Cumulants in Noncommutative Probability Theory

III. Creation and annihilation operators on Fock spaces

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Abstract. Cumulants of noncommutative random variables arising from Fock space constructions are considered. In particular, simplified calculations are given for several known examples on $q$-Fock spaces.

In the second half of the paper we consider in detail the Fock states associated to characters of the infinite symmetric group recently constructed by Bożejko and Guta. We express moments of multidimensional Dyck words in terms of the so called cycle indicator polynomials of certain digraphs.

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Introduction

Fock space constructions like in [GM02] give rise to natural interchangeable families and are thus well suited for cumulant calculations like in part I [Leh04] and part II [Leh03]. In this paper we develop some general formulas and recompute cumulants for generalized Toeplitz operators, notably for $q$-Fock spaces, previously considered by A. Nica [Nic96].

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and M. Anshelevich [Ans01], and the Fock spaces associated to characters of the infinite symmetric group recently introduced by M. Bożejko and M. Guta [BG02]. For the latter we indicate a general formula for mixed moments of creation and annihilation operators in terms of the cycle indicator polynomial of a certain directed graph.

The paper has three sections.

In the first section some general formulae are developed, in particular the cumulants of generalized Toeplitz operators are given by

\[ K_n(L^* + \sum_{k=0}^{\infty} \alpha_{k+1} L^k) = b_n \alpha_n \]

where \( b_n \) is a certain statistic on the symmetric group.

In the second section, we review the \( q \)-cumulants of Nica and Anshelevich in the light of the new theory.

In the final section we study in detail the Fock space associated to irreducible characters on the infinite symmetric group recently introduced by Bożejko and Guta. It turns out that expectations of multidimensional Dyck words are given by the so-called cycle-indicator polynomial of a certain digraph associated to the Dyck word. This polynomial counts the number of coverings of the digraph with Hamiltonian cycles and satisfies a certain cut-and-fuse recursion formula, which is reflected by a certain commutation relation of the creation and annihilation operators. We characterize those Fock states, which only depend on this digraph construction, as averages of the Bożejko-Guta Fock states, that is, states which are associated to not necessarily irreducible characters of the symmetric group.

1. Preliminaries

In this section we collect the necessary definitions and auxiliary results needed later on. For details we refer to part I [Leh04].

1.1. Exchangeability Systems and Cumulants. We recall first that an exchangeability system \( \mathcal{E} \) for a noncommutative probability space \((\mathcal{A}, \varphi)\) consists of another noncommutative probability space \((\mathcal{U}, \tilde{\varphi})\) and an infinite family \( \mathcal{J} = (\iota_k)_{k \in \mathbb{N}} \) of state-preserving embeddings \( \iota_k : \mathcal{A} \to \mathcal{A}_k \subseteq \mathcal{U} \), which we conveniently denote by \( X \mapsto X^{(k)} \), such that the algebras \( \mathcal{A}_j \) are interchangeable with respect to \( \tilde{\varphi} \): for any family \( X_1, X_2, \ldots, X_n \in \mathcal{A} \), and for any choice of indices \( h(1), \ldots, h(n) \) the expectation is invariant under any permutation \( \sigma \in \mathfrak{S}_n \) in the sense that

\[ \tilde{\varphi}(X_1^{(h(1))} X_2^{(h(2))} \cdots X_n^{(h(n))}) = \varphi(X_1^{(\sigma(h(1)))} X_2^{(\sigma(h(2)))} \cdots X_n^{(\sigma(h(n)))}) \]

Throughout this paper we will assume that the algebra \( \mathcal{U} \) is generated by the algebras \( \mathcal{A}_k \) and that the action of \( \mathfrak{S}_\infty \) extends to all of \( \mathcal{U} \). Denote by \( \Pi_n \) the set of partitions (or equivalence relations) of the set \( [n] = \{1, 2, \ldots, n\} \). The value (1.1) only depends on the kernel \( \pi = \ker h \in \Pi_n \) defined by

\[ i \sim_\pi j \Leftrightarrow h(i) = h(j) \]

and we denote it \( \varphi_\pi(X_1, X_2, \ldots, X_n) = \tilde{\varphi}(X_1^{(\pi(1))} X_2^{(\pi(2))} \cdots X_n^{(\pi(n))}) \). Here we consider a partition \( \pi \in \Pi_n \) as a function \( \pi : [n] \to \mathbb{N} \), mapping each element to the number of the block containing it. This is a canonical example of an index function \( h \) with \( \ker h = \pi \).

Subalgebras \( \mathcal{B}, \mathcal{C} \subseteq \mathcal{A} \) are called \( \mathcal{E} \)-exchangeable or, more suggestively, \( \mathcal{E} \)-independent if for any choice of random variables \( X_1, X_2, \ldots, X_n \in \mathcal{B} \cup \mathcal{C} \) and subsets \( I, J \subseteq \{1, \ldots, n\} \)
such that \( I \cap J = \emptyset \), \( I \cup J = \{1, \ldots, n\} \), \( X_i \in B \) for \( i \in I \) and \( X_i \in C \) for \( i \in J \), we have the identity

\[
\varphi_\pi(X_1, X_2, \ldots, X_n) = \varphi_{\pi'}(X_1, X_2, \ldots, X_n)
\]

whenever \( \pi, \pi' \in \Pi_n \) are partitions with \( \pi|_I = \pi'|_I \) and \( \pi|_J = \pi'|_J \). We say that two families of random variables \( (X_i) \) and \( (Y_j) \) are \( \mathcal{E} \)-exchangeable if the algebras they generate have this property.

Then it is possible to define cumulant functionals, indexed by set partitions \( \pi \in \Pi_n \), via

\[
K_{\pi}^\mathcal{E}(X_1, X_2, \ldots, X_n) = \sum_{\sigma \leq \pi} \varphi_\sigma(X_1, X_2, \ldots, X_n) \mu(\sigma, \pi)
\]

where \( \mu(\sigma, \pi) \) is the Möbius function of the lattice of set partitions, cf. part I. Alternatively, the cumulants can be defined by Good’s formula. Given noncommutative random variables \( X_1, X_2, \ldots, X_n \), take an arbitrary partition \( \pi \in \Pi_n \). We choose for each \( k \in \{1, \ldots, n\} \) an exchangeable copy \( \{X_j^{(k)} : j \in \{1, \ldots, n\}\} \) of the given family \( \{X_j : j \in \{1, \ldots, n\}\} \). For each block \( B = \{k_1 < k_2 < \cdots < k_b\} \in \pi \) we pick a primitive root of unity \( \omega_b \) of order \( b = |B| \), and set for each \( i \in B \)

\[
X_i^{\pi,\omega} = \omega_b X_i^{(k_1)} + \omega_b^2 X_i^{(k_2)} + \cdots + \omega_b^b X_i^{(k_b)}
\]

Then we have Good’s formula [Leh04, Prop. 2.8]

\[
K_{\pi}^\mathcal{E}(X_1, X_2, \ldots, X_n) = \frac{1}{|B|} \varphi(X_1^{\pi,\omega}X_2^{\pi,\omega} \cdots X_n^{\pi,\omega});
\]

if \( \pi \in \Pi_n \) consists of one block only, we may abbreviate and write

\[
K_n(X_1, X_2, \ldots, X_n) = \frac{1}{n} \varphi(X_1^{\omega}X_2^{\omega} \cdots X_n^{\omega})
\]

where \( \omega \) is a primitive root of unity of order \( n \) and

\[
X_i^{\omega} = \omega X_i^{(1)} + \omega^2 X_i^{(2)} + \cdots + \omega^n X_i^{(n)}.
\]

The use of the probabilistic termini “independence” and “cumulants” is justified by the following proposition which establishes the analogy to classical probability.

**Proposition 1.1** ([Leh04]). Two subalgebras \( B, C \subseteq A \) are \( \mathcal{E} \)-independent if and only if mixed cumulants vanish, that is, whenever \( X_i \in B \cup C \) are some noncommutative random variables and \( \pi \in \Pi_n \) is an arbitrary partition such that there is a block of \( \pi \) which contains indices \( i \) and \( j \) such that \( X_i \in B \) and \( X_j \in C \), then \( K_{\pi}^\mathcal{E}(X_1, X_2, \ldots, X_n) \) vanishes.

**Remark 1.2.** In the sequel we will sometimes not distinguish between i.i.d. sequences in \( A \) and sequences of the form \( X^{(i)} \). The latter do not belong to \( A \) strictly speaking, but we can replace \( A \) by the algebra \( \tilde{A} \) generated by \( (A_i)_{i \in I} \), where \( I \subseteq \mathbb{N} \) is an infinite subset, and construct an exchangeability system for \( \tilde{A} \) by considering \( N \) as a disjoint union of infinitely many copies of \( I \).

One of our main tools will be the product formula of Leonov and Shiryaev.

**Proposition 1.3** ([Leh04, Prop. 3.3]). Let \( (X_{ij})_{i \in \{1, \ldots, m\}, j \in \{1, \ldots, n_i\}} \subseteq A \) be a family of noncommutative random variables, in total \( n = n_1 + n_2 + \cdots + n_m \) variables. Then every partition \( \pi \in \Pi_m \) induces a partition \( \tilde{\pi} \) on \( \{1, \ldots, n\} \simeq \{(i, j) : i \in [m], j \in [n_i]\} \) with
blocks \( \tilde{B} = \{(i, j) : i \in B, j \in [n_i]\} \), that is, each block \( B \in \pi \) is replaced by the union of the intervals \( \{n_{i-1} + 1, n_{i-1} + 2, \ldots, n_i\}\). Then we have

\[
K^\xi_{\pi}(\prod_{j_1} X_{1,j_1}, \prod_{j_2} X_{2,j_2}, \ldots, \prod_{j_m} X_{m,j_m}) = \sum_{\sigma \in \Pi_n} \sum_{\sigma \setminus 0 = \tilde{\pi}} K^\xi_{\sigma}(X_{1,1}, X_{1,2}, \ldots, X_{m,n_m})
\]

2. Fock spaces

We recall some definitions from [GM02].

**Definition 2.1.** Let \( \mathfrak{A} \) be a real Hilbert space. The algebra \( \mathfrak{A}(\mathfrak{A}) \) is the unital \(*\)-algebra with generators \( \{\omega(\xi) : \xi \in \mathfrak{A}\} \) and relations

\[
\omega(\lambda \xi + \mu \eta) = \lambda \omega(\xi) + \mu \omega(\eta) \quad \omega(\xi)^* = \omega(\xi)
\]

for all \( \xi, \eta \in \mathfrak{A} \) and \( \lambda, \mu \in \mathbb{R} \).

**Definition 2.2.** Let \( \mathfrak{H} \) be a complex Hilbert space. The algebra \( \mathcal{C}(\mathfrak{H}) \) is the unital \(*\)-algebra with generators \( L(\xi) \) and \( L^*(\xi) \) and relations

\[
L(\lambda \xi + \mu \eta) = \lambda L(\xi) + \mu L(\eta) \quad L^*(\xi) = L(\xi)^*
\]

for all \( \xi, \eta \in \mathfrak{H} \) and \( \lambda, \mu \in \mathbb{C} \).

**Definition 2.3.** A Fock state on \( \mathcal{C}(\mathfrak{H}) \) is a state \( \rho \) satisfying

\[
\begin{bmatrix}
\rho(L^*(\xi) L^*(\eta)) & \rho(L^*(\xi) L(\eta)) \\
\rho(L(\xi) L^*(\eta)) & \rho(L(\xi) L(\eta))
\end{bmatrix} = \begin{bmatrix} 0 & \langle \xi, \eta \rangle \\
0 & 0 \end{bmatrix}
\]

and

\[
\rho(L^{\varepsilon_1}(U \xi_1) L^{\varepsilon_2}(U \xi_2) \cdots L^{\varepsilon_n}(U \xi_n)) = \rho(L^{\varepsilon_1}(\xi_1) L^{\varepsilon_2}(\xi_2) \cdots L^{\varepsilon_n}(\xi_n))
\]

for all unitary operators \( U \in \mathcal{U}(\mathfrak{H}) \) and all choices of exponents \( \varepsilon_j \in \{*, 1\} \). In particular, odd moments vanish.

If the Hilbert space \( \mathfrak{H} \) is infinite dimensional, we can decompose it into an infinite direct sum \( \mathfrak{H} = \bigoplus \mathfrak{H}_i \) and the subalgebras \( \mathcal{C}(\mathfrak{H}_i) \) are interchangeable, because any permutation of indices can be implemented by a unitary operator on \( \mathfrak{H}_i \). Therefore Fock spaces are a rich source of exchangeability systems. We proceed by calculating cumulants in certain special cases.

Throughout this paper the exchangeability system will be

\[
\mathcal{E} = (\mathcal{C}(\mathfrak{A}), \rho, (\iota_k))
\]

where \( \mathfrak{A} = \bigoplus \mathfrak{H}_i \) is the direct sum of infinitely many copies \( \mathfrak{H}_i \simeq \mathfrak{H} \), which give rise to the embeddings \( \iota_k : \mathcal{C}(\mathfrak{H}) \to \mathcal{C}(\mathfrak{H}_k) \subseteq \mathcal{C}(\mathfrak{A}) \), which are interchangeable with respect to the Fock state \( \rho \).

**Proposition 2.4.** Any Fock state is given by a function \( t \) on pair partitions

\[
(2.1) \quad \rho(L^{\varepsilon_1}(\xi_1) L^{\varepsilon_2}(\xi_2) \cdots L^{\varepsilon_n}(\xi_n)) = \sum_{\pi \in \Pi_1^{(2)}} t(\pi) \prod_{\{k < l\} \in \pi} Q(\varepsilon_k, \varepsilon_l) \langle \xi_k, \xi_l \rangle
\]

where \( \varepsilon_j \in \{*, 1\} \) and

\[
Q(\varepsilon, \varepsilon') = \begin{cases} 
1 & \text{if } \varepsilon = * \text{ and } \varepsilon' = 1 \\
0 & \text{otherwise}
\end{cases}
\]
The proof is essentially the same as the proof of Theorem II.2.4. Conversely, it was shown in [GM02] that a positive definite function on pair partitions gives rise to a Fock state and a Fock representation of $C(\mathfrak{F})$.

**Corollary 2.5.** For $\varepsilon_j \in \{1, \ast\}$ the cumulants of the creation and annihilation operators are given by

$$K^n_\varepsilon(L^{\varepsilon_1}(\xi_1), L^{\varepsilon_2}(\xi_2), \ldots, L^{\varepsilon_n}(\xi_n)) = \begin{cases} 0 & \text{if } \pi \notin \Pi_n^{(2)} \\ t(\pi) \prod_{\{k<l\} \in \pi} Q(\varepsilon_k, \varepsilon_l) \langle \xi_k, \xi_l \rangle & \text{if } \pi \in \Pi_n^{(2)} \end{cases}$$

**Definition 2.6.** A positive definite function $t$ on pair partitions is called *multiplicative* if it factors with respect to the connected components of $\pi$. (See Definition I.1.5).

The following lemma is immediate from the definition.

**Lemma 2.7.** Pyramidal independence holds if and only if $t$ is multiplicative.

From now on we will work with a fixed orthonormal basis $\{e_i\}$ of $\mathfrak{F}$ and calculate expectation of words in $L_i = L(e_i)$ or creation operators $L = L(h)$ for some fixed unit vector $h \in \mathfrak{F}$.

**Definition 2.8.** A *lattice path* is a sequence of points $((x_i, y_i))_{i=0,1,\ldots,n}$ in $\mathbb{N}_0 \times \mathbb{Z}$ such that $y_0 = y_n = 0$, $y_j \geq 0$ for all $j$ and $x_i = i$. As the $x$-coordinates are redundant, we will also refer to the sequence $(y_i)_{i=0,1,\ldots,n}$ as lattice paths. A lattice path is *irreducible* if $y_j > 0$ for $j = 1, 2, \ldots, n-1$ and reducible otherwise. A Lukasiewicz path is a lattice path $(y_j)$ such that $y_i - y_{i-1} \leq 1$. A *lattice word* is a word $L^{k_1}L^{k_2}\cdots L^{k_n}$ where $k_j \in \mathbb{Z}$ and $L^{-k}$ is interpreted as $L^\ast$ such that the sequence $y_j = k_1 + k_2 + \cdots + k_j$ constitutes a lattice path. A Lukasiewicz word is defined accordingly. A Dyck path is a lattice path $(y_j)$ such that $y_i - y_{i-1} = \pm 1$. A $N$-dimensional lattice path is a sequence of points $\vec{y}_j$ in $\mathbb{Z}^N$ such that $\vec{y}_0 = \vec{y}_n = \vec{0}$ and $\vec{y}_{j+1} - \vec{y}_j$ is a multiple of a basis vector $e_i$ and such that all coordinates of $\vec{y}_j$ are nonnegative. A word $L_{i_1}^{k_1}L_{i_2}^{k_2}\cdots L_{i_n}^{k_n}$ is an $(N$-dimensional) lattice word if the path $\vec{y}_j = k_1e_{i_1} + k_2e_{i_2} + \cdots + k_ne_{i_n}$ is a lattice path. Multidimensional Lukasiewicz words and Dyck words are defined similarly.

**Proposition 2.9.** For $k_j \in \mathbb{Z}$ we have

$$\rho(L_{i_1}^{k_1}L_{i_2}^{k_2}\cdots L_{i_n}^{k_n}) = 0$$

unless the word $L_{i_1}^{k_1}L_{i_2}^{k_2}\cdots L_{i_n}^{k_n}$ is a lattice word.

**Proof.** Indeed if one of the $\vec{y}_j$ in Definition 2.8 has a negative component then the sum \eqref{eq:2.1} is empty, as there is no pairing with nonzero contribution.

It was shown in [GM02] that Fock states can always be modeled on so-called combinatorial Fock spaces as follows. Let $\mathfrak{F}$ be a fixed Hilbert space. Let $V_n$ be a sequence of Hilbert spaces with an action $U_n$ of the symmetric group $\mathfrak{S}_n$ and densely defined intertwining operators $j_n : V_n \to V_{n+1}$ such that

\begin{equation}
\label{eq:2.2}
    j_n \circ U_n(\sigma) = U_{n+1}(\iota_n(\sigma)) \circ j_n
\end{equation}
where $\iota_n : \mathcal{S}_n \rightarrow \mathcal{S}_{n+1}$ is the natural inclusion. Let

$$\mathcal{F}_V(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \frac{1}{n!} V_n \otimes_s \mathcal{S}_{\otimes n}$$

where $V_n \otimes_s \mathcal{S}_{\otimes n}$ is the subspace spanned by the vectors which are invariant under the action $U_n \otimes U_n$ of $\mathcal{S}_n$ on $V_n \otimes \mathcal{S}_{\otimes n}$ given by

$$(U_n(\sigma) \otimes \bar{U}_n(\sigma))(v \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = U_n(\sigma)v \otimes \xi_{\sigma^{-1}(1)} \otimes \xi_{\sigma^{-1}(2)} \otimes \cdots \otimes \xi_{\sigma^{-1}(n)}$$

From now on we will abbreviate $v \otimes \xi_1 \cdot \xi_2 \cdots \xi_n := v \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$;

for $v \in V_n$ and $\eta \in \mathcal{S}_{\otimes n}$ we define the Symmetrisator

$$v \otimes_s \eta = P_n v \otimes \eta = \frac{1}{n!} \sum_{\sigma} U_n(\sigma)v \otimes \bar{U}_n(\sigma)\eta$$

and left and right creation operators

(2.3) $L_{V,j}(h)v \otimes_s \eta = (n+1)(j_n v) \otimes_s (h \otimes \eta)$ \hspace{1cm} $R_{V,j}(h)v \otimes_s \eta = (n+1)(j_n v) \otimes_s (\eta \otimes h)$;

its adjoint is the annihilation operator $L_{V,j}^*$ which is the restriction of

$$L_{V,j}^*(\xi) : V_{n+1} \otimes \mathcal{S}_{\otimes n+1} \rightarrow V_n \otimes \mathcal{S}_{\otimes n}$$

$$v \otimes \xi_1 \cdot \xi_2 \cdots \xi_n \mapsto \langle \xi_1, \xi \rangle j_n^* v \otimes \xi_2 \cdot \xi_3 \cdots \xi_n$$

to $V_{n+1} \otimes_s \mathcal{S}_{\otimes n+1}$. A similar formula holds for $R_{V,j}^*$. On symmetric tensors it is given by

(2.4) $R^*(\xi) v \otimes_s \xi_1 \cdots \xi_{n+1} = \frac{1}{n+1} \sum_{i=1}^{n+1} \langle \xi_i, \xi \rangle j_n^* U_{n+1}(\tau_{i,n+1}) v \otimes_s \xi_1 \cdots \xi_{i-1} \cdot \xi_n \cdot \xi_{i+1} \cdots \xi_n$

where $\tau_{i,j}$ is the transposition which exchanges $i$ and $j$. Left and right creation operators are equivalent and we will consider either of them, whenever it is notationally convenient.

Let $\mathcal{C}_{V,j}(\mathcal{H})$ be the $*$-algebra generated by these creation operators and $\Omega_V \in V_0$ a unit vector. Then

$$\rho_{V,j}(X) = \langle \Omega_V, X \Omega_V \rangle$$

is a Fock state.

**Proposition 2.10.** [GM02, Thm. 2.7] Let $t$ be a positive definite function on pair partitions. Then for any complex Hilbert space $\mathcal{H}$ the GNS-representation of $\mathcal{C}(\mathcal{H}, \rho_t)$ is unitarily equivalent to $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ for some sequence $(V_n, j_n)_{n=0}^{\infty}$ which up to unitary equivalence is uniquely determined by $t$.

Using this model it is now easy to compute cumulants in certain cases.

**Proposition 2.11.** If the lattice word $(L_{k_1}, L_{k_2}, \ldots, L_{k_n})$ is reducible, then

$$K_n^c(L_{k_1}, L_{k_2}, \ldots, L_{k_n}) = 0$$

**Proof.** Indeed for a lattice word $L_{i_1}^{k_1} L_{i_2}^{k_2} \cdots L_{i_m}^{k_m}$ we have

$$L_{i_1}^{k_1} L_{i_2}^{k_2} \cdots L_{i_m}^{k_m} \Omega_V = \rho(L_{i_1}^{k_1} L_{i_2}^{k_2} \cdots L_{i_m}^{k_m}) \Omega_V$$
and therefore if \( m \) is the largest index for which \( k_{m+1} + k_{m+2} + \cdots + k_n = 0 \), then by assumption \( m > 1 \) and Lemma I.2.4 implies that
\[
(L^{k_{m+1}})^\omega (L^{k_2})^\omega \ldots (L^{k_n})^\omega \Omega_V = \rho((L^{k_{m+1}})^\omega (L^{k_2})^\omega \ldots (L^{k_n})^\omega) \Omega_V = 0.
\]

In the case of the free creation operators of Voiculescu we can say even more, see Proposition 3.4 below. There is a special kind of operators for which cumulants can be calculated explicitly in general.

**Definition 2.12.** Let \( h \in \mathcal{H} \) be a fixed unit vector and let \( L = L(h) \) be a creation operator. A **generalized Toeplitz operator** is an operator of the form
\[
L^* + \sum_{k=1}^{\infty} \alpha_k L^{k-1}
\]
Such operators were considered by Voiculescu as a model of free random variables [Voic86] and for \( q \)-deformations of them by Nica [Nic96]. The cumulants of Toeplitz operators are rather restrictive and only allow Lukasiewicz words which are “prime” and consequently can be immediately read off the coefficients.

**Proposition 2.13.** For \( k_j \in \{*, 0, 1, \ldots \} \) we have
\[
K_n^E(L^{k_1}, L^{k_2}, \ldots, L^{k_n}) = 0
\]
unless \( k_1 = k_2 = \cdots = k_{n-1} = * \) and \( k_n = n - 1 \). In the latter case, the cumulant equals the expectation of the product:
\[
K_n^E(L^*, \ldots, L^*, L^{n-1}) = \rho(L^*)^{n-1}L^{n-1} = \sum_{\pi \in \Pi_{2^{n-2}}} K_\pi^E(L^*, L^*, \ldots, L, L) = \sum_{\sigma \in \mathfrak{S}_{n-1}} \nu(\sigma)
\]
where \( \nu \) is a function on the symmetric group, because every contributing 2-partition connects a point of \( \{1, 2, \ldots, n-1\} \) to a point in \( \{n, n+1, \ldots, 2n-2\} \) and can be interpreted as a permutation.

**Proof.** We may assume that all \( k_j \neq 0 \), because otherwise the presence of the identity operator makes the cumulant vanish. Let \( m \) be the total number of factors when decomposing the powers of \( L \) into single creators, i.e. \( m = \sum |k_j| \) where we put \( |*| = 1 \). Let \( \pi \in \Pi_m \) be the interval partition induced by the \( n \) exponents \( k_j \), that is \( \pi = 0_n \) in the notation of the proof of Proposition I.3.3. If the \( k_j \) are not as claimed, then there are at least two monomials \( L^k \) with \( k > 0 \). By the product formula (Proposition I.3.3) we have
\[
K_n^E(L^{k_1}, L^{k_2}, \ldots, L^{k_n}) = \sum_{\sigma \vee \pi = 1_m} K_\sigma^E(L^{\varepsilon_1}, L^{\varepsilon_2}, \ldots, L^{\varepsilon_m})
\]
where \( \varepsilon_j \in \{*, 1\} \). Now \( \sigma \) must be both a pair partition connecting creation operators with annihilation operators and at the same time we must have \( \sigma \vee \pi = 1_m \), but the two different blocks with \( k > 0 \) cannot be connected in this way, because each \( L^* \) can only be connected to one \( L^k \). Therefore the sum is empty. \( \square \)
A similar formula holds for partitioned cumulants.

**Proposition 2.14.** For \( k_j \in \{*, 0, 1, \ldots \} \) we have

\[
K^\xi_\pi(L^{k_1}, L^{k_2}, \ldots, L^{k_n}) = 0
\]

unless \( k_j \) and \( \pi \) are compatible in the sense that

\[
k_j = \begin{cases} 
  b - 1 & \text{if } j \text{ is the last element of a block } B \text{ of length } |B| = b \\
  * & \text{otherwise}
\end{cases}
\]

In that case

\[
K^\xi_\pi(L^{k_1}, L^{k_2}, \ldots, L^{k_n}) = \rho_\pi(L^{k_1}, L^{k_2}, \ldots, L^{k_n})
\]

By multilinear expansion we have the following corollary.

**Corollary 2.15.** Let

\[
b_n = \rho((L^*)^{n-1}L^{n-1}) = \sum_{\sigma \in S_{n-1}} \nu(\sigma)
\]

where \( \nu(\sigma) \) is the statistic on the symmetric group defined in Proposition 2.13. Then the generalized Toeplitz operator

\[
T = L^* + \sum_{k=1}^{\infty} \frac{\alpha_k}{b_k} L^{k-1}
\]

has cumulants

\[
K^\xi_n(T, T, \ldots, T) = \alpha_n
\]

**Example 2.16** \((q\text{-cumulants})\). For Nica’s \(q\)-cumulants [Nic96] we have

\[
b_n(q) = [n]_q!
\]

which includes the classical case \( b_n(1) = n! \) and the free case \( b_n(0) = 1 \) considered by Voiculescu. Moreover, for partitioned cumulants a simple formula holds, too, namely Corollary 3.5 below.

### 3. \(q\)-Fock space

In an attempt to unify bosonic and fermionic Fock space, the deformed \(q\)-Fock spaces were constructed in [FB70], [BS94] and [BKS97].

**Definition 3.1** \((q\text{-Fock space})\). On free Fock space \( \mathcal{F}(\mathcal{F}) = \mathcal{F}_0(\mathcal{F}) = C\Omega \oplus \bigoplus_{n \geq 1} \mathcal{F}^{\otimes n} \) define the \(q\)-symmetrizer on \( n\)-particle space by

\[
P^q_n(\eta_1 \cdot \eta_2 \cdots \eta_n) = \sum_{\sigma \in S_n} q^{\sigma} U_n(\sigma) \eta_1 \cdot \eta_2 \cdots \eta_n
\]

\[
= \sum_{\sigma \in S_n} q^{\sigma} \eta_{\sigma^{-1}(1)} \cdot \eta_{\sigma^{-1}(2)} \cdots \eta_{\sigma^{-1}(n)}
\]

where \(|\sigma|\) is the number of inversions of the partition \( \sigma \). Define the \(q\)-inner product on elementary tensors \( \xi \in \mathcal{F}^{\otimes m}, \eta \in \mathcal{F}^{\otimes n} \) by

\[
\langle \xi, \eta \rangle_q = \delta_{m, n} \langle \xi, P^q_n \eta \rangle
\]
where $(\cdot, \cdot)$ is the inner product on $\mathcal{F}(H)$ (linear in the second variable). $q$-Fock space is the completion with respect to this norm and denoted $\mathcal{F}_q(H)$. On this space we define creation and annihilation operators

\begin{align}
L(\xi) \Omega & = \xi \\
L^*(\xi) \Omega & = 0 \\
L(\xi) \eta_1 \cdots \eta_n & = \xi \cdot \eta_1 \cdots \eta_n \\
L^*(\xi) \eta_1 \cdots \eta_n & = \sum q^{k-1} \langle \xi, \eta_k \rangle \eta_1 \cdots \hat{\eta}_k \cdots \eta_n
\end{align}

These satisfy the $q$-commutation relations

$$L^*(\xi) L(\eta) - q L(\eta) L^*(\xi) = \langle \xi, \eta \rangle I$$

and give rise to canonical interchangeable random variables with respect to the vacuum expectation $\rho(X) = \langle \Omega, X \Omega \rangle$: Given mutually orthogonal and isomorphic subspaces $(E_j)_{j \geq 0}$ of $H$, the $*$-subalgebras $A_j$ generated by \{ $L(f) : f \in E_j$ \} are interchangeable. Moreover, orthogonal subspaces of $E_0$ give rise to interchangeable algebras.

**Proposition 3.2** ([BS94]). The Fock state corresponding to $q$-Fock space is given by the positive definite function

$$t(\pi) = q^{nc(\pi)}$$

where $nc(\pi)$ is the number of crossings of the pair partition $\pi$. In particular, pyramidal independence holds.

The cases $q = -1, 0, 1$ give rise to fermionic, free and bosonic Fock spaces and correspondingly fermionic graded probability theory [MN97] (see section I.4.8), free probability theory [Voi85, Voi86] and classical probability theory with corresponding notions of independence and convolution.

However, for other $q$ there is no $q$-convolution [vLM96]: There exist $q$-independent s.a. variables $X$, $X'$, $Y$ such that $X$ and $X'$ have the same distribution but the distributions of $X + Y$ and $X' + Y$ differ. See also [Gut01], where a $q$-convolution for generalized Gaussians is constructed.

Nevertheless, choosing certain models of random variables various $q$-cumulants have been found.

### 3.1. $q$-Toeplitz operators

Motivated by Speicher's work on free cumulants and Voiculescu's Toeplitz model of free random variables [Voi86], Nica [Nic95] considered $q$-Toeplitz operators

\begin{equation}
T = L_q^* q^{n} + \sum_{n=0}^{\infty} \frac{\alpha_{k+1}}{[k]_q} L_q^k
\end{equation}

with $L = L(h)$ for some fixed unit vector $h$. Here and in the sequel we will use the standard $q$-notations

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q$$

Here exchangeable copies correspond to $L_i = L_q^{(i)} = L_q(e_i)$. We want to show that Nica’s cumulants involving left-reduced crossings coincide with the cumulants of this exchangeability system. Actually technical reasons force us to rather consider right-reduced crossings. We will calculate mixed cumulants of creation- and annihilation operators.
**Theorem 3.3.** For a lattice word $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $\varepsilon_j \in \{\ast, 0, 1, \ldots\}$, the $q$-cumulant $K^q_n(L^1_q, L^2_q, \ldots, L^n_q)$ vanishes unless $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{n-1} = \ast$ and $\varepsilon_n = n - 1$. In the latter case

$$K^q_n(L^*_q, L^*_q, \ldots, L^*_q, L_q^{n-1}) = \rho(L^*_qL^*_q \ldots L^*_qL_q^{n-1}) = [n-1]_q!$$

**Proof.** This follows from (3.2), namely

$$L^* e^{\otimes k} = (1 + q + \cdots + q^{n-1}) e^{\otimes (k-1)} = [n]_q e^{\otimes (k-1)}$$

from which we infer that

$$L^{n-1}L^{n-1} = [n-1]_q! \Omega.$$ 

Now apply Proposition 2.13.

In the case $q = 0$ (free Fock space) we have a complete description of the cumulants.

**Proposition 3.4.** Let $q = 0$. Then for $k_j \in \mathbb{Z}$ the cumulants are

$$K^q_n(L^{k_1}, L^{k_2}, \ldots, L^{k_n}) = \begin{cases} \rho(L^{k_1}L^{k_2} \cdots L^{k_n}) = 1 & \text{if there is no non-trivial lattice subword} \\ 0 & \text{otherwise} \end{cases}$$

Here $L^k$ stands for $(L^*)^{-k}$ if $k < 0$.

**Proof.** If there is a lattice subword $L^{k_p}, L^{k_{p+1}}, \ldots, L^{k_q}$, then

$$L^{k_p}L^{k_{p+1}} \cdots L^{k_q} = \rho(L^{k_p}L^{k_{p+1}} \cdots L^{k_q})i$$

and

$$(L^{k_p})^{\omega}(L^{k_{p+1}})^{\omega} \cdots (L^{k_q})^{\omega} = \rho((L^{k_p})^{\omega}(L^{k_{p+1}})^{\omega} \cdots (L^{k_q})^{\omega}) = 0$$

If there is no such subword, we use the noncrossing moment-cumulant formula

$$K^q_n(L^{k_1}, L^{k_2}, \ldots, L^{k_n}) = \sum_{\pi \in NC_n} \rho(L^{k_{\pi(1)}} \cdots L^{k_{\pi(n)}}) \mu\text{NC}(\pi, \hat{1}_n)$$

and the only partition which gives a nonzero contribution is $\pi = \hat{1}_n$.

Note that for general $q$ this is no longer true.

Let us now compute the partitioned cumulants of Lukasiewicz words, i.e. words of the form $L^{\varepsilon_1}_{\pi(1)} \cdots L^{\varepsilon_n}_{\pi(n)}$ with $\varepsilon_j \in \{\ast, 0, 1, 2, \ldots\}$. By Proposition 2.14 the cumulants coincide with the expectations

$$\rho(L^{\varepsilon_1}_{\pi(1)} \cdots L^{\varepsilon_n}_{\pi(n)})$$

and we only need to compute those, in which each block $B = \{i_1 < i_2 < \cdots < i_b\}$ satisfies $\varepsilon_{i_1} = \varepsilon_{i_2} = \cdots = \varepsilon_{i_{b-1}} = \ast$ and $\varepsilon_{i_b} = b - 1$. For noncrossing $\pi$, it follows by pyramidal independence that

$$\rho(L^{\varepsilon_1}_{\pi(1)} \cdots L^{\varepsilon_n}_{\pi(n)}) = \prod_B \rho(\prod_{i \in B} L^{\varepsilon_i});$$

for more general $\pi$ we must count the right reduced crossings.

**Corollary 3.5.** For a partition $\pi = \{B_1, B_2, \ldots, B_p\}$ the corresponding cumulant of the $q$-Toeplitz operator is

$$K^q_\pi(L^*_q + \sum_{k=0}^{\infty} \frac{\alpha_{k+1}}{|k|_q!} L^k_q) = q^{rrc(\pi)} \alpha_{|B_1|} \alpha_{|B_2|} \cdots \alpha_{|B_p|}$$
where \( rrc(\pi) \) is the number of right reduced crossings
\[
rrc(\pi) = \# \{(i < i' < j < j') : i, j \in B, i', j' \in B', j = \max B, j' = \max B'\}
\]

**Proof.** Let us define an un-crossing map \( \Phi : \Pi \to \Pi \). For \( \pi \in \Pi \), sort its blocks according to their last elements. Let \( B_0 \) be the last block which is not an interval. Choose \( j_2 \in B_0 \) maximal s.t. \( j_2 - 1 \notin B_0 \) and let \( j_1 \in B_0 \) be its predecessor in \( B_0 \). In other words, \( j_1 \) and \( j_2 \) enclose the last “hole” in \( B_0 \). Let \( \Phi(\pi) \) be the partition obtained by cyclically rotating the interval \((j_1, j_1 + 1, \ldots, j_2 - 1)\) to \((j_1 + 1, j_1 + 2, \ldots, j_2 - 1, j_1)\). Then \( rrc(\pi) = rrc(\Phi(\pi)) + c \) where \( c \) is the number of right reduced arcs of \( \pi \) which are crossed by the arc \((j_1, \max B_0)\). This number is equal to
\[
c = \sum_{\substack{B \in \pi \colon j_1 < \max B < j_2}} |B| - |B \cap [j_1 + 1, j_2 - 1]|\]

and it is also immediate that
\[
\rho(L_{\pi(1)}^{\varepsilon_1} L_{\pi(2)}^{\varepsilon_2} \cdots L_{\pi(n)}^{\varepsilon_n}) = q^c \rho(L_{\pi(1)}^{\varepsilon_1} L_{\pi(2)}^{\varepsilon_2} \cdots L_{\pi(1)}^{\varepsilon_{j_1 - 1}} L_{\pi(j_1 + 1)}^{\varepsilon_{j_1 + 1}} \cdots L_{\pi(j_2 - 1)}^{\varepsilon_{j_2 - 1}} L_{\pi(j_2)}^{\varepsilon_{j_2 - 1}} L_{\pi(j_2 - 1)}^{\varepsilon_j} \cdots L_{\pi(n)}^{\varepsilon_n})
\]

\[\square\]

### 3.2. “Reduced” \( q \)-cumulants

In this section we consider the cumulants found in [Ans01] (see also [SY01]) which are weighted by another statistic on partitions, the number of so-called reduced crossings. Instead of taking powers of creation operators, one adds gauge operators to the scenery.

**Definition 3.6.** Let \( T \) be an operator with dense domain \( \mathcal{D} \). The **gauge operator** \( \gamma(T) \) on \( \mathcal{F}_q(\mathfrak{H}) \) with dense domain \( \mathcal{F}_{alg}(\mathcal{D}) \) is defined by
\[
\gamma(T) \Omega = 0 \quad \gamma(T) \eta_1 \cdots \eta_n = \sum q^{k-1} T \eta_k \cdots \eta_1 \cdots \eta_n
\]

**Proposition 3.7 ([Ans01, Prop. 2.2]).** If \( T \) is essential selfadjoint with dense domain \( \mathcal{D} \) and \( T(\mathcal{D}) \subseteq \mathcal{D} \), then \( \gamma(T) \) is essential selfadjoint with dense domain \( \mathcal{F}_{alg}(\mathcal{D}) \).

Then consider processes of the form
\[
\gamma_I(\xi, T, \lambda) = L_I(\xi) + L_I^*(\xi) + \gamma_I(T) + |I| \lambda
\]
where \( \mathfrak{H} = L^2(\mathbb{R}_+) \otimes V \) is the underlying Hilbert space, \( \xi \in V \) is an analytic vector for \( T : \mathcal{D} \subseteq V \to V \) and \( L_I(\xi) = L(\chi_I \otimes \xi) \) etc. In [Ans01, Prop. 2.2] stochastic measures are used to find the cumulants of such processes.

This kind of independence can be reduced to the following symmetry. Let \( \mathfrak{H} = \ell_2 \otimes V \) and denote \( \{e_j\}_{j \geq 0} \) the canonical basis of \( \ell_2 \) and \( P_j \) the one-dimensional projections on \( e_j \). For \( v \in V, T \in B(V) \) (or densely defined), \( \lambda \in \mathbb{C} \) (or \( \mathbb{R} \) to be s.a.,) define
\[
\gamma(v, T, \lambda) = L(e_0 \otimes v) + L^*(e_0 \otimes v) + \gamma(P_0 \otimes T) + \lambda
\]

Interchangeable copies are obtained by replacing \( e_0 \) with \( e_j \):
\[
\gamma_j(v, T, \lambda) = L_j(v) + L_j^*(v) + \gamma_j(T) + \lambda
\]
where
\[
L_j(v) = L(e_j \otimes v) \quad \gamma_j(T) = \gamma(P_j \otimes T)
\]
Remark 3.8. Note that all these operators are infinite divisible. This can be seen by embedding $l_2$ into $L_2(\mathbb{R})$ by sending the basis elements $e_i$ to the characteristic function of unit intervals. Therefore the class of obtainable distributions is the same as in [Ans01].

We chose to work with unit intervals. Therefore the class of obtainable distributions is the same as in [Ans01].

We will consider the unital $\ast$-algebras $A_i$ generated by $X_i = \{L_i(v), L_i^*(v), \gamma_i(T) : v \in V, T \in B(V)\}$ and more generally, for an index set $I$, the algebra $A_I$ generated by $A_i$, $i \in I$. Then these algebras satisfy pyramidal independence. First we need a lemma.

Lemma 3.9 ([Ans01, Lemma 3.2]). Let $E_j = \text{span}\{e_i\}_{i \in I_j} \subseteq l_2$, $j = 1, 2$ with $I_1 \cap I_2 = \emptyset$ (i.e. $E_1 \perp E_2$). Then we have

1. $\mathcal{F}(E_1 \otimes V) \oplus \mathcal{C} \Omega \perp \mathcal{F}(E_2 \otimes V) \oplus \mathcal{C} \Omega$
2. Let $\eta \in \mathcal{F}(E_2 \otimes V)$, $X \in A_1$, then $X\eta = (X - \rho(\eta)) \Omega \otimes \eta + \rho(X) \eta$

Proof. The first part is clear since $E_1 \otimes V \perp E_2 \otimes V$. For the second part, observe that by orthogonality we have for $i \in I_1$

$L_i^*(v) \eta = 0$
$L_i(v) \eta = (e_i \otimes v) \otimes \eta$
$\gamma_i(T) \eta = 0$

More generally, for $\eta_1 \in \mathcal{F}(E_1 \otimes V) \oplus \mathcal{C} \Omega$, $L_i^*(v) \eta_1 \otimes \eta = (L_i^*(v) \eta_1) \otimes \eta$
$\gamma_i(T) \eta_1 \otimes \eta = (\gamma_i(T) \eta_1) \otimes \eta$

Thus $\eta$ is either unchanged or sent to 0 and the claim follows. □

Pyramidal independence still holds.

Proposition 3.10 ([Ans01, Lemma 3.3]). Let $X, X' \in A_I$, $Y \in A_J$ with $I \cap J = \emptyset$, then $\rho(XYX') = \rho(XX') \rho(Y)$.

Proof. By the preceding lemma,

$\langle \Omega, XYX'\Omega \rangle = \langle X^*\Omega, Y \otimes X'\Omega + \rho(Y) X'\Omega \rangle$
$= \rho(Y) \langle \Omega, XX'\Omega \rangle$
$= \rho(Y) \rho(XX')$ □

Now we want to compute cumulants of the generators $L_i(v)$, $L_i^*(v)$, and $\gamma_i(T)$. In the following let $X_i$ denote one of $L_j(v)$, $L_j^*(v)$, $\gamma_j(T)$, $j \in I$, $v \in V$, $T \in B(V)$. First observe that the expectation of a word $X_1X_2 \cdots X_n$ vanishes unless $X_1$ is an annihilator and $X_n$ is a creator. Using lemma 3.9 again one sees the following more general fact. Let $w = X_1X_2 \cdots X_n$ be a word in generators, $X_j \in A_i$, and denote $\pi \in \Pi_n$ the partition induced by the indices $i_j$. Then again $\rho(X_1X_2 \cdots X_n) = 0$ unless each block of $\pi$ starts with an annihilator and ends with a creator. Moreover the number of creators must equal the number of annihilators in each block. The following proposition shows that the cumulants are even more restrictive and behave like those of “gaussian” variables: only one pair of creator/annihilator is allowed in each block.
**Proposition 3.11.** The cumulant $K^n_\pi(X_1, X_2, \ldots, X_n)$ vanishes unless each block of $\pi$ starts with an annihilator, ends with a creator and otherwise only contains gauge operators. If these conditions are satisfied, the cumulant equals the expectation of the “discrete Fourier transform” (I.1.3).

If we abbreviate $X_j$ we have $p = \sum e_\omega = \sum k p_k$, we have

\[
L_0^1(v)^\omega = \sum \omega^k L(e_k \otimes v) = L(e_\omega \otimes v)
\]

\[
L_0^0(v)^\omega = L^*(e_\omega \otimes v)
\]

\[
\gamma_0(T)^\omega = \gamma(p_\omega \otimes T)
\]

Now $p_\omega e_\omega = e_{\omega^m + 1}$ and

\[
\langle e_\omega, e_{\omega^m} \rangle = \sum \omega^k \omega^m \sum \omega^{(m+1)k} = 0
\]

unless $m + 1$ is a multiple of $n$. The action of $X_1^\omega X_2^\omega \cdots X_n^\omega$ on $\Omega$ starts with a creator

\[
X_1^\omega X_2^\omega \cdots X_n^\omega \Omega = X_1^\omega X_2^\omega \cdots X_{n-1}^\omega (e_\omega \otimes v)
\]

Then a mixture of creation, annihilation and gauge operators changes the first component by either tensoring with $e_\omega$ or multiplying with $p_\omega$. If there is an annihilator besides $X_1$, it encounters $e_{\omega^m}$ with $m < n - 1$ and all the inner products vanish. If there is a creator, it must be matched by an annihilator different from $X_1$, and again the inner products vanish. Hence, $X_1, X_2, \ldots, X_n$ must have the structure claimed in the theorem. If this is the case, we have $X_1 = L_0^1(v_1), X_n = L_0^0(v_2)$, and $X_j = \gamma_0(T_j)$ for $2 \leq j \leq n - 1$, and

\[
\rho(X_1^\omega X_2^\omega \cdots X_n^\omega) = \langle e_\omega \otimes v_1, e_{\omega^{n-1}} \otimes T_2 T_3 \cdots T_{n-1} v_2 \rangle = \langle e_\omega, e_{\omega^{n-1}} \rangle \langle v_1, T_2 T_3 \cdots T_{n-1} v_2 \rangle = n \rho(X_1^\omega X_2 \cdots X_n)
\]

\[\square\]

**Definition 3.12** (An un-crossing map [Ans01]). On the set of partitions $\Pi_n$ define a map $\Phi : \Pi_n \to \Pi_n$ which fixes interval partitions and otherwise acts as follows. Let $B$ be the last block which is not an interval (if we sort blocks with respect to their maximal element). Let $j_2 = \max\{s \in B : s - 1 \not\in B\}$ (the start of the last subinterval of $B$) and $j_1$ its predecessor in $B$. These two numbers enclose a hole of $B$ and the map $\Phi$ moves this hole to the end of $B$: Let $\alpha = ((j_1 + 1)(j_1 + 2) \cdots b(B))^{b(B) - j_2 + 1} \in \mathcal{G}_n$ and $\Phi(\pi) = \alpha(\pi)$, i.e., $i \sim_{\Phi(\pi)} j \iff \alpha^{-1}(i) \sim_{\pi} \alpha^{-1}(j)$. We will need the number of blocks which end between $j_1$ and $j_2$ but do not start there,

\[
c_b(\pi) = |\{s : j_1 < b(B_s) < j_2\}| - |\{s : j_1 < a(B_s) < j_2\}|
\]

Then $rc(\pi) = rc(\Phi(\pi)) + c_b(\pi)$ and since iterating the map $\Phi$ ends at an interval partition after at most $n$ steps, we have

\[
rc(\pi) = \sum_{k=0}^{n} c_b(\Phi^k(\pi)).
\]
Theorem 3.13 ([Ans01, Lemma 3.8]). Let $X_j \in X_{i_j}$ be generators and $\pi \in \Pi_n$ be the kernel of the index map $j \mapsto i_j$, i.e., $p \sim_\pi q \iff i_p = i_q$. Assume that each block consists of gauge operators enclosed by an annihilator at the beginning and a creator at the end. Then
\[
\rho(X_1X_2 \cdots X_n) = q^{rc(\pi)} \prod_{B \in \pi} \rho(X_B)
\]
that is, for such words $\rho$ is multiplicative modulo a factor $q^{rc(\pi)}$.

Proof. Let $B$ be the block containing $n$. If $B$ is an interval, we can factor it out by pyramidal independence.
If not, consider the last block and let $j_1, j_2$ be as in the construction of the un-crossing map above. If this hole is a union of intervals, we can factor it out by pyramidal independence.
Otherwise there are crossings. Let
\[
\eta = X_{j_2}X_{j_2+1} \cdots X_n \Omega \in \mathfrak{H} \otimes^1
\]
and
\[
\xi = X_{j_1+1}X_{j_1+2} \cdots X_{j_2-1} \Omega \in \mathfrak{H} \otimes^{c_b(\pi)}
\]
note that $X_{j_1}$ is either an annihilator or a gauge operator, therefore
\[
X_{j_1}X_{j_1+1} \cdots X_n \Omega = X_{j_1}X_{j_1+1} \cdots X_{j_2-1} \eta
\]
\[
= X_{j_1}(\xi \otimes \eta)
\]
\[
= q^{c_b(\pi)} X_{j_1} \eta \otimes \xi
\]
\[
= q^{c_b(\pi)} X_{j_1}X_{j_2}X_{j_2+1} \cdots X_n X_{j_1+1}X_{j_1+2} \cdots X_{j_2-1} \Omega
\]
because $c_b(\pi)$ is the number of creators between $X_{j_1}$ and $X_{j_2}$. Thus
\[
\rho(X_1X_2 \cdots X_n) = q^{c_b(\pi)} \rho(X_1X_2 \cdots X_{j_1}X_{j_2}X_{j_2+1} \cdots X_n X_{j_1+1}X_{j_1+2} \cdots X_{j_2-1})
\]
and the partition determined by the permuted indices is exactly $\Phi(\pi)$. \qed

By multilinear expansion we get from this

Corollary 3.14. For $X_j = L^0_\delta(v_j) + L_0(v_j) + \gamma_0(T_j) + \lambda_j$ the partitioned cumulants are multiplicative modulo a factor $q^{rc(\pi)}$:
\[
K^q_{\pi}(X_1X_2 \cdots X_n) = q^{rc(\pi)} \prod_{B \in \pi} K^q_{|B|}(X_B)
\]
4. Fock spaces associated to characters of the infinite symmetric group

Recently the calculations on another concrete Fock space have been carried out in [BG02]. It was shown that using a certain embedding of pair partitions into symmetric groups, one can evaluate characters of the latter and obtain Fock states in this way.

4.1. A simple case.

**Definition 4.1.** Let \( \pi \in \Pi_{2n}^{(2)} \) be a pair partition. There exists a unique noncrossing pair partition \( \hat{\pi} \in NC_{2n}^{(2)} \) such that the set of left points of the pairs in \( \pi \) and \( \hat{\pi} \) coincide. A cycle in \( \pi \) is a sequence \( ((l_1, r_1), \ldots, (l_m, r_m)) \) of pairs of \( \pi \) such that the pairs \( (l_1, r_2), (l_2, r_3), \ldots, (l_m, r_1) \) belong to \( \hat{\pi} \). The length of this cycle is \( m \). We denote by \( c(\pi) \) the number of cycles of \( \pi \) and \( c_m(\pi) \) the number of cycles of length \( m \).

Let \( \pi = \{(l_1, r_1), \ldots, (l_n, r_n)\} \) and \( \hat{\pi} = \{(l_1, \hat{r}_1), \ldots, (l_n, \hat{r}_n)\} \) then define a permutation \( \sigma \) by its images \( \sigma(i) = j \) if \( r_i = \hat{r}_j \), i.e., \( \hat{r}_i = r_{\sigma^{-1}(i)} \). Then the number of cycles of \( \pi \) is the number of cycles of \( \sigma \).

Here the cumulant function which we want to consider is given by

\[
\tau_N(\pi) = \left( \frac{1}{N} \right)^{|\pi| - c(\pi)}
\]

where \( N \in \mathbb{Z} \setminus \{0\} \) is a fixed integer. The corresponding combinatorial Fock space can be realized as follows. The function \( \phi_N(\cdot) \) actually comes from a character of the infinite symmetric group which is given by

\[
\phi_N(\sigma) = \prod_{m \geq 2} \left( \frac{1}{N} \right)^{(m-1)c_m(\sigma)}
\]

With this function there is associated the symmetrizer

\[
P_N^{(n)} = \sum_{\sigma \in \mathfrak{S}_n} \phi_N(\sigma) \tilde{U}^{(n)}(\sigma)
\]

where

\[
\tilde{U}^{(n)}(\sigma) = \left( e + \frac{1}{N} \tilde{f}^{(n)}((1, 2)) \right) \left( e + \frac{1}{N} (\tilde{U}^{(n)}((1, 3)) + \tilde{U}^{(n)}((2, 3))) \right) \cdots
\]

\[
\cdots \left( e + \frac{1}{N} (\tilde{U}^{(n)}((1, n)) + \tilde{U}^{(n)}((2, n)) + \cdots + \tilde{U}^{(n)}((n-1, n))) \right)
\]

On full Fock space \( \mathcal{F}(\mathfrak{H}) \) consider the deformed inner product

\[
\langle \xi, \eta \rangle_N = \langle \xi, P_N \eta \rangle
\]

i.e., for simple tensors the inner product is

\[
\langle \xi_1 \cdots \xi_m, \eta_1 \cdots \eta_n \rangle_N = \delta_{mn} \sum_{\sigma \in \mathfrak{S}_n} \phi_N(\sigma) \langle \xi_1, \eta_{\sigma^{-1}(1)} \rangle \langle \xi_2, \eta_{\sigma^{-1}(2)} \rangle \cdots \langle \xi_n, \eta_{\sigma^{-1}(n)} \rangle
\]

One can show that this is a positive bilinear form and after dividing through its kernel and completing it we get a Hilbert space \( \mathcal{F}_N(\mathfrak{H}) \). On these we have the following creation and annihilation operators

\[
L_N(\xi) \Omega = \xi \quad \quad L_N^*(\xi) \Omega = 0
\]

\[
L_N(\xi) \xi_1 \cdots \xi_n = \xi \cdot \xi_1 \cdots \xi_n
\]

\[
L_N^*(\xi) \xi_1 \cdots \xi_n = \langle \xi, \xi_1 \rangle \xi_2 \cdots \xi_n
\]
\[
+ \frac{1}{N} \sum_{k=2}^{n} (\xi, \xi_k) \xi_2 \cdots \xi_{k-1} \cdot \xi_1 \cdot \xi_{k+1} \cdots \xi_n
\]

It follows that for a unit vector \( \xi \in \mathcal{S} \) we have the deformed factorial function

\[
b_{n+1} = \rho(L_N^* (\xi)^n L_N(\xi)^n) = \left(1 + \frac{1}{N}\right) \left(1 + \frac{2}{N}\right) \cdots \left(1 + \frac{n-1}{N}\right)
\]

and by Corollary 2.15 we have the following formula for the cumulants.

**Proposition 4.2.**

\[
K_n^\xi (L_N^*) = \sum_{k=1}^{\infty} \frac{\alpha_k}{(1 + \frac{1}{N}) (1 + \frac{2}{N}) \cdots (1 + \frac{k-2}{N})} L_N^{k-1} = \alpha_n
\]

**4.2. Some graph theory: the cycle cover polynomial.** In order to understand the partitioned moments and cumulants we need some graph theory. Given a multidimensional Dyck word

\[
W = L_{i_1}^{\varepsilon_1} L_{i_2}^{\varepsilon_2} \cdots L_{i_{2n}}^{\varepsilon_{2n}}
\]

with \( \varepsilon_j \in \{*, 1\} \), the expectation

\[
\rho(W) = \sum_{\pi \in \Pi_{2n}^2} \frac{t^{n-c(\pi)}}{\pi \leq \alpha}
\]

roughly has the following interpretation. To the word \( W \) we associate a digraph \( \Gamma_W \). First ignore the “colors” \( i_j \) and consider the onedimensional Dyck word \( L_{i_1}^{\varepsilon_1} L_{i_2}^{\varepsilon_2} \cdots L_{i_{2n}}^{\varepsilon_{2n}} \). The number of *’s among the \( \varepsilon_j \) is the same as the number of 1’s. Let \( \tilde{\pi} \in NC_{2n}^{(2)} \) be the unique noncrossing pair partition whose left points are the indices \( j \) with \( \varepsilon_j = * \). We interpret this partition as a bijection \( b \) between the annihilators and creators. The graph \( \Gamma_W \) has vertex set \( V = \{1, \ldots, n\} \) and we put an arrow from \( v_p \) to \( v_{p'} \) if \( L_{i_p}^* \) and \( L_{i_{p'}}^* \) have the same color and \( b(p') > p \).

In other words, we first construct a bipartite graph \( B \) on the two sets of vertices: The set of annihilators \( V_s = \{p(1) < p(2) < \cdots < p(n)\} \) and creators \( V_1 = \{q(1) < q(2) < \cdots < q(n)\} \). The edges are \( E = \{(p(r), q(s)) : i_p(r) = i_q(s) \text{ and } q(s) > p(r)\} \). There is a unique noncrossing pair partition \( \pi \in NC_{2n}^{(2)} \) consisting of pairs \( \{p < q\} \) s.t. \( \varepsilon_p = * \) and \( \varepsilon_q = 1 \). It can be constructed recursively by connecting the rightmost annihilator to its right neighbour, removing both from the word and repeating the procedure on the new word. In particular, considering \( \pi \) as a permutation (product of transpositions), we have \( \pi(p(n)) = p(n) + 1 \). Then

\[
\rho(L_{i_1}^{\varepsilon_1} L_{i_2}^{\varepsilon_2} \cdots L_{i_{2n}}^{\varepsilon_{2n}}) = \sum_{\mu} \frac{1}{N^{n-c(\sigma)}}
\]

where the sum runs over all complete matchings of the bipartite graph \( B \) and \( \sigma \in \mathcal{S}_n \) is the permutation on \( V_s \) which maps \( p(i) \) to \( \pi^{-1} \circ \mu(i) \), if we consider \( \pi \) and \( \mu \) as bijections from \( V_s \) to \( V_1 \).

Further we construct a directed graph \( \Gamma \) with vertex set \( \{p(1), \ldots, p(n)\} \) which we relabel as \( v_1, \ldots, v_n \) and we put a directed edge from \( v_r \) to \( v_s \) if \( \{p(r), \pi(p(s))\} \in E(B) \), where \( B \) is the bipartite graph constructed above. This is the contraction of bipartite graph \( B \) along the bijection \( \pi \), which was also constructed in [Las02].

In other words, \( v_r v_s \in E(\Gamma) \) if and only if \( i_{p(r)} = i_{\pi(p(s))} = i_{q(s)} \) and \( \pi(p(s)) > p(r) \). A complete matching of \( B \) corresponds to a partition of \( \Gamma \) into hamiltonian cycles, that is, a
partition of the vertices into vertex-disjoint cycles, and the number of cycles of $\sigma$ is equal to the number of components of the Hamiltonian partition (also called a cycle cover or 2-factor).

We are therefore led to the so-called cycle cover polynomial $C_c(\Gamma; x)$. This is a specialization of both the cover polynomial $C!(\Gamma; x, y)$ of Chung and Graham [CG95] and the geometric cover polynomial $C(\Gamma; x, y)$ of D’Antona and Munarini [DM00], which have been proposed as a kind of Tutte polynomial for digraphs. Namely we have

\[ C!(\Gamma; x, y) = \sum_{(C, P)} x^{|C|} y^{|P|} \]

here the sum runs over all cycle-path covers $(C, P)$ (cycle covers $C$, respectively) of $\Gamma$ with $|C|$ cycles and $|P|$ paths. Loops are interpreted as cycles of length 1 and single vertices as paths of length 0 and $y_n = y(y-1) \cdots (y-n+1)$ is the falling factorial function.

**Theorem 4.3.**

\[ \rho(L^\varepsilon_1 L^\varepsilon_2 \cdots L^\varepsilon_{2n}) = \sum_{i} \delta(i, i) = \sum_{i} \delta(i, i+1) + d\Gamma_{ji} \]

with $d\Gamma_{ji} = d\Gamma(|e_j\rangle \langle e_i|)$ where for an operator $T \in B(\mathfrak{g})$ we denote by $d\Gamma(T)$ its differential second quantization

\[ d\Gamma(T) = \sum_{i=1}^n \xi_i \cdots T \xi_i \cdots \xi_n. \]

These operators act like derivations on analytic vectors:

\[ d\Gamma_{ji} L_{i_1} \cdots L_{i_n} \Omega = \sum_{k=1}^n \delta(i, i_k) L_{i_1} \cdots L_{i_{k-1}} L_j L_{i_{k+1}} \cdots L_{i_n} \Omega \]

i.e., it replaces $L_i$ by $L_j$ and for any noncommutative polynomial $P$

\[ d\Gamma_{ji} P(L_1, L_2, \ldots) \Omega = D_i P(L_1, L_2, \ldots) |L_j\rangle \Omega, \]

the noncommutative derivative of $P$ with respect to $L_i$ in the direction $L_j$. In particular $d\Gamma_{ii} P \Omega = n_i P \Omega$, where $n_i$ is the total degree of $L_i$ in $P$.

Therefore we have the following recursive procedure to compute the expectation of a multidimensional Dyck word $W = L^\varepsilon_1 L^\varepsilon_2 \cdots L^\varepsilon_{2n}$, namely to look for the rightmost annihilator $L^*_{i_p(n)}$, replace $L^*_{i_p(n)} L_{i_p(n+1)}$ by $\delta(i_p(n), i_p(n+1)) I + \frac{1}{N} d\Gamma_{i_p(n+1) i_p(n)}$ and apply (4.2).

In terms of cover polynomials this corresponds to the cut and fuse recursion, which is a convenient way to compute the cover polynomial. Fix an edge $e = v_1 v_2$ of $\Gamma$. Then it is
The second sum runs over all edges emanating from \( v \) and \( v \).

**Claim 4.4.** There is a one-to-one correspondence between the recursion

\[
\rho(L_{i_1}^* \cdots L_{i_{k+1}}^* L_{i_{k+2}} \cdots L_{i_{2n}}^*) = \rho(L_{i_1}^* \cdots L_{i_{k-1}}^* (\delta(i_k, i_{k+1}) I + \frac{1}{N} d_{i_{k+1}, i_k}) L_{i_{k+2}} \cdots L_{i_{2n}}^*)
\]

and (4.3).

There are two cases to consider, depending on whether \( i_{p(n)} = i_{p(n)+1} \) or not. First assume \( i_{p(n)} = i_{p(n)+1} = i \), then with \( k = p(n) \) we have

\[
\rho(L_{i_1}^* \cdots L_{i_k}^* L_{i_{k+1}} \cdots L_{i_{2n}}^*) = \rho(L_{i_1}^* \cdots L_{i_{k-1}}^* (I + \frac{1}{N} d_{i_{k+1}, i_k}) L_{i_{k+2}} \cdots L_{i_{2n}}^*)
\]

\[= (1 + \frac{n_i}{N})\rho(L_{i_1}^* \cdots L_{i_{k-1}} L_{i_{k+2}} \cdots L_{i_{2n}}^*)
\]

where \( n_i \) is the total degree of \( L_i \) in \( L_{i_{k+2}} \cdots L_{i_{2n}}^* \). On the graph side, there is a loop from \( v_n \) to itself, and therefore removing successively all edges emanating from \( v_n \) we obtain

\[
C_c(\Gamma; x) = x C_c(\Gamma \setminus v_n; x) + C_c(\Gamma \setminus v_n v_n; x)
\]

\[= x C_c(\Gamma \setminus v_n; x) + \sum_{\substack{i_{p(n)} = i_{p(t)} \\pi(p(t)) > p(n)}} C_c(\Gamma/v_n v_t; x)
\]

The second sum runs over all edges emanating from \( v_n \). The remaining term \( C_c(\Gamma \setminus \{v_n v_n, v_n v_t, \ldots, v_n v_n\}; x) \) vanishes because there is no cycle through \( v_n \) anymore.

Now we show that each \( \Gamma/v_n v_t \) is isomorphic to \( \Gamma \setminus v_n \). Indeed \( v_r v_s \in E(\Gamma \setminus v_n) \) if and only if \( v_r v_s \in E(\Gamma) \), while \( v_r v_s \in E(\Gamma/v_n v_t) \) if and only if either \( s \neq t \) and \( v_r v_s \in E(\Gamma) \) or \( s = t \) and \( v_r v_n \in E(\Gamma) \). The first case being trivial, consider the case \( s = t \). Then \( v_n v_t \in E(\Gamma) \) implies \( i_{p(n)} = i_{\pi(p(t))} \) and \( \pi(p(t)) > p(n) \). Now \( v_n v_t \in E(\Gamma) \) if and only if \( i_{p(r)} = i_{\pi(p(t))} \) and \( \pi(p(t)) > p(r) \). The second condition is redundant because already \( p(n) > p(r) \).

On the other hand,

\[
v_r v_n \in E(\Gamma) \iff i_{p(r)} = i_{\pi(p(n))} \text{ and } \pi(p(n)) = p(n) + 1 > p(r)
\]

and again the second condition is automatically satisfied. Moreover \( i_{\pi(p(n))} = i_{p(n)+1} = i_{p(n)} = i_{\pi(p(t))} \). Thus \( v_r v_t \in E(\Gamma) \) if and only if \( v_r v_n \in E(\Gamma) \) and finally \( v_r v_s \in E(\Gamma/v_n v_t) \) if and only if \( v_r v_s \in E(\Gamma) \) for all \( s \).
Now assume that $i_p(n) \neq i_p(n+1)$. Then with $k = p(n)$
\[\rho(L^*_1 \cdots L^*_k L_{i_k+1} \cdots L_{i_{2n}}) = \rho(L^*_1 \cdots L^*_k \frac{1}{N} d\Gamma_{i_{k+1}, i_k} L_{i_{k+2}} \cdots L_{i_{2n}})\]
\[= \frac{1}{N} \sum_{r=k+2}^{2n} \delta(i_k, i_r) \rho(L^*_1 \cdots L^*_k \cdots L_{i_{r-1} L_{i_{k+1}} L_{i_{r+1}} L_{i_{2n}}}\]

On the graph side, this corresponds to the identity
\[C_c(\Gamma; x) = \sum_{e, \epsilon = \epsilon_n} C(\Gamma / e; x) = \sum_{t \neq \pi(n), \pi(n)+1}^t \alpha\pi'(\pi(p(t))) / \pi(p(t)) > p(n)\]
and there is a one to one correspondence between the summands: Let $r = \pi(p(t)) > p(n)$, then the graph $\Gamma_r$ corresponding to $\rho(L^*_1 \cdots L^*_k \cdots L_{i_{r-1} L_{i_{k+1}} L_{i_{r+1}} L_{i_{2n}}})$ has vertices $v_1, \ldots, v_{n-1}$, partition $\pi' = \pi \setminus \{p(n), p(n) + 1\}$ and edges
\[v_x v_y \in E(\Gamma_r) \iff \begin{cases} i_p(x) = i_{\pi(p(y))} \text{ and } \pi'(p(y)) > p(x) & \text{if } y \neq t \\ i_p(x) = i_{p(n)+1} & \text{if } y = t \end{cases}\]
\[\iff \begin{cases} v_x v_y \in E(\Gamma) & \text{if } y \neq t \\ v_x v_n \in E(\Gamma) & \text{if } y = t \end{cases}\]
\[\iff v_x v_y \in E(\Gamma / v_n v_1)\]

**Remark 4.5.** Not every digraph is the digraph of a Dyck word. A necessary condition is, that the vertices with common successors can be linearly ordered. Denote by $R(v)$ the set of successors of a vertex $v$. Given two vertices $v_i$ and $v_j$, if there is a vertex $v_k$ such that both $R(v_i) \cap R(v_k) \neq \emptyset$ and $R(v_j) \cap R(v_k) \neq \emptyset$, then either $R(v_i) \subseteq R(v_j)$ or $R(v_j) \subseteq R(v_i)$.

### 4.3. The Vershik-Kerov construction.
We recall the construction of the representations of Vershik and Kerov [VK81] which give rise to the irreducible characters of the infinite symmetric group $S_\infty$ which were first found by Thoma [Tho64], namely
\[\varphi_{\alpha, \beta}(\sigma) = \prod_{m \geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right)^{\rho_m(\sigma)}\]
where $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ and $\beta_1 \geq \beta_2 \geq \cdots \geq 0$ are given sequences such that $\sum \alpha_i + \sum \beta_i \leq 1$ and $\rho_m(\sigma)$ is the number of cycles of length $m$ of $\sigma$.

To keep notations simple we consider only the case where $\beta_i = 0$ and $\sum \alpha_i = 1$. In this case $\alpha_n$ can be interpreted as a probability measure $m^{(\alpha)}$ on $N$ with $m^{(\alpha)}(\{k\}) = \alpha_k$. We consider the elements of the space $\mathfrak{X}_n = \mathbb{N}^n$ with the product measure $m^{(\alpha)}$ as words $x = x_1 x_2 \cdots x_n$ and abbreviate the value of the measure by
\[\alpha_x = m^{(\alpha)}(\{x\}) = \prod_{i=1}^{n} \alpha_{x_i}\]

$S_n$ acts on $\mathfrak{X}_n$ by $(\sigma x)_i = x_{\sigma^{-1}(i)}$. We write $x \sim y$ if $x$ and $y$ are in the same orbit, i.e., if there exists a permutation $\sigma \in S_n$ such that $y = \sigma x$. Now consider the space of
“rearrangements”
\[ \tilde{X}_n = \{(x, y) \in X_n \times X_n : x \sim y\}. \]

If we equip the space of functions \( f : \tilde{X} \to \mathbb{C} \) with the Hilbert norm
\[ \|f\|_2^2 = \int_{X_n} \sum_{y \sim x} |f(x, y)|^2 \, dm_n(\alpha)(x) = \sum_{x \in X_n} \sum_{y \sim x} |f(x, y)|^2, \]
then the Hilbert space
\[ V_n^{(\alpha)} = \{ f : \tilde{X} \to \mathbb{C} : \|f\|_2^2 < \infty \} \]
carries a unitary representation \( U_n^{(\alpha)} \) of \( \mathcal{G}_n \)
\[ (U_n^{(\alpha)}(\sigma) h)(x, y) = h(\sigma^{-1} x, y) \]
This space has an orthogonal basis \( \{ \delta_{x, y} : (x, y) \in \tilde{X}_n \} \) of functions \( \delta_{x, y}(x', y') = \delta(x, x') \delta(y, y') \).
The action of \( \mathcal{G}_n \) is
\[ U_n^{(\alpha)}(\sigma) \delta_{x, y} = \delta_{\sigma x, y} \]
and their inner products are
\[ \langle \delta_{x', y'}, \delta_{x'', y''} \rangle_{V_n^{(\alpha)}} = \delta(x', x'') \delta(y', y'') \alpha_{x'} \]

**Theorem 4.6 ([VK81]).** Let \( 1_n = \sum_{x \in \tilde{X}_n} \delta_{x, x} \) be the diagonal, then
\[ \langle U_n^{(\alpha)}(\sigma) 1_n, 1_n \rangle = m_n^{(\alpha)}(\{ x : \sigma x = x \}) = \varphi_{\alpha, 0}(\sigma) \]

**Proposition 4.7.** The maps \( j_n : V_n^{(\alpha)} \to V_{n+1}^{(\alpha)} \) defined by
\[ j_n h(x, y) = \delta(x_{n+1}, y_{n+1}) h(x_1 \cdots x_n, y_1 \cdots y_n), \]
i.e.,
\[ j_n \delta_{x, y} = \sum_{z \in \tilde{X}_1} \delta_{xz, yz} \]
clearly satisfy the intertwining relation (2.2) and their adjoints are given by
\[ j_n^* \delta_{xz, yz'} = \delta(z, z') \alpha_z \delta_{x, y}. \]

We can therefore construct the symmetric Fock space spanned by the vectors
\[ \delta_{x, y} \otimes_s \xi_1 \cdot \xi_2 \cdots \xi_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{G}_n} \delta_{\sigma x, y} \otimes \xi_{\sigma^{-1}(1)} \cdot \xi_{\sigma^{-1}(2)} \cdots \xi_{\sigma^{-1}(n)} \]
It will be notationally more convenient to work with the right creation and annihilation operators (2.3)
\[ R(\xi_{n+1}) \delta_{x, y} \otimes_s \xi_1 \cdot \xi_2 \cdots \xi_n = (n + 1) j_n \delta_{xz, yz} \otimes_s \xi_1 \cdot \xi_2 \cdots \xi_{n+1} \]
\[ = (n + 1) \sum_{z \in \tilde{X}_1} \delta_{xz, yz} \otimes_s \xi_1 \cdot \xi_2 \cdots \xi_{n+1} \]

The annihilation operator is given by (2.4) and can be simplified to
\[ R^*(\xi) \delta_{x, y} \otimes_s \xi_1 \cdot \xi_2 \cdots \xi_{n+1} \]
\[ = \frac{1}{n + 1} \sum_{i=1}^{n+1} \langle \xi_i, \xi \rangle j_n^* U_{n+1}(\tau_{i, n+1}) \delta_{x, y} \otimes_s \xi_1 \cdots \xi_{i-1} \cdot \xi_{n+1} \cdot \xi_{i+1} \cdots \xi_n \]
Definition 4.8. For an operator $T \in B(\mathcal{H})$ we define weighted differential second quantisation operators

$$d\Gamma^{(k)}(T) \delta_{x,y} \otimes_s \xi_1 \cdots \xi_n = \sum_{i=1}^{n} \alpha_{\xi_i}^k \delta_{x,y} \otimes_s \xi_1 \cdots \xi_{i-1} \cdot T \xi_i \cdot \xi_{i+1} \cdots \xi_n$$

Then we have the following “commutation relation”.

Lemma 4.9.

(4.4) \[ R^*(\xi) R(\eta) = \langle \eta, \xi \rangle I + d\Gamma^{(1)}(\langle \eta \rangle \langle \xi \rangle) \]

In particular for standard basis vectors $\xi = e_i$ and $\eta = e_j$ we have

(4.5) \[ R_i^* R_j = \delta(i,j) I + d\Gamma^{(1)}_{ji} \]

where $d\Gamma^{(k)}_{ji} = d\Gamma^{(k)}(\langle e_j \rangle \langle e_i \rangle)$.

Proof.

$$R^*(\xi) R(\xi_{n+1}) \delta_{x,y} \otimes_s \xi_1 \cdots \xi_n$$

$$= (n+1) R^*(\xi) \sum_{z \in X_1} \delta_{x,z} \otimes_s \xi_1 \cdots \xi_{n+1}$$

$$= \sum_{z \in X_1} \sum_{i=1}^{n+1} \langle \xi_i, \xi \rangle j_n^* U_{n+1}(\tau_{i,n+1}) \delta_{x,z} \otimes_s \bar{U}_{n+1}(\tau_{i,n+1}) \xi_1 \cdots \xi_{n+1}$$

$$= \sum_{z \in X_1} \sum_{i=1}^{n} \langle \xi_i, \xi \rangle \delta(x_i, z) \alpha_z \delta_{x_1 \cdots x_{i-1} x_i z, y} \otimes_s \xi_1 \cdots \xi_{i-1} \xi_i \cdots \xi_{n+1}$$

$$+ \sum_{z \in X_1} \langle \xi_{n+1}, \xi \rangle \alpha_z \delta_{x,y} \otimes_s \xi_1 \cdots \xi_n$$

$$= \sum_{i=1}^{n} \langle \xi_i, \xi \rangle \delta(x_i, z) \alpha_z \delta_{x_1 \cdots x_{i-1} x_i , y} \otimes_s \xi_1 \cdots \xi_{i-1} \xi_i \cdots \xi_{n+1}$$

$$+ \langle \xi_{n+1}, \xi \rangle \delta_{x,y} \otimes_s \xi_1 \cdots \xi_n$$

$$= \sum_{i=1}^{n} \langle \xi_i, \xi \rangle \delta(x_i, z) \alpha_z \delta_{x,y} \otimes_s \xi_1 \cdots \xi_{i-1} \xi_i \cdots \xi_{n+1}$$

$$+ \langle \xi_{n+1}, \xi \rangle \delta_{x,y} \otimes_s \xi_1 \cdots \xi_n$$

It will also be convenient to have weighted creation and annihilation operators at our disposal.
**Definition 4.10.** For \( k \in \mathbb{N}_0 \) let
\[
R^{(k)}(\xi) \delta_{x,y} \otimes_{s} \xi_1 \cdots \xi_n = (n + 1) \sum_{z \in X_1} \alpha_z^k \delta_{x,z} \otimes_{s} \xi_1 \cdots \xi_n \cdot \xi
\]
in particular, \( R^{(0)}(\xi) = R(\xi) \).

Then the action of \( d\Gamma_{ji} \) on analytic vectors is like a derivation:

**Lemma 4.11.**
\[
d\Gamma^{(k)}_{ji} R^{(k_n)}_{i_n} \cdots R^{(k_1)}_{i_1} \Omega = \sum_{j=1}^{n} \delta(i, i_j) R^{(k_n)}_{i_n} \cdots R^{(k_{j+1})}_{i_{j+1}} R^{(k_{j+1})}_{i_{j+1}} R^{(k_{j-1})}_{i_{j-1}} \cdots R^{(k_1)}_{i_1} \Omega
\]

The adjoint of a weighted creation operator is a weighted annihilation operator, as expected.

**Proposition 4.12.**
\[
R^{(k)}(\xi)^* \delta_{x,y} \otimes_{s} \xi_1 \cdots \xi_n = \frac{1}{n} \sum_{i=1}^{n} \delta(x_i, y_n) \alpha_{x_i}^{k+1} \langle \xi_i, \xi \rangle \delta_{x_1 \cdots x_i-1, y_1 \cdots y_{n-1}} \otimes_{s} \xi_1 \cdots \xi_i \cdots \xi_n
\]

The general “commutation relation” now is as follows.

**Proposition 4.13.**
\[
R^{(k)}(\xi)^* R^{(m)}(\eta) = \langle \xi, \eta \rangle \sum_{z} \alpha_z^{m+k+1} I + d\Gamma^{(m+k+1)}(|\xi \rangle \langle \eta|)
\]

and in particular
\[
R^{(k)}_{i} R^{(m)}_{j} = \delta(i, j) \sum_{z} \alpha_z^{m+k+1} I + d\Gamma^{(m+k+1)}_{ji}
\]

In this case the expectations of multidimensional Dyck words is given by evaluations of the *cycle indicator polynomial* of D’Antona and Munarini.

### 4.4. More graph theory: the cycle indicator polynomial.

**Definition 4.14 ([DM00]).** Let \( \Gamma \) be a digraph. The *cycle-path indicator polynomial* of \( \Gamma \) is the multivariate polynomial
\[
I(\Gamma; x, y) = \sum_{(C, P) \gamma \in C} \prod_{\gamma \in C} x_{|\gamma|} \prod_{\pi \in P} y_{|\pi|}
\]

where the sum runs over all cycle-path coverings \((C, P)\) of \( \Gamma \), and \(|\gamma|\) (resp. \(|\pi|\)) denotes the number of vertices of a cycle \( \gamma \) (resp. a path \( \pi \)).

The cycle-path indicator polynomial satisfies a recursion similar to the recursion (4.3) of the geometric cover polynomial. However in order to do the cut and fuse operations, one somehow has to remember the deleted vertices. This is done by putting weights on the vertices.

**Definition 4.15.** Let \( w : V(\gamma) \to \mathbb{N} \) be a weight function on the vertices of the digraph \( \Gamma \) and denote the weighted graph by \( \Gamma_w = (\Gamma, w) \). For a path \( \pi = v_1 \cdots v_k \) let \( w(\pi) = \)
$w(v_1) + \cdots + w(v_k)$ and similarly the weight of a cycle is the sum of the weights of its vertices. Define the cycle-path indicator polynomial of $\Gamma_w$ as

$$I(\Gamma_w; x, y) = \sum_{(C, P)} \prod_{\gamma \in C} x_{w(\gamma)} \prod_{\pi \in P} y_{w(\pi)}$$

An unweighted graph therefore has the same cycle-path indicator polynomial as the same graph with weight 1 on each vertex.

We define the cut and fuse operations on weighted digraphs as follows.

- $\Gamma_w \setminus v = (\Gamma \setminus v, w|_{\Gamma \setminus v})$ for $v \in V(\Gamma)$
- $\Gamma_w \setminus e = (\Gamma \setminus e, w)$ for $w \in E(\Gamma)$
- $\Gamma_w / e = (\Gamma / e, w^*)$ for $e = v_1v_2 \in E(\Gamma)$

where we denote $v_0$ the collapsed vertex and

$$w^*(v) = \begin{cases} w(v) & v \neq v_0 \\ w(v_1) + w(v_2) & v = v_0 \end{cases}$$

**Proposition 4.16 ([DM00]).** The cycle-path indicator polynomial satisfies the following cut-and-fuse recursion. Fix an edge $e$, then

$$I(\Gamma_w; x, y) = \begin{cases} x_{w(v)} I(\Gamma_w \setminus v; x, y) + I(\Gamma_w \setminus e; x, y) & \text{if } e = vv \text{ is a loop} \\ I(\Gamma_w \setminus e; x, y) + I(\Gamma_w / e; x, y) & \text{if } e = v_1v_2 \text{ is not a loop} \end{cases}$$

If $\Gamma$ has no edges, then

$$I(\Gamma_w; x, by) = \prod_v y_{w(v)}$$

and if $\Gamma$ is empty, then $I(\emptyset; x, y) = 1$.

We are only interested in the cycle indicator polynomial

$$I_c(\Gamma_w; x) = I(\Gamma_w; x, 0).$$

Again the commutation relation (4.5) corresponds to cut and fuse operations and we obtain the following formula. Roughly the expectation of a weighted multidimensional Dyck word is given by the cycle indicator of the weighted graph $\Gamma_w$, where $\Gamma$ is constructed as in section 4.2 and the weight of a vertex is computed from the weight of the corresponding annihilator plus one plus the weight of the creator to which it is connected by $\pi$.

**Theorem 4.17.**

$$\rho(R_{i_1}^{(k_1)} \cdots R_{i_n}^{(k_n)}) = I_c(\Gamma_w; x)$$

where $\Gamma_w$ is the graph with weights $w(v_i) = k_{p(i)} + 1 + k_{\pi(p(i))}$ and $x_1 = 1$,

$$x_k = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k+1} \sum_{i=1}^{\infty} \beta_i^k$$

for $k \geq 2$. 
Theorem 4.18. Any positive definite functional on pair partitions whose values only depend on the graphs \( \Gamma(\pi) \) is given by a (not necessarily irreducible) character on the infinite symmetric group. The multiplicative ones are exactly those which come from irreducible characters of the symmetric group.

Proof. It suffices to show this for pair partitions. We will construct the representation of \( S_n \) as in the proof of [GM02, Theorem 2.7]. Let \( H_n \) be the space spanned by products

\[
R_{\sigma(1)} R_{\sigma(2)} \cdots R_{\sigma(n)} \Omega
\]

this space carries a natural representation \( U_n \) of \( S_n \), namely

\[
U_n(\sigma) R_{\tau(1)} R_{\tau(2)} \cdots R_{\tau(n)} \Omega = R_{\sigma \tau(1)} R_{\sigma \tau(2)} \cdots R_{\sigma \tau(n)} \Omega
\]

and therefore for any partition with given cycle structure the expectation function is given by

\[
\langle U_n(\sigma) R_n R_{n-1} \cdots R_1 \Omega, R_n R_{n-1} \cdots R_1 \Omega \rangle = \langle R_n^* R_2^* \cdots R_n^* R_{\sigma(n)} R_{\sigma(n-1)} \cdots R_{\sigma(1)} \Omega, \Omega \rangle = t(\pi)
\]

where \( \sigma \) has the same cycle structure as \( \pi \). \qed

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