GEOMETRISING THE CLOSED STRING FIELD THEORY ACTION

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ABSTRACT

We complete the set of string vertices of non-negative dimension by introducing in a consistent manner those moduli spaces which had previously been excluded. As a consequence we obtain a ‘geometrised’ string action taking the simple form $S = f(B)$ where ‘$B$’ is the sum of the string vertices. That the action satisfies the B-V master equation follows from the recursion relations for the string vertices which take the form of a ‘geometrical’ quantum master equation.

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1. Introduction and Summary

One hopes eventually to be able to reformulate closed string field theory in a form which is both simplified and which brings out the deeply geometrical underlying basis. There were two main hurdles which needed to be overcome before such a goal could be realised. The first of these was the need to find a geometrical description of the usual string field theory operators $\partial$, $K$ and $I$. The second was the need to complete the set of string vertices $B_{g,n}^{\bar{n}}$ by both introducing those vertices of non-negative dimension which had for various reasons previously been excluded, and extending the set to include the vertices of ‘negative’ dimension $\dagger$.

In an earlier paper [1], some advance was made in this direction. It was shown there that it was consistent to express all three operators, $\partial$, $K$ and $I$ as inner derivations on the B-V algebra of string vertices. Indeed the following identifications were derived,

$$\begin{align*}
\partial A &= \{L, A\} \\
K A &= -\{A, R\} \\
I A &= \{A, l\},
\end{align*}$$

where $L$, $R$ and $l$ are graded-even elements of the B-V algebra satisfying,

$$\begin{align*}
\{L, L\} &= \{L, l\} = \{l, l\} = \{R, R\} = 0.
\end{align*}$$

(1.1)

The recursion relations for the string vertices were then given by a ‘geometrical’ quantum B-V master equation,

$$\frac{1}{2}\{B, B\} + \Delta B = 0,$$

(1.4)

while the action took the form,

$$S = S_{1,0} + f(B).$$

(1.5)

While this signalled firm progress, it was not yet totally satisfactory seeing as $B = \sum_{g,n,\bar{n}} B_{g,n}^{\bar{n}}$ still excluded the non-negative dimensional vertices $B_{1,0}^{1}$, $B_{1,0}^{0}$ and $B_{0,0}^{\bar{n}}$ (where $n \geq 2$). The

$\dagger$ Recall that the dimension of $B_{g,n}^{\bar{n}}$ is $6g - 6 + 2n + 3\bar{n}$. 

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main goal of the current paper will be to include these spaces in a consistent manner ensuring that previous results (particularly quantum background independence [2], the ghost-dilaton theorem [3,4], and the recursion relations [5]), are still satisfied.

The structure of our paper is as follows.

In §2.1 we review the original reasons for excluding the spaces \(B_{1,0}^0, B_{1,0}^1\), and explain why the existence of the space \(B_{0,2}^0\) (which was introduced in [1]) makes it consistent to reintroduce them. As a corollary we also see how a choice of \(B_{0,2}^0\) determines the choice of connection. In §2.2 we briefly explain why it is consistent to introduce the moduli spaces \(B_{0,0}^{\bar{n}}\) and thereby complete the set of string vertices of non-negative dimension.

Armed with the extended \(B\)-complex we then state in §2.3 the expression for the quantum action around arbitrary string backgrounds, generalising the results of [6]. It takes a completely geometrical form expressed solely as a function of the sum of string vertices, \(S = f(B)\). Furthermore, the recursion relations are contained in a quantum master action for the \(B\)-spaces. As an application of these we end by listing expressions for the boundaries of the newly introduced moduli spaces.

In the conclusion in §4 we review what has been achieved. Unless explicitly stated otherwise, we will use units in which \(\hbar = \kappa = 1\) throughout.

2. String Vertices and the Geometrised Action

In this section, we will extend the \(B\)-space complex by introducing the positive-dimensional vertices \(B_{1,0}^0\) (being a moduli space of tori with a single special puncture), \(B_{1,0}^1\) (a space of unpunctured tori), and \(B_{0,0}^{\bar{n}}\) where \(n \geq 2\) (a moduli spaces of genus zero surfaces with no ordinary punctures and at least two special punctures).

2.1. The Moduli Spaces \(B_{1,0}^0\) and \(B_{1,0}^1\)

All of the spaces mentioned above are distinguished by the fact that they are moduli spaces of surfaces containing no ordinary punctures. This means that they cannot be sewn, and hence can take an active part neither in background deformations (in particular they should not affect the proofs of background independence or the ghost-dilaton theorem), nor in the recursion relations, so one would expect that there should be no problem in including them into the \(B\)-space complex. Let us then reconsider the reason for leaving out these spaces,
and readmit them if we find a satisfactory excuse for doing so. We first discuss briefly why
the space $B_{1,0}^1$ was excluded.

When the spaces $B_{g,n}^1$ were first introduced for the proof of quantum background indepen-
dence, they were defined only for $n \geq 2$ at genus zero and for $n \geq 1$ at higher genus (Eqn.(4.19)
of [2]), the remaining spaces $B_{g,0}^1$ (for all genera $g \geq 1$) being set to vanish as they could not
take part in background deformations and were therefore irrelevant to the discussion.

However, their irrelevance to background independence was not a reason for dismissing
them entirely, and this was acknowledged in proving the ghost-dilaton theorem [3,4], where
all these spaces were reinstated, with the exception of the space $B_{1,0}^1$. The reason for this
omission was that the general equation derived in [3] implied that its b oundary was given by
$$\partial B_{1,0}^1 = -\Delta B_{1,2}^0 - iB_{1,1}^0,$$
which was inconsistent as it did not satisfy $\partial \partial B_{1,0}^1 = 0$.

This problem disappears if we assume the existence of the one loop va cuum vertex $B_{1,0}^0$
satisfying the recursion relations Eqn.(1.4) as we can then show that $\partial \partial B_{1,0}^1 = 0$ is in fact
satisfied. In particular, we find the following,
$$\partial B_{1,0}^1 = \kappa B_{1,0}^0 - iB_{1,1}^0 - \Delta B_{0,2}^1. \quad (2.1)$$
Taking the boundary once again and using the usual operator identities,
$$\partial \partial B_{1,0}^1 = \partial \kappa B_{1,0}^0 - \partial iB_{1,1}^0 - \partial \Delta B_{0,2}^1
\quad = \kappa \partial B_{1,0}^0 - i \partial B_{1,1}^0 + \Delta \partial B_{0,2}^1
\quad = -\kappa \Delta B_{0,2}^0 + i \Delta B_{0,3}^0 + \Delta \kappa B_{0,2}^0 - \Delta i B_{0,3}^0
\quad = 0. \quad (2.2)$$
where we have applied the recursion relations to $B_{0}^0$ and $B_{0,2}^1$ to obtain the expressions $\partial B_{1,0}^0 =
-\Delta B_{0,2}^0$ and $\partial B_{0,2}^1 = \kappa B_{0,2}^0 - iB_{0,3}^0$ for their boundaries. The term $\kappa B_{0,2}^0$ above should be
identified with $\tilde{V}_{0,3}$ of §6.2 of [2], the usual $V_{0,3}'$ being a special case. We shall discuss this in
more detail momentarily.

Having assumed the existence of $B_{1,0}^0$ satisfying the recursion relations, let us now see
further evidence to suggest why this is consistent. In particular, we note that the proof
of quantum background independence §6 of [2] for the field-independent $O(\hbar)$ terms was
particularly tricky, and left us with the unusual condition $\partial B_{1,0}^0 = -\pi F \Delta \tilde{V}_{0,3}$ on the boundary
of $B_{1,0}^0$. We will now show how our expectations for the one loop vacuum vertex do away with
these unwanted features.
We recall the field-independent $O(h)$ condition for background independence Eqn.(6.9) of [2],

$$\partial_\mu S_{1,0} = \int_{\mathcal{B}^0_{1,1} + \Delta \mathcal{B}^0_{0,2}} \langle \Omega^{(0)1,1} | \hat{O}_\mu \rangle ,$$  

(2.3)

Having assumed the existence of the vertex $\mathcal{V}_{1,0}$ we can now write this as,

$$\partial_\mu S_{1,0} = \partial_\mu f(\mathcal{V}_{1,0}) = f_\mu(K \mathcal{V}_{1,0}) = \int_{K \mathcal{V}_{1,0}} \langle \Omega^{(0)1,1} | \hat{O}_\mu \rangle .$$  

(2.4)

This allows us to rewrite the condition Eqn.(2.3) as follows,

$$\int_{K \mathcal{V}_{1,0} - T \mathcal{V}_{1,1} - \Delta \mathcal{B}^1_{0,2}} \langle \Omega^{(0)1,1} | \hat{O}_\mu \rangle = 0 .$$  

(2.5)

If we glance at Eqn.(2.1) we see that integration region is simply $\partial \mathcal{B}^1_{1,0}$. A simple application of Stokes’ theorem then explains why the background independence condition is satisfied,

$$\int_{\partial \mathcal{B}^1_{1,0}} \langle \Omega^{(0)1,1} | \hat{O}_\mu \rangle = \int_{\mathcal{B}^1_{1,0}} \langle \Omega^{(0)1,1} | \hat{Q} | \hat{O}_\mu \rangle = 0 .$$  

(2.6)

There is no need for any further analysis or application of auxiliary constraints.

This shows that it is algebraically consistent to introduce $\mathcal{B}^0_{1,0}$, and we now give a geometrical description of what this algebra implies. It was mentioned in [5] that $\mathcal{B}^0_{1,0}$ was ‘not constrained’ by the recursion relations. On assuming the existence of $\mathcal{B}^0_{0,2}$, which is a twice-punctured sphere representing the kinetic term, this is no longer true and, as we have mentioned, the recursion relations imply that $\partial \mathcal{B}^0_{1,0} = -\Delta \mathcal{B}^0_{0,2}$. This is as we would naively expect - as we increase the height of the internal foliation of the vacuum graph to $2\pi$, the diagram should split into the twice-punctured sphere whose punctures are glued together by a propagator [Fig. 1].

![Figure 1. The torus decomposition into a twice punctured sphere and a propagator.](image)
Usually we consider the string vertices to have stubs of length $\pi$, so that the propagator can have any length from 0 to $\infty$. In general it is also possible to use shorter stubs of length $a$ say, where $0 \leq a \leq \pi$, in combination with ‘cutoff propagators’ \cite{5} which would be of length $\geq 2(\pi - a)$. The latter condition would ensure that no nontrivial loops have length less than $2\pi$, as is required by the minimal area metric prescription. If we are to apply the same conditions to the vertex $\mathcal{B}_{0,2}^0$ as are applied to the other vertices, we would expect that it be conformally equivalent to a cylinder of length $2a$. In this case, the general expression for the kinetic term would be,

$$Q = \frac{1}{2}(R_{12}|c_0|^{(2)}Q^{(2)}e^{2aL_0^{(2)}}|\Psi_1\rangle |\Psi_2\rangle .$$

For the usual case $a = \pi$, we require the insertion $e^{2\pi L_0^+}$.

![Figure 2. The auxiliary vertex $\mathcal{V}_{0,3}'$ as a limiting case of $\tilde{\mathcal{V}}_{0,3}$.

This description of $\mathcal{B}_{0,2}^0$ elucidates some points regarding the choice of connection $\Gamma$. In §3.2 of \cite{7} we reviewed the origin of symplectic connections. In particular, a choice of auxiliary three string vertex $\tilde{\mathcal{V}}_{0,3}$ determines the vertex $\mathcal{B}_{1,2}^1$ (which interpolates between $\mathcal{I}\mathcal{B}_{0,3}^0$ and $\tilde{\mathcal{V}}_{0,3}$), and this (or more accurately, $\Delta \tilde{\mathcal{V}}_{0,3}$), in turn determines the choice of connection through Eqn.(3.23) of \cite{7}. The particular case of the canonical connection $\hat{\Gamma}$ follows from choosing $\mathcal{V}_{0,3}'$ (introduced in §3.3 of \cite{8}) as the auxiliary three-string vertex.

Now the recursion relations determine that the boundary of $\mathcal{B}_{0,2}^1$ is given by,

$$\partial \mathcal{B}_{0,2}^1 = \mathcal{K}\mathcal{B}_{0,2}^0 - \mathcal{I}\mathcal{B}_{0,3}^0,$$

which means that $\tilde{\mathcal{V}}_{0,3}$ is identified with $\mathcal{K}\mathcal{B}_{0,2}^0$. Moreover, the choice of auxiliary three-string

* Note that in \cite{1}, we identified $\mathcal{K}\mathcal{B}_{0,2}^0 = -\{L, R\}$ with $\mathcal{V}_{0,3}'$. The identification with $\tilde{\mathcal{V}}_{0,3}$ is more general, and it is $\tilde{\mathcal{V}}_{0,3}$ which should appear in the recursion relations Eqn.(2.37) of \cite{1}.
vertex determining the connection is defined by the choice of $B_{0,2}^0$ which as we saw above encodes the choice of stub length and cutoff propagator. Now, $B_{0,2}^0$ is just a cylinder of length $2a$. The operator $K$ adds a puncture over the entire surface of this cylinder, resulting in the vertex $V_{0,3}$. The vertex $V_{0,3}$ stems from the particular case $\lim_{a \to 0}$ corresponding to a degenerate cylinder (being the overlap surface corresponding to the standard twice-punctured sphere), [Fig. 2]. In this case the special puncture may only be added over the boundary of the coordinate disks. Each of the resulting three-punctured spheres are identical (being related simply by a rotation), and we thereby recover the usual description $V_{0,3}$. It was recognised in §6.2 of [2] that the canonical connection $\hat{\Gamma}$ has some rather singular properties. We are now able to identify the source of these properties with the degeneration of the cylinder described by $B_{0,2}^0$.

The considerations above suffice to show that it is consistent to include the moduli spaces $B_{1,0}^1$ and $B_{1,0}^0$ into the string vertices.

2.2. THE MODULI SPACES $B_{0,0}^3$

The moduli spaces of spheres with at least two special punctures and no ordinary punctures were not considered solely because they contribute ineffectual constants to the action, and were therefore not essential to the usual discussions. Consequently we see no harm in including them and they will be reinstated in what follows.

2.3. THE GEOMETRISED STRING ACTION

Having successfully introduced into $\mathcal{B}$ the missing string vertices of non-negative dimension, we can now state the resulting expression for the action. It takes the elegant ‘geometrised’ form,

$$S = f(\mathcal{B}),$$

(2.9)

where $\mathcal{B} = \sum_{g,n,\bar{n}} B_{g,n}^\bar{n}$, where $(g, n, \bar{n})$ may take all values except when $g = 0$ and $n + \bar{n} \leq 1$. That this action satisfies the master equation is clear in view of the form of the recursion

\[\dagger\] The connection is well-defined as sewing together the two ordinary punctures results in precisely the same once-punctured torus for each surface in $\tilde{V}_{0,3}$ irrespective of the position of the special puncture. This follows from the usual translation symmetry of the torus.
relations, which are still given by a quantum master equation for the string vertices (Eqn.(2.36) of [1]),
\[ \frac{1}{2} \{ B, B \} + \Delta B = 0, \]  
(2.10)

The boundaries of the newly introduced moduli spaces are summarised as follows,
\[ \partial B_{1,0}^1 = \kappa B_{1,0}^0 - TB_{1,1}^0 - \Delta B_{1,2}^1, \]
\[ \partial B_{1,0}^0 = -\Delta B_{2,0}^0, \]
\[ \partial B_{0,0}^0 = 0, \]
\[ \partial B_{0,0}^{\bar{n}} = \kappa B_{0,0}^{\bar{n} - 1} - TB_{0,1}^{\bar{n} - 1}, \quad (\bar{n} > 2). \]  
(2.11)

3. Conclusion

We have seen how the moduli space \( B_{0,2}^0 \) introduced in [1] has provided the key to incorporating into the sum of string vertices \( B \) the spaces \( B_{1,0}^1 \) and \( B_{1,0}^0 \) which, together with the spaces \( B_{0,0}^{\bar{n}} \), complete the set of string vertices of non-negative dimension.

The entire theory has been encoded elegantly in terms of the set of string vertices, and the string fields \( |\Psi\rangle \) and \( |F\rangle \), which define the function mapping the string vertices to the action.

One might note that the three moduli spaces \( B_{0,0}^1 \), \( B_{0,1}^0 \), and \( B_{0,0}^0 \) remain to be defined. These will be discussed in a forthcoming paper [9] which will be concerned mainly with new insights related to background independence stemming from our geometrised formulation.

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