Quantum self-consistency of $AdS \times \Sigma$ brane models.

Antonino Flachi$, Oriol Pujolàs†

IFAE, Campus UAB, 08193 Bellaterra (Barcelona), Spain

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Continuing on our previous work, we consider a class of higher dimensional brane models with the topology of $AdS_{D+1} \times \Sigma$, where $\Sigma$ is a one-parameter compact manifold and two branes of codimension one are located at the orbifold fixed points. We consider a set-up where such a solution arises from Einstein-Yang-Mills theory and evaluate the one-loop effective potential induced by gauge fields and by a generic bulk scalar field. We show that this type of brane models resolves the gauge hierarchy between the Planck and electroweak scales through redshift effects due to the warp factor $a = e^{-kr}$. The value of $a$ is then fixed by minimizing the effective potential. We find that, as in the Randall Sundrum case, the gauge field contribution to the effective potential stabilizes the hierarchy without fine-tuning as long as the laplacian $\Delta_{\Sigma}$ on $\Sigma$ has a zero eigenvalue. Scalar fields can stabilise the hierarchy depending on the mass and the non-minimal coupling. We also address the quantum self-consistency of the solution, showing that the classical brane solution is not spoiled by quantum effects.

Keywords: Extra dimensions; Brane models; Hierarchy stabilisation

I. INTRODUCTION

An original way to address the gauge hierarchy problem has been suggested by Randall and Sundrum, by considering five dimensional anti de Sitter spacetime compactified on $S^{1}/Z_{2}$ and two 3–branes located at the orbifold fixed points [1]. This setup results in a non factorisable geometry, which has the virtue of resolving the large ratio between the Planck and electroweak scales as a geometrical effect: in the four dimensional effective theory the masses on the negative tension brane are redshifted by a factor $a = e^{-kr}$ ($1/k$ is the curvature of anti de Sitter spacetime and $r$ is the radius of the orbifold). Hence, it is possible to generate a TeV mass scale from a Planck sized mass by taking $kr \sim 12$.

Obviously, to make this scenario consistent, the size of the orbifold has to be determined dynamically and not fixed by hand. In this sense, the RS model does not completely solve the hierarchy problem, unless a stabilisation mechanism for the size of the fifth dimension is included. (It is worth noting that this is not a peculiarity of the RS model, rather a well known feature of higher dimensional theories with extra dimensions.)

Within the RS proposal, a way of achieving such a stabilisation was initially suggested by Goldberger and Wise, introducing an appropriate classical interaction between the branes and a bulk scalar field [2]. In this way, it is possible to stabilise the extra fifth dimension without fine tuning, however, the lack of a fundamental origin for such interaction renders such a mechanism artificial.

An alternative to the GW mechanism can, in principle, be the Casimir energy generated by quantum fields. Already in the old Kaluza-Klein theories, Candelas and Weinberg looked at this possibility [3], and, inspired by their work, several authors investigated the role of quantum effects in the new brane models [4–9]. In particular, as a result of such studies, it has recently been realized [10] that the quantum effective potential due to bulk gauge fields can stabilise the radion and generate the hierarchy of scales without fine tuning. This provides a viable alternative to the GW mechanism in the RS model. Further aspects of quantum effects have been investigated in [11–20].

The brane world idea has opened up a range of interesting possibilities in addressing many long standing problems of particle physics and cosmology. As one of the prototypes, the RS scenario can be viewed as a model belonging to a larger class and especially in connection with a possible embedding of such scenarios within string theory, it is worth exploring at more depth extensions of the RS model by considering spacetimes with higher dimensionalities and curved internal spaces (Explicit six dimensional examples can be found in [21–26]). Needless to say, such extensions provide models with a richer structure than the RS one (See [27,28]).

Many generalisations of the Randall-Sundrum model fall in the quite general class of higher dimensional warped solutions studied in [29], where a $D$–dimensional system of gravity plus Yang-Mills is considered. The base spacetime is described by the following line element:

$$ds^2 = e^{2\sigma(y)}g_{\mu\nu}dx^\mu dx^\nu + e^{2\rho(y)}g_{ij}dX^idX^j + dy^2, \quad (1)$$

where the coordinates $x^\mu$ parametrise $D_1$ dimensional Minkowski space $M$, the coordinates $X^i$ cover a
$D_2$-dimensional compact internal manifold $\Sigma$ of radius $R$ and the coordinate $y \in [-\pi, \pi]$ parametrises the orbifold. We define $D = D_1 + D_2 + 1$ and take $D_1 = 4$. Hatted quantities refer to higher dimensional ones.

Depending on the geometry of the internal space, Einstein equations lead to different types of warp factors $\sigma(y)$ and $\rho(y)$: when the internal space is a Ricci flat manifold and the Yang-Mills flux is switched off, the general result for the warp factors is given by a combination of exponentials. Simpler solutions with

$$\sigma(y) = \rho(y) = -k|y|, \quad (2)$$

are found when the bulk cosmological constant is taken to be negative. Additionally, the condition of Ricci flatness of $\Sigma$ can be relaxed at the price of introducing some extra bulk matter, like, for instance, a scalar field with hedgehog configuration [23,24]. In the above cited papers, the set-up allows for the presence of one brane only; two brane models can be constructed by gluing two slices of the previous spacetime and imposing the $Z_2$-identification.

Solutions of the type $AdS_{D_1+1} \times \Sigma$ are, instead, found when the internal space is non Ricci flat without adding any extra bulk matter. In such case the warp factor along $\Sigma$ is constant and we can take

$$\rho(y) = 0, \sigma(y) = -k|y|. \quad (3)$$

We note that in such case the requirement of a negative higher dimensional cosmological constant can be relaxed.

In a previous paper [30], we have considered scenarios of the type (1), (2) and evaluated the effective potential arising from massive, non-minimally coupled scalar fields. We showed that one-loop effects generate a suitable effective potential which can stabilise the hierarchy without fine tuning, provided that the internal space is flat. Aside, we have also seen that considering a warped internal manifold was providing a novel way to solve the hierarchy, which was relying on a mixture of large volume and redshift effects.

Here, we wish to extend the previous results to the second type of spacetimes (1), (3)*.

The plan of the paper is the following. In the next section we briefly discuss the model and comment on the relevant mass scales of our set-up. Section III is devoted to compute the quantum effective potential arising from a quantised bulk scalar and gauge field. The possibility of stabilising the hierarchy by using quantum effects is discussed in section IV, where the problem of the quantum self-consistency of the solution is also addressed. In the last section we report our conclusions. Some results concerning the uniform asymptotic expansion of the Bessel functions (needed in the computation of the effective potential) are collected in appendix.

II. BACKGROUND SOLUTION AND SCALES

In the present section we describe the background solution when branes are included and discuss the relevant scales of the problem.

The spacetime we consider corresponds to the line element (1) where the warpings satisfy the condition (3). In other words, the bulk spacetime we consider is the direct product of five dimensional anti de Sitter space and a compact one-parameter manifold $\Sigma$.

Randjbar-Daemi and Shaposhnikov have considered this type of solutions and showed that they arise from a system of gravity plus Yang-Mills fields [23,29], with bulk action given by

$$S_{BG} = \int d^Dx \sqrt{g} \left\{ M^{D-2} \mathcal{R} - \frac{1}{4g_s^2} \hat{F}_{IJ}\hat{F}^{IJ} \right\}. \quad (4)$$

The equations of motion can be obtained in the standard way, and once the ansatz for the metric tensor (1), (3) is used, the following independent equations are obtained:

$$k^2 = -\frac{\hat{M}^{2-D} \hat{\Lambda}}{D_1(D-2)} + \frac{\hat{M}^{2-D}}{D_1(D-2)} \frac{\hat{\mathcal{F}}^2}{4g_s^2 R^4}, \quad (5)$$

$$\frac{\Omega}{R^2} = \frac{\hat{M}^{2-D}}{D-2} \left\{ \hat{\Lambda} + \frac{2D - D_2 - 4}{D_2} \frac{\hat{\mathcal{F}}^2}{4g_s^2 R^4} \right\}, \quad (6)$$

where we have expressed the curvature $\mathcal{R}$ of the internal manifold in terms of its radius $R$ ($\Omega$ is a constant):

$$\mathcal{R}_\Sigma = \frac{D_2 \Omega}{R^2},$$

and

$$\frac{\hat{\mathcal{F}}^2}{R^4} = \hat{g}^{IJ} \hat{g}^{MN} \hat{F}_{IJ} \hat{F}_{MN}. \quad (7)$$

The previous equations (5), (6) allow us to determine the radius of the internal manifold and the Yang-Mills flux in terms of $\hat{\Lambda}$, $\hat{M}$ and $k$:

$$R^2 = \Omega D_2 P^2, \quad (7)$$

$$\frac{\hat{\mathcal{F}}^2}{4g_s^2} = D_2^2 \Omega^2 P^4 (\hat{\Lambda} + D_1(D-2)k^2 \hat{M}^{D-2}), \quad (8)$$

*In passing, it is worth noting that this type of metrics arise in string theory. Here we won’t be concerned with any string theory application, and simply refer the reader to references [31,32], where this type of solutions are found in the more fundamental context of supergravity or M-theory.
where, for notational convenience, we have defined

\[ P^{-2} = 2\tilde{M}^{2-D}\Lambda + D_1(2D - D_2 - 4)(D - 2)k^2. \]

One immediately notices that the radius of $\Sigma$ is ‘stabilised’ at classical level at the price of tuning the Yang-Mills flux according to (8). In this sense we point out some analogy with the recent work of Carroll and Guica [33], who considered the direct product of Minkowski space and a 2-sphere. In their case the radius of the 2-sphere is stabilised by the flux and a relaxing the tuning of such flux would induce a de Sitter or anti-de Sitter geometry rather than Minkowski. The same is also true in our case with the additional modification of the warp factor.

Since we are considering two branes embedded in such a spacetime, we have to add to the action appropriate brane tension terms. It is easy to see that there are no solutions of the type considered here, if the tension term is isotropic. The requirement of conservation of the higher dimensional energy-momentum tensor along with the junction conditions forces us to introduce such anisotropy\(^1\) [25,26,34].

The brane energy-momentum tensor is then given by:

\[
T_{\mu}^{\nu} = \delta(y) \text{diag}\left(\tau_{-}^{M}\delta_{\mu}^{\nu}, \tau_{+}^{\Sigma}\delta_{i}^{j}\right) + \\
+ \delta(y - \pi r) \text{diag}\left(\tau_{+}^{M}\delta_{\mu}^{\nu}, \tau_{+}^{\Sigma}\delta_{i}^{j}\right).
\]

The spacetime we are considering can then be constructed by gluing two copies of a slice of the bulk space and imposing the $Z_2$–identification. The Israel junction conditions fix the brane tensions to be

\[
\tau_{\pm}^{M} = \frac{D_1 - 1}{D_1}\tau_{\pm}^{\Sigma} = \frac{D_1 - 1}{D_1} \left(\mp 4D_1k\tilde{M}^{D-2}\right). \tag{10}
\]

We can now look at the physical scales to see whether such class of models suggests anything about the gauge hierarchy problem.

By integrating out the extra dimensions we can write a relation between the four- and higher-dimensional Planck scales

\[
m_p^2 = \frac{v_\Sigma}{D_1 - 2}(\hat{M}R)^{D_2}\frac{\hat{M}}{k}\hat{M}^{D_1 - 2}, \tag{11}
\]

where

\[
v_\Sigma R^{D_2} = \int_{\Sigma} \sqrt{g_\Sigma} d^2x
\]

and the EW/Planck hierarchy can then be written, for $D_1 = 4$, as

\[
h^2 = a^2\tilde{M}^2 m_p^2 \sim \frac{a^2}{(RM)^{D_2}} \frac{k}{\hat{M}} \sim 10^{-32}. \tag{12}
\]

We see that, analogously to [30], the hierarchy $h$ is expressed in terms of $a$ and $R$, however in the present case it is not possible to use both the redshift and large volume effects as in our previous work [30]. To see this, we remind that in the case of equal warpings the crucial ingredient was that the internal manifold was growing exponentially away from the negative tension brane located at $y = y_-$ and this was diluting gravity as in models with large extra dimension. On the other hand, gauge interactions, confined on the negative tension brane, were not diluted because the size of $\Sigma$ at $y = y_-$ was of order of the fundamental cut-off.

Here the situation is different as we are considering the direct product $AdS \times \Sigma$. In such case, the size, $R$, of the internal manifold has to be everywhere small, if we require that the extra $\Sigma$-dimensions remain invisible to ordinary matter. Since $R$ is determined at classical level, Einstein equations leave us with a first ‘consistency’ check on such class of models if we were going to construct any (pseudo-)realistic scenario.

If we express the cosmological constant by factoring out two powers of the mass,

\[
\hat{\Lambda} = \lambda^2 \hat{M}^{D-2}, \tag{13}
\]

relations (7), (8) can be recast in the following form:

\[
\frac{F^2}{4g^2} \sim \frac{\hat{M}^{D-2}}{(k^2 + \lambda^2)}, \tag{14}
\]

\[
R^2 \sim \frac{1}{k^2 + \lambda^2}. \tag{15}
\]

Now, a natural assumption is that the bulk cosmological constant is of the same order as the higher dimensional Planck scale, $\lambda \sim \hat{M}$, and $k$ smaller than $\hat{M}$, implying

\[
R \sim \hat{M}^{-1}, \tag{16}
\]

meaning that the size of the internal manifold is of order of the cut-off and thus satisfying the requirement of small $R$. The previous relation also implies that

\[
(kR)^2 \sim \frac{k^2}{\lambda^2 + k^2} < 1. \tag{17}
\]

This last condition will be tacitly used in the subsequent computation of the effective potential.

From the gauge hierarchy point of view, this class of models does not suggest any improvement with respect to the RS model. As one can see from (12) and (16), the hierarchy is resolved only through redshift effects. Obviously, one could relax the condition $\lambda \sim \hat{M}$, but this in turn would interchange the gauge hierarchy problem with the need for an ‘ad hoc’ tuning of the bulk cosmological constant, as we would have to justify a value of $\lambda$ different from its natural value $\hat{M}$.

\(^1\)The source for such anisotropy can be due to different contributions to the vacuum energy or also due to a background three-form field [34].
III. QUANTISED FIELDS

As we pointed out in our previous paper, quantum effects in scenarios with more than one extra dimension can be qualitatively different from models without internal spaces and can, in principle, provide new ways of addressing the hierarchy. It then seems reasonable to ask the same question in relation to the class of models described previously.

Therefore we devote this section to the computation of the one-loop effective potential arising from a massive bulk scalar field \( \Phi(x, X, y) \) coupled non-minimally to the higher dimensional curvature. We also point out that, as noted in [10,35], it is possible to relate the effective potential from a bulk scalar with the one arising from a gauge field, the computation being virtually the same.

It is possible to do so by appropriately fixing the non-minimal coupling and the bulk mass of the scalar field to (we take \( D_1 = 4 \) and work in the physical gauge)

\[
\xi = 1/8 ,
\]
\[
m^2 = -k^2/2 .
\]  
(18)  
(19)

The field equation for \( \Phi(x, X, y) \) is given by the Klein-Gordon equation

\[
\left[ -\Box_D + m^2 + \xi \hat{R} \right] \Phi = 0 ,
\]  
(20)

where \( \hat{R} \) is the higher dimensional curvature and \( \Box_D \) the D’Alembertian, both computed from the metric (1), (3).

Standard Kaluza-Klein theory tells us that such a higher dimensional field can be expressed in terms of a complete set of modes, which describe a tower of fields with masses quantised according to some eigenvalue problem. Such a decomposition is, of course, arbitrary, however a convenient choice is

\[ \Phi(x, X, y) = \sum_{l,n} \Psi_l(X) \varphi_{l,n}(x) Z_{l,n}(y) , \]  
(21)

where the modes \( \Psi_l(X) \) are chosen to be a complete set of solutions of the Klein-Gordon equation on the manifold \( \Sigma \):

\[ P_\Sigma \Psi_l(X) = \left[ -\Delta_\Sigma + \xi R_\Sigma \right] \Psi_l(X) = \frac{1}{R^2} \lambda^2_l \Psi_l(X) , \]  
(22)

with eigenvalues \( \lambda^2_l \) (independent of \( R \)) and degeneracy \( y_l \). If we now require \( \varphi_{l,n}(x) \) to satisfy the Klein-Gordon equation in Minkowski spacetime, \( M \), with masses \( m^2_{l,n} \),

\[ \left[ -\square + m^2_{l,n} \right] \Phi_{l,n}(x) = 0 , \]  
(23)
equation (20) leaves us with a radial equation for the modes \( Z_{l,n}(y) \)

\[
\mathcal{D}^{(l)}_y Z_{l,n} = m^2_{l,n} Z_{l,n} , \]  
(24)

where the differential operator \( \mathcal{D}_y \) is given by

\[
\mathcal{D}^{(l)}_y = e^{2\sigma} \left[ -e^{-D_1 \sigma} \partial_y e^{D_1 \sigma} \partial_y + \mu^2 - 2D_1 \xi \sigma'' \right] , \]  
(25)

and

\[
\mu^2_l = m^2 + \frac{1}{R^2} \xi^2 - D_1 (D_1 + 1) k^2 \xi . \]  
(26)

The most general solution to (24) can be written in terms of Bessel functions and by imposing the appropriate boundary conditions, we find that the eigenvalues \( m_n \) are determined by the transcendental equation:

\[
F^\beta_\nu \left( \frac{m_n k^2}{ka} \right) = 0 . \]  
(27)

The function \( F^\beta_\nu (z) \) is given by

\[
F^\beta_\nu(z) = Y^\beta_\nu(az) J^\beta_\nu(z) - J^\beta_\nu(az) Y^\beta_\nu(z) , \]  
(28)

where \( \nu^2 = \frac{\mu^2}{k^2} + \frac{D_1^2}{4} \) (29)

and

\[
J^\beta_\nu(z) = J_\nu(z) \]  
(30)

for twisted field configurations (\( Z_{n,l}(-y) = -Z_{n,l}(y) \)) or

\[
J^\beta_\nu(z) = j_\nu(z) \]  
(31)

for untwisted ones (\( Z_{n,l}(-y) = Z_{n,l}(y) \)). Analogous expressions are valid also for \( Y^\beta_\nu(z) \). In the following we focus on the case of untwisted fields only.

The one loop effective action \( \Gamma^{(1)} \) can be expressed as the sum over the contributions of each mode [36]:

\[
\Gamma^{(1)} = - \int d^{4-2\epsilon} x \, V(s) , \]  
(32)

with

\[
V(s) = - \frac{2^{2\epsilon}}{(4\pi)^{2\epsilon}} \Gamma(s) \sum_{n,l} g_s m^{2\epsilon}_{n,l} , \]  
(33)

where the prime in the sum assumes that the zero mass mode is excluded and \( s = -2 + \epsilon \). We are using dimensional regularisation and continuing along Minkowski spacetime \( (4 \rightarrow 4-2\epsilon) \) and \( \mu \) is a renormalisation scale introduced for dimensional reasons.

In order to evaluate the sum in (33), we find convenient to separate the \( \lambda_0 \)-mode from the rest of the tower\(^4\):

\[
\]
\(^4\)This procedure is not essential, however, by performing such spitting, the RS contribution comes about explicitly. Moreover, the RS divergence has to cancel when the two contributions are summed and this provides a non-trivial check of the calculation.
\[ V(s) = V_{RS}(s) + V_*(s) . \]  

The first term corresponds to the usual Randall-Sundrum contribution:

\[ V_{RS}(s) = -\frac{(ka)^4}{2(4\pi)^2}(ka/\mu)^{-2s} \Gamma(s) \sum_{n} g_0 x_{n,0}^{-2s} , \]

with \( x_{n,t} = \frac{m_n}{\Lambda_t} \). This term, present only when \( g_0 = 1 \), has been evaluated in [5,4] and we report the result without further comments:

\[ V_{RS} = -g_0 \frac{k^4}{32\pi^2}(k/\mu)^{-2s} \left\{ -d_1 \frac{1}{\epsilon} (1 + a^4 - 2a^2) + c_1 + a^4 c_2 - 2a^4 \mathcal{V}(a) \right\} , \]

where

\[ \mathcal{V}(a) = \int_{0}^{\infty} dz z^3 \ln \left( 1 - \frac{k_\nu(z) i_\nu(az)}{k_\nu(az) i_\nu(z)} \right) \]

and the coefficients \( c_1 \) and \( c_2 \) do not depend on \( a \). The remaining term in (34) is given by

\[ V_*(s) = -\frac{(ka)^4}{2(4\pi)^2}(ka/\mu)^{-2s} \Gamma(s) \sum_{n,l=1}^\infty g_l x_{n,l}^{-2s} , \]

and can be handled in the usual manner by transforming it into a contour integral and by deforming the contour appropriately, according to a general technique developed in [37,38] (See [39] for a comprehensive review). Standard manipulations lead to

\[ V_*(s) = -\frac{(ka)^4}{2(4\pi)^2}(ka/\mu)^{-2s} \Gamma(s) \sum_{l=1}^\infty g_l \int_{0}^{\infty} dz z^{-2s} \frac{d}{dz} \ln P_{\nu_l}(z) \]

where

\[ P_{\nu_l}(z) = F_{\nu_l}(iz) = i_\nu(az) k_{\nu_l}(z) - i_\nu(z) k_{\nu_l}(az) , \]

and

\[ i_\nu(z) = z I'_\nu(z) + \frac{1}{2} D_1 (1 - 4\xi) I_\nu(z) , \]

\[ k_\nu(z) = z K'_\nu(z) + \frac{1}{2} D_1 (1 - 4\xi) K_\nu(z) . \]

Now we have to analytically continue the previous expression (39) to the left of \( \Re(s) < 1/2 \). A possible way of achieving this is to employ the uniform asymptotic expansion (UAE). This is because the order of the Bessel function depends explicitly on the eigenvalues \( \lambda_l \). In order to apply the UAE, we rescale the integral (39), \( z \to \nu_l z \):

\[ V_*(s) = \left(ka\right)^4 \frac{(ka/\mu)^{-2s}}{2(4\pi)^2 \Gamma(1-s)} \sum_{l=1}^\infty g_l \int_{0}^{\infty} dz z^{-2s} \frac{d}{dz} \ln P_{\nu_l}(\nu_l z) , \]

and to isolate the divergent part, we express the integrand as its large \( \nu_l \) portion plus terms leading to finite contributions.

By using (A1), (A2), we can recast (41) as the sum of three terms:

\[ V_*(s) = V_1 + V_2 + V_3 , \]

with

\[ V_j = -\frac{(ka)^4}{2(4\pi)^2 \Gamma(1-s)} \sum_{l=1}^\infty g_l \nu_l^{2s} \]

\[ \int_{0}^{\infty} dz z^{-2s} \frac{d}{dz} \ln H_j(z) , \]

and

\[ H_1(z) = (1 + a^2 z^2)^{1/4} e^{-\nu_l \eta(az)} (1 + z^2)^{1/4} e^{\nu_l \eta(z)} , \]

\[ H_2(z) = \Sigma_1^{(I)}(z) \Sigma_1^{(K)}(az) , \]

\[ H_3(z) = 1 - e^{2\nu_l (\eta(az) - \eta(z))} \Sigma_1^{(I)}(az) \Sigma_1^{(K)}(z) \Sigma_1^{(I)}(z) \Sigma_1^{(K)}(az) , \]

where \( \eta(z) \) is defined in the appendix. The first term is straightforward to evaluate and gives

\[ V_1 = -\frac{(ka)^4}{8(4\pi)^2} (ka/\mu)^{-2s} \left\{ \Gamma(s) \hat{\zeta}(s)(1 + a^2 z^2) - \right. \]

\[ \left. - \frac{1}{2\sqrt{\pi}} \Gamma(s - 1/2) \hat{\zeta}(s - 1/2)(1 - a^2 z^2) \right\} . \]

The second one is slightly more involved to evaluate. The uniform asymptotic expansion (A8), (A9) allows us to write

\[ V_2 = \frac{(ka)^4}{2(4\pi)^2} (ka/\mu)^{-2s} \left\{ \sum_{n=1}^{\infty} \sum_{k=0}^{n} (1 + (-1)^n a^2 z^2) \right. \]

\[ \left. + \sum_{n=1}^{\infty} g_l \nu_l^{-2s} \cdot \right\} . \]

In order to deal with the sum over the eigenvalues \( \nu_l \), we have defined the following base \( \hat{\zeta} \)-function:

\[ \hat{\zeta}(s) = \sum_{l=1}^\infty g_l \nu_l^{-2s} . \]

The last term in (42) is the usual non local contribution and, since it is finite by construction, we can safely put \( \epsilon = 0 \):
\[ V_3 = \frac{(ka)^4}{2(4\pi)^2} \sum_{n=-\infty}^{\infty} \sum_{q=0}^{\infty} \left| g_n \nu_1(a) \right| \]  

where \( \nu_1(a) = \int_0^\infty dz z^3 \ln \left\{ 1 - \frac{iz_n(az)k_n(z)}{iz_n(z)k_n(az)} \right\} \) .

In order to make the \( R \)-dependence in (44) and (45) explicit, it is convenient to rescale the above defined \( \zeta \)-function by expanding the binomial. A simple calculation gives:

\[ \zeta(s) = \frac{(kR)^{2s}}{\Gamma(s)} \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} (kR\nu)^{2q} \Gamma(s + q) \zeta(s + q) \]  

where

\[ \zeta(s) = \sum_{l=1}^{\infty} g_l \lambda_l^{2s} \]  

does not depend on \( R \) and

\[ \nu^2 = \frac{n^2}{k^2} - D_1(1 + D_1)\xi + \frac{D_1^2}{4} \] .

The use of (49) allows us to express the result in terms of the generalised \( \zeta \)-function (50) and the additional (Mittag-Leffler) representation for the \( \zeta \)-function can then be used to deal with the pole structure of (50) and express the residues at the poles in terms of geometrical quantities [40]. The Mittag-Leffler representation for the \( \zeta \)-function associated with the operator \( P_\Sigma \) (see, for example, [30]) is

\[ \zeta(s) = \frac{1}{\Gamma(s)} \left\{ \sum_{p=0}^{\infty} \frac{\tilde{C}_p}{s - D_2/2 + p} + f(s) \right\} \]  

where \( \tilde{C}_p = C_p - g_0\delta_pD_2/2 \) and the \( C_p \) are the heat-kernel coefficients of the operator \( P_\Sigma \), \( p \) runs over the positive half integers and \( f(s) \) is an entire function. As in the case of [30], since the internal space \( \Sigma \) is boundaryless the heat-kernel coefficients of semi-integer order are zero.

Relation (51) can now be used to regulate the effective potential and some calculations lead to

\[ V(s) = \frac{(ka)^4}{2(4\pi)^2} (k\lambda/a)^{-2\epsilon} (kR)^{2s} \sum_{n=-\infty}^{\infty} \sum_{q=0}^{\infty} (1 + (-1)^na^{2s}) \left( \frac{1}{\epsilon} a_{n,q} + b_{n,q} \right) (kR)^{2q+n} \]

\[ - \frac{g_0k^4}{2(4\pi)^2} (k/\mu)^{-2\epsilon} \left\{ - \frac{d_s}{\epsilon} (1 + a^{4-2\epsilon}) + c_1 + a^4c_2 \right\} \]

\[ + \frac{(ka)^4}{2(4\pi)^2} \right\{ g_0\nu(a) + \sum_{l=1}^{\infty} g_l \nu_1(a) \right\} \]

where the coefficients of the previous expression can be written as

\[ a_{n,q} = \frac{(-1)^q}{q!} \nu^2 C_{2D_2/2-n/2} A_n \]

where

\[ A_{-1} = \frac{1}{8\sqrt{\pi}} \] ,

\[ A_q = -\frac{1}{4} \] ,

\[ A_n = \sum_{k=0}^{n} S_{n,k} \] , for \( n > 1 \)

\[ S_{n,k} = \frac{\Gamma(k + n/2 + s)}{\Gamma(k + n/2)\Gamma(n/2 + s)} \sigma_{n,k} \]

The coefficients \( b_{n,q} \) are related to the \( a_{n,q} \) via the following correspondence

\[ b_{n,q} = a_{n,q}(\tilde{C}_p \rightarrow \Omega^{-p}) \] ,

where the \( \Omega_p \) represent the finite part in the power series of \( \Gamma(s)\zeta(s) \) around \( s = p \).

A check on the previous result is provided by the cancellation of the (lower dimensional) RS divergence, given by

\[ g_0 k^2 \frac{(1 + a^4)}{32\pi^2 \epsilon} = g_0 k^4 \frac{(1 + a^4)}{32\pi^2 \epsilon} (\Delta_0 + \Delta_2 \nu^2 + \Delta_4 \nu^4) \],

where

\[ \Delta_0 = \frac{27}{128} + \frac{3}{8} \Delta - \frac{1}{2} \Delta^2 + \frac{1}{2} \Delta^3 - \frac{1}{4} \Delta^4 \]

\[ \Delta_2 = \frac{13}{16} - \frac{1}{2} \Delta^2 \]

\[ \Delta_4 = -\frac{1}{8} \]

\[ \Delta = \frac{1}{2} D_1(1 - 4\xi) \] .

A simple inspection of (52) shows that the relevant terms for such a cancellation are the ones corresponding to the couples \( (n,q) = (0,2) \) , \( (2,1) \) , \( (4,0) \). Such terms can be easily extracted from (52) and the use of the coefficients \( \sigma_{n,k} \) (the relevant ones are reported in the appendix, (A10)) shows that the RS divergence is indeed cancelled.

The result for the vacuum energy (52) is divergent and needs to be renormalised. The counterterm action can be constructed analogously to the case of two equal warappings [30]:

\[ S_{n,q} = \frac{1}{32\pi^2 \epsilon} \int d^4x \Sigma \left( \frac{kR^{2q+n}}{R^4} \{ a^{4-2\epsilon} + (-1)^n \} \right) \kappa_{(n,q)} \]
where we have defined (the factor proportional to \(\nu\) has been reabsorbed in the coefficients \(\kappa_{(n,q)}\))

\[
\kappa_{(n,q)} = (-1)^n \kappa_{(n,q)} = k^{2q+n} \kappa_{(n,q)},
\]

and it is easy to see that all the divergences can be reabsorbed in counterterms of the previous type. Once we subtract the counter-terms, we arrive at the following expression for the renormalised effective potential

\[
V(a) = \frac{(1/R)^4}{2(4\pi)^2} \sum_{n=-1}^{\infty} \sum_{q=0}^{\infty} \frac{a_{n,q} \ln(\mu R)^2 + b_{n,q}}{(a^4 + (-1)^n)^{-1}} (kR)^{2q+n}
- \frac{g_{0}k^4}{(4\pi)^2} \left[ c_1 + a^4 c_2 + (1 + a^4)d_1 \ln(k/\mu)^2 \right]
+ \frac{(ka)^4}{(4\pi)^2} \left[ g_{0}V(a) + \sum_{i=1}^{\infty} g_{i} V_{i}(a) \right].
\]

(57)

IV. RADION STABILISATION AND QUANTUM SELF-CONSISTENCY OF THE SOLUTION

In the previous section we have computed and renormalised the Casimir energy arising from a massive bulk scalar field non-minimally coupled to the curvature and from a massless bulk gauge field. So we are now in the position to see whether or not quantum effects provide a reasonable stabilisation mechanism for the class of models of the type \(AdS \times \Sigma\). To this aim, let us consider the full action \(S\), where we include the contribution \(\Gamma^{(1)}\) arising from a quantised field:

\[
S = S_{BG} + \Gamma^{(1)},
\]

(58)

where we generically write the quantum contribution as

\[
\Gamma^{(1)} = - \int d^4x \sqrt{g} V(a).
\]

(59)

\(S_{BG}\) is the classical background action obtained by using the ansatz for the metric (1) (with \(\eta_{\mu\nu} \rightarrow \eta_{\mu\nu}(x)\)) in (4) and by integrating out the extra \(D_2 + 1\) dimensions. Now, varying the full action \(S\) with respect to \(\tilde{g}_{\mu\nu}(x)\)

\[
\frac{\delta S}{\delta g_{\mu\nu}} = 0,
\]

(60)

and requiring that the minimum is at \(\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu}\) will tell us whether or not the classical solution is spoiled by quantum effects. On the other side, varying \(S\) with respect to the radion \(a\)

\[
\frac{\delta S}{\delta a} = 0,
\]

(61)

at \(\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu}\), will tell us whether we can obtain an exponentially large hierarchy, \(a = e^{-\pi kr}\) (with \(kr \sim 12\)), in which case such solution also solves the hierarchy problem. We want to stress that one can have solutions that satisfy (60) but not (61) and therefore are self-consistent but do not solve the hierarchy problem. A simple calculation shows that by combining the previous requirements (60), (61), the following constraint for the effective potential is obtained:

\[
(1 - a^4)V'(a) + 4a^3V(a) = 0,
\]

(62)

where the prime denotes derivative with respect to \(a\). Equation (62) is exactly the same as the one obtained for the RS model [5].

We have now to specify the matter content of our model and in turn the function \(V(a)\). We consider two possibilities: a minimal model whose action is given by (4) and a non-minimal model where (4) is our classical background theory upon which we quantise a bulk scalar field. In the first case, we assume that the gauge field splits into a classical plus a quantum part:

\[
A_{\mu} = A_{\mu}^{C} + A_{\mu}^{Q},
\]

(63)

and thus the quantum contribution comes from the quantum counterpart of the gauge field. We shall consider the \(AdS\) components only, which have a zero vev and do not couple to the Yang-Mills flux configuration. In the non-minimal case, it is the scalar field that provides us with the quantum effective potential.

We recast the result for the effective potential as follows:

\[
V(a) = \frac{k^4}{32\pi^2} \left\{ \Gamma_1 + a^4 \Gamma_2 + \Gamma_{NL}(a) \right\}
\]

(64)

where \(\Gamma_1\) and \(\Gamma_2\) do not depend on \(a\). The non-local contribution,

\[
\Gamma_{NL}(a) = a^4 V(a) + a^4 \sum_{i=1}^{\infty} g_{i} V_{i}(a),
\]

(65)

is slightly more involved to inspect, however, in our case, it is sufficient to see that the contribution coming from the massive Kaluza-Klein modes (involving the sum over \(l\)) is highly suppressed with respect to the (RS) zero-mode term, proportional to \(V(a)\). This can be shown by noticing that the dominant contribution to the integral in \(V_{i}(a)\) comes from the region \(z \lesssim 1\). Expanding the integrand in such region allows one to see that \(V_{i}(a)\) goes like \(a^{2k_{i}}\) and a simple inspection of the sum tells us that the non-local contribution coming from the massive KK states is proportional to powers of \(a^{1/(kr)}\). The non local contribution can then be approximated as

\[
\Gamma_{NL}(a) \simeq a^4 V(a).
\]

(66)

Fixing the field content of the theory (or the bulk parameters) will uniquely determine the function \(V(a)\). (Such
term has been evaluated for any of \( \nu \) in [10]). By expanding the integrand for small \( a \), one finds that in the minimal case (only with a quantised gauge field)

\[
V(a) = \frac{\gamma}{\ln a}, \quad (67)
\]

with \( \gamma \) being \( a \)-independent. In the non-minimal case one has to distinguish three possibilities: when the order of the Bessel functions is \( \nu = 0 \), this corresponds to taking

\[
\xi = \frac{4m^2 + D_1^2 k^2}{4D_1(D_1 + 1)k^2}, \quad (68)
\]

when \( \nu = 1 \) and this corresponds to fixing the values of \( \xi \) and \( m \) according to (18) and (19), and finally, when \( \nu \) is different from the two previous values\(^8\). In the first case, we find

\[
V(a) = \frac{\beta}{\alpha + \ln a}, \quad (69)
\]

where \( \alpha \) and \( \beta \) do not depend on \( a \). The second case, obviously, gives back relation (67), whereas in the third case \( V(a) \) is proportional to \( a^N \) with \( N \geq 4 \).

The previous relations along with the self-consistency condition (62) allow us to see in which cases we obtain a solution to the hierarchy problem with the bonus for the solution to be self-consistent.

By using the expression for the effective potential (64) and (67), we find that the solution to (62) in the minimal case, in the limit of \( a \ll 1 \), is

\[
a \sim e^{-\gamma/(\Gamma_1 + \Gamma_2)} \quad (70)
\]

which shows that there is no need of any fine tuning in order to get an exponentially small \( a \).

In the non-minimal case, one can easily check that fixing the parameters \( \xi \) and \( m \) according to (19) or (68) provides also a large hierarchy, whereas in the other cases no solution to (62) is found for small values of \( a \).

**V. CONCLUSIONS**

In this article we have investigated the role of quantum effects arising from bulk fields in higher dimensional brane models. Specifically, we have considered a class of warped brane models whose topology is \( AdS_5 \times \Sigma \), where \( \Sigma \) is a \( D_2 \) dimensional one-parameter compact manifold and two branes of codimension one are placed at the orbifold fixed points.

We have seen that such a set-up can be obtained from Einstein-Yang-Mills theory. Contrarily to the case studied in [30], where both the radion \( a \) and the radius of \( \Sigma \) were undetermined classically, here the radius of the internal space \( \Sigma \) is stabilised at a size comparable with the higher dimensional cut-off once the Yang-Mills flux is tuned according to (8). This guarantees that, when matter is placed on the wall, the extra dimensions in the \( \Sigma \)-direction remain invisible, as it must be. On the other hand, the fact that the size of the internal manifold is of order \( 1/M \), does not suggest any new way of addressing the hierarchy, which is resolved only through a redshift effect coming from the \( AdS \) direction.

We considered two possible scenarios: a first one, which we labeled as ‘minimal’, where the action is just the Einstein-Yang-Mills one and only the Yang-Mills field is quantised, and a second one, called ‘non-minimal’, in which we quantised a scalar field on a classical Einstein-Yang-Mills background.

We evaluated the renormalised one-loop effective action at lowest order, namely the Casimir energy in the case of a massive non-minimally coupled scalar field. The resulting scalar effective potential can be related to the one arising from quantised gauge fields by appropriately fixing the parameters \( \xi \) and \( m \).

The computation is similar to the one carried out in [30], with some technical differences due to the explicit presence of the eigenvalues of the scalar operator on the manifold \( \Sigma \) in the order of the Bessel functions. This can be effectively dealt with by using the uniform asymptotic expansion of the modes, which turned out slightly more involved than the corresponding computation in the case of [30]. On the other hand, the Mittag-Leffler expansion for the generalized \( \xi \)-function allowed us to express the Casimir energy in terms of heat-kernel coefficients of the internal space \( \Sigma \) as in the case previously considered in [30]. The same non-trivial check of the cancellation of the RS divergence works. Also the renormalisation is carried out analogously to [30] by subtracting suitable counter-terms proportional to a number of boundary or bulk local operators.

Finally we investigated the self consistency of the model by requiring that the quantum corrected action is minimised by the background solution. As for the stabilization our analysis indicates that the Casimir force can stabilise the radion without fine tuning thanks to any KK mode whose index \( \nu \) is 0 or 1. The latter is reproduced in the ‘minimal’ case by the zero mode of the gauge field, in analogy to what was found in [10] in the Randall Sundrum context. Obviously, the (scalar) laplacian \( \Delta \Sigma \) defined on \( \Sigma \) must have a zero eigenvalue for this to happen.

For scalar fields, a large contribution to the effective potential is produced only at the price of choosing ap-

\(^8\)Fixing the bulk matter content can also be understood as a sort of tuning, which can be removed only by a more fundamental theory that leads to the specified field content. Moreover, special values of the mass of bulk scalars are unstable under quantum corrections unless supersymmetry is present.

8
appropriately the mass and the non-minimal coupling. In such case a large hierarchy is generated, but the masses of the modes are unstable under quantum effects and supersymmetry has to be invoked.

It would be interesting to apply the previous ideas to models where both the \( x \)– and \( X \)–coordinates cover curved internal spaces, as it could, for example, happen in a cosmological scenario. In such case the self-consistency condition is more involved than the one presented here and might allow more interesting conclusions.

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APPENDIX A: UNIFORM ASYMPTOTIC EXPANSION AND COMPUTATION OF THE COEFFICIENTS \( \sigma_{n,k} \)

In the present appendix, we report the relevant formulas concerning the uniform asymptotic expansion (UAE) for the altered Bessel functions, \( i_{\nu_i}(z) \) and \( k_{\nu_i}(z) \), used in the computation of \( V_2 \). By using the results reported in [39,41], we find

\[
i_{\nu_i}(\nu z) = \frac{\nu e^{\nu \eta}}{\sqrt{2\pi \nu_i}} (1 + z^2)^{1/4} \Sigma_{\nu_i}^{(f)}(z),
\]

\[
k_{\nu_i}(\nu z) = -\frac{\pi \nu_i}{2} e^{\nu \eta} (1 + z^2)^{1/4} \Sigma_{\nu_i}^{(K)}(z),
\]

with

\[
\Sigma_{\nu_i}^{(f)}(z) = \frac{1}{2\nu_i \sqrt{1 + z^2}} D_1(1 - 4\xi) \Sigma_1 + \Sigma_2,
\]

\[
\Sigma_{\nu_i}^{(K)}(z) = \frac{1}{2\nu_i \sqrt{1 + z^2}} D_1(1 - 4\xi) \Sigma_3 - \Sigma_4,
\]

where

\[
t = \frac{1}{\sqrt{1 + z^2}},
\]

\[
\eta(z) = \sqrt{1 + z^2} + \ln \left( \frac{z}{\sqrt{1 + z^2}} \right).
\]

The functions \( \Sigma_I \) are given by

\[
\Sigma_1 = \sum_{k=0}^{\infty} \frac{u_k}{\nu_i^k},
\]
\[
\Sigma_2 = \sum_{k=0}^{\infty} \frac{v_k}{\nu_i^k},
\]
\[
\Sigma_3 = \sum_{k=0}^{\infty} (-1)^k \frac{u_k}{\nu_i^k},
\]
\[
\Sigma_4 = \sum_{k=0}^{\infty} (-1)^k \frac{v_k}{\nu_i^k}.
\]

with the coefficients of the previous expansions expressed by the following recursion relations:

\[
u_{k+1}(t) = \frac{1}{2} t^2 (1 - t^2) u_k(t) + \frac{1}{8} \int_0^t (1 - 5z^2) u_k(x) dx,
\]
\[
v_{k+1}(t) = u_{k+1}(t) - \frac{1}{2} t^2 (1 - t^2) u_k(t) - t^2 (1 - t^2) u_k'(t)
\]

with \( u_0(t) = 1 \). It is possible to expand the previous functions in powers of \( \nu_i \):

\[
\Sigma_{\nu_i}^{(f)}(z) = 1 + \sum_{j=1}^{\infty} \frac{p_j(t)}{\nu_i^j},
\]
\[
\Sigma_{\nu_i}^{(K)}(z) = 1 + \sum_{j=1}^{\infty} (-1)^j \frac{p_j(t)}{\nu_i^j},
\]

where

\[
p_j(t) = \frac{D_1(1 - 4\xi)}{2} t_{u_{j-1}} + v_j.
\]

It is now easy to see that, in order to obtain the coefficients \( \sigma_{n,k} \), we only need to expand the logarithm of \( \Sigma_{\nu_i}^{(f,K)}(z) \):

\[
\ln \left( 1 + \sum_{j=1}^{\infty} \frac{p_j(t)}{\nu_i^j} \right) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{\sigma_{n,k} t^{n+2k}}{\nu_i^n}.
\]

\[
\ln \left( 1 + \sum_{j=1}^{\infty} (-1)^j \frac{p_j(t)}{\nu_i^j} \right) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} (-1)^n \sigma_{n,k} t^{n+2k} \nu_i^n.
\]

The coefficients \( \sigma_{n,k} \) can be obtained by using any symbolic manipulation program. We report here only the ones needed to cancel the RS divergence:

\[
\sigma_{4,0} = -\frac{27}{128} + \frac{3}{8} \Delta - \frac{1}{2} \Delta^2 + \frac{1}{2} \Delta^3 - \frac{1}{4} \Delta^4,
\]
\[
\sigma_{2,1} = \frac{5}{8} - \frac{1}{2} \Delta,
\]
\[
\sigma_{2,0} = -\frac{3}{16} + \frac{1}{2} \Delta - \frac{1}{2} \Delta^2,
\]

with \( \Delta \) given by (55).
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