From the icosahedron to natural triangulations of $\mathbb{C}P^2$ and $S^2 \times S^2$

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Abstract. We present two constructions in this paper: (a) A 10-vertex triangulation $\mathbb{C}P^2_{10}$ of the complex projective plane $\mathbb{C}P^2$ as a subcomplex of the join of the standard sphere ($S^2$) and the standard real projective plane ($\mathbb{R}P^2$, the decahedron), its automorphism group is $A_4$; (b) a 12-vertex triangulation $(S^2 \times S^2)_{12}$ of $S^2 \times S^2$ with automorphism group $2S_5$, the Schur double cover of the symmetric group $S_5$. It is obtained by generalized bistellar moves from a simplicial subdivision of the standard cell structure of $S^2 \times S^2$. Both constructions have surprising and intimate relationships with the icosahedron. It is well known that $\mathbb{C}P^2$ has $S^2 \times S^2$ as a two-fold branched cover; we construct the triangulation $\mathbb{C}P^2_{10}$ of $\mathbb{C}P^2$ by presenting a simplicial realization of this covering map $S^2 \times S^2 \rightarrow \mathbb{C}P^2$. The domain of this simplicial map is a simplicial subdivision of the standard cell structure of $S^2 \times S^2$, different from the triangulation alluded to in (b). This gives a new proof that Kühnel’s $\mathbb{C}P^2_9$ triangulates $\mathbb{C}P^2$. It is also shown that $\mathbb{C}P^2_{10}$ and $(S^2 \times S^2)_{12}$ induce the standard piecewise linear structure on $\mathbb{C}P^2$ and $S^2 \times S^2$ respectively.

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1 Introduction and Results

It is well known that the minimal triangulation $\mathbb{R}P^2_6$ of the real projective plane arises naturally from the icosahedron. Indeed, it is the quotient of the boundary complex of the icosahedron by the antipodal map. In this note, we report the surprising result that there is a small triangulation (using only 10 vertices) of the complex projective plane which is

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also intimately related to the icosahedron. Indeed, this simplicial complex $\mathbb{CP}^2_{10}$ occurs as a subcomplex of the simplicial join $S^2_4 \ast \mathbb{R}P^2_5$. Our starting point is the beautiful fact that $\mathbb{CP}^2$ is homeomorphic to the symmetrised square $(S^2 \times S^2)/\mathbb{Z}_2$ of the 2-sphere, where $\mathbb{Z}_2$ acts by co-ordinate flip. So, letting $S^2_4$ denote the 4-vertex triangulation of $S^2$ (i.e., the boundary complex of the tetrahedron), we look for a $\mathbb{Z}_2$-stable simplicial subdivision of the product cell complex $S^2_4 \times S^2_4$, without introducing extra vertices. In order to ensure that the quotient complex (after quotienting by $\mathbb{Z}_2$) does triangulate the quotient space $(S^2 \times S^2)/\mathbb{Z}_2 = \mathbb{CP}^2$, the $\mathbb{Z}_2$ action on this simplicial subdivision must be “pure” (cf. Definition 2 and Lemma 7 in Section 5). It turns out that the following $(S^2 \times S^2)_{16}$ is the unique 16-vertex triangulation satisfying these requirements.

**Description of $(S^2 \times S^2)_{16}$:** The vertices are $x_{ij}$, $1 \leq i, j \leq 4$. The full automorphism group is $A_4 \times \mathbb{Z}_2$, where $A_4$ acts on the indices and $\mathbb{Z}_2$ acts by $x_{ij} \leftrightarrow x_{ji}$. Modulo this group the facets (maximal simplices) are the following:

$$x_{11}x_{22}x_{33}x_{12}x_{13}, \ x_{11}x_{22}x_{12}x_{14}x_{34}, \ x_{11}x_{22}x_{14}x_{24}x_{34}, \ x_{11}x_{22}x_{21}x_{24}x_{31}, \ x_{11}x_{22}x_{24}x_{31}x_{34}.$$ The full list of facets of $(S^2 \times S^2)_{16}$ may be obtained from these five basic facets by applying the group $A_4 \times \mathbb{Z}_2$. Under this group, the first three basic facets form orbits of length 24 each, while each of the last two forms an orbit of length 12, yielding a total of $3 \times 24 + 2 \times 12 = 96$ facets. It may be verified that the face vector of $(S^2 \times S^2)_{16}$ is $(16, 84, 216, 240, 96)$.

**Description of $\mathbb{CP}^2_{10}$:** Quotienting the above $(S^2 \times S^2)_{16}$ by the group $\mathbb{Z}_2$ generated by the automorphism $x_{ij} \leftrightarrow x_{ji}$, we get the $\mathbb{CP}^2_{10}$ mentioned above. Its vertices are $x_{ij}$, $1 \leq i \leq j \leq 4$. Let $\alpha, \beta$ be the generators of the alternating group $A_4$ given by $\alpha = (123)$,

$$\beta = (12)(34).$$

Then $\alpha, \beta$ act on the vertices of $\mathbb{CP}^2_{10}$ by:

$$\alpha \equiv (x_{11}x_{22}x_{33})(x_{23}x_{13}x_{12})(x_{24}x_{34}x_{14}), \ \beta \equiv (x_{11}x_{22})(x_{33}x_{44})(x_{24}x_{13})(x_{14}x_{23}).$$

The following are the basic facets of $\mathbb{CP}^2_{10}$ modulo $A_4 = \langle \alpha, \beta \rangle$:

$$x_{11}x_{22}x_{33}x_{12}x_{13}, \ x_{11}x_{22}x_{12}x_{14}x_{34}, \ x_{11}x_{22}x_{14}x_{24}x_{34}, \ x_{11}x_{22}x_{12}x_{13}x_{24}, \ x_{11}x_{22}x_{13}x_{24}x_{34}.$$ The full list of facets of $\mathbb{CP}^2_{10}$ may be obtained from these five basic facets by applying the group $A_4$. Under this group, the first three basic facets form orbits of length 12 each, while each of the last two forms an orbit of length 6, yielding a total of $3 \times 12 + 2 \times 6 = 48$ facets. The complex $\mathbb{CP}^2_{10}$ is 2-neighbourly and its face vector is $(10, 45, 110, 120, 48)$.

Here we prove the following:

**Theorem 1.** There are exactly two 16-vertex simplicial complexes which (i) are simplicial subdivisions of the cell complex $S^2_4 \times S^2_4$, (ii) retain the self-homeomorphism $\alpha : (x, y) \mapsto (y, x)$ of $|S^2_4| \times |S^2_4|$ as a simplicial automorphism, and (iii) the action of $\mathbb{Z}_2 = \langle \alpha \rangle$ is pure (cf. Definition 2). These two complexes are isomorphic and one of them is $(S^2 \times S^2)_{16}$.

**Corollary 2.** The complex $\mathbb{CP}^2_{10} := (S^2 \times S^2)_{16}/\mathbb{Z}_2$ is a 10-vertex triangulation of $\mathbb{CP}^2$. Its full automorphism group is $A_4$.
Let $T$ and $\mathcal{I}$ denote the solid tetrahedron and the icosahedron in $\mathbb{R}^3$ respectively. Thus, the cell complex $S^2_4 \times S^2_4$ alluded to above is a subcomplex of the boundary complex of the product polytope $T \times T$ in $\mathbb{R}^6$. Although we do not present the details in this paper, Theorem 1 can be strengthened (following the same line of arguments) to show that there is a unique simplicial subdivision $S^5_{16}$ of the cell complex $\partial(T \times T)$ which is $\mathbb{Z}_2$-stable with a pure $\mathbb{Z}_2$-action. To our utter surprise, it turns out that as an abstract simplicial complex, $S^5_{16}$ is isomorphic to the combinatorial join $S^2_4 \ast S^2_4$ of the boundary complexes of $T$ and $\mathcal{I}$ respectively.

**Remark 1.** This last fact has the following geometric interpretation. Let $T \oplus \mathcal{I}$ denote the convex hull of $T \cup \mathcal{I}$, where $T$ and $\mathcal{I}$ sit in two (three-dimensional) affine subspaces of $\mathbb{R}^6$ meeting at a point which is in the interior of both polyhedra. Then $T \oplus \mathcal{I}$ is a simplicial 6-polytope and the boundary complex of this polytope is combinatorially isomorphic to a simplicial subdivision of the boundary complex of $T \times T$. This geometric result cries out for a geometric explanation; but we have none.

By the construction, $(S^2 \times S^2)_{10}$ is a subcomplex of $S^2_4 \ast S^2_4$. Since the decahedron $\mathbb{R}P^2_6$ is the quotient of $S^2_4$ by $\mathbb{Z}_2$, and $\mathbb{Z}_2$ acts trivially on $S^2_4$ (the latter being the combinatorial child of the “diagonal” $S^2$ in $S^2 \times S^2$, i.e., the $S^2_4$ in Figure 1), on passing to the quotient, we find the surprising inclusion

$$\mathbb{C}P^2_{10} \subseteq S^2_4 \ast \mathbb{R}P^2_6.$$ 

Indeed, $S^2_4$ and $\mathbb{R}P^2_6$ occur as induced subcomplexes of $\mathbb{C}P^2_{10}$ on a complementary pair of vertex sets. Since both $S^2_4$ and $\mathbb{R}P^2_6$ are classical objects, and the combinatorial join is such a well known operation on simplicial complexes, this inclusion says that $\mathbb{C}P^2_{10}$ was all along sitting there right before our eyes!

The number 10 obtained here is not optimal. It is well known (cf. [10, 2, 3, 4]) that any triangulation of $\mathbb{C}P^2$ requires at least nine vertices, and there is a unique 9-vertex triangulation $\mathbb{C}P^2_9$ of this manifold, obtained by Kühnel ([11, 12]). But, our construction is natural in that it is obtained by a combinatorial mimicry of a topological construction of $\mathbb{C}P^2$. It shares this naturalness with another 10-vertex triangulation, say $K^4_{10}$, of $\mathbb{C}P^2$ available in the literature, namely the “equilibrium” triangulation of Banchoff and Kühnel ([8]). Here we prove the following:

**Theorem 3.** The simplicial complex $\mathbb{C}P^2_{10}$ is bistellar equivalent to both $\mathbb{C}P^2_9$ and $K^4_{10}$.

**Corollary 4.** (a) Kühnel’s 9-vertex simplicial complex $\mathbb{C}P^2_9$ triangulates $\mathbb{C}P^2$. (b) Both $\mathbb{C}P^2_{10}$ and $\mathbb{C}P^2_9$ induce the standard pl-structure on $\mathbb{C}P^2$.

Of course, in principle these ideas generalize to arbitrary dimensions. In general, the $d$-dimensional complex projective space $\mathbb{C}P^d$ is the symmetric $d$-th power of $S^2$, i.e., the quotient of $(S^2)^d$ by the symmetric group $S_d$ acting by co-ordinate permutations. Unfortunately, even in the next case $d = 3$, it is not possible to subdivide the cell complex $S^3_4 \times S^3_4 \times S^3_4$ into a simplicial complex, with a pure $S_3$-action, without adding more vertices. Indeed, we found that we need to add 60 more vertices to obtain an $(S^2 \times S^2 \times S^2)_{124}$. On quotienting, we obtain a $\mathbb{C}P^3_{30}$ - again with full automorphism group $A_4$. The details
are so complicated that we decided to postpone publication. We are presently trying to see if one can apply bistellar moves to this $CP^3_1$ to reduce the number of vertices. It is known that any triangulation of $CP^3$ requires at least 17 vertices (cf. [2]).

After we submitted a preliminary version of this paper to arXiv (arXiv:1004.3157v1, 2010), Ulrich Brehm ([9]) communicated to us that he had the idea of obtaining $CP^3_1$ as a quotient of a 16-vertex $\tilde{S}$ in the 1980’s; however he never published the details.

We obtain a second simplicial subdivision $(S^2 \times S^2)'_{16}$ of $S^4_4 \times S^4_4$.

**Description of $(S^2 \times S^2)'_{16}$:** This is a second simplicial subdivision of the cell complex $S^4_4 \times S^4_4$. It has the same vertex-set and automorphism group $A_4$. Modulo the group $A_4$, its basis facets are:

\[ x_{11}x_{12}x_{13}x_{21}x_{31}, x_{11}x_{12}x_{14}x_{21}x_{31}, x_{11}x_{13}x_{14}x_{21}x_{31}, x_{12}x_{13}x_{23}x_{31}x_{32}, \\
  x_{12}x_{14}x_{21}x_{24}x_{31}, x_{12}x_{14}x_{24}x_{31}x_{34}, x_{12}x_{21}x_{24}x_{31}x_{32}, x_{12}x_{24}x_{31}x_{32}x_{34}. \]

Each facet is in an orbit of length 12, yielding a total of $8 \times 12 = 96$ facets. The complex $(S^2 \times S^2)'_{16}$ has the same face vector as $(S^2 \times S^2)_{16}$, namely, $(16, 84, 216, 240, 96)$.

We perform a finite sequence of generalized bistellar moves on $(S^2 \times S^2)'_{16}$ and obtain the following 12-vertex triangulation $(S^2 \times S^2)_{12}$ of $S^2 \times S^2$.

**Description of $(S^2 \times S^2)_{12}$:** The vertices are $x_{ij}$, $1 \leq i \neq j \leq 4$. Its automorphism group $2S_5$ is generated by the two automorphisms $h = (x_{12}x_{14}x_{21}x_{24}x_{31})(x_{13}x_{42}x_{43}x_{32}x_{34})$ and $g = (x_{12}x_{21}x_{24}x_{42}x_{43}x_{41}x_{34}x_{13}x_{31}x_{32}x_{23})$. Modulo this group, $(S^2 \times S^2)_{12}$ is generated by the following two basic facets:

\[ x_{12}x_{14}x_{21}x_{24}x_{31}, x_{12}x_{13}x_{14}x_{21}x_{31}. \]

The first basic facet is in an orbit of size 12, while the second is in an orbit of size 60, yielding a total of 72 facets. Its face vector is $(12, 60, 160, 180, 72)$.

**Theorem 5.** The simplicial complex $(S^2 \times S^2)_{12}$ is a triangulation of $S^2 \times S^2$. Its full automorphism group is $2S_5$, the non-split extension of $\mathbb{Z}_2$ by $S_5$.

The complex $(S^2 \times S^2)_{12}$ has many remarkable properties. Its automorphism group is transitive on its vertices and edges. All its vertices have degree 10 and all its edges have degree 8. Indeed, the link of each edge is isomorphic to the 2-sphere $S^2$ obtained from the boundary complex of the octahedron by starring two vertices in a pair of opposite faces. Also, all triangles of $(S^2 \times S^2)_{12}$ are of degree 3 or 5. The automorphism group is transitive on its triangles of each degree. The degree 3 triangles constitute a weak pseudomanifold whose strong components are two icosahedra. Thus, we find a pair $I_1$, $I_2$ of icosahedra sitting canonically inside the 2-skeleton of $(S^2 \times S^2)_{12}$. These two icosahedra are “antimorphic” in the sense that the identity map is an antimorphism between them (cf. Definition 1 below). The structure of $(S^2 \times S^2)_{12}$ is completely described in terms of this antimorphic pair of icosahedra. The full automorphism group $2S_5$ of $(S^2 \times S^2)_{12}$ is a double cover of the common automorphism group of these two icosahedra.
Again, the number 12 here is not optimal. In [12], Kühnel and Laßmann have shown that any triangulation of \( S^2 \times S^2 \) needs at least 11 vertices, and in [14], Lutz finds (via computer search) several 11-vertex triangulations of \( S^2 \times S^2 \), all with trivial automorphism groups. Surprisingly, even though \( (S^2 \times S^2)_{12} \) is not minimal, it does not admit any proper bistellar moves. Thus, there is no straightforward way to obtain a minimal triangulation of \( S^2 \times S^2 \) starting from \( (S^2 \times S^2)_{12} \).

In [17], Sparla proved two remarkable inequalities on the Euler characteristic \( \chi \) of a combinatorial 4-manifold \( M \) satisfying certain conditions. His first result is that if there is a centrally symmetric simplicial polytope \( P \) of dimension \( d \geq 6 \) such that \( M \subseteq \partial P \) and \( \text{skel}_2(M) = \text{skel}_2(P) \), then \( 10(\chi - 2) \geq 4^d (d-1)/3 \). Equality holds here if and only if \( P \) is a cross polytope (i.e., dual of a hypercube). His second result is: if \( M \) has \( 2d \) vertices and admits a fixed point free involution then \( 10(\chi - 2) \leq 4^d (d-1)/2 \). Equality holds if and only if \( M \) also satisfies the hypothesis of the first result for a cross polytope \( P \). Notice that, in view of the Dehn-Sommerville equations, equality in either inequality determines the face vector of \( M \) in terms of \( d \) alone. To obtain an example of equality (in both results) with \( d = 6 \), Sparla searched for (and found) a 4-manifold with the predicted face vector under the assumption of an automorphism group \( A_5 \times Z_2 \). To determine the topological type of the resulting 12-vertex 4-manifold, he had to compute its intersection form and then appeal to Freedman’s classification of simply connected smooth 4-manifolds. We believe that our approach to Sparla’s complex not only elucidates its true genesis, but also reveals its rich combinatorial structure and contributes to an elementary determination of its topological type. Note, however, that Sparla’s approach reveals yet another remarkable property of \( (S^2 \times S^2)_{12} \). It provides a tight rectilinear embedding of \( S^2 \times S^2 \) in \( \mathbb{R}^6 \).

**Remark 2.** If \( X \) is a triangulated 4-manifold on at most 12 vertices, then its vertex-links are homology 3-spheres on at most 11 vertices, and hence (cf. [5]) are combinatorial spheres. Thus all triangulated 4-manifolds on at most 12 vertices are combinatorial manifolds. (More generally, this argument yields: All triangulated \( d \)-manifolds on at most \( d + 8 \) vertices are combinatorial manifolds.) In particular, both \( CP_{10}^2 \) and \( (S^2 \times S^2)_{12} \) are combinatorial manifolds. Actually, an old result of Bing ([7]) says that all the vertex links of any triangulated 4-manifold are simply connected triangulated 3-manifolds. Therefore, in view of Perelman’s theorem (Poincaré conjecture) ([15]), all triangulated 4-manifolds are combinatorial manifolds, irrespective of the number of vertices.

**Remark 3.** In [1], Akhmedov and Park have shown that \( S^2 \times S^2 \) has countably infinite number of distinct smooth structures. Since there is an one to one correspondence between the smooth structures and pl-structures on a 4-manifold (cf. [16, page 167]), it follows that \( S^2 \times S^2 \) has infinitely many distinct pl-structures. Since \( (S^2 \times S^2)_{16} \) and \( (S^2 \times S^2)'_{16} \) are simplicial subdivisions of \( S^2_4 \times S^2_4 \), it follows that the pl-structures given by \( (S^2 \times S^2)_{16} \) and \( (S^2 \times S^2)'_{16} \) are standard. Again, \( (S^2 \times S^2)_{12} \) is combinatorially equivalent to \( (S^2 \times S^2)'_{16} \) (cf. Remark 5) and hence gives the same pl-structure as \( (S^2 \times S^2)'_{16} \). So, all the triangulations of \( S^2 \times S^2 \) discussed here give the standard pl-structure on \( S^2 \times S^2 \).

## 2 Preliminaries

All simplicial complexes considered here are finite and the empty set is a simplex (of dimension \(-1\)) of every simplicial complex. We now recall some definitions here.
For a finite set $V$ with $d + 2$ ($d \geq 0$) elements, the set $\partial V$ (respectively, $\bar{V}$) of all the proper (resp. all the) subsets of $V$ is a simplicial complex and triangulates the $d$-sphere $S^d$ (resp. the $(d + 1)$-ball). The complex $\partial V$ is called the standard $d$-sphere and is also denoted by $S^d_{d+2}(V)$ (or simply by $S^d_{d+2}$). The complex $\bar{V}$ is called the standard $(d + 1)$-ball and is also denoted by $D^{d+1}_{d+2}(V)$ (or simply by $D^{d+1}_{d+2}$). (Generally, we write $X = X_n^d$ to indicate that $X$ has $n$ vertices and dimension $d$.)

For simplicial complexes $X$, $Y$ with disjoint vertex-sets, their join $X \ast Y$ is the simplicial complex whose simplices are all the disjoint unions $A \cup B$ with $A \in X$, $B \in Y$.

If $\sigma$ is a simplex of a simplicial complex $X$ then the link of $\sigma$ in $X$, denoted by $lk_X(\sigma)$, is the simplicial complex whose simplices are the simplices $\tau$ of $X$ such that $\tau \cap \sigma = \emptyset$ and $\sigma \cup \tau$ is a simplex of $X$. The number of vertices in the link of $\sigma$ is called the degree of $\sigma$. Also, the star of $\sigma$, denoted by $star_X(\sigma)$ or $star(\sigma)$, is the subcomplex $\sigma \ast lk_X(\sigma)$ of $X$.

For a simplicial complex $X$, $|X|$ denotes the geometric carrier. It may be described as the subspace of $[0, 1]^V(X)$ (where $V(X)$ is the vertex set of $X$) consisting of all functions $f: V(X) \to [0, 1]$ satisfying (i) $\text{Support}(f) \subseteq X$ and (ii) $\sum_{x \in V(X)} f(x) = 1$. If a space $Y$ is homeomorphic to $|X|$ then we say that $X$ triangulates $Y$. If $|X|$ is a topological manifold (respectively, $d$-sphere) then $X$ is called a triangulated manifold (resp. triangulated $d$-sphere). If $|X|$ is a pl manifold (with the pl structure induced by $X$) then $X$ is called a combinatorial manifold. For $1 \leq d \leq 4$, $X$ is a combinatorial $d$-manifold if and only if the vertex links are triangulated $(d - 1)$-spheres.

The face vector of a $d$-dimensional simplicial complex is the vector $(f_0, f_1, \ldots, f_d)$, where $f_i$ is the number of $i$-dimensional simplices in the complex.

If $X$ is a $d$-dimensional pure simplicial complex (i.e., every maximal simplex is $d$-dimensional) and $D, \bar{D}$ are triangulations of the $d$-ball such that (i) $\partial D = \partial \bar{D} = \bar{D} \cap X$, and (ii) $D \subseteq X$, then the simplicial complex $\bar{X} := (X \setminus D) \cup \bar{D}$ is said to be obtained from $X$ by a generalized bistellar move (GBM) with respect to the pair $(D, \bar{D})$. Clearly, in this case, $\bar{X}$ and $X$ triangulate the same topological space and if $u$ is a vertex in $\partial D$ then $lk_X(u)$ is obtained from $lk_{\bar{X}}(u)$ by a GBM (cf. [6]).

In particular, let $A$ be a simplex of $X$ whose link in $X$ is a standard sphere $\partial B$. Suppose also that $B \not\subseteq X$. Then, we may perform the GBM with respect to the pair of balls $(A \ast \partial B, B \ast \partial A)$. Such an operation is called a bistellar move, and will be denoted by $A \mapsto B$. Also, if $C$ is any simplex of $X$ and $x$ is a new symbol, then we may perform the GBM on $X$ with respect to the pair $(C \ast \bar{X}(C), (\bar{x} \ast \partial C) \ast \bar{X}(C))$. The resulting simplicial complex $\tilde{X}$ is said to be obtained from $X$ by starring the vertex $x$ in the simplex $C$. In case $C$ is a facet, this is a bistellar move - the only sort of bistellar move which increases the number of vertices. All other kinds of bistellar moves are said to be proper.

Two pure simplicial complexes are called bistellar equivalent if one is obtained from the other by a finite sequence of bistellar moves. If $X$ is obtained from $Y$ by the bistellar move $A \mapsto B$ then the complex $Z$ obtained from $Y$ by starring a new vertex $v$ in $B$ is a subdivision of both $X$ and $Y$. This implies that bistellar equivalent complexes induce same pl-structure on their common geometric carrier.

The group $\mathbb{Z}_2$ acts on $S^2 \times S^2$ by co-ordinate flip. The following proposition is well known to algebraic geometers (cf. [13]):

**Proposition 6.** The quotient space $(S^2 \times S^2)/\mathbb{Z}_2$ is homeomorphic to the complex projective plane $\mathbb{C}P^2$. 
3 Relations with the icosahedron

Emergence of the icosahedron: Let $T_0$ be the tetrahedron with vertex-set $V = \{x_1, x_2, x_3, x_4\}$. Then, viewed abstractly, the boundary complex of the product polytope $T_0 \times T_0$ has vertex-set $V \times V$, and faces $A \times B$, where $A$ and $B$ range over all the subsets of $V$. The product cell complex for $S^2 \times S^2 = (\partial T_0) \times (\partial T_0)$ is the subcomplex consisting of cells $A \times B$, where $A$ and $B$ range over all the proper subsets of $V$. We use the notation $x_{ij}$ to denote the vertex $(x_i, x_j)$ of $T_0 \times T_0$. For $i \neq j$, $k \neq l$, $x_{ij}x_{kl}$ forms an edge of $T_0 \times T_0$ if and only if it is one of the solid edges of the icosahedron in Figure 1. (This picture is a Schlegel diagram obtained by projecting the boundary of the icosahedron on one of its faces. Thus, there is only one “hidden” face (namely, $x_{41}x_{42}x_{43}$) in the picture. What is important for us is the label given to the vertices.)

Notice that the broken edges in the icosahedron are precisely the edges $x_{ij}x_{kl}$ where \{i, j, k, l\} is an even permutation of \{1, 2, 3, 4\}.

To obtain the appropriate triangulation of $S^2 \times S^2$, we join $x_{ii}$ to all vertices for all $i$ and also introduce the broken edges of the icosahedron. Thus viewed, one sees the simplicial subdivision $(S^2 \times S^2)_16$ of the cell complex $(\partial T_0) \times (\partial T_0)$ as a subcomplex of $(\partial T) \ast (\partial \mathcal{I})$, where $T$ is the tetrahedron with vertex-set \{x_{ii} : 1 \leq i \leq 4\} and $\mathcal{I}$ is the icosahedron depicted in Figure 1.

Notice also that the $\mathbb{Z}_2$-action $x_{ij} \leftrightarrow x_{ji}$ fixes the vertices of $T$ and acts on $\mathcal{I}$ as the antipodal map. Thus, going modulo $\mathbb{Z}_2$, we find $\mathbb{C}P^2_{10}$ as a subcomplex of the 5-dimensional simplicial complex $S^2_4 \ast \mathbb{R}P^2_6$, where $S^2_4$ is the 4-vertex 2-sphere given by the boundary complex of $T$ and $\mathbb{R}P^2_6$ is the (minimal) triangulation of the real projective plane (with vertices of the same name being identified) given in Figure 1.

From our nomenclature for the vertices, the inclusion $\mathbb{C}P^2_{10} \subseteq S^2_4 \ast \mathbb{R}P^2_6$ is obvious, as is the fact that $(\partial T) \ast (\partial \mathcal{I})$ is a simplicial subdivision of the boundary complex of $T_0 \times T_0$.

Finally, note that $\Delta_i = \{x_{ij} : j \neq i\}$ and $\Delta^i = \{x_{ji} : j \neq i\}$ are triangles of the
icosahedron, and \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\} and \{\Delta^1, \Delta^2, \Delta^3, \Delta^4\} are antipodal pairs of quadruples (consisting of triangles) partitioning the vertex-set of the icosahedron. It is easy to see that there are exactly five such pairs in the icosahedron, and the automorphism group \(A_5 \times \mathbb{Z}_2\) of \(I\) acts transitively on them. The stabilizer of each such pair is \(A_4 \times \mathbb{Z}_2\), and \(A_4\) acts regularly on the vertex-set of \(I\). Our choice of nomenclature for the vertices of \(I\) amounts to choosing one such antipodal pair of quadruples. This is because we have \(\Delta_i \cap \Delta^j = \emptyset\) if \(i = j\) and \(= \{x_{ij}\}\) if \(i \neq j\). Viewed dually, one sees Kepler’s regular tetrahedra embedded in the dodecahedron. Namely, the centres of \(\Delta_i, 1 \leq i \leq 4\) (as well as of \(\Delta^i, 1 \leq i \leq 4\)) are the vertices of a regular tetrahedron inscribed in the dual dodecahedron.

The 12-vertex triangulation \((S^2 \times S^2)_{12}\) of \(S^2 \times S^2\) is obtained from \((S^2 \times S^2)_{16}\) by a sequence of bistellar moves (cf. proof of Theorem 5). However, its most elegant description requires the introduction of the following definition.

**Definition 1.** Let \(I_1\) and \(I_2\) be two copies of the icosahedron. A bijection \(f: V(I_1) \rightarrow V(I_2)\) is said to be an **antimorphism** if, for all vertices \(x, y\) of \(I_1\), we have (a) \(x\) and \(y\) are at distance one in \(I_1\) if and only if \(f(x)\) and \(f(y)\) are at distance two in \(I_2\), and (b) \(x\) and \(y\) are at distance two in \(I_1\) if and only if \(f(x)\) and \(f(y)\) are at distance one in \(I_2\). (It follows that \(x\) and \(y\) are at distance 3 (antipodal) in \(I_1\) if and only if \(f(x)\) and \(f(y)\) are at distance 3 (antipodal) in \(I_2\).) Here distance refers to the usual graphical distance on the respective edge graph. In case \(V(I_1) = V(I_2)\) and the identity map is an antimorphism between \(I_1\) and \(I_2\), then we say that \(I_1\) and \(I_2\) are **antimorphic**. Thus, the two icosahedra in Figure 2 below are antimorphic (the map, taking each vertex of the left icosahedron in Figure 2 to the vertex of the same name in the right icosahedron, is an antimorphism).

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**Figure 2:** An antimorphic pair of icosahedra

**Another description of \((S^2 \times S^2)_{12}\):** Take an antimorphic pair of icosahedra, say \(I_1\) and \(I_2\) (with common vertex set \(V\)). It turns out that \(I_1\) and \(I_2\) have the identical automorphism group \(A_5 \times \mathbb{Z}_2\) (not merely isomorphic, cf. Lemma 9 below). Also, there is a bijection \(\varphi\) from the triangles of \(I_1\) to the triangles of \(I_2\) such that for each triangle \(\Delta = abc\) of \(I_1\), \(\varphi(\Delta) = ijk\) is the only triangle of \(I_2\) for which \(aij, bjk\) and \(cik\) are triangles
of $I_2$ (cf. Lemma 9). Now, the vertex-set of $(S^2 \times S^2)_{12}$ is $V = V(I_1) = V(I_2)$ and it has two types of facets. (i) For each vertex $x$, the neighbors of $x$ in $I_1$ form facets. (ii) For each triangle $\Delta$ of $I_1$ and each vertex $y$ in $\Delta' = \varphi(\Delta)$, $(\Delta \cup \Delta') \setminus \{y\}$ is a facet. Thus $(S^2 \times S^2)_{12}$ has 12 facets of the first type and 20 \times 3 = 60 facets of the second type. From the description, it is clear that the common automorphism group $A_5 \times \mathbb{Z}_2$ of $I_1$ and $I_2$ is an automorphism group of $(S^2 \times S^2)_{12}$. It turns out that its full automorphism group is $2S_5$ generated by the two automorphisms $g = (x_{12}x_{21}x_{24}x_{42}x_{43}x_{41}x_{14}x_{34}x_{31}x_{32}x_{33})$ and $h = (x_{12}x_{14}x_{21}x_{24}x_{31})(x_{13}x_{42}x_{43}x_{32}x_{34})$. The automorphism $g$ interchanges $I_1$ and $I_2$.

Remark 4. It should be emphasized that the existence of an antimorphic pair of icosahedra (exploited in the above construction of $(S^2 \times S^2)_{12}$) is a minor miracle, and only an empirically verified fact. Its deeper geometric significance, if any, remains to be understood.

4 A self-dual CW decomposition of $\mathbb{C}P^2$

Here we have taken the cell complex $\partial T_0 \times \partial T_0$, and triangulated it to obtain the simplicial complex $(S^2 \times S^2)_{16}$ and finally quotiented this simplicial complex by $\mathbb{Z}_2$ to obtain $\mathbb{C}P^2_{10}$. This procedure reflects our obsession with simplicial complexes. However, one may straightforwardly quotient the cell complex by $\mathbb{Z}_2$ to obtain a (non-regular) CW decomposition of $\mathbb{C}P^2$. This CW complex is self-dual in the sense that its face-vector $(10, 24, 31, 24, 10)$ exhibits a curious palindromic symmetry. We proceed to describe it in some details. Consider the $\mathbb{Z}_2$ action on $\mathbb{R}^6 \equiv \mathbb{R}^3 \times \mathbb{R}^3$ given by $(x, y) \leftrightarrow (y, x)$. Let $\eta: \mathbb{R}^6 \to \mathbb{R}^6/\mathbb{Z}_2$ be the quotient map. We know that $\eta(S^2 \times S^2) = \mathbb{C}P^2$. We give a CW decomposition $W$ of the space $\eta(\partial T_0 \times \partial T_0)$.

For $0 \leq i \leq 4$, let $W^i$ denote the set of $i$-cells in $W$. For $i \neq 2$ the $i$-cells in $W$ are the images (under the map $\eta$) of $i$-cells in $\partial T_0 \times \partial T_0$. A 2-cell in $W$ is the image of a 2-cell $F$ in $\partial T_0 \times \partial T_0$ which is not of the form $E \times E$ for some edge $E$ in $\partial T_0$. More explicitly

$W^0 = V(\mathbb{C}P^2_{10})$,

$W^1 = \{\eta(E) : E$ is an edge of $\partial T_0 \times \partial T_0\}$,

$W^2 = \{\eta([x_{ij}x_{ik}x_{il}]) : 1 \leq j < k < l \leq 4, 1 \leq i \leq 4\}$

$\cup \{\eta([x_{ij}x_{ik}] \times |x_k x_l|) : i < j, k < l$ and either $i < k$ or $i = k$ and $j < l\}$,

$W^3 = \{\eta(A) : A$ is a 3-cell of $\partial T_0 \times \partial T_0\}$ and

$W^4 = \{\eta(B) : B$ is a 4-cell of $\partial T_0 \times \partial T_0\}$.

Then, $W^1$ contains 24 cells, $W^2$ contains 16 + 15 = 31 cells, $W^3$ contains 4 \times 6 = 24 cells and $W^4$ contains 10 cells. Clearly, each 1-cell in $W$ is regular (i.e., homeomorphic to a closed interval). Since all the 2-cells are homeomorphic images of the corresponding 2-cells in $\partial T_0 \times \partial T_0$, it follows that all the 2-cells in $W$ are regular.

For $0 \leq i \leq 4$, let $X_i = \bigcup_{\beta \in W^i \cup \ldots \cup W^4} \beta$. Then $\partial \alpha \subseteq X_{i-1}$ if $\alpha \in W^i$ for $i \neq 3$. Let $\gamma$ be a 3-cell in $W$. If $\gamma = \eta([x_i x_j x_k] \times |x_i x_j|)$, $i < j < k$, then $\gamma$ is obtained from $|x_i x_j x_k| \times |x_i x_j|$ by identifying $|x_{ii} x_{ij} x_{ij}|$ with $|x_{ii} x_{jj} x_{ij}|$ (by the identification given by $x_{ij} \leftrightarrow x_{ji}$). Thus, $\gamma$ is a regular 3-cell and $\partial \gamma = \eta([x_i x_k] \times |x_i x_j|) \cup \eta([x_j x_k] \times |x_i x_j|) \cup$
η([x_ii x_{ij} x_{jkl}] \cup [x_{ij} x_{jkl} x_{kjl}]). (Now, it is clear why we do not have to take \( \eta([x_i x_j] \times [x_i x_j]) \) in \( W^2 \). In fact, \( \eta([x_i x_j] \times [x_i x_j]) \) is inside of \( \gamma \).) Therefore, \( \partial \gamma \subseteq X_2 \). Same things are true if \( \gamma = \eta([x_i x_j x_k] \times [x_i x_k]) \) or \( \eta([x_i x_j x_k] \times [x_j x_k]) \). On the other hand, if \( \gamma = \eta(F \times E) \), where \( E \) is an edge and \( F \) is a 2-simplex and \( E \not\subseteq F \), then \( \gamma \) is homeomorphic to \( F \times E \) and hence is a regular 3-cell. In this case, it follows from the definition of \( W^2 \) that \( \partial \gamma \subseteq X_2 \).

Thus \( W \) is a CW complex.

If \( \sigma \) is a 4-cell in \( W \) then, either \( \sigma = \eta([x_i x_j x_k] \times [x_i x_j x_k]) \), for some \( i < j < k \) or \( \sigma = \eta([x_i x_j x_k] \times [x_i x_j x_l]) \), where \( \{i, j, k, l\} \) is an even permutation of \( \{1, 2, 3, 4\} \). In the first case, \( \sigma \) is homeomorphic to \( [x_i x_j x_k x_{kl}] \cup [x_i x_j x_{kk} x_{ij}] \cup [x_i x_j x_{kk} x_{jk}] \) and hence \( \sigma \) is a regular 4-cell. In the second case, \( \sigma \) is obtained from \( [x_i x_j x_k] \times [x_i x_j x_l] \) by identifying \( [x_i x_{ij} x_{ij}] \) with \( [x_i x_{ij} x_{ji}] \) (by the identification given by \( x_{ij} \leftrightarrow x_{ji} \)). So, \( \sigma \) is not a regular cell. Thus \( W^4 \) contains four regular 4-cells and six singular 4-cells.

Since each cell in \( W \) is the quotient of a cell in \( S^2_4 \times S^2_4 \), \( (S^2 \times S^2)_{16} \) is a simplicial subdivision of \( S^2 \times S^2 \) and \( \mathbb{C}P^2_1 \) is the quotient of \( (S^2 \times S^2)_{16} \), it follows that \( \mathbb{C}P^2 \) is a simplicial subdivision of \( W \).

5 Proofs

**Definition 2.** Let \( G \) be a group of simplicial automorphisms of a simplicial complex \( X \) with vertex set \( V(X) \). We shall say that the action of \( G \) on \( X \) is pure if it satisfies: \((a)\) whenever \( u, v \) are distinct vertices from the same \( G \)-orbit, \( uv \) is a non-edge of \( X \), and \((b)\) for each \( G \)-orbit \( \theta \subseteq V(X) \) and each \( \alpha \in X \), the stabiliser \( G_\alpha \) of \( \alpha \) in \( G \) acts transitively on \( \theta \cap V(\text{lk}_X(\alpha)) \).

**Lemma 7.** Let \( G \) be a group of simplicial automorphisms of a simplicial complex \( X \). Let \( q: V(X) \rightarrow V(X)/G \) denote the quotient map, and \( X/G := \{q(\alpha) : \alpha \in X\} \). If the action of \( G \) on \( X \) is pure then \( X/G \) is a simplicial complex which triangulates \( |X|/G \) (where the action of \( G \) on \( V(X) \) is extended to an action of \( G \) on \( |X| \) piecewise linearly, i.e., affinely on the geometric carrier of each simplex). That is, we have \( |X/G| = |X|/G \).

**Proof.** The condition \((a)\) ensures that the quotient map \( q \) is one-one on each simplex of \( X \). The simplicial map \( q: X \rightarrow X/G \) induces a piecewise linear continuous map \( |q| \) from \( |X| \) onto \( |X/G| \).

**Claim.** The fibres of \( q: X \rightarrow X/G \) are precisely the \( G \)-orbits on simplices of \( X \) (that is, if \( \alpha, \alpha' \in X \) are such that \( q(\alpha) = q(\alpha') \) then there exists \( g \in G \) such that \( g(\alpha) = \alpha' \)).

We prove the claim by induction on \( k = \dim(\alpha) = \dim(\alpha') \). The claim is trivial for \( k = -1 \). So, assume \( k \geq 0 \), and the claim is true for all smaller dimensions. Choose a simplex \( \beta \subseteq \alpha \) of dimension \( k-1 \), and let \( \beta' \subseteq \alpha' \) be such that \( q(\beta') = q(\beta) \). By induction hypothesis, \( \beta' \) and \( \beta \) are in the same \( G \)-orbit. Therefore, applying a suitable element of \( G \), we may assume, without loss of generality, that \( \beta' = \beta \). Let \( \alpha = \beta \cup \{x\} \), \( \alpha' = \beta \cup \{x'\} \). Then \( q(x) = q(x') \), i.e., \( x \) and \( x' \) are in the same \( G \)-orbit. Now, by assumption \((b)\), there is a \( g \in G_\beta \) such that \( g(x) = x' \). Then \( g(\alpha) = \alpha' \). This proves the claim.

In the presence of condition \((a)\), the claim ensures that the fibres of \( |q| \) are precisely the \( G \)-orbits on points of \( |X| \). Hence \( |q| \) induces the required homeomorphism between \( |X|/G \) and \( |X/G| \). \(\Box\)
Up to isomorphism, there are exactly two 6-vertex 2-spheres, namely, $S_1$ and $S_2$ given in Figure 3. We need the following lemma to prove Theorem 1.

**Lemma 8.** Let $C$ be the triangular prism given in Figure 3(b) (i.e., $C$ is the product of a 2-simplex and an edge). Up to isomorphism, there exists a unique 6-vertex simplicial subdivision $\hat{C}$ of $C$. The facets (tetrahedra) in $\hat{C}$ are $a_1b_1b_2b_3, a_1a_2b_3, a_1a_2a_3b_3$. Moreover, $\partial \hat{C}$ is isomorphic to $S_2$ of Figure 3(a) and determines $\hat{C}$ uniquely.

**Proof.** Let $\hat{C}$ be a 6-vertex subdivision of $C$. Then there exists a 3-simplex $\sigma$ in $\hat{C}$ which contains the 2-simplex $b_1b_2b_3$. Without loss of generality, we may assume that $\sigma = a_1b_1b_2b_3$. Then $C$ is the union of $\sigma$ and the pyramid $P$ given in Figure 4. Since we are not allowed to introduce new vertices, clearly the rectangular base of $P$ must be triangulated using two triangles, in one of two isomorphic ways, and the remaining tetrahedra in $\hat{C}$ must have the apex of $P$ as a vertex and one of these two triangles as base. Thus, without loss of generality, $P = a_1a_2b_2b_3 \cup a_1a_2a_3b_3$. This proves the first part.

The last part follows from the fact that the facets of $\hat{C}$ are the maximal cliques in the 1-skeleton of $\partial \hat{C}$.

**Proof of Theorem 1.** Let $X$ be a 16-vertex simplicial subdivision of $S_4^2 \times S_4^2$ satisfying (i), (ii) and (iii).

For $i \neq j$, consider the 2-cell $x_ix_j \times x_ix_j$. By (iii), $x_{ij}x_{ji}$ can not be an edge in $X$. This implies that $x_{ii}x_{jj}, x_{ij}x_{ji}, x_{ii}x_{ij}x_{jj} \in X$ and $x_ix_j \times x_ix_j = x_{ii}x_{jj}x_{ij} \cup x_{ij}x_{ji}x_{jj}$ (cf. Figure 5(a)).

\[ C = a_1b_1b_2b_3 \cup P \]
\[ P = a_1a_2b_2b_3 \cup a_1a_2a_3b_3 \]
\[ \hat{C} \]

**Figure 4: Simplicial subdivision of the triangular prism**
For \(i, j, k\) distinct, consider the 2-cell \(x_i x_j \times x_k\). Since \(X\) satisfies (iii), both \(x_{ij}\) and \(x_{ji}\) can’t be in \(\text{lk}_X(x_{ik})\). Now, \(x_{ik} x_{ij}\) is an edge in the cell complex \(S_4^2 \times S_4^2\) and hence is an edge in \(X\). Thus, \(x_{ik} x_{ji}\) can not be an edge in \(X\). This implies that \(x_{ii} x_{jk}, x_{ii} x_{ji} x_{jk}\), \(x_{ii} x_{ik} x_{jk} \in X\) and \(x_{ij} \times x_{ik} x_{jk} = x_{ii} x_{ij} x_{jk} \cup x_{ii} x_{ik} x_{jk}\) (cf. Figure 5 (b)).

![Figure 5: Simplicial subdivisions of rectangular 2-cells of \(S_4^2 \times S_4^2\)](image)

Consider the 2-cell \(x_1 x_3 \times x_2 x_4\). Clearly, \(x_1 x_3 \times x_2 x_4 = x_{12} x_{32} x_{34} \cup x_{12} x_{14} x_{34}\) or \(= x_{12} x_{32} x_{14} \cup x_{32} x_{14} x_{34}\).

**Case 1.** \(x_1 x_3 \times x_2 x_4 = x_{12} x_{32} x_{34} \cup x_{12} x_{14} x_{34}\) (cf. Figure 5 (c)). So, \(x_{12} x_{34} \in X\). Then, by (ii), \(x_{21} x_{43} \in X\) and, by (iii), \(x_{12} x_{43}, x_{21} x_{34} \notin X\). This implies that \(x_{23} x_{34} \in X\). Clearly, \(x_{12} x_{34} \notin X\). Then, by (ii), \(x_{12} x_{34} \in X\) and, by (iii), \(x_{12} x_{34} \notin X\). Hence is the 2-skeleton of \(X\). Observe that we have already 84 edges as mentioned in the construction of \((S^2 \times S^2)_{16}\) and, since \(X\) satisfies (iii), all the 36 remaining 2-sets are non-edges in \(X\).

Observe that any 3-cell in \(S_4^2 \times S_4^2\) is the product of a 2-simplex and an edge. For \(i, j, k\) distinct, consider the 3-cell \(x_i x_j x_k \times x_i x_j\). Since \(x_{ii} x_{ij}, x_{ii} x_{ki}\) and \(x_{ij} x_{ki}\) are edges, by Lemma 8, \(x_i x_j x_k \times x_i x_j = x_{ii} x_{ij} x_{kj} x_{jj} \cup x_{ii} x_{ki} x_{kj} x_{jj} \cup x_{ii} x_{ki} x_{ji} x_{jj}\) is the unique subdivision of \(x_i x_j x_k \times x_i x_j\) (cf. Figure 6 (a)). Similarly, \(x_i x_j \times x_i x_j x_k = x_{ii} x_{ij} x_{jk} x_{jj} \cup x_{ii} x_{ik} x_{jk} x_{jj} \cup x_{ii} x_{ik} x_{ij} x_{jj}\) is the unique subdivision of \(x_i x_j \times x_i x_j x_k\) (cf. Figure 6 (b)).

![Figure 6: Simplicial subdivisions of 3-cells of \(S_4^2 \times S_4^2\)](image)

For \(i, j, k, l\) distinct, consider the 3-cell \(x_i x_j x_k \times x_i x_l\). Here \(x_{ii} x_{ij} x_{kl}\) and \(x_{ii} x_{kl}\) are edges. By interchanging \(j\) and \(k\) (if required) we may assume that \(\{i, j, k, l\}\) is an even permutation of \(\{1, 2, 3, 4\}\). Then \(x_{ij} x_{kl}\) is an edge and hence, by Lemma 8, \(x_i x_j x_k \times
Figure 7. \( \gamma \) neighborly links. Therefore, the full automorphism group must fix this set of four vertices.

Clearly, \( \text{lk}(B) \) inside \( X \) is an even permutation of \( \{i,j,k,l\} \). This completes the proof.

Proof of Corollary 2. From Proposition 6, Lemma 7 and Theorem 1, it is immediate that \( \mathbb{C}P^2 \) triangulates \( \mathbb{C}P^2 \).

Since the automorphism group \( A_4 = \langle \alpha, \beta \rangle \) of \( (S^2 \times S^2)_{16} \) commutes with \( \mathbb{Z}_2 \), it descends to an automorphism group \( A_4 = \langle \alpha, \beta \rangle \) of \( \mathbb{C}P^2 \). We need to show that there are no other automorphisms.

It is easy to check that the four vertices \( x_i, 1 \leq i \leq 4 \), are the only ones with 2-neighborly links. Therefore, the full automorphism group must fix this set of four vertices. Since \( A_4 \) is 2-transitive on this 4-set, it suffices to show that there is no non-trivial automorphism \( \gamma \) fixing both \( x_1 \) and \( x_2 \). Suppose the contrary. Then \( \gamma \) is a non-trivial automorphism of \( \text{lk}(x_1x_2) \). But \( \text{lk}(x_1x_2) \) is the 8-vertex triangulated 2-sphere given in Figure 7.
From the picture, it is apparent that \( \text{lk}(x_{11}x_{22}) \) has only one non-trivial automorphism, namely \((x_{13}, x_{24})(x_{14}, x_{23})(x_{33}, x_{44})\). Therefore, \( \gamma = (x_{13}, x_{24})(x_{14}, x_{23})(x_{33}, x_{44}) \) and hence \( \gamma \) fixes the 3-simplex \( x_{11}x_{33}x_{44}x_{34} \). Then \( \gamma \) must either fix or inter-change the two vertices \( x_{13} \) and \( x_{14} \) in the link of this 3-simplex, a contradiction. This completes the proof. \( \square \)

**Proof of Theorem 3.** Consider the following sequence of bistellar moves on \( \mathbb{CP}^2 \) (performed one after the other):

(i) \( x_{22}x_{33}x_{44} \leftrightarrow x_{23}x_{24}x_{34} \), (ii) \( x_{11}x_{33}x_{44} \leftrightarrow x_{13}x_{14}x_{34} \), (iii) \( x_{11}x_{22}x_{44} \leftrightarrow x_{12}x_{14}x_{24} \),

(iv) \( x_{14}x_{33}x_{44} \leftrightarrow x_{12}x_{13}x_{34} \), (v) \( x_{22}x_{34}x_{44} \leftrightarrow x_{13}x_{23}x_{24} \),

(vi) \( x_{23}x_{33}x_{44} \leftrightarrow x_{12}x_{24}x_{34} \),

(vii) \( x_{12}x_{22}x_{44} \leftrightarrow x_{13}x_{14}x_{24} \), (viii) \( x_{33}x_{44} \leftrightarrow x_{12}x_{13}x_{24}x_{34} \),

(ix) \( x_{22}x_{44} \leftrightarrow x_{13}x_{14}x_{23}x_{24} \).

At the end of these moves, we get a 10-vertex triangulation \( K \) of \( \mathbb{CP}^2 \).

On \( K \) we perform the following sequence of bistellar moves one after another.

(x) \( x_{11}x_{24}x_{44} \leftrightarrow x_{12}x_{14}x_{23} \), (xi) \( x_{11}x_{13}x_{44} \leftrightarrow x_{14}x_{23}x_{34} \), (xii) \( x_{11}x_{44} \leftrightarrow x_{12}x_{14}x_{23}x_{34} \),

(xiii) \( x_{44}x_{14}x_{24} \leftrightarrow x_{12}x_{13}x_{23} \), (xiv) \( x_{44}x_{14} \leftrightarrow x_{34}x_{12}x_{13}x_{23} \), (xv) \( x_{44} \leftrightarrow x_{23}x_{44}x_{12}x_{13}x_{23} \).

(Note that the last three bistellar moves together is same as the GBM with respect to \( \{x_{14}, x_{24}, x_{34}\} \).

Let \( \mathbb{CP}_9^2 \) be as described in [11] with vertex-set \( \{1, 2, \ldots, 9\} \). Consider the map \( \varphi: L \to \mathbb{CP}_9^2 \) given by: \( \varphi(x_{11}) = 1 \), \( \varphi(x_{23}) = 2 \), \( \varphi(x_{24}) = 3 \), \( \varphi(x_{34}) = 4 \), \( \varphi(x_{22}) = 5 \), \( \varphi(x_{13}) = 6 \), \( \varphi(x_{14}) = 7 \), \( \varphi(x_{33}) = 8 \), \( \varphi(x_{12}) = 9 \). It is easy to see that \( \varphi \) is an isomorphism. Thus, \( \mathbb{CP}_9^2 \) is bistellar equivalent to \( \mathbb{CP}_9^2 \).

Now, on \( K \) we perform the following sequence of bistellar moves:

(xvi) \( x_{11}x_{22}x_{33} \leftrightarrow x_{12}x_{13}x_{23} \), (xvii) \( x_{22}x_{33}x_{24} \leftrightarrow x_{14}x_{23}x_{34} \),

(xviii) \( x_{22}x_{33}x_{13} \leftrightarrow x_{12}x_{14}x_{34} \), (xix) \( x_{22}x_{33} \leftrightarrow x_{12}x_{14}x_{23}x_{34} \).

We obtain a 10-vertex triangulation \( M \) of \( \mathbb{CP}^2 \). Let \( K^4_{10} \) be as described in [8] with vertex-set \( \{X, Y, Z, 0, 1, \ldots, 6\} \). Consider the map \( \psi: M \to K^4_{10} \) given by \( \psi(x_{33}) = X \),
\(\psi(x_2) = Y, \psi(x_{44}) = Z, \psi(x_{11}) = 0, \psi(x_{13}) = 1, \psi(x_{12}) = 2, \psi(x_{23}) = 3, \psi(x_{14}) = 4, \psi(x_{24}) = 5, \psi(x_{24}) = 6.\) It is easy to see that \(\psi\) is an isomorphism. Thus, \(\mathbb{C}P^2_{10}\) is bistellar equivalent to \(K_{10}^4\). This completes the proof. \(\square\)

**Proof of Corollary 4.** Part (a) follows from Corollary 2 and Theorem 3.

In [8], explicit coordinates for simplices of \(\mathbb{C}P^2\) by \(K_{10}^4\) is the standard one. Part (b) now follows from Theorem 3. \(\square\)

**Lemma 9.** Let \(I_1\) and \(I_2\) be an antimorphic pair of icosahedra. Then we have:

(a) \(\text{Aut}(I_1) = \text{Aut}(I_2) = A_5 \times \mathbb{Z}_2.\)

(b) For each triangle \(\Delta\) of \(I_1\), there is a unique triangle \(\Delta'\) of \(I_2\) such that each of the three triangles of \(I_2\) sharing an edge with \(\Delta\) has its third vertex in \(\Delta\). Further, the map \(\varphi: \Delta \mapsto \Delta'\) is a bijection from the triangles of \(I_1\) to the triangles of \(I_2\). There is a similarly defined bijection \(\psi\) from the triangles of \(I_2\) to the triangles of \(I_1\), and

(c) Every isomorphism \(f: I_1 \rightarrow I_2\) intertwines \(\varphi\) and \(\psi\).

(Warning: The maps \(\varphi\) and \(\psi\) are not induced by any vertex - to - vertex map!)

**Proof.** Recall that \(I_1\) and \(I_2\) have the same vertex set and the same pairs of antipodal vertices. Thus, they have the same antimodal map (sending each vertex \(x\) to its antipode \(\bar{x}\)). Now, the full automorphism group of the icosahedron is generated by its rotation group \(A_5\) and the antipodal map. So, to prove Part (a), it suffices to show that \(I_1\) and \(I_2\) share the same rotation group. For each pair \(x, \bar{x}\) of antipodes, \(I_1\) has a rotation symmetry \(\alpha_{x, \bar{x}}^1\) which fixes \(x\) and \(\bar{x}\) and rotates the remaining vertices along the 5-cycles \(\text{lk}_{I_1}(x)\) and \(\text{lk}_{I_2}(\bar{x})\). The rotation group of \(I_1\) is generated by these automorphisms of order five. But, \(\text{lk}_{I_2}(x)\) (respectively, \(\text{lk}_{I_2}(\bar{x})\)) is the graph theoretic complement of the pentagon \(\text{lk}_{I_1}(x)\) (respectively, \(\text{lk}_{I_1}(\bar{x})\)). Therefore, \(\alpha_{x, \bar{x}}^2\) is the square of \(\alpha_{x, \bar{x}}^1\). This proves Part (a).

Notice that if \(f_1, f_2: I_1 \rightarrow I_2\) are two antimorphisms, then \(f_1 \circ f_2^{-1} \in \text{Aut}(I_2)\) and \(f_2^{-1} \circ f_1 \in \text{Aut}(I_1)\). Thus, the antimorphism is unique up to right multiplication by elements of \(\text{Aut}(I_1)\) (or left multiplication by elements of \(\text{Aut}(I_2)\)). Therefore, there is no loss of generality in taking the antimorphic pair of icosahedra as the one given in Figure 2.

Since the common automorphism group is transitive on the triangles of \(I_1\) (and of \(I_2\)), it is enough to look at the triangle \(\Delta = x_{12}x_{13}x_{14}\) of \(I_1\). From Figure 2, we see that the links in \(I_2\) of two vertices of \(\Delta\) have exactly two vertices in common. Namely, we have \(V(\text{lk}_{I_2}(x_{12})) \cap V(\text{lk}_{I_2}(x_{13})) = \{x_{21}, x_{32}\}, V(\text{lk}_{I_2}(x_{12})) \cap V(\text{lk}_{I_2}(x_{14})) = \{x_{24}, x_{41}\}, V(\text{lk}_{I_2}(x_{13})) \cap V(\text{lk}_{I_2}(x_{14})) = \{x_{31}, x_{43}\}\). Therefore, any triangle \(\Delta'\) of \(I_2\) satisfying the requirement must be contained in the vertex set \(\{x_{21}, x_{32}, x_{24}, x_{41}, x_{31}, x_{43}\}\). But one sees that this set of six vertices contains a unique triangle in \(I_2\), namely \(\Delta' = x_{21}x_{31}x_{41}\). Thus the map \(\varphi: \Delta \rightarrow \Delta'\) is well defined. Similarly, there is a well defined map \(\psi\) from the triangles of \(I_2\) to the triangles of \(I_1\). The map \(\psi \circ \varphi\) is the antipodal map on the triangles of \(I_1\) to themselves. Similarly, \(\varphi \circ \psi\) is the antipodal map on triangles of \(I_2\). Hence \(\varphi\) (as well as \(\psi\)) is a bijection. This proves Part (b).

To prove Part (c), let \(f\) be any isomorphism from \(I_1\) to \(I_2\). Since \(I_1\) and \(I_2\) are antimorphic, it is immediate that \(f\) also defines an isomorphism from \(I_2\) to \(I_1\). Let \(\Delta\) be
any triangle of \( I_1 \) and let \( \Delta' = \varphi(\Delta) \). By definition, there are three triangles \( \Delta'_1, \Delta'_2, \Delta'_3 \) of \( I_2 \) each of which shares a vertex with \( \Delta \) and an edge with \( \Delta' \). Then \( f(\Delta) \) and \( f(\Delta') \) are triangles of \( I_2 \) and \( I_1 \), respectively. Also, \( f(\Delta'_1), f(\Delta'_2), f(\Delta'_3) \) are three triangles of \( I_1 \) each of which shares a vertex with \( f(\Delta) \) and an edge with \( f(\Delta') \). Therefore, by definition of \( \psi \), \( \psi(f(\Delta)) = f(\Delta') = f(\varphi(\Delta)) \). 

\[ \square \]

**Proof of Theorem 5.** As in the proof of Theorem 1, one may verify that \((S^2 \times S^2)'_{16}\) is a simplicial subdivision of \( S^2 \times S^2 \), and hence it triangulates \( S^2 \times S^2 \). We apply the following sequence of bistellar moves to \((S^2 \times S^2)'_{16}\) to create a second 16-vertex triangulation \((S^2 \times S^2)_{16}\) of \( S^2 \times S^2 \):

\[
\begin{align*}
x_{12}x_{13}x_{14} & \mapsto x_{23}x_{34}x_{42}, \quad x_{21}x_{23}x_{24} \mapsto x_{14}x_{31}x_{43}, \\
x_{31}x_{32}x_{34} & \mapsto x_{12}x_{24}x_{41}, \quad x_{41}x_{42}x_{43} \mapsto x_{13}x_{21}x_{32}.
\end{align*}
\]

Since this set of bistellar moves is stable under the automorphism group \( A_4 \) of \((S^2 \times S^2)'_{16}\), it follows that \((S^2 \times S^2)_{16}\) inherits the group \( A_4 \). Also, both complexes have \( \text{lk}(x_{11}) = S_3^3(\{x_{12}, x_{13}, x_{14}\}) \times S_3^3(\{x_{21}, x_{31}, x_{41}\}) \). However, while \((S^2 \times S^2)'_{16}\) has both \( x_{12}x_{13}x_{14} \) and \( x_{21}x_{31}x_{41} \) as triangles, we have chosen the bistellar moves judiciously to ensure that \((S^2 \times S^2)_{16}\) does not have the triangle \( x_{12}x_{13}x_{14} \). Therefore, we may apply the following four GBM’s (one after the other) to \((S^2 \times S^2)'_{16}\) to delete the four vertices \( x_{ii}, 1 \leq i \leq 4 \):

\[
\begin{align*}
(\text{st}(x_{11}), D_3^2(\{x_{12}, x_{13}, x_{14}\}) & \ast S_3^1(\{x_{21}, x_{31}, x_{41}\}), \\
(\text{st}(x_{22}), D_3^2(\{x_{21}, x_{23}, x_{24}\}) & \ast S_3^1(\{x_{12}, x_{32}, x_{42}\}), \\
(\text{st}(x_{33}), D_3^2(\{x_{31}, x_{32}, x_{34}\}) & \ast S_3^1(\{x_{13}, x_{23}, x_{43}\}), \\
(\text{st}(x_{44}), D_3^2(\{x_{41}, x_{42}, x_{43}\}) & \ast S_3^1(\{x_{14}, x_{24}, x_{34}\}).
\end{align*}
\]

The resulting complex \( X \) is therefore a 12-vertex triangulation of \( S^2 \times S^2 \). So, to confirm the first statement of this theorem, it suffices to show that \( X \) is isomorphic to the complex \((S^2 \times S^2)_{12}\) described in Section 3. Indeed, with the antimorphic pair of icosahedra (and their vertex names) as in Figure 2, we shall show that we actually have \( X = (S^2 \times S^2)_{12}\).

Notice that \( X \) inherits the automorphism group \( A_4 \) from \((S^2 \times S^2)_{16}\), and modulo this group, the following six are basic facets of \( X \):

\[
\begin{align*}
x_{12}x_{14}x_{21}x_{24}x_{31}, \quad x_{12}x_{13}x_{14}x_{21}x_{31}, \quad x_{12}x_{23}x_{31}x_{13}x_{32}, \\
x_{12}x_{31}x_{34}x_{14}x_{24}, \quad x_{24}x_{31}x_{32}x_{12}x_{21}, \quad x_{24}x_{31}x_{32}x_{12}x_{41}.
\end{align*}
\]

Each basic facet is in an \( A_4 \)-orbit of size 12, yielding a total of \( 6 \times 12 = 72 \) facets of \( X \). Since \((S^2 \times S^2)_{12}\) also has 72 facets and since the group \( A_4 \) (acting on subscripts) is a subgroup of the automorphism group \( A_5 \times \mathbb{Z}_2 \) of \((S^2 \times S^2)_{12}\), it suffices to observe that all six basic facets of \( X \) listed above are also facets of \((S^2 \times S^2)_{12}\). Indeed, the first facet \( x_{12}x_{14}x_{21}x_{24}x_{31} \) is in \((S^2 \times S^2)_{12}\) since these five vertices are the neighbors of \( x_{23} \) in \( I_1 \) (and of \( x_{21} \) in \( I_2 \)). In each of the remaining five basic facets of \( X \), the first three vertices constitute a triangle \( \Delta \) of \( I_1 \) with the last two vertices in the corresponding triangle \( \Delta' = \varphi(\Delta) \) of \( I_2 \) (cf. Lemma 9). (For instance, \( \Delta = x_{12}x_{13}x_{14} \) is a triangle of \( I_1 \), with corresponding triangle \( \Delta' = x_{21}x_{31}x_{41} \) of \( I_2 \). Therefore, the second basic facet of \( X \) is a facet of \((S^2 \times S^2)_{12}\).) This shows that \((S^2 \times S^2)_{12} = X \), so that \((S^2 \times S^2)_{12}\) triangulates \( S^2 \times S^2 \).
To compute the full automorphism group of $(S^2 \times S^2)_{12}$, notice that it has exactly 40 triangles of degree 3 (the rest are of degree 5), namely the twenty triangles of $I_1$ and the twenty triangles of $I_2$. Consider the graph whose vertices are these forty triangles, two of them being adjacent if and only if they share an edge. This graph has exactly two connected components, of size 20 each, namely the triangles of $I_1$ and $I_2$. This shows that any automorphism $f$ of $(S^2 \times S^2)_{12}$ either fixes both $I_1$ and $I_2$ or interchanges them. So, \text{Aut}(I_1) = \text{Aut}(I_2) = A_5 \times \mathbb{Z}_2$ is a subgroup of index at most two in the full automorphism group of $(S^2 \times S^2)_{12}$.

Let $f: I_1 \rightarrow I_2$ be any isomorphism. Since $I_1$ and $I_2$ are amorphic, it is immediate that $f$ is also an isomorphism from $I_2$ to $I_1$. Since the five neighbors in $I_1$ of any vertex are also the neighbors in $I_2$ of the antipodal vertex, it is immediate that $f$ maps each of the 12 facets of the first kind in $(S^2 \times S^2)_{12}$ to a facet of the same kind. Also, for any triangle $\Delta$ of $I_1$, the construction of $(S^2 \times S^2)_{12}$ shows that $\text{lk}(\Delta) = S_2^1(\varphi(\Delta))$, and also, for any triangle $\Delta'$ of $I_2$, $\text{lk}(\Delta') = S_2^1(\psi(\Delta'))$. Since $f$ intertwines $\varphi$ and $\psi$ (Lemma 9), we also have $\text{lk}(f(\Delta)) = S_2^1(\psi(f(\Delta))) = S_2^1(f(\varphi(\Delta))) = f(S_2^1(\varphi(\Delta))) = f(\text{lk}(\Delta))$. Similarly, for any triangle $\Delta'$ of $I_2$, $\text{lk}(f(\Delta')) = f(\text{lk}(\Delta'))$. Thus, $f$ also maps all sixty facets of the second type in $(S^2 \times S^2)_{12}$ to facets of the same type. Thus, any isomorphism between $I_1$ and $I_2$ is also an automorphism of $(S^2 \times S^2)_{12}$. Therefore, the full automorphism group $G$ of $(S^2 \times S^2)_{12}$ has $H = A_5 \times \mathbb{Z}_2$ as an index two subgroup. Thus, $G$ is of order 240. Indeed, $G$ consists of the 120 common automorphisms of $I_1$ and $I_2$, and the 120 isomorphisms between $I_1$ and $I_2$. In particular, take $g = (x_{12}x_{21}x_{24}x_{42}x_{14}x_{41}x_{34}x_{13}x_{31}x_{32}x_{23})$, which is an isomorphism between $I_1$ and $I_2$. Note that $g^6$ is the common antipodal map of $I_1$ and $I_2$, hence it is in the center of $G$. Thus, $G/(g^6)$ is the extension of $A_5$ by the involution $\alpha = g \mod g^6$. But $A_5$ has only one non-trivial extension by an involution, namely $S_5$. So, $G$ is an extension of a central involution by $S_5$. It can not be the split extension $S_5 \times \mathbb{Z}_2$ since this has no element of order 12. Therefore, $G$ is the unique non-split extension $2S_5$ of $\mathbb{Z}_2$ by $S_5$. \hfill $\square$

Remark 5. If the link of a vertex $u$ in a triangulated 4-manifold $X$ is $S_2^1(\{x, y, z\}) \ast S_2^3(\{a, b, c\})$ and $xyz$ is not a simplex in $X$ then the GBM $(\text{st}_X(u), D_2^3(\{x, y, z\}) \ast S_2^3(\{a, b, c\}))$ is equivalent to the sequence of the following three bistellar moves: $uab \mapsto xyz$, $ua \mapsto cxyz$, $u \mapsto bxyz$. Thus, from the proof of Theorem 5, $(S^2 \times S^2)_{12}$ can be obtained from $(S^2 \times S^2)_{16}$ by a sequence of bistellar moves only.

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