Diagrammatic sets and rewriting in weak higher categories

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There is a draft, but I am rewriting it from scratch. Some definitions have changed. Some results I will mention do not hold with the old definitions. The new version should be out before the end of the month.
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- Everything must be weak. $n$-categories in this world are $(\infty, n)$-categories.
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Higher categories for all

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Segal spaces, complicial sets... pick your favourite.
Higher categories for all

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- *(In higher dimensions...)*?
2014
Bialgebra equation
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An interaction of *planar* (2d) diagrams, producing a transformation of 3d diagrams (a 4d diagram)
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_How do we interpret this?_
Pasting theorems

The foundation of diagrammatic reasoning is a pasting theorem:
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the statement that we can univocally interpret
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There is a lack of pasting theorems
for models of weak higher categories.
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The golden age of strict $\omega$-categories

- **1987**: Ross Street’s *The algebra of oriented simplexes* is out, sparking an interest in the combinatorics of higher-dimensional categorical diagrams.
The golden age of strict $\omega$-categories

- **1987**: Ross Street’s *The algebra of oriented simplexes* is out, sparking an interest in the combinatorics of higher-dimensional categorical diagrams.

Then several works on the combinatorics of *pasting diagrams* and their *pasting theorems* in strict $n$-categories:

- **1988**: John Power
- **1989**: Michael Johnson
- **1991**: Ross Street, John Power
- **1993**: Richard Steiner
Directed complexes

We can associate to a cell complex its face poset...

![Directed complex diagram]

![Pastig diagram]
Directed complexes

We can associate to a cell complex its face poset...

and to a pasting diagram its oriented face poset.
An orientation on a finite poset $P$ is an edge-labelling $o : \mathcal{H}P_1 \to \{+, -\}$ of its Hasse diagram.
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An oriented graded poset is a finite graded poset with an orientation.
An *orientation* on a finite poset $P$ is an edge-labelling $o : \mathcal{HP}_1 \rightarrow \{+, -\}$ of its Hasse diagram.

An *oriented graded poset* is a finite graded poset with an orientation.

If $U \subseteq P$ is (downward) closed, $\alpha \in \{+, -\}$, $n \in \mathbb{N}$,

$$\Delta^\alpha_n U := \{x \in U \mid \dim(x) = n \text{ and if } y \in U \text{ covers } x, \text{ then } o(y \rightarrow x) = \alpha\},$$

$$\partial^\alpha_n U := \text{cl}(\Delta^\alpha_n U) \cup \{x \in U \mid \text{for all } y \in U, \text{ if } x \leq y, \text{ then } \dim(y) \leq n\},$$

$$\Delta_n U := \Delta^+_n U \cup \Delta^-_n U, \quad \partial_n U := \partial^+_n U \cup \partial^-_n U.$$
If $U$ is a closed subset of $P$, then $U$ is a *molecule* if either

- $U$ has a greatest element, in which case we call it an *atom*, or
- there exist molecules $U_1$ and $U_2$, both properly contained in $U$, and $n \in \mathbb{N}$ such that $U_1 \cap U_2 = \partial^n_+ U_1 = \partial^n_- U_2$ and $U = U_1 \cup U_2$. 

An oriented graded poset $P$ is a *directed complex* if, for all $x \in P$ and $\alpha, \beta \in \{+, -\}$, if $n = \dim(x)$,

1. $\partial_\alpha x$ is a molecule, and
2. $\partial_\alpha (\partial_\beta x) = \partial_\alpha n - 2 x$. 

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Directed complexes

Steiner 1993 (rephrased)

*Every molecule in a directed complex is the oriented face poset of a pasting diagram.*
Directed complexes

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Under certain conditions, the pasting diagram can be uniquely reconstructed from its oriented face poset.
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All directed complexes present ω-categories — fewer present polygraphs, that is, ω-categories that are freely generated by some of their cells.
Let $P, Q$ be oriented graded posets. We can take their cartesian product as posets.
Directed complexes

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The product of two directed complexes is still a directed complex $P \otimes Q$, the (lax) Gray product of $P$ and $Q$. 
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If $P$ has dim $n$ and $Q$ has dim $k$, $P \otimes Q$ has dim $n + k$. 
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A variant of this was used to define the Gray product of $\omega$-categories (Steiner 2004, Ara-Maltsiniotis 2017)
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Gray products and diagrammatic algebra

2d + 2d = 4d

Around this time, I start seeing Gray products everywhere in diagrammatic algebra
Gray products and diagrammatic algebra

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(Fortunately I was not the only one)

\[
2d + 2d = 4d
\]
Gray products and diagrammatic algebra

Example: Biunitary equations

Used by Jamie Vicary and Mike Stay to unify quantum and encrypted communication protocols. They are models of a Gray product of 2-categories.
Example: Distributive laws of monads

They are models in $\textbf{Cat}$ of a Gray product of 2-categories.
The original example is not simply a Gray product.
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**monoidal category** $\rightsquigarrow$ 2-category with one 0-cell
Gray products and diagrammatic algebra

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\[
\begin{align*}
\text{monoidal category} & \rightsquigarrow 2\text{-category with one 0-cell} \\
\text{PRO} & \rightsquigarrow 2\text{-cat with one 0-cell, one 1-generator}
\end{align*}
\]

With pointed objects, it is natural to take smash products \(\wedge\).

\[
\text{PRO} \wedge \text{PRO} \rightsquigarrow 4\text{-cat with one 0-cell, one 2-generator}
\]

Morally this should be a braided monoidal category. But in strict \(\omega\)-categories, it is a commutative monoidal category. This breaks everything.
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Gray products and diagrammatic algebra

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1991: Mikhail Kapranov and Vladimir Voevodsky publish $\infty$-groupoids and homotopy types, claiming a proof that strict higher categories model all homotopy types in the sense of the homotopy hypothesis.
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1998: Carlos Simpson proves that the result is false (without pointing to a specific mistake).
KV’s non-proof...

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1998: Carlos Simpson proves that the result is false (without pointing to a specific mistake).

The core of the argument relies on the fact that “doubly monoidal” degenerates to “commutative” in strict 3-categories (strict Eckmann-Hilton).
...still contained some good ideas

Good takeaway #1 from Kapranov-Voevodsky:

\[ \text{homotopy types may have } \textit{semistrict algebraic models} \]
\[ \text{with weak units} \]

- **2006**: André Joyal and Joachim Kock in dim 3
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- **2017**: Simon Henry and I come up independently with the *regularity* constraint as a way of avoiding the pitfall of strict Eckmann-Hilton
- **2018**: Henry proves the homotopy hypothesis for “regular \(\omega\)-groupoids”.
Regularity: only $n$-diagrams with spherical boundary have a composite
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Diagrams with spherical boundary

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\( \sim \) “are homeomorphic to \( n \)-balls”
Diagrams with spherical boundary

but not
An $n$-dimensional molecule $U$ in a directed complex has spherical boundary if, for all $k < n$,

$$\partial_k^+ U \cap \partial_k^- U = \partial_{k-1} U.$$
Technical interlude #2: Spherical boundary

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A directed complex is regular if all atoms have spherical boundary.
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A directed complex is *regular* if all atoms have spherical boundary.

- The geometric realisation* of a regular directed complex $P$ is a regular CW complex with one cell for each atom of $P$. 

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*simplicial nerve of poset + realisation of simplicial sets
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*simplicial nerve of poset + realisation of simplicial sets
More in general, let $\mathcal{C}$ be a class of molecules closed under isomorphism, boundaries, and inclusion of atoms, and included in the class $\mathcal{S}$ of (regular) molecules with spherical boundary.
More in general, let $C$ be a class of molecules closed under isomorphism, boundaries, and inclusion of atoms, and included in the class $S$ of (regular) molecules with spherical boundary.

- **$C$-directed complex** is a directed complex whose atoms are all in $C$. 
...and more good ideas

Good takeaway #2 from Kapranov-Voevodsky:

Diagrammatic sets

Kapranov-Voevodsky pass from spaces to $\omega$-categories through an intermediate notion of "spaces locally modelled on combinatorial pasting diagrams", they call diagrammatic sets.
...and more good ideas

Good takeaway #2 from Kapranov-Voevodsky:

Diagrammatic sets
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Kapranov-Voevodsky pass from spaces to $\omega$-categories through an intermediate notion of “spaces locally modelled on combinatorial pasting diagrams”, they call diagrammatic sets.
2019: Kapranov-Voevodsky’s equivalence of “Kan diagrammatic sets” and spaces is “morally correct”
Diagrammatic sets

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...except they chose the wrong class of combinatorial diagrams, not closed under most of the operations they perform.
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...except they chose the wrong class of combinatorial diagrams, not closed under most of the operations they perform.

Regular molecules with spherical boundary works.
But we take a more axiomatic approach.
A map \( f : P \to Q \) of \( C \)-directed complexes is a function that satisfies

\[
\partial_n^\alpha f(x) = f(\partial_n^\alpha x)
\]

for all \( x \in P \), \( n \in \mathbb{N} \), and \( \alpha \in \{+, -\} \).
A map $f : P \to Q$ of $C$-directed complexes is a function that satisfies

$$\partial_n^\alpha f(x) = f(\partial_n^\alpha x)$$

for all $x \in P$, $n \in \mathbb{N}$, and $\alpha \in \{+, -\}$.

A map factors essentially uniquely as a \textit{surjection} followed by an \textit{inclusion}.
A map \( f : P \to Q \) of \( \mathcal{C} \)-directed complexes is a function that satisfies

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\]

for all \( x \in P, \ n \in \mathbb{N}, \) and \( \alpha \in \{+, -\} \).

A map factors essentially uniquely as a surjection followed by an inclusion.

Let \( f : P \to Q \) be a map. Then \( f \) is a closed, order-preserving, dimension-non-increasing function of the underlying posets.
A $\mathcal{C}$-functor $f : P \to Q$ of $\mathcal{C}$-directed complexes is a function $f : \mathcal{C}\ell(P) \to \mathcal{C}\ell(Q)$ such that

1. $f$ preserves all unions and binary intersections,
2. $\partial^\alpha_n f(\text{cl}\{x\}) = f(\partial^\alpha_n x)$, and
3. $f(\text{cl}\{x\})$ is a $\mathcal{C}$-molecule

for all $x \in P$, $n \in \mathbb{N}$, and $\alpha \in \{+,-\}$. 
A $C$-functor $f : P \rightsquigarrow Q$ of $C$-directed complexes is a function $f : \mathcal{C}\ell(P) \to \mathcal{C}\ell(Q)$ such that

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for all $x \in P$, $n \in \mathbb{N}$, and $\alpha \in \{+, -\}$.

A class $C$ is algebraic if $C$-functors compose. We assume that $C$ is algebraic.
Technical interlude #3a: Morphisms of directed complexes

A $\mathcal{C}$-functor $f : P \hookrightarrow Q$ of $\mathcal{C}$-directed complexes is a function $f : \mathcal{C}\ell(P) \rightarrow \mathcal{C}\ell(Q)$ such that

1. $f$ preserves all unions and binary intersections,

2. $\partial_n^\alpha f(\cl\{x\}) = f(\partial_n^\alpha x)$, and

3. $f(\cl\{x\})$ is a $\mathcal{C}$-molecule

for all $x \in P$, $n \in \mathbb{N}$, and $\alpha \in \{+, -\}$.

A class $\mathcal{C}$ is algebraic if $\mathcal{C}$-functors compose. We assume that $\mathcal{C}$ is algebraic.

A $\mathcal{C}$-functor factors e.u. as a subdivision followed by an inclusion.
A span of inclusions of subcategories:

\[
\begin{array}{c}
\text{DCpx}_C^{\text{in}} \\
\downarrow \\
\text{DCpx}_C^C \\
\downarrow \\
\text{DCpx}_C^{\text{fun}}
\end{array}
\]
Let \( C \subseteq S \) be an algebraic class of molecules with spherical boundary.

We say that \( C \) is a *convenient* if it satisfies the following axioms:

1. \( C \) contains \( \bullet \);
2. if \( U \in C \) and \( J \subseteq N \setminus \{0\} \), then \( D^J U \in C \);
3. if \( U, V \in C \) and \( U \Rightarrow V \) is defined, then \( U \Rightarrow V \in C \);
4. if \( U_1, U_2 \in C \) and the pasting \( U_1 \sqcup U_2 \) along \( V \subseteq \partial U_2 \) is defined, then \( U_1 \cup U_2 \in C \);
5. if \( U \in C \) and \( V \subseteq \partial U \) is a closed subset, then \( O^1 \odot U / \sim V \in C \);
6. if \( U, V \in C \), then \( U \otimes V \in C \) and \( U \ast V \in C \).
Technical interlude #3b: Convenient classes

Let $C \subseteq S$ be an algebraic class of molecules with spherical boundary.

We say that $C$ is a *convenient* if it satisfies the following axioms:

1. $C$ contains $\bullet$;

2. if $U \in C$ and $J \subseteq N \setminus \{0\}$, then $D^J U \in C$;

3. if $U, V \in C$ and $U \Rightarrow V$ is defined, then $U \Rightarrow V \in C$;

4. if $U_1, U_2 \in C$ and the pasting $U_1 \cup U_2$ along $V \subseteq \partial U_2$ is defined, then $U_1 \cup U_2 \in C$;

5. if $U \in C$ and $V \subseteq \partial U$ is a closed subset, then $O_1 \otimes U / \sim V \in C$;

6. if $U, V \in C$, then $U \otimes V \in C$ and $U \star V \in C$.

The class $S$ is convenient!
Technical interlude #3b: Convenient classes

Let \( C \subseteq S \) be an algebraic class of molecules with spherical boundary.

We say that \( C \) is a convenient if it satisfies the following axioms:

1. \( C \) contains \( \bullet \);
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Diagrammatic sets

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We write $\odot$ for a skeleton of the full subcategory of $\mathbf{DCpx}^\mathcal{C}$ on the atoms of every dimension.
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The Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{C}\text{Set}$ extends to an embedding $\text{DCpx}^\mathcal{C} \hookrightarrow \mathcal{C}\text{Set}$.

- A *diagram* in $X$ is a morphism $x : U \to X$ where $U$ is a molecule.
- It is *composable* if $U \in \mathcal{C}$, and a *cell* if $U$ is an atom.
Fixing half of KV’s proof

- A Kan diagrammatic set has fillers of all “horns of atoms”.
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There is a realisation of Kan diagrammatic sets that is surjective on homotopy types, together with natural isomorphisms between the homotopy groups of a pointed Kan diagrammatic set and those of its realisation.
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This started the French school of rewriting with polygraphs (Yves Lafont, Philippe Malbos, Yves Guiraud, Samuel Mimram...) and related work on $\omega$-categories (François Métayer, Georges Maltsiniotis, Dimitri Ara...)
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polygraphs and CW complexes, “presented $\omega$-categories” and “presented spaces”.
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Many of the core ideas in polygraphic rewriting rest on an analogy between

polygraphs and CW complexes,
“presented $\omega$-categories” and “presented spaces”.

This analogy is limited by the fact that strict $\omega$-categories do not model all spaces.
A suggestion: rewriting in diagrammatic sets

A similar feel to working with polygraphs, but:

1. Better combinatorial grip on rewriting operations like substitution, surgery of diagrams, etc.
2. "Essential" separation between diagrams and cells.
3. Analogy with CW complexes becomes a functor.
4. Diagrams can be interpreted in models of all homotopy types, for rewriting homotopies.
5. Gray products and joins are easily defined and computed.
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The smash product of pointed diagrammatic sets produces this equation, the way it should.
Equivalences and weak composites

Need a model of weak higher categories as “semantic universe”.

There is a natural coinductive definition of equivalence diagram in a diagrammatic set. A diagrammatic set where every composable diagram is connected by an equivalence to a single cell—its “weak composite”—is a reasonable notion of weak $\omega$-category.

If $C = S$, we can interpret every regular diagram and compose every diagram with spherical boundary. “Stuff” a diagram with units and it becomes regular.
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“Stuff” a diagram with units and it becomes regular.
If \((x_1, x_2) \Rightarrow [x_1, x_2]\) exhibits \([x_1, x_2]\) as a weak composite:

\[\forall \exists \]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
y \\
\end{array}
\]

\[
\begin{array}{c}
x_1 \\
\bullet \\
x_2 \\
\end{array}
\]

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\]

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If \((x_1, x_2) \Rightarrow [x_1, x_2]\) exhibits \([x_1, x_2]\) as a weak composite:

\[
\forall x_1 x_2 y \cong \exists z \uparrow [x_1, x_2] \\
\]

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Properties of equivalences:

- All *degenerate* composable diagrams are equivalences.
Equivalences and weak composites

Properties of equivalences:

- All *degenerate* composable diagrams are equivalences.
- Equivalences are closed under higher equivalence.

\[
\text{The relation } x \cong y \text{ iff there is an equivalence } e : x \Rightarrow y
\]

Equivalences coincide with weakly invertible diagrams.

Morphisms of diagrammatic sets preserve equivalences.

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Properties of equivalences:

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A semistrict algebraic model

In the span

\[
\begin{array}{c}
\text{DCpx}^C_{\text{in}} \\
\text{DCpx}^C \\
\text{DCpx}^C_{\text{fun}}
\end{array}
\]

the two functors preserve the set $\Gamma$ of colimit diagrams containing the initial object and all pushouts of inclusions.
A semistrict algebraic model

In the span

\[
\begin{array}{c}
\text{Set} \\
\xrightarrow{\text{Set}} \\
\downarrow \quad \downarrow \\
\text{DCpx}^C_{\text{in}} \\
\xleftarrow{\text{DCpx}^C_{\text{fun}}} \\
\text{DCpx}^C \\
\end{array}
\]

the two functors preserve the set $\Gamma$ of colimit diagrams containing the initial object and all pushouts of inclusions.

\text{Set} is equivalent to the category $\text{PSh}_\Gamma(\text{DCpx}^C_{\text{fun}})$ of $\Gamma$-continuous presheaves on $\text{DCpx}^C$. 
A semistrict algebraic model

Applying $\text{PSh}_\Gamma(-)$, we obtain a cospan
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$$
P \quad \text{Pol}^C
\downarrow \quad \uparrow
\quad \text{Set}
\quad \leftarrow \quad \omega \text{Cat}_{nu}^C
$$

of restriction functors, where $\text{Pol}^C := \text{PSh}_\Gamma(\text{DCpx}_{\text{in}}^C)$ and $\omega \text{Cat}_{nu}^C := \text{PSh}_\Gamma(\text{DCpx}_{\text{fun}}^C)$. 
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\[
\begin{array}{ccc}
\text{Pol}^C & & \text{Set} \\
\downarrow & & \downarrow \circ \\
\omega \text{Cat}_{nu}^C & & \omega \text{Cat}_{nu}^C
\end{array}
\]

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- $\text{Pol}^C$ is a category of “combinatorial $C$-polygraphs” (only faces, no units or compositions)
A semistrict algebraic model

Applying $\text{PSh}_\Gamma(\mathcal{C})$, we obtain a cospan

\[
\begin{array}{c}
\mathbf{Pol}^C \\
\downarrow \\
\circ \text{Set} \\
\uparrow \\
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- $\mathbf{Pol}^C$ is a category of “combinatorial $\mathcal{C}$-polygraphs” (only faces, no units or compositions)
- $\omega \text{Cat}^C_{nu}$ is a category of “non-unital $\mathcal{C}$-$\omega$-categories” (only faces and compositions, no units)
Units and compositions interact nicely separately with faces. If they are let to interact fully with each other, they produce strict Eckmann-Hilton.
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- Idea: put them together with only a modicum of interaction.
A semistrict algebraic model

A *diagrammatic* $\omega$-*category* has a separate “diagrammatic set” and “non-unital $\omega$-*category*” structure on the same underlying combinatorial polygraph, with a compatibility condition ensuring that certain composites of units are units on composites.
A semistrict algebraic model

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- $\otimes\mathbf{Cat}$, $\otimes\mathbf{Set}$, $\omega\mathbf{Cat}_{nu}$ are all Eilenberg-Moore categories of finitary monads on $\mathbf{Pol}^C$, and all the restriction functors have left adjoints.
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- $\odot \mathbf{Cat}$, $\odot \mathbf{Set}$, $\omega \mathbf{Cat}^c_{nu}$ are all Eilenberg-Moore categories of finitary monads on $\mathbf{Pol}^C$, and all the restriction functors have left adjoints.
- The underlying diagrammatic set of a diagrammatic $\omega$-category has weak composites.
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- $\mathcal{C}at$, $\mathcal{O}Set$, $\omega\mathcal{C}at^C_{nu}$ are all Eilenberg-Moore categories of finitary monads on $\mathcal{P}ol^C$, and all the restriction functors have left adjoints.
- The underlying diagrammatic set of a diagrammatic $\omega$-*category* has weak composites.

Idea: take a unit on a composable diagram, and fully compose the boundary only on one side.
A semistrict algebraic model

Say that $C$ is *algebraically free* if all $C$-directed complexes present polygraphs.
A semistrict algebraic model

Say that $\mathcal{C}$ is algebraically free if all $\mathcal{C}$-directed complexes present polygraphs.

*If $\mathcal{C}$ is algebraically free, then $\omega\text{Cat}$ embeds as a full subcategory into $\otimes\text{Cat}$.*
Two conjectures

1. Conjecture: If $X$ is a diagrammatic set with weak composites, its inclusion in the free diagrammatic $\omega$-category on $X$ is a weak equivalence.
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1. **Conjecture:** If \( X \) is a diagrammatic set with weak composites, its inclusion in the free diagrammatic \( \omega \)-category on \( X \) is a weak equivalence.

2. **Conjecture:** Every convenient class \( C \) is algebraically free.
Directed homotopy theory: a tinkerer’s approach

Higher-dimensional rewriting is packed with notions suggestive of a directed homotopy theory.
Directed homotopy theory: a tinkerer’s approach

Higher-dimensional rewriting is packed with notions suggestive of a *directed homotopy theory*.

The appearance of smash products in diagrammatic algebra seems to me another piece of a puzzle.
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My hope is that diagrammatic sets can make the link between rewriting and homotopy theory tighter, on our way to figuring out what the right notions are.
Higher-dimensional rewriting is packed with notions suggestive of a directed homotopy theory. The appearance of smash products in diagrammatic algebra seems to me another piece of a puzzle.

My hope is that diagrammatic sets can make the link between rewriting and homotopy theory tighter, on our way to figuring out what the right notions are.

Work in progress: a model of computation in diagrammatic sets based on a “directed homotopy extension property”.
Thanks for listening!
