Minimal cover of high-dimensional chaotic attractors by embedded coherent structures

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We propose a general method for constructing a minimal cover of high-dimensional chaotic attractors by embedded coherent structures, such as (but not limited to) periodic orbits. By minimal cover we mean a subset of available coherent structures such that the approximation of chaotic dynamics by a minimal cover with a predefined proximity threshold is as good as the approximation by the full available set. The proximity measure can be chosen with considerable freedom and adapted to the properties of any given chaotic system. In the context of a Kuramoto-Sivashinsky chaotic attractor, we demonstrate that the minimal cover can be faithfully constructed even when the proximity measure is defined within a subspace of dimension much smaller than the dimension of space containing the attractor. We discuss how the minimal cover can be used to provide a reduced description of the attractor structure and the dynamics on it.

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Unstable periodic orbits form the skeleton of chaotic attractors, with shorter orbits giving the overall structure, and longer orbits refining this skeleton in smaller neighbourhoods. This property was first brought to light when Poincaré conjectured that any motion of a dynamical system can be approximated by means of those of periodic type[1]. This conjecture opened up a previously inaccessible avenue for breaching the frontier of chaotic dynamics using a periodic orbit-centric approach. In low-dimensional systems, this approach has seen considerable success, culminating in the development of the periodic orbit theory (aka cycle expansions [2]). The approach can also work well for low-dimensional attractors of high-dimensional systems [3, 4]; however when the dimensionality of the attractor is large (several positive Lyapunov exponents), even a qualitative description of the structure of the attractor becomes challenging.

In recent years a considerable amount of progress has been made in locating periodic orbits and other coherent structures in high dimensional chaotic systems. (We use the term ‘coherent structures’, borrowed from the turbulence research community [5], as a collective name for all types of special solutions embedded in the chaotic dynamics of high-dimensional systems, including ‘exact coherent structures’ [6] such as equilibria, travelling waves, periodic and relative periodic orbits, as well as other identifiable solutions (e.g., vortices) that can be used to describe the structure of chaotic dynamics.) Examples include the work of López et al.[7], who presented a method for finding relative periodic solutions for differential equations with continuous symmetries, and used this method to find relative periodic solutions of the Complex Ginzburg-Landau equation. Hof et al.[8] found experimental evidence of the existence of travelling waves in turbulent pipe flow, in agreement with the numerical studies of Faisst et al.[9] and Wedin et al.[10]. Zoldi and Greenside[11] used a damped-Newton method to find unstable periodic orbits in the Kuramoto-Sivashinsky (KS) equation. More recently, Cvitanović et al.[12] used multiple shooting and the Levenberg-Marquardt algorithm to locate over 60 000 coherent structures in the KS equation on a periodic domain. Due to the presence of discrete and continuous symmetries, these structures include not only the traditional equilibria and periodic orbits, but also travelling waves and relative periodic orbits[7, 12] (also called ‘modulated travelling waves’ in [13]).

Now that large numbers of such coherent structures can be located in high-dimensional chaotic systems, the question is: how can we use this information to describe the structure of the attractor and dynamics on it? It is clear that there is a lot of redundancy in the data: coherent structures which are ‘close’ to one another essentially give the same information about the structure and dynamics of the attractor in their neighbourhood.

In this work we adopt the following pragmatic approach: From the set of all available coherent structures, we select a subset which maximally covers the chaotic attractor with minimal redundancy. We call such a subset a minimal cover of the attractor. The elements of this subset can then be used to describe the overall structure of the attractor. In order to perform the selection, we need to define the notion of ‘closeness’ (or proximity) of coherent structures to one another and to points on the attractor. Generally, the specific choice of metric is not important, since the metric of the embedding space changes under topologically conjugate transformations. The key feature that distinguishes ‘close’ points from ‘distant’ points in phase space is that orbits originating from ‘close’ points will remain close for some time (albeit exponentially diverging in the unstable manifold). This is true even if we view the dynamics in projection on a low-dimensional subspace. Therefore, it is preferable to measure distance not between individual points in space, but between segments of orbits. In this context, as we demonstrate below, the notion of the Hausdorff distance [13] between sets (in this case between segments of
trajectories) is useful.

**Minimal cover algorithm.** Let \( P = \{p_i\}_{i=1}^{N_P} \) be a set of coherent structures listed in order of increasing complexity (e.g. equilibria followed by periodic orbits of increasing period). Starting with \( W = \emptyset \), iteratively proceed through the structures in order of complexity, adding into the cover those structures whose distance to \( W \) is above some threshold, \( \varepsilon \).

For \( i = 1, 2, \ldots, N_P \):

\[
W \leftarrow \begin{cases} 
W \cup \{p_i\} & \text{if } d(p_i, W) > \varepsilon \\
W & \text{if } d(p_i, W) \leq \varepsilon 
\end{cases} 
\tag{1}
\]

where \( d(p_i, W) \) denotes the distance between \( p_i \) and \( W \), which will be defined later. Intuitively speaking, we only include a new coherent structure into the set \( W \) if it explores a part of the attractor that is not yet covered by the current \( W \).

**Directed Hausdorff Distance.** During this process we are primarily concerned with calculating the distance from a coherent structure to the cover \( W \). For this, the notion of the directed Hausdorff distance\(^{[15]}\) between sets is useful:

\[
d(p_i, W) = \max_{t \in [0,T^{p_i}]} \min_{w_j \in W} \| p_i(t) - w_j(t) \| \tag{2}
\]

where \( T^{p_i} \) and \( T^{w_j} \) are the periods of coherent structures \( p_i \) and \( w_j \) respectively; \( p_i \) is a given coherent structure in the set \( P \); \( w_j \) is a coherent structure in the cover \( W \); and \( \| \cdot \| \) is the chosen norm in the dynamical system’s phase space.

Intuitively, the directed Hausdorff distance \( d(p_i, W) \) takes the point on the coherent structure \( p_i \) that is farthest from the set \( W \), and gives the distance between this point and its closest neighbour in \( W \)\(^{[10]}\). As such, upon the completion of the above process of constructing \( W \), all coherent structures contained in \( P \) will necessarily be within \( \varepsilon \) distance of \( W \).

**Distance in lower-dimensional projections.** In order to distinguish close and distant orbit segments using the directed Hausdorff distance it is not necessary to work in the full phase space, which could be very high dimensional (as in the case of the discretised Navier-Stokes flow). A reliable distinction can be made in a low-dimensional projection of the space. For example, one could take the first \( n \) principal components of the attractor, and/or work in a symmetry-reduced space.

For continuous-time systems and \( n \geq 2 \), \( n \)-dimensional projections of non-steady orbits are curves, and therefore they are infinite-dimensional objects. Generically, they do not coincide for two different orbits (for coincidence, a special “non-generic” choice of projection is needed, e.g. with respect to some symmetries). For discrete-time systems, we have to consider sufficiently long orbits to guarantee this property. If projections of two orbits never coincide then the Hausdorff distance between projections is, at the same time, a distance in the space of orbits. This equivalence of spaces of orbits and their projections is the essence of the famous Takens embedding theorem\(^{[17]}\). The relationship between the dimension of the approximating attractor and the necessary length of the time series was considered by Eckmann and Ruelle\(^{[18]}\). Also, the Johnson-Lindenstrauss lemma\(^{[19]}\), which is widely used in computer science and signal processing, gives the boundaries for the preservation of the distances in random projections (up to a scalar scaling factor)\(^{[20]}\).

Note that the values of distances differ in different projections, therefore, depending on which projection is used, both the number of structures needed to cover the attractor, and the specific structures chosen (for a given \( \varepsilon \)) will differ relative to the full-dimensional counterpart. However, as we will demonstrate in the rest of this Letter, a reliable reduced representation of the chaotic attractor by a minimal cover can be achieved with a relatively small number of dimensions.

**The Kuramoto-Sivashinsky Equation.** To demonstrate the proposed method, we will use the set of coherent structures found by Cvitanović, Davidchack, and Siminos\(^{[12]}\) for the Kuramoto-Sivashinsky (KS) equation

\[
u_t = -u_{xxxx} - u_{xx} - \frac{1}{2}(u^2)_{xx}, \quad x \in \left[ -L/2, L/2 \right], \quad \tag{3}
\]

with periodic boundary conditions \( u(x + L, t) = u(x, t) \) and system size \( L = 22 \) - a system with numerically stable chaotic dynamics. Since this equation is equivariant under translations and reflections\(^{[12]}\), all solutions related by these symmetry transformations are equivalent. These symmetries also extend the set of possible coherent structures to include, in addition to the usual equilibria and periodic orbits, travelling waves and relative periodic orbits\(^{[12]}\).

 Due to the periodic boundary conditions, it is convenient to work with the Fourier representation of the solutions of\(^{[3]}:\)

\[
u(x,t) = \sum_{k=-\infty}^{+\infty} a_k(t) e^{2\pi i k x}
\tag{4}
\]

where \( a_k \in \mathbb{C} \) and \( a_{-k} = a^*_k \). As shown in\(^{[12]}\), a sufficiently accurate solution of\(^{[3]}\) can be obtained by the spectral method using a truncated Fourier series with \( k \leq 15 \), resulting in a 30-dimensional dynamical system (\( a_0(t) = 0 \) for all \( t \) by Galilean invariance\(^{[12]}\)).

Excluding the trivial equilibrium \( u(x, t) = 0 \), the set of detected coherent structures for this system consists of three equilibria, two travelling waves, and over 60000 relative periodic orbits and pre-periodic orbits with periods \( T^p < 200 \)\(^{[12]}\). This will be the set \( P = \{p_i\} \) from which we will select a subset \( W \) of coherent structures using\(^{[1]}\).
A unique representation of the KS dynamics requires the use of symmetry reduced coordinates. Even though construction of a good symmetry-reduced representation for a given system is not straightforward\(^{21}\), the use of the Hausdorff distance allows us to use any convenient invariant coordinates. In the current work, we will use the magnitudes of the complex Fourier modes, \(|a_k|\). Even though in this representation we do not distinguish points in phase space with the same Fourier mode magnitudes and different phases, the fact that we are looking at segments of orbits, rather than orbit points, allows us to faithfully distinguish any segments of orbits starting from different initial points in the full phase space. In fact, as we demonstrate below, we can even take a subset of \(|a_k|\) with selected values of \(k\) and still make such a distinction.

In order to decide which \(k\) values to use, we determine the size of the chaotic attractor along each of the Fourier modes by measuring \(m_k = \max_t |a_k(t)|\) along a long ergodic trajectory in the attractor, \(t \in [0, 10^7]\). The values for the first seven Fourier modes are

\[m_1,...,7 = \{0.572, 0.981, 1.238, 0.579, 0.327, 0.224, 0.076\}\]

with the values of \(m_k\) for \(k > 7\) exponentially decreasing. For the \(n\)-dimensional projection of the phase space we use the \(n\) Fourier modes with the largest values of \(m_k\).

In what follows, in (2) we use the Euclidean norm in the \(n\)-dimensional projection, \(||\cdot||_n\). The directed Hausdorff distance with associated \(n\) will be denoted \(d_n(\cdot, \cdot)\). In the construction of the minimal cover, \(\{P\}\), the threshold parameter \(\varepsilon\) determines the resolution of the minimal cover. In the \(n\)-dimensional projection, we will use the value of \(\varepsilon_n\) equal to 1% of the length of the main diagonal of the hypercuboid with sides \(m_{kj}\), \(j = 1, \ldots, n\), where \(k_j = \{3, 2, 4, 1, 5, 6, 7, \ldots\}\); that is, \(\varepsilon_n = 0.01 \sqrt{\sum_{j=1}^{n} m_{kj}^2}\), resulting in the values \(\varepsilon_2, \ldots, 7 \approx \{0.01579, 0.01682, 0.01776, 0.01807, 0.01820, 0.01822\}\), and \(\varepsilon_{15} \approx 0.01822\) in the full space of the numerical approximation of the KS solutions.

The application of the above outlined construction procedure to the set \(P\) in the \(n\)-dimensional symmetry-reduced projections resulted in the minimal cover sets \(W_n\) containing 91, 513, 747, 809, 825, and 830 coherent structures for \(n = 2, \ldots, 7\), respectively. We can see that the minimal cover sets with 1% resolution contain only a small fraction of the over 60000 available coherent structures from the set \(P\); in other words, all coherent structures in \(P\) can be approximated (or ‘shadowed’) with \(\varepsilon_n\) resolution by those within the small subset, \(W_n\). An example of such shadowing is shown in Fig. 1 where a relative periodic orbit \(p_{137}\) not in \(W_4\) with period 76.64 is shadowed by two relative periodic orbits in \(W_4\) with periods 62.95 and 14.33.

In order to ascertain how well each \(W_n\) covers the attractor in the full 15-dimensional space, we calculated what percentage of coherent structures in the sets \(P \setminus W_n\) can be fully covered by \(W_n\) with distances calculated in the 15-dimensional space. In other words, what fraction of structures \(p_i \in P \setminus W_n\) satisfy: \(d_{15}(p_i, W_n) \leq \varepsilon_{15}\). By doing so, we find that for \(n = 2, \ldots, 7\) the fraction of structures satisfying this condition are: 0.008, 0.785, 0.995, 0.998, 0.999, 1. Upon further investigation, we also found that for \(n = 4\) and above, this fraction increases to 1 if we also take into account structures that have at least 99% of their trajectory covered. From this we can conclude that the smallest dimension of the projection that can be used to construct a reliable minimal cover is \(n = 4\). We will use \(W_4\) for further analysis of shadowing of a chaotic attractor by the minimal cover set of coherent structures.

**Shadowing of the chaotic attractor.** With the help of the constructed minimal cover set \(W_4\), we can begin to put Poincaré’s conjecture into practice by representing a chaotic trajectory as a sequence of segments shadowed by coherent structures in \(W_4\).

Recall that our goal is to construct a cover of the chaotic set by \(W_n\) that is as good as the cover by the full available set \(P\). Note that \(P\) covers with \(\varepsilon_n\) resolution all regions of phase space which are within the \(\varepsilon_n\)-ball of \(P\): \(B_{\varepsilon_n}(P) = \{x \in \mathbb{R}^n : d_n(x, P) < \varepsilon_n\}\). By construction, \(P \subseteq B_{\varepsilon_n}(W_n)\). Therefore, \(B_{2\varepsilon_n}(P) \subseteq B_{2\varepsilon_n}(W_n)\), i.e., the minimal cover \(W_n\) with resolution \(2\varepsilon_n\) provides the same cover as \(P\) with resolution \(\varepsilon_n\).

There are many possible ways to associate a segment of a chaotic trajectory with a particular coherent structure within \(W_4\). For example, we can use the ‘nearest’ representation, where each point on the chaotic trajectory is associated with the nearest structure in \(W_4\). However, because we are working with the lower dimensional projections of the dynamics, such representation may not always find the nearest structure in the full space. Another approach would be to find all coherent structures which are within \(2\varepsilon_n\) of a chaotic trajectory segment and...
In the present work, we adopt a ‘greedy’ representation, where from among all the coherent structures in $W_4$, shown in a two-dimensional projection. Bottom: Distance between trajectory and chosen structure in $W_4$ at time $t$. The chosen structures for each respective segment are $w_{04}, \ w_{237}, \ w_{342}, \ w_{70}$, and $w_{319}$; segment transitions are indicated by black dots and a change in line colour and style.

associate the segment with one of such structures chosen at random. Such a ‘random’ representation would be more appropriate for systems with noise, for example when analysing experimental time series data.

In the present work, we adopt a ‘greedy’ representation, where from among all the coherent structures in $W_4$ that are within $2\varepsilon_4$ of a chaotic trajectory segment, we associate the segment with the structure which stays within $2\varepsilon_4$ for the longest period of time. This approach is consistent with our earlier observation that segments of trajectories, which remain close to one another in a low-dimensional projection for some time, are likely to be close, or ‘dynamically linked’, in the full phase space.

More specifically, given a chaotic trajectory $\mathbf{a}(t)$, $t \geq t_0$ with starting point $\mathbf{a}(t_0)$, find the structure $w_{s_0} \in W_4$ that stays within $2\varepsilon_4$ of the trajectory for the longest period of time, say until $t = t_1$. Then we associate the segment of trajectory $\phi_0 = \{ \mathbf{a}(t), \ t \in [t_0, t_1) \}$ with the coherent structure $w_{s_0}$. Repeating this process from the point $\mathbf{a}(t_1)$, we associate the segment $\phi_1 = \{ \mathbf{a}(t), \ t \in [t_1, t_2) \}$ with the coherent structure $w_{s_1}$. And so on. As a result, the chaotic trajectory is subdivided into finite time segments $\{\phi_0, \phi_1, \ldots\}$ labelled with indices $\{s_0, s_1, \ldots\}$ such that segment $\phi_i$ is shadowed by $w_{s_i} \in W_4$.

In Fig. 2, we show an example of the encoding of a short trajectory $\mathbf{a}(t), t \in [0, 22.21]$ by $W_4$ using a ‘greedy’ shadowing algorithm. In the upper part of the figure, $\mathbf{a}(t)$ is depicted with the different segments marked by different line styles and colours; in the lower part of the figure is the distance between the orbit $\mathbf{a}(t)$ and the assigned structure in $W_4$. We see that, as expected, when $d_s(\mathbf{a}(t), w_{s_i}) = 2\varepsilon_4$ the algorithm finds the next structure in $W_4$ that stays within $2\varepsilon_4$ for the longest period of time.

**Markov model approximation.** Representation of the dynamics on the attractor by a sequence of transitions between cover elements opens up the possibility of approximating the dynamics by a Markov process. In order to do so, we followed a long trajectory on the attractor until $t = 10^{10}$, segmenting it into transitions and waiting times as per the above shadowing algorithm. In order to verify that our system can be appropriately modelled by a Markov chain, we can check to what extent the model exhibits the memoryless property [22], which manifests itself in the exponential distribution of waiting times [23, 24].

It can be seen from Fig. 3 that up until waiting times of approximately 45 the exponential distribution isn’t apparent, however after this we do see an approximately exponential distribution appear. For small waiting times this result is to be expected due to the fact that we’re dealing with a deterministic, not stochastic, system. However, due to sensitive dependence of chaotic dynamics on initial condition, the memory of the initial condition is lost after a sufficiently long time interval. This particular issue is discussed in detail in many different contexts, including research into distinguishing chaotic processes from stochastic processes/noise [25, 26], and also in the context of the applicability of Markov models to chaotic systems [23, 27].

**Conclusion.** In summary, we have presented a general method for constructing a minimal cover of a high-dimensional attractor from the known coherent structures embedded in the attractor. We have given an example of such a construction for a Kuramoto-Sivashinsky chaotic attractor, and explored the effect that dimension
of projection during construction has on the covering of the attractor. We show that a reliable minimal cover can be constructed in a low-dimensional projection of the attractor, thus substantially reducing the complexity and computational cost of such a construction compared to the full phase space representation. We have also demonstrated how any trajectory in the attractor can be encoded as a sequence of cover elements which shadow this trajectory, which opens up the possibility of describing the dynamics on the attractor as a Markov-type model. Such a model allows for deeper analysis into the structure of the dynamics, which will be the subject of future research.

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