ON THE JORDAN-MOORE-GIBSON-THOMPSON EQUATION: WELL-POSEDNESS WITH QUADRATIC GRADIENT NONLINEARITY AND SINGULAR LIMIT FOR VANISHING RELAXATION TIME

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Abstract. In this paper, we consider the Jordan-Moore-Gibson-Thompson equation, a third order in time wave equation describing the nonlinear propagation of sound that avoids the infinite signal speed paradox of classical second order in time strongly damped models of nonlinear acoustics, such as the Westervelt and the Kuznetsov equation. We show well-posedness in an acoustic velocity potential formulation with and without gradient nonlinearity, corresponding to the Kuznetsov and the Westervelt nonlinearities, respectively. Moreover, we consider the limit as the parameter of the third order time derivative that plays the role of a relaxation time tends to zero, which again leads to the classical Kuznetsov and Westervelt models. To this end, we establish appropriate energy estimates for the linearized equations and employ fixed-point arguments for well-posedness of the nonlinear equations. The theoretical results are illustrated by numerical experiments.

Key words. nonlinear acoustics, energy estimates, singular limit

AMS subject classifications. 35L72, 35L77, 35L80, 35B40, 49K20, 49Q10

1. Introduction. Nonlinear propagation of sound arises in numerous applications. We here especially mention high-intensity ultrasound used in medical imaging and therapy, but also for industrial purposes, such as ultrasound cleaning or welding; see, e.g., [1, 6, 21] and the references therein. For the physical fundamentals of nonlinear acoustics, we refer to, e.g., [3, 7, 12, 22, 28, 30, 29, 38].

Its physical and mathematical description involves the acoustic particle velocity \( \vec{v} \), the acoustic pressure \( p \), as well as the mass density \( \varrho \), which can be decomposed into constant and fluctuating components

\[
\vec{v} = \vec{v}_0 + \vec{v}_\sim, \quad p = p_0 + p_\sim, \quad \varrho = \varrho_0 + \varrho_\sim,
\]

where in the applications mentioned above, the ambient flow vanishes; i.e., \( \vec{v}_0 = 0 \). Furthermore, we have the balances of mass, momentum and sometimes of energy, complemented with an equation of state that relates the mass density to the pressure. Combination of these balance and material laws yields wave-type partial differential equations that are often second order in space and time, but also higher order in time equations play an important role. It is one of these third order in time equations that we focus on in this paper.

One of the most established models of nonlinear acoustics is Kuznetsov’s equation [22, 26]

\[
p_{\sim tt} - c^2 \Delta p_\sim - \delta \Delta p_{\sim t} = \left( \frac{1}{\varrho_0} \frac{B}{2 A} p_\sim^2 + \varrho_0 |\vec{v}_\sim|^2 \right)_{tt},
\]

where \( c \) is the speed of sound, \( \delta \) is the diffusivity of sound

\[
\delta = \frac{1}{\varrho_0} \left( \frac{4 \mu V}{3} + \zeta V \right) + \frac{\kappa}{\varrho_0} \left( \frac{1}{c_V} - \frac{1}{c_p} \right),
\]

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and the velocity is related to the pressure via some balance of forces,

\( g_0 \vec{v}_t = -\nabla p \).

By ignoring local nonlinear effects modeled by the quadratic velocity term, we arrive at the Westervelt equation

\[
\rho_0 \vec{v}_t = -\nabla p, \quad p \sim \partial_{tt} p - c^2 \Delta p - \delta \Delta p_t = \frac{\beta_a}{\rho_0 c^2} p_{\sim tt},
\]

with \( \beta_a = 1 + B/(2A) \); cf. [43]. In terms of the acoustic velocity potential \( \psi \) satisfying \( \vec{v} = -\nabla \psi \) and \( p = \rho_0 \psi_t \), these equations can be rewritten as

\[
\psi_{tt} - c^2 \Delta \psi - \delta \Delta \psi_t = \left( \frac{1}{c^2} \frac{B}{2A} (\psi_t)^2 + |\nabla \psi|^2 \right) t,
\]

and

\[
\psi_{tt} - c^2 \Delta \psi - \delta \Delta \psi_t = \left( \frac{\beta_a}{c^2} (\psi_t)^2 \right) t,
\]

respectively.

As has been observed, e.g., in [14] (see also the references therein), the use of classical Fourier’s law leads to an infinite signal speed paradox, which appears to be unnatural in wave propagation. Therefore in [14], several other constitutive relations for the heat flux within the derivation of nonlinear acoustic wave equations are considered. Among these is the Maxwell-Cattaneo law, whose combination with the above mentioned balance equations and the equation of state leads to the third order in time PDE model:

\[
\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = f,
\]

where \( \tau \) is a positive constant accounting for relaxation (the relaxation time), and

\[
b = \delta + \tau c^2.
\]

If one neglects local nonlinear effects modeled by the quadratic velocity term in (1.6), one arrives at

\[
\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \left( \frac{\beta_a}{c^2} (\psi_t)^2 \right) t,
\]

analogously to the reduction of the Kuznetsov to the Westervelt equation; cf. [18].

Obviously, (1.6) and (1.8) formally reduce to (1.4) and (1.5) upon setting \( \tau = 0 \). The present work is in part devoted to the justification of the limit \( \tau \to 0 \) in (1.6) and (1.8).

In [17] and much more comprehensively in [31], as well as later on in [5, 37], the linearized equation

\[
\tau \psi_{ttt} + \alpha \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = f,
\]

where
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also called the Moore-Gibson-Thompson equation, is studied using semigroup techniques. It turns out that exponential stability of the trajectories depends on the critical parameter

\begin{equation}
\gamma := \alpha - \frac{\tau c^2}{b}.
\end{equation}

In the case of a constant coefficient \(\alpha\), exponential decay of the energy function

\begin{equation}
E[\psi](t) = \frac{1}{2} \left\{ |\psi_t(t)|^2 + |\nabla \psi(t)|^2 + |\psi_{tt}(t)|^2 + |\psi_{ttt}(t)|^2 + |\Delta \psi(t)|^2 \right\},
\end{equation}

requires \(\gamma\) to be strictly positive, whereas the case \(\gamma < 0\) is unstable and the case \(\gamma = 0\) marginally stable. An intuitive explanation for this phenomenon is the following:

According to the linear wave part of the equation, we can trade \(\alpha \psi_{tt}\) for \(c^2 \Delta \psi\), thus also \(\tau \psi_{ttt}\) for \(\tau c^2 \alpha \Delta \psi \) to relate (1.9) back to the linearization of (1.5)

\[\alpha \psi_{tt} - c^2 \Delta \psi - \frac{b}{\alpha} \gamma \Delta \psi_t = f,\]

which is a strongly damped wave equation.

The Moore-Gibson-Thompson equation (1.9) has also been studied in [27], where the problem of identifying \(\gamma(t)\) from boundary measurements has been considered, and in [24, 23], where the effect of additive convolution memory terms acting on \(\Delta u\), \(\Delta u_t\) and their combination, respectively, have been investigated.

For the nonlinear equation (1.8), more precisely for a reformulation of the equation in terms of the acoustic pressure \(p = \varrho_0 \psi_t\), in [18] global in time well-posedness and exponential decay of the energy \(E[p](t)\) defined in (1.11) has been proven for small initial data \((p_0, p_1, p_2) \in H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)\).

One of the key elements in the above cited papers on the analysis of (1.8) and (1.9) is introduction of the auxiliary state

\begin{equation}
z := \psi + \frac{c^2}{b} \psi.
\end{equation}

Indeed, the third order in time equation (1.9) becomes a linear (weakly) damped wave equation for \(z\)

\begin{equation}
\tau z_{tt} + \gamma z_t - b \Delta z - \gamma \frac{c^2}{b} z + \gamma \frac{c^4}{b^2} \psi = f,
\end{equation}

where \(-\gamma \frac{c^2}{b} z + \gamma \frac{c^4}{b^2} \psi\) is a lower order term. This, first of all, illustrates the fact that \(\gamma\) should be nonnegative in order to guarantee a damping behaviour of the term \(\gamma z_t\), and secondly, the key difference to the strongly damped second order equations (1.4), (1.5). As pointed out in [31, Subsection 6.2.1] (see also [17, Remark 1.3]) (1.9) does not give rise to an analytic semigroup. Consequently the operator driving the evolution does not exhibit maximal parabolic regularity [25] and the Implicit Function Theorem argument from [33] cannot be transferred to the present situation.

2. Main results. This paper contributes to the analysis of the JMGT equation in two ways. Firstly, we prove well-posedness with a quadratic gradient nonlinearity arising when taking into account local nonlinear effects (cf. the additional \(|\nabla \psi|^2\) term on the right hand side in (1.6) as compared to (1.8)). Since maximal parabolic
regularity and Implicit Function Theorem arguments do not apply here, we base our arguments on energy estimates for a fixed-point reformulation

\[ \tau \psi_{ttt} + (1 - k\psi_t)\psi_{tt} - c^2 \Delta \psi - b\Delta \psi_t = 2\nabla \psi \cdot \nabla \psi_t, \]

of \((1.6)\) where we use the abbreviation \(k = \frac{2B}{c^2 2A}\), whose sign will not matter, whereas, in what follows we will always assume the coefficients \(b\) and \(c^2\) to be strictly positive. Moreover, we rely on the formulation in terms of the acoustic velocity potential \(\psi\) which more easily than the acoustic pressure formulation from [18] allows to include the quadratic velocity term \((|\nabla \psi|^2)_t\) on the right hand side of \((1.6)\). The energy estimates required for this purpose differ from those provided in [18] for the equation \((1.8)\) without this quadratic gradient nonlinearity.

Secondly, we consider the limit \(\tau \to 0\) and prove that solutions of \((1.6)\) converge to a solution of \((1.4)\) as \(\tau \to 0\). Also for this purpose, the previously derived energy estimates are crucial. These estimates differ for the Westervelt-type version \((1.8)\) and for the Kuznetsov-type version \((1.6)\) of the JMGT equation, which is why we treat them in separate sections.

The paper is organized as follows. In Section 3, we consider \((1.9)\) with a fixed positive coefficient \(\alpha\) (possibly space and time dependent), bounded away from zero, and an inhomogeneity \(f\), as well as a fixed positive \(\tau\). We prove well-posedness of this linearized model together with an energy estimate. Section 4 contains a well-posedness proof of the Westervelt version \((1.8)\) of the equation by setting \(\alpha = 1 - k\psi_t\) and \(f = 0\). The proof is based on the fixed-point equation \((2.1)\), but with zero right hand side, as the gradient nonlinearity is not present in \((1.8)\). This fact allows to prove local in time well-posedness for small initial data, even without any sign condition on \(\gamma\); cf.\((1.10)\). However, the energy estimates from Section 3 do not cover the gradient nonlinearity in the Kuznetsov type version \((1.6)\), so that higher order energy estimates are needed. We derive them in Section 5. These involve the auxiliary function \(z\) according to \((1.12)\) and require positivity of both \(\alpha\) and \(\gamma = \alpha - \frac{c^2}{b}\), where the latter follows from positivity and boundedness away from zero of \(\alpha\) for \(\tau\) sufficiently small. Section 6 provides the corresponding well-posedness result for the equation \((1.6)\) based on \((2.1)\), i.e., setting \(\alpha = 1 - k\psi_t\) and \(f = 2\nabla \psi \cdot \nabla \psi_t\). Starting from a sufficiently small positive value and letting \(\tau\) tend to zero clearly preserves the sign structure of the coefficients, in particular of \(\gamma\) and \(b\), cf. \((1.10)\), \((1.7)\), so that the energy bounds from Sections 4 and 6 can be used for justifying the limiting process \(\tau \to 0\) in Section 7. There we also provide a brief comparison of the resulting regularity for the limiting equations \((1.4)\), \((1.5)\) to those from [16, 33, 34] and [15, 32], respectively. Finally, numerical experiments in Section 8 illustrate the theoretical findings.

2.1. Theoretical preliminaries. We set here the notation and collect some useful theoretical results that we often use in the analysis.

Throughout the paper, we consider the PDEs on a bounded space time cylinder \(\Omega \times (0, T)\) and impose homogeneous Dirichlet boundary conditions on \(\partial \Omega\) for simplicity

\[ \psi = 0 \quad \text{on} \quad \partial \Omega \times (0, T); \]

i.e., \((-\Delta) : H^1_0(\Omega) \to H^{-1}(\Omega)\) is the Laplace operator equipped with homogeneous Dirichlet boundary conditions. We expect that Neumann and impedance (absorbing) boundary conditions can be treated analogously, but lead to modifications in the energy estimates.
The third order in time equations (1.6) and (1.8) are complemented with initial conditions

\[ \psi(0) = \psi_0, \quad \psi_t(0) = \psi_1, \quad \psi_{tt}(0) = \psi_2, \]

whereas for the limiting second order in time equations (1.4), (1.5) as \( \tau \to 0 \), the initial condition on \( \psi_{tt} \) naturally disappears, which also be seen in the energy estimates.

The spatial domain \( \Omega \subset \mathbb{R}^d \), where \( d \in \{1, 2, 3\} \), is assumed to be sufficiently smooth to admit integration by parts as well as second order elliptic regularity.

### 2.1.1. Notation.

For simplicity of notation, we often omit the time interval and the spatial domain when writing norms, i.e., \( \| \cdot \|_{L^p} \) denotes the norm on \( L^p(0, T; L^q(\Omega)) \). We also abbreviate the \( L^2(\Omega) \) inner product by \( (\cdot, \cdot)_{L^2} \) and the \( L^2(\Omega) \) norm (as well as the absolute value) by \( | \cdot | \).

#### 2.1.2. Helpful inequalities.

We often employ Young’s \( \varepsilon \)-inequality

\[ xy \leq \frac{\varepsilon}{2} x^2 + \frac{1}{2\varepsilon} y^2, \quad \text{where } x, y > 0, \ \varepsilon > 0; \]

see [8, Appendix B]. Furthermore, we employ the continuous embeddings \( H^1(\Omega) \hookrightarrow L^6(\Omega), \ H^2(\Omega) \hookrightarrow L^\infty(\Omega) \)

\[ \|v\|_{L^6(\Omega)} \leq C_{H^1, L^6}^\Omega \|v\|_{H^1(\Omega)} , \quad \|v\|_{L^\infty(\Omega)} \leq C_{H^2, L^\infty}^\Omega \|v\|_{H^2(\Omega)} \]

as well as boundedness of \((-\Delta)^{-1} : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega) \) and the Poincaré-Friedrichs inequality,

\[ \|v\|_{H^2(\Omega)} \leq C_{(-\Delta)^{-1}} \|v\|_{L^2(\Omega)} , \quad \|v\|_{H^1(\Omega)} \leq C_{PF}^\Omega \|\nabla v\|_{L^2(\Omega)}, \]

and the trace theorem

\[ \|\nu \cdot \nabla v\|_{H^{-1/2}(\partial \Omega)} \leq C_{tr}^\Omega \|v\|_{H^1(\Omega)} , \quad \|v\|_{H^{1/2}(\partial \Omega)} \leq C_{tr}^\Omega \|v\|_{H^1(\Omega)} \]

#### 3. Analysis of the linear damped wave equation (1.13).

We now turn our attention again to the equation (1.9) which we restate here for convenience:

\[ \tau \psi_{tt} + \alpha \psi_t - c^2 \Delta \psi - b \Delta \psi_t = f \text{ in } \Omega \times (0, T). \]

We assume that for some \( \underline{\alpha} > 0 \),

\[ \alpha(t) \geq \underline{\alpha} \text{ on } \Omega \text{ for a.e. } t \in (0, T), \]

as well as that the regularity

\[ \alpha \in X^W_\alpha := L^\infty(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; W^{1,3}(\Omega)), \]

\[ f \in H^1(0, T; L^2(\Omega)) \]

holds. Moreover, \( \psi \) is assumed to satisfy the initial conditions (2.3) with

\[ \psi_0 \in H^1_0(\Omega) \cap H^2(\Omega) , \quad \psi_1 \in H^1_0(\Omega) \cap H^2(\Omega) , \quad \psi_2 \in H^1_0(\Omega). \]

For an analysis of (1.9) with constant \( \alpha \), under the assumptions

\[ (\psi_0, \psi_1, \psi_2) \in (H^1_0(\Omega) \cap H^2(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega) \]

and \( f \in L^1(0, T; L^2(\Omega)) \), we refer to [17, Corollary 1.2]. There it was shown that \( \psi \in C^0(0, T; H^1_0(\Omega) \cap H^2(\Omega)) \cap C^1(0, T; H^1_0(\Omega)) \cap C^2(0, T; L^2(\Omega)) \) by means of semigroup techniques.
Theorem 3.1. Let $c^2, b, \tau > 0, k \in \mathbb{R}$, and let $T > 0$. Assume that
- $\alpha \in L^\infty_\text{loc}((0,T); L^\infty(\Omega)) \cap L^\infty(0,T; H^1_0(\Omega))$,
- $\exists \alpha > 0 : \alpha(t) \geq \alpha \text{ a.e. } \Omega \times (0,T)$,
- $f \in H^1(0,T; L^2(\Omega))$,
- $(\psi_0, \psi_1, \psi_2) \in X^W := H^1_0(\Omega) \cap H^2(\Omega)$.

Then there exists a unique weak solution $\psi$ of the problem

$$
\begin{cases}
   \tau \psi_{tt} + \alpha(x,t)\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = f & \text{in } \Omega \times (0,T), \\
   \psi = 0 & \text{on } \partial \Omega \times (0,T), \\
   (\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) & \text{in } \Omega \times \{0\},
\end{cases}
$$

(3.3)

that satisfies

$$
\psi \in X^W := W^{1,\infty}(0,T; H^1_0(\Omega) \cap H^2(\Omega)) \cap W^{2,\infty}(0,T; H^1_0(\Omega)) \cap H^3(0,T; L^2(\Omega)).
$$

Furthermore, the solution fulfills the estimate

$$
\begin{multline}
\tau^2 \|\psi_{tt}\|^2_{L^2 L^2} + \tau \|\psi_{tt}\|^2_{L^\infty H^1} + \|\psi_{tt}\|^2_{L^2 H^1} + \|\psi_t\|^2_{L^\infty H^2} \\
\leq C(\alpha, T, \tau) \left( |\psi_1|^2_{H^2} + |\psi_2|^2_{H^1} + \tau |\psi_2|^2_{H^1} + \|f\|^2_{L^\infty L^2} + \|f_t\|^2_{L^2 L^2} \right).
\end{multline}

(3.5)

The constant above is given by

$$
C(\alpha, T, \tau) = C_1 \left( 1 + T^3 + \|\alpha\|_{L^\infty L^\infty} \right) \exp \left( C_2 \left( \frac{1}{\tau} \|\nabla \alpha\|^2_{L^\infty L^3} + 1 + T \right) T \right),
$$

where $C_1, C_2 > 0$ do not depend on $\tau, T$, and $\alpha$.

If additionally

$$
\|\nabla \alpha\|_{L^\infty L^3} < \frac{\alpha}{C_{H^1,L^6}}
$$

(3.6)

holds, then (3.5) is valid with an upper bound that is independent of $\tau$, i.e.,

$$
C(\alpha, T, \tau) = C(\alpha, T) = C_1 \left( 1 + T^3 + \|\alpha\|_{L^\infty L^\infty} \right) \exp \left( C_2 \left( 1 + T \right) T \right).
$$

(3.7)

Proof. We carry out the proof by via Galerkin approximations in space, relying on certain energy estimates; cf. [8, 40]. Note that the initial data are meaningful since (3.4) implies

$$
\psi \in C([0,T]; H^1_0(\Omega) \cap H^2(\Omega)), \\
\psi_t \in C_w([0,T]; H^1_0(\Omega) \cap H^2(\Omega)), \\
\psi_{tt} \in C_w([0,T]; H^1_0(\Omega)),
$$

where $C_w$ denotes the space of weakly continuous functions; see [41, Lemma 3.3].

Step 1: Discretization in space. Let $\{w_i\}_{i \in \mathbb{N}}$ denote the eigenfunctions of the Dirichlet-Laplacian operator $-\Delta$. Then $\{w_i\}_{i \in \mathbb{N}}$ can be normalized to form an orthogonal basis of $H^1_0(\Omega) \cap H^2(\Omega)$ and to be orthonormal with respect to the $L^2(\Omega)$ scalar product.
Fix \( n \in \mathbb{N} \) and denote \( V_n = \text{span}\{w_1, \ldots, w_n\} \). We seek an approximate solution in the form of

\[
\psi^n = \sum_{i=1}^{n} \xi_i(t)w_i(x),
\]

where \( \xi_i : (0, T) \to \mathbb{R}, i \in [1, n] \). The initial data are chosen as

\[
\psi^n_0(x) = \sum_{i=1}^{n} \xi_{i,0} w_i(x), \quad \psi^n_1(x) = \sum_{i=1}^{n} \xi_{i,1} w_i(x), \quad \psi^n_2(x) = \sum_{i=1}^{n} \xi_{i,2} w_i(x),
\]

where the coefficients \( \xi_{i,0}, \xi_{i,1}, \xi_{i,2} \in \mathbb{R} \) are given by

\[
\xi_{i,0} = (\psi_0, w_i)_{L^2}, \quad \xi_{i,1} = (\psi_1, w_i)_{L^2}, \quad \xi_{i,2} = (\psi_2, w_i)_{L^2},
\]

for \( i \in [1, n] \). In this way, we have by construction that

\[
\begin{align*}
\|\psi^n_0\|_{H^2} & \leq \|\psi_0\|_{H^2} \quad \text{and} \quad \psi^n_0 \to \psi_0 \text{ in } H^1_0 \cap H^2, \\
\|\psi^n_1\|_{H^2} & \leq \|\psi_1\|_{H^2} \quad \text{and} \quad \psi^n_1 \to \psi_1 \text{ in } H^1_0 \cap H^2, \\
\|\psi^n_2\|_{H^1} & \leq \|\psi_2\|_{H^1} \quad \text{and} \quad \psi^n_2 \to \psi_1 \text{ in } H^1_0;
\end{align*}
\]

see [39, Lemma 7.5]. We then consider the following approximation of our problem

\[
\begin{aligned}
&\left\{ \tau \psi^n_{ttt} + \alpha \psi^n_t - c^2 \Delta \psi^n - b \Delta \psi^n_t, \phi \right\}_{L^2} = (f, \phi)_{L^2}, \\
&\text{for every } \phi \in V_n \text{ pointwise a.e. in } (0, T), \\
&(\psi^n(0), \psi^n_t(0), \psi^n_{tt}(0)) = (\psi^n_0, \psi^n_1, \psi^n_2).
\end{aligned}
\]

We introduce matrices \( I^n = [I_{ij}], M^n = [M_{ij}], K^n = [K_{ij}], C^n = [C_{ij}] \), and vector \( F^n = [F_i] \), where

\[
\begin{align*}
I^n_{ij} & = (w_i, w_j)_{L^2} = \delta_{ij}, \quad M^n_{ij}(t) = (\alpha w_i, w_j)_{L^2}, \\
K^n_{ij} & = -c^2(\Delta w_i, w_j)_{L^2}, \quad D^n_{ij} = -b(\Delta w_i, w_j)_{L^2}, \\
F^n_i & = (f, w_i),
\end{align*}
\]

\( i, j \in [1, n] \). By setting \( \xi^n = [\xi_1 \ldots \xi_n]^T \), problem (3.10) can be rewritten as

\[
\begin{cases}
\tau I^n \xi^n_{ttt} + M^n \xi^n_t + D^n \xi^n + K^n \xi^n = F^n(t), \\
(\xi^n(0), \xi^n_t(0), \xi^n_{tt}(0)) = (\xi^n_0, \xi^n_1, \xi^n_2).
\end{cases}
\]

Existence of a solution \( \xi^n \in H^3(0, T_n) \) of (3.12) now follows from standard ODE theory; see, for example, [40, Chapter 1]. We, therefore, conclude that (3.10) has a solution \( \psi^n \in H^3(0, T_n; V_n) \). The upcoming energy estimate will allow us to extend the existence interval to \([0, T]\).

**Step 2: Energy estimates.** Our next goal is to obtain a bound for \( \psi^n \) that is uniform with respect to \( n \). To this end, we test our problem (3.10) with a suitable test function.
First estimate. Testing the first equation in (3.10) with $\phi = -\Delta \psi^n_{tt} \in V_n$ and integrating over $(0, t)$, $t \leq T_n$, yields the energy identity

$$
\begin{align*}
\frac{1}{2} \| \nabla \psi^n_{tt}(t) \|^2 + \frac{1}{2} | - \Delta \psi^n_{tt}(t) |^2 + c \| \nabla \psi^n_{tt} \|_{L^2}^2 \\
= \frac{1}{2} \| \nabla \psi^n_{tt}(0) \|^2 + \frac{1}{2} | - \Delta \psi^n_{tt}(0) |^2 - \int_0^t (\psi^n_{tt} \nabla \alpha, \nabla \psi^n_{tt})_{L^2} ds \\
= c^2 (-\Delta \psi^n_{tt}, -\Delta \psi^n_{tt})_{L^2} \bigg|_0^t + c^2 \int_0^t (-\Delta \psi^n_{tt}, -\Delta \psi^n_{tt})_{L^2} ds \\
+ (f, -\Delta \psi^n_{tt})_{L^2} \bigg|_0^t - \int_0^t (f_t, -\Delta \psi^n_{tt})_{L^2} ds,
\end{align*}
$$

(3.13)

where we have skipped the argument ($s$) under the time integral for notational simplicity and used the abbreviation $L^2_tL^2_n$ for $L^2(0, t; L^2(\Omega))$. To derive (3.13), we have used the following three identities:

$$(\alpha \psi^n_{tt}, -\Delta \psi^n_{tt})_{L^2} = (\alpha \nabla \psi^n_{tt}, \nabla \psi^n_{tt} \cdot \nabla)_{L^2} + (\psi^n_{tt} \nabla \alpha, \nabla \psi^n_{tt}),$$

and

$$c^2 \int_0^t (-\Delta \psi^n_{tt}, -\Delta \psi^n_{tt})_{L^2} ds = c^2 (-\Delta \psi^n_{tt}, -\Delta \psi^n_{tt})_{L^2} \bigg|_0^t - c^2 \int_0^t (-\Delta \psi^n_{tt}, -\Delta \psi^n_{tt})_{L^2} ds,$$

as well as

$$\int_0^t (f, -\Delta \psi^n_{tt})_{L^2} ds = (f, -\Delta \psi^n_{tt})_{L^2} \bigg|_0^t - \int_0^t (f_t, -\Delta \psi^n_{tt})_{L^2} ds.$$

We note that $f \in H^1(0, T; L^2(\Omega)) \hookrightarrow C([0, T]; L^2(\Omega))$. We next estimate $\text{rhs}_1(t)$ from above. We introduce here a constant that depends on the initial data to simplify the notation:

$$C_1(\psi_0, \psi_1, \psi_2; \tau) = \frac{1}{2} \| \nabla \psi^n_{tt}(0) \|_{L^2}^2 + \frac{b}{2} - \Delta \psi^n_{tt}(0) \bigg|_{L^2}^2 + |f(0)|_{L^2} - \Delta \psi^n_{tt}(0) \bigg|_{L^2}^2$$

$$+ c^2 | - \Delta \psi^n(0) |_{L^2} - | - \Delta \psi^n(0) |_{L^2}$$

$$= \frac{1}{2} \| \nabla \psi^n_{tt}(0) \|_{L^2}^2 + \frac{b}{2} - \Delta \psi^n_{tt} \bigg|_{L^2}^2 + |f(0)|_{L^2} - \Delta \psi^n_{tt} \bigg|_{L^2}^2 + c^2 | - \Delta \psi^n |_{L^2} - \Delta \psi^n \bigg|_{L^2}^2.$$

By applying Hölder’s inequality, we get

$$\text{rhs}_1(t) \leq C_1(\psi_0, \psi_1, \psi_2; \tau) + \| \nabla \alpha \|_{L^\infty} \| \psi^n_{tt} \|_{L^2} \| \nabla \psi^n_{tt} \|_{L^2}$$

$$+ c^2 | - \Delta \psi^n(t) |_{L^2} - | - \Delta \psi^n(t) |_{L^2} + c^2 | - \Delta \psi^n(t) |_{L^2}$$

$$+ |f|_{L^\infty} \| \psi^n_{tt} \|_{L^2} + \| f_t \|_{L^\infty} \| \psi^n_{tt} \|_{L^2}.$$

We further estimate the right-hand side with the help of Young’s $\varepsilon$-inequality (2.4)
with \( \varepsilon = \frac{1}{4} \) and \( \varepsilon = 1 \), respectively and the embedding results to obtain

\[
\text{rhs}_1(t) \leq C_1(\psi_0, \psi_1, \psi_2; \tau) + C_{H^1, L^6}^0 \| \nabla \alpha \|_{L^{\infty} L^3} \| \nabla \psi^n \|_{L^2 L^2}^2
\]

\[+ \frac{2c^4}{b} | - \Delta \psi^n(t) |^2_{L^2} + \frac{b}{8} | - \Delta \psi^n(t) |^2_{L^2} + \frac{1}{2} \| f \|_{L^2 L^2}^2 \]

\[+ \frac{2}{b} \| f \|_{L^2 L^2} + \frac{b}{8} - \Delta \psi^n(t) |^2_{L^2} + \frac{1}{2} \| f \|_{L^2 L^2}^2 + \frac{1}{2} \| - \Delta \psi^n \|_{L^2 L^2}^2,
\]

We can estimate the term \( \| - \Delta \psi^n(t) \|_{L^2} \) appearing on the right-hand side as follows

\[
\| - \Delta \psi^n \|_{L^\infty L^2} \leq \sqrt{t} \| - \Delta \psi^n \|_{L^2 L^2} + | - \Delta \psi_0 |_{L^2}.
\]

Altogether, we get

\[
\tau \| \nabla \psi^n \|_{L^\infty L^2} + \| \nabla \psi^n \|_{L^2 L^2} + \| - \Delta \psi^n \|_{L^\infty L^2} \leq C_1(\psi_0, \psi_1, \psi_2; \tau) + C_{H^1, L^6}^0 \| \nabla \alpha \|_{L^{\infty} L^3} \| \nabla \psi^n \|_{L^2 L^2}^2
\]

\[+ \frac{2c^4}{b} T \| - \Delta \psi^n \|_{L^2 L^2}^2 + \frac{2c^4}{b} T \| - \Delta \psi_0 \|_{L^2}^2 + \frac{1}{2} \| f \|_{L^2 L^2}^2 + \frac{1}{2} \| - \Delta \psi^n \|_{L^2 L^2}^2
\]

If (3.6) holds, then the term containing \( \| \nabla \alpha \|_{L^{\infty} L^3} \) can be absorbed into the left hand side.

**A priori bound for \( \psi^n \).** Applying Gronwall’s inequality to (3.16), and taking the supremum over \( t \in (0, T_n) \) then yields

\[
\tau \| \nabla \psi^n \|_{L^\infty L^2} + \| \nabla \psi^n \|_{L^2 L^2} + \| - \Delta \psi^n \|_{L^\infty L^2} \leq \tilde{C}(\alpha, T_n, \tau) \left( \| \psi_0 \|_{H^2} + \| \psi_1 \|_{H^2} + \| \psi_2 \|_{H^2} + \| f \|_{L^\infty L^2} + \| f \|_{L^2 L^2} \right).
\]

By employing the upper bounds for the approximate initial data stated in (3.9) and the inequality \( T_n \leq T \), we further have

\[
\tau \| \nabla \psi^n \|_{L^\infty L^2} + \| \nabla \psi^n \|_{L^2 L^2} + \| - \Delta \psi^n \|_{L^\infty L^2} \leq \tilde{C}(\alpha, T_n, \tau) \left( \| \psi_0 \|_{H^2} + \| \psi_1 \|_{H^2} + \| \psi_2 \|_{H^2} + \| f \|_{L^\infty L^2} + \| f \|_{L^2 L^2} \right).
\]

The constant above is given by

\[
\tilde{C}(\alpha, T, \tau) = \tilde{C}_1 \exp \left( \tilde{C}_2 \left( \frac{1}{b} \| \nabla \alpha \|_{L^{\infty} L^3} + 1 + T \right) T \right),
\]

or by

\[
\tilde{C}(\alpha, T, \tau) = \tilde{C}(T) = \tilde{C}_1 \exp \left( \tilde{C}_2 \left( 1 + T \right) T \right),
\]

if (3.6) holds, where \( \tilde{C}_1, \tilde{C}_2 > 0 \) do not depend on \( n \) or \( \tau \). Since the right-hand side of (3.18) does not depend on \( T_n \), we can extend the existence interval to \([0, T]\), i.e. \( T_n = T \).
Second estimate. By testing (3.10) with $\phi = \tau \psi_{ttt}^n \in V_n$ and integrating over $(0,T)$, we obtain

$$
\tau^2 \| \psi_{ttt}^n \|^2_{L^2 L^2} \leq \| - \alpha \psi_{ttt}^n + c^2 \Delta \psi^n + b \Delta \psi_t^n + f \|_{L^2 L^2} \| \tau \psi_{ttt}^n \|_{L^2 L^2},
$$

from which we have

$$
\tau \| \psi_{ttt}^n \|_{L^2 L^2} \leq \| \alpha \|_{L^\infty L^\infty} \| \psi_{ttt}^n \|_{L^2 L^2} + \| - \Delta \psi^n \|^2_{L^2 L^2} + b \| - \Delta \psi_t^n \|_{L^2 L^2} + \| f \|_{L^2 L^2}.
$$

The terms $\| - \Delta \psi^n \|^2_{L^2 L^2}$, $\| - \Delta \psi_t^n \|^2_{L^2 L^2}$ can be further estimated as in (3.15)

$$
\| - \Delta \psi^n \|_{L^2 L^2} \leq \sqrt{T} | - \Delta \psi_0^n |_{L^2(\Omega)} + \sqrt{\frac{T^3}{n}} - \Delta \psi_1^n \|_{L^\infty L^2},
$$

$$
\| - \Delta \psi_t^n \|_{L^2 L^2} \leq \sqrt{T} | - \Delta \psi_0^n |_{L^\infty L^2},
$$

and the term $\| \psi_{ttt}^n \|_{L^2 L^2}$ by means of the Poincaré-Friedrichs inequality, so that by using (3.18) we obtain (3.5) with $\psi^n$ in place of $\psi$.

Step 3: Passing to the limit. On account of estimate (3.18) and standard compactness results, we know that there exist a subsequence, denoted again by $\{ \psi^n \}_{n \in \mathbb{N}}$, and a function $\psi$ such that

$$
\psi_{ttt}^n \rightharpoonup \psi \text{ weakly in } L^2(0,T;L^2(\Omega)),
$$

$$
\psi_{tt}^n \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;H^1_0(\Omega)),
$$

$$
\psi_t^n \rightharpoonup \psi_t \text{ weakly-* in } L^\infty(0,T;H^1_0(\Omega) \cap H^2(\Omega)),
$$

$$
\psi^n \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;H^1_0(\Omega) \cap H^2(\Omega)).
$$

Our next task is to prove that $\psi$ solves (3.3). We test (3.10) with $\eta \in C^\infty_c(0,T)$ and integrate over time to obtain

$$
\tau^2 \int_0^T (\tau \psi_{ttt}^n, w_i)_{L^2} \eta(t) \, dt
$$

$$
= \int_0^T (\alpha \psi_{ttt}^n - c^2 \Delta \psi^n - b \Delta \psi_t^n - f, w_i)_{L^2} \eta(t) \, dt,
$$

for all $i \in [1,n]$. Thanks to (3.21), letting $n \to \infty$ in (3.22) leads to

$$
\int_0^T (\tau \psi_{ttt}, w_i)_{L^2} \eta(t) \, dt
$$

$$
= \int_0^T (\gamma \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t - f, w_i)_{L^2} \eta(t) \, dt,
$$

for all $i \in \mathbb{N}$ and $\eta \in C^\infty_c(0,T)$. By construction, $\cup_{n \in \mathbb{N}} V_n$ is dense in $L^2(\Omega)$, so $\psi$ solves the PDE in (3.3) in the $L^2(0,T;L^2(\Omega))$ sense. Due to the embeddings

$$
\psi^n \in W^{1,\infty}(0,T;H^1_0(\Omega) \cap H^2(\Omega)) \hookrightarrow C([0,T];H^1_0(\Omega) \cap H^2(\Omega)),
$$

$$
\psi_t^n \in W^{1,\infty}(0,T;H^1_0(\Omega)) \hookrightarrow C([0,T];H^1_0(\Omega)),
$$

$$
\psi_{tt}^n \in H^1(0,T;L^2(\Omega)) \hookrightarrow C([0,T];L^2(\Omega)),
$$

and a function $\psi$ such that

$$
\psi_{ttt}^n \rightharpoonup \psi \text{ weakly in } L^2(0,T;L^2(\Omega)),
$$

$$
\psi_{tt}^n \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;H^1_0(\Omega)),
$$

$$
\psi_t^n \rightharpoonup \psi_t \text{ weakly-* in } L^\infty(0,T;H^1_0(\Omega) \cap H^2(\Omega)),
$$

$$
\psi^n \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;H^1_0(\Omega) \cap H^2(\Omega)).
$$
we know that
\[
\begin{align*}
\psi^n(0) & \to \psi(0) \quad \text{in} \ H^4_{\Omega} \cap H^2(\Omega), \\
\psi^n(0) & \to \psi_t(0) \quad \text{in} \ H^3_{\Omega}, \\
\psi^n(0) & \to \psi_{tt}(0) \quad \text{in} \ L^2(\Omega).
\end{align*}
\]

Thanks to (3.9), we can then infer that \(\psi(0) = \psi_0, \psi_t(0) = \psi_1, \psi_{tt}(0) = \psi_2\). Altogether, we conclude that \(\psi\) is a solution of the initial-boundary value problem (3.3).

**Step 4: Energy inequality for \(\psi\).** We can take the limit inferior as \(n \to \infty\) of (3.18), (3.20), and via the weak and the weak-* lower semi-continuity of norms obtain the final estimate (3.5). Uniqueness of a solution follows by the linearity of the equation.

4. **Well-posedness of the nonlinear Westervelt-type wave equation (1.8)** for \(\tau > 0\). After having studied the linearized equation, we now proceed to the nonlinear model (1.8). For proving well-posedness of (1.8), we consider the fixed-point operator \(T\) that maps \(\phi\) to a solution \(\psi\) of
\[
\tau \psi_{ttt} + (1 - k\phi_t)\psi_{tt} - c^2 \Delta \psi - b\Delta \psi_t = 0,
\]
whose well-DEFINEDNESS is based on Theorem 3.1, on some ball
\[
B^X_\rho = \{\psi \in X^W : \psi(0) = \psi_0, \psi_t(0) = \psi_1, \psi_{tt}(0) = \psi_2, \psi_{ttt}(0) = \psi_{ttt}(0), \psi_{t,ttt}(0) = \psi_{t,ttt}(0)\}
\]
in the space \(X^W\) as defined in (3.4). Thus, for establishing \(T\) as a self-mapping on \(B^X_\rho\), it is crucial to prove that \(\alpha = 1 - k\phi_t\) is in \(X^W\) and that the smallness condition (3.6) holds, provided \(\phi \in B^X_\rho\). Smallness of \(\phi\) will also be required for verifying the nondegeneracy condition \(\alpha(t) \geq \alpha > 0\). Note that the radius of the neighborhood in which the self-mapping property holds is independent of \(\tau\); in particular, it holds for arbitrarily small \(\tau\) and therefore allows for taking limits as \(\tau \to 0\) later on.

Contractivity of \(T\), based on the fact that \(\psi = \psi_1 - \psi_2 = T(\phi_1) - T(\phi_2)\) solves
\[
\tau \hat{\psi}_{ttt} + (1 - k\phi_t)\hat{\psi}_{tt} - c^2 \Delta \hat{\psi} - c^2 \Delta \hat{\psi}_t = k\phi_t \psi_{ttt},
\]
with homogeneous initial and boundary conditions (where \(\hat{\phi} = \phi_t - \phi_1\)), would require to prove that \(\alpha_1 = 1 - k\phi_t\) and \(f_2 = k\phi_t \psi_{ttt}\) are in \(X^W_\rho\) and \(H^1(0, T; L^2(\Omega))\), respectively. This, however, would only be possible in an \(O(\sqrt{\tau})\) neighborhood because of the \(\phi_{ttt}\) term arising in \(f_2\). Therefore, we do not prove contractivity, but base our existence proof on Schauder’s Theorem, similarly to [19].

**Theorem 4.1.** Let \(c^2, b, T > 0\), and \(k \in \mathbb{R}\). Then there exist \(\tau, \rho > 0\), \(\rho_0 > 0\) such that for all \((\psi_0, \psi_1, \psi_2) \in X^W_\rho\) satisfying
\[
\|\psi_0\|_{H^2(\Omega)}^2 + \|\psi_1\|_{H^2(\Omega)}^2 + \|\psi_2\|_{H^2(\Omega)}^2 \leq \rho_0^2,
\]
and all \(\tau \in (0, \bar{\tau})\), there exists a solution \(\psi \in X^W\) of
\[
\begin{cases}
\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - b\Delta \psi_t = \left(\frac{\psi_t}{\sqrt{\tau}}\right)^2, & \text{in } \Omega \times (0, T), \\
\psi = 0 & \text{on } \partial \Omega \times (0, T), \\
(\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) & \text{in } \Omega \times \{0\},
\end{cases}
\]
and
\[ \tau^2 \| \psi_{tt} \|^2_{L^2 L^2} + \tau \| \psi_{tt} \|^2_{L^\infty H^1} + \| \psi_{tt} \|^2_{L^2 H^1} + \| \psi \|^2_{W^{1, \infty} H^2} \leq \rho^2. \]

**Proof.** To obtain the self-mapping property of \( T \) and to verify the condition (3.6) as well as \( \alpha(t) \geq \alpha > 0 \), we estimate the \( X^W_\alpha \) norm of \( \alpha = 1 - k \phi_t \). In view of the estimates
\[ \| \nabla \alpha \|_{L^\infty L^3} = |k| \| \nabla \phi_t \|_{L^\infty L^3} \leq |k| C_{H^2, W^{1,3}}^\infty \| \phi \|_{W^{1, \infty} H^2} \leq |k| C_{H^2, W^{1,3}}^\infty \rho, \]
\[ \| \alpha - 1 \|_{L^\infty L^\infty} = |k| \| \phi_t \|_{L^\infty L^\infty} \leq |k| C_{H^2, L^\infty}^\infty \| \phi \|_{W^{1, \infty} H^2} \leq |k| C_{H^2, L^\infty}^\infty \rho, \]
(3.6) and \( \alpha(t) \geq \alpha > 0 \) can be achieved by choosing
\[ \rho < \left( 2 |k| \max \left\{ C_{H^2, L^\infty}^\infty, C_{H^2, L^6}^\infty C_{H^2, W^{1,3}}^\infty \right\} \right)^{-1}, \]
and the self-mapping property follows from the estimate (3.5), with
\[ \rho_0^2 \leq \left( C_1 \left( \frac{13}{4} + T^3 \right) \exp \left( C_2 \left( 1 + T \right) T \right) \right)^{-1} \rho^2 \]
and \( f = 0 \).

The set \( B^{X^W}_\rho \) is a weak* compact and convex subset of the Banach space \( X^W \) and weak* continuity of \( T \) can be established as follows: For any sequence \( (\phi_n)_{n \in \mathbb{N}} \subseteq B^{X^W}_\rho \) that weakly* converges to \( \phi \in B^{X^W}_\rho \) in \( X^W \), we also have \( (T(\phi_n))_{n \in \mathbb{N}} \subseteq B^{X^W}_\rho \). Thus, by compactness of the embedding \( X^W \to W^{1, \infty}(0, T; L^\infty(\Omega)) \), there exists a subsequence \( (\phi_{n_k})_{k \in \mathbb{N}} \) such that \( 1 - k \phi_{n_k t} \) converges strongly in \( L^\infty(0, T; L^\infty(\Omega)) \) and the self-mapping property follows from the estimate (3.5), with
\[ \psi \rightarrow \frac{1}{\sqrt{\rho}}, \]
and it might be large as \( \tau \to 0 \). The latter does not matter for existence, according to Theorem 3.1, but excludes a fixed-point argument for proving well-posedness of the nonlinear equation (1.6) in this setting.

We can therefore conclude existence of a fixed point of \( T \) in \( B^{X^W}_\rho \) from the general version of Schauder’s Fixed Point Theorem in locally convex topological spaces; see [9].

**5. Higher energy estimates.** Due to the appearance of \( \| f_t \| \) on the right hand side of (3.5), the use for the Kuznetsov version (1.6) of the JMGT equation, where \( f_t = 2\nabla \psi \cdot \nabla \phi_t + 2|\nabla \phi_t|^2 \) is not possible with the energy estimate (3.5) only. Therefore, we need different energy estimates here.

Existence of solutions can still be based on Theorem 3.1, (case \( \| \nabla \alpha \|_{L^\infty L^3} < \frac{\sqrt{\rho}}{C_1} \) with a \( \tau \)-independent bound on the energy) since \( f \) is still in the right space. However, \( \| f_t \|_{L^2 L^2} = 2\| \nabla \psi \cdot \nabla \phi_t + |\nabla \psi_t|^2 \|_{L^2 L^2} \) can only be shown to be bounded by \( \frac{1}{\sqrt{\rho}} \), so it might be large as \( \tau \to 0 \). The latter does not matter for existence, according to Theorem 3.1, but excludes a fixed-point argument for proving well-posedness of the nonlinear equation (1.6) in this setting.

To be able to take limits as \( \tau \to 0 \), we thus need higher order energy estimates. In particular, we derive a \( \tau \)-independent bounds on \( \| \psi \|_{L^\infty H^3} \) which enable us to estimate \( f = 2\nabla \psi \cdot \nabla \phi_t \) in the required norms. As already mentioned, we cannot make use of the energy estimate (3.5) at all since it would yield a bound that tends to infinity as \( \tau \to 0 \), due to \( \| f_t \|_{L^2 L^2} = O(\frac{1}{\sqrt{\rho}}) \). Therefore, we replace it by an estimate
on the auxiliary function \( z \) and complement this with a higher order in space estimate on \( \psi \).

In order to carry out these new error estimates, we now turn our attention to studying the equation (1.9), restated again here for convenience:

\[
\tau \psi_{tt} + \alpha \psi_{tt} - \varepsilon^2 \Delta \psi - b \Delta \psi_t = f \text{ in } \Omega \times (0, T),
\]

together with its equivalent reformulation (1.13) using (1.12), (1.10), i.e.,

\[
\tau z_{tt} + \gamma z_t - b \Delta z - \gamma \frac{\varepsilon^2}{\tau} z + \gamma \frac{\varepsilon^4}{\tau^2} \psi = f \text{ in } \Omega \times (0, T),
\]

with

\[
z = \psi_t + \frac{\varepsilon^4}{\tau^2} \psi.
\]

We assume that for some \( \underline{\alpha} > 0, \underline{\gamma} > 0 \)

\[
\alpha(t) \geq \underline{\alpha}, \quad \gamma(t) = \alpha(t) - \tau \frac{\varepsilon^2}{\tau} \geq \underline{\gamma} \quad \text{on } \Omega \text{ for a.e. } t \in (0, T),
\]
as well as that the regularity

\[
\alpha, \gamma \in X^K_\alpha := W^{1,1}(0, T; H^1(\Omega)) \cap C(0, T; W^{1,3}(\Omega)) \cap L^1(0, T; H^2(\Omega)),
\]

\[
f \in L^2(0, T; H^1(\Omega)),
\]

holds. We also assume that all appearing derivatives of \( \alpha \) and \( \gamma \) can be made small.

**Theorem 5.1.** Let \( \varepsilon^2, b, \tau > 0, k \in \mathbb{R}, \) and let \( T > 0. \) Assume that

- \( \alpha \in X^K_\alpha := W^{1,1}(0, T; H^1(\Omega)) \cap L^\infty(0, T; W^{1,3}(\Omega)) \cap L^1(0, T; H^2(\Omega)), \)
- \( \exists \underline{\alpha}, \underline{\gamma} > 0 : \alpha(t) \geq \underline{\alpha}, \gamma(t) \geq \underline{\gamma} \text{ a.e. in } \Omega \times (0, T), \)
- \( f \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \)
- \( (\psi_0, \psi_1, \psi_2) \in X^K_0 := H^1_0(\Omega) \cap H^3(\Omega) \times H^3_0(\Omega) \cap H^2(\Omega) \times H^3_0(\Omega). \)

Then for sufficiently small \( ||\alpha - 1||_{X^K_\alpha} \), there exists a unique solution \((\psi, z)\) of the problem

\[
\begin{align*}
&\tau z_{tt} + (\alpha - \tau \frac{\varepsilon^2}{\tau}) z_t - b \Delta z - \gamma \frac{\varepsilon^2}{\tau} z + \gamma \frac{\varepsilon^4}{\tau^2} \psi = f \quad \text{in } \Omega \times (0, T), \\
&z = \psi_t + \frac{\varepsilon^4}{\tau^2} \psi \quad \text{in } \Omega \times (0, T), \\
&\psi = 0 \quad \text{on } \partial \Omega \times (0, T), \\
\end{align*}
\]

(3.3)

that satisfies \((\psi, z) \in L^\infty(0, T; H^3(\Omega)) \times (L^\infty(0, T; H^2(\Omega)) \cap W^{1,\infty}(0, T; H^1_0(\Omega))))\), i.e.,

\[
\psi \in X^K := L^\infty(0, T; H^3(\Omega)) \cap W^{1,\infty}(0, T; H^2(\Omega)) \cap W^{2,\infty}(0, T; H^1_0(\Omega)).
\]

Furthermore, there exists a \( C(\gamma, T) > 0 \), which does not depend on \( \tau \), such that

\[
\begin{align*}
||\psi||^2_{L^\infty H^3} + \left|\left|\psi_t + \frac{\varepsilon^2}{\tau^2} \psi \right|\right|^2_{L^\infty H^2} + \left|\left|\psi_{tt} + \frac{\varepsilon^4}{\tau^2} \psi \right|\right|^2_{L^2 H^1_0} + \tau \left|\left|\psi_{tt} + \frac{\varepsilon^4}{\tau^2} \psi_t \right|\right|^2_{L^\infty H^1_0} \\
\leq C(\gamma, T) \left( ||\psi_0||^2_{H^3} + ||\psi_1||^2_{H^2} + \tau ||\psi_2||^2_{H^1} + ||f||^2_{L^2 H^1} \right).
\end{align*}
\]

\(^1\)more precisely, of \( ||\nabla \gamma||_{L^\infty L^3} \)
Note that while \( f \in H^1(0,T; L^2_2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \) is required, in the right hand side of the energy estimate only \( \|f\|_{L^2_2} \), but not \( \|f\|_{H^1_1} \) appears.

**Proof. Step 1: Existence of a solution.** Theorem 3.1 implies existence of a solution \((\psi, z)\) of (5.3) with (3.4) and

\[
z \in L^\infty(0,T; H^1_1(\Omega) \cap H^2_2(\Omega)) \cap W^{1,\infty}(0,T; H^1_0(\Omega)) \cap H^2(0,T; L^2(\Omega)).
\]

Note that \( z_t \) inherits the homogeneous Dirichlet boundary conditions from \( \psi \). Thus it only remains to establish the higher order energy estimates. For this purpose, we return to the Galerkin approximation (3.10) and define \( z^n = \psi^n + \frac{\delta}{\beta} \psi^n \).

**Step 2: A priori estimates.** As in Section 3, our goal is to obtain a bound for \( \psi^n \) that is uniform with respect to \( n \). To this end, we test the spatially discretized version of our problem (5.3) with two test functions.

**The first energy identity.** Problem (3.10) can be equivalently rewritten as

\[
\begin{cases}
(\tau z^n_t + \gamma z^n_t - b\Delta z^n - \frac{c^2}{b} \gamma z^n + \gamma \frac{c^2}{b} \psi^n, \phi)_{L^2} = (f, \phi)_{L^2}, \\
\text{for every } \phi \in V_n \text{ pointwise a.e. in } (0,T), \\
z^n = \psi^n + \frac{\delta}{\beta} \psi^n, \\
(\psi^n(0), \psi^n(0), \psi^n(0)) = (\psi^n_0, \psi^n_1, \psi^n_2).
\end{cases}
\]

Multiplying the first equation in (5.6) by \(-\Delta z^n_t \in V_n\) and integrating over \( \Omega \) and \((0,t)\) yields the energy identity

\[
\begin{align*}
\frac{\delta}{2} |\nabla z^n_t(t)|^2 + \int_0^t |\sqrt{\gamma} \nabla z^n_t|^2 \, ds + \frac{\delta}{2} |z^n(t)|^2 &
\leq \frac{\delta}{2} |\nabla z^n_t(0)|^2 - \int_0^t (z^n_t \nabla \gamma, \nabla z^n_t)_{L^2} \, ds + \frac{\delta}{2} |z^n(0)|^2 \\
&\quad - \frac{c^2}{b} \int_0^t (\nabla f - \frac{c^2}{b} \psi^n, \nabla z^n_t)_{L^2} \, ds + \frac{c^2}{b} \int_0^t (\nabla \gamma, \nabla z^n_t)_{L^2} \, ds \\
&\quad + \int_0^t (\nabla f - \frac{c^2}{b} \psi^n, \nabla \gamma, \nabla z^n_t)_{L^2} \, ds \\
&\quad + \int_0^t (\nu \cdot \nabla z^n_t, f)_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)} \, ds \\
&=: \text{rhs}_1(t),
\end{align*}
\]

where we have skipped the argument \((s)\) under the time integral for notational simplicity. To derive (5.7), we have used the identity

\[
(\gamma z^n_t, -\Delta z^n_t)_{L^2} = |\sqrt{\gamma} \nabla z^n_t|^2 + (z^n_t \nabla \gamma, \nabla z^n_t)_{L^2}.
\]

Furthermore, we have made use of

\[
-\int_0^t (\gamma \frac{c^2}{b} z^n, -\Delta z^n_t)_{L^2} \, ds
\]

\[
= -\int_0^t (\gamma \frac{c^2}{b} z^n, \nabla z^n_t)_{L^2} \, ds - \int_0^t (z^n \nabla \gamma \frac{c^2}{b}, \nabla z^n_t)_{L^2} \, ds
\]
as well as the identity
\[
\int_0^t \left( f - \gamma \frac{\partial}{\partial t} \psi^n, -\Delta z^n_t \right)_{L^2} \, ds
= \int_0^t \left( \nabla f - \gamma \frac{\partial}{\partial t} \nabla \psi^n - \frac{\partial}{\partial t} \psi^n \nabla \gamma, \nabla z^n_t \right)_{L^2} \, ds
+ \int_0^t (\nu \cdot \nabla z^n_t, f)_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)} \, ds.
\]

The second energy identity. Our aim is to obtain a bound on \( \psi \) in the \( H^3(\Omega) \) norm. To this end, we test (3.10) with \( \phi = (-\Delta)^2 \psi^n \in V_n \) (due to the fact that \( \psi^n \) is a linear combination of eigenfunctions of \( -\Delta \)) which yields the second energy identity
\[
c^2 \int_0^t |\nabla (-\Delta) \psi^n|^2 \, ds + \frac{2}{t^2} |\nabla (-\Delta) \psi^n(t)|^2
= \frac{\partial}{\partial t} [(-\Delta)^2 \psi^n(t)] - \frac{1}{4} |\nabla \psi^n|^2,
\]
and the fact that \(-\Delta \psi^n = 0\) on \( \partial \Omega \). Moreover, we rewrote the \( \alpha \) term as follows
\[
\int_0^t (\alpha \psi^n_t, (-\Delta)^2 \psi^n_t)_{L^2} \, ds
\]
\[
= \int_0^t \left( \nabla [\alpha \psi^n_t] - \alpha \psi^n_t, \nabla (-\Delta) \psi^n \right)_{L^2} \, ds
\]
\[
= (\nabla [\alpha(t) \psi^n_t(t)], \nabla (-\Delta) \psi^n(t))_{L^2} - \left( \nabla (\alpha(0) \psi^n_t(0)), \nabla (-\Delta) \psi^n(0) \right)_{L^2}
\]
\[
- \int_0^t (\nabla [\alpha_t \psi^n_t], \nabla (-\Delta) \psi^n)_{L^2} \, ds - \int_0^t (\nabla [\alpha_t \psi^n_t], \nabla (-\Delta) \psi^n)_{L^2} \, ds
\]
\[
= (\nabla [\alpha(t) \psi^n_t(t)], \nabla (-\Delta) \psi^n(t))_{L^2} - \left( \nabla [\alpha(0) \psi^n_t(0)], \nabla (-\Delta) \psi^n(0) \right)_{L^2}
\]
\[
- \int_0^t (-\Delta [\alpha \psi^n_t], -\Delta \psi^n_t)_{L^2} \, ds - \int_0^t (\nabla [\alpha_t \psi^n_t], \nabla (-\Delta) \psi^n)_{L^2} \, ds,
\]
where we used again that \(-\Delta \psi^n = 0\) on \( \partial \Omega \). Note that under the assumptions made on \( \alpha \), we have that \( (\alpha \psi^n_t)(t) \in H^1_0(\Omega) \cap H^2(\Omega) \) for almost every \( t \in (0, T) \), since \( \psi^n(t) \in H^1_0(\Omega) \cap H^2(\Omega) \).
The left hand sides of our two energy identities (5.7), (5.8) can be estimated from below by
\begin{equation}
\frac{\tau}{2} |\nabla z^n(t)|^2 + \int_0^t \left| \sqrt{\gamma} \nabla z^n_s(0) \right|^2 ds + \frac{b}{2} \left| -\Delta z^n(t) \right|^2
\geq \frac{1}{8} \left( \tau \|\nabla z^n\|_{L^\infty L^2}^2 + \gamma \|\nabla z^n\|_{L^2 L^2}^2 + b \| -\Delta z^n \|_{L^\infty L^2}^2 \right),
\end{equation}
and
\begin{equation}
\frac{c^2}{4} \int_0^t |\nabla (-\Delta) \psi^n|^2 ds + \frac{d}{2} |\nabla (-\Delta) \psi^n(t)|^2
\geq \frac{1}{4} \left( 2c^2 \|\nabla (-\Delta) \psi^n\|_{L^2 L^2}^2 + b \|\nabla (-\Delta) \psi^n\|_{L^\infty L^2}^2 \right),
\end{equation}
respectively.

We will consider the weighted sum (5.7) plus \( \lambda \) times (5.8) which therefore can be bounded from below by
\begin{equation}
\text{lhs}(t) = \frac{1}{8} \left( \tau \|\nabla z^n\|_{L^\infty L^2}^2 + \gamma \|\nabla z^n\|_{L^2 L^2}^2 + b \| -\Delta z^n \|_{L^\infty L^2}^2 \right)
+ \frac{d}{4} \left( 2c^2 \|\nabla (-\Delta) \psi^n\|_{L^2 L^2}^2 + b \|\nabla (-\Delta) \psi^n\|_{L^\infty L^2}^2 \right).
\end{equation}
It then remains to estimate the right-hand sides, \( \text{rhs}_1(t) \) and \( \lambda \text{rhs}_2(t) \).

**Estimates of the right-hand sides.** For estimating the right-hand sides in (5.7), (5.8), we can then use the norms of \( z \), \( z_t \) and \( \psi \) appearing in the lower bounds (5.10), (5.11). Furthermore, we can employ the continuous embeddings (2.5) as well as boundedness of \((-\Delta)^{-1} : L^2(\Omega) \rightarrow H_0^1(\Omega) \cap H^2(\Omega)\) and the Poincaré-Friedrichs inequality, (2.6). Additionally, we employ the identities \( \psi_t = z - \frac{\varepsilon}{2} \psi \) and \( \psi_{tt} = z_t - \frac{\varepsilon^2}{2} z + \varepsilon^2 \psi \).

We introduce two constants depending on the initial data
\begin{align*}
C_1(\psi_0, \psi_1, \psi_2; \tau) &= \frac{\tau}{2} |\nabla z_t(0)|^2 + \frac{b}{2} \left| -\Delta z(0) \right|^2 \\
&= \frac{\tau}{2} |\nabla \psi_2 + \frac{\varepsilon^2}{2} \nabla \psi_1|^2 + \frac{d}{2} \left| -\Delta \psi_1 - \frac{\varepsilon^2}{2} \Delta \psi_0 \right|^2,
\end{align*}
\begin{align*}
C_2(\psi_0, \psi_1, \psi_2; \tau) &= \frac{b}{2} |\nabla (-\Delta) \psi_0|^2 + \tau |(\nabla \psi_2, \nabla (-\Delta) \psi_0)_{L^2} + \frac{\tau}{2} \left| -\Delta \psi_1 \right|^2 \\
&+ (\alpha(0) \nabla \psi_1 + \psi_1 \nabla \alpha(0), \nabla (-\Delta) \psi_0)_{L^2}.
\end{align*}

By applying Young’s inequality (2.4) and the embedding results as well as the trace theorem we get for the right hand side in (5.7),
\begin{align*}
\text{rhs}_1(t) \
&\leq C_1(\psi_0, \psi_1, \psi_2; \tau) + \|\nabla z^n_t\|_{L^2 L^2} \| z^n_t \|_{L^\infty L^2} \| \nabla \gamma \|_{L^\infty L^3} \\
&+ \|\nabla z^n_t\|_{L^2 L^2} \|\nabla z^n\|_{L^\infty L^2} \| \gamma \|_{L^\infty L^3} + \|\nabla z^n_t\|_{L^2 L^2} \| z^n \|_{L^\infty L^2} \| \nabla \gamma \|_{L^\infty L^2} \\
&+ \|\nabla z^n_t\|_{L^2 L^2} \left( \| \nabla \ell f \|_{L^2 L^2} + \frac{c^2}{2} \| \gamma \|_{L^\infty L^2} \| \nabla \psi^n \|_{L^\infty L^2} + \frac{d^2}{2} \| \nabla \gamma \|_{L^\infty L^2} \| \psi^n \|_{L^\infty L^2} \right) \\
&+ \| \nu \cdot \nabla z^n_t\|_{L^2 H^{-1/2}(\partial \Omega)} \| f \|_{L^2 H^{1/2}(\partial \Omega)}.
\end{align*}
We further obtain

\[
\text{rhs}_1(t) \\
\leq C_1(\psi_0, \psi_1, \psi_2; \tau) + C_{H^1, L^p}^\Omega \| \nabla \gamma \|_{L^\infty L^1} \| \nabla \psi_n \|_{L^2 L^2}^2 + \frac{\gamma}{16} \| \nabla \psi_n \|_{L^2 L^2}^2 \\
+ \frac{\gamma}{2} (C_{\Delta}^\Omega - 1) C_{H^2, W^{1,1}} \| - \Delta \psi_n \|_{L^2 L^2}^2 \frac{\gamma}{16} \| \gamma \|_{L^\infty L^3} \\
+ C_{\Delta}^\Omega \| \nabla \gamma \|_{L^\infty L^2}^2 + \| \nabla f \|_{L^2 L^2} \\
+ \frac{\gamma}{6} (\| \nabla \gamma \|_{L^2 L^2}^2 + C_{P,F} \| \nabla \gamma \|_{L^\infty L^2}) C_{\Delta}^\Omega \| \nabla (-\Delta) \psi_n \|_{L^2 L^2} \\
+ (C_{\Delta}^\Omega)^2 C_{P,F}^3 \| f \|_{L^2 L^2}^2 )^2,
\]

since for the Galerkin discretization by eigenfunctions of the Laplacian, we have \((-\Delta) \psi_n \in H_0^1(\Omega) \cap H^2(\Omega)\) for smooth \(\Omega\). All terms on the right hand side except for

\[
\tilde{\text{rhs}}_1 := C_1(\psi_0, \psi_1, \psi_2; \tau) + \frac{4\tau}{2} \left( 1 + (C_{\Delta}^\Omega)^2 C_{P,F}^3 \| f \|_{L^2 L^2}^2 \\
+ C_{\Delta}^\Omega \| \nabla \gamma \|_{L^\infty L^2}^2 C_{\Delta}^\Omega \| \nabla (-\Delta) \psi_n \|_{L^2 L^2} \right)^2.
\]

can be absorbed into the left hand side (5.12) by making \(\| \nabla \gamma \|_{L^\infty L^3}\) small. The right-hand side in (5.8) can be estimated as follows

\[
\text{rhs}_2(t) \\
\leq C_2(\psi_0, \psi_1, \psi_2; \tau) + \tau \| (\nabla (-\Delta) \psi_n) \|_{L^\infty L^2} \| \nabla \psi_n \|_{L^\infty L^2} \| \nabla \psi_n \|_{L^\infty L^2}^2 \\
+ \frac{\gamma}{4} \| -\Delta \psi_n + \frac{\gamma}{6} - \nabla \psi_n \|_{L^\infty L^2}^2 \\
+ \frac{\gamma}{2} \| -\Delta \psi_n + \frac{\gamma}{6} - \nabla \psi_n \|_{L^\infty L^2} \left( \| -\Delta \psi_n + \frac{\gamma}{6} - \nabla \psi_n \|_{L^\infty L^2} \| \alpha \|_{L^1 L^\infty} \\
+ 2 \| \nabla \psi_n - \frac{\gamma}{6} - \nabla \psi_n \|_{L^\infty L^2} \| \nabla \alpha \|_{L^1 L^2} \right) \\
+ \| \nabla (-\Delta) \psi_n \|_{L^\infty L^2} \left( \| \nabla f \|_{L^1 L^2} + \| \nabla \psi_n - \frac{\gamma}{6} - \nabla \psi_n \|_{L^\infty L^2} \left( \| \alpha \|_{L^\infty L^3} + \| \alpha \|_{L^1 L^3} \right) \\
+ \| \nabla \alpha \|_{L^\infty L^2} + \| \nabla \alpha \|_{L^1 L^2} \right) + (C_{\Delta}^\Omega)^2 C_{P,F}^3 \| f \|_{L^2 L^2}^2 \right),
\]

where we have estimated the boundary term by means of the trace theorem

\[
- \int_0^t \langle \nu \cdot \nabla (-\Delta) \psi_n, f \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)} ds \\
\leq (C_{\Delta}^\Omega)^2 \| (-\Delta) \psi_n \|_{L^\infty H^1} \| f \|_{L^1 H^1} \\
\leq (C_{\Delta}^\Omega)^2 C_{P,F}^3 \| \nabla (-\Delta) \psi_n \|_{L^\infty L^2} \| f \|_{L^1 H^1}.
\]
We further have
\[
\begin{align*}
\text{Arhs}_2(t) & \leq \lambda C_2(\psi_0, \psi_1, \psi_2; \tau) + \lambda \frac{\alpha}{\beta} \|\nabla(-\Delta)\psi^n\|_{L^2}^2 \\
& \quad + \lambda \frac{\alpha}{\beta} \tau \left\| \nabla z^n_t - \frac{c}{b} \nabla z^n + \frac{c}{b} \nabla \psi^n \right\|^2_{L^2} + \lambda \frac{\alpha}{\beta} \left\| -\Delta z^n + \frac{c}{b} \Delta \psi^n \right\|^2_{L^2} \\
& \quad + \lambda \left\| -\Delta z^n - \frac{c}{b} \nabla \psi^n \right\|^2_{L^2} \left( \|\alpha\|_{L^1}^2 + 2C_{\Omega(\nabla(-\Delta))} \|\nabla\alpha\|_{L^1}^2 \right) \\
& \quad + \lambda \left\| -\Delta z^n - \frac{c}{b} \nabla \psi^n \right\|^2_{L^2} \left( \|\alpha\|_{L^1}^2 + 2C_{\Omega(\nabla(-\Delta))} \|\nabla\alpha\|_{L^1}^2 \right) \\
& \quad + \|\alpha_t\|_{L^1}^2 + C_{\Omega H^3, L^\infty}(\|\nabla\alpha\|_{L^\infty}^2 + \|\nabla\alpha_t\|_{L^1}^2) \right),
\end{align*}
\]  
where by making \(\tau\) and \(\lambda\) small, all terms except for those containing the initial data and the inhomogeneity,
\[
\begin{align*}
\lambda \tilde{\text{Arhs}}_2 & := \lambda C_2(\psi_0, \psi_1, \psi_2; \tau) + \lambda \frac{\alpha}{\beta} \left( 1 + (\frac{C_{\Omega}}{C_{\Omega F}})^2 \right) \|f\|_{L^1 H^1}^2,
\end{align*}
\]  
can be absorbed into the left hand side (5.12).

We now combine the energy estimates obtained from (5.7), and \(\lambda\) times (5.8) with a small constant \(\lambda > 0\) and absorb the indicated terms from the right-hand side estimates (5.13), (5.15) into the left hand side so that only \(\tilde{\text{Arhs}}_1\) and \(\lambda \tilde{\text{Arhs}}_2\) remain on the right hand side, cf. (5.12), (5.14), (5.16). Therewith, we end up with an inequality of the form
\[
\begin{align*}
\eta(t) & \leq C \left( \int_0^t \eta(s) \text{d}s + \|\psi_0\|_{L^1}^2 + \|\psi_1\|_{L^2}^2 + \tau \|\psi_2\|_{H^1}^2 + \|f\|_{L^2}^2 \right),
\end{align*}
\]  
for
\[
\begin{align*}
\eta(t) & = \frac{1}{2} \left( \tau \|\nabla z^n_t\|_{L^\infty(0,t;L^2)}^2 + \gamma \|\nabla z^n_t\|_{L^2(0,t;L^2)}^2 + b \|\Delta z^n\|_{L^\infty(0,t;L^2)}^2 \right) \\
& \quad + \lambda \left( 2c^2 \|\nabla(-\Delta)\psi^n\|_{L^2(0,t;L^2)}^2 + b \|\nabla(-\Delta)\psi^n\|_{L^\infty(0,t;L^2)}^2 \right),
\end{align*}
\]  
to which we employ Gronwall’s lemma.

To obtain a uniform bound on the full \(H^3(\Omega)\) norm of \(\psi^n\), we combine the \(|\nabla(-\Delta)\psi^n|_{L^2}\) term with \(|-\Delta z^n|_{L^2}\) and the fact that
\[
\psi^n(x, t) = e^{-(c^2/b)t} \psi_0(x) + \int_0^t e^{-(c^2/b)(t-s)} z^n(x, s),
\]  
\(t \in (0, T)\). In this way, we have
\[
\begin{align*}
|\psi^n(t)|_{H^3(\Omega)} & \leq C_{\Omega(\nabla(-\Delta))} \left( |\nabla(-\Delta)\psi^n(t)|_{L^2} + |(-\Delta)\psi^n(t)|_{L^2} \right) \\
& \leq C_{\Omega(\nabla(-\Delta))} \left( |\nabla(-\Delta)\psi^n(t)|_{L^2} + |e^{-(c^2/b)t}(-\Delta)\psi_0|_{L^2} \right. \\
& \quad \left. + \left| \int_0^t e^{-(c^2/b)(t-s)}(-\Delta)z(s) \text{d}s \right|_{L^2} \right),
\end{align*}
\]
for a.e. \( t \in (0, T) \). Altogether, we get the estimate
\[
\|\psi^n\|_{L^\infty H^3}^2 + \|\psi^n_t + \frac{c^2}{T} \psi^n\|_{L^\infty H^2}^2 + \|\psi_{tt} + \frac{c^2}{T} \psi^n_t\|_{L^2 H_0^1}^2 \\
+ \tau \|\psi^n_{tt} + \frac{c^2}{T} \psi^n_t\|_{L^\infty H_0^1}^2 \\
\leq C(\gamma, T) \left( |\psi(0)|_{H^3}^2 + |\psi(t)|_{H^2}^2 + \tau |\psi(t)|_{H^1}^2 + \|f\|_{L^2 H^1}^2 \right)
\]
with a constant \( C(\gamma, T) > 0 \) independent of \( \tau \), provided \( \|\nabla \gamma\|_{L^\infty L^3} \) is sufficiently small.

**Step 3: Passing to the limit.** On account of estimate (5.18) and the Banach-Alaoglu theorem, we know that there exists a subsequence, denoted again by \( \{\psi^n\}_{n \in \mathbb{N}} \), and a function \( \tilde{\psi} \) such that
\[
\psi^n \rightharpoonup \tilde{\psi} \text{ weakly-} \ast \text{ in } L^\infty(0, T; H_0^1(\Omega) \cap H^3(\Omega)), \\
\psi^n_t \rightharpoonup \tilde{\psi}_t \text{ weakly-} \ast \text{ in } L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)), \\
\psi^n_{tt} \rightharpoonup \tilde{\psi}_{tt} \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \\
\psi^n_{ttt} \rightharpoonup \tilde{\psi}_{ttt} \text{ weakly-} \ast \text{ in } L^\infty(0, T; H_0^1(\Omega)).
\]

By uniqueness of limits \( \tilde{\psi} \) has to coincide with the solution \( \psi \) according to Theorem 3.1, which thus satisfies (5.5).

### 6. Well-posedness of the nonlinear Kuznetsov-type wave equation (1.6)

**for \( \tau > 0 \).** We text intend to employ the Banach fixed point theorem to prove well-posedness of (1.6). To this end, we introduce the operator \( T \) that maps \( \phi \) to a solution \( \psi \) of
\[
\tau \psi_{ttt} + (1 - k\phi_t)\psi_{tt} - c^2\Delta \psi - b\Delta \psi_t = 2\nabla \phi \cdot \nabla \phi_t,
\]
on some ball
\[
B^{X^K}_\rho = \left\{ \psi \in X^K : \psi(0) = \psi_0, \psi_t(0) = \psi_1, \psi_{tt}(0) = \psi_2, \text{ and} \right. \\
\left. \|\psi\|_{X^K}^2 := |\tau| \|z_t\|_{L^\infty H^1} + \|z_t\|_{L^2 H^1}^2 \\
+ \|z\|_{L^2 H^2}^2 + \|\psi_t\|_{L^\infty H^3}^2 \leq \rho^2, \right.
\]
(6.1)
\[
\text{for } z = \psi_t + \frac{c^2}{b} \psi
\]
in the space \( X^K \) defined in (5.4) for proving well-posedness. Thus, for establishing \( T \) as a self-mapping on \( B^{X^K}_\rho \), it is crucial to prove that \( \alpha = 1 - k\phi_t \) and \( f = 2\nabla \phi \cdot \nabla \phi_t \) are in \( X^K_\alpha \) and \( L^2(0, T; H^1(\Omega)) \), respectively, and that the derivatives of \( \alpha \) are small, provided \( \phi \in B^{X^K}_\rho \).

Concerning non-degeneracy (5.1), we assume that \( \tau \in (0, \bar{\tau}] \) with \( \bar{\tau} < \frac{b}{k} \) so that
\[
\gamma^* := 1 - \bar{\tau} \frac{c^2}{b} > 0.
\]

Therefore, keeping
\[
\|\gamma - \gamma^*\|_{L^\infty L^\infty} = \|\alpha + 1\|_{L^\infty L^\infty} = k\|\phi_t\|_{L^\infty L^\infty} \leq m
\]
(6.3)
with \( m < \gamma^* \) allows to choose

\[
\gamma := 1 - \tau \frac{\rho^2}{\bar{T}} - m > 0, \quad \alpha := 1 - m > 0
\]

in (5.1) independently of \( \tau \), which will also be important for the considerations in Section 7. Thus we also need to verify that (6.3) follows from \( \phi \in B^\infty_K \).

To additionally obtain contractivity, based on the fact that the difference \( \hat{\psi} = \psi_1 - \psi_2 = T(\phi_1) - T(\phi_2) \) solves

\[
\tau \hat{\psi}_{tt} + (1 - k \phi_1 t) \hat{\psi}_t - c^2 \Delta \hat{\psi} - b \Delta \hat{\psi}_t = k \hat{\phi}_t \hat{\psi}_{tt} + 2\nabla \hat{\phi} \cdot \nabla \phi_1 t + 2\nabla \phi_2 \cdot \nabla \hat{\phi}_t,
\]

with homogeneous initial and boundary conditions (where \( \hat{\phi} = \phi_1 - \phi_1 \)), we need to prove that \( \alpha_1 = 1 - k \phi_1 t \) and \( f_2 = k \hat{\phi}_t \hat{\psi}_{tt} + 2\nabla \hat{\phi} \cdot \nabla \phi_1 t + 2\nabla \phi_2 \cdot \nabla \hat{\phi}_t \) are in \( X^K \) and \( L^2(0, T; H^1(\Omega)) \), and that the derivatives of \( \alpha_1 \) are small, provided \( \phi_1, \phi_2 \in B^\infty_K \). Moreover, \( \|f_2\|_{L^2 H^1} \) needs to be estimated by a multiple of \( \|\hat{\phi}\|_{X^K} \) with a small factor.

**Theorem 6.1.** Let \( c^2, b, T > 0, k \in \mathbb{R} \). Then there exist \( \tilde{\tau}, \rho > 0, \rho_0 > 0 \) such that for all \((\psi_0, \psi_1, \psi_2) \in X^K_0 \) satisfying

\[
\|\psi_0\|_{H^3(\Omega)}^2 + \|\psi_1\|_{H^2(\Omega)}^2 + \|\psi_2\|_{H^1(\Omega)}^2 \leq \rho_0^2,
\]

and all \( \tau \in (0, \tilde{\tau}) \), there exists a unique solution \( \psi \in X^K \) of

\[
\begin{cases}
\tau \psi_{tt} + \psi_t - c^2 \Delta \psi - b \Delta \psi_t = \left( \frac{1}{c^2} B(\psi) + |\nabla \psi|^2 \right)_t & \text{in } \Omega \times (0, T), \\
\psi = 0 & \text{on } \partial \Omega \times (0, T), \\
(\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) & \text{in } \Omega \times \{0\},
\end{cases}
\]

and

\[
\tau \|z_t\|_{L^\infty H^1}^2 + \|z_t\|_{L^2 H^1}^2 + \|z\|_{L^\infty H^2}^2 + \|\psi\|_{L^\infty H^3}^2 \leq \rho^2.
\]

**Proof.** We first prove that \( T \) is a self-mapping on \( B^\infty_K \). For this purpose, we estimate \( \alpha \) and \( f \), assuming that \( \phi \in B^\infty_K \) and abbreviating \( w = \phi_t + \frac{c^2}{b} \phi \):

\[
\begin{align*}
\|\nabla \alpha\|_{L^1 L^2} &= |k| \|\nabla w_t - \frac{c^2}{b} \nabla w + \frac{c^2}{b} \nabla \phi\|_{L^1 L^2} \\
&\leq |k| \sqrt{T(1 + \frac{c^2}{b} \sqrt{T} + \frac{c^2}{b})} \rho, \\
\|\nabla \alpha\|_{L^\infty L^3} &= |k| \|\nabla w - \frac{c^2}{b} \nabla \phi\|_{L^\infty L^3} \\
&\leq |k| C \|\phi\|_{H^2 \rightarrow L^3} (1 + \frac{c^2}{b}) \rho, \\
\|\Delta \alpha\|_{L^1 L^2} &= |k| \|\Delta w + \frac{c^2}{b} \Delta \phi\|_{L^1 L^2} \\
&\leq |k| \sqrt{T(1 + \frac{c^2}{b})} \rho, \\
\|\gamma - \gamma^*\|_{L^\infty L^\infty} &= \|\alpha - 1\|_{L^\infty L^\infty} = |k| \|w - \frac{c^2}{b} \phi\|_{L^\infty L^\infty} \\
&\leq |k| C \|\phi\|_{H^2 \rightarrow L^\infty} (1 + \frac{c^2}{b}) \rho.
\end{align*}
\]
Moreover, we find that
\[
\|\nabla f\|_{L^2 L^2} = 2 \|\nabla^2 \phi \nabla \phi_t + \nabla^2 \phi_t \nabla \phi\|_{L^2 L^2}
\leq 2 \left( \|\nabla^2 \phi\|_{L^6} \|\nabla \psi - \frac{c^2}{b} \nabla s\|_{L^2 L^3} + \|\nabla^2 w - \frac{c^2}{b} \nabla^2 \phi\|_{L^2 L^2} \|\nabla \phi\|_{L^\infty L^\infty} \right)
\leq 2 (C_{H^2 \to W^{2,6}}(Q, \Omega) + C_{H^2 \to W^{1,3}}(Q, \Omega)) \left( 1 + \frac{c^2}{b} \right) \rho^2 =: C_f \rho^2.
\]
Therefore, energy estimate (5.5) yields
\[
\|\hat{\psi}\|_{X^K} \leq C(\gamma, T) \left( C_f \rho^4 + \|\psi_0\|_{H^2(\Omega)}^2 + \|\psi_1\|_{H^2(\Omega)}^2 + \tau \|\psi_2\|_{H^1(\Omega)}^2 \right) \leq \rho^2,
\]
provided that (6.6) holds with
\[
\rho_0 \leq \min \left\{ \frac{1}{4C(\gamma, T)C_f}, \frac{m}{kC_{H^2 \to L^\infty}} \right\},
\]
which implies that \( T \) maps \( B^X_{\rho} \) into itself.

For proving contractivity of \( T \), we estimate \( \alpha_1 \) analogously to (6.8), and with abbreviations \( \hat{w} := \phi_t + \frac{c^2}{b} \hat{\phi} \), \( \hat{w}_1 := \phi_{tt} + \frac{c^2}{b} \phi_t \), \( z_i := \psi_{tt} + \frac{c^2}{b} \psi_{tt}, i \in \{1, 2\} \). We have
\[
\|\nabla f_2\|_{L^2 L^2} \\
\leq |k| \left( \|\nabla \hat{w} - \frac{c^2}{b} \nabla \phi_t\|_{L^\infty L^3} \|\hat{w}_2\|_{L^6 L^6} \right)
+ \|\hat{w} - \frac{c^2}{b} \hat{\phi}\|_{L^\infty L^\infty} \|\nabla \hat{w}_2\|_{L^6 L^6}
+ 2 \left( \|\nabla^2 \hat{\phi}\|_{L^\infty L^6} \|\nabla \hat{w}_1\|_{L^\infty L^3} + \|\nabla^2 \hat{w}\|_{L^2 L^2} \|\nabla \phi_t\|_{L^\infty L^\infty} \right)
+ 2 \left( \|\nabla^2 \phi\|_{L^6 L^6} \|\nabla \hat{w}_1\|_{L^\infty L^3} + \|\nabla^2 \phi_t\|_{L^2 L^2} \|\nabla \hat{\phi}\|_{L^\infty L^\infty} \right)
\leq (|k| \|\hat{\psi}\|_{X^K} + 2 \|\phi\|_{X^K} + 2 \|\phi\|_{X^K}) (C_{H^2 \to W^{1,3}}(Q, \Omega) + C_{H^2 \to L^\infty}(Q, \Omega))
\times \left( 1 + \frac{c^2}{b} + \frac{c^4}{b^2} \right) \|\phi\|_{X^K}
\leq \hat{C}_f \rho \|\phi\|_{X^K}.
\]
By applying estimate (5.5) to equation (6.5) with homogeneous initial conditions, we obtain
\[
\|\hat{\psi}\|_{X^K} \leq \sqrt{C(\gamma, T) \hat{C}_f \rho} \|\hat{\phi}\|_{X^K},
\]
which after possibly decreasing \( \rho \) yields contractivity.

Since \( B^X_{\rho} \) is closed, we can make use of Banach’s Contraction Principle to conclude existence and uniqueness of a solution.

Remark 6.2. Note that \( C(\gamma, T) \) in (5.5) depends on the final time due to the use of Gronwall’s inequality. Actually, we do not expect that global in time well-posedness can be proven in the nonlinear case due to the fact that we must deal with a quadratic nonlinearity and only have weak damping in the equation for \( z \).
7. Singular limit for vanishing relaxation time. We next focus on proving a limiting result for (1.6) and (1.8) as $\tau \to 0$. Recall that $b = \delta + \tau c^2$ cf. (1.7) and the norms on the spaces $X^W$, $X^K$ depend on $\tau$, whereas the radius $\rho$ in (4.2), (6.1) is independent of $\tau$. We denote by $\bar{X}^W$, $\bar{X}^K$ the respective $\tau$-independent part of the norms defined in (4.2), (6.1), respectively, i.e.,

\begin{align}
\|\psi\|^2_{\bar{X}^W} &:= \|\psi_t\|^2_{L^2 H^1} + \|\psi\|^2_{W^{1,\infty} H^2}, \\
\|\psi\|^2_{\bar{X}^K} &:= \|\psi_t + \frac{c^2}{b}\psi_t\|^2_{L^2 H^1} + \|\psi_t + \frac{c^2}{b}\psi\|^2_{L^\infty H^2} + \|\psi\|^2_{L^\infty H^3}.
\end{align}

Moreover we recall the spaces for the initial data

\begin{align*}
X_0^W &:= H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega), \\
X_0^K &:= H^1_0(\Omega) \cap H^3(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega),
\end{align*}

(the only difference is in the regularity of $\psi^0$). Therewith, we can formulate a limiting result for (1.6) and (1.8).

**Theorem 7.1.** Let $c^2$, $b$, $T > 0$, and $k \in \mathbb{R}$. Then there exist $\bar{\tau}$, $\rho_0 > 0$ such that for all $(\psi^0, \psi_1, \psi_2) \in X_0^W$, the family $((\psi^\tau)_{\tau \in (0, \bar{\tau})}$ of solutions to (4.5) according to Theorem 4.1 converges weakly* in $\bar{X}^W$ to a solution $\check{\psi} \in \bar{X}^W$ of (1.5) with homogeneous Dirichlet boundary conditions (2.2) and initial conditions $\check{\psi}(0) = \psi_0$, $\check{\psi}_t(0) = \psi_1$.

The statement remains valid with the equations (4.5), (1.5), the spaces $X_0^W$, $\bar{X}^W$ and Theorem 4.1 replaced by the equations (6.7), (1.4), the spaces $X_0^K$, $\bar{X}^K$ and Theorem 6.1, respectively.

**Proof.** From the energy estimates in Theorems 4.1 (or 6.1), we have uniform boundedness of $(\psi^\tau)_{\tau \in (0, \bar{\tau})}$ in $\bar{X}^W$ (or in $\bar{X}^K$) and therefore existence of a weakly* $\bar{X}^W$ (or $\bar{X}^K$) convergent sequence $(\psi^\tau_{\ell})_{\ell \in \mathbb{N}}$. By compactness of embeddings, this sequence also converges strongly in $C^1(0, T; L^4(\Omega)) \cap C(0, T; W^{1,4}(\Omega))$. Its limit $\check{\psi}$ therefore lies in $\bar{X}^W$ (or in $\bar{X}^K$) and satisfies the initial conditions $\check{\psi}(0) = \psi_0$, $\check{\psi}_t(0) = \psi_1$.

To prove that $\check{\psi}$ also satisfies the respective PDEs, we test with arbitrary functions $v \in C_0^\infty(0, T; C_0^\infty(\Omega))$ and invoke the Fundamental Lemma of Calculus of Variations. To this end, we introduce an abbreviation for the nonlinear term in the respective equations, namely

\[ \mathcal{N}(\psi) = \begin{cases} \frac{\rho}{2\tau} (\psi_t)^2 & \text{for (1.5), (1.8)} \\
\frac{1}{c^2} \frac{\rho}{2\tau} (\psi_t)^2 + |\nabla \psi|^2 & \text{for (1.4), (1.6)}, \end{cases} \]

so that the $\tau$-dependent and the limiting equation can be rewritten in both Westervelt and Kuznetsov cases as

\begin{align}
\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - (\delta + \tau c^2) \Delta \psi_t - (\mathcal{N}(\psi))_t &= 0 \\
\psi_{tt} - c^2 \Delta \psi - \delta \Delta \psi_t - (\mathcal{N}(\psi))_t &= 0,
\end{align}

respectively. Note that by the regularity inherent in the spaces $X^W$, $X^K$ and $\bar{X}^W$, $\bar{X}^K$ respectively, $\psi^\tau$ satisfies (7.2) in $L^2(0; T; L^2(\Omega))$ and inserting $\check{\psi}$ into the left hand
side of (7.3) yields an \( L^2(0; T; L^2(\Omega)) \) function. Therewith, we get, for \( \hat{\psi}_t := \bar{\psi} - \psi^\ell \) and any \( v \in C_0^\infty(0, T; C_0^\infty(\Omega)) \)

\[
\int_0^T \int_\Omega \left( \psi_{ttt} - c^2\Delta \bar{\psi} - \delta \Delta \hat{\psi}_t - N(\bar{\psi})_t \right) v \, dx \, dt \\
= \int_0^T \int_\Omega \left( \psi_{tttt} - c^2\Delta \hat{\psi}_t - \delta \Delta \hat{\psi}_{tt} - (N(\bar{\psi})_t - N(\psi^\ell)_t) - \tau_t\psi_{ttt} - \tau_t c^2 \Delta \psi_{ttt} \right) v \, dx \, dt \\
= I - II - III.
\]

Above, we have that

\[ I = \int_0^T \int_\Omega \left( \psi_{ttt} - c^2\Delta \bar{\psi} - \delta \Delta \hat{\psi}_t \right) v \, dx \, dt \rightarrow 0 \text{ as } \ell \rightarrow \infty \]

due to the weak* convergence to zero of \( \hat{\psi}_t \) in \( \bar{X}^W \) (or \( \bar{X}^K \)),

\[ II = \int_0^T \int_\Omega \left( (N(\bar{\psi}) - N(\psi^\ell))_t \right) v \, dx \, dt \\
= -\int_0^T \int_\Omega \left( (N(\bar{\psi}) - N(\psi^\ell)) \right) v_t \, dx \, dt \\
= -\int_0^T \int_\Omega \left( k(\bar{\psi}_t + \psi^\ell)_t + \sigma(\nabla \bar{\psi} + \nabla \psi^\ell) \cdot \nabla \bar{\psi}_t \right) v_t \, dx \, dt \rightarrow 0 \text{ as } \ell \rightarrow \infty \]

due to the boundedness of \( (\psi_t)_{t\in\mathbb{N}} \) in \( \bar{X}^W \) (or \( \bar{X}^K \)) by \( \rho \), and the strong convergence to zero of \( \bar{\psi}_t \) in \( C^1(0, T; L^4(\Omega)) \cap C(0, T; W^{1,4}(\Omega)) \), and

\[ III = \tau_t \int_0^T \int_\Omega \left( \psi_{tttt} + c^2\Delta \psi_{tt} \right) v \, dx \, dt \\
= \tau_t \int_0^T \int_\Omega \left( \psi_{ttt} + c^2\Delta \psi_t \right) v_t \, dx \, dt \rightarrow 0 \text{ as } \ell \rightarrow \infty \]

due to the boundedness of \( (\psi_t)_{t\in\mathbb{N}} \) in \( \bar{X}^W \) (or \( \bar{X}^K \)), and \( \tau_t \rightarrow 0 \).

A subsequence-subsequence argument, together with uniqueness of the solution to (7.3) according to results in, e.g., [15, 16, 32, 33] yields convergence of the whole family \( (\psi^\tau)_{\tau\in(0,T)} \).

Remark 7.2. Note that, in contrast to [19], no compatibility condition on \( \psi_2 \) is needed, since no continuity of the limit \( \tilde{\psi}_t \) with respect to time arises and in the used energy estimates the \( \psi_2 \) term vanishes as \( \tau \rightarrow 0 \).

Remark 7.3. Theorem 7.1 also contains a regularity result on the respective solution \( \bar{\psi} \in \bar{X}^W \) and \( \psi \in \bar{X}^K \) of (1.5) and (1.4) with homogeneous Dirichlet boundary conditions (2.2) and initial conditions \( \bar{\psi}(0) = \psi_0 \in H^1_0(\Omega) \cap H^2(\Omega) \) or \( H^3(\Omega) \), \( \psi_t(0) = \psi_1 \in H^1_0(\Omega) \).

By comparing this regularity with the regularity results on the Westervelt equation from [15, 32] and with those for the Kuznetsov equation from [16, 33, 34], we get
• Westervelt equation:
  - \[15\] \((u_0, u_1) \in (H^1_0(\Omega) \cap H^2(\Omega))^2\)
  \[\Rightarrow u \in C^2(0,T; L^2(\Omega)) \cap H^2(0,T; H^1_0(\Omega)) \cap C(0,T; H^2(\Omega));\]
  - \[32\] \((u_0, u_1) \in (H^1_0(\Omega) \cap H^2(\Omega))^2\)
  \[\Rightarrow u \in H^2(0,T; L^2(\Omega)) \cap H^1(0,T; H^1_0(\Omega) \cap H^2(\Omega));\]
  - here: \((\psi_0, \psi_1) \in (H^1_0(\Omega) \cap H^2(\Omega))^2\)
  \[\Rightarrow u \in H^2(0,T; H^1_0(\Omega)) \cap W^{1,\infty}(0,T; H^2(\Omega));\]

• Kuznetsov equation:
  - \[34\] \((\psi_0, \psi_1) \in H^3(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega)\)
  \[\Rightarrow \psi \in C^1(0,T; H^1(\Omega)) \cap H^1(0,T; H^2(\Omega));\]
  - \[16\] \((u_0, u_1) \in (H^1_0(\Omega) \cap H^2(\Omega))^2\)
  \[\Rightarrow u \in C^2(0,T; L^2(\Omega)) \cap H^2(0,T; H^1_0(\Omega)) \cap C(0,T; H^2(\Omega));\]
  - \[33\] \((u_0, u_1) \in (H^1_0(\Omega) \cap H^2(\Omega))^2\)
  \[\Rightarrow u \in H^{4/3}(0,T; H^1_0(\Omega)) \cap H^2(0,T; H^2(\Omega));\]
  - here: \((\psi_0, \psi_1) \in H^1(\Omega) \cap H^3(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega) \cap H^2(\Omega)\)
  \[\Rightarrow u \in H^2(0,T; H^1_0(\Omega)) \cap W^{1,\infty}(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^3(\Omega));\]

We point out that these papers also contain results on global in time existence and exponential decay of solutions, as well as, in case of \([4, 34]\), on the Cauchy problem, and, in case of \([32, 33]\), in general, non-Hilbert \(L^p(\Omega)\) and \(W^{n,p}(\Omega)\) spaces.

8. Numerical results. As an illustration of our theoretical findings, we solve and compare numerically equations (1.5) and (1.8) in a 1D channel geometry. For the medium, we choose water with parameters

\[c = 1500 \text{ m/s}, \quad \delta = 6 \cdot 10^{-9} \text{ m}^2/\text{s}, \quad \rho = 1000 \text{ kg/m}^3, \quad B/A = 5;\]

cf. \([20, \text{ Chapter 5}]\). Discretization in space is performed by employing B-splines as basis functions within the framework of Isogeometric Analysis (IGA); see \([2, 13]\). For a detailed insight into the application of Isogeometric Analysis in nonlinear acoustics, we refer to \([11, 35]\). We use quadratic basis functions with maximum \(C^1\) global regularity and have 251 degrees of freedom for the channel length \(l = 0.2\) m. The nonlinearities are resolved by a fixed-point iteration with the tolerance set to \(\text{TOL} = 10^{-8}\).

After discretizing in space, we end up with a semi-discrete matrix equation and proceed with a time-stepping scheme. For the Westervelt equation (1.8), we employ the standard Newmark relations \([36]\) for second-order equations:

\[
\begin{align*}
\psi^{n+1} &= \psi^n + \Delta t \dot{\psi}^n + \frac{(\Delta t)^2}{2\gamma} \left((1 - 2\beta)\ddot{\psi}^n + 2\beta \dot{\psi}^{n+1}\right), \\
\dot{\psi}^{n+1} &= \dot{\psi}^n + \Delta t \left((1 - \gamma)\ddot{\psi}^n + \gamma \dot{\psi}^{n+1}\right),
\end{align*}
\]

\[1\] note that \(u\) there is the acoustic pressure, i.e., related to \(\psi\) by \(u = \varrho_0 \psi\);

\[2\] and additionally \((1 - k)u_0)^{-1}[c^2\Delta u_0 + b\Delta u_1 + ku_1^2] \in L^2(\Omega);

\[3\] and additionally \((1 - k)\psi)^{-1}[c^2\Delta \psi_0 + b\Delta \psi_1] \in H^1_0(\Omega);

\[4\] and additionally \((1 - k)\psi)^{-1}[c^2\Delta u_0 + b\Delta u_1 + ku_1^2 + 2|\nabla u_0|^2] \in L^2(\Omega);

\[5\] and additionally \((1 - k)\psi)^{-1}[c^2\Delta \psi_0 + b\Delta \psi_1 + \nabla \psi_0 \cdot \nabla \psi_1] \in H^1_0(\Omega)\]
realized through a predictor-corrector scheme analogously to [35, Algorithm 1]. In (8.1), $\Delta t$ denotes the time step size and $n$ and $n + 1$ indicate the current and the following time step, respectively.

For the Jordan-Moore-Gibson-Thompson equation with Westervelt-type nonlinearity (1.8), we use an extension of the Newmark relations to third-order models similar to the one employed in [10, Appendix B.2]:

\[
\begin{align*}
\psi_{n+1} &= \psi_n + \Delta t \dot{\psi}_n + \frac{(\Delta t)^2}{2!} \ddot{\psi}_n + \frac{(\Delta t)^3}{3!} \left( (1 - 6\beta) \dddot{\psi}_n + 6\beta \dddot{\psi}_n + \dddot{\psi}_{n+1} \right), \\
\dot{\psi}_{n+1} &= \dot{\psi}_n + \Delta t \ddot{\psi}_n + \frac{(\Delta t)^2}{2!} \left( (1 - 2\gamma) \dddot{\psi}_n + 2\gamma \dddot{\psi}_n + \dddot{\psi}_{n+1} \right), \\
\ddot{\psi}_{n+1} &= \ddot{\psi}_n + \Delta t \left( (1 - \eta) \dddot{\psi}_n + \eta \dddot{\psi}_{n+1} \right).
\end{align*}
\]

The average acceleration scheme corresponds to taking the Newmark parameters $(\beta, \gamma) = (1/4, 1/2)$ in (8.1) for the Westervelt equation and $(\beta, \gamma, \eta) = (1/12, 1/4, 1/2)$ in (8.2) for the Jordan-Moore-Gibson-Thompson equation (1.8), which is what we use in all the experiments.

We set the initial conditions to

\[
(\psi_0, \psi_1, \psi_2) = \left( 0, A \exp \left( -\frac{(x - 0.1)^2}{2\sigma^2} \right), 0 \right),
\]

with $A = 8 \cdot 10^4 \text{m}^2/\text{s}^2$ and $\sigma = 0.01$. Discretization in time is performed with 800 time steps for the final time $T = 45 \mu\text{s}$. The spatial and temporal refinement always remain the same for both equations and different values of the relaxation time $\tau$. All the numerical results are obtained with the help of the GeoPDEs package in MATLAB [42].

Figure 8.1 displays on the left side snapshots of the pressure wave $u = \rho \psi_t$ for a fixed relaxation time $\tau = 0.1 \mu\text{s}$ (right) Pressure wave for different relaxation parameters $\tau$ at final time.

Figure 8.1 displays on the left side snapshots of the pressure wave $u = \rho \psi_t$ obtained by employing equation (1.8) with the relaxation time set to $\tau = 0.1 \mu\text{s}$. We observe the nonlinear steepening of the wave as it propagates. On the right, we see how the pressure profile changes with decreasing relaxation time. The pressure wave for $\tau = 0 \mu\text{s}$ is computed by solving the Westervelt equation.
To further illustrate the results from Section 7, we solve equation (1.8) with the relaxation time varying over \( \tau \in [10^{-4}, 1] \mu s \) and compute the difference to the solution of the Westervelt equation (1.5). We plot the relative errors in the \( \bar{X}^W \) norm and in the \( C([0,T];H^1(\Omega)) \) norm, i.e.,

\[
\text{error}_{\bar{X}^W}(\tau) = \frac{\|\psi^T - \bar{\psi}\|_{\bar{X}^W}}{\|\bar{\psi}\|_{\bar{X}^W}}, \quad \text{error}_{C^1}(\tau) = \frac{\|\psi^T - \bar{\psi}\|_{C^1}}{\|\bar{\psi}\|_{C^1}};
\]

see Figure 8.2. The numerical errors decrease with the parameter \( \tau \), in agreement with the theoretical results of Theorem 7.1. Figure 8.2 even indicates a stronger result, i.e., strong convergence in the \( \bar{X}^W \) norm. For \( \tau = 10^{-10} \) s, the errors amount to \( \text{error}_{C^1}(\tau) \approx 6.43 \cdot 10^{-5} \) and \( \text{error}_{\bar{X}^W}(\tau) \approx 8.5 \cdot 10^{-4} \). The error plots also suggest a lower rate of convergence in the \( \bar{X}^W \) norm.

\begin{figure}[h]
\centering
\begin{tabular}{cc}
\includegraphics[width=0.5\textwidth]{error_C1.png} & \includegraphics[width=0.5\textwidth]{error_barXW.png}
\end{tabular}
\caption{Relative errors for varying relaxation time in (left) \( C([0,T];H^1(\Omega)) \) and (right) \( \bar{X}^W \).}
\end{figure}

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