1. Introduction

Shellability is a property satisfied by two important families of objects in combinatorics, namely, polytope boundaries [19] and order complexes of geometric lattices [5]. Moreover, skeleta of shellable complexes are themselves shellable [7]. Extendable shellability is the stronger demand that any shelling of any subcomplex may be continued into a shelling of the whole complex. This property is less understood than shellability, and much less common. It is easy to construct polytopes that are not extendably shellable [19]. In 1994 Simon conjectured that for any integers $0 \leq d \leq n$ the $d$-skeleton of the $n$-simplex is extendably shellable [15, Conjecture 4.2.1]. For $d \leq 2$ this was soon proven by Björner and Eriksson [6], but for $3 \leq d \leq n - 4$ the conjecture remains open.

Recently Bigdeli et al. [4] and Dochtermann et al. [10, Conjecture 4.8] [11] have proposed a new approach based on the graph-theoretic notion of chordality, which allowed them to establish Simon’s conjecture for $d \geq n - 3$, and even to show that shellability is equivalent to extendable shellability for $d$-complexes with up to $d + 3$ vertices [11].

Given a $d$-dimensional pure simplicial complex $\Delta$, recall that a clique of $\Delta$ is any subset $V \subseteq [n]$ such that all subsets of $V$ of size $d + 1$ appear among the facets of $\Delta$. For example, if $\Delta$ is the graph $\{12, 23, 13, 14\}$, then $1$, $12$ and $123$ are cliques, whereas $124$ is not.

A pure $d$-dimensional simplicial complex $\Delta$ is called ridge-chordal if it can be reduced to the empty set by repeatedly deleting a ridge $r$ (i.e. a $(d - 1)$-face) such that the vertices of the star of $r$ form a clique [3]. One can see that “ridge-chordal 1-complexes” are precisely the graphs admitting a perfect elimination ordering, i.e. graphs in which every minimal vertex cut is a clique; by Dirac’s theorem, these are precisely the “chordal graphs”, the graphs where every cycle of length at least four has a chord [9].

Now, let $\text{Cl}(\Delta)$ be the “clique complex” of $\Delta$, i.e., the simplicial complex whose faces are the cliques of $\Delta$. This $\text{Cl}(\Delta)$ is a simplicial complex with the same $d$-faces of $\Delta$ and the same $(d - 1)$-faces of the $n$-simplex. The following conjecture appeared naturally, in several recent works:

**Conjecture A** (2, Question 6.3], [10, Conjecture 4.8], [14, Statement A]).

If the Alexander dual of $\text{Cl}(\Delta)$ is shellable, then $\Delta$ is ridge-chordal.

There are three reasons why Conjecture A is natural and of interest:
As explained by Bigdeli et al. [4, Corollary 3.7] and [14, Corollary 4.16], Conjecture A directly implies Simon’s conjecture, cf. Remark 4.

The conjecture is true if one slightly strengthens the assumption “shellable” into “vertex-decomposable”, see [2, Theorem 5.2] and Remark 3 below.

Some partial converse holds: If $\Delta$ is ridge-chordal, then the Alexander dual of $\text{Cl}(\Delta)$ is Cohen–Macaulay over any field, although not necessarily shellable [3, Theorem 3.2].

The purpose of this short note is to disprove Conjecture A:

**Theorem A.** There is a (non–shellable) constructible 2-dimensional complex $\Delta$ that is not ridge-chordal, such that the Alexander dual of $\text{Cl}(\Delta)$ is shellable and even 4-decomposable.

The complex we construct does not disprove Simon’s conjecture, because the shelling of the Alexander dual of $\text{Cl}(\Delta)$, which is 8-dimensional on 12 vertices, does extend to a shelling of the 8-skeleton of the 11-simplex. However, it suggests that possible counterexamples to Simon’s conjecture could be searched among (Alexander duals of clique complexes of) simplicial $d$-complexes $\Delta$ such that $\text{Cl}(\Delta)$ has no free $(d-1)$-faces, for $d \geq 3$.

**2. Construction of the counterexample**

Recall that the link and the deletion of a face $\sigma \in \Delta$ are defined respectively by

$$\text{link}_\Delta(\sigma) := \{ \tau \in \Delta : \sigma \cap \tau = \emptyset, \sigma \subseteq F \supseteq \tau \text{ for some facet } F \}$$

and

$$\text{del}_\Delta(\sigma) := \{ \tau \in \Delta : \sigma \nsubseteq \tau \}.$$

We say that a face $\sigma$ in a pure simplicial complex $\Delta$ is *shedding* if $\text{del}_\Delta(\sigma)$ is pure. A pure simplicial complex $\Delta$ is *k-decomposable* if $\Delta$ is a simplex or if there exists a shedding face $\sigma \in \Delta$ with $\dim \sigma \leq k$ such that $\text{link}_\Delta(\sigma)$ and $\text{del}_\Delta(\sigma)$ are both $k$-decomposable. It is easy to see that if $\Delta$ is $k$-decomposable then it is also $t$-decomposable, for every $k \leq t \leq \dim \Delta$. The notion of $k$-decomposable interpolates between vertex-decomposable complexes (which are the same as 0-decomposable complexes) and shellable complexes (which are the same as $d$-decomposable complexes, where $d$ is their dimension).

We start with a Lemma that is implicit in the work of Bigdeli–Faridi [2].

**Lemma 1.** Let $r$ be a ridge of a pure $d$-dimensional simplicial complex $\Delta$, with $d \geq 1$. Let $S$ be the set of vertices of $\text{Star}(r, \Delta)$. Then $S \subseteq \text{Cl}(\Delta) \iff r$ is a free face in $\text{Cl}(\Delta)$.

**Proof.** $\Rightarrow$: If $r$ lies in two facets $F_1$ and $F_2$ of $\text{Cl}(\Delta)$, then $F_1 = r \cup S_i$ for some $S_i \subseteq [n]$. Since $F_1, F_2 \in \text{Cl}(\Delta)$, for every $s \in S_1 \cup S_2$ we have $r \cup \{s\} \in \Delta$. So $r \cup (S_1 \cup S_2) \subseteq S$ is a clique of $\Delta$. Since $r \cup S_1$ and $r \cup S_2$ are both facets of $\text{Cl}(\Delta)$, we have $S_1 = S_2$, whence $F_1 = F_2$.

$\Leftarrow$: Let $F$ be the unique facet of $\text{Cl}(\Delta)$ that contains $r$. Were there a vertex $s$ of $S$ outside $F$, we would have $r \cup \{s\} \in \Delta \subseteq \text{Cl}(\Delta)$; so there would be $G \in \text{Cl}(\Delta)$, $G \neq F$, such that $r \cup \{s\} \subseteq G$, a contradiction. Hence $S \subseteq F$. \hfill $\Box$

**Lemma 2.** Let $\Delta$ be a pure simplicial complex. If $\Delta$ is ridge-chordal and $\dim \Delta = \dim \text{Cl}(\Delta)$, then $\Delta$ has at least one free codimension-one face.

**Proof.** If $\Delta$ is ridge-chordal, then it must have a ridge $r$ such that the vertices of $\text{Star}(r, \Delta)$ form a clique. By Lemma 1 this $r$ is a free face of $\text{Cl}(\Delta)$. But since $\dim \Delta = \dim \text{Cl}(\Delta)$, the complexes $\Delta$ and $\text{Cl}(\Delta)$ have the same free $(d-1)$-faces, since the $(d-1)$-faces we add when passing from $\Delta$ to $\text{Cl}(\Delta)$ belong to no $d$-face. \hfill $\Box$
Figure 1. A constructible complex $\Delta$ that is not ridge-chordal, because it lacks free edges.

**Proof of Theorem A.** For any $d \geq 2$, there exists a shellable simplicial $d$-complex $C_d$ that has only one free $(d - 1)$-face \[1\]. Let $\Delta$ be the 2-complex obtained from two copies of the complex $C_2$ by identifying the two free edges, as in Figure 1. By definition, $\Delta$ is constructible. By Van Kampen’s theorem, $\Delta$ is contractible. Since $\Delta$ has no free edge and of the same dimension of its clique complex, it is neither ridge-chordal (by Lemma 2) nor shellable (because all shellable contractible complexes are collapsible).

Now, let $A$ be the Alexander dual of $\text{Cl}(\Delta)$. This $A$ is 8-dimensional, with 12 vertices and 194 facets. We claim that $A$ is shellable and even 4-decomposable. We used the following trick to break the claim into five claims that are computationally easy to verify (using, for instance, \[8\]):

- $\sigma = [3, 4, 5, 6, 7]$ is a shedding face of $A$;
- $\text{link}_A(\sigma)$ is shellable 3-dimensional, hence 3-decomposable;
- If $D_1 = \text{del}_A(\sigma)$, then $\tau = [8, 9, 10, 11, 12]$ is a shedding face of $D_1$;
- $\text{link}_{D_1}(\tau)$ is shellable 3-dimensional, hence 3-decomposable;
- $D_2 = \text{del}_{D_1}(\tau)$ is 8-dimensional vertex-decomposable, so in particular 4-decomposable. \[\square\]

**Remark 3.** The 4-decomposability of the example $A$ above is close to being optimal, because by the work of Bidgeli and Faridi there cannot be any 0-decomposable (i.e. vertex-decomposable) counterexample to Conjecture \[\overline{A}\]. To see this, recall that the $d$-closure of a pure $d$-dimensional simplicial complex $\Delta$ (see \[2\] Definition 2.1) is exactly the clique complex $\text{Cl}(\Delta)$. Hence, by \[2\] Proposition 2.7 and \[2\] Theorem 3.4, the following properties are equivalent:

- $\Delta$ is ridge-chordal;
- $\text{Cl}(\Delta)$ is $d$-chordal, in the sense of Bigdeli-Faridi \[2\] Definition 2.6];
- $\text{Cl}(\Delta)$ is $d$-collapsible, in the sense of Wegner \[17\].

Now, let $\Delta$ be a complex such that the Alexander dual of $\text{Cl}(\Delta)$ is 0-decomposable. By \[2\] Theorem 5.2], the complex $\text{Cl}(\Delta)$ is $d$-chordal; so by the equivalence above, $\Delta$ is ridge-chordal and Conjecture $\overline{A}$ holds. En passant, this also explains why Conjecture $\overline{A}$ is equivalent to \[2\] Question 6.3]. Our complex $\Delta$ for which $\text{Cl}(\Delta)^*$ is shellable, is not ridge-chordal, so in particular $\text{Cl}(\Delta)$ is not $d$-chordal.

**Remark 4.** Often in the literature the problems we discussed are phrased in terms of “clutters”. Let $d \geq 1$ be an integer. A $d$-uniform clutter $C$ on $n$ vertices is the collection of the facets of a pure $(d - 1)$-dimensional simplicial complex $\Gamma_C$ on $[n]$. Denote by $I(C)$ the edge ideal of $C$. Let $\overline{C}$ be the clutter on $[n]$ whose edges are the $d$-dimensional non-faces of $\Gamma_C$. It is easy to see that the edge ideal of $\overline{C}$ is the Stanley–Reisner ideal of $\text{Cl}(\Gamma_C)$. Moreover, the ridge-chordality of $\Gamma_C$ is equivalent to the chordality of $C$, as defined in \[3\]. With this terminology, Conjecture $\overline{A}$ can be rephrased as
“if $C$ is a $d$-uniform clutter such that $I(\overline{C})$ has linear quotients, then $C$ is chordal.”

Theorem A, forgetting the 4-decomposability claim, can be stated as

“there exists a 3-uniform clutter $C$ such that $I(\overline{C})$ has linear quotients, but $C$ is not chordal.”

Remark 5. Ridge-chordality was introduced in [3] with the goal to extend Fröberg’s characterization of the squarefree monomial ideals with 2-linear resolution [13]. Several other higher-dimensional extensions of graph chordality exist in the literature: See for instance [12], [16], [18]. An interesting weakening of ridge-chordality is the demand that $I(\overline{\Delta})$ have a linear resolution over any field [3, Theorem 3.2], where $\overline{\Delta}$ is the complex whose facets are the $d$-dimensional non-faces of $\Delta$. As shown by [2, Example 4.7], or by our counterexample above, some complexes $\Delta$ satisfying this property are not ridge-chordal.

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