THE GREEN-TAO THEOREM FOR AFFINE CURVES OVER \( \mathbb{F}_q \)

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Abstract. Green and Tao famously proved in a 2008 paper that there are arithmetic progressions of prime numbers of arbitrary lengths. Soon after, analogous statements were proved by Tao for the ring of Gaussian integers and by Lê for the polynomial rings over finite fields. In 2020 this was extended to orders of arbitrary number fields by Kai-Mimura-Munemasa-Seki-Yoshino. We settle the case of the coordinate rings of affine curves over finite fields. The main contribution of this paper is subtle choice of a polynomial subring of the given ring which plays the role of \( \mathbb{Z} \) in the number field case. This choice and the proof of its pleasant properties eventually depend on the Riemann-Roch formula.

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1. Introduction

In this paper we prove the following:

**Theorem 1.1.** Let \( p \) be a prime. Let \( \mathcal{O}_0 \) be an integral domain finitely generated over \( \mathbb{F}_p \) and whose fraction field has transcendence degree 1 over \( \mathbb{F}_p \). Then for any positive integer \( k \geq 1 \), the set of prime elements of \( \mathcal{O}_0 \) contains a \( k \)-dimensional affine subset.

Recall that an affine subset of a vector space is by definition a translate of a vector subspace (necessarily unique). Its dimension is defined to be that of the corresponding vector subspace.

We actually prove a density version of the theorem which we now formulate. Let \( \mathcal{O} \) be the integral closure of \( \mathcal{O}_0 \) in its fraction field. It is a Dedekind domain finite over \( \mathcal{O}_0 \).

There is a canonical linear norm, defined in \( \S 2 \)

\[
\| - \| : \mathcal{O} \to \mathbb{R}_{\geq 0},
\]

which gives an increasing exhaustive filtration by finite subsets

\[
\mathcal{O}_{\leq N} := \{ \alpha \in \mathcal{O} \mid \| \alpha \| \leq N \} \quad (N \geq 0).
\]

For an inclusion \( A \subset X \neq \emptyset \) of subsets of \( \mathcal{O} \), one can consider the upper relative density:

\[
\delta_X(A) := \limsup_{N \to +\infty} \frac{|A \cap \mathcal{O}_{\leq N}|}{|X \cap \mathcal{O}_{\leq N}|}.
\]

The density statement is formulated as follows.

**Theorem 1.2** (Green-Tao theorem in positive characteristic; see Theorem 5.2). Let \( \mathcal{O} \) be a Dedekind domain finitely generated over \( \mathbb{F}_p \) and \( \mathcal{P}_\mathcal{O} \) be the set of its prime elements. Then every subset \( A \subset \mathcal{P}_\mathcal{O} \) with \( \delta_{\mathcal{P}_\mathcal{O}}(A) > 0 \) contains a \( k \)-dimensional affine subset for an arbitrary \( k \geq 0 \).

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This implies Theorem 1.1 because the prime elements of $O_0$ have positive upper density in $P_O$ as we recall in §5.2.

In fact, in our proof of Theorem 1.2 we search for $k$-dimensional affine subsets of a very specific form.

**Definition 1.3.** For a subring $\mathfrak{a} \subset O$ and a finite subset $S \subset O$, by an $\mathfrak{a}$-homothetic copy of $S$ let us mean a subset of $O$ of the form

$$a \cdot S + \beta = \{a\alpha + \beta \mid \alpha \in S\}$$

with $a \in \mathfrak{a}$ and $\beta \in O$. Let us say it is non-trivial if we can take $a \neq 0$.

For later use, note that these notions make perfect sense for any integral domain $\mathfrak{a}$, a torsion-free $\mathfrak{a}$-module $a$ and any subset $S \subset a$.

For a suitable $\mathfrak{a} \subset O$ and an arbitrary $S$, we shall prove in Theorem 5.2 that any subset of $P_O$ with positive upper relative density contains a non-trivial $\mathfrak{a}$-homothetic copy of $S$. Theorem 1.2 then follows because if we take $S$ to be a $k$-dimensional linear subspace of $O$ then every non-trivial $\mathfrak{a}$-homothetic copy of it is a $k$-dimensional affine subset. This is why we propose to call Theorem 1.2 the Green-Tao theorem in positive characteristic, as the Green-Tao theorem for number fields is commonly formulated as follows.

**Theorem 1.4 (Green-Tao theorem for number fields: [3], [9], [6]).** Let $K$ be a number field and $O_K$ the ring of its integers. Denote by $P_K$ the set of prime elements of $O_K$. Then any set $A \subset P_K$ with positive upper relative density contains a non-trivial $\mathbb{Z}$-homothetic copy of $S$ for an arbitrary finite subset $S \subset O_K$.

In [6] they also prove a variant of this statement for the “prime elements” (in an appropriate sense) in a given non-zero ideal $a \subset O_K$. While we will also state and prove Theorem 5.2 in this generality, the reader is advised to assume $a = O$ in the first reading. We will stick to the case $a = O$ in the rest of Introduction.

1.1. Overview of the proof. Theorem 1.2 for polynomial rings $O = \mathbb{F}_q[t]$ is due to Lê [7]. For Theorem 1.4 the case of $\mathbb{Z}$ is the renowned theorem of Green and Tao [3] and the case of $\mathbb{Z}[\sqrt{-1}]$ is due to Tao [9]. For the ring $O_K$ of integers in a general number field $K$, it is a result of Mimura, Munemasa, Seki, Yoshino and the present author [6]. See Table 1. All of them follow the strategy of Green-Tao [3].

Details of our arguments are closest to those of [6]. The new issue we have to face is that while the number ring $O_K$ has a canonical base ring $\mathbb{Z}$ which is simple enough and such that $O_K \cong \mathbb{Z}^{\mathfrak{n}}$ compatibly with the metrics on both sides, there is no canonical one for $O$ in positive characteristic.

In §2 we use the Riemann-Roch formula to find an appropriate subring $\mathfrak{a}$ of $O$, which is isomorphic to the polynomial ring and over which $O$ is finite. The subtlety of our choice is that, moreover, the $\mathfrak{a}$-linear isomorphism $O \cong \mathfrak{a}^\mathfrak{n}$ (given once we choose a basis) is compatible with the metrics on both sides; see Proposition 2.2. Once we have done this, everything in [6] goes through. So we refer the reader to [6 §§1–2] for a detailed overview.

Let us just recall the three main ingredients:

- the relative Szemerédi theorem (recalled in §3);
- the construction of a pseudorandom measure $\lambda: O \to \mathbb{R}_{\geq 0}$ (§4 completed in §5);
- the prime elements $P_O$ have positive density with respect to the measure $\lambda$ (§5).
Here, the relative Szemerédi theorem (Theorem 3.2) roughly asserts the following: suppose we are given a function $\lambda: O \to \mathbb{R}_{\geq 0}$ which is a pseudorandom measure—this condition says $\lambda$ is close enough to the constant function 1 in a certain measure (see Definition 3.1). Suppose also that a subset $A \subset O$ has positive (upper) density with respect to $\lambda$ in that

$$\limsup_{N \to +\infty} \frac{\sum_{\alpha \in A \cap O_{\leq N}} \lambda(\alpha)}{|O_{\leq N}|} > 0.$$

Then $A$ contains a non-trivial $\epsilon$-homothetic copy of $S$ for every finite subset $S \subset O$.

The construction of $\lambda: O \to \mathbb{R}_{\geq 0}$ is ideal-theoretic in nature. The proof that it is pseudorandom ultimately relies on the knowledge that the zeta function $\zeta_{\mathcal{O}}$ has a simple pole at 1 with a positive residue.

That the prime elements have positive density with respect to $\lambda$ will be deduced from the Chebotarëv density theorem, an analog of Prime Number Theorem in our setting.

A pitfall in the construction of $\lambda$ is that there is a somewhat natural function $\lambda': O \to \mathbb{R}_{\geq 0}$ but we are not going to use it directly because we do not know if it is pseudorandom in our sense. Instead, we choose an element $W \in \mathfrak{a}$ with sufficiently many different prime factors and $b \in O$ coprime to $W$, and define the function $\lambda$ as the composite $O \xrightarrow{W(-)+b} O \xrightarrow{\lambda'} \mathbb{R}_{\geq 0}$ times a normalizing factor. This enables us to prove the pseudorandomness. Thus, the set $A \subset O$ in the relative Szemerédi theorem is going to be taken as the inverse image of $\mathcal{P}_O$ by the affine linear map $W(-)+b$; this is of course equivalent to considering the prime elements which are congruent to $b$ modulo $W$. The reader will see how this trick (so-called $W$-trick) works as the proof unrolls in §§4–5.

While the translation of the arguments in the number field case [6] into our situation is straightforward in many places, one cannot formally apply the results in loc. cit. because of the slight difference of languages between number fields and algebraic curves. So we include full proofs for the convenience of the reader. Some details get simpler—as the reader might naturally expect—partly thanks to the fact that the canonical norm $\|\cdot\|: O \to \mathbb{R}_{\geq 0}$ is ultrametric, meaning that $\|\alpha + \beta\| \leq \max\{\|\alpha\|,\|\beta\|\}$ for all $\alpha, \beta \in O$, so that the subsets $O_{\leq N}$ are in fact subgroups.

Notation. For a function on a non-empty finite set $f: X \to \mathbb{C}$, we use the standard expectation notation:

$$\mathbb{E}(f | X) = \mathbb{E}(f(x) \mid x \in X) := \frac{1}{|X|} \sum_{x \in X} f(x).$$

From §4 onwards, we will make extensive use of the big-$O$ notation with dependence parameters as in [6]. Notation in §2: let $f$ and $g$ be $\mathcal{C}$-valued functions on a set $X$ which depend on additional parameters $a, b, c, \ldots$. Assume that the values of $g$ are positive real numbers. We write

$$f(x) = O_{b,c,\ldots}(g(x)) \quad (x \in X)$$

to mean that there is a positive constant $C = C_{b,c,\ldots} > 0$ depending only on the parameters in the subscript such that the inequality $|f(x)| < Cg(x)$ holds for all $x \in X$. (Thus in this case the implied constant $C$ can be taken independent of $a$.) An expression like $O_{b,c,\ldots}(1)$ would mean a positive constant depending only on the parameters $b, c, \ldots$ in the subscript. Note in particular that $O(1)$ without subscript would mean an absolute constant depending on nothing at all. When $g(x)$ is a heavy formula, the notation $f(x) = O_{b,c,\ldots}(1) \cdot g(x)$ is sometimes preferred. A quantity written in the form $1 + O_{b,c,\ldots}(g(x))$ is one whose difference with 1 is $O_{b,c,\ldots}(g(x))$. When we say something like “$f(x,y) = O_{b,c,\ldots}(g(x,y))$ for all sufficiently large $x, y \in \mathbb{R}$” we are taking the domain $X$ to be a subset of $\mathbb{R}^2$ consisting of the pairs of real numbers larger than certain thresholds. In practice, the thresholds often depend on the parameters $a, b, c, \ldots$. We usually indicate how the thresholds depend on the parameters, especially when that piece of information is relevant.
2. Choice of a polynomial subring

The purpose of this section is to define the canonical norm \( \| - \| : \mathcal{O} \to \mathbb{R}_{\geq 0} \), set up necessary algebraic terminology and find an appropriate subring \( o \cong \mathbb{F}_q[t] \) of \( \mathcal{O} \).

We use [8] as the main reference about algebraic background. We assume the reader is familiar with advanced undergraduate commutative algebra as in [1] and basic notions of algebraic curves (equivalently function fields in one variable) e.g. as in [8, Chapter 5] [5, Chapter I, Section 6] but they do not have to know more than the Riemann-Roch theorem [8, Theorem 5.4, p.49] [5, Theorem 1.3 in Chapter IV, p.295].

2.1. The canonical linear norm. Let \( \mathcal{O} \) be a Dedekind domain finitely generated over \( \mathbb{F}_p \). The purpose of this subsection is to describe the canonical submultiplicative linear norm on the ring \( \mathcal{O} \).

By a linear norm or an ultrametric norm on an abelian group \( G \) let us mean a non-negatively valued function \( \| - \| : G \to \mathbb{R}_{\geq 0} \) which is ultrametric:

\[
\| a + b \| \leq \max \{ \| a \|, \| b \| \} \quad \text{for all } a, b \in G
\]

and non-degenerate in that the only element with norm 0 is the zero element. A submultiplicative linear norm on an (always commutative) ring \( A \) is a linear norm on the abelian group \( A \) which moreover satisfies

\[
\| \alpha \beta \| \leq \| \alpha \| \cdot \| \beta \|.
\]

In our examples the multiplicative unit will always have norm 1. A norm \( \| - \| \) is said to be multiplicative if the above inequality is always an equality.

Let \( \mathbb{F}_q \) be the integral closure of \( \mathbb{F}_p \) in \( \mathcal{O} \). Let \( \overline{X} \) be the complete non-singular curve over \( \mathbb{F}_q \) which contains \( X := \text{Spec} \mathcal{O} \) as an open subscheme. Let \( n \) be the cardinality of the complement:

\[
n := |\overline{X} \setminus X|.
\]

Regard each \( v \in \overline{X} \) as a discrete valuation \( \text{Frac}(\mathcal{O})^* \to \mathbb{Z} \). Let \( \mathbb{F}(v) \) be its residue field and \( \text{deg}(v) := |\mathbb{F}(v) : \mathbb{F}_q| \) its degree. Define a linear norm \( \| - \|_v \) on \( \mathcal{O} \) by:

\[
\| - \|_v : \mathcal{O} \to \mathbb{R}: \alpha \mapsto \left( \frac{1}{|\mathbb{F}(v)|} \right)^{v(\alpha)} = q^{-v(\alpha)\text{deg}(v)}.
\]

The value \( \| 0 \|_v \) is understood to be 0. The canonical norm \( \| - \| \) on \( \mathcal{O} \) is defined by

\[
(1) \quad \| \alpha \| := \max_{v \in \overline{X} \setminus X} \{\| \alpha \|_v \}.
\]

This is a submultiplicative linear norm because the following formulas hold for all \( v \in \overline{X} \) and \( \alpha, \beta \in \text{Frac}(\mathcal{O})^* \):

\[
v(\alpha + \beta) \geq \min \{v(\alpha), v(\beta)\}, \quad v(\alpha \beta) = v(\alpha) + v(\beta).
\]

Define the norm of a non-zero ideal \( \alpha \subset \mathcal{O} \) as the cardinality of the quotient \( \mathcal{N}(\alpha) := |\mathcal{O}/\alpha| \), and for \( \alpha \in \mathcal{O} \setminus \{0\} \) write \( \mathcal{N}(\alpha) := \mathcal{N}(\alpha \mathcal{O}) \) for short. By convention we define \( \mathcal{N}(0) := 0 \). We call it the ideal norm of \( \alpha \) to avoid confusion with the linear norm \( \| - \| \). For \( \alpha \neq 0 \) we know \( \mathcal{N}(\alpha) = \prod_{v \in \overline{X}} |\mathbb{F}(v)|^{v(\alpha)} \) say by prime decomposition of ideals in \( \mathcal{O} \). It follows by the product formula for complete algebraic curves (e.g. [8, Proposition 5.1, p.47]) that the following equality holds for all \( \alpha \in \mathcal{O} \):

\[
(2) \quad \mathcal{N}(\alpha) = \prod_{v \in \overline{X} \setminus X} \| \alpha \|_v.
\]

In particular \( \alpha \) is in \( \mathcal{O}^* \) if and only if \( \prod_{v \in \overline{X} \setminus X} \| \alpha \|_v = 1 \).

For positive real numbers \( N > 0 \), set:

\[
\mathcal{O}_{\leq N} := \{ \alpha \in \mathcal{O} \mid \| \alpha \| \leq N \}
\]

\[
= \left\{ \alpha \in \mathcal{O} \mid v(\alpha) + \frac{\log_q N}{\text{deg}(v)} \geq 0 \text{ for all } v \in \overline{X} \setminus X \right\}.
\]
By the Riemann-Roch theorem for curves \cite[Corollary 4 of Theorem 5.4, p.49]{[8]}, we know that \(|O_{\leq N}|\) is approximately proportional to \(N^n\). To be more precise, consider the following invariants:

\[ g := \text{the genus of } X, \]
\[ d_0 := \text{the least common multiple of } \deg(v)'s \text{ for } v \in \overline{X} \setminus X. \]

Let us denote by \([-\ ]\) the floor function \(x \mapsto (\text{the largest integer not exceeding } x)\) and consider the next divisor on \(X\) for \(N \geq 1\):

\[ D_N := \sum_{v \in \overline{X} \setminus X} \left\lfloor \frac{\log_q N}{\deg(v)} \right\rfloor v. \]

Its degree is \(\sum_{v \in \overline{X} \setminus X} \left\lfloor \frac{\log_q N}{\deg(v)} \right\rfloor \deg(v)\) which is \(\leq \log_q N \cdot |\overline{X} \setminus X| = n \log_q N\). We have by definition \(O_{\leq N} = \Gamma(X, O_X(D_N))\), or \(O_{\leq N} = L(D_N)\) in the notation of \cite{[8]}. Therefore by Riemann-Roch \cite[Corollary 4 of Theorem 5.4, p.49]{[8]} we get \(|O_{\leq N}| = N^n/q^{g-1}\) for every \(N \geq q^{(2g-1)/n}\) which is a power of \(q^{d_0}\). From this we also get the following bound valid for all real numbers \(N \geq q^{(2g-1)/n}\):

\[ \left(\frac{N}{q^{d_0}}\right)^n / q^{g-1} < |O_{\leq N}| \leq N^n/q^{g-1}. \]

When we consider a non-zero ideal \(a \subset O\) (which is relevant only if the reader is interested in the case \(a \neq O\) of Theorem \[5.2\]), we endow \(a\) the induced linear norm and write

\[ a_{\leq N} := \{ \alpha \in a \mid \|\alpha\| \leq N \} = \left\{ \alpha \in a \mid v(\alpha) + \frac{\log_q N}{\deg(v)} \geq 0 \text{ for all } v \in \overline{X} \setminus X \right\}. \]

By the Riemann-Roch formula again (or from (3)) we get

\[ \frac{N^n}{N(a)q^{nd_0+g-1}} < |a_{\leq N}| \leq \frac{N^n}{N(a)q^{g-1}}, \text{ if } N^n \geq N(a)q^{2g-1}. \]

\textbf{Remark 1.} The use of the canonical norm among other norms is not essential. We could have chosen an arbitrary positive integer \(d_v \geq 1\) for each \(v \in \overline{X} \setminus X\) and defined \(\|\alpha\|_v := q^{-d_v \cdot v(\alpha)}\). The content of this paper would remain valid with minor modifications. However, it did not seem appealing to the author to allow the freedom of this choice at the cost of heavier notation.

\textbf{Remark 2.} The \(A = P_O\) case of Theorem \[1.2\] can be reduced to the case where \(n = |\overline{X} \setminus X| = 1\) (with a general \(A \subset P_O\)), for which the treatment in the rest of \[2\] can be much simpler because then we have \(\|\cdot\|_\omega = \|\cdot\|_v = N(-)\) (where \(v \in \overline{X} \setminus X\) is the unique element). Since we eventually prove Theorem \[1.2\] in full strength, we only give a sketch of this reduction argument. Take any point \(v \in \overline{X} \setminus X\) and set \(X' := \overline{X} \setminus \{v\}\) and \(O' := \Gamma(X', O_{\overline{X}})\). We have a canonical injection \(i : O' \hookrightarrow \overline{O}\) and know that all but finitely many associate classes (corresponding to a subset of \(X' \setminus X\)) of prime elements of \(O'\) remain prime elements in \(O\). Since \(O'^* = F_q^*\), those exceptional prime elements are finite in number, so in particular have density zero in \(P_O\). Let \(P_{O'}\) be the set of remaining prime elements. Now we apply Theorem \[1.2\] to \(O'\) and \(P_{O'} \subset P_{O'}\) to find a \(k\)-dimensional affine subset contained in \(P_{O'}\). Since the canonical injection \(i\) (which is of course \(F_q\)-linear) carries \(P_{O'}\) into \(P_O\), we have found a \(k\)-dimensional affine subset in \(P_O\). This proves Theorem \[1.2\] for \(O\) in the special case \(A = P_O\).

\textbf{2.2. The choice of a subring.} The constellation theorem \cite{[6]} for the ring of integers \(O_K\) of a number field \(K\) ensures that the set of prime elements of \(O_K\) contains a \(Z\)-homothetic copy of any given finite subset \(S \subset O_K\). In this subsection we choose a subring \(o \cong F_q[t]\) of \(O\) which plays the role of \(Z\) in \(O_K\).

Recall the definition \(X = \text{Spec}O\) and that \(\overline{X}\) is the complete non-singular curve over \(F_q\) containing \(X\) as an open subscheme. Also \(n = |\overline{X} \setminus X|\).
Proposition 2.1. There exists an element $t \in \mathcal{O}$ such that the following two conditions are satisfied:

1. $\mathcal{O}$ is finite over $\mathbb{F}_q[t]$, say of rank $r$;
2. the value $\|t\|_v$ is independent of $v \in \mathbb{X} \setminus X$, say $q^d$.

Furthermore we have $r = nd$ and the canonical linear norms of $\mathcal{O}$ and $\mathfrak{o} := \mathbb{F}_q[t]$ satisfy the following compatibility: if we write $\|\cdot\|_\mathfrak{o}$ for the canonical norm of $\mathcal{O}$ and $\|\cdot\|_\mathfrak{o}$ for that of $\mathfrak{o}$, then we have $\|\cdot\|_\mathfrak{o}^d = \|\cdot\|_\mathfrak{o}$ as functions on $\mathfrak{o}$ for all $v \in \mathbb{X} \setminus X$ and in particular $\|\cdot\|_\mathfrak{o}^d = \|\cdot\|_\mathfrak{o}$.

Proof. Let $d$ be a large enough common multiple of $\deg(v)$ ($v \in \mathbb{X} \setminus X$). We claim that there exists a function $\phi \in \mathcal{O}$ which has a pole at each $v \in \mathbb{X} \setminus X$ of order exactly $d/\deg(v)$. For this, for each $v$ consider the set $\Gamma(\mathbb{X}, \mathcal{O}((d/\deg(v)))$ of rational functions on $\mathbb{X}$ whose only possible pole is $v$ with order $\leq d/\deg(v)$. By the Riemann-Roch formula [8 Corollary 4 to Theorem 5.4, p.49], if $d$ is large enough the inclusion

$$\Gamma\left(\mathbb{X}, \mathcal{O}\left(\frac{d}{\deg v} - 1\right) v\right) \subseteq \Gamma\left(\mathbb{X}, \mathcal{O}\left(\frac{d}{\deg v} v\right)\right)$$

is a proper one so there is a function $\phi_v$ whose only pole is at $v$ and of order exactly $d/\deg(v)$. Choose one such $\phi_v$ for each $v$ with a common $d$. Then the function $\phi := \sum_{v \in \mathbb{X} \setminus X} \phi_v$ has the claimed property.

Denote also by $\phi$ the corresponding finite map of curves $\phi : \mathbb{X} \to \mathbb{P}^1$. By the choice of $\phi$ we have an equality of divisors on $\mathbb{X}$:

$$(5) \quad \phi^*(\infty) = \sum_{v \in \mathbb{X} \setminus X} \frac{d}{\deg v} v := D.$$

One can also see that the degree $r$ of the map $\phi$ equals $\deg(D) = nd$. Let $t$ be the coordinate of $\mathbb{A}^1 \subset \mathbb{P}^1$. We have $\phi^{-1}(\mathbb{A}^1) = \mathbb{X} \setminus \{D\} = X$ so $t$ can be seen as an element of $\mathcal{O}$ and assertion (1) holds.

Next, let $f(t) \in \mathbb{F}_q[t]$ be a polynomial of degree $e$. Since the valuation $v_{\infty} : \mathbb{F}_q(t)^* \to \mathbb{Z}$ at $\infty \in \mathbb{P}^1$ agrees with the degree function when restricted to $\mathbb{F}_q[t]$, we have $\|f\|_\mathfrak{o} := \|f\|_{v_{\infty}} = q^e$. By (5) we know $v(f) = -\frac{de}{\deg(v)}$ for each $v \in \mathbb{X} \setminus X$. It follows that

$$(6) \quad \|f\|_v = \left(q^{\deg(v)}\right)^{\frac{de}{\deg(v)}} = q^{d} = \|f\|_\mathfrak{o}^d$$

for all $v$. Therefore the assertion (2) and the compatibility assertion holds. \(\square\)

We fix an $\mathfrak{o} \subset \mathcal{O}$ as in Proposition 2.1 throughout the paper. To avoid overloaded notation, we will avoid the use of the canonical norm of $\mathfrak{o}$ as much as possible and reserve the symbol $\|\cdot\|$ for the canonical norm of $\mathcal{O}$. As a consequence we use the following potentially confusing piece of notation:

$$(7) \quad \mathfrak{o}_{\leq N} := \{f \in \mathfrak{o} \|f\| \leq N\} = \{f \in \mathfrak{o} \|f\|_\mathfrak{o} \leq N^{1/d}\}.$$

Note that therefore the cardinality $|\mathfrak{o}_{\leq N}|$ is equal, up to a bounded constant, to $N^{1/d}$. Despite this potential confusion, this notation is convenient in the bulk of our discussion.

2.3. Equivalence of linear norms. Let $\mathfrak{o} \subset \mathcal{O}$ be as in Proposition 2.1. We know $\mathcal{O}$ is a free $\mathfrak{o}$-module of rank $r$. Let $\alpha_1, \ldots, \alpha_r \in \mathcal{O}$ be a basis. One can consider the max norm on $\mathcal{O}$ with respect to this basis:

$$\left\|\sum_{i=1}^r f_i \alpha_i\right\|_\alpha := \max_i \{\|f_i\|_\mathfrak{o}\}.$$

It is an ultrametric norm on the abelian group $\mathcal{O}$.

Let us recall that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on an abelian group $G$ are said to be equivalent if there are positive real numbers $c, C > 0$ such that the next inequality holds on $G$:

$$c \|\cdot\|_2 \leq \|\cdot\|_1 \leq C \|\cdot\|_2.$$
It is easy to see that the equivalence class of the norm \( \| - \|_\alpha \) is independent of the choice of the basis \( \alpha = (\alpha_1, \ldots, \alpha_r) \).

**Proposition 2.2.** For any given \( \mathfrak{o}\)-basis \( \alpha \) of \( \mathcal{O} \), the associated norm \( \| - \|_\alpha \) is equivalent to the canonical norm \( \| - \|_\mathcal{O} \).

**Proof.** First, if \( \| f_i \|_\mathcal{O} \leq N \) holds for all \( i \), then by the ultrametricity and submultiplicativity of \( \| - \|_\mathcal{O} \) we have

\[
\sum_i f_i \alpha_i \leq N \cdot \left( \max_i \| \alpha_i \|_\mathcal{O} \right)
\]

so that we have \( \| - \|_\mathcal{O} \leq \| - \|_\alpha \cdot (\max_i \| \alpha_i \|_\mathcal{O}) \).

The inequality in the other direction is slightly harder. For a notational reason, let us introduce the degree function \( \deg : \mathcal{O} \to \mathbb{Z} \cup \{-\infty\} \) defined by

\[
\deg(\alpha) := \log_q(\| \alpha \|_\mathcal{O}).
\]

For integers \( M \geq 0 \), denote by \( \mathcal{O}_{\deg \leq M} \) the \( \mathbb{F}_q^* \)-vector subspace of elements with degree \( \leq M \); of course one has \( \mathcal{O}_{\deg \leq M} = \mathcal{O}_{\leq (q^M)} \). Note that by Proposition 2.1, the element \( t \in \mathcal{O} \) is multiplicative in the sense that the equality

\[
\| t \alpha \|_\mathcal{O} = \| t \|_\mathcal{O} \cdot \| \alpha \|_\mathcal{O}, \quad \text{i.e.,} \quad \deg(t \alpha) = \deg(t) + \deg(\alpha)
\]

holds for all \( \alpha \in \mathcal{O} \) rather than a mere inequality. (Actually, all elements of \( \mathfrak{o} \) are multiplicative by (6).) It follows that the following multiplication by \( t \) map is injective for all \( M \geq 0 \), where we recall from Proposition 2.1 that \( d = \deg(t) \):

\[
\frac{\mathcal{O}_{\deg \leq M}}{\mathcal{O}_{\deg \leq M - d}} \times_{\mathcal{O}_{\deg \leq M}} \frac{\mathcal{O}_{\deg \leq M + d}}{\mathcal{O}_{\deg \leq M}}.
\]

We claim that it is also surjective for all sufficiently large \( M \geq 0 \). There are at least two ways to see this. One is to use the Riemann-Roch theorem which tells us that both sides of (8) have the same dimension for \( M \) large enough.

The second is more down-to-earth. Let \( M_0 := \max_{1 \leq i \leq r} \{ \deg(\alpha_i) \} \) and suppose \( M \geq M_0 \).

Write an arbitrary element \( \alpha \in \mathcal{O}_{\deg \leq M_0} \) in the form

\[
\alpha = \sum_i f_i(t) \alpha_i \mod \mathcal{O}_{\deg \leq M}, \quad f_i(t) \in \mathfrak{o}.
\]

Since \( \alpha_i \in \mathcal{O}_{\deg \leq M_0} \subset \mathcal{O}_{\deg \leq M} \) is zero in the group in question, we may assume \( f_i(t) \) has no constant term. Then we have a well-defined element \( \alpha/t := \sum_i f_i(t) \alpha_i \) which is in \( \mathcal{O}_{\deg \leq M} \) because \( t \) is a multiplicative element. Then \( \alpha \) is the image of \( \alpha/t \) under the map (8).

In any case let \( M_0 \) be such that (8) is surjective for all \( M \geq M_0 \). Now since \( \mathcal{O}_{\deg \leq M_0} \) is a finite set, there trivially exists an \( e_0 \geq 0 \) such that all \( \alpha \in \mathcal{O}_{\deg \leq M_0} \) can be written in the form

\[
\alpha = \sum_i f_i(t) \alpha_i \quad \text{with} \quad \deg(f_i(t)) \leq \deg(\alpha) + e_0.
\]

By induction on \( \deg(\alpha) \) using the bijection (8), the same holds for all \( \alpha \in \mathcal{O} \). In multiplicative terms, this precisely says there is a positive constant \( C = q^{-e_0} \) such that

\[
\| \alpha \|_\alpha \leq C \cdot \| \alpha \|_\mathcal{O} \quad \text{for all} \quad \alpha \in \mathcal{O}.
\]

This completes the proof of Proposition 2.2. \( \square \)

For the “prime elements in an ideal” case of Theorem 5.2, let \( \mathfrak{a} \subset \mathcal{O} \) be a non-zero ideal. It is also a rank \( r \) free \( \mathfrak{o} \)-module. We can consider the restriction of the canonical norm \( \| - \|_\mathcal{O} \) to \( \mathfrak{a} \) and the max norm \( \| - \|_\mathfrak{a} \) with respect to an \( \mathfrak{o} \)-basis \( \beta = (\beta_1, \ldots, \beta_r) \) of \( \mathfrak{a} \).

**Corollary 2.3.** The two linear norms \( \| - \|_\mathcal{O} \) and \( \| - \|_\mathfrak{a} \) on \( \mathfrak{a} \) are equivalent.
Proof. While the proof of Proposition 2.2 works for this case just as well, here we present a proof using the proposition. Let $\alpha$ continue to be an $o$-basis of $O$. By Proposition 2.2 it suffices to show that the restriction of $\|\cdot\|_\alpha$ to $a$ and $\|\cdot\|_\beta$ are equivalent.

Each $\beta_i$ can be written (uniquely) as $\beta_i = \sum_{1 \leq j \leq r} g_{ij} \alpha_j$. Take a positive number $C$ such that $C \geq \|g_{ij}\|_\alpha$ for all $i, j$. For an element $x = \sum_i f_i \beta_i = \sum_j (\sum_i f_i g_{ij}) \alpha_j$, by the ultrametricity of $\|\cdot\|_\alpha$ and the choice of $C$ we have:

$$\|x\|_\alpha = \max_j \left\{ \left\| \sum_i f_i g_{ij} \right\|_\alpha \right\} \leq \max_{i,j} \left\{ \|f_i g_{ij}\|_\alpha \right\} \leq C \max_i \{ \|f_i\|_\alpha \} = C \|x\|_\beta.$$

Next, by the theory of finitely generated modules over a principal ideal domain (say), we know that there is a non-zero element $f(t) \in o = \mathbb{F}_q[t]$ such that $f(t)O \subset a$. The previous argument applied to the element $f(t)x \in f(t)O$ gives $\|f(t)x\|_\beta \leq C \|f(t)x\|_{f(t)\alpha}$. By the definition of the max norm we can isolate the $f(t)$-factors so that:

$$\|x\|_\beta \leq \frac{C}{\|f(t)\|_O} \cdot \|x\|_\alpha.$$

This completes the proof. \qed

Remark 3. Proposition 2.2 fails if $o$ is not chosen as in Proposition 2.1 even if $O$ is finite over $o$. For example, set $o := \mathbb{F}_q[t]$ and $O := \mathbb{F}_q[s, s^{-1}]$ and consider the homomorphism $\varphi : o \to O$:

$$t \mapsto \frac{(s-1)^3}{s}.$$

If we let $f$ be the induced finite map $f : \mathbb{P}^1 \to \mathbb{P}^1$, we have $f^*(\{ \infty \}) = 2 \{ \infty \} + \{ 0 \}$ as divisors. The element $t$ is not a multiplicative element for the canonical norm $\|\cdot\|$ because for example $\|t/s\| = \|(s-1)^3/s^2\| = q^2$ which is not equal to $\|t\| \cdot \|1/s\| = q^2 \cdot q^1$. Instead $t$ is a multiplicative element for the following linear norm $\|\cdot\|' : O \to \mathbb{R}_{\geq 0}$:

$$\|a\|' := \max \left\{ q^{-v_o(a)}, q^{-2v_o(a)} \right\}.$$

The arguments of the proof of Proposition 2.2 show that the max norm on $O \cong o^3$ is equivalent to $\|\cdot\|'$. However, one easily sees that $\|\cdot\|'$ is not equivalent to $\|\cdot\|$; for example, one has $\|1/s^a\| = q^a$ and $\|1/s^b\|' = q^{2a}$ so their ratio is not bounded.

Recall from (7) that we endow $o$ with the induced norm from $O$ and so we have $N^{1/d} < |o_{\leq N}| \leq q \cdot N^{1/d}$ where the right hand inequality becomes an equality when $N$ is a power of $q^d$. Let us note the following simple observation.

Lemma 2.4. Let $t, r \geq 1$ be positive integers and $\phi : o^r \to o^r$ a surjective $o$-linear map. Then there exists a positive number $U > 0$ such that for all $N > 0$ and $x \in (o_{\leq N})^r$, the following set $\phi^{-1}(x) \cap (o_{\leq U N})^t$ contains at least $|o_{\leq N}|^{1-r}$ elements.

Proof. Let us denote by $\|\cdot\|_{o^r}$ and $\|\cdot\|_{o^r}$ the max norms on $o^r$ and $o^r$ with respect to the standard bases. Choose an $o$-linear section $\sigma : o^r \to o^r$ to $\phi$. Let $e_i \in o^r$ be the standard basis (1 $\leq i \leq r$) and set $f_i := \sigma(e_i)$. We know $\ker(\phi)$ is a free $o$-module of rank $t-r$; choose a basis $f_{r+1}, \ldots, f_t$ of $\ker(\phi)$. Let $U > 0$ be larger than $\|f_i\|_{o^r}$ for all $1 \leq i \leq t$. We claim this $U$ works.

Suppose we are given $N > 0$ and $x \in (o_{\leq N})^r$. Writing $x$ in the form $x = \sum_{i=1}^r a_i e_i$, we see

$$\|\sigma(x)\|_{o^r} = \left\| \sum_{i=1}^r a_i f_i \right\|_{o^r} \leq \left( \max_i \|a_i\| \right) \cdot U \leq NU.$$

Also, for each choice of $a_{r+1}, \ldots, a_t \in o_{\leq N}$ we have by the same reasoning: $\|\sum_{i=r+1}^t a_i f_i\|_{o^r} \leq UN$. Therefore we get $|\sigma(x) + \sum_{i=r+1}^t a_i f_i|_{o^r} \leq UN$ for all choices of $a_i$'s as above. Since this last element is in $\phi^{-1}(x)$, our claim follows. \qed
Corollary 2.5. Let $\phi: \sigma^t \to a$ be a surjective $o$-linear map. Then there exists a positive number $U > 0$ such that for all $N > 0$ and $x \in O_{\leq N}$, the following set

$$\phi^{-1}(x) \cap (a_{\leq UN})^t$$

contains at least $|a_{\leq N}|^{t-r}$ elements.

Proof. This immediately follows from Proposition 2.2 Corollary 2.3 and Lemma 2.4. 

2.4. “Geometry of numbers”. Consider the group homomorphism $\phi: O^* \to \bigoplus_{v \in X \setminus X} \mathbb{Z}$ defined by $\alpha \mapsto (v(\alpha))_v$. It is known that its image has rank $n - 1$:

$$\text{rank}(\text{image}(\phi)) = n - 1.$$  

For a proof, see [8, Proposition 14.2, p.243]. For the convenience of the reader let us recall a proof in the language of sheaves. Let $j: X = \text{Spec} O \to X$ be the open immersion. By the short exact sequence of sheaves on $X$ (where $i_v: \{v\} \to X$ is the closed immersion):

$$1 \to O^*_X \to j_*O^*_X \to \bigoplus_{v \in X \setminus X} i_v Z \to 0,$$

we get an exact sequence $\mathbb{F}_q^* \to O^* \to \bigoplus_{v \in X \setminus X} \delta \to \text{Pic}(X)$. Since $\text{Pic}(X) = \mathbb{Z} \oplus \text{finite}$ and the connecting map $\delta$ has nontrivial image into the $\mathbb{Z}$-part, the claim (9) follows.

Now consider the map $\mathcal{L}: O \setminus \{0\} \to \bigoplus_{v \in X \setminus X} \mathbb{R} \cong \mathbb{R}^n$ defined by

$$\mathcal{L}(\alpha) := (\log \|\alpha\|_v)_v.$$  

It is an obvious analog of the multiplicative Minkowski map $\mathcal{L}$ that was also used in [6, §4]. By (2) we know that $\mathcal{L}$ maps the subset $O^*$ into the hyperplane $H$ of $\mathbb{R}^n$ defined by:

$$H = \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\} \subset \mathbb{R}^n$$

and by (9) that $\mathcal{L}(O^*) \subset H$ is a full-rank lattice.

Let us say a subset $D \subset O \setminus \{0\}$ is norm-length compatible if the set

$$\left\{ \frac{\|\alpha\|^n}{\mathcal{N}(\alpha)} \mid \alpha \in D \right\} \subset \mathbb{R}$$

is bounded from above. Note from (1) and (2) that this set is always bounded from below by 1. As usual, a subset $D \subset O \setminus \{0\}$ is called an $O^*$-fundamental domain of $O \setminus \{0\}$ (or of $O$ by slight abuse of terminology) if the composite map $D \to O \setminus \{0\} \to (O \setminus \{0\})/O^*$ is a bijection. As in [6, §4.3], the following statement holds.

Proposition 2.6. There exist norm-length compatible $O^*$-fundamental domains of $O$.

Proof. Consider the function $f: \mathbb{R}^n \to \mathbb{R}; (x_1, \ldots, x_n) \mapsto n \max_i x_i - \sum_i x_i$. It fits into the following commutative diagram:

$$\begin{array}{ccc}
O \setminus \{0\} & \xrightarrow{\mathcal{L}} & \mathbb{R}^n \\
\|\cdot\|^n \downarrow & & \downarrow f \\
\mathbb{R}^n & \xrightarrow{\log} & \mathbb{R}.
\end{array}$$

Let $H \subset \mathbb{R}^n$ be as in (10) and $\pi: \mathbb{R}^n \to H$ the projection along the vector $(1, \ldots, 1)$. Since the value of $f$ is unchanged by the translation along the vector $(1, \ldots, 1) \in \mathbb{R}^n$, it follows that the norm-length compatibility of a subset $D \subset O \setminus \{0\}$ is equivalent to the boundedness of the non-negatively valued function $(x_1, \ldots, x_n) \mapsto \max_i x_i$ on the set $\pi(\mathcal{L}(D)) \subset H$. This is equivalent to the boundedness of the set $\pi(\mathcal{L}(D))$ itself; recall that one can define the notion of boundedness of a subset of $\mathbb{R}^n$ using any choice of a linear norm and the resulting notion is independent of the choice. It follows that for any given bounded subset $\Delta \subset H$, the set $\mathcal{L}^{-1}(\pi^{-1}(\Delta))$ is a norm-length compatible subset of $O \setminus \{0\}$. 

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Recall by (2) and (9) that \( L(O^*) \) is a full-rank lattice of \( H \). Let \( \Delta \subset H \) be any bounded complete set of representatives for the quotient \( H/L(O^*) \) (say a fundamental parallelogram). The inverse image \( \pi^{-1}(\Delta) \subset \mathbb{R}^n \) is a complete set of representatives for the quotient \( \mathbb{R}^n/L(O^*) \). By the previous paragraph, the set \( L^{-1}(\pi^{-1}(\Delta)) \subset O \) is norm-length compatible. It is acted on by the group \( \mathbb{F}_q^n = \ker(L: O^* \to \mathbb{R}^n) \) and the natural map of quotient sets \( L^{-1}(\pi^{-1}(\Delta))/\mathbb{F}_q^n \to (O \setminus \{0\})/O^* \) is a bijection. Therefore, any choice of an \( \mathbb{F}_q^n \)-fundamental domain of \( L^{-1}(\pi^{-1}(\Delta)) \) gives a norm-length compatible \( O^* \)-fundamental domain of \( O \).

For \( \alpha \in O \), one can consider its \( O^* \)-orbit \( \alpha O^* \). We will need the following bound. For the big-\( O \) notation, see Notation at the end of Introduction.

**Proposition 2.7.** For all \( N \geq q \) and \( \alpha \in O_{\leq N} \), we have

\[
|\langle \alpha O^* \rangle \cap O_{\leq N}| = O_\Theta \left( (\log N - \frac{1}{n} \log N(\alpha))^{n-1} + 1 \right).
\]

The term “+1” is there only to cover the rare case \( N(\alpha) = N^n \) (the largest possible). For the application in this paper, its corollary \( |\langle \alpha O^* \rangle \cap O_{\leq N}| = O_\Theta ((\log N)^{n-1}) \) suffices and saves space.

**Proof.** We may assume \( \alpha \neq 0 \). For a real number \( C \), let \( \mathbb{R}^n_{\leq C} \) be the set of points \( (x_1, \ldots, x_n) \) with \( x_i \leq C \) for all \( i \). Let \( H \subset \mathbb{R}^n \) as in (10) and define \( \Delta_{\leq C} := \mathbb{R}^n_{\leq C} \cap H \), which is an \((n-1)\)-dimensional simplex. Let us use the symbol \( \Delta_{\leq C} \) to denote translations of \( \Delta_{\leq C} \) inside \( H \). Write \( \Gamma := L(O^*) \). Sending the set in question by \( L \), we see:

\[
|\langle \alpha O^* \rangle \cap O_{\leq N}| = |F_q^n| \cdot |L(\alpha) + \Gamma| \cap \mathbb{R}^n_{\leq \log N}.
\]

So it suffices to bound the size of the set on the right hand side. By the translation “\(-L(\alpha)\)” we have a bijection

\[
(L(\alpha) + \Gamma) \cap \mathbb{R}^n_{\leq \log N} \cong \Gamma \cap (\mathbb{R}^n_{\leq \log N} - L(\alpha)).
\]

Next \( \mathbb{R}^n_{\leq \log N} - L(\alpha) \) is the translation of \( \mathbb{R}^n_{\leq \log N - \frac{1}{n} \log N(\alpha)} \) by the vector \(-L(\alpha) + \frac{1}{n} \log N(\alpha) \cdot (1, \ldots, 1) \in H \). Therefore the intersection \( H \cap (\mathbb{R}^n_{\leq \log N - L(\alpha)}) \) is a \( \Delta'_{\leq \log N - \frac{1}{n} \log N(\alpha)} \). Since \( \Gamma \) is contained in \( H \) anyway, we can write

\[
\Gamma \cap (\mathbb{R}^n_{\leq \log N} - L(\alpha)) = \Gamma \cap \Delta'_{\leq \log N - \frac{1}{n} \log N(\alpha)}.
\]

Now since \( \Gamma \) is a lattice in \( H \), the cardinality of a set of the form \( \Gamma \cap \Delta'_{\leq C} \) is asymptotically proportional to \( C^{n-1} \) as \( C \to +\infty \) with error bounded independently of the specific translation \( \Delta_{\leq C} \sim \Delta'_{\leq C} \); see for example [4] Appendix A. This completes the proof. \( \square \)

### 3. Szemerédi’s theorem

Let us keep the notation \( O \) and \( o \) from the previous section. Namely \( O \) is a Dedekind domain finitely generated over \( \mathbb{F}_q \) in which \( \mathbb{F}_q \) is integrally closed and \( o = \mathbb{F}_q[l] \) is a subring of \( O \) as in Proposition [2.1]. In case the reader is interested in the “prime elements of ideals” case of Theorem [5.2] let \( a \subset O \) be a non-zero ideal. (If not, they can always assume \( a = O \).) Recall that an \( o \)-homothetic copy of a (finite) subset \( S \subset a \) is by definition a subset of \( a \) of the form

\[
a \cdot S + \beta = \{a\alpha + \beta \mid \alpha \in S\}
\]

with \( a \in o \) and \( \beta \in a \). It it said to be non-trivial if we can take \( a \neq 0 \).

The next definition is a rather straightforward translation of [6] Definition 5.3.

**Definition 3.1.** (I) A finite subset \( S \subset a \) is said to be a standard shape if it contains \( 0 \) and generates \( a \) as an \( o \)-module. (The word “shape” is used because \( \mathbb{Z} \)-homothetic copies of \( S \) in an abelian group are also called constellations with shape \( S \) especially in the context of torsion-free abelian groups.) In the rest of this Definition, assume \( S \) is a standard shape.
(2) Write $k = |S|$ and give $S$ a numbering $S = \{s_1, \ldots, s_{k-1}, s_k = 0\}$ for the ease of notation. Let 

$$\phi_S\colon \mathbb{R}^{|S|} \to a$$

be the surjective $o$-linear map which sends the standard vector $e_i$ ($1 \leq i \leq k-1$) to $s_i$.

Let us denote elements of the sum $\mathbb{R}^k \oplus \mathbb{R}^k$ by symbols like $(x_i^1)_{1 \leq i \leq k}$. Given an index $1 \leq j \leq k$ and a function $\omega\colon \{1, \ldots, j, \ldots, k\} \to \{\pm\}$, we obtain an element $(x_i^{\omega(i)})_{i \in \{1, \ldots, j\} \setminus \{j\}}$ of $\bigoplus_{i \in \{1, \ldots, j\} \setminus \{j\}} o \cong \mathbb{R}^{k-1}$. Let us call this map the restriction along $\omega$ and denote it by $\text{res}_\omega\colon \mathbb{R}^k \oplus \mathbb{R}^k \to \mathbb{R}^{k-1}$.

For $1 \leq j \leq k$, we define a surjective $o$-linear map

$$\psi_{S,j}\colon \bigoplus_{i \in \{1, \ldots, k\} \setminus \{j\}} o \to a$$

as follows: we define $\psi_{S,k} = \phi_S$. For $1 \leq j \leq k-1$ we define $\psi_{S,j}(x_1, \ldots, \hat{x}_j, \ldots, x_k) := x_k s_j + \sum_{i \in \{1, \ldots, k-1\} \setminus \{j\}} x_i (s_i - s_j)$.

(3) Let $\rho > 0$ be a positive real number and $N_0 > 0$ a positive integer. A non-negatively valued function $\lambda\colon \mathcal{O} \to \mathbb{R}_{\geq 0}$ is said to be an $(S, N_0, \rho, o)$-pseudorandom measure if for every choice of the data below:

- a subset $B \subset \mathbb{R}^k$ which is the product of translates of $o \leq N$ with $N \geq N_0$,
- a subset $\Omega \subset \bigcup_{1 \leq j \leq k} \{\pm\}^{\{1, \ldots, j, \ldots, k\}}$ (namely a set of pairs $(j, \omega)$ of an index $1 \leq j \leq k$ and a function $\omega\colon \{1, \ldots, j, \ldots, k\} \to \{\pm\}$),

we have the inequality:

$$\left| \mathbb{E}\left( \prod_{(j, \omega) \in \Omega} (\lambda \circ \psi_{S,j} \circ \text{res}_\omega) \right| B \times B \right) - 1 < \rho.$$  

We will need the following form of relative Szemerédi theorem.

**Theorem 3.2.** Let $S \subset a$ be a standard shape and $\delta > 0$ a positive real number. Then there exist positive real numbers $\rho = \rho(a, a, S, \delta)$ and $\gamma = \gamma(a, a, S, \delta) > 0$ such that the following holds. Let $\lambda\colon \mathcal{O} \to \mathbb{R}_{\geq 0}$ satisfy the $(S, N_0, \rho, o)$-pseudorandomness condition for some $N_0 \geq N(a)q^{2q-1}$ (see Definition 3.1 for this condition). Let $N \geq N_0$ and $A \subset a \leq N$. Suppose the following two inequalities are satisfied:

$$\mathbb{E}(\lambda \cdot 1_A | a \leq N) \geq \delta,$$

$$\mathbb{E}(\lambda^k \cdot 1_A | a \leq N) \leq \gamma N, \quad \text{where } k := |S|.$$  

Then the following inequality holds:

$$\mathbb{E}\left( \prod_{s \in S} (\lambda \cdot 1_A)(as + \beta) \left| (a, \beta) \in o \leq N \times a \leq N \right. \right) > \gamma.$$  

In particular there exist non-trivial $o$-homothetic copies of $S$ contained in $A$.

This statement and its proof are an immediate analog of [6, Theorem 5.4]. Nonetheless we write down the proof for the convenience of the reader. For this we have to recall the notion of weighted hypergraphs and borrow results on them from combinatorics. We will content ourselves with the following narrower definition than usual.

**Definition 3.3.** An $r$-uniform weighted hypergraph consists of the following data:

- a finite set $J$;
- a finite set $V_i$ of vertices given for each $i \in J$;
- for each subset $e \subset J$ with cardinality $r$, a weight function $\nu_e\colon \prod_{i \in e} V_i \to \mathbb{R}_{\geq 0}$.
The case where \( \nu_e \) have values in \( \{0, 1\} \) recovers the notion of an \( r \)-uniform hypergraph by interpreting the value 0 as “no \( r \)-edge” and 1 as “an \( r \)-edge.” The case \( r = 2 \) corresponds to classical (weighted, \( |J| \)-partite) graphs.

Consider the product \( \prod_{i \in J} V_i \times \prod_{i \in J} V_i \) and denote its elements by symbols like \( (x_i^\pm) \). Parallelly to the above, for a subset \( e \subset J \) and a function \( \omega: e \to \{\pm\} \) we get an element \( (x_i^\omega(i))_{i \in e} \in \prod_{i \in e} V_i \). This defines the restriction along \( \omega \), denoted by \( \text{res}_\omega: \prod_{i \in J} V_i \times \prod_{i \in J} V_i \to \prod_{i \in e} V_i \).

**Definition 3.4.** For a positive real number \( \rho > 0 \), an \( r \)-uniform weighted hypergraph as above is said to be \( \rho \)-pseudorandom if for all choices of a subset \( \Omega \subset \bigcup_{e \subset J | |e| = r} \{\pm\}^e \) (namely a set of pairs \((e, \omega)\) of a subset \( e \subset J \) with \(|e| = r \) and a function \( \omega: e \to \{\pm\} \)), the following estimate holds:

\[
(15) \quad \left| \mathbb{E} \left( \prod_{(e, \omega) \in \Omega} (\nu_e \circ \text{res}_\omega) \left| \prod_{i \in J} V_i \times \prod_{i \in J} V_i \right. \right) - 1 \right| < \rho.
\]

The next theorem is a deep result from combinatorics.

**Theorem 3.5 (Relative Hypergraph Removal Lemma [2]).** Let \( 0 < r \leq k \) be positive integers and \( \varepsilon > 0 \) be a positive real number. Then there exist positive real numbers \( \gamma = \gamma(r, k, \varepsilon) \) and \( \rho = \rho(r, k, \varepsilon) > 0 \) such that the following holds.

Let \( ((V_i)_{i \in J}, (\nu_e)_{e \subset J, |e| = r}) \) be a \( \rho \)-pseudorandom \( r \)-uniform weighted hypergraph. Suppose given a subset \( E_e \subset \prod_{i \in e} V_i \) for each \( e \subset J \) with \(|e| = r \) and suppose that the following inequality holds:

\[
(16) \quad \mathbb{E} \left( \prod_{e \subset J, |e| = r} (\nu_e 1_{E_e})(x_{(i)_i \in e}) \left| (x_{(i)_i \in J})_{i \in J} \in \prod_{i \in J} V_i \right. \right) \leq \gamma.
\]

Then there is a family of subsets \( E'_e \subset E_e \) for \( e \subset J \) with \(|e| = r \) such that:

\[
(17) \quad \bigcap_{e \subset J, |e| = r} E'_e \times V_{J \setminus e} = \emptyset, \quad \text{and} \quad \mathbb{E}(\nu_e 1_{E'_e \setminus E_e} | V_e) \leq \varepsilon \quad \text{for all} \ e.
\]

**Proof.** Conlon-Fox-Zhao [2, Theorem 2.12] state this in a slightly different way but their proof actually shows our current statement. See [6, Theorem 5.10] for detail. \( \square \)

Recall that we write \( X = \text{Spec} \mathcal{O} \) and let \( \overline{X} \) be the complete non-singular curve containing \( X \) as an open subscheme. Also let us recall some integer quantities:

\[
\begin{align*}
n &= |\overline{X} \setminus X|, \\
d_0 &= \lcm \{ \deg(v) \mid v \in \overline{X} \setminus X \}, \\
d &= \nu(t) \quad \text{(independent of} v \in \overline{X} \setminus X), \\
r &= nd = \text{rank}_k(\mathcal{O}), \\
g &= \text{the genus of} \overline{X}.
\end{align*}
\]

**Proof of Theorem 3.2.** Let \( S \subset a \) and \( \delta > 0 \) be as in the statement. Recall \( k = |S| \). Consider the \( \phi \)-linear map \( \phi_S: \alpha^{k-1} \to a \) in Definition 3.1. By Corollary 2.3 there is a constant \( U > 0 \) such that for every \( N \geq 1 \) and \( \alpha \in a_{\leq N} \), the set \( \phi_S^{-1}(\alpha) \cap a_{\leq U} \) contains at least \( (N^{1/d})^{k-1} \) elements. Using Theorem 3.3 we set positive numbers \( \varepsilon, \gamma, \rho > 0 \) as:

\[
\begin{align*}
\varepsilon &= \delta/(N(a)kq^{nd_0 + d + k - 2U^{(k-1)/d}}), \\
\gamma &= \gamma(k - 1, k, \varepsilon), \\
\rho &= \rho(k - 1, k, \varepsilon),
\end{align*}
\]

whose motivation will only be clear later.

Now suppose we are given an \((\mathcal{S}, N_0, \rho, \phi)\)-pseudorandom function \( \lambda: a \to \mathbb{R}_{\geq 0} \) with the above \( \rho \) and some \( N_0 \geq 0 \). Also let \( N \geq N_0 \) and suppose the subset \( A \subset \mathcal{O}_{\leq N} \) satisfies
and (13). Out of these data, we construct a \((k - 1)\)-uniform weighted hypergraph \((V_i)_{1 \leq i \leq k}, \nu_j: \prod_{i \neq j} V_i \to \mathbb{R}_{\geq 0}\) as follows. The vertex sets are:

\[ V_i := \{\text{hyperplanes } H \subset \sigma^{k-1} \mid H \text{ is defined by } x_i = h \text{ with } h \in \sigma_{UN} \} \quad \text{if } 1 \leq i \leq k - 1, \]

\[ V_k := \{\text{hyperplanes } H \subset \sigma^{k-1} \mid H \text{ is defined by } \sum_{i=1}^{k-1} x_i = h \text{ with } h \in \sigma_{UN} \}. \]

To define the weight functions, note that for any index \(1 \leq j \leq k\) and tuple \((H_i)_{i \in \{1, \ldots, k, \ldots, k\}}\), the intersection \(\bigcap_{i \neq j} H_i\) consists of exactly one point of \(\sigma^{k-1}\). Let \(T: \bigcup_{j=1}^{k} \left(\prod_{i \neq j} V_i\right) \to \sigma^{k-1}\) be the map sending a given tuple to this point. Its restriction to the \(j\)th summand shall be denoted by \(T_j\) if we need to emphasize the domain of definition. By abuse of notation, also denote by \(T_j\) the composite map \(\prod_{i=1}^{k} V_i \xrightarrow{\nu_j} \prod_{i \neq j} V_i \xrightarrow{T_j} \sigma^{k-1}\) where \(\nu_j\) is the projection dropping the \(j\)th entry. We define the weight functions for \(1 \leq j \leq k\):

\[ \nu_j: \prod_{i \neq j} V_i \xrightarrow{T_j} \sigma^{k-1} \xrightarrow{\phi_S} a \xrightarrow{\lambda} \mathbb{R}_{\geq 0}. \]

We can specify tuples \((H_i)_{1 \leq i \leq k}\) of hyperplanes by tuples \((h_i)_{1 \leq i \leq k} \in (\sigma_{UN})^k\) of scalars appearing in their defining equations. This gives us the left-hand vertical map in the following commutative diagram, where the map \(\psi_{S,j}\) was defined in Definition 3.1.

\[ \begin{array}{ccc}
V_i & \xrightarrow{T} & \sigma^{k-1} \\
\bigoplus_{i \in \{1, \ldots, j, \ldots, k\}} & \psi_{S,j} & \xrightarrow{\phi_S} a \\
\end{array} \]

for all \(j\).

It follows that the estimate (11) implies the estimate (15) for the weighted hypergraph at hand. Thus it is \(\rho\)-pseudorandom.

Define a subset \(E_j \subset \prod_{i \neq j} V_i\) for each \(j\) by:

\[ E_j := \{(H_i)_{i \neq j} \mid \bigcap_{i \neq j} H_i \subset \phi_S^{-1}(A)\} \]

\[ = T_j^{-1}\phi_S^{-1}(A), \]

and set \(\overline{E}_j := E_j \times V_j \subset \prod_{i=1}^{k} V_i\). The significance of these sets is as follows: given an element \(H \equiv (H_i)_{i \in \{1, \ldots, j, \ldots, k\}}\), the subset \(\{\phi_S T_i(H)\}_{i=1}^{k} \subset a\) is an \(\sigma\)-homothetic copy of \(S\), nontrivial if and only if if \(\bigcap_{i=1}^{k} H_i = \emptyset\).

Now, specifying a tuple \((h_i)_{1 \leq i \leq k} \in (\sigma_{UN})^k\) is equivalent to specifying its first \((k - 1)\) entries \(h = (h_i)_{1 \leq i \leq k-1} \in (\sigma_{UN})^{k-1}\) and a scalar \(a = h_k - \sum_{i=1}^{k-1} h_i \in \sigma_{UN}\). For \(1 \leq i \leq k - 1\), let

\[ e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \sigma^{k-1} \]

be the \(i\)-th standard vector and let \(e_k = 0\) be the zero vector. We claim the following inequality:

\[ \mathbb{E}\left(\prod_{i=1}^{k} (\lambda 1_A)(\phi_S(h + ae_i)) \mid (h, a) \in (\sigma_{UN})^{k-1} \times (\sigma_{UN} \setminus \{0\})\right) > \gamma. \]
Toward contradiction, suppose otherwise. The assumption \[13\] is equivalently formulated as

\[
\mathbb{E}\left(\prod_{i=1}^{k}(\lambda_{A})(\phi_B(h + ae_i)) \mid (h, a) \in (\alpha_{\leq UN})^{k-1} \times \{0\}\right) \leq \gamma.
\]

It follows that the expectation computed on \((\alpha_{\leq UN})^{k-1} \times \alpha_{\leq UN}\) is also \(\leq \gamma\). By the definitions of \(\nu_j\) and \(E_j\) as pullbacks, we know that the following commutes:

\[
\prod_{i \neq j} V_i \xrightarrow{\phi_B T} a \quad \quad \nu_j E_j \xrightarrow{\phi_B} \mathbb{R}_{\geq 0}.
\]

Also recalling the definition of \(T\) we find that this last inequality precisely says that the hypothesis \[16\] of Theorem \[3.5\] is satisfied for our situation. Therefore there is a family of subsets \(E'_i \subset E_i\) as in Theorem \[3.5\].

We claim that the existence of such \(E'_i\) leads to the negation of \[12\]. Define a map

\[
\iota_0: \phi_B^{-1}(A) \cap \alpha_{\leq UN}^{k-1} \to \prod_{i=1}^{k} V_i \quad \text{by} \quad a \mapsto \text{(the hyperplane } \in V_i \text{ passing through } a)_{i}.
\]

We have an equality of maps \(T_j \circ \iota_0 = \text{id}\) from \(\phi_B^{-1}(A) \cap \alpha_{\leq UN}^{k-1}\) to itself for all \(j\). It follows \(\iota_0\) maps into \(\bigcap_{j=1}^{k} E_j\) (recall \(E_j := E_j \times V_j \subset \prod_{i=1}^{k} V_i\)). Endow this set with the following filtration; for the sake of space, we write also \(E'_i := E'_i \times V_i:\)

\[
\bigcap_{i=1}^{k} E_i \supset \cdots \supset \left(\bigcap_{i=1}^{l-1} E_i\right) \cap \left(\bigcap_{i=l+1}^{k} E_i\right) \supset \cdots \supset \bigcap_{i=1}^{k} E'_i = \emptyset.
\]

Hence \(\bigcap_{i=1}^{k} E_i\) is the disjoint sum of the successive complements so we can define a map \(\text{pr}: \bigcap_{i=1}^{k} E_i \to \bigcup_{i=1}^{k} (E_i \setminus E'_i)\) by the condition:

the restriction \(\text{pr}: \left(\bigcap_{i=1}^{l-1} E_i\right) \cap \left(\bigcap_{i=l+1}^{k} E_i\right) \to E_i \setminus E'_i\) is the projection \(\text{pr}_i\).

Then define a map \(\iota: \phi_B^{-1}(A) \cap \alpha_{\leq UN}^{k-1} \to \bigcup_{i=1}^{k} (E_i \setminus E'_i)\) by \(\text{pr} \circ \iota_0\). We see that \(T \circ \iota = \text{id}\) and in particular \(\iota\) is injective. So far we have obtained the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_B} & \phi_B^{-1}(A) \cap \alpha_{\leq UN}^{k-1} \setminus E'_i \\
\downarrow{T} & & \downarrow{\lambda} \\
\bigcup_{i=1}^{k} E_i \setminus E'_i & \xrightarrow{\lambda} & \mathbb{R}_{\geq 0}
\end{array}
\]

where we know the fibers of the map \(\phi_B\) have cardinality \(> (N^{1/d})^{k-1-r}\) by the choice of \(U > 0\). It follows that

\[
\sum_{H \in \bigcup_{i}(E_i \setminus E'_i)} \nu_i(H) \geq \sum_{a \in \phi_B^{-1}(A) \cap \alpha_{\leq UN}^{k-1}} \lambda(\phi_B(a)) > (N^{1/d})^{k-1-r} \cdot \sum_{a \in A} \lambda(a).
\]
By (17) we know that the left hand side is bounded by
\[
\leq \sum_{i=1}^{k} e \cdot |a_{\leq N}|^{k-1} \leq k e \cdot (q(U N)^{1/d})^{k-1}.
\]
By these inequalities and the fact (4) (or (3)) that \( |a| \geq N/(N(a)q^{N_0+g-1}) \) for \( N \geq N(a)q^{2g-1} \) we get:
\[
\mathbb{E}(1_A \lambda | a_{\leq N} ) < N(a)q^{N_0+g+k-2}(U^{1/d})^{k-1} \cdot e \leq \delta,
\]
contradicting (12). This shows the claimed inequality (20).

To conclude, let us deduce (14) from (20). First, if the elements \( a \in o \) and \( \beta \in a_{\leq N} \) satisfy \( as + \beta \in A \subset a_{\leq N} \) for an \( s \in S \setminus \{0\} \), it is necessarily true that \( as \in a_{\leq N} \) because \( a_{\leq N} \) is a subgroup. Since \( ||s|| \geq 1 \), we necessarily have \( |a| \leq N \). It follows that the terms with \( |a| > N \) do not contribute to the expectation (20) so that we obtain
\[
\mathbb{E}\left( \prod_{i=1}^{k}(\lambda_1 A)(\phi_S(h + ae_i)) \mid (h, a) \in o_{U \leq N}^{k-1} \times (a_{\leq N} \setminus \{0\}) \right) > \gamma.
\]
Note that the fibers of \( \phi_S : o_{U \leq N}^{k-1} \to o_{\leq N} \) have cardinality \( |o_{U \leq N}^{k-1}|^{k-1-r} \) and so the same is true for the vertical map in the next commutative diagram:
\[
\begin{array}{ccc}
o_{U \leq N}^{k-1} \times o_{\leq N} & \xrightarrow{\phi_S \times \text{id}} & (h, a) \mapsto \prod_{i=1}^{k}(\lambda_1 A)(\phi_S(ae_i + h)) \\
\end{array}
\]
It follows that
\[
|o_{U \leq N}^{k-1}|^{k-1-r} \sum_{\beta \in a_{\leq N}} \prod_{\beta + as \in S} (\lambda_1 A)(\beta + as) \geq \sum_{h \in o_{U \leq N}^{k-1}} \prod_{h + as \in S} (\lambda_1 A)(\phi_S(h) + as).
\]
If we divide both sides by \( |o_{U \leq N}^{k-1}| \cdot |a_{\leq N}| \) this precisely says:
\[
\frac{1}{|o_{U \leq N}^{k-1}|} \cdot \text{L.H.S. of (14)} \geq \text{L.H.S. of (20)} > \gamma.
\]
Note by (4) that the first factor on the left hand side is smaller than 1 at least if \( g \geq 1 \) or \( U \geq q \). It follows that the asserted inequality (14) holds. This completes the proof of Theorem 3.2. \( \square \)

4. The von Mangoldt function

The definitions and results are parallel to those in [6, §6].

**Definition 4.1.** Let \( \chi : \mathbb{R} \to [0, 1] \) be a \( C^\infty \) function with support \( \subset [-1, 1] \) and \( \chi(0) = 1 \), which we fix throughout the paper. Let \( R > 0 \) be a positive number. In the proof of the main result we will need to take \( R \) very large. Let \( \text{Ideals}_O \) be the multiplicative monoid of non-zero ideals of \( O \). We define the function \( \Lambda_{R, \chi} : \text{Ideals}_O \to \mathbb{R} \) by
\[
\Lambda_{R, \chi}(b) := \log R \sum_{c \in \text{Ideals}_O \text{ with } \|c\| \leq b} \mu(c) \chi \left( \frac{\log N(c)}{\log R} \right).
\]

Given a non-zero ideal \( a \) of \( O \), we define the function \( \Lambda_{R, \chi}^a : a \to \mathbb{R} \) by the composition:
\[
\Lambda_{R, \chi}^a : a \xrightarrow{\alpha \mapsto \alpha a^{-1}} \text{Ideals}_O \xrightarrow{\Lambda_{R, \chi}} \mathbb{R}.
\]
Note that the membership \( \alpha \in a \) implies that \( \alpha a^{-1} \) is a (non-zero) ideal of \( O \) so that the above composition is well defined.

Below we use the notation \( N(-) \) also for non-zero ideals of \( o \). This does not cause confusion because a non-zero ideal of \( o \) is never an ideal of \( O \) and vice versa. Of course
every non-zero ideal of $\mathfrak{a} = \mathbb{F}_q[t]$ is a principal ideal $\mathfrak{f}_0$ and if $f$ has degree $e$ as a polynomial then we have $N(\mathfrak{f}_0) = q^e (\| f \|_\mathfrak{a})$. For $N > 0$ we have $\mathfrak{a}_{\leq N} = \{ f \in \mathfrak{a} \mid N(\mathfrak{f}_0) \leq N \}$ by Proposition 2.1. For non-zero ideals $\mathfrak{b} \subset \mathcal{O}$, by the canonical injection $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{b} \to \mathcal{O}/\mathfrak{b}$, we know $N(\mathfrak{a} \cap \mathfrak{b}) \leq N(\mathfrak{b})$.

For $\mathfrak{b} \in \text{Ideals}_\mathcal{O}$, define $\varphi_\mathcal{O}(\mathfrak{b}) := |(\mathcal{O}/\mathfrak{b})^\times|$. By Chinese Remainder Theorem we know $\varphi_\mathcal{O}(\mathfrak{b}) = N(\mathfrak{b}) \prod_{p|\mathfrak{b}} \left(1 - \frac{1}{N(p)}\right)$ where $p$ runs through the prime ideal divisors of $\mathfrak{b}$. For elements $\alpha \in \mathcal{O} \smallsetminus \{0\}$ let us write $\varphi_\mathcal{O}(\alpha) := \varphi_\mathcal{O}(\alpha \mathcal{O})$.

**Theorem 4.2** (of Goldston-Yıldırım type). Let $\alpha$ be a non-zero ideal of $\mathcal{O}$. Let $m, t \geq 0$ be non-negative integers. Let

$$\phi_1, \ldots, \phi_m : \alpha^t \to \mathfrak{a}$$

be $\alpha$-linear maps whose cokernels are finite and such that $\ker(\phi_i)$ does not contain $\ker(\phi_j)$ whenever $i \neq j$.

Then there are large positive numbers $R_0 > 1$, $w_0 > 1$ and a small one $0 < f_0 < 1$ such that for every choice of the quantities below:

- real numbers $R \geq R_0$ and $w \geq w_0$ such that $\frac{\log w}{\sqrt{\log R}} < f_0$,
- an element $W \in \mathfrak{o}$ whose prime factors are exactly $\{ \pi, \text{prime element} \mid N(\pi) \leq w \}$,
- a translate $B \subset \alpha^t$ of a product $\mathfrak{o}_{\leq N_1} \times \cdots \times \mathfrak{o}_{\leq N_t}$ with $N_i \geq R^{2m}/q$ for all $1 \leq i \leq t$,
- $b_1, \ldots, b_m \in \mathfrak{a} \smallsetminus \bigcup_{p|W} p\mathfrak{a}$,

we have an estimate

$$\mathbb{E} \left( \prod_{i=1}^{m} \Lambda_{R, \chi}(W \phi_i(x) + b_i)^2 \mid x \in B \right) = \left(1 + O_{\chi, m, r} \left(\frac{\log w}{\sqrt{\log R}}\right) + O_{\chi, m, r, \mathcal{O}} \left(\log w \log \log R\right) \right)^m \left(\log R \cdot \frac{N(W \mathcal{O})}{\varphi(W) \kappa_\mathcal{O}} C_\chi \right),$$

where $C_\chi$ and $\kappa_\mathcal{O}$ are positive constants associated with $\chi$ and $\mathcal{O}$ which are defined in (53) and (40).

That the error terms can be taken entirely independent of the choice of $B$, $b_1, \ldots, b_m$ is part of the statement of Theorem 3.2. Also, the ratio $N(W \mathcal{O})/\varphi(W) = \prod_{p|W} \left(1 - \frac{1}{N(p)}\right)$ depends only on the real number $w$ and not on the specific $W$.

The proof of this theorem occupies the rest of this section.

Toward the proof of Theorem 4.2, first we unfold the relevant definitions to get

$$\text{L.H.S. of (21)} = \sum_{(b_i, \mathfrak{c}_i) \in \text{Ideals}_\mathcal{O}^m} (\log R)^{2m} \mu(b_i) \mu(\mathfrak{c}_i) \chi \left(\frac{\log N(b_i)}{\log R}\right) \chi \left(\frac{\log N(\mathfrak{c}_i)}{\log R}\right) \mathbb{E} \left( \prod_{i=1}^{m} 1_{\mathfrak{a}(b_i \cap \mathfrak{c}_i)}(W \phi_i(x) + b_i) \mid x \in B \right).$$

Note that only those terms with $N(b_i) \leq \log R$ and $N(\mathfrak{c}_i) \leq \log R$ for all $i$ contribute to the sum because $\text{Supp} \chi \subset [-1, 1]$. Define an $\alpha$-linear map $\bar{\phi}$ by

$$\bar{\phi} : \alpha^t \to \bigoplus_i \mathfrak{a} / \mathfrak{a} \cdot (b_i \cap \mathfrak{c}_i) ; \quad x \mapsto (\phi_i(x) \mod \mathfrak{a} \cdot (b_i \cap \mathfrak{c}_i)).$$

For a given tuple of ideals $(\mathfrak{b}, \mathfrak{c})$, let $I \subset \mathfrak{a}$ be the ideal $I = \mathfrak{a} \cap \bigcap_i b_i \cap \mathfrak{c}_i$. The map $\bar{\phi}$ factors through $(\mathfrak{a}/I)^t$. Also we write $b = (b_i)_1 \in \bigoplus_{i=1}^m \mathfrak{a}$ and $\bar{b}$ for its residue class in $\bigoplus_i \mathfrak{a} / \mathfrak{a} \cdot (b_i \cap \mathfrak{c}_i)$. Then for $x \in \alpha^t$, the condition that

$$F(x) := \prod_{i=1}^{m} 1_{\mathfrak{a}(b_i \cap \mathfrak{c}_i)}(W \phi_i(x) + b_i) = 1$$
is equivalent to the equality $W\tilde{\phi}(x) + \tilde{b} = 0$, namely the equality $\tilde{F}(x) := 1_{\{0\}}(W\tilde{\phi}(x) + \tilde{b}) = 1$. It follows that we have a commutative diagram:

$$\begin{array}{ccc}
\sigma' & \xrightarrow{\phi} & \color{red}{(\sigma/I)^t} \\
\downarrow^{F} & & \downarrow^{0,1} \\
\color{red}{F} & \rightarrow & \color{red}{\{0,1\}}.
\end{array}$$

The next assertion paves the way for the computation of the $E(-)$ term in (22).

**Lemma 4.3.** Let $(b, \xi) \in \text{Ideals}^{2m}_O$ be a tuple with $N(b_i), N(\xi_i) \leq \log R$ for all $i$. Let $I$, $F$ and $\tilde{F}$ as in (24). Then we have an equality

$$E(F | B) = E(\tilde{F} | (\sigma/I)^t).$$

**Proof.** Note that $N(I) \leq N(\bigcap_i (b_i \cap \xi_i)) \leq \prod_i N(b_i) \cdot N(\xi_i) \leq R^{2m}$. As an elementary fact in $O = \mathbb{F}_q[t]$, we know that for any $l \geq N(I)/q$ the composite map $\sigma \colon o \rightarrow o/I$ is surjective. (In other words, if $f \in \mathbb{F}_q[t]$ is a polynomial of degree $e$, the polynomials of degree $\leq e - 1$ form a set of representatives for the quotient ring $\mathbb{F}_q[t]/(f)$.) It follows that the composite homomorphism $B \rightarrow \sigma' \rightarrow (\sigma/I)^t$ is surjective. By the commutative diagram (24) our claim follows. \hfill $\square$

Now let us move on to compute the right hand side of (25).

4.1. The $\pi$-parts. Let $\pi \in o$ be a prime element. Let us call a non-zero ideal $b$ of $O$ a $\pi$-ideal if the $\sigma$-module $O/b$ is annihilated by a power of $\pi$. By the prime decomposition of ideals of $O$, every $b \in \text{Ideals}_O$ is uniquely written as a product

$$b = \prod \pi b^{(\pi)}$$

where $\pi$ runs through the associate classes of prime elements of $o$ and $b^{(\pi)}$ is a $\pi$-ideal. We call $b^{(\pi)}$ the $\pi$-part of $b$. The $\pi$-part of a tuple of ideals $(b, \xi) = (b_i, \xi_i)_{1 \leq i \leq m} \in \text{Ideals}^{2m}_O$ shall mean the tuple of the $\pi$-parts of its entries: $(b, \xi)^{(\pi)} := (b_i^{(\pi)}, \xi_i^{(\pi)})_{1 \leq i \leq m}$.

Let us write

$$E((b, \xi); W, b)$$

for the quantity in (25). It depends also on the data of $\phi_i$ but we do not include it in the notation.

**Lemma 4.4.** The quantity $E((b, \xi); W, b)$ decomposes into the product of its $\pi$-parts; namely,

$$E((b, \xi); W, b) = \prod \pi E((b, \xi)^{(\pi)}; W, b),$$

where $\pi$ runs through the associate classes of the prime elements of $o$.

**Proof.** By Chinese Remainder Theorem, the map $\tilde{\phi} : (\sigma/I)^t \rightarrow \bigoplus_i o/a(b_i \cap \xi_i)$ decomposes to its $\pi$-parts, that is:

$$\tilde{\phi} = \prod \pi \tilde{\phi}^{(\pi)} : \prod \pi (\sigma/I(\pi))^t \rightarrow \prod \pi \bigoplus_i o/a(b_i^{(\pi)} \cap \xi_i^{(\pi)}),$$

where $\tilde{\phi}^{(\pi)}$ is the map defined by (23) with $(b, \xi)^{(\pi)}$ in place of $(b, \xi)$. Then in (25), the $(0,1)$-valued function $\tilde{F}$ decomposes into the product of functions $\tilde{F}^{(\pi)} : \sigma/I(\pi) \rightarrow \{0,1\}$ which are defined exactly as $\tilde{F}$ with $(b, \xi)^{(\pi)}$ in place of $(b, \xi)$. By a Fubini type computation our assertion now follows. \hfill $\square$

Our next task is to evaluate $E((b, \xi)^{(\pi)}; W, b)$. The computation is divided to two cases: when $\|\pi\|_o$ is small and when it is large. Let us use Greek letters $\alpha, \beta, \gamma$ to denote ideals when they are assumed to be $\pi$-ideals for a fixed prime element $\pi \in o$.

**Lemma 4.5.** Consider a tuple of ideals $(\beta, \gamma) = (\beta_i, \gamma_i)_{1 \leq i \leq m}$ and suppose $\beta_i$ and $\gamma_i$ are all $\pi$-ideals for a common $\pi$. Then, the quantity $E((\beta, \gamma); W, b)$ equals 1 if $(\beta_i, \gamma_i) = (O, O)$ for all $i$. Assuming otherwise in the following, we have:
(1) If \( N(\pi o) \leq w \), then \( E((\beta, \gamma); W, b) = 0 \).
(2) Assume \( N(\pi o) > w \) and \( w \) is large enough depending on \( \{\phi_i\} \). Then:
   (a) if there is only one \( i \) with \( \beta_i \cap \gamma_i \subseteq \mathcal{O} \), then \( E((\beta, \gamma); W, b) = \frac{1}{N(\beta_i \cap \gamma_i)} \).
   (b) if there are at least two \( i \)'s with \( \beta_i \cap \gamma_i \subseteq \mathcal{O} \), then \( E((\beta, \gamma); W, b) \leq \frac{1}{N(\pi o)^2} \).

**Proof.** Let us consider the case (1). In this case, we know \( W \in \pi o \) by our assumption on the prime factors of \( W \) in Theorem 1.2 We claim the value \( W \phi(x) + b \) is never 0 in \( \bigoplus_o a/\pi o(\beta_i \cap \gamma_i) \). Indeed, choose any \( i \) with \( \beta_i \cap \gamma_i \subseteq \mathcal{O} \) (which we are assuming to exist) and any of its prime factors \( p \). It is a prime ideal over \( \pi o \), and hence \( W \in \pi o \) always. It follows that \( W \phi_i(x) \in \pi o \) for all \( x \in \phi \). Meanwhile, \( b_i \notin \pi o \) by assumption. It follows \( W \phi_i(x) + b_i \notin \pi o \) and in particular \( \notin (\beta_i \cap \gamma_i) \). Our claim follows.

Next we consider the case (2). In this case the ideals \( \beta_i, \gamma_i \) are all coprime to \( W \) in \( \mathcal{O} \). For the case (2a), it suffices to show that the map \( W \phi_i(-) + b_i: (\pi o) \to a/\pi o(\beta_i \cap \gamma_i) \) is surjective. Since the translation \( +b_i \) and the multiplication by \( W \) map on \( a/\pi o(\beta_i \cap \gamma_i) \) are both bijective, it suffices to show that the map

\[
\phi_i \mapsto a/\pi o(\beta_i \cap \gamma_i)
\]

is surjective when \( w \) is large enough. By assumption \( \text{coker}(\phi_i) \) is an \( o \)-module which is a finite abelian group. Hence there are only finitely many prime ideals \( (\pi o) \subseteq o \) satisfying \( \pi \cdot \text{coker}(\phi_i) \subseteq \text{coker}(\phi_i) \). Now assume \( w \) exceeds the norms of those \( \pi \)'s. Then as long as \( \beta_i \) and \( \gamma_i \) are \( \pi \)-ideals and \( N(\pi o) > w \), we have \( (\beta_i \cap \gamma_i) \cdot \text{coker}(\phi_i) = \text{coker}(\phi_i) \), i.e., the map (26) is surjective.

Let us consider the case (2b). First we specify how large \( w \) should be. We are assuming that \( \ker(\phi_i) \)'s do not contain each other. For each pair of distinct indices \( i, j \), choose an element \( x_{ij} \in \ker(\phi_i) \setminus \ker(\phi_j) \). Since \( \phi_j(x_{ij}) \in a \) is non-zero, there are only finitely many prime ideals \( p \subseteq \mathcal{O} \) with \( \phi_j(x_{ij}) \in \pi o \). Let \( w \) exceed the norms of all the \( p \)'s appearing this way.

To show (2b) it suffices to verify that the image of the map \( \phi \) has cardinality \( \geq N(\pi o)^2 \). Suppose \( i \neq j \) are among the indices with \( \beta_i \cap \gamma_i \subseteq \mathcal{O} \) and let \( p_i, p_j \) be prime ideals containing them. We show that the image of the next further composition

\[
\phi \mapsto \bigoplus_{i=1}^{m} \frac{a}{a(\beta_i \cap \gamma_i)} \to (a/\pi o_i) \oplus (a/\pi o_j)
\]

has cardinality \( \geq N(\pi o)^2 \). The images of the two elements \( x_{ij}, x_{ji} \) are respectively the residue classes of \((0, \phi_j(x_{ij})) \) and \((\phi_i(x_{ij}), 0) \), and both are non-zero by the very choice of \( w \). It follows that their \( (\pi o)^2 \)-linear combinations are all distinct (note that the target is an \( o/(\pi o)^2 \)-vector space). Therefore the image of the map (27) contains at least \(|o/(\pi o)^2| \) distinct elements. □

Now we want to plug our results here into (22), but to proceed further, we need the help of Fourier analysis.

### 4.2. Fourier transform

Let \( \hat{\chi} \) be the Fourier transform of the function \( x \mapsto e^x \chi(x) \) so that by inverse Fourier transform:

\[
e^x \chi(x) = \int_{\mathbb{R}} \hat{\chi}(\xi) e^{\xi x} \sqrt{-1} d\xi.
\]

It follows that \( \chi(\log N(b)) = \int_{\mathbb{R}} N(b) \frac{1}{\log N(b)} (1+\xi \sqrt{-1}) \hat{\chi}(\xi) \xi \sqrt{-1} d\xi \) for ideals \( b \). By the theory of Fourier analysis, we know that \( \hat{\chi} \) decays rapidly:

**Lemma 4.6.** For any given positive numbers \( A \) and \( b \geq 1 \), we have \( \hat{\chi}(\xi) = O_{A,\chi}(1 + |\xi|^{-A}) \) and hence

\[
\int_{\mathbb{R}} |\hat{\chi}(\xi)|d\xi < \infty \quad \text{and} \quad \left(\int_{-\infty}^{-b} + \int_{b}^{+\infty}\right) |\hat{\chi}(\xi)|d\xi = O_{A,\chi}(b^{-A}).
\]

**Proof.** See any textbook on Fourier analysis or [6] Lemma 6.15 and its corollary. □
The right hand side of (22) is written as:

\[
(\log R)^{2m} \sum_{(b_i, c_i) \in \text{Ideals}^2_{O^m}} (\log R)^m \prod_{i=1}^m \mu(b_i) \mu(c_i) \int_{\mathbb{R}} N(b_i)^{\frac{1}{m} \log \pi(-1+\xi_i \sqrt{-1})} \hat{\chi}(\xi_i) d\xi_i \int_{\mathbb{R}} N(c_i)^{\frac{1}{m} \log \pi(-1+\eta_i \sqrt{-1})} \hat{\chi}(\eta_i) d\eta_i 
\cdot E((b, c); W, b).
\]

Let \(I\) be the interval \(I := [-\sqrt{\log R}, \sqrt{\log R}]\). For tuples \((\xi, \eta) \in \mathbb{R}^{2m}\), consider the infinite sum

\[
E((\xi, \eta), R, w, b) := \sum_{(b_i, c_i) \in \text{Ideals}^2_{O^m}} E((b, c); W, b) \cdot \prod_{i=1}^m \mu(b_i) \mu(c_i) N(b_i)^{\frac{1}{m} \log \pi(-1+\xi_i \sqrt{-1})} N(c_i)^{\frac{1}{m} \log \pi(-1+\eta_i \sqrt{-1})}.
\]

This subsection is devoted to the proof of:

**Proposition 4.7.** The sum \(E((\xi, \eta), R, w, b)\) converges absolutely and uniformly in \((\xi, \eta) \in \mathbb{R}^{2m}\).

For any given \(A > 0\), the quantity (28) is equal, up to an error \(\pm O_{A, \chi, m, r}((\log R)^{-A})\), to:

\[
(\log R)^{2m} \int_{\mathbb{R}^{2m}} E((\xi, \eta), R, w, b) \left( \prod_{i=1}^m \hat{\chi}(\xi_i) \hat{\chi}(\eta_i) \right) d\xi d\eta.
\]

4.2.1. **Convergence.** Note that by the presence of the M"obius function, only those terms where all \(b_i\) and \(c_i\) are square-free contribute to the sum (29) and that the sum decomposes into the product of its \(\pi\)-parts by the multiplicativity of the functions involved (see Lemma 4.4) for the multiplicativity of \(E((b, c); W, b)\). Namely:

\[
E((\xi, \eta), R, w, b) = \prod_{(\pi)} E^{(\pi)}((\xi, \eta), R, w, b), \quad \text{where}
E^{(\pi)}((\xi, \eta), R, w, b) := \sum_{(\beta_i, \gamma_i), \text{\pi-ideals, square-free}} E((\beta, \gamma); W, b) \left( \prod_{i=1}^m \mu(\beta_i) \mu(\gamma_i) N(\beta_i)^{\frac{1}{m} \log \pi(-1+\xi_i \sqrt{-1})} N(\gamma_i)^{\frac{1}{m} \log \pi(-1+\eta_i \sqrt{-1})} \right).
\]

Note that there are at most \(r\) prime ideals of \(O\) over a given \((\pi)\) and hence there are at most \(2^r\) square-free \(\pi\)-ideals.

By Lemma 4.5 if \(N(\pi o) \leq w\) then \(E^{(\pi)}((\xi, \eta), R, w, b) = 1\); we also know the following when \(N(\pi o) > w\), supposing (as we shall always do) that \(w\) is large enough to invoke the lemma:

- Unless \((\beta_i, \gamma_i) = (O, O), (O, p), (p, O)\) or \((p, p)\) for some prime \(\pi\)-ideal \(p\) with \(N(p) = N(\pi o)\), we have \((0) \subseteq E((\beta, \gamma); W, b) \leq 1/N(\pi o)^2\);
- For those exceptional cases in the previous item, we know \(E((\beta, \gamma); W, b) = 1\) for the first case and \(= 1/N(\pi o)\) for the others.

This gives us the following crude estimate, where \(O_r(1)\) can be taken to be \(2^r\):

\[
E^{(\pi)}((\xi, \eta), R, w, b) = 1 + \sum_{p, \deg = 1} \left( -N(\pi o)^{-1+\frac{1}{m} \log \pi(-1+\xi_i \sqrt{-1})} - N(\pi o)^{-1+\frac{1}{m} \log \pi(-1+\eta_i \sqrt{-1})} + N(\pi o)^{-1+\frac{1}{m} \log \pi(-2+(\xi_i+\eta_i) \sqrt{-1})} \right)
+ O_r(1) \cdot \frac{1}{N(\pi o)^2}.
\]

In particular we have \(E((\xi, \eta), R, w, b) = 1 + O_r \left( N(\pi o)^{-1+\frac{1}{m} \log \pi} \right)\) uniformly in \(\xi, \eta\). Therefore by basic facts on Euler products (such as \[\text{Lemma 6.19}\]) we conclude that the product \(\prod_{(\pi)} E^{(\pi)}((\xi, \eta), R, w, b)\) converges absolutely. As a result, the sum of absolute values
associated with the sum $E((\xi, \eta), R, w, b)$ \textcolor{red}{(29)} can be estimated as:

$$
\sum_{(b_i, c_i), b_i \leq c_i} E((b, \xi); W, b) \cdot \left( \prod_{i=1}^{m} N(b_i)^{1 \omega^{-1}} N(c_i)^{1 \omega^{-1}} \right)
= \prod_{(\pi)} \left( 1 + O_{r} \left( N(\pi o)^{-1 \omega^{-1}} \right) \right) = (\log R + O(1))^{O_{r}(1)},
$$

proving the convergence claim so that the value \textcolor{red}{(34)} equals \textcolor{red}{(30)}.

\textcolor{red}{4.2.2. Summation and integration.} We want to replace the domain $\mathbb{R}$ of integration in \textcolor{red}{(28)} by the bounded interval $I = [-\sqrt{\log R}, \sqrt{\log R}]$.

\textbf{Lemma 4.8.} Regarding the expression \textcolor{red}{(28)},

(1) We have the following estimate:

$$
\int_{\mathbb{R}} N(b)^{-1 \omega^{-1}} (-1+\xi \sqrt{-1}) \hat{\chi}(\xi) d\xi = \int_{I} N(b)^{-1 \omega^{-1}} (-1+\xi \sqrt{-1}) \hat{\chi}(\xi) d\xi \pm O_{A, b} \left( N(b)^{-1 \omega^{-1}} (\log R)^{-A} \right)
$$

for all ideals $b \subset O$.

(2) Let $d\xi \, d\eta$ be a shorthand symbol for $d\xi_1 \ldots d\xi_m \, d\eta_1 \ldots d\eta_m$. For each $(b, \xi) \in (\text{Ideals}_{O})^{2m}$ and positive number $A > 0$, we have:

$$
\int_{I^{2m}} \left( \prod_{i=1}^{m} N(b_i)^{-1 \omega^{-1}} (-1+\xi_i \sqrt{-1}) \right) \hat{\chi}(\xi_i) d\xi_i \cdot \left( \prod_{i=1}^{m} N(c_i)^{-1 \omega^{-1}} (-1+\eta_i \sqrt{-1}) \right) \hat{\chi}(\eta_i) d\eta_i \pm O_{A, b, m} \left( (\log R)^{-A} \prod_{i=1}^{m} N(b_i)^{-1 \omega^{-1}} N(c_i)^{-1 \omega^{-1}} \right).
$$

\textbf{Proof.} (1) The error $\int_{\mathbb{R} \setminus I} N(b)^{-1 \omega^{-1}} (-1+\xi \sqrt{-1}) \hat{\chi}(\xi) d\xi$ is bounded in magnitude by

$$
N(b)^{-1 \omega^{-1}} \int_{\mathbb{R} \setminus I} |\hat{\chi}(\xi)| d\xi
$$

which has at most the claimed size by Lemma \textcolor{red}{4.6}.

(2) Apply (1) to each of $b_i$ and $c_i$ and take the product, taking into account the bound

$$
\int_{I} N(b)^{-1 \omega^{-1}} (-1+\xi \sqrt{-1}) \hat{\chi}(\xi) d\xi = N(b)^{-1 \omega^{-1}} O_{\chi}(1).
$$

This completes the proof. \hfill $\Box$

Apply the operation $(\log R)^{2m} \sum_{(b, \xi)} \prod_{i=1}^{m} \mu(b_i) \mu(c_i) \left( - \right) E((b, \xi); W, b)$ to the estimate of Lemma \textcolor{red}{4.8} (2). The left hand side becomes precisely \textcolor{red}{(28)}. The main term of the right hand side becomes

$$
(\log R)^{2m} \sum_{(b, \xi)} \int_{I^{2m}} F((b, \xi), \xi, \eta) d\xi d\eta, \quad \text{where}
$$

$$
F((b, \xi), \xi, \eta) := \left( \prod_{i=1}^{m} \mu(b_i) \mu(c_i) N(b_i)^{-1 \omega^{-1}} N(c_i)^{-1 \omega^{-1}} \hat{\chi}(\xi_i) \hat{\chi}(\eta_i) \right) E((b, \xi); W, b).
$$

We claim that we can interchange the sum and integral here. Indeed, by the convergence part of Proposition \textcolor{red}{4.7}, i.e., formula \textcolor{red}{(33)}, the sum $\sum_{(b, \xi)} F((b, \xi), \xi, \eta)$ converges absolutely and uniformly in $\xi$ and $\eta$ to a continuous function. Since $I$ is a bounded closed interval, our claim follows so that the value \textcolor{red}{(34)} equals \textcolor{red}{(30)}. \hfill $\Box$
The error term is at most the following, which we can bound again by Equation (33):

\[ O_{A,\chi,m}((\log R)^{2m-A}) \sum_{\langle b \rangle} \left( \prod_{i=1}^m N(b_i)^{\frac{1}{\log R} - N(c_i)^{\frac{1}{\log R}}} \right) E((b,\varepsilon);W,b) = O_{A,\chi,m}((\log R)^{2m-A} + O(1)). \]

The proof of Proposition 4.7 is now complete.

4.3. Intermission. Before proceeding further, let us recall basic facts from elementary calculus and the theory of the zeta function. The absolute constants \( c_i \) and \( C_i \) appearing below can be made explicit, but we do not seek to do so because their precise values are not important. Potentially big constants are denoted in upper case and potentially small ones in lower case. From §4.4 on, when we say some quantities should be small or large enough, we will be implicitly using these constants to specify the thresholds.

4.3.1. Some calculus. There is a positive real number \( c_1 > 0 \) such that for all \( \varepsilon \in \mathbb{C} \) with \( |\varepsilon| \leq c_1 \) one has

\[
e^\varepsilon = 1 + O(1) \cdot \varepsilon \quad \text{and} \quad \log(1 + \varepsilon) = 1 + O(1) \cdot \varepsilon
\]

with both \( O(1) \leq 2 \). (Actually one can take \( c_1 := 1/2 \), say.) Next, for real numbers \( A \geq 2 \) we have Taylor expansion

\[ A^{-1+\varepsilon} = A^{-1} + \frac{\log A}{A} \varepsilon + \frac{1}{2} \frac{\log(A)^2}{A} \varepsilon^2 + \cdots. \]

Noting that the positively valued function \( A \to \frac{\log A}{A} \) has bounded range, there are \( c_2 > 0 \) and \( C_3 > 0 \) such that for all \( A \geq 2 \) and \( \varepsilon \in \mathbb{C} \) with \( |\varepsilon| \leq \frac{c_2}{\log A} \), we have

\[ 1 - A^{-1+\varepsilon} = 1 - A^{-1} + O(1) \cdot \frac{\log A}{A} \varepsilon \]

with \( O(1) \leq C_3 \). It follows that

\[ \frac{1 - A^{-1+\varepsilon}}{1 - A^{-1}} = 1 + \frac{O(1)}{1 - A^{-1}} \frac{\log A}{A} \varepsilon = 1 + O(1) \frac{\log A}{A} \varepsilon \]

with a new \( O(1) \) constant \( \leq 2C_3 \). Also, since the function \( A \to \frac{\log A}{A} \) decreases for \( A \geq e \), for prime ideals \( p \subset \mathcal{O} \) with \( p \cap \mathcal{O} = \langle \pi \rangle \) we have

\[ \frac{\log N(p)}{N(p)} \leq \frac{\log N(\pi \mathcal{O})}{N(\pi \mathcal{O})} \]

at least if \( N(\pi \mathcal{O}) \geq 3 \). This inequality happens to be true even when \( N(\pi \mathcal{O}) = 2 \) thanks to the equality \( \frac{\log 2}{2} = \frac{\log 4}{4} \).

4.3.2. The zeta function. We need to recall the zeta function of \( \mathcal{O} \). For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), we set:

\[ \zeta_{\mathcal{O}}(s) := \prod_{\pi \in X} \left( 1 - \frac{1}{N(\pi)^s} \right)^{-1} = \sum_{a \in \text{Ideals}_{\mathcal{O}}} \frac{1}{N(a)^s}. \]

It is known that \( \zeta_{\mathcal{O}}(s) \) extends to a meromorphic function on \( \mathbb{C} \). Actually we know [8, Theorem 5.9, p.53] that there is a polynomial \( L(u) \in \mathbb{Z}[u] \) of degree \( 2g \), with \( L(q^{-1}) = |\text{Pic}^0(\mathcal{X})|/q^g \), such that:

\[ \zeta_{\mathcal{O}}(s) \cdot \prod_{\pi \in X_{\mathcal{O},X}} \left( 1 - \frac{1}{N(\pi)^s} \right)^{-1} = \frac{L(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}. \]

It follows that \( \zeta_{\mathcal{O}}(s) \) has a simple pole at \( s = 1 \) with positive residue, say \( \kappa_{\mathcal{O}} > 0 \):

\[ \zeta_{\mathcal{O}}(s) = \frac{\kappa_{\mathcal{O}}}{s-1} + O(1) \quad \text{as} \ s \to 1. \]
Moreover, by Weil’s Riemann Hypothesis for algebraic curves [8, Theorem 5.10, p.55], we know that all the roots of \( L(u) \) in \( \mathbb{C} \) have magnitude \( q^{-1/2} \). Chebotarëv’s density theorem [5.3] below is an important consequence of this.

By explicit computation we know \( \zeta_{\mathcal{O}}(s) = 1/(1 - q^{1-s}) \) (see [8, p.11]). From this it follows for integers \( i \geq 1 \):

\[
|\{ \text{maximal ideals } \pi \sigma \subset \sigma \mid N(\pi \sigma) = q^i \}| = q^i + O\left(\frac{q^{i/2}}{i}\right),
\]

see [8, Theorem 2.2, p.14]. This gives the following. The sums are over maximal ideals \( \pi \sigma \subset \sigma \) satisfying the indicated conditions:

\[
\sum_{N(\pi \sigma) > w} \frac{1}{N(\pi \sigma)^2} \leq C_4 \frac{1}{w \log_q w} \quad \text{and}
\]

\[
\sum_{N(\pi \sigma) \leq w} \frac{\log_q N(\pi \sigma)}{N(\pi \sigma)} \leq C_5 \log_q w
\]

for some \( C_4, C_5 > 0 \) and all \( w > 1 \).

4.4. Euler product. Now we compute the main term [30] using the Euler product presentation [31] and estimate [32].

We start with some detailed estimate of the Euler product. Recall that the latter estimate requires \( N(\pi \sigma) > w \) and that \( w \) be large enough. Assume \( w \) is large enough to match this requirement. Then by [32] and basic facts like \((1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_1 + \varepsilon_2)^{-1} = 1 + O(\varepsilon_1 \varepsilon_2)\), we have for \( N(\pi \sigma) > w \):

\[
E(\pi)(\xi, \eta, R, w, b) = \frac{1 - N(\pi \sigma)^{-1 + \frac{1}{\log R}(-1 + \xi_i \sqrt{-1})}}{1 - N(\pi \sigma)^{-2 + \frac{1}{\log R}(-2 + (\xi_i + \eta_i) \sqrt{-1})}} \cdot (1 + \mathcal{O}_r(\log(\pi \sigma)^{-2})).
\]

Take the product of (43) over all \( \pi \sigma \) with \( N(\pi \sigma) > w \). By the definition of the zeta function \( \zeta_\mathcal{O} \) in (39), we get the following. There, the symbol \( \Pi_p \) with \( N(\pi \sigma) \leq w \) means the product over prime ideals \( p \) of \( \sigma \) such that \( \pi \sigma := p \cap \sigma \) satisfies the indicated condition:

\[
E(\xi, \eta, R, w, b) = \prod_{i=1}^{m} \left( \frac{1 - N(p)^{-1 + \frac{1}{\log R}(-1 + \xi_i \sqrt{-1})}}{1 - N(p)^{-2 + \frac{1}{\log R}(-2 + (\xi_i + \eta_i) \sqrt{-1})}} \cdot \frac{\zeta(1 + \frac{1}{\log R}(2 - (\xi_i + \eta_i) \sqrt{-1}))}{\prod_{p \text{ with } N(p) \leq w} \left( 1 - N(p)^{-1 + \frac{1}{\log R}(-2 + (\xi_i + \eta_i) \sqrt{-1})} \right)} \right).
\]

Here the last factor has been obtained using (33), (36) via \( \exp \circ \log = \text{id} \) and (41) as follows:

\[
\prod_{N(\pi \sigma) > w} \left( 1 + \mathcal{O}_r(\log(\pi \sigma)^{-2}) \right) = \exp \left( \sum_{N(\pi \sigma) > w} \mathcal{O}_r(\log(\pi \sigma)^{-2}) \right) = \exp(\mathcal{O}_r(1/(w \log_q w))) = 1 + \mathcal{O}_r(1/(w \log_q w)),
\]

where for the last estimate we have to assume \( \mathcal{O}_r(1)/(w \log_q w) \) is small enough. Since we have \( m \) such factors, we get the factor \( 1 + m \mathcal{O}_r(1/w \log_q w) \).

Formula (40) can be written as \( \zeta_\mathcal{O}(1 + \varepsilon) = \frac{\mathcal{O}_\mathcal{O}(1)}{\varepsilon}(1 + O(\varepsilon)) \) with \( \varepsilon \in \mathbb{C} \) close to 0. Applying this to \( \varepsilon = \frac{1}{\log R}(1 - \xi_i \sqrt{-1}), \frac{1}{\log R}(1 - \eta_i \sqrt{-1}) \) and \( \frac{1}{\log R}(2 - (\xi_i + \eta_i) \sqrt{-1}) \) all of size \( O(\sqrt{\log R}) \),
we find that the product of zeta functions in \((44)\) has the following form when \(R\) is large enough:

\[
\left( \frac{1}{\kappa_{\mathcal{O}} \log R} \right)^m \cdot (1 + mO_{\mathcal{O}} \left( \frac{1}{\sqrt{\log R}} \right)) \cdot \prod_{i=1}^m \frac{1 - \xi_i \sqrt{-1}(1 - \eta_i \sqrt{-1})}{2 - (\xi_i + \eta_i) \sqrt{-1}}.
\]

We have to compute the products \(\prod_p\) with \(N(\pi \sigma) \leq w\) in \((44)\) as well. By \((36)\) we know for small complex numbers \(\varepsilon\):

\[
\prod_{p \text{ with } N(\pi \sigma) \leq w} (1 - N(p)^{-1 + \varepsilon}) = \prod_{p \text{ with } N(\pi \sigma) \leq w} (1 - N(p)^{-1}) \left( 1 + O(1) \frac{\log N(p)}{N(p)} \varepsilon \right).
\]

The product of the first factors is \(\frac{\varphi(W)}{N(W \mathcal{O})}\). For the second factors, by \((37), (42)\) and the fact that the number of prime ideals \(p\) over a given \(\pi \sigma\) is at most \(r\), for small \(\varepsilon\) we have:

\[
(\text{product of the second factors in } (45)) = \prod_{(\pi) \text{ with } N(\pi) \leq w} \left( 1 + O(1) \frac{\log N(\pi \sigma)}{N(\pi \sigma)} \varepsilon \right)^r
\]

We apply this to \(\varepsilon = \frac{1}{\log R}(-1 + \xi \sqrt{-1})\), with \(\xi = \xi_i\) or \(\eta_i\) and to \(\frac{1}{\log R}(-2 + (\xi_i + \eta_i) \sqrt{-1})\) which are of size \(O(\frac{1}{\sqrt{\log R}})\). It follows that if \(\frac{1}{\sqrt{\log R}}\) is smaller than \(\frac{w \log q}{\log R}\) times an absolute constant, then the product of the products \(\prod_p\) with \(N(\pi \sigma) \leq w\) in \((44)\) is of the form:

\[
\left( \frac{N(W \mathcal{O})}{\varphi(W)} \right)^m \left( 1 + 3mrO(1) \frac{\log q}{\log R} \right),
\]

with \(O(1)\) an absolute constant. By \((44) - (46)\), we get an estimate:

\[
(47) \quad E(\xi, \eta; R, w, b) = (1 + mO_{\mathcal{O}} \left( \frac{1}{\sqrt{\log R}} \right)) (1 + mO_{\mathcal{O}} \left( \frac{1}{w \log w} \right)) (1 + 3mrO(1) \frac{\log q}{\log R}) \left( \frac{1}{\kappa_{\mathcal{O}} \log R \varphi(W)} \right)^m \prod_{i=1}^m \frac{1 - \xi_i \sqrt{-1}(1 - \eta_i \sqrt{-1})}{2 - (\xi_i + \eta_i) \sqrt{-1}}.
\]

The error factor above is a \(1 + O_{m,r} \left( \frac{1}{w \log R} \right) + O_{\mathcal{O},r,m} \left( \frac{\log q}{\log R} \right)\).

We are ready to compute \((30)\). Let us recall for the convenience of reference:

\[
(30) := (\log R)^{2m} \int_{I^{2m}} E(\xi, \eta; R, w, b) \left( \prod_{i=1}^m \hat{\chi}(\xi_i) \hat{\chi}(\eta_i) \right) d\xi d\eta.
\]

**Proposition 4.9.** We have

\[
(48) \quad (\log R)^{2m} \int_{I^{2m}} E(\xi, \eta; R, w, b) \left( \prod_{i=1}^m \hat{\chi}(\xi_i) \hat{\chi}(\eta_i) \right) d\xi d\eta =
\]

\[
C \hat{\chi}^{- \frac{\log R N(W \mathcal{O})}{\kappa_{\mathcal{O}} \varphi(W)}} \left( \int_{I^{2m}} \prod_{i=1}^m \frac{1 - \xi_i \sqrt{-1}(1 - \eta_i \sqrt{-1})}{2 - (\xi_i + \eta_i) \sqrt{-1}} \hat{\chi}(\xi_i) \hat{\chi}(\eta_i) d\xi d\eta \right) \left( 1 + O_{m,r,\chi} \left( \frac{1}{w \log w} \right) \right).
\]

**Proof.** By substituting \((47)\) we get

\[
\text{L.H.S. of } (48) = \left( \frac{\log R N(W \mathcal{O})}{\kappa_{\mathcal{O}} \varphi(W)} \right)^m \left( \int_{I^{2m}} \prod_{i=1}^m \frac{1 - \xi_i \sqrt{-1}(1 - \eta_i \sqrt{-1})}{2 - (\xi_i + \eta_i) \sqrt{-1}} \hat{\chi}(\xi_i) \hat{\chi}(\eta_i) d\xi d\eta \right) \left( 1 + O_{m,r} \left( \frac{1}{w \log w} \right) + O_{\mathcal{O},r,m} \left( \frac{\log q}{\log R} \right) \right).
\]
Write $F(\xi, \eta) := \frac{(1-\xi, \sqrt{1-(1-\eta)^2})}{2-\xi(1+\eta)\sqrt{1}}\chi(\xi)\overline{\chi}(\eta)$ and $E(\xi, \eta) := \prod_{i=1}^{m} F(\xi, \eta_i)$ for short. The value above is estimated as:

\[
(49) \quad \left(\frac{\log R N(W)}{\varphi(\varphi(W))}\right)^m \left[\int_{I_2m} E(\xi, \eta) d\xi d\eta + \int_{I_2m} |E(\xi, \eta)| d\xi d\eta \right] + \int_{I_2m} |F(\xi, \eta)| d\xi d\eta (O_{m,r} \left(\frac{1}{w \log_q w}\right) + O_{\log R} \left(\frac{\log w}{\sqrt{\log R}}\right)) \right].
\]

We want to bound the integral $\int_{I_2m} |F(\xi, \eta)| d\xi d\eta$. By Lemma 4.6, we know for any $A > 0$

\[
\chi(\xi) = O_{X,A}((1 + |\xi|)^{-A})
\]

and therefore for any $A > 0$

\[
(50) \quad |F(\xi, \eta)| \leq \frac{1}{2} O_{X,A}((1 + |\xi|)^{-A}(1 + |\eta|)^{-A}).
\]

It follows that (by taking $A := 2$ for example)

\[
(51) \quad \int_{I_2} |F(\xi, \eta)| d\xi d\eta = O_{X}(1) \int_{\mathbb{R}^2} |F(\xi, \eta)| d\xi d\eta = O_{X}(1) \int_{\mathbb{R}^2} |(1 + |\xi|)^{-2}(1 + |\eta|)^{-2}| d\xi d\eta
\]

is a finite value. Hence $\int_{I_2m} |F(\xi, \eta)| d\xi d\eta = \prod_{i=1}^{m} \int_{I} |F(\xi, \eta_i)| d\xi d\eta_i = O_{X}(1)$ is also a finite value.

Next we consider the integral $\int_{I_2m} E(\xi, \eta) d\xi d\eta$. We want to replace $\int_{I_2m}$ by $\int_{\mathbb{R}^{2m}}$. Consider the following partition of the domain of integral:

\[
\mathbb{R}^{2m} = (I \cup (\mathbb{R} \setminus I))^{2m} = I^{2m} \sqcup \bigcup_{J} \Omega_J
\]

where $J$ runs through maps $\{1, \ldots, 2m\} \rightarrow (I, \mathbb{R} \setminus I)$ except the constant map into the one-point set $\{I\}$, and $\Omega_J$ denotes the corresponding product $\Omega_J := J_1 \times \cdots \times J_{2m} \subset \mathbb{R}^{2m}$.

**Lemma 4.10.** For any $J$ as above and any $A > 0$, we have the estimate $\left|\int_{\Omega_J} E(\xi, \eta) d\xi d\eta\right| \leq O_{X,A}((\log R)^{-A})$.

**Proof of Lemma 4.10** Since $J$ is not the constant function at $\{I\}$, there is an index $1 \leq k \leq 2m$ such that $J_k = \mathbb{R} \setminus I$. By symmetry, we may assume $k = 1$. By (50) (51), we have for any $A > 2$

\[
\left|\int_{\Omega_J} E(\xi, \eta) d\xi d\eta\right| \leq \int_{\Omega_J} d\xi d\eta \left[O_{X,A}((1 + |\xi|)^{-A}(1 + |\eta|)^{-2}) \prod_{i=2}^{m} O_{X}((1 + |\xi_i|)^{-2}(1 + |\eta_i|)^{-2})\right]
\]

\[
= \left(\int_{\mathbb{R} \setminus I} O_{X,A}((1 + |\xi|)^{-A}) d\xi\right) \cdot O_{X}(1)
\]

\[
= O_{X,A}\left((\sqrt{\log R})^{-A+1}\right).
\]

This completes the proof of Lemma 4.10.

By Lemma 4.10 we can proceed as:

\[
(52) \quad \int_{\mathbb{R}^{2m}} E(\xi, \eta) d\xi d\eta = \left(\int_{I_2m} + \sum_{J} \int_{\Omega_J} E(\xi, \eta) d\xi d\eta\right)
\]

\[
= \int_{I_2m} E(\xi, \eta) d\xi d\eta + (2^m - 1) O_{X,A}((\log R)^{-A})
\]

\[
= \int_{I_2m} E(\xi, \eta) d\xi d\eta + O_{X,A,m}(\log R)^{-A})
\]
for any \( A > 0 \).

Now that we have estimated the integrals in (49) in (51) and (52), we obtain for any
\( A > 0 \):

\[
\text{L.H.S. of (48)} = \left( \frac{\log R \mathcal{N}(W\mathcal{O})}{\kappa_\mathcal{O} \varphi_\mathcal{O}(W)} \right)^m \left[ \int_{\mathbb{R}^2m} E(\xi, \eta) d\xi d\eta + O_{X,A,m} \left( (\log R)^{-A} \right) + O_{m,r,\chi} \left( \frac{1}{w \log q w} \right) + O_{\mathcal{O},r,m,\chi} \left( \frac{\log w}{\sqrt{\log R}} \right) \right].
\]

If we set \( A = 1/2 \), the term \( O_{X,A,m} \left( (\log R)^{-A} \right) \) can be absorbed into \( O_{\mathcal{O},r,m,\chi} \left( \frac{\log w}{\sqrt{\log R}} \right) \). The main term \( \int_{\mathbb{R}^2m} E(\xi, \eta) d\xi d\eta = \left( \int_{\mathbb{R}^2} F(\xi, \eta) d\xi d\eta \right)^m \) can be evaluated by a standard Fourier analysis computation (e.g. [9, p.170] or [6, Lemma 6.29]):

\[
\int_{\mathbb{R}^2} \frac{(1 - \xi \sqrt{-1})(1 - \eta \sqrt{-1})}{2 - (\xi + \eta) \sqrt{-1}} \tilde{\chi}(\xi) \tilde{\chi}(\eta) d\xi d\eta = \int_0^{+\infty} \chi'(x)^2 dx =: C_\chi.
\]

We conclude that

\[
\text{L.H.S. of (48)} = \left( \frac{\log R \mathcal{N}(W\mathcal{O})}{\kappa_\mathcal{O} \varphi_\mathcal{O}(W)} \right)^m \left[ \mathcal{C}_\chi^m + O_{m,r,\chi} \left( \frac{1}{w \log q w} \right) + O_{\mathcal{O},r,m,\chi} \left( \frac{\log w}{\sqrt{\log R}} \right) \right].
\]

This completes the proof of Proposition 4.9. \( \square \)

Let us collect the computations we have done and finish the proof of the main result of this section.

Proof of Theorem 4.2. We wanted to evaluate up to error the average:

\[
\mathbb{E} \left( \prod_{i=1}^m \Lambda_{R,\chi}^a(W\phi_i(x) + b_i)^2 \middle| x \in B \right).
\]

By (22) and (28), it equals:

\[
\sum_{(b_i, c_i), i \in \text{Ideals}^a_G} (\log R)^{2m} \left( \prod_{i=1}^m \mu(b_i) \mu(c_i) \int_{\mathbb{R}} \mathcal{N}(b_i) \frac{1}{\log R} \tilde{\chi}(\xi_i) d\xi_i \right)
\]

\[
\cdot \int_{\mathbb{R}} \mathcal{N}(c_i) \frac{1}{\log R} \tilde{\chi}(\eta_i) d\eta_i \cdot \mathbb{E}(\mathbb{E}(\mathfrak{b}, \mathfrak{c}); W, b).
\]

By Proposition 4.7, this has been estimated as

\[
(\log R)^{2m} \int_{\mathbb{R}^2m} E(\xi, \eta, R, w, b) \left( \prod_{i=1}^m \tilde{\chi}(\xi_i) \tilde{\chi}(\eta_i) \right) d\xi d\eta + O_{A,X,m,r}((\log R)^{-A})
\]

for any \( A > 0 \). By Proposition 4.9, this is further estimated as:

\[
\left( \frac{\log R \mathcal{N}(W\mathcal{O})}{\kappa_\mathcal{O} \varphi_\mathcal{O}(W)} \right)^m \left[ 1 + O_{m,r,\chi} \left( \frac{1}{w \log q w} \right) + O_{\mathcal{O},r,m,\chi} \left( \frac{w \log w}{\sqrt{\log R}} \right) \right] + O_{A,X,m,r}((\log R)^{-A}).
\]

By setting \( A := 1 \), the last error term can be absorbed in \( O_{\mathcal{O},m,r,\chi} \left( \frac{w \log w}{\sqrt{\log R}} \right) \) in the parentheses. (Note that \( \mathcal{N}(W\mathcal{O}) / \varphi_\mathcal{O}(W) \leq 1 \) regardless of the specific \( W \).)

The proof of Theorem 4.2 is thus completed. \( \square \)

5. Chebotarëv and the end of proof

As always, let \( \mathcal{O} \) continue to be a Dedekind domain finitely generated over \( \mathbb{F}_p \). We restate Theorem 1.2 in a slightly broader generality. In the number field case \( [6] \), the extra generality allowed one to prove a constellation theorem for prime-valued points on a binary quadratic form \( ax^2 + bxy + cy^2 \) over \( \mathbb{Z} \).

Definition 5.1. Let \( a \subset \mathcal{O} \) be a non-zero ideal. Let us define the set \( \mathcal{P}_a \) of prime elements of \( \mathcal{O} \) by

\[
\mathcal{P}_a := \{ \alpha \in a \mid a/\alpha \mathcal{O} \cong \mathcal{O}/p \text{ as an } \mathcal{O}-\text{module for some maximal ideal } p \}.
\]
Note that $\mathcal{P}_a$ is exactly the set of prime elements of $O$ because an isomorphism $O/\alpha O \cong O/p$ of $O$-modules forces the equality $\alpha O = p$.

Now we can state our main theorem in its proper generality.

**Theorem 5.2.** Let $O$ be a Dedekind domain finitely generated over $\mathbb{F}_p$, and $\alpha \subset O$ as in Proposition 2.1. Let $a \subset O$ be a non-zero ideal and $S \subset a$ a finite subset of it. Then any subset $A \subset \mathcal{P}_a$ of positive relative density contains a non-trivial $\sigma$-homothetic copy of $S$.

For the proof, we need to recall Chebotarëv’s density theorem. This is by far the deepest input from algebraic geometry in this work. Let $\mathbb{X}$ be a complete non-singular geometrically irreducible curve over $\mathbb{F}_q$. Let $\text{Pic}(\mathbb{X}) \xrightarrow{\phi} G$ be a finite quotient of $\text{Pic}(\mathbb{X})$. The restriction of the degree map $\deg : \ker(\phi) \to \mathbb{Z}$ is necessarily non-trivial. Let $D \geq 1$ be the order of its cokernel. (Let us always use $D$ in this sense when $G$ is understood.) It follows that we have the degree map of the following form:

$$\deg : G \to \mathbb{Z}/D\mathbb{Z}.$$

The next result is a consequence of Weil’s Riemann Hypothesis for algebraic curves over finite fields.

**Theorem 5.3 (Chebotarëv’s density theorem).** Let $G$ be a finite quotient of $\text{Pic}(\mathbb{X})$ and $P \in G$. Then for positive integers $n > 0$, we have the following cardinality estimate:

$$\left| \left\{ x \in \mathbb{X} \mid \deg(x) = n \text{ in } \mathbb{Z} \text{ and } [x] = P \text{ in } G \right\} \right| = \begin{cases} \frac{D}{|G|} \frac{L}{q^n} + O_X \left( \frac{q^{n/2}}{n} \right) & \text{if } n \equiv \deg(P) \text{ in } \mathbb{Z}/D\mathbb{Z}, \\ O_X \left( \frac{q^{n/2}}{n} \right) & \text{else}. \end{cases}$$

**Proof.** A slightly weaker version of this statement can be found in [8, Theorem 9.13B, p.125]. Our statement can be obtained by the same argument by using [8, Proposition 9.21, p.137] and [8, Theorem 9.24, p.141] in place of [8, Theorem 9.16B, p.129].

Via the transition of parameters from the degree $n = \deg(x)$ to the norm $L = q^n =: N(x)$, we get the following:

**Corollary 5.4.** For all sufficiently large positive numbers $L > 0$, we have the following estimate:

$$\frac{1}{q^n} \frac{D}{|G|} \frac{L}{\log_q L} \leq \left| \left\{ x \in \mathbb{X} \mid N(x) \leq L \text{ and } [x] = P \text{ in } G \right\} \right| \leq \frac{2D}{|G|} \frac{L}{\log_q L}.$$

The following is clear from definitions.

**Lemma 5.5.** If $\alpha \in a$ is a prime element, then:

$$0 \leq \Lambda_{\mathcal{R}_a}(\alpha) \leq \log R.$$

The right-hand ‘$\leq$’ is an equality if $N(\alpha) \geq N(a) R$.

When we apply Lemma 5.5 it will be convenient to have a bound for the number of elements $\alpha \in a$ with $N(\alpha) < N(a) R$. For $L \geq 1$, let us write

$$a(L) := \{ \alpha \in a \mid N(\alpha) \leq L \}.$$

**Corollary 5.6.** For positive real numbers $L > 1$ and $M > 1$, we have the bound

$$|\mathcal{P}_a \cap a(N(a)L) \cap a \leq M| = O_\mathcal{O} \left( (\log M)^{n-1} \frac{L}{\log_q L} \right).$$

**Proof.** Given $\alpha \in \mathcal{P}_a \cap a(N(a)L)$, the ideal $p := \alpha \alpha^{-1} \subset O$ is a prime ideal whose class in $\text{Pic}(O)$ equals $-a$ and norm equals $N(\alpha)/N(a) (\leq L)$. Two elements $\alpha, \alpha'$ give the same $p$ if and only if they are associate to each other. By Corollary 5.4 it follows that there are at most $2 \frac{D}{|\text{Pic}(O)|} \frac{L}{\log_q L}$ associate classes inside $\mathcal{P}_a \cap a(N(a)L) \cap a \leq M$. By Proposition 2.7 applied to $a \leq M \subset O \leq M$, each associate class contains at most $O_\mathcal{O}(\log M)^{n-1}$ elements. This completes the proof.
5.1. Proof of the main result. Now we are ready to prove our main result.

Proof of Theorem 5.2. As always set $X = \text{Spec}O$ and let $\overline{X}$ be its non-singular compactification. Let $\mathcal{O} = \mathbb{F}_q[t] \subset O$ be as in Proposition 2.1. Recall from (18) the definitions of some integer quantities:

$$
n = |\overline{X} \setminus X|,
$$
$$
d_0 = \text{lcm}\{\text{deg}(v) \mid v \in \overline{X} \setminus X\},
$$
$$
d = v(t) \quad (\text{independent of } v \in \overline{X} \setminus X),
$$
$$
r = nd = \text{rank}_\mathcal{O}(O),
$$
$$
g = \text{the genus of } \overline{X}.
$$

Also, let $\chi : \mathbb{R} \to [0, 1]$ be a compactly supported $C^\infty$ function as in Definition 4.1. Take a norm-length compatible $O^*$-fundamental domain $D$ of $a \setminus \{0\}$ which exists thanks to Proposition 2.6. This means that there is a small positive number $c_\mathcal{O} > 0$ such that the following inclusion holds for all $M > 0$:

$$a(c_\mathcal{O}M^n) \cap D \subset a_{\leq M}.
$$

Let $\delta > 0$ be any positive number smaller than the upper density $\bar{\delta}_{\mathcal{O}}(A)$ of $A \subset \mathcal{P}_a$. Let $\delta_1$ be the positive number defined by (64) below (which is not very motivating) depending only on the preliminary data $O, a, \chi, S, \delta$ and $D$ that are already available. Using the relative Szemerédi theorem 3.2 we fix the following positive numbers:

$$\rho := \rho(a, a, S, \delta_1), \quad \gamma := \gamma(a, a, S, \delta_1).
$$

Let $w > 1$ be a large integer to be specified in a moment and $R > 1$ be a large real number to be specified much later, satisfying

$$R > w^r.
$$

Recall $k := |S|$ and consider the maps in Definition 3.1: $\psi_{S,j} \circ \text{res}_\omega : \mathfrak{a}^k \oplus \mathfrak{a}^k \to \mathcal{O}$ for $1 \leq j \leq k$ and $\omega : \{1, \ldots, j, \ldots, k\} \to \{\pm\}$. The number of indices $(j, \omega)$ is

$$m := k2^{k-1}.
$$

It is routine to check that this family of maps satisfies the hypothesis of Theorem 4.2; see [6] Lemma 5.8 for details. Hence if $w$ is large enough depending on $S \subset a$ and $r$, and if $R$ is large enough depending in addition on $\chi$ and $w$, then the error terms in Theorem 4.2 can be made smaller than $\rho$:

$$O_{m,r} \left(\frac{1}{w \log w}\right) + O_{\chi,m,r,\mathcal{O}} \left(\frac{\log w}{\sqrt{\log R}}\right) < \rho.
$$

We fix such $w$. The value of $R$ is yet to be fixed.

Set $W := \prod_{N(\pi) \leq w} \pi \in \mathcal{O}$, where the product $\prod_{N(\pi) \leq w}$ is taken over the monic irreducible polynomials $\pi$ satisfying the indicated condition.

Let $e > 1$ be a large positive integer to be specified toward the end of the proof. We consider the following positive real numbers determined by $e$:

$$M = q^e, \quad L = \frac{c_\mathcal{O}M^n}{N(a)}, \quad N = M/\|W\|, \quad R = N^{1/(2m+1)}.
$$

Since $\delta < \bar{\delta}_{\mathcal{O}}(A)$ by our choice, for infinitely many $e \in \mathbb{N}$ the following inequality holds:

$$|A \cap a_{\leq M}| > \delta \cdot |\mathcal{P}_a \cap a_{\leq M}|.
$$

By (54), the set $a_{\leq M}$ contains $a(N(a)L) \cap \mathcal{D}$. Hence the right hand side is at least:

$$\geq \delta \cdot |\mathcal{P}_a \cap a(N(a)L) \cap \mathcal{D}|.
$$

For every element $a \in \mathcal{P}_a \cap a(N(a)L) \cap \mathcal{D}$, the ideal $\mathfrak{a}a^{-1} \subset \mathcal{O}$ is a prime ideal with norm $N(\mathfrak{a})/N(a)$ and whose class in $\text{Pic}(\mathcal{O})$ equals $-\mathfrak{a}$. Therefore the association $\mathfrak{a} \mapsto \mathfrak{a}a^{-1}$ establishes a bijection from $\mathcal{P}_a \cap a(N(a)L) \cap \mathcal{D}$ to the following set:

$$\left\{p \in |\text{Spec}(\mathcal{O})| \mid N(p) \leq L \text{ and } |p| + |a| = 0 \text{ in } \text{Pic}(\mathcal{O})\right\}.$$
Its cardinality is already estimated in Corollary \ref{cor5.4}. As a result we get:

\begin{equation}
|A \cap a_{\leq M}| > \delta \cdot \frac{1}{q^L} \frac{D}{|\Pic(O)|} \frac{L}{\log_q L} =: \delta \cdot C_O \frac{L}{\log_q L},
\end{equation}

where we have written $C_O := \frac{D}{|\Pic(O)|}$ for short.

Since we want to use Lemma \ref{lem5.5} later, we want to consider only those elements with ideal norm $> N(a)R$. By Corollary \ref{cor5.6} we know

$$|a(N(a)R) \cap a_{\leq M}| = O_\mathcal{O} \left( \left( \log M \right)^n \frac{R}{\log R} \right).$$

By \ref{lem5.7} the right hand side has the order of $(\log L)^n \frac{L^{1/(2m+1)n}}{\log R}$ or less as a function of $e$, which is smaller than the right-most term of \ref{eq59}. Hence by replacing $\delta$ by a slightly smaller value if necessary, we see that the following variant of \ref{eq59} is valid:

\begin{equation}
|A \cap (a_{\leq M} \setminus a(N(a)R))| > \delta \cdot C_O \frac{L}{\log_q L}.
\end{equation}

\textbf{Lemma 5.7.} For every $\alpha \in \mathcal{P}_a \setminus a(N(a)R)$, the residue class of $\alpha$ in $a/Wa$ generates it as an $\mathcal{O}$-module.

\textbf{Proof of Lemma.} By Chinese Remainder Theorem for $\mathcal{O}$-modules, the assertion is equivalent to that $\alpha \in a \setminus (\bigcup_{p|W} p\alpha)$. Suppose there is a $p|W$ such that $\alpha \in p\alpha$. Since $\alpha \in \mathcal{P}_a$ it follows that $\alpha = \alpha p$. By the definition of $W$ the ideal $\pi \sigma = p \cap \sigma$ has norm $\leq w$. It follows that $N(p) \leq w^r$ and hence

$$N(a)R < N(\alpha) \leq N(a)w^r.$$ 

This contradicts the assumption \ref{eq55}. This proves Lemma \ref{lem5.7}. \hfill \Box

As $a$ is a rank 1 projective $\mathcal{O}$-module, we have an isomorphism of $\mathcal{O}$-modules $a/Wa \cong \mathcal{O}/W\mathcal{O}$. The generators of $a/Wa$ correspond to the elements of $(\mathcal{O}/W\mathcal{O})^*$. It follows that:

$$\{\alpha \in a/Wa \mid \alpha \text{ generates } a/Wa\} = \varphi(\mathcal{O})(W).$$

By Lemma \ref{lem5.7} we see that the set $A \cap (a_{\leq M} \setminus a(N(a)R))$ decomposes into the sum of $\varphi(\mathcal{O})(W)$ disjoint subsets according to the mod $W$ classes. By the pigeonhole principle, it follows that for some residue class $[b] \in a/Wa$ we have

\begin{equation}
|\{\alpha \in A \cap (a_{\leq M} \setminus a(N(a)R)) \mid \alpha = [b] \text{ in } a/Wa\}| \geq \frac{1}{\varphi(\mathcal{O})(W)} \cdot \text{(R.H.S of \ref{eq60})}.
\end{equation}

Choose one such $[b] \in a/Wa$. Let us fix a $C > 0$ depending only on $a$ and $W$ such that the projection $a_{\leq C} \to a/Wa$ is surjective (which exists because the target is a finite set) and choose a lift $\tilde{b} \in a_{\leq C}$ of $[b]$. Let $\text{Aff}_{W,b} : a \to a$ be the affine linear map $\alpha \mapsto W\alpha + b$. Set

$$B := \text{Aff}_{W,b}^{-1}(A \setminus a(N(a)R)) \subset a.$$ 

We have the following inclusion if $e > 1$ is large enough:

\begin{equation}
\text{Aff}_{W,b}(a) \cap (a_{\leq M}) \subset \text{Aff}_{W,b}(a_{\leq N}).
\end{equation}

Indeed, suppose $\alpha \in a$ satisfies $\|W\alpha + b\| \leq M = N\|W\|$. Since $\|b\| \leq C < N\|W\|$ for $e$ large enough, by the ultrametricity of $\|\cdot\|$ this implies $\|W\alpha\| \leq N\|W\|$. We get $\|\alpha\| \leq N$ because $W$ is a multiplicative element for the norm $\|\cdot\|$. This proves the inclusion \ref{eq62}.

In particular the set on the left hand side of \ref{eq61} is contained in $\text{Aff}_{W,b}(B \cap a_{\leq N})$.

Having fixed $W$ and $b$, we can finally define a function $\lambda : a \to \mathbb{R}_{\geq 0}$ by the formula:

\begin{equation}
\lambda(\alpha) := \frac{1}{\log R} \frac{\varphi(\mathcal{O})(W)}{\|W\|^n} \frac{C_\alpha}{C_\chi} A_{R,\lambda}^a(W\alpha + b)^2.
\end{equation}

By \ref{eq66} the function $\lambda$ is $(R^{2m}/q, \rho, S, \sigma)$-pseudorandom. Let us verify the other hypotheses in the Szemerédi theorem \ref{thm3.2}. 

By Lemma 5.5, the restriction of $\lambda$ to $B$ equals the constant function $\frac{\varphi(W) \kappa_\mathcal{O}}{||W||^\alpha} \log R$. This together with (60) and (62) implies:

$$\mathbb{E} \left( \lambda 1_B \mid a \leq N \right) \geq \frac{1}{\varphi(W)} \delta \kappa_\mathcal{O} \frac{L}{\log_q L} \left( \frac{\varphi(W) \kappa_\mathcal{O}}{||W||^\alpha} \log R \right) / |a \leq N|.$$ 

We have $|a \leq N| \leq M/e^{n / ||W||^\alpha} N(a) q^{\gamma - 1}$ (which is an equality if $N$ happens to be a power of $q^\delta$). By the definition (57) of our parameters we get for all sufficiently large $e > 1$:

$$\geq \frac{1}{2} \delta \kappa_\mathcal{O} c \rho q^{\gamma - 1} \log(q) \frac{1}{C_N} \log R \geq \delta_1.$$ 

This establishes one of the two requirements in the relative Szemerédi theorem [3.2].

We have to establish one more inequality to invoke the relative Szemerédi theorem. By Lemma 5.5 we have $\lambda^k 1_B \leq \text{const.} \cdot (\log R)^{2k}$ where the constant comes from the coefficient in the definition of $\lambda$ in (63). So:

$$\mathbb{E} \left( \lambda^k 1_B \mid \mathcal{O} \leq N \right) \leq \text{const.} \cdot \left( \frac{1}{2m + 1} \log N \right)^{2k},$$

which is $< \gamma N$ for $e$ sufficiently large.

Now fix $e$ so that it satisfies (58) and is large enough to make all the above inequalities true. We can apply the relative Szemerédi theorem [3.2] to the current situation by (64), (65), and the pseudorandomness of $\lambda$. It follows that $B$ contains an $e$-homothetic copy of $S$. Sending it by the affine $e$-linear map $\text{Aff}_{W,b} : B \to A$, we get an $e$-homothetic copy of $S$ in $A$. This completes the proof of Theorem 5.2. \[\square\]

**Remark 4.** The above proof actually shows a finitary version of the theorem as in [6, Theorem A] because the dependence of the threshold for $e$ on the set $A$ is via its density $\delta$ (though of course the specific value of $e$ should be determined depending on $A$ to ensure (58)).

**Remark 5.** The assumption that $A$ has positive upper density in $\mathcal{P}_a$ was used solely at (58). It follows that we could have assumed more directly that $A \subset \mathcal{P}_a$ satisfies an inequality of the form:

$$|A \cap a \leq M| > \text{const.} \cdot \frac{M^n}{\log M}$$

for arbitrarily large $M$, with the positive constant depending only on $a$ and $A$. See [6, §§8-9] for a fully axiomatic treatment in the number field context. In fact, not surprisingly, this inequality is equivalent to $A$ having positive upper density in $\mathcal{P}_a$; see [6, Proposition 8.14] for the arguments in the number field case, which is also valid here.

### 5.2. Non-normal case

It is routine to deduce Theorem 5.2 from Theorem 5.1. Let $\mathcal{O}_0$ be an integral domain finitely generated over $\mathbb{F}_q$ and of transcendence degree 1. Let $\mathcal{O}$ be its normalization. By Theorem 5.2 and Remark 5, it suffices to show:

**Proposition 5.8.** The following inequality holds for infinitely many $M \in \mathbb{N}$:

$$|\mathcal{P}_{\mathcal{O}_0} \cap \mathcal{O} \leq M| > \text{const.} \cdot \frac{M^n}{\log M}$$

with the positive constant depending only on $\mathcal{O}_0$.

We give only sketches. See also [6, §10] for a detailed account in the setting of number fields. Let $f$ be the conductor:

$$f := \{ \alpha \in \mathcal{O} \mid \alpha \mathcal{O} \subset \mathcal{O}_0 \},$$

which is an ideal of $\mathcal{O}$ contained in $\mathcal{O}_0$. Let $\mathcal{P}_f$ be a temporary notation for the set of prime elements of $\mathcal{O}$ coprime to $f$, which is $\mathcal{P}_{\mathcal{O}}$ minus finitely many associate classes.
One shows that the elements of $\mathcal{P}_{O_0}$ are precisely those elements of $\mathcal{P}_{O}^f$ which are contained in $O_0$. More explicitly, we have the following cartesian diagram:

\begin{equation}
\begin{array}{ccc}
\mathcal{P}_{O_0} & \subset & \mathcal{P}_{O}^f \\
\downarrow & & \downarrow p \\
(\mathcal{O}/f)^* & \subset & (\mathcal{O}/f)^*
\end{array}
\end{equation}

Set $O_f^* := \{ f \in O^* \mid f \mod f = 1 \text{ in } O/f \}$. It is a subgroup of $O^*_f$ which is of finite index in $O^*$. Let $D \subset O \setminus \{0\}$ be a norm-length compatible $O^*_f$-fundamental domain whose existence easily follows from Proposition 2.6. It suffices to show the inequality (66) with $\mathcal{P}_{O_0} \cap D$ in place of $\mathcal{P}_{O_0}$. By the norm-length compatibility of $D$, we are reduced to showing the following inequality for infinitely many $L \in \mathbb{N}$:

$$|\mathcal{P}_{O_0} \cap D \cap \mathcal{O}(L)| > \text{const.} \frac{L}{\log_q L},$$

with the constant depending only on $O_0$ and $D$. Hence it suffices to show the $\alpha_0 = 1$ case (say) of the following claim:

**Claim 5.9.** For each $\alpha_0 \in (\mathcal{O}/f)^*$, write $\mathcal{P}_{O_0,\alpha_0}^f := p^{-1}(\alpha_0)$ where $p$ is the vertical map in diagram (67). Then the following inequality holds for all sufficiently large $L > 1$:

$$|\mathcal{P}_{O_0,\alpha_0} \cap D \cap \mathcal{O}(L)| > \text{const.} \frac{L}{\log_q L},$$

with the positive constant depending only on $f$ and $O$.

Recall the definition of the Picard group $\text{Pic}(\mathcal{O}, f)$ with modulus $f$ as a quotient of a free abelian group:

$$\text{Pic}(\mathcal{O}, f) = \frac{\mathbb{Z}[\mathcal{X} \setminus \text{Spec}(\mathcal{O}/f)\cup \{(\alpha) \mid \alpha \in \mathbb{F}_q(\mathcal{X})_f^*\}]}{\{\mathcal{O}/f \cap \mathcal{O}(\alpha) \mid \alpha \in \mathbb{F}_q(\mathcal{X})_f^*\}},$$

where $\mathbb{F}_q(\mathcal{X})_f^*$ is the subgroup of $\mathbb{F}_q(\mathcal{X})_f$ consisting of $\alpha$ which are regular around $\text{Spec}(\mathcal{O}/f)$ and are equal to 1 in $\mathcal{O}/f$. Given an element $\alpha_0 \in (\mathcal{O}/f)^*$, consider a lift $\tilde{\alpha}_0 \in \mathcal{O}$ and its divisor $(\tilde{\alpha}_0)$. Its class in $\text{Pic}(\mathcal{O}, f)$ does not depend on the choice of $\tilde{\alpha}_0$ so we get a well-defined class $[(\alpha_0)] \in \text{Pic}(\mathcal{O}, f)$. Consider the following set of prime ideals:

$$\text{Spec}(\mathcal{O})_{\alpha_0} := \{ p \in \text{Spec}(\mathcal{O}) \mid p = \tilde{\alpha}_0 \mathcal{O} \text{ for some lift } \tilde{\alpha}_0 \in \mathcal{O} \text{ of } \alpha_0 \}$$

$$= \{ p \in \text{Spec}(\mathcal{O}) \mid [p] = [(\alpha_0)] \text{ in } \text{Pic}(\mathcal{O}, f) \}.$$

The obvious map $\mathcal{P}_{O_0,\alpha_0}^f \cap D \to \text{Spec}(\mathcal{O})_{\alpha_0}; \alpha \mapsto \alpha \mathcal{O}$ is a bijection.

The Chebotarëv Density Theorem 5.3 holds with $\text{Pic}(\mathcal{X}, f)$ replaced by $\text{Pic}(\mathcal{X}, f)$ with the same proof because the result [8 Theorem 9.24, p.141] we cited is stated in this generality. Thus for every finite quotient $G \in \text{Pic}(\mathcal{X}, f)$, an element $P \in G$ and $n > 0$, we have:

$$\left| \{ x \in \mathcal{X} \setminus \text{Spec}(\mathcal{O}/f) \mid \deg(x) = n \text{ and } [x] = P \text{ in } G \} \right|$$

$$= \begin{cases} \frac{D}{|G|} q^n + \mathcal{O}_\mathcal{X} \left( \frac{q^{n/2}}{n} \right) & \text{if } n \equiv \deg(P) \text{ in } \mathbb{Z}/D\mathbb{Z}, \\
\mathcal{O}_\mathcal{X} \left( \frac{q^{n/2}}{n} \right) & \text{else.} \end{cases}$$

We apply this to $G = \text{Pic}(\mathcal{O}, f)$ and its element $[(\alpha_0)]$. It follows for $n \equiv \deg(\alpha_0)$ in $\mathbb{Z}/D\mathbb{Z}$, we have

$$\left| \{ \alpha \in \mathcal{P}_{O_0,\alpha_0}^f \cap D \mid \mathcal{N}(\alpha) = q^n \} \right| = \frac{D}{|\text{Pic}(\mathcal{O}, f)|} q^n + \mathcal{O}_\mathcal{X} \left( \frac{q^{n/2}}{n} \right).$$

This proves Claim 5.9 and hence Proposition 5.8.
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References

[1] M. F. Atiyah, I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Company, Inc.; Massachusetts, 1969, ISBN 978-0201407518

[2] D. Conlon, J. Fox, Y. Zhao, A relative Szemerédi theorem, Geom. Funct. Anal. 25 (3) (2015) 733–762. https://doi.org/10.1007/s00039-015-0324-9

[3] B. J. Green, T. Tao, The primes contain arbitrarily long arithmetic progressions, Ann. of Math. 167(2) (2008), 481–547. https://doi.org/10.4007/annals.2008.167.481

[4] Benjamin Green, Terence Tao, Linear equations in primes, Ann. of Math. 171 (2010), 1753–1850. https://doi.org/10.4007/annals.2010.171.1753

[5] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics 52, Springer Science+Business Media, Inc.; New York, 1977. https://doi.org/10.1007/978-1-4757-3849-0

[6] Wataru Kai, Masato Mimura, Akihiro Munemasa, Shin-ichiro Seki and Kiyoto Yoshino, Constellations in prime elements of number fields, preprint, 2020. https://arxiv.org/abs/2012.15669

[7] Thài Hoàng Lê, Green–Tao theorem in function fields, Acta Arithmetica 147 (2011), 129–152. https://doi.org/10.4064/aa147-2-3

[8] Michael Rosen, Number Theory in Function Fields, Graduate Texts in Mathematics 210, Springer-Verlag New York; New York, 2002. https://doi.org/10.1007/978-1-4757-6046-0

[9] T. Tao, The Gaussian primes contain arbitrarily shaped constellations, J. Anal. Math. 99 (2006), 109–176. https://doi.org/10.1007/BF02789444

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