Adaptive Numerical Simulation of a Phase-field Fracture Model in Mixed Form tested on an L-shaped Specimen with High Poisson Ratios

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Abstract

This work presents a new adaptive approach for the numerical simulation of a phase-field model for fractures in nearly incompressible solids. In order to cope with locking effects, we use a recently proposed mixed form where we have a hydro-static pressure as additional unknown besides the displacement field and the phase-field variable. To fulfill the fracture irreversibility constraint, we consider a formulation as a variational inequality in the phase-field variable. For adaptive mesh refinement, we use a recently developed residual-type a posteriori error estimator for the phase-field variational inequality which is efficient and reliable, and robust with respect to the phase-field regularization parameter. The proposed model and the adaptive error-based refinement strategy are demonstrated by means of numerical tests derived from the L-shaped panel test, originally developed for concrete. Here, the Poisson’s ratio is changed from the standard setting towards the incompressible limit $\nu \rightarrow 0.5$.

Key words. finite elements, phase-field fracture, error estimation, adaptive refinement, mixed system, incompressible solids

1 Introduction

Crack propagation is one of the major research topics in mechanical, energy, and environmental engineering. A well-established variational approach for Griffith’s [5] quasi-static brittle fracture was introduced by Francfort and Marigo [3]. Miehe et al. [10] introduced the name ‘phase-field modeling’ for this variational approach. If the observed solid is assumed to be nearly incompressible, the classical phase-field fracture model fails due to volume-locking. In this work, we combine the mixed problem formulation, recently proposed by the authors in [7], with the adaptive numerical solution based on a residual-type error estimator for the arising phase-field variational inequality [6,11]. This allows to simulate crack propagation on adaptive refined meshes in nearly incompressible materials by using the phase-field method.
2 A Phase-field Model for Nearly Incompressible Solids

2.1 Notation and Spaces

We emanate from a two-dimensional, open and smooth domain \( \Omega \subset \mathbb{R}^2 \). Let \( I \) be a loading interval \([0, T]\), where \( T > 0 \) is the end time value. A displacement function \( u : (\Omega \times I) \rightarrow \mathbb{R}^2 \) is defined on the domain \( \Omega \). On a subset \( \Gamma_D \subset \partial \Omega \) of the boundary, we enforce Dirichlet boundary conditions. For the phase-field variable \( \phi \), hence the irreversibility condition is approximated by

\[
\text{i.e., that } \phi \text{ is monotone non-increasing with respect to } t \in I.
\]

By \( (a, b) := \int_I a \cdot b \, dx \) for vectors \( a, b \), the \( L^2 \) scalar-product is denoted. The Frobenius scalar product of two matrices of the same dimension is defined as \( A : B := \sum_{i,j} a_{ij} b_{ij} \) and therewith the \( L^2 \)-scalar product is given by \( (A, B) := \int_I A : B \, dx \).

For a weak problem formulation, we consider a subdivision \( 0 = t_0 < \ldots < t_N = T \) of the interval \( I \). In each time step, we define approximations \((u^n, \phi^n) = (u(t_n), \phi(t_n))\) and hence the irreversibility condition is approximated by \( \phi^n \leq \phi^{n-1} \) for all \( n = 1, \ldots, N \). To simplify the notation, we omit the superscript \((\cdot)^n\) and set \( u := u^N \) and \( \phi := \phi^N \), whenever it is clear from the context. The phase-field space is \( \mathcal{W} := H^1(\Omega) \) with a feasible set \( \mathcal{K} := \{ \psi \in \mathcal{W} | 0 \leq \psi \leq \psi^N \leq 1 \} \). Further, we define the function spaces \( \mathcal{V} := (H^1_0(\Omega))^2 := \{ \psi \in (H^1(\Omega))^2 | \psi = 0 \text{ a.e. on } \Gamma_D \} \), \( \mathcal{U} := L^2(\Omega) \), and \( \mathcal{X} := \{ \Lambda \in \mathcal{W}^* | \Lambda \geq 0 \} \), where \( \mathcal{W}^* \) is the dual space of \( \mathcal{W} \). Further, let \( u_D \in (H^1(\Omega))^2 \cap C^0(\Gamma_D) \) be a continuation of the Dirichlet-data. For the classical phase-field fracture model, we refer to Miehe et al. \([10]\). In the next section, the mixed form of the phase-field fracture model is formulated.

2.2 Mixed Phase-field Fracture Model

The stress tensor \( \sigma(u) \) is given by \( \sigma(u) := 2\mu E_{\text{lin}}(u) + \lambda \text{tr}(E_{\text{lin}}(u)) \mathbf{I} \) with the Lamé coefficients \( \mu, \lambda > 0 \). The linearized strain tensor therein is defined as \( E_{\text{lin}}(u) := \frac{1}{2}(\nabla u + \nabla u^T) \). By \( \mathbf{I} \), the two-dimensional identity matrix is denoted. For a mixed formulation of the problem, we define

\[
p := A \nabla : u,
\]

with \( p \in \mathcal{U} \), such that the pure elasticity equation reads as follows:

Find \( u \in \mathcal{V} \) and \( p \in \mathcal{U} \) such that

\[
2\mu E_{\text{lin}}(u), E_{\text{lin}}(w)) + (\nabla \cdot w, p) = 0 \quad \forall w \in \mathcal{V},
\]

\[
(\nabla \cdot u, q) - \frac{1}{2} p \langle p, q \rangle = 0 \quad \forall q \in \mathcal{U}.
\]

Following \([7]\), we consider the tensile \( (\sigma^+(u, p)) \) and compressive \( (\sigma^-(u, p)) \) parts of the stress tensor. For this reason, the positive part of the pressure \( p^+ \in L^2(\Omega) \) has to be defined as \( p^+ := \max\{p, 0\} \), and \( E_{\text{lin}}^+(u) \) is given as the projection of \( E_{\text{lin}}(u) \) onto positive semidefinite matrices. Now, we can split the stress tensor \( \sigma(u, p) \) as:

\[
\sigma^+(u, p) := 2\mu E_{\text{lin}}^+(u) + p^+ \mathbf{I},
\]

\[
\sigma^-(u, p) := 2\mu E_{\text{lin}}^-(u) + (p - p^+) \mathbf{I}.
\]

In the following, the critical energy release rate is denoted by \( G_c \), and a degradation function is defined as \( g(\phi) := (1 - \kappa)\phi^2 + \kappa \), with a small regularization parameter \( \kappa > 0 \). Next, we can formulate the mixed phase-field problem in incremental form \([7]\).
Problem 1 (Mixed Phase-field Formulation) Given the initial data $\varphi^{n-1} \in \mathcal{K}$, find $u := u^n \in (u_D + \mathcal{V})$, $p := p^n \in \mathcal{U}$ and $\varphi := \varphi^n \in \mathcal{K}$ for loading steps $n = 1, 2, \ldots, N$ such that

\[
\begin{aligned}
g(\varphi^{n-1})(\sigma^+(u, p), E_{\text{int}}(w)) + (\sigma^-(u, p), E_{\text{int}}(w)) &= 0 \quad \forall w \in \mathcal{V}, \\
(\nabla \cdot u, q) - \frac{1}{2} \lambda(p, q) &= 0 \quad \forall q \in \mathcal{U}, \\
(1 - \kappa)(\varphi^+(u, p) : E_{\text{int}}(w), \psi - \varphi) + G_c(-\epsilon/(1 - \varphi), \psi - \varphi) \\
+ G_c(\nabla \varphi, \nabla(\psi - \varphi)) &\geq 0 \quad \forall \psi \in \mathcal{K} \subset \mathcal{W},
\end{aligned}
\]

where $\epsilon > 0$ describes the bandwidth of the transition zone between broken and unbroken material. This weak formulation in Problem 1 can be reformulated to a complementarity system by introducing a Lagrange multiplier $\Lambda \in \mathcal{X}$, see [6, 7].

The numerical treatment of the phase-field system in a monolithic fashion including the discretization as well as the adaptive refinement strategy are discussed in the following.

2.3 Numerical Treatment and Programming Code

Based on the complementarity formulation of Problem 1 with the help of a Lagrange multiplier, the crack irreversibility constraint is enforced, see [7, Section 4.1]. For the discretization in space, we employ a Galerkin finite element method in each loading step. To this end, the domain $\Omega$ is partitioned into quadrilaterals. To fulfill a discrete inf-sup condition, Taylor-Hood elements with biquadratic shape functions ($Q_2$) for the displacement field $u$ and bilinear shape functions ($Q_1$) for the pressure variable $p$ as well as for the phase-field variable are used. For further details on the stable mixed form of the classical phase-field fracture model as well as the handling of the crack irreversibility condition and the numerical solving steps, we refer to [7].

The overall implementation is done in DOPenLib [2, 3] using the finite element library deal.II [1].

2.4 Adaptive Refinement

A residual-type a posteriori error estimator $\eta$ for the classical phase-field fracture model, presented and tested in [5], provides a robust upper bound. Here, robust means that the unknown constant in the bound does not depend on $\epsilon$ such that the quality of the estimator is independent of $\epsilon$. The mesh adaptation is realized using extracted local error indicators from the a posteriori error estimator in [6, Section 3.2] on the given meshes over all loading steps.

In the following, $\mathcal{M}^n$ denotes the mesh in the incremental step $n$ and $I^n_h$ is the corresponding nodal interpolation operator on the mesh $\mathcal{M}^n$. The searched discrete quantities are denoted by an index $(\cdot)_h$, i.e., the displacement $u^n_h$, the phase-field variable $\varphi^n_h$, the pressure $p^n_h$, and the Lagrange multiplier $\Lambda^n_h$. The adaptive solution strategy is given in the following.

Algorithm 1 Given a partition in time $t_0 < \ldots < t_N$, and an initial mesh $\mathcal{M}^0 = \mathcal{M}$ for all $n = 0, \ldots, N$.

1. Set $\psi^n_h = u^n_h$ and solve the discrete complementarity system to obtain the discrete solutions $u^n_h, \varphi^n_h, p^n_h, \Lambda^n_h$ for all $n = 1, \ldots, N$.
2. Evaluate the error estimator in order to obtain $\eta^n$ for each incremental step.
3. Stop, if $\sum_{n=1}^N (\eta^n)^2$ and $\|p^n_h \varphi^{n-1} - \varphi^{n-1}\|$ are small enough for all $n = 1, \ldots, N$.
4. For each $n = 1, \ldots, N$, mark elements in $\mathcal{M}^n$ based on $\eta^n$ according to an optimization strategy, as implemented in deal.II [7].
5. Refine the meshes according to the marking and satisfaction of the constraints on hanging nodes.
6. Repeat from step 1.
3 Numerical Results

In this section, the mixed phase-field model formulation is applied to simulate crack propagation in an L-shaped specimen with the help of adaptive refined meshes. First, the setup of the L-shaped panel test and the corresponding material and numerical parameters are given. Afterwards, the load-displacement curves and the crack paths are discussed for three different Poisson ratios from the standard setting towards the incompressible limit $\nu \to 0.5$.

3.1 Configuration of the L-shaped Panel Test

The L-shaped panel test was originally developed by Winkler [12] to test the crack pattern of concrete experimentally and numerically. Concrete is compressible with a Poisson ratio of $\nu = 0.18$. To simulate fracture propagation in nearly incompressible materials, within this work, the Poisson’s ratio is increased towards an incompressible solid.

![Figure 1: Geometry and boundary conditions of the L-shaped panel test.](image)

In Figure 1 the test geometry of the L-shaped panel test is declared. In the right corner $\Gamma_{uy}$ on a small stripe of 30mm at the boundary, a special displacement condition is defined as a loading-dependent non-homogeneous Dirichlet condition:

$$u_y = t \cdot \text{mm/s}, \quad \text{for } t \in I := [0; 0.4s],$$

where $t$ denotes the total time and $T = 0.4s$ is the end time which corresponds to a displacement of 0.4mm. The time interval $I$ is divided into steps of the loading size $\delta t$.

**Remark 1** To avoid developing unphysical cracks in the singularity on the boundary $\Gamma_{uy}$, the domain where the phase-field inequality is solved, is constrained to the subset given by $x < 400\text{mm}$ similar to [8]. For $x > 400\text{mm}$ we assume $\varphi = 1$.

In Table 1, the required material and numerical parameters for the L-shaped panel test are listed. Keep in mind, that the given values for $\mu$ and $\lambda$ fit to the original material concrete and are changed for other values of $\nu$ in the following numerical tests, as listed in Table 2.

Further, the discretization parameter $h$ in Table 1 changes within the refinement steps, so $h$ is the starting mesh parameter on the coarsest mesh before adaptive refining.

In Table 3, the minimal and maximal number of degrees of freedom are given for three different test cases $\nu = 0.18$, $\nu = 0.40$ and $\nu = 0.49$. The adaptive computations are based on a three times uniform refined mesh and three adaptive refinement steps. For comparison, also the load-displacement curves for tests, executed on a four times uniform refined mesh, are added in Figure 2. The load-displacement curves in Figure 2 indicate that the higher the Poisson ratio, the higher is the maximal loading value before the crack starts.

| Parameter | Description | Value |
|-----------|-------------|-------|
| $\mu$     | Lamé coefficient | 10.95kN/mm² |
| $\lambda$ | Lamé coefficient | 6.16kN/mm² |
| $\nu$     | Poisson’s ratio | 0.18  |
| $G_c$     | Critical energy rate | $8.9 \times 10^{-5}$kN/mm |
| $h$       | Discretization parameter | 7.289mm |
| $\epsilon$| Bandwidth    | 14.0mm |
| $\delta t$| Incremental size | $10^{-4}$s |
| $T$       | End time     | 0.4s   |
| $\kappa$  | Regularization parameter | $10^{-10}$ |

Table 1: Standard settings of the material and numerical parameters for the L-shaped panel test.

In Table 2, the minimal and maximal number of degrees of freedom are given for three different test cases $\nu = 0.18$, $\nu = 0.40$ and $\nu = 0.49$. The adaptive computations are based on a three times uniform refined mesh and three adaptive refinement steps. For comparison, also the load-displacement curves for tests, executed on a four times uniform refined mesh, are added in Figure 2. The load-displacement curves in Figure 2 indicate that the higher the Poisson ratio, the higher is the maximal loading value before the crack starts.
Table 2: Tests with different Poisson’s ratios.

| υ  | μ       | λ       |
|----|---------|---------|
| 0.18 | 10.95 · 10³ | 6.18 · 10³ |
| 0.40 | 10.95 · 10³ | 42.36 · 10³ |
| 0.49 | 10.95 · 10³ | 51.89 · 10⁴ |

Table 3: The minimal and maximal number of degrees of freedom (DoF) per incremental step on adaptive meshes.

| υ  | min. #DoF | max. #DoF |
|----|-----------|-----------|
| 0.18 | uniform 213,445 | 125,599 |
| 0.40 | uniform 213,445 | 121,709 |
| 0.49 | uniform 213,445 | 91,574 |

Figure 2: Load-displacement curves for the L-shaped panel test with different Poisson ratios and adaptively refined meshes versus uniform refinement. Weighted loading measured on the lower left boundary $\Gamma_{\text{measured}}$ labeled in Figure 1.

4 Conclusion

We have combined and extended [7] and [6] to adaptive refinement based on robust residual-type a posteriori error estimators for phase-field model for fractures in nearly incompressible materials. The method is demonstrated on a numerical test for the L-shaped panel test. Therefore, we proposed three test cases in Section 3 with different Poisson ratios $\nu$ approximating the incompressible limit $\nu = 0.5$. The load-displacement curves of the three tests show a correlation between an increasing Poisson ratio and a stronger loading force. In view of mesh adaptivity, we observed very convincing findings: the mesh refinement is localized in the area of the (a priori unknown) fracture path and allows to resolve the crack tip region. Further, our adaptive refined meshes allow for a faster crack growth compared to uniformly refined meshes.

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Figure 3: Poisson’s ratio $\nu = 0.40$. Snapshots of the phase-field function after three adaptive refinement steps in the incremental steps 0.2082, 0.209, 0.2099, 0.2136, 0.2323 and 0.2997s on the current adaptive mesh.

Figure 4: Poisson’s ratio $\nu = 0.40$. Enhanced extract of the phase-field function in the crack tip after three adaptive refinement steps in the incremental steps 0.2099, 0.2176 and 0.2997s.
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