An explicit expression of the Lerch zeta function on maximal domains of holomorphy

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Abstract
We give two results on the Lerch zeta function $\Phi(z, s, w)$. The first is to give an explicit expression providing both the analytic continuation of $\Phi$ in $n$-variables ($n \in \{1, 2, 3\}$) to maximal domains of holomorphy in $\mathbb{C}^n$ with computable evaluation and an extended formula for the special values of $\Phi$ at non-positive integers in the variable $s$. The second is to show that Lerch’s functional equation is essentially the same as Apostol’s functional equation using the first result.

Keywords: Lerch zeta function, analytic continuation, functional equation
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1. Introduction

The Lerch zeta function (sometimes called the Lerch transcendent)

$$\Phi(z, s, w) = \sum_{n=0}^{\infty} \frac{z^n}{(n + w)^s},$$

introduced by Lerch [9] and Lipschitz [10] in the case $z \in \mathbb{C}, |z| \leq 1$, is a generalization of the Hurwitz zeta function

$$\zeta(s, w) = \sum_{n=0}^{\infty} \frac{1}{(n + w)^s}$$

and the polylogarithm

$$\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$
One can verify that $\Phi$ converges absolutely in either the case

$$|z| < 1, \ s \in \mathbb{C}, \ w \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, \text{ or } |z| = 1, \ \Re(s) > 1, \ w \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}. \quad (1.2)$$

Many individual works on analytic continuation of $\Phi(z, s, w)$ in some of the variables $z, s, w$ with restricted condition have been done using various integral representation technics. Lerch [9] obtained the analytic continuation to the whole $s$-plane with the condition $|z| \leq 1, \ \Re(w) > 0$ according to the line of Riemann’s classic method. Berndt [3] gave a simple proof for the analytic continuation in $s$ with the condition $z = e^{2\pi ia} (0 < a < 1), \ 0 < w \leq 1$ using an original contour integral representation. Ferreira and López [5, Proposition 1] showed the analytic continuation in $s$ with the condition $z \in \Omega_a, \ w \in \mathbb{C} \setminus \mathbb{R} < 0,$ where $\Omega_a$ is a proper subset of $\mathbb{C}$ depending on the sign of $\Re(w)$. Also, as studied by Erdélyi [4, §1.11], the integral representation given by Lerch can be regarded as an analytic function of $z$ which is holomorphic in a suitable cut plane. On the other hand, Kanemitsu, Katsurada and Yoshimoto [6], and Lagarias and Li [7], [8] considered $\Phi(z, s, w)$ as an analytic function in three complex variables $(z, s, w)$ and gave analytic continuations to various large domains in $\mathbb{C}^3$. Especially, Lagarias and Li obtained an analytic continuation to a multivalued function on a maximal domain of holomorphy in three variables [8, Theorem 3.6].

In this paper, for a set of complex variables $V$ on a domain $D \subset \mathbb{C}^n$, we say holomorphic on $D$ in $V$ or simply holomorphic on $D$ by means of the holomorphy of multivariable(one variable) complex functions. i.e. $f$ is holomorphic with respect to each variable in $V$ on $D$ with other fixed variables. Also, we fix a real $\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ (i.e. $\Re(e^{i\varphi}) < 1$) and choose the principal value of the argument (function) $\arg \lambda$ of a complex $\lambda \in \mathbb{C}^*$ by $\varphi \leq \arg \lambda < \varphi + 2\pi$, which defines the principal branch of the logarithm function $\log \lambda$, forces $(n+w)^s = e^{s \log(n+w)}$ to be single-valued, and $\log r$ is not always real for $r \in \mathbb{R}_{>0}$; namely, there exists a unique integer $\nu$ (depending on the choice $\varphi$) such that

$$\log r = \ln r + 2\pi i \nu \quad \text{for any } r \in \mathbb{R}_{>0},$$

where $\ln r \ (r \in \mathbb{R}_{>0})$ denotes the usual logarithm valued in $\mathbb{R}$. Then the infinite series \[1.1\] defines a single-valued function in complex multivariable(one
holomorphic on six simply-connected domains

\begin{align}
(1) \quad |z| < 1, s \in \mathbb{C}, w \in \mathbb{C} \setminus \ell_{\phi} \text{ in } (z, s, w), \\
(2) \quad |z| < 1, s \in \mathbb{C} \text{ in } (z, s) \text{ with fixed } w \in \ell_{\phi} \setminus \mathbb{Z}_{\leq 0}, \\
(1') \quad \Re(s) > 1, w \in \mathbb{C} \setminus \ell_{\phi} \text{ in } (s, w) \text{ with fixed } z \neq 1, |z| = 1, \\
(2') \quad \Re(s) > 1 \text{ in } s \text{ with fixed } z \neq 1, \Re(s) > 1, w \in \ell_{\phi} \setminus \mathbb{Z}_{\leq 0}, \\
(3) \quad \Re(s) > 1, w \in \ell_{\phi} \setminus \mathbb{Z}_{\leq 0}, \text{ and fixed } w \in \ell_{\phi} \setminus \mathbb{Z}_{\leq 0}, \\
(4) \quad \Re(s) > 1 \text{ in } s \text{ with } z = 1 \text{ and fixed } \ell_{\phi} \setminus \mathbb{Z}_{\leq 0}, \\
\end{align}

where \( \ell_{\phi} \) denotes the set

\[ \{ w \in \mathbb{C} \mid w = -n + re^{i\phi} \text{ for } r \geq 0, n \in \mathbb{Z}_{\geq 0} \} \]

of infinite copies of the complex half line \( \{ re^{i\phi} \in \mathbb{C} \mid r \geq 0 \} \) (together with 0). The domains \( (D_0) \) contain the region (1.2), and the function (1.1) is discontinuous for all \( w \in \ell_{\phi} \) with \( s \in \mathbb{C} \setminus \mathbb{Z} \) and all \( z \) with \( |z| = 1 \). Our first result establishes a brief proof for the single-valued analytic continuation of \( \Phi \) on \( (D_0) \) to maximal domains of holomorphy in \( \mathbb{C}^n \) in any \( n \)-variables in \( \{z, s, w\} \) for each \( n \in \{1, 2, 3\} \), including previous results in \[9\], \[1\], \[3\], \[5\], \[6\], \[8\]. Specifically, we have the following theorem.

**Theorem 1.1.** Let \( \epsilon \) be a positive real satisfying

\[ \Re(e^{i\phi}) < \Re(e^{i\arg(1+i\epsilon)}) = 1/\sqrt{1 + \epsilon^2}, \]

and let \( m, N \in \mathbb{Z}_{\geq 0} \). For each \( j \in \{1, 2, 3, 4\} \) (resp. \( j \in \{1, 2\} \)), the Lerch zeta function (1.1) on the domain \( (D_0)(j) \) (resp. \( (D_0)(j') \)) can be analytically continued to the single-valued function

\begin{align}
\sum_{n=0}^{N-1} \frac{z^n}{(n + w)^s} + \frac{z^N}{e^{4\pi i s} \Gamma(s)} \sum_{r=0}^{m} \sum_{r=0}^{m} \frac{\mathcal{B}_r(z, N + w)(-1)^r e^{s+r-1}}{r!} (s + r - 1) \\
+ \frac{z^N}{e^{4\pi i s} \Gamma(s)} \tilde{I}_N + \frac{z^N}{e^{4\pi i s} \Gamma(s)} J_{N, m} 
\end{align}

(1.3)
holomorphic on the domain \( \overline{D}(j) \) (resp. \( \overline{D}(j) \)), where

1. \( \overline{D}_z \setminus \bullet \varphi' \times \overline{D}_s \times \overline{D}_w \setminus \ell_\varphi \) in \((z, s, w)\),
2. \( \overline{D}_z \setminus \bullet \varphi' \times \overline{D}_s \) in \((z, s)\) with fixed \( w \in (\ell_\varphi \setminus \mathbb{Z}_{\leq 0}) \cap \overline{D}_w \),
3. \( \overline{D}_s \setminus P_z \times \overline{D}_w \setminus \ell_\varphi \) in \((s, w)\) with fixed \( z \in -\bullet \varphi' \setminus (1, e') \),
4. \( \overline{D}_s \setminus P_z \) in \( s \) with fixed \( z \in -\bullet \varphi' \setminus (1, e') \), \( w \in (\ell_\varphi \setminus \mathbb{Z}_{\leq 0}) \cap \overline{D}_w \),

\[
\overline{D}_s := \{ z \in \mathbb{C} \mid z \notin [1, e'] \},
\overline{D}_s := \{ s \in \mathbb{C} \mid \Re(s) > -m \},
\overline{D}_w := \{ w \in \mathbb{C} \mid \Re(w) > -N \},
\]

and \( -\bullet \varphi' := \{ z \in \mathbb{C} \mid z = 1 + re^{i\varphi'} \text{ for } r \geq 0 \} \) with any fixed real \( \varphi' \) satisfying \( \Re(e^{i\varphi'}) \geq 0 \), and \( P_z := \{ 1 \} \) when \( z = 1 \), or \( P_z := \emptyset \) when \( z \neq 1 \).

Also, \( B_r(z, w) \) \((r \in \mathbb{Z}_{\geq 0})\) is Apostol’s rational function defined by (2.1), and \( \tilde{I}_N \), \( J_{N, m} \) are holomorphic functions (see (2.3), (2.4), (2.5)) on the domains \( \overline{D} \). Furthermore, for any \((z, s, w)\) in \( \mathbb{C} \times \overline{D}_s \times \overline{D}_w \), if \( \epsilon \) satisfies the additional condition both \( z \neq e^x \) and

\[
0 < \epsilon \leq \begin{cases} 
\frac{1}{4(|N + w| + 1)} & \text{if } z = 1, \\
\frac{1}{4(|z| + 1)(|N + w| + 1)} & \text{if } z \neq 1,
\end{cases} \tag{\epsilon 2}
\]

then the following evaluation holds.

\[
|\tilde{I}_N(z, s, w)| \leq \begin{cases} 
\frac{1}{\epsilon A_c(z)} E_{\epsilon, N}(s, w) & \text{if } z \in \mathbb{C} \setminus [e^x, \infty), \\
(1 + \frac{1}{\epsilon}) \frac{C_c(s, w)}{B_c(z)} E_{\epsilon, N}(s, w) & \text{if } z \in (e^x, \infty).
\end{cases}
\]

\[
|J_{N, m}(z, s, w)| \leq \begin{cases} 
\frac{1}{2^m \Re(s) + m} & \text{if } z = 1, \\
\frac{1}{|z - 1| \Re(s) + m} & \text{if } z \neq 1.
\end{cases}
\]

Here \( A_c, B_c, C_c, E_{\epsilon, N} \) are defined as (2.7). Especially, \( \tilde{I}_N \) (resp. \( J_{N, m} \)) \( \to 0 \) as \( N \) (resp. \( m \)) \( \to \infty \) on any compact subset of \( \mathbb{C} \times \overline{D}_s \times \overline{D}_w \).
From Theorem 1.1, taking the limit $\epsilon \to 0$ and $m, N \to \infty$ in the expression (1.3), we have the following theorem (The special values of $\Phi$ at $s \in \mathbb{Z}_{\leq 0}$ follows from §2.7).

**Theorem 1.2.** For each $j \in \{1, 2, 3, 4\}$ (resp. $j \in \{1, 2\}$), the Lerch zeta function (1.1) on the domain $(D_0)(j)$ (resp. $(D_0)(j')$) can be analytically continued to a single-valued function holomorphic on the domain $(D_1)(j)$ (resp. $(D_1)(j)$), where

1. $\mathbb{C} \setminus \cdot \cdot \cdot \times \mathbb{C} \times \mathbb{C} \setminus \ell_{\phi}$ in three variables $(z, s, w)$,
2. $\mathbb{C} \setminus \cdot \cdot \cdot \times \mathbb{C}$ in two variables $(z, s)$ with fixed $w \in \ell_{\phi} \setminus \mathbb{Z}_{\leq 0}$, (D.1)
3. $\mathbb{C} \setminus P_z \times \mathbb{C} \setminus \ell_{\phi}$ in two variables $(s, w)$ with fixed $z \in \cdot \cdot \cdot \cdot \cdot$, (D.2)
4. $\mathbb{C} \setminus P_z$ in one variable $s$ with fixed $z \in \cdot \cdot \cdot \cdot \cdot, w \in \ell_{\phi} \setminus \mathbb{Z}_{\leq 0}$.

Especially, for any fixed $(z, w) \in \mathbb{C} \times \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, The function (1.1) has the analytic continuation to $\mathbb{C} \setminus P_z$ in the variable $s$, having a simple pole at $s = 1$ with residue 1 if and only if $z = 1$. Furthermore, for any positive integer $r$, by means of analytic continuation, we have

$$\Phi(z, 1 - r, w) = -\frac{B_r(z, w)}{r}.$$  

Theorem 1.2 would be applicable to explicit evaluation of the Lerch zeta function at any specific point $(z, s, w) \in \mathbb{C} \times \mathbb{C} \setminus P_z \times \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ by taking sufficiently small $\epsilon > 0$ and large $m, N$. It also follows from Theorem 1.2 that by gluing the simply-connected sheets $\mathbb{C} \setminus \cdot \cdot \cdot \cdot \cdot$ and $\mathbb{C} \setminus \ell_{\phi}$ with $\phi \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, $\phi' \in \mathbb{R}$, $\Re(e^{i\phi'}) \geq 0$ together, we obtain the single-valued analytic continuation to the universal covering of the multiply-connected region $\mathbb{C} \setminus \{1\} \times \mathbb{C} \times \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with infinitely many punctures, which corresponds to the result [8, Theorem 3.6]. Note that the expression (1.3) enables us to evaluate the Lerch zeta function for any $z \in \mathbb{R}_{\geq 1}$, which has not been explicitly discussed in the literature. Also, we obtain the formula for the special value $\Phi(e^{2\pi ia}, 1 - r, w)$ with $a \in \mathbb{R}$ for a positive integer $r$, which is originally due to Apostol [1]. See Remark 2.10 for a specific example of Theorem 1.2.

Next we focus on functional equations for $\Phi(z, s, w)$ with the variable change $z = e^{2\pi ia}$. In the paper [9], Lerch proved the functional equation (called Lerch’s transformation formula)

$$\Phi(e^{2\pi ia}, 1 - s, w) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{\pi i(\frac{3}{4} - 2aw)}\Phi(e^{-2\pi iw}, s, a) + e^{\pi i(-\frac{3}{4} + 2(1-a)w)}\Phi(e^{2\pi iw}, s, 1 - a) \right\} \quad (1.4)$$
for $\Im(a) > 0$ or $0 < a < 1$, $0 < w < 1$, and $\Re(s) > 0$ (actually proved for all $s \in \mathbb{C}$ though it was not explicitly mentioned in [9]) using contour integral technic appeared as in Riemann’s first proof of the functional equation for the Riemann zeta function. On the other hand, Apostol [1] proved, according to the line of Riemann’s second proof, there exists another symmetric functional equation

$$\Lambda(a, 1 - s, w) = 2(2\pi)^{-s} \left( \cos \frac{\pi s}{2} \right) \Gamma(s)e^{-2\pi i aw} \Lambda(-w, s, a)$$

(1.5)

for $0 < a < 1$, $0 < w < 1$ (not explicitly written in [1]), and all $s \in \mathbb{C}$, where

$$\Lambda(a, s, w) := \Phi(e^{2\pi ia}, s, w) + e^{-2\pi ia}\Phi(e^{-2\pi ia}, s, 1 - w).$$

This equation was also found by Weil [11, p.57]. As a consequence, Apostol showed that if $0 < a < 1$ and $0 < w < 1$ then Lerch’s equation (1.4) is obtained from the equation (1.5).

In §3, we show the following theorem using Theorem 1.2. Let

$$\ell_{\varphi}^\text{sym} := \{ w \in \mathbb{C} | w = n + 1 + re^{i(\varphi + \pi)} \text{ for } r \geq 0, n \in \mathbb{Z} \geq 0 \} \cup \ell_{\varphi}.$$  

Theorem 1.3. Fix $\varphi' \in 2\pi\mathbb{Z}$. Let $U$ be the maximal domain in $\mathbb{C} \setminus (\ell_{\varphi}^\text{sym} \cup (\mathbb{Z} + i\mathbb{R}))$ containing the interval $(0, 1) (\subset \mathbb{R})$, and let

$$U \times \mathbb{C} \times U,$$  

(Deq)

be a domain of $(a, s, w)$. Then, Lerch’s equation (1.4) holds on (Deq) if and only if Apostol’s equation (1.5) holds on (Deq).

In fact, Lerch’s and Apostol’s equations hold on (Deq), respectively (Proposition 3.1), and hence these are essentially the same equation on (Deq).

2. Analytic continuation of $\Phi$ and its evaluation

In this section, we prove Theorem 1.1.

2.1. Apostol’s generating function

Apostol [1] introduced the sequence of rational functions $B_r(z, w) \ (r \in \mathbb{Z}_{\geq 0})$ in $z, w$ by means of the generating function

$$\sum_{r=0}^{\infty} \frac{B_r(z, w)}{r!} t^r = \frac{te^{tw}}{e^t z - 1}.$$  

(2.1)
The specialization $z = 1$ together with the variable $w$ in the definition yields the definition of the Bernoulli polynomial in $w$, and for $r \geq 1$, there is the recurrence relation

$$B_r(1, w) = \frac{1}{r + 1} \sum_{k=0}^{r-1} (-1)^{r-k} \binom{r + 1}{k} B_k(1, w) \left( (w - 1)^{r-k+1} - w^{r-k+1} \right).$$

(2.2)

If we regard both $z$ and $w$ as variables, then $B_0(z, w) = 0$, $B_1(z, w) = 1/(z - 1)$, and for $r \geq 2$, there is the recurrence relation

$$B_r(z, w) = \frac{1}{z-1} \sum_{k=0}^{r-1} (-1)^{r-k-1} \binom{r}{k} B_k(z, w) \left( z(w - 1)^{r-k} - w^{r-k} \right).$$

(2.3)

Thus $B_r(z, w) \in \mathbb{Q}[1/(z - 1), w]$, having pole only at $z = 1$ for $r \geq 1$, and holomorphic for $(z, w) \in \mathbb{C} \setminus \{1\} \times \mathbb{C}$ as a function in two complex variables $z, w$. In the rest of the paper, if the value of $z$ has been specified, then we regard the definition (2.1) as that under the specialization of $z$, in order to simplify notation in the case $z = 1$. In this sense, $B_r(1, w)$ is the Bernoulli polynomial.

We will apply the following lemma to the proof of Theorem 1.1 (Proposition 2.5).

**Lemma 2.1.** For any $r \in \mathbb{Z}_{\geq 0}$, we have

$$|B_r(z, w)| \leq \begin{cases} 2^r \cdot r! (|w| + 1)^r & \text{if } z = 1, \\ 2^{r-1} \cdot r! \left( |z| - 1 \right)^r (|w| + 1)^{r-1} & \text{if } z \neq 1. \end{cases}$$

**Proof.** By induction. The case $r = 0$ is clear from

$$B_0(1, w) = 1, \quad B_0(z, w) = 0 \quad (z \neq 1).$$

Consider the case $r \geq 1$. Assume that the statement of the lemma holds for
any $k \in \mathbb{Z}_{\geq 0}$ with $k < r$. Then, by the recurrence relation (2.2) we have
\[
|B_r(1, w)| \leq \frac{1}{r + 1} \sum_{k=0}^{r-1} \binom{r}{k} |B_k(1, w)| \sum_{l=0}^{r-k} \binom{r - k + 1}{l} |w|^l
\]
\[
\leq \frac{1}{r + 1} \sum_{k=0}^{r-1} \binom{r}{k} |B_k(1, w)| (r - k + 1)(|w| + 1)^{r-k}
\]
\[
\leq r! (|w| + 1)^r \sum_{k=0}^{r-1} \frac{2^k}{(r - k)!}
\]
\[
\leq r! (|w| + 1)^r \sum_{k=0}^{r-1} 2^k
\]
\[
\leq 2^r \cdot r! (|w| + 1)^r.
\]

Also, using the recurrence relation (2.3) yields
\[
|B_r(z, w)| \leq \frac{1}{|z - 1|} \sum_{k=0}^{r-1} \binom{r}{k} |B_k(z, w)| (|z| + 1)(|w| + 1)^{r-k}
\]
\[
\leq \frac{1}{|z - 1|} \sum_{k=0}^{r-1} \binom{r}{k} 2^{k-1} \cdot k! (|z| + 1)^k (|w| + 1)^{r-1}
\]
\[
\leq \frac{1}{|z - 1|^r} \sum_{k=0}^{r-1} \binom{r}{k} 2^{k-1} \cdot k! (|z| + 1)^{r-1} (|w| + 1)^{r-1}
\]
\[
\leq \frac{r!}{|z - 1|^r} (|z| + 1)^{r-1} (|w| + 1)^{r-1} \sum_{k=0}^{r-1} \frac{2^{k-1}}{(r - k)!}
\]
\[
\leq \frac{2^{r-1} \cdot r!}{|z - 1|^r} (|z| + 1)^{r-1} (|w| + 1)^{r-1}.
\]
We start with six simply-connected domains

(1) \( D_z \times D_s \times \tilde{D}_w \setminus \ell_\varphi \),
(2) \( D_z \times D_s \) with fixed \( w \in (\ell_\varphi \setminus \mathbb{Z}_{\leq 0}) \cap \tilde{D}_w \),
(1') \( D_s \times \tilde{D}_w \setminus \ell_\varphi \) with fixed \( z \neq 1, |z| = 1 \),
(3) \( D_s \times \tilde{D}_w \setminus \ell_\varphi \) with \( z = 1 \),
(2') \( D_s \) with fixed \( z \neq 1, |z| = 1, w \in (\ell_\varphi \setminus \mathbb{Z}_{\leq 0}) \cap \tilde{D}_w \),
(4) \( D_s \) with \( z = 1 \) and fixed \( w \in (\ell_\varphi \setminus \mathbb{Z}_{\leq 0}) \cap \tilde{D}_w \).

For each \( j \in \{1, 2, 1', 3, 2', 4\} \), since \( \mathcal{D}(j) \subset \mathcal{D}_0(j) \), the infinite series \( \sum_{n=0}^{\infty} \) defines a single-valued function holomorphic on \( \mathcal{D}(j) \) in three variables for \( j = 1 \), two variables for \( j \in \{2, 1', 3\} \), and one variable for \( j \in \{2', 4\} \). We regard the real interval \([a, b] (\subset \mathbb{R})\) as the oriented line segment on \( \mathbb{R} \) from \( a \) to \( b \), also \([a, \infty)\) as the oriented half line on \( \mathbb{R} \) from \( a \) to \( \infty \), and so on. On each domain of \( \mathcal{D} \) with restricted \( w \in \mathbb{R}_{>0} \), consider the typical integral representation

\[
\frac{z^n}{(n+w)^s} \Gamma(s) = e^{-4\pi iws} \int_{[0, \infty)} e^{-t} t^n \left( \frac{t}{n+w} \right)^s \frac{dt}{t}, \quad n \geq N,
\]

where \( \Gamma(s) \) denotes the gamma function and \( \left( \frac{t}{n+w} \right)^s = e^{s \log \frac{t}{n+w}} \). Since \( n+w \in \mathbb{R}_{>0} \), the substitution \( t/(n+w) \to t \) yields the analytic continuation

\[
\frac{z^n}{(n+w)^s} \Gamma(s) = e^{-4\pi iws} \int_{[0, \infty)} e^{-t(n+w)} t^n t^{s-1} dt, \quad n \geq N,
\]

where the right hand side of this equality is holomorphic for all \( (z, s, w) \in \mathbb{C} \times D_s \times \tilde{D}_w \). Multiplying \( \Gamma(s)^{-1} \), taking the sum with respect to \( n \), and using the condition of \( D_z, D_s, \tilde{D}_w \), we have

\[
\Phi(z, s, w) = \sum_{n=0}^{N-1} \frac{z^n}{(n+w)^s} + \frac{1}{e^{4\pi iws} \Gamma(s)} \int_{[0, \infty)} \left( \sum_{n=N}^{\infty} e^{-t(n+w)} z^n \right) t^{s-1} dt
\]

\[
= \sum_{n=0}^{N-1} \frac{z^n}{(n+w)^s} + \frac{z^N}{e^{4\pi iws} \Gamma(s)} \int_{[0, \infty)} \frac{e^{-t(N+w)}}{1 - e^{-t} z} t^{s-1} dt
\]

\[
= \sum_{n=0}^{N-1} \frac{z^n}{(n+w)^s} + \frac{z^N}{e^{4\pi iws} \Gamma(s)} (I_N + J_{N, m} + K_{N, m}) \quad (2.4)
\]

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on each domain of \([D]\), where \(m\) denotes any non-negative integer, and

\[
I_N(z, s, w) := \int_{[\epsilon, \infty)} \phi_N(t)t^{s-2}dt,
\]

\[
J_{N, m}(z, s, w) := \int_{[0, \epsilon]} (\phi_N(t) - \phi_{N, m}(t))t^{s-2}dt,
\]

\[
K_{N, m}(z, s, w) := \int_{[0, \epsilon]} \phi_{N, m}(t)t^{s-2}dt,
\]

\[
\phi_N(z, w, t) := te^{-t(N+w)}(1 - e^{-t})z\quad (=: \phi_N(t)),
\]

\[
\phi_{N, m}(z, w, t) := \sum_{r=0}^{m} \frac{B_r(z + N + w)}{r!}(-1)^r t^r\quad (=: \phi_{N, m}(t)).
\]

Here \(\epsilon\) denotes any positive real, and \(\phi_{N, m}(z, w, t)\) is a rational function in either \(\mathbb{Q}[w, t]\) or \(\mathbb{Q}[1/(z-1), w, t]\) according to \(z = 1\) or not (Recall that \(B_r(1, N + w)\) stands for the Bernoulli polynomial, to ease the notation).

On the domains \([D]\), we have

\[
K_{N, m}(z, s, w) = \int_{[0, \epsilon]} \sum_{r=0}^{m} \frac{B_r(z + N + w)}{r!}(-1)^r t^{r+s-2}dt
\]

\[= \sum_{r=0}^{m} \frac{B_r(z + N + w)}{r!}(-1)^r \epsilon^{s+r-1}\frac{1}{s + r - 1},\]

which is a meromorphic function in \(\mathbb{Q}[\log \epsilon, 1/(z-1), w, 1/((s-1)s(s+1) \cdots (s+m-1))]\) when \(z \neq 1\), or \(\mathbb{Q}[\log \epsilon, w, 1/((s-1)s(s+1) \cdots (s+m-1))]\) when \(z = 1\), having only simple poles, and hence the function \(K_{N, m}/\Gamma(s)\) appearing in \(\mathbb{D}\) is holomorphic for any \((z, s, w) \in \mathbb{C} \setminus \{1\} \times \mathbb{C} \times \mathbb{C}\) except for \(s = 1\) by the fact that the entire function \(1/\Gamma(s)\) has simple zeros only at \(s \in \mathbb{Z}_{\leq 0}\). The (non-)holomorphy at \(s = 1\) follows from the fact that \(B_0(z, N + w)\) is equal to 0 when \(z \neq 1\), or 1 when \(z = 1\), which leads us to the definition of \(P_z\).

Let

\[
\tilde{D}_z := \{z \in \mathbb{C} \mid z \notin [1, \epsilon]\},
\]

\[
\tilde{D}_s := \{s \in \mathbb{C} \mid \Re(s) > -m\}.
\]

Now we take a real \(\epsilon > 0\) satifying the condition \((\epsilon 1)\). Then, by the results in \(\S\,2.3\,\S\,2.4\,\S\,2.5\) there exists a single-valued holomorphic function \(\tilde{I}_N\) on
the domains (2.6) giving analytic continuations of $I_N$ on each domain of $(D)$. Also, the function $J_{N,m}$ is single-valued and holomorphic on (2.6).

Summarize the discussion above. For any $m, N \in \mathbb{Z}_{\geq 0}$, $\epsilon \in \mathbb{R}_{>0}$ under the condition (1), we have the expression

$$
\Phi(z, s, w) = \sum_{n=0}^{N-1} \frac{z^n}{(n+w)^s} + \frac{z^N}{e^{4\pi i s} \Gamma(s)} \sum_{r=0}^{m} \frac{B_r(z, N+w) (-1)^r \epsilon^{s+r-1}}{r! \ s+r-1} + \frac{z^N}{e^{4\pi i s} \Gamma(s)} \bar{I}_N(z, s, w) + \frac{z^N}{e^{4\pi i s} \Gamma(s)} J_{N,m}(z, s, w)
$$

on the domains $(\bar{D})$, whose right hand side is both single-valued and holomorphic on four domains

1. $\bar{D} \setminus \cdot \varphi \times \bar{D}_s \times \bar{D}_w \setminus \ell \varphi,$
2. $\bar{D} \setminus \cdot \varphi \times \bar{D}_s$ with fixed $w \in (\ell \varphi \setminus \mathbb{Z}_{\leq 0}) \cap \bar{D}_w,$
3. $\bar{D}_s \setminus P_z \times \bar{D}_w \setminus \ell \varphi$ with fixed $z \in \cdot \varphi \setminus (1, 1'),$
4. $\bar{D}_s \setminus P_z$ with fixed $z \in \cdot \varphi \setminus (1, 1'),$ $w \in (\ell \varphi \setminus \mathbb{Z}_{\leq 0}) \cap \bar{D}_w.$

Then

i) $(D)(j) \subset (\bar{D})(j)$ for $j \in \{1, 2, 3, 4\},$

ii) $(D)(j') \subset (\bar{D})(j)$ for $j \in \{1, 2\}.$

Since $(D)(j) \subset (D_{0a})(j)$ for each $j \in \{1, 2, 1', 3, 2', 4\},$ the right hand side of (2.5) gives analytic continuations of $\Phi$ (i.e. (1.1)) on $(D_{0a})$ to $(D)$ in accordance with i), ii). Here we need to separate the domains (2.6) (with $P_z$) into four domains $(\bar{D})$ because the function $(n+w)^s = e^{s \log(n+w)}$ in $w$ is discontinuous on the half line $-n + e^{i\varphi} \mathbb{R}_{\geq 0} (\subset \ell \varphi)$ when $s \in \mathbb{C} \setminus \mathbb{Z}$. The evaluation of $\bar{I}_N, J_{N,m}$ follows from Proposition 2.4, 2.5 in §2.6. This proves Theorem 1.1.

**Remark 2.2.** It turns out that the multivaluedness of the Lerch zeta function $\Phi$ on the domain $\bar{D}_z \times \bar{D}_s \times \bar{D}_w$ arises from $(n+w)^s = e^{s \log(n+w)}$ in the terms

$$
\sum_{n=0}^{N-1} \frac{z^n}{(n+w)^s}
$$

of (2.4) and the (non-trivial multivalued) functions $\bar{I}_N, J_{N,m}$ (§2.6).
2.3. Holomorphy of $J_{N, m}(z, s, w)$

Consider the function

$$J_{N, m}(z, s, w) = \int_{[0, \epsilon]} \left( \phi_N(t) - \phi_{N, m}(t) \right) t^{s-2} dt,$$

where $t^{s-2} = e^{(s-2)\log t}$ with $\varphi \leq \arg t < \varphi + 2\pi$ and $\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Especially $\log t$ is holomorphic on the path $[0, \epsilon]$.

Since the function $\phi_N(t)$ is holomorphic for $(z, w, t)$ in any compact subset of $\tilde{D}_z \times \tilde{D}_s \times \tilde{D}_w$ (resp. for all $(w, t) \in \tilde{D}_w \times [0, \epsilon]$ with $z = 1$), (Note that $\phi_N$ in two variables $z, t$ is discontinuous on those $(z, t)$ with $z = e^t$, in particular at $(z, t) = (e^t, t)$ for any $t \in [0, \epsilon]$), by the definition of $B_r(z, N + w)$ we can expand $\phi_N$ to the Taylor series

$$\phi_N(t) = \sum_{r=0}^\infty \frac{B_r(z, N + w)}{r!} (-1)^r t^r$$

around $t = 0$, and hence $|\phi_N(t) - \phi_{N, m}(t)| = O(t^{m+1})$ as $t \to 0$ on any compact subset $T \subset \tilde{D}_z \times \tilde{D}_s \times \tilde{D}_w$ (resp. $T \subset \tilde{D}_s \times \tilde{D}_w$ with $z = 1$).

This implies that the absolute value of the integrand of $J_{N, m}$ is equal to $O(t^{m+\Re(s)-1})$ as $t \to 0$ on the set $T$. Thus the integral $J_{N, m}$ converges and is holomorphic for all $(z, s, w) \in \tilde{D}_z \times \tilde{D}_s \times \tilde{D}_w$ (resp. for all $(s, w) \in \tilde{D}_s \times \tilde{D}_w$ with $z = 1$).

2.4. Analytic continuation of $I_N(z, s, w)$ to $\mathbb{C} \setminus \{e^t\} \times \mathbb{C} \times \mathbb{C}$

Let $0 < \epsilon < \alpha$. For any oriented path $L \subset \mathbb{C}$ from $\epsilon$ to $\alpha$, let

$$I_N^L(z, s, w) := \int_L \phi_N(t) t^{s-2} dt,$$

where $t^{s-2} = e^{(s-2)\log t}$ with $\varphi \leq \arg t < \varphi + 2\pi$ and $\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. One has $I_N^{[\epsilon, \alpha]} \to I_N$ as $\alpha \to \infty$. The function $\log t$ is holomorphic for any $t \in \mathbb{C} \setminus e^{i\varphi}\mathbb{R}_{\geq 0}$. Assume

$$L \subset H_t := \mathbb{C} \setminus e^{i\varphi}\mathbb{R}_{\geq 0}.$$

Since the function $\phi_N(t)$ is holomorphic for $(z, w, t)$ in any compact subset of $\mathbb{C} \setminus e^L \times \mathbb{C} \times L$ (resp. for all $(w, t) \in \mathbb{C} \times L$ with $z = 1$), where $e^L = \{e^t \mid t \in L\}$ (Note that $\phi_N$ in two variables $z, t$ is discontinuous on those $(z, t)$ with $z = e^t$,
in particular at \((z, t) = (e^t, t)\) for any \(t \in L\), the integral \(I_N^L\) converges and is holomorphic for all \((z, s, w) \in \mathbb{C} \setminus e^L \times \mathbb{C} \times \mathbb{C}\) (resp. for all \((s, w) \in \mathbb{C} \times \mathbb{C}\) with \(z = 1\)).

In the case \(\alpha = \infty\), if \(L\) satisfies the condition
\[
\inf \{\Re(t) \in \mathbb{R} \mid t \in L\} > -\infty
\]
\[
\exists \gamma_0 \in \mathbb{R} \text{ such that } \Im(t) \text{ is constant for any } t \in L \text{ with } \Re(t) > \gamma_0
\]
then the integral \(I_N^L\) still converges because for any compact subset \(T\) of \(\{(z, s, w) \in \mathbb{C} \setminus e^L \times \mathbb{C} \times \mathbb{C}\}\) there exists a positive constant \(C \in \mathbb{R}\) depending on \(N, L, T\) such that
\[
|\phi_N(t)|^{t_0 - 2} = \frac{e^{-\Re(t)(N + \Re(w)) + \Im(t)\Im(w)}|t|^2 \Re(s) + 1 e^{-\Im(s) \arg t}}{|1 - e^{-\Re(t) - i\Im(t)z}|} |t|^{-2} \leq \frac{C}{|t|^2} \text{ for } t \in L.
\]
Especially, \(I_{N}^{[e^\epsilon, \infty)}\) converges on the domain \(\mathbb{C} \setminus \{e^{\epsilon}, \infty\} \times \mathbb{C} \times \mathbb{C}\) containing any domain of \(\square\).

Now we take a real \(\epsilon > 0\) satisfying
\[
\Re(e^{i\phi}) < \Re(e^{i\arg(1 + i\epsilon)}) (= 1/\sqrt{1 + \epsilon^2}),
\]
which is possible by the assumption \(\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Z}\) implying \(\Re(e^{i\varphi}) < 1\). Then we show that the function
\[
\tilde{I}_N(z, s, w) := \begin{cases} I_N(z, s, w) & \text{on } \mathbb{C} \setminus \{e^{\epsilon}, \infty\} \times \mathbb{C} \times \mathbb{C} \\ I_N^{L_{\infty}}(z, s, w) & \text{on } (e^{\epsilon}, \infty) \times \mathbb{C} \times \mathbb{C} \end{cases}
\]
is holomorphic on the domain \(\mathbb{C} \setminus \{e^{\epsilon}\} \times \mathbb{C} \times \mathbb{C}\), where \(L_{\infty}^\epsilon(\subset H_t)\) denotes the oriented line segment in the complex \(t\)-plane connecting the four complex points \(e^\epsilon, \epsilon(1 + i\epsilon), \alpha + i\epsilon^2, \alpha\), in this order (See Figure 2). By the discussion above, \(I_N(= I_N^{[e^\epsilon, \infty)}\) (resp. \(I_N^{L_{\infty}}\)) converges and is holomorphic on \(\mathbb{C} \setminus \{e^\epsilon, \infty\} \times \mathbb{C} \times \mathbb{C}\) (resp. \(\mathbb{C} \setminus e^{L_{\infty}} \times \mathbb{C} \times \mathbb{C}\), where \((e^\epsilon, \infty) \subset \mathbb{C} \setminus e^{L_{\infty}}\). Let
\[
\square_{t, \alpha} := \{t \in \mathbb{C} \mid \epsilon \leq \Re(t) \leq \alpha, 0 \leq \Im(t) \leq \epsilon^2\},
\]
\[
R_z := \left\{ re^{i\theta} \in \mathbb{C} \mid r > 0, \frac{\pi}{2} < \theta < \frac{3\pi}{2} \right\}.
\]
Then \(\square_{t, \alpha} \subset H_t\) (by the condition \((\square)\)), \(e^{[\epsilon, \epsilon]} \cap \overline{R_z} = \emptyset\), where \(\overline{R_z}\) denotes the closure of \(R_z\), and hence the integrand of \(I_N^{L_{\infty}}\) is holomorphic for any \((z, s, w, t) \in R_z \times \mathbb{C} \times \mathbb{C} \times \square_{t, \alpha}\). Thus, by Cauchy-Goursat theorem, we have \(I_N^{L_{\infty}} = I_N^{[e^\epsilon, \alpha]}\) (and hence \(I_N^{L_{\infty}} = I_N\)) on the domain \(R_z \times \mathbb{C} \times \mathbb{C}\). Therefore, since \(R_z \subset (\mathbb{C} \setminus \{e^\epsilon, \infty\}) \cap (\mathbb{C} \setminus e^{L_{\infty}})\), \(\tilde{I}_N\) is holomorphic on \(\mathbb{C} \setminus \{e^\epsilon\} \times \mathbb{C} \times \mathbb{C}\) by analytic continuation.
2.5. Multivaluedness of $\tilde{I}_N$, $J_{N,m}$ on $\tilde{D}_z \times \tilde{D}_s \times \tilde{D}_w$

When a complex $z \in \mathbb{C}$ moves continuously on the multiply-connected domain $\mathbb{C} \setminus \{1\}$ and returns to original position, the oriented line segment $e^{-t}z$ ($t \in [0, \infty)$) on $\mathbb{C}$ could pass through the point $1 \in \mathbb{R}$ and returns to the original position. This implies that a non-trivial multivaluedness of the functions $\tilde{I}_N$, $J_{N,m}$ on the domain $\tilde{D}_z \times \tilde{D}_s \times \tilde{D}_w$ in the variable $z$ occurs in a loop around the point 1, because the function $\phi_N$ has simple poles at $t \in \mathbb{C}$ with $e^t = z$ for any fixed $z$ in $\mathbb{C} \setminus \{1\}$. Therefore, in order to make both $\tilde{I}_N$ and $J_{N,m}$ single-valued for $z \in \tilde{D}_z$, we need a cut

$$\rightarrow \varphi = \{ z \in \mathbb{C} \mid z - 1 = re^{i\varphi'} \text{ for } r \geq 0 \} \quad \text{for some } \varphi' \in \mathbb{R}$$

in the $z$-plane (See Figure 3). Since the infinite series (1.1) is no longer holo-

![Figure 2: the paths $L_\alpha^\epsilon$, $e^{L_\alpha^\epsilon}$ and the regions $\square_{t,\alpha}$, $e^{\square_{t,\alpha}}$](image2)

![Figure 3: A movement of the line segment $e^{-t}z$ ($t \in [0, \epsilon] \cup [\epsilon, \infty)$)](image3)
morphic on the initial domain \( D_z \) when \( \Re(e^{i\varphi'}) < 0 \), the condition \( \Re(e^{i\varphi'}) \geq 0 \) is necessary. Thus, we obtain single-valued holomorphic functions \( \tilde{I}_N, J_{N,m} \) on two domains

\[
\begin{align*}
(1) \quad & \tilde{D}_z \setminus \bullet \varphi' \times \tilde{D}_s \times \tilde{D}_w, \\
(3) \quad & \tilde{D}_s \times \tilde{D}_w \text{ with fixed } z \in \varphi' \setminus (1, e^\varepsilon] (= (-\bullet \varphi' \cap \tilde{D}_z) \cup \{1\}),
\end{align*}
\]

giving single-valued analytic continuation of \( I_N, J_{N,m} \) on the domains \((\mathcal{D})\) in accordance with

i) \((\mathcal{D})(j) \subset (2.6)(1)\) for \( j \in \{1, 2, 1', 2'\}\),
ii) \((\mathcal{D})(j) \subset (2.6)(3)\) for \( j \in \{3, 4\}\),
iii) \( \tilde{I}_N = I_N \) on each domain of \((\mathcal{D})\).

2.6. Evaluation of \( \tilde{I}_N, J_{N,m} \)

Let

\[
A_\varepsilon(z) := \begin{cases} 
1 & \text{if } \Re(z) \leq 0, \\
1 - \Re(z) \frac{z}{|z|^2} & \text{if } 0 < \frac{\Re(z)}{|z|^2} \leq e^{-\varepsilon}, \\
1 - e^{-\varepsilon}z & \text{if } e^{-\varepsilon} < \frac{\Re(z)}{|z|^2},
\end{cases}
\]

\[
B_\varepsilon(z) := \min \left\{ e^{-\varepsilon}z - 1, |\sin \varepsilon^2| \right\},
\]

\[
C_\varepsilon(s, w) := \sqrt{2} \frac{N(s)+1}{|s|} \exp(e^2|\Im(w)| + |\Im(s)|((\tan^{-1}e + 2\pi|\nu|)),
\]

\[
E_{\varepsilon, N}(s, w) := \begin{cases} 
\frac{\Re(s)+1}{e(\Re(w)+N)} & \text{if } \frac{\Re(s)+1}{\Re(w)+N} > \varepsilon, \\
\frac{e^{\Re(s)+1}}{\Re(s)+1} & \text{if } \frac{\Re(s)+1}{\Re(w)+N} \leq \varepsilon.
\end{cases}
\]

Lemma 2.3. For any \( z \in \mathbb{C} \setminus \{e^\varepsilon\} \),

\[
|1 - e^{-t}z| \geq \begin{cases} 
A_\varepsilon(z) > 0 & \text{if } t \in [\varepsilon, \infty), \ z \in \mathbb{C} \setminus [e^\varepsilon, \infty), \\
B_\varepsilon(z) & \text{if } t \in L_\varepsilon^\infty, \ z \in (e^\varepsilon, \infty).
\end{cases}
\]

Under the condition \((0 < \varepsilon < \sqrt{\pi})\), we have \( B_\varepsilon(z) > 0 \).

Proof. In the case \( t \in [\varepsilon, \infty), \ z \in \mathbb{C} \setminus [e^\varepsilon, \infty) \), differentiating the real function

\[
|1 - e^{-t}z|^2 = (e^{-t}\Re(z) - 1)^2 + (e^{-t}\Im(z))^2
\]
in the variable $t$ yields $|1 - e^{-t}z| \geq A_{\epsilon}(z) > 0$.

Next assume that $t \in L_{\infty}([\epsilon, \infty))$. If $t = \epsilon + i\delta$ ($0 \leq \delta \leq \epsilon^2$) then
\[
|1 - e^{-t}z| \geq |e^{-t}z| - 1 = |e^{-\Re(t)}z| - 1 = e^{-\epsilon z} - 1 > 0.
\]

Otherwise, $t = \gamma + i\epsilon^2$ ($\gamma \geq \epsilon$), then differentiating the real function
\[
|1 - e^{-t}z|^2 = (e^{-\gamma}z - 1)^2 + 2e^{-\gamma}z(1 - \cos\epsilon^2).
\]

in the variable $\gamma$ yields
\[
(e^{-\gamma}z - 1)^2 + 2e^{-\gamma}z(1 - \cos\epsilon^2) \geq \sin^2\epsilon^2 \quad \text{for } \gamma \geq \epsilon,
\]

and hence $|1 - e^{-t}z| \geq \sin\epsilon^2$ ($> 0$ if $0 < \epsilon < \sqrt{\pi}$). This proves $|1 - e^{-t}z| \geq B_{\epsilon}(z)$.

\textbf{Proposition 2.4.} Let $\epsilon$ be a positive real satisfying
\[
\Re(e^{i\varphi}) < \Re(e^{i\arg(1+i\epsilon)}) \ (= 1/\sqrt{1+\epsilon^2}). \quad (\epsilon 1)
\]

For any $(z, s, w) \in \mathbb{C} \setminus \{e^{\epsilon}\} \times \mathbb{C} \times \widetilde{D}_w$, we have
\[
|\widetilde{I}_N(z, s, w)| \leq \begin{cases} 
\frac{1}{\epsilon A_{\epsilon}(z)} E_{\epsilon, N}(s, w) & \text{if } z \in \mathbb{C} \setminus [e^{\epsilon}, \infty), \\
(1 + \frac{1}{\epsilon}) \frac{C_{\epsilon}(s, w)}{B_{\epsilon}(z)} E_{\epsilon, N}(s, w) & \text{if } z \in (e^{\epsilon}, \infty).
\end{cases}
\]

Especially, $\widetilde{I}_N \to 0$ as $N \to \infty$ on any compact subset of $\mathbb{C} \setminus \{e^{\epsilon}\} \times \mathbb{C} \times \widetilde{D}_w$.

\textit{Proof.} In the case $z \in \mathbb{C} \setminus [e^{\epsilon}, \infty)$, $\widetilde{I}_N = I_N$, and differentiating the real function $e^{-t(N+\Re(w))|t^{s+1}|}$ in the variable $t \geq \epsilon$ yields
\[
e^{-t(N+\Re(w))|t^{s+1}|} \leq E_{\epsilon, N}(s, w).
\]

By Lemma \textbf{2.3} we have the evaluation
\[
|I_N| \leq \int_{\epsilon}^{\infty} \frac{e^{-t(N+\Re(w))|t^{s+1}|}}{|1 - e^{-t}z|} \frac{dt}{t^2} \leq \frac{1}{\epsilon A_{\epsilon}(z)} E_{\epsilon, N}(s, w).
\]
Consider the case \( z \in (e^\epsilon, \infty) \). For any \( t \in L^\infty \), it follows from the assumption \( (\epsilon \cdot 1) \) that \( 0 \leq \arg t - 2\pi \nu \leq \tan^{-1} \epsilon \), and one can verify
\[
|t| < \sqrt{2} \Re(t), \quad e^{\Im(t) \Im(w)} \leq e^{2\Im(w)}, \quad |t^{s+1}| \leq |t|^{\Re(s)+1} e^{\Im(s) (\tan^{-1} \epsilon + 2\pi |\nu|)},
\]
and hence
\[
|t^{s+1} e^{-t(N+w)}| \leq |t^{s+1} e^{-\Re(t)(\Re(w)+N)} e^{\Im(t) \Im(w)}|
\leq \sqrt{2}^{\Re(s)+1} e^{2\Im(w)} |e^{\Im(s)} (\tan^{-1} \epsilon + 2\pi |\nu|) \Re(t)\Re(s)+1 e^{-\Re(t)(\Re(w)+N)}
\leq C_\epsilon(s, w) E_\epsilon, N(s, w).
\]
Here the last evaluation for the case \( \Re(s) + 1 > 0 \) follows from the differential of the real function \( x^{\Re(s)+1} e^{-x(\Re(w)+N)} \) in the variable \( x = \Re(t) \geq \epsilon \). Using Lemma \[2.3\] yields
\[
|\tilde{I}_N| \leq \int_{L^\infty} \frac{|t^{s+1} e^{-t(N+w)}| |dt|}{1 - e^{-t z}} \frac{|t^2|}{|t^2|}
\leq C_\epsilon(s, w) E_\epsilon, N(s, w) \int_{L^\infty} |dt| |t^2|.
\]
The statement of the proposition follows from
\[
\int_{L^\infty} |dt| |t^2| = \int_0^{\epsilon^2} \frac{d\delta}{\epsilon^2 + \delta^2} + \int_\epsilon^{\infty} \frac{d\gamma}{\gamma^2 + \epsilon^4} \quad (t = \gamma + i\delta)
\leq \int_0^{\epsilon^2} \frac{d\delta}{\epsilon^2} + \int_\epsilon^{\infty} \frac{d\gamma}{\gamma^2}
= 1 + \frac{1}{\epsilon}.
\]

\[\square\]

**Proposition 2.5.** Let \( N, m \in \mathbb{Z}_{\geq 0} \), \((z, s, w) \in \mathbb{C} \times \tilde{D}_s \times \mathbb{C}\), and let \( \epsilon \) be a real satisfying
\[
0 < \epsilon \leq \begin{cases} 
\frac{1}{4(|N + w| + 1)} & \text{if } z = 1, \\
\frac{|z - 1|}{4(|z| + 1)(|N + w| + 1)} & \text{if } z \neq 1.
\end{cases}
\]
Then,

$$|J_{N,m}(z, s, w)| \leq \begin{cases} 
1 & \text{if } z = 1, \\
\frac{2^m \Re(s) + m}{2^{-m+1} \epsilon \Re(s)} & \text{if } z \neq 1.
\end{cases}$$

Especially, $J_{N,m} \to 0$ as $m \to \infty$ on any compact subset of $\mathbb{C} \times \tilde{D}_s \times \mathbb{C}$.

**Proof.** We show the statement by Lemma 2.1. In the case $z \neq 1$, let

$$F(z, w) := \frac{2}{|z-1|}(|z|+1)(|w|+1) \quad (\geq 2).$$

Then, for any $t \in \mathbb{C}$ with $|t| < F(z, N + w)^{-1}$, we have

$$|\phi_N(t) - \phi_{N,m}(t)| \leq \sum_{r=m+1}^{\infty} \frac{|B_r(z, N+w)|}{r!} |t|^r$$

$$\leq \sum_{r=m+1}^{\infty} \frac{F(z, N+w)^{r-1}}{|z-1|} |t|^r$$

$$= \frac{F(z, N+w)^m |t|^{m+1}}{|z-1| (1 - F(z, N+w)|t|)} ,$$

and hence

$$\left| \int_0^\epsilon (\phi_N(t) - \phi_{N,m}(t)) t^{s-2} dt \right| \leq \int_0^\epsilon \frac{F(z, N+w)^m |t|^{m+1}}{|z-1| (1 - F(z, N+w)|t|)} \cdot |t^{s-2}| dt$$

$$\leq \left| \int_0^\epsilon \frac{F(z, N+w)^m}{|z-1| (1 - F(z, N+w)|t|)} dt \right|$$

$$\leq 2F(z, N+w)^m \frac{\epsilon \Re(s) + m}{|z-1| \Re(s) + m} \quad (\to 0 \text{ as } \epsilon \to 0)$$

$$\leq 2^{-m+1} \frac{\epsilon \Re(s)}{|z-1| \Re(s) + m} \quad (\to 0 \text{ as } m \to \infty).$$

Similarly, one can prove the case $z = 1$. \qed
2.7. The special values of $\Phi(z, s, w)$ at non-positive integers in $s$

For any $r \in \mathbb{Z}_{>0}$, taking an integer larger than or equal to $r$ as the integer $m$ in the expression (1.3) yields the special value

$$
\Phi(z, 1-r, w) = \sum_{n=0}^{N-1} \frac{z^n}{(n+w)^{1-r}} + \lim_{s \to 1-r} \frac{z^N}{\Gamma(s)} \frac{B_r(z, N+w)}{r!} \frac{(-1)^r \alpha^{s+r-1}}{s+r-1}
$$

For the first and second equalities we use the fact that $\Gamma(s)$ has simple poles only at $s = 1-r$ with residue $(-1)^{1-r}/(r-1)!$. The last equality follows from Lemma 2.6.

**Lemma 2.6.** For any positive integer $r$ and any non-negative integer $N$,

$$z \cdot B_r(z, N+1+w) - B_r(z, N+w) = r(N+w)^{r-1}.$$

Equivalently,

$$\frac{B_r(z, N+w)z^N}{r} - \frac{B_r(z, w)}{r} = \sum_{n=0}^{N-1} \frac{z^n}{(n+w)^{1-r}}.$$

**Proof.** It is easily seen from definition that

$$z \cdot \sum_{r=0}^{\infty} \frac{B_r(z, N+1+w)}{r!} t^r - \sum_{r=0}^{\infty} \frac{B_r(z, N+w)}{r!} t^r = \frac{z e^{t(N+1+w)} - t e^{t(N+w)}}{e^{tz} - 1} = \frac{e^{t(N+w)}}{e^{tz} - 1} = \sum_{r=1}^{\infty} \frac{r(N+w)^{r-1}}{r!} t^r.$$

**Remark 2.7.** The specialization $w = 0$ in Lemma 2.6 yields the result [2].
2.8. Corollaries and Remarks of Theorem 1.1

By the specialization \( z = 1 \) or \( w = 1 \) in Theorem 1.2, we have results on the Hurwitz zeta function or the polylogarithm, respectively.

**Corollary 2.8.** The Hurwitz zeta function \( \zeta(s, w) = \Phi(1, s, w) \) has single-valued analytic continuations holomorphic on two domains

1. \( \mathbb{C} \setminus \{1\} \times \mathbb{C} \setminus \ell_{\varphi} \) in two variables \((s, w)\),
2. \( \mathbb{C} \setminus \{1\} \) in one variable \( s \) with fixed \( w \in \ell_{\varphi} \setminus \mathbb{Z}_{\leq 0} \).

This function has a simple pole at \( s = 1 \) with residue 1.

**Corollary 2.9.** The polylogarithm \( \text{Li}_s(z) = z \Phi(z, s, 1) \) has single-valued analytic continuations holomorphic on two domains

1. \( \mathbb{C} \setminus \rightarrow_{\varphi} \times \mathbb{C} \) in two variables \((z, s)\),
2. \( \mathbb{C} \setminus \{1\} \) in one variable \( s \) with fixed \( z \in \rightarrow_{\varphi} \setminus \mathbb{C} \).

In particular, if \( z = 1 \) then the second case corresponds to the Riemann zeta function.

**Remark 2.10.** We give an explicit example of \( \text{Li}_1(z) = z \Phi(z, 1, 1) \). Taking \( N = 0, m = 1 \) for the equation (1.3) yields the analytic continuation

\[
\Phi(z, 1, 1) = -\epsilon \cdot B_1(z, 1) + \tilde{I}_0(z, 1, 1) + J_{0,1}(z, 1, 1),
\]

where the right hand side is both single-valued and holomorphic on the domain \( \overline{D}(1) \) with fixed \( s = w = 1 \). In the case \( z \in \overline{D}_z \setminus (\rightarrow_{\varphi} \cup [e^\epsilon, \infty)) \) we have \( \tilde{I}_0 = I_0 \), and hence

\[
z\Phi(z, 1, 1) = z \int_{[0,\infty)} \frac{e^{-t}}{1 - e^{-t}z} \, dt = z \int_{[0,\infty)} \frac{-e^{-t}e^{i(\varphi - \varphi')}}{(e^{-t}z - 1)e^{i(\varphi - \varphi')}} \, dt = \left[ \log((e^{-t}z - 1)e^{i(\varphi - \varphi')}) \right]_0^\infty = i \arg(-e^{i(\varphi - \varphi')} - \log((z - 1)e^{i(\varphi - \varphi')})),
\]

which is both single-valued and holomorphic on \( \mathbb{C} \setminus \rightarrow_{\varphi} \) with principal value \( \varphi \leq \arg \lambda < \varphi + 2\pi \) for \( \lambda \in \mathbb{C}^* \). In particular, the substitution \( \varphi = -\pi, \varphi' = 0 \) yields a usual single-valued function \( z\Phi(z, 1, 1) = -\log(1-z) \), holomorphic on \( \mathbb{C} \setminus [1, \infty) \) as in [3], [4].
Remark 2.11. In general, by the condition $\varphi \leq \arg \lambda < \varphi + 2\pi$ for $\lambda \in \mathbb{C}^*$, the function $z \mapsto \arg((z-1)e^{i(\varphi-\varphi')})$ is both single-valued and holomorphic on $\mathbb{C} \setminus -\phi'$. Especially, we can take $\varphi'$ as $\varphi'$ when $\Re(e^{i\varphi}) \geq 0$, or $\varphi' = \varphi + \pi$ when $\Re(e^{i\varphi}) < 0$, which means

$$-\phi' = \begin{cases} \{z \in \mathbb{C} \mid z - 1 = re^{i\varphi} \text{ for } r \geq 0\} & \text{if } \Re(e^{i\varphi}) \geq 0 \\ \{z \in \mathbb{C} \mid 1 - z = re^{i\varphi} \text{ for } r \geq 0\} & \text{if } \Re(e^{i\varphi}) < 0. \end{cases}$$

3. Equivalence of Lerch’s equation and Apostol’s equation

In this section, on a large region of $(a, s, w)$, we show that Lerch’s equation (1.4) holds if and only if Apostol’s equation (1.5) holds (Theorem 1.3), meaning that these two equations are essentially the same. We first extend both domains of Lerch’s equation and Apostol’s equation using Theorem 1.2. For simplicity, let $\varphi' \in 2\pi \mathbb{Z}$. Then $-\phi' = [1, \infty)$.

3.1. Lerch’s and Apostol’s equations on extended domains

If $\Phi(z, s, w)$ is holomorphic at $z = e^{2\pi i \gamma}$ ($\gamma \in \mathbb{C}$) then $\Phi(e^{2\pi ia}, s, w)$ is holomorphic at all $a \in \gamma + \mathbb{Z}$ because the complex exponential function $z = e^{2\pi ia}$ is an entire function, and thus we can apply Theorem 1.2 to $\Phi(e^{2\pi ia}, s, w)$.

Proposition 3.1. The left and right hand side of Lerch’s equation (1.4) are holomorphic on two open sets

1. $(a, s, w) \in \mathbb{C} \setminus (\ell^\text{sym} \cup (\mathbb{Z} + i\mathbb{R}_{\leq 0})) \times \mathbb{C} \times \mathbb{C} \setminus (\ell_{\varphi} \cup (\mathbb{Z} + i\mathbb{R})),$
2. $s \in \mathbb{C}$ with fixed $a \in (\mathbb{Z} + i\mathbb{R}_{\leq 0}) \setminus \ell^\text{sym}$, $w \in \mathbb{Z}_{> 0}$.

In particular, Lerch’s equation (1.4) holds on the domain

$$(a, s, w) \in U_a \times \mathbb{C} \times U_w,$$  \hspace{1cm} (D_L)$$

where $U_a$ denotes a domain in $\mathbb{C} \setminus (\ell^\text{sym} \cup (\mathbb{Z} + i\mathbb{R}_{\leq 0}))$ satisfying one of two conditions

i) $U_a$ contains the interval $(0, 1) \subset \mathbb{R}$,
ii) $U_a \cap \{a \in \mathbb{C} \mid \Im(a) > 0\} \neq \emptyset,$

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and $U_w$ the maximal domain in $\mathbb{C} \setminus (\mathfrak{L}_\varphi \cup (\mathbb{Z} + i\mathbb{R}))$ containing $(0, 1)$. Also, the left and right hand side of Apostol’s equation (1.5) are holomorphic on the open set

$$(a, s, w) \in \mathbb{C} \setminus (\mathfrak{L}_\varphi^\text{sym} \cup (\mathbb{Z} + i\mathbb{R})) \times \mathbb{C} \times \mathbb{C} \setminus (\mathfrak{L}_\varphi^\text{sym} \cup (\mathbb{Z} + i\mathbb{R})).$$

In particular, Apostol’s equation (1.5) holds on the domain

$$(a, s, w) \in U \times \mathbb{C} \times U,$$

where $U$ denotes the maximal domain in $\mathbb{C} \setminus (\mathfrak{L}_\varphi^\text{sym} \cup (\mathbb{Z} + i\mathbb{R}))$ containing $(0, 1)$. Especially, both Lerch’s and Apostol’s equations hold on the domain $[D_A]$ ($= [D_1]$ by taking $U_a = U_w = U$).

**Proof.** For Lerch’s equation, it suffices to find the intersection of domains of \(\Phi(e^{2\pi ia}, 1 - s, w), \Phi(e^{-2\pi iw}, s, a), \text{ and } \Phi(e^{2\pi iw}, s, 1 - a)\) on each domain of $[D_1]$ in Theorem 1.2.

$[D_1](1)$ On the domain $\mathbb{C} \setminus \mathfrak{L}_\varphi \times \mathbb{C} \times \mathfrak{L}_\varphi$ in $(z, s, w)$.

The domains for \(\Phi(e^{2\pi ia}, 1 - s, w), \Phi(e^{-2\pi iw}, s, a), \text{ and } \Phi(e^{2\pi iw}, s, 1 - a)\) are

1. $e^{2\pi ia} \notin [1, \infty), 1 - s \in \mathbb{C}, w \notin \mathfrak{L}_\varphi,$
2. $e^{-2\pi iw} \notin [1, \infty), s \in \mathbb{C}, a \notin \mathfrak{L}_\varphi,$
3. $e^{2\pi iw} \notin [1, \infty), s \in \mathbb{C}, 1 - a \notin \mathfrak{L}_\varphi,$

respectively. The condition $e^{2\pi ia} \in [1, \infty)$ holds if and only if $a \in \mathbb{Z} + i\mathbb{R}_{\leq 0}$. Thus the intersection in $(a, s, w)$ is $a \in \mathbb{C} \setminus (\mathfrak{L}_\varphi^\text{sym} \cup (\mathbb{Z} + i\mathbb{R}_{\leq 0})), s \in \mathbb{C}, w \in \mathbb{C} \setminus (\mathfrak{L}_\varphi \cup (\mathbb{Z} + i\mathbb{R}))$.

$[D_1](2)$ On the domain $\mathbb{C} \setminus \mathfrak{L}_\varphi \times \mathbb{C} \times \mathfrak{L}_\varphi$ in $(z, s)$ with fixed $w \in \mathfrak{L}_\varphi \setminus \mathbb{Z}_{\leq 0}$.

The domains for \(\Phi(e^{2\pi ia}, 1 - s, w), \Phi(e^{-2\pi iw}, s, a), \text{ and } \Phi(e^{2\pi iw}, s, 1 - a)\) are

1. $e^{2\pi ia} \notin [1, \infty), 1 - s \in \mathbb{C}$ with fixed $w \in \mathfrak{L}_\varphi \setminus \mathbb{Z}_{\leq 0},$
2. $e^{-2\pi iw} \notin [1, \infty), s \in \mathbb{C}$ with fixed $a \in \mathfrak{L}_\varphi \setminus \mathbb{Z}_{\leq 0},$
3. $e^{2\pi iw} \notin [1, \infty), s \in \mathbb{C}$ with fixed $1 - a \in \mathfrak{L}_\varphi \setminus \mathbb{Z}_{\leq 0},$

respectively. The condition $a, 1 - a \in \mathfrak{L}_\varphi \setminus \mathbb{Z}_{\leq 0}$ implies $\varphi \in 2\pi\mathbb{Z}$, which contradicts to the condition $\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

$[D_1](3)$ On the domain $\mathbb{C} \setminus P_z \times \mathbb{C} \setminus \mathfrak{L}_\varphi$ in $(s, w)$ with fixed $z \notin \mathfrak{L}_\varphi$.

The domains for \(\Phi(e^{2\pi ia}, 1 - s, w), \Phi(e^{-2\pi iw}, s, a), \text{ and } \Phi(e^{2\pi iw}, s, 1 - a)\) are

1. $1 - s \in \mathbb{C} \setminus P_{e^{2\pi ia}}, w \notin \mathfrak{L}_\varphi, e^{2\pi ia} \in [1, \infty)$ with fixed $a \in \mathbb{C},$
2. $s \in \mathbb{C} \setminus P_{e^{-2\pi iw}}, a \notin \mathfrak{L}_\varphi, e^{-2\pi iw} \in [1, \infty)$ with fixed $w \in \mathbb{C},$
3. $s \in \mathbb{C} \setminus P_{e^{2\pi iw}}, 1 - a \notin \mathfrak{L}_\varphi, e^{2\pi iw} \in [1, \infty)$ with fixed $w \in \mathbb{C},$
respectively. The condition \(e^{2\pi ia} \cdot e^{\pm 2\pi iw} \in [1, \infty)\) is equivalent to \(a \in \mathbb{Z} + i\mathbb{R}_{\leq 0}, w \in \mathbb{Z}\). Since \(w, a, 1 - a \notin \ell_\varphi \supset \mathbb{Z}_{\leq 0}\), we have \(w \in \mathbb{Z}_{>0}, a \notin \mathbb{Z}\). Thus the intersection in \((a, s, w)\) is \(s \in \mathbb{C} \setminus \{1\}\) with fixed \(a \in (\mathbb{Z} + i\mathbb{R}_{\leq 0}) \setminus \ell_{\varphi_{\text{sym}}}, w \in \mathbb{Z}_{>0}\).

**Remark 3.2.** In the context of the proposition above, Lerch chose the cut \(\ell_\varphi\) with angle \(\varphi = -\pi\) in the proof of Lerch’s equation [9] (In this case \(\ell_\varphi\) is exactly the negative real axis together with 0). Also, Apostol seems to have chosen the cut with angle \(\varphi = -\pi\), but not explicitly mentioned in [1].

### 3.2. Proof of Theorem 1.3

We first show that if Apostol’s equation (1.5) holds on the domain \((D_{\text{eq}})\) then Lerch’s equation (1.4) holds on the same domain, which was originally proved by Apostol [1] for a smaller region of \(a, w\). Let

\[
\Omega(s) := 2(2\pi)^{-s} \left(\cos \frac{\pi s}{2}\right) \Gamma(s),
\]

\[
\Lambda^{-}(a, s, w) := \Phi(e^{2\pi ia}, s, w) - e^{-2\pi ia} \Phi(e^{-2\pi ia}, s, 1 - w).
\]

By Proposition 3.1 Apostol’s equation is differentiable in \(a, s, w\), and thus \(\Lambda^{-}(a, s, w) = 2\Phi(e^{2\pi ia}, s, w) - \Lambda(a, s, w)\) is also
differentiable. It follows from Lemma 3.3 that
\[
\frac{\partial}{\partial a} \Lambda(a, 1 - s, w) = 2\pi i \left\{ \Lambda^{-}(a, -s, w) - w\Lambda(a, 1 - s, w) \right\},
\]
\[
\frac{\partial}{\partial a} \Lambda(-w, s, a) = -s\Lambda^{-}(-w, s + 1, a),
\]
(3.1)
\[
\frac{\partial}{\partial a} e^{-2\pi iaw}\Omega(s)\Lambda(-w, s, a) = -e^{-2\pi iaw}\Omega(s)
\]
\[
\times \left\{ s\Lambda^{-}(-w, s + 1, a) + 2\pi iw\Lambda(-w, s, a) \right\}.
\]
Using Apostol’s equation \(\Lambda(a, 1 - s, w) = e^{-2\pi iaw}\Omega(s)\Lambda(-w, s, a)\) yields
\[
\Lambda^{-}(a, -s, w) = \frac{is}{2\pi} e^{-2\pi iaw}\Omega(s)\Lambda^{-}(-w, s + 1, a).
\]
By the formula \(\Gamma(s + 1) = s\Gamma(s)\) and the substitution \(s \to s - 1\), we have the functional equation
\[
\Lambda^{-}(a, 1 - s, w) = 2i(2\pi)^{-s} \left( \sin \frac{\pi s}{2} \right) \Gamma(s)e^{-2\pi iaw}\Lambda^{-}(-w, s, a),
\]
(3.2)
which was originally given by Apostol [1, p.164] for a small region of \(a, w\) (However, it seems that there is a misprint: not “\(\exp(-2\pi ia(1 - x))\)” but “\(\exp(2\pi ia(1 - x))\)” in the functional relation in [1, p.164]). Then, adding two equations (1.5), (3.2) gives Lerch’s equation (1.4).
Conversely, assume that Lerch’s equation (1.4) holds on the domain \(D_{eq}\) (= \(D_L\) by taking \(U_a = U_w = U\)). Using Lerch’s equation (1.4) with the substitution \((a, s, w) \to (1 - a, s, 1 - w)\) (the domain \(D_{eq}\) is invariant under this substitution) yields
\[
\frac{(2\pi)^s}{\Gamma(s)} e^{2\pi iaw} \Lambda(a, 1 - s, w)
\]
\[
= \frac{(2\pi)^s}{\Gamma(s)} e^{2\pi iaw} \left\{ \Phi(e^{2\pi ia}, 1 - s, w) + e^{-2\pi iaw}\Phi(e^{2\pi i(1-a)}, 1 - s, 1 - w) \right\}
\]
\[
= e^{\pi i\frac{s}{2}} \Phi(e^{-2\pi iw}, s, a) + e^{\pi i(\frac{s}{2}+w)}\Phi(e^{2\pi iw}, s, 1 - a)
\]
\[
+ e^{-\pi i\frac{s}{2}} \Phi(e^{2\pi i(1-w)}, s, a) + e^{\pi i(\frac{s}{2}+w)}\Phi(e^{-2\pi i(1-w)}, s, 1 - a)
\]
\[
= \left( e^{\pi i\frac{s}{2}} + e^{-\pi i\frac{s}{2}} \right) \left\{ \Phi(e^{-2\pi iw}, s, a) + e^{2\pi iw}\Phi(e^{2\pi iw}, s, 1 - a) \right\}
\]
\[
= 2 \left( \cos \frac{\pi s}{2} \right) \Lambda(-w, s, a).
\]
This is Apostol’s equation (1.5).
Lemma 3.3. The differential-difference equations

\[
\frac{\partial}{\partial a} \Phi(e^{2\pi ia}, 1 - s, w) = 2\pi i \left\{ \Phi(e^{2\pi ia}, -s, w) - w\Phi(e^{2\pi ia}, 1 - s, w) \right\},
\]
\[
\frac{\partial}{\partial a} \Phi(e^{-2\pi ia}, 1 - s, 1 - w) = -2\pi i \left\{ \Phi(e^{-2\pi ia}, -s, 1 - w) + (w - 1)\Phi(e^{-2\pi ia}, 1 - s, 1 - w) \right\},
\]
\[
\frac{\partial}{\partial a} \Phi(e^{-2\pi iw}, s, a) = -s\Phi(e^{-2\pi iw}, s + 1, a),
\]
\[
\frac{\partial}{\partial a} \Phi(e^{2\pi iw}, s, 1 - a) = s\Phi(e^{2\pi iw}, s + 1, 1 - a)
\]
hold for all \((a, s, w) \in \{D_{eq}\}\).

Proof. We can differentiate the series (1.1) by terms on the first domain of \((D_0)\) with the substitution \((z, s, w) \rightarrow (e^{2\pi ia}, 1 - s, w)\).

\[
\frac{\partial}{\partial a} \Phi(e^{2\pi ia}, 1 - s, w) = 2\pi i \left\{ \Phi(e^{2\pi ia}, -s, w) - w\Phi(e^{2\pi ia}, 1 - s, w) \right\}.
\]

By Theorem 1.2 and the identity theorem for holomorphic functions, this equation holds for all \((a, s, w) \in \mathbb{C} \setminus (\mathbb{Z} + i\mathbb{R}_{\leq 0}) \times \mathbb{C} \times \mathbb{C} \setminus \ell_\varphi\), which contains \(\{D_{eq}\}\). Similarly, we have the other equations in the statement of the lemma.

Remark 3.4. By the substitutions \((a, s, w) \rightarrow (\mp w, 1 - s, \pm a)\) in the equations (3.1), we obtain the differential-difference equations

\[
\frac{\partial}{\partial w} \Lambda(-w, s, a) = -2\pi i \left\{ \Lambda(-w, s - 1, a) - a\Lambda(-w, s, a) \right\},
\]
\[
\frac{\partial}{\partial w} \Lambda(a, 1 - s, w) = (s - 1)\Lambda(a, 2 - s, w).
\]

In the same manner as above, we have

\[
\frac{\partial}{\partial a} \Lambda^-(a, 1 - s, w) = -s\Lambda^-(a, s + 1, a),
\]
\[
\frac{\partial}{\partial a} \Lambda^-(a, 1 - s, w) = 2\pi i \left\{ \Lambda(a, -s, w) - w\Lambda^-(a, 1 - s, w) \right\},
\]
\[
\frac{\partial}{\partial w} \Lambda^-(a, 1 - s, w) = (s - 1)\Lambda(a, 2 - s, w),
\]
\[
\frac{\partial}{\partial w} \Lambda^-(w, s, a) = -2\pi i \left\{ \Lambda(w, s - 1, a) - a\Lambda^-(w, s, a) \right\}.
\]
One can deduce from these calculations that we cannot find new functional equations by the differentiation in $a, w$ of Apostol’s equation (and Lerch’s equation).

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