Why would multiplicities be log-concave?

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Abstract

It is a basic property of the entropy in statistical physics that is concave as a function of energy. The analog of this in representation theory would be the concavity of the logarithm of the multiplicity of an irreducible representation as a function of its highest weight. We discuss various situations where such concavity can be established or reasonably conjectured and consider some implications of this concavity. These are rather informal notes based on a number of talks I gave on the subject, in particular, at the 1997 International Press lectures at UC Irvine.

0 Introduction

The aim of these notes is to discuss some heuristic arguments, conjectures, and rigorous results related to the following phenomenon. Physical analogy, explained in Section 1, suggests that under certain circumstances the logarithms of multiplicities of irreducible representations can be expected to be concave as a function of the highest weight. In Section 2 we discuss some cases when this is known or expected to be the case and explore various implications of this concavity. In Section 3 we discuss the classical limit, in which much more general results can be established.

This text is not a survey. It is based on several talks I gave on the subject on various occasions and represents only my personal point of view. I hope that the somewhat informal style of these notes will make the basic ideas easier to explain. For missing details, the reader is referred to the original papers [16, 8, 17, 12, 18]. For surveys on log-concavity in general, see [4, 22].

I very much benefited from the discussions with a number of people, first of all, with my colleagues from the Institute of Problems of Information Transmission, especially R. Dobrushin, G. Olshanski, and S. Pirogov,
and also with V. Ginzburg, W. Graham, A. Khovanskii, and A. Kirillov. In particular, the results of [16] lead V. Ginzburg to conjecture that the push-forward of the Liouville measure on an arbitrary symplectic manifold under the moment map for a compact group action should be log-concave. Same conjecture, independently of [16], was proposed by A. Knuts on (later, a counterexample to this conjecture was found in [12]; for positive results see [8, 17, 18]).

I would like to thank A. Buch for providing me with a program for computation of tensor product multiplicities.

1 Physical motivation: entropy and its concavity

1.1

Consider a quantum mechanical system, that is, a selfadjoint operator $H$ in a Hilbert space $V$. For simplicity, we assume that $V$ is spanned by the eigenvectors of $H$.

The multiplicity $\Omega(E)$ of an eigenvalue $E \in \text{spec}(H)$ measures how many states of our system have the energy $E$. In other words, fixing an energy level $E$ this does not determine the state of the system uniquely: there remain $\Omega(E)$ possibilities. The size of this indeterminacy equals $\log_2 \Omega(E)$ bits of information.

In statistical physics, there is the basic relation

$$S = k \log \Omega,$$

where $\Omega$ is the number of states with given values of macroscopic parameters such as energy, $k$ is the Boltzmann constant, and $S$ is the entropy, which measures the degree of disorder in the system or, in other words, the lack of information about the precise state of the system. We are thus led to think of

$$S(E) = \log \Omega(E)$$

as of the entropy of the energy level $E$.

\footnote{This relation, in the form $S = k \log W$, is written on Boltzmann’s tombstone.}
1.2

In statistical mechanics, the entropy is always a concave function of all additive macroscopic parameters such as energy $E$, volume $V$, or the number of particles $N$. There is a simple physical argument for this concavity and it goes as follows. Suppose we have two systems with parameters $(E_1, V_1, \ldots)$ and $(E_2, V_2, \ldots)$, respectively, contained in two reservoirs separated by an impervious wall:

$$E_1, V_1, \ldots$$

$$E_2, V_2, \ldots$$

Let us now bring them in contact by removing this wall. The energy and the volume of the new system will be $E_1 + E_2$ and $V_1 + V_2$, respectively, whereas the entropy will increase

$$S(E_1 + E_2, \ldots) \geq S(E_1, \ldots) + S(E_2, \ldots) \quad (1.1)$$

because of the additional disorder introduced by allowing the systems to mix. The net increase in entropy is called the entropy of mixing and its positivity reflects the irreversibility of mixing.

There is, however, one case when the mixing is clearly reversible and that is when the two systems were identical to begin with, that is, when

$$(E_1, V_1, \ldots) = (E_2, V_2, \ldots),$$

in which case we can simply insert back the wall to recover the original situation. Thus, in this case the entropy of mixing vanishes. In other words,

$$S(2E, \ldots) = 2S(E, \ldots). \quad (1.2)$$

Combining (1.1) with (1.2) we get the concavity of the entropy.$^2$

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$^2$ In thermodynamics, one has the relation $\frac{\partial^2 S}{\partial E^2} = \frac{1}{T}$ where $T$ is the temperature. Therefore $\frac{\partial^2 S}{\partial E^2} < 0$ means that temperature rises when energy increases.
1.3

Of course, in order to apply statistical considerations one needs the system in question to have a very large or infinite number of degrees of freedom. Still, it is natural to ask whether in some interesting cases one can expect or, even better, prove the concavity of $S(E)$. Fortunately, interesting examples do exist.

An obvious limitation for the entropy concavity principle is that the concavity of $S(E)$ is clearly not preserved under direct sums. Hence our system has be in some sense irreducible. The concrete meaning of this irreducibility will be different in different context. In Section 2, the space $V$ will be an irreducible module of some ambient group. In Section 3, we will be dealing with group actions on irreducible algebraic varieties.

1.4

First, however, one has to modify the definition of concavity. Indeed, the support $\text{spec}(H)$ of the function $\Omega(E)$ is countable and hence $S(E)$ cannot be concave in the usual sense.

We suppose that $\text{spec}(H)$ is contained in a lattice, which without loss of generality we can take to be $\mathbb{Z} \subset \mathbb{R}$, and we define concavity to mean

$$S(\alpha E_1 + (1 - \alpha) E_2) \geq \alpha S(E_1) + (1 - \alpha) S(E_2), \quad \alpha \in [0, 1],$$

whenever the middle point $\alpha E_1 + (1 - \alpha) E_2$ lies in the lattice $\mathbb{Z}$.

The abstract form of this convention is the following:

**Definition 1.** Let $F : A \to O$ be a function from a Abelian semigroup $A$ to an ordered Abelian semigroup $O$. We say that this function is **concave** if

$$(p + q) F(C) \geq p F(A) + q F(B)$$

for any $A, B, C \in A$ satisfying

$$(p + q) C = p A + q B, \quad p, q \in \mathbb{Z}_{\geq 0}.$$  

In the case when

$$O = (\mathbb{R}_{\geq 0}, \times)$$

is the multiplicative semigroup of nonnegative real numbers with the usual ordering, we also call the function $F$ logarithmically concave, or log-concave for short.

In our examples, $A$ will be usually isomorphic to $\mathbb{Z}^n$ or $\mathbb{R}^n$, whereas the target semigroup $O$ will occasionally be something more interesting.
1.5

Since the eigenvalues of $H$ are now integers, the time evolution $e^{itH}$ defines a representation of the standard circle $\mathbb{T}^1$ on $V$.

More generally, for any compact group $^3K$, one can ask whether for some interesting representation $V$ of $K$ the multiplicities $\Omega(\lambda)$ of irreducible representations $V^\lambda$

$$V = \bigoplus_{\lambda \in K^\wedge} \Omega(\lambda) V^\lambda$$

form a log-concave function on the weight lattice of $K^\wedge$.

Examples of such representations will be discussed in Section 2. They are, in a sense, related to the “thermodynamics” of classical groups.

1.6

Now consider a Hamiltonian system of classical mechanics, that is, a manifold $M^{2n}$ with symplectic form $\omega$ and with an energy function

$$h : M^{2n} \to \mathbb{R}.$$  

The form $\frac{\omega^n}{n!}$ is a volume form on $M^{2n}$ which defines a measure (called the Liouville measure). Let $\Omega(E)$ be the density of the push-forward of this measure under $h$

$$h_* \left( \frac{\omega^n}{n!} \right) = \Omega(E) \, dE, \quad E \in \mathbb{R}.$$  

In other words, $\Omega(E)$ tells us how many states of our system have the energy $E$. Again, we think of

$$S(E) = \log S(E)$$

as of the entropy $^4$ of the energy level $E$.

As in Section 1.5, this can be generalized to the situation of a Hamiltonian action of a compact group $K$ action on $M^{2n}$. Let

$$\phi : M^{2n} \to \text{Lie}(K)^*$$
be the moment map for this action. For any $\xi \in \text{Lie}(K)^*$ the volume
\[
\Omega(\xi) = \text{Vol} \phi^{-1}(\xi)
\]
measures how many points of $X$ have the energy $\xi$. This function is clearly invariant under the coadjoint action of $K$ on Lie($K$)$^*$, so we can and will assume that $\xi$ lies in the positive Weyl chamber $h_+$. We can ask whether for some actions the function $\log \Omega(\xi)$ is concave on $h_+$. Observe that such a concavity implies, in particular, that the set
\[
\text{supp} \Omega(\xi) = \phi(M^{2n}) \cap h_+,
\]
is convex, which is a famous classical result [4, 10, 11, 14].

It turns out that the supply of cases where $\log \Omega(\xi)$ is concave is now much richer than in the quantum situation. As shown by W. Graham in [8], it includes all torus actions on compact Kähler manifolds. It also includes [12] all actions on projective varieties, possibly singular. It was conjectured by V. Ginzburg and A. Knutson that it is true for any symplectic $M^{2n}$. This was shown to be not the case by Y. Karshon in [12].

We will discuss this classical situation in algebraic setting in Section 3.

2 Some results and conjectures on log-concavity of multiplicities

2.1

Again, we begin with a motivation, this time a historical one. Here is how the question of logarithmic concavity of multiplicities arose in the “thermo-dynamics” of classical groups.

Let $U(\infty)$ denote the inductive limit of $U(n)$ with respect to standard embeddings $U(n) \subset U(n + 1)$ which can be visualized as follows:
The description of the characters of $U(\infty)$ is a fundamental result with nontrivial history. Voiculescu in [25] proved that functions of the form

$$g \mapsto \det \left[ e^{\gamma^+(g-1) + \gamma^-(g-1)} \prod \frac{1 + \beta_i^+(g-1)}{1 - \alpha_i^+(g-1)} \frac{1 + \beta_i^-(g^{-1} - 1)}{1 - \alpha_i^-(g^{-1} - 1)} \right]$$  \hspace{1cm} (2.1)

where

$$0 \leq \alpha_i^\pm, \quad 0 \leq \beta_i^\pm \leq 1, \quad 0 \leq \gamma^\pm$$

are parameters, are characters of $U(\infty)$ and conjectured that there are no other characters. It was observed by Boyer [3] and, independently, by Vershik and Kerov [24] that this conjecture is equivalent to the Schoenberg’s conjecture about the so-called totally positive sequences (see below) which was already established by Edrei in [6] using some deep results about entire functions.

2.2

Vershik and Kerov also outlined a different and more direct proof which uses approximation of characters of $U(\infty)$ by normalized characters of $U(n)$. It follows from a general principle due to Vershik (see [23] and also [20]), that any character $\chi$ of $U(\infty)$ is a limit of a sequence of normalized characters $\chi_n$ of $U(n)$ as $n \to \infty$ in the sense that

$$\chi_n \bigg|_{U(k)} \xrightarrow{\text{uniformly}} \chi \bigg|_{U(k)}$$  \hspace{1cm} (2.2)

for any fixed $k = 1, 2, 3, \ldots$. In [24], Vershik and Kerov gave necessary and sufficient conditions for the convergence of $\{\chi_n\}$ and identified the corresponding limits with functions (2.1).

This approximation principle is a materialization of certain general ergodic theory ideas and is closely akin to some standard constructions in statistical physics such as construction of Gibbsian measures in an infinite volume by a thermodynamic limit transition. In that case, one chooses a

5 An abstract definition of characters is: indecomposable central continuous positive definite functions. More concretely, they are spherical functions of the Gelfand pair

$$U(\infty) \times U(\infty) \supset \text{diag} U(\infty)$$

or, equivalently, traces of factor representations of type $I_n$ or $\Pi_1$.  

7
sequence of boxes which fill up the space (just as in the above visualization of \( U(\infty) \)), for each box one picks some boundary condition which specifies a Gibbsian measure (in our case, \( \chi_n \)), and one requires convergence of the induced measures on all compact sets.

### 2.3

Note that the formula (2.1) is multiplicative in the eigenvalues of \( g \in U(\infty) \). This multiplicativity can be established a priori; as shown by Olshanski, see for example [21], such and more general multiplicativity are very characteristic for representations of infinite-dimensional classical groups.

Any character \( \chi \) of \( U(\infty) \) is therefore uniquely determined by its restriction to \( U(1) \)

\[
x(z) = \chi\left( \begin{bmatrix} z & 1 & \cdots \\ \end{bmatrix} \right) = \sum_{k \in \mathbb{Z}} x_k z^k.
\]

Conversely, any function \( g \mapsto \det x(g) \) is a character of \( U(\infty) \) provided that it is positive definite, which means that its restriction to any \( U(n) \) is a non-negative linear combination of the characters of \( U(n) \), that is, of the rational Schur functions \( s_\lambda \).

The identity

\[
\prod_{i=1}^{n} \left( \sum_{k \in \mathbb{Z}} x_k z_i^k \right) = \sum_{\lambda=(\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n} \det \left[ x_{\lambda_i-i+j} \right]_{i,j=1 \ldots n} \ s_\lambda(z_1, \ldots, z_n),
\]

shows that this positivity is equivalent to the positivity of some (in fact, all) minors of the infinite Toeplitz matrix \( [x_{j-i}]_{i,j \in \mathbb{Z}} \), which is precisely Schoenberg’s definition of a totally positive sequence.

In particular, the positivity of \( 2 \times 2 \) minors means that

\[
x_n^2 \geq x_{n-1} x_{n+1}.
\]

Thus, one knows a priori that the restriction of any character of \( U(\infty) \) to \( U(1) \) has log-concave multiplicities.

### 2.4

The question whether the same is true before the limit, that is, whether the restriction of any irreducible representation of \( U(n) \) to standard \( U(1) \) has
log-concave multiplicities, surfaced when we were working with G. Olshanski on a generalization of the Vershik-Kerov theorem [24]. Originally, this log-concavity was needed to replace the uniform convergence in (2.2) by convergence of Taylor series, see Section 3 in [16]. Eventually, in [19] it was replaced by a more elementary argument, but nonetheless this log-concavity is a valid question with interesting answer.

As it turns out, for any representation $V^\lambda$ of $U(n)$ the multiplicity of the irreducible representation $V^\mu$ of the standard $U(k) \subset U(n)$ is a log-concave function of the pair $(\lambda, \mu) \in U(n)^\wedge \oplus U(k)^\wedge$. In fact, one can say more. Without loss of generality, let us assume that $\lambda$ is a partition and consider the space $V^{\lambda/\mu} = \text{Hom}_{U(k)}(V^\mu \to V^\lambda)$ whose dimension is the multiplicity in question. The space $V^{\lambda/\mu}$ is an $U(n-k)$ module with character given by the skew Schur function $s_{\lambda/\mu}$.

$$\text{tr}_{V^{\lambda/\mu}}\left(\begin{bmatrix} z_1 & z_2 & \cdots \\ \vdots \end{bmatrix}\right) = s_{\lambda/\mu}(z_1, z_2, \ldots).$$

One has the following

**Theorem 1 ([16]).** Suppose $(\lambda_i, \mu_i)$, $i = 1, 2, 3$, are partitions such that

$$(\lambda_2, \mu_2) = \frac{1}{2}(\lambda_1, \mu_1) + \frac{1}{2}(\lambda_3, \mu_3).$$

Then the following polynomial has nonnegative coefficients:

$$s_{\lambda_2/\mu_2}^2 - s_{\lambda_1/\mu_1} s_{\lambda_3/\mu_3} \in \mathbb{Z}_{\geq 0}[z_1, z_2, \ldots]. \quad (2.3)$$

The coefficients of the polynomial $s_{\lambda/\mu}$ correspond to standard tableaux of shape $\lambda/\mu$. In the proof of Theorem [16], one constructs a certain transformation on pairs of standard tableaux and proves that it is injective.

Similar results for orthogonal and symplectic groups are also established in [16].
2.5

It is likely that (2.3) is actually a nonnegative linear combination of Schur functions. One can propose a conjecture which would, among other things, imply this property.

Recall that the Littlewood-Richardson coefficients $c_{\lambda \mu \nu}$ are defined by

$$c_{\lambda \mu \nu} = \dim \left( V^\lambda \otimes V^\mu \otimes V^\nu \right)^G$$

where the superscript $G$ stands for invariants of $G = U(n)$. If either of the arguments of $c_{\lambda \mu \nu}$ is not a dominant weight, we set $c_{\lambda \mu \nu} = 0$ by definition. Often, one uses the numbers

$$c^\lambda_{\mu \nu} = c_{\lambda^* \mu \nu}$$

where $\lambda^*$ is the highest weight of the dual module $(V^\lambda)^*$

$$(\lambda_1, \ldots, \lambda_n)^* = (-\lambda_n, \ldots, -\lambda_1).$$

The numbers $c^\lambda_{\mu \nu}$ are coefficients in the expansions

$$V^{\lambda/\mu} = \sum_\nu c^\lambda_{\mu \nu} V^\nu,$$

$$V^\mu \otimes V^\nu = \sum_\lambda c^\lambda_{\mu \nu} V^\lambda.$$

**Conjecture 1.** The function

$$(\lambda, \mu, \nu) \rightarrow \log c_{\lambda \mu \nu}$$

is concave.

If true, this concavity would have some interesting applications. In particular, since

$$c_{\lambda \mu \nu} = c_{\lambda \nu \mu}$$

we conclude that

$$c_{\lambda, \mu + \nu, \mu + \nu} \geq c_{\lambda \nu \mu}.$$
provided \( \frac{\mu + \nu}{2} \) is an integral weight. This is equivalent to the inclusion of representations

\[
V^{\nu} \otimes V^{\mu} \subset \left( V^{\frac{\mu + \nu}{2}} \right) \otimes^{2} ,
\]

which can be interpreted as saying that the representation valued function

\[
V : \lambda \mapsto V^{\lambda}
\]

is concave with respect to the natural ordering and tensor multiplication of representations.

If (2.4) is true then we certainly have the following inclusion of \( U(n) \)-modules

\[
(V^{\lambda_{1}} \otimes V^{\mu_{1}}) \otimes (V^{\lambda_{3}} \otimes V^{\mu_{3}}) \subset (V^{\lambda_{2}} \otimes V^{\mu_{2}}) \otimes^{2}
\]

for \((\lambda_{i}, \mu_{i})\) as in Theorem 1. The last inclusion is equivalent to

\[
V^{\lambda_{1}/\mu_{1}} \otimes V^{\lambda_{3}/\mu_{3}} \subset (V^{\lambda_{2}/\mu_{2}}) \otimes^{2} .
\]

Indeed, the equation (2.7) is equivalent to

\[
\left( \sum_{\nu_{1}} c_{\lambda_{1} \mu_{1} \nu_{1}} V^{\nu_{1}} \right) \otimes \left( \sum_{\nu_{3}} c_{\lambda_{3} \mu_{3} \nu_{3}} V^{\nu_{3}} \right) \subset \left( \sum_{\nu_{2}} c_{\lambda_{2} \mu_{2} \nu_{2}} V^{\nu_{2}} \right) \otimes \left( \sum_{\nu_{4}} c_{\lambda_{4} \mu_{4} \nu_{4}} V^{\nu_{4}} \right) ,
\]

whereas (2.6) says that

\[
\left( \sum_{\nu_{1}} c_{\lambda_{1} \mu_{1} \nu_{1}} V^{\nu_{1}} \right) \otimes \left( \sum_{\nu_{3}} c_{\lambda_{3} \mu_{3} \nu_{3}} V^{\nu_{3}} \right) \subset \left( \sum_{\nu_{2}} c_{\lambda_{2} \mu_{2} \nu_{2}} V^{\nu_{2}} \right) \otimes \left( \sum_{\nu_{4}} c_{\lambda_{4} \mu_{4} \nu_{4}} V^{\nu_{4}} \right) .
\]

Remark that it follows from Weyl’s dimension formula that

\[
\lambda \mapsto \log \dim V^{\lambda}
\]

is a concave function. That is, the function (2.5) considered as a function into just vector spaces without group action is concave with respect to the tensor product.
To get (2.8) from (2.9), take the dual space of everything, which will replace \(V^\nu\) by \(V^{\nu^*}\), and then replace \(\lambda_i\) by \(\lambda_i^*\) and \(\nu_i^*\) by \(\nu_i\). The inclusion (2.7) is equivalent to Schur-positivity of (2.3).

Similarly, the conjecture and the symmetry

\[
c_{\lambda\mu\nu} = c_{\nu\lambda\mu}
\]

imply that

\[
c_{\lambda',\mu',\nu'} \geq c_{\lambda\mu\nu}
\]

provided the weight

\[
\begin{pmatrix}
\lambda' \\
\mu' \\
\nu'
\end{pmatrix} = \begin{pmatrix}
\alpha & 0 & 1 - \alpha \\
1 - \alpha & \alpha & 0 \\
0 & 1 - \alpha & \alpha
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\mu \\
\nu
\end{pmatrix},
\]

is an integral weight and \(0 \leq \alpha \leq 1\).

2.6

Here is another implication of Conjecture [4] which is actually known to be true. Concavity of \(\log c_{\lambda\mu\nu}\) implies that the support

\[
\text{supp } c_{\lambda\mu\nu} = \{ (\lambda, \mu, \nu), c_{\lambda\mu\nu} \neq 0 \}
\]

is convex. In particular, since it contains the origin \((0, 0, 0)\), it is saturated, meaning that

\[
c_{k\lambda,k\mu,k\nu} \neq 0 \Rightarrow c_{\lambda,\mu,\nu} \neq 0,
\]

for any \(k = 2, 3, \ldots\) In fact, since \(c_{0,0,0} = 1\), Conjecture [4] implies that

\[
c_{k\lambda,k\mu,k\nu} \geq (c_{\lambda,\mu,\nu})^k.
\]

The saturation (2.10) turns out to be a very important property, see [4]. It has been recently established by A. Knutson and T. Tao in [15], see also [2].
2.7

As already pointed out in Section 1.3, log-concavity is (in contrast to so many things in representation theory) not an additive property: it is totally destroyed by direct sums.

It seems however likely that log-concavity should be a *multiplicative* property, that is, it should behave nicely with respect to tensor products. For example, recall that it is well known and easy to prove that the convolution of two log-concave sequences is again log-concave. This is equivalent to saying that the set of $U(1)$-modules with log-concave multiplicities is closed under tensor products.

This multiplicativity principle fits together nicely with the above conjecture about tensor product multiplicities.

3 Log-concavity in the classical limit

3.1

Dealing with actual multiplicities may be a subtle business. Fortunately, many of these subtleties disappear in the classical limit and much more general results can be established.

Let us assume that the phase space of our classical system is an irreducible projective algebraic variety $X \in \mathbb{P}^N$ over $\mathbb{C}$ which is stable under the action of a compact group $K \subset GL(N + 1)$. We write $X$ in place of $M^{2n}$ to stress the fact that we are now working with projective algebraic varieties which are allowed to be singular.

Even for singular $X$, the moment map

$$\phi : X \rightarrow \text{Lie}(K)^*$$

is still well defined as the restriction of the moment map for the $K$-action on $\mathbb{P}^N$. It is well known (see e.g. Theorem 6.5 in [9]) that the function $\Omega(\xi)$ from Section 1.6 describes the asymptotics of the multiplicities of $K$-modules in polynomials of very large degree on $X$.

More concretely, let

$$\mathbb{C}[X] = \bigoplus_{d=0}^{\infty} \mathbb{C}[X]_d$$

be the homogeneous coordinate ring of $X$. The space $\mathbb{C}[X]_d$ of degree $d$
polynomials on $X$ decomposes as a $K$-module
\[ \mathbb{C}[X]_d = \bigoplus_{\lambda \in K^\wedge} \Omega_d(\lambda) V^\lambda. \]

One can view $\Omega_k(\lambda)$ as a measure on $K^\wedge \subset \mathfrak{h}_+$. After proper normalization, the measures $\Omega_k(k\lambda)$ converge weakly to $\Omega(\xi) \, d\xi$ as $k \to \infty$ where $d\xi$ is the Lebesgue measure on $\mathfrak{h}_+$.

In other words, for any $A \subset \mathfrak{h}_+$, the integral $\int_A \Omega(\xi) \, d\xi$ describes the leading asymptotics of the sum $\sum_{\lambda \in kA} \Omega_k(\lambda)$ as $k \to \infty$. Hence, informally, $\Omega(\xi)$ is the multiplicity $\Omega_k(k\lambda)$ averaged over some infinitesimal neighborhood of $\xi$. Such an averaging over infinitesimally close energy levels is a very natural thing to do from the statistical physics perspective.

### 3.2

The function $\Omega(\xi)$ depends not only on the $K$-action on $X$ as such but also on the embedding $X \subset \mathbb{P}^N$, where $\mathbb{C}^{N+1}$ is a representation space of $K$ or, equivalently, of the complexification $G$ of $K$.

In intrinsic terms, such an embedding is a very ample invertible sheaf $L$ in the $G$-linearized Picard group $\text{Pic}^G(X)$ of $X$. Write $\Omega(\xi, L)$ to stress the dependence on both $\xi$ and $L$. Because $L$ enters the definition of $\Omega(\xi, L)$ only via $L^{\otimes n}$, $n \to \infty$, the function $\Omega(\xi, L)$ is well-defined for any
\[ L \in \text{Pic}^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \]

Since ample sheafs form a semigroup in $\text{Pic}^G(X)$ it makes sense to ask whether $\log \Omega(\xi, L)$ is concave as a function of the pair $(\xi, L)$.

In fact, we already saw an example of such a bivariate concavity in Section 2.4 where the multiplicities for restrictions from $U(n)$ to $U(k)$ turned out to be concave in the pair of highest weights.

### 3.3

In this setting, the log-concavity of $\Omega(\xi)$ and $\Omega(\xi, L)$ was established in [17] and [18], respectively, by using the classical Brunn-Minkowski inequality of convex analysis.

Here we want to use the same ideas to approach the problem from a slightly different angle. Instead of looking at the weak limit of measures
\(\Omega_k(k\lambda)\), which involves averaging over infinitesimally close energy levels, we want to look at the asymptotics of the sequence \(\Omega_k(k\lambda)\) for some fixed \(\lambda\). This is a natural thing to do from the representation theory point of view.

### 3.4

There is a standard trick which allows to dispose of the first variable \(\xi\) in \(\Omega(\xi, L)\) by enlargeing the variety \(X\). Indeed, by definition of the multiplicities \(\Omega_k\) we have

\[
\Omega_k(k\lambda, L) = \dim \left( H^0(X, L^{\otimes k}) \otimes V^{k\lambda^*} \right)^G,
\]

\[
= \dim H^0 \left( X \times G/B, (L \boxtimes L_{\lambda^*})^{\otimes k} \right)^G,
\]

where \(\lambda^* \in K^\wedge\) is the highest weight of \((V^\lambda)^*\), \(G/B = K/T\) is the flag variety of \(K\), the sheaf \(L_{\lambda^*} \in \text{Pic}^G(G/B)\) corresponds to the map of \(G/B\) onto the orbit of the highest vector in \(P(V^\lambda)\), and superscript \(G\) denotes \(G\)-invariants. So, without loss of generality, we can assume that \(\xi = 0\).

### 3.5

Recall that, by definition,

\[
\mathbb{C}[X//L G] = \bigoplus_k H^0(X, L^{\otimes k})^G
\]

is the homogeneous coordinate ring of the Geometric Invariant Theory quotient \(X//L G\) corresponding to \(L \in \text{Pic}^G(X)\). We write \(X//L G\) in place of the standard \(X//G\) to stress the dependence on \(L\).

The sequence

\[
\Omega_k(0, L) = \dim H^0(X, L^{\otimes k})^G, \quad k = 0, 1, 2, \ldots
\]

may fail to have a \(k \to \infty\) asymptotics for the following trivial reason. Consider the set

\[
\{k, H^0(X, L^{\otimes k})^G \neq 0\} \subset \mathbb{Z}_{\geq 0}.
\]

Since \(X\) is irreducible, it is a semigroup and it either contains all sufficiently large integers or lies in a proper subgroup if \(\mathbb{Z}\). We want to avoid the latter case because in that case the sequence (3.1) does not have any asymptotics. So we will replace \(L\) by a suitable power of \(L\) in that case.
By replacing $L$ by its power one can also achieve that $\mathbb{C}[X//_LG]$ is generated by its degree 1 graded component and so we can assume this as well. Thus, we have an embedding

$$X//_LG \subset \mathbb{P} \left( (H^0(X, L)^G)^* \right)$$

and we denote by $\deg X//_LG$ the degree of this embedding. It follows that in this case

$$\Omega_k(0, L) \sim \deg X//_LG \frac{k^d}{d!}, \quad d = \dim X//_LG,$$

as $k$ goes to $\infty$. We now want to show that $\log \deg X//_LG$ is a concave function of $L$.

**3.6**

In fact, one can establish a more general fact. Let $Y$ be an irreducible algebraic variety of dimension $d$ and let $K = \mathbb{C}(Y)$ be the field of rational functions on $Y$. Let $S \subset K$ be a $\mathbb{C}$-linear subspace such that $1 \in S$ and which generates $K$ as a field. The embedding $S \subset K$ corresponds to a subvariety $Y_S \subset S^*$ which is birationally isomorphic to $Y$. Let $\deg Y_S$ denote its degree.

Given two such subspaces $S_1$ and $S_2$, denote by $S_1 S_2$ the subspace generated by all products $f_1 f_2$, where $f_i \in S_i$. We will show that

$$\sqrt[d]{\deg Y_{S_1 S_2}} \geq \sqrt[d]{\deg Y_{S_1}} + \sqrt[d]{\deg Y_{S_2}} \quad (3.2)$$

for any such pair $S_1$ and $S_2$. Since, clearly,

$$\deg Y_{S^2} = 2^d \deg Y_S$$

the inequality $(3.2)$ implies that $\sqrt[d]{\deg Y_S}$, and consequently, $\log \deg Y_S$ is a concave function of $S$.

**3.7**

In particular, $(3.2)$ would imply the concavity of $\log \deg X//_LG$. Indeed, although the varieties $X//_LG$ may not be isomorphic for different $L$, they are always birationally isomorphic. Their common field of fractions is the field $K = \mathbb{C}(X)^G$ of rational $G$-invariants.

Given some $L_1$ and $L_2$, pick some $\phi_i \in H^0(X, L_i)^G$. Replacing the $L_i$’s if necessary by their multiples, we can assume that $\mathbb{C}[X//_{L_i}G]$ is generated
by $S_i = \phi_i^{-1}H^0(X, L_i)^G \subset k$. Since the algebra $\mathbb{C}[X/\bigoplus L_i G]$ contains the algebra generated by $S_1 S_2$, we get from (3.2) the desired lower bound on the asymptotics of the dimensions of the graded components of $\mathbb{C}[X/\bigoplus L_i G]$.

3.8
Now, in order to establish (3.2), we will construct convex sets $\Delta_S \in \mathbb{R}^d$ of dimension $d = \dim Y$ such that

$$\deg Y_S = d! \ vol \Delta_S$$

and

$$\Delta_{S_1 S_2} \supset \Delta_{S_1} + \Delta_{S_2}.$$ 

The inequality (3.2) will then follow immediately from the classical Brunn-Minkowski inequality, see e.g. [3]. See also e.g. the appendix by A. Khovanskii in [3] for a discussion of the relationship between classical inequalities of the convex analysis and algebraic geometry. For example, the Alexandrov-Fenchel inequality, which is stronger than the Brunn-Minkowski inequality, corresponds to the Hodge index theorem for surfaces.

3.9
The construction of $\Delta_S \in \mathbb{R}^d$ is similar to the definition of a Newton polytope.

Choose a smooth point $y \in Y$ which lies away from the singularities of the maps from $Y$ to $Y_S$ and their inverses. Choose a flag of subvarieties

$$Y \supset Y^1 \supset \cdots \supset Y^d = y, \quad \text{codim} \ Y^k = k,$$

which are all smooth at $y$. Fix some local equation $u_k$ of $Y^k$ in $Y^{k-1}$.

This data give rise to a map

$$\mathbb{K} \setminus 0 \ni f \mapsto \nu(f) = (\nu_1(f), \ldots, \nu_d(f)) \in \mathbb{Z}^d,$$

where

$$\nu_1(f) = \text{ord}_{Y^1} f,$$

$$\nu_2(f) = \text{ord}_{Y^2} \left( f u_1^{-\nu_1(f)} \right)_{Y^1},$$

$$\nu_3(f) = \text{ord}_{Y^3} \left( f u_1^{-\nu_1(f)} u_2^{-\nu_2(f)} \right)_{Y^2}. \ldots$$
and so on. It is clear that $v$ is a valuation, that is,

$$v(fg) = v(f) + v(g),$$

$$v(f + g) \geq \min\{v(f), v(g)\},$$

where the ordering on $\mathbb{Z}^d$ is lexicographic.

It is also clear that the residue field of $v$ is isomorphic to $\mathbb{C}$ and hence for any $\mathbb{C}$-linear subspace of $S \subset \mathbb{K}$ we have

$$\dim_C S = |v(S \setminus 0)|.$$  \hspace{1cm} (3.4)

**3.10**

By definition, set

$$\Gamma_S = \{(k, v(f)), f \in S^k \setminus 0\} \subset \mathbb{Z}^{1+d}.$$  \hspace{1cm} (3.3)

It follows from (3.3) that $\Gamma_S$ is a semigroup.

Denote by $\Lambda_S \subset \mathbb{Z}^{1+d}$ the lattice generated by $\Gamma_S$. Let $\nabla_S \subset \mathbb{R}^{1+d}$ be the closed convex cone generated by $\Gamma_S$ and let $\Delta_S$ be the intersection of $\nabla_S$ with the subspace $(1, \mathbb{R}^d) \subset \mathbb{R}^{1+d}$. It is clear that

$$\Delta_S = \left\{ \frac{v(f)}{k}, f \in S^k \setminus 0 \right\},$$

where bar denotes closure.

Since the point $y$ corresponds to a smooth point of $Y_S$, there exist

$$f_0, f_1, \ldots, f_d = 1 \in \mathbb{C}[Y_S]$$

such that

$$v(f_k) = (0, \ldots, 0, 1, \ldots).$$

Thus, $\Delta_S$ contains a $d$-dimensional simplex and so

$$\dim \Delta_S = d.$$  \hspace{1cm} (3.5)
3.11

Let us now prove that

\[ \text{vol } \Delta_S = d! \deg Y_S, \]

On the one hand, it follows that (3.4) and the definition of \( \Gamma_S \) that

\[ \left| \Gamma_S \cap (k, \mathbb{Z}^d) \right| = \dim_C S^k \sim \deg Y_S \frac{k^d}{d!}, \quad k \to \infty. \]

Since \( \Gamma_S \subset \nabla_S \cap \Lambda_S \) we have

\[ \frac{\deg Y_S}{d!} \leq \text{vol } \Delta_S, \]

where the normalization of Lebesgue measure is given by the intersection of the lattice \( \Lambda_S \) with the subspace \( (1, \mathbb{R}^d) \subset \mathbb{R}^{1+d} \).

The inverse inequality will be deduced from the following results of Khovanskii [13]. Choose a sequence of finitely generated subgroups

\[ \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_S \]

such that \( \Gamma_S = \bigcup \Gamma_i \) and each \( \Gamma_i \) generates the lattice \( \Lambda_S \). Let \( \nabla_i \) denote the cone generated by \( \Gamma_i \) and let \( \Delta_i \) be the corresponding hyperplane section of \( \nabla_i \). It is clear that

\[ \Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_S = \bigcup \Delta_i. \quad (3.6) \]

It is a theorem of Khovanskii, see Proposition 3 in Section 3 of [13], that there exists vectors \( \gamma_i \in \Gamma_i \) such that

\[ (\nabla_i + \gamma_i) \cap \Lambda_S \subset \Gamma_i. \]

It follows that

\[ \left| \frac{\Gamma_i \cap (k, \mathbb{Z}^d)}{k^d} \right| \to \text{vol } \Delta_i, \quad k \to \infty. \]

Hence \( \text{vol } \Delta_i \leq (d!)^{-1} \deg Y_S \) and (3.6) implies that

\[ \text{vol } \Delta_S \leq \frac{\deg Y_S}{d!}, \]

as was to be shown.
The relation
\[ \Delta_{S_1S_2} \supseteq \Delta_{S_1} + \Delta_{S_2}. \] (3.7)
follows immediately from (3.3) and the definition of \( \Delta_S \). Now we are almost in position to finish the proof of (3.2) by applying the Brunn-Minkowski inequality. One remaining detail is that our normalization of the volume \( \text{vol} \Delta_S \) depends on the lattice \( \Lambda_S \). It is, however, clear that
\[ \Lambda_{S_1S_2} \subset \Lambda_{S_1}, \Lambda_{S_2}. \]
Therefore, if we normalize the volume according to the lattice \( \Lambda_{S_1S_2} \) we have
\[ \text{vol} \Delta_{S_1S_2} = \frac{\deg Y_{S_1S_2}}{d!}, \quad \text{vol} \Delta_{S_i} \geq \frac{\deg Y_{S_i}}{d!}. \]
This and the Brunn-Minkowski inequality applied to (3.7)
\[ d\sqrt{\text{vol} \Delta_{S_1S_2}} \geq d\sqrt{\text{vol} \Delta_{S_1}} + d\sqrt{\text{vol} \Delta_{S_2}} \]
completes the proof of (3.2).

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