An $A_\infty$-structure on the cohomology ring of the symmetric group $S_p$ with coefficients in $\mathbb{F}_p$

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February 4, 2013

Abstract

Let $p$ be a prime. Let $\mathbb{F}_p S_p$ be the group algebra of the symmetric group over the finite field $\mathbb{F}_p$ with $|\mathbb{F}_p| = p$. Let $\mathbb{F}_p$ be the trivial $\mathbb{F}_p S_p$-module. We present a projective resolution $\text{PRes}_{\mathbb{F}_p}$ of the module $\mathbb{F}_p$ and equip the Yoneda algebra $\text{Ext}^*_{\mathbb{F}_p S_p}(\mathbb{F}_p, \mathbb{F}_p)$ with an $A_\infty$-structure such that $\text{Ext}^*_{\mathbb{F}_p S_p}(\mathbb{F}_p, \mathbb{F}_p)$ becomes a minimal model of the dg-algebra $\text{Hom}^*_{\mathbb{F}_p S_p}(\text{PRes}_{\mathbb{F}_p}, \text{PRes}_{\mathbb{F}_p})$.

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MSC 2010: 18G15.
0.1 Introduction

$A\otimes\infty$-algebras  Let $R$ be a commutative ring. Let $A$ be a $\mathbb{Z}$-graded $R$-module. Let $m_1 : A \to A$ be a graded map of degree $1$ with $m_1^2 = 0$, i.e. a differential on $A$. Let $m_2 : A \otimes A \to A$ be a graded map of degree $0$ satisfying the Leibniz rule, i.e.

$$m_1 \circ m_2 = m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1).$$

The map $m_2$ is in general not required to be associative. Instead, we require that for a morphism $m_3 : A^{\otimes 3} \to A$, the following identity holds.

$$m_2 \circ (m_2 \otimes 1 - 1 \otimes m_2) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes 1^{\otimes 2} + 1 \otimes m_1 \otimes 1 + 1^{\otimes 2} \otimes m_1).$$

Following Stasheff, cf. [20], this can be continued in a certain way with higher multiplication maps to obtain a tuple of graded maps $(m_n : A^{\otimes n} \to A)_{n \geq 1}$ of certain degrees satisfying the Stasheff identities, cf. e.g. (5). The tuple $(A, (m_n)_{n \geq 1})$ is then called an $A\otimes\infty$-algebra.

A morphism of $A\otimes\infty$-algebras from $(A', (m'_n)_{n \geq 1})$ to $(A, (m_n)_{n \geq 1})$ is a tuple of graded maps $(f_n : A'^{\otimes n} \to A)_{n \geq 1}$ of certain degrees satisfying the identities (6). The first two of these are

$$(6)[1] : \quad f_1 \circ m'_1 = m_1 \circ f_1,$$

$$(6)[2] : \quad f_1 \circ m'_2 - f_2 \circ (m_1 \otimes 1 + 1 \otimes m'_1) = m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1).$$

So a morphism $f = (f_n)_{n \geq 1}$ of $A\otimes\infty$-algebras from $(A', (m'_n)_{n \geq 1})$ to $(A, (m_n)_{n \geq 1})$ contains a morphism of complexes $f_1 : (A', m'_1) \to (A, m_1)$. We say that $f$ is a quasi-isomorphism of $A\otimes\infty$-algebras if $f_1$ is a quasi-isomorphism. Furthermore, there is a concept of homotopy for $A\otimes\infty$-morphisms, cf. e.g. [10, 3.7] and [14, Définition 1.2.1.7].

History  The history of $A\otimes\infty$-algebras is outlined in [10] and [11].

As already mentioned, Stasheff introduced $A\otimes\infty$-algebras in 1963.

If $R$ is a field, $\mathbb{F} := R$, we have the following basic results on $A\otimes\infty$-algebras, which are known since the early 1980s.

- Each quasi-isomorphism of $A\otimes\infty$-algebras is a homotopy equivalence, cf. [18], [9], . . .
- The minimality theorem: Each $A\otimes\infty$-algebra $(A, (m_n)_{n \geq 1})$ is quasi-isomorphic to an $A\otimes\infty$-algebra $(A', (m'_n)_{n \geq 1})$ with $m'_1 = 0$, cf. [8], [7], [18], [4], [6], [16], . . . . The $A\otimes\infty$-algebra $A'$ is then called a minimal model of $A$.

Suppose given an $\mathbb{F}$-algebra $B$ and suppose given a $B$-module $M$ together with a projective resolution $\text{PRes}(M)$. The homology of the dg-algebra $\text{Hom}_B^\bullet(\text{PRes}(M), \text{PRes}(M))$ is the Yoneda algebra $\text{Ext}_B^\bullet(M, M)$. By the minimality theorem, it is possible to construct an $A\otimes\infty$-structure on $\text{Ext}_B^\bullet(M, M)$ such that $\text{Ext}_B^\bullet(M, M)$ becomes a minimal model of the dg-algebra $\text{Hom}_B^\bullet(\text{PRes}(M), \text{PRes}(M))$. For the purpose of this introduction, we will
call such an $A_\infty$-structure on $\text{Ext}^*_B(M, M)$ the canonical $A_\infty$-structure on $\text{Ext}^*_B(M, M)$, which is unique up to isomorphisms of $A_\infty$-algebras, cf. [10, 3.3].

This structure has been calculated or partially calculated in several cases.

Let $p$ be a prime.

For an arbitrary field $F$, Madsen computed the canonical $A_\infty$-structure on $\text{Ext}^*_F(\alpha/(\alpha^n), (\alpha))_0(F, F)$, where $\alpha$ is the trivial $F[\alpha]/(\alpha^n)$-module, cf. [15, Appendix B.2]. This can be used to compute the canonical $A_\infty$-structure on the group cohomology $\text{Ext}^*_F C_m(F_p, F_p)$, where $m \in \mathbb{Z}_{\geq 1}$ and $C_m$ is the cyclic group of order $m$, cf. [21, Theorem 4.3.8].

Vejdemo-Johansson developed algorithms for the computation of minimal models [21]. He applied these algorithms to compute large enough parts of the canonical $A_\infty$-structures of the group cohomologies $\text{Ext}^*_F D_8(F_2, F_2)$ and $\text{Ext}^*_F D_{16}(F_2, F_2)$ to distinguish them, where $D_8$ and $D_{16}$ denote dihedral groups. He stated a conjecture on the complete $A_\infty$-structure on $\text{Ext}^*_F D_8(F_2, F_2)$. Furthermore, he computed parts of the canonical $A_\infty$-structure on $\text{Ext}^*_F Q_8(F_2, F_2)$ for the quaternion group $Q_8$. He conjecturally stated the minimal complexity of such a structure. Based on this work, there are now built-in algorithms for the Magma computer algebra system. These are capable of computing partial $A_\infty$-structures on the group cohomology of $p$-groups.

In [22], Vejdemo-Johansson examined the canonical $A_\infty$-structure $(m_n)_{n \geq 1}$ on the group cohomology $\text{Ext}^*_F(C_k \times C_l)_0(F_p, F_p)$ of the abelian group $C_k \times C_l$ for $k, l \geq 4$ such that $k, l$ are multiples of $p$. He showed that for infinitely many $n \in \mathbb{Z}_{\geq 1}$, the operation $m_n$ is non-zero.

In [12], Klamt investigated canonical $A_\infty$-structures in the context of the representation theory of Lie-algebras. In particular, given certain direct sums $M$ of parabolic Verma modules, she examined the canonical $A_\infty$-structure $(m'_k)_{k \geq 1}$ on $\text{Ext}^*_F(M, M)$. She proved upper bounds for the maximal $k \in \mathbb{Z}_{\geq 1}$ such that $m'_k$ is non-vanishing and computed the complete $A_\infty$-structure in certain cases.

The result  For $n \in \mathbb{Z}_{\geq 1}$, we denote by $S_n$ the symmetric group on $n$ elements.

The group cohomology $\text{Ext}^*_F(F_p, F_p)$ is well-known. For example, in [1, p. 74], it is calculated using group cohomological methods.

Here, we will construct the canonical $A_\infty$-structure on $\text{Ext}^*_F(F_p, F_p)$.

We obtain homogeneous elements $\iota, \chi \in \text{Hom}^*_F(\text{PRes} F_p, \text{PRes} F_p) =: A$ of degree $|\iota| = 2(p - 1) =: l$ and $|\chi| = l - 1$ such that $\iota^j, \chi \circ \iota^j =: \chi \iota^j$ are cycles for all $j \in \mathbb{Z}_{\geq 0}$ and such that their set of homology classes $\{\iota^j \mid j \in \mathbb{Z}_{\geq 0}\} \cup \{\chi \iota^j \mid j \in \mathbb{Z}_{\geq 0}\}$ is an $F_p$-basis of $\text{Ext}^*_F(F_p, F_p) = H^*A$, cf. Proposition 20.

For all primes $p$, the canonical $A_\infty$-structure $(m'_n : (H^*A)^{\otimes n} \to H^*A)_{n \geq 1}$ on $H^*A$ is given as follows.

On the elements $\chi^{a_1} \iota^{j_1} \otimes \cdots \otimes \chi^{a_n} \iota^{j_n}, n \in \mathbb{Z}_{\geq 1}, a_i \in \{0, 1\}$ and $j_i \in \mathbb{Z}_{\geq 0}$ for $i \in \{1, \ldots, n\}$,
the maps \( m'_n \) are given as follows, cf. Definition 23 and Remark 37.

If there is an \( i \in \{1, \ldots, n\} \) such that \( a_i = 0 \), then
\[
\begin{align*}
\chi_j^{a_i} \otimes \cdots \otimes \chi_j^{a_n} = 0 & \quad \text{for } n \neq 2 \\
\chi_j^{a_1} \otimes \chi_j^{a_2} = \chi_j^{a_1 + a_2} & \quad \text{for } n = 2.
\end{align*}
\]

If all \( a_i \) equal 1, then
\[
\begin{align*}
\chi_j^{a_1} \otimes \cdots \otimes \chi_j^{a_n} = 0 & \quad \text{for } n \neq p \\
\chi_j^{a_1} \otimes \cdots \otimes \chi_j^{a_p} = (-1)^{n-1} & \quad \text{for } n = p.
\end{align*}
\]

In particular, we have \( m'_n = 0 \) for all \( n \in \mathbb{Z}_{\geq 1} \setminus \{2, p\} \).

### 0.2 Outline

**Section 1** The goal of section 1 is to obtain a projective resolution of the trivial \( \mathbb{F}_p S_p \)-Specht module \( \mathbb{F}_p \). A well-known method for that is "Walking around the Brauer tree", cf. [3]. Instead, we use locally integral methods to obtain a projective resolution in an explicit and straightforward manner.

Over \( \mathbb{Q} \), the Specht modules are absolutely simple. Therefore we have a morphism of \( \mathbb{Z}_p \)-algebras \( r : \mathbb{Z}_p S_p \to \prod_{\lambda \vdash p} \text{End}_{\mathbb{Z}_p} S^\lambda_p \) : \( \Gamma \) induced by the operation of the elements of \( \mathbb{Z}_p S_p \) on the Specht modules \( S^\lambda \) for partitions \( \lambda \) of \( p \), which becomes a Wedderburn isomorphism when tensoring with \( \mathbb{Q} \). So \( \Gamma \) is a product of matrix rings over \( \mathbb{Z}_p \). There is a well-known description of \( \text{im } r =: \Lambda \), which we use for \( p \geq 3 \) to obtain projective \( \Lambda \)-modules \( P_k \subseteq \Lambda, k \in [1, p - 1] \), and to construct the indecomposable projective resolution \( \text{PRes} \mathbb{Z}_p \) of the trivial \( \mathbb{Z}_p S_p \)-Specht module \( \mathbb{Z}_p \). The non-zero parts of \( \text{PRes} \mathbb{Z}_p \) are periodic with period length \( l = 2(p - 1) \). In section 1.3, we reduce \( \text{PRes} \mathbb{Z}_p \) modulo \( p \) to obtain a projective resolution \( \text{PRes} \mathbb{F}_p \) of the trivial \( \mathbb{F}_p S_p \)-Specht module \( \mathbb{F}_p \).

**Section 2** The goal of section 2 is to compute a minimal model of the dg-algebra \( \text{Hom}_{\mathbb{F}_p S_p} (\text{PRes} \mathbb{F}_p, \text{PRes} \mathbb{F}_p) =: A \) by equipping its homology \( \text{Ext}_{\mathbb{F}_p S_p}^\ast (\mathbb{F}_p, \mathbb{F}_p) = H^\ast A \) with a suitable \( A_\infty \)-structure and finding a quasi-isomorphism of \( A_\infty \)-algebras from \( H^\ast A \) to \( A \).

Towards that end, we recall the basic definitions concerning \( A_\infty \)-algebras and some general results in section 2.1.

While there does not seem to be a substantial difference between the cases \( p = 2 \) and \( p \geq 3 \), we separate them to simplify notation and argumentation. Consider the case \( p \geq 3 \). In section 2.2, we obtain a set of cycles \( \{v^j \mid j \in \mathbb{Z}_{\geq 0}\} \cup \{v^j \mid j \in \mathbb{Z}_{\geq 0}\} \) in \( A \) such that their homology classes are a graded basis of \( H^\ast A \). In section 2.3, we obtain a suitable \( A_\infty \)-structure on \( H^\ast A \) and a quasi-isomorphism of \( A_\infty \)-algebras from \( H^\ast A \) to \( A \). For the prime 2, both steps are combined in the short section 2.4.
0.3 Notations and conventions

Stipulations

- For the remainder of this document, \( p \) will be a prime with \( p \geq 3 \).
- Write \( l := 2(p-1) \). This will give the period length of the constructed projective resolution of \( \mathbb{F}_p \) over \( \mathbb{F}_p S_p \), cf. e.g. (1), Lemma 6.

Miscellaneous

- Concerning "\( \infty \)”, we assume the set \( \mathbb{Z} \cup \{ \infty \} \) to be ordered in such a way that \( \infty \) is greater than any integer, i.e. \( \infty > z \) for all \( z \in \mathbb{Z} \), and that the integers are ordered as usual.
- For \( a \in \mathbb{Z} \), \( b \in \mathbb{Z} \cup \{ \infty \} \), we denote by \([a, b] := \{ z \in \mathbb{Z} \mid a \leq z \leq b \} \subseteq \mathbb{Z} \) the integral interval. In particular, we have \([a, \infty] = \{ z \in \mathbb{Z} \mid z \geq a \} \subseteq \mathbb{Z} \) for \( a \in \mathbb{Z} \).
- For \( n \in \mathbb{Z}_{\geq 0} \), \( k \in \mathbb{Z} \), let the binomial coefficient \( \binom{n}{k} \) be defined by the number of subsets of the set \( \{1, \ldots, n\} \) that have cardinality \( k \). In particular, if \( k < 0 \) or \( k > n \), we have \( \binom{n}{k} = 0 \). Then the formula \( \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \) holds for all \( k \in \mathbb{Z} \).
- For a commutative ring \( R \), an \( R \)-module \( M \) and \( a, b \in M \), \( c \in R \), we write \( b \equiv_c a \iff a - b \in cM \).

Often we have \( M = R \) as module over itself.
- Modules are right-modules unless otherwise specified.
- For sets, we denote by \( \uplus \) the disjoint union of sets.
- |·|: For a homogeneous element \( x \) of a graded module or a graded map \( g \) between graded modules, we denote by \(|x| \) resp. \(|g| \) their degrees (This is not unique for \( x = 0 \) resp. \( g = 0 \)). For \( y \) a real number, \(|y| \) denotes its absolute value.

Symmetric Groups

Let \( n \in \mathbb{Z}_{\geq 1} \). We denote the symmetric group von \( n \) elements by \( S_n \). For a partition \( \lambda \vdash n \), we denote the corresponding Specht module by \( S^\lambda \).

Complexes

Let \( R \) be a commutative ring and \( B \) an \( R \)-algebra.
- For a complex of \( B \)-modules
\[
\cdots \rightarrow C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \rightarrow \cdots ,
\]
its \( k \)-th boundaries, cycles and homology groups are defined by \( B^k := \text{im} \, d_{k+1} \), \( Z^k := \ker d_k \) and \( H^k := Z^k / B^k \).

For a cycle \( x \in Z^k \), we denote by \( \overline{x} := x + B^k \in H^k \) its equivalence class in homology.
Let
\[ C = (\cdots \to C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \to \cdots) \]
\[ C' = (\cdots \to C'_{k+1} \xrightarrow{d'_{k+1}} C'_k \xrightarrow{d'_k} C'_{k-1} \to \cdots) \]
be two complexes of \( B \)-modules.

Given \( z \in \mathbb{Z} \), let
\[
\text{Hom}_B^z(C, C') := \prod_{i \in \mathbb{Z}} \text{Hom}_B(C_{i+z}, C'_i).
\]

For an additional complex \( C'' = (\cdots \to C''_{k+1} \xrightarrow{d''_{k+1}} C''_k \xrightarrow{d''_k} C''_{k-1} \to \cdots) \) and maps \( h = (h_i)_{i \in \mathbb{Z}} \in \text{Hom}_B^m(C, C') \), \( h' = (h'_i)_{i \in \mathbb{Z}} \in \text{Hom}_B^n(C', C'') \), \( m, n \in \mathbb{Z} \), we define the composition by component-wise composition as
\[
h' \circ h := (h'_i \circ h_{i+j})_{i \in \mathbb{Z}} \in \text{Hom}_B^{m+n}(C, C'').
\]

We will assemble elements of \( \text{Hom}_B^z(C, C') \) as sums of their non-zero components, which motivates the following notations regarding ”extensions by zero” and sums.

For a map \( g : C_x \to C'_y \), we define \( [g]^y_x \in \text{Hom}_B^{x-y}(C, C') \) by
\[
([g]^y_x)_i := \begin{cases} g & \text{for } i = y \\ 0 & \text{for } i \in \mathbb{Z} \setminus \{y\} \end{cases}.
\]

Let \( k \in \mathbb{Z} \). Let \( I \) be a (possibly infinite) set. Let \( g_i = (g_{i,j})_{j} \in \text{Hom}_B^k(C, C') \) for \( i \in I \) such that \( \{i \in I \mid g_{i,j} \neq 0\} \) is finite for all \( j \in \mathbb{Z} \).

We define the sum \( \sum_{i \in I} g_i \in \text{Hom}_B^k(C, C') \) by
\[
\left( \sum_{i \in I} g_i \right)_j := \sum_{i \in I, g_{i,j} \neq 0} g_{i,j}.
\]

The graded \( R \)-module \( \text{Hom}_B^*(C, C') := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_B^k(C, C') \) becomes a complex via the differential \( d_{\text{Hom}_B^*}(C, C') \), which is defined on elements \( g \in \text{Hom}_B^k(C, C') \), \( k \in \mathbb{Z} \) by
\[
d_{\text{Hom}_B^*}(C, C')(g) := d' \circ g - (-1)^k g \circ d \in \text{Hom}_B^{k+1}(C, C'),
\]
where \( d := (d_{i+1})_{i \in \mathbb{Z}} = \sum_{i \in \mathbb{Z}} d_{i+1} \) \( \in \text{Hom}_B^1(C, C) \) and analogously \( d' := (d'_{i+1})_{i \in \mathbb{Z}} = \sum_{i \in \mathbb{Z}} d'_{i+1} \) \( \in \text{Hom}_B^1(C', C'') \).

An element \( h \in \text{Hom}_B^0(C, C') \) is called a complex morphism if it satisfies \( d_{\text{Hom}_B^*}(C, C')(h) = 0 \), i.e. \( d' \circ g = g \circ d \).
1 The projective resolution of $\mathbb{F}_p$ over $\mathbb{F}_pS_p$

1.1 A description of $\mathbb{Z}_{(p)}S_p$

Recall that $p \geq 3$ is a prime.

For $R$ a ring and $\lambda \vdash p$ a partition of $p$, the $RS_p$-Specht module $S^\lambda$ is finitely Generatey free over $R$ with dimension independent of $R$, cf. [5, 8.1, proof of 8.4]. We denote this dimension by $n_\lambda$.

A partition of the form $\lambda^k := (p - k + 1, 1^{k-1})$, $k \in [1, p]$ is called a hook partition of $p$.

Over the valuation ring $\mathbb{Z}_{(p)}$, there is a well-known description of the group algebra $\mathbb{Z}_{(p)}S_p$, cf. e.g. [13, Corollary 4.2.8] (using [17]), cf. also [19, Chapter 7]:

**Proposition 1.** Set $n^k_b = \left(\frac{p-k-1}{k-1}\right)$, $n^k_c = \left(\frac{p}{k-1}\right)$. Then $n^k_b + n^k_c = \left(\frac{p}{k-1}\right) = n_\lambda$. Set $\Gamma := \prod_{\lambda \vdash p} \mathbb{Z}_{(p)}^{n_\lambda \times n_\lambda}$. For $\rho \in \Gamma$, and $\lambda \vdash p$, we denote by $\rho^\lambda$ the $\lambda$-th component of $\rho$. For $\lambda = \lambda^k$, $k \in [1, p]$, a hook partition, we name certain subblocks of $\rho^\lambda$ as follows.

$$\rho^\lambda = \begin{pmatrix}
\rho^\lambda_{cb} & \rho^\lambda_{cc} \\
\rho^\lambda_{bc} & \rho^\lambda_{bb}
\end{pmatrix}
\begin{pmatrix}
n^k_c & \text{n}_b^k \\
n^k_b & \text{n}_b^k
\end{pmatrix}
$$

We have the following $\mathbb{Z}_{(p)}$-subalgebra $\Lambda$ of $\Gamma$.

$$\Lambda := \{ \rho \in \Gamma \mid \rho^\lambda_{bb} \equiv_p \rho^\lambda_{cc}^{k+1} \text{ for } k \in [1, p-1] \text{ and } \rho^\lambda_{bc} \equiv_p 0 \text{ for } k \in [1, p] \}$$

Now there is an isomorphism of $\mathbb{Z}_{(p)}$-algebras

$$r : \mathbb{Z}_{(p)}S_p \xrightarrow{\sim} \Lambda.$$ 

such that $\rho \in \Lambda$ acts on the trivial $\mathbb{Z}_{(p)}S_p$-module $\mathbb{Z}_{(p)}$ by multiplication with its $(1 \times 1$ / scalar-)component $\rho^\lambda$, i.e. for $x \in \mathbb{Z}_{(p)}S_p$ and for $y \in \mathbb{Z}_{(p)}$, we have $yx = y \cdot r(x)^\lambda$.

**Example 2.** For $p = 5$, the $\mathbb{Z}_{(p)}$-algebra $\mathbb{Z}_{(5)}S_5$ is isomorphic to the subalgebra $\Lambda$ of $\Gamma = \mathbb{Z}_{(5)}^{1\times 1} \times \mathbb{Z}_{(5)}^{4\times 4} \times \mathbb{Z}_{(5)}^{6\times 6} \times \mathbb{Z}_{(5)}^{4\times 4} \times \mathbb{Z}_{(5)}^{1\times 1} \times \mathbb{Z}_{(5)}^{5\times 5} \times \mathbb{Z}_{(5)}^{5\times 5}$ described as
We define corresponding projective right $\Lambda$-modules $\hat{P}_i$ for $i = 1, 2, 3, 4$. An entry in this tuple of matrices indicates that an element of $\Lambda$ must have its corresponding entry in the indicated set. A relation 
$\begin{array}{c}
\lambda \mapsto \\
\eta
\end{array}$
between (equal sized) subblocks indicates that these subblocks are equivalent modulo 5, i.e. the difference of corresponding entries is an element of $5\mathbb{Z}_5$. The blocks are labeled with the diagrams of the corresponding partitions. The right ideals $\hat{P}_i = \hat{e}_i \Lambda$, $i \in [1, 4] = [1, p - 1]$ (cf. the definitions below) are framed with red lines.

### 1.2 A projective resolution of $\mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)} S_p$

Recall that $p \geq 3$ is a prime.

Recall from Proposition 1 that $\Lambda$ is a subring of $\Gamma = \prod_{\lambda \vdash p} \mathbb{Z}_{(p)}^{n_{(\lambda)} \times n_{\lambda}}$.

For $\lambda \vdash p$ and $i, j \in [1, n_{(\lambda)}]$, we set $\eta_{\lambda, i, j}$ to be the element of $\Gamma$ such that $(\eta_{\lambda, i, j})^{\lambda} = 0$ for $\lambda \neq \lambda$ and $(\eta_{\lambda, i, j})^\lambda \in \mathbb{Z}_{(p)}^{n_{(\lambda)} \times n_{\lambda}}$ has entry 1 at position $(i, j)$ and zeros elsewhere.

Let $k \in [1, p - 1]$. We obtain the idempotent

$$
\hat{e}_k := \eta_{\lambda^{k}, n_{(\lambda)}^{k}+1, n_{(\lambda)}^{k}+1} + \eta_{\lambda^{k+1}, 1, 1} \in \Lambda.
$$

We define corresponding projective right $\Lambda$-modules

$\hat{P}_k := \hat{e}_k \Lambda$ for $k \in [1, p - 1]$. 


1.2 A projective resolution of $\mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)}S_p$

**Remark 3.** Let $A$ be an $R$-algebra and let $e, e' \in A$ be two idempotents. For the right modules $eA, e'A$, we have the isomorphism of $R$-modules

$$\text{Hom}_A(eA, e'A) \xrightarrow{T_{e',e}} e'Ae$$

Thus given idempotents $eA, e'A$, we have

$$T_{e',e}(f) := f(e), \quad f \mapsto (ea \mapsto e'be) \mapsto e'be.$$

Given idempotents $e, e', e'' \in A$, and elements $f \in \text{Hom}_A(eA, e'A)$, $g \in A(e'A, e''A)$, we have $T_{e',e}(g \circ f) = g(f(e)) = g(e'f(e)) = g(e') \cdot f(e) = T_{e',e}(g) \cdot T_{e',e}(f)$.

**Definition 4.** We define via Remark 3

$$\hat{e}_k := T_{\hat{e}_k, \hat{e}_k}^{-1}(\hat{e}_k) \in \text{Hom}_A(\hat{P}_k, \hat{P}_k) \quad \text{for } k \in [1, p - 1]$$

$$\hat{e}_{p-1, p-1} := T_{\hat{e}_{p-1, \hat{e}_{p-1}}}^{-1}(\hat{p}_\lambda, 1, 1) \in \text{Hom}_A(\hat{P}_{p-1}, \hat{P}_{p-1})$$

$$\hat{e}_{k+1, k} := T_{\hat{e}_{k+1, \hat{e}_k}, \hat{e}_k}^{-1}(\hat{p}_{\lambda, k+1, k} + 1, 1) \in \text{Hom}_A(\hat{P}_{k+1}, \hat{P}_k) \quad \text{for } k \in [1, p - 2]$$

Note that $\hat{e}_k$ is the identity map on $\hat{P}_k$ for $k \in [1, p - 1]$. Moreover, we define the $\mathbb{Z}_{(p)}S_p$-linear map $\hat{e} : \hat{P}_1 \to \mathbb{Z}_{(p)}$, $\hat{e}(\rho) := \rho^{\lambda^1}$.

It is straightforward to show the following lemma.

**Lemma 5.** We have

$$\hat{e}_{1,1} + \hat{e}_{1,2} \circ \hat{e}_{2,1} = p\hat{e}_1$$

$$\hat{e}_{k,k-1} \circ \hat{e}_{k-1,k} + \hat{e}_{k,k+1} \circ \hat{e}_{k+1,k} = p\hat{e}_k \quad \text{for } k \in [2, p - 2]$$

$$\hat{e}_{p-1,p-2} \circ \hat{e}_{p-2,p-1} + \hat{e}_{p-1,p-1} = p\hat{e}_{p-1}$$

$$\hat{e} \circ \hat{e}_{1,1} = p\hat{e}.$$

Furthermore, it is straightforward to check that we obtain a projective resolution of $\mathbb{Z}_{(p)}$ as follows. We set

$$\hat{P}_i := \begin{cases} 
\hat{P}_{\omega(i)} & i \geq 0 \\
0 & i < 0 
\end{cases},$$

where the integer $\omega(i)$ is given by the following construction: Recall the stipulation $l := 2(p - 1)$. We have $i = jl + r$ for some $j \in \mathbb{Z}$ and $0 \leq r \leq l - 1$. Then

$$\omega(i) := \begin{cases} 
 r + 1 & \text{for } 0 \leq r \leq p - 2 \\
l - r = 2(p - 1) - r & \text{for } p - 1 \leq r \leq 2(p - 1) - 1 = l - 1.
\end{cases} \quad (1)$$
1 The projective resolution of $F_p$ over $F_pS_p$

So $\omega(i)$ increases by steps of one from 1 to $p - 1$ as $i$ runs from $jl$ to $jl + (p - 2)$ and $\omega(i)$ decreases from $p - 1$ to 1 as $i$ runs from $jl + (p - 1)$ to $jl + (l - 1)$. Finally we set

$$d_i := \begin{cases} 
\hat{e}_{\omega(i-1),\omega(i)} : \hat{P}_{\omega(i)} \to \hat{P}_{\omega(i-1)} & i \geq 1 \\
0 & i \leq 0
\end{cases}.$$ 

Now we have the projective resolution of $\mathbb{Z}_{(p)}$

$$\cdots \overset{d_3}{\to} \hat{P}_3 \overset{d_2}{\to} \hat{P}_2 \overset{d_1}{\to} \hat{P}_1 \overset{0 = d_0}{\to} 0 \to \cdots,$$

written more explicitly as

$$\cdots \overset{\hat{e}_{1,2}}{\to} \hat{P}_2 \overset{\hat{e}_{1,1}}{\to} \hat{P}_1 \overset{\hat{e}_{2,1}}{\to} \hat{P}_2 \to \cdots \overset{\hat{e}_{p-1,p-2}}{\to} \hat{P}_{p-2} \overset{\hat{e}_{p-1,p-1}}{\to} \hat{P}_{p-1} \to \cdots \overset{\hat{e}_{1,2}}{\to} \hat{P}_2 \overset{\hat{e}_{1,1}}{\to} \hat{P}_1 \to 0 \to \cdots,$$

with augmentation $\hat{e} : \hat{P}_1 \to \mathbb{Z}_{(p)}$.

1.3 A projective resolution of $F_p$ over $F_pS_p$

The isomorphism $r : \mathbb{Z}_{(p)}S_p \to \Lambda$ from Proposition 1 induces an isomorphism of $F_p$-algebras $F_pS_p = \mathbb{Z}_{(p)}S_p/(p\mathbb{Z}_{(p)}S_p) \xrightarrow{\tilde{r}} \Lambda/(p\Lambda) =: \tilde{\Lambda}$.

For the sake of simplicity in the next step, we identify $\tilde{\Lambda}$ and $F_pS_p$ along $\tilde{r}$.

Lemma 6. Recall that $p \geq 3$ is a prime. Applying the functor $- \otimes_{\Lambda} \tilde{\Lambda} \Lambda$, we obtain

- the projective modules $P_k := \hat{P}_k \otimes_{\Lambda} \tilde{\Lambda} \Lambda$ for $k \in [1, p - 1]$,
- $F_p := \mathbb{Z}_{(p)} \otimes_{\Lambda} \tilde{\Lambda} \Lambda$ (the trivial $F_pS_p$-module),
- $e_k := \hat{e}_k \otimes_{\Lambda} \tilde{\Lambda} \Lambda \in \text{Hom}_{F_pS_p}(P_k, P_k)$ for $k \in [1, p - 1]$,
- $e_{1,1} := \hat{e}_{1,1} \otimes_{\Lambda} \tilde{\Lambda} \Lambda \in \text{Hom}_{F_pS_p}(P_1, P_1)$,
- $e_{p-1,p-1} := \hat{e}_{p-1,p-1} \otimes_{\Lambda} \tilde{\Lambda} \Lambda \in \text{Hom}_{F_pS_p}(P_{p-1}, P_{p-1})$,
- $e_{k+1,k} := \hat{e}_{k+1,k} \otimes_{\Lambda} \tilde{\Lambda} \Lambda \in \text{Hom}_{F_pS_p}(P_{k+1}, P_k)$ for $k \in [1, p - 2]$,
- $e_{k,k+1} := \hat{e}_{k,k+1} \otimes_{\Lambda} \tilde{\Lambda} \Lambda \in \text{Hom}_{F_pS_p}(P_k, P_{k+1})$ for $k \in [1, p - 2]$,
- $\varepsilon := \hat{e} \otimes_{\Lambda} \tilde{\Lambda} \Lambda \in \text{Hom}_{F_pS_p}(P_1, F_p)$.

The complex

$$\text{PRes}_F := (\text{PRes} \mathbb{Z}_{(p)}) \otimes_{\Lambda} \tilde{\Lambda} \Lambda = (\cdots \to \text{Pr}_2 \overset{d_2}{\to} \text{Pr}_1 \overset{d_1}{\to} \text{Pr}_0 \overset{0 = d_0}{\to} 0 \to \cdots),$$

$$\text{Pr}_i := \begin{cases} 
P_{\omega(i)} & i \geq 0 \\
0 & i < 0
\end{cases} \quad d_i := \begin{cases} 
e_{\omega(i-1),\omega(i)} : P_{\omega(i)} \to P_{\omega(i-1)} & i \geq 1 \\
0 & i \leq 0
\end{cases}.$$
is a projective resolution of \( \mathbb{F}_p \) with augmentation \( \varepsilon : P_1 \to \mathbb{F}_p \). More explicitly, \( \text{PRes} \mathbb{F}_p \) is

\[
\ldots \to P_2 \xrightarrow{e_{1,2}^{l+1}} P_1 \xrightarrow{e_{1,1}^{l+1}} P_1 \xrightarrow{e_{2,1}^{l-1}} P_2 \xrightarrow{e_{p-1,p-2}^{p-2}} P_{p-2} \to \ldots \to P_3 \to 0.
\]

**Lemma 7.** Recall that \( p \geq 3 \) is a prime.

(a) We have the relations

\[
\begin{align*}
    &e_{1,1} + e_{1,2} \circ e_{2,1} = 0 \\
    &e_{k,k-1} \circ e_{k-1,k} + e_{k,k+1} \circ e_{k+1,k} = 0 \quad \text{for } k \in [2, p-2] \\
    &e_{p-1,p-2} \circ e_{p-2,p-1} + e_{p-1,p-1} = 0 \\
    &\varepsilon \circ e_{1,1} = 0
\end{align*}
\]

and \( e_k \) is the identity on \( P_k \) for \( k \in [1, p-1] \).

(b) Given \( k \in [2, p-1] \), we have \( \text{Hom}_{\mathbb{F}_p}(P_k, \mathbb{F}_p) = \{0\} \).

(c) Given \( k, k' \in [1, p-1] \) such that \( |k - k'| > 1 \), we have \( \text{Hom}_{\mathbb{F}_p}(P_k, P_{k'}) = \{0\} \).

(d) The set \( \{\varepsilon\} \) is an \( \mathbb{F}_p \)-basis of \( \text{Hom}_{\mathbb{F}_p}(P_1, \mathbb{F}_p) \).

Assertion (a) results from Lemma 5.

Assertions (b), (c) and (d) are derived from corresponding assertions over \( \mathbb{Z}_{(p)} \mathbb{S}_P \) using \( \text{Hom}_{\mathbb{F}_p}(P/pP, M/pM) \simeq \text{Hom}_{\mathbb{Z}_{(p)} \mathbb{S}_P}(P, M)/p \text{Hom}_{\mathbb{Z}_{(p)} \mathbb{S}_P}(P, M) \) for \( \mathbb{Z}_{(p)} \mathbb{S}_P \)-modules \( P \) and \( M \), where \( P \) is projective.

## 2 A\(_\infty\)-algebras

### 2.1 Definitions, General theory

In this subsection, we review results presented in [10] and we fix notation.

Let \( R \) be a commutative ring. We understand linear maps between \( R \)-modules to be \( R \)-linear. Tensor products are tensor products over \( R \). By graded \( R \)-modules we understand \( \mathbb{Z} \)-graded \( R \)-modules.

**Definition 8.** In the definition of the tensor product of graded maps, we implement the Koszul sign rule: Let \( A_1, A_2, B_1, B_2 \) be graded \( R \)-modules and \( g : A_1 \to B_1, h : A_2 \to B_2 \) graded maps. Then we set for homogeneous elements \( x \in A_1, y \in A_2 \)

\[
(g \otimes h)(x \otimes y) := (-1)^{|h||x|} g(x) \otimes h(y).
\]
Concerning the signs in the definition of $A_\infty$-algebras and $A_\infty$-morphisms, we follow the variant given e.g. in [14] and [7].

**Definition 9.** Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(i) Let $A$ be a graded $R$-module. A *pre-$A_n$-structure on $A$* is a family of graded maps $(m_k : A^\otimes k \to A)_{k \in [1,n]}$ with $|m_k| = 2 - k$ for $k \in [1,n]$. The tuple $(A,(m_k)_{k \in [1,n]})$ is called a *pre-$A_n$-algebra*.

(ii) Let $A$, $A'$ be graded $R$-modules. A *pre-$A_n$-morphism from $A'$ to $A$* is a family of graded maps $(f_k : A'^\otimes k \to A)_{k \in [1,n]}$ with $|f_k| = 1 - k$ for $k \in [1,n]$.

**Definition 10.** Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(i) An *$A_n$-algebra* is a pre-$A_n$-algebra $(A, (m_k)_{k \in [1,n]})$ such that for $k \in [1,n]$

\[
\sum_{k=r+s+t, \ r,t \geq 0, \ s \geq 1} (-1)^{rs+it} m_{r+1+t} \circ (1^\otimes r \otimes m_s \otimes 1^\otimes t) = 0.
\] (5)[k]

In abuse of notation, we sometimes abbreviate $A = (A, (m_k)_{k \geq 1})$ for $A_\infty$-algebras.

(ii) Let $(A', (m'_k)_{k \in [1,n]})$ and $(A, (m_k)_{k \in [1,n]})$ be $A_n$-algebras. An *$A_n$-morphism or morphism of $A_n$-algebras* from $(A', (m'_k)_{k \in [1,n]})$ to $(A, (m_k)_{k \in [1,n]})$ is a pre-$A_n$-morphism $(f_k)_{k \in [1,n]}$ such that for $k \in [1,n]$, we have

\[
\sum_{k=r+s+t, \ r,t \geq 0, \ s \geq 1} (-1)^{rs+it} f_{r+1+t} \circ (1^\otimes r \otimes m'_s \otimes 1^\otimes t) = \sum_{1 \leq r \leq k, \ i_1 + \ldots + i_r = k, \ i_s \geq 1} (-1)^r m_r \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r}),
\] (6)[k]

where $v := \sum_{1 \leq t < s \leq r} (1 - i_s)i_t$.

**Example 11 (dg-algebras).** Let $(A, (m_k)_{k \geq 1})$ be an $A_\infty$-algebra. If $m_n = 0$ for $n \geq 3$ then $A$ is called a *differential graded algebra* or *dg-algebra*. In this case the equations (5)[n] for $n \geq 4$ become trivial: We have $(r + 1 + t) + s = n + 1 \Rightarrow (r + 1 + t) + s \geq 5 \Rightarrow m_{r+1+t} = 0$ or $m_s = 0$. So all summands in (5)[n] are zero for $n \geq 4$. Here are the equations for $n \in \{1,2,3\}$:

\[
\begin{align*}
(5)[1] & : \quad 0 = m_1 \circ m_1 \\
(5)[2] & : \quad 0 = m_1 \circ m_2 - m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1) \\
(5)[3] & : \quad 0 = m_1 \circ m_3 + m_2 \circ (1 \otimes m_2 - m_2 \otimes 1) \\
& \quad + m_3 \circ (m_1 \otimes 1^\otimes 2 + 1 \otimes m_2 \otimes 1 + 1^\otimes 2 \otimes m_1) \\
& \quad m_3 = 0 = m_2 \circ (1 \otimes m_2 - m_2 \otimes 1)
\end{align*}
\]

So (5)[1] ensures that $m_1$ is a differential. Moreover, (5)[3] states that $m_2$ is an associative binary operation, since for homogeneous $x, y, z \in A$ we have $0 = m_2 \circ (1 \otimes m_2 - m_2 \otimes 1)(x \otimes y \otimes z) = m_2(x \otimes m_2(y \otimes z) - m_2(x \otimes y) \otimes z)$, where because of $|m_2| = 0$ there are no additional signs caused by the Koszul sign rule. Equation (5)[2] is the Leibniz rule.
We define \( m \). Then \( n \).

Thus \( f_1 : (A', m'_1) \rightarrow (A, m_1) \) is a complex morphism.

By (5)[1], \( (A', m'_1) \) and \( (A, m_1) \) are complexes. Equation (6)[1] is

\[
f_1 \circ m'_1 = m_1 \circ f_1.
\]

Thus \( f_1 : (A', m'_1) \rightarrow (A, m_1) \) is a complex morphism.

Recall the conventions concerning \( \text{Hom}^k_B(C, C') \).

**Lemma 13** (cf. e.g. [10, Section 3.3]). Let \( B \) be an (ordinary) \( R \)-algebra and \( M = ((M_i)_{i \in \mathbb{Z}}, (d_i)_{i \in \mathbb{Z}}) \) a complex of \( B \)-modules, that is a sequence \( (M_i)_{i \in \mathbb{Z}} \) of \( B \)-modules and \( B \)-linear maps \( d_i : M_i \rightarrow M_{i-1} \) such that \( d_{i-1} \circ d_i = 0 \) for all \( i \in \mathbb{Z} \). Let

\[
\text{Hom}^i_B(M, M) := \prod_{z \in \mathbb{Z}} \text{Hom}_B(M_{z+i}, M_z)
\]

\[
= \{ g = (g_z)_{z \in \mathbb{Z}} \mid g_z \in \text{Hom}_B(M_{z+i}, M_z) \text{ for } z \in \mathbb{Z} \}.
\]

Then

\[
A = \text{Hom}^*_B(M, M) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}^i_B(M, M)
\]

is a graded \( R \)-module. We have \( d := (d_{z+1})_{z \in \mathbb{Z}} = \sum_{i \in \mathbb{Z}} [d_{i+1}]^i_{i+1} \in \text{Hom}^1_B(M, M) \). We define \( m_1 := d_{\text{Hom}^*(M, M)} : A \rightarrow A \), that is for homogeneous \( g \in A \) we have

\[
m_1(g) = d \circ g - (-1)^{|g|} g \circ d.
\]

We define \( m_2 : A \otimes A \rightarrow A \) for homogeneous \( g, h \in A \) to be composition, i.e.

\[
m_2(g \otimes h) := g \circ h.
\]

For \( n \geq 3 \) we set \( m_n : A \otimes A \rightarrow A, m_n = 0 \). Then \( (m_n)_{n \geq 1} \) is an \( A_\infty \)-algebra structure on \( A = \text{Hom}^*_B(M^*, M^*) \). More precisely, \( (A, (m_n)_{n \geq 1}) \) is a dg-algebra.

**Remark 14.** In \( \text{Hom}^*(\text{PRes} \mathbb{F}_p, \text{PRes} \mathbb{F}_p) \) we have (cf. (3))

\[
d = \sum_{i \geq 0} [e_{\omega(i), \omega(i+1)}]_{i+1}^i.
\]
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**Definition 15** (Homology of $A_\infty$-algebras, quasi-isomorphisms, minimality, minimal models). As $m_1^2 = 0$ (cf. (5)[1]) and $|m_1| = 1$, we have the complex

$$\cdots \to A^{i-1} \xrightarrow{m_1|_{A^{i-1}}} A^i \xrightarrow{m_1|_{A^i}} A^{i+1} \to \cdots$$

We define $H^kA := \ker(m_1|_{A^k})/\im(m_1|_{A^{k-1}})$ and $H^*A := \bigoplus_{k \in \mathbb{Z}} H^kA$, which gives the homology of $A$ the structure of a graded $R$-module.

A morphism of $A_\infty$-algebras $(f_k)_{k \geq 1} : (A', (m'_k)_{k \geq 1}) \to (A, (m_k)_{k \geq 1})$ is called a quasi-isomorphism if the morphism of complexes $f_1 : (A', m'_1) \to (A, m_1)$ (cf. Example 12) is a quasi-isomorphism.

An $A_\infty$-algebra is called minimal, if $m_1 = 0$. If $A$ is an $A_\infty$-algebra and $A'$ is a minimal $A_\infty$-algebra quasi-isomorphic to $A$, then $A'$ is called a minimal model of $A$.

The existence of minimal models is assured by the following theorem.

**Theorem 16.** (minimality theorem, cf. [11] (history), [8], [7], [18], [4], [6], [16], . . . ) Let $(A, (m_k)_{k \geq 1})$ be an $A_\infty$-algebra such that the homology $H^*A$ is a projective $R$-module. Then there exists an $A_\infty$-algebra structure $(m'_k)_{k \geq 1}$ on $H^*A$ and a quasi-isomorphism of $A_\infty$-algebras $(f_k)_{k \geq 1} : (H^*A, (m'_k)_{k \geq 1}) \to (A, (m_k)_{k \geq 1})$, such that

- $m'_1 = 0$ and
- the complex morphism $f_1 : (H^*A, m'_1) \to (A, m_1)$ induces the identity in homology. I.e. each element $x \in H^*A$, which is a homology class of $(A, m_1)$, is mapped by $f_1$ to a representing cycle.

For constructing $A_\infty$-structures induced by another $A_\infty$-algebra, we have the following

**Lemma 17** (cf. [7, Proof of Theorem 1]). Let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Let $(A', (m'_k)_{k \in [1,n]})$ be a pre-$A_n$-algebra. Let $(A, (m_k)_{k \in [1,n]})$ be an $A_n$-algebra. Let $(f_k)_{k \in [1,n]}$ be a pre-$A_n$-morphism from $A'$ to $A$ such that (6)[$k$] holds for $k \in [1,n]$. Suppose $f_1$ to be injective. Then $(A', (m'_k)_{k \in [1,n]})$ is an $A_n$-algebra and $(f_k)_{k \in [1,n]}$ is a morphism of $A_n$-algebras from $(A', (m'_k)_{k \in [1,n]})$ to $(A, (m_k)_{k \in [1,n]})$.

This results from the bar construction and a straightforward induction on $n$.

**Lemma 18** ([23, Theorem 5]). Let $R$ be a commutative ring and $(A, (m_n)_{n \geq 1})$ be a dg-algebra (over $R$). Suppose given a graded $R$-module $B$ and graded maps $f_n : B^\otimes n \to A$, $m'_n : B^\otimes n \to B$ for $n \geq 1$. Suppose given $k \geq 1$ such that we have $f_i = 0$ for $i \geq k$, we have $m'_i = 0$ for $i \geq k + 1$, and (6)[$n$] is satisfied for $n \in [1, 2k - 2]$. Then (6)[$n$] is satisfied for all $n \geq 1$.

2.2 The homology of $\text{Hom}_{\mathbb{F}_p^*}^n(\text{PRes} \mathbb{F}_p, \text{PRes} \mathbb{F}_p)$

We need a well-known result of homological algebra in a particular formulation:
Lemma 19. Let $F$ be a field. Let $B$ be an $F$-algebra. Let $M$ be a $B$-module. Let $Q = (\cdots \to Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \to 0 \to \cdots)$ be a projective resolution of $M$ with augmentation $\varepsilon : Q_0 \to M$. Then we have maps for $k \in \mathbb{Z}$

$$\Psi_k : \operatorname{Hom}^k_B(Q, Q) \to \operatorname{Hom}^k_B(Q, M) := \operatorname{Hom}_B(Q_k, M)$$

$$(g_i : Q_{i+k} \to Q_i)_{i \in \mathbb{Z}} \mapsto \varepsilon \circ g_0$$

The right side is equipped with the differentials (dualization of $d_k$)

$$(d_k)^* : \operatorname{Hom}_B(Q_k, M) \to \operatorname{Hom}_B(Q_{k+1}, M)$$

$$(g) \mapsto (-1)^k g \circ d_k$$

and the left side is equipped with the differential $m_1$ of its dg-algebra structure, cf. Lemma 13.

Then $(\Psi_k)_{k \in \mathbb{Z}}$ becomes a complex morphism from the complex $\operatorname{Hom}^*_B(Q, Q)$ to the complex $\operatorname{Hom}^*_B(Q, M)$ that induces isomorphisms $\Psi_k$ of $F$-vector spaces on the homology

$$\Psi_k : H^k \operatorname{Hom}^*_B(Q, Q) \xrightarrow{\cong} H^k \operatorname{Hom}^*_B(Q, M)$$

$$(g_i : Q_{i+k} \to Q_i)_{i \in \mathbb{Z}} \mapsto \varepsilon \circ g_0$$

Lemma 19 is [2, §5 Proposition 4a)] applied to the quasi-isomorphism induced by the augmentation, cf. [2, §3 Définition 1].

Recall the notation $[x]_l^y$ for the description of elements of $\operatorname{Hom}^l_B(C, C')$.

Proposition 20. Recall that $p \geq 3$ is a prime and $l = 2(p - 1)$.

Write $A := \operatorname{Hom}^*_B(P_{\text{Res}} F, P_{\text{Res}} F)$. Let

$$\iota := \sum_{i \geq 0} [e_{\omega(i)}]_{i+1}^i = \sum_{i \geq 0} \sum_{k=0}^{i-1} [e_{\omega(k)}]_{i+1-l+k}^i \in A^l$$

$$\chi := \sum_{i \geq 0} \left( [e_1]_{d+i+1}^{d+i} + \left( \sum_{k=1}^{p-2} [e_{k+1,k}]_{d+i+1-k}^{d+i+1+k} \right) + [e_{p-1}]_{d+i+1-(p-1)}^{d+i+1} \right) \in A^{l+1}$$.  

(a) For $j \geq 0$, we have $\iota^j = \sum_{i \geq 0} [e_{\omega(i)}]_{i+j}^{i+j} = \sum_{i \geq 0} \sum_{k=0}^{i-1} [e_{\omega(k)}]_{i+j}^{i+j+k}$.  

(b) Suppose given $y \geq 0$. Let $h \in A^y$ be $l$-periodic, that is $h = \sum_{i \geq 0} \sum_{k=0}^{i-1} [h_k]_{i+j}^{i+j+k}$. Then for $j \geq 0$, we have

$$h \circ \iota^j = \iota^j \circ h = \sum_{i \geq 0} \sum_{k=0}^{i-1} [h_k]_{i+j}^{i+j+k+y} \in A^{y+j}$.  

(c) Suppose given $y \in \mathbb{Z}$. For $h \in A^y$ and $j \geq 0$, we have $m_1(h \circ \iota^j) = m_1(h) \circ \iota^j$.  

(d) For $j \geq 0$, we have $m_1(\iota^j) = 0$. Thus $\iota^j$ is a cycle.
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(e) For $j \geq 0$, we have

$$
\chi^{i,j} := \chi \circ \iota^j = \iota^j \circ \chi \\
= \sum_{i \geq 0} \left( [\varepsilon_1]^i_{(i+j+1)i-1} + \left( \sum_{k=1}^{p-2} [\varepsilon_{k+1,k}]^i_{(i+j+1)i-1+k} + [\varepsilon_{p-1,j}]^i_{(i+j+1)i-1+(p-1)} + \left( \sum_{k=1}^{p-2} [\varepsilon_{p-1-k,1-p-k}]^i_{(i+j+1)i-1+(p-1)+k} \right) \right) \right) \in A^{i+l-1}.
$$

Thus the proof by induction is complete.

For convenience, we also define $\chi^0 \iota^j := \iota^j$ and $\chi^1 \iota^j := \chi \circ \iota^j$ for $j \geq 0$.

(f) For $j \geq 0$, we have $m_1(\chi^{i,j}) = 0$. Thus $\chi^{i,j}$ is a cycle.

(g) Suppose given $k \in \mathbb{Z}$. A $\mathbb{F}_p$-basis of $H^k A$ is given by

$$
\{ \overline{\iota^j} \} if k = jl for some j \geq 0 \\
\{ \overline{\chi \iota^j} \} if k = jl + l - 1 for some j \geq 0 \\
\emptyset else.
$$

Thus the set $\mathcal{B} := \{ \overline{\iota^j} \mid j \geq 0 \} \cup \{ \overline{\chi \iota^j} \mid j \geq 0 \}$ is an $\mathbb{F}_p$-basis of $H^* A = \bigoplus_{z \in \mathbb{Z}} H^z A$.

Proof. The element $\iota$ is well-defined since $\omega(y) = \omega(l + y)$ for $y \geq 0$.

In the definition of $\chi$ we need to check that the "$[\ast]^*_{\ast}$" are well defined. This is easily proven by calculating the $\omega(y)$ where $y$ is the lower respective upper index of "$[\ast]^*_{\ast}$".

(a): As $\text{Pr}_i = \{0\}$ for $i < 0$, the identity element of $A$ is given by $\iota^0 = \sum_{i \geq 0} [\varepsilon_{\omega(i)}]^i_i$, which agrees with the assertion in case $j = 0$. So we have proven the induction basis for induction on $j$. So now assume that for some $j \geq 0$ the assertion holds. Then

$$
\iota^{j+1} = \iota \circ \iota^j = \left( \sum_{i \geq 0} [\varepsilon_{\omega(i)}]^i_{i+l} \right) \circ \left( \sum_{i' \geq 0} [\varepsilon_{\omega(i')}]^i_{i'+j+l} \right) = \sum_{i \geq 0} [\varepsilon_{\omega(i)} \circ \varepsilon_{\omega(i+l)}]^i_{i+l+j+l} = \sum_{i \geq 0} [\varepsilon_{\omega(i)}]^i_{i+(j+1)l}.
$$

Thus the proof by induction is complete.

(b): We have

$$
\iota^j \circ h = \left( \sum_{i \geq 0} \sum_{k=0}^{l-1} [\varepsilon_{\omega(i+k)}]^i_{i+l+k} \right) \circ \left( \sum_{i' \geq 0} \sum_{k'=0}^{l-1} [h_{i+k'}]^i_{i'+l+k'} \right) = \sum_{i \geq 0} \sum_{k=0}^{l-1} [h_{i+k}]^i_{i+l+k+k+y} \\
h \circ \iota^j = \left( \sum_{i \geq 0} \sum_{k=0}^{l-1} [h_{i+k}]^i_{i+l+k+y} \right) \circ \left( \sum_{i' \geq 0} [\varepsilon_{\omega(i')}]^i_{i'+l+j} \right) = \sum_{i \geq 0} \sum_{k=0}^{l-1} [h_{i+j+k}]^i_{i+l+k+y}.
$$

So we have proven (b).

(c): The differential $d$ of $\text{PRes}_p$ is $l$-periodic (cf. Remark 14) and thus

$$
m_1(h) \circ \iota^j = (d \circ h - (-1)^y h \circ d) \circ \iota^j
$$
(b), $|\iota|=0 \Rightarrow d \circ h \circ \iota - (-1)^{|\iota|} h \circ \iota \circ d = m_1(h \circ \iota).

(d): We have $m_1(\iota^j) = m_1(\iota^0) \circ \iota^j = (d \circ \iota^0 - (-1)^0 \iota^0 d) \circ \iota^j = (d - d) \circ \iota^j = 0.$

(e) is implied by (b) using the fact that $\chi$ is $l$-periodic.

(f): Because of (c) we have $m_1(\chi \iota^j) = m_1(\chi) \circ \iota^j$. Because $|\chi| = l - 1$ is odd we have

$$m_1(\chi) = d \circ \chi = -\chi \circ d = \chi \circ d + d \circ \chi.$$

In the step marked by "*" we sort the summands by their targets. Note that when splitting sums of the form $\sum_{k=1}^{p-2} [\epsilon_i, \iota]_{\iota^i} \in \mathbb{F}$ into $(...)_1 + \sum_{k=2}^{p-2} (...)_k$ or into $(...)_{p-2} + \sum_{k=1}^{p-3} (...)_k$, the existence of the summand that is split off is ensured by $p \geq 3$.

(g): We first show that the differentials of the complex Hom$^*(\text{PRes}_p, \mathbb{F}_p)$ (cf. Lemma 19) are all zero: By Lemma 7, $\{\iota\}$ is an $\mathbb{F}_p$-basis of Hom$^*_{\text{PRes}_p}(P_1, \mathbb{F}_p)$, and for $k \in [2, p-1]$ we have Hom$^*_{\text{PRes}_p}(P_k, \mathbb{F}_p) = 0$. So the only non-trivial $(d_k)^*$ are those where $\text{Pr}_k = \text{Pr}_{k+1} = P_1$. This is the case only when $k = lj + l - 1$ for some $j \geq 0$. Then $d_k = e_{1,1}$. For $\iota \in \text{Hom}(P_1, \mathbb{F}_p)$, we have $(d_k)^*(\iota) = (-1)^k \iota \circ e_{1,1} \overset{L7(a)}{=} 0$. As Hom$(P_1, \mathbb{F}_p) = \langle \iota \rangle_{\mathbb{F}_p}$, we have $(d_k)^* = 0.$

So $H^* \text{Hom}^*_{\text{PRes}_p, \mathbb{F}_p} = \text{Hom}^*_{\text{PRes}_p, \mathbb{F}_p}$. We use Lemma 19.
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For \( k = jl, j \geq 0 \), we have \( \tilde{\Phi}^k(\langle j \rangle) \sim^{(a)} \varepsilon \), and \( \{ \varepsilon \} \) is a basis of \( H^k \operatorname{Hom}^*(\operatorname{Pres} F_p, F_p) \).

For \( k = jl + l - 1, j \geq 0 \), we have \( \tilde{\Phi}^k(\langle j \rangle) \sim^{(a)} \varepsilon \), and \( \{ \varepsilon \} \) is a basis of \( H^k \operatorname{Hom}^*(\operatorname{Pres} F_p, F_p) \).

Finally, for \( k = jl + r \) for some \( j \geq 0 \) and some \( r \in [1, l - 2] \) and for \( k < 0 \), we have \( H^k \operatorname{Hom}^*(\operatorname{Pres} F_p, F_p) = \{ 0 \} \).

\[ \square \]

### 2.3 An \( A_\infty \)-structure on \( \operatorname{Ext}_{F_p,S_p}^*(F_p, F_p) \) as a minimal model of \( \operatorname{Hom}_{F_p,S_p}^*(\operatorname{Pres} F_p, \operatorname{Pres} F_p) \)

Recall that \( p \geq 3 \) is a prime. Write \( A := \operatorname{Hom}_{F_p,S_p}^*(\operatorname{Pres} F_p, \operatorname{Pres} F_p) \), which becomes an \( A_\infty \)-algebra \( (A, (m_n)_{n \geq 1}) \) over \( R = F_p \) via Lemma 13. We implement \( \operatorname{Ext}_{F_p,S_p}^*(F_p, F_p) \) as \( \operatorname{Ext}_{F_p,S_p}^*(F_p, F_p) := \operatorname{H}^*A \).

Our goal in this section is to construct an \( A_\infty \)-structure \( (m'_n)_{n \geq 1} \) on \( \operatorname{H}^*A \) and a morphism of \( A_\infty \)-algebras \( f = (f_n)_{n \geq 1} : (\operatorname{H}^*A, (m'_n)_{n \geq 1}) \rightarrow (A, (m_n)_{n \geq 1}) \) which satisfy the statements of Theorem 16. I.e. we will construct a minimal model of \( A \). In preparation of the definitions of the \( f_n \) and \( m'_n \), we name and examine certain elements of \( A \):

**Lemma 21.** Suppose given \( k \in [2, p - 1] \). We set

\[ \gamma_k := \sum_{i \geq 0} \left( [e_k^1]_{k(l - 1) + li} + [e_{p-k}^1]_{k(l - 1) + (p-1) + li} \right) \in A^{k(l-2)+1} \]

For \( j \geq 0 \), we have

\[ \gamma_k \cdot t^j := \gamma_k \cdot t^j = t^j \cdot \gamma_k = \sum_{i \geq 0} \left( [e_k^1]_{k(l - 1) + (i + j)} + [e_{p-k}^1]_{k(l - 1) + (p-1) + (i + j)} \right) \in A^{k(l-2)+1+jl} \]

and

\[ m_1(\gamma_k t^j) = \sum_{i \geq 0} \left( [e_{k-1,k}^1]_{k(l - 1) + li} + [e_{p-k+1,k-1}^1]_{k(l - 1) + (p-1) + li} + [e_{k,k-1}^1]_{k(l - 1) + (p-1) + li} + [e_{p-k,p-k}^1]_{k(l - 1) + (p-1) + li} \right). \]

**Proof.** We need to prove that \( \gamma_k \) is well-defined. Let \( i \geq 0 \).

We consider the first term. The complex \( \operatorname{Pres} F_p \) (cf. (3), (1)) has entry \( P_k \) at position \( k(l - 1) + li \) and at position \( k - 1 + li \). We have \( k(l - 1) + li = (k - 1 + i)l + l - k \). Since \( (k - 1 + li) \in \{ [k-1]+1 \} \) since \( 0 \leq k - 1 \leq p - 2 \). \( k(l - 1) + li, k - 1 + li \geq 0 \), we have \( \operatorname{Pr}_{k(l - 1) + li} = P_{\omega(k(l - 1) + li)} = P_k \) and \( \operatorname{Pr}_{k-1 + li} = P_{\omega(k-1 + li)} = P_k \). So the first term is well-defined.

Now consider the second term. The complex \( \operatorname{Pres} F_p \) has entry \( P_{p-k} \) at position \( k(l - 1) + (p-1) + li \) and \( p-k \). We have \( k(l - 1) + (p-1) + li = (i + k)l + (p-1) - k, \) so \( \omega(k(l - 1) + (p-1) + li) = (p-1) - k + 1 = p - k \). Since \( 0 \leq (p-1) - k \leq p - 2 \). We have \( \omega(k-1+(p-1)+li) = 2(p-1)-(k-1)-(p-1) = p-k \) since \( p-1 \leq k-1+(p-1) \leq 2(p-1) - 1 \). As \( k(l - 1) + (p-1) + li, k-1+(p-1) + li \geq 0 \), we have
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$Pr_{k(l-1)+(p-1)+li} = P_{\omega(k(l-1)+(p-1)+li)} = P_{p-k}$ and $Pr_{k-1+(p-1)+li} = P_{\omega(k-1+(p-1)+li)} = P_{p-k}$. So the second term is well-defined.

The degree of the tuple of maps is computed to be $(k(l-1) + li) - (k - 1 + li) = k(l-2) + 1 = (k(l-1) + (p-1) + li) - (k - 1 + (p-1) + li)$.

The explicit formula for $\gamma_{k,l^j}$ is an application of Proposition 20(b).

The degree $|\gamma_{k,l^j}| = k(l-2) + 1$ is odd, so

$$m_1(\gamma_{k,l^j}) \overset{R.14}{=} d \circ \gamma_{k,l^j} + \gamma_{k,l^j} \circ d$$

Note that in the second line $k - 2 + li \geq 0$ as $i \geq 0$ and $k \geq 2$.

**Lemma 22.** For $j, j' \geq 0$, we have $\chi_{l^j} \circ \chi_{l^{j'}} = m_1(\gamma_{2,l^{j+j'}})$.

**Proof.** It suffices to prove that $\chi \circ \chi = m_1(\gamma_2)$ since then $\chi_{l^j} \circ \chi_{l^{j'}} \overset{P.20(e)}{=} \chi \circ \chi \circ l^{j+j'} = m_1(\gamma_{2,l^{j+j'}}) \overset{P.20(e)}{=} m_1(\gamma_{2,l^{j+j'}})$.

To determine when a composite is zero, we will need the following. For $0 \leq k, k' < l$, we examine the condition

$$il + l - 1 + k = i'l + k'. \quad (8)$$

If $k = 0$ then $(8)$ holds iff $i = i'$ and $k' = l - 1$.

If $k \geq 1$ then $(8)$ holds iff $i + 1 = i'$ and $k' = k - 1$.

So

$$\chi \circ \chi = \left( \sum_{i \geq 0} \left[ e_{1,i}^d l^d + l - 1 + [e_{2,1}^d + 1 l^d + l + \left( \sum_{k=2}^{p-2} [e_{k+1,k}^d l^d + l - 1 + k] \right) + \left( e_{p-1,1}^d l^{d+p-1} + [e_{p-2,0}^d l^{d+p-1} + \left( \sum_{k=2}^{p-2} [e_{p-k-1,p-k}^d l^{d+p-1} + k] \right) \right) \right] l^{i+l-1} + \left( \sum_{k'=1}^{p-3} [e_{p-k',1,k'}^d l^{i+l-1} + k'] + [e_{p-1,2}^d l^{i+l-2} + \left( \sum_{k'=1}^{p-3} [e_{p-k'-1,p-k'}^d l^{i+l-1} + k'] \right) \right) \right) \right)$$

$$\circ \left( \sum_{i' \geq 0} \left[ e_{1,i'}^d l^{d+i} + l - 1 + \left( \sum_{k'=1}^{p-3} [e_{k'+1,k'}^d l^{i'+l-1} + k'] + [e_{p-1,2}^d l^{i'+l-2} + \left( \sum_{k'=1}^{p-3} [e_{p-k'-1,p-k'}^d l^{i'+l-1} + k'] \right) \right] l^{i+l-1} + \left( \sum_{k'=1}^{p-3} [e_{p-k'-1,p-k'}^d l^{i'+l-1} + k'] + [e_{1,2}^d l^{i'+l-2} \left( \sum_{k'=1}^{p-3} [e_{p-k'-1,p-k'}^d l^{i'+l-1} + k'] \right) \right) \right) \right) $$
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\[
\sum_{i \geq 0} \left( [e_1 \circ e_{1,2}]_{il+p}^{il} + [e_{2,1} \circ e_1]_{il+p}^{il+1} + \sum_{k=2}^{p-2} [e_{k+1,k} \circ e_{k,k-1}]_{il+2l-1}^{il+k} \right) + [e_p-1 \circ e_{p-1,p-2}]_{il+2l-1}^{il+p-3} \\
= \sum_{i \geq 0} \left( [e_1 \circ e_{1,2}]_{il+p-2}^{il} + [e_{2,1} \circ e_1]_{il+p-1}^{il+1} + [e_{p-1,p-2} \circ e_{(i+2)l+p-3}]_{il+2l-1}^{il+p} \right)
\]

\[\text{by L.21} = m_1(\gamma_2)\]

Below are the definitions which will give a minimal \(A_\infty\)-algebra structure on \(H^*A\) and a quasi-isomorphism of \(A_\infty\)-algebras \(H^*A \to A\).

**Definition 23.** Recall from Proposition 20 that \(\mathfrak{B} = \{\chi^j | j \geq 0\} \cup \{\chi^j | j \geq 0\} = \{\chi^{a_lj} | j \geq 0, a \in \{0, 1\}\}\) is a basis of \(H^*A\). For \(n \in \mathbb{Z}_{\geq 1}\), we set

\(\mathfrak{B}^{\otimes n} := \{\chi^{a_1j_1} \otimes \ldots \otimes \chi^{a_nj_n} \in (H^*A)^{\otimes n} | a_i \in \{0, 1\} \text{ and } j_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in [1, n]\}\),

which is a basis of \((H^*A)^{\otimes n}\) consisting of homogeneous elements.

For \(n \geq 1\), we define the \(\mathbb{F}_p\)-linear map \(f_n : (H^*A)^{\otimes n} \to A\) as follows:

**Case** \(n = 1\): \(f_1\) is given on \(\mathfrak{B}\) by \(f_1(\chi^j) := \iota^j\) and \(f_1(\chi^{a_lj}) := \chi^{a_lj}\).

**Case** \(n \in [2, p-1]\): \(f_n\) is given on elements of \(\mathfrak{B}^{\otimes n}\) by

\[
f_n(\chi^{a_1j_1} \otimes \ldots \otimes \chi^{a_nj_n}) := \begin{cases} 0 & \text{if } \exists i \in [1, n] : a_i = 0 \\ (-1)^{n-1} \gamma_{n^{j_1+\ldots+j_n}} & \text{if } 1 = a_1 = a_2 = \ldots = a_n \end{cases}
\]

**Case** \(n \geq p\): We set \(f_n := 0\).

For \(n \geq 1\), we define the \(\mathbb{F}_p\)-linear map \(m'_n : (H^*A)^{\otimes n} \to H^*A\) by defining it on elements \(\chi^{a_1j_1} \otimes \ldots \otimes \chi^{a_nj_n} \in \mathfrak{B}^{\otimes n}\):

**Case** \(\exists i \in [1, n] : a_i = 0\):

\[
m'_n(\chi^{a_1j_1} \otimes \ldots \otimes \chi^{a_nj_n}) := 0 \text{ for } n \not= 2 \text{ and } m'_2(\chi^{a_1j_1} \otimes \chi^{a_2j_2}) := \chi^{a_1+a_2j_1+j_2} \text{ (Note that } a_1 + a_2 \in \{0, 1\}).
\]

**Case** \(a_1 = a_2 = \ldots = a_n = 1\):

\[
m'_n(\chi^{j_1} \otimes \ldots \otimes \chi^{j_n}) := 0 \text{ for } n \not= p \text{ and } m'_p(\chi^{j_1} \otimes \ldots \otimes \chi^{j_p}) := (-1)^{p-1+j_1+\ldots+j_p} = -\chi^{p-1+j_1+\ldots+j_p}.
\]

Note that since \(p \geq 3\), we have \(m'_2(\chi^{j_1} \otimes \chi^{j_2}) = 0\) for \(j_1, j_2 \geq 0\).
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Theorem 24. The pair $(H^*A, (m'_n)_{n \geq 1})$ is a minimal $A_\infty$-algebra. The tuple $(f_n)_{n \geq 1}$ is an quasi-isomorphism of $A_\infty$-algebras from $(H^*A, (m'_n)_{n \geq 1})$ to $(A, (m_n)_{n \geq 1})$. More precisely, $f_1 : (H^*A, m'_1) \to (A, m_1)$ induces the identity in homology.

The proof of Theorem 24 will take the remainder of section 2.3. We will use Lemma 17.

Lemma 25. The maps $f_n$ and $m'_n$ have degree $|f_n| = 1 - n$ and $|m'_n| = 2 - n$. I.e. $(f_n)_{n \geq 1}$ is a pre-$A_\infty$-morphism from $H^*A$ to $A$, and $(H^*A, (m'_n)_{n \geq 1})$ is a pre-$A_\infty$-algebra.

Proof. We have $|f_1| = 0$ as $|\chi^2| = |\chi^4|$ and $|\chi^4| = |\chi^4|$. For $n \geq p$ the map $f_n$ is of degree $1 - n$ as $f_n = 0$. For $n \in [2, p-1]$ the statement $|f_n| = 1 - n$ is proven by checking the degrees for the elements of the basis $B \otimes n$ whose image under $f_n$ is non-zero:

$$|f_n(\chi^{2j_1} \otimes \cdots \otimes \chi^{2j_n})| = |(-1)^{n-1} \gamma_n^{2j_1+\cdots+j_n}| = 1 - n + \sum_{x=1}^{n} |\chi^{2j_x}| = 1 - n + |\chi^{2j_1} \otimes \cdots \otimes \chi^{2j_n}|$$

Thus $|f_n| = 1 - n$ for all $n$ and we have proven the first statement.

Now we show $|m'_n| = 2 - n$. As before, we only need check the degrees for basis elements whose image is non-zero: For $\chi^{a_1 \cdot 2j_1} \otimes \chi^{2a_2 \cdot 2j_2}$, $j_1, j_2 \geq 0$, $a_1, a_2 \in \{0, 1\}$, $0 \in \{a_1, a_2\}$, we have

$$|m'_2(\chi^{a_1 \cdot 2j_1} \otimes \chi^{2a_2 \cdot 2j_2})| = |\chi^{a_1 + 2a_2 \cdot 2j_1 + 2j_2}| = a_1(l - 1) + a_2(l - 1) + j_2l = |\chi^{a_1 \cdot 2j_1} \otimes \chi^{2a_2 \cdot 2j_2}| + (2 - 2).$$

For $\chi^{j_1} \otimes \cdots \otimes \chi^{j_p}$, $j_x \geq 0$ for $x \in [1, p]$, we have

$$|m'_p(\chi^{j_1} \otimes \cdots \otimes \chi^{j_p})| = |p^{-1} + j_1 + \cdots + j_p| = l(p - 1 + j_1 + \cdots + j_p)$$

$$= lp - l + l(j_1 + \cdots + j_p) = lp - 2p + l(j_1 + \cdots + j_p)$$

$$= p(l - 1) + l(j_1 + \cdots + j_p) + 2 - p = |\chi^{j_1} \otimes \cdots \otimes \chi^{j_p}| + 2 - p$$

Lemma 26. We have $m'_1 = 0$. The equation (6)$[1]$ holds. The complex morphism $f_1 : (A', m'_1) \to (A, m_1)$ is a quasi-isomorphism inducing the identity in homology.

Proof. The equality $m'_1 = 0$ follows immediately from the definition. Thus $m_1 \circ f_1 = 0 = f_1 \circ m'_1$. Moreover $f_1$ is a quasi-isomorphism inducing the identity in homology by construction, cf. Proposition 20(g).

Lemma 27. The map $f_1$ is injective.

Proof. The set $X := \{x^a \mid a \in \{0, 1\}, j \in \mathbb{Z}_{\geq 1}\} \subseteq A$ is linearly independent, since it consists of nonzero elements of different summands of the direct sum $A = \bigoplus_{k \in \mathbb{Z}} \text{Hom}^k(\text{PRes}\mathbb{F}_p, \text{PRes}\mathbb{F}_p)$. The set $\mathcal{B}$, which is a basis of $H^*A$, is mapped bijectively to $X$ by $f_1$, so $f_1$ is injective.
Lemma 28. The equation \((6)[2]\) holds.

Proof. As \(m'_1 = 0\), equation \((6)[2]\) is equivalent to (cf. \((7))

\[ f_1 \circ m'_2 = m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1). \]

We check this equation on \(\mathfrak{B}^{\otimes 2}\): Recall Proposition 20 and Definition 23.

\[
\begin{align*}
 f_1m'_2(i^j \otimes i^{j'}) &= i^{j+j'} = m_2(f_1 \otimes f_1)(i^j \otimes i^{j'}) = (m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1))(i^j \otimes i^{j'}) \\
 f_1m'_2(i^j \otimes \chi i^{j'}) &= \chi i^{j+j'} = m_2(f_1 \otimes f_1)(i^j \otimes \chi i^{j'}) \\
 &= (m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1))(\chi i^j \otimes i^{j'}) \\
 f_1m'_2(\chi i^j \otimes i^{j'}) &= \chi i^{j+j'} = m_2(f_1 \otimes f_1)(\chi i^j \otimes i^{j'}) \\
 &= (m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1))(\chi i^j \otimes \chi i^{j'}) \\
 f_1m'_2(\chi i^j \otimes \chi i^{j'}) &= 0 \leq 22 \quad -m_1(\gamma_{2t} i^{j+j'}) + m_2(f_1 \otimes f_1)(\chi i^j \otimes \chi i^{j'}) \\
 &= (m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1))(\chi i^j \otimes \chi i^{j'})
\end{align*}
\]

Note that there are no additional signs due to the Koszul sign rule since \(|f_1| = 0\). \(\square\)

The following results directly from Definition 23.

Corollary 29. For \(n \geq 2\) and \(a_1, \ldots, a_n \in \{0, 1\}\), \(j_1, \ldots, j_n \geq 0\), we have

\[ f_n(\chi^{a_1j_1} \otimes \ldots \otimes \chi^{a_nj_n}) = f_n(\chi^{\alpha_1} \otimes \ldots \otimes \chi^{\alpha_n}) \circ \iota^{j_1+\ldots+j_n}. \]

If there is additionally an \(x \in [1, n]\) with \(a_x = 0\) then

\[ f_n(\chi^{a_1j_1} \otimes \ldots \otimes \chi^{a_nj_n}) = 0. \]

Equation \((6)[n]\) can be reformulated as

\[
\begin{align*}
 f_1 \circ m'_n + \sum_{n=r+s+t, \ r,i \geq 0, s \geq 1, \ s \leq n-1} (-1)^{rs+t} f_{r+1+t} \circ (1^{\otimes r} \otimes m'_s \otimes 1^{\otimes t}) \\
 &= m_1 \circ f_n + \sum_{2 \leq r \leq n, \ i_1 + \ldots + i_r = n, \ i_\geq 1} (-1)^v m_r \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_\geq 1}) \\
 &=: \Phi_n
\end{align*}
\]

where \(v = \sum_{1 \leq t < s \leq r} (1 - i_s)i_t.\)
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A term of the form $f_{r+t} \circ (1^{\otimes r} \otimes m'_t \otimes 1^{\otimes t})$, $s \geq 3$, $r + t \geq 1$, is zero because of Corollary 29 and the definition of $m'_t$. Also recall $m'_t = 0$. Thus

$$\Phi_n = \sum_{n=r+2+t}^{n-2} (-1)^{2r+t} f_{n-1} \circ (1^{\otimes r} \otimes m'_t \otimes 1^{\otimes t}) = \sum_{r=0}^{n-2} (-1)^{n-r} f_{n-1} \circ (1^{\otimes r} \otimes m'_t \otimes 1^{\otimes n-r-2}).$$

(9)

Because of $m_k = 0$ for $k \geq 3$, we have

$$\Xi_n = \sum_{i_1 + i_2 = n} (-1)^{(1-i_2)i_1} m_2 \circ (f_{i_1} \otimes f_{i_2}) = \sum_{i=1}^{n-1} (-1)^{n_i} m_2 \circ (f_i \otimes f_{n-i}).$$

(10)

We have proven:

**Lemma 30.** For $n \geq 1$, condition (6)[n] is equivalent to $f_1 \circ m'_t + \Phi_n = m_1 \circ f_n + \Xi_n$ where $\Phi_n$ and $\Xi_n$ are as in (9) and (10).

**Lemma 31.** Condition (6)[n] holds for $n \geq 3$ and arguments $\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n} \in \mathfrak{B}_{\otimes n} = \{ \chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n} \in (H^*A)^{\otimes n} | a_i \in \{0, 1\} \text{ and } j_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in [1, n] \}$ where $0 \in \{ a_1, \ldots, a_n \}$.

**Proof.** Because of Lemma 30 and Definition 23 it is sufficient to show that

$$\Phi_n(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n}) = \Xi_n(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n})$$

if at least one $a_x$ equals 0.

**Case 1** At least two $a_x$ equal 0:

To show $\Phi_n(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n}) = 0$, we show $f_{n-1}(1^{\otimes r} \otimes m'_t \otimes 1^{\otimes n-r-2})(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n}) = 0$ for $r \in [0, n-2]$: In case both components of the argument of $m'_t$ are of the form $\chi^{\ell j}$, the result of $m'_t$ is of the form $\chi^{\ell'}$ (see Definition 23). Since $2 \leq n-1$, Corollary 29 implies the result of $f_{n-1}$ is zero. Otherwise at least one of the components of the argument of $f_{n-1}$ must be of the form $\chi^{\ell j}$ and the result of $f_{n-1}$ is zero as well. So $\Phi_n(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n}) = 0$.

To show $\Xi_n(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n}) = 0$, we show $m_2(f_i \otimes f_{n-i})(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n}) = 0$ for $i \in [1, n-1]$: \begin{itemize} 
  \item Suppose given $i \in [2, n-2]$: The statements $a_1 = \ldots = a_i = 1$ and $a_{i+1} = \ldots = a_n = 1$ cannot be true at the same time, so $f_i(\ldots) = 0$ or $f_{n-i}(\ldots) = 0$ and we have $m_2(f_i \otimes f_{n-i})(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n}) = 0$.
  \item Suppose that $i = 1$. Because at least two $a_x$ equal 0 the statement $a_2 = \ldots = a_n = 1$ cannot be true. Since $n-1 \geq 2$, we have $f_{n-1}(\ldots) = 0$ and $m_2(f_1 \otimes f_{n-1})(\chi^{a_1 \ell j_1} \otimes \ldots \otimes \chi^{a_n \ell j_n}) = 0$.
  \item The case $i = n-1$ is analogous to the case $i = 1$.
\end{itemize}
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So we have $\Phi_n(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = 0 = \Xi_n(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n})$.

**Case 2a** Exactly one $a_x$ equals 0, where $x \in [2, n-1]$. We have $\Phi_n(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = 0$: In case $n \geq p + 1$, it follows from $f_{n-1} = 0$. Let us check the case $n \in [3, p]$: Because of Definition 23, we have $f_{n-1}(1^{\otimes r} \otimes m'_2 \otimes 1^{\otimes n-r-2})(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = 0$ unless $r \in \{x - 2, x - 1\}$. So

$$\Phi_n(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = (-1)^{n-x+2} f_{n-1}(1^{\otimes x-2} \otimes m'_2 \otimes 1^{\otimes n-x} - 1^{\otimes x-1} \otimes m'_2 \otimes 1^{\otimes n-x-1})$$

$$f(t^j_1) = \chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}$$

$$f_{n-1}(1^{\otimes x-2} \otimes m'_2 \otimes 1^{\otimes n-x} - 1^{\otimes x-1} \otimes m'_2 \otimes 1^{\otimes n-x-1})$$

$$f_{n-1}(1^{\otimes x-2} \otimes m'_2 \otimes 1^{\otimes n-x} - 1^{\otimes x-1} \otimes m'_2 \otimes 1^{\otimes n-x-1})$$

To show $\Xi_n(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = 0$, we show $m_2(f \otimes f_{n-1})(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = 0$ for $i \in \left[1, n - 1\right]$: The element $\chi_{a_1 t^j_1}$ is a tensor factor of the argument of $f_i$ or of $f_{n-1}$. We write $y = i$ or $y = n - i$ accordingly. Then $y \geq 2$ since $x \notin \{1, n\}$, so $f_y(\ldots) = 0$ and thus $m_2(f \otimes f_{n-1})(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = 0$. So $\Phi_n(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = 0 = \Xi_n(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n})$.

**Case 2b** Only $a_1 = 0$, all other $a_x$ equal 1. We have $f_{n-1}(1^{\otimes r} \otimes m'_2 \otimes 1^{\otimes n-r-2})(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = 0$ unless $r = 0$. So

$$\Phi_n(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n}) = (-1)^n f_{n-1}(1^{\otimes 0} \otimes m'_2 \otimes 1^{\otimes n-2})(\chi_{a_1 t^j_1} \otimes \ldots \otimes \chi_{a_n t^j_n})$$

$$= (-1)^n f_{n-1}(m'_2(\chi_{a_1 t^j_1} \otimes \chi_{a_2 t^j_2} \otimes \chi_{a_3 t^j_3} \otimes \ldots \otimes \chi_{a_n t^j_n})$$

$$= (-1)^n f_{n-1}(\chi_{a_1 t^j_1} \otimes \chi_{a_2 t^j_2} \otimes \chi_{a_3 t^j_3} \otimes \ldots \otimes \chi_{a_n t^j_n})$$

$$= \left\{ \begin{array}{ll}
\gamma_{n-1 t^{j_1 + \ldots + j_n}} & 3 \leq n \leq p \\
0 & n \geq p + 1
\end{array} \right.$$
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So $\Phi_n(\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j}) = \Xi_n(\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j})$.

**Case 2c** Only $a_n = 0$, all other $a_x$ equal 1.

Argumentation analogous to case 2b gives

$$\Phi_n(\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j}) = (-1)^2 f_{n-1}(1 \otimes \chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j})$$

$$= f_{n-1}(\chi^{l_j} \otimes \cdots \otimes \chi^{a_nl_j} \otimes m'_{l_j}(\chi^{a_nl_j} \otimes l_j))$$

$$= \begin{cases} (-1)^{n-2} \gamma_{n-1l_j+\cdots+j_n} & 3 \leq n \leq p \\ 0 & n \geq p + 1 \end{cases}$$

and

$$\Xi_n(\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j}) = (-1)^{n(n-1)} m_2(f_{n-1} \otimes f_1)(\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j})$$

$$= m_2(f_{n-1} \otimes f_1)(\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_{n-1}l_j})$$

$$= \begin{cases} (-1)^{n-2} \gamma_{n-1l_j+\cdots+j_n} & 3 \leq n \leq p \\ 0 & n \geq p + 1 \end{cases}$$

So $\Phi_n(\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j}) = \Xi_n(\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j})$.

Now we examine the cases where $a_1 = \ldots = a_n = 1$:

**Lemma 32.** For $n \geq 3$, we have $\Phi_n(\chi^{l_j} \otimes \cdots \otimes \chi^{l_j}) = 0$ for $\chi^{l_j} \otimes \cdots \otimes \chi^{l_j} \in \mathcal{B}^{\otimes n} = \{\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j} \in (\text{Hom} F_p, F_p)^{\otimes n} \mid a_i \in \{0, 1\} \text{ and } j_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in [1, n]\}$.

**Proof.** We have $\Phi_n(\chi^{l_j} \otimes \cdots \otimes \chi^{l_j}) = 0$ since $\Phi_n = \sum_{r=0}^{n-2} (-1)^{n-r} f_{n-1}(1 \otimes \chi^{l_j} \otimes 1^{\otimes n-r})$ and the argument of $m'_2$ is always of the form $\chi^{l_j} \otimes \chi^{l_j}$, whence its result is zero.

**Lemma 33.** Condition (6)[n] holds for $n \in [3, p-1]$ and arguments $\chi^{l_j} \otimes \cdots \otimes \chi^{l_j} \in \mathcal{B}^{\otimes n} = \{\chi^{a_1l_j} \otimes \cdots \otimes \chi^{a_nl_j} \in (\text{Hom} F_p, F_p)^{\otimes n} \mid a_i \in \{0, 1\} \text{ and } j_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in [1, n]\}$.

**Proof.** For computing $\Xi_n$, we first show that $m_2(f_k \otimes f_{n-k})(\chi^{l_j} \otimes \cdots \otimes \chi^{l_j}) = 0$ for $k \in [2, n-2]$. We will need the following congruence.

$$\begin{align*}
(k(l-1) + l(i+x)) - (n-k-1 + l') & \equiv_{p-1} -k + k - n + 1 = -(n-1) \\
\equiv_{p-1} 0
\end{align*}$$

The last statement results from $2 \leq n \leq p - 1$. We set “±” as a symbol for the (a posteriori irrelevant) signs in the following calculation. For $k \in [2, n-2]$, we have

$$m_2(f_k \otimes f_{n-k})(\chi^{l_j} \otimes \cdots \otimes \chi^{l_j})$$
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\[ \Xi_n(\chi_{t^{j_1}} \otimes \ldots \otimes \chi_{t^{j_n}}) = m_2((-1)^{n-1} \gamma_{k^{l_1} \ldots + j_k} \otimes (-1)^{n-1} \gamma_{n-k^{l_{k+1} + \ldots + j_n}}) \]

\[ j_1 + \ldots + j_k = x, \]
\[ j_{k+1} + \ldots + j_n = y \]

\[ \Xi_n(\chi_{t^{j_1}} \otimes \ldots \otimes \chi_{t^{j_n}}) = \pm \gamma_{k^x} \otimes \gamma_{n-k^y} \]

\[ = \pm \left( \sum_{i \geq 0} |e_{k-i}|_{k(l-1)+i(i+x)} + \sum_{i \geq 0} |e_{p-k-i}|_{k(l-1)+(p-1)+i(i+x)} \right) \]
\[ \otimes \left( \sum_{i' \geq 0} |e_{n-k-i}|_{(n-k)(l-1)+i(i'+y)} + \sum_{i' \geq 0} |e_{p-n-k-k}|_{(n-k)(l-1)+(p-1)+i(i'+y)} \right) \]

\[ (11) \equiv 0. \]

So

\[ \Xi_n(\chi_{t^{j_1}} \otimes \ldots \otimes \chi_{t^{j_n}}) = m_2((-1)^{n} f_1 \otimes f_{n-1} + (-1)^{n(n-1)} f_{n-1} \otimes f_1) \]

\[ = m_2((-1)^{n+n(n-1)} f_1 \otimes f_{n-1} \otimes f_1) \]

\[ + f_{n-1} \otimes f_1 \otimes f_{n-1} \]

\[ = m_2(\chi_{t^{j_1}} \otimes (-1)^{n-2} \gamma_{n-1} t^{j_2 + \ldots + j_n} + (-1)^{n-2} \gamma_{n-1} t^{j_1 + \ldots + j_n-1} \otimes \chi_{t^{j_n}}) \]

\[ = (-1)^n (\chi_{t^{j_1}} \otimes \gamma_{n-1} t^{j_2 + \ldots + j_n} + \gamma_{n-1} t^{j_1 + \ldots + j_n-1} \otimes \chi_{t^{j_n}}) \]

\[ p.20(c), l.21 \]

\[ (11)^n (\chi \circ \gamma_{n-1} + \gamma_{n-1} \circ \chi) \otimes t^{j_1 + \ldots + j_n} \]

\[ \chi \circ \gamma_{n-1} = \left( \sum_{i \geq 0} |e_{n-1} \circ e_{n-1}|_{n-1}(l-1)+i(i+1) \right) \]

\[ + \sum_{i \geq 0} |e_{p-n,p+n-1} \circ e_{p-n+n-1}|_{n-1}(l-1)+i(i+1) \]

\[ = \sum_{i \geq 0} \left( |e_{n-1} \circ e_{n-1}|_{n-1}(l-1)+i(i+l) + |e_{p-n,p+n-1} \circ e_{p-n+n-1}|_{n-1}(l-1)+i(i+l) \right) \]

\[ \gamma_{n-1} \circ \chi = \left( \sum_{i' \geq 0} |e_{n-1} \circ e_{n-1}|_{n-1}(l-1)+i(l-1)+(p-1) \right) \]

\[ + \sum_{i' \geq 0} |e_{p-n,p+n-1} \circ e_{p-n+n-1}|_{n-1}(l-1)+i(l-1)+(p-1) \]

\[ = \sum_{i' \geq 0} \left( |e_{n-1} \circ e_{n-1}|_{n-1}(l-1)+i(l-1)+(p-1) + |e_{p-n,p+n-1} \circ e_{p-n+n-1}|_{n-1}(l-1)+i(l-1)+(p-1) \right) \]

\[ \eta_{p-n} = \sum_{i' \geq 0} |e_{n-1} \circ e_{n-1}|_{n-1}(l-1)+i(l-1)+(p-1) \]

\[ + \sum_{i' \geq 0} |e_{p-n,p+n-1} \circ e_{p-n+n-1}|_{n-1}(l-1)+i(l-1)+(p-1) \]

\[ = \sum_{i' \geq 0} \left( |e_{n-1} \circ e_{n-1}|_{n-1}(l-1)+i(l-1)+(p-1)+i(l) \right) \]

\[ + \sum_{i' \geq 0} |e_{p-n,p+n-1} \circ e_{p-n+n-1}|_{n-1}(l-1)+i(l-1)+(p-1)+i(l) \]

\[ = \sum_{i' \geq 0} \left( |e_{n-1} \circ e_{n-1}|_{n-1}(l-1)+i(l-1)+(p-1)+i(l) + |e_{p-n,p+n-1} \circ e_{p-n+n-1}|_{n-1}(l-1)+i(l-1)+(p-1)+i(l) \right) \]

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So $\chi \circ \gamma_{n-1} = \gamma_{n-1} \circ \chi = m_1(\gamma_n)$ by Lemma 21. Therefore

$$\Xi_n(\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_n}}) = (-1)^n m_1(\gamma_n) \circ l_{j_1+\ldots+j_n} \overset{P.20(c)}{=} (-1)^n m_1(\gamma_n l_{j_1+\ldots+j_n})$$

$$= - m_1(((-1)^{n-1} \gamma_{n-1} l_{j_1+\ldots+j_n}))$$

$$= - m_1 \circ f_n(\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_n}}).$$

We conclude using Lemma 30 by

$$(f_1 \circ m'_n + \Phi_n)(\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_n}}) = 0 = (m_1 \circ f_n + \Xi_n)(\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_n}}).$$

\[\square\]

**Lemma 34.** Condition (6)[p] holds for arguments $\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_p}} \in \mathfrak{B}^{\otimes p} = \{\chi^{a_1 l_{j_1}} \otimes \ldots \otimes \chi^{a_p l_{j_p}} \in (H^*A)^{\otimes p} | a_i \in \{0, 1\} \text{ and } j_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in [1, p]\}.$

**Proof.** Recall that $|l| = l = 2(p-1)$ is even, $|\chi| = l - 1$ is odd and $|f_i| = 1 - i$ by Lemma 25. We have

$$\Xi_p(\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_p}}) = \sum_{i=1}^{p-1}(-1)^{p_i} m_2(f_i \otimes f_{p-i})(\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_p}})$$

$$= \sum_{i=1}^{p-1}(-1)^{p_i + i(1 - (p - i))} m_2(f_i(\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_i}}) \otimes f_{p-i}(\chi^{l_{j_i+1}} \otimes \ldots \otimes \chi^{l_{j_p}}))$$

$$= \sum_{i=1}^{p-1} f_i(\chi^{l_{j_1}} \otimes \ldots \otimes \chi^{l_{j_i}}) \circ f_{p-i}(\chi^{l_{j_i+1}} \otimes \ldots \otimes \chi^{l_{j_p}})$$

$$p \geq 3 \Rightarrow \chi^{l_{j_1}} \circ ((-1)^{p_{i-2}} \gamma_{p_1} l_{j_2+\ldots+j_p} + (-1)^{p_{i-2}} \gamma_{p_1} l_{j_1+\ldots+j_p-1} \circ \chi^{l_{j_p}})$$

$$+ \sum_{i=2}^{p-2}(-1)^{i-1} \gamma_{p_{i-2}} l_{j_1+\ldots+j_i} \circ ((-1)^{p_{i-1}} \gamma_{p_1} l_{j_1+\ldots+j_p})$$

$$\overset{P.20(b)}{=} (-1)^p \left(\chi \circ \gamma_{p-1} + \gamma_{p-1} \circ \chi + \sum_{k=2}^{p-2} \gamma_k \circ \gamma_{p-k}\right) \circ l_{j_1+\ldots+j_p}$$

$$\chi \circ \gamma_{p-1} = \left(\sum_{i \geq 0} \left[ e_1 \right]_{(i+1)l-1}^{i+1} + \left(\sum_{k=1}^{p-2} \left[ e_{k+1, k} \right]_{(i+1)l-1+k}^{i+k}\right)\right) \circ \left(\sum_{\nu \geq 0} \left[ e_{p-1} \right]_{(p-1)(l-1)+\nu'}^{l-2(p-1)+\nu'} + \sum_{\nu \geq 0} \left[ e_1 \right]_{(p-1)(l-1)+\nu'}^{l-1+2(p-1)+\nu'}\right)$$

$$\gamma_{p-1} \circ \chi = \left(\sum_{\nu \geq 0} \left[ e_{p-1} \right]_{(p-1)(l-1)+\nu'}^{l+1+\nu'} + \sum_{\nu \geq 0} \left[ e_1 \right]_{(p-1)(l-1)+\nu'}^{l-1+2(p-1)+\nu'}\right) \circ \left(\sum_{i \geq 0} \left[ e_{i+1, i} \right]_{i+l-1}^{i+l} + \left(\sum_{k=1}^{p-2} \left[ e_{k+1, k} \right]_{(i+l-1)+k}^{i+k}\right)\right)$$

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We have
\[ \sum_{i \geq 0} \left[ e_{p-1} \right]^{i+l+(p-1)}_{(i+1)l+1+(p-1)} + \left( \sum_{k=1}^{p-2} \left[ e_{p-k-1} \right]^{i+l+(p-1)+k}_{(i+1)l+1+(p-1)+k} \right) \]
\[ = \sum_{i' \geq 0} \left[ e_{p-1} \right]^{(p-1)-l+i'}_{(p+l')-1+(p-1)} + \sum_{i' \geq 0} \left[ e_{1} \right]^{(p+l')-1}_{(p+l')-1-l-i'} \]
\[ = \sum_{i' \geq 0} \left[ e_{p-1} \right]^{p-2+i' l}_{(p+i')l+1} + \sum_{i' \geq 0} \left[ e_{1} \right]^{i'l+l-1}_{(p+i')l+1-l-i} \]
\[ \gamma_k \circ \gamma_{p-k} = \left( \sum_{i_0 \geq 0} \left[ e_{k} \right]^{k-1+l+i+k}_{(i+k)l+k+1} + \sum_{i_0 \geq 0} \left[ e_{p-k} \right]^{k-1+(p-1)+l+i}_{(i+k)l+k+1} \right) \]
\[ \circ \left( \sum_{i' \geq 0} \left[ e_{p-k} \right]^{k-2+p-l+i}_{(i+k)l+k+1} + \sum_{i' \geq 0} \left[ e_{k} \right]^{k+2+(p-1)+l+i}_{(i+k)l+k+1} \right) \]
\[ = \sum_{i_0 \geq 0} \left[ e_{k} \right]^{k-1+l+i+k}_{(i+k)l+k+1} + \sum_{i_0 \geq 0} \left[ e_{p-k} \right]^{k-1+(p-1)+l+i}_{(p+i)l+k+1} \]
\[ = \sum_{i_0 \geq 0} \left[ e_{k} \right]^{k-1+i+k}_{(i+k)l+k+1} + \sum_{i_0 \geq 0} \left[ e_{p-k} \right]^{k-1+(p-1)+l+i}_{(p+i+k)l+k+1} \]
\[ = \sum_{i_0 \geq 0} \left[ e_{k} \right]^{k-1+i+k}_{(i+k)l+k+1} + \sum_{i_0 \geq 0} \left[ e_{p-k} \right]^{k-1+(p-1)+l+i}_{(p+i+k)l+k+1} \cdot \]

Thus
\[ \chi \circ \gamma_{p-1} + \gamma_{p-1} \circ \chi + \sum_{k=2}^{p-2} \gamma_k \circ \gamma_{p-k} \]
\[ = \sum_{i_0 \geq 0} \sum_{p-2}^{k=0} \left( \left[ e_{k+1} \right]^{k+i}_{(i+1)l+k} + \left[ e_{p-k-1} \right]^{k+(p-1)+i}_{(p+i)l+k+1} \right) \]
\[ \sum_{i_0 \geq 0} \sum_{p-2}^{k=0} \left( \left[ e_{k+1} \right]^{k+i}_{(i+1)l+k} + \left[ e_{p-k-1} \right]^{k+(p-1)+i}_{(p+i)l+k+1} \right) \]
\[ \Xi_p(\chi_{j_1} \otimes \ldots \otimes \chi_{j_p}) = (-1)^{p} \ell^{p-1+j_1+\ldots+j_p} . \]

So we conclude using Lemma 30 by
\[ (f_1 \circ m'_{p} + \Phi_p)(\chi_{j_1} \otimes \ldots \otimes \chi_{j_p}) \]
\[ \Xi_p(\chi_{j_1} \otimes \ldots \otimes \chi_{j_p}) = (-1)^{p} \ell^{p-1+j_1+\ldots+j_p} . \]
\[ \Xi_p(\chi_{j_1} \otimes \ldots \otimes \chi_{j_p}) = (-1)^{p} \ell^{p-1+j_1+\ldots+j_p} . \]

**Lemma 35.** Condition (6) holds for \( n \in [p+1, 2(p-1)] \) and arguments
\( \chi_{j_1} \otimes \ldots \otimes \chi_{j_n} \in \mathfrak{B}^{\otimes n} = \{ \chi^{a_1}_{j_1} \otimes \ldots \otimes \chi^{a_n}_{j_n} \in (H^*A)^{\otimes n} \mid a_i \in \{0, 1\} \text{ and } j_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in [1, n] \} \).

**Proof.** As \( f_k = 0 \) for \( k \geq p \), we have
\[ \Xi_n(\chi_{j_1} \otimes \ldots \otimes \chi_{j_n}) = \sum_{k=n-p+1}^{p-1} \left(-1\right)^{nk} m_2(f_k \otimes f_{n-k})(\chi_{j_1} \otimes \ldots \otimes \chi_{j_n}) \]

The right side is a linear combination of terms of the form \( \gamma_k \circ \gamma_{n-k} \) for \( k \in [n-p-1, p-1] \). We have
\[ \gamma_k \circ \gamma_{n-k} = \left( \sum_{i_0 \geq 0} \left[ e_k \right]^{k-1+i+k}_{(i+k)l+k+1} + \sum_{i_0 \geq 0} \left[ e_{p-k} \right]^{k-1+(p-1)+i+k}_{(p+i)l+k+1} \right) \]

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We see that $F$ is a quasi-isomorphism and the proof of Theorem 24 is complete.

One could formulate a lemma similar to Lemma 35 for the case $n > 2(p - 1)$ as then the sum $\sum_{k=n-p+1}^{n-1} (-1)^{nk} m_2(f_k \otimes f_n) \otimes \mathcal{L}^j$ is in fact empty. Instead we use Lemma 18 to prove $(6)[n]$ for $n > 2p - 2$.

Proof of Theorem 24. Lemmas 26, 28, 31 and 33 to 35 ensure that $(6)[n]$ holds for $n \in [1, 2p - 2]$. Then Lemma 18 with $k = p$ proves that $(6)[n]$ holds for all $n \in [1, \infty]$, cf. also Definition 23. By Lemma 27, $f_1$ is injective. By Lemma 25, the degrees are as required in Lemma 17. Lemma 17 proves that $(H^*A, (m'_n)_{n\geq1})$ is an $A_\infty$-algebra and $(f_n)_{n\geq1}$ is an $A_\infty$-morphism from $(H^*A, (m'_n)_{n\geq1})$ to $(A, (m_n)_{n\geq1})$. By Lemma 26, we have $m'_1 = 0$. Thus $(H^*A, (m'_n)_{n\geq1})$ is a minimal $A_\infty$-algebra. By Lemma 26, the complex morphism $f_1 : (H^*A, m'_1) \to (A, m_1)$ is a quasi-isomorphism which induces the identity in homology. So the $A_\infty$-morphism $(f_n)_{n\geq1} : (H^*A, (m'_n)_{n\geq1}) \to (A, (m_n)_{n\geq1})$ is a quasi-isomorphism and the proof of Theorem 24 is complete.

### 2.4 At the prime 2

We examine the case at the prime 2. We use a direct approach. Note that $S_2$ is a cyclic group so the theory of cyclic groups applies as well.

We have $\mathbb{F}_2S_2 = \{0, (id), (1, 2), (id) + (1, 2)\}$. We have maps given by

\[
\begin{align*}
\varepsilon : & \mathbb{F}_2S_2 \rightarrow \mathbb{F}_2 \\
\quad a(id) + b(1, 2) & \mapsto a + b \\
D : & \mathbb{F}_2S_2 \rightarrow \mathbb{F}_2S_2 \\
\quad a(id) + b(1, 2) & \mapsto (a + b) ((id) + (1, 2)).
\end{align*}
\]

We see that $\varepsilon$ is surjective and $\ker \varepsilon = \ker D = \text{im} D = \{0, (id) + (1, 2)\}$. The maps $\varepsilon$ and $D$ are $\mathbb{F}_2S_2$-linear, where $\mathbb{F}_2$ is the $\mathbb{F}_2S_2$-module that corresponds to the trivial representation of $S_2$. So we have a projective resolution of $\mathbb{F}_2$ with augmentation $\varepsilon$ by

\[
\text{PRes} \mathbb{F}_2 := (\cdots \xrightarrow{D} \mathbb{F}_2S_2 \xrightarrow{D} \mathbb{F}_2S_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{-1} \cdots),
\]

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where the degrees are written below.

We set $e_1$ to be the identity on $\mathbb{F}_2 S_2$.

Let $A := \Hom_{\mathbb{F}_2 S_2}(\text{PRes} \mathbb{F}_2, \text{PRes} \mathbb{F}_2)$ and let the $A_\infty$-structure on $A$ be $(m_n)_{n \geq 1}$ (cf. Lemma 13). Recall the conventions concerning $\Hom^{k}_{\mathbb{F}_2}(C, C')$ for complexes $C, C'$ and $k \in \mathbb{Z}$.

**Lemma 36.** An $\mathbb{F}_2$-basis of $H^* A$ is given by $\{\overline{e^j} | j \geq 0\}$ where

$$\xi := \sum_{i \geq 0} [e_1]_{i+1}^i \in A.$$ 

**Proof.** Straightforward induction yields, for $j \geq 0$,

$$\xi^j = \sum_{i \geq 0} [e_1]_{i+j}^i.$$ 

We have

$$m_1(\xi^j) = d \circ \xi^j - (-1)^j \xi^j \circ d = d \circ \xi^j + \xi^j \circ d$$

$$= \left( \sum_{i \geq 0} |D|_{i+1}^i \right) \circ \left( \sum_{i \geq 0} [e_1]_{i+j}^i \right) + \left( \sum_{i \geq 0} [e_1]_{i+j}^i \right) \circ \left( \sum_{i \geq 0} |D|_{i+1}^i \right)$$

$$= \sum_{i \geq 0} |D|_{i+j+1}^i + \sum_{i \geq 0} |D|_{i+j+1}^i = 0,$$

so $\xi^j$ is a cycle. As $\Hom_{\mathbb{F}_2 S_2}(\text{PRes} \mathbb{F}_2, \mathbb{F}_2) = \{0, \varepsilon\}$ and $\varepsilon \circ D = 0$, the differentials of $\Hom^*(\text{PRes} \mathbb{F}_2, \mathbb{F}_2)$ (cf. Lemma 19) are all zero. So $\{\varepsilon\}$ is an $\mathbb{F}_2$-basis of $H^k \Hom^*(\text{PRes} \mathbb{F}_2, \mathbb{F}_2)$ for $k \geq 0$. Since in the notion of Lemma 19, $\overline{\Psi}_k(\xi^j) = \varepsilon$, the set $\{\overline{\varepsilon^j}\}$ is an $\mathbb{F}_2$-basis of $H^k \Hom^*(\text{PRes} \mathbb{F}_2, \text{PRes} \mathbb{F}_2)$ for $k \geq 0$. For $k < 0$ we have $H^k \Hom^*(\text{PRes} \mathbb{F}_2, \text{PRes} \mathbb{F}_2) \cong H^k \Hom^*(\text{PRes} \mathbb{F}_2, \mathbb{F}_2) = 0$. So $\{\overline{\varepsilon^j} | j \geq 0\}$ is an $\mathbb{F}_2$-basis of $H^* A$. $\square$

We define families of maps $(f_n : (H^*)^\otimes n \to A)_{n \geq 1}$ and $(m'_n : (H^*)^\otimes n \to H^* A)_{n \geq 1}$ as follows. $f_1$ and $m'_2$ are given on a basis by

$$f_1(\overline{\varepsilon^j}) := \xi^j \quad \text{for } j \geq 0$$

$$m'_2(\overline{\xi^j} \otimes \overline{\xi^k}) := \overline{\xi^{j+k}} \quad \text{for } j, k \geq 0.$$ 

All other maps are set to zero.

It is straightforward to check that $(H^* A, (m'_n)_{n \geq 1})$ is a pre-$A_\infty$-algebra and $(f_n)_{n \geq 1}$ is a pre-$A_\infty$-morphism from $H^* A$ to $A$. As $m'_2$ is associative, $(H^* A, (m'_n)_{n \geq 1})$ is a dg-algebra, so in particular an $A_\infty$-algebra. As $f_k = 0$ for $k \neq 1$, (6)[n] simplifies to

$$f_1 \circ m'_n = m_n \circ (f_1 \otimes \cdots \otimes f_1).$$

As $m'_n = 0$ and $m_n = 0$ for $n \geq 3$, (6)[n] is satisfied for all $n \geq 3$. For $n \in \{1, 2\}$, we have

$$f_1 \circ m'_1 = m_1 \circ f_1$$

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\[ f_1 \circ m'_2 = m_2(f_1 \otimes f_1). \]

The second equation follows immediately from the definition of \( m'_2 \) and \( f_1 \). The first equation holds as \( m'_1 = 0 \) and the images of \( f_1 \) are all cycles. So (6) holds for all \( n \) and \( (f_n)_{n \geq 1} \) is an \( A_\infty \)-mor- phism from \( (H^*A,(m'_n)_{n \geq 1}) \) to \( (A,(m_n)_{n \geq 1}) \). By the construction of \( f_1 \), it induces the identity on homology. Thus \( (H^*A,(m'_n)_{n \geq 1}) \) is a minimal model of \( (A,(m_n)_{n \geq 1}) \).

**Remark 37** (Comparison with primes \( p \geq 3 \)). At a prime \( p \geq 3 \), we have constructed a projective resolution with period length \( l = 2(p - 1) \) in (2). If one constructs a projective resolution of \( \mathbb{Z}(2) \) analogous to the case \( p \geq 3 \), we have a sequence of the form

\[ \cdots \rightarrow \mathbb{Z}(2) S_2 \xrightarrow{\hat{e}^*_2} \mathbb{Z}(2) S_2 \xrightarrow{\hat{e}^*_{2,2}} \mathbb{Z}(2) S_2 \xrightarrow{\hat{e}^*_{2,2}} \mathbb{Z}(2) S_2 \xrightarrow{\hat{e}_{2,2}} \mathbb{Z}(2) S_2 \rightarrow 0 \rightarrow \cdots \]

with a period length of 2, where

\[ \hat{e}_{2,2} : (\text{id}) \mapsto (\text{id}) - (1,2) \]
\[ \hat{e}^*_{2,2} : (\text{id}) \mapsto (\text{id}) + (1,2). \]

However, modulo 2 the differentials \( \hat{e}_{2,2} \) and \( \hat{e}^*_{2,2} \) reduce to the same map \( D : \mathbb{F}_2 S_2 \rightarrow \mathbb{F}_2 S_2 \), so we obtain a period length of 1.

The maps \( \iota \) resp. \( \chi \) from Proposition 20 may be identified with \( \xi^2 \) resp. \( \xi \). This way, the definition of \( m'_2 \) at the prime 2 is readily compatible with Definition 23.

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