Finite - temperature quantum phase transition in d - waves superconductors

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Abstract

The zero temperature d - wave superconductor phase transition theory given in the case of $T = 0$ for two - dimensional superconductors (I. Herbut, PRL 85, 1532 (2000)) is generalized for finite temperatures. The Gaussian behavior of the system is associated with a non - Fermi behavior of the normal state observed in the resistivity of cuprate superconductors.
I. MODEL AND SCALING EQUATIONS

In a fermionic system disorder and the attractive interaction on the d-wave channel at \( T = 0 \) give rise to the superconductor pairing. The two-dimensional model \( (d = 2) \) has been studied in [1] and the main result of this paper is the mapping of the fermionic system in a dissipative bosonic system with a wide crossover regime controlled by the fluctuations of the order parameter. This system is similar to the insulator-superconductor at \( T = 0 \) and finite temperature reconsidered by the Urbana group [2–6], in study of the transport properties near the critical point of this transition. Following classical results [7] from the superfluid-insulator transition they showed that the insulator-superconductor transition is characterized by the dynamical critical exponent \( z = 2 \) and used the renormalization-group method (RG) to describe the critical behavior of the conductivity near the transition point. The model used in [1] contain an action with dissipative term which is a relevant perturbation with the particle-hole asymmetry. In fact, the analogy between the d-wave superconductivity and this system was mentioned in connection with the contribution of the different regimes to the conductivity.

This report is complementary to papers [1] and [5] considering the crossover effects as well as the case of the finite value of the damping parameter from the dissipative term of action.

We start with the action from Ref. [1] defined using the field operators \( \phi_i(k) \) written as:

\[
S[\phi] = \sum_i \sum_k \left[ \phi_i^\dagger(k) \left( \frac{\hbar^2 k^2}{2m} + \frac{\left| \omega_n \right|}{\Gamma} - \mu \right) \phi_i(k) \right] \\
+ \frac{u}{4} \sum_i \sum_{k_1} \ldots \sum_{k_4} \delta \left( \sum_{l=1}^4 k_l \right) \phi_i(k_1) \ldots \phi_i(k_4)
\]

(1)

In Eq. (1) we have contributions of the n-independent replicas and \( \phi_i(k) = \phi(k) \). The other parameters are related with those from Ref. [1] as: \( \mu = \mu_b m^{1/2} \), \( m = m + b^{1/2} \), \( u = 4 \lambda \) and \( \Gamma \) is an energy parameter which controls the strength of the quantum fluctuations. The effective chemical potential \( \mu_b \) which is negative for the normal phase (see Ref. [1]) is expressed by the relation \( \mu_b = (\ln \tau E_F - g^{-1})/\tau \), \( \tau \) being the scattering time, proportional to the impurities concentration and \( g \) the superconductor coupling constant. If we introduce the critical scattering time \( \tau_c = 2/\Delta(0) \) \( (\Delta(0) \) is the \( T = 0 \) order parameter) above which the order parameter vanishes, the chemical potential \( \mu_b \) depends of the impurities concentration, \( x \), as \( \mu_b(x) \sim |x - x_c| \). This dependence is obtained using the wellknown result that the scattering on a non-magnetic impurities in d-wave superconductors has a similar effect with the scattering on the magnetic impurities in the s-wave superconductors, and also using \( 1/\tau \sim x \).

In this model the Eq. (1) describe a quantum phase transition controlled by non-magnetic impurities in d-wave superconductors.

The effect of finite temperature on this quantum phase transition can be studied using the RG method and has been applied by different authors [7–15] to study the influence of temperature on the quantum phase transition in the interacting bosonic systems. Following this method we consider the case of finite temperatures and \( \Gamma \neq 0 \) and using the scaling \( k = k'/b \) and \( \omega_n = \omega_n'/b^z \) \( (b = \ln l \) and \( z \) is the critical exponent) we get the scaling equations:
\[
\frac{dT(l)}{dl} = 2T(l) \tag{2}
\]

\[
\frac{d\Gamma(l)}{dl} = 0 \tag{3}
\]

\[
\frac{d\mu(l)}{dl} = 2\mu(l) - f^{(2)}[T(l), \mu(l)] u(l) \tag{4}
\]

\[
\frac{du(l)}{dl} = (2 - z)u(l) - f^{(4)}[T(l), \mu(l)] u^2(l) \tag{5}
\]

and for the free energy

\[
\frac{dF(l)}{dl} = 4F(l) + f^{(0)}[T(l), \mu(l)] \tag{6}
\]

Functions \( f^{(2)}, f^{(4)} \) and \( f^{(0)} \) are given for \( d = 2 \) system by:

\[
f^{(2)}[T(l), \mu(l)] = \int_0^\Lambda \frac{d^2k}{(2\pi)^2} k_B T \sum_n \left[ \frac{\hbar^2 k^2}{2m} + \frac{\mid\omega_n\mid}{\Gamma} - \mu \right]^{-1} \tag{7}
\]

\[
f^{(4)}[T(l), \mu(l)] = \int_0^\Lambda \frac{d^2k}{(2\pi)^2} k_B T \sum_n \left[ \frac{\hbar^2 k^2}{2m} + \frac{\mid\omega_n\mid}{\Gamma} - \mu \right]^{-2} \tag{8}
\]

\[
f^{(0)}[T(l), \mu(l)] = \int_0^\Lambda \frac{d^2k}{(2\pi)^2} k_B T \sum_n \arctan \frac{Im \chi(k, i\omega_n)}{Re \chi(k, i\omega_n)} \tag{9}
\]

where

\[
\chi^{-1}(k, i\omega_n) = \frac{\hbar^2 k^2}{2m} + \frac{\mid\omega_n\mid}{\Gamma} - \mu
\]

and \( \Lambda \) is the momentum cutoff.

We have to mention that these equations have been obtained from Eq. (1) neglecting the quartic term which describes the interaction between fluctuations from different replicas.

### II. FIXED POINTS

In order to study the influence of the temperature on the quantum effects, we have to calculate the fixed points of the scaling equations.

As in the case of the bosons, (see Ref. [12]) we will analyze the two relevant cases: the low temperature and high temperature regimes.
A. Low temperature regime

In the low temperature regime $z = 2$ and from Eqs. (2-5) we get:

\[
\frac{du(l)}{dl} = -\frac{m}{4\pi \bar{h}} u^2(l) \quad (10)
\]

\[
\frac{d\mu(l)}{dl} = 2\mu(l) + f^{(2)}[T(l), \mu(l)] u(l) \quad (11)
\]

where we used $\mu < 0$ for the normal phase.

The function $f^{(2)}$ has been calculated taking the $\omega_n = 0$ and $\omega_n \neq 0$ frequencies at finite temperatures as:

\[
f^{(2)}[T(l), \mu(l)] \simeq \frac{\Lambda^2}{4\pi} + \frac{m k_B T}{2\pi \bar{h}^2} \int_0^\infty \frac{dy}{\exp[y + \frac{\mu}{T}] - 1} \quad (12)
\]

From Eq. (10) we obtain:

\[
u(l) = \frac{4\pi \bar{h}^2}{m} \frac{1}{l + l_0} \quad (13)
\]

where $l_0 = 4\pi \bar{h}^2/mu$.

In order to get the fixed point of Eq. (11) we use the condition $d\mu(l)/dl = 0$ and we obtain:

\[2\mu^*_0 + \frac{\Lambda^2 u}{4\pi} + \frac{m k_B T}{2\pi \bar{h}^2} \int_0^\infty \frac{dy}{\exp[y + \frac{\mu}{T}] - 1} u(l) = 0 \quad (14)\]

From this equation we can see that for finite temperature there is no phase transition because the last integral from Eq. (14) is divergent. At $T = 0$ we get:

\[\mu^*_0 = -\frac{\Lambda^2 u}{8\pi} \quad (15)\]

which correspond to the quantum phase transition.

B. High temperatures regime

In this case the scaling equations are obtained from general equations (2-5) by taking $z = 2$ and using the approximation $\coth x/2T \simeq 2T/x$. The scaling equations (see also Ref. [13,14]) becomes:

\[
\frac{du(l)}{dl} = 2u(l) - \frac{5}{2} K_k \frac{\Lambda^2 k_B T(l)}{[\epsilon_A + \mu]^2} \quad (16)
\]

\[
\frac{d\mu(l)}{dl} = 2\mu(l) - \frac{1}{2} K_k \frac{\Lambda^2 k_B T(l)}{[\epsilon_A + \mu]^2} \quad (17)
\]

where $\epsilon_A = \hbar^2 \Lambda^2/2m$ and $K_k = 1/2\pi$. From these equations we obtain the fixed points $\mu^* = -5\epsilon_A/2$ and $u^* = 2\pi \epsilon_A^2/45\Lambda^2 k_B T$. This is the $T = \infty$ fixed point.

The crossover between the two regimes is equivalent to the crossover between the quantum and classical critical behavior and we will define a parameter $l^*$ for the two regimes.
III. CROSSOVER BETWEEN QUANTUM AND CLASSICAL REGIMES

The quantum behavior is defined by the fixed point $T = 0$. The effect of the temperature can be considered if we determine the temperature scale $\tilde{l}$ defined by:

$$T(\tilde{l}) = T_0$$

where $T_0$ is a characteristic temperature. Using the solution $T(l) = T e^{2l}$ we get the parameter $\tilde{l}$ as:

$$\tilde{l} = \frac{1}{2} \ln \frac{T_0}{T}$$

Using Eqs. (18-19) we define the parameter $l^*_0$ by:

$$\mu(l^*_0) = -\alpha \frac{\hbar^2 \Lambda^2}{2m}$$

being the energy parameter which define the low temperature critical region, and $\alpha \leq 1$.

From Eq. (11) we calculate (see also Ref. [12]) $\mu(l)$ as:

$$\mu(l) = -4\epsilon \Lambda e^{2l} \left\{ k_B T \left[ \frac{1}{2l_0} \ln \left( 1 - e^{-\epsilon \Lambda / k_B T} \right) \right] - \frac{1}{2l_0} \left( 1 + \frac{l}{l_0} \right)^{-1} \ln \left( 1 - e^{-\epsilon \Lambda / k_B T} \right) \right\} - \frac{k_B T}{\epsilon \Lambda} F(l)$$

This expression can be transformed in:

$$\mu(l) = -4\epsilon \Lambda e^{2l} \left\{ k_B T \left[ \frac{1}{2l_0} \ln \left( 1 - e^{-\epsilon \Lambda / k_B T} \right) \right] - \frac{1}{2l_0} \left( 1 + \frac{l}{l_0} \right)^{-1} \ln \left( 1 - e^{-\epsilon \Lambda / k_B T} \right) \right\} - \frac{k_B T}{\epsilon \Lambda} F(l)$$

where

$$F(l) = \int_0^{2l} \frac{dx}{(x + 2l_0)^2} \ln \left[ 1 - \exp \left( -\frac{\epsilon \Lambda}{k_B T} e^{-2l} \right) \right]$$

The energy scale will be fixed using the Eqs. (12) and (21) and following Ref. [12] we calculate

$$e^{-2l^*} \simeq 16 \frac{T}{T_0} \frac{1}{\ln \frac{T}{T_0}}$$

where $T_0 = \alpha \hbar^2 \Lambda^2 / 8mk_B$. Using this result we obtain

$$l^*_0 = \frac{1}{2} \ln \left( \frac{T_0}{T} \ln \frac{T_0}{T} \right)$$

Let us consider the high temperatures regime, describe by the Eqs. (16-17), which will be rewritten using the new interaction $v(l) = k_B T(l) u(l)$, as
\[
\frac{dv(l)}{dl} = 2v(l) - \frac{5}{\pi \hbar^4 \Lambda^2} v^2(l) \quad (26)
\]

\[
\frac{d\mu(l)}{dl} = 2\mu(l) - \frac{m}{\pi \hbar^2 \Lambda^2} v(l) \quad (27)
\]

The exact solution of Eq. (26),
\[
v(l) = \frac{2v(\tilde{l})}{Bv(\tilde{l}) + [2 - v(\tilde{l})]e^{-2(l-\tilde{l})}} \quad (28)
\]

where \( B = \frac{5m^2}{\pi \hbar^4 \Lambda^2} \) will be approximated as
\[
v(l) \simeq v(\tilde{l}) e^{-2(l-\tilde{l})} \quad (29)
\]

and from Eq. (27) we get
\[
\mu(l) = -\frac{4(l - \tilde{l})}{l + l_0} e^{2(l-\tilde{l})} \quad (30)
\]

The new energy scale will be fixed by \( l_1^* \) as
\[
\mu(l_1^*) = -\frac{\hbar^2 \Lambda^2}{2m} \quad (31)
\]

and from Eqs. (30) and (31) we express
\[
l_1^* = \frac{1}{2} \ln \left( \frac{T_0}{T} \ln \frac{T_0}{T} \right) \quad (32)
\]

The Eqs. (25) and (32) show that we can perform a matching between the two regimes and:
\[
l_0^* = l_1^* = l^* \quad (33)
\]

Physically this result can be regarded as following: In the \( T - l \) plane we can reach the critical region starting from the quantum regime or from the classical regime. The characteristic temperature in this point is
\[
T^* = T_0 \ln \frac{T_0}{T} \quad (34)
\]

and \( u^*(l = l^*) \) calculated from Eq. (13) is
\[
u^* = \frac{4\pi \hbar^2}{m} \frac{1}{l_0 + \ln \left( \frac{T_0}{T} \ln \frac{T_0}{T} \right)} \quad (35)
\]

This value is very small for \( T < T_0 \) and in this region the perturbation theory is valid. In the next section we calculate the specific heat in these two regimes.


**IV. FREE ENERGY AND SPECIFIC HEAT**

The free energy will be calculated from Eqs. (6) and (9) as:

\[
\frac{dF}{dl} = 4F(l) + f^{(0)}[T(l), \mu(l)]
\]  

(36)

where

\[
f^{(0)}[T(l), \mu(l)] = -\frac{K_2\Lambda^2}{\pi} \int_0^T d\omega \coth \frac{\omega}{2k_BT(l)} \tan^{-1} \frac{\omega}{\frac{\hbar^2\Lambda^2}{2m} - \mu(l)}
\]  

(37)

In order to get the temperature dependence of the free energy and the specific heat we divide the temperature interval in two regimes: \(0 \leq x \leq \frac{1}{2}\ln \frac{T_0}{T}\) (quantum regime) and \(\frac{1}{2}\ln \frac{T_0}{T} \leq x \leq l_M\), where \(l_M\) is temperature independent in the classical regime. The general solution of Eq. (36) has the form:

\[
F[T(l)] = \int_0^l dx e^{-4x} f^{(0)}[Te^{2x}]
\]  

(38)

and following [10] we expand \(f^{(0)}(T)\) as \(\lim_{T \to 0}[f^{(0)}(T) - f^{(0)}(0)] \sim T^2\). In the first regime the contribution of \(f^{(0)}(0)\) is negligible. In the second regime one may approximate \(f^{(0)}(T) \sim T\) and from Eq. (28) we get

\[
F(T) = F_1 T^2 \ln \frac{T_0}{T} - F_2 T
\]  

(39)

where \(F_1\) and \(F_2\) are constants.

Using now for the specific heat the relation \(C_v = -k_B\partial^2 F/\partial T^2\) we obtain

\[
C_v(T) = \gamma_0 T + \gamma_1 T \ln \frac{T}{T_0}
\]  

(40)

where the first term gives the classic contribution and the last term is the contribution of the non-Fermi excitations.

**V. DISCUSSIONS**

The results obtained can be discussed referring to the experimental result obtained on the \(Zn\) substitution in cuprate superconductors. The measurements [16] predicted for zero resistivity a value close to the universal \(2D\) \(\rho_0 \sim \hbar/4e^2\) and a superconductor - insulator transition. More recent experiments [17] showed that the suppression of \(d\) - wave superconductivity leads to a metallic non-superconductor phase and the metal - insulator transition is suggested at \(k_F l \sim 1\). At low temperatures \(\rho_{ab} \sim 1/T\) and \(\rho(0)\) is finite.

The temperature dependence of the specific heat is also different from the behavior of the natural phase [18], but this was explained by an energy dependent electronic density of states. However, in the very low temperature domain, \(T < 1K\) this dependence can be described by the quantum contribution \(T \ln T/T_0\) from Eq. (40).
The possibility of a non-Fermi metallic state has been also predicted recently [19] for a 2D system with a field-tuned superconductor-insulator transition. Just above the transition, the phase appears to be metallic, but with strong deviation from the Fermi-liquid behavior. Recently [20], the thermal conductivity measurements in these systems showed a metal-insulator crossover for the normal state and the existence of a low-energy scale in these materials.

The transport properties for such a model, but taking the dissipative term zero, have been performed using RG in [5]. The difference between our results and the results from Ref. [5] are given by the approximation in the calculation of $l^*$. The normal state has been considered in [1] as a metal state and the conductivity of this system behaves like in the two-dimensional metal insulator transition. The dissipation term has in this case the leading role and the singular part of the conductivity calculated in [21–23] is not universal just for the case of the particle-hole symmetry. However, the conductivity can be interpolated between this regime and the non-universal case observed by different authors [17,18].

An interesting physical picture of the transport was developed by Dalidovich and Philips [2–6] using the standard theory [21,22] in the critical region. Such a theory is very close to our picture excepting the technical aspects. However, we have to mention that there is an important difference between our RG calculation of $l^*$ and that from Ref. [5]. It is given by the fact that in Ref. [5] this quantity has been obtained using the results from [15] where all the calculations have been performed with a singular coupling constant. We showed in [12] that the calculation of $l^*$ has to be done using $u(l) = 1/(l + l_0)$, a value which gives a correct critical value for the superfluid temperature. The effect of this difference on the values of the physical observable as conductivity, will be evaluated in a future paper, but preliminary numerical calculation showed that in the low temperature regime the difference is very small.
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