ON THE NEGATIVE K-THEORY OF SINGULAR VARIETIES

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Abstract. Let $X$ be an $n$-dimensional variety over a field $k$ of characteristic zero, regular in codimension 1 with singular locus $Z$. In this paper we study the negative $K$-theory of $X$, showing that when $Z$ is sufficiently nice, $K_{1-n}(X)$ is an extension of $KH_{1-n}(X)$ by a finite dimensional vector space, which we compute explicitly. We also show that $KH_{1-n}(X)$ almost has a geometric structure. Specifically, we give an explicit 1-motive $[L \to G]$ and a map $G(k) \to KH_{1-n}(X)$ whose kernel and cokernel are finitely generated abelian groups.

1. Introduction

Historically, the computation of the algebraic $K$-theory of schemes has been a difficult problem. Progress has steadily been made over the past few decades, and in this paper we focus on the negative $K$-theory of varieties in characteristic 0. These groups tend to be more accessible than those in positive degree. For example, it is well known that the negative $K$-theory of regular schemes vanish. For singular schemes, some progress has been made for schemes in low dimension – for example, see Weibel [19] for the case of normal surfaces.

Let $X$ be integral $n$-dimensional scheme ($n \geq 3$) of finite type over an algebraically closed field $k$ of characteristic zero, such that $X$ has only isolated singularities. In this paper, we give a full description of $K_{-2}(X)$ when $n = 3$, and partially generalize our findings to a description of $K_{1-n}(X)$ when $Z = \text{Sing}(X)$ is either smooth or of codimension greater than 2.

With a little work, we establish an exact sequence

$$NK_{1-n}(X) \longrightarrow K_{1-n}(X) \longrightarrow KH_{1-n}(X) \longrightarrow 0$$

which computes $K_{1-n}(X)$. We compute the contributions $NK_{1-n}(X)$ and $KH_{1-n}(X)$ separately, and then determine how they fit together. More specifically, we compute $KH_{1-n}(X)$ and determine the image of $NK_{1-n}(X)$ in $K_{1-n}(X)$.

When $X$ is a irreducible $n$-dimensional scheme of finite type over a field $k$ (not necessarily algebraically closed) of characteristic zero, with at most isolated singularities, we show that the image of $NK_{1-n}(X)$ in $K_{1-n}(X)$ is a finite-dimensional $k$-vector space.

On the other hand, when $X$ is an integral $n$-dimensional scheme of finite type over an algebraically closed field $k$ of characteristic zero, such that $Z = \text{Sing}(X)$ is either smooth over $k$ or of codimension greater than 2, we relate $KH_{1-n}(X)$ to the largest

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torsion-free mixed Hodge structure $H$ in $H^n(X, \mathbb{Z})$ of type $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ such that $\text{Gr}^W_1 H$ is polarizable.

The main theorem, in the case that $X$ is a complex threefold with isolated singularities, exemplifies almost all of the interesting phenomena that occur in the general case.

**Theorem 1.1** (Main theorem for $K_{-2}(X)$ of a complex threefold with isolated singularities). Let $X$ be an integral three-dimensional variety of finite type over $\mathbb{C}$ with only isolated singularities. Then there is a short exact sequence

$$0 \rightarrow V \rightarrow K_{-2}(X) \rightarrow KH_{-2}(X) \rightarrow 0,$$

where $V$ is a finite-dimensional $\mathbb{C}$-vector space (explicitly computed in section 6), and $KH_{-2}(X)$ has the following description. Let $H \subseteq H^3(X, \mathbb{Z})$ be the largest torsion-free mixed Hodge structure of type $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ such that $\text{Gr}^W_1 H$ is polarizable, and let $M = [L \rightarrow G]$ be a 1-motive corresponding to $H$ under Deligne’s equivalence between 1-motives and mixed Hodge structures of the given type. Then there is a left-exact sequence

$$0 \rightarrow L(\mathbb{C}) \rightarrow G(\mathbb{C}) \xrightarrow{\alpha} KH_{-2}(X)$$

such that $\text{coker}(\alpha)$ is a finitely generated abelian group.

Morally, the main theorem says that $K_{-2}(X)$ is an extension of something that, up to some finitely generated abelian groups, is isomorphic to the $\mathbb{C}$-points of a 1-motive, by a finite-dimensional vector space.

### 1.1. Notation and outline of paper

In this section, we introduce the problem and give a brief history. We then state the main result, establish notation, and give an outline of the paper. The computation of $KH_{1-n}(X)$ will take up the majority of this paper, and sections 2 through 5 are dedicated to this computation. In section 2, we take a (good) resolution of singularities for $X$, and then apply a descent argument to establish an exact sequence (9) computing $KH_{1-n}(X)$. We then compute each of the contributions in section 3. The main piece of $K_{1-n}(X)$ is a 1-motive that arises out of this computation, which is the focus of section 4. These computations are done after picking a resolution of singularities for $X$, and it is natural to ask what parts, if any, are independent of the choice of resolution. We address this question in section 5. In section 6, we compute $NK_{1-n}(X)$ and describe its image in $K_{1-n}(X)$ under the map given in (2.1), wrapping up the computation of $K_{1-n}(X)$.

Throughout this paper, $k$ will denote a field of characteristic 0. $\text{Sch}/k$ will denote the category of separated schemes over $k$ of finite type, and $Z = \text{Sing}(X)$ will denote the singular locus of $X$. Starting in section 4, we will need to refer to both Picard groups and the schemes that represent them; to avoid confusion, we will let $\text{Pic}(X)$ denote the Picard scheme of $X$, whenever it exists, and similarly for $\text{Pic}^0(X)$.  

2. Calculation of $KH_{1-n}(X)$

In this section, in addition to our assumptions about $X$ laid out in 1.1, we will also assume that $X$ reduced, and either codim $Z > 2$ or that $Z$ is smooth. We begin by considering a good resolution of singularities $p: \tilde{X} \to X$, i.e. a proper birational map which is an isomorphism outside of $Z$, and such that the exceptional divisor $E$ is a simple normal crossing divisor. $KH$ satisfies cdh-descent [7], and applying it to our good resolution yields a long exact sequence of $KH$ groups

$$
\cdots \to KH_{-q}(\tilde{X}) \to KH_{-q}(X) \to KH_{-q}(E) \to \cdots
$$

Since $\tilde{X}$ is smooth, its $KH$-groups agree with its $K$-groups, which vanish in negative degree (and the same if $Z$ is smooth). If codim $Z > 2$, then the $K$-dimension theorem asserts that $K_{-q}(Z) = KH_{-q}(Z) = 0$ for all $q > n - 2$. In either case, we obtain natural isomorphisms $K_{1-q}(E) \cong K_{-q}(X)$ for all $q > n - 2$.

When $Z = \coprod_i Z_i$ has more than one connected component, we will have $KH_{1-n}(X) \cong \oplus_i KH_{2-n}(E_i)$, where $E_i$ is the total transform of $Z_i$. We can compute each of these groups separately, so we may assume that $Z$ is connected. By Zariski’s main theorem, $E$ will also be connected, so we will also assume that $E$ has $r$ irreducible components. In any case, to compute $K_{1-n}(X)$, we will compute $K_{2-n}(E)$ instead, using the fact that it has simple normal crossings.

There is a cdh-descent spectral sequence for $KH$, which we apply to $E$: [7]

$$E_2^{pq} = H_{cdh}^p(E, aKH_{-q}) \Rightarrow KH_{-p-q}(E),$$

where $a$ denotes sheafification in the cdh-topology. The first thing to note is that since schemes are locally smooth in the cdh-topology, the natural map $K \to KH$ induces an isomorphism of sheaves $aK_q \cong aKH_q$. We now have a short lemma.

**Lemma 2.1.** For $q \leq 1$, we have the following isomorphism of sheaves on $Sch/k$:

$$a_{cdh}K_q = \begin{cases} a_{cdh}\mathbb{G}_m, & q = 1 \\ a_{cdh}\mathbb{Z}, & q = 0 \\ 0, & q < 0. \end{cases}$$

**Proof.** First, since every cdh-cover has a refinement by smooth schemes, and $K_q(U) = 0$ whenever $U$ is regular and $q < 0$, $a_{cdh}K_q = 0$ when $q < 0$.

We can sheafify both $K_0$ and $\mathbb{Z}$ in two steps, as follows:

$$
\begin{array}{c}
K_0 \to a_{Zar}K_0 \to a_{cdh}a_{Zar}K_0 = a_{cdh}K_0 \\
\downarrow \text{rank} \downarrow \text{rank} \\
\mathbb{Z} \to a_{Zar}\mathbb{Z} \to a_{cdh}a_{Zar}\mathbb{Z} = a_{cdh}\mathbb{Z}
\end{array}
$$
The rank map \( a_{\text{Zar}} K_0 \to \mathbb{Z} \) is an isomorphism on local rings, so \( a_{\text{cdh}} K_0 \to a_{\text{cdh}} \mathbb{Z} \) is an isomorphism. We have a similar argument for the sheaf \( a_{\text{cdh}} K_1 \):

\[
\begin{array}{ccc}
K_1 & \longrightarrow & a_{\text{Zar}} K_1 \\
\downarrow & & \downarrow \\
\mathbb{G}_m & \longrightarrow & G_m
\end{array}
\]

\( d_{2}^{n-3,0} : H_{\text{cdh}}^{n-3}(E, \mathbb{G}_m) \to KH_{2-n}(E) \to H_{\text{cdh}}^{n-2}(E, \mathbb{Z}) \to 0. \)

To calculate \( KH_{2-n}(E) \), we need to know about the map \( d_{2}^{n-3,0} : H_{\text{cdh}}^{n-3}(E, \mathbb{Z}) \to H_{\text{cdh}}^{n-1}(E, \mathbb{G}_m) \). We mentioned in the introduction that the case \( n = 3 \) differs from the general case \( n > 3 \); we see an example of this below.

**Lemma 2.2.** In the case \( n = 3 \), the differential \( d_{2}^{0,0} \) is zero.

**Proof.** Let \( P \) be a closed point of \( E \), and consider the diagram

\[
\begin{array}{ccc}
K_0(E) & \longrightarrow & KH_0(E) \\
\downarrow & & \downarrow \\
K_0(P) & \longrightarrow & KH_0(P)
\end{array}
\]

obtained from naturality of both the map \( K \to KH \) and the descent spectral sequence. Since we assume \( E \) is connected, the vertical map on the right, \( H_{\text{cdh}}^{0}(E, \mathbb{Z}) \to H_{\text{cdh}}^{0}(P, \mathbb{Z}) \) is an isomorphism. Furthermore, the rank map on the left is surjective since there are vector bundles on \( E \) of any rank. A diagram chase shows that the map \( E_{\to}^{0,0}(E) \to E_{\to}^{0,0}(E) \) is an isomorphism.

Thus when \( n = 3 \), the descent spectral sequence reduces to a short exact sequence

\[
0 \longrightarrow H_{\text{cdh}}^{2}(E, \mathbb{G}_m) \longrightarrow KH_{-1}(E) \longrightarrow H_{\text{cdh}}^{1}(E, \mathbb{Z}) \longrightarrow 0.
\]

To compute the \( \text{cdh} \)-cohomology groups appearing in (9), we take a small detour to recall several constructions associated to the simple normal crossing divisor \( E \).
The simplicial and semisimplicial schemes associated to $E$, denoted $\Delta_\ast E$ and $\Delta^\text{alt}_\ast E$ respectively, are constructed as follows.

\[
\Delta_\ast E = \prod_{i_0, \ldots, i_p} (E_{i_0} \times_E \cdots \times_E E_{i_p}) \\
\Delta^\text{alt}_\ast E = \prod_{i_0 < \cdots < i_p} (E_{i_0} \times_E \cdots \times_E E_{i_p}),
\]

with the face maps $d_{p,j}$ given by the natural projections from the fiber products and the degeneracy maps $s_{p,j}$ induced by the diagonal maps (isomorphisms) $E_j \to E_j \times_E E_j$. The simplicial scheme $\Delta_\ast E$ has both face and degeneracy maps, whereas the semisimplicial scheme $\Delta^\text{alt}_\ast E$ has only face maps.

Let $\mathcal{V}_k$ denote the additive category of $k$-varieties, where the objects are $k$-varieties, and the morphisms are formal $\mathbb{Z}$-linear combinations of actual morphisms of varieties. From $\Delta^\text{alt}_\ast E$ we can construct a complex of varieties $C_\ast(\Delta^\text{alt}_\ast E)$ in $\mathcal{V}_k$ in the standard way by taking the differentials to be the alternating sum of the face maps, i.e. $d_p = \sum_i (-1)^i d_i$ (and similarly for $\Delta_\ast E$).

A related construction is that of the dual complex associated to $E$ which we denote, following [10], by $\mathcal{D}(E)$. It is a CW-complex constructed as follows. For each component $E_i$ of $E$ we have a vertex, which we label $i$. Then for each (connected) component of each intersection $E_i \cap E_j$, we glue in a 1-cell between vertices $i$ and $j$ – this is the 1-skeleton of $\mathcal{D}(E)$. Proceeding inductively, we attach an $m$-cell onto the $(m-1)$-skeleton for each connected component of each $m$-fold intersection. Since $E$ has finitely many components, we will eventually stop gluing, and will be left with the CW-complex $\mathcal{D}(E)$.

Additionally, different resolutions of $X$ yield different dual complexes $\mathcal{D}(E)$, but it turns out that the homotopy type of $\mathcal{D}(E)$ is independent of the choice of good resolution [12, Theorem 1.2]. This fact is reflected in our use of the notation $\mathcal{DR}(X)$ to denote the homotopy type of $\mathcal{D}(E)$. On the other hand $\mathcal{D}(E)$ is in general only a cell complex and need not be a simplicial complex.

It is convenient for simplifying upcoming calculations to investigate for which $X$ we can find a resolution whose exceptional divisor $E$ has $\mathcal{D}(E)$ a simplicial complex. Luckily, the answer turns out to be the best possible: such a resolution always exists. We begin with establishing an obvious criterion on $E$ to have $\mathcal{D}(E)$ be a simplicial complex. An intersection $\cap_{i \in I} E_i$ is by definition smooth in a simple normal crossing divisor, so it is the disjoint union of its components. We will call $\cap_{i \in I} E_i$ a bad intersection if it is not irreducible. It turns out that bad intersections are the only obstruction for $\mathcal{D}(E)$ to be a simplicial complex.

**Lemma 2.3.** Given a good resolution $p: \tilde{X} \to X$, the dual complex $\mathcal{D}(E)$ is a simplicial complex if and only if each of the intersections $\cap_{i \in I} E_i$ is irreducible.

**Proof.** Suppose $E$ has $m$ irreducible components, and that $\cap_{i \in I} E_i$ is a bad intersection. Then in the construction of the dual complex, we will have multiple $|I|$-cells glued in the same place, so $\mathcal{D}(E)$ cannot be simplicial.

Conversely, suppose $\cap_{i \in I} E_i$ is irreducible, and consider the corresponding $|I|$-simplex $D_I$ in $\mathcal{D}(E)$. All of the faces of $D_I$ are in $\mathcal{D}(E)$, because any such face
corresponds to a smaller intersection of the $E_i$, which must be nonempty. Furthermore, for any other subset $J$ of $\{1, \ldots, m\}$, we have $D_I \cap D_J = D_{I \cup J}$, which is a face of both.

The preceding criterion eliminates disconnected intersections (which correspond to multiple cells glued to the same vertices). We now proceed with the proof that a resolution of $X$ always exists with dual complex $\mathcal{D}(E)$ a simplicial complex. The idea behind the proof was communicated to us by János Kollár.

**Proposition 2.4.** There exists a resolution of $X$ with exceptional divisor $E$ for which $\mathcal{D}(E)$ is a simplicial complex. Moreover, such a resolution can be obtained from any good resolution by further blowups.

**Proof.** Let $p: \tilde{X} \to X$ be a good resolution with exceptional divisor $E$. We will iteratively blow up enough closed subschemes so that the conditions of Lemma 2.3 are satisfied. We begin blowing up components of bad intersections of the smallest dimension, then move up in dimension.

Write $E = \bigcup_{i=1}^m E_i$ as the union of its irreducible components, and suppose $E$ has no bad intersections of codimension $> r$. We will blow up components of bad intersections of codimension $r$ one by one; we claim that when we have blown them all up, the resulting divisor will not have any bad intersections of codimension $r$. Write $E_I = \bigcap_{i \in I} E_i$, and let $B_r(E_I)$ be the number of connected components $E_I^{(j)}$ that belong to some bad intersection $E_I$ (i.e. $E_I$ has more than one connected component). Fix a bad intersection $E_I$ of codimension $r$, and without loss of generality, blow up $\tilde{X}$ along the smooth irreducible center $Z = E_I^{(1)}$. We claim that if $p': \text{Bl}_{E_I^{(1)}} \tilde{X} \to X$, then $B_r(p') = B_r(p) - 1$. It is easy to verify that if we continue in this manner, we will eventually remove all of the bad codimension $r$ intersections of $E$. This finishes the proposition. □

**Definition 2.5.** We call a strong resolution $p: \tilde{X} \to X$ an excellent resolution if the exceptional divisor $E$ is a simple normal crossing divisor and $\mathcal{D}(E)$ is a simplicial complex.

One way that the above constructions are relevant to us is that the collection of maps $\{E_i \to E\}_i$ is a cdh-cover, so we get a Čech-to-derived spectral sequence

$$E_1^{p,q} = H^q_{\text{cdh}}(\Delta_p^* E, K_m) \Longrightarrow H^{p+q}_{\text{cdh}}(E, K_m),$$

for each $m$. We may replace the rows $E_1^{*,q}$ in the $E_1$ page of this spectral sequence with the quasi-isomorphic complexes consisting of only the non-degenerate parts [21, Lem. 8.3.7]; that is, we may replace $H^q_{\text{cdh}}(\Delta_p^* E)$ with $H^q_{\text{cdh}}(\Delta^q_{alt} E)$. By a result of Voevodsky [17], we may also replace the cdh-cohomology groups with Zariski cohomology groups, since the sheaves $a_{\text{cdh}} K_m$ are homotopy invariant sheaves with transfers [16, Sec. 3.4]. So we obtain

$$E_1^{p,q} = H^q_{\text{Zar}}(\Delta_p^* E, K_m) \Longrightarrow H^{p+q}_{\text{cdh}}(E, K_m).$$
For this first quadrant spectral sequence, many terms are zero. First, $\Delta_p^{alt}E = \emptyset$ for $p > n - 1 \geq \dim E$, so $E_{p,q}^{1} = 0$ for $p > n - 1$. Additionally, since $\dim \Delta_p^{alt}E \leq \dim E - p = n - 1 - p$, we have $E_{1,q}^{p,q} = 0$ for $p + q > n - 1$.

We first use this spectral sequence to compute the groups $H^i_{cdh}(E, \mathbb{Z})$.

**Lemma 2.6.** $H^i_{cdh}(E, \mathbb{Z}) \cong H^i(D(E), \mathbb{Z})$. In particular, these groups are finitely generated.

**Proof.** In addition to the observations we have already made about the spectral sequence, we also have $E_{1,q}^{p,q} = 0$ for $q > 0$ since $\mathbb{Z}$ is flasque as a Zariski sheaf. So $H^i_{cdh}(E, \mathbb{Z})$ is just the cohomology of the complex

$$0 \rightarrow H^0_{\text{Zar}}(\Delta_{i-1}E, \mathbb{Z}) \rightarrow H^0_{\text{Zar}}(\Delta_i^{alt}E, \mathbb{Z}) \rightarrow H^0_{\text{Zar}}(\Delta_{i+1}E, \mathbb{Z}) \rightarrow \cdots$$

in degree $i$. Since $H^0_{\text{Zar}}(Y, \mathbb{Z}) = \mathbb{Z}$ for $Y$ smooth and connected, this complex is isomorphic to the cellular chain complex of $D(E)$.

Furthermore, since the homotopy type of $D(E)$ is independent of the choice of resolution [12], we also have that $H^i_{cdh}(E, \mathbb{Z}) \cong H^i(DR(X), \mathbb{Z})$ is also independent of the choice of resolution. \hfill $\Box$

**Example 2.7.** This same approach allows us to calculate $KH_{-n}(X)$. Applying cdh-descent for $KH$ to our resolution of singularities of $X$ yields $KH_{1-n}(E) \cong KH_{1-n}(X)$; application of the descent spectral sequence then yields $KH_{1-n}(E) \cong E_{2}^{n-1,0} = H^{n-1}_{cdh}(E, \mathbb{Z})$. The above lemma then tells us that $KH_{1-n}(E) \cong H^i(D(E), \mathbb{Z}) = H^i(DR(X), \mathbb{Z})$. \hfill $\Box$

Applying Lemma 2.6 to equation (9), we see that the kernel and cokernel of the map $H^i_{cdh}(E, \mathbb{G}_m) \rightarrow KH_{1-n}(E)$ are finitely generated. We may rephrase this result by saying that the cohomology group $H^{n-1}_{cdh}(E, \mathbb{G}_m)$, which is generally large as we will see, approximates $KH_{1-n}(E)$, up to some finitely generated groups.

We would now like to compute $H^{n-1}_{cdh}(E, \mathbb{G}_m)$. We begin by analyzing the spectral sequence (14).

3. **Simplifying the simplicial spectral sequence**

We begin with a small, well-known fact.

**Lemma 3.1.** Let $Y$ be a smooth scheme over $k$. Then $H^q_{\text{Zar}}(Y, \mathbb{G}_m) = 0$ whenever $q > 1$.

**Proof.** The claim follows immediately from the explicit flasque resolution

$$0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{X}^\times \rightarrow \text{CaDiv} \rightarrow 0,$$

where $\mathcal{X}$ is the sheaf of total quotient rings on $Y$, and CaDiv is the sheaf of Cartier divisors on $Y$. \hfill $\Box$
Since $\Delta_{alt}^n E$ is a smooth semisimplicial scheme, Lemma 3.1 tells us that the $E_1$ page of the spectral sequence (14) with $m = 1$ only has two rows, and we need only compute $E_{3}^{n-1,0}$ and $E_{3}^{n-2,1}$. The $E_1$ page of this spectral sequence looks like

\[
\cdots \longrightarrow \text{Pic}(\Delta_{alt}^{n-3} E) \xrightarrow{d_{n-3,1}^{n-3,1}} \text{Pic}(\Delta_{alt}^{n-2} E) \longrightarrow 0 \longrightarrow 0
\]

(17)

\[
\cdots \longrightarrow k(\Delta_{alt}^{n-3} E)^\times \xrightarrow{d_{n-3,1}^{n-3,1}} k(\Delta_{alt}^{n-2} E)^\times \longrightarrow 0
\]

In order to compute $E_{n_\infty}^{n-1,0} = E_{3}^{n-1,0}$, we need to determine the possibly nonzero differential $d_{3}^{n-3,1}$, which we have denoted using a dashed arrow in the diagram above. Applying the global sections of the resolution (16) for each $\Delta_{alt}^p E$ in each column yields the following diagram.

\[
\cdots \longrightarrow \text{Div}(\Delta_{alt}^{n-3} E) \xrightarrow{d_{n-3,1}^{n-3,1}} \text{Div}(\Delta_{alt}^{n-2} E) \longrightarrow 0
\]

(18)

\[
\cdots \longrightarrow k(\Delta_{alt}^{n-3} E)^\times \xrightarrow{d_{n-3,1}^{n-3,1}} k(\Delta_{alt}^{n-2} E)^\times \longrightarrow k(\Delta_{alt}^{n-1} E)^\times
\]

The face maps $d_i$ induce pullback maps on Picard groups, and the $E_1$ differentials are the alternating sum of these pullback maps. The dashed horizontal maps in the above diagram are the alternating sum of the pullbacks on the divisors themselves. They are dashed because they are not necessarily defined on all of the source, only on those divisors $\text{Div}(\Delta_{alt}^m E)$ which intersect $\text{Div}(\Delta_{alt}^{m+1} E)$ transversally. To remedy this, we will find a quasi-isomorphic subcomplex for which the horizontal maps are defined, then use this subcomplex to show that the map $d_{3}^{n-3,1}$ in (17) is the zero map. Our current discussion motivates the following definition.

**Definition 3.2.** For each $p$, we define the group of good divisors

\[(19) \quad \text{Div}_g(\Delta_{alt}^p E) = \{ D \in \text{Div}(\Delta_{alt}^p E) \mid D \text{ intersects } \Delta_{alt}^m E \text{ transversally for all } m > p \}.
\]

**Remark 3.3.** By Bertini’s theorem, this definition is equivalent to the one that instead requires the image of $D$ under any composition of any of the face maps $d_j$ to be defined. In addition, while the notation $\text{Div}_g$ comes from Carlson [2], our definitions are slightly different. Carlson only requires that the image of $d_j$ is contained in $\text{Div}(\Delta_{alt}^{m+1} E)$, instead of requiring that any composable composition of the $d_j$ is defined. Furthermore, Carlson’s definition applies to a more general class of semisimplicial schemes, as we only define $\text{Div}_g$ for semisimplicial schemes associated to the special class of simple normal crossing schemes.

We will now prove the following:

**Lemma 3.4 (Moving Lemma).** For each $p$, let $A_p$ be the pullback
\[
\begin{array}{c}
A_p \longrightarrow \text{Div}_g(\Delta_{alt}^p E) \\
\ker(\alpha_p) \downarrow \quad \downarrow v_k \quad \downarrow v_{\text{Div}} \quad \downarrow v_{\text{coker}} \\
\ker(\beta_p) \longrightarrow k(\Delta_{alt}^p E)^\times \longrightarrow \text{Div}(\Delta_{alt}^p E) \longrightarrow \text{Pic}(\Delta_{alt}^p E)
\end{array}
\]

where \( \beta_p \) is the rational function-to-divisor map. Then the vertical maps are a quasi-isomorphism of complexes.

**Proof.** We add in the horizontal kernels and cokernels to the diagram above, and label the vertical maps:

\[
\begin{array}{c}
A_p \longrightarrow \text{Div}_g(\Delta_{alt}^p E) \\
\ker(\alpha_p) \downarrow \quad \downarrow \alpha_p \\
\ker(\beta_p) \longrightarrow k(\Delta_{alt}^p E)^\times \longrightarrow \text{Div}(\Delta_{alt}^p E) \\
\end{array}
\]

First, injectivity of \( v_{\text{coker}} \) follows from a diagram chase and the fact that the middle square is cartesian. To establish surjectivity of \( v_{\text{coker}} \), let \( t \in \text{Div}(\Delta_{alt}^p E) \). We would like to lift \( t \) to a good divisor on \( \Delta_{alt}^p E \). In order to do so, we would like to wiggle \( t \) by a principal divisor so that it meets \( \Delta_{alt}^p E \) transversally for \( q > p \). But since \( \Delta_{alt}^q E = \emptyset \) for sufficiently large \( q \) and each \( \Delta_{alt}^p E \) has a a finite number of components, we may apply Bertini’s theorem to find a lift \( s \in \text{Div}_g(\Delta_{alt}^p E) \) of \( t \).

By replacing each column \( k(\Delta_{alt}^p E)^\times \longrightarrow \text{Div}(\Delta_{alt}^p E) \) with the quasi-isomorphic complex obtained from \( \text{Div}_g(\Delta_{alt}^p E) \) as in the lemma above, we can replace the diagram (18) with the following diagram

\[
\begin{array}{c}
\cdots \longrightarrow \text{Div}_g(\Delta_{alt}^{n-3} E) \longrightarrow \text{Div}_g(\Delta_{alt}^{n-2} E) \longrightarrow 0 \longrightarrow 0 \\
\cdots \longrightarrow A_{n-3} \longrightarrow A_{n-2} \longrightarrow k(\Delta_{alt}^{n-1} E)^\times \longrightarrow 0
\end{array}
\]

where all of the horizontal maps are indeed defined. We may then use this diagram to calculate the differential \( d_2^{n-3,1} \) that appears in the spectral sequence (14). We claim this map is zero.

**Lemma 3.5.** The differential \( d_2^{n-3,1} \) appearing in the spectral sequence (14) is the zero map.

**Proof.** Recall that \( E \) was assumed to have \( r \) irreducible components. We regret having to introduce the following notation. Let \( r = \{1 < \cdots < r\} \), and let us also set

\[
\begin{array}{c}
I = \{\{i_0 < \cdots < i_{n-1}\} | i_1, \ldots, i_{n-1} \in r\}.
\end{array}
\]
$I$ denotes the set of all ordered subsets of \{1, \ldots, r\} that have length \(n\). We will also let \(i\) and \(j\) denote ordered subsets of \(m\) of length \(n - 2\) and \(n - 1\), respectively. Keeping tight track of the indices would be a notational burden and detracts from the main thrust of the proof, so there will be some looseness in our usage of \(i\) and \(j\).

If \(i = \{i_0, \ldots, i_{n-2}\}\) and \(i \not\subseteq j = \{i_0, \ldots, i_m, a, i_{m+1}, \ldots, i_{n-2}\}\) so that \(\hat{j}\) is obtained from \(i\) by inserting \(a\) after the \(m^{th}\) element of \(i\), then we define \(\text{sign}(i, j) = (-1)^m\).

Consider \(D \in \text{Div}_U(\Delta_{n-3}^alt E)\) that represents an element of \(E_2^{n-3, 1} = \text{ker}(d_1^{n-3, 1})\) (see (17)). The image of \(D\) in \(\text{Pic}(\Delta_{n-2}^alt E)\) is zero, so it pulls back to a rational function \(g = (g_j) \in A_{n-2}\).

Write \(D = (D_i)\) and \(D_i = D_i' - D_i''\) such that \(D_i'\) and \(D_i''\) are effective divisors whose supports intersect in codimension at least two. The divisors \(D_i', D_i''\) are defined locally on open \(U\) by the vanishing of sections \(f_i', f_i'' \in \Gamma(U, \mathcal{O}_U)\), respectively.

On \(E_j\), the divisor \(\sum_{i \subseteq j} (-1)^{\text{sign}(i, j)} (D_i \cap E_j)\) has degree zero, and is locally defined by \(f_j := \prod_{i \subseteq j} (f_i')^{(-1)^{\text{sign}(i, j)}}\). Then the divisor defined locally by \(g_j / f_j\) has no zeroes or poles hence is constant on \(E_j\). This shows that the function defined locally by the \(f_j\) is in fact actually a rational function, and gives the same divisor class as \(g\). Since \(D\) is a good divisor, it meets \(\Delta_{n-1}^alt E\) transversally, i.e. the support of \(D\) does not intersect \(\Delta_{n-1}^alt E\). Then the zeros of the \(f_j', f_j''\) do not intersect \(\Delta_{n-2}^alt E\), so we can use the \(f_j\) to evaluate \(d_2^{n-3, 1}(D)\). But then \(d_2^{n-3, 1}(D)\) is gotten by the composite of two face maps, so it must be trivial. \(\square\)

Consequently, \(E_{\infty}^{n-1, 0} = H^{n-1}(\mathcal{D}(E), k^\times)\), and we have the following corollary.

**Corollary 3.6.** Writing \(\text{coker}(\text{Pic})\) for the cokernel of \(d_1^{n-3, 1}: \text{Pic}(\Delta_{n-3}^alt E) \to \text{Pic}(\Delta_{n-2}^alt E)\), we have a short exact sequence

\[
0 \longrightarrow H^{n-1}(\mathcal{D}(E), k^\times) \longrightarrow H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m) \longrightarrow \text{coker}(\text{Pic}) \longrightarrow 0.
\]

**Example 3.7.** If \(H_{n-2}(\mathcal{D}(E), \mathbb{Z})\) is torsion-free, or if \(k\) contains all roots of unity (e.g. when \(k\) is algebraically closed), then we also have \(H^{n-1}(\mathcal{D}(E), k^\times) = H^{n-1}(\mathcal{D}(E), \mathbb{Z}) \otimes k^\times\), via the universal coefficient theorem. In particular, \(H^{n-1}(\mathcal{D}(E), k^\times) \cong (k^\times)^r = T_E(k)\), for some \(r\), is the \(k\)-points of some split torus \(T_E\).

The thrust of the next subsection is to show that a 1-motive naturally arises out of the spectral sequence (14).

4. **Computation of Picard groups**

Since all of the schemes \(\Delta_p^alt E\) are projective, the Picard functor is representable for these schemes; in particular, \(\text{Pic}^0(\Delta_p^alt E)\), the connected component of the Picard scheme \(\text{Pic}(\Delta_p^alt E)\), exists. Let the Néron-Severi group functor, \(\text{NS}\), be the presheaf cokernel defined by \(\text{Pic}/\text{Pic}^0\). As in Corollary 3.6, whenever there is no ambiguity, we will write \(\text{ker}(\text{NS})\) for the kernel of the induced map on Néron-Severi groups \(\text{NS}(\Delta_{n-3}^alt E) \to \text{NS}(\Delta_{n-2}^alt E)\), and similarly for the kernels and cokernels of other such maps induced by \(\Delta_{n-3}^alt E \to \Delta_{n-2}^alt E\).
We are interested in the group $\text{coker}(\text{Pic})$. Writing $\text{Pic}$ as an extension of $\text{NS}$ by $\text{Pic}^0$ and then applying the snake lemma to the resulting diagram obtained from the map $\Delta_{n-3}^{alt} E \to \Delta_{n-2}^{alt} E$ gives us an exact sequence ending in

$$
\cdots \to \text{ker}(\text{NS}) \to \text{coker}(\text{Pic}^0) \to \text{coker}(\text{Pic}) \to \text{coker}(\text{NS}) \to 0.
$$

Taking (24) and pulling back along the map $\text{coker}(\text{Pic}^0) \to \text{coker}(\text{Pic})$ yields the diagram

$$
\begin{array}{ccccccc}
0 & \to & H^{n-1}(D(E), k^\times) & \to & G_E(k) & \to & \text{coker}(\text{Pic}^0) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{n-1}(D(E), k^\times) & \to & H^{n-1}_{cdh}(E, \mathbb{G}_m) & \to & \text{coker}(\text{Pic}) & \to & 0
\end{array}
$$

Applying the snake lemma to this diagram, we see that the two vertical maps on the right have the same kernel and cokernel, and that $\text{ker}(\text{NS})$ surjects onto $\text{ker}(\beta)$:

$$
\begin{array}{ccccccc}
\text{ker}(\text{NS}) & \to & \text{ker}(\text{NS}) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{ker}(\beta) & \to & \text{ker}(\beta) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{n-1}(D(E), k^\times) & \to & G_E(k) & \to & \text{coker}(\text{Pic}^0) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{n-1}(D(E), k^\times) & \to & H^{n-1}_{cdh}(E, \mathbb{G}_m) & \to & \text{coker}(\text{Pic}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{coker}(\text{NS}) & \to & \text{coker}(\text{NS})
\end{array}
$$

Furthermore, since $\text{NS}(\Delta_{n-4}^{alt} E)$ is a complex, the map $\text{NS}(\Delta_{n-4}^{alt} E) \to \text{NS}(\Delta_{n-3}^{alt} E)$ factors via $\text{ker}(\text{NS})$. We note that the composite

$$
\begin{array}{ccccccc}
\text{NS}(\Delta_{n-4}^{alt} E) & \to & \text{ker}(\text{NS}) & \to & \text{coker}(\text{Pic}^0) \\
& & \downarrow & & \downarrow & & \\
& & \text{ker}(\text{Pic}) & \to & \text{ker}(\text{NS})
\end{array}
$$

is zero; this follows immediately from the square

$$
\begin{array}{ccccccc}
\text{Pic}(\Delta_{n-4}^{alt} E) & \to & \text{NS}(\Delta_{n-4}^{alt} E) \\
\downarrow & & \downarrow & & \\
\text{ker}(\text{Pic}) & \to & \text{ker}(\text{NS})
\end{array}
$$
For the rest of this section, let \( k \) be algebraically closed and of sufficiently small cardinality so that there is an embedding \( k \to \mathbb{C} \). We will show that the diagrams

\[
\begin{array}{c}
\ker(\text{NS}) \\
0 \longrightarrow H^{n-1}(\mathcal{D}(E), k^\times) \longrightarrow G_E(k) \longrightarrow \text{coker}(\text{Pic}^0) \longrightarrow 0
\end{array}
\]

are isomorphic to the \( k \)-points of 1-motives \( M'_E \) and \( M_E \), respectively, over \( k \).

Since \( \text{Pic}^0(\Delta_{alt}E) \) is representable and \( k \) is algebraically closed, the functor of taking \( k \)-points is exact. In particular, the \( k \)-points of the cokernel of the map \( \text{Pic}^0(\Delta_{alt}E) \to \text{Pic}^0(\Delta_{alt-2}E) \) is the cokernel of the \( \text{Pic}^0 \) groups, that is, \( \text{coker}(\text{Pic}^0) \). In other words, \( \text{coker}(\text{Pic}^0) \) is the \( k \)-points of the corresponding abelian variety \( \text{coker} (\text{Pic}^0) \).

Similarly, the group \( H^{n-3}(\text{NS}(\Delta_{alt}E)) \) is isomorphic to the \( k \)-points of a torus, as in Example 3.7. Therefore, for ease of notation and for suggestiveness, let \( T_E \) be a (split) torus so that \( T_E(k) = H^{n-1}(\mathcal{D}(E), k^\times) \).

We may compose the map \( G_E(k) \to H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m) \) with the edge map \( H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m) \to KH_{-2}(E) \), coming from the descent spectral sequence, to get a map \( G_E(k) \to H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m) \to KH_{-2}(E) \), which we will denote \( \alpha \); we can write \( \text{coker}(\alpha) \) in the following short exact sequence:

\[
\begin{array}{c}
0 \longrightarrow \text{coker}(\text{NS}) \longrightarrow \text{coker}(\alpha) \longrightarrow H^{n-2}(\mathcal{D}(E), \mathbb{Z}) \longrightarrow 0.
\end{array}
\]

Similarly, \( \ker(\alpha) \) can also be given by a short exact sequence:

\[
\begin{array}{c}
0 \longrightarrow \ker(\beta) \longrightarrow \ker(\alpha) \longrightarrow \text{im}(d_{2}^{n-3,0}) \longrightarrow 0,
\end{array}
\]

where \( d_{2}^{n-3,0} \) is the \( E_2 \) differential \( H^{n-3}_{\text{cdh}}(E, \mathbb{Z}) \to H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m) \) in the descent spectral sequence (5). Since the Néron-Severi groups are finitely generated, \( \ker(\text{NS}) \) is finitely generated, so the quotient \( \ker(\beta) \) is finitely generated as well. Furthermore, Lemma 2.6 tells us that the term \( \text{im}(d_{2}^{n-3,0}) \) is also finitely generated, so \( \ker(\alpha) \) is also finitely generated.

When \( n = 3 \), Lemma 2.2 implies that the edge map \( H^{2}_{\text{cdh}}(E, \mathbb{G}_m) \to KH_{-1}(E) \) is an injection (see (11)), so \( \ker(\alpha) = \ker(\beta) \). In particular, the sequence
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(34) \[ \text{ker}(NS) \longrightarrow G_E(k) \longrightarrow KH_{-2}(X) \]

is exact. All in all, for general $n$, both $\text{ker}(\alpha)$ and $\text{coker}(\alpha)$ are finitely generated, so that $KH_{1-n}(X)$ is isomorphic to the $k$ points of some group scheme, up to some finitely generated groups.

The rest of this section will be dedicated to showing the following.

**Proposition 4.1.** The diagrams (30) and (31) are isomorphic to the $k$-points of 1-motives $M'_E, M_E$, respectively, over $k$.

**Proof.** To the semisimplicial scheme $\Delta_{alt}^\bullet E$, we may associate a complex $C^\bullet(\Delta_{alt}^\bullet E)$ of schemes, following [1, Sec. 2]. Make $\text{Sch}/k$ into an additive category by modifying the morphisms to be formal $\mathbb{Z}$-linear combinations of actual $k$-scheme morphisms, and then construct $C^\bullet(\Delta_{alt}^\bullet E)$ in the usual way, by taking the differentials to be alternating sums of face maps.

For ease of notation, we will write $A^\bullet := C^\bullet(\Delta_{alt}^\bullet E)$. We now check that our construction agrees with [1]. As it would be redundant to set up our own notation, we will merely follow theirs. We apply their construction to $X^\bullet = A^\bullet$. In addition, $\Delta_{alt}^\bullet E$ is already projective, so there is no need to take a compactification. So we have

\[
\begin{align*}
^0W^0(A^\bullet) &= A^\bullet \\
^0W^1(A^\bullet) &= E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_2 \longrightarrow E_1 \\
^0W^{n-1}(A^\bullet) &= E_n \longrightarrow E_{n-1} \\
^0W^n(A^\bullet) &= E_n \\
^0W^{n+1}(A^\bullet) &= \emptyset
\end{align*}
\]

and

\[
\begin{align*}
^0W'^{-1}(A^\bullet) &= A^\bullet \\
^0W'^m(A^\bullet) &= A^\bullet \\
^0W''^1(A^\bullet) &= \emptyset
\end{align*}
\]

so that $W$, the convolution of $W'$ and $W''$, is given by
\[ oW''-1(A_\bullet) = \Delta_{\bullet}^{alt} E \]
\[ oW''0(A_\bullet) = \Delta_{\bullet}^{alt} E \]
\[ oW''1(A_\bullet) = E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_2 \rightarrow E_1 \]
\[ \vdots \]
\[ oW''n-1(A_\bullet) = E_n \rightarrow E_{n-1} \]
\[ oW''n(A_\bullet) = E_n \]
\[ oW''n+1(A_\bullet) = \emptyset \]

(37)

where the chain maps are the alternating sum of the face maps. In addition, in any explicitly written-out complexes, the leftmost term has homological degree zero. Then the spectral sequence \([1, 3.1.3]\) with \(r = 0\) is

\[ E_1^{p,q} = H^q(\Delta_{\bullet}^{alt} E, G_m) \Rightarrow H^{p+q}(\mathcal{K}') \]

(38)

which is the spectral sequence (14). Next, we claim that the 1-motives \(M_{n-1}'(A_\bullet) = [\Gamma_{n-1}'(A_\bullet) \rightarrow G_{n-1}(A_\bullet)]\) and \(M_{n-1}''(A_\bullet) = [\Gamma_{n-1}'(A_\bullet) \rightarrow G_{n-1}(A_\bullet)]\) are the 1-motives \(M'_{E}\) and \(M_{E}\) referred to earlier, where

\[ \Gamma_{n-1}'(A_\bullet) = \ker (\partial : P_{n-3}(A_\bullet)/P_{n-3}(A_\bullet)^0 \rightarrow P_{\geq n-2}(A_\bullet)/P_{\geq n-2}(A_\bullet)^0) \]
\[ \Gamma_{n-1}(A_\bullet) = \text{coker } (\text{NS}(\Delta_{\bullet}^{alt} E \rightarrow \Gamma_{n-1}'(A_\bullet)) \]

and

\[ G_{n-1}(A_\bullet) = \text{coker } (\partial : P_{n-3}(A_\bullet) \rightarrow P_{\geq n-2}(A_\bullet)), \]

(39)

and \(\mathcal{K}'\) is as in spectral sequence (38). To check this, we first compute the lattices \(\Gamma_{n-1}'(A_\bullet)\) and \(\Gamma_{n-1}(A_\bullet)\). We can see that \(P_{n-3}(A_\bullet) = \text{Pic}(\Delta_{\bullet}^{alt} A)\); we still need to determine \(P_{\geq n-2}(A_\bullet)\). For the latter, the above spectral sequence (38) gives a short exact sequence

\[ 0 \rightarrow H^{n-1}(\mathcal{D}(E), k^\infty) \rightarrow P_{\geq n-2}(A_\bullet) \rightarrow \text{Pic}(\Delta_{\bullet}^{alt} E) \rightarrow 0. \]

(42)

Consider the pullback of the diagram along the inclusion \(\text{Pic}^{0}(\Delta_{\bullet}^{alt} E) \rightarrow \text{Pic}(\Delta_{\bullet}^{alt} E)\). The pullback of this square is \(P_{\geq n-2}(A_\bullet)^0\) \([1, \text{Lemma 3.3}]\). We indicate this in the diagram below.
Applying the snake lemma to the above diagram, we obtain

\[ P_{\geq n-2}(A_\bullet)/P_{\geq n-2}(A_\bullet)^0 \cong \text{NS}(\Delta_{n-2}^\text{alt} E), \]

so that \( \Gamma_{n-1}(A_\bullet) = \ker(\text{NS}(\Delta_{n-3}^\text{alt} E) \to \text{NS}(\Delta_{n-2}^\text{alt} E)) \), which agrees with our lattice \( L_E = \ker(\text{NS}) \). The lattice \( \Gamma_{n-1}(A_\bullet) \) is just the cokernel

\[ \Gamma_{n-1}(A_\bullet) = \text{coker}(\text{NS}(\Delta_{n-4}^\text{alt} E) \to \Gamma_{n-1}'(A_\bullet)) = H^{n-3}(\text{NS}(A_\bullet)) \]

which agrees with our other lattice term in (31). It remains to check that the semiabelian variety \( G_{n-1}(A_\bullet) \) agrees with our \( G_E \). Using the short exact sequence above that calculates \( P_{\geq n-2}(A_\bullet) \), we get

\[ \text{Pic}^0(\Delta_{n-3}^\text{alt} E) \]

\[ \downarrow g \]

\[ 0 \to T_E \to P_{\geq n-2}(A_\bullet)^0 \to \text{Pic}^0(\Delta_{n-2}^\text{alt} E) \to 0 \]

where \( G_{n-1}(A_\bullet) \) is the cokernel of the vertical map \( g \). We take the pullback of the first horizontal map \( T_E \to P_{\geq n-2}(A_\bullet)^0 \) along \( g \).

\[ \text{coker } f \]

\[ \to G_{n-1}(A_\bullet) \]

\[ \to \text{coker } h \]

\[ \to 0 \]

Since \( T_E \) is a closed subgroup of \( P_{\geq n-2}(A_\bullet)^0 \), we see that \( W \) is a closed subgroup of \( \text{Pic}^0(\Delta_{n-3}^\text{alt} E) \). Furthermore, because the square is Cartesian, the induced map on cokernels is injective. We add these observations to the diagram (46). To finish, we need the following lemma:
Lemma 4.2. The $k$-points of the bottom row of (47) isomorphic to the short exact sequence in the top row of the diagram (26).

Proof. Let $W' = \text{im } f$ denote the image of $W$ in $T_E$. Since $\text{Pic}^0(\Delta_{n-3}^\text{alt}E)$ is proper over $k$, so $g$ is also proper. We have already observed that $W$ is proper over $k$ as well. Furthermore, the map $W \to T_E$ is also proper, so $W'$ is a closed subvariety of $T_E$ that is proper over $k$ [8, II, Exercise 4.4]. On the other hand, $T_E$ is affine, and $W'$, being closed in $T_E$, is also affine. But then $W'$ is finite over $k$, as it is affine and proper over $k$[8, II, Exercise 4.6].

In addition, since $W'$ is a finite subgroup of $T_E$, we claim that $\text{coker } f$ is isomorphic to $T_E$. $T_E$ is a group of multiplicative type, and since all finite subgroups of a group of multiplicative type are also of multiplicative type, $W'$ is of multiplicative type [18, 2.2]. There is an anti-equivalence between group schemes of multiplicative type over $k$ and finite abelian groups [9, Proposition 20.17]. Here, the map $W' \to T_E$ corresponds to a surjective map of a lattice onto a finite abelian group. The kernel of this map must also be finitely generated free abelian of the same rank, so that $\text{coker } f$ must be isomorphic to a copy of $T_E$.

Finally, since the top right horizontal map is surjective, $\text{coker } h$ is the same as the cokernel of the map $\text{Pic}^0(\Delta_{n-3}^\text{alt}E) \to \text{Pic}^0(\Delta_{n-2}^\text{alt}E)$, as in our case. $\square$

Applying the snake lemma and making the identification $\text{coker } f \cong T_E$ yields a short exact sequence of commutative group schemes

$$
0 \to T_E \to G_{n-1}(A_\bullet) \to \text{coker}(\text{Pic}^0) \to 0,
$$

which agrees with our construction. $\square$

Now that we have established that $M_E$ is a 1-motive, we are interested in how to calculate it. In the landmark paper [6], Deligne established an equivalence between torsion-free 1-motives and torsion-free mixed Hodge structures of a given type; we state the version given in [1, 1.5].

Theorem 4.3. Let $\mathcal{M}_1(\mathbb{C})$ denote the category of 1-motives over $\mathbb{C}$, and let $\text{MHS}_1$ denote the category of mixed Hodge structures of type $\{(0,0),(0,1),(1,0),(1,1)\}$, such that $\text{Gr}_1^W H$ is polarizable. Then we have an equivalence of categories

$$
(49) \quad r_H : \mathcal{M}_1 \to \text{MHS}_1.
$$

The main theorem [1, Theorem 0.1] asserts that the free part 1-motive $(M_E)_f$, after base extending to $\mathbb{C}$, corresponds to a unique mixed Hodge structure $H_E$ in $W_2 H^{n-1}(E(\mathbb{C}), \mathbb{Z})$. (More specifically, $H_E$ is the unique largest torsion-free mixed Hodge structure of type $\{(0,0),(0,1),(1,0),(1,1)\}$ in $W_2 H^{n-1}(E(\mathbb{C}), \mathbb{Z})$ such that $\text{Gr}_1^W H_E$ is polarizable.) This gives us a concrete way of computing the free part of the 1-motive $M_E$ arising from the computation of $H^{n-1}_\text{cdh}(E, \mathbb{G}_m).$
5. Independence of the choice of resolution

Now that we have constructed a 1-motive \( M_E = [L_E \rightarrow G_E] \), we wish to determine to what extent it is independent of the choice of resolution. Under Deligne’s equivalence of 1-motives and mixed Hodge structures, we get another 1-motive, which we denote \( M = [L \rightarrow G] \), that comes from a unique mixed Hodge structure \( H \) in \( W_2 H^n(X(\mathbb{C}), \mathbb{Z}) \), of the considered type. We will not only establish to what extent \( M_E \) is independent of the choice of resolution, but also we will establish a relation between \( M_E \) and \( M \). The precise statement is given below.

**Proposition 5.1.** For each resolution \( p: \widetilde{X} \rightarrow X \), there exists a morphism \( M_E \rightarrow M \) which is an isomorphism unless \( n = 3 \), in which case we have an isomorphism on the non-lattice parts and a surjection on the lattices.

**Proof.** Taking the long exact sequence in singular cohomology (of the \( \mathbb{C} \)-points) induced by the blowup square resolving the singularities of \( X \) via \( p \), we obtain

\[
\cdots \rightarrow H^{r-1}(\widetilde{X}, \mathbb{Z}) \oplus H^{r-1}(\mathbb{Z}, \mathbb{Z}) \rightarrow H^{r-1}(E, \mathbb{Z}) \rightarrow H^r(X, \mathbb{Z}) \rightarrow \cdots
\]

From this long exact sequence, we get a map \( H_E \rightarrow H \) of mixed Hodge structures, since the weights are functorial with respect to morphisms. Since the groups \( H^i(Z, \mathbb{Z}) \) vanish for \( i > n - 2 \) and the groups \( H^i(\widetilde{X}, \mathbb{Z}) \) are pure of weight \( i \), and \( n \geq 3 \), taking the weight \( 2 \) part of the sequence yields an isomorphism \( W_2 H^{n-1}(E, \mathbb{Z}) \cong W_2 H^n(X, \mathbb{Z}) \) unless \( n = 3 \), in which case we only have a surjection. Similarly, taking the weight \( 1 \) part of the above sequence yields an isomorphism \( W_1 H^{n-1}(E, \mathbb{Z}) \cong W_1 H^n(X, \mathbb{Z}) \). The weight \( 2 \) part contains the lattice, and the weight \( 1 \) part contains the rest of the 1-motive, proving the claim. \( \square \)

**Remark 5.2.** Because the map \( L_E \rightarrow G \) factors through \( L \), we see from the composite

\[
L_E(k) \longrightarrow L(k) \longrightarrow G(k) \longrightarrow KH_{1-n}(E)
\]

that the images of \( L_E \) and \( L \) in \( G \) are the same. So when \( n = 3 \), the sequence

\[
L(k) \longrightarrow G(k) \longrightarrow KH_{-2}(E)
\]

is still exact.

Another way to see that the torus \( H^{n-1}(D(E), k^\times) \) is independent of the resolution is to see that the homotopy type of \( D(E) \) is independent of the choice of resolution [13]. So all of the cohomology groups \( H^i(D(E), \mathbb{Z}) \) (in particular, \( i = n - 3, n - 2 \) coming out of the exact sequence (9)) are independent of the choice of resolution, and thus \( H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m) \) is independent of the choice of resolution as well. More directly, we can apply \( cdh \)-descent to the cohomology groups themselves; we get a long exact sequence

\[
\cdots \rightarrow H^{n-1}_{\text{cdh}}(\mathbb{Z}, \mathbb{G}_m) \oplus H^{n-1}_{\text{cdh}}(\widetilde{X}, \mathbb{G}_m) \rightarrow H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m) \rightarrow H^{n-1}_{\text{cdh}}(X, \mathbb{G}_m) \rightarrow \cdots
\]
Since $Z$ and $\tilde{X}$ are smooth, their cdh-cohomology groups agree with their Zariski cohomology groups:

**Theorem 5.3.** Let $Y$ be smooth over $k$, and $\mathcal{F}$ a homotopy invariant sheaf with transfers on the cdh-site over $X$. Then the change of topology morphism induces an isomorphism $H^p_{\text{cdh}}(X, \mathcal{F}) \cong H^p_{\text{Zar}}(X, \mathcal{F})$.

**Proof.** [17]

This result is quite useful, because on smooth schemes, all of the sheaves $a_{\text{Zar}}K_n$ are homotopy invariant sheaves with transfers [16, Section 3.4]. Furthermore, $H^i_{\text{Zar}}(Y, \mathbb{G}_m) = 0$ whenever $Y$ is smooth over $k$ and $i > 1$, so we obtain an isomorphism $H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m) \cong H^n_{\text{cdh}}(X, \mathbb{G}_m)$.

Now that we know that $H^{n-1}_{\text{cdh}}(E, \mathbb{G}_m)$ is independent of the choice of resolution, the exact sequence (24) shows that the group $\text{coker}$(Pic) is also independent of the choice of resolution. Furthermore, since $\text{coker}$(Pic$^0$) was independent of the choice of resolution, the cokernel of $\text{coker}$(Pic$^0$) $\rightarrow$ $\text{coker}$(Pic), which is $\text{coker}$(NS), is also independent of the choice of resolution, as is the kernel of that map. In summary, all of the various groups appearing in the diagram (25) are independent of the choice of resolution except possibly the group $\text{ker}$(NS), and only in the case $n = 3$. We give an example to show that indeed this is the case, that $\text{ker}$(NS) is not independent of the choice of resolution when $n = 3$.

**Example 5.4.** Let $X$ be an integral 3-fold $X$ with a smooth singular locus $Z$ of dimension $\leq 1$. Suppose we have an excellent resolution $p: \tilde{X} \rightarrow X$ with exceptional divisor $E$ that has at least two irreducible components $E_1, E_2$ that have a nonempty intersection $E_{12}$ (which, by assumption, must be a smooth curve). Let the other irreducible components of $E$ be $E_3, \ldots, E_m$. Take a closed point $x$ that lies in $E_{12}$ but does not lie in any of the other $E_i$. Blowing up along $x$, we obtain a diagram

\[
\begin{array}{ccc}
\text{Bl}_x E & \rightarrow & \text{Bl}_x \tilde{X} \\
\downarrow & & \downarrow \\
E & \rightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
Z & \rightarrow & X
\end{array}
\]

so that $\text{Bl}_x \tilde{X} \rightarrow X$ is also an excellent resolution. $\text{Bl}_x E$ then has $m + 1$ irreducible components: the two blown-up components $E'_1 = \text{Bl}_x E_1$ and $E'_2 = \text{Bl}_x E_2$; the “untouched” components $E_3, \ldots, E_m$; and a new component $E'$ that is the exceptional divisor of $\text{Bl}_x \tilde{X}$. The relationships between the intersections of the various components are given below.
ON THE NEGATIVE $K$-THEORY OF SINGULAR VARIETIES

(55)\[
\begin{align*}
E'_1 \cap E'_2 &= \text{Bl}_x E_{12} \cong E_{12} \\
E'_i \cap E &= \text{exceptional divisor of } \text{Bl}_x E_i, \quad i = 1, 2 \\
E_i \cap E' &= \emptyset, \quad i > 2 \\
E_i \cap E'_j &= E_i \cap E_j, \quad i > 2, j = 1, 2
\end{align*}
\]

In general, for a smooth surface $S$ that contains a point $y$, we will have $\text{NS}(\text{Bl}_y S) = \text{NS}(S) \oplus \mathbb{Z}$ [8, V, Theorem 5.8], so that from the following diagram obtained from blowing up along $x$

\[
\begin{array}{ccc}
\text{NS}(\Delta_{alt}^0 E) & \rightarrow & \text{NS}(\Delta_{alt}^0 E) \oplus \text{NS}(E') \oplus \mathbb{Z}^2 \\
\downarrow & & \downarrow \\
\text{NS}(\Delta_{alt}^1 E) & \rightarrow & \text{NS}(\Delta_{alt}^1 E) \oplus \mathbb{Z}^2
\end{array}
\]

we see that $\text{NS}(\Delta_{alt}^0 E)$ has changed by $\text{NS}(E') \oplus \mathbb{Z}^2$ and that $\text{NS}(\Delta_{alt}^1 E)$ has changed by $\mathbb{Z}^2$. Since $E'$ is projective, $\text{NS}(E')$ has rank at least 1, so that $\text{ker}(\text{NS})$ must become strictly bigger, and in particular depends on the choice of resolution of $X$.

This makes sense, because by Proposition 5.1, we have in general only a surjection $H_{n-3}(\text{NS}(\Delta_{alt}^E)) \rightarrow L$ and not an isomorphism.

Remark 5.5. Proposition 5.1 tells us that when $X$ is projective, the 1-motive $M_E$ is independent of the choice of good resolution $p$ unless $n = 3$, in which case the non-lattice parts of the 1-motive are independent of the choice of good resolution $p$. Therefore, to calculate $KH_{1-n}(X)$ when $X$ is not projective, we need only take an algebraic compactification $\tilde{X}$ of $X$, smooth along the boundary, and then compute $KH_{1-n}(\tilde{X})$, as $KH_{1-n}(\tilde{X}) \cong KH_{2-n}(E) \cong KH_{1-n}(X)$. This shows that $KH_{1-n}(X)$ is independent of the choice of algebraic compactification $\tilde{X}$. This result makes sense in light of the observation that negative $KH$ vanishes for smooth schemes, and we compactify away from the singular locus. In some sense, we are computing, $KH_{1-n}$ of the singularity $x_0$ locally sitting inside $X$.

We wrap things up by putting together everything we have proven so far.

Theorem 5.6 (Main Theorem for $KH_{1-n}(X)$). Let $X$ be an normal, integral $n$-fold over an algebraically closed field $k$ of characteristic zero, with singular locus $Z = \text{Sing}(X)$ such that $Z$ is smooth or codim $Z > 2$. Then there exists a 1-motive

\[
M = \begin{bmatrix}
    & L \\
0 & T & G & A & 0
\end{bmatrix}
\]

and a map $\alpha: G(k) \rightarrow KH_{1-n}(X)$, natural in $X$, whose kernel and cokernel are finitely generated. If $p: \tilde{X} \rightarrow X$ is any good resolution of singularities, then $\text{ker}(\alpha)$ and $\text{coker}(\alpha)$ have the more explicit descriptions (33) and (32), respectively.
In particular, the descriptions of \( \ker(\alpha) \) and \( \ker(\beta) \) are independent of the choice of resolution of \( X \).

Furthermore, if \( X \longrightarrow \overline{X} \) is an algebraic compactification of \( X \), then after base extending to \( \mathbb{C} \), the (torsion-free) 1-motive \( (M_{\mathbb{C}})_{fr} \) corresponds, under the equivalence (49), to the unique largest torsion-free mixed Hodge structure \( H \) of type \( \{(0,0), (0,1), (1,0), (1,1)\} \) in \( W_2H^n(\overline{X}(\mathbb{C}), \mathbb{Z}) \) such that \( \text{Gr}_{W}^1H \) is polarizable. Moreover, the non-lattice parts of \( M \), and hence the map \( \alpha \), are independent of the choice of algebraic compactification \( X \longrightarrow \overline{X} \).

Finally, when \( n = 3 \), then we have the additional property that the sequence (52) is exact.

6. Calculation of \( NK_{1-n}(X) \)

We now turn our attention towards \( NK_{1-n}(X) \), the other remaining contribution to \( K_{1-n}(X) \). For this section, let \( k \) be a field of characteristic zero (not necessarily algebraically closed) and \( X \) be a (not necessarily irreducible) \( n \)-dimensional variety over \( k \) with isolated singularity \( x_0 \). We first establish the exact sequence (1) referred to in the introduction.

**Lemma 6.1.** There is an exact sequence

\[
NK_{1-n}(X) \xrightarrow{d_{1}^{1-n}} K_{1-n}(X) \longrightarrow KH_{1-n}(X) \longrightarrow 0.
\]

**Proof.** There is a strongly convergent, homological right half-plane spectral sequence [20]

\[
E_1^{p,q} = H^p_{\text{Zar}}(X, a_{\text{Zar}}^nK_q) \implies KH_{p+q}(X)
\]

The K-dimension theorem [3, Conjecture 0.1] implies that the groups \( N^pK_{-q}(X) \) are zero whenever \( q \geq n \) and \( p \geq 1 \).

So we can see that there are no nonzero differentials coming into or going out of each \( E_m^{0,1-n} \) after the first page, so that \( E_\infty^{0,1-n} = E_2^{0,1-n} \). In addition, all of the groups \( E_\infty^{p,1-n-p} \) are zero, except when \( p = 0 \). This gives us the exact sequence we are looking for. \( \square \)

We now reduce to the case when \( X \) is affine.

**Lemma 6.2.** \( N^tK_{-q}(X) \cong N^tK_{-q}(U) \) for any \( q \in \mathbb{Z} \), any \( t \geq 1 \), and any open \( U \subset X \) containing the isolated singularity \( x_0 \).

**Proof.** We have a spectral sequence [15, Theorem 10.3]

\[
E_2^{p,q} = H^p_{\text{Zar}}(X, a_{\text{Zar}}N^tK_q) \implies N^tK_{-p-q}(X)
\]

We apply the spectral sequence (60) to \( X \). Because smooth schemes are \( K_{-q} \)-regular, it follows that for any smooth open subscheme \( U \subset X \), we have \( N^tK_{-q}(U) = 0 \) whenever \( t \geq 1 \), as we have indicated above. Since \( X \) has only a singularity at \( x_0 \), we have \( N^tK_{-q}(U) = 0 \) whenever \( x_0 \notin U \). It follows that the Zariski sheaf \( aN^tK_{-q} \) is a skyscraper sheaf supported at \( x_0 \). In particular, \( aN^tK_{-q} \) is flasque,
In particular, we may choose $U = \text{Spec } R$ to be an open affine neighborhood of $x_0$. The intuition here is that since the $N^qK_{-q}$-groups are zero on smooth schemes, they only detect singularities, and their value depends only on the type of singularity involved.

Recall that we are interested in the case $q = n - 1$. Cortiñas, et al. [4, Example 3.5, Proposition 4.1] elucidates the structure of the $N^pK_q$ groups, which, specializing to $p = 1$ and $q = n - 1$, gives

$$NK_{1-n}(X) \cong NK_{1-n}(U) \cong H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \otimes \mathbb{Q} t\mathbb{Q}[t].$$

The maps in the spectral sequence (59) are induced by the maps on the simplicial structure of $X \times \mathbb{A}^\bullet$; in particular,

$$NK_{-q}(X) = \ker(\partial_0 : K_{-q}(X \times \mathbb{A}^1) \xrightarrow{t=0} K_{-q}(X)),$$

where $t$ is the parameter of $\mathbb{A}^1$ — the same $t$ as in (61). The decomposition (61), found in [4], boils down to applying the Künneth formula for Hochschild homology [21, Proposition 9.4.1]

$$HH_n(R[t]) \cong \oplus_{i+j=n} HH_i(R) \otimes \mathbb{Q} HH_j(\mathbb{Q}[t]),$$

from which we see that the $t$ in the $\mathbb{Q}[t]$ is indeed the parameter $t$ in the copy of $\mathbb{A}^1$ when computing the $N$-functors.

The differential $\partial_0 - \partial_1 : K_{1-n}(X \times \mathbb{A}^1) \rightarrow K_{1-n}(X)$ reduces to just $-\partial_1$ on $NK_{1-n}(U) = \ker(\partial_0)$, and $\partial_1$ just sets $t = 1$. Therefore, the image of $NK_{1-n}(X)$ in $K_{1-n}(X)$ is isomorphic to $H^{n-1}_{\text{cdh}}(U, \mathcal{O})$. In summary, we have proven that

**Proposition 6.3.** There is a short exact sequence

$$0 \longrightarrow H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \longrightarrow K_{1-n}(X) \longrightarrow KH_{1-n}(X) \longrightarrow 0.$$  

**Remark 6.4.** The observation here that the maps in the spectral sequence come from the simplicial structure on $X \times \mathbb{A}^\bullet$ can be taken further. For example, we can say something about $K_{2-n}(X)$. Proceeding as in the computation of $NK_{1-n}(X)$, we have, via [4, Corollary 4.2],

$$N^2K_{1-n}(U) \cong NK_{1-n}(U) \otimes Q s_1 Q[s_1]$$

$$\cong H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \otimes Q s_0 Q[s_0] \otimes Q s_1 Q[s_1].$$

The top face map from $K_{1-n}(X \times \mathbb{A}^2) \rightarrow K_{1-n}(X \times \mathbb{A}^1)$ sends $1 - s_0 - s_1$ to zero, so it sends $s_0$ to $t$ and $s_1$ to $1 - t$. Therefore, the image of $d^{2,1-n}_1$ in $NK_{1-n}(X)$ is just $H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \otimes Q t(1-t)\mathbb{Q}[t]$, which is precisely the kernel of the
map $\partial_1 = (t \mapsto 1)$. The $E_1$ page of the spectral sequence is therefore exact at $(1, 1 - n)$, and so $E_2^{1,1-n} = 0$. We may make the same argument for the map $d_1^{3,1-n} : N^3K_{1-n}(X) \longrightarrow N^2K_{1-n}(X)$. Let us write

$$N^3K_{1-n}(X) \cong NK_{1-n}(X) \otimes r_1\mathbb{Q}[r_1] \otimes r_2\mathbb{Q}[r_2]$$

(66)

$$N^2K_{1-n}(X) \cong H^{n-1}_{\text{cdh}}(U,\mathcal{O}) \otimes r_0\mathbb{Q}[r_0] \otimes r_1\mathbb{Q}[r_1] \otimes r_2\mathbb{Q}[r_2]$$

$$NK_{2-n}(X) \longrightarrow K_{2-n}(X) \longrightarrow KH_{2-n}(X) \longrightarrow 0.$$ 

In particular, the map $K_{2-n}(X) \longrightarrow KH_{2-n}(X)$ is surjective.

As we have already noted, the group $H^{n-1}_{\text{cdh}}(U,\mathcal{O})$ in (64) is independent of the choice of open affine neighborhood $U$ of the singularity $x_0$. The following lemma makes this statement precise.

**Lemma 6.5.** Let $V \subset U$ be an open affine neighborhood of $x_0$. Then the inclusion $V \hookrightarrow U$ induces an isomorphism $H^{n-1}_{\text{cdh}}(U,\mathcal{O}) \cong H^{n-1}_{\text{cdh}}(V,\mathcal{O})$.

**Proof.** Take a Nisnevich cover $\{V \rightarrow U, V' \rightarrow U\}$, and then cover $V'$ by open affines $V'_1$. Since $V'$ is smooth, so are all of the $V'_1$, and in particular, they have no higher cdh-cohomology groups (Theorem 5.3). A standard Čech spectral sequence argument then shows that the induced map is an isomorphism.

Alternatively, this isomorphism can be obtained directly from Proposition 6.3, by seeing that the kernel $H^{n-1}_{\text{cdh}}(U,\mathcal{O})$ of the map $K_{1-n}(X) \longrightarrow KH_{1-n}(X)$ is independent of the choice of open affine neighborhood $U$ containing the isolated singularity $x_0$.

The discussion using the decomposition (61) yielding the short exact sequence (64) is a reasonable description of $K_{1-n}(X)$, but cdh-cohomology groups are often difficult to compute. It turns out that we can be more explicit in our description of $K_{1-n}(X)$ in the exact sequence (60) by identifying the term $H^{n-1}_{\text{cdh}}(U,\mathcal{O})$ in terms of known invariants of the singularity $x_0$.

**Definition 6.6.** Let $R$ be a finite type $k$-algebra such that $U = \text{Spec} R$ has only isolated singularities. The generalized Du Bois invariants $b^{p,q}$ for $p \geq 0, q \geq 1$ are

$$b^{p,q} = \text{length } H^q_{\text{cdh}}(U,\Omega^p).$$

(69)
These invariants are finite by [5]. Du Bois invariants were introduced by Steenbrink [11]. By [5, Lemma 2.1, Equation 2.7], we see that $H_{\text{cdh}}^{n-1}(U, \mathcal{O})$ is a $k$-vector space of dimension $b^0 n^{-1}$. In particular, its dimension is finite.

Finally, in the case of $n = 3$, we have a full computation of $K_{-2}(X)$.

**Corollary 6.7.** Let $X$ be an integral threefold with an isolated singularity $x_0$. Then for any open affine $U$ containing $x_0$, $K_{-2}(X)$ is an extension of $KH_{-2}(X)$ by $H^2_{\text{cdh}}(U, \mathcal{O})$, where $KH_{-2}(X)$ has the description given by Theorem 5.6, and $H^2_{\text{cdh}}(U, \mathcal{O})$ is a $k$-vector space of finite dimension $b^0 2$.

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