The Adelic Grassmannian and Exceptional Hermite Polynomials

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Abstract
It is shown that when dependence on the second flow of the KP hierarchy is added, the resulting semi-stationary wave function of certain points in George Wilson's adelic Grassmannian are generating functions of the exceptional Hermite orthogonal polynomials. This surprising correspondence between different mathematical objects that were not previously known to be so closely related is interesting in its own right, but also proves useful in two ways: it leads to new algorithms for effectively computing the associated differential and difference operators and it also answers some open questions about them.

Keywords Bispectrality · KP hierarchy · Exceptional orthogonal polynomials · Generating function · Hermite polynomials · Adelic Grassmannian

Mathematics Subject Classification (2010) 33C45 · 33C47

1 Introduction

Suppose that operators $L$ and $\Lambda$ share an eigenfunction $\psi(x, z)$ so that

$$L\psi = p(z)\psi \quad \text{and} \quad \Lambda\psi = \pi(x)\psi,$$

where $L$ is an operator acting on functions of $x$ that is independent of $z$, $\Lambda$ is an operator on functions of $z$ that is independent of $x$, and the eigenvalues $p(z)$ and $\pi(x)$ are both non-constant functions. In this case we say that the operators $L$ and $\Lambda$ are bispectral and that $(L, \Lambda, \psi)$ is a bispectral triple.

The term “bispectrality” was used to describe this situation in [4] where the authors identified all bispectral Schrödinger operators of the form $L = \partial_x^2 + u(x)$
sharing an eigenfunction with an ordinary differential operator $\Lambda$ in $z$. Bispectrality has since been considered in much greater generality, allowing $L$ and $\Lambda$ to be any sorts of operators acting on functions of variables that can be either scalar or vector valued and either discrete or continuous.

A fundamental breakthrough in the study of bispectral operators was George Wilson’s 1993 paper [26] in which he made use of the Sato Grassmannian $\text{Gr}$ which was developed for producing solutions to the KP hierarchy [24, 25]. That construction normally associates to a point $W \in \text{Gr}$ a pseudo-differential operator $L_W$ depending on time variables $t_i$ for $i = 1, 2, 3 \ldots$ and a “wave function” $\psi_W(t_1, t_2, \ldots, z) = (1 + O(z^{-1})) \exp \sum_{i=1}^{\infty} t_i z^i$ satisfying the equations

$$L_W \psi_W = z \psi_W \quad \text{and} \quad \partial_i \psi_W = (L_W^i)_+ \psi_W$$

which together are equivalent to a hierarchy of nonlinear evolution equations for the coefficients of $L_W$. The connection to bispectrality is most apparent when one “turns off” all but one of the time variables. By setting $t_1 = x$ and $t_i = 0$ for $i > 1$ the wave function becomes a function of only the variables $x$ and $z$ as in the definition of bispectrality. Wilson showed that if $W$ is in the “adelic Grassmannian” $\text{Gr}^{\text{ad}} \subset \text{Gr}$ then there is a non-trivial ring of ordinary differential operators in $x$ having $\psi_W$ as an eigenfunction and there exists another point $\beta(W) \in \text{Gr}^{\text{ad}}$ such that $\psi_W(x, z) = \psi_{\beta(W)}(z, x)$. Wilson generalized the notion of a bispectral triple to allow for $L$ and $\Lambda$ to be elements of commutative algebras, rather than fixed operators, and showed that (up to trivial renormalizations) $\text{Gr}^{\text{ad}}$ is the moduli space of bispectral ordinary differential operators that commute with operators of relatively prime order [26].

Following Wilson’s seminal paper, it is now recognized that it is more natural to define a bispectral triple as two commutative algebras of operators and a common eigenfunction. Furthermore, nearly all papers on this subject now produce bispectral algebras using versions of Wilson’s construction that have been suitably modified to different settings. For example, to consider the case of differential operators that do not commute with operators of relatively prime order the scalar eigenfunction is replaced with a vector eigenfunction and to study bispectral difference operators the (pseudo)-differential operators are replaced with infinite matrices (e.g. see [17] and [14]).

Classical orthogonal polynomials are families of orthogonal polynomials that are the eigenfunction of a second-order Sturm-Liouville eigenvalue problem. As such, they may be considered as the eigenfunctions of a differential-difference bispectral triple, where the 3-term recurrence plays the role of the differential equation in spectral parameter. This idea can be naturally connected to the adelic grassmannian [15]. See also [13, 16] and the references therein for an application of such ideas to Krall polynomials, and the Askey-Wilson scheme.

Exceptional orthogonal polynomials [12, 21] generalize classical orthogonal polynomials, because they are the eigenfunctions of a second-order Sturm-Liouville problem, but fall outside the Askey-Wilson scheme by allowing for polynomial sequences that omit a finite number of “exceptional” degrees [2]. This relaxed assumption implies that exceptional polynomials cannot satisfy a 3-term recurrence relations. Indeed, unlike the differential-differential bispectral problem investigated by Duistermaat and Grünbaum [4], the dual eigenvalue problem for exceptional Hermite polynomials
consists of an algebra of commuting difference operators [10]. A similar situation seems to hold for the case of exceptional Laguerre and Jacobi polynomials [3, 5] and for discrete exceptional polynomials [6, 22]. This observation suggests that the ensemble consisting of (i) a family of exceptional orthogonal polynomials, (ii) the corresponding exceptional second-order operator, and (iii) higher order recurrences should also be regarded as an instance of a differential-difference bispectral triple.

In the case of exceptional Hermite polynomials, the second-order exceptional operator in question is known to have trivial monodromy [20]. It is also known that every exceptional operator is related by a Darboux transformation to a classical operator [9]. All of this is a further indication that bispectrality should be a key concept in the theory of exceptional polynomials.

The present paper grew out of an investigation of questions surrounding the bispectrality of the exceptional Hermite orthogonal polynomials [5, 7, 10]. Since the differential operators that have these as eigenfunctions all have even degree and since the operator in the other variable is a difference operator, one might expect that Wilson’s construction should be suitably modified to address these questions. However, the most surprising result to be presented below is the fact that essentially no modification is needed; the exceptional Hermite orthogonal polynomials were already present (but unnoticed) in Wilson’s original construction. This observation turns out to be quite useful, greatly simplifying the construction of the exceptional Hermite orthogonal polynomials and the associated operators, and providing answers to some open questions surrounding them.

The organization of the paper is as follows.

- Section 2 collects some background material on partitions, Maya diagrams and Schur functions.
- Section 3 is a quick review of $\text{Gr}^{\text{ad}}$, Wilson’s adelic grassmannian and of the bispectral involution.
- Section 4 reviews classical and exceptional Hermite polynomials and shows how these objects are naturally associated with certain points in $\text{Gr}^{\text{ad}}$. In particular, Theorem 4.10 shows that the wave function corresponding to self-dual points $W^{(\lambda)} \in \text{Gr}^{\text{ad}}$ labelled by a partition $\lambda$ serve as generating functions for the family of exceptional Hermite orthogonal polynomials (cf. [7, 10]) associated with the same partition. The only modification needed to Wilson’s original construction is that rather than setting all time variables $t_i$ with $i > 1$ equal to zero, one must instead only set time variables with index $i > 2$ equal to zero. The second time variable $y = t_2$ plays the role of a scaling parameter for the exceptional orthogonal polynomials.
- Theorem 4.15 gives a useful formula for the exceptional Hermite polynomials as linear combinations of the classical Hermite polynomials with coefficients derived from wave functions in $\text{Gr}^{\text{ad}}$. This formulation has a significant computational and conceptual advantage over the usual formulation in terms of Wronskians.
- Section 4.5: The exceptional Hermite orthogonal polynomials are known to be annihilated by point supported distributions, however it was not previously known how to determine which distributions annihilated a given instance of the
exceptional polynomials. In the context of $\text{Gr}^{\text{ad}}$, this question is answered by Theorem 4.19 using Wilson’s bispectral involution $W \mapsto \beta(W)$.  

- Section 5 introduces the bispectral triple associated with a given family of exceptional Hermite polynomials. In this context, it is natural to introduce and to study non-commutative stabilizer algebras corresponding to the points in $\text{Gr}^{\text{ad}}$. Proposition 5.6 shows that the bispectral involution defines an anti-isomorphism between two such algebras. The eigenvalue relations engendered by the bispectral triple can then be conveniently constructed as the restriction of this anti-isomorphism to the commutative subalgebras corresponding to the eigenvalues.  

- In Section 6 we realize the above bispectral triple as lowering relations [11] and recurrence relations [10] obeyed by exceptional Hermite polynomials. In Section 6.1 we show that the algebra of lowering operators is naturally isomorphic to the stabilizer algebra of $W^{(\lambda)}$. In Section 6.2, the ring of the corresponding higher-order Jacobi operators is shown to be the stabilizer algebra of $\beta(\hat{W}^{(\lambda)})$, the bispectral dual of a deformation of $W^{(\lambda)}$ under the second KP flow. The $\text{Gr}^{\text{ad}}$-based construction allows for a significant computational advantage in determining the form of these exceptional Jacobi operators.  

- Section 7 collects some examples and details of calculations related the intertwining relations, lowering operators and exceptional recurrence relations.  

1.1 Conventions and Notations

Let $\mathcal{P}$ denote the ring of complex valued univariate polynomials regarded as mappings without reference to any particular variables. If $U \subset \mathcal{P}$ is a polynomial subspace and $x$ is an indeterminate, we will employ the notation $U(x)$ to denote a subspace of the corresponding $\mathbb{C}[x] \approx \mathcal{P}$. Most of the functions encountered below will depend on several variables, and so we will use $\partial_x$ to indicate denote the elementary partial-derivative operator $\frac{\partial}{\partial x}$. The symbol $\text{Wr}_x$ denotes the usual Wronskian determinant with respect to the indeterminate $x$. The symbol $\text{Wr}$, without any subscript, denotes the Wronskian taken with respect to the first argument. The symbol $\mathcal{S}$ denotes the unit shift operator. If $f_n, \ n \in \mathbb{N}_0$ is a sequence indexed by an indeterminate $n$, we will write $\mathcal{S}_n f_n := f_{n+1}$.  

For later convenience, we also adopt the following unusual convention. When a differential operator is constructed by the substitution of an elementary partial-derivative operator into a multi-variable function, then it is understood that the derivatives all appear to the right of all other variables — regardless of whether they commute as operators. In contrast, when substituting a differential operator into a univariate polynomial, it is understood that powers of the operator are computed through composition. For example, let $\pi(x, y, z) = x^2 y^{-1} z - x y^3 z^4$. Then, according to this convention we have  

$$
\pi(\partial_z, y, z) = y^{-1} z \partial_z^2 - y^3 z^4 \partial_z
$$

$$
\pi(x, y, \partial_x) = x^2 y^{-1} \partial_x - x y^3 \partial_x^4.
$$
In particular, applying either of those differential operators to the function $e^{xz}$ results in $\pi(x, y, z)e^{xz}$. On the other hand, if $\gamma(z) = z^3$ then we define

$$\gamma(z\partial_z) = (z\partial_z) \circ (z\partial_z) \circ (z\partial_z) = z^3 \partial_z^3 + 3z^2 \partial_z^2 + z\partial_z.$$

The following notations are all rigorously introduced as needed later in the text, but are briefly summarized here for the reader’s convenience. The symbol $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers, while $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is the set of non-negative integers. The usual univariate classical Hermite polynomials will be denoted by $h_n(x)$, $n \in \mathbb{N}_0$ while $H_n(x, y)$, $n \in \mathbb{N}_0$ are the same polynomials converted to a bivariate form through the inclusion of a scaling parameter. Similarly, $h_{(\lambda)}_n(x)$ will denote the exceptional Hermite polynomials in their univariate form, $H_{(\lambda)}_n(x, y)$, where $n$ is the degree, are the bivariate exceptional Hermites, and finally $R_{(\lambda)}_m(x, y)$, where $m$ is the difference of the degrees of the numerator and denominator, are rational functions made by dividing the bivariate exceptional Hermite polynomials by a denominator polynomial $\tau_{(\lambda)}(x, y)$. The span of the former will be denoted by $\hat{U}_{(\lambda)} \subset \mathcal{P}$ while the span of the latter will be denoted by $\hat{W}_{(\lambda)} = \tau_{(\lambda)}^{-1} \hat{U}_{(\lambda)}$.

The symbol $\lambda$ will denote a partition and $\mathcal{M}(\lambda)$ the corresponding Maya diagram. For each $\lambda$, there are certain naturally defined sets $\mathcal{I}(\lambda)$ and $\mathcal{J}(\lambda) \subset \mathbb{Z}$ that serve as the index sets of $H_{(\lambda)}_n$ and $R_{(\lambda)}_m$, respectively. The symbols $K_{(\lambda)}(m), G_{q(\lambda)}(m), q \in \mathbb{Z}$ denote finite sets of integers that encode various combinatorial properties of $\lambda$. The symbols $\kappa_{(\lambda)}(m), \gamma_{q(\lambda)}(m), q \in \mathbb{Z}$ denote the corresponding monic polynomials whose roots are precisely these sets.

The symbol $\mathbb{D}$ will denote the vector space of distributions generated by 1-point functionals.¹ The symbol $\mathbb{D}_\zeta$ refers to the subspace of functionals with support at a particular $\zeta \in \mathbb{C}$. The symbol $W_{(\lambda)} \in \text{Gr}^{\text{ad}}$ refers to a point in the adelic Grassmannian that is canonically associated to a partition $\lambda$. These are the points whose $\tau$-function is a Schur polynomial; they are discussed by Wilson in Section 10, Example 2 of [26]. The bispectral involution will be denoted by $\beta : \text{Gr}^{\text{ad}} \to \text{Gr}^{\text{ad}}$. As in Wilson, the application of a functional to a function will be denoted using angle brackets, as in $\langle c, f \rangle$. We will also use angle brackets to denote the inner product relative to which the Hermite polynomials are orthogonal. To avoid any possible confusion, we will add a subscripted $H$ and write $\langle \cdot, \cdot \rangle_H$ in such cases.

## 2 Partitions and Schur Functions

### 2.1 Partitions and Maya Diagrams

A *partition* $\lambda$ is a decreasing, non-negative integer sequence $\lambda_1 \geq \lambda_2 \geq \cdots$ such that

$$|\lambda| := \sum_{i=1}^{\infty} \lambda_i < \infty.$$

¹Wilson in [26] refers to these as 1-point conditions; our $\mathbb{D}$ is Wilson’s $\mathcal{C}$. 
Implicit in this definition is the assumption that \( \lambda_i = 0 \) for \( i \) sufficiently large. The length of \( \lambda \), which we will denote by \( \ell(\lambda) \), is the number of non-zero elements of the sequence.

A closely related concept is that of a **Maya diagram**\(^2\). A Maya diagram is a subset of \( \mathbb{Z} \) that contains a finite number of positive integers and excludes a finite number of negative integers. For a given \( \lambda \), define the strictly decreasing sequence

\[
m_i(\lambda) = \lambda_i - i, \quad i = 1, 2, \ldots
\]

The set

\[
\mathcal{M}^{(\lambda)} = \{m_i(\lambda) : i = 1, 2, \ldots\}
\]

is a Maya diagram because \( m_{i+1}(\lambda) = m_i(\lambda) - 1 \) for \( i \geq \ell(\lambda) + 1 \). Conversely, if \( M \subset \mathbb{Z} \) is a Maya diagram, then

\[
M = \mathcal{M}^{(\lambda)} + l = \{m_i(\lambda) + l : i = 1, 2, \ldots\}
\]

for some partition \( \lambda \) and \( l \in \mathbb{Z} \).

To a partition \( \lambda \) and an integer \( l \geq \ell(\lambda) \), define the index set of length \( l \) associated to \( \lambda \) to be

\[
\mathcal{K}_{i}^{(\lambda)} = \{m_1(\lambda) + l, \ldots, m_l(\lambda) + l\}. \tag{5}
\]

Since \( l \geq \ell(\lambda) \), it follows that \( m_l(\lambda) + l \geq 0 \) and \( m_{l+1}(\lambda) + l < 0 \). Hence \( \mathcal{K}_{i}^{(\lambda)} \) consists precisely of the non-negative elements of \( \mathcal{M}^{(\lambda)} + l \).

Observe that when \( l = \ell(\lambda) \) then \( m_l(\lambda) + \ell(\lambda) > 0 \) and \( m_j(\lambda) + \ell(\lambda) < 0 \), \( j > \ell(\lambda) \), by definition. In this case, it is convenient to drop the subscript and write

\[
\mathcal{K}^{(\lambda)} := \mathcal{K}_{\ell(\lambda)}^{(\lambda)} = \{m_1(\lambda) + \ell(\lambda), \ldots, m_{\ell}(\lambda) + \ell(\lambda)\}. \tag{6}
\]

Thus, \( \mathcal{K}^{(\lambda)} \) is the smallest index set, and also the only index set consisting of strictly positive elements. The correspondence \( \lambda \mapsto \mathcal{K}^{(\lambda)} \) is a bijection between the set of partitions and the set of finite subsets of \( \mathbb{N} \).

For a partition \( \lambda \), let

\[
\mathcal{J}^{(\lambda)} := \mathbb{Z} \setminus \mathcal{M}^{(\lambda)} \tag{7}
\]

denote the complement of the corresponding Maya diagram.\(^3\) An integer can be “inserted” into the partition \( \lambda \) to produce a new partition as follows. For \( m \in \mathcal{J}^{(\lambda)} \) let \( m \triangleright \lambda \) denote the partition

\[
\lambda_1 - 1, \ldots, \lambda_j - 1, m + j, \lambda_{j+1}, \lambda_{j+2}, \ldots, \tag{8}
\]

where \( j \) is the smallest natural number such that \( m + j \geq \lambda_{j+1} \). The sequence (8) is a partition because \( m \in \mathcal{J}^{(\lambda)} \) implies that either \( j = 0 \) and \( m > m_1(\lambda) \), or that

\[
\lambda_{j} - j = m_j(\lambda) > m > m_{j+1}(\lambda) = \lambda_{j+1} - j - 1.
\]

\(^2\)For our purposes, Maya diagrams are more convenient than Young diagrams, but they are merely two ways of representing partitions. See [8] for a rigorous presentation of the correspondence.

\(^3\)Note that \( -\mathcal{J}^{(\lambda)} \) is itself a Maya diagram.
Another way to understand the transformation $\lambda \mapsto m \triangleright \lambda$ is to observe that it adds one element to the corresponding index sets. To wit,

$$K_{l+1}^{(m \triangleright \lambda)} = K_l^{(\lambda)} \cup \{m + l\}, \quad l \geq \ell(\lambda). \quad (9)$$

### 2.2 Schur Functions

For every $k \in \mathbb{N}$, define the ordinary Bell polynomials $B_k(t_1, \ldots, t_k) \in \mathbb{C}[t_1, \ldots, t_k]$ as the coefficients of the power generating function

$$\psi_0(t; z) := \exp\left( \sum_{k=1}^{\infty} t_k z^k \right) = \sum_{k=0}^{\infty} B_k(t_1, \ldots, t_k) z^k, \quad t = (t_1, t_2, \ldots). \quad (10)$$

Since the above generating function can also be written as

$$\psi_0(t; z) = \sum_{j=0}^{\infty} \frac{1}{\mu!} \left( \sum_{k=0}^{\infty} t_k z^k \right)^\mu,$$

the multinomial formula implies that

$$B_k(t_1, \ldots, t_k) = \sum_{\|\mu\| = k} \frac{t_1^{\mu_1} t_2^{\mu_2} \cdots t_k^{\mu_k}}{\mu_1! \mu_2! \cdots \mu_k!}, \quad \|\mu\| = \mu_1 + 2\mu_2 + \cdots + k\mu_k$$

$$= \frac{t_1^k}{k!} + \frac{t_1^{k-2} t_2}{(k-2)!} + \cdots + t_1 t_{k-1} + t_k. \quad (11)$$

For any partition $\lambda$, define the Schur function $S^{(\lambda)}(t_1, \ldots, t_N) \in \mathbb{Q}[t_1, \ldots, t_N]$, $N = |\lambda|$ to be the multivariate polynomial

$$S^{(\lambda)} = \det(B_{\lambda_i-i+j})_{i,j=1}^l \quad (12)$$

where $l$ is any integer satisfying $l \geq \ell(\lambda)$ and $B_k = 0$ when $k < 0$.

Moreover, since

$$\partial_{t_j} B_j(t_1, \ldots, t_j) = B_{j-1}(t_1, \ldots, t_{j-1}), \quad j \geq i,$$

we may re-express (12) in terms of a Wronskian determinant:

$$S^{(\lambda)} = \text{Wr}[B_{k_1}, \ldots, B_{k_l}]. \quad (13)$$

where $k_1 > \cdots > k_l$ are the elements of the index set $K_l^{(\lambda)}$ defined in (5). It is important to note that (12) and (13) yield the same polynomial $S^{(\lambda)}$ regardless of the value of $l \geq \ell(\lambda)$.\(^4\)

The Schur functions, $S^{(\lambda)}$, are closely related to the representation theory of the symmetric group on $n$ elements. Such irreducible representations are labelled by

\(^4\)This is because increasing the value of $n$ by one has the effect of increasing the size of the matrix, adding a new first row and column, but since there is necessarily a 1 in the top-left corner and zeroes below it, the determinant is unchanged.
partitions \( \lambda \) such that \(|\lambda| = N\). The conjugacy classes of the symmetric group correspond to cycle types \( c_{\mu} = (1^{\mu_1}, 2^{\mu_2}, \ldots) \) where
\[
\|\mu\| := \sum_j j \mu_j = N.
\]

It is known [18, Section I.7] that
\[
S(\lambda)(t_1, \ldots, t_N) = \sum_{\|\mu\| = N} \chi(\lambda)(c_{\mu}) t_{\mu_1}^{\mu_1} \dots t_{\mu_\ell(\lambda)}^{\mu_\ell(\lambda)}
\]
where \(\chi(\lambda)\) denotes the character of the representation labelled by \(\lambda\). By the hook-length formula, the coefficient of \(t_N^N\) in \(S(\lambda)\), is equal to
\[
\frac{\chi^{(\lambda)}(1^N)}{N!} = \frac{d_\lambda}{N!} = \frac{\prod_{i<j}(k_i - k_j)}{\prod_i k_i!},
\]
where \(d_\lambda\) is the dimension of the irreducible representation corresponding to \(\lambda\), and where \(k_1 > \cdots > k_\ell\) are the elements of the index set \(K_\ell(\lambda)\) defined in (5).

3 Sato Theory and the Adelic Grassmannian

3.1 The Adelic Grassmannian

For \(k \in \mathbb{N}\) and \(\zeta \in \mathbb{C}\) we let \(\text{ev}(k, \zeta) : \mathcal{P} \to \mathcal{P}\) denote the evaluation functional composed with the \(k\)th-order derivative:
\[
\langle \text{ev}(k, \zeta), f \rangle = f^{(k)}(\zeta), \quad f \in \mathcal{P}.
\]

Let
\[
\mathbb{D}_{\zeta} = \text{span}\{\text{ev}(k, \zeta) : k \in \mathbb{N}_0\}
\]
do not the vector space of all 1-point functionals with support at a fixed \(\zeta \in \mathbb{C}\) and let
\[
\mathbb{D} = \bigoplus_{\zeta \in \mathbb{C}} \mathbb{D}_{\zeta}
\]
be the vector space spanned by 1-point functionals with arbitrary support. As the need arises, we proceed with the understanding that functionals in \(\mathbb{D}\) are also allowed to act on rational and analytic functions (with the appropriate domain safeguards). In situations where a functional acts on a multi-variable function, we adopt the convention that \(c(z)\) indicates that \(c \in \mathbb{D}\) acts on a function of the variable \(z\).

We will say that a subspace \(C \subset \mathbb{D}\) is homogeneous if it has a basis of one-point functionals. Thus, \(C \subset \mathbb{D}\) is homogeneous if and only if
\[
C = \bigoplus_\zeta (C \cap \mathbb{D}_{\zeta}).
\]

Let \(C \subset \mathbb{D}\) be a finite-dimensional subspace of differential functionals, and let
\[
\text{Ker} \ C = \{f \in \mathcal{P} : \langle c, f \rangle = 0 \text{ for all } c \in C\}
\]
be the joint kernel of the elements of $C$. It is easy to show that \( \dim C = \text{codim} \ker C \), where the latter denotes the codimension of $\ker C \subset P$. Dually, if $U = \ker C$ for some finite-dimensional $C \subset \mathbb{D}$, then

$$C = \text{Ann} U = \{ c \in \mathbb{D} : \langle c, f \rangle = 0 \text{ for all } f \in U \}.$$

Let $C \subset \mathbb{D}$ be a homogeneous, finite-dimensional subspace of functionals. Define

$$q_C(z) := \prod_{i=1}^{n} (z - \zeta_i), \quad W_C := q_C^{-1} \ker C \quad (18)$$

where $c_i \in \mathbb{D}_{\zeta_i}, \ i = 1, \ldots, n, \ \zeta_i \in \mathbb{C}$ is a choice of basis for $C$. It is evident that $q_C$ and $W_C$ are independent of the choice of basis.

**Definition 3.1** We define $\text{Gr}^{\text{ad}}$, the adelic Grassmannian [26], to be the set of all subspaces of the form $W_C$ where $C \subset \mathbb{D}$ is homogeneous. We will say that $C \subset \mathbb{D}$ is reduced if for all $\zeta \in \mathbb{C}$ we have $\text{ev}(0, \zeta) \neq C$. Equivalently, $C$ is reduced if and only if the elements of $\ker C$ do not have a shared root.

**Proposition 3.2** For every $W \in \text{Gr}^{\text{ad}}$ there exists a unique reduced homogeneous $\hat{C} \subset \mathbb{D}$ such that $W = W_{\hat{C}}$.

### 3.2 KP Wave functions and Wilson’s Bispectral Algebras

Sato theory associates a wave function and a rational solution of the KP hierarchy to each point in $\text{Gr}^{\text{ad}}$ as follows [24] (see also [25, 26]).

Let $W_C \in \text{Gr}^{\text{ad}}$, $C \in \mathbb{D}$ as per (18), and let $c_i \in \mathbb{D}_{\zeta_i}, \ i = 1, \ldots, l$ be a basis of $C$. Let

$$K_C = \partial_{t_1}^l + \sum_{i=1}^{l} a_i(t) \partial_{t_1}^{l-i}$$

denote the monic differential operator whose action on an arbitrary function $f$ is

$$K_C f(t) = \frac{\text{Wr}[\phi_1(t), \ldots, \phi_l(t), f(t)]}{\text{Wr}[\phi_1(t), \ldots, \phi_l(t)]}.$$

where

$$\phi_i(t) := \langle c_i, \psi_0(t, z) \rangle$$

with $\psi_0(t, z)$ the generating function of the Bell polynomials previously introduced in (10).

The dynamical wave function associated to $W = W_C$ is defined to be

$$\psi_W(t; z) = \frac{1}{q_C(z)} K_C \psi_0(t, z) = \frac{\text{Wr}_{t_1} [\phi_1(t), \ldots, \phi_l(t), \psi_0(t, z)]}{q_C(z) \tau_C(t)} \quad (19)$$

where

$$\tau_C(t) = \text{Wr}_{t_1} [\phi_1(t), \ldots, \phi_l(t)]. \quad (20)$$
The dynamical wave function can also be derived from the \( \tau \)-function using the so-called Miwa shift [25, Equation (5.16)]:
\[
\psi_{W}(t, z) = \frac{\phi_{W}(t, z)}{\tau_{C}(t)} \psi_{0}(t; z)
\]  
(21)
where
\[
\phi_{W}(t, z) = \tau_{C}(t_{1} - z^{-1}, t_{2} - 1/2z^{-2}, \ldots).
\]

Even though the definition of \( \psi_{W} \) depends on a choice of functionals \( C \), the correspondence \( W \mapsto \psi_{W} \) is well-defined as a consequence of the following; c.f., Proposition 3.2.

**Proposition 3.3** Let \( C \subset \mathbb{D} \) be a homogeneous subspace of functionals with \( U = \text{Ker} \, C \) the corresponding polynomial subspace. Let \( r \in \mathcal{P} \) be a monic polynomial and let
\[
C_{r} = \{ c \in \mathbb{D} : c \circ r \in C \} = \text{Ann}(rU).
\]  
(22)
Then, \( W_{C_{r}} = W_{C} \) and \( K_{C_{r}} = K_{C} \circ r(\partial_{t_{1}}) \)

**Corollary 3.4** The definition (19) of the wave function \( \psi_{W} \) is independent of the choice of \( C \).

The dynamical wave function is fully characterized by the following properties.

**Proposition 3.5** For an \( l \)-dimensional homogeneous \( C \subset \mathbb{D} \) and \( W = W_{C} \in \text{Gr}^{\text{ad}} \), the corresponding wave function has the form
\[
\psi_{W}(t, z) = \frac{1}{q_{C}(z)} \left( z^{l} + \sum_{i=1}^{l} \phi_{i}(t) z^{l-i} \right) \psi_{0}(t, z),
\]  
(23)
where the coefficients \( \phi_{i}(t) \) are rational functions, and where
\[
\langle c(z), q_{C}(z) \psi_{W}(t, z) \rangle = 0,
\]  
(24)
for all \( t \) and \( c \in C \). Moreover, if (24) holds for some \( c \in \mathbb{D} \), then necessarily \( c \in C \).

The connection to the KP hierarchy takes the form of the following observations [25]. The pseudo-differential operator
\[
L_{W}(t, z) = K_{C}(t, z) \circ \partial_{t_{1}} \circ K_{C}^{-1}(t, z)
\]
satisfies the nonlinear evolution (2) of the KP hierarchy. The ring
\[
\mathcal{R}_{W} := \{ p \in \mathcal{P} : pW \subset W \}
\]  
(25)
is called the stabilizer of \( W \). Dually, the stabilizer ring may be characterized as
\[
\mathcal{R}_{W} := \{ p \in \mathcal{P} : c \circ p \in C \text{ for all } c \in C \}.
\]  
(26)
It follows that for every \( p \in \mathcal{R}_{W} \) we have that
\[
L_{p} := p(L_{W}) = K_{C}(t, z) \circ p(\partial_{t_{1}}) \circ K_{C}^{-1}(t, z)
\]  
(27)
is a differential operator. Moreover, by construction, $L_p$ satisfies the eigenvalue equation

$$L_p(t, z)\psi_W(t, z) = p(z)\psi_W(t, z).$$

(28)

Since any polynomial with a factor of $(q_C(z))^N$ is in $\mathcal{R}_W(z)$ for sufficiently high powers of $N$, this construction produces an algebra of differential operators that is non-empty and includes every sufficiently high order.

Although the construction above was initially created to study the dynamics of the KP hierarchy, the seminal paper by Wilson [26] used it to address the bispectral problem in the following elegant way. Rename the first time variable by setting $x = t_1$ and “turn off” all of the other time variable by setting $t_i = 0$ for $i > 1$. Then the stationary wave function

$$\psi_W(x, z) = \psi_W(x, 0, 0, \ldots; z)$$

(29)

is an eigenfunction for a ring of ordinary differential operators in $x$ with eigenvalues depending polynomially on $z$; this follows by (28). Thus, [26, Proposition 5.1]

$$\psi_W(x, z) = q_C(z)^{-1}\left(z^l + \sum_{i=1}^{l} \phi_i(x)z^{l-i}\right)e^{xz},$$

(30)

where $\phi_i(x), \ i = 1, \ldots, n$ are rational functions uniquely determined by the conditions

$$\langle c_i(z), (z^l + \sum_{i=1}^{l} \phi_i(x)z^{l-i})e^{xz}\rangle = 0, \ \ i = 1, \ldots, l,$$

where $c_1, \ldots, c_l$ are a basis of $C$. Wilson also showed [26, Theorem 2] that the relation

$$\psi_W(z, x) = \psi_{\beta(W)}(x, z)$$

(31)

defines an involution $W \mapsto \beta(W)$ on $\text{Gr}^{ad}$.

It follows that $\psi_W(x, z)$ is part of a bispectral triple in that it is also the eigenfunction for differential operators in $z$ with eigenvalues depending polynomially on $x$.

4 \text{Gr}^{ad} and Exceptional Hermite Polynomials

4.1 Classical Hermite Polynomials

Classical Hermite polynomials are orthogonal polynomials defined by the recurrence relation

$$h_0 = 1, \quad xh_n(x) = \frac{1}{2}h_{n+1}(x) + nh_{n-1}(x), \quad n = 1, 2, \ldots$$

(32)

They are orthogonal with respect to the following inner product:

$$\int_{\mathbb{R}} h_m(x)h_n(x)e^{-x^2}dx = \sqrt{\pi} 2^n n!\delta_{n,m}$$

(33)
and satisfy the following second-order eigenvalue equation
\[ h_n'' - 2xh_n' = -2nh_n, \quad n = 0, 1, \ldots \] (34)

The Hermite polynomials may also be defined in terms of the Rodrigues formula
\[ h_n(x) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2}, \quad n = 0, 1, 2, \ldots \] (35)

Relation (35) entails the following representation of the Hermite polynomials in terms of an exponential generating function
\[ \sum_{n=0}^{\infty} \frac{h_n(x)}{n!} \left( -\frac{z}{2} \right)^n = e^{x^2} \exp \left( -\frac{z}{2} \partial_x \right) e^{-x^2} = e^{x^2 - (x/z)^2} = e^{xz - \frac{1}{4}z^2}. \] (36)

Let us introduce a bivariate version of the Hermite polynomials, defined as
\[ H_n(x, y) := (-y)^{n/2} h_n \left( x \sqrt{-4y} \right). \] (37)

The univariate Hermite polynomials can be recovered as
\[ h_n(x) = 2^n H_n(x, -1/4). \]

The generating function for the bivariate polynomials takes the form:
\[ \Psi_0(x, y, z) := \exp(xz + yz^2) = \sum_{n=0}^{\infty} H_n(x, y) \frac{z^n}{n!}. \] (38)

It follows from relation (38) that \( H_n(x, y) \) is monic in \( x \) and is weighted degree homogeneous relative to the grading
\[ \deg x = 1, \quad \deg y = 2. \] (39)

A number of fundamental identities involving Hermite polynomials follow from (38). For example, mirroring the argument of (36), the bivariate version of the Rodrigues formula takes the form
\[ H_n(2xy, y) = e^{-y^2} \frac{\partial^n}{\partial x^n} e^{y^2}, \quad n = 0, 1, 2, \ldots . \] (40)

By inspection, \( \Psi_0(x, y, z) \) is annihilated by the operator \( 2y \partial_x^2 + x \partial_x - z \partial_z \). Since
\[ z \partial_z \Psi_0(x, y, z) = \sum_{n=0}^{\infty} n H_n(z^n) \frac{z^n}{n!}, \]
this observation entails the following, scaled version of the Hermite differential equation:
\[ T(x, y, \partial_x) H_n(x, y) = n H_n(x, y), \quad \text{where} \quad T(x, y, z) = 2yz^2 + xz. \] (41)

Applying the scaling transformation (37) to the classical orthogonality relation (33) yields the following scaled orthogonality relation. For fixed \( y < 0 \), we have
\[ \langle H_n_1(x, y), H_n_2(x, y) \rangle_H = v_n(y) \delta_{n_1,n_2}, \quad n \in \mathbb{N}_0. \] (42)

where
\[ \langle f(x), g(x) \rangle_H = \int_{\mathbb{R}} f(x) g(x) e^{\frac{y^2}{2x^2}} dx, \quad y < 0 \] (43)
and where
\[ v_n(y) = 2(-\pi y)^{1/2} (-2y)^n n! \]  \hfill (44)

Specializing the generating function for Bell polynomials (10), gives the following, well-known, representation of Hermite polynomials as a finite sum:
\[ H_n(x, y) = n! B_n(x, y, 0, \ldots) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2j)!j!} x^{n-2j} y^j \]  \hfill (45)

Next, consider the 1st order eigenvalue relation:
\[ \partial_x \Psi_0(x, y, z) = z \Psi_0(x, y, z) \]
This relation entails the well-known lowering identity
\[ \partial_x H_n(x, y) = n H_{n-1}(x, y). \]  \hfill (46)

In more combinatorial language, we can say that \( H_n(x, y), \; n \in \mathbb{N}_0 \) forms an Appell sequence [23].

Similarly, the relation
\[ (\partial_z - 2yz) \Psi_0(x, y, z) = x \Psi_0(x, y, z). \]  \hfill (47)
entails the bivariate version of the recurrence relation (32), namely:
\[ \Theta_1(n, y, S_n) H_n(x, y) = x H_n(x, y), \quad \text{where} \quad \Theta_1(n, y, z) = z - 2ynz^{-1} \]  \hfill (48)
where \( S \) is the unit right-shift operator.

### 4.2 Exceptional Polynomials

Exceptional Hermite polynomials [7] are a far ranging generalization of the classical Hermite polynomials. Just like their classical counterparts, exceptional polynomials satisfy a second-order eigenvalue equation. The key difference is that the resulting polynomial family has a finite number of missing, exceptional degrees.

Let \( \lambda \) be a fixed partition and set \( N = |\lambda|, \; \ell = \ell(\lambda) \). Let \( k_1 > \cdots > k_\ell \) be the elements of the corresponding index set \( \mathcal{K}(\lambda) \) as per (6). In the existing literature, exceptional Hermite polynomials associated to the partition \( \lambda \) are defined as the Wronskian of classical Hermite polynomials,
\[ h^{(\lambda)}_{k+N-\ell} = \text{Wr}[h_{k_\ell}, \ldots, h_{k_1}, h_k], \quad k \notin \mathcal{K}(\lambda). \]

As was the case with classical Hermite polynomials, we define a bivariate version of exceptional Hermite polynomials. Observe that the map
\[ H_k \mapsto \text{Wr}[H_{k_\ell}, \ldots, H_{k_1}, H_k], \quad k \notin \mathcal{K}(\lambda), \]  \hfill (49)
changes the degree by \( N - \ell \). Set
\[ \mathcal{I}(\lambda) = \mathcal{J}(\lambda) + N, \]  \hfill (50)
and observe that if \( n \in \mathcal{J}(\lambda) \), then \( k = n - N + \ell \notin \mathcal{K}(\lambda) \) is a valid index for the Wronskian in (49). We are thus able to define a non-zero polynomial

\[
H_n^{(\lambda)} := \frac{\text{Wr}[H_{k_\ell}, \ldots, H_{k_1}, H_{n-N+\ell}]}{\prod_{i<j} (k_i - k_j) \prod_i (n - N + \ell - k_i)}, \quad n \in \mathcal{I}(\lambda)
\]  

(51)

Observe that the “degree shift” of the index in (51) ensures precisely that the exceptional polynomial \( H_n^{(\lambda)}(x, y) \) has degree \( n \) in \( x \). Furthermore, \( H_n^{(\lambda)}(x, y) \) is weighted-homogeneous relative to (39) and monic in \( x \).

**Proposition 4.1** The polynomial family \( \{H_n^{(\lambda)} : n \in \mathcal{I}(\lambda)\} \) is missing the exceptional degrees

\[
\mathcal{K}_N^{(\lambda)} = \{0, 1, \ldots, N - \ell - 1\} \cup \{\lambda_\ell + N - \ell, \ldots, \lambda_1 + N - 1\}.
\]  

(52)

**Proof** By the remark just after (13), we have

\[
H_n^{(\lambda)} := \frac{\text{Wr}[H_{k_N}, \ldots, H_{k_1}, H_n]}{\prod_{i<j} (k_i - k_j) \prod_i (n - k_i)}, \quad n \in \mathcal{I}(\lambda)
\]  

(53)

where \( k_1, \ldots, k_N \) is an enumeration of \( \mathcal{K}_N^{(\lambda)} \). By (5) and the remark that follows, \( \mathcal{K}_N^{(\lambda)} \) consists of non-negative elements of \( \mathcal{M}^{(\lambda)} + N \). Conclusion (52) follows because

\[
\lambda_1 - 1 > \lambda_2 - 2, \ldots > \lambda_\ell - \ell > -\ell - 1 > \ell - 2 > \cdots > -N, \ldots
\]

is a decreasing enumeration of \( \mathcal{M}^{(\lambda)} \).

Analogously to (37), the bivariate and univariate Wronskians are related by

\[
\text{Wr}[H_{k_\ell}, \ldots, H_{k_1}, H_k](x, y) = 2^{\frac{1}{2}(\ell+1)}(-y)^{(k+N-\ell)/2} \text{Wr}[h_{k_\ell}, \ldots, h_{k_1}, h_k]\left(\frac{x}{\sqrt{-4y}}\right)
\]  

(54)

Thus, one could define \( H_n^{(\lambda)}(x, y) \) by appropriately scaling and normalizing the univariate Wronskian \( \text{Wr}[h_{k_\ell}, \ldots, h_{k_1}, h_{n-\delta}] \), but (51) is more direct.

For notational convenience, let

\[
\tau^{(\lambda)}(x, y) := \frac{N!}{d_\lambda} S^{(\lambda)}(x, y, 0, \ldots),
\]  

(55)

with \( d_\lambda \) as per the hook-length formula (15). Inspection of (14) shows that \( \tau^{(\lambda)}(x, y) \) is a monic polynomial in \( x \) of degree \( N \) and weighted-homogeneous relative to (39). Hence, \( \tau^{(\lambda)}(x, y) \) is nothing other than the Schur function \( S^{(\lambda)} \) with all but the first two variables set to zero, renormalized so as to be monic. The notation was chosen to hint at the connection to the \( \tau \)-functions of integrable systems, but the main point here is the observation that the exceptional Hermite polynomials associated to a partition can be expressed simply in terms of the Schur functions produced from that partition via insertion:
Theorem 4.2 The exceptional Hermite polynomials are given by:

\[ H_n^{(\lambda)} = \tau_{(n-N)\lambda}, \quad n \in \mathcal{I}(\lambda). \]  

(56)

Proof By (13), (15) and (45), we have

\[ \tau^{(\lambda)} = \frac{\text{Wr}[H_{k_\ell}, \ldots, H_{k_1}]}{\prod_{i<j \leq n}(k_i - k_j)}. \]  

(57)

The desired conclusion now follows by (9).

Before continuing let us also note the following generalization of (45).

Corollary 4.3 Let \( \chi^{(\lambda)} \) be the character of the \( \lambda \)-irrep of the symmetric group on \( N \) objects, and \( c_j := (2^j, 1^{N-2j}) \) the indicated cycle type. Then,

\[ \text{Wr}[H_{k_\ell}, \ldots, H_{k_1}](x, y) = \prod_{i<j \leq n}(k_i - k_j) \sum_{i=0}^{[N/2]} \chi^{(\lambda)}(c_j) \frac{N!}{(N-2j)!j!} x^{N-2j} y^j \]  

(58)

This result was first announced in [3], where it was proved using a different method.

Next, define the differential operator

\[ T^{(\lambda)}(x, y, \partial_x) = 2y \partial_x^2 + \left(x - 4y \frac{\tau^{(\lambda)}_x(x, y)}{\tau^{(\lambda)}(x, y)}\right) \partial_x + \left(2y \frac{\tau^{(\lambda)}_{xx}(x, y)}{\tau^{(\lambda)}(x, y)} - x \frac{\tau^{(\lambda)}_x(x, y)}{\tau^{(\lambda)}(x, y)}\right). \]  

(59)

The above expression is called an exceptional operator because it admits polynomial eigenfunctions for all but the finite number of exceptional degrees in (52).

Proposition 4.4 The exceptional Hermite polynomials, \( H_n^{(\lambda)}, \quad n \in \mathcal{I}(\lambda) \) are eigenfunction of \( T^{(\lambda)} \) with

\[ T^{(\lambda)} H_n^{(\lambda)} = (n - N) H_n^{(\lambda)}. \]  

(60)

We postpone the proof until Proposition 4.7, below. Note that, since \( \tau^{(\emptyset)} = 1 \), the classical Hermite differential (41) is the particular case of the above result corresponding to the trivial partition.

Modulo certain regularity assumptions, the polynomials \( H_n^{(\lambda)}(x, y), \quad n \in \mathcal{I}(\lambda) \) constitute an orthogonal family. Say that \( \lambda \) is an even partition if \( \ell \) is even and if \( \lambda_{2i-1} = \lambda_{2i} \) for every \( i = 1, \ldots, \ell/2 \). Equivalently, \( \lambda \) is even if and only if \( \kappa^{(\lambda)}(m) \geq 0 \) for all \( m \in \mathcal{J}(\lambda) \). The following result was proved by Krein and Adler (see [7]).

Proposition 4.5 If \( \lambda \) is an even partition and \( y < 0 \) is fixed, then \( \tau^{(\lambda)}_y(x) := \tau^{(\lambda)}(x, y) \) has no real zeros.
Moreover, we have the following, proved in [7]. Set

\[ w_y^{(\lambda)}(x) = \frac{e^{\frac{x^2}{2y}}}{\tau_y^{(\lambda)}(x)^2}, \quad (61) \]

\[ \nu_m^{(\lambda)}(y) = 2(-\pi y)^{1/2}(-2y)^m (m + \ell)! \frac{\tau^{(\lambda)}(y)}{\kappa^{(\lambda)}(m)} = 2^{-\ell} \nu_{m+\ell}(y) \frac{\tau^{(\lambda)}(y)}{\kappa^{(\lambda)}(m)}, \quad m \in J^{(\lambda)}. \quad (62) \]

where

\[ \kappa^{(\lambda)}(m) = \prod_{i=1}^{\ell}(m - m_i(\lambda)). \quad (63) \]

**Proposition 4.6** If \( \lambda \) is an even partition and \( y < 0 \) is fixed, then the corresponding sequence of polynomials \( H_m^{(\lambda)}(x, y), \quad n \in J^{(\lambda)} \) is complete and orthogonal relative to \( w_y^{(\lambda)}(x)dx \). Indeed, for \( n_1, n_2 \in I^{(\lambda)} \), we have

\[ \int_{-\infty}^{\infty} H_{n_1}^{(\lambda)}(x, y)H_{n_2}^{(\lambda)}(x, y)w_y^{(\lambda)}(x)dx = \nu_{n_2-N}^{(\lambda)}(y)\delta_{n_1n_2}. \quad (64) \]

It turns out that the eigenvalue relation (60) and the orthogonality relation (64) are easier to express and understand if we change gauge and consider the following exceptional rational functions. Let

\[ \kappa_I^{(\lambda)}(m) = \prod_{k \in K^{(\lambda)}}(m - k) = \kappa^{(\lambda)}(m - l)F_{l-\ell}(m), \quad l \geq \ell, \quad (65) \]

denote the monic polynomial with simple zeroes precisely at the elements of \( K_I^{(\lambda)} \) as defined in (5). We can now define

\[ R_m^{(\lambda)} = \frac{\tau^{(m+\lambda)}}{\tau^{(\lambda)}} = \frac{H_m^{(\lambda)}}{\tau^{(\lambda)}}, \quad m \in J^{(\lambda)}. \quad (66) \]

Equivalently, for \( l \geq \ell \), we have

\[ R_k^{(\lambda)} = \frac{1}{\kappa_I^{(\lambda)}(k)} \frac{\text{Wr}[H_k, \ldots, H_k, H_k]}{\text{Wr}[H_{k_1}, \ldots, H_{k_1}]}, \quad k \notin K_I^{(\lambda)}. \quad (67) \]

The eigenvalue and orthogonality relations are now easier to formulate. Set

\[ \tilde{T}^{(\lambda)}(x, y, \partial_x) = (\tau^{(\lambda)})^{-1} \circ T^{(\lambda)} \circ \tau^{(\lambda)} = 2y\partial_x^2 + x\partial_x + 4y \left( \log \tau^{(\lambda)}(x, y) \right)_{xx} \quad (68) \]

**Proposition 4.7** With the above definitions, we have

\[ \tilde{T}^{(\lambda)}R_m^{(\lambda)} = mR_m^{(\lambda)}, \quad m \in J^{(\lambda)}. \quad (69) \]

This result was proved in [7] and [10], but we will give a novel, simplified proof in Section 4.4 once we introduce the intertwining operator.
Note that (64) may be restated quite simply as
\[ \left\langle R^{(\lambda)}_{m_1}(x, y), R^{(\lambda)}_{m_2}(x, y) \right\rangle_H = \delta_{m_1, m_2} v^{(\lambda)}_{m_1}(y), \quad m_1, m_2 \in J^{(\lambda)}, \] (70)
where the inner product is the same as in (43). The orthogonality of \( R^{(\lambda)}_m, m \in J^{(\lambda)} \) stems from the fact \( \tilde{T}^{(\lambda)} \) is a symmetric operator relative to the above inner product. This, in turn, is a consequence of the fact that the classical \( T(x, y, \partial_x) \) is symmetric relative to the same inner product, and the fact that \( \tilde{T} \) is a modification of \( T \) by a zeroth order term.

### 4.3 Semi-Stationary Wave Functions as Generating Functions

Thus far we have considered dynamical wave functions depending on the infinitely many variables of the KP hierarchy and stationary wave functions obtained from them by setting all time variables except the first equal to zero. It turns out that exceptional Hermite polynomials are best studied in the intermediate case in which the first \textit{and} second KP time variables are retained.

Note, for example, that the generating function (38) for the bivariate form of the classical Hermite polynomials is a restricted vacuum wave function in which all time variables \( t_i \) for \( i > 2 \) have been set to zero:
\[ \psi_0(x, y, 0, 0, \ldots, z) = \exp(xz + yz^2) = \Psi_0(x, y, z). \] (71)

The main result of this section is to demonstrate the exceptional Hermite polynomials are similarly generated by the wave functions of certain points in \( \text{Gr}^{\text{ad}} \) indexed by partitions. Many of their known properties and answers to some open questions concerning them can be derived from the bispectrality of these generating functions and Wilson’s bispectral involution. We will return to this point in the sections to follow.

Fix a partition \( \lambda \), and let \( N = |\lambda|, \ell = \ell(\lambda) \). Define \( W^{(\lambda)} \in \text{Gr}^{\text{ad}} \) as
\[ W^{(\lambda)}(z) := \text{span}\{z^m : m \in J^{(\lambda)}\}, \] (72)
where \( J^{(\lambda)} \) is the complement of the corresponding Maya diagram as per (7). Set
\[ C^{(\lambda)} = \text{span}\{\text{ev}(k, 0) : k \in K^{(\lambda)}\} \] (73)

**Proposition 4.8** We have \( W^{(\lambda)} = W_{C^{(\lambda)}} \).

**Proof** By construction, \( \ker C^{(\lambda)}(z) = \{z^k : k \in \mathbb{N}_0 \setminus K^{(\lambda)}\} \). By (7) and (72), it follows that
\[ W_{C^{(\lambda)}}(z) = z^{-\ell} \ker C^{(\lambda)}(z) = W^{(\lambda)}(z). \]
\[ \square \]

**Definition 4.9** We refer to
\[ \Psi^{(\lambda)}(x, y, z) := \psi_{W^{(\lambda)}}(x, y, 0, \ldots ; z), \] (74)
obtained by letting $x = t_1$, $y = t_2$ and $t_i = 0$ for $i > 2$ in the dynamical wave functions the semi-stationary wave function associated to $W^{(\lambda)}$.

Just as relation (38) shows that the vacuum wave function serves as a generating function for the classical Hermite polynomials, the semi-stationary wave function $\Psi^{(\lambda)}(x, y, z)$ serves as a generating function for the corresponding exceptional Hermite rational functions. To be more precise, we have the following.

**Theorem 4.10** We have

$$\Psi^{(\lambda)}(x, y, z) = \sum_{m \in \mathcal{J}^{(\lambda)}} \frac{\kappa^{(\lambda)}(m)}{(m + \ell)!} R_m^{(\lambda)}(x, y) z^m, \quad (75)$$

$$= \sum_{m = -\ell}^{\infty} \frac{\kappa^{(\lambda)}(m)}{(m + \ell)!} R_m^{(\lambda)}(x, y) z^m, \quad (76)$$

$$z^N \tau^{(\lambda)}(x, y) \Psi^{(\lambda)}(x, y, z) = \sum_{n \in \mathcal{I}^{(\lambda)}} \frac{\kappa_N^{(\lambda)}(n)}{n!} H_n^{(\lambda)}(x, y) z^n, \quad (77)$$

$$= \sum_{n = 0}^{\infty} \frac{\kappa_N^{(\lambda)}(n)}{n!} H_n^{(\lambda)}(x, y) z^n, \quad (78)$$

with $\kappa^{(\lambda)}(m), \kappa_N^{(\lambda)}(n)$ the polynomials defined in (63) and (65).

Note that $\kappa^{(\lambda)}(m) = 0$ precisely for those $m \geq -\ell$ for which $m \notin \mathcal{J}^{(\lambda)}$. Thus (76) is sensible despite the fact that $R_m$ is not defined when $m \notin \mathcal{J}^{(\lambda)}$. A similar remark applies to (78).

**Proof** Let $k_1 > \cdots > k_\ell$ be the elements of $\mathcal{K}^{(\lambda)}$ as per (6). By (73), $\text{ev}(k_i, 0), \ i = 1, \ldots, \ell$ is a basis for the annihilator of $z^\ell W^{(\lambda)}(z)$. By (38),

$$H_k(x, y) = \langle \text{ev}(k, 0)(z), \Psi_0(x, y, z) \rangle, \quad k \in \mathbb{N}_0.$$ 

Hence, by (19)

$$\Psi^{(\lambda)}(x, y, z) = \frac{\text{Wr}_x[H_{k_\ell}(x, y), \ldots, H_{k_1}(x, y), \Psi_0(x, y, z)]}{\text{Wr}_x[H_{k_\ell}(x, y), \ldots, H_{k_1}(x, y)] z^\ell} = \sum_{n = 0}^{\infty} \frac{\text{Wr}_x[H_{k_\ell}(x, y), \ldots, H_{k_1}(x, y), H_n(x, y)]}{\text{Wr}_x[H_{k_\ell}(x, y), \ldots, H_{k_1}(x, y)]} z^{n-\ell} \quad (79)$$

By (66), for $m \in \mathcal{I}^{(\lambda)}$ and $n = m + N$, we have

$$\frac{\text{Wr}_x[H_{k_\ell}(x, y), \ldots, H_{k_1}(x, y), H_n(x, y)]}{\text{Wr}_x[H_{k_\ell}(x, y), \ldots, H_{k_1}(x, y)]} = \kappa^{(\lambda)}(m) R_m^{(\lambda)}(x, y),$$

which entails (75). Relation (77) follows by (56), (66) and (65). \qed
By (21) and (55), the semi-stationary wave function can also be given as
\[
\Psi^{(\lambda)}(x, y, z) = \frac{\Phi^{(\lambda)}(x, y, z)}{\tau^{(\lambda)}(x, y)} \Psi_0(x, y, z) \tag{80}
\]
where
\[
\Phi^{(\lambda)}(x, y, z) = \frac{N!}{d^\lambda} S^{(\lambda)}(x - z^{-1}, y - 2z^{-2}, -3z^{-3}, \ldots, -Nz^{-N}), \tag{81}
\]
and where \(S^{(\lambda)}\) is the Schur function defined in (13). By (14), \(S^{(\lambda)}(t_1, \ldots, t_N)\) is weighted-homogeneous of degree \(N\) relative to the grading \(\text{deg } t_i = i\). It follows that \(\Phi^{(\lambda)}(x, y, z)\) is weighted-homogeneous of degree \(N\), relative to the grading
\[
\text{deg } x = 1, \quad \text{deg } y = 2, \quad \text{deg } z = -1. \tag{82}
\]

Let \(\tilde{U}^{(\lambda)}\) denote the \(\mathcal{P}\)-module spanned by exceptional Hermite polynomials:
\[
\tilde{U}^{(\lambda)}(x, y) = \text{span}\{H_n^{(\lambda)}(x, y) : n \in \mathcal{I}^{(\lambda)}\} \otimes \mathbb{C}[y], \tag{83}
\]
and let
\[
\tilde{W}^{(\lambda)} = (\tau^{(\lambda)})^{-1} \tilde{U}^{(\lambda)}. \tag{84}
\]

We will derive a number of results regarding exceptional Hermite polynomials by manipulating meromorphic generating functions that have a Laurent series expansion of the form
\[
\Psi(x, y, z) = \sum_{m \in \mathcal{J}^{(\lambda)}} F_m(x, y) z^m, \quad F_m \in \tilde{W}^{(\lambda)}. \tag{85}
\]

**Definition 4.11** For a given partition \(\lambda\), we will call \(\Phi(x, y, z) \in \mathbb{C}[x, y, z, z^{-1}]\) a \(\lambda\)-generator if
\[
\Psi(x, y, z) = \frac{\Phi(x, y, z)}{\tau^{(\lambda)}(x, y)} e^{xz + yz^2}
\]
has the form shown in (85). We will use \(\mathcal{F}^{(\lambda)}\) to denote the set of all \(\lambda\)-generators.

The semi-stationary wave function is the canonical example of a \(\lambda\)-generator with \(\Phi = \Phi^{(\lambda)}\). Also, observe that multiplication by a polynomial in \(y\) preserves (85). For this reason we regard \(\mathcal{F}^{(\lambda)}\) as a \(\mathcal{P}\)-module rather than a vector space.

In Section 4.5, we will characterize \(\mathcal{F}^{(\lambda)}\) in term of 1-point functionals.

### 4.4 The Intertwiner

Let \(\lambda\) be a partition. Let \(N = |\lambda|, \ell = \ell(\lambda)\), and let \(\Phi^{(\lambda)}\) be as in (81). Set
\[
\hat{K}^{(\lambda)}(x, y, z) = \frac{z^\ell \Phi^{(\lambda)}(x, y, z)}{\tau^{(\lambda)}(x, y)} = z^\ell + \sum_{i=1}^\ell \hat{K}_i(x, y) z^\ell - i, \tag{86}
\]
and observe that by (79), the coefficients \(\hat{K}_i(x, y) \in \mathbb{C}[x, y]\) are weighted-homogeneous of degree \(i = 1, \ldots, \ell\). Recalling the convention set forth in
Section 1.1 regarding the substitution of elementary derivative operators into multi-variable polynomials we then have the operator

$$K(\lambda)(x, y, \partial_x) := \partial_x^\ell + \sum_{i=1}^\ell K_i(x, y) \partial_x^{\ell-i},$$

which we refer to as the semi-stationary intertwining operator. The choice of terminology is justified by the following.

**Proposition 4.12** We have,

$$K(\lambda)(x, y, \partial_x)\Psi_0(x, y, z) = z^\ell \Psi(\lambda)(x, y, z).$$

**Proof** It suffices to observe that $\partial_x \Psi_0(x, y, z) = z \Psi_0(x, y, z)$. \qed

By (79), an equivalent definition of the intertwiner is

$$K(\lambda) f = \frac{\text{Wr}[H_{k_1}, \ldots, H_{k_\ell}, f]}{\text{Wr}[H_{k_1}, \ldots, H_{k_\ell}]}$$

where $k_1, \ldots, k_\ell$ enumerate the index set $K(\lambda)$.

By (14), we may write

$$\Phi(\lambda)(x, y, z) = \sum_{i=0}^N \Phi_i(y, z)x^{N-i},$$

where the coefficients

$$\Phi_i(y, z) \in \mathbb{C}[y, z, z^{-1}]$$

are weighted-homogeneous of degree $i$.

Also note that $\Phi_0(y, z) = 1$ as a consequence of (15); that is $\Phi(\lambda)(x, y, z)$ is monic in $x$.

**Lemma 4.13** Let $H(y, z)$ denote the umbral operator [23] whose action on a polynomial $\phi(x) = \sum_i \phi_i x^i$ is

$$H(y, z)\phi(x) := \sum_i \phi_i H_i(x + 2yz, y).$$

Then, $H(y, z)$ is a 1-parameter transformation group with respect to $y$; that is,

$$H(y_1 + y_2, z) = H(y_1, z) \circ H(y_2, z), \quad H(y, z)^{-1} = H(-y, z).$$

Moreover, for $\pi \in \mathbb{C}[x, y, z]$ and $\hat{\pi}(x, y, z) = H(y, z)\pi(x, y, z)$, we have

$$\pi(\partial_z, y, z)e^{yz+yz^2} = \hat{\pi}(x, y, z)e^{yz+yz^2}.$$  

**Proof** By (46), $H_n(x, y)$ is a Appell sequence. Hence,

$$H_j(x + 2yz, y) = \sum_{k=0}^j \binom{j}{k} H_{j-k}(2yz, y)x^k.$$
By (40), we have
\[ e^{-yz^2} \circ \partial_z \circ e^{yz^2} = \sum_{k=0}^{j} \binom{j}{k} H_{j-k}(2yz, y) \partial_z^k. \]
Hence,
\[ \hat{\pi}(\partial_y, y, z) = e^{-yz^2} \circ \pi(\partial_z, y, z) \circ e^{yz^2}. \]
Relation (91) follows. Moreover,
\[ \pi(\partial_z, y, z)e^{xz+yz^2} = e^{yz^2} \hat{\pi}(\partial_z, y, z)e^{xz} = \hat{\pi}(x, y, z)e^{xz+yz^2}. \]

Using (90) we now define the dual intertwining operator,
\[ K^{(\lambda)}(x, y, z) = H(-y, z)\Phi^{(\lambda)}(x, y, z) \]
\[ = H_N(x - 2yz, -y) + \sum_{i=1}^{N} \Phi_i(y, z)H_{N-i}(x - 2yz, -y) \] (93)

**Proposition 4.14** The dual intertwiner \( K^{(\lambda)}(\partial_z, y, z) \) is a monic differential operator of order \( N \). Moreover,
\[ K^{(\lambda)}(\partial_z, y, z)\Psi_0(x, y, z) = \tau^{(\lambda)}(x, y)\Psi^{(\lambda)}(x, y, z). \] (94)

**Proof** The first assertion now follows directly from the definition (93). By (91), we have \( \Phi(x, y, z) = H(y, z)K^{(\lambda)}(x, y, z) \). The second assertion now follows by (92). \( \square \)

One useful consequence of (94), is a formula giving the exceptional Hermite polynomials as a linear combination of the classical Hermite polynomials whose coefficients are obtained straightforwardly from \( K^{(\lambda)} \). Let \( K^\natural(n, y) \) be the unique polynomial characterized by the relation
\[ (K^{(\lambda)}(z, yz^2, \partial_z) \circ \partial_z^N)z^n = K^\natural(n, y)z^n. \] (95)

Note that \( \partial_z^k z^n = F_k(n)z^{n-k} \) where
\[ F_k(x) = \frac{\Gamma(x + 1)}{\Gamma(x - k + 1)} = \begin{cases} 1 & k = 0 \\ x(x - 1) \cdots (x - k + 1) & k = 1, 2, \ldots \\ ((x + 1)(x + 2) \cdots (x + k))^{-1} & k = -1, -2, \ldots \end{cases} \] (96)
denotes the generalized falling factorial. Thus,
\[ K^\natural(n, y) = \sum_{j=0}^{N} K^\natural_j(n)y^j = \sum_{j=0}^{N} \sum_{i=0}^{N-j} K_{ij} y^j F_{i+j}(n). \] (97)
where $K_{ij}$ are the coefficients of $K$ as per

$$K^{(\lambda)}(x, y, z) = \sum_{j=0}^{N} \sum_{i=0}^{N-j} K_{ij} x^i y^j z^{i+2j-N}. \quad (98)$$

**Theorem 4.15** The expression $K^{\sharp}(n, y)/\kappa_{N}^{(\lambda)}(n)$, where the denominator is the polynomial defined in (65), is a monic $N^{th}$-degree polynomial in $y$ whose coefficients are polynomials in $n$. The difference operator $K^{\sharp}(n, yS_{n}^{-2})$, maps sequences with support in $\mathbb{N}_0$ to sequences with support in $I^{(\lambda)}$. Moreover,

$$\kappa_{N}^{(\lambda)}(n)H^{(\lambda)}_{n}(x, y) = K^{\sharp}(n, yS_{n}^{-2})H_{n}(x, y), \quad n \in I^{(\lambda)}. \quad (99)$$

It will be instructive to reformulate this result in a more explicit manner. The Theorem claims that

$$\upsilon^{(\lambda)}_{j}(n) := \frac{K^{\sharp}_{j}(n)}{\kappa_{N}^{(\lambda)}(n)}, \quad j = 0, \ldots, N \quad (100)$$

are polynomials with $\upsilon^{(\lambda)}_{0}(n) = 1$. It also claims that

$$H^{(\lambda)}_{n}(x, y) = H_{n}(x, y) + \sum_{j=1}^{N} \upsilon^{(\lambda)}_{j}(n)y^{j}H_{n-2j}(x, y), n \in I^{(\lambda)}. \quad (101)$$

and that $\upsilon^{(\lambda)}_{j}(n) = 0$ if $n \in I^{(\lambda)}$ but $n - 2j < 0$.

**Lemma 4.16** The operator $K^{(\lambda)}(z, yz^{2}, \partial_{z}) \circ \partial_{z}^{N}$ maps $C[z]$ into $z^{N}W^{(\lambda)} \otimes C[y]$.

**Proof** Let

$$K_{j}(x, z) := \sum_{i=0}^{N-j} K_{ij} x^i z^{i+2j} \quad (102)$$

denote the coefficients of $z^{N}K^{(\lambda)}(x, y, z)$. Observe that

$$K^{(\lambda)}(z, yz^{2}, \partial_{z}) \circ \partial_{z}^{N} = \sum_{j=0}^{N} \sum_{i=0}^{N-j} K_{ij} y^{j}z^{i+2j} \partial_{z}^{i+2j} = \sum_{j=0}^{N} y^{j}z^{2j}K_{j}(z, \partial_{z})$$

By (75), (94), $K^{(\lambda)}(\partial_{z}, y, z)$ maps $C[z]$ into $W^{(\lambda)} \otimes C[y]$. It follows that each $K_{j}(\partial_{z}, z), \quad j = 0, \ldots, N$ maps $C[z]$ into $z^{N}W^{(\lambda)}$. Observe that $K_{j}(z, x)z^{2j} = K_{j}(x, z)x^{2j}$. Consequently,

$$z^{2j}K_{j}(z, \partial_{z}) = K_{j}(\partial_{z}, z) \circ \partial_{z}^{2j}, \quad j = 0, \ldots, N \quad (103)$$

also maps $C[z]$ into $z^{N}W^{(\lambda)}$. 

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Proof of Theorem 4.15. Let $K_j(x, z), j = 0, \ldots, N$ be as in (102). By the preceding Lemma, for each $j = 0, \ldots, N$, the operator $z^{2j} K_j(\partial_z, z)$ maps $\mathbb{C}[z]$ into $z^N W^{(\lambda)} = \{z^n : n \in \mathcal{I}(\lambda)\}$. Observe that
$$z^{2j} K_j(z, \partial_z) z^n = K_j^{\#}(n) z^n, \quad j = 0, \ldots, N.$$ Since $\mathbb{N}_0 = K_{N}^{(\lambda)} \cup \mathcal{I}(\lambda)$, it follows that $K_j^{\#}(n) = 0$ if and only if $n \in K_{N}^{(\lambda)}$. This proves that each $\upsilon^{(\lambda)}_j(n), j = 0, \ldots, N$ is a polynomial. We already remarked that $K^{(\lambda)}_j(\partial_z, y, z)$ is a monic operator of order $N$. By (103), we have $K_0^{\#}(n) z^n = K_0(\partial_z, z) z^n$. Hence $K_0^{\#}(n)$ is a monic polynomial of degree $N$. From this, it follows that $\upsilon_0^{(\lambda)}(n) = 1$. By (103), $K_j^{\#}(n) = 0$ if $n < 2j$. Therefore, $\upsilon_j^{(\lambda)}(n) = 0$ if $n \in \mathcal{I}(\lambda)$ and $n - 2j < 0$.

To prove (101), observe that

$$z^N \tau^{(\lambda)}(x, y) \Psi^{(\lambda)}(x, y, z) = z^N K^{(\lambda)}(\partial_z, y, z) \Psi_0(x, y, z) = \sum_{n=0}^{\infty} \sum_{j=0}^{N} y^j H_n(x, y) K_j^{\#}(n) \frac{z^n}{n!} \quad (104)$$

Hence,

$$\sum_{n\in \mathcal{I}(\lambda)} \kappa^{(\lambda)}_N(n) \left( \sum_{j=0}^{N} y^j H_n(x, y) \upsilon^{(\lambda)}_j(n) \frac{z^n}{n!} \right) = \sum_{n\in \mathcal{I}(\lambda)} H_n^{(\lambda)}(x, y) \kappa^{(\lambda)}_N(n) \frac{z^n}{n!}.$$  

4.5 Exceptional One-point Functionals

Recall from (72) and (84) that $W^{(\lambda)}$ is the span of monomials corresponding to the Maya diagram $\mathcal{M}^{(\lambda)}$, and that $W^{*\lambda}$ is the span of the exceptional Hermite rational functions. In this section we will show that $W^{(\lambda)}$ and $W^{*\lambda}$ have a dual relation with respect to the bispectral involution on $\text{Grad}$. In effect, this serves as a characterization of the 1-point functionals that annihilate the exceptional Hermite polynomials. As a byproduct, we will obtain a characterization of $\mathcal{F}^{(\lambda)}$, the module of $\lambda$-generators, in terms of 1-point functionals.

Although the semi-stationary $\Psi^{(\lambda)}(x, y, z)$ depends only on three variables, it is also possible to write it in terms of the stationary wave function depending only on two variables — provided we interpret the dependence on $y$ as a curve in $\text{Grad}$. For $c \in \mathbb{D}$, let $\hat{c}_y$ denote the 1-parameter family of functionals defined by

$$\langle \hat{c}_y(z), f(z) \rangle = \left\langle c(z), e^{yz^2} f(z) \right\rangle. \quad (105)$$
In general, the coefficients of $\hat{c}_y$ involve exponential functions of $y$. However, for $c \in \mathbb{D}_0$ the coefficients are polynomials; that is, if $c = \text{ev}(n, 0)$, then $\hat{c}_y \in \mathbb{C}[y] \otimes \mathbb{D}_0$. Explicitly, by (40) and (45),

$$\text{ev}(n, 0)_y = \sum_{k=0}^{n} \binom{n}{k} H_k(0, y) \text{ev}(n-k, 0)$$

(106)

$$= \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2j)!j!} y^j \text{ev}(n-2j, 0)$$

One can then extend (106) by linearity to all $\mathbb{D}_0$. For $C \subset \mathbb{D}_0$, let $\hat{C} \subset \mathbb{C}[y] \otimes \mathbb{D}_0$ be the corresponding 1-parameter family of functionals. Let

$$\hat{W}_y^{(\lambda)} := W_{\hat{C}_y^{(\lambda)}}$$

where $\hat{C}_y = \{\hat{c}_y : c \in C\}$. (107)

be the corresponding curve in $\text{Gr}^\text{ad}$.

**Proposition 4.17** With the above definitions, we have

$$\Psi^{(\lambda)}(x, y, z) = \psi_{\hat{W}_y^{(\lambda)}}(x, z)e^{yz^2}. \quad (108)$$

**Proof** By definition, $C_\lambda$ is spanned by $\text{ev}(k_i, 0), \ i = 1, \ldots, \ell$. By (38) and (105),

$$H_n(x, y) = \left\{c(z), \exp(xz + yz^2)\right\} = \{\hat{c}_y(z), e^{xz}\}, \quad c = \text{ev}(n, 0).$$

Hence, by (29),

$$e^{yz^2} \psi_{\hat{W}_y^{(\lambda)}}(x, z) = \frac{\text{Wr}_x[H_{k_\ell}(x, y), \ldots, H_{k_1}(x, y), e^{xz}e^{yz^2}]}{\text{Wr}_x[H_{k_\ell}(x, y), \ldots, H_{k_1}(x, y)]z^\ell} = \Psi^{(\lambda)}(x, y, z).$$

\[\Box\]

Our next observation is the following characterization of $W^{(\lambda)}$ as a point in $\text{Gr}^\text{ad}$.

**Proposition 4.18** We have $\beta(W^{(\lambda)}) = W^{(\lambda)}$; i.e., $W^{(\lambda)} \in \text{Gr}^\text{ad}$ is a fixed point of the bispectral involution.

**Proof** By definition of the stationary wave function; c.f., (19) and (29),

$$\psi_{W^{(\lambda)}}(x, z) = z^{-\ell} \frac{\text{Wr}_x[x^{k_1}, \ldots, x^{k_\ell}, e^{xz}]}{\text{Wr}_x[x^{k_1}, \ldots, x^{k_\ell}]}$$

$$= \prod_{i<j} (k_i - k_j)x^{-|\lambda|}z^{-\ell}e^{xz}$$

$$\begin{vmatrix} x^{k_1} & k_1x^{k_1-1} & \ldots & F_{\ell}(k_1)x^{k_1-\ell} \\ \vdots & \vdots & \ddots & \vdots \\ x^{k_\ell} & k_\ell x^{k_\ell-1} & \ldots & F_{\ell}(k_\ell)x^{k_\ell-\ell} \\ 1 & z & \ldots & z^\ell \end{vmatrix}$$

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where \( F_j(a) := a(a-1) \cdots (a-j+1) \). By inspection, the coefficient of \( z^j \) in the above determinant is a constant times \( x^p \) where
\[
p = \sum_{i=1}^{\ell} k_i - \frac{1}{2} \ell (\ell + 1) + j = |\lambda| - \ell + j.
\]
It follows that \( \psi_{W(\lambda)}(x, z) \) is a linear combination of monomials of the form \( (xz)^j e^{yz^2}, \quad j = 0, \ldots, \ell \). Therefore, \( \psi_{W(\lambda)}(x, z) = \psi_{W(\lambda)}(z, x) \).

Recall that \( \hat{W}^{(\lambda)}(x, y) = \text{span}\{R^{(\lambda)}_m(x, y) : m \in J^{(\lambda)}(x, y)\} \otimes \mathbb{C}[y] \). For a fixed \( y \in \mathbb{C} \), let \( W_y^{(\lambda)} \) denote the vector space obtained by restricting \( \hat{W}^{(\lambda)} \) to that particular value of \( y \). Starting from (58), a straightforward calculation shows that
\[
R^{(\lambda)}_m(x, 0) = x^m, \quad m \in J^{(\lambda)}(x, 0).
\]
Thus, by proposition 4.18,
\[
W_0^{(\lambda)} = \hat{W}_0^{(\lambda)} = \beta(\hat{W}_0^{(\lambda)}).
\]
We now show that \( W_y^{(\lambda)} \in \text{Gr}^{ad} \) for all \( y \).

**Theorem 4.19** The exceptional Hermite polynomials are determined by conditions that are dual under the bispectral involution. To be more precise:
\[
U_y^{(\lambda)} = \tau_y^{(\lambda)} \beta(\hat{W}_y^{(\lambda)}), \quad y \in \mathbb{C}.
\] (109)
where \( U_y^{(\lambda)} \subset \mathcal{P} \) denotes the restriction to a particular value of \( y \in \mathbb{C} \).

**Proof** It suffices to show that \( W_y^{(\lambda)} = \beta(\hat{W}_y^{(\lambda)}) \) for all \( y \in \mathbb{C} \). Fix a \( y \in \mathbb{C} \). By (31) and (108),
\[
\psi_{\beta(\hat{W}_y^{(\lambda)})}(z, x) = \psi^{(\lambda)}(x, y, z) e^{-yz^2}.
\]
By (81) and (80), for \( y, z \) fixed \( \tau^{(\lambda)}(x, y) \psi^{(\lambda)}(x, y, z) \) is either regular or has removable singularities for all \( x \in \mathbb{C} \). Hence, \( \tau_y^{(\lambda)} \beta(\hat{W}_y^{(\lambda)}) \) is a polynomial subspace for every value of \( y \). Thus, it becomes possible to define \( \check{C}^{(\lambda)} \) as the space of continuous curves in \( c_y \in \mathbb{D} \) such that \( c_y \in \text{Ann} \tau_y^{(\lambda)} \beta(\hat{W}_y^{(\lambda)}) \) for all \( y \). Hence, \( \psi_y^{(\lambda)}(x, y) e^{-yz^2}, \quad y \in \mathbb{C} \) has the form shown in (30) with \( C = \check{C}_y^{(\lambda)} \) and \( q_C(z) = \tau^{(\lambda)}(z, y) \). Moreover, by Proposition 3.5, \( c \in \check{C}_y^{(\lambda)} \) if and only if
\[
\left\langle c_y(x), \Phi^{(\lambda)}(x, y, z) e^{xz+yz^2} \right\rangle = 0, \quad y \in \mathbb{C}.
\]
By (77), for every \( c \in \mathbb{D} \), we have
\[
z^N \left\langle c(x), \Phi^{(\lambda)}(x, y, z) e^{xz+yz^2} \right\rangle = \sum_{n \in \mathcal{I}^{(\lambda)}} \kappa^{(\lambda)}_n(n) \left\langle c(x), H_n^{(\lambda)}(x, y) \right\rangle \frac{z^n}{n!}.
\]
By (81) and (80), the above relation is the Taylor series of an entire function. Hence, 
\[ c \in \hat{C}^{(\lambda)} \text{ if and only if } \left\langle c_y(x), H_n^{(\lambda)}(x, y) \right\rangle = 0, \quad n \in J^{(\lambda)}, \ y \in \mathbb{C}. \]

It follows that \( U_y^{(\lambda)} = \text{Ker} \hat{C}_y^{(\lambda)}, \ y \in \mathbb{C} \), as was to be shown.

As a particular case, by (54), the elements of \( \hat{C}^{(\lambda)}(-1/4) \) are the one-point differential functionals that annihilate the univariate exceptional Hermite polynomials \( h_n^{(\lambda)}, \ n \in J^{(\lambda)}. \)

We are now able to provide the following alternate characterization of \( J^{(\lambda)}, \) the module of \( \lambda \)-generators.

**Proposition 4.20** A \( \Phi(x, y, z) \in \mathbb{C}[x, y, z, z^{-1}] \) is a \( \lambda \)-generator if and only if \( z^\ell \Phi(x, y, z) \) is a polynomial and if

\[
\left\langle c(z), z^\ell \Phi(x, y, z) e^{xz+yz^2} \right\rangle \equiv 0 \quad \text{for all } c \in C^{(\lambda)}; \quad \text{and} \\
\left\langle c_y(x), \Phi(x, y, z) e^{xz+yz^2} \right\rangle \equiv 0 \quad \text{for all } c \in \hat{C}^{(\lambda)}, \ y \in \mathbb{C}. \tag{111}
\]

**Proof** Suppose that

\[
\Phi(x, y, z) e^{xz+yz^2} = \sum_{m \in J^{(\lambda)}} F_m(x, y) z^m, \quad F_m \in \hat{U}^{(\lambda)}. \tag{112}
\]

Then, for \( c \in C^{(\lambda)} \) we have

\[
\left\langle c(z), z^\ell \Phi(x, y, z) e^{xz+yz^2} \right\rangle = \sum_{m \in J^{(\lambda)}} F_m(x, y) \left\langle c(z), z^{m+\ell} \right\rangle = 0.
\]

Similarly, for \( c \in \hat{C}^{(\lambda)}, \)

\[
\left\langle c_y(x), \Phi(x, y, z) e^{xz+yz^2} \right\rangle = \sum_{m \in J^{(\lambda)}} \left\langle c_y(x) F_m(x, y) \right\rangle z^{m+\ell} = 0
\]

Conversely, suppose that \( z^\ell \Phi(x, y, z) \) is a polynomial and that (110) and (111) hold. By (110) it follows that

\[
\Phi^{(\lambda)}(x, y, z) e^{xz+yz^2} = \sum_{m \in J^{(\lambda)}} F_m(x, y) z^m
\]

where \( F_n(x, y) \) are polynomials. By Theorem 4.19 and by (111), each \( F_m \in \hat{U}^{(\lambda)}. \)
5 Bispectrality

5.1 The Stabilizer Algebras

For a partition $\lambda$, let $\mathcal{A}^{(\lambda)}$ be the algebra of differential operators that preserve $W^{(\lambda)}$. Let $\mathcal{A}^{*\lambda}$ be the dual algebra of differential operators that preserve $W^{*\lambda}$. To be more precise, $\pi \in \mathcal{A}^{(\lambda)}$ if and only if

$$\pi(\partial_z, z) W^{(\lambda)}(z) \subset W^{(\lambda)}(z)$$

and $\sigma \in \mathcal{A}^{*\lambda}$ if and only if

$$\sigma(x, y, \partial_x) W^{*\lambda}(x, y) \subset W^{*\lambda}(x, y).$$

In the subsequent sections we will see that the operators and eigenvalues associated with the exceptional Hermite bispectral triple belong to certain commutative subalgebras of $\mathcal{A}^{(\lambda)}$ and $\mathcal{A}^{*\lambda}$.

To gain a better understanding of $\mathcal{A}^{(\lambda)}$, we introduce basic homogeneous operators whose action on a monomial either annihilates that monomial or shifts its degree. Set

$$\mathcal{G}^{(\lambda)}_k := (\mathcal{M}^{(\lambda)} + k) \cap \mathcal{J}^{(\lambda)}, \quad k \in \mathbb{Z}$$

(113)

$$\gamma^{(\lambda)}_k (m) := \prod_{i \in \mathcal{G}^{(\lambda)}_k} (m - i),$$

(114)

Let $G^{(\lambda)}_k$ be the differential operator defined by

$$G^{(\lambda)}_k (\partial_z, z) := z^{-k} \gamma^{(\lambda)}_k (z \partial_z), \quad k \in \mathbb{Z},$$

(115)

where

$$E(\partial_z, z) = z \partial_z,$$

is the Cauchy-Euler operator in $z$.

**Proposition 5.1** The operator algebra $\mathcal{A}^{(\lambda)}$ is generated by $E$ (the Cauchy-Euler operator) and by $G^{(\lambda)}_k$, $k \in \mathbb{Z}$.

**Proof** By construction,

$$E(\partial_z, z) z^m = mz^m$$

(116)

$$G^{(\lambda)}_k (\partial_z, z) z^m = \gamma^{(\lambda)}_k (m) z^{m-k}.$$  

(117)

The Cauchy-Euler operator preserves $W^{(\lambda)}$ because the latter is generated by monomials, and because (116) holds. By (117), we have $z^m \in W^{(\lambda)}(z)$ and $z^{m-k} \notin W^{(\lambda)}(z)$ if and only if $m \notin \mathcal{J}^{(\lambda)}$ and $m - k \notin \mathcal{M}^{(\lambda)}$. The latter is true if and only if $m \in \mathcal{G}^{(\lambda)}_k$, if and only if $\gamma^{(\lambda)}_k (m) = 0$. For this reason, $G^{(\lambda)}_k$ restricts to a well-defined linear transformation of $W^{(\lambda)}$. 


Conversely, since $W(\lambda)$ is generated by monomials, $A(\lambda)$ is generated by homogeneous operators $G(\partial_z, z)$ with the property that $G(\partial_z, z)z^m \in W(\lambda)(z)$, $m \in J(\lambda)$. If $G(x, z)$ has weighted-degree $k$, then it must be of the form

$$G(\partial_z, z) = z^{-k}G^z(z\partial_z),$$

where $G^z(m)$ is a polynomial that vanishes on all $m \in G^{(\lambda)}_k$. It follows that $G^z(m) = y_k^{(\lambda)}(m)\alpha(m)$ for some polynomial $\alpha(m)$, and hence that

$$G(\partial_z, z) = G^{(\lambda)}_k(\partial_z, z) \circ \alpha(z\partial_z),$$

as was to be shown.

The sequence of integers

$$g^{(\lambda)}_q := \#G^{(\lambda)}_q = \deg y_q^{(\lambda)}(m) \quad (118)$$

is a key combinatorial signature of the partition $\lambda$ and is intimately connected with the structure of the stabilizer rings $R^{(\lambda)}$ and $\mathcal{R}^{(\lambda)}$ that will be introduced in the following section. For now, we note the following symmetry property.

**Proposition 5.2** For $q \in \mathbb{Z}$, we have

$$g^{(\lambda)}_q = g^{(\lambda)}_{-q} + q. \quad (119)$$

**Proof** Without loss of generality, suppose that $q \geq 1$.

Since $\lambda_i = 0$ for $i > \ell$ we have that

$$m_{\ell+q+j}(\lambda) + q = -\ell - j = m_{\ell+j}(\lambda), \quad j = 1, 2, \ldots.$$

Hence, by (63), (65) and (113), we have

$$G^{(\lambda)}_q = \{m_1 + q, \ldots, m_{\ell+q} + q, \ldots, m_{\ell+q+j} + q, \ldots\} \setminus \{m_1, \ldots, m_{\ell}, \ldots, m_{\ell+j}, \ldots\} = \{m_1 + q, \ldots, m_{\ell+q} + q\} \setminus \{m_1, \ldots, m_{\ell}\} = (\{m_1 + q, \ldots, m_{\ell} + q\} \setminus \{m_1, \ldots, m_{\ell}\}) \sqcup \{m_{\ell+1} + q, \ldots, m_{\ell+q} + q\},$$

where $\sqcup$ denotes a disjoint union and $m_i = m_i(\lambda)$. By similar reasoning,

$$G^{(\lambda)}_{-q} = \{m_1 - q, \ldots, m_{\ell} - q\} \setminus \{m_1, \ldots, m_{\ell+q}\} = \{m_1 - q, \ldots, m_{\ell} - q\} \setminus \{m_1, \ldots, m_{\ell}\}$$

$$G^{(\lambda)}_{-q} + q = \{m_1, \ldots, m_{\ell}\} \setminus \{m_1 + q, \ldots, m_{\ell} + q\}$$

If $A, B$ are finite sets of equal cardinality, then $A \setminus B$ and $B \setminus A$ also have equal cardinality. It follows that

$$\#G^{(\lambda)}_{-q} = \#(G^{(\lambda)}_{-q} + q) = \#(\{m_1 + q, \ldots, m_{\ell} + q\} \setminus \{m_1, \ldots, m_{\ell}\}) = \#G^{(\lambda)}_{-q} - q.$$

The following technical result is fundamental in the manipulation of generating functions based on $\mathcal{F}^{(\lambda)}$. 

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Lemma 5.3 (Reindexing Lemma) We have

\[ \kappa^{(\lambda)}(m)\gamma^{(\lambda)}_k(m) = \kappa^{(\lambda)}(m-k)\gamma^{(\lambda)}_{-k}(m-k)F_k(m+\ell), \quad k \in \mathbb{Z}. \]  

(120)

Proof Fix \( \lambda \) and let \( m_i = m_i(\lambda), \quad i \in \mathbb{N} \) be as per (3). By inspection, (120) holds for \( k = 0 \). By the definition of the generalized falling factorial (96),

\[ F_{-k}(m)F_k(m+\ell) = 1. \]  

(121)

Thus (120) is equivalent to

\[ \kappa^{(\lambda)}(m-k)\gamma^{(\lambda)}_{-k}(m-k) = \kappa^{(\lambda)}(m)\gamma^{(\lambda)}_k(m)F_{-k}(m+\ell), \quad k \in \mathbb{Z}. \]

Hence, no generality is lost if we assume that \( k \geq 1 \). Set

\[ I_k = \{m_1 + k, \ldots, m_\ell + k\} \cap \{m_1, \ldots, m_\ell\} \]

As was shown in the Proof to Proposition 5.2,

\[ G^{(\lambda)}_k = (\{m_1 + k, \ldots, m_\ell + k\} \setminus \{m_1, \ldots, m_\ell\}) \cup \{m_{\ell+1} + k, \ldots, m_{\ell+k} + k\} \]

\[ G^{(\lambda)}_{-k} = \{m_1 - k, \ldots, m_\ell - k\} \setminus \{m_1, \ldots, m_\ell\} \]

Hence,

\[ \gamma^{(\lambda)}_k(m) = \prod_{k \in G^{(\lambda)}_k} (m-k) = \frac{\prod_{i=1}^{\ell+k} (m-m_i - k)}{\prod_{k \in I_q} (m-k)} = \frac{\kappa^{(\lambda)}_{\ell+k}(m+\ell)}{\prod_{k \in I_q} (m-k)} \]

\[ \gamma^{(\lambda)}_{-k}(m-k) = \prod_{k \in G^{(\lambda)}_{-k}} (m-k) = \frac{\prod_{i=1}^{\ell} (m-m_i)}{\prod_{k \in I_q} (m-k)} = \frac{\kappa^{(\lambda)}(m)}{\prod_{k \in I_q} (m-k)} \]

\[ \kappa^{(\lambda)}(m-k)F_k(m+\ell) = \kappa^{(\lambda)}_{\ell+k}(m+\ell) \]

The desired relation (120) follows immediately. \( \square \)

The algebras \( A^{(\lambda)} \) and \( \tilde{A}^{(\lambda)} \) are closely related to \( F^{(\lambda)} \). For \( \pi(x,y,z) \) that is polynomial in \( x \) set

\[ \Phi_{\pi}(x,y,z) := \tau^{(\lambda)}(x,y)e^{-xz-yz^2}\pi(\partial_z, y, z)\Psi^{(\lambda)}(x,y,z). \]  

(122)

Equivalently,

\[ \pi(\partial_z, y, z)\Psi^{(\lambda)}(x,y,z) = \frac{\Phi_{\pi}(x,y,z)}{\tau^{(\lambda)}(x,y)}e^{xz+y^2z^2}. \]

Recall that a \( \lambda \)-generator is a function of three variables: \( \Phi(x,y,z) \). However a \( \pi(x,z) \in A^{(\lambda)}(x,z) \) is a function of only two variables. Thus, we need to consider linear combinations of operators in \( A^{(\lambda)} \) with polynomial coefficients to establish the isomorphism \( F^{(\lambda)}(x,y,z) \cong A^{(\lambda)}(x,z) \otimes \mathbb{C}[y] \). For the sake of convenience, we will denote the latter simply as \( F^{(\lambda)} \cong A^{(\lambda)} \otimes \mathcal{P} \).

Proposition 5.4 If \( \pi \in A^{(\lambda)} \), then \( \Phi_{\pi} \in F^{(\lambda)} \). Conversely, for each \( \Phi \in F^{(\lambda)} \) there exists a \( \pi \in A^{(\lambda)} \otimes \mathcal{P} \) such that \( \Phi = \Phi_{\pi} \).
Proof Suppose that $\pi = E$ is the Cauchy-Euler operator. We have
\[
E(\partial_z, z)\Psi^{(\lambda)}(x, y, z) = \frac{\Phi_E(x, y, z)}{\tau^{(\lambda)}(x, y)}e^{xz+yz^2} = \sum_{m=-\ell}^{\infty} \frac{\kappa^{(\lambda)}(m) m!}{(m + \ell)!} R^{(\lambda)}(x, y)z^m
\]
Hence, $\Phi_E \in \mathcal{F}^{(\lambda)}$ by inspection. Next, suppose that $\pi = G_k^{(\lambda)}$ for some $k \in \mathbb{Z}$. By (117) and the reindexing Lemma,
\[
G_k^{(\lambda)}(\partial_z, z)\Psi^{(\lambda)}(x, y, z) = \sum_{m=-\ell}^{\infty} \frac{\kappa^{(\lambda)}(m - k) \gamma^{(\lambda)}(-k) F_k(m + \ell)}{(m + \ell)!} R^{(\lambda)}(x, y)z^{m-k}
\]
The last step is justified by the fact that if $m-k < -\ell$, then $F_k(m + \ell) = 0$. Hence, $\Phi_\pi \in \mathcal{F}^{(\lambda)}$ in this case also. The case of the general $\pi \in \mathcal{A}^{(\lambda)}$ now follows by Proposition 5.1.

Conversely, let $\Phi \in \mathcal{F}^{(\lambda)}$ be given. Set $\hat{\Phi}(x, y, z) = H(-y, z)\Phi(x, y, z)$. Hence, by Lemma 4.13, $\hat{\Phi}(\partial_z, y, z)e^{xz+yz^2} = \Phi(x, y, z)e^{xz+yz^2}$. By Proposition 4.20,
\[
\left\{ c_y(x), \hat{\Phi}(x, y, z)e^{xz+yz^2} \right\} \equiv 0
\]
for all $c \in \mathcal{C}^{(\lambda)}$. The kernel of $K^{(\lambda)}(\partial_z, y, z)$ consists of
\[
\psi_c(y, z) := \left\{ c_y(x), e^{xz+yz^2} \right\}, \quad c \in \mathcal{C}^{(\lambda)}.
\]
Observe that
\[
\hat{\Phi}(\partial_z, y, z)\psi_c(y, z) = \left\{ c_y(x), \hat{\Phi}(\partial_z, y, z)e^{xz+yz^2} \right\} = 0, \quad c \in \mathcal{C}^{(\lambda)}.
\]
Hence, there exists a $\pi(\partial_z, y, z)$ such that $\hat{\Phi}(\partial_z, y, z) = \pi(\partial_z, y, z) \circ K^{(\lambda)}(\partial_z, y, z)$. Hence, by (94),
\[
\tau^{(\lambda)}(x, y)\pi(\partial_z, y, z)\Psi^{(\lambda)}(x, y, z) = \pi(\partial_z, y, z) \circ K^{(\lambda)}(\partial_z, y, z)e^{xz+yz^2}
\]
\[
= \Phi(x, y, z)e^{xz+yz^2}
\]
Similarly, for $\sigma(x, y, z)$ that is polynomial in $z$, set
\[
\Phi_\sigma(x, y, z) := \tau^{(\lambda)}(x, y)e^{-xz-yz^2} \sigma(x, y, \partial_x)\Psi^{(\lambda)}(x, y, z).
\]

Proposition 5.5 If $\sigma \in \mathcal{A}^{(\lambda)}$, then $\Phi_\sigma \in \mathcal{F}^{(\lambda)}$. Conversely for each $\Phi \in \mathcal{F}^{(\lambda)}$ there exists a $\sigma \in \mathcal{A}^{(\lambda)}$ such that $\Phi = \Phi_\sigma$. 

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Proof Let \( \sigma \in \hat{A}^{(\lambda)} \) be given. Hence,

\[
\sigma(x, y, \partial_x) \Psi(\lambda)(x, y, z) = \frac{\Phi_\sigma(x, y, z)}{z^\ell \tau(\lambda)(x, y)} e^{xz+yz^2}
\]

\[
= \sum_{m \in J(\lambda)} \sigma(x, y, \partial_x) R^{(\lambda)}_m(x, y) \kappa(\lambda)(m) \frac{z^m}{(m+\ell)!}.
\]

By assumption, \( \sigma(x, y, \partial_x) R^{(\lambda)}_m(x, y) \in \hat{W}^{(\lambda)}(x, y) \). Therefore, \( \Phi_\sigma \in \mathcal{F}(\lambda) \).

Conversely, let \( \Phi \in \mathcal{F}(\lambda) \) be given. By Proposition 4.20,

\[
\left\{ c(z), z^\ell \Phi(x, y, z) e^{xz+yz^2} \right\} = 0
\]

for all \( c \in C(\lambda) \). The kernel of \( \tilde{K}^{(\lambda)}(x, y, \partial_x) \) is spanned by \( H_{k_1}(x, y), \ldots, H_{k_\ell}(x, y) \). Let \( c_i = ev(k_i, 0), \ i = 1, \ldots, \ell \) and recall that

\[
H_{k_i}(x, y) = \left\{ c_i(z), e^{xz+yz^2} \right\}.
\]

Hence,

\[
\Phi(x, y, \partial_x) H_{k_i}^{(\lambda)}(x, y) = \left\{ c_i(z), \Phi(x, y, \partial_x) e^{xz+yz^2} \right\} = 0, \ i = 1, \ldots, \ell.
\]

Hence, there exists a \( \sigma(x, y, \partial_x) \) such that

\[
\Phi(x, y, \partial_x) = \tau(\lambda)(x, y) \sigma(x, y, \partial_x) \circ \tilde{K}^{(\lambda)}(x, y, \partial_x).
\]

Hence, by (94),

\[
\tau^{(\lambda)}(x, y) z^\ell \sigma(x, y, \partial_x) \Psi^{(\lambda)}(x, y, z) = \tau(\lambda)(x, y) \sigma(x, y, \partial_x) \tilde{K}^{(\lambda)}(x, y, \partial_x) e^{xz+yz^2}
\]

\[
= \Phi(x, y, \partial_x) e^{xz+yz^2} = \Phi(x, y, z) e^{xz+yz^2} \quad \square
\]

Proposition 5.6 For every \( \pi \in A^{(\lambda)} \otimes \mathcal{P} \), there exists a \( \pi^b \in \hat{A}^{(\lambda)} \) such that

\[
\pi(\partial_z, y, z) \Psi^{(\lambda)}(x, y, z) = \pi^b(x, y, \partial_x) \Psi^{(\lambda)}(x, y, z).
\] (124)

The corresponding mapping \( A^{(\lambda)} \otimes \mathcal{P} \rightarrow \hat{A}^{(\lambda)} \) is an algebra anti-isomorphism.

Proof Let \( \pi \in A^{(\lambda)} \otimes \mathcal{P} \) and let \( \Phi_\pi \) be the polynomial given by (122). By Proposition 5.5, there exists a \( \pi^b \in \hat{A}^{(\lambda)} \) such that

\[
\pi^b(x, y, \partial_x) \Psi^{(\lambda)}(x, y, z) = \frac{\Phi(x, y, z)}{z^\ell \tau^{(\lambda)}(x, y, z)} \Psi_0(x, y, z) = \pi(\partial_z, y, z) \Psi^{(\lambda)}(x, y, z).
\]

Given, \( \pi_1, \pi_2 \in A^{(\lambda)} \otimes \mathcal{P} \) observe that

\[
(\pi_1(\partial_z, y, z) \circ \pi_2(\partial_z, z)) \Psi^{(\lambda)}(x, y, z) = \pi_1(\partial_z, y, z) \left( \pi_2^b(x, y, \partial_x) \Psi^{(\lambda)}(x, y, z) \right)
\]

\[
= \pi_2^b(x, y, \partial_x) \left( \pi_1(\partial_z, y, z) \Psi^{(\lambda)}(x, y, z) \right)
\]

\[
= (\pi_2^b(x, y, \partial_x) \circ \pi_1^b(x, y, \partial_x)) \Psi^{(\lambda)}(x, y, z)
\]
Therefore $\pi \mapsto \pi^b$ is an anti-homomorphism. By Proposition 5.4 this anti-homomorphism is onto. Suppose that $\pi(\partial_z, y, z)$ is an operator that annihilates $\Psi^{(\lambda)}(x, y, z)$. Hence, it must annihilate every $z^m$, $m \in J^{(\lambda)}$. This implies that $\pi = 0$, and therefore $\pi \mapsto \pi^b$ is also one-to-one. 

See [1] for a similar use of anti-isomorphisms in the study of bispectrality. A fundamental instance of the anti-isomorphism (124) is the relation

$$E(\partial_z, z)\Psi^{(\lambda)}(x, y, z) = ˜T(x, y, \partial x)\Psi^{(\lambda)}(x, y, z)$$

where $E$ is the Cauchy-Euler operator and where $\tilde{T}$ is the exceptional operator given in (68). An examination of the generating function (75) shows that (125) is equivalent to the eigenvalue (69). We will prove the former and thereby establish the latter.

**Proof of Proposition 4.7** Set $u^{(\lambda)}(x, y) = \left(\log \tau^{(\lambda)}(x, y)\right)_{xx}$. We wish to show that

$$E^b(x, z) = yz^2 + xz + 4yu^{(\lambda)}(x, y).$$

Since $E(x, z) = xz$, we have

$$\hat{E}(x, y, z) = H(y, z)(xz) = zH_1(x + 2yz, y) = 2yz^2 + xz.$$

Hence,

$$\hat{E}(x, y, \partial_x) = 2y\partial_x^2 + x\partial_x = T(x, y, \partial_x)$$

is the classical Hermite operator. Hence, it suffices to show that

$$\tilde{T}(x, y, \partial_x) \circ \dot{K}^{(\lambda)}(x, y, \partial_x) = \dot{K}^{(\lambda)}(x, y, \partial_x) \circ T(x, y, \partial_x),$$

which is equivalent to

$$[T, \dot{K}^{(\lambda)}] = -4yu^{(\lambda)}\dot{K}^{(\lambda)}.$$

It is well known that

$$\text{Wr}[f_1, \ldots, f_l, f] = \tau f^{(l)} - \tau' f^{(l-1)} + \ldots,$$

where $\tau = \text{Wr}[f_1, \ldots, f_l]$. For the case at hand, $\dot{K}^{(\lambda)} = \partial_x^\ell - \frac{\tau^{(\lambda)}}{\tau^{(\lambda)}} \partial_x^{\ell-1} + \ldots$. Hence, $[T, \dot{K}^{(\lambda)}]$ is an operator of order $\ell$ — the same as the order of $\dot{K}^{(\lambda)}$. By (88), the kernel of $\dot{K}^{(\lambda)}$ is generated by $H_{k_1}, \ldots, H_{k_l}$. These are all eigenfunctions of $T$, and hence annihilated by the commutator. Hence, $[T, \dot{K}^{(\lambda)}] = \mu \dot{K}^{(\lambda)}$. By inspection, $\mu$ is the leading coefficient of

$$- \left[2y\partial_x^2, \frac{x}{\tau^{(\lambda)}} \partial_x^{\ell-1}\right] = -4yu^{(\lambda)}\partial_x^\ell.

\[\square\]

### 5.2 The Bispectral Triple

For a partition $\lambda$ and $W^{(\lambda)} \in \text{Gr}^{\text{ad}}$ as per (72), let $R^{(\lambda)} \subset \mathcal{P}$ denote the ring that preserves $W^{(\lambda)}$. 

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Let $\hat{R}(\lambda)$ denote the ring of bivariate polynomials that preserve $\hat{\Psi}^{(\lambda)}$. Thus, $\pi \in \hat{R}(\lambda)$ if and only if
\[
\pi(z)\hat{W}^{(\lambda)}(z) \subset W^{(\lambda)}(z)
\]
and $\sigma \in \hat{R}(\lambda)$ if and only if
\[
\sigma(x,y)\hat{W}^{(\lambda)}(x,y) \subset \hat{W}^{(\lambda)}(x,y)
\]

It is natural to regard $R(\lambda)$ as a commutative subalgebra of $A(\lambda)$ and to regard $\hat{R}(\lambda)$ as a commutative subalgebra of $\hat{A}(\lambda)$. Indeed, as a direct consequence of Proposition 5.6 we have
\[
\pi^{\flat}(x,y,\partial_x)/\Psi_1^{(\lambda)}(x,y,z) = \pi(z)/\Psi_1^{(\lambda)}(x,y,z), \quad \pi \in R(\lambda), \tag{126}
\]
\[
\sigma^{\sharp}(\partial_z,y,z)/\Psi_1^{(\lambda)}(x,y,z) = \sigma(x,y)/\Psi_1^{(\lambda)}(x,y,z), \quad \sigma \in \hat{R}(\lambda). \tag{127}
\]

Let $S^{(\lambda)} = \{\pi^{\flat}: \pi \in R(\lambda)\}$, and $\hat{S}^{(\lambda)} = \{\sigma^{\sharp}: \sigma \in \hat{R}(\lambda)\}$ denote the corresponding commutative subalgebras of $A^{(\lambda)}$ and $A^{(\lambda)} \otimes P$, respectively. Thus (126) and (127) should be regarded as the eigenvalue equations of the bispectral triple $(S^{(\lambda)}, \hat{S}^{(\lambda)}, \Psi^{(\lambda)})$. This is, essentially, a parameterized version of Wilson’s construction in [26].

**Definition 5.7** We say that $q \in \mathbb{N}_0$ is a critical degree of $R(\lambda)$ if $z^q \in R(\lambda)(z)$. Analogously, we say that $q \in \mathbb{N}_0$ is a critical degree of $\hat{R}(\lambda)$ if there exists a $\sigma(x,y) \in \hat{R}(\lambda)(x,y)$ such that $\deg_x \sigma(x,y) = q$. Let $D(\lambda)$ denote the set of critical degrees of $R(\lambda)$ and $\hat{D}(\lambda)$ denote the set of critical degrees of $\hat{R}(\lambda)$.

Observe that since $R(\lambda)$, $\hat{R}(\lambda)$ are closed under composition, both $D(\lambda)$ and $\hat{D}(\lambda)$ are additive subsets of $\mathbb{N}_0$.

In Section 6.1, below, we will show that the operators in $\hat{S}^{(\lambda)}$ are lowering operators for exceptional Hermite polynomials with (126) understood as a lowering relation. In Section 6.2 we will exhibit a homomorphism that maps $S^{(\lambda)}$ into a certain commutative algebra of difference operators $\mathcal{S}^{(\lambda)}$. This homomorphism transforms the differential eigenvalue (127) into the discrete eigenvalue equation
\[
\Theta_q^{(\lambda)}(m,y,S_m)R^{(\lambda)}_m(x,y) = \sigma_q^{(\lambda)}(x,y)R^{(\lambda)}_m(x,y), \quad q \in \hat{D}(\lambda). \tag{128}
\]
where $\sigma_q \in \hat{R}(\lambda)$ is a monic polynomial of degree $q \in \hat{D}(\lambda)$, and $\Theta_q \in \mathcal{S}^{(\lambda)}$ is the corresponding monic difference operator of order $2q$ obtained by applying the above

---

5Strictly speaking, $\mathcal{R}(\lambda)$ has the structure of a $\mathbb{C}$-algebra and $\hat{\mathcal{R}}(\lambda)$ the structure of a $P$ algebra, but the accepted custom in Sato theory seems to be to refer to these objects as stabilizer rings.
homomorphism to $\sigma_q^z$. Up to an index shift, (128) can be regarded as a recurrence relation for exceptional Hermite polynomials:

$$\Theta_q^{(\lambda)}(n - N, y, S_n)H_n^{(\lambda)}(x, y) = \sigma_q^{(\lambda)}(x, y)H_n^{(\lambda)}(x, y), \quad q \in \mathcal{D}^{(\lambda)}.$$  \hspace{1cm} (129)

However it is more instructive to couple (129) with the eigenvalue (60) into a differential-difference bispectral triple $(\tilde{T}^{(\lambda)}, \mathcal{S}^{(\lambda)}, R^{(\lambda)})$ involving exceptional Hermite rational functions.

**Proposition 5.8** For a monic $\pi \in \mathcal{R}^{(\lambda)}$, the operator $\pi^\lambda(\partial_x, y, z)$ is a monic differential operator whose order is equal to the degree of $\pi$. Dually, for a monic $\sigma \in \mathcal{D}^{(\lambda)}$, the expression $\sigma^\lambda(\partial_x, y, z)$ is a monic differential operator whose order is equal to the $x$-degree of $\sigma(x, y)$.

**Proof** These assertions follow by inspection of the construction of $\Phi_\pi$ and $\Phi_\sigma$ utilized in the proofs to Proposition 5.4 and 5.5. \hfill $\square$

Going forward, the following combinatorial description of $\mathcal{R}^{(\lambda)}$ will prove useful.

**Definition 5.9** For $q \in \mathbb{N}$, a Maya diagram $M \subset \mathbb{Z}$ is a $q$-core [18, p. 12] [19, p. 123] if and only if if there exist $n_0, \ldots, n_{q-1} \in \mathbb{Z}$ such that

$$M = \bigcup_{i=0}^{q-1} \{mq + i : m \leq n_i\},$$  \hspace{1cm} (130)

We will also say that a partition $\lambda$ is a $q$-core if the corresponding $\mathcal{M}^{(\lambda)}$ is a $q$-core.

The following alternative characterization of a $q$-core is useful.

**Proposition 5.10** A Maya diagram $M \subset \mathbb{Z}$ is a $q$-core if and only if $M \subset M + q$.

**Proof** Let $M \subset \mathbb{Z}$ be a Maya diagram and let $q \in \mathbb{N}$. Define the Maya diagrams

$$M_i = \{m \in \mathbb{Z} : qm + i \in M\}, \quad i = 0, \ldots, q - 1.$$  

By definition, $M$ is a $q$-core if and only if each $M_i$ is a trivial Maya diagram; i.e., if $M_i \subset M_i + 1$. Observe that

$$M = \bigcup_{i=0}^{q-1} (qm_i + i), \quad \text{and} \quad M + q = \bigcup_{i=0}^{q-1} (q(M_i + 1) + i).$$

Hence, $M$ is a $q$-core if and only if each $M_i \subset M_i + 1$ if and only if $M \subset M + q$. \hfill $\square$

**Proposition 5.11** Let $g_q^{(\lambda)}, \ q \in \mathbb{Z}$ be the integer sequence defined in (118). A partition $\lambda$ is a $q$-core if and only if $g_{-q}^{(\lambda)} = 0$, or equivalently, if and only if $g_q^{(\lambda)} = q$. 
Proof. The first criterion is a direct consequence of the definition (113). The second criterion follows by (119).

Proposition 5.12 We have \( q \in D^{(\lambda)} \) if and only if \( \lambda \) is a \( q \)-core.

Proof. By definition (72), \( z^q \in R^{(\lambda)}(z) \) if and only if \( J^{(\lambda)} + q \subset J^{(\lambda)} \). The latter is true if and only if \( M^{(\lambda)} \subset M^{(\lambda)} + q \).

The result below is a characterization of the inclusion \( S^{(\lambda)} \subset A^{(\lambda)} \otimes P \). This result will play a key role in our study of exceptional recurrence relations in the sections that follow.

Proposition 5.13 For \( \pi \in A^{(\lambda)} \otimes P \), define

\[
\hat{\pi}(x, y, z) = H(y, z)\pi(x, y, z).
\] (131)

Then \( \pi \in S^{(\lambda)} \) if and only if

\[
\deg_z \hat{\pi}(x, y, z) \leq 0. \tag{132}
\]

Moreover, if condition (132) holds, then \( \pi = \sigma^{\#} \), where \( \sigma(x, y) \) is the leading coefficient of \( \hat{\pi}(x, y, z) \); i.e.,

\[
\sigma(x, y) = \lim_{z \to \infty} \hat{\pi}(x, y, z). \tag{133}
\]

Proof. Let \( \pi(x, y, z) \in A^{(\lambda)}(x, z) \otimes \mathbb{C}[y] \) be given. By Lemma 4.13,

\[
\pi(\partial_z, y, z)e^{xz + yz^2} = \hat{\pi}(x, y, z)e^{xz + yz^2} = \hat{\pi}(x, y, \partial_x)e^{xz + yz^2}.
\]

Consequently,

\[
K^{(\lambda)}(x, y, \partial_x)\hat{\pi}(x, y, \partial_x) = \sigma(x, y)K^{(\lambda)}(x, y, \partial_x).
\]

By definition, \( \pi \in S^{(\lambda)} \) if and only if \( \sigma(x, y) = \pi^\flat(x, y, z) \) is \( z \)-independent. It follows that \( \pi^\flat(x, y, \partial_x) \) is a zeroth order differential operator if and only if (132) holds.

In Proposition 5.12, above, we showed that \( q \) is a critical degree of \( R^{(\lambda)} \) if and only if the corresponding partition is a \( q \)-core. A priori, there does not seem to be a simple criterion that describes the critical degrees of \( R^{(\lambda)}(\partial_x) \). However, it is possible to give some necessary conditions.

Proposition 5.14 If for fixed \( y \neq 0 \), the polynomial \( \tau_y^{(\lambda)}(x) \) has only simple zeroes, then \( \sigma \in \hat{R}^{(\lambda)} \) if and only if \( \sigma(x, y) \) is weighted-homogeneous and \( \partial_x \sigma(x, y) \) is divisible by \( \tau_y^{(\lambda)}(x, y) \).
The proof can be found in [10]. Thus, if \( \tau(\lambda) \) has only simple roots, then the critical degrees of \( \hat{R}^{(\lambda)} \) consists of all \( q \geq N + 1 \) where \( N = |\lambda| = \deg \tau(\lambda) \). In this case, \( \hat{R}^{(\lambda)} \) is the span of the following monic polynomials:

\[
\sigma_q(x, y) := \frac{1}{q + N} \int x^q \tau(\lambda)(x, y), \quad q = 0, 1, 2, \ldots
\]

(134)

**Proposition 5.15** We have \( \hat{D}^{(\lambda)} \subset D^{(\lambda)} \). In other words, if \( q \) is a critical degree of \( \hat{R}^{(\lambda)} \), then necessarily \( \lambda \) is a \( q \)-core.

**Proof** Let \( q \in \hat{D}^{(\lambda)} \) and \( \sigma_q(x, y) \in \hat{R}^{(\lambda)}(x, y) \) a corresponding eigenvalue such that \( \deg_x \sigma_q(x, y) = q \). Fix an \( m \in J^{(\lambda)} \). By definition, \( \sigma_q(x, y)R_m^{(\lambda)}(x, y) \) is a \( \mathbb{C}[y] \)-linear combination of \( R_k^{(\lambda)}(x, y), k \in J^{(\lambda)} \). By definition (66), \( R_m^{(\lambda)}(x, y) \) is monic in \( x \). By assumption, \( \sigma_q(x, y) \) is also monic in \( x \). Hence \( m + q \in J^{(\lambda)} \) also. Hence \( J^{(\lambda)} + q \subset J^{(\lambda)} \). Therefore \( M^{(\lambda)} \) is a \( q \)-core, by Proposition 5.12.

The converse need not hold. In Example 7.1, below, we will demonstrate that the partition \( \lambda = (2, 2, 0, \ldots) \) is a 4-core, but that 4 is not a critical degree of \( \hat{R}^{(\lambda)} \).

### 6 Lowering and Recurrence Relations for the Exceptional Hermites

In Section 6.1, we show that relation (126) of the bispectral triple corresponds to a lowering relation for exceptional Hermite polynomials. In the following section, we exhibit a homomorphism from differential operators to difference operators that will allow us to transform (127) into recurrence relations. Thus, the existence of lowering operators in \( \partial_x \) and difference operators in \( n \) sharing the exceptional Hermites as eigenfunctions follows naturally from the bispectrality of the wave functions in the adelic Grassmannian. Furthermore, it is notable that the correspondence is constructive in nature. Consequently, we will emphasize not only that the existence of the operators follows from Wilson’s construction but moreover that this provides a convenient way to actually compute all of the corresponding operators.

#### 6.1 Lowering Operators

We are now ready to describe the lowering operators for the exceptional Hermite rational functions \( R_n^{(\lambda)}(x, y), n \in J^{(\lambda)} \). A conjugation of the lowering relation (135) below by \( \tau^{(\lambda)}(x, y) \) then yields the corresponding lowering relations for the exceptional polynomials \( H_n^{(\lambda)}(x, y), n \in I^{(\lambda)} \).

**Theorem 6.1** Let \( q \in \mathbb{N} \) be a critical degree of \( R^{(\lambda)} \). Let \( L_q^{(\lambda)}(x, y, \partial_x) = \pi^b(x, y, \partial_x) \) be the the corresponding monic operator of order \( q \) corresponding to \( \pi(z) = z^q \). Then

\[
L_q^{(\lambda)}(x, y, \partial_x)R_m^{(\lambda)}(x, y) = \gamma_q^{(\lambda)}(m)R_{m-q}^{(\lambda)}(x, y), \quad m \in J^{(\lambda)}.
\]

(135)
where $R_m^{(\lambda)}(x, y), m \in \mathcal{J}^{(\lambda)}$ are the exceptional rational functions \((66)\). Moreover,

$$L_q^{(\lambda)} f = \frac{\text{Wr}[R_{k_1}^{(\lambda)}, \ldots, R_{k_q}^{(\lambda)}]}{\text{Wr}[R_{k_1}^{(\lambda)}, \ldots, R_{k_q}^{(\lambda)}]} f,$$

where $k_1, \ldots, k_q$ is an enumeration of $\mathcal{G}_q^{(\lambda)} \subset \mathcal{J}^{(\lambda)}$.

**Proof** Using \((65)\), \((63)\) and \((75)\) we have:

$$L_q^{(\lambda)}(x, y, \partial_x) \Psi^{(\lambda)}(x, y, z) = z^q \Psi^{(\lambda)}(x, y, z)$$

$$= \sum_{m=-\ell}^{\infty} \kappa^{(\lambda)}(m) R_m^{(\lambda)}(x, y, z) \frac{z^{m+q}}{(m+\ell)!}$$

$$= \sum_{m=-\ell}^{\infty} \kappa_q^{(\lambda)}(m) R_{m-q}^{(\lambda)}(x, y, z) \frac{z^m}{(m+\ell)!},$$

where $\ell = \ell(\lambda)$. By \((113)\), we have

$$\{m_1 + q, \ldots, m_q + q\} = \{m_1, \ldots, m_\ell\} \cup \mathcal{G}_q^{(\lambda)}.$$

Hence, $\kappa_q^{(\lambda)}(m) = \kappa^{(\lambda)}(m) \gamma_q^{(\lambda)}(m)$, which implies \((135)\). Furthermore, since $\gamma_q^{(\lambda)}(m)$ vanishes precisely at $k_1, \ldots, k_q$, it follows that $R_{k_1}^{(\lambda)}, \ldots, R_{k_q}^{(\lambda)}$ are in the kernel of $L_q^{(\lambda)}$. Since $L_q^{(\lambda)}$ is a monic differential operator, \((136)\) follows. \hfill \Box

**Proposition 6.2** The commutative algebra $\hat{S}^{(\lambda)}$ is generated by the lowering operators $L_q^{(\lambda)}, q \in \mathcal{D}^{(\lambda)}$.

**Proof** Since $W^{(\lambda)}$ is spanned by monomials, the same is true for $\mathcal{R}^{(\lambda)}$. \hfill \Box

It is also worthwhile to note the following, alternate, characterization of the lowering operators.

**Proposition 6.3** Let $L_q^{(\lambda)}(x, y, \partial_x), q \in \mathcal{D}^{(\lambda)}$ be a lowering operator \((136)\) and $K^{(\lambda)}(x, y, \partial_x)$ the intertwiner as per \((86)\). Then,

$$K^{(\lambda)}(x, y, \partial_x) \circ \partial_x^q = L_q^{(\lambda)}(x, y, \partial_x) \circ K^{(\lambda)}(x, y, \partial_x).$$

**Proof** It suffices to observe that

$$z^{q+\ell} \Psi^{(\lambda)}(x, y, z) = K^{(\lambda)}(x, y, \partial_x)(z^q e^{xz+y^2z^2})$$

$$= K^{(\lambda)}(x, y, \partial_x) \circ \partial_x^q e^{xz+y^2z^2}$$

$$= L_q(x, y, \partial_x) K^{(\lambda)}(x, y, \partial_x) e^{xz+y^2z^2}. \hfill \Box$$

In this way, we recover the interpretation of $\mathcal{R}^{(\lambda)}$ and of critical degrees presented in \((27)\).
6.2 Recurrence Relations

In this section we describe the recurrence relations satisfied by exceptional Hermite polynomials. As a motivation, it is instructive to revisit the connection between the classical 3-term recurrence relation (48) and the first order differential relation (47). Set

\[ \pi_1(x, y, z) := x - 2yz \]

and express (47) as

\[ \pi_1(\partial_z, y, z) \psi_0(x, y, z) = x \psi_0(x, y, z). \]

Set

\[ \pi^{\natural}_1(n, y, z) = -2yz + nz^{-1} \tag{138} \]

so that

\[ \pi_1(\partial_z, y, z) z^n = \pi^{\natural}_1(n, y, z) z^n. \]

The classic recurrence relation (48) can then be derived as follows:

\[
x \psi_0(x, y, z) = \sum_{n=0}^{\infty} H_n(x, y) \pi^{\natural}_1(n, y, z) \frac{z^n}{n!} \tag{139}
\]

\[
= \sum_{n=0}^{\infty} H_n(x, y) \left( -2y(n + 1) \frac{z^{n+1}}{(n+1)!} + \frac{z^{n-1}}{(n-1)!} \right)
\]

\[
= \sum_{n=0}^{\infty} \Theta_1(n, y, S_n) H_n(x, y) \frac{z^n}{n!} \tag{140}
\]

where

\[ \Theta_1(n, y, z) = -2y \pi^{\natural}_1(n, (4y)^{-1}, z) = z - 2ynz^{-1}. \]

As we will see below, the construction \( \pi \rightarrow \pi^{\natural} \rightarrow \Theta \) generalizes to the case of \( \Psi^{(\lambda)}(x, y, z) \) and leads to an explicit formula for exceptional recurrence relations.

Let \( \lambda \) be a partition, \( N = |\lambda| \) and \( \ell = \ell(\lambda) \). As we will show below, the recurrence relations corresponding to \( \lambda \) take the form

\[
\Theta^{(\lambda)}_q(m, y, S_m) R^{(\lambda)}_m(x, y) = \sigma^{(\lambda)}_q(x, y) R^{(\lambda)}_m(x, y), \tag{141}
\]

where \( q \in \mathbb{N}_0 \) is a critical degree of \( R^{(\lambda)}_m \), where \( \sigma^{(\lambda)}_q \in \mathcal{R}^{(\lambda)}_m \) is a monic, homogeneous polynomial of degree \( q \), and where \( \Theta^{(\lambda)}_q(m, y, S_m) \) is a monic difference operator of order \( 2q \) derived from the action of the corresponding \( \pi_q = \sigma_q^{\natural} \) on monomials \( z^m \). The transformation \( \pi_q \mapsto \Theta_q \) doubles the order because the action of a \( \pi_q(\partial_z, y, z), \pi \in \mathcal{S}^{(\lambda)}, q \in \mathcal{D}^{(\lambda)} \) on \( z^m \) involves degree shifts \( k \in \{-q, -q + 2, \ldots, q\} \). In the classical case, this phenomenon is illustrated by relation (138).

Thus, the operators \( \Theta^{(\lambda)}_q \) generate an algebra of difference operators \( \mathcal{G}^{(\lambda)} \) which is naturally isomorphic to the algebra of differential operators \( \mathcal{S}^{(\lambda)} \). This isomorphism is best understood as the restriction of an algebra homomorphism \( \mathcal{A}^{(\lambda)} \rightarrow \mathcal{G}^{(\lambda)} \), where the latter is the algebra of difference operators that preserves sequences with support in \( \mathcal{J}^{(\lambda)} \). This homomorphism from differential to difference operators effectively
transforms the differential eigenvalue relation (127) into the difference eigenvalue relation (141).

Just like in the classical case, the exceptional Jacobi operator, relative to a basis of normalized \( R_m^{(\lambda)}(x, y) \), is represented by a symmetric matrix. This is a consequence of the fact that multiplication by the corresponding eigenvalue is a symmetric operator relative to (43). This symmetry imposes a certain relation between the coefficients of the exceptional Jacobi operator and the exceptional norming constants. We will derive and make use of this observation below.

We begin by describing the homomorphism \( \mathcal{A}^{(\lambda)} \rightarrow \mathfrak{A}^{(\lambda)} \). For a given partition \( \lambda \), let \( J^{(\lambda)} \) be the vector space of sequences supported on \( J^{(\lambda)} \subset \mathbb{Z} \). Let \( \varepsilon \) denote the multiplication operator

\[
\varepsilon(m) f_m = m f_m, \tag{142}
\]

where \( f_m, m \in J^{(\lambda)} \) is a sequence. Evidently, \( \varepsilon \in \text{End} \, \ast J^{(\lambda)} \). For \( q \in \mathbb{Z} \), define the weighted shift operator

\[
\Gamma_q^{(\lambda)}(m, S_m) := \gamma_q^{(\lambda)}(m) S_m^q. \tag{143}
\]

Observe that \( \Gamma_q^{(\lambda)} \in \text{End} \, \ast J^{(\lambda)} \) because

\[
\Gamma_q^{(\lambda)}(m, S_m) f_m = \gamma_q^{(\lambda)}(m) f_{m+q},
\]

and because \( \gamma_q^{(\lambda)}(m) = 0 \) precisely when \( m \in J^{(\lambda)} \) but \( m + q \notin J^{(\lambda)} \). Let \( \mathfrak{A}^{(\lambda)} \subset \text{End} \, \ast J^{(\lambda)} \) be the algebra of difference operators generated by \( \varepsilon \) and \( \Gamma_q^{(\lambda)}, q \in \mathbb{Z} \).

Recall that \( \mathcal{F}^{(\lambda)} \) is isomorphic to the module of Laurent series

\[
\frac{\Phi(x, y, z)}{\tau^{(\lambda)}(x, y)} e^{x z + y z^2} = \sum_{m \in J^{(\lambda)}} F_m(x, y) z^m, \quad F_m \in \ast W^{(\lambda)}, \quad \Phi \in \mathcal{F}^{(\lambda)}. 
\]

Thus, \( \mathfrak{A}^{(\lambda)} \) also acts on \( \mathcal{F}^{(\lambda)} \). This gives the isomorphism \( \mathcal{A}^{(\lambda)} \simeq \mathfrak{A}^{(\lambda)} \) with \( E(\partial_z, z) \mapsto \varepsilon(m) \) and \( G_q^{(\lambda)}(\partial_z, z) \mapsto \Gamma_q^{(\lambda)}(m, S_m) \).

The homomorphism \( \mathcal{A}^{(\lambda)} \rightarrow \mathfrak{A}^{(\lambda)} \) can also be described as a mapping \( \pi(\partial_z, y, z) \mapsto \Theta^{(\lambda)}(m, y, S_m) \) where the latter will be defined in (148), below.

**Proposition 6.4** For every \( \pi(x, y, z) \in \mathbb{C}[x, y, z, z^{-1}] \) there exists a \( \pi^\natural(m, y, z) \in \mathbb{C}[m, y, z, z^{-1}] \) such that

\[
\pi(\partial_z, y, z) z^m = \pi^\natural(m, y, z) z^m \tag{144}
\]

**Proof** It suffices to observe that \( \partial_z^i z^m = F_i(m) z^{m-i} \). Thus, the mapping \( \pi \mapsto \pi^\natural \) is described by

\[
x^i \mapsto F_i(m) z^{-i}, \quad y^j \mapsto y^j, \quad z^k \mapsto z^k,
\]

where \( F_i \) is the falling factorial (96). \( \square \)
Thus, if \( \pi(x, y, z) \in \mathbb{C}[x, y, z, z^{-1}] \) is weighted-homogeneous of degree \( q \in \mathbb{Z} \) with \( d = \deg_y \pi(x, y, z) \), then it can be given as

\[
\pi(x, y, z) = y^{q/2} \sum_k \pi_k(xz)y^{-k/2}z^{-k} = \sum_{j=0}^{d} \pi_{q-2j}(xz)y^j z^{2j-q}
\]  

(145)

where \( \pi_{q-2j}(x), \ j = 0, \ldots, d \) are polynomials. In this way,

\[
\pi^z(m, y, z) = y^{q/2} \sum_k \pi_k^z(m)y^{-k/2}z^{-k} = \sum_{j=0}^{d} \pi_{q-2j}(m)y^j z^{2j-q}, \quad k = q - 2j.
\]  

(146)

**Proposition 6.5** For a weighted-homogeneous \( \pi(x, y, z) \) we have \( \pi \in A^{(\lambda)} \otimes P \) if and only if every \( \pi_k^z(m) \) is divisible by \( \gamma_k^{(\lambda)}(m) \).

**Proof** By Proposition 5.1 there exist polynomials \( \alpha_k(m), \ k = q - 2d, q - 2d + 2, \ldots, q \) such that

\[
\pi(\partial_z, y, z) = y^{q/2} \sum_k \alpha_k(z\partial_z)G_k^{(\lambda)}(\partial_z, z)y^{-k/2}.
\]

Hence,

\[
\pi^z(m, y, z) = y^{q/2} \sum_k \alpha_k(m)\gamma_k^{(\lambda)}(m)y^{-k/2}z^{-k},
\]

and \( \pi_k^z(m) = \gamma_k^{(\lambda)}(m)\alpha_k(m) \). \( \square \)

**Proposition 6.6** For every \( \pi \in A^{(\lambda)} \otimes P \) we have

\[
\pi^z(x, y, \partial_x) R_m^{(\lambda)}(x, y) = \Theta^{(\lambda)}_\pi(m, y, S_m) R_m^{(\lambda)}(x, y).
\]  

(147)

where the difference operator \( \Theta^{(\lambda)}_\pi \in A^{(\lambda)} \otimes P \) is given by

\[
\Theta^{(\lambda)}_\pi(m, y, S_m) = y^{q/2} \sum_k \gamma_k^{(\lambda)}(m)\alpha_k(m + k)y^{-k/2}S_m^k,
\]  

(148)

and where

\[
\alpha_k(m) = \frac{\pi_k^z(m)}{\gamma_k^{(\lambda)}(m)}, \quad k = q - 2d + q, q - 2d + 2, \ldots, q.
\]  

(149)
Proof of Proposition 6.6 Using (75), (124) and (120), we have

\[
\sum_{m=-\ell}^{\infty} \pi^0(x, y, \partial_x) R_m^{(\lambda)}(x, y) \frac{k^{(\lambda)}(m)}{(m + \ell)!} z^m
\]

\[
= y^{q/2} \sum_{m=-\ell}^{\infty} \sum_{k} \gamma_k^{(\lambda)}(m) \alpha_k^{(\lambda)}(m) R_m^{(\lambda)}(x, y) y^{-k/2} z^{m-k},
\]

\[
= y^{q/2} \sum_{m=-\ell}^{\infty} \sum_{k} \gamma_k^{(\lambda)}(m) \alpha_k^{(\lambda)}(m + k) R_{m+k}^{(\lambda)}(x, y) \frac{k^{(\lambda)}(m)}{(m + \ell)!} y^{-k/2} z^m,
\]

where the last step is justified by the fact that if \( m - k < -\ell \), then \( F_k(m + \ell) = 0 \).

Matching coefficients yields (148).

Note that the coefficients of \( \Theta^{(\lambda)}_\pi \) are non-singular because, by Proposition 6.5, \( \alpha_k^{(\lambda)} \) are polynomial. Also note that \( \Theta^{(\lambda)}_\pi \) is an endomorphism of \( \mathcal{J}^{(\lambda)} \), because \( m \in \mathcal{J}^{(\lambda)} \) and \( m + k \notin \mathcal{J}^{(\lambda)} \) is precisely the condition \( \gamma_{-k}^{(\lambda)}(m) = 0 \).

Let \( \mathfrak{S}^{(\lambda)} \subset \mathfrak{A}^{(\lambda)} \otimes \mathcal{P} \) be the commutative subalgebra corresponding to the image of \( \mathfrak{S}^{(\lambda)} \subset \mathfrak{A}^{(\lambda)} \otimes \mathcal{P} \) under the above isomorphism. The elements of \( \mathfrak{S}^{(\lambda)} \) are precisely the exceptional Jacobi operators. To be more precise, let \( q \) be a critical degree of \( \mathcal{R}^{(\lambda)} \) and \( \sigma_q^{(\lambda)}(x, y) \in \mathcal{R}^{(\lambda)}(x, y) \) a corresponding weighted homogeneous, \( x \)-monic polynomial of degree \( q \). Let \( \pi_q = \sigma_q^{(\lambda)} \), so that (127) holds. We will refer to the corresponding difference operator \( \Theta_q^{(\lambda)} := \Theta_q^{(\lambda)}_\pi \) as a \( q \)th order exceptional Jacobi operator\(^6\). We are now able to assert the following.

**Theorem 6.7** Let \( q \) be a critical degree of \( \mathcal{R}^{(\lambda)} \), and \( \sigma_q^{(\lambda)}, \Theta_q^{(\lambda)}, \pi_q \) as above so that, by definition,

\[
\sigma_q^{(\lambda)}(x, y) R_m^{(\lambda)}(x, y) = \Theta_q^{(\lambda)}(m, y, S_m) R_m^{(\lambda)}(x, y).
\]

Then, \( \deg_y \pi_q(x, y, z) \leq q \) and

\[
\Theta_q^{(\lambda)}(m, y, z) = (-2y)^q \pi_q^\sharp(m, (4y)^{-1}, z)
\]

where \( \pi_q^\sharp \) is related to \( \pi_q \) by (144). Explicitly,

\[
\sigma_q(x, y) R_m^{(\lambda)}(x, y) = y^{q/2} \sum_k (-2)^{-k} \pi_{q, -k}^\sharp(m) y^{-k/2} R_{m+k}^{(\lambda)}(x, y)
\]

where \( \pi_{q, k}^\sharp(m), k = -q, -q + 2, \ldots, q \) are the coefficients of \( \pi_q^\sharp \) as per (146).

**Lemma 6.8** Let \( d = \deg_y \pi_q(x, y, z) \) and let \( \pi_{q, k}^\sharp(m) \) be the coefficients of \( \pi_q^\sharp(m, y, z) \) as per (146). Then, necessarily \( d \leq q \) and \( \deg \pi_{q, k}^\sharp(m) \leq (q + k)/2 \) for all \( k \).

\(^6\)There isn’t a unique \( q \)th order Jacobi operator, because one can modify \( \sigma_q(x, y) \) by adding eigenvalues of lower degree.
Proof By assumption, $\pi_q \in \hat{A}^{(\lambda)} \otimes P$. Let $\hat{\pi}_q(x, y, z) = H(y, z)\pi_q(x, y, z)$.

Let $\hat{\pi}_{ij}$ and $\pi_{ij}$ denote the corresponding coefficients so that

$$\pi_q(x, y, z) = \sum_{i,j \geq 0} \pi_{ij} x^i y^j z^{i+2j-q} = \sum_{i,j \geq 0} \hat{\pi}_{ij} y^j z^{i+2j-q} H_i(x - 2yz, -y).$$

By Proposition 5.13, $\deg_z \hat{\pi}_q(x, y, z) \leq 0$. Hence, there are no terms of positive $z$-degree in the last sum. Hence, $i + 2j - q \geq j$ for all non-zero terms in the first sum.

Hence, by (145),

$$\deg \pi_{q,k} \leq (q - k)/2$$

for all $k \in \{q - 2d, q - 2d + 2, \ldots, q\}$. In particular $2d - q \leq d$, which means that $d \leq q$.

As was mentioned earlier, the symmetry of the exceptional Jacobi operator imposes a certain relation between the coefficients $\pi_{q,k}(m)$, $k = -q, -q + 2, \ldots, q$ and the exceptional norming constants $v^{(\lambda)}_m(y)$, as defined in (62).

Lemma 6.9 Let $q, \pi_q$ be as above, and let $\alpha_{q,k}(m)$ be as per (149). Then,

$$\alpha_{q,-k}(m) = (-2)^k \alpha_{q,k}(m + k), \quad k = -q, -q + 2, \ldots, q. \quad (153)$$

Proof By (148) and (120), for $m \in J^{(\lambda)}$, we have

$$\sigma_q(x, y) R^{(\lambda)}_m(x, y) = y^{q/2} \sum_k \gamma^{(\lambda)}_{-k}(m) \alpha_{q,k}(m + k) y^{-k/2} R^{(\lambda)}_{m+k}(x, y).$$

Hence, setting $n = m + k$ and using (70), we have

$$\left\{ \sigma_q R^{(\lambda)}_m, R^{(\lambda)}_n \right\}_H = \gamma^{(\lambda)}_{-k}(m) \alpha_{q,k}(m + k) y^{(q-k)/2} v^{(\lambda)}_n(y)$$

$$= \gamma^{(\lambda)}_{n-m}(m) \alpha_{q,n-m}(m) y^{(q-m-n)/2} \frac{(n + \ell)!}{\kappa^{(\lambda)}(n)} (-2)^n v^{(\lambda)}_0(y)$$

Since multiplication by $\sigma_q$ is a symmetric operator, the above expression is symmetric in $m, n$. Hence,

$$\gamma^{(\lambda)}_{n-m}(m) \alpha_{q,n-m}(m) \frac{(n + \ell)!}{\kappa^{(\lambda)}(n)} (-2)^n = \gamma^{(\lambda)}_{n-m}(m) \alpha_{q,m-n}(m) \frac{(m + \ell)!}{\kappa^{(\lambda)}(m)} (-2)^m$$

$$\alpha_{q,-k}(m) = (-2)^k \alpha_{q,k}(m + k) \frac{\kappa^{(\lambda)}(m) \gamma^{(\lambda)}_{-k}(m)}{\kappa^{(\lambda)}(n) \gamma^{(\lambda)}_{k}(m + k)} F_k(m + k + \ell)$$

The desired relation now follows by (120).

Proof of Theorem 6.7 By (148) and (153), for $m \in J^{(\lambda)}$ we have

$$\sigma_q(x, y) R^{(\lambda)}_m(x, y) = y^{q/2} \sum_k \gamma^{(\lambda)}_{-k}(m) \alpha_{q,k}(m + k) y^{-k/2} R_{m+k}(x, y)$$

$$= y^{q/2} \sum_k (-2)^{-k} \gamma^{(\lambda)}_{-k}(m) \alpha_{q,-k}(m) y^{-k/2} R_{m+k}(x, y). \quad (154)$$
where the sum is over \( k = -q, -q + 2, \ldots, q \). On the other hand,
\[
(-2y)^q \pi_q^\nu(m, (4y)^{-1}, z) = (-2y)^q \sum_k \pi_q^\nu_{q,k}(m)(4y)^{(k-q)/2}z^{-k}
\]
\[
= y^{q/2} \sum_k (-2)^{-k} \gamma_{-k}^{(\lambda)}(m)\alpha_{q,-k}(m)y^{-k/2}z^k
\]

A direct comparison of the last line and of (154) establishes (150).

\[
\square
\]

7 Algorithms and Examples

7.1 Intertwiners

Recall that, by (66) and (88),
\[
R_m^{(\lambda)}(x, y) = \frac{H_m^{(\lambda)}(x, y)}{\tau^{(\lambda)}(x, y)}, \quad m \in J^{(\lambda)}
\]
may be given in terms of a Wronskian as
\[
\kappa^{(\lambda)}(m)R_m^{(\lambda)}(x, y) = K^{(\lambda)}(x, y, \partial x)H_{m+\ell}(x, y), \quad m \in I^{(\lambda)}.
\]
Theorem 4.15 exhibits a constructive procedure for giving exceptional Hermite polynomials as linear combinations of classical polynomials by using the dual intertwiner \( K^{(\lambda)}(\partial_z, y, z) \). Let us illustrate the calculations with an example.

**Example 7.1** Consider the partition \( \lambda = (2, 2, 0, \ldots) \). Correspondingly, \( \ell = 2, N = 4, \) and
\[
\mathcal{M}^{(\lambda)} = \{1, 0, -3, -4, \ldots\}, \quad \mathcal{I}^{(\lambda)} = \{2, 3, 6, 7, 8, \ldots\}, \quad \mathcal{K}^{(\lambda)}(\lambda) = \{2, 3\}.
\]

Using (13), (45), and (81), we have
\[
S^{(\lambda)}(t_1, t_2, \ldots) = \frac{t_1^4}{12} + t_2^2 - t_1 t_3
\]
\[
\Phi^{(\lambda)}(x, y, z) = (x - z^{-1})^4 + 12 \left(y - \frac{1}{2}z^{-2}\right) - (x - z^{-1}) \left(-\frac{1}{3}z^{-3}\right)
\]
\[
= x^4 + 12y^2 - 4x^3z^{-1} + (6x^2 - 12y)z^{-2},
\]
\[
\tau^{(\lambda)}(x, y) = x^4 + 12y^2.
\]

Applying (93) gives
\[
K^{(\lambda)}(\partial_z, y, z) = \partial_z^2 - \left(8yz + 4z^{-1}\right)\partial_z + \left(24y^2z^2 + 12y + 6z^{-2}\right)\partial_z^2
\]
\[
-32y^3z^3\partial_z + \left(16y^4z^4 - 16y^3z^2 - 24yz^{-2}\right).
\]
Applying (97) gives
\[ \kappa^{(\lambda)}_N(n) = n(n - 1)(n - 4)(n - 5) \]
\[ \upsilon_1^{(\lambda)}(n) = -4(2n - 3) \]
\[ \upsilon_2^{(\lambda)}(n) = 24(n - 2)(n - 3) \]
\[ \upsilon_3^{(\lambda)}(n) = -16(n - 2)(n - 3)(2n - 11) \]
\[ \upsilon_4^{(\lambda)}(n) = 16(n - 2)(n - 3)(n - 6)(n - 7) \]

The corresponding exceptional polynomials
\[ H^{(\lambda)}_n = \frac{\mathrm{Wr}[H_2, H_3, H_{n-2}]}{(n - 4)(n - 5)}, \quad n \in \mathcal{I}^{(\lambda)}, \]
may therefore be given as
\[ H^{(\lambda)}_n(x, y) = H_n(x, y) - 4(2n - 3)yH_{n-2}(x, y) + 24(n - 2)(n - 3)y^2H_{n-4}(x, y) - 16(n - 2)(n - 3)(2n - 11)y^3H_{n-6}(x, y) + 16(n - 2)(n - 3)(n - 6)(n - 7)y^4H_{n-8}(x, y), \quad n \in \mathcal{I}^{(\lambda)}. \]

**Example 7.2** Next, consider the partition \( \lambda = (2, 1, 0, \ldots) \). Correspondingly,
\[ \mathcal{M}^{(\lambda)} = \{\ldots, -4, -3, -1, 1\}, \quad \mathcal{I}^{(\lambda)} = \{1, 3, 5, 6, 7, \ldots\}, \quad K^{(\lambda)} = \{1, 3\}. \ (155) \]

By (51), the corresponding exceptional polynomials are
\[ H^{(\lambda)}_n = \frac{\mathrm{Wr}[H_1, H_3, H_{n-1}]}{2(n - 2)(n - 4)}, \quad n \in \mathcal{I}^{(\lambda)}. \]

Using the same formulas as above, we have
\[ S^{(\lambda)}(t_1, t_2, \ldots) = \frac{t_1^3}{3} - t_3 \]
\[ \Phi^{(\lambda)}(x, y, z) = x^3 - 3x^2z^{-1} + 3xz^{-2} \]
\[ \tau^{(\lambda)}(x, y) = x^3. \ (156) \]

The \( \tau \)-function of this example is degenerate because it corresponds to a solution of KdV; the corresponding \( \tilde{W}^{(\lambda)} \) is a stationary point of the second KP flow. Applying (93) gives
\[ K^{(\lambda)}(x, y, z) = x^3 - \left(6yz + 3z^{-1}\right)x^2 + \left(12y^2z^2 + 6y + 3z^{-2}\right)x - 8y^3z^3 \]

Note that since \( \tilde{W}^{(\lambda)} \) is stationary, we have
\[ K^{(\lambda)}_z(x, y, z) = -z^{-1}K^{(\lambda)}(z, 0, x) = z^2 - 3x^{-1}z + 3x^{-2} \]

Applying (97) and (99) gives
\[ H^{(\lambda)}_n = H_n + 6yH_{n-2} - 12(n - 1)(n - 3)y^2H_{n-4} + 8(n - 1)(n - 3)(n - 5)y^3H_{n-6}. \]
7.2 Lowering Operators

In this section, we collect some calculations related to Theorem 6.1.

Example 7.2 (continued) Recall that \( \lambda = (2, 1, 0, \ldots) \) with the corresponding Maya diagram given in (155). We will use Proposition 5.12 to determine the critical degrees of \( R^{(\lambda)} \). The index set for \( R^{(\lambda)}_{m}(x, y), m \in J^{(\lambda)} \) is

\[
J^{(\lambda)} = \{-2, 0, 2, 3, 4, 5, 6, \ldots\}.
\]

\[
R^{(\lambda)}_{-2} = x^{-2}, \quad R^{(\lambda)}_{0} = 1 - 6x^{-2}y, \quad R^{(\lambda)}_{2} = x^2 - 4y + 12x^{-2}y^2, \quad R^{(\lambda)}_{3} = x^3, \ldots
\]

By inspection of Fig. 1, \( D^{(\lambda)} = \{2, 4, 5, 6, \ldots\} \), which means that the ring of lowering operators \( S^{(\lambda)} \) is generated by \( L_{2} \) and \( L_{5} \).

Applying (136) gives

\[
L_{2} = \partial_{x}^{2} - 6x^{-2}
\]

\[
L_{5} = \partial_{x}^{5} - 15x^{-2}\partial_{x}^{3} + 45x^{-3}\partial_{x}^{2} - 45x^{-4}\partial_{x}
\]

Because \( W^{(\lambda)} \) is stationary under the 2nd KP flow, the lowering operators are independent of \( y \). The corresponding lowering relations are:

\[
L_{2}R^{(\lambda)}_{m} = (m + 2)(m - 3)R^{(\lambda)}_{m-2}, \quad m \in J^{(\lambda)}
\]

\[
L_{5}R^{(\lambda)}_{m} = (m + 2)m(m - 2)(m - 4)(m - 6)R^{(\lambda)}_{m-5}, \quad m \in J^{(\lambda)}
\]

Note that the above relations are sensible, because the polynomial \( \gamma_{q}^{(\lambda)}(m) \) on the RHS annihilates precisely those indices \( m \in J^{(\lambda)} \) for which \( m - q \notin J^{(\lambda)} \).

7.3 Critical Degrees and Recurrence Relations

The explicit construction of an exceptional recurrence relation (150) requires knowledge of the critical degrees \( q \) of \( \hat{R}^{(\lambda)} \). For each such \( q \in D^{(\lambda)} \), one also requires

\[
M^{(\lambda)} - 1
\]

\[
M^{(\lambda)} + 1
\]

\[
M^{(\lambda)} + 2
\]

\[
M^{(\lambda)} + 3
\]

\[
M^{(\lambda)} + 4
\]

\[
M^{(\lambda)} + 5
\]

\[
M^{(\lambda)} + 6
\]
the eigenvalue $\sigma_q(x, y)$ and the sequence of polynomials $\pi_{q,k}^\natural(m)$, $k = -q, -q + 2, \ldots, q$, which serve as the coefficients of the recurrence relation. From an algorithmic standpoint, the determination of $q$, $\sigma_q$ and the $\pi_{q,k}^\natural(m)$ is a combined calculation. By Proposition 6.5,

$$\pi_{q,k}^\natural(m) = \gamma_k^{(\lambda)}(m)\alpha_k^{(q)}(m), \quad k = -q, -q + 2, \ldots, q. \tag{157}$$

where the $\alpha_k^{(q)}(m)$ are polynomials, with $\gamma_k^{(\lambda)}(m)$ fixed as per (114). Thus, for a given $q \geq 1$, one has to consider a certain homogeneous linear system whose unknowns are the $q + 1$ polynomials $\alpha_k^{(q)}(m)$. If the system has a non-trivial solution, then the corresponding $q$ is a critical degree. One can extract the eigenvalue, and the coefficients of the recurrence relation from the corresponding solution.

By Lemma 6.8, $\deg \pi_{q,k}^\natural(m) \leq (q + k)/2$. Hence, by (114),

$$\deg \alpha_k^{(q)}(m) \leq b_k^{(q)} := \frac{1}{2}(q + k) - g_k, \tag{158}$$

where $g_k = g_k^{(\lambda)} = \deg \gamma_k(m)$ for notational convenience. In other words, $b_k^{(q)}$ is an upper bound for the degrees of freedom inherent in the choice of $\pi_{q,k}^\natural(m)$.

We represent the level $q$ variables using the truncated list

$$\alpha^{(q)} = (\alpha_{-q}(m), \alpha_{-q+2}(m), \ldots, \alpha_{q}(m))$$

and set

$$\pi^\natural(x, y, z; \alpha^{(q)}) = y^{q/2} \sum_{k \in \mathbb{Z}} \gamma_k^{(\lambda)}(m)\alpha_k^{(q)}(m)y^{-k/2}z^{-k}. \tag{159}$$

As per Proposition 6.4, let $\pi(x, y, z; \alpha^{(q)})$ be such that

$$\pi(\partial z, y, z; \alpha)z^m = \pi^\natural(m, y, z; \alpha)z^m. \tag{160}$$

Set

$$\hat{\pi}(x, y, z; \alpha^{(q)}) := H(y, z)\pi(x, y, z; \alpha^{(q)}), \quad \sigma(x, y; \alpha^{(q)}) := y^{q/2} \sum_k \hat{\pi}_{kk}(\alpha^{(q)})x^iy^{-k/2}z^{-k} \tag{162}$$

By Proposition 5.13, $\pi(\partial z, y, z; \alpha) \in S^{(\lambda)}$ if and only if

$$\hat{\pi}_{ik}(\alpha^{(q)}) = 0, \quad i > k \tag{163}$$

If that is the case, then

$$\pi^\natural(x, y, z; \alpha^{(q)}) = \sigma(x, y; \alpha^{(q)}).$$

Thus, (163) constitutes the linear system for the recurrence relations.

Let us write

$$\alpha_k^{(q)}(m) = \sum_{a=0}^{b_k^{(q)}} \alpha_{ka}m^a, \quad k = -q, -q + 2, \ldots, q \tag{164}$$
where \( \alpha_{ka} \) are lexicographically ordered indeterminates. This means that \( \alpha_{k1a1} \preceq \alpha_{k2a2} \) if and only if \( k_1 < k_2 \), or if \( k_1 = k_2 \) and \( a_1 \leq a_2 \). Let us also say that \( \alpha_{k1a1} \) and \( \alpha_{k2a2} \) have the same parity if \( k_1 \equiv k_2 \pmod{2} \). One can show that

\[
\hat{\pi}_{a+g_k,k}(\alpha^{(q)}) = \alpha_{ka} + \ldots,
\]

where the \( \ldots \) indicates terms of higher lexicographic order and equal parity. Thus, the system (163) is quasi-triangular, because \( \alpha_{ka} \) can be eliminated provided \( a + g_k > k \).

Also, by (153) we have

\[
\alpha_{-ka} = (-2)^k \left( \alpha_{ka} + \sum_{i=a+1}^{b_k} \left( \frac{i}{a} \right) \alpha_{ki} \right), \quad a = 0, \ldots, b_k^{(q)}.
\]

Thus, for \( k < 0 \), the row-reduction may be improved by employing the universal (166) in place of the more computationally demanding (163).

**Example 7.2** (continued) Let us determine the critical degrees for the partition \( \lambda = (2, 2, 0, \ldots) \). The Maya diagram and its translates are shown in the figure below. The black-filled boxes belong to \( \mathcal{M}(\lambda) + q \), the empty boxes below to \( \mathcal{J}(\lambda) \); the red-filled boxes belong to \( \mathcal{G}(\lambda) = (\mathcal{M}(\lambda) + q) \cap \mathcal{J}(\lambda) \).

The critical degrees of \( \mathcal{R}(\lambda) \) are the shifts \( q \) for which \( \mathcal{M}(\lambda) \subset \mathcal{M}(\lambda) + q \). These are also the shifts for which \( g^{(\lambda)}_q = \# \mathcal{G}_q^{(\lambda)} = q \). The above table indicates that the set of all such shifts is \( \mathcal{D}(\lambda) = \{0, 4, 5, 6, \ldots\} \). These are also the orders of the lowering operators for this partition. Not all of these are critical degrees of \( \hat{\mathcal{R}}^{(\lambda)} \). Since \( \tau^{(\lambda)}_y(x) = x^4 + 12y^2 \) has simple zeros for \( y \neq 0 \), Proposition 5.14 may be applied to conclude that \( \hat{\mathcal{D}}^{(\lambda)} = \{5, 6, 7, \ldots\} \). This can also be established using a direct calculation using criterion (163) (Fig. 2).

**Fig. 2** Translates of \( \mathcal{M}^{(\lambda)} \) where \( \lambda = (2, 2, 0, \ldots) \).
We now illustrate the relevant procedure by determining the recurrence relation for \( q = 6 \). By (159), the generic operator that preserves \( W^{(\lambda)} \) and has shifts \(-6, -4, \ldots, 6\) is given by

\[
\pi^5(m, y, z, \alpha^{(6)}) = \alpha_{-6,0} y^6 z^6 + (\alpha_{-4,0} + \alpha_{-4,1}) y^5 z^4 + \alpha_{-2,0} (m + 2)(m + 1) y^4 z^2 + (\alpha_{00} + \alpha_{01} m + \alpha_{02} m^2) y^3 + \alpha_{20} (m + 2)(m + 1)(m - 2)(m - 3) y^2 z^2 + (\alpha_{40} + \alpha_{41} m)(m + 2)(m + 1)(m - 4)(m - 5) y z^4 + \alpha_{60} (m + 2)(m + 1)(m - 2)(m - 3)(m - 6)(m - 7) z^6
\]

By (163), we will have \( \pi \in S^{(\lambda)} \) provided \( \hat{\pi}_{ij}(\alpha^{(6)}) = 0 \) for all \( i > j \). In that case, by (162),

\[
\sigma(x, y; \alpha^{(6)}) = (-6x^2 y^2 - 12y^3)\alpha_{20} + (x^4 y + 12x^2 y^2 - 52y^3)\alpha_{40} + (4x^4 y - 48x^2 y^2 - 144y^3)\alpha_{41} + (x^6 + 30x^4 y - 396x^2 y^2 - 264y^3)\alpha_{60}
\]

will be the corresponding eigenvalue.

Applying (160) and (161), the linear system in question has the following matrix:

| \( \hat{\pi}_{ij}(\alpha^{(6)}) \) | \( \alpha_{01} \) | \( \alpha_{02} \) | \( \alpha_{03} \) | \( \alpha_{20} \) | \( \alpha_{41} \) | \( \alpha_{40} \) | \( \alpha_{60} \) |
|---|---|---|---|---|---|---|---|
| \( \hat{\pi}_{32} \) | 0 | 0 | 0 | 4 | 52 | 8 | 240 |
| \( \hat{\pi}_{10} \) | 1 | 1 | 1 | 0 | 60 | 48 | -48 |
| \( \hat{\pi}_{20} \) | 0 | 1 | 3 | 36 | 216 | 24 | 720 |
| \( \hat{\pi}_{30} \) | 0 | 0 | 1 | 8 | 40 | 0 | 160 |
| \( \hat{\pi}_{42} \) | 0 | 0 | 0 | 1 | 10 | 0 | 60 |
| \( \hat{\pi}_{54} \) | 0 | 0 | 0 | 0 | 1 | 0 | 12 |

The symmetry relations (166) give

\[
\alpha_{-2,0} = 4\alpha_{2,0}, \quad \alpha_{-4,0} = 16(\beta_{4,0} + 4\alpha_{4,1}), \quad \alpha_{-4,1} = 16\alpha_{4,1}, \quad \alpha_{-6,0} = 64\alpha_{6,0}
\]

Setting \( \alpha_{0,0} = 0, \alpha_{6,0} = 1 \), solving the above relations, and using (151) gives the following recurrence relation of order 12:

\[
(x^6 + 36x^2 y^2 - 192y^3) R^{(\lambda)}_m
= R_{m+6} - 6(2m + 5)y R_{m+4} + 60(m + 1)(m + 2) y^2 R_{m+2} + (304m - 240m^2 - 160m^3) y^3 R_m + 240(m - 3)(m - 2)(m + 1)(m + 2) y^4 R_{m-2} - 96(m - 5)(m - 4)(m + 1)(m + 2)(2m - 3) y^5 R_{m-4} + 64(m - 7)(m - 6)(m - 3)(m - 2)(m + 1)(m + 2) y^6 R_{m-6}
\]

This corresponds to the eigenvalue equation

\[
\pi_6(\partial_z, y, z) \Psi^{(\lambda)}(x, y, z) = (x^6 + 36x^2 y^2 - 192y^3) \Psi^{(\lambda)}(x, y, z),
\]

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where
\[
\pi_6(\partial_z, y, z) = \partial_z^6 - 12yz\partial_z^5 + \left(60y^2z^2 - 30y - 24z^{-2}\right)\partial_z^4 \\
+ \left(-160y^3z^3 + 240y^2z + 192yz^{-1} + 96z^{-3}\right)\partial_z^3 \\
+ \left(240y^4z^4 - 720y^3z^2 - 360y^2 - 288yz^{-2} - 108z^{-4}\right)\partial_z^2 \\
+ \left(-192y^5z^5 + 960y^4z^3 - 96y^3z - 288yz^{-3} - 144z^{-5}\right)\partial_z \\
+ \left(64y^6z^6 - 480y^5z^4 + 480y^4z^2 + 720y^2z^{-2} + 720yz^{-4} + 504z^{-6}\right)
\]

Let us consider the similar calculation for \( q = 4 \). Generically,
\[
\pi^q(m, y, z, \alpha^{(4)}) = \alpha_{-4,0}y^4z^4 + (\alpha_{00} + \alpha_{01}m + \alpha_{02}m^2)y^2 \\
+ \alpha_{40}(m + 2)(m + 1)(m - 4)(m - 5)yz^{-4}
\]
The linear system \( \hat{\pi}_{ik}(\alpha^{(4)}) = 0, \ i > k \) is a truncation of the \( \alpha^{(6)} \) system shown above. The corresponding matrix

\[
\begin{pmatrix}
\hat{\pi}_{32} & \alpha_{01} & \alpha_{02} & \alpha_{40} \\
\hat{\pi}_{10} & 0 & 0 & 8 \\
\hat{\pi}_{20} & 1 & 1 & 48 \\
& 0 & 1 & 24
\end{pmatrix}
\]

has maximal rank, which means that \( 4 \notin \mathcal{D}^{(\lambda)} \). In other words, just as predicted by Proposition 5.14, there is no recurrence relation of order 8.

8 Conclusions and Remarks

Both the wave functions in the adelic Grassmannian and the exceptional Hermite polynomials exhibit bispectrality. However, it was not previously recognized that some of those wave functions were generating functions for the exceptional Hermite polynomials. That this fundamental connection previously went unnoticed may be a consequence of the fact that Wilson’s bispectral wave functions were obtained by setting all higher KP variables \( t_i \) for \( i > 1 \) to zero while this correspondence holds only when \( y = t_2 \) is non-zero.

Stating the correspondence precisely required the use of new notation and some technical lemmas. It is also stated most naturally not in terms of the exceptional Hermite polynomials \( H_n^{(\lambda)} \) but rather through their rational counterparts, \( R_m^{(\lambda)} \). Nevertheless, the rewards are worth these efforts. Many of the known properties of the exceptional Hermite polynomials are easily rederived from the bispectrality of these generating functions. Moreover, utilizing this connection also leads to new results and more effective algorithms for computing the associated algebras of operators.

One of the key benefits to situating exceptional polynomials within \( \text{Gr}^{\text{ad}} \) is the realization that there are two relevant notions of bispectrality: differential-differential and differential-difference. A consequence of this remark is the existence of a difference intertwiner that serves to give exceptional polynomials as a canonical linear
combination of their classical counter-parts. The other consequence, of course, is the re-interpretation of exceptional recurrence relations in terms of the commutative algebra of operators canonically associated to every point in $\text{Gr}^{\text{ad}}$.

This paper considered only the wave functions associated to a collection of points in $\text{Gr}^{\text{ad}}$ indexed by partitions and their flows under the second flow of the KP hierarchy. These wave functions are precisely the generating functions for the exceptional Hermite functions with a scaling parameter. It is our intention to consider in a future paper how this construction generalizes to other points in the adelic Grassmannian and to their dependences on the higher KP time variables.

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