Convergence Analysis of the Particle Method for the Geometric Thin-Film Equation

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Abstract

In this work, we characterize weak (particle) solutions of the Geometric Thin-Film Equation. The Geometric Thin-Film Equation is a mathematical model of droplet spreading in the long-wave limit, which includes a regularization of the contact-line singularity. The model admits weak solutions which comprise a weighted sum of delta functions whose centres evolve over time – the weights and the centres are referred to as particles. In the weak formulation of the problem, the Geometric Thin-Film Equation reduces to a set of ordinary differential equations (ODEs) for the particle trajectories. Therefore, our first main result involves characterizing the solutions of this system of ODEs: under certain mild assumptions on the initial conditions and the convolution kernel, we show that the system of ODEs is globally defined for all time. Our second main result is to show that such particle solutions converge to a weak solution of the Geometric Thin-Film Equation. Finally, we demonstrate that the weak solution constructed in this way is 1/2-Hölder continuous in time.

1 Introduction

When a fluid rests on a substrate, the free-surface height is a key variable which characterizes the fluid mechanics. In the longwave limit, where the horizontal length scale of the fluid is much larger than the vertical length scale, the Navier-Stokes equations reduce to a fourth-order nonlinear parabolic equation for the free-surface height $h(x, t)$, this is the thin-film equation
(in this work, we have only one spatial variable $x$). In the simplest setting where the thin film is deposited on a flat non-inclined surface without external forces and only driven by surface tension, the evolution of $h(x, t)$ satisfies the thin-film equation
\[ \partial_t h = -\partial_x (h^3 \partial_{xxx} h), \] (1)
on the domain $\Omega_T := \mathbb{R} \times [0, \infty)$ subject to the initial condition $h(x, 0) = h_0(x)$.

The thin-film equation can be used to describe not only extended films deposited on substrates, but also, droplets provided the contact angle is small. In particular, Equation (1) has admits a piecewise parabolic equilibrium solution $h(x) = \max(0, A - Bx^2)$, where $A$ and $B$ are positive constants, corresponding to a stationary droplet which extends from $x = -\sqrt{A/B}$ to $x = \sqrt{A/B}$. In this context, $x = \pm \sqrt{A/B}$ are triple points, which are points where the fluid, the substrate, and the gas in the surrounding atmosphere meet (such points are also referred to as contact lines, which makes geometric sense in the context of a three-dimensional droplet).

The thin-film equation as stated in Equation (1) fails to properly capture the passage to equilibrium, in particular, the motion of the contact line [2]. The moving contact line violates the no-slip assumption of viscous fluid flow and manifests as a stress singularity at the contact line. A typical occurrence of this scenario is the droplet spreading. To resolve the paradox, the Equation (1) needs to be modified in such a way as to resolve the singularity at the contact line while keeping the macro scale dynamics of the thin-film equation. This step is often referred to as regularization. Multiple regularization methods are proposed in the literature [2, 3, 4, 5, 6, 7, 8, 9]. While these approaches differ, they all rely on the parametrizing of physics at small scales (typically, on the nanometre scale), and give the same qualitative results on the droplet scale (typically, on scales from microns to millimetres).

The present work is concerned with the Geometric Thin-Film Equation, a novel regularization of Equation (1) introduced previously by the present authors in Reference [9]. This regularization involves a smoothened free-surface height $\bar{h}(x, t)$, which is connected to the basic free-surface height $h(x, t)$ via convolution:
\[ \bar{h} = K \ast h. \] (2a)

By requiring that the dynamics of $h(x, t)$ reduce the surface energy
\[ E = \frac{1}{2} \int_{-\infty}^{\infty} h_x \bar{h}_x \, dx \] (2b)
over time, it has been shown that \( h(x,t) \) must satisfy the following equation:

\[
\partial_t h + \partial_x [h\mu(\delta h)\partial_{xxx}\delta h] = 0. \tag{2c}
\]

Furthermore, in Reference [9] the authors justify the choice of \( K \) as the Green’s function for the bi-Helmholtz problem \( (1 - \alpha^2 \partial_{xx})^2 K(x) = \delta(x) \), hence \( K \) is given by:

\[
K(x) = \frac{1}{4\alpha^2(\alpha + |x|)}e^{-|x|/\alpha}. \tag{2d}
\]

Equation (2) is solved on the domain \( \Omega_T \), subject to initial condition \( h(x,0) = h_0(x) \geq 0 \). Also, \( \mu(h) = h^2 \) is the mobility, and \( \alpha \) is a small positive length-scale (the smoothing lengthscale).

In [9], the present authors have explored Equation (2) numerically. There, it was demonstrated that the regularized model reproduces the known spreading behaviour of droplets with small equilibrium contact angles, specifically, the Cox-Voinov law and the Tanner’s Law of spreading. As such, the aim of the present work is to further characterize solutions of Equation (2), this time using analytical techniques.

The Geometric Thin-Film Equation can be viewed as a special case of a mechanical model for energy-dissipation on general configuration spaces—the derivation of the general model involves methods from Geometric Mechanics such as Lie Derivatives and Momentum Maps [10] – hence the name Geometric Thin-Film Equation. The main advantage of this new method so far has been the non-stiff nature of the differential equations in the model, which leads to robust numerical simulation results. A second advantage (the main focus of the present work) is that the Geometric Thin-Film Equation admits so-called particle solutions. These give rise to an efficient and accurate numerical method (the particle method) for solving the model equations.

**Particle solutions**

One of the key properties of this regularization is that Equation (2) admits weak solutions of the form

\[
h^N(x,t) = \sum_{i=1}^{N} w_i \delta(x - x_i(t)). \tag{3}
\]

Each weight \( w_i \) and position \( x_i(t) \) is identified with a pseudoparticle – hence, solutions of this form are referred to as ‘particle solutions’. Substituting
Equation (3) into a weak form of Equation (2) yields a system of $N$ first order ODEs given by

$$\begin{aligned}
\dot{x}_i &= \left[ \mu(\bar{h}^N) \partial_{xxx} \bar{h}^N \right]_{x=x_i(t)}, \\
x_i(0) &= x_i^0, \\
i = 1, \ldots, N.
\end{aligned} \tag{4}$$

where we have used a compact notation for

$$\bar{h}^N(x_i(t), t) = \sum_{j=1}^{N} w_j K(x_i(t) - x_j(t)), \tag{5}$$

$$\partial_{xxx} \bar{h}^N(x_i(t), t) = \sum_{j=1}^{N} w_j K'''(x_i(t) - x_j(t)). \tag{6}$$

We note that $K$ is only twice differentiable in the classical sense and that $K'''$ is not defined at the origin. However, one of the key results in the present article is that a solution that avoids this discontinuous point at $t = 0$ stays away from this discontinuous point for all later times.

We are left with the freedom to choose the initial particle positions $x_i^0$ and the particle weights $w_i$ but a natural choice is to ensure the particle solution is “close” to the initial condition given in Equation (2). To be more precise, $(w_i, x_i^0)_{i=1}^{N}$ is chosen such that for all test functions $\phi(x)$, we have

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} \bar{h}^N(x, 0) \phi(x) dx = \int_{-\infty}^{\infty} h_0(x) \phi(x) dx. \tag{7}$$

Furthermore, since $h_0(x) \geq 0$ and $\|h_0\|_1 < \infty$, we shall also have $w_i \geq 0$ and $W := \sum_{i=1}^{N} w_i < \infty$. These choices will be justified in later sections. We refer the reader to [9] for a specific choice of $(x_i^0, w_i)_{i=1}^{N}$.

**Literature Review**

In the present work, we prove the global existence of solutions of the system of ODEs given in Equation (4). Also, using the theory developed in [11], we will prove that such weak particle solutions converge to a solution of the geometric thin-film equation. We, first of all, put our results in the context of existing literature on particle solutions of continuum models.

Particle solutions are a key feature of the Camassa–Holm (CH) equation, a partial differential equation for water waves. In this context, the particles
are referred to as ‘peakons’ and are analogous to solitons found in other water-wave models. Peakons are weak solutions of the CH equation; the peakon positions and momenta satisfy a set of ODEs with a Hamiltonian structure. As such, the authors of Reference [12] have shown the convergence of particle solutions to a weak solution of the CH equation, the main theoretical tool used here is Helly’s selection theorem. The present authors have generalized this approach using a metric Arzelà-Ascoli compactness theorem [11]. Other continuum models also admit particle solutions. For instance, in [13], the author analysed the interval of existence of an aggregation equation and provides an “acceptability” condition for the kernel function for which the solution exhibits finite time blow-up behaviour. Our approach to the present problem is similar to these existing works.

The Geometric Thin-Film Equation (2) has already been characterized numerically by the present authors. In particular, in [9], we introduced a fast summation algorithm to reduce the numerical cost of evaluating the system of ODEs in Equation (4) from $O(N^2)$ to $O(N)$. The algorithm relies on an assumption that the relative ordering of the particles is preserved throughout the interval of the existence of the solution, in other words, crossings (or collisions) are not permitted between the particles. Indeed, one of the main aims of the present work is to prove the no-crossing theorem rigorously.

We emphasize that other models which admit particle solutions do possess a no-crossing theorem (e.g. the CH equation [14, 15]). In the case of the CH equation, the no-crossing theorem is proved using arguments about the conservation of total momentum, these arguments rely on the underlying Hamiltonian structure of the CH equation. However, in the present case, there is no such Hamiltonian structure, and a new approach to proving the no-crossing theorem must be developed.

**Plan of the paper**

Motivated by this review of prior work, we show that the particle solutions defined in Equation (3) are weak solutions of the geometric thin-film equation (2) (Section 2). We introduce a geometrical argument on the manifold in which the particle crossings occur to prove a no-crossing theorem for the particle solution of the Geometric Thin-Film Equation (Section 3), this allows us to conclude that such particle solutions exist globally in time. Finally, in Section 4 we show that the sequence of the particle solutions converges as $N \to \infty$ and the limiting function is also a weak solution of Equation (2c).
The solution is shown to be contained in the space $C_b^{0,\frac{1}{2}}(\mathbb{R}^+; H^3(\mathbb{R}))$, which is the space of $1/2$-Hölder continuous functions $\hat{h}: \mathbb{R}^+ \to H^3(\mathbb{R})$ that satisfy $\sup_{t \in \mathbb{R}^+} \|\hat{h}(t)\| < \infty$.

\section{Weak formulation of the Geometric Thin-Film Equation}

In this section, we show that the particle solutions (Equation (3)) are weak solutions of the Geometric Thin-Film Equation (2).

**Definition.** Let $\bar{h} \in C(\mathbb{R}^+; W^{3,1}_{loc}(\mathbb{R}))$, $h = (1 - \alpha^2 \partial_{xx})^2 \bar{h}$, with the latter regarded as a distribution representable by a measure. We say that the pair $(h, \bar{h})$ is a weak solution of Equation (2c) if it satisfies

\[ \int_{-\infty}^{\infty} (h\phi)_{t=0} \, dx + \int_{-\infty}^{\infty} \bar{h}(1 - \alpha^2 \partial_{xx})^2 \phi \, dx \, dt + \int_{-\infty}^{\infty} \bar{h}^3 \partial_{xxx} \bar{h} \phi_x \, dx \, dt \]
\[ - 2\alpha^2 \int_{0}^{\infty} \int_{-\infty}^{\infty} \bar{h}^2 \partial_{xx} \bar{h} \partial_{xxx} \bar{h} \phi_x \, dx \, dt - \alpha^4 \int_{0}^{\infty} \int_{-\infty}^{\infty} \bar{h} \partial_x \bar{h} (\partial_{xxx} \bar{h})^2 \phi_x \, dx \, dt \]
\[ - \frac{1}{2} \alpha^4 \int_{0}^{\infty} \int_{-\infty}^{\infty} (\bar{h} \partial_{xxx} \bar{h})^2 \phi_{xx} \, dx \, dt = 0, \quad (8) \]

for all test functions $\phi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^+)$. 

**Theorem 2.1.** The pair of functions $(h^N, \bar{h}^N)$ defined in Equation (3) is a weak solution of Equation (2c) for all $N \in \mathbb{N}$.

**Proof.** We label each term in Equation (8): the first term is labelled as $T_1$, the second term as $T_2$, and the remaining terms as $T_3$. The aim of the proof is then to show that $T_1 + T_2 + T_3 = 0$. The first term is:

\[ T_1 = \sum_{i=1}^{N} w_i \int_{-\infty}^{\infty} \delta(x - x_i^0) \phi(x, 0) \, dx = \sum_{i=1}^{N} w_i \phi(x_i^0, 0). \quad (9) \]

Since $\langle \bar{h}, (1 - \alpha^2 \partial_{xx})^2 \phi_i \rangle = \langle h, \phi_i \rangle$, the second term becomes

\[ T_2 = \sum_{i=1}^{N} w_i \int_{0}^{\infty} \int_{-\infty}^{\infty} \delta(x - x_i(t)) \phi_i(x, t) \, dx \, dt = \sum_{i=1}^{N} w_i \int_{0}^{\infty} \phi_i(x_i(t), t) \, dt. \quad (10) \]
Here and below, we use \(\langle \cdot, \cdot \rangle\) to denote the natural pairing of two functions, or the pairing of a function and a distribution. Using the identity
\[
\frac{d}{dt} \phi(x_i(t), t) = \dot{x}_i(t) \phi_x(x_i(t), t) + \phi_t(x_i(t), t),
\]  
(11)

\(T_2\) then becomes
\[
T_2 = - \sum_{i=1}^{N} w_i \phi(x_i^0, 0) - \sum_{i=1}^{N} w_i \int_{0}^{\infty} \dot{x}_i(t) \phi_x(x_i(t), t) \, dt.
\]  
(12)

For the remaining terms, we consider the pairing
\[
\langle h, \bar{h}^2 \partial_{xxx} \bar{h} \phi_x \rangle = \langle \bar{h}, (1 - \alpha^2 \partial_{xx})^2 (\bar{h}^2 \partial_{xxx} \bar{h} \phi_x) \rangle,
\]  
(13)

\[
= \langle \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_x \rangle - 2\alpha^2 \langle \bar{h}, \partial_{xx} (\bar{h}^2 \partial_{xxx} \bar{h} \phi_x) \rangle + \alpha^4 \langle \bar{h}, \partial_{xxxx} (\bar{h}^2 \partial_{xxx} \bar{h} \phi_x) \rangle,
\]  
(14)

\[
= \langle \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_x \rangle - 2\alpha^2 \langle \partial_{xx} \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_x \rangle - \alpha^4 \langle \partial_{xxx} \bar{h}, \partial_x (\bar{h}^2 \partial_{xxx} \bar{h} \phi_x) \rangle.
\]  
(15)

Expanding \(I\) we get
\[
I = 2 \langle \partial_{xxx} \bar{h}, \bar{h} \partial_x \bar{h} \partial_{xxx} \bar{h} \phi_x \rangle + \langle \partial_{xxx} \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_x \rangle
\]
\[
+ \langle \partial_{xx} \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_{xx} \rangle,
\]  
(16)

\[
2I = 2 \langle \partial_{xxx} \bar{h}, \bar{h} \partial_x \bar{h} \partial_{xxx} \bar{h} \phi_x \rangle + \langle \partial_{xxx} \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_{xx} \rangle.
\]  
(17)

So Expression (15) becomes
\[
\langle h, \bar{h}^2 \partial_{xxx} \bar{h} \phi_x \rangle = \langle \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_x \rangle - 2\alpha^2 \langle \partial_{xx} \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_x \rangle
\]
\[
- \alpha^4 \langle \partial_{xxx} \bar{h}, \partial_x (\bar{h}^2 \partial_{xxx} \bar{h} \phi_x) \rangle - \frac{1}{2} \alpha^4 \langle \partial_{xxx} \bar{h}, \bar{h}^2 \partial_{xxx} \bar{h} \phi_{xx} \rangle.
\]  
(18)

Thus the rest of the terms in Equation (8) is precisely
\[
T_3 = \int_{0}^{\infty} \int_{-\infty}^{\infty} \bar{h} \partial_{xxx} \bar{h} \phi_x \, dx \, dt,
\]  
(19)

\[
= \sum_{i=1}^{N} w_i \int_{0}^{\infty} \int_{-\infty}^{\infty} \delta(x - x_i(t)) \bar{h} \partial_{xxx} \bar{h} \phi_x \, dx \, dt,
\]  
(20)

\[
= \sum_{i=1}^{N} w_i \int_{0}^{\infty} (\bar{h}^2 \partial_{xxx} \bar{h} \phi_x)_{x=x_i(t)} \, dt.
\]  
(21)
Putting these terms together gives
\[ T_1 + T_2 + T_3 = -\sum_{i=1}^{N} w_i \int_{0}^{\infty} \phi_x(x_i(t), t) \left[ \dot{x}_i(t) - (\bar{h}^2 \partial_{xxx} \bar{h})_{x=x_i(t)} \right] dt = 0. \]

(22)

So \((h, \bar{h})\) is a weak solution of Equation (2c).

\[\square\]

3 Global existence of particle solutions and a no crossing theorem

In this section, we show that the system of ODEs (3) has a unique, globally defined solution – for suitable initial conditions. The strategy of the proof is to show that particles that are initially separated stay separated (do not cross). This ensures that the trajectories avoid regions where the function on the right-hand side of the ODE system is non-Lipschitz. Hence, a unique, globally-defined solution is guaranteed. The key insight which enables us to complete the proof is to demonstrate that the set
\[ D = \left\{ x \in \mathbb{R}^N : x_i < x_j, i < j \right\} \]

is a trapping region for Equation (4). We start with the following definitions:

Definition (Crossing). Let \( U \subseteq \mathbb{R}^N \), and let \( v : U \rightarrow \mathbb{R}^N \). Consider the initial value problem
\[
\begin{align*}
\dot{x}(t) &= v(x(t)) \quad t \geq 0, \\
x(0) &= x_0.
\end{align*}
\]

(23)

We say that Equation (23) has no-crossing in finite time if the solution exists for all \( t \geq 0 \) and
\[ x_i(t) \neq x_j(t) \quad \text{for all } i \neq j \text{ and } t \geq 0. \]

(24)

Let \( \Delta_{ij} = \{ x \in \mathbb{R}^N : x_i = x_j \} \) and define \( \Delta = \bigcup_{i \neq j} \Delta_{ij} \). Thus, \( \Delta \) is the set where crossings between two or more variables/particles occur. We note that the solution of an autonomous system has no crossing if and only if \( x(t) \notin \Delta \) for all \( t \geq 0 \).
Definition (Trapping region). Let $U \subseteq \mathbb{R}^N$ be a domain with $\partial U$ piecewise smooth, and let $\mathbf{n}(\mathbf{x})$ be the inward pointing normal to $\partial U$ defined at regular points of $\partial U$. Let $\mathbf{v} : U \to \mathbb{R}^N$ be bounded and Lipschitz. We say that $U$ is a trapping region for $\dot{x} = \mathbf{v}(\mathbf{x})$ if for all $\mathbf{x}$ regular points of $\partial U$,

$$n(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) > 0. \quad (25)$$

Note that here we define $\mathbf{v}(\mathbf{x})$ for $\mathbf{x} \in \partial U$ by

$$\mathbf{v}(\mathbf{x}) := \lim_{n \to \infty} \mathbf{v}(\mathbf{x}_n), \quad (26)$$

for any sequence $(\mathbf{x}_n) \subseteq U$ with $\mathbf{x}_n \to \mathbf{x}$. This limit exists because of the Lipschitz assumption of $\mathbf{v}$ on $U$.

We will also use the following theorem from which we take from Reference [16]:

**Theorem 3.1** (Sufficient condition for global existence and uniqueness [16]). Let $U \subseteq \mathbb{R}^N$ be a domain with piecewise smooth boundary. Let $\mathbf{v} : U \to \mathbb{R}^N$ be bounded and Lipschitz on $U$. If $U$ is a trapping region for $\dot{x} = \mathbf{v}(\mathbf{x})$ and $\mathbf{x}(0) \in U$ then there exists a unique solution for the initial value problem which exists for all $t \geq 0$. Furthermore, the solution is contained in $U$ for all $t \geq 0$.

**Corollary 3.2** (No-crossing theorem). Consider again the assumption of Theorem 3.1. If $U \cap \Delta = \emptyset$, then for any initial data $\mathbf{x}(0) \in U$, the system has no crossing in finite time.

**Proof.** The proof follows from the fact that (the image of) $\mathbf{x}(t)$ is contained in $U$. \qed

**Lemma 3.3.** Let $D = \{ \mathbf{x} \in \mathbb{R}^N : x_i < x_j, i < j \}$, and let $f : (0, \infty) \to \mathbb{R}$ and $g : (-\infty, 0) \to \mathbb{R}$ be Lipschitz continuous. Let $\mathbf{v} : D \to \mathbb{R}^N$ be defined by

$$v_i(\mathbf{x}) = \sum_{j=1}^{i-1} w_j f(x_i - x_j) + \sum_{j=i+1}^{N} w_j g(x_i - x_j). \quad (27)$$

Then $\mathbf{v}$ is Lipschitz on $D$. 


Proof. Let \( x, y \in D \),

\[
\|v(x) - v(y)\|_1 = \sum_{i=1}^{N} |v_i(x) - v_i(y)|, \quad (28)
\]

\[
\leq \sum_{i=1}^{N} \sum_{j=1}^{i-1} w_j |f(x_i - x_j) - f(y_i - y_j)|
+ \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_j |g(x_i - x_j) - g(y_i - y_j)|. \quad (29)
\]

Since \( g \) and \( f \) are Lipschitz on their respective domains, there exists \( L > 0 \) such that

\[
\|v(x) - v(y)\|_1 \leq \sum_{i=1}^{N} \sum_{j=1}^{N} w_j L |x_i - x_j - y_i + y_j|, \quad (30)
\]

\[
\leq \sum_{i=1}^{N} \sum_{j=1}^{N} w_j L (|x_i - y_i| + |x_j - y_j|), \quad (31)
\]

\[
\leq \sum_{i=1}^{N} \sum_{j=1}^{N} w_j 2L \|x - y\|_1, \quad (32)
\]

\[
\leq 2LNW \|x - y\|_1. \quad (33)
\]

Example 3.4. The velocity field defined by

\[
v_i(x) = \partial_{xxx} \bar{h}^N \big|_{x=x_i} = \sum_{j=1}^{N} w_j K'''(x_i - x_j). \quad (34)
\]

is Lipschitz on \( D \) because \( K''' \) can be written as

\[
K'''(x) = f(x) \chi_{(0,\infty)}(x) + g(x) \chi_{(-\infty,0)}(x), \quad (35)
\]

where \( \chi_A \) is the indicator function on the set \( A \), for some Lipschitz function \( f \) and \( g \). It follows that since \( \bar{h}^N(x) \) and \( \partial_{xxx} \bar{h}^N(x) \) is bounded and Lipschitz on \( D \), the velocity field defined in Equation (34) is bounded and Lipschitz on \( D \).
Theorem 3.5 (Global existence of weak solution). Consider the initial value problem defined in Equation (4). If the initial condition satisfies

$$x_i^0 < x_j^0 \quad \text{for all } i < j,$$

then there exists a unique global solution to the initial value problem. Furthermore, the system has no crossing in finite time.

Proof. We show that $D = \{ \mathbf{x} \in \mathbb{R}^N : x_i < x_j, i < j \}$ is a trapping region for the system. From Example 3.4, $\mathbf{v}$ defined in Equation (4) is bounded and Lipschitz on $D$. Let $\Gamma$ be the set of regular points of $\partial D$ and let $n_{ij} \in \mathbb{R}^N, i \neq j$ with $-1$ for the $i$-th component, $+1$ for the $j$-th component, and $0$ everywhere else. Note that for all $\mathbf{x} \in \Gamma$, $\mathbf{x} \in \Delta_{i,i+1}$ for some unique $i = 1, \ldots, N-1$. So the inward pointing normal vector at $\mathbf{x} \in \Gamma \cap \Delta_{i,i+1}$ is given by $n_{i,i+1}$ and Equation (25) corresponds to

$$-v_i(\mathbf{x}) + v_{i+1}(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \Gamma \cap \Delta_{i,i+1}. \quad (37)$$

for all $i = 1, \ldots, N-1$. To shorten the equation, fix $\mathbf{x} \in D$ and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \bar{h}^N(x)^2 = \sum_{i,j=1}^{N} w_iw_jK(x-x_i)K(x-x_j), \quad (38)$$

then $f$ is continuous and positive on $\mathbb{R}$. We compute the quantity

$$-v_i(\mathbf{x}) + v_{i+1}(\mathbf{x})$$

$$= \sum_{j=1}^{N} w_j f(x_{i+1})K''''(x_{i+1} - x_j) - \sum_{j=1}^{N} w_j f(x_i)K''''(x_i - x_j), \quad (39)$$

$$= \sum_{j=1}^{i-1} w_j [f(x_{i+1})K''''(x_{i+1} - x_j) - f(x_i)K''''(x_i - x_j)]$$

$$+ w_i f(x_{i+1})K''''(x_{i+1} - x_i) - w_{i+1} f(x_i)K''''(x_i - x_{i+1})$$

$$+ \sum_{j=i+2}^{N} w_j [f(x_{i+1})K''''(x_{i+1} - x_j) - f(x_i)K''''(x_i - x_j)]. \quad (40)$$
To evaluate at $x \in \partial D \cap \Delta_{i,i+1}$, we take $d(x, \Delta_{i,i+1}) \to 0$, equivalently $x_{i+1} - x_i \downarrow 0$. Because $f$ is continuous on $\mathbb{R}$ and $K'''$ is continuous when restricted on $(-\infty, 0)$ and $(0, \infty)$ separately, we have

$$\lim_{x_{i+1} \to x_i} [f(x_i)K'''(x_i - x_j) - f(x_{i+1})K'''(x_{i+1} - x_j)] = 0, \quad (41)$$

for $j = 1, \ldots, i - 1, i + 2, \ldots, N$. So

$$\lim_{d(x,\Delta_{i,i+1}) \to 0} [-v_i(x) + v_{i+1}(x)] = w_i f(x_i)K'''(0_+) - w_{i+1} f(x_{i+1})K'''(0_-). \quad (42)$$

Given that $K'''(0_+) > 0$ and $K'''(0_-) < 0$, we conclude

$$-v_i(x) + v_{i+1}(x) > 0 \quad \forall x \in \Gamma \cap \Delta_{i,i+1}. \quad (43)$$

Therefore $D$ is a trapping region for Equation (4). By Theorem 3.1 and Corollary 3.2, the solution of Equation (4) is unique and exists globally for all initial data $x_0 \in D$. Furthermore, the solution has no crossing in finite time.

We remark that for systems with symmetric velocity fields

$$x_i = x_j \implies v_i(x) = v_j(x), \quad \forall i, j \quad (44)$$

and Lipschitz continuous on the whole of $\mathbb{R}^N$, then the no-crossing property follows from the fact that (i) two different trajectories on the phase space cannot intersect, and (ii) if the initial condition $x_0$ is contained in $\Delta_{ij}$ for some $i \neq j$, then the solution of Equation (44) is contained in $\Delta_{ij}$. Thus, the set $\Delta$ partitions $\mathbb{R}^N$ into $N!$ regions and any solution starting in one region is not allowed to move to another region as that would require the intersection of solutions. Furthermore, if the velocity field is bounded, then each of the $N!$ partitions is a trapping region.

4 Convergence of particle solution and existence of a weak solution

In this section, we show that a subsequence of the family of particle solutions $(h^N)$ converges and classify the regularity of the limiting function. Then we conclude by showing that the limiting function is a weak solution of Equation (2c). The starting point is the following result, which is a minor variant of a previous result taken from [17]
Theorem 4.1. (A metric Arzelà-Ascoli theorem [17]) Let \((X, \tau)\) be a sequentially compact Hausdorff topological space, and let \(d\) be a \(\tau\)-lower semicontinuous metric on \(X\). Let \(f_n : \mathbb{R}^+ \to X\) such that
\[
\limsup_{n \to \infty} d(f_n(s), f_n(t)) \leq L|s - t| \quad \forall s, t \in \mathbb{R}^+,
\]
for some \(L \geq 0\). Then there exists a subsequence of \((f_n)\), labelled in the same way, and \(f : \mathbb{R}^+ \to X\) such that
\[
f_n(t) \overset{\tau}{\to} f(t) \quad \forall t \in \mathbb{R}^+,
\]
and \(f\) is \(d\)-Lipschitz with Lipschitz constant \(L\).

In order to apply Theorem 4.1 we first introduce some notation. Let \(C_0(\mathbb{R})\) denote the Banach space of continuous functions \(f : \mathbb{R} \to \mathbb{R}\) that vanish at infinity, equipped with the supremum norm. Let \(\mathcal{M}(\mathbb{R}) \equiv C_0(\mathbb{R})^*\) and \(\mathcal{M}^+(\mathbb{R})\) denote the spaces of Radon and positive Radon measures on \(\mathbb{R}\), respectively, equipped with the usual variation norm \(\| \cdot \|_1\). For our purpose, we set \((X, \tau) = (B_{\mathcal{M}^+(\mathbb{R})}, w^*)\), where \(B_{\mathcal{M}^+(\mathbb{R})} = \{m \in \mathcal{M}^+(\mathbb{R}) : \|m\|_1 \leq 1\}\), and \(w^*\) denotes the weakstar topology with respect to \(C_0(\mathbb{R})\). As \(C_0(\mathbb{R})\) is separable, \((X, \tau)\) is metrizable and compact, hence sequentially compact. Consider the set of Lipschitz functions
\[
A = \{f \in C_0(\mathbb{R}) : \|f\|_\infty, \text{Lip}(f) \leq 1\},
\]
and a norm \(\|\cdot\|\) on \(\mathcal{M}(\mathbb{R})\) defined by
\[
\|m\| = \sup_{f \in A} m(f).
\]

We refer the reader to [11, Section 3] for the justification of our choice of norm. As the pointwise supremum of a family of \(w^*\)-continuous functions, \(\|\cdot\|\) is \(w^*\)-lower semicontinuous. It follows that \(d(m, m') := \|m - m'\|\) defines a \(w^*\)-lower semicontinuous metric on \(B_{\mathcal{M}^+(\mathbb{R})}\). Finally, a function \(f : \mathbb{R} \to \mathbb{R}\) having finite essential variation shall be called a BV function and the Banach space of (equivalence classes of) integrable BV functions is denoted \(BV(\mathbb{R})\).

From here on, \(\delta\) denotes Dirac measure instead of a distribution and we assume by rescaling that \(W \leq 1\). Thus, the particle solutions \(h^N : \mathbb{R}^+ \to B_{\mathcal{M}^+(\mathbb{R})}\) are given by
\[
h^N(t) = \sum_{i=1}^N w_i \delta_{x_i(t)}.
\]
Define the space of functions $\mathcal{X} = \{ h : \mathbb{R}^+ \to B_{M^+}(\mathbb{R}) : h \text{ is } d\text{-continuous} \}$. In the next proposition, we show that the family of particle solutions satisfy the assumption of Theorem 4.1. This result echoes [11, Proposition 3.1].

**Proposition 4.2.** We have $h^N \in \mathcal{X}$ for all $N \in \mathbb{N}$ and $\sup_N \text{Lip}(h^N) < \infty$ with respect to $d$.

**Proof.** Let $s, t \geq 0$, and let $f \in A$. Then

$$
|(h^N(t) - h^N(s))(f)| \leq \sum_{i=1}^{N} w_i |(\delta_{x_i(t)} - \delta_{x_i(s)})(f)|;
$$

(50)

$$
= \sum_{i=1}^{N} w_i |f(x_i(t)) - f(x_i(s))|,
$$

(51)

$$
\leq \sum_{i=1}^{N} w_i |x_i(t) - x_i(s)|.
$$

(52)

On the other hand, we have the following bounds for the kernel defined in Equation (2d):

$$
0 \leq |\tilde{h}^N(x)| \leq W\|K\|_\infty = \frac{1}{4\alpha}, \quad 0 \leq |\partial_{xxx}\tilde{h}^N(x)| \leq W\|K''\|_\infty = \frac{1}{2\alpha^4},
$$

(53)

for all $x \in \mathbb{R}$. By the Mean Value Theorem,

$$
|x_i(t) - x_i(s)| \leq \| (\tilde{h}^N)^2 \partial_{xxx}\tilde{h}^N \|_\infty |t - s| \leq \frac{1}{32\alpha^6}|t - s|.
$$

(54)

Thus

$$
d(h^N(t), h^N(s)) = \sup_{f \in A} |(h^N(t) - h^N(s))(f)| \leq \frac{1}{32\alpha^6}|t - s|,
$$

(55)

giving $\sup_N \text{Lip}(h^N) < \infty$ with respect to $d$. \hfill \Box

By Theorem 4.1 there exists a subsequence of $(h^N)$, labelled in the same way, and $h \in \mathcal{X}$ such that

$$
h^N(t) \xrightarrow{w^*} h(t) \quad \forall t \geq 0.
$$

(56)
To show that $\bar{h}(t) := K \ast h(t) \in H^3(\mathbb{R})$ for all $t \geq 0$, we define a bounded linear map $T : H^3(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ by

$$(Tu)(x) = \langle K(\cdot - x)|u \rangle_{H^3(\mathbb{R})}. \tag{57}$$

Since $K \in H^3(\mathbb{R})$, and is even, by [11, Proposition 2.2], $T$ is well-defined and the dual operator $T^* : M(\mathbb{R}) \rightarrow H^3(\mathbb{R})$ is given by

$$T^* m = K \ast m. \tag{58}$$

Thus $\bar{h}(t) \in H^3(\mathbb{R})$ for all $t \geq 0$. Also note that since $K \in W^{3,1}(\mathbb{R})$ and $K^{(3)} \in BV(\mathbb{R})$, and $h$ is $d$-Lipschitz, by [11, Proposition 4.4], $\bar{h} \in C^0_{b}((\mathbb{R}^+; H^3(\mathbb{R})))$.

**Theorem 4.3.** $(h, \bar{h})$ defined above is a weak solution of Equation (2c).

**Proof.** We show that $(h, \bar{h})$ satisfy Equation (8). The first term follows directly from the fact that $h^N(0) \rightarrow h_0$ in $(B_{M^*(\mathbb{R})}, w^*)$ and $\phi|_{t=0}$ is in the predual space $C_0(\mathbb{R})$. From [11, Proposition 5.2], we have that $\partial^k_x \bar{h}^N \rightarrow \partial^k_x \bar{h}$ in $L^1_{loc}(\Omega)$, for $k = 0, 1, 2, 3$, so the linear term converges as $N \rightarrow \infty$. Furthermore, since for all $k = 0, 1, 2, 3$, the functions $(\partial^k_x \bar{h}^N), \partial^k_x \bar{h} \in L^\infty(\Omega)$ are uniformly bounded with respect to $\| \cdot \|_{\infty}$, the nonlinear terms in Equation (8) also converge as $N \rightarrow \infty$. \hfill $\Box$

5 Conclusion

In this work, we have shown that particle solutions provide a sequence of weak solutions to the Geometric Thin-Film Equation. The exposition consisted of a number of steps. First, we demonstrated that the particle solutions exist globally and are unique. We also showed that the particle solutions have a no-crossing property for all $t \geq 0$ provided the initial condition is contained in $D = \{x \in \mathbb{R} : x_i < x_j, i < j\}$. Then, using a compactness result applied to the measure norm $\| \cdot \|$, we have shown that the sequence of particle solutions converges and the limiting function is a weak solution to Equation (2c) and satisfies the initial condition $h_0(x)$. The solution is also shown to be $1/2$-Hölder continuous in time.
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