THE JACOBI FLOW

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For Włodek Tulczyjew, on the occasion of his 65th birthday.

It is well known that the geodesic flow on the tangent bundle is the flow of a certain vector field which is called the spray \( S : TM \to TTM \). It is maybe less well known that the flow lines of the vector field \( \kappa_T : TTM \to TTTM \) project to Jacobi fields on \( TM \). This could be called the ‘Jacobi flow’. This result was developed for the lecture course [5], and it is the main result of this paper. I was motivated by the paper [6] of Urbanski in these proceedings to publish it, as an explanation of some of the uses of iterated tangent bundles in differential geometry.

1. The tangent bundle of a vector bundle. Let \((E, p, M)\) be a vector bundle with fiber addition \( +_E : E \times_M E \to E \) and fiber scalar multiplication \( m^E : E \to E \). Then \((TE, \pi_E, E)\), the tangent bundle of the manifold \( E \), is itself a vector bundle, with fiber addition denoted by \( +_T \) and scalar multiplication denoted by \( m^T \).

If \((U_\alpha, \psi_\alpha : E \mid U_\alpha \to U_\alpha \times V)_{\alpha \in A}\) is a vector bundle atlas for \( E \), such that \((U_\alpha, u_\alpha)\) is a manifold atlas for \( M \), then \((E \mid U_\alpha, \psi'_\alpha)_{\alpha \in A}\) is an atlas for the manifold \( E \), where
\[
\psi'_\alpha := (u_\alpha \times \text{Id}_V) \circ \psi_\alpha : E \mid U_\alpha \to U_\alpha \times V \to u_\alpha(U_\alpha) \times V \subset \mathbb{R}^m \times V.
\]
Hence the family \((T(E \mid U_\alpha), T\psi'_\alpha : T(E \mid U_\alpha) \to T(u_\alpha(U_\alpha) \times V) = u_\alpha(U_\alpha) \times V \times \mathbb{R}^m \times V)_{\alpha \in A}\) is the atlas describing the canonical vector bundle structure of \((TE, \pi_E, E)\). The transition functions are in turn:

\[
(\psi_\alpha \circ \psi^{-1}_\beta)(x, v) = (x, \psi_{\alpha \beta}(x)v) \quad \text{for } x \in U_{\alpha \beta}
\]

\[
(u_\alpha \circ u^{-1}_\beta)(y) = u_{\alpha \beta}(y) \quad \text{for } y \in u_\beta(U_{\alpha \beta})
\]

\[
(T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(y, v; \xi, w) = (u_{\alpha \beta}(y), \psi_{\alpha \beta}(u^{-1}_{\alpha \beta}(y))v; d(u_{\alpha \beta})(y)\xi,
\]

\[
(d(\psi_{\alpha \beta} \circ u^{-1}_{\beta})(y))\xi + \psi_{\alpha \beta}(u^{-1}_{\beta}(y))w.
\]

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So we see that for fixed \((y, v)\) the transition functions are linear in \((\xi, w) \in \mathbb{R}^m \times V\). This describes the vector bundle structure of the tangent bundle \((TE, \pi_E, E)\).

For fixed \((y, \xi)\) the transition functions of \(TE\) are also linear in \((v, w) \in V \times V\). This gives a vector bundle structure on \((TE, Tp, TM)\). Its fiber addition will be denoted by \(T(+_E) : T(E \times_M E) = TE \times_T M \rightarrow TE\), since it is the tangent mapping of \(+_E\). Likewise its scalar multiplication will be denoted by \(T(m^E_t)\). One may say that the second vector bundle structure on \(TE\), that one over \(TM\), is the derivative of the original one on \(E\).

The space \(\{\Xi \in TE : Tp.\Xi = 0 \text{ in } TM\} = (Tp)^{-1}(0)\) is denoted by \(VE\) and is called the vertical bundle over \(E\). The local form of a vertical vector \(\Xi\) is \(T\psi'_{\alpha},\Xi = (y, v; 0, w)\), so the transition functions are \((T\psi'_{\alpha} \circ T(\psi'_{\beta})^{-1})(y, v; 0, w) = (u_{\alpha,\beta}(y), \psi_{\alpha,\beta}(u^{-1}_\beta(y))v; 0, \psi_{\alpha,\beta}(u^{-1}_\beta(y))w)\). They are linear in \((v, w) \in V \times V\) for fixed \(y\), so \(VE\) is a vector bundle over \(M\). It coincides with \(0^v_M(TE, Tp, TM)\), the pullback of the bundle \(TE \rightarrow TM\) over the zero section. We have a canonical isomorphism \(V_1E : E \times_M E \rightarrow VE\), called the big vertical lift, given by \(V_1E(u_x, v_x) := \partial_0[0(t_x + t_{v_x})]\), which is fiber linear over \(M\). We will mainly use the small vertical lift \(vlE : E \rightarrow TE\), given by \(vlE(v_x) = \partial_0[0(t_x + t_{v_x})]\). The local representation of the vertical lift is \((T\psi'_{\alpha} \circ Vl_E \circ (\psi'_{\beta})^{-1})(y, v) = (y, 0; 0, v)\).

If \(\varphi : (E, p, M) \rightarrow (F, q, N)\) is a vector bundle homomorphism, then we have \(vlF \circ \varphi = T\varphi \circ Vl_E : E \rightarrow VF \subset TF\). So \(vl\) is a natural transformation between certain functors on the category of vector bundles and their homomorphisms. The mapping \(v_{rp}E := pr_2 \circ Vl^{-1}_E : VE \rightarrow E\) is called the vertical projection.

2. The second tangent bundle of a manifold. All of 1 is valid for the second tangent bundle \(TTM\) of a manifold, but here we have one more natural structure at our disposal. The canonical flip or involution \(\kappa_M : TTM \rightarrow TTM\) is defined locally by

\[(TTu \circ \kappa_M \circ TTu^{-1})(x, \xi; \eta, \zeta) = (x, \eta; \xi, \zeta),\]

where \((U, u)\) is a chart on \(M\). Clearly this definition is invariant under changes of charts \((Tu_\alpha\) equals \(\psi'_{\alpha}\) from 1).

The flip \(\kappa_M\) has the following properties:

1. \(\kappa_N \circ TTf = TTf \circ \kappa_M\) for each \(f \in C^\infty(M, N)\).
2. \(T(\pi_M) \circ \kappa_M = \pi_{TTM}\) and \(\pi_{TTM} \circ \kappa_M = T(\pi_M)\).
3. \(\kappa^{-1}_M = \kappa_M\).
4. \(\kappa_M\) is a linear isomorphism from the vector bundle \((TTM, T(\pi_M), TM)\) to the bundle \((TTM, \pi_{TTM}, TM)\), so it interchanges the two vector bundle structures on \(TTM\).
5. It is the unique smooth mapping \(TTM \rightarrow TTM\) which satisfies

\[\partial_t \partial_s c(t, s) = \kappa_M \partial_t \partial_s c(t, s)\]

for each \(c : \mathbb{R}^2 \rightarrow M\).

All this follows from the local formula given above. A quite early use of \(\kappa_M\) is in [4].
3. Lemma. For vector fields $X, Y \in \mathfrak{X}(M)$ we have

\[ [X,Y] = \text{spr}_{TM} \circ (TY \circ X - \kappa_M \circ TX \circ Y), \]

\[ TY \circ X - TM \kappa_T \circ TX \circ Y = \text{vl}_{TM}(Y,[X,Y]) \]

\[ = (\text{vl}_{TM} \circ [X,Y]) T^+(TM) (0_{TM} \circ Y). \]

See [3] 6.13, 6.19, or 37.13 for different proofs of this well known result.

4. Linear connections and their curvatures. Let $(E,p,M)$ be a vector bundle. Recall that a linear connection on the vector bundle $E$ can be described by specifying its connector $K : TE \to E$. This notions seems to be due to [2]. Any smooth mapping $K : TE \to E$ which is a (fiber linear) homomorphism for both vector bundle structures on $TE$,

\[
\begin{array}{ccc}
TE & \xrightarrow{K} & E \\
\pi_E & & \downarrow p \\
E & \xrightarrow{p} & M
\end{array}
\quad
\begin{array}{ccc}
TE & \xrightarrow{K} & E \\
Tp & & \downarrow p \\
TM & \xrightarrow{\pi_M} & M
\end{array}
\]

and which is a left inverse to the vertical lift, $K \circ \text{vl}_E = \text{Id}_E : E \to TE \to E$, specifies a linear connection. Namely: The inverse image $H := K^{-1}(0_E)$ of the zero section $0_E \subset E$, it is a subvector bundle for both vector bundle structures, and for the vector bundle structure $\pi_E : TE \to E$ the subbundle $H$ turns out to be a complementary bundle for the vertical bundle $VE \to E$. We get then the associated horizontal lift mapping

\[ C : TM \times_M E \to TE, \quad C(\cdot,u) = \left( (Tp|\ker(K : T_uE \to E_{p(u)})) \right)^{-1} \]

which has the following properties

\[ (Tp,\pi_E) \circ C = \text{Id}_{TM \times_M E}; \]

\[ C(\cdot,u) : T_{p(u)}M \to T_uE \text{ is linear for each } u \in E, \]

\[ C(X_x,\cdot) : E_x \to (Tp)^{-1}(X_x) \text{ is linear for each } X_x \in T_xM. \]

Conversely given a smooth horizontal lift mapping $C$ with these properties one can reconstruct a connector $K$.

For any manifold $N$, smooth mapping $s : N \to E$ along $f = p \circ s : N \to M$, and vector field $X \in \mathfrak{X}(N)$ a connector $K : TE \to E$ defines the covariant derivative of $s$ along $X$ by

\[ \nabla_X s := K \circ Ts \circ X : N \to TN \to TE \to E. \]

See the following diagram for all the mappings.

\[ \begin{array}{ccc}
TE & \xrightarrow{K} & E \\
\pi_E & \downarrow & \\
TN & \xrightarrow{s} & E \\
\downarrow \nabla_X s & & \\
N & \xrightarrow{f} & M
\end{array} \]
In canonical coordinates as in 1 we have then
\[ C((y, \xi), (y, v)) = (y, v; \Gamma_y(v, \xi)), \]
\[ K(y, v; \xi, w) = (y, w - \Gamma_y(v, \xi)), \]
\[ \nabla_{(y, \xi)}(\mathrm{Id}, s) = (\mathrm{Id}, ds(y)\xi - \Gamma_y(s(y), \xi)), \]
where the Christoffel symbol \(\Gamma_y(v, \xi)\) is smooth in \(y\) and bilinear in \((v, \xi)\). Here the sign is the negative of the one in many more traditional approaches, since \(\Gamma\) parametrizes the horizontal bundle.

Let \(C^\infty_f(N, E)\) denote the space of all sections along \(f\) of \(E\), isomorphic to the space \(C^\infty_f(f^*E)\) of sections of the pullback bundle. The covariant derivative may then be viewed as a bilinear mapping \(\nabla : \mathfrak{X}(N) \times C^\infty_f(N, E) \to C^\infty_f(N, E)\). It has the following properties which follow directly from the definitions:

1. \(\nabla_X s\) is \(C^\infty(N, \mathbb{R})\)-linear in \(X \in \mathfrak{X}(N)\). For \(x \in N\) also we have \(\nabla_{X(x)} s = K.Ts.X(x) = (\nabla_X s)(x) \in E\).
2. \(\nabla_X (h.s) = dh(X)s + h.\nabla_X s\) for \(h \in C^\infty(N, \mathbb{R})\).
3. \(\nabla_{TX,Y}s = \nabla_{T\xi,s}(s \circ g)\). If \(Y \in \mathfrak{X}(Q)\) and \(X \in \mathfrak{X}(N)\) are \(g\)-related, then we have \(\nabla_Y s \circ g = (\nabla_Y s) \circ g\).

For vector fields \(X, Y \in \mathfrak{X}(M)\) and a section \(s \in C^\infty(E)\) the curvature \(R \in \Omega^2(M, L(E, E))\) of the connection is given by

\[ R(X, Y)s = ([\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s) \]

**Theorem.** Let \(K : TE \to E\) be the connector of a linear connection on a vector bundle \((E, p, M)\). If \(s : N \to E\) is a section along \(f := p \circ s : N \to M\) then we have for vector fields \(X, Y \in \mathfrak{X}(N)\)

\[ \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}s = \]
\[ = (K \circ TK \circ \kappa_E - K \circ TK) \circ TTS \circ TX \circ Y = \]
\[ = R(\circ (Tf \circ X, Tf \circ Y))s : N \to E, \]

where \(R \in \Omega^2(M; L(E, E))\) is the curvature.

**Proof.** Let first \(m_i^E : E \to E\) denote the scalar multiplication. Then we have \(\partial |_{m_i^E} = vl_E\) where \(vl_E : E \to TE\) is the vertical lift. We use then lemma 3 and some obvious commutation relations to get in turn:

\[ \nabla_X vl_E \circ K = \partial |_{m_i^E} \circ K = \partial |_{m_i^E} \circ m_i^{TE} = TK \circ \partial |_{m_i^{TE}} = TK \circ vl_T(TE, E, E). \]
\[ \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}s = \]
\[ \nabla_X vl_E \circ K \circ T(K \circ Ts \circ Y) \circ X - K \circ T(K \circ Ts \circ X) \circ Y - K \circ Ts \circ [X,Y] \]
\[ K \circ Ts \circ [X,Y] = K \circ vl_E \circ K \circ Ts \circ [X,Y] \]
\[ = K \circ TK \circ vl_E \circ Ts \circ [X,Y] = K \circ K \circ TTS \circ vl_T \circ [X,Y] \]
\[ = K \circ T \circ TTS \circ ((TY \circ X - \kappa_N \circ TX \circ Y) \circ T \circ Ts \circ [X,Y] \circ Y) \]
\[ = K \circ T \circ TTS \circ TY \circ X - K \circ TK \circ TTS \circ \kappa_N \circ TX \circ Y = 0. \]
Now we sum up and use $T Ts \circ \kappa N = \kappa E \circ T Ts$ to get the first result. If in particular we choose $f = \text{Id}_M$ so that $s$ is a section of $E \to M$ and $X, Y$ are vector fields on $M$, then we get the curvature $R$.

To see that in the general case $(K \circ TK \circ \kappa E - K \circ TK) \circ T Ts \circ TX \circ Y$ coincides with $R(Tf \circ X, Tf \circ Y)$ is one has to write out (1) and $(TTs \circ TX \circ Y)(x) \in TTE$ in canonical charts induced from vector bundle charts of $E$. □

5. Torsion. Let $K : TTM \to M$ be a linear connector on the tangent bundle, let $X, Y \in \mathfrak{X}(M)$. Then the torsion is given by

$$\text{Tor}(X, Y) = (K \circ \kappa_M - K) \circ TX \circ Y.$$ 

If moreover $f : N \to M$ is smooth and $U, V \in \mathfrak{X}(N)$ then we get also

$$\text{Tor}(Tf \circ U, Tf \circ V) = \nabla_U(Tf \circ V) - \nabla_V(Tf \circ U) - Tf \circ [U, V] = (K \circ \kappa_M - K) \circ TTf \circ TU \circ V.$$ 

Proof. (9) We have in turn

$$\text{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$= K \circ TY \circ X - K \circ TX \circ Y - K \circ vT_L \circ [X, Y]$$

$$K \circ vT_L \circ [X, Y] = K \circ ((TY \circ X - \kappa_M \circ TX \circ Y) (T- \circ 0T_L \circ Y)$$

$$= K \circ TY \circ X - K \circ \kappa_M \circ TX \circ Y = 0.$$ 

An analogous computation works in the second case, and that $(K \circ \kappa_M - K) \circ TTf \circ TU \circ V = \text{Tor}(Tf \circ U, Tf \circ V)$ can again be checked in local coordinates. □

6. Sprays. Given a linear connector $K : TTM \to M$ on the tangent bundle with its horizontal lift mapping $C : TM \times_M TM \to TTM$, then $S := C \circ \text{diag} : TM \to TM \times_M TM \to TTM$ is called the spray. This notion is due to [1]. The spray has the following properties:

$$\pi_T \circ S = \text{Id}_T \quad \text{a vector field on } TM,$$

$$T(\pi_T) \circ S = \text{Id}_T \quad \text{a second order differential equation},$$

$$S \circ m_T^{TM} = T(m_T^{TM}) \circ m_T^{TTM} \circ S \quad \text{‘quadratic’},$$

where $m_t^E$ is the scalar multiplication by $t$ on a vector bundle $E$. From $S$ one can reconstruct the torsion free part of $C$. The following result is well known:

Lemma. For a spray $S : TM \to TTM$ on $M$, for $X \in TM$

$$\text{geo}^S(X)(t) := \pi_M(\text{Fl}^S(X))$$

defines a geodesic structure on $M$, where $\text{Fl}^S$ is the flow of the vector field $S$.

The abstract properties of a geodesic structure are obvious:

$$\text{geo} : TM \times \mathbb{R} \supset U \to M$$

$$\text{geo}(X)(0) = \pi_M(X), \quad \partial_0 \text{geo}(X)(t) = X$$

$$\text{geo}(tX)(s) = \text{geo}(X)(ts)$$

$$\text{geo}(\text{geo}(X)'(t))(s) = \text{geo}(X)(t + s)$$

From a geodesic structure one can reconstruct the spray by differentiation.
7. Theorem. Let $S : TM \to TTM$ be a spray on a manifold $M$. Then $\kappa_{TM} \circ TS : TTM \to TTTM$ is a vector field. Consider a flow line
\[ Y(t) = \text{Fl}_{t}^{\kappa_{TM} \circ TS}(Y(0)) \]
of this field. Then we have:
\[ c := \pi_{M} \circ \pi_{TM} \circ Y \text{ is a geodesic on } M. \]
\[ \dot{c} = \pi_{TM} \circ Y \text{ is the velocity field of } c. \]
\[ J := T(\pi_{M}) \circ Y \text{ is a Jacobi field along } c. \]
\[ \dot{J} = \kappa_{M} \circ Y \text{ is the velocity field of } J. \]
\[ \nabla_{\dot{c}} J = K \circ \kappa_{M} \circ Y \text{ is the covariant derivative of } J. \]
The Jacobi equation is given by:
\[ 0 = \nabla_{\dot{c}} \nabla_{\dot{c}} J + R(J, \dot{c}) \dot{c} + \nabla_{\dot{c}} \text{Tor}(J, \dot{c}) \]
\[ = K \circ TK \circ TS \circ Y. \]
This implies that in a canonical chart induced from a chart on $M$ the curve $Y(t)$ is given by
\[ (c(t), c'(t); J(t), J'(t)). \]

Proof. Consider a curve $s \mapsto X(s)$ in $TM$. Then each $t \mapsto \pi_{M}(\text{Fl}_{t}^{S}(X(s)))$ is a geodesic in $M$, and in the variable $s$ it is a variation through geodesics. Thus $J(t) := \partial_{s}|_{0} \pi_{M}(\text{Fl}_{t}^{S}(X(s)))$ is a Jacobi field along the geodesic $c(t) := \pi_{M}(\text{Fl}_{t}^{S}(X(0)))$, and each Jacobi field is of this form, for a suitable curve $X(s)$. We consider now the curve $Y(t) := \partial_{s}|_{0} \text{Fl}_{t}^{S}(X(s))$ in $TTM$. Then by 2.6 we have
\[ \partial_{t} Y(t) = \partial_{t} \partial_{s}|_{0} \text{Fl}_{t}^{S}(X(s)) = \kappa_{TM} \partial_{s}|_{0} \partial_{t} \text{Fl}_{t}^{S}(X(s)) = \kappa_{TM} \partial_{s}|_{0} S(\text{Fl}_{t}^{S}(X(s))) = (\kappa_{TM} \circ TS)(\partial_{s}|_{0} \text{Fl}_{t}^{S}(X(s))) = (\kappa_{TM} \circ TS)(Y(t)), \]
so that $Y(t)$ is a flow line of the vector field $\kappa_{TM} \circ TS : TTM \to TTTM$. Moreover using the properties of $\kappa$ from section 2 and of $S$ from section 6 we get
\[ T\pi_{M}.Y(t) = T\pi_{M}.\partial_{s}|_{0} \text{Fl}_{t}^{S}(X(s)) = \partial_{s}|_{0} \pi_{M}(\text{Fl}_{t}^{S}(X(s))) = J(t), \]
\[ \pi_{M}T\pi_{M}Y(t) = c(t), \]the geodesic,
\[ \partial_{t} J(t) = \partial_{t} T\pi_{M}.\partial_{s}|_{0} \text{Fl}_{t}^{S}(X(s)) = \partial_{t} \partial_{s}|_{0} \pi_{M}(\text{Fl}_{t}^{S}(X(s))), \]
\[ = \kappa_{M} \partial_{s}|_{0} \partial_{t} \pi_{M}(\text{Fl}_{t}^{S}(X(s))) = \kappa_{M} \partial_{s}|_{0} \partial_{t} \pi_{M}(\text{Fl}_{t}^{S}(X(s))) \]
\[ = \kappa_{M} \partial_{s}|_{0} T\pi_{M}.\partial_{t} \text{Fl}_{t}^{S}(X(s)) = \kappa_{M} \partial_{s}|_{0} (T\pi_{M} \circ S) \text{Fl}_{t}^{S}(X(s)) \]
\[ = \kappa_{M} \partial_{s}|_{0} \text{Fl}_{t}^{S}(X(s)) = \kappa_{M} Y(t), \]
\[ \nabla_{\dot{c}} J = K \circ \dot{c} J = K \circ \kappa_{M} \circ Y. \]
Finally let us express the well known Jacobi expression, where we put $\gamma(t, s) := \pi_{M}(\text{Fl}_{t}^{S}(X(s)))$ for short and use most of the expressions from above:
\[ \nabla_{\dot{c}} \nabla_{\dot{c}} J + R(J, \dot{c}) \dot{c} + \nabla_{\dot{c}} \text{Tor}(J, \dot{c}) = \]
\[ = \nabla_{\dot{c}} \nabla_{\dot{c}} \gamma_{T}, \gamma_{\partial_{s}} + R(\gamma_{T}, \partial_{s}, \gamma_{\partial_{s}}) \gamma_{T}, \partial_{s} + \nabla_{\partial_{s}} \text{Tor}(\gamma_{T}, \partial_{s}, \gamma_{\partial_{s}}) \]
\[ = K.T(K.T(\gamma_{T}, \partial_{s}).\partial_{t}), \partial_{t} \]
\[ + (K.T K.\kappa_{TM} - K.T K).T T(\gamma_{T}).\partial_{s}, \partial_{t} \]
\[ + K.T((K.\kappa_{M} - K).T T(\gamma_{T}).\partial_{s}, \partial_{t}).\partial_{t}. \]
Note that for example for the term in the second summand we have

\[ TTT\gamma.TT\partial_t.T\partial_s.\partial_t = T(T(\partial_t\gamma)\partial_s).\partial_t = \partial_t\partial_s\partial_t\gamma = \partial_t.\kappa_M.\partial_t.\partial_s\gamma = T\kappa_M.\partial_t.\partial_t.\partial_s\gamma \]

which at \( s = 0 \) equals \( T\kappa_M\bar{J} \). Using this we get for the Jacobi expression at \( s = 0 \):

\[
\nabla_\partial_t\nabla_\partial_t J + R(J,\dot{c})\dot{c} + \nabla_\partial_t \text{Tor}(J,\dot{c}) = \\
= (K.TK + K.TK.\kappa_M.T\kappa_M - K.TK.T\kappa_M + K.TK.T\kappa_M - K.TK).\partial_t\partial_t J = \\
= K.TK.\kappa_M.T\kappa_M.\partial_t\partial_t J = K.TK.T\kappa_M.\partial_t Y = K.TK.TSY, \\
\]

where we used \( \partial_t\partial_t J = \partial_t(\kappa_M.Y) = T\kappa_M\partial_t Y = T\kappa_M.\kappa_M.TSY \). Finally the validity of the Jacobi equation \( 0 = K.TK.TSY \) follows trivially from \( K \circ S = 0_{TM} \).

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