Linearization method for constant thrust control

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Abstract. Low-thrust control technologies display advantages in space missions. In this study, a linearization method for constant-thrust control is proposed. Based on elliptic integrals, the constant-thrust linear equations are introduced. Furthermore, an analytical two-section linear equations are derived. In addition, for considering $J_2$ perturbation, semi-analytical linear equations are also presented. Numerical simulations are conducted to demonstrate the validation of the proposed method, which proves to be a practical choice for engineering.

1. Introduction

In recent years, low-thrust control technologies are widely used in aerospace engineering [1-3] such as formation maintenance [4,5], orbital rendezvous [6] and station keeping [7]. Generally, these missions require the spacecraft achieve an orbital transfer to arrive the specific position or state, which can be summed up in two-point boundary value problems. The Lambert method, as a simple method which provides a rapid double-impulse solution, has aroused wide attention and been extended to multiple-revolution and perturbation cases [8,9].

However, to face the complex engineering requirements, the study of continuous-thrust control methods is necessary, which depends sensitively on the boundary conditions, transfer time, and the engine of the thrust. Therefore, there are no analytical method in general. Fortunately, for special cases, a series of approximate or semi-analytical methods have been proposed to obtain the rapid solution for evaluating the transfer strategy. Applying first-order extension method, Avanzini [10] transformed the original equations into algebraic equations. However, it was often used in short-time case for reducing position and velocity errors. Docherty [11] derived analytical solution of Sun-Synchronous transfer problem by average-integral method. However, it is only suitable for circular reference orbit. Based on semi-analytical sensitive matrix, Bai [12,13] proposed a linear two-stage control strategy for the formation maintenance. The main contribution of this paper is to propose an analytical and linearized method for solving the transfer problems, which could reduce the demand for the engine in the space missions. Firstly, based on elliptic integrals, the constant thrust control linear equations are introduced. Furthermore, the analytical two-section linear equations are derived considering, and $J_2$ perturbation can be considered.

This paper is organized as follows. In Sec. II, the mathematical model for constant thrust control and constant thrust control linear equations are introduced. In Sec. III, the two-section linear equations are derived, and $J_2$ perturbation is also considered. In Sec. IV, the effectiveness of the methods above is examined through several numerical simulations. Finally, Sec. V summarizes the paper.
2. The conception and mathematical model of constant thrust control

The classical perturbation equations related to \( X=(a,e,i,\Omega,\omega,M) \) for two-body case can be expressed as:

\[
\begin{align*}
\frac{da}{dt} &= 2a^2\frac{v}{\mu} f_e, \\
\frac{de}{dt} &= \frac{1}{e} \left[ 2(e + \cos \theta) f_e - \frac{r}{a} \sin \theta f_e \right], \\
\frac{di}{dt} &= \frac{e^2}{h} \cos(\omega + \theta) f_e, \\
\frac{d\Omega}{dt} &= \frac{r}{h \sin i} \sin(\omega + \theta) f_e, \\
\frac{do}{dt} &= \frac{1}{e} \left[ 2 \sin \theta f_e + \left( \frac{2e}{v} + \frac{r \cos \theta}{av} \right) f_e \right] - \frac{r \sin(\omega + \theta) \cos i}{h \sin i} f_e, \\
\frac{dM}{dt} &= -n \frac{r}{dE} \left[ 2(e^2 + r) \sin \theta f_e + \frac{r \cos \theta}{a} f_e \right],
\end{align*}
\]

where \( p = a(1-e^2), h = \sqrt{\mu p}, b = a\sqrt{1-e^2}, r = a(1-e \cos E), v = \sqrt{\mu(2I/r-1/a)}, \) and \( f = (f_e, f_i, f_\Omega)^T \) denotes perturbation acceleration in the TNH frame [14]. To obtain the approximate analytical equations instead of non-linear equations (1), elliptic integrals related to eccentric anomaly \( E \) are introduced:

\[
\begin{align*}
\int_{\kappa_1}^{\kappa_2} \sqrt{1-e^2 \cos^2 E} dE &= I_1(e, E)|_{\kappa_2}^{\kappa_1}, \\
\int_{\kappa_1}^{\kappa_2} \frac{1}{\sqrt{1-e^2 \cos^2 E}} dE &= I_2(e, E)|_{\kappa_2}^{\kappa_1}, \\
\int_{\kappa_1}^{\kappa_2} \frac{\cos^2 E}{\sqrt{1-e^2 \cos^2 E}} dE &= I_3(e, E)|_{\kappa_2}^{\kappa_1}. \\
\end{align*}
\]

Based on the hypothesis of low thrust and calculation of elliptic integrals [14], the constant thrust linearization formulas can be obtained: \( X|^{\kappa_2}_{\kappa_1} = T_x f_e + N_x f_i + H_x f_\Omega \):

\[
\begin{align*}
\int_{\kappa_1}^{\kappa_2} \frac{dE}{dE} &= \frac{2a^3}{\mu} f_e \int_{\kappa_1}^{\kappa_2} \sqrt{1-e^2 \cos^2 E} dE = \frac{2a^3}{\mu} I_1 f_e |_{\kappa_2}^{\kappa_1} \\
\int_{\kappa_1}^{\kappa_2} \frac{de}{dE} &= \frac{2a^3(1-e^2)}{\mu} \left[ \ln(e \sin E + \sqrt{1-e^2 \cos^2 E}) - e f_e \right] |_{\kappa_2}^{\kappa_1} \\
\int_{\kappa_1}^{\kappa_2} \frac{di}{dE} &= \frac{2a^3}{\mu e} \left[ 3 \sin^{-1}(e \cos E) + (2 - e \cos E) \sqrt{1-e^2 \cos^2 E} \right] |_{\kappa_2}^{\kappa_1} \\
\int_{\kappa_1}^{\kappa_2} \frac{d\Omega}{dE} &= \frac{1}{\mu e \sin i} \left[ -e \sin \theta \left( \frac{\sin 2E}{4} + \frac{3E}{2} \right) + e \sin \theta (1+e^2) \sin E - \sqrt{1-e^2} \sin \omega \cos E - \sqrt{1-e^2} \sin \omega \cos 2E \right] |_{\kappa_2}^{\kappa_1} \\
\int_{\kappa_1}^{\kappa_2} \frac{do}{dE} &= \frac{2a^3}{\mu e} \left[ \ln(e \sin E + \sqrt{1-e^2 \cos^2 E}) - \sin E \sqrt{1-e^2 \cos^2 E} \right] |_{\kappa_2}^{\kappa_1} \\
\int_{\kappa_1}^{\kappa_2} \frac{dM}{dE} &= \frac{a^3}{e \mu} \left[ 3 \sin^{-1}(e \cos E) + \sqrt{1-e^2 \cos^2 E} \right] f_e |_{\kappa_2}^{\kappa_1} \\
\end{align*}
\]
3. The two-section control method considering $J_2$ perturbation

For fixed $f = (f_1, f_2, f_3)^T$, equations (3-8) only determine a three-dimensional manifold, but orbital elements are six-dimensional. Therefore, at least two or more sections of constant thrust are needed to achieve orbital transfer. In this section, based on linear equations (3-8), a linear two-section control method is proposed:

$$
X_{f_{ek}}^{e} = \int_{e_k}^{e} \frac{dX}{dE} \frac{dE}{dt} + \int_{e_k}^{e} \frac{dX}{dE} \frac{dE}{dt} = T_{x_1}f_{s_1} + N_{x_1}f_{s_1} + T_{x_2}f_{s_2} + N_{x_2}f_{s_2} + H_{x_3}f_{s_3}
$$

where $(f_{s_1}, f_{s_1}, f_{s_3})$ and $(f_{s_2}, f_{s_2}, f_{s_2})$ represent the constant thrust vector in sections $[t_0, t_n]$ and $[t_n, t_f]$, $t_n$ and $E_n$ denote the mid of time and its corresponding eccentric anomaly. Furthermore, to consider $J_2$ perturbation, we assume that $J_2$ term is a small thrust the equations (1) can be split into two parts

$$
\frac{dX}{dE} = \begin{pmatrix}
-\cos \gamma & \sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{pmatrix} \left(3(3\sin^2 i \sin^2 u - 1) - 6\sin i \sin u \cos u - 6\sin i \cos i \sin u \right)
$$

where $R_e$ is earth’s equator radius and $u$ is latitude argument. Thus, $J_2$ perturbation acceleration can be expressed as a function of the eccentric anomaly $E$, and then integrate each item:

$$
\int_{e_k}^{e} \frac{d\Omega}{dE} = \frac{3\mu J_2 R_e^2}{a^3} \sin i \cos i \sin u \int_{e_k}^{e} dE
$$

The first part represents the change of orbital elements by constant thrust, which can be calculated by analytical linear equations (9). The second part means the increment by analytical linear equations (9).
where latitude argument $u$ also can be expressed as a function of eccentric anomaly $E$:

\[
\sin u = \sin \omega - \frac{\cos E - e}{1 - e \cos E} \sqrt{1 - e^2} \sin E
\]

(19)

These integral terms can be simplified to analytical expressions or obtained directly by numerical calculation. Thus, the variations of orbital elements are as follows:

\[
X_{\text{or}}^{(E_i)} = \int_{E_i}^{E_f} \frac{dX}{dE} dE + \int_{E_i}^{E_f} dE = T_x f_x + N_x f_{x_1} + H_x f_{x_2} + T_{x_2} f_{x_2} + N_{x_2} f_{x_2} + H_{x_2} f_{x_2} + dX_{J_2}
\]

(20)

where $dX_{J_2}$ denotes the increments of orbital elements caused by $J_2$ perturbation.

### 4. Simulation results

To demonstrate the accuracy of the linear equations proposed in this paper, two numerical simulations are carried out. First, for the same initial orbital elements, the final orbital elements controlled by constant thrust in two-body case are obtained by numerical integration method and linear equations respectively, and then use the former as a standard value to examine the accuracy of the latter. Second, the accuracy of the linear equations for $J_2$ case are also examined compared with the numerical integration method. Numerical simulations demonstrate the validation of the proposed method:

#### 4.1 Linear equations for two-body case

This numerical simulation is presented to validate the effectiveness of the linear equations for two-body case. Suppose that the eccentric anomaly changes from $E_i$ to $E_f$ during $\Delta t = t_2 - t_1$. The results obtained by linear equations are compared with those calculated by numerical integration. Tables 1 and 2 show the initial orbit elements and parameters of the constant thrust acceleration. Table 3 shows the errors compared with true values, which mainly concentrates on the semi-long axis, eccentricity and the mean anomaly, which can be ignored for the problem of orbit transfer.

#### 4.2 Linear equations for $J_2$ case

Another numerical simulation is presented to validate the effectiveness of the linear equations for $J_2$ case. Suppose that the eccentric anomaly changes from $E_i$ to $E_f$ during $\Delta t = t_2 - t_1$. The results obtained by linear equations are compared with those calculated by numerical integration. Tables 4 and 5 show the initial orbit elements and parameters of the constant thrust acceleration. Table 6 shows the errors compared with true values, which mainly concentrates on the mean anomaly, which can be ignored for the problem of orbit transfer.
5. Conclusions
The linear equations are validated by a sequence of numerical simulations, where errors mainly concentrate on the semi-long axis, eccentricity and the mean anomaly. For the problem of orbit transfer, the error of the mean anomaly can be ignored, and the accuracy of other terms is enough. Compared with the conventional continuous-thrust control strategy, the proposed constant control strategy is much more convenient because in each section the thrust is a constant vector. Thus, these proposed linear equations can provide practical choice for engineering applications.

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