A proof of the log-concavity conjecture related to the computation of the ergodic capacity of MIMO channels

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Abstract

An upper bound on the ergodic capacity of MIMO channels was introduced recently in [1]. This upper bound amounts to the maximization on the simplex of some multilinear polynomial \( p(\lambda_1, ..., \lambda_n) \) with non-negative coefficients. In general, such maximizations problems are NP-HARD. But if say, the functional \( \log(p) \) is concave on the simplex and can be efficiently evaluated, then the maximization can also be done efficiently. Such log-concavity was conjectured in [1]. We give in this paper self-contained proof of the conjecture, based on the theory of \textbf{H-Stable} polynomials.

1 The conjecture

Let \( B \) be \( M \times M \) matrix. Recall the definition of the \textbf{permanent} :

\[
\text{Per}(B) = \sum_{\sigma \in S_M} \prod_{1 \leq i \leq M} A(i, \sigma(i)).
\]

The following Conjecture was posed in [1].

\textbf{Conjecture 1.1}: Let \( A \) be \( M \times N, M < N \) matrix with non-negative entries. We denote as \( A_S \) a submatrix

\[ A_S = \{ A(i, j) : 1 \leq i \leq m; j \in S \subset \{1, ..., N\} \}. \]

Define the following multi-linear polynomial with non-negative coefficients

\[
F_A(\lambda_1, ..., \lambda_N) = \sum_{|S| = M, S \subset \{1, ..., N\}} \text{Per}(A_S) \prod_{j \in S} \lambda_j. \tag{1}
\]

Then the functional \( \log(F_A) \) is concave on \( R^+_N = \{ (\lambda_1, ..., \lambda_N) : \lambda_j \geq 0, 1 \leq j \leq N \} \).

We present in this paper a proof of Conjecture[1,1]. Actually we prove that the polynomial \( F_A \) is either zero or \textbf{H-Stable}.

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2 H-Stable polynomials

To make this note self-contained, we present in this section proofs of a few necessary results. The reader may consult [5] and [3] for the further reading and references.

We denote as $\text{Hom}_+^{m,n}$ a convex closed cone of homogeneous polynomials with non-negative coefficients of degree $n$ in $m$ variables and with non-negative coefficients; as $\mathbb{R}_m^+$ a convex closed cone of non-negative vectors in $\mathbb{R}^m$ and as $\mathbb{R}_m^{++}$ a convex open cone of positive vectors in $\mathbb{R}^m$.

**Definition 2.1:** A homogeneous polynomial $p \in \text{Hom}_+^{m,n}$ is called $\textbf{H-Stable}$ if

$$|p(z_1, \ldots, z_m)| > 0; \text{Re}(z_i) > 0, 1 \leq i \leq m;$$

is called $\textbf{H-SStable}$ if $|p(z_1, \ldots, z_m)| > 0$ provided that $\text{Re}(z_i) \geq 0, 1 \leq i \leq m$ and $0 < \sum_{1 \leq i \leq m} \text{Re}(z_i)$.

**Example 2.2:** Consider a bivariate homogeneous polynomial $p \in \text{Hom}_+^{2,n}$, $p(z_1, z_2) = (z_2)^n P(z_1 z_2)$, where $P$ is some univariate polynomial. Then $p$ is $\textbf{H-Stable}$ iff the roots of $P$ are non-positive real numbers. This assertion is just a reformulating of the next set equality:

$$\mathcal{C} = \{ \frac{z_1}{z_2} : \text{Re}(z_1), \text{Re}(z_2) > 0 \} = \{ x \in \mathbb{R} : x \leq 0 \}.$$

In other words

$$P(t) = a \prod_{1 \leq i \leq k \leq n} (t + a_i) : a_i \geq 0, 1 \leq i \leq k; a > 0.$$

Which gives the following expression for the bivariate homogeneous polynomial $p$:

$$p(z_1, z_2) = a z_2^{n-k} \prod_{1 \leq i \leq k \leq n} (z_1 + a_i z_2)$$

**Fact 2.3:** Let $p \in \text{Hom}_+^{m,n}$ be $\textbf{H-Stable}$. Then $\log(p)$ is concave on $\mathbb{R}_m^+$.

**Proof:** Consider two vectors $X, Y \in \mathbb{R}_m^+$ such that their sum $X + Y \in \mathbb{R}_m^+$ has all positive coordinates. It is sufficient to prove that the bivariate homogeneous polynomial $q \in \text{Hom}_+^{2,n}$

$$q(t, s) = p(tX + sY),$$

is log-concave on $\mathbb{R}_+^2$. Clearly, the polynomial $q$ is $\textbf{H-Stable}$. Therefore, using Example 2.2, we get that

$$\log(q(t, s)) = \log(a) + (n-k) \log(s) + \sum_{1 \leq i \leq k \leq n} \log(t + a_i s) : a_i \geq 0, 1 \leq i \leq k; a > 0.$$

The log-concavity of $q$ follows now from the concavity of the logarithm on $[0, \infty)$.
Remark 2.4: Since the polynomial $p$ is homogeneous of degree $n$ hence, by the standard argument, the function $p^\frac{1}{n}$ is concave on $R^m_+$ as well.

Fact 2.5: Let $p \in \text{Hom}_+(m, n)$ be H-Stable and $x_i \geq 0, 1 \leq i \leq m$ then the following inequality holds

$$|p(x_1 + iy_1, ..., x_m + iy_m)| \geq p(x_1, ..., x_m) \quad (2)$$

Proof: Consider without loss of generality the positive case $x_i > 0, 1 \leq i \leq m$. Then there exists a positive real number $\mu > 0$ such that $y_i + \mu x_i > 0, 1 \leq i \leq m$. It follows from Example(2.2) that for all complex numbers $z \in C$

$$p(zx_1 + (y_1 + \mu x_1), ..., zx_m + (y_m + \mu x_m) = p(x_1, ..., x_m) \prod_{1 \leq i \leq n} (z + a_i); a_i > 0, 1 \leq i \leq m.$$ 

Thus

$$p(zx_1 + y_1, ..., zx_m + y_m) = p(x_1, ..., x_m) \prod_{1 \leq i \leq n} (z + a_i) - \mu)$$

We get, using the homogeniuty of the polynomial $p$, that

$$p(x_1 + iy_1, ..., x_m + iy_m) = p(x_1, ..., x_m) \prod_{1 \leq j \leq n} (1 + i(a_j - \mu)).$$

As $|\prod_{1 \leq j \leq n}(1 + i(a_j - \mu))| \geq 1$ this proves that the inequality (2) holds.

Corollary 2.6: A nonzero polynomial $p \in \text{Hom}_+(m, n)$ is H-Stable if and only the inequality (2) holds.

Corollary 2.7: Let $p_i \in \text{Hom}_+(m, n)$ be a sequence of H-Stable polynomials and $p = \lim_{i \to \infty} p_i$. Then $p$ is either zero or H-Stable.

Some readers might recognize Corollary (2.7) as a particular case of A. Hurwitz’s theorem on limits of sequences of nowhere zero analytical functions. Our proof below is elementary.

Proof: Suppose that $p$ is not zero. Since $p \in \text{Hom}_+(m, n)$ hence $p(x_1, ..., x_m) > 0$ if $x_j > 0: 1 \leq j \leq m$. As the polynomials $p_i$ are H-Stable therefore $|p_i(Z)| \geq |p_i(Re(Z))| : Re(Z) \in R^m_+$. Taking the limits we get that $|p(Z)| \geq |p(Re(Z))| > 0 : Re(Z) \in R^m_+$, which means that $p$ is H-Stable.

We need the following simple yet crucial result.

Proposition 2.8: Let $p \in \text{Hom}_+(m, n)$ be H-Stable. Then the polynomial $p_{(1)} = \text{Hom}_+(m - 1, n - 1)$,

$$p_{(1)}(x_2, ..., x_m) =: \frac{\partial}{\partial x_1}p(0, x_2, ..., x_m),$$

is either zero or H-Stable.

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Proof: Fix complex numbers $z_i, 2 \leq i \leq m$ and define the following univariate polynomial

$$R(t) = p(t, z_2, ..., z_m).$$

It follows that $R'(0) = p_{(1)}(z_2, ..., z_m)$. We consider two cases.

First case: the polynomial $p \in \text{Hom}_+(m, n)$ is \textbf{H-Stable}. In this case the polynomial $p_{(1)} \in \text{Hom}_+(m-1, n-1)$ is \textbf{H-Stable} as well. Indeed, in this case if the real parts $Re(z_i) \geq 0, 2 \leq i \leq m$ and $\sum_{2 \leq i \leq m} Re(z_i) > 0$ then all the roots $v_1, ..., v_{n-1}$ of the univariate polynomial $R$ have strictly negative real parts:

$$R(t) = h \prod_{2 \leq i \leq n-1} (t - v_i), 0 \neq h \in C.$$

Therefore

$$p_{(1)}(z_2, ..., z_m) = R'(0) = h(-1)^{n-2} \left( \prod_{2 \leq i \leq n-1} v_i \right) \left( \sum_{2 \leq i \leq n-1} (v_i)^{-1} \right) \neq 0$$

as the real part

$$Re\left( \sum_{2 \leq i \leq n-1} (v_i)^{-1} \right) = \sum_{2 \leq i \leq n-1} \frac{Re(v_i)}{|v_i|^2} > 0.$$

Second case: the polynomial $p \in \text{Hom}_+(m, n)$ is \textbf{H-Stable} but not \textbf{H-SStable}. We need to approximate $p$ by a sequence of \textbf{H-SStable} polynomials. Here is one natural approach: let $A$ be any $m \times m$ matrix with positive entries. Define the following polynomials:

$$p_{I+\epsilon A}(Z) =: p((I + \epsilon A)Z), Z \in C^m.$$

Clearly, the for all $\epsilon > 0$ the polynomials $p_{I+\epsilon A} \in \text{Hom}_+(m, n)$ and are \textbf{H-SStable}. It follows that polynomials $\frac{\partial}{\partial x_1} p_{I+\epsilon A}(0, x_2, ..., x_m)$ are \textbf{H-SStable} as well. Note that

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial x_1} p_{I+\epsilon A}(0, z_2, ..., z_m) = p_{(1)}(z_2, ..., z_m).$$

Using Corollary(2.7) we get that the polynomial $p_{(1)}$ is either \textbf{H-Stable} or zero. \[\square\]

3 Proof of the conjecture

Proof: We will need a few auxiliary polynomials:

$$P(x_1, ..., x_M; \lambda_1, ..., \lambda_N) = \prod_{1 \leq j \leq N} (\lambda_j + \sum_{1 \leq i \leq m} A(i, j)x_i). \hspace{1cm} (3)$$

Clearly, the polynomial $P \in \text{Hom}_+(M + N, N)$ is \textbf{H-Stable} if the entries of the matrix $A$ are non-negative. Applying Proposition(2.8) inductively, we get that the following polynomial

$$R(\lambda_1, ..., \lambda_N) = \frac{\partial^m}{\partial x_1 \partial x_2 ... \partial x_m} P(X = 0; \lambda_1, ..., \lambda_N) \hspace{1cm} (4)$$
is either zero or **H-Stable** as well. It is easy to see that

$$R(\lambda_1,\ldots,\lambda_N) = \sum_{|S|=M,S \subset \{1,\ldots,N\}} \text{Per}(A_S) \prod_{j \in S} \lambda_j,$$

(5)

where $\bar{S} = \{1,\ldots,N\} - S$ is the compliment of the set $S$. Now everything is ready for the punch line: the **multilinear homogeneous polynomial**, defined in (1),

$$F_A(\lambda_1,\ldots,\lambda_N) = \left( \prod_{1 \leq i \leq N} \lambda_i \right) R((\lambda_1)^{-1},\ldots,(\lambda_N)^{-1}).$$

(6)

Recall that the real part $Re(z^{-1}) = \frac{Re(z)}{|z|^2}$ for all non-zero complex numbers $z \in C$. Therefore, if the real parts $Re(\lambda_i) > 0$, $1 \leq i \leq n$ then the same is true for the inverses:

$$Re((\lambda_i)^{-1}) > 0, 1 \leq i \leq n.$$

This proves that the polynomial $F_A$ is either zero or **H-Stable**. The log-concavity follows from Fact(2.3). □

### 4 Conclusion

The reader should not be deceived by the simplicity of our proof: very similar arguments are behind the breakthrough results in [5], [4], [6]. The reader is advised to read very nice exposition in [3].

Conjecture (1.1) is actually a very profound question. Had it been asked and properly answered in 1960-70s, then the theory of permanents (and of related things like mixed discriminants and mixed volumes [6]) could have been very different now.

Though the “permanental” part in [1] is fairly standard (the authors essentially rediscovered so called Godsil-Gutman Formula [8]) it is quite amazing how naturally the permanent enters the story. Switching the expectation and the logarithm can be eventful indeed. The log-concavity comes up really handily in the optimization context of [1]. The thing is that maximization on the simplex of $\sum_{1 \leq i \leq j \leq N} b(i,j) x_i x_j$ is **NP-COMPLETE** even when $b(i,j) \in \{0,1\}$, $1 \leq i \leq j \leq N$.

Our proof is yet another example on when the best answer to a question posed in the real numbers domain lies in the complex numbers domain. Yet, we don’t exclude a possibility of a direct “monstrous” proof.

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