Neutral Bound States in Kink-like Theories

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Abstract

In this paper we present an elementary derivation of the semi-classical spectrum of neutral particles in a field theory with kink excitations. In the non-integrable cases, we show that each vacuum state cannot generically support more than two stable particles, since all other neutral excitations are resonances, which will eventually decay. A phase space estimate of these decay rates is also given. This shows that there may be a window of values of the coupling constant where a particle with higher mass is more stable than the one with lower mass. We also discuss the crossing symmetry properties of the semiclassical form factors and the possibility of extracting the elastic part of the kink $S$-matrix below their inelastic threshold. We present the analysis of theories with symmetric and asymmetric wells, as well as of those with symmetric or asymmetric kinks. Illustrative examples of such theories are provided, among others, by the Tricritical Ising Ising, the Double Sine Gordon model and by a class of potentials recently introduced by Bazeira et al.

\begin{flushleft}
\textsuperscript{1}On leave of absence from International School for Advanced Studies, Trieste (Italy)
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1 Introduction

The aim of this paper is to explain in the easiest possible way the presence of neutral bound states in two dimensional field theories with kink topological excitations. The theories that we will consider are those described by a scalar real field $\varphi(x)$, with a Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \varphi)^2 - U(\varphi) ,$$

(1.1)

where the potential $U(\varphi)$ possesses several degenerate minima at $\varphi^0_a (a = 1, 2, \ldots, n)$, as the one shown in Figure 1. These minima correspond to the different vacua $|a\rangle$ of the associate quantum field theory.

Figure 1: Potential $U(\varphi)$ of a quantum field theory with kink excitations.

The basic excitations of this kind of models are kinks and anti-kinks, i.e. topological configurations which interpolate between two neighbouring vacua. Semiclassically they correspond to the static solutions of the equation of motion, i.e.

$$\partial^2_x \varphi(x) = U''[\varphi(x)] ,$$

(1.2)

with boundary conditions $\varphi(-\infty) = \varphi^0_a$ and $\varphi(+\infty) = \varphi^0_b$, where $b = a \pm 1$. This equation can be equivalently expressed in terms of a first order differential equation

$$\frac{d\varphi}{dx} = \pm \sqrt{2U(\varphi)} .$$

(1.3)

Denoting by $\varphi_{ab}(x)$ the solutions of this equation, their classical energy density is given by

$$\epsilon_{ab}(x) = \frac{1}{2} \left( \frac{d\varphi_{ab}}{dx} \right)^2 + U(\varphi_{ab}(x)) ,$$

(1.4)
and its integral provides the classical expression of the kink masses

$$M_{ab} = \int_{-\infty}^{\infty} \epsilon_{ab}(x) .$$

(1.5)

As a rule of thumb, it is useful to notice that the classical masses of the kinks $\varphi_{ab}(x)$ are simply proportional to the heights of the potential between the two minima $\varphi_a^{(0)}$ and $\varphi_b^{(0)}$.

The classical solutions can be set in motion by a Lorentz transformation, i.e. $\varphi_{ab}(x) \rightarrow \varphi_{ab}[(x \pm vt)/\sqrt{1 - v^2}]$. In the quantum theory, these configurations describe the kink states $| K_{ab}(\theta) \rangle$, where $a$ and $b$ are the indices of the initial and final vacuum, respectively. The quantity $\theta$ is the rapidity variable which parameterises the relativistic dispersion relation of these excitations, i.e.

$$E = M_{ab} \cosh \theta , \quad P = M_{ab} \sinh \theta .$$

(1.6)

Conventionally $| K_{a,a+1}(\theta) \rangle$ denotes the kink between the pair of vacua $\{|a\rangle, |a+1\rangle\}$ while $| K_{a+1,a} \rangle$ is the corresponding anti-kink. For the kink configurations it may be useful to adopt the simplified graphical form shown in Figure 2.

![Figure 2: Kink and antikink configurations.](image)

The multi-particle states are given by a string of these excitations, with the adjacency condition of the consecutive indices for the continuity of the field configuration

$$| K_{a_1,a_2}(\theta_1) K_{a_2,a_3}(\theta_2) K_{a_3,a_4}(\theta_3) \ldots \rangle , \quad (a_{i+1} = a_i \pm 1)$$

(1.7)

Although very convenient, one should keep in mind that this graphical representation oversimplifies the exponential approaching to the vacua, which can be different for the one to the right and for the one to the left.
In addition to the kinks, in the quantum theory there may exist other excitations in the guise of ordinary scalar particles (breathers). These are the neutral excitations $| B_c(\theta) \rangle_a$ ($c = 1, 2, \ldots$) around each of the vacua $| a \rangle$. For a theory based on a Lagrangian of a single real field, these states are all non-degenerate: in fact, there are no extra quantities which commute with the Hamiltonian and that can give rise to a multiplicity of them. The only exact (alias, unbroken) symmetries for a Lagrangian as (1.1) may be the discrete ones, like the parity transformation $P$, for instance, or the charge conjugation $C$. However, since they are neutral excitations, they will be either even or odd eigenvectors of $C$.

The neutral particles must be identified as the bound states of the kink-antikink configurations that start and end at the same vacuum $| a \rangle$, i.e. $| K_{ab}(\theta_1) K_{ba}(\theta_2) \rangle$, with the “tooth” shapes shown in Figure 3.

$$| K_{ab}(\theta_1) K_{ba}(\theta_2) \rangle \simeq i g^{c}_{ab} \theta - i u^{c}_{ab} \langle B_c \rangle_a . \quad (1.8)$$

In this expression $g^{c}_{ab}$ is the on-shell 3-particle coupling between the kinks and the neutral particle. Moreover, the mass of the bound states is simply obtained by substituting the resonance value $i u^{c}_{ab}$ within the expression of the Mandelstam variable $s$ of the two-kink channel

$$s = 4 M^2_{ab} \cosh^2 \frac{\theta}{2} \quad \rightarrow \quad m_c = 2 M_{ab} \cos \frac{u^{c}_{ab}}{2} . \quad (1.9)$$

Since the kink $\varphi_{a-1,a}(x)$ that interpolates to the vacuum on the left of $| a \rangle$ may be different from the kink $\varphi_{a,a+1}(x)$ which interpolates to the vacuum on the right,
a-priori there could be two different towers of breathers that pile up in each vacuum: the first coming from the poles of \( | K_{a,a-1}(\theta_1)K_{a-1,a}(\theta) \rangle \) while the second one from the poles of \( | K_{a,a+1}(\theta)K_{a+1,a}(\theta_2) \rangle \). However, as we will discuss in the following, this situation cannot occur.

Concerning the vacua themselves, as well known, in the infinite volume their classical degeneracy is removed by selecting one of them, say \( | k \rangle \), out of the \( n \) available. This happens through the usual spontaneously symmetry breaking mechanism, even though – strictly speaking – there may be no internal symmetry to break at all. This is the case, for instance, of the potential shown in Figure 1, which does not have any particular invariance. In the absence of a symmetry which connects the various vacua, the world – as seen by each of them – may appear very different: they can have, indeed, different particle contents. The problem we would like to examine in this paper concerns the neutral excitations around each vacuum, in particular the question of the existence of such particles and of the value of their masses.

The answer provided by the perturbation theory to this question is straightforward: after defining \( \eta(x) \equiv \phi(x) - \phi_a(0) \) and making a Taylor expansion of \( U(\phi) \) near \( \phi_a(0) \)

\[
U(\phi_a(0) + \eta) = \frac{1}{2} \omega_a^2 \eta^2 + \frac{1}{3} \lambda_3 \eta^3 + \frac{1}{4} \lambda_4 \eta^4 + \cdots \quad (1.10)
\]

one identifies the mass \( m \) of the fundamental particle around the vacuum \( | a \rangle \) and \( \omega_a \), while the rest of the expansion with its interaction terms. The quantity \( \omega_a \) is of course the zero-order value of \( m \) of such a particle, but the crucial point is another one: according to the perturbation theory, as far as the potential has a quadratic curvature at its minimum, there is always a neutral excitation above the corresponding vacuum state. This conclusion is, unfortunately, false.

A famous counter-example is given by the Sine-Gordon model, i.e. the quantum field theory associated to the potential

\[
U_{SG}(\varphi) = \frac{m_0^2}{\beta^2} \left[ 1 - \cos(\beta \varphi) \right] , \quad (1.11)
\]

where \( m_0 \) is a mass-like parameter and \( \beta \) is a coupling constant. Such a theory has an infinite number of degenerate vacua \( | a \rangle \) \( (a = 0, \pm 1, \pm 2, \ldots) \), localised at \( \phi_a(0) = 2\pi a / \beta \), each of them with the same curvature \( \omega^2 = m_0^2 \beta^2 \). Through the above perturbative argument, one would conclude that each vacuum has always, at least, one neutral excitation. On the other hand, the exact S-matrix of this model \( \Pi \) shows that the situation is rather different: indeed, such a particle does not exist if \( \beta^2 > 4\pi \).

As in the case of the Sine-Gordon model, the knowledge of the exact S-matrix of a quantum problem would obviously provide a clear cut answer to the question of the
particle content of a theory: a proper identification of its poles gives its spectrum. This has been amply proved by the large number of the exact S-matrices associated to the integrable deformations of Conformal Field Theories \[2\] (for a review, see \[3\]). But, what happens if the theory is not integrable? How can we proceed if the exact S matrix is not known?

2 A semiclassical formula

The particle content of certain non-integrable models can be studied by using the so-called Form Factor Perturbation Theory \[4, 5, 6\]. As shown in \[7\], this approach can be also extended to compute, in a reliable way, the decay widths of the unstable particles of the theory\(^2\). In addition to this approach, another interesting route for investigating the non-integrable models comes from semi-classical methods. Originally proposed by Dashen-Hasslacher-Neveu \[9\] and by Goldstone-Jackiw \[10\], this approach has been recently applied either to study quantum field theories on a finite volume \[12, 13, 14\] or to obtain their spectrum at the semiclassical level \[15\].

The main difference between the two approaches is the following. The Form Factor Perturbation Theory is a formalism based on the S-matrix theory. More precisely, it moves its first steps with the exact scattering amplitudes of the integrable models, reached as a limit of the non-integrable ones. On the contrary, the semi-classical methods build their analysis on the Lagrangian density of the model, irrespectively whether it describes an integrable system or not. As shown in \[15\], the two approaches sometimes coincide while, in some other cases, they complement each other, i.e. one needs both methods to recover the whole spectrum of the theory.

For the problem that concerns this paper – to find the neutral spectrum of the Lagrangian theory \[1.1\] – the semiclassical methods are the obvious choice. Their correct implementation may need, though, important pieces of information coming from the S-matrix theory or from the Form Factor Perturbation Theory. The starting point of our analysis is a remarkably simple formula due to Goldstone-Jackiw \[10\]. In its refined version, given in \[11\] and rediscovered in \[12\], it reads as

\(^2\)The analytic prediction of the paper \[7\] for the decay widths of the unstable particles of the Ising model has been confirmed by the numerical analysis done in \[8\].
follows \(^3\) (Figure 4)

\[
f_{ab}^2(\theta) = \langle K_{ab}(\theta_1) \mid \varphi(0) \mid K_{ab}(\theta_2) \rangle \simeq \int_{-\infty}^{\infty} dx e^{iM_{ab}\theta x} \varphi_{ab}(x) ,
\]

where \(\theta = \theta_1 - \theta_2\).

\[\text{Figure 4: Matrix element between kink states.}\]

Notice that, if we substitute in the above formula \(\theta \rightarrow i\pi - \theta\), the corresponding expression may be interpreted as the following Form Factor

\[
F_{ab}^\varphi(\theta) = f(i\pi - \theta) = \langle a \mid \varphi(0) \mid K_{ab}(\theta_1) K_{ab}(\theta_2) \rangle .
\]

In this matrix element, it appears the neutral kink states around the vacuum \(|a\rangle\) we are interested in.

By following the references \([10, 11, 12, 18]\), let’s firstly recall the main steps that lead to this formula and let’s make the first comments on its content. Denoting the adimensional coupling constant of the theory generically by \(g\), we will assume that the mass of the kink becomes arbitrarily large when \(g \rightarrow 0\), say as \(M_{ab} \simeq 1/g\). Consider now the Heisenberg equation of motion satisfied by the field \(\varphi(x)\)

\[
\left(\partial_t^2 - \partial_x^2\right) \varphi(x, t) = -U'[\varphi(x, t)] ,
\]

and sandwich it between the kink states of momentum \(p_1\) and \(p_2\). By using

\[
\langle K_{ab}(p_1) \mid \varphi(x, t) \mid K_{ab}(p_2) \rangle = e^{-ip_1 \cdot x} \langle K_{ab}(p_1) \mid \varphi(0) \rangle \langle K_{ab}(p_2) \rangle
\]

\[\text{3The matrix element of the field } \varphi(y) \text{ is easily obtained by using } \varphi(y) = e^{-ip_\nu y_\nu} \varphi(0) e^{ip_\nu y_\nu} \text{ and by acting with the conserved energy-momentum operator } P_\mu \text{ on the kink state. Moreover, for the semiclassical matrix element } F_{ab}^\varphi(\theta) \text{ of the operator } G[\varphi(0)], \text{ one should employ } G[\varphi_{ab}(x)]. \text{ For instance, the matrix element of } \varphi^2(0) \text{ are given by the Fourier transform of } \varphi_{ab}(x).\]

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we have

\[-(p_1 - p_2)_\mu (p_1 - p_2)^\mu] \langle K_{ab}(p_1) \mid \varphi(0) \mid K_{ab}(p_2) \rangle
\]
\[= -\langle K_{ab}(p_1) \mid U'[\varphi(0)] \mid K_{ab}(p_2) \rangle . \tag{2.16}\]

Once it has been extracted the $x^\mu$-dependence of the matrix element (2.15), the remaining expression $\langle K_{ab}(p_1) \mid \varphi(0) \mid K_{ab}(p_2) \rangle$ should depend on the relativistic invariants of the channel of the two kinks. Since these invariants can be expressed in terms of difference of the rapidities of the two kinks, this suggests to adopt the rapidity variables and write eq. (2.16) as

\[2 M_{ab}^2 (\cosh \theta - 1) \langle K_{ab}(\theta_1) \mid \varphi(0) \mid K_{ab}(\theta_2) \rangle
\]
\[= -\langle K_{ab}(\theta_1) \mid U'[\varphi(0)] \mid K_{ab}(\theta_2) \rangle , \tag{2.17}\]

where $\theta = \theta_1 - \theta_2$. Let’s now assume that the kinks are sufficiently slow, so that their dispersion relations can be approximated by the non-relativistic expressions

\[E = M \cosh \theta \simeq M \left(1 + \frac{\theta^2}{2}\right) , \quad P = M \sinh \theta \simeq M \theta \ll M . \tag{2.18}\]

In this quasi-static regime, we can define the matrix element of the field $\varphi(0)$ between the kink states as the Fourier transform with respect to the Lorentz invariant difference $\theta = \theta_1 - \theta_2$

\[f_{ab}^\varphi(\theta) = \langle K_{ab}(\theta_1) \mid \varphi(0) \mid K_{ab}(\theta_2) \rangle \simeq \int_{-\infty}^{\infty} dx \, e^{iM_{ab} \theta x} \hat{f}(x) , \tag{2.19}\]

with the inverse Fourier transform given by

\[\hat{f}(x) = \int \frac{d\theta}{2\pi} e^{-iM_{ab} \theta x} f_{ab}^\varphi(\theta) . \tag{2.20}\]

In the quasi-static limit, the left hand side of equation (2.17) becomes

\[M_{ab}^2 \theta^2 f_{ab}^\varphi(\theta) , \tag{2.21}\]

which, in real space, corresponds to

\[-\frac{d^2}{dx^2} \hat{f}_{ab}(x) . \tag{2.22}\]

Concerning the right hand side, let’s assume that $U'[\varphi(0)]$ can be expressed in terms of powers of the field $\varphi(x)$ (either as a finite sum or an infinite series)

\[U'[\varphi] = \sum_{n=1}^{\infty} \alpha_n \varphi^n . \tag{2.23}\]
Consider now the matrix elements of the generic term of this expression between the kink states
\[
\langle K_{ab}(\theta_1) | \varphi^n(0) | K_{ab}(\theta_2) \rangle .
\] (2.24)

By inserting \((n - 1)\) times a complete set of state, we have
\[
\sum_{m_1, \ldots, m_{n-1}} \langle K_{ab}(\theta_1) | \varphi(0) | m_1 \rangle \langle m_1 | \varphi(0) | m_2 \rangle \ldots \langle m_{n-1} | \varphi(0) | K_{ab}(\theta_2) \rangle .
\] (2.25)

The only states which are involved in the above sums are those having the same topological charge of the kink \(K_{ab}\), with the lowest mass states given precisely by the kinks \(K_{ab}(\theta_i)\) themselves. By truncating the sums just on these states and using the definition (2.19), we have then
\[
\langle K_{ab}(\theta_1) | \varphi^n(0) | K_{ab}(\theta_2) \rangle \approx \int_{-\infty}^{\infty} dx e^{iM_{ab} \theta x} \left( \hat{f}(x) \right)^n .
\] (2.26)

Hence, at the leading order \(1/g\), the function \(\hat{f}(x)\) satisfies the same differential equation (1.2) satisfied by the static kink solution, i.e.
\[
\frac{d^2}{dx^2} \hat{f}_{ab}(x) = U'[f_{ab}(x)] ,
\] (2.27)

arriving then to the result (2.12).

The appealing aspect of the formula (2.12) stays in the relation between the Fourier transform of the classical configuration of the kink, – i.e. the solution \(\varphi_{ab}(x)\) of the differential equation (1.3) – to the quantum matrix element of the field \(\varphi(0)\) between the vacuum \(|a\rangle\) and the 2-particle kink state \(|K_{ab}(\theta_1) K_{ba}(\theta_2)\rangle\). Once the solution of eq. (1.3) has been found and its Fourier transform has been taken, the poles of \(F_{ab}(\theta)\) within the physical strip of \(\theta\) identify the neutral bound states which couple to \(\varphi\). Then, their mass can be extracted by using eq. (1.9), while the on-shell 3-particle coupling \(g_{ab}^c\) can be obtained from the residue at these poles (Figura 5).

\[
\lim_{\theta \rightarrow i u_{ab}^c} \left( \theta - i u_{ab}^c \right) F_{ab}(\theta) = i g_{ab}^c \langle a | \varphi(0) | B_c \rangle .
\] (2.28)

It is important to stress that, for a generic theory, the classical kink configuration \(\varphi_{ab}(x)\) is not related in a simple way to the anti-kink configuration \(\varphi_{ba}(x)\). It is precisely for this reason that neighbouring vacua may have a different spectrum of neutral excitations, as shown in the examples discussed in the following sections.

It is also worth noting that this procedure for extracting the bound states masses permits in many cases to avoid the semiclassical quantization of the breather solutions [2], making their derivation much simpler. The reason is that, the classical
breather configurations depend also on time and have, in general, a more complicated structure than the kink ones. Yet, in non–integrable theories these configurations do not exist as exact solutions of the partial differential equations of the field theory. On the contrary, in order to apply eq. (2.12), one simply needs the solution of an ordinary differential equation, the one given by (1.3): in the absence of an exact expression, one could even conceive the idea of employing the solution extracted by a numerical integration of the equation (1.3), an operation which requires few seconds on any laptop4.

Let’s now add a remark of technical nature: the Fourier transform (3.32) has always a singular part, due to the constant asymptotic behaviours of $\varphi(x)$ at $x \to \pm \infty$. This piece can be easily isolated by splitting the integral as follows

$$f_{ab}(k) = \int_{-\infty}^{\infty} dx \, e^{ikx} \varphi_{ab}(x) = \int_{-\infty}^{0} dx \, e^{ikx} \varphi_{ab}(x) + \int_{0}^{\infty} dx \, e^{ikx} \varphi_{ab}(x)$$

$$= \int_{-\infty}^{0} dx \, e^{ikx} [\varphi_{ab}(x) - \varphi_a^{(0)}] + \int_{0}^{\infty} dx \, e^{ikx} [\varphi_{ab}(x) - \varphi_b^{(0)}]$$

$$+ \varphi_a^{(0)} \int_{-\infty}^{0} dx \, e^{ikx} + \varphi_b^{(0)} \int_{0}^{\infty} dx \, e^{ikx} .$$

The singular part is encoded in the last two terms, for which we have

$$\varphi_a^{(0)} \int_{-\infty}^{0} dx \, e^{ikx} + \varphi_b^{(0)} \int_{0}^{\infty} dx \, e^{ikx} = i(\varphi_a^{(0)} - \varphi_b^{(0)}) \, P \left( \frac{1}{k} \right) + \pi (\varphi_a^{(0)} + \varphi_b^{(0)}) \delta(k) ,$$

4One might doubt about the possibility to locate the poles of $F_{ab}(\theta)$ by performing a Fourier transform of the numerical solution. However, their position can be easily determined in a different way, i.e. by looking at the exponential behavior of the solutions at $x \to \pm \infty$, as discussed in the next section. This behavior can be quickly extracted by numerical solutions and it is also, analytically, perfectly under control.
where \( P \) stays for the Cauchy principal value. Since these singular terms do not contain information on the pole structure of the matrix elements, they will always be discarded from now on. Concerning the regular part, it can be computed by means of the derivative of the kink solutions, by using \( \mathcal{F}\left(\frac{df}{dx}\right) = -ik\mathcal{F}(f) \), where

\[
\mathcal{F}(f) = \int_{-\infty}^{\infty} dk e^{ikx} f(x) .
\]

Additional comments on the nature of the poles of the semiclassical form factors will be given in the next section, with the help of an explicit example.

The range of validity of the formula (2.12) is a more delicate issue. As discussed above, its derivation relies on the basic hypothesis that the kink momentum is very small compared to its mass, and also on the possibility of neglecting intermediate higher particle contributions \([10, 11, 12]\). These two assumptions usually translate into the combined condition \( \theta \approx O(g) \ll 1 \), where \( g \) is the adimensional coupling constant of the theory. This authorises, for instance, to substitute in the result of the Fourier transform, \( \theta \to \sinh \theta \) (since \( \theta \) is infinitesimal), but keeping untouched all expressions containing \( \theta/g \). But, the above constraint may result in a different level of accuracy on various physical quantities, with the precision that may also depend on the model under investigation.

Consider, for instance, the Sine-Gordon model, a theory that will be studied in details in Section 5. Its semiclassical mass formula turns out to be an exact expression, i.e. valid at all orders in the coupling constant till its critical value \( \beta^2_c = 4\pi \), beyond which, the bound states disappear. Moreover, the numerical values of the semiclassical Form Factors do not significantly differ from the exact ones for all real values of \( \theta \). On the wave of these success, one may dare to extract the \( S \)-matrix by using the semiclassical Form Factors. If one tries to do so, the outcoming expression turns out to be remarkably close to the exact one but, it always fails to meet an important crossing symmetric factor. The same happens in other models too, as it will be explained later.

All this to say that, some caution is necessary in handling the results obtained by using the formula (2.12). Morally it has the same status of the WKB approximation in quantum mechanics: this usually provides very accurate results for the discrete states, remaining nevertheless a poor approximation for those of the continuum. But even for the bound states, the formula (2.12) seems sometimes to lead to some puzzling conclusions. This happens, for instance, in the case of the Double Sine-Gordon model \([15, 16, 17]\). Fortunately, there is a way out from the possible paradox that involves the bound states, the solution of this problem being one of the main motivations of this paper. In summary, properly handled, the semiclassical
formalism remains one of the most powerful method for extracting the spectrum of
the bound states in kink-like theories.

In the next two sections we will first analyse a class of theories with only two
vacua, which can be either symmetric or asymmetric ones. After the analysis of
the Sine-Gordon model, we will proceed to discuss the interesting case of a vacuum
state, in communication through asymmetric kinks, with two of its neighbouring
ones. The simplest example of this kind of situation is provided by the Double Sine
Gordon model. As we shall see, the conclusions drawn from all the above cases
enable us to address the study of the most general theories with kink excitations,
as shown by the examples discussed in the last section.

3 Symmetric wells

A prototype example of a potential with two symmetric wells is the $\varphi^4$ theory in its
broken phase. The potential is given in this case by

$$U(\varphi) = \frac{\lambda}{4} \left( \varphi^2 - \frac{m^2}{\lambda} \right)^2 .$$  

(3.30)

Let us denote with $| \pm 1 \rangle$ the vacua corresponding to the classical minima
$\varphi_\pm^{(0)} = \pm \frac{m}{\sqrt{\lambda}}$. By expanding around them, $\varphi = \varphi_\pm^{(0)} + \eta$, we have

$$U(\varphi_\pm^{(0)} + \eta) = m^2 \eta^2 \pm m \sqrt{\lambda} \eta^3 + \frac{\lambda}{4} \eta^4 .$$  

(3.31)

Hence, perturbation theory predicts the existence of a neutral particle for each of
the two vacua, with a bare mass given by $m_b = \sqrt{2} m$, irrespectively of the value of
the coupling $\lambda$. Let’s see, instead, what is the result of the semiclassical analysis.

The kink solutions are given in this case by\(^5\)

$$\varphi_{-a,a}(x) = a \frac{m}{\sqrt{\lambda}} \tanh \left( \frac{mx}{\sqrt{2}} \right) , \quad a = \pm 1$$  

(3.32)

and their classical mass is

$$M_0 = \int_{-\infty}^{\infty} \epsilon(x) \, dx = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} .$$  

(3.33)

The value of the potential at the origin, which gives the height of the barrier between
the two vacua, can be expressed as

$$U(0) = \frac{3m}{8\sqrt{2}} M_0 ,$$  

(3.34)

\(^5\)In the following we will always discard the integration constant $x_0$ on which is localised the
kink solution.
and, as noticed in the introduction, is proportional to the classical mass of the kink.

If we take into account the contribution of the small oscillations around the classical static configurations, the kink mass gets corrected as

\[
M = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} - m \left( \frac{3}{\pi \sqrt{2}} - \frac{1}{2\sqrt{6}} \right) + \mathcal{O}(\lambda) .
\]  

(3.35)

It is convenient to define

\[ c = \left( \frac{3}{2\pi} - \frac{1}{4\sqrt{3}} \right) > 0 , \]

and also the adimensional quantities

\[ g = \frac{3\lambda}{2\pi m^2} ; \quad \xi = \frac{g}{1 - \pi cg} . \]  

(3.36)

In terms of them, the mass of the kink can be expressed as

\[ M = \frac{\sqrt{2}m}{\pi \xi} = \frac{m_b}{\pi \xi} . \]  

(3.37)

Since the kink and the anti-kink solutions are equal functions (up to a sign), their Fourier transforms have the same poles. Hence, the spectrum of the neutral particles will be the same on both vacua, in agreement with the \( Z_2 \) symmetry of the model.

For explicitly computing it, let’s first consider

\[
\left( \frac{d\varphi}{dx} \right)_{-a,a} = a \frac{m^2}{\sqrt{2} \lambda} \frac{1}{\cosh^2 \frac{mx}{\sqrt{2}}} ,
\]  

(3.38)

and then apply the prescription for the Fourier transform given in the previous section. As a result, we have

\[
f_{-a,a}(\theta) = \int_{-\infty}^{\infty} dx \, e^{iM\theta x} \varphi_{-a,a}(x) = i a \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh \left( \frac{\pi M}{2m} \theta \right)} .
\]

By making now the analytical continuation \( \theta \to i\pi - \theta \) and using the above definitions (3.36), we arrive to

\[
F_{-a,a}(\theta) = \langle a \mid \varphi(0) \mid K_{-a,a}(\theta_1)K_{a,-a}(\theta_2) \rangle = A_a \frac{1}{\sinh \left( \frac{(i\pi - \theta)}{2} \right)} ,
\]  

(3.39)

where \( A_a \) is a constant, given by

\[
A_a = i a m \left( \frac{3(1 + \pi c \xi)}{(\pi \xi)} \right)^{1/2} .
\]
The poles of the above expression are located at

\[ \theta_n = i\pi (1 - \xi n) \quad , \quad n = 0, \pm 1, \pm 2, \ldots \quad (3.40) \]

and, if

\[ \xi \geq 1 \quad , \quad (3.41) \]

none of them is in the physical strip \( 0 < \text{Im}\, \theta < \pi \). Consequently, in the range of the coupling constant

\[ \frac{\lambda}{m^2} \geq \frac{2\pi}{3} \frac{1}{1 + \pi c} = 1.02338... \quad (3.42) \]

the theory does not have any neutral bound states, neither on the vacuum to the right nor on the one to the left. Viceversa, if \( \xi < 1 \), there are \( n = \left\lfloor \frac{1}{\xi} \right\rfloor \) neutral bound states, where \( \lfloor x \rfloor \) denote the integer part of the number \( x \). Their semiclassical masses are given by

\[ m_b^{(n)} = 2M \sin \left( \frac{n\pi\xi}{2} \right) = n \, m_b \left[ 1 - \frac{3}{32} \frac{\lambda^2}{m^4} n^2 + \ldots \right] . \quad (3.43) \]

Note that the leading term is given by multiples of the mass of the elementary boson \( |B_1\rangle \). Therefore the \( n \)-th breather may be considered as a loosely bound state of \( n \) of it, with the binding energy provided by the remaining terms of the above expansion.

But, for the non-integrability of the theory, all particles with mass \( m_n > 2m_1 \) will eventually decay. It is easy to see that, if there are at most two particles in the spectrum, it is always valid the inequality \( m_2 < 2m_1 \). However, if \( \xi < \frac{1}{3} \), for the higher particles one always has

\[ m_k > 2m_1 \quad , \quad \text{for } k = 3, 4, \ldots n \quad . \quad (3.44) \]

According to the semiclassical analysis, the spectrum of neutral particles of \( \varphi^4 \) theory is then as follows: (i) if \( \xi > 1 \), there are no neutral particles; (ii) if \( \frac{1}{2} < \xi < 1 \), there is one particle; (iii) if \( \frac{1}{3} < \xi < \frac{1}{2} \) there are two particles; (iv) if \( \xi < \frac{1}{3} \) there are \( \left\lfloor \frac{1}{\xi} \right\rfloor \) particles, although only the first two are stable, because the others are resonances.

The decay processes of the higher particles, \( B_k \rightarrow r \, B_1 + s \, B_2 \), where \( r \) and \( s \) are all those integers which satisfy

\[ m_k \geq r \, m_1 + s \, m_2 \quad , \quad r + s = n \quad (3.45) \]

can be computed by Fermi golden rule

\[ d\Gamma = (2\pi)^2 \delta^2(P - p_1 - \cdots - p_n) \quad |T_{fi}|^2 \quad \frac{1}{2E} \prod_{i=1}^{n} \frac{dp_i}{(2\pi)^2 E_i} . \quad (3.46) \]
Figure 6: Neutral bound states of $\varphi^4$ theory for $g < 1$. The lowest two lines are the stable particles whereas the higher lines are the resonances.

In this formula, $P$ denote the 2-momentum of the decay particle whereas the amplitude $T_{fi}$ is given by the matrix element

$$T_{fi} = \langle B_k(P) \mid B_1(p_1) \ldots B_1(p_s) B_2(p_{s+1}) \ldots B_2(p_n) \rangle.$$  \hspace{1cm} (3.47)

At the moment it is difficult to have control of this matrix element, so the best we can do is to estimate some universal ratios of the decay rates by phase space alone, assuming that the matrix elements are of the same order of magnitude for the various processes. For instance, taking $\xi < \frac{1}{5}$, we can estimate the decay rates

$$\Gamma_{11}^5 \quad \ldots \ldots \quad B_5 \to B_1 + B_1$$
$$\Gamma_{12}^5 \quad \ldots \ldots \quad B_5 \to B_1 + B_2$$
$$\Gamma_{22}^5 \quad \ldots \ldots \quad B_5 \to B_2 + B_2$$  \hspace{1cm} (3.48)

with respect to the decay rate $\Gamma_{11}^3$ of the process $B_3 \to B_1 + B_1$. We have

$$A = \frac{\Gamma_{11}^5}{\Gamma_{11}^3} \approx \frac{m_3}{m_5} \sqrt{\frac{m_3^2 - 4m_1^2}{m_5^2 - 4m_1^2}},$$

$$B = \frac{\Gamma_{12}^5}{\Gamma_{11}^3} \approx \sqrt{\frac{m_2^2 (m_3^2 - 4m_1^2)}{[m_5^2 - (m_1 - m_2)^2][m_5^2 - (m_1 + m_2)^2]}},$$  \hspace{1cm} (3.49)

$$C = \frac{\Gamma_{22}^5}{\Gamma_{11}^3} \approx \frac{m_3}{m_5} \sqrt{\frac{m_3^2 - 4m_1^2}{m_5^2 - 4m_2^2}}.$$

Notice that at $\xi = \xi_{c1} \approx 0.1558...$ the 5-particle has its mass exactly equal to $2m_2$. This makes the amplitude ratio $C$ divergent at this point. For $\xi > \xi_c$, the decay process $B_5 \to B_2 + B_2$ is, instead, obviously forbidden. The plot of the quantities $A, B$ and $C$ in the range $0 < \xi < \frac{1}{5}$ (for the first two) and for $0 < \xi < \xi_{c1}$ for the latter, is shown in Figure 7.
Figure 7: (a): plot of decay ratios $A$ (lower curve) and $B$ (middle curve) as a function of $\xi \in [0, \frac{1}{5}]$. (b): plot of the decay ratio $C$ for $\xi \in [0, \xi_c)$.

Since $\Gamma_c^t = \sum_{a,b} \Gamma_{a,b}^c$ is the inverse of the life-time of the particle $B_c$, from Figure 8a one can see that the higher particle $B_5$ is only slightly more stable than the lower particle $B_3$ for $\xi < \xi_c$. It is only around the critical value $\xi_c$ that there is an enhancement of $C$ and, correspondingly, the life-time of $B_5$ becomes much smaller than the one of $B_3$. For $\xi > \xi_c$, the ratio of the total life-time of the particles $B_5$ and $B_3$ is given only by $(A + B)^{-1}$. Plotting this quantity as a function of $\xi$, one discovers that there is a narrow window of values of $\xi$, given by the interval $[\xi_{c_1}, \xi_{c_2}]$, with $\xi_{c_2} \simeq 0.1612..$, where the life-time of $B_5$ is larger than the one of $B_3$ (Figura 8b). This counter-intuitive behavior of the life-time ratios is a simple consequence of the peculiar properties of the phase space in two dimensions. However, in the decay processes analysed in \[7\], where the transition amplitudes were exactly computed, the tendency of the higher particles to be more stable than the lower ones was shown to be further enhanced by the dynamics. It would be interesting to study whether this is also the case of $\varphi^4$ theory, using, perhaps, the numerical methods introduced in \[8\].

Let us now comment the general scenario emerging from the semiclassical analysis. One could be obviously suspicious about the conclusion of the absence of the bound states for $\lambda > \lambda_c$, with the critical value $\lambda_c$ given in eq. (3.42): after all, this value is not infinitesimal and it might be, in fact, very well out of the realm of validity of the semiclassical approximation. This is a legitimate suspect and, in the absence of an exact solution which either confirms or disproves it, it is fair to say that the spectrum of the theory in such a strong coupling regime essentially remains an open question. Nevertheless, we would like to draw the attention on the following
arguments in favour of the semiclassical analysis:

1. Concerning $\lambda_\epsilon$, in other models the critical value of the coupling is definitely not small either. In Sine-Gordon for instance, the critical value is given by $\beta_\epsilon^2 = 4\pi$.

2. The possibility of having bound states in this theory is related to the cubic interaction into the potential (3.31). This provides, in fact, an effective attractive interaction between the particles which, for low values of $\lambda/m^2$, overcomes the repulsive interaction given by the $\phi^4$ term. However, the coefficient in front of $\phi^3$ scales as $\sqrt{\lambda}$, whereas the one in front of $\phi^4$ as $\lambda$. Therefore, it seems plausible that there should exist a sufficient large value of $\lambda/m^2$ where the repulsive force prevails on the attractive one.

3. There is an additional little piece of information encoded in the semiclassical formula of the poles (3.40) which can be further exploited. Namely, suppose that the adimensional coupling constant $g$ is very small, so that both the formula of the poles and the corresponding one of the masses (3.43) can be trusted. These expressions show that, by increasing $g$, the common tendency of all particles is to move forward the threshold of the two kink state and to decay afterward. If nothing will occur to stop this motion of the poles by increasing the coupling constant, the actual scenario of the theory will be then the one predicted by the semiclassical analysis, i.e. the existence of a critical value $g_\epsilon$ (surely different from the one given by the semiclassical approximation) but with no neutral particles beyond it.
4. Finally, there is an exact mapping (which will be discussed in Section 5) between the kinks of $\varphi^4$ and those of the Sine-Gordon model. Hence, one could argue that, as there is a critical value of the coupling in Sine-Gordon model in order to have bound states, the same should also happen for $\varphi^4$ theory.

### 3.1 Simple but useful observations

In this section we will discuss some general features of the semiclassical methods which will be useful in the study of other models.

We will firstly present an equivalent way to derive the Fourier transform of the kink solution. To simplify the notation, let’s get rid of all possible constants and consider the Fourier transform of the derivative of the kink solution, expressed as

$$G(k) = \int_{-\infty}^{\infty} dx \, e^{ikx} \frac{1}{\cosh^2 x}. \quad (3.50)$$

We split the integral in two terms

$$G(k) = \int_{-\infty}^{0} dx \, e^{ikx} \frac{1}{\cosh^2 x} + \int_{0}^{\infty} dx \, e^{ikx} \frac{1}{\cosh^2 x}, \quad (3.51)$$

and we use the following series expansion of the integrand, valid on the entire real axis (except the origin)

$$\frac{1}{\cosh^2 x} = 4 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-2n|x|}. \quad (3.52)$$

Substituting this expression into (3.51) and computing each integral, we have

$$G(k) = 4i \sum_{n=1}^{\infty} (-1)^{n+1} n \left[ -\frac{1}{k-2n} + \frac{1}{k+2n} \right]. \quad (3.53)$$

Obviously it coincides with the exact result, $G(k) = \pi k / \sinh \frac{\pi}{2} k$, but this derivation permits to easily interpret the physical origin of each pole. In fact, changing $k$ to the original variable in the crossed channel, $k \rightarrow (i\pi - \theta) / \xi$, we see that the poles which determine the bound states at the vacuum $|a\rangle$ are only those relative to the exponential behaviour of the kink solution at $x \rightarrow -\infty$. This is precisely the point where the classical kink solution takes values on the vacuum $|a\rangle$. In the case of $\varphi^4$, the kink and the antikink are the same function (up to a minus sign) and therefore they have the same exponential approach at $x = -\infty$ at both vacua $|\pm 1\rangle$.

Mathematically speaking, this is the reason for the coincidence of the bound state spectrum on each of them: this does not necessarily happens in other cases, as we will see in the next section, for instance.
The second comment concerns the behavior of the kink solution near the minima of the potential. In the case of \( \varphi^4 \), expressing the kink solution as

\[
\varphi(x) = \frac{m}{\sqrt{\lambda}} \tanh \left( \frac{m x}{\sqrt{2}} \right) = \frac{m}{\sqrt{\lambda}} \frac{e^{\sqrt{2}x} - 1}{e^{\sqrt{2}x} + 1},
\]

(3.54)

and expanding around \( x = -\infty \), we have

\[
\varphi(t) = -\frac{m}{\sqrt{\lambda}} \left[ 1 - 2t + 2t^2 - 2t^3 + \cdots 2 (-1)^n t^n \cdots \right],
\]

(3.55)

where \( t = \exp[\sqrt{2}x] \). Hence, all the sub-leading terms are exponential factors, with exponents which are multiple of the first one. Is this a general feature of the kink solutions of any theory? The answer is positive. To prove it, consider the equation

\[
\frac{d\varphi}{dx} = \sqrt{2U(\varphi)},
\]

(3.56)

in the limit in which \( x \to -\infty \) and \( \varphi \to \varphi_a \) (the same reasoning can be done, as well, in the other limit \( x \to +\infty \) and \( \varphi \to \varphi_b \)). To simplify the expression, let’s make the shift \( \eta(x) = \varphi(x) - \varphi_a \). Assuming that the potential \( U(\varphi) \) is regular near \( \varphi_a \), we can expand the right hand side of (3.56) in power of \( \eta \)

\[
\frac{d\eta}{dx} = \alpha_1 \eta + \alpha_2 \eta^2 + \alpha_3 \eta^3 + \cdots
\]

(3.57)

It is now easy to see that the nature of the solution strongly depends on the presence or on the absence of the first term, which express the square root of the curvature of the potential \( U(\varphi) \) at \( \varphi_a \). In fact, if \( \alpha_1 = \omega \neq 0 \), there is an exponential approach to the minimum, with all sub-leading terms given by multiples of the same exponential. To show this, let’s introduce \( t = e^{\omega x} \) and express \( \eta \) in power series of \( t \), \( \eta(t) = \sum_{n=1}^{\infty} \mu_n t^n \). Substituting this expression into (3.57), we have the following recursive equations for the coefficients \( \mu_n \)

\[
\sum_{n=1}^{\infty} n \mu_n t^n = \sum_{n=1}^{\infty} \mu_n t^n + \frac{\alpha_2}{\omega} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \mu_{n-k} \mu_k \right) t^n + \frac{\alpha_3}{\omega} \sum_{n=1}^{\infty} \left( \sum_{k_1,k_2,k_3} \mu_{k_1} \mu_{k_2} \mu_{k_3} \right) t^n + \cdots
\]

(3.58)

which iteratively permit to determine uniquely all of them. Explicitly, with the normalization \( \mu_1 = 1 \), we have

\[
\begin{align*}
2\mu_2 &= \mu_2 + \frac{\alpha_2}{\omega} \mu_1^2 \\
3\mu_3 &= \mu_3 + 2 \frac{\alpha_2}{\omega} \mu_1 \mu_2 + \frac{\alpha_3}{\omega} \mu_1^3 \\
\cdots &= \cdots
\end{align*}
\]

(3.59)
By summoning to well known theorems of uniqueness of the solution of the differential equation (3.56), the determination of these coefficients uniquely defines the kink configuration near the minimum.

When $\omega = 0$, the approach to the minimum is instead no longer exponential but is developed through a power-law. For instance, if the first non-zero coefficient is $\alpha_2$, by posing $x = -1/(\alpha_2 t)$, $\eta(t) = \sum_{n=1}^{\infty} \gamma_n t^n$ and substituting into (3.57), we have the recursive equations for the coefficients $\gamma_n$

\[
\sum_{n=1}^{\infty} n\gamma_n t^{n+1} = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \gamma_{n-k} \gamma_k \right) t^n + \frac{\alpha_3}{\alpha_2} \sum_{n=1}^{\infty} \left( \sum_{k_1+k_2+k_3=n} \gamma_{k_1} \gamma_{k_2} \gamma_{k_3} \right) t^n + \cdots \quad (3.60)
\]

which, as before, iteratively permit to fix all of them.

The fact that the approach to the minimum of the kink solutions is always through multiples of the same exponential (when the curvature $\omega$ at the minimum is different from zero) implies that the Fourier transform of the kink solution has poles regularly spaced by $\xi_a \equiv \frac{\omega}{\pi M_{ab}}$ in the variable $\theta$. If the first of them is within the physical strip, the semiclassical mass spectrum derived from the formula (2.12) near the vacuum $|a\rangle$ has therefore the universal form

\[
m_n = 2M_{ab} \sin \left( \frac{n \pi \xi_a}{2} \right) . \quad (3.61)
\]

As we have previously discussed, this means that, according to the value of $\xi_a$, we can have only the following situations at the vacuum $|a\rangle$: (a) no bound state if $\xi_a > 1$; (b) one particle if $\frac{1}{2} < \xi_a < 1$; (c) two particles if $\frac{1}{3} < \xi_a < \frac{1}{2}$; (d) $\left\lfloor \frac{1}{\xi_a} \right\rfloor$ particles if $\xi_a < \frac{1}{3}$, although only the first two are stable, the others being resonances. So, semiclassically, each vacuum of the theory cannot have more than two stable particles above it. Viceversa, if $\omega = 0$, there are no poles in the Fourier transform of the kink and therefore there are no neutral particles near the vacuum $|a\rangle$.

Finally, we would like to comment on the mass formula of the first neutral state, given by

\[
m_1 = 2M \sin \left( \frac{\pi \xi_a}{2} \right) \approx m \left[ 1 - \frac{1}{24} \left( \frac{m}{M} \right)^2 + \cdots \right] . \quad (3.62)
\]

The first term coincides with the curvature of the minimum. Is there a way to understand the presence of the subleading correction? Although it is well known (see, for instance, the discussion in [18]) that this term cannot be correctly reproduced by
semiclassical perturbation theory\textsuperscript{6}, it is worth showing a simple calculation which indicates its relation with the dynamics of the kinks. To this aim, let’s assume that the propagator of the kink (and the anti-kink) can be written as

\[
G(k) \simeq \frac{i}{k^2 - M^2}, \tag{3.63}
\]

while for the propagator of the neutral particle we take

\[
G_0(k) \simeq \frac{i}{k^2 - m^2}, \tag{3.64}
\]

where \(m^2\) is the curvature of the potential at the minimum. If the kink-antikink has \(B_1\) as bound state, there exists a non-zero 3 particle coupling \(c_{K,\bar{K}}^1\) and the possibility of a virtual process as the one shown in Figure \(\text{9}\), in which the neutral particle \(B_1\) splits in a couple of kink-antikink and recombines afterward.

\[\begin{array}{c}
K \\
\hline
\hline
B \\
\hline
\hline
\bar{K} \\
\hline
\end{array}\]

\[=\]

\[\bullet\]

\[(-i \Sigma)\]

\textbf{Figure 9: Self-energy due to the loop of the kink-antikink.}

The above diagram gives rise to the self-energy of the neutral particle. If \(p\) is its momentum, its explicit expression is given by

\[
(-i \Sigma(p^2)) = (ic_{K,\bar{K}}^1)^2 J(p^2), \tag{3.65}
\]

where

\[
J(p^2) = \int \frac{d^2q}{(2\pi)^2} \frac{i}{q^2 - M^2 + i\epsilon} \frac{i}{(p-q)^2 - M^2 + i\epsilon}. \tag{3.66}
\]

We are interested in evaluating this expression on mass-shell, i.e. at \(p^2 = m^2\). The two denominators in \(J(p^2)\) can be combined as

\[
\frac{1}{q^2 - M^2 + i\epsilon} \frac{1}{(p-q)^2 - M^2 + i\epsilon} = \int_0^1 dx \frac{1}{[q^2 - 2xq \cdot p + xp^2 - M^2 + i\epsilon]^2}. \]

\textsuperscript{6}The reason is related to the intrinsic ambiguity of the normal-ordering procedure, which can arbitrarily alter the value of the mass scale.
Making the change of variable \( l = q - xp \) and introducing \( \Delta(x) = -x(1-x)m_1^2 + M^2 \), we have

\[
J(m^2) = \int_0^1 dx \int \frac{d^2l}{(2\pi)^2} \frac{1}{((l^2 - \Delta + i\epsilon)^2}
= \frac{i}{4\pi} \int_0^1 \frac{1}{\Delta(x)} = \frac{i}{\pi m} \arctan \frac{m}{\sqrt{4M^2 - m^2}},
\]

where the factor \( i \) comes from the analytical continuation to the euclidean variables in the integral on \( l \). In the limit \( \xi_a \to 0 \), the mass of the kink is much larger than the mass of the neutral particle, so that

\[
J(m^2) \simeq i \frac{1}{4\pi M^2},
\]

and therefore, for the value of the self-energy on mass-shell, we have

\[
(-i\Sigma(m^2)) \simeq i (c_{K,\bar{K}}^1)^2 \frac{1}{4\pi M^2}.
\]

Figure 10: Propagator of the scalar particle in the ladder approximation.

In the ladder approximation shown in Figure 10 the propagator of the neutral particle becomes

\[
G(k) \simeq G_0(k) + G_0(k)(-i\Sigma(m_1^2))G_0(k)
+ G_0(k)(-i\Sigma(m_1^2))G_0(k)(-i\Sigma(m_1^2))G_0(k) + \cdots = \frac{i}{k^2 - m^2 - \Sigma(m_1^2)}
\]

and, correspondingly, the mass of the particle changes as

\[
m^2 \to m_1^2 = m^2 + \Sigma(m_1^2) = m^2 - (c_{K,\bar{K}}^1)^2 \frac{1}{4\pi M^2}.
\]

Notice that this correction is always negative, i.e. the presence of the kink tends to decrease the value of the mass of the neutral particle, initially expressed by the curvature of the potential. If it happens that, by varying the coupling constant, the second term exceeds the first, there is an imaginary value in the mass \( m_1 \).
This implies that the particle disappears from the stable part of the spectrum. Considering the coefficient $c_{K,K}^1$ as fixed, this occurs for sufficiently large values of the coupling constant. Viceversa, decreasing the coupling, the mass of the kink becomes larger and the correction gets consequently smaller, making the mass of the neutral particle closer to its perturbative value.

### 3.2 Watson’s equation and the $S$-matrix

As a final topic of this section, we now consider the issue of the Watson’s equation satisfied by the Form Factor. This concerns, in particular, the interesting possibility of extracting the $S$-matrix of the kinks, at least in certain regimes of rapidity and coupling constant. The basic idea behind the Watson’s equation is simply the completeness of the asymptotic states: given a matrix element of a local operator $\mathcal{G}$ on a given $\text{in}$ state $\langle n \rangle_{in}$, i.e. $F_{in}^\mathcal{G} = \langle 0 \mid \mathcal{G}(0) \mid n \rangle_{in}$, one can employ the completeness relation of the $\text{out}$ states to get the following relation\footnote{This is a scheleton form of the Watson’s equation: the matrix elements depend on the momenta of the particles and therefore the sum of the intermediate states stays also for a multiple integral on these variables.}

$$F_{in}^{(n)} = \langle 0 \mid \mathcal{G} \mid n \rangle_{in} = \sum_m \langle 0 \mid \mathcal{G} \mid m \rangle_{out} \langle m \mid n \rangle_{in}$$

$$\sum_m \langle 0 \mid \mathcal{G} \mid m \rangle_{out} S_{n \rightarrow m}^{m} = \sum_m S_{n \rightarrow m} F_{out}^{m} \ . \quad (3.72)$$

In this equation $S_{n \rightarrow m}$ is the $S$-matrix amplitude relative to the scattering of the $n$ initial particles into $m$. Let’s suppose now that the momenta of the incoming particles are so small that it is impossible to open higher inelastic channels. Moreover, let’s assume that are also absent “decay” processes in lower mass particle states. If this kinematical regime exists, then the remaining non-zero scattering amplitudes in (3.72) are nothing else but elastic, so that the Watson’s equations become similar to those employed in the integrable models \cite{19}, i.e.

$$\langle 0 \mid \mathcal{G}(0) \mid n \rangle_{in} \simeq S_{n \rightarrow n} \langle 0 \mid \mathcal{G}(0) \mid n \rangle_{out} \ . \quad (3.73)$$

In this case, from the ratio of the $F_{in}^{(n)}/F_{out}^{(n)}$, one could get the elastic part of the $S$-matrix $S_{n \rightarrow n}$, an expression obviously valid only in the kinematical region below the lowest threshold of the $n$-particle channel. Concerning the two-body $S$-matrix, it is important to notice that its elastic part is a pure phase only for real values of $\theta$ below threshold. In the following, however, we enforce it to be a phase also for complex values of the rapidity since, anyhow, this is the best we can do to obtain an estimate of this quantity.
Let’s follow this suggestion to see whether it would be possible to determine the elastic part of the $S$ matrix of the two kink states in the $\phi^4$ theory. First of all, we have to establish that there are no neutral particles $B_n$ with mass $m_n < M$ otherwise, for the non integrability of the theory, the “decay” channel $| K \bar{K} \rangle \to | B_n B_n \rangle$ will always be open, even if the kinks are at rest (here $\bar{K}$ denotes the anti-kink). Since $m_n > m_1$, it is sufficient to impose $m_1 > M$ in order to prevent such decays. This gives rise to the following condition on the coupling constant

$$\sin \frac{\pi \xi}{2} \geq \frac{1}{2}, \quad i.e. \quad \xi \geq \frac{1}{3}. \quad (3.74)$$

Once we are in this range of the coupling constant, the absence of the higher mass thresholds is ensured by taking sufficiently small values of the rapidity difference of the two kinks.

Notice that the Watson’s equations are valid irrespectively of the operator $G$, an important point on which we shall come back later. Taking for granted this insensitivity to the operator $G$, then we can take the Form Factor of the field $\phi(x)$ computed in (3.39). The Form Factor of the out state is simply obtained by substituting in (3.39) $\theta \to -\theta$. By the ratio of these quantities, we arrive to the putative expression

$$S^a_{-a,-a}(\theta) = S^{-a,-a}_{-a,a}(\theta) \equiv S(\theta) \simeq \frac{\sinh \left( \frac{(i\pi + \theta)}{\xi} \right)}{\sinh \left( \frac{(i\pi - \theta)}{\xi} \right)}. \quad (3.75)$$

If correct, this formula should describe the elastic scattering of the kinks

$$| K_{-a,a}(\theta_1) K_{a,-a}(\theta_2) \rangle \to | K_{-a,a}(\theta_2) K_{a,-a}(\theta_1) \rangle$$

below their inelastic threshold.

The two amplitudes of the kink scattering can be represented by the diagrams of Figure 11, where the indices on the left and on the right are those relative to the initial and final vacua respectively, whereas the indices on the top and on the bottom are the other vacua “visited” during the scattering.$^8$

The amplitudes obtained in eq. (3.75) satisfy (by construction) the unitarity equation

$$S(\theta) S(-\theta) = 1. \quad (3.76)$$

However, they fail to satisfy the crossing relation which is expected for the correct $S$-matrix

$$S^a_{-a,-a}(i\pi - \theta) = S^{-a,-a}_{-a,a}(\theta). \quad (3.77)$$

$^8$In this particular theory, it is impossible to change the intermediate vacua in the scattering process: this means that the kinks of $\phi^4$ essentially behave as ordinary particles.
Figure 11: Elastic scattering amplitudes of the kinks and their poles in the s-channel and in the t-channel. The dots at the vertices are the on-shell 3-particle couplings.

The validity of this relation simply follows by turning of $90^0$ one of the amplitudes of Figure 11 and comparing with the other. In addition to this problem, the above expression for $S$ has another drawback, as it becomes evident by a closer look at its analytic structure: even though it has all the poles of the bound states in the s-channel correctly localised at $\theta = i\pi (1 - n\xi)$, their residue

$$\text{Res } S(i\pi (1 - n\xi)) = -i \sin \left( \frac{2\pi}{\xi} \right),$$

is not always (imaginary) positive as, instead, it should be. In the vicinity of these poles the correct $S$-matrix should indeed reduce to

$$S(\theta) = i \frac{(g_{a,a}^n)^2}{\theta - i u_{a,a}^n},$$

and, for unitary theories, the 3-particle coupling $g_{a,a}^n$ is real.

To make a further progress toward the correct identification of the elastic $S$ matrix of the kinks, let’s now explore the arbitrariness of the operator $\mathcal{G}(x)$ entering the Watson’s equation. Semiclassically, for any operator which is a reasonable function $G(\varphi)$ of the field $\varphi(x)$, its Form Factor is given by

$$f_{ab}^G = \langle K_{ab}(\theta_1) \mid G[\varphi(0)] \mid K_{ab}(\theta_2) \rangle = \int_{-\infty}^{\infty} dx e^{i M_{ab} \theta x} G[\varphi_{ab}(x)].$$  

(3.80)
To be defined, let’s consider a class of operators expressed by power series in $\varphi$

$$
G = a_1 \varphi + a_2 \varphi^2 + \cdots a_n \varphi^n + \cdots \tag{3.81}
$$

The regular part of the above Form Factor is obtained in terms of the derivative of the function inside the integral which, in a simplified notation, is given by

$$
H(k) = \int_{-\infty}^{\infty} dx \, e^{ikx} \frac{dG}{d\varphi} \left( \frac{d\varphi}{dx} \right), \tag{3.82}
$$

where $\varphi$ denotes here the kink solution. It is easy to see that $H(k)$ has always the same poles of the Fourier transform of $\left( \frac{d\varphi}{dx} \right)$ (just using the previous argument on the asymptotic behavior of the integrand). However, its residues can be arbitrarily varied by changing the coefficients $a_n$ of the expansion (3.81). Said in another way, at the semiclassical level, the most general expression of the Form Factors for the $\varphi^4$ theory is given by

$$
H(k) = k \sum_{n=-\infty}^{\infty} \frac{s_n}{k + 2n}, \tag{3.83}
$$

where the coefficients $s_n$ are arbitrary numbers. By varying them, the only effect is to change the position of the zeros of the function $H(k)$, obviously leaving the position of its poles untouched.

Reestablishing now the original variable $k \to (i\pi - \theta)/\xi$ and taking the ratio of the Form Factors of an operator of the above class

$$
R(\theta) = \frac{H(i\pi - \theta)}{H(i\pi + \theta)}, \tag{3.84}
$$

we see that the resulting function can be any function which fulfills the unitarity equation

$$
R(\theta)R(-\theta) = 1, \tag{3.85}
$$

and which has the correct poles in the s-channel. Arbitrary other poles of $R(\theta)$ may come from the zeros of $H(i\pi + \theta)$. In conclusion, the farthest we can go in the use of the semiclassical form factors, is to fix the poles of the $S$-matrix in the s-channel alone. To find its actual expression, one necessarily needs to integrate this information with the others coming from the physical nature of the problem at hand.

For the $\varphi^4$ theory this is easy. In fact, an $S$-matrix which must simultaneously satisfy the two equations

$$
S(\beta)S(-\beta) = 1; \quad S(\beta) = S(i\pi - \theta), \tag{3.86}
$$

can only be expressed as a product of the elementary functions

\[ \eta = [1 - \eta] \equiv \frac{\tanh \frac{\theta}{2}(\theta + i\pi \eta)}{\tanh \frac{\theta}{2}(\theta - i\pi \eta)}. \]  

(3.87)

These functions have two poles, with residues of opposite sign: one at \( \theta = i\pi \eta \), the other at its crossing symmetric position \( \theta = i\pi(1 - \eta) \). Hence, the natural proposal for the \( S \)-matrix of the kinks below their inelastic threshold is

\[ S_{a,a}^{a,a}(\theta) = (-1)^{n+1} \prod_{k=1}^{n} [k \xi], \]  

(3.88)

an expression which holds for the following values of the coupling constant

\[ \frac{1}{n+1} < \xi < \frac{1}{n}. \]  

(3.89)

By construction, it satisfies both the unitarity and the crossing symmetry equations. It has the right poles in correspondence with those expected from Figure 11. It is also easy to check that it has the correct positive residue at all poles in the \( s \)-channel (and a negative one at all the \( t \)-channel poles).

In view of the condition (3.74), the only physical values that \( n \) can take in the above expression are \( n = 0, 1, 2 \). Somehow surprisingly, it keeps its validity (for instance, the positivity of the residues at all poles) even for other values of \( \xi \), where there are decay processes into lighter breathers. Notice that, when there are no breathers in the theory \( (n = 0) \), the \( S \)-matrix of the kinks simply coincides with the one of the Ising model in its low temperature phase, \( S = -1 \) \[20\].

### 3.3 Finite volume

A way to investigate the spectrum of a quantum field theory is by studying its euclidean version on a finite volume, say on an cylinder of width \( R \) along the space direction and infinitely long in the euclidean time direction \( \tau \). Assuming periodic boundary conditions \( \varphi(0, \tau) = \varphi(R, \tau) \), how the spectrum \( E_i(R) \) of the finite-volume Hamiltonian would look like?

For \( \varphi^4 \) theory, the answer is as follows. First of all, for dimensional reasons, the energy levels can be cast in the form

\[ E_i(R) = \frac{2\pi}{R} e_i(MR), \]  

(3.90)

where \( e_i(MR) \) are the scaling functions\(^9\). They depends on the adimensional parameter \( MR \), where \( M(\lambda) \) is the mass of the kink, which we assume to be always

\(^9\)The semiclassical expression of the scaling functions of \( \varphi^4 \) theory with anti-periodic boundary conditions has been studied in \[14\]. An issue of a particular interest is their crossover from the conformal regime to the massive behavior.
finite. Secondly, it is easy to foresee their behavior in two limits, $R \to 0$ and $R \to \infty$.

For $R \to 0$, the theory presents a conformal invariance. The scaling functions becomes then $e_i(0) = (2\Delta_i - c/12)$, where $c = 1$ is the central charge of the bosonic theory, whereas $\Delta_i$ are the conformal dimensions of the various conformal fields present in this limit. In the other case, $R \to \infty$, the theory displays instead its massive behavior. Therefore all levels go to a multi-particle state, with a mass gap $M$ given by the sum of the masses of the excitations entering this state. They can be either breathers or kinks (for periodic boundary conditions, there are only couples of kink-antikink states). Taking into account a possible bulk energy term $e_0(\lambda)$, the energy levels are then expected to go asymptotically as

$$E_i(R) \simeq \epsilon_0 R + M_i, \quad R \to \infty$$

(3.91)

![Figure 12: Energy levels of the bound states, as functions of $MR$, for the symmetric well potential in a finite volume with periodic boundary conditions.](image)

At a finite volume, however, there is a finite energy barrier (order $R$) between the two vacua and they will be in contact each other through the tunneling of the kink states. Correspondingly, the energies of the vacua, together with all the energies of the excitations above them, have an asymptotical exponential splitting: they come in pairs and become doubly degenerate only in the infinite volume limit. A typical outcoming of this circumstance is shown in Figura 12.
To check whether the semiclassical prediction is correct or not, it would be sufficient to study the movement of the energy levels by increasing $\lambda$. If the prediction of a critical value beyond which there are no longer bound states is correct, one should observe a progressive approach of all their energy lines toward the two-kink threshold $2M$ and their disappearance into the continuum once $\lambda > \lambda_c$.

4 Asymmetric wells

In order to have a polynomial potential with two asymmetric wells, one must necessarily employ higher powers than $\varphi^4$. The simplest example of such a potential is obtained with a polynomial of maximum power $\varphi^6$, and this is the example discussed here. Apart from its simplicity, the $\varphi^6$ theory is relevant for the class of universality of the Tricritical Ising Model. As we can see, the information available on this model will turn out to be a nice confirmation of the semiclassical scenario.

A class of potentials which may present two asymmetric wells is given by

$$U(\varphi) = \frac{\lambda}{2} \left( \varphi + a \frac{m}{\sqrt{\lambda}} \right)^2 \left( \varphi - b \frac{m}{\sqrt{\lambda}} \right)^2 \left( \varphi^2 + c \frac{m^2}{\lambda} \right),$$

(4.92)

with $a, b, c$ all positive numbers. To simplify the notation, it is convenient to use the dimensionless quantities obtained by rescaling the coordinate as $x^\mu \to mx^\mu$ and the field as $\varphi(x) \to \sqrt{\lambda}/m\varphi(x)$. In this way the lagrangian of the model becomes

$$\mathcal{L} = \frac{m^6}{\lambda^2} \left[ \frac{1}{2} (\partial \varphi)^2 - \frac{1}{2} (\varphi + a)^2 (\varphi - b)^2 (\varphi^2 + c) \right].$$

(4.93)

The minima of this potential are localised at $\varphi_0^{(0)} = -a$ and $\varphi_1^{(0)} = b$ and the corresponding ground states will be denoted by $|0\rangle$ and $|1\rangle$. The curvature of the potential at these points is given by

$$U''(-a) \equiv \omega_0^2 = (a + b)^2 (a^2 + c) ;$$
$$U''(b) \equiv \omega_1^2 = (a + b)^2 (b^2 + c).$$

(4.94)

For $a \neq b$, we have two asymmetric wells, as shown in Figure 13. To be definite, let’s assume that the curvature at the vacuum $|0\rangle$ is higher than the one at the vacuum $|1\rangle$, i.e. $a > b$.

The problem we would like to examine is whether the spectrum of the neutral particles $|B\rangle_s (s = 0, 1)$ may be different at the two vacua, in particular, whether it would be possible that one of them (say $|0\rangle$) has no neutral excitations, whereas the
other has just one neutral particle. The ordinary perturbation theory shows that both vacua has neutral excitations, although with different value of their mass:

\[
m^{(0)} = (a + b) \sqrt{2(a^2 + c)} \, , \quad m^{(1)} = (a + b) \sqrt{2(b^2 + c)} .
\] (4.95)

Let’s see, instead, what is the semiclassical scenario. The kink equation is given in this case by

\[
\frac{d\varphi}{dx} = \pm (\varphi + a)(\varphi - b) \sqrt{\varphi^2 + c} .
\] (4.96)

We will not attempt to solve exactly this equation but we can present nevertheless its main features. The kink solution interpolates between the values \(-a\) (at \(x = -\infty\)) and \(b\) (at \(x = +\infty\)). The anti-kink solution does viceversa, but with an important difference: its behaviour at \(x = -\infty\) is different from the one of the kink. As a matter of fact, the behaviour at \(x = -\infty\) of the kink is always equal to the behaviour at \(x = +\infty\) of the anti-kink (and viceversa), but the two vacua are approached, in this theory, differently. This is explicitly shown in Figure 14 and proved in the following.

Let us consider the limit \(x \to -\infty\) of the kink solution. For these large values of \(x\), we can approximate eq. (4.96) by substituting, in the second and in the third term of the right-hand side, \(\varphi \simeq -a\), with the result

\[
\left( \frac{d\varphi}{dx} \right)_{0,1} \simeq (\varphi + a)(a + b) \sqrt{a^2 + c} , \quad x \to -\infty
\] (4.97)

This gives rise to the following exponential approach to the vacuum \(|0\rangle\)

\[
\varphi_{0,1}(x) \simeq -a + A \exp(\omega_0 x) , \quad x \to -\infty
\] (4.98)

where \(A > 0\) is a arbitrary constant (its actual value can be fixed by properly solving the non-linear differential equation). To extract the behavior at \(x \to -\infty\) of the

Figure 13: Example of \(\varphi^6\) potential with two asymmetric wells and a bound state only on one of them.
anti-kink, we substitute this time $\varphi \simeq b$ into the first and third term of the right hand side of (4.96), so that

\[
\left(\frac{d\varphi}{dx}\right)_{1,0} \simeq (\varphi - b)(a + b)\sqrt{b^2 + c} , \quad x \to -\infty
\]  

(4.99)

This ends up in the following exponential approach to the vacuum $|1\rangle$

\[
\varphi_{1,0}(x) \simeq b - B \exp(\omega_1 x) , \quad x \to -\infty
\]  

(4.100)

where $B > 0$ is another constant. Since $\omega_0 \neq \omega_1$, the asymptotic behaviour of the two solutions gives rise to the following poles in their Fourier transform

\[
\mathcal{F}(\varphi_{0,1}) \to \frac{A}{\omega_0 + ik}
\]  

(4.101)

\[
\mathcal{F}(\varphi_{1,0}) \to \frac{-B}{\omega_1 + ik}
\]

In order to locate the pole in $\theta$, we shall reintroduce the correct units. Assuming to have solved the differential equation (4.96), the integral of its energy density gives the common mass of the kink and the anti-kink. In terms of the constants in front of the Lagrangian (4.93), its value is given by

\[
M = \frac{m^5}{\lambda^2} \alpha ,
\]  

(4.102)

where $\alpha$ is a number (typically of order 1), coming from the integral of the adimensional energy density (1.5). Hence, the first pole\(^{10}\) of the Fourier transform of the

\(^{10}\text{In order to determine the others, one should look for the subleading exponential terms of the solutions.}\)
kink and the antikink solution are localised at
\[
\theta^{(0)} \simeq i\pi \left(1 - \omega_0 \frac{m}{\pi M}\right) = i\pi \left(1 - \frac{\omega_0 \lambda^2}{\alpha m^4}\right)
\]
(4.103)
\[
\theta^{(1)} \simeq i\pi \left(1 - \omega_1 \frac{m}{\pi M}\right) = i\pi \left(1 - \frac{\omega_1 \lambda^2}{\alpha m^4}\right)
\]

If we now choose the coupling constant in the range
\[
\frac{1}{\omega_0} < \frac{\lambda^2}{m^4} < \frac{1}{\omega_1},
\]
(4.104)
the first pole will be out of the physical sheet whereas the second will still remain inside it! Hence, the theory will have only one neutral bound state, localised at the vacuum \(|1\rangle\). This result may be expressed by saying that the appearance of a bound state depends on the order in which the topological excitations are arranged: an antikink-kink configuration gives rise to a bound state whereas a kink-antikink does not.

Finally, notice that the value of the adimensional coupling constant can be chosen so that the mass of the bound state around the vacuum \(|1\rangle\) becomes equal to mass of the kink. This happens when
\[
\frac{\lambda^2}{m^4} = \frac{\alpha}{3\omega_1}.
\]
(4.105)

Strange as it may appear, the semiclassical scenario is well confirmed by an explicit example. This is provided by the exact scattering theory of the Tricritical Ising Model perturbed by its sub-leading magnetization. Firstly discovered through a numerical analysis of the spectrum of this model \[21\], its scattering theory has been discussed later in \[22\]. It involves several amplitudes but, for our purposes, it is enough to focus only on those given below. With the same meaning of the diagrams as in the previous section, their exact expression is given by

\[
\begin{align*}
0 \times 1 & \times 0 = S_{01}^{11}(\theta) = \frac{i}{2} S_0(\theta) \sinh \left(\frac{9}{5} \theta - i \frac{\pi}{5}\right) \\
1 \times 0 & \times 1 = S_{10}^{00}(\theta) = \frac{i}{2} S_0(\theta) \frac{\sin \left(\frac{\pi}{5}\right)}{\sin \left(\frac{2\pi}{5}\right)} \sinh \left(\frac{9}{5} \theta + i \frac{2\pi}{5}\right)
\end{align*}
\]
The function $S_0(\theta)$ ensures the unitarity condition of the whole set of amplitudes and it is given by

$$S_0(\theta) = \frac{w(\theta, -\frac{1}{9}) w(\theta, \frac{1}{10}) w(\theta, \frac{3}{10}) t(\theta, \frac{2}{9}) t(\theta, -\frac{8}{9}) t(\theta, \frac{7}{9}) t(\theta, -\frac{1}{9})}{\sinh \frac{9}{10}(\theta - i\pi) \sinh \frac{9}{10}(\theta - \frac{2\pi i}{3})}$$

where

$$w(\theta, x) = \frac{\sinh \left( \frac{9}{10}\theta + i\pi x \right)}{\sinh \left( \frac{9}{10}\theta - i\pi x \right)}; \quad t(\theta, x) = \frac{\sinh \frac{1}{2}(\theta + i\pi x)}{\sinh \frac{1}{2}(\theta - i\pi x)}.$$

The structure of poles and zeros of the $S$-matrix of this problem is quite rich but, on the physical sheet, $0 \leq \text{Im} \theta \leq i\pi$, the only poles of the $S$-matrix are located at $\theta = \frac{2\pi i}{3}$ and $\theta = \frac{i\pi}{3}$.

\begin{align*}
S_{00}^{11}(\theta) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \theta = \frac{2\pi i}{3} \\
S_{11}^{00}(\theta) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ Amplitude s–channel t–channel }
\end{align*}

Figure 15: Elastic scattering amplitudes of the kinks in an asymmetric wells potential and their intermediate states in the s-channel and in the t-channel.

The first pole corresponds to a bound state in the s-channel whereas the second one is the singularity due to the particle exchanged in the crossed t-channel. However, the residues of the two amplitudes at $\theta = \frac{2\pi i}{3}$ are quite different! In fact, for the first amplitude we have

$$\text{Res}_{\theta=\frac{2\pi i}{3}} S_{00}^{11}(\theta) = 0; \quad (4.107)$$

while for the second

$$\text{Res}_{\theta=\frac{2\pi i}{3}} S_{11}^{00}(\theta) = i \frac{s \left( \frac{2}{5} \right)}{s \left( \frac{1}{5} \right)} \omega; \quad (4.108)$$
where
\[ \omega = \frac{5}{9} s\left(\frac{1}{9}\right) s\left(\frac{1}{18}\right) s\left(\frac{1}{9}\right) s\left(\frac{1}{18}\right) s^2\left(\frac{5}{18}\right), \] (4.109)
\[ s(x) \equiv \sin(\pi x). \]

Hence, in the s-channel of the amplitude \( S_{11}^{11} \), there is no bound state related to the vacuum \( |0\rangle \): its only singularity comes from the bound state on the vacuum \( |1\rangle \), exchanged in the t-channel. In the amplitude \( S_{11}^{00} \) the situation is reverted (the two amplitudes are related by crossing): there is the s-channel singularity due to the bound state present on the vacuum \( |1\rangle \) while the one of the t-channel is absent. This is easily seen in Figure 15 where the original amplitudes are stretched along the vertical direction (s-channel) and along the horizontal one (t-channel).

A simple way to check the above scenario would be to study the finite volume energy spectrum with periodic boundary conditions for the field \( \varphi(x, \tau) \) at the edge of the cylinder of width \( R \), as was done, in fact, for the Tricritical Ising Model in [21]. At a finite \( R \) the energies of the two vacua are exponentially split through the tunnelling process of the kinks. However, this does not occur for their excitation. If we consider, for simplicity, the case of only one bound state, its energy level is a single, isolated curve placed between the vacua energy and the threshold lines (Figure 16). This situation has to be contrasted with the one shown in Figure 12 related to a potential with two symmetric wells.

Figure 16: Energy levels of the asymmetric well potential in a finite volume with periodic boundary conditions.
5 Sine-Gordon model in the semiclassical limit

In this section we will use the exact solution of the Sine-Gordon model in order to check the semiclassical approximation and to learn some important lessons from this comparison. The potential is given in this case by

\[ U(\varphi) = \frac{m^2}{\beta^2}(1 - \cos \beta \varphi) . \]  

(5.110)

There is an infinite number of minima, \( \varphi_n^{(0)} = 2\pi n/\), which correspond to the quantum vacua \( |n\rangle \). For their equivalence, one can choose to study the excitations on one of them, say the vacuum \( |0\rangle \).

5.1 Exact scattering theory

The exact scattering theory of this model has been discussed in \( \Xi \) and it will be briefly summarised below. The kink of the Sine-Gordon model interpolate between the vacuum \( |0\rangle \) and its neighbouring ones \( |\pm 1\rangle \), and their scattering processes are described by the amplitudes

\[
\begin{align*}
| K_{0,a}(\theta_1)K_{a,0}(\theta_2) \rangle &= S_{0a}^{0a}(\theta) \, | K_{0,a}(\theta_2)K_{a,0}(\theta_1) \rangle + S_{a0}^{a0}(\theta) \, | K_{0,-a}(\theta_2)K_{-a,0}(\theta_1) \rangle \\
| K_{a,0}(\theta_1)K_{0,-a}(\theta_2) \rangle &= S_{a0}^{a0}(\theta) \, | K_{a,0}(\theta_2)K_{0,-a}(\theta_1) \rangle 
\end{align*}
\]

(5.111)

with \( a = \pm 1 \). In the neutral kink-antikink channel, the amplitude \( S_{00}^{00}(\theta) = S_R(\theta) \) describes their reflection process while \( S_{00}^{0a}(\theta) = S_T(\theta) \) describes their transmission. In the kink-kink scattering there is only the transmission amplitude \( S_{a0}^{a0}(\theta) = S(\theta) \). The reason of this terminology stays in the identification of the states, due to the equivalence of the various vacua: for instance, the kink \( |K_{-1,0}\rangle \) must be identified with \( |K_{0,1}\rangle \) and similar identification can be also established for the others. The above amplitudes, represented as in Figure 17, satisfy the unitarity equations

\[
\begin{align*}
S_R(\theta)S_R(-\theta) + S_T(\theta)S_T(-\theta) &= 1 ; \\
S_R(\theta)S_T(-\theta) + S_T(\theta)S_R(-\theta) &= 0 ; \\
S(\theta)S(-\theta) &= 1 ,
\end{align*}
\]

(5.112)

and the crossing symmetry relations

\[
S_R(i\pi - \theta) = S_R(\theta) \quad ; \quad S_T(i\pi - \theta) = S(\theta) .
\]

(5.113)

Their closed solution is given by \( \Xi \)

\[
S_T(\theta) = \frac{\sinh \frac{4\theta}{\xi}}{\sinh \frac{4}{\xi}(i\pi - \theta)} S(\theta) \quad ; \quad S_R(\theta) = \frac{i\sin \frac{4\theta}{\xi}}{\sinh \frac{4}{\xi}(i\pi - \theta)} S(\theta) ,
\]

(5.114)
where $\xi$ is the so-called renormalised coupling constant

\begin{equation}
\xi = \frac{\beta^2}{8\pi} \frac{1}{1 - \frac{\beta^2}{8\pi}}, \quad (5.115)
\end{equation}

while the amplitude $S(\theta)$ is given by

\begin{equation}
S(\theta) = -\exp \left[ -i \int_0^{\infty} \frac{dt}{t} \frac{\sinh \frac{\xi}{2} (\pi - \xi)}{\sinh \frac{\xi}{2} \cosh \frac{\pi}{2} \sin \theta t} \right]. \quad (5.116)
\end{equation}

The pole of $S(\theta)$ are all outside the physical sheet, as it can be read from its equivalent infinite-product representation

\begin{equation}
S(\theta) = \prod_{n=0}^{\infty} \frac{\Gamma \left( \frac{(n+1)\xi}{2\pi} + i \frac{\theta}{2\pi} \right) \Gamma \left( \frac{1}{2} + (n+1)\xi \right)}{\Gamma \left( \frac{(n+1)\xi}{2\pi} - i \frac{\theta}{2\pi} \right) \Gamma \left( \frac{1}{2} + (n+1)\xi \right) \Gamma \left( \frac{1}{2} + \frac{n\xi}{2\pi} + i \frac{\theta}{2\pi} \right) \Gamma \left( \frac{1}{2} + \frac{n\xi}{2\pi} - i \frac{\theta}{2\pi} \right)} \frac{\Gamma \left( \frac{1}{2} + \frac{n\xi}{2\pi} - i \frac{\theta}{2\pi} \right) \Gamma \left( \frac{1}{2} + \frac{n\xi}{2\pi} + i \frac{\theta}{2\pi} \right)}{\Gamma \left( \frac{1}{2} + \frac{n\xi}{2\pi} + i \frac{\theta}{2\pi} \right) \Gamma \left( \frac{1}{2} + \frac{n\xi}{2\pi} - i \frac{\theta}{2\pi} \right)} \cdot \quad (5.117)
\end{equation}

Hence, the bound states can be obtained from the poles inside the physical strip of the amplitudes $S_R$ and $S_T$. Since they are located at

\begin{equation}
\theta = i\pi (1 - k\xi) \quad , \quad k = 1, 2, \ldots \quad (5.118)
\end{equation}

$\xi$ must be less than 1 in order to have a bound state. This leads to the critical value of the coupling constant given by

\begin{equation}
\beta_c^2 = 4\pi, \quad (5.119)
\end{equation}

beyond which, there are no bound states. For $\beta < \beta_c$, their number $N$ is given by $N = \left\lfloor \frac{\pi}{\xi} \right\rfloor$. Calling $M$ the mass of the kink, their mass is expressed as

\begin{equation}
m_n = 2M \sin \left( n \frac{\xi}{2} \right) \quad , \quad n = 1, 2, \ldots N \quad (5.120)
\end{equation}

Obviously, the stability of the particles with mass $m_n > 2m_1$ is ensured in this case by the integrability of the model. Concerning the residues of the amplitude $S_R$ and $S_T$ in the $s$-channel, they are alternately equal or opposite, depending on $n$

\begin{equation}
\text{Res } S_R[i\pi(1 - n\xi)] = (-1)^n \text{ Res } S_T[i\pi(1 - n\xi)] = (-1)^{n+1} \frac{\xi}{\pi} \sin \frac{\pi^2}{\xi} S[i\pi(1 - n\xi)] \cdot \quad (5.121)
\end{equation}

To appreciate the meaning of this result, let’s introduce the combinations which diagonalise the $S$-matrix and which display, in this case, also the charge conjugation.
\[
S^{1,1}_{00}(\theta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & -1 & 1
\end{pmatrix}
\]

\[
S^{-1,1}_{00}(\theta) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & -1 & 1
\end{pmatrix}
\]

\[
S^{1,-1}_{-1,1}(\theta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & -1
\end{pmatrix}
\]

\[
S^{-1,-1}_{-1,1}(\theta) = \begin{pmatrix}
0 & -1 & 1 \\
-1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

Figure 17: Elastic scattering amplitudes of the kinks and their poles in the s-channel and in the t-channel.

Symmetry of the kinks

\[
S^{(+)}(\theta) = S_R(\theta) + S_T(\theta) = \frac{\sinh \frac{\pi}{2\xi} (i\pi + \theta)}{\sinh \frac{\pi}{2\xi} (i\pi - \theta)} S(\theta),
\]

\[
S^{(-)}(\theta) = S_R(\theta) - S_T(\theta) = \frac{\cosh \frac{\pi}{2\xi} (i\pi + \theta)}{\cosh \frac{\pi}{2\xi} (i\pi - \theta)} S(\theta).
\]

In view of (5.121), the bound states with \( n \) odd appear as poles only in \( S^{(-)}(\theta) \) whereas those with \( n \) even appear as poles only in \( S^{(+)}(\theta) \). The two sets couple, respectively, to the combination of the kink states

\[
| K_{0,1}K_{0,1} \rangle \pm | K_{0,-1}K_{-1,0} \rangle.
\]

The above result permits to check explicitly the non-degeneracy of each mass level
As a matter of fact, the equal or opposite value of the residues of $S_R$ and $S_T$ is a general feature, which is valid each time that a vacuum state is in communication with two symmetric neighbouring vacua. It is worth spending few words on its derivation and on its possible generalization.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{potential.png}
\caption{Portion of a potential with three neighbouring vacua, two of them symmetrically placed with respect to the central one, which supports a certain number of bound states.}
\end{figure}

### 5.2 Residue relations

Consider a potential which has, locally, a situation like the one shown in Figure 18. The vacuum in the middle, which we denote by $|0\rangle$, exchanges kinks with the two neighbouring ones, here labelled by $|\pm 1\rangle$. The latter vacua are equal between each other, but not necessarily equal to the central one. This means that the kinks $| K_{01} \rangle$ and $| K_{0,-1} \rangle$ (as well as their anti-kinks) have the same masses, and that $| K_{01} K_{01} \rangle$ has the same properties of $| K_{0,-1} K_{-1,0} \rangle$. In this situation, even if the theory may be not-integrable, one can define an (elastic) reflection and transmission amplitudes for the scattering of the above states, just as in the case of Sine-Gordon. Obviously, in the non-integrable case, this picture will only be valid in the kinematical region below their inelastic threshold (and also with the lowest mass of the bound states higher than the mass of the kink themselves).

Suppose now that the vacuum $|0\rangle$ has certain bound states $| B_n \rangle_0$. Let’s call $g_+^{(n)}$ and $g_-^{(n)}$ the coupling to the $n$-th bound state to the kinks $| K_{01} K_{10} \rangle$ and $| K_{0,-1} K_{-1,0} \rangle$, respectively. The bound states show up in the poles of both $S_R$ and $S_T$. Strenching the corresponding amplitudes along the $s$-channel, the residue in $S_T$ is proportional to $(g_-^{(n)} g_+^{(n)})$, whereas the residue in $S_R$ is proportional either to $(g_+^{(n)})^2$.
Figure 19: Reflection and transmission amplitudes for the kinks $| K_{0a}K_{a0} \rangle$ ($a = \pm 1$). The black dot represents $g_+^{(n)}$ while the white one $g_-^{(n)}$.

or to $(g_-^{(n)})^2$, depending on which amplitude one is looking at (Figure 19). However, for the equivalence of the two vacua $| \pm 1 \rangle$, the last two quantities must be equal. Hence

$$g_+^{(n)} = \pm g_-^{(n)},$$

(5.123)

i.e. the residues of the $S_R$ and $S_T$ are either equal or opposite. Correspondingly, at the pole, the even, or the odd combination of the amplitudes, becomes a one-dimensional projector: the neutral particle state is not degenerate, as we already know.

The reasoning can be further generalised in the case of three neighbouring vacua, one different from the other, but with kinks of the same masses. This is not impossible: suppose, in fact, that we deform the potential of Figure 19 but always keeping equal the integrals (1.5) of the right and left kink. Notice that, by using the equation (1.3), $\epsilon_{ab}(x)$ can be equivalently expressed as

$$\epsilon_{ab}(x) = \begin{cases} \left( \frac{d\varphi_{ab}}{dx} \right)^2, \\ U(\varphi_{ab}) \end{cases}.$$
So, we are simply looking for a potential which supports two kink configurations \( \varphi_{0, \pm 1}(x) \) that, although different, satisfy however

\[
\int_{-\infty}^{+\infty} \left( \frac{d\varphi_{01}}{dx} \right)^2 dx = \int_{-\infty}^{+\infty} \left( \frac{d\varphi_{0,-1}}{dx} \right)^2 dx .
\]  

(5.124)

There is, of course, no mathematical obstacles in fulfilling this request\(^\text{11}\). Concerning the shape of the potential, it may look like the one shown in Figure 20.

![Figure 20](image)

Figure 20: Portion of a potential with three neighbouring vacua of different shape but with kinks of equal mass interpolating between them.

Under this deformation, the masses of the bound states may change but their values can never cross, for the non-degeneracy of these particle. The kinks, on the contrary, are still degenerate and therefore they can mix each other under the scattering processes. Due to their asymmetric shape, the elastic part of their scattering is described, in this case, by 3 amplitudes. Using the short notation \( K_{\pm} \) and \( \overline{K}_{\pm} \) to denote the left and right kink/antikinks, we have

\[
|\overline{K}_{-}(\theta_1)K_{-}(\theta_2)\rangle = S_1(\theta) \ |K_{-}(\theta_2)K_{-}(\theta_1)\rangle + S_2(\theta) \ |K_{+}(\theta_2)\overline{K}_{+}(\theta_1)\rangle ,
\]

\[
|K_{+}(\theta_1)\overline{K}_{+}(\theta_2)\rangle = S_2(\theta) \ |\overline{K}_{-}(\theta_2)K_{-}(\theta_1)\rangle + S_3(\theta) \ |K_{+}(\theta_2)\overline{K}_{+}(\theta_1)\rangle .
\]

(5.125)

Calling as before \( g_{-}^{(n)} \) and \( g_{+}^{(n)} \) the on-shell coupling of the left and right kinks to the bound states, the residue \( r_2 \) of \( S_2 \) will be proportional to \( (g_{-}^{(n)} g_{+}^{(n)}) \), whereas those of \( S_1 \) and \( S_3 \) (\( r_1 \) and \( r_3 \)) will be proportional to \( (g_{-}^{(n)})^2 \) and \( (g_{+}^{(n)})^2 \), respectively. However, the residues \( r_1 \) and \( r_3 \) are not equal in this case, although they satisfy the relation

\[
r_1 r_3 = (r_2)^2 .
\]

(5.126)

\(^{11}\)Such a potential can be found, for instance, by employing the deformation procedure described in \[29\].

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Diagonalising the above $S$-matrix, we have two different phase shifts

$$e^{i\delta_{\pm}(\theta)} = \frac{1}{2} \left( S_1 + S_3 \pm \sqrt{(S_1 - S_3)^2 + 4(S_2)^2} \right) . \quad (5.127)$$

Accordingly to the relative sign of $g_+^{(n)}$ and $g_-^{(n)}$, one of the two will have a vanishing residue of the pole of the bound state, as it is easily checked by using eq. (5.126). The other amplitude will act, then, as a one-dimensional projector.

An explicit example of an asymmetric 3-vacua kink scattering is provided, for instance, by the following set of functions

$$S_1(\theta) = \frac{i}{2} S_0(\theta) \frac{\sin \left( \frac{\pi}{5} \right)}{\sin \left( \frac{2\pi}{5} \right)} \sinh \left( \frac{9}{5} \theta - \frac{2\pi}{5} i \right) ,$$

$$S_2(\theta) = -\frac{i}{2} S_0(\theta) \left( \frac{\sin \left( \frac{\pi}{5} \right)}{\sin \left( \frac{2\pi}{5} \right)} \right)^{1/2} \sinh \frac{9}{5} \theta , \quad (5.128)$$

$$S_3(\theta) = -\frac{i}{2} S_0(\theta) \frac{\sin \left( \frac{\pi}{5} \right)}{\sin \left( \frac{2\pi}{5} \right)} \sinh \left( \frac{9}{5} \theta + \frac{2\pi}{5} i \right) ,$$

where $S_0(\theta)$ is the same of eq. (4.106) and ensures the unitarity condition of these amplitudes. It is easy to see that, at $\theta = 2\pi i/3$, the residues of $S_1$ and $S_2$ are different, although they satisfy the relation (5.126).

### 5.3 Semiclassical analysis

Before proceeding with the analysis of the classical kink configurations of the Sine-Gordon theory, let’s underline one fact that will be important later. Namely, if we take the limit $\xi \to 0$ in the exact scattering amplitudes, the $S$-matrix develops an essential singularity in $\xi$. Moreover, it is no longer a meromorphic function of $\theta$. The term responsible for this singular behavior is $S_0(\theta)$, with the breaking of the analyticity of this function due to the accumulation of its infinite number of poles and zeros when $\xi \to 0$. In this limit, its explicit expression can be obtained by using the infinite product representation (5.117). It reads as follows [11, 23]

$$S(\theta) = \exp \left[ \frac{\pi}{\xi} \int_0^{\pi} \log \left[ \frac{e^{\theta-i\eta} + 1}{e^{\theta} + e^{-i\eta}} \right] d\eta \right] , \quad \xi \to 0 . \quad (5.129)$$

Turning now the attention to the semi-classical data of the kinks, the solutions of (1.3) are given by

$$\phi_{a,a\pm1}(x) = \pm \frac{4}{\beta} \arctan(e^{mx}) , \quad (5.130)$$

with the usual identification of the vacua, modulo $2\pi/\beta$. Since the kink and antikink solutions are equal (up to a sign), the semiclassical spectrum on $|0\rangle$ and on
its neighbouring vacua are the same, as it is expected from their equivalence. The classical mass of the soliton is given by $8m_\beta/\beta^2$. Including its first quantum corrections, one has

$$M = \frac{8m}{\beta^2} - \frac{m}{\pi} = \frac{m}{\pi\xi}.$$  \hspace{1cm} \text{(5.131)}

Let’s now compute the semiclassical Form Factor of an even and an odd operator. For the even one, we can choose, for instance, $G_+ = A_+ \cos[\beta \varphi(x)]$

$$f^{(+)}(\theta) = A_+ \langle K_{01} | \cos[\beta \varphi(0)] | K_{01} \rangle = A_+ \int_{-\infty}^{\infty} dx \ e^{iM \theta x} \cos[\beta \varphi_0(x)]$$

$$= - \pi A_+ \left( \frac{M}{m} \right) \frac{\theta}{\sinh \frac{\theta}{2\xi}},$$ \hspace{1cm} \text{(5.132)}

while for the odd one, $G_- = A_- \sin[\beta \varphi(x)]$

$$f^{(-)}(\theta) = A_- \langle K_{01} | \sin[\beta \varphi(0)] | K_{01} \rangle = A_- \int_{-\infty}^{\infty} dx \ e^{iM \theta x} \sin[\beta \varphi_0(x)]$$

$$= -i \pi A_- \left( \frac{M}{m} \right) \frac{\theta}{\cosh \frac{\theta}{2\xi}}.$$ \hspace{1cm} \text{(5.133)}

In the final expressions we have isolated the term $\theta/2$. The reason comes from a close comparison with the exact expression of the Form Factors of the above operators, which is available in the literature [19]. By choosing a proper normalization $A_\pm$ of the operators, they can be written as

$$f^{(+)}(\theta) = \frac{\theta}{2} \frac{1}{\sinh \frac{\theta}{2\xi}} \quad \leftrightarrow \quad f^{(+)}_{\text{exact}}(\theta) = \sinh \frac{\theta}{2} \frac{1}{\sinh \frac{\theta}{2\xi}} F_{\text{min}}(\theta)$$

$$f^{(-)}(\theta) = \frac{\theta}{2} \frac{1}{\cosh \frac{\theta}{2\xi}} \quad \leftrightarrow \quad f^{(+)}_{\text{exact}}(\theta) = \sinh \frac{\theta}{2} \frac{1}{\cosh \frac{\theta}{2\xi}} F_{\text{min}}(\theta)$$ \hspace{1cm} \text{(5.134)}

where

$$F_{\text{min}}(\theta) = \exp \left[ \int_0^{\infty} dt \ \frac{\sinh \frac{t}{2}(1 - \xi) \sin^2 \frac{\theta t}{2\pi}}{t \ \sinh \frac{\xi t}{2} \ \cosh \frac{\theta t}{2} \ \sinh t} \right].$$ \hspace{1cm} \text{(5.135)}

We see that, a part of the function $F_{\text{min}}(\theta)$, on which we are going to comment soon, the remaining expressions can be made equal by substituting $\theta/2 \rightarrow \sinh \theta/2$, an operation which is definitely permitted within the semiclassical approximation.

From a numerical point of view, as far as $\xi < 1$, the above substitution is really harmless. In fact, at fixed $\xi$, the function $F_{\text{min}}(\theta)$ asymptotically goes as

$$F_{\text{min}}(\theta) \simeq \exp \left[ \frac{\theta}{4} \frac{(1 - \xi)}{\xi} \right], \quad \theta \rightarrow \infty$$ \hspace{1cm} \text{(5.136)}
and, therefore, for $\xi < 1$, the ratio $F_{\text{min}}(\theta) / \sinh \frac{\theta}{2\xi}$ becomes negligible before one has the possibility to appreciate the difference between the term $\theta/2$ and $\sinh(\theta/2)$: plotting together the semiclassical and the exact expressions, one can hardly distinguish them on the entire infinite range of $\theta$ (Figure 21).

Figure 21: Comparison at $\xi = 0.3$, between the exact form factor of $\cos \beta \varphi(x)$ (continuous line) and its semiclassical expression (dotted line).

One may have noticed that the form factors $f^{(\pm)}(\theta)$ of the operators $\cos \beta(\varphi)$ and $\cos \beta(\varphi)$ has a finite value at $\theta = 0$. The reason is that they are local fields with respect to the kinks. The same is true for all integer harmonics $\cos(n\beta \varphi(x))$ and $\sin(n\beta \varphi(x))$. In fact, the semilocal index $\gamma_\alpha$ of the exponential operator $e^{i\alpha \varphi(x)}$ with respect to the kink is [5]

$$\gamma_\alpha = \frac{\alpha}{\beta}.$$  \hspace{1cm} (5.137)

For the form factor $g^\alpha(\theta) = \langle K(\theta_1) \mid e^{i\alpha \varphi(0)} \mid K(\theta_2) \rangle$ of these operators, $\gamma$ is the quantity that rules their residue at $\theta = 0$

$$\text{Res}_{\theta=0} g^\alpha(\theta) = (1 - e^{2\pi i \gamma_\alpha}) \langle 0 \mid e^{i\alpha \varphi(0)} \mid 0 \rangle.$$  \hspace{1cm} (5.138)

However, taking the operators $\cos \frac{\beta}{2} \varphi(x)$ and $\sin \frac{\beta}{2} \varphi$, it is easy to see that the form factor of the first operator should have a pole at $\theta = 0$, whereas the other should not. This is, indeed, well confirmed by the semiclassical expression of these quantities.
For the first, we have in fact
\[
\begin{align*}
 f^{(+)}_{1/2}(\theta) &= \langle K_{01}(\theta_1) | \cos \left( \frac{\beta}{2} \varphi(0) \right) | K_{01} \rangle = \int_{-\infty}^{\infty} dx \, e^{iM \theta x} \cos \left( \frac{\beta}{2} \varphi_{01}(x) \right) \\
 &= -i \pi \frac{M}{m} \frac{1}{\sinh \frac{\pi M \theta}{2m}} = -i \frac{1}{\xi} \frac{1}{\sinh \frac{\theta}{2\xi}},
\end{align*}
\]
while for the second
\[
\begin{align*}
 f^{(-)}_{1/2}(\theta) &= \langle K_{01}(\theta_1) | \sin \left( \frac{\beta}{2} \varphi(0) \right) | K_{01} \rangle = \int_{-\infty}^{\infty} dx \, e^{iM \theta x} \sin \left( \frac{\beta}{2} \varphi_{01}(x) \right) \\
 &= \pi \frac{M}{m} \frac{1}{\cosh \frac{\pi M \theta}{2m}} = \frac{1}{\xi} \frac{1}{\cosh \frac{\theta}{2\xi}}.
\end{align*}
\]

Concerning the bound states, making the analytic continuation \( \theta \rightarrow i\pi - \theta \), it is easy to see that one obtains the exact spectrum of the bound states \((5.120)\), with \( n \) even, from the poles of the form factors of the even operators, as for instance \( F^{(+)}(\theta) = f^{(+)}(i\pi - \theta) \). Vice versa, from the poles of the odd operator form factors, like \( F^{(-)}(\theta) = f^{(-)}(i\pi - \theta) \), one obtains the exact mass of the particles with \( n \) odd. This happens because the neutral bound states are eigenvectors of the charge conjugation of the model, with eigenvalues \((-1)^n\) and, therefore, the two sets couple only to operators with the same parity.

Let’s now discuss the issue of the \( S \)-matrix. Taking, for instance, the ratios of the semiclassical form factors of the operators \( \cos \frac{\beta}{2} \varphi \) and \( \sin \frac{\beta}{2} \varphi \) computed \( i\pi \mp \theta \), one obtains
\[
\begin{align*}
 S^{(+)} &= \frac{F^{(+)}_{1/2}(i\pi - \theta)}{F^{(+)}_{1/2}(i\pi + \theta)} = \frac{\sinh \frac{\pi M \theta}{2\xi}(i\pi + \theta)}{\sinh \frac{\pi M \theta}{2\xi}(i\pi - \theta)} , \quad (5.141) \\
 S^{(-)} &= \frac{F^{(-)}_{1/2}(i\pi - \theta)}{F^{(-)}_{1/2}(i\pi + \theta)} = \frac{\cosh \frac{\pi M \theta}{2\xi}(i\pi + \theta)}{\cosh \frac{\pi M \theta}{2\xi}(i\pi - \theta)} .
\end{align*}
\]
Comparing these results with eq. \((5.122)\), we see that they are remarkably close to the exact expressions but, nevertheless, they miss the important term \( S(\beta) \). This is related to the problem, previously discussed, of the impossibility of recovering the correct crossing symmetry of the \( S \) by using the semiclassical Form Factors of a generic operator. In this case, however, this problem seems to get even worse for the mathematical nature of the function which is missing, eq. \((5.129)\). In fact, this is a function with a branch cut in \( \theta \), that can only be obtained by an accumulation of infinite number of poles and zeros. Together with its essential singularity at \( \xi = 0 \), its non-analytic behavior makes \( S(\theta) \) completely invisible to the semiclassical formula.

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we are using. Saying differently, the term \( S(\theta) \) could only be recovered by the ratio of the functions \( F_{\text{min}}(i\pi \pm \theta) \). By the exact solution of theory \([19]\), we have indeed

\[
F_{\text{min}}(i\pi + \theta) = S(\theta) F_{\text{min}}(i\pi - \theta), \quad F_{\text{min}}(\theta) = F_{\text{min}}(-\theta). \tag{5.142}
\]

However, in order to obtain the expression \([5.129]\), such function should behave, for \( \xi \to 0 \), as

\[
F_{\text{min}}(\theta) = \exp \left[ \frac{\pi}{\xi} \sum_{k=1}^{\infty} k \int_0^{\pi} d\eta \log \left[ \frac{1 + \left( \frac{\theta}{\pi(2k+2-\eta)} \right)^2}{1 + \left( \frac{\theta}{\pi(2k+2+\eta)} \right)^2} \right] \right]. \tag{5.143}
\]

It is easy to convince oneself that the semiclassical formula \([2.12]\) can never produce such an expression for the form factor of any operator \( G(\varphi(x)) \) which is an analytic function of the field \( \varphi(x) \).

In the end, we would like to close this section with a remark that may lighten the bound states of \( \varphi^4 \) theory, analysed in Section \([3]\). To do so, it is convenient to rescale the coordinates \( x^\mu \to mx^\mu \) and the field \( \varphi \to \beta \varphi \) of the Sine-Gordon theory, so that its Lagrangian becomes

\[
L = \frac{m^2}{\beta^2} \left[ \frac{1}{2} (\partial \varphi)^2 + (\cos \phi - 1) \right]. \tag{5.144}
\]

Under the substitution

\[
\varphi_{\pm}(x,t) = \pi \pm 4 \arctan \Phi(x,t), \tag{5.145}
\]

the Lagrangian of the Sine-Gordon model becomes

\[
L = \frac{m^2}{\beta^2} \frac{1}{(1 + \Phi^2)^2} \left[ \frac{1}{2} (\partial \Phi)^2 - \frac{1}{8} (\Phi^2 - 1)^2 \right]. \tag{5.146}
\]

The static solutions of this Lagrangian coincide with those of \( \varphi^4 \), once the coupling constants of the two theories are related as

\[
\frac{\lambda}{m^2} = \frac{\beta}{2}. \tag{5.147}
\]

By inserting in this relation the critical value of the coupling of the Sine-Gordon, \( \beta_c = \sqrt{4\pi} \), we may get a different estimate of the critical value of \( \varphi^4 \) theory

\[
\frac{\lambda_c}{m^2} = \sqrt{\pi} = 1.77245... \tag{5.148}
\]

This value is of the same order of magnitude of \([5.12]\), previously obtained. It would have been, probably, too ambitious to search a better agreement between the two values. After all, even though the static solutions of the two models are similar, their time dependent solutions are instead rather different (the Sine-Gordon has an integrable dynamics, whereas \( \varphi^3 \) does not).
6 Double Sine-Gordon wells done doubly well

In this section we consider the situation of a vacuum state in communication with two neighboring ones through kinks of different masses. A prototype of this situation is given by the Double Sine-Gordon model, with potential given by

\[ V(\phi) = -\frac{\mu}{\beta^2} \cos \beta \phi - \frac{\lambda}{\beta^2} \cos \left( \frac{\beta}{2} \phi + \delta \right) + C. \]  

(6.149)

By choosing

\[ \delta = \frac{\pi}{2}, \quad C = \frac{1}{\beta^2} \left( \mu + \frac{\lambda^2}{8\mu} \right), \]  

(6.150)

and by varying \( \lambda \), the shape of the potential changes as shown in Figure 22.

![Figure 22: Shapes of the Double Sine Gordon potential by varying \( \lambda \).](image)

There are two regions, qualitatively different, in the space of parameters, the first given by \( 0 < \lambda < 4\mu \) and the second given by \( \lambda > 4\mu \). They are separated by the value \( \lambda = 4\mu \) (where the curvature at the minima vanishes) which can be identified as a phase transition point [5].

Let’s focus our analysis in the coupling constant region where \( \lambda < 4\mu \). Switching on \( \lambda \), there are several effects on the potential: the original minima of the Sine-Gordon, located at \( \phi_{\text{min}} = 0, \frac{2\pi}{\beta} \) (mod \( \frac{4\pi}{\beta} \)), remain degenerate but they move to \( \phi_{\text{min}} = -\phi_0, \frac{2\pi}{\beta} + \phi_0 \) (mod \( \frac{4\pi}{\beta} \)), with \( \phi_0 = \frac{\pi}{\beta} \arcsin \frac{\lambda}{4\mu} \). The different minima has, however, the same curvature, given by

\[ m^2 = \mu - \frac{1}{16} \frac{\lambda^2}{\mu}. \]  

(6.151)

In correspondance with the above shifts, there are two different types of kinks, one called “large kink” and interpolating through the higher barrier between \( -\phi_0 \) and \( \frac{2\pi}{\beta} + \phi_0 \), the other called “small kink” and interpolating through the lower barrier between \( -\frac{2\pi}{\beta} + \phi_0 \) and \( -\phi_0 \). Their classical expressions were explicitly given in the
\[ \varphi_L(x) = \frac{\pi}{\beta} + \frac{4}{\beta} \arctan \left[ \frac{\sqrt{4\mu + \lambda}}{4\mu - \lambda} \tanh \left( \frac{m}{2} x \right) \right] \quad \left( \text{mod} \frac{4\pi}{\beta} \right), \quad (6.152) \]

\[ \varphi_S(x) = -\frac{\pi}{\beta} + \frac{4}{\beta} \arctan \left[ \frac{\sqrt{4\mu - \lambda}}{4\mu + \lambda} \tanh \left( \frac{m}{2} x \right) \right] \quad \left( \text{mod} \frac{4\pi}{\beta} \right). \quad (6.153) \]

For the following considerations, we can neglect the periodic structure of this potential\(^{12}\) and concentrate our attention only on the three vacua around the origin, here denoted by \(| 0 \rangle\) (the one near the origin) and \(| \pm 1 \rangle\) (the other two). Around the vacuum \(| 0 \rangle\), the admitted quantum kink states are

\[ | L \rangle = | K_{0,1} \rangle \quad \text{and} \quad | S \rangle = | K_{0,-1} \rangle, \]

together with the corresponding antikink states \(| \bar{L} \rangle = | K_{1,0} \rangle\) and \(| S \rangle = | K_{-1,0} \rangle\). The topological charges of these kinks are different, and given by

\[ Q_L = -Q_L = 1 + \frac{\beta \phi_0}{\pi}, \quad Q_S = -Q_S = 1 - \frac{\beta \phi_0}{\pi}. \quad (6.154) \]

The classical masses of the large and small kink get splitted when we switch on \(\lambda\), and their exact value can be easily computed

\[ M_{L,S} = \frac{8 m}{\beta^2} \left\{ 1 \pm \frac{\lambda}{\sqrt{16 \mu^2 - \lambda^2}} \left( \frac{\pi}{2} \pm \arcsin \frac{\lambda}{4\mu} \right) \right\}. \quad (6.155) \]

The expansion of this formula for small \(\lambda\) is given by

\[ M_{L,S} \rightarrow \frac{8\sqrt{\mu}}{\beta^2} \pm \frac{\lambda}{\beta^2} \frac{\pi}{\sqrt{\mu}} + O(\lambda^2), \quad (6.156) \]

where the first term is the classical mass of the unperturbed Sine-Gordon kink.

### 6.1 The embarassment of the riches

Let’s now apply the semiclassical formula \(^{27,12}\) for obtaining the spectrum of the neutral particles at the vacuum \(| 0 \rangle\). For the form factor \(F_{LL}^\varphi(\theta) = f_{LL}^\varphi(i\pi - \theta)\) of the large kink \(^{6.152}\) (with \(\lambda < 4\mu\)) we have

\[ F_{LL}^\varphi(\theta) = i \frac{4\pi}{\beta} \frac{1}{i\pi - \theta} \frac{\sinh \left[ \alpha \frac{M}{m} (i\pi - \theta) \right]}{\sinh \left[ \beta \frac{M}{m} (i\pi - \theta) \right]}, \quad (6.157) \]

\(^{12}\)The periodicity of the potential obviously implies that the following analysis applied as well to any other vacuum of this theory.
where
\[ \alpha = 2 \arctan \sqrt{\frac{4\mu + \lambda}{4\mu - \lambda}} \]
while \( m \) and \( M_L \) are given by (6.151) and (6.155), respectively. For the form factor
\[ F_{SS}^\varphi (\theta) = f_{SS}^\varphi (i\pi - \theta) \]
(6.158)
where
\[ \alpha = 2 \arctan \sqrt{\frac{4\mu - \lambda}{4\mu + \lambda}} \]
while \( m \) and \( M_S \) are given by (6.151) and (6.155), respectively.

By looking at the poles of these expressions within the physical strip, it seems that there are two towers of neutral particles at the vacuum \( |0\rangle \): the one coming from the bound states of the \( |L \bar{L}\rangle \)
\[ m^{(n)}_{(L)} = 2M_L \sin \left( n_L \frac{m}{2M_L} \right) , \quad 0 < n_L < \frac{M_L}{m} , \]
(6.159)
the other coming from the bound states of \( |\bar{S} S\rangle \)
\[ m^{(n)}_{(S)} = 2M_S \sin \left( n_S \frac{m}{2M_S} \right) , \quad 0 < n_S < \frac{M_S}{m} . \]
(6.160)
As a matter of fact, this situation is not peculiar of the Double Sine Gordon model but it occurs each time that there are kinks of different masses originating from the same vacuum. Consider, for instance, a simplified version of a three vacua configuration, realised by the potential (Figure 23)
\[ V(\varphi) = \frac{m^2}{2} \left\{ \begin{array}{ll} (\varphi + 2b)^2 , & \varphi \leq -b ; \\ \varphi^2 , & -b \leq \varphi \leq a ; \\ (\varphi - 2a)^2 , & \varphi > a . \end{array} \right. \]
(6.161)
The explicit configurations of the long and short kink of this potential are pretty simple
\[ \varphi_L(x) = \left\{ \begin{array}{ll} a e^{mx} , & x \leq 0 ; \\ 2a - a e^{-mx} , & x \geq 0 , \end{array} \right. \quad \varphi_S(x) = \left\{ \begin{array}{ll} -b e^{mx} , & x \leq 0 , \\ -2b + b e^{-mx} , & x \geq 0 . \end{array} \right. \]
(6.162)
and their classical masses are
\[ M_L = ma^2 , \quad M_S = mb^2 . \]
The form factors can be easily computed

\[
F_{\phi L L}(\theta) = \frac{i}{M_L(i\pi - \theta)} \left[ \frac{1}{\theta - i\pi(1 - \xi_L)} - \frac{1}{\theta - i\pi(1 + \xi_L)} \right],
\]  
\[
F_{\phi S S}(\theta) = -\frac{i}{M_S(i\pi - \theta)} \left[ \frac{1}{\theta - i\pi(1 - \xi_S)} - \frac{1}{\theta - i\pi(1 + \xi_S)} \right],
\]

where

\[
\xi_L = \frac{m}{\pi M_L} = \frac{1}{\pi a^2}, \quad \xi_S = \frac{m}{\pi M_S} = \frac{1}{\pi b^2}.
\]

By looking at the pole in the physical strip of the above amplitudes, it seems then that there are two particles on the vacuum \(|0\rangle\), whose masses are expressed by the formulas

\[
m_L = 2ma^2 \sin \frac{\pi}{2a^2}, \quad m_S = 2mb^2 \sin \frac{\pi}{2b^2}.
\]  

When \(a = b\), the two masses coincide but, as we discussed in the introduction, the corresponding state cannot be degenerate. Hence, one of the two spectra in (6.166) has to be spurious. The same conclusion applies, as well, to the mass formulas (6.159) and (6.160) of the Double Sine Gordon model\(^{13}\). But, what went wrong in this case with the semiclassical formula?

To understand the origin of this discrepancy, notice that each kink solution knows only half of the shape of the vacuum state from which it originates: for instance, the long kink starts its “motion” from the minimum at the origin, but its next values are determined only by the shape of the potential on its right. As far as this kink solution is concerned, the shape of the potential to the left of the origin could be arbitrarily changed without effecting the behavior of this solution. The same

\(^{13}\)The argument concerning the non-degeneracy of the neutral states invalidates the conclusions previously reached on the spectrum of the Double Sine Gordon model\(^{15}\), and it makes somehow obvious the finding of the numerical analysis on this model presented in\(^{17}\).
considerations also apply to the short kink, which is determined only by the shape of the potential on the left of the origin.

The above observation means that, when we employ the long-kink solution to extract the mass spectrum, it is as we are referring to a potential which is not the actual one. It is rather a potential \( U_L(\varphi) \), whose values for \( \varphi < 0 \) are obtained by the specular image of those for \( \varphi > 0 \) of the original potential. Vice versa, when we employ the short-kink solution, it is as we are referring to a potential \( U_R(\varphi) \), whose values for \( \varphi > 0 \) are the specular image of those for \( \varphi < 0 \), which determine the short-kink solution. For the long and short kinks of the above example, the fictitious potentials \( V_L(\varphi) \) and \( V_S(\varphi) \) reconstructed by the semiclassical solutions are the ones shown in Figure 24. Similar fictitious potentials can be extracted, as well, for the Double Sine-Gordon model. Hence, no wonder that employing eq. (2.12) in the case of kinks with different mass, each of them gives rise to a different spectrum of bound states on the same vacuum.

Saying the things differently, at the leading order in the coupling constant in which the semiclassical form factor (2.12) was computed, the short and long kink states are invisible each other. They start to be aware of the existence of the other only at the next leading order in the coupling constant. For instance, in the expression (2.25) involving the long-kink, the first subleading terms are given by the matrix elements with a couple of short kink and antikink state, as

\[
\langle L(\theta_1) | \varphi(0) \rangle L \bar{S}S \langle L \bar{S}S | \varphi(0) \rangle L \ldots \langle L | \varphi(0) \rangle L(\theta_2) \rangle .
\]

With \( M_L > M_S \), these terms (as well as all the others, obtained by inserting more times the couples \( \bar{S}S \)) are always present, no matter which are the values of the external rapidities \( \theta_1 \) and \( \theta_2 \) of the long kink. In this case, it becomes then rather artificial to pin down their presence by simply appealing to the perturbation expansion in the coupling constant. Their presence, however, spoil the possibility of obtaining a close differential equation for the form factors of the kink, as the one that has led to the semiclassical expression (2.12). In principle, they can be taken in account by employing the path integral formalism discussed, for instance, in [25] but, in practise, this can be a rather painful and ferociously complicated procedure. So, for the purpose of this paper, much better to content ourselves with the possibility of identifying the mass spectrum according to the heuristic considerations of the next section.
6.2 The importance of being small

Once it has been clarified the origin of the discrepancy of the spectra coming from the two kink solutions, it remains to understand what is the correct spectrum of bound states. Although the exact expression of the mass formula has remained elusive to us, we would like to show that the spectrum can be studied, in a relatively simple way, at least in two different cases: (a) when the asymmetric kinks have approximately the same mass; (b) when the mass of one of them is much smaller than the mass of the other.

Let’s consider first the case (a). This situation can be realised starting by a symmetric configuration of the potential (which we assume to be an even function $V(\varphi) = V(-\varphi)$) and slightly deforming it by an infinitesimal deformation $\lambda \delta V(\varphi)$, with $\delta V(\varphi)$ odd under $\varphi \to -\varphi$. Switching on $\lambda$, the effect of the new term is to decrease one the maxima of the potential and to increase the other. This is, for instance, what happens in the Double Sine-Gordon for small value of $\lambda$. Under this deformation, the masses of the kinks changes as $M_{L,S} \simeq M \pm \lambda M$. Denote by $|b\rangle_L$ and by $|b\rangle_S$ the bound state of the long and the short kinks which, in the unperturbed theory, have equal mass. The actual breather of the unperturbed theory is a linear combination of these two “half-breathers” $|b\rangle_{L,S}$ – a combination that can be determined by imposing that the state is an eigenvector of the parity transformation $P$ ($P^2 = 1$). Let’s assume, for instance, that the bound state is expressed by the combination

$$|B\rangle = \frac{|b\rangle_L - |b\rangle_R}{\sqrt{2}}.$$ 

(6.168)

In the unperturbed potential, its mass can be equivalently written as

$$m_B = \frac{1}{2}(m_L + m_S).$$ 

(6.169)
Switching now \( \delta V \), at the first order in \( \lambda \) the state \( (6.168) \) does not change. The masses \( m_L \) and \( m_R \) receive, instead, a linear correction of opposite sign,

\[
\delta_{L,S} \simeq \pm \left( 2 \sin \left( \frac{m}{2M} \right) - \frac{m}{M} \cos \left( \frac{m}{2M} \right) \right) .
\] (6.170)

Plugging these corrections into \( (6.169) \), the mass of the breather remains then unchanged. This result matches with the first order Form Factor Perturbation Theory \[16, 17\], as it can be seen by employing the parity operator \( P \)

\[
\lambda \langle B | \delta V(\varphi) | B \rangle = \lambda \langle B | P \left( P \delta V(\varphi) P \right) P | B \rangle = -\lambda \langle B | \delta V(\varphi) | B \rangle = 0 .
\] (6.171)

Let’s consider now the case (b), i.e. when one of the kink is much heavier than the other. We would like to present a series of arguments in favour of the thesis that, in this circumstance, the semiclassical spectrum is essentially determined by the short-kink solution, i.e.

\[
m^{(n)}_{(s)} \simeq 2M_S \sin \left( n_S \frac{m}{2M_S} \right) , \quad 0 < n_S < \frac{\pi M_S}{m} ,
\] (6.172)

with the stable part of the spectrum obtained only for \( n_S \leq 2 \). The arguments are the following

1. The actual mass of the bound state depends on both \( M_S \) and \( M_L \). However, repeating the argument presented in Section 3.1, the correction induced by the long kink is expected to be suppressed with respect to the one of the short kink by the ratio \( \left( \frac{M_S}{M_L} \right)^2 \). Therefore, making heavier the mass of the long kink, it can be forseen that its influence on the mass of the neutral particle should becomes less and less relevant. In particular, when \( M_L \to \infty \), the long kink decouples from the theory and the masses of the breathers are only determined by the dynamics of the remaining short kink.

2. Another argument in favour of the formula \( (6.172) \) directly comes from the Watson equations satisfied by the form factors of the short and long kink solutions. Let’s first consider the scattering process of the small kink-antikink state \( | \bar{S}(\hat{\theta}_1) S(\hat{\theta}_2) \rangle \). As far as we have the inequality

\[
M_S \cosh \hat{\theta} < M_L \cosh \theta ,
\] (6.173)

in the center of mass (defined by \( \frac{\theta}{2} \equiv \theta_1 = -\theta_2 \) and \( \frac{\hat{\theta}}{2} \equiv \hat{\theta}_1 = -\hat{\theta}_2 \)), there is no possibility of converting the above state into a \( | L(\theta_1) \bar{L}(\theta_2) \rangle \). Therefore the scattering process of this short kink-antikink state can only be elastic

\[
| \bar{S}(\hat{\theta}_1) S(\hat{\theta}_2) \rangle \longrightarrow | \bar{S}(\hat{\theta}_1) S(\hat{\theta}_2) \rangle .
\] (6.174)
Figure 25: Elastic scattering amplitude of the short kink-antikink state.

The elastic range of $\hat{\theta}$ obviously enlarges by making the mass of the long kink heavier. In this region, the Watson equation which is satisfied by any form factor of the short kink-antikink state becomes then

$$F^O_{SS}(\hat{\theta}) = S_{SS}(\hat{\theta}) F^O_{SS}(-\hat{\theta}),$$  \hspace{1cm} (6.175)

where $S_{SS}(\hat{\theta})$ is the elastic $S$-matrix of the process (6.174). Assuming that this elastic process leads through the exchange of the scalar particles, we have the diagram of Figure 25.

Eq. (6.175) implies that the ratio $F^O_{SS}(\hat{\theta})/F^O_{SS}(-\hat{\theta})$ is a pure phase for the real values of $\hat{\theta}$ below the inelastic threshold given by eq. (6.173), perfectly in agreement with the semiclassical result. If we now trust the semiclassical result of the form factors also for complex values of the rapidity (in particular for those concerning the location of their poles), we arrive to the mass spectrum (6.172).

Consider now the long kink-antikink scattering. First of all, notice that for the non-integrability of the theory, the scattering processes of the state $| L(\theta_1) \bar{L}(\theta_2) \rangle$ always involve, as a final state, $| \bar{S}(\hat{\theta}_1) S(\hat{\theta}_2) \rangle$

$$| L(\theta_1) \bar{L}(\theta_2) \rangle \rightarrow | \bar{S}(\hat{\theta}_1) S(\hat{\theta}_2) \rangle.$$

(6.176)

i.e. the production process is always present.
Going to the center of mass, the rapidity of the outcoming small kink-antikink state is determined by

\[ M_S \cosh \hat{\theta} = M_L \cosh \theta. \]  \hspace{1cm} (6.177)

In addition to the production process, in the scattering of the long kink-antikink state, there is also its elastic part

\[ |L(\theta_1) \bar{L}(\theta_2)\rangle \rightarrow |L(\theta_1) \bar{L}(\theta_2)\rangle. \] \hspace{1cm} (6.178)

For values of \( \theta \) below other inelastic thresholds, the Watson equation satisfied by the form factors of the long kink-antikink state is then

\[ F_{\text{OL}}^{\text{OF}}(\theta) = S_{\text{LL}}(\theta) F_{\text{LL}}^{\text{CO}}(-\theta) + S_{\text{LR}}(\theta) F_{\text{SS}}^{\text{CO}}(-\hat{\theta}) , \hspace{1cm} (6.179)\]

where \( S_{\text{LL}}(\theta) \) is the S-matrix relative to the process \( (6.178) \), whereas \( S_{\text{LR}}(\theta) \) is the one of \( (6.176) \). Assuming that, also in this case, the scattering processes are dominated by the exchange of the scalar particles, we have the diagrams of Figure 26. In contrast with the short kink case, this time the ratio \( F_{\text{LL}}^{\text{CO}}(\theta)/F_{\text{LL}}^{\text{CO}}(-\theta) \) can never be a pure phase, not even for real values of \( \theta \). Hence the semi-classical result cannot be the correct one: in fact, from the ratio of the long kink form factors, one always gets a pure phase expression.

Notice that in the production diagram \( |L \bar{L}\rangle \rightarrow |\bar{S} S\rangle \) it appears the same 3-particle coupling than in the elastic scattering \( |\bar{S} S\rangle \rightarrow |\bar{S} S\rangle \). This implies
that the intermediate particles of the two processes are the same. Assuming that their masses are those extracted by the short kink-antikink bound states, we can predict where the correct position of the poles of the long kink-antikink form factors should be: denoting the position of these poles by \( iu^n_{SS} = i\pi(1 - \xi^n_{SS}) \), with \( \xi^n_{SS} = n\frac{m}{M_S} \), and \( iu^n_{LL} = i\pi(1 - \xi^n_{LL}) \), the resonance value \( \xi^n_{LL} \) is determined by the relation

\[
M_L \sin\left(\frac{u^n_{LL}}{2}\right) = M_R \sin\left(\frac{u^n_{SS}}{2}\right),
\]

shown in Figure 27. It is easy to see that this relation is similar to the law of refraction of light between two media of refraction indices \( M_L \) and \( M_R \).

3. In order to extract the spectrum of the neutral particles at a given vacuum \( |a\rangle \), the actual thing to do is, of course, to quantize the time-dependent solution nearby the corresponding minimum \( \varphi^{(0)}_a \) of the potential. To this aim, one initially needs to solve the equation of motion

\[
\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = -\frac{dU}{d\varphi},
\]

by requiring the following properties of the solution:

- To be a periodic function in time, with a certain frequency \( \omega \).
- To be localised around the minimum \( \varphi^{(0)}_a \) of the potential.
To have a finite value of its energy, given by

\[ E = \int_{-\infty}^{\infty} \left[ \frac{1}{2} \left( \frac{\partial \varphi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \varphi}{\partial x} \right)^2 + U(\varphi) \right] dx . \] (6.182)

The finiteness of the energy implies that, at \( x \to \pm \infty \), the solution should necessarily go to \( \varphi_a^{(0)} \), where the potential vanishes. However, we must also require that \( E \leq 2M_S \), which is the lowest energy threshold in the neutral sector. Without this condition, in fact, the asymptotical time evolution of the solution will consist of a state of small kink and small antikink, moving a part with respect each other, with a velocity fixed by the excess of energy with respect to their rest mass. This is the classical equivalence of a decay process.

Having stated these conditions, the problem of solving eq. (6.181) is equivalent to find the small oscillation of a string in the funnel of the potential \( U(\varphi) \) relative to the minimum \( \varphi_a^{(0)} \), with its ends (at \( x = \pm \infty \)) kept fixed at the value of its bottom (Figure 28). During its swinging, the string explores and probes the shape of the potential nearby the valley of the minimum \( \varphi_a^{(0)} \). The oscillations, however, cannot be too wide, otherwise the motion will violate the energy condition \( E < 2M_S \). Notice that, in this analogy, the static kink solutions are nothing else than strings which, at \( x = -\infty \) are at the bottom of one valley, whereas at \( x = \infty \), after overpassing one of the peaks of the potential landscape, are in a neighbouring one.

In order to solve (6.181), let’s pose \( \varphi(x,t) = \varphi_a^{(0)} + \eta(x,t) \) and expand correspondingly the potential near this minimum

\[ U(\varphi_a^{(0)} + \eta) = \frac{1}{2} \omega_0^2 \eta^2 + \frac{\lambda_3}{3} \eta^3 + \frac{\lambda_4}{4} \eta^4 + \frac{\lambda_5}{5} \eta^5 + \cdots . \] (6.183)

Notice that the asymmetry of the potential with respect to its two barriers (i.e. the mere fact that there exist the small and the long kink) is encoded in the non-zero coefficients of the odd powers of \( \eta \). Any smooth, real-valued solution of (6.181) that is periodic in time with frequency \( \omega > 0 \) has a Fourier representation

\[ \eta(x,t) = \sum a_n(x) \exp(in\omega t) , \quad a_{-n}(x) = a_n^*(x) . \] (6.184)

All the coefficients \( a_n(x) \) must be localised, i.e. they should vanish when \( x \to \pm \infty \). Substituting this expansion into (6.181) and using eq. (6.183), one
reaches the non-linear equations for $a_n$’s

$$
(\partial_x^2 + n^2 \omega^2 - \omega_0^2) a_n = -\lambda_3 \sum_m a_m a_{n-m} - \lambda_4 \sum_k \sum_m a_k a_m a_{n-k-m} + \cdots
$$

(6.185)

Spatially uniform infinitesimal solution of (6.181) oscillate with frequency $\omega_0$, and it is possible to prove that (6.181) admits no breather solution for $\omega > \omega_0$ [26]. Classically, this result is expressing the fact that the actual mass is always smaller than the one fixed by the curvature of the potential. Hence, one can focus on the interval $0 < \omega < \omega_0$. In order to find a small parameter $\epsilon$ to make a reasonable expansion, previous studies of this equation\textsuperscript{14} (see, for instance, [9, 27, 28]) suggest to use

$$
\epsilon \equiv \left(1 - \omega^2/\omega_0^2\right)^{1/2},
$$

(6.186)

and to introduce a slightly modified asymptotic expansion in terms of the

\textsuperscript{14}It is interesting to observe that in [27, 28], the authors actually argue about the \textit{non-existence}, strictly speaking, of a stable solution of eq. (6.181). However, the stability breaking of the solution occurs through exponentially small terms, invisible to any perturbative orders, which correspond to a radiation rate of the perturbative solution. Due to the extremely slow decay of these processes, for any practical purpose, one can safely ignore this mathematical subtlety, as also admitted by the same authors.
dimensionless variables:

\[ \xi = \frac{\epsilon \omega_0 x}{\sqrt{1 + \epsilon^2}}, \quad \tau = \frac{\omega_0 t}{\sqrt{1 + \epsilon^2}}. \]  

(6.187)

The solution of (6.185) will be expressed in terms of a series expansion in \( \epsilon \) of all the terms \( a_n \). For instance, in the case of Sine-Gordon model, the breather mode (at the leading order in \( \epsilon \)) is given by \[18, 28\]

\[ \varphi(x,t) = \frac{4}{\beta} \arctan \left( \frac{\epsilon \sin(\omega_0 t/\sqrt{1 + \epsilon^2})}{\cosh(\epsilon \omega_0 x/\sqrt{1 + \epsilon^2})} \right) \]

\[ \simeq \frac{4}{\beta} \sech \xi \sin[\tau(1 - \epsilon^2/2)] + \cdots. \]

(6.188)

Once such solution of (6.185) has been found, the next steps will be to compute its classical energy \( E \) and its action

\[ W = \int_{-\infty}^{\infty} dx \int_0^T \left( \frac{\partial \varphi}{\partial t} \right)^2 dt, \]

(6.189)

along one period. Referring once again to the Sine-Gordon model, the results are

\[ E = 2M \left( \epsilon - \frac{\epsilon^3}{3} + \cdots \right), \]

\[ W = \frac{4\pi M}{m} \left( \epsilon - \frac{\epsilon^3}{3} + \cdots \right), \]

(6.190)

where \( M \) is the mass of the soliton, alias the heights (per unit length) of the barriers of the potential landspace. Eliminating now \( \epsilon \), one has

\[ E = m \left( \frac{W}{2\pi} - \frac{1}{24} \left( \frac{m}{M} \right)^2 \left( \frac{W}{2\pi} \right)^3 + \cdots \right). \]

(6.191)

The familiar mass spectrum of the Sine-Gordon model is finally obtained by imposing the quantization condition

\[ W = 2\pi n. \]

(6.192)

If the same calculations are repeated for \( \varphi^4 \) theory \[9, 28\], one finds

\[ \varphi(x,t) \simeq \frac{m}{\sqrt{\lambda}} \left[ \frac{2\epsilon}{\sqrt{3}} \sech \xi \cos \tau - \epsilon^2 \sech^2 \xi \right.

\[ + \left. \frac{\epsilon^3}{3} \sech^2 \xi \cos 2\tau + \frac{\epsilon^4}{6\sqrt{3}} \sech^4 \xi \cos 3\tau + \cdots \right], \]

(6.193)
with the classical energy and the action of the swinging string in the valley of this potential given by

\[ E = M \left( 2\epsilon + \frac{37\epsilon^3}{27} + \cdots \right), \quad (6.194) \]

\[ W = \frac{2\pi M}{\sqrt{2m}} \left( 2\epsilon + \frac{46\epsilon^3}{27} + \cdots \right) \]

As in the Sine-Gordon model, also in this case the energy is expressed in terms of the mass of the soliton, alias in terms of the height of the barrier of the potential. Eliminating \( \epsilon \) and imposing the quantization condition \( (6.192) \), one arrives, as before, to the series expansion of the usual formula \( m_n = 2M \sin \left( \frac{n \pi}{2M} \right) \).

The two examples above should have made clear that, in the general case, the energy and the action of the classical solution will be expressed in terms of the lowest heights of the potential, a feature which is pretty intuitive. In the case in which these heights are the same (like the Sine-Gordon case), there is only one energy scale, given by the mass of the soliton. But, also in the case of \( \varphi^4 \), there is only one energy scale, given by the height of the potential between the two vacua: on the left (or on the right) of each of them, there is in fact an infinite barrier.

The same one-scale situation is expected to be valid when the potential has a barrier much higher than the other: in this case, the small oscillations of the string will be essentially determined by the barrier given by the lowest peak (the other barrier will only induce corrections at higher order in \( \epsilon \)). Correspondingly, the spectrum of the neutral bound states should essentially coincide with the one extracted by the lowest kink.

This discussion should also clarify the reason of the difficulty to find an exact expression of the mass spectrum when the kinks have a comparable mass. In this case, in fact, there are two scales in the problem, i.e. \( M_L \) and \( M_S \), and the energy together with the action of the string will be non-trivial functions thereof. To find their expression, at least in a concrete example, is an interesting open problem on which we hope to come back in the future. From a practical point of view, notice that, when \( M_L \) is not much larger than \( M_S \), the masses obtained by employing either eq. \( (6.159) \) or eq. \( (6.160) \) are always very close each other and they can provide an indication on the actual value of the masses of the bound states.

Even though it seems rather difficult, in general, to determine where the crossover from the two-scale scenario to the one-scale scenario takes place,
when $M_L/M_S > 2$ one should be relatively safe by taking $M_S$ as the only scale of the problem\(^{15}\).

## 7 The BLLG potentials

In this section we will briefly discuss the particle content of an interesting class of potentials, introduced by Bazeia et al. in [29]. These potentials, which can be expressed in terms of the Chebyshev polynomials of second kind, are closely related to the $\varphi^4$ potential (in its broken phase). In fact, they are obtained from this theory by using the so-called deformation procedure, explained in [29]. In the following we denote them as BLLG potentials. In terms of the dimensionless coordinates and the dimensionless field $\varphi$ previously used ($x^\mu \to mx^\mu$, $\varphi \to \sqrt{\lambda/m} \varphi$), the Lagrangian of the BLLG models is given by

$$L = \frac{m^4}{\lambda} \left[ \frac{1}{2} (\partial_\mu \varphi)^2 - U_a(\varphi) \right],$$

where

$$U_a(\varphi) = \frac{1}{2a^2} (1 - \varphi^2)^2 V_{a-1}^2(\varphi),$$

$$V_a(\varphi) = \frac{\sin[(a+1) \arccos \varphi]}{\sin(\arccos \varphi)}.$$

The potentials of these theories fall in two classes, depending whether $a$ is an odd or an even integer: those with $a$ odd are like $\varphi^4$ in its broken phase, i.e. with a maximum at the origin, whereas those with $a$ even are like $\varphi^6$ potential, with a zero at the origin. Some of these potentials are drawn in Figure 29. Following the analysis done in [29], let’s briefly summarise their main properties.

The minima of the above potentials are localised at

$$\varphi_k^{(0)} = \cos \left( \frac{k-1}{a} \pi \right), \quad k = 1, 2, \ldots, 2a + 1.$$  

By taking into account the periodicities $\varphi_k^{(0)} = -\varphi_{a+2-k}^{(0)}$ and $\varphi_{a+k}^{(0)} = \varphi_{a+2-k}^{(0)}$, it is easy to see that the number of distinct (positive) values are $a+1$, all negative values obtained by reflection. For $a$ even, one has this set of zeros

$$\{ \varphi_1^{(0)} = 1, \varphi_2^{(0)}, \ldots, \varphi_{\frac{a}{2}}^{(0)}, \varphi_{\frac{a}{2}+1}^{(0)} = 0, -\varphi_{\frac{a}{2}}^{(0)}, \ldots, -\varphi_2^{(0)}, -\varphi_1^{(0)} = -1 \}.$$  

\(^{15}\)The string oscillations are localised in a region order $\epsilon$. It is hard to immagine a string oscillating twice higher than the first barrier, whose height for unit length is expressed by $M_S$, still keeping its energy lower than $2M_S$. 

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whereas for \( a \) odd

\[
\{ \varphi_1^{(0)} = 1, \varphi_2^{(0)}, \ldots, \varphi_{\frac{a}{2}}^{(0)} \varphi_{\frac{a}{2}+1}^{(0)}, \ldots, -\varphi_{\frac{a}{2}}^{(0)}, -\varphi_1^{(0)} = -1 \}. \quad (7.199)
\]

These minima are the vacuum states \( |k\rangle (k = 1, 2, \ldots, a+1) \) of the corresponding quantum theory.

The nice thing about these potentials is that, thanks to the deformation procedure, all kink solutions are explicitly known. They are expressed as

\[
\varphi_k(x) = \cos \frac{\psi(x) + (k - 1)\pi}{a},
\]

where \( k = 1, 2, \ldots, a \) and \( \psi(x) \in [0, \pi] \) is the principal determination of \( \arccos \tanh(x) \).

These topological configurations interpolate between the vacua \( \varphi_{k+1}^{(0)} \) (reached at \( x = -\infty \)) and \( \varphi_k^{(0)} \) (reached at \( x = \infty \)). The solutions \( \varphi_{k+a}(x) \) are the corresponding anti-kink configurations. In order to compute their mass, it is useful to introduce the so-called super-potential \( W_a(\varphi) \) and write the potential \( U_a(\varphi) \) as

\[
U_a(\varphi) = \frac{1}{2} \left( \frac{dW_a}{d\varphi} \right)^2.
\]

This is always possible since \( U_a(\varphi) \geq 0 \). For \( a \neq 2 \), the explicit expression of \( W_a(\varphi) \) is

\[
W_a(\varphi) = \frac{1}{a^2(a^2-4)} \left[ (a^2(1 - \varphi^2) - 2) \cos(a \arccos \varphi) -2a \varphi \sqrt{1 - \varphi^2} \sin(a \arccos \varphi) \right ],
\]

whereas, for \( a = 2 \),

\[
W_2(\varphi) = \frac{1}{4} \varphi^2 (2 - \varphi^2).
\]
In terms of the super-potential, the mass of the $k$-th kink is given by

$$M_k = \frac{m^3}{\lambda} \left| W_a(\varphi_k) - W_a(\varphi_{k+1}) \right|. \quad (7.204)$$

By using (7.203), we arrive to the final expression of the kink masses

$$M_k = \frac{m^3}{\lambda} \frac{1}{a^2(a^2 - 4)} \left| a^2 \left( \sin^2 \left( \frac{(k - 1)\pi}{a} \right) + \sin^2 \left( \frac{k\pi}{a} \right) \right) - 4 \right|, \quad a \neq 2 \quad (7.205)$$

whereas $M_1 = \frac{1}{4}m^3/\lambda$ for $a = 2$.

The kink with the lower mass is the one connecting the farthest vacua of the potential, either on the left or on the right. The kinks with the higher mass are, instead, the ones related to the vacua nearby the origin. Since the masses of the kinks are proportional to the heights of the potential, by making an histogram of the their values – each of them placed at the middle of the two vacua interpolated by the corresponding kink – one expects to get a skeleton form of the original potential, as it is indeed the case, see Figure 30.

Let’s discuss now the spectrum of the neutral particles at each vacuum $| k \rangle$. Notice that all vacua, except the most external ones, have the same curvature. More precisely, reintroducing the correct dimensional units and denoting the curvature by $\omega^2$, one has

$$\omega_1^2 = \omega_{a+1}^2 = 4m^2, \quad \omega_k^2 = m^2. \quad (7.206)$$

The perturbative mass of the neutral particles at the vacua $| 1 \rangle$ and $| a + 1 \rangle$ is twice larger than the one at the other vacua. Together with the lowest value of the mass of the kink that originates from the most external minima, it is clear that the most sensitive situation for the existence of the neutral particles happens at the two farthest vacua.

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$^{16}$In this formula we have restored the original dimensional quantities.
Referring to the discussion of the previous section, as far as the mass of the large kink is much higher than the mass of the short kink, the spectrum of the neutral particles at the corresponding vacuum is determined by the smallest one. If these masses are comparable, if one prefers, can use instead their average. The conditions on the relative weights of the masses can be easily checked for the BLLG potential. Since, in both cases, it is always the lowest mass that matters, we can simplify the notation, at least, by always employing the masses of the smaller kinks. To have neutral particles at the vacuum $\vert k \rangle$, one has then to check the condition

$$\xi_k = \frac{\omega_k}{M_k^{(S)}} < \pi,$$

(7.207)

where $M_k^{(S)}$ is the mass of the small kink (or antikink) which has the vacuum $\vert k \rangle$ as asymptotic limit at $x \to -\infty$. Specializing the above formula to the vacua $\vert 1 \rangle$ and $\vert a + 1 \rangle$, one has

$$\xi_1 = \frac{2a^2(a^2 - 4)}{4 - a^2 \sin^2 \frac{\pi}{a}} \frac{\lambda}{m^2}.$$

(7.208)

Concerning the values of $\xi_k$ at the other vacua, notice that for all vacua $\vert k \rangle$ to the left of the origin, the small kink are always the antikink of $\varphi_k(x)$. For those to the right of the origin, the small kink is instead the kink $\varphi_k(x)$ itself. We have

$$\xi_k = \frac{a^2(a^2 - 4)}{4 - a^2 \left( \sin^2 \frac{(k-1)\pi}{a} + \sin^2 \frac{(k-2)\pi}{a} \right)} \frac{\lambda}{m^2}, \quad k = 2, \ldots, \tilde{a}$$

(7.209)

$$\xi_k = \frac{a^2(a^2 - 4)}{4 - a^2 \left( \sin^2 \frac{k\pi}{a} + \sin^2 \frac{(k-1)\pi}{a} \right)} \frac{\lambda}{m^2}, \quad k = \tilde{a} + 1, \ldots, a + 1$$

where $\tilde{a} = \frac{a}{2} + 1$ for $a$ even, whereas $\tilde{a} = \frac{a + 1}{2}$ for $a$ odd. For instance, in the case of the potential $U_9(\varphi)$, with the notation $\tilde{\xi} = \xi m^2/(\pi \lambda)$, one finds

$$\tilde{\xi}_1 = 725.2.$$  

$$\tilde{\xi}_2 = 362.6.$$  

$$\tilde{\xi}_3 = 50.98.$$  

$$\tilde{\xi}_4 = 22.06.$$  

$$\tilde{\xi}_5 = 14.67.$$  

$$\tilde{\xi}_6 = 14.67.$$  

$$\tilde{\xi}_7 = 22.06.$$  

$$\tilde{\xi}_8 = 50.98.$$  

$$\tilde{\xi}_9 = 362.6.$$  

$$\tilde{\xi}_{10} = 725.2.$$  

(7.210)
Hence, there are a series of nested equations relative to the bound states on the various vacua: for instance, if
\[ \frac{\lambda}{m^2} < \tilde{\xi}_1^{-1}, \]
there is one particle on the vacua \( |1\rangle \) and \( |10\rangle \) and two particles on all the others. Increasing the value of \( \tilde{\xi} \) to the interval
\[ \tilde{\xi}_1^{-1} < \frac{\lambda}{m^2} < \tilde{\xi}_2^{-1}, \]
the farthest vacua do not have neutral excitations any longer, the vacua \( |2\rangle \) and \( |9\rangle \) have one particle, while all the other have two bound states. Increasing \( \tilde{\xi} \), there is a progressive emptying of particles on the various vacua, so that, when
\[ \frac{\lambda}{m^2} > \tilde{\xi}_5^{-1}, \]
there are no more neutral particles on all vacua.

8 Conclusions

In this paper we have used simple arguments of the semi-classical analysis to investigate the spectrum of neutral particles in a quantum field theory with kink excitations. Leaving apart the exact values of the quantities extracted by the semi-classical methods, it is perhaps more important to underline some general features which have emerged through this analysis. One of them concerns, for instance, the existence of a critical value of the coupling constant, beyond which there are no neutral bound states. Another result is about the maximum number \( n \leq 2 \) of neutral particles living on a generic vacuum of a non-integrable theory. An additional aspect is the role played by the asymmetric vacua and by the asymmetric kinks.

There are several interesting open problems which deserve a further investigation. An important open question is to find the exact mass formula (if it exists) when the asymmetric kinks have a comparable value of their masses. This goes together with the problem of finding a convenient way of taking into account higher order terms in the form factor expression of the kinks. Another challenging aspect concerns the refinement of the analysis of the resonances and of the corresponding decay processes. Under this respect, it may be possible that useful insights will come in the future from a numerical analysis.
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