Resolutions of Orbifold Singularities and the Transportation Problem on the McKay Quiver

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Abstract. Let $\Gamma$ be a finite group acting linearly on $\mathbb{C}^n$, freely outside the origin, and let $N$ be the number of conjugacy classes of $\Gamma$ minus one.

In \cite{SI94, SI96b} a generalisation of Kronheimer’s construction \cite{Kro89} of moduli of Hermitian-Yang-Mills bundles with certain invariance properties was given. This produced varieties $X_\zeta$ (parameterised by $\zeta \in \mathbb{Q}^N$) which are partial resolutions of $\mathbb{C}^n/\Gamma$.

In this article, it is shown that $X_\zeta$ can be described as moduli spaces of representations of the McKay quiver associated to the action of $\Gamma$, subject to certain natural commutation relations.

This allows a complete description of these varieties in the case when $\Gamma$ is abelian. They are shown to be toric varieties corresponding to convex polyhedra which are the solution sets for a generalisation of the transportation problem on the McKay quiver.

The generalised transportation problem is solved for a general quiver to give a description of the extreme points, faces, and tangent cones to the solution polyhedra in terms of certain distinguished trees in the underlying graph to the quiver. Applied to the case of the McKay quiver, this gives an explicit computational procedure for calculating $X_\zeta$, its Euler number, and giving a complete list of the singularities which can occur for all $\zeta$. The $\zeta$-parameter-space $\mathbb{Q}^N$ is thus partitioned into a finite disjoint union of cones inside which the biregular type of $X_\zeta$ remains constant. Passing from one cone to the other corresponds to a birational transformation.

The example $\mathbb{C}^3/\mathbb{Z}_5$ (weights $1, 2, 3$) is worked out in detail: there are two types of $\zeta$-cones: ones for which $X_\zeta$ is a smooth resolution, and others where it has a singularity isomorphic to a cone over a quadric in $\mathbb{C}^4$. This gives some evidence for the conjecture expressed in \cite{SI96b} according to which the singularities of $X_\zeta$ are at most quadratic for a generic $\zeta$. Computer calculations also show that the cases where $\Gamma \subset SU(3)$, and $|\Gamma| \leq 11$ yield crepant $X_\zeta$. For generic $\zeta$, the Euler number of $X_\zeta$ is also equal to $|\Gamma|$, and the $X_\zeta$ give smooth crepant resolutions for the singularity.

A further example of a picture of the polyhedron corresponding to $X_\zeta$ is drawn for the singularity $\frac{1}{11}(1, 4, 6)$.

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0. Introduction

This paper is concerned with affine orbifold singularities, namely with singularities of the type \( X = \mathbb{C}^n / \Gamma \) for \( \Gamma \) a finite group acting linearly on \( \mathbb{C}^n \).

In [SI94, SI96b] a method using moduli of invariant Hermitian-Yang-Mills bundles was given for constructing partial resolutions of \( X \) carrying natural asymptotically locally Euclidean (ALE) metrics. In this article, the same construction is described in terms of representation moduli of quivers. This allows a complete description of the moduli to be given for the case of abelian groups using the language of toric varieties. It turns out that the convex polyhedra describing the moduli are the solutions of a generalisation of a well-known network optimization problem (the transportation problem) on the McKay quiver, in which the quiver plays the role of a network where commodities are transported, and the parameter \( \zeta \) specifies the supplies and demands at each vertex.

0.1. Background. The resolution of the Kleinian singularities \( \mathbb{C}^2 / \Gamma \) for \( \Gamma \) a finite subgroup of \( \text{SL}(2) \) is a classical subject; their minimal resolution \( \widetilde{X} \) was first constructed by Du Val, and Brieskorn [Bri71] showed that the components of the exceptional divisors form graphs which are dual to the homogeneous Dynkin diagrams for the Lie algebras \( A, D, E \). In 1980, McKay [McK80] remarked that this establishes a correspondence between the extended homogeneous Dynkin diagrams \( \overline{A}, \overline{D}, \overline{E} \) and the irreducible representations of \( \Gamma \). More recently [IR94] another correspondence was constructed between conjugacy classes “of weight 1” and crepant divisors for any quotient singularity generated by a finite subgroup of \( \text{SL}(n) \).
In the case where $\Gamma \subset \text{SL}(3)$, Dixon, Harvey, Vafa and Witten proposed a definition of the orbifold Euler characteristic $[\text{DHVW85, DHVW86}]$ and conjectured the existence of smooth resolutions with trivial canonical bundle (i.e. which are crepant) and whose Euler number is given by the DHVW orbifold Euler number. The final cases in the proof of this conjecture were only recently completed $[\text{MOP87, Roa91, Mar93, Roa93, itô94, Roa94}]$.

For the case $\Gamma \subset \text{SL}(4)$, the author has obtained some interesting analogous results if one considers terminalisations rather than resolutions $[\text{SI96a}]$.

A promising link between the representation theory of $\Gamma$ and the construction of resolutions was established in 1986; Kronheimer $[\text{Kro86a, Kro89}]$ constructed a family of hyper-Kähler quotients $X_\zeta$ by looking at a dimensional reduction of the Hermitian Yang-Mills (HYM) equations on a $\Gamma$-equivariant bundle over $\mathbb{C}^2$. He used the representation theory of $\Gamma$ and McKay’s observation to prove that, for generic $\zeta$, these quotients are isomorphic to the minimal resolution $\tilde{X}$. The author’s thesis $[\text{SI94}]$ (from which the present paper and its companion $[\text{SI96b}]$ are in large part extracted) grew out of the ambition to generalise the results as far as possible to dimensions higher than two.

Despite the fact that there is little hope for the results for GL($n$) and general $n$ to be as strong as those for SL(2) groups, some of the nice properties of the SL(2) case generalise to the GL($n$) case.\footnote{Some interesting new aspects are also present for the SL(3) case $[\text{SI94}]$.}

In $[\text{SI94}]$ several descriptions were given of the generalised construction. Probably the most concise is the one given in $[\text{SI96b}]$: construct moduli spaces $X_\zeta$ of instantons on the trivial bundle $\mathbb{C}^n \times R \to \mathbb{C}^n$. Here $R$ denotes the regular representation space for the group $\Gamma$, and $\zeta$ is a linearisation of the bundle action. The instantons are required to satisfy Hermitian-Yang-Mills-type equations, as well as additional $\Gamma$-equivariance and translation-invariance properties.

In the present paper, an entirely different viewpoint is adopted, namely that of considering $X_\zeta$ are representation moduli of the McKay quiver. More precisely, one considers representations $\text{Rep}_{Q,K}(R)$ of the McKay quiver $Q$ into the multiplicity space $R$ of the regular representation of $\Gamma$, subject to certain commutation relations $K$. A reductive group $\text{PGL}(R)$ acts on the above space, and one obtains GIT quotients $X_\zeta$ which depend on a rational parameter $\zeta$. 

\footnote{Some interesting new aspects are also present for the SL(3) case $[\text{SI94}]$.}
0.2. **Main Results.** This paper assumes that $\Gamma \subset \text{GL}(n)$ acts on $\mathbb{C}^n$ freely outside the origin for any $n \geq 2$, which means that $X = \mathbb{C}^n/\Gamma$ has an isolated singularity. The main results are as follows.

Firstly, $X_\zeta$ are identified with representation moduli of the McKay quiver. This is fairly straightforward and involves mostly translating quiver concepts over to the language used in [SL96b]. There are two further sets of results.

0.2.1. *Toric description of $X_\zeta$.** The first is a toric description of the representation variety $\text{Rep}_{Q, K}(\mathbb{R})$ and its moduli $X_\zeta$. The convex polyhedron $C_\zeta$ corresponding to $X_\zeta$ is identified with the projection to $\mathbb{R}^n$ of the solution polyhedron for the transportation problem on the McKay quiver. The transportation problem is a well-known linear network optimization problem [KH80, GM84]: given an assignment of real numbers to the vertices of a quiver $Q$ (thought of as representing the demand and supply of certain commodities), the aim is to find an assignment of non-negative real numbers to the arrows (a *flow* on $Q$), in such a way that the demands and supplies at each vertex are satisfied (i.e. the equation $\partial f = \zeta$ holds). For any given assignment of weights $\zeta$, the solution set is a convex polyhedron inside $\mathbb{R}^{Q_1}$ denoted by $F_\zeta$.

The notation in use is as follows:

**Notation 0.1.** Let $\Gamma$ be the cyclic group of order $r$ acting freely outside the origin in $Q$ with weights $w_1, \ldots, w_n \in \mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$ i.e. via

$$\rho: \mu_r \subset \mathbb{C}^* \longrightarrow \mathbb{C}^{*n}$$

$$\lambda \longmapsto \begin{pmatrix} \lambda^{w_1} \\ \lambda^{w_2} \\ \vdots \\ \lambda^{w_n} \end{pmatrix}.$$

The McKay quiver $Q$ of $\Gamma$ has vertices $Q_0 = \mathbb{Z}_r$, and arrows $a_v^i := v \to v - w_i \pmod{r}$ for each $v \in \mathbb{Z}_r$ and $i = 1, \ldots, n$.

Let $\pi: Q_1 \to \{1, \ldots, n\}$ be the map defined by $\pi(a_v^i) := i$ and let $\pi: \mathbb{R}^{Q_1} \to \mathbb{R}^n$ be the induced linear map defined by mapping the standard bases. (Here and elsewhere we use the exponential notation $\mathbb{R}^{Q_1} := \text{Map}(Q_1, \mathbb{R})$. The standard basis of $\mathbb{R}^{Q_1}$ is given by the “indicator” functions $\chi_a$ defined by $\chi_a(a') = 1$ if $a = a'$ and $\chi_a(a') = 0$ otherwise). Let $\Pi$ be the sub-lattice of $\mathbb{Z}^n$ of index $r$ defined by

$$\Pi := \ker \hat{\rho} = \{x \in \mathbb{Z}^n | \sum x_i w_i \equiv 0 \pmod{r}\}.$$

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2This is for the purpose of simplicity — the method would seem to be applicable to the general case with some modifications.

3The term *network* is usually used in this context instead of quiver.
For each element $\zeta \in \mathbb{Z}^{Q_0}$ such that $\sum_{v \in Q_0} \zeta(v) = 0$, consider the corresponding solution polyhedron $F_\zeta$ to the transportation problem on $Q$, and let $C_\zeta = \pi F_\zeta$ be its projection to $\mathbb{R}^n$.

Adopting the above notation, one can state the first main theorem.

**Theorem** (c.f. Thms. 5.4 and 5.5). The moduli $X_\zeta$ are isomorphic to the toric varieties

$$X_\zeta \cong T^\dim C_\zeta,$$

where $C_\zeta$ correspond to the projection to $\mathbb{R}^n$ of the solution polyhedra to the transportation problem on $Q$.

0.2.2. **Generalised Transportation Problem.** The problem of determining $C_\zeta$ can be considered as a generalised transportation problem. The solution to the classical transportation problem is well known:

**Theorem** (Classical Transportation Problem (c.f. Theorem 10)). The extreme points of $F_\zeta$ are precisely those flows in $F_\zeta$ whose supports are trees, i.e. contain no cycles.

The basic idea behind the proof of this theorem is to associate to each cycle $c$ a corresponding flow $\tilde{\chi}_c$ as follows. A cycle in $Q$ is a sequence $c_1, \ldots, c_m$ of arrows such that they form a cycle when their orientation is disregarded. The disjoint union of the arrows $c_i$ whose direction agrees (resp. disagrees) with the ordering $c_1, \ldots, c_k$ is called the positive (resp. negative) part of $c$ and is denoted it by $c^+$ (resp. $c^-$).

For each arrow $a \in Q_1$, let $\chi_a$ denote the flow which takes the value 1 on the arrow $a$ and zero elsewhere. To each cycle $c$, one can define the basic flow associated to $c$ by

$$\tilde{\chi}_c := \sum_{c_i \in c^+} \chi_{c_i} - \sum_{c_i \in c^-} \chi_{c_i} \in \mathbb{Z}^n.$$

Note that for any cycle $c$, $\partial \tilde{\chi}_c = 0$, i.e. the associated flow does not contribute anything at any vertex. If $f \in F_\zeta$ contains $c$ in its support, then one can add $\pm \epsilon \tilde{\chi}_c$ to $f$ for some small $\epsilon$ and still remain in $F_\zeta$. Hence $f$ cannot be an extreme point. Vice-versa, if $f$ is not extreme, then one can reconstruct a cycle in its support.

The generalisation of this theorem to describe the projection $C_\zeta = \pi(F_\zeta)$ is quite straight-forward. One begins by defining the type of any given cycle as the element $\pi(\tilde{\chi}_c) \in \mathbb{Z}^n$. Cycles of type $0 \in \mathbb{Z}^n$ play a special role, much as ordinary cycles do in the classical case.

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4The support of a flow is the set of arrows on which it is non-zero.
Definition 0.2. The closure of $S \subset Q_1$ is defined to be the smallest over-set $\overline{S} \supset S$ such that
\[ c^- \subseteq \overline{S} \iff c^+ \subseteq \overline{S}, \]
for all cycles $c$ of type zero. Two configurations $S, S'$ will be called equivalent (written $S \sim S'$) if $\overline{S} = \overline{S'}$.

The same methods as in the classical case allows one to prove the following generalisation.

**Theorem** (Extreme Points of $C_\zeta$). The extreme points of $C_\zeta$ are the images under $\pi$ of precisely those flows in $F_\zeta$ whose support contains no cycles of non-zero type.

In fact, by taking into account all cycles in the support of a given flow, it is possible to determine the dimension of the face which $\pi(f)$ belongs to. Some additional terminology will be convenient to state the results.

Let $C$ denote the set of all configurations, i.e. the set of non-empty subsets of $Q_1$. For a configuration $S \subset Q_1$, let $F_0(S)$ be the cone generated by the flows $\tilde{\chi}_c$ for the cycles $c$ whose negative part is included in $S$ and let $Z_0(S)$ be its maximal vector subspace (i.e. the subspace generated by the flows $\tilde{\chi}_c$ for the cycles $c \subseteq S$). The rank of $S$ is defined to be the dimension of $\pi Z_0(\overline{S})$. The set of configurations (resp. trees) of rank $k$ is denoted $C^k$ (resp. $T^k$). Also, write $C_\zeta$ (resp. $T_\zeta$) for the subset of configurations (resp. trees) which are admissible for $\zeta$, namely configurations (resp. trees) $S \in C$ which arise as the support of some element in $F_\zeta$.

**Theorem** (c.f. Theorem 7.4). For all $\zeta$, the map
\[ \text{Face}_\zeta : C_\zeta \rightarrow \text{Faces of } \pi F_\zeta \]
\[ S \mapsto \pi F_\zeta \cap (\pi f + \pi Z_0(\overline{S}))) \]
is independent of the choice of $f \in F_\zeta \cap \text{supp}^{-1}(S)$, and induces a bijection
\[ \text{Face}_\zeta : C^k_\zeta / \sim \rightarrow k\text{-faces of } \pi F_\zeta. \]
Furthermore, for all $[S] \in C^k_\zeta / \sim$,
\[ T_{\text{Face}_\zeta(S)} \pi F_\zeta = \pi F_0(\overline{S}), \]
where the left-hand side denotes the tangent cone to the polyhedron $\pi F_\zeta$ at the face $\text{Face}_\zeta(S)$.

In other words, $\pi F_0(\overline{S})$ gives the tangent cone corresponding to the configuration $S$ (which is independent of the value of $\zeta$) and $\text{Face}_\zeta(S)$
gives the corresponding face of $C_\zeta$ (whose direction is also independent of $\zeta$).

This theorem has a number of important corollaries.

**Corollary** (c.f. Cor. 8.3). If $\zeta$ and $\zeta'$ have the same admissible configurations ($C_\zeta = C_{\zeta'}$) or even just the same admissible trees ($T_\zeta = T_{\zeta'}$) then the corresponding polyhedra $C_\zeta$ and $C_{\zeta'}$ are geometrically isomorphic. Two polyhedra are said to be geometrically isomorphic if they are combinatorially isomorphic and their tangent cones at the corresponding faces are identical. In particular, their associated fans and toric varieties must be identical.

**Corollary** (c.f. Cor. 8.2). Let $T \subset C$ denote the configurations which are trees, and let $\text{ext } C_\zeta$ denote the extreme points of $C_\zeta$. Then

$$\# \text{ ext } C_\zeta = \# C_0^\zeta / \sim = \# T_0^\zeta / \sim.$$ 

**Remark** 0.3. Since any tree admits a unique flow such that $\partial f = \zeta$, it is very easy to determine $T_\zeta$. Furthermore, if $\zeta$ is generic, then $C_\zeta \subset C_{\text{span}}$, the subset of spanning configurations, i.e. configurations whose arrows join any two vertices of the quiver (not necessarily in an oriented way). In this case, $T \in T_0^\zeta$ if and only if the tree $T$ admits an assignment of elements of $\mathbb{Z}^n$ to the vertices which satisfies particular properties (a so-called $n$-weighting in the terminology of Section 12.2.1). This condition can again be checked by a very simple algorithm.

To sum up: the above corollary allows one to determine the extreme points of $C_\zeta$ for any $\zeta$ very easily.

**Corollary** (c.f. Cor. 8.3 and Lemma 12.4). The extreme points of the polyhedron $\pi F_\zeta$ correspond to the trees $T$ in $T_0^\zeta$ and the tangent cone to $\pi F_\zeta$ at the point corresponding to $T$ is $\pi F_0(T) = \pi F_0(T)$. Thus the fan associated to the polyhedron $\pi F_\zeta$ is given by the dual cones $\pi F_0(T)^\vee$ for the trees $T \in T_0^\zeta$ and all their faces.

The theorem and corollaries above allow a complete understanding of the extreme points, faces, and tangent cones to the solution polyhedra of a generalised transportation problem. Applied to the case of the McKay quiver, they allow one to determine the Euler number and the singularities of $X_\zeta$ as well as giving a complete list of the singularities which can occur for all $\zeta$.

Several interesting questions regarding these moduli nevertheless remain, such as whether they produce smooth (resp. terminal) resolutions in the SL(3) (resp. SL(4)) case. A conjecture is given in the
companion paper [SI96b] and several examples are computed in Section 12 of this paper.

Depending on the point of view, isomorphic objects are denoted by different notations. For ease of reference, Table 1 gives a correspondence table between the point of view in [SI96b] and in Parts 1 and 2 of this paper.

| Moduli of Bundles                  | Moduli of Representations          | Toric Varieties                  |
|------------------------------------|------------------------------------|----------------------------------|
| $M^\Gamma$                         | $\text{Rep}_Q(R)$                  | $T^{\Lambda^0,0} = CQ_1$        |
| $N^\Gamma$                         | $\text{Rep}_{Q,K}(R)$              | $T^{\Lambda,0}$                  |
| $G^\Gamma = \text{PGL}^\Gamma(R)$ | $\text{PGL}(R)$                   | $T^{\Lambda,0} = T^{0,0}$        |
| $X_\zeta$                          | $\mathcal{M}_\zeta$               | $T^{\Pi,C_\zeta}$               |

**Table 1.** Correspondence between the notations in use in this paper and in the companion paper [SI96b].

0.3. **Methods and Outline.** The methods used to prove the main results are notationally cumbersome due to the quiver notation, but on the whole straightforward. A familiarity with toric geometry will make reading easier, although all the required facts and notation are explained when needed.

The paper is divided into two parts; the first dealing with general groups, and the second specializing to the case of abelian ones. Each part begins with a section summarizing the notation in use.

0.3.1. **Part 1.** Section 1 summarises the notation.

Section 2 outlines the basic facts and notation concerning quivers, their representations and their moduli. The necessary details regarding geometric invariant theory (GIT) quotients are also given.

Section 3 explains the case of the McKay quiver associated to the representation of a finite group. The McKay correspondence is briefly mentioned, and the “canonical” commutation relations $K$ are defined.

0.3.2. **Part 2.** Section 4 summarises the notation.

Section 5 specialises further the discussion from Part 1 to the case of abelian groups. The McKay quiver and commutation relations for these can be described quite simply and some specialised notation is
introduced for this purpose. The representation variety and its moduli are proved to be toric varieties corresponding to an \((n + \frac{1}{2} \Gamma - 1)\)-dimensional convex cone \(C\) and to its \(n\)-dimensional convex polyhedral slices \(C_\zeta\) respectively.

From Section 6 onwards, the focus is on the transportation problem; its solution polyhedra coincide with \(C_\zeta\) in the case of the McKay quiver.

Section 7 explains the ordinary transportation problem on a network.

Section 8 explains the generalised transportation problem and states the theorems describing its solution polyhedra.

An example of an application to the McKay quiver and several important corollaries are given in Section 8.

The proofs of the theorems are given in Section 9, except for the proof of some technical lemmas regarding flows which are left until Section 10.

Section 11 contains a discussion of the singularities of \(X_\zeta\).

The paper concludes in Section 12 with some practical examples and computations.

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1. Summary of Notation for Part I

1.1. General.

\(k\) Algebrically closed field.
\(\mathbb{Z}_+\) Non-negative integers.
\(\mathbb{R}_+\) Non-negative reals.

\(^5\)Minor portions have been rewritten to include references to advances in the field made since then (notably [Itô94, Roa94, IR94]).
1.2. **Quivers, Representations.**

- **Q** 
  - n-dimensional complex vector-space.
- **Γ** 
  - Finite sub-group of SL(Q)
- **Q** 
  - Generic quiver $Q = (Q_0, Q_1)$.
  - From Section $3$ onwards, $Q = Q_{\Gamma,Q}$; the McKay quiver associated to $(\Gamma, Q)$.
- **Q_0** 
  - Vertices of $Q$; in the case of the McKay quiver, elements of $Q_0$ index the irreducible representations $R_v$ of $\Gamma$.
- **Q_1** 
  - Arrows of $Q$.
- **R_v** 
  - Irreducible representation of $\Gamma$ corresponding to vertex $v$ of the McKay quiver.
- **R** 
  - Regular representation of $\Gamma$; $R = \bigoplus_{v \in Q_0} R_v \otimes R_v$.
- **R_v** 
  - The trivial $\Gamma$-module giving the multiplicity of $R_v$ in $R$.
- **V** 
  - Multiplicity space for $R$; $V = \bigoplus_{v \in Q_0} V_v$.
- **R_V** 
  - The $\Gamma$-module $\bigoplus_{v \in Q_0} V_v \otimes R_v$ corresponding to $V$.
- **α** 
  - $\Gamma$-invariant element of $(Q \otimes \text{End } R)$.
- **q_i** 
  - Basis of $Q$ ($i = 1, \ldots, n$).
- **α_i** 
  - Component of $\alpha$ with respect to $\{q_i\}$: $\alpha = \sum_{i=1}^n q_i \otimes \alpha_i$.
- **\tilde{\alpha}** 
  - Corresponding representation of the McKay quiver into $R$.
- **a** 
  - Arrow of $Q$; also written $t(a) \to h(a)$, where $t(a)$ denotes the tail and $h(a)$ the head of $a$.
- **\tilde{\alpha}_a** 
  - Value of $\tilde{\alpha}$ on the arrow $a$; $\tilde{\alpha}_a: R_{t(a)} \to R_{h(a)}$.
- **K** 
  - Relations on $Q$.
- **Rep_{Q,K}(R)** 
  - Space of all representations of $Q$ into $R$ satisfying the relations $K$.
- **PGL(R)** 
  - The isomorphism group for representations of $Q$ into $R$
  
  $$PGL(R) := \times_{v \in Q_0} GL(R_v) / \mathbb{C}^*.$$ 
- **\xi** 
  - The Lie algebra of $PU(R)$ (a real form of $PGL(R)$).
- **M_{Q,K,\zeta}** 
  - Representation moduli of $(Q, K)$
  
  $$M_{Q,K,\zeta} := \text{Rep}_{Q,K}(R) \sslash \chi PGL(R).$$
  
  The linearisation $\chi$ is related to $\zeta$ by $\zeta = d\chi(1)|_{\Gamma}$.
- **X_\zeta** 
  - The moduli above in the case when $Q$ is the McKay quiver, and $K$ are the commutation relations defined in Section $3.2$.
- **\rho_\zeta** 
  - The natural projective morphism $X_\zeta \to X_0$ (a partial resolution).

**Part 1. General Groups**

2. **Quivers, Relations and Representation Moduli**
2.1. Quivers. A quiver is an oriented graph, possibly with multiple arrows between the same vertices and with loops (arrows which begin and end at the same vertex). Formally, a quiver $Q$ consists of a pair of finite sets $Q = (Q_0, Q_1)$ with two maps $Q_1 \xrightarrow{t} Q_0$; the elements of $Q_0$ are called vertices and those of $Q_1$ are called arrows. The elements $t(a)$ and $h(a)$ are called the tail and head of the arrow $a \in Q_1$ respectively. The arrow $a$ is also sometimes denoted $t(a) \to h(a)$.

2.2. Representations. Let $k$ be an algebraically closed field.

A representation of a quiver is a realization of its diagram of vertices and arrows in some category: it corresponds to replacing the vertices by objects and the arrows by morphisms between the objects. From now on, only the category of $k$-vector-spaces is considered; a representation $V$ of a quiver $Q$ is thus taken to be a collection of finite dimensional vector-spaces $V_v$, indexed by the vertices $v \in Q_0$, and of linear maps $V_{v \to v'} : V_v \to V_{v'}$, indexed by the arrows $v \to v' \in Q_1$.

The set of all representations of $Q$ into a fixed $Q_0$-graded vector-space $V = \oplus_{v \in Q_0} V_v$ forms a vector-space, denoted $\text{Rep}_Q V$.

Example 2.1. Representations of the quiver $Q$ with one vertex and one loop are just endomorphisms of a vector-space.

Example 2.2. A primitive representation of $Q$ is an element of $\text{Rep}_Q (kQ_0)$, where $kQ_0$ denotes the free $k$-vector-space on the vertices. A primitive representation thus corresponds to an assignment of an element of $k$ to each arrow in $Q$, i.e. to an element of the vector-space $kQ_1$.

2.3. Relations. If the morphisms $V_{v \to v'}$ are required to satisfy relations between them, then one talks about a representation of a quiver with relations.

More formally, a relation is defined as a formal sum of paths in a quiver. A (non-trivial) path $p$ is a sequence $a_1 \ldots a_n$ of arrows which compose, i.e. such that $t(a_i) = h(a_{i+1})$ for $1 \leq i < n$:

$$h(p) \leftarrow a_1 \leftarrow a_2 \leftarrow \ldots \leftarrow a_n \leftarrow t(p) .$$

The vertex $h(a_1)$ ($t(a_n)$) is called the head (tail) of the path $p$ and denote it by $t(p)$ ($h(p)$). If $V$ is a representation of $Q$, then there is an induced morphism

$$V(p) = V_{a_1} V_{a_2} \ldots V_{a_n} : V_{t(p)} \to V_{h(p)}$$
corresponding to each path. The trivial path \( e_v \) consists of a single vertex \( v \) and no arrows; \( V_{e_v} \) is of course the identity endomorphism of \( V_v \).

A representation is said to satisfy \( r = \sum \lambda_i p^i \) if \( \sum \lambda_i V(p^i) = 0 \). If \( \mathcal{K} \) denotes a set of relations, then the set of representations of \( Q \) into \( V \) satisfying the relations \( \mathcal{K} \) will be denoted \( \text{Rep}_{Q, \mathcal{K}}(V) \). It is an affine variety inside \( \text{Rep}_Q(V) \).

**Remark 2.3.** There are in general many classes of relations one can consider, but in this paper, only commutation relations will be considered, namely relations generated by differences of two paths, both paths having the same tail and head.

2.3.1. Morphisms of Representations. Let \( V, V' \) be two representations of the same quiver \( Q \) (possibly with relations). A morphism of representations \( \theta: V \to V' \) is given by morphisms \( \theta_v: V_v \to V'_v \) for each vertex \( v \), which satisfy \( V'_a \theta_{t(a)} = \theta_{h(a)} V_a \) for each arrow \( a \). If \( V_v = V'_v \) and the linear maps \( \theta_v \) are all isomorphisms, then \( \theta \) is called an isomorphism of representations.

2.3.2. Representation Moduli. Given a quiver \( Q \), some relations \( \mathcal{K} \) and a representation space \( V \), one is naturally interested in the moduli space of representations. Loosely speaking, this is the set of isomorphism classes of representations of \( Q \) into \( V \). More precisely, the group

\[
\text{GL}(V) := \times_{v \in Q_0} \text{GL}(V_v)
\]

acts on \( \text{Rep}_{Q, \mathcal{K}}(V) \) and its orbits consist of equivalence classes of representations. The scalar subgroup \( \mathbb{C}^* \subset \text{GL}(V) \) acts trivially, and one is left with a free action of the quotient \( \text{PGL}(V) := \text{GL}(V)/\mathbb{C}^* \).

Since \( \text{PGL}(V) \) is not compact, one must resort to geometric invariant theory (GIT) in order to obtain quotients which are well-defined as quasi-projective varieties. The “canonical” quotient is the affine GIT quotient

\[
\mathcal{M}_{Q, \mathcal{K}, 0} := \text{Rep}_{Q, \mathcal{K}}(V) \sslash \text{PGL}(V).
\]

(2.1)

There are other possible quotients however and their existence is in fact the basis of this paper. In general, given a complex affine variety \( X \) acted upon by a reductive group \( G \), and given a character \( \chi: G \to \mathbb{C}^* \), one defines

\[
X/\chi G := \text{Proj} \bigoplus_{r \in \mathbb{N}} \mathcal{O}_X(rL_\chi)^G,
\]

where \( L_\chi \) is the trivial bundle over \( X \), on which \( G \) acts via \( \chi \). The reader is referred to [SI96b, SI94] for a detailed treatment.
When the complex reductive group $G$ has a real form $K$, rather than specifying the character $\chi$ of $G$ one can specify instead its derivative at the identity $\text{Lie} G \to \mathbb{R}$, or even the restriction of the latter to the Lie algebra of the real form, giving an element of $\zeta = d\chi(1)_{|\text{Lie} K} \in (\text{Lie} K)^\ast$.

For the moduli of representations of $Q$ into $V$, a real form of $\text{PGL}(V)$ is given by $\text{PU}(V) := \times_{v \in Q_0} U(V_v)/U(1)$, and the dual of its Lie algebra is the subspace

$$\mathfrak{k} := \{ \zeta \in \times_{v \in Q_0} \text{End}_\mathbb{R}(V_v) | \sum_{v \in Q_0} \text{trace} \zeta_v = 0 \},$$

where $\zeta_v$ denotes the restriction of $\zeta$ to $\text{End}_\mathbb{R}(V_v)$. For $\zeta$ an integral element of $\mathfrak{k}$, define

$$\mathcal{M}_{Q,K,\zeta_\chi} := \text{Rep}_{Q,K}(V) /_{\chi} \text{PGL}(V), \quad \text{where } \zeta_\chi = d\chi(1)_{|\text{Lie} K} \quad (2.2)$$

This is seen to coincide with Definition (2.1) in the case $\zeta_\chi = 0$ (i.e. $\chi = 1$).

**Remark 2.4.** In the above, “$\zeta$ integral” means integral with respect to the natural lattice in $\mathfrak{k}$ (i.e. the kernel of the exponential map). Strictly speaking, one is not restricted to integral values, any rational value will do, provided one makes sense of “fractional linearisations” in the obvious manner. We do not bother, as nothing substantially new is gained from this approach (the moduli for $k\zeta$ are isomorphic to those for $\zeta$).

### 3. The McKay Quiver

The McKay quiver $Q_{\Gamma, Q}$ is a quiver which is naturally associated to a representation $Q$ of a finite group $\Gamma$. As explained below, its vertices are the irreducible representations of $\Gamma$ and the arrows describe how the tensor product of $Q$ with each irreducible decomposes into a sum of irreducibles.

It seems natural to expect a relation between $Q_{\Gamma, Q}$ and the quotient singularity $X = Q/\Gamma$. In fact, for the case of a finite subgroup $\Gamma \subset \text{SU}(2)$ there is a remarkable relation between $Q_{\Gamma, Q}$ and the minimal resolution $\widetilde{X} \to X$. McKay remarked [McK80] that the quivers $Q_{\Gamma, C^2}$ are precisely the extended Dynkin diagrams of type $\mathbb{A}$, $\mathbb{D}$, and $\mathbb{E}$. The ordinary Dynkin diagrams of type $A, D$ and $E$ had previously been shown by Brieskorn [Bri71] to be the dual graphs to the graphs of rational curves in the exceptional fibre of $\widetilde{X} \to X$.

Kronheimer [Kro86a, Kro86b, Kro89] showed that one can construct the minimal resolution $X$ by considering what turn out to be, upon
closer inspection, moduli spaces of representations of $Q_{\Gamma, C^2}$ into the regular representation space of $\Gamma$. The representations are also required to satisfy some commutation relations, denoted $K$.

In this section, the generalisation of these commutation relations to any McKay quiver is described.

3.1. The McKay Quiver. Let $\Gamma$ be a finite group and let $\{R_i, i \in I\}$ be the set of irreducible representations of $\Gamma$. Any $\Gamma$-module $R_V$ decomposes into a sum of irreducibles:

$$R_V = \bigoplus_{i \in I} V_i \otimes R_i,$$

(3.1)

and gives an $I$-graded vector-space (trivial as a $\Gamma$-module) $V = \oplus_{i \in I} V_i$, called the multiplicity space for $R_V$. The dimension of $V_i$ is called the multiplicity of $R_i$ in $R_V$ and the vector $\dim V = (\dim V_i)_{i \in I}$ is called the dimension vector of the $\Gamma$-module $R_V$. Conversely, given any $I$-graded vector-space $V$, one can construct a corresponding $\Gamma$-module $R_V$ by formula (3.1).

The McKay quiver $Q = Q_{\Gamma, Q}$ of a representation $Q$ is constructed as follows. The vertices $Q_0 = I$ are the irreducible representations of $\Gamma$, and there are $a_{ij}$ (possibly zero) arrows from vertex $i$ to vertex $j$. The non-negative integers $a_{ij}$ (which form what is called the adjacency matrix of $Q$) are defined by the following irreducible decompositions (for each $i \in I$)

$$Q \otimes R_i = \bigoplus_j a_{ji} R_j.$$

(3.2)

More invariantly, one may write

$$Q \otimes R_i = \bigoplus_j A_{ji}^* \otimes R_j$$

(3.3)

where

$$A_{ji} : = \text{Hom}_\Gamma(Q \otimes R_i, R_j),$$

(3.4)

and think of the arrows from $i$ to $j$ as giving a basis for the $a_{ij}$-dimensional vector-space $A_{ij}$.

If $V$ is a $Q_0$-graded vector-space, then the representations of $Q$ into $V$ correspond to the $\Gamma$-module endomorphisms $R_V \rightarrow Q \otimes R_V$. In fact, $\text{Hom}_\Gamma(R_V, Q \otimes R_V) = \text{Hom}_\Gamma(\oplus_i V_i \otimes R_i, Q \otimes (\oplus_j V_j \otimes R_j))$

$$= \bigoplus_{i,j} A_{ij}^* \otimes \text{Hom}(V_i, V_j),$$

(3.5)
so given $\alpha \in \text{Hom}_\Gamma(R_V, Q \otimes R_V)$, one can pair it with an arrow $a = i \to j$ (considered as a basis element of $A_{ij}$) and obtain a map $V_i \to V_j$. Thus

$$\text{Hom}_\Gamma(R_V, Q \otimes R_V) = \text{Rep}_Q(V).$$

The representation of $Q$ which corresponds to $\alpha \in \text{Hom}_\Gamma(R_V, Q \otimes R_V)$ will be denoted $\tilde{\alpha}$.

**Remark 3.1.** Note that the orientation of $Q$ is reversed when the representation $Q$ is replaced by its dual $Q^*$, so when $Q$ is self-dual all the arrows $i \to j$ have opposite arrows $j \to i$. The pair $i \rightleftharpoons j$ is usually denoted $i \leftrightarrow j$.

**Example 3.2** (The McKay Correspondence). For a subgroup $\Gamma < \text{SU}(2)$, the standard 2-dimensional representation $Q$ is always self-dual, since $\Lambda^2 Q \cong \mathbb{C}$ as $\Gamma$-modules. McKay’s observation [McK80] is that the quivers $Q_{\Gamma, Q}$ coincide with the extended homogeneous† Dynkin diagrams $\overline{A}_k, \overline{D}_k, \overline{E}_6, \overline{E}_7, \overline{E}_8$ represented in Figure 1.

![Figure 1. The extended homogeneous Dynkin diagrams $\overline{A}_k, \overline{D}_k, \overline{E}_6, \overline{E}_7, \overline{E}_8$ (The extra vertex is indicated marked •.)](image)

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3.2. **The Commutation Relations.** Recall from Section 3.1 that if $R_V$ is any $\Gamma$-module, the $\Gamma$-module homomorphisms $R_V \to Q \otimes R_V$ correspond to representations of the McKay quiver into the $I$-graded vector-space $V$ which is the multiplicity space for $R_V$.

One natural $\Gamma$-module to consider as a candidate for $R_V$ is the “canonical one” given by the regular representation space $R$ of $\Gamma$. Its

†A Dynkin diagram is called *homogeneous* if it has no multiple bonds.
multiplicity space will be denoted \( R = \bigoplus_{v \in Q_0} R_v \). Its components satisfy the well-known equalities \( \dim R_v = \dim R_v \).

The simplest way to define the commutation relations \( \mathcal{K} \) is to use the isomorphism (3.6). Let \( q_i \) be a basis of \( Q \) which is orthonormal with respect to some positive definite \( \Gamma \)-invariant hermitian inner product. Let \( \tilde{\alpha} \in \text{Rep}_Q(R) \) be a representation of \( Q \) into \( R \), and let \( \alpha \in Q \otimes \text{End}_C R \) be the \( \Gamma \)-invariant element which corresponds via the isomorphism (3.6). The group \( \Gamma \) acts on \( \text{End}_C R \) in the natural way, i.e. by conjugation, and the element \( \alpha \) decomposes with respect to the basis \( q_i \) into endomorphisms \( \alpha_i \in \text{End}_C R \) for \( i = 1, \ldots, n \) which satisfy the following equivariance condition

\[
\sum_l \gamma_{kl} \alpha_l = \varphi(\gamma) \alpha_k \varphi(\gamma)^{-1}, \quad \forall k, \gamma, \tag{3.7}
\]

where \( \gamma = (\gamma_{kl}) \) is the matrix corresponding to the action of the element \( \gamma \) on \( Q \) with respect to the basis \( \{q_i\}_{i=1}^n \).

The commutation relations are defined by the condition

\[
[\alpha_i, \alpha_j] = 0. \tag{3.8}
\]

Thus the variety \( \mathcal{M}_{Q,K}(R) \) of representations of \( Q \) into \( R \) satisfying these relations coincides with the variety \( \mathcal{N}^T \) defined in [S196b, S1994].

3.3. Representation Moduli. Given \( \zeta \in \mathfrak{f} \), the representation moduli of the McKay quiver of \((\Gamma, Q)\) in the multiplicity space \( R \) for the regular representation \( R \) of \( \Gamma \) are defined by the GIT quotients

\[
\mathcal{M}_\zeta = \text{Rep}_{Q,K}(R)/\zeta \text{ PGL}(R).
\]

Since the group \( \text{PGL}(R) \) coincides with the group \( \text{PGL}^\Gamma(R) \) (the projectivization of the group of \( \Gamma \)-invariant linear endomorphisms of \( R \)), one sees that the moduli \( \mathcal{M}_\zeta \) coincide with the moduli \( X_\zeta \) of Hermitian-Yang-Mills type bundles defined and studied in [S196b].

Part 2. Abelian Groups

In Part 1 the moduli \( X_\zeta \) were shown to coincide with the moduli of Hermitian-Yang-Mills bundles defined in [S196b] and hence \((\Gamma \text{ acts freely outside the origin)} \) to provide partial resolutions \( \rho_\zeta : X_\zeta \to X_0 = Q/\Gamma \) of the isolated quotient singularity.

The focus from now on will be the moduli \( X_\zeta \) in the case where \( \Gamma \) is an abelian group (of order \( r \)) acting linearly on \( Q \cong \mathbb{C}^n \). (Later we shall specialise to the cyclic group of order \( r \).)

When \( \Gamma \) is abelian, the singularity \( X \) is toric and can be resolved within the toric category, in general in many ways. In fact, as shown in
the first section below, the space of representations $\text{Rep}_{Q,K}(\mathbb{R})$ and its quotients $X_\zeta$ are also toric varieties; in toric notation (reviewed in 5.3)

$$\text{Rep}_{Q,K}(\mathbb{R}) = \mathcal{T}^{\Lambda,C} \text{ and } X_\zeta = \mathcal{T}^{\Pi,C_\zeta},$$

where $\Lambda, \Pi$ are certain lattices, and where the $n$-dimensional convex polyhedra $C_\zeta$ are obtained by ‘slicing’ the $n + r - 1$-dimensional cone $C$ by affine $n$-planes.

The rest of this paper is devoted to describing the slices $C_\zeta$, and in particular their extreme points and their tangent cones. The number of extreme points is the Euler characteristic of $X_\zeta$, and the geometry of the tangent cones of $C_\zeta$ at these points describes the singularities of $X_\zeta$.

There are two main steps in the study of the slices $C_\zeta$. The first step is to reduce their description to a network flow problem on the McKay quiver. The particular problem turns out to be a generalisation of the classical transportation problem \cite{KH80, GM84}. This is done in Section 5.3.

The second step is to describe the polyhedra which solve the generalised transportation problem. A solution for a general quiver is given in Section 6. The main result is Theorem 7.3, which says that the flows which correspond to the extreme points of $C_\zeta$ are precisely the ones whose support contains no cycles of non-zero type in its “closure” (see §2.7.1 for the definition). This is then applied to the case of the McKay quiver in Sections 10 and 11 to describe the slices $C_\zeta$ for all $\zeta$.

The last section in this part chapter concerns various conjectures, in particular for singularities of dimension 3. One hope is that, in general, the quotients $X_\zeta$ provide a class of partial desingularisations of the isolated quotient singularity $X$, which will be in some sense “natural,” and, in good cases, non-singular and minimal with respect to this property. This is motivated by the fact that Kronheimer \cite{Kro86a} has shown that for the case $\Gamma \subset \text{SU}(2)$, the $X_\zeta$ coincide with the minimal smooth resolution of the singularity for generic values of $\zeta$. One candidate for a “good case” is the case when $\Gamma \subset \text{SU}(3)$.

**Conjecture 1.** If $\Gamma \subset \text{SU}(3)$ is abelian and acts on $\mathbb{C}^3$ freely outside the origin then $\rho_\zeta: X_\zeta \to X_0 = \mathbb{C}^3/\Gamma$ is crepant and is a smooth resolution for generic values of $\zeta$.

Using the description of $C_\zeta$, this conjecture is reduced in Section 12.4 to a statement concerning the existence and combinatorial properties of certain trees in the McKay quiver. Brute-force computer calculations show the conjecture to be true for the singularities $\frac{1}{6}(1, 2, 3), \frac{1}{7}(1, 2, 4), \ldots$.
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\(\frac{1}{8}(1, 2, 5), \frac{1}{9}(1, 2, 6), \frac{1}{10}(1, 2, 7)\) and \(\frac{1}{11}(1, 2, 8)\), but there is as yet no general proof. Other conjectures regarding, for example, the Euler number of the resolutions also translate to graph theoretical and combinatorial statements on the McKay quiver.

4. Summary of Notation for Part

\(\Gamma\) Finite abelian group \(\Gamma\) of order \(r\) (usually \(\mu_r\)).

\(\mu_r\) Group of \(r\)-th roots of unity in \(\mathbb{C}^*\).

\(\hat{\Gamma}\) Character group \(\hat{\Gamma} := \text{Hom}(\Gamma, \mathbb{C}^*)\).

\(\rho\) The morphism giving the action of \(\Gamma\) on \(\mathbb{C}^n\):

\[\rho : \Gamma \rightarrow \mathbb{C}^n\]

\(w_i\) Weights \((w_i \in \mathbb{Z}; i = 1, \ldots, n)\) for the action of \(\Gamma\) on \(\mathbb{C}^n\).

\(\hat{\rho}\) Dual morphism \(\hat{\rho} : \mathbb{Z}^n \rightarrow \hat{\Gamma}\).

\(\Pi\) A sub-lattice of \(\mathbb{Z}^n\) of index \(r\) given by

\[\Pi := \ker \hat{\rho} = \{x \in \mathbb{Z}^n\mid \sum x_iw_i \equiv 0 \pmod{r}\}\].

\(\frac{1}{r}(w_1, \ldots, w_n)\) The quotient singularity \(\mathbb{C}^n/\mu_r\), where \(\mu_r\) acts on \(\mathbb{C}^n\) with weights \((w_1, \ldots, w_n)\).

\(R_v\) One-dimensional representation on which \(\Gamma\) acts by \(\lambda \rightarrow \lambda^v\).

\(\nu\) Natural morphism \(\nu : \Gamma \rightarrow \text{PGL}(\mathbb{R})\).

\(\hat{\nu}\) Dual morphism \(\hat{\nu} : \Lambda^{0,0} \rightarrow \hat{\Gamma}\).

\(a^i_v\) Arrow of type \(i\):

\[a^i_v := v \rightarrow v - w_i \ (v \in \mathbb{Q}, \ i \in \{1, \ldots, n\}\).

\(\tilde{\alpha}(a^i_v)\) Value of \(\tilde{\alpha}\) on \(a^i_v\); Equal to the \((v - w_i, v)\)-th entry of the matrix \(\alpha\).

4.1. Toric Geometry.

\(M\) Generic lattice.

\(M_{\mathbb{Q}}\) Associated rational vector-space.

\(T^M\) Algebraic torus \(T^M := \text{Spec} \mathbb{C}[M]\).

\(P\) Generic polyhedron in \(M_{\mathbb{Q}}\).

\(\mathcal{T}^{M,P}\) Quasi-projective toric variety \(\text{Proj} \mathbb{C}[\tilde{P} \cap \tilde{M}]\) defined by \(M\) and \(P\).

\(\tilde{M}\) Product lattice \(\tilde{M} := \mathbb{Z} \times M\).

\(\tilde{P}\) (Closure of) the cone over \(P\):

\[\tilde{P} := \overline{\mathbb{Q}_{\geq 0}(\{1\} \times P)} \subset M_{\mathbb{Q}}\].

\(T_F P\) Tangent cone of \(P\) at face \(F\).

4.2. Flows.

\(k^A\) Set (lattice, vectorspace, cone) of maps \(A \rightarrow k\). Used for \(k = \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}, \mathbb{R}_+, \mathbb{C}\), and \(A = \mathbb{Q}_0, \mathbb{Q}_1, S\).

\(f\) Generic flow (element of \(\mathbb{R}^{Q_1}\)).

\(f^\pm\) Positive/negative part of \(f\).

\(\Lambda^1\) Lattice \(\mathbb{Z}^{Q_1}\) of integer-valued flows on \(Q\).

\(\Lambda^1_R\) Vectorspace of real-valued flows on \(Q\).
\[ \Lambda^1_{\mathbb{R}^+} \] First quadrant in \( \Lambda^1_{\mathbb{R}} \) (non-negative real-valued flows).

\( \chi_a \) Flow taking the value 1 on \( a \) and zero elsewhere. Basis element of \( \Lambda^1 \).

\( \chi^i_v \) Alternative notation for \( \chi_{a^i_v} \).

\( \Lambda^0 \) Lattice \( \mathbb{Z}^{Q_0} \) of assignments of integers to the vertices of \( Q \); \( \Lambda^0_{\mathbb{R}} \) coincides with Lie \( GL(\mathbb{R}) \).

\( \Lambda^{0,0} \) Sub-lattice \( \{ \zeta \in \mathbb{Z}^{Q_0} | \sum_{v \in Q_0} \zeta(v) = 0 \} \) of \( \Lambda^0 \); \( \Lambda^{0,0}_{\mathbb{R}} \) coincides with Lie \( PGL(\mathbb{R}) \).

\( \zeta \) Element of \( \Lambda^{0,0} \).

\( \text{In}(v) \) Set of arrows \( a \) such that \( h(a) = v \).

\( \text{Out}(v) \) Set of arrows \( a \) such that \( t(a) = v \).

\( \partial \) Natural map \( \partial: \Lambda^1 \to \Lambda^{0,0} \) which calculates the net contribution of a flow at each vertex:

\[ \partial f(v) = \sum_{a \in \text{In}(v)} f(a) - \sum_{a \in \text{Out}(v)} f(a). \]

\( \chi_v \) Function taking the value 1 on \( v \) and zero elsewhere. Basis element of \( \Lambda^0 \).

\( \Lambda^2 \) Sub-lattice of \( \Lambda^1 \) generated by the elements corresponding to the commutation relations

\[ \chi^i_v + \chi^j_v - \chi^i_{v-w_i} - \chi^j_{v-w_j}, \quad \text{for } i, j = 1, \ldots, n \]

\( \Lambda \) Quotient lattice \( \Lambda^1/\Lambda^2 \).

\( C \) Image of cone \( \Lambda^1_{\mathbb{R}^+} \) inside the quotient \( \Lambda \). Also written \( \Lambda_{\mathbb{R}^+} \).

\( C_\zeta \) Convex polyhedron obtained by slicing \( C \) with the hyper-plane \( \partial f = \zeta \). Equal to the image of \( F_\zeta \) under \( \pi \).

\( \pi \) Map \( \pi: Q_1 \to \{1, \ldots, n\} \) which assigns to each arrow of the McKay quiver its type: \( \pi(a^i_v) := i \). Induces a map \( \pi: \Lambda^1_{\mathbb{R}^+} \to \mathbb{R}^n \) on the flows:

\[ \pi(f) = \left( \sum_{v \in Q_0} f(a^1_v), \ldots, \sum_{v \in Q_0} f(a^n_v) \right). \]

4.3. Configurations.

\( S \) Configuration of arrows (non-empty subset of \( Q_1 \)).

\( \text{supp } f \) Support of the flow \( f \) (arrows on which \( f \) takes a non-zero value).

\( \mathcal{C} \) Set of all configurations of arrows.

\( \mathcal{T} \) Set of all configurations of arrows which are trees (i.e. which contain no cycles).

\( F_\zeta \) Admissible (i.e. non-negative) flows which satisfy \( \partial f = \zeta \).

\( F_\zeta(S) \) Flows satisfying \( \partial f = \zeta \) which are non-negative outside \( S \). (A convex cone if \( \zeta = 0 \)).
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\( Z_\zeta(S) \) Flows satisfying \( \partial f = \zeta \) which are zero outside \( S \). (Equal to the maximal subspace contained in \( F_0(S) \) if \( \zeta = 0 \).

\( f_\zeta \) Map \( f_\zeta : T \to \Lambda^1_\mathbb{R} \) assigning to each tree \( T \) the unique flow \( f = f_\zeta(T) \) such that \( \partial f = \zeta \).

\( \Lambda^{0,0}_\mathbb{R}(T) \) Open convex cone in \( \Lambda^{0,0}_\mathbb{R} \) of values of \( \zeta \) for which the tree \( T \) is the support of some flow in \( F_\zeta \).

\( \mathcal{D} \) Generic subset of \( \mathcal{C} \).

\( \mathcal{D}_\zeta \) Elements of \( \mathcal{D} \) which are the support of some \( f \in F_\zeta \).

\( \mathcal{D}_{\text{span}} \) Elements of \( \mathcal{D} \) which span all vertices of \( \mathcal{Q} \).

\( \mathcal{D}^k \) Elements of \( \mathcal{D} \) of rank \( k \) (i.e. elements \( S \) such that rank \( \pi F_0(S) = k \).

4.4. Cycles and Paths.

\( p \) Generic path in \( \mathcal{Q} \).

\( p^+ \) Positive part of \( p \).

\( p^- \) Negative part of \( p \).

\( \tilde{\chi}_p \) Basic flow associated to the path \( p \).

\( c \) Generic cycle in \( \mathcal{Q} \).

\( \{v\}(j_0, \ldots, j_k) \) Notation for basic flows.

4.5. Miscellaneous.

\( \mathcal{W}_S \) Weighting of the vertices of \( \mathcal{Q}_0 \) (a map \( \mathcal{Q}_0 \to \mathbb{Z}^n \) with special properties).

5. Representations and Relations in the Abelian Case

5.1. The McKay Quivers. In the abelian case, all the irreducible representations are one-dimensional and are determined by an element of the dual group \( \hat{\Gamma} \), which is a finite abelian group isomorphic to \( \Gamma \). Thus the vertex set of \( \mathcal{Q} \) is nothing but \( \hat{\Gamma} \). The group \( \Gamma \) will always be identified with a subgroup of \( \mathbb{C}^\times \subset \mathbb{C}^n \), and thus \( \hat{\Gamma} \) will be thought of as a product of finite groups \( \mathbb{Z}_{r_i} \) with the group operation denoted additively.

Example 5.1. Let \( \Gamma = \mu_5 \subset \mathbb{C}^\times \), the group of 5-th roots of unity, acting on \( \mathbb{C}^3 \) with weights 1, 2 and 3, i.e. via

\[
\lambda \to \begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix}.
\]

This action is denoted symbolically by \( 1/5 (1, 2, 3) \). The character group of \( \mu_5 \) is \( \hat{\mu}_5 = \mathbb{Z}_5 := \mathbb{Z}/5\mathbb{Z} \), so \( \mathcal{Q} \) has 5 vertices. For each \( i \in \mathbb{Z}_5 \),
let $\chi_i: \lambda \mapsto \lambda^i$ be the corresponding character, so that $\chi_i \chi_j = \chi_{i+j}$. Writing $R_i$ for the irreducible representations, one has $R_i \otimes R_j = R_{i+j}$.

The representation $Q$ is just $R_1 \oplus R_2 \oplus R_3$, so that

$$Q \otimes R_i = R_{i+1} \oplus R_{i+2} \oplus R_{i+3}.$$  

Thus, the quiver has arrows from the vertex $i$ to the vertices $i-1, i-2$ and $i-3$ for each $i$ (see Figure 2 below).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{mckay_quiver}
\caption{The McKay quiver for the action $\frac{1}{5}(1,2,3)$}
\end{figure}

In fact, it is easy to see that the McKay quiver for the action of any cyclic group has a similar appearance. To see what the arrows are, decompose $Q$ into a sum of one-dimensional irreducibles

$$Q = \bigoplus_{i=1}^{n} R_{w_i},$$

where $w_i \in \widehat{\Gamma}$ for $i = 1, \ldots, n$ are the weights of the action of $\Gamma$ on $Q$. Using (3.1), one sees easily that

$$Q \otimes R_v = R_{v+w_1} \oplus \cdots \oplus R_{v+w_n},$$

so there is one arrow from $v$ to $v-w_i$ for each vertex $v$ and each weight $w_i$, giving a total of $nr$ arrows. The arrows corresponding to the weight $w_i$ are written

$$a^i_v := v \rightarrow v-w_i, \text{ for } v \in Q_0$$

and are said to be of type $i$.

The McKay quiver for a general abelian group $\Gamma$ can be obtained by decomposing $\Gamma$ into products of cyclic groups. Define the product of $Q$ and $Q'$ to be the quiver with vertices $Q_0 \times Q'_0$ and with arrows

$$\{(v, t(a')) \rightarrow (v, h(a')) : v \in Q_0, a' \in Q'_1\}$$

$$\cup \{(t(a), v') \rightarrow (h(a), v') : v' \in Q'_0, a \in Q_1\}.$$  

Then the McKay quiver for $\Gamma$ is given by taking the product of the quivers for the cyclic factors.

\footnote{The additions are modulo 5.}
5.2. **Commutation Relations.** Pick a basis $q_i$ of $Q$ such that the action of $\Gamma$ on $Q$ is diagonal, with $q_i$ corresponding to the irreducible component $R_{w_i}$. Then decomposing $R$ into irreducibles and using Shur’s Lemma

$$\text{Hom}_\Gamma(R, R_{w_i} \otimes R) = \bigoplus_{v, v' \in Q_0} \text{Hom}_\Gamma(R_v \otimes R_{w_i} \otimes R_{w_i} \otimes R_{v'})$$

$$= \bigoplus_{v \in Q_0} \text{Hom}(R_v, R_{v-w_i})$$

This means that the scalar map $\tilde{\alpha}(a^i_v) : R_v \rightarrow R_{v-w_i}$ is multiplication by the $(v-w_i, v)$-th entry of the $i$-th component matrix $\alpha_i \in \text{End} R$. The condition $[\alpha_i, \alpha_j] = 0$ therefore translates to the commutation relation

$$\tilde{\alpha}(a^i_v)\tilde{\alpha}(a^j_{v-w_i}) - \tilde{\alpha}(a^j_v)\tilde{\alpha}(a^i_{v-w_i}) = 0$$

for all $v \in Q_0$ and $i, j \in \{1, \ldots, n\}$.

![Figure 3](image-url)  

**Figure 3.** A picture of the commutation relation (5.5) for $v = 0$, $i = 1$ and $j = 3$ in the McKay quiver for $\frac{1}{7}(1, 2, 3)$. (The product of the representation along the continuous arrows must equal the product along the dotted arrows.)

5.3. **Toric Descriptions of the Representation Moduli.**

5.3.1. **Toric Varieties.** Let $M$ be an integral lattice isomorphic to $\mathbb{Z}^n$, let $M_\mathbb{Q} := M \otimes \mathbb{Q}$ be the associated rational vector space, and let $C \subset M_\mathbb{Q}$ be a convex $n$-dimensional cone.

Let $T^M := \text{Spec } \mathbb{C}[M]$ be the algebraic torus whose character group is $M$. Then

$$T^{M,C} := \text{Spec } \mathbb{C}[C \cap M]$$

is a compactification of $T^M$, called the toric variety associated to the cone $C$ with respect to the lattice $M$. The covariant notation $T^{M',C'}$
which uses the dual objects is widely in use in the literature, but the contravariant version is more convenient for the purposes of this paper. This definition can be extended from cones to general convex polyhedra as follows. If $P$ is any convex polyhedron in $M$, denote by $\tilde{P}$ the closure of the cone over $P$, namely

\[ \tilde{P} := \overline{\{1\} \times P} \subset \tilde{M} = \mathbb{Q} \times M. \] (5.6)

Then one can define:

\[ T_{M,P} := \Proj \mathbb{C}[\tilde{P} \cap \tilde{M}], \] (5.7)

and this is easily seen to extend the previous definition for cones.

5.3.2. GIT quotients of Toric Varieties. Note that there is a natural line bundle $L_{M,P} := \mathcal{O}(1)$ on $T_{M,P}$ on which $T_M$ acts and whose sections are given simply by evaluation on the lattice points of $P$. If one translates $P$ by an element $\zeta \in \tilde{M}$, one gets the same toric variety $(T_{M,P} + \zeta = T_{M,P})$ and the same line bundle $L$, but with a different linearisation of the action of $T_M$, differing from the old by the character $\zeta$.

The consequence of this for GIT quotients is that if $M'$ is any sublattice of $M$, and if $\zeta$ an element of $M'$ (and hence a character of $T_{M'}$) one can consider either the quotient $T_{M,P} / / T_{M'}$ with $T_{M'}$ taken to act on $L_{M,P}$ via multiplication by the section $\zeta$ or, equivalently, the quotient

\[ T_{M',\zeta + P} / / T_{M'} \]

where $T_{M'}$ now acts on $L_{M',\zeta + P}$ in the ordinary way.

Using this notation, one can state the following proposition which describes how toric varieties behave under GIT quotients (c.f. [Tha93]).

**Proposition 5.2.** Let $0 \to M'' \to M \to M' \to 0$ be an exact sequence of lattices and $P \subset M$ be a convex polyhedron. Write

\[ (M, P) / / M' := (M'', P \cap M''). \]

Then

\[ T_{M,P} / / T_{M'} = T^{(M,P) / / M'}. \]

**Proof.** An element $x^m \in \mathbb{C}[M \cap P]$ is invariant under $T_{M'}$ if $m$ belongs to the stabiliser of $T_{M'}$ in $T_{M}$, which is nothing but $M''$. Thus $\mathbb{C}[\tilde{P} \cap \tilde{M}]^{T_{M'}} = \mathbb{C}[\tilde{P} \cap \tilde{M}'' \cap \tilde{M}'] = \mathbb{C}[(\tilde{P} \cap \tilde{M}'') \cap \tilde{M}']$ and the result follows by taking $\Proj$. 

---

\(^8\)The dual of a cone $C$ is the cone $C^\vee := \{n \in M' | n(c) \geq 0, \forall c \in C\}$
5.3.3. The Representation Variety. The description of the previous sections shows that $\text{Rep}_{Q,K}(R)$ is a subvariety of $\mathbb{C}^{Q_1}$ defined by the equations (5.5). The goal is now to describe the quotients by the group $\text{PGL}(R)$. It turns out that $\text{Rep}_{Q,K}(R)$ is a toric variety and that $\text{PGL}(R)$ is an algebraic sub-torus, so that one is able to use Proposition 5.2.

Consider the lattice $\Lambda_1^1 := \mathbb{Z}^{Q_1}$ and the first quadrant $\Lambda_1^{1+} := \mathbb{R}^{Q_1}_+$ inside its associated real vector-space $\Lambda_1^1$. Recall that the toric variety which corresponds to $\mathbb{C}^{Q_1}$ is

$$T_{\Lambda_1^1,\Lambda_1^{1+}} := \text{Spec } \mathbb{C}[\Lambda_1^1],$$

where $\Lambda_1^1$ is the semi-group $\mathbb{Z}^{Q_1}_+$ and $\mathbb{C}[\Lambda_1^1]$ denotes its group algebra. More precisely, the $\mathbb{C}$-points of the scheme $\text{Spec } \mathbb{C}[\Lambda_1^1]$ are into one-one correspondence with the points of $\mathbb{C}^{Q_1}$ as follows. If $x: \mathbb{C}[\Lambda_1^1] \to \mathbb{C}$ is an algebra homomorphism representing a point of $\text{Spec } \mathbb{C}[\Lambda_1^1]$ then, evaluating it on the generators of $\Lambda_1^1$ gives a map $Q_1 \to \mathbb{C}$, i.e. a point of $\mathbb{C}^{Q_1}$. Conversely, if $\tilde{\alpha} \in \mathbb{C}^{Q_1}$, there is an induced morphism of semi-groups

$$(\Lambda_1^1, +) \to (\mathbb{C}, \cdot)$$

$$f \mapsto \prod_a (\tilde{\alpha}(a))^f(a),$$

and this induces an algebra morphism $\mathbb{C}[\Lambda_1^1] \to \mathbb{C}$.

Under this identification, the points of $\text{Rep}_{Q,K}(R)$ correspond to the semi-group morphisms $\Lambda_1^1 \to \mathbb{C}$ which are the identity when restricted to the sub-semi-group $\Lambda_1^2 \cap \Lambda_1^{1+}$ of $\Lambda_1^{1+}$. Here $\Lambda_1^2$ is the sub-lattice of $\Lambda_1^1$ generated by the elements

$$\chi_i^v + \chi_j^{v-w_i} - \chi_j^v - \chi_i^{v-w_j},$$

for $i, j = 1, \ldots, n$ (and $\chi_i^v$ denotes the indicator function $\chi_{a_i^v}$ for the element $a_i^v$, i.e. the basis element of $\Lambda_1^1$ which takes the value 1 on the arrow $a_i^v = v \to v - w_i \in Q_1$ and zero elsewhere).

**Proposition 5.3.** The quotient $\Lambda := \Lambda_1^1/\Lambda_1^2$ is a lattice (i.e. is torsion free). If one denotes by $\Lambda_{1+}$ (resp. $C = \Lambda_{1+}$) the lattice (resp. the cone) generated by the image of the elements of $\Lambda_1^1$ in $\Lambda$, then the variety of representations of the McKay quiver with relations into $R$ is given by

$$\text{Rep}_{Q,K}(R) = T_{\Lambda, C} = \text{Spec } \mathbb{C}[\Lambda_{1+}].$$

**Proof.** The only fact which has to be proved is that $\Lambda$ is a lattice. This will be postponed till Lemmas [10.3] [10.5].

The group $\text{GL}(R)$ on the other hand coincides with $\mathbb{C}^*Q_0$. If $\Lambda_0$ denotes the lattice $\mathbb{Z}^{Q_0}$, then one can write $\text{GL}(R) = T_{\Lambda_0}$ in the notation
of Section 5.3. Similarly, writing $\Lambda^{0,0}$ for the sub-lattice of $\Lambda^0$ whose elements $\zeta$ satisfy $\sum_{v \in Q_0}(v) = 0$, one has $\text{PGL}(R) = T^{\Lambda^{0,0}}$.

Applying Proposition 5.2, one obtains the following toric description of the moduli $X_\zeta$.

**Theorem 5.4.** The moduli spaces $X_\zeta$ are quasi-projective toric varieties

$$X_\zeta = T^{\Pi, C_\zeta},$$

where $\Pi := \ker \hat{\rho} = \pi \times \partial(\Lambda)/\Lambda^{0,0}$ is a sub-lattice of $\mathbb{Z}^n$ of index $\sharp \Gamma$ and $C_\zeta$ are convex polyhedra in $\Pi_R$ in obtained by slicing the cone $C$ with the affine planes $\zeta + \Lambda^{0,0}_R$.

**Proof.** By definition

$$X_\zeta = T^{\Lambda, C} / / \zeta T^{\Lambda^{0,0}},$$

and this coincides with

$$T^{\Lambda, C} / / T^{\Lambda^{0,0}}.$$

If $\Lambda' := \pi \times \partial(\Lambda)/\Lambda^{0,0}$ was a lattice then applying Proposition 5.2 would give $X_\zeta = T^{\Lambda', C_\zeta}$.

It therefore remains to prove that $\Lambda' = \Pi$. This is done in Section 10: Lemmas 10.3–10.5 show that there is an exact sequence of abelian groups

$$0 \to \Lambda \xrightarrow{\pi \times \partial} \mathbb{Z}^n \times \Lambda^{0,0} \xrightarrow{\hat{\rho} - \hat{\nu}} \mathbb{Z}^n / \Pi \simeq \hat{\Gamma} \to 0.$$ (5.8)

Note that the morphisms appearing in the exact sequence in the above proof are:

- The projection $\pi: \Lambda^1 \to \mathbb{Z}^n$ (or better, its descent to $\Lambda \to \mathbb{Z}^n$).
- The (descent to $\Lambda$) of the morphism of lattices $\partial: \Lambda^1 \to \Lambda^{0,0}$ dual to the action of $\text{PGL}(R) = T^{0,0}$ on $\text{Rep}_{Q}(R) = \mathbb{C}Q_1$ given by

$$\hat{\partial}: T^{0,0} \longrightarrow T^1$$

$$\lambda \longmapsto a \longmapsto \lambda_{t(a)}^{-1} \lambda_{h(a)}.$$

Here $\lambda_v$ denotes the component of $\lambda \in T^{0,0}$ corresponding to the vertex $v \in Q_0$.

- The morphism of lattices

$$\hat{\nu}: \Lambda^{0,0} \longrightarrow \hat{\Gamma}$$

$$\zeta \longmapsto \sum_{v \in Q_0} \zeta(v) v$$

dual to the action of $\Gamma$ on $\text{End} R$ by conjugation.
The morphism of lattices
\[ \hat{\rho} : \mathbb{Z}^n \longrightarrow \hat{\Gamma} \]
\[ \zeta \mapsto \sum_{v \in Q_0} x_i w_i \]
dual to the action \( \rho : \Gamma \to \mathbb{C}^* \).

The fact that \( \pi \times \partial \) descends to \( \Lambda \) corresponds to the fact that \( \text{GL}_{\mathbb{Q}}(\Gamma) = \mathbb{C}^* \) and \( \text{PGL}_{\mathbb{R}}(\mathbb{Q}) = T_0 \), which act on \( \text{Rep}_{\mathbb{Q}}(\mathbb{Q}, K) \) rather than simply on \( \text{Rep}_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{C}Q_1 \). On the other hand, the fact that their image maps into \( \text{ker}(\hat{\rho} - \hat{\nu}) \) corresponds to the fact that \( \text{Rep}_{\mathbb{Q}}(\mathbb{Q}, K) \) can be identified with the \( \Gamma \)-invariant part of \( \text{Hom}(R, Q \otimes R) \).

The action of \( T_{\Pi} \) arises from the existence of the map
\[ \pi : Q_1 \to \{1, \ldots, n\}, \]
which assigns to each arrow its type. This induces a \( \mathbb{Z} \)-linear map \( \pi : \Lambda^1 \to \mathbb{Z}^n \), which is denoted by the same letter, and one obtains an action of \( \mathbb{C}^* \) on \( \mathbb{C}Q_1 \) via the corresponding morphism of algebraic tori:
\[ \tau \cdot \hat{\alpha} := \hat{\pi}(\tau)\hat{\alpha}. \]
Explicitly, for \( \tau \in \mathbb{C}^n \), \( \hat{\alpha} \in \mathbb{C}Q_1 \), and \( a \in Q_1 \),
\[ (\tau \cdot \hat{\alpha})(a) = \tau_{\pi(a)}\hat{\alpha}(a), \]
where \( \tau \in \mathbb{C}^n \) has components \( \tau_i := \tau(e_i) \) with respect to the standard basis \( e_i \) of \( \mathbb{Z}^n \), and \( \chi_a \) denotes the basis element of \( \Lambda^1 \) which is the indicator function of the singleton \( \{a\} \subset Q_1 \). The action of \( \mathbb{C}^* \) corresponds to multiplying the value of \( \hat{\alpha} \) on all arrows \( a \) with the same \( \pi(a) \) by the same factor \( \tau_{\pi(a)} \). Since \( \Gamma \) is abelian, \( \rho(\Gamma) \subset \mathbb{C}^n \) acts trivially on the representations, and one has an action of \( T_{\Pi} := \mathbb{C}^n / \rho(\Gamma) \).

According to the toric formalism, one has
\[ T_{\Pi} = \mathbb{C}^n / \text{Im} \rho = T^{\ker \hat{\rho}}, \]
and thus \( \Pi = \ker \hat{\rho} \).

Note also that the exact sequence implies that the cone \( C := \Lambda_{\mathbb{R}^+} \) is isomorphic to \( (\pi \times \partial)(\Lambda^1_{\mathbb{R}^+}) \). Therefore, one has the following theorem.

**Theorem 5.5.** The slices \( C_\zeta \) (which describe the toric moduli \( X_\zeta \) when \( \zeta \) is integral) are given by
\[ C_\zeta = \pi_{\mathbb{R}}(F_\zeta), \]
where
\[ F_\zeta := \mathbb{R}Q_1 \cap \partial^{-1}_\mathbb{R}(\zeta). \]

The next section is devoted to the study of the polyhedra \( F_\zeta \) and \( C_\zeta \).
6. Flow Polyhedra

The question of determining the polyhedra $C_\zeta$ is in fact a generalisation of a basic problem, well-known to network optimization specialists:

**Problem 6.1** (The Transportation Problem ([KH80, GM84])). Given a quiver — a “network” in optimization parlance — $Q$ and an element $\zeta \in \mathbb{R}^{Q_0}$ specifying given demands and supplies of some commodity at the vertices, find all the non-negative elements $f \in \mathbb{R}^{Q_1+}$, representing flows of commodities along each arrow, whose net contribution at each vertex balances the demands and supplies specified by $\zeta$.

Writing $\text{In}(v)$ (Out$(v)$) for the arrows which have their head (tail) at a given vertex $v$, then $\partial : \mathbb{R}^{Q_1} \to \mathbb{R}^{Q_0}$ is given by

$$\partial f(v) = \sum_{a \in \text{In}(v)} f(a) - \sum_{a \in \text{Out}(v)} f(a).$$

The element $\partial f$ is called the net contribution of $f$. The flow $f \in \mathbb{R}^{Q_1}$ is said to be a $\zeta$-flow if

$$\partial f = \zeta. \tag{6.1}$$

If $f$ is non-negative, it is called an admissible $\zeta$-flow.

![Figure 4](image)

**Figure 4.** An admissible flow on the McKay quiver for $\frac{1}{2}(1,2,3)$. (The numbers along the edges indicate the values of $f$ and the numbers at the vertices indicate the values of $\partial f = \zeta$.)

The problem is therefore to determine the set $F_\zeta$ of all admissible flows for a given $\zeta$. Note that the solution set is empty unless $\zeta$ satisfies $\sum_{v \in Q_0} \zeta(v) = 0$ (i.e. supply = demand). Recall that the sub-space for which this holds was denoted $\Lambda_{\mathbb{R},0}$. The set $F_\zeta$ is the intersection of the cone $\mathbb{R}^{Q_1+}$ (the first quadrant) with an affine translate of the

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9Actually, the interest in the transportation problem usually centers around finding the extreme flow which minimizes a certain cost function, not in finding all extreme flows. Furthermore, there are usually capacity constraints for the maximal flow allowed along each arrow.
vector-space $\ker \partial$, so $F_\zeta$ is a convex polyhedron; the interest lies in determining its extreme points. For instance, if one is trying to minimize a convex cost function of the flows, then the minimum will be attained at one of these extreme points. A solution to this problem is given by the easy:

**Theorem 6.2.** The extreme points of $F_\zeta$ are precisely the admissible $\zeta$-flows whose support contains no cycles.

Here, the *support* of a flow is the set of arrows on which it takes non-zero values. A *cycle* means a sequence of arrows in $Q_1$ which form a cycle in the underlying graph to $Q$, when their orientation is disregarded.

*Figure 5. A cycle in the McKay quiver for $\frac{1}{7} (1, 2, 3)$*

### 7. Generalised Transportation Problem

In the present paper, the interest is in the image $C_\zeta$ of $F_\zeta$ under the projection $\pi: \mathbb{R}^{Q_1} \to \mathbb{R}^n$, and one is therefore led to the following slightly more general problem.

**Problem 7.1.** Given a quiver $Q$, a natural integer $n$ and a projection $\pi: \mathbb{R}^{Q_1} \to \mathbb{R}^n$, characterize the (supports of the) flows which, under $\pi$, map to an extreme point of $\pi F_\zeta$ for a given $\zeta$.

**Remark 7.2.** This problem can be thought of as a transportation problem with some extra structure: instead of associating a cost with each arrow, one supposes that there are linear relations between the arrow costs which are parametrised by $n$ independent variables — in other words, suppose that the cost function really is a (convex) function of these $n$ variables, so that the cost-minimizing flows will correspond to extreme points of $\pi F_\zeta$.

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Theorem 6.2 is the basis for the “Simplex on a graph” algorithm [Lue69]. See Proposition 3.16 in [KH80], noting that trees correspond to what is called a “basic solution” or a “basis” for the linear programming problem [Lue69, §2.3, Defn., p.17]. For the case of the permutation polytope, see [YKK84, §5, Th. 1.1].
If \( f \) is any flow with \( \pi(f) = x \in \pi F_\zeta \), then the support \( S \) of \( f \) will be called a \( \zeta \)-configuration (of arrows) for \( x \), or simply a configuration of \( x \) (there will in general be many \( \zeta \)-configurations for a given \( x \)). If \( x \) is an extreme point of \( \pi F_\zeta \), then any configuration \( S \) of \( x \) will be called an extreme configuration. The problem is thus to determine all the possible extreme configurations.

7.1. **Cycle Type and Closure.** In order to state the solution to the problem, one needs to introduce some concepts relating to cycles. Suppose that \( c = (c_1, \ldots, c_k) \) is a cycle, i.e. a sequence of arrows \( c_i \in Q_1 \) such that they form a circuit, when suitably oriented. Consecutive arrows \( c_i \)'s are not allowed to be the same (although they can join the same vertices). An arrow \( c_i \) is said to agree with the ordering \( c_1, \ldots, c_k \) if the tail of \( c_i \) is an extremity of \( c_{i-1} \) and the head of \( c_i \) is an extremity of \( c_{i+1} \), and is said to disagree otherwise. Call the disjoint union of the arrows \( c_i \) whose direction agrees (disagrees) with the ordering \( c_1, \ldots, c_k \) the positive (negative) part of \( c \) and denote it by \( c^+ \) (\( c^- \)).

For each arrow \( a \in Q_1 \), let \( \chi_a \) denote the basis element of \( \mathbb{Z}^{Q_1} \) which takes the value 1 on the arrow \( a \) and zero elsewhere. To each cycle \( c \), one can define the basic flow associated to \( c \) by

\[
\tilde{\chi}_c := \sum_{a \in c^+} \chi_a - \sum_{a \in c^-} \chi_a \in \mathbb{Z}^n.
\]

The type of \( c \) is defined to be element \( \pi(\tilde{\chi}_c) \in \mathbb{Z}^n \). Cycles of type zero are of special interest. They correspond to cycles having zero total number of arrows of any given type, the number being counted algebraically, according to whether the arrow agrees or disagrees with the orientation of \( c \).

![Figure 6](image)

**Figure 6.** A cycle of type zero in the McKay quiver for \( \frac{1}{3}(1, 2, 3) \). (The numbers next to the arrows indicate their type.)

\(^{11}\)Note that it is necessary to state this explicitly for quivers, due to the possibility of several arrows joining the same two vertices.

\(^{12}\)The disjoint union is used in case some cycles contain the same arrow twice.
The cycle closure of a configuration $S$ is defined to be the smallest over-set $\overline{S}$ of $S$ such that, for any cycle $c$ in $Q$ of type 0,

$$c^+ \subseteq \overline{S} \iff c^- \subseteq \overline{S}.$$  \hspace{1cm} (7.1)

The closure of $S$ can be computed by searching for all the cycles $c$ of type 0 satisfying $c^+ \subseteq S$, adjoining $c^-$ to $S$, and repeating this procedure until (7.1) is satisfied. Note that, even if $S$ contains no cycles of non-zero type (for instance, if $S$ is a tree), the arrows one adjoins may create such cycles in $\overline{S}$.

Two configurations $S, S'$ will be called equivalent (written $S \sim S'$) if $S = S'$.

7.2. Statements of the Generalised Theorems. The generalised version of Theorem 10 can now be stated.

Theorem 7.3 (Generalised Extreme Flows). The extreme points of the polyhedron $\pi F_\zeta$ are the images under $\pi$ of the admissible $\zeta$-flows whose supports have no cycles of non-zero type in their closures. If $S$ is a such a configuration and if there is a $\zeta$-flow with support $S$, then the image of that flow under $\pi$ is an extreme point of $\pi F_\zeta$.

Note that if $\pi$ is the identity map, then any cycle is of non-zero type and one recovers Theorem 10.

The above theorem can be generalised to get a description of the faces of $\pi F_\zeta$ of all dimensions. Recall that the tangent cone of $P$ at one of its faces $F$ is the convex cone

$$T_F P := \mathbb{R}_+(P - F) := \mathbb{R}_+(P - f),$$

for any $f \in \text{int } F$,

where $\text{int } F$ denotes the relative interior of the face $F$.

For any configuration of arrows $S$ define

$$F_\zeta(S) := \{ f \in \partial^{-1}(\zeta) : a \not\in S \implies f(a) \geq 0 \}$$

and

$$Z_\zeta(S) := \{ f \in \partial^{-1}(\zeta) : a \not\in S \implies f(a) = 0 \}.$$ 

One has $F_\zeta(\emptyset) = F_\zeta$ and $Z_\zeta(S) = F_\zeta \cap \text{supp}^{-1}(S)$. When $\zeta = 0$, $F_0(S)$ is a cone and $Z_0(S)$ is its maximal vector subspace. The cone $F_0(S)$ (resp. the vector-space $Z_0(S)$) is generated over $\mathbb{R}_+$ by the flows $\tilde{\chi}_c$ for cycles $c$ such that $c^- \subseteq S$ (resp. $c \subseteq S$).

Let $\mathcal{C}$ denote the set of all configurations, i.e. the set of non-empty subsets of $Q_1$. The rank of $S$ is defined to be the rank of $\pi Z_0(S)$. It is trivial to see that the rank function determines a partition of $\mathcal{C}$ into non-empty disjoint sets $\mathcal{C} = \mathcal{C}^0 \cup \mathcal{C}^1 \cup \cdots \cup \mathcal{C}^n$ which respects the equivalence relation $\sim$ induced by $S \mapsto \overline{S}$. As a further piece of notation, for any subset of $D \subseteq \mathcal{C}$, denote by $D^k$ the subset of configurations in $D$ which
have rank $k$. Also, write $D_\zeta$ for the subset of configurations which are admissible for $\zeta$, namely configurations $S \in D$ which arise as the support of some element in $F_\zeta$.

Theorem 7.3 says that the configurations corresponding to the extreme points of $F_\zeta$ are precisely those belonging to $C_0^{\zeta}$. In general one has the following complete description of the extreme faces and tangent cones of $C_\zeta$:

**Theorem 7.4 (Extreme Faces and Tangent Cones).** For all $\zeta$, the map

$$\text{Face}_\zeta : C_\zeta \to \text{Faces of } \pi F_\zeta$$

$$S \mapsto \pi F_\zeta \cap (\pi f + \pi Z_0(S))$$

is independent of the choice of $f \in F_\zeta \cap \text{supp}^{-1}(S)$, and induces a bijection

$$\text{Face}_\zeta : C^k_\zeta / \sim \to k\text{-faces of } \pi F_\zeta.$$ 

Furthermore, for all $[S] \in C_k^k / \sim$,

$$T_{\text{Face}_\zeta(S)} \pi F_\zeta = \pi F_0(S),$$

where the left-hand side denotes the tangent cone to the polyhedron $\pi F_\zeta$ at the face $\text{Face}_\zeta(S)$. In other words, $\pi F_0(S)$ gives the tangent cone corresponding to the configuration $S$ (which is independent of the value of $\zeta$) and $\text{Face}_\zeta$ gives the corresponding face of $C_\zeta$ (whose direction is also independent of $\zeta$).

**Remark 7.5.** Note that both the direction of the face corresponding to $S$ and the tangent cone of $\pi F_\zeta$ at this face are independent of the value of $\zeta$.

In fact, (see Lemma 9.2) $Z_0(S)$ is generated by the flows for the cycles supported in $S$, so that Theorem 7.3 corresponds to the case when $S$ has rank zero.

The concepts needed for the proof of Theorem 7.4 are developed in Section 9.1. The theorem is then proved in Section 9.2 for the classical case (where $\mathbb{R}^{Q_1} = \mathbb{R}^n$ and $\pi : Q_1 \to Q_1$ is the identity). The general case follows easily from this and is treated in Section 9.3.

Before this, an example of an application and several important corollaries are given.

**8. Application to the McKay Quiver**

8.1. **Example.** Consider the McKay quiver for the action $\frac{1}{5}(1,2,3)$. Recall that identifying $Q_0$ with $\mathbb{Z}_5$, the arrows are $a^i_v := v \to v - i$ for $v \in \mathbb{Z}_5$ and $i \in \{1,2,3\}$. Let $\pi$ be the map $Q_1 \to \{1,2,3\}$ which assigns
to each arrow $a_i$ its type $i$. This induces a projection $\pi : \mathbb{R}^Q_1 \to \mathbb{R}^3$ which assigns to the basis element $\chi_i := \chi_{a_i}$ the basis element $e_i$ of $\mathbb{R}^3$. With respect to this map, the cycles of type zero are those with total number of arrows of any given type equal to zero, where the number of arrows is counted algebraically according to the orientation of the cycle.

Consider the configuration $T \subset Q_1$ represented in Figure 7. What is the closure of $T$? A little thought shows that $T$ is simply $T$ itself. Since $T$ has no cycles, $Z_0(T) = 0$, so $T \in C^0$. If $T$ is admissible for $\zeta$, and $f$ is any $\zeta$-flow with support $T$, then the theorem says that $\pi(f)$ is a 0-face of $\pi F_\zeta$, i.e. an extreme point. Furthermore, the tangent cone to $\pi F_\zeta$ at $\pi(f)$ is the cone $\pi F_0(T)$; this is generated by the types of the cycles whose negative part is contained in $T$. These are listed in Figure 8.

There are four different types

- $v_1 = (1, 1, -1)$
- $v_2 = (1, -2, 1)$
- $v_3 = (-1, 0, 2)$
- $v_4 = (-1, 3, 0),$

and they generate $\pi F_0(T)$ as a cone. Any three of these form a basis for the lattice $\Pi \subset \mathbb{Z}^3$ of index 5 given by

$$\Pi := \ker \hat{\rho} = \{(a, b, c) : a + 2b + 3c \equiv 0 \pmod{5}\},$$

and they satisfy the single relation $v_1 + v_3 = v_2 + v_4$. This corresponds via the usual toric formalism to a singularity of $X_\zeta$ of the type $xw = yz \subset \mathbb{C}^4$. The variety $X_\zeta$ therefore has such a singularity whenever the tree $T$ is admissible for $\zeta$, i.e. whenever the flow $f_\zeta(T)$ is positive. Writing this out explicitly, one sees that the admissible cone for $T$.

\[\text{Figure 7. A configuration } T \in C^0 \text{ in the McKay quiver for } \frac{1}{5}(1, 2, 3).\]

\[\text{Figure 8. The types of cycles whose negative part is contained in } T.\]

\[\text{Figure 8. The tangent cone to } \pi F_\zeta \text{ at } \pi(f) \text{ is the cone } \pi F_0(T).\]
Figure 8. The different types of cycles which generate the cone $\pi F_0(T)$ for the configuration $T$ in Figure 7.

$\Lambda_{R}^{0,0}(T)$ is given by the following inequalities:\footnote{See \[8.2.2\] for the exact definition of $\Lambda_{R}^{0,0}(T)$.}

\[-\zeta_0 > 0\]
\[-\zeta_2 > 0\]
\[-\zeta_4 > 0\]
\[-\zeta_4 - \zeta_1 > 0\]

8.2. Corollaries.

8.2.1. $k$-Adjacency. The “simplex on a graph” algorithm \cite{KHS80, Alg. 3.3} can be interpreted as moving from one extreme point of $F_\zeta$ to an adjacent one by varying the flow along a single cycle in the network. Geometrically this says that two extreme flows are joined by an edge of $F_\zeta$ if and only if the union of their supports contains only one cycle (up to multiples obtained by going around the cycle $k \in \mathbb{Z}$ times).

In order to generalize the notion of being joined by an edge, the following terminology is introduced: two points of a polyhedron are called $k$-adjacent if they are contained in a $k$-dimensional face, but in no face of smaller dimension. Note that by convexity, two points are contained in the same face if and only if their midpoint is also in that face.

\footnote{See \cite[§5.4]{EM92}, or, for the special case of the “permutation polytope,” \cite[§5, Th. 1.3]{YKK84}.}
face. Since a configuration of the midpoint is obtained by taking the union of configurations of the two points, the answer to the question is deduced as a corollary to Theorem 7.4.

**Corollary 8.1 (k-Adjacency).** Two points \(x, x' \in \pi F_\zeta\) are \(k\)-adjacent if and only if for some (and hence for any) configurations \(S\) of \(x\) and \(S'\) of \(x'\),

\[S \cup S' \in \mathcal{C}^k,\]

i.e. their union has rank \(k\).

Note that the case \(k = 0\) gives the condition for two configurations to give rise to the same extreme point. The case \(k = 1\) says that two configurations are joined by an edge of \(\pi F_\zeta\) if and only if their union only contains cycles whose types are multiples of a fixed element \(v \in \mathbb{R}^n\). For the case when \(\pi\) is the identity, this reduces to the classical statement that the union \(S \cup S'\) contains only one cycle (up to multiples obtained by going around the cycle \(k \in \mathbb{Z}\) times).

8.2.2. **Trees.** It can be shown easily (see Corollary 9.10) that any extreme point of \(\pi F_\zeta\) has at least one configuration \(T\) which is a tree, i.e. it has no cycles. Thus, to know the extreme points and the tangent cones of \(\pi F_\zeta\) at these points, one need only answer the question of which trees occur as extreme configurations. Denoting the set of trees in \(\mathcal{C}\) by \(\mathcal{T}\), the generalised theorem implies that the extreme points of \(\mathcal{C}_\zeta\) are given by the images of the flows whose supports are members of \(\mathcal{T}_\zeta^0\). This set is easy to determine: for a given \(\zeta\), there is, on each tree \(T\), a unique flow \(f_\zeta(T) \in \Lambda^1\) which meets the demand \(\zeta\) and is supported on \(T\); \(f_\zeta(T)\) is called the \((\zeta-)\)flow admitted by the tree \(T\). (See 12 for an example). This defines a map

\[f_\zeta: \mathcal{T} \to \Lambda^1.\]

The set \(f_\zeta^{-1}(\Lambda^1_+)\) of trees \(T\) which admit non-negative \(\zeta\)-flows is of course the set \(\mathcal{T}_\zeta\) of admissible trees for \(\zeta\).

The classical Theorem 10 says that

\[\text{ext } F_\zeta = f_\zeta(\mathcal{T}_\zeta).\]

Similarly, the generalised Theorem 7.3 gives the following:

**Corollary 8.2.** Let \(f_\zeta\) be the map which assigns to each tree \(T\) the unique \(\zeta\)-flow with support equal to \(T\). Then the extreme points of \(\mathcal{C}_\zeta\) are given by

\[\text{ext } \pi F_\zeta = \pi f_\zeta(\mathcal{T}_\zeta^0),\]
where $\mathcal{T}^0 := C^0 \cap \mathcal{T}$ is the set of trees whose closures have type zero. Furthermore, Theorem 7.4 implies that
\[ \sharp \text{ ext } C_\zeta = \sharp C_\zeta^0 / \sim = \sharp T_\zeta^0 / \sim, \]
where $\sim$ is the equivalence relation induced by closure.

Note (see Lemma 9.9 for a proof) that for the sets $S \in C_\zeta^0$ one has $\pi F_0(S) = \pi F_0(S)$. This implies the following corollary:

**Corollary 8.3.** The extreme points of the polyhedron $\pi F_\zeta$ correspond to the trees $T$ in $T_\zeta^0$ and the tangent cone to $\pi F_\zeta$ at the point corresponding to $T$ is $\pi F_0(T) = \pi F_0(T)$. Thus the fan $\Sigma_\zeta$ associated to the polyhedron $\pi F_\zeta$ is given by the dual cones $\pi F_0(T)^\vee$ for the trees $T \in T_\zeta^0$ and all their faces. In fact, its $k$-skeleton, i.e. the set of its $k$-dimensional cones, is $\Sigma_\zeta^{(k)} := \{ \pi F_0(T)^\vee : S \in C_\zeta^{n-k} \}$.

**Remark 8.4.** This corollary says that the singularities of $C_\zeta$ are precisely those given (with respect to the lattice $\Pi$) by the cones $\{ \pi F_0(T) | T \in T_\zeta^0 \}$.

8.2.3. **Variation of the Flow Polyhedra with $\zeta$.** The following corollary of Theorem 7.4 describes when two different values of $\zeta$ give isomorphic polyhedra:

**Corollary 8.5.** If $\zeta$ and $\zeta'$ have the same admissible configurations ($\mathcal{C}_\zeta = \mathcal{C}_{\zeta'}$) or even just the same admissible trees ($\mathcal{T}_\zeta = \mathcal{T}_{\zeta'}$) then the corresponding polyhedra $C_\zeta$ and $C_{\zeta'}$ are geometrically isomorphic. Two polyhedra are said to be geometrically isomorphic if they are combinatorially isomorphic and their tangent cones at the corresponding faces are identical. In particular, their associated fans are identical, and their corresponding toric varieties are isomorphic.

Note that when one multiplies $\zeta$ by a non-zero number, the polyhedron $C_\zeta$ is simply scaled-up. Fixing a tree $T \in T^0$ determines an open convex cone $\Lambda^0_\mathbb{R}(T) \subseteq \Lambda^0$ of values of $\zeta$ for which $T$ is $\zeta$-admissible. The cone $\Lambda^0_\mathbb{R}(T)$ is called the admissible cone of $T$. If $\mathcal{E}$ is a fixed subset of $T^0$, then the condition $T_\zeta^0 = \mathcal{E}$ defines an open cone in the $\zeta$-parameter space $\Lambda^0_\mathbb{R}$. As $\mathcal{E}$ varies, one obtains a partition of $\Lambda^0_\mathbb{R}$ into a union of open cones inside which the polyhedra $C_\zeta$ are geometrically isomorphic.

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\[16\] Recall that the fan associated to a convex polyhedron $P$ is the collection of dual cones to the tangent cones of $P$ at all its faces.
8.2.4. Degeneracies. Another fact which will be of interest to us is that, for generic values of $\zeta$, any $\zeta$-flow has a support which is a spanning subgraph of the quiver, i.e. which connects any two vertices. This is written $C_\zeta \subset C_{\text{span}}$.

For instance, the faces of $\Lambda^{0,0}_q(T)$ consist of degenerate values of $\zeta$ for which some extreme $\zeta$-flows have supports which are strict subsets of $T$ and so cannot be spanning subsets.

In the next section, basic flows associated to paths and cycles are studied in more detail. They provide the key to the proofs of the other results.

9. Proofs

9.1. Basic Flows. Many proofs in the context of network flows use the basic technique of decomposing a flow into certain basic components associated to paths in $Q$. A path in $Q$ means a sequence $p = (p_1, \ldots, p_k)$ of arrows in $Q_1$ which form a connected path in the underlying graph to $Q$, once their orientation has been disregarded. Consecutive $p_i$’s are not allowed to be the same (although they can join the same vertices). As for cycles, the disjoint union of the arrows of the path $p = (p_1, \ldots, p_k)$ which agree (resp. disagree) with the sense of traversal specified by the sequence $p_1, \ldots, p_k$ are called the positive (resp. negative) arrows of $p$ and denoted $p^+$ (resp. $p^-$). A path will sometimes be confused with its set of arrows, for instance in statements such as “a path $p$ is in a set $S \subset Q_1$ (written $p \subseteq S$)” which means of course that all its arrows belong to the set $S$.

To each path $p$ the basic flow $\tilde{\chi}_p : Q_1 \rightarrow \mathbb{Z}$ is defined by

$$\tilde{\chi}_p(a) = \sum_{a \in p^+} \chi_a - \sum_{a \in p^-} \chi_a \in \mathbb{Z}^n. \quad (9.1)$$

Note that if $p$ is a path from $v$ to $v'$, then $\partial \tilde{\chi}_p = \chi_{v'} - \chi_v$. Conversely, one has the following lemma.

Lemma 9.1. If $f$ is an integral flow with $\partial f = \chi_{v'} - \chi_v$ for two vertices $v, v'$ of $Q$, then there exists a path $p$ from $v$ to $v'$ such that $p^\pm \subseteq \text{supp } f^\pm$.

Proof. By induction on the 1-norm of $f$: $\|f\|_1 := \sum_{a \in Q_1} |f(a)|$. If $\|f\|_1 = 1$ then, obviously, $f = \chi_a$ or $-\chi_a$, for some arrow $a \in Q_1$ which is either $v \rightarrow v'$ or $v' \rightarrow v$ respectively. Either way, $f = \tilde{\chi}_p$ for the corresponding one-arrow path $p$ from $v$ to $v'$. Now suppose that

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17See Section 7.1 for the definition of agree.
18Basic flows are also termed simple flows \cite{BS65} or elementary flows by other authors.
\[ \|f\|_1 > 1 \] and that \( \partial f = \chi_{v'} - \chi_v \). There must be an arrow \( a \) such that one of the following statements holds

1. \( t(a) = v \) and \( f(a) > 0 \), or
2. \( h(a) = v \) and \( f(a) < 0 \).

If (1) holds then the flow \( f' = f - \chi_a \) satisfies the induction hypothesis with \( \partial f' = \chi_{v'} - \chi_{h(a)} \), so there exists a path \( p' \subseteq \text{supp} f' \) from \( h(a) \) to \( v' \). But then \( p = ap' \) is a path from \( v \) to \( v' \) with \( p^+ = p'^+ \cup \{a\} \subseteq \text{supp} f'^+ \cup \{a\} = \text{supp} f^+ \). Case (2) follows in a similar way setting \( f' = f + \chi_a \) and \( p = (-a)p \).

Obviously, if \( p \) is actually a cycle, then the basic flow associated to \( p \) is a 0-flow (it satisfies \( \partial \tilde{\chi}_p = 0 \)), and is supported in \( p \). Figure 9 gives an example of such a flow.

![Figure 9. A basic 0-flow associated to the cycle in Figure 5](image)

The converse to this statement is given by the following lemma [BS65, Th. 7.2], [GM84, §5.1.6, Th. 3].

**Lemma 9.2** (Decomposition into basic flows). Any 0-flow \( f \) can be decomposed into a positive linear combination of basic flows for finitely many cycles \( c_i \) such that \( c_i^\pm \subseteq \text{supp} f^\pm \).

**Proof.** Let \( f \in \ker \partial \) and \( S \) be its support. If \( S \) contains a cycle \( c \) then let \( v \) be a vertex such that \( c^+ \) has an arrow \( a \) with \( t(a) = v \). Then \( f' := f - f(a)\tilde{\chi}_c \in \ker \partial \) and \( \text{supp} f' = \text{supp} f \setminus \{a\} \) is strictly smaller than \( \text{supp} f \). Repeating this reasoning finitely many times, one obtains a flow \( f' \in \ker \partial \) such that \( S' = \text{supp} f' \) contains no cycles. Suppose \( S' \) is non-empty, and let \( p \) be a path in \( S' \) which is maximal, i.e. is not contained in any strictly larger path. Let \( p \) go from vertex \( v \) to vertex \( v' \) and \( a \) denotes the first arrow of \( p \) (the one joined to \( v \)). Then this is the only arrow in the quiver which is joined to \( v \) and on which \( f' \) is non-zero, so \( \partial f'(v) \) must be non-zero. Since this is impossible, \( S' \) must be empty, and \( f = \sum_i x_i \tilde{\chi}_{c_i} \) for some cycles \( c_i \subseteq S \) and real numbers \( x_i \). Replacing \( c_i \) by \( -c_i \) if necessary, one may assume \( x_i > 0 \). (Note also that if \( f \) is integral to start with, the \( x_i \) are also integral.)
I claim that these cycles may be chosen in such a way that their orientations are conformal, i.e. such that any two cycles $c_i, c_j$ satisfy $c_i^+ \cap c_j^- = \emptyset$. Indeed suppose that $c_i$ and $c_j$ are not conformal, and suppose, say, $x_i \geq x_j$. On the subset $U \subseteq Q_1$ on which their orientations disagree, the sum of $x_i c_i$ and $x_j c_j$ will cancel each other out, and only $x_i - x_j$ units of flow will survive. The complement $c_i \cup c_j \setminus U$ will consist of one or more disjoint cycles $d_1, \ldots, d_k$, all conformal to $c_i$. Thus upon adding the two basic flows

\[ x_i \bar{\chi}_{c_i} + x_j \bar{\chi}_{c_j} = (x_i - x_j) \bar{\chi}_{c_i} + x_j (\bar{\chi}_{d_1} + \cdots + \bar{\chi}_{d_k}), \]

giving $k + 1$ conformal cycles with all coefficients positive or zero. Repeating this procedure for all the pairs of non-conformal cycles gives the desired conformal decomposition.

Now writing

\[ f = \sum_i \bar{\chi}_{c_i} = \sum_i (\chi_{c_i^+} - \chi_{c_i^-}), \]

and since $c_i^+ \cap c_j^- = \emptyset$, one has

\[ f^\pm = \sum_i \chi_{c_i^\pm}, \]

which is the desired result. \qed

Remark 9.3. The decomposition lemma above can be viewed as a consequence of the following homological argument. Regard $Q$ as a CW-complex, and for each cycle $p$ in $Q$, adjoin a 2-cell whose boundary is the element $\bar{\chi}_p$. Then the resulting CW-complex $\bar{Q}$ has $H^1(\bar{Q}) = 0$. An elementary proof was given because it illustrates the basic technique of decomposing a flow into basic flows.

9.2. Proof of Theorem 7.4 — Classical Case. The classical analog of Theorem 7.4 (i.e. the case when $\pi$ is the identity map) can now be proved. The statement says that the $k$-dimensional faces of $F_\zeta$ correspond to the $\zeta$-configurations $S$ such that rank $Z_0(S) = k$, i.e. whose cycles span a lattice of rank $k$. This theorem follows from the following four facts:

Fact 9.4. If $f \in F_\zeta$ then the cone $\mathbb{R}_+(F_\zeta - f)$ contains a subspace of dimension $k$ if and only if $f$ is contained in the (relative) interior of a face of dimension $k$ (and hence in no lower dimensional face).

Fact 9.5. For any $f \in F_\zeta$, one has $\mathbb{R}_+(F_\zeta - f) = F_0(\text{supp } f)$.

Fact 9.6. The maximal vector subspace in the cone $F_0(S)$ is $Z_0(S)$. 
Fact 9.7. The subspace $Z_0(S)$ is generated by the basic flows for the cycles supported in $S$.

Fact 9.4 follows from the definition of a $k$-dimensional face, 9.6 is trivial and 9.7 follows from Lemma 9.2. Fact 9.5 is proved in the following lemma.

Lemma 9.8. If $f \in F_\zeta$, then $\mathbb{R}_+(F_\zeta - f) = F_0(\text{supp} f)$.

Proof. Let $f \in F_\zeta$, so that $F_\zeta = (f + \ker \partial) \cap \mathbb{R}_+^{Q_1}$. The non-negative multiples of elements in $(f + \ker \partial) \cap \mathbb{R}_+^{Q_1} - f$ belong to $\ker \partial$ and are non-negative outside $\text{supp} f$, so belong to $F_0(\text{supp} f)$. Conversely, suppose $m \in F_0(\text{supp} f)$. Then $\partial(f + \epsilon m) = \zeta$ for all $\epsilon \in \mathbb{R}$, and it suffices to show that there exists $\epsilon > 0$ such that $f + \epsilon m \in \mathbb{R}_+^{Q_1}$. But this follows because $f$ is bounded below by a positive number on $\text{supp} f$, whereas $m \geq 0$ outside $\text{supp} f$. \qed

9.3. Proof of Theorem 7.4 — General Case. In order to prove the general case of Theorem 7.4, a bit more has to be said about the various configurations which can occur for a point $x \in \pi F_\zeta$.

Lemma 9.9. If $S$ is a $\zeta$-configuration for $x$ then so is $\overline{S}$. Furthermore, all $\zeta$-configurations for $x$ have the same closure.

Proof. Let $f \in F_\zeta$ be a flow such that $\pi f = x$ and let $S = \text{supp} f$. The proof begins by constructing a flow $\overline{f} \in F_\zeta \cap \pi^{-1}(x)$ whose support is $\overline{S}$. If $c$ is a cycle of type 0 in $Q$ such that $c^- \subseteq S$ and $c^+ \nsubseteq S$, one can add a small positive multiple of $\tilde{\chi}_c$ to $f$ and obtain a non-negative flow $f'$. Since $c$ is a cycle, $\partial \tilde{\chi}_c = 0$, and since $c$ has type 0, $\pi f = \pi f'$ and hence $f' \in F_\zeta$. Continuing in this way until all cycles of type 0 satisfy (7.1), one obtains the required flow $\overline{f}$.

For the second statement of the lemma, note that if $f, f'$ are two elements of $F_\zeta \cap \pi^{-1}(x)$, then $\partial(f - f') = 0$, so by Lemma 9.2, has a decomposition into basic flows for cycles $c_i$:

$$f - f' = \sum_i x_i \tilde{\chi}_{c_i},$$

with $c_i^+ \subseteq f, c_i^- \subseteq f'$ and $x_i > 0$. Now $c_i^+ \subseteq S \implies c_i^- \subseteq \overline{S}$, so

$$f' = f + \sum_i \tilde{\chi}_{c_i}$$

implies that $S' \subseteq \overline{S}$. By symmetry, one also has $S \subseteq \overline{S}'$, and so $\overline{S} = \overline{S}'$. \qed

A little corollary needed later is
Corollary 9.10. Any $S \in C^0$ contains a tree $T$ such that $\overline{T} = \overline{S}$.

Proof. Let $S = \text{supp } f$ be a configuration for $x$. Eliminate any cycles in the support of $f$ by adding basic flows corresponding to those cycles, as in the proof of Lemma 9.9. The resulting flow $f'$ is supported in a tree $T$ and, since all the basic flows have type zero it satisfies $\pi(f') = x$. Lemma 9.9 shows that $\overline{T} = \overline{S}$.

Proof of the general Theorem 7.4. Consider a point $x \in \pi F_\zeta$. The cone $\mathbb{R}_+ (\pi F_\zeta - x)$ gives the tangent cone to $\pi F_\zeta$ at the minimal face of $F_\zeta$ containing $x$. It is obtained by taking the union over all $f \in F_\zeta \cap \pi^{-1}(x)$ of $\pi \mathbb{R}_+ (F_\zeta - f)$. By Lemma 9.8 this gives

$$\bigcup \{ \pi F_0(S) : S \text{ a } \zeta\text{-configuration for } x \},$$

which by Lemma 9.3 is $\sigma(S) = \pi F_0(\overline{S})$, for any $S = \text{supp } f$ and $f \in F_\zeta \cap \pi^{-1}(x)$. The minimal face of $\pi F_\zeta$ containing $x$ is given by intersecting $\pi Z_0(\overline{S})$, the largest subspace in the cone $\pi F_0(\overline{S})$, with the tangent cone of $\pi F_\zeta$ at $x$ and then translating by $x$. This gives

$$x + (\pi F_\zeta)_x \cap Z_0(\overline{S}),$$

which is precisely the face $\text{Face}_\zeta(S)$ mentioned in the theorem and all faces of $\pi F_\zeta$ are obtained in this way. It is obvious that equivalent configurations have the same rank and give the same face, so the partition of $C$ has the stated properties and $\text{Face}_\zeta$ is a bijection $C^0_\zeta / \sim \rightarrow k$-faces.

10. Exactness Results

The exactness of the sequence (5.8) is proven in this section.

10.1. Basic Flows: Sequential Notation. We introduce some notation which is convenient to describe basic flows. This is used in Section 10.2 to give an elementary proof of the exactness of (5.8).

For $v \in Q_0$ and $j \in \{1, \ldots, n\}$, write $\chi^j_v$ for the basis element $\Lambda^1$ which is the indicator function of the singleton $\{a^j_v\} \subset Q_1$. Define the following symbols:

$$\{v\}(j) := \chi^j_v \quad \text{(10.1)}$$

$$\{v\}(-j) := -\chi^j_{v+w_j}. \quad \text{(10.2)}$$

For $k > 0$, and $j_0, \ldots, j_k \in \pm\{1, \ldots, n\}$, define

$$\{v\}(j_0, \ldots, j_k) := \{v\}(j_0) + \{v - w_{j_0}\}(j_1, \ldots, j_k).$$

Also define $(j)\{v\} := \{v + w_j\}(j)$ and

$$(j_k, \ldots, j_0)\{v\} = (j_k, \ldots, j_1)\{v + w_{j_0}\} + (j_0)\{v\}.$$
This sequential notation is designed especially for representing basic flows and has the advantage of including only the relevant information. For instance, if \( p = (p_1, \ldots, p_k) \) is a path in \( Q \), then the basic flow associated to \( p \) is, in this notation, 
\[
\tilde{\chi}_p = \{t(p_1)\}(j_1, \ldots, j_k),
\]
where
\[
\tilde{j}_i := \begin{cases} 
\pi(p_i), & p_i \in p^+ \\
-\pi(p_i), & p_i \in p^- 
\end{cases}
\]

For example, the basic flow represented in Figure 9 can be written as 
\[
\{0\}(1, 1, 2, 1, 1, -3, -3)
\]
in this notation. Figure 10 shows yet another example.

![Figure 10](image_url)

**Figure 10.** The basic flow \( \{0\}(1, 1, 3, -2, 1, 3, -1, -1) \) in the McKay quiver for \( \mathfrak{p}(1, 2, 4) \).

Often, when there is a need to specify both endpoints, the notation \( \{v\}(j_0, \ldots, j_k) =: \{v\}(j_0, \ldots, j_k)\{v'\} \) with \( v' = v - \sum w_{j_i} \) will be adopted. The following identities are easily checked for any \( v \in Q_0 \) and \( j, j_i \in \pm\{1, \ldots, n\} \):
\[
(j)\{v\} + \{v\}(-j) = 0, \quad (10.3)
\]
\[
\{v\}(j_0, \ldots, j_k, -j_k, \ldots, -j_0) = 0, \quad (10.4)
\]
\[
\{v\}(j_0, \ldots, j_k)\{v'\} + \{v'\}(j_{k+1}, \ldots, j_0) = \{v\}(j_{k+1}, \ldots, j_0), \quad (10.5)
\]
where the last identity holds of course for \( v' = v - \sum_{i=0}^k w_{j_i} \).

If \( \sum w_{j_i} = 0 \), i.e. the corresponding flow \( \{v\}(j_1, \ldots, j_k) \) is a \( \partial \)-closed flow associated to a cycle, there is another identity which allows us to cyclically permute the indices:
\[
\{v\}(j_0, \ldots, j_k) = \{v - w_{j_0}\}(j_1, \ldots, j_k, j_0). \quad (10.6)
\]

Recall that \( \Lambda^2 \) is the sub-lattice of \( \Lambda^1 \) generated by the elements corresponding to the commutation relations. These can be written in various notations:
\[
r^{ij}_v = \chi^i_v + \chi^{i}_{v-w_{i}} - \chi^j_v - \chi^{j}_{v-w_{j}} = \{v\}(i, j, -i, -j).
\]
Putting these identities together gives the following lemma.

**Lemma 10.1.** For any permutation \( \sigma \) of \( \{0, \ldots, k\} \) one has

\[
\{v\}(j_{\sigma(0)}, \ldots, j_{\sigma(k)}) = \{v\}(j_0, \ldots, j_k) \mod \Lambda^2.
\]

**Proof.** It is enough to show that the equality holds for any transposition \( \sigma = (q, q+1) \) of consecutive elements. Let \( j_q = i \) and \( j_{q+1} = l \). By the identities (10.4)–(10.5) one has

\[
\{v\}(j_0, \ldots, j_k) = \{v\}(j_0, \ldots, j_{q-1}) + \{v_q\}(i, l) + \{v_q\}(j_q+2, \ldots, j_k),
\]

where \( v_q := v - \sum_{i=0}^{a-1} w_{j_i} \) for any \( a \in \{0, \ldots, k\} \). Using \( \{v_q\}(l, i, -l, -i) + \{v_q\}(i, l) = \{v_q\}(l, i) \), one sees that

\[
\{v\}(j_{\sigma(0)}, \ldots, j_{\sigma(k)}) - \{v\}(j_0, \ldots, j_k) = \{v_q\}(l, i, -l, -i).
\]

Now if \( i, l \in \{1, \ldots, n\} \) then the result follows because \( \{v_q\}(l, i, -l, -i) = r_{v_q}^{-l} \). If on the other hand \( -i, l \in \{1, \ldots, n\} \), then the result is true because

\[
\{v_q\}(l, i, -l, -i) = \{v_q + w_i\}(-i, l, i, -l), \quad \text{ (by equation (10.6)}
\]

\[
= r_{v_q+w_i}^{-l}.
\]

The other two possibilities (\((i, -l)\) and \((-i, -l)\)) also follow in this way. \(\square\)

Another useful result follows from these identities and Corollary 9.2:

**Lemma 10.2.** Any \( \partial \)-closed flow can be written as the basic flow associated to a single cycle in \( Q \) (not necessarily contained in its support).

**Proof.** Note that in the proof of Lemma 9.2, \( f \in \ker \partial \) was decomposed into a sum of basic flows for cycles \( c \). The sum of two basic flows corresponding to cycles can be written, using identities (10.4)–(10.6) as

\[
\{v\}(j_1, \ldots, j_k) + \{v'\}(j'_1, \ldots, j'_{k'}) =
\]

\[
\{v\}(j_1, \ldots, j_k, a_1, \ldots, a_1, j'_1, \ldots, j'_{k'}, -a_1, \ldots, -a_1),
\]

(10.7)

where \( a_i \in \pm\{1, \ldots, n\} \) are such that \( v - \sum_i w_{a_i} = v' \), and one has \( \sum_i w_{j_i} = \sum_i w_{j'_i} = 0 \) since the basic flows on the left-hand side correspond to cycles. This proves the lemma. \(\square\)
10.2. **Exactness Results.** The promised proof of the exactness of (5.8) can now be given. Recall the morphism of lattices $\pi \times \partial: \Lambda_1 \to \mathbb{Z}_n \times \Lambda^0_0$ defined in section 5.3.

**Lemma 10.3.** One has $\ker \pi \times \partial = \Lambda^2$, i.e.

$$\Lambda^2 \to \Lambda_1 \xrightarrow{\pi \times \partial} \mathbb{Z}_n \times \Lambda^0_0$$

is exact.

**Proof.** Since $\pi \times \partial(r_i^j) = 0$ one only has to prove that $\ker \pi \times \partial \subseteq \Lambda^2$. Let $f \in \ker \pi \times \partial$, and suppose that $c'$ is a cycle such that $f = \tilde{\chi} c'$, as described in Lemma 10.2. Then $f$ is of the form

$$f = \{v\}(j_0, \ldots, j_k),$$

for some $v \in \mathbb{Q}_o$ and $j_i \in \pm \{1, \ldots, n\}$. Since $\pi f = 0$, one has $\#\{i : j_i = j\} = \#\{i : j_i = -j\}$ for any $j \in \{1, \ldots, n\}$. Thus, up to permutations, $(j_0, \ldots, j_k)$ can be rewritten as $(j_1, -j_1, j_2, -j_2, \ldots, j_l, -j_l)$ for some elements $j_k \in \{1, \ldots, n\}$. Since $\{v\}(j_1, -j_1, j_2, -j_2, \ldots, j_l, -j_l) = 0$, the result follows by Lemma 10.1.

From this lemma it follows that (5.8) is exact, and induces an inclusion $\Lambda \hookrightarrow \mathbb{Z}_n \times \Lambda^0_0$. Recall that $\Pi = \ker \hat{\rho}$, where

$$\hat{\rho}: \mathbb{Z}_n \longrightarrow \hat{\Gamma}$$

$$x \longmapsto \sum_i x_i w_i.$$

**Lemma 10.4.** One has $\Pi = \pi(ker \partial)$.

**Proof.** Suppose $x \in \Pi$. One wants to find $f \in \ker \partial$ such that, for all $i \in \{1, \ldots, n\}$, $f$ satisfies

$$\sum_{a: \pi(a) = i} f(a) = x_i.$$

This is easy: just take the basic flow given by

$$f = \{v\}(1, \ldots, 1, 2, \ldots, 2, \ldots, n, \ldots, n),$$

for any $v \in \mathbb{Q}_o$. (If $x_i$ is negative under any brace, the notation is taken to mean $-x_i$ copies of $-i$.) Now $\sum_{i=1}^n w_i x_i = 0$ in $\hat{\Gamma}$ is equivalent to the fact that the basic flow corresponds to a cycle, and so $\partial f = 0$. The converse follows from Lemma 10.2 and the preceding sentence. □
Lemma 10.5. There is an exact sequence of abelian groups
\[ 0 \to \Lambda \xrightarrow{\pi \times \partial} \mathbb{Z}^n \times \Lambda^{0,0} \xrightarrow{\hat{\nu} - \hat{\rho}} \mathbb{Z}^n / \Pi \cong \hat{\Gamma} \to 0, \]

where
\[ \hat{\nu} : \Lambda^{0,0} \to \hat{\Gamma} \]
\[ \zeta \mapsto \sum_{v \in \mathbb{Q}_0} \zeta(v) v \]
is the morphism of lattices dual to the action of \( \Gamma \) on \( \text{End} R \) by conjugation.

Proof. One needs to show that if \((x, \zeta) \in \mathbb{Z}^n \times \Lambda^{0,0}\) satisfies
\[ \sum_{i=1}^{n} x_i w_i - \sum_v \zeta(v) v = 0, \]
then there exists a flow \( f \in \Lambda^1 \) such that \( \pi(f) = x \) and \( \partial f = \zeta \).

One can construct this flow in two steps. For convenience, suppose that the weights of the action of \( \Gamma \) on \( Q \) have been normalised so that \( w_1 = 1 \). First construct a flow \( g \) such that \( \partial g = \zeta \) with only arrows of type 1: start at vertex 0 and let \( g(1 \to 0) = \zeta(0) \). Next, let \( g(2 \to 1) = \zeta(1) + \zeta(0) \), and continue in this way, setting
\[ g(k + 1 \to k) = \sum_{j=0}^{k} \zeta(j), \]
and \( g = 0 \) on all the other arrows. Then \( \partial g = \zeta \) by construction, and \( \pi(g) = X_1 e_1 \), where \( X_1 = \sum_j \zeta(j) j \). Now add on any flow of the form
\[ g' = \{v\}^{(1, \ldots, 1, 2, \ldots, 2, \ldots, n, \ldots, n)} \]
using the same conventions as in the previous lemma for the case when the integers under the braces are negative. Equation (10.9) means that \( g' \) is the basic flow associated to a cycle, and so \( \partial g' = 0 \). The flow \( f = g + g' \) has the required properties. \( \square \)

11. Singular Configurations

In this section some comments are made regarding the singularities of \( C_{\zeta} \).

Recall that the tangent cone to \( C_{\zeta} = \pi F_{\zeta} \) at a point \( x \) which has a configuration \( S \in C^k \), is the cone \( \pi F_0(\mathcal{S}) \). This is singular if its dual \( (\pi F_0(\mathcal{S}))^\vee \) is generated by a part of a basis of \( \Pi^* \). In this case, \( S \) is called a singular configuration. The set of singular configurations is denoted \( \mathcal{S} \).
Proposition 11.1. The following statements are true.

1. $C_\zeta$ is singular at the faces corresponding to configurations $S \in S_\zeta$.
2. $C_\zeta$ is non-singular in co-dimension $k$ if and only if $S_{n-k}^\zeta = \emptyset$.
3. $C_\zeta$ is generically non-singular in co-dimension $k$ if and only if $S_{\text{span}}^\zeta = \emptyset$.
4. $C_\zeta$ is generically non-singular if and only if $S_{\text{span}} = \cup_{\zeta \text{ generic}} S_\zeta$.

Proof. Part (1) of the following proposition follows from Theorem [7.4] and Lemma [12.4]. Part (2) follows because $S \in S_{n-k} \iff \dim \sigma_S = k$. The last two statements follow because $S_{\text{span}} = \cup_{\zeta \text{ generic}} S_\zeta$.

Question 1. What is the lowest $k$ for which $S^k_\zeta$ is empty (i.e. in what co-dimension is $C_\zeta$ smooth)? The cone $C_0$ is smooth in co-dimension 1 (it has an isolated singularity), so it seems likely that $S^k = \emptyset$ for $k \geq 1$. Of course, this is equivalent to the statement that $S^1 = \emptyset$.

The statements about singular trees are translated here for the record.

Conjecture 2. The polyhedra $C_\zeta$ are non-singular in co-dimension $n-1$, i.e. $S^1 = \emptyset$. If $\Gamma \subset \text{SU}(3)$, then $C_\zeta$ are non-singular for generic values of $\zeta$, i.e. $S^0_{\text{span}} = \emptyset$. Furthermore, the Euler number of $C_\zeta$ for generic $\zeta$ is equal to the order of $\Gamma$, i.e. $\# T^0_\zeta/\sim = \# \Gamma$.

Let us look at the case of singular points. Suppose that $S \in C^0$ is a singular configuration. This means that the primitive generators $\rho(S) = (\rho^1_S, \ldots, \rho^k_S)$ of $\pi F_0(S)$ do not form a basis of $\Pi$. If $k = n$, $\pi F_0(S)$ corresponds to a finite abelian quotient singularity, whereas, if $k > n$, the singularity is determined by the linear relations holding between the $\rho^i_S$. To determine whether a given set $S \in C^0$ is singular or not, one must find all the cycles $c$ such that $c^- \subseteq S$, calculate their type, and see what primitive vectors of $\Pi$ one obtains. It is sufficient to restrict one’s attention to the cycles which are not decomposable into a union of cycles. Let us look at some examples.

12. Examples and Computations

12.1. Commutators. In this section, the commutator of a configuration $S$ is defined; this is specific to the McKay quiver and its purpose is purely computational: it gives a necessary criterion for determining when $S \in C^0$ which is extremely useful in practical calculations.

Let us begin by working out what the closure of a subset $S \subseteq Q_1$ corresponds to in the case of the McKay quiver for the action $\frac{1}{r}(w_1, \ldots, w_n)$. Let $\pi$ be the map $Q_1 \to \{1, \ldots, n\}$ which assigns to each
arrow $a_v^i = v \rightarrow v - w_i$ its type $i$. This induces a map $\pi: \mathbb{R}Q_1 \rightarrow \mathbb{R}^n$ which assigns to the basis element $\chi_v^i = \chi a_v^i$ the basis element $e_i$ of $\mathbb{R}^n$.

Recall that to calculate the closure of $S$, one must find the smallest over-set of $S$ which contains the positive part of cycles of type zero if and only if it contains the negative part. The simplest cycles which have type zero are cycles with only four arrows: starting from vertex $v$, go forward along arrow $a_v^i$ to $v - w_i$, forward again along $a_v^j$ to $v - w_i - w_j$, back along $a_v^j$ to $v - w_j$ and back along $a_v^i$ to $v$. This cycle is denoted by $c_{ij}^v$. The basic flow corresponding to this cycle is $\{v\}_{ij} = \{a_v^i, a_v^j\}$. If $S$ contains $c_{ij}^v = \{a_v^i, a_v^j\}$ for some $v, i, j$ the closure of $S$ must contain $c_{ij}^v = \{a_v^i, a_v^j\}$. The two pairs of arrows will be called complementary pairs.

**Definition 12.1.** Denote by $p_{ij}^v$ the pair of arrows $\{a_v^i, a_v^j\}$. If $S$ is a subset of $Q_1$, the smallest over-set $S^{\text{opt}} \supseteq S$ satisfying the "commutation condition"

$$p_{ij}^v \subseteq S^{\text{opt}} \iff p_{ji}^v \subseteq S^{\text{opt}}$$

(12.1)

is called the commutator of $S$.

![Figure 11. The commutator of the set in Figure 3.](image)

Notice that this is not an invariant set: there is no way to assign elements of $\mathbb{Z}^n$ to the vertices in such a way that equation (12.3) is satisfied.

The fact that $\overline{S} \supseteq S^{\text{opt}}$ is extremely useful in practical computations, because it gives the following easily checked necessary condition for $S$ to be an IC-set.

**Lemma 12.2.** If $S$ is a spanning IC-set then the morphism $W_S: Q_0 \rightarrow \mathbb{Z}^n$ defined by equations (12.4) satisfies the following conditions for all $v \in Q_0$, and $i, j \in \{1, \ldots, n\}$:

$$p_{ij}^v \subseteq S \implies \begin{cases} W_S(v - w_j) - W_S(v) = e_j \\ W_S(v - w_i - w_j) - W_S(v - w_i) = e_i \end{cases}$$

(12.2)

where $\{e_i\}$ denotes the standard basis of $\mathbb{Z}^n$. 
It is rather surprising that this condition is in fact not sufficient. Figure 12 gives an example of a set $S$ for which $S = S^\infty \neq \overline{S}$.

![Figure 12. A configuration $S$ for which $S = S^\infty \neq \overline{S}$. The dotted arrows indicate the arrows in $\overline{S} \setminus S^\infty$.](image)

### 12.2. Weightings and Stabilisers

There is another way of understanding the condition $S \in C^k$ for a set $S \subseteq Q_1$ which relates directly to the toric geometry of $X_\zeta = T^{\Pi,C_\zeta}$. The basic idea is that a $k$-dimensional face of $C_\zeta$ corresponds to elements of $X_\zeta$ which are fixed by a torus of codimension $k$. For instance, the extreme points correspond to fixed points of $T^{\Pi}$. Let us begin with this case for simplicity.

#### 12.2.1. Fixed Points and $n$-weightings

Recall that the lattice $\mathbb{Z}^{Q_0}$ is denoted by $\Lambda^0$. The sub-lattice of co-rank 1 defined by the equation $\sum_{v \in Q_0} \zeta(v) = 0$ is denoted $\Lambda^{0,0}$.

For any subset of arrows $S \subseteq Q_1$ one attempts to find a morphism $W_S: \Lambda^{0,0} \to \mathbb{Z}^n$ satisfying

$$W_S(\partial \chi_a) = \pi(\chi_a), \quad \text{for } a \in S. \tag{12.3}$$

If $W_S$ exists, it satisfies

$$W_S(\partial \tilde{\chi}_p) = \pi \tilde{\chi}_p \tag{12.4}$$

for any path $p = (p^1, \ldots, p^k)$ in $S$. In particular, this equation implies that $\pi \tilde{\chi}_c = 0$ must hold for all cycles $c \subseteq S$. In other words, if $W_S$ exists, then rank $\pi Z_0(S) = 0$. Conversely, if all cycles in $S$ have zero type then one can find a morphism $W_S$ satisfying (12.3). Note that this is possible if and only if there is morphism $W_S': \Lambda^0 \to \Pi$ which extends it. The latter corresponds to a labeling $W_S': Q_0 \to \mathbb{Z}^n$ of the vertices of the quiver by elements of $\mathbb{Z}^n$ in such a way that equation 12.3 holds. This is called an $n$-weighting of $S$, and $S$ is called invariant if it admits such a weighting. In summary, $S$ is an extreme configuration if and only if $\overline{S}$ is invariant, if and only if $\pi Z_0(S) = 0$. Such $S$ are called invariant closure configurations or IC-configurations.
for short. Particularly important is the set $T^0$ of IC-trees; it is easy to determine whether $T \in T$ is an IC-tree: just check whether the arrows $a \in T$ all satisfy $W_T \partial \chi_a = \pi \chi_a$.

To see the relationship to the toric geometry of $X_{\xi}$, recall that $X_{\xi}$ is the GIT quotient of a certain affine variety in $C^{Q_1}$ by $T^{0,0}$. The extreme points of $C_{\xi}$ therefore correspond to elements $\tilde{\alpha} \in C^{Q_1}$ which are mapped to the same $T^{0,0}$-orbit under the morphism of algebraic tori

$$\tilde{W}_S : C^{*n} \to T^1.$$ 

This is the case for any $\tilde{\alpha}$ such that $\text{supp} \, \tilde{\alpha} \subseteq S^{\Box}$, where $S^{\Box}$ is the set of all arrows in $Q_1$ which satisfy (12.3) (this includes of course $\overline{S}$).

This discussion can be extended to higher dimensional faces, although it is not as practical for concrete calculations.

12.2.2. Higher Dimensional Faces. The sub-lattice of integral flows on $Q$ which are zero outside $S$ will be denoted $Z^S \subseteq Z^{Q_1}$. Recall that $Z_0(S)$ is the set of 0-flows which are zero outside $S$: $Z_0(S) = Z^S \cap \ker \partial$.

Define, for each $S \subseteq Q_1$, a morphism of lattices

$$W_S : \partial Z^S \to \pi Z^S / \pi Z_0(S)$$

defined by

$$\partial \chi_a := \pi Z_0(S) + \pi \chi_a, \quad a \in S.$$ 

(12.5)

This fits into a commutative diagram

$$\begin{array}{c}
\partial Z^S \\
\downarrow \partial \\
Z^S
\end{array} \xrightarrow{W_S} \begin{array}{c}
\pi Z^S / \pi Z_0(S) \\
\uparrow \text{pr} \\
\pi Z^S
\end{array}.$$
Taking the image of this under the functor $\hat{\cdot} = \text{Hom}(\cdot, \mathbb{C}^*)$, one obtains a corresponding diagram of algebraic tori:

\[
\begin{array}{cccc}
T^{0,0} & \xleftarrow{\hat{W}_S} & T^{\pi Z^S/\pi Z_0(S)} \\
\downarrow{\hat{\partial}} & & \downarrow{\hat{\pi}} \\
T^1 & \xleftarrow{\pi|Z^S} & T^{\pi Z^S} \subseteq \mathbb{C}^n.
\end{array}
\]

This diagram shows that the action of the sub-torus $T^{\pi Z^S/\pi Z_0(S)}$ of $T^{\pi Z^S}$ on $\tilde{\alpha} \in \mathbb{C}^S$ via $\hat{\pi}|Z^S\hat{\pi}$ leaves $\tilde{\alpha}$ in the same orbit of $T^{0,0}$. In fact, since the torus corresponding to $Z^S_c$ (where $S_c$ denotes the complement of $S$ in $Q_1$) acts trivially on $\mathbb{C}^S$, one sees that $T^{\pi Z^Q_1/\pi Z_0(S)}$ acts on $\mathbb{C}^S$ fixing the $T^{0,0}$-orbits and is a sub-torus of $T^{\pi Z^Q_1} = \mathbb{C}^n$ of codimension rank $\pi Z_0(S)$.

12.2.3. Existence and Uniqueness of $n$-weightings.

Definition 12.3. Two vertices in $v, v' \in Q$ are said to be connected by a subset $S \subseteq Q_1$ if there is a path $p$ in $S$ whose endpoints are $\{v, v'\}$. A subset $S \subseteq Q_1$ is called a spanning set if it connects any two vertices in $Q_0$.

One sees easily that the condition that $S$ be a spanning subset is equivalent to the statement $\partial Z^S = \Lambda^{0,0}$. Thus if $S$ is a spanning subset, any $n$-weighting $W_S$ of $S$ is completely determined by (12.4), and so is unique. In fact, if $T \subseteq S$ is any spanning tree in $S$, then $T$ determines a unique morphism $W_T: \Lambda^{0,0} \to \mathbb{Z}^n$, and one sees that $S$ is invariant if and only

\[ W_T \partial \chi_a = \pi \chi_a, \quad \forall a \in S \setminus T. \quad (12.6) \]

The following lemma shows that, for generic values of $\zeta$, one can effectively use the condition above to check the invariance of configurations.

Lemma 12.4. If $\zeta$ is generic in $\Lambda^0 = \Lambda^0 \otimes_{\mathbb{Z}} \mathbb{R}$ and $f$ is a $\zeta$-admissible flow then $\text{supp}(f)$ is a spanning subset.

Proof. If $\text{supp}(f)$ is not a spanning subset then there exists a partition of $Q_0$ into disjoint non-empty subsets $S_0$ and $S_1$ such that no element of $S_0$ is connected to any element of $S_1$. Consider the restriction of $f$
to the $S_i$: the previous statement implies that $\partial(f|_{S_i}) = (\partial f)|_{S_i}$. Since, for any $\zeta$-flow $g$, “the flow is conserved,” i.e. $\sum_{v \in \mathbb{Q}_0} (\partial g)_v = 0$, one has

$$\sum_{v \in S_i} \zeta_v = \sum_{v \in S_i} (\partial f)_v$$

$$= \sum_{v \in \mathbb{Q}_0} (\partial (f|_{S_i}))_v$$

$$= 0,$$

but this does not happen for a generic $\zeta$ in $\mathbb{R}^{\mathbb{Q}_0}$.  

First, an example which has some singular cones: the action of the group of fifth roots of unity on $\mathbb{C}^3$ with weights 1, 2 and 3. In this case, there are whole cones of values of $\zeta$ for which $X_\zeta$ is smooth, and others where $X_\zeta$ is singular.

**Remark 12.5 (About the computations).** The computations were done using several computer programs. A Pascal program was used to produce a list of all the IC-trees for any action of a cyclic group. Then, for each value of $\zeta$, another Pascal program was used to determine which trees were admissible and to work out the corresponding flows and extreme points. Then a Mathematica program was run to draw the pictures of the polyhedra and of the extreme flows.

12.3. **Example: the action $\frac{1}{5}(1,2,3)$.** Recall the action $\frac{1}{5}(1,2,3)$ of the group $\mu_5$ of fifth roots of unity on $\mathbb{C}^3$ with weights 1, 2 and 3 considered in example 5.1. The McKay quiver in this case is the regular oriented graph with 5 vertices and an arrow between each vertex $v$ and $v - 1$, $v - 2$ and $v - 3$ (mod 5).

The set $\mathcal{T}^0$ of IC-trees contains a total of 55 trees, once one has factored out by the symmetry which consists in permuting the vertices cyclicly (the action of $\hat{\Gamma}$ on itself). To draw $C_\zeta$, choose values of $\zeta$, calculate the $\zeta$-flows on all the IC-trees, discard those which are negative, and project the resulting points to $\mathbb{R}^3$ via the map $\pi$. The resulting convex polyhedra $C_\zeta$ are the intersection of the positive orthant with a finite number of half-spaces. The polyhedron $C_\zeta$ for the value $\zeta = (-1, -1, -1, -1, 4)$ is shown in Figure 14. The trees and flows corresponding to the extreme points appear in Figure 15.

One sees immediately from the figure that $X_\zeta$ has a singularity at the point corresponding to the extreme point $(1, 3, 1)$: the tangent cone there has four generators. The other extreme points are the intersection of three faces: in order to determine whether they are in fact singular or not one must check whether the primitive generators in $\Pi \subset \mathbb{Z}^3$ of
Figure 14. $C_\zeta$ for $\frac{1}{5}(1, 2, 3)$, $\zeta = (-1, -1, -1, -1, 4)$. (The view is from “behind”, from the point with coordinates $(-1.3, -1, -1)$. The vertices have been numbered (where possible) in order of increasing $z$ coordinate.)

the tangent cone actually generate $\Pi$. In this case it turns out that they do, so they are smooth points.

In fact, there are a total of 7 singular non-isomorphic IC-trees. These have been listed in Figure 16. The first tree (rotated by $\frac{2\pi}{5}$) corresponds to the singular point $(1, 3, 1)$ above. The corresponding tangent cone to $C_\zeta$ was described in Section 7.2 — it has four generators $v_1, \ldots, v_4$, any three of which generate $\Pi$, and satisfying a single relation of the form $v_1 + v_3 = v_2 + v_4$. The corresponding singularity is therefore of the type $xw = yz \subset \mathbb{C}^4$: a cone over a quadric surface.

Checking the cases listed in Figure 16, one sees that it is possible to find generic values of $\zeta$ for which none of these trees (nor their rotations by elements of $\hat{\Gamma}$) are admissible: for instance, $\zeta = (9, 8, -3, -2, -12)$ is such a value; this gives a smooth resolution of $\mathbb{C}^3/\mathbb{Z}_5$ which has Euler number 9. The corresponding polyhedron and flows are shown in Figures 17 and 18.
Figure 15. Extreme flows for $\frac{1}{5}(1, 2, 3)$, $\zeta = (-1, -1, -1, -1, 4)$. (The values of the flows are indicated to the right of the arrows. The numbers in the top left-hand corners correspond to the vertex numbers in Figure 14.)

Figure 16. Non-isomorphic singular IC-trees in $T^0$ for the action $\frac{1}{5}(1, 2, 3)$. 
Remark 12.6. In fact, all the singular trees in Figure 16 correspond to the same type of singularity, namely a cone over a quadric surface. All the IC-trees whose cones are simplicial correspond to non-singular points. Thus one can tell whether \( X_\zeta \) is non-singular simply by checking whether all extreme points have three edges emanating from them. This is also the case for other singularities: for instance, \( \frac{1}{3}(1, 2, 5) \), \( \frac{1}{8}(1, 2, 6) \), \( \frac{1}{9}(1, 2, 7) \); it is not always the case however: for instance, for \( \frac{1}{7}(1, 2, 3) \) and \( \frac{1}{10}(1, 2, 8) \) where there exists elements of \( T^0 \) which correspond to \( \mathbb{Z}_2 \)-quotient singularities.

12.4. Crepant Resolutions. Let \( \Sigma_\zeta \) denote the fan determined by the polyhedron \( C_\zeta \). By Corollary 3.3, the one-skeleton of \( \Sigma_\zeta \) is given by

\[
\Sigma^{(1)}_\zeta = \{ \sigma(S)^{\vee} : S \in C_\zeta^{n-1} \},
\]

Note that this is consistent with the conjecture in [S1960] regarding the quadratic nature of the singularities of \( X_\zeta \).
so $X_\zeta$ has trivial canonical bundle if the primitive generators $v_S$ of the cones $\sigma(S)^\vee$ for $S \in C^{n-1}$ all belong to the hyper-plane defined by the equation $\sum n_i = 1$ in $\Pi^* = \mathbb{Z}^3 + \frac{2}{7}(w_1, w_2, w_3)$.

**Example 12.7.** Consider the action $\frac{1}{3}(1, 1, 1)$. This has only three IC-trees up to isomorphism. They consist of trees with two arrows of the same type. For generic values of $\zeta$, a little thought shows that the polyhedron $C_\zeta$ is the positive quadrant with a small equilateral triangle chopped off. The dual fan is the barycentric subdivision of the positive quadrant by the ray passing through the point $v = \frac{1}{3}(1, 1, 1) \in \Pi^*$. This is a non-singular fan, and since the point $v$ belongs to the hyper-plane $\sum n_i = 1$, this has trivial canonical bundle. The variety $X_\zeta$ is the total space of the bundle $\mathcal{O}(-3)$ over $\mathbb{P}^2$.

To conclude this section, a picture of a more complicated example is draw. Note that $\Gamma \subset SU(3)$ and that the variety $X_\zeta$ is a smooth crepant resolution with Euler number 11.
Figure 19. An example of $C_\zeta$ for the action $\frac{1}{11}(1, 4, 6)$.

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