Averages of exponential twists of the von Mangoldt function

Xiumin Ren and Wei Zhang

Abstract In this paper, we obtain some improved results for the exponential sum
\[ \sum_{x<n \leq 2x} \Lambda(n) e(\alpha kn^\theta) \] with \( \theta \in (0, 5/12) \), where \( \Lambda(n) \) is the von Mangoldt function. Such exponential sums have relations with the so-called quasi-Riemann hypothesis and were considered by Vinogradov \[8\] and Murty-Srinivas \[5\].

Keywords Exponential sums over primes, zero-density estimates

2000 Mathematics Subject Classification 11L20, 11M26

1. Introduction

In this paper, we are interested in the exponential sum
\[ S(k, x, \theta) := \sum_{x<n \leq 2x} \Lambda(n) e(\alpha kn^\theta), \]
where \( x \geq 2 \) and \( k \in \mathbb{Z}^+ \) are the main parameters, \( \alpha \neq 0 \) and \( 0 < \theta < 1 \) are fixed, \( \Lambda(n) \) is the von Mangoldt function, and \( e(z) = e^{2\pi i z} \).

We call \( S(k, x, \theta) \) Vinogradov’s exponential sum, since it was first considered by I. M. Vinogradov \[8\] in the special case \( \theta = 1/2 \). Actually, he proved in \[8\] that, for \( k \leq x^{1/10} \),
\[ S(k, x, 1/2) \ll k^{1/4} x^{7/8+\varepsilon}, \]
where the implied constant may depend on \( \alpha \) and \( \varepsilon \). Iwaniec and Kowalski (see (13.55) in \[4\]) remarked that the stronger inequality
\[ S(1, x, 1/2) \ll x^{5/6} \log^4 x \]
follows from an application of Vaughan’s identity. For general \( \theta \) and \( k \), Murty and Srinivas \[5\] proved that
\[ S(k, x, \theta) \ll k^{1/8} x^{(7+\theta)/8} \log(xk^3), \]
where the implied constant may depend on \( \alpha \) and \( \theta \). In 2006, Ren \[6\] proved that
\[ S(k, x, \theta) \ll (k^{1/2} x^{(1+\theta)/2} + x^{4/5} + k^{-1/2} x^{1-\theta/2}) \log^A x, \] for arbitrary \( A > 0 \), and for \( \theta \leq 1/2 \) and \( k < x^{1/2-\theta} \),
\[ S(k, x, \theta) \ll (k^{1/10} x^{3/4+\theta/10} + k^{-1/2} x^{1-\theta/2}) \log^{11} x. \] (1.2)

In this paper, we will prove the following Theorem 1.1, which is new for \( \theta \in (0, 5/12) \). In \[3\], Iwaniec, Luo and Sarnak showed that such type exponential sums are connected to the quasi-Riemann Hypothesis (or the existence of zero-free region) for \( L(s, f) \), where \( f \) is any holomorphic cusp form of integral weight for \( SL(2, \mathbb{Z}) \).

---

\(^1\)School of Mathematics, Shandong University, Jinan, Shandong, 250100, China, xmren@sdu.edu.cn
\(^2\)School of Mathematics and Statistics, Henan University Kaifeng, Henan 475004 China, zhangweimath@126.com
**Theorem 1.1.** For $0 < \theta < 5/12$ and $1 \leq k < x^{5/12-\theta-\varepsilon}$, there exists an absolute constant $c_0 > 0$ such that

$$S(k, x, \theta) \ll k^{-1/2} x^{1-\theta/2} \exp(-c_0(\log x)^{1/3-\varepsilon}),$$

where the implied constant may depend on $\alpha$, $\theta$, and $\varepsilon$, which denotes an arbitrarily small positive constant.

Obviously, when $\theta < 5/12$ and $k < x^{5/12-\theta-\varepsilon}$, Theorem 1.1 improve (1.2). Some much sharper estimates can be obtained if one assumes the zero-density hypothesis, i.e,

$$N(\sigma, T) \ll T^{2(1-\sigma)} \log^B T, \quad \sigma \geq 1/2,$$

(1.3)

where $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ in the region $\{\sigma \leq \Re s \leq 1, |t| \leq T\}$ and $B$ is some positive constant. In fact, under (1.3), it is proved in [6] that

$$S(k, x, \theta) \ll (k^{1/2} x^{(1+\theta)/2} + k^{-1/2} x^{1-\theta/2}) \log^{B+2} x,$$

(1.4)

where the implied constant may depend on $\alpha$, $\varepsilon$ and $\theta$.

It is worth pointing out that, comparing with Theorem 1.1, the ranges of $\theta$ and $k$ have been extended in Theorem 1.2.

**2. Proof of Theorem 1.1**

To prove Theorem 1.1, we will borrow the idea in [6] by using the results related to zeros of Riemann zeta function. The following lemma will be used in the proof of Theorem 1.1 and Theorem 1.2.

**Lemma 2.1** (see page 71 of [7]). Let $F(u)$ and $G(u)$ be real functions in $[a, b]$, satisfying $|G(u)| \leq M$ and that $G(u)$ and $1/F'(u)$ are monotone.

1. If $F''(u) \geq m > 0$ or $F''(u) \leq -m < 0$, then
   $$\int_a^b G(u)e(F(u))du \ll \frac{M}{m};$$

2. If $F''(u) \geq r > 0$ or $F''(u) \leq -r < 0$, then
   $$\int_a^b G(u)e(F(u))du \ll \frac{M}{\sqrt{r}}.$$

**Proof of Theorem 1.1** Using partial summation and the explicit formula (see (5.53) in [4]): for $1 \leq T \leq x$,

$$\sum_{n \leq x} \Lambda(n) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T} (\log xT)^2\right),$$
we have
\[ \sum_{x<n<2x} \Lambda(n)e(k\alpha n^\theta) = \int_x^{2x} e(k\alpha u^\theta)d \sum_{n\leq u} \Lambda(n) \]
\[ = \int_x^{2x} e(k\alpha u^\theta)du - \sum_{|\gamma| \leq T} \int_x^{2x} u^{\theta-1}e(k\alpha u^\theta)du \]
\[ + O \left( (1 + k|\alpha|x^\theta) \frac{x \log^2 x}{T} \right). \tag{2.1} \]

Here \( \rho = \beta + i\gamma \) denotes a zero of \( \zeta(s) \) with \( 0 < \beta < 1, \, |\gamma| \leq T \). Set \( T = T_0 = x \), then the error-term is \( O(((1 + k|\alpha|x^\theta) \log^2 x) = O(kx^\theta \log^2 x) \). Moreover, we have
\[ \int_x^{2x} e(k\alpha u^\theta)du = \frac{1}{\theta} \int_{x^\theta}^{(2x)^\theta} u^{1/\theta-1}e(k\alpha u^\theta)du \ll_{\alpha, \theta} k^{-1}x^{1-\theta}. \tag{2.2} \]
Making the change of variable \( u^\theta = v \), we get
\[ \int_x^{2x} u^{\theta-1}e(k\alpha u^\theta)du = \frac{1}{\theta} \int_{x^\theta}^{(2x)^\theta} v^{\frac{\theta}{\theta-1}}e(f(v))dv, \]
where
\[ f(v) = k\alpha v + \frac{\gamma}{2\pi \theta} \log v. \]
Trivially one has
\[ \int_x^{2x} u^{\theta-1}e(k\alpha u^\theta)du \ll x^\beta. \tag{2.3} \]

On the other hand we have
\[ |f'(v)| = \left| k\alpha + \frac{\gamma}{2\pi \theta v} \right| \geq \min_{v \in [x^\theta, (2x)^\theta]} \left| \gamma + 2\theta \pi k\alpha v \right|, \]
\[ |f''(v)| = \frac{|\gamma|}{2\pi \theta v^2}. \]
By Lemma 2.1 and (2.3) we get
\[ \int_{x^\theta}^{(2x)^\theta} v^{\frac{\theta}{\theta-1}}e(f(v))dv \ll \begin{cases} \frac{x^\beta}{\sqrt{1+\theta k|\alpha|x^\theta}} & \text{for } |\gamma| \leq 4(1 + \theta \pi k|\alpha|(2x)^\theta), \\ \frac{x^\beta}{1+|\gamma|} & \text{for } 4(1 + \theta \pi k|\alpha|(2x)^\theta) \leq |\gamma| \leq T_0. \end{cases} \]
Therefore
\[ \sum_{|\gamma| \leq T} \int_x^{2x} u^{\theta-1}e(k\alpha u^\theta)du \ll \frac{1}{\sqrt{1 + \theta k|\alpha|x^\theta}} \sum_{|\gamma| \leq 4(1 + \theta \pi k|\alpha|(2x)^\theta)} x^\beta + \sum_{4(1 + \theta \pi k|\alpha|(2x)^\theta) \leq |\gamma| \leq T_0} x^\beta \frac{1}{1 + |\gamma|}. \]
Assume that, for some positive constant $C$,
\[ N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)} \log^{C} T. \]
Then by the Riemann-Von Mangoldt formula, for $2 \leq U \leq T_0$ we have
\[
\sum_{1 \leq \gamma \leq U} x^{\beta} = - \int_{0}^{1} x^{\sigma} dN(\sigma, U) \ll x^{1/2} U \log U + (\log U)^{C} \log x \sup_{1/2 \leq \sigma \leq \sigma_0} U^{A(\sigma)(1-\sigma)} x^{\sigma},
\]
where
\[ \sigma_0 = 1 - c_0 (\log T)^{-2/3} (\log \log T)^{-1/3} \]
with $c_0$ an absolute positive constant. Here we have used the well known zero-free region results (for example, see [4, 7]) which states that $\zeta(s) \neq 0$ for $\sigma > \sigma_0$.

Let $x$ be sufficiently large such that $\theta \pi k|\alpha|(2x)^{\theta} \gg 1$, then we have
\[
\frac{1}{\sqrt{1 + \theta k|\alpha| x^{\theta}}} \sum_{|\gamma| \leq 4(1 + \theta \pi k|\alpha| (2x)^{\theta})} x^{\beta} \ll \left( \log x \right)^{C+1} \left( k^{1/2} x^{(1+\theta)/2} + \max_{1/2 \leq \sigma \leq \sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{\sigma+\theta A(\sigma)(1-\sigma)-\theta/2} \right),
\]
and
\[
\sum_{4(1 + \theta \pi k|\alpha| (2x)^{\theta}) \leq |\gamma| \leq T_0} \frac{x^{\beta}}{1 + |\gamma|} \ll \left( \log x \right)^{C+2} \left( x^{1/2} + \max_{1/2 \leq \sigma \leq \sigma_0} k^{A(\sigma)(1-\sigma)-1} x^{\sigma+\theta A(\sigma)(1-\sigma)-\theta} \right).
\]
Writing
\[ g(\sigma) = \sigma + \theta A(\sigma)(1 - \sigma) - \frac{\theta}{2}, \]
and collecting the above estimates we get
\[
\sum_{1 \leq \gamma \leq T} \int_{x}^{2x} u^{\gamma-1} e(k\alpha u^{\theta}) du \ll \left( \log x \right)^{C+2} \left( k^{1/2} x^{(1+\theta)/2} + \max_{1/2 \leq \sigma \leq \sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{g(\sigma)} \right).
\]
By the well known result of Ingham [2] and Huxley [1], we can choose $A(\sigma) = 12/5$. Thus we have
\[
\max_{1/2 \leq \sigma \leq \sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{g(\sigma)} \ll \left( \log x \right)^{C_1} \sup_{1/2 \leq \sigma \leq \sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{\sigma+12\theta(1-\sigma)/5-\theta/2} \ll k^{-1/2} x^{1-\theta/2} \left( \log x \right)^{C_1} \sup_{1/2 \leq \sigma \leq \sigma_0} \left( k^{12/5} x^{12\theta/5-1} \right)^{1-\sigma}.
\]
Thus for $\theta < 5/12$ and $k < x^{5/12-\theta-\varepsilon}$, we get
\[
\max_{1/2 \leq \sigma \leq \sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{\theta(\sigma)} 
\ll k^{-1/2} x^{1-\theta/2}(\log x)^C \sup_{1/2 \leq \sigma \leq \sigma_0} x^{-c_0(\log x)^{-2/3}(\log \log x)^{-1/3}} 
\ll k^{-1/2} x^{1-\theta/2} \exp(-c_0(\log x)^{1/3}(\log x \log x)^{-1/3}) 
\ll k^{-1/2} x^{1-\theta/2} \exp(-c_0(\log x)^{1/3-\varepsilon}).
\]
This together with (2.1) and (2.2) shows that, for $\theta \in (0, 5/12)$ and $1 \leq k < x^{5/12-\theta-\varepsilon}$,
\[
\sum_{x < n \leq 2x} \Lambda(n)e(\alpha n^\theta)
\ll k^{1/2} x^{(1+\theta)/2}(\log x)^C + k^{-1/2} x^{1-\theta/2} \exp(-c_0(\log x)^{1/3-\varepsilon}) + k^{-1} x^{1-\theta} + k x^\theta 
\ll k^{-1/2} x^{1-\theta/2} \exp(-c_0(\log x)^{1/3-\varepsilon}).
\]
This finishes the proof of Theorem 1.1. □

Acknowledgement This work was supported by National Natural Science Foundation of China (Grant No. 11871307).

References

[1] M.N. Huxley, On the difference between consecutive primes. Invent. Math. 15 (1972), 164-170.
[2] A.E. Ingham, On the estimation of $N(\sigma, T)$. Q. J. Math. 15 (1940), 291-292.
[3] H. Iwaniec, W.Z. Luo and P. Sarnak, Low lying zeros of families of L-functions, Extrait Publ. Math. 91 (2000) 55-131.
[4] H. Iwaniec and E. Kowalski, Analytic Number Theory, Am. Math. Soc. Colloquium Publ. vol.53. Am. Math. Soc., Providence, 2004.
[5] M.R. Murty and K. Srinivas, On the uniform distribution of certain sequences. Ramanujan J. 7 (2003), 185-192.
[6] X.M. Ren, Vinogradov’s exponential sum over primes, Acta Arith. 124(2006) 269-285.
[7] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd. edn University Press, Oxford 1986.
[8] I.M. Vinogradov, Special Variants of the Method of Trigonometric Sums, Nauka, Moscow, 1976 (in Russian); English transl.: I. M. Vinogradov, Selected Works, Springer, Berlin, 1985.

Xiumin Ren, School of Mathematics, Shandong University, Jinan, Shandong 250100, China
Email address: xmren@sdu.edu.cn

Wei Zhang, School of Mathematics and Statistics, Henan University, Kaifeng 475004, Henan, China
Email address: zhangweimath@126.com