Linear cosmological perturbations in almost scale-invariant fourth-order gravity

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We study a class of almost scale-invariant modified gravity theory, using a particular form of
\( f(R,G) = \alpha R^2 + \beta G \log G \) where \( R \) and \( G \) are the Ricci and Gauss-Bonnet scalars, respectively and \( \alpha, \beta \) are arbitrary constants. We derive the Einstein-like field equations to first order in cosmological perturbation theory in longitudinal gauge.

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I. INTRODUCTION

Recent observations show that the universe is expanding at an accelerated rate, at the moment the cause of this is still unknown. In the cosmological standard model, one can assume different mechanisms to describe dark energy (DE) such as the cosmological constant \( \Lambda \), or scalar fields: quintessence, \( k \)-essence, and many other alternatives [1–3]. Current surveys are being planned like DES [4], DESI [5], Euclid [6], and LSST [7] to probe large scales in order to find an answer to this problem.

Dark energy in general relativity (GR) is usually considered as a change in the energy momentum tensor, \( T_{\mu\nu} \), however one can change the left hand side of the Einstein field equations and take the accelerated expansion as an effect coming from the geometry of spacetime, this is usually called modified gravity. Research has focused until recently in models with \( f(R) \), a function of the Ricci scalar [8–12], but one can also focus on more complex models like \( f(R,T) \), where \( T \) is the trace of the energy momentum tensor [13]. Among this modified theories of gravity, Gauss-Bonnet (GB) gravity has been widely studied in its \( f(G) \) approach [14, 15], where gravity is required to couple with some scalar field \( G \) [16–18]. Recent work [19] has focused on the study of “almost scale-invariant theories” where \( f(R,G) \) is a function of the Ricci scalar and the GB term

\[
G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},
\]

(1.1)

where \( R_{\mu\nu} \) is the Ricci tensor and \( R_{\mu\nu\rho\sigma} \) is the Riemann tensor. This model was initially proposed as a gravitational alternative for DE and inflation in Ref. [20] and its applications to late-time cosmology have been studied in [20–22].

Modified theories of gravity do have their problems, such as Ostrogradsky instabilities [23, 24] and ghosts [25], but with the particular choice of the a Lagrangian one can avoid this typical problems and find cosmological solutions as power-law inflation and local attractors [19]. Motivated by this, our work is focused in the special choice of \( f(R,G) \) that gives

\[
\mathcal{L} = \frac{1}{2}m_p^2 \sqrt{-g} \left( \alpha R^2 + \beta G \log G \right),
\]

(1.2)

for the Lagrangian with constants \( \alpha, \beta \), where \( m_p \) is Planck’s mass, \( g \) is determinant of the metric tensor \( g_{\mu\nu} \), \( R \) is Ricci scalar, and \( G \) is Gauss-Bonnet invariant defined above.

The action is, as usual, defined as

\[
S = \int \mathcal{L} dx^4.
\]

(1.3)

The paper is organised as follows: In Section II we give a comparison on how the Einstein tensor is written in GR, \( f(R) \) and \( f(R,G) \) in general, then we present the background field equations for the Lagrangian (1.2). In Section III we present the linear order perturbed Einstein-like tensor that describes the geometry of the universe, in longitudinal gauge for scalar perturbations. In Section IV we conclude and give an outlook on future work. The expression for the Einstein-like tensor, derived from a general Lagrangian as a function \( f(R,G) \) is given in Appendix A.

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Notation. The sign convention is \((- + + +)\). Greek indices, such as \(\{\alpha, \beta, \ldots, \mu, \nu, \ldots\}\), run from 0 to 3. Latin indices, such as \(\{a, b, \ldots, i, j, k, \ldots\}\), run from 1 to 3, that is only over spatial dimensions. Throughout this cork we use the units \(c = \hbar = 1\). We use prime to denote derivatives with respect to conformal time, and we use a comma to denote partial derivatives with respect to comoving spatial coordinates, i.e.,

\[ X' \equiv \frac{\partial X}{\partial \eta}, \quad X, i \equiv \frac{\partial X}{\partial x^i}. \]  

(1.4)

For simplicity we work with a flat background spatial metric which is compatible with current observations.

II. GOVERNING EQUATIONS

In standard GR, the Einstein-Hilbert action is given by integrating the Lagrangian,

\[ \mathcal{L}_{GR} = \frac{1}{2} m_p^2 \sqrt{-g} R, \]  

(2.1)

and the field equations are obtained by varying the action with respect to the metric tensor \(g_{\mu\nu}\), in GR one has only the Einstein tensor

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \]  

(2.2)

and one can then relate the geometry with the matter content of the universe by making it equal to the energy-momentum tensor and obtaining the usual equations of motion (EOM), as

\[ G_{\mu\nu} = \kappa^2 T_{\mu\nu}, \]  

(2.3)

where \(\kappa^2 = 8\pi G_N\), \(G_N\) is Newton’s constant, and \(T_{\mu\nu}\) the energy-momentum tensor. Similarly, in the case of the most popular versions of modified gravity, one can replace the Ricci tensor, \(R\), in the Einstein-Hilbert action (2.1) with a function \(f(R)\) and get a modified Einstein tensor,

\[ \hat{G}_{\mu\nu} = R_{\mu\nu} \partial R f - (\nabla_{\mu} R)(\nabla_{\nu} R) \partial^2 R f - \frac{1}{2} R \partial R f \]  

(2.4)

where we dropped the dependency on \(R\) on \(f\) to keep the equation more compact. We can see that (2.4) reduces to the usual Einstein tensor if we set \(f(R) = R\).

In principle one can use a function \(f(R)\) as complex as one likes, however, in this paper we concentrate on the modified gravity function that includes the Gauss-Bonnet term \(G\) defined in the previous section in Eq. (1.1), and therefore we work with \(f(R, G)\). The full expression for the general Einstein-like tensor is rather lengthy and is given in Appendix. Once more, if one drops the dependency on \(G\) and makes \(f(R, G) = R\) only, one recovers the usual Einstein tensor from GR. In this paper we use a particular function \(f(R, G)\) studied in Ref. [19] given by

\[ f(R, G) = \alpha R^2 + \beta G \log G, \]  

(2.5)

where \(\alpha\) and \(\beta\) are arbitrary constants. This function leads to a particular Einstein-like tensor

\[ \mathcal{G}_{\mu\nu} = 2\alpha (R_{\mu\nu} R + g_{\mu\nu} \nabla^2 R - \nabla_{\mu} \nabla_{\nu} R) - \frac{1}{2} (\alpha R^2 + \beta G \log G) g_{\mu\nu} + \frac{\beta}{2} (1 + \log G) \left[ C^1_{\mu\nu} \right] + \frac{\beta}{2G} \left[ C^2_{\mu\nu} \right] - \frac{\beta}{2G^2} \left[ C^3_{\mu\nu} \right], \]  

(2.6)

where the \(C^i_{\mu\nu}\)’s are the coefficients corresponding to the derivatives \(\partial C_i f\) from Eq. (A1) *, we will be working with this description of the universe geometry.

By modifying the Lagrangian and adding the Gauss-Bonnet term, the governing equations are also modified,

\[ \mathcal{G}_{\mu\nu} = \kappa^2 T_{\mu\nu}, \]  

(2.7)

*Note that \(\partial R \partial G f(R, G) = \partial G \partial R f(R, G) = 0\) with the choice for \(f(R, G)\) made in Eq. (2.5).
analogous to Eq. (2.3) where the right hand side stays the same, since we are only modifying the way we describe the geometry of the universe.

The perturbed Friedmann-Lemaitre-Robertson-Walker (FLRW) metric for a universe with a flat background is given by

\[
ds^2 = a^2 \left[ - (1 + 2\phi) d\eta^2 + 2 (B_i - S_i) d\eta dx^i + \left\{ (1 - 2\psi) \delta_{ij} + E_{ij} + F_{(i;j)} + \frac{1}{2} h_{ij} \right\} dx^i dx^j \right].
\]

where \( a = a(\eta) \) is the scale factor, \( \phi, B, \psi \) and \( E \) are scalar metric perturbations, \( S_i \) and \( F_i \) are vector metric perturbations, and \( h_{ij} \) is a tensor metric perturbation. The reason for splitting the metric perturbation into these three types is that the governing equations decouple at linear order, and hence we can solve each perturbation type separately. At higher order this is no longer true. In this paper we only study scalar perturbations, postponing the discussion of vector and tensor perturbations for future work.

The general energy-momentum tensor for a fluid with density \( \rho \), isotropic pressure \( P \) and fluid 4-velocity \( u^\mu \), is defined as

\[
T_{\mu\nu} = (\rho + P) u_\mu u_\nu + Pg_{\mu\nu} + \pi_{\mu\nu}.
\]

The fluid 4-velocity is subject to the constraint

\[
u = -a(1 + \phi), \quad u_i = a(v_i + B_i),
\]

where \( v \) is the scalar velocity perturbation.

With the above definitions, the components of the stress energy tensor in the background are given by

\[
T^0_0 = -\rho_0, \quad T^i_i = 0, \quad T^i_j = \delta^i_j P_0,
\]

and at first order,

\[
\delta T^0_0 = -\delta \rho, \quad \delta T^i_i = (\rho_0 + P_0)(v_i + B_i), \quad \delta T^i_j = \delta P \delta^i_j + \Pi^i_j,
\]

where \( \Pi \) is the scalar, anisotropic stress perturbation.

A. Background

In standard Einstein gravity, Eq. (2.3), the governing equations in component form can also be rewritten as the Friedmann equations,

\[
\mathcal{H}^2 = \frac{\kappa^2}{3} a^2 \rho_0, \quad \mathcal{H}' = -\frac{\kappa^2}{6} a^2 (\rho_0 + 3P_0),
\]

where \( \mathcal{H} = a'/a \) is the Hubble parameter. Note that in this work we use conformal time, related to cosmic time, \( t \), by \( d\tau = ad\eta \).
Using the definition of the modified governing equations, Eqn. 2.7 we can also find Friedmann-like equations for \( f(R, G) \) gravity. In the background these equations of motion are given, for our choice of theory and hence \( g_{\mu\nu} \), by

\[
\frac{18}{a^2} \left[ \alpha \left( H H'' - H'^2 - H^4 \right) + \frac{2}{3} \beta \left( 66 H H'' - 20 H'^2 - 44 H^4 - 31 H^2 H' \right) \right] + \frac{1}{H'} \left[ 35 H^3 H' - 52 H^6 - 6 H'^2 \right] - \frac{1}{H^2} \left[ 4 H^3 + 18 H'^2 \right] + \frac{32}{H} H'H'' + \frac{16}{H^4} H^2 H'' - \frac{13}{H^4} H'H''^2 \right] = \kappa^2 T_{00},
\]

\[
\frac{6}{a^2} \delta_{ij} \left[ \alpha \left( H H'' - H''^2 + 6 H^2 H' - H^4 \right) + \frac{2}{3} \beta \left( 9 H^2 H' - 2 H H'' + \frac{10}{3} H'^2 \right) \right] - \frac{2}{H} H'H'' + \frac{1}{H'} \left[ H^3 H'' - H^2 H''' - \frac{4}{3} H^6 \right] + \frac{1}{H' H^2} H^2 H''^2 \right] = \kappa^2 T_{ij}.
\]

The off-diagonal components of the field equations vanish in the background. Also note that the logarithmic dependence \( \log G \) cancels out. Already at the background level in this theory the field equations, (2.18) and (2.19), are more complicated than the ones in the standard GR case, Eqns. (2.16) and (2.17). In particular the equations now contain time derivatives up to fourth-order, instead of up to second-order in the standard case. The above equations have been previously derived in Ref. [38], and our results agree with the results found there.

### III. PERTURBED GOVERNING EQUATIONS

Due to the complexity of the governing equations in fourth-order gravity, we derived the components of the Einstein-like tensor in longitudinal gauge, instead of leaving the gauge unspecified. We can easily reconstruct the tensor components for an arbitrary gauge, by substituting in the definitions of the variables in longitudinal gauge given in the following.

Longitudinal gauge is widely used in the literature, it has also proven useful for calculations on small scales, since it gives evolution equations closest to the Newtonian ones and it has also been used in backreaction studies. The longitudinal gauge is completely determined by the spatial gauge choice \( \tilde{E}_\ell = 0 \) and \( \tilde{B}_\ell = 0 \). The remaining scalar metric perturbations, \( \phi \) and \( \psi \), are given as

\[
\tilde{\phi}_\ell = \phi + H (B + E') + (B - E')',
\]

\[
\tilde{\psi}_\ell = \psi - H (B - E'),
\]

note that \( \tilde{\phi}_\ell \) and \( \tilde{\psi}_\ell \) are identical to the Bardeen potentials \( \Phi \) and \( \Psi \) [30].

The fluid density perturbation, \( \delta \rho \), and scalar velocity, \( v \), are given by

\[
\tilde{\delta \rho}_\ell = \delta \rho + \rho_0^0 (B - E')',
\]

\[
\tilde{v}_\ell = v + E'.
\]

These gauge-invariant quantities are simply a gauge-invariant definition of the perturbations in the longitudinal gauge. Using these definitions one can get the general form of the field equations. From here onwards we drop the tilde and the subscript so the notation does not get cluttered.
A. Scalars

The scalar perturbations give the biggest contribution to the components of the Einstein-like tensor, and this is given by

$$
\delta g_{00} = -\frac{2}{a^2} \left[ \alpha \left\{ 18 \left( H^4 + H'^2 - H H'' \right) \phi - 9 H H' \phi' - 9 H'^2 \phi'' + 9 \left( 4 H^3 - H H' - H'' \right) \psi' \\
+ 9 \left( 2 H' - H^2 \right) \psi'' - 9 H \psi''' + 12 \left( H^2 + H' \right) \nabla^2 \phi + 15 H \nabla^2 \psi'' + 3 \nabla^2 \nabla^2 \left( \phi - 2 \psi \right) \right\} \\
+ \beta \left\{ 24 H'^2 \left[ 26 H^6 + 19 H^4 H' + 20 H^2 H'^2 + 7 H^2 H'^3 + 2 H'^4 \right] - 6 \left( 73 H^6 + 126 H^4 H' \\
+ 64 H^2 H'^2 + 32 H'^3 \right) \nabla^2 \phi + 12 \left[ 6 H^4 + 18 H^2 H' + 13 H'^2 \right] \nabla^2 \phi \right\} \\
+ \left( 6 \left[ 13 H^2 - 6 H^4 \right] H'' - 3 H \left[ 104 H^4 + 4 H^2 H' + 139 H^6 H'^2 + 322 H^4 H'^3 + 168 H^2 H'^4 \\
+ 96 H^5 \right] + 6 \left[ 35 H^8 + H^6 H' + 4 H^4 H'^2 + 76 H^2 H'^3 + 78 H^4 \right] H'' \right\} \frac{\phi'}{H^6 H'^2} \\
- \left( 71 H'^2 + 130 H^5 H' + 64 H^3 H'^2 + 32 H H'^3 - 24 H^4 H'' - 72 H^2 H' H'' - 52 H^2 H'' \right) \frac{3 \psi''}{H^3 H'} \\
- \left( 104 H'^2 - 620 H^4 H' - 189 H^8 H'^2 + 174 H^6 H'^3 + 176 H^4 H'^4 + 112 H^2 H'^5 - 70 H^3 H'' \\
+ 213 H^2 H' H'' + 1187 H^5 H'^2 H'' - 216 H^3 H'^3 H'' - 252 H^4 H'^4 H'' + 12 H^6 H''^2 + 46 H^2 H'^2 H''^2 \\
+ 104 H^4 H'' \right) \frac{3 \psi'}{H^5 H'^2} \\
- \left( 104 H^4 + 75 H^8 H' + 56 H^6 H'^2 - 4 H^4 H'^3 + 8 H^2 H'^4 - 70 H^7 H'' - 26 H^5 H' H'' - 8 H^3 H'^2 H'' \\
+ 12 H^6 H'^2 H'' + 26 H^2 H'' \right) \frac{3 \psi''}{H^6 H'^2} \right\} \\
- \left( 71 H'^2 + 130 H^5 H' + 64 H^3 H'^2 + 32 H H'^3 - 24 H^4 H'' - 72 H^2 H' H'' - 52 H^2 H'' \right) \frac{3 \psi''}{H^4 H'} \\
+ \left( 4 H'^2 \left[ 18 H^4 + 44 H^2 H'^2 + 77 H^4 H'^2 + 51 H^6 H' - 26 \right] + 2 \left[ 13 H'^2 - 6 H^4 \right] H'' \right\} \frac{\nabla^2 \phi}{H^4 H'^2} \\
+ 2 H \left[ 35 H^6 - 34 H^4 H' - 109 H^2 H'^2 - 68 H^6 \right] H'' \right\} \frac{\nabla^2 \phi}{H^4 H'^2} \\
- \left( 70 H'^2 + 130 H^5 H' + 66 H^3 H'^2 + 32 H H'^3 - 24 H^4 H'' - 73 H^2 H' H'' - 52 H^2 H'' \right) \frac{\nabla^2 \phi'}{H^3 H'} \\
- \left( 104 H'^2 + 42 H^4 H' + 2 H^6 H'^2 - 20 H^4 H'^3 - 8 H^2 H'^4 - 36 H^7 H'' + H^5 H' H'' + 32 H^3 H'^2 H'' \\
+ 32 H^5 H'' - 18 H^2 H'' \right) \frac{3 \nabla^2 \phi}{H^5 H'} \\
+ \left( 209 H'^2 + 236 H^4 H' + 100 H^3 H'^2 + 48 H^2 H'^3 - 70 H^4 H'' - 152 H^2 H' H'' - 72 H^2 H'' \right) \frac{3 \nabla^2 \phi'}{H^4 H'} \\
+ \left( \frac{H^2}{H} \right) \frac{\nabla^2 \phi'' + \left( \frac{H^2}{3 H} \right) \nabla^2 \nabla^2 \phi - \left( \frac{2}{3} \right) \nabla^2 \nabla^2 \psi}{H^4 H'} \right\} \right].
$$
\[
\delta G_{0i} = \delta G_{i0} = \frac{2}{\alpha^2} \left[ \frac{3}{6} H'' + 2H'H' - 4H^3) \phi - 3 (H^2 - 3H') \phi' + 3H\phi'' + 3 (7H' - 5H^2) \psi' + 3\psi''

- 3H\nabla^2 (\phi - 2\psi) + \nabla^2 (\phi' - 2\psi') \right] + \frac{\beta}{6H^4H'^2} \left( (96H^6 + 156H^7H' + 84H^5H'^2 + 83H^4H'^3 + 24H^4H'^4 - 24H^4H'' - 82H^4H''')

- (1880H^8 + 2156H^7H'^2 + 1710H^6H'^3 + 992H^5H'^4 + 432H^4H'^5 + 576H^6H'' + 178H^5H'H'' - 1231H^4H'^2H'' - 984H^3H'^3H'' - 360H^2H'^4H'' - 144H^4H''')^2

- 288H^2H'^2H''^2 \phi/3H^4H'^2

- (576H^6 + 1566H^5H'H' + 733H^4H'^3 + 504H^3H'^4 + 444H^2H'^5 + 49H^4H'' - 56H^5H'H'' - 691H^4H'^2H'' - 180H^3H'^3H'' - 144H^2H'^4H'' - 444H^2H''H'^2 - 288H^2H''^2) \frac{\psi'}{6H^4H'^2}

- (144H^6 + 208H^7H' + 166H^5H'^2 + 88H^4H'^3 + 54H^3H'^4 - 33H^2H'^5 - 129H^4H'H'' - 123H^2H'^2H'' - 45H^4H''') \frac{2\psi''}{3H^4H'^2}

+ \left( \frac{H^2}{3H'} \nabla^2 \phi' - \left( \frac{2}{3} \right) \nabla^2 \psi' + \left( \frac{H^3}{H'} \right) \phi'' + \left( \frac{H^2}{H'} \right) \psi'' \right) \right],
\]

(3.6)
\[ \delta G_{ij} = \frac{2}{a^2} \left[ \alpha \left\{ \delta_{ij} \left[ 12 \left( H^4 - 6H^2\dot{H}' + \dot{H}'^2 - 2H'' + H''' + H'''' \right) \phi - 3 \left( 6\dot{H}^3 + H\dot{H}' - 6H'' \right) \phi' - 3 \left( \dot{H}^2 - 4\dot{H}' \right) \phi'' + 3\phi''' \right] + 3\phi'' + \left( \frac{3\phi'}{\dot{H}'^2} - 2\phi \right) \right] \right\} \]
The trace of the spatial part of the Einstein-like tensor is given by

\[
\delta G^k_k = \frac{2}{a^2} \left[ \alpha \left( 36 \left( H^4 - 6H^2H' + H'^2 - HH'' + H''' \right) \phi - 9 \left( 6H^3 + HH' - 6H'' \right) \phi' - 9 \left( H^2 - 4H' \right) \phi'' \\
+ 9H\phi'' + 18 \left( H^4 - 6H^2H' + H'^2 - HH'' + H''' \right) \psi - 9 \left( 2H^3 + 13H'H' - 5H'' \right) \psi' \\
- 9 \left( 7H^2 - 6H' \right) \psi'' + 9 \psi''' - 6 \left( H^2 + 2H' \right) \nabla^2 \phi - 15H\nabla^2 \phi' + 3\nabla^2 \phi'' + 12H'\nabla^2 \psi \\
- 12\nabla^2 \psi'' - 2\nabla^2 \nabla^2 \phi + 4\nabla^2 \nabla^2 \psi \right) + \beta \left( 8H^3H'H'' - 16H^4H'^2 - 4H^4 - 4H^2H' + H^2 \left[ 13H^3 - H'H^2 \right] \right) \frac{2\phi}{H'^2} \\
- \left( 8H^8H' + 48H^6H'^2 + 64H^4H'^4 - 18H^5 - 6H^5H'H'' + 36H^3H'^2H'' \\
- 3H^4 \left( 47H'^3 + 4H'H'^2 - 2H'H'' \right) \phi \frac{H'H^3}{H'^2} \\
+ \left( 7H'H' + 2H^2H'^2 + 2H^3 - 4H^3H' \right) \frac{3\phi'}{H'^2} \\
+ \left( 4H^7H' + 48H^5H'^2 - 22H^5H'^4 + 21H^4H'H'' - 6H^2H'^2 + 6H^3H'' \\
+ 3H^3 \left( 4H^3 - 2H'^2 + H'H'' \right) \right) \frac{2\psi}{H'H'^2} \\
- \left( 8H^6H' + 27H^4H'^2 + 40H^4 + 6H^3H'H'' + 18H^2H'^2 \right) \\
- 6H^2 \left[ 14H^3 + 2H'^2 - H'H'' \right] \frac{\psi''}{H'^3} + \left( 4H^4H' - H^2H'^2 + H^3 - 2H^3H'' \right) \frac{6\psi'''}{H'H^2} \\
- \left( 8H^4H' + 30H^3H'^2 - 27H^3H'^3 + 31H^2H'^4 + 6H' \left[ H'^2 + H^2H' - H^4 \right] \\
- 12H^4H'^2 + 6H^3H'H'' \right) \frac{\nabla^2 \phi}{3H'H'^3} + \left( H' \left[ 7H^4 - 6H^2H' + 2H'^2 \right] - 4H^3H'' \right) \nabla^2 \phi' \\
+ \left( 2H^7H' + 187H^5H'^2 - 52H^4H'^4 - 96H^4H'H'' + 36H^2H'^2H'' + 24H^3H'' \\
+ H^3 \left[ 12H'^2 - 77H' \right] \right) \frac{\nabla^2 \psi}{3H^3H'^2} \\
+ \left( 2H \left[ 2H^4 - 16H^2H' + 11H' \right] + \left[ 6H^2 - 3H' \right] \frac{\psi''}{3H^2H'} \right) \frac{2\nabla^2 \psi'}{3H^2H'} \\
+ \left( \frac{H^3}{H'} \phi'' + \left( \frac{3H^2}{H'} \right) \psi''' + \left( \frac{H^2}{H'} \right) \nabla^2 \phi'' - 4\nabla^2 \psi'' - \left( \frac{2}{3} \right) \nabla^2 \nabla^2 \phi + \left( \frac{4H'}{3H^2} \right) \nabla^2 \nabla^2 \psi \right) \right]. \tag{3.8}
\]

To construct the traceless Einstein-like tensor we need to subtract the trace from the spatial part given by Eq. (3.8). In order to do this, we apply two spatial derivatives to \( \delta G_{ij} \) (see e.g. Ref. [24]). We then use the linearity of the equation to simplify it even more, and we are left with single Laplacian operators instead of double Laplacians, simplifying
calculations,

\[
\int \int (\delta \phi) \frac{\partial j}{\partial \phi} = \frac{2}{a^2} \left\{ \left( H^4 - 6H^2H' \phi - H\phi' + \phi'' \right) + 3H\phi'' + 6 \left( H^4 - 6H^2H' + H/2 - H\phi' \right) \psi - 3 \left( 2H^3 + 13H^3 - 5H\phi' \right) \psi'
\]

\[
- 3 \left( 7H^2 - 6H \phi \right) \psi + 3\psi''' - 2 \left( H^2 + 2H' \right) \nabla^2 \left( \phi - 2\psi \right) - 3H\nabla^2 \left( \phi' - 2\psi' \right) + \nabla^2 \left( \phi'' - 2\psi'' \right) \right\}
\]

\[
+ \beta \left\{ \left( 8H^4H'' - 16H^2H^2 - 4H^4 - 4H^4H'' + H^2 \left[ 13H^2 - H'' \right] \right) \frac{8\phi}{H''^2} - \left( 8H^8H' + 48H^6H^2 + 64H^2H^4 - 18H^5 - 6H^3H'' + 36H^3H^2H'' \right) \right. 
\]

\[
- 3H^4 \left[ 47H^3 + 4H'' + 2H' \right] + \left( 7H^4H' + 2H^2H'' + 2H' - 4H^3H'' \right) \frac{\phi''}{H''^2} + \left( 4H^2H' - 48H^5H - 22H^4 + 21H^4H'' - 6H^3H'' \right) + \frac{2\psi}{3H''^2} 
\]

\[
- \left( 8H^9H' + 32H^7H^2 - 10H^5 - 14H^4H'' - 8H^4H'' + 8H^3H^2H'' + 42H^2H^3H'' \right) + \left( 4H^7H' + 30H^9H^2 - 10H^3H'' + 3H^4H'^2 + 6H^2H'' + 6H^3H'' + 12H^3H'' \right) + \frac{\psi''}{3H''^2} 
\]

\[
- 6H^2 \left[ 14H^3 + 2H'' - 2H' \right] + \frac{2\psi''}{H''^2} 
\]

\[
- \left( 4H^4H' - H^2H^2 + H^3 - 2H^3H'' \right) + \frac{2\psi''}{H''^2} \right. 
\]

\[
+ \left( 4H^2H' + 30H^3H^2 - 19H^3H + 3H^5H'' + 6H^2H'' + 12H^3H'' \right) + \frac{2\psi''}{H''^2} 
\]

\[
- \left( 7H^4H' + 2H^2H'' + 2H' - 4H^3H'' \right) + \frac{\psi''}{3H''^2} \right. 
\]

\[
+ \left( 2^2 + 54H^4H' - 44H^3H'' - 27H^3H'' + 15H^2H'' + 12H^2H'' \right) + \frac{\psi''}{2H''^2} \right. 
\]

\[
+ \left( 3H^3 \left[ H'' - 4H'' \right] \right) + \frac{\psi''}{8H''^2} \right. 
\]

\[
- \left( 11H^3H' - 16H^2H^2 - 2H^2H'' + 3H'' \right) + \frac{\psi''}{2H''^2} \right. 
\]

\[
+ \left( H^3 + H^2 \right) \frac{\psi''}{H''^2} + \left( H^2 + H'' \right) + \frac{\psi}{H''} \right. \right\} \frac{\nabla^2 \phi'' - \nabla^2 \psi''}{27H''^4} \right. \right\} . \tag{3.9}
\]

Subtracting (3.9) and (3.8) from Eq. (3.7) one gets the trace free Einstein-like tensor.

**IV. CONCLUSION AND FUTURE WORK**

In this paper, we have provided a derivation of the governing equations for a fourth order \( f(R, G) \) theory. Throughout we assumed a flat FRLW universe. We rederived the governing equations in the background, and, using cosmological perturbation theory to linear order, derived the governing equations for the scalar perturbations. Since the equations are rather complex, we only present them in longitudinal gauge. However, we also provide the definitions
of the gauge-invariant variables in this gauge, both for the metric and the matter perturbations. One can therefore easily rewrite the governing equations for any other gauge.

Here we only studied scalar perturbations, leaving the discussion of vector and tensor perturbations for future work. The recent detection of gravitational waves and present and future gravitational wave observatories like LIGO, Virgo, KAGRA, and LISA makes calculating the tensor perturbation evolution equations an exciting future project.

We also need to find solutions to the governing equations, since only then can we calculate observational signatures that can be compared to the observational data. The equations presented in this work are rather complex and hence difficult to solve. In order to make solving the equations less time consuming, we have also provided the Mathematica sheets in Github.

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Assumption, we do take into account the fact that the Gauss-Bonnet term dependance of $G$ reduces to the known cases of GR and
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Appendix A: Einstein-like tensor

The choice of a function of the Ricci scalar is arbitrary, so we keep the function only as $f(R, G)$ without any assumption, we do take into account the fact that the Gauss-Bonnet term $G$, given in Eq. (1.1), also has a dependency on the metric, so it has to be varied with respect of the metric itself, from where the equation becomes rather lengthy, this general equation, reduces to the known cases of GR and $f(R)$ when we choose $f(R) = R$ and we drop the dependance of $G$ respectively.

In general, the Einstein-like tensor, without choosing any particular $f(R, G)$, is given by

$$G_{\mu\nu} = g_{\mu\nu} \left\{ \partial_R f \left[ \frac{1}{8} \right] + \partial_R^2 f \left[ \nabla^2 R \right] + \partial_R^3 f \left[ (\nabla_\alpha R) (\nabla^\alpha R) \right] + \partial_R f \left[ 2 \nabla^2 R - 4 \nabla_\alpha \nabla_\beta R^\alpha\beta \right] - \frac{1}{2} f \right\}$$

$$+ \partial_R^3 f \left[ 16 R^2 (\nabla^2 R) - 32 R^2 \beta (\nabla_\alpha R_{\beta\sigma}) (\nabla^\alpha R) + 12 R (\nabla_\alpha R) (\nabla^\alpha R) + 8 R^2 \beta \sigma \lambda \rho (\nabla_\alpha R_{\beta\sigma\lambda\rho}) (\nabla^\alpha R) \right]$$

$$- 16 R (\nabla_\alpha R) (\nabla^\alpha R) + 32 R^2 \beta (\nabla_\alpha R^\lambda) (\nabla^\alpha R_{\beta\lambda}) - 16 R^2 \beta \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma\lambda\rho\tau})$$

$$- 8 R \beta (\nabla_\alpha R^\lambda \rho \tau \sigma) (\nabla^\alpha R_{\beta\lambda \rho \tau \sigma}) - 8 R^2 \beta (\nabla_\alpha R^\lambda \rho \tau \sigma) (\nabla^\alpha R_{\beta\lambda \rho \tau \sigma}) - 8 R_{\alpha \beta} (\nabla^\alpha R) (\nabla^\beta R)$$

$$- 16 R^2 \beta (\nabla_\alpha R_{\beta\sigma}) - 16 R (\nabla_\sigma R_{\alpha \beta}) (\nabla^\sigma R^\alpha \beta) + 64 R^2 \beta (\nabla^\sigma R_{\alpha \beta}) (\nabla^{\sigma \lambda} R_{\alpha \beta}) + 32 R^2 \beta \sigma \lambda \rho \tau \sigma (\nabla_\alpha R_{\beta\sigma \lambda\rho\tau\sigma})$$

$$+ 4 R^2 \beta \sigma \lambda \rho \tau \sigma \lambda \rho \tau (\nabla^\sigma R_{\beta\lambda \rho \tau \sigma \lambda \rho \tau})$$

$$+ \partial_R f \left[ 8 R^2 (\nabla^2 R) - 64 R^2 \beta \sigma \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) + 16 R^2 \beta \sigma \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) \right]$$

$$+ 128 R_{\alpha \beta} R^\lambda \rho \tau \sigma R (\nabla^\alpha R) (\nabla^\beta R) - 32 R_{\alpha \beta} R^\lambda \rho \tau \sigma R (\nabla^\alpha R) (\nabla^\beta R)$$

$$- 16 R^2 \beta \sigma \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) (\nabla^\beta R)$$

$$- 64 R^2 \beta \sigma \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) (\nabla^\beta R)$$

$$- 16 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) (\nabla^\beta R)$$

$$+ 128 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) (\nabla^\beta R)$$

$$+ 8 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) (\nabla^\beta R)$$

$$+ \partial_R g_{\mu\nu} \left\{ 4 R (\nabla^2 R) + 6 (\nabla_\alpha R) (\nabla^\alpha R) - 8 (\nabla_\alpha R) (\nabla^\alpha R) R_{\alpha \beta} - 8 R_{\alpha \beta} (\nabla^2 R) \right\}$$

$$- 8 (\nabla_\alpha R) (\nabla^\alpha R) R_{\alpha \beta} + 2 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) (\nabla^\beta R)$$

$$+ \partial_R g_{\mu\nu} \left\{ 12 R^2 (\nabla_\sigma R) (\nabla^\alpha R) - 64 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) + 16 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\alpha R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\alpha R) \right\}$$

$$+ 64 R_{\alpha \beta} R^\lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\sigma R_{\beta\sigma} R_{\alpha \beta}) - 16 R_{\alpha \beta} R^\lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\sigma R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\beta R)$$

$$- 32 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\sigma R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\beta R) + 64 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\sigma R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\beta R)$$

$$+ 4 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\sigma R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\beta R)$$

$$+ 4 R^2 \beta \sigma \lambda \rho \tau \sigma \beta \lambda \rho \tau \sigma \lambda \rho \tau (\nabla_\sigma R_{\beta\sigma} R_{\alpha \beta}) (\nabla^\beta R)$$
\[\begin{align*}
+ \partial Rf \left[ R_{\mu \nu} \right] - \partial^2 f \left[ \nabla_\mu \nabla_\nu R \right] & - \partial Rf \left[ (\nabla_\mu R) (\nabla_\nu R) \right] \\
+ \partial_\alpha f \left[ 2R_{\mu \nu} - 8R_\mu^\alpha R_{\nu \alpha} + 2R_\mu^\alpha R_{\nu \alpha} B - 4(\nabla_\alpha R_{\nu \mu}^\alpha) + 4(\nabla_\mu R_{\nu \alpha}^\alpha) + 4(\nabla_\alpha R_{\nu \mu}^\alpha) \\
& + 2(\nabla_\beta R_{\nu \alpha}^\beta - 2(\nabla_\nu R_{\mu \alpha}) \right] \\
+ \partial R_\alpha f \left[ 4(\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) - 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) + 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) + 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) + 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) \\
& + R_{\mu \nu} (\nabla_\alpha R_{\nu \mu}^\alpha) + 4(\nabla_\alpha R_{\nu \mu}^\alpha) + 4(\nabla_\mu R_{\nu \alpha}^\alpha) + 4(\nabla_\mu R_{\nu \alpha}^\alpha) \\
& + 2(\nabla_\beta R_{\nu \alpha}^\beta - 2(\nabla_\nu R_{\mu \alpha}) \right] \\
+ \partial_\alpha \partial_\beta f \left[ 4R_{\mu \nu \beta} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) - 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) + 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) + 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) \\
& + R_{\mu \nu} (\nabla_\alpha R_{\nu \mu}^\alpha) + 4(\nabla_\alpha R_{\nu \mu}^\alpha) + 4(\nabla_\mu R_{\nu \alpha}^\alpha) + 4(\nabla_\mu R_{\nu \alpha}^\alpha) \\
& + 2(\nabla_\beta R_{\nu \alpha}^\beta - 2(\nabla_\nu R_{\mu \alpha}) \right] \\
+ \partial_\alpha \partial_\beta \partial_\gamma f \left[ 128R_{\alpha \beta \rho \sigma} R (\nabla_\rho R_{\sigma \alpha}) (\nabla_\beta R_{\nu \mu}^\beta) - 32R_{\mu \nu} R (\nabla_\alpha R_{\beta \sigma \lambda}) (\nabla_\beta R_{\nu \mu}^\beta) - 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) + 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) + 4R_{\mu \nu} (\nabla_\alpha R) (\nabla_\beta R_{\nu \mu}^\beta) \\
& + R_{\mu \nu} (\nabla_\alpha R_{\nu \mu}^\alpha) + 4(\nabla_\alpha R_{\nu \mu}^\alpha) + 4(\nabla_\mu R_{\nu \alpha}^\alpha) + 4(\nabla_\mu R_{\nu \alpha}^\alpha) \\
& + 2(\nabla_\beta R_{\nu \alpha}^\beta - 2(\nabla_\nu R_{\mu \alpha}) \right] \\
\end{align*}\]
\[ + \partial_R \partial_G \left[ 64 R^{\beta \sigma} R_{\mu \nu} (\nabla_\alpha R_{\beta \sigma}) (\nabla^\alpha R) - 16 R_{\mu \nu} (\nabla_\alpha R) (\nabla^\alpha R) - 16 R_{\mu \nu} R^{\beta \sigma \lambda \rho} (\nabla_\alpha R_{\beta \sigma \lambda \rho}) (\nabla^\alpha R) \right. \\
+ \left. 8 R^{\alpha \beta \lambda \rho} R_{\mu \nu \alpha} (\nabla_\beta R_{\sigma \lambda \rho}) + 8 R^{\alpha \beta \lambda \rho} R_{\mu \nu \alpha} (\nabla_\beta R_{\sigma \lambda \rho}) + 16 R_{\mu \nu \alpha} R (\nabla^\alpha R) (\nabla^\beta R) \right. \\
- \left. 32 R^{\beta \sigma} R_{\mu \nu \alpha} (\nabla^\alpha R) (\nabla^\beta R_{\beta \sigma}) - 32 R^{\beta \sigma} R_{\mu \nu \alpha} (\nabla^\alpha R) (\nabla^\beta R_{\beta \sigma}) - 32 R^{\beta \sigma} R_{\mu \nu \alpha} (\nabla^\alpha R) (\nabla^\beta R_{\beta \sigma}) \right. \\
- \left. 32 R^{\beta \sigma} R_{\mu \nu \alpha} (\nabla^\alpha R) (\nabla^\beta R_{\beta \sigma}) + 8 R_{\mu \nu} R^{\beta \sigma \lambda \rho} (\nabla_\alpha R_{\beta \sigma \lambda \rho}) (\nabla^\alpha R) + 16 R_{\mu \nu \alpha} R (\nabla^\alpha R) (\nabla^\beta R) \right. \\
- \left. 32 R^{\beta \sigma} R_{\mu \nu \alpha} (\nabla^\alpha R) (\nabla^\beta R_{\beta \sigma}) - 64 R^{\alpha \beta \sigma \lambda} R_{\mu \nu \alpha} (\nabla^\alpha R) (\nabla^\beta R_{\sigma \lambda}) - 32 R^{\beta \sigma} R_{\mu \nu \alpha} (\nabla^\alpha R) (\nabla^\beta R_{\beta \sigma}) \right. \\
+ \left. 8 R_{\mu \nu} R^{\beta \sigma \lambda \rho} (\nabla^\alpha R) (\nabla_\alpha R_{\beta \sigma \lambda \rho}) + 16 R_{\mu \nu} R (\nabla^\alpha R) (\nabla_\alpha R_{\beta \sigma \lambda \rho}) + 32 R_{\mu \nu} R (\nabla^\alpha R) (\nabla_\alpha R_{\beta \sigma \lambda \rho}) + 16 R_{\mu \nu} R (\nabla^\alpha R) (\nabla_\alpha R_{\beta \sigma \lambda \rho}) \right. \\
- \left. 8 R^{\alpha \beta \sigma \lambda} R (\nabla_\alpha R_{\beta \sigma \lambda \rho}) (\nabla^\alpha R) - 8 R^{\alpha \beta \sigma \lambda} R (\nabla_\alpha R_{\beta \sigma \lambda \rho}) (\nabla^\alpha R) - 8 R^{\alpha \beta \sigma \lambda} R (\nabla_\alpha R_{\beta \sigma \lambda \rho}) (\nabla^\alpha R) \right. \\
+ \left. 8 R_{\mu \nu} R^{\beta \sigma \lambda \rho} (\nabla^\alpha R) (\nabla_\alpha R_{\beta \sigma \lambda \rho}) + 16 R^{\alpha \beta \sigma \lambda \rho} R (\nabla_\alpha R_{\beta \sigma \lambda \rho}) (\nabla^\alpha R) - 4 R^{\alpha \beta \sigma \lambda} R^{\rho \tau \xi} (\nabla_\mu R_{\alpha \beta \sigma \lambda}) (\nabla^\mu R^{\rho \tau \xi}) \right] \\
(A1) \]

where \( \partial_R \) and \( \partial_G \) are the derivatives with respect to the Ricci scalar and the Gauss-Bonnet term, respectively, i.e. \( \partial_R = \partial / \partial R \) and \( \partial_G = \partial / \partial G \). From here also one can identify the \( C^\mu_{\alpha \nu} \) present in Eq. (A.1).