Calabi-Yau Manifolds of Cohomogeneity One
as Complex Line Bundles

Kiyoshi Higashijima\textsuperscript{1}, Tetsuji Kimura\textsuperscript{1} and Muneto Nitta\textsuperscript{2}

\textsuperscript{1} Department of Physics, Graduate School of Science, Osaka University,
Toyonaka, Osaka 560-0043, Japan
\textsuperscript{2} Department of Physics, Purdue University, West Lafayette, IN 47907-1396, USA

Abstract

We present a simple derivation of the Ricci-flat Kähler metric and its Kähler potential on the canonical line bundle over arbitrary Kähler coset space equipped with the Kähler-Einstein metric.

\textsuperscript{*} E-mail: higashij@phys.sci.osaka-u.ac.jp
\textsuperscript{†} E-mail: t-kimura@het.phys.sci.osaka-u.ac.jp
\textsuperscript{‡} E-mail: nitta@physics.purdue.edu
1 Introduction

A lot of attention has been paid to compact Calabi-Yau manifolds, since they give candidates of compactification of superstring theory \[1\]. Calabi-Yau manifolds are defined as Kähler manifolds satisfying the Einstein equation with zero cosmological constant, namely the Ricci-flat condition. It is, however, quite difficult to obtain explicit metrics of Calabi-Yau manifolds in general, since the Ricci-flat condition is a highly non-trivial partial differential equation. No explicit metric is known for compact Calabi-Yau manifolds. We would like to discuss non-compact Calabi-Yau manifolds.

Isometry of the manifold plays a crucial role to solve the Einstein equation, since it often reduces the equation to an ordinary differential equation (or a set of algebraic equations). When the isometry group \(G\) is transitive, the manifold is called homogeneous and can be written as a coset space \(G/H\), where \(H\) is the isotropy group. However coset spaces cannot have Ricci-flat metric, since their scalar curvature is positive. When the isometry is not transitive but generic orbits have dimensions less than the whole space by one, the manifold is called cohomogeneity one and can be written as \(\mathbb{R} \times G/H\) at least locally. Hyper-Kähler manifolds, which belong to a class of Calabi-Yau manifolds, of cohomogeneity one were completely classified by Dancer and Swann \[2\]: the only one manifold is the Calabi metric \[3\] on the cotangent bundle over the projective space \(\mathbb{C}P^{N-1} = SU(N)/[SU(N-1) \times U(1)]\), \(T^*\mathbb{C}P^{N-1}\). Cohomogeneity one manifolds with holonomy of \(Spin(7)\) or \(G_2\) have been recently studied extensively by many authors in connection with the compactification of string theory or M-theory (see, for example, \[4, 5\]).

General discussion of Calabi-Yau manifolds of cohomogeneity one has been recently given by Dancer and Wang \[6\]. A class of Calabi-Yau manifolds of cohomogeneity one is given by the cotangent bundles over rank one symmetric spaces, \(T^*(G/H)\), obtained by Stenzel \[7\]. In particular, the cotangent bundle over \(S^{N-1} = SO(N)/SO(N-1)\), gives the higher dimensional generalization \[8, 9\] of the Eguchi-Hanson space \[10\] and the deformed conifold \[11\]. An another important class of the Calabi-Yau manifolds of cohomogeneity one is given by the complex line bundles over coset spaces with Kähler-Einstein metrics. They were firstly constructed by Page and Pope \[12\] in real coordinates. In connection with supersymmetric nonlinear sigma models \[13\] on Calabi-Yau manifolds, it is important to give such manifolds in complex coordinates for the manifest supersymmetry on the string world sheet. This has been done in a series of the papers given by the present authors \[14, 15, 16\] for the canonical line bundles over the Hermitian symmetric spaces, using the gauge theoretical (Kähler quotient) construction of the base spaces \[17\]. (See also \[18\].) A new way of a resolution of the conical singularity of the conifold is given in \[14\], which is achieved by the line bundle over the quadric surface \(SO(N)/[SO(N-2) \times U(1)]\). The generalization of the conifold to the one with the isometry of \(E_6\) or \(E_7\) is given in \[16\], which is identified with the line bundle over \(E_6/[SO(10) \times U(1)]\) or \(E_7/[E_6 \times U(1)]\).

In this paper, we give the Calabi-Yau metric and its Kähler potential on the canonical line bundle
over an arbitrary Kähler-Einstein manifold of a coset space $G/H$, using the method of the supersymmetric nonlinear realization developed in [19]. We extensively use the Kähler-Einstein structures of the base manifolds to calculate the determinants of the metrics.

This paper is organized as follows. In section 2, we discuss the Kähler coset spaces $G/H$ and their Einstein metrics. The useful formula for the determinant of the metric is given. In section 3, we present the explicit expressions of the Ricci-flat Kähler metric and its Kähler potential on the canonical line bundle over an arbitrary Kähler coset space endowed with the Kähler-Einstein metric. Some examples are given in section 4. Section 5 is devoted to discussion. In Appendix A, we construct the Ricci tensors of the Riemann and Kähler coset spaces $G/H$ in terms of the structure constants of $G$. In Appendix B, we demonstrate that the Einstein condition on Kähler coset spaces reduces to algebraic equations using some examples.

2 Kähler-Einstein Metrics on Kähler Coset Spaces

A coset space $G/H$ has the unique Kähler metric up to scale constants [20], when the isotropy $H$ is in the form of

$$H = H_{ss} \times U(1)^k,$$

(2.1)

where $H_{ss}$ is the semi-simple subgroup of $H$, and $k \equiv \text{rank } G - \text{rank } H_{ss}$ is the dimension of the torus in $H$. All of the Kähler coset spaces and their complex structures were classified by Bordemann, Forger and Römer [21] in terms of the painted Dynkin diagram. The Kähler potential of arbitrary Kähler coset space was given by Itoh, Kugo and Kunitomo [22], using the supersymmetric nonlinear realization [19]. We briefly discuss their construction of Kähler potentials in this section.

We denote the Lie algebra of $G$ by its Calligraphic font $\mathcal{G}$. The generators of $\mathcal{G}$ can be divided into those of $\mathcal{H}$ and their orthonormal complements: $\{ S_\alpha \} = \mathcal{H}$ ($\alpha = 1, \cdots, \text{dim } H$) and $\{ X_I \} = \mathcal{G} - \mathcal{H}$ ($I = 1, \cdots, \text{dim } G/H$), with $\text{tr} (S_\alpha X_I) = 0$. A complex structure on $G/H$ can be defined by dividing $X_I$ into two sets of non-Hermitian generators, $X_I$ and $X_{I^*}$, and by regarding the complex isotropy algebra and its complement as $\mathcal{H}^C \oplus \{ X_I \} = \hat{\mathcal{H}}$ and $\{ X_{I^*} \} = G^C - \hat{\mathcal{H}}$, respectively. Under these definitions, there exists a homeomorphism between real and complex coset spaces: $G/H \simeq G^C/\hat{\mathcal{H}}$.

The representative of the complex coset space $G^C/\hat{\mathcal{H}}$ is given by

$$\xi(\varphi) = \exp(i \varphi^i X_{I^*} \delta_i^I),$$

(2.2)

in which the complex coordinates $\varphi^i$ parametrize $G^C/\hat{\mathcal{H}}$ ($i = 1, \cdots, \text{dim } G/H$). Here we have used the matrices of the fundamental representation of $G$ for the generators, but we do not write it explicitly. The transformation law of the coordinates under $g \in G$ is given by

$$\xi \rightarrow \xi' = g \xi \hat{h}^{-1}(g, \xi), \quad \hat{h}' \in \hat{\mathcal{H}},$$

(2.3)
where \( \xi' = \exp(i\varphi^i X_I^* \delta^I_j) \) and \( \hat{h}' \) is needed to project \( g \xi \) onto the coset representative. With preparing \( k \) projection matrices \( \eta_\alpha \) (\( \alpha = 1, \cdots, k \)), satisfying the projection condition \[19\]
\[
\eta \hat{H} \eta = \hat{H} \eta , \quad \eta^2 = \eta , \quad \eta^\dagger = \eta ,
\]
the Kähler potential of \( G/H \) is given by \[22\]
\[
\Psi(\varphi, \varphi^*) = \sum_{\alpha=1}^{k} v_\alpha \log \det \eta_\alpha \xi^\dagger \xi ,
\]
where \( \det_\eta \) denotes the determinant of the subspace projected by \( \eta \), and \( v_\alpha \) are real positive constants. This transforms under \( g \in G \) by
\[
\Psi \rightarrow \Psi' = \Psi + \gamma(\varphi) + \gamma^*(\varphi^*) , \quad \gamma(\varphi) \equiv \sum_{\alpha=1}^{k} v_\alpha \log \det \eta_\alpha \hat{h}'^{-1}(g, \xi) .
\]

The Kähler metric is given by \( g_{ij^*} = \partial_i \partial_{j^*} \Psi \), where \( \partial_i \) denotes differentiation with respect to \( \varphi^i \). With a suitable choice of \( v_\alpha \), any Kähler coset space \( G/H \) becomes Einstein \[23, 21, 24\] (see Appendices A and B):
\[
R_{ij^*} = h g_{ij^*} ,
\]
where \( R_{ij^*} \equiv -g^{kl^*} R_{kl^* ij^*} \) is the Ricci-tensor with \( R_{ij^* kl^*} \) being the curvature tensor, and \( h \) is a real positive constant. Using an another expression of the Ricci tensor, we obtain
\[
R_{ij^*} = -\partial_i \partial_{j^*} \log \det g_{kl^*} = h \partial_i \partial_{j^*} \Psi .
\]

This can be integrated to give the determinant formula \[3\],
\[
\det g_{ij^*} = e^{-h \Psi |\text{hol.}|^2} ,
\]
where “hol.” denotes a holomorphic function. We extensively use this formula in the next section.\[4\]

3 Ricci-flat Metrics and Kähler Potentials on Line Bundles

In this section we construct a Ricci-flat Kähler metric and its Kähler potential on canonical line bundle over an arbitrary Kähler coset space as a base manifold. We consider a direct product of \( N \) Kähler coset spaces \( G_a/H_a \) (\( a = 1, \cdots, N \)) as a base manifold:
\[
M = (G_1/H_1) \times (G_2/H_2) \times \cdots \times (G_N/H_N) ,
\]
\[1\] In our previous papers \[14, 15, 16\], we used complex isotropy transformations, \( g_{ij^*} \rightarrow g'_{ij^*} = (\hat{h} h'^{-1})_{ij^*} \) to evaluate determinants of metrics of hermitian symmetric spaces. In some cases, the coordinates \( \varphi^i \) can be brought to a single component while this is not generally possible. The use the determinant formula \[2.9\] improves this flaw and offers the far simpler method.
and we label each quantity in the last section by the index $a$. The isometry of $M$ is $G = \prod_{a=1}^{N} G_a$. The Kähler potential for the invariant metric on $M$ is given by

$$\Psi = \sum_{a=1}^{N} \Psi_a, \quad \Psi_a(\varphi, \varphi^*) = \sum_{\alpha=1}^{k_a} v_{\alpha a} \log \det \eta_{\alpha a} \xi^\alpha_a, \quad (3.2)$$

where $k_a$ is the dimension of the torus in $H_a$. This transforms under the isometry $g = (g_1, \ldots, g_N) \in G$ by

$$\Psi \to \Psi' = \Psi + \sum_{a=1}^{N} \left[ \gamma_a(\varphi) + \gamma_a^*(\varphi^*) \right], \quad \gamma_a(\varphi) = \sum_{\alpha=1}^{k_a} v_{\alpha a} \log \det \eta_{\alpha a} {h_a'}^{-1}. \quad (3.3)$$

We introduce a fiber $\sigma$ of a complex line bundle over $M$ whose transformation law under $g \in G$ is defined by

$$\sigma \to \sigma' = \exp \left( - \sum_{a=1}^{N} h_a \gamma_a(\varphi) \right) \sigma, \quad (3.4)$$

where $\gamma_a(\varphi)$ is defined in (3.3). This transformation law is the same with the one of [25] except for a factor of $h_a$. The reason of the inclusion of $h_a$ in the transformation law is clarified below.

In [25], the consistency of the global definition of the manifold is discussed. The $G$-invariant is found as

$$X \equiv \log \left( |\sigma|^2 e^{\sum_{a=1}^{N} h_a \Psi_a} \right) = \log |\sigma|^2 + \sum_{a=1}^{N} h_a \Psi_a = \log |\sigma|^2 + \hat{\Psi}, \quad (3.5)$$

where $\hat{\Psi} \equiv \sum_{a=1}^{N} h_a \Psi_a$ can be regarded as a potential for the Ricci tensor. We assume the Kähler potential of the line bundle over $M$ as a function of $X$:

$$K = K(X). \quad (3.6)$$

We write the coordinates for the total space as $z^\mu = \{\sigma, \varphi^i_a\}$, in which the index $i$ runs through the dimension of $M_a$ for each factor of $M_a$, but we often omit $a$ if there is no confusion. Components of the Kähler metric $g_{\mu\nu} = \partial_\mu \partial_\nu K$ are given by

$$g_{\mu\nu} = \begin{pmatrix} g_{\sigma\sigma} & g_{\sigma j^*} \\ g_{i^*} & g_{ij^*} \end{pmatrix}, \quad (3.7)$$

in which each block can be written as

$$g_{\sigma\sigma} = K' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \sigma^*}, \quad g_{\sigma j^*} = K'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \varphi^{*j}}, \quad g_{i^*} = K' \frac{\partial X}{\partial \varphi^i} \frac{\partial X}{\partial \varphi^{*j}} + K'' \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}}, \quad (3.8)$$

where the prime denotes the differentiation with respect to the argument $X$. Using equations $\partial X/\partial \sigma = 1/\sigma$ ($\sigma \neq 0$) and

$$\frac{\partial^2 X}{\partial \varphi^{*i} \partial \varphi^{*j}} = \begin{pmatrix} h_1 g_{ij^*}^1 & 0 \\ 0 & \ddots \end{pmatrix}, \quad (3.9)$$

4
the determinant of the metric can be calculated, to give
\[
\det g_{\mu\nu} = g_{\sigma\sigma} \cdot \det(g_{ij} - g^{-1}_{\sigma\sigma} g_{i\sigma} g_{\sigma j}) = \frac{1}{|\sigma|^2} \mathcal{K}'(\mathcal{K}')^d \cdot \prod_{\alpha=1}^N \det(h_\alpha g^\alpha_{ij}) ,
\]
where \( d \equiv \dim C \). Using the determinant formula (2.9) for each \( M_a \), we obtain
\[
\det g_{\mu\nu} = e^{-X} \mathcal{K}'(\mathcal{K}')^d |\text{hol.|}^2 .
\]
Since the Ricci tensor is defined by \( R_{\mu\nu} = -\partial_{\mu} \partial_{\nu} \log \det g_{\kappa\lambda} \), the Ricci-flat condition \( R_{\mu\nu} = 0 \) implies
\[
\det g_{\mu\nu} = (\text{constant}) \times |\text{hol.|}^2 .
\]
We thus obtain the Ricci-flat condition for the line bundles over \( M \) as an ordinary differential equation:
\[
e^{-X} \frac{d}{dX}(\mathcal{K}')^D = a ,
\]
where \( a \) is a constant, and \( D \equiv d + 1 \) is complex dimension of the total space. The solution for \( \mathcal{K}' \) can be obtained as
\[
\mathcal{K}' = (\lambda e^X + b)\frac{1}{D} ,
\]
where \( \lambda \) is a constant related to \( a \) and \( D \), and \( b \) is an integration constant. The Kähler potential can be integrated, to yield
\[
\mathcal{K}(X) = D(\lambda e^X + b)\frac{1}{D} + b \frac{1}{\pi} \cdot I(b^{-\frac{1}{\pi}}(\lambda e^X + b)\frac{1}{\pi} ; D) ,
\]
where the function \( I(y; n) \) is defined by
\[
I(y; n) \equiv \int_y^1 \frac{dt}{t^n - 1} = \frac{1}{n} \left[ \log (y - 1) - \frac{1 + (-1)^n}{2} \log (y + 1) \right] + \frac{1}{n} \sum_{r=1}^{\frac{n-1}{2}} \cos 2r\pi \cdot \log \left( y^2 - 2y \cos \frac{2r\pi}{n} + 1 \right)
+ \frac{2}{n} \sum_{r=1}^{\frac{n-1}{2}} \sin 2r\pi \cdot \arctan \left[ \frac{\cos(2r\pi/n) - y}{\sin(2r\pi/n)} \right] .
\]
The components of the metric can be calculated, to give
\[
g_{\sigma\sigma} = \frac{\lambda}{D}(\lambda e^X + b)\frac{1}{D} e^\bar{\Psi} ,
\]
\[
g_{\sigma j} = \frac{\lambda}{D}(\lambda e^X + b)\frac{1}{D} e^\bar{\Psi} \sigma \cdot \partial_j \bar{\Psi} ,
\]
\[
g_{ij} = \frac{\lambda}{D}(\lambda e^X + b)\frac{1}{D} e^\bar{\Psi} |\sigma|^2 \cdot \partial_i \bar{\Psi} \partial_j \bar{\Psi} + (\lambda e^X + b)\frac{1}{D} \cdot \partial_i \partial_j \bar{\Psi} .
\]
The metric of the submanifold of the surface defined by $\sigma = 0$ ($d\sigma = 0$) is

$$g_{ij} \big|_{\sigma=0} (\varphi, \varphi^*) = b^\dagger \partial_i \partial_j \widehat{\Psi},$$

(3.18)

which is the Kähler-Einstein metric of the Kähler coset space $M$, whose potential is given by $\widehat{\Psi}$. Therefore the total space is the canonical line bundle over $M$, whose base manifold is endowed with the Kähler-Einstein metric. The total manifold can be locally written as

$$\mathbb{R} \times G/H', \quad \text{with } H' = H_{ss} \times U(1)^{k-1}.$$  

(3.19)

Here we make a comment. If we take the constants in the transformation law (3.4) as arbitrary values instead of $h_a$, the invariant $X'$ is different from $X$ defined in (3.5). However we obtain the same determinant with (3.11) with $X$ of (3.5), in which their difference is included in $|\text{hol.}|^2$. Hence (3.4) is necessary to obtain the ordinary differential equation (3.13).

4 Examples

In this section, we give some examples for definiteness. We discuss line bundles over hermitian symmetric spaces and a non-symmetric space $SU(l + m + n)/S[U(l) \times U(m) \times U(n)]$. Then the line bundle over $\mathbb{C}P^{N-1} \times \mathbb{C}P^{M-1}$ is considered.

4.1 Line Bundles over Hermitian Symmetric Spaces

The hermitian symmetric spaces (HSS) $G/H$ are

$$\mathbb{C}P^{N-1} = \frac{SU(N)}{SU(N-1) \times U(1)}, \quad G_{N,M} = \frac{SU(N)}{SU(N-M) \times U(M)}, \quad \frac{SO(2N)}{U(N)}, \quad \frac{Sp(N)}{U(N)}, \quad Q^{N-2} = \frac{SO(N)}{SO(N-2) \times U(1)}, \quad \frac{E_6}{SO(10) \times U(1)}, \quad \frac{E_7}{E_6 \times U(1)}.$$  

(4.1)

The Kähler potential of HSS can be written as

$$\Psi = v \log \det \eta \xi \xi^\dagger, \quad \xi \in G^C/\widehat{H},$$

(4.2)

with a suitable $\eta$ (see [17]). For HSS, the constant in the Einstein condition (2.7) can be calculated as

$$h = \frac{1}{2v} C_2(G),$$

(4.3)

where $C_2(G)$ is the eigenvalue of the quadratic Casimir operator in the adjoint representation of $G$ [see (B.2) in Appendix B.1]. Its value is $C_2(G) = N, N, N-1, N+1, N-2, 12$ or 18 for each HSS in (4.1), respectively. We obtain the solutions (3.15) with $X$ being

$$X = \log |\sigma|^2 + \frac{1}{2} C_2(G) \log \det \eta \xi \xi^\dagger,$$

(4.4)
which coincide with our previous results \[14, 15, 16\], in which we used the gauge theoretical construction. This clarifies the origin of the coefficients in \(X\) as the coefficients in the Einstein condition for HSS. The total spaces can be locally regarded as \(\mathbb{R} \times G/H'\) with \(H' = H/U(1)\) in (4.1).

In \[14, 15, 16\], there is the coordinate singularity in the original coordinates, and a coordinate transformation is needed to obtain a regular metric. Different from these papers, the coordinate singularity has been avoided from beginning by the definition of the transformation law (3.4) of \(\sigma\).

### 4.2 Line Bundle over \(SU(l + m + n)/S[U(l) \times U(m) \times U(n)]\)

As an example of a line bundle over Kähler \(G/H\), which is not a HSS, we consider \(SU(l + m + n)/S[U(l) \times U(m) \times U(n)]\). There exists two kinds of complex structures on this coset space \[26\] (see Appendix B.2).

I) In one of the two inequivalent complex structures, the complex broken generators becomes like [see (B.6)]

\[
i\varphi^I X_I \delta^I_i = \begin{pmatrix} 0_l & A & B \\ 0 & 0_m & C \\ 0 & 0 & 0_n \end{pmatrix},
\]

(4.5)

where \(A, B\) and \(C\) are \(l \times m\), \(l \times n\) and \(m \times n\) matrices, belonging to the \((l, \overline{m}, 1)\), \((l, 1, \overline{m})\) and \((1, m, \overline{m})\) representations of \(SU(l) \times SU(m) \times SU(n)\), respectively. The coset representative can be calculated as

\[
\xi = \exp(i\varphi^I X_I \delta^I_i) = \begin{pmatrix} 1_l & A & B + \frac{1}{2}AC \\ 0 & 1_m & C \\ 0 & 0 & 1_n \end{pmatrix}.
\]

(4.6)

Using this representative, the Kähler potential is obtained as

\[
\Psi_1 = \sum_{\alpha=1}^2 v_\alpha \log \det \eta_\alpha \xi^\dagger \xi = v_1 \log \det \begin{pmatrix} 1_n + C^\dagger C + \left(B^\dagger + \frac{1}{2}C^\dagger A^\dagger\right)\left(B + \frac{1}{2}AC\right) \\ 1_m + A^\dagger A \\ 1_n + C^\dagger C \end{pmatrix} \left(C + A^\dagger(B + \frac{1}{2}AC)\right) \
\end{pmatrix},
\]

(4.7)

where the projection operators \(\eta_\alpha\) are given by

\[
\eta_1 = \begin{pmatrix} 0_l \\ 0_m \\ 1_n \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0_l \\ 1_m \\ 1_n \end{pmatrix}.
\]

(4.8)
The metric and the Ricci tensor at the origin \( A = B = C = 0 \) can be calculated, to give\(^2\)

\[
g_{ij}|_0 = \begin{pmatrix} v_2 \delta_{ef} & 0 & 0 \\ 0 & (v_1 + v_2) \delta_{pq} & 0 \\ 0 & 0 & v_1 \delta_{uv} \end{pmatrix},
\]

(4.9a)

\[
\mathcal{R}_{ij}|_0 = \begin{pmatrix} (l + m) \delta_{ef} & 0 & 0 \\ 0 & (l + 2m + n) \delta_{pq} & 0 \\ 0 & 0 & (m + n) \delta_{uv} \end{pmatrix},
\]

(4.9b)

where we have used the indices summarized in Table 1 in Appendix B. We thus obtain \( h = (l + m)/v_2 = (m + n)/v_1 = (l + 2m + n)/(v_1 + v_2) \) from the Einstein condition (2.7). The line bundle over \( SU(l + m + n)/SU(l) \times SU(m) \times SU(n) \) endowed with the complex structure (4.13) can be obtained by the solution (3.15) with \( X \) being

\[
X = \log |\sigma|^2 + \Psi|_{v_1 = m + n, v_2 = l + m}.
\]

(4.10)

The manifold can be locally written as \( \mathbb{R} \times SU(l + m + n)/SU(l) \times SU(m) \times SU(n) \).

II) The other complex structure of \( SU(l + m + n)/SU(l) \times SU(m) \times SU(n) \) is given by

\[
i \varphi^i X_{I^*} \delta^I_i = \begin{pmatrix} 0_l & 0 & B \\ D & 0_m & C \\ 0 & 0 & 0_n \end{pmatrix}, \quad \xi = \begin{pmatrix} 1_l & 0 & B \\ D & 1_m & C + \frac{1}{2} DB \\ 0 & 0 & 1_n \end{pmatrix},
\]

(4.11)

where \( B \) and \( C \) are the same matrices in (4.14), but \( D \) is an \( m \times l \) matrix, belonging to the \((\mathbb{I}, m, 1)\) representation of \( SU(l) \times SU(m) \times SU(n) \). The Kähler potential can be calculated as

\[
\Psi_{II} = \sum_{\alpha=1}^{2} v_\alpha \log \det \eta_\alpha \xi^\dagger \xi
\]

\[
= v_1 \log \det_{n \times n} \left[ 1_n + B^\dagger B + \left( C^\dagger + \frac{1}{2} B^\dagger D^\dagger \right) \left( C + \frac{1}{2} DB \right) \right]

+ v_2 \log \det_{(l+n) \times (l+n)} \left( \begin{array}{cc} 1_l + D^\dagger D & B + D^\dagger (C + \frac{1}{2} DB) \\ B^\dagger + (C^\dagger + \frac{1}{2} B^\dagger D^\dagger) D & 1_n + B^\dagger B + (C^\dagger + \frac{1}{2} B^\dagger D^\dagger) (C + \frac{1}{2} DB) \end{array} \right),
\]

(4.12)

where \( \eta_\alpha \) are given by

\[
\eta_1 = \begin{pmatrix} 0_l \\ 0_m \\ 1_n \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1_l \\ 0_m \\ 1_n \end{pmatrix}.
\]

(4.13)

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\(^2\) Since the manifold is homogeneous, it is sufficient to calculate quantities at the origin. These can be calculated easily using so-called Kähler normal coordinates.\(^\ddagger\).
The metric and the Ricci tensor at the origin \( D = B = C = 0 \) are obtained, to yield

\[
g_{ij^*}|_0 = \begin{pmatrix} v_2 \delta_{ef^*} & 0 & 0 \\ 0 & v_1 \delta_{pq^*} & 0 \\ 0 & 0 & (v_1 + v_2) \delta_{uv^*} \end{pmatrix}, \tag{4.14a}
\]

\[
\mathcal{R}_{ij^*}|_0 = \begin{pmatrix} (l + m) \delta_{e^*} & 0 & 0 \\ 0 & (l + n) \delta_{p^*} & 0 \\ 0 & 0 & (2l + m + n) \delta_{u^*} \end{pmatrix}. \tag{4.14b}
\]

We thus obtain \( h = (l + m)/v_2 = (l + n)/v_1 = (2l + m + n)/(v_1 + v_2) \) from the Einstein condition (2.7). The line bundle over \( SU(l+m+n)/S[U(l) \times U(m) \times U(n)] \) endowed with the complex structure (4.11) can be obtained by the solution (3.15) with \( X \) being

\[
X = \log |\sigma|^2 + \Psi \big|_{v_1 = l + n, v_2 = l + m}. \tag{4.15}
\]

The manifold can be also locally written as \( \mathbb{R} \times SU(l+m+n)/SU(l) \times SU(m) \times SU(n) \times U(1) \).

These two complex structures coincide in the simplest case of \( l = m = n = 1 \): \( G/H = SU(3)/U(1)^2 \). In this case, the manifold is locally \( \mathbb{R} \times SU(3)/U(1) \).

### 4.3 Line Bundle over \( CP^{N-1} \times CP^{M-1} \)

As an example of the base manifold of the product of two Kähler coset spaces, we consider \( CP^{N-1} \times CP^{M-1} \). The Kähler potential for the Fubini-Study metric on this manifold is

\[
\Psi = v_1 \log (1 + |\phi|^2) + v_2 \log (1 + |\chi|^2), \tag{4.16}
\]

where \( \phi = \{ \phi^a \} (a = 1, \cdots, N - 1) \) and \( \chi = \{ \chi^\alpha \} (\alpha = 1, \cdots, M - 1) \) are coordinates of \( CP^{N-1} \times CP^{M-1} \), respectively. The metric and the Ricci tensor can be calculated, to yield

\[
g_{ij^*} = \left( \begin{array}{ccc} \frac{1}{v_1} g_{ab^*} & 0 & 0 \\ 0 & \frac{1}{v_2} g_{\alpha\beta^*} & 0 \\ 0 & 0 & \frac{M}{M} g_{\alpha\beta^*} \end{array} \right), \tag{4.17a}
\]

\[
\mathcal{R}_{ij^*} = \left( \begin{array}{ccc} \frac{1}{v_1} \delta_{ab}(1 + |\phi|^2) - \phi^a \phi^b & 0 & 0 \\ 0 & \frac{1}{v_2} \delta_{\alpha\beta}(1 + |\chi|^2) - \chi^\alpha \chi^\beta & 0 \\ 0 & 0 & \frac{M}{M} g_{\alpha\beta^*} \end{array} \right). \tag{4.17b}
\]

which satisfy the Einstein condition (2.7) with \( h_1 = N/v_1 \) and \( h_2 = M/v_2 \). The quantity

\[
X = \log |\sigma|^2 + \hat{\Psi}, \quad \hat{\Psi} = N \log (1 + |\phi|^2) + M \log (1 + |\chi|^2), \tag{4.18}
\]

is invariant under the transformation of \( G = U(N) \times U(M) \). We thus have obtained the Kähler potential of the line bundle over \( CP^{N-1} \times CP^{M-1} \), substituting (4.15) into (3.15) with \( D = N + M - 1 \). For definiteness, the metric can be written as

\[
g_{\sigma\sigma^*} = \frac{\lambda}{N + M - 1} (\lambda e^X + b)^{\frac{2(N-M)}{N+M-1}} e^{\hat{\Psi}}, \tag{4.19}
\]
\[
g_{\sigma^* j} = \frac{\lambda}{N + M - 1} \left( \lambda e^X + b \right) \frac{2^{-N-M}}{N+M-1} e^\tilde{\Psi} \sigma^* \frac{V_j \varphi^j}{1 + |\varphi|^2},
\]
\[
g_{ij^*} = \frac{\lambda}{N + M - 1} \left( \lambda e^X + b \right) \frac{2^{-N-M}}{N+M-1} e^\tilde{\Psi} |\sigma|^2 \frac{V_i \varphi^i}{1 + |\varphi|^2} 1 + |\varphi|^2
\]
\[
+ \left( \lambda e^X + b \right) \frac{1}{N+M-1} \frac{V_i \delta_{ij}}{(1 + |\varphi|^2)^2},
\]
(4.19)

where \(\varphi^i = \{\phi^a, \chi^\alpha\}\) and \(V_i = N\) (or \(M\)) when \(i\) runs over \(a\) (or \(\alpha\)) and we take no sum over the index \(i\) or \(j\). The metric of the submanifold of \(\sigma = 0\) is
\[
g_{ij^*}|_{\sigma=0}(\varphi, \varphi^*) = \begin{pmatrix}
 b \frac{1}{N+M-1} N (1 + |\phi|^2)^{-2} & 0 \\
 0 & b \frac{1}{N+M-1} M (1 + |\chi|^2)^{-2}
\end{pmatrix},
\]
(4.20)

which is the metric of \(\mathbb{C}P^{N-1} \times \mathbb{C}P^{M-1}\) of definite ratio of radii.

The total space can be locally written as \(\mathbb{R} \times \frac{SU(N) \times SU(M)}{SU(N-1) \times SU(M-1) \times U(1)}\). In the case of \(N = M = 2\), the isomorphism \(G/H = \mathbb{C}P^1 \times \mathbb{C}P^1 \simeq Q^2 = SO(4)/U(1)^2\) holds, and the total space coincides with the conifold whose conical singularity is resolved by \(Q^2\) \cite{14}, which is a special case of the solution obtained by Pando Zayas and Tseytlin \cite{28}.

5 Discussion

We have used only the Einstein condition on a base manifold, but not the symmetry as in \cite{3, 14, 15, 16}. This implies the existence of a Calabi-Yau metric on a canonical line bundle on a base manifold without any isometry, but the only known Kähler-Einstein spaces with positive scalar curvature are Kähler coset spaces \cite{24} discussed in this paper. On the other hand, a Calabi-Yau metric of cohomogeneity one which is not in a form of a bundle can be constructed \cite{1} as the Stenzel metric \cite{7}. In terms of the nonlinear realization, the total space in this paper can be understood by the existence of the so-called quasi-Nambu-Goldstone (QNG) boson \cite{23}. If there is only one QNG boson in generic points of the manifold, it is cohomogeneity one. The generalization to a vector bundle can be also considered, using the matter coupling method in the nonlinear realization \cite{13, 23}. We consider that almost all Calabi-Yau manifolds of cohomogeneity one can be constructed using these methods. Explicit metrics of Calabi-Yau manifolds would provide more precise information about superstring propagating on these manifolds than topological structures.

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Appendix

A Ricci Tensor of Riemann and Kähler Coset Spaces

In this appendix we calculate Ricci tensors of Riemann and Kähler coset spaces. Let us first consider Riemann coset spaces $G/H$ \cite{31, 32}. After we construct the Ricci tensor by using the structure constants of $G$, we require the coset to be Kähler.

A.1 Riemann Coset Spaces

Here we assume the group $G$ is semi-simple. We prepare a set of generators $S_\hat{a}$ spanning the algebra $H$ and a set of generators $X_\hat{I}$ generating the coset $G/H$:

\[
H = \{S_\hat{a}\} : \text{unbroken generators}, \quad (A.1a)
\]

\[
G - H = \{X_\hat{I}\} : \text{broken generators}. \quad (A.1b)
\]

We consider reductive coset spaces $G/H$: \[[X_\hat{I}, S_\hat{a}] \propto X_\hat{J},\] in which case $X_\hat{I}$ belong to, in general, reducible representations of $H$. The algebra can be written as

\[
[X_\hat{I}, X_\hat{J}] = i f_{\hat{I}\hat{J}\hat{K}} X_\hat{K} + i f_{\hat{I}\hat{J}\hat{a}} S_\hat{a}, \quad (A.2a)
\]

\[
[X_\hat{I}, S_\hat{a}] = i f_{\hat{I}\hat{a}} X_\hat{J}, \quad (A.2b)
\]

\[
[S_\hat{a}, S_\hat{b}] = i f_{\hat{a}\hat{b}} \tilde{S}_\hat{c}. \quad (A.2c)
\]

Here we have taken generators as hermitian. When there exists an automorphism, defined by $S \to S$ and $X \to -X$, a coset space is called a symmetric space and $f_{\hat{I}\hat{J}\hat{a}}$ vanishes.

Using real coordinates $\varphi^i$, the coset representative of $G/H$ can be written as $U(\varphi) = \exp(i \varphi^i \tilde{e}_i^\hat{I}).$ The $G$-valued 1-form,

\[
\alpha(\varphi) \equiv \frac{1}{i} U^{-1} dU, \quad (A.3)
\]

called the Maurer-Cartan 1-form, can be decomposed like

\[
\alpha = \tilde{e}_i^\hat{I}(\varphi) X_\hat{I} + \omega^\hat{a}(\varphi) S_\hat{a} = \tilde{e}_i^\hat{I}(\varphi) X_\hat{I} + \omega^\hat{a}(\varphi) S_\hat{a} d\varphi^i, \quad (A.4)
\]

where $\omega^\hat{a}$ is called a $H$-connection and $\tilde{e}_i^\hat{I}$ is a (rescaled) vielbein. The most general $G$-invariant metric $g_{ij}(\varphi)$ on $G/H$ contains real positive scale constants $r_I$ associated with $H$-irreducible representations of broken generators $X_\hat{I}$ ($r_I = r_J$ if $X_\hat{I}$ and $X_\hat{J}$ belong to the same representation of $H$):

\[
g_{ij}(\varphi) = (r_I)^{-2} \delta_\hat{I}\hat{J} \tilde{e}_i^\hat{I}(\varphi) \tilde{e}_j^\hat{J}(\varphi) = \delta_\hat{I}\hat{J} \tilde{e}_i^\hat{I}(\varphi) \tilde{e}_j^\hat{J}(\varphi). \quad (A.5)
\]
Here the metric of the tangent space is normalized in the ordinary basis of $e^\hat{i}$, defined by rescaling $e^\hat{i} = (r_I)^{-1} \tilde{e}^\hat{i}$. They can be expanded in the coset coordinates, like

$$
\tilde{e}^\hat{i}(\varphi) = \delta^\hat{i}_1 + O(\varphi), \quad e^\hat{i}_1(\varphi) = (r_I)^{-1} \delta^\hat{i}_1 + O(\varphi). \quad (A.6)
$$

The Maurer-Cartan 1-form satisfies an equation, $d\alpha = -i\alpha \wedge \alpha$, called the Maurer-Cartan equation. Under the decomposition (A.4), this equation becomes

$$
de\tilde{e} = \frac{1}{2} r_J r_K f_{JK} \tilde{e}^\hat{j} \wedge \tilde{e}^\hat{k} + \frac{r_K}{r_I} f_{Kb} \tilde{e}^\hat{b} \wedge \alpha^\hat{c}, \quad d\alpha^\hat{a} = \frac{1}{2} f_{Jb} \tilde{e}^\hat{j} \wedge \tilde{e}^\hat{b} + \frac{1}{2} f_{Kc} \omega^\hat{b} \wedge \omega^\hat{c}. \quad (A.7)
$$

In ordinary basis of vielbeins, these equations can be rewritten as

$$
de\hat{e} = \frac{1}{2} r_J r_K f_{JK} \hat{e}^\hat{j} \wedge \hat{e}^\hat{k} + \frac{r_K}{r_I} f_{Kb} \hat{e}^\hat{b} \wedge \alpha^\hat{c}, \quad d\alpha^\hat{a} = \frac{1}{2} f_{Jb} \hat{e}^\hat{j} \wedge \hat{e}^\hat{b} + \frac{1}{2} f_{Kc} \omega^\hat{b} \wedge \omega^\hat{c}. \quad (A.8)
$$

The connection 1-form $\Omega^\hat{j}_\hat{j}$ and the curvature 2-form $R^\hat{i}_\hat{j}$ can be read off from the Cartan’s structure equations:

$$
de\hat{e} = -\Omega^\hat{j}_\hat{j} \wedge e^\hat{j}, \quad R^\hat{i}_\hat{j} = d\Omega^\hat{i}_\hat{j} + \Omega^\hat{i}_\hat{K} \wedge \Omega^\hat{K}_\hat{j} = \frac{1}{2} R^\hat{i}_\hat{L}\hat{M} e^\hat{i} \wedge e^\hat{M}, \quad (A.9)
$$

where we have imposed the torsion-free condition, and $R^\hat{i}_\hat{L}\hat{M}$ is the Riemann tensor. From equations (A.8) and (A.9), we can write down the Riemann tensor of the coset space $G/H$ endowed with the $G$-invariant metric (A.3):

$$
R^\hat{i}_\hat{L}\hat{M} = r_LR_M f_{bL} \hat{f}_{bM} + \frac{1}{2} r_L r_M \left( \begin{array}{cc} I & J \\ K & L \end{array} \right) f_{JK} \hat{f}_{LM} \hat{K} \\
+ \frac{1}{4} \left( \begin{array}{cc} I & K \\ J & L \end{array} \right) \left( \begin{array}{cc} K & J \\ L & M \end{array} \right) f_{L\hat{M}} \hat{f}_{\hat{K}\hat{L}} - \frac{1}{4} \left( \begin{array}{cc} I & K \\ M & L \end{array} \right) f_{L\hat{M}} \hat{f}_{\hat{K}\hat{L}} \hat{M}, \quad (A.10a)
$$

$$
\left( \begin{array}{cc} I & J \\ K & L \end{array} \right) \equiv \frac{r_J r_K}{r_I} + \frac{r_I r_K}{r_J} - \frac{r_I r_J}{r_K}. \quad (A.10b)
$$

The Ricci tensor $\mathcal{R}_{\hat{J}\hat{M}} = \frac{\hat{R}}{\hat{J}\hat{M}}$ can be calculated, to yield

$$
\mathcal{R}_{\hat{J}\hat{M}} = r_J r_M f_{b\hat{L}} \hat{f}_{b\hat{M}} + \frac{1}{4} f_{Jb} \hat{f}_{b\hat{M}} \hat{K} \left( \begin{array}{cc} M & K \\ I & L \end{array} \right) \left( \begin{array}{cc} J & I \\ K & L \end{array} \right). \quad (A.11)
$$

### A.2 Kähler Coset Spaces

Now let us consider the Kähler coset spaces imposing conditions on the Riemann coset spaces following Ref. [22]. We rewrite the broken and unbroken generators as

$$\mathcal{H} = \{S_{\alpha}\} = \{S_{\alpha}, Y_{\alpha}\}, \quad \text{: unbroken generators}, \quad (A.12a)$$

$$\mathcal{G} - \mathcal{H} = \{X_I\} = \{X_I, X_I^r\}, \quad \text{: broken generators}. \quad (A.12b)$$
Here $X_I$ and $X_I^*$ are non-hermitian and hermitian conjugate to each other: $X_{I^*} = (X_I)\dagger$, $S_{a}$ generate the semi-simple subgroup $H_{ss}$ of $H$, and $Y_\alpha$ is a generator commuting with any generators of $\mathcal{H}$, that is, $Y_\alpha$ is a $U(1)$ generator of the torus in $H$. We assume that $Y_\alpha$ is orthogonal to each other: $\text{tr} (Y_\alpha Y_\beta) = N_\alpha \delta_{\alpha\beta}$, with a normalization constant $N_\alpha$. In this subsection, we denote holomorphic coordinates by $\varphi^i$, and replace real coordinates in the last subsection by $\varphi^i$ and $\varphi^*$. Hence, for instance, vielbein is not holomorphic: $e^I(\varphi, \varphi^*) = e^I_1(\varphi, \varphi^*) d\varphi^i [e^I_r(\varphi, \varphi^*) = e^I_r(\varphi, \varphi^*) d\varphi^*_i]$.

We define the $Y$-charge by

$$Y \equiv \sum_{\alpha=1}^k v_\alpha Y_\alpha ,$$

where $v_\alpha$ is a real positive constant and $k$ is the dimension of the torus of $H$ ($1 \leq k \leq \text{dim} G$). The generators with the negative $Y$-charges are defined as the complex broken generators, generating the complex coset space $G/H$ [22]: the $Y$-charges of the generators, $X_I$ and $X_{I^*}$, and $\mathcal{H}$ are

$$[Y, X_I] = x_I X_I , \quad [Y, X_{I^*}] = -x_I X_{I^*} , \quad [Y, \mathcal{H}] = 0 ,$$

where $x_I$ is a real positive constant related to $r_I$ by $x_I = (r_I)^{-2}$. From these commutation relations, the structure constants are constrained by

$$f_{IJ}J^K = f_{I^*J^*}K^* = f_{IJ}A = f_{I^*J^*}A^* = f_{I^*}J = f_{I^*K^*}J^* = 0 .$$

We would like to obtain further constraints from the Kähler condition: the Kähler form $\Omega$ defined by

$$\Omega = ig_{ij}d\varphi^i \wedge d\varphi^*j = i\delta_{ij}e^{I} \wedge e^{*J} = ix_I\delta_{IJ}e^{I} \wedge e^{*J} \quad (\text{A.16})$$

must be closed: $d\Omega = 0$. This condition gives

$$f_{IJ}K \neq 0 , \quad f_{I^*J^*}K^* \neq 0 \quad \text{if} \quad x_I + x_J = x_K , \quad (\text{A.17a})$$
$$f_{IJ}K \neq 0 \quad \text{if} \quad x_I - x_J = x_K , \quad (\text{A.17b})$$
$$f_{IJ}K^* \neq 0 \quad \text{if} \quad x_I - x_J = -x_K , \quad (\text{A.17c})$$
$$f_{I^*a}J \neq 0 , \quad f_{I^*a}J^* \neq 0 \quad \text{if} \quad x_I = x_J . \quad (\text{A.17d})$$

Using these constraints, we obtain the Ricci tensor of Kähler coset spaces $G/H$ as

$$\mathcal{R}_{JM^*} = r_{J^*M}f_{J^*b}f_{JM^*b} + \frac{1}{4} (f_{JK}l f_{JM^*K} + f_{JK}l f_{JM^*K^*} + f_{JK}l^* f_{JM^*K^*} + f_{JK}l^* f_{JM^*K^*}^*) \left[ \begin{array}{c} M \\ K \\ I \\ J \\ K \end{array} \right] .$$

(\text{A.18})

The Einstein condition [2.7] in the vielbein basis is

$$\mathcal{R}_{IJ^*} = h\delta_{IJ^*} .$$

(\text{A.19})

We explicitly solve this equation for some examples in the next section.
B Examples

In this appendix we consider two examples of the Kähler \( G/H \). We discuss the hermitian symmetric spaces (HSS) \([16]\) followed by a non-symmetric space \( SU(l + m + n)/S[U(l) \times U(m) \times U(n)] \) \([22]\). We see that the HSS are Einstein; on the other hand, the Einstein condition on the non-symmetric space reduces to algebraic equations among scale constants in the metric.

B.1 Hermitian Symmetric Spaces

In this subsection let us consider symmetric spaces. The HSS are Kähler, since the Kähler condition is automatically satisfied.

The HSS have constraints on the structure constants given by \( f_{IJ}^K = f_{I^*J^*}^{K^*} = f_{I^*J}^{K^*} = f_{IJ}^{K^*} = 0 \), and each metric contains only one scale constant: \( r_I \equiv r \left( x_I = r^{-2} \equiv v \right) \) for all \( I \). Therefore the Ricci tensor (A.18) is reduced to

\[
R_{JM^*} = \frac{r^2}{2} f_{J\hat{b}} f_{I\hat{M}^*} \hat{b} = \frac{r^2}{2} \left( f_{J\hat{b}} f_{I\hat{M}^*} \hat{b} + f_{J^*\hat{b}} f_{I^*\hat{M}^*} \hat{b} \right) = -\frac{r^2}{2} f_{\hat{A}J} \hat{B} f_{\hat{B}M^*} \hat{A}
\]

where the index \( \hat{A} \) represents the whole generators of \( G \): \( \hat{A} \in \{ I, I^*, \hat{a} \} \). Here we have adopted the adjoint representation \( \text{ad}(X_I) \hat{B} = i f_{\hat{A}I} \hat{B} \), and defined \( C_2(G) \) as the eigenvalue of the quadratic Casimir operator in the adjoint representation of \( G \).

From (B.1), the HSS are Einstein (A.19) in which \( h \) is given by

\[
h = \frac{1}{2v} C_2(G) .
\]

B.2 A Non-symmetric Space \( SU(l + m + n)/S[U(l) \times U(m) \times U(n)] \)

In this subsection, let us consider the coset space \( G/H = SU(l + m + n)/S[U(l) \times U(m) \times U(n)] \), which is non-symmetric.

B.2.1 Complex Structures

First we discuss the complex structures of the coset space \( SU(l + m + n)/S[U(l) \times U(m) \times U(n)] \). This coset admits two kinds of inequivalent complex structures \([23, 26]\). The unbroken group \( H = \)
SU(l) × SU(m) × SU(n) × U(1)^2 has two U(1) generators Y_1 and Y_2, which can be chosen as

\[
Y_1 = \begin{pmatrix} n1_l \\ 0_m \\ -l1_n \end{pmatrix}, \quad Y_2 = \begin{pmatrix} m1_l \\ -(l+n)1_m \\ m1_n \end{pmatrix},
\]

embedded into the fundamental representation of G = SU(l+m+n). These generators are orthogonal to each other: \(\text{tr}(Y_\alpha Y_\beta) = N_\alpha \delta_{\alpha\beta}\). The whole generators of SU(l+m+n) can be decomposed as

\[
\begin{pmatrix}
SU(l) & A & B \\
\bar{A} & SU(m) & C \\
\bar{B} & \bar{C} & SU(n)
\end{pmatrix}.
\]

(B.4)

Here SU(l), SU(m) and SU(n) are the blocks of the unbroken subalgebra \(\mathcal{H}\), and A, B and C are blocks of the broken generators belonging to irreducible representations of \(\mathcal{H}\) (we simply call them “irreducible blocks”). We explain these irreducible blocks: the irreducible block A consists of \(l \times m\) generators denoted by \(X_e\) \((e = 1, \cdots , lm)\). These generators belong to the \((l, m, 1)\)-representation of SU(l) × SU(m) × SU(n), with their \(Y_1\)- and \(Y_2\)-charges being \(n\) and \(l+m+n\), respectively. The block \(\bar{A}\) is hermitian conjugate of A. Our notation and \(H\)-representations of irreducible blocks A, B and C are summarized in Table 1.

| irreducible block | A          | B          | C          |
|-------------------|------------|------------|------------|
| index             | e, f, \cdots | p, q, \cdots | u, v, \cdots |
| matrix size       | \(l \times m\) | \(l \times n\) | \(m \times n\) |
| \(H\)-representations | \((l, m, 1)^n l + m + n\) | \((l, 1, m)^{l+n, 0}\) | \((1, m, \bar{m})^l -(l+m+n)\) |

Table 1: The broken generators of SU(l+m+n)/S[U(l) × U(m) × U(n)]. We denote complex broken generators belonging to A, B or C by \(X_e, X_p, X_u\). The \(H\)-representations are denoted by the representations of SU(l) × SU(m) × SU(n) in the braces and the \(Y_1\)- and \(Y_2\)-charges as the indices.

Depending on real constants \(v_1\) and \(v_2\) in the definition (A.13) of \(Y\)-charge, the complex structure of \(G/H\) continuously varies. There exist, however, two classes of inequivalent complex structures, which are not continuously deformed to each other, represented, for instance, by \[26\]

\[
\text{I}) \quad Y = -Y_1 ,
\]

\[
\text{II}) \quad Y = -mY_1 + nY_2 .
\]

(B.5)

I) In the case of \(Y = -Y_1\), the broken generators \(X_{l^*}\) of \(G^C - \hat{\mathcal{H}}\) with negative \(Y\)-charges are given by

\[
G^C - \hat{\mathcal{H}} = \{A, B, C\} ,
\]

(B.6)
and the scale parameters $x_I$ are related as

$$x_A + x_C = x_B .$$  \hfill (B.7)

II) In the case of $Y = -m Y_1 + n Y_2$, the broken generators and the relation among scale parameters are given by

$$g^C - \hat{H} = \{ \hat{A}, B, C \}, \quad x_A + x_B = x_C . \hfill (B.8)$$

### B.2.2 Ricci Tensor and the Einstein Condition

We show that the Einstein condition reduces to a set of algebraic equations in cases of Kähler coset spaces. To this end, let us calculate the Ricci tensor in both cases.

I) In the case of $Y = -Y_1$, the non-zero components of the Ricci tensor are given by

$$R_{ef} = \frac{r}{r} f_{e\alpha}^g f_{g\beta} \hat{\alpha} + \frac{r}{r} f_{e\alpha}^g f_{gf}^\beta \hat{\beta} + \frac{r}{r} f_{ei}^g f_{g\beta} e^\alpha + \frac{r}{r} f_{ei}^g f_{gf}^\beta e^\beta$$

\hfill (B.9a)

$$R_{pq} = \frac{r}{r} f_{p\alpha}^g f_{q\beta} \hat{\alpha} + \frac{r}{r} f_{p\alpha}^g f_{qf}^\beta \hat{\beta} + \frac{r}{r} f_{pe}^g f_{q\beta} e^\alpha + \frac{r}{r} f_{pe}^g f_{qf}^\beta e^\beta$$

\hfill (B.9b)

$$R_{uv} = \frac{r}{r} f_{u\alpha}^g f_{v\beta} \hat{\alpha} + \frac{r}{r} f_{u\alpha}^g f_{vf}^\beta \hat{\beta} + \frac{r}{r} f_{ue}^g f_{v\beta} e^\alpha + \frac{r}{r} f_{ue}^g f_{vf}^\beta e^\beta$$

\hfill (B.9c)

Due to the constraint (B.7), we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = 0,$$

\hfill (B.10)

In order to evaluate the various terms in (B.9), we use the normalization of generators $T_A$ of $SU(N)$ in the fundamental representation, given by

$$\text{tr} (T_A T_B) \equiv \delta_{AB} .$$

\hfill (B.11)

In this normalization, the eigenvalues of the quadratic Casimir operators $C_2(N)$ and $C_2(\text{ad.})$ in the fundamental and the adjoint representations, respectively, are given by

$$C_2(N) = \frac{N^2 - 1}{N}, \quad C_2(\text{ad.}) = 2N .$$

\hfill (B.12)

We also fix the normalization the $U(1)$ generators $Y_\alpha$ in the same way.

Using these equations, we can calculate the structure constants in (B.9) explicitly. From the algebra (A.2), for instance, we can obtain the following relation:

$$f_{e\alpha}^g f_{gf}^\beta = C_2(A) \delta_{e f} , \quad \text{etc.}$$

\hfill (B.13)
Here $C_2(A)$ is the eigenvalue of the quadratic Casimir operator of the irreducible block $A$ in (B.4), which can be calculated, using the relations (B.13), to give

$$C_2(A) = \frac{l^2 - 1}{l} + \frac{m^2 - 1}{m} + \frac{n}{l(l+n)} + \frac{m+n}{m(l+n)} = l + m. \tag{B.14}$$

In the same way, we obtain

$$f_{\hat{a}r_{\hat{a}}} = C_2(B)\delta_{pq}, \quad f_{\hat{a}u_{\hat{a}}w_{\hat{a}}} = C_2(C)\delta_{uv} = (m+n)\delta_{uv}. \tag{B.15}$$

Moreover, the fact of $G = SU(N)$ provides us the relations, given by

$$f_{eu}^pf_{pf^*}^u = n\delta_{e f^*}, \quad f_{pe^*}^uf_{q^*}^e = m\delta_{pq^*}, \quad f_{ue^*}^pf_{pv^*}^e = l\delta_{uv^*}. \tag{B.16}$$

Therefore the Ricci tensor (B.9) can be explicitly calculated, to yield

$$\mathcal{R}_{IJ^*} = \begin{pmatrix}
\frac{1}{x_A}(l+m)\delta_{ef^*} & 0 & 0 \\
0 & \frac{1}{x_B}(l+2m+n)\delta_{pq^*} & 0 \\
0 & 0 & \frac{1}{x_C}(m+n)\delta_{uv^*}
\end{pmatrix}. \tag{B.17}$$

This concides with the direct calculation (4.9b) at the origin due to (A.6).

The Einstein condition (A.19) reduces to a set of algebraic equations given by

$$h = \frac{l+m}{x_A} = \frac{l+2m+n}{x_B} = \frac{m+n}{x_C}. \tag{B.18}$$

II) In the case of $Y = -mY_1 + nY_2$, the components of the Ricci tensor are given by

$$\mathcal{R}_{ef^*} = r_{e\hat{e}}^rf_{\hat{e}a}^gf_{gf^*}^\hat{a} + \frac{r_{e\hat{e}}^rf_{\hat{e}f}^p}{4}(f_{ep}^uf_{pf^*}^p + f_{eu^*}^{p^*}f_{pf^*}^u)\begin{pmatrix} A & B \\ C & B \end{pmatrix}, \tag{B.19a}$$

$$\mathcal{R}_{pq^*} = r_{p\hat{p}}^rq_{\hat{q}f}^{\hat{r}}f_{\hat{r}q^*}^\hat{a} + \frac{r_{p\hat{p}}^rq_{\hat{r}q}}{4}(f_{pe^*}^uf_{q^*}^e + f_{pu^*}^{e^*}f_{q^*}^{e^*})\begin{pmatrix} B & A \\ C & A \end{pmatrix}, \tag{B.19b}$$

$$\mathcal{R}_{uv^*} = r_{u\hat{u}}^rv_{\hat{v}a}^w f_{\hat{v}w^*}^\hat{a} + \frac{r_{u\hat{u}}^rv_{\hat{v}w}}{4}(f_{ue^*}^pf_{pv^*}^{e^*} + f_{up^*}^{e^*}f_{pv^*}^{e^*})\begin{pmatrix} C & A \\ B & A \end{pmatrix}. \tag{B.19c}$$

By the same discussion from (B.10) to (B.15), we obtain the Ricci tensor, given by

$$\mathcal{R}_{IJ^*} = \begin{pmatrix}
\frac{1}{x_A}(l+m)\delta_{ef^*} & 0 & 0 \\
0 & \frac{1}{x_B}(l+n)\delta_{pq^*} & 0 \\
0 & 0 & \frac{1}{x_C}(2l+m+n)\delta_{uv^*}
\end{pmatrix}. \tag{B.20}$$

This also concides with the direct calculation (4.14b) at the origin due to (A.6).

The Einstein condition (A.19) reduces to a set of algebraic equations:

$$h = \frac{l+m}{x_A} = \frac{l+n}{x_B} = \frac{2l+m+n}{x_C}. \tag{B.21}$$
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