On the prime distribution

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In this paper, the estimation formula of the number of primes in a given interval is obtained by using the prime distribution property. For any prime pairs \( p > 5 \) and \( q > 5 \), construct a disjoint infinite set sequence \( A_1, A_2, \ldots, A_i, \ldots \), such that the number of prime pairs \( (p_i, q_i, p_i - q_i = p - q) \) in \( A_i \) increases gradually, where \( i > 0 \). So twin prime conjecture is true. We also prove that for any even integer \( m > 2700 \), there exist more than 10 prime pairs \( (p, q) \), such that \( p + q = m \). Thus Goldbach conjecture is true.

**Keywords**: Prime number; Prime distribution; Twin prime conjecture; Goldbach conjecture; Triple prime conjecture

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1 Introduction

Like Goldbach conjecture, twin prime conjecture is also one of the famous unsolved problems in number theory. In 1973, Chen [1] proved that for any even number \( h \), there are infinite prime numbers \( p \), so that the number of prime factors of \( p + h \) does not exceed 2. In 2008, Green and Tao [2] proved the existence of arbitrarily long arithmetic progressions in the primes. In 2014, Zhang [3] proved that bounded gaps between primes are all less than 70 million.

In this paper, the estimation formula of the number of primes in a given interval is given by using the prime distribution property. For any prime pairs \( p > 5 \) and \( q > 5 \), construct a disjoint infinite set sequence \( A_1, A_2, \ldots, A_i, \ldots \), such that the number of prime pairs \( (p_i, q_i, p_i - q_i = p - q) \) in \( A_i \) increases gradually, where \( i > 0 \). So the original conjecture is true. We also prove that for any even integer \( m > 2700 \), there exist more than 10 prime pairs \( (p, q) \), such that \( p + q = m \). Thus Goldbach conjecture is true. For any triple primes \( p > 7, q > 7 \) and \( r > 7 \), construct a disjoint infinite set sequence \( A_1, A_2, \ldots, A_i, \ldots \), such that the number of triple primes \( (p_i, q_i, r_i, p_i - q_i = p - q, p_i - r_i = p - r) \) in \( A_i \) increases gradually, where \( i > 0 \). So triple primes conjecture is true.

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2 Three lemmas

In this section, three lemmas are proved.

**Lemma 1.** Given two coprime natural numbers $p$ and $q$. If the remainder of natural numbers in the set $A$ with respect to $p$ is evenly distributed, then the remainder of natural numbers in the set $\{aq + c | a \in A\}$ is still evenly distributed, where $c \geq 0$ is an integer.

**Proof.** Prove by contradiction.
Without loss of generality, suppose that $a_i \equiv i \mod p, a_i \in A, 0 \leq i < p$, but the remainder of natural numbers in the set $\{aq + c | a \in A, 0 \leq i < p\}$ is not evenly distributed. Let’s say that $a_iq + c$ and $a_jq + c$ have the same remainder about $p$,
$$
\implies a_iq + c - (a_jq + c) = kp, k \text{ is an integer}. \implies (a_i - a_j)q = kp.
$$
As $p$ and $q$ are coprime natural numbers, $\implies a_i \equiv a_j \mod p$. This is a contradiction. \hfill \Box

Based on Lemma 1, we prove the following Lemma 2.

**Lemma 2.** Given natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ pairwise prime, and the remainder of natural numbers in the set $A$ with respect to $\alpha_i, 1 \leq i \leq n$, are evenly distributed, where $|A| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_n$. We conclude that
$$
|\{a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq n\}| = (\alpha_1 - 1) \times (\alpha_2 - 1) \times \ldots \times (\alpha_n - 1)
$$

**Proof.** If $n = 1$, then $|A| = \alpha_1$. Obviously, $|\{a \in A \text{ and } a \not\equiv 0 \mod \alpha_1\}| = \alpha_1 - 1$.
If $n = 2$, then $|A| = \alpha_1 \times \alpha_2$.
Without loss of generality, suppose that

\[
A = \begin{pmatrix} 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 \\
\alpha_1 + 1 & \alpha_1 + 2 & \cdots & 2\alpha_1 - 1 & 2\alpha_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(\alpha_2 - 1)\alpha_1 + 1 & (\alpha_2 - 1)\alpha_1 + 2 & \cdots & (\alpha_2 - 1)\alpha_1 - 1 & (\alpha_2 - 1)\alpha_1 \\
(\alpha_2 - 1)\alpha_1 + 1 & (\alpha_2 - 1)\alpha_1 + 2 & \cdots & \alpha_2\alpha_1 - 1 & \alpha_2\alpha_1 \end{pmatrix}
\]

\[\implies\]

\[
\{a \in A \text{ and } a \not\equiv 0 \mod \alpha_1\} = \begin{pmatrix} 1 & 2 & \cdots & \alpha_1 - 1 \\
\alpha_1 + 1 & \alpha_1 + 2 & \cdots & 2\alpha_1 - 1 \\
\vdots & \vdots & \ddots & \vdots \\
(\alpha_2 - 1)\alpha_1 + 1 & (\alpha_2 - 2)\alpha_1 + 2 & \cdots & (\alpha_2 - 1)\alpha_1 - 1 \\
(\alpha_2 - 1)\alpha_1 + 1 & (\alpha_2 - 1)\alpha_1 + 2 & \cdots & \alpha_2\alpha_1 - 1 \end{pmatrix}
\]

As $\alpha_1$ and $\alpha_2$ are coprime natural numbers, and by Lemma 1, $\{i\alpha_1 + j | 0 \leq i < \alpha_2 \} \equiv \{0, 1, 2, \ldots, \alpha_2 - 1\} \mod \alpha_2$, where $0 < j < \alpha_1$.
$$
\implies |\{a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq 2\}| = (\alpha_1 - 1) \times (\alpha_2 - 1).
$$

If $n = k + 1$, then $|A| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k \times \alpha_{k+1}$. 
As $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$ are pairwise primes, and by Lemma 1, the remainder of natural numbers in the set $\{i \times \alpha_{k+1} \mid 0 < i \leq \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k\}$ with respect to $\alpha_i, 1 \leq i \leq k$, is evenly distributed.

$\implies$ the remainder of natural numbers in the set $\{a \mid a \in A \text{ and } a \not\equiv 0 \mod \alpha_{k+1}\}$ with respect to $\alpha_i, 1 \leq i \leq k$, is evenly distributed and $|\{a \mid a \in A \text{ and } a \not\equiv 0 \mod \alpha_{k+1}\}| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k \times (\alpha_{k+1} - 1)$.

Similarly, as $\alpha_1, \alpha_2, \ldots, \alpha_k$ are pairwise primes, and by Lemma 1, the remainder of natural numbers in the set $\{i \times \alpha_k \mid 0 < i \leq \alpha_1 \times \alpha_2 \times \ldots \times \alpha_{k-1} \times (\alpha_{k+1} - 1)\}$ with respect to $\alpha_i, 1 \leq i \leq k-1$, is evenly distributed.

$\implies$ the remainder of natural numbers in the set $\{a \mid a \in A \text{ and } a \not\equiv 0 \mod \alpha_x, x = k, k+1\}$ with respect to $\alpha_i, 1 \leq i \leq k-1$, is evenly distributed and $|\{a \mid a \in A \text{ and } a \not\equiv 0 \mod \alpha_x, x = k, k+1\}| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_{k-1} \times (\alpha_k - 1) \times (\alpha_{k+1} - 1)$.

\ldots.

Finally, we have

$|\{a \mid a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq k+1\}| = (\alpha_1 - 1) \times (\alpha_2 - 1) \times \ldots \times (\alpha_{k+1} - 1)$

$\square$

The following examples is helpful to understand Lemma 2.

It is known that the natural numbers 2, 3, 5 are mutually prime. Let $A = \{1, 2, 3, \ldots, 29, 30\}$. Obviously the remainder of the natural numbers in $A$ about 2, 3, 5 is evenly distributed. According to Lemma 2, after removing all natural numbers with 0 remainder about 2, 3, 5, the number of natural numbers in $A$ becomes $2 \times 3 \times 5 = 30$. Namely, \{1, 7, 11, 13, 17, 19, 23, 29\}.

Let $A = \{7 \times i + 1 \mid 1 \leq i \leq 30\}$. According to Lemma 1, the remainder of natural numbers about 2, 3, 5 in $A$ is still evenly distributed. Then, according to Lemma 2, after removing all natural numbers with 0 remainder about 2, 3, 5, the number of natural numbers in $A$ becomes $2 \times 3 \times 5 = 30$.

Based on Lemma 2, the number of primes in 30 consecutive natural numbers \{6, 7, \ldots, 34, 35\} is $(2 - 1)(3 - 1)(5 - 1) = 30(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = 8$. As $30 = 2 \times 3 \times 5$, the formula is accurate. These specific prime numbers are: 7, 11, 13, 17, 19, 23, 29, 31.

We are concerned about the following two issues here.

1. Number counting formula of primes $(2 - 1)(3 - 1)(5 - 1)$ is valid for 30 consecutive natural numbers less than 49. Because 49 is not a multiple of 2, 3, 5, but 49 = 7 \times 7. For example, the number of primes among 30 consecutive natural numbers \{20, 21, \ldots, 48, 49\} is 7. These specific prime numbers are: 23, 29, 31, 37, 41, 43, 47. That is, the actual number of primes is less than $(2 - 1)(3 - 1)(5 - 1)$.

2. Note that number counting formula of primes

$$30\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)(1 - \frac{1}{7}) \approx 6.86$$

is less than the actual number 8 of primes in 30 consecutive natural numbers \{6, 7, \ldots, 34, 35\}, and also less than the actual number 7 of primes in 30 consecutive natural numbers \{20, 21, \ldots, 48, 49\}.

\[\text{On the prime distribution}\]
The number counting formula of primes $30(1 - \frac{1}{3})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})$ is valid for $30$ consecutive natural numbers less than $11^2$. However, it is well known that the prime distribution is not even, and the prime numbers in the front part of the effective range are dense and the prime numbers behind are sparse. For example, the actual number $6$ of primes in $30$ consecutive natural numbers $\{71, 72, \ldots, 99, 100\}$ is less than $6.86$. These specific prime numbers are: $71, 73, 79, 83, 89, 97$.

This paper mainly considers the case that the estimation formula for the number of primes lower than the actual number of primes.

The following proves Lemma 3

**Lemma 3.**

\[
(1 - \frac{d}{30 \times x + c}) < (1 - \frac{d}{30 \times (2x + e + 1) + c})(1 - \frac{d}{30 \times (2x + e + 2) + c})
\]

$x \geq 0, e \geq 0, 0 < c < 32, 0 < d < 30x$.

**Proof.** First prove:

\[
\frac{d}{30 \times (2x + 1) + c} + \frac{d}{30 \times (2x + 2) + c} < \frac{d}{30 \times x + c}
\]

\[
\iff (b + c)(2b + c + 30 + 2b + c + 60) < (2b + c + 30)(2b + c + 60), b = 30x
\]

\[
\iff (b + c)(4b + 2c + 90) < (2b + c + 30)(2b + c + 60)
\]

\[
\iff 4b^2 + 6bc + 90b + 90c + 2c^2 < 4b^2 + 4bc + 180b + c^2 + 90c + 1800
\]

From both sides of the inequality remove $4b^2 + 4bc + 90b + 90c + c^2$

The original inequality $\iff c^2 + 2bc < 90b + 1800 \iff 0 < (90 - 2c)b + 1800 - c^2$.

As $c < 32$, thus

\[
\frac{d}{30 \times (2x + e + 1) + c} + \frac{d}{30 \times (2x + e + 2) + c} < \frac{d}{30 \times x + c}
\]

\[
\iff (1 - \frac{d}{30 \times (2x + e + 1) + c})(1 - \frac{d}{30 \times (2x + e + 2) + c}) > 1 - \frac{d}{30 \times x + c}
\]

\[\square\]

### 3 Possible form of prime numbers $\{11 + 30 * x | x \geq 0\}$ and $\{13 + 30 * x | x \geq 0\}$

Introduce some basic properties of prime numbers. All prime numbers are in odd numbers with single digits of $1, 3, 7$ and $9$ (except $2$ and $5$). Now let’s take a look at the prime number whose single digit is $1$. It is easy to find that there are only two possible forms:
\{11 + 30 \times x | x \geq 0\} and \{31 + 30 \times x | x \geq 0\}

For prime number whose single digit is 3, there are only two possible forms:
\{13 + 30 \times x | x \geq 0\} and \{23 + 30 \times x | x \geq 0\}

For prime number whose single digit is 7, there are only two possible forms:
\{7 + 30 \times x | x \geq 0\} and \{17 + 30 \times x | x \geq 0\}

For prime number whose single digit is 9, there are only two possible forms:
\{19 + 30 \times x | x \geq 0\} and \{29 + 30 \times x | x \geq 0\}

Among possible form of prime numbers \{11 + 30 \times x | x \geq 0\}, if any 11 + 30 \times x is not a prime number, then there are only four possible decomposition forms:
\[H_1 = [7 + 30a] [23 + 30b], a \geq 0, b \geq 0; \]
\[H_2 = [13 + 30a] [17 + 30b], a \geq 0, b \geq 0; \]
\[H_3 = [11 + 30a] [31 + 30b], a \geq 0, b \geq 0; \]
\[H_4 = [19 + 30a] [29 + 30b], a \geq 0, b \geq 0. \]

As \((19 + 30a)^2 = 30c + 1\), thus \((19 + 30a)^2\) is not a possible decomposition form of 11 + 30 \times x.

Table 1: The possible form of prime numbers \{11 + 30 \times x | 0 \leq x < 210\}

| x | 11 | 41 | 71 | 101 | 131 | 161 | 191 | 221 | 251 | 281 | 311 | 341 | 371 | 401 | 431 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

In Table 1, the number in the cell indicates that the corresponding 11 + 30 \times x is not a prime number, and the number in the cell is a factor.

For example, 11 + 30 \times 33 = 1001 = 11 \times 7 \times 13;
11 + 30 \times 47 = 1421 = 29 \times 49 = 7 \times 203.

Based on Lemma 2, we consider a formula for estimating the number of primes. In Table 1, consider first 30 natural numbers \{11 + 30 \times x | 0 \leq x < 30\}. As 30 is not the multiple of 7, 11, 13, 17, 19, 23, 29, 31, thus consider an estimation formula lower than the actual number of primes:

\[
30(1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31}) \approx 17.20
\]

The actual number of primes in \{11 + 30 \times x | 0 \leq x < 30\} is 19.
If \( a \in \{11 + 30 \times x | 0 \leq x < 30\} \) and \( a \) is a composite number, then \( a \) must have one factor in \( \{7, 11, 13, 17, 19, 23\} \), so the above estimation formula is lower than the actual number of primes.

As \( (a + 30)(b + 30) = 30(30 + a + b) + ab \), the above estimation formula is valid for \( 11 + 30 \times x < 53 \times 37 = 1961 \approx 11 + 30 \times 65 \). So the above estimation formula is applied to the front part of the effective range, which is the reason that the above estimation formula is lower than the actual number of primes.

Table 2: The possible form of prime numbers \( \{13 + 30 \times x | 0 \leq x < 210\} \)

| \( x \) | \( 11 \) | \( 17 \) | \( 19 \) | \( 23 \) | \( 29 \) | \( 31 \) | \( 37 \) |
|-------|-------|-------|-------|-------|-------|-------|-------|
| 13    | 43    | 71    | 101   | 131   | 163   | 193   | 223   |
| 463   | 923   | 1423  | 1923  | 2423  | 2923  | 3423  | 3923  |
| 17    | 31    | 61    | 91    | 121   | 151   | 181   | 211   |
| 311   | 341   | 371   | 401   | 431   | 461   | 491   | 521   |
| 23    | 11    | 41    | 71    | 101   | 131   | 161   | 191   |
| 19    | 29    | 59    | 89    | 119   | 149   | 179   | 209   |
| 29    | 49    | 79    | 109   | 139   | 169   | 199   | 229   |
| 31    | 61    | 91    | 121   | 151   | 181   | 211   | 241   |
| 271   | 301   | 331   | 361   | 391   | 421   | 451   | 481   |
| 31    | 61    | 91    | 121   | 151   | 181   | 211   | 241   |
| 37    | 67    | 97    | 127   | 157   | 187   | 217   | 247   |

Among possible form of prime numbers \( \{13 + 30 \times x | x \geq 0\} \), if any \( 13 + 30 \times x \) is not a prime number, then there are only four possible decomposition forms:

\[
H_4 = \left[7 + 30a\right] \left[19 + 30b\right], a \geq 0, b \geq 0; 
H_5 = \left[13 + 30a\right] \left[31 + 30b\right], a \geq 0, b \geq 0; 
H_6 = \left[11 + 30a\right] \left[23 + 30b\right], a \geq 0, b \geq 0; 
H_7 = \left[17 + 30a\right] \left[29 + 30b\right], a \geq 0, b \geq 0.
\]

In Table 2, consider first 30 natural numbers \( \{11 + 30 \times x | 0 \leq x < 30\} \). As 30 is not the multiple of 7, 11, 13, 17, 19, 23, 29, 31, thus consider an estimation formula lower than the actual number of primes:

\[
30(1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31}) \approx 17.20
\]

The actual number of primes in \( \{13 + 30 \times x | 0 \leq x < 30\} \) is 20.

If \( a \in \{13 + 30 \times x | 0 \leq x < 30\} \) and \( a \) is a composite number, then \( a \) must has one factor in \( \{7, 11, 13, 17, 19\} \), so the above estimation formula is lower than the actual number of primes. As 19 + 30 is a composite number, the above estimation formula is valid for \( 13 + 30 \times x < 41 \times 53 = 2173 = 13 + 30 \times 72 \). So the above estimation formula is applied to the front part of the effective range, which is the reason that the above estimation formula is lower than the actual number of primes.
4 Twin prime conjecture

We further consider possible form of twin prime numbers \{(11+30\cdot x, 13+30\cdot x)|x \geq 0\}. From Table 1 and Table 2, \(6131, 6133\) is twin prime numbers.

Consider first 30 pairs of natural numbers \{(11 + 30 \cdot x, 13 + 30 \cdot x)|0 \leq x < 30\}. As 30 is not the multiple of 7, 11, 13, 17, 19, 23, 29, 31, thus consider an estimation formula lower than the actual number of twin prime numbers \(C(30)\):

\[
30(1 - \frac{2}{7})(1 - \frac{2}{11})(1 - \frac{2}{13})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31}) \approx 9.31
\]

The actual number of twin prime numbers in \{(11 + 30 \cdot x, 13 + 30 \cdot x)|0 \leq x < 30\} is 13.

The above estimation formula can be understood in this way (take prime number 7 as an example): for every 7 consecutive cells, there must be one cell of 11 + 30 \cdot x_1 can be divided by 7, and another cell of 13 + 30 \cdot x_2 can be divided by 7, where \(x_1 \neq x_2\). Therefore, one term in the above formula is \(\frac{7}{2}\).

When \(x = 19\), \(11 + 30 \cdot 19 = 581\) can be divided by 7, and \(13 + 30 \cdot 19 = 583\) can be divided by 11. Therefore, this cell is counted twice, so the estimation formula \(C(30)\) is lower than the actual number of twin primes.

The estimation formula for the number of twin prime numbers in \{(11 + 30 \cdot x, 13 + 30 \cdot x)|30 \leq x < 90\} is \(C(60)\):

\[
60(1 - \frac{2}{7})(1 - \frac{2}{11})(1 - \frac{2}{13})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31})
\]

\[
(1 - \frac{2}{37})(1 - \frac{2}{41})(1 - \frac{2}{43})(1 - \frac{2}{47})(1 - \frac{2}{49})(1 - \frac{2}{53})(1 - \frac{2}{59})(1 - \frac{2}{61}) \approx 10.72
\]

The actual number of twin prime numbers in \{(11 + 30 \cdot x, 13 + 30 \cdot x)|30 \leq x < 90\} is 15.

In the above formula, 77, 49 may not appear in the formula because they are multiples of 7. For the completeness of the formula, 77, 49 are still retained, which only make the valuation smaller.

Because \(91^2 > 13 + 270 \cdot 30\), the effective range of the above estimation formula is: \(13 + 30 \cdot x \leq 13 + 30 \cdot 30\), where 270 > 90. So the above estimation formula is applied to the front part of the effective range, which is the main reason that the above estimation formula is lower than the actual number of primes.

The actual number of twin prime numbers in \{(11 + 30 \cdot x, 13 + 30 \cdot x)|90 \leq x < 210\} is 24. The estimation formula for the number of twin prime numbers in \{(11 + 30 \cdot x, 13 + 30 \cdot x)|90 \leq x < 210\} is \(C(120)\):

\[
120(1 - \frac{2}{7}) \cdots (1 - \frac{2}{31})
\]

\[
(1 - \frac{2}{37})(1 - \frac{2}{41})(1 - \frac{2}{43})(1 - \frac{2}{47})(1 - \frac{2}{49})(1 - \frac{2}{53})(1 - \frac{2}{59})(1 - \frac{2}{61}) \approx 10.72
\]
The effective range of the above estimation formula is: $211^2 > 30 \times 70 \times 210 > 30 \times 450$.

More generally,

$$(30 \times 2^k - 30)^2 - 30 \times (30 \times 2^{k+1} - 30) = 900 \times (2^{2k} - 2^{k+2} + 2) > 0$$

$$\Rightarrow (30 \times 2^k - 30)^2 > 30 \times (30 \times 2^{k+1} - 30)$$

where $k \geq 2$.

When $k = 2$, the inequality means $90^2 > 30 \times 210$.

When $k = 3$, the inequality means $210^2 > 30 \times 450$.

When $k = 4$, the inequality means $450^2 > 30 \times 930$.

\[ \cdots \cdots \]

Therefore, the above estimation formula is applied to the front part of the effective range, which is the main reason that the above estimation formula is lower than the actual number of primes.

The actual number of twin prime numbers in $\{(11 + 30 \times x, 13 + 30 \times x) | 210 \leq x < 450\}$ is 29. The estimation formula for the number of twin prime numbers in $\{(11 + 30 \times x, 13 + 30 \times x) | 210 \leq x < 450\}$ is $C(240)$ (only the part about $\{7 + 30 \times x | x \geq 0\}$ is given here):

\[
\begin{align*}
(1 &- \frac{2}{7}) \\
(1 &- \frac{2}{37})(1 - \frac{2}{67}) \\
(1 &- \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187}) \\
(1 &- \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427})
\end{align*}
\]

It is easy to obtain the following (only the part about $\{7 + 30 \times x | x \geq 0\}$ is given here):

\[
\begin{align*}
\frac{C(60)}{C(30)} &= 2(1 - \frac{2}{37})(1 - \frac{2}{67}) \\
\frac{C(120)}{C(60)} &= 2(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187}) \\
\frac{C(240)}{C(120)} &= 2(1 - \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427}) \\
\vspace{10pt}
\cdots \cdots \\
\end{align*}
\]

By Lemma 3,

\[
(1 - \frac{2}{217})(1 - \frac{2}{247}) > (1 - \frac{2}{97}), (1 - \frac{2}{277})(1 - \frac{2}{307}) > (1 - \frac{2}{127}), \cdots \cdots ,
\]

Thus

\[
\cdots \cdots > \frac{C(240)}{C(120)} > \frac{C(120)}{C(60)} > \frac{C(60)}{C(30)}
\]
The complete \( \frac{C(60)}{(60)} \) is as follows,

\[
2(1 - \frac{2}{3})(1 - \frac{2}{11})(1 - \frac{2}{13})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31}) = 2 \times 0.575 > 1
\]

Therefore

\[
\cdots \cdots > C(240) > C(120) > C(60) > C(30) \approx 9.31
\]

In fact, the actual number of twin prime numbers in \( \{(11 + 30 \times x, 13 + 30 \times x)\} \) is 71, and the actual number of twin prime numbers in \( \{(11 + 30 \times x, 13 + 30 \times x)\} \) is 113. \ldots

Similarly, we can always find another larger interval, which has more than 9.31 twin prime numbers. Therefore, there are infinite twin primes.

Among possible form of prime numbers \( \{17 + 30 \times x | x \geq 0\} \), if any \( 17 + 30 \times x \) is not a prime number, then there are only four possible decomposition forms:

- \( H_0 = [7 + 30a][11 + 30b] \), \( a \geq 0, b \geq 0 \);
- \( H_1 = [13 + 30a][29 + 30b] \), \( a \geq 0, b \geq 0 \);
- \( H_2 = [17 + 30a][23 + 30b] \), \( a \geq 0, b \geq 0 \).

Consider an estimation formula lower than the actual number of primes in \( \{17 + 30 \times x | 0 \leq x < 30\} \):

\[
30(1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31}) \approx 17.20
\]

By considering an estimation formula lower than the actual number of prime pairs in \( \{(11 + 30 \times x, 17 + 30 \times x) | x \geq 0\} \), similarly we can prove that there are infinite prime pairs in \( \{(11 + 30 \times x, 17 + 30 \times x) | x \geq 0\} \).

To sum up, we get the following theorem.

**Theorem 1.** For any two prime numbers \( p_0 > 5 \) and \( q_0 > 5 \), there are infinite prime pairs \( p_i \) and \( q_i \), \( i \geq 1 \), such that \( p_i - q_i = p_0 - q_0 \).

## 5 Goldbach conjecture

Very similar to the case of twin prime conjecture, we further consider possible form of prime pairs \( \{(11 + 30 \times x, 13 + 30 \times (n - x)) | x \geq 0\} \). From Table 1 and Table 2 (131, 6163) is a pair of primes, where \( n = 209, x = 4 \).

Consider 30 pairs of natural numbers \( \{(13 + 30 \times x, 39 + 30 \times (29 - x)) | 0 \leq x < 30\} \). As 30 is not the multiple of 7, 11, 13, 17, 19, 23, 29, 31, thus consider an estimation formula lower than the actual number of twin prime numbers \( C(30) \):

\[
30(1 - \frac{2}{7})(1 - \frac{2}{11})(1 - \frac{2}{13})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31}) \approx 9.31
\]

The actual number of prime pairs in \( \{(11 + 30 \times x, 13 + 30 \times (29 - x)) | 0 \leq x < 30\} \) is 11.
The above estimation formula can be understood in this way (take prime number 7 as an example): for every 7 consecutive cells, there must be one cell of 11 + 30 * x1 can be divided by 7, and another cell of 13 + 30 * (29 - x2) can be divided by 7, where x1 ≠ x2. Therefore, one term in the above formula is \(7 - \frac{2}{7}\).

When \(x = 11\), 11 + 30 * 11 = 341 can be divided by 11, and 13 + 30 * (29 - 11) = 553 can be divided by 7. Therefore, this cell is counted twice, so the estimation formula \(C(30)\) is lower than the actual number of prime pairs.

The estimation formula for the number of prime pairs in \(\{11 + 30 * x, 13 + 30 * (89 - x)\} | 30 \leq x < 90\}\) is \(C(60)\):

\[
60(1 - \frac{2}{7})(1 - \frac{2}{11})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31}) = 10.72
\]

The actual number of prime pairs in \(\{11 + 30 * x, 13 + 30 * (89 - x)\} | 30 \leq x < 90\}\) is 16.

In the above formula, 77, 49 may not appear in the formula because they are multiples of 7. For the completeness of the formula, 77, 49 are still retained, which only make the valuation smaller.

The effective range of the above estimation formula is: 91^2 > 30 * 270 > 30 * 210.

The actual number of prime pairs in \(\{11 + 30 * x, 13 + 30 * (209 - x)\} | 90 \leq x < 210\}\) is 29. The estimation formula for the number of prime pairs in \(\{11 + 30 * x, 13 + 30 * (209 - x)\} | 90 \leq x < 210\}\) is \(C(120)\):

\[
120(1 - \frac{2}{7}) \cdots (1 - \frac{2}{31})
\]

\[
(1 - \frac{2}{37})(1 - \frac{2}{67}) \cdots (1 - \frac{2}{91})
\]

\[
(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187}) \cdots (1 - \frac{2}{211})
\]

The effective range of the above estimation formula is: 211^2 > 30 * 70 * 210 > 30 * 450.

More generally,

\[
(30 * 2^k - 30)^2 - 30 * (30 * 2^{k+1} - 30) = 900 * (2^{2k} - 2^{k+2} + 2) > 0
\]

\[
\implies (30 * 2^k - 30)^2 > 30 * (30 * 2^{k+1} - 30)
\]

where \(k \geq 2\).

When \(k = 2\), the inequality means 90^2 > 30 * 210.

When \(k = 3\), the inequality means 210^2 > 30 * 450.

Therefore, the above estimation formula is applied to the front part of the effective range, which is the main reason that the above estimation formula is lower than the actual number of primes.

The actual number of prime pairs in \(\{11 + 30 * x, 13 + 30 * (449 - x)\} | 210 \leq x < 450\}\) is 44. The estimation formula for the number of prime pairs in \(\{11 + 30 * x, 13 + 30 * (449 - x)\} | 210 \leq x < 450\}\) is 40.
On the prime distribution

$(449 - x)|210 \leq x < 450$ is $C(240)$(only the part about $\{7 + 30 \cdot x|x \geq 0 \}$ is given here):

$$
(1 - \frac{2}{7}) \\
(1 - \frac{2}{37})(1 - \frac{2}{67}) \\
(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187}) \\
(1 - \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427})
$$

It is easy to obtain the following (only the part about $\{7 + 30 \cdot x|x \geq 0 \}$ is given here):

$$
\frac{C(60)}{C(30)} = 2(1 - \frac{2}{37})(1 - \frac{2}{67}) \\
\frac{C(120)}{C(60)} = 2(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187}) \\
\frac{C(240)}{C(120)} = 2(1 - \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427})
$$

By Lemma 3,

$$
(1 - \frac{2}{217})(1 - \frac{2}{247}) > (1 - \frac{2}{97}), (1 - \frac{2}{277})(1 - \frac{2}{307}) > (1 - \frac{2}{127})
$$

Thus

$$
\ldots \ldots > \frac{C(240)}{C(120)} > \frac{C(120)}{C(60)} > \frac{C(60)}{C(30)}
$$

The complete $\frac{C(60)}{C(30)}$ is as follows,

$$
2(1 - \frac{2}{37})(1 - \frac{2}{67})(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187})(1 - \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427})(1 - \frac{2}{43})(1 - \frac{2}{49})(1 - \frac{2}{53})(1 - \frac{2}{61})
$$

$$
= 2 \times 0.575 > 1
$$

Therefore

$$
\ldots \ldots > C(240) > C(120) > C(60) \approx 10.72
$$

In fact, the actual number of prime pairs in $\{(11 + 30 \cdot x, 13 + 30 \cdot (929 - x))|450 \leq x < 930\}$ is 73, and the actual number of prime pairs in $\{(11 + 30 \cdot x, 13 + 30 \cdot (1889 - x))|930 \leq x < 1890\}$ is 136, . . . .

Similarly, we can always find another larger interval, which has more than 10.72 prime pairs $(p, q)$, such that $p + q = 24 + 30 \cdot (30 \cdot 2^k - 30 - 1)$, where $k > 1$.

For any $24 + 30 \cdot a, a \geq 90$, if $30 \cdot 2^k - 30 - 1 < a < 30 \cdot 2^{k+1} - 30 - 1$, by Lemma 2, the estimation formula for the number of prime pairs in $\{(11 + 30 \cdot x, 13 + 30 \cdot (a - x))|0 \leq x \leq a\}$ is greater than $C(30 \cdot 2^{k-1}) \geq C(60)$. 
Since we can also consider the prime pairs in \(\{(7+30x, 17+30*(a-x))|0 \leq x \leq a\}\), thus there exist much more than 10 prime pairs \((p,q)\), such that \(p+q = 24 + 30a\).

For example, for \(24 + 30 \times 99\), since \(30 \times 2^2 - 30 - 1 < 99 < 30 \times 2^{2+1} - 30 - 1\), so the estimation formula for the number of prime pairs in \(\{(11 + 30x, 13 + 30*(99-x))|0 \leq x \leq 99\}\) is greater than \(C(30 \times 2^{2-1}) = C(60)\). In fact, the actual number of prime pairs in \(\{(11 + 30x, 13 + 30*(99-x))|0 \leq x \leq 99\}\) is 27, and the actual number of prime pairs in \(\{(7 + 30x, 17 + 30*(99-x))|0 \leq x \leq 99\}\) is 32.

It is easy to show that for \(a > 0\),
\[
\begin{align*}
0 + 30a &= 7 + 30(a-1) + 23; \\
2 + 30a &= 13 + 30(a-1) + 19; \\
4 + 30a &= 11 + 30(a-1) + 23; \\
6 + 30a &= 13 + 30(a-1) + 23; \\
8 + 30a &= 19 + 30(a-1) + 19; \\
10 + 30a &= 17 + 30(a-1) + 23; \\
12 + 30a &= 19 + 30(a-1) + 23; \\
14 + 30a &= 13 + 30(a-1) + 31; \\
16 + 30a &= 17 + 30(a-1) + 29; \\
18 + 30a &= 7 + 30a + 11; \\
20 + 30a &= 7 + 30a + 13; \\
22 + 30a &= 11 + 30a + 11; \\
26 + 30a &= 13 + 30a + 13; \\
28 + 30a &= 11 + 30a + 17.
\end{align*}
\]

Very similar to the case of \(\{24 + 30a = 11 + 30a + 13\}\), we can prove that for any \(2l + 30a, 0 \leq l < 15, a \geq 90\), there exist more than 10 prime pairs \((p,q)\), such that \(p+q = 2l + 30a\).

To sum up, we get the following theorem.

**Theorem 2.** For any even integer \(m > 30 \times 90\), there exist more than 10 prime pairs \((p,q)\), such that \(p+q = m\).

### 6 Triple prime conjecture

Based on Lemma 2, we consider a formula for estimating the number of primes. Consider an estimation formula for the number of primes in first 210 natural numbers \(\{11 + 210x|0 \leq x < 210\}\):

\[
\begin{align*}
210(1 - \frac{1}{11})(1 - \frac{1}{15})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31})(1 - \frac{1}{37})(1 - \frac{1}{41})(1 - \frac{1}{43}) \\
(1 - \frac{1}{47})(1 - \frac{1}{53})(1 - \frac{1}{59})(1 - \frac{1}{61})(1 - \frac{1}{67})(1 - \frac{1}{71})(1 - \frac{1}{73})(1 - \frac{1}{79})(1 - \frac{1}{83})(1 - \frac{1}{89}) \\
(1 - \frac{1}{97})(1 - \frac{1}{101})(1 - \frac{1}{103})(1 - \frac{1}{107})(1 - \frac{1}{109})(1 - \frac{1}{113})(1 - \frac{1}{127})(1 - \frac{1}{131})(1 - \frac{1}{137})(1 - \frac{1}{139})(1 - \frac{1}{149}) \\
(1 - \frac{1}{151})(1 - \frac{1}{157})(1 - \frac{1}{163})(1 - \frac{1}{167})(1 - \frac{1}{173})(1 - \frac{1}{179})(1 - \frac{1}{181})(1 - \frac{1}{191})(1 - \frac{1}{193})(1 - \frac{1}{197})(1 - \frac{1}{199})(1 - \frac{1}{211})
\end{align*}
\]

\(~\approx~95.00\)
On the prime distribution

The actual number of primes in \{11 + 210 \cdot x | 0 \leq x < 210\} is 98.

Similarly, consider an estimation formula for the number of primes in first 210 natural numbers \{17 + 210 \cdot x | 0 \leq x < 210\}:

\[
210(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31})(1 - \frac{1}{37})(1 - \frac{1}{41})(1 - \frac{1}{43})
\]

\[
(1 - \frac{1}{47})(1 - \frac{1}{53})(1 - \frac{1}{59})(1 - \frac{1}{61})(1 - \frac{1}{67})(1 - \frac{1}{71})(1 - \frac{1}{73})(1 - \frac{1}{79})(1 - \frac{1}{83})(1 - \frac{1}{89})
\]

\[
(1 - \frac{1}{97})(1 - \frac{1}{101})(1 - \frac{1}{103})(1 - \frac{1}{107})(1 - \frac{1}{109})(1 - \frac{1}{113})(1 - \frac{1}{127})(1 - \frac{1}{131})(1 - \frac{1}{137})(1 - \frac{1}{139})(1 - \frac{1}{149})
\]

\[
(1 - \frac{1}{151})(1 - \frac{1}{157})(1 - \frac{1}{163})(1 - \frac{1}{167})(1 - \frac{1}{173})(1 - \frac{1}{179})(1 - \frac{1}{181})(1 - \frac{1}{191})(1 - \frac{1}{193})(1 - \frac{1}{197})(1 - \frac{1}{199})(1 - \frac{1}{211})
\]

\[
\approx 95.00
\]

The actual number of primes in \{17 + 210 \cdot x | 0 \leq x < 210\} is 96.

Further consider an estimation formula for the number of primes in first 210 natural numbers \{23 + 210 \cdot x | 0 \leq x < 210\}:

\[
210(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31})(1 - \frac{1}{37})(1 - \frac{1}{41})(1 - \frac{1}{43})
\]

\[
(1 - \frac{1}{47})(1 - \frac{1}{53})(1 - \frac{1}{59})(1 - \frac{1}{61})(1 - \frac{1}{67})(1 - \frac{1}{71})(1 - \frac{1}{73})(1 - \frac{1}{79})(1 - \frac{1}{83})(1 - \frac{1}{89})
\]

\[
(1 - \frac{1}{97})(1 - \frac{1}{101})(1 - \frac{1}{103})(1 - \frac{1}{107})(1 - \frac{1}{109})(1 - \frac{1}{113})(1 - \frac{1}{127})(1 - \frac{1}{131})(1 - \frac{1}{137})(1 - \frac{1}{139})(1 - \frac{1}{149})
\]

\[
(1 - \frac{1}{151})(1 - \frac{1}{157})(1 - \frac{1}{163})(1 - \frac{1}{167})(1 - \frac{1}{173})(1 - \frac{1}{179})(1 - \frac{1}{181})(1 - \frac{1}{191})(1 - \frac{1}{193})(1 - \frac{1}{197})(1 - \frac{1}{199})(1 - \frac{1}{211})
\]

\[
\approx 95.00
\]

The actual number of primes in \{23 + 210 \cdot x | 0 \leq x < 210\} is 94.

We further consider possible form of triple primes \{(11 + 210 \cdot x, 17 + 210 \cdot x, 23 + 210 \cdot x)|x \geq 0\}. For example, when \(x = 100, 21011, 21017\) and 21023 are all primes.

Consider an estimation formula \(C(210)\) lower than the actual number of triple primes in first 210 triplets of natural numbers \{(11 + 210 \cdot x, 17 + 210 \cdot x, 23 + 210 \cdot x)|0 \leq x < 210\}:

\[
210(1 - \frac{3}{11})(1 - \frac{3}{13})(1 - \frac{3}{17})(1 - \frac{3}{19})(1 - \frac{3}{23})(1 - \frac{3}{29})(1 - \frac{3}{31})(1 - \frac{3}{37})(1 - \frac{3}{41})(1 - \frac{3}{43})
\]

\[
(1 - \frac{3}{47})(1 - \frac{3}{53})(1 - \frac{3}{59})(1 - \frac{3}{61})(1 - \frac{3}{67})(1 - \frac{3}{71})(1 - \frac{3}{73})(1 - \frac{3}{79})(1 - \frac{3}{83})(1 - \frac{3}{89})
\]

\[
(1 - \frac{3}{97})(1 - \frac{3}{101})(1 - \frac{3}{103})(1 - \frac{3}{107})(1 - \frac{3}{109})(1 - \frac{3}{113})(1 - \frac{3}{127})(1 - \frac{3}{131})(1 - \frac{3}{137})(1 - \frac{3}{139})(1 - \frac{3}{149})
\]

\[
(1 - \frac{3}{151})(1 - \frac{3}{157})(1 - \frac{3}{163})(1 - \frac{3}{167})(1 - \frac{3}{173})(1 - \frac{3}{179})(1 - \frac{3}{181})(1 - \frac{3}{191})(1 - \frac{3}{193})(1 - \frac{3}{197})(1 - \frac{3}{199})(1 - \frac{3}{211})
\]

\[
\approx 17.46
\]
The actual number of triple primes in \( \{(11 + 210 \times x, 17 + 210 \times x, 23 + 210 \times x) | 0 \leq x < 210\} \) is 19.

Based on Lemma 1, the above estimation formula can be understood in this way (take prime number 11 as an example): for every 11 consecutive cells, there must be one cell of \( 11 + 210 \times x_1 \) can be divided by 11, another cell of \( 17 + 210 \times x_2 \) can be divided by 11 and the third cell of \( 23 + 210 \times x_3 \) can be divided by 11, where \( x_1, x_2, x_3 \) are distinct numbers. Therefore, one term in the above formula is \( \frac{11}{3} \cdot \frac{1}{11} \).

However, when \( x = 23, 11 + 210 \times 23 = 4841 \) can be divided by 47, \( 17 + 210 \times 23 = 4847 \) can be divided by 37, and \( 23 + 210 \times 23 = 4853 \) can be divided by 23. Therefore, this cell is counted three times, so the estimation formula \( C(210) \) is lower than the actual number of triple primes.

The estimation formula for the number of triple prime numbers in \( \{(11 + 210 \times x, 17 + 210 \times x, 23 + 210 \times x) | 210 \leq x < 630\} \) is \( C(420) \):

\[
\begin{align*}
420(1 - \frac{3}{11})(1 - \frac{3}{13})\cdots(1 - \frac{3}{211}) \\
(1 - \frac{3}{221})(1 - \frac{3}{431})(1 - \frac{3}{433})\cdots(1 - \frac{3}{421})(1 - \frac{3}{631}) & \approx 17.66
\end{align*}
\]

The actual number of triple prime numbers in \( \{(11 + 210 \times x, 17 + 210 \times x, 23 + 210 \times x) | 210 \leq x < 630\} \) is 22.

In the above formula, 221, 247, 253, \cdots may not appear in the formula because \( 221 = 13 \times 17, 247 = 13 \times 19, 253 = 23 \times 11 \cdots \). For the completeness of the formula, 221, 247, 253, \cdots are still retained, which only make the valuation further smaller.

The actual number of triple prime numbers in \( \{(11 + 210 \times x, 17 + 210 \times x, 23 + 210 \times x) | 630 \leq x < 1470\} \) is 35.

The estimation formula for the number of triple prime numbers in \( \{(11 + 210 \times x, 17 + 210 \times x, 23 + 210 \times x) | 630 \leq x < 1470\} \) is \( C(840) \):

\[
\begin{align*}
840(1 - \frac{3}{11})(1 - \frac{3}{13})\cdots(1 - \frac{3}{211}) \\
(1 - \frac{3}{221})(1 - \frac{3}{431})(1 - \frac{3}{433})\cdots(1 - \frac{3}{421})(1 - \frac{3}{631})(1 - \frac{3}{841})(1 - \frac{3}{1051})(1 - \frac{3}{1261})(1 - \frac{3}{1471}) & \approx 20.96
\end{align*}
\]

The effective range of the above estimation formula is: \( 1471^2 > 210 \times 7 \times 1470 > 210 \times 3150 \).

More generally,

\[
(210 \times 2^k - 210)^2 - 210 \times (210 \times 2^{k+1} - 210) = 210^2 \times (2^{2k} - 2^{k+2} + 2) > 0
\]

\[
\implies (210 \times 2^k - 210)^2 > 210 \times (210 \times 2^{k+1} - 210)
\]

where \( k \geq 2 \).
When \( k = 2 \), the inequality means \( 630^2 > 210 \times 1470 \).
When \( k = 3 \), the inequality means \( 1470^2 > 210 \times 3150 \).
When \( k = 4 \), the inequality means \( 3150^2 > 210 \times 6510 \).

Therefore, the above estimation formula is applied to the front part of the effective range, which is the main reason that the above estimation for formula is lower than the actual number of primes.

It is easy to obtain the following:

\[
\frac{C(420)}{C(210)} = 2(1 - \frac{3}{221})(1 - \frac{3}{431})(1 - \frac{3}{223})(1 - \frac{3}{433}) \cdots (1 - \frac{3}{421})(1 - \frac{3}{631})
\]

\[
\frac{C(840)}{C(420)} = 2(1 - \frac{3}{641})(1 - \frac{3}{851})(1 - \frac{3}{1061})(1 - \frac{3}{1271}) \cdots (1 - \frac{3}{841})(1 - \frac{3}{1051})(1 - \frac{3}{1261})(1 - \frac{3}{1471})
\]

\[
\frac{C(1680)}{C(840)} = 2(1 - \frac{3}{1481})(1 - \frac{3}{1691})(1 - \frac{3}{1901})(1 - \frac{3}{2111}) \cdots (1 - \frac{3}{2521})(1 - \frac{3}{2731})(1 - \frac{3}{2941})(1 - \frac{3}{3151})
\]

Similar to the proof of Lemma 3, it is easy to prove that,

\[
(1 - \frac{d}{210 \times x + c}) < (1 - \frac{d}{210 \times (2x + e + 1) + c})(1 - \frac{d}{210 \times (2x + e + 2) + c})
\]

\[
x \geq 0, e \geq 0, 0 < c < 212, 0 < d < 210x.
\]

Therefore,

\[
(1 - \frac{3}{1481})(1 - \frac{3}{1691}) > (1 - \frac{3}{641})(1 - \frac{3}{1901})(1 - \frac{3}{2111}) > (1 - \frac{3}{851}), \ldots
\]

Thus

\[
\cdots > \frac{C(1680)}{C(840)} > \frac{C(840)}{C(420)} > \frac{C(420)}{C(210)}
\]

Since

\[
\frac{C(420)}{C(210)} = 2(1 - \frac{3}{221})(1 - \frac{3}{431})(1 - \frac{3}{223})(1 - \frac{3}{433}) \cdots (1 - \frac{3}{421})(1 - \frac{3}{631})
\]

\[
= 2 \times 0.506 > 1
\]

Therefore

\[
\cdots > C(1680) > C(840) > C(420) > C(210) \approx 17.46
\]

Similarly, we can always find another larger interval, which has more than 17.46 triple prime numbers. Therefore, there are infinite triple primes.

Very similarly, we can further consider possible form of triple primes \( \{(17 + 210 \times x, 23 + 210 \times x, 29 + 210 \times x)|x \geq 0\}, \{(31 + 210 \times x, 37 + 210 \times x, 43 + 210 \times x)|x \geq 0\}, \)
{(47 + 210 * x, 53 + 210 * x, 59 + 210 * x) | x ≥ 0}, · · · · · ·, and prove that there are infinite triple primes.

To sum up, we get the following theorem.

**Theorem 3.** For any triple primes \( p > 7, q > 7 \) and \( r > 7 \), there are infinite triple primes \( (p_i, q_i, r_i) \), such that \( p_i - q_i = p - q \), \( p_i - r_i = p - r \), where \( i > 0 \).

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