The entropy of dense non-commutative fermion gases

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Abstract. We investigate the properties of two- and three-dimensional non-commutative fermion gases with fixed total z-component of angular momentum, $J_z$, and at high density for the simplest form of non-commutativity involving constant spatial commutators. Analytic expressions for the entropy and pressure are found. The entropy exhibits non-extensive behaviour while the pressure reveals the presence of incompressibility in two, but not in three dimensions. Remarkably, for two dimensional systems close to the incompressible density, the entropy is proportional to the square root of the system size, i.e., for such systems the number of microscopic degrees of freedom is determined by the circumference, rather than the area (size) of the system. From a commutative perspective these results can also be interpreted in terms of an effective excluded volume. Possible generalizations to three dimensions, which restore rotational symmetry, are briefly discussed.

1. Introduction

The realization that gravitational instabilities make the absolute localization of events impossible, which in turn may be captured through a non-commutative space-time structure [1], has led to considerable interest in the formulation of non-commutative quantum field theories [2], quantum mechanics [3] and the possible physical consequences of non-commutativity.

One aspect of non-commutative systems that has received very little attention in the literature is the thermodynamics. The first attempt to understand the thermodynamics of such systems was made in [4]. However, the computation could still only be carried out numerically and thus the thermodynamic limit could not be probed reliably. The results of these calculations were quite striking and showed strong deviations from commutative behaviour at high densities and temperatures. The most important feature is the appearance of incompressibility at high densities for fixed angular momentum. This reflects the effective excluded volume implied by the non-commutative length scale. Under these conditions several other thermodynamic features also deviate strongly from the commutative case and, in particular, the entropy is no longer extensive.

This sets the scene for the present paper, which aims at understanding the thermodynamics of non-commutative fermion gases in two and three dimensions analytically, particularly at high densities and low temperature.

2. The two-dimensional non-commutative fermion gas

We begin by summarizing the key results of [4, 5]. Consider a system of $N$ spinless particles of mass $m_0$ which are confined to a disc with area $A = \pi R^2$. The particle coordinates satisfy $[x, y] = i \theta \ (\theta > 0)$ and we set $R^2 = \theta (2M + 1)$ with $M$ a positive integer. The latter serves as a dimensionless measure of the system size. The single particle energies are then
$E_{r,m} = \hbar^2 x_{r,m}/(\theta m_0)$ with $x_{r,m}$ the zeros of the Laguerre polynomial $L_{M+1}^m(x)$ ($m \geq 0$) or $L_{M+1}^{m+1}(x)$ ($-M \leq m < 0$). Here $m$ is an angular momentum label while $r$ labels the $M + 1$ ($m \geq 0$) or $M + m + 1$ ($-M \leq m < 0$) states within a specific angular momentum sector. Note that $m$ is bounded from below by $-M$ and so there are only a finite number of negative angular momentum states.

The $q$-potential of the grand canonical ensemble is

$$q(A, \beta, \mu, \omega) = \sum_{m=-M}^{\infty} \sum_{r} \log[1 + e^{-\beta(E_{r,m} - \mu - \hbar \omega m)}].$$

(1)

We set $E_0 = \hbar^2 / (\theta m_0)$ and introduce the dimensionless parameters $\tilde{\beta} = E_0 \beta$, $\tilde{\mu} = \mu / E_0$ and $\tilde{\omega} = \hbar \omega / E_0$ in terms of which the $q$-potential becomes

$$q(M, \tilde{\beta}, \tilde{\mu}, \tilde{\omega}) = \sum_{m=-M}^{\infty} \sum_{r} \log[1 + e^{-\tilde{\beta}(x_{r,m} - \tilde{\mu} - \tilde{\omega} m)}].$$

(2)

The central thermodynamic quantities can be calculated from $q$ using

$$N = \frac{1}{\tilde{\beta}} \frac{\partial q}{\partial \tilde{\mu}}, \quad L = \frac{1}{\hbar} \frac{\partial q}{\partial \tilde{\omega}}, \quad S = q - \tilde{\beta} \frac{\partial q}{\partial \tilde{\beta}}$$

and we also define dimensionless measures of the density and pressure by $\tilde{\rho} = N/(2M + 1)$ and $\tilde{P} = q/(\tilde{\beta}(2M + 1))$.

In the thermodynamic limit (2) can be replaced by

$$q(M, \tilde{\beta}, \tilde{\mu}, \tilde{\omega}) = \int_{-M}^{\infty} dm \int_{x_{r,m}}^{x_{r,m}} D(x, m) \log[1 + e^{-\tilde{\beta}(x - \tilde{\mu} - \tilde{\omega} m)}]$$

(4)

where $D(x, m)$ is the asymptotic density of the zeros of the Laguerre polynomial $L_{M+1}^m(x)$ ($m \geq 0$) or $L_{M+1}^{m+1}(x)$ ($-M \leq m < 0$) in the limit where $M$ and $m$ tend to infinity in a fixed ratio. It is known that $D(x, m)$ is given by [6]

$$D(x, m) = \frac{\sqrt{4Mx - (m - x)^2}}{2\pi x}$$

(5)

for $x$ in the interval on which the argument of the radical is non-negative.

The low density limit of (4) can be explored quite easily and one finds that it yields precisely the commutative density of states and thermodynamics. More interesting is the high density limit, which we investigate next.

To see the effects of non-commutativity it is necessary to consider densities which scale with the system size. This will render the chemical potential extensive and ensure that $c = \tilde{\mu}/M$ remains finite as $M$ tends to infinity. A more appropriate measure of the density is therefore $\nu \equiv \tilde{\rho}/M \approx N/(2M^2)$, which will be kept fixed when taking $M \to \infty$. We also define the scaled angular momentum by $\ell \equiv \tilde{L}/(M^3 \hbar)$.

The integrals defining the $q$-potential can be approximated at low temperatures using the Sommerfeld expansion [7]. The final result, up to linear order in temperature, reads

$$q(M, b, c, w) = \frac{M^2}{6b} \left[ \frac{c (b^2 c^2 + \pi^2 \Theta(-c))}{w^2} - \frac{(w - 1)^2 (3b^2 c^2 + \pi^2) - 3b^2 c (w - 1) w^2 + b^2 w^4}{(w - 1)^3} \right]$$

(6)
where \( \Theta(\cdot) \) is the step function. This result is valid for \( w < 1 \) and \( c \geq w^2/(w - 1) \).

Transforming from \((b, c, w)\) back to \((\tilde{\beta}, \tilde{\mu}, \tilde{\omega})\) yields the desired expression for \( q(M, \tilde{\beta}, \tilde{\mu}, \tilde{\omega}) \) in (2) in the high density, low temperature limit.

Our next task is to combine (3) and (6) and solve for \( \tilde{\mu} \) and \( \tilde{\omega} \) as functions of the angular momentum and number of particles. We find that for solutions to exist, the angular momentum is bounded from below by \(-M^2/6\); a reflection of the finite number of negative angular momentum states. A further, more striking feature, is that for each \( \nu \), the entropy vanishes as the square root of \( \nu \), indicating that \( \tilde{\mu} \to \infty \) and \( \tilde{\omega} \to -\infty \).

Substituting the expressions for \( \tilde{\mu} \) and \( \tilde{\omega} \) in (8) into the relations (3) yields expressions for the entropy and pressure of the non-commutative gas close to the critical density:

\[
\frac{\mu}{M} = 1 - \frac{7 + 24\ell - 24\nu(6\nu - 1)}{24[(8\nu + 2)(\nu - \nu_0)(\nu_c - \nu)]^{1/2}}
\]

\[
\omega = 1 - \frac{1}{4} \left[ \frac{8\nu + 2}{(\nu - \nu_0)(\nu_c - \nu)} \right] \]

where \( \nu_0 \approx (3 - 2\sqrt{18\ell + 3})/12 \). Note that as \( \nu \) approaches \( \nu_c \) these quantities diverge: \( \tilde{\mu} \to \infty \) and \( \tilde{\omega} \to -\infty \).

In general the exact expressions for \( \tilde{\mu} \) and \( \tilde{\omega} \) are quite cumbersome and so we will focus only on the physically interesting region close to the critical density. Here we find

\[
\tilde{\mu} = M \left[ \frac{\nu_c - \nu_0}{1 + 4\nu_c} \right]^{1/2} \frac{2^{3/2}\pi^2}{3\beta} \sqrt{\nu_c - \nu}
\]

and

\[
\tilde{P} = M^2 \left[ \frac{1 + 4\nu_c}{\nu_c - \nu_0} \right]^{1/2} \frac{1}{96\sqrt{2}} \frac{1}{\sqrt{\nu_c - \nu}}.
\]

In terms of the dimensionless system size \( M \) and particle density \( \tilde{\rho} \approx N/(2M) \) this implies, for example, that at \( \ell = 0 \)

\[
S \sim \sqrt{M} \sqrt{M\nu_c - \tilde{\rho}}
\]

and

\[
\tilde{P} \sim M^{5/2}/\sqrt{M\nu_c - \tilde{\rho}}.
\]

We see that as \( \nu \) approaches \( \nu_c \), the entropy vanishes as the square root of \( \nu_c - \nu \), indicating that the “tight packing” limit has been reached and only a single microstate is accessible to the system. This coincides with a divergence in the pressure which signals that the system is incompressible at the critical density. Due to their entropic nature these phenomenon are expected to persist at higher temperatures. In particular, the value of \( \nu_c \) and the \( \sqrt{\nu_c - \nu} \) dependence of the entropy and pressure should remain unaffected provided that the system is sufficiently close to its maximum density.

The next question we address is the following: How will the entropy of systems close to the incompressible density \( \nu_c \) scale with system size? The physical motivation for this question is that we would like to know how the entropy of a very dense gas, close to its maximum density, behaves as a function of the size or number of particles when more particles are added and the size of the object increased, but in such a way that the density is always kept very close to the maximal density. For this purpose consider a system containing \( N \) particles with a minimum size \( M_c \) given by \( M_c^2 = N/(2\nu_c) \). The density can then be expressed as \( \nu = N/(2M^2) = \nu_c M_c^2/M^2 \).

The limit that interest us is when the system size exceeds the minimum size by a small fixed area \( 2\pi\theta\Delta M \). Then \( M = M_c + \Delta M \) with \( \Delta M \) a non-negative integer such that \( \Delta M << M_c \) and \( \nu \approx \nu_c(1 - 2\Delta M/M_c^2) \), i.e., the density is close to the incompressible density. Crucial here is that the quantization of the system size implies that \( \Delta M \sim 1 \). With \( \Delta M \) held fixed we now consider
the dependence of the entropy on the system size $M \approx M_c$. Substituting in (9) the entropy close to the incompressible point reads

$$S(M_c) \sim M \sqrt{\nu_c} \sqrt{1 - \frac{M^2}{M_c^2}} \approx \sqrt{2 \nu_c \Delta M} \sqrt{M_c}.$$  \hfill (11)

As $R^2 \approx 2 M \theta$ it follows that the entropy is proportional to the circumference, rather than the area, of the system. When $M \gg M_c$, i.e. far from the incompressible point, normal extensive behaviour is recovered.

3. The three-dimensional non-commutative fermion gas

We assume the following commutation relations in three dimensions

$$[x_i, x_j] = i \theta_{i,j}$$ \hfill (12)

with $\theta_{ij}$ a real anti-symmetric matrix. Through an appropriate choice of coordinates it is always possible to restrict the non-commutativity to two of the spacial coordinates with the third coordinate being commutative. We consider a cylindrical geometry with radius $R$ and length $L$ with its central axis orientated along the $z$-direction. The $x$- and $y$-directions are taken to be non-commutative and at a fixed $z$ the problem therefore reduces to the two-dimensional case considered earlier. In the $z$-direction we have the regular one dimensional free particle problem with vanishing boundary conditions. In the notation of (2) the $q$-potential reads

$$q_{3D}(M, \tilde{\beta}, \tilde{\mu}, \tilde{\omega}, \tilde{\gamma}) = \sum_{n=1}^{\infty} \sum_{m,r} \log[1 + e^{-\tilde{\beta}(x_r,m + \gamma n^2 - \tilde{\mu} \cdot \tilde{\omega})}]$$

$$= \sum_{n=1}^{\infty} q_{2D}(M, \tilde{\beta}, \tilde{\mu} - \gamma n^2, \tilde{\omega})$$ \hfill (13)

$$= \sum_{n=1}^{\infty} q_{2D}(M, \tilde{\beta}, \tilde{\mu} - \gamma n^2, \tilde{\omega})$$ \hfill (14)

where $\tilde{\gamma} = \gamma/E_0$ and $\gamma = \hbar^2 \pi^2/(2m_0 L^2)$. In the thermodynamic limit $R$ and $L$ will tend to infinity in a fixed ratio. This can be made explicit by writing $\tilde{\gamma} = G/M$ where $G = (\pi/2)^2 (R/L)^2$ is a dimensionless constant defining the shape of the cylinder. As is evident from (14) the $q_{3D}$-potential is just a sum of two-dimensional potentials with shifted chemical potentials. The sum terminates automatically once $\tilde{\mu} - \gamma n^2$ drops below $M \tilde{\omega}^2/(\tilde{\omega} - 1)$, as discussed after equation (6). Under the same assumptions that led to $q_{2D}$ in (6) we can now express $q_{3D}$ as

$$q_{3D}(M, \tilde{\beta}, \tilde{\mu}, \tilde{\omega}, \tilde{\gamma}) = \int_{0}^{n_+} dn \ q_{2D}(M, \tilde{\beta}, \tilde{\mu} - \tilde{\gamma} n^2, \tilde{\omega})$$ \hfill (15)

where $n_+ > 0$ is such that $\tilde{\mu} - \gamma n_+^2 = M \tilde{\omega}^2/(\tilde{\omega} - 1)$. This integral is straightforward to perform since $q_{2D}$ is a piecewise polynomial in its third argument.

In three dimensions it is convenient to use $\nu = N/M^3$ and $\ell = L/(\hbar M^4)$ as measures of the density and angular momentum. Combining (15) and (3) leads to a set of equations for $\tilde{\mu}$ and $\tilde{\omega}$ which can be solved numerically for given $\nu$ and $\ell$. Unlike the two-dimensional case there is no maximum density at which the entropy vanishes and pressure diverges, i.e. no incompressible point. This reflects the fact that the excluded area effect is restricted to the $x - y$ plane and does not place any restriction on how the commutative modes in the $z$-direction are occupied. However, the effects of non-commutativity are still present and responsible for modifying the scaling behaviour of the entropy as we progress from low to high densities. We find that $S$ grows
like $\nu^{1/3}$ at low densities and decreases like $1/\nu$ at high densities. In these two limits we are able to solve for $\tilde{\mu}$ and $\tilde{\omega}$ when $\ell = 0$ to find

$$\frac{S}{kM^2} = \left[\frac{\pi^6}{18G\beta^3}\right]^{1/3} \nu^{1/3} \quad \text{and} \quad \frac{S}{kM^2} = \frac{(4.2397\ldots) 1}{G\beta} \frac{1}{\nu} \quad (16)$$

for low and high densities respectively. In terms of the dimensionless volume $\tilde{V} = V/\theta^{3/2}$ and density $\tilde{\rho} = N/\tilde{V}$ these expressions become

$$\frac{S}{k} = \frac{1}{\beta} \left(\frac{\pi}{6}\right)^{2/3} \tilde{V} \tilde{\rho}^{1/3} \quad \text{and} \quad \frac{S}{k} = \frac{\eta G^{2/3} \tilde{V}^{7/3}}{\tilde{\rho}}. \quad (17)$$

with $\eta \approx 6.4757 \times 10^{-4}$. The low density result agrees exactly with that of a commutative fermion gas [7]. It is independent of both the shape of the cylinder (i.e. $G$) as well as the non-commutative parameter $\theta$. In contrast, the high density result depends on both these parameters and also exhibits an unusual non-extensive volume dependence. Furthermore, this entropy is inversely proportional to the particle density as is the case for a one-dimensional ideal gas. This observation that some aspects of the high density thermodynamics mimic that of a one-dimensional system is also observed in the relation between pressure and energy density. For a non-interacting gas in $d$ dimensions it holds quite generally that $P = (2/d)(E/V)$. We indeed find that $P = (2/3)(E/V)$ and $P = 2(E/V)$ in the low and high densities limits respectively.

These results hint that this type of non-commutativity and constraint on $J_z$ lead to a high density limit that behaves as a commutative one dimensional system. This counters our expectations for the real world and we can therefore only assume that this form of non-commutativity, the symmetry breaking it implies and the constraint on $J_z$ are inappropriate. A natural alternative is to change the non-commutative relations from the simple form (12) to those of a fuzzy sphere $[x_i, x_j] = i\theta\epsilon_{ijk}x_k$. In this way $R^3$ can be realized as an “onion structure” with quantized radius and the two dimensional program of [4] can in principle be executed to compute the spectrum of a three dimensional well.

4. Conclusions
The thermodynamic behaviour of two- and three-dimensional non-commutative gases was investigated analytically in the high density region. Strong deviations from conventional commutative behaviour was found. In particular an incompressible point was found in two dimensions and the entropy exhibits in both cases non-extensive behaviour. More profoundly, for two dimensional systems with density close to the incompressible density the entropy exhibits a very particular dependence on the system size in that it is proportional to the square root of the system size, i.e. the circumference rather than the area. Incompressibility is absent in three dimensions and the scaling of entropy at high density is unconventional, showing rather a crossover to one dimensional commutative thermodynamics. This is probably due to the particular form of non-commutativity used here in three dimensions and the symmetry breaking it implies. In particular the constraint on $J_z$, rather than $J$, is problematic and cannot be implemented correctly in the current framework.

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