Non-abelian Berry’s phase and Chern numbers in higher spin pairing condensates

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We show that the non-Abelian Berry phase emerges naturally in the s-wave and spin quintet pairing channel of spin-3/2 fermions. The topological structure of this pairing condensate is characterized by the second Chern number. This topological structure can be realized in ultra-cold atomic systems and in solid state systems with at least two Kramers doublets.

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I. INTRODUCTION

Topological gauge structure and Berry’s phase play an increasingly important role in condensed matter physics. The quantized Hall conductance can be deeply understood in terms of the first Chern class. The fractional quantum Hall effect can be fundamentally described by the U(1) topological Chern-Simons gauge theory. The effective action for ferromagnets and one dimensional antiferromagnets contain Berry phase terms which fundamentally determine the low energy dynamics. More recently, Berry’s phase associated with the BCS quasi-particles in pairing condensates has also been studied extensively.

However, most of the well known applications of the geometrical phase in condensed matter physics involves Berry’s original abelian U(1) phase factor. More recently, the non-abelian SU(2) Berry’s phase (or holonomy, to be precise) has been systematically investigated in the context of condensed matter systems. Denler and Zhang investigated the quasi-particle wave functions in the unified SO(5) theory of antiferromagnetism and superconductivity, and found that the SDW and the BCS quasi-particle states accumulate an SU(2) Berry’s phase (or holonomy) when the order parameter returns to itself after an adiabatic circuit. Zhang and Hu found a higher dimensional generalization of the quantum Hall effect based on a topologically non-trivial SU(2) background gauge field. Rather surprisingly, the non-abelian SU(2) holonomy also found its deep application in the technologically relevant field of quantum spintronics. All these condensed matter applications are underpinned by a common mathematical framework, which naturally generalizes the concept of Berry’s U(1) phase factor. This new class of applications is topologically characterized by the second Chern class and the second Hopf map, and applies to fermionic systems with time reversal invariance.

In this paper, we investigate the non-trivial topological structures associated with the higher spin condensates. We first review the momentum space gauge structure of the spin one condensate, namely the \(A\) phase of \(^3He\). As it has been pointed out, the momentum space gauge structure of the pairing condensate is given by that of the t’Hooft-Polyakov monopole. We then investigate the new system of spin 2 (quintet) pairing condensate of the underlying spin 3/2 fermions. The most general Hubbard model of spin 3/2 fermions has recently been introduced and investigated extensively by Wu et al., who found that the model always has a generic SO(5) symmetry in the spin sector. Building on this work, we show here that the fermionic quasi-particles of the quintet pairing condensate can be described by the second Hopf map. Similar to the SDW+BCS system investigated by Demler and Zhang, the quasi-particles of the quintet pairing condensate also accumulate an SU(2) holonomy. The quintet pairing condensate can be experimentally realized in a number of systems. Cold atoms with spin 3/2 in the continuum or on the optical lattice can be accurately described by the model of local contact interactions, and the quintet pairing condensate can be experimentally tuned over a wide range, including the range for stable quintet condensates. Effective spin 3/2 fermions can also be realized in solid state systems with at least two Kramers doublets, for example in bands formed by \(P_3/2\) orbitals.

In the rest of this paper, we shall use spin-1/2 system to be short for the spin-1/2 superfluid \(^3He-A\) and spin-3/2 system for the s-wave spin-3/2 superconductor in the quintet channel. The repeated indices are summed assumingly throughout this paper.

II. SUPERFLUID \(^3He-A\)

A. Goldstone manifold and the 1st Hopf map

The general form of the equilibrium order parameter in the \(^3He-A\) phase can be written as

\[
\langle \Delta(k)_{a} \rangle = \Delta \hat{d}_a \left( \hat{e}_i^{(1)} + i \hat{e}_i^{(2)} \right),
\]

where the spin index \((a)\) and orbital index\((i)\) run from 1 to 3. \(\Delta\) is a complex number that contains the information of the magnitude and the \(U(1)\) phase. \(\hat{d}\) is the normal vector of the plane to which the spin direction is restricted. The orthogonal vectors \(\hat{e}_i^{(1)}\) and \(\hat{e}_i^{(2)}\) and \(\hat{l} = \hat{e}_i^{(1)} \times \hat{e}_i^{(2)}\) form a local physical coordinate frame.
The Goldstone manifold of the order parameter is given by

\[ R_a = G/H = \frac{U(1) \times SO(3)^{(L)} \times SO(3)^{(S)}}{SO(2)^{(S)} \times U(1)^{\text{combined}} \times Z_2^{\text{combined}}} \]

\[ = S^2 \times SO(3)^{\text{relative}}/Z_2. \quad (2) \]

Here \(SO(3)^{\text{relative}}\) denotes such rotations about the axis \(l\) that lead to new degenerate states which are relative towards gauge transformations. The \(U(1)^{\text{combined}}\) comes from the fact that the A-phase state is invariant under combined transformations of the gauge transformation with the parameter \(\phi\) from the \(U(1)\) group and the orbital rotation of \(\hat{c}^{(1)}, \hat{c}^{(2)}\) and \(l\) about axis \(l\) by the same angle \(\phi\). The \(Z_2^{\text{combined}}\) denotes the combined operation that \(\hat{d} \rightarrow -\hat{d}, \Delta_k \rightarrow -\Delta_k\). This combined discrete symmetry leads to the existence of half-quantum vortices. Around a half-quantum vortex, the vector field \(\hat{d}\) is continuously rotated into \(-\hat{d}\) and the \(U(1)\) phase of \(\Delta_k\) continuously evolves from 0 to \(\pi\) when the order parameter returns to itself after an adiabatic circuit.

If we fix the local orthogonal frame in an arbitrary direction and adiabatically move the quasi-particle around a line defect of half-quantum vortex, the trajectory of the order parameter is a closed loop on the \(S^2/Z_2\) space. On the other hand, the degrees of freedom of the quasi-particle (a 2-dimensional spinor) forms a 3-dimensional sphere \(S^3\) and the trajectory of the quasi-particle on \(S^3\) is not closed. This adiabatic evolution defines the following map

\[ S^3 \rightarrow S^2/Z_2. \quad (3) \]

In the topological terminology, Eq. (3) is determined by the third homotopic group denoted by \(\pi_3(S^2/Z_2)\). Due to a theorem in the Homotopy theory, \(\pi_k(S^n/Z_2) = \pi_k(S^n)\), for \(k \geq 2\).

\[ \text{Eq. (3) is homotopically equivalent to the first Hopf map} \]

\[ S^3 \rightarrow S^2 \text{ i.e., } U(1) \text{ Berry phase in the FQHE and other nano-structure in the semiconductors}.\]

B. Berry connection, 1st Chern number, t’Hooft-Polyakov monopole and Dirac monopole

If we define the spinor as

\[ \Psi_k = \left( c_{-k \frac{1}{2}}, \frac{1}{\sqrt{2}} c_{-k \frac{1}{2}} c_{-k \frac{1}{2}}, \frac{1}{\sqrt{2}} c_{-k \frac{1}{2}} c_{-k \frac{1}{2}} \right), \quad (5) \]

the mean field Hamiltonian for \( \text{\text{3He-A}}\) is given by

\[ \mathcal{H} = \sum_k \Psi^\dagger_k H_k \Psi_k \quad (6) \]

with

\[ H_k = \begin{pmatrix} \epsilon_k \sigma^0 & \Delta_k \\ \Delta_k^\dagger & -\epsilon_k \sigma^0 \end{pmatrix}, \quad (7) \]

where \(\Delta_k = -\Delta_k \hat{d}_a \sigma^a R\) and \(R = -i a^2\). \(\epsilon_k\) is the kinetic energy on the lattices referenced from the Fermi surface and the summation of momentum \(k\) is over half of the Brillouin zone to avoid the double counting. Here, \(\sigma^0\) is the \(2 \times 2\) identity matrix and \(\sigma^{1,2,3}\) are Pauli matrices.

The Berry phase connection (BPC) is defined by the differential change of states projecting to themselves. In this paper, it will be illustrated by using the state with a positive eigenvalue. BPC obtained from the state with a negative eigenvalue is simply the complex conjugate to the one with the eigenvalue of a different sign. The BPC and its field strength can be obtained respectively.

\[ A_a = \frac{-i A_a^c \sigma_c}{2}, \quad A_a^c = \varepsilon_{abc} \hat{d}_b \hat{d}_c. \quad (8) \]

and

\[ F_{bc}^a = \partial_b A_a^c - \partial_c A_a^b + \epsilon_{abc} A_b^d A_d^e \]

\[ = -\frac{1}{d^2} \varepsilon_{abc} \hat{d}_c \hat{d}_a, \quad (9) \]

The gauge invariant magnetic field can be defined as

\[ B^a = \frac{1}{2} \varepsilon_{abc} F_{bc}^d \hat{d}_c = -\hat{d}_a. \quad (10) \]

This is a U(1) magnetic-monopole like field in the \(d\)-space. It emerges when there are line defects in the \(d\) field, e.g., half-quantum vortices in the superfluid \(He - 3A\) phase. If we transport the spin-1/2 fermion adiabatically around the vortex, the electronic wavefunction gains the phase accumulated due to the \(\hat{d}\) field, as we discussed previously. Moreover, the first Chern number can be computed easily.

\[ C_1 = \frac{1}{4\pi} \oint B \cdot dS = -1. \quad (11) \]

This is the famous t’Hooft-Polyakov monopole (TPM). Different from the Dirac monopole, the gauge field of TPM is non-abelian and finite everywhere over \(S^2/Z_2\) while the Dirac magnetic monopole is abelian and has a singularity string. There is a deep and direct relation between them, which can be achieved by a singular gauge transformation.

The present \(SO(3)\) Berry phase defines a \(SO(3)\) gauge theory on \(S^2/Z_2\). Using the covariant derivative \(D_a = \partial_a + A_a\), the \(SO(3)\) generators in the presence of t’Hooft-Polyakov monopole can be written as

\[ L_{ab} = \Lambda_{ab} - id^2 f_{ab}, \quad a = 1, 2, 3 \quad (12) \]

where \(\Lambda_{ab} = -i d_a \hat{D}_b + i d_b \hat{D}_a\) and \(f_{ab} = -i F_{ab}^c 2\). Defining \(I_a = \frac{1}{2} \epsilon_{abc} L_{bc}\), one finds easily that \([I_a, I_b] = i \epsilon_{abc} I_c\) satisfying the \(SO(3)\) algebra. Using Eq. (8) and Eq. (9), one can show

\[ L_{ab} = L_{ab}^{(0)} + \epsilon_{abc} \sigma_c 2 \quad (13) \]
where $L^{(0)}$ is the orbital angular momentum, defined by $L^{(0)}_{ab} = -id_a \partial_b + id_b \partial_a$. Define \[ V = \exp \left[ i \frac{\partial a_3 d_a}{\sqrt{d^2 - d_3^2}} \right] \] \hspace{2cm} (14)

where $\sigma_{ab} = \epsilon_{abc} c_e / 2$ and $\cos \vartheta = d_3 / d$. One can perform a singular $SO(3)$ gauge transformation on $L_{ab}$ such that $J_{ab} = VL_{ab}V^\dagger$ where

\[ J_{\mu\nu} = -i d_\mu \partial_\nu + i d_\nu \partial_\mu + \epsilon_{\mu\nu} \frac{\sigma_3}{2} \] \hspace{2cm} (15)

\[ J_{\mu3} = -i d_\mu \partial_3 + i d_3 \partial_\mu - \epsilon_{\mu\nu} \frac{d_\nu}{d + d_3} \sigma_3 \] \hspace{2cm} (16)

with $\mu, \nu = 1, 2$. $\epsilon_{\mu\nu}$ is the antisymmetry tensor which has only one component $\epsilon_{12} = 1$ in this case. It is obvious that $J_{\mu\nu} = J_{12}$ forms the $U(1)$ generator on $S^2/Z_2$. From the definition of Eq. (15), one can extract the $U(1)$ BPC regardless the unnecessary $\frac{\sigma_3}{2}$.

\[ a_\mu = -i \epsilon_{\mu\nu} \frac{d_\nu}{d(d + d_3)}, \hspace{0.5cm} a_3 = 0 \] \hspace{2cm} (17)

and the finite $U(1)$ field strength over $S^2/Z_2$

\[ F_{ab} = i \epsilon_{abc} \frac{d_c}{d^2} \] \hspace{2cm} (18)

We should notice that the singular gauge transformation we used has a singularity string along the negative $z$-axis. Therefore, while the covariant $SO(3)$ BPC is finite over the whole $d$ field, the $U(1)$ BPC has a singularity string which is reflected through the transformation. This transformation is only valid on the northern hemisphere including the equator of the $S^2/Z_2$. One is able to choose another gauge which has the singularity along the positive $z$-axis to describe the transformation on the southern hemisphere.

The role of this singular gauge transformation is very intriguing. We can view the covariant $SO(3)$ gauge potential $A^3_\alpha$ in Eq. (5) as a vector $d_\alpha$ pointing in the isospin space. The singular gauge transformation is nothing but the rotation of the spin vector from $d_\alpha$ to $d_\beta$. Therefore, the invariant subgroup of $SO(3)$ is emerged from the isometry group of the equator of $S^2/Z_2$, which is $U(1)$. This mechanism accounts for the appearance of the $U(1)$ Berry phase in this problem. As a result, the $SO(3)$ Berry phase in this system is essentially equivalent to the $U(1)$ Berry phase.

III. $s$-WAVE QUINTET PAIRING CONDENSATE IN SPIN-3/2 SYSTEM

A. Goldstein manifold and the 2nd Hopf map

Another candidate for non-trivial gauge structures is the spin-3/2 fermionic system with contact interaction, in which an exact $SO(5)$ symmetry was identified recently. It may be studied in the ultra-cold atomic systems, like $^9$Be, $^{132}$Cs, $^{135}$Ba, $^{137}$Ba. The four-component spinor $(c_\alpha, c_{\beta}, c_{-\alpha}, c_{-\beta})^T$ forms the spinor representations of the $SU(4)$ group which is the unitary transformation of the four-component complex spinor. The kinetic energy term has the explicit $SU(4)$ symmetry. However, the $s$-wave contact interaction term breaks the $SU(4)$ symmetry to $SO(5)$. Because of the $s$-wave scattering, there are only the singlet and quintet channels as required by the Pauli’s exclusion principle. Interestingly, the spin $SU(2)$ singlet and quintet channels interaction can also be interpreted as $SO(5)$ group’s singlet and 5-vector representations.

The Cooper pair structures has also been studied in spin-3/2 system. The singlet and quintet pairing channel operators are described by

\[ \eta^\dagger (r) = \frac{1}{2} c_{\alpha}(r) \eta^\dagger(r) \] \[ \chi^\dagger (r) = -i \frac{1}{2} c_{\alpha}(r) (\Gamma^a R)_{\alpha\beta} c_{\beta}(r) \]

where $\Gamma^a$ are the $SO(5)$ Gamma matrices which takes the form

\[ \Gamma^1 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \hspace{0.5cm} \Gamma^4 = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, \hspace{0.5cm} \Gamma^5 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \] \hspace{2cm} (19)

satisfying the Clifford algebra $\{ \Gamma^a, \Gamma^b \} = 2\delta^{ab}$. The $SO(5)$ charge conjugate matrix $R$ is given by

\[ R = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \] \hspace{2cm} (20)

The quintet pairing structure is spanned by the five polar like operators $\chi^\dagger_{1~5}$, whose expectation value has a 5-vector and a phase structure as $d_\alpha e^{i\phi}$. The Goldstone manifold for the quintet pairing is

\[ R_{3/2} = \begin{pmatrix} SO(5)_s \otimes SO(3)_L \otimes U(1) \\ SO(4)_s \otimes SO(3)_L \otimes Z_2 \end{pmatrix} = S^4 \otimes U(1)/Z_2, \] \hspace{2cm} (21)

where the $Z_2$ symmetry comes from the combined operations $d_\alpha \rightarrow -d_\alpha, \phi \rightarrow \phi + \pi$.

Because the 4-component spinor forms the 7-dimensional sphere, similar to the spin 1/2 case, the adiabatic transportation of the quasi-particle around a half-quantum vortex in our spin-3/2 system defines a map

\[ S^7 \rightarrow S^4/Z_2, \] \hspace{2cm} (22)

which is homotopically equivalent to the second Hopf map i.e. $S^7 \rightarrow S^4$.

B. Berry connection, 2nd Chern number, $SO(4)$ monopole and Yang monopole

Let us introduce the spinor
\[ \Psi_k^\dagger = \left( c_{k,\uparrow}, c_{k,\downarrow}, c_{-k,\uparrow}, c_{-k,\downarrow}, c_{k,\uparrow}, c_{k,\downarrow}, c_{-k,\uparrow}, c_{-k,\downarrow} \right), \]  

where \( c_{k\alpha}^\dagger \) is the creation operator of an electron with the spin component \( \sigma \) and momentum \( k \). The mean field Hamiltonian can be written as

\[ H = \sum_k \Psi_k^\dagger H_k \Psi_k \]  

with

\[ H_k = \begin{pmatrix} \epsilon_k \Gamma^0 & \Delta_k \\ \Delta_k^\dagger & -\epsilon_k \Gamma^0 \end{pmatrix} \]  

where \( \epsilon_k \) is the kinetic energy on the lattices referenced from the Fermi surface. \( \Delta_k = -\Delta_k d_a \Gamma^a R \) while \( \Delta_k \) contains the magnitude and the phase of the superconducting (SC) order parameter. The momentum \( k \) is summed only over half of the Brillouin zone to avoid the double counting. The subscript \( a \) runs from 1 to 5. \( d_a \) forms a 4-dimensional sphere \( S^4 \). \( \Gamma^0 \) is the 4 \times 4 identity matrix, and \( \Gamma^a \) are given by Eq. (19). The eigenvalues of Eq. (25) are

\[ \lambda = \pm E_k = \pm \sqrt{\epsilon_k^2 + |\Delta_k|^2} \]  

and their corresponding eigenvectors are

\[ \psi_k^\dagger (k) = \frac{1}{\sqrt{(E_k + \epsilon_k)^2 + |\Delta_k|^2}} \begin{pmatrix} (E_k + \epsilon_k) |\alpha\rangle \\ \Delta_k^\dagger |\alpha\rangle \end{pmatrix} \]  

\[ \psi_k (k) = \frac{1}{\sqrt{(E_k + \epsilon_k)^2 + |\Delta_k|^2}} \begin{pmatrix} \Delta_k |\alpha\rangle \\ (E_k + \epsilon_k) |\alpha\rangle \end{pmatrix} \]  

where \( |\alpha\rangle \) are \( SU(4) \) spinors.

We are interested in the system with the presence of half-quantum vortices. The formation of this kind of vortices is very similar to the ones in the spin-1/2 system. If we transport the spin-3/2 fermion adiabatically around one of them, a nontrivial phase is accumulated due to the \( \delta \) field. The BPC and the covariant field strength can be obtained respectively by

\[ A_a = \frac{i}{d^2} d_c \Gamma^{ca}, \quad a = 1, 2, 3, 4, 5 \]  

and

\[ F_{abc} = -\frac{i}{d^2} (d_a \Gamma^{bc} + d_b \Gamma^{ca} + d_c \Gamma^{ab}) \]  

where \( \Gamma^{ab} = \frac{1}{4} [\Gamma^a, \Gamma^b] \) making up of the \( SO(5) \) generators. Similar non-abelian gauge structures also appeared in the pseudoparticle field in high dimensions \( 28,29,30 \). Instead of the first Chern number, we have the non-vanishing second Chern number.

\[ C_2 = -\frac{1}{96 \pi^2} \oint d\Omega_4 \text{Tr} (F_{abc} F_{abc}) = 1. \]  

where \( d\Omega_4 \) denotes the integration over the angular part of \( d_a \). The field strength on \( S^4 \), \( f_{ab} = [D_a, D_b] = \partial_a A_b - \partial_b A_a + [A_a, A_b] \), can be obtained

\[ f_{ab} = -i \frac{1}{d^2} P_{ab,cd} \Gamma^{cd} \]  

where \( P_{ab,cd} = \frac{1}{2} (\delta_{ad} \delta_{bc} - \delta_{bd} \delta_{ac} + \delta_{ad} \delta_{bc} - \delta_{bd} \delta_{ac}) \). \( P_{ab,cd} \) is a 10 \times 10 matrix because \( a \) and \( b \) are anti-symmetric as well as \( c \) and \( d \). Similar to the projection operator \( \delta_{ab} - q \mu \gamma_\mu \delta_{ab} \) in QED, \( P_{ab,cd} \) is the transverse projection operator from 10-dimensional space to 6-dimensional space satisfying \( d_a P_{ab,cd} = 0 \). The relation between \( F_{abc} \) and ordinary field strength \( f_{ab} \) can be revealed if we define \( G_{ab} \) which is dual to \( F_{abc} \) by \( G_{ab} = \epsilon_{abde} F_{a e d c} \). Then, one can show

\[ f_{ab} = \frac{1}{2} \epsilon_{abcdef} d_c |G_{de}| \]  

Because of the projection operator \( P_{ab,cd} \) in \( f_{ab} \), the fundamental degrees of freedom of the gauge structure in this problem is not \( SO(5) \) but \( SO(4) \), because \( SO(4) \) has six generators. We shall show that it is able to make a route from \( f_{ab} \) to the \( SO(4) \) gauge field strength using a singular gauge transformation. Similar to the analysis in the second section, the transformation operator in this \( SO(5) \) case has the following form

\[ U = \exp \left[ -\frac{i}{\sqrt{d^2 - d_5^2}} \Sigma_{\mu \nu} d_{\mu} \right], \quad \mu = 1, 2, 3, 4 \]  

where \( \cos \vartheta = d_5/d \). Then, Eq. (20) becomes

\[ a_\mu = \frac{-i}{d(d + d_5)} \Sigma_{\mu \nu} d_\nu, \quad \mu = 1, 2, 3, 4 \]  

\[ a_5 = 0, \]  

where \( \Sigma_{\mu \nu} \) are the \( SO(4) \) generators in the \( (1, 0) \oplus (0, 1) \) representation which have the following form

\[ \Sigma_{\mu \nu} = \begin{pmatrix} \eta^i_{\mu \nu} \sigma_i^0 & 0 \\ 0 & \eta^i_{\mu \nu} \sigma_i^0 \end{pmatrix} \]  

where \( \eta^i_{\mu \nu} = \epsilon_{i\mu\nu4} + \delta_{i\mu} \delta_{4\nu} - \delta_{i\nu} \delta_{4\mu} \) is the t'Hooft symbol, \( \mu \) and \( \nu \) runs from 1 to 4. In this reducible representation.
of SO(4) gauge group, one can easily distillate the SU(2) ingredients because SO(4) = SU(2) ⊗ SU(2). The self-dual SU(2) gauge field is given by

\[ a_\mu = -\frac{i}{d(d+d_5)} \eta_{\mu\nu} \sigma_i \frac{\sigma_i}{2} \mu = 1, 2, 3, 4 \]
\[ a_5 = 0. \]  

(38)

Similar to the spin-1/2 system, we obtain the SO(4) BPC which is only defined on the northern hemisphere with the equator. The singularity string along the negative z-axis inherits from the singular gauge transformation. The role of the singular gauge transformation by U can be also interpreted as the rotation of a 5-dimensional vector from an arbitrary direction \( \hat{d}_a \) to \( \hat{d}_5 \) in the 5-dimensional isospin space. Therefore, the invariant subgroup is the isometry group of the equator \( S^3/Z_3 \), which is SO(4). Surprisingly, the representation we achieve in SO(4) gauge theory is the reducible \( (\frac{5}{2}, 0) \oplus (0, \frac{5}{2}) \), which is the direct sum of two SU(2) gauge theory. Thus, the SU(2) Berry phase naturally arises in this system.

This SU(2) nature of the Berry phase in the spin-3/2 system is manifested if we choose a special spinor |\( \alpha \rangle \) such that \( \Delta_\mu |\alpha \rangle = \Delta_\mu |e^{i\psi}\alpha \rangle \), which is studied by Demler and Zhang. In this representation, |\( \alpha \rangle \) is not only a SU(4) spinors but also a SO(5) one. Then, the BPC is given by

\[ A_{\pm}^{\pm} = \langle \psi_{\alpha}^{\pm} | \partial_\mu | \psi_{\beta}^{\pm} \rangle = \langle \alpha | \partial_\mu | \beta \rangle. \]  

(39)

This is exactly the SU(2) holonomy in the context of Demler and Zhang. The special choice of the spinors |\( \alpha \rangle \) is equivalent to fixing \( \hat{d}_a = \hat{d}_5 \) in our notion.

IV. CONCLUSION AND DISCUSSION

In summary, we found that the SU(2) non-Abelian Berry phase emerges naturally in quintet condensates of spin 3/2 fermions. The underlying algebraic structures for the \( ^3He \) and the spin-3/2 system are the \( 1^\text{st} \) and the \( 2^\text{nd} \) Hopf map respectively. The Chern numbers for both cases were obtained in a standard manner. In the previous case, only the first Chern number is non-vanishing, while in the later case, only second Chern number is nonzero. Both systems appear to have finite gauge potential, which means that the BPC can be defined covariantly everywhere over the d-field. However, the corresponding U(1) and SU(2) gauge connections can only be defined patch by patch in the d-space. The bridge across between the finite gauge connection and the one with singularity is constructed by the singular gauge transformation, which can only be defined patch by patch in the d-space as well. The singular gauge transformation also bear with some physical meaning. It can be understood as the rotation in the spin (isospin) space, namely \( \hat{d}_a \). When the spin (isospin) points to the north pole, the invariant subgroup becomes the isometry group of the equator. When it is rotated by the gauge transformation, the gauge structure becomes finite and covariant over the whole d space.

To experimentally manifest this topological effect, the spin-3/2 ultra-cold atomic systems may serve as a promising candidate. It may also shed some light on measuring the second Chern number, which has not be revealed by any system so far. Furthermore, our calculation can also be generalized to consider the spin-\( \frac{5}{2} \) superconductors in which the algebraic structure is suggested to be the third Hopf map.22

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