\section*{\Large $\Lambda(p)$-sets and the limit order of operator ideals}

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\begin{abstract}
Given an infinite set $\Lambda$ of characters on a compact abelian group we show that $\Lambda$ is a $\Lambda(p)$-set for all $2 < p < \infty$ if and only if the limit order of the ideal $\Pi_{\Lambda}$ of all $\Lambda$-summing operators coincides with that of the ideal $\Pi_\gamma$ of all Gaussian-summing operators, i.e. $\lambda(\Pi_{\Lambda}, u, v) = \lambda(\Pi_\gamma, u, v)$ for all $1 \leq u, v \leq \infty$. This is a natural counterpart to a recent result of Baur which says that $\Lambda$ is a Sidon set if and only if $\Pi_\Lambda = \Pi_\gamma$. Furthermore, our techniques, which are mainly based on complex interpolation, lead us to exact asymptotic estimates of the Gaussian-summing norm $\pi_\gamma(id : S^n_u \hookrightarrow S^n_v)$ of identities between finite-dimensional Schatten classes $S^n_u$ and $S^n_v$, $1 \leq u, v \leq \infty$.
\end{abstract}

\section{Introduction and results}

We use standard notation and notions from Banach space theory, as presented e.g. in [DJT95], [LT79] and [TJ89]. If $E$ is a Banach space, then $B_E$ is its (closed) unit ball and $E'$ its dual; we consider complex Banach spaces only. As usual $L(E, F)$ denotes the Banach space of all (bounded and linear) operators from $E$ into $F$ endowed with the operator norm.

For an infinite orthonormal system $B \subset L_2(\mu)$ (over some probability space $(\Omega, \mu)$) an operator $T : E \to F$ between Banach spaces $E$ and $F$ is said to be $B$-summing if there exists a constant $c > 0$ such that for all finite sequences $b_1, \ldots, b_n$ in $B$ and $x_1, \ldots, x_n$ in $E$

$$
\left( \int_{\Omega} \left( \sum_{i=1}^n b_i \cdot T x_i \right)^2 d\mu \right)^{1/2} \leq c \cdot \sup_{x' \in B_{E'} \cap \ell^p} \left( \sum_{i=1}^n \langle x', x_i \rangle^2 \right)^{1/2}, \quad (1.1)
$$

we write $\pi_B(T)$ for the smallest constant $c$ satisfying (1.1). In this way we obtain the injective and maximal Banach operator ideal $(\Pi_B, \pi_B)$, which became of interest recently in the theses of Baur [Bau97] and Seigner [Sei95]. For a sequence of independent standard Gaussian random variables the associated Banach operator ideal $\Pi_\gamma$ of all Gaussian-summing operators was introduced by Linde and Pietsch [LP77], and for operators acting on finite-dimensional Hilbert spaces $\pi_\gamma$ is also known as the $\ell$-norm, which turned out to be important for the study of the geometry of Banach spaces (see e.g. [TJ89]).

For an infinite subset $\Lambda$ of the character group $\Gamma$ of some compact abelian group $G$ (which can be viewed as an orthonormal system in $L_2(G, m_G)$, where $m_G$ denotes the normalized Haar measure on $G$) Baur in [Bau97, 9.5] (see also [Bau99, 4.2]) gave the following characterization:

\begin{quote}
$\Lambda$ is a Sidon set if and only if $\Pi_\Lambda = \Pi_\gamma$.
\end{quote}
In particular, our proof uses the close relationship between the Gaussian-summing norm of the identity operator id and the Dvoretzky dimension of a finite-dimensional Banach space E due to Pisier.

As a natural counterpart of Baur’s result we prove the following characterization of sets which are \( \Lambda(p)\)-sets for all \( 2 < p < \infty \), with no control of \( K_p(\Lambda) \) as in Pisier’s characterization of Sidon sets above:

**Theorem 1.** For every infinite subset \( \Lambda \subset \Gamma \) the following are equivalent:

(a) \( \Lambda \) is a \( \Lambda(p)\)-set for all \( 2 < p < \infty \).

(b) \( \lambda(\Pi_\Lambda, u, v) = \lambda(\Pi_{\gamma}, u, v) \) for all \( 1 \leq u, v \leq \infty \).

Here, the limit order \( \lambda(A, u, v) \) of a Banach operator ideal \((A, A)\) for \( 1 \leq u, v \leq \infty \) is defined as usual (see e.g. [Pie88, 14.4]):

\[
\lambda(A, u, v) := \inf \{ \lambda > 0 \mid \exists \rho > 0 \forall n \in \mathbb{N} : A(id : \ell_u^n \rightarrow \ell_v^n) \leq \rho \cdot n^\lambda \}.
\]

Note that there exist sets which are \( \Lambda(p)\)-sets for all \( 2 < p < \infty \) but fail to be Sidon sets (see e.g. [LR73, 5.14]). Our proof is mainly based on complex interpolation techniques, in particular on formulas for the complex interpolation of spaces of operators due to Pisier [Pi90] and Kouba [Kou91] (see also [DM98]). These techniques also yield asymptotic estimates of the Gaussian-summing norm of identities between finite-dimensional Schatten classes:

**Theorem 2.** For \( 1 \leq u, v \leq \infty \)

\[
\pi_{\gamma}(S_u^n \rightarrow S_v^n) \asymp \begin{cases} n^{1/2+1/v} & \text{if } 2 \leq u \leq \infty, \\ n^{1/2+\max(0,1/2+1/v-1/u)} & \text{if } 1 \leq u \leq 2. \end{cases}
\]

In particular,

\[
\pi_{\gamma}(S_u^n \rightarrow S_v^n) \asymp n^{1/2+\lambda(\Pi_{\gamma}, u, v)}.
\]

Here, \( S_u^n \) denotes the space of all linear operators \( T : \ell_u^n \rightarrow \ell_2^n \) endowed with the norm \( \|T\|_{S_u^n} := \|(s_k(T))_{k=1}^n\|_{\ell_2} \), where \( (s_k(T))_{k=1}^n \) is the sequence of singular numbers of \( T \). Besides the interpolation argument, our proof uses the close relationship between the Gaussian-summing norm of the identity operator id_E and the Dvoretzky dimension of a finite-dimensional Banach space E due to Pisier.

## 2 Complex interpolation of B-summing operators

Our main tool will be an “interpolation theorem” for the \( B\)-summing norm of a fixed operator acting between finite-dimensional complex interpolation spaces; a similar approach for the \((s,2)\)-summing norm was used in [DM98] to study the well-known “Bennett–Carl inequalities" within the context of interpolation theory.

For all information on complex interpolation we refer to [BL78]. Given an interpolation couple \([E_0, E_1]\) of complex Banach spaces and \( 0 < \theta < 1 \), the associated complex interpolation space is
denoted by \([E_0, E_1]_\theta\). If \(E_0\) and \(E_1\) are finite-dimensional Banach spaces with the same dimensions, we speak of a finite-dimensional interpolation couple, and in this case we define
\[
d_\theta[E_0, E_1] := \sup_m \|\mathcal{L}(\ell_p^n, [E_0, E_1]_\theta) \rightarrow [\mathcal{L}(\ell_p^n, E_0), \mathcal{L}(\ell_p^n, E_1)]_\theta\|.
\]

Pisier [Pi98] and Kouba [Kou91] derived upper estimates for \(d_\theta[E_0, E_1]\) for particular situations (see also [DM99]); we will use the fact that for \(1 \leq p_0, p_1 \leq 2\)
\[
d_\theta[\ell_p^n, \ell_p^n] \leq \sqrt{2}, \tag{2.1}
\]
in particular, \(\sup_n d_\theta[\ell_p^n, \ell_p^n] < \infty\). Junge [Jun96, 4.2.6] gave an analogue for Schatten classes:
\[
\sup_n d_\theta[S_p^n, S_p^n] < \infty. \tag{2.2}
\]

These “uniform” estimates will be crucial for the applications of the following result:

**Proposition 3.** Let \(0 < \theta < 1\). Then for two finite-dimensional interpolation couples \([E_0, E_1]\) and \([F_0, F_1]\), each \(T \in \mathcal{L}([E_0, E_1]_\theta, [F_0, F_1]_\theta)\) and each orthonormal system \(B \subset L_2(\mu)\)
\[
\pi_B(T : [E_0, E_1]_\theta \rightarrow [F_0, F_1]_\theta) \leq d_\theta [E_0, E_1] \cdot \pi_B(T : E_0 \rightarrow F_0)^{1-\theta} \cdot \pi_B(T : E_1 \rightarrow F_1)^\theta.
\]

**Proof.** For the moment set \(E_\theta := [E_0, E_1]_\theta, F_\theta := [F_0, F_1]_\theta\), and consider for \(\eta = 0, \theta, 1\) and \(\mathcal{F} = \{b_1, \ldots, b_m\} \subset B\) the mapping
\[
\Phi_{\eta, \mathcal{F}}^m : \ell_p^n, E_\eta) \rightarrow L_2(\mu, F_\eta)
\]

we obviously get that
\[
\pi_B(T : E_\eta \rightarrow F_\eta) = \sup\{\|\Phi_{\eta, \mathcal{F}}^m\| \mid m \in \mathbb{N}, \mathcal{F} \subset B \text{ with } |\mathcal{F}| = m\}.
\]

For the interpolated mapping
\[
[\Phi_{0, \mathcal{F}}^m, \Phi_{1, \mathcal{F}}^m]_{\theta} : [\mathcal{L}(\ell_p^n, E_0), \mathcal{L}(\ell_p^n, E_1)]_{\theta} \rightarrow [L_2(\mu, F_0), L_2(\mu, F_1)]_{\theta}
\]

by the usual interpolation theorem
\[
\|\Phi_{0, \mathcal{F}}^m, \Phi_{1, \mathcal{F}}^m\|_{\theta} \leq \|\Phi_{0, \mathcal{F}}^m\|^{1-\theta} \cdot \|\Phi_{1, \mathcal{F}}^m\|^{\theta}.
\]

Since \([L_2(\mu, F_0), L_2(\mu, F_1)]_{\theta} = L_2(\mu, [F_0, F_1]_{\theta})\) (isometrically, see [BL78, 5.1.2]) we obtain
\[
\|\Phi_{\theta, \mathcal{F}}^m\| \leq \|\mathcal{L}(\ell_p^n, [E_0, E_1]_{\theta}) \rightarrow [\mathcal{L}(\ell_p^n, E_0), \mathcal{L}(\ell_p^n, E_1)]_{\theta}\| \cdot \|\Phi_{0, \mathcal{F}}^m, \Phi_{1, \mathcal{F}}^m\|_{\theta}.
\]

Consequently
\[
\pi_B(T : [E_0, E_1]_{\theta} \rightarrow [F_0, F_1]_{\theta})
\]

are finite-dimensional Banach spaces with the same dimensions.

Since for \(0 < \theta < 1\) and \(1 \leq p_0, p_1, p_\theta \leq \infty\) such that \(1/p_\theta = (1-\theta)/p_0 + \theta/p_1\) it holds \([\ell_p^{n_0}, \ell_p^{n_1}]_{\theta} = \ell_p^{n_\theta}\) isometrically (see [BL78, 5.1.1]), we obtain together with (2.1) the following corollary:
Corollary 4. For $0 < \theta < 1$ let $1 \leq u_0, u_1, u_\theta \leq 2$ and $1 \leq v_0, v_1, v_\theta \leq \infty$ such that $1/u_\theta = (1-\theta)/u_0 + \theta/u_1$ and $1/v_\theta = (1-\theta)/v_0 + \theta/v_1$. Then

$$\lambda(\Pi_B, u_\theta, v_\theta) \leq (1-\theta) \cdot \lambda(\Pi_B, u_0, v_0) + \theta \cdot \lambda(\Pi_B, u_1, v_1).$$

3 The proof of Theorem 1

As a generalization of the notion of $\Lambda(p)$-sets, an orthonormal system $B \subseteq L_2(\mu)$ is said to be a $\Lambda(p)$-system if $B \subseteq L_p(\mu)$ and there exists a constant $c > 0$ such that for all $f \in \text{span}B$ we have $\|f\|_{L_p(\mu)} \leq c \cdot \|f\|_{L_2(\mu)}$; the infimum over all such constants $c$ is denoted by $K_p(B)$. Now one direction of the equivalence in Theorem 1 can be formulated for general orthonormal systems:

Proposition 5. Let $B \subseteq L_2(\mu)$ be a $\Lambda(p)$-system for all $2 < p < \infty$. Then for all $1 \leq u, v \leq \infty$

$$\lambda(\Pi_B, u, v) = \lambda(\Pi_\gamma, u, v) = \begin{cases} 1/v & \text{if } 2 \leq u \leq \infty, \\ \max(0, 1/2 + 1/v - 1/u) & \text{if } 1 \leq u \leq 2. \end{cases}$$

Proof. Although the limit order of $\Pi_\gamma$ is already known by the results of [PW98, 4.15], the following proof may also be used to compute it independently (at least the upper estimates; the lower ones are somehow simple), but for simplicity we fall back upon this knowledge.

Since $\Pi_\gamma \subseteq \Pi_B \subseteq \Pi_\ell$ (see [PW98, 4.15]; $\Pi_2$ denotes the Banach operator ideal of all 2-summing operators), we only have to show that $\lambda(\Pi_B, u, v) \leq \lambda(\Pi_\gamma, u, v)$, and moreover, we conclude that $\lambda(\Pi_B, u, v) \leq \lambda(\Pi_2, u, v) = \lambda(\Pi_\gamma, u, v)$ for all $1 \leq u \leq \infty$ and $1 \leq v \leq 2$. For $2 < v < \infty$ and $2 \leq u \leq \infty$ it can be easily seen that

$$\Pi_B(\ell_u^m \hookrightarrow \ell_v^m) \leq K_v(B) \cdot m^{1/v}$$

(just copy the proof of [PW98, 3.11.11]), hence—together with the continuity of the limit order, see [Pie80, 14.4.8]—we obtain

$$\lambda(\Pi_B, u, v) \leq 1/v = \lambda(\Pi_\gamma, u, v)$$

for all $2 \leq u, v \leq \infty$. Now the case $1 \leq u \leq 2 \leq v \leq \infty$ follows from Corollary 4. For $1 \leq u \leq 2$ choose $u_0 := 1, u_1 := 2, v_0 := 2, v_1 := \infty, \theta := 2/u'$ and $v_u$ such that $1/v_u = 1/u - 1/2$. Then

$$\lambda(\Pi_B, u, v_u) \leq (1-\theta) \cdot \lambda(\Pi_B, 1, 2) + \theta \cdot \lambda(\Pi_B, 2, \infty) = 0.$$

For arbitrary $1 \leq u \leq 2 \leq v \leq \infty$ factorize through $\ell_v^m$:

$$\pi_B(\ell_u^m \hookrightarrow \ell_v^m) \leq m^{\max(0,1/v + 1/2 - 1/u)} \cdot \pi_B(\ell_u^m \hookrightarrow \ell_v^m),$$

hence $\lambda(\Pi_B, u, v) \leq \max(0, 1/v + 1/2 - 1/u) = \lambda(\Pi_\gamma, u, v)$.

The reverse implication in Theorem 1 follows from [Bau97]: (b) trivially implies $\lambda(\Pi_B, \infty, \infty) = 0$, and the comments after [Bau97, 7.12] then tell us that $\Pi_B \subseteq \Pi_\Lambda$ for all $2 < p < \infty$. This in turn gives by [Bau97, 9.6] (see also [Bau99, 5.1]) that $\Lambda$ is a $\Lambda(p)$-set for all $2 < p < \infty$.

Note that the last argument requires the setting of characters on a compact abelian group; Baur has recently informed us that her results are also valid for the non-abelian case, and therefore our Theorem 1 as well.
4 The proof of Theorem 2

Proof. For $1 \leq v \leq 2$ by (7.174) $S_u$ is of cotype 2, hence

$$n^{1/v + \min(1, 2-1/u)} = \pi_2(S_u^n \hookrightarrow S_v^n) \geq \pi_7(S_u^n \hookrightarrow S_v^n) \geq C_2(S_v)^{-1} \cdot \pi_2(S_u^n \hookrightarrow S_v^n)$$

(see e.g. [DM98, Corollary 3]) for $1 \leq u \leq \infty$ and $1 \leq v \leq 2$; here $C_2(S_v)$ denotes the Gaussian cotype 2 constant of $S_v$. We are left with the case $2 \leq v \leq \infty$; first let $u = v = \infty$. Then by [PW98, 4.15.18] (a result of Pisier, see also [Pi89, 4.4] and [Pi86]) and [FLM77, 3.3] for each $\varepsilon > 0$

$$\pi_7(S_u^n \hookrightarrow S_v^n) \approx \sqrt{D(S_u^n, \varepsilon)} \approx n^{1/2},$$

where $D(X, \varepsilon)$ denotes the Dvoretzky dimension of a Banach space $X$, i.e. the largest $m$ such that there exists an $m$-dimensional subspace $X_m$ of $X$ with Banach–Mazur distance $d(X_m, \ell^2_m) \leq 1 + \varepsilon$ (see [PW98, 4.15.15]). Now the general case $2 \leq u, v \leq \infty$ follows by factorization:

$$\pi_7(S_u^n \hookrightarrow S_v^n) \leq n^{1/v} \cdot \pi_7(S_u^n \hookrightarrow S_v^n) \sim n^{1/v + 1/2},$$

and conversely

$$\pi_7(S_u^n \hookrightarrow S_v^n) \geq n^{1/v - 1/2} \cdot \pi_7(S_u^n \hookrightarrow S_v^n) = n^{1/v + 1/2}.$$

The case $1 \leq u \leq 2 \leq v \leq \infty$ is done by interpolation: We have (recall that $\pi_2(S_u^n \hookrightarrow S_v^n) = n^{1/2}$)

$$\pi_7(S_u^n \hookrightarrow S_v^n) \leq \pi_7(S_u^n \hookrightarrow S_v^n) \approx \pi_7(S_u^n \hookrightarrow S_v^n) \approx n^{1/2},$$

hence for $1 \leq u < 2 < v_u < \infty$ and $0 < \theta < 1$ such that $1/v_u = 1/u - 1/2$ and $\theta = 2/w'$

$$\pi_7(S_u^n \hookrightarrow S_v^n) \leq d_0[S_u^n, S_v^n] \cdot \pi_7(S_u^n \hookrightarrow S_v^n) \cdot \pi_7(S_u^n \hookrightarrow S_v^n) \approx n^{1/2};$$

recall that $\sup_u d_0[S_u^n, S_v^n] < \infty$ by (7.2.3), and that $[S_u^n, S_v^n] \theta = S_u^n$ and $[S_v^n, S_v^n] \theta = S_v^n$ hold isometrically (this can be deduced from e.g. [PT68, Satz 8] and the complex reiteration theorem [BL78, 4.6.1]). The remaining estimates now follow easily from

$$\pi_7(S_u^n \hookrightarrow S_v^n) \geq \pi_7(\ell_2^n \hookrightarrow \ell_2^n) = n^{1/2},$$

and

$$\pi_7(S_u^n \hookrightarrow S_v^n) \leq \pi_7(S_u^n \hookrightarrow S_v^n) \cdot \pi_7(S_u^n \hookrightarrow S_v^n).$$

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