Current mass dependence of the quark condensate and the constituent quark mass

M. Musakhanov

Theoretical Physics Dept, Uzbekistan National University,
Tashkent 7000174, Uzbekistan, e-mail: yousuf@iaph.silk.org

We discuss the current mass dependence of the basic quantities of the quark models—constituent quark mass \( M \) and quark condensate \( i < \psi^\dagger \psi > \). The framework of the consideration is QCD instanton vacuum model.

I. INTRODUCTION

Among the quark models, instanton vacuum based Effective Action approach is a most promising since in this model the hadron properties and their interactions features are closely related to the properties of QCD vacuum.

Without any doubts instantons are a very important component of the QCD vacuum. Their properties are described by the average instanton size \( \rho \) and inter-instanton distance \( R \). In 1982 Shuryak fixed them phenomenologically as

\[
\rho = \frac{1}{3} \text{fm}, \quad R = 1 \text{ fm}.
\]

From that time the validity of such parameters was confirmed by theoretical variational calculations and recent lattice simulations of the QCD vacuum (see recent review [3]). The presence of instantons in QCD vacuum very strongly affects light quark properties, owing consequent generation of quark-quark interactions. These effects lead to the formation of the massive constituent interacting quarks. This implies spontaneous breaking of chiral symmetry (SBCS), which leads to the collective massless excitations of the QCD vacuum—pions. The most important degrees of freedom in low-energy QCD are these quasiparticles. So instantons play a leading role in the formation of the lightest hadrons and their interactions, while the confinement forces are rather unimportant, probably. The properties of the hadrons and their interactions are concentrated in the QCD Effective Action in terms of quasiparticles. The features of light quarks placed into instanton vacuum are concentrated in the fermionic determinant \( \det \tilde{N} \) (in the field of \( N_+ \) instantons and \( N_- \) antiinstantons) calculated by Lee and Bardeen(LB) in 1979:

\[
det \tilde{N} = \det B, \quad B_{ij} = im\delta_{ij} + a_{ji},
\]

and \( a_{ij} \) is the overlapping matrix element of the quark zero-modes \( \Phi_{\pm,0} \) generated by instantons(antiinstantons). This matrix element is nonzero only between instantons and antiinstantons (and vice versa) due to specific chiral properties of the zero-modes and equal to

\[
a_{-+} = - < \Phi_{-,0} | i\hat{\partial} | \Phi_{+,0} >.
\]

The overlapping of the quark zero-modes provides the propagating of the quarks by jumping from one instanton to another one. So, the determinant of the infinite matrix was reduced to the determinant of the finite matrix in the space of only zero-modes. From Eqs. (2), (3) it is clear that for \( N_+ \neq N_- \)

\[
det_N \sim m|N_+ - N_-| \quad \text{which will strongly suppress the fluctuations of } |N_+ - N_-|.
\]

Therefore in final formulas we will assume \( N_+ = N_- = N/2 \).

In (2) we observe the competition between current mass \( m \) and overlapping matrix element \( a \sim \rho^2 R^{-3} \). With typical instanton sizes \( \rho \sim 1/3 \text{fm} \) and inter-instanton distances \( R \sim 1 \text{ fm} \), \( a \) is of the order of the strange current quark mass, \( m_s = 150 \text{ MeV} \). So in this case it is very important to take properly into account the current quark mass.

The fermionic determinant \( \det \tilde{N} \) averaged over instanton/anti-instanton positions, orientations and sizes can be considered as a partition function of light quarks \( Z_N \). Then the properties of the hadrons

*talk presented at the International Symposium on Hadrons and Nuclei, Seoul, February 20-22, 2001.
and their interactions are concentrated in the QCD Effective Action. We calculate this one via fermionic representation of $\det_N$ which provide easy way for the averaging over instanton collective coordinates – positions and orientations [9]. This approach leads to the Diakonov-Petrov (DP) Effective Action [3] with a specific choice of the degrees of freedom [2]. It was shown that DP Effective Action is a good tool in the chiral limit but failed beyond this limit, checked by the calculations of the axial-anomaly low energy theorems [6]. The solution of this puzzle related with the observation that the fermionisation of $\det_N$ is not unique procedure and another fermionic representation of $\det_N$ leads to a different choice of the degrees of freedom in the Effective Action. Within this approach it was proposed so called Improved Effective Action (IA) which is more properly taken into account current quark masses and satisfied axial-anomaly low energy theorems also beyond the chiral limit [8] at least at $O(m)$ order.

Completely another approach to the same problem was developed by Pobylitsa [9]. He directly summed up planar diagrams for the propagator in the instanton medium in large $N_c$ limit for two extreme cases: $N/VN_c^{-} > 0$ and $N/VN_c^{-} > \infty$. We will compare his result for constituent mass with our one and will calculate quark condensate within this approach too.

In the present case we concentrate on the calculation of the current mass dependence of the quark condensate $i < \bar{\psi} \psi >$. As a byproduct we find also current mass dependence of the constituent quark mass $M$. Since the quark condensate does not depend on the specific choice of the degrees of freedom in the Effective Action and entirely is defined by the current mass dependence of the partition function $Z_N$, it gives important information on the accuracy of the fundamental LB result by comparison with phenomenological data. LB result (2) itself has accuracy which is $O(m^2)$ order. We consider here the $m$ dependencies of quark condensate $i < \bar{\psi} \psi >$ and constituent quark mass $M$ within DP and IA and compare with results of slightly modified version of Improved Action (MIA), which has difference from IA on of $O(m^2)$ order terms. So, both DP and IA approaches must lead to the same $m$-dependence of the quark condensate and must coincide with MIA approach, at least within $O(m^2)$ accuracy. We consider first modification of Improved Action, and further calculate the current mass dependencies of abovementioned quantities within all variants of Effective Action. The comparison of these results with the result of the calculations within Pobylitsa approach provide independent test of the calculations. Another test of the results is provided by heavy quark limit, under the assumption that the gluon field strength $G^a_{\mu \nu}$ is much less than the square of quark mass $m^2$. We find that the quark condensate in all approaches based on instanton vacuum model has almost the same rather strong $m$ dependence and in the region $m > 0.3$ GeV they are in accordance with heavy quark approximation. As example, the strange quarks condensate $< s^0 s > \sim 0.5 < u^0 u >$ at $m_s \sim 0.15$ GeV. We find also rather strong $m$ dependence of the constituent quark mass $M$. Since in Pobylitsa, IA and MIA cases the total quark mass is $m + M$, the total mass is almost constant in the region $m < 0.2$ GeV. This dependencies contrasted very much naive expectations and need detailed phenomenological analysis.

II. MODIFICATION OF IMPROVED EFFECTIVE ACTION

First, accordingly Eqs. (2) and (3) and by introducing the Grassmanian $(N_+, N_-)$ variables $\Omega$, $\hat{\Omega}_j$ we represent

$$\det B = \int d\Omega d\hat{\Omega} \exp(\bar{\Omega}B\Omega),$$

where

$$\Omega B \Omega = \hat{\Omega}_i (im + a^T)_{ij} \Omega_j = -\Omega_j (\bar{\Phi}_{j,0}^+ (-i\hat{\partial} + im)_{i,j} \Phi_{i,0}) \hat{\Omega}_i$$

This formula is transformed to:

$$\bar{\Omega} B \Omega = \Omega_j (\bar{\Phi}_{j,0}^+ (i\hat{\partial}(i\hat{\partial} + im)^{-1} i\hat{\partial} + m^2 (i\hat{\partial} + im)^{-1})_{i,j} \Phi_{i,0}) \bar{\Omega}_i$$

$$= \Omega_j (\bar{\Phi}_{j,0}^+ (i\hat{\partial}(i\hat{\partial} + im)^{-1} i\hat{\partial})^{-1}_{i,j} \Phi_{i,0}) \bar{\Omega}_i + ...$$

where we are neglecting by $O(m^2)$ term, since Lee-Bardeen result for $\det_N$ itself was derived within this accuracy. The next step is to introduce $N_+, N_-$ sources $\eta_i$ and $\bar{\eta}_j$ defined as:
\[
\tilde{\eta}_i = -\Phi_{i,0}^\dagger \Omega_i \hat{\theta}_i, \eta_j = i \hat{\theta}_j \Phi_{j,0} \tilde{\Omega}_j
\]

Then \((\tilde{\Omega} B \Omega)\) can be rewritten as

\[
(\tilde{\Omega} B \Omega) = -\tilde{\eta}(i \hat{\theta} + im)^{-1} \eta = -\sum_{ij} \tilde{\eta}_i (i \hat{\theta} + im)^{-1} \eta_j
\]

and \(\text{det } B\) can be rewritten as

\[
\text{det } B = \int d\Omega d\tilde{\Omega} \exp(\Omega B \Omega)
\]

\[
= \left( \text{det}(i \hat{\theta} + im) \right)^{-1} \int d\Omega d\tilde{\Omega} D\psi D\psi^\dagger \exp \int dx (\psi^\dagger(x)(i \hat{\theta} + im)\psi(x)
\]

\[
+ \sum_{i} (\tilde{\eta}_i(x)\psi(x) + \psi^\dagger(x)\eta_i(x)))
\]

The integration over Grassmanian variables \(\Omega\) and \(\tilde{\Omega}\) (with the account of the \(N_f\) flavors \(\text{det}_N = \prod_f \text{det } B_f\)) provides finally the fermionized representation of fermionic determinant \([\text{F}]\) in the form:

\[
\text{det } B = \int D\psi D\psi^\dagger \exp \left( \int d^4x \sum_f \psi^\dagger_f(i \hat{\theta} + im_f)\psi_f \right)
\]

\[
\times \prod_f \left\{ \prod_{+} V_+[\psi^\dagger_f, \psi_f] \prod_{-} V_-[\psi^\dagger_f, \psi_f] \right\}
\]

where

\[
V_{\pm}[\psi^\dagger_f, \psi_f] = \int d^4x \left( \psi^\dagger_f(x)i \hat{\theta}\Phi_{\pm,0}(x; \xi_{\pm}) \right) \int d^4y \left( \Phi_{\pm,0}^\dagger(y; \xi_{\pm})(i \hat{\theta}\psi_f(y)) \right).
\]

Now the averaging over collective coordinates \(\xi_{\pm}\) become trivial problem. The further steps are the exponentiation and the bosonization of the integrand \([\text{F}]\). Finally, the corresponding partition function in terms of constituent quarks has a form \([\text{F}]\):

\[
Z_N = \int d\lambda_+ d\lambda_- D\Phi_+ D\Phi_- \exp \left( -S[\lambda_+, \Phi_+; \lambda_-, \Phi_-] \right),
\]

where

\[
S[\lambda_+, \Phi_+; \lambda_-, \Phi_-] = -\sum_{\pm} \left( N_{\pm} \ln \left( \frac{4\pi^2 \rho_{\pm}^2}{N_{\pm}} \right) - N_{\pm} \right) - S_\Phi + S_\psi,
\]

\[
S_\Phi = \int d^4x \sum_{\pm} (N_{f} - 1) \lambda_{\pm}^{-\frac{N_{f} - 1}{2}} (\text{det } \Phi_{\pm})^{-\frac{1}{2N_{f} - 1}},
\]

\[
S_\psi = -\text{Tr} \ln \left( \left( -\hat{k} + im_f \delta_{fg} + iF(k_1)F(k_2) \sum_{\pm} \Phi_{\pm,fg}(k_1 - k_2) \frac{1 + \gamma_5}{2} \right) \right)^{-1}.\]

Variation of the total action \(S[\lambda_+, \Phi_+; \lambda_-, \Phi_-]\) over \(\lambda_{\pm}\) leads to the saddle-point:

\[
\lambda_{\pm} = \left( N_{\pm}^{-1} \int d^4x (\text{det } \Phi_{\pm})^{-\frac{1}{2N_{f} - 1}} \right)^{(N_{f} - 1)},
\]

The additional variation over \(\Phi_{\pm}\) must vanish in the common saddle-point. Since we take \(N_+ = N_- = N/2\), this one is

\[
\Phi_{\pm,fg} = \Phi_{\pm,fg}(0) = M_f \delta_{fg},
\]
and
\[ \lambda_\pm = \lambda = \frac{2V}{N} \prod_f M_f, \]

This condition leads to the saddle-point equation for the momentum dependent constituent mass \( M_f(k) \), i.e.,
\[ M_f(k) = M_f F^2(k). \tag{14} \]

The contribution of the quark loop to the saddle-point equation is
\[ \text{Tr} \ln[(-\hat{k} + im + iF^2(k) \sum_{\pm} \Phi_{\pm} \frac{1 \pm \gamma_5}{2}(-\hat{k} + im)^{-1}]. \tag{15} \]

Then we get the saddle-point equation
\[ \frac{N}{V} = 4N_c \int d^4k \frac{M_f F^2(k)(m_f + M_f F^2(k))}{(2\pi)^4 \hat{k}^2 + (m_f + M_f F^2(k))^2} \tag{16} \]

The form-factor \( F(k) \) is related to the zero–mode wave function in momentum space \( \Phi_{\pm}(k; \xi_{\pm}) \) [6]. We use simplified expression for this form-factor:
\[ F(k) = \frac{L^2}{L^2 + k^2}, \tag{17} \]

where \( L^2 \sim 2/\rho^2 = 0.72 \text{GeV}^2 \), which was proposed in [11].

The important steps in the derivation of these formulas (12) and (13) are:
1. The fermionisation of (6) (which is in fact not unique procedure);
2. Independent averaging over positions and orientations of the instantons, due to the small packing parameter of the instanton media – \( (\rho/R)^4 \sim (1/3)^4 \);
3. The exponentiation and the bosonization of the partition function, described in [6].

The matrices \( \Phi_{\pm} \), whose usual decomposition is \( \Phi_{\pm} = \exp(\pm \frac{i}{2} \phi) \text{MS exp}(\pm \frac{i}{2} \phi) \), \( \phi \) and \( \sigma \) being \( N_f \times N_f \) matrices, describes mesons and \( M_{fg} = M_f \delta_{fg} \). At the saddle-point \( \sigma = 1, \phi = 0 \). The usual decomposition for the pseudoscalar fields \( \phi = \sum_8 \lambda_i \phi_i \) may be used. These mesons are considered as a small fluctuation near the saddle point.

The account of the fluctuations of number of instantons \( N \) can be easily done. Let \( N_{\pm} = 0.5(N \pm \Delta), \Delta << N \). Then assuming the saddle-points \( \Phi_{\pm,fg} = \delta_{fg} M_{f\pm}, M_{f\pm} = M_f(1 \pm \delta_f), \delta_f << 1 \) we find additional to (16) another saddle-point equation
\[ \Delta/V = 4N_c \delta_f \int d^4k \frac{m_f M_f F^2(k)}{(2\pi)^4 \hat{k}^2 + (m_f + M_f F^2(k))^2} \tag{18} \]

Taking into account the definition of the condensate \( i < \psi^\dagger \psi > \) [27], we find
\[ m_f \delta_f = \frac{\Delta}{V i < \psi^\dagger \psi_f > } (1 + O(m_f^2)) \tag{19} \]

This formula leads to the \( \Delta \)-distribution, which is in accordance with general theorems [6].

### III. CURRENT MASS DEPENDENCE OF CONSTITUENT MASS

The saddle-point equation (16) leads to the momentum dependent constituent mass \( M_f(k) = M_f F^2(k) \). The constituent quark propagator, as it is follows from (12), has a form:
\[ S = (-\hat{k} + i(m_f + M_f F^2(k)))^{-1}, \tag{20} \]

In IA approach analogous saddle-point condition:
\[ \frac{N}{V} = 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M_f^{IA}F^2(k)(1 + m_f^2/k^2)(m_f + M_f^{IA}F^2(k)(1 + m_f^2/k^2))}{k^2 + (m_f + M_f^{IA}F^2(k)(1 + m_f^2/k^2))^2} \]  \hfill (21)

also define \( M_f^{IA} \) and the constituent quark propagator has a form:

\[ S_f^{IA} = (-\hat{k} + i(m_f + M_f^{IA}F^2(k)(1 + m_f^2/k^2)))^{-1}, \hfill (22) \]

On the other hand, DP Effective Action leads to the propagator:

\[ S_f^{DP} = (-\hat{k} + M_f^{DP}(k))^{-1}, \hfill (23) \]

where \( M_f^{DP}(k) = M_f^{DP}F^2(k) \) is followed from analogous saddle-point equation:

\[ \frac{4VN_c}{N} \int \frac{d^4k}{(2\pi)^4} \frac{M_f^{DP}F^2(k)}{k^2 + M_f^{DP}F^2(k)} = 1 - \frac{M_f^{DP}m_fVN_c}{2\pi^2\rho^2}, \hfill (24) \]

As was mentioned in the Introduction, Pobylitsa \[9\] in quenched approximation directly summed up planar diagrams for the propagator in the instanton medium in large \( N_c \) limit for two extreme cases: \( N/VN_c^- > 0 \) and \( N/VN_c^- > \infty \). In the first (and most interesting) case his result can be summarized in the form:

\[ S_f = (-\hat{k} + i(m + M_f(k)))^{-1}, \hfill (25) \]

where

\[ M_f(k) = M_0F^2(k)[(1 + m^2/d^2)^{1/2} - m/d], \]

\[ d = \left( \frac{0.08385}{2N_c} \right)^{0.5} \frac{8\pi\rho}{R^2} = 0.198 \text{ GeV}. \hfill (26) \]

Fig.1 represent different versions of the current mass dependence of constituent mass derived from saddle-point equations (16), (21), (24) and Pobylitsa result (26).

FIG. 1. Current mass dependence of the constituent mass: Solid line – Modified Improved Action calculations. Long dashed line – Pobylitsa approach calculations. Dashed line – Improved Action calculations. Short dashed line – DP Action calculations.
IV. CURRENT MASS DEPENDENCE OF THE QUARK CONDENSATE

First, we calculate the quark condensate by using the evident formula

\[ i<\psi^\dagger \psi_f> = V^{-1} Z_N^{-1} \frac{\partial Z_N}{\partial m_f} \]

\[ = \text{Tr} \left[ (\hat{k} + im_f + iM_f(k))^{-1} - (\hat{k} + im_f)^{-1} \right]. \] (27)

In (27) the saddle-point condition was taken into account. It is evident, the condensates in IA and Pobylitsa approaches are calculated with similar formula.

With DP Action \[6\] the condensate is

\[ i<\psi^\dagger \psi_f>_{DPW} = \frac{N_c M_f^{DP}}{2\pi^2 \rho^2}. \] (28)

Simple numerical calculations leads to the condensate as a function of current mass presented in Fig.2. We present here also heavy quark approximation result for the quark condensate. In this limit, for heavy quarks we have to use the expansion over small parameter \( G/m^2 \) under the assumption that the gluon field strength \( G_{\mu\nu} \) is much less than the square of quark mass \( m^2 \) \([12,13]\). Then

\[ \int d^4x i<\psi^\dagger \psi> = \text{Tr} \left( \frac{i}{\hat{P} + im_c} - \frac{i}{\hat{p} + im} \right) = m \text{Tr} \left( \frac{1}{\hat{P}^2 + m_c^2 + \frac{g^2}{24\pi^2m^2} \text{tr}_c G_{\alpha\beta} G_{\alpha\gamma} \gamma_\gamma + ...} \right) \] (29)

Here \( \sigma G \equiv \sigma_{\mu\nu} G_{\mu\nu}, \sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu], \hat{P} \equiv P_\mu \gamma_\mu \) and \( P_\mu = iD_\mu = i(\partial_\mu - igA_\mu^a t^a). \)

In the instanton vacuum:

\[ \int d^4x r g^2 G_{\alpha\beta}^2 = (4\pi)^2 N, \int d^4x r g^3 G_{\alpha\beta} G_{\alpha\gamma} G_{\beta\gamma} = -i(4\pi)^2 \frac{6}{5\rho^2} N \]

and

\[ i<\psi^\dagger \psi> = \frac{2}{3mR^4} + \frac{4}{75\rho^2 m^3 R^4} = \frac{2}{3mR^4} \left( 1 + \frac{12}{150\rho^2 m^2} \right) \] (30)

FIG. 2. Current mass dependence of the quark condensate: Solid line – Modified Improved Action calculations. Long dashed line – Pobylitsa approach calculations. Dashed line – Improved Action calculations. Short dashed line – DP Action calculations. Dashed-dot line – Heavy quark approximation \([30]\).
We see that MIA, IA, DP and Pobylitsa results almost coincide with each other and in the good correspondence with heavy quark approximation at $m > 0.3 \text{GeV}$.

V. CONCLUSION

It were investigated the current quark mass dependencies of quark condensate and constituent quark mass in QCD instanton vacuum model. It were considered different approaches:

1. Diakonov&Petrov effective action (see recent papers [6]);
2. Improved Action [8] and presented here Modified Improved Action.
   They are essentially based on Lee& Bardeen result for the quark determinant in instanton vacuum background [9].
3. Direct summation of planar diagrams for the propagator in the instanton medium in quenched approximation [10].

All of these approaches leads to rather fast dependencies of quark condensate and constituent quark mass on current quark mass, as were demonstrated in Figs 1,2. Then, the strange quarks condensate $<s'^s> \sim 0.5 <u'^u>$ at $m_s \sim 0.15 \text{GeV}$ and total quark mass $m + M$ is almost constant in the region $m < 0.2 \text{GeV}$.

We conclude that strange quark physics might be very different from usual expectation based on old-fashioned quark model and demand careful phenomenological reanalysis.

Acknowledgments

I am very grateful to M.Birse, D.Diakonov, V.Petrov, P.Pobylitsa and M.Polyakov for useful discussions.

[1] E. V. Shuryak, Nucl. Phys. B 203, 93, 116 (1982)
[2] D. Diakonov and V. Petrov, Nucl. Phys. B 245, 259 (1984)
[3] T. De Grand, A. Hasenfratz, T. Kovacs, Progr. Theor. Phys. Suppl.131, 573 (1998)
[4] C. Lee, W. A. Bardeen, Nucl. Phys. B 153, 210 (1979)
[5] M. M. Musakhanov, F. C. Khanna, Phys. Lett. B 395, 298 (1997)
[6] D.I. Diakonov, M.V. Polyakov, C. Weiss, Nucl. Phys. B 461, 539 (1996);
   Dmitri Diakonov, hep-ph/9602375
[7] E. Di Salvo, M.M. Musakhanov, Europ. Phys. J. C 5, 501 (1998)
[8] M.M. Musakhanov, Europ. Phys. J. C9, 235 (1999), hep-ph/9810295
[9] P. Pobylitsa, Phys. Lett. B 226, 387 (1989);
   Quarks in the instanton vacuum I. Quark propagator, preprint 1598, LNP, 1990
[10] V.F. Tokarev, preprint INP P-0406, Moscow 1985;
    Soviet J. Teor. Math. Phys., 73, 223 (1987)
[11] V.Yu. Petrov, M.V. Polyakov, R. Ruskov, C. Weiss and K. Goeke, hep-ph/9807229
[12] A.I.Vainshtein, V.I.Zakharov, V.A.Novikov, M.A.Shifman, Sov. J. Nucl. Phys. 39, 77 (1984).
[13] F. Araki, M. Musakhanov, H. Toki, Axial Currents of Virtual Charm in Light Quark Processes, hep-ph/9808290.