AN ENERGY GAP PHENOMENON FOR THE WHITNEY SPHERE

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Abstract. For an immersed Lagrangian submanifold, let \( \hat{A} \) be the Lagrangian trace-free second fundamental form. In this note we consider the equation \( \nabla^* T = 0 \) on Lagrangian surfaces immersed in \( \mathbb{C}^2 \), where \( T = -2\nabla^* (\hat{A}, \omega) \), and we prove a gap theorem for the Whitney sphere as a solution to this equation.

1. Introduction

Gap phenomena are common in differential geometry hence are classical and prolific throughout the literature till now. Generally speaking, there are two categories of gap phenomena. One of them includes those rigidity theorems in submanifolds theory and we can take some examples from [1,6,8]. Another type occurs when we conduct the blow-up analysis for geometric flows, such as the well known Sacks-Uhlenbeck energy gap result in [12] for harmonic maps. The method to deal with the second type is usually evolved from PDE combined with related Bochner identity.

Let \( \Psi \) be any given differential operator that acts on immersions between two manifolds, consider a tensor field \( T \) on domain manifold. Then there exists a universal constant \( \epsilon > 0 \) such that \( T = 0 \) if \( \| T \|_{L^2} \leq \epsilon \) and \( \Psi = 0 \).

In this paper we will establish a gap theorem for Lagrangian immersions (see section 2 as definition). The following result of Kuwert-Schätzle’s Gap lemma for the Willmore flow in [7] (or see Bernard-Rivière’s alternative proof in [3]) is our main motivation:

**Theorem.** Let \( f : \Sigma \to \mathbb{R}^n \) be a properly immersed (compact or non-compact) Willmore surface, and let \( \Sigma_\rho(0) := f^{-1}(B_\rho(0)) \). Then there exists \( \epsilon_0(n) > 0 \) such that if

\[
\liminf_{\rho \to \infty} \frac{1}{\rho^n} \int_{\Sigma_\rho(0)} |\hat{A}|^2 d\mu = 0
\]

and

\[
\int_{\Sigma} |\hat{A}|^2 d\mu < \epsilon_0(n),
\]

then \( f \) is an embedded plane or sphere.

The small energy condition here is quite natural in the sense of variation since they can be interpreted geometrically by stating that the deviation of the immersion from being simplest geometric objects such as planes and spheres is sufficiently small in an averaged sense. If we turn to the Lagrangian geometry case, we may consider the objects being the Lagrangian planes, the Clifford torus

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and the Whitney sphere. Luo and Wang have already considered the case when the second fundamental form is small and they obtained the following results:

**Theorem.** Let $l : \Sigma \to \mathbb{C}^2$ be a properly immersed HW surface, then there exists $\epsilon_0(n) > 0$ such that if $\|A\|_{L^2} < \epsilon_0(n)$, then it must be a Lagrangian plane.

Instead of considering $\tilde{A}$ in the Lagrangian frame, the Lagrangian trace-free second fundamental form $\tilde{A}$ (see definition in section 2) seems to be a better candidate. As our personal interests, we introduce a $(0,2)$-tensor $T := \nabla(H \cdot \omega) - \frac{1}{2} \text{div} JH \cdot g$ and consider the equation $\nabla^* T = 0$. In the point of view of geometric, the tensor $T$ measures the deviation of the mean curvature vector field to be a conformal vector field. We can check by a straightforward calculation that the equation $\nabla^* T = 0$ bears the Whitney sphere as one of its solution, hence it’s reasonable to consider if the Whitney sphere is its only solution given $\tilde{A}$ small enough. We obtain the following result:

**Theorem 1.1.** Assume $f : \Sigma \to \mathbb{C}^2$ is a properly immersed Lagrangian surface (compact or non-compact) such that $\nabla^* T = 0$, given $\gamma \in C^1_c(\Sigma)$ is a cut-off function satisfies $|\nabla \gamma| \leq \frac{C \gamma}{R}$, then there exists $\epsilon_0 > 0$ such that if

$$\int_{\{\gamma > 0\}} |\tilde{A}|^2 d\mu \leq \epsilon_0,$$

then we have

$$\int_{\Sigma} (|\nabla \tilde{A}|^2 + |H|^2 |\tilde{A}|^2) \gamma^2 d\mu \leq \frac{C}{R^2} \int_{\{\gamma > 0\}} |\tilde{A}|^2 d\mu.$$

As a corollary we obtained the following gap results:

**Corollary 1.1.** Assume $f : \Sigma \to \mathbb{C}^2$ is a properly immersed Lagrangian surface (compact or noncompact) satisfies $\nabla^* T = 0$, and let $\Sigma_\varrho(0) = f^{-1}(B_\varrho(0))$, then there exists $\epsilon_0 > 0$ such that if

$$\int_{\Sigma} |\tilde{A}|^2 d\mu \leq \epsilon_0 \quad \text{and} \quad \liminf_{\varrho \to \infty} \frac{1}{\varrho^2} \int_{\Sigma_\varrho(0)} |A|^2 d\mu = 0,$$

then $f$ is a Lagrangian plane or the Whitney sphere.

We expect that there is a gap theorem for the Whitney sphere as the HW surface given the smallness of $\tilde{A}$ and our results can be used to provide some idea in proving it in the future. One of the difficulties here is the smallness pf $\tilde{A}$ can’t provide us any information on the mean curvature $H$ which is much different as the standard Euclidean case. Hence instead of writing the Bochner type identity of $\tilde{A}$ in terms of the mean curvature itself, we write it in terms of the tensor $T$ which also allows us utilizing the information of the equation better.

This note is organized as follows: in Section 2 we introduce some elementary notions on Lagrangian submanifolds as well as the Willmore functional. Section 3 is devoted to an curvature estimate for Lagrangian surfaces which is essential for us to get the main gap theorem.

## 2. Preliminary

In this section, we will introduce some elementary notions in the Lagrangian geometry and the Willmore functional. Let $\mathbb{C}^2 = \mathbb{R}^4$ be the 2-dimensional complex
plane with standard metric $ds^2 = dx_i^2 + dy_i^2$ and the standard symplectic structure
$\omega = dx_i \wedge dy_i$ in complex coordinates $(z_1, z_2)$, where $z_i = x_i + \sqrt{-1}y_i$, $i = 1, 2$.

**Definition 2.1.** Let $\Sigma$ be a surface in $\mathbb{C}^2$, with tangent and normal bundles, $T\Sigma$ and $N\Sigma$, respectively. Then $\Sigma$ is Lagrangian if and only if one of the following equivalent conditions holds:

1. $\omega$ restricted to $\Sigma$ is zero,
2. $JT\Sigma = N\Sigma$, where $J$ is the standard complex structure on $\mathbb{C}^2$,
3. Liouville form $\alpha = \frac{1}{2}(x_i dy_i - y_i dx_i)$ restricted to $\Sigma$ is closed.

**Definition 2.2.** An immersion $l$ from a surface $\Sigma$ into $\mathbb{C}^2$ is called a Lagrangian immersion if $l^* \omega = 0$.

The simplest Lagrangian surfaces of $\mathbb{C}^2$ are the totally geodesic ones, i.e. the Lagrangian subspaces or planes, another important family of such immersion are the Whitney spheres.

**Example 2.1** (Whitney immersion in $\mathbb{C}^2$).

\[
\psi : \mathbb{S}^2 \longrightarrow \mathbb{C}^2
\]

\[
(x_1, x_2, x_3) \longmapsto \frac{r}{1 + x_3^2}(x_1, x_1 x_3, x_2, x_2 x_3) + \overrightarrow{C}
\]

is a family of Lagrangian immersion. Here we embed $\mathbb{S}^2$ into $\mathbb{R}^3$ with center at the origin. The image of $\Phi$ in $\mathbb{C}^2$ is called the Whitney sphere $\mathbb{S}_W$, and the constants $r$ and $\overrightarrow{C}$ will be referred as the radius and the center receptively.

In view of topology, there is no embedded sphere in $\mathbb{C}^2$ as a Lagrangian submanifold. Whitney spheres have the best possible behavior among them because their only non-embedding points are the poles. On the other hand their second fundamental forms also have some simple symmetric property:

\[
A(v, w) = \frac{1}{4}\{\langle v, w \rangle H + \langle Jv, H \rangle Jw + \langle Jw, H \rangle Jv\}.
\]

Denote

\[
\hat{A}(v, w) := A(v, w) - \frac{1}{4}\{\langle v, w \rangle H + \langle Jv, H \rangle Jw + \langle Jw, H \rangle Jv\}.
\]

We call $\hat{A}$ the Lagrangian trace-free second fundamental form. Several characterizations results have been proven by Castro-Urbano [4] and Ros-Urbano [11]: The Whitney sphere has least Willmore energy in the class of Lagrangian spheres.

Precisely, $W(\Sigma) \geq W(\mathbb{S}_W)$ for all Lagrangian sphere $\Sigma$ and equality holds if and only if $\Sigma$ is a Whitney sphere. Moreover they proved that $\Sigma$ is a Whitney sphere if and only if $\hat{A} = 0$ which is similar to the classical umbilical case in $\mathbb{R}^{n+1}$ which says $\hat{A}(\Sigma) = 0$ if and only if $\Sigma$ is standard sphere $\mathbb{S}^n$. Under this point of view we would like to call $\hat{A} = 0$ the Lagrangian umbilical condition while the Whitney sphere plays a similar role of the round sphere in Lagrangian surfaces. In order to introduce variational problem for Lagrangian submanifolds, we recall there are two types of variational vector fields:

**Definition 2.3.** Let $V$ be a smooth vector field along some Lagrangian surface, then $l : \Sigma \to \mathbb{C}^2$.
(1) $V$ is called a Lagrangian vector field if the associated 1-form $V^\sharp := l^*(V \cdot \omega)$ is closed.

(2) $V$ is called a Lagrangian vector field if $V^\sharp$ is exact.

For immersed surfaces $f : \Sigma \to \mathbb{R}^n$, the Willmore functional is defined as

$$(2.3) \quad W(f) = \int_{\Sigma} |\mathring{\mathcal{A}}|^2 d\mu,$$

where $\mathring{\mathcal{A}} := A - \frac{1}{2} g \otimes H$ denotes the trace-free part of the second fundamental form $A = (D^2 f)^\perp$ and $\mu$ is the induced area measure from the target manifold $\mathbb{R}^n$ by $f$. For any normal vector field $\phi$, we recall the Laplace operator on the normal bundle as $\Delta^\perp \phi = -\nabla^\perp \cdot \nabla^\perp \phi$, where $\nabla^\perp \phi = (D_X \phi)^\perp$ is the covariant derivative and $\nabla^\perp^* \phi := -e_i \nabla^\perp e_i \phi$ is the adjoint covariant derivative on the normal bundle. Without further notice, we will use $\nabla$ to simplify $\nabla^\perp$ in the following. Then the Euler-Lagrange operator of $(2.3)$ is

$$(2.4) \quad W = \Delta^\perp H + Q(\mathring{\mathcal{A}}) H,$$

by some calculations, where $H = g^{ij} A_{ij}$ is the mean curvature vector field while $Q(\mathring{\mathcal{A}})$ given by the formula:

$$(2.5) \quad Q(\mathring{\mathcal{A}}) \phi := \mathring{\mathcal{A}}(e_i, e_j)(\mathring{\mathcal{A}}(e_i, e_j), \phi),$$

where we use Einstein summation convention and g-orthonormal basis $\{e_1, e_2\}$.

Now if we consider the Willmore functional for Lagrangian surfaces, we shall search the minimizer for $(2.3)$ among all Lagrangian immersions from $\Sigma$ to $\mathbb{R}^n$. Since we have two kinds of ways of variation by definition 2.3, there will be two kinds of minimizers which are called LW-surface and HW-surface respectively in Luo-Wang [9].

**Definition 2.4.** Let $l : \Sigma \to \mathbb{C}^2$ be a smooth Lagrangian immersion, then

(1) $l$ is called a Lagrangian stationary Willmore surface if it’s a critical point of the Willmore functional under compactly supported Lagrangian variations,

(2) $l$ is called a Hamiltonian stationary Willmore surface if it’s a critical point of the Willmore functional under compactly supported Hamiltonian variations.

It’s straightforward to obtain Euler-Lagrange equations for them.

(i) $l : \Sigma \to \mathbb{C}^2$ is a LW-surface if

$$(2.6) \quad P(W(l)^\sharp) = 0,$$

holds, where $P$ is the projection map from the space of 1-forms $\Omega^1(\Sigma)$ to the space of closed 1-forms $Z^1(\Sigma)$.

(ii) $l : \Sigma \to \mathbb{C}^2$ is a HW-surface if

$$(2.7) \quad d^*(W(l)^\sharp) = 0,$$

holds, where $d^*$ is induced by the metric on $l$.

3. **Estimates for Lagrangian surfaces with locally small $L^2$-norm of $\mathring{\mathcal{A}}$**

3.1. **Preparations.** We will derive some elementary formulas first in this section. Denote $h_i = (H, J e_i) = H \omega(e_i)$ and $\mathring{A}_{ijk} = (\mathring{A}(e_i, e_j), J e_k) = \mathring{A} \omega(e_i, e_j, e_k)$ in local normal coordinates, we have to point out that it doesn’t matter which
two of the three indices of $\hat{A}$ are contracted here thanks to its full symmetric property under Lagrangian settings:

$$\hat{A}_{ijk} = \langle De_i e_j, Je_k \rangle = -\langle e_j, JD_{e_i} e_k \rangle = \langle Je_j, De_i e_k \rangle = \hat{A}_{ikj}.$$  

**Proposition 3.1.**

\begin{align}
|\hat{A}|^2 &= |A|^2 - \frac{3}{4}|H|^2, \\
-2\nabla^* (\hat{A}_{\omega}) &= \nabla (H_{\omega}) - \frac{1}{2} \text{div} JH \cdot g.
\end{align}

**Proof.** By definition of $\hat{A}$,

$$|\hat{A}|^2 = \hat{A}_{ijk} \hat{A}^{ijk} = [A_{ijk} - \frac{1}{4}(\delta_{ij} h_k + \delta_{ik} h_j + \delta_{jk} h_i)]^2 = A_{ijk} A^{ijk} - \frac{3}{4} h_i h^i,$$

and by Codazzi property of second fundamental form $A$ and mean curvature $H$ (or $H_{\omega}$) we have

$$-\nabla^* (\hat{A}_{\omega})(e_i, e_j) = \hat{A}_{ijl, l} = A_{ijl, l} - \frac{1}{4}(\delta_{ij} \text{div} JH + h_{i,j} + h_{j,i})$$

$$= A_{ilj, l} - \frac{1}{4}(\delta_{ij} \text{div} JH + 2h_{i,j})$$

$$= h_{i,j} - \frac{1}{4}(\delta_{ij} \text{div} JH + 2h_{i,j}).$$

□

The next lemma is on the relationship of $T$ and Willmore functional.

**Lemma 3.1.** If $f : \Sigma \rightarrow \mathbb{C}^2$ is a properly immersed Lagrangian surface,  

$$W = \Delta H + \frac{1}{8}|H|^2 H + Q(A) H + \hat{A}(JH, JH).$$

Furthermore we have dual version in the sense of symplectic form $\omega$:

\begin{equation}
W_{\omega} = -2\nabla^* T + \frac{1}{2}|\hat{A}|^2 H_{\omega} + Q(A) H + \hat{A}_{\omega}(JH, JH).
\end{equation}

**Proof.** By (2.4) and (2.5), we have

$$Q(A)$$

$$= A(e_i, e_j) \langle A(e_i, e_j), H \rangle$$

$$= [\hat{A}_{ijh} + \frac{1}{4}(\delta_{ij} h_k + \delta_{ik} h_j + \delta_{jh} h_i)] [\hat{A}_{js} + \frac{1}{4}(\delta_{js} h_k + \delta_{is} h_j + \delta_{js} h_i)] \langle Je_s, H \rangle J e_h$$

$$= \hat{A}_{ijh} \langle A(e_i, H) \langle J e_s, H \rangle J e_h + \hat{A}_{ijh} h_i h_j J e_h + \frac{5}{8}|H|^2 H$$

$$= Q(\hat{A}) H + \hat{A}(JH, JH) + \frac{5}{8}|H|^2 H.$$
On the other hand, we use Ricci identity, Gauss equation and codazzi property to obtain:

\[ \nabla^* T(e_i) = -T_{ki,k} \]

\[ = -h_{i,kk} + \frac{1}{2} \delta_{ki} h_{l,lk} \]

\[ = -h_{i,kk} + \frac{1}{2} (h_{i,kl} + h_s R_{l|l}^s) \]

\[ = -\frac{1}{2} \Delta(H, \omega)(e_i) - \frac{K}{2} h_i \]

\[ = -\frac{1}{2} \Delta(H, \omega)(e_i) - \frac{1}{2} \left| H^2 - \frac{1}{2} \omega^2 \right| H, \omega(e_i), \]

where \( K \) is the Gaussian curvature of the surface. Substitute with the above two formulas in (2.4) we complete the proof. \( \square \)

**Proposition 3.2** (A Bochner type identity). If \( f : \Sigma \to \mathbb{C}^2 \) is a properly immersed Lagrangian surface, then under local normal coordinates,

\[
(3.4) \quad \dot{A}_{ijk,mm} = 3K \dot{A}_{ijk} + \frac{T_{ij,k} - \frac{1}{2} \delta_{ij} T_{km,m}}{3} + \frac{T_{jk,i} - \frac{1}{2} \delta_{jk} T_{im,m}}{3} + \frac{T_{ik,j} - \frac{1}{2} \delta_{ik} T_{jm,m}}{3}
\]

**Proof.** Using the Ricci identity and the relationship between \( T \) and the mean curvature \( H \) we have

\[
\dot{A}_{ijk,mm} = \dot{A}_{ijm,km} + \frac{1}{4} (\delta_{ij} h_{m,km} - \delta_{ij} h_{k,mm} + \delta_{im} h_{j,km} - \delta_{ik} h_{j,mm})
\]

\[ + \delta_{jm} h_{i,km} - \delta_{jk} h_{i,mm} \]

\[ = \dot{A}_{ijm,km} + \frac{\dot{A}_{ijm} R_{km}^l}{4} + \frac{\dot{A}_{ijm} R_{jkm}^l}{4} + \frac{\dot{A}_{ijm} R_{km}^l}{4} + \frac{1}{4} (h_{j,km} + h_{i,km}) \]

\[ - \delta_{jk} h_{i,mm} \]

\[ = \dot{A}_{imm,jk} + \dot{A}_{imj,km} + \dot{A}_{imj} R_{km}^l + \dot{A}_{imj} R_{jkm}^l + \dot{A}_{imj} R_{km}^l + \frac{1}{4} (h_{j,km} + h_{i,km}) \]

\[ - \delta_{jk} h_{i,mm} \]

\[ = \dot{A}_{ijkl} (\delta_{ik} \delta_{lm} - \delta_{im} \delta_{jk}) + \dot{A}_{ijkl} (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jk}) + \dot{A}_{ijl} (\delta_{ik} \delta_{lm} - \delta_{im} \delta_{jk}) + \dot{A}_{iij} (\delta_{ik} \delta_{lm} - \delta_{im} \delta_{jk}) \]

\[ - \delta_{im} \delta_{km} + \frac{1}{4} (h_{j,ki} + h_{i,ij} - \delta_{ik} h_{j,m} - \delta_{jk} h_{i,m} + \frac{1}{4} (h_{i,jm} + h_{j,im}) \]

\[ - \delta_{jk} h_{i,mm} \]

\[ = \dot{A}_{ijkl} (\delta_{ik} \delta_{lm} - \delta_{im} \delta_{jk}) + \dot{A}_{ijkl} (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jk}) + \dot{A}_{ijl} (\delta_{ik} \delta_{lm} - \delta_{im} \delta_{jk}) + \dot{A}_{iij} (\delta_{ik} \delta_{lm} - \delta_{im} \delta_{jk}) \]

\[ - \delta_{im} \delta_{km} + \frac{1}{4} (h_{j,ki} + h_{i,ij} - \delta_{ik} h_{j,m} - \delta_{jk} h_{i,m} + \frac{1}{4} (h_{i,jm} + h_{j,im}) \]

\[ - \delta_{jk} h_{i,mm} \]

\[ = 3K \dot{A}_{ijk} + \frac{1}{4} (2h_{i,jk} + h_{i,ki} + h_{i,ij}) - \frac{1}{4} (\delta_{ik} h_{j,mm} + \delta_{jk} h_{i,mm} + \delta_{ij} h_{m,km}) \]

\[ = 3K \dot{A}_{ijk} + \frac{2T_{ij,k} + T_{jk,i} + T_{ik,j}}{4} - \delta_{jk} h_{i,km} - \delta_{ik} h_{j,mm} - \delta_{ij} h_{m,km} \]

\[ = 3K \dot{A}_{ijk} + \frac{2T_{ij,k} + T_{jk,i} + T_{ik,j}}{4} - \delta_{ik} T_{jm,m} - \delta_{jk} T_{im,m} \]
where $\hat{A}_{imn,jk} = 0$ in the 4-th equality due to the fact that $\hat{A}$ is trace-free, and we substituted $h_{i,jk}$ with $(T_{ij,k} + \frac{1}{2}\delta_{ij}h_{l,lk})$ in the 6th equality to get the 7th. Then we apply the method of symmetrization to get (3.4) as desired. □

3.2. Curvature estimates.

**Lemma 3.2.** Assume $f : \Sigma \to \mathbb{C}^2$ is a properly immersed Lagrangian surface (compact or noncompact), cut-off function $\gamma \in C^1_c(\Sigma)$ satisfies $|\nabla \gamma| \leq \frac{C_0}{R}$, then we have:

\[
\int_{\Sigma} (|\nabla \hat{A}|^2 + |H|^2|\hat{A}|^2) \gamma^2 d\mu 
\leq c \int_{\Sigma} \langle \nabla^* T, H \rangle \gamma^2 d\mu + c \int_{\Sigma} |\hat{A}|^4 \gamma^2 d\mu + \frac{C}{R^2} \int_{\{\gamma > 0\}} |A|^2 d\mu.
\]

**Proof.** Multiplying (3.4) by $\hat{A}$ then by integrating we get

\[
\int_{\Sigma} (\Delta \hat{A}, \gamma^2 \hat{A}) d\mu = \int_{\Sigma} \langle \nabla T, \gamma^2 \hat{A} \rangle d\mu + \frac{3}{8} \int_{\Sigma} |H|^2|\hat{A}|^2 \gamma^2 d\mu - \frac{3}{2} \int_{\Sigma} |\hat{A}|^4 \gamma^2 d\mu
\]

For the L.H.S., just integrating by parts:

\[
L.H.S. = - \int_{\Sigma} \langle \nabla^* \nabla \hat{A}, \gamma^2 \hat{A} \rangle d\mu
= - \int_{\Sigma} |\nabla \hat{A}|^2 \gamma^2 d\mu - 2 \int_{\Sigma} \langle \nabla \hat{A}, \gamma \nabla \gamma \otimes \hat{A} \rangle d\mu
\]

now for the first term of R.H.S. we use definition of $T$ and integrate by parts again to get:

\[
\int_{\Sigma} \langle \nabla T, \gamma^2 \hat{A} \rangle d\mu = \int_{\Sigma} \langle T, \gamma^2 \nabla^* \hat{A} \rangle d\mu - 2 \int_{\Sigma} \langle T \otimes \nabla \gamma, \gamma \hat{A} \rangle d\mu
= - \frac{1}{2} \int_{\Sigma} \langle T, T \rangle \gamma^2 d\mu - 2 \int_{\Sigma} \langle T \otimes \nabla \gamma, \gamma \hat{A} \rangle d\mu
= - \frac{1}{2} \int_{\Sigma} \langle T, \nabla H - \frac{1}{2} \text{div} J H \cdot g \rangle \gamma^2 d\mu - 2 \int_{\Sigma} \langle T \otimes \nabla \gamma, \gamma \hat{A} \rangle d\mu
= - \frac{1}{2} \int_{\Sigma} \langle T, \nabla H \rangle \gamma^2 d\mu - 2 \int_{\Sigma} \langle T \otimes \nabla \gamma, \gamma \hat{A} \rangle d\mu
= - \frac{1}{2} \int_{\Sigma} \langle \nabla^* T, H \rangle \gamma^2 d\mu + \int_{\Sigma} \langle T, H \otimes \nabla \gamma \rangle \gamma d\mu - 2 \int_{\Sigma} \langle T \otimes \nabla \gamma, \gamma \hat{A} \rangle d\mu.
\]
Hence we achieve by $|T| \leq c|\nabla \hat{A}|$ and $|\nabla \gamma| \leq \frac{C_0}{R^2}$

\[
\int \Sigma \left( (|\nabla \hat{A}|^2 + \frac{3}{8}|H|^2|\hat{A}|^2) \gamma^2 \right) d\mu
\]

\[
= \frac{1}{2} \int \Sigma \langle \nabla^* T, H \rangle \gamma^2 d\mu + \frac{3}{2} \int \Sigma |\hat{A}|^4 \gamma^2 d\mu - 2 \int \Sigma \langle \nabla \hat{A}, \gamma \nabla \gamma \otimes \hat{A} \rangle d\mu
\]

\[
- \int \Sigma \langle T, H \otimes \nabla \gamma \rangle \gamma d\mu + 2 \int \Sigma \langle T \otimes \nabla \gamma, \gamma \hat{A} \rangle d\mu
\]

\[
\leq \frac{1}{2} \int \Sigma \langle \nabla^* T, H \rangle \gamma^2 d\mu + \frac{3}{2} \int \Sigma |\hat{A}|^4 \gamma^2 d\mu + \frac{C}{R^2} \int_{\gamma>0} |\hat{A}|^2 d\mu + \frac{1}{2} \int \Sigma |\nabla \hat{A}|^2 \gamma^2 d\mu
\]

\[
+ \frac{C}{R^2} \int_{\gamma>0} |H|^2 d\mu
\]

\[
\leq \frac{1}{2} \int \Sigma \langle \nabla^* T, H \rangle \gamma^2 d\mu + \frac{3}{2} \int \Sigma |\hat{A}|^4 \gamma^2 d\mu + \frac{1}{2} \int \Sigma |\nabla \hat{A}|^2 \gamma^2 d\mu + \frac{C}{R^2} \int_{\gamma>0} |\hat{A}|^2 d\mu.
\]

The following theorem from [10] allows us absorbing the highest order term of $\hat{A}$ above:

**Theorem** (Michael-Simon Sobolev inequality). Let $f : \Sigma \to \mathbb{C}^2$ be an immersion and $\nu$ be a non-negative $C^1$ function on $\Sigma$, where $U \subseteq \mathbb{C}^2$ is a domain contains $f(\Sigma)$. Then

\[
\int \Sigma \nu^2 d\mu \leq c \left( \int \Sigma |\nabla \nu| d\mu \right)^2 + c \left( \int \Sigma |H| d\mu \right)^2,
\]

where $H$ is mean curvature vector and $c$ is a constant independent of $f$.

**Lemma 3.3.** Under the same assumption of Lemma 2.1:

\[
\int \Sigma |\hat{A}|^4 \gamma^2 d\mu \leq C \int \Sigma |\hat{A}|^2 d\mu \int \Sigma (|\nabla \hat{A}|^2 + |H|^2|\hat{A}|^2) \gamma^2 d\mu + \frac{C}{R^2} \left( \int_{\gamma>0} |\hat{A}|^2 d\mu \right)^2
\]

holds.

**Proof.** Substitute $\nu = |\hat{A}|^2 \gamma$ in (3.7), we have:

\[
\int \Sigma |\hat{A}|^4 \gamma^2 d\mu
\]

\[
\leq c \left( \int \Sigma |\hat{A}|^2 |\nabla \hat{A}| \gamma d\mu \right)^2 + c \left( \int \Sigma |\hat{A}| |\nabla \gamma| d\mu \right)^2 + c \left( \int \Sigma |\hat{A}|^2 |H| \gamma d\mu \right)^2
\]

\[
\leq C \int \Sigma |\hat{A}|^2 d\mu \int \Sigma (|\nabla \hat{A}|^2 + |H|^2|\hat{A}|^2) \gamma^2 d\mu + \frac{C}{R^2} \left( \int_{\gamma>0} |\hat{A}|^2 d\mu \right)^2.
\]

\]

Combining the above two lemmas we get the following gradient estimate for $\hat{A}$:
Theorem 3.1. Assume $f: \Sigma \to \mathbb{C}^2$ is a properly immersed Lagrangian surface (compact or noncompact), cut-off function $\gamma \in C^1_c(\Sigma)$ satisfies $|\nabla \gamma| \leq \frac{C_0}{R}$, then there exists $\varepsilon_0 > 0$ such that if
\[
\int_{\{\gamma > 0\}} |\dot{A}|^2 d\mu \leq \varepsilon_0,
\]
then we have
\[
\int_{\Sigma} (|\nabla \dot{A}|^2 + |H|^2 |\dot{A}|^2) d\mu \leq C \int_{\Sigma} \langle \nabla^* T, H \rangle \gamma^2 d\mu + \frac{C}{R^2} \int_{\{\gamma > 0\}} |A|^2 d\mu.
\]
As the first application we can deduce the Gap theorem for the equation $\nabla^* T = 0$ with small $\|\dot{A}\|_{L^2(\Sigma)}$ for immersed Lagrangian surface. First we need a classification theorem by Castro-Urbano [4] or Ros-Urbano [11] as following:

Theorem (Lagrangian umbilical surfaces, [4, 11]). Let $\Psi: M \to \mathbb{C}^n$ be a Lagrangian immersion of an $n$-dimensional manifold $M$, then
\[
\dot{A}(v, w) = A(v, w) - \frac{1}{4} \{\langle v, w \rangle H + \langle Jv, H \rangle Jw + \langle Jw, H \rangle Jv \} = 0
\]
holds for any vectors $v$ and $w$ tangent to $M$ if and only if $\Psi(M)$ is either an open set of the Whitney sphere or is totally geodesic.

Now if we take $\gamma(p) = \phi(\frac{1}{\rho} |f(p)|)$ in Proposition 2.4, where the non-negative function $\phi \in C^1_c(\mathbb{R})$ satisfies $\phi(s) = 1$ for $s \leq \frac{1}{2}$ while $\phi(s) = 0$ for $s \geq 1$, we obtained:

Theorem 3.2 (Gap Lemma). Assume $f: \Sigma \to \mathbb{C}^2$ is a properly immersed Lagrangian surface (compact or noncompact) satisfies $\nabla^* T = 0$, and let $\Sigma_\phi(0) = f^{-1}(B_\phi(0))$, then there exists $\varepsilon_0 > 0$ such that if
\[
\int_{\Sigma} |\dot{A}|^2 d\mu \leq \varepsilon_0, \quad \text{and} \quad \liminf_{\rho \to \infty} \frac{1}{\rho^2} \int_{\Sigma_\phi(0)} |A|^2 d\mu = 0
\]
then $f$ is a Lagrangian plane or the Whitney sphere.

Remark 3.1. If we only consider closed surfaces category, the gap phenomenon holds for Willmore equation as well. It’s well known that Urbano and Castro proved in [5] that the Whitney sphere is the only Lagrangian sphere solution to $W = 0$ and they also classified Willmore Lagrangian tori. Now with the help of (3.1), Gauss equation and Gauss-Bonnet formula, we have:
\[
\int_{\Sigma} |\dot{A}|^2 d\mu = \frac{1}{2} \int_{\Sigma} |\dot{A}|^2 d\mu - 2\pi \chi(\Sigma)
\]
hence if $\int_{\Sigma} |\dot{A}|^2 d\mu$ is sufficiently small, the Whitney sphere is the only Willmore Lagrangian surface. Since the small energy condition implies that $\chi(\Sigma)$ to be non-negative, it’s either a sphere or a torus. Applying the results of [7] it must be in the sphere category.

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