An extension of Ramo’s theorem for signals in silicon sensors

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Abstract

We discuss a further extension of Ramo’s theorem that allows the calculation of signals in detectors that contain non-linear materials of arbitrary permittivity and finite conductivity (volume resistivity) as well as a static spacecharge. The readout-electrodes can be connected by an arbitrary impedance network. This extension is needed for the treatment of semiconductor sensors where the finite volume resistivity in the sensitive detector volume cannot be neglected. The signals are calculated by means of time dependent weighting fields. An explicit example is given and it is shown that this theorem has a very practical application when using semiconductor device simulation programs.

1. Introduction

The currents induced on grounded electrodes by moving charges can be calculated with static weighting fields using the Shockley-Ramo theorem [1][2]. The extension of the theorem for the presence of a static space-charge in silicon sensors is treated in [3]. In case the electrodes are not grounded but connected with linear impedance elements, the voltages and currents can be calculated by time dependent weighting fields as shown in [4] or by application of an equivalent circuit diagram as shown in [5]. The presence of dielectric and nonlinear media in the detector is treated in [6][7]. The case where the volume between the electrodes contains conductive material in addition is treated in [8][9]. In this report we present another extension of the theorem where we consider static spacecharge, arbitrary permittivity and finite conductivity of a non-linear detector medium and a connection of the electrodes with arbitrary impedance elements. This extension is needed for semiconductor sensors that are only partially depleted or where in general the finite volume resistivity cannot be neglected. In this case the signals can be calculated by means of time dependent weighting fields.

The static drift fields and time dependent weighting fields can be calculated with TCAD device simulation programs, and the transport of charges and calculation of the induced signal can then be performed with programs like Garfield [10][11] by convoluting the velocity vectors of the moving charges with the time dependent weighting fields. We will first discuss the theorems and then give an example of a partially depleted silicon sensor.

2. Theorems

In [9] it is shown how to calculate signals on electrodes that are connected by arbitrary impedance elements and that are embedded in a medium with position and frequency dependent permittivity and conductivity. The signals are calculated by time dependent weighting fields, and an equivalent circuit representing the medium can be used to relate the induced voltages to currents induced on the electrodes in case they are grounded. A constant static spacecharge in the detector volume is only affecting the electric field that causes the drift of the charges, but not the weighting fields, as shown in [3], so the theorems also hold for this case. The weighting fields are therefore defined by removing the static spacecharge, removing the charges produced by the passage of the particle and placing a current pulse
Figure 1: The electric field $\vec{E}_D(\vec{x})$ in the detector volume that is responsible for the movement of the charge. The Laplace parameter $s = i\omega$ represents the frequency dependence of the material properties and the impedance elements.

Figure 2: a) Two point charges $q, -q$ placed in the detector at $t = 0$. Their movement induces voltages $V_n(t)$ on the electrodes. b) The electric field $\vec{K}_{D1}(\vec{x}, t)$ due to placing a charge $Q(t) = Q_0 \Theta(t)$ on electrode 1. This is equivalent to placing a current $I(t) = Q_0 \delta(t)$ on the electrode.

(for calculation of the induced voltage) or voltage pulse (for calculation of the induced current) on the electrode in question. The resulting time dependent weighting fields are then convoluted with the velocity vector of the charges (Eq. 14 and Eq. 18 of [9]).

From linear superposition it follows that this same weighting field can be calculated by leaving the system with the applied static voltages and the static spacecharge, adding the current or voltage pulse to the electrode in question and calculating the difference of the resulting field from the static field, as discussed in [7]. This is particularly useful for nonlinear media where the material properties and spacecharge depend on the bias voltages. By applying an ‘infinitesimal’ current or voltage pulse to the electrode in question one can ‘linearize’ the system around this point to find the weighting field. The calculation of the signals then applies to situations where the electric fields from the moving charges are small enough in order not to affect the material properties. From these considerations the theorems follow directly and a formulated in the following.

In Fig. 1 we see a system of metal electrodes set to voltages $V_n$ and connected with an arbitrary impedance network. The applied voltages together with the detector material cause a static electric field $\vec{E}_D(\vec{x})$. 


a static spacecharge distribution $\rho_0(x)$ and a position and frequency dependent permittivity $\varepsilon(x, s)$ and volume resistivity $r(x, s)$ (or conductivity $\sigma(x, s) = 1/r(x, s)$). In these expressions, $s$ represents the Laplace parameter $s = \omega i$ and defines the frequency dependence of the material properties and discrete impedances. A pair of charges $q, -q$ is created in the detector at $t = 0$ and these charges are moving in the electric field $E_D(x, t)$ along trajectories $\vec{x}_1(t)$ and $\vec{x}_2(t)$.

**Theorem 1:**

The voltages $V_n^{\text{ind}}(t)$ induced on the electrodes (Fig. 3a) can be calculated by

$$V_n^{\text{ind}}(t) = -\frac{q}{Q_2} \int_0^t K_n(\vec{x}_1(t'), t - t') \, d\vec{x}_1(t') \, dt'$$

$$+ \frac{q}{Q_2} \int_0^t K_n(\vec{x}_2(t'), t - t') \, d\vec{x}_2(t') \, dt'$$

where the weighting fields $K_n(\vec{x}, t)$ are defined the following way (Fig. 3b): the charges $q, -q$ are removed, an 'infinitesimal' charge $Q(t) = Q_n \Theta(t)$ is added to electrode $n$, which results in a field $K_{D_n}(\vec{x}, t)$, from which the weighting field $K_n(\vec{x}, t)$ is derived as

$$K_n(\vec{x}, t) = K_{D_n}(\vec{x}, t) - E_D(\vec{x})$$

(1)

The Heaviside step function $\Theta(t)$ is defined as $\Theta(t) = 0$ for $t < 0$ and $\Theta(t) = 1$ for $t > 0$. Since $I(t) = d(Q_n \Theta(t))/dt = Q_n \delta(t)$, the weighting field can be understood as being the result of a delta current on the electrode in question.

In case an electrode is grounded or kept at a fixed potential, as shown in Fig. 3b), the voltage on the electrode stays unchanged and there is no 'induced voltage'. Still, the movement of the charges induces a current that is flowing between the electrode and ground, which we calculate in the following. Let us assume that the electrode is connected to ground through a resistor $z_{11} = R$, that is very small compared to all other impedances in the circuit. The current flowing between the electrode and ground is then $I_1^{\text{ind}}(t) = V_1^{\text{ind}}(t)/R$. The charge $Q(t) = Q_n \Theta(t)$ that defines the weighting field will on the other hand result in a voltage $V(t) = RQ_n \delta(t)$ on the electrode. In turn, placing the voltage $V(t) = V_n \delta(t)$ on the electrode will result in a charge $Q_n(t) = Q_n R \Theta(t)$ on the electrode. Inserting these expressions in the above equation and taking the limit of $R \to 0$ we have the second theorem:
Theorem 2:
The currents \( I_{\text{ind}}(t) \) induced on grounded electrodes (Fig. 3a) can be calculated by

\[
I_{\text{ind}}(t) = -\frac{q}{V_\varepsilon} \int_0^t \vec{E}_n[\vec{x}_1(t'), t - t'] \vec{x}_1(t') dt' + \frac{q}{V_\varepsilon} \int_0^t \vec{E}_n[\vec{x}_2(t'), t - t'] \vec{x}_2(t') dt'
\]

(3)

where the weighting fields \( E_n(\vec{x}, t) \) are defined the following way (Fig. 3b): the charges \( q, -q \) are removed, an 'infinitesimal' voltage \( V(t) = V_\varepsilon \delta(t) \) is added to electrode \( n \), which results in a field \( E_{Dn}(\vec{x}, t) \), from which the weighting field \( E_n(\vec{x}, t) \) is derived as

\[
\vec{E}_n(\vec{x}, t) = \vec{E}_{Dn}(\vec{x}, t) - \vec{E}_D(\vec{x})
\]

(4)

In practice one cannot apply a true delta pulse in a device simulation program, so one applies a 'short' pulse \( I_\varepsilon(t) \) or \( V_\varepsilon(t) \) of duration \( T \) and then replaces \( Q_\varepsilon \) and \( V_\varepsilon \) in Eq. 1 and Eq. 3 by \( \int I_\varepsilon(t) dt \) and \( \int V_\varepsilon(t) dt \). Since the typical 'time constant' of the detector medium is \( \tau \approx \varepsilon(x, s)/\sigma(x, s) \), the duration \( T \) has to be smaller than \( \tau \). For highly conductive materials where \( \tau \) approaches zero, \( T \) has to be smaller than the integration time of the readout electronics, or equivalently \( 1/T \) has to be larger than the bandwidth of the readout circuitry.

The first theorem allows the precise calculation of the signals with numeric device simulation programs when connecting the sensor electrodes to the correct biasing circuits and readout electronics. Since the detector 'sees' only the input impedance of an amplifier, it is most practical to connect an element representing this input impedance to the electrode, then calculate the induced voltage and use this induced voltage as a source term for the detailed readout circuit simulation in a dedicated analog circuit simulation program. The second theorem can be very useful when assuming the input impedance (resistance) of the readout electronics to be negligible with respect to the electrode impedances and the impedance of the biasing network. We discuss an example in the following.

3. Example

To illustrate the above theorems we discuss a silicon sensor described in Example 5.3 from [12] at an applied voltage \( V \) that is smaller than the depletion voltage \( V_{\text{dep}} \) (Fig. 4). An n-type bulk material with doping concentration \( N_D \) is equipped with a layer of highly doped p-type material. This p\(^+\) layer is connected to the supply voltage \(-V\) via a loading resistor \( R_L \) and the signal is read out from this layer by an amplifier with input resistance \( R_{\text{amp}} \) through a decoupling capacitor \( C_3 \). The other face of the sensor consists of a highly doped n\(^+\) layer that is connected to ground. The depletion voltage \( V_{\text{dep}} \) for the sensor and the thickness \( d_0 \) of the depleted layer for an applied voltage of \(-V\) are given by

\[
V_{\text{dep}} = \frac{qN_Dd^2}{2\varepsilon_1} \quad d_0 = d \sqrt{\frac{V}{V_{\text{dep}}}}
\]

(5)

where \( q \) is the elementary charge and \( \varepsilon_1 = \varepsilon_\varepsilon_0 \) is the dielectric permittivity of silicon. The static space charge density \( \rho_0 \) of the depleted layer and the volume resistivity \( \sigma \) of the un-depleted bulk layer are given by

\[
\rho_0 = qN_D = \frac{2V_{\text{dep}}\varepsilon_1}{d^2} \quad \frac{1}{\sigma} = \sigma = q\mu_nN_D
\]

(6)

where \( \mu_n \) is the electron mobility.
3.1. Movement of the charges

For the following calculation we assume the n\textsuperscript{+} and p\textsuperscript{+} layers to have infinite conductivity and assume the 'built in voltage' \( V_{bi} \) and the intrinsic thickness of the depletion layer to be negligible i.e. we assume the boundary between the depleted and un-depleted layer at \( z = d_0 \) to be 'sharp'. The drift field is defined by the potential \(-V\) at \( z = 0 \), a constant spacecharge \( \rho_0 \) in \( 0 < z < d_0 \) as well as zero potential at \( z = d_0 \), which gives

\[
E_D(z) = -\frac{2V}{d_0} \left( 1 - \frac{z}{d_0} \right) \quad z < d_0
\]  

(7)

The magnitude of the electric field decreases linearly from the value \( E_D = -2V/d_0 \) at \( z = 0 \) to zero on the interface to the un-depleted layer at \( z = d_0 \). The velocity of the electrons and holes is proportional to the electric field according to \( v = \mu E \), so the movement of a single electron and a single hole deposited at \( z = z_0 \) is defined by the following differential equations

\[
\frac{dz_e(t)}{dt} = -\mu_e E_D(z_e(t)) \quad \frac{dz_h(t)}{dt} = \mu_h E_D(z_h(t)) \quad z_e(0) = z_h(0) = z_0
\]  

(8)

with the solution

\[
z_e(t) = d_0 - (d_0 - z_0) e^{-t/\tau_e} \quad \tau_e = \frac{d^2}{2\mu_e V_{dep}} \quad 0 < t < \infty
\]  

(9)

\[
z_h(t) = d_0 - (d_0 - z_0) e^{t/\tau_h} \quad \tau_h = \frac{d^2}{2\mu_h V_{dep}} \quad 0 < t < t_h
\]  

(10)

The holes take the time \( t_h(z_0) = -\tau_h \ln \left( 1 - \frac{z_0}{d_0} \right) \) to arrive at \( z = 0 \), while the electrons take an infinite amount of time to arrive at \( z = d_0 \) since the electric field is zero at this position. The related velocities
are
\[ v_e(t) = \frac{dz_e(t)}{dt} = \frac{d_0 - z_0}{\tau_e} e^{-t/\tau_e} \]  
\[ v_h(t) = \frac{dz_h(t)}{dt} = -\frac{d_0 - z_0}{\tau_h} e^{t/\tau_h} \Theta(t_h - t) \] (11) (12)

3.2. Induced currents on the grounded electrode (Theorem 2)
As an illustration of Theorem 2 we calculate the induced current as outlined in Fig. 3. Adding the voltage pulse \( V_\varepsilon \delta(t) \) to the applied constant voltage \( -V \) defines the field \( E_{D1}(t) \) which in the time domain reads as

![Figure 5: Definition of the weighting field for calculation of the current induced on the grounded electrode.](image)

\[ E_{D1}(t,z) = \frac{V_\varepsilon}{d} \left( \delta(t) + \frac{d - d_0}{d_0} \frac{1}{\tau} e^{-t/\tau} \right) - \frac{2V}{d_0} \left( 1 - \frac{z_0}{d_0} \right) \tau = \frac{\varepsilon_1 d}{d_0 \sigma} \; z < d_0 \] (13)

The weighting field \( E_1(t) \) is then given by

\[ E_1(t) = E_{D1}(z,t) - E_D(z) = \frac{V_\varepsilon}{d} \left( \delta(t) + \frac{d - d_0}{d_0} \frac{1}{\tau} e^{-t/\tau} \right) \] (14)

The current induced by the electrons and holes is then

\[ i_e(t) = \frac{q}{V_\varepsilon} \int_0^t E_1(t - t') v_e(t') dt' \]

\[ i_h(t) = -\frac{q}{V_\varepsilon} \int_0^t E_1(t - t') v_h(t') dt' \; \; t < t_h \] (15)

\[ = -\frac{q}{V_\varepsilon} \int_0^{t_h} E_1(t - t') v_h(t') dt' \; \; t > t_h \]

which evaluates to

\[ i_e(t, z_0) = q \frac{d_0 - z_0}{d} \left[ \frac{d - d_0}{d_0(\tau - \tau_e)} \left( e^{-t/\tau} - e^{-t/\tau_e} \right) + \frac{1}{\tau_e} e^{-t/\tau_e} \right] \] (16)

\[ i_h(t, z_0) = q \frac{d_0 - z_0}{d} \left[ \frac{d - d_0}{d_0(\tau + \tau_h)} \left( e^{t/\tau_h} - e^{-t/\tau} \right) + \frac{1}{\tau_h} e^{t/\tau_h} \right] \; \; t < t_h \] (17)

\[ = q \frac{(d - d_0)(d_0 - z_0)}{d_0 d(\tau + \tau_h)} \left( e^{t_h(z_0)/\tau + t_h(z_0)/\tau_h} - 1 \right) e^{-t/\tau} \; \; t > t_h \] (18)
The charge induced by the electrons and the holes is given by

\[ Q_e = \int_0^\infty i_e(t)dt = q \left( 1 - \frac{z_0}{d_0} \right) \quad Q_h = \int_0^\infty i_h(t)dt = q \frac{z_0}{d_0} \]  (19)

so the total induced charge is \( Q_{\text{tot}} = Q_e + Q_h = q \), as expected. In the limit of very small values of \( \tau \), where the un-depleted layer acts like a perfect conductor, or very large values of \( \tau \), where the un-depleted layer acts like a perfect insulator, the expressions for the induced currents are

\[ \lim_{\tau \to 0} i_e(t) = q \frac{d_0 - z_0}{d_0 \tau_e} e^{-t/\tau_e} \quad \lim_{\tau \to 0} i_h(t) = q \frac{d_0 - z_0}{d_0 \tau_h} e^{t/\tau_h} \Theta(t_h - t) \]  (20)

\[ \lim_{\tau \to \infty} i_e(t) = q \frac{d_0 - z_0}{d_0 \tau_e} e^{-t/\tau_e} \quad \lim_{\tau \to \infty} i_h(t) = q \frac{d_0 - z_0}{d_0 \tau_h} e^{t/\tau_h} \Theta(t_h - t) \]  (21)

As an example we discuss a sensor with \( d = 300 \mu m, N_D = 8.3 \times 10^{11} \text{cm}^{-3} \) at a voltage of 25.2 V and a single e-h pair placed at \( z_0 = 150 \mu m \). With the silicon parameters \( \varepsilon_1 = 11.8 \varepsilon_0, \mu_e = 500 \text{cm}^2/\text{Vs}, \mu_h = 1500 \text{cm}^2/\text{Vs} \) we have \( 1/\sigma = r = 5 \text{kJ}\Omega \text{cm}, V_{\text{dep}} = 56.8 \text{V}, d_0 = 200 \mu m \). The time constants evaluate to \( \tau = 7.9 \text{ns}, \tau_e = 5.3 \text{ns}, \tau_h = 15.8 \text{ns} \) and the signals are shown in Fig. 6.

Figure 6: Induced currents from a) a single electron b) a single hole c) a single electron-hole pair, for a sensor of 300 \( \mu m \) thickness with a depletion voltage of 56.8 V. The applied voltage is \( V=-25.2 \text{V} \) resulting in a depleted region of \( d_0 = 200 \mu m \) thickness. The e-h pair is deposited at \( z_0 = 150 \mu m \) at \( t = 0 \). The dotted line assumes zero volume resistivity of the un-depleted layer and the dashed line assumes infinite volume resistivity.

Next we consider a continuous charge deposit approximating the ionization of a charged particle crossing
the silicon sensor. We assume a linear charge density of $\lambda$ [C/cm], so we have to replace $q$ by $dq = \lambda dz_0$ and integrate over $0 < z_0 < d_0$

$$I_e(t) = \int_0^{d_0} \frac{\lambda}{q} i_e(t, z_0) dz_0 = \frac{d_0 \lambda}{2d(\tau - \tau_e)} \left[(d - d_0)e^{-t/\tau} + (d_0 \tau - d_\tau_e \frac{1}{\tau_e} e^{-t/\tau_e})\right]$$

$$I_h(t) = \int_0^{d_0} \frac{\lambda}{q} i_h(t, z_0) dz_0 = \frac{d_0 \lambda}{2d(\tau - \tau_h)} \left[(d - d_0)e^{-t/\tau} + (d_0 \tau - d_\tau_h \frac{1}{\tau_h} e^{-t/\tau_h})\right]$$

The total induced charge is then

$$Q_{\text{tot}}^{\text{ind}} = \int_0^{\infty} [I_e(t) + I_h(t)] dt = \frac{\lambda d_0}{2} + \frac{\lambda d_0}{2} = \lambda d_0$$

as required. In the limit of very small values of $\tau$ or very large values of $\tau$ the expressions for the induced currents become

$$\lim_{\tau \to 0} I_e(t) = \frac{d_0 \lambda}{2\tau_e} e^{-t/\tau_e} \quad \lim_{\tau \to 0} I_h(t) = \frac{d_0 \lambda}{2\tau_h} e^{-t/\tau_h}$$

$$\lim_{\tau \to \infty} I_e(t) = \frac{d_0^2 \lambda}{2d \tau_e^2} e^{-t/\tau_e} \quad \lim_{\tau \to \infty} I_h(t) = \frac{d_0^2 \lambda}{2d \tau_h^2} e^{-t/\tau_h}$$

As an example we use a most probable number of 14200 e-h pairs in 200 $\mu$m of silicon i.e. $\lambda = 14200q/0.02$ C/cm, and the results are shown in Fig. 7.

The entire calculation from above can actually also be performed in the Laplace domain. The convolutions in Eq. [15] are simple products when working with the Laplace transforms of $E_1(t), v_e(t), v_h(t)$, so we have

$$E_1(s) = V_e \frac{(d + sd_0 \tau)}{d_0 d(1 + s \tau)} \quad v_e(s) = \frac{d_0 - z_0}{1 + s \tau_e} \quad v_h(s) = \frac{d_0 - z_0}{1 - s \tau_h} (1 - e^{(1/\tau_h - s)t_h})$$

$$i_e(s, z_0) = \frac{q}{V_w} E_1(s) v_e(s) = \frac{q(d + sd_0 \tau)}{d_0 d(1 + s \tau)} \frac{d_0 - z_0}{1 + s \tau_e}$$

$$i_h(s, z_0) = -\frac{q}{V_w} E_1(s) v_h(s) = -\frac{q(d + sd_0 \tau)}{d_0 d(1 + s \tau)} \frac{d_0 - z_0}{1 - s \tau_h} (1 - e^{(1/\tau_h - s)t_h(z_0)})$$

Integrating over $z_0$ for the continuous charge deposit we have

$$I_e(s) = \int_0^{d_0} \frac{\lambda}{q} i_e(s, z_0) dz_0 = \frac{d_0 \lambda(d + sd_0 \tau)}{2d(1 + s \tau)(1 + s \tau_e)}$$

$$I_h(s) = \int_0^{d_0} \frac{\lambda}{q} i_h(s, z_0) dz_0 = \frac{d_0 \lambda(d + sd_0 \tau)}{2d(1 + s \tau)(1 + s \tau_h)}$$

The expression $i_e(s, z_0), i_h(s, z_0), I_e(s), I_h(s)$ are the correct Laplace transforms of Eqs. [16] [17] [18] [22]

3.3. Induced voltage on the electrode connected by impedance elements (Theorem 1)

To illustrate Theorem 1 we calculate the induced voltage on the electrode when connected to the readout circuit and biasing network. To find the weighting field $K_1(z, t)$ we have to place a charge $Q(t) = Q_e \Theta(t)$ or equivalently a current $I(t) = Q_e \delta(t)$ on the electrode, as indicated in Fig. 8. As shown in [3] the
medium can be represented by an equivalent circuit. According to Eqs. 5 and 6 of \cite{9} the admittance ‘matrix’ of the electrode is given by
\begin{equation}
    y_{11}(s) = \frac{s}{V_w} \varepsilon_1 E_1(s) A = \frac{\varepsilon_1 s (1 + \varepsilon_1 \rho s)}{d_0 + d \varepsilon_1 \rho s} A
\end{equation}
and the related impedance is given by
\begin{equation}
    z_{11}(s) = \frac{1}{y_{11}(s)} = \frac{d_0 + d \varepsilon_1 \rho s}{\varepsilon_1 s (1 + \varepsilon_1 \rho s)}
\end{equation}
The equivalent circuit of the detector is therefore equal to a capacitor $C_1$ representing the depleted layer and a capacitor $C_2$ in parallel with a resistor $R$ representing the un-depleted layer (Fig.\[\text{8b}\]):
\begin{equation}
    C_1 = \varepsilon_1 \frac{A}{d_0} \quad C_2 = \varepsilon_1 \frac{A}{d - d_0} \quad R = \rho \frac{d - d_0}{A} \quad z_{11}(s) = \frac{1}{sC_1} + \frac{R/(sC_2)}{R + 1/(sC_2)}
\end{equation}
Connecting this equivalent circuit to the discrete readout elements, as shown in Fig.\[\text{8b}\] we can calculate the weighting field $K_1(t)$ by placing a current pulse $I(t) = Q_\varepsilon \delta(t)$ i.e. $I(s) = Q_\varepsilon$ on the electrode. The impedance $Z_2$ of the network connected to the electrode is (Fig.\[\text{8b}\])
\begin{equation}
    Z_2(s) = \frac{R_L (1 + sC_3 R_{amp})}{1 + sC_3 (R_L + R_{amp})}
\end{equation}
The potential \( V(s) \) of the electrode and the electric field \( K_1(s) = \Delta V/d_0 \) inside the capacitor i.e in the depleted region, are then

\[
V(s) = Q \frac{z_{11}(s)Z_2(s)}{z_{11}(s) + Z_2(s)}, \quad K_1(s) = \frac{Q}{d_0 sC_1} \frac{Z_2(s)}{z_{11}(s) + Z_2(s)}
\]  

(36)

This is the required weighting field \( K_1(s) \), and the voltages induced by the electrons and holes for the setup in Fig. 4 are finally given by

\[
u_e(s, z_0) = \frac{q}{Q_e} K_1(s) v_e(s, z_0) \quad u_h(s, z_0) = -\frac{q}{Q_e} K_1(s) v_h(s, z_0)
\]  

(37)

From [9] we know that this voltage can also be calculated by placing the currents \( i_e(s) \) and \( i_h(s) \) on the equivalent circuit, giving

\[
u_e(s, z_0) = \frac{Z_2(s)}{z_{11}(s) + Z_2(s)} z_{11}(s)i_e(s, z_0) \quad u_h(s, z_0) = \frac{Z_2(s)}{z_{11}(s) + Z_2(s)} z_{11}(s)i_h(s, z_0)
\]  

(38)

Inserting the expressions from Eq. 28 and 29 shows that this is equivalent to the above expressions. The induced voltage for the uniform charge deposit is then

\[
V(s) = \frac{z_{11}(s)Z_2(s)}{z_{11}(s) + Z_2(s)} [I_e(s) + I_h(s)]
\]  

(39)

Assuming \( R_L \) to be chosen large enough to be negligible the current into the amplifier is

\[
I_{\text{amp}}(s) \approx \frac{z_{11}(s)}{z_{11}(s) + Z_2(s)} [I_e(s) + I_h(s)]
\]  

(40)

As an example we use a sensor area of \( A = 1 \text{ cm}^2 \), which gives \( R = 50 \Omega, C_1 = 52 \text{ pF}, C_2 = 105 \text{ pF} \). For an amplifier input resistance \( R_{\text{amp}} = 50 \Omega \) and a coupling capacitor of \( C_3 = 100 \text{ pF} \) we get the result shown in Fig. 9.

4. Summary

A generalization of Ramo’s theorem that allows the calculation of signals in detectors that contain non-linear materials of arbitrary permittivity and finite conductivity (volume resistivity), static spacecharge

Figure 8: a) Definition of the field \( K_{D1}(z, t) \) b) Equivalent circuit for calculation of the weighting field \( K_1(t) \).
Figure 9: Current induced on the grounded electrode (dashed line) and the current flowing through the amplifier (solid line) for the geometry from Fig. 4 for $R = 50 \Omega, C_1 = 52 \text{pF}, C_2 = 105 \text{pF}, R_{amp} = 50 \Omega$ and $C_3 = 100 \text{pF}$.

as well as readout-electrodes connected by an arbitrary impedance network, was given. The induced voltages and currents can be calculated by convolution of the velocity vectors of the moving charges with time dependent weighting fields. These time dependent weighting fields are defined by removing these charges and adding an infinitesimal voltage or current pulse to the electrode in question, while leaving all other applied potentials an impedance elements in place. The difference of these time dependent fields and the static drift field define the weighting fields.

This theorem is very well suited for calculation of signals with numeric device simulation programs, specifically for silicon sensors with un-depleted or partially depleted regions. One can connect the biasing networks and impedances of the readout circuits and find the time dependent weighting field with numeric simulation programs. The statistics of primary ionization and the drift and diffusion of the charges can then be calculated with a separate dedicated program and the movements of these charges are then simply convoluted with these weighting fields.

An analytic example for a simple silicon sensor, that allows benchmarking the use of device simulation programs, was given.

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5. Bibliography

References

[1] S. Ramo, Currents induced in electron motion, PROC. IRE 27, 584 (1939)
[2] W. Shockley, "Currents to Conductors Induced by a Moving Point Charge". Journal of Applied Physics. 9 (10): 635 (1938)
[3] G. Cavalleri, E. Gatti et al., Extension of Ramo’s theorem as applied to induced charge in semiconductor detectors, NIM 92 (1971) 137-140
[4] E. Gatti, G. Padovini and V. Radeka, Signal evaluation in multielectrode radiation detectors by means of a time dependent weighting vector, NIM 193 (1982) 651-653
[5] W. Blum, W. Riegler, L. Rolandi, Particle detection with drift chambers; 2nd ed., Springer (2008)
[6] L.A. Hamel, M. Julien, Comments on Ramo’s theorem, in: Ralph B. James (Ed.), Hard X-Ray and Gamma-Ray Detector Physics III, Proceedings of SPIE vol. 4507, 2001, pp. 255-263.
[7] L. Hamel, M. Julien, Generalized demonstration of Ramo’s theorem with space charge and polarization effects, NIMA 597 (2008), 207-211

[8] W. Riegler, Induced signals in resistive plate chambers, NIMA 491 (2002) 258-217

[9] W. Riegler, Extended theorems for signal induction in particle detectors, NIMA 535 (2004) 287-293

[10] H. Schindler, Microscopic Simulation of Particle Detectors, CERN thesis (2012) https://cds.cern.ch/record/1500583

[11] H. Schindler, R. Veenhof et al., Garfield++ simulation of tracking detectors, https://garfieldpp.web.cern.ch/garfieldpp/

[12] G. Lutz, Semiconductor Radiation Detectors, Springer, 1999