Extragradient with Positive Momentum is Optimal for Games with Cross-Shaped Jacobian Spectrum

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Abstract

The extragradient method has recently gained increasing attention, due to its convergence behavior on smooth games. In \(n\)-player differentiable games, the eigenvalues of the Jacobian of the vector field are distributed on the complex plane, exhibiting more convoluted dynamics compared to classical (i.e., single player) minimization. In this work, we take a polynomial-based analysis of the extragradient with momentum for optimizing games with cross-shaped Jacobian spectrum on the complex plane. We show two results. First, based on the hyperparameter setup, the extragradient with momentum exhibits three different modes of convergence: when the eigenvalues are distributed \(i\) on the real line, \(ii\) both on the real line along with complex conjugates, and \(iii\) only as complex conjugates. Then, we focus on the case \(ii\), i.e., when the eigenvalues of the Jacobian have cross-shaped structure, as observed in training generative adversarial networks. For this problem class, we derive the optimal hyperparameters of the momentum extragradient method, and show that it achieves an accelerated convergence rate.

1 INTRODUCTION

While most of the machine learning models are formulated as the minimization of a single loss involving data, a growing number of works adopt game formulations that involve multiple players and objectives. For example, generative adversarial networks (GANs) (Goodfellow et al. 2014), actor-critic models (Pfau and Vinyals 2016), and automatic curricula (Sukhbaatar et al. 2017) all can be formulated as a two-player game, to name a few. Naturally, there is increasing interest in the theoretical understanding of differentiable games.

Due to the interaction of multiple players and objectives, there are issues in optimizing games that are not present in minimizing a single objective. Most notably, the eigenvalues of the game Jacobian lie on the complex plane, unlike single-objective minimization, where the eigenvalues of the Hessian lie on the real line. Further, even for simple bilinear games, standard algorithms—like the gradient method—fail to converge due to the presence of rotational dynamics (Mescheder, Geiger, et al. 2018; Balduzzi et al. 2018; Gidel, Hemmat, et al. 2019; Berard et al. 2020).

On the other hand, the extragradient method (EG), originally introduced in Korpelevich (1976) for saddle-point problems, was proven to converge for bilinear games (Tseng 1995). As such, many recent works theoretically analyze EG from different angles, such as variational inequality (Gidel, Berard, et al. 2018; Gorbunov et al. 2022), stochastic (Mishchenko et al. 2020; Li et al. 2021), and distributed (Liu et al. 2020; Beznosikov et al. 2021) perspectives.

Most existing works, including some of the aforementioned ones, analyze EG by assuming some structure on the objectives, such as (strong) monotonicity or Lipschitzness (Solodov and Svaiter 1999; Tseng 1995; Ryu et al. 2019). Such assumptions, in the context of differentiable games, subsequently confine the distribution of the eigenvalues of the game Jacobian. For instance, the strong monotonicity and Lipschitzness assumption translates to the game Jacobians with their eigenvalues in the intersection between a circle and a half plane (Azizian, Scieur, et al. 2020). However, such assumptions rarely hold in applications such as training GANs (Goodfellow et al. 2014), which is often an empirical motivation to study differentiable games and convergent algorithms therein, including EG. Thus, a significant gap between theory and practice has emerged.

In this work, we take a different approach, and analyze the momentum extragradient method (EGM) through the lens of (residual) polynomials, inspired by recent works that

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...take a similar approach to analyze (single-objective) optimization algorithms (Lacotte and Pilanci 2020; Pedregosa and Scieur 2020; Scieur and Pedregosa 2020; Goujaud, Scieur, et al. 2022). Such approach enables us to consider a setting with a linear vector field where the distribution of the eigenvalues of the game Jacobian has a cross-shape, as illustrated in Figure 1, and empirically observed in Berard et al. (2020) in training GANs.

Our contributions can be summarized as follows:

• We derive the residual polynomials of EGM, and identify its three different modes of convergence depending on the hyperparameter setup, that can be applied to different eigenvalue structures of the game Jacobian (c.f., Theorem 2).

• We then focus on the case where the eigenvalues of the game Jacobian have a cross-shaped structure, and derive the (asymptotic) accelerated convergence rate of EGM, as well as its optimal hyperparameters for the considered class of games (c.f., Theorem 3 and Corollary 1).

• We compare the convergence rate with other first order methods including the gradient method (GD), the momentum gradient method (GDM), and the extragradient method (EG). For the class of games we consider with the cross-shaped eigenvalue structure, we show that none of the aforementioned methods (asymptotically) converge at an accelerated rate (c.f., Remarks 2 and 3).

1.1 Problem Setup

Following Letcher et al. (2019) and Balduzzi et al. (2018), we define the $n$-player differentiable game as a family of twice continuously differentiable losses $\ell_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}$, for $i = 1, \ldots, n$. The player $i$ controls the parameter $w(i) \in \mathbb{R}^{d_i}$. We denote the concatenated parameters by $w = [w(1), \ldots, w(n)] \in \mathbb{R}^d$, where $d = \sum_{i=1}^n d_i$.

For this problem, a Nash equilibrium satisfies:

$$w^{(i)}* \in \arg \min_{w(i) \in \mathbb{R}^{d_i}} \ell_i(w^{(i)}, w^{(-i)*}) \quad \forall i \in \{1, \ldots, n\},$$

where the notation $^{(-i)}$ denotes all indices except for $i$. We also define the vector field of the game $v$ as the concatenation of the individual gradients:

$$v(w) = [\nabla w(1)\ell_1(w), \ldots, \nabla w(n)\ell_n(w)]^T,$$

and its associated Jacobian $\nabla v$ as:

$$\nabla v(w) = \begin{bmatrix}
\nabla^2 w(1)\ell_1(w) & \cdots & \nabla w(n)\nabla w(1)\ell_1(w) \\
\vdots & \ddots & \vdots \\
\nabla w(1)\nabla w(n)\ell_n(w) & \cdots & \nabla^2 w(n)\ell_1(w)
\end{bmatrix}.$$  

Unfortunately, finding Nash equilibria for general games is impractical to solve (Shoham and Leyton-Brown 2008; Letcher et al. 2019). Instead, we focus on finding a stationary point of the vector field $v$, since a Nash equilibrium is always a stationary point of the gradient dynamics. That is, we want to solve:

$$\text{Find } w^* \in \mathbb{R}^d \text{ such that } v(w^*) = 0. \quad (1)$$

**Notation.** We denote the spectrum of a matrix $M$ by $\text{Sp}(M)$, and its spectral radius by $\rho(M) := \max \{ |\lambda | : \lambda \in \text{Sp}(M) \}$. $M \succ 0$ denotes that $M$ is a positive-definite matrix. $\Re(z)$ and $\Im(z)$ respectively denote the real and the imaginary part of a complex number $z$.

1.2 Related Work

The above game formulation is strictly more general than minimization, and includes many problems such as saddle-point problems (Nicolaides 1982), bilinear games (Liang and Stokes 2019; Mokhtari et al. 2020), and variational inequality problems (Tseng 1995; Facchinei and Pang 2003). For minimization, the vector field $v$ and the Jacobian $\nabla v$ are, respectively, the gradient and the Hessian of the objective function. The Hessian is a symmetric matrix, so its eigenvalues lie on the real line; the ratio of the largest and the smallest eigenvalues is called the condition number. It is now a common wisdom in numerical linear algebra and optimization theory how the notion of condition number characterizes the difficulty of a (strongly convex) minimization problem (Nesterov 2003). In conjunction with provable lower bounds within various function classes (Nemirovskij and Yudin 1983), an algorithm is considered “optimal” for a class of functions when its attainable upper bound on the convergence rate matches the lower bound of that function class (Nesterov 1983; Nesterov 2003).

Differentiable games, on the other hand, have much more convoluted dynamics, as the eigenvalues of the Jacobian at the solution are distributed over the complex plane (Meschder, Nowozin, et al. 2017; Balduzzi et al. 2018). To address this, recent line of works proposes new algorithms with better upper bounds on the convergence rate (Nemirovski 2004; Palaniappan and Bach 2016; Mescheder, Nowozin, et al. 2017; Daskalakis, Ilyas, et al. 2017; Gidel, Berard, et al. 2018; Gidel, Hemmat, et al. 2019; Mokhtari et al. 2020). Vast majority of the aforementioned literature proves convergence based on two approaches: either bounding $i)$ the magnitude, or $ii)$ the real part of the eigenvalues of the (submatrices of the) Jacobian. In this paper we make instead a more fine-grained assumption on these eigenvalues, to obtain faster algorithms and convergence rates.

To understand the difficulty of a differentiable game, Az-
cross-shaped spectrum model above would include some of the observed GAN spectra.

The cross-shaped spectrum in (2) is also observed in a class of quadratic games. Specifically, consider the following two player quadratic game, where \( x \in \mathbb{R}^{d_1} \) and \( y \in \mathbb{R}^{d_2} \) are the parameters controlled by each player, whose loss function respectively is:

\[
\ell_1(x, y) = \frac{1}{2} x^\top S_1 x + x^\top M_{12} y + x^\top b_1 \quad \text{and} \quad (3)
\]

\[
\ell_2(x, y) = \frac{1}{2} y^\top S_2 y + y^\top M_{21} x + y^\top b_2, \quad (4)
\]

where \( S_1, S_2 > 0 \). Then, the vector field can be written as:

\[
v(x, y) = \begin{bmatrix} S_1 x + M_{12} y + b_1 \\ M_{21} x + S_2 y + b_2 \end{bmatrix} = Aw + b, \quad \text{where}
\]

\[
A = \begin{bmatrix} S_1 & M_{12} \\ M_{21} & S_2 \end{bmatrix}, \quad w = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \quad (5)
\]

If \( S_1 = S_2 = 0 \) and \( M_{12} = -M_{21}^\top \), the game Jacobian \( \nabla v = A \) has only purely imaginary eigenvalues (Azizian, Scieur, et al. 2020, Lemma 7), recovering bilinear games.

Further, when \( M_{12} = -M_{21}^\top \), and they share a common basis with \( S_1 \) and \( S_2 \) as in the below proposition, then \( A \) has a cross-shaped spectrum. We utilize this proposition in designing our experiments in Section 4.

**Proposition 1.** Let \( A \) be a matrix of the form \( \begin{bmatrix} S_1 & B \\ -B^\top & S_2 \end{bmatrix} \), where \( S_1, S_2 > 0 \). Without loss of generality, assume that \( \dim(S_1) > \dim(S_2) = d \). Then, \( \text{Sp}(A) \) has a cross-shape if there exists orthonormal matrices \( U, V \) and diagonal matrices \( D_1, D_2 \) such that

\[
S_1 = U \text{diag}(a, \ldots, a, D_1) U^\top, \quad S_2 = \text{diag}(a, \ldots, a), \quad \text{and} \quad B = U D_2 V^\top.
\]

Note that all proofs can be found in the supplementary material. To solve (1) for games with Jacobian eigenvalue structure in (2), we use the extragradient with momentum, which we expound in the next section.

**2 ALGORITHM**

**2.1 Momentum Extragradient via Chebyshev Polynomials**

To solve (1) for games with Jacobian eigenvalue structure in (2), we focus on the extragradient method with momentum (EGM):

\[
w_{t+1} = w_t - hv(w_t - \gamma v(w_t)) + m(w_t - w_{t-1}), \quad (6)
\]

where \( h \) is the step size, \( \gamma \) is the extrapolation step size, and \( m \) is the momentum parameter. The extragradient (EG) method (without momentum) was first proposed in Korpelevich (1976) for saddle point problems, and recently received much attention due to its convergence behavior for
some class of differentiable games such as bilinear games, for which the standard gradient method diverges (Gidel, Hemmat, et al. 2019; Azizian, Scieur, et al. 2020; Azizian, Mitliagkas, et al. 2020). For completeness, we remind the gradient method (GD) and the momentum gradient method (GDM):

\[
\text{(GD)} \quad w_{t+1} = w_t - h v(w_t), \quad \text{and} \\
\text{(GDM)} \quad w_{t+1} = w_t - h v(w_t) + m (w_t - w_{t-1}).
\]

(7)

We take a polynomial-based approach to analyze EGM. Such analysis dates back to the development of the conjugate gradient method (Hestenes and Stiefel 1952), and is still actively used, for instance, to derive lower bounds (Arjevani and Shamir 2016), develop accelerated decentralized algorithms (Berthier et al. 2020), and analyze average-case performance (Pedregosa and Scieur 2020).

On that end, we use the following lemma (Chihara 2011), which elucidates the intimate connection between first-order methods\(^2\) and (residual) polynomials, when the vector field satisfies \(v(w) = A w + b\):

**Lemma 1** (Chihara 2011). There exists a real polynomial \(p_t\) of degree at most \(t\) satisfying

\[
w_t - w^* = p_t(A)(w_0 - w^*),
\]

where \(p_t(0) = 1\), and \(v(w^*) = A w^* + b = 0\).

By taking norms, (8) further implies the following worst-case convergence rate:

\[
\|w_t - w^*\| = \| p_t(A)(w_0 - w^*)\| \\
\leq \|p_t(A)\| \cdot \|w_0 - w^*\|.
\]

(9)

As EGM in (6) is a first-order method (Arjevani and Shamir 2016; Azizian, Scieur, et al. 2020), we can study the associated residual polynomials, as we summarize in the following theorem:

**Theorem 1** (Residual polynomials of EGM). Consider the momentum gradient method in (6). Its associated residual polynomials are:

\[
\tilde{P}_0(\lambda) = 1, \quad \tilde{P}_1(\lambda) = 1 - \frac{h \lambda (1 - \gamma)}{1 + m}, \quad \text{and} \\
\tilde{P}_{t+1}(\lambda) = (1 + m - h \lambda (1 - \gamma) \tilde{P}_t(\lambda) - m \tilde{P}_{t-1}(\lambda).
\]

Further, let \(T_t(\cdot)\) and \(U_t(\cdot)\) be the Chebyshev polynomials of the first and the second kind, respectively. Then, the above expressions simplify to

\[
P_t(\lambda) = m^{t/2} \left( \frac{2m}{1 + m} T_t(\sigma(\lambda)) + \frac{1 - m}{1 + m} U_t(\sigma(\lambda)) \right)
\]

(10)

with \(\sigma(\lambda) = \frac{1 + m - h \lambda (1 - \gamma)}{2 \sqrt{m}}\),

(11)

where we refer to the term \(\sigma(\lambda)\) as the link function.

As can be seen in (11), the link function for EGM is quadratic with respect to \(\lambda\). As we show below, GDM admits an almost identical expression, but with a different link function. For completeness, we write below the residual polynomials of GDM, expressed in terms of the Chebyshev polynomials (Pedregosa 2020):

\[
P_t^{\text{GDM}}(\lambda) = m^{t/2} \left( \frac{2m}{1 + m} T_t(\xi(\lambda)) + \frac{1 - m}{1 + m} U_t(\xi(\lambda)) \right)
\]

(12)

with \(\xi(\lambda) = \frac{1 + m - h \lambda}{2 \sqrt{m}}\).

(13)

Notice that the Chebyshev polynomial-based expression of EGM in (10) and that of GDM in (12) are identical except for the link functions \(\sigma(\lambda)\) and \(\xi(\lambda)\), which enter as arguments in \(T_t(\cdot)\) and \(U_t(\cdot)\). This difference in link functions is crucial, as the Chebyshev polynomials behave very differently based on the domain, as shown in the following lemma.

**Lemma 2** (Goujaud and Pedregosa 2022). Let \(z\) be a complex number. The sequence

\[
\left( \left| \frac{2m}{1 + m} T_t(z) + \frac{1 - m}{1 + m} U_t(z) \right| \right)_{t \geq 0}
\]

grows exponentially in \(t\) for \(z \notin [-1, 1]\), while in that interval, the following bounds hold:

\[
|T_t(z)| \leq 1 \quad \text{and} \quad |U_t(z)| \leq t + 1.
\]

(14)

Therefore, to study the convergence behavior of EGM, we are interested in the case where the set of step sizes and the momentum parameters lead to \(|\sigma(\lambda)| \leq 1\), so that we can use the bounds in (14). We call the set of hyperparameters such that the image of the link function lies in the interval \([-1, 1]\) as the robust region, i.e., \(\sigma^{-1}([-1, 1])\).

### 2.2 Three Modes of the Momentum Extragradient

Within the robust region of EGM, we can compute its worst-case convergence rate as follows:

\[
r_t := \max_{\lambda \in S^*} |P_t(\lambda)|
\]

\[
= \max_{\lambda \in S^*} \left| m^{t/2} \left( \frac{2m}{1 + m} T_t(\sigma(\lambda)) + \frac{1 - m}{1 + m} U_t(\sigma(\lambda)) \right) \right|
\]

\[
\leq m^{t/2} \left( \frac{2m}{1 + m} \max_{\lambda \in S^*} |T_t(\sigma(\lambda))| + \frac{1 - m}{1 + m} \max_{\lambda \in S^*} |U_t(\sigma(\lambda))| \right)
\]

\[
\leq m^{t/2} \left( \frac{2m}{1 + m} + \frac{1 - m}{1 + m} (t + 1) \right) \leq m^{t/2}(t + 2),
\]

(15)

where the last inequality uses \(m \in (0, 1)\). Note that \(r_t\) denotes the worst-case convergence rate by definition. Indeed, in conjunction with (9), it follows \(\left\|w_t - w^*\right\| \leq r_t\). To get the asymptotic rate, we take the limit supremum of the \(2t\)-th root of \(r_t\):

\[
\limsup_{t \to \infty} \sqrt[2t]{r_t} = \limsup_{t \to \infty} \left( m^{t/2}(t + 2) \right)^{1/2} = \sqrt{m}.
\]

(16)

The reason why we take the \(2t\)-th root of \(r_t\) is to normalize by the number of vector field computations, as we compare

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\(^2\)We consider first-order methods as those that only use first-order information in each iteration.
in Section 3 the asymptotic rate of EGM in (16) with other
gradient methods that use a single vector field computation
in the recurrences, such as GD and GDM.

Recall that the Chebyshev polynomial expressions of EGD
in (10) and that of GD^3 are identical except for the link
functions. Hence, the convergence rate in (15) applies to
both EGM and GDM, as long as the link functions |σ(λ)|
and |ξ(λ)| are bounded by 1.

Yet, due to the difference in link functions, the robust re-

gion of EGM and GDM are drastically different. Specifi-
cally, due to the quadratic link function of EGM, the robust
region is more flexible; in particular, this link function can
still satisfy the upper bounds (14) in Lemma 2 for a sub-
set of complex values, resulting in the convergence rate of
(15). Hence, compared to GDM whose link function is lin-
ear in λ and so can only satisfy (14) for a subset of real val-
ues, EGM enjoys the convergence rate in (15) for a much
wider set of λ values.

The robust region of EGM can be described with the four
extreme points below:

\[
\begin{align*}
\sigma^{-1}(-1) &= \frac{1}{2^\gamma} \pm \frac{1}{4^\gamma} - \frac{(1+\sqrt{m})^2}{h\gamma}, \text{ and } (17) \\
\sigma^{-1}(1) &= \frac{1}{2^\gamma} \pm \frac{1}{4^\gamma} - \frac{(1-\sqrt{m})^2}{h\gamma}. \quad (18)
\end{align*}
\]

Depending on the choice of hyperparameters, the above
four extreme points can be distributed differently on three
different modes, which we summarize in the next theorem.

**Theorem 2.** Consider the extragradient with momentum in
(6), expressed with the Chebyshev polynomials as in (10).

Then, the robust region summarized in (17) and (18) have
the following three modes:

1. If \( \frac{h}{2^\gamma} \geq (1 + \sqrt{m})^2 \), then \( \sigma^{-1}(-1) \) and \( \sigma^{-1}(1) \) are all
real numbers;

2. If \( (1 - \sqrt{m})^2 \leq \frac{h}{2^\gamma} < (1 + \sqrt{m})^2 \), then \( \sigma^{-1}(-1) \) are complex, and \( \sigma^{-1}(1) \) are real;

3. If \( (1 - \sqrt{m})^2 > \frac{h}{2^\gamma} \), then \( \sigma^{-1}(-1) \) and \( \sigma^{-1}(1) \) are all
complex numbers.

We empirically verify Theorem 2 in Figure 2. Note that the
hyperparameters are set up so that each case in Theorem 2
is covered. In the remainder of the paper, we specifically
focus on the second case, which can be modeled via the
Jacobian spectrum model in (2).

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3 OPTIMAL PARAMETERS AND
CONVERGENCE RATES

3.1 Optimal Parameters for Cross-Shaped Games

We now focus on solving (1) for games with cross-shaped
Jacobian eigenvalue structure as in (2). Therefore, in the
remainder of the paper, we focus on the second case of
Theorem 2, which is illustrated in the middle panel of Fig-
ure 2. As mentioned in Section 1.3, the cross-shaped eigen-
value structure of game Jacobians have been empirically
observed in [Berard et al. 2020, Figure 4] while training
GANs in various settings. Therefore, we analyze the more
realistic and strictly more general setting than the purely
imaginary interval \([-bi, bi]\) commonly considered in the
bilinear game literature ([Liang and Stokes 2019; Azizian,
Scieur, et al. 2020; Mokhtari et al. 2020]).

We now connect the cross-shaped spectrum model in (2)
and the robust region of EGM described with four extra
points in (17) and (18). As we focus on the second case
of Theorem 2, the hyperparameters satisfy the condition
\( (1 - \sqrt{m})^2 \leq \frac{h}{2^\gamma} < (1 + \sqrt{m})^2 \). In this case, we can write
the robust region of EGM as:

\[
\begin{align*}
\sigma^{-1}([-1, 1]) &= \left[ \frac{1}{2^\gamma} - \sqrt{\frac{1}{4^\gamma} \left(1 - \frac{(1-\sqrt{m})^2}{h\gamma}\right)}, \frac{1}{2^\gamma} + \sqrt{\frac{1}{4^\gamma} \left(1 - \frac{(1-\sqrt{m})^2}{h\gamma}\right)} \right] \\
&\cup \left[ \frac{1}{2^\gamma} - \sqrt{\frac{1}{4^\gamma} \left(1 - \frac{(1+\sqrt{m})^2}{h\gamma}\right)}, \frac{1}{2^\gamma} + \sqrt{\frac{1}{4^\gamma} \left(1 - \frac{(1+\sqrt{m})^2}{h\gamma}\right)} \right]. \quad (19)
\end{align*}
\]

Here, the first interval lies on \( \mathbb{R} \), as the square root term is
real; conversely, in the second interval, the square root term
is imaginary, with the fixed real component: \( \frac{1}{2^\gamma} \).

The optimal hyperparameters of EGM in terms of the worst-case asymptotic convergence rate in (16) for this
problem occurs when the robust region in (19) and the spec-
trum model in (2) coincide. This condition can be summa-
rized in the following three equalities:

\[
\begin{align*}
\frac{1}{2^\gamma} - \sqrt{\frac{1}{4^\gamma} \left(1 - \frac{(1-\sqrt{m})^2}{h\gamma}\right)} &= \mu, \\
\frac{1}{2^\gamma} + \sqrt{\frac{1}{4^\gamma} \left(1 - \frac{(1-\sqrt{m})^2}{h\gamma}\right)} &= L, \text{ and} \\
\sqrt{\frac{1}{4^\gamma} \left(1 + \sqrt{m}\right)^2} - \frac{1}{2^\gamma} &= c.
\end{align*}
\]

Based on the above equalities, we can compute the optimal
hyperparameters, as summarized in the next theorem:

**Theorem 3 (Optimal hyperparameters for EGM).** Con-
sider solving (1) for games where the Jacobian has cross-
shaped spectrum as in (2). For this problem, the optimal
hyperparameters for the momentum extragradient method
in (6) can be set as follows:

\[
\begin{align*}
h &= \frac{16(\mu + L)}{(\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L})^2}, \quad \gamma = \frac{1}{\mu + L}, \text{ and} \\
m &= \left( \frac{\sqrt{4c^2 + (\mu + L)^2} - \sqrt{4\mu L}}{\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L}} \right)^2.
\end{align*}
\]
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We summarize EGM with the above optimal hyperparameters in Algorithm 1. Recalling (16), we immediately get the asymptotic convergence rate from Theorem 3. Further, this formula can be simplified in the ill-conditioned regime, where the inverse condition number $\tau := \mu/L \to 0$:

$$\sqrt{m} = \left( \frac{\sqrt{4c^2 + (\mu + L)^2} - \sqrt{4\mu L}}{\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L}} \right)^{1/2} = \left( \frac{(2c/L)^2 + (1 + \tau)^2 - 2\sqrt{\tau}}{2c/L)^2 + (1 + \tau)^2 + 2\sqrt{\tau}} \right)^{1/2} \approx 1 - \frac{2\sqrt{\tau}}{2(2c/L)^2 + 1} + o(\sqrt{\tau}).$$

(20)

From (20), we can see that EGM achieves accelerated convergence rate $1 - O(\sqrt{\mu/L})$, as long as $c = O(L)$.

A special case of the spectrum model in (2) is when the length of the first set and the second equals, i.e., when $c = \frac{L - \mu}{2}$. Then, the optimal hyperparameters of EGM in Theorem 3 further simplify, as in the next corollary:

**Corollary 1.** Consider solving (1) for games where the Jacobian has cross-shaped spectrum as in (2). Further, assume that $c = \frac{L - \mu}{2}$. Then, the optimal hyperparameters for the momentum extragradient method in (6) can be set as follows:

$$h = \frac{8(\mu + L)}{(\sqrt{\mu^2 + L^2} + \sqrt{4\mu L})^2}, \quad \gamma = \frac{1}{\mu + L}, \quad \text{and}$$

$$m = \left( \frac{\sqrt{\mu^2 + L^2} - \sqrt{4\mu L}}{\sqrt{\mu^2 + L^2} + \sqrt{4\mu L}} \right)^2.$$

Again, from Corollary 1, we can immediately get the asymptotic convergence rate via (16), which can be interpreted further in the ill-conditioned regime where $\tau \to 0$:

$$\sqrt{m} = \left( \frac{\sqrt{\mu^2 + L^2} - \sqrt{4\mu L}}{\sqrt{\mu^2 + L^2} + \sqrt{4\mu L}} \right)^{1/2} \approx 1 - \sqrt{2\sqrt{\tau}} + o(\sqrt{\tau}) \approx 1 - \sqrt{2} \sqrt{\frac{\mu}{L}}. \quad (21)$$

Similarly to (20), in the simplified setting of the Jacobian spectrum model in (2) with $c = \frac{L - \mu}{2}$, EGM with optimal hyperparameters achieve an accelerated convergence rate.

**Remark 1.** Notice that the optimal momentum $m$ in both Theorem 3 and Corollary 1 are positive. This is in contrast with Gidel, Hemmat, et al. (2019) where the gradient method with negative momentum is studied. This difference elucidates the different dynamics of how momentum interacts with the gradient and the extragradient methods.

**Algorithm 1:** Momentum extragradient with optimal hyperparameters for cross-shaped spectrum in (2)

**Input:** Initialization $w_0$, problem constants $\mu, L, c$.

**Set:**

$$h = \frac{16(\mu + L)}{(\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L})^2}, \quad \gamma = \frac{1}{\mu + L}, \quad \text{and}$$

$$m = \left( \frac{\sqrt{4c^2 + (\mu + L)^2} - \sqrt{4\mu L}}{\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L}} \right)^2.$$

$$w_1 = w_0 - \frac{h}{1 + m} v(w_0 - \gamma v(w_0))$$

for $t = 1, 2, \ldots$ do

$$w_{t+1} = w_t - h v(w_t - \gamma v(w_t)) + m(w_t - w_{t-1})$$

end

### 3.2 Comparison with Other Methods

We now compare the asymptotic rates of EGM we obtained in the previous subsection with other first order methods, including GD, GDM, and EG. Note that for simplicity we focus on the setting when $c = \frac{L - \mu}{2}$ in (2), i.e., when the two intervals in (2) have the same length. The optimal hyperparameters for this setting was presented in Corollary 1, and the asymptotic convergence rate was obtained in (21). The analysis of the general case with any $c$ can be extended with similar approach, which we leave for future work.

In Azizian, Mitliagkas, et al. (2020), GD and EG are interpreted as fixed-point iterations, following Polyak (1987). In this framework, the iterates of a method are considered to be generated by the following scheme:

$$w_{t+1} = F(w_t), \quad \forall t \geq 0,$$

(22)

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is an operator representing the method. Analyzing the above scheme in general settings
is usually challenging as $F$ can be any nonlinear operator. However, if $F$ is twice differentiable and $w$ is near a stationary point $w^*$, the analysis can be simplified by linearizing $F$ at the stationary point, i.e.,

$$ F(w) \approx F(w^*) + \nabla F(w^*)(w - w^*). $$

Then, for $w_0$ in a neighborhood of $w^*$, Polyak (1987) shows one can obtain an asymptotic convergence rate of (22) by $\rho(\nabla F(w^*)) \leq \rho^* < 1$, meaning that (22) converges linearly to $w^*$ at the rate $O((\rho^* + \varepsilon)^t)$ for $\varepsilon \geq 0$. Further, if $F$ is linear, $\varepsilon = 0$.

Under this scheme, the fixed point operators $F_{h}^{GD}$ and $F_{h}^{EG}$ of GD and EG, respectively, can be written as follows:

$$ (GD) \quad w_{t+1} = w_t - hv(w_t) = F_{h}^{GD}(w_t), \quad \text{and} $$

$$ (EG) \quad w_{t+1} = w_t - hv(w_t - hv(w_t)) = F_{h}^{EG}(w_t). $$

The local convergence rate can now be obtained by bounding the spectral radius of the Jacobian of the operators under certain assumptions. We summarize the relevant theorems from Gidel, Hemmat, et al. (2019) and Azizian, Mittiagkas, et al. (2020) below:

**Theorem 4** (Azizian, Mittiagkas, et al. 2020). Let $w^*$ be a stationary point of $v$. Denote by $\Delta^* := \text{Sp}(\nabla v(w^*))$ for notational brevity. Further, assume the eigenvalues of $\nabla v(w^*)$ all have positive real parts. Then,

1. **(Gidel, Hemmat, et al. 2019)** For the gradient method in (23) with step size $h = \min_{\lambda \in \Delta^*} \Re(1/\lambda)$, it satisfies:

$$ \rho(\nabla F_{h}^{GD}(w^*))^2 \leq 1 - \min_{\lambda \in \Delta^*} \Re(1/\lambda) \min_{\lambda \in \Delta^*} \Re(\lambda). $$

2. For the extragradient method in (24) with step size $h = (4 \max_{\lambda \in \Delta^*} |\lambda|^{-1})^{-1}$, it satisfies:

$$ \rho_{EG}^2 \leq 1 - \frac{1}{4} \left( \min_{\lambda \in \Delta^*} \frac{\Re(\lambda)}{\max_{\lambda \in \Delta^*} |\lambda|} + \min_{\lambda \in \Delta^*} \frac{|\lambda|^2}{|\lambda|^2} \right), $$

where we denoted $\rho_{EG} := \rho(\nabla F_{h}^{EG}(w^*))$ for brevity.

Since our spectrum model in (2) satisfies the condition that the eigenvalues of $\nabla v(w^*)$ all have positive real parts, we can obtain the convergence rate of GD and EG, based on Theorem 4. With our (simplified) spectrum model with $c = \frac{L-\mu}{2}$, for GD, it follows that $\min_{\lambda \in \Delta^*} \Re(\lambda) = \mu$, and $\min_{\lambda \in \Delta^*} \Re(1/\lambda) = 1/L$. Similarly, for EG, it follows that $\max_{\lambda \in \Delta^*} |\lambda| = \mu$, and $\max_{\lambda \in \Delta^*} |\lambda|^2 = L^2$. Furthermore, $\min_{\lambda \in \Delta^*} |\lambda|^2 = \mu^2$ if $L \geq (\sqrt{2} + 1)\mu$, and $\min_{\lambda \in \Delta^*} |\lambda|^2 = (L - \mu)^2/2$ otherwise.

With the above convergence rates of GD and EG in (25) and (26), respectively, for solving (1) with the Jacobian spectrum structure in (2), we summarize this in the next remark.

**Remark 2.** With the conditions in **Theorem 4**, for games with cross-shaped Jacobian spectrum as in (2) with $c = \frac{L-\mu}{2}$, the above analysis leads to the following local convergence rates of GD and EG:

$$ \rho(\nabla F_{h}^{GD}(w^*))^2 \leq 1 - \frac{1}{L}, \quad \text{and} $$

$$ \rho_{EG}^2 \leq \begin{cases} 1 - \frac{1}{4} \left( \frac{\mu + \mu^2}{\mu L^2} \right) & \text{if } L \geq (\sqrt{2} + 1)\mu, \\ 1 - \frac{1}{4} \left( \frac{\mu + (L-\mu)^2}{\mu L^2} \right) & \text{otherwise}. \end{cases} $$

We see that both GD and EG have non-accelerated convergence rates, unlike the rate of EGM we obtained in (21), for the same problem class.

We now compare the convergence rate of EGM with that of GDM, which iterates as in (7). In Azizian, Scieur, et al. (2020), it was shown that GD is optimal for the class of games where the eigenvalues of the Jacobian is included in a disc in the complex plane, indicating acceleration is not possible for this problem class. However, it is well-known that GDM achieves the optimal convergence rate when the the eigenvalues of the Jacobian lie on the (strictly positive) real line segment, which amounts to (strongly-convex) minimization. Hence, Azizian, Scieur, et al. (2020) studies the intermediate case, when the eigenvalues of the Jacobian are included in an ellipse, which can be thought of as the real segment $[\mu, L]$ perturbed with $\varepsilon$ in an elliptic way. That is, they consider the following spectral shape:

$$ K_\varepsilon = \begin{cases} z \in \mathbb{C} : \left( \frac{|z| - \mu + L}{L - \mu} \right)^2 + \left( \frac{2\varepsilon}{\tau} \right)^2, & \text{if } \varepsilon > 0 \text{ and } \frac{\mu}{L} \leq \frac{1}{2}. \end{cases} $$

Similarly to GD and EG above, in Azizian, Scieur, et al. 2020, GDM is interpreted as a fixed point iteration, as in (7),

$$ w_{t+1} = w_t - hv(w_t) + m(w_t - w_{t-1}) = F_{GD}(w_t, w_{t-1}). $$

To study the convergence rate of GDM, we use the following theorem from Azizian, Scieur, et al. (2020):

**Theorem 5** (Azizian, Scieur, et al. 2020). Define $\epsilon(\mu, L)$ as $\epsilon(\mu, L) = \frac{\mu}{L}^\theta$ with $\theta > 0$ and $a \land b = \min(a, b)$. If $\text{Sp}(\nabla F_{GD}(w^*, w^*)) \subset K_\varepsilon$, and when $\tau \to 0$, it satisfies:

$$ \rho_{GD} \leq \begin{cases} 1 - 2\sqrt{\tau} + O(\tau^{\theta/4}), & \text{if } \theta > 1, \\ 1 - 2(\sqrt{\theta^2 - 1} + \sqrt{\tau} + O(\tau)) & \text{if } \theta = 1, \\ 1 - \tau^{\theta-\theta} + O(\tau^{\theta/4}), & \text{if } \theta < 1, \end{cases} $$

where we denoted $\rho_{GD} := \rho(\nabla F_{GD}(w^*, w^*))$ for notational brevity. Note that the hyperparameters $h$ and $m$ are functions of $\mu$, $L$, and $\varepsilon$ only.

---

4[Note: A visual illustration of this ellipse can be found in Azizian, Scieur, et al. 2020, Figure 2.]

5[As the GDM recursion for $w_{t+1}$ depends on $w_t$ and $w_{t-1}$, Azizian, Scieur, et al. 2020 uses an augmented operator; see Lemma 2 in that work for details.]
From Theorem 5, we see that GDM achieves the accelerated rate, i.e., \(1 - O(\sqrt{\theta})\), until \(\theta = \frac{1}{2}\). In other words, the biggest elliptic perturbation \(\epsilon\) where GDM permits the accelerated rate is \(\epsilon = \sqrt{\mu L}\). We interpret Theorem 5 for games with cross-shaped Jacobian spectrum in the following remark:

**Remark 3.** With the conditions in Theorem 5, for games with cross-shaped Jacobian spectrum in (2) with \(c = \frac{L - \mu}{2}\), GDM cannot achieve an accelerated rate when \(\frac{L - \mu}{2} = c > \epsilon = \sqrt{\mu L}\).

Using \(L > \mu\), the above inequality implies \(\frac{L}{\mu} > \sqrt{5}\). That is, when the condition number exceeds \(\sqrt{5} \approx 2.236\), GDM cannot be accelerated; on the contrary, as we showed in (20) and (21), EGM converges at the accelerated rate in the ill-conditioned regime.

## 4 EXPERIMENTS

In this section, we perform synthetic experiments on optimizing a game where the Jacobian has a cross-shaped spectrum in (2). Specifically, we focus on two-player quadratic games, where player 1 controls \(x \in \mathbb{R}^{d_1}\) with loss function (3), and player 2 controls \(y \in \mathbb{R}^{d_2}\) with loss function (4).

In our setting, the corresponding vector field in (5) satisfies \(M_{12} = -M_{21}^*\), but \(S_1\) and \(S_2\) can be nonzero symmetric matrices. Further, the Jacobian \(\nabla v = A\) has the cross-shape eigenvalue structure in (2), with \(c = \frac{L - \mu}{2}\) to reflect the analyses we provided in Section 3.2.

We utilize Proposition 1 to design our experiments. We briefly describe how we generate \(A\) with the desired spectrum. Notice that \(\text{Sp} \left( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right) = \{a \pm bi\}\). Hence, we generate several \(2 \times 2\) matrices of the above form with the desired eigenvalues that align with (2), and put them in a block diagonal matrix. We then perform a change of basis to keep the same eigenvalues, but make the experiments more realistic.

For the problem constants, we use \(\mu = 1\), and \(L = 200\). The optimum \(x^* y^*\) = \(w^* \in \mathbb{R}^{200}\) is generated using the standard normal distribution. For simplicity, we assume \(b = [b_1 b_2]^T = [0 0]^T\).

We compare GD in (23), GDM in (7), EG in (24), and EGM in (6). All algorithms are initialized with 0. We plot the experimental results in Figure 3.

For EGM (optimal), we set the hyperparameters using Corollary 1. For GD (theory) and EG (theory), we set the hyperparameters using Theorem 4. For GDM (grid search), as Theorem 5 does not give a specific form for hyperparameter setup, we perform grid search of \(h^\text{GDM}\) and \(m^\text{GDM}\), and choose the best performing ones. Specifically, we consider \(0.005 \leq h^\text{GDM} \leq 0.015\) with \(10^{-3}\) increment, and \(0.01 \leq m^\text{GDM} \leq 0.09\) with \(10^{-2}\) increment. Further, as Theorem 4 might be conservative, we perform grid searches for GD and EG as well. For GD (grid search), we use the same setup as \(h^\text{GDM}\). For EG (grid search), we use \(0.001 \leq h^\text{EG} \leq 0.05\) with \(10^{-4}\) increment.

There are several remarks to make. First, EGM (optimal) with the hyperparameters obtained in Corollary 1 indeed exhibit orders of magnitude faster rate of convergence, even compared to other methods with grid-search hyperparameter tuning, corroborating our theoretical findings in Section 3. Second, while EG (theory) is slower than GD (theory), confirming Remark 2, GD (grid search) can be tuned to converge faster via grid search. Third, even though the best performance of GDM (grid search) is obtained through grid search, one can see the GD (grid search) obtains slightly faster convergence rate than GDM (grid search), confirming Remark 3.

## 5 CONCLUSION

In this work, we focused on finding a stationary point of games. Via polynomial-based analysis, we showed the extragradient with momentum converges in three different modes of eigenvalue structure of the game Jacobian, depending on the hyperparameters. Then, we focused on a more specific problem class, where the eigenvalues of the game Jacobian are distributed in the complex plane with a cross-shape. For this problem class, we obtained the optimal hyperparameters for the momentum extragradient method, which enjoys an accelerated linear asymptotic convergence rate. We compared the obtained rates of the momentum extragradient method with other first order methods, and showed that the considered methods do not achieve the accelerated convergence rate.
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A Missing Proofs

A.1 Proof of Proposition 1

Proof. Define $D_3 = \text{diag}(a, \ldots, a)$ of dimensions $d \times d$. Let us prove that if there exists $U, V$ orthonormal matrices and $D_1, D_2$ matrices with non-zeros coefficients only on the diagonal such that (with a slight abuse of notation)

\[ S_1 = U \text{diag}(D_3, D_1)U^\top, S_2 = V D_3 V^\top, \quad \text{and} \quad B = U D_2 V^\top, \]

then the spectrum of $A$ is cross-shaped. In that case, we have

\[
A = \begin{bmatrix}
U [D_3; D_1] U^\top & U D_2 V^\top \\
-V D_2 U^\top & V D_3 V^\top
\end{bmatrix}
= \begin{bmatrix}
U & 0 \\
0 & V
\end{bmatrix}
\begin{bmatrix}
[D_3; D_1] & D_2 \\
-D_2^\top & D_3
\end{bmatrix}
\begin{bmatrix}
U & 0 \\
0 & V
\end{bmatrix}^\top.
\]

Now by considering the basis $W = ((U_1, 0), (0, V_1), \ldots, (U_{d_v}, 0), (0, V_{d_v}), (U_{d_v+1}, 0), \ldots, (U_d, 0))$ we have that $A$ can be block diagonalized in that basis as

\[
A = W \text{diag} \left( \begin{bmatrix}
\vdots & \vdots \\
[D_2]_{11} & a \\
\vdots & \vdots
\end{bmatrix}, \ldots, \begin{bmatrix}
\vdots & \vdots \\
[D_2]_{d_v,d_v} & a \\
\vdots & \vdots
\end{bmatrix}, [D_1], \ldots, [D_1]_{d_d-d_v,d_d-d_v} \right) W^\top.
\]

(28)

Now, notice that

\[
\text{Sp} \left( \begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} \right) = \{a \pm bi\},
\]

(29)

since the associated characteristic polynomial of the above matrix is:

\[
(a - \lambda)^2 + b^2 = 0 \implies a - \lambda = \pm bi \implies \lambda = a \pm bi.
\]

Hence, using (29) in the formulation of $A$ in (28), we have that the spectrum of $A$ is cross-shaped.

\[ \square \]

A.2 Proof of Lemma 1

Proof of Lemma 1 can be found for example in Azizian, Scieur, et al. 2020, Section B (appendix).

A.3 Proof of Theorem 1

Derivation of the first part

Proof. We want to find the residual polynomial $\tilde{P}_1(A)$ of the extragradient with momentum (EGM) in (6). That is, we want to find

\[
w_1 - w^* = \tilde{P}_1(A)(w_0 - w^*),
\]

(30)

where $\{w_t\}_{t=0}^\infty$ is the iterates generated by EGM, which is possible by Lemma 1, as EGM is a first-order method (Arjevani and Shamir 2016; Azizian, Scieur, et al. 2020). We now prove this by induction. To do so, we will use the following properties. First, note that as we are looking for a stationary point, it holds that $v(w^*) = 0$. Further, as $v$ is linear by the assumption of Lemma 1, it holds that $v(w) = A(w - w^*)$.

Base case. For $t = 0$, $\tilde{P}_1(A)$ is a degree-zero polynomial, and hence equals $I$, which denotes the identity matrix. Thus, $w_0 - w^* = \|w_0 - w^*\|$ holds true.

For completeness, we also prove when $t = 1$. In that case, observe that EGM in Algorithm 1 proceeds as $w_1 = w_0 - \frac{h}{1+\gamma} v(w_0 - \gamma v(w_0))$. Subtracting $w^*$ on both sides, we have:

\[
w_1 - w^* = w_0 - w^* - \frac{h}{1+\gamma} v(w_0 - \gamma v(w_0))
\]
where in the third equality, we used the induction hypothesis in (30).

Subtracting $w$ on both sides, we have:

$$w_{t+1} - w^* = w_t - w^* - hA(I - \gamma A)(w_t - w^*) + m(w_t - w_{t-1})$$

Again, for completeness, we prove when $t = 1$ as well. In that case, by the definition of Chebyshev polynomials of the first and the second kinds, we have $T_1(\lambda) = \lambda$, and $U_1(\lambda) = 2\lambda$. Therefore,

$$P_1(\lambda) = m^{t/2} \left( \frac{2m}{1+m} T_1(\sigma(\lambda)) + \frac{1-m}{1+m} U_1(\sigma(\lambda)) \right)$$

Induction step. As the induction hypothesis, assume $\tilde{P}_t$ satisfies (30). We want to prove this holds for $t + 1$. We have:

$$w_{t+1} = w_t - hv(w_t - \gamma v(w_t)) + m(w_t - w_{t-1})$$

$$w_{t+1} = w_t - hv(w_t - \gamma A(w_t - w^*)) + m(w_t - w_{t-1})$$

$$w_{t+1} = w_t - hA(w_t - \gamma A(w_t - w^*)) + m(w_t - w_{t-1})$$

$$w_{t+1} = w_t - hA(w_t - w^*) + h\gamma A^2(w_t - w^*) + m(w_t - w_{t-1})$$

$$w_{t+1} = w_t - hA(I - \gamma A)(w_t - w^*) + m(w_t - w_{t-1}).$$

Subtracting $w^*$ on both sides, we have:

$$w_{t+1} - w^* = w_t - w^* - hA(I - \gamma A)(w_t - w^*) + m(w_t - w_{t-1})$$

$$w_{t+1} - w^* = (I - hA(I - \gamma A))(w_t - w^*) + m(w_t - w^* - (w_{t-1} - w^*))$$

$$w_{t+1} - w^* = (I - hA(I - \gamma A)) \tilde{P}_t(A)(w_0 - w^*) + m(\tilde{P}_t(A)(w_0 - w^*) - \tilde{P}_{t-1}(A)(w_0 - w^*))$$

$$w_{t+1} - w^* = (I + mI - hA(I - \gamma A)) \tilde{P}_t(A)(w_0 - w^*) - m\tilde{P}_{t-1}(A)(w_0 - w^*)$$

$$w_{t+1} - w^* = \tilde{P}_{t+1}(A)(w_0 - w^*),$$

where in the third equality, we used the induction hypothesis in (30).

\[ \square \]

Derivation of the second part in (10)

Proof. We show $P_t = \tilde{P}_t$ for all $t$ via induction.

Base case. For $t = 0$, by the definition of Chebyshev polynomials of the first and the second kinds, we have $T_0(\lambda) = U_0(\lambda) = 1$. Thus,

$$P_0(\lambda) = m^0 \left( \frac{2m}{1+m} T_0(\sigma(\lambda)) + \frac{1-m}{1+m} U_0(\sigma(\lambda)) \right)$$

$$P_0(\lambda) = \frac{2m}{1+m} + \frac{1-m}{1+m} = 1 = \tilde{P}_0(\lambda).$$

Again, for completeness, we prove when $t = 1$ as well. In that case, by the definition of Chebyshev polynomials of the first and the second kinds, we have $T_1(\lambda) = \lambda$, and $U_1(\lambda) = 2\lambda$. Therefore,

$$P_1(\lambda) = m^{t/2} \left( \frac{2m}{1+m} T_1(\sigma(\lambda)) + \frac{1-m}{1+m} U_1(\sigma(\lambda)) \right)$$

$$P_1(\lambda) = m^{t/2} \left( \frac{2m}{1+m} \sigma(\lambda) + \frac{1-m}{1+m} \cdot 2 \cdot \sigma(\lambda) \right)$$

$$P_1(\lambda) = m^{t/2} \left( \frac{2\sigma(\lambda)}{1+m} \right).$$
Induction step. As the induction hypothesis, assume that $P_t = \hat{P}_t$ for $t$. We want to show this holds for $t + 1$.

$$P_{t+1} = m^{(t+1)/2} \left[ \frac{2m}{1 + m} T_{t+1}(\sigma(\lambda)) + \frac{1 - m}{1 + m} U_{t+1}(\sigma(\lambda)) \right]$$

$$= m^{(t+1)/2} \left[ \frac{2m}{1 + m} \left( 2\sigma(\lambda) T_t(\sigma(\lambda)) - T_{t-1}(\sigma(\lambda)) \right) 
+ \frac{1 - m}{1 + m} \left( 2\sigma(\lambda) U_t(\sigma(\lambda)) - U_{t-1}(\sigma(\lambda)) \right) \right]$$

$$= 2\sigma(\lambda) \cdot m^{1/2} \cdot m^{t/2} \left[ \frac{2m}{1 + m} T_t(\sigma(\lambda)) + \frac{1 - m}{1 + m} U_t(\sigma(\lambda)) \right]$$

$$= 2\sigma(\lambda) \cdot m^{1/2} \cdot m^{t/2} \left[ \frac{2m}{1 + m} T_t(\sigma(\lambda)) + \frac{1 - m}{1 + m} U_t(\sigma(\lambda)) \right]$$

$$= 2\sigma(\lambda) \cdot m \cdot \hat{P}_t(\lambda) - m \cdot \hat{P}_{t-1}(\lambda)$$

$$= (1 + m - h\lambda(1 - \gamma\lambda)) \hat{P}_t(\lambda) - m \hat{P}_{t-1}(\lambda),$$

where in the second to last equality we use the induction hypothesis.

A.4 Proof of Lemma 2

Proof of Lemma 2 can be found in Goujaud and Pedregosa 2022.

A.5 Proof of Theorem 2

Proof. We analyze each case separately.

Case 1: There are two square roots: $\sqrt{\frac{1}{4\gamma^2} - (1 - \sqrt{m})^2}$ and $\sqrt{\frac{1}{4\gamma^2} - (1 + \sqrt{m})^2}$. The second one is real if:

$$\frac{1}{4\gamma^2} \geq \frac{(1 + \sqrt{m})^2}{h\gamma} \implies \frac{h\gamma}{4\gamma^2} = \frac{h}{4} \geq (1 + \sqrt{m})^2,$$

which implies the first is real, as $(1 + \sqrt{m})^2 \geq (1 - \sqrt{m})^2$.

Case 3: There are two square roots: $\sqrt{\frac{1}{4\gamma^2} - (1 - \sqrt{m})^2}$ and $\sqrt{\frac{1}{4\gamma^2} - (1 + \sqrt{m})^2}$. The first one is complex if:

$$\frac{1}{4\gamma^2} < \frac{(1 - \sqrt{m})^2}{h\gamma} \implies \frac{h\gamma}{4\gamma^2} = \frac{h}{4} < (1 - \sqrt{m})^2,$$

which implies the second is complex, as $(1 + \sqrt{m})^2 \geq (1 - \sqrt{m})^2$.

Case 2: This case follows automatically from the above two cases.

A.6 Proof of Theorem 3

Proof. We recall the conditions required for Theorem 3 below:

$$\frac{1}{2\gamma} - \sqrt{\frac{1}{4\gamma^2} - \frac{(1 - \sqrt{m})^2}{h\gamma}} = \mu,$$

(31)
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\[ \frac{1}{2\gamma} + \sqrt{\frac{1}{4\gamma^2} - \frac{(1 - \sqrt{m})^2}{h\gamma}} = L, \quad \text{and} \]
\[ \sqrt{\frac{(1 + \sqrt{m})^2}{h\gamma} - \frac{1}{4\gamma^2}} = c. \quad (33) \]

First, by adding (31) and (32), we get:

\[ \frac{1}{\gamma} = \mu + L \implies \gamma = \frac{1}{\mu + L}. \quad (34) \]

Plugging (34) back into (31), we have:

\[ \frac{1}{2\gamma} - \sqrt{\frac{1}{4\gamma^2} - \frac{(1 - \sqrt{m})^2}{h\gamma}} = \mu \]
\[ \frac{\mu + L}{2} - \mu = \sqrt{\left(\frac{\mu + L}{2}\right)^2 - \frac{(1 - \sqrt{m})^2(\mu + L)}{h}} \]
\[ \left(\frac{L - \mu}{2}\right)^2 = \left(\frac{\mu + L}{2}\right)^2 - \frac{(1 - \sqrt{m})^2(\mu + L)}{h} \]
\[ \frac{(1 - \sqrt{m})^2(\mu + L)}{h} = \left(\frac{\mu + L}{2}\right)^2 - \left(\frac{L - \mu}{2}\right)^2 = \mu L \]
\[ h = \frac{(1 - \sqrt{m})^2(\mu + L)}{\mu L}. \quad (35) \]

Plugging (34) and (35) into (33), we have:

\[ \sqrt{\frac{(1 + \sqrt{m})^2}{h\gamma} - \frac{1}{4\gamma^2}} = c \]
\[ \sqrt{\frac{(1 + \sqrt{m})^2 \cdot \mu L}{(1 - \sqrt{m})^2} - \left(\frac{\mu + L}{2}\right)^2} = c \]
\[ \frac{(1 + \sqrt{m})^2 \cdot \mu L}{(1 - \sqrt{m})^2} = c^2 + \left(\frac{\mu + L}{2}\right)^2 = \frac{4c^2 + (\mu + L)^2}{4} \]
\[ \frac{(1 + \sqrt{m})^2}{(1 - \sqrt{m})^2} = \frac{4c^2 + (\mu + L)^2}{4\mu L} \]
\[ \frac{1}{\sqrt{m}} \sqrt{4c^2 + (\mu + L)^2 + 4\mu L} = (1 - \sqrt{m}) \sqrt{4c^2 + (\mu + L)^2} \]
\[ \sqrt{m} \left(\frac{\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L}}{\sqrt{4c^2 + (\mu + L)^2 + 4\mu L}}\right) = \frac{\sqrt{4c^2 + (\mu + L)^2 - \sqrt{4\mu L}}}{\sqrt{4c^2 + (\mu + L)^2 + 4\mu L}} \quad (36) \]

Finally, to simplify (35) further, from (36), we have:

\[ 1 - \sqrt{m} = \frac{4\sqrt{\mu L}}{\sqrt{4c^2 + (\mu + L)^2 + 4\mu L}}. \]

Hence, from (35),

\[ h = \frac{(\mu + L)(1 - \sqrt{m})^2}{\mu L} = \frac{16\mu L(\mu + L)}{(\sqrt{4c^2 + (\mu + L)^2 + 4\mu L})^2} = \frac{16(\mu + L)}{(\sqrt{4c^2 + (\mu + L)^2 + 4\mu L})^2}. \quad (37) \]
A.7 Proof of Corollary 1

Proof. This result can be obtained either by plugging in $c = \frac{L - \mu}{2}$ to (36) and (37), or by using the same technique as the proof of Theorem 3. For completeness, we prove by plugging in $c = \frac{L - \mu}{2}$ to (36) and (37) here.

First, for $\sqrt{m}$, from (36), we have:

$$\sqrt{m} = \frac{\sqrt{4c^2 + (\mu + L)^2} - \sqrt{4\mu L}}{\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L}} = \frac{\sqrt{4\left(\frac{L - \mu}{2}\right)^2 + (\mu + L)^2} - \sqrt{4\mu L}}{\sqrt{4\left(\frac{L - \mu}{2}\right)^2 + (\mu + L)^2} + \sqrt{4\mu L}}$$

$$= \frac{\sqrt{2\mu^2 + 2L^2} - \sqrt{4\mu L}}{\sqrt{2\mu^2 + 2L^2} + \sqrt{4\mu L}} = \frac{\mu^2 + L^2 - \sqrt{2\mu L}}{\mu^2 + L^2 + \sqrt{2\mu L}}.$$

Now, for $h$, focusing on the denominator of (37):

$$\left(\frac{\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L}}{2}\right)^2 = \left(\frac{\sqrt{4\left(\frac{L - \mu}{2}\right)^2 + (\mu + L)^2} + \sqrt{4\mu L}}{2}\right)^2$$

$$= \left(\sqrt{2\mu^2 + 2L^2} + \sqrt{2\mu L}\right)^2 = 2\mu^2 + 2L^2 + 4\mu L + 4\sqrt{\mu L(2\mu^2 + 2L^2)}.$$

Plugging this back into (37), we have:

$$\frac{16(\mu + L)}{\left(\sqrt{4c^2 + (\mu + L)^2} + \sqrt{4\mu L}\right)^2} = \frac{16(\mu + L)}{2\mu^2 + 2L^2 + 4\mu L + 4\sqrt{\mu L(2\mu^2 + 2L^2)}}$$

$$= \frac{8(\mu + L)}{\mu^2 + L^2 + 2\mu L + 2\sqrt{2\mu L(\mu^2 + L^2)}}$$

$$= \frac{8(\mu + L)}{(\sqrt{\mu^2 + L^2} + \sqrt{2\mu L})^2}.$$

\[\square\]

B Details on Remarks

B.1 Details on Remark 2

To compute the convergence rates of GD and EG from Theorem 4 applied to the cross-shaped spectrum of game Jacobian in (2), we need to compute $\min_{\lambda \in \Delta^*} \Re(1/\lambda)$ and $\min_{\lambda \in \Delta^*} \Re(\lambda)$ for GD. For a complex number $z = a + bi \in \mathbb{C}$, we can compute $\Re(1/z)$ as:

$$\frac{1}{z} = \frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i \quad \Rightarrow \quad \Re\left(\frac{1}{z}\right) = \frac{a}{a^2 + b^2}.$$

The four extreme points of the cross-shaped spectrum model in (2) with $c = \frac{L - \mu}{2}$ are:

$$\mu = \mu + 0i, \quad L = L + 0i, \quad \text{and} \quad \frac{L - \mu}{2} = \frac{L - \mu}{2} + 0i.$$

Hence, $\Re(1/z)$ for each of the above points is:

$$\Re\left(\frac{1}{\mu}\right) = \frac{\mu}{\mu^2} = \frac{1}{\mu},$$

$$\Re\left(\frac{1}{L}\right) = \frac{L}{L^2} = \frac{1}{L}, \quad \text{and}$$
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\[ \mathfrak{R} \left( \frac{1}{L - \mu + \frac{L - \mu}{2} i} \right) = \frac{L - \mu}{\frac{L - \mu}{2} + \frac{(L - \mu)^2}{2}} = \frac{\frac{L - \mu}{2}}{2L^2 + 2\mu^2 - 4L \mu} = \frac{L - \mu}{L - \mu} = \frac{1}{L - \mu} \]

Therefore, \( \min_{\lambda \in \Delta^*} \mathfrak{R} \left( \frac{1}{\lambda} \right) = \frac{1}{L} \).

For \( \min_{\lambda \in \Delta^*} \mathfrak{R}(\lambda) \), we have
\[ \mathfrak{R}(\mu) = \mu, \quad \mathfrak{R}(L) = L, \quad \text{and} \quad \mathfrak{R} \left( \frac{L - \mu}{2} \pm \frac{L - \mu}{2} i \right) = \frac{L - \mu}{2}. \]

Therefore,
\[ \min_{\lambda \in \Delta^*} \mathfrak{R}(\lambda) = \begin{cases} \mu & \text{if } 3\mu < L \\ \frac{L - \mu}{2} & \text{otherwise}. \end{cases} \]

As we consider the ill-conditioned regime where \( \frac{\mu}{L} \to 0 \), we have \( \min_{\lambda \in \Delta^*} \mathfrak{R}(\lambda) = \mu \). Thus, we get
\[ \rho(\nabla F^\text{GD}(w^*))^2 \leq 1 - \min_{\lambda \in \Delta^*} \mathfrak{R} \left( \frac{1}{\lambda} \right) \min_{\lambda \in \Delta^*} \mathfrak{R}(\lambda) = 1 - \frac{\mu}{L}. \]

Similarly for EG, we need to compute \( \min_{\lambda \in \Delta^*} |\lambda|, \min_{\lambda \in \Delta^*} |\lambda|^2 \), and \( \max_{\lambda \in \Delta^*} |\lambda|^2 \). For \( z = a + bi \in \mathbb{C}, |z| = \sqrt{a^2 + b^2} \). Hence, we have
\[ |\mu + 0i| = \mu, \quad |L + 0i| = L, \quad \text{and} \quad \left| \frac{L - \mu}{2} \pm \frac{L - \mu}{2} i \right| = \frac{\sqrt{2}(L - \mu)}{2}. \]

Thus, for \( \min_{\lambda \in \Delta^*} |\lambda| \), we have
\[ \min_{\lambda \in \Delta^*} |\lambda| = \begin{cases} \mu & \text{if } \mu < \frac{\sqrt{2}}{2 + \sqrt{2}} L \\ \frac{\sqrt{2}L}{\mu} & \text{otherwise.} \end{cases} \]

Similarly to the above argument, in the ill-conditioned regime, we have \( \min_{\lambda \in \Delta^*} |\lambda| = \mu \).

For \( \max_{\lambda \in \Delta^*} |\lambda| \), we have:
\[ (2 - \sqrt{2})L > L > \mu > -\sqrt{2}\mu \]
\[ 2L - \sqrt{2}L > -\sqrt{2}\mu \]
\[ 2L > \sqrt{2}(L - \mu) \]
\[ L > \frac{\sqrt{2}(L - \mu)}{2} \]
\[ \therefore \max_{\lambda \in \Delta^*} |\lambda| = L. \]

Finally, for \( \min_{\lambda \in \Delta^*} |\lambda|^2 \), we have \( |\lambda|^2 \in \{ \mu^2, L^2, (L - \mu)^2 \} \). Thus, we have
\[ \min_{\lambda \in \Delta^*} |\lambda|^2 = \begin{cases} \mu^2 & \text{if } L \geq (\sqrt{2} + 1)\mu \\ (L - \mu)^2 & \text{otherwise.} \end{cases} \]

Combining all three, we get:
\[ \rho^2_{\text{EG}} := \rho(\nabla F^\text{EG}(w^*))^2 \leq 1 - \frac{1}{4} \left( \frac{\min_{\lambda \in \Delta^*} \mathfrak{R}(\lambda)}{\max_{\lambda \in \Delta^*} |\lambda|} + \frac{\min_{\lambda \in \Delta^*} |\lambda|^2}{\max_{\lambda \in \Delta^*} |\lambda|^2} \right) = \begin{cases} 1 - \frac{1}{4} \left( \frac{\mu}{L} + \frac{\mu^2}{16L^2} \right) & \text{if } L \geq (\sqrt{2} + 1)\mu, \\ 1 - \frac{1}{4} \left( \frac{l}{L} + \frac{(L - \mu)^2}{16L^2} \right) & \text{otherwise.} \end{cases} \]
B.2 Details on Remark 3

Per Theorem 5, the largest $\epsilon$ that permits acceleration for GDM is $\epsilon = \sqrt{\mu L}$. Therefore, in the special case of (2) we consider, i.e., when $c = \frac{L - \mu}{2}$, GDM cannot achieve acceleration if $\frac{L - \mu}{2} > \sqrt{\mu L}$. Hence, we have:

\[
\begin{align*}
\frac{L - \mu}{2} &> \sqrt{\mu L} \\
L - \mu &> 2\sqrt{\mu L} \\
L^2 + \mu^2 &> 6\mu L > 6\mu^2 \quad (\because L > \mu) \\
L &> \sqrt{5\mu}.
\end{align*}
\]