Inference of collective Gaussian hidden Markov models

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Abstract—We consider inference problems for a class of continuous state collective hidden Markov models, where the data is recorded in aggregate (collective) form generated by a large population of individuals following the same dynamics. We propose an aggregate inference algorithm called collective Gaussian forward-backward algorithm, extending recently proposed Sinkhorn belief propagation algorithm to models characterized by Gaussian densities. Our algorithm enjoys convergence guarantee. In addition, it reduces to the standard Kalman filter when the observations are generated by a single individual. The efficacy of the proposed algorithm is demonstrated through multiple experiments.

I. INTRODUCTION

Filtering and smoothing problems in dynamic systems have been studied in a variety of applications including navigation, object tracking, robot localization, and control [1], [2]. These problems can be more generally posed as a Bayesian inference problem in probabilistic graphical models (PGMs) [3]. For instance, the Kalman filter model is a special type of PGMs known as hidden Markov model (HMM) with linear Gaussian state and observation densities, which we call Gaussian HMM in this paper. The traditional filtering and smoothing methods are based on the observations recorded by a single individual.

The problem of modeling population-level observations has gained much attention in recent years [4], [5]. The observations are often recorded in aggregate form in terms of population counts due to various reasons such as privacy concerns, cost of data collection, and measurement fidelity. A timely example is modeling the spread of COVID-19 in a geographical area over time. In this example, the observation sequence may represent aggregate features including number of deaths and number of positives, based on which one might be interested in predicting number of infections in future. Collective filtering have emerged as a method to model such aggregate observations. Some more example applications of collective filtering include bird migration analysis [6] and estimating the crowd flow in an urban environment [7].

Traditional filtering algorithms such as belief propagation [8] and the Kalman filter are not applicable to collective settings due to data aggregation.

Collective filtering has been studied under the umbrella of recently proposed more general framework known as collective graphical models (CGMs) [5], [4]. Various aggregate inference algorithms under the CGM framework have been proposed including approximate MAP inference [9], non-linear belief propagation [10], and Sinkhorn belief propagation (SBP) [4]. The collective forward-backward algorithm (CFB) [4] is a special case of SBP for aggregate inference in CGMs. However, all of these algorithms are only applicable to CGMs with discrete states. A very recent work on filtering for continuous time HMMs is collective feedback particle filter [11].

In this paper, we are interested in inference problem in collective HMMs with continuous states. The continuous observations recorded by a large number of agents, with each agent following a certain underlying model, are aggregated such that the individuals are indistinguishable from the measurements. In such data aggregation, the individual’s association with the measurements remains unknown. We propose an algorithm for estimating aggregate state distributions in collective HMMs with Gaussian densities, which we call the collective Gaussian forward-backward (CGFB) algorithm. Our algorithm is based on the recently proposed CFB algorithm [4], [6], which has convergence guarantees. We also propose a sliding window filter for faster inference extending SW-SBP [6] to Gaussian HMMs. Furthermore, we show that in case of a single individual, our algorithm naturally reduces to the standard Kalman filter and smoother. We demonstrate the performance of algorithm via multiple numerical experiments.

Rest of the paper is organized as follows. In Section II we briefly discuss related background including collective HMMs. We present our main results and algorithm in Section III. Next, we discuss the connections of our algorithm with the standard Kalman filter in Section IV. This is followed by experimental results in Section V and conclusion in Section VI.

II. BACKGROUND

In this section, we briefly discuss related background including HMMs, collective HMMs, and collective forward-backward algorithm.

A. Hidden Markov Models

Hidden Markov models (HMMs) consist of a Markov process describing the true (hidden) state of a dynamic system and an observation process corrupted by noise. Let us denote the state variables $X_1, X_2, \ldots$ and observation variables $O_1, O_2, \ldots$. An HMM is parameterized by the initial distribution $p(X_1)$, the state transition probabilities $p(X_{t+1} | X_t)$, and the observation probabilities $p(O_t | X_t)$ for each time step $t = 1, 2, \ldots$. An HMM of length $T$ is represented
graphically as shown in Figure 1. The joint distribution of an HMM with length $T$ can be factorized as

$$p(x, o) = p(x_1) \prod_{t=1}^{T-1} p(x_{t+1} \mid x_t) \prod_{t=1}^T p(o_t \mid x_t),$$  \quad (1)

where $x = \{x_1, x_2, \ldots, x_T\}$ and $o = \{o_1, o_2, \ldots, o_T\}$ represent a particular assignment of hidden and observation variables, respectively.

The state and observation variables can take either discrete or continuous values. In this paper, we assume that both state as well as observation variables take continuous values unless otherwise specified. In standard inference problems over HMMs, the observations from a single individual $\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_T$ are recorded over time, based on which the hidden state distributions $p(x_t \mid \hat{o}_1:T)$ are estimated. The standard forward-backward algorithm [12] is a popular method for inference over HMMs. Note that when the estimation of state distributions at time step $t$ is made based on only the current and past observations $\hat{o}_{1:t}$, it is called filtering and when the future observations are also taken into account, it is more generally termed as inference or smoothing.

![Fig. 1: Graphical representation of a length T HMM.](image)

### B. Collective Hidden Markov Models

Collective (aggregate) HMMs refer to the scenario when the data is generated by $M$ individuals independently transitioning from state to state according to the same Markov chain and the corresponding noisy observations are recorded in aggregate form such that the association to the corresponding state is not known. Let $X_t^{(m)}$ be the random variable representing the state of $m^{th}$ individual at time $t$ and $O_t^{(m)}$ be the observation variable of $m^{th}$ individual at time $t$. The observations are made in aggregate form $y_t(o_t)$ representing the distribution of the collective observations for each time step $t = 1, 2, \ldots, T$. The goal of inference in collective HMMs is to estimate the aggregate state distributions $n_t(x_t)$ given all the aggregate observation distributions. A pictorial representation of a collective HMM is depicted in Figure 2. Here, $n_t(x_t)$ is an estimate of the state distribution of the $M$ agents at time step $t$.

Inference in aggregate HMMs aims to estimate the aggregate hidden distributions based on indistinguishable aggregate measurements. Traditional inference algorithms such as forward-backward algorithms can not be used here due to data aggregation. The collective forward-backward (CFB) algorithm [4] was recently proposed for aggregate inference in HMMs. It is a special case of more general aggregate inference algorithm known as Sinkhorn belief propagation (SBP) [4], which has convergence guarantee.

The update sequence of these messages has a two-pass structure: a forward pass followed by a backward pass. In forward pass, $\gamma_t$, $\alpha_t$, and $\xi_t$ are updated $\forall t = 1, 2, \ldots, T$; and in backward pass, $\gamma_t$, $\beta_t$, and $\xi_t$ are updated $\forall t = T, T - 1, \ldots, 1$. The algorithm is guaranteed to converge upon which, the required aggregate hidden state distributions are estimated as

$$n_t(x_t) \propto \alpha_t(x_t)\beta_t(x_t)\gamma_t(x_t), \quad \forall t = 1, 2, \ldots, T.$$

### III. Main Results

We propose collective Gaussian forward-backward (CGFB) algorithm for inference in aggregate Gaussian hidden Markov models:...
models (GHMMs). GHMMs are special case of continuous state HMMs where the model densities take the form

\[ p(X_{t+1}|X_t, \theta_X) = \mathcal{N}(x_{t+1}; Ax_t, Q) \]  
\[ p(O_t|X_t, \theta_O) = \mathcal{N}(o_t; Cx_t, R) \]  
\[ p(X_1|\theta_1) = \mathcal{N}(x_1; \pi, \Pi) \]

where \( \theta_X = \{A, Q\}, \theta_O = \{C, R\} \), and \( \theta_1 = \{\pi, \Pi\} \) are the parameters characterizing model densities in the HGMM model. Alternatively, the model densities for \( t = 1, 2, \ldots \) can be characterized via the following set of equations.

\[ X_{t+1} = AX_t + W_t \] 
\[ O_t = CX_t + V_t \]

where \( W_t \sim \mathcal{N}(w_t; 0, Q), V_t \sim \mathcal{N}(v_t; 0, R) \), and \( X_1 \sim \mathcal{N}(x_1; \pi, \Pi) \).

### A. Collective GHMM Inference: Collective Gaussian Forward-Backward Algorithm

We have a total of \( M \) trajectories of continuous observations over a single GHMM characterized by \( \{ \theta \} \). The recorded observations are \( \{ o_1^{(m)}, o_2^{(m)}, \ldots, o_T^{(m)} \}, \forall m = 1, 2, \ldots, M \) with \( o_t^{(m)} \) being the continuous observations of the \( m \)th trajectory at time \( t \). The objective of the collective GHMM inference is to estimate the distributions \( n_t(x_t) \), \( \forall t \). We assume that the aggregate observation is approximated by a Gaussian density at each time step \( t \), i.e.,

\[ y_t(o_t) \sim \mathcal{N}(o_t; \hat{\mu}_t, \hat{P}_t) \]

where \( \hat{\mu}_t \) and \( \hat{P}_t \) can be estimated from the recorded observations.

For continuous states and continuous observations, the four messages in the CFB algorithm take the form

\[ \alpha_t(x_t) \propto \int p(x_t|x_{t-1})\alpha_{t-1}(x_{t-1})\gamma_{t-1}(x_{t-1}) dx_{t-1} \]  \tag{6a} 
\[ \beta_t(x_t) \propto \int p(x_{t+1}|x_t)\beta_{t+1}(x_{t+1})\gamma_{t+1}(x_{t+1}) dx_{t+1} \]  \tag{6b} 
\[ \gamma_t(x_t) \propto \int p(o_t|x_t)\gamma_t(o_t)\delta_t(o_t) do_t \]  \tag{6c} 
\[ \xi_t(o_t) \propto \int p(o_t|x_t)\alpha_t(x_t)\beta_t(x_t) dx_t \]  \tag{6d} 

with boundary conditions

\[ \alpha_1(x_1) = p(x_1) \quad \text{and} \quad \beta_T(x_T) = 1. \]

Based on above, the messages in collective GHMMs are characterized by Theorem 1.

**Theorem 1:** The forward, backward, upward, and downward messages in collective GHMM take the following form,

\[ \alpha_t(x_t) \propto \exp \left( -\frac{1}{2}x_t^TA^t(f)x_t + x_t^T\eta_t(f) \right), \]  \tag{7a} 
\[ \beta_t(x_t) \propto \exp \left( -\frac{1}{2}x_t^TA^t(b)x_t + x_t^T\eta_t(b) \right), \]  \tag{7b} 
\[ \gamma_t(x_t) \propto \exp \left( -\frac{1}{2}x_t^TA^t(u)x_t + x_t^T\eta_t(u) \right), \]  \tag{7c} 
\[ \xi_t(o_t) \propto \exp \left( -\frac{1}{2}x_t^TA^t(d)x_t + x_t^T\eta_t(d) \right), \]  \tag{7d} 

for \( t = 1, 2, \ldots, T \). Here, the message parameters are the fixed points of the following recursive updates

\[ \Lambda_t(f) = Q^{-1} - Q^{-1}A(A^TQ^{-1}A + \Lambda_t(f) + \Lambda_{t-1}(u))^{-1}A^TQ^{-1} \]  
\[ \eta_t(f) = \Lambda_t(f) - \Lambda_t(f)A(A^TQ^{-1}A + \Lambda_t(f) + \Lambda_{t-1}(u))^{-1}(\eta_{t-1}(f) + \eta_{t-1}(u)) \]  
\[ \Lambda_t(b) = A^TQ^{-1}(Q^{-1} + \Lambda_{t+1}(f) + \Lambda_{t+1}(u))^{-1}(\eta_{t+1}(f) + \eta_{t+1}(u)) \]  
\[ \eta_t(b) = A^TQ^{-1}(Q^{-1} + \Lambda_{t+1}(f) + \Lambda_{t+1}(u))^{-1}(\eta_{t+1}(f) + \eta_{t+1}(u)) \]  
\[ \Lambda_t(u) = C^T(R + (\hat{P}_t - \Lambda_t(d))^{-1}C^{-1})^{-1}C^T \]  
\[ \eta_t(u) = C^T(R + (\hat{P}_t - \Lambda_t(d))^{-1}C^{-1}(\hat{P}_t - \hat{\mu}_t - \eta_t(d)) \]

with boundary conditions

\[ \Lambda_1(f) = \Pi^{-1}, \quad \eta_1(f) = \Pi^{-1}\pi, \quad \Lambda_T(b) = 0, \quad \eta_T(b) = 0. \]

Moreover, the required marginals can be computed as

\[ n_t(x_t) \propto \alpha_t(x_t)\beta_t(x_t)\gamma_t(x_t) \]
\[ \propto \mathcal{N}(x_t; \mu_t, \Sigma_t) \]

where

\[ P_t = (\Lambda_t(f) + \Lambda_t(b) + \Lambda_t(u))^{-1} \]
\[ \mu_t = \Sigma_t(\eta_t(f) + \eta_t(b) + \eta_t(u)) \]

**Proof:** See Appendix A.
Algorithm 1: Collective Gaussian Forward-Backward (CGFB) Algorithm

Initialize all the message parameters

while not converged do

Forward pass:
for \( t = 2, 3, \ldots, T \) do
i) Update upward parameters \( \Lambda^{(u)}_{t+1} \) and \( \eta^{(u)}_{t-1} \)
ii) Update forward parameters \( \Lambda^{(f)}_{t} \) and \( \eta^{(f)}_{t} \)
ii) Update downward parameters \( \Lambda^{(d)}_{t} \) and \( \eta^{(d)}_{t} \)
end for

Backward pass:
for \( t = T-1, \ldots, 1 \) do
i) Update upward parameters \( \Lambda^{(u)}_{t+1} \) and \( \eta^{(u)}_{t+1} \)
ii) Update backward parameters \( \Lambda^{(b)}_{t} \) and \( \eta^{(b)}_{t} \)
ii) Update forward parameters \( \Lambda^{(f)}_{t} \) and \( \eta^{(f)}_{t} \)
end for

end while

Estimate required state density parameters \( \mu_t \) and \( P_t \)

B. Sliding Window Filter for Fast Inference

The size of the underlying GHMM increases with time and filtering becomes more and more expensive computationally. Therefore for real-time aggregate filtering, it is not suitable to consider the full length model. Similar to the recently proposed sliding window Sinkhorn belief propagation (SW-SBP) algorithm [6], the inference in collective GHMM can be performed in an incremental (online) fashion. Based on SW-SBP, we propose sliding window collective Gaussian forward backward (SW-CGFB) algorithm that employs a sliding window filter of length \( K \) such that at each time step, only the most recent \( K \) (aggregate) observations are taken into account for estimating the current state density. Moreover in order to encode the discarded information, the forward messages are utilized as the prior.

![Fig. 5: Sliding window filter for incremental inference in collective GHMMs.](image)

The sliding window filter scheme for inference in collective GHMMs is illustrated in Figure 5 with window size of \( K \). When the new observation comes in at time \( t+1 \), the left-most state variable is assigned a prior same as the forward message to the variable at time \( t \) which is indicative of the discarded information. The CGFB algorithm is run in the \( K \) length GHMM with this prior represented by the forward message.

IV. CONNECTIONS TO STANDARD KALMAN FILTER

Suppose we know the estimated state distribution \( n_t(x_t) \propto N(x_t; \mu_t, P_t) \) and as the single new observation \( \hat{\alpha}_{t+1} \) is recorded at time \( t+1 \), the goal is to estimate the state distribution \( n_{t+1}(x_{t+1}) = p(x_{t+1}\mid \hat{\alpha}_{t+1}) \). This is the setting of the standard Kalman filter. We show that the standard Kalman filter is equivalent to the SW-CGFB algorithm for the case of a single individual (\( M = 1 \)) and unit window size (\( K = 1 \)).

The sliding window filter scheme with window size \( K = 1 \) for a single individual is shown in Figure 6. Note that here the backward messages do not contribute to the state density estimation due to the boundary condition in Algorithm 1. Moreover, the upward messages do not depend on the downward message and reduce to the corresponding observation probabilities of the underlying GHMM [4].

![Fig. 6: Sliding window filter with window size \( K = 1 \) and number of agents \( M = 1 \).](image)

Identifying the forward message \( \alpha_{t+1}(x_{t+1}) \) as prediction step and then incorporating the upward message from the node observation as correction step, the state density can be estimated as

\[
n_{t+1}(x_{t+1}) \propto \alpha_{t+1}(x_{t+1}) \gamma_{t+1}(x_{t+1}).
\]

Here the predicted state density takes the form

\[
\alpha_{t+1}(x_{t+1}) \propto \int p(x_{t+1}\mid x_t) \alpha_t(x_t) \gamma_t(x_t) \, dx_t
\]

\[
\propto \int p(x_{t+1}\mid x_t) n_t(x_t) \, dx_t
\]

\[
= \int N(x_{t+1}; A x_t, Q) N(x_t; \mu_t, P_t) \, dx_t
\]

\[
\propto N(x_{t+1}; \mu_{t+1}\mid x_{t+1}, P_{t+1}\mid x_{t+1}),
\]

with

\[
P_{t+1}\mid x = Q + A P_t A^T
\]

\[
\mu_{t+1}\mid x = A \mu_t.
\]
Remark 1: Similarly, it can be shown that the message
its full derivation can be found in Appendix B.

equations updates of the standard Kalman filter, respectively, and
Equations (10) and (11) describe the prediction and correc-
tion updates for the case of a single individual, i.e.,
updates in Algorithm 1 reduce to the standard Kalman

\[
\begin{align*}
\pi & = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\Pi & = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \\
A & = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 - 0.5\Delta t \end{bmatrix}, \\
Q & = \Delta t \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \\
R & = \Delta t \begin{bmatrix} 0 & 0.7 \end{bmatrix}, \\
\end{align*}
\]

where \(\Delta t\) is set to 0.05 for all our experiments. Based
on the above GHMM parameters, we generate \(M\) number
of trajectories and record the aggregate observations
\(\{o_1^m, o_2^m, \ldots, o_T^m\}, \forall m = 1, 2, \ldots, M\). We then use
CGFB algorithm to estimate the aggregate state distributions
\(n_t(x_t), \forall t = 1, 2, \ldots, T\), which are compared against
the ground truth distributions in terms of quadratic error
averaged over all the time steps \(t\). Figure 7 depicts error
curves for mean and covariance estimates with respect to the
corresponding ground truth means and covariances. It can be

clearly observed that the algorithm converges and estimated
parameters are very close to the ground truth.

Fig. 7: Convergence of CGFB with respect to the correspond-
ing ground truths with \(M = 200\) and \(T = 100\). Different

colors represent different random seeds.

Next, we study the effect of number of agents \(M\) on the es-
estimation of aggregate state distributions. Figure 8 shows
the error values for different number of agents. We observe that
errors in both mean and covariance estimates get smaller as
the number of agents \(M\) increase. Furthermore, we evaluate
the performance of CGFB against \(M\) independent standard
Kalman filters assuming the identities of each particle and
measurement are known. Let the Kalman filter mean and
covariance estimates for \(M\) agents at time \(T\) be \(\mu_T^m\) and \(P_T^m\)
for \(m = 1, 2, \ldots, M\). We compute the mean and variances
of these \(M\) Kalman estimations as

\[
\begin{align*}
\mu_T^{KF} &= \frac{1}{M} \sum_{m=1}^{M} \mu_T^m, \\
P_T^{KF} &= \frac{1}{M} \sum_{m=1}^{M} P_T^m + (\mu_T^m - \mu_T^{KF})(\mu_T^m - \mu_T^{KF})^T,
\end{align*}
\]

and use them to compare against the estimates obtained by
the CGFB algorithm. This comparison in terms of quadratic
error is shown in Figure 9.

V. NUMERICAL EXPERIMENTS

We conduct multiple experiments to evaluate the perfor-
mance of our CGFB algorithm. We simulate a system with
the GHMM parameters:

\[
\begin{align*}
\pi & = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\Pi & = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \\
A & = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 - 0.5\Delta t \end{bmatrix}, \\
Q & = \Delta t \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \\
R & = \Delta t \begin{bmatrix} 0 & 0.7 \end{bmatrix}, \\
\end{align*}
\]

Note that in general, the parameters \(\mu_t\) and \(P_t\) of
the estimated aggregate state distribution \(n_t(x_t)\) also depend on
the future observation. However, in the setting of sliding
window filter, they are same as \(\mu_t\) and \(P_t\) taking only
the current and past observations into account. After the
prediction step, the correction step aims to estimate the
required state distribution as

\[
\begin{align*}
n_{t+1}(x_{t+1}) & \propto \alpha_{t+1}(x_{t+1})\gamma_{t+1}(x_{t+1}) \\
& \propto N(x_{t+1}; \mu_{t+1|t}, P_{t+1|t}) p(\delta_{t+1|t}) \\
& \propto N(x_{t+1}; \mu_{t+1|t}, P_{t+1|t}) N(\delta_{t+1|t}; Cx_{t+1}, R) \\
& \propto \exp \left( -\frac{1}{2}x_{t+1}^T(P^{-1}_{t+1|t} + C^T R^{-1} C)x_{t+1} \right) \\
& \quad + x_{t+1}^T(P^{-1}_{t+1|t}\mu_{t+1|t} + C^T R^{-1}\delta_{t+1|t}) \\
& \propto \exp \left( -\frac{1}{2}(x_{t+1} - \mu_{t+1|t})^T(P^{-1}_{t+1|t} \\
& \quad + C^T R^{-1} C)(x_{t+1} - \mu_{t+1|t}) \right) \\
& \propto N(x_{t+1}; \mu_{t+1|t+1}, P_{t+1|t+1}),
\end{align*}
\]

with

\[
\begin{align*}
P_{t+1|t+1} &= (I - K_{t+1} \Pi)P_{t+1|t} (11a) \\
\mu_{t+1|t+1} &= \mu_{t+1|t} + K_{t+1}(\delta_{t+1|t} - C\mu_{t+1|t}) (11b) \\
K_{t+1} &= P_{t+1|t} C^T (R + CP_{t+1|t}C)^{-1}. (11c)
\end{align*}
\]

Equations (10) and (11) describe the prediction and correc-
tion updates of the standard Kalman smoother updates for the case of a single individual, i.e.,
\(M = 1\).
Finally, we evaluate the performance of our algorithm with sliding window filter. First, we present the comparison of time complexity between the full length graph that we refer as baseline CGFB and SW-CGFB in Figure 10. It is evident that the time taken by the baseline CGFB increases linearly with time, while SW-CGFB takes a constant amount of time at each step. Now we compare the SW-CGFB algorithm against the baseline. The naive version of SW-CGFB refers to the sliding window filter without considering the prior from the previous step. Figures 11 and 12 depict the error performance for different values of window sizes. In both the figures, the results are averaged over 10 different seeds and the shaded areas represent the corresponding standard deviations. It can be observed that the naive sliding window filter without considering the prior performs poorly as compared to the one with prior. Moreover, the errors in the mean estimates decrease with the increase in window size.

VI. CONCLUSION

In this paper, we proposed an algorithm for filtering and inference in collective GHMMs, where the data from several individuals is recorded in aggregate form such that the observations are indistinguishable. The algorithm enjoys convergence guarantees and naturally reduces to the standard Kalman filter for the case of a single individual. We also proposed a sliding window filter scheme for fast inference in collective GHMMs.

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Above equation simplifies in canonical form as where

\[
\Lambda = D. \text{Sheldon}, T. \text{Sun}, A. \text{Kumar}, \text{and T. Dietterich, “Approximate inference in collective graphical models, in International Conference on Machine Learning, 2013, pp. 1004–1012.}
\]

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\]

\[
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\]

**APPENDIX**

### A. Proof of Theorem 7

**Proof:** The product messages take the form

\[
\alpha_t(x_t)\gamma_t(x_t) \propto \exp \left( -\frac{1}{2} x^T (\Lambda_t^{(f)} + \Lambda_t^{(u)}) x + x^T (\eta_t^{(f)} + \eta_t^{(u)}) \right)
\]

\[
\beta_t(x_t)\gamma_t(x_t) \propto \exp \left( -\frac{1}{2} x^T (\Lambda_t^{(b)} + \Lambda_t^{(u)}) x + x^T (\eta_t^{(b)} + \eta_t^{(u)}) \right)
\]

\[
\alpha_t(x_t)\beta_t(x_t) \propto \exp \left( -\frac{1}{2} x^T (\Lambda_t^{(f)} + \Lambda_t^{(b)}) x + x^T (\eta_t^{(f)} + \eta_t^{(b)}) \right)
\]

**Forward Messages:**

The forward messages can be written as

\[
\alpha_t(x_t) \propto \int p(x_t|x_{t-1})\alpha_{t-1}(x_{t-1})\gamma_{t-1}(x_{t-1}) \, dx_{t-1}
\]

\[
= \int \exp \left( -\frac{1}{2} (x_t - Ax_{t-1})^T Q^{-1} (x_t - Ax_{t-1}) \right)
\]

\[
\exp \left( -\frac{1}{2} x_{t-1}^T (\Lambda_{t-1}^{(f)} + \Lambda_{t-1}^{(u)}) x_{t-1} + x_{t-1}^T (\eta_{t-1}^{(f)} + \eta_{t-1}^{(u)}) \right) \, dx_{t-1}
\]

\[
\propto \mathcal{N} \left( x_t; \mu_{t-1}^{(f)} + \Lambda_{t-1}^{(u)} x_{t-1}, A + \Lambda_{t-1}^{(f)} + \Lambda_{t-1}^{(u)} \right)
\]

where \( \mu_{t-1}^{(f)} \) and \( P_{t-1}^{(f)} \) are the mean and covariance of the product message \( \alpha_{t-1}(x_{t-1})\gamma_{t-1}(x_{t-1}) \), respectively. The above equation simplifies in canonical form as

\[
\Lambda_t^{(f)} = (A + \Lambda_t^{(f)} A^T)^{-1}
\]

\[
= Q^{-1} Q^{-1} A (A^T) Q^{-1} A + (\Lambda_t^{(f)})^{-1} A T (A T)^{-1} Q^{-1}
\]

\[
= Q^{-1} A (A^T) Q^{-1} A + (\Lambda_t^{(f)} + \Lambda_t^{(u)})^{-1} A T (A T)^{-1} Q^{-1}
\]

\[
\eta_t^{(f)} = \Lambda_t^{(f)} \mu_{t-1}^{(f)}
\]

\[
= (Q + A P_{t-1}^{(f)} A^T)^{-1} (A P_{t-1}^{(f)} \eta_{t-1}^{(f)})
\]

\[
= Q^{-1} A (A^T) Q^{-1} A + (\Lambda_t^{(f)} + \Lambda_t^{(u)})^{-1} (\eta_t^{(f)} + \eta_t^{(u)})
\]

**Backward Messages:**

\[
p(x_{t+1}|x_t)\beta_{t+1}(x_{t+1})\gamma_{t+1}(x_{t+1})
\]

\[
\propto \exp \left( -\frac{1}{2} (x_{t+1} - Ax_t)^T Q^{-1} (x_{t+1} - Ax_t) \right)
\]

\[
- \frac{1}{2} (x_{t+1} - \mu_{t+1}^{(b)})^T (\Lambda_{t+1}^{(b)}) (x_{t+1} - \mu_{t+1}^{(b)})
\]

\[
- \frac{1}{2} (x_{t+1} - \mu_{t+1}^{(u)})^T (\Lambda_{t+1}^{(u)}) (x_{t+1} - \mu_{t+1}^{(u)})
\]

\[
\propto \exp \left( -\frac{1}{2} (x_{t+1} - \nu)^T \Sigma_1 (x_{t+1} - \nu) \right)
\]

\[
- \frac{1}{2} (Ax_t - \mu_{t+1}^{(b)})^T \Sigma_2 (Ax_t - \mu_{t+1}^{(b)})
\]

\[
- \frac{1}{2} (Ax_t - \mu_{t+1}^{(u)})^T \Sigma_3 (Ax_t - \mu_{t+1}^{(u)})
\]

where

\[
\Sigma_1 = (Q^{-1} + \Lambda_{t+1}^{(b)} + \Lambda_{t+1}^{(u)})^{-1}
\]

\[
\Sigma_2 = Q^{-1} \Sigma_{t+1} \Lambda_{t+1}^{(b)}
\]

\[
\Sigma_3 = Q^{-1} \Sigma_{t+1} \Lambda_{t+1}^{(u)}
\]

Therefore,

\[
\beta_t(x_t) \propto \int p(x_{t+1}|x_t)\beta_{t+1}(x_{t+1})\gamma_{t+1}(x_{t+1}) \, dx_{t+1}
\]

\[
\propto \exp \left( -\frac{1}{2} x_t^T A^T (\Sigma_2 + \Sigma_3) A x_t 
\]

\[
+ x_t^T A^T \Sigma_2 \mu_{t+1}^{(b)} + A^T \Sigma_3 \mu_{t+1}^{(u)} \right)
\]

\[
\propto \exp \left( -\frac{1}{2} x_t^T \Lambda_t^{(b)} x + x_t^T \eta_t^{(b)} \right)
\]

with

\[
\Lambda_t^{(b)} = A^T (\Sigma_2 + \Sigma_3) A
\]

\[
= A^T Q^{-1} (Q^{-1} + \Lambda_{t+1}^{(b)} + \Lambda_{t+1}^{(u)})^{-1} (\Lambda_{t+1}^{(b)} + \Lambda_{t+1}^{(u)}) A
\]

\[
\eta_t^{(b)} = A^T \Sigma_2 \mu_{t+1}^{(b)} + A^T \Sigma_3 \mu_{t+1}^{(u)}
\]

\[
= A^T Q^{-1} (Q^{-1} + \Lambda_{t+1}^{(b)} + \Lambda_{t+1}^{(u)})^{-1} (\Lambda_{t+1}^{(b)} + \Lambda_{t+1}^{(u)})^{-1} (\eta_{t+1}^{(b)} + \eta_{t+1}^{(u)})
\]

**Downward Messages:**

The downward density can be simplified as

\[
\xi_t(o_t) \propto \int p(o_t|x_t)\alpha_t(x_t)\beta_t(x_t) \, dx_t
\]

\[
\propto \int \mathcal{N}(o_t; C x_t, R) \mathcal{N}(x_t; \mu_{t}^{(f)}, P_{t}^{(f)}) \, dx_t
\]

\[
\propto \mathcal{N} \left( o_t; C \mu_{t}^{(f)}, R + C P_{t}^{(f)} C^T \right)
\]

where \( \mu_{t}^{(f)} \) and \( P_{t}^{(f)} \) are the mean and covariance of product message \( \alpha_{t}(x_{t})\beta_{t}(x_{t}) \), respectively. In canonical form, it further reduces to
\[ \Lambda_t^{(d)} = (R + CP_t^{(fb)}C^T)^{-1} \]
\[ = R^{-1} - R^{-1}C(C^TR^{-1}C + \Lambda_t^{(fb)})^{-1}C^TR^{-1} \]
\[ = R^{-1} - R^{-1}C(C^TR^{-1}C + \Lambda_t^{(i)} + \Lambda_t^{(b)})^{-1}C^TR^{-1} \]

\[ \eta_t^{(d)} = \Lambda_t^{(d)}C\tilde{\mu}_t^{(fb)} \]
\[ = (R + CP_t^{(fb)}C^T)^{-1}(CP_t^{(fb)}\eta_t^{(fb)}) \]
\[ = R^{-1}C(C^TR^{-1}C + \Lambda_t^{(fb)})^{-1}\eta_t^{(fb)} \]
\[ = R^{-1}C(C^TR^{-1}C + \Lambda_t^{(i)} + \Lambda_t^{(b)})^{-1}(\eta_t^{(i)} + \eta_t^{(b)}) \]

**Upward Messages:**

Assuming that the output samples take the form of a Gaussian density
\[ y_t(o_t) = \mathcal{N}(o_t; \tilde{\mu}_t, \tilde{P}_t), \quad (13) \]

\[ p(o_t|x_t) \frac{y_t(o_t)}{\xi_t(o_t)} \propto \exp \left( -\frac{1}{2} (o_t - Cx_t)^T R^{-1} (o_t - Cx_t) \right) \]
\[ - \frac{1}{2} (o_t - \tilde{\mu}_t)^T \tilde{P}_t^{-1} (o_t - \tilde{\mu}_t) \]
\[ + \frac{1}{2} (o_t - \tilde{\mu}_t)^T \Lambda_t^{(d)} (o_t - \tilde{\mu}_t) \]
\[ \propto \exp \left( -\frac{1}{2} (o_t - \nu)^T \Sigma_1 (o_t - \nu) \right) \]
\[ - \frac{1}{2} (Cx_t - \tilde{\mu}_t)^T \Sigma_2 (Cx_t - \tilde{\mu}_t) \]
\[ - \frac{1}{2} (Cx_t - \tilde{\mu}_t)^T \Sigma_3 (Cx_t - \tilde{\mu}_t) \]

where
\[ \Sigma_1 = (R^{-1} + \tilde{P}_t^{-1} - \Lambda_t^{(d)})^{-1} \quad (14a) \]
\[ \Sigma_2 = R^{-1} \Sigma_1 \tilde{P}_t^{-1} \quad (14b) \]
\[ \Sigma_3 = -R^{-1} \Sigma_1 \Lambda_t^{(d)} \quad (14c) \]

Therefore,
\[ \gamma_t(x_t) \propto \int p(o_t|x_t) \frac{y_t(o_t)}{\xi_t(o_t)} \, do_t \]
\[ \propto \exp \left( -\frac{1}{2} x_t^T \left( \Sigma_2 + \Sigma_3 \right) C x_t \right) \]
\[ + x_t^T \left( C^T \Sigma_2 \tilde{\mu}_t + C^T \Sigma_3 \eta_t^{(u)}(t) \right) \]
\[ \propto \exp \left( -\frac{1}{2} x_t^T \Lambda_t^{(u)} x + x_t^T \eta_t^{(u)} \right) \]

with
\[ \Lambda_t^{(u)} = C^T (\Sigma_2 + \Sigma_3) C \]
\[ = C^T R^{-1}(R^{-1} + \tilde{P}_t^{-1} - \Lambda_t^{(d)})^{-1}(\tilde{P}_t^{-1} - \Lambda_t^{(d)})^T C \]
\[ = C^T (R + (\tilde{P}_t^{-1} - \Lambda_t^{(d)})^{-1} C \]
\[ \eta_t^{(u)} = C^T \Sigma_2 \tilde{\mu}_t + C^T \Sigma_3 \eta_t^{(u)} \]
\[ = C^T R^{-1}(R^{-1} + \tilde{P}_t^{-1} - \Lambda_t^{(d)})^{-1}(\tilde{P}_t^{-1} \mu_t - \Lambda_t^{(d)} \mu_t) \]
\[ = C^T R^{-1}(R^{-1} + \tilde{P}_t^{-1} - \Lambda_t^{(d)})^{-1}(\tilde{P}_t^{-1} \mu_t - \eta_t^{(d)}) \]

**B. Proof of Kalman updates** \[10\] and \[11\]

**Proof:**

\[ P_{t+1|t}^{-1} = (P_{t+1|t}^{-1} + C^T R^{-1} C)^{-1} \]
\[ = P_{t+1|t}^{-1} - P_{t+1|t}^{-1}C^T(R + CP_{t+1|t}C^T)^{-1}CP_{t+1|t} \]
\[ = (I - P_{t+1|t}^{-1}C^T(R + CP_{t+1|t}C^T)^{-1})P_{t+1|t} \]
\[ = (I - K_{t+1}C)P_{t+1|t} \]

where
\[ K_{t+1} = P_{t+1|t}C^T(R + CP_{t+1|t}C)^{-1}. \]

Moreover,
\[ \mu_{t+1|t+1} = P_{t+1|t+1}^{-1}(P_{t+1|t+1}^{-1} + C^T R^{-1})^{-1} \]
\[ = (I - K_{t+1}C)P_{t+1|t}^{-1} + K_{t+1}(R + CP_{t+1|t}C)^{-1} \]
\[ = (I - K_{t+1}C)P_{t+1|t}^{-1} + (I - K_{t+1}C)P_{t+1|t}^{-1}C^T R^{-1} \]
\[ = K_{t+1} + K_{t+1}(R + CP_{t+1|t}C)^{-1} \]
\[ = K_{t+1} + K_{t+1}(R + CP_{t+1|t}C)^{-1} \]
\[ = K_{t+1}(\hat{\theta}_{t+1} - C\mu_{t+1|t}). \]

\[ \]