Dynamic analysis of elastic rubber tired car wheel breaking under variable normal load

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Abstract. The purpose of the paper is to analyze the dynamics of the braking of the wheel under normal load variations. The paper uses a mathematical simulation method according to which the calculation model of an object as a mechanical system is associated with a dynamically equivalent schematic structure of the automatic control. Transfer function tool analyzing structural and technical characteristics of an object as well as force disturbances were used. It was proved that the analysis of dynamic characteristics of the wheel subjected to external force disturbances has to take into account amplitude and phase-frequency characteristics. Normal load variations impact car wheel braking subjected to disturbances. The closer slip to the critical point is, the higher the impact is. In the super-critical area, load variations cause fast wheel blocking.

1. Introduction

Motor vehicle is the most dangerous mode of transport. In Russia, every year about 25,000-30,000 people die on the roads, more than 250,000 people sustain injuries. It equals the population of a town. Only for three days, the number of people who die on the roads of Russia is higher than for the whole year as a result of other traffic accidents [1]. More than 98% of road traffic accidents occur when drivers brake or when accidents are associated with braking [2].

Braking is closely related to tire grip characteristics. Because of tire traction, the car changes its direction and speed. To improve the quality of braking the car is equipped with anti-blocking systems (ABS), electronic stability programs (ESP), etc. However the efficiency of these systems decreases significantly when braking wheels are influenced by road irregularities. For example, the variations of normal load $R_z$ upon the wheel increase its braking path by 37-40% [3].

So, the efficient control of wheel braking subjected to normal load disturbances is a topical issue. The paper presents research results for the dynamics of wheel braking under normal load $R_z$.

2. Materials and methods

At the first stage, let us construct a diagram (Fig. 1) of braking under the variable normal load.
Braking process is shown as a single car wheel travelling at the velocity of $V$ and rotating at the velocity of $\omega_k$ under a variable basic loading $R_z$ and constant braking moment $M_f$.

A mathematical model for a car wheel braking process can be presented as a system of equations (1). Detailed description of the parameters of the equations is given in [4]. For example, $r_{ko}$ is a force wheel radius, which is a rolling radius in a controlled mode, $r_o$ is a radius of a nonloaded wheel, $J_k$ is a moment of wheel inertia. $C_1$ and $C_2$ adapt the characteristics of $r_{ko}$ to $R_z$. $S$ is a tire slip ($0 < S < 1.0$), $R_x$ is a longitudinal road reaction to the braking force. The resisting rolling moment $M_f$ represents friction losses, and $\varphi_{max}$ is a tire grip.

\[
\begin{align*}
S &= 1 - \frac{\omega_k \cdot r_{ko}}{V} \\
r_{ko} &= r_o \left( 1 - C_1 \cdot \sqrt{R_z + C_2 \cdot R_z} \right) \\
d\omega_k &= \frac{R_x \cdot r_k - M_t - M_f}{J_k} \\
R_x &= R_z \cdot \varphi_{max} \cdot f(s) \\
f(s) &= R_z \cdot \varphi_{max} \cdot \sin \left[ a \cdot \arctg(b \cdot s) \right]
\end{align*}
\]

The experimental results show that the nature of the normal load and value of the normalized braking function $f_{(s)}$, determining the mode of wheel braking:

\[
f_{(s)} = \frac{M_t + M_f}{r_{ko} \cdot R_z \cdot \varphi_{max}} = \frac{1}{s}
\]

Let us use the method of study of complex dynamic systems gaining widespread in the control theory to analyze in detail the dynamic characteristics of a braking wheel under the variable normal load.

Let us assume that the wheel is a dynamic system under the normal load, with a braking moment on entry, an angular velocity and acceleration, a slip and a longitudinal reaction on return. At the first stage, we can analyze dynamic characteristics of the wheel in a relatively small range of changes under the normal load $R_z$ in a steady state ($R_z = R_{zo}$) when the braking moment is kept constant.
when the nonlinear nature of the system of equations describing wheel braking (1) has not manifested yet. Then the dependences $f(S)$ and $r_{ko}(R_z)$ can be linearized and presented as

$$f(S) = f_0(S) + \partial_s \Delta S;$$

$$r_{ko} = r_{ko0} + \partial_{R_z} \Delta R_z$$

where $f_0(S)$ and $r_{ko0}$ correspond to the steady state of wheel braking when $R_z = R_{zo};$

$$\partial_s = \frac{\partial f(S)}{\partial S} \quad \text{and} \quad \partial_{R_z} = \frac{\partial r_{ko}}{\partial R_z}$$

are partial derivatives of the dependences in points of the steady state.

Let us expand the dependences forming the system of equations describing the wheel braking process (1) in a Taylor series using only linear terms. The changes of $M_f$ are neglected ($M_f = \text{const}$). In the steady state, the angular acceleration $\omega_k$ and moments impacting the wheel are equal to zero:

$$-\left(\frac{M_t + M_f}{J_k}\right) + R_{zo} \cdot \frac{r_{ko0}}{J_k} = 0$$

After rearrangement, there is the following system of equations for increments:

$$\begin{align*}
\Delta R_z &= \frac{\partial R_z}{\partial R_z} \cdot \Delta R_z + \frac{\partial R_z}{\partial S} \cdot \Delta S \\
\Delta \omega_k &= \frac{\partial \omega_k}{\partial R_x} \cdot \Delta R_x + \frac{\partial \omega_k}{\partial r_{ko}} \cdot \Delta r_{ko} \\
\Delta S &= \frac{\partial S}{\partial \omega_k} \cdot \Delta \omega_k + \frac{\partial S}{\partial r_{ko}} \cdot \Delta r_{ko} \\
\Delta r_{ko} &= \frac{\partial r_{ko}}{\partial R_z} \cdot \Delta R_z
\end{align*}$$

Let us determine the partial derivatives in this system by differentiating the basic equations describing the wheel braking process (1) when the values of variables correspond to the steady state:

$$\begin{align*}
\frac{\partial R_z}{\partial f(S)} &= f_{max} \cdot f(S); \\
\frac{\partial R_z}{\partial S} &= R_{zo} \cdot f_{max} \cdot \Delta S; \\
\frac{\partial \omega_k}{\partial R_x} &= R_{ki} \cdot \frac{\partial \omega_k}{\partial R_x}; \\
\frac{\partial \omega_k}{\partial r_{ko}} &= \frac{\partial \omega_k}{\partial r_{ko}}; \\
\frac{\partial S}{\partial \omega_k} &= \frac{\partial S}{\partial \omega_k}; \\
\frac{\partial S}{\partial r_{ko}} &= \frac{\partial S}{\partial r_{ko}}; \\
\frac{\partial r_{ko}}{\partial R_z} &= \frac{\partial r_{ko}}{\partial R_z}
\end{align*}$$

Let us plug (4) in (5). After rearrangement, it is necessary to turn to the operator notation of variables:
where $\Delta R_x(p)$, $\Delta R_z(p)$, $\Delta \omega(p)$ and $\Delta S(p)$ are Laplace transforms of corresponding variables;

$$p = \frac{d}{dt}$$ is a differentiation operator (a Laplas operator).

Based on the system of equations, let us construct a structural diagram of dynamic characteristics of the braking wheel subjected to disturbances of the variable normal load (Fig. 2).

**Figure 2.** The structural scheme of dynamic characteristics of the car wheel according to the normal load as an input parameter when the braking moment is kept constant

The diagram consists of amplifier elements 1-6, an integrator 7 and comparative elements 8 - 10. Transfer ratios of amplifier elements are characterized as follows:

$$K_1 = -\frac{(1-S_o)}{r_{ko}} \cdot \frac{\partial r}{\partial \omega} \text{ of the normal load } R_z \text{ to a slip } S \text{ occurred due to the changes in a force arm, wheel rolling radius } r_{ko} \text{ in a controlled mode;}$$

$$K_2 = \frac{r_{ko}}{V} \text{ of the angular velocity } \omega_k \text{ to a slip } S;$$
К₃ = \frac{M_i + M_f}{r_{kvo} \cdot R_{z_o}} \text{ of the normal reaction } R_\text{z} \text{ to the longitudinal reaction } R_\text{x} \text{ caused by the changes in a cohesion limit force;}

K₄ = \frac{R_{z_o} \cdot \varphi_{\max} \cdot \partial S}{M_i + M_f} \text{ of the slip } S \text{ to the longitudinal reaction } R_\text{x};

K₅ = \frac{M_i + M_f}{r_{kvo} \cdot J_k} \cdot \partial r \text{ of the normal reaction } R_\text{x} \text{ to the angular acceleration } \omega_k \text{ caused by the changes in a force arm } R_\text{x};

K₆ = \frac{r_{kvo}}{J_k} \text{ of the longitudinal reaction } R_\text{x} \text{ to the acceleration } \omega_k .

3. Results and discussion

Some of the mentioned transfer coefficients depend heavily on the mode of wheel rolling (Fig. 2) characterized for a given value \( R_{z_o} \) and levels of the braking moment \( M_t \) by \( f_o \) or \( S_o \).

In a free mode, the normal reaction impacts the dynamic system only through element 1 (\( K_3 = K_5 = 0 \)), in other words, due to the changes in a rolling radius \( r_{kvo} \) as kinematic parameters influencing a slip \( S \).

With rising braking moment and slipping, the impact \( R_\text{z} \) through element 3 becomes predominant, and at a critical slip (\( \partial S = 0 \)), the impact \( R_\text{z} \) on \( R_\text{x} \) causing the changes of kinematic parameters of the wheel (feedback formed by elements 2, 4, 6, 7) discontinues, and changes affect only a cohesion limit force.

As the impact of transfer functions of the elements on dynamic characteristics of output parameters of the system manifests in their entirety, let us write the transfer functions for each output parameter (6) on an individual basis:

\[ W_s = K_1 + K_2 \cdot W_{\omega}; \quad W_x = K_3 + K_4 \cdot W_s \]  \hspace{1cm} \text{(7)}

\[ W_{\omega_k} = K_5 + K_6 \cdot W_x \quad W_{\omega} = W_{\omega_k} \cdot \frac{1}{p} \]

where \( W_s, W_x, W_{\omega_k} \) and \( W_{\omega} \) are the transfer functions of corresponding parameters under the normal load \( R_\text{x} \).

Solving this equation, let us obtain an expression for the transfer functions:

\[ W_x = \left( K_3 + K_4 \cdot K_1 \right) \cdot T_j p - \frac{K_5}{K_6} \cdot \frac{1}{1 + T_j p} \]  \hspace{1cm} \text{(8)}

\[ W_{\omega_k} = K_6 \cdot \left( K_5 \frac{K_5}{K_6} + (K_3 + K_4 \cdot K_1) \right) \cdot T_j p \cdot \frac{1}{1 + T_j p} \]  \hspace{1cm} \text{(9)}

\[ W_{\omega} = K_6 \cdot \left( K_5 \frac{K_5}{K_6} + (K_3 + K_4 \cdot K_1) \right) \cdot T_j \cdot \frac{1}{1 + T_j p} \]  \hspace{1cm} \text{(10)}

\[ W_s = \left( - \frac{K_5}{K_6} + K_3 \right) \cdot \frac{1}{K_4} \cdot K_1 \cdot T_j p \cdot \frac{1}{1 + T_j p} \]  \hspace{1cm} \text{(11)}

where \( T_j \) is the time constant of a feedback formed with elements 2, 4, 6, 7 (see Fig. 2), representing the impact of the wheel inertia on its dynamic characteristics.
With rising slip, the time constant $T_j$ increases due to a decrease in the rate of rise $f(S)$ of the diagram, and at the approach to the critical point of a tire slip $S > S_{cr}$, the time constant $T_j$ becomes negative suggesting that processes in this area are unstable.

Thus, expressions (8)...(11) enable to evaluate the contribution of transfer coefficients $K_1 - K_6$ to the formation of corresponding transfer functions and identify the areas of predominate impact of transforming properties of the elastic rubber tired wheel on its dynamic characteristics.

For example, for the areas of a significant slip $K_3 >> K_4 \cdot K_1$, and relation $K_5 / K_6 << K_3$, the changes in radius $r_{ko}$ of wheel rolling can be neglected when describing cohesion characteristics of its tires.

Plugging $P = i \cdot \Omega$ where ($\Omega = 2 \pi v$ is a circle frequency, $i = \sqrt{-1}$) in the expression for transfer functions and separating real and imaginary components, after rearrangement there is the formulas for calculation of amplitude frequency characteristics (AFC) and phase frequency characteristics (FFC) of:

- longitudinal reaction:
  \[ H_{\Omega X} = K_{xo} \sqrt{\frac{1 + (K_{zx} / K_{xo})^2 \cdot T_j^2 \cdot \Omega^2}{1 + T_j^2 \cdot \Omega^2}}, \quad \Theta_{\Omega X} = -\arctg \left[ \frac{T_j \cdot \Omega \cdot 1 - K_{zx} / K_{xo}}{1 + T_j^2 \cdot \Omega^2 \cdot (K_{zx} / K_{xo})^2} \right]; \]
- angular velocity:
  \[ H_{\Omega \omega} = K_{\omega o} \cdot \frac{1}{\sqrt{1 + T_j^2 \cdot \Omega^2}}, \quad \Theta_{\Omega \omega} = -\arctg \cdot T_j \cdot \Omega; \]
- angular acceleration:
  \[ H_{\Omega \varepsilon} = K_{\varepsilon o} \cdot \frac{\Omega}{\sqrt{1 + T_j^2 \cdot \Omega^2}}, \quad \Theta_{\Omega \varepsilon} = \arctg \cdot \frac{1}{T_j \cdot \Omega}; \]
- slip:
  \[ H_{\Omega S} = K_{So} \sqrt{\frac{1 + (K_1 / K_{So})^2 \cdot T_j^2 \cdot \Omega^2}{1 + T_j^2 \cdot \Omega^2}}, \quad \Theta_{\Omega S} = -\arctg \cdot T_j \cdot \Omega \cdot \frac{1 - K_1 / K_{So}}{1 + T_j^2 \cdot \Omega^2 \cdot (K_1 / K_{So})^2}; \]

where $K_{zx} = K_3 + K_1 \cdot K_4$ is a transfer coefficient of the direct relation of $R_z$ to $R_x$;

$K_{xo}$, $K_{\\omega o}$, $K_{\varepsilon o}$ are transfer coefficients characterizing corresponding transfer functions in a steady state:

\[ K_{xo} = \frac{K_5}{K_6}; \quad K_{\\omega o} = (K_5 + K_6 \cdot K_{zx}) \cdot \frac{1}{K_4} \]

Figures 3 and 4 show AFCs and FFCs calculated by the mentioned above formula for two braking modes: with a minor slip ($S_o = 0.05$) and a significant slip ($S_o = 0.3$).
Amplitude frequency and phase frequency characteristics of the braking wheel influenced by variable basic load:

Figure 3.

- a) the longitudinal reaction;
- b) the angular acceleration,

- — — — when \( S_0 = 0.05 \);
- — — — when \( S_0 = 0.3 \)

Figure 4.

- a) the angular velocity;
- b) the slip,

- — — — when \( S_0 = 0.05 \);
- — — — when \( S_0 = 0.3 \)

In these modes, the normalized braking function \( f \) is equal to 0.6 and 0.995 correspondingly.

The values of parameters determining the characteristics were calculated for the above-mentioned tire Hankook 175/70 R13 with an inertia moment \( J_k = 0.75 \, \text{kg} \cdot \text{m}^2 \), air pressure \( P_w = 170 \, \text{kPa} \); when the average value of a normal load \( R_z = 3.9 \, \text{kN} \) and velocity \( V = 8.76 \, \text{m/sec} \) (31.5 km/h). The values of transfer coefficients are shown in Fig. 1.

### Table 1.
The values of transfer coefficients in points corresponding to the steady-state braking process when the slip is kept constant

| Transfer coefficients | Values of transfer coefficients |
|-----------------------|-------------------------------|
|                       | when \( S_0 = 0.05 \)         | when \( S_0 = 0.3 \)         |
| \( K_1 \)             | 0.000095                      | 0.000074                     |
| \( K_2 \)             | -0.0338                       | -0.03377                     |
| \( K_3 \)             | 0.51                          | 0.865                        |
The results show that a minor braking slip followed by increase in normal load variation frequency causes gradual increase in rangeability of an angular acceleration $\dot{\omega}_k$ of the wheel and longitudinal reaction $R_z$ (see Fig. 3), but their values remain small.

The rangeability of velocity $\omega$ and slip $S$ of the braking wheel tire are also small (see Fig. 4). Their dependencies have a weak extremum at frequency $\nu \approx 3$ Hz.

Increase in frequency $\nu$ which stabilizes at ~ 1.25 rad (~70°) when the frequency is ~ 4 Hz (see Fig. 3, a) causes intensive rise in a phase change of the longitudinal reaction $R_z$.

The phase change of the angular acceleration $\dot{\omega}_k$ varies only slightly with rising frequency $\nu$ and varying normal load. It amounts to 1.45 rad (~80°). The phase delay of the angular velocity and slip is insignificant (see Fig. 4).

For that braking mode, the variable normal load impacts the dynamic characteristics of the wheel due to the changes in a rolling radius $r_k$, changing under the variable normal load $R_z$.

When a wheel braking slip is near-critical ($S_0 < S_{cr}$), the time constant of a feedback is rising, and amplitude frequency and phase frequency characteristics of wheel parameters at variations $R_z$ are getting similar to the characteristics for the variable braking moment [2, 5, 6, 7, 8, 9, 10]. However PFCs are reversal. The dynamic characteristics are determined by inertia properties of the wheel and changes in a cohesion limit longitudinal reaction $R_z$.

When a wheel braking slip is super-critical ($S_0 = 0.3$), the time constant of a feedback $T_j$ is negative, and the wheel braking mode is unstable. The exceptions are small frequencies ($\nu < 1$ Hz) of normal load variations when one can observe a significant rangeability of the longitudinal reaction $R_z$ and angular wheel acceleration $\dot{\omega}_k$ at constant AFCS and minor phase changes (see Fig. 3).

The angular velocity and slip reaction to normal load variations only at small frequencies and high amplitudes (see Fig. 4).

With rising frequency of normal load variations, the impact $R_z$ on these parameters decreases, but the phase delay goes up to about 90°. At low frequencies, there exists a significant phase change of the angular acceleration. In this mode, changes in force and kinematic parameters of the wheel occur due to the changes in a cohesion limit longitudinal reaction under normal load.

4. Conclusion

At low values of the braking moment, the impact of disturbances of the variable normal load on wheel parameters is determined by variations of its force radius – a rolling radius in a driven mode; at high values – by variations of a cohesion limit longitudinal reaction.

Normal load variations impact force and kinematic parameters of the braking elastic rubber tired wheel when its potential cohesion properties manifest themselves under the interaction with a support surface.

The dynamic properties of the wheel manifest themselves when frequencies of normal load variations are $\nu = 3$ Hz. At low frequencies, they are weak.

At high rangeability of normal load variations $\Delta R_z$, because of the nonlinearity $f(S_0)$ of a diagram, wheel parameters depend on the rangeability $\Delta R_z$, rather than on their frequency $\nu$.

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