EXAMPLES OF SUBFACTORS WITH PROPERTY T STANDARD INVARIENT

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Abstract. Let $H$ and $K$ be two finite groups with a properly outer action on the II\textsubscript{1} factor $M$. We prove that the group type inclusions $M^H \subset M \rtimes K$, studied in detail in [BH], have property $T$ in the sense of [Po6] if and only if the group generated by $H$ and $K$ in the outer automorphism group of $M$ has Kazhdan’s property $T$ [K]. This construction yields irreducible, infinite depth subfactors with small Jones indices and property T standard invariant.

0. Introduction

If $H$ and $K$ are two finite groups with a properly outer action on the II\textsubscript{1} factor $M$, we can compose the two subfactors $M^H \subset M$ and $M \subset M \rtimes K$ to obtain a new inclusion $M^H \subset M \rtimes K$. While the Jones index of $M^H \subset M \rtimes K$ is finite, being equal to $|H| \cdot |K|$, in general this inclusion can no longer be obtained as a fixed point algebra (or a crossed product) by a group action, or more generally a coaction of a Kac algebra. It was shown in [BH] that several interesting properties of these group type inclusions are determined by the group $G$ generated by $H$ and $K$ in the outer automorphism group of the factor $M$. For instance, it was shown there that $M^H \subset M \rtimes K$ is amenable (resp. has finite depth) if and only if $G$ is an amenable (resp. finite) group. Since any second countable discrete group has a properly outer action on the hyperfinite II\textsubscript{1} factor $R$, numerous examples of subfactors, whose standard invariants are essentially as badly or as well behaved as the group $G$ itself, can be constructed in this way.

The second author introduced recently a concept of property T for the standard invariant of a subfactor and used certain locally trivial, diagonal subfactors associated to a $G$-kernel to construct reducible, infinite depth subfactors with finite Jones index whose standard invariant has property T ([Po5], [Po6]). We provide in this note new examples of property T subfactors by showing that the group type

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inclusions $M^H \subset M \rtimes K$ have property T in the sense of [Po6] if and only if the group $G$ has Kazhdan’s property T. Once this result is established, every infinite property T group, which is a quotient of a free product of two finite groups, provides an example of an irreducible, infinite depth subfactor of the hyperfinite II$_1$ factor with finite Jones index whose standard invariant has property T. The lowest possible index of an irreducible, infinite depth subfactor with property T standard invariant that this construction can give is index 6 and we give an example of such a subfactor in section 2, based on work by Conder.

Here is a more detailed description of the two sections below. In section 1 we discuss briefly the different notions of property T used in this paper and recall a few facts that are needed in the subsequent section. In section 2 we prove that the the standard invariant of the group type subfactors $M^H \subset M \rtimes K$ has property T if and only if the group $G$, generated by $H$ and $K$ in Out $M$ has Kazhdan’s property T. This is done without explicitly computing the symmetric enveloping inclusion associated to $M^H \subset M \rtimes K$. The key fact here is that property T has certain hereditary properties, which are established in theorem 2.6. The result regarding $M^H \subset M \rtimes K$ as mentioned above follows then from these hereditary properties and the fact that a certain locally trivial subfactor associated to the kernel $G$ has property T in the sense of [Po6] if and only if the group $G$ has Kazhdan’s property T. We finish section 2 with a number of explicit examples of subfactors with property T standard lattice (see corollary 2.9 and remarks).

1. Property T

We present in this section three related concepts of property T used in this paper and fix the notation. Recall first that a countable discrete group $G$ has Kazhdan’s property T [K] if, roughly speaking, every unitary representation $\pi : G \to U(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ that has almost invariant vectors has nonzero invariant vectors. More precisely, $G$ has property T, if there is a finite set $F \subset G$ and there is an $\epsilon > 0$ such that whenever $\pi : G \to U(\mathcal{H})$ is a unitary representation of $G$ on the Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a unit vector with $\|\pi(s)\xi - \xi\| < \epsilon$, for all $s \in F$, then there is a nonzero vector $\eta \in \mathcal{H}$ with $\pi(g)\eta = \eta$, for all $g \in G$ (see for instance [HV] for other equivalent definitions).

Next we explain two notions of property T related to inclusions of von Neumann algebras. Throughout this paper $N \subset M$ will denote an extremal inclusion of II$_1$ factors with finite Jones index unless otherwise stated. As usual, we denote by $e_N$ the orthogonal projection $L^2(M) \to L^2(N)$ and by $J : L^2(M) \to L^2(M)$ the modular conjugation. The standard invariant of $N \subset M$, consisting of the system of higher relative commutants associated to $N \subset M$, will be denoted by $\mathcal{G}_{N,M}$ (see for instance [Po2]). We refer to $\mathcal{G}_{N,M}$ also as the standard lattice associated to $N \subset M$ (see [Po4]). Inclusions whose index is not necessarily finite will usually be denoted by $N \subset \mathcal{M}$.

**Definition 1.1.** ([A-D], [Po1]) Let $N \subset \mathcal{M}$ be an inclusion of von Neumann algebras and let $\mathcal{M}$ be a II$_1$ factor. We say that $\mathcal{M}$ has property T relative to $N$, or that the pair $\mathcal{N} \subset \mathcal{M}$ has property T, if there is an $\epsilon > 0$ and operators $x_1, x_2, \ldots, x_n \in \mathcal{M}$ such that if $H$ is an $\mathcal{M}$-$\mathcal{M}$ bimodule and $\xi \in H$ is a vector satisfying $\|\xi\| = 1$, $\|\pi(g)\xi - \xi\| < \epsilon$, $1 \leq i \leq n$, $[\xi, x_i] = 0$, for all $\pi \in \mathcal{N}$, then there is a nonzero
vector \( \eta \in \mathcal{H} \) such that \([\eta, x] = 0\) for all \( x \in \mathcal{M} \). \( \eta \) is called an \( \mathcal{M} \)-central vector. We call \( \{ \epsilon, x_1, \ldots, x_n \} \) a critical set for \( \mathcal{N} \subset \mathcal{M} \).

Note that we do not assume that \( \mathcal{N} \subset \mathcal{M} \) has finite Jones index [Jo]. In fact, if \( \mathcal{N} \subset \mathcal{M} \) is an inclusion of II\(_1\) factors with finite index, then, using an orthonormal basis of \( \mathcal{M} \) over \( \mathcal{N} \) [PiPo], one sees easily that \( \mathcal{M} \) has property T relative to \( \mathcal{N} \). Let us also mention that if \( \mathcal{G} \) is a countable discrete group with a properly outer action on the II\(_1\) factor \( \mathcal{N} \), then the crossed product \( \mathcal{N} \rtimes \mathcal{G} \) has property T relative to \( \mathcal{N} \) if and only if the group \( \mathcal{G} \) has Kazhdan’s property T ([A-D], [Po1]).

We recall next the notion of property T for the standard invariant of an extremal subfactor \( \mathcal{N} \subset \mathcal{M} \) as introduced in [Po5], [Po6]. If \( \mathcal{N} \subset \mathcal{M} \) is an inclusion of II\(_1\) factors with finite Jones index, we associate to it the symmetric enveloping inclusion \( \mathcal{N} \vee \mathcal{M}^{\mathcal{G}} \subset \mathcal{M} \rtimes \mathcal{M}^{\mathcal{G}} \), where \( \mathcal{M} \rtimes \mathcal{M}^{\mathcal{G}} \) is the II\(_1\) factor obtained in the following way (see [Po3], [Po6] for details): We consider the C*-algebra \( C^*(\mathcal{M}, e_N, JM J) \subset B(L^2(\mathcal{M})) \), generated by \( \mathcal{M} \), \( JM J \) and \( e_N \) on \( L^2(\mathcal{M}) \). It can be shown that \( C^*(\mathcal{M}, e_N, JM J) \) has a unique trace \( \text{tr} \) and one defines \( \mathcal{M} \rtimes \mathcal{M}^{\mathcal{G}} \) to be the von Neumann algebra obtained via the GNS construction with respect to \( \text{tr} \) from \( C^*(\mathcal{M}, e_N, JM J) \). We can now give the desired definition:

**Definition 1.2.** [Po6] Let \( \mathcal{N} \subset \mathcal{M} \) be an extremal inclusion of II\(_1\) factors with finite Jones index and with standard lattice (or standard invariant) \( \mathcal{G}_{\mathcal{N}, \mathcal{M}} \). We say that \( \mathcal{G}_{\mathcal{N}, \mathcal{M}} \) has property T if the symmetric enveloping II\(_1\) factor \( \mathcal{M} \rtimes \mathcal{M}^{\mathcal{G}} \) has property T relative to the subfactor \( \mathcal{N} \vee \mathcal{M}^{\mathcal{G}} \) (in the sense of definition 1.1). Equivalently, we will say that the standard invariant (or the standard lattice) of \( \mathcal{N} \subset \mathcal{M} \) has property T.

It is easy to see that the index of the inclusion \( \mathcal{N} \vee \mathcal{M}^{\mathcal{G}} \subset \mathcal{M} \rtimes \mathcal{M}^{\mathcal{G}} \) is finite if and only if \( \mathcal{N} \subset \mathcal{M} \) has finite depth. Thus the standard lattice of finite depth subfactors has property T as expected.

Observe that the definition of property T for \( \mathcal{G}_{\mathcal{N}, \mathcal{M}} \) depends a priori on the subfactor \( \mathcal{N} \subset \mathcal{M} \). We have however the following result [Po6, section 9]:

**Theorem 1.3.** Suppose \( \mathcal{G}_{\mathcal{N}, \mathcal{M}} \) is the standard lattice of an extremal subfactor. The property T of \( \mathcal{G}_{\mathcal{N}, \mathcal{M}} \) depends only on the (trace preserving) isomorphism class of \( \mathcal{G}_{\mathcal{N}, \mathcal{M}} \), not on the extremal subfactor from which it was constructed.

If we let \( \mathcal{P} \) be a II\(_1\) factor and \( \theta_0 = \text{id}, \theta_1, \ldots, \theta_n \) automorphism of \( \mathcal{P} \), then we can consider the locally trivial inclusion of II\(_1\) factors \( \mathcal{N}^\theta \subset \mathcal{M}^\theta \), where \( \mathcal{N}^\theta = \{ \sum_{i=0}^n \theta_i(x)e_{ii} \mid x \in \mathcal{P} \} \), \( \mathcal{M}^\theta = \mathcal{P} \otimes M_{n+1}(\mathbb{C}) \), \( \{e_{ij}\}_{0 \leq i,j \leq n} \), the matrix units in \( M_{n+1}(\mathbb{C}) \). These reducible inclusions of II\(_1\) factors provide examples of subfactors with property T standard invariant [Po6, proposition 9.7]:

**Proposition 1.4.** The standard lattice of the subfactor \( \mathcal{N}^\theta \subset \mathcal{M}^\theta \) has property T if and only if the group \( \mathcal{G} \) generated by the \( \theta_i \)'s, \( 0 \leq i \leq n \), in the outer automorphism group of \( \mathcal{P} \) has Kazhdan’s property T.

The above proposition is proved by computing the symmetric enveloping inclusion associated to \( \mathcal{N}^\theta \subset \mathcal{M}^\theta \) explicitly. It turns out that it is a crossed product by the group \( \mathcal{G} \).
2. Group type inclusions with property T

We study in this section the group type inclusions $M^H \subset M \rtimes K$, where $H$ and $K$ are finite groups with a properly outer action on the $\text{II}_1$ factor $M$ [BH]. $M^H$ denotes as usual the fixed point algebra under the $H$-action and $M \rtimes K$ the crossed product. Note that $M^H \subset M \rtimes K$ has Jones index $|H| \cdot |K|$ and is extremal. It was shown in [BH] that various analytical properties of these subfactors, such as finite depth, amenability and strong amenability (in the sense of [Po2]), are determined by the group $G = \langle H, K \rangle$, generated by $H$ and $K$ in the outer automorphism group of $M$. Furthermore, $M^H \subset M \rtimes K$ is irreducible if and only if $H \cap K = \{e\}$ in $\text{Out } M = \text{Aut } M/\text{Int } M$. Recall that if $M$ is the hyperfinite $\text{II}_1$ factor $R$, then any group $G$, which is a quotient of the free product of $H$ and $K$ gives rise to an irreducible subfactor of the above type. We will show in this section the following theorem:

**Theorem 2.1.** Let $H$ and $K$ be finite groups with properly outer actions $\sigma$, resp. $\rho$ on the $\text{II}_1$ factor $M$. The standard invariant of the subfactor $M^H \subset M \rtimes K$ has property T if and only if the group $G$ generated by $\sigma(H)$ and $\rho(K)$ in the outer automorphism group of $M$ has Kazhdan’s property T.

This theorem will follow from a more general result, theorem 2.6 below. We will also present at the end of this section some explicit examples of irreducible, infinite depth subfactors whose standard lattice has property T.

The next result can be found in [Po1]. We include the proof here for the convenience of the reader:

**Proposition 2.2.**

i) Let $Q \subset N \subset M$ be an inclusion of $\text{II}_1$ factors with $[N : Q] < \infty$. Then $M$ has property T relative to $Q$ if and only if $M$ has property T relative to $N$.

ii) Let $N \subset M$ be an inclusion of $\text{II}_1$ factors and let $p \in N$ be a nonzero projection. Then $M$ has property T relative to $N$ if and only if $pMp$ has property T relative to $pNp$.

**Proof.** Let us first prove i). If $M$ has property T relative to $Q$, then it is trivial that $M$ has property T relative to $N$ - the same critical set works. Conversely, suppose $M$ has property T relative to $N$, let $\{n_i\}_{i \in I}$ be a (finite) orthonormal basis of $N$ over $Q$. Let $\{\epsilon, x_1, \ldots, x_n\}$, $x_i \in M$, $1 \leq i \leq n$, be a critical set for $N \subset M$ (definition 1.1). Suppose $H$ is an $M$-$M$-bimodule, $\xi \in H$, $\|\xi\| = 1$ with $\langle \xi, Q \rangle = 0$. Set $\eta = \sum_{i \in I} n_i \xi n_i^* \in H$. Then we have for all $x \in N$

$$x\eta = \sum_i x n_i \xi n_i^* = \sum_j \sum_i n_j E_Q(n_j^* x n_i) \xi n_i^* = \sum_j \sum_i n_j \xi E_Q(n_j^* x n_i) n_i^*$$

$$= \sum_j n_j \xi (\sum_i E_Q(n_j^* x n_i) n_i^*) = \sum_j n_j \xi n_j^* x = \eta x.$$

Furthermore $\|n_i \xi n_i^* - \xi n_i n_i^*\| \leq \|n_i \xi - \xi n_i\| \|n_i^*\| \leq [N : Q] \|n_i \xi - \xi n_i\|$, for all $i$. Thus, if we let $\delta = \max_i \|n_i \xi - \xi n_i\|$, then
\[
\|\eta - [N : Q]\xi\| = \| \sum_i n_i \xi n_i^* - \sum_i \xi n_i n_i^* \| \leq \| n_i \xi n_i^* - \xi n_i n_i^* \|
\]
\[
\leq [N : Q] \sum_i \| n_i \xi - \xi n_i \| \leq [N : Q]\|I\|\delta
\]

Hence \(\|\eta\| \geq [N : Q] (1 - |I|\delta)\), so that \(\eta \neq 0\), if \(0 < \delta < \frac{1}{|I|}\). Fix such a \(\delta\) and pick an \(\epsilon' > 0\) with \(\epsilon' < \min\{\delta, \frac{(1-|I|\delta)}{3m}\}\). Then \(\{\epsilon', x_i, x_i n_j, n_j^* x_i, n_j\}_{1 \leq i, j \leq I}\) is a critical set for \(Q \subset M\), since we obtain \(\|x_i n_j' - \eta' x_i\| < \epsilon', 1 \leq i \leq n\), where \(\eta' = \frac{\eta}{\|\eta\|}\) and \(\{x, x_1, \ldots, x_n\}\) was the critical set for \(N \subset M\).

We prove next ii). Let us assume first that \(pMp\) has property T relative to \(pNp\). Let \(\{\epsilon, y_1, \ldots, y_n\}\), \(y_i \in pMp\), be a critical set. We may assume \(\|y_i\| \leq 1\), \(1 \leq i \leq n\). Since \(M\) is a factor, we can find isometries \(v_1, \ldots, v_m \in M\) such that \(v_1 v_1^* \leq p\), \(v_i v_i^* = p\), \(2 \leq i \leq m\), and \(\sum_{i=1}^m v_i v_i^* = 1, M\). Let \(\delta = \min\{\epsilon/6, \frac{1}{3m}\}\) and set \(\{\delta, x_i\} = \{y_j\}_{1 \leq j \leq n} \cup \{v_j\}_{1 \leq j \leq m} \cup \{p\} \subset M\). We will show that this constitutes a critical set for \(N \subset M\). Let \(H\) be an \(M\)-\(M\) bimodule, \(\xi \in H\), \(\|\xi\| = 1\) with \([\xi, N] = 0\) and \(\|\xi, x_i\| < \delta\), for all \(i\). Set \(\xi_0 = p\xi\). Clearly, \(\xi_0\) is central for \(pNp\). Note that \(\|y_j \xi_0 - \xi_0 y_j\| = \|y_j p\xi p - p\xi y_j\| \leq \|y_j \xi - \xi y_j\| + \|\xi - \xi y_j\| + \|\xi y_j - p\xi y_j\| \leq 2\|\xi - p\xi\| + \|\xi - \xi y_j\| < 3\delta = \epsilon\).

Furthermore \(\|\xi\|^2 = \sum_{i=1}^m \|v_i^* \xi v_i\|^2 \leq \sum_i (\|v_i^* \xi - v_i^* \xi v_i\|^2 + 2\|v_i^* \xi v_i\|^2)\|v_i - \xi v_i\| + \|v_i^* \xi v_i\|^2\|v_i - \xi v_i\| + \|\xi v_i\|^2\|v_i - \xi v_i\|^2\|v_i - \xi v_i\|^2\). Hence \(\|\xi\|^2 \leq \delta(\delta + 2)m + \sum_i \|v_i^* \xi v_i\|^2\|v_i - \xi v_i\|^2\). But \(\|v_i^* \xi v_i\|^2 \leq \|\xi\|^2\), for all \(i\), so that \(\|\xi\|^2 \leq \frac{\delta(\delta + 2)m + \sum_i \|v_i^* \xi v_i\|^2\|v_i - \xi v_i\|^2 + \|\xi\|^2\|v_i - \xi v_i\|^2}{m}\) is also small.

Since \(\|y_j \xi_0 - \xi_0 y_j\| = \|y_j p\xi p - p\xi y_j\| \leq \|y_j \xi - \xi y_j\| + \|\xi - \xi y_j\| + \|\xi y_j - p\xi y_j\| \leq 2\|\xi - p\xi\| + \|\xi - \xi y_j\| < 3\delta = \epsilon\).

Conversely, suppose that \(M\) has property T relative to \(N\). Let \(\{\epsilon, x_1, \ldots, x_n\}\), \(x_i \in M\), \(1 \leq i \leq n\), be a critical set for \(N \subset M\). Let \(H_0\) be an \(nMn\)-\(nMn\) bimodule and let \(\xi_0 \in H_0\), \(\|\xi_0\| = 1\), with \([\xi_0, pNp] = 0\). As above, we can find partial isometries \(v_1, \ldots, v_m \in M\) with \(v_i v_i^* = p\), \(2 \leq i \leq m\), \(v_1 v_1^* \leq p\) and \(\sum_{i=1}^m v_i v_i^* = 1\). Consider the set \(\{\delta, v_x, v_y\}_{1 \leq x \leq n, 1 \leq y \leq m} - \) it will be a critical set for \(pNp \subset pMn\), if \(\delta\) is small enough. Set \(\alpha = \text{tr}(p)^{-1}\) and let \(H\) be the \(\alpha\)-amplification of \(H_0\). The vector \(\xi = \sum_{i=1}^m v_i^* \xi v_i \in H\) is \(N\)-central and \(\neq 0\). We leave it as an exercise to check that \(\|\xi v_k - x_k \xi\| < \epsilon, 1 \leq k \leq n\), if \(\delta\) is chosen small enough. Thus, since \(M\) has property T relative to \(N\), there is an \(M\)-central vector \(\eta \in H\), \(\eta \neq 0\). But then \(pMn \neq 0\) and \(pMn \subset pNp = H_0\) is the desired \(pMn\)-central vector in \(H_0\).

Proposition 2.2 implies the following corollary (see [Po6] for a slightly more general statement):

**Corollary 2.3.** Let \(N \subset M\) be an extremal inclusion of \(II_1\) factors with finite index and let \(N \subset M \subset M_1\) be the basic construction. Then the standard invariant of \(N \subset M\) has property T if and only if the standard invariant of \(M \subset M_1\) has property T.

**Proof.** Consider the inclusions \(M \vee N_{op} \subset M_1 \vee N_{op} \subset M \otimes M_{op}\) and \(M \vee N_{op} \subset M \vee M_{op} \subset M \otimes M_{op}\). By proposition 2.2, i) we have that \(M \otimes M_{op}\) has property T relative to \(M_1 \vee N_{op}\) if and only if it has property T relative to \(M \vee N_{op}\). Using
the second inclusion, we get similarly that $M \& N$ has property T relative to $M \vee N$ if and only if it has property T relative to $M \vee M$. Thus $M \& M$ has property T relative to $M \vee M$ (i.e. $G_{N,M}$ has property T) if and only if $M \& M$ has property T relative to $M_1 \vee N$. But $M_1 \vee N \subset M \& M$ is a reduced inclusion of the symmetric enveloping inclusion for $M \subset M_1$, reduced by $e_{1}^{\text{op}}$ [Po6, section 2]. By proposition 2.2 ii) we are done. □

We also need the following observation from [Po6]:

**Lemma 2.4.** Let $Q \subset N \subset M$ be an extremal inclusion of II$_1$ factors with finite index and let $\{m_j\}_{j \in J}$ be an orthonormal basis of $N$ over $Q$. The map which sends $e_N$ to $\sum_j m_je_Q m_j^{*}$ and which is the identity on $\text{Alg}(M,M^{\text{op}})$ implements a unital embedding of $C^{*}(M,e_{N},JMJ)$ into $C^{*}(M,e_{Q},JMJ)$ and thus a unital embedding of $M \& M^{\text{op}}$ into $M \& M^{\text{op}}$.

**Proof.** If $x \in N$, then we have $x = \sum_j m_je_Q(m_j^{*}x)$ and hence $\sum_j m_je_Q m_j^{*}(\hat{x}) = \sum_j m_je_Q(m_j^{*}x)^{\wedge} = \hat{x}$. Also, if $x \in M$, $\hat{x} \perp N$, then $\hat{m}_j^{*}x \perp N$ as well, so $\sum_j m_je_Q m_j^{*}(\hat{x}) = \sum_j m_je_Q(m_j^{*}x) = \sum_j m_je_Q e_N(m_j^{*}x) = 0$. Hence $\sum_j m_je_Q m_j^{*} = e_N$ in $B(L^{2}(M))$. Thus we get $C^{*}(M,e_{N},JMJ) \subset C^{*}(M,e_{Q},JMJ)$. By [Po6] it follows that this implements an inclusion $M \& M^{\text{op}} \subset M \& M^{\text{op}}$ as well. □

**Lemma 2.5.** Let $N \subset P \subset M$ be an extremal inclusion of II$_1$ factors with finite index and let $P_{-1} \subset N \subset P \subset M$ be such that $N \subset P$ is the basic construction for $P_{-1} \subset N$. Then $G_{N,M}$ has property T if and only if $G_{P_{-1},M}$ has property T.

**Proof.** By lemma 2.4 we have the inclusion $(M \vee M^{\text{op}} \subset M \& M^{\text{op}}) \subset (M \vee M^{\text{op}} \subset M \& M^{\text{op}})^{\text{op}}$. We will show that it is in fact an equality. Clearly, the statement follows then by applying proposition 2.2 i). Let $N_1 \subset P_{-1} \subset N \subset P \subset M$ be such that $N \subset M$ is the basic construction for $N_1 \subset N$ and $N \subset P$ is the basic construction for $P_{-1} \subset N$ (see for instance [Bi]). Then by lemma 2.4 we have the inclusions $C^{*}(M,e_{N},JM_{M}JM_{M}) \subset C^{*}(M,e_{P_{-1}},JM_{M}JM_{M}) \subset C^{*}(M,e_{N_{1}},JM_{M}JM_{M})$. By [Po6, section 2] the first algebra actually equals the last (using the fact that the Jones projection $M \rightarrow N_1$ can be written in terms of the $e_i$’s). □

We show next the main theorem of this section.

**Theorem 2.6.** Let $N \subset P \subset M$ be an extremal inclusion of II$_1$ factors with finite index and let $P_{-1} \subset N \subset P \subset M \subset P_1$ be such that $N \subset P$ is a basic construction for $P_{-1} \subset N$ and $M \subset P_1$ is a basic construction for $P \subset M$. Then the standard invariant of $P_{-1} \subset P_1$ has property T if and only if the standard invariant of $N \subset M$ has property T. The same statement holds if “property T” is replaced by “amenable”.

**Proof.** Let $P_{-2} \subset N_1 \subset P_{-1} \subset N \subset P \subset M \subset P_1$ be such that $P_{-2} \subset N \subset P_1$ and $N_1 \subset N \subset M$ are basic constructions. Thus the standard invariant of $P_{-2} \subset N$ has property T if and only if the standard invariant of $N \subset P_1$ has property T (corollary 2.3). Furthermore, by lemma 2.5 $G_{N_{1},M}$ has property T if and only if $G_{N_{1},M}$ has property T. □
property T. By corollary 2.3 we have that $G_{N,M}$ has property T if and only if $G_{N_1,N}$ has property T and the latter has property T if and only if $G_{P_{-2},N}$ does (lemma 2.5 again). Hence, $G_{N,M}$ has property T if and only if $G_{N,P_1}$ has property T.

Now consider the inclusions $P_{-1} \subset N \subset P \subset P_1$ and apply lemma 2.5 again. Thus $G_{P_{-1},P_1}$ has property T if and only if $G_{N,P_1}$ has property T. Together with the above equivalences we have therefore that the standard invariant of $P_{-1} \subset P_1$ has property T if and only if the one of $N \subset M$ has property T.

Using ([Po2], [Po6]) the same proof goes through if “property T” of the standard invariant is replaced by “amenability” of the standard invariant (assuming of course in addition that the II_1 factors appearing are hyperfinite, if one replaces “property T” by “amenable inclusions” rather than “amenable standard invariant”). □

To prove theorem 2.1 we also need the following proposition.

**Proposition 2.7.** Let $H$ and $K$ be finite groups with properly outer actions $\sigma$, resp. $\rho$ on the II_1 factor $P$. Denote $N = P^H$ and $M = P \rtimes K$ and let $P_{-1} \subset N \subset P \subset M \subset P_1$ be such that $N \subset P$ is the basic construction for $P_{-1} \subset N$ and $M \subset P_1$ is the basic construction for $P \subset M$. Then $P_{-1} \subset P_1$ is a locally trivial inclusion given by $\{\rho_k \sigma_h\}_{h \in H, k \in K}$.

**Proof.** Note that $P \subset P_1$ can be described as follows: $P_1 = P \otimes B(L^2(K))$ with matrix units $\{e_{k,k'}\}_{k,k' \in K}$ and the inclusion $P \hookrightarrow P_1$ is given by $P \ni x \mapsto \sum_{k \in K} \rho_k(x)e_{k,k}$. By perturbing each $\sigma_h$ and $\rho_k$ by some inner automorphism of $P$ we can assume that they all leave the same $B \subset P$ invariant, with matrix factor $B \cong B(L^2(H))$ and matrix units $\{f_{h,h'}\}_{h,h' \in H}$. But then $P_{-1} \subset P$ is isomorphic to the inclusion $P_{-1} \hookrightarrow P_{-1} \otimes B(L^2(H))$, given by $P_{-1} \ni x \mapsto \sum_{h \in H} \sigma_h(x)f_{h,h}$. Altogether this gives that $P_{-1} \subset P_1$ is isomorphic to the inclusion $P_{-1} \hookrightarrow P_{-1} \otimes B(L^2(H)) \otimes B(L^2(K))$ given by $P_{-1} \ni x \mapsto \sum_{h \in H, k \in K} \rho_k \sigma_h(x)f_{h,h}e_{k,k}$. □

Theorem 2.1 follows now immediately from proposition 1.4 and theorem 2.6.

**Remark 2.8.** Observe that we recovered in theorem 2.6 and proposition 2.7 the result [BH, theorem 4.9]) showing that $P^H \subset P \times K$ has amenable principal graph if and only if the group $G$ generated by $\sigma(H)$ and $\rho(K)$ in $\text{Out}(P)$ is amenable.

Numerous explicit examples of infinite groups $G$ that have property T and are quotients of the free product of two finite groups $H$ and $K$ can be found in [BS] (we would like to thank Pierre de la Harpe for pointing out this reference to us). The construction of these groups is based on Sarnak’s examples of Ramanujan graphs [Sa, chapter 3] and an explicit presentation of this class of property T groups can be found in [BS, theorem 2 and corollary 2]. Since any discrete group has a properly outer action on the hyperfinite II_1 factor $R$, we obtain therefore (by applying theorem 2.1) irreducible, infinite depth subfactors of the form $R^H \subset R \times K$ whose standard invariant has property T.

We will construct next irreducible, infinite depth subfactors with Jones index 6 whose standard invariant has property T. By theorem 2.1 we need to find an infinite property T group which is a quotient of $PSL(2, \mathbb{Z}) = \mathbb{Z}_2 \ast \mathbb{Z}_3$.

**Corollary 2.9.** There are irreducible, infinite depth subfactors of the hyperfinite II_1 factor with Jones index 6 whose standard lattice has property T.
Proof. Let $G$ be the group given by the presentation $\langle x, y \mid x^2 = y^3 = (xy)^{12} = (xy^{-1}xy^{-1}xyxyxy^{-1}xy)^2 = 1 \rangle$. It is shown in [Co] that this group is a finite index subgroup of $SL(3, \mathbb{Z}) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $SL(3, \mathbb{Z})$ as the inverse-transpose automorphism. Since $SL(3, \mathbb{Z})$ has property T ([K], see also [HV]) and property T is preserved under finite extensions and finite index subgroups (see for instance [HV]), the group $G$ must have property T as well. $G$ is the desired infinite quotient of $PSL(2, \mathbb{Z})$ with property T - apply then theorem 2.1. \qed

Remark 2.10.

i) In [CRW] $SL(3, \mathbb{Z})$ is written as a quotient of $A_4 \rtimes \mathbb{Z}_3$. By theorem 2.1 we obtain therefore an irreducible, infinite depth subfactor with index 36 and property T standard invariant.

ii) In [CRW] $SL(3, \mathbb{Z}) \rtimes \mathbb{Z}_2$ is written as a quotient of $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$. Again by theorem 2.1, we obtain an irreducible, infinite depth subfactor with index 12 and property T standard invariant and intermediate subfactors of index 2 and 6.

We end this paper by mentioning a few problems that are closely related to the above work and seem to be of interest at this point.

a) Are there irreducible, infinite depth subfactors with property T standard lattice and Jones index $< 6$? Observe that such a subfactor would have necessarily index $> 3 + \sqrt{3}$ by [Ha] and [Po6, section 9].

b) The group type inclusions $M^H \subset M \rtimes K$ have intermediate subfactors. Construct irreducible, infinite depth subfactors whose standard lattice has property T that do not have intermediate subfactors.

c) Construct irreducible, infinite depth subfactors with noninteger Jones index whose standard lattice has property T.

d) It would be interesting to compute the symmetric enveloping inclusion associated to $M^H \subset M \rtimes K$ and to give then a different proof of our theorem 2.1.

e) Suppose $N \subset P \subset M$ is an extremal inclusion of II$_1$ factors with finite index such that the standard lattice of $N \subset M$ has property T. Suppose furthermore that one of the intermediate inclusions has finite depth. Does it then follow that the standard lattice of the other intermediate inclusion has property T? The answer may very well be negative.
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