The Logic of Consistent Histories: A Reply to Maudlin

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Abstract

The relationship between quantum logic, standard propositional logic, and the (consistent) histories rules for quantum reasoning is discussed. It is shown that Maudlin’s claim [1], that the histories approach is inconsistent, is incorrect. The histories approach is both internally consistent and adequate for discussing the physical situations considered by Maudlin.

Contents

1 Introduction
2 Logic and Quantum Mechanics
  2.1 Propositional logic
  2.2 Quantum Logic
  2.3 Histories Logic and Its Consistency
  2.4 Role of Logic in Quantum Theory
3 Spins Prepared in Boxes
4 GHZ Paradox
5 Conclusion

1 Introduction

The argument for the locality of quantum mechanics in [2] based on the (“consistent” or “decoherent”) histories interpretation has been criticized by Maudlin [1] who claims that the histories approach is inconsistent. It will be shown here that Maudlin’s arguments are incorrect: the histories approach is both internally consistent as a form of quantum logic, and also an appropriate tool for consistently analyzing the two specific physical situations mentioned in [1]. Naturally, a system of reasoning that is internally consistent may not be appropriate or adequate when applied to a particular problem, and it could be that Maudlin’s complaint about inconsistency is really about, or perhaps includes, something which might better be called inadequacy. In what follows some attempt will be made to address adequacy along with consistency in the narrower sense, though the emphasis will be on the latter.

In Sec. 2 a brief comparison is made between (classical) propositional (or sentential) logic, the quantum logic first proposed by Birkhoff and von Neumann [3], and the histories approach, with particular attention on the consistency of these three systems. Following that, Secs. 3 and 4 address two physical situations mentioned in [1]: spins prepared and placed in labeled boxes, and, briefly, the GHZ paradox. Section 5 contains a short summary.

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2 Logic and Quantum Mechanics

2.1 Propositional logic

In discussing the consistency of the histories approach to quantum mechanics it is helpful to first review a few features of formal logic. We need only consider propositional (sentential) logic\footnote{This branch of logic is discussed in many books. A readable (albeit longwinded) internet resource will be found at \cite{4}.} not the more complicated (first-order) predicate logic. The rules of the former are of two types. First come rules for putting together formulas (also called sentences or statements or propositions) in a meaningful way using logical connectives. Thus $\neg A$ (read as “NOT A”), $A \land B$ (“$A$ AND $B$”), and $A \lor B$ (“A OR $B$”) are allowed by these rules, provided $A$ and $B$ (think of them as assertions which could be true or false) are themselves propositions or symbols in an approved list. A combination such as $A \land \neg B$ is not allowed; it is not meaningful. Sometimes one refers to expressions which have been properly constructed as “well-formed formulas.” Second come rules that govern logical inference: the validity of an argument which begins with a set premises, each of which must itself be a well-formed formula, and ends in a conclusion, also a well-formed formula. The basic idea behind the rules is to make sure that the truth of the conclusion follows from the truth of (at least some of) the premises, based solely on the logical structure of the argument; i.e., that the argument is valid.

Logicians are not in the business of guaranteeing the truth of the premises that enter an argument; instead their goal is identify valid arguments. For example, from the truth of $A \land B$ as a premise it follows as a conclusion that $A$ is true, but this is not so if the premise is, instead, $A \lor B$.

A system of propositional logic is considered inconsistent if its rules allow one to reason to the truth of both a proposition and its negation, to the truth of $A$ and of $\neg A$, from the same premises. Standard propositional logic is known to be consistent provided one follows the rules worked out for it by logicians. The consistency of alternative logics must, of course, be assessed on the basis of their own rules.

2.2 Quantum Logic

Quantum logic was discussed briefly in \cite{2} and \cite{1}; see \cite{5} for an accessible treatment with more details. Propositions such as $A$ and $B$ correspond to subspaces of a complex Hilbert space, which for present purposes can be considered finite dimensional. Negation $\neg A$ corresponds to the orthogonal complement of a subspace, $A \land B$ to the intersection $A \cap B$ of two subspaces, and $A \lor B$ to their direct sum $A \oplus B$. The resulting logic resembles but is not identical to standard propositional logic; in particular the distributive laws of the latter no longer hold for quantum logic. This has as a consequence the fact that if one insists on applying the rules of (standard) propositional logic to the formulas of quantum logic it is possible to derive contradictory results. See, for example, the discussion of Eq. (2) in \cite{2}. This does not imply that quantum logic is inconsistent: its consistency must be checked using its own rules, and assessed by these rules it provides a consistent system of reasoning \cite{6}.

It is worth noting that whereas the connectives $\land$ and $\lor$ are defined differently in quantum and in ordinary propositional logic, the two are closely connected in the following sense. If the projectors $A$ and $B$ for two subspaces commute, $AB = BA$, then (using the same symbols for a space and its projector) $A \land B$ and $A \lor B$ behave very much like their standard counterparts, both formally and intuitively, when understood as representing quantum properties. Indeed, if one starts with any set of commuting projectors and repeatedly applies the quantum logical connectives (including $\neg$) to these projectors and to those that result from earlier applications, the result is a collection of projectors that continue to commute with each other and form what is called a Boolean algebra (or Boolean lattice). All the ideas of propositional logic then apply without any change to this “commuting fragment” (as we might call it) of quantum logic. In brief, as long as one fixes one’s attention on a collection of commuting projectors, quantum logic is the same as standard propositional logic; differences only appear when projectors fail to commute. And since the physical properties considered in classical mechanics correspond, when viewed quantum mechanically, as a collection of commuting projectors, see Sec. 26.6 of \cite{7}, there is, contrary to \cite{1}, a very good reason to refer to ordinary propositional logic as “classical logic” when distinguishing it from alternatives used in quantum theory.

2.3 Histories Logic and Its Consistency

The logic used in histories quantum mechanics, as explained in Sec. II of \cite{2}, employs subspaces (equivalently, their projectors) and negation $\neg$ in exactly the same way as in quantum logic. The crucial difference
between the two is the single framework rule of the histories approach. A framework consists of a set of commuting projectors that form a Boolean algebra. Within such a framework the connectives are the same as those used in quantum logic, and as noted above, correspond exactly to those in ordinary propositional logic as it is employed in classical physics. The single framework rule states that in forming propositions only projectors (equivalently, quantum properties or subspaces) belonging to a fixed framework may be employed, and in constructing a logical argument all the premises and also the conclusion must belong to the same framework. This rule is best thought of as analogous to the rules of propositional logic that define meaningful propositions; what it excludes are the quantum counterparts of things like \( A \land \lor B \).

As a consequence of the single framework rule, in all situations in which histories logic permits the use of connectives their meaning coincides with the same symbols whether used in quantum or in (classical) propositional logic, a point overlooked in [1]. Its logical rules, always applied within the given framework (but remember that combinations using different frameworks are not allowed by these rules) thus coincide with those of quantum logic as applied in this limited context, and also with the usual rules of standard propositional logic. Consequently, a properly constructed argument that follows the rules of histories quantum mechanics, and is thus necessarily restricted to a single framework, has precisely the same structure as an argument in standard propositional logic, and can never lead to a contradiction. Any proof of the consistency of propositional logic applies equally to the logic used in consistent histories. In particular, Maudlin’s arguments in [1] for the consistency of classical logic imply the consistency of the histories approach, contradicting his claim that the latter is inconsistent.

Note that, as discussed in more detail in [2], the single framework rule is not at all a prohibition against the physicist constructing various distinct and mutually incompatible frameworks in the course of analyzing some quantum mechanical problem. What the rule forbids is combining conclusions from different frameworks. It is this that prevents the inference of logical contradictions, and ensures the consistency of this kind of quantum reasoning. For more details, including the consistency of arguments involving time development (for which there are additional complications, though the basic principles are the same) see Ch. 16 of [7].

### 2.4 Role of Logic in Quantum Theory

Proponents of quantum logic and proponents of the histories approach do not think of these as replacements for standard propositional logic in terms of everyday reasoning, or the logic required to write scientific papers, both of which are in any case only partially represented in propositional logic. Instead, the idea is that when dealing with specifically quantum phenomena one needs a mode of reasoning for a situation for which ordinary logic is inadequate. With reference to quantum logic see the very helpful discussion at the beginning of [8]. Of course, if the whole world is quantum mechanical it is reasonable to expect propositional logic as applied in classical physics to emerge from an appropriate quantum logic as an adequate approximation to the latter in those situations in which classical mechanics provides a good approximation to quantum mechanics. And there is a sound basis for this expectation in the fact that, as noted above, both quantum and histories logic are identical with propositional logic, formally and in terms of their intuitive contents, in physical situations where one is only interested in a Boolean subalgebra of quantum projectors.

Thus both quantum logic and histories logic are best viewed as extensions of ordinary propositional logic to a domain in which quantum phenomena are important, where one might plausibly expect that modifications of the traditional rules of reasoning are required. Just as, to use an analogy, the rules of ordinary arithmetic for products of numbers, where the order does not matter, have to be modified if one is considering noncommuting operators. This point seems to have been overlooked by Maudlin in [1]. What he refers to as “Bell’s logic” can be impeccable but still fail to apply to something in the quantum domain if for the latter a different mode of reasoning is more appropriate. The issue is whether such an alternative system of reasoning can clear up the the well-known conceptual difficulties of quantum theory, not whether its rules agree with propositional logic. Bell was himself well aware [9] that he had not solved the measurement problem, and a similar humility might be appropriate on the part of others who are in the same situation.

We are now in a position to respond to objections by Maudlin [1] to the histories approach framed in terms of two specific physical situations.

### 3 Spins Prepared in Boxes

According to the histories approach the question “Is \( S_z = +1/2 \) or is \( S_z = +1/2? \)” referring to a particular spin-half particle at a particular moment in time is meaningless because the projectors, which we
shall denote by \(|x^+\rangle = |x^+\rangle\langle x^+| \) and \(|z^+\rangle\), referring these two possibilities do not commute. Maudlin claims that this represents an inconsistency in histories quantum mechanics. Imagine, he says, spin half particles prepared either in the state \(S_x = +1/2\) or in the state \(S_x = +1/2\), and placed in boxes with each box carrying a label indicating the preparation procedure. If a label falls off a box it is, he asserts, obviously meaningful to ask the question whether the particle in this particular box is in the state \(S_x = +1/2\) or \(S_x = +1/2\).

To see what is problematical about this assertion, ask the question: “How might we distinguish by means of a measurement whether the particle in this box is in the state \(S_x = +1/2\) or in the state \(S_x = +1/2\)?” All students of quantum mechanics know that no measurement can give a definite answer to this question. If, for example a measurement of \(S_z\) is carried out by means of a Stern-Gerlach apparatus and the result is \(+1/2\), this may be because the particle was prepared in the state \(S_z = +1/2\), but even if it was prepared in the state \(S_x = +1/2\) there is a probability of \(1/2\) that an \(S_z\) measurement will yield \(+1/2\) rather than \(-1/2\).

One does not have to be a logical positivist to be suspicious about a supposedly obvious distinction which cannot be subjected to experimental test. Should we perhaps distinguish between a particle being in a state with some definite value of some component of spin, and its being prepared in such a state, even in a situation in which there is no magnetic field acting to cause spin precession after the measurement? Indeed we should, and the histories approach provides the tools needed to make this distinction, through its concept of a dependent or contextual state, Ch. 14 of [7]. Suppose the particle was prepared in the state \(|x^+\rangle\), \(S_x = +1/2\), and a record, Maudlin’s label, was made. We can represent the situation by a quantum projector \(|x^+\rangle \otimes L_x\), where \(L_x\) projects on the quantum state of the label. Similarly \(|z^+\rangle \otimes L_z\) is the state of the particle-plus-label corresponding to a preparation of \(S_z = +1/2\). Since \(L_x\) and \(L_z\) refer to distinct macroscopic states, \(L_x L_z = 0\) (at least to an excellent approximation). Consequently, \(|x^+\rangle \otimes L_x\) and \(|z^+\rangle \otimes L_z\) are also orthogonal. Thus in the histories approach \(|x^+\rangle \otimes L_x\) OR \(|z^+\rangle \otimes L_z\)” makes sense, even though “\(|x^+\rangle\) OR \(|z^+\rangle\)” does not.

Thus quantum mechanics itself, even in the inadequate presentation found in current textbooks, with measurements an unanalyzed black box, undermines Maudlin’s claim that his spin-half example demonstrates something wrong with the histories approach. And it is precisely the histories approach that provides the sorts of distinctions, along with the logical tools needed to make sense of them, required for a fully quantum-mechanical analysis of preparations, measurements, and microscopic quantum states.

4 GHZ Paradox

For an accessible account of the Greenberger-Horne-Zeilinger or GHZ paradox [11] see the article by Mermin [12] that appeared shortly afterwards, or his later review paper [13]. As discussed in [1], it involves three spin-half particles in a particular quantum state, at three separate locations so that they do not interact with each other, each of which can be subjected to two types of measurement, of either \(S_x\) or \(S_y\). The measurement apparatuses and the means of deciding which spin component is to be measured are located close to the respective particles and sufficiently far apart (e.g., at spacelike separation in the relativistic sense) that they cannot influence each other. If one assumes that each measurement reveals a pre-existing property, e.g., \(S_y = +1/2\) for particle 2, then it is impossible to assign simultaneous \(S_x\) and \(S_y\) values to each particle in such a way that the measurement statistics will agree with those predicted by quantum theory for the specified initial quantum state. This is the paradox. Maudlin claims that any local theory must arrive at this paradox, and hence a theory which agrees with the predictions of quantum mechanics for the statistics of measurement outcomes is necessarily nonlocal.

The histories approach does not allow for simultaneous assignment of \(S_z\) and \(S_y\) values to a single particle, and thus blocks this route to a paradox by applying its single framework rule. Maudlin acknowledges this when he asserts (Sec. V) that “the single framework rule is supposed to prevent us from making this calculation,” by which he means the calculation that leads to the paradox. Indeed, the single framework

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2Maudlin at the end of Sec. II of [1] incorrectly replaces noncommuting projectors with the condition that the projectors have no common eigenstate. The two are not equivalent in a Hilbert space of dimension 3 or more. However, the distinction is not critical for the following discussion.

3While no measurement on a single particle can reliably distinguish \(S_x = +1/2\) from \(S_x = +1/2\), the situation is different if a large number \(N\) of particles are either (i) all prepared in the same state \(S_z = +1/2\), or (ii) all prepared in the same state \(S_x = +1/2\). When \(N\) is large these two situations can be reliably distinguished, i.e., with little probability of error, because the quantum states corresponding to (i) and (ii) in the total tensor product Hilbert space of dimension \(2^N\) are nearly orthogonal, and thus distinguishable, even though for a single particle the states \(|x^+\rangle\) and \(|z^+\rangle\) are very far from being orthogonal.

4For more on these topics, see [10].
rule does precisely that, and in this respect the histories approach is perfectly consistent according to its own rules. Maudlin has not located any inconsistency in the histories approach. Instead he is pointing out that its rules are different from those he wishes to apply to the problem at hand. His claim that the histories approach is inconsistent is like claiming that the use of Riemannian geometry in general relativity is inconsistent because its rules are different from those of Euclidean geometry.

Many other issues could be discussed with reference to the GHZ paradox, such as the relationship of measurement outcomes, represented in proper quantum mechanical fashion, to prior states of the measured particles. They require the use of specific frameworks, and since there are a large number of possibilities their analysis lies outside the scope of the present paper. The interested reader may wish to look at the very detailed analysis, using the histories approach, of Hardy’s paradox, which poses essentially the same conceptual issues (and has led to similar claims of nonlocality) as does the GHZ paradox, in Ch. 25 of [7].

5 Conclusion

Maudlin’s claim in [1], that the histories approach discussed in [2] is inconsistent, is incorrect. He has not located any inconsistency in the logical rules employed in the histories approach. Instead, his own arguments for the consistency of propositional logic imply the consistency of the logic of histories, as explained in Sec. 2.3. What he appears to find objectionable is that the histories rules are not identical to those of classical propositional logic. This difference, however, is not grounds for declaring the histories rules inconsistent. The rules of the older quantum logic also differ from those of classical logic, something to which Maudlin does not seem to object. The consistency of each system needs to be evaluated by its own rules, not by whether it can be combined with something else. Naively mixing the rules of quantum logic with those of classical logic can lead to inconsistencies, and the same is true if one mixes the histories rules with the classical rules.

In support of his claim for the inconsistency of the histories approach Maudlin uses two examples: spins in boxes, and the GHZ paradox. His analysis of the former is flawed, for he does not properly distinguish the preparation of quantum states from their measurement. It is the histories approach that provides the precise analysis needed to make sense of this situation. In the case of GHZ, Maudlin correctly notes that the histories approach blocks the attempt to construct a paradox. In doing so it (once again) pulls the rug out from under an attempt to demonstrate nonlocality in the quantum world. But that does not indicate any inconsistency in the histories analysis.

The main problem with quantum logic is not any internal inconsistency, or that it leads to results that contradict the empirically well-established formulas of textbook quantum theory. Instead, it is that this approach has not managed to resolve the conceptual difficulties of quantum mechanics, something even even its proponents admit; for a recent assessment see [8].

The histories approach is also internally consistent, and agrees with all experiments that confirm the validity of quantum theory. It provides a fully quantum-mechanical description of how physical measurements actually work, thus resolving the measurement problem. In addition it can handle numerous paradoxes, see Chs. 20 to 25 of [7], that cause difficulties for other interpretations of quantum mechanics. Its success in resolving these problems suggests that it deserves much more serious study on the part of physicists and philosophers interested in quantum foundations than it has hitherto received. Needless to say, such study must begin with a careful effort to understand the rules of reasoning used in the histories approach and how these are applied to various examples. There may be fundamental inconsistencies in the histories approach, and the community would benefit from having them pointed out. However, those who suspect that something is amiss might wish to first pay attention to previous claims of this sort, and the manner in which they have been refuted; see Sec. 8.2 of [10] for references.

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5This analogy was used by Putnam [14] when he was advocating the use of quantum logic.
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