A general theory for anisotropic Kirchhoff-Love shells with embedded fibers and in-plane bending

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Abstract:

In this work we present a generalized Kirchhoff-Love shell theory that can capture anisotropy not only in stretching and out-of-plane bending, but also in in-plane bending. This setup is particularly suitable for heterogeneous and fibrous materials such as textiles, biomaterials, composites and pantographic structures. The presented theory is a direct extension of existing Kirchhoff-Love shell theory to incorporate the in-plane bending resistance of fibers. It also extends existing high gradient Kirchhoff-Love shell theory for initially straight fibers to initially curved fibers. To describe the additional kinematics of multiple fiber families, a so-called in-plane curvature tensor – which is symmetric and of second order – is proposed. The effective stress tensor and the in-plane and out-of-plane moment tensors are then identified from the mechanical power balance. These tensors are all second order and symmetric for general materials. The constitutive equations for hyperelastic materials are derived from different expressions of the mechanical power balance. The weak form is also presented as it is required for computational shell formulations based on rotation-free finite element discretizations.

Keywords: anisotropic bending; fibrous composites; in-plane bending; Kirchhoff-Love shells; nonlinear gradient theory; textiles

1 Introduction

Fiber reinforced composites have become an important material in the sports, automotive, marine, and aerospace industry owing to their high specific stiffness-to-weight ratio, which allows for lightweight designs. To produce such composites, fabric sheets are formed by warp and weft yarns (i.e. in a bundle of fibers) loosely linked together by different technologies, resulting for example in woven fabrics or non-crimp fabrics. The fabrics are then draped (molded) into desired shapes before injecting liquid resin (adhesives). After the resin solidifies, the fibers are strongly bonded together in the final product.

In this work, we are interested in continuum models for fabric sheets both with and without matrix material. Such models are required for the description and simulation of fiber-reinforced shell structures and draping processes. Geometrically, fabric structures can be modeled by a surface with embedded curves representing yarns. From the microscopic point of view, the

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resistance (in-plane and out-of-plane) of fabrics results from the deformation of yarns and their interaction, i.e. their linkage and contact. In particular, axial stretching of fibers is associated with (anisotropic) membrane resistance, and the linkage between yarns offers shear resistance. Twisting of a yarn can be assumed to be fully associated with the second fundamental form of the yarn-embedding surface (Steigmann and Dell’Isola, 2015). Bending of a yarn can have both in-plane and out-of-plane components and is characterized by the corresponding curvatures. The out-of-plane curvature is associated with the second fundamental form, while the in-plane curvature is associated with the gradient of the surface metric.

In the context of classical Cauchy continuum theory, fabric sheets can be modeled as thin shells from the macroscopic point of view. For these, Kirchhoff-Love kinematics is usually adopted. Membrane deformation is characterized by stretching and shearing, while out-of-plane deformation is characterized by bending and twisting. In Kirchhoff-Love shell models, two kinematical quantities – the surface metric and the second fundamental form – are used for the two deformation types. In the literature, the general case of arbitrarily large deformations and nonlinear material behavior of shells has been treated extensively. See e.g. the texts of Naghdi (1982); Pietraszkiewicz (1989); Libai and Simmonds (1998) and references therein. Here, we refer to these works as the classical nonlinear Kirchhoff-Love shell theory. Note that although Kirchhoff-Love shell theory is mostly discussed for solids, its application can also be extended to liquid shells (Steigmann, 1999a). The incorporation of material anisotropy in classical Kirchhoff-Love shells due to embedded fibers – for both stretching and out-of-plane bending – is straightforward, see e.g. Tepole et al. (2015), Wu et al. (2018) and references therein. As shown by Roohbakhshan and Sauer (2017), classical shell theory also admits complex anisotropic bending models, e.g. due to fibers not located at the mid-surface.

While Kirchhoff-Love theory can also be regarded as a special case of Cosserat theory (Steigmann, 1999b), it has its own development history and has the advantage of simplicity and intuitiveness when following its argument structure. Therefore, it facilitates building corresponding computational models as well as physically-based constitutive models. This motivated Sauer and Duong (2017) to formulate a unified Kirchhoff-Love theory for both liquid and solid shells. Their work aimed at providing a concise, yet general theoretical framework for corresponding computational rotation-free shell formulations (Duong et al., 2017; Sauer et al., 2017).

In draping simulations, the classical Cauchy continuum model is efficient and is capable of predicting the overall fabric behavior (see e.g. Khiem et al. (2018)). However, since the Cauchy continuum assumes no bending moment at a material point, it fails to capture deformations governed by in-plane fiber bending. An example of these are the localized shear bands (Boisse et al., 2017) in the bias-extension test. In this case, simulations with finite element shell models based on the Cauchy continuum will fail to converge to a finite width of the shear band under mesh refinement. Another example is the asymmetric deformation of woven fabrics with different in-plane fiber bending stiffness for different fiber families (Madeo et al., 2016; Barbagallo et al., 2017).

Similar effects of the in-plane bending stiffness can also be found in pantographic structures (Dell’Isola et al., 2016b; Placidi et al., 2016; Dell’Isola et al., 2019) and fibrous composites, such as biological tissues (Gasser et al., 2006). Thus, a more general model that considers the in-plane bending response is required for fabrics, fibrous composites and pantographic structures.

The first theoretical work considering in-plane bending was presented by Wang and Pipkin (1986, 1987) to model cloth and cable networks. Their theory is a special form of finite-deformation fibrous plate theory with inextensible fibers that contain bending couples which are

\footnote{From another point of view, classical Kirchhoff-Love shell theory also falls in the category of high gradient theories due to the high gradient term in the bending energy.}
proportional to their curvature. Following another approach, general gradient theory (e.g. of Mindlin and Tiersten (1962); Koiter (1963); Toupin (1964); Mindlin (1965); Germain (1973)) can also be used to incorporate in-plane bending. In this context, Spencer and Soldatos (2007), Soldatos (2010), and Steigmann (2012) introduced explicitly the in-plane bending of fibers in the continuum theory for fiber-reinforced solids. Later, Steigmann and Dell’Isola (2015) as well as Asmanoglo and Menzel (2017) extended the theory to fiber-reinforced composites. Following this, Steigmann (2018) developed a shell theory that includes general fiber bending, twisting, and stretching. This theory is formulated as a special case of Cosserat theory and it treats fibers as Kirchhoff-Love rods that are distributed continuously across the shell surface. However, to the best of our knowledge, the theory has not yet been fully formulated for the general case of more than two fiber families with initially curved fibers. Recently, Balobanov et al. (2019) presented a gradient theory for Kirchhoff-Love shells. However explicit in-plane fiber bending is not considered and the weak form requires at least $C^2$-continuity of the geometry.

Many works also consider computational models for gradient shell theory. Among others, see Ferretti et al. (2014); D’Agostino et al. (2015); Dell’Isola et al. (2016b,a); Placidi et al. (2016) and Balobanov et al. (2019). Recently, Schulte et al. (2020) presented the first rotation-free computational shell formulation using $C^1$-continuous discretization.

In this contribution, we propose a general Kirchhoff-Love shell theory that incorporates fiber bending (both in-plane and out-of-plane), geodesic fiber twisting, and stretching. We show that the in-plane fiber bending contribution can be formulated analogously to its out-of-plane counterpart. The corresponding weak form, therefore, requires at least $C^1$ continuous surface discretizations in the framework of rotation-free finite element formulation (Duong et al., 2021). We are restricting ourselves to hyperelastic material models for the fabrics. The inelastic behavior, e.g. due to inter- and intra-ply fiber sliding, will be considered in future work. Compared to earlier works, our approach contains the following novelties and merits:

- A concise shell theory that is a direct extension of the classical nonlinear Kirchhoff-Love thin shell theory based on the Cauchy continuum,
- a general shell theory that can admit a wide range of different constitutive models for straight or initially curved fibers with no limitation on the number of fiber families and angles between them,
- introduction of a new symmetric in-plane curvature tensor,
- identification of the work-conjugated pairs of symmetric stress and symmetric strain measures,
- the weak form as it is required for finite element formulations,
- the analytical solution for several nonlinear benchmark examples that include different modes of fiber deformation and are useful for verifying computational formulations.

The following presentation is structured as follows: Sec. 2 summarizes the kinematics of thin shells with embedded curves and introduces the in-plane curvature tensor to capture the in-plane curvature of fibers. With this, the balance laws are presented in Sec. 3. Different choices of work-conjugation pairs are then discussed in Sec. 4. Sec. 5 gives some examples of constitutive models for the proposed theory. Sec. 6 then presents the weak form. Several analytical benchmark examples supporting the proposed theory are presented in Sec. 7. The paper is concluded by Sec. 8.
2 Kinematics for thin shells with embedded curves

In this section, the nonlinear Kirchhoff-Love shell kinematics (Naghdi, 1982) is extended to deforming surfaces with embedded curves. The extended kinematics allows to capture not only stretching and out-of-plane bending, but also in-plane bending of the embedded curves. Anisotropy is allowed in all these modes of deformation. The description is presented fully in the general framework of curvilinear coordinates. The variation of different kinematical quantities can be found in Appendix A.

2.1 Geometric description

The mid-surface of a thin shell at time \( t \) is modeled in three dimensional space as a 2D manifold, denoted by \( S \). In curvilinear coordinates, \( S \) is described by the one-to-one mapping of a point \((\xi^1, \xi^2)\) in parameter space \( P \) to the point \( x \in S \) as
\[
x = x(\xi^\alpha, t) , \quad \text{with} \quad \alpha = 1, 2 .
\]

The (covariant) tangent vectors along the convective coordinate curves \( \xi^\alpha \) at any point \( x \in S \) can be defined by
\[
a_\alpha := \frac{\partial x}{\partial \xi^\alpha} = x_\alpha ,
\]
where the comma denotes the parametric derivative. The unit normal vector can then be defined by
\[
n := \frac{a_1 \times a_2}{\|a_1 \times a_2\|} .
\]

From the tangent vector \( a_\alpha \) in Eq. (2), the so-called dual tangent vectors, denoted by \( a^\alpha \), are defined by
\[
a^\alpha \cdot a^\beta = \delta^\alpha_\beta ,
\]
with \( \delta^\alpha_\beta \) being the Kronecker delta. The covariant and dual tangent vectors are related to each other by\footnote{In this paper, the summation convention is applied for repeated Greek indices taking values from 1 to 2.}
\[
a_{\alpha\beta} := a_\alpha \cdot a_\beta , \quad a^{\alpha\beta} := a^\alpha \cdot a^\beta ,
\]
de note the co- and contra-variant surface metric, respectively.

Further, in order to model fibrous thin shells, a fiber (or a bundle of fibers) is geometrically represented by a curve \( \mathcal{C} \) defined by the points \( x = x(s, t) \), with \( ds = \|dx\| \), embedded in surface \( S \) (see Fig. 1). The normalized tangent vector of \( \mathcal{C} \), defined by
\[
\ell := \frac{\partial x}{\partial s} = \ell_\alpha a^\alpha = \ell^\alpha a_\alpha ,
\]
represents the fiber direction at location \( s \). Here, \( \ell_\alpha := \ell \cdot a_\alpha \) and \( \ell^\alpha := \ell \cdot a^\alpha \) denote the covariant and contra-variant components of vector \( \ell \) in the convective coordinate system, respectively. Assuming that the deformation of fibers satisfies Euler-Bernoulli kinematics, the in-plane director for the fiber \( \mathcal{C} \) can be defined by
\[
c := n \times \ell = c_\alpha a^\alpha = c^\alpha a_\alpha .
\]
Figure 1: A fiber bundle represented by curve $C$ is embedded in shell surface $S$. The red planes illustrate tangent planes.

With this, bases $\{a_1, a_2, n\}$ and $\{a^1, a^2, n\}$ can be represented by the local Cartesian basis $\{\ell, c, n\}$ as

$$a_\alpha = \ell_\alpha \ell + c_\alpha c,$$

and

$$a^\alpha = \ell^\alpha \ell + c^\alpha c.$$  \hspace{1cm} (8)

The surface identity tensor $i$ and the full identity $1$ in $\mathbb{R}^3$ can then be written as

$$i := a_\alpha \otimes a^\alpha = a^\alpha \otimes a_\alpha = c \otimes c + \ell \otimes \ell,$$  \hspace{1cm} (9)

and

$$1 = i + n \otimes n.$$  \hspace{1cm} (10)

2.2 Surface curvature

The curvature of surface $S$ can be described by the symmetric second order tensor

$$b := b_{\alpha \beta} a^\alpha \otimes a^\beta = b^\beta_{\alpha} a^\alpha \otimes a_\beta = b^{\alpha \beta} a_\alpha \otimes a_\beta.$$  \hspace{1cm} (11)

The components $b_{\alpha \beta}$ can be computed from the derivative of surface normal $n$ as

$$b_{\alpha \beta} := -n_\alpha \cdot a_\beta,$$  \hspace{1cm} (12)

which leads to Weingarten’s formula

$$n_\alpha = -b_{\alpha \beta} a^\beta.$$  \hspace{1cm} (13)

Alternatively, components $b_{\alpha \beta}$ can be extracted from the derivatives of tangent vectors $a_\alpha$ as

$$b_{\alpha \beta} := n \cdot a_{\alpha;\beta} = n \cdot a_{\alpha;\beta},$$  \hspace{1cm} (14)

where

$$a_{\alpha;\beta} := \frac{\partial a_\alpha}{\partial \xi^\beta} = x_{\alpha;\beta}, \text{ and } a_{\alpha;\beta} := (n \otimes n) a_{\alpha;\beta} = b_{\alpha \beta} n.$$  \hspace{1cm} (15)

They only differ for quantities with free indices. Index-free quantities, such as $n$, satisfy $n_\alpha = n_{,\alpha}$.\footnote{They only differ for quantities with free indices. Index-free quantities, such as $n$, satisfy $n_\alpha = n_{,\alpha}$.}
Unlike $n,\alpha$, which always lies in the tangent plane, $a,\alpha,\beta$ can have tangential and normal components. With respect to basis $\{a_1, a_2, n\}$ the four vectors $a,\alpha,\beta$ can be expressed as

$$a,\alpha,\beta = \Gamma^\gamma_{\alpha\beta} a_\gamma + b_{\alpha\beta} n,$$

(16)

where the tangential components

$$\Gamma^\gamma_{\alpha\beta} := a,\alpha,\beta \cdot a^\gamma$$

are known as the surface Christoffel symbols. They are symmetric in indices $\alpha$ and $\beta$. Using transformation (8.2), they can be expressed as

$$\Gamma^\gamma_{\alpha\beta} = c^\gamma \Gamma^\gamma_{c\alpha\beta} + \ell^\gamma \Gamma^\gamma_{\ell\alpha\beta},$$

(18)

where we have defined

$$\Gamma^\gamma_{c\alpha\beta} := c^\gamma \Gamma^\gamma_{c} + \Gamma^\gamma_{\ell\alpha\beta} := \ell^\gamma \Gamma^\gamma_{\ell\alpha\beta}.$$

(19)

The curvature tensor $b$ defined in Eq. (11) has the two invariants

$$H := \frac{1}{2} \text{tr}_s b = \frac{1}{2} b^\alpha_{\alpha}, \quad \text{and} \quad \kappa = \det_s b := \det[b^\beta_{\alpha}].$$

(20)

They correspond to the mean and Gaussian curvatures, respectively.

**Remark 2.1:** The curvature tensor $b$ can fully describe any out-of-plane curvature of surface $S$. Thus, we can extract from it the curvature of $S$ along any direction. For example,

$$\kappa_n := b : \ell \otimes \ell = b_{\alpha\beta} \ell^\alpha \ell^\beta,$$

(21)

with

$$\ell^\alpha := \ell^\alpha \ell^\beta, \quad \ell^\alpha \ell^\beta := b_{\alpha\beta} \ell^\alpha \ell^\beta,$$

(22)

gives the curvature of $S$ in direction $\ell$. Hence, $\kappa_n$ also expresses the so-called normal curvature of the curve $C$ embedded in $S$. Further,

$$\tau_g := b : \ell \otimes c = b : c \otimes \ell = b_{\alpha\beta} \ell^\alpha c^\beta,$$

(23)

denotes the so-called geodesic torsion of $C$.

The mentioned invariants $H, \kappa, \kappa_n, \tau_g$ are included in Tab. 1. They are useful in constructing material models for (both isotropic and anisotropic) out-of-plane bending.

### 2.3 Curvature of embedded curves

We aim at capturing any geodesic twisting, normal and in-plane curvature of an embedded curve $C \in S$. As noted in Remark 2.1, the normal curvature (21) and geodesic twisting (22) of $C$ can already be described via the out-of-plane curvature tensor (11). The in-plane curvatures, on the other hand, does not follow from tensor $b$. Instead, it can be extracted from the so-called curvature vector of $C$. It is defined by the directional derivative of $\ell$ in direction $\ell$, i.e.

$$\ell_\alpha := \frac{\partial \ell}{\partial s} = (\nabla_s \ell) \ell = \ell^\alpha \ell_\alpha.$$

(23)

Here and henceforth, $\nabla_s \bullet := \bullet_\alpha \otimes a^\alpha$ denotes the surface gradient operator. Following Eq. (6) and considering Eqs. (4) and (16), derivative $\ell_\alpha$ can be expressed as

$$\ell_\alpha = \ell^\beta_{\alpha\beta} a_\beta + \ell^\beta b_{\beta\alpha} n = \ell_{\beta\alpha} a^\beta + \ell^\beta b_{\beta\alpha} n,$$

(24)

Note, that $\nabla_s \bullet$ can have both in-plane and out-of-plane components.
**Table 1:** Various curvature measures of the surface $S$ and of a fiber family $C$ embedded in $S$. Note, that the magnitude of these measures is invariant, but their sign (except for $\kappa$ and $\kappa_p$) depends on the direction of directors $n$ and/or $c$.

where the semicolon denotes the covariant derivatives

$$
\ell^\beta_{\alpha} := \ell_{\alpha} \cdot a^\beta = \ell^\beta_{\alpha} + \ell^\gamma \Gamma^\beta_{\gamma \alpha}, \quad \ell_{\beta;\alpha} := \ell_{\alpha} \cdot a_\beta = \ell_{\beta,\alpha} - \ell_\gamma \Gamma^\beta_{\gamma \alpha}.
$$

The magnitude $||\ell_s|| =: \kappa_p$ is called the principal curvature of $C$, and the direction $\ell_s/\kappa_p =: n_p$ is referred to as the principal normal to $C$. Note that $n_p$ is normal to the curve but not necessarily normal to the surface $S$.

In principle, vector $\ell_s$ can be expressed in any basis, which then induces different curvatures from the corresponding components. Here, we express $\ell_s$ in the basis $\{ \ell, c, n \}$, i.e.

$$
\ell_s = \kappa_p n_p = \kappa_n c + \kappa_n n,
$$

where

$$
\kappa_n := n \cdot \ell_s
$$

6 since it is the sole principal invariant of the first order tensor $\ell_s$. 

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| Invariant | tensor notation | index notation | geometrical meaning |
|-----------|----------------|----------------|---------------------|
| out-of-plane and in-plane curvature tensors: $b = b_{\alpha \beta} a^\alpha \otimes a^\beta$ and $\bar{b} = \bar{b}_{\alpha \beta} a^\alpha \otimes a^\beta$ | $H := \frac{1}{2} \text{tr}_b b$ | $\frac{1}{2} b^\alpha_{\alpha}$ | mean curvature of $S$ |
| $\kappa := \det b$ | $\det[b^\beta_{\alpha}]$ | Gaussian curvature of $S$ |
| $\kappa_n := b : \ell \otimes \ell = \ell_{,\ell} \cdot n$ | $b_{\alpha \beta} \ell^\alpha \ell^\beta$ | normal curvature of $C \in S$ |
| $\tau_g := b : c \otimes c = b : c \otimes \ell$ | $b_{\alpha \beta} \ell^\alpha c^\beta$ | geodesic torsion of $C \in S$ |
| $\kappa_s := \bar{b} : \ell \otimes \ell$ | $\bar{b}_{\alpha \beta} \ell^\alpha \ell^\beta$ | geodesic curvature of $C \in S$ |
| $\kappa_p := ||\ell_s|| = \sqrt{\kappa_n^2 + \kappa_s^2}$ | principal curvature of $C \in S$ |

measures for the change in normal curvature $\kappa_n$ of $C \in S$

| $k_n := b : \ell \otimes \ell - b_0 : L \otimes L$ | $\kappa_n - \kappa_n^0$ | absolute change |
| $k_n := b : \lambda \ell \otimes \ell - b_0 : L \otimes L$ | $\kappa_n \lambda - \kappa_n^0$ | stretch-excluded change |
| $K_n := K : L \otimes L$ | $\kappa_n \lambda^2 - \kappa_n^0$ | nominal change |

measures for the change in geodesic torsion $\tau_g$ of $C \in S$

| $t_g := b : \ell \otimes c - b_0 : L \otimes c_0$ | $\tau_g - \tau_g^0$ | absolute change |
| $t_g := b : \lambda \ell \otimes c - b_0 : L \otimes c_0$ | $\tau_g \lambda - \tau_g^0$ | stretch-excluded change |
| $T_g := K : L \otimes c_0$ | $b_{\alpha \beta} c_\beta L^\beta - \tau_g^0$ | nominal change |

measures for the change in geodesic curvature $\kappa_g$ of $C \in S$

| $k_g := b : \ell \otimes \ell - b_0 : L \otimes L$ | $\kappa_g - \kappa_g^0$ | absolute change |
| $k_g := b : \lambda \ell \otimes \ell - b_0 : L \otimes L$ | $\kappa_g \lambda - \kappa_g^0$ | stretch-excluded change |
| $K_g := K : L \otimes L$ | $\kappa_g \lambda^2 - \kappa_g^0$ | nominal change |
denotes the normal curvature of $C$. By inserting (23) and (24), $\kappa_n$ can be shown to be identical to Eq. (21). On the other hand,
\[ \kappa_g := c \cdot \ell_s \] (28)
is the in-plane (i.e. geodesic) curvature of $C$. It can be computed by inserting (23) and (24) into Eq. (28), giving
\[ \kappa_g = \ell_{\alpha \beta} c^\alpha \ell^\beta = \ell_{\alpha \beta}^c c^\alpha \ell^\beta = (\nabla_s \ell) : c \otimes \ell , \] (29)
where
\[ \nabla_s \ell := i \nabla \ell = \ell_{\alpha \beta} a^\alpha \otimes a^\beta = \ell_{\alpha \beta}^c a^\alpha \otimes a^\beta \] (30)
denotes the projected surface gradient of $\ell$.

Remark 2.2: From Eq. (26) follows
\[ \kappa_n^2 = \kappa_g^2 + \kappa_n^2 . \] (31)

Remark 2.3: The three scalars $\kappa_n$, $\kappa_g$, and $\tau_g$ are associated with the three bending modes of $C \in S$: normal (i.e. out-of-plane) bending, geodesic (i.e. in-plane) bending, and geodesic twisting, respectively.

2.4 Definition of the in-plane curvature tensor

In principle, we can use the second order tensor $\nabla_s \ell$ to characterize the in-plane curvature as the counterpart to tensor $b$ from Eq. (11), which characterizes the out-of-plane curvature. Tensor $\nabla_s \ell$ is, however, unsymmetric. We thus construct an alternative tensor by rewriting Eq. (29) and using identity $c_{\alpha} \cdot \ell = -c \cdot \ell_s$ that follows from $c \cdot \ell = 0$, so that
\[ \kappa_g := -\ell^\alpha_c c^\beta_{\alpha \beta} = -\ell^\alpha_{\alpha \beta} c_{\alpha \beta} = b : \ell \otimes \ell , \] (32)
where
\[ c^\beta_{\alpha \beta} := c^\beta_{\alpha} + c^\gamma \Gamma^\beta_{\gamma \alpha} , \quad \text{and} \quad c_{\beta \alpha} := c_{\beta \alpha} - c_{\gamma} \Gamma^\gamma_{\beta \alpha} , \] (33)
similar to Eq. (25). Here, we have defined the so-called in-plane curvature tensor of $C$
\[ b := -\frac{1}{2} \left[ \nabla_s c + (\nabla_s c)^\Gamma \right] = -\frac{1}{2} (c_{\alpha \beta} + c_{\beta \alpha}) a^\alpha \otimes a^\beta = \bar{b}_{\alpha \beta} a^\alpha \otimes a^\beta . \] (34)
Components $\bar{b}_{\alpha \beta}$ thus can be computed from
\[ \bar{b}_{\alpha \beta} = \bar{b} : a_\alpha \otimes a_\beta = -\frac{1}{2} (c_{\alpha \beta} + c_{\beta \alpha}) = -\frac{1}{2} (c_{\cdot} \cdot a_\beta + c_{\cdot \beta} \cdot a_\alpha) , \] (35)
where
\[ c_{\cdot \alpha} = c_{\alpha} = c^\beta a_{\beta \alpha} + c^\gamma \Gamma^\beta_{\gamma \alpha} \]
\[ = c_{\beta} a_{\gamma \alpha} + c_{\gamma} \alpha a_{\beta} \]
\[ = b_{\alpha \beta} \ell^\beta \ell_{\alpha} - c^\beta \ell_{\beta \alpha} \ell . \] (36)
The last equation is obtained from identities $c \cdot n = 0$, $\ell \cdot n = 0$, Eqs. (24) and (13).

Remark 2.4: Since vector $\ell$ is normalized, we have the identity
\[ \ell \cdot \ell_{\alpha} = \ell_{\beta} \ell^\beta_{\alpha} = \ell^\beta \ell_{\beta \alpha} = 0 , \] (37)
due to Eq. (24). Further, equating Eqs. (29) and (32) gives the relation
\[ \ell^\beta c_{\beta \alpha} = -c^\beta \ell_{\beta \alpha} , \] (38)
and from Eq. (36), we find
\[ c_{\beta \alpha} = a_{\beta} \cdot c_{\alpha} = -c^\gamma \ell_{\beta \gamma \alpha} \]
\[ c^\beta_{\alpha} = a^\beta \cdot c_{\alpha} = -c^\gamma \ell_{\gamma \alpha} . \] (39)

\[ ^7\text{Unlike } \nabla_s \bullet, \nabla_s \bullet \text{ has only in-plane components.} \]
2.5 Shell deformation

To characterize the shell deformation, a reference configuration $\mathcal{S}_0$ is chosen. For this, the tangent vectors $A_\alpha$, the normal vector $N$, the surface metric $A_{\alpha\beta}$, the out-of-plane curvature tensor $b_0 := B_{\alpha\beta} A^\alpha \otimes A^\beta$ are defined similarly to Eqs (2), (3), (5), and (11), respectively. Also, the normalized fiber direction $L = L^\alpha A_\alpha = L_\alpha A^\alpha$, the in-plane fiber director $c_0 = c_0^\alpha A^\alpha$, and the in-plane curvature tensor $\tilde{b}_0 := \tilde{B}_{\alpha\beta} A^\alpha \otimes A^\beta = -\frac{1}{2} (c_0^0 + c_0^\alpha) A^\alpha \otimes A^\beta$ are defined similarly to Eqs. (6), (7), and (34), respectively.

Having $a_{\alpha\beta}$, $b_{\alpha\beta}$, and $\tilde{b}_{\alpha\beta}$ the deformation of a shell can now be characterized by the following three tensors:

1. The surface deformation gradient:
   \[ F := a_\alpha \otimes A^\alpha. \] (40)

   This tensor can be used to map the reference fiber direction $L$ to $\ell$ as
   \[ \lambda \ell = FL = L^\alpha a_\alpha, \] (41)

   where $\lambda$ is the fiber stretch. Comparing (6) and (41) gives
   \[ \ell^\alpha = L^\alpha \lambda^{-1}. \] (42)

   From (25) and (42) follows that
   \[ \ell_{\alpha;\beta} = \hat{L}_{\alpha,\beta} - \ell_{\alpha\gamma} (\hat{L}_{\beta}^\gamma + \ell^\delta \Gamma^\gamma_{\delta\beta}) + \ell^\gamma a_\gamma \cdot a_{\gamma;\beta}, \] (43)

   where
   \[ \hat{L}_{\alpha,\beta} := \lambda^{-1} L_{\beta}^\alpha, \quad \text{and} \quad \hat{L}_{\alpha,\beta} := a_{\alpha\gamma} \hat{L}_{\gamma,\beta}. \] (44)

   Inserting (43) into Eq. (39) gives
   \[ c_{\beta;\alpha} = -\ell_{\beta} (c_0^\gamma \hat{L}_{\gamma,\alpha} + \ell^\gamma \Gamma^\gamma_{\gamma\alpha}), \]
   \[ c_{\alpha} = -\ell^\beta (c_0^\gamma \hat{L}_{\gamma,\alpha} + \ell^\gamma \Gamma^\gamma_{\gamma\alpha}). \] (45)

   With the surface deformation gradient, the right Cauchy-Green surface tensor is defined by
   \[ C := F^T F = a_{\alpha\beta} A^\alpha \otimes A^\beta. \] Tab. 2 lists some invariants induced by $C$ that can be useful to construct material models. The Green-Lagrange surface strain tensor, which represents the change of the surface metric, is then defined by
   \[ E := \frac{1}{2} (C - I) = \frac{1}{2} (a_{\alpha\beta} - A_{\alpha\beta}) A^\alpha \otimes A^\beta = E_{\alpha\beta} A^\alpha \otimes A^\beta. \] (46)

2. The change of the out-of-plane curvature tensor:
   \[ K := F^T b F - b_0 = (b_{\alpha\beta} - B_{\alpha\beta}) A^\alpha \otimes A^\beta = K_{\alpha\beta} A^\alpha \otimes A^\beta. \] (47)

3. The change of the in-plane curvature tensor:
   \[ \tilde{K} := F^T \tilde{b} F - \tilde{b}_0 = (\tilde{b}_{\alpha\beta} - \tilde{B}_{\alpha\beta}) A^\alpha \otimes A^\beta = \tilde{K}_{\alpha\beta} A^\alpha \otimes A^\beta. \] (48)

   Here, $\tilde{b}_{\alpha\beta}$ can be computed from Eq. (35) taking into account Eq. (45). This gives
   \[ \tilde{b}_{\alpha\beta} = \frac{1}{2} \ell^\gamma (\ell_{\alpha} \Gamma^\gamma_{\gamma\beta} + \ell_{\beta} \Gamma^\gamma_{\gamma\alpha}) + \frac{1}{2} \ell^\gamma (\ell_{\alpha} \hat{L}_{\gamma,\beta} + \ell_{\beta} \hat{L}_{\gamma,\alpha}). \] (49)
With this, the geodesic curvature follows from Eq. (32) as
\[ \kappa_g := \bar{b}_{\alpha\beta} \ell_{\alpha\beta} = \ell_{\alpha\beta} \Gamma^\gamma_{\alpha\beta} c_\gamma + \lambda^{-1} c_\alpha \ell_\beta L^\alpha_{\beta,\gamma} . \] (50)

Further, by using relation \( L^\alpha_{\beta} = L_{,\beta} L^\alpha - L_{,\alpha} L^\beta_{,\beta} \), similar to Eq. (25.1) – where \( \Gamma^\gamma_{\alpha\beta} := A^\gamma \cdot A_{,\alpha\beta} \) denote the Christoffel symbols of the initial configuration – Eq. (50) can be rewritten as
\[ \kappa_g = \ell_{\alpha\beta} c_\gamma S^\gamma_{\alpha\beta} + \lambda^{-1} c_\alpha \ell_\beta L^\alpha_{,\beta} , \]
\[ =: \kappa^L_g , \]
\[ =: \kappa^L_{\gamma\beta} \]
\[ \text{where } S^\gamma_{\alpha\beta} := \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta} \text{ and } L^\alpha_{,\beta} := A^\alpha \cdot L_{,\beta} \ . \]

**Remark 2.5:** As seen in Eq. (51), the geodesic curvature involves not only the change in the Christoffel symbols (term \( \kappa^L_{\gamma\beta} \)), but also the gradient of \( L \) (term \( \kappa^L_g \)). For initially straight fiber, the geodesic curvature becomes
\[ \kappa_g := \kappa^L_g = \ell_{\alpha\beta} c_\gamma S^\gamma_{\alpha\beta} \]
(52)
as \( \kappa^L_{\gamma\beta} \) vanishes. But for initially curved fibers, where \( \kappa^L_{\gamma\beta} \neq 0 \), expression (51) should be used. This point will be demonstrated in Sec. 7.1.

**Remark 2.6:** To measure the change in the curvatures, one can use the invariants
\[ k_n := \langle b : \ell \otimes \ell - b_0 : L \otimes L \rangle = k_n - \kappa^0_n , \]
\[ k_g := \langle \bar{b} : \ell \otimes \ell - \bar{b}_0 : L \otimes L \rangle = k_g - \kappa^0_g , \]
\[ t_g := \langle b : \ell \otimes c - b_0 : L \otimes c_0 \rangle = t_g - \tau^0_g , \]
(53)
called the absolute change in the normal curvature, geodesic curvature, and geodesic torsion. Here, \( *^0 \) denotes the corresponding quantities in the initial configuration.

**Remark 2.7:** However, the curvature changes (53) are not ideal for constitutive models, as they can lead to bending moments that respond to fiber stretching even when there is no bending. An example is pure dilatation, e.g. due to thermal expansion or hydrostatic stress states as is discussed e.g. in example 7.4. To exclude fiber stretching, we define the invariants
\[ k_n := \langle b : \lambda \ell \otimes \ell - b_0 : L \otimes L \rangle = k_n \lambda - \kappa^0_n , \]
\[ k_g := \langle \bar{b} : \lambda \ell \otimes \ell - \bar{b}_0 : L \otimes L \rangle = k_g \lambda - \kappa^0_g , \]
\[ t_g := \langle b : \lambda \ell \otimes c - b_0 : L \otimes c_0 \rangle = t_g \lambda - \tau^0_g , \]
(54)
called the stretch-excluded change in the normal curvature, geodesic curvature, and geodesic torsion.

**Remark 2.8:** Both curvature changes (53) and (54) can cause fiber tension apart from bending moments. Therefore, one can use the so-called nominal change in the normal curvature, geodesic curvature, and geodesic torsion, defined by
\[ K_n := \langle K : L \otimes L \rangle = k_n \lambda^2 - \kappa^0_n , \]
\[ K_g := \langle \bar{K} : L \otimes L \rangle = k_g \lambda^2 - \kappa^0_g , \]
\[ T_g := \langle K : L \otimes c_0 \rangle = b_{\alpha\beta} c_\beta^0 \ell^\beta - \tau^0_g , \]
(55)
where \( K \) and \( \bar{K} \) are defined by (47) and (48), respectively. These invariants can also be found e.g. in Steigmann and Dell’Isola (2015) and Schulte et al. (2020). Since the measures (55) do not cause an axial tension in the fibers, their material tangents simplify significantly. But note that, like (53), they can cause bending moments due to fiber stretching.

**Remark 2.9:** The mentioned curvature measures (53), (54), and (55) are also listed in Tab. 1. It should be noted that all these measures are equivalent for inextensible fibers, i.e. \( \lambda = 1 \), which is usually assumed for textile composites.
### 3 Balance laws

In this section, we discuss the balance laws for fibrous thin shells taking into account not only in-plane stretching and out-of-plane bending, but also in-plane bending. We first discuss in details the theory with one embedded fiber family $C_i$. The extension to multiple fiber families is straightforward and will be discussed subsequently.

In order to define internal stresses and internal moment tensors, the shell $S$ is virtually cut into two parts at position $x$ as depicted in Fig. 2. On the parametrized cut, denoted by $I(s)$, we define the unit tangent vector $\tau := \partial x/\partial s$ and the unit normal $\nu := \tau \times n = \nu_\alpha a^\alpha$ at $x$.

#### 3.1 Stress tensor

The traction vector $T$ appearing on the cut can have arbitrary direction, but we can generally express it with respect to the basis $\{a_1, a_2, n\}$ as

$$ T = T^\alpha a_\alpha + T^3 n. \quad (56) $$

This traction induces the Cauchy stress tensor $\sigma$. Due to the plane-stress assumption, $\sigma$ can be expressed in the form

$$ \sigma = N^{\alpha\beta} a_\alpha \otimes a_\beta + S^\alpha a_\alpha \otimes n. \quad (57) $$

Accordingly, the components of $\sigma^T$ in direction $a^\alpha$ are defined by

$$ T^\alpha := \sigma^T a^\alpha = N^{\alpha\beta} a_\beta + S^\alpha n, \quad (58) $$

which is illustrated in Fig. 2b for an infinitesimal triangular element on $S$. With this, we can write

$$ \sigma = a_\alpha \otimes T^\alpha. \quad (59) $$

Further, the force equilibrium of the triangle (see Fig. 2b) gives

$$ T \, ds = (\nu_\alpha N^{\alpha\beta}) a_\beta \, ds + (\nu_\alpha S^\alpha) n \, ds = T^\alpha \nu_\alpha \, ds, \quad (60) $$

which implies Cauchy’s formula

$$ T = (T^\alpha \otimes a_\alpha) \nu = \sigma^T \nu, \quad (61) $$

since $\nu_\alpha = \nu \cdot a_\alpha$. By comparing Eq. (61) with Eq. (56), we can further write

$$ T^\alpha = \nu_\beta N^{\beta\alpha}, \quad T^3 = \nu_\alpha S^\alpha. \quad (62) $$

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Invariant & tensor notation & index notation & geometrical relevance \\
\hline
$I_1$ := tr$_s C$ & $A^{\alpha\beta} a_\alpha a_\beta$ & surface shearing of $S$ \\
$I_2 = J^2$ := det$_s C = (\text{det}_s F)^2$ & det$[a_\alpha a_\beta]/\text{det}[A_{\alpha\beta}]$ & surface area change of $S$ \\
$A_i = \lambda_i^2 := C : L_i \otimes L_i$ (no sum in $i$) & $a_{\alpha\beta} L_i^\alpha L_i^\beta$ & stretching of $C_i$ \\
$\gamma_{ij} := C : L_i \otimes L_j$ & $a_{\alpha\beta} L_i^\alpha L_j^\beta$ & nominal angle between $C_i$ & $C_j$, $i \neq j$ \\
$\hat{\gamma}_{ij} := \ell_i \cdot \ell_j$ & $a_{\alpha\beta} \ell_i^\alpha \ell_j^\beta$ & absolute angle between $C_i$ & $C_j$, $i \neq j$ \\
\hline
\end{tabular}
\caption{Various induced invariants of the right surface Cauchy-Green tensor $C$, considering multiple fiber families $C_i$ ($i = 1, \ldots, n_f$). Here, $L_i^\alpha := L_\alpha^i L_\beta^i$ (no sum in $i$).}
\end{table}
Figure 2: Internal stresses and moments: (a) depiction of physical traction vector $T$ and moment vector $\hat{m}$ acting on the cut $I$ (red curve) of surface $S$ with its embedded fiber $C$ (blue curve). Both $T$ and $\hat{m}$ are general vectors in 3D space. The moment vector $\hat{m}$ can be decomposed into the component $\bar{m}$, causing in-plane bending, and the component $m$, causing out-of-plane bending and twisting. Vectors $\ell$, $c$, $\nu$, $\tau$, and $m$ lie in the tangent plane of $S$. (b) Stress and (c) moment components acting on a triangular element.

Remark 3.1: In general, $N^{\alpha\beta}$ is not symmetric, and due to the presence of transverse shear components $S^\alpha$, the traction vector $T$ has both in-plane and out-of-plane components as is seen from Eq. (56) and (62).

Remark 3.2: All the stresses appearing in the fibers are contained in expression (57). Indeed, consider a fiber $\hat{C}$ described by beam theory. Accordingly, the stress tensor $\sigma_{\text{fib}}$ in the fiber $\hat{C}$ can be written as

$$\sigma_{\text{fib}} = \sigma \ell \otimes \ell + s_c \ell \otimes c + s_n \ell \otimes n,$$

where $\sigma$, $s_c$, and $s_n$ denotes the axial stress and the two shear stresses. By inserting the basic vectors $\ell = \ell^\alpha a_\alpha$, and $c = c^\alpha a_\alpha$, Eq. (63) becomes

$$\sigma_{\text{fib}} = N^{\alpha\beta} a_\alpha \otimes a_\beta + S^\alpha a_\alpha \otimes n,$$

with

$$N^{\alpha\beta} = \sigma \ell^\alpha \ell^\beta + s_c \ell^\alpha c^\beta,$$
$$S^\alpha = s_n \ell^\alpha.$$

3.2 Generalized moment tensor

At position $x$ on each side of cut $I(s)$, the bending moment vector, denoted by $\hat{m}$ (with unit moment per unit length), is allowed to have both in-plane and out-of-plane components. It can be expressed with respect to basis $\{\tau, \nu, n\}$ as

$$\hat{m} := m_\tau \tau + m_\nu \nu + \hat{m} n,$$
where $\bar{m} \mathbf{n} =: \bar{m}$ is the moment causing in-plane bending and
\[ m_r \tau + m_\nu \nu =: \mathbf{m} , \]
denotes the combined moment causing out-of-plane bending and twisting.

Similar to the Cauchy stress tensor (57), the total moment tensor $\hat{\mathbf{\mu}}$ induced by $\bar{m}$ can be expressed in the form
\[ \hat{\mathbf{\mu}} := m^{\alpha \beta} a_\alpha \otimes a_\beta + \bar{m}^{\alpha} a_\alpha \otimes \mathbf{n} . \]  
In this form, the first term $m^{\alpha \beta} a_\alpha \otimes a_\beta$ combines both out-of-plane bending and twisting in response to a change in the out-of-plane curvature of $S$, while the second term $\bar{m}^{\alpha} a_\alpha \otimes \mathbf{n}$ is the response to a change in the in-plane curvature of fiber $C$.

The components of $\hat{\mathbf{\mu}}^T$, when projected to directions $a_\alpha$, thus read (see Fig. 2c)
\[ \hat{m}^{\alpha} := \hat{\mathbf{\mu}}^T a_\alpha = m^{\alpha \beta} a_\beta + \bar{m}^{\alpha} \mathbf{n} . \]  
Here, moment vectors $m^{\alpha \beta} a_\beta$ and $\bar{m}^{\alpha} \mathbf{n}$ are associated with the angular velocity vector around the in-plane axis
\[ \mathbf{n} \times \dot{\mathbf{n}} = (\mathbf{n} \cdot \dot{a}^\alpha) a_\alpha \times \mathbf{n} , \]
and the out-of-plane axis $\mathbf{n} = \ell \times c = \ell^\alpha a_\alpha \times c$, respectively. Therefore, it is mathematically convenient to express $\hat{m}^{\alpha}$ with respect to the basis $\{a_1 \times \mathbf{n}, a_2 \times \mathbf{n}, \mathbf{n}\}$, i.e.
\[ \hat{m}^{\alpha} = M^{\beta \alpha} (a_\beta \times \mathbf{n}) + \bar{M}^{\beta \alpha} (a_\beta \times c) = \mathbf{n} \times M^{\alpha} + c \times \bar{M}^{\alpha} . \]

Here, we have defined the so-called stress couple vectors for out-of-plane and in-plane bending
\[ M^{\alpha} := -M^{\alpha \beta} a_\beta , \quad \text{and} \quad \bar{M}^{\alpha} := -\bar{M}^{\alpha \beta} a_\beta , \quad \text{with} \quad \bar{M}^{\alpha \beta} := \bar{m}^{\alpha} \ell^\beta . \]  
Eq. (68) thus becomes
\[ \hat{\mathbf{\mu}} := M^{\alpha \beta} a_\alpha \otimes (a_\beta \times \mathbf{n}) + \bar{M}^{\alpha \beta} a_\alpha \otimes (a_\beta \times c) . \]

Similar to Eq. (61), moment equilibrium of the triangle (see Fig. 2c) gives Cauchy’s formula
\[ \hat{\mathbf{m}} = \hat{\mathbf{m}}^{\alpha} \nu_\alpha = \mathbf{n} \times \mathbf{M} + c \times \bar{\mathbf{M}} = \hat{\mathbf{\mu}}^T \nu , \]
where
\[ \mathbf{M} := M^{\alpha} \nu_\alpha , \quad \text{and} \quad \bar{\mathbf{M}} := \bar{M}^{\alpha} \nu_\alpha \]
are referred to as the stress couple vectors associated with out-of-plane and in-plane bending, respectively. By comparing Eq. (74) and (66), the stress couple vectors $\mathbf{M}$ and $\bar{\mathbf{M}}$ can be related to their moment vector counterparts by
\[ \mathbf{m} = \mathbf{n} \times \mathbf{M} , \quad \text{and} \quad \bar{\mathbf{m}} = c \times \bar{\mathbf{M}} , \quad \text{with} \quad \bar{\mathbf{M}} = -\bar{\mathbf{m}} \ell . \]

**Remark 3.3:** In line with previous works, see e.g. Sauer and Duong (2017), we can also define the so-called stress couple tensors,
\[ \mathbf{\mu} := -M^{\alpha \beta} a_\alpha \otimes a_\beta , \quad \text{and} \quad \bar{\mathbf{\mu}} := -\bar{M}^{\alpha \beta} a_\alpha \otimes a_\beta , \]
associated with out-of-plane and in-plane bending, respectively. In view of (74), we obtain the mapping
\[ \mathbf{M} = \mathbf{\mu}^T \nu , \quad \text{and} \quad \bar{\mathbf{M}} = \bar{\mathbf{\mu}}^T \nu . \]
Remark 3.4: In order to relate the components of the moment vector (66) to the components of the stress couple tensors, we can equate equations (66) and (74). This results in

\[ m^\nu := \hat{m} \cdot \nu = m \cdot \nu = M^{\alpha \beta} \nu_\alpha \tau_\beta, \]
\[ m^\tau := \hat{m} \cdot \tau = m \cdot \tau = -M^{\alpha \beta} \nu_\alpha \nu_\beta, \]
\[ \bar{m} := \hat{m} \cdot n = \bar{m} \cdot n = \bar{M}^{\alpha \beta} \nu_\alpha \ell_\beta = \bar{m}^\alpha \nu_\alpha, \]

where we have used the relations \( a_\beta \times n = \tau_\beta \nu - \nu_\beta \tau \) due to \( a_\alpha = \tau_\alpha + \nu_\alpha \nu. \)

Remark 3.5: Similar to Eq. (63), the moment tensor for a fiber modeled by beam theory can also be written in the form of Eq. (68), i.e.

\[ \hat{\mu}_{fib} := \mu_\ell \ell \otimes \ell + \mu_c \ell \otimes c + \bar{\mu} \ell \otimes n = m^{\alpha \beta} a_\alpha \otimes a_\beta + \bar{m}^\alpha a_\alpha \otimes n. \]

Here, \( \mu_\ell, \mu_c \), and \( \bar{\mu} \) denote twisting, out-of-plane bending, and in-plane bending moments in fiber \( C \), and we have identified

\[ m^{\alpha \beta} = \mu_\ell \ell^{\alpha \beta} + \mu_c \ell^{\alpha} c^\beta; \]
\[ \bar{m}^\alpha = \bar{\mu} \ell^\alpha. \]

Remark 3.6: Further, inserting \( \bar{m}^\alpha \) from Eq. (81) into (72.3) and (79.3) gives

\[ \bar{M}^{\alpha \beta} = \bar{\mu} \ell^{\alpha \beta} \]
\[ \bar{m} = \bar{\mu} \ell \cdot \nu. \]

Therefore, we can conclude that \( \bar{M}^{\alpha \beta} \) is symmetric. Note, that \( \bar{\mu} \) does not depend on the cut \( I \) since it is a component of the internal moment tensor, but \( \bar{m} \) does depend on \( I \) as seen in Eq. (82.2). For example, \( \bar{m} = 0 \) when the cut is parallel to the fiber, i.e. when \( \ell \cdot \nu = 0. \)

3.3 Balance of linear momentum

Consider body forces \( f \) acting on an arbitrary simply-connected region \( P \subset S \). The balance of linear momentum implies that the temporal change of the linear momentum is equal to the acting external forces. That is,

\[ \frac{D}{Dt} \int_P \rho \mathbf{v} \, da = \int_P f \, da + \int_{\partial P} T \, ds, \quad \forall P \in S, \]

where \( \mathbf{v} \) denotes the material velocity of the surface. Inserting \( T \) from Eq. (61), and applying the surface divergence theorem and conservation of mass, one obtains the local form of linear momentum balance

\[ \text{div}_s \sigma^T + f = \rho \dot{v}, \]

where \( \text{div}_s \) denotes the surface divergence operator, defined by \( \text{div}_s \mathbf{a} := \mathbf{e}_\alpha \cdot a^\alpha. \) Note that Eq. (84) can also be written in the form (see Sauer and Duong (2017))

\[ T^\alpha_{\beta} + f = \rho \dot{v}, \]

since

\[ \text{div}_s \sigma^T = \sigma^T_{\nu_\beta} \cdot a^\beta = T^\alpha_{\nu_\alpha} + \Gamma^\beta_{\alpha \beta} T^\alpha = T^\alpha_{\nu_\alpha}, \]

follows from Eq. (59).
3.4 Balance of angular momentum

The balance of angular momentum implies that the temporal change of angular momentum is equal to all external moments. That is,

\[ \frac{D}{Dt} \int_P \rho \mathbf{x} \times \mathbf{v} \, da = \int_P \mathbf{x} \times \mathbf{f} \, da + \int_{\partial P} \mathbf{x} \times \mathbf{T} \, ds + \int_{\partial P} \mathbf{m} \, ds , \]  

(87)

where \( \mathbf{m} \) includes both in-plane and out-of-plane bending moments acting on \( \partial P \). From Eqs. (61) and (74), the surface divergence theorem gives

\[ \int_{\partial P} \mathbf{x} \times \mathbf{T} \, ds = \int_P (a_\alpha \times \mathbf{T}^\alpha + \mathbf{x} \times \nabla_s \mathbf{\sigma}^T) \, da \]  

(88)

and

\[ \int_{\partial P} \mathbf{m} \, ds = \int_P \nabla_s \mathbf{\mu}^T \, da . \]  

(89)

Inserting these equations into (87) and applying local mass conservation gives

\[ \int_P (a_\alpha \times \mathbf{T}^\alpha + \mathbf{x} \times (\nabla_s \mathbf{\sigma}^T + \mathbf{f} - \rho \mathbf{\dot{v}})) \, da = 0 . \]  

(90)

The second integral in Eq. (90) vanishes due to (84). This leads to the local form of the angular momentum balance,

\[ a_\alpha \times \mathbf{T}^\alpha + \mathbf{x} \times (\nabla_s \mathbf{\sigma}^T + \mathbf{f} - \rho \mathbf{\dot{v}}) = 0 . \]  

(91)

Keeping Eq. (82.1) in mind, the surface divergence of tensor \( \mathbf{\mu}^T \) follows from Eq. (73) as

\[ \nabla_s \mathbf{\mu}^T = (- M^{\alpha\beta} b^\gamma_\alpha + \overline{m}^\alpha_\alpha b^\beta_\gamma c^\gamma) a_\beta \times a_\gamma + \left( M^{\alpha\beta}_\gamma + \overline{\mu} \overline{\tau}_\gamma \ell^\alpha - \overline{\mu} \kappa_\gamma c^\alpha \right) a_\alpha \times \mathbf{n} , \]  

(92)

where we have used \( a_\alpha = \ell_\alpha \ell + c_\alpha c = (\ell_\alpha c^\gamma - c_\alpha \ell^\gamma) a_\gamma \times \mathbf{n} \). Inserting Eqs. (92) and (58) into Eq. (91) implies that the stress

\[ \tilde{\mathbf{\sigma}}^{\alpha\beta} := N^{\alpha\beta} - M^{\gamma\alpha} b^\beta_\gamma + \overline{m}^\gamma_\gamma \ell^\alpha c^\beta \]  

(93)

is symmetric, and that the shear stress is given by

\[ S^\alpha = - M^{\beta\alpha}_\beta + \overline{\mu} (\kappa_\alpha c^\alpha - \overline{\tau}_\gamma \ell^\alpha) . \]  

(94)

Remark 3.7: The last term in stress expression (93) relates to the in-plane shear force in the fiber, which, in accordance with the assumed Euler-Bernoulli kinematics, follows as the derivative of the in-plane bending moment.

3.5 Mechanical power balance

The mechanical power balance can be obtained from local momentum balance (84). To this end, we can write

\[ \int_P \mathbf{v} \cdot (\nabla_s \mathbf{\sigma}^T + \mathbf{f} - \rho \mathbf{\dot{v}}) \, da = 0 . \]  

(95)

Here, the divergence term can be transformed by the identity

\[ \mathbf{v} \cdot \nabla_s \mathbf{\sigma}^T = \nabla_s (\mathbf{v} \mathbf{\sigma}^T) - \mathbf{\sigma}^T : \nabla_s \mathbf{v} , \]  

(96)

the divergence theorem

\[ \int_P \nabla_s (\mathbf{v} \mathbf{\sigma}^T) \, da = \int_{\partial P} \mathbf{v} \cdot \mathbf{\sigma}^T \mathbf{v} \, ds = \int_{\partial P} \mathbf{v} \cdot \mathbf{T} \, ds , \]  

(97)
\[ \boldsymbol{\sigma}^T \cdot \nabla_s \mathbf{v} = N^{\alpha\beta} a_{\alpha} \cdot \dot{a}_{\alpha} + S^\alpha \mathbf{n} \cdot \dot{\mathbf{a}}_\alpha . \]  

(98)

Inserting Eq. (93) and (94) into this gives

\[ \boldsymbol{\sigma}^T \cdot \nabla_s \mathbf{v} = \frac{1}{2} \tilde{\sigma}^{\alpha\beta} \dot{a}_{\alpha\beta} - M^{\alpha\beta} n_{\alpha} \cdot \dot{a}_\beta - \tilde{\mu} (\kappa_n \mathbf{c} \cdot \dot{\mathbf{n}} - \tau_g \ell \cdot \dot{\mathbf{n}}) - M^{\alpha\beta} n \cdot \dot{\mathbf{a}}_\alpha + \bar{m}_\alpha^{\alpha} \ell \cdot \dot{\mathbf{c}} . \]  

(99)

Considering Eq. (15.2), (36), and (81.2), the last two terms can be written as

\[ M^{\alpha\beta} n \cdot \dot{a}_\alpha = (\dot{\mathbf{n}} \cdot M^{\alpha})_{,\alpha} + M^{\alpha\beta} n_{\alpha} \cdot \dot{a}_\beta , \]

\[ \bar{m}_\alpha^{\alpha} \ell \cdot \dot{\mathbf{c}} = (\bar{m}^{\alpha} \ell \cdot \dot{\mathbf{c}})_{,\alpha} + \bar{m}^{\alpha} (c^{\beta} \ell_{\beta\alpha}) + \tilde{\mu} (\kappa_n \mathbf{c} \cdot \dot{\mathbf{n}} - \tau_g \ell \cdot \dot{\mathbf{n}}) . \]  

(100)

With these and local mass conservation, Eq. (95) becomes

\[ \dot{K} + P_{\text{int}} = P_{\text{ext}} , \]  

(101)

where

\[ \dot{K} = \int_P \rho \mathbf{v} \cdot \dot{\mathbf{v}} \, da \]  

(102)

is the rate of kinetic energy,

\[ P_{\text{int}} = \frac{1}{2} \int_P \tilde{\sigma}^{\alpha\beta} \dot{a}_{\alpha\beta} \, da + \int_P M^{\alpha\beta} \dot{b}_{\alpha\beta} \, da + \int_P \bar{m}^{\alpha} (c^{\beta} \ell_{\beta\alpha}) \, da , \]  

(103)

is the internal power, and

\[ P_{\text{ext}} = \int_P \mathbf{v} \cdot \mathbf{f} \, da + \int_{\partial P} \mathbf{v} \cdot \mathbf{T} \, ds + \int_{\partial P} \dot{\mathbf{n}} \cdot \mathbf{M} \, ds + \int_{\partial P} \dot{\mathbf{c}} \cdot \bar{\mathbf{M}} \, ds \]  

(104)

denotes the external power. The last term in \( P_{\text{int}} \) can be written in alternative forms, as is discussed in the following section. The mechanical power balance (101) simplifies to the expression in Sauer and Duong (2017), for \( \bar{m}^{\alpha} \equiv 0 \) and \( \bar{M} = 0 \).

4 Work conjugate variables and constitutive equations

As seen from the expression (103), the internal stress power per current area reads

\[ \dot{w}_{\text{int}} := \frac{1}{2} \tilde{\sigma}^{\alpha\beta} \dot{a}_{\alpha\beta} + M^{\alpha\beta} \dot{b}_{\alpha\beta} + \bar{m}^{\alpha} (c^{\beta} \ell_{\beta\alpha}) , \]  

(105)

which straightforwardly shows work conjugate pairs. Accordingly, \( c^{\beta} \ell_{\beta\alpha} a^{\alpha} = c \bar{\nabla}_\mathbf{c} \ell \) is a strain measure for the change in the in-plane curvatures. However, it is somewhat unintuitive and thus inconvenient for the construction of material models. In the following, we will present two approaches with alternative definitions of this strain measure.

4.1 Geodesic curvature-based approach

This approach uses the geodesic curvature \( \kappa_g \) instead of \( c^{\beta} \ell_{\beta\alpha} \) for in-plane bending power. Namely, inserting \( \bar{m}^{\alpha} \) from (81.2) into Eq. (105) and rearranging terms gives

\[ \dot{w}_{\text{int}} := \frac{1}{2} \tilde{\sigma}^{\alpha\beta} \dot{a}_{\alpha\beta} + M^{\alpha\beta} \dot{b}_{\alpha\beta} + \bar{\mu} \dot{\kappa}_g , \]  

with \( \sigma^{\alpha\beta} := \tilde{\sigma}^{\alpha\beta} + \bar{\mu} \kappa_g \ell^{\alpha\beta} . \)  

(106)
Here, \( \sigma^{\alpha\beta} \) represents the components of an effective membrane stress tensor\(^\text{10} \). It is symmetric as both \( \tilde{\sigma}^{\alpha\beta} \) and \( \ell^{\alpha\beta} \) are symmetric. The internal power (103) then becomes

\[
P_{\text{int}} = \frac{1}{2} \int_{\mathcal{P}_0} \tau^{\alpha\beta} \dot{a}_{\alpha\beta} \, dA + \int_{\mathcal{P}_0} M_0^{\alpha\beta} \dot{b}_{\alpha\beta} \, dA + \int_{\mathcal{P}_0} \bar{\mu}_0 \kappa_\gamma \, dA ,
\]

where \( \mathcal{P}_0 \in \mathcal{S}_0 \) and

\[
\tau^{\alpha\beta} := J \sigma^{\alpha\beta} , \quad M_0^{\alpha\beta} := J M^{\alpha\beta} , \quad \bar{\mu}_0 := J \bar{\mu} .
\]

They are the nominal quantities corresponding to the physical quantities \( \sigma^{\alpha\beta} \), \( M^{\alpha\beta} \), and \( \bar{\mu} \), respectively. Accordingly, we can assume a stored energy function of an hyperelastic shell in the form

\[
W = W(a_{\alpha\beta}, b_{\alpha\beta}, \kappa_\gamma; h^{\alpha\beta}) ,
\]

where \( h^{\alpha\beta} \) collectively represents the components of the structural tensor(s), e.g. \( \ell^{\alpha\beta}, c^{\alpha\beta}, \) or, \( c^{\alpha\beta} \ell^{\beta} \), that characterize material anisotropy. Eq. (109) can equivalently be expressed in terms of invariants. Using the usual arguments of Coleman and Noll (1964), the constitutive equations can be written as

\[
\tau^{\alpha\beta} = 2 \frac{\partial W}{\partial a_{\alpha\beta}} , \quad M_0^{\alpha\beta} = \frac{\partial W}{\partial b_{\alpha\beta}} , \quad \bar{\mu}_0 = \frac{\partial W}{\partial \kappa_\gamma} .
\]

\[\text{Remark 4.1:} \quad \text{Compared to classical shell theory (see e.g. Naghdi (1982); Sauer and Duong (2017), the effective membrane stress } \sigma^{\alpha\beta} \text{ in (106.2) additionally contains the high order bending term } \bar{\mu} \text{ and the in-plane fiber shear term } m_\gamma \gamma \text{ (see Eq. (93)). For slender fibers, they are negligible since the in-plane bending stiffness is usually much smaller than the membrane stiffness. However, they may becomes significant when there is a large (usually local) change in curvatures (e.g. at shear bands).}
\]

\[\text{Remark 4.2:} \quad \text{Although a constitutive formulation following from (107) is elegant, expression (110.3) is restricted to a material response expressible in terms of the geodesic curvature } \kappa_\gamma \text{. Therefore, this setup might be unsuited for complex material behavior, e.g. due to fiber dispersion (Gasser et al., 2006). In such cases, a more sophisticated structural tensor is usually desired for the in-plane bending response and it thus may not always be possible to express } W \text{ in terms of } \kappa_\gamma . \text{ This motivates the following director gradient-based approach.}
\]

\[\text{4.2 Director gradient-based approach}
\]

In this approach, the power expression (105) is rewritten by employing relation (38) and definitions (72.3) and (34) as

\[
\dot{w}_{\text{int}} := \frac{1}{2} \sigma^{\alpha\beta} \dot{a}_{\alpha\beta} + M^{\alpha\beta} \dot{b}_{\alpha\beta} + M_0^{\alpha\beta} \dot{b}_{\beta\alpha} ,
\]

where \( \sigma^{\alpha\beta} \) is now defined by\(^\text{10} \)

\[
\sigma^{\alpha\beta} := \tilde{\sigma}^{\alpha\beta} + (M^{\gamma\delta} c^{\delta\gamma}) \ell^{\alpha\beta} .
\]

Thus, the internal power can be written as

\[
P_{\text{int}} = \frac{1}{2} \int_{\mathcal{P}_0} \tau^{\alpha\beta} \dot{a}_{\alpha\beta} \, dA + \int_{\mathcal{P}_0} M_0^{\alpha\beta} \dot{b}_{\alpha\beta} \, dA + \int_{\mathcal{P}_0} M_0^{\alpha\beta} \dot{b}_{\beta\alpha} \, dA ,
\]

where we have again used (108) and defined

\[
M_0^{\alpha\beta} := J M^{\alpha\beta} .
\]

\[\text{\footnote{Note that, generally } \sigma^{\alpha\beta} \neq N^{\alpha\beta} = a^\alpha \sigma a^\beta . \sigma^{\alpha\beta} = N^{\alpha\beta} \text{ only in special cases, i.e. when both in-plane and out-of-plane bending are negligible.}}\]
Accordingly, the stored energy function can now be given in the form
\[
W = W(a_{\alpha\beta}, b_{\alpha\beta}; \bar{h}^{\alpha\beta}) .
\] (115)
This functional can also be expressed equivalently in terms of invariants. The corresponding constitutive equations now read
\[
\tau^{\alpha\beta} = 2 \frac{\partial W}{\partial a_{\alpha\beta}}, \quad M_{\alpha\beta} = \frac{\partial W}{\partial b_{\alpha\beta}}, \quad \bar{M}_{0}^{\alpha\beta} = \frac{\partial W}{\partial \bar{a}_{\alpha\beta}}. \quad (116)
\]

**Remark 4.3:** Compared to (109), expression (115) allows to model more complex in-plane bending behavior using a generalized structural tensor applied to $\bar{a}_{\alpha\beta}$. We therefore consider this setup in the following sections.

**Remark 4.4:** Eq. (113) can also be written in tensor notation as
\[
P_{\text{int}} = \int_{P_{0}} S : \dot{E} \ dA - \int_{P_{0}} \mu_{0} : \dot{K} \ dA - \int_{P_{0}} \bar{\mu}_{0} : \dot{K} \ dA ,
\] (117)
where $E$, $K$, and $\bar{K}$ are the strain tensors defined by (46), (47), and (48), respectively, and
\[
S := \tau^{\alpha\beta} A_{\alpha} \otimes A_{\beta}, \quad \mu_{0} := -M_{\alpha\beta}^{0} A_{\alpha} \otimes A_{\beta}, \quad \bar{\mu}_{0} := -\bar{M}_{0}^{\alpha\beta} A_{\alpha} \otimes A_{\beta},
\] (118)
are the effective second Piola-Kirchhoff surface stress tensor, and the nominal stress couple tensors associated with out-of-plane and in-plane bending, respectively. They are symmetric and follow from the pull back of tensors $J \sigma^{\alpha\beta} a_{\alpha} \otimes a_{\beta}$, $J \mu$ and $J \bar{\mu}$ in Eqs. (112) and (77).

### 4.3 Comparison with existing gradient theory of Kirchhoff-Love shells

To show the consistency of our proposed theory with the existing strain gradient theory of Steigmann (2018), we insert $\kappa_{g}$ obtained from (51) into the power expression (107). This results in the expression (see Appendix B)
\[
P_{\text{int}} = \int_{P_{0}} \tau^{\alpha} : \dot{a}_{\alpha} \ dA + \int_{P_{0}} M_{\alpha\beta}^{0} \dot{b}_{\alpha\beta} \ dA + \int_{P_{0}} \bar{M}_{0 \gamma} \dot{S}_{\alpha\beta}^{\gamma} \ dA ,
\] (119)
where $S_{\alpha\beta}^{\gamma} := \Gamma_{\alpha\beta}^{\gamma} - \bar{\Gamma}_{\alpha\beta}^{\gamma}$. The last term in (119) is the power due to in-plane fiber bending. Here, the relative Christoffel symbol $S_{\alpha\beta}^{\gamma}$ is the chosen strain measure for the in-plane curvature, and $\bar{M}_{0 \gamma} := J \bar{\mu} \ell^{\alpha\beta} c_{\gamma}$ is the in-plane bending moment corresponding to a change in $S_{\alpha\beta}^{\gamma}$.

In the first term of (119), $\tau^{\alpha} := J \sigma^{\alpha\beta} a_{\beta}$ denotes the effective stress vectors that are work conjugate to $\dot{a}_{\alpha}$, with $\sigma^{\alpha\beta}$ now being defined by
\[
\sigma^{\alpha\beta} := \tilde{\sigma}^{\alpha\beta} - \bar{\mu} \kappa_{g} \ell^{\alpha\beta} + \bar{\mu} (\lambda^{-1} L_{\gamma}^{\alpha} \ell^{\gamma} + S_{\gamma \delta}^{\alpha} \ell^{\gamma} \ell^{\delta}) \ell^{\alpha} \ell^{\beta} - \mu (\lambda^{-1} L_{\gamma}^{\alpha} \ell^{\gamma} + S_{\gamma \delta}^{\alpha} \ell^{\gamma} \ell^{\delta}) \ell^{\alpha} c^{\beta},
\] (120)
which is generally unsymmetric. Therefore, in contrast to expressions (107) and (113), the effective stress $\tau^{\alpha\beta} := \tau^{\alpha} : a^{\beta}$ is here generally unsymmetric. With a possible loss of generality, its symmetrization is adopted in Steigmann (2018), i.e. $\tau^{\alpha\beta} := \frac{1}{2}(\tau^{\alpha} : a^{\beta} + \tau^{\beta} : a^{\alpha})$, so that the internal power reads
\[
P_{\text{int}} = \frac{1}{2} \int_{P_{0}} \tau^{\alpha\beta} \dot{a}_{\alpha\beta} \ dA + \int_{P_{0}} M_{\alpha\beta}^{0} \dot{b}_{\alpha\beta} \ dA + \int_{P_{0}} \bar{M}_{0 \gamma} \dot{S}_{\alpha\beta}^{\gamma} \ dA ,
\] (121)
which is equivalent to the expression (63) in Steigmann (2018).\(^{11}\)

\(^{11}\)adapted to the notations used in this paper.
Remark 4.5: As shown in Appendix B, the asymmetry of the effective stress (120) is due to the fact that it still contains the in-plane bending apart from surface stretching. It is unsymmetric even for initially straight fibers in a general setting. Therefore the symmetrization employed in Steigmann (2018) is valid only for special cases.

Remark 4.6: The power term $\bar{M}^{\alpha\beta}_{\gamma} \dot{\gamma}^{\alpha\beta}$ in the gradient theory of Steigmann (2018) can become ill-defined for initially curved fibers when $\dot{\gamma}^{\alpha\beta}$ approaches zero even though there is a change in geodesic curvature. This can solely result from the choice of parametrization, as seen e.g. in Sec. 7.1. In contrast, the internal power expressions (103), (107) and (113) presented above overcome this limitation.

4.4 Extension to multiple fiber families

In case of $n_f$ fiber families $C_i, i = 1, ..., n_f$, we define the tangent $\ell_i$ and director $c_i$ for each $C_i$. The in-plane curvatures (48) and the moment $\bar{m}$ in (66) are then defined for each fiber family $C_i$. Then $P_{\text{int}}$ in Eq. (113) simply becomes

$$P_{\text{int}} = \frac{1}{2} \int_{P_0} \tau^{\alpha\beta} a_{\alpha\beta} \, dA + \int_{P_0} M^{\alpha\beta}_0 b_{\alpha\beta} \, dA + \sum_{i=1}^{n_f} \int_{P_i} \bar{M}^{\alpha\beta}_0 i_{\alpha\beta} \, dA .$$

(122)

In accordance with Eq. (122), the form of the stored energy function is extended from (115) to

$$W = W(a_{\alpha\beta}, b_{\alpha\beta}, \bar{b}_{\alpha\beta}; h^{\alpha\beta}_i ,) ,$$

(123)

or equivalently in terms of the invariants, e.g.

$$W = W(I_1, J, \Lambda_i, \gamma_{ij}, H, \kappa, \kappa^i_n, \tau^i_k, \kappa^i_g) .$$

(124)

The constitutive equations thus read

$$\tau^{\alpha\beta} = 2 \frac{\partial W}{\partial a_{\alpha\beta}} , \quad M^{\alpha\beta}_0 = \frac{\partial W}{\partial b_{\alpha\beta}} , \quad \bar{M}^{\alpha\beta}_0 = \frac{\partial W}{\partial \bar{b}_{\alpha\beta}} ,$$

(125)

where $\bar{M}^{\alpha\beta}_0$ are the components of the nominal stress couple tensor associated with in-plane bending of fiber $i$. They correspond to the change in the in-plane curvature $\bar{b}^{\alpha\beta}_i$ of fiber $i$.

5 Constitutive examples

This section presents constitutive examples for the presented theory considering unconstrained and constrained fibers. We restrict ourselves here to two families of fibers. Note however that our approach allows for any number of fiber families.

5.1 A simple generalized fabric model

A simple generalized shell model for two-fiber-family fabrics that are initially curved and bonded to a matrix is given by

$$W = W_{\text{matrix}} + W_{\text{fib-stretch}} + W_{\text{fib-bending}} + W_{\text{fib-torsion}} + W_{\text{fib-angle}} ,$$

(126)
where

\[ W_{\text{matrix}} = U(J) + \frac{1}{2} \mu (J_1 - 2 - 2 \ln J) , \]

\[ W_{\text{fib-stretch}} = \frac{1}{8} \varepsilon_L \sum_{i=1}^{2} (\Lambda_i - 1)^2 , \]

\[ W_{\text{fib-bending}} = \frac{1}{2} \sum_{i=1}^{2} \left[ \beta_n (K^n_i)^2 + \beta_g (K^g_i)^2 \right] , \tag{127} \]

\[ W_{\text{fib-torsion}} = \frac{1}{2} \beta \tau \sum_{i=1}^{2} (T^g_i)^2 , \]

\[ W_{\text{fib-angle}} = \frac{1}{4} \epsilon_a (\gamma_{12} - \gamma_{012})^2 \]

are the strain energy for matrix deformation, fiber stretching, out-of-plane and in-plane fiber bending, fiber torsion, and the linkage between the two fiber families, respectively. \( U(J) \) is the surface dilatation energy. In the above expression, \( \sqrt{\Lambda} , T_g , K_n , K_g , \) and \( \gamma_{12} \) denote the fiber stretch, the nominal change in geodesic fiber torsion, normal fiber curvature and geodesic fiber curvature, and the relative angle between fiber families, respectively (see Tabs. 1 and 2). Symbols \( \mu , \varepsilon , \) and \( \beta \) are material parameters. \( \varepsilon_L \) can be taken as zero during fiber compression (\( \Lambda_i < 1 \)) to mimic buckling phenomenologically.

From Eq. (116) and (127), we then find the effective stress and moment components for the director gradient-based formulation (Sec. 4.2), (see Appendix A and Sauer and Duong (2017) for the required variations of kinematical quantities)

\[ \tau^{\alpha \beta} = J \frac{\partial U}{\partial J} a^{\alpha \beta} + \mu (A^{\alpha \beta} - a^{\alpha \beta}) + \frac{1}{2} \varepsilon_L \sum_{i=1}^{2} (\Lambda_i - 1) L^{\alpha \beta}_i + \epsilon_a (\gamma_{12} - \gamma_{012}) (L^a_1 L^b_2)^\text{sym} , \]

\[ M_{0 \alpha}^\beta = \beta_n \sum_{i=1}^{2} K_{n_i} L_i^{\alpha \beta} + \beta \tau \sum_{i=1}^{2} T_i^g (\epsilon_{0h}^\alpha L_i^\beta)^\text{sym} , \]

\[ \bar{M}_{0 \alpha}^\beta = \beta_g \sum_{i=1}^{2} K_{g_i} L_i^{\alpha \beta} , \tag{128} \]

where \( (\bullet^{\alpha \beta})^\text{sym} = \frac{1}{2} (\bullet^{\alpha \beta} + \bullet^{\beta \alpha}) \) denotes symmetrization.

### 5.2 Fiber inextensibility constraints

For most textile materials, the deformation is usually characterized by very high tensile stiffness in fiber direction and low in-plane shear stiffness and bending stiffness. In this case, one may model the very high tensile stiffness along the fiber direction \( i \) by the inextensibility constraint

\[ g_i := \Lambda_i - 1 = 0 , \quad \Lambda_i > 1 , \tag{129} \]

where \( \Lambda_i \) is defined in Tab. 2. This constraint then replaces the \( W_{\text{fib-stretch}} \) term in (126). To enforce this constraint, we can employ the Lagrange multiplier method in the strain energy function

\[ \tilde{W} := W + \sum_{i=1}^{n_f} q_i g_i , \tag{130} \]
where \( q_i \) \((i = 1, \ldots, n_f)\) denote the corresponding Lagrange multipliers. The stress components in this case become
\[
\tau^{\alpha\beta} = 2 \frac{\partial \tilde{W}}{\partial a^{\alpha\beta}} = 2 \frac{\partial W}{\partial a^{\alpha\beta}} + \sum_{i=1}^{n_f} 2 q_i L_i^{\alpha\beta}.
\] (131)
This leads to the same stress and moment components as in (128.1) with the exception that \( \epsilon_L (\Lambda_i - 1)/2 \) is now replaced by \( 2 q_i \).

6 Weak form

This section presents the weak form for the generalized Kirchhoff-Love shell. The weak form is obtained by the same steps as the mechanical power balance in Sec. 3.5 simply by replacing velocity \( v \) by variation \( \delta x \). This gives
\[
G_{\text{in}} + G_{\text{int}} - G_{\text{ext}} = 0 \quad \forall \delta x \in \mathcal{V},
\] (132)
where according to Eq. (101), (102), (104), and (122)
\[
G_{\text{in}} = \int_{S_0} \delta \mathbf{x} \cdot \rho_0 \dot{v} \, dA,
\]
\[
G_{\text{int}} = \frac{1}{2} \int_{S_0} \tau^{\alpha\beta} \delta a^{\alpha\beta} \, dA + \int_{S_0} M_{0}^{\alpha\beta} \delta b_{\alpha\beta} \, dA + \sum_{i=1}^{n_f} \int_{S_0} \bar{M}_{0i}^{\alpha\beta} \delta \bar{b}_{\alpha\beta} \, dA,
\] (133)
\[
G_{\text{ext}} = \int_S \delta \mathbf{x} \cdot \mathbf{f} \, da + \int_{\partial S} \delta \mathbf{x} \cdot \mathbf{T} \, ds + \int_{\partial S} \delta \mathbf{n} \cdot \mathbf{M} \, ds + \sum_{i=1}^{n_f} \int_{\partial S} \delta \mathbf{c}_i \cdot \bar{M}_i \, ds.
\]
Using (125), \( G_{\text{int}} \) can also be expressed as the variation of potential (123) w.r.t. its arguments,
\[
G_{\text{int}} = \int_{S_0} \delta \tilde{W} \, dA = \int_{S_0} \frac{\partial \tilde{W}}{\partial a^{\alpha\beta}} \delta a^{\alpha\beta} \, dA + \int_{S_0} \frac{\partial \tilde{W}}{\partial b_{\alpha\beta}} \delta b_{\alpha\beta} \, dA + \sum_{i=1}^{n_f} \int_{S_0} \frac{\partial \tilde{W}}{\partial \bar{b}_{\alpha\beta}} \delta \bar{b}_{\alpha\beta} \, dA.
\] (134)
For constrained materials, e.g. (130), \( G_{\text{int}} \) becomes
\[
G_{\text{int}} = \int_{S_0} \delta \tilde{W} \, dA = \int_{S_0} \left( \frac{1}{2} \tau^{\alpha\beta} \delta a_{\alpha\beta} + M_{0}^{\alpha\beta} \delta b_{\alpha\beta} + \sum_{i=1}^{n_f} \bar{M}_{0i}^{\alpha\beta} \delta \bar{b}_{\alpha\beta} \right) \, dA + \sum_{i=1}^{n_f} \int_{S_0} \frac{\partial \tilde{W}}{\partial q_i} \delta q_i \, dA.
\] (135)
For the constitutive example in Sec. 5, \( \tau^{\alpha\beta} \) is then given by (131), \( M_{0}^{\alpha\beta} \) and \( \bar{M}_{0i}^{\alpha\beta} \) are given by (128.2) and (128.3), while \( \partial \tilde{W}/\partial q_i = g_i \).

The linearization of weak form (133) and its discretization can be found in Duong et al. (2021).

7 Analytical solutions

This section illustrates the preceding theory by several analytical examples considering simple homogeneous deformation states. They are useful elementary test cases for the verification of computational formulations.
7.1 Geodesic curvature of a circle embedded in an expanding flat surface

The first example presents the computation of geodesic curvature $\kappa_g$ from Eqs. (51) and (52) and confirms that only Eq. (51) gives the correct value for initially curved fibers. This illustrates the limitation of the existing Kirchhoff-Love gradient theory for cases where $\dot{\Gamma}_{\alpha\beta} = 0$ due to the choice of surface parametrization.

![Figure 3: Computation of geodesic curvature $\kappa_g$: A circular fiber $C$ is embedded in the planar surface $\mathcal{S}$ that is expanded by the homogeneous deformation $F = \lambda I$.](image)

To this end, we consider a circular fiber $C_0 \in \mathcal{S}_0$ with initial radius $R$ expanding according to the deformation gradient $F = \lambda I$ to $C \in \mathcal{S}$ with the current radius $r$ as shown in Fig. 3. Therefore, the geodesic curvatures of $C_0$ and $C$ are expected to simply be $\kappa_0^g = 1/R$ and $\kappa_g = 1/r$, respectively.

With respect to the convective coordinates $(\xi^1 = X, \xi^2 = Y)$ shown in Fig. (3a), a position vector on any point of $\mathcal{S}_0$ and $\mathcal{S}$ can be represented by

\[
X_s = X(\xi^1, \xi^2) := \xi^1 e_1 + \xi^2 e_2, \\
x_s = x(\xi^1, \xi^2) := \lambda \xi^1 e_1 + \lambda \xi^2 e_2.
\]

The surface tangent and normal vectors follow from Eq. (136) as

\[
A_1 = e_1, \quad A_2 = e_2, \quad \text{and} \quad N = e_3,
\]

\[
a_1 = \lambda e_1, \quad a_2 = \lambda e_2, \quad \text{and} \quad n = e_3,
\]

so that

\[
A_1^i = e_1, \quad A_2^i = e_2,
\]

\[
a_1^i = \frac{1}{\lambda} e_1, \quad a_2^i = \frac{1}{\lambda} e_2.
\]

The surface deformation gradient then reads

\[
F = a_\alpha \otimes A^\alpha = \lambda \left( e_1 \otimes e_1 + e_2 \otimes e_2 \right),
\]

and the surface Christoffel symbols take the form

\[
\bar{\Gamma}^\gamma_{\alpha\beta} := A_{\alpha,\beta} \cdot A^\gamma = 0, \quad \Gamma^\gamma_{\alpha\beta} := a_{\alpha,\beta} \cdot a^\gamma = 0.
\]

In the reference configuration, the fiber is parametrized by the arc-length coordinate $s$ as

\[
X_c = X(s) := R \cos \frac{s}{R} e_1 + R \sin \frac{s}{R} e_2.
\]
From this and Eq. (139) follows
\[ L := X_{c,s} = -\sin \frac{s}{R} e_1 + \cos \frac{s}{R} e_2 = -\frac{Y}{R} A_1 + \frac{X}{R} A_2, \]
\[ \lambda \ell := F L = -\frac{Y}{R} a_1 + \frac{X}{R} a_2, \tag{142} \]
\[ \ell := \frac{F L}{\lambda} = -\frac{Y}{R} e_1 + \frac{X}{R} e_2, \]
\[ c := n \times \ell = -\frac{X}{R} e_1 - \frac{Y}{R} e_2. \]

In this equation \((\xi^1, \xi^2) = (X, Y) \in C_0\). Thus,
\[ [\hat{L}]_{\alpha \beta} := \frac{1}{\lambda} [L^\alpha_{\beta}] = \frac{1}{\lambda R} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{143} \]
and
\[ [\ell^\alpha] := [\ell \cdot a^\alpha] = \frac{1}{\lambda R} \begin{bmatrix} -Y \\ X \end{bmatrix}, \quad [c^\alpha] := [c \cdot a^\alpha] = \frac{\lambda}{R} \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{144} \]

Inserting these expressions into Eq. (51), and using the identity \(X^2 + Y^2 = R^2\) gives the geodesic curvature
\[ \kappa^g_G = 0, \quad \kappa^L_G = \frac{1}{\lambda R} = \frac{1}{r}, \quad \kappa_g = \kappa^G_G + \kappa^L_G = \frac{1}{r}. \tag{145} \]

For \(C_0\), setting \(\lambda = 1\) directly yields \(\kappa_g^0 = 1/R\).

In contrast, Eq. (52) obviously fails to reproduce the correct geodesic curvatures since the Christoffel symbols are zero everywhere solely due to the choice of the surface parametrization. Furthermore, since \(\Gamma^\alpha_{\beta \gamma} = 0\), the in-plane bending term of the internal power is ill-defined in gradient theory (121), whereas we get a well-defined power from (51) with (103), (107) and (113). It correctly captures the change in geodesic curvature.

### 7.2 Biaxial stretching of a sheet containing diagonal fibers

The second example presents an analytical solution for homogeneous biaxial stretching of a rectangular sheet from dimension \(L \times H\) to \(\ell \times h\), such that \(\ell = \lambda \ell L\) and \(h = \lambda h H\). The sheet contains matrix material and two fiber families distributed diagonally as shown in Fig. 4. The strain energy function in this example is taken from (127) as
\[ W = \frac{\mu}{2} I_1 - 2 - \ln J + \frac{1}{8} \epsilon \sum_{i=1}^2 (\Lambda_i - 1)^2 + \frac{1}{4} \epsilon a (\gamma_{12} - \gamma_{12}^0)^2. \tag{146} \]
Therefore the stress components follow as

\[ \tau^{\alpha\beta} = \mu (A^{\alpha\beta} - a^{\alpha\beta}) + \sum_{i=1}^{2} \tau_i L_i^{\alpha\beta} + \epsilon_a (\gamma_{12} - \gamma_{12}^0) (L_i^1 L_i^2)^{\text{sym}}, \]

(147)

where \( \tau_i := \frac{1}{2} \epsilon_L (\Lambda_i - 1) \) denotes the nominal fiber tension. The parameterization can be chosen such that the surface tangent vectors are \( A_1 = L e_1, A_2 = H e_2, a_1 = \ell e_1, \) and \( a_2 = h e_2, \)

where \( e_\alpha \) are the basis vectors shown in Fig. 4. The fiber directions are \( L_1 = (L e_1 + H e_2)/D, L_2 = (L e_1 - H e_2)/D, \) where \( D^2 := L^2 + H^2. \) We thus find

\[
[A^{\alpha\beta}] = \left[ \begin{array}{ccc} 1/L^2 & 0 & 0 \\ 0 & 0 & 1/H^2 \end{array} \right], \quad [a^{\alpha\beta}] = \left[ \begin{array}{ccc} 1/\ell^2 & 0 & 0 \\ 0 & 0 & 1/h^2 \end{array} \right], \quad [L_1^{\alpha\beta}] = \frac{1}{D^2} \left[ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right],
\]

\[
[L_2^{\alpha\beta}] = \frac{1}{D^2} \left[ \begin{array}{ccc} 1 & -1 & 1 \\ -1 & 0 & 1 \end{array} \right], \quad \text{and} \quad [L_1^0 L_2^0]^{\text{sym}} = \frac{1}{D^2} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right].
\]

(148)

Further, we find the fiber stretch and angles

\[
\Lambda_\ell := \Lambda_1 = \Lambda_2 = (\ell^2 + h^2)/D^2, \\
\gamma_{12}^0 = A_{\alpha\beta} L_1^\alpha L_2^\beta = (L^2 - H^2)/D^2, \\
\gamma_{12} = a_{\alpha\beta} L_1^\alpha L_2^\beta = (\ell^2 - h^2)/D^2.
\]

(149)

Inserting these into Eq. (147) gives the stress tensor \( \sigma = J^{-1} \tau^{\alpha\beta} a_\alpha \otimes a_\beta, \) where \( J := \lambda_\ell \lambda_h. \)

The resultant reaction forces at the boundaries then follow as

\[
F_1 = h e_1 \sigma e_1 = \frac{h}{J} \left[ \mu (\lambda_\ell^2 - 1) + \frac{2 \ell^2}{D^2} \tau + \epsilon_a \frac{\ell^2}{D^2} (\gamma_{12} - \gamma_{12}^0) \right],
\]

\[
F_2 = \ell e_2 \sigma e_2 = \frac{\ell}{J} \left[ \mu (\lambda_h^2 - 1) + \frac{2 h^2}{D^2} \tau - \epsilon_a \frac{h^2}{D^2} (\gamma_{12} - \gamma_{12}^0) \right],
\]

(150)

where \( \tau := \tau_1 = \tau_2 = \frac{1}{2} \epsilon_L (\Lambda_\ell - 1). \)

**Remark 7.1:** The presented solution (150) includes pure shear by simply setting \( \lambda_h = 1/\lambda_\ell. \)

**Remark 7.2:** For uniaxial tension e.g. in the \( e_1 \) direction and with free horizontal boundaries, condition \( F_2 = 0 \) in Eq. (150.2) gives the solution of \( \lambda_h, \) which in turn can be inserted to Eq. (150.1) for the resultant reaction force \( F_1(\lambda_\ell) \) as

\[
F_1 = \frac{H}{\lambda_\ell D^4} \left[ D^4 \mu (\lambda_\ell^2 - 1) + (\epsilon_\ell + \epsilon_a) \lambda_\ell^2 + (\epsilon_\ell - \epsilon_a) \lambda_h^2 \right] + (\epsilon_\ell - \epsilon_a) \lambda_h^2 H^2 L^2 + (\epsilon_\ell - \gamma_{12}^0 \epsilon_a) \lambda_h^2 L^2 D^2, \]

(151)

where

\[
\lambda_h^2 = \frac{1}{2a} \left( -b + \sqrt{b^2 + 4a\mu} \right),
\]

(152)

with \( a := \frac{H^2}{D^4} (\epsilon_\ell + \epsilon_a), \) and \( b := \mu + (\epsilon_\ell - \epsilon_a) \frac{\lambda_\ell^2 L^2 H^2}{D^4} - \epsilon_\ell \frac{H^2}{D^2} + \gamma_{12}^0 \epsilon_a H^2/D^2. \)

**Remark 7.3:** Solution (150) also captures the inextensibility of fibers. In this case either the vertical, or the horizontal boundaries have to be stress free. In the latter case, \( F_2 = 0 \) and

\[
\lambda_h = \sqrt{\frac{D^2}{H^2} - \frac{\lambda_\ell^2 L^2}{H^2}},
\]

(153)

due to \( D^2 = \ell^2 + h^2. \) In this case the deformation is limited by \( \lambda_\ell^{\text{max}} = D/L. \) From \( F_2 = 0, \) the nominal fiber tension now follows as

\[
\tau = \frac{1}{2} \epsilon_a (\gamma_{12} - \gamma_{12}^0) - \frac{1}{2} \frac{D^2}{h^2} \mu (\lambda_h^2 - 1),
\]

(154)
Inserting (154) into (150.1) then gives

\[
F_1 = \frac{h}{J} \left[ \mu (\lambda_1^2 - 1) - \frac{\ell^2}{h^2} \mu (\lambda_2^2 - 1) + 2 \epsilon_a \frac{\ell^2}{J \ell^2} (\gamma_{12} - \gamma_{012}) \right].
\]  

(155)

7.3 Picture frame test

![Figure 5: Picture frame test: (a.) Initial and (b.) deformed configurations containing two fiber families. (c.) Exact solution of the shear force vs. shear angle \( \theta := 2 \varphi - 90^\circ \).](image)

The third example computes the analytical solution for the shear force in the picture frame test of an \( L_0 \times L_0 \) square sheet with two fiber families as shown in Fig. 5a-b. From the figure, we find the surface tangent vectors

\[
A_1 = L_0 \cos \varphi_0 e_1 + L_0 \sin \varphi_0 e_2, \quad a_1 = L_0 \cos \varphi e_1 + L_0 \sin \varphi e_2,
\]

and the fiber directions

\[
L_1 = \cos \varphi_0 e_1 + \sin \varphi_0 e_2, \quad \ell_1 = \cos \varphi e_1 + \sin \varphi e_2,
\]

\[
L_2 = -\cos \varphi_0 e_1 + \sin \varphi_0 e_2, \quad \ell_2 = -\cos \varphi e_1 + \sin \varphi e_2,
\]

(156)

(157)

where \( \varphi_0 = \pi/4 \). With these, the components of tensors \( C \) and \( (L_1 \otimes L_2)^{\text{sym}} \) read

\[
[C_{\alpha}] = a_{\alpha\gamma} A^{\gamma\beta} = \begin{bmatrix}
1 & -\cos(2 \varphi) \\
-\cos(2 \varphi) & 1
\end{bmatrix}, \quad \text{and} \quad [L_1^\alpha L_2^\beta]^{\text{sym}} = \frac{1}{2 L_0^2} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix},
\]

(158)

respectively. From Eq. (158.1), the surface stretch is found as

\[
J = \sqrt{\det[C_{\alpha}]} = \sin(2 \varphi).
\]

(159)

Further, the strain energy function in this example is taken from (127) as

\[
W = \frac{1}{4} \epsilon_a (\gamma_{12} - \gamma_{012})^2,
\]

so that the Cauchy stress components are

\[
[\sigma^{\alpha\beta}] = \frac{1}{J} \epsilon_a (\gamma_{12} - \gamma_{012}) [L_1^\alpha L_2^\beta]^{\text{sym}} = -\frac{1}{2 L_0^2} \epsilon_a \cot(2 \varphi) \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

(160)

Here, we have used Eq. (158.2), Eq. (159), \( \gamma_{12} = L_1 \cdot L_2 = 0 \), and \( \gamma_{012} = \ell_1 \cdot \ell_2 = -\cos(2 \varphi) \).
Consider the upper left edge with normal vector \( \mathbf{\nu} = -\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2 = \nu_1 \mathbf{a}_1 = \nu_2 \mathbf{a}_2 = L_0 \sin(2\varphi) \). The traction components on this edge can be computed from

\[
[t^\alpha] = [\sigma^{\alpha\beta} \nu_\beta] = -\frac{1}{2L_0} \epsilon_\alpha^{12} \cos(2\varphi) \begin{bmatrix} 1 \\ 0 \end{bmatrix} .
\] (161)

Therefore, the traction vector solely contains shear contribution

\[
t = t^\alpha \mathbf{a}_\alpha = t^1 \mathbf{a}_1 = -\frac{1}{2} \epsilon_\alpha^{12} \cos(2\varphi) \left( \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2 \right) ,
\] (162)

so that the shear force (i.e. the tangent reaction) at the edge of the sheet is

\[
R_s = t \cdot \frac{\mathbf{a}_1}{||a_1||} L_0 = -\frac{1}{2} \epsilon_\alpha^{12} \cos(2\varphi) L_0 = \frac{1}{2} \epsilon_\alpha^{12} \sin(\theta) L_0 ,
\] (163)

where \( \theta := 2\varphi - 90^\circ \) denotes the so-called shear angle. This solution is plotted in Fig. 5c.

7.4 Annulus expansion

The fourth example presents an analytical solution for the expansion of an annulus containing distributed circular fibers and matrix material as depicted in Fig. 6. The inner and the outer rings with radius \( R_i \) and \( R_o \), respectively, are expanded to \( r_i \) and \( r_o \) by the constant stretch \( \lambda = r_i/R_i = r_o/R_o \). The strain energy density (per reference area) is taken as

\[
W = W_{\text{matrix}} + W_{\text{fib-bend}} + W_{\text{fib-stretch}} ,
\] (164)

where

\[
W_{\text{matrix}} = \frac{1}{2} K (J - 1)^2 ,
\]

\[
W_{\text{fib-bend}} = \frac{1}{2} \beta k_g^2 ,
\] (165)

\[
W_{\text{fib-stretch}} = \frac{1}{8} \epsilon_L (\Lambda - 1)^2 .
\]

Here, \( K(R) \), \( \beta \), and \( \epsilon_L \) are material parameters for matrix dilatation, fiber bending, and fiber stretching, respectively, and \( J, \Lambda \), and \( k_g \) are invariants induced by tensors \( C \) and \( K \) as listed in Tab. 1 and 2.
7.4.1 Kinematical quantities

According to Fig. 6, the initial and current configurations as well as initial fiber direction can be described by

\[ X = X(R, \phi) := R \cos \phi \, e_1 + R \sin \phi \, e_2 , \]
\[ x = x(R, \phi) := r \cos \phi \, e_1 + r \sin \phi \, e_2 , \quad (166) \]
\[ L = L(\phi) := -\sin \phi \, e_1 + \cos \phi \, e_2 . \]

Here \( r = \lambda R \), due to the homogeneous deformation, with \( \lambda \) being fiber stretch. From this, we find the covariant tangent vectors

\[ A_1 = \frac{\partial X}{\partial R} = \cos \phi \, e_1 + \sin \phi \, e_2 , \]
\[ A_2 = \frac{\partial X}{\partial \phi} = -R \sin \phi \, e_1 + R \cos \phi \, e_2 , \quad (167) \]
\[ a_1 = \frac{\partial x}{\partial R} = \lambda \cos \phi \, e_1 + \lambda \sin \phi \, e_2 , \]
\[ a_2 = \frac{\partial x}{\partial \phi} = -r \sin \phi \, e_1 + r \cos \phi \, e_2 , \]
and the constant surface normal \( n = N = e_3 \) during deformation. From the tangent vectors, we get

\[ [A_{\alpha \beta}] = \begin{bmatrix} 1 & 0 \\ 0 & R^2 \end{bmatrix}, \quad [A^{\alpha \beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1/R^2 \end{bmatrix}, \quad (168) \]
\[ [a_{\alpha \beta}] = \begin{bmatrix} \lambda^2 & 0 \\ 0 & r^2 \end{bmatrix}, \quad [a^{\alpha \beta}] = \begin{bmatrix} 1/\lambda^2 & 0 \\ 0 & 1/r^2 \end{bmatrix}, \]

and thus we find the contra-variant tangent vectors

\[ A^1 = \cos \phi \, e_1 + \sin \phi \, e_2 , \]
\[ A^2 = -\frac{1}{R} \sin \phi \, e_1 + \frac{1}{R} \cos \phi \, e_2 , \quad (169) \]
\[ a^1 = \frac{1}{\lambda} \cos \phi \, e_1 + \frac{1}{\lambda} \sin \phi \, e_2 , \]
\[ a^2 = -\frac{1}{r} \sin \phi \, e_1 + \frac{1}{r} \cos \phi \, e_2 . \]

The initial and current fiber direction thus can be expressed as

\[ L = L^\alpha A_\alpha , \]
\[ \ell = \frac{1}{\lambda} F L = \ell^\alpha a_\alpha , \quad (170) \]

with

\[ [L^\alpha] := [L \cdot A^\alpha] = \begin{bmatrix} 0 \\ 1/R \end{bmatrix}, \quad \text{and} \quad [\ell^\alpha] := [L^\alpha / \lambda] = \begin{bmatrix} 0 \\ 1/r \end{bmatrix}. \quad (171) \]

The components of the structural tensors \( L \otimes L \) and \( \ell \otimes \ell \) thus read

\[ [L^{\alpha \beta}] = \begin{bmatrix} 0 & 0 \\ 0 & 1/R^2 \end{bmatrix}, \quad [\ell^{\alpha \beta}] = \begin{bmatrix} 0 & 0 \\ 0 & 1/r^2 \end{bmatrix}, \quad [\ell^{\alpha \beta}] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (172) \]
Further, the fiber director is obtained as

\[ c = n \times \ell = -\cos \phi e_1 - \sin \phi e_2 = c_\alpha a^\alpha , \tag{173} \]

with

\[ [c_\alpha] := [c \cdot a_\alpha] = \begin{bmatrix} -\lambda \\ 0 \end{bmatrix}. \tag{174} \]

Therefore, its derivatives read

\[ c_1 := \frac{\partial c}{\partial R} = 0 , \]
\[ c_2 := \frac{\partial c}{\partial \phi} = \sin \phi e_1 - \cos \phi e_2 . \tag{175} \]

From Eq. (35) then follows the components of the in-plane curvature tensor in the current configuration as

\[ [\bar{b}_{\alpha\beta}] = -\frac{1}{2} [c_\alpha \cdot a_\beta + c_\beta \cdot a_\alpha] = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}. \tag{176} \]

Similarly, we find the in-plane curvature tensor in the initial configuration as

\[ [\bar{B}_{\alpha\beta}] = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}. \tag{177} \]

With this, the current geodesic curvature can be computed from Eq. (32) as

\[ \kappa_g = \bar{b}_{\alpha\beta} \ell_{\alpha\beta} = 1/r . \tag{178} \]

Similarly, we can verify the initial geodesic curvature \( \kappa^0_g = 1/R \). These results lead to \( k_g = \kappa_g \lambda - \kappa^0_g = 0 \), and thus \( W_{\text{fib-bend}} = 0 \) due to the particular choice of strain energy (165.2). This means that the change in the geodesic curvature \( \kappa_g \) in this example is purely due to fiber stretching and not due to fiber bending.

### 7.4.2 Analytical expression for the reaction forces

From Eq. (164), we find the effective membrane stress

\[ J \sigma^{\alpha\beta} = 2 \frac{\partial W}{\partial a_{\alpha\beta}} = K (J - 1) J a^{\alpha\beta} + \frac{1}{2} \epsilon_L (\lambda^2 - 1) L^{\alpha\beta} . \tag{179} \]

Here, the component of the Cauchy stress tensor is \( N^{\alpha\beta} = \sigma^{\alpha\beta} \) since \( M^{\alpha\beta} = M^{\alpha\beta} = 0 \) (see Eq. (112) with (93)). Its mixed components thus read

\[ [N^{\alpha\beta}] = K (J - 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\epsilon_L}{2 J} (\lambda^2 - 1) \begin{bmatrix} 0 & 0 \\ 0 & \lambda^2 \end{bmatrix} , \tag{180} \]

where we have inserted \( A^{\alpha\beta} \) and \( L^{\alpha\beta} \) from (168) and (172). It can be verified that with a homogeneous deformation \( J = \lambda^2 \), the local equilibrium equation \( \text{div}_s \sigma = \text{div}_s (N^{\alpha\beta} a_\alpha \otimes a_\beta) = 0 \) is satisfied for the functionally graded surface bulk modulus

\[ K(R) = \frac{1}{2} \epsilon_L \ln R . \tag{181} \]
Further, the resultant reaction force at the cut in Fig. 6 (right) can be computed by

\[ R_\nu = \int_{r_i}^{r_o} (\nu \otimes \nu) T \, dr = R_\nu \ell, \]

\[ R_\tau = \int_{r_i}^{r_o} (\tau \otimes \tau) T \, dr = R_\tau c, \]

since \( \nu = \ell \) and \( \tau = c \). Here, \( T = N^\alpha\beta \nu_\beta a_\alpha \) denotes the traction acting on the cut. Therefore, we get

\[ T \cdot \ell = N^\alpha\beta \ell_\beta = \left( \frac{1}{2} \epsilon_L + K \right) (J - 1) , \]

\[ T \cdot c = N^\alpha\beta c_\alpha \ell^\beta = 0 . \]

Inserting this into (182) and taking (181) into account, finally results in

\[ R_\nu = \frac{1}{2} \epsilon_L (J - 1) \bar{\lambda} \left( R_o \ln R_o - R_i \ln R_i \right) , \]

\[ R_\tau = 0 . \]

7.5 Pure bending of a flat rectangular sheet

The last example presents the analytical solution for the pure bending of a flat rectangular sheet \( S_0 \) with dimension \( S \times L \), subjected to an external bending moment \( M_{ext} \) along the two shorter edges, as shown in Fig. 7. This problem was solved by Sauer and Duong (2017) for an isotropic material, and here we additionally consider the influence of fiber bending. For simplicity the fibers are aligned along the bending direction. The strain energy function is taken from (127), which reduces to

\[ W = \frac{\mu}{2} (I_1 - 2 - \ln J) + \frac{\beta}{2} K_n^2 \]

in this example.

![Figure 7: Pure bending: Deformation of a rectangular sheet (left) into a circular arc (right).](image)

7.5.1 Kinematical quantities

We extend the kinematical quantities derived in Sauer and Duong (2017) to account for embedded fibers. The sheet is parametrized by \( \xi^1 \in [0, S] \) and \( \xi^2 \in [0, L] \). Bending induces (high-order) in-plane deformation, such that the current configuration \( S \) has dimension \( s \times \ell \).
The stretches along the longer and shorter edges are $\lambda = \ell / L$ and $\lambda_s = s / S$, respectively, and we have relations $\theta := \kappa_{\ell} \xi$ and $r := 1 / \kappa_{\ell}$, where $\kappa_{\ell}$ is the homogenous curvature of the sheet.

With this, the sheet in the initial and current configurations and the initial fiber direction can be described by

$$X(\xi, \eta) = \xi e_1 + \eta e_2 ,$$
$$x(\xi, \eta) = r \sin \theta e_1 + \lambda_s \eta e_2 + r (1 - \cos \theta) e_3 ,$$
$$L = e_1 .$$

Therefore, the initial and current tangent vectors and current normal vector are

$$A_1 = \frac{\partial X}{\partial \xi} = e_1 ,$$
$$A_2 = \frac{\partial X}{\partial \eta} = e_2 ,$$
$$a_1 = \frac{\partial x}{\partial \xi} = \lambda_{\ell} \left( \cos \theta e_1 + \sin \theta e_3 \right) ,$$
$$a_2 = \frac{\partial x}{\partial \eta} = \lambda_s e_2 ,$$
$$n = - \sin \theta e_1 + \cos \theta e_3 .$$

From these, we find the components of the structural tensor

$$[L^{\alpha\beta}] = [A^\alpha (L \otimes L) A^\beta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

the surface metrics

$$[A_{\alpha\beta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$
$$[A^{\alpha\beta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$
$$[a_{\alpha\beta}] = \begin{bmatrix} \lambda_{\ell}^2 & 0 & 0 \\ 0 & \lambda_s^2 & 0 \end{bmatrix} ,$$
$$[a^{\alpha\beta}] = \begin{bmatrix} \lambda_{\ell}^{-2} & 0 & 0 \\ 0 & \lambda_s^{-2} & 0 \end{bmatrix} ,$$

the surface stretch $J = \lambda_{\ell} \lambda_s$, and the components of the curvature tensor

$$[b_{\alpha\beta}] = \begin{bmatrix} \kappa_{\ell} \lambda_{\ell}^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$
$$[b^{\alpha\beta}] = \begin{bmatrix} \kappa_{\ell} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$
$$[b^{\alpha\beta}] = \begin{bmatrix} \kappa_{\ell} \lambda_{\ell}^{-2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$
$$H = \frac{\kappa_{\ell}}{2} .$$

Therefore, the nominal change in the normal curvature is

$$K_n = b_{\alpha\beta} L^{\alpha\beta} = \kappa_{\ell} \lambda_{\ell}^2 .$$

### 7.5.2 Analytical external moment and mean curvature relation

From Eq. (185), we find

$$J \sigma^{\alpha\beta} = 2 \frac{\partial W}{\partial a_{\alpha\beta}} = \mu \left( A^{\alpha\beta} - a^{\alpha\beta} \right) ,$$
$$J M^{\alpha\beta} = \frac{\partial W}{\partial b_{\alpha\beta}} = \beta K_n L^{\alpha\beta} .$$

According to Eqs. (93) and (112) with $\tilde{M}^{\alpha\beta} = \tilde{m}^{\alpha}_{\gamma\alpha} = 0$, the components of the Cauchy stress tensors read

$$N^{\alpha}_{\beta} = \sigma^{\alpha\gamma} a_{\gamma\beta} + M^{\alpha\gamma} b_{\gamma\beta} .$$

Inserting $A^{\alpha\beta}$, $a^{\alpha\beta}$, and $L^{\alpha\beta}$ from Eqs. (189), (190), and (188) gives

$$[N^{\alpha}_{\beta}] = \frac{\mu}{J} \begin{bmatrix} \lambda_{\ell}^2 - 1 & 0 & 0 \\ 0 & \lambda_s^2 - 1 & 0 \end{bmatrix} + \frac{\beta}{J} \kappa_{\ell}^2 \lambda_{\ell}^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$
By assuming free edges, we have $N_1 = N_2 = 0$. That is

\[ N_1 = \frac{\mu}{J} (\lambda_1^2 - 1) + \frac{\beta}{J} \kappa_1^2 \lambda_1^4 = 0, \]
\[ N_2 = \frac{\mu}{J} (\lambda_2^2 - 1) = 0. \]  

From the second equation follows $\lambda_s = 1$. The distributed external moment $M_{\text{ext}}$ is in equilibrium to the distributed internal moment $m_\tau$ – see Eq. (79.2) – on any cut parallel to the shorter edge. We therefore have

\[ M_{\text{ext}} = -m_\tau = M^{\alpha\beta} \nu_\alpha \nu_\beta = \frac{\beta}{J} \kappa_\ell \lambda_\ell^4, \]

where $\nu = a_1/\lambda_\ell$, $\nu_1 = \lambda_\ell$, and $\nu_2 = 0$. By combining Eqs. (197) and (196.1), we get an expression for $\lambda_\ell$ in terms of $M_{\text{ext}}$,

\[ \lambda_\ell = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{4 \mu \beta} M_{\text{ext}}^2}}, \text{ with the condition } M_{\text{ext}}^2 \leq \frac{1}{4} \mu \beta. \]

With this result and (197), we obtain the relation between external moment and the mean curvature,

\[ H = \frac{M_{\text{ext}}}{2 \beta \lambda_\ell^4}. \]

8 Conclusion

We have presented a generalized Kirchhoff-Love shell theory capable of capturing in-plane bending of fibers embedded in the surface. The formulation is an extension of classical shell theory and can handle multiple fiber families, possibly with initial curvature. The thin shell kinematics is fully characterized by the change of three symmetric second order tensors: $a_{\alpha\beta}$, $b_{\alpha\beta}$, and the newly added $\bar{b}_{\alpha\beta}$ – denoted the in-plane curvature tensor. These tensors capture the change of the tangent vectors $a_1$, $a_2$, and the in-plane fiber director $c$. Tensor $b_{\alpha\beta}$ and director vector $c$ are defined for each fiber family separately. The corresponding stress and moments work-conjugate to $a_{\alpha\beta}$, $b_{\alpha\beta}$, and $\bar{b}_{\alpha\beta}$, are the effective membrane stress $\sigma_{\alpha\beta}$, the out-of-plane bending moment $M^{\alpha\beta}$, and the in-plane bending moment $\bar{M}^{\alpha\beta}$. Their symmetry follows from angular momentum balance. With these work-conjugated pairs, general constitutive equations for hyperelastic materials are derived. Induced invariants and their physical meaning can be identified in a straightforward manner from the defined kinematics. The weak form is also provided. It is required for a finite element formulation based on $C^1$-continuous surface discretizations that are presented in future work (Duong et al., 2021). Several analytical examples are presented to serve as useful elementary test cases for the verification of nonlinear computational formulations. The presented analytical examples also confirm that the model correctly describes the mechanics of shells with initially curved fibers.

A Variation of various kinematical quantities

This section provides the variation of several kinematical quantities defined in Sec. 2. They are required for the derivation of the mechanical work balance (Sec. 4), the stresses and moments from a stored energy function (Sec. 5), the weak form (Sec. 6), and the linearization of the weak
form (Duong et al., 2021). We focus here mostly on the new variables introduced above, while the expressions for existing ones can be found elsewhere, e.g. in Sauer and Duong (2017).

Consider a variation of position \( \mathbf{x} \) by \( \delta \mathbf{x} \). Accordingly, the variation of the tangent vectors reads \( \delta \mathbf{a}_\alpha = \delta \mathbf{x}_\alpha \). From Eq. (5), the variation of \( \delta \mathbf{a}_{\alpha\beta} \) is

\[
\delta \mathbf{a}_{\alpha\beta} = \delta \mathbf{a}_\alpha \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \delta \mathbf{a}_\beta .
\]

(200)

Further, from Eq. (15.2), we find

\[
\delta \mathbf{a}_{\beta;\alpha} = \mathbf{n} \otimes \mathbf{n} \left( \delta \mathbf{a}_{\alpha;\beta} - \Gamma^\gamma_{\alpha\beta} \delta \mathbf{a}_\gamma \right),
\]

(201)

where we have used the variation (see e.g. Wriggers (2006)),

\[
\delta \mathbf{n} = - \mathbf{a}^\alpha \left( \mathbf{n} \cdot \delta \mathbf{a}_\alpha \right).
\]

(202)

From Eq. (41) we have

\[
\lambda^2 = (\mathbf{F} L) \cdot (\mathbf{F} L) = L_{\alpha\beta} a_{\alpha\beta}.
\]

It then follows that,

\[
\delta \lambda = \frac{1}{2 \lambda} L_{\alpha\beta} \delta a_{\alpha\beta} .
\]

(203)

With \( \delta \lambda \), the variation of \( \ell \) can be derived from Eq. (41) as

\[
\delta \ell = (1 - \ell \otimes \ell) \ell^\alpha \delta \mathbf{a}_\alpha .
\]

(204)

The variation of components \( \ell_\alpha = \ell \cdot \mathbf{a}_\alpha \) and \( \ell^\alpha = \ell \cdot \mathbf{a}^\alpha \) read

\[
\delta \ell_\alpha = \ell^\beta c_\alpha \cdot \delta \mathbf{a}_\beta + \ell \cdot \delta \mathbf{a}_\alpha ,
\]

(205)

and

\[
\delta \ell^\alpha = - \ell^\alpha \cdot \ell^\beta \delta \mathbf{a}_\beta ,
\]

(206)

where we have used (204) and (see e.g. Sauer and Duong (2017))

\[
\delta \mathbf{a}^\alpha = (a^\alpha \mathbf{n} \otimes \mathbf{n} - a^\beta \otimes \mathbf{a}^\alpha) \delta \mathbf{a}_\beta .
\]

(207)

With \( \ell^\alpha \ell^\beta = \ell^\alpha \ell^\beta \) and Eq. (206), we further find

\[
\delta \ell^\alpha \ell^\beta = - \ell^\alpha \ell^\beta \delta \alpha \beta .
\]

(208)

The variation of the in-plane director \( \mathbf{c} \) is

\[
\delta \mathbf{c} = - (\mathbf{n} \otimes \mathbf{c}) \delta \mathbf{n} - (\ell \otimes \mathbf{c}) \delta \ell ,
\]

(209)

which follows from identities \( \mathbf{c} \cdot \mathbf{n} = 0 \) and \( \ell \cdot \mathbf{n} = 0 \). Inserting Eq. (202) and (204) into Eq. (209) gives

\[
\delta \mathbf{c} = (c^\alpha \mathbf{n} \otimes \mathbf{n} - \ell^\alpha \ell \otimes \mathbf{c}) \delta \mathbf{a}_\alpha .
\]

(210)

Using this result and Eq. (207), the variation of the components of \( \mathbf{c} \) read

\[
\delta c_\alpha = - \ell^\alpha \mathbf{c} \cdot \delta \mathbf{a}_\beta + \mathbf{c} \cdot \delta \mathbf{a}_\alpha ,
\]

(211)

and

\[
\delta c^\alpha = - \ell^\alpha \mathbf{c} \cdot \delta \mathbf{a}_\beta - c^\beta \mathbf{a}^\alpha \cdot \delta \mathbf{a}_\beta .
\]

(212)

From Eq. (19) follows

\[
\delta \Gamma^c_{\alpha\beta} = \mathbf{c} \cdot \delta \mathbf{a}_{\alpha\beta} + \mathbf{a}_{\alpha\beta} \cdot \delta \mathbf{c} .
\]

(213)

\(^{12}\)Variation \( \delta \bullet \) can be replaced by \( \dot{\bullet} \) and \( \Delta \bullet \) denoting material time derivatives and linearization, respectively.
Further, taking the variation of $\hat{L}_{\alpha \beta}$ (see Eq. (44)) and using result (203) gives

$$\delta \hat{L}_{\alpha \beta} = -\hat{L}_{\alpha \beta} \ell^\gamma \ell^\gamma \cdot \delta a_\gamma = \hat{L}_{\alpha \beta} \ell^\gamma \delta \ell^\gamma .$$  \hfill (214)

From Eq. (45.2), the variation of $c_{\alpha \beta}$ reads

$$\delta c_{\alpha \beta} = -\ell^\gamma \delta \Gamma^\gamma_{\alpha \gamma} - \ell^\beta (\Gamma^\gamma_{\alpha \gamma} + \ell_\gamma c_\delta \ell^\delta) \delta \ell^\gamma - (c_\gamma \hat{L}_{\alpha \gamma} + \ell^\gamma \Gamma^\gamma_{\alpha \gamma}) \delta \ell^\beta - \ell^\beta \hat{L}_{\alpha \gamma} \delta c_\gamma .$$  \hfill (215)

From Eq. (36.1) follows

$$\delta c_{\alpha \beta} = a_\beta \delta c_{\alpha \gamma} + c_\beta \delta a_\alpha + \delta a_{\beta ; \alpha} + a_{\beta ; \alpha} \delta c^\beta .$$  \hfill (216)

The variation of the out-of-plane curvature is (see e.g. Sauer and Duong (2017))

$$\delta b_{\alpha \beta} = n \cdot \delta a_{\alpha \beta} - \Gamma_{\alpha \gamma} \cdot n \cdot \delta a_\gamma .$$  \hfill (217)

The variation of the in-plane curvature follows from Eq. (35) as

$$\delta \bar{b}_{\alpha \beta} = -\frac{1}{2} (\delta a_\alpha \cdot c_{\beta, \gamma} + a_\alpha \cdot \delta c_{\beta} + \delta a_{\beta} \cdot c_{\alpha} + a_{\beta} \cdot \delta c_{\alpha}) .$$  \hfill (218)

The variation of the normal curvature follows from Eq. (21) and (208) as

$$\delta \kappa_n = \frac{\partial \kappa_n}{\partial a_{\alpha \beta}} \delta a_{\alpha \beta} + \frac{\partial \kappa_n}{\partial b_{\alpha \beta}} \delta b_{\alpha \beta} + \frac{\partial \kappa_n}{\partial \bar{b}_{\alpha \beta}} \delta \bar{b}_{\alpha \beta} ,$$  \hfill (219)

with

$$\frac{\partial \kappa_n}{\partial a_{\alpha \beta}} = -\kappa_n \ell_{\alpha \beta} , \quad \frac{\partial \kappa_n}{\partial b_{\alpha \beta}} = \ell_{\alpha \beta} , \quad \text{and} \quad \frac{\partial \kappa_n}{\partial \bar{b}_{\alpha \beta}} = 0 .$$  \hfill (220)

Similarly, from Eqs. (50.1) and (208), the variation of the geodesic curvature gives

$$\delta \kappa_g = \frac{\partial \kappa_g}{\partial a_{\alpha \beta}} \delta a_{\alpha \beta} + \frac{\partial \kappa_g}{\partial b_{\alpha \beta}} \delta b_{\alpha \beta} + \frac{\partial \kappa_g}{\partial \bar{b}_{\alpha \beta}} \delta \bar{b}_{\alpha \beta} ,$$  \hfill (221)

with

$$\frac{\partial \kappa_g}{\partial a_{\alpha \beta}} = -\kappa_g \ell_{\alpha \beta} , \quad \frac{\partial \kappa_g}{\partial b_{\alpha \beta}} = 0 , \quad \text{and} \quad \frac{\partial \kappa_g}{\partial \bar{b}_{\alpha \beta}} = \ell_{\alpha \beta} .$$  \hfill (222)

From Eq. (22), (206), and (212), the variation of the geodesic torsion reads

$$\delta \tau_g = \frac{\partial \tau_g}{\partial a_{\alpha \beta}} \delta a_{\alpha \beta} + \frac{\partial \tau_g}{\partial b_{\alpha \beta}} \delta b_{\alpha \beta} + \frac{\partial \tau_g}{\partial \bar{b}_{\alpha \beta}} \delta \bar{b}_{\alpha \beta} ,$$  \hfill (223)

with $\frac{\partial \tau_g}{\partial \bar{b}_{\alpha \beta}} = 0$, and

$$\frac{\partial \tau_g}{\partial a_{\alpha \beta}} = -\frac{1}{2} \kappa_n (\ell^\alpha c^\beta + \ell^\beta c^\alpha) - \frac{1}{2} \tau_g (c^\alpha + \ell^\alpha) ,$$

$$\frac{\partial \tau_g}{\partial b_{\alpha \beta}} = \frac{1}{2} (\ell^\alpha c^\beta + \ell^\beta c^\alpha) .$$  \hfill (224)

Therefore, from Eqs. (31), (219) and (221), the variation of the principal curvature of the fiber is

$$\delta \kappa_p = \frac{\partial \kappa_p}{\partial a_{\alpha \beta}} \delta a_{\alpha \beta} + \frac{\partial \kappa_p}{\partial b_{\alpha \beta}} \delta b_{\alpha \beta} + \frac{\partial \kappa_p}{\partial \bar{b}_{\alpha \beta}} \delta \bar{b}_{\alpha \beta} ,$$  \hfill (225)

with

$$\frac{\partial \kappa_p}{\partial a_{\alpha \beta}} = -\kappa_p \ell_{\alpha \beta} , \quad \kappa_p \frac{\partial \kappa_p}{\partial b_{\alpha \beta}} = \kappa_n \ell_{\alpha \beta} , \quad \text{and} \quad \kappa_p \frac{\partial \kappa_p}{\partial \bar{b}_{\alpha \beta}} = \kappa_g \ell_{\alpha \beta} .$$  \hfill (226)
B Effective membrane stress in the existing gradient theory of Kirchhoff-Love shells

This section presents the intermediate steps to derive the effective membrane stress (120) appearing in the existing gradient theory of Kirchhoff-Love shells (119), where the change in the relative Christopher symbol $S^\gamma_{\alpha \beta}$ is used for the strain measures of in-plane bending. It also shows that the resulting effective membrane stress is unsymmetric for general materials even for initially straight fibers.

To this end, we first take the time derivative of the geodesic curvature (51). This gives

$$\dot{\kappa}_g = \dot{\kappa}^\Gamma_g + \dot{\kappa}^L_g,$$  \hspace{1cm} (227)

where

$$\dot{\kappa}^\Gamma_g = \dot{\ell}^{\alpha \beta} c_\gamma S^\gamma_{\alpha \beta} + \ell^{\alpha \beta} \dot{c}_\gamma S^\gamma_{\alpha \beta} + \ell^{\alpha \beta} c_\gamma \dot{S}^\gamma_{\alpha \beta},$$

$$\dot{\kappa}^L_g = \dot{\lambda} - 1 (\ell^{\gamma \delta} L^{\gamma}_{\alpha \beta} - \ell^{\delta \gamma} L^{\delta}_{\alpha \beta}) c_\alpha \ell^{\beta} L^{\alpha}_{\gamma \beta} + \lambda c_\alpha \dot{\ell}^{\beta} L^{\alpha}_{\gamma \beta}.$$ \hspace{1cm} (228)

Here, the time derivatives $\dot{\ell}^{\alpha \beta}, \dot{c}_\alpha, \dot{\lambda},$ and $\dot{\ell}^{\alpha}$ can be obtained from replacing $\delta(\bullet)$ with $\dot{\bullet}$ from expressions (208), (211), (203), and (206), respectively. By taking these into account, Eq. (228) becomes

$$\dot{\kappa}^\Gamma_g = -2 \kappa^\Gamma_g \ell^{\alpha \beta} a_\beta \cdot \dot{a}_\alpha + (\ell^{\gamma \delta} S^\gamma_{\alpha \beta} - \ell^{\delta \gamma} \ell^{\theta}_{\alpha \beta} (\ell^{\varepsilon}) c_\beta \cdot \dot{a}_\alpha + \ell^{\alpha \beta} c_\gamma \dot{S}^\gamma_{\alpha \beta},$$

$$\dot{\kappa}^L_g = -2 \kappa^L_g \ell^{\alpha \beta} a_\beta \cdot \dot{a}_\alpha + \lambda^{-1} (\ell^{\gamma \delta} L^{\gamma}_{\alpha \beta} - \ell^{\delta \gamma} L^{\delta}_{\alpha \beta}) c_\beta \cdot \dot{a}_\alpha.$$ \hspace{1cm} (229)

With this, inserting (227) into (106) gives

$$\dot{w}_{\text{int}} := \sigma^{\alpha \beta} a_\beta \cdot \dot{a}_\alpha + M^{\alpha \beta} b_\alpha + \bar{\mu} \ell^{\alpha \beta} c_\gamma \dot{S}^\gamma_{\alpha \beta},$$ \hspace{1cm} (230)

where $\sigma^{\alpha \beta}$ is defined by Eq. (120). It is valid for initially curved fibers. For initially straight fibers, $\kappa^L_g$ in (227) vanishes, so that $\sigma^{\alpha \beta}$ becomes

$$\sigma^{\alpha \beta} := \overline{\sigma}^{\alpha \beta} + \mu (\kappa^L_g - \kappa^\Gamma_g) \ell^{\alpha \beta} + \bar{\mu} \left[ (\ell^{\gamma \delta} S^\gamma_{\alpha \beta}) c_\beta - (\ell^{\delta \gamma} S^\delta_{\gamma \beta} \ell^{\theta}_{\alpha \beta}) \ell^{\alpha} \ell^{\beta} \right],$$ \hspace{1cm} (231)

which is unsymmetric for general materials. The asymmetry is due to the fact that the derivative $\dot{c}_\alpha$ of fiber director components $c_\alpha$ appearing in (228.1) contains not only surface stretching, but also in-plane bending. In other words, $\dot{c}_\alpha \neq \frac{\partial c_\alpha}{\partial a_{\beta \gamma}} \dot{a}_{\beta \gamma}$.

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