Kantowski-Sachs Universe Models in $f(T)$ Theory of Gravity

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Abstract

The $f(T)$ theory is recently proposed to explain the present cosmic accelerating expansion of the universe. $f(T)$ theory is an extension of Teleparallel theory of gravity, where $T$ is the torsion scalar. This paper contains the construction of $f(T)$ models within the Kantowski-Sachs universe. For this purpose, we use conservation equation and equation of state parameter, which represents the different phases of the universe. We discuss possible cases for the matter dominated era, radiation dominated era, present dark energy phase and their combinations. Particularly, a constant solution has been obtained which may correspond to the cosmological constant. Further, we consider two well known $f(T)$ models and derive the equation of state parameter and discuss the cosmic acceleration. Also, the Hubble parameter and average scale factor have been evaluated.

Keywords: $f(T)$ Gravity, Kantowski-Sachs Universe, Torsion.

1 Introduction

Various cosmological observations, including the type Ia Supernova [1], the cosmic microwave background radiation [2] and the large scale structure [3,4], have shown that the universe is undergoing an accelerating expansion and

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it entered this accelerating phase only in the near past. The unexpected observed phenomenon poses one of the most puzzling problems in cosmology today. Usually, it is assumed that there exists, in our universe, an exotic energy component with negative pressure, named dark energy (DE), which dominates the universe and drives it to an accelerating expansion at recent times.

Many candidates of DE have been proposed such as the cosmological constant, quintessence, phantom, quintom as well as the (generalized) Chaplygin gas, and so on. However, alternatively, we can take this observed accelerating expansion as a signal of the breakdown of our understanding to understand the laws of gravitation. Thus, a modified theory of gravity is needed. Modified theories of gravity, e.g., Scalar tensor theory, Brans-Dick theory, String theory, Gauss-Bonnet theory, \( f(R) \) theory, \( f(T) \) gravity etc. have recently gained a lot of interest during the last decade. These theories provide the very natural gravitational alternative for the DE. The modification of gravitational action may resolve cosmological problems, paradigm DE and DM issues. In this paper we focus our attention only on \( f(T) \) theory of gravity. This theory of gravity is the generalization of teleparallel theory of gravity [5].

\( f(T) \) theory is proposed best to account for the present accelerating expansion [6-9]. In teleparallel gravity (TPG), we use the Weitzenböck connection instead of using the Levi-Civita connection, which we usually used in \( GR \). As a result, in TPG, the Weitzenböck spacetime has only non-zero torsion and is curvature free. Similar to \( GR \), where the action involves the curvature scalar \( R \), the action of TPG is obtained by simply replacing \( R \) with torsion scalar \( T \). In analogy to the \( f(R) \) theory, Bengochea and Ferraro suggested [6] a modified TPG theory, named \( f(T) \) theory, by generalizing the action of TPG, i.e., by replacing \( T \) with \( f(T) \). They found that it can explain the observed acceleration of the universe. It is worth mentioning here that the field equations of \( f(R) \) theory are of fourth order while the field equations of \( f(T) \) theory are of second order, which seem easier to be solved.

Linder proposed two new \( f(T) \) models in order to explain the present cosmic accelerating expansion [7]. He said that \( f(T) \) theory could unify a number of interesting extensions of gravity beyond \( GR \). He investigated that the power law and exponential models depending upon torsion might give the de-Sitter fate of the universe. Wu and Yu [10] analyzed the dynamical property of this theory by using a concrete power law model and showed that
the universe could evolve from radiation dominated era to matter dominated era and finally enter in an exponential expansion era.

Yang [11] introduced some new $f(T)$ models and gave their physical implications and cosmological behavior. Wu and Yu [12] discussed two new $f(T)$ models and showed how the crossing of phantom divide line takes place. They also explained the observation constraints corresponding to these models. Karami and Abdolmaleki [13] found that equation of state $EoS$ parameter of holographic and new age graphic $f(T)$ models always cross the phantom divide line where entropy connected model has to experience some conditions on parameters model. The same author [14] obtained $EoS$ parameter of polytropic, standard, generalized and modified Chaplygin gas in this modified scenario. Dent, et al. [15] investigated this theory at the background and perturbed level and also explored it for quintessence scenarios. Li, et al. [16] explored local Lorentz invariance and remarked that $f(T)$ theory is not local Lorentz invariant.

Chen, et al. [17] investigated expressions for growth factor, stability and vector-tensor perturbations. Bamba, et al. [18] studied the cosmological equations of $EoS$ in exponential, logarithmic and their combined $f(T)$ models. Wang [19] searched spherically symmetric static solution of $f(T)$ models with a Maxwell term and demonstrated that in conformal Cartesian coordinates the Reissner-Nordstrom solution does not exist in this theory. Myrzakulov [20] discussed different $f(T)$ models including scalar fields and gave analytical solutions for scale factors and scalar fields.

Sharif and Rani explored Bianchi type-1 universe using different $f(T)$ gravity models [21]. They also discussed K-essence models in the framework of $f(T)$ gravity. Recently, we explored Kantowski-Sachs universe models in $f(T)$ theory of Gravity [22]. Recently, some interesting $f(T)$ models have been explored by different authors in [23]-[25]. In this paper, we explore some $f(T)$ models within the Kantowski-Sachs universe. For this purpose, we use conservation equation and equation of state parameter, which represent the different phases of the universe. Also, we discuss the cosmic acceleration of the universe and $EoS$ parameter by considering two particular $f(T)$ models.

The structure of the paper is as follows. In section 2, we shall present some basics of the $f(T)$ theory of gravity and the corresponding field equations for Kantowski-Sachs spacetime. Section 3 contains a detailed construction of $f(T)$ models by using two different approaches. Section 4 is devoted to study the $EoS$ parameter for two particular models and also a discussion on cosmic acceleration is provided. In the last section, we summarize and conclude the
results.

2 An Overview of Generalized Teleparallel Theory \( f(T) \)

In this section, we introduce briefly the teleparallel theory of gravity and its generalization to \( f(T) \) theory. The Lagrangian density for teleparallel and \( f(T) \) gravity are, respectively, given as follows [22]:

\[
L_T = \frac{h}{16\pi G} T, \quad (1)
\]
\[
L_{f(T)} = \frac{h}{16\pi G} F(T), \quad (2)
\]

where \( T \) is the torsion scalar, \( f(T) \) is a general differentiable function of torsion, \( G \) is the gravitational constant and \( h = det(h^i_{\mu}) \). Mathematically, the torsion scalar is defined as

\[
T = S_{\rho}^{\mu\nu} T^\rho_{\mu\nu}, \quad (3)
\]

where \( S_{\rho}^{\mu\nu} \) is antisymmetric in its upper indices while \( T^\rho_{\mu\nu} \) is antisymmetric torsion tensor in its lower indices. Here \( S_{\rho}^{\mu\nu} \) is determined by the relation

\[
S^{\mu\rho\sigma} = \frac{1}{4}(T^{\mu\rho\sigma} + T^{\rho\mu\sigma} - T^{\sigma\mu\rho}) - \frac{1}{2}(g^{\mu\sigma}T_{\lambda\rho}^\lambda - g^{\rho\mu}T_{\sigma\lambda}^\lambda) \quad (4)
\]

and \( T_{\lambda}^{\rho\mu} \) is defined as [26]

\[
T_{\lambda}^{\rho\mu} = \Gamma_{\nu\mu}^{\lambda} - \Gamma_{\mu\nu}^{\lambda} = h_{i}^{\lambda}\left(\partial_{\mu}h_{\nu}^{i} - \partial_{\nu}h_{\mu}^{i}\right). \quad (5)
\]

Here \( h_{i}^{\mu} \) are the components of the non-trivial tetrad field \( h_{i} \) in the coordinate basis. It is an arbitrary choice to choose the tetrad field related to the metric tensor \( g_{\mu\nu} \) by the following relation

\[
g_{\mu\nu} = \eta_{ij} h_{\mu}^{i} h_{\nu}^{j}, \quad (6)
\]

where \( \eta_{ij} \) is the Minkowski spacetime for the tangent space such that \( \eta_{ij} = diag(+1, -1, -1, -1) \). For a given metric there exists infinite different tetrad fields \( h_{i}^{\mu} \) which satisfy the following properties:

\[
h_{i}^{\mu} h_{j}^{\mu} = \delta_{ij}^{\mu}; h_{i}^{\mu} h_{i}^{\nu} = \delta_{\mu}^{\nu}. \quad (7)
\]
In this paper, the Latin alphabets \((i, j, \ldots = 0, 1, 2, 3)\) will be used to denote the tangent space indices and the Greek alphabets \((\mu, \nu, \ldots = 0, 1, 2, 3)\) to denote the spacetime indices. The variation in the indices other than the above mentioned range will be specified when needed. The variation of Eq.(2) with respect to the vierbein field leads to the following field equations
\[
\left[ e^{-1}\partial_\mu \left( e_i^{\lambda\mu} \right) + h_i^\lambda T^\rho_\mu S_\rho^{\lambda\mu} \right] F_T + S_i^{\mu\nu} \partial_\mu (T) F_{TT} \\
+ \frac{1}{4} h_i^{\nu} F = \frac{1}{2} \kappa^2 h_i^\rho T^\nu.
\] (8)

Here \(f_T = \frac{df}{dT}, f_{TT} = \frac{d^2 f}{dT^2}, \kappa^2 = 8\pi G, S_i^{\mu\nu} = h_i^\rho S_\rho^{\mu\nu}, \) and \(T_\mu^{\nu}\) is the energy-momentum tensor, given as
\[
T_\nu^{\mu} = \text{diag} \left( \rho_m, -p_m, -p_m, -p_m \right),
\] (9)
where \(\rho_m\) is the density while \(p_m\) is the pressure of the matter inside the universe.

**The Field Equations**

The line element for a flat, homogeneous and anisotropic Kantowski-Sachs spacetime is
\[
d s^2 = -A^2(t)dt^2 - B^2(t) \left( d\theta^2 + \sin^2\theta d\phi^2 \right),
\] (10)
where the scale factors \(A\) and \(B\) are functions of cosmic time \(t\) only. Using Eqs.(6) and (10), we obtain tetrad components as follows [27]:
\[
h^i_\mu = \text{diag} \left( 1, A, B, B \sin\theta \right),
\]
\[
h^\mu_i = \text{diag} \left( 1, A^{-1}, B^{-1}, (B \sin\theta)^{-1} \right),
\] (11)
which obviously satisfy Eq.(7). Substituting Eqs.(4) and (5) in (3) and using (10), it follows after some manipulation
\[
T = -2 \left( 2\dot{A}\dot{B} + \ddot{B}^2 \right) B^2.
\] (12)

The field equations (8), for \(i = 0 = \nu\) and \(i = 1 = \nu\), turn out to be
\[
F - 4 \left( \frac{2\dot{A}\dot{B}}{AB} + \frac{\ddot{B}^2}{B^2} \right) F_T = 2\kappa^2 \rho_m,
\] (13)
\[
4 \left( \frac{\dot{A}\dot{B}}{B} + \frac{A\dot{B}^2}{B^2} + \frac{\dot{A}\ddot{B}}{B} + \frac{\ddot{A}\dot{B}}{AB} \right) F_T - 16 \frac{A\dot{B}}{B} \left( \frac{\ddot{A}}{AB} + \frac{\ddot{B}}{AB} \right)
- \frac{\dot{A}^2\dot{B}}{A^2 B} - \frac{\dot{A}\ddot{B}^2}{AB^2} + \frac{\ddot{B}}{B^2} \frac{\ddot{B}^3}{B^3} \right) F_{TT} - F = 2\kappa^2 p_m.
\] (14)

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The conservation equation takes the form

\[ \rho_m' + \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) (\rho_m + p_m) = 0. \]  \( (15) \)

The average scale factor \( R \), the mean Hubble parameter \( H \) and the anisotropy parameter \( \Delta \) of the expansion respectively become

\[ \rho_m' + \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) (\rho_m + p_m) = 0. \]  \( (16) \)

where \( H_i \) are the directional parameters in the direction \( x, y \) and \( z \) respectively given as

\[
H_1 = \frac{\dot{A}}{A}, \\
H_2 = \frac{\dot{B}}{B} = H_3. 
\]  \( (17) \)

It is mentioned here that the isotropic expansion of the universe is obtained for \( \Delta = 0 \) which further depends upon the values of unknown scale factors and parameters involved in the corresponding models [28]-[30].

The equation (12) can be written as

\[ 2T = J - 9H^2, \quad J = \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B}. \]  \( (18) \)

which implies that

\[ H = \frac{1}{3} \sqrt{J - 2T}. \]  \( (19) \)

If we take \( F(T) = T \) then Eqs.(13) and (14) will reduce to

\[
\rho_m + \rho_T = \frac{1}{2}\kappa^2 \left[ -4 \left( \frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right) + T \right], \\
p_m + p_T = \frac{1}{2}\kappa^2 \left[ \frac{\dot{A}\dot{B}}{B} + \frac{A\dot{B}^2}{B^2} + \frac{A\dot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} \right] - T \left[ -1 + F_T \right] - F - T. 
\]  \( (20) \)

\[
\rho_T = \frac{1}{2}\kappa^2 \left[ -4 \left( \frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right) \left( 1 - F_T \right) + T - F \right], 
\]  \( (22) \)
and

\[
pT = \frac{1}{2\kappa^2} \left[ 4 \left( \frac{\dot{A}B}{B} + \frac{\dot{A}B^2}{B^2} + \frac{\ddot{A}B}{B} + \frac{\dot{A}B}{AB} \right)^2 (1 - FT) \\
+ 16 \frac{\dot{A}B}{B} \left( \frac{\dot{A}B}{AB} + \frac{\dot{A}B}{AB} - \frac{\dot{A}^2B}{AB^2} - \frac{\dot{A}B^2}{AB^2} \right) \\
+ \frac{\ddot{B}B}{B^2} - \frac{\dot{B}^3}{B^3} \right] F_{TT} - T + F \right].
\] (23)

The relationship between energy density \( \rho \) and pressure of matter \( p \) is described by EoS, \( p = \omega \rho \) where \( \omega \) is the EoS parameter. For normal, relativistic and non-relativistic matters, EoS parameter has different corresponding values. Using Eqs.(13) and (14), the EoS parameter is obtained as follows

\[
\omega = -1 + \frac{4 (E - U) F_T - 16 Z F_{TT}}{-4UF_T + F},
\] (24)

where

\[
E = \frac{\dot{A}B}{B} + \frac{\dot{A}B^2}{B^2} + \frac{\ddot{A}B}{B} + \frac{\dot{A}B}{AB},
\] (25)

\[
U = \frac{2\dot{A}B}{AB} + \frac{\dot{B}^2}{B^2},
\] (26)

\[
Z = \frac{\dot{A}B}{B} \left[ \frac{\ddot{A}B}{AB} + \frac{\dot{A}B}{AB} - \frac{\dot{A}^2B}{AB^2} - \frac{\dot{A}B^2}{AB^2} + \frac{\ddot{B}B}{B^2} - \frac{\dot{B}^3}{B^3} \right].
\] (27)

It is mentioned here that the homogeneous part of Eq.(13) yields the following solution

\[
F(T) = \frac{C_0}{\sqrt{T}},
\] (28)

where \( C_0 \) is an integration constant. Using this equation in Eq.(14), we obtain

\[
p_m = -\frac{c_0}{2\kappa^2} \left( \frac{2E}{T} + \frac{12Z}{T^2} + 1 \right) \frac{1}{\sqrt{T}}.
\] (29)

It is mentioned here that the \( p_m \) vanishes for the FRW spacetime [31].
3 Construction of Some $F(T)$ Models

Here we construct some $F(T)$ models with different cases of perfect fluid by using two approaches. In the first approach we use the continuity equation (15) while in the second approach, EoS parameter (26) will be used. As the constituents of the universe are non-relativistic matter, radiation and DE, we consider the corresponding values of $\omega$ in the following subsections.

3.1 Using Continuity Equation

In this approach, we use the following relation [32] for Kantowski- Sachs spacetime

$$\frac{1}{9} \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right)^2 = H_0^2 + \frac{\kappa^2 \rho_0}{3AB^2 \sin \theta} \cdot \frac{1}{9},$$

(30)

where $H_0$ is the Hubble constant having primary implication in cosmology and $\rho_0$ is an integration constant. The value of $H_0$ corresponds to the rate at which the universe is expanding today. This equation implies that

$$\left( AB^2 \sin \theta \right)^{-1} = \frac{3}{\kappa^2 \rho_0} \left( H^2 - H_0^2 \right).$$

(31)

Using EoS in Eq.(15), it follows that

$$\frac{\rho_m}{\dot{\rho}_m} + 3H (1 + \omega) = 0.$$

(32)

The components of the universe are described by the terms dark matter($DM$) and dark energy($DE$). We consider different cases of fluids and their combination to construct corresponding $F(T)$ models. For example, for relativistic matter, $\omega = \frac{1}{3}$, for non-relativistic matter, it is zero and for $DE$ era, it is equal to $-1$ [33].

Case 1 ($\omega = 0$):

This is the case of non-relativistic matter, like cold dark matter ($CDM$) and baryons. It is well approximated as pressureless dust and called the matter dominated era. Inserting $\omega = 0$ in Eq.(34) and using Eq.(33), we have

$$\rho_m = \frac{\rho_c}{AB^2} = \frac{3\rho_c \sin \theta}{\kappa^2 \rho_0} \left( H^2 - H_0^2 \right),$$

(33)
where $\rho_c$ is an integration constant. In terms of torsion scalar, the above equation becomes

$$
\rho_m = \frac{\rho_c \sin \theta}{3\kappa^2 \rho_0} \left( J - 9H_0^2 - 2T \right). \tag{34}
$$

Substituting the values of $\rho_m$ from Eq.(36) in Eq.(13), we have

$$
2TF_T + F = \frac{2\rho_c \sin \theta}{3\kappa^2 \rho_0} \left( J - 9H_0^2 - 2T \right), \tag{35}
$$

which has the solution

$$
F(T) = \frac{\rho_c \sin \theta}{3\rho_0 \sqrt{T}} \int \frac{J - 9H_0^2 - 2T}{\sqrt{T}} dT. \tag{36}
$$

This will have a unique solution if the value of $J$ is known which corresponds to the unknown scale factors. Thus for matter dominated era, we obtain a model in the form of torsion scalar and Hubble constant.

**Case 2 ($\omega = \frac{1}{3}$):**

Here we consider the relativistic matter, like photons and massless neutrinos with $EoS$ parameter $\omega = \frac{1}{3}$. This case represents the radiation dominated era of the universe. Substituting $\omega = \frac{1}{3}$ in Eq.(13) and making use of Eqs.(20) and (33), we obtain

$$
\rho_m = \frac{\rho_r \sin \frac{4}{3} \theta}{3^{\frac{4}{3}} \kappa^{\frac{2}{3}} \rho_0^{\frac{2}{3}}} \left( J - 9H_0^2 - 2T \right)^{\frac{4}{3}}, \tag{37}
$$

where $\rho_r$ is another integration constant. Inserting this value of $\rho_m$ in Eq.(13), we get

$$
2TF_T + F = \frac{2\rho_r \sin \frac{4}{3} \theta}{3^{\frac{4}{3}} \kappa^{\frac{2}{3}} \rho_0^{\frac{2}{3}}} \left( J - 9H_0^2 - 2T \right)^{\frac{4}{3}}, \tag{38}
$$

which has solution

$$
F(T) = \frac{\rho_r \sin \frac{4}{3} \theta}{3^{\frac{4}{3}} \kappa^{\frac{2}{3}} \rho_0^{\frac{2}{3}} \sqrt{T}} \int \frac{(J - 9H_0^2 - 2T)^{\frac{4}{3}}}{\sqrt{T}} dT. \tag{39}
$$

This also depends upon the value of $J$ as well as torsion scalar and Hubble constant.
Case 3 ($\omega=-1$):
This case represents the present $DE$ constituting 74 percent of the universal density. $DE$ is assumed to have a large negative pressure in order to explain the observed acceleration of the universe. It is also termed as energy density of vacuum or cosmological constant $\Lambda$. Replacing $\omega=-1$ in Eq. (34), we get

$$\rho_m = \rho_d,$$  (40)

where $\rho_d$ is an integration constant. Consequently, Eq.(13) takes the form

$$2TF_T + F = 2\kappa^2 \rho_d$$  (41)

with solution

$$F(T) = \frac{\kappa^2 \rho_d}{\sqrt{T}} \int \frac{1}{\sqrt{T}}dT.$$  (42)

Case 4 (Combination of $\omega = 0$ and $\omega = \frac{1}{3}$):
Let us now consider the case when the energy density is a combination of different fluids, the dust fluid and the radiations. Adding Eqs.(36) and (39), after simplification, it follows that

$$\rho_m = \frac{\rho_c \sin \theta}{6\kappa^2 \rho_0} \left( J - 9H_0^2 - 2T \right) + \frac{\rho_r \sin \frac{4}{3} \theta}{2.3 \frac{4}{3} \kappa^2 \rho_0^\frac{4}{3}} \left( J - 9H_0^2 - 2T \right)^\frac{4}{3}.$$  (43)

Substituting this value of $\rho_m$ in Eq.(13), we get

$$2TF_T + F = \frac{\rho_c \sin \theta}{3\rho_0} \left( J - 9H_0^2 - 2T \right) + \frac{\rho_r \sin \frac{4}{3} \theta}{3^\frac{4}{3} \kappa^\frac{4}{3} \rho_0^\frac{4}{3}} \left( J - 9H_0^2 - 2T \right)^\frac{4}{3}$$  (44)

and its solution is

$$F(T) = \frac{\rho_c \sin \theta}{6\rho_0 \sqrt{T}} \int \frac{(J - 9H_0^2 - 2T)}{\sqrt{T}}dT$$

$$+ \frac{\rho_r \sin \frac{4}{3} \theta}{2.3^\frac{4}{3} \kappa^\frac{4}{3} \rho_0^\frac{4}{3} \sqrt{T}} \int \frac{(J - 9H_0^2 - 2T)^\frac{4}{3}}{\sqrt{T}}dT.$$  (45)
Case 5 (Combination of $\omega = 0$ and $\omega = -1$):
The combination of EoS parameters for matter dominated era and DE yields

$$\rho_m = \frac{\rho_c \sin \theta}{6\kappa^2 \rho_0} \left(J - 9H_0^2 - 2T\right) + \frac{\rho_d}{2}. \quad (46)$$

Inserting this value of $\rho_m$ in Eq.(13), we get

$$2TF_T + F = \frac{\rho_c \sin \theta}{3\rho_0} \left(J - 9H_0^2 - 2T\right) + \kappa^2 \rho_d, \quad (47)$$

yielding

$$F(T) = \frac{\rho_c \sin \theta}{6\rho_0 \sqrt{T}} \int \frac{(J - 9H_0^2 - 2T)}{\sqrt{T}} dT + \frac{\kappa^2 \rho_d}{2\sqrt{T}} \int \frac{1}{\sqrt{T}} dT. \quad (48)$$

Case 6 (Combination of $\omega = -1$ and $\omega = \frac{1}{3}$):
This case gives the following form of the energy density

$$\rho_m = \frac{\rho_r \sin^{\frac{4}{3}} \theta}{2.3^{\frac{5}{2}} \kappa^{\frac{8}{3}} \rho_0^\frac{1}{3}} \left(J - 9H_0^2 - 2T\right)^\frac{4}{3} + \frac{\rho_d}{2}. \quad (49)$$

Substituting this value in Eq.(13), we get

$$2TF_T + F = \frac{\rho_r \sin^{\frac{4}{3}} \theta}{3^{\frac{5}{2}} \kappa^{\frac{8}{3}} \rho_0^\frac{1}{3}} \left(J - 9H_0^2 - 2T\right)^\frac{4}{3} + \kappa^2 \rho_d, \quad (50)$$

which gives

$$F(T) = \frac{\rho_r \sin^{\frac{4}{3}} \theta}{2.3^{\frac{5}{2}} \kappa^{\frac{8}{3}} \rho_0^\frac{1}{3} \sqrt{T}} \int \frac{(J - 9H_0^2 - 2T)^{\frac{4}{3}}}{\sqrt{T}} dT + \frac{\kappa^2 \rho_d}{2\sqrt{T}} \int \frac{1}{\sqrt{T}} dT. \quad (51)$$

It is mentioned here that the cases 4-6 provide $F(T)$ models for combination of different matters. Normally, the dark matter and DE developed independently. However, there are attempts [34] to include an interaction amongst them so that one can get some insights and see the combined effect of different fluids. Dark matter plays a central role in galaxy evolution and has measurable effects on the anisotropies observed in the cosmic microwave background. Although, matter made a large fraction of total energy of the universe but
its contribution would fall in the far future as DE becomes more dominated. It may provide an interaction between dark matter and DE and drive transition from an early matter dominated era to a phase of accelerated expansion. Using the same phenomenon, DE and different forms of matter are discussed in the framework of $F(T)$ theory which may help to discuss accelerated expansion of the universe.

### 3.2 Using EoS Parameter

Here we formulate some $F(T)$ models in a slightly different way. We substitute different values of parameter $\omega$ in Eq.(13) and solve it accordingly. The Eq.(26) can be written as

$$16ZF_{TT} - 4(E + \omega U)F_T + (1 + \omega)F = 0.$$  \hspace{1cm} (52)

Now, we construct $F(T)$ models in the following cases:

**Case 1:**

When we put $\omega = \frac{1}{3}$ in Eq.(54), we obtain

$$ZF_{TT} - \left( E + \frac{U}{3} \right) F_T + \frac{1}{3}F = 0.$$  \hspace{1cm} (53)

This has the following general solution.

$$F(T) = C_1 \exp \left[ \left\{ \frac{(3E + U) + \sqrt{(3E + U)^2 - 12Z}}{6Z} \right\} T \right] + C_2 \exp \left[ \left\{ \frac{(3E + U) - \sqrt{(3E + U)^2 - 12Z}}{6Z} \right\} T \right],$$  \hspace{1cm} (54)

where $C_1$ and $C_2$ are constants.

**Case 2:**

Here we consider the dust case when pressure is zero, that is, $\omega = 0$. Then the Eq.(52) takes the form

$$16ZF_{TT} - 4EF_T + F = 0.$$  \hspace{1cm} (55)

It has the following general solution

$$F(T) = C_3 \exp \left[ \left\{ \frac{E + \sqrt{E^2 - 4Z}}{8Z} \right\} T \right].$$
where $C_3$ and $C_4$ are constants.

**Case 3:**
For $\omega = -1$, Eq.(54) becomes

$$4ZF_{TT} - (E - U) F_T = 0,$$

(57)

whose general solution is

$$F(T) = C_5 + C_6 \exp \left[ \left( \frac{E - U}{4Z} \right) T \right],$$

(58)

where $C_5$ and $C_6$ are constants. The Eqs.(54), (56) and (58) represent $F(T)$ models corresponding radiation, matter and DE phases respectively. The exponential form of $F(T)$ models represents a universe which always lies in phantom or non-phantom phase depending on parameters of the models [35].

### 4 Construction of EoS Parameters and Cosmic Acceleration

In this section we derive EoS parameter by using two different $F(T)$ models and also investigate cosmic acceleration. For this purpose, we evaluate $\rho_m$ and $p_m$ using the field equations and then construct the corresponding EoS parameters.

#### 4.1 The First Model

Consider the following $F(T)$ model [31]

$$F = \alpha T + \frac{\beta}{T},$$

(59)

where $\alpha$ and $\beta$ are positive real constants. Inserting this value of $F$ in Eqs.(13) and (14), it follows that
\[2\kappa^2 \rho_m = (-4U + T) \alpha + \beta \left(1 + 4UT^{-1}\right) T^{-1},\]  
\[2\kappa^2 p_m = (4E - T) \alpha - \beta \left(4ET^{-1} + 32ZT^{-2} + 1\right) T^{-1}.\]  

Dividing Eq.(61) by (60), the EoS parameter is obtained as follows

\[
\omega = -1 + \frac{4(E - U) \alpha - \beta (4(E - U) T^{-1} + 32ZT^{-1}) T^{-2}}{(-4U + T) \alpha + \beta (1 + 4UT^{-1}) T^{-1}}.\]  

Now, we would like to discuss the last equation for particular values of \(\alpha\) and \(\beta\). For \(\alpha \neq 0\), \(\beta = 0\), we obtain

\[
\omega = -1 + \frac{2}{3} \left(1 - \frac{E}{U}\right).\]  

This leads to three different cases of \(\omega\) representing different phases of the evolution of the universe as follows:

- If \(\frac{E}{U} > 1\) then \(\omega < -1\), which corresponds to the phantom accelerating universe.
- When \(\frac{E}{U} < 1\) then \(\omega > -1\), slightly which corresponds to the quintessence region.
- When \(\frac{E}{U} = 1\), we obtain a universe whose dynamics is dominated by cosmological constant with \(\omega = -1\) which corresponds to the phantom accelerating universe.

It is interesting to mention here that model (59) reduces to GR spatially flat Friedmann equation in the limiting case when anisotropy vanishes. Also, for the case, when \(\alpha \neq 0\), \(\beta \neq 0\), we obtain no physical results.

### 4.2 The Second Model

Assume the \(F(T)\) has the form [31]

\[F = \alpha T + \beta T^n,\]  

where \(n\) is a positive real number. The corresponding field equations become

\[2\kappa^2 \rho_m = (-4U + T) \alpha + \beta \left(-4nUT^{-1} + 1\right) T^n,\]  
\[2\kappa^2 p_m = (4E - T) \alpha + 4n\beta ET^{n-1} - 16n(n - 1)\beta ZT^{n-2} - \beta T^n.\]
Consequently, the EoS parameter takes the form

$$\omega = -1 + \frac{4(-U + E)\alpha + 4n\beta (-U + E) T^{n-1} - 16n(n-1)\beta Z T^{n-2}}{(-4U + T)\alpha + \beta (-4nUT^{-1} + 1) T^n}. \quad (67)$$

The case $\alpha \neq 0, \beta = 0$, leads to the same discussion as in the first case. For $\alpha = 0, \beta \neq 0$, we have

$$\omega = -1 + \frac{2n}{2n + 1} \left[ 1 - \left\{ \frac{E}{U} + \frac{8n(n-1)Z}{U^2} \right\} \right]. \quad (68)$$

For any positive real number $n$, we can discuss as follows:

- When $\left[ \frac{E}{U} + \frac{8n(n-1)Z}{U^2} \right] < 1$, the Eq.(70) gives $\omega < -1$ which represents the phantom accelerating universe.
- For $\left[ \frac{E}{U} + \frac{8n(n-1)Z}{U^2} \right] = 1$, we obtain $\omega = -1$ and hence the universe rests in DE era dominated by cosmological constant.
- The case $\left[ \frac{E}{U} + \frac{8n(n-1)Z}{U^2} \right] < -1$, corresponds to the quintessence era because $\omega > -1$.

Assuming $n = 1$ as a particular case in Eqs.(67) and (68), we have

$$\rho_m = \frac{(\alpha + \beta)(-4U + T)}{2\kappa^2}, \quad (69)$$
$$p_m = \frac{(\alpha + \beta)(4E - T)}{2\kappa^2}. \quad (70)$$

In the following, we discuss the evolution of the scale factor for Kantowski-Sachs universe. For this purpose, we assume [31]

$$p_m = \frac{A_{-1}(T)}{\rho_m} + A_0(T) + A_1(T)\rho_m, \quad (71)$$

such that $A_{-1}, A_0, A_1$ are constants. Substituting Eqs.(71) and (72) in the above equation, it follows that

$$4E - T = \frac{a}{-4U + T} + b + c (-4U + T), \quad (72)$$

where

$$a = \frac{4\kappa^4 A_{-1}}{(\alpha + \beta)^2}, \quad b = \frac{2\kappa^2 A_0}{\alpha + \beta}, \quad c = A_1. \quad (73)$$
This leads to
\[
T = -\frac{4U + 4E - b + 8Uc}{2(1 + c)} \\
\pm \frac{1}{2(1 + c)} \left[ (4U + 4E - b + 8Uc)^2 - 4(1 + c) \left( 16cU^2 - 4bU + a + 16UE \right) \right]^{\frac{1}{2}}.
\]
\[(74)\]

Substituting this value of torsion in Eq.(21) we have
\[
H = \frac{1}{3} \left[ J - \frac{4U + 4E - b + 8Uc}{1 + c} \right] \pm \frac{1}{1 + c} \left\{ (4E + 4U - b + 8Uc)^2 - 4(1 + c) \left( 16cU^2 - 4bU + a + 16UE \right) \right\}^{\frac{1}{2}}
\]
\[(75)\]

The corresponding average scale factor becomes
\[
R = R_0 \exp \left\{ \frac{1}{3} \int \left[ J - \frac{4U + 4E - b + 8Uc}{1 + c} \right] \pm \frac{1}{1 + c} \left\{ (4E + 4U - b + 8Uc)^2 - 4(1 + c) \left( 16cU^2 - 4bU + a + 16UE \right) \right\}^{\frac{1}{2}} \right\} \right] dT.
\]
\[(76)\]

As a special case of model (73), if we take \(A_{-1}\) as a constant while \(A_0 = 0 = A_1\), we obtain standard Chaplygin gas EoS [36]. In this respect, Eqs.(74) and (75) give the following results respectively.
\[
T = 2(E + U) \pm \sqrt{\{2(E + U)\}^2 - (a + 16UE)},
\]
\[(77)\]
\[
H = \frac{1}{3} \left[ J - 4(E + U) \pm 2\sqrt{\{2(E + U)\}^2 - (a + 16UE)} \right].
\]
\[(78)\]

The average scale factor for Chaplygin gas has the form
\[
R = R_0 \exp \left\{ \frac{1}{3} \int \left[ J - 4(E + U) \pm 2\sqrt{\{2(E + U)\}^2 - (a + 16UE)} \right] \right\} dT.
\]
\[(79)\]
This represents an exponential expansion which may result a rapid increment between the distance of the two non-accelerating observers as compared to the speed of light. As a result, both observers are unable to contact each other. Thus if our universe is forthcoming to a de-Sitter Universe [7], then we would not be able to observe any galaxy other than our own Milky way system.

5 Summary and Conclusion

The study of cosmological models has become a burning issue since the last decade. Much interest has been given by the researchers to resolve the cosmological problems including the existence of \( DE \) and \( DM \) in the universe. As \( GR \) can not explain the rushing growth of the universe so we need some other framework of gravity, which may resolve this issue. There are many alternate theories of gravity among which \( F(T) \) theory of gravity is one of the candidates.

The purpose of this paper is to investigate the recently developed \( F(T) \) gravity. For this purpose we have taken Kantowski-Sachs spacetime model describing anisotropic and spherically homogeneous universe. Some \( F(T) \) models have been constructed by using two different approaches. In the first approach, we have used the continuity equation while in the second method, \( EoS \) is used. The results obtained so far in these approaches are given in the following tables (1 – 2):

Table 1. Expressions for \( F(T) \) using Continuity Equation

| CASES | \( F(T) \) |
|------|-------------|
| 1    | \( \frac{\rho_c \sin \theta}{3 \rho_0 \sqrt{T}} \int \frac{J-9H_0^2-2T}{\sqrt{T}} dT \) |
| 2    | \( \frac{\rho_c \sin \frac{\theta}{3}}{\kappa \rho_0 \sqrt{T}} \int \frac{J-9H_0^2-2T}{\sqrt{T}} dT \) |
| 3    | \( \frac{\kappa^2 \rho_d}{\sqrt{T}} \int \frac{1}{\sqrt{T}} dT \) |
| 4    | \( \frac{\rho_c \sin \theta}{6 \rho_0 \sqrt{T}} \int \frac{(J-9H_0^2-2T)}{\sqrt{T}} dT + \frac{\rho_c \sin \frac{\theta}{3}}{2 \kappa \rho_0 \sqrt{T}} \int \frac{(J-9H_0^2-2T)}{\sqrt{T}} dT \) |
| 5    | \( \frac{\rho_c \sin \theta}{6 \rho_0 \sqrt{T}} \int \frac{(J-9H_0^2-2T)}{\sqrt{T}} dT + \frac{\kappa^2 \rho_d}{2 \sqrt{T}} \int \frac{1}{\sqrt{T}} dT \) |
| 6    | \( \frac{\rho_c \sin \frac{\theta}{3}}{2 \kappa \rho_0 \sqrt{T}} \int \frac{(J-9H_0^2-2T)}{\sqrt{T}} dT + \frac{\kappa^2 \rho_d}{2 \sqrt{T}} \int \frac{1}{\sqrt{T}} dT \) |
Table 2. Expressions for $F(T)$ using EoS Parameter

| CASES | $F(T)$ |
|-------|--------|
| 1     | $C_1 \exp\left\{\frac{(3E+U)+\sqrt{(3E+U)^2-12Z}}{6Z} T\right\} + C_2 \exp\left\{\frac{(3E+U)-\sqrt{(3E+U)^2-12Z}}{6Z} T\right\}$ |
| 2     | $C_3 \exp\left\{\frac{E+\sqrt{E^2-4Z}}{4Z} T\right\} + C_4 \exp\left\{\frac{E-\sqrt{E^2-4Z}}{4Z} T\right\}$ |
| 3     | $C_5 + C_6 \exp\left\{\frac{E-U}{4Z} T\right\}$ |

These $F(T)$ gravity models represent three different eras of the universe corresponding to different values of EoS parameter. These are the matter, radiation and DE dominated eras corresponding to $\omega = 0$, $\omega = \frac{1}{3}$ and $\omega = -1$ respectively, given in table 1 as cases 1-3. If we consider combination of radiation and matter, we may have more interesting results to study the developing universe. Using different combinations of EoS parameter we obtain three more models, given in table 1 as cases 4-6. Also we have obtained $F(T)$ models in exponential form for some particular values of EoS parameter, given in table 2.

It is well known that the evolution of EoS parameter is one of the biggest efforts in the observational cosmology today. We have considered two well known $F(T)$ models, given in Eqs.(61) and (66) and found the corresponding expressions for EoS parameter $\omega$. These expressions have been investigated for some particular values of the parameters $\alpha$ and $\beta$ which yield fruitful results corresponding to realistic situations. Further, we discuss the cosmic acceleration for these models. We conclude that our universe would approach to de-Sitter universe in the infinite future. The isotropic expansion of the universe is obtained for $\Delta = 0$ which depends upon the values of unknown scale factors and parameters involved in the corresponding models.

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