A Lecture Hall Theorem for $m$-Falling Partitions

To our mentor and friend, George E. Andrews, on his 80th birthday

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Abstract. For an integer $m \geq 2$, a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called $m$-falling, a notion introduced by Keith, if the least non-negative residues mod $m$ of $\lambda_i$’s form a nonincreasing sequence. We extend a bijection originally due to the third author to deduce a lecture hall theorem for such $m$-falling partitions. A special case of this result gives rise to a finite version of Pak–Postnikov’s $(m, c)$-generalization of Euler’s theorem. Our work is partially motivated by a recent extension of Euler’s theorem for all moduli, due to Xiong and Keith. We note that their result actually can be refined with one more parameter.

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1. Introduction

A partition $\lambda$ of a positive integer $n$ is a nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$, such that $\sum_{i=1}^{r} \lambda_i = n$. The $\lambda_i$’s are called the parts of $\lambda$, and $n$ is called the weight of $\lambda$, usually denoted as $|\lambda|$. For convenience, we often allow parts of size zero and append as many zeros as needed.

Being widely perceived as the genesis of the theory of partitions, Euler’s theorem asserts that the set of partitions of $n$ into odd parts and the set of partitions of $n$ into distinct parts are equinumerous. Equivalently

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$$\prod_{i=1}^{\infty} (1 + q^i) = \prod_{i=1}^{\infty} \frac{1}{1 - q^{2i-1}}.$$ 

Among numerous generalizations and refinements of Euler’s theorem \([1, 2, 5–10, 12, 14–19, 21, 23]\), the one that arguably attracted the most attention is the following finite version, called the Lecture Hall Theorem, discovered by Bousquet-Mélou and Eriksson \([5]\). Andrews \([4]\) gave a proof using the method of partition analysis.

If \(\lambda = (\lambda_1, \ldots, \lambda_n)\) is a partition of length \(n\) with some parts possibly zero, such that
\[
\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0, \tag{1.1}
\]
then \(\lambda\) is called a lecture hall partition of length \(n\). Let \(\mathcal{L}_n\) be the set of lecture hall partitions of length \(n\).

**Theorem 1.1.** ([5, Theorem 1.1]) For \(n \geq 1\)
\[
\sum_{\lambda \in \mathcal{L}_n} q^{\mid \lambda \mid} = \prod_{i=1}^{n} \frac{1}{1 - q^{2i-1}}. \tag{1.2}
\]

It can be easily checked that any partition \(\lambda\) into distinct parts less than or equal to \(n\) satisfies the inequality condition in (1.1). That is, \(\lambda \in \mathcal{L}_n\) as long as \(n \geq \lambda_1\), which shows that (1.2) indeed yields Euler’s theorem when \(n \to \infty\).

Glaisher \([11]\) found a purely bijective proof of Euler’s theorem and was able to extend it to the equinumerous relationship between partitions with parts repeated less than \(m\) times and partitions into non-multiples of \(m\) for any \(m \geq 2\). That is
\[
\prod_{i=1}^{\infty} \left(1 + q^i + \cdots + q^{(m-1)i}\right) = \prod_{i \not\equiv 0 \mod m} \frac{1}{1 - q^i}.
\]

Recently, Xiong and Keith \([24]\) obtained a substantial refinement of Glaisher’s result with respect to certain partition statistics, which we define next.

Throughout this paper, we will assume that \(m \geq 2\). For any partition \(\lambda = (\lambda_1, \lambda_2, \ldots)\), let
\[
s_i(\lambda) = \lambda_i - \lambda_{i+1} + \lambda_{m+i} - \lambda_{m+i+1} + \lambda_{2m+i} - \lambda_{2m+i+1} + \cdots, \quad 1 \leq i \leq m.
\]
We define its \(m\)-alternating sum type to be the \((m-1)\)-tuple \(s(\lambda) := (s_1(\lambda), \ldots, s_{m-1}(\lambda))\) and its \(m\)-alternating sum
\[
s(\lambda) := \sum_{i=1}^{m-1} s_i(\lambda).
\]
We note that the \(m\)-alternating sum type of \(\lambda\) does not put any restriction on \(s_m\).

Similarly, let
\[
\ell_i(\lambda) = \#\{j : \lambda_j \equiv i \pmod{m}\}, \quad 1 \leq i \leq m.
\]
We define its \( m \)-length type to be the \( (m - 1) \)-tuple \( \ell(\lambda) := (\ell_1(\lambda), \ell_2(\lambda), \ldots, \ell_{m-1}(\lambda)) \) and its \( m \)-length \( \ell(\lambda) = \sum_{i=1}^{m-1} \ell_i(\lambda) \). Note that the \( m \)-length type of \( \lambda \) is independent of the parts in \( \lambda \) that are multiples of \( m \).

Let us define the following two subsets of partitions:

- \( D_m \): the set of partitions in which each non-zero part can be repeated at most \( m - 1 \) times;
- \( O_m \): the set of partitions in which each non-zero part is not divisible by \( m \), called \( m \)-regular partitions.

**Theorem 1.2.** ([24, Theorem 2.1]) For \( m \geq 2 \),

\[
\sum_{\mu \in D_m} z^{s_1(\mu)} \cdots z^{s_{m-1}(\mu)} q^{\mu} = \sum_{\lambda \in O_m} z^{\ell_1(\lambda)} \cdots z^{\ell_{m-1}(\lambda)} q^{\lambda}.
\]

The natural desire to find certain “lecture hall version” for the result of Xiong and Keith motivated us to take on this investigation. While the version with full generality matching their result is yet to be found, we do obtain a lecture hall theorem for \( m \)-falling partitions.

A partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is called \( m \)-falling, which was introduced by Keith [13], if the least non-negative residues mod \( m \) of \( \lambda_i \)'s form a nonincreasing sequence. We denote the set of \( m \)-falling and \( m \)-regular partitions (\( m \)-falling regular partitions for short) as \( O_{m\searrow} \).

For \( n \geq 1 \), let

\[
O^n_{m\searrow} := \{ \lambda \in O_{m\searrow} : \lambda_1 < nm \}
\]

and \( L^n_{m\searrow} \) be a subset of \( D_m \) with certain ratio conditions between parts. Due to the complexity of the conditions, the definition of \( L^n_{m\searrow} \) is postponed to Sect. 3. A partition in \( L^n_{m\searrow} \) is called an \( m \)-falling lecture hall partition of order \( n \).

We now state the main result of this paper.

**Theorem 1.3.** (\( m \)-falling lecture hall theorem) For \( m \geq 2 \) and \( n \geq 1 \),

\[
\sum_{\mu \in L^n_{m\searrow}} z^{s_1(\mu)} \cdots z^{s_{m-1}(\mu)} q^{\mu} = \sum_{\lambda \in O^n_{m\searrow}} z^{\ell_1(\lambda)} \cdots z^{\ell_{m-1}(\lambda)} q^{\lambda}.
\]

Moreover, we obtain the following generating function of \( m \)-falling regular partitions with the largest part less than \( nm \). We shall adopt the common notation \( (q; q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n) \) for \( n \geq 1 \) with \( (q; q)_0 = 1 \). As usual, the Gaussian coefficients are given by

\[
\left[ \begin{array}{c} m+n \\ m \end{array} \right]_q = \frac{(q; q)_{m+n}}{(q; q)_m (q; q)_n}.
\]

**Theorem 1.4.** We have

\[
\sum_{\lambda \in O^n_{m\searrow}} z^{\ell(\lambda)} q^{\lambda} = \sum_{i=0}^{\infty} \left[ \begin{array}{c} m-2+i \\ i \end{array} \right]_q \left[ \begin{array}{c} n-1+i \\ i \end{array} \right] q^m z^i q^i.
\]
The final result of this paper is a refinement of Theorem 1.2. Let us consider the residue sequence of a partition. Namely, for \( \lambda = (\lambda_1, \lambda_2, \ldots) \), we take for each part the least non-negative residue mod \( m \) and denote the resulting sequence as \( v(\lambda) = v_1v_2 \cdots \). Recall the permutation statistic ascent:

\[
asc(w) = \#\{i : 1 \leq i < n, w_i < w_{i+1}\},
\]

for any word \( w = w_1 \cdots w_n \), which consists of totally ordered letters. We extend this statistic to partitions via their residue sequences and let \( asc(\lambda) = asc(v(\lambda)) \).

We have the following refinement of Theorem 1.2.

**Theorem 1.5.** For \( m \geq 2 \),

\[
\sum_{\mu \in \mathcal{D}_m} z_{s_1(\mu)}^{s_1(\mu)} \cdots z_{s_{m-1}(\mu)}^{s_{m-1}(\mu)} z_{s_m(\mu)}^{s_m(\mu)} q^{\mu|\mu|} = \sum_{\lambda \in \mathcal{O}_m} z_{\ell_1(\lambda)}^{\ell_1(\lambda)} \cdots z_{\ell_{m-1}(\lambda)}^{\ell_{m-1}(\lambda)} z_{\ell_m(\lambda)}^{\ell_m(\lambda)} q^{-asc(\lambda)} |\lambda|.
\]

(1.4)

To make this paper self-contained, in the next section we first recall the Stockhofe–Keith map and then prove Theorem 1.5. In Sect. 3, we define \( m \)-falling lecture hall partitions and prove Theorem 1.3, one special case of which gives rise to a lecture hall theorem (see Theorem 3.1) for Pak–Postnikov’s \((m, c)\)-generalization [17] of Euler’s theorem. In the end, we sketch a proof of Theorem 1.4.

**2. Preliminaries and a Proof of Theorem 1.5**

In this section, we first recall further definitions and notions involving partitions for later use. After that, we will review the Stockhofe–Keith map and prove Theorem 1.5.

Given two (infinite) sequences \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \), we define the usual linear combination \( k\lambda + l\mu \) as

\[
k\lambda + l\mu = (k\lambda_1 + l\mu_1, k\lambda_2 + l\mu_2, \ldots)
\]

for any two non-negative integers \( k \) and \( l \).

For a partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \), its conjugate partition \( \lambda' = (\lambda'_1, \ldots, \lambda'_s) \) is a partition resulting from choosing \( \lambda'_i \) as the number of parts of \( \lambda \) that are not less than \( i \) [3, Definition 1.8].

The following lemma (see for instance [24, Lemma 1]) follows via the conjugation of partitions.

**Lemma 2.1.** The conjugation map \( \lambda \mapsto \lambda' \) is a weight-preserving bijection, such that

1. \( s(\lambda) = l(\lambda') \),
2. \( \lambda_1 - s(\lambda) = \ell_m(\lambda') \).

**Proof.** 1. This immediately follows via the conjugation of partitions, so we omit the details.
2. Again, by conjugation, we see that \( s_m(\lambda) = \ell_m(\lambda') \). In addition, by the definition
\[
\lambda_1 = s_1(\lambda) + \cdots + s_m(\lambda) = s(\lambda) + s_m(\lambda).
\]
Thus \( \lambda_1 - s(\lambda) = \ell_m(\lambda') \).
□

Using conjugation, we can derive an interesting set of partitions that are equinumerous to \( D_m \), namely, \( m \)-flat partitions:

- \( F_m \): the set of partitions in which the differences between consecutive parts are at most \( m - 1 \) and the smallest positive part must also be less than \( m \), called \( m \)-flat partitions.

**Remark 2.2.** The two sets \( D_m \) and \( F_m \) are clearly in one-to-one correspondence via conjugation.

### 2.1. Stockhofe–Keith Map \( \phi : O_m \to D_m \)

Given any partition \( \lambda \), we define its base \( m \)-flat partition, denoted as \( \beta(\lambda) \), as follows. Whenever there are two consecutive parts \( \lambda_i \) and \( \lambda_{i+1} \) with \( \lambda_i - \lambda_{i+1} \geq m \), we subtract \( m \) from each of the parts \( \lambda_1, \lambda_2, \ldots, \lambda_i \). We repeat this until we reach a partition in \( F_m \), which is taken to be \( \beta(\lambda) \).

Suppose we are given a partition \( \lambda \in O_m \). We now describe step-by-step how to get a partition \( \phi(\lambda) = \mu \in D_m \) via the aforementioned Stockhofe–Keith map \( \phi \).

**Step 1:** Decompose \( \lambda = m\sigma + \beta(\lambda) \).

**Step 2:** Insert each part in \( m\sigma' \), from the largest one to the smallest one, into \( \beta(\lambda) \) according to the following insertion method. Note that after each insertion, we always arrive at a new \( m \)-flat partition. In particular, the final partition we get, say \( \tau \), is in \( F_m \) as well.

**Step 3:** Conjugate \( \tau \) to get \( \mu = \tau' \in D_m \).

**Insertion method to get \( \tau \in F_m \)**

Initiate \( \tau = (\tau_1, \tau_2, \ldots) = \beta(\lambda) \). Note that parts in \( m\sigma' \) are necessarily multiples of \( m \). Suppose we currently want to insert a part \( km \) into \( \tau \).

If \( km - \tau_1 \geq m \), then find the unique integer \( i, 1 \leq i \leq k \), such that

\[
(\tau_1 + m, \tau_2 + m, \ldots, \tau_i + m, (k - i)m, \tau_{i+1}, \ldots)
\]
is still a partition in \( F_m \). Replace \( \tau \) with this new partition.

Otherwise, we simply insert \( km \) into \( \beta(\lambda) \) as a new part and replace \( \tau \) with this new partition.

For example, let us take \( m = 3 \) and \( \lambda = (19, 17, 14, 13, 13, 8, 1) \in O_3 \). We use 3-modular Ferrers graphs [3] to illustrate the process of deriving \( \mu \), see Fig. 1, where the inserted entries are displayed in boldface in Step 2.

We should remark that the original description of the Stockhofe–Keith map [13,22] consists of only Steps 1 and 2. Thus, the map [13,22] accounts for the following theorem.
Theorem 2.3. ([24, Theorem 3.1]) $m$-Regular partitions of any given $m$-length type are in bijection with $m$-flat partitions of the same $m$-length type.

Theorem 2.3 with Remark 2.2 proves Theorem 1.2, so we will adopt the modified definition of the Stockhofe–Keith map which consists of the above three steps.

2.2. Proof of Theorem 1.5
In view of the proof of Theorem 1.2 in [24], all it remains is to examine the map $\phi$ with respect to the extra parameter $z_m$. Suppose $\lambda \in \mathcal{O}_m$, and the largest part of $\beta(\lambda)$ is $b_1$, then we have \( \frac{\lambda_1 - b_1}{m} = \text{asc}(\lambda) \), according to the definition of $m$-flat partitions. Next, during Step 2, we insert columns of $m\sigma$ into $\tau$, and each insertion will give rise to a new part in $\tau$ that is divisible by $m$; therefore, we see that \( \frac{\lambda_1 - b_1}{m} = s_m(\mu) \). The above discussion gives
\[
s_m(\mu) = \frac{\lambda_1 - b_1}{m} = \left\lfloor \frac{\lambda_1}{m} \right\rfloor - \left\lfloor \frac{b_1}{m} \right\rfloor = \left\lfloor \frac{\lambda_1}{m} \right\rfloor - \text{asc}(\lambda),
\]
as desired. \( \square \)

Remark 2.4. For $\lambda \in \mathcal{O}_m$, let $\phi(\lambda) = \mu$. Then, the extra parameter tracked by $z_m$ gives
\[
\mu_1 = s(\mu) + s_m(\mu) = s(\mu) + \left\lfloor \frac{\lambda_1}{m} \right\rfloor - \text{asc}(\lambda) = \ell(\lambda) + \left\lfloor \frac{\lambda_1}{m} \right\rfloor - \text{asc}(\lambda),
\]
which has previously been derived by Keith [13, Theorem 6] as well (he used $f_\lambda$ instead of $\text{asc}(\lambda)$). Moreover, this refinement is reminiscent of Sylvester’s bijection for proving Euler’s theorem, in which case $m = 2$ and we always have $\text{asc}(\lambda) = 0$, see, for example, Theorem 1 (item 4) in [26].

3. A Lecture Hall Theorem for $m$-Falling Partitions
We will first handle the case with a single residue class. Let us fix $c, 1 \leq c \leq m - 1$. For $n \geq 1$, let
\[
\mathcal{O}_{c,m} := \{ \lambda \in \mathcal{O}_m : \lambda_i \equiv c \pmod{m}, \text{ for all } i \},
\]
\[
\mathcal{O}_{c,m}^n := \{ \lambda \in \mathcal{O}_{c,m} : \lambda_1 < nm \},
\]
\[
\mathcal{D}_{c,m} := \left\{ \lambda \in \mathcal{D}_m : s(\lambda) = (0, \ldots, 0, a, 0 \ldots, 0) \text{ for some } a > 0, \text{ or } |\lambda| = 0 \right\},
\]
\[
\mathcal{L}_{c,m}^n := \left\{ \lambda \in \mathcal{D}_{c,m} : l(\lambda) \leq \left\lfloor \frac{n + 1}{2} \right\rfloor (m - 2) + n \text{ and } \frac{\lambda_{km+c}}{n-2k} \geq \frac{\lambda_{km+m}}{n-2k-1} \geq \frac{\lambda_{(k+1)m+c}}{n-2k-2} \text{ for } 0 \leq k < \left\lfloor \frac{n}{2} \right\rfloor \right\},
\]
where $l(\lambda)$ is the number of non-zero parts in $\lambda$, and we make the convention that for a fraction $\frac{p}{q}$, if $q = 0$ we replace the relevant fraction with 0.
Figure 1. Three steps to get $\phi(\lambda) = \mu$
Theorem 3.1. For any \( n \geq 1 \)
\[
\sum_{\lambda \in \mathcal{O}_{c,m}^{n}} z^{\ell(\lambda)} q^{|\lambda|} = \sum_{\mu \in \mathcal{L}_{c,m}^{n}} z^{s(\mu)} q^{|\mu|}
= \frac{1}{(1-zq^{c})(1-zq^{m+c})\cdots(1-zq^{n-1)m+c})}. \tag{3.1}
\]

The above result can be viewed as a finite (or “lecture hall”) version of the following Pak–Postnikov’s \((m, c)\)-generalization \([17, \text{Theorem 1}]\) of Euler’s partition theorem since \( \lim_{n \to \infty} \mathcal{O}_{c,m}^{n} \to \mathcal{O}_{c,m} \) and \( \lim_{n \to \infty} \mathcal{L}_{c,m}^{n} \to \mathcal{D}_{c,m} \).

Theorem 3.2. For \( n \geq 1, m \geq 2 \) and \( 1 \leq c \leq m-1 \), the number of partitions of \( n \) into parts congruent to \( c \mod m \) equals the number of partitions of \( n \) with exactly \( c \) parts of maximal size, \( m-c \) (if any) second by size parts, \( c \) (if any) third by size parts, etc.

Proof of Theorem 3.1. Since a partition \( \lambda \in \mathcal{O}_{c,m}^{n} \) has all its parts congruent to \( c \mod m \), with the largest part \( \lambda_{1} < n/m \), we see that \( \ell(\lambda) = \ell_{c}(\lambda) = l(\lambda) \) and we have
\[
\sum_{\lambda \in \mathcal{O}_{c,m}^{n}} z^{\ell(\lambda)} q^{|\lambda|} = \frac{1}{(1-zq^{c})(1-zq^{m+c})\cdots(1-zq^{n-1)m+c})}.
\]

It remains to prove the first equality. We achieve this by constructing a weight-preserving bijection \( \varphi_{n} \) from \( \mathcal{O}_{c,m}^{n} \) to \( \mathcal{L}_{c,m}^{n} \), such that \( \ell(\lambda) = s(\varphi_{n}(\lambda)) \). We extend the bijection from \( \mathcal{O}_{n} \) to \( \mathcal{L}_{n} \) constructed in \([25]\) to deal with the \( m \)-falling partitions considered here.

Define the maps \( \varphi_{n} : \mathcal{O}_{c,m}^{n} \to \mathcal{L}_{c,m}^{n} \) as follows. For \( \lambda \in \mathcal{O}_{c,m}^{n} \), let \( \mu \) be the sequence obtained from the empty sequence \((0,0,\ldots)\) by recursively inserting the parts of \( \lambda \) in nonincreasing order according to the following insertion procedure. We define \( \varphi_{n}(\lambda) = \mu \).

**Insertion procedure**

Let \((\mu_{1}, \mu_{2}, \ldots) \in \mathcal{L}_{c,m}^{n} \). To insert \( km + c \) into \((\mu_{1}, \mu_{2}, \ldots) \), set \( j = 0 \).

If \( j < k \) and \( \frac{\mu_{mj+c} + 1}{n-2j} \geq \frac{\mu_{mj+m} + 1}{n-2j-1} \) \hspace{1cm} \text{(Test I)}
then increase \( j \) by 1 and go to \( \text{(Test I)} \);
otherwise stop testing and return
\[ (\mu_{1}, \mu_{2}, \ldots) + \left( \underbrace{1,\ldots,1}_{jm}, k-j+1, \ldots, k-j+1 \right), \underbrace{m-c}_{k-j, \ldots, k-j, 0, 0, \ldots}. \]

The effect of this insertion is that we use up a complete part \( km + c \), so the weight of the sequence \((\mu_{1}, \mu_{2}, \ldots) \) and its \( m \)-alternating sum are increased by \( km + c \) and 1, respectively. In addition, it can be checked easily that the returned sequence satisfies the condition for \( \mathcal{L}_{c,m}^{n} \). We omit the details.
The map $\varphi_n$ is indeed invertible, since the parts of $\lambda$ were inserted in nonincreasing order, i.e., from the largest to the smallest. If the parts are not inserted in this order, $\varphi_n$ is not necessarily invertible. The inverse of $\varphi_n$, namely, $\psi_n$, can be described similarly in this algorithmic fashion. For a given partition $\mu \in \mathcal{L}_{c,m}^n$ with $s(\mu) = k$, define $\psi_n(\mu)$ to be the sequence $\lambda = (\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$ obtained from the empty sequence $(0, 0, \ldots)$ by adding nondecreasing parts one at a time that are derived from peeling off partially or entirely certain parts of $\mu$ according to the following deletion procedure.

**Deletion procedure**

Let $(\mu_1, \mu_2, \ldots) \neq (0, 0, \ldots)$ be in $\mathcal{L}_{c,m}^n$. Set $k = 0$ and $j = 0$.

If $(\mu_1, \mu_2, \ldots) - \begin{pmatrix} jm \cr c \cr k-j, k-j+1, \ldots, k-j, 0, 0, \ldots \end{pmatrix} \in \mathcal{L}_{c,m}^n$,

then stop testing and return $km + c$ with $\mu = (\mu_1, \mu_2, \ldots) - \begin{pmatrix} jm \cr c \cr k-j, k-j+1, \ldots, k-j+1, \cr k-j, \ldots, k-j, 0, 0, \ldots \end{pmatrix}$;

otherwise, if $j < k$, then increase $j$ by 1 and go to (Test D);

otherwise increase $k$ by 1, set $j = 0$ and go to (Test D).

The effect of this deletion is that the weight of the sequence $(\mu_1, \mu_2, \ldots)$ and its $m$-alternating sum are decreased by $km + c$ and 1, respectively. In addition, it should be noted that this deletion process must stop after a finite number of steps. Since $(\mu_1, \mu_2, \ldots) \neq (0, 0, \ldots)$ belongs to $\mathcal{L}_{c,m}^n$, there must be $i$, such that $\mu_{im+c} > \mu_{im+m}$.

Let $j$ be the largest such $i$. Then, $\mu_l = 0$ for any $l > jm + m$ and

$(\mu_1, \mu_2, \ldots) - \begin{pmatrix} jm \cr c \cr k-j, k-j+1, \ldots, k-j+1, \cr k-j, \ldots, k-j, 0, 0, \ldots \end{pmatrix} \in \mathcal{L}_{c,m}^n$,

which shows such $j$ must pass (Test D).

To finish the proof, we make the following claims about $\varphi_n$ and $\psi_n$ without giving the proofs, since all of them are essentially the same as those found in [25], which is the case when $m = 2$ and $c = 1$.

- Each insertion outputs a new $\mu \in \mathcal{L}_{c,m}^n$, and in particular, $\varphi_n$ is well-defined.
- Each deletion outputs a new $\lambda \in \mathcal{O}_{c,m}^n$, and in particular, $\psi_n$ is well-defined.
Figure 2. $\varphi_5((11, 11, 8, 8, 8, 5, 5)) = (13, 13, 10, 7, 7, 4, 1, 1)$

- The deletion procedure reverses the insertion procedure, consequently $\psi_n$ is the inverse of $\varphi_n$.

Before we move on, we provide an example for the insertion procedure.

Example 3.3. Let $m = 3$, $c = 2$, $n = 5$, and $\mu = (3, 3, 2, 0, 0, \ldots) \in \mathcal{L}_{2,3}^5$. We insert 8 into $\mu$ as follows. Note that

$$\frac{\mu_2 + 1}{5} = \frac{4}{5} \geq \frac{\mu_3 + 1}{4} = \frac{3}{4}$$

but

$$\frac{\mu_5 + 1}{3} = \frac{1}{3} \not\geq \frac{\mu_6 + 1}{2} = \frac{1}{2}.$$

Therefore, we get

$$(3, 3, 2, 0, 0, \ldots) + (1, 1, 1, 2, 2, 1) = (4, 4, 3, 2, 2, 1).$$

In Fig. 2, we illustrate the process of getting $\mu$ by applying $\varphi_5$ to $\lambda = (11, 11, 8, 8, 8, 5, 5) \in \mathcal{O}_{2,3}^5$ using 3-modular Ferrers graphs. Newly inserted entries after each step are displayed in boldface.

Remark 3.4. In general, our bijection $\varphi_n$ works on $m$-modular Ferrers graphs. For the special case when $m = 2$ and $c = 1$, $\varphi_n$ actually reduces to the original bijection constructed in [25]. Another notable generalization can be found in [20].
As we will see, the bijection \( \varphi_n \) plays a crucial role in our proof of Theorem 1.3. We need a few more definitions.

**Definition 3.5.** For a partition \( \lambda \in \mathcal{D}_m \), we define two local statistics “first bigger” and “last bigger”. For each \( i, 0 \leq i \leq \left\lfloor \frac{l(\lambda) - 1}{m} \right\rfloor \), where \( l(\lambda) \) is the number of non-zero parts in \( \lambda \), suppose
\[
\lambda_{im+1} = \cdots = \lambda_{im+j} > \lambda_{im+j+1} \geq \cdots \geq \lambda_{im+k} > \lambda_{im+k+1} = \cdots = \lambda_{im+m}.
\]
Then, we let \( fb_i = j, lb_i = k \).

Note that since \( \lambda \in \mathcal{D}_m \), such \( j \) and \( k \) must exist and \( j \leq k \), so \( fb_i \) and \( lb_i \) are well-defined.

**Definition 3.6.** Fix a positive integer \( n \). For a partition \( \lambda \in \mathcal{D}_m \), we call it an \( m \)-falling lecture hall partition of order \( n \), if \( l(\lambda) \leq \left\lfloor \frac{n+1}{2} \right\rfloor (m - 2) + n \), and the following two conditions hold.

1. For \( i, 0 \leq i \leq \left\lfloor \frac{l(\lambda) - 1}{m} \right\rfloor \), \( lb_i \leq fb_{i+1} \).
2. \[
\frac{\lambda_1}{n} \geq \frac{\lambda_m}{n-1} \geq \frac{\lambda_{m-1}}{n-2} \geq \frac{\lambda_{2m}}{n-3} \geq \cdots \geq \frac{\lambda_{(k-1)m+1}}{2} \geq \frac{\lambda_{km}}{1}, \quad \text{if } n = 2k,
\]
\[
\frac{\lambda_1}{n} \geq \frac{\lambda_m}{n-1} \geq \frac{\lambda_{m+1}}{n-2} \geq \frac{\lambda_{2m}}{n-3} \geq \cdots \geq \frac{\lambda_{km+1}}{0} \geq \frac{\lambda_{km+m}}{1}, \quad \text{if } n = 2k+1.
\]

We denote the set of all \( m \)-falling lecture hall partitions of order \( n \) as \( \mathcal{L}_{n,m}^n \).

**Remark 3.7.** Partitions in \( \mathcal{D}_m \) satisfying the above condition (1) are said to be of \( m \)-alternating type in [13].

Recall the definition of \( m \)-falling regular partitions. A partition is \( m \)-falling regular if the parts are not multiples of \( m \) and their positive residues are nonincreasing.

For a chosen vector \( \mathbf{v} = (v_1, v_2, \ldots, v_{m-1}) \), let
\[
\mathcal{O}_{m}^\mathbf{v} \setminus := \{ \lambda \in \mathcal{O}_{m}^n : l(\lambda) = \mathbf{v} \},
\]
\[
\mathcal{L}_{m}^\mathbf{v} \setminus := \{ \mu \in \mathcal{L}_{m}^n : s(\mu) = \mathbf{v} \}.
\]

**Proof of Theorem 1.3.** For any \( c, 1 \leq c \leq m - 1 \) and a given vector \( \mathbf{v} = (v_1, v_2, \ldots, v_{m-1}) \), we consider two embeddings:
\[
f_\mathbf{v} : \mathcal{O}_{m}^\mathbf{v} \setminus \hookrightarrow \mathcal{O}_{c,m}^n \quad \text{and} \quad g_\mathbf{v} : \mathcal{L}_{m}^\mathbf{v} \setminus \hookrightarrow \mathcal{L}_{c,m}^n,
\]
such that \( \ell(\lambda) = \ell(f_\mathbf{v}(\lambda)) \) and \( s(\mu) = s(g_\mathbf{v}(\mu)) \). To be precise, for a given partition \( \lambda \), both \( f_\mathbf{v} \) and \( g_\mathbf{v} \) change the residue of each part of \( \lambda \mod m \) to be uniformly the predetermined value \( c \). In terms of the corresponding \( m \)-modular Ferrers graph, the two maps keep all the cells labelled \( m \), but relabel all the remaining cells by \( c \). Therefore, in general, neither of these two maps preserves the weight of the partition, but they do keep the number of cells in their \( m \)-modular Ferrers graphs unchange.
Table 1. \(\phi_3: O_{3,\lambda}^{(3,2)} \rightarrow L_{3,\mu}^{(3,2)}\)

| \(O_{3,\lambda}^{(3,2)}\) | \(L_{3,\mu}^{(3,2)}\) |
|--------------------------|--------------------------|
| 8, 8, 7, 7, 7           | 15, 12, 10               |
| 8, 8, 7, 7, 4           | 14, 11, 9                |
| 8, 8, 7, 7, 1           | 13, 10, 8                |
| 8, 8, 7, 4, 4           | 12, 9, 8, 1, 1           |
| 8, 8, 7, 4, 1           | 12, 9, 7                |
| 8, 8, 7, 1, 1           | 11, 8, 6               |
| 8, 8, 4, 4, 4           | 11, 8, 7, 1, 1           |
| 8, 8, 4, 4, 1           | 10, 7, 6, 1, 1           |
| 8, 8, 4, 1, 1           | 10, 7, 5                |
| 8, 8, 1, 1, 1           | 9, 6, 4               |
| 8, 5, 4, 4, 4           | 9, 6, 6, 2, 2           |
| 8, 5, 4, 4, 1           | 9, 6, 5, 1, 1           |
| 8, 5, 4, 1, 1           | 8, 5, 4, 1, 1           |
| 8, 5, 1, 1, 1           | 8, 5, 3                |
| 8, 2, 1, 1, 1           | 7, 4, 2                |
| 5, 5, 4, 4, 4           | 8, 5, 5, 2, 2           |
| 5, 5, 4, 4, 1           | 7, 4, 4, 2, 2           |
| 5, 5, 4, 1, 1           | 7, 4, 3, 1, 1           |
| 5, 5, 1, 1, 1           | 6, 3, 2, 1, 1           |
| 5, 2, 1, 1, 1           | 6, 3, 1                |
| 2, 2, 1, 1, 1           | 5, 2               |

Moreover, the given vector \(v\) and the \(m\)-falling condition uniquely determine the preimage of any partition in \(f_v(O_{m,\lambda}^n)\). Similarly, the condition (1) in Definition 3.6 together with \(v\) dictates the preimage of any partition in \(g_v(L_{m,\mu}^n)\). This enables us to define a bijection

\[
\phi_n = g_v^{-1} \circ \varphi_n \circ f_v: O_{m,\lambda}^n \rightarrow L_{m,\mu}^\mu,
\]

where \(v\) is the \(m\)-length type of the partition it acts on.

It has been proved in Theorem 3.1 that \(\varphi_n\) is a bijection satisfying \(\ell(\lambda) = s(\varphi_n(\lambda))\), and the discussion above shows that both \(f_v\) and \(g_v\) are invertible. Consequently, we see that \(\phi_n\) is indeed a bijection such that \(l(\mu) = s(\phi_n(\lambda))\) for any \(\lambda \in O_{m,\lambda}^n\), and we complete the proof. \(\square\)

Example 3.8. As an illustrative example, we take \(m = 3, n = 3\) and fix the vector \(v = (3, 2)\). In Table 1, we list out all the 21 partitions \(\lambda\) in \(O_{3,\lambda}^{(3,2)}\) with \(\lambda_1 < nm = 9\), as well as all the 21 partitions \(\mu\) in \(L_{3,\mu}^{(3,2)}\) with \(l(\mu) \leq 5\). They are matched up via our map \(\phi_3\). The derivation of one particular partition \((8, 5, 5, 2, 2)\) from \((5, 5, 4, 4, 4)\) using 3-modular Ferrers graphs is detailed in Fig. 3.

Now, we turn to the proof of Theorem 1.4.
Step 1: \( \lambda = \begin{array}{ccc}
3 & 2 \\
3 & 2 \\
3 & 1 \\
3 & 1 \\
3 & 1 \\
\end{array} \xrightarrow{f_{(3,2)}} \begin{array}{ccc}
3 & c \\
3 & c \\
3 & c \\
3 & c \\
3 & c \\
\end{array} \\

Step 2: \( \emptyset \xrightarrow{\text{insert}} \begin{array}{c}
3 \\
3 \\
3 \\
3 \\
3 \\
\end{array} \xrightarrow{\text{insert}} \begin{array}{c}
3 \\
3 \\
3 \\
3 \\
3 \\
\end{array} \xrightarrow{\text{insert}} \begin{array}{c}
3 \\
3 \\
3 \\
3 \\
3 \\
\end{array} \xrightarrow{\text{insert}} \begin{array}{c}
3 \\
3 \\
3 \\
3 \\
3 \\
\end{array} \\

Step 3: \( \begin{array}{ccc}
3 & c \\
3 & c \\
3 & 3 \\
3 & 3 \\
3 & c \\
\end{array} \xrightarrow{g_{(3,2)}^{-1}} \begin{array}{ccc}
3 & 2 \\
3 & 2 \\
3 & 3 \\
3 & 3 \\
3 & 3 \\
\end{array} \xrightarrow{\text{conjugate}} \mu = (8, 5, 5, 2, 2) \\

Figure 3. \( \phi_3((5, 5, 4, 4, 4)) = (8, 5, 5, 2, 2) \)
Proof of Theorem 1.4. Based on the $m$-modular Ferrers graphs of $m$-regular falling partitions, we obtain

$$
\sum_{\mu \in O_m^{\ell}} z_1^{\ell_1(\mu)} \cdots z_{m-1}^{\ell_{m-1}(\mu)} q^{\mu} = \sum_{i=0}^{\infty} h_i(z_1 q, z_2 q^2, \ldots, z_{m-1} q^{m-1}) \left[ n - \frac{1}{i} \right]_{q^m}, \quad (3.2)
$$

where the $i$th homogeneous symmetric function $h_i(x_1, \ldots, x_k)$ is defined by

$$
h_i(x_1, \ldots, x_k) = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_i \leq k} x_{j_1} x_{j_2} \cdots x_{j_i}.
$$

Setting $z_1 = \cdots = z_{m-1} = z$ in (3.2), we get

$$
\sum_{\mu \in O_m^{\ell}} z^{\ell(\mu)} q^{\mu} = \sum_{i=0}^{\infty} h_i(q, q^2, \ldots, q^{m-1}) \left[ n - \frac{1}{i} \right]_{q^m} z^i. \quad (3.3)
$$

Since

$$
h_i(q, q^2, \ldots, q^{m-1}) = q^i \left[ m - \frac{2}{i} \right]_{q},
$$

it follows from (3.3) that

$$
\sum_{\mu \in O_m^{\ell}} z^{\ell(\mu)} q^{\mu} = \sum_{i=0}^{\infty} \left[ m - \frac{2}{i} + \frac{1}{i} \right]_{q} \left[ n - \frac{1}{i} + \frac{1}{i} \right]_{q^m} z^i q^i,
$$

as claimed. \qed

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