DILATION BOOTSTRAP

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ABSTRACT. We propose a methodology for constructing confidence regions with partially identified models of general form. The region is obtained by inverting a test of internal consistency of the econometric structure. We develop a dilation bootstrap methodology to deal with sampling uncertainty without reference to the hypothesized economic structure. It requires bootstrapping the quantile process for univariate data and a novel generalization of the latter to higher dimensions. Once the dilation is chosen to control the confidence level, the unknown true distribution of the observed data can be replaced by the known empirical distribution and confidence regions can then be obtained as in Galichon and Henry, 2008 and Beresteanu et al., 2008.

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Introduction

In several rapidly expanding areas of economic research, the identification problem is steadily becoming more acute. In policy and program evaluation Manski, 1990 and more general contexts with censored or missing data Shaikh and Vytlacil, 2010, Magnac and Maurin, 2008 and measurement error Chen et al., 2005, ad hoc imputation rules lead to fragile inference. In demand estimation based on revealed preference Blundell et al., 2008 the data is generically insufficient for identification. In the analysis of social interactions Brock and Durlauf, 2001, Manski, 2004, complex strategies to reduce the large dimensionality of the correlation structure are needed. In the estimation of models with complex strategic interactions and multiple equilibria Bjorn and Vuong, 1985, Tamer, 2003, assumptions on equilibrium selection mechanisms may not be available or acceptable.

More generally, in all areas of investigation with structural data insufficiencies or incompletely specified economic mechanisms, the hypothesized structure fails to identify a unique possible data generating mechanism for the data that is actually observed. In such cases, many traditional estimation and testing techniques become inapplicable and a framework for inference in incomplete models is developing, with an initial focus on estimation of the set of structural parameters compatible with true data distribution (hereafter identified set). A question of particular relevance in applied work is how to construct valid confidence regions for the identified set.
Formal methodological proposals abound since the seminal work of Chernozhukov et al., 2007, but computational efficiency is still a major concern.

In the present work, we propose a methodology that clearly distinguishes how to deal with sampling uncertainty on the one hand, and model uncertainty on the other, so that unlike previous methodological proposals, search in the parameter space is conducted only once, thereby greatly reducing the computational burden. The key to this separation is to deal with sampling variability without any reference to the hypothesized structure, using a methodology we call the *dilation method*. This consists in dilating each point in the space of observable variables in such a way that the empirical probability (which is known) of a dilated set dominates the true probability (which is unknown) of the original set (before dilation). The unknown true probability (i.e. the true data generating mechanism) is then removed from the analysis, and we can proceed as if the problem were purely deterministic, hence apply the methods proposed in Galichon and Henry, 2008 and Beresteanu et al., 2008.

To construct confidence regions of level $1 - \alpha$ for the identified set, such a dilation $y \mapsto J(y)$ (where $\mapsto$ denotes a one-to-many map) must satisfy $\tilde{Y}^* \in J(\tilde{Y})$ a.s. for some pair of random vectors $(\tilde{Y}^*, \tilde{Y})$, with probability $1 - \alpha$, where $\tilde{Y}$ is drawn from the true distribution of observable variables and $\tilde{Y}^*$ is drawn from the empirical distribution relative to the observed sample. We propose a dilation bootstrap procedure to construct $J$, in which bootstrap realizations $Y^b_j, j = 1, \ldots, n$ are matched one-to-one with the original sample points $Y_j, j = 1, \ldots, n$ so as to minimize
\[ \eta_n^b = \max_{j=1,\ldots,n} \| Y_j^b - Y_{\sigma(j)} \|, \] where the permutation \( \sigma \) defines the matching. The \( \alpha \) quantile of the distribution of \( \eta_n^b \) then defines the radius of the dilation.

When the observable \( Y \) is a random variable, the dilation bootstrap relies on bootstrapping the quantile process, as proposed by Doss and Gill, 1992. However, bootstrapping the quantile process relies on order statistics and had no higher dimensional generalization to date. This is now provided by the dilation bootstrap, which removes the constraint on dimension through the appeal to optimal matching. Although the problem of finding minimum cost matchings (called the assignment or marriage problem) is very familiar to economists, as far as we know, its application within an inference procedure is unprecedented.

The rest of the paper is organized as follows. The next section describes the econometric framework and introduces the Composition Theorem and the dilation method the latter justifies. Section 2.1 discusses the application of the Composition Theorem to constructing confidence regions for partially identified parameters. Section 2.3 presents the bootstrap feasible dilation and its theoretical underpinnings. Section 3 presents simulation evidence on the performance of the dilation bootstrap in comparison with alternative methods. Section 4 explains how the method extends to higher dimensions and discrete choice and the last section concludes.
1. Dilation method and Composition Theorem

We consider the problem of inference on the structural parameters of an economic model, when the latter are (possibly) only partially identified. The economic structure is defined as in Jovanovic, 1989, which generalizes Koopmans and Reiersøl, 1950. Variables under consideration are divided into two groups. Latent variables $U$ capture unobserved heterogeneity in the model. They are typically not observed by the analyst, but some of their components may be observed by the economic actors. Observable variables $Y$ include outcome variables and other observable heterogeneity. They are observed by the analyst and the economic actors. We call observable distribution $P$ the true probability distribution generating the observable variables, and denote by $\nu$ the probability distribution that generated the latent variables $U$.

The econometric structure under consideration is given by a binary relation between observable and latent variables, i.e. a subset of $\mathcal{Y} \times \mathcal{U}$, which can be written without loss of generality as a correspondence from $\mathcal{U}$ to $\mathcal{Y}$.

**Assumption 1** (Econometric specification). Observable variables $Y$, with realizations $y \in \mathcal{Y} \subseteq \mathbb{R}^{d_y}$ and latent variables $U$, with realizations $u \in \mathcal{U} \subseteq \mathbb{R}^{d_u}$, are defined on a common probability space $(\Omega, \mathcal{F}, P)$ and satisfy the relation: $Y \in G(U) \subseteq \mathcal{Y}$ almost surely.

**Example 1** (Revealed Preferences). This approach is particularly well suited to revealed preference analysis. Suppose $X$ is the vector of observed choices made by an
agent, possibly over several periods. Let $Z$ be a vector of observable variables defining the environment in which the agent made their choices. Call $Y = (X, Z)$ the vector of all observable variables. Suppose the agent maximized a utility $u(X, Z, U|\theta)$ under constraints $g(X, Z, U|\theta) \leq 0$ (budget constraints, etc...), where $\theta$ is a vector of structural parameters (including elasticities, risk aversion, etc...) and $U$ a random vector describing unobserved heterogeneity. Call $D(U, X|\theta)$ the demand correspondence, i.e. the set of utility maximizing choices. Then we can define $G(U|\theta)$ by $Y \in G(U|\theta)$ if and only if $X \in D(U, X|\theta)$, and $G(U|\theta)$ exhausts all the information embodied in the utility maximization model.

**Example 2** (Games). Another family of examples of our framework arises with parametric games. Let $N$ players with observable characteristics $X = (X_1, \ldots, X_N)$ and unobservable characteristics $U = (U_1, \ldots, U_N)$ have strategies $Z = (Z_1, \ldots, Z_N)$ and payoffs parameterized by $X, U, Z$ and $\theta$. For a given choice of equilibrium concept in pure strategies, call $C(X, U, \theta)$ the equilibrium correspondence, i.e. the set of pure strategy equilibrium profiles. Then the empirical content of the game is characterized by $Z \in C(X, U, \theta)$, which can be equivalently rewritten $Y \in G(U; \theta)$ with $Y = (Z, X)$.

We assume a parametric structure for the unobserved heterogeneity and the model linking unobserved heterogeneity variables to observable ones.

**Assumption 2** (Correspondence). The correspondence $G : U \rightrightarrows \mathcal{Y}$ is known by the analyst up to a finite dimensional vector of parameters $\theta \in \Theta \subseteq \mathbb{R}^{d_\theta}$. It is denoted
For all $\theta \in \Theta$, $G(\cdot; \theta)$ is measurable (i.e. the set $\{u : G(u; \theta) \cap A \neq \emptyset\}$ is measurable for each open subset $A$ of $\mathcal{Y}$) and has non empty and closed values.

Note that the measurability and closed values assumptions are very mild conditions. The assumption that the correspondence is non-empty, however, may be restrictive. In the revealed preferences example, we require that the demand correspondence be non empty. In the games example, we require existence of equilibrium.

**Assumption 3 (Latent variables).** The distribution $\nu$ of the unobservable variables $U$ is assumed to belong to a parametric family $\nu(\cdot|\theta)$, $\theta \in \Theta$. The same notation is used for the parameters of $\nu$ and $G$ to highlight the fact that they may have components in common.

The pair of random vectors $(Y, U)$ involved in the model is generated by a probability distribution, that we denote $\pi$. Since the vector $U$ is unobservable, the probability distribution $\pi$ is not directly identifiable from the data. However, the econometric model imposes restrictions on $\pi$. The distribution of its component $Y$ is the observable distribution $P$. The distribution of its component $U$ is the hypothesized probability distribution $\nu(\cdot|\theta)$. Finally, the joint distribution is further restricted by the fact that it gives mass 0 to the event that the relation $Y \in G(U|\theta)$ is violated. For any given value of the structural parameter vector $\theta$, a joint distribution satisfying all these restrictions may or may not exist. If it does, it is generally non unique. The
identified set \( \Theta_I \) is the collection of values of the structural parameter vector \( \theta \) for which such a joint probability distribution does indeed exist.

- If \( \Theta_I = \emptyset \), the model is rejected.
- If \( \Theta_I \) is a singleton, the parameter vector \( \theta \) is point identified.
- Otherwise, the parameter \( \theta \) is set identified.

The set \( \Theta_I \), first formalized in this way in Galichon and Henry, 2006 is sometimes called “sharp identification region” to emphasize the fact that it exhausts all the information on the parameter available in the model. No value \( \theta \in \Theta_I \) could be rejected on the basis of the knowledge of the model and the observable distribution \( P \) only. Take a parameter value \( \theta \in \Theta \). It belongs to the identified set \( \Theta_I \) if and only if there exists a joint distribution satisfying the required restrictions, in other words, if and only if there exists a “version” of \( U \), i.e. a random vector \( \tilde{U} \) with the same distribution as \( U \), namely \( \nu(\cdot|\theta) \), such that \( Y \in G(\tilde{U}|\theta) \) with probability 1. Hence, denoting by \( X \sim \mu \) the statement “the random vector \( X \) has probability distribution \( \mu \),” we can characterize the identified set in the following way, which we take as our formal definition.

**Definition 1** (Identified set).

\[
\Theta_I = \left\{ \theta \in \Theta \mid \exists \tilde{Y} \sim P, \tilde{U} \sim \nu(\cdot|\theta) : P(\tilde{Y} \notin G(\tilde{U}|\theta)) = 0 \right\}.
\]
Our inference method on the identified set will be based on a general way of combining sources of uncertainty (sampling uncertainty or data incompleteness) by composition of correspondences. Suppose the probability measure $Q$ on $Y$ is the known distribution of a random vector $Z$ and that it is related to the true unknown distribution $P$ of the observed variables $Y$ by the following relation:

**Assumption 4** (Dilation). There exists a correspondence $J : Y \Rightarrow Y$ such that $P(\tilde{Z} \notin J(\tilde{Y})) \leq \beta$ for some $\tilde{Z} \sim Q$, $\tilde{Y} \sim P$ and $0 \leq \beta < 1$.

Assumption 4 characterizes the additional level of indeterminacy the analyst faces. The structural model is incomplete in the sense that the relation between unobserved heterogeneity $U$ and outcomes $Y$ is a many-to-many mapping. In addition, due to observability issues or sampling uncertainty, the distribution of outcomes $P$ is unknown and the relation between true outcome $Y$ and a variable $Z$ that we can simulate is also many-to-many.

**Example 3** (Measurement error). Suppose true outcome $Y$ is mismeasured as $Z = Y + \epsilon$ and nothing is known about measurement error $\epsilon$ except that it is small, i.e. $\|\epsilon\| \leq \eta$ for some $\eta > 0$, with a degree of confidence $1 - \beta$. In that case, Assumption 4 holds with the correspondence $J$ defined by $J(y) = B(y, \eta)$ for all $y \in Y$, where $B(y, \eta)$ is the closed ball centered at $y$ with radius $\eta$. 
Example 4 (Censored outcomes). Suppose the true outcome $Y$ is reported with censoring as $Z = J(Y)$, where $J(y)$ returns the minimum of $y$ and an upper bound $B > 0$. Assumption 4 is satisfied with $\beta = 0$.

The following theorem shows how the two levels of uncertainty can be combined without loss of information.\textsuperscript{1}

**Theorem 1** (Composition Theorem). Under assumptions 1 to 4, there exist $\tilde{Z} \sim Q$ and $\tilde{U} \sim \nu$ such that $\mathbb{P}(\tilde{Z} \notin J \circ G(\tilde{U}|\theta)) \leq \beta$.

Theorem 1 implies that when the distribution $P$ of outcomes is unknown, the infeasible identified set $\Theta_I$ can be replaced by a feasible identified set

$$\tilde{\Theta}_I = \left\{ \theta \in \Theta \mid \exists \tilde{Z} \sim Q, \tilde{U} \sim \nu(\cdot|\theta) : \mathbb{P}(\tilde{Z} \notin J \circ G(\tilde{U}|\theta)) \leq \beta \right\}.$$

**Proof of Theorem 1.** Under Assumptions 1 and 3, there is a pair $(Y,U)$ such that $Y \sim P$ and $U \sim \nu(\cdot|\theta)$ and $Y \in G(U|\theta)$ almost surely. Equivalently, the minimum over all pairs $(\tilde{Y}, \tilde{U})$, with $\tilde{Y} \sim P$ and $\tilde{U} \sim \nu(\cdot|\theta)$, of the quantity $\mathbb{E}(1\{\tilde{Y} \notin G(\tilde{U}|\theta))\}$ is zero. By proposition 1 of Galichon and Henry, 2009 (hereafter denoted P1), the latter is equivalent to

$$\sup (P(A) - \nu(\{ u \in U : G(u|\theta) \cap A \neq \varnothing \}|\theta)) = 0, \quad (1.1)$$

\textsuperscript{1}The current proof of Theorem 1, suggested by Alexei Onatski, is shorter and simpler than our original proof in previous versions of the paper. We are responsible for any remaining errors.
where the sup is over all Borel subsets $A$ of $Y$. Similarly, by Assumption 4, the minimum over all pairs $(\tilde{Z}, \tilde{Y})$, with $\tilde{Z} \sim Q$ and $\tilde{Y} \sim P$, of the quantity $\mathbb{E}(1\{\tilde{Z} \notin J(\tilde{Y})\})$ is smaller than or equal to $\beta$. By P1 the latter is equivalent to

$$\sup (Q(A) - P(\{y \in Y : J(y) \cap A \neq \emptyset\} | \theta)) \leq \beta. \quad (1.2)$$

Denote $J^{-1}(A) = \{y \in Y : J(y) \cap A \neq \emptyset\}$. By (1.1), we have $P(J^{-1}(A)) \leq \nu(\{u \in \mathcal{U} : G(u|\theta) \cap J^{-1}(A) \neq \emptyset\} | \theta)$ for all Borel subsets $A$ of $Y$. Hence, (1.2) yields

$$\sup (Q(A) - \nu(\{u \in \mathcal{U} : G(u|\theta) \cap J^{-1}(A) \neq \emptyset\} | \theta)) \leq \beta,$$

Hence

$$\sup (Q(A) - \nu(\{u \in \mathcal{U} : J \circ G(u|\theta) \cap A \neq \emptyset\} | \theta)) \leq \beta, \quad (1.3)$$

since $G(u|\theta) \cap J^{-1}(A) \neq \emptyset$ and $J \circ G(u|\theta) \cap A \neq \emptyset$ are equivalent. Finally, by a third application of P1, (1.3) is equivalent to $\beta$ weakly dominating the minimum of the quantity $\mathbb{E}(1\{\tilde{Z} \notin J \circ G(\tilde{U}|\theta)\})$ over all pairs $(\tilde{Z}, \tilde{U})$ with $\tilde{Z} \sim Q$ and $\tilde{U} \sim \nu(\cdot | \theta)$ and the result follows.

To illustrate the composition theorem, consider a special case of the revealed preference example 1 combined with measurement error, as in example 3. Suppose we observe the share $Y$ of risky assets in the portfolio of investors, who are assumed to maximize the expectation of a CARA utility function $u(Y, A; U) = \exp(-U[(1-Y) + YA])$, hence they are assumed to maximize $Y \mathbb{E}(A) - UY^2 \text{var}(A)/2$, where $\mathbb{E}(A)$ is the perceived mean of the risky asset $A$ and $\text{var}(A)$ its perceived variance. We further suppose
investors differ by their risk aversion $U$, for which the analyst hypothesizes an exponential distribution $(F_U(u; \theta) = \mathbb{P}(U \leq u; \theta) = 1 - e^{-\theta u})$ and by their perception of the riskiness of the asset, and all the analyst knows is a pair of bounds $(\Lambda, \bar{\Lambda})$ such that $\mathbb{E}(A)/\text{var}(A) \in [\Lambda, \bar{\Lambda}]$. The investor’s maximization yields $Y = \mathbb{E}(A)/U \text{var}(A)$, so that the model can be summarized by $Y \in G(U) = [\Lambda/U, \bar{\Lambda}/U]$. Values $\underline{\Lambda} = 50\%$ and $\bar{\Lambda} = 200\%$ can be calibrated according to Weitzman, 2007. The true distribution of income $Y$ is unknown, but the true cumulative distribution of a mismeasured version $Z = Y + \epsilon$, with $\|\epsilon\| \leq \eta$ a.s., is $F_Z(y) = \mathbb{P}(Z \leq z) = \exp(-1/z)$. By Theorem 1, the identified set $\tilde{\Theta}_I$ can be derived from the composed correspondence $J \circ G : u \mapsto J \circ G(u) = [\Lambda/u - \eta, \bar{\Lambda}/u + \eta]$, where $J : y \mapsto J(y) = B(y, \eta)$ is a dilation satisfying Assumption 4. The cumulative distribution of risk aversion satisfies $1 - e^{-\theta u} = \mathbb{P}(U \leq u) \in [\mathbb{P}(\bar{\Lambda}/u + \eta \leq Z), \mathbb{P}(\Lambda/u - \eta \leq Z)] = [1 - e^{-(\bar{\Lambda}/u + \eta)^{-1}}, 1 - e^{-(\Lambda/u - \eta)^{-1}}]$. Hence, for all $u > 0$, $(\bar{\Lambda} + \eta u)^{-1} \leq \theta \leq (\Lambda - \eta u)^{-1}$. Therefore, the identified set can be derived as $\tilde{\Theta}_I = [1/\bar{\Lambda}, 1/\Lambda]$.

2. Dilation method and sampling uncertainty

2.1. Confidence regions. The main application of the Composition Theorem 1 that we consider here is the construction of valid confidence regions for partially identified models, based on a sample of realizations of the observable variables.
Assumption 5 (Sampling). Let \((Y_1, \ldots, Y_n)\) be a sample of independent and identically distributed random vectors with distribution \(P\) and let \(P_n = \sum_{j=1}^{n} \delta_{Y_j}\) be the empirical distribution associated with the sample.

We propose a new method to construct a confidence region for the identified set \(\Theta_I\) of definition 1.

Definition 2 (Confidence region). A valid \(\alpha\)-confidence region for the identified set \(\Theta_I\) is a sequence of random regions \(\Theta_n^\alpha\) satisfying

\[
\liminf_n \mathbb{P}(\Theta_I \subseteq \Theta_n^\alpha) \geq 1 - \alpha.
\]

As noted in Imbens and Manski, 2004, this is not the only way to define confidence regions in a partially identified setting, as one might also consider coverage (point wise or uniform) of each value within the identified set. Here we concentrate on a situation where one cannot assume that any value within the identified set can be construed as the true value, so that the whole set is the object of interest. Moreover, a confidence region for the identified set is also a uniform confidence region for each of its elements. The construction of the confidence region is based on a new nonparametric way of controlling sampling uncertainty and its validity relies on a corollary to the Composition Theorem (Theorem 1 of Section 1). We construct sample based sets \(J_n^\alpha\), where \(\alpha \in (0, 1)\) is the desired confidence level, to account for the discrepancy between the empirical distribution \(P_n\) associated with the sample and the true observable distribution \(P\). We thereby obtain an analogue of Assumption 4:
Assumption 6 (Sample dilation). With probability $1 - \alpha_n$ such that $\limsup_n \alpha_n \leq \alpha$, conditionally on the sample $(Y_1, \ldots, Y_n)$, the sequence of correspondences $J_n^\alpha$ satisfies $Y \in J_n^\alpha(\tilde{Y}^*)$ almost surely for some $\tilde{Y} \sim P$, $\tilde{Y}^* \sim P_n$.

Heuristically, the region $J_n^\alpha$ satisfying Assumption 6 ensures that with suitable confidence, the realizations of the empirical distribution are caught by the enlarged realizations of the true distribution $J_n^\alpha(\tilde{Y})$. Once the dilation $J_n^\alpha$ is obtained, the Composition Theorem can be applied to prove the following:

**Theorem 2.** Under assumptions 1, 2, 3, 5 and 6, then $\Theta_n^\alpha := \{ \theta \in \Theta \mid \exists \tilde{Y}^* \sim P_n, \tilde{U} \sim \nu(\cdot|\theta) : P(\tilde{Y}^* \notin J_n^\alpha(\tilde{U}|\theta)) = 0 \}$ is a valid $\alpha$-confidence region for the identified set $\Theta_I$.

The dilation $J_n^\alpha$ is chosen to control the confidence level: indeed, by Proposition 1 of Galichon and Henry, 2009 (called P1 in the proof of Theorem 1), the statement $\exists \tilde{Y}^* \sim P_n, \tilde{U} \sim \nu(\cdot|\theta) : P(\tilde{Y}^* \notin J_n^\alpha(\tilde{U}|\theta)) = 0$ is equivalent to $P(A) \leq P_n(J_n^\alpha(A))$, for all Borel subset $A$ of $\mathcal{Y}$. Hence, the unknown distribution $P$ of an event $A$ is dominated by the empirical distribution of the dilation $J_n^\alpha(A)$ of the event $A$. As both $P_n$ and $\nu(\cdot|\theta)$ are known, the construction of $\Theta_n^\alpha$ is feasible and efficient methods to compute it were proposed in Galichon and Henry, 2008 and Beresteanu et al., 2008.

2.2. Oracle dilation. We now turn to the question of how to construct the dilation $J_n^\alpha$ that satisfies Assumption 6. When $Y$ is a random variable, such dilation will be obtained from uniform confidence bands for the quantile process.
Definition 3 (Quantile process). Let $F$ be the cumulative distribution of $Y$. Let $Q(t)$, $t \in [0, 1]$ be the quantile function of $Y$, defined by $Q(t) = \inf\{y \in [0, 1] : F(y) \geq t\}$. Call $Q_n$ the empirical quantile relative to the sample $(X_1, \ldots, X_n)$. It is defined by $Q_n(t) = Y(j)$ for $j - 1 < nt \leq j$ for each $j$, with $Y(j)$ denoting the $j$th order statistic.

The quantile process is defined as $q_n(t) := \sqrt{n}(Q_n(t) - Q(t))$.

The idea of the construction of dilations satisfying Assumption 6 is based on the quantile transformation. Indeed, letting $Z$ be a uniform random variable on $[0, 1]$ and defining $\tilde{Y} = Q(Z)$ and $\tilde{Y}^* = Q_n(Z)$, we have a pair of random variables $\tilde{Y}$ and $\tilde{Y}^*$ with respective probability distributions $P$ and $P_n$. Suppose a uniform confidence band is available for the quantile function of the form $\mathbb{P}(\eta_n := \sup_{0 \leq t \leq 1}|q_n(t)| \leq \tilde{c}_n(\alpha)) = 1 - \alpha_n$. Then, with probability $1 - \alpha_n$, we have $|\tilde{Y}^* - \tilde{Y}| = |Q_n(Z) - Q(Z)| \leq \tilde{c}_n(\alpha)/\sqrt{n}$ almost surely. Hence, the dilation $J_n^\alpha$ defined for all $y$ by $J_n^\alpha(y) = B(y, \tilde{c}_n(\alpha)/\sqrt{n})$ satisfies Assumption 6. Moreover, the choice of dilation $J_n^\alpha(y) = B(y, \tilde{c}_n(\alpha)/\sqrt{n})$ is optimal in the sense that, under the regularity conditions of Assumption 7, $|Q_n(Z) - Q(Z)|$ achieves the minimum of $|\tilde{Y}^* - \tilde{Y}|$ when $\tilde{Y}^*$ (respectively $\tilde{Y}$) ranges over the set of random variables with distribution $P_n$ (respectively $P$). Note that smaller dilations are desirable, as they maximize informativeness of the resulting confidence region.

The following conditions guarantee the existence of such uniform confidence bands for the quantile process.
Assumption 7 (Uniform quantile bands). The sample \( \{Y_1, \ldots, Y_n\} \) is an iid sample of random variables with cumulative distribution function \( F \) satisfying:

(i) \( F(y) \) is twice continuously differentiable on its support \( (a, b) \).

(ii) \( F' = f > 0 \) on \( (a, b) \).

(iii) For some \( \gamma > 0 \), \( \sup_{y \in (a, b)} F(y)(1 - F(y))|f'(y)|/f(y)^2 \leq \gamma \).

(iv) \( \limsup_{y \downarrow a} f(y) < \infty \) and \( \limsup_{y \uparrow b} f(y) < \infty \).

(v) \( f \) is nondecreasing (resp. nonincreasing) on an interval to the right of \( a \) (resp. to the left of \( b \)).

A distribution function \( F \) satisfying Assumption 7 is called tail monotonic with index \( \gamma \) by Parzen, 1979. To indicate the mildness of Assumption 7, Parzen, 1979 gives the following example where it fails: \( 1 - F(y) = \exp(-y - C \sin y) \) with \( 0.5 < C < 1 \). As shown below, under Assumption 7, asymptotic results on the empirical quantile process allow us to derive a dilation \( J_n^\alpha \) that satisfies Assumption 6 for all \( \alpha \in (0, 1) \). Define \( c(\alpha) \) implicitly by \( \mathbb{P}(\sup_{0 \leq t \leq 1} |B(t)| \leq c(\alpha)) = 1 - \alpha \), where \( B(t) \) is a Gaussian process called a Brownian bridge. For any \( \alpha \in (0, 1) \), we then have the following result.

Proposition 1 (Oracle dilation). Under assumptions 5 and 7, the dilation \( J_n^\alpha \) defined for each \( y \) by \( J_n^\alpha(y) = [y - c(\alpha)/\sqrt{n} f(y), y + c(\alpha)/\sqrt{n} f(y)] \) satisfies Assumption 6.
Proof of Proposition 1. Under Assumption 7, we have the following strong approximation result in Csörgő, 1983, theorem 4.1.2 page 31:

\[
\sup_{0 \leq t \leq 1} |f(Q(t))q_n(t) - B_n(t)| = O(n^{-1/2+\varepsilon}), \quad \text{a.s.}
\]

for \( \varepsilon > 0 \) arbitrary, where \( B_n(t) \) is a sequence of Brownian bridges. Hence, the interval

\[
Q_n(t) - c(\alpha)/\sqrt{n}f(Q(t)) \leq Q(t) \leq Q_n(t) - c(\alpha)/\sqrt{n}f(Q(t))
\]

is an asymptotically valid uniform confidence band for \( Q(t), 0 \leq t \leq 1, \) of level \( 1 - \alpha. \)

Take \( Z \) a uniform random variable on \([0,1]\). Define \( \tilde{Y}^* := Q_n(Z) \) and \( Y := Q(Z) \). By the quantile transform, \( \tilde{Y}^* \) has distribution \( P_n \) and \( \tilde{Y} \) has distribution \( P \). Therefore, with probability tending to \( 1 - \alpha \), there exists \( \tilde{Y}^* \sim P_n \) and \( \tilde{Y} \sim P \) such that

\[
\tilde{Y} - c(\alpha)/\sqrt{n}f(\tilde{Y}) \leq \tilde{Y}^* \leq \tilde{Y} + c(\alpha)/\sqrt{n}f(\tilde{Y}) \quad \text{almost surely, and the result follows.} \quad \Box
\]

The dilation in proposition 1 is infeasible, as it depends on the unknown \( f \) and it relies on quantiles \( c(\alpha) \) that are difficult to compute. We develop a feasible alternative in our dilation bootstrap procedure in section 2.3. We resort to a bootstrap matching algorithm to construct feasible versions of the dilation above.

2.3. Bootstrap dilation. To introduce the simple idea underlying the method, consider the sample \( (Y_1, \ldots, Y_n) \) and a given bootstrap realization \( (Y^b_1, \ldots, Y^b_n) \) as in figure 1. As before, \( (Y_{(1)}, \ldots, Y_{(n)}) \) are the order statistics associated with the sample and \( (Y^b_{(1)}, \ldots, Y^b_{(n)}) \) are the order statistics associated with the bootstrap realization.
(with arbitrary ranking of the ties). In the illustrative example of figure 1, the smallest observation of the initial sample $Y_{(1)}$ was drawn once in the bootstrap sample, the second smallest was not drawn, the third smallest was drawn once, the fourth smallest twice, and the largest $Y_{(n)}$ was drawn twice. The arrows in the figure represent the bijection that matches the $j$’th order statistic of the initial sample $Y_{(j)}$ with the $j$’th order statistic of the bootstrap sample $Y_{(j)}^b$ for each $j = 1, \ldots, n$.

![Figure 1. Bootstrap Quantile Matching.](image)

To achieve a bootstrap analog of Assumption 6, we need a dilation $J_n^b$ and a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $Y_{(j)} \in J_n^b(Y_{\sigma(j)}^b)$ for all $j = 1, \ldots, n$. One such permutation matches the order statistics of the initial sample with the order statistics of the bootstrap sample. In this matching in the example of figure 1, $Y_{(1)}$ is matched with $Y_{b(1)}$, namely with itself. $Y_{(2)}$ was not drawn in the bootstrap sample, so it is matched with $Y_{b(2)}$, which is equal to $Y_{(3)}$, for whom $Y_{(2)}$ is the second closest neighbor in Euclidian distance. $Y_{(3)}$ is the nearest neighbor of its match $Y_{b(3)} = Y_{(4)}$. 
Y_{(4)} is matched with itself, Y_{(n-1)} is the nearest neighbor of its match $Y_{(n-1)}^b = Y_{(n)}$ and finally $Y_{(n)}$ is matched with itself. The longest distance between two matches is 

$$\eta_n^b = |Y_{(n-1)}^b - Y_{(n-1)}| = |Y_{(n)} - Y_{(n-1)}|.$$ 

Hence, if $J_n^b(y) = B(y, \eta_n^b)$, $Y_{(j)}^b \in J_n^b(Y_{(j)})$ will be satisfied for all $j = 1, \ldots, n$ in this particular bootstrap sample realization. The chosen matching in figure 1 characterizes the bootstrap quantile process (see Definition 4) and it minimizes the largest deviation $\eta_n^b$, and hence produces the smallest dilation.

**Definition 4** (Bootstrap quantile process). A bootstrap sample is a sample $(Y_1^b, \ldots, Y_n^b)$ of i.i.d. variables with distribution $P_n$. The quantile function of the distribution of the bootstrap sample (bootstrap quantile) is defined for each $t \in [0, 1]$ by $Q_n^b(t) = Y_{(j)}^b$ for $j - 1 < nt \leq j$. The bootstrap quantile process is defined as $q_n^b(t) := \sqrt{n}(Q_n^b(t) - Q_n(t))$. Call $\eta_n^b$ the maximum of the bootstrap quantile process. 

In the illustrative example of figure 1, the bootstrap quantile process attains its maximum over $t \in [0, 1]$ at $t$ such that $n - 2 < nt \leq n - 1$ and $\eta_n^b = Y_{(n-1)}^b - Y_{(n-1)} = Y_{(n)} - Y_{(n-1)}$. In the population of bootstrap realizations, $\eta_n^b$ has distribution with $1 - \alpha$ quantile $c_n^*(\alpha)$. The latter can be approximated by simulation with a large number $B$ of bootstrap replications. We obtain $\eta_n^b$ for each $b = 1, \ldots, B$. Call $\hat{c}_n^*(\alpha)$ the $[B\alpha]$-th largest among the $\eta_n^b$’s (where $[\cdot]$ denoted integer part) and $\hat{J}_n^{\alpha,*}(y) = B(y, \hat{c}_n^*(\alpha))$, then by construction, a proportion $1 - \alpha$ of the bootstrap samples indexed by $b = 1, \ldots, n$ will satisfy $Y_{(j)}^b \in J_n^{\alpha,*}(Y_{(j)})$ for all $j = 1, \ldots, n$. By Theorem 2 of Singh, 1981
(see also Theorem 5.1 of Bickel and Freedman, 1981), the bootstrap quantile process 
\( (q_n^b(t))_{t \in [0,1]} \) has almost surely the same uniform weak limit as the empirical quantile process 
\( (q_n(t))_{t \in [0,1]} \) and we therefore have the following result on the validity of the bootstrap dilation.

Proposition 2 (Bootstrap dilation). Let \( c_n^*(\alpha) \) be the \( 1 - \alpha \) quantile of the supremum
\( \eta_n^b \) of the bootstrap quantile process \( (q_n^b(t))_{t \in [0,1]} \). Under assumptions 5 and 7, the dilation defined for each \( y \) by 
\[ J_{n,\alpha}^*(y) = B(y, c_n^*(\alpha) / \sqrt{n}) \]
satisfies Assumption 6 almost surely.

Note that in the univariate case, the simulation approximation \( \hat{c}_n^*(\alpha) \) to the quantile \( c_n^*(\alpha) \) is very simple to derive. The simplest algorithm requires ordering the initial sample and each of the bootstrap samples and computing the maximum of \( |Y_{(j)}^b - Y_{(j)}| \) over \( j = 1, \ldots, n \). However, we have introduced, with figure 1 and the discussion above, an equivalent algorithm, which runs as follows: for each \( b = 1, \ldots, B \), find the permutation \( \sigma \) over \( \{1, \ldots, n\} \), which minimizes the quantity \( \max_j |Y_{(j)}^b - Y_{\sigma(j)}| \). Unlike the algorithm based on the order statistics, such an optimal matching or optimal assignment procedure can be performed regardless of dimension and efficient algorithms and implementations are available.
3. Simulation evidence

We assess the small sample performance of the dilation bootstrap on the following simulation design. Observable variables $Y$ have a standard normal distribution, while unobserved heterogeneity variable $U$ is assumed to follow a normal distribution with mean $\theta$ and variance 1. The cumulative distribution of $U$ is denoted $F_U$. The model correspondence $G$ is defined for each $u$ by $G(u) = [u - 1, u + 1]$, so that the model is characterized by the relation $Y \in G(U) = [U - 1, U + 1]$. Therefore, the identified set can be immediately derived as $\Theta_I = [-1, 1]$. For 5,000 initial samples of size $n = 50, 100, 500$, with empirical distributions $P_n$, we compute $\hat{c}_n^*(\alpha)$ with 5,000 bootstrap replications, and use the dilation $\hat{J}_n^{\alpha,*}(y) = B(y, \hat{c}_n^*(\alpha))$, so that a parameter value $\theta$ belongs to the $(1 - \alpha)$-confidence region $\Theta_{CR}$ for $\Theta_I$ if and only if there exist $\tilde{Y}^* \sim P_n$ and $\tilde{U} \sim F_U(.; \theta)$ such that $\mathbb{P}^*(\tilde{Y}^* \in \hat{J}_n^{\alpha,*} \circ G(\tilde{U})) = 1$. Since $P_n$ and $F_U$ are known, the latter condition can be checked efficiently with the core determining class method of Galichon and Henry, 2008, section 2.3. We report Monte Carlo coverage probabilities in case of significance level $\alpha = 0.01, 0.05$ and 0.1 in Table 1.

| Sample Size | 50  | 100 | 500 |
|-------------|-----|-----|-----|
| $\alpha = 0.01$ | 0.0122 | 0.0118 | 0.0108 |
| $\alpha = 0.05$ | 0.0324 | 0.0364 | 0.0438 |
| $\alpha = 0.10$ | 0.0590 | 0.0648 | 0.0754 |
The most notable feature to note is the tendency to under reject in small samples, especially for true size $\alpha = 0.10$ but also for true size $\alpha = 0.05$. For true size $\alpha = 0.01$ on the other hand, the procedure displays slight over rejection in small samples. For comparison purposes, we also report coverage probabilities from the generic subsampling procedure for set coverage in Chernozhukov et al., 2007 based on the criterion function $\sqrt{n} \max(\max_{j=1,\ldots,n}[F_n(Y_j) + F_U(Y_j + 1)], \max_{j=1,\ldots,n}[-F_n(Y_j) + F_U(Y_j - 1)])$. In order to avoid artificially favoring our results, we report coverage probabilities under several subsample sizes and when the true identified set is known and no initial estimate is needed in the Chernozhukov et al., 2007 procedure. The Monte Carlo coverage probabilities for 500 initial samples and 500 subsamples of sizes 40, 45, 48 when $n = 50$, 85, 92, 95 for $n = 100$ and 425, 450, 475 for $n = 500$ are reported in Table 2. We find the procedure over rejects in all but one case, and there is moderate dependence in the choice of subsample size.

4. Extensions

The dilation method and dilation bootstrap have natural extensions to the cases, where observable variables $Y$ are multivariate and to the case, where $Y$ is discrete. We consider both extensions in the following subsections.

4.1. Multivariate extension. Consider first the case, where the random vector of observable variables $Y$ has dimension $d \geq 2$. This extension allows the consideration of multiple equations models. Moreover, it is particularly relevant in this partially
Table 2. Rejection levels from the infeasible CHT procedure.

| Sample | Subsample | $\alpha = 0.01$ | $\alpha = 0.05$ | $\alpha = 0.10$ |
|--------|-----------|-----------------|-----------------|-----------------|
|        |           | 40              | 0.022           | 0.086           |
| 50     | 45        | 0.026           | 0.060           | 0.098           |
|        | 48        | 0.030           | 0.058           | 0.130           |
|        | 85        | 0.040           | 0.100           | 0.116           |
| 100    | 92        | 0.034           | 0.082           | 0.156           |
|        | 95        | 0.056           | 0.080           | 0.138           |
|        | 425       | 0.068           | 0.098           | 0.140           |
| 500    | 450       | 0.042           | 0.084           | 0.116           |
|        | 475       | 0.062           | 0.118           | 0.138           |

identified framework, as it also allows the consideration of single equations models with endogenous regressors.

**Example 5** (Single equation model with endogeneity). *Suppose the econometric model under consideration is $Z = f(X, U; \theta)$, where $Z$ and $X$ are observed random variables, $U$ is unobserved heterogeneity and $f$ is a function parameterized by $\theta$. Suppose no assumption is made on the dependence between $X$ and $U$. Define $Y = (X, Z)'$. Define the correspondence $G$ for each $u$ by $(x, z) \in G(u; \theta)$ if and only if $z = f(x, u; \theta)$. Then the model can be rewritten $Y \in G(U; \theta)$ as in Assumption 1.*

In case $Y$ is multivariate, although Theorem 2 holds irrespective of dimension, the construction of a dilation satisfying Assumption 6 can no longer rely on the traditional quantile process as in Propositions 1 and 2. However, the quantity $\eta_n =$
\[
\inf\{\|\tilde{Y}^* - \tilde{Y}\|_\infty : \tilde{Y}^* \sim P_n, \tilde{Y} \sim P\}
\]
is still well defined. When attained, it is achieved by a pair of random vectors \((\tilde{Y}^*, \tilde{Y})\) with marginal distributions \(P_n\) and \(P\), which minimizes the largest deviation. Equivalently, there exist \(\tilde{Y}^* \sim P_n\) and \(\tilde{Y} \sim P\) such that \(\tilde{Y}^*\) belongs to a closed ball \(B(\tilde{Y}, \eta_n)\) centered on \(\tilde{Y}\) and with radius \(\eta_n\), i.e. such that \(\mathbb{E}[1\{\tilde{Y}^* \notin B(\tilde{Y}, \eta_n)\}] = 0\).

When \(Y\) is uniformly distributed on the unit cube \([0, 1]^d\), the quantity \(\eta_n\) is well studied in the probability literature. Hence, using asymptotic results on the quantity \(\eta_n\) in the literature, specifically Leighton and Shor, 1989 for the case \(d = 2\) and Shor and Yukich, 1991 for the case \(d \geq 3\), we can derive analytical formulae for the dilation \(J_n\):

**Proposition 3** (Minimax matchings). The exist a constant \(c > 0\) and a function \(c_d > 0\) of the dimension \(d\) of \(Y\) such that \(J_n(y) = B(y, c_2(\ln n)^{3/4}/\sqrt{n})\) satisfies Assumption 6 with \(\alpha_n = n^{-c\sqrt{\ln n}}\) when \(d = 2\) and \(J_n(y) = B(y, c_d(\ln n/n)^{1/d})\) satisfies Assumption 6 for any \(\alpha \in [0, 1)\) when \(d \geq 3\).

However, the results of Proposition 3 only pertain to the uniform case and produce conservative confidence regions. More generally, we propose constructing suitable dilations based on the distribution of \(\eta_n\).

**Definition 5** (Minimax matching). Call \(c_n(\alpha)\) the \(1 - \alpha\) quantile of the distribution of \(\eta_n = \inf\{\|\tilde{Y}^* - \tilde{Y}\|_\infty : \tilde{Y}^* \sim P_n, \tilde{Y} \sim P\}\).
By construction, we then see that the ball $B(y, c_n(\alpha))$ is a suitable dilation, in the sense that it satisfies Assumption 6.

**Proposition 4** (Multivariate oracle dilation). The dilation $J_n^\alpha$ defined for each $y$ by $J_n^\alpha(y) = B(y, c_n(\alpha))$ satisfies Assumption 6.

As for the approximation of $c_n(\alpha)$ to obtain a feasible dilation, once again, although the quantile process is no longer defined, the matching algorithm described in Section 2.3 is easily generalizable and delivers a bootstrap dilation approximation of $J_n^\alpha$. The general procedure is described as follows.

**Bootstrap Algorithm:**

- Consider bootstrap samples $(Y_1^b, \ldots, Y_n^b), \ b = 1, \ldots, B$ drawn from $P_n$ and call $P_n^b$ the empirical distribution of sample $b$.
- For each bootstrap replication $b$, define

  $$\eta_n^b = \min_{\sigma} \max_{j \in \{1, \ldots, n\}} \|Y_j^b - Y_{\sigma(j)}\|,$$

  where $\sigma$ ranges over all permutations of $\{1, \ldots, n\}$.
- Let $\hat{c}_n^*(\alpha)$ be the $[B\alpha]$ largest among the $\eta_n^b, \ b = 1, \ldots, B$, and for each $y$, set $\hat{J}_n^{\alpha,*}(y) = B(y, \hat{c}_n^*)$.

The problem of finding the permutation that achieves $\eta_n^b$ is called *bottleneck bipartite matching* in the combinatorial optimization and operations research literature.
4.2. **Case of discrete choice.** We now turn to the case of aggregate data from discrete choice. To fix ideas, consider a voting model, where $K$ parties are represented in $n$ electoral districts and observations $\hat{p}_{i,k}$, $i = 1, \ldots, n$ and $k = 1, \ldots, K$, are reported shares of votes for party $k$ in district $i$. Voter $l$ chooses the party that maximizes their utility $u_{i,k}^l(\theta) + \rho_{i,k} + \epsilon_{i,k}^l$, where $u_{i,k}(\theta)$ is a deterministic function of (observed covariates and) the unknown parameter $\theta$, $\rho_{i,k}$ are random district-party effects (independent of voters) and the $\epsilon_{i,k}^l$'s are i.i.d. type I extreme value random utilities. True vote shares for party $k$ in district $i$ satisfy $\ln p_{i,k}^*(\rho_{i,k}) = u_{i,k}(\theta) + \rho_{i,k} + \ln \sum_k \exp(u_{i,k} + \rho_{i,k})$. True shares $p_{i,k}^*$ are unobserved, however, due to the possibility of electoral fraud. Reported shares $p_{i,k}$ are assumed to satisfy $p_{i,k} \geq p_{i,k}^*$ when a representative of party $k$ is present during the vote count in district $i$. In districts, where no party representative is present, the situation is equivalent to missing data on vote shares. Let $X_{i,k}$ be equal to 1 if a representative of party $k$ is present in district $i$ during vote count, and zero otherwise. We assume $X = (X_{i,k})_{i=1,\ldots,n; k=1,\ldots,K}$ is exogenous. The correspondence characterizing the model is

$$G \left( \left( \rho_{i,k} \right)_{k=1}^K | X; \theta \right) =$$

$$\left\{ \left( p_{i,k} \right)_{k=1}^K : \sum_{k=1}^K p_{i,k} = 1; \ p_{i,k} \geq p_{i,k}^*(\rho_{i,k})X_{i,k}, \text{ each } k \right\}.$$ 

District $i$ has $n_i$ voters. Call $\hat{p}_{i,k}$ the proportion of votes in district $i$ reported as going to party $k$ and write $\hat{p}_i = (\hat{p}_{i,k})_{k=1,\ldots,K}$. By the central limit theorem, $\sqrt{n_i}(\hat{p}_i - p_i)$ has Gaussian limiting distribution with zero mean and covariance matrix $V_i$, with
diagonal elements \( p_{i,k}(1 - p_{i,k}) \) and off-diagonal elements \(-p_{i,k}p_{i,k'}\). Call \( Z_i \) a random vector with distribution \( N(0, V_i/n_i) \) and let \( \eta_i \) be such that \( \mathbb{P}(Z_i \notin B(0, \eta_i)) = \alpha_i \), where \( B(0, \eta_i) \) is the open ball centered at zero with radius \( \eta_i \). Define the dilation \( J_{n_i}^{\alpha_i} \) defined for each \( p \) by \( J_{n_i}^{\alpha_i}(p) = B(p, \eta_i) \). Then \( J_n^{\alpha}(p) = \bigcup_i J_{n_i}^{\alpha_i}(p) = B(p, \max_i \eta_i) \) satisfies Assumption 6 for \( \alpha = \limsup_n \prod_{i=1}^n \alpha_i \).

In the two-party case, call \( \hat{p}_i \) the reported share of votes for party 1, \( p_i \) the true or population reported share and \( p_i^* \) the true share (absent reporting fraud). The true share satisfies
\[
\ln p_i^*(\rho_i) = u_{i,1}(\theta) + \rho_{i,1} + \ln(\exp(u_{i,1}(\theta) + \rho_{i,1}) + \exp(u_{i,2}(\theta) + \rho_{i,2})).
\]
Because of fraud issues, all we know about the relation between \( p_i \) and \( p_i^* \) is the following:

- \( p_i \geq p_i^* \) if party 1 places an observer in district \( i \).
- \( p_i \leq p_i^* \) if party 2 places an observer in district \( i \).

Note that reported vote shares are equal to true vote shares in case both parties have observers present for vote count. Letting \( X_{i,k} \) take value 1 if party \( k \) places an observer in district \( i \) and zero otherwise, the correspondence characterizing the model is

\[
G(p_i|X, \theta) = \{ p_i : p_i \geq p_i^*(\rho_i)X_{i,1} \text{ and } (1 - p_i) \geq (1 - p_i^*(\rho_i))X_{i,2} \} = \left[ \frac{X_{i,1} \exp(u_{i,1}(\theta) + \rho_{i,1})}{\exp(u_{i,1}(\theta) + \rho_{i,1}) + \exp(u_{i,2}(\theta) + \rho_{i,2})}, 1 - \frac{X_{i,2} \exp(u_{i,1}(\theta) + \rho_{i,2})}{\exp(u_{i,1}(\theta) + \rho_{i,1}) + \exp(u_{i,2}(\theta) + \rho_{i,2})} \right].
\]
By the central limit theorem, $\sqrt{n_i}(\hat{p}_i - p_i)$ has Gaussian limiting distribution with zero mean and variance $p_i(1 - p_i)$. Call $c_{\alpha_i/2}$ the quantile of level $1 - \alpha_i/2$ of the standard normal distribution. Call $\eta = \max_i \eta_i$ with $\eta_i = c_{\alpha_i/2}\sqrt{p_i(1 - p_i)/n_i}$. Then the dilation defined for each $p$ by $J_n^\alpha(p) = [p - \eta, p + \eta]$ satisfies Assumption 6 with $\alpha = \limsup_n \Pi_i \alpha_i$. The composition of the dilation $J_n^\alpha$ and the correspondence $G$ yields

$$J_n^\alpha \circ G(\rho|X; \theta) = [X_1 p^*(\rho) - \eta, X_2(1 - p^*(\rho)) + \eta].$$

The region $\tilde{\Theta}_I$ containing all $\theta$ such that $\hat{p} \in J_n^\alpha \circ G(\rho|X, \theta)$ a.s. is therefore a valid confidence region for the identified set and can be computed efficiently using methods proposed in Galichon and Henry, 2008.

**Conclusion**

We have proposed a method to combine several sources of uncertainty, such as missing or corrupted data and structural incompleteness in the model through a composition of correspondences. We show that our composition theorem applies in particular to the construction of confidence regions in partially identified models of general form. In that case, the composition theorem is applied to the composition of the correspondence that defines the econometric structure and a dilation of the sample space that controls the significance level and allows to replace the unknown distribution of observable data by the empirical distribution of the sample in the characterization of compatibility between model and data. An important computational
advantage of this method over previous proposed confidence regions for partially identified parameters is that the dilation is performed independently of the structural parameter, hence needs to be performed only once. The remaining search over the parameter space is purely deterministic. The dilation is obtained through a minimax matching procedure. It is equivalent to a uniform confidence band for the quantile process when the dimension of the endogenous variable is one, however, it has no parallel in higher dimensions. The method is shown to perform well in simulation experiments.

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