Approximating Sparse Quadratic Programs

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Abstract. Given a matrix $A \in \mathbb{R}^{n \times n}$, we consider the problem of maximizing $x^T Ax$ subject to the constraint $x \in \{-1, 1\}^n$. This problem, called MaxQP by Charikar and Wirth [FOCS’04], generalizes MaxCut and has natural applications in data clustering and in the study of disordered magnetic phases of matter. Charikar and Wirth showed that the problem admits an $\Omega(1/\lg n)$ approximation via semidefinite programming, and Alon, Makarychev, Makarychev, and Naor [STOC’05] showed that the same approach yields an $\Omega(1)$ approximation when $A$ corresponds to a graph of bounded chromatic number. Both these results rely on solving the semidefinite relaxation of MaxQP, whose currently best running time is $\tilde{O}(n^{1.5} \cdot \min\{N, n^{1.5}\})$, where $N$ is the number of nonzero entries in $A$ and $\tilde{O}$ ignores polylogarithmic factors.

In this sequel, we abandon the semidefinite approach and design purely combinatorial approximation algorithms for special cases of MaxQP where $A$ is sparse (i.e., has $O(n)$ nonzero entries). Our algorithms are superior to the semidefinite approach in terms of running time, yet are still competitive in terms of their approximation guarantees. More specifically, we show that:

- Unit MaxQP, where $A \in \{-1, 0, 1\}^{n \times n}$, admits an $(1/3d)$-approximation in $O(n^{1.5})$ time, when the corresponding graph has no isolated vertices and at most $dn$ edges.
- MaxQP admits an $\Omega(1/\lg a_{\text{max}})$-approximation in $O(n^{1.5} \lg a_{\text{max}})$ time, where $a_{\text{max}}$ is the maximum absolute value in $A$, when the corresponding graph is $d$-degenerate.
- MaxQP admits a $(1 - \epsilon)$-approximation in $O(n)$ time when the corresponding graph has bounded local treewidth.
- Unit MaxQP admits a $(1 - \epsilon)$-approximation in $O(n^2)$ time when the corresponding graph is $H$-minor-free.

1 Introduction

In this paper we are interested in the following (integer) quadratic problem which was coined MaxQP by Charikar and Wirth [14]. Given an $n \times n$ symmetric matrix with zero valued diagonal entries $A$, $a_{i,j} \in \mathbb{Q}$ for all $i, j \in \{1, \ldots, n\}$, we want to maximize

$$\text{val}_x(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i x_j \quad \text{s.t. } x_i \in \{-1, 1\} \text{ for all } i \in \{1, \ldots, n\}. \quad (1)$$

Observe that the requirement that all diagonal values of $A$ are zero is to avoid the term $\sum_i a_{i,i}$ which is constant in (1). Furthermore, a non-symmetric matrix $A$ can be replaced with an equivalent symmetric $A'$ by setting $a'_{i,j} = a_{j,i} = \frac{1}{2} \cdot (a_{i,j} + a_{j,i})$, and so the requirement that $A$ is symmetric is just for convenience sake.

Our interest in MaxQP lies in the fact that it is a generic example of integer quadratic programming which naturally appears in different contexts. Below we review three examples:
– **Graph cuts:** Readers familiar with the standard quadratic program formulation of MAXCUT [23] will notice the similarity to (1). Indeed, given a graph \( G = (V, E) \) with vertex set \( V = \{1, \ldots, n\} \) and edge weights \( a_{i,j} \geq 0 \) for each \( \{i,j\} \in E \), the corresponding MAXQP instance on \(-1 \cdot A\) has an optimum solution of value \( 2k - \sum_{i,j} a_{i,j} \) iff \( G \) has a maximum cut of total weight \( k \). Thus, MAXQP with only negative entries can be used to solve MAXCUT exactly, implying that even this special case is NP-hard. Furthermore, this special case translates to the closely related MAXCUT GAIN problem [14, 29].

– **Correlation clustering:** In correlation clustering [6, 13, 15, 35], we are provided with pairwise judgments of the similarity of \( n \) data items. In the simplest version of the problem, there are three possible inputs for each pair: similar (i.e. positive), dissimilar (i.e. negative), or no judgment. In a given clustering of the \( n \) items, a pair of items is said to be in agreement (disagreement) if it is a positive (negative) pair within one cluster or a negative (positive) pair across two distinct clusters. In MAXCORR, the goal is to maximize the correlation of the clustering; that is, the absolute difference between the number of pairs in agreement and the number of pairs in disagreement, across all clusters. Note that when only two clusters are allowed, this directly corresponds to UNIT MAXQP, the variant of MAXQP where \( a_{i,j} \in \{-1, 0, 1\} \) for each entry \( a_{i,j} \) of \( A \).

– **Ising spin glass model:** Spin glass models are used to in physics to study disordered magnetic phases of matter. Such system are notoriously hard to solve, and various techniques to approximate the free energy were developed. In the Ising spin-glass model [9, 36], each node in the graph represents a single spin which can either point up (+1) or down (-1), and neighboring spins \( (i, j) \) may have either positive or negative coupling energy \( a_{i,j} \) between them. The energy of this system (when there is no external field) is given by its Hamiltonian \( H = -1 \cdot \sum_{i,j} a_{i,j} \alpha(i) \alpha(j) \), where \( \alpha(i) \in \{-1, 1\} \) is the spin at site \( i \). A famous problem in the physics of spin-glasses is the characterization of the ground state — the state that minimizes the energy of the system. This problem is precisely MAXQP.

It is convenient to view MAXQP in graph-theoretic terms. Let \( G = (V, E) \) be the graph associated with \( A \), where \( V = \{1, \ldots, n\} \) and \( E = \{\{i,j\} : a_{i,j} \neq 0\} \). The first algorithmic result for MAXQP was from Bieche et al. [12] and Barahona et al. [11] who studied the problem in the context of the Ising spin glass model. They showed that when \( G \) is restricted to be planar, the problem is polynomial-time solvable via a reduction to maximum weight matching. At the same time, Barahona proved that the problem is NP-hard for three-dimensional grids [9], or apex graphs (graphs with a vertex whose removal leaves the graph planar) [10].

### 1.1 Approximation Algorithms

As MAXQP is NP-hard, even for restricted instances, our focus is naturally on polynomial-time approximation algorithms. We note that the fact that the values of \( A \) are allowed to be both positive and negative makes MAXQP quite unique in this context. First of all, there is an immediate equivalence between the maximization version of MAXQP and its minimization version, as maximizing \( \text{val}_x(A) \) is the same as minimizing \( \text{val}_x(-1 \cdot A) \). Furthermore, solutions might have negative values. This poses an extra challenge since a solution with a non-positive value is not an \( f(n) \)-approximate solution, for any function \( f \), in case the optimum is positive (which it always is whenever \( A \neq 0 \), see [14] and Lemma 6). In particular, a uniformly at random chosen solution \( x \) has \( \text{val}_x(A) = 0 \) on expectation, and unlike MAXCUT, such a solution is unlikely to be useful as any kind of approximation.

Alon and Naor [2] were the first to show that these difficulties can be overcome by carefully rounding a semidefinite relaxation of MAXQP. In particular, they studied the problem when \( G \) is restricted to a bipartite graph, and showed that using a rounding technique that relies on the famous Grothendieck inequality, one can obtain an approximation factor guarantee of \( \approx 0.65 \) for the bipartite case. Later, together with Makarychev and Makarychev [1], they showed that the integrality gap of the semidefinite relaxation is \( \Omega(\log \chi(G)) \) and \( \Omega(\log \omega(G)) \), where \( \chi(G) \) and \( \omega(G) \) are the chromatic and clique numbers of \( G \) respectively. In particular, this gap is constant for several interesting graph classes such as \( d \)-degenerate graphs and \( H \)-minor-free graphs, and it generalizes the previous result in [2] as \( \chi(G) \leq 2 \) when \( G \) is bipartite.
Theorem 1 ([1, 2]). MaxQP restricted to graphs of $O(1)$ chromatic number can be approximated within a factor of $\Omega(1)$ in polynomial time.

Regarding the general version of the problem, where $G$ can be an arbitrary graph, an $O(\lg n)$ integrality gap for the semidefinite relaxation was first shown by Nesterov [33]. However, his proof was non-constructive. Charikar and Wirth [14] made his proof constructive, and provided a rounding procedure for the relaxation that guarantees $\Omega(1/\lg n)$-approximate solutions regardless of the structure of $G$.

Theorem 2 ([14, 33]). MaxQP can be approximated within a factor of $\Omega(1/\lg n)$ in polynomial time.

As for the time complexity of the algorithm in Theorem 1 and 2 above, Arora, Hazan, and Kale [4] provided improved running times for several semidefinite programs, including the relaxation of MaxQP. They showed that this relaxation can be solved (to within any constant factor) in time $O(n^{1.5} \cdot \min\{N, n^{1.5}\})$, where $N$ is the number of nonzero entries in $A$ and $\tilde{O}$ ignores polylogarithmic factors. Thus, for general matrices $A$ this running time is $O(n^3)$, and for matrices with $O(n)$ nonzero entries this is $O(n^{2.5})$.

There has also been work on approximation lower bounds for MaxQP. Alon and Naor [2] showed that MaxQP restricted to bipartite graphs cannot be approximated within $16/17 + \varepsilon$ unless $P=NP$, while Charikar and Wirth [14] showed that, assuming $P \neq NP$, the problem admits no $(11/13 + \varepsilon)$-approximation when $G$ is an arbitrary graph. Both these results follow somewhat directly from the $16/17 + \varepsilon$ lower bound for MAXCUT [27]. In contrast, Arora et al. [3] showed a much stronger lower bound by proving that there exists a constant $c > 0$ such that MaxQP cannot be approximated within $\Omega(1/\lg^c n)$, albeit under the weaker assumption that $NP \not\subseteq DTime(n^{O(1)} n)$.

1.2 Our results

In this paper we focus on sparse graphs, i.e. graphs where the number of edges $m$ is $O(n)$. This corresponds to matrices $A$ having $O(n)$ nonzero entries. Note that MaxQP remains APX-hard in this case as well (see Theorem 8 in Appendix B). Nevertheless, we show that one can abandon the semidefinite approach in favor of simpler “purely combinatorial” algorithms, while still maintaining comparable performances. In particular, our algorithms are faster than those obtained from the semidefinite approach whose fastest known implementation requires $O(n^{2.5})$ time [4]. Furthermore, most of them are quite easy to implement.

Our first result concerns Unit MaxQP, the special case of MaxQP where $a_{i,j} \in \{-1, 0, 1\}$ for each entry of $A$. Here we obtain an $\Omega(1)$-approximation algorithm for general sparse graphs that do not have any particular structure.

Theorem 3. Let $d > 0$. Then Unit MaxQP restricted to graphs $G = (V, E)$ with no isolated vertices and $|E| \leq d \cdot n$ can be approximated within a factor of $1/3d$ in $O(n^{1.5})$ time.

Note that there are several interesting graph classes included in the theorem above but excluded by Theorem 1. For instance, consider a graph consisting of a clique of size $\sqrt{n}$ together with a perfect matching on the remaining vertices. The result of Alon et al. [1] implies that the semidefinite relaxation has an integrality gap of $\Omega(\lg n)$ on such a graph, while the algorithm in Theorem 3 provides an $\Omega(1)$-approximation.

Our next result extends Theorem 3 to the weighted case, but at a cost to the approximation factor guarantee. Furthermore, it applies for a less general graph class, namely the class of $d$-degenerate graphs. Recall that a graph is $d$-degenerate if each of its subgraphs has a vertex of degree at most $d$. Let $a_{\max} = \max_{i,j} |a_{i,j}|$ denote the maximum absolute value in $A$.

Theorem 4. Let $d > 0$. Then MaxQP restricted to $d$-degenerate graphs can be approximated within a factor of $\Omega(1/\lg a_{\max})$ in $O(n^{1.5} \lg a_{\max})$ time.

Note that Theorem 1 provides an $\Omega(1)$-approximation for $d$-degenerate graphs, yet the algorithm in Theorem 4 is faster by a factor of $n$.

We next consider sparse graph classes with additional structure. We show that one can obtain a $(1 - \varepsilon)$-approximation algorithm, for any $\varepsilon > 0$, if the given graph has bounded local treewidth [20] (see precise definition in Section 4.1). Well-known graph classes that are known to have bounded local treewidth are planar and bounded genus graphs, as well as bounded degree graphs.
Theorem 5. Let $\varepsilon > 0$. Then MAXQP restricted to graphs of bounded local treewidth can be approximated within a factor of $(1 - \varepsilon)$ in $O(n)$ time.

This theorem is in particular relevant for the problem of finding the ground state of Ising spin-glass models on finite graphs, since typically these graphs are embeddable on a fixed surface and/or have bounded degree. As an example, the popular Edwards-Anderson model [19] is concerned with three dimensional cubic lattices, and Theorem 5 above gives a $(1 - \varepsilon)$-approximation in linear-time for such graphs (see also [22, 26, 37] for other examples).

Recall that $G$ is $H$-minor free if one cannot obtain in $G$ an isomorphic copy of $H$ by a series of edge contractions, edge deletions, and vertex deletions. For $H$-minor-free graphs we present an algorithm with a $(1 - \varepsilon)$ factor guarantee, for any $\varepsilon > 0$, but only for the unit weight case.

Theorem 6. For $\varepsilon > 0$ and any graph $H$ there is an $O(n^2)$ time $(1 - \varepsilon)$-approximation algorithm for UNIT MAXQP restricted to $H$-minor free graphs.

Finally, we note that our results have direct consequences for the MAXCORR problem: Cherikar and Wirth [14] proved that an $\alpha$-approximation algorithm for MAXQP implies an $\alpha/(2 + \alpha)$-approximation algorithm for MAXCORR. Combining this with Theorems 3, 5 and 6 directly gives us the following:

Corollary 1. MAXCORR can be approximated within a factor of

- $1/(6d + 1) - \varepsilon$ on graphs $G = (V, E)$ with $|E| \leq d \cdot |V|$ in $O(n^{1.5})$ time;
- $1/3 - \varepsilon$ on graphs of bounded local treewidth in $O(n)$ time;
- $1/3 - \varepsilon$ on $H$-minor free graphs in $O(n^2)$ time.

2 Preliminaries

Throughout the paper we use $G = (V, E)$ to denote the graph associated with our input matrix $A$; that is, $V = \{1, \ldots, n\}$ and $E = \{(i, j) : a_{ij} \neq 0\}$. Thus, $n = |V|$ and we let $m = |E|$. We slightly abuse notation by allowing a solution $x$ to denote either a vector in $\{-1, 1\}^n$ indexed by $V$ or a function $x : V \to \{-1, 1\}$. For a solution $x$, we let $\text{val}_x(G) = \sum_{(i, j) \in E} a_{ij} x_i x_j$, and we let $\text{opt}(G) = \max_x \text{val}_x(G)$. We use $||A||$ to denote the sum of absolute values in $A$, i.e. $||A|| = \sum_{i,j} |a_{ij}|$. Note that $\text{opt}(G) \leq ||A||$.

We use standard graph-theoretic terminology when dealing with the graph $G$, as in e.g. [17]. In particular, for a subset $V' \subseteq V$, we let $G[V']$ denote the subgraph of $G$ induced by $V'$; i.e., the subgraph with vertex set $V'$ and edge-set $\{(u, v) \in E : u, v \in V'\}$. We let $G - V'$ denote $G[V \setminus V']$, and for a subset of edges $E' \subseteq E$ we let $G - E'$ denote the graph $(V, E')$ without isolated vertices. For a pair of disjoint subsets $V_1, V_2 \subseteq V$, we let $E(V_1, V_2)$ denote the set of edges $E(V_1, V_2) = \{(u, v) \in E : u \in V_1, v \in V_2\}$. Finally, we use $N(v) = \{u : (u, v) \in E\}$ to denote the neighborhood of a vertex $v \in V$ of $G$.

2.1 Useful observations

Note that for a uniformly chosen at random solution $x$, the value $a_{i,j} x_i x_j$ is zero in expectation for any edge $(i, j) \in E$. This implies that $\text{opt}(G) \geq 0$. Moreover, a solution $x$ with $\text{val}_x(G) \geq 0$ can be computed in linear time:

Lemma 1. One can compute in $O(m + n)$ time a solution $x$ for which $\text{val}_x(G) \geq 0$.

Proof. For each vertex $i \in V$, define the subset of edges $E(i) = \{(i, j) \in E : j < i\}$. Consider an arbitrary initial solution $x$, and let $z_i = \sum_{(i, j) \in E(i)} x_i x_j a_{i,j}$. Then $z_i$ is the contribution of edges in $E(i)$ to $\text{val}_x(G)$. We compute a solution $x^*$ by scanning the vertices from 1 to $n$. For a given vertex $i$, we check whether $z_i < 0$. If so, we set $x^*_i = -x_i$, and otherwise we set $x^*_i = x_i$. Note that $z^*_i = \sum_{(i, j) \in E(i)} x^*_i x^*_j a_{i,j}$ must now be positive. As the value of $x^*_i$ does not change $z^*_j$ for any $j < i$, when we finish our scan we have $z^*_i \geq 0$ for each $i \in \{1, \ldots, n\}$. Thus, $\text{val}_{x^*}(G) = \sum_i z^*_i \geq 0$. \qed
Lemma 2. Let $V_1, V_2 \subseteq V$ be two disjoint subsets of vertices, and let $x_1$ and $x_2$ be two solutions for $G[V_1]$ and $G[V_2]$ of value $z_1$ and $z_2$ respectively. Then at least one of the solutions $x_1 \cup x_2$ and $-x_1 \cup x_2$ has value $z_1 + z_2$ for $G[V_1 \cup V_2]$.

Proof. Suppose $x_1 \cup x_2$ has value less than $z_1 + z_2$. This means that the total contribution of the edges in $E(V_1, V_2)$ is negative in this solution. Observe that in $-x_1 \cup x_2$ each edge of $E(V_1, V_2)$ with negative contribution under $x_1 \cup x_2$ now has positive contribution, and vice versa. The lemma thus follows.

Combining Lemma 1 and Lemma 2 above, we get an important property of $\text{val}(G)$, namely that it is monotone with respect to induced subgraphs.

Lemma 3. Let $H$ be an induced subgraph of $G$. Then given a solution $x$, one can compute a solution $x^*$ for $G$ with $\text{val}_x(G) \geq \text{val}_x(H)$.

Proof. Let $V_0 \subseteq V$ be the vertices of $G$ which are not present in $H$. According to Lemma 1 we can compute a solution $x_0$ for $G[V_0]$ with value at least zero in linear time. According to Lemma 2 either $x_0 \cup x$ or $x_0 \cup -x$ have value at least $\text{val}_{x_0}(G[V_0]) + \text{val}_{x_0}(H) \geq \text{val}_x(H)$. Thus, taking $x^*$ to be the solution with higher value out of $x_0 \cup x$ or $x_0 \cup -x$ proves the lemma.

3 A Lower Bound

In this section we present approximation algorithms for MAXQP using a lower bound we develop for the value of the optimum solution. In particular, we provide complete proofs for Theorem 3 and Theorem 4.

Beginning with the unit weight case, i.e. the case when $A \in \{-1, 0, 1\}^{n \times n}$, we obtain a lower bound analogous to the classical MAXCUT bound of Edwards [18], although our approach follows the later proof of Erdős et al. [21]. This will directly imply Theorem 3. We then show how to extend our lower bound to general weights in case $G$ is triangle-free, i.e. the case where $G$ contains no three pairwise mutually adjacent vertices. In the last subsection we show how to remove the triangle-freeness restriction in case $G$ is $d$-degenerate, providing a proof for Theorem 4.

3.1 Unit weights

A set $S \subseteq V$ of vertices is a star in $G$ if $|S| \geq 2$ and $G[S]$ is connected and has at most one vertex of degree greater than 1 (called the center of $S$). We say a star $S$ is uniform if the edges of $G[S]$ are either all positive or all negative. A star packing of $G$ is a family of pairwise disjoint subsets of vertices $S = \{S_1, \ldots, S_s\}$ such that each $S_i$ is a uniform star in $G$. We let $V_S = \bigcup_{S_i \in S} S_i$, and $I_S = V \setminus V_S$. We refer to $s = |S|$ as the size of $S$, and to the value $m(S) = \sum_{i}(|S_i| - 1)$ (the total number of edges in $S$), as the magnitude of $S$.

Star packings will be useful throughout the section for showing lower bounds on $\text{opt}(G)$. The direct connection between these two concepts is given in the lemma below.

Lemma 4. Given a star packing $S$ of magnitude $m(S)$, one can compute in linear time a solution $x$ with $\text{val}_x(G) \geq m(S)$.

Proof. By Lemma 3 it suffices to compute a solution $x$ for $V_S$ with $\text{val}_x(G[V_S]) \geq m(S)$. Let $S = \{S_1, \ldots, S_s\}$. We construct such a solution $x$ by induction on $s$. For $s = 1$, we assign the vertices of $S_1$ the same value in $\{-1, 1\}$ in case all edges of $G[S_1]$ are positive, and we assign the leaves and the center vertex of $S_1$ opposing values if all edges of $G[S_1]$ are negative. Thus, $\text{val}_x(G[S_1]) = |S_1| - 1 = m(S)$. Suppose then that $s > 1$, and let $S_0 = S \setminus \{S_s\}$. By induction, we have a solution $x_0$ for $V_{S_0}$ with $\text{val}_{x_0}(G[V_{S_0}]) \geq m(S_0)$. Let $x_s : S_s \rightarrow \{-1, 1\}$ be such that $\text{val}_{x_s}(G[S_s]) = |S_s| - 1$, as in the case of $s = 1$. Then, by Lemma 2, either $x_0 \cup x_s$ or $x_0 \cup -x_s$ have value at least $m(S_0) + |S_s| - 1 = m(S)$, and we are done.

We construct a particular star packing $S^*$ for $G$. We begin by first letting $S^*$ be any matching of maximum size in $G$. Observe that since $S^*$ is a maximum matching, the set $I_{S^*}$ is independent in $G$, and there are no other star packings in $G$ of greater size. Both of these invariants will be maintained throughout
our construction. We next greedily add edges to $S^*$ by exhaustively applying the following rule as long as possible:

**Rule 1**: If $S_i \cup \{v\}$ is a uniform star for some $v \in I_{S^*}$ and some $S_i \in S$, then add $v$ to $S_i$.

Once Rule 1 cannot be applied, every edge between a vertex in a positive star and $I_{S^*}$ is negative, and vice versa. We then exhaustively apply Rule 2:

**Rule 2**: If there is a center $c_i$ of a star $S_i \in S^*$, $|S_i| \geq 3$, which has more than $|S_i| - 1$ neighbors $N \subseteq I_{S^*}$, then replace $S_i$ with $\{c_i\} \cup N$ in $S^*$.

It is clear that $S^*$ remains a star packing after we finish applying both rules above. We next provide a lower bound on the magnitude of $S^*$. For each star $S \in S^*$, let $N_S \subseteq I_{S^*}$ denote the set of neighbors of $S$ in $I_{S^*}$; that is, $N_S = \{u \in I_{S^*} : \{u, v\} \in E, v \in S\}$. Then we have:

**Lemma 5.** $|N_S| \leq |S| - 1$ for each $S \in S^*$.

**Proof.** Suppose $|S| \geq 3$, and let $c \in S$ be the center of $S$. First observe that any edge between $S \setminus \{c\}$ and $I_{S^*}$ can be used to create a new star $S_{i+1}$, contradicting the fact that $S$ is of maximum size. Assume that all edges in $G[S_i]$ are positive (the negative case is symmetric). Then, since Rule 1 cannot be applied, all edges between $c_i$ and $I_{S^*}$ are negative. Furthermore, since Rule 2 cannot be applied, there are no more than $|S| - 1$ of these edges. Thus, $|N_S| \leq |S| - 1$ in this case.

Suppose then that $S = \{u, v\}$, and that $\{u, v\}$ is positive (again, the case where $\{u, v\}$ is negative is symmetric). Since Rule 1 cannot be applied, all edges between $u$ or $v$ and $I_{S^*}$ are negative. Moreover, since Rule 2 cannot be applied, neither $u$ nor $v$ can be adjacent to more than one vertex in $I_{S^*}$. Finally, if $u$ is adjacent to $u' \in I_{S^*}$ and $v$ is adjacent to $v' \in I_{S^*}$ with $u' \neq v'$, then we can replace $S$ with $\{u, u'\}$ and $\{v, v'\}$, contradicting the fact that $S$ is of maximum size. Thus, $|N_S| \leq |S| - 1$ in this case as well.

**Lemma 6.** $|E| \leq d \cdot n$. Then $m(S^*) \geq m/3d$.

**Proof.** We present a mapping from $V$ to the edges of $S^*$ which maps at most three vertices to a single edge, proving that $m(S^*) \geq n/3$. The lemma will then follow immediately from the fact that $n \geq m/d$.

For a vertex $v$ belonging to some star $S_i$ of $S^*$, we map $v$ to any edge incident to $v$ in $G[S_i]$. Thus, exactly one edge of $G[S_i]$ will have two vertices in its preimage, while the remaining edges have only one. After mapping all vertices in the stars of $S^*$, we proceed to map the remaining vertices in $I_{S^*}$ as follows. We go through each star $S_i$ at a time, and map the vertices in $I_{S^*}$ that are connected to vertices in $S_i$. There are at most $|S_i| - 1$ such vertices according to Lemma 5, so we can map each such vertex to a unique edge in $G[S_i]$. This increases the size of the preimage of each edge in $G[S_i]$ to at most three. After going over all stars $S_i$, we map all vertices of $I_{S^*}$, as $G$ has no isolated vertices, and so we obtain a mapping from $V$ to the edges of $S^*$ with the promised property.

**Proof of Theorem 3.** Due to Lemmata 4 and 6, the star packing $S^*$ yields a solution $x$ with $\text{val}_x(G) \geq m/3d$. Since $\text{opt}(G) \leq ||A|| = m$, this solution is $1/3d$-approximate. The running time for computing $S^*$ is dominated by the computation of the maximum matching for the initial star packing, taking $O(m \sqrt{n}) = O(n^{1.5})$ time [32]; exhaustive application of Rules 1 and 2 and the computation of the solution from the star packing both take $O(m + n) = O(n)$ time.

### 3.2 Triangle-free with general weights

As an intermediate step towards the proof of Theorem 4 we extend the lower bound of the previous subsection to arbitrary weights in case $G$ is triangle-free. For weighted graphs, we let the magnitude of a star packing $S$ be the total absolute value of edges in $S$, i.e. $m(S) = \sum_{S \in S} \sum_{(i,j) \in E(G[S])} |a_{i,j}|$.

Let $a_{\min} = \min_{i,j} |a_{i,j}|$ and $a_{\max} = \max_{i,j} |a_{i,j}|$ denote the minimum and maximum absolute values in $A$ respectively. Let us first consider the case where the ratio between these two values is at most 2, i.e. $a_{\max} \leq 2 \cdot a_{\min}$. Observe that in this case the lower bound given in Lemma 6 can be easily be
extended to \(m/6d\). This is because the star packing \(S^*\) has magnitude at least \(m(S^*) \geq n/3 \cdot a_{\min}\), while \(\text{opt}(G) \leq ||A|| \leq dn \cdot a_{\max} \leq 2dn \cdot a_{\min}\).

**Lemma 7.** \(m(S^*) \geq ||A||/6d\) in case \(a_{\max} \leq 2 \cdot a_{\min}\).

Next, consider the general weight case. We partition the edges of \(G\) into \(\ell = \lfloor \lg |a_{\max}| \rfloor\) subsets \(E_0, \ldots, E_{\ell-1}\), where \(E_i\) contains all edges \((u, v)\) with \(|a_{u,v}| \in [2^i, 2^{i+1} - 1]\). For each \(i \in \{0, \ldots, \ell - 1\}\), let \(A_i\) denote the submatrix of \(A\) corresponding to \(E_i\), and let \(G_i = G[E_i]\). Then, as shown in the previous subsection, we can compute in polynomial time a star packing \(S^*_i\) with \(m(S^*_i) \geq ||A_i||/6c\). By the pigeonhole principle this gives us:

**Lemma 8.** \(m(S^*_i) \geq ||A||/6d \lg a_{\max}\) for some \(i \in \{0, \ldots, \ell - 1\}\).

Now the crucial observation here is that, as \(G\) is triangle-free, each \(S^*_i\) is also a star packing in \(G\). Indeed, if \(S \in S^*_i\) is a star on at least three vertices, then there can be no edges in \(G\) between degree 1 vertices of \(G_i[S]\). Thus, by Lemma 4, each \(S^*_i\) corresponds to a solution \(x_i\) with \(\text{val}_{x_i}(G) \geq m(S^*_i)\). Combining this with Lemma 8 above proves that the solution \(x_i\) of maximum value is an \(\Omega(1/\lg a_{\max})\)-approximate solution.

### 3.3 Triangle deletion

Towards proving Theorem 4 we show how to obtain a triangle-free subgraph of \(G\), the total edge weights of which are a constant fraction of \(||A||\). For this we utilize the local-ratio technique [7] commonly used in approximation algorithms [8].

Recall that if \(G\) is a \(d\)-degenerate graph, then there exists an ordering \(v_1, \ldots, v_n\) of the vertices of \(G\) such \(|\{v_i, v_j\} \in E : i < j| \leq d\) for each \(i \in \{1, \ldots, n\}\) (and this ordering can be computed in linear time). To simplify notation, we assume the natural ordering on the vertices \(\{1, \ldots, n\}\) of \(G\) satisfies this property. We let \(\bar{N}_i = \{j : i, j \in E\}\) and \(\bar{N}_i\). Furthermore, we let \(G_i = G[N_i]\) for each \(i \in \{1, \ldots, n\}\), and use \(E_i\) to denote the edge set of \(G_i\).

We present an algorithm which we call the **triangle deletion algorithm**. The algorithm recursively computes a triangle traversal set \(F \subseteq E\), that is, an edge set \(F\) such \(G - F\) is triangle-free. We use \(w(i,j)\) to denote \(|a_{i,j}|\) for each \(i, j \in \{1, \ldots, n\}\), and we let \(W(E') = \sum_{(i,j) \in E'}w(i,j)\) for any subset of edges \(E' \subseteq E\). The algorithm starts with \(F = \emptyset\).

**TRIANGLEDELETION**\((G, w)\):

1. Let \(F \subseteq E\) be all edges \((i, j)\) with \(w(i, j) = 0\).
2. Find the smallest \(i \in \{1, \ldots, n\}\) such that \(G_i - F\) contains a triangle.
   - if no such \(i\) exists, then return \(F\).
3. Let \(\varepsilon\) be the minimum \(w(i, j)\) of any edge \((i, j)\) \(\in E_i\).
4. Set \(w_i(i, j) = \varepsilon\) if \((i, j)\) is an edge in \(G_i\), and otherwise \(w_i(i, j) = 0\).
5. Let \(F = \text{TRIANGLEDELETION}(G, w - w_i)\).
6. If \(E_i \subseteq F\), then remove some edge \((i, j)\) \(\in E_i\) from \(F\).
7. Return \(F\).

**Lemma 9.** The triangle deletion algorithm returns a set of edges \(F \subseteq E\) such that:

1. \(G - F\) contains no triangle.
2. \(w(E \setminus F) \geq 2/(d^2 + d) \cdot w(E)\).

**Proof.** First observe that the algorithm is guaranteed to terminate, as in each recursive step at least one edge gets its weight decreased to zero. We prove the two properties in the lemma via induction on the recursive steps of the algorithm. So the base case is the case in which no graph \(G_i - F\) contains a triangle (see step 2), so clearly, \(G - F\) is triangle-free. Furthermore, \(w(F) = 0\) and so \(w(E \setminus F) = W(E)\), and the second condition is satisfied as well.

Consider a recursive call in which the algorithm does not terminate at step 2, i.e. in which there exists a smallest \(i \in \{1, \ldots, n\}\) where \(G_i - F\) contains a triangle. Let \(F\) be the set computed in step 5 of the
algorithm. Then by induction we know that $G - F$ contains no triangle. Furthermore, by definition of $i$, any triangle in $G$ containing vertex $i$ must be completely included in $G_i$. Thus, if $E_i \subseteq F$, removing some edge $\{i, j\} \in F \cap E_i$ from $F$ in step 6 does not add a triangle to $G - F$. It follows that the set $F$ returned at step 7 satisfies the first condition of the lemma.

To see that it also satisfies the second condition, let $w_2 = w - w_1$, where $w_1$ is the weight function constructed at step 4 of the algorithm. By induction we have

$$w_2(E \setminus F) \geq 2/(d^2 + d) \cdot w_2(E)$$

after step 5 of the algorithm, and this also holds at step 7 since we do not add edges to $F$ at step 6. Now, observe that by construction of $w_1$, we have $w_1(E) = \varepsilon \cdot |E_i| \leq \varepsilon \cdot (d^2 + d)/2$. Furthermore, at step 7 the set $F$ contains at most one edge of $E_i$, and so $w_1(F) \geq \varepsilon$. Together this implies that

$$w_1(E \setminus F) \geq 2/(d^2 + d) \cdot w_1(E).$$

Thus, from the two inequalities above we get

$$w(E \setminus F) = w_1(E \setminus F) + w_2(E \setminus F) \geq 2/(d^2 + d) \cdot (w_1(E) + w_2(E)) = 2/(d^2 + d) \cdot w(E),$$

and so $F$ satisfies the second condition of the lemma as well.

Proof of Theorem 4. First observe that as $G$ is $d$-degenerate we have $m \leq dn$. Further, we may assume that $G$ has no isolated vertices since deleting them does not affect the degeneracy. Our algorithm obtains a triangle-free subgraph $G'$ of $G$ using the triangle deletion algorithm above. Letting $A'$ denote the matrix corresponding to $G'$, we have $|A'| \geq 2/(d^2 + d)|A|$ by Lemma 9. Next, our algorithm uses Lemma 8 to obtain a star packing $S^*_i$ of magnitude

$$m(S^*_i) \geq |A'|/6d\lg a_{\max} \geq 2|A|/(6d^3 + 6d^2) \lg a_{\max}.$$  

Finally, using Lemma 4, the algorithm computes a solution $x$ with $\val_x(G) \geq m(S^*_i)$. As $\opt(G) \leq |A|$, this solution has an approximation ratio of $2/(6d^3 + 6d^2) \lg a_{\max} = \Omega(1/\lg a_{\max})$.

As for the time complexity of our algorithm, observe that the triangle deletion algorithm runs in $O(n + m) = O(n)$ time. The next step of the algorithm requires computing $O(\lg a_{\max})$ star packings, each taking $O(n^{1.5})$ time to compute. Altogether this gives us a running time of $O(n^{1.5} \lg a_{\max})$.

4 Treewidth Partitions

In this section we present approximation algorithm for sparse MaxQP instances that have some additional structure. Namely, we prove Theorems 5 and 6. Our algorithms all evolve around the Baker technique for planar graphs [5] and its generalizations [16, 20, 24], all using what we refer to here as a treewidth partition — a partition of the vertices of $G$ into $V_0, \ldots, V_{k-1}$ such that $G - V_i$ has bounded treewidth for any subset $V_i$ in the partition. As treewidth plays a central role here, we begin with formally defining this notion.

A tree decomposition is a pair $(T, \mathcal{X})$ where $\mathcal{X}$ is a family of vertex subsets of $G$, called bags, and $T$ is a tree with $\mathcal{X}$ as its node set. The decomposition is required to satisfy (i) $\{X \in \mathcal{X} : v \in X\}$ is connected in $T$ for each $v \in V$, and (ii) for each $\{u, v\} \in E$ there is a bag $X \in \mathcal{X}$ that contains both $u$ and $v$. The width of a tree decomposition $(T, \mathcal{X})$ is $\max_{X \in \mathcal{X}} |X| - 1$, and the treewidth of $G$ is the smallest width amongst all its tree decompositions. The proof of the following lemma is deferred to Appendix A.

Lemma 10. MaxQP restricted to graphs of treewidth at most $k$ can be solved in $O(2^k k \cdot n)$ time.
4.1 Locally bounded treewidth

For a vertex \( v \in V \) and a fixed positive integer \( r \), let \( B_r(v) \) denote the \textit{ball of radius} \( r \) \textit{around} \( v \); that is, \( B_r(v) = \{ u \in V : d(v, u) \leq r \} \) where \( d(v, u) \) is the number of edges in a shortest path from \( u \) to \( v \) in \( G \). Graph \( G \) is said to have \textit{locally bounded treewidth} \cite{20, 24} if there exists some function \( f : \mathbb{N} \rightarrow \mathbb{N} \) such that \( tw(G[B_r(v)]) \leq f(r) \) for any \( v \) and \( r \). As mentioned in Section 1.2, notable examples of graphs of locally bounded treewidth are planar graphs, bounded genus graphs, and bounded degree graphs.

Our starting point is a layer decomposition \( L_0, \ldots, L_\ell \subseteq V \) of \( G \), where \( L_0 = \{ v \} \) for some arbitrary vertex \( v \in V \), and \( L_i = \{ u : d(v, u) = i \} \) are all vertices at distance \( i \) from \( v \), for each \( i \in \{ 1, \ldots, \ell \} \). This is the standard starting point of all Baker type algorithms, and can be computed via breadth-first search from \( v \) in linear time. Note that \( L_0, \ldots, L_\ell \) is a partition of \( V \), and that for each \( i \in \{ 0, \ldots, \ell \} \), each vertex in \( L_i \) has neighbors only in \( L_{i-1} \cup L_i \cup L_{i+1} \) (here and elsewhere in this section we set \( L_{\ell+1} = L_0 = \emptyset \) when necessary).

Given \( 0 < \varepsilon \leq 1 \), we let \( k \) be the smallest integer such that \( 4/k \leq \varepsilon \). For each \( i \in \{ 0, \ldots, k-1 \} \), let \( L_i \) denote the union of all vertices in layers with index equal to \( i \mod k \); that is, \( L_i = \bigcup_{j \equiv i \mod k} L_j \). Define two subgraphs of \( G \): The graph \( G_i \) is the graph induced by \( V - L_i \), and the graph \( H_i \) is the graph induced by \( N[L_i] \). Note that there is some overlap between the vertices of \( G_i \) and \( H_i \), but each edge of \( G \) appears in exactly one of these subgraphs. Also note that since \( G \) has bounded local treewidth, both \( G_i \) and \( H_i \) are bounded treewidth graphs \cite{24}.

Our algorithm computes \( k \) different solutions for \( G \), and selects the best one (i.e. the one which maximizes \((1)\)) as its solution. For \( i \in \{ 0, \ldots, k-1 \} \), we first compute an optimal solution for \( G_i \) in linear time using the algorithm given in Lemma 10. We then extend this solution to a solution \( x_i \) for \( G \) as is done in Lemma 3. In this way we obtain in linear time \( k \) solutions \( x_0, \ldots, x_{k-1} \) with \( \text{val}_{x_i}(G) \geq \text{opt}(G_i) \) for each \( i \in \{ 0, \ldots, k-1 \} \). In Lemma 11 we argue that the solution of maximum objective value is \((1-\varepsilon)\)-approximate to the optimum of \( G \); the proof of Theorem 5 will then follow as a direct corollary.

\textbf{Lemma 11.} There is a solution \( x \in \{ x_0, \ldots, x_{k-1} \} \) with \( \text{val}_{x}(G) \geq (1-\varepsilon) \cdot \text{opt}(G) \).

\textit{Proof.} Let \( x^* \) denote the optimal solution for \( G \). Then, as the edge set of \( G \) is partitioned into the edges of \( G_i \) and \( H_i \), we have

\[
\text{opt}(G_i) + \text{opt}(H_i) \geq \text{val}_{x^*}(G_i) + \text{val}_{x^*}(H_i) = \text{val}_{x^*}(G) = \text{opt}(G)
\]

for each \( i \in \{ 0, \ldots, k-1 \} \). Next observe that any two subgraphs \( H_{i_1} \) and \( H_{i_2} \) with \(|i_1 - i_2| \geq 4\) do not have vertices in common, nor are there any edges between these two subgraphs in \( G \). It follows that for any \( j \in \{0, 1, 2, 3\} \), the graph \( \bigcup_{i \equiv j \mod 4} H_i \) is an induced subgraph in \( G \), and so \( \text{opt}(G) \geq \sum_{i \equiv j \mod 4} \text{opt}(H_i) \) by Lemma 3. Thus, we have

\[
4 \cdot \text{opt}(G) \geq \sum_j \sum_{i \equiv j \mod 4} \text{opt}(H_i) = \sum_i \text{opt}(H_i).
\]

Combining the two inequalities above we get

\[
\sum_{i=0}^{k-1} \text{val}_{x_i}(G) \geq \sum_{i=0}^{k-1} \text{opt}(G_i) \geq k \cdot \text{opt}(G) - \sum_{i=0}^{k-1} \text{opt}(H_i) \\
\geq k \cdot \text{opt}(G) - 4 \cdot \text{opt}(G) = (k - 4) \cdot \text{opt}(G).
\]

It follows that the best solution out of \( x_0, \ldots, x_{k-1} \) has value at least \((1 - 4/k) \cdot \text{val}(G)\), which is at least \((1 - \varepsilon) \cdot \text{val}(G)\), since \( 4/k \leq \varepsilon \).

4.2 \textit{H}-minor-free instances

Let \( G \) be an \textit{H}-minor free graph for some fixed graph \( H \), that is, \( H \) cannot be formed from \( G \) by deleting edges and vertices and by contracting edges. To obtain the \((1-\varepsilon)\)-approximation for \textsc{Unit MaxQP} on \textit{H}-minor free graphs (Theorem 6) we make use of the lower bound obtained in Section 3.1, our algorithm for \textsc{MaxQP} restricted to bounded-treewidth graphs, and the following theorem by Demaine \textit{et al.} \cite{16}:
Theorem 7 ([16]). For a fixed graph $H$, there is a constant $c_H$ such that, for any integer $k \geq 1$ and for every $H$-minor-free graph $G$, the vertices of $G$ can be partitioned into $k$ sets such that the graph obtained by taking the union of any $k - 1$ of these sets has treewidth at most $c_H \cdot k$. Furthermore, such a partition can be found in polynomial time.

Note that this theorem gives a similar partition to the one used in the previous subsection, albeit slightly weaker. In particular, there is no restriction on the edges connecting vertices in different subsets of the partition as was the case in the previous subsection. It is for this reason that arbitrary weights are difficult to handle, and we need to resort to the lower bound of Lemma 6. Fortunately, for the unweighted case, we can use the fact that there exists some constant $h$ depending only on $H$ such that $G$ has at most $hn$ edges (see e.g. [17]). In particular, it can be shown that $h = O(n'\sqrt{\lg n'})$ [31], where $n'$ is the number of vertices of $H$. Combining this fact with Lemma 6 we get:

Lemma 12. $\text{opt}(G) \geq m/3h$.

Our algorithm proceeds as follows. Fix $k \geq 6h/\varepsilon$, and let $V_0, \ldots, V_{k-1}$ denote the partition of $V$ computed by the algorithm from Theorem 7. For each $i \in \{0, \ldots, k - 1\}$, let $E_i$ denote the set of edges $E(V_i \cup \bigcup_{j \neq i} V_j)$, and let $m_i = |E_i|$. Furthermore, let $G_i = G - E_i$. As both $G[V_i]$ and $G[V \setminus V_i]$ have bounded treewidth, we can compute an optimal solution for each of these subgraphs (and therefore also for $G_i$) using the algorithm in Lemma 10. Using Lemma 2, we can extend the optimal solutions for $G[V_i]$ and $G[V \setminus V_i]$ to a solution $x_i$ for $G$ with value

$$\text{val}_{x_i}(G) \geq \text{opt}(G_i).$$

On the other hand, the optimal solution of $G$ cannot do better than

$$\text{opt}(G_i) + m_i \geq \text{opt}(G).$$

Combining the two inequalities above, we can bound the sum of the objective values obtained by all our solutions by

$$\sum_{i=0}^{k-1} \text{val}_{x_i}(G) \geq \sum_{i=0}^{k-1} \text{opt}(G_i) \geq \sum_{i=0}^{k-1} (\text{opt}(G) - m_i)$$

$$= \sum_{i=0}^{k-1} \text{opt}(G) - 2m \geq (k - 6h) \cdot \text{opt}(G),$$

where the last inequality follows from Lemma 12. Thus at least one of these solutions has value at least $(k - 6h)/k \cdot \text{opt}(G)$, which is greater than $(1 - \varepsilon)\text{opt}(G)$ by our selection of parameter $k$.

To analyze the time complexity of our algorithm, observe that computing each solution $x_i$ requires $O(n)$ time according to Lemma 10 and Lemma 2. Thus, the time complexity of the algorithm is dominated by the time required to compute the partition promised by Theorem 7. Demaine et al. [16] showed that this partition can be computed in linear time given the graph decomposition promised by Robertson and Seymour’s graph minor theory [34]. In turn, Grohe et al. [25] presented an $O(n^2)$ time algorithm for this decomposition, improving earlier constructions [16, 28]. Thus, the total running time of our algorithm can also be bounded by $O(n^2)$. This completes the proof of Theorem 6.

5 Conclusion

We presented efficient combinatorial approximation algorithms for sparse instances of MAXQP without resorting to the semidefinite relaxation, as done by Alon and Naor [2] and Charikar and Wirth [14]. From a theoretical perspective, we still leave open whether there is a combinatorial algorithm approximating $d$-degenerate MAXQP instances in polynomial time. Further, is it possible to approximate sparse $\text{UNIT MAXQP}$ instances up to a constant factor in linear time? Finally, the simplicity of our algorithms compels the study of their usability in practice, especially for characterizations of ground states of spin glass models.
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12
A An Exact Algorithm for Bounded Treewidth Graphs

We next prove Lemma 10 by presenting an algorithm for MAXQP restricted to graphs of treewidth at most \( k \) running in \( O(2^k k \cdot n) \) time. For this we require the concept of nice tree decompositions [30].

A tree decomposition \((T, \mathcal{X})\) is rooted if there is a designated bag \( R \in \mathcal{X} \) being the root of \( T \). A rooted tree decomposition is nice if each bag \( X \in \mathcal{X} \) is either (i) a leaf node \( (X \) contains exactly one vertex and has no children in \( T \)), (ii) an introduce node \( (X \) has one child \( Y \in T \) with \( Y \subset X \) and \( |X \setminus Y| = 1 \)), (iii) a forget node \( (X \) has one child in \( T \) with \( Y \subset X \) and \( |Y \setminus X| = 1 \)), or (iv) a join node \( (X \) has two children \( Y, Z \in T \) with \( X = Y = Z \)). Given a tree decomposition, one can compute a corresponding nice tree decomposition with the same width in linear time [30].

Our algorithm employs the standard dynamic programming technique on nice tree decompositions.

Proof of Lemma 10. Let \((T, \mathcal{X})\) be a nice tree decomposition of \( G \) of width \( k \) with root bag \( R \). For a node \( X \in \mathcal{X} \) let \( T_X \) be the subtree of \( T \) rooted at \( X \). Furthermore, let \( G_X \) be the subgraph of \( G \) induced by the vertices in the bags of \( T_X \) (while \( G[X] \) is the subgraph of \( G \) induced only by the vertices in \( X \)). We describe a table in which we have an entry \( D[X, x] \) for each bag \( X \in \mathcal{X} \) and for each solution \( x : X \to \{-1, 1\} \). The entry \( D[X, x] \) contains the value of an optimum solution for \( G_X \), where the values of the vertices in \( X \) are fixed by the solution \( x \).

If \( X \) is a leaf node, then \( G_X \) contains no edges and so \( D[X, x] = 0 \). If \( X \) is an introduce node, then let \( v \in X \setminus Y \) be the introduced vertex, where \( Y \) is the child of \( X \) in \( T \), and let \( x \setminus x_v \) be the solution \( x \) restricted to the vertices of \( Y \). Then \( D[X, x] \) additionally contains the value of all edges incident to \( v \), that is,

\[
D[X, x] = D[Y, x \setminus x_v] + \sum_{u \in N(v)} x_u x_v a_{u,v}.
\]

If \( X \) is a forget node, then let \( v \in Y \setminus X \) be the forgotten vertex, where \( Y \) is the child of \( X \) in \( T \). Then, every value except for \( x_v \) is set in \( x \), so we must choose it so that the value is maximized. Then

\[
D[X, x] = \max_{x_v : x_v \to \{-1, 1\}} D[Y, x \cup x_v].
\]

Finally, if \( X \) is a join node, then let \( Y \) and \( Z \) be the children of \( X \) in \( T \). Note that \( D[Y, x] + D[Z, x] \) contains the value of \( G[X] \) twice, so

\[
D[X, x] = D[Y, x] + D[Z, x] - \text{val}_x(G[X]).
\]

The tree decomposition contains \( O(n) \) nodes, and for each node there are at most \( O(2^k) \) solutions; thus we need to compute \( O(2^k \cdot n) \) entries \( D[\cdot, \cdot] \), each of which can be computed in \( O(k) \) time. The optimum value is the maximum over all \( O(2^k) \) solutions for the root bag \( R \). So we can compute \( \text{opt}(A) \) in \( O(2^k k \cdot n) \) time.
B A Hardness Result

Alon and Naor [2] show that MAXQP restricted to bipartite graphs is not approximable in polynomial time with a ratio of $16/17 + \varepsilon$ unless P=NP. Using the same idea, we show that this approximation lower bound also holds for UNIT MAXQP on 2-degenerate bipartite graphs.

**Theorem 8.** If UNIT MAXQP on 2-degenerate bipartite graphs admits a polynomial-time $(16/17 + \varepsilon)$-approximation, then P=NP.

**Proof.** We reduce from unweighted MAXCUT which does not admit a $(16/17 + \varepsilon)$-approximation unless P=NP [27]. Given an undirected unweighted graph $G = (V, E)$, we compute a graph $G' = (V \cup V', E')$ by subdividing each edge in $E$, that is, for every edge $\{u, w\} \in E$ we add a vertex $v$ to $V'$ and the edges $\{u, v\}, \{v, w\}$ to $E'$. One edge has weight 1 while the other edge has weight $-1$. Clearly, $G'$ is bipartite; $V'$ is one bipartition. As all vertices in $V'$ have degree two, $G'$ is 2-degenerate as well.

Let $x$ be a solution for $G'$. Observe that for every vertex $v \in V'$ we can assume that at least one of its incident edges contributes positively to $\text{val}_x(G')$; otherwise multiply $x_v$ by $-1$. Furthermore, note that the cut in $G$ corresponding to $x$ (restricted to $V$) is of size $\text{val}_x(G')/2$: If both edges incident to $v$ contribute positively to $\text{val}_x(G')$, then the edge in $G$ subdivided by $v$ is cut. Otherwise, the two edges contribute 0 to $\text{val}_x(G')$, and the corresponding edge in $G$ is not cut.

It follows that if there is a $(16/17 + \varepsilon)$-approximation for MAXQP, then there is one for MAXCUT as well, implying P=NP by [27].