0\(^+\) dominance with random interactions

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Abstract. The ground states of all even-even nuclei have angular momentum equal to zero, \(I = 0\), and positive parity, \(\pi = +\). This feature was believed to be a consequence of the attractive short-range interaction between nucleons. However, a predominance of \(I^\pi = 0^+\) ground states was discovered in 1998 using the two-body random ensemble. Since then many efforts have been devoted to understand and solve this problem from a lot of view points. Still, the underlying physical origin of the \(I^\pi = 0^+\) dominance has not been fully understood. Our recent progress is shown in understanding the 0\(^+\) dominance of many body systems.

1. Introduction
It was discovered in 1998 by Johnson and his collaborators [1] that the dominance of the 0\(^+\) ground state of even fermion systems is obtained by diagonalizing a Hamiltonian which consists of scalar two-body random interactions. Many works have been done in the context of this interesting discovery [2].

In this paper we first express the lowest (or highest) eigenenergy in terms of the random interaction by its mean energy and variance. Here we need to introduce a factor, which depends only on the dimension of the system. Then we apply the method to estimate the probability \(P(I)\), which is the probability that the \(I\) state to be the ground state.

2. The 0\(^+\) ground state dominance under two-body random interactions
2.1. Hamiltonian of the shell model and two-body random ensemble
In order to simplify the argument, we take a single \(j^n\) configuration, but the following argument is valid for many-\(j\) shells. The Hamiltonian that we use is as follows:

\[
\hat{H} = \sum_{J=0}^{2j-1} \sqrt{2J+1} G_J \left[ A^{\dagger(J)} \tilde{A}^{(J)} \right]^{(0)},
\]

with \(A^{\dagger(J)} = \frac{1}{\sqrt{2}} \left[ a^\dagger_j a_j \right]^{(J)}\) and \(\tilde{A}^{(J)} = -\frac{1}{\sqrt{2}} \left[ \tilde{a}_j \tilde{a}_j \right]^{(J)}\), where \(G_J\)'s are the strengths of two-body interactions between fermions or bosons, and are assumed to follow a two-body random ensemble (TBRE), i.e., the \(G_J\)'s are a set of random numbers with a distribution function

\[
\rho(G_J) = \frac{1}{\sqrt{2\pi}} \exp \left( -G_J^2/2 \right).
\]
Matrix elements of $\hat{H}$ among spin $I$ states can be expressed as follows:

$$H_{I\beta\gamma} = \langle j^n I\beta | \hat{H} | j^n I\gamma \rangle \equiv \sum_{J=0}^{2j-1} \alpha_{I,J}^I G_J. \tag{3}$$

For later use, the matrix $\alpha_{I}^J$ is defined as $\left(\alpha_{I}^J\right)_{\beta\gamma} \equiv \alpha_{I,J}^I, (\beta, \gamma = 1, \ldots, d_I)$, with $d_I$ being the dimension of spin $I$ states.

2.2. Probabilities of $I^+$ ground states $P(I)$ and an empirical approach to predict $P(I)$

All probabilities are obtained by diagonalizing 1000 times the TBRE Hamiltonians for $j^n$ systems. The results are shown in Fig. 1, in which the probability of $I^+$ being the ground state, $P(I)$, is shown as a function of $j$. Except for the cases $j = 7/2$ and 13/2, we clearly see the $0^+$ dominance.

Let us set only one of the $G_J$ to be $-1$ and all others equal to zero. Diagonalizing the Hamiltonian for all possible $I$ states, we can find which angular momentum $I$ takes the lowest energy eigenvalue. We repeat this process for all $J = 0, 2, \ldots, 2j - 1$. The result is shown in Table 1. We then count how many times a certain $I$ spin state becomes the ground state. This number is denoted as $N_I$. For example, when $j = 13/2$, we find that the $0^+$ ground state appears twice, the $0^+$ ground state twice, and the $4^+, 12^+$ and $20^+$ ground states once. We then obtain an empirical formula $P(I)^{\text{emp}} = N_I / N$ to predict the probability for $I^+$ to be the ground state. In this formula, $N$ stands for the total number of possible $J$’s. (In the present case $N = (2j + 1)/2$.) Comparing them with $P(I)$ obtained by diagonalization, we see good agreements. The empirical formula predicts the $0^+$ dominance very well. We have, however, the following self-criticism;

- Diagonalization, which is very tedious, is unavoidable to get $N_I$ for each $G_J$ (that is $(2j - 1)/2$ times except the trivial case of $J = 0$), although $(2j - 1)/2$ is much smaller than 1000.
- The physical reason why this empirical formula works reasonably well is not clear.
Table 1. Angular momenta which give the lowest energy eigenvalues when $G_J = -1$ and all other two-body matrix elements are zero, for the case of four fermions in a single-$j$ shell.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $G_{20}$ | $G_{22}$ | $G_{24}$ | $G_{26}$ | $G_{28}$ | $G_{30}$ |
|------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 7    | 0     | 4     | 2     | 8     |       |         |         |         |         |         |         |         |         |         |         |         |
| 9    | 0     | 4     | 0     | 0     | 12    |         |         |         |         |         |         |         |         |         |         |         |
| 11   | 0     | 4     | 0     | 4     | 8     | 16      |         |         |         |         |         |         |         |         |         |         |
| 13   | 0     | 4     | 0     | 2     | 2     | 12      | 20      |         |         |         |         |         |         |         |         |         |
| 15   | 0     | 4     | 0     | 2     | 0     | 0       | 16      | 24      |         |         |         |         |         |         |         |         |
| 17   | 0     | 4     | 6     | 0     | 4     | 2       | 0       | 20      | 28      |         |         |         |         |         |         |         |
| 19   | 0     | 4     | 8     | 0     | 2     | 8       | 2       | 16      | 24      | 32      |         |         |         |         |         |         |
| 21   | 0     | 4     | 8     | 0     | 2     | 0       | 0       | 20      | 28      | 36      |         |         |         |         |         |         |
| 23   | 0     | 4     | 8     | 0     | 2     | 0       | 10      | 2       | 24      | 32      | 40      |         |         |         |         |         |
| 25   | 0     | 4     | 8     | 0     | 2     | 4       | 8       | 10      | 6       | 28      | 36      | 44      |         |         |         |         |
| 27   | 0     | 4     | 8     | 0     | 2     | 4       | 2       | 0       | 4       | 20      | 32      | 40      | 48      |         |         |         |
| 29   | 0     | 4     | 8     | 0     | 0     | 2       | 6       | 8       | 12      | 8       | 24      | 36      | 44      | 52      |         |         |
| 31   | 0     | 4     | 8     | 0     | 0     | 2       | 0       | 8       | 14      | 16      | 6       | 32      | 40      | 48      | 56      |         |

- $P(I_{\text{max}})_{\text{emp}} = 1/N$ is alright for fermions, but not a very good prediction for boson systems. In this case $P(I_{\text{max}})$ calculated by diagonalizing the Hamiltonian is about two times as large as $P(I_{\text{max}})_{\text{emp}}$.
- Results should depend on the shape of the ensemble, but the empirical method ignores it.

3. Average energies

3.1. Average of $\alpha_I^j$ and average energies

We then proceed to discuss the average of the energies of $E_{I\beta}$ ($\beta = 1, \ldots, d_I$), which is written as

$$\bar{E}_I = \frac{1}{d_I} \sum_{\beta=1}^{d_I} E_{I\beta} = \frac{1}{d_I} \text{Tr} \left( \hat{H} \right) = \sum_{J=0}^{2j-1} \bar{\alpha}_I^J G_J,$$

where

$$\bar{\alpha}_I^J = \frac{1}{d_I} \sum_{\beta} \alpha_{I\beta\beta} = \frac{1}{d_I} \text{Tr} \left( \alpha_I^J \right),$$

and $d_I$ is the dimension with angular momentum $I$.

By taking 1000 random ensembles of $\{G_J\}$, we can calculate $P(I)$ directly by using the above formula for the average $\bar{E}_I$. The results show that $P(I_{\text{max}} = 24)$ is the largest probability, $P(3)$ is the second, $P(2)$ the third, $P(0)$ the fourth, and others are very small when $j = 15/2$ and $n = 4$. More generally, $P(0) \sim P(2) \sim P(4) \ll P(I_{\text{max}})$, and the other $P(I) \approx 0$. Namely $P(0)$ is not the largest, and this means that, the $0^+$ states do not necessarily have the lowest average energy.

However, the dispersion of $E_{I=0}$ around $\bar{E}_{I=0}$ seems to be very large, and therefore the lowest energy state among many others with $I = 0$ has a large possibility to be the ground state. By the way, the dispersion of $E_{I_{\text{max}}}$ is zero because there is only one $I_{\text{max}}$ state. In the next subsection, let us take into account the dispersion of $\bar{E}_I$ around $\bar{E}_I$. 
3.2. Square of average energies \((\bar{E}_I)^2\) and average of squared energies \((E_I)^2\)

The squared average energies \((\bar{E}_I)^2\) are written as

\[
(\bar{E}_I)^2 = \left( \sum_{J}^{2j-1} \bar{\alpha}_I^J G_J \right)^2 = \sum_{J}^{2j-1} \sum_{K}^{2j-1} \bar{\alpha}_I^J \bar{\alpha}_I^K G_J G_K,
\]

and the average of the squared energies is expressed as

\[
\langle (E_I)^2 \rangle = \frac{1}{d_I} \sum_{J=1}^{d_I} \langle (E_{I\beta})^2 \rangle = \frac{1}{d_I} \text{Tr} \left( \hat{H}^2 \right).
\]

Using these \((\bar{E}_I)^2\) and \((E_I)^2\), the dispersion of \(E_{I\beta}\) around \(\bar{E}_I\) which is denoted as \((\sigma_I \{G_J\})^2\), can be expressed as

\[
(\sigma_I \{G_J\})^2 = \frac{1}{d_I} \text{Tr} \left[ \left( \hat{H} - \bar{E}_I \right)^2 \right] = \frac{1}{d_I} \text{Tr} \left[ \left( \hat{H} \right)^2 \right] - (\bar{E}_I)^2
\]

\[
= \frac{1}{d_I} \sum_{J,K}^{2j-1} \text{Tr} \left[ \left( \alpha_I^J - \bar{\alpha}_I^J \right) \left( \alpha_I^K - \bar{\alpha}_I^K \right) \right] G_J G_K,
\]

where \(I\) is the unit matrix with dimension \(d_I\). Here arises a problem: What is an appropriate way to estimate \(E_I^{(\text{Max})}\) or \(E_I^{(\text{Min})}\) by using \(\sigma_I \{G_J\}\)? Later we will try to answer this question.

3.3. Ensemble averages

Assuming the Gaussian ensemble for the strengths of two-body interactions, we have for the ensemble averages of \(G_J\) and \(G_J G_K\): \langle G_J \rangle = 0, \langle G_J G_K \rangle = \delta_{JK}. \) We then obtain easily the ensemble averages of \((\bar{E}_I)^2\) and \((E_I)^2\). They are

\[
\langle (\bar{E}_I)^2 \rangle = \sum_{J}^{2j-1} \sum_{K}^{2j-1} \bar{\alpha}_I^J \bar{\alpha}_I^K \langle G_J G_K \rangle = \frac{1}{d_I} \sum_{J=0}^{2j-1} \text{Tr} \left[ (\bar{\alpha}_I^J I)^2 \right],
\]

and

\[
\langle (E_I)^2 \rangle = \frac{1}{d_I} \sum_{J=1}^{d_I} \sum_{\beta=1}^{d_I} \sum_{\gamma=1}^{d_I} \alpha_I^{J\beta} \alpha_I^{J\gamma} \alpha_I^{J\gamma} \alpha_I^{J\beta} = \frac{1}{d_I} \sum_{J=0}^{2j-1} \text{Tr} \left[ (\alpha_I^J I)^2 \right].
\]

We then define the dispersion \((\sigma_I)^2\) as follows:

\[
(\sigma_I)^2 \equiv \langle (E_I)^2 \rangle - \langle (\bar{E}_I)^2 \rangle = \frac{1}{d_I} \sum_{J=0}^{2j-1} \text{Tr} \left[ (\alpha_I^J I - \bar{\alpha}_I^J I)^2 \right].
\]

This \((\sigma_I)^2\) is nothing but the ensemble average of \((\sigma_I \{G_J\})^2\) defined in sec. 3-2.

We denote the dispersion of \(\alpha_I^J\) around \(\bar{\alpha}_I^J\) for each \(J\) by \((\sigma_I^J)^2\), and express it as follows:

\[
(\sigma_I^J)^2 = \frac{1}{d_I} \text{Tr} \left( (\alpha_I^J - \bar{\alpha}_I^J I)^2 \right).
\]

In terms of \((\sigma_I^J)^2\), \((\sigma_I)^2\) can be written as

\[
(\sigma_I)^2 = \frac{1}{d_I} \sum_{J=0}^{2j-1} (\sigma_I^J)^2.
\]
4. A guess to estimate the maximum (or minimum) energy \(E_i^{(\text{Max})}\) (or \(E_i^{(\text{Min})}\))

Let us take the following guess to estimate the largest (or smallest) eigenvalue of the matrix \(H_{I,\beta}\). We assume that the eigenvalues \(E_i, (\beta = 1, \ldots, d_I)\) are distributed with Gaussian shape

\[
\rho(E_i) = \frac{d_I}{\sqrt{2\pi} \sigma_I} \exp\left[-\frac{(E_i - \bar{E}_I)^2}{2(\sigma_I)^2}\right].
\]

To estimate \(E_i^{(\text{Max})}\) (or \(E_i^{(\text{Min})}\)), we need to solve the following equation

\[
\int_{\bar{E}_I} \rho(E_i) dE_i = \frac{d_I}{2} - 1,
\]

which means that all \(E_i\) except the maximum one can be found below \(E_i^{(\text{Max})}\). Let us change the variable \(E_i\) to \(t\) which is defined as \(t = (E_i - \bar{E}_I) / (\sqrt{2}\sigma_I)\). Then the above equation is converted to \(\text{Erfc}(t^M) = \sqrt{\pi} / d_I\), where the error function is defined as \(\text{Erfc}(x) = \int_x^\infty \exp[-t^2] dt\)

and \(t^M = \left(E_i^{(\text{Max})} - \bar{E}_I\right) / (\sqrt{2}\sigma_I)\). We cannot solve this equation analytically, although we are able to solve numerically. But for large \(d_I\) we can get \(t^M \approx \sqrt{\ln d_I - \frac{1}{2} \ln (4\pi \ln d_I)}\) by using the asymptotic expansion of the error function for large argument. Then \(E_i^{(\text{Max})} \approx \bar{E}_I + \Phi_G(d_I)\sigma_I \{G_J\}\), where \(\Phi_G(d_I) = \sqrt{2\ln d_I - \ln (4\pi \ln d_I)}\). Thus we have,

\[
E_i^{(\text{Max})} = \bar{E}_I + \Phi_G(d_I) \sqrt{\frac{1}{d_I} \sum_{J,K}^{2j-1} \text{Tr} \left[\left(\alpha_I^J - \bar{\alpha}_I^J\right) \left(\alpha_I^K - \bar{\alpha}_I^K\right)\right]} G_J G_K
\]

To easily see the microscopic origin of the \(0^+\) dominance, this equation is further simplified using the random phase approximation:

\[
\text{Tr} \left[\left(\alpha_I^J - \bar{\alpha}_I^J\right) \left(\alpha_I^K - \bar{\alpha}_I^K\right)\right] \approx \delta_{JK} \text{Tr} \left(\left(\alpha_I^J - \bar{\alpha}_I^J\right)^2\right).
\]

Then we have

\[
E_i^{(\text{Max})} = \sum_{J=0}^{2j-1} \bar{\alpha}_I^J G_J + \Phi_E(d_I) \sum_{J=0}^{2j-1} \sigma_I^J G_J
\]

For \(E_i^{(\text{Min})}\) and \(E_i^{(\text{Min})}\), the sign in the second term should be \(-\) instead of \(+\). Note that this guess is only valid when \(d_I \gg 1\). Instead of the Gaussian distribution, we may have another guess by assuming that the eigenvalues of the matrix distribute equidistantly. Then we have \(E_i^{(\text{Max})} = \sum_{J=0}^{2j-1} \bar{\alpha}_I^J G_J + \Phi_E(d_I) \sum_{J=0}^{2j-1} \sigma_I^J G_J\), where \(\Phi_E(d_I) = \sqrt{3(d_I - 1)/(d_I + 1)}\). Using these two equations to estimate \(E_i^{(\text{Max})}\) or \(E_i^{(\text{Min})}\), we can calculate the probability that the spin \(I\) state is the ground state. The results for \(I = 0\) are shown in fig. 2, and compared to the \(P(I)\) obtained by diagonalizing \(\hat{H}\) for a TBRE, and to \(P(I)_{\text{emp}} = N_I/N\). These predictions for \(P(I)\) agree reasonably well with each other. For boson systems, \(P(I_{\text{max}})_{\text{emp}}\) is almost two times larger than \(P(I_{\text{max}})\) obtained by diagonalization, but the new methods based on the above formulae predict the results of the diagonalization better than \(P(I_{\text{max}})_{\text{emp}}\). We have to mention that a
Figure 2. Probabilities of $I = 0$ ground states for different $j$ shells with four fermions: *TBRE* probabilities obtained from 1000 runs of the TBRE Hamiltonian, *eigen* empirical probability $P(I)_{emp}$, *Approx.A* approximate method assuming Gaussian distribution of eigenenergies, where the modified formula $\Phi(d_I) = \sqrt{2 \ln(d_I + 8) - \ln(4\pi \ln(d_I + 8))}$ is used in order to have a better fit for $d_I \approx 1$, *Approx.B* approximate method assuming eigenenergies distribute equidistantly.

very similar method has been reported by Papenbrock and Weidenmüller [3]. Their assumption is that the maximum energy is approximated as $E_I^{(Max)} = r_I \sigma_I \{G_J\}$, which is the same as our $\sigma_I \{G_J\}$ when $\bar{\alpha}_I^J$ is assumed to be 0. They obtained the numerical factors $r_I$ phenomenologically by the least square fit method. Our criticism to their interesting method is as follows:

- There is no microscopic method to calculate $r_I$;
- The origin of the $0^+$ dominance is not clear.

5. More rigorous treatment without taking ensemble average
In the previous section, we have assumed the *random phase approximation* to estimate the largest or the smallest eigenenergies. This is certainly not a good approximation, when the $\{G_J\}$ are large. In order to take into account the non-linear effect on $\{G_J\}$, we can generalize the previous estimate to a more general one,

$$E_I^{(Max)} = \bar{E}_I + \Phi(d_I) \sqrt{\frac{1}{d_I} \sum_{J,K} \text{Tr} \left[(\alpha_I^J - \bar{\alpha}_I^J) (\alpha_I^K - \bar{\alpha}_I^K)\right] G_J G_K}$$  \hspace{1cm} (20)

where $\Phi(d_I)$ represents a factor which depends on the dimension $d_I$ and should approach $\Phi_G(d_I)$ when $d_I \gg 1$. We are currently examining this approach to solve the $0^+$ dominance problem more generally [4].

6. Conclusion
In order to understand the origin of the $0^+$ dominance, we averaged the energies of spin $I$ states and estimated the largest (or smallest) energy among each, using the dispersion of energy
eigenvalues around their average values. Then we found an empirical formula to predict the probability for spin $I$ states to be the ground state. By using average energies and the prediction of the largest (or lowest) eigenenergy, we predicted $P(I)$ and found that it agrees reasonably well with the $P(I)$ obtained by solving secular equations with TBRE, and with $P(I)_{\text{emp}}$ which were obtained by diagonalizing the Hamiltonian for each independent two-body interaction. This approach can provide a microscopic origin of the $0^+$ dominance and a physical reason why $P(I)_{\text{emp}}$ works well.

References
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