A note on the elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity

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Abstract

In this note we complete the study made in [1] on a Kirchhoff type equation with a Berestycki-Lions nonlinearity. We also correct Theorem 0.6 inside.

Introduction

In this note we consider the nonlinear Kirchhoff equation

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \right) \Delta u = g(u) \text{ in } \mathbb{R}^N, \quad N \geq 3,$$

where we assume general hypotheses on $g$. We will investigate the existence of a solution depending on the value of the positive parameter $a$ and $b$. We will fix an uncorrect sentence contained in [1] and complete that paper with additional results.

We refer to [1] and the references within for a justification of our study and a bibliography on the problem.
1 Existence and characterization of the solutions

Assume that

$(g_1)$ \( g \in C(\mathbb{R}, \mathbb{R}), g(0) = 0; \)

$(g_2)$ \( -\infty < \liminf_{s \to 0^+} g(s)/s \leq \limsup_{s \to 0^+} g(s)/s = -m < 0; \)

$(g_3)$ \( -\infty \leq \limsup_{s \to +\infty} g(s)/s^{2^*-1} \leq 0; \)

$(g_4)$ there exists \( \zeta > 0 \) such that \( G(\zeta) := \int_0^\zeta g(s) \, ds > 0. \)

It is well known that the previous assumptions coincide with that in [2], where the problem

\[- \Delta v = g(v) \text{ in } \mathbb{R}^N, \quad N \geq 3, \]

was studied and solved.

First of all, we present the following general result which provides a characterization of the solutions of (1)

**Theorem 1.1.** \( u \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N) \) is a solution to (1) if and only if there exists \( v \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N) \) solution to (2) and \( t > 0 \) such that \( t^2a + t^{4-N}b \int_{\mathbb{R}^N} |\nabla v|^2 = 1 \) and \( u(\cdot) = v(t\cdot). \)

**Proof** We first prove the “if” part. Suppose \( v \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N) \) and \( t > 0 \) are as in the statement of the theorem and set \( u(\cdot) = v(t\cdot) = v_t(\cdot) \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N). \) We compute

\[- \Delta u(x) = -\Delta v_t(x) = -t^2 \Delta v(tx) \]

\[ = t^2g(v(tx)) = t^2g(u(x)) = \frac{g(u(x))}{a + bt^{2-N} \int_{\mathbb{R}^N} |\nabla v|^2} \]

\[ = \frac{g(u(x))}{a + b \int_{\mathbb{R}^N} |\nabla u|^2}. \]

Now we prove the “only if” part. Suppose \( u \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N) \) is a solution of (1) and set \( h = \sqrt{a + b \int_{\mathbb{R}^N} |\nabla u|^2}, \quad v(\cdot) = u(h\cdot) = u_h(\cdot). \) Of course \( v \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N) \) and \( u(\cdot) = v(\frac{1}{h}\cdot). \) Moreover

\[- \Delta v(x) = -h^2 \Delta u(hx) = h^2 \frac{g(u(hx))}{a + b \int_{\mathbb{R}^N} |\nabla u|^2} = g(u_h(x)) = g(v(x))\]
and, if we set \( t = \frac{1}{\lambda} \),
\[
t^2 = \frac{1}{a + b \int_{\mathbb{R}^N} |\nabla u|^2} = \frac{1}{a + b t^{2-N} \int_{\mathbb{R}^N} |\nabla v|^2}.
\]

\( \blacksquare \)

**Remark 1.2.** Assume (g1–g4). By the existence result contained in [2], it is evident that for \( N = 3 \) there exists a solution of (1) for any \( a \) and \( b \) positive numbers.

For \( N = 4 \), we should have a solution if and only if there exists \( v \) solution of (2) such that \( b \int_{\mathbb{R}^N} |\nabla v|^2 < 1 \). Taking into account the computations in [2, Section 4.3], we know that the ground state solution of equation (2) has the minimal value of the \( D^{1,2}(\mathbb{R}^N) \) norm among all the solutions of the equation. Then, for \( N = 4 \) we conclude that equation (1) has a solution if and only if the ground state solution \( \bar{v} \) of (2) is such that \( b \int_{\mathbb{R}^N} |\nabla \bar{v}|^2 < 1 \).

Consider the functional of the action related with equation (1)
\[
I(u) = \frac{1}{2} \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \right) \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u),
\]
where \( G(s) = \int_0^s g(z) \, dz \), being \( g \) possibly modified as in [2] in order to make \( I \) a \( C^1 \) functional on \( H^1(\mathbb{R}^N) \). We observe that, for small dimensions, the value of the action computed in the solutions increases as the \( D^{1,2}(\mathbb{R}^N) \) norm increases according to the following

**Proposition 1.3.** Assume \( N = 3 \) or \( N = 4 \). If \( v_1 \) and \( v_2 \) are solutions of (2) and \( \int_{\mathbb{R}^N} |\nabla v_1|^2 < \int_{\mathbb{R}^N} |\nabla v_2|^2 \) and, for \( N = 4 \), we also have \( b \int_{\mathbb{R}^N} |\nabla v_2|^2 < 1 \), then, calling \( t_1 \) and \( t_2 \) the positive numbers such that respectively \( v_1(t_1) \) and \( v_2(t_2) \) are solutions of (1), we have \( t_2 < t_1 \) and \( I(v_1(t_1)) < I(v_2(t_2)) \).

**Proof** By Theorem 1.1, it is immediate to see that \( t_2 < t_1 \). Now observe that any solution of (1) satisfies the Pohozaev identity
\[
a \frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla u|^2 + b \frac{N-2}{2N} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^N} G(u) = 0. \tag{3}
\]
As a consequence the action computed in any solution of (1) is
\[
I(u) = a \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^2 + b \frac{4 - N}{4N} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^2
\]
and then, if \( v \) and \( t > 0 \) are related with \( u \) as in Theorem 1.1, we have that

\[
I(u) = a \frac{t^{2-N}}{N} \int_{\mathbb{R}^N} |\nabla v|^2 + b \frac{4-N}{4N} t^{4-2N} \left( \int_{\mathbb{R}^N} |\nabla v|^2 \right)^2.
\]

Since \( t^2 a + t^{4-N} b \int_{\mathbb{R}^N} |\nabla v|^2 = 1 \), we can cancel the dependence of \( I \) from \( b \)

\[
I(u) = \frac{a}{4} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{4 - N}{4N} \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{t^N}.
\]

(4)

The conclusion easy follows from (4). \( \square \)

In the following Corollary we establish the conditions which guarantee the existence of a ground state solution for \( N \geq 3 \). In particular we correct Theorem 0.6 in [1] for what concerns the dimension \( N = 4 \).

Corollary 1.4. Assume (g1 \ldots g4).

If \( N = 3 \) then equation (1) has a ground state solution. If \( N = 4 \) then equation (1) has a ground state solution if and only if \( b \int_{\mathbb{R}^N} |\nabla v|^2 < 1 \), being \( \bar{v} \) a ground state solution of (2). If \( N \geq 5 \) then equation (1) has a solution if and only if

\[
a \leq \left( \frac{N-4}{N-2} \right)^{\frac{N-2}{N-4}} \left( \frac{2}{(N-4)b \int_{\mathbb{R}^N} |\nabla \bar{v}|^2} \right)^{\frac{N-2}{N-4}},
\]

(5)

being \( \bar{v} \) a ground state solution of (2). Moreover the functional attains the infimum.

**Proof** Since the functional of the action related with equation (2), when computed in the solutions of the equation, is directly proportional to the \( \mathcal{D}^{1,2}(\mathbb{R}^N) \) norm of the solutions (see Remark 1.2), the conclusion for cases \( N = 3 \) and \( N = 4 \) follows immediately by Proposition 1.3 and [2]. If \( N \geq 5 \), by Theorem 1.1 we have to show that there exists a solution \( v \) of equation (2) and \( t > 0 \) such that \( t^2 a + t^{4-N} b \int_{\mathbb{R}^N} |\nabla v|^2 = 1 \). Of course such a couple \((v, t)\) exists if only if there exists \( t_0 > 0 \) such that

\[
t_0^2 a + t_0^{4-N} b \int_{\mathbb{R}^N} |\nabla \bar{v}|^2 = 1
\]

(6)

for \( \bar{v} \) ground state solution of (2). By studying the function \( f(t) = at^2 + b_0 t^{4-N} \) for \( t > 0 \), being \( b_0 = b \int_{\mathbb{R}^N} |\nabla \bar{v}|^2 \), we observe that (6) holds for some \( t_0 \) if and only if

\[
\min_{t>0} f(t) \leq 1.
\]

(7)
An easy computation leads to (5). As a remark we point out that if (7) holds with the strict inequality, then we can find two values \( t_1 < t_2 \) which solve (6) and two corresponding distinct solutions \( \tilde{v}_{t_1} \) and \( \tilde{v}_{t_2} \) to equation (1).

Now we prove that the functional \( I \) attains the minimum. For \( i = 1, 2 \), define \( g_i \) and \( G_i \) as in [2]. Observe that, by (3.4) and (3.5) in [2],

\[
I(u) = \frac{1}{2} \left( a + \frac{b}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \right) \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} G_2(u) - \int_{\mathbb{R}^N} G_1(u)
\]

\[
\geq \frac{1}{2} \left( a + \frac{b}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \right) \int_{\mathbb{R}^N} |\nabla u|^2 + (1 - \varepsilon) \int_{\mathbb{R}^N} G_2(u) - C_\varepsilon \int_{\mathbb{R}^N} |u|^{2N/N-2},
\]

where \( \varepsilon < 1 \) and \( C_\varepsilon > 0 \) are suitable constants.

Since \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2N/N-2}(\mathbb{R}^N) \), for a suitable positive constant \( C \) we have

\[
I(u) \geq \frac{1}{2} \left( a + \frac{b}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \right) \int_{\mathbb{R}^N} |\nabla u|^2
\]

\[
+ \frac{1 - \varepsilon}{2} m \int_{\mathbb{R}^N} |u|^2 - C \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{N}{N-2}},
\]

and, since for \( N \geq 5 \) we have \( 1 < \frac{N}{N-2} < 2 \), we deduce that the functional is bounded below and coercive with respect to \( H^1(\mathbb{R}^N) \) norm.

Now, since for every \( u \in H^1(\mathbb{R}^N) \) and its corresponding Schwarz symmetrization \( u^* \) we have

\[
\int_{\mathbb{R}^N} |\nabla u^*|^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2, \quad \int_{\mathbb{R}^N} G(u^*) = \int_{\mathbb{R}^N} G(u),
\]

we can look for a minimizer of \( I \) in \( H^1_1(\mathbb{R}^N) \), the set of radial functions in \( H^1(\mathbb{R}^N) \). As in [2], we can prove that the functional

\[
u \in H^1_1(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} G_1(u) \in \mathbb{R}
\]

is compact, so, by standard arguments based on Weierstrass Theorem, \( I \) attains the infimum. \( \Box \)

**Remark 1.5.** Suppose \( N \geq 5 \). By previous Corollary we have that if (5) does not hold, then \( I \) is nonnegative in \( H^1(\mathbb{R}^N) \).
References

[1] Azzollini, A.: The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity. *Differential and Integral equations*, 25: 543–554 (2012).

[2] Berestycki, H., Lions, P.L.: Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.*, 82: 313–345 (1983).