3.12). This condition is equivalent to demanding that

$$
P \epsilon \equiv \frac{1}{2}(1 - i\gamma_{23}) \epsilon = 0,
$$

(3.15)

where $P$ is a projection operator satisfying $P^2 = P$. We have thereby imposed a $d = 2$ worldsheet chirality constraint on the spinor parameter $\epsilon$ which projects out half of its components. We are thus left with just 2 supercharges as expected, thus preserving half of the 4-dimensional $\mathcal{N} = 1$ supersymmetry. Equation (3.11) is solved by virtue of the projection condition and holomorphicity of the field $\tau$.

Since we have a consistent truncation from $D = 5$ to $d = 4$, the solitonic string solution can be oxidized straightforwardly back up to 5 dimensions, giving the resulting metric

$$
\text{ds}_5^2 = \frac{H(y)}{E(z, \bar{z})}[-dt^2 + dx^2 + E^4(z, \bar{z}) dz d\bar{z}] + E^2(z, \bar{z}) H^4(y) dy^2.
$$

(3.16)

In order to interpret this solution [10], we should rewrite it as

$$
\text{ds}_5^2 = \frac{H}{E_1}(-dt^2 + dx^2) + HE_1^2 E_2 dz d\bar{z} + E_1^2 H^4 dy^2,
$$

(3.17)

with $E_1 = E_2 = E(z, \bar{z})$. Now taking limits in which we set the harmonic functions $E_1, E_2$ and $H$ pairwise equal to 1, we recover the metrics of the 3–brane ($E_1 = E_2 = 1$), a membrane ($E_1 = H = 1$) and a string ($E_2 = H = 1$) in 5 dimensions, with the string being delocalized in the $y$–direction.\(^2\)

Comparing with the full 5-dimensional action (2.1) we can verify that the membrane is supported electrically by $G_{\alpha\beta\gamma\delta}$ and the string is supported magnetically by $F_{\alpha\beta}$. Thus our 5-dimensional interpretation of the solitonic string solution is as an intersection of one 3–brane and a 2–brane, which then stretches between it and the other orbifold 3–brane, and with the intersection string delocalized over the worldvolume of the 2–brane. This setup is sketched in Figure 1. Note that we have

$$
\tau_2 = e^{\frac{b}{2}} = \frac{1}{2} [h(z) + \bar{h}(\bar{z})] = E(z, \bar{z}).
$$

(3.18)

We can draw two observations from this equation, remembering that $e^{\frac{b}{2}}$ governs the distance between the orbifold branes (cf. Equation (2.7)): i) the distance grows as $\ln |z|$ for large $|z|$ and ii)

\(^2\)Such semi-localized intersecting brane solutions have been discussed, e.g. in Ref. [14].
since $\tau$ takes its values only in the fundamental domain $F$, there is a non–zero minimal distance between the branes at which

$$\epsilon^2 = \frac{\sqrt{3}}{2}.$$  \hspace{1cm} (3.19)

Thus the reduction of the target space from $SL(2,\mathbb{R})/U(1)$ to $SL(2,\mathbb{Z})/SL(2,\mathbb{R})/U(1)$ prevents the collision of the two boundary branes. This is of significance since the full quantum string theory is expected to exhibit such a local $SL(2,\mathbb{Z})$ symmetry [15]. In this light, cosmological scenarios relying on a collision between the boundary branes would need to be revised. Moreover, one might speculate that this reduction of the target space could provide a mechanism for the two branes to bounce after reaching their minimum separation in the presence of an attractive potential.\(^3\)

However, for the rest of this paper, we will proceed by considering the case of an unreduced target space $SL(2,\mathbb{R})/U(1)$.

\(^3\)The scale of the minimum separation would be set by the tension of the 3-brane worldvolume string, similarly to the way in which $\alpha' = 1/(2\pi T)$ sets the minimum length $\sqrt{2\pi\alpha'}$ as a consequence of T–duality, under which $r \to 2\pi\alpha'/r$. 

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Figure 1: The five-dimensional interpretation of the solitonic string as the intersection of a membrane with the two boundary 3-branes, with the intersecting string delocalized along the membrane.
4 Truncating the Pseudoscalar

The pseudoscalar $\chi$ can be set to zero consistently if the equations of motion for the metric and $b$ can still be satisfied at all times when $\chi = 0$. In the absence of a potential, this is possible since $\chi$ does not have any $\chi$–independent source terms. Then the radion $b$ becomes a free massless scalar satisfying $\Box b = 0$. But a potential $V(\phi, \bar{\phi})$ is, in general, a function of $b$ and $\chi$. And if it contains terms of the form $\chi f(b)$, where $f(b)$ is an arbitrary function of $b$, then the resulting equation of motion for $\chi$ will be

$$D^{(2)} \chi = f(b),$$

where $D^{(2)}$ is a second order differential operator. Clearly in this case it is inconsistent to neglect $\chi$. Thus we can conclude that the condition for the truncatability of $\chi$ reads

$$\frac{\partial V}{\partial \chi} |_{\chi=0} = 0.$$  

Any potential $V(\phi, \bar{\phi})$ in supergravity arises from a superpotential $W(\phi)$ (which is a holomorphic function of $\phi$) by the formula

$$V(\phi, \bar{\phi}) = e^K [K_{\phi\bar{\phi}} D_\phi D_{\bar{\phi}} W - 3 W \bar{W}],$$

with $D_\phi = \partial_\phi + \frac{\partial K}{\partial \phi}$, where $K$ is the Kähler potential and $K_{\phi\bar{\phi}} = (\frac{\partial^2 K}{\partial \phi \partial \bar{\phi}})^{-1}$. $K$ was given above in equation (2.12) and so we obtain in the present theory

$$V(\phi, \bar{\phi}) = \frac{1}{4(\phi + \bar{\phi})^2} \frac{\partial W}{\partial \phi} \frac{\partial \bar{W}}{\partial \bar{\phi}} - \frac{1}{(\phi + \bar{\phi})^3} \left( \frac{\partial W}{\partial \phi} \bar{W} + W \frac{\partial \bar{W}}{\partial \bar{\phi}} \right) + \frac{1}{(\phi + \bar{\phi})^4} W \bar{W}. \quad (4.4)$$

Thus we have

$$\frac{\partial V}{\partial \chi} = \frac{i \sqrt{2}}{4(\phi + \bar{\phi})^2} (W_{,\phi} \bar{W}_{,\bar{\phi}} - W_{,\bar{\phi}} \bar{W}_{,\phi})$$

$$- \frac{i \sqrt{2}}{(\phi + \bar{\phi})^3} (W_{,\phi} \bar{W} - W \bar{W}_{,\bar{\phi}})$$

$$+ \frac{i \sqrt{2}}{(\phi + \bar{\phi})^4} (W_{,\phi} \bar{W} - W \bar{W}_{,\phi}) \quad (4.5).$$

Setting $\chi = 0$, we have $\phi = \bar{\phi}$. Hence we see that the condition for the truncatability of $\chi$ becomes (as can be verified using a series expansion of $W$)

$$W = \bar{W} \quad \text{for} \quad \phi = \bar{\phi},$$

up to an irrelevant overall phase. Thus, if the superpotential is real when $\chi$ is set to zero, then $\chi$ can be truncated and the formula for the potential becomes

$$V = \frac{1}{16 \phi^2} (W, \phi - 2 W, \phi^2) - \frac{3}{16} W^2. \quad (4.7)$$
But for a general complex superpotential we are not allowed to disregard $\chi$ and it may not be dynamically sensible to do so even if it is mathematically consistent. This could be important for cosmology. We in general have two interacting scalars and both scalars need to be taken into account when studying the consequences of inflation, density perturbations or quintessence.

For future reference, let us pause here to calculate the solutions to (4.7) leading to a flat potential $V = V_0$ representing a cosmological constant. Setting $\tilde{W} = \phi^{-2}W$ simplifies the equation, since we can rewrite it as

$$\phi^2\tilde{W}^2 = 3\tilde{W}^2 + 16V_0.$$  \hfill (4.8)

For $V_0 < 0$, the solutions are

$$W = 4\sqrt{-\frac{V_0}{3}}\phi^2$$  \hfill (4.9)

$$W = 2\sqrt{-\frac{V_0}{3}}(\phi^2 + \sqrt{3} + \phi^2 - \sqrt{3})^2,$$  \hfill (4.10)

whereas for $V_0 > 0$ the solution is

$$W = 2\sqrt{\frac{V_0}{3}}(\phi^2 + \sqrt{3} - \phi^2 - \sqrt{3}).$$  \hfill (4.11)

### 5 A Note on the Cyclic Universe Potential

The potential proposed in the cyclic universe scenario [5] is of the form

$$V(\phi) = V_0e^{-\frac{1}{\phi}}(1 - \phi^{-10}).$$  \hfill (5.1)

The radion is the only scalar being considered: $\chi$ has been set to zero and accordingly $\phi = e^b$. A serious criticism of this conjectured potential is that it cannot in fact be derived from a superpotential, and therefore cannot exist in heterotic M–theory. For its existence one would need to solve

$$16V_0e^{-\frac{1}{\phi}}(1 - \phi^{-10}) = (\frac{W_\phi}{\phi} - \frac{2W}{\phi^2})^2 - \frac{3W^2}{\phi^4}.$$  \hfill (5.2)

To see why this cannot be solved for $W$, given $V$ in (5.1), let us try to solve this equation as $\phi \rightarrow 0$. Then the dominant term on the left is $-16V_0e^{-\frac{1}{\phi}}\phi^{-10}$ and because it is negative we require on the right that the negative second term be dominant in this limit. This gives $W \rightarrow 4\sqrt{\frac{V_0}{3}}e^{-\frac{1}{\phi}}\phi^{-3}$ as $\phi \rightarrow 0$. But for this $W$ it is in fact the first (positive) term on the right that dominates, in contradiction with our assumptions. A last possibility would be that both terms on the right are equally dominant. This would be the case if $W$ were a power of $\phi$. But a polynomial $W$ cannot reproduce the non-perturbative $e^{-\frac{1}{\phi}}$ factor in (5.1). Note that this also applies for any negative power of $\phi$ in (5.1); the power $-10$ is of no special importance for this argument. We conclude that the cyclic universe potential cannot exist in heterotic M–theory.
However, to some extent the potential (5.1) can be supersymmetrically approximated, and in particular for the region in $e^{\frac{b}{2}}$ that is relevant for calculating the spectrum of density fluctuations. (We note also that in this region the cyclic potential coincides closely with the ekpyrotic one.) We take the approximate superpotential to be

$$W(\phi) = e^{-\frac{b}{2}}c\phi^5 e^{-\phi} + W_{flat},$$

(5.3)

where $W_{flat}$ denotes the flat solution found above in (4.11). This gives a potential of the shape plotted in Figure 2; $c$ determines the depth of the dip. The shape is similar to that of the cyclic potential (5.1) except for the (inevitable) little positive bump near the origin. Now the branes can only collide if the scalar field picks up enough kinetic energy to overcome this bump.

We can restore the second scalar and plot the resulting two–field potential in the steep region where the density fluctuations are evaluated (see Figure 3). Note that, as one comes in from large $b$ values, the potential falls in the $e^{\frac{b}{2}}$ direction, but the transverse curvature becomes larger and larger, thus tending to confine the scalar field to the $\chi = 0$ plane. However, there is an instability at the minimum of the original one-field potential: at the saddle point, the field will generically roll off to one of the true minima of the potential where $\chi$ is equal to plus or minus a non-zero constant. It might be interesting to see what implications this has for cosmology, bearing in mind that in this model the adiabatic modes of density fluctuations will be complemented by isocurvature modes.

*En passant*, we note that the shape of the potential shown in Figure 3 is reminiscent of the potentials considered in hybrid inflation scenarios [16], so the cosmological applications of the radion dynamics considered in this paper could perhaps also be extended to inflationary models.
6 Kaluza–Klein Corrections to Potentials

In the Kaluza–Klein reduction from eleven to five dimensions on a Calabi–Yau manifold, there can be couplings of massive graviton supermultiplets (the most significant of which will have a mass roughly equal to the inverse radius of the Calabi–Yau manifold) to the massless scalar \( \phi \), couplings which can be linear in the massive supermultiplet fields. Not enough is known about Calabi–Yau manifolds to enable a detailed calculation of these couplings, so this should be considered as an indicative study of eventual couplings of this type. Given a coupling linear in the massive graviton supermultiplet fields, the massive modes cannot be ignored and must be properly integrated out, as discussed in Ref. [17], the relevant results of which we now review.

The analysis is done in superspace. The radion and its pseudoscalar partner \( \chi \) plus their fermionic superpartner \( \lambda \) together with the associated auxiliary fields are component fields of a chiral superfield that we shall also denote by \( \phi \). When we come to compare the forms of corrected and uncorrected potentials, we can set \( \chi = \lambda = 0 \) and eliminate the auxiliary fields, taking once again \( \phi = e^{\frac{b}{2}} \); but for now we shall remain in superspace. Since we consider here the coupling problem to lowest order in a massive graviton supermultiplet, it is sufficient to restrict attention to the linearized level. Linearized supergravity is represented by a vector superfield \( V_{\alpha \bar{\alpha}} \), and in order to be able to write an action invariant under both linearized super–Poincaré and super–Weyl
transformations, one must also include a chiral conformal compensating multiplet, represented here by a prepotential superfield $X$. The mass term for this supermultiplet is then given by

$$\frac{m^2}{2} \int d^4x d^4\theta V_{a\dot{a}} \bar{V}^{a\dot{a}} - \frac{m^2}{2} \int d^4x d^4\theta X\bar{X};$$ (6.1)

this combination is needed to reproduce the Pauli-Fierz mass term for a ghost-free massive spin two field. Kaluza-Klein couplings of the massive graviton supermultiplet to massless fields must be gauge–invariant in form (since gauge invariance is broken only in the mass term), and therefore the massive graviton supermultiplet must couple to a supercurrent $J_{a\dot{a}}$. The available supercurrent must be constructed out of the massless scalar superfield $\phi$ and gauge invariance implies that it must obey the conservation law

$$D^a J_{a\dot{a}} = D_{\dot{a}} \mathcal{S},$$ (6.2)

where

$$\mathcal{S} = -\frac{1}{2} D^2 K + 3 \mathcal{W}.$$ (6.3)

Here $K$ is the Kähler potential as given before and $\mathcal{W}$ is the posited superpotential. The kinetic action for the massless scalars is given by the superspace integral of the Kähler potential, i.e.

$$\int d^4xd^4\theta K.$$ (6.4)

The non-truncatable couplings to the massive supermultiplet can be written as

$$\int d^4xd^4\theta V_{a\dot{a}} \bar{J}^{a\dot{a}} - \int d^4xd^2\theta (XS + \bar{X}\mathcal{S}).$$ (6.5)

Upon integrating out $V_{a\dot{a}}$ and $X$ while keeping only lowest order correction terms, we may drop the kinetic action terms for the massive multiplet (which we have not written out here) and retain only the mass term (6.1) and the coupling term (6.5). All the correction terms so obtained are of higher derivative structure except for

$$\frac{18}{m^2} \int d^4xd^4\theta W\bar{W},$$ (6.6)

which modifies the Kähler potential by

$$K \rightarrow K_{corr} = K + \frac{18}{m^2} \mathcal{W}\bar{W}.$$ (6.7)

Thus, after higher Kaluza-Klein corrections are taken into account, the presence of a superpotential modifies the kinetic structure of the massless scalars in the theory. The significance of this correction depends very much on the scale of compactification and on the shape of the superpotential considered.
Recall from equation (4.3) that the potential $V$ depends on both the superpotential $W$ and on the Kähler potential $K$. Now, since $K$ has been modified, the potential $V$ acquires corrections as well and we then have

$$V_{\text{corr}} = e^K + \frac{18}{m^2} W \bar{W} [(K + \frac{18}{m^2} W \bar{W}) D_\phi \bar{W} D_\bar{\phi} W - 3W \bar{W} ],$$

with the covariant derivative $D_\phi = \partial_\phi + \frac{\partial K}{\partial \phi} + \frac{18}{m^2} \frac{\partial W}{\partial \phi} \bar{W}$. However, since the correction to $K$ is of the form $W \bar{W}$ this in fact does not change the condition on truncatability of the pseudoscalar $\chi$ (i.e. that $W$ should be real when $\phi = \bar{\phi}$), so we can still set $\chi$ to zero consistently if we assume that $W$ has real expansion coefficients. This gives us the following expression for the potential:

$$V_{\text{corr}} = (2\phi)^{-4} e^{\frac{18}{m^2} W^2} \left\{\left(\frac{1}{\phi^2} + \frac{18}{m^2} W_{\phi}^2 \right)^{-1} [W_{,\phi} (1 + \frac{18}{m^2} W^2) - 2 \frac{W}{\phi}]^2 - 3W^2 \right\}.$$

Superpotentials of non–perturbative origin have been investigated in this context by Lima et al. [18] and Moore et al. [19]. They arise from membrane instantons and are generically of the form $W = e^{-\phi}$, when no additional matter is considered on the brane worldvolume. They lead to an exponentially falling potential $V$ which therefore acts as a repulsive force between the branes. Moreover this potential gets corrected very little by the massive Kaluza–Klein modes.

![Graph of an originally constant cosmological potential after inclusion of massive Kaluza-Klein supermultiplet corrections. The height $V_0$ of the original uncorrected potential determines the range of $\phi = e^{b/2}$ values where the corrections eventually become important.](image)

Figure 4: More dramatic effects can be observed for potentials whose uncorrected forms do not go to zero as $\phi \to \infty$. Indeed, let us see what the fate of a constant potential is when the corrections are
taken into account. We should note that it is not clear how such potentials could be generated in the underlying microscopic theory, but many cosmological models can be approximated in this way, at least asymptotically, and so they are of interest to consider. A flat potential is obtained by considering one of the superpotentials (4.9–4.11). Now when we look at what the Kaluza–Klein corrections do to this initially flat potential, the effect is eventually drastic. As we go to large values of $\phi$, we notice an initial dip and then an exponential rise (see Figure 4). This is due to the $e^{15m^2W^2}$ prefactor in formula (6.9). This behavior is generic for potentials that do not tend to zero at large values of $\phi$. However, carrying out the same analysis for, e.g., the cyclic universe potential, which is asymptotically flat, one notes that when plugging in phenomenologically reasonable values of the parameters involved, the Kaluza–Klein corrections only become important at values of $\phi$ so large that the corrections can be ignored in the region of cosmological interest.

7 Conclusions

We have investigated the rôle of the chiral radion supermultiplet in heterotic M–theory. The existence of a consistent truncation from 5 to 4 dimensions is of central importance in our work. This opens up the possibility of constructing cosmological scenarios with dynamical boundary branes.

A prominent rôle in this paper is played by the pseudoscalar $\chi$. We showed that it cannot be ignored unless the superpotential can be written as an expansion with only real coefficients, and even then ignoring it may not be dynamically justified. In cases where it can be truncated we estimated the effect of Kaluza–Klein corrections due to heavy modes on the potential of the theory. They can be safely ignored for all potentials that approach zero fast enough asymptotically for large radion values; otherwise they tend to make the potential rise exponentially for large interbrane distances, thus effectively putting an upper bound on the distance between the boundary branes.

We constructed a solitonic string solution in $d = 4$ dimensions that is supported by the pseudoscalar $\chi$. From the standpoint of 5 dimensions, the string appears as the intersection of a membrane with a boundary brane. An important ingredient in consideration of the solitonic string is the assumption that the full quantum theory should exhibit an unbroken $SL(2, \mathbb{Z})$ symmetry that is interpreted locally. As a consequence, the distance between the boundary branes cannot reach zero, and thus the singularity at which the orbifold and Calabi–Yau volumes vanish is avoided. Virtual exchanges of membranes between the boundary branes may lead to forces between the branes. It would be interesting to calculate a potential for these forces from a purely 5–dimensional perspective. This potential should of course respect the $SL(2, \mathbb{Z})$ invariance [20]. But $V$ is given by

$$V(\tau, \bar{\tau}) = e^K [K^{\tau\bar{\tau}} D_\tau W \overline{D_\tau W} - 3W \overline{W}]$$
and we have $K(\tau, \bar{\tau}) = -4\ln(\tau - \bar{\tau})$ if we write $K$ in terms of $\tau$. Let $M$ denote an $SL(2, \mathbb{Z})$ transformation:

$$M \tau \equiv \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1.$$ 

For invariance of the potential, we want $V(M \tau, M \bar{\tau}) = V(\tau, \bar{\tau})$. Now

$$K(M \tau, M \bar{\tau}) = K(\tau, \bar{\tau}) + 4\ln(c\tau + d) + 4\ln(c\bar{\tau} + d).$$

This puts the following condition on the superpotential $W(\tau)$:

$$W(\tau) = (c\tau + d)^{-4}W(\tau).$$

Noting that Dedekind’s $\eta$-function satisfies

$$\eta(M \tau)^{24} = (c\tau + d)^{12}\eta(\tau)^{24},$$

we can see that, for example, taking $W(\tau) \propto \eta(\tau)^{-8}$ (multiplied by an arbitrary modular invariant function) will give us an $SL(2, \mathbb{Z})$ invariant potential. One would like to see which such non-perturbative superpotentials can arise in the theory, and what their implications would be on the dynamics of the radion mode.

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