More on the convergence of Gaussian convex hulls

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Abstract

A "law of large numbers" for consecutive convex hulls for weakly dependent Gaussian sequences $\{X_n\}$, having the same marginal distribution, is extended to the case when the sequence $\{X_n\}$ has a weak limit. Let $\mathbb{B}$ be a separable Banach space with a conjugate space $\mathbb{B}^*$. Let $\{X_n\}$ be a centered $\mathbb{B}$-valued Gaussian sequence satisfying two conditions: 1) $X_n \Rightarrow X$ and 2) For every $x^* \in \mathbb{B}^*$

$$
\lim_{n,m,|n-m|\to\infty} E\langle X_n,x^*\rangle\langle X_m,x^*\rangle = 0.
$$

Then with probability 1 the normalized convex hulls

$$W_n = \frac{1}{(2\ln n)^{1/2}} \text{conv}\{X_1,\ldots,X_n\}
$$

converge in Hausdorff distance to the concentration ellipsoid of a limit Gaussian $\mathbb{B}$-valued random element $X$. In addition, some related questions are discussed.

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1 Introduction and formulation of results

Let $\mathcal{B}$ be a separable Banach space with a norm $|| \cdot ||$ and let $\mathcal{B}^*$ and $\langle \cdot, \cdot \rangle$ denote its conjugate space and the corresponding inner product, respectively. For $A \in \mathcal{B}$ the notation $\text{conv}\{A\}$ is used for the closed convex hull of $A$. If $X$ is a $\mathcal{B}$-valued centered Gaussian random element with a distribution $\mathcal{P}$ then by $\mathcal{H}$ we denote its reproducing kernel Hilbert space and $E$ will stand for the closed unit ball in $\mathcal{H}$, see, e.g., [8], p. 207. The set $E$ is also called concentration ellipsoid of $X$.

Finally, we introduce the separable complete metric space $\mathcal{K}_\mathcal{B}$ of all nonempty compact subsets of a Banach space $\mathcal{B}$ equipped with the Hausdorff distance $d_\mathcal{B}$:

$$d_\mathcal{B}(A, B) = \max\{\inf\{\epsilon \mid A \subset B^\epsilon\}, \inf\{\epsilon \mid B \subset A^\epsilon\}\},$$

where $A^\epsilon$ is the open $\epsilon$-neighborhood of $A$. Convergence of compact sets in $\mathcal{B}$ always will be considered in this metric.

Investigation of the asymptotic behavior of convex hulls

$$W_n = \text{conv}\{X_1, \ldots, X_n\}$$

of multivariate Gaussian random variables is an important part of Extreme Value Theory and has various applications, see for example, [9] and reference list, containing 160 items, in it. In 1988 Goodman [7] proved a fundamental result that the normalized set $\{X_1, \ldots, X_n\}$ of independent and identically distributed $\mathcal{B}$-valued centered Gaussian random elements with a distribution $\mathcal{P}$ is approaching the concentration ellipsoid of $\mathcal{P}$ as $n$ grows to infinity. From this result one can immediately derive that a.s.

$$\frac{1}{b(n)}W_n \overset{\mathcal{K}_\mathcal{B}}{\rightarrow} \mathcal{E}, \quad n \to \infty, \quad (1)$$

where $b(t) = \sqrt{2 \ln(t)}$, $t > e$. Moreover, the rate of convergence in this relation is of the order $o(b(n)^{-1})$.

Later the convergence of the type (1) was proved first for stationary $d$-dimensional weakly dependent Gaussian sequences in [2], and then the similar result was proved for $\mathcal{B}$-valued Gaussian random fields on $\mathbb{R}^m$ or $\mathbb{Z}^m$ in [3]. Although at the introduction of the paper [3] it was said that only the case $m > 1$ is considered, inspection of the proof of the main result -Theorem 1.1
in [3] - shows that the result holds for \( m = 1 \), too. In particular, this result states that if a \( \mathbb{B} \)-valued centered Gaussian sequence \( \{X_k, k \in \mathbb{N}\} \) has the same marginal distribution \( \mathcal{P} \) and satisfies the following condition

\[
E \langle X_n, x^* \rangle \langle X_m, x^* \rangle \to 0, \quad \text{for all } x^* \in \mathbb{B}^* \text{ as } n, m, |n - m| \to \infty. \tag{2}
\]

then (1) holds.

In the paper we show that the condition of equality of marginal distributions can be essentially relaxed substituting it by the weak convergence of the sequence \( \{X_n\} \), and for weak convergence we use the sign \( \Rightarrow \).

**Theorem 1.** Suppose that a centered Gaussian sequence of \( \mathbb{B} \)-valued random elements \( \{X_k, k \in \mathbb{N}\} \) satisfies (2) and the following condition:

\[
X_n \Rightarrow X. \tag{3}
\]

Then a.s.

\[
\frac{1}{b(n)} W_n \xrightarrow{k \mathbb{B}} \mathcal{E}, \quad \text{as } n \to \infty, \tag{4}
\]

where \( \mathcal{E} \) is concentration ellipsoid of \( X \).

Since the proof of this theorem will be carried in two steps, and in the first step we consider the case \( \mathbb{B} = \mathbb{R} \), we look more closely what is the meaning of the result in this particular case. Let \( N(0, \sigma^2) \) stand for a Gaussian random variable with mean zero and variance \( \sigma^2 \), and \( \{X_k, k \in \mathbb{Z}_+\} \), is a sequence of \( N(0, \sigma_k^2) \) random variables. Without loss of generality we can assume that \( X \) in (3) is \( N(0, 1) \). We have

\[
W_n = \left[ \min\{X_1, X_2, \ldots, X_n\}, \max\{X_1, X_2, \ldots, X_n\} \right]
\]

and \( \mathcal{E} = [-1, 1] \). If the covariance function \( \rho(m, n) := EX_m X_n \to 0 \) as \( n, m, |n - m| \to \infty \), then we have the relation (4). It is clear, that if the dependence between elements \( X_k \) of the sequence is stronger, the sequence of their convex hulls is more concentrated. One can consider the extreme case, when \( X_k \equiv X_0 \) for all \( k \geq 1 \), then \( W_n = \{X_0\} \) is one point and \( \lim_{n \to \infty} (g(n))^{-1} W_n = \{0\} \) for any sequence \( g(n) \to \infty \). The following example gives us additional information in this question.

**Remark 2.** Let us consider the sequence of i.i.d. \( N(0, 1) \) random variables \( \{\xi_j\}, j \geq 1 \), and let \( S_k = k^{-1/2} \sum_{j=1}^k \xi_j \). Taking \( X_k = S_k \), we are in the
setting of Theorem \(H\), but the condition \(2\) is not satisfied, since, if \(n = m + k, k > 0\), then

\[
\rho(m, n) = \frac{ES_m S_{m+k}}{(m(m+k))^{1/2}} = \frac{m}{(m(m+k))^{1/2}} = \left(1 + \frac{k}{m}\right)^{-1/2}. \tag{5}
\]

Thus, in order to get \(\rho(m, n) \to 0\), it is not sufficient to require \(m \to \infty, k \to \infty\), but stronger condition is required \(m \to \infty, k/m \to \infty\). On the other hand, \(S_k\) is a sum of i.i.d.random variables, therefore, denoting \(c(t) = (2 \ln \ln t)^{1/2}, t > e\), the classical LIL gives us that the cluster set for the sequence \(\{X_n/c(n)\}\) is \([-1, 1]\), while for the sequence \(\{X_n/b(n)\}\) with probability one limit is zero. We shall prove that for this example we have the following result:

**Proposition 3.** With probability one

\[
\frac{1}{c(n)} W_n \xrightarrow{K_B} [-1, 1], \quad \text{as } n \to \infty, \tag{6}
\]

where \(W_n = [-V(n), V(n)]\), and \(V(n) := \max\{X_1, X_2, \ldots, X_n\}\)

In connection with this example it is possible to formulate the following problem.

Suppose that a sequence \(\{X_n\}\) has standard normal marginal distributions and covariance function \(\rho(m, n)\). For which functions \(g(n)\) and under what conditions for covariance function \(\rho\) we can get the relation \(6\) with function \(g\) instead of \(c\)?

This Proposition and Theorem \(H\) give us two examples of such functions \(g\). What other normalizing functions are possible in relation \(6\)?

Let us make three final remarks.

**Remark 4.** As in \(B\), having \(H\), we can get some information on asymptotic behavior of \(Ef(W_n)\) for some functions defined on \(K_B\). In case \(B = \mathbb{R}^m\) typical examples of such functions are diameter, volume or surface measure.

**Remark 5.** From the proof of Theorem \(H\) we can extract some information about the rate of convergence in \(H\). Namely, in the proof we have the equality

\[
\mathbb{P}\{d_B(X, b(n)\mathcal{E}) > \varepsilon\} = \mathbb{P}\{\exp \frac{1}{2} \psi_\varepsilon \geq n\},
\]
and since these probabilities are monotonically non-increasing and
\[ \sum_n \mathbb{P}\{\exp \frac{\psi_\varepsilon}{2} \geq n\} = E\{\exp \frac{\psi_\varepsilon}{2}\} < \infty \]
we get
\[ \mathbb{P}\left\{ \mathbb{d}\left( \frac{X_n}{b_n}, \mathcal{E} \right) > \varepsilon \right\} = o(n^{-1}). \]

Let us note that this result cannot be compared with the result from \[7\],
where it is proved that with probability 1
\[ \mathbb{d}_\mathbb{B}\left( \frac{X_n}{b_n}, \mathcal{E} \right) = o(b_n^{-1}). \]

\textbf{Remark 6.} In Theorem \[1\] and in previous results for Gaussian sequences
limit set of convex hulls was ellipsoid of some Gaussian measure. If we
dismiss the condition of weak convergence of Gaussian sequence \{X_k\}, the
limit set may exist, but not necessarily will be an ellipsoid. For example, the
following statement holds.

\textbf{Proposition 7.} Let \( \mathbb{B} \) be a separable Banach space. Let \( V \subset \mathbb{B} \) be a central
symmetric polytope, \( V = \text{conv}\{a_k, -a_k, k = 1, \ldots, m\} \). Then there exists a
sequence of independent Gaussian vectors \( \{X_k\} \) such that a.s.
\[ \frac{1}{b(n)} W_n \rightarrow V. \]

\section{Proofs}

\textit{Proof of Theorem 1\textsuperscript{1}} As it was mentioned above, the proof will be carried in
two steps, and in the first step we consider the case \( \mathbb{B} = \mathbb{R} \).

I. Without loss of generality we can suppose that \( X \) has a standard Gauss-
ian distribution; then \( \mathcal{E} = [-1, 1] \).

The condition \( 2 \) transforms now in
\[ EX_n X_m \rightarrow 0 \quad \text{as} \quad n, m, |n - m| \rightarrow \infty. \]

From weak convergence of \( X_n \) to \( X \) follows that \( \sigma_n^2 := EX_n^2 \rightarrow 1 \) as \( n \rightarrow \infty \).

For r.v. \( Y_n = X_n / \sigma_n \) the conditions \( 3 \) and \( 2 \) are fulfilled. Since \( Y_n \)
are identically distributed, setting \( U_n = \text{conv}\{Y_1, \ldots, Y_n\} \), by Theorem 1.1.
from \( 3 \)
\[ \frac{1}{b(n)} U_n \xrightarrow{\mathcal{K}_1} [-1, 1], \text{ a.s. as } n \rightarrow \infty. \]
Let us show that a.s.

\[ \Delta_n := d_{\mathbb{R}} \left( \frac{1}{b(n)} W_n, \frac{1}{b(n)} U_n \right) \to 0. \]  

(8)

We have

\[ \Delta_n \leq \frac{1}{b(n)} \max_{k \leq n} \{ |X_k - Y_k| \} = \frac{1}{b(n)} \max_{k \leq n} \{ |Y_k| |1 - \sigma_k| \}. \]  

(9)

Let \( Z_n = \max_{k \leq n} \{ |Y_k| \}; \) \( M_n = \max_{k \leq n} \{ |1 - \sigma_k| \}. \) For \( \varepsilon > 0 \) find \( n_0 \) such that \( \sup_{k > n_0} |1 - \sigma_k| < \varepsilon. \) Then for \( n \geq n_0 \) by (9)

\[ \Delta_n \leq \frac{1}{b(n)} \max_{n_0 \leq k \leq n} \{ |Y_k| \} \varepsilon + \frac{M_{n_0} Z_{n_0}}{b(n)}. \]

It follows from (7) that \( \limsup_n \Delta_n \leq \varepsilon. \) Hence we get (8) and (4) is proved for \( \mathbb{B} = \mathbb{R}. \)

II. General case. We shall show that with the probability 1 the sequence \( \{ b(n)^{-1} W_n \} \) is relatively compact in \( K_{\mathbb{B}}. \) Due to Lemmas 2.2 and 2.3 from [3] it is sufficient to prove that there exists a compact set \( K \) such that for every \( \varepsilon > 0 \) with probability 1 for all sufficiently large \( n \) we have the following inclusion \( \{ b(n)^{-1} W_n \} \subset K^\varepsilon. \) We take \( K = \mathcal{E}. \) Since the space \( \mathbb{B} \) is fixed, instead of \( d_{\mathbb{B}} \) we shall write simply \( d. \) It is clear that this inclusion will follow from the following relation: for every \( \varepsilon > 0 \) a.s.

\[ \limsup_n d(X_n, (1 + \varepsilon)b(n) \mathcal{E}) = 0. \]  

(10)

By Skorokhod representation theorem we can suppose that \( X_n \to X \) a.s.

Let \( \sigma^2 = E\|X\|^2, \delta^2_{\max} = \sup_n E\|X_n\|^2. \) It follows from Fernique’s theorem about integrability of exponential moments (see [4]) that for every \( \gamma, 0 < \gamma < (2\sigma^2)^{-1}, \)

\[ E \exp\{\gamma\|X\|^2\} < \infty. \]

Moreover, from the proof of Fernique’s theorem one can deduce that for every \( \gamma, 0 < \gamma < (2\delta^2_{\max})^{-1}, \)

\[ \limsup_n E \exp\{\gamma\|X_n\|^2\} < \infty. \]

It means, in particular, that for every \( \gamma, 0 < \gamma < (2\delta^2_{\max})^{-1}, \) the family \( \exp\{\gamma\|X_n\|^2\} \) is uniformly integrable. Therefore

\[ \delta_n^2 := E\|X_n - X\|^2 \to 0 \text{ and } E \exp\{\gamma\|X_n - X\|^2\} \to 1. \]
For $\varepsilon > 0$ let $n_\varepsilon$ be such that $\sigma^2_\varepsilon := \sup_{n>n_\varepsilon} \delta_n^2 < \varepsilon^2$.

We have

$$P\{d(X_n, (1 + \varepsilon)b(n)E) > \varepsilon\} = A_n + B_n,$$

where

$$A_n = P\{d(X_n, (1 + \varepsilon)b(n)E) > \varepsilon, \|X_n - X\| < \varepsilon b(n)\},$$

$$B_n = P\{d(X_n, (1 + \varepsilon)b(n)E) > \varepsilon, \|X_n - X\| \geq \varepsilon b(n)\}.$$

Evidently

$$A_n \leq P\{d(X, b(n)E) > \varepsilon\}.$$

Now we formulate Talagrand's lemma \cite{Talagrand} as it is formulated in \cite{Ledoux}, see Lemma 3.1 therein.

**Lemma 8.** Let $X$ be a $B$-valued centered Gaussian random element with a concentration ellipsoid $E$. Then for any $\varepsilon > 0$ there is a random variable $\psi_\varepsilon$ such that

$$E \{\exp \left\{ \frac{1}{2}\psi_\varepsilon \right\} \} < \infty$$

and for all $l > 0$

$$P\{d(X, lE) \leq \varepsilon\} = P\{\psi_\varepsilon < l^2\}.$$

We apply this Lemma taking $l = b(n)$ and obtain

$$A_n \leq P\{d(X, b(n)E) > \varepsilon\} = P\{\psi_\varepsilon \geq 2 \ln n\} = P\{\exp \left\{ \frac{1}{2}\psi_\varepsilon \right\} \geq n\}$$

Hence $\sum_n A_n < \infty$.

For $0 < \gamma < (2\sigma^2_\varepsilon)^{-1}$ we have $E \exp\{\gamma \|X_n - X\|^2\} < \infty$ for each $n > n_\varepsilon$, therefore, denoting

$$L_\varepsilon(a) = \sup_{n>n_\varepsilon} \{E \exp\{a \|X_n - X\|^2\}\},$$

for $a \in \left(\frac{1}{2\sigma^2_\varepsilon}, \frac{1}{2\sigma^2_\varepsilon}\right)$ and $n \geq n_\varepsilon$ we apply once more Fernique's theorem and get

$$B_n \leq P\{\|X_n - X\| > \varepsilon b(n)\} \leq \frac{E \exp\{a \|X_n - X\|^2\}}{n^{2\sigma^2_\varepsilon}} \leq \frac{L_\varepsilon(a)}{n^{2\sigma^2_\varepsilon}}.$$

Since $2\sigma^2_\varepsilon > 0$, this estimate gives the convergence of the series $\sum_n B_n$. 

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Therefore we see that for every $\varepsilon > 0$

$$\sum_n \mathbb{P}\{d(X_n, (1 + \varepsilon)b(n)\mathcal{E}) > \varepsilon\} < \infty.$$ 

Then the Borel-Cantelli lemma gives us (10), which shows that for every $\delta > 0$ with probability 1 for all sufficiently large $n$

$$\frac{1}{b(n)}W_n \subset \mathcal{E}\delta.$$ 

This proves the relative compactness of $\{b(n)^{-1}W_n\}$.

It follows from Lemma 2.7 [3] that now it is sufficient to prove the convergence for every $\theta \in S_1^*(0) := \{x^* \in \mathbb{B}^* : ||x^*|| = 1\}$

$$M_n(\theta) \xrightarrow{a.s.} M_\mathcal{E}(\theta), \quad n \to \infty,$$ 

where $M_n, M_\mathcal{E}$ are support functions for $b(n)^{-1}W_n$ and $\mathcal{E}$, respectively. We recall that a function $M_A(\theta)$, defined by the relation

$$M_A(\theta) := \sup_{x \in A} \langle x, \theta \rangle, \quad A \in K_{\mathbb{B}}, \ \theta \in S_1^*(0),$$ 

is called the support function of a set $A \in K_{\mathbb{B}}$. Since

$$M_n(\theta) = \frac{1}{b(n)} \max_{k \leq n} \{\langle X_k, \theta \rangle\},$$

the convergence (11) follows from the first part of the proof. \qed

**Proof of Proposition** Let us denote

$$c_1 := \lim_{n} \inf \frac{V(n)}{c(n)}.$$ 

Let $l \geq 2$ be a fixed integer, then we have $c(l^m) \sim b(m)$, as $m \to \infty$. $V(n)$ is non-decreasing, therefore, for $n \in [l^k, l^{k+1}]$, we have

$$\frac{V(n)}{c(n)} \geq \frac{c(l^k) V(l^k)}{c(n) c(l^k)}.$$
Since
\[
\frac{c(l^k)}{c(n)} \geq \frac{c(l^k)}{c(l^k+1)} \to 1, \quad \text{as } n \to \infty,
\]
we get
\[
c_1 \geq \liminf_m \left\{ \frac{V(l^m)}{c(l^m)} \right\} \geq \liminf_m \left\{ \frac{\max\{X_l, X_{l^2}, \ldots, X_{l^m}\}}{c(l^m)} \right\}.
\]
In our example we have (5), therefore \( r := \sup_{i \neq j} EX_lX_{lj} = l^{-1/2} \). Due to Lemma 2.5 from [3] we get
\[
\liminf_m \left\{ \frac{X_l, X_{l^2}, \ldots, X_{l^m}}{c(l^m)} \right\} \geq \sqrt{1 - r} = \sqrt{1 - l^{-1/2}}.
\]
This quantity can be made close to 1 if we choose \( l \) sufficiently large, therefore with probability one we have
\[c_1 \geq 1. \tag{12}\]
In order to get the estimate from above for
\[
c_2 := \limsup_n \left\{ \frac{V(n)}{c(n)} \right\}
\]
we shall need the following lemmas.

**Lemma 9.** Suppose that a sequence of random variables \( \{Y_k\} \) satisfies the following condition: for all \( \gamma < (2\sigma^2)^{-1}, \sigma > 0, \)
\[
\sup_n E \exp\{\gamma Y_n^2\} < \infty. \tag{13}
\]
Then
\[
\limsup_n \left\{ \frac{\max_{k \leq n}\{Y_k\}}{b(n)} \right\} \leq \sigma.
\]

The proof of this lemma coincides with the proof of Lemma 1 in [1], despite of the fact that in this paper the variables \( \{Y_k\} \) was i.i.d. It turns out that independence is not used at all and condition of identical distributions of \( Y_k \) can be replaced by condition (13).

**Lemma 10.** ([11], Thm. 2.2) Let \( \{\xi_k\}, \ k \geq 1, \) be independent symmetric random variables, \( S_n = \sum_{k=1}^n \xi_k. \) Then for every \( x \geq 0 \)
\[
P\left( \max_{1 \leq k \leq n} |S_k| \geq x \right) \leq 2P(|S_n| \geq x).
\]
Lemma 11. Let $X$ and $Y$ be two non-negative random variables with distribution functions $F$ and $G$. If for some $a \geq 1$ and for all $x \geq 0$ we have $1 - F(x) \leq a(1 - G(x))$. Then for every $\gamma > 0$

$$E \exp\{\gamma X^2\} \leq a E \exp\{\gamma Y^2\}.$$  

Elementary proof of this statement follows from equalities

$$E \exp\{\gamma X^2\} = \sum_{k=0}^{\infty} \gamma^k (k!)^{-1} E(X^{2k}), \quad E(X^{2k}) = \int_0^{\infty} (1 - F(x))d(x^{2k}).$$

Let us fix a non-integer (we have in mind that we shall choose $a$ close to 1) number $a > 1$ and let us denote

$$\Delta_j = \{i \in N : a^j \leq i \leq a^{j+1}\}, \quad Y_j = \max_{i \in \Delta_j} X_i.$$

As in the case of lower bound we can prove that

$$c_2 \leq \limsup_m \left\{ \frac{V([a^m])}{c([a^m])} \right\}.$$  

Note that

$$\Delta_j = \{i \in N : [a^j] + 1 \leq i \leq [a^{j+1}]\}$$

and $V([a^m]) = \max_{0 \leq j \leq m-1} Y_j$, therefore

$$|Y_j| \leq \max_{i \in \Delta_j} \{|X_i|\} = \max_{i \in \Delta_j} \left\{ \frac{|S_i|}{\sqrt{i}} \right\} \leq \frac{1}{\sqrt{|a^j| + 1}} \max_{i \in \Delta_j} \{|S_i|\} \leq \frac{1}{\sqrt{|a^j| + 1}} \max_{i \in \{[a^j], [a^j+1]\}} \{|S_i|\}.$$  

Applying Lemma 10 we have

$$P\{\max_{i \in \Delta_j} |S_i| \geq x\} \leq 2P\{|S_{[a^{j+1}]}| \geq x\},$$  

whence

$$P\{|Y_j| \geq x\} \leq 2P\left\{ \left| \frac{S_{[a^{j+1}]}'}{\sqrt{a^j}} \right| \geq x \right\}.$$  

Let $\xi(j, a) = (a^j)^{-1}S_{[a^{j+1}]}$. It is easy to see that $\xi(j, a)$ has distribution $N(0, \sigma^2(j, a))$ with

$$\sigma^2(j, a) = \frac{|a^{j+1}|}{a^j} \to a, \text{ as } j \to \infty, \text{ and } \sigma^2(j, a) \leq a. \quad (14)$$
Applying Lemma 11 with \( \gamma < \frac{1}{2a} \), we get
\[
E \exp\{\gamma Y_j^2\} \leq 2E \exp\{\gamma \xi(j, a)^2\}.
\]

Due to (14) we have \( \sup_j E \exp\{\gamma \xi(j, a)^2\} := C(a) < \infty \), therefore, using Lemma 9 with \( \sigma^2 = a \) and recalling that \( c([a^m]) \sim b(m) \), as \( m \to \infty \), we get that with probability 1
\[
c_2 \leq \limsup_m \frac{1}{b(m)} V([a^m]) \leq \sqrt{a}.
\]

Since the last estimate holds for any \( a > 1 \), we get that with probability 1
\[
c_2 \leq 1. \tag{15}
\]

Estimates (12) and (15) prove (6).

\[\square\]

Proof of Proposition 7. Let \( \mathbb{N} = \bigcup_{k=1}^m T_k \), where the sets \( T_k, k = 1, \ldots, m \), are disjoint and have positive densities \( p_k \). Let \( \{X_n\} \) be a sequence of independent random vectors such that for each \( k \) and \( j \in T_k \), \( X_j \) has Gaussian distribution concentrated on the line \( \{ta_k, t \in \mathbb{R}^1\} \) with zero mean and variance \( \sigma_k^2 = \|a_k\| \). We denote \( W_n^{(k)} = \text{conv}\{X_j \leq n, j \in T_k\} \). Since for any \( p > 0 \),
\[
\lim_n b(np)/b(n) = 1,
\]

Theorem 1 implies that a.s. for any \( k = 1, \ldots, m \),
\[
\frac{1}{b(n)} W_n^{(k)} \to \text{conv}\{-a_k, a_k\}.
\]

Clearly, we have \( W_n = \text{conv}\{W_n^{(1)}, \ldots, W_n^{(m)}\} \), therefore a.s.
\[
\frac{1}{b(n)} W_n \to \text{conv}\{\cup_{k=1}^m \text{conv}\{-a_k, a_k\}\} = V.
\]

\[\square\]
References

1. Davydov, Yu., On convex hull of Gaussian samples, Lith. Math. J., 2011, 51, 171–179

2. Davydov Yu. and Domby, C., Asymptotic behavior of the convex hull of a stationary Gaussian process, Lith. Math. J., 2012, 52(4), 363–368

3. Yu. Davydov and V. Paulauskas, On the asymptotic form of convex hulls of Gaussian random fields, Cent. Eur. J. Math., 2014, 12, 5, 711–720

4. U. Einmahl, Law of the iterated logarithm type results for random vectors with infinite second moment, Matematica Applicanda, 2016, 44, 1, 167–181

5. U. Einmahl and D. Li, some results on two-sided LIL behavior, Ann. Probab., 2005, 33, 4, 1601–1624

6. Fernique, X., Régularité de processus gaussiens, Inventiones Mathematicae, 1971, 12, 304–320

7. Goodman, V., Characteristics of normal samples, Ann. Probab., 1988, 16, 3, 1281–1290

8. Ledoux, M. and Talagrand, M., Probability in Banach Spaces, Springer, 1991

9. Majumdar, S. N., Comptet, A., and Randon-Furling, J., Random convex hulls and extreme value statistics, J. Stat. Phys., 2010, 138, 955–1009

10. V.V. Petrov, Limit Theorems of Probability Theory. Sequences of Independent Random Variables, Clarendon Press, Oxford, 1995

11. Talagrand M., Sur l’integrabilité des vecteurs gaussiens. Z. Wahrscheinlichkeit. Verw. Geb. 1984, 68, 1–8.