THE MOMENT PROBLEM FOR CONTINUOUS POSITIVE SEMIDEFINITE LINEAR FUNCTIONALS

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Abstract. Let \( \tau \) be a locally convex topology on the countable dimensional polynomial \( \mathbb{R} \)-algebra \( \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n] \). Let \( K \) be a closed subset of \( \mathbb{R}^n \), and let \( M := M_{\{g_1, \ldots, g_s\}} \) be a finitely generated quadratic module in \( \mathbb{R}[X] \). We investigate the following question: When is the cone \( \text{Psd}(K) \) (of polynomials nonnegative on \( K \)) included in the closure of \( M \)? We give an interpretation of this inclusion with respect to representing continuous linear functionals by measures. We discuss several examples; we compute the closure of \( M = \sum \mathbb{R}[X]^2 \) with respect to weighted norm-\( p \) topologies. We show that this closure coincides with the cone \( \text{Psd}(K) \) where \( K \) is a certain convex compact polyhedron.

1. Introduction

Given a finite subset \( S := \{g_1, \ldots, g_s\} \) of the polynomial ring \( \mathbb{R}[X] \), the question of approximating polynomials nonnegative on the basic closed semialgebraic set \( K_S \) via elements of the quadratic module \( M_S \) (see Definition 2.10) is a main topic in real algebraic geometry and has many applications in optimization and functional analysis (see Lasserre [10]). Putinar’s Archimedean Positivstellensatz [13] brought important improvements to the Positivstellensatz [8]: for \( K \) a compact basic closed semialgebraic set, and \( S \) any representation of \( K \) (containing the inequality \( N - \sum x_i^2 \geq 0 \) expressing that \( K := K_S \) is bounded, for some \( N \in \mathbb{N} \)), any polynomial \( f > 0 \) on \( K_S \) belongs to \( M_S \). The above results have direct applications to the moment problem for semialgebraic sets. Given a closed set \( K \subseteq \mathbb{R}^n \), the \( K \)-moment problem is the question of when a linear functional \( \ell : \mathbb{R}[X] \to \mathbb{R} \) is representable as integration with respect to a positive Borel measure on \( K \).

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Denoting the set of all nonnegative polynomials on $K$ by $\text{Psd}(K)$, a necessary condition is that $\ell(f) \geq 0$, for $f \in \text{Psd}(K)$. In [6, 7], Haviland proved that this necessary condition is sufficient. However in general $\text{Psd}(K)$ is not finitely generated, so Haviland’s result may be impractical. Fortunately, it follows from Putinar’s Archimedean Positivstellensatz that nonnegativity of $\ell$ on $\text{Psd}(K_S)$ is ensured once nonnegativity of $\ell$ on the finitely generated archimedean quadratic module $M_S$ (Definition 2.10) is established. Thus one is reduced to checking $s + 2$ many systems of inequalities:

\begin{align*}
\ell(h^2 g_i) &\geq 0 \quad \text{for } h \in \mathbb{R}[X], \; i = 0, \ldots, s + 1, \\
g_0 &:= 1, \quad g_{s+1} := (N - \sum x_i^2).
\end{align*}

Thus the $K_S$-moment problem is “finitely solvable”. This can be summarized in a topological statement: If $\text{Psd}(K_S) \subseteq \overline{M_S}^\varphi$, then the $K_S$-moment problem for $M_S$ is finitely solvable. Here, $\varphi$ denotes the finest locally convex topology on $\mathbb{R}[X]$, and $\overline{M_S}^\varphi$ denotes the closure with respect to that topology. In [1, 2, 12] coarser topologies are considered: it is shown that $\sum \mathbb{R}[X]^2$ is dense in $\text{Psd}([-1,1]^n)$ for the $\ell_1$ topology on $\mathbb{R}[X]$. In the language of moments, this result means that every $\ell_1$ continuous linear functional, non-negative on $\sum \mathbb{R}[X]^2$ (i.e. a positive semidefinite functional), is representable by a positive Borel measure on $[-1,1]^n$.

In this paper, we generalize this setting to arbitrary locally convex topologies on $\mathbb{R}[X]$. In Section 2 we review background on topological vector spaces. In Section 3 we present our setting as a threefold statement about a locally convex topology $\tau$, a closed subset $K$ of $\mathbb{R}^n$, and a cone $C$ in $\mathbb{R}[X]$. We observe that if $\text{Psd}(K) \subseteq \overline{C}^\varphi$ then any $\tau$-continuous functional, non-negative on $C$, is integration with respect to a positive Borel measure on $K$ (see Proposition 3.1). We apply the above setting to the example of the weighted norm-$p$ topologies, see Remark 3.2 and Theorems 3.12, 3.13, 3.14 and 3.15. We compute the closure of the cone $C := \sum \mathbb{R}[X]^2$ in the topology of coefficientwise convergence (see Proposition 3.16).

Finally, we mention that this point of view has been recently revisited: in [11] Lasserre proves that for a certain fixed norm $\| \cdot \|_w$ defined in [11], and any finite $S$, $\overline{M_S}^\| \cdot \|_w = \text{Psd}(K_S)$. Also, it follows from recent work [5] that Theorem 3.8 and Theorem 3.12 carry over with the cone $\sum \mathbb{R}[X]^2$ replaced by the cone of sums of $2d$-powers, $\sum \mathbb{R}[X]^{2d}$, for any integer $d \geq 1$. In a forthcoming paper [4], we compute the closure of $\sum \mathbb{R}[X]^{2d}$ with respect to other locally convex topologies.
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2. Preliminaries

In this section, we give some background from functional analysis. To make the paper self-contained and for the sake of the reader, we have given a thorough explanation in this section.

2.1. Background on Topological Vector Spaces. In the following, all vector spaces are over the field of real numbers (unless otherwise specified).

A vector space topology on a vector space $V$ is a topology $\tau$ on $V$ such that every point of $V$ is closed and the vector space operations, i.e., vector addition and scalar multiplication are $\tau$-continuous. A topological vector space is a pair $(V, \tau)$ where $V$ is a vector space and $\tau$ is a vector space topology on $V$. A standard argument shows that $\tau$ is Hausdorff.

A subset $A \subseteq V$ is said to be convex if for every $x, y \in A$ and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in A$. A locally convex topology is a vector space topology which admits a neighbourhood basis of convex open sets at each point. A norm on $V$ is a function $\| \cdot \| : V \rightarrow \mathbb{R} \geq 0$ satisfying

1. $\|v\| = 0 \Leftrightarrow x = 0$,
2. $\forall \lambda \in \mathbb{R}, \|\lambda v\| = |\lambda|\|v\|$,
3. $\forall v_1, v_2 \in V, \|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$.

A topology $\tau$ on $V$ is said to be normable (respectively metrizable), if there exists a norm (respectively metric) on $V$ which induces the same topology as $\tau$. Every norm induces a locally convex metric topology on $V$ where the induced metric is defined by $d(v_1, v_2) = \|v_1 - v_2\|$.

Remark 2.1. For two normed space $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$, a linear operator $T : X \rightarrow Y$ is said to be bounded if there exists $N \geq 0$ such that for all $x \in X$, $\|Tx\|' \leq N\|x\|$. A standard result states that boundedness and continuity in normed spaces are equivalent.

For a topological vector space $(V, \tau)$ we denote the set of all continuous linear functionals $\ell : V \rightarrow \mathbb{R}$ by $V^*$.

Definition 2.2. For $C \subseteq V$, let

$$C^\vee_\tau = \{\ell \in V^* : \ell \geq 0 \text{ on } C\}$$

be the first dual of $C$ and define the second dual of $C$ by

$$C^{\vee\vee}_\tau = \{a \in V : \forall \ell \in C^\vee_\tau, \ell(a) \geq 0\}.$$
The following is immediate from the definition:

**Proposition 2.3.** For a locally convex topological vector space \((V, \tau)\) and \(C, D \subseteq V\) the following holds

1. \(C \subseteq D \Rightarrow D^\vee_\tau \subseteq C^\vee_\tau\),
2. \(C \subseteq C^\vee_\tau^\vee\),
3. \(C^\vee_\tau^\vee^\vee = C^\vee_\tau\).

A subset \(C\) of \(V\) is called a cone if \(C + C \subseteq C\) and \(R^+ C \subseteq C\). It is clear that \(C\) is convex.

**Theorem 2.4 (Separation).** Suppose that \(A\) and \(B\) are disjoint nonempty convex sets in \(V\). If \(A\) is open, then there exists \(\ell \in V^*\) and \(\gamma \in \mathbb{R}\) such that \(\ell(x) < \gamma \leq \ell(y)\) for every \(x \in A\) and \(y \in B\). Moreover, if \(B\) is a cone, then \(\gamma\) can be taken to be 0.

**Proof.** For the first part, see [15, Theorem 3.4]. Suppose that \(B\) is a cone and suppose that \(\ell\) and \(\gamma\) are given by first part. If \(\gamma > 0\), then \(\ell(y) > 0\) for all \(y \in B\). Therefore \(\forall \epsilon > 0\) \(\epsilon y \in B\) so \(0 < \gamma \leq \ell(\epsilon y) = \epsilon \ell(y) \xrightarrow{\epsilon \to 0} 0\). This implies that \(\gamma \leq 0\). Note also that \(\ell \geq 0\) on \(B\). Otherwise, \(\ell(x) < \gamma \leq \ell(y) < 0\) for any \(x \in A\) and some \(y \in B\). Then for \(r > 0\), \(ry \in B\) and \(\ell(x) < \gamma \leq \ell(ry) = r\ell(y) \xrightarrow{r \to \infty} -\infty\) which is impossible. Therefore \(\ell(x) < \gamma \leq 0 \leq \ell(y)\) \(\forall x \in A, \forall y \in B\). Hence \(\gamma\) can be chosen to be 0. \(\square\)

Below \(\overline{C}\) denotes the closure of \(C\) with respect to \(\tau\). It follows that:

**Corollary 2.5 (Duality).** For any nonempty cone \(C\) in \((V, \tau)\), \(\overline{C}^\tau = C^\vee_\tau^\vee\).

**Proof.** Since each \(\ell \in C^\vee_\tau\) is continuous, for any \(a \in \overline{C}^\tau\), \(\ell(a) \geq 0\), so \(\overline{C}^\tau \subseteq C^\vee_\tau^\vee\). Conversely, if \(a \notin \overline{C}^\tau\) then since \(\tau\) is locally convex, there exists an open convex set \(U\) of \(V\) containing \(a\) with \(U \cap C = \emptyset\). By [2.4] there exists \(\ell \in C^\vee_\tau\) such that \(\ell(a) < 0\), so \(a \notin C^\vee_\tau^\vee\). \(\square\)

### 2.2. Finest Locally Convex Topology on \(\mathbb{R}[X]\)

Let \(V\) be any vector space over \(\mathbb{R}\) of countable infinite dimension. We define the direct limit topology \(\varphi\) on \(V\) as follows: \(U \subseteq V\) is open if and only if \(U \cap W\) is open in \(W\) for each finite dimensional subspace \(W\) of \(V\).

**Theorem 2.6.** The open sets in \(V\) which are convex form a basis for the direct limit topology. Moreover \((V, \varphi)\) is a topological vector space and \(\varphi\) is the finest locally convex topology on \(V\).

**Proof.** [13, Section 3.6 and Theorem 3.6.1]. \(\square\)
Remark 2.7. (i) The vector space \((V, \varphi)\) is not metrizable. Let \(U\) be a neighbourhood of 0 in \(V\). From the proof of [13, Theorem 3.6.1], there exist \(a_i \in \mathbb{R}^+\), \(i = 1, 2, \ldots\), such that \(\prod_{i=1}^{\infty} \langle -a_i, a_i \rangle \subseteq U\), where

\[
\prod_{i=1}^{\infty} \langle -a_i, a_i \rangle = \left\{ \sum_i t_i e_i : -a_i < t_i < a_i \right\},
\]

and \(\{e_i\}_{i=1}^{\infty}\) forms a basis for \(V\) and all summands are 0 except for finitely many \(i\). If there exists a countable neighbourhood basis at 0 then there exist real numbers \(a_{ij}, i, j = 1, 2, \ldots\) such that \(\prod_{i=1}^{\infty} \langle -a_{1i}, a_{1i} \rangle, \prod_{i=1}^{\infty} \langle -a_{2i}, a_{2i} \rangle, \ldots\) forms a neighbourhood basis at 0. Take 0 < \(b_i < a_{ii}\) for each \(i\), then \(\prod_{i=1}^{\infty} \langle -b_i, b_i \rangle\) is a neighbourhood of 0 which does not contain any of the above basic open sets, a contradiction.

(ii) Every linear functional is continuous with respect to \(\varphi\). For the weak topology (induced by the set of all linear functionals), convex sets have the same closure as they have under \(\varphi\) [15, Theorem 3.12].

(iii) Direct limit topology and finest locally convex topology are defined even when \(V\) is uncountably infinite dimensional. But they only coincide when the space is countable dimensional.

2.3. Moment Problem. In analogy to the classical Riesz Representation Theorem, Haviland considered the problem of representing linear functionals on the algebra of polynomials by measures. The question of when, given a closed subset \(K\) in \(\mathbb{R}^n\), a linear map \(\ell : \mathbb{R}[X] \to \mathbb{R}\) corresponds to a finite positive Borel measure \(\mu\) on \(K\) is known as the Moment Problem.

Definition 2.8. For a subset \(K \subseteq \mathbb{R}^n\), define the cone of nonnegative polynomials on \(K\) by

\[
\text{Psd}(K) = \{ f \in \mathbb{R}[X] : \forall x \in K f(x) \geq 0 \}.
\]

Theorem 2.9 (Haviland). For a linear function \(\ell : \mathbb{R}[X] \to \mathbb{R}\) and a closed set \(K \subseteq \mathbb{R}^n\), the following are equivalent:

1. There exists a positive regular Borel measure \(\mu\) on \(K\) such that,

\[
\forall f \in \mathbb{R}[X] \quad \ell(f) = \int_K f \, d\mu.
\]

2. \(\forall f \in \text{Psd}(K)\) \(\ell(f) \geq 0\).

The main challenge in applying Haviland’s Theorem is verifying its condition (2). We analyse this problem for a certain class of closed subsets.
Definition 2.10. A subset $K \subseteq \mathbb{R}^n$ is called a basic closed semialgebraic set if there exists a finite set of polynomials $S = \{g_1, \ldots, g_s\}$ such that $K = K_S := \{ x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, \ldots, s \}$. Note that we can define $K_C = \{ x \in \mathbb{R}^n : \forall f \in C f(x) \geq 0 \}$ for any subset $C \subseteq \mathbb{R}[X]$, but this set may not be a semialgebraic set. A subset $M$ of $\mathbb{R}[X]$ is called a quadratic module if $1 \in M$, $M + M \subseteq M$, and for each $h \in \mathbb{R}[X]$, $h^2 M \subseteq M$. For $S = \{ g_1, \ldots, g_s \}$, let

$$M_S := \left\{ \sum_{i=0}^{s} \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[X]^2 \ \text{for} \ i = 0, \ldots, s \ \text{and} \ g_0 = 1 \right\}.$$ 

One can check that $M_S$ is the smallest quadratic module of $\mathbb{R}[X]$ containing $S$. Clearly $M_S \subseteq \text{Psd}(K_S)$.

A quadratic module $M$ of $\mathbb{R}[X]$ is said to be archimedean if for a sufficiently large integer $N$, $(N - \sum x_i^2) \in M$.

Remark 2.11. Note that to check whether a given linear functional $\ell : \mathbb{R}[X] \to \mathbb{R}$ is nonnegative on the quadratic module $M_S$, it suffices to verify the following:

$$(2) \quad \ell(h^2 g_i) \geq 0 \quad \text{for} \ h \in \mathbb{R}[X], \ i = 0, \ldots, s, \ g_0 := 1.$$ 

If $\text{Psd}(K_S) \subseteq (M_S)_{\text{cv}}$, then, by Haviland’s Theorem, every linear functional nonnegative on $M_S$ corresponds to a measure on $K_S$. Since $M_S$ is a cone in $\mathbb{R}[X]$ it follows by Corollary 2.5 that $(M_S)_{\text{cv}} = M_S^{\text{cv}}$. Therefore we are interested in the inclusion

$$(3) \quad \text{Psd}(K_S) \subseteq M_S^{\text{cv}}.$$ 

In other words, for a given basic closed semialgebraic set $K$, if one can find a finite $S \subseteq \mathbb{R}[X]$ such that $K = K_S$ and at the same time inclusion (3) holds, then the problem of representing a functional by a measure on $K$ is reduced to verifying that conditions (2) hold.

3. The Moment Problem for functionals continuous with respect to a locally convex topology

We have seen that every linear functional is continuous with respect to $\varphi$. We now consider a linear functional $\ell$, continuous with respect to an arbitrary locally convex topology on $\mathbb{R}[X]$. We further consider an arbitrary closed subset $K \subseteq \mathbb{R}^n$, and an arbitrary cone $C \subseteq \mathbb{R}[X]$.

Proposition 3.1. For a locally convex topology $\tau$ on $\mathbb{R}[X]$, a closed subset $K \subseteq \mathbb{R}^n$ and a cone $C \subseteq \mathbb{R}[X]$, the following are equivalent:
\( C^\vee_\tau \subseteq \text{Psd}(K)^\vee_\tau \),
(2) \( \text{Psd}(K) \subseteq C^\vee_\tau \),
(3) \( \forall \ell \in C^\vee_\tau \) there exists a positive Borel measure on \( K \) such that.
\[
\forall f \in \mathbb{R}[X] \quad \ell(f) = \int_K f \, d\mu.
\]

Proof. (1) \( \iff \) (2) is clear by Proposition 2.3. For (1) \( \Rightarrow \) (3), note that \( \text{Psd}(K)^\vee_\tau \subseteq \text{Psd}(K)^\vee_\phi \) then apply Haviland’s Theorem 2.9. (3) \( \Rightarrow \) (1) is clear. \( \square \)

Remark 3.2. As explained in the introduction, given an arbitrary compact semialgebraic set \( K \), to determine whether a linear functional comes from a measure on \( K \), we need to check the \( s+2 \) conditions given in (1). In the next two subsections, we show that for certain compact convex sets \( K \) and functionals continuous in weighted norm \( p \) topologies, we just need to check a single condition, namely \( \ell(h^2) \geq 0 \) (see Corollary 3.10 and Theorems 3.12, 3.13, 3.14 and 3.15).

3.1. Norm-\( p \) Topologies. We are interested in computing the closure of the cone \( \sum \mathbb{R}[X]^2 \) in \( \mathbb{R}[X] \) under certain norm topologies. We start by reviewing some basic facts about \( \| \cdot \|_p \) norms (see [3]). In what follows, \( N := \{0, 1, \cdots, \} \) the set of natural numbers, \( X := (X_1, \cdots, X_n) \), \( X_\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n} \) where \( \alpha = (\alpha_1, \cdots, \alpha_n) \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

Let \( 1 \leq p < \infty \), and define the mapping \( \| \cdot \|_p : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{\infty\} \) for each \( s : \mathbb{N}^n \rightarrow \mathbb{R} \) with
\[
\|s\|_p = \left( \sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^p \right)^{\frac{1}{p}} = \left( \sum_{d=0}^{\infty} \sum_{|\alpha|=d} |s(\alpha)|^p \right)^{\frac{1}{p}}.
\]
For \( p = \infty \), define \( \|s\|_\infty = \sup_{\alpha \in \mathbb{N}^n} |s(\alpha)| \). For \( 1 \leq p < \infty \), we let
\[
\ell_p(\mathbb{N}^n) = \{s \in \mathbb{R}^\mathbb{N}^n : \|s\|_p < \infty\},
\]
and
\[
c_0(\mathbb{N}^n) = \{s \in \mathbb{R}^\mathbb{N}^n : \lim_{\alpha \in \mathbb{N}^n} |s(\alpha)| = 0\}.
\]
It is well-known that \( \| \cdot \|_p \) is a norm on \( \ell_p(\mathbb{N}^n) \) and \( (\ell_p(\mathbb{N}^n), \| \cdot \|_p) \) forms a Banach space. Moreover, if \( 1 \leq p < q \leq \infty \) then \( \ell_p(\mathbb{N}^n) \subseteq \ell_q(\mathbb{N}^n) \). Let \( V_p \) be the set of all finite support real \( n \)-sequences, equipped with \( \| \cdot \|_p \). Fixing the monomial basis \( \{X_\alpha : \alpha \in \mathbb{N}^n\} \), we identify the space of real polynomials \( \mathbb{R}[X] \), endowed by \( \| \cdot \|_p \)-norm, with \( V_p \).

For \( 1 \leq p \leq \infty \), define the conjugate \( q \) of \( p \) as follows:
- If \( p = 1 \), let \( q = \infty \),
• If }p = \infty, let }q = 1,
• if }1 < p < \infty, let }q be the real number satisfying }1 \frac{1}{p} + \frac{1}{q} = 1.
For a proof of Proposition 3.3, Lemma 3.4 and Lemma 3.5 see [3] or [15].

**Proposition 3.3.** For }1 \leq p < \infty, }V_p is a dense subspace of }\ell_p(N^n), and }V_\infty is dense in }c_0(N^n). A linear functional }\ell on }V_p is continuous if and only if }\|\(\ell(V^\alpha)\)\|_q < \infty where }q is the conjugate of }p.

**Lemma 3.4.** For }1 \leq p \leq q \leq \infty, the identity map }id_{pq} : }V_p \to }V_q is continuous, i.e., }\|\cdot\|_p induces a finer topology than }\|\cdot\|_q on }R[X]. Therefore, for any }C \subseteq }R[X] we have }C^{\|\cdot\|_p} \subseteq }C^{\|\cdot\|_q}.

**Lemma 3.5.** (Hölder’s inequality) Let }1 \leq p \leq \infty and let }q be the conjugate of }p. Let }a \in }\ell_p(N^n) and }b \in }\ell_q(N^n). Then }ab \in }\ell_1(N^n) and }\|ab\|_1 \leq }\|a\|_p} \|b\|_q, where we define }ab(\alpha) := a(\alpha)b(\alpha) for every }\alpha \in N^n.

In [11, Theorem 9.1], Berg, Christensen and Ressel showed that the closure of }\sum }R[X]^2 in the }\|\cdot\|_1\)-topology is }Psd([-1,1]^n). The proof given by Berg, Christensen and Ressel in [11,2] is based on techniques from harmonic analysis on semigroups, whereas in [12], Lasserre and Netzer produce a concrete sequence in }\sum }R[X]^2 converging to each }f \in }Psd([-1,1]^n) in }\|\cdot\|_1. The above result of Berg et al. easily extends to all }\|\cdot\|_p\)-topologies, see Theorem 3.8 below. We first need the following.

**Proposition 3.6.** Let }1 \leq p \leq \infty and }x \in }R^n, and let }e_x : }V_p \to }R be the evaluation homomorphism on }V_p defined by }e_x(f) := f(x). Then the following statements are equivalent:

1. }e_x is continuous.
2. }\|\(x^\alpha)_{\alpha \in N^n}\|_q < \infty, where }q is the conjugate of }p.
3. }x \in (-1,1)^n if }1 \leq p < \infty, and }x \in [-1,1]^n if }p = \infty.

**Proof.** (2)\iff (3) First assume that }1 \leq p < \infty. Let }x = (x_1, \ldots, x_n) \in }R^n. Then

\[
(\|x^\alpha\|_{\alpha \in N^n})^p = \sum_{\alpha \in N^n} |x^\alpha|^p = \sum_{\alpha_1, \ldots, \alpha_n = 0} |x_1|^{p\alpha_1} \cdots |x_n|^{p\alpha_n} = \left(\sum_{\alpha_1 = 0}^\infty |x_1|^{p\alpha_1}\right) \cdots \left(\sum_{\alpha_n = 0}^\infty |x_n|^{p\alpha_n}\right),
\]

a product of geometric series. It follows that }\|x^\alpha\|_{\alpha \in N^n} < \infty if and only if }|x_i| < 1 for }i = 1, \ldots, n. For }p = \infty, }\|x^\alpha\|_{\infty} = \sup_{\alpha \in N^n} |x^\alpha| is finite if and only if }|x_i| \leq 1, for each }1 \leq i \leq n.
(2)⇒(1) First suppose that $1 \leq p < \infty$. Assuming $f(X) = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} X^\alpha$,
\[
\|e_\perp\| = \sup\|f\|_p = 1 |f(x)| = \sup\|f\|_p = 1 |\sum_{\alpha \in \mathbb{N}^n} f_{\alpha} x^\alpha| \\
\leq \sup\|f\|_p = 1 \sum_{\alpha \in \mathbb{N}^n} |f_{\alpha}| |x^\alpha| \\
= \sup\|f\|_p = 1 \sum_{\alpha \in \mathbb{N}^n} |x^\alpha| \\
= \|\sum_{\alpha \in \mathbb{N}^n} |x^\alpha|,\|_q.
\]

(By Hölder’s inequality) Therefore if $\|\sum_{\alpha \in \mathbb{N}^n} |x^\alpha|,\|_q < \infty$, the $e_\perp$ is continuous.

For $p = \infty$,
\[
\|e_\perp\| = \sup\|f\|_\infty = 1 |f(x)| = \sup\|f\|_\infty = 1 |\sum_{\alpha \in \mathbb{N}^n} f_{\alpha} x^\alpha| \\
\leq \sum_{\alpha \in \mathbb{N}^n} |f_{\alpha}| |x^\alpha| \\
= \|\sum_{\alpha \in \mathbb{N}^n} |x^\alpha|,\|_1.
\]

So, if $\|\sum_{\alpha \in \mathbb{N}^n} |x^\alpha|,\|_1 < \infty$, then $e_\perp$ is continuous.

(1)⇒(2) First consider the case where $1 \leq p < \infty$. Suppose that $e_\perp$ is continuous on $V_p$. By Proposition 3.3, $V_p$ is a dense subspace of $(\ell_p(\mathbb{N}^n), \|\cdot\|_p)$ which is a Banach space. Therefore $e_\perp$ has a continuous extension to $(\ell_p(\mathbb{N}^n), \|\cdot\|_p)$ denoted again by $e_\perp$. Using the fact that $\ell_p(\mathbb{N}^n)^* = \ell_q(\mathbb{N}^n)$, continuity of $e_\perp$ implies that $\|\sum_{\alpha \in \mathbb{N}^n} |x^\alpha|,\|_q < \infty$.

Now suppose that $p = \infty$ and $\|\sum_{\alpha \in \mathbb{N}^n} |x^\alpha|,\|_1 = \infty$. Then, by part (3), for some $1 \leq i \leq n$, $|x_i| \geq 1$. For any $k \in \mathbb{N}$, $k \geq 1$, let $f_k(X) = \frac{1}{k^2}(1 + X + X^2 + \cdots + X^k)$ and $g_k(X) = \frac{1}{k}(1 - X + X^2 - \cdots + (-X)^k)$. Clearly $f_k, g_k \to 0$ in $\|\cdot\|_\infty$, but
\[
|e_\perp(f_k)| \geq \frac{k+1}{k}, \quad \text{if } x_i \geq 1, \\
|e_\perp(g_k)| \geq \frac{k+1}{k}, \quad \text{if } x_i \leq -1,
\]

Therefore in either cases at least one of $(e_\perp(f_k))$ or $(e_\perp(g_k))$ does not converge to 0. Hence, for $x \notin (-1, 1)^n$, $e_\perp$ is not continuous. This proves the result for $p = \infty$. \hfill \Box

**Corollary 3.7.** For $1 \leq p \leq \infty$, $\text{Psd}([-1, 1]^n)$ is a closed subset of $V_p$.

**Proof.** We first note that
\[
\text{Psd}([-1, 1]^n) = \text{Psd}((-1, 1)^n) = \bigcap_{x \in (-1, 1)^n} e_\perp^{-1}([0, +\infty]).
\]

However, by Proposition 3.6(iii), for every $x \in (-1, 1)^n$, $e_\perp$ is continuous on $V_p$. Hence the result follows. \hfill \Box

**Theorem 3.8.** For $1 \leq p \leq \infty$, \[\sum_{x \in [-1, 1]^n} |X|^p \|x\|^p = \text{Psd}([-1, 1]^n).\]
Proof. Corollary 3.7 implies that $\sum \mathbb{R}[X]|h|^p \subseteq \text{Psd}([-1,1]^n)$. The other inclusion follows from Lemma 3.4 and [1, Theorem 9.1]. □

Definition 3.9. Let $\ell : \mathbb{R}[X] \to \mathbb{R}$ be a linear functional. We say that $\ell$ is positive semidefinite if $\ell(h^2) \geq 0$ for every $h \in \mathbb{R}[X]$.

Proposition 3.1, Corollary 2.5 and Theorem 3.8 have an important consequence for the moment problem:

Corollary 3.10. Let $1 \leq p \leq \infty$, and let $\ell : \mathbb{R}[X] \to \mathbb{R}$ be a linear functional on $\mathbb{R}[X]$ such that $\|(\ell(X^\alpha))_{\alpha \in \mathbb{N}^n}\|_q < \infty$ where $q$ is the conjugate of $p$. If $\ell$ is positive semidefinite, then there exists a positive Borel measure $\mu$ on $[-1,1]^n$ such that $\forall f \in \mathbb{R}[X]$ $\ell(f) = \int_{[-1,1]^n} f \, d\mu$.

3.2. Weighted Norm-$p$ Topologies. We further extend Theorem 3.8 to weighted norm $p$-topologies. Let $r = (r_1,\ldots, r_n)$ be a $n$-tuple of positive real numbers and $1 \leq p < \infty$. The vector space

$$\ell_{p,r}(\mathbb{N}^n) := \{ s \in \mathbb{R}^{\mathbb{N}^n} : \sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^{p_1^{\alpha_1} \cdots r_n^{\alpha_n}} \alpha_n < \infty \}$$

is a Banach space with respect to the norm $\|s\|_{p,r} = (\sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^{p_1^{\alpha_1} \cdots r_n^{\alpha_n}})^{\frac{1}{p}}$. Similarly,

$$\ell_{\infty,r}(\mathbb{N}^n) := \{ s \in \mathbb{R}^{\mathbb{N}^n} : \sup_{\alpha \in \mathbb{N}^n} |s(\alpha)|^{r_1^{\alpha_1} \cdots r_n^{\alpha_n}} \alpha_n < \infty \}$$

is a Banach space with respect to the norm $\|s\|_{\infty,r} = \sup_{\alpha \in \mathbb{N}^n} |s(\alpha)|^{r_1^{\alpha_1} \cdots r_n^{\alpha_n}}$. Set

$$c_{0,r}(\mathbb{N}^n) := \{ s \in \mathbb{R}^{\mathbb{N}^n} : \lim_{\alpha \in \mathbb{N}^n} |s(\alpha)|^{r_1^{\alpha_1} \cdots r_n^{\alpha_n}} = 0 \}.$$

Then $c_{0,r}(\mathbb{N}^n)$ is a closed subspace of $\ell_{\infty,r}(\mathbb{N}^n)$ with respect to the norm $\| \cdot \|_{\infty,r}$. Denote by $V_{p,r}$ the set of all finite support real $n$-sequences (which we naturally identify with $\mathbb{R}[X]$), equipped with $\| \cdot \|_{p,r}$. Analogous to Proposition 3.3, the completion of $V_{p,r}$ is $\ell_{p,r}(\mathbb{N}^n)$ when $1 \leq p < \infty$ and $c_{0,r}(\mathbb{N}^n)$ when $p = \infty$. We now determine the continuous linear functionals on $\ell_{p,r}(\mathbb{N}^n)$.

Lemma 3.11. Let $1 \leq p \leq \infty$, and $q$ be the conjugate of $p$.

Then $\ell_{1,r}(\mathbb{N}^n)^* = \ell_{\infty,r-1}(\mathbb{N}^n)$, and $c_{0,r}(\mathbb{N}^n)^* = \ell_{1,r-1}(\mathbb{N}^n)$.

For $1 < p < \infty$, $\ell_{p,r}(\mathbb{N}^n)^* = \ell_{q,r-\frac{1}{p}}(\mathbb{N}^n)$.
Proof. The map defined by
\[
T_{p,r} : \ell_p(\mathbb{N}^n) \rightarrow \ell_p(\mathbb{N}^n)
(\langle s(\alpha) \rangle_{\alpha \in \mathbb{N}^n} \mapsto (\langle s(\alpha) r_1^{\frac{\alpha_1}{p}} \cdots r_n^{\frac{\alpha_n}{p}} \rangle_{\alpha \in \mathbb{N}^n})
\]
is an isometric isomorphism with inverse
\[
T_{p,r}^{-1} : \ell_p(\mathbb{N}^n) \rightarrow \ell_p(\mathbb{N}^n)
(\langle t(\alpha) \rangle_{\alpha \in \mathbb{N}^n} \mapsto (\langle t(\alpha) r_1^{\frac{\alpha_1}{p}} \cdots r_n^{\frac{\alpha_n}{p}} \rangle_{\alpha \in \mathbb{N}^n})
\]

Let \( f \in \ell_p(\mathbb{N}^n)^* \). Then \( f \circ T_{p,r} \in \ell_p(\mathbb{N}^n)^* = \ell_q(\mathbb{N}^n) \). Hence there exist \( t \in \ell_q(\mathbb{N}^n) \) such that \( t = f \circ T_{p,r} \).

Define the function \( t' : \mathbb{N}^n \rightarrow \mathbb{R} \) by \( t'(\alpha) = r_1^{\alpha_1} \cdots r_n^{\alpha_n} t(\alpha), \alpha \in \mathbb{N}^n \). It is straightforward to verify that \( t' \in \ell_{q,r^{-1}}(\mathbb{N}^n) \) if \( 1 \leq p < \infty \), and \( t' \in \ell_{\infty,r^{-1}}(\mathbb{N}^n) \) if \( p = 1 \). Moreover \( t'(\alpha) = f(\delta_\alpha) \), where \( \delta_\alpha \) is the Kroneker function at the point \( \alpha \in \mathbb{N}^n \). The proof of \( c_{0,r}(\mathbb{N}^n)^* = \ell_{1,r^{-1}}(\mathbb{N}^n) \) is similar. 

\[
\square
\]

**Theorem 3.12.** Let \( 1 \leq p \leq \infty \). Then:

1. For \( 1 \leq p < \infty \), \( \sum \mathbb{R}[X]^2 ||p,r|| = \text{Psd}(\prod_{i=1}^n [-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}}]) \).
2. \( \sum \mathbb{R}[X]^2 ||\infty,r|| = \text{Psd}(\prod_{i=1}^n [-r_i, r_i]) \).

**Proof.** (1) We first establish \( \text{Psd}(\prod_{i=1}^n [-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}}]) \subseteq \sum \mathbb{R}[X]^2 ||p,r|| \). Suppose that \( f \in \mathbb{R}[X] \) and \( f \geq 0 \) on \( \prod_{i=1}^n [-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}}] \). Since the polynomial \( \tilde{f}(X) = f(r_1^{\frac{1}{p}} X_1, \cdots, r_n^{\frac{1}{p}} X_n) \) is a nonnegative polynomial on \([-1,1]^n \), by Theorem 3.8, there exist a sequence \( (g_i)_{i \in \mathbb{N}} \) in \( \sum \mathbb{R}[X]^2 \) which approaches \( \tilde{f} \) in \( ||\cdot||_p \). On the other hand, a straightforward computation obtains \( ||g_i - \tilde{f}||_p = ||\tilde{g}_i - f||_p \), where \( \tilde{g}_i(X) = g_i(r_1^{\frac{1}{p}} X_1, \cdots, r_n^{\frac{1}{p}} X_n) \). Since \( (\tilde{g}_i)_{i \in \mathbb{N}} \) is a sequence in \( \sum \mathbb{R}[X]^2 \), the result follows. For the other inclusion, we first note that
\[
\text{Psd}(\prod_{i=1}^n [-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}}]) = \text{Psd}(\prod_{i=1}^n (-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}})) = \bigcap_{x \in \prod_{i=1}^n (-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}})} e^{-1}_{\infty}(\{0, +\infty\}).
\]
A routine computation shows that for every \( x \in \prod_{i=1}^n (-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}}), \)
\[
(x^\alpha)_{\alpha \in \mathbb{N}^n} \in \ell_{\infty,p^{-1}}(\mathbb{N}^n) \quad \text{if} \quad p = 1,
\]
and
\[
(x^\alpha)_{\alpha \in \mathbb{N}^n} \in \ell_{q,r^{-p}}(\mathbb{N}^n) \quad \text{if} \quad 1 < p < \infty,
\]
where $q$ is the conjugate of $p$. It follows from Lemma 3.11 that $e_x$ is continuous on $V_{p,r}$. Hence $\text{Psd}(\prod_{i=1}^{n}[-r_i, r_i])$ is a closed subset of $V_{p,r}$ containing $\sum R[X]^2$, and the inclusion follows.

(2) Again, a routine computation shows that for every $x \in \prod_{i=1}^{n}(-r_i, r_i)$, $(x^\alpha)_{\alpha \in \mathbb{N}^n} \in \ell_{1,r^{-1}}(\mathbb{N}^n)$ and the rest follows similar to part (i). \hfill \Box

We now apply Theorem 3.12 to obtain the $K$ moment property for $\sum R[X]^2$ for certain convex compact polyhedron and weighted norm-$p$ topologies as we summarize below in the following three theorems:

**Theorem 3.13.** Let $r = (r_1, \ldots, r_n)$ with $r_i > 0$ for $i = 1, \ldots, n$, and let $\ell : \mathbb{R}[X] \to \mathbb{R}$ be a linear functional such that the sequence $s(\alpha) = \ell(X^\alpha)$ satisfies

$$\sup_{\alpha \in \mathbb{N}^n} |s(\alpha)| r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} < \infty.$$  

Then $\ell$ is positive semidefinite if and only if there exists a positive Borel measure $\mu$ on $K = \prod_{i=1}^{n}[-r_i, r_i]$ such that

$$\forall f \in \mathbb{R}[X] \quad \ell(f) = \int_{K} f \, d\mu.$$  

**Theorem 3.14.** Let $1 < p < \infty$, $q$ the conjugate of $p$, and $r = (r_1, \ldots, r_n)$ with $r_i > 0$ for $i = 1, \ldots, n$. Suppose that $\ell : \mathbb{R}[X] \to \mathbb{R}$ is a linear functional such that the sequence $s(\alpha) = \ell(X^\alpha)$ satisfies

$$\sum_{\alpha \in \mathbb{N}^n} |s(\alpha)| r_1^{-\frac{\alpha_1}{p}} \cdots r_n^{-\frac{\alpha_n}{p}} < \infty.$$  

Then $\ell$ is positive semidefinite if and only if there exists a positive Borel measure $\mu$ on $K = \prod_{i=1}^{n}[-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}}]$ such that

$$\forall f \in \mathbb{R}[X] \quad \ell(f) = \int_{K} f \, d\mu.$$  

**Theorem 3.15.** Let $r = (r_1, \ldots, r_n)$ with $r_i > 0$ for $i = 1, \ldots, n$, and let $\ell : \mathbb{R}[X] \to \mathbb{R}$ be a linear functional such that the sequence $s(\alpha) = \ell(X^\alpha)$ satisfies

$$\sum_{\alpha \in \mathbb{N}^n} |s(\alpha)| r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} < \infty.$$  

Then $\ell$ is positive semidefinite if and only if there exists a positive Borel measure $\mu$ on $K = \prod_{i=1}^{n}[-r_i, r_i]$ such that

$$\forall f \in \mathbb{R}[X] \quad \ell(f) = \int_{K} f \, d\mu.$$
3.3. **Closure of $\sum \mathbb{R}[x]^2$ in the Topology of Coefficientwise Convergence.** In this final section, we characterize the closure of $\sum \mathbb{R}[x]^2$ in the coefficientwise convergent topology. A net $\{f_i\} \in \mathbb{R}[x]$ converges in the coefficientwise convergent topology to $f \in \mathbb{R}[x]$ if for every $\alpha \in \mathbb{N}^n$, the coefficients of $x^\alpha$ in $f_i$ converges to the coefficient of $x^\alpha$ in $f$.

**Proposition 3.16.** For a polynomial $f(x) \in \mathbb{R}[x]$, the followings are equivalent:

1. $f(0) \geq 0$.
2. $f$ is coefficientwise limit of squares in $\mathbb{R}[x]$.
3. $f$ is coefficientwise limit of elements of $\sum \mathbb{R}[x]^2$.

**Proof.** (2)$\Rightarrow$(3) and (3)$\Rightarrow$(1) are clear. It remains to prove (1)$\Rightarrow$(2). For each $i \geq 1$ let $g_{i0} + g_{i1} + g_{i2} + \cdots$ be the power series expansion of $\sqrt{\frac{1}{i} + f}$ in the ring of formal power series $\mathbb{R}[[x]]$ where $g_{ij}$ is a form of degree $j$ in $\mathbb{R}[x]$. Let $h_i = g_{i0} + g_{i1} + \cdots + g_{ii}$, then $h_i \xrightarrow{i \to \infty} f$ in the topology of coefficientwise convergence. □

**References**

[1] C. Berg, J. P. R. Christensen, P. Ressel, *Positive definite functions on abelian semigroups*, Math. Ann. 223, 253-274, (1976).

[2] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups, Theory of Positive Definite and Related Functions*, Springer-Verlag, (1984).

[3] J. B. Conway *A Course in Functional Analysis, 2nd ed.*, GTM. 96, Springer (1990).

[4] M. Ghasemi, S. Kuhlmann, *Closure of the cone of sums of 2d-powers in real topological algebras*, to appear.

[5] M. Ghasemi, M. Marshall, S. Wagner, *Closure of the cone of sums of 2d-powers in certain weighted $\ell_1$-seminorm topologies*, to appear.

[6] E. K. Haviland, *On the momentum problem for distribution functions in more than one dimension*, Amer. J. Math. 57, 562-572, (1935).

[7] E. K. Haviland, *On the momentum problem for distribution functions in more than one dimension II*, Amer. J. Math. 58, 164-168, (1936).

[8] J. L. Krivine, *Anneaux prédordonnés*, J. Analyse Math. 167, 160-196, (1932).

[9] S. Kuhlmann, M. Marshall, *Positivity, Sums of Squares and The Multi-Dimensional Moment Problem*, Trans. Amer. Math. Soc. 354, 4285-4301, (2002).

[10] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optimization 11, pp 796–817. (2001).

[11] J. B. Lasserre, *The K-Moment problem for continuous linear functionals*, To appear in Trans. Amer. Math. (2011).

[12] J. B. Lasserre, T. Netzer, *SOS approximation of nonnegative polynomials via simple high degree perturbations*, Math. Zeitschrift 256, 99-112, (2006).
[13] M. Marshall, *Positive Polynomials and Sum of Squares*, Mathematical Surveys and Monographs, Vol 146. (2007).

[14] M. Putinar, *Positive polynomials on compact semialgebraic sets*, Indiana Univ. Math. J. (3) 43 (969-984), (1993).

[15] W. Rudin, *Functional Analysis, 2nd ed.*, International series in pure and applied mathematics. (1991).

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