Evolution of metric perturbations in a model of nonsingular inflationary cosmology

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Abstract. In this paper we study the primordial perturbations generated in an inflation model in which the period of inflation is preceded by a cosmological bounce. After reviewing the background evolution in this model we study the resulting curvature perturbations, entropy fluctuations and non-Gaussianities. Our results show that both the power spectrum of curvature perturbations and the spectra describing the non-Gaussianities obtain oscillatory features. Next, we calculate the tensor perturbations in this model and find a valley in its spectrum. This is a generic feature of inflationary models with a bounce. Further observational signatures and theoretical implications of the model are discussed.

Keywords: gravity waves / theory, cosmological perturbation theory, physics of the early universe

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1 Introduction

One problem of the scalar field-driven inflationary scenario [1] (see also [2]), is the presence of an initial space-like singularity. Such a singularity is unavoidable if inflation is realized by making use a scalar field while the background spacetime is described by the standard Einstein action [3]. As a consequence, there has been a lot of interest in resolving this singularity by means of quantum gravity effects or effective field theory techniques.

A potential solution to the cosmological singularity problem may be provided by a non-singular bouncing cosmology. Such a scenario has been studied in models motivated by various approaches to quantum gravity, such as the Pre-Big-Bang [4] and the cyclic/Ekpyrotic [5] scenarios, gravity actions with higher order corrections [6, 7], braneworld scenarios [8] or loop quantum cosmology [9]. Non-singular bounces may be studied using effective field theory techniques by introducing a curvature term [10] or matter fields violating the null energy condition [11–13].

In the context of bouncing cosmologies, it has been realized that perturbations which are generated as quantum vacuum fluctuations and exit the Hubble radius before the bounce might lead to a scale-invariant spectrum of density fluctuations today. Examples are the Ekpyrotic scenario [14], models with a matter-dominated contracting phase [15–18], or a bounce inflation scenario [19, 20]. These cases yield alternatives to inflation for explaining the current observational data. Recently, ref. [21] has provided a good review on various models of bouncing cosmology.
In the current paper, we focus on a model of inflationary cosmology with a preceding bounce.\footnote{To obtain such a scenario in the framework of standard Einstein gravity, a null-energy-condition (NEC) violating degree of freedom is usually required which might be obtained by introducing a phantom field \cite{phantom}. Thus, the models often suffer from the problem of quantum instability as shown in refs. \cite{instability1,instability2}. However, it is argued that this problem might be avoided in some specific models, such as in the Lee-Wick theory \cite{LeeWick} or in p-adic string theory \cite{p-adic}. Here we introduce a phantom field in order to obtain NEC violation.} This model has been previously considered in ref. \cite{previous} where it was shown that a scale-invariant spectrum in the ultra-violet regime results. It also predicts an oscillatory signature in the spectrum which could be seen in forthcoming observations. It is important to obtain more predictions of this bouncing model in order to be able to distinguish it from the standard inflationary paradigm. This is the aim of the current study. In this paper, we first calculate the entropy perturbations caused by the fluctuations in the subdominant field. Next we compute the non-Gaussianities, and finally the relic gravitational wave background (GWB). From the detailed calculations, we find that the entropy perturbations are not scale-invariant and that it is hard to convert them into curvature perturbations. The non-Gaussianities are suppressed by slow-roll parameters, but there is an interesting oscillatory signal in the nonlinearity parameter. The maximal value of the nonlinearities is slightly larger than that in the usual slow-roll inflation models. More interestingly, we find that the tensor power spectrum is nearly scale-invariant, but there is a distinct valley in the spectrum. A crude understanding of this feature is as follows. Usually, the amplitude of gravity waves is roughly proportional to the expansion rate of the universe at the time when the modes are set up. Since at the time of the bounce the Hubble parameter approaches zero, there will be a range of modes for which the amplitude of the GWB will be suppressed a lot. This feature is quite generic to models with a bounce, and thus can be important in distinguishing bounce models from inflation.

The outline of this paper is as follows. In section 2, we briefly review the nonsingular inflation model with a Coleman-Weinberg potential. In section 3 we study the linear cosmological perturbations including fluctuations of the subdominant field. In section 4, we calculate the non-Gaussianities in the model. In section 5, we study the tensor perturbations and obtain the corresponding power spectrum and spectral index. Considering the influence of possible factors during the late time evolution, we finally obtain the energy spectrum of GWB. Section 6 contains a discussion and our conclusions.

2 A review of the background evolution

The model we consider can be described in terms of two scalar fields which minimally couple to four dimensional Einstein gravity. The Lagrangian is given by

\[
\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - V(\phi, \psi),
\]  

(2.1)

in a spatially flat Friedmann-Robertson-Walker (FRW) universe where the signature of the metric is taken to be \((-,-,+,+).\) As in previous work performed in collaboration with Qiu, Xia and Li \cite{previous}, we take the potential to be a function of only the field \(\phi\), and specifically assume a Coleman-Weinberg form \cite{ColemanWeinberg}:

\[
V = \frac{1}{4} \lambda \phi^4 \left( \ln \frac{|\phi|}{v} - \frac{1}{4} \right) + \frac{1}{16} \lambda v^4,
\] 

(2.2)

which has a local maximum with value \(\lambda v^4/16\) at \(\phi = 0\) and vanishes at the minima when \(\phi = \pm v\). The scalar \(\psi\) only affects the evolution around the bounce but decays quickly away from the bounce point. Thus, it is a subdominant field.
Figure 1. A plot of the evolution of the EoS $w$ (the ratio of pressure divided by energy density), the Hubble parameter $H$ and the canonical field $\phi$ in the model we considered. In the numerical calculation we choose the parameters $\lambda = 8.0 \times 10^{-14}$, $v = 0.82M_{\text{pl}}$, and the initial conditions as: $\phi = -0.82M_{\text{pl}}$, $\dot{\phi} = 3.0 \times 10^{-10}M_{\text{pl}}^2$, $\psi = -0.72M_{\text{pl}}$, $\dot{\psi} = 5.0 \times 10^{-13}M_{\text{pl}}^2$ where $M_{\text{pl}} \equiv 1/\sqrt{G}$.

This background evolution in this model is shown in figures 1 and 2. We start in the contracting phase. As initial conditions we choose that $\phi$ is in its left vacuum $-v$ and that the kinetic term of $\psi$ is sufficiently small. Note that no field has a value larger than the Planck scale, so the model does not suffer from the initial condition problem which appears in usual inflation model [28]. At the beginning, the field $\phi$ oscillates around the vacuum $-v$. Hence, the equation of state (EoS) oscillates about $w = 0$ and, averaging over time, the universe looks similar to a matter dominated one. Since the universe is contracting, the amplitude of oscillation of the canonical field increases in proportion to $a^{-3/2}$. During the contracting phase, the contribution of $\psi$ to the energy density also increases as $a^{-3}$. When $\phi$ reaches the plateau, the oscillations of $\phi$ will freeze out. Hence, there will be a moment when $\dot{\phi}^2 = \dot{\psi}^2$. At this moment, the bounce takes place. The universe arrives at the bounce point when the energy density of $\phi$ is cancelled by that of $\psi$. After the bounce, as the field $\phi$ moves forward slowly along the plateau, the universe enters into a slow-roll inflationary phase. Finally, when the field $\phi$ drops into the right vacuum $+v$, it will oscillate again and reheat the universe.

3 Curvature perturbations and entropy perturbations

In ref. [20], the authors considered the metric perturbations of the model described above, but neglected the existence of entropy perturbations. In this section we study the linear curvature and isocurvature perturbations both analytically and numerically. Specifically, we will determine the evolution of the subdominant field fluctuations $\delta \psi$ in the model.

3.1 Basic perturbation equations

Under the longitudinal (conformal-Newtonian) gauge, the metric perturbations are given by

$$ds^2 = a^2(\tau) \left[-(1 + 2\Phi) d\tau^2 + (1 - 2\Phi) dx^i dx^i\right],$$

(3.1)

where we introduce the comoving time $\tau$ defined by $d\tau = dt/a$. The metric field $\Phi$ depends on space and time and contains the information about the scalar metric fluctuations (see
Figure 2. A plot of the evolution of the time derivatives $\dot{\phi}$ and $\dot{\psi}$ in the model. The background parameter is the same as in figure 1.

ref. [29] for a comprehensive review). It is the so-called Bardeen potential. To linear order, we can expand the Einstein equations and obtain the equation of motion

$$
\Phi'' + 2\left(\mathcal{H} - \frac{\Phi''}{\dot{\phi}}\right)\Phi' + 2\left(\mathcal{H}' - \mathcal{H}\frac{\Phi''}{\dot{\phi}}\right)\Phi - \nabla^2 \Phi = 8\pi G \left(2\mathcal{H} + \frac{\Phi''}{\dot{\phi}}\right)\psi'\delta\psi.
$$

(3.2)

By varying the matter action with respect to the subdominant field, we obtain the equation of motion for its perturbations,

$$
\delta\psi'' + 2\mathcal{H}\delta\psi' + k^2\delta\psi = 4\psi'\Phi',
$$

(3.3)

which combined with eq. (3.2) can describe all the perturbation modes consistently. Note that until now we have not used any approximations.

Before we start with the actual calculations we would like to explain why we use the Bardeen potential $\Phi$ to study the metric perturbation but not other variables. Another frequently used variable is the curvature perturbation in comoving coordinates $\zeta$ which is defined as,

$$
\zeta \equiv \Phi + \frac{\mathcal{H}}{H^2 - \mathcal{H}^2}(\Phi' + \mathcal{H}\Phi).
$$

(3.4)

This variable can be calculated from $\Phi$ and the background parameters. For example, when the universe is in a nearly de-Sitter like expansion, there is a simple relation $\zeta \simeq \Phi/\epsilon$ with the slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$. Usually this variable is well known to describe the adiabatic perturbations on large scales in an expanding universe since it is conserved on super-Hubble scales. However, it is ill-defined both at the bounce point with $H = 0$ and the cosmological constant boundary $w = -1$. This issue has been remarked in ref. [17, 30, 31]. Consequently, we study the evolution of the Bardeen potential $\Phi$ directly, and then infer the value $\zeta$ after the universe enters the expanding phase.
3.2 Evolution of perturbations

In the following calculation we will see that large \( k \) modes are able to obtain a scale-invariant spectra but small ones are not. In order to present our analytical calculations more clearly, we denote the time corresponding to the beginning of the bouncing phase by \( t_{B-} \), the end of that by \( t_{B+} \), and the moment of the bounce by \( t_B \).

3.2.1 Contracting phase

In the initial stage of the background evolution when \( t \ll t_{B-} \), the average value of the EoS is zero. The universe behaves like a matter dominated one and the contribution of \( \psi \) can be neglected. Thus we have the expressions

\[
\mathcal{H} = \frac{2}{\tau - \tilde{\tau}_{B-}}, \quad \mathcal{H}' = -\frac{2}{(\tau - \tilde{\tau}_{B-})^2}, \quad \mathcal{H}'' = \frac{4}{(\tau - \tilde{\tau}_{B-})^3},
\]

and \( \tilde{\tau}_{B-} = \tau_{B-} - \frac{2}{\mathcal{H}_{B-}} \) is the time that corresponds to when the singular bounce would occur if the universe were to remain matter-dominated. Further we have the relation

\[
\frac{\phi''}{\phi'} = 2\frac{\mathcal{H}'}{\mathcal{H}^2 - \mathcal{H}''}. \tag{3.5}
\]

Using these relations, we obtain the simplified equations for \( \Phi \) and \( \delta\psi \) in the contracting phase

\[
\Phi''_k + \frac{6}{\tau - \tilde{\tau}_{B-}} \Phi'_k + k^2 \Phi_k \simeq 0, \quad (3.6)
\]

\[
\delta\psi''_k + \frac{4}{\tau - \tilde{\tau}_{B-}} \delta\psi'_k + k^2 \delta\psi_k \simeq 0, \quad (3.7)
\]

in Fourier space.

Before solving these two equations, we discuss the initial conditions for the perturbations. One may notice that long before the bounce, i.e. for \( \tau \to -\infty \), the effects of gravity on the above equations are very weak. Thus, provided the universe is empty and adiabatic, the two perturbation variables are oscillating as plane waves, and thus one possible choice on initial conditions is that they both satisfy Bunch-Davies initial conditions, i.e.,

\[
\Phi_k = \frac{4\pi G}{k^2} \frac{\phi''}{\mathcal{H}} \left( \frac{v_k}{z} \right), \quad \delta\psi_k = \frac{v_k}{a}, \quad (3.8)
\]

where \( z = a\phi'/\mathcal{H} \) and \( v_k \) is the variable in terms of which the kinetic term in the action for cosmological perturbations has canonical kinetic term. Assuming an initial vacuum state \( v_k = e^{ik\tau}/\sqrt{2k} \). We argue that this choice is quite reasonable for to the following reasons. Long before the bounce, the energy density and curvature are quite small and thus the temperature is quite low. In such a cold (and also empty) universe it is hard to excite any particles Therefore, the background will be very close to the vacuum.

With these initial conditions, the analytic solutions for the fluctuation variables in the contracting phase are

\[
\Phi^c_k = \frac{A}{(\tau - \tilde{\tau}_{B-})^3} \frac{e^{ik(\tau - \tilde{\tau}_{B-})}}{\sqrt{2k^3}} \left[ 1 + \frac{3i}{k(\tau - \tilde{\tau}_{B-})} - \frac{3}{k^2(\tau - \tilde{\tau}_{B-})^2} \right], \quad (3.9)
\]

\[
\delta\psi^c_k = i \frac{A}{\sqrt{6\pi G}} \frac{e^{ik(\tau - \tilde{\tau}_{B-})}}{\sqrt{2k}} \left[ 1 + \frac{i}{k(\tau - \tilde{\tau}_{B-})} \right], \quad (3.10)
\]

with

\[
A = i4\pi G \rho_{B-}(\tau_{B-} - \tilde{\tau}_{B-})^3, \quad (3.11)
\]

where the subscript "c" indicates the contracting phase.
3.2.2 Bouncing phase

Since the contribution of the field $\psi$ becomes more and more important, the contraction will stop when $w = -1$ and then the universe will enter the bouncing phase at some moment $\tau_B$. Correspondingly the EoS of the universe will fall to negative infinity rapidly. During this process, the Hubble parameter will shrink soon and arrive at zero at $\tau_B$. Equivalently, the Hubble radius $1/H$ approaches infinity, and all the perturbations are in the sub-Hubble region. Therefore, the perturbations will perform oscillations in the bouncing phase. Interestingly, for sub-Hubble modes from the solution (3.9) which consists of pure outgoing modes ($\propto e^{i k \tau}$) entering the bouncing phase, there would be some ingoing modes ($\propto e^{-i k \tau}$) generated. This important feature leads to an oscillatory feature in the primordial spectrum as we obtain below.

In order to get some analytic insight into the evolution of the fluctuations in the bouncing phase, we can use two approximations here. First, we parameterize the Hubble parameter close to the bounce point as

$$
H(t) = \alpha (t - t_B),
$$

(3.12)

where we choose $t_B$ to be the time when the bounce takes place and the coefficient $\alpha$ is a positive constant which can be determined from numerical calculations. Second, there is a useful relation

$$
\phi'' \simeq -2 \mathcal{H} \phi',
$$

(3.13)

around the bounce, since when $\phi$ arrives at the plateau of the potential, it satisfies that

$$
\phi'' + 2 \mathcal{H} \phi' = a^2 V_{,\phi} = a^2 \lambda \phi^3 \ln \frac{\phi}{v} \simeq 0,
$$

(3.14)

with $\phi \simeq 0$. Thus we can see that $\delta \psi$ almost decouples from the perturbation equation for the Bardeen potential.

Substituting the above two approximations into eqs. (3.2) and (3.3), we then obtain the perturbation equations in the bouncing phase:

$$
\Phi_k'' + 3 y_1 (\tau - \tau_B) \Phi_k' + \left( k^2 + y_1 \right) \Phi_k \simeq 0,
$$

(3.15)

$$
\delta \psi''_k + y_1 (\tau - \tau_B) \delta \psi'_k + k^2 \delta \psi_k \simeq 4 \psi' \Phi_k',
$$

(3.16)

where we define the parameter $y_1 \equiv \frac{8}{\pi} \alpha a_B^2$.

The solutions of the first equation are linear combinations of a Hermite polynomial and of a confluent hyper-geometric function [17]. These two functions are linearly independent, and when the comoving wavenumber $k$ is large enough, both of them oscillate. In this limit, we can give a simpler expression for the Bardeen potential,

$$
\Phi^b_k \simeq e^{- \frac{3}{2} y_1 (\tau - \tau_B)^2} \left\{ C_k e^{i k (\tau - \tau_B)} + D_k e^{-i k (\tau - \tau_B)} \right\},
$$

(3.17)

where the subscript "b" represents the bouncing phase.

Note that the perturbations in the contracting phase and the bouncing phase are linked through matching $\Phi$ and $\zeta$ according to the matching condition derived in refs. [32, 33] (also see [34] for a recent study). Moreover, in our model the background evolution is smooth, and the matching conditions imply that $\Phi$ and $\Phi'$ are also continuous.
We focus our interest on the sub-Hubble modes which keep oscillating during the whole contracting phase and the bouncing phase. The matching conditions require,
\[ \Phi^c = \Phi^{b^i}|_{B^-}, \quad \Phi^{c'} = \Phi^{b^i'}|_{B^-}, \]
and thus determine the coefficients \( C_k^\Phi \) and \( D_k^\Phi \) as follows,
\[
C_k^\Phi \simeq e^{\frac{3}{2}y_1(\tau_B - \tau_B^-)} e^{\frac{ik}{2}k^3} \frac{A}{\sqrt{2k^3}} \left( \frac{\tau_B - \tau_B^-}{\tau_B - \tau_B^-} \right)^3 \left[ 1 + \frac{3iy_1}{4k} (\tau_B - \tau_B^-) \right], \quad (3.19)
\]
\[
D_k^\Phi \simeq e^{\frac{3}{2}y_1(\tau_B - \tau_B^-)} e^{-ik(\tau_B - \tau_B^-)} \frac{A}{\sqrt{2k^3}} \left( \frac{\tau_B - \tau_B^-}{\tau_B - \tau_B^-} \right)^3 \left[ - \frac{3iy_1}{4k} (\tau_B - \tau_B^-) \right]. \quad (3.20)
\]
From eq. (3.19), we find that (roughly speaking) the initial Bunch-Davies state has been inherited in the bouncing phase. However, \( D_k^\Phi \neq 0 \) implies that a mode proportional to \( e^{-ik\tau} \) can be generated when the perturbations pass through the bouncing phase.

Now we move to the analysis of the subdominant field fluctuations \( \delta \psi \). Eq. (3.16) looks quite complicated, but we still can learn some qualitative features from this equation. Since the r.h.s. of the source term in eq. (3.16) does not vanish but is proportional to \( k \Phi_k, \delta \psi \) could be amplified by metric perturbations during the bouncing phase. So there would be a peak in the evolution of \( \delta \psi_k \) around the bounce point. Moreover, on the l.h.s. of eq. (3.16) the term \( k^2 \delta \psi_k \) dominates over the other ones, which leads to an approximate solution \( \delta \psi_k \sim \frac{\dot{\psi}^*}{k} \Phi_k \).

### 3.2.3 Inflationary phase

After the bounce, the scalar \( \phi \) begins to slowly roll down to the vacuum \( v \). We denote the beginning of this phase as \( \tau_{B^+} \). Since the kinetic term of \( \psi \) decays rapidly in an expanding universe, the EoS of the universe evolves from negative infinity to a value close to but slightly larger than \( -1 \). After the slow-roll phase has ended, the universe is then able to transit to the normal expanding history. The perturbations \( \Phi \) and \( \delta \psi \) are again decoupled, and will quickly freeze after they exit the Hubble radius. In the following we focus on the metric perturbations.

During the period of slow-rolling, the well-known solution of the metric perturbations takes the form,
\[
\Phi_\tau^i \simeq i \sqrt{\frac{\pi}{2}} [k(\tau - \tilde{\tau}_{B^+})]^{\frac{3}{2}} \left[ E_k^\Phi H_{\frac{1}{2}}^{(1)}(k(\tau - \tilde{\tau}_{B^+})) - E_k^{\Phi'} H_{\frac{1}{2}}^{(2)}(k(\tau - \tilde{\tau}_{B^+})) \right], \quad (3.21)
\]
where \( \tilde{\tau}_{B^+} \equiv \tau_{B^+} + \frac{1}{H_{B^+}} \) and the subscript "i" represents the inflationary phase. In the above solution, \( H_{\frac{1}{2}}^{(1,2)} \) represent the \( \frac{1}{2} \)th Hankel functions of the first and second kinds. It has the following asymptotic form in the super-Hubble region
\[ \Phi_\tau^i \simeq E_k^\Phi + F_k^\Phi. \quad (3.22) \]

We again match this solution with the perturbations in the bouncing phase in the sub-Hubble region \( k > H_{B^+} \), and obtain
\[
E_k^\Phi \simeq e^{-\frac{3}{2}y_1(\tau_{B^+} - \tau_B)^2} C_k^e e^{ik(\tilde{\tau}_{B^+} - \tau_B)}, \quad (3.23)
\]
\[
F_k^\Phi \simeq e^{-\frac{3}{2}y_1(\tau_{B^+} - \tau_B)^2} D_k^e e^{-ik(\tilde{\tau}_{B^+} - \tau_B)}. \quad (3.24)
\]
In the solution \( E_k^\Phi \) is proportional to the coefficient of \( e^{ik\tau} \) and \( F_k^\Phi \) term is proportional to that of \( e^{-ik\tau} \). If an inflationary phase begins with a pure Bunch-Davies initial condition, the...
From the above results, we find that the dominant mode of perturbations is nearly independent of $k$ (except for a phasic angle). When modes escape outside the Hubble radius during the inflationary stage, they will give rise to a scale-invariant spectrum and thus can explain CMB observations. Interestingly we notice that there exists an exponential factor $e^{-\frac{3}{4}y(\tau_{B+} - \tau_B)}$ when the perturbations in the slow-roll-expanding phase are matched with those in the bouncing phase. A similar factor already appeared when the perturbations transit from the contracting phase to the bounce, but these two factors lead to opposite effects on the fluctuations because of a negative sign. Therefore, the amplitude of the perturbations in the slow-roll-expanding phase is suppressed by the bounce. When we combine these two factors, the final effect on the spectrum becomes less important. For simplicity, we will assume that the universe undergoes the bounce very fast with $|\tau_{B+} - \tau_B|$ very small (relative to the characteristic time of oscillation of the fluctuation modes we are interested in).

### 3.2.4 Final result and numerical calculations

As studied in the above sections, we obtain the following form for the leading term in the Bardeen potential [20],

$$\Phi_k \simeq \frac{4\pi G}{\sqrt{2k^3}}|\dot{\phi}|e^{ik\tau} \times \left\{ 1 - \frac{3\mathcal{H}_B - e^{ik/H_B}}{2k} \sin\left[k/H_B\right] \right\}, \quad (3.25)$$

where $\mathcal{H}_B$ represents the comoving Hubble parameter at the beginning and the end of the bouncing phase, respectively. By comparing the coefficient (3.25) with the initial form (3.8), we find the dominant mode has deviated from the Bunch-Davies initial condition by the time the inflationary phase takes place. In addition, we have an approximate relation $\zeta \simeq \Phi/\epsilon$ where $\epsilon$ is defined as the usual slow-roll parameter. Finally, we obtain the following expression for the primordial power spectrum of curvature perturbation,

$$P_\zeta \simeq \frac{8}{3}G^2 \rho \left\{ 1 - \frac{3\mathcal{H}_B - e^{ik/H_B}}{2k} \sin\left[k/H_B\right] \right\}. \quad (3.26)$$

From this result, one find that the first term provides a nearly scale-invariant spectrum which is consistent with current observations. However, the second term shows that there is a wiggle on the spectrum due to the effects of the bounce.

The previous analytic studies involve some approximations. In order to make the analysis exact, we also solved the perturbation equations numerically. We provide the numerical results in figures 3, 4 and 5. In the numerical calculations, we normalize the current scale factor to be $a_0 = 1$ and choose the current Hubble parameter as $H_0 = 72\text{km}s^{-1}\text{Mpc}^{-1} = 1.536 \times 10^{-42}\text{GeV}$. The background parameters are taken to be the same as shown in the caption of figure 1. Thus, the Hubble parameter during inflation is $H \simeq 1.68 \times 10^{12}\text{GeV}$, and the $e$-folding number of the period of inflation is $N \simeq 60$. We choose the above set of background parameters for two reasons. One is that these background parameters can provide a long enough inflationary period and so can satisfy the current observational constraints very well; the other is that with these parameters we expect the effects of a bounce could be observable in current or future experiments.

In figure 3, we plot the absolute value of the Bardeen potential $|\Phi|$ as a function of the comoving wavenumber $k = 8 \times 10^{-4}\text{Mpc}^{-1}$. We see that, in the period long before the bounce,
Figure 3. The evolution of the absolute value of the Bardeen potential $k^3 |\Phi_k|$ as a function of cosmic time $t$ in our model. The background parameters are the same as in figure 1. We take the initial condition of the metric perturbation to be the Bunch-Davies state and consider the comoving wavenumber $k = 8 \times 10^{-4}\text{Mpc}^{-1}$.

Figure 4. A plot of the power spectrum of the Bardeen potential $P_\Phi$ as a function of comoving wavenumber $k$ in our model. The background parameters are the same as in figure 1. The Bardeen potential is oscillating and its amplitude increases during the contraction of the background. It reaches its largest value when the bounce occurs, and then decreases after the bounce. After the bounce the universe enters an inflationary phase, and the Bardeen potential freezes at the moment of Hubble radius crossing.

Figure 4 shows the power spectrum of the Bardeen potential $P_\Phi$ as a function of comoving wavenumber $k$. One finds that the spectrum is oscillating while its central value is fixed, and the amplitude of the oscillations becomes small on large $k$ region. This feature is consistent with the analytic calculations.

We also evaluate $\delta\psi$ as shown in figure 5 for different comoving numbers $k$. Its amplitude initially grows in the contracting phase. However, the slope of the curve is different from that for the metric perturbations. Since in the bouncing phase the metric perturbation
Figure 5. The evolution of the absolute value of the subdominant field perturbation $k^{3/2} |\delta \psi_k|$ as a function of cosmic time $t$ in our model. The parameters are the same as above. In the plot we consider different comoving wavenumbers $k = 8 \times 10^{-4}, 2.4 \times 10^{-3}, 4 \times 10^{-3} \text{Mpc}^{-1}$ respectively.

is able to amplify the perturbation of the subdominant field, $\delta \psi$ reaches its peak around the bounce point. After that, the amplitude soon freezes when inflation begins. Moreover, after the bounce $\Phi$ and $\delta \psi$ will decouple and the background trajectory is along the direction of the normal scalar, so we can conclude that the subdominant field perturbations can seed the entropy perturbations.

4 Bispectrum and the nonlinear parameter

One important aspect in the study of the primordial curvature perturbations is the analysis of the non-Gaussianities [35]. In simple inflation model, non-Gaussianities are suppressed by slow-roll parameters [36, 37]. However, in the Ekpyrotic/cyclic [38], matter bounce [39], and non-canonical inflation [40] models, large non-Gaussianities are predicted. Observationally, current cosmological data [41, 42] is consistent with a Gaussian distribution. However, forthcoming observations such as Planck satellite [43] will be sensitive to the non-Gaussianities with much higher precision.

Given the considerations above, now we investigate the leading nonlinear curvature perturbations in our model. As we have discussed previously, the universe experiences a nearly exponential expansion after the bounce, and so we can analyze the curvature perturbations of our nonsingular inflation model as in usual inflation theory except for the fact that the initial condition are modified. Thus, the slow-roll parameters can be defined in our model, as $\epsilon \equiv -\dot{H}/H^2$ (already defined previously) and $\eta \equiv -\frac{\ddot{\phi}\phi}{H^2} + \frac{\dot{\phi}^2}{2H^2}$.

We expand the action to third order and drop the terms suppressed by slow-roll parameters $\epsilon$ and $\eta$. Following the analysis of [36] the cubic action becomes,

$$S_3 = \int dt d^3x \left[ g a^2 \dot{\zeta}^2 \dot{\zeta}^2 + 2f(\zeta) \frac{\delta L}{\delta \zeta} \right], \quad (4.1)$$

where

$$f(\zeta) = \frac{3\epsilon - 2\eta}{4} \zeta^2 + \frac{\epsilon}{2} \zeta^2 \zeta^2 \frac{\partial^2}{\partial \zeta^2} \zeta. \quad (4.2)$$
and \( g = 4a^3 e^2 H \).

Note that the last term \( \frac{\delta L}{\delta \zeta} |_{1} \) in the third-order action can be absorbed by a field redefinition of \( \zeta \)

\[
\zeta = \tilde{\zeta} + f(\tilde{\zeta}) .
\]  

(4.3)

This field redefinition does not affect any other terms in the cubic action since \( f \) is quadratic in \( \zeta \). Moreover, the last term in the cubic action can be cancelled by an extra quadratic part of the field redefinition which is proportional to the first-order equations of motion \( \frac{\delta L}{\delta \zeta} |_{1} \).

With this third-order action, we can now compute the three point function of our model. It can be obtained by making use of the path integral formalism in the interaction picture as follows

\[
\langle \zeta(t, k_1) \zeta(t, k_2) \zeta(t, k_3) \rangle = i \int_{t_i}^{t_f} d\tilde{t} \langle [\zeta(t, k_1) \zeta(t, k_2) \zeta(t, k_3), L_3(\tilde{t})] \rangle,
\]  

(4.4)

where \( t_i \) denotes the initial time for the modes deep inside the Hubble radius, and \( L_3 \) is the third order perturbative Lagrangian.

Recall that in eq. (3.26) we obtained linear curvature perturbations which relates to those in the standard inflation theory by multiplication with the factor

\[
C(k) = 1 - \frac{3H_B}{2k} \sin \frac{2k}{H_B} + \cdots .
\]  

(4.5)

Taking into account this factor in the calculations, we obtain the following leading order contributions to the three point function:

(a): the contribution from \( \dot{\zeta}^2 \partial^{-2} \zeta \). To simplify the expression, we define \( \bar{P}_\zeta = \frac{8}{3} G^2 \bar{P}_\zeta \), which is just the curvature spectrum in inflation models, and \( K = \sum_i k_i \). Then we have,

\[
(2\pi)^7 \delta^{(3)} \left( \sum_i k_i \right) \bar{P}_\zeta^2 \left( \prod_i C(k_i) \right) \times \frac{\epsilon}{K} \sum_{i < j} k_i^2 k_j^2 .
\]  

(4.6)

(b): contribution from field redefinition,

\[
(2\pi)^7 \delta^{(3)} \left( \sum_i k_i \right) \bar{P}_\zeta^2 \left( \prod_i C(k_i) \right) \times \left[ \frac{3\epsilon - 2\eta}{8} \sum_i \frac{k_i^3}{C(k_i)} + \frac{\epsilon}{8} \sum_i \sum_{i \neq j} \frac{k_i k_j^2}{C(k_i)} \right] .
\]  

(4.7)

Since the non-Gaussianities measure the deviations of the CMB power spectrum from a Gaussian distribution, we can define the following nonlinear parameter \( f_{NL} \)

\[
\zeta = \zeta_g + \frac{3}{5} f_{NL} (\zeta_g^2 - \langle \zeta_g^2 \rangle).
\]  

(4.8)

Thus, eventually it is \( f_{NL} \) which characterizes the size of non-Gaussianities, and we obtain

\[
f_{NL} = \frac{10}{3} \left[ \frac{\epsilon}{K} \sum_{i < j} k_i^2 k_j^2 + \frac{3\epsilon - 2\eta}{8} \sum_i \frac{k_i^3}{C(k_i)} + \frac{\epsilon}{8} \sum_i \sum_{i \neq j} \frac{k_i k_j^2}{C(k_i)} \right] \left/ \left( \sum_i \frac{k_i^3}{C(k_i)} \right) \right. .
\]  

(4.9)
There are two limiting cases of non-Gaussianities, which are of particular interests for observations. They are the equilateral shape \((k_1 \sim k_2 \sim k_3)\) and the local shape \((k_1 \sim k_2 \gg k_3)\). For the case of equilateral shape, \(f_{\text{NL}}\) is given by

\[
f_{\text{NL}}^{\text{equil}} \simeq \frac{10}{9} \left[ \frac{15}{8} + C(k) \right] \epsilon - \frac{3}{4} \eta, \tag{4.10}
\]

with \(k = k_1 = k_2 = k_3\). For the local shape, which corresponds to the \(k_3\) mode exiting the Hubble radius much earlier than the other two, we get

\[
f_{\text{NL}}^{\text{local}} \simeq \frac{5}{3} \left[ 2 + C(k) \right] \epsilon - \eta, \tag{4.11}
\]

with \(k = k_1 = k_2 \gg k_3\). One may notice that the above results reduce to those obtained in the single scalar slow-roll inflation when \(C(k) \to 1\). However, since the factor \(C(k)\) has modified the boundary conditions of curvature perturbations when the universe enters inflationary stage, it can lead to an oscillatory signature in the size of the non-Gaussianities. In order to compare our result with those predicted by usual inflation model, we plot \(f_{\text{NL}}\) of equilateral and local shapes in figure 6. In the figure, we take the same normalization as in previous sections Thus, the physical scales correspond to scales which are observable in the CMB. Thus the clear oscillatory signatures could be detected in future observations once their precision becomes high enough.

Our results show that the non-Gaussianities in this model are still suppressed by slow-roll parameters. However, there is an oscillatory signature in \(f_{\text{NL}}\) and the maximal value of \(f_{\text{NL}}\) is bigger than that in usual slow-roll inflation models. The reason for this effect is that the dominant modes of the curvature perturbations have deviated from the Bunch-Davies form when they pass through the bounce and enter the inflationary stage. This is similar to the cases with non-Gaussianity generated from modified initial conditions, see for example refs. [44].
Note that we have neglected the role of the subdominant field in yielding non-Gaussianities, and thus ignored the contribution of non-gravitational interactions in the three point function. This can be justified on two grounds. First, in the specific model we considered, there is no coupling between the two fields and even no potential for the sub-dominant field $\psi$, and so the non-gravitational interactions are weak. Second, according to ref. [45], $f_{\text{NL}}$ can only be affected if there is nonlinear evolution outside the Hubble radius during inflation, and this only takes place if $\psi$ can dominate during the late time evolution of the universe, which obviously does not happen in our model. In ref. [45], the authors have applied a so-called $\delta N$ formalism, and the conclusions obtained are quite generic even for a phantom field.

5 Gravitational wave background

Another important clue to discover the information of the very early universe is the relic GWB formed by primordial tensor fluctuations. The basic mechanism for the generation of primordial GWB in cosmology has been discussed in refs. [46], and a usual inflation model predicts that the power spectrum of primordial tensor fluctuations is scale-invariant and its value is roughly equal to scalar spectrum times the slow-roll parameter $\epsilon$ [47]. However, this relation is not valid in a bouncing cosmology, as shown for instance in ref. [48]. Besides, GWB has also attracted intensive interest of experimenters. A number of detectors are searching for its signals, e.g. Planck [43], Big Bang Observer (BBO) [49], LIGO [50] (see e.g. [51] for an overview).

In this section we consider the evolution of gravitational waves of our model. In order to standardize the calculation, we use the same convention as in ref. [52]. To begin, we write down the metric containing tensor perturbations in a flat FRW background as follows,

$$ds^2 = a(\tau)^2[-d\tau^2 + (\delta_{ij} + \bar{h}_{ij})dx^idx^j],$$

(5.1)

where the Latin indexes represent spatial coordinates. Here the tensor perturbation $\bar{h}_{ij}$ satisfies the following constraints:

$$\bar{h}_{ij} = \bar{h}_{ji}; \quad \bar{h}_{ii} = 0; \quad \bar{h}_{ij,j} = 0.$$  

(5.2)

Due to these constraints, we only have two degrees of freedom in $\bar{h}_{ij}$ which correspond to two polarizations.

Note that we are interested in the spectrum of gravitational waves and the spectral index. Based on the above formalism, the tensor power spectrum can be written as,

$$P_T(k, \tau) \equiv \frac{d(d0|k|^2|0)}{d\ln k} = 32\pi G \frac{k^3}{(2\pi)^3} |H_{ij}(\tau, k)|^2,$$

(5.3)

where we have taken the Fourier transformations

$$\bar{h}_{ij}(\tau, x) = \sqrt{16\pi G} \int \frac{d^3k}{(2\pi)^3} H_{ij}(\tau, k)e^{ikx},$$

(5.4)

and the definition of tensor spectral index $n_T$ is given by

$$n_T \equiv \frac{d\ln P_T}{d\ln k}.$$  

(5.5)

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The GWB at the present time can be characterized by the energy spectrum,
\[ \Omega_{GW}(k, \tau) \equiv \frac{1}{\rho_c(\tau)} \frac{d(0|\rho_{GW}(\tau)|0)}{d \ln k}, \] (5.6)
where \( \rho_{GW}(\tau) \) is the energy density of gravitational waves, and the parameter \( \rho_c(\tau) \) is the critical density of the universe. Since the modes of the GWB which we aim to observe have already reentered the horizon, they should oscillate. Consequently, we can make use of the Friedmann equation \( H^2(\tau) = \frac{8\pi G}{3} \rho_c(\tau) \) and then deduce the relation between the power spectrum and the energy spectrum,
\[ \Omega_{GW}(k, \tau) \simeq \frac{1}{12} \frac{k^2}{a^2(\tau)H^2(\tau)} P_T(k, \tau), \] (5.7)
which will be used in the following calculations.

### 5.1 Tensor perturbations

Now we follow each Fourier mode of the tensor perturbations, labelled by its comoving wavenumber \( k \), and find that there are two regions of values of \( k \) which differ in the number of times of Hubble radius crossing. The evolution of tensor perturbations is sketched in figure 7. Initially all perturbation modes stay inside the Hubble radius in the far past. Since the Hubble radius shrinks in the contracting phase, those modes with a small comoving wavenumber exit the Hubble radius while the large \( k \) scales remain inside. When the bounce takes place, all perturbations are inside the Hubble radius because at that moment the Hubble radius diverges. After that the bounce is followed by a slow-roll expanding phase, so the Fourier modes will once again escape out of the Hubble radius provided that the number of e-foldings of the post-bounce inflationary period is large enough. After that, the modes will reenter the Hubble radius at late times after the slow-roll expanding phase has finished.

Therefore, we can classify the tensor perturbations with different comoving wavenumbers \( k \) into two categories, as the two lines sketched in figure 7 represent. The blue line denotes a mode with large value of \( k \) which does not exit the Hubble radius until the post-bounce inflationary phase; the green line consists of a mode with a small value of \( k \) which exits the Hubble radius in the contracting phase.

Due to the symmetry of \( H_{ij} \), we can express the two polarizations as one function \( v \equiv \frac{a}{\sqrt{2}}(H_{11} + iH_{22}) \). Neglecting the anisotropic stress tensor in the very early universe, we obtain the equation of motion for \( v \):
\[ v'' + k^2 v - \frac{a''}{a} v = 0. \] (5.8)

Following the background evolution of the universe, we obtain three solutions of gravitational waves similar to what we did with scalar perturbations. When the universe is contracting, we have
\[ v = (\tau - \tilde{\tau}_{B-})^\frac{1}{2} \left\{ A_{k}^{T} H_{\frac{3}{2}}^{(1)}[k(\tau - \tilde{\tau}_{B-})] + B_{k}^{T} H_{\frac{3}{2}}^{(2)}[k(\tau - \tilde{\tau}_{B-})] \right\}. \] (5.9)

Here \( H_{\nu}^{(1)} \) and \( H_{\nu}^{(2)} \) are the \( \nu \)-th Hankel function of the first kind and second kind, respectively. Besides, by choosing the initial Bunch-Davies conditions, we obtain \( A_{k}^{T} = 0 \) and
Figure 7. A sketch of the evolution of tensor perturbations with different comoving wavenumbers $k$ in our model.

$B_k = -\sqrt{\pi}/2$. Therefore, the asymptotic forms of the solution to the tensor perturbation in the contracting phase is

$$v(k, \tau) = \begin{cases} 
-\frac{i}{\sqrt{2}}k^{-\frac{3}{2}}(\tau - \tilde{\tau}_{B-})^{-\frac{1}{2}}, & \text{outside horizon;} \\
\frac{1}{\sqrt{2k}}e^{-i k (\tau - \tilde{\tau}_{B-})}, & \text{inside horizon.}
\end{cases} \quad (5.10)$$

When the universe undergoes the bouncing phase, we have the approximate relation that $\frac{\alpha}{a} \simeq \frac{1}{\pi} \alpha a_B^2 = \frac{y}{2}$. To solve eq. (5.8), we have

$$v(k, \tau) = \begin{cases} 
C_k^T \cos[l(\tau - \tau_B)] + D_k^T \sin[l(\tau - \tau_B)], & k^2 \geq \frac{y}{2}; \\
C_k^T e^{l(\tau - \tau_B)} + D_k^T e^{-l(\tau - \tau_B)}, & k^2 < \frac{y}{2},
\end{cases} \quad (5.11)$$

where we define $l^2 = |k^2 - \frac{y}{2}|$. Since the Hubble parameter approaches zero when the universe evolves from the contracting to the expanding phase, all perturbation modes will be sub-Hubble at the bounce point. However, from the above solution we interestingly find that, $k_{ph}^2 (\sim k^2/a_B^2)$ and $H (\sim \alpha)$ are comparable.

After the bounce, the slow-roll expanding phase takes place which drives the universe to inflate like a de-Sitter spacetime. In this case, the solution to the gravitational waves is given by

$$v = (\tau - \tilde{\tau}_{B+})^{\frac{1}{2}} \times \left\{ E_k^T H_\nu^{(1)}[k(\tau - \tilde{\tau}_{B+})] + F_k^T H_\nu^{(2)}[k(\tau - \tilde{\tau}_{B+})] \right\}, \quad (5.12)$$

where $\nu = \frac{1}{2} + \frac{1}{\frac{1}{2} + 1} = \frac{3}{2}$. This solution has an asymptotic form,

$$v \simeq -i \sqrt{\frac{2}{\pi}}k^{-\frac{3}{2}}(\tau - \tilde{\tau}_{B+})^{-1}(E_k^T - F_k^T), \quad (5.13)$$
after the modes exit the horizon.

Having obtained the solutions of the tensor perturbations in different phases, now we need to match these solutions and determine the coefficients $C_T^k$, $D_T^k$, $E_T^k$, and $F_T^k$ respectively. This procedure is similar to the matching process of scalar perturbations as done in the previous section. For a non-singular bounce scenario such as the model we considered, the continuity of the background evolution implies that both $v$ and $v'$ are able to pass through the bounce smoothly. So we match $v$ and $v'$ in (5.10) and (5.11) on the surface $\tau_{B-}$, and those in (5.11) and (5.12) on the surface $\tau_{B+}$. With these matching conditions, we can determine all the coefficients and finally get $E_T^k - F_T^k$.

However, as what we have analyzed at the beginning of this section, there are two paths for the tensor perturbations to evolve from a contracting phase to an expanding phase. So there are two possible results for $E_T^k - F_T^k$. In the first case, the comoving wavenumber is large enough so that the tensor perturbations have never escaped outside the Hubble radius. Thus we have

$$|E_T^k - F_T^k| \simeq \frac{\sqrt{\pi}}{2} \left[ 1 - \frac{e^{-2kH_b}}{4\pi^2} \left( 1 - e^{2ik\delta\tau_B} \right) \right], \quad (5.14)$$

where $\sigma \equiv \frac{y^2}{2k^2}$, and $\delta\tau_B = \tau_{B+} - \tau_{B-}$. In the second case where the modes of gravitational waves are in small $k$ region, the expression $E_T^k - F_T^k$ is given by

$$E_T^k - F_T^k \simeq -\frac{\sqrt{\pi}}{8} \frac{\mathcal{H}_{B-}(2l + 1) + \mathcal{H}_{B-}}{\mathcal{H}_{B+}(l - \mathcal{H}_{B+})} e^{-l\delta\tau_B}. \quad (5.15)$$

Based on the above analysis, we are able to derive the primordial power spectrum of gravitational waves. From the definition of eq. (5.3), the primordial power spectrum is given by

$$P_T(k) = \frac{64GH^2}{\pi^2} |E_T^k - F_T^k|^2. \quad (5.16)$$

From eqs. (5.14) and (5.15), we can read off that the spectrum is scale-invariant both in the large $k$ and small $k$ regions, but oscillates when $k$ is near by a critical value $\sqrt{\frac{y}{2}}$. To illustrate the above analysis clearly, we solve the equations numerically and plot the primordial tensor power spectrum and the corresponding spectral index in figure 8.

One can see from figure 8 that when the value of $k$ is large enough, the red solid lines converge to the blue dash lines with a superimposed oscillation. The amplitude of the oscillation is largest when $k$ approaches the neighborhood of the critical value $\sqrt{\frac{y}{2}}$, and soon drops to a minimal value when $k$ gets smaller. This damping effect is caused by the modified dispersion relation of the tensor perturbations when they pass through the bouncing phase.\(^2\) However, when the comoving wavenumber $k$ gets even smaller, the power spectrum is able to climb up and finally reaches a certain value with its spectral index returning to zero again.

### 5.2 Energy spectrum of today’s GWB

To relate the power spectrum observed today to the primordial one, one can define a transfer function $T(k, \tau)$, given by refs. [52, 53]:

$$T(k, \tau) \simeq \frac{0.80313}{2\pi} \left( \frac{1 + z(\tau)}{1 + z_k} \right)^2 \Gamma^2 \left( \alpha + \frac{1}{2} \right) \left( \frac{2}{\alpha} \right)^{2\alpha}, \quad (5.17)$$

\(^2\) A similar scenario of the primordial gravitational waves has been considered in ref. [52], where the authors have considered the damping effects from spacetime noncommutativity.
Figure 8. The red solid curve represents the primordial power spectrum $P_T$ and the spectral index $n_T$ of tensor perturbations in our model. The blue dashed curve give the primordial power spectrum and the spectral index in a single scalar inflation model. In the figure, we take the values of parameters the same as in figure 1.

where $\alpha = \frac{2}{1 + 3w}$ is determined by the EoS of the universe, $z(\tau)$ is the redshift at the moment $\tau$ and $z_k$ is the redshift when the $k$ mode of gravitational wave reenters the horizon. Here the factor 0.80313 comes from the damping effect of freely streaming neutrinos [54]. Moreover, the factor $(\frac{1+z(\tau)}{1+z_k})^2$ describes the redshift-suppressing effect on the primordial gravitational waves. The rest factor shows that, when the gravitational waves reenter the Hubble radius, there is a “wall” lying at the Hubble radius which affects the tensor spectrum.

Taking into account that our present universe is dominated by dark energy for which the EoS is $w \simeq -1$, we are able to obtain today’s transfer function. Then we can get today’s tensor power spectrum

$$P_T(k, \tau_0) = \frac{0.80313}{2(1 + z_k)^2} \frac{64G H_i^2}{\pi^2} |E^T_k - F^T_k|^2,$$

where $H_i$ represents the Hubble parameter in the inflationary stage. Eventually, the present energy spectrum of GWB is given by $\Omega_{\text{GW}}(k, \tau_0) = \frac{1}{12} \frac{k^2}{(aH_0)^2} P_T(k, \tau_0)$.

In figure 9 we plot the numerical results of the energy spectrum in our model. One can see that, when the frequency of GWB is large enough, the tensor energy spectrum of our model agrees with the prediction of the single scalar inflation model. However, when the frequency becomes smaller, the physics of a bounce begins to affect the behavior of the GWB. Moreover, there is an interesting valley feature at intermediate wavenumbers. This feature results from the suppression of the Hubble parameter around the bounce.

6 Discussion and conclusions

Bouncing cosmologies, due to the avoidance of the cosmological singularity, have attracted a lot of interest in the literature [55–57]. However, since the bounce happens in the extremely high energy regime, it is difficult to observe it directly by experiment. To find evidence of a
Figure 9. The red solid curve represents the energy spectrum $\Omega_{GW}$ in our model. The blue dashed curve gives the energy spectrum in a single scalar inflation model. In the figure, we take the values of parameters the same as in figure 1.

bounce, we need to understand the observational signatures of a bounce. One potential clue is to study the primordial gravitational fluctuations. In the context of the Pre-Big-Bang and Ekpyrotic cosmologies, the primordial curvature perturbations strongly depends on the physics at the time of the singularity, and thus a large uncertainty is involved $[58–60]$. In the framework of loop quantum cosmology, it is argued that fluctuations before and after the bounce are largely independent $[61]$ (yet see ref. $[62]$ for some criticisms).

As shown in ref. $[20]$ and in the current paper, an inflation model with a preceding bounce could give rise to a roughly scale-invariant curvature spectrum which has features which are different from that in the standard inflation model. We expect to see the deviations in observations. In ref. $[20]$ we have done a detailed simulation on CMB temperature power spectrum and the Large Scale Structure (LSS) matter power spectrum. The results show that, in some specific models, this signal might be detectable in forthcoming observations, such as PLANCK $[43]$ and LAMOST $[63]$. If this signal were detected in the near future, it would support the physics of a bouncing cosmology.$^3$ Moreover, the non-Gaussianities are important since they may be able to lead to a discrimination between our model and others which provide a spectrum with small amplitude oscillations. Our analysis has shown that there is an oscillatory behavior in the small $k$ regions for the non-Gaussianity. As far as we know, it is a new feature which has not yet been obtained in others models. Thus we expect this may be an important evidence for a bounce. To complete the analysis of gravitational perturbations of the model, we also have studied the GWB and the corresponding energy spectrum. From the analysis, we find that the spectrum of tensor perturbations is quite different from the usual one. In particular, there is a valley in the spectrum for intermediate wavenumbers between the high-frequency and the low-frequency regimes. We wish to point out that any model of a bounce could produce a valley in the spectrum of gravitational waves, for a simple reason that the amplitude of gravity waves is roughly proportional to the expansion rate at the time when the scale exits the Hubble radius, and so the amplitude of GWB would be suppressed for modes which exit around the bounce point. Therefore, our

$^3$A similar but more phenomenological scenario has been studied in refs. $[19, 64]$ as a possible solution to the suppressed low $l$ multi-poles of the CMB anisotropies.
analysis of the GWB can be applied to other bounce models, and the valley feature might act as a smoking gun for a bouncing cosmology.

Finally let us comment on the theoretical implications of the model we considered. One may notice that, the behavior of spectrum predicted by the bounce model is quite similar to what is obtained in certain approaches to the effects of trans-Planckian physics on inflation [65]. In inflation models combined with certain trans-Planckian corrections there are wiggles in the scalar spectrum, since in those models the initial conditions for scalar perturbations are modified for a certain range of energy scales \( \Lambda \) [66, 67]. However, those models did not tell us what is the physics behind \( \Lambda \) and there are trans-Planckian models where the oscillatory features disappear [68, 69]. The comprehensive study of this paper implies that the scale \( \Lambda \) which appears in certain trans-Planckian models might be the scale of a bounce. Even if a perturbation mode is generated before the bounce from the pure Bunch-Davies state, its sub-Hubble behavior could be changed when it passes through the bouncing phase. When this mode enters the inflationary period, it has mixture of positive frequency (\( \propto e^{ik\eta} \)) and negative frequency (\( \propto e^{-ik\eta} \)) parts, and thus some wiggles can form on the spectrum. The same argument applies to the nonlinear perturbations and tensor modes.

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