A Mean Value Theorem for Closed Geodesics on Congruence Surfaces

Vladimir Lukianov
Tel-Aviv University
November 10, 2018

Abstract. We define a weighted multiplicity function for closed geodesics of given length on a finite area Riemann surface. These weighted multiplicities appear naturally in the Selberg trace formula, and in particular their mean square plays an important role in the study of statistics of the eigenvalues of the Laplacian on the surface.

In the case of the modular domain, E. Bogomolny, F. Leyvraz and C. Schmit gave a formula for the mean square, which was rigorously proved by M. Peter. In this paper we calculate the mean square of weighted multiplicities for some surfaces associated to congruence subgroups of the unit group of a rational quaternion algebra, in particular for congruence subgroups of the modular group. Remarkably, the result turns out to be a rational multiple of the mean square for the modular domain.

1 Introduction

The aim of the present work is to give a tool for finding the eigenvalue statistics of the hyperbolic Laplacian

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

defined on a Riemann surface $\Omega = \Gamma \backslash \mathbb{H}$, for a discrete cofinite subgroup $\Gamma$ of $\text{PSL}_2(\mathbb{R})$ of "congruence type" (see below).

Let $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq ...$ be the discrete spectrum of the hyperbolic Laplacian on $\Omega$. It is known [6] that it satisfies the Weyl’s law:

$$N(T) := \# \{ 0 \leq r_j \leq T \} = \frac{\text{vol}(\Omega)}{4\pi} T^2 + O(T \ln T),$$

where $\lambda_j = \frac{1}{4} + r_j^2$. We define a smooth version of the counting function as follows. Let $f$ be an even test function with the compactly supported smooth
Fourier transform $\hat{f}$. Define

$$N_f(\tau) = \sum_{j \geq 0} f(L(r_j - \tau)) + f(L(-r_j - \tau)).$$

For example, if $f$ is the characteristic function of the interval $(-1/2, 1/2)$, then $N_f(\tau)$ counts the number of $r_j$ lying in the intervals $\pm(\tau - 1/2L, \tau + 1/2L)$.

Put

$$h(r) = f(L(r - \tau)) + f(L(-r - \tau))$$

then we can use the Selberg Trace Formula to express $N_f(\tau)$ as:

$$N_f(\tau) = \sum_{j \geq 0} h(r_j) = \{\text{identity contribution}\} +$$

\begin{align*}
+ \{\text{hyperbolic contribution}\} + \\
+ \{\text{elliptic contribution}\} + \\
+ \{\text{parabolic and continuous spectrum contribution}\}
\end{align*}

We say that the element $T$ of $\Gamma$ is hyperbolic if $|\text{tr}T| > 2$. Such an element is conjugate in $\text{PSL}_2(\mathbb{R})$ to a diagonal matrix $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, where $\lambda > 1$ is real.

For the element $T$, such that $|\text{tr}T| = t$ we define the norm\(^1\) of $T$ to be

$$\mathcal{N}(T) = \lambda^2 = \left(\frac{t + \sqrt{t^2 - 4}}{2}\right)^2.$$

For us the most important term in formula (1) is the hyperbolic contribution term, which is defined explicitly as:

$$\sum_{t > 2(T)} \sum_{\text{hyperbolic} \atop |\text{tr}T|=t} \frac{\ln \mathcal{N}(T_0)}{\mathcal{N}(T)^{1/2} - \mathcal{N}(T)^{-1/2}} g(\ln \mathcal{N}(T)).$$

There are in general several conjugacy classes with $|\text{tr}T| = t$. Define the weighted multiplicity function $\beta_t(t)$ by:

$$\beta_t(t) = \frac{1}{4} \sum_{\{T\} \subset \Gamma \atop T = T_0^k \text{ is hyperbolic} \atop |\text{tr}T|=t} \frac{\ln \mathcal{N}(T_0)}{\mathcal{N}(T)^{1/2} - \mathcal{N}(T)^{-1/2}},$$

where $T_0$ is a primitive hyperbolic element, that is $T_0$ is not a power of any other hyperbolic element. In this notation we can rewrite the hyperbolic contribution term as follows

$$\sum_{t > 2} \beta_t(t) g(\ln \mathcal{N}(T)),$$

\(^1\)The number $\lambda^2$ is called also a multiplier of $T$.\)
so the information about the weighted multiplicities will be useful for understanding the behavior of this term.

From the prime geodesic theorem it follows that the mean value of $\beta_\Gamma(t)$ is unity:
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{2 < t \leq N} \beta_\Gamma(t) = 1.
\]

Our main result is a computation of the mean square of the weighted multiplicities for the case that $\Gamma$ is congruence subgroup of the unit group of a quaternion algebra over the rationals.

Let $B$ be a quaternion algebra over $\mathbb{Q}$. Let
\[
N_B : B \to \mathbb{Q} \quad \text{and} \quad Tr_B : B \to \mathbb{Q}
\]
denote respectively the reduced norm and the reduced trace of the elements of $B$. If $\alpha \in B$, then
\[
N_B(\alpha) = \alpha + \overline{\alpha} \quad \text{and} \quad Tr_B(\alpha) = \alpha \overline{\alpha}
\]
where $\overline{\alpha}$ is the conjugate of $\alpha$.

We recall that quaternion algebra over field $F$ is either division algebra, or is isomorphic to $M_2(F)$.

Let $d_B$ be the reduced discriminant of $B$, that is $d_B$ is the product of all primes $p$, such that $B_p = B \otimes \mathbb{Q}_p$ is a division algebra. Note that $d_B > 1$ if and only if $B$ is a division algebra. We assume that $B$ is indefinite, that is $B_\infty := B \otimes \mathbb{R} \cong M_2(\mathbb{R})$.

Let $R$ be an order in $B$, then for $\alpha \in R$ we have that both the reduced trace and the reduced norm of $\alpha$ are integers.

Fix now the isomorphism $B \otimes \mathbb{R} \cong M_2(\mathbb{R})$, and set
\[
\Gamma_R = \{ \alpha \in R \mid N_B(\alpha) = 1 \}.
\]
Under above isomorphism $\Gamma_R$ is identified with a subgroup of $SL_2(\mathbb{R})$, which is a cofinite Fuchsian group, (and is co-compact if $d_B > 1$). Moreover, the traces of its elements are integers. In particular, one can consider $B = M_2(\mathbb{Q})$, and for the given natural $Q$ the set
\[
R = \left\{ \begin{bmatrix} a & b \\ Qc & d \end{bmatrix} \in M_2(\mathbb{Z}) \right\}.
\]
Then $B$ is indefinite quaternion algebra over $\mathbb{Q}$ and $R$ is an order in $B$. Here
\[
\Gamma_R = \Gamma_0(Q) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0(\text{mod} \ Q) \right\}.
\]

Remark 1.1 We will often identify $\Gamma_R$ with its image in the projective group $PSL_2(\mathbb{R})$. 

Mean Square of Weighted Multiplicities
We are now able to state our main theorems:

**Theorem 1.2** Let

\[ \beta_Q(t) = \frac{1}{4} \sum_{\{T\} \subset \Gamma_0(Q) \atop T = T_0^k \text{ is hyperbolic} \atop |\text{tr}T| = t} \frac{\ln \mathcal{N}(T_0)}{\mathcal{N}(T)^{1/2} - \mathcal{N}(T)^{-1/2}}, \]

where \( \Gamma_0(Q) \) is the congruence subgroup of \( SL_2(\mathbb{Z}) \), defined in (2). Then for the squarefree odd \( Q \)

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{2 < t \leq N} \beta_Q^2(t) = C_1 \prod_{q \mid Q} 2(q^2 - q - 1)(q + 1)^2 \frac{q(q^3 + q^2 - q - 3)}{q(q^3 + q^2 - q - 3)} = \kappa_Q. \]

Here

\[ C_1 = \lim_{N \to \infty} \frac{1}{N} \sum_{2 < t \leq N} \beta_1^2(t) = \frac{1015}{864} \prod_{p \not=} 2(p^3 + p^2 - p - 3) \frac{(p^2 - 1)^2(p + 1)}{(p^2 - 1)^2} = 1.328... \]

which is the mean square of the weighted multiplicities for the modular domain \((Q = 1)\), proved by M. Peter [10], following a conjecture of Bogomolny et al [3].

**Theorem 1.3** Let \( B \) be an indefinite division quaternion algebra over \( \mathbb{Q} \) with the maximal order \( R \), and discriminant \( d_B \). Then for

\[ \beta_R(t) = \frac{1}{4} \sum_{\{T\} \subset \Gamma_R \atop T = T_0^k \text{ is hyperbolic} \atop |\text{tr}T| = t} \frac{\ln \mathcal{N}(T_0)}{\mathcal{N}(T)^{1/2} - \mathcal{N}(T)^{-1/2}}, \]

the mean square of weighted multiplicities is

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{2 < t \leq N} \beta_R^2(n) = C_1 \cdot \prod_{p \mid d_B} \frac{2(p^3 - 1)(p - 1)}{p(p^3 + p^2 - p - 3)}. \]

We now explain the method of proof. As a first step, we express the weighted multiplicities in terms of Dirichlet’s \( L \)-functions. For \( \Gamma_0(Q) \) the expression is

\[ \beta_Q(t) = \frac{1}{v} L(1, \chi_D) \cdot \prod_{q \mid Q} \left\{ 1 + \frac{2}{\left( \frac{D}{q} \right)} \frac{q^2 \mid D}{q^2 \mid D} \right\}. \quad (3) \]
This is proved in section 3 by connecting the weighted multiplicities with class numbers and using Dirichlet’s class number formula.

The principal tool in our approach, following Peter, is that the formula (3) displays the weighted multiplicity $\beta_Q(t)$ as a "limit periodic function" in a suitable sense (see section 5 for background on these). To show this, we approximate the $L$-functions by a finite Euler product using a zero-density theorem, in a certain semi-norm coming from the theory of limit periodic functions (section 6). For computing mean squares, this suffices and allows us to use Parseval’s equality in this setting to express the mean square as

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq t \leq N} \beta_Q^2(t) = \sum_{b \geq 1} \sum_{1 \leq a \leq b \atop \gcd(a,b) = 1} \left| \hat{\beta}_Q \left( \frac{a}{b} \right) \right|^2 = \\
= \prod_{p \text{ prime}} \left( 1 + \sum_{c \geq 1} \sum_{1 \leq a \leq p^c \atop a \equiv 0 \pmod{p}} \left| \hat{\beta}_{(p,Q)} \left( \frac{a}{p^c} \right) \right|^2 \right),
$$

where $\hat{\beta}$ are the Fourier coefficients of $\beta$, defined in section 5.

We then carry out a length calculation of the Fourier coefficients in section 7.2, finally ending up with rather complicated expressions described in Theorem 7.2.

The result is that the mean square is an Euler product

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta_Q^2(n) = \prod_{p \text{ prime}} M_p(Q),
$$

with

$$
M_p(Q) = 1 + \sum_{c \geq 1} A_Q(p^c),
$$

and $A_Q(p^c)$ is given by (12), section 7.3. We evaluate the sum $M_p(Q)$ over prime powers as a rational function of $p$ and find that it depends on divisibility of $Q$ by $p$, in particular, for $p \nmid Q$, $p \neq 2$,

$$
M_p(Q) = M_p(1) = \frac{p^2(p^3 + p^2 - p - 3)}{(p^2 - 1)^2(p + 1)}.
$$

This will prove Theorem 1.2.

In section 8 we sketch this procedure for the case of the unit group of the maximal order of a quaternion algebra (theorem 1.3).

**An application:**

The computation of the mean square is used to study the statistics of $N_f(\tau)$: by Weyl’s law we expect $N_f(\tau)$ to be asymptotically equal to a multiple of $\tau/L$. To
study the fluctuations around this expectation one may consider $\tau$ as a random variable.

Then define an averaging operator:

$$\langle F \rangle_T = \frac{1}{T} \int_0^T F(\tau) d\tau$$

Studying the moments of $N_f$ shows that if $L \to \infty$, but $L = o(\ln T)$, then its limiting value distribution is Gaussian with the mean

$$\langle N_f(\tau) \rangle_T = \frac{\tau 2 \text{vol}(\Omega)}{L} \int_{-\infty}^{\infty} f(x) dx$$

and the variance

$$\left\langle \left( N_f(\tau) - \langle N_f(\tau) \rangle_T \right)^2 \right\rangle_T = \frac{4\kappa}{\pi L} \int_0^\infty f^2(u) e^{\pi Lu} du,$$

where $\kappa$ is the mean square of the weighted multiplicities. See [14], [9]. Thus our computation of $\kappa$ for $\Gamma$ of congruence type yields the value distribution of $N_f$.

Acknowledgement. This work was supported in part by Gutwirth Scholarship and by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities. It was carried out as part of the author’s PHD thesis at Tel Aviv University, under the supervision of prof. Ze’ev Rudnick.

2 Preliminaries on orders in quaternion algebras

First of all we recall that a ring $B$ with unity is called an algebra of dimension $n$ over a field $F$, if the following three conditions are satisfied:

1° $B \supset F$, and $1_B = 1_F$;
2° any element of $F$ commutes with all elements of $B$;
3° $B$ is a vector space over $F$ of dimension $n$.

Definition 2.1 Let $B$ be a finite dimensional algebra over $\mathbb{Q}$ with identity element $1_B$. A subset $R \subset B$ is called an order in $B$ if the following conditions are satisfied:

1° $R$ is a finitely generated $\mathbb{Z}$-module;
2° $R$ contains basis of $B$ over $\mathbb{Q}$;
3° $R$ is a subring of $B$ and $1_B \in R$. 

For any finite dimensional algebra $B$ over $\mathbb{Q}$ and for each place $\nu$ of $\mathbb{Q}$ (i.e. $\nu = \infty$ or $\nu = p$ a prime) we define $B_{\nu} := B \otimes_{\mathbb{Q}} \mathbb{Q}_{\nu}$. $B_{\nu}$ is an algebra over $\mathbb{Q}_{\nu}$ of dimension $\dim_{\mathbb{Q}} B$. For $\nu = p$ a prime, orders in $B_p$ defined as in the previous definition, replacing $\mathbb{Q}$ and $\mathbb{Z}$ by $\mathbb{Q}_p$ and $\mathbb{Z}_p$. If $R$ is an order of $B$, we get an order in $B_p$ by the definition

$$R_p := R \otimes_{\mathbb{Z}} \mathbb{Z}_p = \text{[the closure of } R \text{ in } B_p].$$

We also recall the definition of adele ring $B_A$ [13, pp. 197-198] and [17, p. 62]. As a set this is

$$B_A = \{(b_\nu) \in \prod_{\nu} B_\nu \mid b_p \in R_p \text{ for almost all primes } p\},$$

where $R$ is any order in $B$.

**Definition 2.2** A nonzero integer $d$ is called a fundamental discriminant if $d \equiv 1(\text{mod } 4)$, $d$ square free, $d \neq 1$; or $d \equiv 0(\text{mod } 4)$, $d^4 \neq 1(\text{mod } 4)$, $d^4$ squarefree.

Consider now a quadratic extension of $\mathbb{Q}$ arising as the splitting field over $\mathbb{Q}$ of a polynomial $P(X) = X^2 - tX + 1$, $t \in \mathbb{Z}$. Notice that $P(X)$ is irreducible iff $\sqrt{t^2 - 4} \notin \mathbb{Q}$. In this case we can write $t^2 - 4 = l^2d$, $l \in \mathbb{Z}^+$, $d$ is a fundamental discriminant. The splitting field of $P(X)$ is $\mathbb{Q}(\sqrt{d})$ and the zeros of $P(X)$ are

$$x_1 = \frac{t}{2} + \frac{l \sqrt{d}}{2}, \quad x_2 = \frac{t}{2} - \frac{l \sqrt{d}}{2}.$$

For each fixed fundamental discriminant $d$ we let for each $f \in \mathbb{Z}^+$

$$\tau[f] := \mathbb{Z} + f\omega\mathbb{Z}, \quad \text{where } \omega = \frac{d + \sqrt{d}}{2}.$$ 

Then orders in $\mathbb{Q}(\sqrt{d})$ are precisely $\tau[f]$, $f = 1, 2, 3, ...$ [2, pp. 48-49], and in particular $\tau[1]$ is the unique maximal order [5, pp. 146-147]. The index $[\tau[1] : \tau[f]] = f$, and we call $\tau[f]$ the order of index $f$ in $\mathbb{Q}(\sqrt{d})$. Note that $\tau[f_1] \subset \tau[f_2]$ iff $f_2 | f_1$.

For $P(X)$ as above, the roots $x_1, x_2$ of $P(X)$ satisfy

$$\mathbb{Z} + x_1\mathbb{Z} = \mathbb{Z} + x_2\mathbb{Z} = \tau[l] \quad (l^2 - 4 = l^2d).$$

Let $K = \mathbb{Q}(\sqrt{d})$, $d$ as above, then

$$K_\infty = \mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \left\{ \frac{\mathbb{R} \oplus \mathbb{R}}{\mathbb{C}}, d > 0 \right\},$$

$$\left\{ \frac{\mathbb{R} \oplus \mathbb{R}}{\mathbb{C}}, d < 0 \right\},$$
\[ K_p = \mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \left\{ \begin{array}{c} \mathbb{Q}_p \oplus \mathbb{Q}_p, \quad \left( \frac{p}{r} \right) = 1 \\ \mathbb{Q}_p(\sqrt{d}), \quad \left( \frac{d}{p} \right) = -1 \end{array} \right\}. \]

The distinct orders in \( K_p \) are

\[ \mathfrak{o}[p^n] := \mathbb{Z}_p + p^n \mathbb{Z}_p, n = 0, 1, 2, \ldots \text{ and } \omega = \frac{d + \sqrt{d}}{2} \text{ as above.} \]

We have

\[ \mathfrak{o}[1] \supset \mathfrak{o}[p] \supset \mathfrak{o}[p^2] \supset \ldots \text{ and } [\mathfrak{o}[1] : \mathfrak{o}[p^n]] = p^n. \]

Combining the definitions of order in \( K \) and in \( K_p \), we get

\[ (\mathfrak{r}[f])_p = \mathfrak{o}[p^n], \text{ where } n = \text{ord}_pf. \]

For any order \( \mathfrak{r} \) in \( K = \mathbb{Q}(\sqrt{d}) \) we define

\[ \mathfrak{r}_\infty := K_\infty \]
\[ \mathfrak{r}_A := \prod \mathfrak{r}_v \subset K_A \]
\[ \mathfrak{r}_A^\times := \{ x \in K_\infty^\times \mid N(x) > 0 \} \]
\[ \mathfrak{r}_A^\times_+ := \{ (\alpha_v) \in \mathfrak{r}_A^\times \mid N(\alpha_\infty) > 0 \} = \mathfrak{r}_\infty^\times_+ \times \prod_p \mathfrak{r}_p^\times \subset \mathfrak{r}_A^\times. \]

Here \( N \) denotes the extension of the norm \( N : K \longrightarrow \mathbb{Q} \) to the regular norm in the algebra \( K_\infty \) over \( \mathbb{Q}_\infty = \mathbb{R} \). [17, p.53].

The subgroup \( \mathfrak{r}_A^\times_+ \cdot K^\times \) has finite index in \( \mathfrak{r}_A^\times \), and we set

\[ h(\mathfrak{r}) := [K_A^\times : (\mathfrak{r}_A^\times_+ \cdot K^\times)], \]

the class number of \( \mathfrak{r} \).

For \( K = \mathbb{Q}(\sqrt{d}) \) as above one can also show [1, Chapter 5.2] that \( h(\mathfrak{r}[f]) \) is equal to the number of inequivalent primitive (and if \( d < 0 \), positive) quadratic forms \( ax^2 + bxy + cy^2 \) with discriminant \( b^2 - 4ac = df^2 \). In other words \( h(\mathfrak{r}[f]) = h(df^2) \), where the right-hand side is as in [8, vol I, pp. 127-].

### 3 Expression of weighted multiplicities function in the terms of Dirichlet’s \( L \)-functions

In this section we define the weighted multiplicities function \( \beta_{Q}(t) \) and express it in the terms of Dirichlet’s \( L \)-functions. It is a necessary step, which allows to analyze the behavior of \( \beta_{Q}(t) \). The result is stated in Theorem 3.1, proof of which takes the rest of the section.
Let $T \in SL_2(\mathbb{Z})$, such that $|t| = |tr(T)| > 2$. For such $T$ the equation $\det(T - \lambda I) = 0$ has two different solutions: $\lambda_{1,2} = \frac{t \pm \sqrt{t^2 - 4}}{2}$, and so $T$ is diagonalizable in $\mathbb{R}$. Such $T$ is called hyperbolic. Define $\mathcal{N}(T) := \lambda^2$, where $|\lambda| > 1$. Hence $\mathcal{N}(T) = \frac{(|t| + \sqrt{t^2 - 4})^2}{4}$.

Let $Q$ be an odd squarefree integer, and let

$$\Gamma_0(Q) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{Q} \right\}$$

be the congruence subgroup of $SL_2(\mathbb{Z})$. Define the weighted multiplicities function

$$\beta_Q(t) := \frac{1}{4} \sum_{\substack{T \in \Gamma_0(Q) \\
 hyperbolic, doesn't \\
 fix \ cusp \ \text{cusp}} \mid tr(T) = t} \ln \mathcal{N}(T_0) \frac{1}{\mathcal{N}(T)^{\frac{1}{2}} - \mathcal{N}(T)^{-\frac{1}{2}}},$$

where the sum is taken over conjugacy classes of $\Gamma_0(Q)$.

**Theorem 3.1** For $2 < |t| \in \mathbb{Z}$

$$\beta_Q(t) = \sum_{D, v \geq 1 \atop D \text{ is a discriminant} \atop Dv^2 = t^2 - 4} \frac{1}{v} L(1, \chi_D) \prod_{q | Q} \left\{ 1 + \left( \frac{D}{q} \right) \frac{q^2 | D}{q^2 | D} \right\},$$

and $\beta_Q(t) = 0$, otherwise. Here $\left( \cdot \right)$ is a Legendre’s symbol and $\chi_D$ is a quadratic character.

We start from noting that

$$\Gamma_0(Q) = \bigsqcup_{t \in \mathbb{Z}} H_t,$$

where

$$H_t = \{ T \in \Gamma_0(Q) \mid tr(T) = t \}.$$

So we can write (the change from $1/4$ to $1/2$ comes from not collecting together $t$ and $-t$)

$$\beta_Q(t) = \frac{1}{2} \sum_{\substack{T \in H_t/\Gamma_0(Q) \\
 \text{hyperbolic, doesn't} \\
 \text{ fix \ cusp}}} \ln \mathcal{N}(T_0) \frac{1}{\mathcal{N}(T)^{\frac{1}{2}} - \mathcal{N}(T)^{-\frac{1}{2}}}}$$

(4)

Here $H_t/\Gamma_0(Q)$ is the set of $\Gamma_0(Q)$-conjugacy classes of $H_t$. 
Lemma 3.2  For a hyperbolic element $T \in H_t$ (i.e. $t^2 - 4 > 0$) no fixed point of $T$ is a cusp of $\Gamma_0(Q)$.

Proof. Note, that the cusps of $\Gamma_0(Q)$ are exactly the points in $\{i\infty\} \cup \mathbb{Q}$, by [13, cor. 1.5.5, th. 4.1.3(2)]. Assume $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. First, $c \neq 0$, since we will get $T$ parabolic. So, the fixpoints of $T$ are the two real solutions to $\frac{ax + b}{cx + d} = x \Leftrightarrow x = \frac{a - d}{2c} \pm \frac{\sqrt{t^2 - 4}}{2|c|}$, which are both irrational, since $\sqrt{t^2 - 4} \notin \mathbb{Q}$.

Now we want to know when in the sum (4), $H_t$ is not the empty set.

Lemma 3.3  Let $t \in \mathbb{Z}$, such that $\sqrt{t^2 - 4} \notin \mathbb{Q}$, and write $t^2 - 4 = l^2d$, $l \in \mathbb{Z}^+$, $d$ is a fundamental discriminant. Then $H_t$ is not empty iff for all primes $q$ divide $Q$ we have $q | l$ or $\left(\frac{d}{q}\right) \neq -1$.

Proof. $H_t \neq \emptyset$ iff there some $a, b, c \in \mathbb{Z}$, such that $\det \begin{bmatrix} a & b \\ Qc & t - a \end{bmatrix} = 1$. Hence $H_t \neq \emptyset$ iff there is some $a \in \mathbb{Z}$, such that $a(t - a) \equiv 1 \pmod{Q} \Leftrightarrow (2a - t)^2 \equiv t^2 - 4 \pmod{Q} \Leftrightarrow (2a - t)^2 \equiv l^2d \pmod{Q}$. Since $Q$ is odd squarefree, the last congruence is solvable iff $(2a - t)^2 \equiv l^2d \pmod{q}$ for all primes $q | Q$. Or equivalently: the last congruence is solvable iff $q | l$ or $\left(\frac{d}{q}\right) \neq -1$, for all $q | Q$.

Note, that for $T \in H_t$

$$N(T)^\frac{1}{2} - N(T)^{-\frac{1}{2}} = \frac{|t| + \sqrt{t^2 - 4}}{2} - \frac{2}{|t| + \sqrt{t^2 - 4}} = \sqrt{t^2 - 4}. $$

So, for fixed $t \in \mathbb{Z}$, such that $\sqrt{t^2 - 4} \notin \mathbb{Q}$, $t^2 - 4 > 0$, $H_t \neq \emptyset$ we can write (4) in the form

$$\beta_Q(t) = \frac{1}{2} \sum_{T \in H_t/\Gamma_0(Q)} \frac{\ln N(T_0)}{\sqrt{t^2 - 4}}$$

(5)

We will enumerate the $\Gamma_0(Q)$-conjugacy classes in $H_t$, and for each conjugacy class will get the number $N(T_0)$.

Fix some $T_i \in H_t$, and take the polynomial $P(X) = X^2 - tx + 1$, i.e. the characteristic polynomial of $T_i$, and so $P(T_i) = 0$. $P(X)$ is irreducible, since $\sqrt{t^2 - 4} \notin \mathbb{Q}$. Hence $Q[T_i] \cong \mathbb{Q}(\sqrt{d})$, where $t^2 - 4 = l^2d$. 
Define the set $C(T_i) := \{\delta T_i \delta^{-1} | \delta \in GL_2(\mathbb{Q})\}$. Then $H_t = R \cap C(T_i)$, where $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) | Q \mid c \right\}$ is an order in $M_2(\mathbb{Q})$.

For any $\delta \in GL_2(\mathbb{Q})$, $(\mathbb{Q}[T_i] \cap \delta R \delta^{-1})$ is an order in $\mathbb{Q}[T_i]$, by the fact, that if $A'$ is a subalgebra of $A$, and $O$ is an order in $A$, then $A' \cap O$ is an order in $A'$. For any order $r$ in $\mathbb{Q}[T_i]$ define $C(T_i, r) := \{\delta T_i \delta^{-1} | \delta \in GL_2(\mathbb{Q})\}$, $\mathbb{Q}[T_i] \cap \delta^{-1} R \delta = r$. Following [16], [13] we have a

**Lemma 3.4**

$$C(T_i) = \bigcup_{f=1}^{\infty} C(T_i, r[f]),$$

$$H_t = \bigcup_{f \mid t} C(T_i, r[f]),$$

and each $C(T_i, r[f])$ is closed under $\Gamma_0(Q)$-conjugation.$^2$

**Proof.** Clearly, $C(T_i) = \bigcup_{f=1}^{\infty} C(T_i, r[f])$, since there are no other orders than $r[1]$, $r[2]$, ... in $\mathbb{Q}[T_i]$. To prove disjointedness, we must show that if $\delta_1 T_i \delta_1^{-1} = \delta_2 T_i \delta_2^{-1}$ ($\delta_1, \delta_2 \in GL_2(\mathbb{Q})$), then $\delta_1^{-1} R \delta_1$ and $\delta_2^{-1} R \delta_2$ have the same intersection with $\mathbb{Q}[T_i]$. From $\delta_1 T_i \delta_1^{-1} = \delta_2 T_i \delta_2^{-1}$ we get $\delta_1^{-1} \delta_2 \in \mathbb{Q}[T_i]$, since if an element of $M_2(\mathbb{Q})$ commutes with $T_i$, it is in $\mathbb{Q}[T_i]$, by [13, Lemma 5.2.2(3)], and thus

$$\mathbb{Q}[T_i] \cap \delta_2^{-1} R \delta_2 = (\delta_1^{-1} \delta_2) (\mathbb{Q}[T_i] \cap \delta_2^{-1} R \delta_2) (\delta_1^{-1} \delta_2)^{-1} = \mathbb{Q}[T_i] \cap \delta^{-1} R \delta_1.$$

This prove the first relation.

Next, for any order $r$ in $\mathbb{Q}[T_i]$ and any $\delta \in GL_2(\mathbb{Q})$, such that $\delta T_i \delta^{-1} \in C(T_i, r)$, we have:

$$T_i \in r \Leftrightarrow T_i \in \mathbb{Q}[T_i] \cap \delta^{-1} R \delta \Leftrightarrow T_i \in \delta^{-1} R \delta \Leftrightarrow \delta T_i \delta^{-1} \in R \Leftrightarrow \delta T_i \delta^{-1} \in H_t.$$

In other words:

- if $T_i \in r$, then $C(T_i, r) \subset H_t$;
- if $T_i \not\in r$, then $C(T_i, r) \cap H_t = \emptyset$.

Note that $T_i \in r[l]$, where $l^2 - 4 = l^2 d$, $l \in \mathbb{Z}^+$. So the orders of $r$ in $\mathbb{Q}[T_i]$ that contain $T_i$ are exactly $r[f]$, such that $r[f] \supset r[l]$, that is, such that $f \mid l$. By definition $H_t \subset C(T_i)$, and so from the first relation the second relation follows.

---

$^2$ $r[f] := \mathbb{Z} + fw\mathbb{Z}$, is an all orders in $\mathbb{Q}(\sqrt{d})$, where $f \in \mathbb{Z}^+, w = \frac{d + \sqrt{d}}{2}$. Since $\mathbb{Q}(\sqrt{d}) \cong \mathbb{Q}[T_i]$ we use the symbol $r[f]$ for the corresponding order in $\mathbb{Q}[T_i]$.
Finally, \( \gamma^{-1} R \gamma = R \), for any \( \gamma \in R^x \). So we get that each \( C(T_\gamma, \tau) \) is closed under \( R^x \)-conjugation, and in particular under \( \Gamma_0(Q) \)-conjugation. \( \blacksquare \)

Let \( \varepsilon_d = \frac{x + y \sqrt{d}}{2} \) be the proper fundamental unit in \( \mathbb{Q}(\sqrt{d}) \) (\((x, y)\) is the positive integer solution to \( x^2 - dy^2 = 4 \), for which \( y > 0 \) is minimal). Define

\[
\mathfrak{v}[f]^1 := \{ \alpha \in \mathfrak{v}[f] \mid N(\alpha) = 1 \},
\]

the units of the order \( \mathfrak{v}[f] \). Since

\[
\mathfrak{v}[1]^1 = \{ \pm \varepsilon_d^k \mid k \in \mathbb{Z} \},
\]

we have

\[
\mathfrak{v}[f]^1 = \left\{ \pm \left( \varepsilon_d^{[\mathfrak{v}[f] \cdot \mathfrak{v}[f]^1]} \right)^k \mid k \in \mathbb{Z} \right\}.
\]

**Lemma 3.5** If \( T \) is hyperbolic, i.e. \( d > 0 \), then \( N(T_0) = \varepsilon_d^{[\mathfrak{v}[f] \cdot \mathfrak{v}[f]^1]} \).

**Proof.** Choose some \( \delta \in GL_2(\mathbb{Q}) \), such that \( T = \delta T_\delta^{-1} \), and \( \mathbb{Q}[T_\delta] \cap \delta R \delta^{-1} = \tau \). Then \( Z_{\Gamma_0(Q)}(T) \), the centralizer of \( T \) in \( \Gamma_0(Q) \) is

\[
Z_{\Gamma_0(Q)}(T) = \mathbb{Q}[T] \cap \Gamma_0(Q) = \delta(\mathbb{Q}[T_\delta] \cap \delta^{-1}\Gamma_0(Q)\delta^{-1}) = \delta\mathfrak{v}[f]\delta^{-1}
\]

by [13, lemma 5.2.2(3)]. Since

\[
\mathfrak{v}[f]^1 = \left\{ \pm \left( \varepsilon_d^{[\mathfrak{v}[f] \cdot \mathfrak{v}[f]^1]} \right)^k \mid k \in \mathbb{Z} \right\},
\]

for \( d > 0 \) we can take for \( T_0 \) the image of \( \varepsilon_d^{[\mathfrak{v}[f] \cdot \mathfrak{v}[f]^1]} \) under the composite of the two isomorphisms \( \mathbb{Q}[T_\delta] \cong \mathbb{Q}(\sqrt{d}) \) and \( \mathbb{Q}[T_\delta] \ni S \mapsto \delta S \delta^{-1} \in \mathbb{Q}[T] \). Then

\[
tr_{M_2(\mathbb{Q})}(T_0) = tr \left( \varepsilon_d^{[\mathfrak{v}[f] \cdot \mathfrak{v}[f]^1]} \right).
\]

If we will denote \( \varepsilon_d^{[\mathfrak{v}[f] \cdot \mathfrak{v}[f]^1]} = \alpha + \beta \sqrt{d} \in \mathbb{Q}(\sqrt{d}) \), note, that \( \alpha, \beta \geq 0 \), and \( N(\alpha + \beta \sqrt{d}) = \alpha^2 - d\beta^2 = 1 \). We have \( tr(\alpha + \beta \sqrt{d}) = t \), and so

\[
N(T_0) = \left( \frac{|t| + \sqrt{t^2 - 4}}{4} \right)^2 = \left( \frac{|2\alpha| + \sqrt{4\alpha^2 - 4}}{4} \right)^2 = \left( \frac{2|\alpha| + \sqrt{4(1 + d\beta^2) - 4}}{4} \right)^2 = \left( \frac{2|\alpha| + 2|\beta| \sqrt{d}}{4} \right)^2 = \left( \frac{|\alpha| + |\beta| \sqrt{d}}{4} \right)^2 = \left( \alpha + \beta \sqrt{d} \right)^2 = \varepsilon_d^{[\mathfrak{v}[f] \cdot \mathfrak{v}[f]^1]}.
\]
Now we can rewrite (5) in the form

$$
\beta_Q(t) = \sum_{f|l} |C(T_t, v[f])|/\Gamma_0(Q) \frac{\ln [\varepsilon_d^{c^2(t^2)}]}{\sqrt{t^2 - 4}},
$$

where we write $t^2 - 4 = dl^2$ with $d$ a fundamental discriminant and $l \geq 1$.

For the quantities $|C(T_t, v[f])/\Gamma_0(Q)|$ we refer to the [13, §6.6], [16]:

**Proposition 3.6** Let $Q$ be squarefree and $h(df^2)$ the narrow class number of $\mathbb{Q}(\sqrt{d})$. Then

$$
|C(T_t, v[f])/\Gamma_0(Q)| = h(df^2) \cdot \left\{ \begin{array}{ll}
2, & \text{if } d < 0 \\
1, & \text{if } d > 0
\end{array} \right\} \cdot \prod_{p|Q} \left\{ 2, \frac{p}{p} \right\} \frac{d}{q} \cdot \prod_{q|Q} \left\{ 2, \frac{q}{q} \right\}.
$$

**Corollary 3.7**

$$
\beta_Q(t) = \sum_{f|l} h(df^2) \frac{\ln [\varepsilon_d^{c^2(t^2)}]}{\sqrt{t^2 - 4}} \cdot \prod_{q|Q} \left\{ 2, \frac{q}{q} \right\} \cdot \prod_{q|Q} \left\{ 2, \frac{d}{q} \right\}.
$$

where we write $t^2 - 4 = dl^2$ with $d$ a fundamental discriminant and $l \geq 1$.

**Proof.** Immediately from proposition 3.6. ■

We can continue the process

$$
\beta_Q(t) = \sum_{f|l} h(df^2) \frac{\ln [\varepsilon_d^{c^2(t^2)}]}{\sqrt{t^2 - 4}} \cdot \prod_{q|Q} \left\{ 2, \frac{q}{q} \right\} =
$$

$$
= \sum_{f|l} h(df^2) \frac{\ln \varepsilon_d^{c^2(t^2)}}{l^2 \cdot \sqrt{t^2 - 4}} \cdot \prod_{q|Q} \left\{ 2, \frac{q}{q} \right\} =
$$

$$
= \sum_{f|l} h(D) \frac{\ln \varepsilon_D^{c^2(t^2)}}{l \cdot \sqrt{t^2 - 4}} \cdot \prod_{q|Q} \left\{ 2, \frac{q}{q} \right\}.
$$
Now using Dirichlet’s class number formula \( h(D) \ln \varepsilon_D = \sqrt{D}L(1, \chi_D) \), we get

\[
\beta_Q(t) = \sum_{f \mid t \atop (D=\pm f^2)} L(1, \chi_D) \frac{f}{T} \prod_{q \mid Q} \left\{ \frac{2,}{1 + \left(\frac{D}{q}\right) q \mid f} \right\} = \\
= \sum_{D,v \geq 1 \atop Dv^2=r^2-4} \frac{1}{v} L(1, \chi_D) \prod_{q \mid Q} \left\{ \frac{2,}{1 + \left(\frac{D}{q}\right) q \mid f} \right\} = \\
= \sum_{D,v \geq 1 \atop Dv^2=r^2-4} \frac{1}{v} L(1, \chi_D) \prod_{q \mid Q} \left\{ \frac{2,}{1 + \left(\frac{D}{q}\right) q^2 \mid D} \right\},
\]

and the theorem 3.1 follows.

**Remark 3.8** Here \( D \) is a discriminant, i.e. \( D \equiv 0, 1(\mod 4) \). We assume it from now on.

### 4 Factorization of weighted multiplicities

Here we factorize (lemma 4.1) the weighted multiplicities function as a finite products of a local terms, defined below.

We have

\[
\beta_Q(n) = \sum_{D,v \geq 1 \atop Dv^2=n^2-4} \frac{1}{v} L(1, \chi_D) \prod_{q \mid Q} \left\{ \frac{2,}{1 + \left(\frac{D}{q}\right) q^2 \mid D} \right\}.
\]

We use now the Euler product formula for \( L(1, \chi_D) \). For \( n \geq 3, P \geq Q, \) define

\[
\beta_{P,Q}(n) := \sum_{D,v \geq 1 \atop Dv^2=n^2-4 \atop p \mid v \implies p \leq P} \left( \frac{1}{v} \prod_{p \leq P} \left( 1 - \frac{\chi_D(p)}{p} \right)^{-1} \right) \prod_{q \mid Q} \left\{ \frac{2,}{1 + \left(\frac{D}{q}\right) q^2 \mid D} \right\}.
\]

Note, that

\[
q^2 \mid D \iff q^2 \mid (n^2 - 4)v^{-2} \iff q^2 \mid (n^2 - 4) \prod_{p \leq P} p^{-2b_p}
\]

\[
\iff \begin{cases} 
q^2 \mid n^2 - 4, \ b_q = 0 \\
q^2 \mid (n^2 - 4)q^{-2b_q}, \ b_q \neq 0 
\end{cases} \iff q^2 \mid (n^2 - 4)q^{-2ord_qv}.
\]
Hence

$$\beta_{P,Q}(n) = \sum_{D,n \geq 1 \atop Dn^2 = n^2 - 4 \atop p|n \Rightarrow p \leq P} \left( \frac{1}{v} \prod_{p \leq P} \left( 1 - \frac{\chi_D(p)}{p} \right)^{-1} \right) \times$$

$$\times \prod_{q|Q} \left\{ 1 + \frac{2}{q} \right\}, \quad q^2 \mid (n^2 - 4)q^{-2ord_qv}$$

Define a function

$$\beta_{(p,Q)}(n) := \sum_{b \geq 0} \frac{1}{p^b} \left( 1 - \frac{1}{p} \chi_{(n^2 - 4)p^{-2b}}(p) \right)^{-1} \cdot I_{p^b}(n),$$

where

$$I_{p^b}(n) := \begin{cases} 1, & n^2 = 4 \mod p^{2b} \\ 0, & \text{else} \end{cases}, \text{ for } p \neq 2, p \nmid Q$$

$$I_{2^b}(n) := \begin{cases} 1, & n^2 = 4 \mod 2^{2b}, \text{ } (n^2 - 4)2^{-2b} \text{ is a discriminant} \\ 0, & \text{else} \end{cases},$$

and for \( q \mid Q \)

$$I_{q^b}(n) := \begin{cases} 1, & q^2 \mid (n^2 - 4)q^{-2b} \\ 0, & \text{else} \end{cases}.$$

Lemma 4.1 For \( Q \) odd squarefree and \( P \geq Q \) we have

$$\beta_{P,Q}(n) = \prod_{p \leq P \atop p \nmid Q} \beta_{(p,Q)}(n) \cdot \prod_{q|Q} \beta_{(q,Q)}(n).$$

Proof.

$$\prod_{p \leq P \atop p \nmid Q} \beta_{(p,Q)}(n) \cdot \prod_{q|Q} \beta_{(q,Q)}(n) = \prod_{p \leq P \atop p \nmid Q} \left( \sum_{b \geq 0} \frac{1}{p^b} \left( 1 - \frac{1}{p} \chi_{(n^2 - 4)p^{-2b}}(p) \right)^{-1} \cdot I_{p^b}(n) \right) \times$$

$$\prod_{q|Q} \left( \sum_{b \geq 0} \frac{1}{q^b} \left( 1 - \frac{1}{q} \chi_{(n^2 - 4)q^{-2b}}(q) \right)^{-1} \cdot I_{q^b}(n) \right).$$

After opening the brackets, we will get the sum of terms of the form:
\[
\frac{1}{p_1^{b_1}} \left( 1 - \frac{1}{p_1} \chi_{(n^2-4)p_1^{-2b_1}}(p_1) \right)^{-1} I_{b_1}(n) \cdot \frac{1}{p_2^{b_2}} \left( 1 - \frac{1}{p_2} \chi_{(n^2-4)p_2^{-2b_2}}(p_2) \right)^{-1} I_{b_2}(n) \ldots \\
\frac{1}{p_k^{b_k}} \left( 1 - \frac{1}{p_k} \chi_{(n^2-4)p_k^{-2b_k}}(p_k) \right)^{-1} I_{b_k}(n) \cdot \frac{1}{q_1^{b_1}} \left( 1 - \frac{1}{q_1} \chi_{(n^2-4)q_1^{-2b_1}}(q_1) \right)^{-1} I_{b_1}(q_1) \ldots \\
\frac{1}{q_l^{b_l}} \left( 1 - \frac{1}{q_l} \chi_{(n^2-4)q_l^{-2b_l}}(q_l) \right)^{-1} I_{b_l}(q_l).
\]

Therefore, since \( P \geq Q \) we have

\[
\prod_{p \leq P}^\beta_{(p,Q)}(n) \cdot \prod_{q \mid Q}^\beta_{(q,Q)}(n) = \sum_{n^2=4 \text{ (mod } p^{2b})}^{\text{ord } n^2=4 \text{ (mod } 2^b) \text{ is a discriminant}} \times \prod_{q \mid Q} \left\{ 1 + \left( \frac{(n^2-4)q^{-2ord_q v}}{q} \right) \mid q^2 \mid (n^2-4)q^{-2ord_q v} \right\} = 0.
\]

5 Limit periodic functions and Fourier analysis

Let \( s \geq 1 \). For \( f : \mathbb{N} \rightarrow \mathbb{C} \), define the seminorm

\[
\|f\|_s := \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} |f(n)|^s \right)^{1/s} \in [0, \infty).
\]

A function \( f \) is called \( s \)-limit periodic if for every \( \varepsilon > 0 \) there is a periodic function \( h \) with \( \|f - h\|_s \leq \varepsilon \). The set \( D^s \) of all \( s \)-limit periodic functions becomes a Banach space with \( \|\cdot\|_s \) if functions \( f_1, f_2 \) with \( \|f_1 - f_2\|_s = 0 \) are
identified. If $1 \leq s_1 < s_2 < \infty$, we have $\mathcal{D}^1 \supseteq \mathcal{D}^{s_1} \supseteq \mathcal{D}^{s_2}$ as sets (but they are endowed with different norms). For all $f \in \mathcal{D}^1$, the mean value

$$M(f) := \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} f(n)$$

exists. The space $\mathcal{D}^2$ is a Hilbert space with inner product

$$\langle f, h \rangle := M(f \overline{h}), \quad f, h \in \mathcal{D}^2.$$

For $u \in \mathbb{R}$, define $e_u(n) := e^{2\pi i un}$, $n \in \mathbb{N}$. In $\mathcal{D}^2$, we have canonical orthonormal base $\{e_{a/b}\}$, where $1 \leq a \leq b$ and $\gcd(a, b) = 1$.

For all $f \in \mathcal{D}^1$, the Fourier coefficients $\hat{f}(u) := M(f e^{-u})$, $u \in \mathbb{R}$, exist.

**Lemma 5.1** For $f \in \mathcal{D}^1$, $u \notin \mathbb{Q}$, we have $\hat{f}(u) = 0$.

**Proof.** Let $f \in \mathcal{D}^1$. For any $\varepsilon > 0$ there is a linear combination $\sum_{1 \leq v \leq V} e_v(n)$, such that $\left\| f - \sum_{1 \leq v \leq V} e_v(n) \right\|_1 < \varepsilon$, where $v \in \mathbb{Q}$. So $|f(n) - \sum_{1 \leq v \leq V} e_v(n)| < \varepsilon$, for all $1 \leq n \leq N$. Therefore we have

$$\left| \hat{f}(u) - \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \sum_{1 \leq v \leq V} e_v(n) e_{-u}(n) \right| =$$

$$= \lim_{N \to \infty} \frac{1}{N} \left| \sum_{1 \leq n \leq N} \left( f(n) - \sum_{1 \leq v \leq V} e_v(n) \right) e_{-u}(n) \right| \leq$$

$$\leq \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \varepsilon |e_{-u}(n)| \leq \varepsilon.$$

Let $u \notin \mathbb{Q}$, since $v - u \notin \mathbb{Q}$ we have

$$\left| \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \sum_{1 \leq v \leq V} e_v(n) e_{-u}(n) \right| = \left| \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \sum_{1 \leq v \leq V} e^{2\pi i n(v-u)} \right| =$$

$$= \left| \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq v \leq V} \frac{1 - e^{2\pi i N(v-u)}}{1 - e^{2\pi i (v-u)}} \right| \leq \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq v \leq V} \frac{2}{\text{const}} = 0,$$

and the lemma follows. ■
6 Limit periodicity of weighted multiplicities

In this section we will prove that the weighted multiplicities function $\beta_Q(n)$ is limit periodic (prop. 6.1), and as consequence, we will obtain the formula for calculating its mean square (end of the section).

**Proposition 6.1** The functions $\beta_Q(n) \in D^1$, and for $1 \leq s \leq 2$,

$$\lim_{P \to \infty} \| \beta_Q - \beta_{P,Q} \|_s = 0.$$ 

Write $\beta_Q(n) - \beta_{P,Q}(n) = \Delta_P^{(1)}(n) + \Delta_P^{(2)}(n)$, where

$$\Delta_P^{(1)}(n) := \sum_{D,v \geq 1; Dv^2 = n^2 - 4 \atop p \mid v \text{ for some } p > P} \frac{1}{v} \prod_{q \mid Q} \left\{ 1 + \left( \frac{D}{q} \right) \frac{q^2}{q^2} \right\} L(1, \chi_D)$$

and

$$\Delta_P^{(2)}(n) := \sum_{D,v \geq 1; Dv^2 = n^2 - 4 \atop p \mid v \Rightarrow p \leq P} \frac{1}{v} \prod_{q \mid Q} \left\{ 1 + \left( \frac{D}{q} \right) \frac{q^2}{q^2} \right\} \times$$

$$\times \left( L(1, \chi_D) - \prod_{p \leq P} \left( 1 - \frac{\chi_D(p)}{p} \right)^{-1} \right).$$

**Lemma 6.2** For $P \geq Q$ we have

$$\frac{1}{x} \sum_{2 < n \leq x} \left| \Delta_P^{(1)}(n) \right|^2 \ll \sum_{v > P} \frac{1}{v^2}.$$ 

**Proof.** Note that

$$\Delta_P^{(1)}(n) \leq \sum_{D,v \geq 1; Dv^2 = n^2 - 4 \atop p \mid v \text{ for some } p > P} \frac{2^{\omega(Q)}}{v} L(1, \chi_D), \quad (6)$$

where $\omega(Q)$ is the number of prime divisors of $Q$. Cauchy’s inequality gives

$$\left| \Delta_P^{(1)}(n) \right| \leq \left( \sum_{D,v \geq 1; Dv^2 = n^2 - 4 \atop p \mid v \text{ for some } p > P} \frac{2^{2\omega(Q)}}{v^2} \right)^{1/2} \left( \sum_{D,v \geq 1; Dv^2 = n^2 - 4 \atop p \mid v \text{ for some } p > P} L(1, \chi_D)^2 \right)^{1/2}.$$
For \( x \geq 1 \), this gives
\[
\sum_{2 < n \leq x} \left| \Delta_{P}^{(1)}(n) \right|^2 \leq \sum_{v > P} \frac{2^{2\omega(Q)}}{v^2} \sum_{2 < n \leq x} L(1, \chi_D)^2.
\]

M. Peter shows [10], [11] that the last sum is
\[
\sum_{D, v \geq 1; Dv^2 = n^2 - 4} L(1, \chi_D)^2 \sim \text{const} \cdot x,
\]
as \( x \to \infty \). Therefore we have the claim of the lemma. ■

In order to estimate \( \Delta_{P}^{(2)}(n) \) we must compare \( L(1, \chi_D) \) with a partial product of its Euler products. This is done by comparing both terms with a smoothed version of the Dirichlet series for \( L(1, \chi_D) \). Let \( N \geq 1 \). Then
\[
\Delta_{P}^{(2)}(n) = \Delta_{P, N}^{(2.1)}(n) + \Delta_{P, N}^{(2.2)}(n) + \Delta_{P, N}^{(2.3)}(n),
\]
where
\[
\Delta_{P, N}^{(2.1)}(n) := \sum_{D, v \geq 1; Dv^2 = n^2 - 4} \frac{1}{v} \prod_{q|Q} \left\{ 1 + \left( \frac{2}{q} \right) \frac{q^2 | D}{q^2 \nmid D} \right\} \left( L(1, \chi_D) - \sum_{l \geq 1} \frac{\chi_D(l)}{l} e^{-l/N} \right),
\]
\[
\Delta_{P, N}^{(2.2)}(n) := \sum_{D, v \geq 1; Dv^2 = n^2 - 4} \frac{1}{v} \prod_{q|Q} \left\{ 1 + \left( \frac{2}{q} \right) \frac{q^2 | D}{q^2 \nmid D} \right\} \sum_{l \geq 1; p|l \text{ for some } p > P} \frac{\chi_D(l)}{l} e^{-l/N},
\]
\[
\Delta_{P, N}^{(2.3)}(n) := \sum_{D, v \geq 1; Dv^2 = n^2 - 4} \frac{1}{v} \prod_{q|Q} \left\{ 1 + \left( \frac{2}{q} \right) \frac{q^2 | D}{q^2 \nmid D} \right\} \sum_{l \geq 1; p|l, p \leq P} \frac{\chi_D(l)}{l} \left( e^{-l/N} - 1 \right).
\]

For the following approximations we will use two lemmas:

**Lemma 6.3 (Sarnak)**
\[
\sum_{2 < n \leq x} 1 \sim \text{const} \cdot x.
\]

**Proof.** [15, Lemma 4.2] ■
Lemma 6.4 (Peter) For \( l, v \in \mathbb{N} \) and \( x \geq 3 \), we have
\[
\sum_{2 < n \leq x \atop d \geq 1 \atop dv^2 = n^2 - 4} x \chi_d(l) \ll \frac{x}{v^2 - \varepsilon K(l)} + v^\varepsilon l,
\]
where \( K(l) \) is the squarefree kernel of \( l \) and \( \varepsilon > 0 \) is arbitrary.

**Proof.** See [12], estimate (2.7). 

Lemma 6.5 For \( P \geq Q \) and \( x, N \geq 1 \), we have
\[
\frac{1}{x} \sum_{2 < n \leq x} \left| \Delta_{P,N}^{(2,3)}(n) \right|^2 \ll \left( \frac{N^{-1/2} + \sum_{l > \sqrt{N}; l \mid p \Rightarrow p \leq P} \frac{1}{l}}{N} \right)^2.
\]

**Proof.** Since \( |e^{-u} - 1| \ll u \) for \( 0 \leq u \leq 1 \), we see that for \( n > 2 \) the inner sum in \( \Delta_{P,N}^{(2,3)}(n) \) is
\[
\ll \sum_{l \geq 1; p \mid l \Rightarrow p \leq P} \frac{1}{l} \left| e^{-l/N} - 1 \right| \ll \sum_{l > \sqrt{N}; p \mid l \Rightarrow p \leq P} \frac{2}{l} + \sum_{1 < l \leq \sqrt{N}; p \mid l \Rightarrow p \leq P} \frac{1}{l/N} \ll \sum_{l > \sqrt{N}; p \mid l \Rightarrow p \leq P} \frac{1}{l} + N^{-1/2} =: c_1(P, N).
\]

Cauchy’s inequality and (6) give
\[
\sum_{2 < n \leq x} \left| \Delta_{P,N}^{(2,3)}(n) \right|^2 \ll \sum_{2 < n \leq x} \left( \sum_{D, v \geq 1; Dv^2 = n^2 - 4} \frac{\omega(Q)}{v} \right)^2 c_1(P, N)^2 \ll c_1(P, N)^2 \sum_{2 < n \leq x} \left( \sum_{D, v \geq 1; Dv^2 = n^2 - 4} \frac{\omega(Q)}{v^2} \right) \left( \sum_{D, v \geq 1; Dv^2 = n^2 - 4} 1 \right) \ll c_1(P, N)^2 \sum_{2 < n \leq x \atop D, v \geq 1; Dv^2 = n^2 - 4} 1.
\]

By using 6.3 the result follows. 

Lemma 6.6 For \( P \geq Q \) and \( x, N \geq 1 \), we have
\[
\frac{1}{x} \sum_{2 < n \leq x} \left| \Delta_{P,N}^{(2,2)}(n) \right|^2 \ll \sum_{l > P^2} \frac{\tau(l)}{lK(l)} + \frac{1}{x^{1/3 - \varepsilon}} N^2
\]
where \( \tau(l) \) is the number of positive divisors of \( l \).
Proof. Let $\alpha > \frac{1}{2}$. We write

$$\Delta_{P,N}^{(2,2)}(n) = \sum_{D,v \geq 1 \atop v > n^\alpha} \frac{1}{v} \prod_{q\mid Q} \left\{ 1 + \left( \frac{D}{q} \right) \frac{q^2 \mid D}{q^2 \mid D} \right\} \sum_{l \geq 1; p|l \text{ for some } p > P} \frac{\chi_D(l)}{l} e^{-l/N} +$$

$$+ \sum_{D,v \geq 1 \atop Dv^2 = n^2 - 4 \atop v > n^\alpha} \frac{1}{v} \prod_{q\mid Q} \left\{ 1 + \left( \frac{D}{q} \right) \frac{q^2 \mid D}{q^2 \mid D} \right\} \sum_{l \geq 1; p|l \text{ for some } p > P} \frac{\chi_D(l)}{l} e^{-l/N} =$$

$$\Delta_{P,N}^{(2,2,1)}(n) + \Delta_{P,N}^{(2,2,2)}(n).$$

A trivial estimate gives

$$\Delta_{P,N}^{(2,2,2)}(n) \leq \sum_{D,v \geq 1 \atop Dv^2 = n^2 - 4 \atop v > n^\alpha} \frac{2^{\omega(Q)}}{v} \sum_{l \geq 1; p|l \text{ for some } p > P} \frac{\chi_D(l)}{l} e^{-l/N} \leq \frac{1}{v} \sum_{l \geq 1; p|l \text{ for some } p > P} \sum_{l \geq 1; p|l \text{ for some } p > P} \frac{1}{l} e^{-l/N} \ll$$

$$\log N \sum_{D,v \geq 1 \atop Dv^2 = n^2 - 4 \atop v > n^\alpha} \frac{1}{v} \ll \log N \cdot \frac{1}{n^\alpha} \tau(n^2 - 4) \ll \log N \cdot \frac{1}{n^{\alpha - \varepsilon}}.$$

Thus, since $\alpha > \frac{1}{2}$, we have

$$\sum_{2 < n \leq x} \left| \Delta_{P,N}^{(2,2,2)}(n) \right|^2 \ll (\log N)^2 \cdot \sum_{2 < n \leq x} \frac{1}{n^{2(\alpha - \varepsilon)}} \ll (\log N)^2.$$  \hfill (7)

By Cauchy’s inequality

$$\left| \Delta_{P,N}^{(2,2,1)}(n) \right| \leq \sum_{D,v \geq 1 \atop Dv^2 = n^2 - 4 \atop v \leq n^\alpha} \frac{2^{\omega(Q)}}{v^2} \left| \sum_{l \geq 1; p|l \text{ for some } p > P} \frac{\chi_D(l)}{l} e^{-l/N} \right| \leq$$

$$\left( \sum_{D,v \geq 1; Dv^2 = n^2 - 4} \frac{2^{\omega(Q)}}{v^2} \right)^{\frac{1}{2}} \cdot \left( \sum_{D,v \geq 1; Dv^2 = n^2 - 4; v \leq n^\alpha} \left| \sum_{l \geq 1; p|l \text{ for some } p > P} \frac{\chi_D(l)}{l} e^{-l/N} \right|^2 \right)^{\frac{1}{2}}.$$

Thus for $x \geq 1$,

$$\sum_{2 < n \leq x} \left| \Delta_{P,N}^{(2,2,1)}(n) \right|^2 \ll \sum_{2 < n \leq x} \sum_{D,v \geq 1; Dv^2 = n^2 - 4; v \leq n^\alpha} \left| \sum_{l \geq 1; p|l \text{ for some } p > P} \frac{\chi_D(l)}{l} e^{-l/N} \right|^2 =$$
\[
\sum_{l_1, l_2 > p, |l_i| \text{ for some } p > P} \frac{1}{l_1 l_2} e^{-(l_1 + l_2)/N} \sum_{1 \leq v \leq x^{\alpha}} \sum_{D \geq 1} \chi_D(l_1 l_2).
\]

Applying Peter’s lemma 6.4 to the innermost sum gives the estimate

\[
\sum_{2 < n \leq x} |\Delta_{p, N}^{(2, 2, 1)}(n)|^2 \ll \sum_{l > P^2} \frac{\tau(l)}{l K(l)} + \sum_{l_1, l_2 \geq 1} \frac{l_1 l_2}{l_1 l_2} e^{-(l_1 + l_2)/N} \sum_{1 \leq v \leq x^{\alpha}} v^\varepsilon
\]

\[
\ll x \sum_{l > P^2} \frac{\tau(l)}{l K(l)} + N^2 x^{\alpha(1 + \varepsilon)}.
\]

Thus together with (7), for \(\alpha = 2/3 > 1/2\) we have

\[
\frac{1}{x} \sum_{2 < n \leq x} |\Delta_{p, N}^{(2, 2)}(n)|^2 \ll \sum_{l > P^2} \frac{\tau(l)}{l K(l)} + \frac{1}{x} (\log N)^2 + \frac{1}{x^{1/3 - \varepsilon}} N^2.
\]

In order to estimate \(\Delta_{p, N}^{(2, 1)}(n)\) we must show that the error

\[
I(D, N) := L(1, \chi_D) - \sum_{l \geq 1} \frac{\chi_D(l)}{l} e^{-l/N},
\]

which comes from smoothing Dirichlet series expansion of \(L(1, \chi_D)\), is small for large \(N\).

**Lemma 6.7** For \(1/2 < \sigma_0 < 1\) define the rectangle

\[
R_x := \{ s \in \mathbb{C} \mid \sigma_0 \leq \text{Re}(s) \leq 1, |\text{Im}(s)| \leq \log^2 x \}.
\]

(a) If \(L(s, \chi_D)\) has no zeros in \(R_x\) and \(D \leq x^2\), then

for \(\text{Re}(s) = \kappa, |\text{Im}(s)| \leq (\log x)^2/2\)

\[
I(D, N) \ll x^{\varepsilon} N^{(\kappa - 1)};
\]

(b) If \(L(s, \chi_D)\) has zeros in \(R_x\), then

\[
\# \{ (n, v, D) \mid 2 < n \leq x, D, v \geq 1, n^2 - Dv^2 = 4, L(s, \chi_D) \text{ has zeros in } R_x \} \ll x^\mu + \varepsilon,
\]

where \(\mu := 8(1 - \sigma_0)/\sigma_0 < 1, \sigma_0 < \kappa < 1\)
Proof. See [10, Lemma 3.6].

**Lemma 6.8** There are $0 < \kappa, \mu < 1$ such that for $P \geq Q, x, N \geq 1$ and $\varepsilon > 0$ we have

$$
\frac{1}{x} \sum_{2 < n \leq x} \left| \triangle_{P,N}^{(2,1)} (n) \right|^2 \ll x^\varepsilon N^{2(\kappa-1)} + x^{\mu-1+\varepsilon} (\log(x^2 N))^2.
$$

**Proof.** Note that a trivial estimation gives $I(D, N) \ll \log(DN)$. Cauchy’s inequality, previous lemma and (6) give

$$
\sum_{2 < n \leq x} \left| \triangle_{P,N}^{(2,1)} (n) \right|^2 \ll \sum_{2 < n \leq x} \left( \sum_{D,v \geq 1; Dv^2 = n^2 - 4} \frac{2^{2\omega(Q)}}{v^2} \right) \sum_{D,v \geq 1; Dv^2 = n^2 - 4} \left| I(D, N) \right|^2 \ll
$$

$$
\ll \sum_{2 < n \leq x; D,v \geq 1; Dv^2 = n^2 - 4} \left( x^\varepsilon N(\kappa-1) \right)^2 + \sum_{2 < n \leq x; D,v \geq 1; Dv^2 = n^2 - 4} \log^2(DN) \ll
$$

$$
\ll x \left( x^\varepsilon N(\kappa-1) \right)^2 + x^{\mu+\varepsilon} (\log(x^2 N))^2,
$$

which proves the lemma.

Now the results are collected.

**Lemma 6.9** For $P \geq Q$, we have

$$
\| \beta_Q - \beta_{P,Q} \|_2 \ll \left( \sum_{v > P} \frac{1}{v^2} \right)^{1/2} + \left( \sum_{l > P^2} \frac{\tau(l)}{lK(l)} \right)^{1/2}.
$$

**Proof.** For $x \geq 1$ choose $N := x^{1/8}$. Then previous lemmas show that

$$
\frac{1}{x} \sum_{2 < n \leq x} \left| \triangle_{P}^{(2)} (n) \right|^2 \ll \left( x^{-1/16} + \sum_{l > x^{1/16} ; l \equiv \pm P} \frac{1}{l} \right)^2 + \sum_{l > P^2} \frac{\tau(l)}{lK(l)} +
$$

$$
+ \frac{1}{x} (\log x)^2 + \frac{1}{x^{1/12 - \varepsilon}} + x^{(\kappa-1)/4 + \varepsilon} + x^{\mu-1+\varepsilon} (\log x)^2.
$$

Since the series

$$
\sum_{l \geq 1 ; l \equiv \pm P} \frac{1}{l}
$$
converges, we have for $P \geq Q$ fixed
\[ \left\| \triangle^2_P(n) \right\|_2 \ll \sum_{l > P^2} \frac{\tau(l)}{IK(l)}. \]

Together with Lemma 6.2 this proves the claim. ■

Now we are able to prove the proposition 6.1

**Proof of the proposition 6.1.** By previous lemma we have
\[ \| \beta_Q - \beta_{P,Q} \|_2 \ll \left( \sum_{v > P} \frac{1}{v^2} \right)^{1/2} + \left( \sum_{l > P^2} \frac{\tau(l)}{IK(l)} \right)^{1/2}; \]
here
\[ \sum_{v > P} \frac{1}{v^2} \to 0, \]
as $P \to \infty$, since the series $\sum_{v \geq 1} \frac{1}{v^2}$ converges. Furthermore,
\[ \sum_{l > P^2} \frac{\tau(l)}{IK(l)} \to 0, \]
as $P \to \infty$, since
\[ \sum_{l > P^2} \frac{\tau(l)}{IK(l)} \ll \sum_{a, b \geq 1; a \text{ squarefree}} \frac{\tau(ab^2)}{ab^2 \cdot a} \ll \sum_{a \geq 1} \sum_{b \geq 1} \frac{b^{2s}}{b^2} < \infty. \]

Thus $\lim_{P \to \infty} \| \beta_Q - \beta_{P,Q} \|_2 = 0$. For $f : \mathbb{N} \to \mathbb{C}$ arbitrary and $1 \leq s \leq 2$ we have $\| f \|_s \leq \| f \|_2$ by Hölder’s inequality. Thus $\lim_{P \to \infty} \| \beta_Q - \beta_{P,Q} \|_s = 0$, for all $1 \leq s \leq 2$ and, in particular $\lim_{P \to \infty} \| \beta_Q - \beta_{P,Q} \|_1 = 0$.

Since the $b$-th summand of $\beta_{p,Q}$ is $p^{2b+1}$-periodic for $p \nmid Q$, $2^{2b+3}$-periodic in case $p = 2$, and $p^{2b+2}$-periodic in case $p \mid Q$, and the series representing $\beta_{p,Q}$ is uniformly convergent, the function $\beta_{p,Q}$ is uniformly limit periodic, i.e. $\beta_{p,Q} \in \mathcal{D}^u$; here $\mathcal{D}^u$ is the set of all functions which can be approximated to an arbitrary accuracy by periodic functions with respect to the supremum norm. Since $\mathcal{D}^u$ is closed under multiplication it follows from Lemma 4.1, that $\beta_{p,Q} \in \mathcal{D}^u$ for all $P \geq Q$. This gives $\beta_Q \in \mathcal{D}^u$ for all $1 \leq s \leq 2$. ■

So we have now
\[ \hat{\beta}_Q(0) := M(\beta_Q) := \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \beta_Q(n). \]

One can prove the
Lemma 6.10 For $b \in \mathbb{N}$, $a \in \mathbb{Z}$, $\gcd(a, b) = 1$, choose $a_p \in \mathbb{Z}$ for all $p \mid b$ such that $\sum_{p \mid b} a_p p^{-\ord_p b} \equiv ab^{-1} \pmod{1}$. Then

$$\hat{\beta}_Q(a b) = \prod_{p \mid b} \hat{\beta}_{p,Q}(a p^{\ord_p b}).$$

(8)

Proof. Word by word the proof of the same fact in [10, Lemma 4.3]

Corollary 6.11

$$\lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \beta_Q(n) = \hat{\beta}_Q(0) = 1$$

By Parseval’s equality and by previous lemma and corollary

$$M(\hat{\beta}) := \lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta_Q^2(n) = \sum_{b \geq 1} \sum_{1 \leq a \leq b \atop \gcd(a, b) = 1} \left| \hat{\beta}_Q \left( \frac{a}{b} \right) \right|^2 = \prod_{p \text{ prime}} \left( 1 + \sum_{c \geq 1} \sum_{1 \leq a \leq p^c \atop a \equiv 0 \pmod{p}} \left| \hat{\beta}_{p,Q} \left( \frac{a}{p^c} \right) \right|^2 \right).$$

(9)

Here the term 1 in a brackets is a contribution of $c = 0$, that is $\left| \hat{\beta}_{p,Q}(0) \right|^2$.

7 Calculating the mean square of weighted multiplicities $\beta_Q(n)$

In this section we will prove Theorem 1.2.

Define the functions

$$\beta_{p,Q,b}(n) := \left( 1 - \frac{1}{p} \chi(n^2 - 4p^{-2b})(p) \right)^{-1} \cdot I_{p^b}(n),$$

and calculate the Fourier coefficients of the $\beta_{p,Q}(n)$ by the Fourier coefficients of the $\beta_{p,Q,b}(n)$.

$$\hat{\beta}_{p,Q}(r) = \sum_{b \geq 0} \frac{1}{p^b} \beta_{p,Q,b}(r)$$

(10)

In [10] was proved that for all $p \nmid Q$,

$$\hat{\beta}_{p,Q}(0) = 1$$

(11)

We will prove that $\hat{\beta}_{q,Q}(0) = 1$ holds as well, for all $q \mid Q$. 
7.1 Calculation of the period of $\beta_{(q,Q,b)}(n)$

Let us calculate now the minimal period of the function defined above:

$$
\beta_{(q,Q,b)}(n) := \left(1 - \frac{1}{q^{(n^2-4)q^{-2b}(q)}}\right)^{-1} \cdot \mathbb{I}_q(n)
$$

**Lemma 7.1** For $q \mid Q$ the minimal period of the $\beta_{(q,Q,b)}(n)$ is $q^{2b+2}$.

**Proof.** a) we will find a period of a $\chi$. We will look for a minimal $k$ such that $\chi(n + k) = \chi(n)$ for all $n$. That is

$$
\left(\frac{(n^2 - 4) q^{-2b}}{q}\right) = \left(\frac{(n + k)^2 - 4}{q} q^{-2b}\right) = \left(\frac{(n^2 - 4) q^{-2b} + (2nk + k^2) q^{-2b}}{q}\right)
$$

and it’s true for $k = q^{2b+1}$.

b) we will find a period of the $\mathbb{I}_q(n)$. We will look for a minimal $k$ such that $\mathbb{I}_q(n + k) = \mathbb{I}_q(n)$ for all $n$.

$$
n^2 - 4 \equiv 0(\text{mod } q^{2b}) \iff (n + k)^2 - 4 \equiv 0(\text{mod } q^{2b}) \iff 
$$

$$
n^2 - 4 + 2nk + k^2 \equiv 0(\text{mod } q^{2b}) \iff 2nk + k^2 \equiv 0(\text{mod } q^{2b}),
$$

and it’s true for $k = q^{2b}$.

$$
\left(\frac{n^2 - 4}{q} q^{-2b}\right) \equiv 0(\text{mod } q^2) \iff \left(\frac{(n + k)^2 - 4}{q} q^{-2b}\right) \equiv 0(\text{mod } q^2) \iff 
$$

$$
\left(\frac{n^2 - 4}{q} q^{-2b} + (2nk + k^2) q^{-2b}\right) \equiv 0(\text{mod } q^2) \iff \left(\frac{2nk + k^2}{q^2}\right) q^{-2b} \equiv 0(\text{mod } q^2),
$$

and it’s true for $k = q^{2b+2}$.

So the minimal period of the $\beta_{(q,Q,b)}(n)$ is $q^{2b+2}$ that is $\beta_{(q,Q,b)}(n + q^{2b+2}) = \beta_{(q,Q,b)}(n)$ for all $n$. 

7.2 Calculation of the Fourier coefficients $\hat{\beta}_{(q,Q,b)}(r)$ and $\hat{\beta}_{(q,Q)}(r)$

**Theorem 7.2** For any prime $q \mid Q$ the Fourier coefficients $\hat{\beta}_{(q,Q,b)}(\frac{a}{q^e})$ are:
\[ c = 0, \quad b = 0, \quad \widehat{\beta}_{(q,Q,0)}(0) = 1 - \frac{2}{q^2(q-1)} \]

\[ c = 0, \quad b \neq 0, \quad \widehat{\beta}_{(q,Q,b)}(0) = \frac{2(q^2 + q + 1)}{q^{2b+2}} \]

\[ c = 2b + 2, \quad b \neq 0, \quad \widehat{\beta}_{(q,Q,b)}\left(\frac{a}{q^c}\right) = \frac{2}{q^{2b+2}} \cos\left(\frac{4\pi a}{q^c}\right) \]

\[ c = 2b + 1, \quad b \neq 0, \quad \widehat{\beta}_{(q,Q,b)}\left(\frac{a}{q^c}\right) = \frac{1}{q^{2b+2}} \left(1 - \frac{1}{q}\right)^{-1} q^\frac{3}{2} \epsilon_q \times \]

\[ e^{-4\pi i \frac{a}{q}} \left[ e^{-2\pi i \frac{a}{q}} - e^{-2\pi i \frac{a}{q}} \cos\left(\frac{4\pi a}{q^c}\right) \right] - \frac{2}{q-1} \cos\left(\frac{4\pi a}{q^c}\right), \]

where \( \epsilon_q = \begin{cases} 1, & q \equiv 1 \text{(mod 4)} \\ i, & q \equiv 3 \text{(mod 4)} \end{cases} \)

\[ c \leq 2b, \quad b \neq 0, \quad \widehat{\beta}_{(q,Q,b)}\left(\frac{a}{q^c}\right) = \frac{2}{q^{2b+2}} \cos\left(\frac{4\pi a}{q^c}\right) (q^2 + q + 1) \]

\[ c = 1, \quad b = 0, \quad \widehat{\beta}_{(q,Q,0)}\left(\frac{a}{q}\right) = -\frac{2}{q^2(q-1)} \cos\left(\frac{4\pi a}{q}\right) + \frac{1}{q-1} \sum_{n \equiv 0 \text{(mod } q)} \left(\frac{n^2 - 4}{q}\right) e^{-2\pi i n \frac{a}{q}} \]

\[ c = 2, \quad b = 0, \quad \widehat{\beta}_{(q,Q,0)}\left(\frac{a}{q^2}\right) = \frac{2}{q^2} \cos\left(\frac{4\pi a}{q^2}\right). \]

First we calculate \( \widehat{\beta}_{(q,Q,b)}(0) \).

**Remark 7.3** In all of the sums below we want to be sure that the function \( \beta_{(q,Q,b)}(n) \) is defined at \( n \), that is \( n \) is a trace of some element of \( \Gamma_0(Q) \). The necessary and sufficient condition for \( n \) to be a trace of some element of \( \Gamma_0(Q) \) is the condition that \( \left(\frac{n^2 - 4}{q}\right) \neq -1 \) for all \( q \mid Q \). We will easily see that those \( n \), for which \( \left(\frac{n^2 - 4}{q}\right) \neq -1 \) do not contribute to the sum. That is why we can sum over all \( n \) of the given range without any restriction.

a) \( b = 0 \),
\[ \hat{\beta}_{(q,Q)}(0) = \frac{1}{q^2} \sum_{n \pmod{q^2}} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right) \right)^{-1} \left\{ \begin{array}{ll}
2, & q^2 \mid n^2 - 4 \\
1, & q^2 \nmid n^2 - 4 \end{array} \right\} \]

\[ = \frac{1}{q^2} \sum_{n \pmod{q^2}} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right) \right)^{-1} \cdot \left\{ \begin{array}{ll}
2, & n = \pm 2, \text{ or } \left( \frac{n^2 - 4}{q} \right) = 1 \\
1, & n \neq \pm 2, q \mid n^2 - 4 \end{array} \right\} \]

\[ = \frac{1}{q^2} \left( 2 \cdot 2 + \# \{ n \pmod{q^2} \mid \left( \frac{n^2 - 4}{q} \right) = 1 \} \right) \left( 1 - \frac{1}{q} \right)^{-1} + \]
\[ \# \{ n \neq \pm 2 \pmod{q^2} \mid q \mid n^2 - 4 \} = \frac{1}{q^2} \left( 2 \cdot 2 + \frac{2q - 3}{2} \cdot q \left( 1 - \frac{1}{q} \right)^{-1} + 2(q - 1) \right) \]
\[ = \frac{1}{q^2} (4 + q(q - 3) - \frac{q}{q - 1} + 2(q - 1)) = 1 - \frac{2}{q^2(q - 1)}. \]

b) \( b \neq 0, \)

\[ \hat{\beta}_{(q,Q,b)}(0) = \frac{1}{q^{2b+2}} \sum_{n \pmod{q^{2b+2}}} \left( 1 - \frac{1}{q} \left( \frac{(n^2 - 4)q^{-2b}}{q} \right) \right)^{-1} \times \]
\[ \times \left\{ \begin{array}{ll}
2, & n^2 = 4 \pmod{q^{2b}}, q^2 \mid (n^2 - 4)q^{-2b} \\
0, & n^2 = 4 \pmod{q^{2b}}, q^2 \nmid (n^2 - 4)q^{-2b} \end{array} \right\} = \]
\[ = \frac{1}{q^{2b+2}} \left( 2 \cdot 2 + \left( 1 - \frac{1}{q} \right)^{-1} \right) \# \{ n \pmod{q^{2b+2}} \mid \left( \frac{(n^2 - 4)q^{-2b}}{q} \right) = 1 \} \] + \# \{ n \neq \pm 2 \pmod{q^{2b+2}} \mid \left( \frac{(n^2 - 4)q^{-2b}}{q^{2b+1}} \right) = 1 \} \}

**Lemma 7.4** The cardinality of the set

\[ \left\{ n \pmod{q^{2b+2}} \mid \left( \frac{n^2 - 4}{q^{2b+1}} \right) = 1 \right\} \]

is \( q(q - 1), \)

and the cardinality of the set

\[ \left\{ n \neq \pm 2 \pmod{q^{2b+2}} \mid \left( \frac{n^2 - 4}{q^{2b+1}} \right) = 1 \right\} \]

is \( 2q - 2. \)
**Proof.** a) There are $2q^2$ numbers $n$ modulo $q^{2b+2}$ such that $n^2 - 4 \equiv 0 \pmod{q^{2b}}$. They are of the form $kq^b$, where $k = \pm 1, \pm 2, \ldots, \pm q^2$. So it’s need to check how many of the $k$’s are squares modulo $q$. There are $(q - 1)/2 = q(q - 1)$ numbers in the first set.

b) The number of $n \neq \pm 2$ modulo $q^{2b+2}$ such that $n^2 = 4 \pmod{q^{2b+1}}$ is $2q - 2$.

So

$$\hat{\beta}_{(q,Q,b)}(0) = \frac{1}{q^{2b+2}} \left( 4 + \frac{2}{q-1}q(q-1) + 2q - 2 \right) = \frac{2(q^2 + q + 1)}{q^{2b+2}}.$$ 

From the relation (10) it follows that

$$\hat{\beta}_{(q,Q)}(0) = \sum_{b \geq 0} \frac{1}{q^b} \hat{\beta}_{(q,Q,b)}(0) = 1 - \frac{2}{q^2(q-1)} + \sum_{b \geq 1} \frac{1}{q^b} \frac{2(q^2 + q + 1)}{q^{2b+2}} = 1 - \frac{2}{q^2(q-1)} + \frac{2(q^2 + q + 1)}{q^2} \frac{1}{q^3 - 1} = 1.$$ 

Now we will compute the Fourier coefficients $\hat{\beta}_{(q,Q,b)}(\frac{a}{q^b})$.

a) $b \neq 0$, $c = 2b + 2$.

We will need a lemmas before we will start.

**Lemma 7.5** If $q \nmid a$, then

$$\sum_{k=1}^{q^2} \left( \frac{k}{q} \right) e^{-2\pi i \frac{k^a}{q^b}} = 0,$$

for $q$ prime.

**Proof.** Note that

$$\left( \frac{m}{q} \right) = \left( \frac{m + lq}{q} \right),$$

for $l \in \mathbb{Z}$. Now we can write

$$\sum_{k=1}^{q^2} \left( \frac{k}{q} \right) e^{-2\pi i \frac{k^a}{q^b}} = \sum_{m=1}^{q} \sum_{l=0}^{q-1} \left( \frac{m + lq}{q} \right) e^{-2\pi i (m + lq) \frac{a}{q^b}} =$$
\[
\sum_{m=1}^{q} \sum_{l=0}^{q-1} \left( \frac{m}{q} \right) e^{-2\pi im\frac{c}{q}} e^{-2\pi il\frac{2b}{q}} = \sum_{m=1}^{q} \left( \frac{m}{q} \right) e^{-2\pi im\frac{c}{q}} \sum_{l=0}^{q-1} e^{-2\pi il\frac{2b}{q}}.
\]

But
\[
\sum_{l=0}^{q-1} e^{-2\pi il\frac{2b}{q}} = 0
\]

and hence
\[
\sum_{k=1}^{q} \left( \frac{k}{q} \right) e^{-2\pi ik\frac{c}{q}} = 0,
\]
as desired. ■

**Lemma 7.6** For \( c = 2b + 2 \)

\[
\sum_{n \neq \pm 2 (\text{mod } q^{2b+2})} \frac{(n^2 - 4)q^{-2b}}{q^{2b+1}n^2 - 4 q^{2b}n^2 - 4} e^{-2\pi in\frac{c}{q}} = 0.
\]

**Proof.**

\[
\sum_{n \neq \pm 2 (\text{mod } q^{2b+2})} \frac{(n^2 - 4)q^{-2b}}{q^{2b+1}n^2 - 4} e^{-2\pi in\frac{c}{q}} =
\]

\[
e^{-4\pi i\frac{c}{q}} \sum_{k=1, \frac{k}{q} \neq k}^{q-1} \left( 1 + \frac{k}{q} \right) \left( 1 - \frac{1}{q} \frac{k}{q} \right) e^{-2\pi ikq^{2b-c}} +
\]

\[
e^{4\pi i\frac{c}{q}} \sum_{k=1, \frac{k}{q} \neq k}^{q-1} \left( 1 - \frac{1}{q} \frac{k}{q} \right) e^{-2\pi ikq^{2b-c}} = \{ c = 2b + 2 \} =
\]

\[
e^{-4\pi i\frac{c}{q}} \sum_{k=1, \left( \frac{k}{q} \right) = 1}^{q-1} 2 \left( 1 - \frac{1}{q} \right) e^{-2\pi ik\frac{c}{q}} +
\]

\[
e^{4\pi i\frac{c}{q}} \sum_{k=1, \left( \frac{k}{q} \right) = 1}^{q-1} 2 \left( 1 - \frac{1}{q} \right) e^{-2\pi ik\frac{c}{q}}
\]
Mean Square of Weighted Multiplicities

\[ = e^{-4\pi i \frac{a}{q}} \cdot 2 \left( 1 - \frac{1}{q} \right)^{-1} \cdot \frac{1}{2} \sum_{\substack{k=1 \\ q \Vert k}} q^{2-1} e^{-2\pi i k \frac{a}{q^2}} \left( 1 + \left( \frac{k}{q} \right) \right) + \]

\[ e^{4\pi i \frac{a}{q}} \cdot 2 \left( 1 - \frac{1}{q} \right)^{-1} \cdot \frac{1}{2} \sum_{\substack{k=1 \\ q \Vert k}} q^{2-1} e^{-2\pi i k \frac{a}{q^2}} \left( 1 + \left( \frac{-k}{q} \right) \right) \]

\[ = e^{-4\pi i \frac{a}{q}} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{\substack{k=1 \\ q \Vert k}} \left( \frac{k}{q} \right) e^{-2\pi i k \frac{a}{q^2}} + e^{4\pi i \frac{a}{q}} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{\substack{k=1 \\ q \Vert k}} e^{-2\pi i k \frac{a}{q^2}} + \]

\[ e^{4\pi i \frac{a}{q}} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{\substack{k=1 \\ q \Vert k}} \left( -\frac{k}{q} \right) e^{-2\pi i k \frac{a}{q^2}} + e^{4\pi i \frac{a}{q}} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{\substack{k=1 \\ q \Vert k}} e^{-2\pi i k \frac{a}{q^2}}. \]

By the previous lemma and the fact that

\[ \sum_{\substack{k=1 \\ q \Vert k}} q^{2-1} e^{-2\pi i k \frac{a}{q^2}} = 0 \]

the expression we want to compute is equal to 0. ■

Now it will be much easier to compute what we want to compute:

\[ \hat{\beta}(q,Q,b) \left( \frac{a}{q^2} \right) = \frac{1}{q^{2b+2}} \sum_{n \equiv q^{2b+2} \mod q} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4q^{-2b}}{q} \right) \right)^{-1} \times \]

\[ \times \begin{cases} 2, & n^2 = 4 \mod q^{2b}, \quad q^2 \mid (n^2 - 4q^{-2b}) \\ \left( \frac{n^2 - 4q^{-2b}}{q} \right), & n^2 = 4 \mod q^{2b}, \quad q^2 \nmid (n^2 - 4q^{-2b}) \\ 0, & \text{else} \end{cases} e^{-2\pi i n \frac{a}{q^2}} \]

\[ = \frac{1}{q^{2b+2}} \sum_{n=2 \equiv q^{2b+2}} 2 e^{-2\pi i n \frac{a}{q^2}} + \]

\[ \sum_{\substack{n \equiv q^{2b+2} \mod q^2 \nmid (n^2 - 4q^{-2b}) \mid (n^2 - 4q^{-2b})}} \left( 1 - \frac{1}{q} \left( \frac{(n^2 - 4q^{-2b})}{q} \right) \right)^{-1} \left( 1 + \left( \frac{(n^2 - 4q^{-2b})}{q} \right) \right) e^{-2\pi i n \frac{a}{q^2}} \]
\[ = \frac{1}{q^{2b+2}} \left( \sum_{n=-2+q^{2b+2}}^{n=2} 2e^{-2\pi in\frac{a}{q}} + \sum_{n \neq \pm 2 (\text{mod } q^{2b+2}) \atop q^{2b+1}|n^2-4} e^{-2\pi in\frac{a}{q}} + \sum_{n \neq \pm 2 (\text{mod } q^{2b+2}) \atop q^{2b+1}|n^2-4} \left( 1 - \frac{1}{q} \left( \frac{(n^2-4)q^{-2b}}{q} \right) \right)^{-1} \left( 1 + \left( \frac{(n^2-4)q^{-2b}}{q} \right) \right) e^{-2\pi in\frac{a}{q}} \right) \]

\[ = \frac{1}{q^{2b+2}} \left( \sum_{n=-2+q^{2b+2}}^{n=2} 2e^{-2\pi in\frac{a}{q}} + 2\cos \left( \frac{4\pi a}{q^c} \right) \sum_{k=1}^{q-1} e^{-2\pi ikaq^{2b-c+1}} + 0 \right). \]

by the previous lemma. Note that
\[ \sum_{k=1}^{q-1} e^{-2\pi ikaq^{2b-c+1}} = -1 \]
when \( c = 2b + 2 \). Therefore
\[ \hat{\beta}(q,Q,b) \left( \frac{a}{q^c} \right) = \frac{1}{q^{2b+2}} \left( 2e^{-4\pi i\frac{a}{q}} + 2e^{4\pi i\frac{a}{q}} + 2\cos \left( \frac{4\pi a}{q^c} \right) \cdot (-1) \right) = \frac{1}{q^{2b+2}} \cos \left( \frac{4\pi a}{q^c} \right) \cdot (4 - 2) = \frac{2}{q^{2b+2}} \cos \left( \frac{4\pi a}{q^c} \right). \]

b) \( b \neq 0, c = 2b + 1 \).

The same way as before we will need two lemmas:

**Lemma 7.7**
\[ \sum_{k=1}^{q^2} \left( \frac{k}{q} \right) e^{-2\pi ika} = q^2 \left( \frac{-a}{q} \right) \epsilon_q, \text{ where } \epsilon_q = \begin{cases} 1, & q \equiv 1 \pmod{4} \\ i, & q \equiv 3 \pmod{4} \end{cases} \].
Proof.

\[
\sum_{k=1}^{q^2} \left( \frac{k}{q} \right) e^{-2\pi i k \frac{a}{q}} = \sum_{m=1}^{q} \sum_{l=0}^{q-1} \left( \frac{m + lq}{q} \right) e^{-2\pi i (m + lq) \frac{a}{q}} = \\
= \sum_{m=1}^{q} \sum_{l=0}^{q-1} \left( \frac{m}{q} \right) e^{-2\pi i m \frac{a}{q}} e^{-2\pi i l \frac{a}{q}} = q \sum_{m=1}^{q} \left( \frac{m}{q} \right) e^{-2\pi i m \frac{a}{q}} = \\
= q \left( \frac{-1}{q} \right) \sum_{m=1}^{q} \left( \frac{-m}{q} \right) e^{2\pi i (-m) \frac{a}{q}} = q \left( \frac{-1}{q} \right) \sum_{m=1}^{q} \left( \frac{m}{q} \right) e^{2\pi i m \frac{a}{q}} = \\
= q \left( \frac{-1}{q} \right) \frac{a}{q} \sum_{m=1}^{q} \left( \frac{m}{q} \right) e^{2\pi i m \frac{a}{q}}.
\]

The equality holds by using of property of the Gaussian sum. The value of the last sum is \( \epsilon_q \sqrt{q} \), where \( \epsilon_q \) defined as above. For this see, for example, [7, chapter 6]. Hence

\[
\sum_{k=1}^{q^2} \left( \frac{k}{q} \right) e^{-2\pi i k \frac{a}{q}} = q \left( \frac{-1}{q} \right) \frac{a}{q} \epsilon_q \sqrt{q} = q \left( \frac{-a}{q} \right) \epsilon_q,
\]

and we have the claim of the lemma. \( \blacksquare \)

Lemma 7.8 For \( c = 2b + 1 \), and \( \epsilon_q \) defined as before

\[
\sum_{n \neq \pm 2(\mod q^{2b+2})} \left( \frac{n^2 - 4q^{-2b}}{q^{2b}|n|^2 - 4} \right) \epsilon_{q^{2b+1}|n|^2 - 4} e^{-2\pi i c \frac{a}{q}} = \\
= \left( 1 - \frac{1}{q} \right)^{-1} q^2 \epsilon_q \left[ e^{-4\pi i \frac{q}{q} \left( \frac{-a}{q} \right)} + e^{4\pi i \frac{q}{q} \left( \frac{a}{q} \right)} \right] - 2q \left( 1 - \frac{1}{q} \right)^{-1} \cos \left( \frac{4\pi a}{q^2} \right).
\]

Proof.

\[
\sum_{n \neq \pm 2(\mod q^{2b+2})} \left( \frac{n^2 - 4q^{-2b}}{q^{2b}|n|^2 - 4} \right) \epsilon_{q^{2b+1}|n|^2 - 4} e^{-2\pi i c \frac{a}{q}} = \\
= e^{-4\pi i \frac{q}{q}} \sum_{k=1}^{q^2-1} \left( 1 + \left( \frac{k}{q} \right) \right) \left( 1 - \frac{1}{q} \left( \frac{k}{q} \right) \right)^{-1} e^{-2\pi i kaq^{2b-c}} + 
\]
\[ e^{4\pi i \frac{a}{q}} \sum_{k=1}^{q^2-1} \left( 1 + \left( \frac{-k}{q} \right) \right) \left( 1 - \frac{1}{q} \left( \frac{-k}{q} \right) \right)^{-1} e^{-2\pi i a q^{2b-c}} = \{c = 2b + 1\} = \]

\[ = e^{-4\pi i \frac{a}{q}} \sum_{k=1, (\frac{k}{q})=1}^{q^2-1} 2 \left( 1 - \frac{1}{q} \right)^{-1} e^{-2\pi i k \frac{a}{q}} + \]

\[ e^{4\pi i \frac{a}{q}} \sum_{k=1, (\frac{k}{q})=1}^{q^2-1} 2 \left( 1 - \frac{1}{q} \right)^{-1} e^{-2\pi i k \frac{a}{q}} = \]

\[ = e^{-4\pi i \frac{a}{q}} \cdot 2 \left( 1 - \frac{1}{q} \right)^{-1} \cdot \frac{1}{2} \sum_{k=1}^{q^2-1} e^{-2\pi i k \frac{a}{q}} \left( 1 + \left( \frac{k}{q} \right) \right) + \]

\[ e^{4\pi i \frac{a}{q}} \cdot 2 \left( 1 - \frac{1}{q} \right)^{-1} \cdot \frac{1}{2} \sum_{k=1}^{q^2-1} e^{-2\pi i k \frac{a}{q}} \left( 1 + \left( \frac{k}{q} \right) \right) = \]

\[ = e^{-4\pi i \frac{a}{q}} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{k=1}^{q^2-1} \left( \frac{k}{q} \right) e^{-2\pi i k \frac{a}{q}} + e^{-4\pi i \frac{a}{q}} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{k=1}^{q^2-1} e^{-2\pi i k \frac{a}{q}} + \]

\[ e^{4\pi i \frac{a}{q}} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{k=1}^{q^2-1} \left( \frac{k}{q} \right) e^{-2\pi i k \frac{a}{q}} + e^{4\pi i \frac{a}{q}} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{k=1}^{q^2-1} e^{-2\pi i k \frac{a}{q}}. \]

By the previous lemma and the fact that

\[ \sum_{k=1}^{q^2-1} e^{-2\pi i k \frac{a}{q}} = -q \]

we can write that our expression is equal to

\[ \left( 1 - \frac{1}{q} \right)^{-1} q^2 \epsilon_q \left[ e^{-4\pi i \frac{a}{q}} \left( \frac{-a}{q} \right) + e^{4\pi i \frac{a}{q}} \left( \frac{a}{q} \right) \right] - 2q \left( 1 - \frac{1}{q} \right)^{-1} \cos \left( \frac{4\pi a}{q} \right), \]

proving the claim. ■

By using the same arguments as in the case a) we will get
\[
\hat{\beta}(q,Q,b)(\frac{a}{q^c}) = \frac{1}{q^{2b+2}} \left( \sum_{n=2}^{q^{2b+2}} 2e^{-2\pi i n \frac{a}{q^c}} + \sum_{n \not\equiv \pm 2 (mod\ q^{2b+2})} e^{-2\pi i n \frac{a}{q^c}} + \sum_{n=-2+q^{2b+2}}^{q^{2b+2}+1} e^{-2\pi i n \frac{a}{q^c}} \right)
\]

\[
= \frac{1}{q^{2b+2}} \left( \sum_{n=2}^{q^{2b+2}} 2e^{-2\pi i n \frac{a}{q^c}} + 2\cos \left(\frac{4\pi a}{q^c}\right) \sum_{k=1}^{q-1} e^{-2\pi i k a} + \left(1 - \frac{1}{q}\right)^{-1} q^2 \epsilon_q \left[ e^{-4\pi i \frac{a}{q^c}} \left(\frac{-a}{q}\right) + e^{4\pi i \frac{a}{q^c}} \left(\frac{a}{q}\right) \right] - 2q \left(1 - \frac{1}{q}\right)^{-1} \cos \left(\frac{4\pi a}{q^c}\right) \right).
\]

by using previous lemma. Note that
\[
\sum_{k=1}^{q-1} e^{-2\pi i k a} = q - 1,
\]
so
\[
\hat{\beta}(q,Q,b)(\frac{a}{q^c}) = \frac{1}{q^{2b+2}} \left(4\cos \left(\frac{4\pi a}{q^c}\right) + 2(q-1) \cos \left(\frac{4\pi a}{q^c}\right) + \left(1 - \frac{1}{q}\right)^{-1} q^2 \epsilon_q \left[ e^{-4\pi i \frac{a}{q^c}} \left(\frac{-a}{q}\right) + e^{4\pi i \frac{a}{q^c}} \left(\frac{a}{q}\right) \right] - 2q \left(1 - \frac{1}{q}\right)^{-1} \cos \left(\frac{4\pi a}{q^c}\right) \right)
\]

\[
= \frac{1}{q^{2b+2}} \left( \left(1 - \frac{1}{q}\right)^{-1} q^2 \epsilon_q \left[ e^{-4\pi i \frac{a}{q^c}} \left(\frac{-a}{q}\right) + e^{4\pi i \frac{a}{q^c}} \left(\frac{a}{q}\right) \right] - \frac{2}{q-1} \cos \left(\frac{4\pi a}{q^c}\right) \right).
\]

c) \(b \neq 0, 2b \geq c.\)

By doing the same steps as before we can write
Lemma 7.9  For $2b \geq c$,

\[
\sum_{n \neq \pm 2 \pmod{q^{2b+2}} \atop q^{2b+1} \mid n^2 - 4 \atop q^n \mid n^2 - 4} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right)^{2b} \right)^{-1} \left( 1 + \frac{1}{q} \left( \frac{n^2 - 4}{q} \right)^{2b} \right) e^{-2\pi in \frac{a}{q^c}} =
\]

\[
= 2q^2 \cos \left( \frac{4\pi a}{q^c} \right). 
\]

Proof.

\[
\sum_{n \neq \pm 2 \pmod{q^{2b+2}} \atop q^{2b+1} \mid n^2 - 4 \atop q^n \mid n^2 - 4} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right)^{2b} \right)^{-1} \left( 1 + \frac{1}{q} \left( \frac{n^2 - 4}{q} \right)^{2b} \right) e^{-2\pi in \frac{a}{q^c}} =
\]

\[
eq e^{-4\pi i \frac{a}{q^c}} \sum_{k=1 \atop q \nmid k}^{q^2-1} \left( 1 + \left( \frac{k}{q} \right) \left( 1 - \frac{1}{q} \left( \frac{k}{q} \right) \right)^{-1} e^{-2\pi i k q^{2b-c}} \right) = \{2b \geq c\} =
\]

\[
eq e^{-4\pi i \frac{a}{q^c}} \sum_{k=1 \atop q \nmid k}^{q^2-1} 2 \left( 1 - \frac{1}{q} \right)^{-1} + e^{4\pi i \frac{a}{q^c}} \sum_{k=1 \atop q \nmid k}^{q^2-1} 2 \left( 1 - \frac{1}{q} \right)^{-1} =
\]

\[
eq 2q \frac{q^2-1}{q-1} \# \{ k \pmod{q^2} \mid \left( \frac{k}{q} \right) = 1 \} 2 \cos \left( \frac{4\pi a}{q^c} \right) = 4 \frac{q}{q-1} q(q-1) \cos \left( \frac{4\pi a}{q^c} \right) =
\]

\[
= 2q^2 \cos \left( \frac{4\pi a}{q^c} \right). 
\]

And gathering all together we have

\[
\beta_{\widetilde{\beta}(q,Q,b)} \left( \frac{a}{q^c} \right) = \frac{1}{q^{2b+2}} \left( \sum_{n=2 \atop n \neq \pm 2 \pmod{q^{2b+2}}}^{q^{-1}} 2e^{-2\pi in \frac{a}{q^c}} + 2 \cos \left( \frac{4\pi a}{q^c} \right) \sum_{k=1}^{q-1} e^{-2\pi i k q^{2b-c+1}} +
\right)
\]
\[
\sum_{n \neq \pm 2 (\text{mod } q^{2b+2}) \atop q^{2b+1} | n^2 - 4} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right) q^{-2b} \right)^{-1} \left( 1 + \left( \frac{n^2 - 4}{q} \right) q^{-2b} \right) e^{-2\pi i n \frac{a}{c}} \right)
\]

\[
= \frac{1}{q^{2b+2}} \left( 4 \cos \left( \frac{4\pi a}{q^c} \right) + 2(q - 1) \cos \left( \frac{4\pi a}{q^c} \right) + 2q^2 \cos \left( \frac{4\pi a}{q^c} \right) \right)
\]

\[
= \frac{1}{q^{2b+2}} \cos \left( \frac{4\pi a}{q^c} \right) \left[ 4 + 2(q - 1) + 2q^2 \right] = \frac{2}{q^{2b+2}} \cos \left( \frac{4\pi a}{q^c} \right) (q^2 + q + 1).
\]

d) \( b = 0 \), \( c = 1 \).

As always we need two lemmas:

**Lemma 7.10**

\[
\sum_{n \pmod{q^2}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi i n \frac{a}{q^c}} = q \sum_{n \pmod{q}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi i n \frac{a}{q^c}}.
\]

**Proof.**

\[
\sum_{n \pmod{q^2}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi i n \frac{a}{q^c}} = \sum_{m=1}^{q} \sum_{l=0}^{q-1} \left( \frac{(m+ql)^2 - 4}{q} \right) e^{-2\pi i (m+ql) \frac{a}{q^c}} =
\]

\[
= \sum_{m=1}^{q} \sum_{l=0}^{q-1} \left( \frac{m^2 - 4}{q} \right) e^{-2\pi i m \frac{a}{q^c}} = q \sum_{m=1}^{q} \left( \frac{m^2 - 4}{q} \right) e^{-2\pi i m \frac{a}{q^c}} =
\]

\[
= q \sum_{n \pmod{q}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi i n \frac{a}{q^c}}.
\]

**Lemma 7.11**

\[
\sum_{n \neq \pm 2 (\text{mod } q^2) \atop q | n^2 - 4} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right) \right)^{-1} \left( \left( \frac{n^2 - 4}{q} \right) + 1 \right) e^{-2\pi i n \frac{a}{q^c}} =
\]

\[
= q \left( 1 - \frac{1}{q} \right)^{-1} \left[ -2 \cos \left( \frac{4\pi a}{q^c} \right) + \sum_{n \pmod{q}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi i n \frac{a}{q^c}} \right].
\]
Proof.

\[
\sum_{n \not\equiv \pm 2 (\text{mod } q^2)} \left(1 - \frac{1}{q} \left(\frac{n^2 - 4}{q}\right)\right)^{-1} \left(\frac{n^2 - 4}{q}\right) + 1 \right) e^{-2\pi i n \frac{a}{q}} =
\]

= \sum_{n \equiv \pm 2 (\text{mod } q)} \left(1 - \frac{1}{q} \left(\frac{n^2 - 4}{q}\right)\right)^{-1} \left(\frac{n^2 - 4}{q}\right) + 1 \right) e^{-2\pi i n \frac{a}{q}} =

= \sum_{n \equiv \pm 2 (\text{mod } q)} 2 \left(1 - \frac{1}{q}\right)^{-1} e^{-2\pi i n \frac{a}{q}} =

= 2 \cdot \left(1 - \frac{1}{q}\right)^{-1} \sum_{n \equiv \pm 2 (\text{mod } q)} \left(\frac{n^2 - 4}{q}\right) + 1 \right) e^{-2\pi i n \frac{a}{q}} =

\left(1 - \frac{1}{q}\right)^{-1} \left[ \sum_{n \equiv \pm 2 (\text{mod } q)} e^{-2\pi i n \frac{a}{q}} + \sum_{n \equiv \pm 2 (\text{mod } q)} \left(\frac{n^2 - 4}{q}\right) e^{-2\pi i n \frac{a}{q}} \right] =

= \left(1 - \frac{1}{q}\right)^{-1} \left[ -2q \cos \left(\frac{4\pi a}{q}\right) + \sum_{n \equiv \pm 2 (\text{mod } q)} \left(\frac{n^2 - 4}{q}\right) e^{-2\pi i n \frac{a}{q}} \right] =

\left(1 - \frac{1}{q}\right)^{-1} \left[ -2q \cos \left(\frac{4\pi a}{q}\right) + \sum_{n \equiv \pm 2 (\text{mod } q)} \left(\frac{n^2 - 4}{q}\right) e^{-2\pi i n \frac{a}{q}} - 0 \right] =

= \left(1 - \frac{1}{q}\right)^{-1} \left[ -2q \cos \left(\frac{4\pi a}{q}\right) + q \sum_{n \equiv \pm 2 (\text{mod } q)} \left(\frac{n^2 - 4}{q}\right) e^{-2\pi i n \frac{a}{q}} \right].

by previous lemma, and that’s it. ■

Now we can gather the results

\[
\beta_{(q,Q,0)}(\frac{a}{q}) = \frac{1}{q^2} \sum_{n \equiv \pm 2 (\text{mod } q)} \left(1 - \frac{1}{q} \left(\frac{n^2 - 4}{q}\right)\right)^{-1} \left\{ \left(\frac{n^2 - 4}{q}\right) + 1, \quad q^2 \mid n^2 - 4 \right\} e^{-2\pi i n \frac{a}{q}} =
\]
\[ \frac{1}{q^2} \left( \sum_{\substack{n=2 \\ n=2, q^2}} 2e^{-2\pi inq^2} + \sum_{\substack{n\neq 2 \pmod{q^2} \\ q \mid n^2 - 4}} e^{-2\pi inq^2} + \right. \\
\left. \sum_{\substack{n\neq \pm 2 \pmod{q^2} \\ q \mid n^2 - 4}} \left(1 - \frac{1}{q} \left(\frac{n^2 - 4}{q}\right)\right)^{-1} \left(\left(\frac{n^2 - 4}{q}\right) + 1\right) e^{-2\pi inq^2} \right) = \\
\frac{1}{q^2} \left(4 \cos \left(\frac{4\pi a}{q}\right) + 2(q-1) \cos \left(\frac{4\pi a}{q}\right) + \right. \\
\left. q \left(1 - \frac{1}{q}\right)^{-1} \left[-2 \cos \left(\frac{4\pi a}{q}\right) + \sum_{n \pmod{q}} \left(\frac{n^2 - 4}{q}\right) e^{-2\pi inq^2} \right] \right) = \\
\left[-2 \cos \left(\frac{4\pi a}{q^2}\right) + \sum_{n \pmod{q}} \left(\frac{n^2 - 4}{q}\right) e^{-2\pi inq^2} \right], \\
\text{c) } b = 0, \ c = 2. \\
\text{Lemma 7.12} \\
\sum_{\substack{n \neq \pm 2 \pmod{q^2} \\ q \mid n^2 - 4}} e^{-2\pi in\frac{q^2}{q}} = -2 \cos \left(\frac{4\pi a}{q^2}\right). \\
\text{Proof.} \\
\text{The condition } \{ n \neq \pm 2 \pmod{q^2}, \ q \mid n^2 - 4 \} \text{ we can rewrite in the form } \{ n = \pm 2 + kq, \ k = 1, \ldots, q - 1 \}. \text{ Thus} \\
\sum_{\substack{n \neq \pm 2 \pmod{q^2} \\ q \mid n^2 - 4}} e^{-2\pi in\frac{q^2}{q}} = \sum_{k=1}^{q-1} e^{-2\pi i(\pm 2 + kq)\frac{q}{q^2}} = \\
2 \cos \left(\frac{4\pi a}{q^2}\right) \sum_{k=1}^{q-1} e^{-2\pi ik\frac{q}{q^2}} = -2 \cos \left(\frac{4\pi a}{q^2}\right), \\
\text{and we are done.} \]
Lemma 7.13

\[ \sum_{\substack{n \neq \pm 2 \pmod{q^2} \\ q \nmid n^2 - 4}} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right) \right)^{-1} \left( \frac{n^2 - 4}{q} + 1 \right) e^{-2\pi in \frac{a}{q^2}} = 0. \]

Proof.

\[ \sum_{\substack{n \neq \pm 2 \pmod{q^2} \\ q \nmid n^2 - 4}} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right) \right)^{-1} \left( \frac{n^2 - 4}{q} + 1 \right) e^{-2\pi in \frac{a}{q^2}} = \]

\[ = \sum_{\substack{n \pmod{q^2} \\ n \neq \pm 2 \pmod{q} \left( \frac{n^2 - 4}{q} \right) = 1}} \left( 1 - \frac{1}{q} \left( \frac{n^2 - 4}{q} \right) \right)^{-1} \left( \frac{n^2 - 4}{q} + 1 \right) e^{-2\pi in \frac{a}{q^2}} = \]

\[ = 2 \cdot \frac{1}{2} \left( 1 - \frac{1}{q} \right)^{-1} \sum_{\substack{n \pmod{q^2} \\ n \neq \pm 2 \pmod{q}}} \left( \frac{n^2 - 4}{q} + 1 \right) e^{-2\pi in \frac{a}{q^2}} = \]

\[ = \left( 1 - \frac{1}{q} \right)^{-1} \left[ \sum_{\substack{n \pmod{q^2} \\ n \neq \pm 2 \pmod{q}}} e^{-2\pi in \frac{a}{q^2}} + \sum_{\substack{n \pmod{q^2} \\ n \neq \pm 2 \pmod{q}}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi in \frac{a}{q^2}} \right]. \]

The first sum in the brackets is 0, and so we need to compute an expression

\[ \left( 1 - \frac{1}{q} \right)^{-1} \sum_{\substack{n \pmod{q^2} \\ n \neq \pm 2 \pmod{q}}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi in \frac{a}{q^2}} = \]

\[ = \left( 1 - \frac{1}{q} \right)^{-1} \sum_{n \pmod{q^2}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi in \frac{a}{q^2}} = \]

\[ = \left( 1 - \frac{1}{q} \right)^{-1} \sum_{m=l=0}^{q-1} \left( \frac{(m + lq)^2 - 4}{q} \right) e^{-2\pi im \frac{a}{q^2}} = \]

\[ = \left( 1 - \frac{1}{q} \right)^{-1} \sum_{m=1}^{q-1} \sum_{l=0}^{q-1} \left( \frac{m^2 - 4}{q} \right) e^{-2\pi im \frac{a}{q^2}} e^{-2\pi il \frac{a}{q^2}} = \]
Mean Square of Weighted Multiplicities

\[
\left(1 - \frac{1}{q}\right)^{-1} \sum_{m=1}^{q} \left(\frac{m^2 - 4}{q}\right) e^{-2\pi im \theta} \sum_{l=0}^{q-1} e^{-2\pi il \theta} = 0,
\]
since the last sum is 0.

\[
\hat{\beta}_{(q,Q,0)}(\frac{a}{q^2}) = \frac{1}{q^2} \sum_{n (\text{mod } q^2)} \left(1 - \frac{1}{q} \left(\frac{n^2 - 4}{q}\right)\right)^{-1} \left\{ \left(\frac{n^2-4}{q}\right) + 1, \begin{array}{c}
q^2 | n^2 - 4
q^2 \nmid n^2 - 4
\end{array}\right\} e^{-2\pi in \theta} =
\]

\[
= \frac{1}{q^2} \left( \sum_{n=2}^{q^2} 2e^{-2\pi in \theta} + \sum_{n \equiv \pm 2 (\text{mod } q^2)} e^{-2\pi in \theta} \right)
\]

\[
= \frac{1}{q^2} \left( 4 \cos \left(\frac{4\pi a}{q^2}\right) - 2 \cos \left(\frac{4\pi a}{q^2}\right) + 0 \right) = \frac{2}{q^2} \cos \left(\frac{4\pi a}{q^2}\right)
\]

Summarize the above results:

\[
c = 0, \quad b = 0, \quad \hat{\beta}_{(q,Q,0)}(0) = 1 - \frac{2}{q^2(q - 1)}
\]

\[
c = 0, \quad b \neq 0, \quad \hat{\beta}_{(q,Q,b)}(0) = \frac{2(q^2 + q + 1)}{q^{2b+2}}
\]

\[
c = 2b + 2, b \neq 0, \quad \hat{\beta}_{(q,Q,b)}(\frac{a}{q^c}) = \frac{2}{q^{2b+2}} \cos \left(\frac{4\pi a}{q^c}\right)
\]

\[
c = 2b + 1, b \neq 0, \quad \hat{\beta}_{(q,Q,b)}(\frac{a}{q^c}) = \frac{1}{q^{2b+2}} \left(1 - \frac{1}{q}\right)^{-1} q^3 \epsilon_q \times
\]

\[
\times \left[ e^{-4\pi i \frac{\theta}{q}} \frac{-a}{q} + e^{4\pi i \frac{\theta}{q}} \frac{a}{q} \right] - \frac{2}{q - 1} \cos \left(\frac{4\pi a}{q^c}\right)
\]
Theorem 7.14

The Fourier coefficients \( \hat{\beta}_{(q,Q,b)}(a/q^c) \) are:

\[
\begin{align*}
\hat{\beta}_{(q,Q)}(a/q) &= \frac{1}{q-1} \sum_{n \equiv b \pmod{q}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi in q} \\
\hat{\beta}_{(q,Q)}(a/q) &= \frac{2}{q(q-1)} \cos \left( \frac{4\pi a}{q^2} \right) \\
\hat{\beta}_{(q,Q)}(a/q) &= \frac{2}{q-1} \cos \left( \frac{4\pi a}{q^2} \right) \frac{1}{q\frac{2\pi}{q}} \text{ for } c > 2, c \text{ even} \\
\hat{\beta}_{(q,Q)}(a/q) &= \frac{1}{q-1} \frac{1}{q^{n(q-1)}} \left( \frac{a}{q} \right) \left[ e^{-4\pi i \frac{a}{q}} \left( -1 \right) + e^{4\pi i \frac{a}{q}} \right] \text{ for } c > 2, c \text{ odd}
\end{align*}
\]

Proof. By using (10) we get

a) for \( c = 1 \),

\[
\begin{align*}
\hat{\beta}_{(q,Q)}(a/q) &= \sum_{b \geq 0} \frac{1}{q^b} \hat{\beta}_{(q,Q,b)}(a/q) = -\frac{2}{q^2(q-1)} \cos \left( \frac{4\pi a}{q} \right) + \frac{1}{q-1} \sum_{n \equiv b \pmod{q}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi in q} + \\
\sum_{b \geq 1} \frac{1}{q^b} \hat{\beta}_{(q,Q,b)}(a/q) &= -\frac{2}{q^2(q-1)} \cos \left( \frac{4\pi a}{q} \right) + \frac{1}{q-1} \sum_{n \equiv b \pmod{q}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi in q} + \\
\sum_{b \geq 1} \frac{1}{q^b} \frac{2}{q^{2b+2}} \cos \left( \frac{4\pi a}{q} \right) \left( q^2 + q + 1 \right) &= -\frac{2}{q^2(q-1)} \cos \left( \frac{4\pi a}{q} \right) + \frac{1}{q-1} \sum_{n \equiv b \pmod{q}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi in q} +
\end{align*}
\]

The theorem is completely proved.

Now we can compute the \( \hat{\beta}_{(q,Q)}(r) \).

Theorem 7.14

The Fourier coefficients \( \hat{\beta}_{(q,Q)}(a/q^c) \) are:

\[
\begin{align*}
\hat{\beta}_{(q,Q)}(a/q^c) &= \frac{2}{q^{2c-2}} \cos \left( \frac{4\pi a}{q^c} \right) \\
\hat{\beta}_{(q,Q)}(a/q^c) &= \frac{1}{q-1} \sum_{n \equiv a \pmod{q}} \left( \frac{n^2 - 4}{q^c} \right) e^{-2\pi in \frac{a}{q}} \\
\hat{\beta}_{(q,Q)}(a/q^c) &= \frac{1}{q-1} \frac{1}{q^{n(q-1)}} \left( \frac{a}{q} \right) \left[ e^{-4\pi i \frac{a}{q}} \left( -1 \right) + e^{4\pi i \frac{a}{q}} \right] \text{ for } c > 2, c \text{ odd}
\end{align*}
\]
\[ 2 \cos \left( \frac{4\pi a}{q} \right) (q^2 + q + 1) \frac{1}{q^2(q^3 - 1)} = \frac{1}{q - 1} \sum_{n \pmod{q}} \left( \frac{n^2 - 4}{q} \right) e^{-2\pi in \frac{a}{q}}. \]

b) for \( c = 2 \),

\[ \beta(q,Q) \left( \frac{a}{q^2} \right) = \sum_{b \geq 0} \frac{1}{q^b} \beta(q,Q,b) \left( \frac{a}{q^2} \right) = \frac{2}{q^2} \cos \left( \frac{4\pi a}{q^2} \right) + \sum_{b \geq 1} \frac{1}{q^{2b+2}} \cos \left( \frac{4\pi a}{q^2} \right) (q^2 + q + 1) = \]

\[ = \frac{2}{q^2} \cos \left( \frac{4\pi a}{q^2} \right) + 2 \cos \left( \frac{4\pi a}{q^2} \right) (q^2 + q + 1) = \frac{2}{q(q-1)} \cos \left( \frac{4\pi a}{q^2} \right). \]

c) for \( c > 2 \),

\[ \beta(q,Q) \left( \frac{a}{q^c} \right) = \sum_{0 \leq b < \frac{c}{2}} \frac{1}{q^b} \cdot 0 + \sum_{b = \frac{c}{2}} \frac{1}{q^b} \cdot 2 \cos \left( \frac{4\pi a}{q^c} \right) + \]

\[ \sum_{b = \frac{c}{2}} \frac{1}{q^b} \left[ \frac{1}{q^{2b+2}} \left( \left( 1 - \frac{1}{q} \right)^{-1} q^\frac{c}{2} \epsilon_q \left[ e^{-4\pi i \frac{a}{q}} \left( \frac{-a}{q} \right) + e^{4\pi i \frac{a}{q}} \left( \frac{a}{q} \right) \right] - \frac{2}{q - 1} \cos \left( \frac{4\pi a}{q^c} \right) \right] \right] + \]

\[ \sum_{b \geq \frac{c}{2}} \frac{2}{q^{2b+2}} \cos \left( \frac{4\pi a}{q^c} \right) (q^2 + q + 1). \]

For \( c \) even we have

\[ \beta(q,Q) \left( \frac{a}{q^c} \right) = \frac{1}{q^{\frac{c}{2}}} \cdot 2 \cos \left( \frac{4\pi a}{q^c} \right) + \frac{2}{q^2} \cos \left( \frac{4\pi a}{q^c} \right) (q^2 + q + 1) \sum_{b \geq \frac{c}{2}} \frac{1}{q^{3b}} = \]

\[ = 2 \cos \left( \frac{4\pi a}{q^c} \right) \left[ \frac{1}{q^{\frac{3c}{2}}} + \frac{1}{q^2(q^2 + q + 1) \frac{1}{q^{\frac{c}{2}}} \frac{1}{q^{\frac{c}{2}}} q^3 - 1} \right] = \frac{2}{q - 1} \cos \left( \frac{4\pi a}{q^c} \right) \frac{1}{q^{\frac{3c}{2}}}. \]

And for \( c \) odd

\[ \frac{1}{q^{\frac{c}{2}+1}} \left[ \frac{1}{q^{\frac{c}{2}+1}} \left( \left( 1 - \frac{1}{q} \right)^{-1} q^\frac{c}{2} \epsilon_q \left[ e^{-4\pi i \frac{a}{q}} \left( \frac{-a}{q} \right) + e^{4\pi i \frac{a}{q}} \left( \frac{a}{q} \right) \right] - \frac{2}{q - 1} \cos \left( \frac{4\pi a}{q^c} \right) \right] \right] + \]
7.3 Calculating the mean square of weighted multiplicities function

In this subsection we will calculate the mean-square of the weighted multiplicities \( \beta_Q(n) \).

To calculate the limit

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta^2_Q(n)
\]

we will use (9). Let us define the function

\[
A_Q(p^c) := \sum_{1 \leq a \leq p^c} \left| \overline{\beta(p,Q)} \left( \frac{a}{p^c} \right) \right|^2.
\]

So we have that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta^2_Q(n) = \prod_{p \text{ prime}} \left( 1 + \sum_{c \geq 1} A_Q(p^c) \right).
\]
The values of the $A_Q(p^c)$ for $p \nmid Q$ were calculated by M. Peter [10]. They are:

$$A_Q(p) = \frac{p^2 - 2p - 1}{(p^2 - 1)^2}, \quad A_Q(p^c) = \frac{2(p - 1)}{(p^2 - 1)^2 p^{2c - 3}}$$

$$p = 2; \quad A_Q(2) = \frac{1}{9}, \quad A_Q(4) = \frac{1}{18},$$

$$A_Q(8) = 0, \quad A_Q(16) = \frac{1}{9 \cdot 16},$$

$$A_Q(32) = 0, \quad A_Q(2^c) = \frac{1}{9 \cdot 2^{2c - 5}}, c \geq 6.$$ We just need to complete his work by adding the case $q \mid Q$.

a) $c = 1$;

$$A_Q(q) = \sum_{\substack{1 \leq a \leq q \mid q \mid a \ a \equiv q \ (\mod \ q) \ a \neq q \ (\mod \ q)}} \left| \hat{\beta_{q, Q}}(\frac{a}{q}) \right|^2$$

$$= \frac{1}{(q - 1)^2} \sum_{\substack{1 \leq a \leq q \ n_1, n_2 \ (\mod \ q) \ a \neq q \ (\mod \ q)}} \left( \frac{n_1^2 - 4}{q} \right) \left( \frac{n_2^2 - 4}{q} \right) e^{2\pi i (n_1 - n_2) \frac{q}{q}} =$$

$$= \frac{1}{(q - 1)^2} \sum_{\substack{n_1, n_2 \ (\mod \ q) \ a \neq q \ (\mod \ q)}} \left( \frac{n_1^2 - 4}{q} \right) \left( \frac{n_2^2 - 4}{q} \right) \sum_{a = 1}^{q - 1} e^{2\pi i (n_1 - n_2) \frac{q}{q}}.$$

The sum

$$\sum_{a = 1}^{q - 1} e^{2\pi i (n_1 - n_2) \frac{q}{q}} = \begin{cases} q - 1, & n_1 = n_2 = n \ (\mod \ q) \\ -1, & \text{else} \end{cases}.$$

Note that

$$\sum_{n \ (\mod \ q)} \left( \frac{n^2 - 4}{q} \right) = \frac{q - 3}{2} - \frac{q - 1}{2} = -1,$$

hence

$$A_Q(q) = \frac{1}{(q - 1)^2} \left( (q - 1) \sum_{n \ (\mod \ q)} \left( \frac{n^2 - 4}{q} \right)^2 - \sum_{n_1 \ (\mod \ q) \ n_2 \neq n_1 \ (\mod \ q) \ n_2 \neq n_1 \ (\mod \ q)} \left( \frac{n_1^2 - 4}{q} \right) \left( \frac{n_2^2 - 4}{q} \right) \left( \frac{n_1^2 - 4}{q} \right)^2 \right) =$$

$$= \frac{1}{(q - 1)^2} \left( (q - 1)(q - 2) - \sum_{n_1 \ (\mod \ q)} \left( \sum_{n_2 \ (\mod \ q) \ n_2 \neq n_1 \ (\mod \ q)} \left( \frac{n_1^2 - 4}{q} \right) \left( \frac{n_2^2 - 4}{q} \right) \left( \frac{n_1^2 - 4}{q} \right)^2 \right) \right) =$$
\[
\frac{1}{(q-1)^2} \left( (q-1)(q-2) - \left( 1 - \sum_{n_1 \equiv q \mod q} \left( \frac{n_1^2 - 4}{q} \right)^2 \right) \right) = \\
\frac{1}{(q-1)^2} \left( q^2 - 3q + 2 - (1 - (q-2)) \right) = \frac{q^2 - 2q - 1}{(q-1)^2}.
\]

b) \( c = 2; \)

\[
A_Q(q^2) = \sum_{1 \leq a \leq q^2} \left| \overline{\beta(q,Q)} \left( \frac{a}{q^2} \right) \right|^2 = \sum_{1 \leq a \leq q^2} \left| \frac{2}{q(q-1)} \cos \left( \frac{4\pi a}{q^2} \right) \right|^2 = \\
\frac{4}{q^2(q-1)^2} \sum_{1 \leq a \leq q^2} \left| \cos \left( \frac{4\pi a}{q^2} \right) \right|^2 = \frac{1}{q^2(q-1)^2} \sum_{1 \leq a \leq q^2} \left( e^{8\pi i \frac{a}{q^2}} + e^{-8\pi i \frac{a}{q^2}} + 2 \right) = \\
= \frac{1}{q^2(q-1)^2} \left( 2(q^2 - q) + \sum_{1 \leq a \leq q^2} \left( e^{8\pi i \frac{a}{q^2}} + e^{-8\pi i \frac{a}{q^2}} \right) \right) = \frac{2(q^2 - q)}{q^2(q-1)^2} = \frac{2}{q(q-1)}.
\]

c1) \( c > 2, \) \( c \) even;

\[
A_Q(q^c) = \sum_{1 \leq a \leq q^c} \left| \overline{\beta(q,Q)} \left( \frac{a}{q^c} \right) \right|^2 = \sum_{1 \leq a \leq q^c} \left| \frac{2}{q(q-1)} \cos \left( \frac{4\pi a}{q^c} \right) \right|^2 = \\
\frac{1}{q^{2c-4}(q-1)^2} \sum_{1 \leq a \leq q^c} \left| \cos \left( \frac{4\pi a}{q^c} \right) \right|^2 = \frac{1}{q^{2c-4}(q-1)^2} \sum_{1 \leq a \leq q^c} \left( e^{8\pi i \frac{a}{q^c}} + e^{-8\pi i \frac{a}{q^c}} + 2 \right) = \\
= \frac{1}{q^{2c-4}(q-1)^2} \left( 2(q^c - q^{c-1}) + \sum_{1 \leq a \leq q^c} \left( e^{8\pi i \frac{a}{q^c}} + e^{-8\pi i \frac{a}{q^c}} \right) \right) = \frac{2q^{c-1}(q-1)}{q^{2c-4}(q-1)^2} = \frac{2}{q^{2c-3}(q-1)}.
\]
c2) \( c > 2, c \) odd;

\[
A_Q(q^c) = \sum_{1 \leq a \leq q^c} \left| \overline{\beta(q,Q)} \left( \frac{a}{q^c} \right) \right|^2 = \sum_{1 \leq a \leq q^c} \frac{1}{q - 1} \frac{1}{q^{c - 2}} e_q \left( \frac{a}{q} \right) \left[ e^{-4\pi i q^c} \left( -\frac{1}{q} \right) + e^{4\pi i q^c} \right] =
\]

\[
= \frac{1}{(q - 1)^2} \frac{1}{q^{3c-4}} \prod_{1 \leq a \leq q^c} \left( 1 + \frac{1}{q} \right) \left[ e^{-4\pi i q^c} \left( -\frac{1}{q} \right) + e^{4\pi i q^c} \right] =
\]

\[
= \frac{1}{(q - 1)^2} \frac{1}{q^{3c-4}} \prod_{1 \leq a \leq q^c} \left( 2 + \left( -\frac{1}{q} \right) \left[ e^{8\pi i q^c} + e^{-8\pi i q^c} \right] \right) = \frac{2q^{c-1}(q - 1)}{q^{2c-4}(q - 1)^2} = \frac{2}{q^{2c-4}(q - 1)}.
\]

And we can see that for \( c > 2 \), \( A_Q(q^c) \) does not depend on parity of \( c \). Now we have by [10]

\[
p \neq 2, p \nmid Q; \quad A_Q(p) = \frac{p^2 - 2p - 1}{(p^2 - 1)^2}, \quad A_Q(p^c) = \frac{2(p - 1)}{(p^2 - 1)^2 p^{2c-3}},
\]

\[
p = 2; \quad A_Q(2) = \frac{1}{9}, \quad A_Q(4) = \frac{1}{18},
\]

\[
A_Q(8) = 0, \quad A_Q(16) = \frac{1}{9 \cdot 16},
\]

\[
A_Q(32) = 0, \quad A_Q(2^c) = \frac{1}{9 \cdot 2^{2c-5}}, c \geq 6.
\]

and what we have find, for \( q \mid Q \),

\[
A_Q(q) = \frac{q^2 - 2q - 1}{(q - 1)^2}, \quad A_Q(q^2) = \frac{2}{q(q - 1)}, \quad A_Q(q^c) = \frac{2}{q^{2c-3}(q - 1)}.
\]

Now we can calculate

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta_Q^2(n) = \prod_{p \text{ prime}} \left( 1 + \sum_{c \geq 1} A_Q(p^c) \right) = \left( 1 + \frac{1}{9} \right) \left( 1 + \frac{1}{16} \right) \left( 1 + \frac{1}{16} \right) \left( 1 + \frac{1}{2^{2c-5}} \right) \times
\]

\[
\times \prod_{q \mid Q} \left( 1 + \frac{q^2 - 2q - 1}{(q - 1)^2} + \frac{2}{q(q - 1)} + \sum_{c > 2} \frac{2}{q^{2c-3}(q - 1)} \right) \prod_{p \nmid Q} \left( 1 + \frac{p^2 - 2p - 1}{(p^2 - 1)^2} + \sum_{c > 2} \frac{2(p - 1)}{(p^2 - 1)^2 p^{2c-3}} \right) =
\]
That is

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta^2_Q(n) = \frac{1015}{864} \prod_{q | Q} \frac{2q(q^2 - q - 1)}{(q + 1)(q - 1)^2} \prod_{p \neq 2, p \mid Q} \frac{p^2(p^3 + p^2 - p - 3)}{(p^2 - 1)^2(p + 1)} = \]

\[
= \frac{1015}{864} \prod_{q | Q} \frac{2q(q^2 - q - 1)}{(q + 1)(q - 1)^2} \frac{(q^2 - 1)^2(q + 1)}{q^2(q^3 + q^2 - q - 3)} \prod_{p \neq 2} \frac{p^2(p^3 + p^2 - p - 3)}{(p^2 - 1)^2(p + 1)} = \]

\[
= \left( \prod_{q | Q} \frac{2(q^2 - q - 1)(q + 1)^2}{q(q^3 + q^2 - q - 3)} \right) \cdot 1.328... = C_1 \prod_{q | Q} \frac{2(q^2 - q - 1)(q + 1)^2}{q(q^3 + q^2 - q - 3)},
\]

proving the result pointed out in introduction.

8 Proof of the Theorem 1.3. Main points

Let \( B \) be an indefinite division quaternion algebra over \( \mathbb{Q} \) with discriminant \( d_B \), and \( R \) be the maximal order in \( B \). Let

\[
\Gamma_R = \{ \alpha \in R \mid N_B(\alpha) = 1 \}
\]

be the unit group of \( R \). Define the weighted multiplicities function in the similar way as for \( \Gamma_0(\mathbb{Q}) \):

\[
\beta_R(n) = \frac{1}{4} \sum_{\{T\} \subseteq \Gamma_R} \ln N(T_0) \frac{N(T)}{N(T)^{1/2} - N(T)^{-1/2}}.
\]

The first step is to express weighted multiplicities in the terms of Dirichlet’s \( L \)-functions. The analog of the theorem 3.1 is

**Theorem 8.1** Let \( B \) be indefinite division quaternion algebra over \( \mathbb{Q} \), and \( d_B \) be its reduced discriminant. Then

\[
\beta_R(n) = \sum_{D, v \geq 1 \atop Dv^2 = n^2 - 4} \frac{1}{v} L(1, \chi_D) \prod_{p \mid d_B} \left\{ 1 - \left( \frac{D}{p} \right), \frac{p^2}{p^2 \mid D} \right\},
\]

where \( D \) is a discriminant, i.e. \( D \equiv 0, 1(\mod 4) \).
Proof. One uses exactly the same techniques as for the $\Gamma_0(Q)$ case. ■

In the second step we change $L$-function in the last formula by the Euler’s product formula, and define

$$
\beta_{P,R}(n) := \sum_{D,s \geq 1 \atop D^2-s^2=4} \left( \frac{1}{v} \prod_{p \leq P} \left( 1 - \frac{\chi_D(p)}{p} \right)^{-1} \right) \prod_{p|d_B} \left\{ 1 - \left( \frac{D}{p} \right) \cdot \frac{p^2 | D}{0} \right\}.
$$

We have the local factor decomposition for this function [10], [9]:

**Lemma 8.2** Let $P \geq d_B$, then

$$
\beta_{P,R}(n) = \prod_{p \leq P} \beta_{(p,R)}(n),
$$

where

$$
\beta_{(p,R)}(n) := \sum_{b \geq 0} \frac{1}{p^b} \left( 1 - \frac{1}{p} \chi(n^2-4p^{-2b}_p) \right)^{-1} \cdot I_{p^b}(n).
$$

**Remark 8.3** Here the definition of $I_{p^b}(n)$ is

$$
\mathbb{I}_{2b}(n) = \begin{cases} 
1, & n^2 = 4(\text{mod} \ 2^{2b}), \ (n^2 - 4)2^{-2b} \text{ is a discriminant} \\
0, & \text{else}
\end{cases};
$$

$$
I_{p^b}(n)_{p|d_B} = \begin{cases} 
1, & n^2 = 4(\text{mod} \ p^{2b}), \ p \neq 2 \\
0, & \text{else}
\end{cases};
$$

$$
I_{p^b}(n)_{p \nmid d_B} = \begin{cases} 
1 - \left( \frac{n^2-4}{p} \right) p^{-2b}, & n^2 = 4(\text{mod} \ p^{2b}), \ p^2 \nmid (n^2 - 4)p^{-2b} \\
0, & \text{else}
\end{cases}.
$$

For the next result we use the uniform limit periodicity of $\beta_{(p,R)}$ and once again (see lemma 6.10) get the formula

$$
\hat{\beta}_{R}(\frac{a}{b}) = \prod_{p|b} \hat{\beta}_{(p,R)}(\frac{\alpha_p}{p^{\text{ord}_{d_B}}b}).
$$
for the Fourier coefficients of $\beta_R$. It implies

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta_R^2(n) = \sum_{b \geq 1} \sum_{1 \leq a \leq b} \left| \hat{\beta_R} \left( \frac{a}{b} \right) \right|^2 = \prod_{p \text{ prime}} \left( 1 + \sum_{c \geq 1} \sum_{1 \leq a \leq p^c} \left| \hat{\beta_{(p,R)}} \left( \frac{a}{p^c} \right) \right|^2 \right),
$$

by using Parseval’s equality.

Defining

$$\beta_{(p,R,b)}(n) := \left( 1 - \frac{1}{p} \chi_{(n^2 - 4)p - 2b}(p) \right)^{-1} \cdot \mathbb{I}_{p^b}(n),$$

one obtain [10]

$$\hat{\beta}_{(p,R)}(r) = \sum_{b \geq 0} \frac{1}{p^b} \beta_{(p,R,b)}(r).$$

For the Fourier coefficients in the right-hand side we have

**Theorem 8.4** For any prime $p \mid d_B$ the Fourier coefficients $\hat{\beta}_{(p,R,b)} \left( \frac{a}{p^c} \right)$ are:

- $c = 0, \quad b = 0,$
  $$\hat{\beta}_{(p,R,0)}(0) = \frac{(p - 1)(p^2 + 2p + 2)}{p^2(p + 1)}$$

- $c = 0, \quad b \neq 0,$
  $$\hat{\beta}_{(p,R,b)}(0) = \frac{2(p^3 - 1)}{p^{2b + 2}(p + 1)}$$

- $c = 2b + 2, \quad b \neq 0,$
  $$\hat{\beta}_{(p,R,b)} \left( \frac{a}{p^c} \right) = -\frac{2}{p^{2b + 2}(p + 1)} \cos \left( \frac{4\pi a}{p^c} \right)$$

- $c = 2b + 1, \quad b \neq 0,$
  $$\hat{\beta}_{(p,R,b)} \left( \frac{a}{p^c} \right) = -\frac{2}{p^{2b + 2}(p + 1)} \cos \left( \frac{4\pi a}{p^c} \right)$$

  $$- \frac{p^{1/2} \epsilon_p}{p^{2b}(p + 1)} \left[ e^{-4\pi i \frac{a}{p^c}} \left( \frac{-a}{p} \right) + e^{4\pi i \frac{a}{p^c}} \left( \frac{a}{p} \right) \right],$$

  where $\epsilon_p = \begin{cases} 1, & p \equiv 1 (\text{mod } 4) \\ i, & p \equiv 3 (\text{mod } 4) \end{cases}$

- $c \leq 2b, \quad b \neq 0,$
  $$\hat{\beta}_{(p,R,b)} \left( \frac{a}{p^c} \right) = \frac{2(p^3 - 1)}{p^{2b + 2}(p + 1)} \cos \left( \frac{4\pi a}{p^c} \right)$$
Mean Square of Weighted Multiplicities

\[ c = 1, \quad b = 0, \quad \widehat{\beta}_{(p,R,0)}(\frac{a}{p}) = -\frac{2}{p^2(p+1)} \cos \left( \frac{4\pi a}{p} \right) - \frac{1}{p+1} \sum_{n \mod p} \left( \frac{n^2 - 4}{p} \right) e^{-2\pi in\frac{a}{p}} \]

\[ c = 2, \quad b = 0, \quad \widehat{\beta}_{(p,R,0)}(\frac{a}{p^2}) = -\frac{2}{p^2} \cos \left( \frac{4\pi a}{p^2} \right). \]

**Remark 8.5** The case \( p \nmid d_B \) is done by M. Peter [10]. We will use these results later.

**Corollary 8.6** For any prime \( p \mid d_B \) the Fourier coefficients \( \widehat{\beta}_{(p,R)}(\frac{a}{p^c}) \) are:

\[ \widehat{\beta}_{(p,R)}(0) = 1 \]

\[ \widehat{\beta}_{(p,R)}(\frac{a}{p}) = -\frac{1}{p+1} \sum_{n \mod p} \left( \frac{n^2 - 4}{p} \right) e^{-2\pi in\frac{a}{p}} \]

\[ \widehat{\beta}_{(p,R)}(\frac{a}{p^2}) = -\frac{2}{p(p+1)} \cos \left( \frac{4\pi a}{p^2} \right) \]

\[ \widehat{\beta}_{(p,R)}(\frac{a}{p^c}) = \frac{-2}{(p+1)p^{2c-2}} \cos \left( \frac{4\pi a}{p^c} \right), \text{ for } 2 < c \text{ even} \]

\[ \widehat{\beta}_{(p,R)}(\frac{a}{p^c}) = \frac{-\epsilon_p}{(p+1)p^{2c-2}} \left( \frac{a}{p} \right) \left[ e^{-4\pi i\frac{\frac{a}{p^c} - 1}{p}} + e^{4\pi i\frac{\frac{a}{p^c}}{p}} \right], \text{ for } 2 < c \text{ odd} \]

Now we able to calculate

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta_R^2(n). \]

Let us define the function

\[ A_R(p^c) := \sum_{1 \leq a \leq p^c \atop p \nmid a} \left| \widehat{\beta}_{(p,R)}(\frac{a}{p^c}) \right|^2. \]

Using (13) we find that

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta_R^2(n) = \prod_{p \text{ prime}} \left( 1 + \sum_{c \geq 1} A_R(p^c) \right). \]
The values of the $A_R(p^c)$ for $p \nmid d_B$ were calculate by M.Peter [10]. They are:

$$p \neq 2, p \nmid d_B; \quad A_R(p) = \frac{p^2 - 2p - 1}{(p^2 - 1)^2}, \quad A_R(p^c) = \frac{2(p - 1)}{(p^2 - 1)^2 p^{2c-3}}$$

$$p = 2; \quad A_R(2) = \frac{1}{9}, \quad A_R(4) = \frac{1}{18}, \quad A_R(8) = 0, \quad A_R(16) = \frac{1}{9 \cdot 16}, \quad A_R(32) = 0, \quad A_R(2^c) = \frac{1}{9 \cdot 2^{2c-3}}, c \geq 6.$$  

We complete his results by adding the case $p \mid d_B$. By simple calculations we have

**Theorem 8.7** Let $p \mid d_B$, then

$$A_R(p) = \frac{p^2 - 2p - 1}{(p + 1)^2}; \quad A_R(p^2) = \frac{2(p - 1)}{p(p + 1)^2}; \quad A_R(p^c) = \frac{2(p - 1)}{(p + 1)^2 p^{2c-3}}.$$  

Gathering the results we conclude

$$\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta_R^2(n) = \prod_{p \text{ prime}} \left(1 + \sum_{c \geq 1} A_R(p^c) \right) =$$

$$\left(1 + \frac{1}{9} + \frac{1}{18} + \frac{1}{9 \cdot 16} + \sum_{c \geq 6} \frac{1}{9 \cdot 2^{2c-5}} \right) \times$$

$$\times \prod_{p \mid d_B} \left(1 + \frac{p^2 - 2p - 1}{(p + 1)^2} + \frac{2(p - 1)}{p(p + 1)^2} + \sum_{c \geq 2} \frac{2(p - 1)}{(p + 1)^2 p^{2c-3}} \right) \times$$

$$\times \prod_{p \nmid 2} \left(1 + \frac{p^2 - 2p - 1}{(p^2 - 1)^2} + \sum_{c \geq 2} \frac{2(p - 1)}{(p^2 - 1)^2 p^{2c-3}} \right)$$

$$= \frac{1015}{864} \prod_{p \mid d_B} \frac{2p(p^2 + p + 1)}{(p + 1)^3} \prod_{p \nmid 2} \frac{p^2(p^3 + p^2 - p - 3)}{(p^2 - 1)^2(p + 1)}.$$
And finally we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{2 \leq n \leq N} \beta_R^2(n) = \frac{1015}{864} \prod_{p | d_B} 2p(p^2 + p + 1) \prod_{p \neq 2, p | d_B} \frac{p^2(p^3 + p^2 - p - 3)}{(p^2 - 1)^2(p + 1)}
\]

\[
= \frac{1015}{864} \prod_{p | d_B} \frac{2p(p^2 + p + 1)}{(p + 1)^3} \cdot \frac{(p^2 - 1)^2(p + 1)}{p^2(p^3 + p^2 - p - 3)} \times
\]

\[
\prod_{p \neq 2} \frac{p^2(p^3 + p^2 - p - 3)}{(p^2 - 1)^2(p + 1)}
\]

\[
= C_1 \cdot \prod_{p | d_B} \frac{2(p^3 - 1)(p - 1)}{p(p^3 + p^2 - p - 3)}, \quad \text{where } C_1 = 1.328\ldots
\]

This calculation concludes the proof.
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