Analysis of the limiting behavior of a biological neurons system with delay

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Abstract. In this work an analytical and numerical analysis of the limiting behaviors of a system consisting of a pair of biological neurons was carried out. In this case connection between neurons will occur with a delay. As a neuron model, the FitzHugh-Nagumo model was chosen as a model that can reproduce many dynamic behaviors of a real neuron and, at the same time, is not very complex computationally.

1. Introduction
For different neurons and in different external conditions, the time for impulse transmission can differ significantly. In this case the behavior of the entire network as a whole can significantly depend on the strength of the connection and the delay time during the transmission of the impulse. These parameters correspond to the properties of the synapse and the speed of excitation propagation along the axon (see, e.g., [1]). By identifying dynamic behavior and controlling the dynamic behavior of the network, it is possible to identify and predict unwanted behaviors, as well as to restore the normal rhythms of the network’s functioning.

To describe a biological neuron, one of the most complete models at the moment is the Hodgkin-Huxley model [2]. Later many other models were proposed to simplify the Hodgkin-Huxley model. One of these models was the FitzHugh-Nagumo model [3, 4], which retains the basic types of neuron behavior, but is much simpler from a computational point of view. In this work this model is used to describe a neuron and an analysis of possible types of behavior of a network of two connected neurons is carried out, depending on the main parameters of the model.

2. Model description
We will consider two neurons connected to each other through a sigmoidal connection (independent of the potential of the postsynaptic neuron) with a delay. Each neuron is described by the FitzHugh-Nagumo model. Then we can write down the final model of the considered system of neurons [5]:

\[
\begin{align*}
\dot{u}_1 &= -a u_1 + (a + 1) u_1^2 - u_1^3 - v_1 + c \tanh(u_2) \\
\dot{v}_1 &= b u_1 - \gamma v_1 \\
\dot{u}_2 &= -a u_2 + (a + 1) u_2^2 - u_2^3 - v_2 + c \tanh(u_1) \\
\dot{v}_2 &= b u_2 - \gamma v_2
\end{align*}
\]
where \( i = 1,2 \), functions \( u_i = u_i(t) \) and \( v_i = v_i(t) \) describe the state of the corresponding neuron at time \( t \), \( a, b, \gamma \) are positive parameters, \( u_i^\tau = u_i(t - \tau) \), \( c \) -- a positive parameter that corresponds to the strength of the connection between neurons, \( \tau \) -- time delay in the transmission of an impulse between two neurons.

3. Analytical results

Equilibrium points are analyzed for the described system. As a result, the number (from 1 to 9) of equilibrium points was determined depending on the parameter values. Also stability areas are determined for the found equilibrium points.

According to the results obtained, if \( a \leq 1 \) and \( 3b \geq \gamma \), then there are no more than 3 equilibrium states in the system (in this case the neurons have the same potential values in the equilibrium state). Otherwise the system can have up to 9 states of equilibrium. In what follows, we will consider only the case \( a \leq 1 \) and \( 3b \geq \gamma \).

Now if \( c > a + \frac{b}{\gamma} \), then the system will have exactly 3 equilibrium states (with a potential greater than 0, equal to 0 and less than 0). If \( c < a + \frac{b}{\gamma} \), then there is one equilibrium state in the system before the critical value of \( c \), and then three (with a potential equal to 0 and two states with a positive potential).

Further the analysis of local bifurcations in the system is carried out. As a result many transitions were found from stable equilibrium points to stable periodic synchronous and asynchronous oscillations.

Series were obtained for the value of the delay, candidates for a change in the type of behavior. These values distinguish stable equilibrium from oscillation.

\[
\begin{align*}
\omega^2 &= \frac{1}{2} \left( -\left( a^2 + \gamma^2 - c^2 - 2b \right) \pm \left( \left( a^2 - \gamma^2 - c^2 \right)^2 - 4b \left( (\alpha + \gamma)^2 - c^2 \right) \right)^{0.5} \right), \\
\tau &= \frac{1}{\omega} \arctan\left( \frac{(a^2 + \gamma^2) + b\omega}{(a^2 + \gamma^2) - b\gamma} \right) + \frac{\pi k}{\omega}, \quad k \in \mathbb{Z}
\end{align*}
\]

where \( \alpha = 3u^2 - 2u(a + 1) + a, c' = c \cosh(u)^{-2}, u \) -- potential of the corresponding state of equilibrium.

For candidates for local bifurcations a criterion was obtained to determine the effect on the number of eigenvalues with positive real parts for the linearized system:

\[
(\omega^2 + \gamma^2) > b^2 + 2b\gamma(a + 1).
\]

Figure 1. \( \{u_1(t)\}, \{u_2(t)\} \) for \( a = 0.15, b = \gamma = 0.02, c = 0.2, \tau = 11, u_1(t, t < 0) = 0.01, u_2(t, t < 0) = 0. \)
4. Numerical result

All analytical results are confirmed numerically. The global bifurcations in the system are also numerically researched. As a result, for this system of a pair of neurons without external influences, such areas of parameter values have been found for which global stability, periodic oscillations, multistability, simultaneous synchronous and asynchronous stable oscillations, and chaotic oscillations are observed in the system.

Below are examples for these types. Global stability (figure 1); periodic oscillations (figure 2); multistability, simultaneous synchronous and asynchronous stable oscillations (figure 3); chaotic oscillations (figure 4).

**Figure 2.** a) $\{u_1(t), u_2(t)\}$ for $a = 0.15, b = \gamma = 0.02, c = 0.2, \tau = 3, u_1(t, t < 0) = 0.01, u_2(t, t < 0) = 0$; b) $\{u_1(t), u_2(t)\}$ for $a = 0.15, b = \gamma = 0.02, c = 1.5, \tau = 10, u_1(t, t < 0) = 0.01, u_2(t, t < 0) = 0$.

**Figure 3.** a) $\{u_1(t), u_2(t), u_1'(t), u_2'(t)\}$ for $a = 0.15, b = \gamma = 0.02, c = 0.5, \tau = 7, u_1(t, t < 0) = 0.5, u_2(t, t < 0) = 0, u_1'(t, t < 0) = 0.4, u_2'(t, t < 0) = 0$; b) $\{u_1(t), u_2(t), 5v_1(t), u_1'(t), u_2'(t), 5v_2'(t)\}$ for $a = 0.15, b = \gamma = 0.02, c = 0.5, \tau = 7, u_1(t, t < 0) = 0.5, u_2(t, t < 0) = 0, u_1'(t, t < 0) = 0.4, u_2'(t, t < 0) = 0$. 
Figure 4. a) \( \{u_1(t)\}, \{u_2(t)\} \) for \( a = 0.15, b = \gamma = 0.02, c = 0.2, \tau = 11.7, u_1(t, t < 0) = 0.01, u_2(t, t < 0) = 0; \) b) \( \{u_1(t), u_2(t), 5v_1(t)\} \) for \( a = 0.15, b = \gamma = 0.02, c = 0.2, \tau = 11.7, u_1(t, t < 0) = 0.01, u_2(t, t < 0) = 0. \)

5. Conclusion
A model of a pair of neurons was built based on the FitzHugh-Nagumo model. Analytical and numerical studies have been carried out for this model. This made it possible to identify the main types of dynamic behavior, as well as criteria for determining the types.

The main controllable parameters in such systems are the strength of the connection and the transmission rate of the impulse or the delay in transmission between neurons. Thus, based on the results obtained, undesirable behaviors in the system can be diagnosed, and then, by influencing the strength of the connection and the delay in transmission, the system can be brought to the desired state.

Acknowledgments
The reported study was funded by RFBR, project number 20-31-90036

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