Movement of time-delayed hot spots in Euclidean space

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Abstract We investigate the shape of the solution of the Cauchy problem for the damped wave equation. In particular, we study the existence, location and number of spatial maximizers of the solution. Studying the shape of the solution of the damped wave equation, we prepare a decomposed form of the solution into the heat part and the wave part. Moreover, as its another application, we give \( L^p-L^q \) estimates of the solution.

Keywords Damped wave equation · Diffusion phenomenon · Hot spot · \( L^p-L^q \) estimate

Mathematics Subject Classification 35L15 · 35B38 · 35C15 · 35B40 · 35K05

1 Introduction

Let \( f \) and \( g \) be real-valued smooth functions defined on \( \mathbb{R}^n \). We consider the damped wave equation with initial data \( (f, g) \),

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{\partial^2}{\partial t^2} - \Delta + \frac{\partial}{\partial t} \right) u(x, t) = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\
\left( u, \frac{\partial u}{\partial t} \right)(x, 0) = (f, g)(x), \quad x \in \mathbb{R}^n.
\end{array} \right.
\end{align*}
\]  

(1.1)

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In the one-dimensional case, the damped wave equation is known as the *telegrapher’s equation* introduced by Oliver Heaviside and describes the current and voltage in an electrical circuit with resistance and inductance. More generally, the Eq. (1.1) is a model of the propagation of the wave with friction or resistance.

The damped wave equation is also known as the *hyperbolic heat conduction equation* introduced in \([1,2,15,17,22]\). The classical model of the heat equation admits the infinite speed of the propagation of heat conduction, which is physically inadmissible. Therefore, the Eq. (1.1) was introduced to modify the model of heat conduction with finite speed of the propagation.

In order to derive the Eq. (1.1) as the hyperbolic heat conduction equation, along Li’s framework in \([15]\), let us consider the one-dimensional case as below: Let \(v(x,t)\) be the temperature at a point \(x \in \mathbb{R}\) and at time \(t\); Let \(q(x,t)\) be the heat flux at a point \(x \in \mathbb{R}\) and at time \(t\); Then, the heat balance law implies

\[
\frac{\partial v}{\partial t}(x,t) + \frac{\partial q}{\partial x}(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0; \tag{1.2}
\]

From the time-delayed Fourier’s law with a small enough positive parameter \(\tau\)

\[
q(x,t+\tau) \approx q(x,t) + \tau \frac{\partial q}{\partial t}(x,t) = -\frac{\partial v}{\partial x}(x,t), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.3}
\]

instead of the usual Fourier’s law

\[
q(x,t) = -\frac{\partial v}{\partial x}(x,t), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.4}
\]

we get the damped wave equation

\[
\left(\tau \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)v(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.5}
\]

with delay time \(\tau\); The additional term \(\tau \partial^2 v/\partial t^2\) brings the finite propagation speed property to the Eq. (1.5); By the scale transformation

\[
u(x,t) = \frac{1}{\tau} v(\sqrt{\tau}x, \tau t), \tag{1.6}
\]

we obtain

\[
\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)u(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0. \tag{1.7}
\]

From such a background, as \(t\) goes to infinity in (1.7) (corresponding to the case where \(\tau\) tends to \(0^+\) in (1.5)), it is expected that the solution of the damped wave equation approaches to that of the (usual) heat equation, which is called the diffusion phenomenon and has been studied by many researchers \([5,9,16,18,20,23]\).

In this paper, we investigate the relation between the damped wave and heat equations in view of the study on the shapes of the solutions. Precisely, we give correspondence to Chavel and Karp’s results in \([3]\). Let us review their works as below: Let \(P_n(t)\phi(x)\) be the unique bounded solution of the Cauchy problem for the heat equation with bounded initial datum \(\phi\), that is,

\[
P_n(t)\phi(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{r^2}{4t}\right) \phi(y)dy, \quad x \in \mathbb{R}^n, \quad t > 0, \quad r = |x - y|; \tag{1.8}
\]
When $\phi$ is a non-zero non-negative bounded function with compact support, they studied the behavior of the set of hot spots

$$H_\phi(t) = \left\{ x \in \mathbb{R}^n \left| P_n(t)\phi(x) = \max_{\xi \in \mathbb{R}^n} P_n(t)\phi(\xi) \right. \right\}; \quad (1.9)$$

They showed that hot spots exist at each time $t$, that all of them are contained in the convex hull of the support of $\phi$ for any time $t$, and that the set $H_\phi(t)$ converges to the one-point set of the centroid (the center of mass) of $\phi$ as $t$ goes to infinity; Furthermore, calculating the Hessian of $P_n(t)\phi$, in [14], Jimbo and Sakaguchi indicated that the set of hot spots $H_\phi(t)$ consists of one point after a large time $t$.

There are many results on the study of hot spots besides [3]. As examples, we introduce [6,7,11–14,21] as below: In [14], Jimbo and Sakaguchi studied the large time behavior of hot spots in unbounded domains. For example, they considered the exterior domain of a ball with a radially symmetric initial datum. In this case, the explicit representation of solutions like (1.8) was still useful; In [11,12], Ishige studied the large time behavior of hot spots in the exterior domain of a ball without the radially symmetric assumption of [14] by using the self-similar transformation and the eigenfunction expansion; In [13], the similar approach to [11,12] was also applicable for the heat equation with a potential; In [6,7], Fujishima and Ishige studied the large time behavior of hot spots in $\mathbb{R}^n$ without the non-negativity of the initial datum $\phi$ and applied their investigation to the blow-up set of a semi-linear heat equation; In [21], the first author generalized the results in [3,14] in terms of a potential with a radially symmetric kernel.

In view of the diffusion phenomena, it is expected that spatial maximizers of the solution of (1.1) have similar properties to hot spots shown in [3,14,21]. To this aim, when $f$ and $g$ are compactly supported, and when $h := f + g$ is non-zero and non-negative, we study the behavior of the set of time-delayed hot spots

$$H(t) = \left\{ x \in \mathbb{R}^n \left| u(x, t) = \max_{\xi \in \mathbb{R}^n} u(\xi, t) \right. \right\}. \quad (1.10)$$

Precisely, we show the following properties:

1. After a large enough time, the set $H(t)$ is contained in the convex hull of the support of $h$. Furthermore, for some small time, we give some examples of $(f, g)$ such that the set $H(t)$ escapes from the convex hull of the support of $h$ (the first assertion of Theorem 4.10, Examples 4.14, 4.15, 4.16 and 4.17).
2. The set $H(t)$ converges to the one-point set of the centroid of $h$ as $t$ goes to infinity (the second assertion of Theorem 4.10).
3. After a large enough time, the set $H(t)$ consists of one point (Proposition 4.13).

In order to understand the meaning of the above statements, we, for example, consider the case where $f$ is non-zero and non-negative, the maximum value of $f$ is greater than that of $h$, and the supports of $f$ and $h$ are separated. Since $u(x, 0) = f(x)$, for any sufficiently small $t > 0$, all of the time-delayed hot spots are “close” to maximum points of $f$, that is, they are contained in the support of $f$ and not contained in the support of $h$. Roughly speaking, our main results claim that time-delayed hot spots move from the set of maximum points of $f$ to the centroid of $h$.

The above properties of time-delayed hot spots are due to the decomposition of the solution operator (fundamental solution) $S_n(t)$ into the heat part and the wave part as

$$S_n(t)g(x) = J_n(t)g(x) + e^{-t/2}W_n(t)g(x). \quad (1.11)$$
Here, $J_n(t)g(x)$ and $W_n(t)g(x)$ are suitable functions behaving like as the function $P_n(t)g(x)$ and the solution of the free wave equation with initial datum $(0, g)$, respectively. The decomposition (1.11) was firstly discovered by Nishihara in [20] in the three-dimensional case and so-called the Nishihara decomposition. In this paper, we give the generalization of the Nishihara decomposition in higher dimensional cases to study the behavior of time-delayed hot spots. Moreover, as its another application, we slightly improve Narazaki’s $L^p-L^q$ estimates given in [18]. As a by-product of the $L^p-L^q$ estimates, we obtain

$$
\|u(\cdot, t) - P_n(t)h\|_{L^\infty} \leq Ct^{-n/2-1} \left( \|f\|_{L^1} + \|g\|_{L^1} + \|f\|_{W^\ast,\infty} + \|g\|_{W^\ast,\infty} \right). 
$$

(1.12)

One may consider that the large time behavior of time-delayed hot spots can be easily investigated by combining the results in [3] and the estimate (1.12). But the expectation is incorrect. This is because the difference between the values of $P_n(t)h$ in the convex hull of support of $h$ and its outside can have the order worse than $t^{-n/2-1}$. Hence the above estimate cannot exclude the possibility that time delayed hot spots escape from the convex hull of support of $h$ by the effect of the wave part. Therefore, we should obtain more precise information about the solution of the damped wave equation.

This paper is organized as follows. In Sect. 2, we give a representation of the solution of (1.1) and its Nishihara decomposition. In Sect. 3, we give preliminary estimates for our investigation. In Sect. 4, we investigate the movement of time-delayed hot spots. In Sect. 5, we give $L^p-L^q$ estimates of the difference between the damped wave and the heat by using the Nishihara decomposition described in Sect. 2.

**Notation 1.1** For the end of this section, we explain our notation.

- The letter $C$ indicates the generic constant which may change from line to line.
- We denote the usual $L^p$ norm by $\| \cdot \|_{L^p}$, that is,

$$
\|\phi\|_{L^p} = \begin{cases} 
\left( \int_{\mathbb{R}^n} |\phi(x)|^p \, dx \right)^{1/p} & (1 \leq p < \infty), \\
\text{ess sup}_{x \in \mathbb{R}^n} |\phi(x)| & (p = \infty).
\end{cases}
$$

(1.13)

- For a natural number $\ell$, we denote the Sobolev norm by $\| \cdot \|_{W^{\ell,\infty}}$, that is,

$$
\|\phi\|_{W^{\ell,\infty}} = \sum_{|\alpha| \leq \ell} \| \partial^{\alpha} \phi \|_{L^\infty}.
$$

(1.14)

- Let $B^n$ and $S^{n-1}$ be the $n$-dimensional unit closed ball and the $(n-1)$-dimensional unit sphere, respectively.
- For real numbers $a$ and $b$, and for two sets $X$ and $Y$ in $\mathbb{R}^n$, we use the notation (Minkowski sum) $aX + bY = \{ax + by \mid x \in X, \ y \in Y \}$. In particular, we write $B^n_t(x) = tB^n + \{x\}$ and $S^{n-1}_t(x) = tS^{n-1} + \{x\}$, that is, the $n$-dimensional closed ball with radius $t$ centered at $x$ and the $(n-1)$-dimensional sphere with radius $t$ centered at $x$, respectively.
- Let us denote by $CS(\phi)$ and $d_\phi$ the convex hull and the diameter of the support of a function $\phi$, respectively.
- Let $\sigma_n$ denote the $n$-dimensional Lebesgue surface measure.
- We understand that the letter $r$ is always used for $r = |x - y|$.
2 Decomposition of the solution

2.1 Solution formula of the damped wave equation

In this subsection, we prepare the explicit form of the solution of the Cauchy problem \((1.1)\). For a smooth function \(g\), let us denote by \(S_n(t)g\) the solution of the Cauchy problem

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} - \Delta u(x, t) = 0, & x \in \mathbb{R}^n, t > 0, \\
\left( u, \frac{\partial u}{\partial t} \right)(x, 0) = (0, g)(x), & x \in \mathbb{R}^n.
\end{cases}
\] (2.1)

The symbol \(S_n(t)\) is called the solution operator of \((2.1)\).

Let \(I_\nu(s)\) be the modified Bessel function of order \(\nu\),

\[
I_\nu(s) = \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(j + \nu + 1)} \left( \frac{s}{2} \right)^{2j + \nu}. \tag{2.2}
\]

We denote by \(c_n\) the constants

\[
\begin{cases}
1 & (n = 1), \\
(n - 1)! \sigma_{n-1}(S^{n-1})^{-1} = 2^{-(n+1)/2} \pi^{-(n-1)/2} & (n \in 2\mathbb{N} + 1), \\
(n - 1)! \sigma_n(S^n)^{-1} = 2^{-(n+2)/2} \pi^{-n/2} & (n \in 2\mathbb{N}),
\end{cases} \tag{2.3}
\]

where \((2\ell - 1)!! = (2\ell - 1) \cdot (2\ell - 3) \cdots 3 \cdot 1\) and \((2\ell)!! = 2\ell \cdot (2\ell - 2) \cdots 4 \cdot 2\).

**Proposition 2.1** Let \(g\) be a smooth function. For any natural number \(n\), the function \(S_n(t)g(x)\) is given by

\[
\begin{cases}
\frac{e^{-t/2}}{2} \int_{x-t}^{x+t} I_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) dy & (n = 1), \\
c_n e^{-t/2} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-1)/2} \int B_n^2(x) I_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) dy & (n \in 2\mathbb{N} + 1), \\
2c_n e^{-t/2} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2} \int B_n^2(x) \cosh \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) dy & (n \in 2\mathbb{N}).
\end{cases}
\]

**Proof** The argument is due to the method of descent described in [4].

It is well known that the solution of the free wave equation

\[
\begin{cases}
\frac{\partial^2 w}{\partial t^2} - \Delta w(x, t) = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\
\left( w, \frac{\partial w}{\partial t} \right)(x, 0) = (0, g)(x), & x \in \mathbb{R}^n
\end{cases}
\]

is given by

\[
W_n(t)g(x) = \begin{cases}
\frac{1}{2} \int_{x-t}^{x+t} g(y) dy & (n = 1), \\
c_n \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-3)/2} \left( \frac{1}{t} \int S_n^{1-1}(x) I_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) dy \sigma_{n-1}(y) \right) & (n \in 2\mathbb{N} + 1), \\
2c_n \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2} \int B_n^2(x) \cosh \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) dy & (n \in 2\mathbb{N}).
\end{cases}
\]
For a point \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and a number \( \xi \in \mathbb{R} \), we write \( z = (x, \xi) \in \mathbb{R}^{n+1} \). Let \( u(x, t) \) be the solution of (1.1), and \( w(z, t) = \exp((\xi + t)/2)u(x, t) \). Then, \( w(z, t) \) satisfies the \((n + 1)\)-dimensional free wave equation with initial datum \( (w, \partial w/\partial t)(z, 0) = (0, e^{\frac{\chi}{2}}g(x)) \).

We first consider the one-dimensional case. Using the fact (5.15), we have

\[
 u(x, t) = e^{-t^2/2} w(z, t) 
\]

\[
 = \frac{1}{2} \int_{B_1^2(z)} e^{(t^2 - |y|^2)/2} g(y_1) \frac{e^{(t^2 - |y|^2)/2}}{\sqrt{t^2 - |x - y_1|^2 - |\xi - y_2|^2}} dy
\]

\[
 = \frac{1}{2} \int_{B_n^2(x)} l_0 \left( \frac{1}{2} \sqrt{t^2 - |x - y'|^2} \right) g(y') dy',
\]

where \( y = (y', y_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \), we obtain the conclusion.

Finally, we consider even-dimensional cases. As we have

\[
 \frac{1}{t} \int_{S_n^1(c)} e^{-t^2/2} g(y) d\sigma_n(y) 
\]

\[
 = \frac{1}{t} \left( \int_{S_n^1(c) \cap \{y_{n+1} \geq \xi\}} + \int_{S_n^1(c) \cap \{y_{n+1} \leq \xi\}} \right) e^{-t^2/2} g(y) d\sigma_n(y) 
\]

\[
 = \int_{B_n^0(x)} \left( e^{t^2 - |x-y'|^2/2} + e^{-t^2 - |x-y'|^2/2} \right) \frac{g(y')}{\sqrt{t^2 - |x - y'|^2}} dy' 
\]

\[
 = 2 \int_{B_n^0(x)} \frac{\cosh \left( \frac{t}{2} \sqrt{t^2 - |x - y'|^2} \right)}{\sqrt{t^2 - |x - y'|^2}} g(y') dy',
\]

the proof is completed. \( \square \)

**Example 2.2** We have the following formulas:

\[
 S_1(t)g(x) = \frac{e^{-t^2/2}}{2} \int_{x-t}^{x+t} l_0 \left( \frac{1}{2} \sqrt{t^2 - x^2} \right) g(y) dy,
\]

\[
 S_2(t)g(x) = \frac{e^{-t^2/2}}{2\pi} \int_{B_n^2(x)} \frac{\cosh \left( \frac{t}{2} \sqrt{t^2 - r^2} \right)}{\sqrt{t^2 - r^2}} g(r) dr,
\]
\[ S_3(t)g(x) = \frac{e^{-t/2}}{4\pi} \left( \frac{1}{t} \frac{\partial}{\partial t} \right) \int_{B^3_t(x)} I_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y)dy. \]

These were given in [4] in the same manner as in Proposition 2.1.

**Remark 2.3** Let \( f \) and \( g \) be smooth functions. Using the solution operator \( S_n(t) \), we can express the solution of the Cauchy problem (1.1) as

\[ u(x, t) = S_n(t) (f + g) (x) + \frac{\partial}{\partial t} S_n(t) f(x). \]

### 2.2 Decomposition of the fundamental solution

In this subsection, using Proposition 2.1, we give the Nishihara decomposed form of the solution operator \( S_n(t) \).

**Theorem 2.4** Let \( g \) be a smooth function.

1. Let \( n \) be an odd number greater than one. Put

\[ k_\ell(s) = \frac{1}{2^\ell} \sum_{j=0}^\infty \frac{1}{j!(j + \ell)!} \left( \frac{s}{2} \right)^{2j} = \frac{I_\ell(s)}{s^\ell}, \]

\[ J_n(t)g(x) = \frac{c_n e^{-t/2}}{2^{n-1}} \int_{B^n_t(x)} k_{n-1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y)dy, \]

\[ W_n(t)g(x) = c_n \sum_{j=0}^{(n-3)/2} \frac{1}{8^j j!} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-3)/2-j} \left( \frac{1}{t} \int_{S_{n-1}^\ell(x)} g(y)d\sigma_{n-1}(y) \right). \]

Then, we have

\[ S_n(t)g(x) = J_n(t)g(x) + e^{-t/2} W_n(t)g(x). \]

Moreover, we have

\[ k_\ell(s) = \frac{1}{s^\ell} \frac{e^s}{\sqrt{2\pi s}} \left( 1 - \frac{(\ell - 1/2)(\ell + 1/2)}{2s} + O \left( \frac{1}{s^2} \right) \right) \]

as \( s \) goes to infinity.

2. Let \( n \) be an even number. Put

\[ k_\ell(s) = \sum_{j=0}^\infty \frac{1}{(2(j + \ell))!(2j + 1)!!} s^{2j+1}, \]

\[ J_n(t)g(x) = \frac{c_n e^{-t/2}}{2^{n-2}} \int_{B^n_t(x)} k_{n/2} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y)dy, \]

\[ W_n(t)g(x) = 2c_n \sum_{j=0}^{(n-2)/2} \frac{1}{8^j j!} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2-j} \int_{B^n_t(x)} \frac{1}{\sqrt{t^2 - r^2}} g(y)dy. \]

Then, we have

\[ S_n(t)g(x) = J_n(t)g(x) + e^{-t/2} W_n(t)g(x). \]
Moreover, we have
\[ k_\ell(s) = \frac{e^s}{2s^\ell} \left( 1 - \frac{\ell(\ell - 1)}{2s} + O \left( \frac{1}{s^2} \right) \right) \]

as \( s \) goes to infinity.

**Remark 2.5**

1. Let \( n \) be an odd number greater than one. From the fact (5.16), the kernel \( k_\ell(s) \) has the following properties:

\[ k_{\ell+1}(s) = \frac{k'_\ell(s)}{s}, \quad k_\ell(0) = \frac{1}{2\ell!}, \quad k'_\ell(0) = 0. \]

2. Let \( n \) be an even number. In the proof of Theorem 2.4, we will show that the kernel \( k_\ell(s) \) is defined by the following recursion:

\[ k_1(s) = \frac{\cosh(s) - 1}{s}, \quad k_\ell(s) = \frac{k'_{\ell-1}(s) - k'_{\ell-1}(0)}{s}. \]

In particular, we have the following properties:

\[ k_\ell(0) = 0, \quad k'_\ell(0) = \frac{1}{2\ell!}. \]

**Proof of Theorem 2.4**

1. The solution formula in Proposition 2.1 implies

\[ S_n(t)g(x) = c_ne^{-t/2} \left( \frac{1}{t} \sqrt{t^2 - r^2} \right)^{(n-3)/2} \left( \frac{1}{t} \int_{S} g(y)d\sigma_{n-1}(y) \right) + c_ne^{-t/2} \left( \frac{1}{t} \sqrt{t^2 - r^2} \right)^{(n-3)/2} \int_{B_n^g(x)} I_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y)dy. \]

Note that the second term of the right-hand side can be written as

\[ \frac{c_ne^{-t/2}}{4} \left( \frac{1}{t} \sqrt{t^2 - r^2} \right)^{(n-3)/2} \int_{B_n^g(x)} k_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y)dy. \]

Since we have

\[ \frac{1}{t} \left( \frac{\partial}{\partial t} k_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right) = \frac{k'_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right)}{2\sqrt{t^2 - r^2}} = \frac{1}{4} k_2 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right), \]

we obtain

\[ \frac{c_ne^{-t/2}}{4} \left( \frac{1}{t} \sqrt{t^2 - r^2} \right)^{(n-3)/2} \int_{B_n^g(x)} k_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y)dy = \frac{c_ne^{-t/2}}{4} \left( \frac{1}{t} \sqrt{t^2 - r^2} \right)^{(n-5)/2} \left( \frac{k_1(0)}{t} \int_{S} g(y)d\sigma_{n-1}(y) \right) \]

\[ + \frac{c_ne^{-t/2}}{4^2} \left( \frac{1}{t} \sqrt{t^2 - r^2} \right)^{(n-5)/2} \int_{B_n^g(x)} k_2 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y)dy. \]

Continuing this argument, we obtain the decomposed form of \( S_n(t)g \).

The asymptotic expansions are direct consequences from \( k_{\ell+1}(s) = k'_\ell(s)/s \) and (5.17).
2. By the solution formula in Proposition 2.1, we have

\[ S_n(t)g(x) = 2c_n e^{-t/2} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2} \int_{B^n_t(x)} \frac{1}{\sqrt{t^2 - r^2}} g(y) \, dy \]

\[ + 2c_n e^{-t/2} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2} \int_{B^n_t(x)} \frac{\cosh\left( \frac{1}{2} \sqrt{t^2 - r^2} \right) - 1}{\sqrt{t^2 - r^2}} g(y) \, dy. \]

\[ =: V_1 + K_1. \]

Here, we note that \( K_1 \) can be written as

\[ K_1 = c_n e^{-t/2} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-4)/2} \int_{B^n_t(x)} k_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) \, dy \]

\[ = c_n e^{-t/2} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-4)/2} \int_{B^n_t(x)} k_1' \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) \, dy \]

\[ = \frac{c_n e^{-t/2}}{2} k_1'(0) \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-4)/2} \int_{B^n_t(x)} \frac{1}{\sqrt{t^2 - r^2}} g(y) \, dy \]

\[ + \frac{c_n e^{-t/2}}{4} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-4)/2} \int_{B^n_t(x)} k_2 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) \, dy \]

\[ =: V_2 + K_2. \]

In the same manner as in the calculation of \( K_1 \), we have

\[ K_2 = c_n e^{-t/2} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-6)/2} \int_{B^n_t(x)} \frac{1}{\sqrt{t^2 - r^2}} g(y) \, dy \]

\[ + \frac{c_n e^{-t/2}}{4^2} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-6)/2} \int_{B^n_t(x)} k_3 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) g(y) \, dy \]

\[ =: V_3 + K_3. \]

Continuing this argument, we obtain

\[ V_j = \frac{c_n e^{-t/2}}{2^{2j-3}} k_{j-1}'(0) \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{n/2-j} \int_{B^n_t(x)} \frac{1}{\sqrt{t^2 - r^2}} g(y) \, dy \]

and the decomposed form of \( S_n(t)g \).

The asymptotic expansions follow from

\[ k_1(s) = \frac{e^s}{2s} + \frac{e^{-s}}{2s} - \frac{1}{s} \]

and the recursion of \( k_{\ell}(s) \).

\[ \square \]

**Remark 2.6** Let \( g \) be a smooth function. Put

\[ J_1(t)g(x) = \frac{e^{-t/2}}{2} \int_{x-t}^{x+t} l_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) - 1 \right) g(y) \, dy, \]

\[ W_1(t)g(x) = W_1(t)g(x) = \frac{1}{2} \int_{x-t}^{x+t} g(y) \, dy. \]
Thanks to Proposition 2.1, we get the Nishihara decomposed form of \( S_1(t) \) as

\[
S_1(t)g(x) = J_1(t)g(x) + e^{-t/2}W_1(t)g(x)
\]

which was given in [16].

**Example 2.7** We have the following formulas:

\[
J_2(t)g(x) = \frac{e^{-t/2}}{2\pi} \int_{B_1^2(x)} \frac{\cosh \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) - 1}{\sqrt{t^2 - r^2}} g(y)dy,
\]

\[
J_3(t)g(x) = \frac{e^{-t/2}}{4\pi} \int_{B_1^2(x)} I_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \frac{1}{2\sqrt{t^2 - r^2}} g(y)dy,
\]

\[
W_2(t)g(x) = W_2(t)g(x) = \frac{1}{2\pi} \int_{B_1^2(x)} \frac{1}{\sqrt{t^2 - r^2}} g(y)dy,
\]

\[
W_3(t)g(x) = W_3(t)g(x) = \frac{1}{4\pi t} \int_{S_1^2(x)} g(y)d\sigma_2(y).
\]

These were given in [9,20] using the explicit form of \( S_n(t)g(x) \) given in Example 2.2.

In [9,16,20], it was shown that \( J_n(t)g \) behaves like \( P_n(t)g \) as \( t \) goes to infinity. More precisely, in the case of \( 1 \leq n \leq 3 \), for \( t > 0 \) and \( 1 \leq q \leq p \leq \infty \), the \( L^p-L^q \) estimate

\[
\| J_n(t)g - P_n(t)g \|_{L^p} \leq C t^{-\frac{n}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \| g \|_{L^q}
\]

was shown.

The Nishihara decomposition and the estimate (2.4) imply the following properties of the damped wave equation (see [20, p. 632]):

- The damped wave equation does not have the smoothing effect, and the singularity of the initial datum propagates along the light cone by the wave property. However, the strength of the singularity decays exponentially by the damping effect.
- If the initial datum is sufficiently smooth, then the damped wave equation may have the same properties as those to parabolic equations under some suitable situations.

For the case of \( n \geq 4 \), in [18], Narazaki proved a similar decomposition to Theorem 2.4 in the Fourier space. However, the explicit form of the decomposition in the configuration space was not known. In Sect. 5, using Theorem 2.4, we will give an estimate of the difference \( S_n(t)g - P_n(t)g - e^{-t/2}W_n(t)g \), which is slightly sharper than that of [18].

### 2.3 Decomposition of the solution with general initial datum

In this subsection, we give the Nishihara decomposed form of the solution of (1.1).

**Proposition 2.8** Let \( f \) and \( g \) be smooth functions, \( h = f + g \), and \( u \) the unique classical solution of the Cauchy problem (1.1).

1. Let \( n = 1 \). Put

\[
\tilde{J}_1(t) f(x) = \frac{e^{-t/2}}{4} \int_{x-t}^{x+t} \left( \frac{t}{\sqrt{t^2 - r^2}} I_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) - I_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right) f(y)dy,
\]

\[
\tilde{W}_1(t) f(x) = \frac{1}{2} \left( f(x + t) + f(x - t) \right),
\]

\[
\tilde{W}_1(t; f, g)(x) = W_1(t)h(x) + \tilde{W}_1(t) f(x).
\]
Then, we have
\[
\frac{\partial}{\partial t} S_1(t) f(x) = \tilde{J}_1(t) f(x) + e^{-t/2} \hat{W}_1(t) f(x).
\]

In other words, the solution \( u(x, t) \) is expressed as
\[
u(x, t) = J_1(t) h(x) + \tilde{J}_1(t) f(x) + e^{-t/2} \hat{W}_1(t; f, g)(x).
\]

2. Let \( n \) be an odd number greater than one. Put
\[
\tilde{J}_n(t) f(x) = \frac{c_n e^{-t/2}}{2^{n+1} } \int_{B_{\gamma}(x)} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \frac{1}{2} \sqrt{t^2 - r^2} \bigg) f(y) dy,
\]
\[
\hat{W}_n(t) f(x) = \frac{c_n}{2^{\frac{3(n-1)}{2}} \left( \frac{n-1}{2} \right)! } \int_{s_{n-1}(x)} f(y) d\sigma_{n-1}(y),
\]
\[
\bar{W}_n(t; f, g)(x) = \frac{1}{2} W_n(t) f(x) + W_n(t) g(x) + \hat{W}_n(t) f(x) + \frac{\partial}{\partial t} W_n(t) f(x).
\]

Then, the derivative of \( S_n(t) f(x) \) with respect to \( t \) is given by
\[
\tilde{J}_n(t) f(x) + e^{-t/2} \hat{W}_n(t) f(x) - \frac{e^{-t/2}}{2} W_n(t) f(x) + e^{-t/2} \frac{\partial}{\partial t} W_n(t) f(x).
\]

In other words, the solution \( u(x, t) \) is expressed as
\[
u(x, t) = J_n(t) h(x) + \tilde{J}_n(t) f(x) + e^{-t/2} \bar{W}_n(t; f, g)(x).
\]

3. Let \( n \) be an even number. Put
\[
\tilde{J}_n(t) f(x) = \frac{c_n e^{-t/2}}{2^n } \int_{B_{\gamma}(x)} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \frac{1}{2} \sqrt{t^2 - r^2} \bigg) f(y) dy,
\]
\[
\hat{W}_n(t) f(x) = \frac{c_n t}{2^{\frac{3n-2}{2}} \left( \frac{n}{2} \right)! } \int_{B_{\gamma}(x)} \frac{1}{\sqrt{t^2 - r^2}} f(y) dy,
\]
\[
\bar{W}_n(t; f, g)(x) = \frac{1}{2} W_n(t) f(x) + W_n(t) g(x) + \hat{W}_n(t) f(x) + \frac{\partial}{\partial t} W_n(t) f(x).
\]

Then, the derivative of \( S_n(t) f(x) \) with respect to \( t \) is given by
\[
\tilde{J}_n(t) f(x) + e^{-t/2} \hat{W}_n(t) f(x) - \frac{e^{-t/2}}{2} W_n(t) f(x) + e^{-t/2} \frac{\partial}{\partial t} W_n(t) f(x).
\]

In other words, the solution \( u(x, t) \) is expressed as
\[
u(x, t) = J_n(t) h(x) + \tilde{J}_n(t) f(x) + e^{-t/2} \bar{W}_n(t; f, g)(x).
\]
Lemma 3.1  Let $f$ and $g$ be smooth bounded functions defined on R^n. This is because their proofs consist of tedious calculations.

In this section, we prepare some estimates which will be frequently used for studying the behavior of time-delayed hot spots. We postpone the proofs of the following lemmas until “Appendix”. This is because their proofs consist of tedious calculations.

Lemma 3.1  Let $f$ and $g$ be smooth bounded functions defined on $\mathbb{R}^n$.

1. For any $x \in \mathbb{R}$, $t \geq 0$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, we have the following inequalities:

$$\left| \frac{\partial^{\alpha}}{\partial x^{\alpha_1} \partial t^{\alpha_2}} W_1(t) g(x) \right| \leq (1 + t) \| g \|_{W^{\alpha_1, \infty}},$$

$$\left| \frac{\partial^{\alpha}}{\partial x^{\alpha_1} \partial t^{\alpha_2}} \tilde{W}_1(t) f(x) \right| \leq \| f \|_{W^{\alpha_1, \infty}},$$

$$\left| \frac{\partial^{\alpha}}{\partial x^{\alpha_1} \partial t^{\alpha_2}} \tilde{W}_1(t; f, g)(x) \right| \leq 2(1 + t) \left( \| f \|_{W^{\alpha_1, \infty}} + \| g \|_{W^{\alpha_1, \infty}} \right),$$

where $|\alpha| = \alpha_1 + \alpha_2$. 

Proof 1. From Proposition 2.1, direct computation shows

$$\frac{\partial}{\partial t} S_1(t) f(x) = \frac{e^{-t/2}}{4} \int_{x-t}^{x+t} \left( \frac{t}{\sqrt{t^2 - r^2}} I_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) - I_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right) f(y) dy$$

$$+ \frac{e^{-t/2}}{2} (f(x + t) + f(x - t))$$

$$= \tilde{J}_1(t) f(x) + e^{-t/2} \tilde{W}_1(t) f(x).$$

Here, we used $I_0'(s) = I_1(s)$ (see (5.16)). From Remark 2.3, we get the conclusion.

2. From Remark 2.5, we have

$$\frac{\partial}{\partial t} J_n(t) f(x) = \frac{c_n}{2^{n-1}} \int_{B_n(t)} \frac{\partial}{\partial t} \left( e^{-t/2} k_{n-1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right) f(y) dy$$

$$+ \frac{c_n e^{-t/2}}{2^{n-1} \left( \frac{n-1}{2} \right)!} \int_{S_n^t(x)} f(y) d\sigma_{n-1}(y)$$

$$= \tilde{J}_n(t) g(x) + e^{-t/2} \tilde{W}_n(t) g(x).$$

From Remark 2.3, we get the conclusion.

3. From Remark 2.5, we have

$$\frac{\partial}{\partial t} J_n(t) g(x) = \frac{c_n}{2^{n-2}} \int_{B_n(t)} \frac{\partial}{\partial t} \left( e^{-t/2} k_n \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right) g(y) dy$$

$$= \tilde{J}_n(t) g(x) + e^{-t/2} \tilde{W}_n(t) g(x).$$

From Remark 2.3, we get the conclusion. 

3 Preliminary estimates

In this section, we prepare some estimates which will be frequently used for studying the behavior of time-delayed hot spots. We postpone the proofs of the following lemmas until “Appendix”. This is because their proofs consist of tedious calculations.
Lemma 3.2
Let \( n \) be an odd number greater than one. There exists a positive constant \( C = C(n) \) such that, for any \( x \in \mathbb{R}^n \), \( t > 0 \) and \( \alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1} \), we have the following inequalities:

\[
\begin{align*}
\left| \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial t^{\alpha_{n+1}}} W_n(t) g(x) \right| & \leq C (1 + t)^{n-2} \|g\|_{W^{(n-3)/2+|\alpha|, \infty}}, \\
\left| \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial t^{\alpha_{n+1}}} \tilde{W}_n(t) f(x) \right| & \leq C (1 + t)^{n-1} \|f\|_{W^{|\alpha|, \infty}}, \\
\left| \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial t^{\alpha_{n+1}}} \tilde{W}_n(t; f, g)(x) \right| & \leq C (1 + t)^{n-1} \left( \|f\|_{W^{(n-1)/2+|\alpha|, \infty}} + \|g\|_{W^{(n-3)/2+|\alpha|, \infty}} \right),
\end{align*}
\]

where \(|\alpha| = \alpha_1 + \cdots + \alpha_{n+1}|.

3. Let \( n \) be an even number. There exists a positive constant \( C = C(n) \) such that, for any \( x \in \mathbb{R}^n \), \( t > 0 \) and \( \alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1} \), we have the following inequalities:

\[
\begin{align*}
\left| \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial t^{\alpha_{n+1}}} W_n(t) g(x) \right| & \leq C (1 + t)^{n-1} \|g\|_{W^{n/2-1+|\alpha|, \infty}}, \\
\left| \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial t^{\alpha_{n+1}}} \tilde{W}_n(t) f(x) \right| & \leq C (1 + t)^n \|f\|_{W^{|\alpha|, \infty}}, \\
\left| \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial t^{\alpha_{n+1}}} \tilde{W}_n(t; f, g)(x) \right| & \leq C (1 + t)^n \left( \|f\|_{W^{n/2+|\alpha|, \infty}} + \|g\|_{W^{n/2-1+|\alpha|, \infty}} \right),
\end{align*}
\]

where \(|\alpha| = \alpha_1 + \cdots + \alpha_{n+1}|

Lemma 3.2 Let \( h : \mathbb{R}^n \to \mathbb{R} \) be a smooth function.

1. Let \( n = 1 \). We have the following identities:

\[
\begin{align*}
\frac{\partial}{\partial x} J_1(t)h(x) &= -\frac{e^{-t/2}}{4} \int_{x-t}^{x+t} \frac{1}{\sqrt{t^2 - r^2}} I_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) h(y)(x - y) dy, \\
\frac{\partial^2}{\partial x^2} J_1(t)h(x) &= \frac{t e^{-t/2}}{8} \left( h(x + t) + h(x - t) \right) \\
&\quad - \frac{e^{-t/2}}{4} \int_{x-t}^{x+t} \frac{1}{\sqrt{t^2 - r^2}} I_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) h(y) dy \\
&\quad + \frac{e^{-t/2}}{8} \int_{x-t}^{x+t} \frac{1}{t^2 - r^2} I_2 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) h(y)(x - y)^2 dy.
\end{align*}
\]

2. Let \( n \) be an odd number greater than one. For any direction \( \omega \in S^{n-1} \), we have the following identities:

\[
\nabla J_n(t)h(x) = \frac{c_n e^{-t/2}}{2^{n-1}} k_{n-1}^{(0)}(0) t^{n-1} \int_{S^{n-1}} h(x + t\theta) d\sigma_{n-1}(\theta)
\]
Lemma 3.3

Let $h$ be a smooth function with compact support. Put

$$E_n(r, t) = \begin{cases} 
  \frac{e^{-t/2}}{4} \frac{1}{\sqrt{t^2 - r^2}} l_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) & (n = 1), \\
  \frac{c_n e^{-t/2}}{2n+1} k_{n+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) & (n \in 2\mathbb{N} + 1), \\
  \frac{c_n e^{-t/2}}{2n} k_{n+1} + 1 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) & (n \in 2\mathbb{N}).
\end{cases}$$

If $x \in CS(h) + (t - d_h)B^n$ and $t \geq d_h$, then we have

$$\nabla J_n(t)h(x) = - \int_{B^n_t(x)} E_n(r, t)h(y)(x - y)dy.$$
Lemma 3.4 Let \( \varphi : [0, +\infty) \to [0, +\infty) \) be a non-decreasing function with \( \varphi(0) = 0 \).

1. If \( \sqrt{t^2 - \varphi(t)^2} \) diverges as \( t \) goes to infinity, then we have

\[
E_n(\varphi(t), t) = \frac{1}{2(4\pi)^{n/2} (t^2 - \varphi(t)^2)^{n/4+1/2}} \times \exp\left(-\frac{t + \sqrt{t^2 - \varphi(t)^2}}{2}\right) \left(1 + O\left(\frac{1}{\sqrt{t^2 - \varphi(t)^2}}\right)\right)
\]

as \( t \) goes to infinity.

2. If \( \varphi(t) \) is of small order of \( t \) as \( t \) goes to infinity, then we have

\[
E_n(\varphi(t), t) = \frac{1}{2(4\pi)^{n/2} t^{n/2+1}} \times \exp\left(-\frac{t + \sqrt{t^2 - \varphi(t)^2}}{2}\right) \left(1 + O\left(\frac{1}{t}\right) + O\left(\frac{\varphi(t)^2}{t}\right)\right)
\]

as \( t \) goes to infinity.

3. If \( \varphi(t) \) is of small order of \( \sqrt{t} \), then we have

\[
E_n(\varphi(t), t) = \frac{1}{2(4\pi)^{n/2} t^{n/2+1}} \left(1 + O\left(\frac{1}{t}\right) + O\left(\frac{\varphi(t)^2}{t}\right)\right)
\]

as \( t \) goes to infinity.

Lemma 3.5 Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a smooth function.

1. Let \( n = 1 \). We have the following identities:

\[
\frac{\partial}{\partial x} \tilde{J}_1(t, f(x)) = \frac{(t - 4)e^{-t/2}}{16} (f(x + t) - f(x - t)) - \frac{e^{-t/2}}{8} \int_{x-t}^{x+t} \frac{1}{\sqrt{t^2 - r^2}} \left(1 + O\left(\frac{1}{\sqrt{t^2 - r^2}}\right)\right) f(y) dy,
\]

\[
\frac{\partial^2}{\partial x^2} \tilde{J}_1(t, f(x)) = \frac{(t - 4)e^{-t/2}}{16} (f'(x + t) - f'(x - t))
\]

\[
+ \frac{(t^2 - 8t)e^{-t/2}}{256} (f(x + t) + f(x - t)) - \frac{e^{-t/2}}{8} \int_{x-t}^{x+t} \frac{1}{\sqrt{t^2 - r^2}} \left(1 + O\left(\frac{1}{\sqrt{t^2 - r^2}}\right)\right) f(y) dy
\]

\[
+ \frac{e^{-t/2}}{16} \int_{x-t}^{x+t} \frac{1}{(t^2 - r^2)} \left(1 + O\left(\frac{1}{\sqrt{t^2 - r^2}}\right)\right) f(y) dy
\]

\[
\times \left(1 + O\left(\frac{1}{\sqrt{t^2 - r^2}}\right)\right)
\]

\[
+ \frac{e^{-t/2}}{16} \int_{x-t}^{x+t} \frac{1}{(t^2 - r^2)} \left(1 + O\left(\frac{1}{\sqrt{t^2 - r^2}}\right)\right) f(y) (x - y)^2 dy.
\]
2. Let \( n \) be an odd number greater than one. For any direction \( \omega \in S^{n-1} \), we have the following identities:

\[
\nabla \tilde{J}_n(t) f(x) = \frac{c_n e^{-t/2}}{2^{n+1}} \left( tk'_{2n+1}(0) - 2k'_{2n}(0) \right) t^{n-1} \int_{S^{n-1}} f(x + t\theta) \theta d\sigma_{n-1}(\theta) \\
- \frac{c_n e^{-t/2}}{2^{n+3}} \int_{B_n^t(x)} \left( tk'_{n+3} \left( \frac{1}{2} t^2 - r^2 \right) - 2k'_{n+1} \left( \frac{1}{2} t^2 - r^2 \right) \right) \times f(y)(x - y) dy,
\]

\[
(\omega \cdot \nabla)^2 \tilde{J}_n(t) f(x) = \frac{c_n e^{-t/2}}{2^{n+1}} \left( tk'_{2n+1}(0) - 2k'_{2n}(0) \right) t^{n-1} \int_{S^{n-1}} \omega \cdot \nabla f(x + t\theta) \omega \cdot \theta d\sigma_{n-1}(\theta) \\
+ \frac{c_n e^{-t/2}}{2^{n+3}} \int_{B_n^t(x)} \left( tk'_{n+3} \left( \frac{1}{2} t^2 - r^2 \right) - 2k'_{n+1} \left( \frac{1}{2} t^2 - r^2 \right) \right) \times f(y) dy \\
- \frac{c_n e^{-t/2}}{2^{n+3}} \left( tk'_{2n+3}(0) - 2k'_{2n+1}(0) \right) t^{n+1} \\
\times \int_{B_n^t(x)} \left( \frac{1}{2} t^2 - r^2 \right) - 2k'_{n+3} \left( \frac{1}{2} t^2 - r^2 \right) \right) \times f(y)(x - y) dy.
\]

3. Let \( n \) be an even number. For any direction \( \omega \in S^{n-1} \), we have the following identities:

\[
\nabla \tilde{J}_n(t) f(x) = \frac{c_n e^{-t/2}}{2^{n+1}} \left( tk'_{2n+1}(0) - 2k'_{2n}(0) \right) t^{n-1} \int_{B^n} \frac{1}{\sqrt{1 - |z|^2}} f(x + tz) dz \\
- \frac{c_n e^{-t/2}}{2^{n+2}} \int_{B_n^t(x)} \left( tk'_{2n+2} \left( \frac{1}{2} t^2 - r^2 \right) - 2k'_{2n+1} \left( \frac{1}{2} t^2 - r^2 \right) \right) \times f(y)(x - y) dy,
\]

\[
(\omega \cdot \nabla)^2 \tilde{J}_n(t) f(x) = \frac{c_n e^{-t/2}}{2^{n+1}} \left( tk'_{2n+1}(0) - 2k'_{2n}(0) \right) t^{n-1} \int_{B^n} \frac{1}{\sqrt{1 - |z|^2}} \omega \cdot \nabla f(x + tz) \omega \cdot z dz \\
+ \frac{c_n e^{-t/2}}{2^{n+3}} \left( tk'_{2n+3}(0) - 2k'_{2n+1}(0) \right) t^{n+1} \\
\times \int_{B_n^t(x)} \frac{1}{\sqrt{1 - |z|^2}} f(x + tz) (\omega \cdot z)^2 dz \\
- \frac{c_n e^{-t/2}}{2^{n+2}} \int_{B_n^t(x)} \left( tk'_{2n+2} \left( \frac{1}{2} t^2 - r^2 \right) - 2k'_{2n+1} \left( \frac{1}{2} t^2 - r^2 \right) \right) \times f(y) dy.
\]
Let $f$ be a non-zero smooth function with compact support.

**Notation 4.1** Let us list up our notation for this section.

1. There exists a positive constant $C = C(n)$ such that, for any $x \in \mathbb{R}^n$ and $t > 0$, we have
   \[ \left| \tilde{J}_n(t) f(x) \right| \leq C(1 + t)^{-n/2 - 1} \| f \|_{L^1}. \]

2. Let $\psi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function with $\psi(0) = 0$. Suppose that $\psi(t)$ is of small order of $\sqrt{t}$. Let
   \[ T_0(\psi) = \min \left\{ T > 0 \left| \forall t \geq T, \ t \geq \psi(t) + d_f \right. \right\}. \]
   There exists a positive constant $C = C(n, d_f, \psi)$ such that, for any $x \in CS(f) + \psi(t)B^n$ and $t \geq T_0(\psi)$, we have
   \[ \left| \nabla \tilde{J}_n(t) f(x) \right| \leq C(1 + t)^{-n/2 - 3} \left( 1 + t + \psi(t)^2 \right) (1 + \psi(t)) \| f \|_{L^1}. \]

3. Let $R$ be a positive constant. There is a positive constant $C = C(n, d_f, R)$ such that, for any $x \in CS(f) + RB^n$, $\omega \in S^{n-1}$ and $t \geq R + d_f$, we have
   \[ \left| (\omega \cdot \nabla) \tilde{J}_n(t) f(x) \right| \leq C(1 + t)^{-n/2 - 2} \| f \|_{L^1}. \]

### 4 Movement of the time-delayed hot spots

Let $u$ denote the classical solution of the Cauchy problem (1.1). In this section, we investigate the asymptotic behavior of maximizers of the function $u(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$.

**Notation 4.1** Let us list up our notation for this section.

(fg) Let $f$ and $g$ be compactly supported smooth functions such that the sum of them $h := f + g$ is non-zero and non-negative.

(H) For a function $\phi : \mathbb{R}^n \to \mathbb{R}$, we denote by $\mathcal{M}(\phi)$ and $\mathcal{C}(\phi)$ the set of maximum and critical points of $\phi$, respectively:
\[
\mathcal{M}(\phi) = \left\{ x \in \mathbb{R}^n \left| \phi(x) = \max_{\xi \in \mathbb{R}^n} \phi(\xi) \right. \right\}, \quad \mathcal{C}(\phi) = \left\{ x \in (\text{supp}\ \phi) \nabla \phi(x) = 0 \right\}.
\]
In particular, we write
\[
\mathcal{H}(t) = \mathcal{M}(u(\cdot, t))
\]
and call a point $p \in \mathcal{H}(t)$ a *time-delayed hot spot* at time $t$. We remark that $\mathcal{H}(t)$ is always contained in $\mathcal{C}(u(\cdot, t))$.

(m) Under the condition (fg), we investigate the distance between time-delayed hot spots and the centroid (center of mass) of $h$:
\[
m_h = \int_{\mathbb{R}^n} h(y) dy / \int_{\mathbb{R}^n} h(y) dy.
\]
We remark that the centroid $m_h$ is in the interior of the convex hull of supp $h$. 

\[ \text{Springer} \]
(δ) Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$. Let

$$\delta(K, L) = \sup_{\eta \in L} \text{dist}(\eta, K).$$

We remark that the parallel body $K + \delta(K, L)B^n$ contains the convex body $L$.

(ψ) Let $\psi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that $\psi(0) = 0$ and $\psi(t)$ is of small order of $\sqrt{t}$ as $t$ goes to infinity.

(φ) Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that $\varphi(0) = 0$ and $\varphi(t)$ is of small order of $t$ as $t$ goes to infinity.

$(T_0)$ Let $f$ and $g$ be as in $(fg)$. For a non-decreasing function $\phi : [0, +\infty) \to [0, +\infty)$, let

$$T_0(\phi) = T_0(\phi; d_h, \delta(CS(f), CS(h)), df)$$

$$= \min \left\{ T \geq 0 \mid \forall t \geq T, t \geq \phi(t) + \max \{d_h, \delta(CS(f), CS(h)) + df\} \right\}.$$ 

We remark that, if $t \geq T_0(\psi)$, then, for any $x \in CS(h) + \phi(t)B^n$, the ball $B^n_t(x)$ contains the union of $CS(h)$ and $CS(f)$.

### 4.1 Asymptotic behavior of time-delayed hot spots

In this subsection, we are interested in the behavior of $C(u(\cdot, t))$ and $H(t)$. As we mentioned in the introduction, in [3], Chavel and Karp showed that, for each $t$, the non-empty set $H_\phi(t) := \mathcal{M}(P_n(t)\phi)$ is contained in the convex hull of the support of $\phi$, and that

$$\sup \{ |x - m_\phi| \mid x \in H_\phi(t) \} = O \left( \frac{1}{t} \right) \quad (4.1)$$
as $t$ goes to infinity. Let us show that similar results hold for the damped wave equation (1.1).

Remark 4.2 Under the condition $(fg)$ in Notation 4.1, for each $t > 0$, the support of $u(\cdot, t)$ is compact. More precisely, the support of $u(\cdot, t)$ is contained in the union of two parallel bodies $CS(h) + tB^n$ and $CS(f) + tB^n$. Hence we always have a time-delayed hot spot.

Lemma 4.3 We use Notation 4.1. There exist a positive constant $C$ and a time $T \geq T_0(\psi)$ such that, for any $t \geq T$, we have

$$\sup \left\{ |x - m_h| \mid x \in C(u(\cdot, t)) \cap (CS(h) + \psi(t)B^n) \right\}$$

$$\leq C \left( \frac{1 + \psi(t)^2}{t} + \frac{1 + \varphi(t)}{t} \right) \times$$

$$\left[ \frac{\| f \|_{L^1} + e^{-t/2}t^{3n/2} \| f \|_{W^{\infty, \infty}} + \| g \|_{W^{\infty, \infty}}}{\| h \|_{L^1}} \right].$$

Proof We give a proof for even dimensional cases. The other cases go parallel.

Let $x$ be a point in $C(u(\cdot, t)) \cap (CS(h) + \psi(t)B^n)$. We remark that, from Proposition 2.8 and Lemma 3.3, we have

$$x = \left( \nabla \tilde{J}_n(t) f(x) + e^{-t/2} \nabla \tilde{W}_n(t; f, g)(x) + \int_{\mathbb{R}^n} E_n(r, t) h(y) dy \right)$$

$$+ \int_{\mathbb{R}^n} E_n(r, t) h(y) dy.$$
Since the function $E_n(\cdot, t)$ is strictly decreasing, we obtain
\[
2(4\pi)^{n/2} t^{n/2+1} E_n(\psi(t) + d, t) \| h \|_{L^1} |x - m_h| \\
\leq 2(4\pi)^{n/2} t^{n/2+1} \left( \int_{\mathbb{R}^n} E_n(r, t) h(y) dy \right) (x - m_h) \\
\leq 2(4\pi)^{n/2} t^{n/2+1} \left( \int_{\mathbb{R}^n} E_n(r, t) h(y) (y - m_h) dy \right) \\
+ \left| \nabla \tilde{J}_n(t) f(x) + e^{-t/2} \nabla \tilde{W}_n(t; f, g)(x) \right|.
\]

Applying the third assertion in Lemma 3.4 to the first term, there exists a constant $C$ such that, for any sufficiently large $t$, we have
\[
2(4\pi)^{n/2} t^{n/2+1} \left( \int_{\mathbb{R}^n} E_n(r, t) h(y) (y - m_h) dy \right) \leq C \frac{1 + \psi(t)^2}{t} \| h \|_{L^1}.
\]

Applying Lemmas 3.1 and 3.6 to the second term, there exists a constant $C$ such that, for any sufficiently large $t$, we have
\[
2(4\pi)^{n/2} t^{n/2+1} \left| \nabla \tilde{J}_n(t) f(x) + e^{-t/2} \nabla \tilde{W}_n(t; f, g)(x) \right| \\
\leq C \left( \frac{1 + \psi(t)}{t} \| f \|_{L^1} + e^{-t/2} 3^{n/2} (\| f \|_{W^{n/2, \infty}} + \| g \|_{W^{n/2, \infty}}) \right)
\]

From the third assertion in Lemma 3.4, the function that assigns the value $2(4\pi)^{n/2} t^{n/2+1} E_n(\psi(t) + d, t)$ for each $t$ is bounded from below. Hence we obtain the conclusion. \qed

**Corollary 4.4** We use Notation 4.1.

1. There exists a time $T \geq T_0(\psi)$ such that, for any $t \geq T$, the intersection $C(u(\cdot, t)) \cap (CS(h) + \psi(t)B^n)$ is contained in the convex hull of the support of $h$.

2. We have

\[
\sup \left\{ |x - m_h| \mid x \in C(u(\cdot, t)) \cap (CS(h) + \psi(t)B^n) \right\} = O\left( \frac{1}{t} \right)
\]

as $t$ goes to infinity.

**Proof** 1. Since the centroid of $h$ is in the interior of the convex hull of $\text{supp} \, h$, Lemma 4.3 guarantees the conclusion.

2. From the first assertion, after a large time, we have

\[
\sup \left\{ |x - m_h| \mid x \in C(u(\cdot, t)) \cap (CS(h) + \psi(t)B^n) \right\} \\
= \sup \left\{ |x - m_h| \mid x \in C(u(\cdot, t)) \cap CS(h) \right\}.
\]

Applying Lemma 4.3 to the case of $\psi = 0$, we get the conclusion. \qed

**Lemma 4.5** We use Notation 4.1. Suppose that the function $\psi(t)$ diverges as $t$ goes to infinity. If $\varphi(t) \geq \psi(t)$, then there exists a time $T \geq T_0(\varphi)$ such that, for any $t \geq T$, the gradient of $u(\cdot, t)$ does not vanish on the region $(CS(h) + \varphi(t)B^n) \setminus (CS(h) + \psi(t)B^n)$.

**Proof** We give a proof for even dimensional cases. The other cases go parallel.

From Proposition 2.8, the solution $u(x, t)$ is expressed as
\[
u(x, t) = J_n(t)h(x) + \tilde{J}_n(t) f(x) + e^{-t/2} \tilde{W}_n(t; f, g)(x).
\]
We remark that, from Lemma 3.1, we have
\[
|\nabla \tilde{W}_n(t; f, g)(x)| \leq C(1 + t)^n \left( \|f\|_{W^{n/2+1,\infty}} + \|g\|_{W^{n/2,\infty}} \right).
\]

From Lemmas 3.2 and 3.5, we have
\[
\nabla J_n(t) h(x) + \nabla \tilde{J}_n(t) f(x)
= -\frac{c_n e^{-t/2}}{2^{n+2}} \int_{B^*_{n}(x)} \left[ \frac{1}{t^2 - r^2} \right] \exp \left( \frac{\sqrt{t^2 - r^2}}{2} \right) \left( 1 + O \left( \frac{1}{\sqrt{t^2 - r^2}} \right) \right)
\times \left( 2h(y) + \left( \frac{t}{\sqrt{t^2 - r^2}} - 1 \right) f(y) \right) (x - y) dy.
\]

Using the asymptotic expansions of \( k_n \) in Theorem 2.4, we have the expansion
\[
\nabla J_n(t) h(x) + \nabla \tilde{J}_n(t) f(x)
= -\frac{c_n e^{-t/2}}{2^{n+2}} \int_{B^*_{n}(x)} \left[ \frac{1}{t^2 - r^2} \right] \exp \left( \frac{\sqrt{t^2 - r^2}}{2} \right) \left( 1 + O \left( \frac{1}{\sqrt{t^2 - r^2}} \right) \right)
\times \left( 2h(y) + \left( \frac{t}{\sqrt{t^2 - r^2}} - 1 \right) f(y) \right) (x - y) dy
\]
as \( t \) goes to infinity.

In order to complete the proof, we use the contradiction argument. For any natural number \( N \geq T_0(\varphi) \), we assume the existence of \( t_N \geq N \) such that the function \( u(\cdot, t_N) \) has a critical point \( x_N \) in the region \( (CS(h) + \varphi(t)B^n) \setminus (CS(h) + \psi(t)B^n) \). Let \( r_N = |x_N - y| \). Since the unit sphere \( S^{n-1} \) is compact, we may assume that the sequence \( (x_N - m_h)/|x_N - m_h| \) converges to a direction \( \omega \) as \( N \) goes to infinity. Then, we have
\[
0 = -\frac{2^{n/2+1}}{c_n} \exp \left( \frac{t_N - \sqrt{t_N^2 - |x_N - m_h|^2}}{2} \right) t_N^{n/2+1} \frac{\nabla u(x_N, t_N)}{|x_N - m_h|} + \frac{\nabla \tilde{W}_n(t_N; f, g)(x_N)}{|x_N - m_h|}
\]
as \( N \) goes to infinity, which contradicts to the non-negativity of \( h \). \( \square \)
Corollary 4.6 We use Notation 4.1.

1. There exists a time \( T \geq T_0(\varphi) \) such that, for any \( t \geq T \), the intersection \( C(\mu(\cdot, t)) \cap (CS(h) + \varphi(t)B^n) \) is contained in the convex hull of the support of \( h \).

2. We have

\[
\sup \| x - m_h \|_{x \in C(\mu(\cdot, t)) \cap (CS(h) + \varphi(t)B^n)} = O\left(\frac{1}{t}\right)
\]

as \( t \) goes infinity.

Proof Corollary 4.4 and Lemma 4.5 guarantee the conclusion. \( \square \)

Lemma 4.7 We use Notation 4.1. There exists a positive constant \( C \) such that, for any \( x \notin CS(h) + \varphi(t)B^n \) and \( t > 0 \), we have

\[
|u(x, t)| \leq C \exp\left(-\frac{\varphi(t)^2}{4t}\right) \left(\|h\|_{L^1} + \|f\|_{L^1} + \|f\|_{W^{s, \infty}} + \|g\|_{W^{s, \infty}}\right).
\]

Proof We give a proof for even dimensional cases. The other cases go parallel.

From Proposition 2.8, we have

\[
u(x, t) = J_n(t)h(x) + \tilde{J}_n(t)f(x) + e^{-t^2/2}W_n(t; f, g)(x).
\]

Since the function \( k_\ell \) is strictly increasing, for any \( x \notin CS(h) + \varphi(t)B^n \), \( y \in \text{supp} \ h \) and \( t > 0 \), we have

\[
e^{-t^2/2k_\ell^2} \left(\frac{1}{2} \sqrt{t^2 - r^2}\right) \leq e^{-t^2/2k_\ell^2} \left(\frac{1}{2} \sqrt{t^2 - \varphi(t)^2}\right).
\]

Using the asymptotic expansion of \( k_\ell \) in Theorem 2.4, for any \( t > 0 \), we have

\[
e^{-t^2/2k_\ell^2} \left(\frac{1}{2} \sqrt{t^2 - \varphi(t)^2}\right) \leq C \exp\left(-\frac{t + \sqrt{t^2 - \varphi(t)^2}}{2}\right) \leq C \exp\left(-\frac{\varphi(t)^2}{4t}\right).
\]

Hence, from Lemma 3.2, we can take a positive constant \( C \) such that, for any \( x \notin CS(h) + \varphi(t)B^n \) and \( t > 0 \), we have

\[
|J_n(t)h(x)| \leq C \exp\left(-\frac{\varphi(t)^2}{4t}\right) \|h\|_{L^1}.
\]

In the same manner, from Lemma 3.5, we have

\[
\left|\tilde{J}_n(t)f(x)\right| \leq C \exp\left(-\frac{\varphi(t)^2}{4t}\right) \|f\|_{L^1}.
\]

Combining these estimates and Lemma 3.1, we get the conclusion. \( \square \)

Lemma 4.8 We use Notation 4.1. There exist a positive constant \( C \) and a time \( T \geq d_h \) such that, for any \( t \geq T \), we have

\[
\inf_{x \in CS(h)} u(x, t) \geq Ct^{-n/2}\|h\|_{L^\infty}.
\]
Proof We give a proof for even dimensional cases. The other cases go parallel. We remark that, from Lemmas 3.1 and 3.6, we have the following estimates:

\[
|\tilde{W}_n(t; f, g)(x)| \leq C(1 + t)^n \left( \|f\|_{W^{n/2, \infty}} + \|g\|_{W^{n/2-1, \infty}} \right),
\]

\[
|\tilde{J}_n(t) f(x)| \leq C(1 + t)^{-n/2-1} \|f\|_{L^1}.
\]

Let us estimate the function \(J_n(t) h(x)\). Since \(h\) is non-negative, there is a point \(\eta \in CS(h)\) such that, for any \(y \in B^n_{p}(\eta)\), \(h(y) \geq \|h\|_{L^{\infty}}/2\). Using the asymptotic expansion of \(k_{n/2}(s)\) in Theorem 2.4, we have

\[
J_n(t) h(x) \geq \frac{c_n \|h\|_{L^{\infty}}}{2^{n-1}} e^{-t/2} \int_{B^n_{p}(\eta)} k_n \left( \frac{1}{2} \sqrt{1 - r^2} \right) dy
\]

\[
= \frac{c_n \|h\|_{L^{\infty}}}{2^{n-2}} \int_{B^n_{p}(\eta)} t^{-n/2} \left( 1 + O \left( \frac{1}{t^2} \right) \right) \left( 1 + O \left( \frac{1}{t} \right) \right) \left( 1 + O \left( \frac{1}{t} \right) \right) dy
\]

\[
\geq Ct^{-n/2} \|h\|_{L^{\infty}}
\]

for any sufficiently large \(t\).

Hence, for any sufficiently large \(t\), we obtain

\[
|u(x, t)| \geq |J_n(t) h(x)| - |\tilde{J}_n(t) f(x)| - e^{-t/2} |\tilde{W}_n(t; f, g)(x)|
\]

\[
\geq Ct^{-n/2} \|h\|_{L^{\infty}} - Ct^{-n/2-1} \|f\|_{L^1}
\]

\[
- Ce^{-t/2} t^n \left( \|f\|_{W^{n/2, \infty}} + \|g\|_{W^{n/2-1, \infty}} \right)
\]

\[
\geq Ct^{-n/2} \|h\|_{L^{\infty}},
\]

which completes the proof.

\[\square\]

Corollary 4.9 We use Notation 4.1. Suppose that \(\exp(-\varphi(t)^2/(4t))\) is of small order of \(t^{-n/2}\) as \(t\) goes infinity. There exists a constant \(T \geq T_0(\varphi)\) such that, for any \(t \geq T\), all of the time-delayed hot spots are contained in the parallel body \(CS(h) + \varphi(t) B^n\).

Theorem 4.10 Let \(f\) and \(g\) be as in \((fg)\) in Notation 4.1.

1. There exists a time \(T \geq T_0(0)\) such that, for any \(t \geq T\), all of the time-delayed hot spots at time \(t\) are contained in the convex hull of \(h\).

2. We have

\[
\sup \{|x - m_h| : |x| \in \mathcal{H}(t)\} = O \left( \frac{1}{t} \right)
\]

as \(t\) goes to infinity.

Proof We take a function \(\varphi\) as in Notation 4.1 (\(\varphi\)) such that \(\exp(-\varphi(t)^2/(4t))\) is of small order of \(t^{-n/2}\) as \(t\) goes to infinity. Then, Corollary 4.9 guarantees that all of the time-delayed hot spots are contained in the parallel body \(CS(h) + \varphi(t) B^n\) after a large time. Hence, Corollary 4.6 implies the conclusion.

\[\square\]

Remark 4.11 Let \(g\) be a non-zero non-negative smooth function with compact support. If \(n = 1\) and \(f = 0\), then, for any \(t \geq 0\), all of the time-delayed hot spots are contained in the convex hull of \(\text{supp } h = \text{supp } g\). In other words, in this case, we can take \(T = 0\) in the first assertion of Theorem 4.10.
**Proof** Fix a point $x$ in the complement of the convex hull of $\text{supp} \, g$. Let $x'$ be the point that gives the distance between $x$ and the convex hull of $\text{supp} \, g$. We have $|x - y| > |x' - y|$ for any $y$ in $\text{supp} \, g$, and $[x - t, x' + t] \cap \text{supp} \, g$ is contained in $[x', x' + t] \cap \text{supp} \, g$. Hence the strictly increasing behavior of $I_0$ implies

$$S_1(t) g(x) = \frac{e^{-t/2}}{2} \int_{x-t}^{x+t} I_0 \left( \frac{1}{2} \sqrt{t^2 - |x - y|^2} \right) g(y) dy$$

$$\leq \frac{e^{-t/2}}{2} \int_{x'-t}^{x'+t} I_0 \left( \frac{1}{2} \sqrt{t^2 - |x - y|^2} \right) g(y) dy$$

$$< \frac{e^{-t/2}}{2} \int_{x'-t}^{x'+t} I_0 \left( \frac{1}{2} \sqrt{t^2 - |x' - y|^2} \right) g(y) dy$$

$$= S_1(t) g(x'),$$

which completes the proof. \qed

### 4.2 Uniqueness of a time-delayed hot spot

In [14], Jimbo and Sakaguchi showed that the set of hot spots $H_g(t)$ consists of one point for sufficiently large $t$. For the damped wave equation, let us show the corresponding result to [14].

**Lemma 4.12** We use Notation 4.1. There exists a time $T \geq T_0(0)$ such that, for any $t \geq T$, the function $u(\cdot, t)$ becomes strictly concave on the convex hull of the support of $h$.

**Proof** Let us give a proof for even dimensional cases. The other cases go parallel.

In view of Proposition 2.8, we estimate the second derivatives of $J_n(t)h$, $\tilde{J}_n(t)f$ and $e^{-t/2} \tilde{W}_n(t; f, g)$. We fix a point $x \in CS(h)$ and a direction $\omega \in S^{n-1}$.

From the asymptotic expansion of $k_t$ in Theorem 2.4, there exists a positive constant $C$ such that, for any $y \in \text{supp} \, h$ and $t \geq d_h$, we have

$$e^{-t/2} \left[ -4k_{2+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) + k_{2+2} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) (\omega \cdot (x - y))^2 \right]$$

$$= \frac{2^{n/2+2}}{t^{2 - r^2} \sqrt{n/4+1/2}} \exp \left( \frac{-t + \sqrt{t^2 - r^2}}{2} \right) \left( 1 + O \left( \frac{1}{t} \right) \right) \leq -C t^{-n/2-1}.$$  

Hence, from Lemma 3.2, there exists a positive constant $C$ such that, for any $x \in CS(h)$, $\omega \in S^{n-1}$ and $t \geq d_h$, we have

$$(\omega \cdot \nabla)^2 J_n(t)h(x) \leq -C t^{-n/2-1} \|h\|_{L^1}.$$  

In the same manner, from Lemma 3.5, we can obtain the existence of a positive constant $C$ such that, for any $x \in CS(h) \subset CS(f) + \delta(CS(f), CS(h))B^n$, $\omega \in S^{n-1}$ and $t \geq \delta(CS(f), CS(h)) + d_f$, we have

$$| (\omega \cdot \nabla)^2 \tilde{J}_n(t)f(x) | \leq C t^{-n/2-2} \|f\|_{L^1}.$$  

On the other hand, from Lemma 3.1, there exists a positive constant $C$ such that, for any $x \in CS(h)$, $\omega \in S^{n-1}$ and $t > 0$, we have

$$e^{-t/2} \left| (\omega \cdot \nabla)^2 \tilde{W}_n(t; f, g)(x) \right| \leq C e^{-t/2} (1 + t)^n \left( \|f\|_{W^{2,2} \infty} + \|g\|_{W^{n/2+1, \infty}} \right).$$  

Hence we obtain the strict concavity of $u(\cdot, t)$ on $CS(h)$ for any sufficiently large $t$. \qed
Proposition 4.13 We use Notation 4.1. There exists a time $T \geq T_0(\varphi)$ such that, for any $t \geq T$, the set of critical points of $u(\cdot, t)$ contained in the parallel body $CS(h) + \varphi(t)B^n$ consists of one-point.

Proof From Corollary 4.6, it is sufficient to show the strict concavity of the function $u(\cdot, t)$ on the convex hull of the support of $h$. Hence Lemma 4.12 guarantees the uniqueness of a critical point of $u(\cdot, t)$.

4.3 Wave effect of the damped wave in view of time-delayed hot spots

In this subsection, we investigate the wave properties of the damped wave equation in view of the movement of time-delayed hot spots. We give some examples of initial data $(f, g)$ which allow time-delayed hot spots to escape from the convex hull of the support of $h := f + g$ for some small time.

Example 4.14 Let $n = 1$. We consider the Eq. (1.1) with $g = 0$. By Example 2.6 and Proposition 2.8, we have

$$u(x, t) = S_1(t)f(x) + \tilde{J}_1(t)f(x) + e^{-t/2}\tilde{W}_n(t)f(x)$$

$$= \frac{e^{-t/2}}{4} \int_{x-t}^{x+t} \left( \frac{t}{\sqrt{t^2 - r^2}} I_1 \left( \frac{1}{2} \sqrt{t^2 - r^2} + I_0 \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right) f(y)dy \right. + \frac{e^{-t/2}}{2} (f(x + t) + f(x - t))$$

Let us give an example such that if the initial datum $f$ has a sufficiently large maximum value and a constant $L^1$ norm, then, for some $t$, time-delayed hot spots escape from the convex hull of the support of $f$.

Let $\rho$ be a non-negative smooth function such that $\text{supp} \rho = [-2, 2]$, $\|\rho\|_{L^1} = 1$ and $\rho(y) \geq \|\rho\|_{L^\infty}/2$ for $-1 \leq y \leq 1$. For example, if we normalize the function

$$\tilde{\rho}(y) = \begin{cases} \exp \left( -\frac{1}{4 - y^2} \right) & (-2 \leq y \leq 2), \\ 0 & \text{(otherwise),} \end{cases}$$

then the normalized function $\tilde{\rho}/\|\tilde{\rho}\|_{L^1}$ satisfies the conditions of $\rho$. We define $f_\varepsilon(y) = \rho(y/\varepsilon)/\varepsilon$ with a small parameter $\varepsilon > 0$. Then we have $f_\varepsilon(x + t) \geq \|\rho\|_{L^\infty}/(2\varepsilon)$ for $-t - \varepsilon \leq x \leq -t + \varepsilon$ and $f_\varepsilon(x - t) \geq \|\rho\|_{L^\infty}/(2\varepsilon)$ for $t - \varepsilon \leq x \leq t + \varepsilon$.

On the other hand, noting $\|f_\varepsilon\|_{L^1} = 1$, we can choose a constant $C$ independent of $\varepsilon$ such that we have

$$\|S_1(t)f + \tilde{J}_1(t)f\|_{L^\infty} \leq Ce^{-t/2} \left( I_0 \left( \frac{t}{2} \right) + (1 + t) \left( 1 + I_1 \left( \frac{t}{2} \right) \right) \right).$$

Using the facts $I_0(0) = 1$ and $I_1(0) = 0$, we can take the parameter $\varepsilon$ sufficiently small so that there is a time $t \geq 4\varepsilon$ satisfying the inequality

$$\frac{e^{-t/2}}{2} \frac{\|f_\varepsilon\|_{L^\infty}}{2} = \frac{e^{-t/2}}{2} \frac{\|\rho\|_{L^\infty}}{2\varepsilon} > Ce^{-t/2} \left( I_0 \left( \frac{t}{2} \right) + (1 + t) \left( 1 + I_1 \left( \frac{t}{2} \right) \right) \right).$$

Hence if $t \geq 4\varepsilon$ satisfies the above inequality and $x \in [t - \varepsilon, t + \varepsilon] \cup [-t - \varepsilon, -t + \varepsilon]$ then we have $u(x, t) > u(\xi, t)$ for any $\xi \in \text{supp} f_\varepsilon$. 

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Example 4.15 Let \( n = 2 \). We consider the damped wave Eq. (1.1) with \( f = 0 \). By Proposition 2.1, we have
\[
u(x, t) = S_2(t)g(x) = \frac{e^{-t/2}}{2\pi} \int_{B_2^2(x)} \cosh\left(\frac{1}{2}\sqrt{r^2 - r'^2}\right) g(y) dy.
\]
Let us show that, if we choose a clever initial datum \( g \), then, for some \( t \), \( S_2(t)g \) has a (non-trivial) critical point in the complement of the convex hull of \( \text{supp} \ g \).

Let \( s_* \) be the unique critical point of the function \( \cosh(s/2)/s \). Direct computation shows \( 2 < s_* < 3 \). Fix a small positive parameter \( \epsilon \) with \( 2\epsilon < s_* \). Then, we have \( s_* < (s_*^2 + 4\epsilon^2)/(4\epsilon) \). Let \( g_\epsilon \) be a non-zero non-negative radially symmetric smooth function with support \( B_\epsilon^2(0) \). Let us show that, for any \( s_* \leq t \leq (s_*^2 + 4\epsilon^2)/(4\epsilon) \), the function \( S_2(t)g \) has a critical point in the complement of the disk of radius \( \sqrt{t^2 - s_*^2 + \epsilon} \) centered at origin.

Fix a point \( x \) with \( |x| = \sqrt{t^2 - s_*^2} + \epsilon \). We remark that, for any \( y \in B_\epsilon^2(0) \), we have the following inequalities:
\[
|x - y| < t, \quad \sqrt{t^2 - |x - y|^2} \leq s_*.
\]
Let
\[
\delta = \frac{t - \sqrt{t^2 - s_*^2} - 2\epsilon}{2\epsilon}.
\]
Then, for the point \( x' := (1 + \delta)x \) and any \( y \in B_\epsilon^2(0) \), we have the following inequalities:
\[
|x' - y| < t, \quad \sqrt{t^2 - |x' - y|^2} \leq s_*.
\]
Therefore, we have
\[
S_2(t)g_\epsilon(x) = \frac{e^{-t/2}}{2\pi} \int_{B_2^2(x)} \cosh\left(\frac{1}{2}\sqrt{r^2 - |x - y|^2}\right) g_\epsilon(y) dy
\]
\[
< \frac{e^{-t/2}}{2\pi} \int_{B_2^2(x')} \cosh\left(\frac{1}{2}\sqrt{r^2 - |x' - y|^2}\right) g_\epsilon(y) dy
\]
\[
= S_2(t)g_\epsilon(x').
\]
Thanks to the compactness of the support of \( S_2(t)g_\epsilon \), for each direction \( \omega \in S^1 \), we get the existence of a maximal point of the function
\[
\left(\sqrt{t^2 - s_*^2 + \epsilon}, t + \epsilon\right) \ni \rho \mapsto S_2(t)g_\epsilon(\rho \omega) \in \mathbb{R}.
\]
Since the function \( g_\epsilon \) is radially symmetric, the function \( S_2(t)g_\epsilon \) so is, and we obtain the existence of a critical point of \( S_2(t)g_\epsilon \) in the complement of the disk of radius \( \sqrt{t^2 - s_*^2 + \epsilon} \) centered at origin.

Example 4.16 Let \( n = 2 \). We consider the damped wave equation (1.1) with \( f = 0 \) again. Let \( g \) be a non-zero non-negative smooth function with compact support. Suppose \( 2d_g < s_* \).

Let us show that, for any \( 2d_g \leq t \leq s_* \), there exists a point \( x \) in the complement of \( CS(g) \) such that, for any point \( \xi \in CS(g) \), we have \( S_2(t)g(\xi) < S_2(t)g(x) \). In other words, if \( g \) has
a small support so that $d_g < s_* / 2$, then, for any $2d_g \leq t \leq s_*$, time-delayed hot spots escape from the convex hull of the support of $g$.

Fix an arbitrary time $2d_g \leq t \leq s_*$. We can choose a point $x \in CS(g)^c$ which satisfies the following conditions:

$$\max_{y \in \text{supp } g} |x - y| = t, \quad \min_{y \in \text{supp } g} |x - y| \geq t - d_g.$$  

For such a point $x$, any $\xi \in CS(g)$ and $y \in \text{supp } g$, the assumption of $t$ implies

$$0 \leq \sqrt{t^2 - |x - y|^2} \leq \sqrt{(2t - d_g)^2} \leq \sqrt{t^2 - d_g^2} \leq \sqrt{t^2 - |\xi - y|^2} \leq t,$$

and the strictly decreasing behavior of $\cosh(s/2)/s$ for $0 < s < s_*$ implies

$$\frac{\cosh \left( \frac{1}{2} \sqrt{t^2 - |x - y|^2} \right)}{\sqrt{t^2 - |x - y|^2}} \geq \frac{\cosh \left( \frac{1}{2} \sqrt{t^2 - |\xi - y|^2} \right)}{\sqrt{t^2 - |\xi - y|^2}}.$$  

Hence we obtain $S_2(t)g(x) > S_2(t)g(\xi)$ for any $\xi \in CS(g)$.

**Example 4.17**  Let $n = 3$. We consider the damped wave equation (1.1) with $f = 0$. By Example 2.7, we have

$$S_3(t)g = J_3(t)g + e^{-t/2}W_3(t)g.$$  

Let us give an example such that if the initial datum $g$ has a sufficiently large maximum value and a constant $L^1$ norm, then, for some $t$, hot spots escape from the convex hull of the support of $g$.

Let $\rho$ be non-zero non-negative smooth function satisfying $\text{supp } \rho = B_3^3(0)$, $\|\rho\|_{L^1} = 1$ and $\rho(y) \geq \|\rho\|_{L^\infty}/2$ on the unit ball $B_3^3$. We define $g_\varepsilon(y) = \rho(y/\varepsilon)/\varepsilon^3$ with a small parameter $\varepsilon > 0$.

If $t > 2\varepsilon$ and $x \in S_t^2(0)$, then we have

$$\sigma_2(S_t^2(x) \cap B_3^3(0)) = \pi \varepsilon^2,$$

and then, we get

$$e^{-t/2}W_3(t)g_\varepsilon(x) = \frac{e^{-t/2}}{4\pi t} \int_{S_t^2(x)} g_\varepsilon(y) d\sigma_2(y) \geq \frac{\varepsilon^2 e^{-t/2}}{8t} \|g_\varepsilon\|_{L^\infty} = \frac{e^{-t/2}}{8\varepsilon t} \|\rho\|_{L^\infty}.$$  

On the other hand, as we will see in (5.7), $J_3(t)g_\varepsilon$ is estimated by

$$\|J_3(t)g_\varepsilon\|_{L^\infty} \leq C(1 + t)^{-3/2} \|g_\varepsilon\|_{L^1} = C(1 + t)^{-3/2},$$

where $C$ is independent of $\varepsilon$.

We can take the parameter $\varepsilon$ sufficiently small so that there is a time $t \geq 4\varepsilon$ satisfying the inequality

$$\frac{e^{-t/2}}{8\varepsilon t} \|\rho\|_{L^\infty} > C(1 + t)^{-3/2}.$$  

If $t \geq 4\varepsilon$ satisfies the above inequality, then, for any $x \in S_t^2(0)$ and $\xi \in \text{supp } g_\varepsilon = B_3^3(0)$, we have

$$S_3(t)g_\varepsilon(x) \geq e^{-t/2}W_3(t)g_\varepsilon(x) > J_3(t)g_\varepsilon(\xi) = S_3(t)g_\varepsilon(\xi),$$

that is, time-delayed hot spots are not in (the convex hull of) the support of $g_\varepsilon$. 

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5 Application of the Nishihara decomposition: $L^p$-$L^q$ estimates

In this section, as an application of Theorem 2.4, we give $L^p$-$L^q$ estimates for the solution of the damped wave Eq. (1.1). In [9, 16, 20], when $n \leq 3$, the following $L^p$-$L^q$ estimates were shown:

$$
\left\| u(\cdot, t) - P_n(t) \left( f + g \right) - e^{-t/2} \tilde{W}_n(t; f, g) \right\|_{L^p} 
\leq C t^{-\frac{\eta}{2} \left( \frac{1}{q} - \frac{1}{p} \right)^{-1}} \left( \| f \|_{L^q} + \| g \|_{L^q} \right),
$$

(5.1)

where $t > 0$ and $1 \leq q \leq p \leq 2$. In [18], when $n \geq 4$, Narazaki showed the following estimates:

$$
\left\| \mathcal{F}^{-1} \left[ (\hat{u}(\cdot, t) - \hat{v}(\cdot, t)) \chi \right] \right\|_{L^p} 
\leq C \left( 1 + t^{-\frac{\eta}{2} \left( \frac{1}{q} - \frac{1}{p} \right)^{-1+\varepsilon}} \left( \| f \|_{L^q} + \| g \|_{L^q} \right) \right),
$$

(5.2)

where $1 \leq q \leq p \leq \infty$, $\varepsilon$ is an arbitrary small positive number, $C = C(n, p, q, \varepsilon)$ is a positive constant, $\chi$ is a compactly supported radially symmetric smooth function satisfying $\chi = 1$ near the origin, $v(x, t) = P_n(t)(f + g)(x)$, $\hat{u}$ and $\hat{v}$ denote the Fourier transform of $u$ and $v$, respectively, and $\mathcal{F}^{-1}$ is the inverse Fourier transform. Moreover, in the case where $1 \leq q \leq p \leq \infty$, $(p, q) = (2, 2)$ or $(p, q) = (\infty, 1)$, we may take $\varepsilon = 0$. Furthermore, we have

$$
\left\| \mathcal{F}^{-1} \left[ (1 - \chi) \left( \hat{u}(\cdot, t) - e^{-t/2} \left( M_0(\cdot, t) \hat{f}(\cdot, t) + M_1(\cdot, t) \hat{g}(\cdot, t) \right) \right) \right] \right\|_{L^p} 
\leq C e^{-\delta t} \| g \|_{L^q}
$$

(5.3)

for some $\delta > 0$, where $1 \leq q \leq p \leq \infty$, $C = C(n, p, q)$ is a positive constant, and

$$
M_1(\xi, t) = \frac{1}{\sqrt{|\xi|^2 - 1/4}} \left( \sin (t|\xi|) \sum_{0 \leq k < (n-1)/4} \frac{(-1)^k}{(2k)!} t^{2k} \Theta(\xi)^{2k} \right.
- \cos (t|\xi|) \sum_{0 \leq k < (n-3)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Theta(\xi)^{2k+1} \bigg),
$$

(5.4)

$$
M_0(\xi, t) = \cos (t|\xi|) \sum_{0 \leq k < (n+1)/4} \frac{(-1)^k}{(2k)!} t^{2k} \Theta(\xi)^{2k}
+ \sin (t|\xi|) \sum_{0 \leq k < (n-1)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Theta(\xi)^{2k+1} + \frac{1}{2} M_1(\xi, t)
$$

(5.5)

with $\Theta(\xi) = |\xi| - \sqrt{|\xi|^2 - 1/4}$.

The aim of this section is to remove the $\varepsilon$ in the estimate (5.2) and the restriction $q \neq 1$ and $p \neq \infty$.

**Theorem 5.1** Let $1 \leq q \leq p \leq \infty$. Assume that the initial data $f$ and $g$ are $L^q$-integrable smooth functions. Let $u$ be the solution to (1.1). There exists a positive constant $C$ such that, for any $t > 0$, we have

$$
\left\| u(\cdot, t) - P_n(t) \left( f + g \right) - e^{-t/2} \tilde{W}_n(t; f, g) \right\|_{L^p} 
\leq C t^{-\frac{\eta}{2} \left( \frac{1}{q} - \frac{1}{p} \right)^{-1}} \left( \| f \|_{L^q} + \| g \|_{L^q} \right).
$$
The proof of this theorem is almost same as in [20] (see also [10]). We note that the one-dimensional case has already been proved by Marcati and Nishihara in [16], and we give a proof only for the case \( n \geq 2 \). By Proposition 2.8, we have

\[
\begin{align*}
  u(\cdot, t) - P_n(t) (f + g) - e^{-t/2} \tilde{W}_n(t; f, g) \\
  = J_n(t) (f + g) - P_n(t) (f + g) + \tilde{J}_n(t)f.
\end{align*}
\] (5.6)

Therefore, the proof is reduced to the following estimates:

**Lemma 5.2** Let \( 1 \leq q \leq p \leq \infty \), and \( g \) an \( L^q \)-integrable smooth function. There exists a constant \( C > 0 \) such that, for any \( t > 0 \), we have the following inequalities:

\[
\begin{align*}
  \| J_n(t)g \|_{L^p} &\leq C (1 + t)^{-\frac{q}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \| g \|_{L^q}, \quad (5.7) \\
  \| \tilde{J}_n(t)g \|_{L^p} &\leq C (1 + t)^{-\frac{q}{2} \left( \frac{1}{q} - \frac{1}{p} \right)-1} \| g \|_{L^q}, \quad (5.8) \\
  \| J_n(t)g - P_n(t)g \|_{L^p} &\leq C t^{-\frac{q}{2} \left( \frac{1}{q} - \frac{1}{p} \right)-1} \| g \|_{L^q}. \quad (5.9)
\end{align*}
\]

**Proof** Let us give a proof for higher odd dimensional cases. Even dimensional cases go parallel.

We first show (5.7) and (5.9). We assume \( t \geq 1 \) and write \( \tilde{c}_n = 2^{-(n-1)}c_n \). For a constant \( 0 < \varepsilon < 1/2 \), put

\[
X_1 = \int_{(1+\varepsilon)^{1/2}B^n(x)} \left( c_ne^{-t/2k_{n-1}} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) - e^{-r^2/(4t)} \right) g(y)dy,
\]

\[
X_2 = \int_{B^n(x) \setminus (1+\varepsilon)^{1/2}B^n(x)} \left( c_ne^{-t/2k_{n-1}} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) - e^{-r^2/(4t)} \right) g(y)dy,
\]

\[
X_3 = \int_{B^n(x)^c} e^{-r^2/(4t)} g(y)dy.
\]

Then we have

\[
J_n(t)g(x) - P_n(t)g(x) = X_1 + X_2 + X_3.
\]

By the Hausdorff–Young inequality ([8, p.142]), we estimate the integral \( X_3 \) as

\[
\| X_3 \|_{L^p} \leq \left( \int_{B^n(0)^c} e^{-\rho \| y \|^2/(4t)} \frac{dy}{(4\pi t)^n/2} \right)^{1/p} \| g \|_{L^q} \leq e^{-t/8} \| g \|_{L^q},
\]

where \( \rho \) is determined by the relation \( 1/q - 1/p = 1 - 1/\rho \).

In the same manner as in the above estimate, we can obtain

\[
\| X_2 \|_{L^p} \leq e^{-ct^\varepsilon} \| g \|_{L^q}
\]

with some constant \( c > 0 \).

Let us estimate the integral \( X_1 \). By the asymptotic expansion in Theorem 2.4, we have

\[
\tilde{c}_n e^{-t/2k_{n-1}} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) = \frac{1}{(4\pi t)^{n/2}} \left( \frac{1}{t^2 - r^2} \right)^{n/4} \exp \left( -t + \frac{t^2 - r^2}{2} \right) \times \left( 1 - \frac{n(n-2)}{4\sqrt{t^2 - r^2}} + O \left( \frac{1}{t^2 - r^2} \right) \right).
\]
Therefore, we obtain
\[
X_1 = \frac{1}{(4\pi t)^{n/2}} \int_{B^n(x)} e^{-r^2/(4t)} F(r, t) g(y) dy,
\]
where
\[
F(r, t) = \exp \left( \frac{r^2}{4t} + \frac{-t + \sqrt{t^2 - r^2}}{2} \right) \left( \frac{t}{\sqrt{t^2 - r^2}} \right)^{n/2} \times \left( 1 - \frac{n(n-2)}{4\sqrt{t^2 - r^2}} + O \left( \frac{1}{t} \right) \right) - 1.
\]
Hence we have
\[
\|X_1\|_{L^p} \leq C \left( \int_{B^n(x)} e^{-\rho|y|^2/(4t)} |F(|y|, t)|^\rho dy \right)^{1/\rho} \|g\|_{L^q}
\]
with \(1/q - 1/p = 1 - 1/\rho\). Asymptotic expansions (5.11) and (5.13) imply
\[
F(|y|, t) = \left( 1 + \frac{1}{t} O \left( \frac{|y|^2}{t^2} \right) \right) \left( 1 + \frac{1}{t} O \left( \frac{|y|^2}{t} \right) \right)^{n/2} \left( 1 + O \left( \frac{1}{t} \right) + \frac{1}{t} O \left( \frac{|y|^2}{t} \right) \right) - 1
\]
for some large integer \(N\). Consequently, we obtain
\[
\|X_1\|_{L^p} \leq C \left\| g \right\|_{L^q} \left( \int_{B^n(x)} e^{-\rho|y|^2/(4t)} \left( 1 + \frac{|y|^2}{t} + \cdots + \left( \frac{|y|^2}{t} \right)^N \right)^\rho \right) \left( \int_{\mathbb{R}^n} e^{-\rho|z|^2} \left( 1 + |z|^2 + \cdots + |z|^{2N} \right)^\rho \right) \left( \int_{\mathbb{R}^n} e^{-\rho|z|^2} \left( 1 + |z|^2 + \cdots + |z|^{2N} \right)^\rho \right) \leq C \|g\|_{L^q} t^{-\frac{n}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} - 1,
\]
which implies the estimate (5.9) for \(t \geq 1\). Moreover, we recall the well-known fact
\[
\|P_n(t)g\|_{L^p} = C t^{-\frac{n}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \|g\|_{L^q}, \quad t > 0
\]
(see [8, p.8]). Using this fact, we have
\[
\|J_n(t)g\|_{L^p} \leq \|J_n(t)g - P_n(t)g\|_{L^p} + \|P_n(t)g\|_{L^p} \leq C t^{-\frac{n}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \|g\|_{L^q},
\]
which implies (5.7) for \(t \geq 1\). The estimates (5.7) and (5.9) for \(0 \leq t < 1\) are easy, and we omit the proof.

Next, we show the estimate (5.8). We assume \(t \geq 1\). For a constant \(0 < \varepsilon < 1/2\), put
\[
X_4 = \tilde{c}_n \int_{B^n_t(x) \cap t^{1+\varepsilon}/2 B^n(x)} \frac{\partial}{\partial t} \left( e^{-t/2 k_{n-1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right)} \right) g(y) dy,
\]
\[
X_5 = \tilde{c}_n \int_{t^{1+\varepsilon}/2 B^n(x)} \frac{\partial}{\partial t} \left( e^{-t/2 k_{n-1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right)} \right) g(y) dy.
\]
Then we have
\[
\tilde{J}_n(t)g(x) = X_4 + X_5.
\]
Since \( k_{\ell+1}(s) = k'_{\ell}(s)/s \) leads to
\[
\frac{\partial}{\partial t} \left( e^{-t/2} k_{\ell+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right)
= e^{-t/2} \left[ -\frac{1}{2} k_{\ell+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) + \frac{t}{2 \sqrt{t^2 - r^2}} k'_{\ell+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right]
= e^{-t/2} \left[ -\frac{1}{2} k_{\ell+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) + \frac{t}{4} k'_{\ell+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right],
\]
in the same manner as the estimate of \( X_2 \), we can obtain
\[
\| X_4 \|_{L^p} \leq C e^{-ct} \| g \|_{L^q}
\]
with some constant \( c > 0 \).

Now we turn to the estimate for \( X_5 \). By using the asymptotic expansion of \( k_{\ell} \), (5.11) and (5.12) again, we have
\[
\frac{\partial}{\partial t} \left( e^{-t/2} k_{\ell+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right)
= \frac{2^{(n-3)/2}}{\sqrt{\pi}} e^{-r^2/(4t)} \exp \left( \frac{r^2}{4t} + \frac{-t + \sqrt{t^2 - r^2}}{2} \right)
\times (t^2 - r^2)^{-n/4} \left( \frac{t}{\sqrt{t^2 - r^2}} - 1 \right) \left( 1 + O \left( \frac{1}{t} \right) \right)
= \frac{2^{(n-3)/2}}{\sqrt{\pi}} e^{-r^2/(4t)}
\times \left( 1 + \frac{1}{t} O \left( \frac{r^2}{t} \right) \right) t^{-n/2} \left( 1 + \frac{1}{t} O \left( \frac{r^2}{t} \right) \right)^{n/2} \frac{1}{t} O \left( \frac{r^2}{t} \right) \left( 1 + O \left( \frac{1}{t} \right) \right)
\leq C t^{-n/2-1} e^{-r^2/(4t)} \left( 1 + \frac{r^2}{t} + \cdots + \left( \frac{r^2}{t} \right)^N \right)
\]
on the ball \( t^{(1+\epsilon)/2} B^n(x) \) with some large integer \( N \). Consequently, we obtain
\[
\| X_5 \|_{L^p} \leq \frac{C \| g \|_{L^q}}{t^{n/2+1}} \left( \int_{r(t^{1+\epsilon}/2) B^n} e^{-\rho|y|^2/(4t)} \left( 1 + \frac{|y|^2}{t} + \cdots + \left( \frac{|y|^2}{t} \right)^N \right)^{\rho} dy \right)^{1/\rho}
\leq C \| g \|_{L^q} t^{-\frac{2q}{p} - \frac{1}{q} \left( \frac{1}{2} - \frac{1}{p} \right)} \left( \int_{\mathbb{R}^n} e^{-\rho|z|^2/4} \left( 1 + |z|^2 + \cdots + |z|^{2N} \right)^{\rho} dz \right)^{1/\rho}
\leq C \| g \|_{L^q} t^{-\frac{q}{p} \left( \frac{1}{2} - \frac{1}{p} \right)},
\]
with \( 1/q - 1/p = 1 - 1/\rho \), which implies (5.8) for \( t \geq 1 \).

The estimate (5.8) for \( 0 \leq t < 1 \) is easy, and we omit the proof.

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Appendix

Proofs of preliminary estimates

Proof of Lemma 3.1 Let us give a proof for even dimensional cases. The other cases go parallel.

Changing the variable as \( y = x + tz \) with \( z \in B^n \), we have

\[
\int_{B^n_t(x)} \frac{1}{\sqrt{t^2 - r^2}} g(y) dy = t^{n-1} \int_{B^n} \frac{1}{\sqrt{1 - |z|^2}} g(x + tz) dz.
\]

When we estimate the function

\[
\hat{W}_n(t) g(x) = 2c_n \sum_{j=0}^{(n-2)/2} \frac{1}{8^j j!} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2-j} \left( t^{n-1} \int_{B^n} \frac{1}{\sqrt{1 - |z|^2}} g(x + tz) dz \right),
\]

the worst term with respect to the growth order of \( t \) is given by \( j = (n-2)/2 \). We can bound it above by \( C(1 + t)^{n-1} \|g\|_L^\infty \). Furthermore, we can bound the term of \( j = 0 \) above by \( C(1 + t)^{n/2} \|g\|_{W^{n/2-1, \infty}} \). The other terms are bounded above by these two quantities (up to a constant multiple). Hence we obtain the estimate for \( \hat{W}_n(t) g(x) \).

From Proposition 2.8, we have

\[
\hat{W}_n(t) f(x) = \frac{c_n t^n}{2^{3n-2} \pi^{n/2}} \int_{B^n} \frac{1}{\sqrt{1 - |z|^2}} f(x + tz) dz,
\]

which implies the estimate for \( \hat{W}_n(t) f(x) \). Also, we have

\[
\hat{W}_n(t; f, g)(x) = \frac{1}{2} W_n(t) f(x) + W_n(t) g(x) + \hat{W}_n(t) f(x) + \frac{\partial}{\partial t} W_n(t) f(x).
\]

Combining the above estimates for \( W_n(t) g(x) \) and \( \hat{W}_n(t) f(x) \), we obtain the conclusion. \(\square\)

Proof of Lemma 3.2 Using Remark 2.5, integration by parts implies the identities. \(\square\)

Proof of Lemma 3.3 If \( \text{dist}(x, CS(h)) \leq t - d_h \) and \( t \geq d_h \), then the intersection \( S_{t}^{n-1}(x) \cap CS(h) \) is a null set with respect to the \( (n-1) \)-dimensional spherical Lebesgue measure. Hence Lemma 3.2 guarantees the conclusion.

Proof of Lemma 3.4 1. We give a proof for even dimensional cases. The other cases go parallel.

We remark that, from the definition of \( c_n \) (2.3), we have

\[
E_n(r, t) = \frac{e^{-t^2/2}}{2^{3n/2+1} \pi^{n/2}} k_{n+1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right).
\]

From Theorem 2.4, we have

\[
k_{n+1} \left( \frac{1}{2} \sqrt{t^2 - \varphi(t)^2} \right) = \frac{1}{2} \left( \frac{2}{\sqrt{t^2 - \varphi(t)^2}} \right)^{n/2+1} \exp \left( \frac{\sqrt{t^2 - \varphi(t)^2}}{2} \right) \left( 1 + O \left( \frac{1}{\sqrt{t^2 - \varphi(t)^2}} \right) \right)
\]

as \( t \) goes to infinity, which implies the conclusion.
2. Applying the fact (5.11) to the first assertion, we obtain the conclusion.
3. Applying the fact (5.12) to the second assertion, we obtain the conclusion.

**Proof of Lemma 3.5** Using Remark 2.5, integration by parts implies the identities.

**Proof of Lemma 3.6**
1. This is a direct consequence of (5.8) in Lemma 5.2.
2. We give a proof for even dimensional cases. The other cases go parallel. From the asymptotic expansion in Theorem 2.4, we have

\[ e^{-t/2} \left( tk_{\frac{n}{2} + 2} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) - 2k_{\frac{n}{2} + 1} \left( \frac{1}{2} \sqrt{t^2 - r^2} \right) \right) \]
\[ = \frac{2^{n/2+1}}{(t^2 - r^2)^{n/4+1/2}} \exp \left( -t + \frac{\sqrt{t^2 - r^2}}{2} \right) \]
\[ \times \left[ \frac{t}{\sqrt{t^2 - r^2}} \left( 1 + O \left( \frac{1}{\sqrt{t^2 - r^2}} \right) \right) - \left( 1 + O \left( \frac{1}{\sqrt{t^2 - r^2}} \right) \right) \right] . \]

Using the facts (5.11) and (5.12), the above expansion coincides with

\[ 2^{n/2+1} t^{-n/2-1} \left( O \left( \frac{1}{t} \right) + O \left( \frac{\psi(t)^2}{t^2} \right) \right) , \]

which implies the conclusion.
3. Using the asymptotic expansion in Theorem 2.4, in the same manner as in the second assertion, we obtain the conclusion.

\[ \square \]

**Frequently used Taylor’s expansions**

Let us list up frequently used Taylor’s expansions:

As \( s \) tends to zero, we have

\[ (1 - s^2)^\alpha = 1 - \alpha s^2 + \frac{\alpha(\alpha - 1)}{2} s^4 + O \left( s^6 \right). \] (5.10)

If a function \( \varphi(t) \) is of small order of \( t \) as \( t \) goes to infinity, then we have

\[ (t^2 - \varphi(t)^2)^\alpha \]
\[ = t^{2\alpha} \left( 1 - \alpha \left( \frac{\varphi(t)}{t} \right)^2 + \frac{\alpha(\alpha - 1)}{2} \left( \frac{\varphi(t)}{t} \right)^4 + O \left( \left( \frac{\varphi(t)}{t} \right)^6 \right) \right) \] (5.11)

as \( t \) goes to infinity.

If a function \( \varphi(t) \) is of small order of \( \sqrt{t} \) as \( t \) goes to infinity, then, as \( t \) goes to infinity, we have the following expansions:

\[ \exp \left( \frac{-t + \sqrt{t^2 - \varphi(t)^2}}{2} \right) = 1 - \frac{\varphi(t)^2}{4t} + O \left( \frac{\varphi(t)^4}{t^2} \right) , \] (5.12)
\[ \exp \left( \frac{\varphi(t)^2}{4t} + \frac{-t + \sqrt{t^2 - \varphi(t)^2}}{2} \right) = 1 + \frac{1}{t} O \left( \frac{\varphi(t)^4}{t^2} \right) . \] (5.13)
Properties of modified Bessel functions

In this section, we collect some properties of the modified Bessel functions

\[ I_\nu(s) = \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(j + \nu + 1)} \left( \frac{s}{2} \right)^{2j+\nu} \]  

(5.14)

used in this paper from [19]:

For a positive constant \( a \), we have

\[ \int_{-a}^{a} \frac{e^{s/2}}{\sqrt{a^2 - s^2}} ds = \pi I_0 \left( \frac{a}{2} \right). \]  

(5.15)

Direct computation shows the following recursion:

\[ I_0'(s) = I_1(s), \quad I_1'(s) = I_0(s) - \frac{1}{s} I_1(s), \quad \frac{d}{ds} \left( \frac{I_\ell(s)}{s^{\ell+1}} \right) = \frac{I_{\ell+1}(s)}{s^{\ell+1}}. \]  

(5.16)

The modified Bessel function \( I_\nu(s) \) has the expansion

\[ I_\nu(s) = \frac{e^s}{\sqrt{2\pi s}} \left( 1 - \frac{(\nu - 1/2)(\nu + 1/2)}{2s} \right. \]
\[ + \frac{(\nu - 1/2)(\nu - 3/2)(\nu + 3/2)(\nu + 1/2)}{2! 2^2 s^2} - \ldots \]
\[ + (-1)^\ell \frac{1}{\ell! 2^{\ell-1} s^{\ell}} \prod_{j=1}^{\ell} (\nu - (j - 1/2)) (\nu + (j - 1/2)) + O \left( \frac{1}{s^{\ell+1}} \right) \]  

(5.17)

as \( s \) goes to infinity.

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