NEAR-THRESHOLD W-PAIR PRODUCTION IN THE MODEL
OF UNSTABLE PARTICLES WITH SMEARED MASS

V. I. KUKSA∗
Institute of Physics, Southern Federal University, Rostov-on-Don 344090, Russia

R. S. PASECHNIK†
Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna 141980, Russia

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Near-threshold production of charged boson pairs is considered within the framework of the model of unstable particles with smeared mass. The results of calculations are in good agreement with LEP II data and Monte-Carlo simulations. Suggested approach significantly simplifies calculations with respect to standard perturbative one.

Keywords: W-boson pair production; unstable particles

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1. Introduction

The measurements of W-pair production at LEP II provided us with an important information about the mass of W boson and non-abelian triple gauge-boson couplings. To extract the exact information from W-pair production we have to calculate the radiative corrections (RC’s), which give a noticeable contribution to the cross-section. Ideally, one would like to have the full RC’s to the process \( e^+e^- \rightarrow W^+W^- \rightarrow 4f \). In practice, this problem is very complicated and can not be considered analytically. For discussion of the LEP II situation and strategy it is useful to distinguish three levels of sophistication in the description of the W-pair production \([1, 2]\):

1) On-shell W-pair production, \( e^+e^- \rightarrow W^+W^- \), with consequent on-shell W decays. All \( O(\alpha) \) RC’s to these processes are known.
2) Off-shell production of W pairs, which then decay into four fermions. Full set of RC’s is very bulky for the analytical observation and analysis.
3) Full process \( e^+e^- \rightarrow 4f \) with an account of the complete set of the \( O(\alpha) \) corrections. This problem leads to the additional diagrams with the same final

∗kuksa@list.ru
†rpasech@theor.jinr.ru
states, and complete electroweak $O(\alpha)$ corrections are described by many thousands diagrams.

On-shell $W$-pair production was considered in Refs. [1]-[3], where the cross-section of the process $e^+e^- \rightarrow W^+W^-$ was given. At tree level, this process is described by three diagrams shown in Fig. 1. The complete $O(\alpha)$ radiative corrections, comprising the virtual one-loop corrections and real-photon bremsstrahlung, were calculated and represented in Refs. [4]-[11]. The description of the on-shell $W$-pair production and consequent decays with an account of RC’s was fulfilled in Refs. [12]-[18]. Off-shell production of $W$-pairs, which then decay into four fermions, was considered in Ref. [19].

In description of the $W$- and $Z$-pairs production we should take into consideration the fact that the gauge bosons are not stable particles and the real process is not $e^+e^- \rightarrow W^+W^-$. This is only an approximation with a level of goodness, which may depend on several factors, while the real process is $e^+e^- \rightarrow W^+W^-, ZZ \rightarrow 4f$. There are many papers, devoted to comprehensive analysis and description of all possible processes with the four-fermion final states. Because of a large number of diagrams, describing these processes, the classification scheme was applied in Refs. [20]-[23]. The possible processes are divided into three classes: charge current (CC), neutral current (NC) and mixed current (MIX). Born processes $e^+e^- \rightarrow W^+W^-, ZZ$ are designated as CC03 and NC02, which correspond to three charge current and two neutral current diagrams. According to this classification the off-shell $W$-pair production with consequent $W$ decay can be described in the framework of the Double-Pole Approximation (DPA) [23]-[26]. The DPA selects only diagrams with two nearly resonant $W$ bosons and the number of graphs is considerably reduced [23].

Complete description of the total set of $4f$-production processes with an account of RC’s is not analytically available due to a huge number of diagrams and presence of non-factorable corrections. But the complete EW $O(\alpha)$ corrections have been calculated for some exclusive processes, for instance, for the processes $e^+e^- \rightarrow \nu_\tau\tau^+\mu^-\bar{\nu}_\mu, u\bar{d}\mu^-\bar{\nu}_\mu$, and $u\bar{d}s\bar{c}$ [27, 28]. Because of complexity of the problem, some approximation schemes are practically applied, namely, Semi-Analytical Approximation (SAA) [2, 29], improved Born approximation [30], an asymptotic expansion in powers of the coupling constant of the cross-section [31], fermion-loop scheme, etc. (see Introduction in Ref. [27, 28]). There are many computer tools of calculations, for instance, Monte-Carlo (MC) simulations, such as RacoonWW [28, 32, 33] and YFSWW [34, 35, 36]. All above mentioned methods are based on the traditional quantum field theory of unstable particles (UP’s) [2]. At the same time, there are some alternative approaches for description of the UP’s, such as the effective theory of UP’s [37]-[39] and the model of UP with smeared mass [40, 41].

In this paper, we suggest the description of the near-threshold $W$-pairs production within the framework of the model of UP with smeared mass initially proposed in Ref. [40]. The model is based on the time-energy uncertainty relation
\[ \Delta E \cdot \Delta t \sim 1 \ (c = \hbar = 1) \]. It follows from the equation of motion in the Heisenberg representation which describes the evolution of the non-stationary quantum system \[43\]. In the case of the unstable particles, \( \Delta t \) is the lifetime and \( \Delta E \) is the value of the mass smearing \( \Delta m \) in the rest-frame system \[10, 40, 43, 44\]. In the model under consideration the UP is described by a state with smeared (fuzzed) mass in accordance with the uncertainty relation. So, the processes \( e^+e^- \to W^+W^- \), \( ZZ \) are described in a traditional way, i.e., in a stable particle approximation, but the phase space is calculated for the states on the smeared mass-shell. In the framework of the model, full process \( e^+e^- \to W^+W^- \), \( ZZ \rightarrow 4f \) is divided into two stages \( e^+e^- \to W^+W^- \), \( ZZ \) and \( W^+W^- \), \( ZZ \rightarrow 4f \) due to exact factorization at tree level \[45\]-\[47\]. For description of the bosons in the final state we use the model polarization matrix which differs from the standard one \[42, 47\] (see the next section).

The model was applied for description of the Finite-Width Effects (FWE’s) in many low- and high-energy processes involving the UP with large width \[42, 45\]-\[48\]. In particular, the approach was successfully applied to the process \( e^+e^- \to ZZ \) in Ref. \[42\]. In this paper it was shown that the results of the model are in good agreement with the LEP II data and turned out to be very close to the corresponding results of MC simulations. So, it is reasonable to apply the same method for the case of \( W \)-pair production.

It should be also noted that the model under consideration directly leads to calculation schemes which are in close analogy with such standard approaches as the Convolution Method (CM), Narrow-Width Approximation (NWA) and SAA. However, the model treatment has some noticeable distinctions which are discussed in detail in Refs. \[45\]-\[47\] (see also the next section). The principal distinction between the standard and model treatment of the FWE’s takes place in description of UP with large width \[47\] and the mass splitting in the neutral meson systems \[48\].

The paper is organized as follows. In the second section, we give a short description of the model and define the status of our calculations. The cross-section of the process \( e^+e^- \to W^+W^- \) at tree level is derived in the framework of the model \[40\] in this section. Section 3 contains the calculation strategy with taking into account of the radiative corrections. In this section we also represent the results of our calculations, MC simulations and LEP II data. Some conclusions concerning the applicability of the method were made in the last section. We note that the aim of this investigation is to test the model approach for the case of near-threshold boson-pair production and analyze the possibility of its improvement by an accounting of the radiative corrections which are not included into the effective field of UP.
2. The model cross-section of the near-threshold $W$-pair production at tree level

Firstly, we give a short description of the model of UP with smeared mass. The model wave function of the UP is

$$\Phi_a(x) = \int \Phi_a(x, \mu) \omega(\mu) d\mu,$$

where $\Phi_a(x, \mu)$ is the standard spectral component, which defines a particle with a fixed mass squared $m^2 = \mu$ in the Stable Particle Approximation (SPA). The weight function $\omega(\mu)$ is formed by the self-energy interactions of UP with vacuum fluctuations and decay products. This function describes the smeared (fuzzed) mass-shell of UP. Thus, the smearing of mass is caused, on the one hand, by instability according to formal uncertainty relation and, on the other hand, by stochastic interaction of UP with the electro-week vacuum fluctuations [46].

The commutative relations for the model operators have an additional $\delta$-function

$$[\dot{\Phi}_\alpha^- (\vec{k}, \mu), \Phi^+_\beta (\vec{q}, \mu')]_{\pm} = \delta(\mu - \mu') \delta(\vec{k} - \vec{q}) \delta_{\alpha\beta},$$

where subscripts “$\pm$” correspond to fermion and boson fields, respectively. The presence of $\delta(\mu - \mu')$ in Eq. (2) means the following assumption: the acts of creation and annihilation of the particles with various $\mu$ (the random mass squared) do not interfere. Thus, the parameter $\mu$ has the status of physically distinguishable value of a random $m^2$.

The model Green functions for the vector and spinor fields in momentum representation have the convolution form:

$$D_{mn}(k) = -i \int \frac{g_{mn} - k_m k_n / \mu}{k^2 - \mu + i\epsilon} \rho(\mu) d\mu,$$

and

$$\hat{D}(k) = i \int \frac{\vec{k} + \vec{k}}{k^2 - \mu + i\epsilon} \rho(\mu) d\mu,$$

where $\rho(\mu) = |\omega(\mu)|^2$.

Further, we consider the model amplitude for the simplest processes with UP in the initial or final state. The expression for scalar field is

$$\phi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int \omega(\mu) d\mu \int \frac{a^\pm(\vec{q}, \mu)}{\sqrt{2q^0_{\mu}}} e^{\pm iqx} d\vec{q},$$

where $q^0_{\mu} = \sqrt{q^2 + \mu}$ and $a^\pm(\vec{q}, \mu)$ are the creation or annihilation operators of UP with the momentum $q$ and mass squared $m^2 = \mu$. Taking into account Eq. (2), one can get

$$[\hat{a}^- (\vec{k}, \mu), \phi^+ (x)]_-, [\phi^- (x), \hat{a}^+ (\vec{k}, \mu)]_- = \frac{\omega(\mu)}{(2\pi)^{3/2} \sqrt{2k^0_{\mu}}} e^{\pm ikx},$$
where $k^0_\mu = \sqrt{k^2 + \mu}$. The expressions (6) differ from the standard ones by the factor $\omega(\mu)$ only. From this result it follows that, if $\hat{a}^+(k, \mu)|0\rangle$ and $|0\rangle \hat{a}^-(k, \mu)$ define UP with the mass $m = \sqrt{\mu}$ and momentum $k$ in the initial or final states, then the amplitude for the transition $\Phi \to \phi\phi_1$ is

$$A(k, \mu) = \omega(\mu)A^{st}(k, \mu), \quad (7)$$

where $A^{st}(k, \mu)$ is the amplitude in the SPA. This amplitude is calculated in the standard way and can include the higher corrections. Moreover, it can be an effective amplitude for the processes with hadron participation. From Eq. (7) it follows that the differential (over $\mu$) probability of transition is

$$dP(k, \mu) = \rho(\mu)|A(k, \mu)|^2d\mu. \quad (8)$$

In Eq. (8) the differential probability $d\Gamma^{st}(k, \mu)$ is defined in the standard way (the SPA).

If there are two UP’s with large widths in the final state of decay $\Phi \to \phi_1\phi_2$, then in analogy with the previous case one can get the double convolution formula:

$$\Gamma(m_\Phi) = \int \Gamma^{st}(m_\Phi; \mu_1, \mu_2)\rho_1(\mu_1)\rho_2(\mu_2)d\mu_1d\mu_2. \quad (9)$$

The polarization matrix for the case of vector UP in the final state has the form

$$\sum_e e_m(q)e_n^*(q) = -g_{mn} + q_0q_n/\mu, \quad (10)$$

In the case of spinor UP in the final state we have

$$\sum_\nu \gamma_{\nu}^{\alpha\pm}(q)\bar{\gamma}_{\beta\nu}^{\alpha\mp}(q) = \frac{1}{2q^0_\mu}(\tilde{q} \mp \sqrt{\mu})_{\alpha\beta}, \quad (11)$$

where the summation over polarizations is implied and $q^0_\mu = \sqrt{q^2 + \mu}$. The same relations take place for the initial states, however, one have to average over their polarizations.

The most important element of the model is the probability density $\rho(\mu)$ which describes the smearing of UP mass. The various definitions of $\rho(\mu)$ were discussed in Ref. [46], where the Lorentzian (Breit-Wigner type), Gaussian and phenomenological distributions have been considered. The Lorentzian distribution was derived by matching the model propagators (3), (4) and standard dressed ones in the Breit-Wigner form.

One of the important properties of the model is the exact factorization of the processes with the UP in an intermediate state. In the frame of the effective theory of UP, which follows from the model, the factorization leads to the convolution
formula for the decay rate \[41\] and factorized formula for the cross-section \[45\]. These results are derived by straightforward calculations at tree level without any approximations. So, the model provides the formal basis for the CM, SAA and NWA, which are the approximate approaches in the traditional treatment. The generalization of the factorization method to the complicated processes of scattering and decays with two or more UP in intermediate states was considered in Ref. \[47\]. This method can be applied for description of the boson-pair production and decays in the factorized form for double-pole set of diagrams (see comment to tree-level result at the end of this section).

In this work, we use two principal elements of the model \[40, 46\] – the convolution structure of the transition probability (an analog of the expression \[9\]) and the polarization matrix for UP in the final state \[10\]. Using these expressions, we get the model Born cross-section of $W$-pair production in the following form

$$\sigma_{WW}^B(s) = \int \int \sigma_{WW}^B(s; \mu_1, \mu_2) \rho_1(\mu_1) \rho_2(\mu_2) d\mu_1 d\mu_2,$$

where $\sigma_{WW}^B(s; \mu_1, \mu_2)$ is the Born cross-section which is calculated in the standard way for fixed bosons masses $\mu_1 = m_1^2$ and $\mu_2 = m_2^2$ (SPA).

![Feynman diagrams](image)

**Fig. 1.** Feynman diagrams for the process $e^+e^- \rightarrow W^+W^-$. 

The Born cross-section is defined by the sum of two diagrams shown in Fig. 1 and can be represented as

$$\sigma_{WW}^B (s; x_1, x_2) = \frac{\pi \alpha^2}{128 s \sin^2 \theta_W} F(s; x_1, x_2),$$
where dimensionless function $F(s; x_1, x_2)$ is defined by the expression

$$F(s; x_1, x_2) = \frac{16}{3(a^2 - b^2)(1 - x_2)^2} \{(3a^2 - b^2)(a^2 - b^2 + 2(1 + a))(1 - x_2)^2 L(a, b)$$

$$+ x_Z \cos(2\theta_W)(3b^2 - 2ab^2(2 + a) + a^3(4 + a))(1 - x_Z)L(a, b)$$

$$+ 2\lambda(a, b)(2b^2 - 3a^2 - 10a - 1)(b^2(1 - 2x_Z) - a(1 - 3x_Z) - x_Z)\}$$

$$+ \lambda(a, b)[x_Z^2\lambda^2(a, b)\cos(4\theta_W)(2b^2 - 3a^2 - 10a - 1) + 12a^3x_Z^2$$

$$- a^2(3b^2(3x_Z^2 - 2x_Z + 1) - 49x_Z^2 + 30x_Z - 15) - 2a(b^2(19x_Z^2 - 10x_Z + 5)$$

$$+ 8x_Z^2) + 2b^4(3x_Z^2 - 2x_Z + 1) - 2b^2(7x_Z^2 - 16x_Z + 8) - 2x_Z^2]\}. \quad (14)$$

In Eq. (14) the dimensionless variables $a, b, x_1, x_2, x_Z$ and the functions $L(a, b)$ and $\lambda(a, b)$ are defined as follows

$$L(a, b) = \ln \left[ \frac{1 - a - \lambda(a, b)}{1 - a + \lambda(a, b)} \right], \quad \lambda(a, b) = \sqrt{1 - 2a + b^2},$$

$$x_{1,2} = \frac{x_{1,2}}{s}, \quad a = x_1 + x_2, \quad b = x_1 - x_2, \quad x_Z = \frac{M_Z^2}{s}. \quad (15)$$

With the help of the expressions (13)-(15) the model Born cross-section is represented in the following convolution form

$$\sigma_{WW}^B(s) = \frac{\pi a^2}{128s \sin^4 \theta_W} \int_0^1 dx_1 \rho(x_1, s) \int_0^{(1 - \sqrt{\pi})^2} \rho(x_2, s) F(s; x_1, x_2) dx_2, \quad (16)$$

where in analogy with the case of $Z$-pair production [2] we use the redefined dimensionless probability density of Lorentzian type

$$\rho(x, s) = \frac{1}{\pi} \frac{G(x, s)}{(x - x_W)^2 + G^2(x, s)}, \quad G(x, s) = \sqrt{\frac{1}{s}} \frac{M_W^2}{s} = \frac{3ax}{4 \sin^2 \theta_W}. \quad (17)$$

where $x = \mu/s, x_W = M_W^2/s$. The expression (16) turns into the standard expression for the on-shell cross-section $\sigma_{WW}^B(s)$, when $\rho(\mu) \rightarrow \delta(\mu - M_W^2)$, i.e. in the limit of fixed $W$ mass.

In Fig. 2 we represent the Born cross-section in the fixed-mass approach (solid line) and in the smeared-mass approach (dashed line). Besides, in this figure we show the experimental LEP2 data in order to illustrate the necessity of radiative corrections. One can see that the model approach leads to the smearing of the threshold, that is to the result which is similar to the standard one with an account of the Finite-Width Effects (FWE). In other words, the model description of boson-pair production is in close analogy with the standard description of the off-shell boson-pair production. Moreover, the model convolution representation of the cross-section in the form (12) or (16) is formally similar to the SAA [2] [21]. However, the SAA is constructed as an approximation for the exclusive process like $e^+e^- \rightarrow W^+W^- \rightarrow f_1 f_2 f_3 f_4$, which then is generalized to the inclusive process with full set of the 4f final states. This approach is based on the approximate factorization of the total cross-section $\sigma(e^+e^- \rightarrow f_1 f_2 f_3 f_4) \longrightarrow \sigma(e^+e^- \rightarrow W^+W^-) Br(W^+ \rightarrow f_1 f_2) Br(W^- \rightarrow f_3 f_4)$. In the frame of the model [40] [46], the factorization is exact.
(see also Refs. [45, 47]), and the expression (12) can be directly derived [47] for the inclusive process \(e^+e^- \rightarrow \sum_f 4f\) in the double-pole approach without any approximations.

From Fig. 2 it follows that the use of the effective model fields, which describe the UP with an account of the self-energy type corrections, is not sufficient. We have to take into account the rest radiative corrections for the realistic description of the measured cross-section.

Now we estimate the uncertainties of the model calculations at the effective tree level, which are caused mainly by the definition of the function \(\rho(\mu)\). As was shown in the framework of the effective theory of UP [45]-[47], this function results from the factorization of full process of production and decay of UP. Let us define the uncertainty as a deviation of the model calculation from the standard one. The part of the model amplitude which describes the decays \(W \rightarrow l\nu_l\)

\[
M^{\text{mod}} \sim \frac{(-g^{\mu\nu} + \frac{q_1^\mu q_1^\nu}{q_1^2})}{P(q_1^2)} \frac{(-g^{\mu'\nu'} + \frac{q_2^\mu q_2^\nu'}{q_2^2})}{P(q_2^2)} \bar{\nu}_1 \gamma_\nu (1 - \gamma_5) \nu_1 \cdot \bar{\nu}_2 \gamma_{\nu'} (1 - \gamma_5) l_2.
\]

The standard expression for the amplitude follows from Eq. (18) after the change \(q_2^2 \rightarrow M_W^2\) in the numerator of the dressed propagators of unstable bosons \(W\). We assume that the denominators \(P(q_2^2)\) in both cases are the same (in the Breit-Wigner or complex pole form). As was mentioned above, the use of the standard propagators does not lead to the exact factorization even at the tree level. In the standard approach this effect takes place in the Narrow-Width Approximation (NWA), while in the framework of the model under consideration the factorization is exact due to
specific form of propagator’s numerator \cite{47}. Using the equality

$$\bar{q}_a (1 - \gamma_5) \nu_a = m_a (1 - \gamma_5) \nu_a,$$

where \( a = 1, 2 \), we get

$$|M^{\text{mod}}|^2 \sim 1 + 2 \frac{m_1}{q_1} + 2 \frac{m_2}{q_2},$$

(20)

Where \( q = \sqrt{(q \cdot q)} \). The same expression takes place for the standard amplitude squared \( |M^{\text{st}}|^2 \) after the change \( q_a \to M_W \). As a result, we have the relative deviation of the model partial cross-section from the standard one \( (m_1 = m_2 = m_f) \):

$$\epsilon_f \sim 4 \frac{m_f}{M_W} [1 - M_W \int_{m_f^2}^{\infty} \frac{\rho(q^2)}{q^2} dq^2].$$

(21)

From (21) with the help of the Breit-Wigner approximation for the function \( \rho(q^2) \) we find that the maximal deviation is for heavy fermions, for instance, for the \( \tau \)-lepton pair and \( b, c \)-quark pairs. However, in the last case the \( \epsilon \) is suppressed by small CKM elements \( |U_{cb}|^2 \). Thus, from (21) it follows that \( \epsilon_{\text{max}} = \epsilon_{\tau} \sim 10^{-3} \), that is the uncertainty of our approach is an order of 0.1%. From this simple analysis we make a conclusion that an error, which is caused by the model approach at tree level, noticeably less than 1%. So, the main uncertainty can be caused only by the implantation of the radiative corrections into our scheme of calculation (see the next section).

3. The model cross-section of \( W \)-pair production with radiative corrections

In this section we discuss the strategy of the RC’s accounting and represent the final results of calculations. As it was shown in Refs. \cite{45, 46}, the model description of UP is equivalent to some effective theory of UP, which includes the self-energy type RC’s in all orders of perturbation theory. Moreover, the UP is the non-perturbative object in the vicinity of the resonance. So, the traditional program of RC’s calculation is not valid in the framework of the model. We have no well defined set of the diagrams which is gauge invariant and renormalized. The model of UP \cite{40, 46} is effective and not gauge one, and we have no any rigid criteria for definition of such a set. So, we keep the strategy which is based on the simple phenomenology and was successfully applied in the case of \( Z \)-pair production \cite{42}.

We do not take into account any corrections to the final states \( W \), because of the effective nature of these states in the framework of the model. We use the effective coupling \( \alpha(M_W) = 1/127.9 \) in the vertex with the final \( W \)-states and \( \alpha = 1/137 \) in the RC’s. So, the principal part of the vertex corrections is effectively included into the coupling, and the low-energy behavior of the bremsstrahlung and radiative corrections to the initial states is taken into consideration.

The set of corrections, caused by the final state interactions in the two \( s \)-channel diagrams in Fig. 1, is included into the effective coupling \( \alpha(M_W) \). The principal
part of the so-called Coulomb singularity contributions, which were considered in Refs. [1], [27] and [31], can be also absorbed by the effective coupling. The one-loop calculation shows that this correction gives from 5.7% at the threshold to 1.8% at 190 GeV [1], while the total change of the effective coupling $\alpha(M_W)$ with respect to $\alpha$ is near 7%. In the calculation we explicitly take into account the $O(\alpha)$ corrections including soft and hard bremsstrahlung, which are not described by the model and by the effective coupling. The real and virtual electromagnetic radiation should enter into the set of these RC’s and mutually compensates the total IR divergences.

The program of RC’s calculations, which is similar to above discussed one, was fulfilled in the series of papers (see, for example, Ref. [11] and references therein) for the case of the on-shell $W$-pair production (the limit of fixed masses $\mu_1 = \mu_2 = M_W^2$). The analytical expression for these corrections is represented in compact and convenient form in Ref. [11]. We generalized this expression to the case of smeared-shell $W$-pair production, that is for arbitrary values of mass parameters $\mu_k$, and applied it in our calculations. As a result, we get the cross-section $\sigma_{WW}(s;\mu_1,\mu_2)$ for the case of $W(\mu_1)$ and $W(\mu_2)$ production including above described corrections in the following form (see also Ref. [11])

$$\sigma_{WW}(s;\mu_1,\mu_2) = \int_{k_{\text{max}}}^{k_{\text{max}}} \rho_\gamma(k) \sigma_{WW}^0(s(1-k),\mu_1,\mu_2) \, dk,$$

where $\rho_\gamma(k)$ is the photon radiation spectrum [49], $k = E_\gamma/E_0$ is the photon energy in units of beam energy and $s(1-k)$ is the effective $s$ available for the $W$-pair production after the photon has been emitted [11]. In the case of the on-shell $W$-pair production ($\mu_1 = \mu_2 = M_W^2$) the value $k_{\text{max}} = 1 - 4M_W^2/s$ is the maximal part of photon energy. The generalization of this value to the case $\mu_1 \neq \mu_2$ leads to

$$k_{\text{max}} = 1 - 2\frac{\mu_1 + \mu_2}{s} + \frac{(\mu_1 - \mu_2)^2}{s^2} = \lambda^2(\mu_1,\mu_2; s).$$

The photon distribution function is written in the form [11]

$$\rho_\gamma(k) = \beta k^{\beta-1}(1 + \delta_{1}^{v+s} + \ldots) + \delta_{1}^{h} + \ldots,$$

where we keep $O(\alpha)$ corrections only (i.e. $\delta_{n>1} = 0$). The corresponding corrections are given by ($v + s = \text{virtual+soft}, h = \text{hard}$) [11]:

$$\beta = \frac{2 \alpha}{\pi}(L-1), \quad L = \ln \frac{s}{m_e^2}, \quad \alpha = \frac{1}{137};$$

$$\delta_{1}^{v+s} = \frac{\alpha}{\pi} \left( \frac{3}{2} L + \frac{\pi^2}{3} - 2 \right), \quad \delta_{1}^{h} = \frac{\alpha}{\pi} (1 - L)(2 - k).$$

Finally, we get the corrected expression for the cross-section of $W$-pair production in the form

$$\sigma_{WW}(s) = \frac{\pi \alpha^2(M_W)k_{\text{QCD}}}{128 s \sin^4 \theta_W(M_W)} \int_0^1 dx_1 \int_0^{(1-\sqrt{x_1})^2} dx_2 \int_0^{\lambda^2(x_1,x_2)} \frac{dk}{1-k} \rho_\gamma(k)$$

$$\rho(x_1, s(1-k)) \rho(x_2, s(1-k)) F(s(1-k); x_1, x_2).$$

(26)
where the functions $F, \rho, \rho_\gamma$ were defined before in Eqs. (14), (17), (24) and we also take into account the effective QCD correction factor $k_{QCD} = 1 + 0.133/\pi$. 

The model cross-section $\sigma_{WW}(s)$ was calculated numerically and represented in Fig. 3 as a function of $s$ by dashed line. The results of MC simulations, RacconWW [32, 33] and YFSWW [34, 35], are represented for comparison by two barely distinguishable solid lines, and the experimental LEP II data [53] are given with the corresponding error bars. From Fig. 3, one can see that the model cross-section with RC’s is in good agreement with the experimental data. Moreover, the deviation of the model from MC curves is significantly less than the experimental errors ($\lesssim 1\%$).

In the previous section, we have got an estimation of the tree level uncertainty which turned out to be significantly less than the value of radiative corrections. So, the total uncertainty of the model approach mainly depends on the set of the RC’s which we take into account. From the above described strategy of the RC’s accounting, it follows that the principal value of the error can be caused by the part of the Coulomb corrections, which follows from the box diagram [1] and gives most likely less than 1%, and by the non-factorable corrections, which destroy the convolution structure of the total cross-section. A comparison between the DPA and the predictions based on the full $O(\alpha)$ corrections reveals differences in the relative corrections $\lesssim 0.5\%$ [27] and 0.9% [54].

From the results of our calculations it follows, that the model approach provides the accuracy which is sufficient for the LEP II data description in the near-threshold energy range. Besides, the contribution of the non-factorable corrections in the
cross section is less than the experimental errors. However, the value of all non-considered corrections can be maximally an order of LEP II uncertainty of the total cross-section, and this point needs an additional consideration. Moreover, the actual status of the calculations concerns rather the testing of our approach than the tool for precise investigations. But, we believe that the approach due its simplicity and physical transparency can provide the basis for construction of such a tool.

4. Conclusions

A large number of the first order Feynman diagrams, which contribute to the production of four fermions in $e^+e^-$ interactions, depends on the specific final states. So, the detailed classification was suggested for the description of these processes. The most important request for $WW$ physics concerns the $O(\alpha)$ radiative corrections in the DPA. Inclusion of the complete EW corrections significantly complicates the calculations which became not available in the analytical form in the case of the full set of $4f$-processes. So, the various approximation schemes have been worked out together with development of the MC simulations.

In this paper, we applied the model of UP with smeared mass for description of the $W$-pair production. The model describes the process $e^+e^- \rightarrow W^+W^-$ as $W$-pair production, where $W$'s are on the smeared mass-shell. This approach is similar to the standard description of the off-shell $W$-pair production in SAA. We have taken into account the soft and hard initial state radiation and a part of the virtual radiative corrections which are relevant in the framework of the model.

It follows from our results that the model is applicable to description of the near-threshold boson-pair production with LEP II accuracy. We get the total cross-section which is in good accordance with the experimental data; it coincides with the MC calculations with a high precision. At the same time, the model provides a compact analytical expression for the cross-section in terms of convolution of the Born cross section with probability densities (or mass distributions) of $W$ bosons. However, we did not fulfill the detailed analysis of an accounting of the EW corrections, so this rather phenomenological formalism can not be directly applied for the precise description of the boson-pair production at high energies and for future experiments at ILC. It is reasonable to consider the possibility of improvement of the approach and its applicability at the energies far from the near-threshold range. We leave this analysis for a separate study.

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