FALTINGS EXTENSION AND HODGE-TATE FILTRATION FOR ABELIAN VARIETIES OVER $p$-ADIC LOCAL FIELDS WITH IMPERFECT RESIDUE FIELDS

TONGMU HE

Abstract. Let $K$ be a complete discrete valuation field of characteristic 0 with not necessarily perfect residue field of characteristic $p > 0$. We define a Faltings extension of $\mathcal{O}_K$ over $\mathbb{Z}_p$, and we construct a Hodge-Tate filtration for abelian varieties over $K$ by generalizing Fontaine’s construction [Fon82] where he treated the perfect residue field case.

Contents

1. Introduction 1
2. Notation 3
3. Review of Hyodo’s Computation of Galois Cohomology Groups of $C(r)$ 3
4. Faltings Extension 6
5. Fontaine’s Injection 8
6. Weak Hodge-Tate Representations 9
7. Hodge-Tate Filtration for Abelian Varieties 10
References 12

1. Introduction

1.1. Let $K$ be a complete discrete valuation field of characteristic 0, with residue field $k$ of characteristic $p > 0$. Let $\overline{K}$ be an algebraic closure of $K$, $G_K$ the Galois group of $\overline{K}$ over $K$, $C$ the $p$-adic completion of $\overline{K}$. We denote by $C(r)$ the $r$-th Tate twist. For an abelian variety $X$ over $K$, we denote its Tate module by $T_p(X)$. When $k$ is perfect and $X$ has good reduction, Tate [Tat67] constructed a canonical $G_K$-equivariant exact sequence

\begin{equation}
0 \rightarrow H^1(X, \mathcal{O}_X) \otimes_K C(1) \rightarrow \text{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) \rightarrow H^0(X, \Omega^1_{X/K}) \otimes_K C \rightarrow 0.
\end{equation}

(1.1.1)

In the same paper, Tate also computed the Galois cohomology groups of $C(r)$. He proved in particular that $H^1(G_K, C(r)) = 0$ for any $r \neq 0$, which implies that the sequence (1.1.1) has a $G_K$-equivariant splitting, and that $H^0(G_K, C(r)) = 0$ for any $r \neq 0$, which implies that the splitting is unique. Tate conjectured that for any proper smooth scheme $X$ over $K$, there is a canonical $G_K$-equivariant decomposition (called the Hodge-Tate decomposition)

\begin{equation}
H^0_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C = \bigoplus_{i=0}^n H^1(X, \Omega^{n-i}_{X/K}) \otimes_K C(i - n).
\end{equation}

(1.1.2)

Then subsequently Raynaud used the semistable reduction theorem to show that any abelian variety over $K$ admits a Hodge-Tate decomposition ([sga72] IX 3.6, 5.6). Afterwards, Fontaine [Fon82] gave a new proof for general abelian varieties. He constructed a canonical map $H^0(X, \Omega^1_{X/K}) \rightarrow \text{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), C(1))$, by computing $\Omega^1_{\overline{\mathbb{Q}_p}/\mathbb{Q}_p}$ and pulling back differentials. The conjecture of Tate was finally settled by Faltings [Fal88, Fal02] and Tsuji [Tsu99, Tsu02] independently.

When $k$ is not necessarily perfect, Hyodo proved that there is still an exact sequence (1.1.1) for abelian varieties with good reduction, following the same argument as in [Tat67] ([Hyo86] Remark 1). But the sequence does not split in general ([Hyo86] Theorem 3). In this paper, we will construct the exact sequence (1.1.1) for general abelian varieties by generalizing Fontaine’s method to the imperfect residue field case.
We remark that Scholze [Sch13] has generalized the conjecture of Tate to any proper smooth rigid-analytic variety $X$ over $C$. He proved that there is a canonical filtration (called the Hodge-Tate filtration) $\Fil^i$ on $H^i_{\et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$, such that
\begin{equation}
(1.1.3) \quad \Fil^i(H^i_{\et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C)/ \Fil^{i+1}(H^i_{\et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C) = H^i(X, \Omega^{n-i}_{X/C}) \otimes_C C(i - n).
\end{equation}

1.2. For any abelian group $M$, we set
\begin{equation}
(1.2.1) \quad T_p(M) = \text{Hom}_\mathbb{Z}([1/p]/\mathbb{Z}, M), \ V_p(M) = \text{Hom}_\mathbb{Z}([1/p], M).
\end{equation}

In section 4, we construct a Faltings extension of $\mathcal{O}_K$ over $\mathbb{Z}_p$. It is a canonical exact sequence of $C$-$G_K$-modules which splits as a sequence of $C$-modules (cf. 4.4),
\begin{equation}
(1.2.2) \quad 0 \longrightarrow C(1) \overset{\iota}{\longrightarrow} V_p(\Omega^1_{C/\mathbb{Q}_p}) \overset{\nu}{\longrightarrow} C \otimes_{\mathcal{O}_C} (\mathcal{O}_{K/\mathbb{Q}_p} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge \longrightarrow 0,
\end{equation}
where $(-)^\wedge$ denotes the $p$-adic completion. Based on Hyodo’s computation of Galois cohomology (cf. 3.8), we will show that the connecting map of the above sequence
\begin{equation}
(1.2.3) \quad \delta : (C \otimes_{\mathcal{O}_C} (\mathcal{O}_{K/\mathbb{Q}_p} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge)^{G_K} \longrightarrow H^1(G_K, C(1))
\end{equation}
is an isomorphism (cf. 4.5).

Following Fontaine, we deduce from the above sequence and its cohomological properties a canonical $K$-linear injective homomorphism (cf. 5.6)
\begin{equation}
(1.2.4) \quad \rho : H^0(X, \Omega^1_{X/K}) \longrightarrow \text{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), V_p(\Omega^1_{C/\mathbb{Q}_p})).
\end{equation}

The arguments are essentially the same as in [Fon82].

Our main result can be stated as follows (cf. 7.4, 7.5, 7.6):

**Theorem 1.3.** For any abelian variety $X$ over $K$, there is a canonical exact sequence of $C$-$G_K$-modules
\begin{equation}
(1.3.1) \quad 0 \longrightarrow H^1(X, \mathcal{O}_X) \otimes_K C(1) \overset{\psi}{\longrightarrow} \text{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) \overset{\phi}{\longrightarrow} H^0(X, \Omega^1_{X/K}) \otimes_K C \longrightarrow 0
\end{equation}
satisfying the following properties:
(i) Any $C$-linear retraction of $\iota$ in (1.2.2) induces a $C$-linear section of $\phi$. More precisely, we have a commutative diagram
\begin{equation}
(1.3.2) \quad \text{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) \overset{\phi}{\longrightarrow} H^0(X, \Omega^1_{X/K}) \otimes_K C \quad \pi \quad \rho \quad \text{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega^1_{C/\mathbb{Q}_p}))
\end{equation}
where $\rho$ is induced by the map (1.2.4) and $\pi$ is induced by any retraction of $\iota$ in (1.2.2).
(ii) The connecting map $\delta'$ associated to (1.3.1) fits into a commutative diagram
\begin{equation}
(1.3.3) \quad H^0(X, \Omega^1_{X/K}) \overset{\delta'}{\longrightarrow} H^1(G_K, H^1(X, \mathcal{O}_X) \otimes_K C(1)) \quad \rho \quad \text{Hom}_{\mathbb{Z}_p}(T_p(X), \Omega^1_{C/\mathbb{Q}_p} \otimes_{\mathcal{O}_K} \mathcal{O}_{K/\mathbb{Z}_p}) \overset{\psi'}{\longrightarrow} \text{Hom}_{\mathbb{Z}_p}(T_p(X), C \otimes_{\mathcal{O}_C} (\mathcal{O}_{K/\mathbb{Q}_p} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge)
\end{equation}
where $\rho$ is the map (1.2.4), $\pi'$ is induced by $-\nu$ of (1.2.2), and the unlabelled arrow is induced by $\delta^{-1}$ (1.2.3) and $\psi$ of (1.3.1).

**Corollary 1.4.** For any abelian variety $X$ over $K$, the sequence (1.3.1) splits if and only if the image of $\rho$ (1.2.4) lies in $\text{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), C(1))$. In fact, when it splits, the splitting is unique.

**Remark 1.5.** Caraiani and Scholze [CS17] have constructed a relative version of Hodge-Tate filtration for proper smooth morphisms of adic spaces. And recently, Abbes and Gros [AG20] have constructed a relative version of Hodge-Tate spectral sequence for projective smooth morphisms of schemes.
2. Notation

2.1. Let \( K \) be a complete discrete valuation field of characteristic 0, with residue field \( k \) of characteristic \( p > 0 \). Let \( \overline{K} \) be an algebraic closure of \( K \), \( G_K \) the Galois group of \( \overline{K} \) over \( K \). Let \( C \) be the \( p \)-adic completion of \( \overline{K} \), \( v_p \) the valuation on \( C \) such that \( v_p(p) = 1 \), \( \| \cdot \|_p \) the absolute value on \( C \) such that \( |p|_p = 1/p \). We fix a complete discrete valuation subfield \( K_0 \) of \( K \) such that \( \mathcal{O}_K/p\mathcal{O}_K = k \) (by Cohen structure theorem, cf. [Gro64] IV 19.8.6). We remark that \( K/K_0 \) is a totally ramified finite extension.

We fix elements \((u_i)_{i \in I} \) of \( \mathcal{O}_K \) such that the reductions \((\pi_i)_{i \in I} \) form a \( p \)-base of \( k \). For each \( i \in I \), we fix elements \((w_{im})_{m \geq 0} \) of \( \mathcal{O}_K \) such that \( w_{im+1} = w_{im} \) and \( w_{i0} = u_i \). We denote by \((e_i)_{i \in I} \) the standard basis of \( \oplus_{i \in I} \mathbb{Z} \).

2.2. For any discrete valuation field \( L \) of characteristic 0, with residue field of characteristic \( p \), we denote by

\[
\hat{\Omega}^1_{\mathcal{O}_L} = (\Omega^1_{\mathcal{O}_L/p})^\wedge
\]

the \( p \)-adic completion of the module of differentials of \( \mathcal{O}_L \) over \( \mathbb{Z}_p \).

For any algebraic extension \( L' \) over \( L \), we set

\[
\hat{\Omega}^1_{\mathcal{O}_L}(\mathcal{O}_{L'}) = \lim_{L_1/L} \hat{\Omega}^1_{\mathcal{O}_{L_1}},
\]

where \( L_1 \) runs through all finite subextensions of \( L'/L \). We remark that \( \hat{\Omega}^1_{\mathcal{O}_L}(\mathcal{O}_{L'}) = \hat{\Omega}^1_{\mathcal{O}_{L_1}}(\mathcal{O}_{L'}) \) for any finite subextension \( L_1 \) of \( L'/L \), and that \( \hat{\Omega}^1_{\mathcal{O}_L}(\mathcal{O}_L) = \hat{\Omega}^1_{\mathcal{O}_{L_1}}(\mathcal{O}_{L_1}) \).

2.3. For any abelian group \( M \), we define

\[
T_p(M) = \lim_{x \to px} M[p^n] = \text{Hom}_Z(\mathbb{Z}[1/p]/\mathbb{Z}, M),
\]

\[
V_p(M) = \lim_{x \to px} M = \text{Hom}_Z(\mathbb{Z}[1/p], M).
\]

Being an inverse limit of \( \mathbb{Z} \)-modules each killed by some power of \( p \), \( T_p(M) \) is a \( p \)-adically complete \( \mathbb{Z}_p \)-module ([Jan88] 4.4). If \( M \) is \( p \)-primary torsion, then \( V_p(M) = T_p(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \), and thus it has a natural \( \mathbb{Q}_p \)-module structure. If \( M \) is a \( \mathbb{Z}_p \)-module, then \( T_p(M) = \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, M) \), \( V_p(M) = \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, M) \).

We set \( \mathbb{Z}_p(1) = T_p(\mathbb{Q}_p) \), a free \( \mathbb{Z}_p \)-module of rank 1 with continuous \( G_K \)-action. For any \( \mathbb{Z}_p \)-module \( M \) and \( r \in \mathbb{Z} \), we set \( M(r) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes r} \), the \( r \)-th Tate twist of \( M \). Let \( X \) be an abelian variety over \( K \). We set \( T_p(X) = T_p(X(\overline{K})) \) and \( V_p(X) = V_p(X(\overline{K})) \).

3. Review of Hyodo’s Computation of Galois Cohomology Groups of \( C(r) \)

**Lemma 3.1.** Let \( B/A \) be a finite extension of discrete valuation rings, whose fraction field extension and residue field extension are both separable. Let \( R \) be a subring of \( A \). Then the canonical map \( B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R} \) is injective.

**Proof.** After replacing \( A \) by its maximal unramified extension in \( B \), we may assume that \( B \) is totally ramified over \( A \). Hence, \( B \) is of the form \( A[X]/(f(X)) \) for some irreducible polynomial \( f \in A[X] \). Let \( x \) be the image of \( X \) in \( B \). Then we have

\[
(\oplus_{i \in I} \mathcal{O}_{K_0})^\wedge \overset{\sim}{\longrightarrow} \Omega^1_{\mathcal{O}_{K_0}}, \ e_i \mapsto d \log u_i, \ \forall i \in I.
\]

**Lemma 3.2** ([Hyo86] 4-4). There is an isomorphism of \( \mathcal{O}_{K_0} \)-modules

\[
(\oplus_{i \in I} \mathcal{O}_{K_0})^\wedge \overset{\sim}{\longrightarrow} \Omega^1_{\mathcal{O}_{K_0}}, \ e_i \mapsto d \log u_i, \ \forall i \in I.
\]
Proof. As \((\overline{w})_{i \in I}\) form a \(p\)-base of the residue field of \(O_{K_0}\), we have \(\Omega_{O_{K_0}/\mathbb{Z}_p}^1 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p\mathbb{Z}_p = \Omega_{K/p}^1 = \bigoplus_{i \in I} k_i\), where \(e_i\) corresponds to \(d \log \overline{w}_i\). Since \(O_{K_0}\) is flat over \(\mathbb{Z}_p\) and \(k\) is formally smooth over \(\mathbb{F}_p\), \(O_{K_0}/p^nO_{K_0}\) is formally smooth over \(\mathbb{Z}_p/p^n\mathbb{Z}_p\) for each \(n \geq 1\) ([Gro64] 0IV 19.7.1, [Sta20] 031L). In particular, \(\Omega_{O_{K_0}/\mathbb{Z}_p}^1 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p^n\mathbb{Z}_p\) is a projective \(O_{K_0}/p^nO_{K_0}\)-module. Hence, we have an exact sequence

\[
0 \to \Omega_{O_{K_0}/\mathbb{Z}_p}^1 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p\mathbb{Z}_p \to \Omega_{O_{K_0}/\mathbb{Z}_p}^1 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p^n\mathbb{Z}_p \to \Omega_{O_{K_0}/\mathbb{Z}_p}^1 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p^{n-1}\mathbb{Z}_p \to 0,
\]

from which we get isomorphisms \(\bigoplus_{i \in I} O_{K_0}/p^nO_{K_0} \xrightarrow{\sim} \Omega_{O_{K_0}/\mathbb{Z}_p}^1 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p^n\mathbb{Z}_p\) by induction. The conclusion follows by taking limit over \(n\). \(\square\)

**Proposition 3.3** ([Hyo86] 4-2-1). There is an exact sequence of \(O_K\)-modules

\[
0 \to \left( \bigoplus_{i \in I} O_K \right) \overset{\theta}{\to} \hat{\Omega}_{O_K}^1 \to \Omega_{O_K/O_{K_0}}^1 \to 0,
\]

where \(\theta(e_i) = d \log u_i\) for any \(i \in I\).

**Proof.** The sequence of modules of differentials of \(O_K/O_{K_0}/\mathbb{Z}_p\),

\[
0 \to O_K \otimes_{O_{K_0}} \Omega_{O_{K_0}/\mathbb{Z}_p}^1 \to \Omega_{O_K/O_{K_0}}^1 \to 0,
\]

is exact by 3.1. Passing to \(p\)-adic completions, as \(\Omega_{O_K/O_{K_0}}^1\) is killed by a power of \(p\), we still get an exact sequence ([Sta20] 0BNG). The conclusion follows from 3.2 and the isomorphism \(O_K \otimes_{O_{K_0}} \left( \bigoplus_{i \in I} O_K \right) \overset{\sim}{\to} \left( \bigoplus_{i \in I} O_K \right)^\wedge\) as \(O_K\) is finite free over \(O_{K_0}\). \(\square\)

**Lemma 3.4** ([Hyo86] 4-4). Let \(M_0 = \bigcup_{i \in I, m \geq 0} K_0(w_i m) \subseteq \overline{K}\). Then there is an isomorphism of \(O_{M_0}\)-modules

\[
M_0 \otimes_{O_{K_0}} \left( \bigoplus_{i \in I} O_K \right)^\wedge \overset{\sim}{\to} \hat{\Omega}_{O_{K_0}}^1 (O_{M_0}).
\]

**Proof.** For an integer \(N > 0\) and a finite subset \(J \subseteq I\), let \(L_0 = \bigcup_{j \in J} K_0(w_i N)\). Then by 3.2, \(\left( \bigoplus_{i \in I} O_L \right)^\wedge\) is isomorphic to \(\hat{\Omega}_{O_{K_0}}^1\) by sending \(e_i\) to \(d \log w_i N\) if \(i \in J\), and to \(d \log u_i\) if \(i \notin J\). The conclusion follows by taking colimit over \(J\) and \(N\). \(\square\)

**Lemma 3.5** ([Hyo86] 4-7). With the same notation as in 3.4, let \(M\) be a finite extension of \(M_0\). Then there is a canonical exact sequence of \(O_M\)-modules

\[
0 \to O_M \otimes_{O_{M_0}} \hat{\Omega}_{O_{K_0}}^1 (O_{M_0}) \to \hat{\Omega}_{O_{K_0}}^1 (O_M) \to \Omega_{O_M/O_{M_0}}^1 \to 0.
\]

**Proof.** We notice that \(M_0\) has perfect residue field. Let \(M_{\text{ur}}\) be the maximal unramified subextension of \(M/M_0\), \(f \in O_{M_{\text{ur}}}[X]\) the monic minimal polynomial of a uniformizer \(\varpi\) of \(O_M\). Then we have \(O_M = O_{M_{\text{ur}}}[X]/(f(X))\). For a sufficiently large finite subextension \(L_1\) of \(M_{\text{ur}}/K_0\) such that \(f \in O_{L_1}[X]\), \(L_2 = L_1(\varpi)\) is totally ramified over \(L_1\). The same argument as in 3.3 gives us a canonical exact sequence

\[
0 \to O_{L_2} \otimes_{O_{L_1}} \hat{\Omega}_{O_{K_0}}^1 (O_{L_1}) \to \hat{\Omega}_{O_{K_0}}^1 (O_{L_2}) \to \Omega_{O_{L_2}/O_{L_1}}^1 \to 0.
\]

By taking colimit over \(L_1\), we get an exact sequence

\[
0 \to O_M \otimes_{O_{M_{\text{ur}}}} \hat{\Omega}_{O_{K_0}}^1 (O_{M_{\text{ur}}}) \to \hat{\Omega}_{O_{K_0}}^1 (O_M) \to \Omega_{O_M/O_{M_{\text{ur}}}}^1 \to 0.
\]

A similar colimit argument shows that \(\hat{\Omega}_{O_{K_0}}^1 (O_{M_{\text{ur}}}) = O_{M_{\text{ur}}} \otimes_{O_{M_0}} \hat{\Omega}_{O_{K_0}}^1 (O_{M_0})\). The conclusion follows from (3.5.3). \(\square\)

**Proposition 3.6** ([Hyo86] 4-2-2). There is an exact sequence of \(O_{\overline{K}}\)-\(G_K\)-modules which splits as a sequence of \(O_{\overline{K}}\)-modules,

\[
0 \to \overline{K}/a(1) \overset{\vartheta}{\to} \hat{\Omega}_{O_{\overline{K}}}^1 (O_{\overline{K}}) \to \overline{K} \otimes_{O_{\overline{K}}} \left( \bigoplus_{i \in I} O_K \right)^\wedge \to 0,
\]

where \(a = \{ x \in \overline{K} \mid v(x) \geq -1/(p-1) \}\), and \(\vartheta(p^{-k} \otimes (\zeta_n)_m) = d \log \zeta_k\) for any \(k \in \mathbb{N}\). The map \(\overline{K} \otimes_{O_{\overline{K}}} \left( \bigoplus_{i \in I} O_K \right)^\wedge \to \hat{\Omega}_{O_{\overline{K}}}^1 (O_{\overline{K}})\), sending \(p^{-m} \otimes e_i\) to \(d \log w_i\) for any \(i \in I\) and \(m \in \mathbb{N}\), gives a splitting of the sequence.
Proof. With the same notation as in 3.4, let $M$ run through all finite subextensions of $\overline{K}/M_0$. We get from 3.5 an exact sequence of $\mathcal{O}_{\overline{K}}$-modules

$$
\begin{aligned}
0 & \longrightarrow \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_M} \hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_M) \longrightarrow \hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}}) \longrightarrow \Omega_1^{1}(\mathcal{O}_{\overline{K}/\mathcal{O}_M}) \longrightarrow 0.
\end{aligned}
$$

We identify its first term with $\overline{K} \otimes_{\mathcal{O}_K} (\oplus_{i \in I} \mathcal{O}_K)^\wedge$ by 3.4. Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$ in $\overline{K}$, $\mathbb{Z}_p$ the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$. By Fontaine's computation ([Fon82], Théorème 1'), we have an isomorphism of $\mathbb{Z}_p$-modules

$$
\overline{\mathbb{Q}}_p/a_0(1) \xrightarrow{\sim} \Omega_1^{1}(\mathcal{O}_{\overline{K}/\mathcal{O}_M}),
$$

where $a_0 = \{ x \in \overline{\mathbb{Q}}_p \mid v_p(x) \geq -1/(p-1) \}$, and we have an isomorphism of $\mathcal{O}_{\overline{K}}$-modules

$$
\overline{K}/a(1) \xrightarrow{\sim} \Omega_1^{1}(\mathcal{O}_{\overline{K}/\mathcal{O}_M}),
$$

where $a = \{ x \in \overline{K} \mid v_p(x) \geq -1/(p-1) \}$. Hence, the composition of

$$
\overline{K}/a(1) \xrightarrow{\sim} \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_M} \Omega_1^{1}(\mathcal{O}_{\overline{K}/\mathcal{O}_M}) \rightarrow \Omega_1^{1}(\mathcal{O}_{\overline{K}/\mathcal{O}_M}) \rightarrow \hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}})
$$

gives a splitting of (3.6.2). Thus, we obtain the splitting sequence (3.6.1) of $\mathcal{O}_{\overline{K}}$-modules. We notice that the Galois conjugates of $\zeta_n, w_{im}$ are of the form $\zeta_n^a, c_{im}^w w_{im}$ respectively, which implies that (3.6.1) is $G_K$-equivariant. \hfill \Box

3.7. As $\hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}})$ is $p$-divisible, we have an exact sequence $0 \rightarrow T_p(\hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}})) \rightarrow V_p(\hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}})) \rightarrow \hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}}) \rightarrow 0$. After inverting $p$, we get an exact sequence

$$
\begin{aligned}
0 & \longrightarrow C(1) \longrightarrow \overline{K} \otimes_{\mathcal{O}_{\overline{K}}} V_p(\hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}})) \longrightarrow \overline{K} \otimes_{\mathcal{O}_{\overline{K}}} \hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}}) \longrightarrow 0,
\end{aligned}
$$

where we identified $\overline{K} \otimes_{\mathcal{O}_{\overline{K}}} T_p(\hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}}))$ with $C(1)$ by (3.6.1).

**Theorem 3.8** ([Hyo86] Theorem 1).

(i) The composition of

$$
\begin{aligned}
K \otimes_{\mathcal{O}_K} \hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}}) \xrightarrow{\epsilon} (\overline{K} \otimes_{\mathcal{O}_{\overline{K}}} \hat{\Omega}_1^{1}(\mathcal{O}_{\overline{K}}))^G_K \xrightarrow{\delta} H^1(G_K, C(1)),
\end{aligned}
$$

where $\epsilon$ is the canonical map and $\delta$ is the connecting map associated to (3.7.1), is an isomorphism. Moreover, for any integer $q$, the cup product induces an isomorphism

$$
\begin{aligned}
(\wedge^q H^1(G_K, C(1)))^\wedge \xrightarrow{\sim} H^q(G_K, C(q)).
\end{aligned}
$$

(ii) The $K$-module $H^1(G_K, C)$ is free of rank 1. Moreover, for any integer $q$, the cup product induces an isomorphism

$$
\begin{aligned}
H^1(G_K, C) \otimes_K (\wedge^{q-1} H^1(G_K, C(1)))^\wedge \xrightarrow{\sim} H^q(G_K, C(q-1)).
\end{aligned}
$$

(iii) For any integers $r$ and $q$ such that $r \neq q$ or $q - 1$, we have $H^q(G_K, C(r)) = 0$.

**Remark 3.9.** By 3.3 we have an isomorphism

$$
\begin{aligned}
K \otimes_{\mathcal{O}_K} (\oplus_{i \in I} \mathcal{O}_K)^\wedge \xrightarrow{\sim} K \otimes_{\mathcal{O}_K} \hat{\Omega}_1^{1}(\mathcal{O}_K), \quad 1 \otimes e_i \mapsto 1 \otimes d \log u_i, \quad \forall i \in I.
\end{aligned}
$$

By composing it with (3.8.1), we get an isomorphism

$$
\begin{aligned}
K \otimes_{\mathcal{O}_K} (\oplus_{i \in I} \mathcal{O}_K)^\wedge \xrightarrow{\sim} H^1(G_K, C(1)), \quad 1 \otimes e_i \mapsto [f_i],
\end{aligned}
$$

where $f_i$ is a 1-cocycle sending each $\sigma \in G_K$ to $\sigma(1 \otimes (d \log w_{im})_m) - 1 \otimes (d \log w_{im})_m$ in view of (3.7.1).
4. Faltings Extension

**Lemma 4.1.** Let $M = \bigcup_{i \in I, m \geq 0} K(w_{im}) \subseteq \overline{K}$. Then there is an isomorphism of $\mathcal{O}_M$-modules
\[
\oplus_{i \in I} M/O_M \simto \Omega^1_{\mathcal{O}_M/O_K}, \quad p^{-m}e_i \mapsto d\log w_{im}, \quad \forall i \in I, m \in \mathbb{N}.
\]

*Proof.* For any $N \geq 0$, we set $M_N = \bigcup_{i \in I} K(w_{iN})$. Since $(\pi_i)$ form a $p$-base of the residue field $k$, the elements of the form $\prod_{i \in I} w_{iN}^{k_i}$ where $0 \leq k_i < p^N$ with finitely many nonvanishing, are linearly independent over $k$. Therefore, $\mathcal{O}_{M_N} = \mathcal{O}_K[T_i|_{i \in I}/(T_i^p - u_i)$, where $T_i$ maps to $w_{iN}$. Hence,
\[
\Omega^1_{\mathcal{O}_{M_N}/\mathcal{O}_K} = \oplus_{i \in I} \mathcal{O}_{M_N}/p^N \mathcal{O}_{M_N} = \oplus_{i \in I} p^{-N} \mathcal{O}_{M_N}/\mathcal{O}_{M_N},
\]
where $p^{-N}e_i$ corresponds to $d\log w_{iN}$. The conclusion follows by taking colimit over $N$. \hfill \Box

**Proposition 4.2.** With the same notation as in 4.1, there is an exact sequence of $\mathcal{O}_{\overline{K}}$-modules
\[
0 \longrightarrow \oplus_{i \in I} \overline{K}/\mathcal{O}_{\overline{K}} \longrightarrow \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \longrightarrow \overline{K}/b(1) \longrightarrow 0,
\]
where $\theta(p^{-m}e_i) = d\log w_{im}$ for any $i \in I$ and $m \in \mathbb{N}$, and $b = \{x \in \overline{K} \mid v_p(x) \geq -v_p(D_{M/M_i}) - 1/(p-1)\}$, where $M_i$ is the fraction field of the Witt ring with coefficients in the residue field of $M$, and $D_{M/M_i}$ is the different ideal of $M/M_i$.

*Proof.* We notice that $M$ has perfect residue field. Thus, the sequence of modules of differentials of $\mathcal{O}_{\overline{K}}/\mathcal{O}_M/\mathcal{O}_K$,
\[
0 \longrightarrow \mathcal{O}_{\overline{K}} \otimes \mathcal{O}_M \Omega^1_{\mathcal{O}_M/\mathcal{O}_K} \longrightarrow \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \longrightarrow \mathcal{O}^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_M} \longrightarrow 0,
\]
is exact by 3.1. We identify its first term with $\oplus_{i \in I} \overline{K}/\mathcal{O}_{\overline{K}}$ by 4.1. By Fontaine’s computation ([Fon82, Théorème 1’]), we have an isomorphism of $\mathcal{O}_{\overline{K}}$-modules
\[
\overline{K}/b(1) \simto \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_M}, \quad p^{-k} \otimes (\zeta_n) \mapsto d\log \zeta_k, \quad \forall k \in \mathbb{N}.
\]
The conclusion follows from (4.2.2). \hfill \Box

**Lemma 4.3.** The canonical map
\[
K \otimes \mathcal{O}_K (\oplus_{i \in I} \mathcal{O}_K)^\wedge \longrightarrow (C \otimes \mathcal{O}_C (\oplus_{i \in I} \mathcal{O}_C)^\wedge)^{G_K}
\]
is an isomorphism.

*Proof.* It follows from the following descriptions
\[
(4.3.2) \quad C \otimes \mathcal{O}_C (\oplus_{i \in I} \mathcal{O}_C)^\wedge = \{ (x_i) \in \prod_{i \in I} C \mid \forall N > 0, \exists \text{ finite } J \subseteq I, |x_i|_p < 1/N, \forall i \notin J \},
\]
\[
(4.3.3) \quad K \otimes \mathcal{O}_K (\oplus_{i \in I} \mathcal{O}_K)^\wedge = \{ (x_i) \in \prod_{i \in I} K \mid \forall N > 0, \exists \text{ finite } J \subseteq I, |x_i|_p < 1/N, \forall i \notin J \}.
\]

**Theorem 4.4.** There is a canonical exact sequence of $C$-$G_K$-modules which splits as a sequence of $C$-modules,
\[
0 \longrightarrow C(1) \overset{\epsilon}{\longrightarrow} V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \longrightarrow C \otimes \mathcal{O}_C (\mathcal{O}_{\overline{K}} \otimes \mathcal{O}_K \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})^\wedge \longrightarrow 0,
\]
where $\epsilon(1 \otimes (\zeta_n)) = (d\log \zeta_n)$. There is an isomorphism of $C$-$G_K$-modules

**Proof.** We consider the sequence of modules of differentials of $\mathcal{O}_L/\mathcal{O}_K/\mathbb{Z}_p$, where $L/K$ is a finite subextension of $\overline{K}/K$, and pass to $p$-adic completions. Since $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ is killed by a power of $p$, we still get an exact sequence ([Sta20] 0315, 0BN)
\[
0 \longrightarrow \mathcal{O}_L \otimes \mathcal{O}_K \hat{\Omega}_{\mathcal{O}_K} \longrightarrow \hat{\Omega}_{\mathcal{O}_L} \longrightarrow \Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \longrightarrow 0.
\]
By taking colimit over all such $L$, we get an exact sequence

$$
\begin{align*}
(4.4.4) & \quad \mathcal{O}_K \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K} \xrightarrow{\alpha} \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K) \xrightarrow{\beta} \mathcal{O}_K^{1/\mathcal{O}_K} \xrightarrow{\gamma} 0.
\end{align*}
$$

Combining with propositions 3.3, 3.6 and 4.2, we get a commutative diagram:

$$
\begin{align*}
(4.4.5) & \quad 0 \longrightarrow \mathcal{O}_K \otimes_{\mathcal{O}_K} (\oplus_{i \in I} \mathcal{O}_K) \xrightarrow{\alpha} \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K) \xrightarrow{\beta} \mathcal{O}_K^{1/\mathcal{O}_K} \xrightarrow{\gamma} 0
\end{align*}
$$

where the rows and columns are exact, and the middle column splits. We set $D = \text{Ker}(\beta) = \text{Im}(\alpha)$. We see that $\mathcal{O}_K \otimes_{\mathcal{O}_K} (\oplus_{i \in I} \mathcal{O}_K) \xrightarrow{\alpha} \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K)$ and $\mathcal{O}_K^{1/\mathcal{O}_K} \xrightarrow{\beta} \mathcal{O}_K^{1/\mathcal{O}_K}$ are injective, whose cokernel is killed by a power of $p$. Now for any $n > 0$, by applying $\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p/p^n \mathbb{Z}_p, -)$ to (4.4.4), we get an exact sequence of $\mathcal{O}_K^{1/\mathcal{O}_K}$-modules

$$
\begin{align*}
(4.4.6) & \quad 0 \longrightarrow D[p^n] \longrightarrow \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K)[p^n] \longrightarrow \mathcal{O}_K^{1/\mathcal{O}_K} \longrightarrow D/p^n D \longrightarrow \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K)/p^n \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K) = 0.
\end{align*}
$$

We notice that the inverse system $(D[p^n])_n$ is Artin-Rees null, and that $(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K)[p^n])_n$ satisfies Mittag-Leffler condition. Therefore, by taking the inverse limit of (4.4.6), we get an exact sequence of $\mathcal{O}_C$-modules

$$
\begin{align*}
(4.4.7) & \quad 0 \longrightarrow T_p(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K)) \longrightarrow T_p(\mathcal{O}_K^{1/\mathcal{O}_K}) \longrightarrow D^\wedge \longrightarrow 0.
\end{align*}
$$

By applying $T_p(-)$ to the middle column of (4.4.5), we get $T_p(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_K)) = \hat{\alpha}(1)$. On the other hand, we notice that $\oplus_{i \in I} \mathcal{O}_K^{1/\mathcal{O}_K}$ is $p$-divisible, and that $(\oplus_{i \in I} \mathcal{O}_K^{1/\mathcal{O}_K})_n$ satisfies Mittag-Leffler condition. Therefore, by applying $T_p(-)$ to the right column of (4.4.5), we get an exact sequence of $\mathcal{O}_C$-modules

$$
\begin{align*}
(4.4.8) & \quad 0 \longrightarrow (\oplus_{i \in I} \mathcal{O}_C)^\wedge \longrightarrow T_p(\mathcal{O}_K^{1/\mathcal{O}_K}) \longrightarrow \hat{\beta}(1) \longrightarrow 0.
\end{align*}
$$

As $\mathcal{O}_K^{1/\mathcal{O}_K}$ is killed by a power of $p$, the map $\mathcal{O}_K^{1/\mathcal{O}_K} \xrightarrow{\alpha} D^\wedge$ becomes an isomorphism after inverting $p$. Afterwards, we get from (4.4.7) a canonical exact sequence of $C$-modules

$$
\begin{align*}
(4.4.9) & \quad 0 \longrightarrow C(1) \longrightarrow V_p(\mathcal{O}_K^{1/\mathcal{O}_K}) \longrightarrow C \otimes_{\mathcal{O}_C} (\mathcal{O}_K^{1/\mathcal{O}_K})\wedge \longrightarrow 0,
\end{align*}
$$

and from (4.4.8) an exact sequence of $\mathcal{O}_C$-modules

$$
\begin{align*}
(4.4.10) & \quad 0 \longrightarrow C \otimes_{\mathcal{O}_C} (\oplus_{i \in I} \mathcal{O}_C)^\wedge \longrightarrow V_p(\mathcal{O}_K^{1/\mathcal{O}_K}) \longrightarrow C(1) \longrightarrow 0.
\end{align*}
$$

The latter gives a splitting of (4.4.9) and an isomorphism $C \otimes_{\mathcal{O}_C} (\oplus_{i \in I} \mathcal{O}_C)^\wedge \xrightarrow{\sim} C \otimes_{\mathcal{O}_C} (\mathcal{O}_K^{1/\mathcal{O}_K})\wedge$ by sending $1 \otimes e_i$ to $1 \otimes 1 \otimes d\log u_i$ by diagram chasing. We notice that the Galois conjugates of $\zeta_n, u_{im}$ are of the form $\zeta_m, c_m w_{im}$ respectively, which implies that (4.4.9) is $G_K$-equivariant. Hence, (4.4.9) gives us the exact sequence (4.4.1) of $C$-$G_K$-modules which splits as a sequence of $C$-modules.

**Corollary 4.5.** The canonical map $K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \to (C \otimes_{\mathcal{O}_C} (\mathcal{O}_K^{1/\mathcal{O}_K})^\wedge)^{G_K}$ is an isomorphism, and the connecting map of the sequence (4.4.1)

$$
\begin{align*}
\delta : K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \longrightarrow H^1(G_K, C(1))
\end{align*}
$$

is an isomorphism which coincides with (3.8.1). In particular,

$$
\begin{align*}
(4.5.2) & \quad V_p(\mathcal{O}_K^{1/\mathcal{O}_K})^{G_K} = 0.
\end{align*}
$$

**Proof.** By (3.9.1), (4.4.2) and 4.3, we see that the canonical map $K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \to (C \otimes_{\mathcal{O}_C} (\mathcal{O}_K^{1/\mathcal{O}_K})^\wedge)^{G_K}$ is an isomorphism. Now (4.5.1) follows from 3.8 (i) and 3.9. And (4.5.2) follows from the fact that $C(1)^{G_K} = 0$. □

**Definition 4.6.** We call the sequence (4.4.1) the *Faltings extension of $\mathcal{O}_K$ over $\mathbb{Z}_p$*. 
5. Fontaine’s Injection

5.1. For any proper model $\mathcal{X}$ of the abelian variety $X$ over $\mathcal{O}_K$ (i.e., a proper $\mathcal{O}_K$-scheme whose generic fiber is $X$), we identify $\mathcal{X}(\mathcal{O}_K)$ with $X(\overline{K})$ by valuative criterion. Pullback of Kähler differentials defines a map

\[(5.1.1) \quad H^0(\mathcal{X}, \Omega^1_{\mathcal{O}/\mathcal{O}_K}) \rightarrow \text{Map}_{\mathcal{O}_K}(X(\overline{K}), \Omega^1_{\mathcal{O}/\mathcal{O}_K}), \quad \omega \mapsto (u \mapsto u^*\omega).\]

We notice that $H^0(X, \Omega^1_{X/\mathcal{O}_K}) = K \otimes_{\mathcal{O}_K} H^0(\mathcal{X}, \Omega^1_{\mathcal{O}/\mathcal{O}_K})$, and that any differential form over $X$ is invariant under translations. Hence, we can take an integer $r > 0$ big enough, such that for any $\omega \in p^r H^0(X, \Omega^1_{X/\mathcal{O}_K})$ and $u_1, u_2 \in \mathcal{X}(\mathcal{O}_K)$, $(u_1 + u_2)^r \omega = u_1^r \omega + u_2^r \omega$ (cf. [Fon82] Proposition 3). Therefore, (5.1.1) induces a homomorphism of $\mathcal{O}_K$-modules

\[(5.1.2) \quad \rho_1 : p^r H^0(\mathcal{X}, \Omega^1_{\mathcal{O}/\mathcal{O}_K}) \rightarrow \text{Hom}_{\mathcal{O}_K}(X(\overline{K}), \Omega^1_{\mathcal{O}/\mathcal{O}_K}), \quad \omega \mapsto (u \mapsto u^*\omega).\]

We may also assume that $p^r H^0(\mathcal{X}, \Omega^1_{\mathcal{O}/\mathcal{O}_K})$ has no $p$-torsion for further use.

5.2. The functor $V_p(-)$ gives us an injective homomorphism

\[(5.2.1) \quad \rho_2 : \text{Hom}_{\mathcal{O}_K}(X(\overline{K}), \Omega^1_{\mathcal{O}/\mathcal{O}_K}) \rightarrow \text{Hom}_{\mathcal{O}_K}(V_p(X), V_p(\Omega^1_{\mathcal{O}/\mathcal{O}_K}))\]

since $X(\overline{K})$ is $p$-divisible (cf. [Fon82] 3.5 Lemma 1).

5.3. The composition $\rho_2 \circ \rho_1$ induces a homomorphism of $K$-modules

\[(5.3.1) \quad H^0(X, \Omega^1_{X/\mathcal{O}_K}) = K \otimes_{\mathcal{O}_K} p^r H^0(\mathcal{X}, \Omega^1_{\mathcal{O}/\mathcal{O}_K}) \rightarrow \text{Hom}_{\mathcal{O}_K}(V_p(X), V_p(\Omega^1_{\mathcal{O}/\mathcal{O}_K})).\]

As the category of $\mathcal{O}_K$-proper models of $X$ is connected, this composition does not depend on the choice of the model and number $r$ (cf. [Fon82] Proposition 4). We conclude by the following lemma that (5.3.1) is injective.

**Lemma 5.4** ([Fon82] 3.5 Lemma 1). There is a proper model $\mathcal{X}$ of $X$ such that $\rho_1$ is injective.

**Proof.** We follow closely the proof of ([Fon82] 3.5 Lemma 1), which does not essentially use the assumption that the residue field $k$ is perfect. We briefly sketch how to adapt Fontaine’s proof.

1. Let $u$ be the origin of $X$ and $d$ the dimension of $X$. We first take a closed immersion $X \rightarrow \mathbb{P}^n_K$, and then we take an open immersion $\mathbb{P}^n_K \rightarrow \mathbb{P}^n_{\mathcal{O}_K}$ described later (all the morphisms are over $\mathcal{O}_K$). Let $\mathcal{X}$ be the scheme theoretic image of the composition $X \rightarrow \mathbb{P}^n_{\mathcal{O}_K}$, which is thus a proper model of $X$. Let $\overline{u}$ be the special point of the scheme theoretic image of $u$. It is a $k$-point. After a linear transformation of coordinates, we can at first choose an open immersion $\mathbb{P}^n_K \rightarrow \mathbb{P}^n_{\mathcal{O}_K}$ such that $\mathcal{O}_{\mathcal{X}, \overline{u}}$ is a $(d+1)$-dimensional regular local ring (cf. [Fon82] 3.6 Lemma 3).

2. The $m_{\mathcal{X}, \overline{u}}$-adic completion of the local ring $\mathcal{O}_{\mathcal{X}, \overline{u}}$ is isomorphic to $\mathcal{O}_K[[T_1, \ldots, T_d]]$, denoted by $\widehat{\mathcal{O}}_{\mathcal{X}, \overline{u}}$. The $m_{\mathcal{X}, \overline{u}}$-adic completion of $\Omega^1_{\mathcal{O}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K}$ is a free $\widehat{\mathcal{O}}_{\mathcal{X}, \overline{u}}$-module of rank $d$, denoted by $\widehat{\Omega}^1_{\mathcal{O}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K}$. The invariance of differential forms over $X$ and the fact that $p^r H^0(\mathcal{X}, \Omega^1_{\mathcal{O}/\mathcal{O}_K}) \subseteq H^0(X, \Omega^1_{X/\mathcal{O}_K})$ imply that the canonical map $p^r H^0(\mathcal{X}, \Omega^1_{\mathcal{O}/\mathcal{O}_K}) \rightarrow \Omega^1_{\mathcal{O}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K}$ is injective (cf. [Fon82] 3.7). We remark that the canonical map $\Omega^1_{\mathcal{O}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K} \rightarrow \widehat{\Omega}^1_{\mathcal{O}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K}$ is injective as $\Omega^1_{\mathcal{O}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K}$ is of finite type over the Noetherian local ring $\mathcal{O}_{\mathcal{X}, \overline{u}}$.

3. We have the following commutative diagram

\[(5.4.1) \quad \begin{array}{ccc}
p^r H^0(\mathcal{X}, \Omega^1_{\mathcal{O}/\mathcal{O}_K}) & \xrightarrow{\rho_1} & \text{Hom}_{\mathcal{O}_K}(X(\overline{K}), \Omega^1_{\mathcal{O}/\mathcal{O}_K}) \\
\widehat{\Omega}^1_{\mathcal{O}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K} & \xleftarrow{\rho'_1} & \text{Map}(\text{Hom}_{\mathcal{O}_K}(\widehat{\mathcal{O}}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K), \Omega^1_{\mathcal{O}/\mathcal{O}_K})
\end{array}\]

where we identify the set of continuous $\mathcal{O}_K$-algebra homomorphisms from $\widehat{\mathcal{O}}_{\mathcal{X}, \overline{u}}$ to $\mathcal{O}_K$ with a subset of $\mathcal{X}(\overline{K})$. To show the injectivity of $\rho_1$ it suffices to show that of $\rho'_1$. More precisely, we need to show that for any nonzero formal differential form $\sum_{i=1}^d a_i(T_1, \ldots, T_d) dT_i$ where $a_i \in \mathcal{O}_K[[T_1, \ldots, T_d]]$, there are $x_1, \ldots, x_d \in m_{\mathcal{X}, \overline{u}}$ such that $\sum_{i=1}^d a_i(x_1, \ldots, x_d) dx_i$ is not zero in $\widehat{\Omega}^1_{\mathcal{O}_{\mathcal{X}, \overline{u}}/\mathcal{O}_K}$.

4. For $d = 1$, suppose $a(T) = \sum_{k \geq 0} a_k T^k$ where $a_k \in \mathcal{O}_K$ not all zero. Let $k_0$ be the minimal number such that $v_p(a_{k_0})$ is minimal. For a sufficiently large integer $N$, we take $x = \omega^{1/p^N} \in m_{\mathcal{X}, \overline{u}}$, where $\omega$ is a...
uniformizer of $O_K$, such that $v_p(a_{k_0}x^{k_0}) < v_p(a_{k_0}x^k)$ for any $k \neq k_0$. Let $M = \bigcup_{l,m \geq 0} K(w_{lm}) \subseteq \overline{K}$. The annihilator of $dx$ in $\Omega^1_{O_K(x)/O_M}$ is generated by $p^N x^{p^N - 1}$. As $M$ has perfect residue field, lemma 3.1 implies that the annihilator of $dx$ in $\Omega^1_{O_K(x)/O_M}$ is again generated by $p^N x^{p^N - 1}$. When $N$ is big enough, $\alpha(x)dx$ is not zero in $\Omega^1_{O_K(x)/O_K}$ (cf. [Fon82] 3.7 Lemme 4).

(5) As $O_K$ is an infinite domain, there are formal series $\beta_1, \ldots, \beta_d \in O_K [T]$ without constant term, such that $\sum_{i=1}^d \alpha_i (\beta_1, \ldots, \beta_d) \cdot \beta_i' \in O_K [T]$ is still nonzero. Hence, the general case reduces to the case $d = 1$ (cf. [Fon82] 3.7 Lemme 5).

□

5.5. As $X/K$ is $p$-divisible, we have a canonical exact sequence

\[ 0 \longrightarrow T_p(X) \longrightarrow V_p(X) \longrightarrow X(K) \longrightarrow 0. \]

After applying the functor $\text{Hom}_{\mathbb{Z}[G_K]}(-, V_p(\Omega^1_{O_K/O_K}))$, we get an exact sequence

\[ 0 \longrightarrow \text{Hom}_{\mathbb{Z}[G_K]}(X(K), V_p(\Omega^1_{O_K/O_K})) \longrightarrow \text{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{O_K/O_K})) \longrightarrow \text{Hom}_{\mathbb{Z}[G_K]}(T_p(X), V_p(\Omega^1_{O_K/O_K})). \]

Let $f : X(K) \rightarrow V_p(\Omega^1_{O_K/O_K})$ be a $G_K$-equivariant homomorphism. For any finite extension $L/K$, we denote by $G_L = \text{Gal}(\overline{K}/L)$ the absolute Galois group of $L$. Then $f$ maps $X(L)$ to $V_p(\Omega^1_{O_K/O_K})^{G_L}$. We notice that the kernel of the surjection $\Omega^1_{O_K/O_K} \rightarrow \Omega^1_{O_L/O_L}$ is killed by a power of $p$, which indicates that the map $V_p(\Omega^1_{O_K/O_K}) \rightarrow V_p(\Omega^1_{O_K/O_K})$ is an isomorphism. Now, by applying (5.5.2) to $L$, we get

\[ V_p(\Omega^1_{O_K/O_K})^{G_L} = V_p(\Omega^1_{O_K/O_L})^{G_L} = 0. \]

Hence $f(X(K)) = \bigcup_{L/K} f(X(L)) = 0$, which indicates that we have an injective map (cf. [Fon82] 3.5 Lemma 2)

\[ \rho_3 : \text{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{O_K/O_K})) \longrightarrow \text{Hom}_{\mathbb{Z}[G_K]}(T_p(X), V_p(\Omega^1_{O_K/O_K})). \]

Remark that any element in the image of $\rho_3 \circ \rho_2 \circ \rho_1$ is $\mathbb{Z}_p$-linear. All in all, we have generalized Fontaine’s injection ([Fon82] Théorème 2') to the imperfect residue field case.

**Theorem 5.6.** There is a canonical $K$-linear injective homomorphism

\[ \rho : H^0(X, \Omega^1_{X/K}) \longrightarrow \text{Hom}_{\mathbb{Z}[G_K]}(T_p(X), V_p(\Omega^1_{O_K/O_K})). \]

6. **Weak Hodge-Tate Representations**

**Definition 6.1.** For any $C$-$G_K$-module $V$ of finite dimension, let

\[ 0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V \]

be a composition series of $V$, i.e. $V_{i+1}/V_i$ is an irreducible $C$-$G_K$-module for any $i$. The set of factors $\{V_{i+1}/V_i\}_{0 \leq i < n}$ does not depend on the choice of the composition series by Schreier refinement theorem. We call the multiset

\[ \text{wt}(V) = \{ r_i \mid V_{i+1}/V_i \cong C(r_i), \ 0 \leq i < n \} \]

the multiset of weak Hodge-Tate weights of $V$. If all the factors are Tate twists of $C$, i.e. $\dim_C V$ equals the cardinality of $\text{wt}(V)$, then we call $V$ a weak Hodge-Tate $C$-representation of $G_K$. We denote by $\mathcal{C}$ the full subcategory of finite-dimensional $C$-$G_K$-modules formed by weak Hodge-Tate representations.

**Proposition 6.2.** Let $V$ be a finite-dimensional $C$-$G_K$-module.

(i) For any short exact sequence of finite-dimensional $C$-$G_K$-modules $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$, we have $\text{wt}(V) = \text{wt}(V') \sqcup \text{wt}(V'')$. In particular, $\mathcal{C}$ is closed under taking subrepresentation, quotient and extension.

(ii) For the dual representation $V^* = \text{Hom}_C(V, C)$, we have $\text{wt}(V^*) = - \text{wt}(V)$.

Proof. The first assertion follows from the basic properties of composition series. The second assertion follows from the basic fact $C(r)^* = C(-r)$. □

**Proposition 6.3.** For $s \in \mathbb{N}$ and $r \in \mathbb{Z}$, the subrepresentations and quotients of $C(r)^{\oplus s}$ in $\mathcal{C}$ are direct summands of $C(r)^{\oplus s}$ of the form $C(r)^{\oplus t}$ for some $t \in \mathbb{N}$. 

Proof. After twisting by $-r$, we may assume that $r = 0$. For any subrepresentation $V$ of $C^\oplus s$, we set $W = C^\oplus s / V$. Consider the following commutative diagram

\[
\begin{array}{c}
0 & \longrightarrow & V^{G_K} \otimes_K C & \longrightarrow & C^\oplus s & \longrightarrow & W^{G_K} \otimes_K C & \longrightarrow & 0.
\end{array}
\]

We see that the first and third vertical maps are injective, because $K$-linearly independent $G_K$-invariant elements are also $C$-linearly independent. But the middle map is identity, which shows that $V = V^{G_K} \otimes_K C$, $W = W^{G_K} \otimes_K C$. Then any splitting of $0 \to V^{G_K} \to K^\oplus s \to W^{G_K} \to 0$ induces a splitting of $0 \to V \to C^\oplus s \to W \to 0$, which completes our proof. \qed

Proposition 6.4. For $s, t \in \mathbb{N}$ and integers $r_1, r_2$ such that $r_1 - r_2 \neq 1$ or $0$, any extension of $C(r_2)^\oplus s$ by $C(r_1)^\oplus t$ in $\mathcal{G}$ is trivial.

Proof. After twisting by $-r_2$, we may assume that $r_2 = 0$ and $r_1 = r \neq 1$ or $0$. Given an exact sequence $0 \to C(r)^\oplus t \to V \to C^\oplus s \to 0$, take $G_K$-invariants, then we obtain an exact sequence

\[
0 = (C(r)^\oplus t)^{G_K} \longrightarrow V^{G_K} \longrightarrow K^\oplus s \longrightarrow H^1(G_K, C(r)^\oplus t) = 0,
\]

from which we get an isomorphism $V^{G_K} \cong K^\oplus s$. Hence $V = C(r)^\oplus t \oplus C^\oplus s$. \qed

7. Hodge-Tate Filtration for Abelian Varieties

7.1. We keep the following simplified notation in this section:

(7.1.1) $G = G_K$, $\Omega = \Omega_{\mathcal{O}_F/\mathcal{O}_K}$;

(7.1.2) $K_I = K \otimes_{\mathcal{O}_K} \Omega_{\mathcal{O}_K} \cong K \otimes_{\mathcal{O}_K} \prod_{i \in I} \mathcal{O}_K$ (by (3.9.1));

(7.1.3) $C_I = C \otimes_{\mathcal{O}_C} (\Omega_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \prod_{i \in I} \mathcal{O}_K)^\wedge \cong C \otimes_{\mathcal{O}_K} (\prod_{i \in I} \mathcal{O}_C)^\wedge$ (by (4.4.2));

(7.1.4) $E = \text{Hom}_{\mathbb{Z}_p}(T_p(X), C)$, $E^G(1) = \text{Hom}_{\mathbb{Z}_p[G]}(T_p(X), C) \otimes_K C(1) \subseteq E(1)$.

We remark that the Tate module $T_p(X)$ of the abelian variety $X$ is a finite free $\mathbb{Z}_p$-module. By applying the functor $\text{Hom}_{\mathbb{Z}_p}(T_p(X), -) = E \otimes_C -$ to the Faltings extension (4.4.1), we get an exact sequence of $C$-$G_K$-modules

\[
\begin{array}{c}
0 & \longrightarrow & \text{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) & \longrightarrow & \text{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega)) & \longrightarrow & \text{Hom}_{\mathbb{Z}_p}(T_p(X), C_I) & \longrightarrow & 0.
\end{array}
\]

\[
\begin{array}{c}
E(1) & = E \otimes_C V_p(\Omega) & = E \otimes_C C_I.
\end{array}
\]

We choose a $C$-linear retraction of $\iota$ in (4.4.1) and denote by

\[
\pi : \text{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega)) \longrightarrow \text{Hom}_{\mathbb{Z}_p}(T_p(X), C(1))
\]

the induced $C$-linear homomorphism.

We denote by $\hat{\iota}$ the composition of

\[
\begin{array}{c}
H^0(X, \Omega_{X/K}^1)^{\rho} & \longrightarrow & \text{Hom}_{\mathbb{Z}_p[G]}(T_p(X), V_p(\Omega)) & \longrightarrow & E(1) \longrightarrow & E(1)/E^G(1).
\end{array}
\]

where $\rho$ is the Fontaine's injection (5.6.1).

Lemma 7.2. The canonical map

\[
E^G \otimes_K K_I \longrightarrow (E \otimes_C C_I)^G
\]

is an isomorphism.

Proof. Since $E$ is a finite-dimensional $C$-vector space, the complete absolute value on $C$ extends to a complete absolute value on $E$ uniquely up to equivalence. We fix such an absolute value and still denote it by $| |_p$. Following (4.3.2) and (4.3.3), the conclusion follows from the following descriptions

\[
\begin{align*}
E \otimes_C C_I &= \{(x_i) \in \prod_{i \in I} E \mid \forall N > 0, \exists \text{ finite } J \subseteq I, |x_i|_p < 1/N, \forall i \notin J, \\
E^G \otimes_K K_I &= \{(x_i) \in \prod_{i \in I} E^G \mid \forall N > 0, \exists \text{ finite } J \subseteq I, |x_i|_p < 1/N, \forall i \notin J\}.
\end{align*}
\]

\qed
Lemma 7.3. The map \( \tilde{\rho} \) is injective, and its image lies in the \( G \)-invariants of \( E(1)/E^G(1) \). Moreover, \( \tilde{\rho} \) does not depend on the choice of \( \pi \). Hence, we have a canonical \( K \)-linear injective homomorphism

\[
(7.3.1) \quad \tilde{\rho} : H^0(X, \Omega^1_{X/K}) \to (E(1)/E^G(1))^G.
\]

Proof. We take a \( K \)-basis \( \{ h_i \} \) of \( E^G \). For any \( \omega \in H^0(X, \Omega^1_{X/K}) \), thanks to 7.2, we denote by \( \sum h_i \otimes \alpha_i \in E^G \otimes_K \mathbb{K}_l \) the image of \( \omega \) in \( \text{Hom}_{K}(T_p(X), \mathbb{C}) \) via Fontaine’s injection \( \rho \) (5.6.1) and (7.1.5). Take any lifting \( \tilde{\beta}_i \in V_p(\Omega) \) of \( \alpha_i \) in the Faltings extension (4.4.1). Consider the element

\[
(7.3.2) \quad \rho(\omega) - \sum h_i \otimes \tilde{\beta}_i \in \text{Hom}_{Z_p}(T_p(X), V_p(\Omega)) = E \otimes_C V_p(\Omega).
\]

In fact, it lies in \( E(1) \). For any \( \sigma \in G \),

\[
(7.3.3) \quad \sigma(\rho(\omega)) - \sum h_i \otimes \tilde{\beta}_i - (\rho(\omega) - \sum h_i \otimes \tilde{\beta}_i) = \sum h_i \otimes (\beta_i - \sigma(\beta_i)) \in E^G(1).
\]

Therefore, \( \rho(\omega) - \sum h_i \otimes \beta_i \) is \( G \)-invariant modulo \( E^G(1) \), i.e., it defines an element in \( (E(1)/E^G(1))^G \). Moreover, this element does not depend on the choice of the lifting \( \tilde{\beta}_i \). Indeed, suppose \( \tilde{\beta}_i, \tilde{\beta}'_i \) two liftings of \( \alpha_i \), then \( \beta_i' - \beta_i \in C(1) \) which shows that \( (\rho(\omega) - \sum h_i \otimes \beta_i) - (\rho(\omega) - \sum h_i \otimes \beta'_i) \in E^G(1) \). In particular, \( \tilde{\rho} \) does not depend on the choice of \( \pi \).

Now we show the injectivity of \( \tilde{\rho} \). Suppose that \( \rho(\omega) - \sum h_i \otimes \beta_i = \sum h_i \otimes \gamma_i \in E^G(1) \). Then for any \( \sigma \in G \),

\[
(7.3.4) \quad \sum h_i \otimes (\sigma(\beta_i + \gamma_i) - (\beta_i + \gamma_i)) = 0,
\]

which implies that \( \beta_i + \gamma_i \in V_p(\Omega)^G = 0 \) by (4.5.2). Hence \( \rho(\omega) = 0 \), which forces \( \omega \) to be zero since \( \rho \) is injective.

\[ \square \]

Theorem 7.4. There is a canonical exact sequence of \( C-G_K \)-modules

\[
(7.4.1) \quad 0 \to H^1(X, \mathcal{O}_X) \otimes_K C(1) \to \text{Hom}_{Z_p}(T_p(X), C(1)) \to H^0(X, \Omega^1_{X/K}) \otimes_K C \to 0.
\]

Proof. We set \( d = \dim X = \dim_{K} H^0(X, \Omega^1_{X/K}) \). Then \( T_p(X) \) is a free \( \mathbb{Z}_p \)-module of rank \( 2d \). Lemma 7.3 implies that the weak Hodge-Tate weight 0 of \( E(1) \) has multiplicity \( \geq d \). Let \( X' \) be the dual abelian variety of \( X \), and we set \( E' = \text{Hom}_{Z_p}(T_p(X'), C) \). Due to the fact that \( E' = E(1)^* \) (by Weil pairing) and proposition 6.2, the weak Hodge-Tate weight 1 of \( E(1) \) has multiplicity \( \geq d \). But \( \dim_{C} E(1) = 2d \), which forces these inequalities to be equalities. In particular, \( \tilde{\rho} : H^0(X, \Omega^1_{X/K}) \to (E(1)/E^G(1))^G \) is an isomorphism. Since \( C(1) \) has only trivial extension by \( C^{\otimes d} \) (proposition 6.4), we see that \( C^{\otimes d} \) is a quotient representation of \( E(1) \). By duality again, we see that \( C(1)^{\otimes d} \) is a subrepresentation of \( E(1) \), and thus the canonical injection \( (E(1)/E^G(1))^G \otimes_K C \to E(1)/E^G(1) \) is an isomorphism. Therefore, we have a canonical surjection

\[
(7.4.2) \quad E(1) \to H^0(X, \Omega^1_{X/K}) \otimes_K C.
\]

By duality, \( H^1(X, \mathcal{O}_X) \otimes_K C(1) = H^0(X', \Omega^1_{X/K})^* \otimes_K C(1) \) canonically identifies with a subrepresentation of \( E(1) \). Now (7.4.1) follows from the avoidance of \( C(1)^{\otimes d} \) and \( C^{\otimes d} \).

\[ \square \]

7.5. Let’s complete the proof of the main theorem 1.3. We choose a retraction of \( \iota \) in the Faltings extension (4.4.1). By our construction, we have the following commutative diagram

\[
(7.5.1) \quad \text{Hom}_{Z_p}(T_p(X), C(1)) \xrightarrow{\phi} H^0(X, \Omega^1_{X/K}) \otimes_K C \xrightarrow{\rho} \text{Hom}_{Z_p}(T_p(X), V_p(\Omega))
\]

where \( \phi \) is the surjection in the Hodge-Tate filtration (7.4.1), \( \pi \) is induced by the chosen retraction, and \( \rho \) is the Fontaine’s injection (5.6.1). Consider the following diagram

\[
(7.5.2) \quad \text{Hom}_{Z_p}(T_p(X), C(1)) \xrightarrow{\phi} H^0(X, \Omega^1_{X/K}) \xrightarrow{d'} H^1(G, H^1(X, \mathcal{O}_X) \otimes_K C(1)) \xrightarrow{\rho} \text{Hom}_{Z_p}(T_p(X), V_p(\Omega)) \xrightarrow{\pi'} \text{Hom}_{Z_p}(T_p(X), C(1))
\]
where $\delta'$ is the connecting map associated to (7.4.1), where $-\pi'$ is the surjection in (7.1.5), and where we identify $H^1(X, \mathcal{O}_X)$ with $\text{Hom}_{\mathbb{Z}_p[G]}(T_p(X), C)$ by (7.4.1) and identify $H^1(G, C(1))$ with $K_I$ by (4.5.1), which gives the right vertical arrow. Let $\{h_t\}$ be a $K$-basis of $H^1(X, \mathcal{O}_X)$. For any $\omega \in H^0(X, \Omega^1_{X/K})$, we write $-\pi' (\rho(\omega)) = \sum h_t \otimes \alpha_t$ by 7.2, where $\alpha_t \in K_I$. Let $\beta_t \in V_p(\Omega)$ be the lifting of $\alpha_t$ via the chosen splitting of the Faltings extension. We see by the diagram (7.5.1) that $\rho(\omega) - \sum h_t \otimes \beta_t$ is a lifting of $\omega$ via $\phi$. Thus, $\delta'(\omega)$ is represented by the following 1-cocycle
\begin{equation}
(7.5.3) \quad \sigma \mapsto \sum h_t \otimes (\beta_t - \sigma(\beta_t)), \quad \forall \sigma \in G.
\end{equation}

We notice that $\alpha_t \in K_I$ corresponds to a class in $H^1(G, C(1))$ represented by the following 1-cocycle
\begin{equation}
(7.5.4) \quad \sigma \mapsto \sigma(\beta_t) - \beta_t, \quad \forall \sigma \in G.
\end{equation}

In conclusion, the diagram (7.5.2) is commutative.

7.6. Now we can prove the corollary 1.4 to the main theorem. If the sequence (7.4.1) splits, then the $\phi$ in (7.5.2) is surjective. Hence $\delta'$ is zero map, and so is $\pi' \circ \rho$. Thus, the image of the Fontaine's injection $\rho$ lies in $\text{Hom}_{\mathbb{Z}_p}(T_p(X), C(1))$. We easily see that conversely if the image of the Fontaine's injection $\rho$ lies in $\text{Hom}_{\mathbb{Z}_p}(T_p(X), C(1))$, then the sequence (7.4.1) splits. Moreover, the splitting is unique by the avoidance of $C(1)^{\otimes d}$ and $C^{\otimes d}$.

References

[AG20] Ahmed Abbes and Michel Gros, Les suites spectrales de Hodge-Tate, arXiv preprint arXiv:2003.04714 (2020).
[AGT16] Ahmed Abbes, Michel Gros, and Takeshi Tsuji, The $p$-adic Simpson correspondence, Annals of Mathematics Studies, vol. 193, Princeton University Press, Princeton, NJ, 2016. MR 3444777
[CS17] Ana Caraiani and Peter Scholze, On the generic part of the cohomology of compact unitary Shimura varieties, Ann. of Math. (2) 186 (2017), no. 3, 649–766. MR 3702677
[Fal88] Gerd Faltings, $p$-adic Hodge theory, J. Amer. Math. Soc. 1 (1988), no. 1, 255–299. MR 924705
[Fal02] ———, Almost étale extensions, Astérisque (2002), no. 279, 185–270, Cohomologies $p$-adiques et applications arithmétiques, II. MR 1922831
[Fon82] Jean-Marc Fontaine, Formes différentielles et modules de Tate des variétés abéliennes sur les corps locaux, Invent. Math. 65 (1981/82), no. 3, 379–409. MR 643559
[Gro04] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. Inst. Hautes Études Sci. Publ. Math. (1964), no. 20, 259. MR 0173675
[Hyo86] Osamu Hyodo, On the Hodge-Tate decomposition in the imperfect residue field case, J. Reine Angew. Math. 365 (1986), 97–113. MR 826154
[Jan88] Uwe Jannsen, Continuous étale cohomology, Math. Ann. 280 (1988), no. 2, 207–245. MR 929536
[Sch13] Peter Scholze, Perfectoid spaces: a survey, Current developments in mathematics 2012, Int. Press, Somerville, MA, 2013, pp. 193–227. MR 3204346
[sga72] Groupes de monodromie en géométrie algébrique. I, Lecture Notes in Mathematics, Vol. 288, Springer-Verlag, Berlin-New York, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. MR 0354656
[Sta20] The Stacks project authors, The stacks project, http://stacks.math.columbia.edu, 2020.
[Tat67] J. T. Tate, $p$-divisible groups, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, pp. 158–183. MR 0231827
[Tsu99] Takeshi Tsuji, $p$-adic étale cohomology and crystalline cohomology in the semi-stable reduction case, Invent. Math. 137 (1999), no. 2, 233–411. MR 1705837
[Tsu02] ———, Semi-stable conjecture of Fontaine-Jannsen: a survey, no. 279, 2002, Cohomologies $p$-adiques et applications arithmétiques, II, pp. 323–370. MR 1922833

TONGMU HE, INSTITUT DES HAUTES TUDIES SCIENTIFIQUES, 35 ROUTE DE CHARTRES, 91440 BURES-SUR-YVETTE, FRANCE

E-mail address: he@ihes.fr