Leaf of Leaf Foliation and Beltrami Parametrization in $d > 2$ dimensional Gravity

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Abstract

This work shows the existence of a $d > 2$ dimensional covariant “Beltrami vielbein” that generalizes the $d = 2$ situation. Its definition relies on a sub-foliation $\Sigma_{d-1}^{ADM} = \Sigma_{d-3} \times \Sigma_2$ of the Arnowitt–Deser–Misner leaves of $d$-dimensional Lorentzian manifolds $M_d$. $\Sigma_2$ stands for the sub-foliating randomly varying Riemann surfaces in $M_d$. The “Beltrami $d$-bein” associated to any given generic vielbein of $M_d$ is systematically determined by a covariant gauge fixing of the Lorentz symmetry of the latter. It is parametrized by $\frac{d(d+1)}{2}$ independent fields belonging to different categories. Each one has a specific interpretation. The Weyl invariant field sector of the Beltrami $d$-bein selects the $\frac{d(d-3)}{2}$ physical local degrees of freedom of $d > 2$ dimensional gravity. The components of the Beltrami $d$-bein are in a one to one correspondence with those of the associated Beltrami $d$-dimensional metric. The Beltrami parametrization of the Spin connection and of the Einstein action delivers interesting expressions. Its use might easier the search of new Ricci flat solutions classified by the genus of the sub-manifold $\Sigma_2$. A gravitational “physical gauge” choice is introduced that takes advantage of the geometrical specificities of the Beltrami parametrization. Further restrictions simplify the expression of the Beltrami vielbein when $M_d$ has a given spatial holonomy. This point is exemplified in the case of $d = 8$ Lorentzian spaces with $G_2 \subset SO(1,7)$ holonomy. The Lorentzian results presented in this paper can be extended to the Euclidean case.

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1 Introduction

A pillar of $d = 2$ gravity is the holomorphic and antiholomorphic factorization of the Polyakov path integral that sums over all configurations of Riemann surfaces [1]. Euclidean $d = 2$ gravity is a vast subject. Its so-called Beltrami representation expresses the three component of the bidimensional metric $g_{\alpha\beta}$ in function of its Beltrami differential $\mu_\tau^\alpha$ and its conformal factor $\Phi$. The squared infinitesimal line length on Riemann surfaces is represented as the covariantly factorized expression $ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta = \exp \Phi |d\tau + \mu_\tau^\alpha dz|^2$. The covariant change of field variables $g_{\alpha\beta} \rightarrow \mu_\tau^\alpha, \mu_\tau^\beta, \Phi$ enforces ab initio all relevant path integral factorization properties when building the local quantum field theories of theories coupled $d = 2$ gravity. It clarifies the proofs of many of their properties. A transparent BRST invariant construction defines the string theory gauge fixing by equating the Beltrami differential $\mu_\tau^\alpha$ to the moduli of the string worldsheets modulo the modular symmetry. The explicit holomorphic and antiholomorphic factorization of the Beltrami representation is useful for examining the classification of the bidimensional conformal anomalies, the determination of the heterotic theories and of the Liouville theory, etc... See [2] and [3] for the definition and some applications of the Beltrami parametrization in bidimensional gravity and supergravity.

The present work investigates the possibility of generalizing the Beltrami parametrization of the bidimensional gravity in the case of higher dimensional gravitational theories. It shows that the $d = 2$ case is the initial condition of a recurrence that defines such a generalization.

The Arnowitt Deser Misner (ADM) paradigm is of great relevance when studying gravity. It eventually determines the famous ADM metric [1]. Its fundamental principle is to consider any given Lorentzian manifold $M_d$ as a set of $(d - 1)$-dimensional leaves $\Sigma_{d-1}$ covariantly ordered by the Lorentz time rather than as a disordered set of points. The present work extends this framework. It introduces a refined leaf of leaf subfoliation that decomposes $\Sigma_{d-1}$ into $\Sigma_2 \times \Sigma_{d-3}$. It is quite obvious that the topological structure of certain manifolds $M_d$ can globally complicate the definition of such a sub-folation of their ADM leaves but, in this work, one only considers those that admit at least locally a leaf of leaf decomposition. This decomposition is the key to determine the generalized "Beltrami parametrization" for the $d$-bein and the metric of $M_d$ for $d > 2$ modulo the reparametrization invariance. Of course the appellation Beltrami field is taken in an enlarged sense all across this article.

The leaf of leaf subfoliation of $M_d$ covariantly separates the coordinates of $M_d$ as $\{\tau, z, \overline{z}, x^1, \ldots, x^{d-3}\}$. $\tau$ is the real Lorentz time coordinate of $M_d$. $(z, \overline{z})$ is the complex coordinate of $\Sigma_2$ and the $x^i$'s $(i = 3, \ldots, d - 1)$ are the $d - 3$ real spatial coordinates of $\Sigma_{d-3}$. This geometrical decomposition goes one step further than the simpler ADM leaf decomposition that covariantly separates the coordinates of $M_d$ as $\{\tau, \vec{x}\}$ but provides no inner structure for the leaves $\Sigma_{d-1}$. The Beltrami metric provided by the leaf of leaf decomposition is thus a refinement of the ADM metric for any given value of $d > 2$. The fields that determine its components fall in different categories classified according to their Weyl weights, each one with a different gravitational interpretation. The Beltrami $d$-bein is covariantly parametrized by the same $\frac{d(d+1)}{2}$ fields as the Beltrami $d$-dimensional metric. The whole formalism can be adapted to the situation where the primary ADM type foliation is done along a space-like direction. Thus, it might also be of interest in the context of Kaluza–Klein compactifications and also for studying gravitational solutions with various possible choices of the metric signature.

The leaf of leaf framework $\Sigma_{d-1}^{ADM} = \Sigma_2 \times \Sigma_{d-3}$ suggests that the sub-leaves $\Sigma_{d-3}$ can be perhaps used to mathematically concentrate the gravitational physical degrees of freedom of gravity with possible interactions with other states on the boundary $\partial \Sigma_{d-3}$ (if the latter exist). In fact, a well identified set of the Weyl invariant components of the fields that compose the Beltrami $d$-bein covariantly expresses the $\frac{d(d-3)}{2}$ physical gravitational degrees of freedom. Moreover, one can use the conformal gauge to gauge fix the inner metric of $\Sigma_2$. The remaining fields of the Beltrami parametrization can be identified as defining the conformal factor, $d - 3$ rescaling functions for the spatial coordinates of $\Sigma_{d-3}$, both ADM lapse and shift functions and the Beltrami differential of $\Sigma_2$. One can easily check $\frac{d(d+1)}{2} = \frac{d(d-3)}{2} + 1 + (d - 3) + 1 + (d - 1) + 2$.

The quite appealing physical interpretation of all the fields that covariantly compose the $d$-dimensional Beltrami parametrization could help for deciding which field variables should be quantized in non perturbative quantum gravity. Getting ab initio the $\frac{d(d-3)}{2}$ gravitational physical degrees of freedom as a subset of the $\frac{d(d+1)}{2}$ local
fields that parametrize the components of the Beltrami $d$-metric is a progress as compared to their definition in the ADM formulation. York shows in [8] that for the latter the $\frac{(d-3)}{2}$ physical gravitational degrees of freedom are the equivalence classes of the components of the ADM leaf inner $(d-1)$-metric defined modulo reparametrization and Weyl transformations. The difference is striking. In the leaf of leaf framework, a local definition of the gravitational field components that compose the observables is enforced from the beginning as a well identified subset of the fields that parametrize the generalized Beltrami $d$-metric and are considered as fundamental fields. In the ADM formulation, the construction of the observables implies a non trivial BRST invariant quantum field theory gauge fixing process for extracting the relevant Weyl invariant parts of the ADM components of the metric as they are defined by York (for instance by using an unimodular gauge choice [9]).

The paper is written in a bottom to top approach. It first shows how the $d = 2$ Beltrami parametrization generalizes in $d = 3$. Going from the case $d = 3$ to the case $d = 4$ is more involved than going from $d = 2$ to $d = 3$. The technical reason is that the off-diagonal part of the Beltrami vierbein is more complicated than that of the Beltrami dreibein. Once this point is resolved and one succeeds in computing the four dimensional Beltrami parametrization, the logics of the inductive process becomes transparent. It allows to systematize the construction of the Beltrami $d$-bein and the corresponding Beltrami metric for all values $d > 2$.

The $d > 3$ situation is of greater relevance for physicists since both for $d = 2$ and $d = 3$ the little group of $SO(1, d-1)$ is too small to admit the propagation of Spin 2 particles. The four dimensional case opens new perspectives. The sub-foliation of its three dimensional ADM leaves $\Sigma_3$ by a Riemann surface $\Sigma_2$ is simplest though non trivial since it provides a one-dimensional sub-manifold $\Sigma_1$ and one may heuristicly consider that both physical helicity states of the graviton propagate along $\Sigma_1$. In this particular case, the complex coordinates $z$ and $\Sigma$ of the sub-foliating surface $\Sigma_2$ share some resemblance with the light-cone coordinates $x^\pm$ giving a simpler and more symmetrical expression for the Beltrami vierbein as shown in section 5.

In fact, the basic result of this work is the generic Beltrami parametrization of the $d$-dimensional metric that is established in [12]. It reads as follows:

$$ds^2 = -N^2\left(d\tau + \sum_{i=3}^{d-1} \mu_i^j dx^j\right)^2 + \exp \Phi \left|dz + \mu_3^j dx^j + \mu_4^3 dx^3 + \ldots + \mu_{d-1}^3 dt^{d-1} + \mu_2^3 d\tau\right|^2$$

$$+ \sum_{i=3}^{d-1} \sum_{j=3}^{d-1} \sum_{\ell=3}^{d-1} \left(\mu_i^j dx^3 + \ldots + \mu_j^{i-1} dx^{j-1} + dx^i + \mu_j^{i+1} dx^{i+1} + \ldots + \mu_j^3 d\tau\right)^2.$$  (1)

In this expression, one has the antisymmetry properties $\mu_i^j = -\mu_j^i$ and $\mu_i^3 = -\mu_3^i$ and an easy counting shows that the metric (1) is indeed parametrized by $\frac{(d+1)}{2}$ independent fields.

The found metric (1) coincides with the standard Euclidean bidimensional Beltrami metric $ds^2 = \exp \Phi |dz + \mu_3^i dx^i|^2$ when $d\tau = dx^3 = 0$. Some factorization properties occur because of the $z \leftrightarrow \Sigma$ symmetry symmetry of the Riemann surfaces $\Sigma_3$ that sub-foliate the ADM leaves $\Sigma_{d-1}^{ADM}$ of $\mathcal{M}_d$. The paper establishes the transformation laws under the Weyl and reparametrization symmetries of all the Beltrami fields that compose the metric (1). They are expressed under the form of their BRST symmetry transformations. The latter are obtained by generalizing the simpler algebraic technics currently used in the bidimensional case [2].

The $\frac{(d-3)}{2}$ fields $\mu_i^m \equiv (\mu_3^i, \mu_4^3, \mu_5^i)$ build a geometrically meaningful Weyl invariant subset of the fields that figure in the definition of the Beltrami metric (1). The claim is that their excitations can be locally identified (at least perturbatively) as the above mentioned $\frac{(d-3)}{2}$ gravitational physical degrees of freedom that possibly propagate in $d > 2$ dimensional Lorentzian manifolds. It suggests that the dynamical gravity physical

*For $d = 2$ and $d = 3$, the ghost loops of semi-perturbative quantum gravity give opposite contribution to the closed loops of all propagating metric field components and no room is left for the propagation of gravitational physical degrees of freedom. Thanks to the BRST symmetry, this property remains true whichever (consistent) gauge choice one uses to fix the path integral zero modes due to the reparametrization invariance. For $d > 3$, there are extra loop contributions for the $\frac{(d-3)}{2}$ physical degrees of freedoms. The cutting rules of those loops induce the particle interpretation of $d > 3$ gravity theories. The covariant sub-foliation of the ADM leaves as $\Sigma_{d-1}^{ADM} = \Sigma_{d-3} \times \Sigma_3$ directly parametrize these gravitational $\frac{(d-3)}{2}$ physical degrees of freedom. The latter point can be verified by expressing the Spin connection and the Einstein action in function of the Beltrami fields and by checking the resulting propagators. This improves the York classical analysis [8] that identifies the gravity degrees of freedom in a non local way, as the equivalence classes of the ADM leaf metrics defined modulo Weyl invariance. In fact [9] underlines that stochastic quantization of gravity indicates quite naturally the property that the gravity physical observables are defined by the functional of
allows one to bypass this construction.

The generalized Beltrami metric \( \text{(1)} \) and the associated Einstein action is best understood by using the vielbein and Spin connection first order formalism. The paper shows that the number \( d^2 \) of the components of a generic \( d \)-bein \( e_\mu \) can be covariantly (i.e., by preserving the \( \text{Diff}_d \) symmetry) reduced down to \( \frac{d(d+1)}{2} \) independent components by gauge fixing the \( \frac{d(d-1)}{2} \) local freedoms offered by the Lorentz gauge symmetry \( \text{SO}(1, d-1) \subset \text{SO}(1, d-1) \times \text{Diff}_d \) of the complete gravitational local symmetry. By doing this appropriately, one determines the covariant Beltrami \( d \)-bein that is to be displayed in \( \text{(109)-(111)} \). Its generic expression remarkably generalizes the Beltrami zweibein formulae as originally written in \( \text{(2)} \). The \( d \)-Beltrami metric \( \text{(1)} \) is then computed from the standard quadratic relation between a vielbein and a metric. The one to one relation between the \( \frac{d(d+1)}{2} \) components of the Beltrami \( d \)-bein and of the Beltrami \( d \)-metric as well as their relationship with the ADM fields deepen the geometrical understanding of all fields that parametrize the Beltrami metric \( \text{(1)} \).

The Spin connection in the Beltrami parametrization is obtained by solving covariant algebraic constraints on the torsion 2-form \( T = de + \omega \wedge e \) when \( e \) is the Beltrami vielbein. The Einstein action can then be expressed as an algebraic quadratic expression of the Spin connection modulo boundary terms. In fact, the Beltrami parametrization expresses the Einstein action under a form analogous to that of its ADM expression but with some further refinements. This paper concretely computes the Beltrami Spin connection as well as the Einstein action in the three dimensional case and establishes the relevant linear equations satisfied by the Beltrami Spin connection in four dimensions. These results illustrate the general methodology for using the Beltrami parametrization.

The Beltrami metric formula \( \text{(1)} \) further simplifies if the space like part of \( \mathcal{M}_d \) has a given holonomy. Extra freedoms occur in this case and add up to those of the local Lorentz invariance. They offer more possibilities to reduce the number of independent fields involved in the Beltrami parametrization. This paper gives the example of the \( d = 8 \) space-times with holonomy \( G_2 \subset \text{SO}(1, 7) \) where \( G_2 \) is the smallest exceptional rank 2 group. \( G_2 \) has 14 generators, so that the 28 freedoms offered by the \( \text{SO}(1, 7) \) gauge symmetry get enhanced into \( 28 + 14 = 42 \) freedoms, allowing one to express the \( d = 8 \) Beltrami metric in a simpler form. The formal resemblance between this restricted \( d = 8 \) Beltrami metric with that of a generic four dimensional Beltrami metric reminds other similarities known to exist (though in a very different context) between the Beltrami metric (1) when \( \gamma \) is a complex constant and one imposes the existence of two Killing vector fields.

The paper is organized as follows. Sections 2 and 3 are useful for a better self-consistency of the whole presentation. Section 2 summarizes the geometrical BRST methods for a better mastering of the reparametrization symmetry at the quantum level. Section 3 is a reminder of the \( d = 2 \) Euclidean gravity Beltrami parametrization methodology and fixes notations that are to be generalized in higher dimensions. The new results are presented in the further sections.

Section 4 details how the \( d = 2 \) Beltrami parametrization can be generalized in three dimensions. It displays in this case the computation of the Spin connection and of the Einstein action in the Beltrami parametrization.

Section 5 explains the four dimensional case and its new features as compared to the case \( d = 3 \) of section 4. 5.1 and 5.2 also suggest that using of the Beltrami parametrization might easier the search of new four dimensional Ricci flat solutions. When \( \Sigma_2 \) has genus zero the Ricci flat solution of \( \text{(1)} \) is quite obviously the Schwarzschild solution. When the sub-foliating manifold \( \Sigma_2 \) has the topology of a torus, the Beltrami parametrization suggests that the so-called axisymmetric Weyl metric \( \text{(11)} \) is a particular case of the Beltrami metric \( \text{(1)} \) when \( \gamma \) is a complex constant and one imposes the existence of two Killing vector fields.

Section 6 proves \( \text{(1)} \) by computing the generic \( d \)-dimensional covariant Beltrami \( d \)-bein and afterward the Beltrami \( d \)-metric. It expresses various considerations about the physical relevance of the sub-foliation of ADM leaves according to \( \Sigma_{d-1}^{\text{ADM}} = \Sigma_2 \times \Sigma_{d-3} \) and identifies the \( \frac{d(d-3)}{2} \) propagating gravitational physical degrees of freedom as the subset of fields \( \mu_i^a = (\mu_1^a, \mu_2^a, \mu_3^a) \) for \( 3 \leq i, j \leq d - 1 \). It explains that the rest of the Beltrami fields \( \Phi, N \) and \( (\mu_1^e, \mu_2^e, \mu_3^e), N^i \) and \( \mu_2^e, \mu_3^e \) in \( \text{(1)} \) are respectively related to the conformal factor field, the ADM lapse and shift vector, \( d - 3 \) independent dilatation factors for each spatial coordinates \( x^i \) of \( \Sigma_{d-3} \) and the Beltrami differential of \( \Sigma_2 \) at fixed \( x^i \) and \( \tau \). A gravitational "physical gauge choice" for the reparametrization invariance is presented that takes advantage of the geometrical specificities of the metrics defined modulo Weyl transformation and that a BRST invariant gauge fixing of gravity in an unimodular gauge allows one to represent observables as functionals of unimodular metrics. The use of the covariant generalized Beltrami parametrization allows one to bypass this construction.
Beltrami metric. In this gauge, the Einstein action only depends on the \( \frac{d(d-3)}{2} \) gravitational physical fields and the time lapse and shift functions.

Section 7 indicates further simplifications of the Beltrami parametrization that may occur when the Lorentzian manifold \( M_d \) has a spatial holonomy \( G \subset SO(1,d-1) \). This is exemplified in the case \( d = 8 \) with \( G_2 \subset SO(1,7) \) holonomy.

The conclusion speculates about about a possible stringy origin of the Riemann surfaces \( \Sigma_2 \) that subfoliate the ADM leaves of pure gravity for tentatively giving better perspectives about the gravitational path integral definition.

Appendix A computes the linear equations that determine the four dimensional Spin connection \( \omega(e) \) in function of the Beltrami vierbein.

Appendix B solves them for three dimensional case.

2 Reminder of the classical ghost field unification for gravity

This section is a reminder of the geometrical method for determining the gravitational BRST symmetry that is used in the further sections for computing the BRST transformation rules of \( d \)-dimensional Beltrami fields.

The gravity fields are the vielbein \( e^a \) and the Spin connection \( \omega^{ab} \) in first order formulation. (Here and elsewhere the flat indices \( a, b, \ldots \) are Lorentz indices that run from 1 to \( d \) for \( M_d \).) Their field strengths are the torsion \( T = de + \omega \wedge e \) and the Lorentz curvature \( R = d\omega + \omega \wedge \omega \). \( \xi^a \) and \( \Omega^{ab} \) are the anticommuting reparametrization vector ghost and the Lorentz symmetry 0-form ghost, respectively. The exterior derivative is \( d = dx^a \partial_a \). \([5]\) proves that the nilpotent BRST symmetry operation \( s \) of general relativity is defined by three covariant and geometrical “horizontality conditions”. Their dependence on \( e \) and \( \omega \) and on their ghosts \( \xi \) and \( \Omega \) is through the “ghostified” vielbein \( \tilde{e}^a \equiv \exp i_{\xi} e^a = e^a + \xi^a, \omega^a \) and the “ghostified” Spin connection \( \tilde{\omega}^{ab} \equiv \omega^{ab} + \Omega^{ab} \).

The gravitational BRST symmetry is in fact defined as follows:

\[
\tilde{e} = \exp i_{\xi} e \\
\tilde{T} = (d + s)\tilde{e} + (\omega + \Omega)\tilde{e} = \exp i_{\xi} T \\
\tilde{R} = (d + s)(\omega + \Omega) + (\omega + \Omega)(\omega + \Omega) = \exp i_{\xi} R = \exp i_{\xi}(d\omega + \omega \wedge \omega).
\]

The BRST equations \([2]\) hold true independently of the definition of the Einstein action and its possible gauge fixing \([5]\). If one imposes the covariant constraint \( T = d\omega + \omega \wedge e = 0 \) the second line of \([2]\) implies \( \tilde{T} = 0 \).

By combining both constraints \( \tilde{e} = \exp i_{\xi} e \) and \( \tilde{T} = T \), one finds that the ghost number two component of the equation \( \tilde{T} = T \) implies

\[
s\xi = \xi^\nu \partial_\nu \xi = Lie_\xi \xi \equiv \frac{1}{2}\{\xi, \xi\}. \]

One has the following identity \([5]\)

\[
\exp -i_{\xi}(d + s) \exp i_{\xi} = d + s - Lie_\xi + i_{\xi - \xi^\nu \partial_\nu \xi} \quad \text{where} \quad Lie_\xi \equiv [i_{\xi}, d].
\]

and

\[
\exp -i_{\xi}(d + s) \exp i_{\xi} = d + s \quad \text{where} \quad s \equiv s - Lie_\xi.
\]
Both properties \( s^2 = 0 \) and \( s^2 = 0 \) are equivalent since \( s\xi = \xi^\nu \partial_\nu \xi \). The relation \( \exp -i_{\xi}(\omega + \Omega) = \omega + \Omega - \xi^\mu \omega_\mu \) suggests the field redefinitions \( \tilde{\omega} \equiv \omega + \Omega \) with \( \tilde{\Omega} \equiv \Omega - \xi^\mu \omega_\mu \).

Left multiplication of \([2]\) by \( \exp -i_{\xi} \) determines the following equivalent reformulation of the gravitational BRST equations \([2]\) that directly define the action of operation \( \tilde{s} \) on all fields

\[
\tilde{e} = e \\
\tilde{T} = (d + \tilde{s})e + (\omega + \tilde{\Omega}) \wedge e = T = 0 \\
\tilde{R} = (d + \tilde{s})(\omega + \tilde{\Omega}) + (\omega + \tilde{\Omega}) \wedge (\omega + \tilde{\Omega}) = (d\omega + \omega \wedge \omega) = R.
\]
The $\xi$ dependence is hidden in (7) owing to the field redefinition $\Omega \rightarrow \hat{\Omega}$ and $s \rightarrow \hat{s}$. A superficial look at the BRST symmetry operation $\hat{s}$ in (7) indicates that the gravitational BRST algebra is formally that of a flat space gauge symmetry for the Lorentz invariance with the redefined Lorentz ghost $\hat{\Omega} = \Omega - i\xi \omega$. The so-called “covariant BRST equations” of the Diff\times Lorentz symmetry are $s\omega = -D\Omega + i\xi R$ and $s\Omega = -\hat{\Omega}\Omega + \frac{1}{2}i\xi i\xi R$ and they derive from (2). They are equivalent to the simpler ones $\hat{s}\omega = -D\hat{\Omega}$ and $\hat{s}\Omega = -\hat{\Omega}\hat{\Omega}$, that derive from (4).

Proving these properties boils down to an appropriate use of the operation $\exp i\xi$ and of the closure of the Poincaré Lie algebra. The Appendix B of the second reference in [5] (that also considers the determination of the BRST symmetry of gravity possibly coupled to gauge symmetries including the cases of supergravity) analyses the structure of the graded algebra built by the ensemble of the generalized derivation operators $d, s, i\xi, Lie_\xi = [i\xi, d], i_s\xi, \hat{s}, \hat{s}$, etc... . This graded algebra enlightens the role of the operation $\exp i\xi$ when analysing the BRST structure of theories coupled to gravity. When one goes beyond the case of genuine gravity, one gets a gravitational nilpotent operation $\hat{s} = s - Lie_\xi - i_{\phi}$, with $\phi = s\xi - \xi^\alpha \partial_\alpha \xi \neq 0$ and $s\phi = \{\phi, \xi\}$. The definition of the operation $\hat{s}$ is only determined by the internal gauge symmetries of the system including local supersymmetry if it is involved. The reparametrization symmetry dependence is systematically hidden in all formula thank’s to the appropriate use of the graded operation $\exp i\xi$. (7) is in fact for the simpler example of pure gravity for which $\phi = 0$. When supergravity is involved, the non vanishing ghost number 2 vector field $\phi^\mu = i\partial^\mu i\kappa$ occurs where $\kappa$ is the commuting spin 1/2 ghost field of local supersymmetry. The non vanishing of $\phi$ in supergravity is due to the modification of the torsion $T$ by addition of the gravitino field $\Psi$ dependent term $\frac{1}{2}i\bar{\Psi}\gamma^\nu \partial_\nu \Psi$, giving $T \equiv dc + \omega \wedge e + \frac{1}{2}i\bar{\Psi}\gamma^\nu \partial_\nu \Psi$ and one has a ghostified gravitino $\hat{\Psi} = \Psi + \kappa$ [5]. Having $\phi \neq 0$ implies the existence of an extra generator $i_{\phi}$ that enlarges the size of purely gravitational BRST superalgebra. It modifies the BRST transformation of the reparametrization ghost $\xi^\mu$ in agreement with the fact that the anticommutator of two supersymmetries is a reparametrization. In fact, many of the results presented in this paper can be generalized to supergravity ending up with a $d > 2$ “superBeltrami” parametrization of the gravitino that will be presented elsewhere.

So, in general, the set of the graded differential operators, $d, s, i\xi, Lie_\xi$, etc.. gets completed by the contraction operator $i_{\phi}$, $Lie_{\phi} = [i\phi, d]$, etc... The graded differential operator $i_{\phi}$ decreases the form-degree by one and increases the ghost number by two, so the value of its bi-grading is equal to one. Thus $i_{\phi}$ combines consistently with $d, s$ and $Lie_\xi$. One gets the following generalization of (4)

$$s\xi = \xi^\mu \partial_\mu \xi + \phi = \frac{1}{2}\{\xi, \xi\} + \phi = \frac{1}{2}Lie_\xi \xi + \phi$$

$$s\phi = \{\xi, \phi\} \equiv Lie_\xi \phi \iff \hat{s}\phi = 0.$$  

(8)

The relation (5) becomes

$$\exp -i_{\xi}(d + s) \exp i_{\xi} = d + s - Lie_\xi + i_{\phi}.$$  

(9)

One has then in full generality that

$$s^2 = 0 \iff \hat{s}^2 = (s - Lie_\xi)^2 = Lie_\phi$$  

(10)

as analysed in [5].

The fields that define in this work the generalized Beltrami parametrization of the $d$-dimensional metric and vielbein as well as the associated ghosts (that are field redefinitions of the standard reparametrization symmetry ghosts $\xi^\mu$) are to be conventionally called “Beltrami fields”. They undergo the reparametrization BRST symmetry in their own non trivial way. Getting their BRST transformations by blindly combining the standard tensorial gravity field BRST transformations and the mapping between the Beltrami field and the components $g_{\mu\nu}$ of the metric would be an overcomplicated task. In contrast, one can elegantly determine the BRST transformations of all Beltrami fields by genuinely adapting the geometrical BRST horizontality equations (2) and (7) to the context of the Beltrami parametrization. The nilpotency of the BRST transformations acting on the $\frac{d(d+1)}{2}$ fields that parametrize both the $d$-dimensional Beltrami vielbein and Beltrami metric is ab-initio warranted in the geometrical construction. It follows that the BRST transformations of the Beltrami fields so directly determined by the geometrical method are automatically the correct ones.
3 Reminder of the Beltrami parametrization for the Euclidean $d = 2$ gravity

This section is a reminder of the known Euclidean $d = 2$ Beltrami parametrization and its methodology for bidimensional gravity [2]. It defines notations that are useful in the further sections that are devoted to the construction of the $d > 2$ Beltrami parametrization.

3.1 Beltrami zweibein and $d = 2$ metric

The fundamental classical field of Euclidean $d = 2$ gravity is the bidimensional Euclidean metric $g_{\alpha\beta}$. It represents all possible Riemann surfaces that must be also classified according to their genus. In real Euclidean coordinates ($x, y$), the metric $g_{\alpha\beta}$ defines the invariant infinitesimal squared length element

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = Adx^2 + 2Cxdy + Bdy^2$$

where $AB - C^2 > 0$, $A > 0$, $B > 0$. In complex coordinates one has $z = x + iy$ and $\bar{z} = x - iy$ and the “Beltrami zweibein” $e = (e^z, e^{\bar{z}})$ that is used in [2] is defined as

$$(e^z, e^{\bar{z}}) = \left(\exp \varphi \begin{pmatrix} 1 & \mu z \mu \bar{z} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mu \bar{z} & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} \exp \varphi & E^z \\ \exp \varphi & E^{\bar{z}} \end{pmatrix}$$

(11)

so that $E^z \equiv dz + \mu z d\bar{z}$ and $E^{\bar{z}} \equiv d\bar{z} + \mu \bar{z} dz$. The bidimensional field $\mu z$ is the Beltrami differential and $\mu \bar{z}$ is its complex conjugate. In a path integral formulation, $\mu z$ and $\mu \bar{z}$ are treated as independent fields and one can consistently defines the reparametrization invariant path integral measure $|d\varphi||d\bar{\varphi}||d\mu z||d\mu \bar{z}|$ provided no conformal anomaly occurs. (11) determines the following expression of $g_{\alpha\beta}$

$$g_{\alpha\beta} = \frac{1}{2} \exp(\varphi + \bar{\varphi}) \begin{pmatrix} \mu \bar{z} & \mu \bar{z} \\ \mu \bar{z} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu \bar{z} & 0 \\ 0 & 1 \end{pmatrix} = \exp(\varphi + \bar{\varphi}) \begin{pmatrix} \mu \bar{z} & \frac{1 + \mu \bar{z}^2}{2} \\ \mu \bar{z} & \frac{1 + \mu \bar{z}^2}{2} \end{pmatrix}.$$  

(12)

One has therefore the so-called Beltrami expression for the line element

$$ds^2 = \exp(\varphi + \bar{\varphi})|dz + \mu z d\bar{z}|^2.$$  

(13)

One may call

$$\left(\begin{array}{c} 1 \\ \mu \bar{z} \\ 1 \end{array}\right)$$

(14)

the $d = 2 \times 2$ Beltrami matrix. The paper will generalize this matrix and its role in higher dimensions. The unimodular part $\hat{g}_{\alpha\beta}$ of $g_{\alpha\beta}$ and the inverse $g^{\alpha\beta}$ of $g_{\alpha\beta}$ are

$$\hat{g}_{\alpha\beta} = \frac{i}{1 - \mu \bar{z}^2} \begin{pmatrix} 2 \mu \bar{z} & 1 + \mu \bar{z} \mu \bar{z} \\ 1 + \mu \bar{z} \mu \bar{z} & 2 \mu \bar{z} \end{pmatrix}, \quad \det \hat{g}_{\alpha\beta} = 1$$

$$g^{\alpha\beta} = 2 \exp-\frac{(\varphi + \bar{\varphi})}{1 - \mu \bar{z}^2} \begin{pmatrix} 1 & -\mu \bar{z} \\ -\mu \bar{z} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\mu \bar{z} \\ -\mu \bar{z} & 1 \end{pmatrix} = 2 \exp-\frac{(\varphi + \bar{\varphi})}{1 - \mu \bar{z}^2} \begin{pmatrix} -2 \mu \bar{z} & 1 + \mu \bar{z} \mu \bar{z} \\ 1 + \mu \bar{z} \mu \bar{z} & -2 \mu \bar{z} \end{pmatrix}.$$  

(15)

The zweibein (11) transforms tensorially under the $U_{Weyl}(1) \times U_{rotation}(1) \times \text{Diff}_2$ gauge symmetry.

The rotation gauge symmetry $U_{rotation}(1)$ can be gauge fixed by imposing $\varphi = \bar{\varphi} = \Phi/2$ in a way that preserves the $U_{Weyl}(1) \times \text{Diff}_2$ gauge symmetry. Modulo the rotation symmetry, the Beltrami zweibein (11) is thus covariantly determined by the same three fields $\mu z, \mu \bar{z}, \Phi$ as the Beltrami $d = 2$ metric. The factorization between the holomorphic and antiholomorphic sector is obvious in (11) and (12). The transformation laws under both reparametrization and the Weyl symmetries of $\mu z, \mu \bar{z}, \varphi, \bar{\varphi}$ will be shortly displayed.
3.2 \( d = 2 \) Beltrami Spin connection and \( d = 2 \) curvature

The Weyl invariant part of the “Beltrami zweibein” (11) is

\[
\left( \frac{E^z}{E^z} \right) = \left( \frac{\mu_z}{1} \right) \left( \frac{dz}{d^z} \right).
\]

The relevance of the Beltrami matrix \( \left( \frac{1}{\mu_z} \mu^{\overline{z}} \right) \) suggests defining the differential operations

\[
\left( \frac{D_z}{D_{\overline{z}}} \right) = \left( \frac{\partial_z - \mu^z \partial_{\overline{z}}}{\partial_{\overline{z}} - \mu^\overline{z} \partial_z} \right).
\]

\( (D_Z, D_{\overline{Z}}) \) is basically the dual basis of \( (E^z, E^\overline{z}) \) in a Cartan moving frame. The exterior differential operator is

\[
d \equiv dz \partial_z + d\overline{z} \partial_{\overline{z}} = \frac{1}{1 - \mu^{\overline{z}} \mu_z} E^z D_z + \frac{1}{1 - \mu^z \mu_{\overline{z}}} E^\overline{z} D_{\overline{z}}.
\]

The Abelian 1-form Spin connection \( \omega = \omega_z dz + \omega_{\overline{z}} d\overline{z} \) can be equivalently expressed as \( \omega = \omega_Z E^z + \omega_{\overline{Z}} E^\overline{z} \) according to

\[
\left( \begin{array}{c} \omega_Z \\ \omega_{\overline{Z}} \end{array} \right) = \frac{1}{1 - \mu^{\overline{z}} \mu_z} \left( \begin{array}{cc} 1 & -\mu^z \\ -\mu_{\overline{z}} & 1 \end{array} \right) \left( \begin{array}{c} \omega_z \\ \omega_{\overline{z}} \end{array} \right) \leftrightarrow \left( \begin{array}{c} \omega_z \\ \omega_{\overline{z}} \end{array} \right) = \frac{1}{\mu^z} \left( \begin{array}{c} \omega_Z \\ \omega_{\overline{Z}} \end{array} \right).
\]

The \( d = 2 \) gravity Spin connection satisfies the algebraic vanishing torsion condition

\[
\left( \begin{array}{c} T^z \\ T^\overline{z} \end{array} \right) \equiv \left( \begin{array}{c} \frac{d e^z - \omega \wedge e^z}{d e^z + \omega \wedge e^\overline{z}} \\ \frac{d e^\overline{z} - \omega \wedge e^\overline{z}}{d e^\overline{z} + \omega \wedge e^z} \end{array} \right) = 0
\]

whose solution is

\[
\left( \begin{array}{c} \omega_Z \\ \omega_{\overline{Z}} \end{array} \right) = \frac{1}{1 - \mu^{\overline{z}} \mu_z} \left( \begin{array}{c} -D_z \overline{\partial} + \partial_{\overline{z}} \mu^z \\ D_{\overline{z}} \overline{\partial} - \partial_z \mu_{\overline{z}} \end{array} \right)
\]

\[
\left( \begin{array}{c} \omega_z \\ \omega_{\overline{z}} \end{array} \right) = \left( \begin{array}{c} \mu^z \\ 1 \end{array} \right) \left( \begin{array}{c} \omega_Z \\ \omega_{\overline{Z}} \end{array} \right) = -\frac{1}{1 - \mu^{\overline{z}} \mu_z} \left( \begin{array}{c} D_z \overline{\partial} - \mu_{\overline{z}} \partial_z \mu^z - \mu^z \partial_{\overline{z}} \mu_{\overline{z}} \\ -D_{\overline{z}} \overline{\partial} + \mu_z D_z \overline{\partial} + \partial_z \mu_{\overline{z}} - \mu_{\overline{z}} \partial_z \mu_z \end{array} \right).
\]

The Abelian exact 2-form Riemann curvature is

\[
R = d\omega = dz \wedge d\overline{z} \left[ -\partial_z (D_{\overline{z}} \overline{\partial} - \mu_{\overline{z}} \partial_z \mu^z) - \partial_{\overline{z}} (D_z \overline{\partial} - \mu^z \partial_{\overline{z}} \mu_{\overline{z}}) \right] = \partial_z \left( D_{\overline{z}} \overline{\partial} - \mu_{\overline{z}} \partial_z \mu^z \right) + \partial_{\overline{z}} \left( D_z \overline{\partial} - \mu^z \partial_{\overline{z}} \mu_{\overline{z}} \right).
\]

One has \( R_{\overline{z}z} \sim \partial_z \partial_{\overline{z}} (\varphi + \overline{\varphi}) - \partial^2_{\overline{z}} \mu^z - \partial_z \partial_{\overline{z}} \mu_{\overline{z}} \) at the first non trivial order in \( \mu^z \) and \( \mu_{\overline{z}} \).

One can gauge fix the Lorentz \( U(1) \) symmetry and impose \( \varphi = \overline{\varphi} \). This gives

\[
R_{z\overline{z}} = (\partial_z, \partial_{\overline{z}}) \frac{1}{1 - \mu_{\overline{z}} \mu^z} \left( -\frac{\mu_{\overline{z}}}{1 + \mu_{\overline{z}} \mu^z} - \frac{\mu^z}{1 + \mu^z \mu_{\overline{z}}} \right) \left( \frac{\partial}{\partial \varphi} + \phi \partial_{\overline{z}} \left( \frac{\partial_{\overline{z}} \varphi - \mu_{\overline{z}} \partial_z \varphi}{1 - \mu_{\overline{z}} \mu^z} \right) + \partial_z \left( \frac{\partial_z \varphi - \mu^z \partial_{\overline{z}} \varphi}{1 - \mu^z \mu_{\overline{z}}} \right) \right)
\]

where \( \varphi = \overline{\varphi} \equiv \Phi/2 \). The Beltrami Laplacian has surged in the right hand side of the last equation. This expression of the curvature is relevant for a Beltrami formulation of conformal anomalies and Wess and Zumino terms in \( d = 2 \) gravity \[2\].

3.3 \( d = 2 \) gravitational Beltrami BRST symmetry

The standard definition of the action of the nilpotent graded differential \( s \) for the BRST symmetry of bidimensional gravity is \( sg_{\alpha \beta} = \text{Lie}_e g_{\alpha \beta} + 2\Omega_W g_{\alpha \beta} \), \( s\Omega_W = \xi^\beta \partial_\beta \Omega_W \) and \( s\xi^\alpha = \xi^\beta \partial_\beta \xi^\alpha \) where \( \Omega_W \) is the Abelian ghost of the Weyl symmetry. The nilpotency of \( s \) is equivalent to the closure and Jacobi identity of the Lie algebra
of the rotation $\times$ Weyl $\times$ Diff$_2$ local symmetry. The reformulation of the BRST symmetry transformations within the framework of the Beltrami parametrization it to provide a covariant separation between the holomorphic and antiholomorphic sectors that is enforced ab initio.

The determination of the BRST symmetry acting on the $d = 2$ Beltrami fields $\varphi, \overline{\varphi}, \mu_\varphi$ and $\mu_{\overline{\varphi}}$ is directly obtained by adapting the generic method displayed in section 2 to the Euclidean bidimensional case. One defines the following ghost and classical field unification of the Beltrami fields [2] :

$$E^z = dz + \mu_\varphi d\varphi \quad \rightarrow \quad \tilde{E}^z = dz + \mu_\varphi d\varphi + c^z$$
$$E^\overline{z} = d\overline{z} + \mu_{\overline{\varphi}} d\overline{\varphi} \quad \rightarrow \quad \tilde{E}^\overline{z} = d\overline{z} + \mu_{\overline{\varphi}} d\overline{\varphi} + c^\overline{z}$$

$$d \rightarrow \tilde{d} \equiv d + s. \quad (25)$$

Consistency implies that $c^z$ and $c^\overline{z}$ are related to the standard reparametrization ghosts $\xi^z$ and $\xi^\overline{z}$ by the following $\mu_\varphi$ and $\mu_{\overline{\varphi}}$ dependent field redefinition :

$$\begin{pmatrix} c^z \\ c^\overline{z} \end{pmatrix} = \begin{pmatrix} \mu_\varphi \\ 1 \end{pmatrix} \begin{pmatrix} \xi^z \\ \xi^\overline{z} \end{pmatrix}. \quad (26)$$

The zweibein $(e^z, e^\overline{z}) = (\exp \varphi E^z, \exp \overline{\varphi} E^\overline{z})$ transforms under the reparametrization symmetry, the rotation and the Weyl gauge symmetries. Its ghost unification involves the conformal factors as follows :

$$\begin{pmatrix} \tilde{e}^z \\ \tilde{e}^\overline{z} \end{pmatrix} = \begin{pmatrix} \exp \varphi & 0 \\ 0 & \exp \overline{\varphi} \end{pmatrix} \begin{pmatrix} E^z \\ E^\overline{z} \end{pmatrix} = \begin{pmatrix} \exp \varphi & 0 \\ 0 & \exp \overline{\varphi} \end{pmatrix} \exp i\xi \begin{pmatrix} E^z \\ E^\overline{z} \end{pmatrix} \quad (27)$$

The exterior differential $d$ and the graded BRST symmetry operation are unified as :

$$\tilde{d} \equiv \exp -i\xi d \exp i\xi = d + s - \text{Lie}_\xi. \quad (28)$$

This defines the modified nilpotent BRST operator $\tilde{s} \equiv s - \text{Lie}_\xi$ as generally explained in section 2 [1].

The Abelian Spin connection 1-form $\omega$ that is the gauge field for the bidimensional Euclidean rotations is ghostified by addition of its anticommuting 0-form ghost $\Omega$. This defines the graded 1-form $\tilde{\omega} \equiv \omega + \Omega$. Section 2 explains that the existence of the operation $\exp i\xi$ naturally leads to the introduction of the graded 1-form $\tilde{\omega} \equiv \exp -i\xi \tilde{\omega} = \omega + \Omega$, where

$$\tilde{\Omega} = \Omega - i\xi \omega = \Omega - \xi^\alpha \omega_\alpha. \quad (29)$$

Using the redefined ghost $\tilde{\Omega}$ instead of the Lorentz ghost $\Omega$ often simplify formula and defines the covariant graded differential operator $\tilde{D} \equiv \tilde{d} + \tilde{\omega}$. (An analogous treatment applies to the Weyl ghost $\Omega_W$ by generalizing $\tilde{D} = \tilde{d} + \tilde{\omega} \rightarrow \tilde{d} + \tilde{\omega} + \tilde{\omega}_W$, but there is no need to mention the Weyl invariance in what follows.)

As already mentioned, the nilpotency of the BRST symmetry associated to the rotation $\times$ Diff$_2$ symmetry is nothing but the consequence of to the closure and Jacobi identity of its Lie algebra. It can be checked by brute force, but the generic geometrical construction of section 2 warranties that the action of $s^2$ vanishes on all the fields since it imposes conditions that are compatible with Bianchi identities.

The unification between the ghost and classical field within the Beltrami parametrization framework therefore conveniently and consistently defines the BRST symmetry of all the Beltrami fields for the rotation $\times$ Diff$_2$ symmetry. They read as follows

$$\tilde{T}^z \equiv \tilde{D}\tilde{e}^z = \tilde{d}\tilde{e}^z - \tilde{\omega}\land\tilde{e}^z = \exp i\xi T^z = 0$$
$$\tilde{T}^\overline{z} \equiv \tilde{D}\tilde{e}^\overline{z} = \tilde{d}\tilde{e}^\overline{z} + \tilde{\omega}\land\tilde{e}^\overline{z} = \exp i\xi T^\overline{z} = 0$$
$$\tilde{R} \equiv \tilde{d}\tilde{\omega} = R, \quad (30)$$

Let us check the consistency of (30). Both vanishing torsion conditions $T^z = T^\overline{z} = 0$ are covariantly compatible with the Bianchi identities $0 = D\land\tilde{T} = \tilde{R}\land\epsilon$ and $\tilde{D}\land\tilde{R} = 0$. The BRST transformations determined by the ghost number decomposition of (30) remain consistently true when $\omega$ is an independent field or when it is

---

1 One may recall that the equivalence between the nilpotency properties of $s$ and $\tilde{s}$ is obvious from [28] as well as the property $s\xi^\alpha = \xi^\beta \partial_\beta s\xi^\alpha$. See [2] for the supersymmetric generalization of this in the specific two dimensional case.
expressed as the solution $\omega(e)$ of the condition $T = 0$. Both equivalent nilpotency relations $(s + d)^2 = s^2 = 0$ and $(\tilde{s} + d)^2 = \tilde{s}^2 = 0$ are direct consequences of the Bianchi identities

$$\begin{align*}
\tilde{D}\tilde{T} &= \tilde{d}\tilde{e} + \tilde{R}\wedge\tilde{e} \\
\tilde{D}\tilde{R} &= \tilde{d}\tilde{\omega}.
\end{align*}$$

(31)

Since one imposes $\tilde{T} = T = 0$ and $\tilde{R} = R$, the components with ghost number larger than one in the right hand side of (31) must vanish. This implies $\tilde{d}\tilde{e} = \tilde{d}\tilde{\omega} = 0$. Thus $s^2 \tilde{e} = \tilde{s}^2 \tilde{\omega} = 0$ while $s\xi^\alpha = \xi^\beta\partial_\beta\xi^\alpha$ is a consequence of $\tilde{T} = 0$.

Once the action of $s$ has been determined, the reparametrization and rotation symmetry infinitesimal transformations of all the Beltrami classical fields are recovered by replacing all the ghosts by infinitesimal parameters in the BRST transformations of the classical fields.

The terms with ghost number one and two of the horizontality condition $\tilde{R} = R$ directly determine the transformations of $\omega$ and its ghost $\Omega$ under the operation $\tilde{s} = s - \text{Lie}_\xi$. They are

$$\begin{align*}
\tilde{s}\omega &= -\tilde{d}\tilde{\Omega} \\
\tilde{s}\Omega &= 0.
\end{align*}$$

(32)

The latter equations determine the action of $s$ on $\omega$ and $\Omega$ by using the relationship between $\omega$ and $\tilde{\omega}$.

Both $z \leftrightarrow \bar{z}$ symmetric horizontality equations for $\tilde{T}$ in (31) decompose as follows, where one uses the definition (21) of $\tilde{e}$:

$$\begin{align*}
\tilde{T}^z &= \exp\varphi\left((d + s)(E^z + c^z) + ((d + s)\varphi - \omega - \Omega)\wedge(E^z + c^z)\right) = 0 \\
\tilde{T}^\bar{z} &= \exp\bar{\varphi}\left((d + s)(E^\bar{z} + c^\bar{z}) + ((d + s)\bar{\varphi} + \omega + \Omega)\wedge(E^\bar{z} + c^\bar{z})\right) = 0.
\end{align*}$$

(33)

The first and the second equation provide separately the BRST transformations of $\mu^z_\varphi$, $c^z$, $\varphi$ and of $\mu^\bar{z}_\varphi$, $c^\bar{z}$, $\bar{\varphi}$ with a minimum effort. Indeed, both ghost number one components of (33) that are proportional to $dz$ and $d\bar{z}$ give respectively

$$\begin{align*}
s\varphi &= \Omega - c^z\omega_z + c^z\partial_z\varphi + \partial_z c^z \\
s\bar{\varphi} &= -\Omega + c^\bar{z}\omega_{\bar{z}} - c^\bar{z}\partial_{\bar{z}}\bar{\varphi} + \partial_{\bar{z}}c^\bar{z}.
\end{align*}$$

(34)

One has trivially $\tilde{E}^z\wedge\tilde{E}^\bar{z} = \tilde{E}^\bar{z}\wedge\tilde{E}^z = 0$. Thus, the multiplication of (33) by $\tilde{E}^z$ and $\tilde{E}^\bar{z}$ gives a pair of equation with no dependence on $\omega$, $\Omega$ and $\varphi$ and $\bar{\varphi}$. They are

$$\begin{align*}
\tilde{E}^z\wedge(d + s)\tilde{E}^\bar{z} &= 0 & \iff & (d + s)(E^z + c^z) = (E^z + c^z)\wedge\partial_z(E^\bar{z} + c^\bar{z}) \\
\tilde{E}^\bar{z}\wedge(d + s)\tilde{E}^z &= 0 & \iff & (d + s)(E^\bar{z} + c^\bar{z}) = (E^\bar{z} + c^\bar{z})\wedge\partial_{\bar{z}}(E^z + c^z).
\end{align*}$$

(35)

The expansion in all possible form-degrees and ghost numbers of (35) determine the BRST transformations of all fields belonging to the Weyl invariant sector of the zweibein. The terms with ghost number zero express both trivial identities $dE^z = E^z\partial_zE^\bar{z} = 0$ and $dE^\bar{z} = E^\bar{z}\partial_{\bar{z}}E^z = 0$. The terms with ghost number one and two express the $z \leftrightarrow \bar{z}$ factorized BRST transformation laws of the Beltrami differential and the associated ghost :

$$\begin{align*}
s\mu^z_\varphi &= \partial_zc^z + c^z\partial_\varphi\mu^z_\varphi - \mu^z_{\varphi}\partial_zc^z \\
s\varphi &= c^z\partial_\varphi c^z \\
s\mu^\bar{z}_\varphi &= \partial_{\bar{z}}c^\bar{z} + c^\bar{z}\partial_{\bar{\varphi}}\mu^\bar{z}_\bar{\varphi} - \mu^\bar{z}_{\bar{\varphi}}\partial_{\bar{z}}c^\bar{z} \\
s\bar{\varphi} &= c^\bar{z}\partial_{\bar{\varphi}}c^\bar{z}.
\end{align*}$$

(36)

Both equations (35) $(s + d)\tilde{E}^z - \tilde{E}^z\partial_z\tilde{E}^\bar{z} = 0$ and $(s + d)\tilde{E}^\bar{z} - \tilde{E}^\bar{z}\partial_{\bar{z}}\tilde{E}^z = 0$ imply that $(s + d)^2\tilde{E}^z = \tilde{E}^z\wedge\tilde{E}^z = 0$ and $(s + d)^2\tilde{E}^\bar{z} = \tilde{E}^\bar{z}\wedge\tilde{E}^z = 0$ since the exterior product of a 1-form by itself vanishes. This is the simplest explicit check that $s^2 = 0$ on $\mu^z_\varphi, \mu^\bar{z}_\varphi, c^z, c^\bar{z}$. Its brute force verification is by iterating twice the $s$ operation on all components of (36). Needless to say that the computation of the BRST transformations (34) and (35) by combining the Beltrami changes of field variables (12) and (20) with $s\gamma_{\alpha\beta} = \text{Lie}_\xi g_{\alpha\beta} + 2\Omega W_{\alpha\beta}$ and $s\xi^\alpha = \xi^\beta\partial_\beta\xi^\alpha$ is tedious and time consuming as compared to the simplicity of their derivation by the geometrical method.
These notions and in particular the Beltrami ghostified unification will be generalized for $d > 1$ neatly separate both the Weyl invariant and non-invariant sector $s$ of the theory justifies the generalization of the Polyakov classical action function of a string field $X$ of string theory in a way that respects the factorization afterward gauge fix in a BRST invariant way the Beltrami differential equal to some background without giving reference to an invariant Lagrangian. It gives an interesting perspective on the symmetries of asymmetric theories.

What should be remembered from this section is that the Beltrami field decomposition is a covariant revelatory of the $d = 2$ gravity factorization properties. The elegance of BRST symmetry equations that neatly separate both the Weyl invariant and non-invariant sectors of the theory justifies the generalization of the "Beltrami" denomination all around this paper for $d > 2$. The Beltrami differential $\mu$ is to be completed in higher dimensions by generalized entities such as $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$ that parametrize the Weyl invariant part of $d$-beins. The $d = 2$ situation looks eventually as the extremal case with no Lorentz time of a more general Lorentzian $d > 2$-dimensional framework. The redefined ghosts $c^z$ and $c^x$ deserve being referred to as $d = 2$ “Beltrami ghosts” because of ghostified unifications $dz + \mu_1 c^z \rightarrow dz + \mu_1 c^z + c$ and $d \rightarrow d + s$. These notions and in particular the Beltrami ghostified unification will be generalized for $d > 2$ dimensional gravity theories that also involve Beltrami ghosts for the reparametrization symmetry.

The next sections introduce and detail the “Beltrami parametrization” for the cases of 3 and 4 dimensions and afterward for all $d > 4$ dimensions. The generalization for $d > 4$ is quite abruptly formulated since the combination of both cases $d = 3$ and 4 basically solve most of the technical subtleties one must overcome to generalize the two dimensional case. The key point is that the generic possibility of a generalized “Beltrami parametrization” for the metrics of Lorentzian $d$-manifolds $M_d$ is the consequence of the possible sub-foliations $\Sigma_d^{ADM} = \Sigma_2 \times \Sigma_{d-3}$ of the $d - 1$ dimensional ADM leaves $\Sigma_d^{ADM}$ of $M_d$ by Riemann surfaces. This paper calls this description a leaf of leaf structure for the formulation of the gravitational interactions in $M_d$. This quite refined structure cannot be guessed so easily in dimension $d = 2$ since in this limiting case the gravity field has no physical dynamics and there is obviously nothing to be sub-foliated.

4 $d = 3$ Beltrami gravity

The smallest dimension for having a locally non trivial Einstein action is equal to three. For $d = 3$ the submanifold $\Sigma_{d-3}$ of the leaf of leaf foliation reduces to a point. This simpler case is however quite instructive to get a suggestive understanding of the basic features of the Beltrami parametrization for all values of $d > 2$. A deeper insight is to be reached in the next section for the case $d = 4$ that is the lowest dimension for which $\Sigma_{d-3}$ is a non trivial space.

The possible couplings of $d = 3$ gravity to matter are of great relevance although they cannot generate physical propagating gravitons. Moreover, the $d = 3$ genuine quantum gravity QFT has non trivial mathematical interest as beautifully discussed in [17] and many other publications including recent ones. Some new information about the stochastic quantization of (Euclidean) $d = 2$ and $d = 3$ gravity can be reached by using the $d = 3$ Beltrami parametrization determined in this section. This will be explained in a separate paper.

In this section, the Beltrami parametrization of $d = 3$ gravity is built with a unified notation that treats at once both Euclidian and Lorentzian cases. All formula will be indexed by a parameter $\epsilon = \pm 1$. The values $\epsilon = -1$
and $\epsilon = 1$ are for the Lorentz and the Euclidean cases, respectively.

### 4.1 Notations for the $d = 3$ Beltrami gravity and its bidimensional leaves

The flat Lorentz (or Euclidean) indices of $SO(2,1)$ (or $SO(3)$) Lie algebra of the Lorentz (or rotation) gauge symmetry of the $d = 3$ Lorentzian (or Euclidean) gravity can be expressed either with the real indices $a = 0, 1, 2$ or with the complex ones $a = 0, z, \zeta$ according to the relationship $z = x^1 + ix^2$. The couple $(z, \zeta)$ stands for the complex coordinate of the bidimensional ADM leaves $\Sigma_2$ of $M_3$, foliated by the third direction with the real coordinate $x^0$. The latter (that is often denoted as $t$ in this section) is the extrinsic Lorentz time coordinate in the Lorentzian case and the third “spatial extrinsic” coordinate in the Euclidean case.

The fully antisymmetric tensor $\epsilon_{abc}$ allows one to identify any given antisymmetric tensor $M^{ab}$ to its “dual” expression $M_a$ such that $M^{ab} = i\epsilon^{abc}M_c$. Thus, the dreibein $e^a$ and the Spin connection $\omega^{ab}$ can be expressed as a pair of 1-forms $e^a$ and $e^a$ where $\omega^{ab} = i\epsilon^{abc}\omega_c$, both valued in the fundamental representation of $SO(2,1)$ in the Lorentzian case (or $SO(3)$ in the Euclidean case). Upper indices are lowered by the $\epsilon = \pm 1$ dependent invariant flat metric $h_{ab}$.

The 2-form field strengths of $e$ and $\omega$ are the torsion $T \equiv de + g\omega\wedge e$ and the Riemann curvature $R \equiv dw + g\omega\wedge w$. $g^2 > 0$ is basically the gravitational constant. One chooses from now on $g = 1$ (equivalently, one can absorb $g$ in a redefinition of $\omega$).

When one expresses the spin connection as $\omega^{ab} = i\epsilon^{abc}\omega_c$ and one uses real indices $a = 0, 1, 2$ for the Lie algebra of $SO(2,1)$ (or $SO(3)$), the three components of both $T$ and $R$ are

\[
T^0 = de^0 + i(\omega^1\wedge e^2 - \omega^2\wedge e^1) \quad R^0 = dw^0 + i\omega^1\wedge e^2 \\
T^1 = de^1 + i(\omega^2\wedge e^0 - \omega^0\wedge e^2) \quad R^1 = dw^1 + i\omega^2\wedge e^0 \\
T^2 = de^2 + i(\omega^0\wedge e^1 - \omega^1\wedge e^0) \quad R^2 = dw^2 + i\omega^0\wedge e^1.
\]

(The factors $i$ are a mere consequence of the definition of $\omega^a$).

If one uses complex Lie algebra indices $a = 0, z, \zeta$, the dreibein and the Spin connection read respectively as $(e^0, e^z, e^\zeta)$ and $(\omega^0, \omega^z, \omega^\zeta)$. By definition of $z = x^1 + ix^2$ and $\zeta = x^1 - ix^2$ one has

\[
\begin{pmatrix}
\omega^z \\
\omega^\zeta
\end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \omega^1 \\
\omega^2
\end{pmatrix}, \quad \begin{pmatrix}
e^z \\
e^\zeta
\end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} e^1 \\
e^2
\end{pmatrix}.
\]

The torsion and Riemann curvature (37) are then expressed as

\[
T^0 = de^0 - \frac{1}{2}\omega^z\wedge e^\zeta + \frac{1}{2}\omega^\zeta\wedge e^z \quad R^0 = dw^0 - \frac{1}{2}\omega^z\wedge \omega^\zeta \\
T^z = de^z - \epsilon\omega^0\wedge e^z + \epsilon\omega^z\wedge e^0 \quad R^z = dw^z - \epsilon\omega^0\wedge \omega^z \\
T^\zeta = de^\zeta + \epsilon\omega^0\wedge e^\zeta - \epsilon\omega^z\wedge e^0 \quad R^\zeta = dw^\zeta + \epsilon\omega^0\wedge \omega^\zeta.
\]

The Bianchi identities are

\[
dT^0 = \frac{1}{2}\omega^z\wedge T^\zeta - \frac{1}{2}\omega^\zeta\wedge T^z - \frac{1}{2}R^z\wedge e^0 + \frac{1}{2}R^\zeta\wedge e^0 \quad dR^0 = \frac{1}{2}\omega^z\wedge R^\zeta - \frac{1}{2}\omega^\zeta\wedge R^z \\
dT^z = \epsilon(\omega^0T^z - \omega^zT^0 - R^0e^z + R^z e^0) \quad dR^z = \epsilon(\omega^0\wedge R^z - \omega^z\wedge R^0) \\
dT^\zeta = \epsilon(-\omega^0T^\zeta + \omega^\zeta T^0 + R^0 e^\zeta - R^\zeta e^0) \quad dR^\zeta = \epsilon(\omega^0\wedge R^\zeta + \omega^\zeta\wedge R^0).
\]

The Einstein action is the integral of the scalar curvature density $\sqrt{-g}R(g_{\mu\nu})$. In first order formalism, it is a function of the dreibein $e$ and the Spin connection $\omega$. One has the following relations

\[
I_{Einstein} = \int d^3x\sqrt{-g}R(g_{\mu\nu}) \sim \int \epsilon_{abc}R^{ab}(\omega)\wedge e^c = \int 2\epsilon e^0\wedge R^0 + e^z\wedge R^\zeta + e^\zeta\wedge R^z.
\]

The global Lorentz (or rotation) invariance of the last term follows from the fact that the flat metric is

\[
h_{ab} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 2\epsilon
\end{pmatrix},
\]

when one uses complex indices. The invariant scalar product is $A \cdot B \equiv A^aB_a = \eta_{ab}A^aB^b = 2\epsilon A^0B^0 + A^zB^\zeta + B^\zeta A^z$ for both $A$ and $B$ valued in the fundamental representation of either $SO(2,1)$ ($\epsilon = -1$) or $SO(3)$ ($\epsilon = 1$).
The equivalence between the second order and the first order Einstein action in (11) holds true modulo the vanishing torsion condition $T^a = 0$. The $\omega$-linear equation $T^a = de^a + \omega^{ab} \wedge e_b = 0$ determines all components $\omega^{ab}_\mu$ of the Spin connection 1-form $\omega$ as a function of the dreibein components $e^a_\mu$. Then, the relation between $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and $e^a_\mu$ holds true modulo any given Lorentz (or rotation) gauge transformation of $e^a_\mu$. Eventually, any given well-defined and complete gauge choice for the Lorentz (or rotation) gauge symmetry consistently determines $e^a_\mu$ and $\omega^{ab}_\mu$ as functions of the metric $g_{\mu\nu}$ and of its derivatives. For a simpler expression of the $d = 3$ Einstein action, the part integrations of the terms that involve $R = d\omega + \omega \wedge \omega$ in the right hand side of (11), followed by the use of the constraint $de = -\omega \wedge e$, imply $\int e \wedge (d\omega + \omega \wedge \omega) \sim \int d(e \wedge \omega + e \wedge \omega \wedge \omega) \sim \int e \wedge \omega \wedge \omega$, modulo the boundary term $\int d(e \wedge \omega)$. The Lorentz indices must be adequately contracted in these expressions for the various terms related by the symbol $\sim$ such as $e \wedge \omega \wedge \omega$. Once it is done, the $d = 3$ Einstein action $\int e_{abc} R^{ab} \wedge e^c$ can be identified with the following Lorentz (or rotation) invariant quadratic form of the Spin connection

$$I_{Einstein} = \int e_{abc} R^{ab} \wedge e^c \sim \int e^0 \wedge \omega^z(e) \wedge \omega^z(e) + e^z \wedge \omega^z(e) \wedge \omega^0(e) + e^z \wedge \omega^0(e) \wedge \omega^z(e).$$

(43)

has no explicit dependence on the parameter $\epsilon = \pm 1$ that distinguishes the Lorentzian and Euclidean cases. The dependence on $\epsilon$ is in fact hidden in the definition (39) of the curvature $R$ and of the torsion equation $T = 0$ that one must solve to compute the Spin connection $\omega(e)$. The quadratic expression (13) of $I_{Einstein}$ made more explicit in the further equation (14) is of course in agreement with the standard expression of the genuinely metric dependent Einstein action as the integral of a quadratic form in the Christoffel coefficients modulo a boundary term.

### 4.2 Beltrami dreibein

One now defines “Beltrami dreibein” as the following restricted triplet of 1-forms $e^z, e^\overline{z}, e^0$ that are covariantly parametrized by the left and right conformal factors $\varphi$ and $\overline{\varphi}$ and the Weyl invariant fields $\mu_\overline{z}, \mu_z, \mu_0, \mu_\overline{0}$:

$$e^z = \exp \varphi (dz + \mu_\overline{z} d\overline{z} + \mu_0 dt), \quad e^\overline{z} = \exp \overline{\varphi} (d\overline{z} + \mu_z dz + \mu_\overline{0} dt), \quad e^0 = Nd\tau \equiv \exp(\varphi + \overline{\varphi}) N d\tau.$$

(44)

The Weyl symmetry distinguishes the four fields $\mu_\overline{z}, \mu_z, \mu_0, \mu_\overline{0}$ and the three fields $\varphi, \overline{\varphi}, N$ as belonging to two different categories. The former are Weyl invariant and the latter are not. It must be noted that the definition $e^0 = Nd\tau$ for the Beltrami dreibein is an early signal that the first order formalism dreibein parametrization (11) anticipates the ADM paradigm and $N$ is closely related to the ADM lapse.

The Beltrami dreibein that is defined by (11) is parametrized by seven independent fields while a generic and unconstrained dreibein is parametrized by nine independent fields. The definition $e^0 = Nd\tau$ amounts to both gauge fixing conditions $e^0 = 0$ for such a generic dreibein. It can be performed as a partial gauge fixing of the $SO(2,1) \times Diff_3$ (or $SO(3) \times Diff_3$) of the local symmetry that preserves a remaining $U(1) \times Diff_3$ local symmetry. The latter $U(1)$ symmetry can be further fixed by imposing $\varphi = \overline{\varphi} = \Phi/2$. Eventually, the Beltrami dreibein is parametrized by the six fields $\mu_\overline{z}, \mu_z, \mu_0, \mu_\overline{0}, \Phi, N$ with a genuine Diff$_3$ covariance. The fate of these six three dimensional “Beltrami fields” is to parametrize the six independent components of the Beltrami $d = 3$ metric and the Einstein action.

In fact, both conditions $e^0 = 0$ exhaust in a reparametrization invariant way two of the three local freedoms that are allowed by the local Lorentz (or rotation) gauge symmetry of the dreibein. But then, consistency requires that both components of the Lorentz ghost $\Omega^z$ and $\Omega^0$ must be constrained. Indeed $e^0 = 0$ must be imposed in a way that respects the BRST symmetry of the $Diff_3 \times SO(1,2)$ (or $Diff_3 \times SO(3)$) local invariance that determines the classical physics, that is, one must have $se^0 = se^\overline{0} = 0$ where $s$ is the nilpotent BRST symmetry operator that will be shortly computed in section 4.6. The definition of the $s$-transformations of the vielbein is such that the Lorentz ghosts $\Omega^z$ and $\Omega^0$ must equate certain functions of the diffeomorphism
ghosts when \( se^0 = se^0 = 0 \). Analogously, when the third freedom corresponding to the gauge transformations around the \( x^0 \) axis in the tangent space is used to fix covariantly \( \varphi = \varphi \equiv \frac{\partial}{\partial x} \), the consistency with the BRST symmetry of this third constraint (namely the equation \( s(\varphi - \varphi) = 0 \)) fixes the value of \( \Omega^0 \) in function of the diffeomorphism ghosts. The formulae that detail the values of \( \Omega^e \), \( \Omega^x \) and \( \Omega^0 \) in function of the Beltrami reparametrization ghosts are expressed right after (70) in section 4.6. It must be noted that these formulae can be obtained in quantum field theory as the equations of motion of the Lorentz antighost, when the gauge fixing \( e^0 = e^0 = \varphi - \varphi = 0 \) is enforced by adding to the classical gravitational action an s-exact term. Altogether, the symmetry of the first order gravity reduces to the genuine \( \text{Diff}_3 \) symmetry of the second order gravity when it expressed in function of the Beltrami dreibein (44) with \( \varphi = \varphi = \Phi/2 \) but it keeps the memory of the Lorentz gauge symmetry expressed in terms of the effective Lorentz ghosts that are well-defined functions of the reparametrization ghosts. The one to one correspondence between the six fields \( \mu_{\parallel}^{\varphi}, \mu_{\parallel}^{\varphi}, \mu_{\parallel}^{\varphi}, \Phi, N \) and the six components of the \( d = 3 \) metric \( g_{\mu\nu} \) will be shortly established in (53).

The first order formalism that introduces the Spin connection \( \omega^{ab} \) that gauges the Lorentz (or rotation) symmetry and the vielbein \( e^a \) as independent fields greatly enhances the comprehension of the local symmetries of gravity since it enhances the \( \text{Diff}_3 \) symmetry of the Einstein action \( \int d^3 \sqrt{|g|} R \) into \( SO(2,1) \times \text{Diff}_3 \) (or \( SO(3) \times \text{Diff}_3 \)). In particular, it expresses quite simply (13) whose structure would be difficult to derive in the second order formalism by using the relation between the Christoffel coefficients and the Spin connection. The use of the first order formalism doesn’t change the physics due to the covariant constraint \( T = de + \omega \wedge e \), but the vielbein has more field components than the metric, the difference being equal to \( \frac{d(d + 1)}{2} \) that is the number freedom offered by the Lorentz gauge symmetry. A quantum field theory cost is to be paid in first order formalism: a consistent and BRST invariant gauge fixing of the Lorentz (or rotation) invariance. It fits the ADM framework and moreover it provides a quite interesting expression of the Spin connection as it is to be verified shortly.

Having explained these points, one can go on.

Consider the parametrization (44) with \( \varphi \neq \varphi \). No gravity fluctuation can occur that leads to the singular value \( N = 0 \). Therefore the following triplet of one-forms

\[
\left( \frac{dz}{dt}, \frac{\varphi}{dt}, \frac{\varphi}{dt} \right) = \begin{pmatrix} \exp -\varphi \, e^\varphi \\ \exp -\varphi \, e^\varphi \end{pmatrix} = \begin{pmatrix} 1 & \mu_{\parallel}^{\varphi} \\ \mu_{\parallel}^{\varphi} & 1 \end{pmatrix} \begin{pmatrix} \frac{dz}{dt} \\ \frac{\varphi}{dt} \end{pmatrix}.
\]

(45)
determines an alternative and meaningful basis for all \( d = 3 \) differential form that is equivalent to the standard basis made of \( dz, d\varphi \) and \( dt \).

The definition (13) that generalizes the bidimensional case (13) introduces the following generalized three by three “Beltrami matrix” \( \mathcal{M} \) and its inverse \( \mathcal{M}^{-1} \):

\[
\mathcal{M} \equiv \begin{pmatrix} 1 & \mu_{\parallel}^{\varphi} \\ \mu_{\parallel}^{\varphi} & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}^{-1} = \begin{pmatrix} 1 & \frac{1}{1 - \mu_{\parallel}^{\varphi} \mu_{\parallel}^{\varphi}} \left( -\mu_{\parallel}^{\varphi} \right) \\ \frac{1}{1 - \mu_{\parallel}^{\varphi} \mu_{\parallel}^{\varphi}} \left( -\mu_{\parallel}^{\varphi} \right) & 1 \end{pmatrix}.
\]

(46)

One has the following expression of the exterior differential operator \( d = dt\partial_0 + dz\partial_z + d\varphi\partial_{\varphi} \)

\[
d = \mathcal{E}^0 D_0 + \mathcal{E}^z D_z + \mathcal{E}^\varphi D_{\varphi}
\]

(47)
where \( D_0, D_z, D_{\varphi} \) are

\[
\begin{pmatrix} D_z \\ D_{\varphi} \\ D_0 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1 - \mu_{\parallel}^{\varphi} \mu_{\parallel}^{\varphi}}{1 - \mu_{\parallel}^{\varphi} \mu_{\parallel}^{\varphi}} \end{pmatrix} \begin{pmatrix} \varphi \\ \mu_{\parallel}^{\varphi} \mu_{\parallel}^{\varphi} \end{pmatrix} \begin{pmatrix} \partial_0 - \mu_{\parallel}^{\varphi} \partial_z \\ \partial_0 - \mu_{\parallel}^{\varphi} \partial_z \\ \partial_0 \end{pmatrix}.
\]

(48)

Analogously, the Spin connection \( \omega = dt\omega_0 + dz\omega_z + d\varphi\omega_{\varphi} \) decomposes as

\[
\omega = \mathcal{E}^0 \omega_0 + \mathcal{E}^z \omega_z + \mathcal{E}^\varphi \omega_{\varphi}
\]

(49)
where \((\omega_2, \omega_T, \omega_0) = (\omega_z, \omega_T, \omega_0)M^{-1}\). Using the \(d = 3\) basis \(E^z, E^T, E^0\) instead of the basis \(dz, d\tau, dt\) is quite convenient. It corresponds to a specific choice of a Cartan \(d = 3\) moving frame. It eases the computation of the nine components \(\omega^a_\mu(e)\) of \(\omega\) stemming from the three 2-form vanishing torsion conditions \(T^a = de + \omega \wedge e = 0\) if one chooses to express them as the equivalent nine 3-form equation \(\epsilon_{abc}e^a \wedge T^b = 0\). Appendices B displays the resolution of these equations. Once \(\omega(e)\) is determined as a function of the Beltrami dreibein, the quadratic formula \(\mathbf{13}\) advantageously determines the Einstein action without having to compute the derivatives of the Spin connection \(\omega(e)\) or the Christoffel symbols and its derivatives (see both further sections 4.4 and 4.5).

Each one of the fields that parametrize the Beltrami vielbein \(\mathbf{14}\) has its own specificity. Both Weyl and Lorentz (or rotation) gauge symmetries operate on \(\varphi + \varphi_T\) and \(\varphi - \varphi_T\) by shift operations. The former is by the Abelian parameter of the Weyl symmetry and the latter is by the third parameter of the Lorentz (or rotation) symmetry that must be gauge fixed to impose \(\varphi = \varphi_T\). The fields \(\mu^z_\mu(z, \tau, t)\) and \(\mu^T_\mu(z, \tau, t)\) are Weyl and Lorentz (or rotation) invariant and they stand for the Beltrami differential of each bidimensional ADM leaf \(\Sigma_2\) for any given value \(t\) \((t\) is often denoted as \(x^0\) in this section). The \(2 \times 2\) matrix within the upper left part of \(\mathcal{M}\) is a Beltrami \(d = 2\) matrix as in \(\mathbf{16}\) for the \(t\) dependent Beltrami differential \(\mu^z_\mu(z, \tau, t)\). \(N\) and \(\mu^0_\mu, \mu^0_0\) are to be shortly identified as the ADM time lapse function and shift vector of the ADM leaves \(\Sigma_2\) of \(\mathcal{M}\). The BRST transformation for the Lorentz x Diff\(_3\) (or rotation x Diff\(_3\)) local symmetry of the Beltrami fields that is computed in section 4.6 checks that \(N\) and \(\mu^0_\mu, \mu^0_0\) transform at fixed value of \(t\) time \(t\) as a scalar and a vector field in \(\Sigma_2\).

### 4.3 \(d = 3\) Beltrami metric

One may redefine the rescaled function \(\tilde{N} \equiv \exp \ (-\frac{1}{2}\epsilon z T) N\) that is Weyl independent. The relation \(g_{\mu\nu} = \epsilon_{\mu\nu0}e^0_\mu e^0_\nu\) determines the \(d = 3\) “Beltrami” metric in function of the six independent fields \(\mu^z_\mu, \mu^T_\mu, \mu^0_\mu, \mu^0_0, \tilde{N}, \Phi = \varphi + \varphi_T\). One gets

\[
g_{\mu\nu} = 2\epsilon_{\mu}\epsilon_{\nu} + e^z_{\mu}e^z_{\nu} + e^T_{\mu}e^T_{\nu}
\]

\[
= \exp(\varphi + \varphi_T) \left( \begin{array}{ccc} 1+\mu^z_\mu & 2\mu^z_\mu & \mu^T_\mu + \mu^0_\mu \\ \mu^z_\mu & 1+\mu^T_\mu & \mu^0_\mu \\ \mu^0_\mu & \mu^0_\mu & 1+\mu^0_\mu \end{array} \right) \exp(\varphi + \varphi_T) \left( \begin{array}{ccc} 1 & \mu^z_\mu & (\tilde{N}^2) \mu^0_\mu \\ \mu^z_\mu & 1 & \mu^0_\mu \\ \mu^0_\mu & \mu^0_\mu & 1 \end{array} \right).
\]  

(50)

A slightly different (but obviously equivalent) matricial computation directly provides the infinitesimal Lorentzian (or rotation) length of the infinitesimal line element \(ds^2 = g_{\mu\nu}dx^\mu dx^\nu\) according to a formula that neatly separates the fields \(N\) and \(\mu^0_\mu, \mu^0_0\) that compose the lapse function and the shift vector. One has indeed

\[
ds^2 = (dz\ d\tau\ dt) \left( \begin{array}{ccc} 1 & \mu^z_\mu & 0 \\ \mu^z_\mu & 1 & 0 \\ 0 & 0 & N \end{array} \right) \exp(\varphi + \varphi_T) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2\epsilon N^2 & N \end{array} \right) \left( \begin{array}{ccc} \mu^z_\mu & 0 \\ \mu^0_\mu & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & \mu^T_\mu & (\tilde{N}^2) \mu^0_\mu \\ \mu^T_\mu & 1 & \mu^0_\mu \\ \mu^0_\mu & \mu^0_\mu & 1 \end{array} \right) \left( \begin{array}{ccc} dz \\ d\tau \\ dt \end{array} \right).
\]  

(51)

The latter computation implies

\[
\frac{1}{2}ds^2 = \exp(\varphi + \varphi_T) \left( \epsilon N^2 dt^2 + (dz + \mu^z_\mu d\tau + \mu^0_\mu dt)(d\tau + \mu^T_\mu d\tau + \mu^0_\mu dt) \right).
\]  

(52)

The Beltrami metric is independent on \(\varphi - \varphi_T\), as it must be the case. The third Lorentz freedom has been left free in (52). It can be further gauge fixed with \(\frac{1}{2}\varphi = \frac{1}{2}\varphi_T \equiv \Phi\), giving

\[
\frac{1}{2}ds^2 = \epsilon N^2 dt^2 + (dz + \mu^z_\mu d\tau + \mu^0_\mu dt)\exp(\Phi d\tau + \mu^T_\mu d\tau + \mu^0_\mu dt).
\]  

(53)
expresses any given generic three dimensional metric in function of the six Beltrami fields \( N, \Phi, \mu^z, \mu^\alpha, \mu_0, \mu_0^\alpha \). It quite evidently generalizes the bidimensional Beltrami metric \( \exp \Phi||d\bar{x} + \mu^\alpha d\bar{z}||^2 \) that is reproduced for \( dt = 0 \).

Some extra care is needed to precisely check the relation between \( N \) and \( (\mu_0^\alpha, \mu_0) \) with the ADM lapse function and the shift vector field. The standard ADM formula in real coordinates is

\[
d s^2 = \epsilon N^2 d\bar{t}^2 + (d\bar{x}^\alpha + N^\alpha dt) g_{\alpha\beta}(d\bar{x}^\beta + N^\beta dt),
\]

where \( g_{\alpha\beta} \) is the \( d = 2 \) leaf inner metric. The \( d = 3 \) ADM metric is thus

\[
g_{\mu\nu} = \left( \begin{array}{cc} g_{\alpha\beta} & N_\alpha \\ N_\beta & N^2 + N_\gamma N^\gamma \end{array} \right) \quad g^{\mu\nu} = \left( \begin{array}{cc} g^{\alpha\beta} & -\epsilon N^\alpha N^\beta \\ -\epsilon N^\alpha N^\beta & \epsilon N^\gamma N^\gamma \end{array} \right). \tag{55}
\]

Since one uses complex coordinates in the leaves, \( g_{\alpha\beta} \rightarrow \left( \begin{array}{cc} g_{zz} & g_{z\bar{z}} \\ g_{z\bar{z}} & g_{\bar{z}\bar{z}} \end{array} \right) \). The two by two \( g_{\alpha\beta} \) figuring in the upper left corner of \( (50) \) can be written as in section 2 according to

\[
g_{\alpha\beta} = \exp \Phi \left( \begin{array}{cc} \mu_0^2 & 0 \\ 0 & \mu_0^2 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} \mu_0^2 \mu_0^{-2} & 0 \\ 0 & \mu_0^2 \mu_0^{-2} \end{array} \right) \quad \text{with} \quad g^{\alpha\beta} = \frac{\exp -\Phi}{(1 - \mu_0^2 \mu_0^{-2})^2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} -\mu_0^2 & 1 \\ 1 & -\mu_0^2 \end{array} \right). \tag{56}
\]

Then, by comparing of \( (55) \) and \( (51) \) one finds that both ADM shift 1-form and shift vector \( N_\alpha \) and \( N^\alpha \) are the following functions of the dreibein parameters defined by \( (44) \)

\[
N_\alpha = \exp \frac{\varphi + \bar{\varphi}}{2} \left( \begin{array}{cc} 1 \\ \mu_0^2 \mu_0^{-2} \end{array} \right) \quad \mu_0 = \frac{1}{1 - \mu_0^2 \mu_0^{-2}} \quad N^\alpha = g^{\alpha\beta} N_\beta = \frac{1}{1 - \mu_0^2 \mu_0^{-2}} \left( \begin{array}{cc} 1 \\ -\mu_0^2 \end{array} \right) \left( \begin{array}{cc} \mu_0^{-2} \mu_0^2 \end{array} \right). \tag{57}
\]

This explains how the Beltrami type pair \( (\mu_0^0, \mu_0) \) introduced in \( (44) \) can be identified as the (Weyl invariant) shift vector of the \( d = 3 \) gravity ADM leaves. The generic \( d \)-dimensional relation between the ADM lapse and shift functions and the Beltrami fields is in \( (123) \) and \( (124) \).

### 4.4 \( d = 3 \) Beltrami Spin connection

The 2-form torsion free condition \( T^a = 0 \) determine nine covariant linear equations that fix the nine field components \( \omega^{ab} \sim \omega^a \) of the Spin connection \( \omega \) as functions of the dreibein components. Their content can be equivalently expressed as the nine independent 3-form equations \( \epsilon_{abc} e^a \wedge T^b = 0 \). Both systems \( T^a = 0 \) or \( \epsilon_{abc} e^a \wedge T^b = 0 \) can be used to determine the the Spin connection \( \omega^a_0 \) in function of the Beltrami dreibein field components. It is convenient to choose the basis \( \mathcal{E}^a, \mathcal{E}^\alpha, \mathcal{E}^0 \) defined in \( (44) \) for expanding the three components of \( \omega^a \) as in \( (49) \). One thus parametrizes the Spin connection \( \omega \) as

\[
\left( \begin{array}{c} \omega^z \\ \omega^\alpha \\ \omega^0 \end{array} \right) = \left( \begin{array}{ccc} \omega^z_0 & \omega^z_\alpha & \omega^z_0 \\ \omega^\alpha_0 & \omega^\alpha_\alpha & \omega^\alpha_0 \\ \omega^0_0 & \omega^0_\alpha & \omega^0_0 \end{array} \right) \left( \begin{array}{c} \mathcal{E}^z \\ \mathcal{E}^\alpha \\ N dt \end{array} \right) \tag{58}
\]

Appendix B solves the linear \( \omega \) dependence of the three equations \( T^a = de^a + \omega^a_0 \wedge e^b = 0 \) in \( d = 3 \). The result is

\[
\begin{align*}
\omega^z_0 & = \epsilon \frac{\exp \varphi}{2N} (D_\alpha (\varphi + \bar{\varphi}) - \nabla_{\mu_0^2} \nabla_\mu) \\
\omega^z_\alpha & = \epsilon \frac{1}{N^2 - 1 - \mu_0^2 \mu_0^{-2}} (\partial_\alpha \varphi + \nabla \varphi) \\
\omega^z_0 & = \epsilon \frac{1}{N^2 - 1 - \mu_0^2 \mu_0^{-2}} (\partial_\alpha \mu_0 + \nabla \mu_0) \\
\omega^0_0 & = \epsilon (D_\alpha \varphi + \frac{\partial_\alpha \varphi}{1 - \mu_0^2 \mu_0^{-2}}) \\
\omega^0_\alpha & = \epsilon (D_\alpha \mu_0 + \frac{\partial_\alpha \mu_0}{1 - \mu_0^2 \mu_0^{-2}}) \\
\omega^0_0 & = \epsilon \frac{1}{N^2 - 1 - \mu_0^2 \mu_0^{-2}} (D_\alpha (\varphi - \bar{\varphi}) + D_\alpha (\varphi - \bar{\varphi})). \tag{59}
\end{align*}
\]

The Spin connection components \( (59) \) involve the operation \( \nabla \) defined in Eq. \( (140) \) of Appendix B whose action is defined as follows:

\[
\begin{align*}
\nabla_\mu \mu_0 & = \partial_\mu \mu_0 + \mu_0 \partial_\mu \mu_0 - \mu_0 \partial_\mu \mu_0 \\
\nabla_\mu \mu_0 & = \partial_\mu \mu_0 + \mu_0 \partial_\mu \mu_0 - \mu_0 \partial_\mu \mu_0 \\
\nabla_\mu \mu_0 & = \partial_\mu \mu_0 + \mu_0 \partial_\mu \mu_0 - \mu_0 \partial_\mu \mu_0 \\
\nabla_\mu \mu_0 & = \partial_\mu \mu_0 + \mu_0 \partial_\mu \mu_0 - \mu_0 \partial_\mu \mu_0. \tag{60}
\end{align*}
\]
One may observed that $\nabla \phi_0$ and $\nabla \phi_0$ are formally identical to the Diff, BRST transformations $s \mu^s$ and $s \mu^s$ where one replaces the ghosts $c^s$ and $c^s$ by the fields $\mu_0$ and $\mu_0$.

The Spin connection components are independent on the derivatives with respect to $t$ of the lapse $N$ and the shift vector components $\mu_0$ and $\mu_0$. It trivially follows that these field don't have a conjugate momentum at the classical level as can be checked from the expression of the $d = 3$ Einstein action that is to be shortly written in (61). (59) gets simpler by gauge fixing the third freedom of the Lorentz invariance with $\varphi = \overline{\varphi} = \frac{1}{2} \Phi$. Then, the Beltrami Spin connection becomes

$$\omega_\mu = \begin{pmatrix}
\omega_2 &=& \frac{1}{2} \nabla_\mu \Phi \\
\omega_2 &=& -\frac{1}{2} \nabla_\mu \phi \\
\omega_2 &=& -\frac{1}{2} \nabla_\mu \phi \\
\omega_2 &=& \omega_2 = \frac{1}{2} \nabla \cdot \phi_0
\end{pmatrix}$$

where the definition of the operator $\nabla$ is to be read off from (59).

### 4.5 $d = 3$ gravity Einstein action in the Beltrami parametrization

(59) expresses the $d = 3$ Einstein action as

$$I_{Einstein} = \int L_{Einstein} dt \wedge dz \wedge d\overline{\varphi} = \int e^0 \wedge \omega^z \wedge \omega^\overline{\varphi} + e^z \wedge \omega^z \wedge \omega^0 + e^\overline{\varphi} \wedge \omega^z \wedge \omega^0$$

modulo boundary terms proportional to $\int d (e^0 \wedge \omega_a)$. Since $e^0 = N dt$, one has

$$I_{Einstein} = \int dt \, dz \, d\overline{\varphi} (1 - \mu_0) \left[ \exp \frac{\varphi}{N} (\omega_2^z \omega^0_0 - \omega_0^0 \omega^0_0) + \frac{\exp \varphi}{N} (\omega_2^z \omega^0_0 - \omega_0^0 \omega^0_0) \right].$$

For $\varphi = \overline{\varphi}$, one thus gets the following expression of the Beltrami Einstein action modulo the boundary term $\int d (e^0 \wedge \omega_a)$:

$$I_{Einstein} = \int dt \, dz \, d\overline{\varphi} (1 - \mu_0) \left[ \exp \frac{\varphi}{N} (D_\mu \Phi - \frac{\nabla_\mu \mu_0}{\nabla_\mu \Phi}) \nabla_\mu \Phi - \frac{\nabla_\mu \mu_0}{\nabla_\mu \Phi} (D_\mu \phi^z_0 - \nabla \phi_0) (D_\mu \phi^z_0 - \nabla \phi_0)
\right.
$$

$$\left. + e D_\mu \nabla^a \Phi - 2 \frac{\partial \mu_0}{\nabla_\mu \phi^z_0} + e D_\mu \nabla^0 \phi^z_0 \right].$$

This expression has the ADM structure. The terms proportional to $1/N$ and $N$ (after a part integration) correspond respectively to the ADM kinetic energy and potential energy. The latter is the product of the lapse function by the intrinsic curvature of the $d = 2$ leaf modulo boundary terms.

### 4.6 $d = 3$ gravity Beltrami BRST symmetry

One considers the following ghost and classical field unification to derive the BRST transformations of all the fields that compose the Beltrami dreibein:

$$\begin{align*}
\dot{e}^0 &= N \dot{t} + e^0 \\
\dot{\xi}^z &= dz + \mu_0^z d\phi + \mu_0^z dt + e^z \\
\dot{\xi}^\overline{\varphi} &= d\phi + \mu_0^z d\phi + \mu_0^z dt + e^\overline{\varphi} \\
\dot{\omega}^0 &= \omega^a + \Omega^a \\
\dot{d} &= d + s.
\end{align*}$$

(65) generalizes the bidimensional Beltrami classical and ghost field unification of Section 2. It is compatible with the set-up of Section 3 that defines the BRST symmetry under the form of geometrical horizontality equations. The three first equations of (65) define the Beltrami ghosts $\dot{e}^2, \dot{c}^2, \dot{e}^0$ in function of the three standard $d = 3$ reparametrization ghosts $\xi^z, \xi^\overline{\varphi}, \xi^0$ such that $s \xi = \xi^z \partial_0 \xi$. The relation is provided by using equation (2) for $d = 3$. One has indeed

$$\begin{align*}
\dot{e}^0 &= \exp i \xi^0 = N \dot{t} + e^0 + N \xi^0 \\
\dot{\xi}^z &= \exp i \xi^z = \exp \varphi \dot{\xi}^z \\
\dot{\xi}^\overline{\varphi} &= \exp i \xi^\overline{\varphi} = \exp \overline{\varphi} \dot{\xi}^\overline{\varphi}
\end{align*}$$

(66)
provides the relation between the ghosts \( c \) and \( \xi \) by expanding and \( \exp i\xi = 1 + i\xi + \frac{1}{2!}i^2\xi^2 + \ldots \). The generic gravitational BRST horizontality conditions (29) and (37) taken at \( d = 3 \) imply

\[
\begin{align*}
\hat{T}^0 &= (d + s)e^0 - \frac{\exp\varphi}{2}\hat{\omega}\hat{z}\hat{\epsilon}^z + \frac{\exp\varphi}{2}\hat{\omega}\hat{T}\hat{c}^z = 0 \\
\hat{T}^z &= (d + s)e^z - e\hat{\omega}^0\hat{c}^z + e\hat{\omega}^z\hat{c}^0 = 0 \\
\hat{T}^\tau &= (d + s)e^\tau + e\hat{\omega}^0\hat{c}^\tau - e\hat{\omega}^\tau\hat{c}^0 = 0 \\
\hat{R}^0 &= (d + s - \text{Lie}_\xi)\hat{\omega}^0 - \frac{1}{2}\hat{\omega}\hat{\omega}\hat{T} = R^0 \\
\hat{R}^z &= (d + s - \text{Lie}_\xi)\hat{\omega}^z - e\hat{\omega}^0\hat{\omega}\hat{T} = R^z \\
\hat{R}^\tau &= (d + s - \text{Lie}_\xi)\hat{\omega}^\tau + e\hat{\omega}^0\hat{\omega}\hat{T} = R^\tau.
\end{align*}
\] (67)

The following subsections compute the nilpotent BRST \( s \)-transformations of the Spin connection, of all the Beltrami fields that compose the dreibein and of the associated ghosts by expanding the bi-graded equations (67) in form degree and ghost number.

### 4.6.1 BRST transformations of the \( d = 3 \) Spin connection and its Lorentz ghost

The BRST equation \( \hat{R} = R \) ensures that

\[
\begin{align*}
\hat{s}\omega &= -d\hat{\Omega} - [\omega, \hat{\Omega}] \\
\hat{s}\hat{\Omega} &= -\frac{1}{2}[\hat{\Omega}, \hat{\Omega}]
\end{align*}
\] (68)

(Remember that \( \hat{s} = s - \text{Lie}_\xi \)). Both the \( s \) and \( \hat{s} \) BRST transformations of the Spin connection are the same whether \( \omega \) is an independent field or the function of \( \epsilon \omega = \omega(\epsilon) \) that solves the covariant constraint \( T = de + \omega\wedge e = 0 \).

### 4.6.2 BRST symmetry of the \( d = 3 \) lapse \( N \) and its ghost \( c^0 \)

The BRST transformations of \( N \) and \( c^0 \) are determined by the components with form degree equal to zero and ghost numbers equal to one and two, respectively, of the following BRST horizontality constraint:

\[
\hat{T}^0 = (d + s)(Ndt + c^0) - \frac{\exp\varphi}{2}(\omega^z + \Omega^z)\wedge(d\sigma + \mu_\xi dz + \mu_0 dt + c^z) + \frac{\exp\varphi}{2}(\omega^T + \Omega^T)\wedge(dz + \mu_0 dz + \mu_0 dt + c^T) = 0.
\] (69)

One gets in this way

\[
\begin{align*}
sN &= \partial_\epsilon c^0 + \frac{1}{2}(\exp\varphi(\omega_0^z c^z - \Omega^z \mu_0^z) + \exp\varphi(\omega_0^\tau c^\tau - \Omega^\tau \mu_0^\tau)) \\
sc^0 &= \frac{1}{2}(\exp\varphi c^z \Omega^z - \exp\varphi c^\tau \Omega^\tau).
\end{align*}
\] (70)

It is mentioned right after (33) that both Lorentz ghosts \( \Omega^z \) and \( \Omega^T \) in (70) are functions of the Beltrami reparametrization ghosts in order that the Beltrami condition \( c^0 = Ndt \) be compatible with the nilpotent BRST symmetry equations as they are defined in (33). One is now on position to quantitatively detail their expressions.

Indeed, both ghost number one components of \( \hat{T}^0 = 0 \) in (69) that are proportional to \( \mathcal{E}^z \) and \( \mathcal{E}^T \) must vanish giving the constraint \( \Omega^z = \exp -\frac{\varphi}{2}\Phi\partial_\epsilon c^0 + \exp(\varphi - \varphi)\omega_0^z(e)c^z \) and \( \Omega^\tau = \exp -\varphi\partial_\epsilon c^0 - \exp(\varphi - \varphi)\omega_0^\tau(e)c^\tau \). Analogously, one must have \( s\varphi = s\varphi \) when one uses the third Lorentz freedom to impose \( \varphi = \varphi = \Phi/2 \) that gives the simpler formulae \( \Omega^z = \exp -\frac{\varphi}{2}\Phi\partial_\epsilon c^0 + \omega_0^z(e)c^z \) and \( \Omega^\tau = \exp -\varphi\partial_\epsilon c^0 - \omega_0^\tau(e)c^\tau \). Then, both form-degree zero and ghost number one components of the BRST horizontality constraint \( \hat{T}^z = \hat{T}^\tau = 0 \) (71), which are to be shortly displayed in (72), imply that the third Lorentz ghost is constrained by \( 2\Omega^0 = \partial_\epsilon \epsilon^0 - \partial_\epsilon c^z + \frac{1}{2}(c^\tau\partial_\epsilon \Phi - c^z\partial_\epsilon \Phi + c^z\partial_\epsilon \Phi) + \epsilon(\omega_0^0(e)c^2 + \omega_0^\tau(e)c^\tau) - \epsilon\exp -\frac{\varphi}{2}(\omega_0^z(e) + \omega_0^\tau(e)c^z) \). These expressions of the Lorentz ghost components \( \Omega^0, \Omega^z, \Omega^\tau \) in function of the reparametrization ghosts \( c^z, c^\tau \) and \( c^0 \) are to be consistently used within the BRST transformation laws of the Spin-connection (68) in the Beltrami parametrization scheme when \( \varphi = \varphi \).
4.6.3 BRST symmetry of Weyl invariant fields \((\mu_\Phi^z, c^z, \mu_\Phi^\bar{z}, c^\bar{z}, \mu_0, \mu_0^\Phi)\) and of \((\varphi, \bar{\varphi})\)

The BRST variations of the fields \(\mu_\Phi^z, c^z, \mu_\Phi^\bar{z}, c^\bar{z}, \mu_0, \mu_0^\Phi\) and \(\varphi, \bar{\varphi}\) derive from the various term stemming from the decompositions in form degree and ghost number of both BRST horizontality constraints for the \(z\) and \(\bar{z}\) components of the torsion:

\[
\begin{align*}
\tilde{T}^z &= \frac{d\tilde{\varepsilon}^z - \epsilon\omega^0\wedge\varepsilon^z + \epsilon\omega^z\wedge\varepsilon^0}{\exp\varphi((d\varphi - \omega^0)\wedge\tilde{\varepsilon}^z + d\tilde{\varepsilon}^z + \epsilon\exp(-\varphi)\varepsilon^z\wedge\varepsilon^0)} = \exp\varphi\frac{((d+s)\varphi - \epsilon\Omega^0)\wedge(dz + \mu_\Phi d\varepsilon^z + \mu_0 dt + c^z)}{+(d+s)(\mu_\Phi dt + c^z) + \exp(-\varphi)\omega^z\wedge\varepsilon^0)} = 0 \\
\tilde{T}^\bar{z} &= \frac{d\tilde{\varepsilon}^\bar{z} + \epsilon\omega^0\wedge\varepsilon^\bar{z} - \epsilon\omega^z\wedge\varepsilon^0}{\exp\varphi((d\varphi + \omega^0)\wedge\tilde{\varepsilon}^\bar{z} + d\tilde{\varepsilon}^\bar{z} - \epsilon\exp(-\varphi)\omega^z\wedge\varepsilon^0)} = \exp\varphi\frac{((d+s)\varphi + \epsilon\Omega^z)\wedge(d\varepsilon^\bar{z} + \mu_\Phi dz + \mu_0 dt + c^\bar{z})}{+(d+s)(\mu_\Phi dz + \mu_0 dt + c^\bar{z}) - \epsilon\exp(-\varphi)\omega^\bar{z}\wedge\Omega^z)} = 0.
\end{align*}
\]

The components with form-degree 0 and ghost number 1 in \(T^z = 0\) and \(T^\bar{z} = 0\) provide

\[
\begin{align*}
\mathbf{s}\varphi &= \epsilon\Omega^0 + \partial_z c^z + c^z\partial_z\varphi - \epsilon\omega^0 c^z + \epsilon\exp(-\varphi)\omega^z c^0 \\
\mathbf{s}\bar{\varphi} &= -\epsilon\Omega^0 + \partial_\bar{z} \bar{c}^\bar{z} + \bar{c}^\bar{z}\partial_\bar{z}\bar{\varphi} + \epsilon\omega^\bar{z}\bar{c}^\bar{z} - \epsilon\exp(-\bar{\varphi})\omega^\bar{z}\bar{c}^\bar{z}.
\end{align*}
\]

This equation determines \(\Omega^0\) in function of the reparametrization ghosts as displayed at the end of 4.6.2 when \(\varphi = \bar{\varphi}\). In what follows, it must be understood that the expressions of \(\Omega^z\) and \(\Omega^\bar{z}\) are those that are also expressed at the end of 4.6.2.

Since \(\tilde{\varepsilon}^z\wedge\tilde{\varepsilon}^\bar{z} = \tilde{\varepsilon}^{Tz}\wedge\tilde{\varepsilon}^{T\bar{z}} = 0\), the multiplication of \(\tilde{T}^z = 0\) and of \(\tilde{T}^\bar{z} = 0\) by \(\tilde{\varepsilon}^z\) and \(\tilde{\varepsilon}^\bar{z}\) implies

\[
\begin{align*}
(dz + \mu_\Phi d\varepsilon^z + \mu_0 dt + c^z)\wedge(d+s)(\mu_\Phi d\varepsilon^z + \mu_0 dt + c^z) + \epsilon\exp(-\varphi)(dz + \mu_\Phi d\varepsilon^z + \mu_0 dt + c^z)\wedge(\omega^z + \Omega^z)\wedge(N dt + c^0) &= 0 \\
(d\varepsilon^\bar{z} + \mu_\Phi dz + \mu_0 dt + c^\bar{z})\wedge(d+s)(\mu_\Phi dz + \mu_0 dt + c^\bar{z}) - \epsilon\exp(-\varphi)(d\varepsilon^\bar{z} + \mu_\Phi dz + \mu_0 dt + c^\bar{z})\wedge(\omega^\bar{z} + \Omega^\bar{z})\wedge(N dt + c^0) &= 0.
\end{align*}
\]

The components with ghost number 1 and ghost number 2 of \((73)\) imply

\[
\begin{align*}
\mathbf{s}\mu_\Phi^z &= \partial_z c^z + c^z\partial_z\mu_\Phi^z - \mu_\Phi^z\partial_z c^z + \epsilon\exp(-\varphi)(\omega^z - \mu_\Phi\omega^z)c^0 \\
\mathbf{s}\mu_0^z &= \partial_0 c^z + c^z\partial_0\mu_0^z - \mu_0^z\partial_0 c^z - \epsilon\exp(-\varphi)(\Omega^z - c^z\omega^z) - (\omega^z_0 + \mu_0^z\omega^z_0)c^0 \\
\mathbf{s}c^z &= c^z\partial_z c^z - \epsilon\exp(-\varphi)(\Omega^z - c^z\omega^z)c^0 \\
\mathbf{s}\mu_\Phi^\bar{z} &= \partial_\bar{z} c^{\bar{z}} + c^{\bar{z}}\partial_\bar{z}\mu_\Phi^\bar{z} - \mu_\Phi^\bar{z}\partial_\bar{z} c^{\bar{z}} - \epsilon\exp(-\bar{\varphi})(\omega^\bar{z} - \mu_\Phi\omega^\bar{z})c^0 \\
\mathbf{s}\mu_0^\bar{z} &= \partial_0 c^{\bar{z}} + c^{\bar{z}}\partial_0\mu_0^\bar{z} - \mu_0^\bar{z}\partial_0 c^{\bar{z}} + \epsilon\exp(-\bar{\varphi})(\Omega^\bar{z} - c^{\bar{z}}\omega^\bar{z}) - (\omega^\bar{z}_0 + \mu_0^\bar{z}\omega^\bar{z}_0)c^0 \\
\mathbf{s}c^{\bar{z}} &= c^{\bar{z}}\partial_\bar{z} c^{\bar{z}} - (\Omega^\bar{z} - c^{\bar{z}}\omega^\bar{z}) + \epsilon\exp(-\bar{\varphi})(\Omega^\bar{z} - c^{\bar{z}}\omega^\bar{z})c^0.
\end{align*}
\]

One can verify the nilpotency of \(s\) on all the fields by a brute computation. As already mentioned, this nilpotency is warranted by construction due to compatibility of of the geometrical definition \((77)\) of the BRST symmetry with the Bianchi identities \(DR = 0\) and \(DT = R\wedge\Phi\) of both Poincaré curvatures \(R\) and \(T\). The direct verification that \(s^2 = 0\) on all the Beltrami fields from \((74)\), \((70)\) and \((72)\) would be nothing but a (very) brute force method to verify the closure and Jacobi identity of the Lie algebra of the \(SO(1, 2) \times \text{Diff}_3\) or \(SO(3) \times \text{Diff}_3\) local symmetry.

4.7 Possible choices of BRST invariant gauge fixings of the \(d = 3\) Beltrami metric

The above results allows one to investigate various possibilities for the different gauge fixings of \(d = 3\) gravity. Any given choice of consistent gauge functions must be enforced by adding corresponding BRST exact terms to the Einstein action. To build such terms one introduces appropriate BRST trivial pairs of antighost and Lagrange multiplier fields that are adapted to the chosen gauge. All choices of gauge, provided they are
consistent ones, are equivalent, meaning that they provide the same mean values for the physical observables. One given choice can be more convenient than another one depending on the question one wishes to investigate.

A natural choice is inspired by the conformal gauge choice of \( d = 2 \) gravity. By using 2 freedoms of the \( d = 3 \) reparametrization symmetry, one can indeed impose in a BRST invariant way the condition

\[
\mu^z = \gamma \quad \mu_{\bar{z}} = \overline{\gamma}.
\]  

Here and elsewhere in the paper, \( \gamma \) and \( \overline{\gamma} \) denote the moduli of the \( d = 2 \) leaf \( \Sigma_2 \). The expression of the moduli \( \gamma \) depends on the genus \( g \) of \( \Sigma_2 \). In fact, as it is well known in the context of the covariant string quantization, for any given value of \( g \), one can express

\[
\gamma = \sum_{k=1}^{g-3} \lambda_k f^k(z, \overline{z}),
\]

where the \( g-3 \) constants \( \lambda_k \) must be integrated over fundamental domains of \( \Sigma_2 \). The functions \( f^k(z, \overline{z}) \) build a \( g-3 \) dimensional basis of the quadratic differentials for the Riemann surfaces of genus \( g \). In this construction the modular invariance must be taken into account for fixing the range of integration of the constants \( \lambda_k \)'s to fundamental domains and avoid replica of modular copies. (See [12] for some explanations of this way of taking into account in a BRST invariant way the global zero modes of the ghosts that occur when one gauge fix the \( d = 2 \) reparametrization symmetry of a Riemann surface.)

The third freedom of the \( d = 3 \) reparametrization symmetry has to be further gauge fixed to complete (76). One can for instance gauge fix the scale invariant function \( \hat{N} \) in (44) as

\[
\hat{N} = 1.
\]

Since \( N = \exp \Phi \hat{N} \) and \( \sqrt{-g} = \exp \Phi (1 - \mu^z \mu_{\bar{z}}) \), the latter condition amounts to the following constraint between the lapse of the \( d = 2 \) ADM leaves and the volume element of the \( d = 3 \) Lorentzian space

\[
\sqrt{-g} = (1 - \gamma \overline{\gamma})N.
\]

This gauge choice corresponds to a choice of coordinates such that the metric reads as

\[
d s^2 = \exp \Phi \left( \epsilon dt^2 + (dz + \gamma d\overline{z} + \mu^0_0 dt)(d\overline{z} + \gamma dz + \mu_{\overline{0}} d\overline{dt}) \right),
\]

genus by genus. At the quantum level, the three gauge conditions for the reparametrization invariance (77) and (78) can be imposed in a BRST invariant way. To do so, one introduces the relevant BRST trivial doublets of antighosts and Lagrange multipliers for defining and adding to the classical action a ghost number zero \( s \)-exact action according to the standard method \( S_{cl} \to S_{cl} + s(...) \) that expresses the BRST invariant action in this gauge. The corresponding path integral must include an integration over the parameters \( \lambda \) varying in fundamental domains.

The BRST invariant action for the gauge fixed metric (80) defines a \( d = 3 \) QFT where the classical field degrees of freedom that propagate are the field \( \Phi \) that compose the conformal factor and both components of the leaf shift vector field \( \mu^z_0 \) and \( \mu_{\overline{0}} \). The dynamics also involves compensating ghost antighost propagations that maintains the BRST symmetry Ward identities for all correlators.

Another different three dimensional gauge choice that is maybe worth being studied corresponds to the following class of gauge functions

\[
\mu^z_0 = \alpha \partial_2 \mu^z_\bar{2}, \quad \mu_{\overline{0}} = \overline{\alpha} \partial_2 \mu_{\overline{z}}, \quad \hat{N} = 1,
\]

where \( \alpha \) and \( \overline{\alpha} \) are a pair of constant parameters. This gauge choice eliminates the shift fields \( \mu^z_0, \mu_{\overline{0}} \) in function of the field \( \mu^z_{\bar{2}}, \mu_{\overline{z}} \) and provides a QFT where the remaining propagating classical fields are the Beltrami differentials \( \mu^z_{\bar{2}}, \mu_{\overline{z}} \), the field \( \Phi \) that define the conformal factor of the leaves and all ghost and antighost fields that are relevant to maintain the BRST invariance. Such a gauge fixing is the analogous of the Yang-Mills regularized Coulomb gauge. (See the third and fourth reference of [9].)
5 \quad d = 4 \; \text{Beltrami gravity}

There are \( \frac{d(d-3)}{2} \) gravitational physical degrees of freedom that possibly propagate in a \( d \)-dimensional manifold \( M_d \). \( d = 4 \) is thus the lowest possible dimension for the existence of a non trivial Spin 2 propagating graviton with two physical particle excitations. The way to pass from the \( d = 3 \) Beltrami parametrization to that in \( d = 4 \) Lorentzian dimensions must parallel what is done in the previous section to pass from the \( d = 2 \) to the \( d = 3 \) cases. This section confronts however a novel task that is of parametrizing both non trivial physical propagating degrees of freedom of the \( d = 4 \) graviton in function of the Beltrami fields.

The six generators of the \( d = 4 \) Lorentz symmetry \( SO(3,1) \) split into three generators for the rotations and three generators for the Lorentz boosts. It is tempting enough to stick with the notational conventions of the three-dimensional Beltrami parametrization (with \( \epsilon = 1 \)) to parametrize the Euclidean ADM leaves \( \Sigma_3 \) of \( M_4 \). Consequently, one denotes in this section the third spatial dimension of \( \Sigma \) in the selfdual part or in the antiselfdual part of the Spin connection according to

The twenty four components of the Spin connection on the basis of one-forms form Spin connection can be expressed as

\[ e_{\mu
\nu} \equiv (\omega_{\mu
\nu} + \frac{1}{2} \epsilon_{\mu
\nu\rho\sigma} \omega_{\rho\sigma}) \]

modulo the (complex) boundary terms \( \mp \omega^\alpha_{\beta\gamma} \). The four dimensional case is quite special among all other cases (but \( d = 8 \) to some extend) because of the possibility of irreducibly decomposing any given two-form into the sum of its selfdual and anti-selfdual parts, a property that applies for decomposing the upper pair of Lorentz antisymmetric indices of the four dimensional Spin connection. The \( d = 4 \) Lorentzian Einstein action can be consequently expressed as a quadratic form either in the selfdual part or in the antiselfdual part of the Spin connection according to

\[ \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{bc} \sim i \eta_{ab} \int \omega^{ac} \wedge e_c \wedge \omega^{bd} \wedge e_d \sim i \eta_{ab} \int \omega^{ac} \wedge e_c \wedge \omega^{bd} \wedge e_d, \] (82)

modulo the (complex) boundary terms \( \pm i \int d(\omega^{ab} \wedge e_a \wedge e_b) \). The definition of both selfdual and antiselfdual components of the four dimensional Spin connection \( \omega^{ab} = \omega^{ab} \pm \frac{1}{2} \epsilon^{abcd} \omega_{cd} \) such that \( \frac{1}{2} \epsilon^{abcd} \omega^{cd} = \pm i \omega^{ab} \). Each expression in the right hand side of (82) is the Lorentz invariant combination of four independent 4-forms, each one being the exterior square of the 2-form carrying \( \omega \). The Bianchi identity \( DT = R^{ab}e_c = 0 \) and a part integration on the term \( dw \) stemming from \( R = d\omega + \omega \wedge \omega \).

One uses complex coordinates for the Riemann surface \( \Sigma_3 \) but it is often convenient in intermediate computations of the four dimensional case to consider the real indices \( i, j, k \) for the 3 real coordinates of \( \Sigma_3 \) with the convention that the non vanishing components of the fully antisymmetric four dimensional Lorentz invariant tensor \( e_{abcd} \) are such that \( \epsilon^{ijk} \gamma = - \epsilon^{rij} \gamma = - \epsilon^{ijk} \gamma = \epsilon^{ijk} \gamma \) equates 1 (respectively \(-1 \)) if \( ijk \) is an even (respectively odd) permutation of \( 1, 2, 3 \). All other components of the tensor \( e_{abcd} \) such that at least two of their four indices are equal obviously vanish.

The vierbein of \( d = 4 \) gravity is expressed as \( e^a = (e^i, e^r) \) in real coordinates and \( e^a = (e^z, e^\tau, e^0, e^r) \) when one uses complex coordinates for the component \( \omega^a \) of the ADM leaves \( \Sigma_3 = \Sigma_2 \times \Sigma_1 \). The four dimensional 1-form Spin connection can be expressed as \( \omega^{ab} = (\omega^{ij}, \omega^{*ij}) \) or equivalently as \( \omega_{ab} = (\omega^{ij}, \omega^{*ij}) \) with \( \omega^{ij} \equiv \omega^{ij} \). The twenty four components of the Spin connection on the basis of one-forms \( dz^\mu = e_z \mu \) can thus be denoted as

\[ \omega^{ab} = (\omega^{ij}, \omega^{*ij}) \sim (\omega^i, \omega^r) \equiv (\omega^\mu, \omega^\tau, \omega^0, \omega^r, \omega^\tau, \omega^0, \omega^r, \omega^\tau, \omega^0). \] (83)

A quick computation shows that, when one decomposes \( \omega^{ab} \) into its selfdual and anti-selfdual components \( \omega^{ab} = \frac{1}{2} \epsilon^{abcd} \omega_{cd} \), the selfdual and anti-selfdual properties of \( \omega^{ab} \) simply satisfy \( \omega^{ij} = \pm i \epsilon^{ijk} \omega^{*kj} \).

One conveniently defines \( \omega^{ij} \equiv \omega^{ij} \pm i \omega^{ij} \). With the convention \( \omega^{ij} \equiv i \epsilon^{ijk} \omega^{*kj} \) one finds that the selfdual and
anti-selfdual properties of $\omega^{ab\pm}$ simply mean $\omega^{z\pm} = 0$ and the Einstein action $I_{\text{Einstein}} = \frac{1}{2} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{bc}$ can be suggestively expressed either as

$$I_{\text{Einstein}} = \int d^4 x L^-(\omega^{z\pm}, e^z, e^\tau) \equiv i \int \omega^{ac\pm} \wedge e_c \wedge \omega^{d\pm} \wedge e_d = 2i \left( \epsilon_i \wedge e_j \wedge \omega^{i\pm} \wedge \omega^{j\pm} - i \epsilon_{ijk} e^\tau \wedge e^k \wedge \omega^{j\pm} \wedge \omega^{i\pm} \right) \quad (84)$$

modulo the boundary term $-i \int d(\epsilon_{ijk} \omega^{i\pm} \wedge e^j \wedge e^k)$ or as

$$I_{\text{Einstein}} = \int d^4 x L^+(\omega^{z\pm}, e^z, e^\tau) \equiv i \int \omega^{ac\pm} \wedge e_c \wedge \omega^{d\pm} \wedge e_d = 2i \left( -\epsilon^i \wedge e^j \wedge \omega^{i\pm} \wedge \omega^{j\pm} + i \epsilon_{ijk} e^\tau \wedge \omega^{j\pm} \wedge \omega^{i\pm} \wedge e^k \right) \quad (85)$$

modulo the boundary term $i \int \epsilon(\epsilon_{ijk} \omega^{i\pm} \wedge e^j \wedge e^k)$. The presence of the term $\epsilon^i \wedge e^j \wedge \omega^{i\pm} \wedge \omega^{j\pm}$ in both (84) or (85) quite transparently illustrates the intertwining between the $d = 4$ Einstein action and the $d = 3$ Einstein action under the form (43).

Both formula for $L^+$ and $L^-$ in (84) and (85) and their simplicity for covariantly separating the Lorentz time and space directions rely on the generic selfdual and antiselfdual decomposition $6 = 3 \oplus 3$ of any given four dimensional Lorentz antisymmetric tensor $M^{ab}$. They are the gravitational analogs of both Lorentzian Yang-Mills formulae $\int d^4 x F^a \wedge F_a = 2 \int d^4 x (F_{tr} \wedge e_{t})^2$ modulo the (complex) boundary terms $+2i \int \epsilon^{abcd} F_{ab} F_{cd}$.

The perspectives offered by the underlying selfduality properties revealed by (84) and (85) and their simplicity for covariantly separating the Lorentz time and space directions rely on the generic selfdual and antiselfdual decomposition $6 = 3 \oplus 3$ of any given four dimensional Lorentz antisymmetric tensor $M^{ab}$. They are the gravitational analogs of both Lorentzian Yang-Mills formulae $\int d^4 x F^a \wedge F_a = 2 \int d^4 x (F_{tr} \wedge e_{t})^2$ modulo the (complex) boundary terms $+2i \int \epsilon^{abcd} F_{ab} F_{cd}$.

5.1 Beltrami vierbein and $d = 4$ Beltrami metric in $z, \tau, t, \tau$ coordinates

The six local freedoms offered by the $SO(1,3) \subset \text{Diff}_4 \times SO(1,3)$ Lorentz gauge symmetry allow the covariant parametrization of the sixteen components of an arbitrary given vierbein $e^a_{\mu}$ in function of the ten fields that determine its Beltrami expression. The replication of the method used in three dimensional case suggests that one postulates as a first step the following $z, \tau, t, \tau$ dependent field decomposition for the Beltrami vierbein $(e^z, e^\tau, e^0, e^\tau)$ :

$$\left( \begin{array}{c} \mathcal{E}^z \\ \mathcal{E}^\tau \\ \mathcal{E}^t \\ \mathcal{E}^\tau \\ \end{array} \right) \equiv \left( \begin{array}{c} \exp -\varphi \, e^z \\ \exp -\varphi \eta \, e^\tau \\ \eta \, e^t \\ \eta \, e^\tau \\ \end{array} \right) = \left( \begin{array}{cccc} 1 & \mu_0^z & \mu_0^\tau & \mu_0^t \\ -\mu_0^z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \\ \end{array} \right) \left( \begin{array}{c} dz \\ dt \\ d\tau \\ \end{array} \right). \quad (86)$$

This exhausts five among the six local Lorentz freedoms, leaving a $U(1) \times \text{Diff}_4 \subset SO(1,3) \times \text{Diff}_4$ covariance for the eleven Beltrami fields $\mu_0^z, \mu_0^\tau, \varphi, \mu_2^z, \mu_2^\tau, \mu_3^z, \mu_3^\tau, a, N, M$. By the further elimination of the $U(1)$ freedom, one can impose $\varphi = \frac{1}{2} \Phi$ which provides a $\text{Diff}_4$ covariant Beltrami parametrization of the vierbein that only depends on the same ten fields that are to parametrize the four dimensional Beltrami metric.

The decomposition (86) is consistent with that done in section 4 for $d = 3$. The three-dimensional Beltrami matrix (43) is indeed recovered by restricting the $4 \times 4$ Beltrami matrix in (86) to its upper-left cornered $3 \times 3$ matrix ; and the upper-left cornered $2 \times 2$ matrix of the latter is nothing but the bidimensional Beltrami matrix (13). The antisymmetric dependence on the field $a$ in (86) is a further subtleties absent in the three dimensional case. Its origin is rooted in the generic Beltrami formulation of section 6 but it cannot be detected for $d = 3$ where the sub-leaf $\Sigma_{d=3}$ reduces to a point. It is in fact best to denote

$$a \equiv \mu_0^t. \quad (87)$$

Indeed, it is to be shown that the three fields $\mu_0^z, \mu_0^\tau, a = \mu_0^t$ compose the four dimensional ADM shift vector.

The gauge fixing that leads to (86) is nothing but a clear four dimensional generalization of the process done respectively in section 3 and 4 in dimensions two and three for defining the Beltrami parametrization. The gauge
fixing of the $SO(1,3) \subset SO(1,3) \times \text{Diff}_4$ Lorentz gauge symmetry that covariantly imposes the six conditions $c^\tau = N d\tau$ and $\varphi = \varphi$ determines a trivial Faddeev–Popov determinant. The latter generates a Gaussian algebraic dependence on the six Lorentz ghosts and antighosts in quantum field theory. The Lorentz ghost and antighost dependence can then be integrated out from the BRST invariant gauge fixed Einstein action. They play no dynamical role in the path integral computations of correlation functions but they can be used as classical sources to write Ward identities that control the Lorentz gauge symmetry in a reparametrization invariant way, which is useful when the gravitational theory is for instance coupled to spinors. When one proceeds to a BRST invariant gauge fixing for the six constraints $c^\tau = N d\tau$ and $\varphi = \varphi$, the six four-dimensional Lorentz ghosts $\Omega^{ab} \sim (\Omega^1, \Omega^3)$ satisfy the algebraic equations of motion of the Lorentz antighosts that enforces the six constraints between the Lorentz ghost and the $\text{Diff}_4$ ghosts. The $d = 4$ formulae are simple generalisations of those detailed right after (70) for $d = 3$ and there is no need to display them in this paper.

The vierbein Beltrami parametrization therefore introduces the following four dimensional Weyl invariant Beltrami $4 \times 4$ matrix:

$$
\mathcal{M} = \begin{pmatrix}
1 & \mu_0^z & \mu_0^x & \mu_0^y \\
\mu_z^0 & 1 & a & 0 \\
0 & 1 & -a & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Its inverse is

$$
\mathcal{M}^{-1} = \begin{pmatrix}
1 & -\mu_z^0 & -\mu_z^x & -\mu_z^y \\
\mu_z^0 & 1 & -a & 0 \\
\mu_x^0 & a & 1 & 0 \\
\mu_y^0 & 0 & 0 & 1
\end{pmatrix}.
$$

The Weyl non invariant fields that compose the Beltrami vierbein in are $\varphi, \varphi, M$ and $N$. Their dependence can be arranged as the components of a diagonal matrix. It follows that the four dimensional metric $g_{\mu\nu} = \epsilon_{\mu
u}^a \eta_{ab} \epsilon_{\nu}^b$ can be decomposed as the following product of $4 \times 4$ matrices

$$
\begin{pmatrix}
1 & \mu_0^z & \mu_0^x & \mu_0^y \\
\mu_z^0 & 1 & a & 0 \\
\mu_x^0 & 0 & 1 & 0 \\
\mu_y^0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\exp \varphi & 0 & 0 & 0 \\
0 & \exp \varphi & 0 & 0 \\
0 & 0 & N & 0 \\
0 & 0 & 0 & M
\end{pmatrix} \begin{pmatrix}
1 & \mu_z^x & \mu_z^y & \mu_z^0 \\
\mu_z^0 & 1 & \mu_z^x & \mu_z^y \\
\mu_x^0 & 0 & 1 & 0 \\
\mu_y^0 & 0 & 0 & 1
\end{pmatrix}.
$$

The reparametrization invariant infinitesimal line element in the Beltrami parametrization for $d = 4$ is therefore

$$
ds^2 = -2M^2(d\tau - a dt)^2 + 2N^2(dt + a d\tau)^2 + (dz + \mu_0^x d\tau + \mu_0^y dt + \mu_0^z d\tau) \exp \Phi(d\tau + \mu_z^x d\tau + \mu_z^y dt + \mu_z^0 d\tau).
$$

The four dimensional Beltrami metric formula is parametrized by the ten Beltrami fields

$$
\mu_0^x, \mu_0^y, \mu_0^z, \mu_z^0, \mu_z^x, \mu_z^y, a \equiv \mu_0^0, M, N, \Phi.
$$

The antisymmetric dependence of the Beltrami matrix on the field $a \equiv \mu_0^0$ makes the Beltrami metric formula subtly different than the four dimensional ADM formula $ds^2 = -N^2 dt^2 + (dx^1 + \beta^1 d\tau)g_{ij}(dx^i + \beta^j d\tau)$. Setting $a = 0$ from the beginning would be an over-gauge fixing of the Lorentz symmetry for a $\text{Diff}_4$ invariant gauge fixing of the sixteen components of a generic vierbein. One must generically break the four dimensional reparametrization invariance in a BRST invariant way to impose $a = 0$.

In fact the excitations of the Weyl invariant fields $\mu_0^0, \mu_0^z$ describe both degrees of freedom of the propagating traceless and transverse graviton in the case $d = 4$ while the same fields carry no gravitational physical degree of freedom for $d = 3$. Indeed, for $d = 3$, both $\mu_0^0$ and $\mu_0^z$ in have no conjugate momentum as can be observed from the tree dimensional Einstein action. They can be identified as both components of the shift vector of the ADM leaves $\Sigma_2$ of $\mathcal{M}_3$ but for $d = 4$ the shift vector of the leaves $\Sigma_3$ of of the leaves $\Sigma_3$ of $\mathcal{M}_4$.

\[\text{It must be noted that the gauge fixing of the Lorentz gauge invariance of gravity that is used in this work deeply differs from the choice } \epsilon_0^a = \epsilon_0^0 \text{ that is often presented in the literature for defining perturbative gravity.}\]
is composed by the other three fields $μ_2^\gamma, μ_3^\gamma$ and $a ≡ μ_0^0$ in (91).  (The generic d-dimensional formulae for the exact correspondence between the ADM vector shift components and the Beltrami fields $μ_m^n, m = 2, 3, i$ is to be further written in section 6. In fact, (124) indicates that if the $μ_m^m$’s vanish the same happens for the ADM shift vector components.)

One possible gauge fixing of the Diff$_4$ invariance of (91) is by the four conditions $μ_2^\tau = μ_3^\tau = 0$ and $μ_2^z = γ, μ_3^z = τ$, with no prejudice on the genus of $Σ_2$ by using (77) for $γ$. It implies

$$ds^2 = -2(M - μ_0^0)^2dτ^2 + 2(N + μ_0^0)^2dt^2 + 2expΦ|dz + γdτ + μ_0^0dt|^2.$$  (93)

[65] can be presumably used to describe in a compact form systems that consists of perturbative gravitons whose degrees of freedom are both fields $μ_0^z$ and $μ_0^γ$ in a gravitational background.

As for static configurations, one may impose $dz + μ_2^z dτ + μ_0^z dt + μ_2^z dτ = dz + μ_2^z dτ$ and $μ_0^z = 0$. The static Beltrami metrics read as

$$ds^2 = -2M^2dτ^2 + 2N^2dt^2 + (dz + μ_2^z dτ)expΦ(dτ + μ_0^z dτ).$$  (94)

One may compute the Ricci tensor for this metric and examine various possibilities with $μ_2^z = γ$ as in (77).

$$ds^2 = -2M^2dτ^2 + 2N^2dt^2 + (dz + γdτ)expΦ(dτ + γ dτ).$$  (95)

5.2 Examples

The case $μ_2^z = γ = 0$ identifies $Σ_2$ as a sphere. Solving the Einstein equation in this case determines the spherical symmetric Schwarzschild solution with flat space boundary condition at spatial infinity. One gets $MN = 1$ and $Φ = 0$ with $expΦdxdy = t^2(dt^2 + sin^2θdθ^2)$.

Solving the Ricci flat condition for the static metric (95) seems a doable task when the topology of the sub-manifolds $Σ_2$ is restricted to that of a torus. It is so when the moduli $γ$ are complex constants $γ ≠ 0$. An encouraging signal is the existence of the so called axisymmetric Weyl metric [11] with two Killing vector fields $ξ^z = ∂_r$ and $ξ^t = ∂_t$ that read as

$$ds^2 = - exp2Ψ(x,y)dτ^2 + x^2 exp-2Ψ(x,y)dt^2 + exp(2κ(x,y) - 2Ψ(x,y))(dx^2 + dy^2),$$  (96)

so that (90) appears as a particular case of (95) with $γ = cte$. This can be formally seen by locally redefining the complex coordinate $z + γ$ into the real coordinates $x, y$ according to $dx^2 + dy^2 ≡ |dz + γdτ|^2$ and then by expressing the three fields $M, N$ and $Φ$ in (95) as functions of both so-called Weyl metric potentials $Ψ(x, y)$ and $κ(x, y)$ in (91). Another open question is whether analytical solutions of (94) can be found when $Σ_2$ has genus equal to two.

5.3 $d = 4$ Beltrami metric with light cone coordinates $z, τ, τ^+, τ^−$

One can rotate the Lorentz time $τ$ and the third spatial coordinate $x^0 = t$ into the light-cone coordinates

$$τ^± = τ ± t.$$  (97)

The $d = 4$ Beltrami parametrization metric (91) becomes then the sum of two interestingly factorized terms:

$$ds^2 = -N^2(dτ^+ + μ_2^+ dτ^−)(dτ− + μ_2^+ dτ^+ + μ_3^+ dτ^+ + μ_3^− dτ^−)(dτ^+ + μ_3^− dτ^− + μ_2^− dτ^−).$$  (98)

In fact, when one uses the light cone coordinates, the $4 \times 4$ Beltrami matrix (90) is

$$\begin{pmatrix}
1 & μ_2^z & 0 & 0 \\
μ_2^z & 1 & 0 & 0 \\
μ_2^z & μ_3^z & 1 & μ_2^+ \\
μ_3^z & μ_3^z & μ_2^+ & 1 \\
\end{pmatrix} \begin{pmatrix}
exp φ & 0 & 0 & 0 \\
0 & exp η & 0 & 0 \\
0 & 0 & N & 0 \\
0 & 0 & 0 & N \\
\end{pmatrix} \begin{pmatrix}
1, 0, 0, 0 \\
0, 1, 0, 0 \\
0, 0, 0, -1 \\
0, 0, -1, 0 \\
\end{pmatrix} \begin{pmatrix}
exp φ & 0 & 0 & 0 \\
0 & exp η & 0 & 0 \\
0 & 0 & N & 0 \\
0 & 0 & 0 & N \\
\end{pmatrix} \begin{pmatrix}
1 & μ_2^z & μ_3^z & μ_2^− \\
μ_3^z & 1 & μ_2^+ & μ_3^− \\
μ_2^− & μ_3^− & 1 & μ_2^+ \\
μ_2^+ & μ_3^+ & μ_2^− & 1 \\
\end{pmatrix}.$$  (99)
There is a mapping between the Beltrami fields $M, N, a$ and $\mu_1^\tau, \mu_2^\tau, \mu_0^\tau, \mu^\tau_\tau$ in (91) and the fields $\mathcal{N}, \mu_1^\tau, \mu_2^\tau$ and $\mu_1^\tau, \mu_2^\tau, \mu_0^\tau, \mu^\tau_\tau$ in (98). It reads as follows:

$$
\begin{bmatrix}
1, a \\
-a, 1
\end{bmatrix} \rightarrow \begin{bmatrix} 1, \mu_1^\tau \\
\mu_2^\tau
\end{bmatrix}, \quad N \rightarrow \mathcal{N}, \quad M \rightarrow \mathcal{N}
$$

and

$$
\begin{bmatrix}
0, 1, 0, 0 \\
1, 0, 0, 0 \\
0, 2, 0 \\
0, 0, -2
\end{bmatrix} \rightarrow \begin{bmatrix} 0, 1, 0, 0 \\
1, 0, 0, 0 \\
0, 0, -\frac{1}{\tau} \\
0, 0, -\frac{1}{\tau}
\end{bmatrix}.
$$

(The last arrow is for the change of the Lorentz flat metrics when one uses light cone coordinates $(t, \tau) \rightarrow \tau^\pm$.)

The light cone coordinate redefinition of the Beltrami matrix (88) is

$$
\begin{pmatrix}
1 & \mu_1^\tau & \mu_2^\tau & \mu_0^\tau \\
\mu_1^\tau & 1 & \mu_2^\tau & \mu_0^\tau \\
0 & 0 & 1 & \mu_2^\tau \\
0 & 0 & \mu_1^\tau & 1
\end{pmatrix}.
$$

One has therefore the following expression for the light cone Beltrami vierbein

$$
\begin{pmatrix}
\mathcal{E}^z \\
\mathcal{E}^\tau \\
\mathcal{E}^+ \\
\mathcal{E}^-
\end{pmatrix} = \begin{pmatrix}
\exp -\varphi e^\ast \\
\exp -\tilde{\varphi} e^\ast \\
\frac{\exp +e^\ast}{\mathcal{N}} \\
\frac{\exp -e^\ast}{\mathcal{N}}
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & \mu_2^\tau \\
0 & 0 & 1 & \mu_1^\tau \\
0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
dz \\
d\bar{z} \\
d\tau^+ \\
d\tau^-
\end{pmatrix}.
$$

(102)

The above expressions suggest a formal four dimensional parallel between the complex coordinates $z, \bar{z}$ of $\Sigma_2$ and the light cone coordinates $\tau^+, \tau^-$. Indeed, (103) involves the single “light cone lapse function” $\mathcal{N}$, the conformal factor $\exp \Phi$ and the eight Weyl invariant fields $\mu_1^\tau, \mu_2^\tau, \mu_1^\tau, \mu_2^\tau, \mu_1^\tau, \mu_2^\tau, \mu_1^\tau, \mu_2^\tau$ with a quite striking symmetry between the $z, \bar{z}$ and $+, -$ indices.[1]

Moreover, the gauge choice for the $d = 4$ reparametrization symmetry $\mu_1^\tau = \mu_2^\tau = \mu_0^\tau = \mu^\tau_\tau = 0$ provides the following suggestive expression for the $d = 4$ metric

$$
 ds^2 = -\mathcal{N}^2(d\tau^+ + \mu_1^\tau d\tau^-)(d\tau^- + \mu_1^\tau d\tau^+) + \exp \Phi(dz + \mu_2^\tau d\bar{z})(d\bar{z} + \mu_2^\tau dz).
$$

A more mathematically oriented publication will discuss some properties of the four dimensional light cone Beltrami vierbein and metric in (102) and (88). The Beltrami parametrization as expressed in (88) seems however physically handier than that in (102) since it offers a genuine distinction between the space and time coordinates. On the other hand (102) could be of interest for studying gravitational solutions for the $(2,2)$ signature.

5.4 $d = 4$ Beltrami Spin connection

Computing the four dimensional Spin connection is a mere generalization of what is done in section 4 for the three dimensional case. On often chooses $\mathcal{E}^z, \mathcal{E}^\tau, \mathcal{E}^t, \mathcal{E}^r$ as a basis of exterior forms giving the decomposition

$$
d = \mathcal{E}^\tau D_\tau + \mathcal{E}^0 D_0 + \mathcal{E}^z D_z + \mathcal{E}^\tau D_{\tau^\ast}
$$

(104)

for the exterior differential operator $d = d\tau \partial_\tau + dt \partial_t + dz \partial_z + d\bar{z} \partial_{\bar{z}}$ and the identity $\mathcal{E}^\tau \omega_\tau + \mathcal{E}^0 \omega_0 + \mathcal{E}^z \omega_z + \mathcal{E}^\tau \omega_{\tau^\ast}$ for the Spin connection. The derivatives $D_\tau, D_0, D_z, D_{\tau^\ast}$ are made explicit in the formulae (133) of Appendix A.

One defines the following $SO(3) \subset SO(1,3)$ invariant matricial decomposition of the Spin connection $\omega$

$$
\omega^{\text{intrinsic}} = \begin{pmatrix} \omega^z \\
\omega^\tau \\
\omega^0 \end{pmatrix} \equiv \begin{pmatrix} \omega^z \\
\omega^z \\
\omega^z \end{pmatrix} \left( \begin{pmatrix} \mathcal{E}^z \\
\mathcal{E}^\tau \\
\mathcal{E}^t \end{pmatrix} \right), \quad \omega^{\text{extrinsic}} = \begin{pmatrix} \omega^{\tau z} \\
\omega^{\tau z} \\
\omega^{\tau z} \\
\omega^{r 0} \end{pmatrix} \equiv \begin{pmatrix} \omega^{\tau z} \\
\omega^{\tau z} \\
\omega^{\tau z} \\
\omega^{r 0} \end{pmatrix} \left( \begin{pmatrix} \mathcal{E}^z \\
\mathcal{E}^\tau \\
\mathcal{E}^t \end{pmatrix} \right).
$$

(105)

---

[1] The $SO(4)$ versus $SU(2) \times SU(2)$ and $SO(1,3)$ versus $SL(2, C)$ correspondences help to enlighten the four-dimensional Beltrami results.
All components of $\omega$ must be computed as the solution of the vanishing torsion four conditions

\[
\begin{align*}
T^0 &= de^0 - \frac{1}{T} \omega^2 \wedge e^2 + \frac{1}{T} \omega^T \wedge e^2 - \omega^0 \wedge e^0 = 0 \\
T^\tau &= \omega^\tau \wedge e^2 + \frac{1}{T} \omega^T \wedge e^2 + \omega^0 \wedge e^0 = 0 \\
T^z &= \omega^z \wedge e^0 - \omega^2 \wedge e^0 = 0 \\
T^\tau &= \omega^\tau \wedge e^0 - \omega^2 \wedge e^0 = 0.
\end{align*}
\]  

(106)

These provide 24 independent equations linear in the components of $\omega$ displayed in (105). They are displayed in Appendix A with a unified notation that apply for both cases when one uses the $(z, \tau, t, \tau)$ and $(z, \tau, \tau^+, \tau^-)$ coordinates. Their restriction for the simpler case $d = 3$ is solved in Appendix B. The complete four dimensional resolution of (106) and the computation of the $d = 4$ Einstein action as in (84) and (85) is quite more involved than doing the analogous work in the case $d = 3$. It deserves further work and is to be published in a separate article specifically devoted to the various aspects of the four dimensional case.

6 Generic Beltrami parametrization for the leaf of leaf foliated $d$-gravity

Consider now the generic $d$-dimensional case. The experience gained in $d = 2, 3$ and 4 dimensions clearly suggests that, to possibly determine for any given value of $d > 2$ a Beltrami parametrization for the $d^2$ components of the generic $d$-bein of a $d$-dimensional Lorentzian manifolds $M_d$, one must proceed by covariantly gauge fixing its $SO(1, d - 1) \subset SO(1, d - 1) \times Diff_d$ Lorentz gauge symmetry. Since the $SO(1, d - 1)$ gauge symmetry offers $\frac{d(d-1)}{2}$ local freedoms, the number of Beltrami $d$-dimensional fields is expected to be $\frac{d(d+1)}{2}$. Each one of the Beltrami fields is to be classified by its Weyl symmetry and by its significance in gravitational theories, independent of the details of the model one wishes to build. Moreover there must be a recursive inclusion of the $(d-1)$-dimensional Beltrami metric into the the $d$-dimensional one.

Before explaining the details of the generic formulation of the Beltrami parametrization in any given dimension of the manifold $M_d$ one should note that, once the Beltrami $d$-bein is defined, the difficulty for computing the Spin connection associated to the Beltrami parametrization is basically the same whether or not gravity is coupled to matter fields (and possibly to auxiliary fields in supergravity). Indeed, whatever the gravity matter couplings are, the Spin connection $\omega$ is systematically defined as the solution of a linear constraint that differs from the pure gravity vanishing torsion constraint $d \epsilon + \omega(\epsilon) \wedge \epsilon = 0$ by the addition of a Lorentz covariant 2-form. Indeed, the Spin connection equation generally reads as follows

\[
d e^a + (\omega^a_b - \epsilon^c G^a_{cb}(\text{matter and auxiliary fields})) \wedge e^b = 0.
\]  

(107)

$G_{abc}$ is a Lorentz covariant tensor according to the property that $G_a = \frac{1}{6} G_{abc} e^a \wedge e^b \wedge e^c$ is a 3-form and it locally depends on matter and/or auxiliary fields. Thus, if one is able to compute the solution $\omega(\epsilon)$ of the linear equation $d \epsilon + \omega(\epsilon) \wedge \epsilon = 0$ as a function of the $\frac{d(d+1)}{2}$ components of the Beltrami $d$-bein, the solution of (107) for any given $G_{abc} \neq 0$ is the trivial shift of this pure gravity solution $\omega^a_b(\epsilon) \rightarrow \omega^a_b(\epsilon) + \epsilon^c G^a_{cb}$. The 3-form $G_3$ is often a complicated function of the fields that couple to gravity. However, both Lorentz symmetry of $G_3$ and $Diff_d$ covariance of the Beltrami parametrization warrant the consistency of this computation of the Spin connection for $G_3 \neq 0$.

6.1 $d$-dimensional Beltrami vielbein

This section defines a notation for the coordinate indices that is generically more appropriate than that used in sections 4 and 5 for the dimensions $d = 3$ and 4. The $d - 1$ spatial coordinates of the sub-foliated ADM leaf $\Sigma_{d-1} = \Sigma_2 \times \Sigma_{d-3}$ are the complex coordinate $z, \bar{z}$ for $\Sigma_2$ and the $d - 3$ real coordinate of $\Sigma_{d-3}$ that are now denoted as $x^i (i = 3, ..., d - 1)$. The Lorentz time coordinate is called $\tau$.

The $d$ coordinates of the pseudo-Riemannian manifold $M_d$ are thus denoted as $(z, \bar{z}, x^i, \tau)$.

\[For instance, in the new minimal $d = 4$ supergravity with gravitino $\Psi = \Psi_\mu dx^\mu$, $G_3 = \frac{1}{6} G_{abc} e^a \wedge e^b \wedge e^c$ is nothing but the curvature of the auxiliary field 2-form $B_2$, with $G_3 = dB_2 + \frac{1}{2} \Psi \gamma^a \wedge \Psi \wedge e_a$.
Now comes the definition of the $d$ one-forms $e^a$ that compose the “Beltrami $d$-bein”. They are obtained by the action on the $d$-vector $(dz, d\tau, dx^3, \ldots, dx^d, d\tau)$ of a matrix that is the product of two $d \times d$ square matrices $M^{(d)}_{\text{diag}}$ and $M_{(d)}$. Both matrices are of a very different nature. $M^{(d)}_{\text{diag}}$ is a diagonal matrix whose $d-1$ independent elements concentrate the dependence on the non Weyl invariant fields of the Beltrami parametrization; in contrast, the “Beltrami matrix” $M_{(d)}$ is composed of the Weyl invariant fields of the Beltrami parametrization and it reduces to the unit matrix when $\mathcal{M}_d$ is flat. The Beltrami $d$-bein $e^a$ is in fact parametrized by $\frac{d(d+1)}{2}$ fields defined as follows:

$$e^a = \begin{pmatrix} e^z \\ e^\tau \\ e^3 \\ \vdots \\ e^d \\ e^{d-1} \end{pmatrix} = M^{(d)}_{\text{diag}} \begin{pmatrix} E^z \\ E^\tau \\ E^3 \\ \vdots \\ E^d \\ E^{d-1} \end{pmatrix}. \quad (108)$$

The non Weyl invariant diagonal matrix $M^{(d)}_{\text{diag}}$ is parametrized by the $d-1$ fields $\Phi, N^i$ and $N$ and reads as

$$M^{(d)}_{\text{diag}} \equiv \begin{pmatrix} \exp \frac{\Phi}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \exp \frac{\Phi}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & N^3 & \cdots & 0 & 0 \\ \vdots & 0 & 0 & \cdots & N^i & 0 \\ \vdots & 0 & 0 & \cdots & 0 & N^{d-1} \\ 0 & 0 & 0 & \cdots & 0 & N \end{pmatrix}. \quad (109)$$

The alternative basis of one-forms $E^z, E^\tau, E^3, \ldots, E^i, \ldots, E^{d-1}, E^\tau$ in (108) is defined by the $d$-dimensional Weyl invariant Beltrami $d \times d$ matrix $M_{(d)}$ such that

$$\begin{pmatrix} E^z \\ E^\tau \\ E^3 \\ \vdots \\ E^i \\ \vdots \\ E^{d-1} \\ E^\tau \end{pmatrix} = \mathcal{M}_{(d)} \begin{pmatrix} dz \\ d\tau \\ dx^3 \\ \vdots \\ dx^i \\ \vdots \\ dx^{d-1} \\ d\tau \end{pmatrix}. \quad (110)$$

with

$$\mathcal{M}_{(d)} \equiv \begin{pmatrix} 1 & \mu_{i}^{z} & \mu_{i}^{3} & \mu_{i}^{4} & \ldots & \mu_{i}^{d-1} & \mu_{i}^{\tau} \\ \mu_{i}^{z} & 1 & \mu_{i}^{3} & \mu_{i}^{4} & \ldots & \mu_{i}^{d-1} & \mu_{i}^{\tau} \\ 0 & 0 & 1 & \mu_{i}^{3} & \ldots & \mu_{i}^{d-1} & \mu_{i}^{\tau} \\ 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \mu_{i}^{3} & \mu_{i}^{4} & \ldots & 1 & \mu_{i}^{\tau} \\ 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \mu_{i}^{d-1} & \mu_{i}^{d-1} & \ldots & \mu_{i}^{d-1} & 1 \end{pmatrix}. \quad (111)$$

The $(d-2) \times (d-2)$ squared sub-matrix that is cornered in the right bottom sector of the Beltrami matrix $\mathcal{M}_{(d)}$ is antisymmetric, that is,

$$\mu_{i}^{j} = -\mu_{j}^{i}, \quad \mu_{j}^{j} = -\mu_{i}^{i}, \quad 3 \leq i, j \leq d-1. \quad (112)$$

$\mathcal{M}_{(d)}$ is the $d$-dimensional generalization of the $2 \times 2$ matrix (13) for $d = 2$, of the $3 \times 3$ matrix (40) for $d = 3$ and of the $4 \times 4$ matrix (58) for $d = 4$, patiently constructed in sections 3,4 and 5.
The generic antisymmetry property \(112\) cannot be detected for \(d = 3\) since \(\Sigma_{d-3}\) reduces to a point in this case. The field dependence of \(111\) is early signal that reveals the necessity of the antisymmetry condition \(112\) of the Beltrami matrix. \(111\) justifies the consistency of expressing \(a \equiv \mu^0\) in the notation of section 5 where the third spatial coordinate is called \(x^0\).

The generic structure of the Beltrami \(d\)-matrix \(111\) suggests quite evidently that the well known parametrization of Riemann surfaces modulo Weyl transformations by the \(d = 2\) Beltrami differential \(dz + \mu^0 d\tau\) is the tip of an iceberg but its existence is revealed by the not so trivial leaf of leaf foliation process of manifolds \(M_d\) according to \(\Sigma_{d-1} = \Sigma_2 \times \Sigma_{d-3}\) that only makes sense for \(d \geq 3\).

The covariant reduction of the \(d^2\) independent components of a generic \(d\)-bein in function of the \(d(d+1)/2\) Beltrami fields that parametrize the Beltrami \(d\)-bein \(108\) is a covariant gauge fixing of the \(d(d-1)/2\) freedoms offered by the local Lorentz symmetry \(SO(1, d-1) \subset SO(1, d-1) \times Diff_d\). The remaining \(Diff_d\) symmetry transforms consistently all these \(d(d+1)/2\) Beltrami fields. If one uses the language of the BRST invariant quantum field theories, the gauge fixing of a general \(d\)-bein \(e^a_\mu\) down to its associated Beltrami \(d\)-bein \(108\) provides a trivial Faddeev–Popov determinant that is consistent with the \(Diff_d\) symmetry as a mere generalization of what is done in the previous sections for \(d = 2, 3, 4\).

The BRST formalism introduces as many independent anticommuting reparametrization vector ghost fields \(\xi^\mu\) as there are local parameters for the infinitesimal reparametrizations transformations. The point is that the \(d\) independent ghost fields \(\xi^\mu\) can be redefined into the \(d\) Beltrami ghosts \(c^a\), \(a = (z, \tau, i, \tau)\). The generic correspondence between \(\xi^\mu\) and \(c^a\) is to be given in the further written formula \(121\) that generalizes in any given arbitrary dimension both formulae \(20\) and \(60\) for \(d = 2\) and \(d = 3\). The experience gained in these previous sections is that one can thus safely anticipate that using of the Beltrami ghost fields \(c^a\) in place of the ghosts \(\xi^\mu\) greatly simplifies the geometrical determination of the BRST transformation laws of all the classical \(d(d+1)/2\) Beltrami fields and thereby the determination of their infinitesimal reparametrization transformations. Moreover, the BRST invariance of the \(d(d-1)/2\) constraints that define the Beltrami \(d\)-bein \(108\) provide algebraical constraints whose expressions imply that the \(d\)-dimensional Lorentz ghosts are themselves constrained to well-defined local functionals of the reparametrization ghosts \(c^a\) \(1\). All needed technical aspects for computing the BRST transformation laws of the Beltrami fields are cautiously explained in section 4 for \(d = 3\). Their \(d\)-dimensional generalization for \(d > 3\) is quite straightforward so that the further section 6.6 that is concerned by this question is to be very short.

### 6.2 Counting the degrees of freedom of \(d\)-gravity in the Beltrami parametrization

The number of the Weyl invariant fields that compose the \(d\)-dimensional 1-forms \(E^z, E^\tau, E^3, \ldots, E^i, \ldots, E^{d-1}, E^\tau\), that is, the number of fields that parametrize the Beltrami \(d \times d\) matrix \(M_{(d)}\) \(111\), is

\[
N_{\text{beltrami}} = 2(d - 1) + \frac{(d - 2)(d - 3)}{2} = \frac{d^2 - d + 2}{2}.
\]  

(113)

This counting takes into account the antisymmetry relations \(112\). The number of independent Weyl noninvariant fields fields \(\Phi, N^3, \ldots, N^{d-1}, N\) that parametrize \(M_{(d)}^{\text{diag}}\) is

\[
N_{\text{diagonal}} = d - 1.
\]  

(114)

One can thus check the number of the Beltrami fields that parametrize the Beltrami \(d\)-bein \(108\) matches the number of components of a generic \(d\)-metric according to

\[
N_{\text{beltrami}} + N_{\text{diagonal}} = \frac{d^2 - d + 2}{2} + d - 1 = \frac{d(d + 1)}{2} = N_{\text{metric}}.
\]  

(115)

\(11\)These effective Lorentz ghosts can be used to study the local Lorentz invariance in the Beltrami formulation for instance for the study of gravitational anomalies.
6.3 Beltrami expression of the $\frac{d(d-3)}{2}$ physical propagating gravitational degrees of freedom

Classically, the number of physical propagating degrees of freedom of $d > 2$ gravity is well known to be

$$N_{\text{physical}} = \frac{d(d - 3)}{2}. \quad (116)$$

York [8] has proved that these gravitational physical degrees of freedom can be consistently represented at the classical level by the equivalent classes of the $d - 1$ inner metrics of spatial ADM leaves, defined modulo the Weyl $\times \text{Diff}_{d-1}$ symmetry. An elementary consistency check of the correctness of this proposition is by the counting $\frac{d(d-1)}{2} - 1 - (d - 1) = \frac{d(d-3)}{2}$ or $\frac{(d-2)(d-1)}{2} - 1 = \frac{d(d-3)}{2}$.

This paper equivalently postulates that the quantum observables correspond to expectation values of functionals of a given subset of the Weyl invariant fields that parametrize of the Beltrami $d$-bein.

This leaves no choice but to select the fields that compose this subset among the $N_{\text{Beltrami}} = \frac{d^2 - d + 2}{2}$ components of $\mathcal{M}_{(d)}$ while leaving aside the fields that compose the ADM shift vector.

This postulate implies that one checks that all the non Weyl invariant Beltrami fields that that compose the diagonal matrix $\mathcal{M}_{\text{diag}}$ have no physical dynamic. They are $\Phi$, which determines the conformal factor $\exp \Phi$, $N$ and the various rescaling factors $N^i$ (whose interpretation will be further clarified after the writing of the Beltrami metric in (122)).

The number of the Weyl invariant fields that are displayed in $\mathcal{M}_{(d)}$ is such that

$$N_{\text{Beltrami}} - N_{\text{physical}} = \frac{d^2 - d + 2}{2} - \frac{d(d-3)}{2} = d + 1. \quad (117)$$

This indicates that $d + 1$ fields among the Weyl invariant Beltrami fields that parametrize the Beltrami matrix $\mathcal{M}_{(d)}$ must be considered as unphysical ones.

The fact that $d + 1 = d - 1 + 2$ is suggestive enough to identify these $d + 1$ fields. They are respectively the $d - 1$ fields $\mu^i_{\tau}, \mu^i_{\tau}, \mu_i^\tau$ and both components $\mu^\tau_r, \mu^\tau_r$ of the Beltrami differential of $\Sigma_2$.

The QFT explanation of this generalizes that given for the $d = 2, 3$ and $4$ cases. The $d - 1$ fields $\mu^i_{\tau}, \mu^i_{\tau}, \mu_i^\tau$ are nothing but a parametrization of the ADM shift 1-form field, as made explicit in the further equation (124) obtained by comparing the Beltrami metric (122) (to be shortly established) and the ADM formula $ds^2 = -Ndr^2 + (dx^m + \beta^m d\tau)g_{mn}(dx^n + \beta^n d\tau)$, where $m, n = (z, \tau, i)$. It follows that the excitations of the $d - 1$ Weyl invariant fields $\mu^\tau_r, \mu^\tau_r, \mu_i^\tau$ cannot be considered as parts of the physical degrees of freedom since we know from the ADM analysis that the shift fields don’t have canonical momenta at the classical level (as well as the lapse $N$ in (123)).

As for the fields $\mu^\tau_\tau$ and $\mu_i^\tau$, they are the Beltrami differentials of $\Sigma_2$ at fixed $\tau$ and $x^i$. They characterize the Riemann surfaces $\Sigma_2$ that sub-foliate each ADM leaf decomposed according to $\Sigma_{d-1} = \Sigma_2 \times \Sigma_{d-3}$. They can be actually gauge fixed by using two freedoms that one may covariantly choose among those of the Diff$_d$ symmetry, according to the same equation as in (124), namely

$$\mu^\tau_\tau = \gamma, \quad \mu_i^\tau = \tau, \quad (118)$$

where $\gamma = \sum_{k=1}^{d-3} \lambda_k f^k(z, \tau)$. Because the fields $\mu^\tau_\tau$ and $\mu_i^\tau$ can be gauge fixed as the moduli of $\Sigma_2$, it is consistent not to count them as parts of the physical degrees of freedom of $d$-gravity.

The proposal made in this paper is thus that the $\frac{d(d-3)}{2}$ physical propagating degrees of freedom of $d$-dimensional gravity must be identified with the Beltrami fields $\mu^\tau_r, \mu^\tau_r, \mu_i^\tau$ that compose the following $(d - 1) \times (d - 1)$ subpart of the Beltrami matrix (124)

$$M_{\text{Physical dofs}} = \begin{pmatrix} 1 & \mu^\tau_\tau & \mu^\tau_\tau & \ldots & \mu^\tau_i & \ldots & \mu^\tau_{d-1} \\
\gamma & 1 & \mu^\tau_\tau & \mu^\tau_\tau & \ldots & \mu^\tau_i & \ldots & \mu^\tau_{d-1} \\
0 & 0 & 1 & \mu^\tau_\tau & \mu^\tau_\tau & \ldots & \mu^\tau_i & \ldots & \mu^\tau_{d-1} \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \mu^\tau_i & \mu^\tau_i & \ldots & \mu^\tau_i & \ldots & \mu^\tau_{d-1} \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \mu^\tau_{d-1} & \mu^\tau_{d-1} & \ldots & \mu^\tau_{d-1} & \ldots & \ldots & \ldots \\
\end{pmatrix} \quad (119)$$
where $\mu_j^i = -\mu_i^j$ and $\mu_2^{\tau}$ has been gauge fixed equal the moduli $\gamma$ of $\Sigma_2$. One can check that the number of the independent fields in $M_{phys}$ is $\frac{d(d-3)}{2}$ according to $2(d-3) + \frac{(d-3)(d-4)}{2} = \frac{d(d-3)}{2}$ or equivalently $2 + 3 + \ldots + (d - 2) = \frac{(d-2)(d-3)}{2} - 1 = \frac{d(d-3)}{2}$.

The definition of the d-dimensional propagating gravitational physical states as the $\frac{d(d-3)}{2}$ generalized Beltrami differential components $\mu_m^m$, $m = (z, \tau, i, j = 3, \ldots, d - 1)$ that compose the matrix (119) at fixed moduli $\gamma$ is an alternative and interesting proposition. The perturbative excitations of these fields correspond to the traceless and transverse excitation of d-metrics.

An attractive feature of this definition of the physical gravitational degrees of freedom is of relying on a geometrically well-identified subset of the fundamental local fields of the theory. It may render easier a formal definition of the mean values of the gravitational physical observables within a (yet to be defined) path integral formalism where the measure is expressed in terms of the Beltrami fields, as a generalization of the bidimensional case where the classical part of the gravitational measure is simply $[d\Phi][d\mu_2^{\tau}][d\mu_2^{\tau}]$. Moreover, the genus dependence of the sub-foliating Riemann surfaces $\Sigma_2$ is encoded in (118) that expresses the gauge fixing of $\mu_2^{\tau}$ and $\mu_2^{\tau}$. Following this partial gauge fixing of the Diff$_d$ invariance, the left upper $2 \times 2$ sub-matrix $\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}$ is what is left from the internal metric of $\Sigma_2$ modulo Weyl invariance. This partial gauge fixing might be a way to systematically encode the relevant topological information about the sub-manifold $\Sigma_{d-1}$ in a path integral formulation. This point certainly deserves more clarifications.

One can conclude this discussion with a simple remark concerning the dimensional reduction. The Kaluza–Klein compactification of the genuine $d$-gravity defines a $(d - 1)$ dimensional gravitational theory with a $(d - 1)$-dimensional metric coupled to a 1-form gauge field $A_m dx^m$ and a scalar field $\phi$. $\phi$ carries one physical degree of freedom. Since the number of physical degrees of freedom of the $d$-dimensional graviton is $N_{d-1}^{\text{physical graviton}} = \frac{d(d-3)}{2}$ and since the compactification conserves the number of physical degrees of freedom, one has

$$N_{d-1}^{\text{physical graviton}} = d - 2 = N_{d-1}^{\text{physical gauge field}} + 1$$

This implies that $N_{d-1}^{\text{physical gauge field}} = d - 3$. One finds therefore that the counting of the physical degrees of freedom of the graviton consistently predicts that a $d$-dimensional gauge field carries $N_d^{\text{physical gauge field}} = d - 2$ physical degrees of freedom. This simple counting gives the way to express the dimensional reduction of a Beltrami $d$-bein into a Beltrami $(d - 1)$-bein completed with a $(d - 1)$ gauge field and a scalar, both related to the Beltrami fields in $d$ dimensions. The argument can be repeated for further compactifications. 120 is a simplistic but quite physical justification for the consistency of the definition of the gravitational degrees of freedom as the above well-defined subset of the Beltrami fields.

### 6.4 A “physical” gravitational gauge choice

Once $\mu_2^{\tau}$ and $\mu_2^{\tau}$ are gauge fixed to be equal to the moduli $\gamma$ and $\tau_1$, of $\Sigma_2$ one must go on and complete the gauge fixing of the $d - 2$ freedoms of the the Diff$_d$ symmetry of $M_d$. A suggestive completion of the gauge fixing is by further imposing $\Phi = 0$ and the $d - 3$ conditions $N^i = 1$. One then gets

$$M_{d_{\text{diag}}}^{d} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & N \end{pmatrix}.$$  

The remaining field dependence of the gauge fixed Beltrami $d$-beins is through the lapse $N$, the components $\mu_2^{\tau}, \mu_2^{\tau}, \mu_2^{\tau}$ of the ADM shift fields and the $\frac{d(d-3)}{2}$ propagating physical Beltrami fields. The $d$-dimensional Beltrami metric will be shortly made explicit in this gauge.
6.5 Beltrami \(d\)-metric and its possible gauge fixings

The reparametrization invariant \(d\)-metric associated to the \(d\)-bein \(e^a_\mu \) in \(105\) is \(g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu \). This yields the following formula for the Beltrami \(d\)-metric of \(\mathcal{M}_d\):

\[
\frac{1}{2} ds^2 = -N^2 \left( d\tau + \sum_{i=3}^{d-1} \mu_i^2 dx^i \right)^2 + \exp \Phi \left| dz + \mu_2^2 d\tau + \mu_3^2 dx^3 + \ldots + \mu_{d-1}^2 dt^{d-1} + \mu_d^2 d\tau \right|^2 \\
+ \sum_{i=3}^{d-1} \sum_{j \neq i, j=3}^{d-1} N^2 \left( \mu_i^2 dx^3 + \ldots + \mu_{j-1}^2 dt^{j-1} + dt^j + \mu_{j+1}^2 dx^{j+1} + \ldots + \mu_d^i d\tau \right)^2, \tag{122}
\]

(One must enforce the antisymmetric relations \(\mu_i^i = -\mu_i^i\) and \(\mu_i^j = -\mu_i^j\) in this formula.)

\(122\) is quite basic. It allows to precisely relate the Beltrami fields \(N\) and \(\mu^0 = \mu^\tau, \mu^\tau, \mu^i\) to the time lapse \(N\) and the shift one-form \(\beta_m\) defined by the ADM metric. Indeed, the comparison of the Beltrami metric \(122\) to the ADM metric \(ds^2 = -N^2 + (dx^m + \beta^m d\tau)g_{mn}(dx^n + \beta^n d\tau)\) for \(m = z, \tau, i\) provides the following formula for \(N\) and \(\beta_m\)

\[
N^2 = N^2 - \exp \Phi \mu^\tau_\tau - \sum_i N^2 \mu^i_\tau^2 \tag{123}
\]

\[
\beta_z = \exp \Phi(\mu^\tau_\tau + \mu^\tau_2 \mu^2_\tau) \quad \beta_\tau = \exp \Phi(\mu^\tau_\tau + \mu^\tau_3 \mu^3_\tau) \quad \beta_i = (N^2 - N^2 \mu^i_\tau) \mu^i_\tau - \exp \Phi(\mu^i_\tau \mu^\tau_\tau + \mu^i_\tau \mu^i_\tau) - \sum_{j \neq i} N^2 \mu^j_\tau \mu^i_\tau. \tag{124}
\]

These relations between the ADM lapse and shift fields \(N\) and \(\beta_m\) and the Beltrami fields \(N\) and \(\mu^0\) indicate that the possible excitations of the latter carry no propagating physical degrees of freedom. The \(d - 3\) fields \(N^i\) appear as dilatation factors for the \(d - 3\) coordinates in \(\Sigma_{d-3}\), and their excitations are neither expected to be counted as parts of the physical gravitational fields degrees of freedom like the conformal factor \(\Phi\). The fate of the fields \(\mu^\tau\) and \(\mu^\tau_\tau\) is to be gauge fixed equal to the moduli of \(\Sigma_2\) as in \(70\). The remaining \(\frac{d(d-3)}{2}\) Weyl invariant fields \(\mu^i_\tau = (\mu^2_\tau, \mu^3_\tau, \mu^i_\tau)\) that compose the Beltrami metric \(122\) must be therefore identified with the propagating physical degrees of freedom of gravity as already claimed at the level of the Beltrami \(d\)-bein.

One possible way to compute the Einstein action in function of the Beltrami fields is by expressing it as a quadratic form in the Spin connection that generalizes the three and four dimensional formulae \(64\) and \(82\). In a presumably less illuminating way, one might go ahead and compute the Christoffel symbols for the Beltrami action by its standard expression in terms of these entities. It is not obvious that the current algebraic general relativity softwares can be straightforwardly adapted to compute the Einstein action under a satisfying form such as \(64\), with \(122\) as an input.

One can now address the question of the gauge fixing of the Beltrami metric. The understanding of the nature of the Beltrami fields suggests two natural choices for gauge fixing the remaining local \(d\) freedoms under the \(\text{Diff}_d\) symmetry of the Beltrami metric.

The first one is to impose the \(d = (d - 3) + 2 + 1\) gauge conditions \(\mu^i_\tau = -\mu^i_\tau = 0, \mu^\tau_\tau = \gamma, \mu^\tau_2 = \tau, N = 1\). This defines the following gauge fixed metric:

\[
ds^2 = -d\tau^2 + \exp \Phi |dz + \gamma d\tau + \mu^2_\tau dx^3 + \ldots + \mu_{d-1}^2 dt^{d-1} + \mu_d^2 d\tau|^2 \\
+ \sum_{i=3}^{d-1} \sum_{j \neq i, j=3}^{d-1} N^2 \left( \mu_i^2 dx^3 + \ldots + \mu_{j-1}^2 dt^{j-1} + dt^j + \mu_{j+1}^2 dx^{j+1} + \ldots + \mu_d^i d\tau \right)^2. \tag{125}
\]

The spatial lapse interpretation of the \(d - 3\) fields \(N^i\) will be clarified in a separate publication by using this gauge.

Alternatively one can use the \(d\) gauge conditions made of \(\Phi = 0, N^i = 1, \mu^\tau_\tau = \gamma\) and \(\mu^\tau_2 = \tau\). (A possible variant is by replacing the condition \(\Phi = 0\) by the unimodular gauge condition.) This gauge choice is
already suggested at the level of the Beltrami $d$-bein in (121) and provides the following expression for the metric:

\[
\frac{1}{2} ds^2 = -N^2 \left( d\tau + \sum_{i=3}^{d-1} \mu_i^a dx^i \right)^2 + \left| dz + \gamma d\tau + \mu_3^a dx^3 + \ldots + \mu_{d-1}^a dt^{d-1} + \mu_d^a d\tau \right|^2 \\
+ \sum_{i=3}^{d-1} \sum_{j=i+1}^{d-1} \left( \mu_i^a dx^i + \mu_j^a dx^j + \mu_{i+1}^a dx^{i+1} + \ldots + \mu_d^a d\tau \right)^2.
\] (126)

The gauge fixed metric (126) is only function of the $d$-gravitational physical propagating degrees of freedom $\mu_i^M$ and of $\mu_i^\mu$ and $N$. $\mu_i^\mu$ and $N$ compose the ADM shift vector and the lapse function. The Einstein Lagrangian reduces in this gauge to a polynomial function of the physical fields $\mu_i^a, \mu_i^\mu, \mu_j^a$ and their derivatives with an algebraic dependence on both ADM lapse and shift functions $N$ and $\beta_m$ expressed in (123) and (124).

A solution such that $\beta_m = 0$ and $N = 1$ is for $\mu_i^\mu = 0$. If this constraint is a consistent one, one reaches a situation where one ends up with a gauge fixed formulation for the Einstein action whose dynamics only depends on the physical degrees of freedom $\mu_i^a, \mu_i^\mu, \mu_j^a$, geometrically well defined from the beginning, with their own classical momenta $p_i^a, p_i^\mu, p_j^a$. This choice of gauge formally suggest the existence of a covariant physical Hamiltonian phase space with $\frac{d^2(d-3)^2}{4}$ components. This sustains the appellation of a "physical gauge choice" that was already done in section 6.4.

More ingenious gauge fixings of the fields selected by the Beltrami parametrization might suggest new hints for a better definition of the quantum gravity rules keeping in mind that the gravitational physical degrees of freedom are identified as a well-defined subset of the fields that parametrize the metric (119). It is however quite clear that the fundamental questions about a consistent definition of quantum gravity remain as they are within the Beltrami parametrization independent of its mathematical appeal and physical motivations.

### 6.6 BRST symmetry

Terms that involve the ghosts must be systematically added to the gauge fixed Einstein action to ensure that the whole action is BRST invariant for any given gauge choice. The method explained in sections 3 and 4 that geometrically determines the BRST symmetry transformations on all Beltrami fields and their ghosts in the cases $d = 2$ and $d = 3$ generalize straightforwardly for all values of the space dimension $d$. One can indeed generically unify $E^a \rightarrow E^a + \xi^a, a = z, \bar{z}, 3 \leq i \leq d$. The relation between the Beltrami ghost $c^a$ and the $d$-dimensional standard reparametrization ghost $\xi^a$ is

\[c^a \equiv \exp i\xi \ E^a\] (127)

and the determination of the BRST symmetry transformation laws acting all on fields of the generalized Beltrami $d$-dimensional parametrization of the metric (108) and their ghosts proceeds by expanding in form degree and ghost number the $d$-dimensional horizontality equations that simply generalize (67).

### 7 Spaces with special holonomy $\mathcal{G} \subset SO(d - 1, 1)$

The Beltrami $d$-parametrization (122) can be further simplified if the space-time has a spatial holonomy. The resulting larger freedom allows one to further simplify the form of the Beltrami $d$-bein. This section exemplifies the general process with the case of eight dimensional manifolds with holonomy $G_2 \subset SO(7, 1)$. 

33
The formula (108) defines the Beltrami eight-bein for a general eight dimensional manifold \( \mathcal{M}_8 \) as follows:

\[
\begin{pmatrix}
  e^z \\
e^\tau \\
e^3 \\
e^4 \\
e^5 \\
e^6 \\
e^7 \\
e^7
\end{pmatrix} = \begin{pmatrix}
  \exp \Phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & \exp \Phi & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & N^3 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & N^4 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & N^5 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & N^6 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & M & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & N
\end{pmatrix} \begin{pmatrix}
  1 & \mu_1^2 & \mu_3^2 & \mu_4^2 & \mu_5^2 & \mu_6^2 & \mu_7^2 & \mu_8^2 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9
\end{pmatrix} \begin{pmatrix}
  dz \\
  d\tau \\
  dx^3 \\
  dx^4 \\
  dx^5 \\
  dx^6 \\
  dx^7 \\
  dx^\tau
\end{pmatrix}.
\]

(128)

One may consider the class of the eight dimensional manifolds with holonomy \( G_2 \subset SO(1,7) \), \( G_2 \) being the simplest exceptional rank 2 group with 14 generators. This extends the number of covariant constraints that one can impose to the 64 components of a general 8-bein from the 28 coming from the \( SO(1,7) \) Lorentz gauge freedoms and resulting to (128) to the higher value \( 42 = 14 + 28 \). The remaining 14 freedoms that result from the \( G_2 \) holonomy of the manifold allow further covariant constraints for the matrix elements of (128), allowing to express the Beltrami eight-bein as follows:

\[
\begin{pmatrix}
  e^z \\
e^\tau \\
e^3 \\
e^4 \\
e^5 \\
e^6 \\
e^7 \\
e^7
\end{pmatrix} = \begin{pmatrix}
  \exp \Phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & \exp \Phi & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & N^4 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & N^6 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & N^6 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & N^6 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & M & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & N
\end{pmatrix} \begin{pmatrix}
  1 & \mu_1^2 & \mu_3^2 & \mu_4^2 & \mu_5^2 & \mu_6^2 & \mu_7^2 & \mu_8^2 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
  \mu_1 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9
\end{pmatrix} \begin{pmatrix}
  dz \\
  d\tau \\
  dx^3 \\
  dx^4 \\
  dx^5 \\
  dx^6 \\
  dx^7 \\
  dx^\tau
\end{pmatrix}.
\]

(129)

The corresponding Beltrami \( d = 8 \) metric is

\[
\frac{1}{2} ds^2 = -N^2(d\tau - adx^7)^2 + M^2(dx^7 + ad\tau)^2
\]

\[
+ N^4 \left( dx^3 + dx^4 + \mu_4^2 (dx^3 - dx^4) \right) + N^6 \left( dx^5 + dx^6 + \mu_6^2 (dx^5 - dx^6) \right) + \exp \Phi \left| dz + \mu_2 dx^2 + \mu_3 dx^3 + \mu_4 dx^4 + \mu_5 dx^5 + \mu_6 dx^6 + \mu_7 dx^7 + \mu_8 dx^8 \right|^2.
\]

(130)

This formula exhibits some analogy with he generic four dimensional Beltrami metric (31). One can furthermore use the eight gauge freedoms of the remaining \( d = 8 \) reparametrization symmetry. A possible gauge choice for the \( \text{Diff}_8 \) invariance is \( a = 0 \), \( M = N = N^4 = N^6 = 0 \). This defines a coordinate system in which the \( d = 8 \) metric with holonomy \( G_2 \) is

\[
\frac{1}{2} ds^2 = N^2(-dx^2 + dx^7)^2 + (dx^3 + dx^4 + dx^5 + dx^6)^2 + \exp \Phi \left| dz + \gamma dx^2 + \mu_2 dx^3 + \mu_3 dx^4 + \mu_4 dx^5 + \mu_5 dx^6 + \mu_7 dx^7 + \mu_8 dx^8 \right|^2.
\]

(131)

This fully gauge fixed metric can be compared to the gauge fixed four dimensional metrics (53).

8 Conclusion

This work presents a generalization of the bidimensional Beltrami parametrization for gravity and theories coupled to gravity that is valid in all dimensions \( d > 2 \). The Beltrami parametrization for \( d \geq 3 \) is made possible by a covariant sub-foliation of the ADM leaves \( \Sigma_{d-1} \sim \Sigma_{d-3} \times \Sigma_2 \) of Lorentzian \( d \)-dimensional manifolds \( \mathcal{M}_d \). The found expressions of the \( d > 2 \) Beltrami d-bein and associated Beltrami d-metric derive from a covariant gauge fixing of the \( \frac{d(d-1)}{2} \) local freedoms offered by the Lorentz gauge symmetry \( SO(d-1,1) \subset SO(d-1,1) \times \text{Diff}_d \) symmetry in \( \mathcal{M}_d \). The fields that compose the generalized d-dimensional Beltrami vielbein are neatly and covariantly separated according to their Weyl weights. They fall in different categories, each one having
its distinct gravitational interpretation. The generic formula (122) defines the “Beltrami parametrization” that holds true for arbitrary dimension \( d \geq 2 \). It exhibits a non trivial \( z \leftrightarrow \tau \) symmetry where \( z \) and \( \tau \) are the complex coordinates of the sub-foliating Riemann surface \( \Sigma_2 \). As noted in the introduction, the notion of a generalized Beltrami parametrization, as it is defined in this article, must be stricto sensu taken in a local sense.

The sub-foliation of the ADM leaves by the Riemann surfaces \( \Sigma_2 \) provides a suggestive definition of the gravitational propagating physical degrees of freedom as a well identified subset of the Beltrami fields. They arise as a generalization of the bidimensional Beltrami differential according to

\[
\mu^i_\tau, \mu^\tau_i \rightarrow \mu^i_z, \mu^\tau_z, \mu^i_\tau, \mu^{ij}_z, \mu^\tau_\tau, \mu^\tau_i
\]

where \( \mu^i_j = -\mu^j_i, 3 \leq i, j \leq d - 1 \). The excitations of \( \mu^i_z, \mu^\tau_i \) and \( \mu^i_j \) can be at least perturbatively identified as the \( d(d-3)/2 \) physical degrees of freedom of the graviton. \( \mu^i_z \) and \( \mu^\tau_z \) have no physical excitations and can be gauge fixed as moduli of \( \Sigma_2 \). The rest of the fields that parametrize the Beltrami metric are the conformal factor, \( d - 2 \) generalized lapse functions (one time lapse for \( \tau \) and \( d - 3 \) dilatation functions for the coordinates \( x^i \)) and \( d - 1 \) Weyl invariant fields that are related to the ADM shift vector.

The Beltrami \( d \)-bein and the associated metric can be further simplified for spaces with a given holonomy, as exemplified by the \( d = 8 \) formula (130).

The paper is written in a bottom to top approach. It starts from the well-known bidimensional situation that is generalized till one obtains a satisfying formulation for the generic \( d \)-dimensional Beltrami metric. It could have been alternatively presented in a top to bottom approach by putting section 6 in first position and considering afterward the \( d = 2, 3, 4 \) dimensional cases as applications. Some readers might prefer such a more formal presentation. But, the generic case presented in section 6 was truly obtained by a trial and error construction of the \( d = 3 \) and \( d = 4 \) cases as cautious generalizations of the solidly established \( d = 2 \) Euclidean case. The accumulation of all the details gathered in section 4 for the three dimensional case (with zero physical degrees of freedom) and in section 5 for the four dimensional case (with two physical degrees of freedom that have a large enough space to possibly propagate) were in fact necessary to get a global understanding of the general case for all values of \( d \).

One can maybe go further in the use of the \( d \)-dimensional Beltrami parametrization than being only concerned by the mere propagation of gravitational degrees of freedom. Indeed, part of the quantization program consists in functionally integrating over all possibilities for the sub-foliating surfaces \( \Sigma_2 \) with \( \Sigma_2^{ADM} \sim \Sigma_{d-3} \times \Sigma_2 \). To do so one may consider the Beltrami fields as the fundamental gravitational fields for defining the measure of the path integral. This presents a clear advantage since a subset of the Beltrami fields directly defines the physical gravitational degrees of freedom. It might be that one can make firstly this reduced part of the functional integration with an appropriate BRST invariant gauge fixing of \( \Sigma_2 \) that possibly takes into account the number of holes in each ADM leaf \( \Sigma^{ADM}_{d-1} \) and by doing only afterward the remaining part of the path integral that concerns the local dynamic. This picture requires more thinking. One may furthermore question wether the Riemann surfaces \( \Sigma_2 \) that sub-foliate the ADM leaves of \( d \)-gravity can be identified as remnants of the worldsheets of an underlying string theory with \( \mathcal{M}_d \) as a target space. A more refined string theory zero limit might exist that could possibly offer more information than the well-known perturbative diagrammatic link between string theory and perturbative gravity around a classical gravitational background, a possibility that might be the subject of a separate publication. There, an additional path integration over the gravitational background of string theory will be tentatively incorporated in the Polyakov path integral for allowing a more sophisticated string theory gauge fixing that would hopefully consistently identify the string worldsheets with the sub-foliating surfaces \( \Sigma_2 \) of the \( d \)-gravity that the present work introduces from the strict point of view of the Einstein gravity.

**Note:** This paper refers to a very limited number of published works. The author thanks in advance the readers who would inform him about references where ideas similar to those presented in this work have been discussed and plans to add some of them in future versions.

**Acknowledgements:** It is a real pleasure to thank John Iliopoulos for many discussions on the subject and Vadim Braid and Tom Wetzstein for precise and careful readings of this article.
A Appendix : $d = 4$ Beltrami Spin connection equations

This appendix computes the 24 linear equations that determine the Spin connection $\omega(e)$ for $d = 4$ when the vierbein is expressed as in $\text{[132]}$. One has from section 5

$$
\begin{pmatrix}
\mathcal{E}_z^z \\
\mathcal{E}_z^i \\
\frac{dt}{d\tau}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\mu_z} & \frac{\mu_z^z}{\mu_z} & 0 \\
\frac{\mu_z}{\mu_z} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
(zd) \\
(a d\tau) \\
\frac{d\tau}{d\tau}
\end{pmatrix}.
$$

The $d = 3$ case is obtained by restricting the 4X4 matrix $\text{[132]}$ to the $3 \times 3$ matrix in its upper left corner and the $d = 2$ case is when it is reduced to the $2 \times 2$ matrix $\begin{pmatrix}
\frac{1}{\mu_z} & \frac{\mu_z^z}{\mu_z} \\
0 & 1
\end{pmatrix}$ in its left top corner. One uses $d = dz \partial_z + dz \partial_\tau + \partial_0 dt + \partial_\tau d\tau = \mathcal{E}_z^z D_z + \mathcal{E}_z^i D_i + \mathcal{E}_z^0 D_0 + \mathcal{E}_z^\tau D_\tau$ with

$$
\begin{pmatrix}
\frac{\partial_z}{\partial_\tau} \\
\frac{\partial_\tau}{\partial_\tau} \\
\frac{\partial_0}{\partial_\tau} \\
\frac{\partial_\tau}{\partial_\tau}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\frac{1}{\mu_z} - \frac{1}{\mu_z}} & \frac{1}{\mu_z} - \frac{\mu_z^z}{\mu_z} & 0 \\
\frac{\mu_z}{\frac{1}{\mu_z} - \frac{1}{\mu_z}} & \frac{1}{\mu_z} - \frac{\mu_z^z}{\mu_z} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\frac{1}{M - \bar{\mu}} & \frac{-\bar{\mu}}{N} & 0 \\
\frac{1}{M - \bar{\mu}} & \frac{-\bar{\mu}}{N} & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial_z}{\partial_\tau} \\
\frac{\partial_\tau}{\partial_\tau} \\
\frac{\partial_0}{\partial_\tau} \\
\frac{\partial_\tau}{\partial_\tau}
\end{pmatrix}.
$$

The 24 components of the $d = 4$ Beltrami Spin connection $\omega(e)$ solve the torsion free conditions $T^0 = T^z = T^\tau = T^\tau = 0$. Once they are determined, one can compute Einstein action as a quadratic form in the $\omega(e)$'s. The equation $T^\tau = 0$ becomes irrelevant in $d = 3$ as well as $T^0 = T^z = 0$ in $d = 2$. The case $d = 2$ is trivial and directly solved in section 2. The 9 torsion zero equations for the case $d = 3$ are solved in Appendix B.

$\text{[132]}$ implies $\mathcal{E}^z = dz + \mu_z^z d\tau + \mu_0^z dt + \mu_\tau^z d\tau$ and $\mathcal{E}^\tau = d\tau + \mu_\tau^z dz + \mu_\tau^0 dt + \mu_\tau^\tau d\tau$. Since $\mathcal{E}^0 = N dt + a d\tau$ and $\mathcal{E}^\tau = a dt + M d\tau$ one has $\mathcal{E}^0 = dN dt + da d\tau$ and $\mathcal{E}^\tau = dM dt + dM d\tau$.

In the coordinate system $x^\mu = (t, z, \tau, \bar{\tau})$ one has $a = -\bar{\mu} = \mu^0$ and $M \neq \bar{M}$; in the light cone coordinates sytem $x^\mu = (\tau^+, \tau^-, z, \bar{\tau})$ and $a = \mu^- = -\mu^\tau$, $M = \bar{N} = N$. Thus the use of $a$ and $\bar{\mu} \neq a$ covers both cases for the coordinates $(\tau, t, z, \bar{\tau})$ and $(\tau^+, \tau^-, z, \bar{\tau})$.

- **Consequences of $T^\tau = 0$**

The 2 form $T^\tau$ writes

$$
\begin{align*}
T^\tau & \equiv d\mathcal{E}^\tau + \frac{1}{2} \omega^{\tau \tau} \wedge d\mathcal{E}^\tau + \frac{1}{2} \omega^{\tau \tau} \wedge d\mathcal{E}^0 + \omega^{\tau \tau} \wedge d\mathcal{E}^0 \\
& = (\mathcal{E}^z D_z M + \mathcal{E}_z^\tau D_\tau M + D_0 M dt) \wedge d\tau + (\mathcal{E}^z D_z \bar{\mu} + \mathcal{E}^\tau D_\tau \bar{\mu} + D_\tau \bar{\mu} d\tau) \wedge dt \\
& - \frac{1}{2} \exp \bar{\mu} \mathcal{E}^z \wedge \omega^{\tau \tau} - \frac{1}{2} \exp \mu_0 \mathcal{E}^z \wedge \omega^{\tau \tau} + \omega^{\tau \tau} \wedge (N dt + a d\tau) \\
& = \mathcal{E}^z \wedge \mathcal{E}^z \left( \frac{1}{2} \exp \bar{\mu} \omega^{\tau \tau} \right) \\
& + \mathcal{E}^z \wedge \mathcal{E}^z \left( \frac{1}{2} \exp \mu_0 \omega^{\tau \tau} \right) \\
& + \mathcal{E}^z \wedge dt (N \omega^{\tau 0}_z - \frac{1}{2} \exp \mu_0 \omega^{\tau 0}_z a) + \mathcal{E}^\tau \wedge dt (\mathcal{D}_\tau M - \frac{1}{2} \exp \bar{\mu} \omega^{\tau \tau} + \omega^{\tau \tau} a) \\
& + \mathcal{E}^\tau \wedge dt (N \omega^{\tau 0}_z - \frac{1}{2} \exp \mu_0 \omega^{\tau 0}_z + \mathcal{D}_\tau \bar{\mu} + D_\tau \bar{\mu}) \\
& + dt \wedge d\tau (D_0 M - D_\tau \bar{\mu} - N \omega^{\tau 0}_z - a \omega^{\tau 0}_z). \tag{134}
\end{align*}
$$

The condition $T^\tau = 0$ implies

$$
\begin{align*}
\exp \bar{\mu} \omega^{\tau \tau}_z - \exp \mu_0 \omega^{\tau \tau}_z &= 0 \\
N \omega^{\tau \tau}_z + \frac{1}{2} \exp \mu_0 \omega^{\tau \tau}_z &= D_\tau \bar{\mu} \\
N \omega^{\tau \tau}_z + \frac{1}{2} \exp \bar{\mu} \omega^{\tau \tau}_z &= D_\tau \mu_0 \\
\frac{1}{2} \exp \mu_0 \omega^{\tau \tau}_z - \omega^{\tau 0}_0 a &= D_\tau M \\
\frac{1}{2} \exp \bar{\mu} \omega^{\tau \tau}_z - \omega^{\tau 0}_0 a &= D_\tau M
\end{align*}
$$

36
\[ N \omega^{\tau_0} - a \omega_0^{\tau_0} = D_o M - D_r \pi \]  

(135)

**Consequences of \( T^0 = 0. \)**

The 2-form \( T^0 \) writes

\[
T^0 = d\omega^0 + \frac{1}{2} \omega^0 \wedge e^\tau = \frac{1}{2} \omega^0 \wedge e^\tau - \omega^{0r} \wedge e^r
\]

\[
= (E^z D_z N + E^z D_z N + D_r N d\tau) \wedge dt + (E^z D_z a + E^z D_z a + D_o a d\tau) \wedge d\tau + \frac{\exp \varphi}{2} E^z \wedge \omega^z - \frac{\exp \varphi}{2} E^z \wedge \omega^z - \omega^{0r} \wedge (\pi dt + M d\tau)
\]

\[
= \frac{1}{2} E^z \wedge E^z (\exp \varphi \omega^z + \exp \varphi \omega^z) + E^z \wedge dt(D_z N - \frac{\exp \varphi}{2} \omega^z - \pi \omega^0) + E^z \wedge dt(D_z N - \frac{\exp \varphi}{2} \omega^z - \pi \omega^0) + E^z \wedge d\tau(-M \omega^0 - \frac{\exp \varphi}{2} \omega^0 + D_z a) + E^z \wedge d\tau(-M \omega^0 - \frac{\exp \varphi}{2} \omega^0 + D_z a)
\]

\[
+ d\tau \wedge dt(D_r N - D_o a - M \omega^0 - \pi \omega^0).
\]

(136)

The vanishing conditions of the projections of \( T^0 \) on \( E^z \wedge E^z, E^z \wedge dt, E^z \wedge d\tau, E^z \wedge d\tau, \) and \( E^z \wedge dt, d\tau \) imply

\[
\exp -\varphi \omega^z + \exp -\varphi \omega^z = 0
\]

\[
\omega^0 = 2 \exp -\varphi (-D_z N + \pi \omega^0)
\]

\[
\omega^0 = 2 \exp -\varphi (D_z N - \pi \omega^0)
\]

(137)

\[
M \omega^0 + \frac{\exp \varphi}{2} \omega^z = D_z a
\]

\[
M \omega^0 - \frac{\exp \varphi}{2} \omega^z = D_z a
\]

\[
M \omega^0 + \pi \omega^0 = D_r N - D_o a
\]

(138)

**Consequences of \( T^z = T^\tau = 0. \)**

Expanding both 2-form equations \( T^z = T^\tau \) is slightly more involved than doing it for \( T^0 = T^\tau = 0. \) The following relation is useful to relate the components of forms expressed either on the basis of 1-forms \( (dz, d\tau, dt, \tau) \) or on the basis of 1-forms \( (E^z, E^\tau, dt, d\tau) \):

\[
\left( \begin{array}{l}
\frac{dz}{d\tau} \\
\frac{d\tau}{d\tau}
\end{array} \right) = \frac{1}{1 - \mu \mu^\tau} \left( \begin{array}{rrr}
1 & -\mu^z & 1 \\
1 & 1 & 0
\end{array} \right) \left( \begin{array}{rrr}
E^z & 0 \\
E^\tau & d\tau
\end{array} \right).
\]

One has therefore

\[
dz \wedge d\tau = \frac{1}{1 - \mu \mu^\tau} \left[ (E^z \wedge E^\tau) - (\mu \mu^\tau \mu^\tau)(dt \wedge d\tau) \right] \left( \begin{array}{rrr}
1 & -\mu^z & 1 \\
1 & 1 & 0
\end{array} \right) \left( \begin{array}{rrr}
0 & 1 & -\mu^z \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{rrr}
E^z & 0 \\
E^\tau & d\tau
\end{array} \right) \left( \begin{array}{rrr}
\frac{1}{1 - \mu \mu^\tau} \\
\frac{1}{1 - \mu \mu^\tau}
\end{array} \right)
\]

\[
= \frac{1}{1 - \mu \mu^\tau} \left[ (E^z \wedge E^\tau) - (\mu \mu^\tau \mu^\tau)(dt \wedge d\tau) \right] \left( \begin{array}{rrr}
1 & -\mu^z & 1 \\
1 & 1 & 0
\end{array} \right) \left( \begin{array}{rrr}
0 & 1 & -\mu^z \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{rrr}
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{rrr}
E^z & 0 \\
E^\tau & d\tau
\end{array} \right) \left( \begin{array}{rrr}
\frac{1}{1 - \mu \mu^\tau} \\
\frac{1}{1 - \mu \mu^\tau}
\end{array} \right)
\]

\[
= \frac{1}{1 - \mu \mu^\tau} \left[ E^z \wedge E^\tau + (\mu \mu^\tau \mu^\tau)(dt \wedge d\tau) - E^z \wedge d\tau + E^z \wedge (\mu \mu^\tau dt + \mu^z d\tau) + E^\tau \wedge (\mu \mu^\tau dt + \mu^z d\tau) \right].
\]

(139)
The following matricial identities are useful to derive (139)

\[
(A, \bar{A}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (B, \bar{B}) = AB - \bar{A}B \text{ and } \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (ad - bc) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Then, \( T^\varepsilon \) and \( T^\tau \) read

\[
T^\varepsilon = \frac{d\varepsilon - \omega^0 \wedge e^\varepsilon + \omega^\varepsilon \wedge e^0 - \omega^\varepsilon \wedge e^\tau}{\varepsilon} = \exp\varphi \left( (d\varphi - \omega^0) \wedge \mathcal{E}^\varepsilon + d\mathcal{E}^\varepsilon + \omega^\varepsilon \wedge (Ndt + ad\tau) - \omega^\tau \wedge (n\bar{d}t + M\bar{d}\tau) \right)
\]

\[
= \exp\varphi \left( (d\varphi - \omega^0) \wedge \mathcal{E}^\varepsilon + \omega^\varepsilon \wedge (Ndt + ad\tau) - \omega^\tau \wedge (n\bar{d}t + M\bar{d}\tau) \right)
+ dz \wedge dt \partial_\varepsilon \mu^\varepsilon + dt \wedge d\tau (\partial_\varepsilon \mu^\varepsilon_0 - \partial_\tau \mu^\varepsilon_0 - \partial_\varepsilon \mu^\tau_0 + d\tau \wedge d\tau (\partial_\varepsilon \mu^\tau_0 - \partial_\tau \mu^\varepsilon_0 - \partial_\tau \mu^\varepsilon_0))
\]

\[
T^\tau = \frac{d\tau - \omega^\tau \wedge e^\tau}{\tau} = \exp\varphi \left( (d\varphi + \omega^0) \wedge \mathcal{E}^\tau + d\mathcal{E}^\tau - \omega^\varepsilon \wedge (Ndt + ad\tau) - \omega^\tau \wedge (n\bar{d}t + M\bar{d}\tau) \right)
\]

\[
= \exp\varphi \left( (d\varphi + \omega^0) \wedge \mathcal{E}^\tau + d\mathcal{E}^\tau - \omega^\varepsilon \wedge (Ndt + ad\tau) - \omega^\tau \wedge (n\bar{d}t + M\bar{d}\tau) \right)
+ dz \wedge dt \partial_\tau \mu^\tau + dt \wedge d\tau (\partial_\tau \mu^\tau_0 - \partial_\tau \mu^\tau_0 - \partial_\tau \mu^\varepsilon_0 + d\tau \wedge d\tau (\partial_\tau \mu^\varepsilon_0 - \partial_\tau \mu^\varepsilon_0 - \partial_\tau \mu^\varepsilon_0)).
\]

The condition \( T^\tau = 0 \) is

\[
0 = \left( d\varphi - \omega^0 \right) \wedge \mathcal{E}^\varepsilon
+ \exp - \varphi \omega^\varepsilon \wedge (Ndt + ad\tau)
- \exp - \varphi \omega^\tau \wedge (Mdt + n\bar{d}t)
+ \frac{1}{1 - \mu^\varepsilon_0} \left( \mathcal{E}^\tau \wedge \mathcal{E}^\varepsilon + (\mu^\varepsilon_0 - \mu^\varepsilon_0 \mu^\tau_0 - \mu^\tau_0 \mu^\varepsilon_0) dt \wedge d\tau - \mathcal{E}^\varepsilon \wedge (\mu^\varepsilon_0 dt + \mu^\tau_0 d\tau) + \mathcal{E}^\tau \wedge (\mu^\varepsilon_0 dt + \mu^\tau_0 d\tau) \right) \partial_\varepsilon \mu^\tau
\]

\[
+ \frac{dt}{1 - \mu^\varepsilon_0} \left( \mathcal{E}^\tau \wedge \mathcal{E}^\varepsilon - (\mu^\varepsilon_0 - \mu^\varepsilon_0 \mu^\tau_0 - \mu^\tau_0 \mu^\varepsilon_0) d\tau \right) (\partial_\varepsilon \mu^\varepsilon_0 - \partial_\varepsilon \mu^\tau_0)
\]

\[
- \frac{dt}{1 - \mu^\varepsilon_0} \left( \mathcal{E}^\tau \wedge \mathcal{E}^\varepsilon - (\mu^\varepsilon_0 - \mu^\varepsilon_0 \mu^\tau_0 - \mu^\tau_0 \mu^\varepsilon_0) d\tau \right) \partial_\tau \mu^\varepsilon_0
\]

\[
+ \frac{dt}{1 - \mu^\varepsilon_0} \left( \mathcal{E}^\tau \wedge \mathcal{E}^\varepsilon - (\mu^\varepsilon_0 - \mu^\varepsilon_0 \mu^\tau_0 - \mu^\tau_0 \mu^\varepsilon_0) d\tau \right) (\partial_\varepsilon \mu^\tau_0 - \partial_\tau \mu^\varepsilon_0)
\]

\[
- \frac{dt}{1 - \mu^\varepsilon_0} \left( \mathcal{E}^\tau \wedge \mathcal{E}^\varepsilon - (\mu^\varepsilon_0 - \mu^\varepsilon_0 \mu^\tau_0 - \mu^\tau_0 \mu^\varepsilon_0) d\tau \right) \partial_\varepsilon \mu^\tau_0
\]

\[
+ dt \wedge d\tau (\partial_\varepsilon \mu^\tau_0 - \partial_\tau \mu^\tau_0),
\]

that is

\[
0 = \mathcal{E}^\varepsilon \wedge \mathcal{E}^\tau \left( -D\varphi + \omega^0 \frac{\partial_\varepsilon \mu^\tau_0}{1 - \mu^\varepsilon_0 \mu^\tau_0} \right)
+ \mathcal{E}^\varepsilon \wedge dt \left( -D\varphi + \omega^0 + \exp - \varphi (N \omega^\varepsilon_0 - n \omega^\tau_0) \right) + \frac{1}{1 - \mu^\varepsilon_0 \mu^\tau_0} (\partial_\varepsilon \mu^\tau_0 - \mu^\varepsilon_0 \partial_\varepsilon \mu^\tau_0 + \mu^\varepsilon_0 (\partial_\varepsilon \mu^\tau_0 - \partial_\varepsilon \mu^\tau_0))
\]

\[
+ \mathcal{E}^\varepsilon \wedge dt \left( \exp - \varphi (N \omega^\varepsilon_0 - n \omega^\tau_0) \right) + \frac{1}{1 - \mu^\varepsilon_0 \mu^\tau_0} (\partial_\varepsilon \mu^\varepsilon_0 + \partial_\tau \mu^\varepsilon_0 + \mu^\varepsilon_0 \partial_\tau \mu^\tau_0 - \mu^\varepsilon_0 \partial_\tau \mu^\varepsilon_0)
\]

\[
+ \mathcal{E}^\varepsilon \wedge d\tau \left( -D\varphi + \omega^0 + \exp - \varphi (M \omega^\tau_0 + \omega^\tau_0) \right) + \frac{1}{1 - \mu^\varepsilon_0 \mu^\tau_0} (\partial_\varepsilon \mu^\tau_0 - \mu^\varepsilon_0 \partial_\varepsilon \mu^\tau_0 + \mu^\varepsilon_0 \partial_\varepsilon \mu^\tau_0 - \mu^\varepsilon_0 \partial_\varepsilon \mu^\tau_0)
\]

\[
+ \mathcal{E}^\varepsilon \wedge d\tau \left( \exp - \varphi (M \omega^\varepsilon_0 + \omega^\varepsilon_0) \right) + \frac{1}{1 - \mu^\varepsilon_0 \mu^\tau_0} (\partial_\varepsilon \mu^\varepsilon_0 + \partial_\tau \mu^\varepsilon_0 + \mu^\varepsilon_0 \partial_\tau \mu^\tau_0 - \mu^\varepsilon_0 \partial_\tau \mu^\varepsilon_0)
\]

\[
+ dt \wedge d\tau \left( \exp - \varphi (aw^\varepsilon_0 - N \omega^\varepsilon_0 - M \omega^\varepsilon_0 + n \omega^\tau_0) + \partial_\varepsilon \mu^\tau_0 - \partial_\tau \mu^\tau_0 \right)
\]

\[
+ \frac{1}{1 - \mu^\varepsilon_0 \mu^\tau_0} \left( \partial_\varepsilon \mu^\tau_0 (\partial_\varepsilon \mu^\tau_0 - \mu^\varepsilon_0 \mu^\tau_0) - (\partial_\varepsilon \mu^\tau_0 (\partial_\varepsilon \mu^\tau_0 - \mu^\varepsilon_0 \mu^\tau_0)) (\mu^\tau_0 - \mu^\varepsilon_0 \mu^\tau_0) + \partial_\varepsilon \mu^\varepsilon_0 (\mu^\tau_0 - \mu^\varepsilon_0 \mu^\tau_0)
\]

\[
+ (\partial_\varepsilon \mu^\tau_0 - \partial_\tau \mu^\tau_0) (\mu^\varepsilon_0 - \mu^\varepsilon_0 \mu^\tau_0) - \partial_\varepsilon \mu^\varepsilon_0 (\mu^\varepsilon_0 - \mu^\varepsilon_0 \mu^\tau_0)) \right).
\]
Analogously, one has

\[ T^r = \exp(\overline{\varphi}(d\varphi + \omega^0) \wedge \mathcal{E}^r + \frac{1}{2}(d\varphi + \omega^0) \wedge (Ndt + ad\tau) - \exp(-\frac{1}{2}d\varphi \wedge (\tau dt + M dt)) \]

\[ = \exp(\overline{\varphi}(d\varphi + \omega^0) \wedge \mathcal{E}^r - \exp(-\overline{\varphi} \wedge (Ndt + ad\tau) - \exp(-\overline{\varphi} \wedge (\tau dt + M dt)) \]

\[ + (d\tau \wedge dz \partial_z \mu^r_z + dt \wedge dz (\partial_a \mu^r_z - \partial_z \mu^r_0)) - dt \wedge d\tau \partial_z \mu^r_0 + dt \wedge d\tau (\partial_a \mu^r_z - \partial_z \mu^r_0) - d\tau \wedge d\tau \partial_z \mu^r_0 + dt \wedge d\tau (\partial_0 \mu^r_z - \partial_a \mu^r_0), \]

and the condition \( T^r = 0 \) implies

\[ 0 = (d\varphi + \omega^0) \wedge \mathcal{E}^r \]

\[ - \exp(-\overline{\varphi} \wedge (Ndt + ad\tau) - \exp(-\overline{\varphi} \wedge (\tau dt + M dt)) \]

\[ + \frac{1}{1 - \mu^\mu_{\mu^r_z}} \left[ (d\varphi + \omega^0) \wedge \mathcal{E}^r + (\frac{1}{2}(d\varphi + \omega^0) \wedge (d\varphi + \omega^0)) \right] \]

\[ + \frac{1}{1 - \mu^\mu_{\mu^r_z}} (\partial_0 \mu^r_z - \partial_0 \mu^r_0) \]

\[ = \mathcal{E}^r \wedge \mathcal{E}^r (-D_0 \varphi - \omega^0 + \frac{\partial_0 \mu^r_z}{1 - \mu^\mu_{\mu^r_z}}) \]

\[ + \mathcal{E}^r \wedge dt \left( -D_0 \varphi - \omega^0 - \exp(-\overline{\varphi}(N\omega^r + \pi \omega^r)) + \frac{1}{1 - \mu^\mu_{\mu^r_z}} (\partial_0 \mu^r_z - \partial_0 \mu^r_0) \right) \]

\[ + \mathcal{E}^r \wedge dt \left( - \exp(-\overline{\varphi}(M\omega^r + \omega^r)) + \frac{1}{1 - \mu^\mu_{\mu^r_z}} (\partial_0 \mu^r_z + \partial_0 \mu^r_0) \right) \]

\[ + \mathcal{E}^r \wedge dt \left( - \exp(-\overline{\varphi}(a \omega^r + M\omega^r)) + \frac{1}{1 - \mu^\mu_{\mu^r_z}} (\partial_0 \mu^r_z + \partial_0 \mu^r_0) \right) \]

\[ + \mathcal{E}^r \wedge dt \left( - \exp(-\overline{\varphi}(a \omega^r - M\omega^r)) + \frac{1}{1 - \mu^\mu_{\mu^r_z}} (\partial_0 \mu^r_z + \partial_0 \mu^r_0) \right) \]

\[ + \mathcal{E}^r \wedge dt \left( - \exp(-\overline{\varphi}(a \omega^r - N\omega^r)) + \frac{1}{1 - \mu^\mu_{\mu^r_z}} (\partial_0 \mu^r_z + \partial_0 \mu^r_0) \right) \]

The result of this Appendix is thus the following system of twenty four linear independent equations that determine the components of the four dimensional Spin connection when the vierbein is expressed in the Beltrami parametrization.
\[
\begin{align*}
\omega^0_Z &= -D_z \varphi \\
\omega^0_Z &= +D_z \varphi \\
\omega^0_0 + \exp -\varphi(N \omega^0_0 - \varpi \omega^0_0) &= D_0 \varphi - D_z \mu^0 \\
\omega^0_0 + \exp -\varphi(N \omega^0_0 + \varpi \omega^0_0) &= -D_0 \varphi + D_z \mu^0 \\
\exp -\varphi(N \omega^0_0 - \varpi \omega^0_0) &= D_0 \mu^0 \\
\exp -\varphi(N \omega^0_0 + \varpi \omega^0_0) &= -D_0 \mu^0 \\
\omega^r_0 + \exp -\varphi(-M \omega^r_0 + a \omega^0_0) &= D_r \varphi - D_z \mu^0 \\
\omega^r_0 + \exp -\varphi(M \omega^r_0 + a \omega^0_0) &= -D_r \varphi + D_z \mu^0 \\
\exp -\varphi(a \omega^0_0 - M \omega^r_0) &= D_z \mu^0 \\
\exp -\varphi(a \omega^0_0 + M \omega^r_0) &= -D_z \mu^0 \\
\exp -\varphi \omega^r_0 - N \omega^r_0 - M \omega^0_0 + \varpi \omega^r_0 &= D_r \mu^0 + D_0 \mu^0 \\
\exp -\varphi \omega^r_0 - N \omega^r_0 + M \omega^0_0 - \varpi \omega^r_0 &= -D_r \mu^0 + D_0 \mu^0 \\
\end{align*}
\]

(143)

\[
\begin{align*}
\exp -\varphi \omega^r_0 + \exp -\varphi \omega^r_0 &= 0 \\
\omega^0_0 &= 2\exp -\varphi (D_z N - \varpi \omega^0_0) \\
\omega^r_0 &= 2\exp -\varphi (D_z N - \varpi \omega^0_0) \\
M \omega^0_0 + \frac{\exp \varphi}{2} \omega^r_0 &= D_z a \\
M \omega^r_0 - \frac{\exp \varphi}{2} \omega^r_0 &= D_z a \\
M \omega^r_0 + \varpi \omega^r_0 &= D_z N - D_0 a \\
\end{align*}
\]

(144)

The action on all fields of the derivation operation \( \mathbb{D} = \partial + \ldots \) that figures in (143) involves the derivatives \( D_z, D_r, D_0, D_r \) defined in (133). To make it explicit, one must look at the details of the twenty four independent \( d = 4 \) vanishing torsion conditions for \( T^z = T^\varpi = 0 \) that have been computed above in (141) and (142). The method of inverting the above 24 equations to determine the \( d = 4 \) Spin connection is not so simple and will be published elsewhere for both cases \( a = -\varpi = \mu^0, N \neq M \) and \( a = \mu^r, \varpi = \mu^r, N = M \equiv N \) in a paper specifically dedicated to the four dimensional case. In what follows we reduce them and obtain the 9(equations that determine the three dimensional case.

### B Appendix : Computing the \( d = 3 \) Beltrami Spin connection

The 24 four dimensional Spin connection equations can be obviously reduced to the simpler system of the 9 linear equations that determine the expression of the Spin connection in \( d = 3 \) dimensions. Both Euclidean and Lorentz three dimensional cases are respectively determined by the values \( \epsilon = 1 \) or \( \epsilon = -1 \) as expressed in section 4. One gets linear equations for the 9 three dimensional Spin connection components \( \omega^z_0, \omega^r_0, \omega^z, \omega^r_0, \omega^z_0, \omega^r_0, \omega^0_0, \omega^0_0, \omega^0_0 \). They derive from the \( d = 4 \) equations \( \omega^z, \omega^r, \omega^0 \) solve where one
leaves aside the $\tau$ dependance. The Euclidean and Lorentz are solved at once by replacing $\omega$ into $\omega$ in both equations for $T^z = T^\tau = 0$ as it is obvious from (39). It is useful to define the following derivation operation $\nabla$ for $d = 3$:

\[
\begin{align*}
\nabla_\mathcal{Z} \mu_0^z & = \nabla_\mathcal{Z} \mu_0^z + \mu_0^z \partial_z \mu_0^z - \mu_0^z \partial_z \mu_0^z \\
\nabla_z \mu_0^z & = \nabla_z \mu_0^z + \mu_0^z \partial_z \mu_0^z - \mu_0^z \partial_z \mu_0^z \\
\nabla_z \mu_0^\tau & = \nabla_z \mu_0^\tau + \mu_0^\tau \partial_z \mu_0^\tau - \mu_0^\tau \partial_z \mu_0^\tau.
\end{align*}
\]

Notice that $\nabla_\mathcal{Z} \mu_0^z \equiv \partial_z \mu_0^z + \mu_0^z \partial_z \mu_0^z - \mu_0^z \partial_z \mu_0^z$ formally equates a leaf holomorphic reparametrization transformation of the Beltrami differential $\mu_0^\tau$ with a parameter $\mu_0^z$ (as can be verified from (36)).

The $d = 3$ condition $T^0 = 0$ is

\[
\begin{align*}
\exp -\varphi \omega_\mathcal{Z}^z + \exp -\varphi \omega_\mathcal{Z}^\tau & = 0 \\
\omega_0^z & = -\exp -\varphi 2D_\varphi N \\
\omega_0^\tau & = \exp -\varphi 2D_z N.
\end{align*}
\]

The 6 conditions stemming from $T^\tau = T^z = 0$ are

\[
\begin{align*}
\omega_0^\tau & = D_\varphi \phi - \frac{\partial_z \mu_0^z}{1 - \mu_0^z \mu_0^\tau} \\
\omega_0^z & = -D_\varphi \phi + \frac{\partial_z \mu_0^\tau}{1 - \mu_0^z \mu_0^\tau} \\
\omega_0^\mathcal{Z} & = \frac{1}{N} \exp \varphi (\nabla_\mathcal{Z} \mu_0^z - \nabla_\mathcal{Z} \mu_0^\tau) \\
\omega_0^z & = -\frac{1}{N} \exp \varphi (\nabla_\mathcal{Z} \mu_0^z - \nabla_\mathcal{Z} \mu_0^\tau) \\
\omega_0^\mathcal{Z} & = \frac{\exp -\varphi}{N} \left( D_0 (\nabla_\mathcal{Z} \mu_0^z - \nabla_\mathcal{Z} \mu_0^\tau) + \omega_0^\mathcal{Z} \right) \\
\omega_0^z & = \frac{\exp -\varphi}{N} \left( -D_0 (\nabla_\mathcal{Z} \mu_0^z - \nabla_\mathcal{Z} \mu_0^\tau) + \omega_0^z \right).
\end{align*}
\]

Combining the first equation in (147) with the sum and the difference of both last equations in (148) implies

\[
\begin{align*}
\omega_0^z & = -\left( \frac{1}{2} \nabla_\mathcal{Z} \mu_0^z - \nabla_\mathcal{Z} \mu_0^\tau \right) \frac{1}{1 - \mu_0^z \mu_0^\tau} \frac{1}{2} D_0 (\phi - \varphi) \\
\omega_0^\mathcal{Z} & = \frac{\exp \varphi}{2N} \left( D_0 (\phi + \varphi) + \frac{\nabla_\mathcal{Z} \mu_0^z + \nabla_\mathcal{Z} \mu_0^\tau}{1 - \mu_0^z \mu_0^\tau} \right) \\
\omega_0^z & = \frac{\exp \varphi}{2N} \left( -D_0 (\phi + \varphi) + \frac{\nabla_\mathcal{Z} \mu_0^z + \nabla_\mathcal{Z} \mu_0^\tau}{1 - \mu_0^z \mu_0^\tau} \right).
\end{align*}
\]

One gets therefore

\[
\omega_0^z = \begin{pmatrix}
\frac{\exp \varphi}{2N} (D_0 (\phi + \varphi) - \frac{\nabla_\mathcal{Z} \mu_0^z + \nabla_\mathcal{Z} \mu_0^\tau}{1 - \mu_0^z \mu_0^\tau}) \\
\frac{\exp \varphi}{N} (-D_0 (\phi + \varphi) + \frac{\nabla_\mathcal{Z} \mu_0^z + \nabla_\mathcal{Z} \mu_0^\tau}{1 - \mu_0^z \mu_0^\tau}) \\
\frac{\exp \varphi}{2N} (\nabla_\mathcal{Z} \mu_0^z - \nabla_\mathcal{Z} \mu_0^\tau) \\
\frac{\exp \varphi}{N} (D_0 (\phi + \varphi - \nabla_\mathcal{Z} \mu_0^z - \nabla_\mathcal{Z} \mu_0^\tau) + \omega_0^\mathcal{Z}) \\
\frac{\exp \varphi}{N} (D_0 (\phi + \varphi + \nabla_\mathcal{Z} \mu_0^z + \nabla_\mathcal{Z} \mu_0^\tau) - \omega_0^\mathcal{Z}) \\
\frac{\exp \varphi}{2N} (-D_0 (\phi + \varphi + \nabla_\mathcal{Z} \mu_0^z + \nabla_\mathcal{Z} \mu_0^\tau) - \omega_0^\mathcal{Z})
\end{pmatrix}
\]

where $\Phi = \phi + \varphi$. This concludes the quite simple determination proof of the $d = 3$ Spin connection components $\omega_0^z, \omega_0^\mathcal{Z}, \omega_0^\tau, \omega_0^\varphi, \omega_0^\mathcal{Z}, \omega_0^\varphi, \omega_0^\tau, \omega_0^z, \omega_0^\varphi, \omega_0^\tau$ as displayed in (51) under the form

\[
\omega_0^z = \begin{pmatrix}
\frac{\exp \varphi}{2N} D_0 \phi \\
\frac{\exp \varphi}{N} D_0 \mu_0^z \\
\frac{\exp \varphi}{2N} D_0 \mu_0^\tau \\
\frac{\exp \varphi}{N} D_0 \mu_0^z \\
\frac{\exp \varphi}{2N} \mu_0^z \\
\frac{\exp \varphi}{2N} \mu_0^\tau \\
\frac{\exp \varphi}{2N} \mu_0^z \\
\frac{\exp \varphi}{N} \mu_0^\tau \\
\frac{\exp \varphi}{2N} \mu_0^z \\
\frac{\exp \varphi}{2N} \mu_0^\tau
\end{pmatrix}.
\]
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