Canonical Coset Parameterization and the Bures Metric of the Three-level Quantum Systems

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Abstract
An explicit parameterization for the state space of an $n$-level density matrix is given. The parameterization is based on the canonical coset decomposition of unitary matrices. We also compute, explicitly, the Bures metric tensor over the state space of two- and three-level quantum systems.

Keywords: Bures metric; Coset decomposition; Density matrices; Three-level systems
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1 Introduction
In recent years, the Riemannian Bures metric [1] has become an interesting subject for the understanding of the geometry of quantum state space. It is the quantum analog of Fisher information in classical statistics, i.e. in the subspace of diagonal matrices it induces the statistical distance [2]. The Bures measure is monotone in the sense that it does not increase under the action of completely positive, trace preserving maps [3]. It is, indeed, minimal among all monotone metrics and its extension to pure state is exactly the Fubini-Study metric [3]. The Bures distance between any two mixed states $\rho_1$ and $\rho_2$ is a function of their fidelity $F(\rho_1, \rho_2)$ [4, 5]

$$d_B(\rho_1, \rho_2) = \sqrt{2 - 2F(\rho_1, \rho_2)}, \quad F(\rho_1, \rho_2) = \left[\text{Tr}\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right)\right]^2.$$ (1)

Fidelity allows one to characterize the closeness of the pair of mixed states $\rho_1$ and $\rho_2$, so, it is an important concept in quantum mechanics, quantum optics and quantum information theory. An explicit formula for the infinitesimal Bures distance between $\rho$ and $\rho + d\rho$ was found by Hübner [6]

$$d_B(\rho, \rho + d\rho)^2 = \frac{1}{2} \sum_{i,j=1}^{n} \frac{|\langle i |d\rho| j \rangle|^2}{\lambda_i + \lambda_j},$$ (2)

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where $\lambda_j$ and $|j\rangle$, $(j = 1, 2, \ldots, n)$ represent eigenvalues and eigenvectors of $\rho$, respectively. Dittmann has derived several explicit formulas, that do not require any diagonalization procedure, for Bures metric on the manifold of finite-dimensional nonsingular density matrices [7, 8], for instance

$$d_B(\rho, \rho + d\rho) = \frac{1}{4} \Tr \left[ d\rho d\rho + \frac{1}{|\rho|} (d\rho - \rho d\rho) (d\rho - \rho d\rho) \right],$$

and

$$d_B(\rho, \rho + d\rho) = \frac{1}{4} \Tr \left[ d\rho d\rho + \frac{3}{(1 - \Tr \rho^2)} (d\rho - \rho d\rho) (d\rho - \rho d\rho) ight] + \frac{3|\rho|^2}{(1 - \Tr \rho^2)} ((d\rho - \rho^{-1} d\rho) (d\rho - \rho^{-1} d\rho)),$$

for nonsingular $2 \times 2$ and $3 \times 3$ density matrices, respectively.

The probability measure induced by the Bures metric in the space of mixed quantum states has been defined by Hall [9]. The question of how many entangled or separable states there are in the set of all quantum states is considered by Zyczkowski et al in [10, 11]. Sommers et al [12] have computed the volume of the $(n^2 - 1)$-dimensional convex set and $(n^2 - 2)$-dimensional hyperarea of the density matrices of an $n$-level quantum system. In a considerable work, Slater investigated the use of the volume elements of the Bures metric as a natural measure over the $(n^2 - 1)$-dimensional convex set of $n$-level density matrices, to determine or estimate the volume of separable states of the pairs of qubit-qubit [13, 14] and qubit-qutrit [15, 16]. Very recently [17], Slater made use of the Bloore parameterization [18] of density matrices in order to obtain the Hilbert-Schmidt volumes of separable subsets for the two qubit system.

The state space of an $n$-level quantum system is identified with the set of all $n \times n$ Hermitian positive semidefinite complex matrices of trace unity, and comprise $(n^2 - 1)$-dimensional convex set. Due to considerable interest in the use of density matrices, a lot of work has been devoted to describe and parameterize density matrices. Any density matrix of an $n$-level system can be expanded in terms of orthogonal generators $\lambda_i$ of $SU(n)$ as [19]

$$\rho = \frac{1}{n} \left( I_n + \frac{n(n-1)}{2} \overrightarrow{\nu} \cdot \lambda \right),$$

where $\overrightarrow{\nu} = (\nu_1, \nu_2, \ldots, \nu_{n^2-1})$ is a real vector and Lie algebra generators $\lambda_i$ are normalized as $\Tr (\lambda_i \lambda_j) = 2 \delta_{ij}$. The above representation is the generalization of the Bloch or coherence vector representation for two-level systems and gives one of the possible descriptions of a state on the basis of the actual measurements which is an important task both from experimental and theoretical viewpoint [20]. The region of Bloch vector $\overrightarrow{\nu}$ which represents a physical density matrix have been found in [20, 21]. An investigation of the geometrical aspects of the Bloch vector space from the spherical coordinate point of view is also made by Kimura et al in [22].

Boya et al [23] have shown that the mixed state density matrices for $n$-level systems can be parameterized in terms of squared components of an $(n-1)$-sphere and unitary matrices. By using the Euler angle parameterization of $SU(3)$ group [24],

$$U = e^{i\lambda_1} e^{i\lambda_2} e^{i\lambda_3} e^{i\lambda_4} e^{i\lambda_5} e^{i\lambda_6} e^{i\lambda_2} e^{i\lambda_3} e^{i\lambda_4} e^{i\lambda_5} e^{i\lambda_6} / \sqrt{3},$$

where $\lambda_i$ are Gell-Mann matrices, Byrd et al [25] have presented a parameterization for eight-dimensional state space of three-level system as

$$\rho = VDV^\dagger,$$
where $D$ is a diagonal density matrix (with two independent eigenvalues), and $V \in SU(3)$ is given by

$$V = e^{i\lambda_1 \alpha}e^{i\lambda_2 \beta}e^{i\lambda_3 \gamma}e^{i\lambda_4 \delta}e^{i\lambda_5 \epsilon}e^{i\lambda_6 \theta}.$$  

They also gave Bures measure on the space as the product of the measure on the space of eigenvalues and the truncated Haar measure on the space of unitary matrices. An Euler angle-based parameterization for density matrices of four-level system is also introduced in [26]. A generalized Euler angle parametrization for $SU(n)$ and $U(n)$ groups has given by Tilma et al [27, 28]. Tilma et al [29] have also used the parameterization for four-level system (two qubit system) in order to study the entanglement properties of the system.

In a comprehensive analysis [30], Życzkowski et al analyzed the geometrical properties of the set of mixed quantum states for an arbitrary $n$-level system and classified the space of density matrices. Dità [31] has provided a parameterization for general Hermitian operators of $n$-level quantum systems. The parameterization is based on the factorization of $n \times n$ unitary matrices [32] and may be used either for Hamiltonian or density matrices. In [33], the authors have shown that the space of two qubit density matrices (four-level systems), can be characterize with 12-dimensional (as real manifold) space of complex orthogonal group $SO(4, \mathbb{C})$ together with four positive Wootters’s numbers [34], where of course, the normalization condition reduces the number of parameters to 15.

By using the definitions (4) and (7), Slater has computed the Bures metric for the eight-dimensional state space of three-level quantum systems [35]. He showed that all entries of the $8 \times 8$ matrix tensor are independent of the Euler angle $\alpha$, and the matrix tensor decomposes into a $6 \times 6$ block and a $2 \times 2$ one, in corresponding to the six Euler angles of unitary matrix $V$ and the two independent eigenvalues of diagonal matrix $D$.

In this paper we consider a canonical coset parameterization for density matrices of an $n$-level quantum system. The parameterization is based on the coset space decomposition of unitary matrices [31]. This parameterization, as well as the Euler angle parameterization do, eliminates any over-parameterization of the density matrix. It also provides a factorization of the Bures measure on the space of density matrices as the product of the measure on the space of eigenvalues and the truncated Haar measure on the space of unitary matrices. We give explicitly, the parameterization for two- and three-level density matrices, and by application of Dittmann’s formulas, the Bures metric over the spaces of these quantum systems are explicitly computed. It is shown that the coset parameterization gives a compact expression for the all metric elements. The analytical simple expression obtained for the Bures metric of the three-level system enable us to use this parameterization for the problem of Bures metric over the space of two qubit system.

The paper is organized as follows: In section 2, the coset space parameterization of an $n$-level density matrix is introduced. We give also, explicitly, the parameterization of the density matrices of the two- and three-level systems in this section. In section 3 we compute explicitly the Bures metric of the two- and the three-level systems. The paper is concluded in section 4 with a brief conclusion.

## 2 Canonical coset parameterization of density matrices

In this section we review some properties of the set of density matrices of an $n$-level quantum system and, by using a canonical coset parameterization for $n \times n$ unitary matrices, we present a coset parameterization for $n$-level density matrices. The state space of an $n$-level quantum system is identified with the set of all $n \times n$ Hermitian positive semidefinite complex matrices
Coset parameterization and the Bures metric of the three-level system

of trace unity, and comprise \((n^2 - 1)\)-dimensional convex set \(\mathcal{M}_n\). The total number of independent variables needed to parameterize a density matrix \(\rho\) is equal to \(n^2 - 1\), provided no degeneracy occurs.

Let us denote the set of all diagonal density matrices of an \(n\)-level system with \(\mathcal{D}_n\). An arbitrary element \(D \in \mathcal{D}_n\) can be written as

\[
D = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}, \quad 0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^{n} \lambda_i = 1.
\]  

This means that the set of all diagonal density matrices forms an \((n-1)\)-dimensional simplex \(S_{n-1}\). A generic density matrix in an arbitrary basis can be obtained as the orbit of points \(D \in \mathcal{D}_n\) under the action of the unitary group \(U(n)\) as

\[
\rho = UDU^\dagger.
\]  

Let \(H\) be a maximum stability subgroup, i.e. a subgroup of \(U(n)\) that consists of all the group elements \(h\) that will leave the diagonal state \(D\) invariant,

\[
hDh^\dagger = D, \quad h \in H, \quad D \in \mathcal{D}_n,
\]  

that is, \(H\) contains all elements of \(U(n)\) that commute with \(D\). For every element \(U \in U(n)\), there is a unique decomposition of \(U\) into a product of two group elements, one in \(H\) and the other in the quotient \(G/H\) \([36]\), i.e.

\[
U = \Omega h, \quad U \in U(n), \quad h \in H, \quad \Omega \in U(n)/H.
\]  

The above decomposition implies that the action of an arbitrary group element \(U \in U(n)\) on the point \(D \in \mathcal{D}_n\) is given by

\[
\rho = UDU^\dagger = \Omega hDh^\dagger \Omega^\dagger = \Omega D\Omega^\dagger.
\]  

This means that in order to characterize the space \(\mathcal{M}_n\), it is sufficient to consider the orbit of points \(D \in \mathcal{D}_n\) under the action of the quotient \(U(n)/H\). Since \(\mathcal{D}_n\) consists points with different degree of degeneracy, the maximum stability subgroup will differ for different \(D \in \mathcal{D}_n\) \([30]\). Let \(m_i\) denotes degree of degeneracy of eigenvalue \(\lambda_i\) of matrix \(D\). This kind of the spectrum follows that \(D\) remains invariant under the action of arbitrary unitary transformation performed in each of the \(m_i\)-dimensional eigensubspace. Therefore \(H = U(m_1) \otimes U(m_2) \otimes \cdots U(m_k)\) is maximum stability subgroup for \(D\), and the quotient space \(U(n)/H\) is a complex flag manifold

\[
\mathcal{F} = \frac{U(n)}{U(m_1) \otimes U(m_2) \otimes \cdots U(m_k)}, \quad m_1 + m_2 + \cdots + m_k = n.
\]  

Two special kinds for the degeneracy of the spectrum of \(D\) are as follows: i) Let \(D\) represents the maximally mixed state \(\rho_\star = \text{diag}\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\}\). In this case the stability subgroup \(H\) is \(U(n)\), and the orbit of point \(\rho_\star\) is only one point, i.e. \(\rho = \rho_\star\). ii) On the other hand if the spectrum of \(D\) is non-degenerate, then the stability subgroup is \(n\)-dimensional torus \(T^n = U(1)^\otimes n\), and the orbit of the point \(D\) is

\[
\rho = \Omega D\Omega^\dagger, \quad \Omega \in U(n)/T^n.
\]
Coset parameterization and the Bures metric of the three-level system

The maximal torus $T^n$ is itself a subgroup of all maximum stability subgroups, therefore the orbit of points $D \in D_n$ under the action of quotient $U(n)/T^n$ generates all points of the space $M_n$. The diagonal matrix $D$ is defined up to a permutation of its entries and, one can divide the simplex $S_{n-1}$ into $n!$ identical simplexes and take any of them. Each part identify points of $S_{n-1}$ which have the same coordinates, but with different ordering, and can be considered as the homomorphic image of simplex $S_{n-1}$ relative to the discreet permutation group $P_n$, i.e. $S_{n-1}/P_n$. Therefore the points of $M_n$ can be characterize as the orbit of diagonal matrices $D \in S_{n-1}/P_n$ under the action of quotient $U(n)/T^n$.

Further insight into the space of density matrices can be obtained by writing to the arbitrary element $U \in U(n)$ as [36]

$$U = \Omega_1^{(n)} \Omega_2^{(n)} \cdots \Omega_n^{(n)},$$

where $\Omega_1^{(n)} \in T^n$, and

$$\Omega_k^{(n)} = \frac{U(k) \otimes T^{n-k}}{U(k-1) \otimes T^{n-k+1}}, \quad k = 2, \cdots, n.$$  

Comparing this with the decomposition $U = \Omega h$, where $h \in T^n$, leads to the following decomposition for an arbitrary element $\Omega$ of quotient $U(n)/T^n$

$$\Omega = \Omega_1^{(n)} \Omega_2^{(n)} \cdots \Omega_n^{(n)}.$$  

A typical coset representative $\Omega_k^{(n)}$ can be written as [36]

$$\Omega_k^{(n)} = \begin{pmatrix} SU(k)/U(k-1) & O \\ OT & I_{n-k} \end{pmatrix},$$

where $O$, $OT$, and $I_{n-k}$ represent, respectively, the $k \times (n-k)$ zero matrix, its transpose and the $(n-k) \times (n-k)$ identity matrix. The $2(k-1)$-dimensional coset space $SU(k)/U(k-1)$ have the following $k \times k$ matrix representation [36]

$$SU(k)/U(k-1) = \begin{pmatrix} \cos \sqrt{B^{(k)} \| B^{(k)} \|} & B^{(k)} \sin \sqrt{B^{(k)} \| B^{(k)} \|} \\ \sin \sqrt{B^{(k)} \| B^{(k)} \|} \sqrt{B^{(k)} B^{(k)}} & \cos \sqrt{B^{(k)} \| B^{(k)} \|} \sqrt{B^{(k)} B^{(k)}} \end{pmatrix},$$

where $B^{(k)}$ represents $(k-1) \times 1$ complex matrix and $[B^{(k)}]^\dagger$ is its adjoint. In the following we consider the $n = 2, 3$ cases, explicitly.

### 2-1 Two-level system

We begin by giving the coset parameterization for a two-level quantum system. Let us consider a diagonal two-level density matrix $D = \text{diag}\{\lambda_1, \lambda_2\}$. Every $h \in H = T^2$ leaves the density matrix $D$ invariant i.e. $h D h^\dagger = D$ for $h \in H$. Any group element $U \in U(2)$ can be decomposed, uniquely, as $U = \Omega h$ where $\Omega \in U(2)/T^2$ and $h \in T^2$ [36]. The coset space with respect to the stability subgroup $H = T^2$ will provides the unitary transformations to construct a generic density matrix $\rho$ as the orbit of diagonal matrix $D$, i.e.

$$\rho = \Omega D \Omega^\dagger, \quad \Omega \in U(2)/T^2.$$
A typical coset representative in the coset space $U(2)/T^2$ is
\[
\Omega = \begin{pmatrix}
\cos \alpha & e^{i\phi} \sin \alpha \\
-e^{-i\phi} \sin \alpha & \cos \alpha
\end{pmatrix},
\] (22)
where $\alpha, \phi$ are real. The range of parameters $\lambda_1, \lambda_2$ can be determined as follows. The set of all diagonal $2 \times 2$ matrices $D$ forms a 1-dimensional simplex $S_1$, which can be divided into two identical parts $0 \leq \lambda_1 \leq \frac{1}{2}, \frac{1}{2} \leq \lambda_1 \leq 1$ (see figure 1a). It can be easily seen that each part can be obtained as the orbit of the other part under the action of the group element $\Omega = \Omega(\alpha = \phi = \frac{\pi}{2})$ which is an element of the coset space $U(2)/T^2$. This means that we can easily consider the diagonal matrix $D$ as
\[
D = \begin{pmatrix}
\cos^2 \theta & 0 \\
0 & \sin^2 \theta
\end{pmatrix}, \quad 0 \leq \theta \leq \frac{\pi}{4}.
\] (23)
With this parameterization any $2 \times 2$ density matrix can be written explicitly as
\[
\rho = \begin{pmatrix}
\sin^2 \alpha \sin^2 \theta + \cos^2 \alpha \cos^2 \theta & -\frac{1}{2} e^{i\phi} \sin 2\alpha \cos 2\theta \\
-\frac{1}{2} e^{-i\phi} \sin 2\alpha \cos 2\theta & \sin^2 \alpha \cos^2 \theta + \cos^2 \alpha \sin^2 \theta
\end{pmatrix}.
\] (24)

2-2 Three-level system
The density matrices for the three-level system comprise eight-dimensional convex set. Let $D = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ be a $3 \times 3$ diagonal density matrix of a three-level system. The density matrix $D$ is invariant under the action of every group element $h \in H = T^3$, i.e. $h D h^\dagger = D$ for $h \in H$. The coset decomposition $U = \Omega h$ where $\Omega \in U(3)/T^3$, $h \in T^3$ provides a parameterization for a generic $\rho$ as
\[
\rho = U D U^\dagger = \Omega h D h^\dagger \Omega^\dagger = \Omega D \Omega^\dagger.
\] (25)
On the other hand the decomposition
\[
U = \Omega_3^{(3)} \Omega_2^{(3)} \Omega_1^{(3)},
\] (26)
with
\[
\begin{align*}
\Omega_3^{(3)} & \in U(3)/U(2) \otimes U(1), \\
\Omega_2^{(3)} & \in (U(2) \otimes U(1))/U(1) \otimes U(1) \otimes U(1), \\
\Omega_1^{(3)} & \in U(1) \otimes U(1) \otimes U(1),
\end{align*}
\] (27)
follows that
\[
\Omega = \Omega_3^{(3)} \Omega_2^{(3)} \Omega_1^{(3)}.
\] (28)
The coset representatives $\Omega_2^{(3)}$ and $\Omega_3^{(3)}$ can be parameterized respectively as
\[
\Omega_2^{(3)} = \begin{pmatrix}
\cos \alpha & e^{i\phi} \sin \alpha & 0 \\
-e^{-i\phi} \sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix},
\] (29)
and
\[
\Omega_3^{(3)} = \begin{pmatrix}
1 + \frac{\beta^2}{\beta^2} (\cos \beta - 1) & \frac{\beta_1 \beta_2}{\beta^2} e^{i(\psi_1 - \psi_2)} (\cos \beta - 1) & \frac{\beta_1}{\beta} e^{i\psi_1} \sin \beta \\
\frac{\beta_1 \beta}{\beta^2} e^{-i(\psi_1 - \psi_2)} (\cos \beta - 1) & 1 + \frac{\beta^2}{\beta^2} (\cos \beta - 1) & \frac{\beta_2}{\beta} e^{i\psi_2} \sin \beta \\
\frac{\beta_1}{\beta} e^{-i\psi_1} \sin \beta & -\frac{\beta_1}{\beta} e^{-i\psi_2} \sin \beta & \cos \beta
\end{pmatrix}.
\] (30)
where $\beta = \sqrt{\beta_1^2 + \beta_2^2}$.

The two-dimensional simplex $S_2$ of eigenvalues of $\rho$ is divided into $3!$ parts (see figure 1b). Let us take one of the parts (e.g. the shaded one) for illustration. It is easy to see that all other parts can be obtained from this one by applying the permutation group $P_3$. On the other hand the elements of the permutation group $P_3$ can be obtained from the coset representative (28) up to a phase as

$$\Omega^{(3)}_3 (\beta_1 = \beta_2 = 0) \Omega^{(3)}_2 (\alpha = 0) = (\text{Id}),$$
$$\Omega^{(3)}_3 (\beta_1 = \beta_2 = 0) \Omega^{(3)}_2 (\alpha = \phi = \frac{\pi}{2}) = i(12),$$
$$\Omega^{(3)}_3 (\beta_1 = \frac{\pi}{2}, \beta_2 = 0, \psi_1 = \frac{\pi}{2}) \Omega^{(3)}_2 (\alpha = 0) = i(13),$$
$$\Omega^{(3)}_3 (\beta_1 = 0, \beta_2 = \frac{\pi}{2}, \psi_2 = \frac{\pi}{2}) \Omega^{(3)}_2 (\alpha = 0) = i(23),$$
$$\Omega^{(3)}_3 (\beta_1 = \frac{\pi}{2}, \beta_2 = 0, \psi_1 = \frac{\pi}{2}) \Omega^{(3)}_2 (\alpha = \phi = \frac{\pi}{2}) = i(123),$$
$$\Omega^{(3)}_3 (\beta_1 = 0, \beta_2 = \frac{\pi}{2}, \psi_2 = \frac{\pi}{2}) \Omega^{(3)}_2 (\alpha = \phi = \frac{\pi}{2}) = i(321).$$

Therefore the ranges of the eigenvalues of $\rho$ can be determined as $\frac{1}{3} \leq \lambda_1 \leq 1$, $0 \leq \lambda_2 \leq \frac{1}{2}$ and $0 \leq \lambda_3 \leq \frac{1}{3}$, or equivalently, one can parameterize the diagonal matrix $D$ as

$$D = \begin{pmatrix}
\cos^2\theta_1 & 0 & 0 \\
0 & \sin^2\theta_1 \cos^2\theta_2 & 0 \\
0 & 0 & \sin^2\theta_1 \sin^2\theta_2
\end{pmatrix},$$

(31)

where $0 \leq \theta_1 \leq \cos^{-1} \frac{1}{\sqrt{3}}$, $\frac{\pi}{6} \leq \theta_2 \leq \frac{\pi}{4}$.

3 Bures metric

In this section we calculate the Bures metric of the two- and the three-level quantum systems. We will use the canonical coset parameterization of the density matrices introduced in the last section.

3-1 Two-level system

In the two-level systems by using Eqs. (3) and (21) we get

$$d_B(\rho, \rho + d\rho) = \frac{1}{4} \text{Tr} \left[ (dD)^2 + \frac{1}{|D|} (I - D)^2 (dD)^2 \right]$$
$$+ \sum_{i<j} \left[ S_{ij} ((\Omega^\dagger d\Omega)_{ij} (\Omega^\dagger d\Omega)_{ji}) \right],$$

(32)

where $I$ is unit matrix and

$$S_{ij} = \frac{1}{|D|} \left[ (D_{ii} - D_{jj})^2 (D_{ij} + D_{ji} - D_{ii}D_{jj} - |D| - 1) \right].$$

(33)

By defining the ordering $\{\theta, \alpha, \phi\}$ for coordinates, the corresponding Bures metric tensor takes the following diagonal form

$$g = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos^2\theta & 0 \\
0 & 0 & \frac{1}{4} \sin^2\theta \cos^2\phi
\end{pmatrix}.$$
Coset parameterization and the Bures metric of the three-level system

\[ (0, 1) \quad \rho_* \quad (1, 0) \]

(a)

\[ (0, 1, 0) \quad (\frac{1}{2}, \frac{1}{2}, 0) \quad (1, 0, 0) \]

(b)

Figure 1: (a) One-dimensional simplex \( S_1 \) of diagonal density matrices of two-level systems, (b) the two-dimensional simplex \( S_2 \) of diagonal density matrices of three-level systems. The simplex \( S_n \) can be decomposed into \( n! \) parts. The parts can be transformed to each other by applying the elements of the permutation group \( P_n \).

3.2 Three-level system

In the three-level system, by using (25) in (4) we have

\[
\begin{align*}
\text{d}_B(\rho, \rho + d\rho) &= \frac{1}{4} \text{Tr} \left[ (dD)^2 + \frac{3}{(1 - \text{Tr}D^2)} \left( (I - D)^2 (dD)^2 + |D (I - D^{-1}) (dD)|^2 \right) \\
&\quad + \sum_{i<j} T_{ij} \left( (\Omega^i d\Omega)^{ij} (\Omega^i d\Omega)^{ji} \right) \right],
\end{align*}
\]

(35)

where

\[
T_{ij} = \frac{3}{2(1 - \text{Tr}D^2)} \left[ (D_{ii} - D_{jj})^2 (D_{ii} + D_{jj} - D_{ii} D_{jj} - \text{Tr}D^2 - 2) \\
\quad + |D (D_{ii}^{-1} - D_{jj}^{-1})^2 (D_{ii}^{-1} + D_{jj}^{-1} - D_{ii}^{-1} D_{jj}^{-1} - 1) | \right].
\]

(36)
Now defining the ordering of coordinates as \( \{ \theta_1, \theta_2, \alpha, \phi, \beta_1, \beta_2, \psi_1, \psi_2 \} \), the corresponding symmetric matrix for metric tensor takes the following form

\[
g = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sin^2 \theta_1 & g_{\alpha \alpha} & g_{\alpha \phi} & g_{\alpha \beta_1} & g_{\alpha \beta_2} & g_{\alpha \psi_1} & g_{\alpha \psi_2} \\
0 & 0 & g_{\phi \phi} & g_{\phi \beta_1} & g_{\phi \beta_2} & g_{\phi \psi_1} & g_{\phi \psi_2} & g_{\phi \psi_2} \\
g_{\beta_1 \beta_1} & g_{\beta_1 \beta_2} & g_{\beta_1 \psi_1} & g_{\beta_1 \psi_2} & g_{\beta_2 \psi_1} & g_{\beta_2 \psi_2} & g_{\psi_1 \psi_1} & g_{\psi_1 \psi_2} \\
g_{\beta_2 \beta_2} & g_{\beta_2 \psi_2} & g_{\psi_1 \psi_2} & g_{\psi_2 \psi_2} & & & & \\
g_{\beta_1 \psi_1} & g_{\beta_1 \psi_2} & g_{\psi_1 \psi_2} & g_{\psi_2 \psi_2} & & & & \\
g_{\beta_2 \psi_1} & g_{\beta_2 \psi_2} & g_{\psi_1 \psi_2} & g_{\psi_2 \psi_2} & & & & \\
g_{\psi_1 \psi_1} & g_{\psi_1 \psi_2} & g_{\psi_1 \psi_2} & g_{\psi_2 \psi_2} & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]  

(37)

The \( 8 \times 8 \) matrix tensor decomposes into a \( 2 \times 2 \) block and a \( 6 \times 6 \) one, in corresponding to the two independent eigenvalues \( \theta_1, \theta_2 \) and six coset parameters \( \alpha, \phi, \beta_1, \beta_2, \psi_1, \psi_2 \). After some analytical calculations, we can obtain the following expression for the matrix elements of the \( 6 \times 6 \) block

- \( g_{\alpha \alpha} = -T_{12} \)
- \( g_{\alpha \phi} = 0 \)
- \( g_{\alpha \beta_1} = 2T_{12} \beta_2 \cos \gamma \left( \frac{\sin \frac{\phi}{2}}{\beta_2} \right)^2 \)
- \( g_{\alpha \beta_2} = -\frac{\beta_1}{\beta_2} g_{\alpha \beta_1} \)
- \( g_{\alpha \psi_1} = 2T_{12} \beta_1 \beta_2 U_2 \sin \gamma \left( \frac{\sin \frac{\phi}{2}}{\beta} \right)^2 \)
- \( g_{\alpha \psi_2} = \frac{U_1}{U_2} g_{\alpha \psi_1} \)
- \( g_{\phi \phi} = -\frac{1}{4} T_{12} \sin^2 2\alpha \)
- \( g_{\phi \beta_1} = -\frac{1}{2} T_{12} \beta_2 \sin 4 \alpha \left( \frac{\sin \frac{\phi}{2}}{\beta} \right)^2 \)
- \( g_{\phi \beta_2} = -\frac{\beta_1}{\beta_2} g_{\phi \beta_1} \)
- \( g_{\phi \psi_1} = \frac{1}{2} T_{12} \beta_1 \sin 2 \alpha \left( \beta_1 W_2 \sin 2 \alpha + 2 \beta_2 U_2 \cos 2 \alpha \cos \gamma \right) \left( \frac{\sin \frac{\phi}{2}}{\beta} \right)^2 \)
- \( g_{\phi \psi_2} = \frac{1}{2} T_{12} \beta_2 \sin 2 \alpha \left( \beta_2 W_1 \sin 2 \alpha - 2 \beta_1 U_1 \cos 2 \alpha \cos \gamma \right) \left( \frac{\sin \frac{\phi}{2}}{\beta} \right)^2 \)
- \( g_{\beta_1 \beta_1} = -4T_{12} \beta_2^2 \left[ 1 - \sin^2 2 \alpha \sin^2 \gamma \right] \left( \frac{\sin \frac{\phi}{2}}{\beta} \right)^4 \)
- \( T_{13} \left[ X^2 \sin^2 \alpha + V_2^2 \cos^2 \alpha - XV_1 \sin 2 \alpha \cos \gamma \right] \)
- \( T_{23} \left[ X^2 \cos^2 \alpha + V_1^2 \sin^2 \alpha + XV_1 \sin 2 \alpha \cos \gamma \right] \)
- \( g_{\beta_1 \beta_2} = 4T_{12} \beta_1 \beta_2 \left[ 1 - \sin^2 2 \alpha \sin^2 \gamma \right] \left( \frac{\sin \frac{\phi}{2}}{\beta} \right)^4 \)
- \( T_{13} \left[ X (V_1 \cos^2 \alpha + V_2 \sin^2 \alpha) - \frac{1}{2} (V_1 V_2 + X^2) \sin 2 \alpha \cos \gamma \right] \)
- \( T_{23} \left[ X (V_1 \sin^2 \alpha + V_2 \cos^2 \alpha) + \frac{1}{2} (V_1 V_2 + X^2) \sin 2 \alpha \cos \gamma \right] \)
Coset parameterization and the Bures metric of the three-level system

\[ g_{\beta_1\psi_1} = -T_{12}\beta_1\beta_2 \left[ 2\beta_2U_2 \sin^2 2\alpha \sin 2\gamma - \beta_1W_2 \sin 4\alpha \sin \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^4 \\
+ \frac{1}{2} (T_{13} - T_{23}) \beta_1 \sin 2\alpha \sin \gamma \left[ U_2X + V_1Y \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right) \]

\[ g_{\beta_1\psi_2} = -T_{12}\beta_2^2 \left[ 2\beta_1U_1 \sin^2 2\alpha \sin 2\gamma + \beta_2W_1 \sin 4\alpha \sin \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^4 \\
- \frac{1}{2} (T_{13} - T_{23}) \beta_1 \sin 2\alpha \sin \gamma \left[ U_1V_1 + XY \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right) \]

\[ g_{\beta_2\beta_2} = -4T_{12}\beta_2^2 \left[ 1 - \sin^2 2\alpha \sin^2 \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^4 \\
- T_{13} \left[ X^2 \cos^2 \alpha + V_2^2 \sin^2 \alpha - XV_2 \sin 2\alpha \cos \gamma \right] \\
- T_{23} \left[ X^2 \sin^2 \alpha + V_2^2 \cos^2 \alpha + XV_2 \sin 2\alpha \cos \gamma \right] \]

\[ g_{\beta_2\psi_1} = T_{12}\beta_1^2 \left[ 2\beta_2U_2 \sin^2 2\alpha \sin 2\gamma - \beta_1W_2 \sin 4\alpha \sin \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^4 \\
+ \frac{1}{2} (T_{13} - T_{23}) \beta_2 \sin 2\alpha \sin \gamma \left[ U_2V_2 + XY \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right) \]

\[ g_{\beta_2\psi_2} = T_{12}\beta_1\beta_2 \left[ 2\beta_1U_1 \sin^2 2\alpha \sin 2\gamma + \beta_2W_1 \sin 4\alpha \sin \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^4 \\
- \frac{1}{2} (T_{13} - T_{23}) \beta_2 \sin 2\alpha \sin \gamma \left[ U_1X + V_2Y \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right) \]

\[ g_{\psi_1\psi_1} = -T_{12}\beta_1^2 \left[ 4\beta_2^2U_2^2 \left( 1 - \sin^2 2\alpha \cos^2 \gamma \right) \right] \\
+ \beta_1^2W_2^2 \sin^2 2\alpha + 2\beta_1\beta_2U_2W_2 \sin 4\alpha \cos \gamma \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^4 \\
- T_{13}\beta_1^2 \left[ U_2^2 \cos^2 \alpha + Y^2 \sin^2 \alpha + U_2Y \sin 2\alpha \cos \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^2 \\
- T_{23}\beta_1^2 \left[ U_2^2 \sin^2 \alpha + Y^2 \cos^2 \alpha - U_2Y \sin 2\alpha \cos \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^2 \]

\[ g_{\psi_1\psi_2} = -T_{12}\beta_1\beta_2 \left[ 4\beta_1\beta_2U_1U_2 \left( 1 - \sin^2 2\alpha \cos^2 \gamma \right) \right] \\
- \beta_1\beta_2W_1W_2 \sin^2 2\alpha - \sin 4\alpha \cos \gamma \left( \beta_2^2U_2W_1 - \beta_1^2U_1W_2 \right) \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^4 \\
+ T_{13}\beta_1\beta_2 \left[ Y(U_1 \sin^2 \alpha + U_2 \cos^2 \alpha) + \frac{1}{2} \sin 2\alpha \cos \gamma (U_1U_2 + Y^2) \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^2 \\
+ T_{23}\beta_1\beta_2 \left[ Y(U_1 \cos^2 \alpha + U_2 \sin^2 \alpha) - \frac{1}{2} \sin 2\alpha \cos \gamma (U_1U_2 + Y^2) \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^2 \]

\[ g_{\psi_2\psi_2} = -T_{12}\beta_2^2 \left[ 4\beta_1^2U_1^2 \left( 1 - \sin^2 2\alpha \cos^2 \gamma \right) \right] \\
+ \beta_2^2W_1^2 \sin^2 2\alpha - 2\beta_1\beta_2U_1W_1 \sin 4\alpha \cos \gamma \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^4 \\
- T_{13}\beta_2^2 \left[ U_1^2 \sin^2 \alpha + Y^2 \cos^2 \alpha + U_1Y \sin 2\alpha \cos \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^2 \\
- T_{23}\beta_2^2 \left[ U_1^2 \cos^2 \alpha + Y^2 \sin^2 \alpha - U_1Y \sin 2\alpha \cos \gamma \right] \left( \frac{\sin \frac{\theta}{\beta}}{\beta} \right)^2 \]
In the above equations we have used the following definitions

$$\gamma = \phi - \psi_1 + \psi_2,$$  \hfill (38)

and

$$T_{12} = -\frac{1}{2} \left( \cos^2 \theta_1 - \sin^2 \theta_1 \cos^2 \theta_2 \right)^2 \left[ 3 + \frac{(1 - \sin^2 \theta_1 \cos^2 \theta_2)(1 + \cos^2 \theta_1 \cos^4 \theta_2)}{\cos^2 \theta_1 \cos^2 \theta_2(\cos^2 \theta_1 + \sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_2)} \right],$$

$$T_{13} = -\frac{1}{2} \left( \cos^2 \theta_1 - \sin^2 \theta_1 \sin^2 \theta_2 \right)^2 \left[ 3 + \frac{(1 - \sin^2 \theta_1 \sin^2 \theta_2)(\cos^2 \theta_2 + \sin^2 \theta_1 \cos^4 \theta_2)}{\sin^2 \theta_1 \cos^2 \theta_2(\cos^2 \theta_1 + \sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_2)} \right],$$

$$T_{23} = -\frac{1}{2} \left[ \sin^2 \theta_1 \cos^2 \theta_2(1 + 3 \sin^2 \theta_1) + \frac{\cos^2 \theta_1}{\sin^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_2} \right].$$

and

$$U_1 = \frac{\beta_1^2}{\beta^2} + \frac{\beta_2^2}{\beta^2} \cos \beta,$$  \hfill (39)

$$U_2 = \frac{\beta_2^2}{\beta^2} + \frac{\beta_1^2}{\beta^2} \cos \beta,$$

$$V_1 = \frac{\beta_1^2}{\beta^2} + \frac{\beta_2^2}{\beta^2} \left( \sin \frac{\beta}{\beta} \right),$$

$$V_2 = \frac{\beta_2^2}{\beta^2} + \frac{\beta_1^2}{\beta^2} \left( \sin \frac{\beta}{\beta} \right),$$

$$W_1 = (1 + \cos \beta) + 2 \frac{\beta_1^2}{\beta^2}(1 - \cos \beta),$$

$$W_2 = (1 + \cos \beta) + 2 \frac{\beta_2^2}{\beta^2}(1 - \cos \beta),$$

$$X = \frac{\beta_1}{\beta} \frac{\beta_2}{\beta} (1 - \sin \frac{\beta}{\beta}),$$

$$Y = \frac{\beta_1}{\beta} \frac{\beta_2}{\beta} (1 - \cos \beta).$$

It should be stress that all elements $g_{ij}$ are simply products of two independent functions, one of the coset space parameters, and the other of the spherical angles $\theta_1, \theta_2$. It is also worth to note that the three angles $\phi, \psi_1$ and $\psi_2$ appear only in the form $\gamma = \phi - \psi_1 + \psi_2$.

4 Conclusion

We present a coset parameterization for density matrices of an $n$-level quantum system. The parameterization is based on the canonical coset decomposition of unitary matrices. By using the parameterization for two- and three-level quantum systems, the Bures metric over the state space of these systems are computed explicitly. We show that in the canonical coset parameterization the symbolic expression for all tensor elements can be obtained. The problem of computing the Bures metric of a two-qubit (four-level) system, which is important for calculations involving entanglement, is also under consideration.

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Coset parameterization and the Bures metric of the three-level system

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