Global Existence for some Cross Diffusion Systems with Equal Cross Diffusion/Reaction Rates.

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Abstract

We consider some cross diffusion systems which is inspired by models in mathematical biology/ecology, in particular the Shigesada-Kawasaki-Teramoto (SKT) model in population biology. We establish the global existence of strong solutions to systems for multiple species having equal either diffusion or reaction rates. The systems are given on bounded domains of arbitrary dimension.

1 Introduction

In this paper, we study the global existence of following strongly coupled parabolic system of \( m \) equations \(( m \geq 2)\) for the unknown vector \( u = [u_i]_{i=1}^m \)

\[
(u_i)_t = \Delta(u_i p_i(u)) + u_i g_i(u), \quad (x,t) \in \Omega \times (0, \infty).
\]  

(1.1)

Here, \( p_i, g_i : \mathbb{R}^m \to \mathbb{R} \) are sufficiently smooth functions. Namely, \( p_i \in C^2(\mathbb{R}^m) \) and \( g_i \in C(\mathbb{R}^m). \) \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^N, \ N \geq 2. \)

The system is equipped with Dirichlet boundary and sufficiently smooth initial conditions

\[
\begin{align*}
\{ & u_i = 0 \text{ on } \partial\Omega \times (0, \infty), \\
& u_i(x, 0) = u_{i,0}(x), \quad x \in \Omega.
\}
\end{align*}
\]  

(1.2)

The consideration of (1.1) is motivated by the extensively studied model in population biology introduced by Shigesada et al. in [9]

\[
\begin{align*}
u_t &= \Delta(d_i u + \alpha_{1i} u^2 + \alpha_{12} uv) + k_1 u + \beta_{11} u^2 + \beta_{12} uv,

v_t &= \Delta(d_2 v + \alpha_{21} uv + \alpha_{22} v^2) + k_2 v + \beta_{21} uv + \beta_{22} v^2.
\end{align*}
\]  

(1.3)

Here, \( d_i, \alpha_{ij}, \beta_{ij} \) and \( k_i \) are constants with \( d_i > 0. \) Dirichlet or Neumann boundary conditions were usually assumed for (1.3). This model was used to describe the population dynamics of two species densities \( u, v \) which move and interact with each other under the influence of their population pressures.

Of course, (1.3) is a special case of (1.1) with \( m = 2 \) and

\[
p_i(u,v) = d_i + \alpha_{1i} u + \alpha_{12} v, \quad g_i(u,v) = k_i + \beta_{1i} u + \beta_{12} v.
\]

We will refer to the functions \( p_i \)’s (respectively, \( g_i \)’s) as the diffusion (respectively, reaction) rates (see [8] for further discussions)

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Under suitable assumptions on the constant parameters $\alpha_{ij}$'s, $\beta_{ij}$'s and that $\Omega$ is a planar domain ($N = 2$), Yagi proved in [11] the global existence of (strong) positive solutions, with positive initial data. In this paper, we will extend this investigation to multi-species versions of (1.3) for more than two species on bounded domains of arbitrary dimension $N$.

The global existence problem of (1.1), a fundamental problem in the theory of pdes. We can write (1.1) in its general divergence form

$$u_t = \text{div}(A(u)Du) + f(u).$$

This a strongly coupled parabolic system with the diffusion matrix $A(u)$, the Jacobian of $[u,p_i(u)]_1^m$, being a full matrix. We say that the system is weakly coupled if $A(u)$ is diagonal (i.e., $p_i$ depends only on $u_i$).

The key point in the proof of global existence of strong solutions of (1.4) is the a priori estimate of their spatial derivatives. In fact, it was established by Amann in [1] that (1.1) has a global strong solution $u$ if there is some exponent $p > N$ such that for any $T \in (0, \infty)$

$$\limsup_{t \to T^-} \|Du\|_{L^p(\Omega)} < \infty.$$

Thus, we need only prove that $\sup_{t \in (0,T)} \|Du\|_{L^p(\Omega)} < \infty$ for all $T \in (0, \infty)$ and some $p > N$. With this a priori estimate, one can alternatively use the homotopy or fixed point approaches in [5] [6] [7], instead of semigroup theories in [1], to obtain the local/global existence of strong solutions.

The derivation of such estimates for (1.1) is a difficult issue when $A(u)$ is full because the known techniques for scalar equations ($m = 1$) are no longer applicable unless the matrix $A(u)$ are of special form, e.g., diagonal or triangular, these techniques can be partly applied together with some ad hoc arguments (see [10]). In this paper, we will consider (1.1) with full diffusion matrix $A(u)$ of special forms where some nontrivial modifications of the classic methods can apply and yield new affirmative answers to the problem.

Precisely, we study the case when either the diffusion or reaction rates are identical. Being inspired by the standard (SKT) system (1.3) where $p_i$ are a linear function in $u$, we consider a function $\Psi$ on $\mathbb{R}$, a linear combination $L(u)$ of $u_i$'s, $L(u) = \sum a_i u_i$, and assume that for $i = 1, \ldots, m$

$$p_i(u) = \lambda_0 + \Psi(L(u)).$$

We also assume that the reaction rates $g_i$'s satisfy the control growth $|g_i(u)| \leq C + c_0 \Psi(|L(u)|)$ for some positive constants $C, c_0$. We will establish the global existence of nonnegative strong solutions to (1.1) with nonnegative initial data.

Clearly, (1.3) is the case when $d_1 = d_2$, $\alpha_{i1} = \alpha_{j1}$, $\alpha_{i2} = \alpha_{j2}$ and $\Psi(s) = s$.

On the other hand, we can relax the assumption that the diffusion rates are identical as in (1.5). The trade off is that the reaction rates $g_i$'s are identical and satisfying the above control growth.

The paper is organized as follows. In Section 2 we discuss some regularity positivity results for strong solutions to scalar parabolic equations. Our main results on the system (1.1) will be presented and proved in Section 3.
2 Some facts on scalar equations

In this section we consider the following scalar equation

$$v_t = \Delta(P(v)) + \text{div}(vb(v)) + vg(v) \quad (2.1)$$

in $Q = \Omega \times (0, T)$ and and study the smoothness, uniform boundedness and positivity of its strong solution $v$ under some special conditions on $P, g$ which will serve our purpose in discussing cross diffusion systems later.

To proceed, we first need the following parabolic Sobolev embedding inequality.

**Lemma 2.1** Let $r^* = p/N$ if $N > p$ and $r^*$ be any number in $(0, 1)$ if $N \leq p$. For any sufficiently nonegative smooth functions $g, G$ and any time interval $I$ there is a constant $C$ such that

$$\iint_{\Omega \times I} g^{r^*} G^p \, dz \leq C \sup_I \left( \int_{\Omega \times \{t\}} g \, dx \right)^{r^*} \iint_{\Omega \times I} (|DG|^p + G^p) \, dz \quad (2.2)$$

If $G = 0$ on the parabolic boundary $\partial \Omega \times I$ then the integral of $G^p$ over $\Omega \times I$ on the right hand side can be dropped.

Furthermore, if $r < r^*$ then for any $\varepsilon > 0$ we can find a constant $C(\varepsilon)$ such that

$$\iint_{\Omega \times I} g^{\varepsilon} G^p \, dz \leq C \sup_I \left( \int_{\Omega \times \{t\}} g \, dx \right)^{r} \iint_{\Omega \times I} (\varepsilon |DG|^p + C(\varepsilon) G^p) \, dz \quad (2.3)$$

**Proof:** For any $r \in (0, 1)$ and $t \in I$ we have via Hölder’s inequality

$$\int_{\Omega} g^{r} G^p \, dx \leq \left( \int_{\Omega} g \, dx \right)^{r} \left( \int_{\Omega} G^{\frac{p}{1-r}} \, dx \right)^{1-r}. \quad (2.4)$$

If $r = r^*$ then $p/(1-r) = N_s = pN/(N-p)$, the Sobolev conjugate of $p$ if $N > p$ (the case $N \leq p$ is obvious), so that the Sobolev inequality gives

$$\left( \int_{\Omega} G^{\frac{p}{1-r}} \, dx \right)^{1-r} \leq \int_{\Omega} (|DG|^p + G^p) \, dx.$$

Using the above in (2.4) and integrating over $I$, we easily obtain (2.2). On the other hand, if $r < r^*$, then $p/(1-r) < N_s$. A simple contradiction argument and the compactness of the imbedding of $W^{1,p}(\Omega)$ into $L^{p/(1-r)}(\Omega)$ imply that for any $\varepsilon > 0$ there is $C(\varepsilon)$ such that

$$\left( \int_{\Omega} G^{\frac{p}{1-r}} \, dx \right)^{1-r} \leq \varepsilon \int_{\Omega} |DG|^p \, dx + C(\varepsilon) \int_{\Omega} G^p \, dx.$$

We then obtain (2.3). \hfill \Box

We now have the following a priori boundedness of solution of (2.1).
Theorem 2.2 Consider a (weak or strong) solution $v$ to \((2.4)\) in $Q = \Omega \times (0,T)$. Assume that there are a function $\lambda(v)$ and a number $\lambda_0$ such that $\lambda(v) \geq \lambda_0 > 0$ and

$$P_v(v) \geq \lambda(v),$$  \hspace{1cm} (2.5)

$$|b(v)| \leq g_1 \lambda(v),$$  \hspace{1cm} (2.6)

$$|g(v)| \leq g_2 \lambda(v),$$  \hspace{1cm} (2.7)

where $g_1, g_2$ are functions such that $g_1^2 + g_2 \in L^q(Q)$ for some $q > N/2 + 1$.

For $v \in \mathbb{R}$ and $p \geq 1$ consider the function

$$F(v,p) = \int_0^v \lambda^{\frac{1}{p}}(s)s^{p-1} \, ds,$$  \hspace{1cm} (2.8)

and assume that

$$|F(v,p)| \sim C p \lambda^{\frac{1}{p}}(v)|v|^p \text{ for all } p \text{ and } v \in \mathbb{R}.$$  \hspace{1cm} (2.9)

If $\|v\lambda(v)\|_{L^1(Q)}$ is finite then $v, Dv$ are bounded and Hölder continuous in $\Omega \times (\tau,T)$ for any $\tau \in (0,T)$. Their norms depend on $\|v\lambda(v)\|_{L^1(Q)}$.

The condition (2.9) is clearly verified if $\lambda(v)$ has a polynomial growth in $|v|$.

Proof: We test the equation with $|v|^{2p-2}v$ and use integration by parts

$$\int_{\Omega} \Delta(P(v))|v|^{2p-2}v \, dx = - \int_{\Omega} P_v(v)DvD(|v|^{2p-2}v) \, dx,$$

$$\int_{\Omega} \text{div}(vb(v))|v|^{2p-2}v \, dx = - \int_{\Omega} vb(v)D(|v|^{2p-2}v) \, dx.$$

Because $D(|v|^{2p-2}v) = (2p-1)|v|^{2p-2}Dv$ and the assumptions on $Q_v(v)$ and $b(v), g(v)$, we easily get for all $p \geq 1$

$$\sup_{(0,T)} \frac{1}{2p} \int_{\Omega} |v|^{2p} \, dx + (2p-1) \int_Q \lambda(v)|v|^{2p-2}|Dv|^2 \, dz \leq C \int_Q g_1|\lambda(v)||v|^{2p-1}|Dv| \, dz + C \int_Q g_2|\lambda(v)||v|^{2p} \, dz.$$  \hspace{1cm} (2.10)

Applying Young’s inequality $g_1|\lambda(v)||v|^{2p-1}|Dv| \leq \varepsilon|v|^{2p-2}|Dv|^2 + C(\varepsilon)g_1^2|v|^{2p}$ for small, $\varepsilon$,

$$\sup_{(0,T)} \frac{1}{2p} \int_{\Omega} |v|^{2p} \, dx + (2p-1) \int_Q \lambda(v)|v|^{2p-2}|Dv|^2 \, dz \leq C \int_Q (g_1^2 + g_2)|\lambda(v)||v|^{2p} \, dz.$$  

As $\lambda(v)|v|^{2p-2} = F_v^2(v,p)$ by the definition (2.8), for $g_3 = g_1^2 + g_2$ the above is

$$\sup_{(0,T)} \frac{1}{2p} \int_{\Omega} |v|^{2p} \, dx + (2p-1) \int_Q |D(F(v,p))|^2 \, dz \leq C \int_Q g_3|\lambda(v)||v|^{2p} \, dz.$$  

Thus, for $p \geq 1$

$$\sup_{(0,T)} \int_{\Omega} |v|^{2p} \, dx, \quad \int_Q |D(F(v,p))|^2 \, dz \leq C p \int_Q g_3|\lambda(v)||v|^{2p} \, dz.$$
Applying the parabolic Sobolev inequality in Lemma 2.1 with $g = |v|^p$ and $G = F(v, p)$, the above estimate yields for $r = 2/N$

\[
\left( \iint_Q |v|^{2p} |F(v, p)|^2 \, dz \right)^{1/r} \leq C p^{1+\frac{2}{r}+\frac{3}{p}} \iint_Q g_3 |\lambda(v)||v|^{2p} \, dz.
\]

As $F(v) \sim C^{-1} \lambda \frac{2}{r}(v)|v|^{2p}$ by (2.9), we then obtain for $\gamma = 1+2/N$

\[
\left( \iint_Q |v|^{2p\gamma} \lambda(v) \, dz \right)^{1/\gamma} \leq C p^{1+\frac{2}{r}} \iint_Q g_3 \lambda(v)|v|^{2p} \, dz.
\]

Hölder’s inequality yields

\[
\iint_Q g_3 \lambda(v)|v|^{2p} \, dz \leq C \left( \iint_Q g_3^q \lambda(v) \, dz \right)^{1/q} \left( \iint_Q |v|^{2pq'} \lambda(v) \, dz \right)^{1/p'}. \tag{2.11}
\]

Let $d\mu = \lambda(v)dz$. As we assume that $g_1^q, g_2 \in L^q(Q, d\mu)$, $g_3 \in L^q(Q, d\mu)$ and the first factor on the right hand side is finite. The above inequality is

\[
\|v\|_{L^{2q}(Q,d\mu)} \leq (2Cp)^{(1+\frac{2}{r})\frac{1}{q'}} \|v\|_{L^{2pq'}(Q,d\mu)}. \tag{2.12}
\]

Because $q > N/2 + 1$, $q' < \gamma = 1+2/N$. Replacing $p$ by $pq'$ and defining $\gamma_0 = \gamma/q' > 1$.

It follows that

\[
\|v\|_{L^{2pq_0}(Q,d\mu)} \leq (2Cp)^{(1+\frac{2}{r})\frac{1}{q}} \|v\|_{L^{2p}(Q,d\mu)}. \tag{2.12}
\]

Because $\gamma_0 > 1$, we can apply the Moser iteration argument to show that $v$ is bounded. Indeed, by taking $2p = \gamma_0$ with $i = 0, 1, \ldots$ to the above estimate implies

\[
\|v\|_{L^{\gamma_i}(Q,d\mu)} \leq (2C)^{\gamma_1 \gamma_2} \|v\|_{L^{1}(Q,d\mu)},
\]

with $\gamma_1 = (\frac{1}{q} + \frac{1}{q_0}) \sum_{i=0}^\infty \gamma_1^i$, $\gamma_2 = (\frac{1}{q'} + \frac{1}{q_0}) \sum_{i=0}^\infty i \gamma_0^{-i}$. Letting $i \to \infty$ and using the fact that $\lim p \to \infty \|v\|_{L^p(Q,d\mu)} = \|v\|_{L^\infty(Q,d\mu)}$ (we will show that $d\mu$ is finite below) we obtain for some constant $C_0$ that $\|v\|_{L^\infty(Q,d\mu)} \leq C_0 \|v\|_{L^1(Q,d\mu)}$.

As $\lambda(v)$ is bounded below by a positive constant, this implies that $v$ is bounded if $v \in L^1(Q,d\mu)$ is bounded. Furthermore, we now show that $d\mu$ is finite. Because

\[
\iint_{|v| \geq 1} \lambda(v) \, dz \leq \|v\|_{L^1(Q,d\mu)},
\]

and $\lambda(u)$ is bounded on the set $|v| < 1$, we see that $d\mu$ is finite.

Once we show that $v$ is bounded, we obtain the local Harnack inequality (using both positive and negative power $p$ and cutoff functions) and so that $v$ is Hölder continuous. The argument is now classical and we refer the readers to the classical books [3, 4] for details. It also follows that $Dv$ is bounded and Hölder continuous in $\Omega \times (\tau, T)$ for any $\tau \in (0, T)$. Indeed, we can adapt the freezing coefficient method in [2] to establish this fact. \hfill \blacksquare
Remark 2.3 The conditions in the theorem and remarks need only hold only for $|v|$ large. This is easily to see if we make use of the cutoff function

$$
\bar{v}(k) = \begin{cases} 
v & \text{if } |v| \geq k, \\
k & \text{if } 0 < v < k, \\
-k & \text{if } -k < v \leq 0
\end{cases}
$$

(2.13)

with $k$ sufficiently large and observe that $D\bar{v}_k = 0$ on the set $|v| < k$.

Remark 2.4 In connection with the systems considered in the next section, we consider the scalar equation

$$v_t = \lambda_0 \Delta v + \Delta(\Psi(v)v) + vg(v),$$

(2.14)

where $\lambda_0 > 0$ and $\Psi: \mathbb{R} \to \mathbb{R}$ be a $C^1$ function and satisfying for $|v|$ large

$$\Psi(v), \; \Psi'(v)v \geq 0.$$  

(2.15)

Asume also that for $v \in \mathbb{R}$ and $p \geq 1$ the function

$$\hat{F}(v, p) = \int_0^v \Psi^{\frac{1}{p}}(s)s^{p-1}ds$$

satisfies

$$|\hat{F}(v, p)| \sim C_p \Psi^{\frac{1}{p}}(v)|v|^{2p}$$

for all $p$ and $v \in \mathbb{R}$.  

(2.17)

This condition allows us to apply Theorem 2.2 with $P(v) = \lambda_0 v + \Psi(v)v$ and $\lambda(v) = \Psi(v) + \Psi'(v)v + \lambda_0$. Thanks to (2.15), $\lambda(v)$ satisfies (2.9). Also, (2.16) and (2.17) imply that the function $F$ defined by (2.8) satisfies (2.9). We then apply Theorem 2.2 to (2.14) and obtain that $v, Dv$ are bounded in $\Omega \times (\tau, T)$ for any $\tau \in (0, T)$ and their norms are bounded in term of $\|v\|_{L^1(\Omega)}$ and $\|v\Psi(v)\|_{L^1(\Omega)}).

We can also consider the scalar equation

$$v_t = \lambda_0 \Delta v + \Delta(\Psi(|v|)v) + vg(v),$$

(2.18)

and $\Psi: \mathbb{R} \to \mathbb{R}$ be a $C^1$ function and satisfying for $v \geq 0$ and large

$$\Psi(v), \; \Psi'(v) \geq 0.$$  

(2.19)

Indeed, we now define $\psi(v) = \Psi(|v|)$. We then have $\psi'(v)v = \Psi'(|v|)\text{sign} v v = \Psi'(|v|)|v| \geq 0$ because of (2.19) $|v| \geq 0$. Thus, $\psi$ satisfies (2.15) and the theorem applies.

In applications we usually prefer that $v$ is nonnegative if the initial is. The following result serves this purpose.

Theorem 2.5 Let $a, g$ be $C^1$ functions on $\mathbb{R} \times Q$ and $b$ be a bounded $C^1$ map from $Q$ into $\mathbb{R}^N$. Assume that $a(w) \geq \lambda_0$ for $w \geq 0$ and $\lambda_0$ is a positive constant. Also suppose that $a, g$ are bounded by a constant depending on $w$ in $(x, t) \in Q$.

Let $w$ be the strong solution to

$$\begin{cases} 
w_t = \text{div}(a(w, x, t)Dw) + \text{div}(wb) + wg(w, x, t), & \text{in } Q \\
w(x, 0) = w_0(x) & \text{on } \Omega.
\end{cases}$$

(2.20)

If $w_0 \geq 0$ then $w \geq 0$ on $Q$. 

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**Proof:** Because $w$ is a strong solution, there is a constant $M > 0$ such that $|w| \leq M$. We then truncate $a, g$ to $C^1$ function $\hat{a}, \hat{g}$ which are constants for $v$ outside $[-M - 1, M + 1]$ and consider the equation

$$v_t = \text{div}(\hat{a}(|v|, x, t)Dv) + \text{div}(vb(x, t)) + v\hat{g}(v, x, t),$$

(2.21)

with initial data $w_0$.

We have $\hat{a}(|v|, x, t) \geq \lambda_0$ and is bounded from above and $|v\hat{g}(v, x, t)| \leq C|v|$ for some constant $C$. These facts and the classical theory of scalar parabolic equation show that (2.18) has a strong solution $v$.

Let $v^+, v^-$ be the positive and negative parts of $v$. We test the equation with $v^-$. Using the facts that $|v| = v^+ + v^-$, $|v| = v^-$ on the set $v^- > 0$, $v^+Dv^- = Dv^+Dv^- = 0$ on the set $v^- > 0$, we obtain

$$-\frac{d}{dt}\int_\Omega (v^-)^2 \, dx - \int_\Omega \hat{a}|Dv^-|^2 \, dx = \int_\Omega [-bv^-Dv^- + (v^-)^2\hat{g}] \, dx.$$

Because $b$ are bounded by a constant $C(M)$, applying Young’s inequality

$$\int_\Omega |bv^-Dv^-| \, dx \leq \varepsilon \int_\Omega |Dv^-|^2 \, dx + C(\varepsilon, M) \int_\Omega (v^-)^2 \, dx.$$

Because $\hat{g}$ is bounded by a constant $C$ depending on $M$ and $a(v) \geq \lambda_0$, we can choose $\varepsilon$ sufficiently small in the above inequality to arrive at

$$\frac{d}{dt}\int_\Omega (v^-)^2 \, dx + \int_\Omega |Dv^-|^2 \, dx \leq C(M) \int_\Omega (v^-)^2 \, dx.$$

Thus, we see that the function

$$z(t) = \int_\Omega (v^-)^2 \, dx$$

satisfies the differential inequality $z' \leq C_1 z$ and $z(0) = 0$ because the initial data $v_0 \geq 0$. We then apply comparison theorem to the equation $y' = Cy$ with $y(0) = 0$ which has the solution $y(t) = 0$. We then have $z(t) = 0$ for all $t \in (0, T)$. Hence, $v^- = 0$ on $Q$ so that $v \geq 0$. It follows that the solution $v$ of (2.18) also solves (2.20). By the uniqueness of strong solutions, $w = v \geq 0$ in $Q$.

3 Cross diffusion system with equal diffusion/reaction rates

In this section, we consider the system (1.1) and assume either that the diffusion rates $p_i$’s or the reaction rates are equal. We will always assume nonegative initial data $u_{i,0}$.

Throughout this section we will consider a nonnegative $C^1$ function $\Psi$ on $\mathbb{R}$ satisfying

$$\Psi'(s) \geq 0 \text{ for } s \geq 0.$$  

(3.1)
3.1 Equal diffusion rates:

We first consider the following system of \( m \) equations for \( u = [u_i]^m \)

\[
\begin{align*}
(u_i)_t &= \Delta (\lambda_0 u_i + \Psi(L(u)) u_i) + u_i g_i(u) \quad \text{in } \Omega \times (0, \infty), \\
(u_i(x,0)) &= u_{i,0}(x) \quad \text{on } \Omega,
\end{align*}
\]

(3.2)

where \( \lambda_0 > 0 \) and \( L(u) \) is a linear combination of \( u_i \). That is, \( L(u) = \sum_{i=1}^m a_i u_i \) with \( a_i > 0 \). Assume that there are constants \( C_{ij}, c_{ij} \geq 0 \) such that

\[
|g_i(u)| \leq \sum_j (C_{ij} + c_{ij} \Psi(|u_i|)).
\]

(3.3)

We have

**Theorem 3.1** If \( c_0 = \max c_{ij} \) is sufficiently then (3.3) has a unique nonnegative strong solution.

As we explained in the introduction, we need only establish a priori the finiteness of \( \sup_{(0,T_{max})} \|Du\|_{L^p(\Omega)} \), with some \( p > N \), for any strong solution \( u = [u_i]^m \) of (3.2) on \( \Omega \times (0, T_{max}) \) for any \( T_{max} \in (0, \infty) \). We will do this for \( p = \infty \) via several lemmas.

**Lemma 3.2** \( u_i \geq 0 \) on \( \Omega \times (0, T_{max}) \).

**Proof:** We can use Theorem 2.5 to show first that \( u_i \geq 0 \) on \( Q = \Omega \times [0, T] \) for any \( 0 < T < T_{max} \) and all \( i \). We rewrite the equation of \( u_i \) as

\[
\begin{align*}
(u_i)_t &= \text{div}(a_i(u_i, x, t) Du_i) + \text{div}(u_i b_i(x, t)) + u_i g_i(u) \quad \text{in } Q, \\
(u_i(x,0)) &= u_{i,0}(x) \quad \text{on } \Omega,
\end{align*}
\]

(3.4)

where

\[
a_i(u_i, x, t) = \lambda_0 + \Psi(L(u)) + \partial_{u_i} \Psi(L(u)) u_i, \quad b_i(x, t) = \sum_{j \neq i} \partial_{u_j} \Psi(L(u_i(x, t))).
\]

Following the proof of Theorem 2.3, because \( u \) bounded on \( Q, \|L(u)\| \leq M \) for some constant \( M \). We truncate the function \( \Psi \) outside the interval \([-M - 1, M + 1] \) to obtain a bounded \( C^1 \) function \( \psi \) satisfying: \( \psi(s), \psi'(s) \geq 0 \) and \( \psi(s) \) is a constant when \( |s| \geq M + 1 \).

Denoting \( \hat{v} = ||v_i||^m \) for any vector \( v = [v_i]^m \). We consider the system

\[
\begin{align*}
(v_i)_t &= \text{div}(\hat{a}_i(v, x, t) Dv_i) + \text{div}(v_i b_i(x, t)) + v_i g_i(u) \quad \text{in } Q, \\
v_i(x,0) &= u_{i,0}(x) \quad \text{on } \Omega,
\end{align*}
\]

(3.5)

where

\[
\hat{a}_i(v, x, t) = \lambda_0 + \psi(L(\hat{v})) + \partial_{v_i} \psi(L(\hat{v})) v_i.
\]

Because \( \psi'(s) \geq 0 \) for \( s \geq 0 \) and \( L(\hat{v}) \geq 0 \), we have \( \partial_{u_i} \psi(L(\hat{v})) v_i = \psi'(L(\hat{v})) a_i \text{sign}(v_i) v_i = \psi'(L(\hat{v})) a_i |v_i| \geq 0 \). We also have \( \psi(L(\hat{v})) \geq 0 \). Thus \( \hat{a}_i(v, x, t) \geq \lambda_0 \) and bounded from above. The system (3.5) is a diagonal system with bounded continuous coefficients and has a unique strong solution \( v \) according to the classical theory (e.g., see [3, Chapter 7]).
Applying the argument in the proof of Theorem 2.5 to each equation in (3.5), the system (3.5) has a nonnegative strong solution \( v \), so that \( \psi(\hat{v}) = \psi(v) \), which also solves (3.4) by the definition of \( \psi \), an extension of \( \Psi \). By the uniqueness of strong solutions, \( u_i = v_i \geq 0 \) in \( Q \) for all \( i \).

Next, define \( W = L(u) \). The following lemma provides bounds of \( W, DW \) that are independent of the number \( M \), which was used only in establishing that \( u_i \geq 0 \).

**Lemma 3.3** Let \( W = L(u) \geq 0 \). Assume that

\[
F(v, p) := \int_0^v \frac{1}{2} \Psi'(s) s^{p-1} \, ds \sim C p \Psi'(v) |v|^{2p} \quad \text{for all } p \text{ and } v \geq 0. \tag{3.6}
\]

Then \( W, DW \) are bounded in \( \Omega \times (\tau, T) \) for any \( \tau \in (0, T) \) by a constant depending only on \( \|W\|_{L^1(Q)}, \|W \Psi(W)\|_{L^1(Q)} \).

**Proof:** Taking a linear combination of the equations, we obtain

\[
W_i = \lambda_0 \Delta W + \Delta (\Psi(W)W) + f(u), \tag{3.7}
\]

where \( f(u) = \sum_i a_i u_i g_i(u) \). Because \( u_i \geq 0 \) and \( a_i > 0 \), \( W \) is nonnegative and \( |u_i| \leq W \). Since \( \Psi(s) \) is increasing for \( s \geq 0 \), the assumption on \( g_i \)’s (3.3) implies

\[
|g_i(u)| \leq \sum_j (C_{ij} + c_{ij} \Psi(|u_i|)) \leq \sum_j (C_{ij} + c_{ij} \Psi(W)).
\]

Hence, \( f \) satisfies for some positive constants \( C \) and \( c_0 = \max c_{ij} \)

\[
|f(u)| \leq C |W|(1 + c_0 \Psi(W)). \tag{3.8}
\]

We then apply Theorem 2.2 (to be precise, its Remark 2.4 and the equation (2.14)) with \( v = W \), noting that \( v = W \geq 0 \). The assumption (3.6) on \( \Psi \) guarantees that (2.17) is satisfied. We see that the norms of \( W, DW \) are bounded in \( \Omega \times (\tau, T) \) for any \( \tau \in (0, T) \) by constants independent of \( M \) but on \( \|W\|_{L^1(Q)} \) and \( \|W \Psi(W)\|_{L^1(Q)} \). The lemma follows.

**Remark 3.4** If the constant \( c_0 \) in (3.8) is sufficiently small then the norms \( \|W\|_{L^1(Q)} \) and \( \|W \Psi(W)\|_{L^1(Q)} \) are bounded by a constant. Indeed, testing the equation of \( W \) by \( W \) and using (3.8)

\[
\sup_{t \in (0, T)} \int_{\Omega \times \{t\}} W^2 \, dx + \int \int_{\Omega \times (0, t)} \Psi(W)|DW|^2 \, dz \leq C \int \int_{\Omega \times (0, t)} [1 + c_0 \Psi(W)] W^2 \, dz. \tag{3.9}
\]

Applying the Sobolev inequality to the function \( F(W, 1) \) (see (3.6)) we find a constant \( C(N) \) such that

\[
\int_{\Omega \times \{t\}} \Psi(W) W^2 \, dx \leq C(N) \int_{\Omega \times \{t\}} \Psi(W)|DW|^2 \, dx.
\]
Thus, using this, we see that if $c_0$ is sufficiently small then the integral of $C_0 \Psi(W)W^2$ in the inequality (3.9) can be absorbed to the left and we get

$$\sup_{t \in [0,T)} \int_{\Omega \times \{t\}} W^2 \, dx + \int_{\Omega \times (0,t)} \Psi(W)|DW|^2 \, dz \leq C \int_{\Omega \times (0,t)} W^2 \, dz.$$ 

This yields an integral Grönwall inequality for $y(t) = \|W\|_{L^2(\Omega \times \{t\})}$ on $(0,T)$ and shows that this norm is bounded by a universal constant on $(0,T)$. This fact and the above inequality show that the left hand side quantities are bounded. We then make use of the parabolic Sobolev inequality to see that $\|W^2 \Psi(W)\|_{L^1(Q,d\mu)}$ is bounded by a constant. This implies $\|W \Psi(W)\|_{L^1(Q)}$ is bounded because $2\gamma > 1$.

**Proof of Theorem 3.1:** We write the equation of $u_i$ in its divergence form

$$(u_i)_t = \text{div}(aDu_i) + \text{div}(u_i b) + u_i g_i(u),$$

where $a = \lambda + \Psi(W)$ and $b = D(\Psi(W))$.

Using the facts that $\Psi(W) \geq 0$ (because $W \geq 0$) and $W$ is bounded, we have $a \geq \lambda_0$ and bounded from above. Also, $b = D(\Psi(W))$ are bounded. In addition, $u_i g_i(u)$ is bounded because $0 \leq u_i \leq W/a_i$ which is bounded. We then use the standard theory of scalar parabolic equation with bounded coefficients to show that $Du_i$ is bounded and Hölder continuous in $\Omega \times (\tau,T)$ for any $\tau \in (0,T)$.

### 3.2 Equal reaction rates:

We now present two examples which relax the assumption of equal diffusion rates $p_i$'s. However, we have to consider equal reaction rates $g_i$'s and restrict ourselves to the case of systems of two equations.

In the sequel, we will always assume that $\Psi$ is a $C^1$ function on $\mathbb{R}$ such that

$$\Psi(s), \Psi'(s) \geq 0 \text{ and } \Psi(s) \geq s \text{ for } s \geq 0. \quad (3.10)$$

We consider first the following system

$$
\begin{align*}
  u_t &= \Delta (\lambda_0 u + u \Psi(L(u,v))) + \varepsilon_0 a \Delta (uv) + u g(u,v), \\
  v_t &= \Delta (\lambda_0 v + v \Psi(L(u,v))) - \varepsilon_0 b \Delta (uv) + v g(u,v).
\end{align*}
\quad (3.11)
$$

Here, $L(u,v) = bu + av$. $\lambda_0, \varepsilon_0, a, b$ are positive constants. Regarding the reaction term, we also assume that there are positive constants $C, c_0$ such that (compare with (3.3))

$$|g(u,v)| \leq C + c_0 \Psi(|L(u,v)|) \text{ for all } u, v \in \mathbb{R}. \quad (3.12)$$

We consider nonnegative initial data $u_0, v_0$ for $u, v$.

**Theorem 3.5** If $\varepsilon_0, c_0$ are sufficiently small then the system (3.11) has a unique global strong solution $(u,v)$ with $u, v \geq 0$.

We need the following proposition which will be useful later.
Proposition 3.6 We consider a strong solution \((u, v)\) with nonnegative initial data \(u_0, v_0\) to the following system

\[
\begin{aligned}
    u_t &= \Delta (\lambda_0 u + u\Psi(L(u, v))) + \varepsilon_0 a \Delta (u|v|) + ug(u, v), \\
    v_t &= \Delta (\lambda_0 v + v\Psi(L(u, v))) - \varepsilon_0 b \Delta (u|v|) + vg(u, v).
\end{aligned}
\]  

(3.13)

For any \(\varepsilon_0 > 0\) we have that \(u, v\) and \(Du, Dw\) are bounded. Also \(u \geq 0\) in \(Q\).

If \(\varepsilon_0\) are sufficiently small then \(v\) is also nonnegative in \(Q\).

Proof: The proof will be divided into several steps. First of all, taking a linear combination of the above equations, we see that \(W = L(u, v)\) satisfying

\[
W_t = \lambda_0 Dw + \Delta (\Psi(W)W) + Wg(u, v).
\]  

(3.14)

Step 1: We show that \(W, Dw\) are bounded and \(W \geq 0\).

For a given strong solution \((u, v)\) of (3.13) we consider the the equation

\[
w_t = \lambda_0 Dw + \Delta (\Psi(|w|)w) + wg(u, v)
\]  

(3.15)

and the initial data \(w_0 = au_0 + bv_0 \geq 0\). We proved in Theorem 2.2 that this equation has a strong solution \(w\) and, by Theorem 2.5 \(w \geq 0\). By uniqueness of strong solutions, \(W = w\) so that \(W \geq 0\).

Now, from the proof of Lemma 3.3 we see that \(W, Dw\) are bounded in \(\Omega \times (\tau, T)\) for any \(\tau \in (0, T)\) in terms of \(\|W\|_{L^1(Q)}\) and \(\|W\Psi(W)\|_{L^1(Q)}\). The latter two norms can be bounded by a constant if \(c_0\) is sufficiently small (see Remark 3.4).

We should note that because we already proved that \(W \geq 0\), hence we do not need here the fact that \(u, v \geq 0\) (which will be established later) as before in Lemma 3.3 but the conditions \(\Psi(s), \Psi'(s) \geq 0\) for \(s \geq 0\) in (3.10) and that \(|g(u, v)| \leq C + c_0 \Psi(|W|)\) in (3.12) (see equation (2.18) of Remark 2.4).

Step 2: We prove that \(u \geq 0\). We write the equation of \(u\) in its divergence form

\[
u_t = \text{div}(ADu) + \text{div}(uB) + ug(u, v)
\]  

(3.16)

with \(A = \lambda_0 + \Psi(W) + \varepsilon_0 a|v|, B = -\Psi'(W)Dw + \varepsilon_0 aD(|v|)\).

Again, we can assume that \(u, v\) are locally bounded as in the proof of Lemma 3.2. Because \(W, Dw\) are bounded, we apply Theorem 2.5 to prove that \(u \geq 0\).

Step 3: We now prove that \(u\) is bounded by using the iteration argument in Theorem 2.2.

We multiply the above equation (3.16) by \(u^{2p-1}\), recall that \(u \geq 0\), and follows the proof of Theorem 2.2 to get

\[
\frac{d}{dt} \int_{\Omega} u^{2p} \, dx + (2p - 1) \int_{\Omega} Au^{2p-2} |Dw|^2 \, dx \leq \int_{\Omega} \text{div}(uB)u^{2p-1} \, dx + \int_{\Omega} g(u, v)u^{2p} \, dx
\]  

(3.17)

From the definition of \(B\) we need to study the following two terms on the right of (3.17)

\[
- \int_{\Omega} \text{div}(u\Psi'(W)D(W))u^{2p-1} \, dx, \int_{\Omega} a\text{div}(uD(|v|))u^{2p-1} \, dx.
\]  

(3.18)
The first one can be treated easily, using the fact that \( W \) is bounded (see also below). We consider the second term. We have

\[
\int_{\Omega} \text{div}(uD(|v|))u^{2p-1} \, dx = -(2p-1) \int_{\Omega} auD(|v|)u^{2p-2}Dv \, dx.
\]

For each \( t > 0 \) we split \( \Omega = \Omega^+(t) \cup \Omega^-(t) \) where \( \Omega^+(t) = \{ x : v(x,t) \geq 0 \} \). Since \( aDv = DW - bDu \) and on \( \Omega^+(t), D(|v|) = Dv \), we have that the integral over \( \Omega^+(t) \) of

\[
-auD(|v|)u^{2p-2}Du \text{ is } \int_{\Omega^+(t)} u^{2p-1}(-DW + bDu)Du \, dx = -\int_{\Omega^+(t)} u^{2p-1}DWDu \, dx + \int_{\Omega^+(t)} bu^{2p-1}|Du|^2 \, dx.
\]

Because \( DW \) is bounded in \( \Omega \times (\tau, T) \) for any \( \tau \in (0,t) \), it follows that for any \( \varepsilon > 0 \) there is \( c_1(\varepsilon) \) such that

\[
\int_{\Omega} u^{2p-1}|DWDu| \, dx \leq \int_{\Omega} (\varepsilon u^{2p-2}|Du|^2 + c_1(\varepsilon)u^{2p}) \, dx.
\]

Choosing \( \varepsilon \) small, the integral of \( u^{2p-2}|Du|^2 \) can be absorbed to the integral of \( \lambda_0u^{2p-2}|Du|^2 \) in the left of (3.17). This argument also applies to the first integral in (3.18).

Meanwhile, on the set \( v \geq 0, \) as \( W \geq bu \geq 0 \) so that \( \Psi(W) \geq W \geq bu \) (by the assumption (3.10) on \( \Psi \)). Thus, the integral over \( \Omega^+(t) \) of \( bu^{2p-1}|Du|^2 \) can also be absorbed to the integral over \( \Omega^+(t) \) of \( \Psi(W)u^{2p-2}|Du|^2 \) in \( Au^{2p-2}|Du|^2 \) of the left of (3.17).

On \( \Omega^-(t), D(|v|) = -Dv \), we have that the integral over \( \Omega^-(t) \) of \( -auD(|v|)u^{2p-2}Du \) is

\[
\int_{\Omega^-} u^{2p-1}(DW - bDu)Du \, dx = \int_{\Omega^-} u^{2p-1}DWDu \, dx - \int_{\Omega^-} bu^{2p-1}|Du|^2 \, dx.
\]

The first integral on the right hand side can be handled as before. The second integral is nonnegative and can be dropped.

Putting these together, we then obtain for all \( p \geq 1 \)

\[
\frac{d}{dt} \int_{\Omega} u^p \, dx + \int_{\Omega} u^{2p-2}|Du|^2 \, dx \leq C \int_{\Omega} u^p \, dx.
\]

This allows to obtain a bound for \( \|u\|_{L^\infty(Q)} \) in terms of \( \|u\|_{L^1(Q)} \) (see Theorem 2.2). Let \( p = 1 \) in the above inequality to get a Grönewall inequality for \( \|u\|_{L^2(\Omega)}^2 \). We see that \( \|u\|_{L^2(\Omega)}^2 \), so is \( \|u\|_{L^1(\Omega)} \), is bounded on \((0,T)\).

Once we prove that \( u \) and \( W, DW \) are bounded we then use a cutoff function and repeat a similar argument to the above one in order to obtain local strong/weak Harnack inequalities. It follows that \( u \) is Hölder continuos. This is a standard procedure and the readers are referred to the book [4]. It also follows that \( Du \) is bounded.

**Step 4:** We show that \( v \) is bounded. This is easy because \( v = (W - bu)/a \) and \( u, Du \) and \( W, DW \) are bounded. We should note that in the above steps we have not imposed any assumptions on \( \varepsilon_0 \). Thus, the first assertion of the Proposition was proved.

**Step 5:** Finally, we prove that \( v \geq 0 \). First of all, we write the equation of \( v \) in its divergence form

\[
v_t = \text{div}(A_1 Dv) + \text{div}(|v|B_1) + vg,
\]
where $A_1 = \lambda_0 + \Psi(W) - \varepsilon_0 b \text{sign}(v)$, $B_1 = \Psi'(W)DW + \varepsilon_0 b Du$.

Since $u$ is bounded by a constant independent of $\varepsilon_0$ and $\Psi(W) \geq 0$, we can choose $\varepsilon_0$ small such that $A_1 \geq \lambda_0/2$. Also, $B_1$ is bounded because $W, DW$ and $Du$ are. The proof of Theorem 2.5 applies and shows that $v \geq 0$. \qed

**Proof of Theorem 3.5** From Proposition 3.6 the system (3.13) has a strong solution $(u, v)$ which also solves (3.11). By uniqueness of strong solutions, we see that strong solution $(u, v)$, and its spatial derivatives, of (3.11) are bounded uniformly in terms of the data. Because $\|Du\|_{L^\infty(\Omega)}, \|Dv\|_{L^\infty(\Omega)}$ do not blow up in any time interval $(0, T)$, the solution exists globally. \qed

We also consider the following system

$$\begin{cases}
\begin{aligned}
u_t &= \Delta(\lambda_0 u + u\Psi(L(u, v))) + \varepsilon_0 a \Delta(|u|) + u\varepsilon_0 a \Delta(|v|) + u\varepsilon_0 a \Delta(|v|), \\
v_t &= \Delta(\lambda_0 v + v\Psi(L(u, v))) + \varepsilon_0 b \Delta(|u|) + v\varepsilon_0 b \Delta(|v|) + v\varepsilon_0 b \Delta(|v|).
\end{aligned}
\end{cases} \tag{3.19}$$

Here, $L(u, v) = bu - av$ and $\varepsilon_0, a, b$ are positive constants.

We then have the following result similar to Theorem 3.5 without the assumption on the smallness of $\varepsilon_0$. However, we have to strengthen the condition (3.10) by assuming in addition that

$$\Psi(s) \geq 0 \quad \forall s \in \mathbb{R}. \tag{3.20}$$

**Theorem 3.7** If $\varepsilon_0$ in the assumption (3.12) is sufficiently small then the system (3.11) has a unique global strong solution $(u, v)$ with $u, v \geq 0$.

**Proof:** Following the proof of Theorem 3.5 we consider a strong solution $(u, v)$ with the same initial data to the following system

$$\begin{cases}
\begin{aligned}
u_t &= \Delta(\lambda_0 u + u\Psi(L(u, v))) + \varepsilon_0 a \Delta(|u|) + u\varepsilon_0 a \Delta(|v|) + u\varepsilon_0 a \Delta(|v|), \\
v_t &= \Delta(\lambda_0 v + v\Psi(L(u, v))) + \varepsilon_0 b \Delta(|u|) + v\varepsilon_0 b \Delta(|v|) + v\varepsilon_0 b \Delta(|v|).
\end{aligned}
\end{cases} \tag{3.21}$$

For any $\varepsilon_0 > 0$ we will prove that $u, v$ and $Du, Dv$ are bounded. We also show that $u, v \geq 0$ in $Q$. We follow the proof of Proposition 3.6 and provide necessary modifications.

Let $W = bu - av$. Taking a linear combination of the two equations, we can follows Step 1 of the proof of Proposition 3.6 to show that $W, DW$ are bounded. Note that we cannot prove that $W \geq 0$ as before because its initial data $bu_0 - av_0$ is nonnegative.

Similarly, Step 2 also yields that $u \geq 0$. We need to change the argument in Step 3 of the proof to prove that $u, Du$ are bounded. We test the equation of $u$ by $u^{2p-1}$. As in Step 3, we need to consider the following term on the right hand side of (3.17)

$$\int_{\Omega} a \text{div}(uD(|v|))u^{2p-1} \, dx = -(2p - 1) \int_{\Omega} a uD(|v|)u^{2p-2} \, Du \, dx.$$ 

We again split $\Omega = \Omega^+ \cup \Omega^-$ where $\Omega^+ = \{v \geq 0\}$. Because $av = bu - W$ (instead of $av = W - bu$ as before) we need to interchange $\Omega^+, \Omega^-$ in the previous argument. Namely, the integral over $\Omega^+$ now contributes a nonnegative term to the left and an integral of $u^{2p}$ to the right. Meanwhile, on $\Omega^-$ we have $W = bu - av \geq bu \geq 0$ so that $\Psi(W) \geq bu$ and
the integral over $\Omega$ of $bu^{2p-1}|Du|^2$ now can be absorbed to the left hand side. The proof then continues to prove that $u, Du$ are bounded.

Using $v = (bu - W)/a$, we see that $v, Dv$ are bounded.

We now show that $v \geq 0$, without the assumption that $\varepsilon_0$ is small. We slightly modify Step 5 of Corollary 3.6. We write the equation of $v$ as

$$v_t = \text{div}(A_2 Dv) + \text{div}(\varepsilon_0 u D(|v|)) + \text{div}(v B_2) + vg(u,v).$$

Here, $A_2 = \lambda_0 + \Psi(W), B_2 = \varepsilon_0 \text{sign}(v) Du + D\Psi(W)$. We follow the proof of Theorem 2.5 and test the equation with $v^-$. We need to consider the integral of $\text{div}(\varepsilon_0 u D(|v|)) v^-$ on the right hand side. Using integration by parts and the fact that $D(|v|) = Dv^+ + Dv^-$,

$$\int_{\Omega} \text{div}(\varepsilon_0 u D(|v|)) v^- dx = -\int_{\Omega} \varepsilon_0 u D(|v|) Dv^- dx = -\int_{\Omega} \varepsilon_0 u |Dv^-|^2 dx.$$ 

Because $u \geq 0$, the last term provides a nonnegative term on the left hand side. Meanwhile, we have that $A_2 \geq \lambda_0$ and $A_2, B_2$ are bounded (as $u, Du, W, DW$ are bounded). We obtain as in the proof of Theorem 2.5 a Gröwall inequality of $\|v^-\|_{L^2(\Omega)}$ and conclude that $v^- = 0$ on $Q$. Thus, $v$ is nonnegative. The proof is complete. $\blacksquare$

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