Area spectrum and quasinormal modes of black holes

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Abstract

We demonstrate that an equidistant area spectrum for the link variables in loop quantum gravity can reproduce both the thermodynamics and the quasinormal mode properties of black holes.

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1 Introduction

The quantum properties of black holes is a fascinating subject. Starting with the pioneering work of Bekenstein and Hawking [1, 2], and continuing into the realms of string theory and quantum gravity [3, 4], they have been the subject of intense investigation.

An important issue is the extent to which the fundamental quantum and statistical mechanical properties of black holes are fixed by internal consistency and their known macroscopic properties alone; that is, the extent to which these properties are independent of any specific model or theory of quantum gravity.

Interesting steps in this direction were taken recently by semiclassical consideration of the macroscopic oscillation modes of black holes, in an approach reminiscent to Debye’s theory for the heat capacity of solids. Hod [5] has pointed out that the decaying ‘ringing modes’ (or quasinormal modes) of a Schwartzchild black hole conform with the existence of the Bekenstein-Hawking temperature and imply emission in terms of a fixed area quantum. He further noticed that, in Planck units ($c = \hbar = G = 1$), this quantum has the numerical value $4 \ln 3$ (a fact later proved analytically [6]). This suggests a composition of the black hole in terms of three-state objects. Dreyer [7] proposed that this result could be recovered in loop quantum gravity if the relevant group is taken to be $SO(3)$ rather than $SU(2)$. These developments have spurred a lot of subsequent activity [8]–[11].

Although the above considerations are still somewhat speculative, they certainly point to the intriguing possibility of using classical, macroscopic properties of black holes to infer information on their quantum, microscopic nature. It is, therefore, of interest to explore the alternative microscopic scenario in which these properties can be accommodated. This is the subject of this paper. As we shall see, an equidistant spectrum naturally reproduces thermodynamic and ringing mode properties of black holes, still allowing for at least two distinct possibilities, depending on the quantum statistics of their microscopic constituents.

2 The quasinormal mode argument

We summarize here the connection between the decaying ringing modes of a black hole and its area spectrum [5, 7].

Perturbations in the metric of a black hole of mass $M$ decay according to a set of characteristic discrete (complex) frequencies. These frequencies for highly damped modes are known to asymptote to the values [12]–[13, 5, 6]

$$\omega_n = \frac{\ln 3}{8\pi M} - \frac{2\pi}{8\pi M} \left( n + \frac{1}{2} \right)$$

(1)

The spacing of the imaginary part of these frequencies agrees with the Bekenstein-Hawking temperature of the black hole $T_{BH} = 1/(8\pi M)$. Hod suggested that the asymptotically constant real part can be semiclassically interpreted as the quantum
of energy \( \Delta E = \Delta M \) emitted by the black hole. Using the standard expression for the area of the black hole \( A = 16\pi M^2 \), this implies that the area decreases by a quantum \( \Delta A \):

\[
\Delta A = 32\pi M \Delta M = 4 \ln 3
\]  

\( (2) \)

Hod further observed that, if we consider the black hole as composed of \( N \) ‘bits’, each contributing a quantum of area \( 4\ln 3 \) and having 3 internal states, then the total number of states of the black hole would be \( \Omega = 3^N \) and the entropy would calculate to

\[
S = \ln \Omega = N \ln 3 = \frac{1}{4}(N4 \ln 3) = \frac{1}{4}A
\]

\( (3) \)

in agreement with the Bekenstein-Hawking area-entropy result.

Finally, Dreyer suggested that the three-state ‘bits’ can be spin-1 link representations in the loop quantum gravity formulation. To have the black hole dominated by spin-1 links (rather than spin-\( \frac{1}{2} \) ones, which is the standard result), Dreyer proposed a modification of the link group from \( SU(2) \) to \( SO(3) \); then only integer-spin representations are allowed, with the lowest nontrivial one, spin-1, dominating.

Discarding half-integer spins can be unwanted, since they are required if the black hole is to couple to half-integer spin (fermionic) fields. Further, Dreyer’s analysis assumes the standard expression for the area and degeneracy contribution of each link to the black hole. Alternative approaches, suggesting an equidistant link area spectrum and a different state counting have been proposed. We shall revisit this analysis by considering these different possibilities.

3 The composite black hole model

The spectrum of the area of black holes, and its degeneracy \( \Omega \) for a given area, are the important ingredients in their quantum statistics. Their determination involves some version of a quantum gravity model for the black hole.

In a large class of quantum gravity models, black holes are effectively represented as composite systems consisting of a large number of identical components. Each component contributes an elementary area, and the total area of the black hole horizon is the sum of the areas of the components. The full spectrum and degeneracy of the black hole area is determined by the area spectrum and degeneracies of each component, as well as the quantum statistics obeyed by the (identical, but not necessarily indistinguishable) components.

As it turns out, any individual component spectrum and degeneracy will lead to the area-entropy law \( S \sim A \) for a macroscopic black hole, as long as the components are considered to be somewhat distinguishable [14]. Completely indistinguishable statistics for the components (fermionic or bosonic), on the other hand, lead to the weaker entropy dependence \( S \sim A^c \), with the exponent \( c < 1 \) depending on the area spectrum of the components.
In this paper we shall concentrate on the composite model deriving from loop quantum gravity [15, 16, 4]. In it, the components are horizon-piercing links on a random lattice, each carrying a representation of $SU(2)$. As such, their elementary area depends on the quadratic Casimir of the representation, while the degeneracy is the dimensionality of the representation. Labeling the spin $j$ of the representation by the integer $n = 2j = 0, 1, 2, \ldots$, the area spectrum of each component is $a_n$ and its degeneracy is $g_n = n + 1$.

Strictly speaking, the link states are not truly independent, since there is the Gauss law constraint that the total $SU(2)$ state of the black hole should be a singlet. The ensuing reduction of the state space, however, proves to be irrelevant for the thermodynamics of macroscopic black holes; in this limit, the link components can be treated as independent.

There are different proposals for the form of the spectrum $a_n$. The standard result [15] is that the area is proportional to the square root of the Casimir, that is,

$$ a_n = 2\gamma \sqrt{j(j+1)} = \gamma \sqrt{n(n+2)} $$

with $\gamma$ a proportionality constant (related to the so-called Immirzi parameter).

In [14], on the other hand, it was argued that quantum corrections would additively renormalize the Casimir into $j(j+1) + \frac{1}{4} = (j + \frac{1}{2})^2$, which leads to an equidistant spectrum:

$$ a_n = 2\gamma (j + \frac{1}{2}) = \gamma (n+1) $$

This conforms with the original arguments of Bekenstein [4] that the horizon area, behaving as an adiabatic invariant, should be integrally quantized, as well as with various more recent proposals for an equidistant area spectrum [17]-[26].

There are also differing considerations of the statistics obeyed by the components. One of them considers them as partially distinguishable, while another considers them as fully distinguishable. These lead to different statistical ensembles with important physical implications. We shall analyze the alternatives in the sequel.

4 Partially distinguishable components

The degeneracy of states of a black hole consisting of many components, according to one counting [4], is

$$ \Omega = \prod_{n=0}^{\infty} g_n^{N_n} $$

where $N_n$ is the number of components with spin $j = n/2$ and $g_n = n + 1$ the number of internal states of each such component. As we shall explain in the next section, this assigns the components a peculiar partial distinguishability.

The statistical mechanics of a macroscopic black hole can be obtained in a standard way by considering a statistical ensemble of black holes, corresponding to
an intensive ‘temperature’ parameter $\beta$ dual to the area (this is not the standard temperature parameter, which is dual to the energy). The area partition function becomes

$$Z = \sum_A \Omega(A)e^{-\beta A} = \sum_{\{N_n\}} \prod_{n=0}^{\infty} g_n^{N_n} e^{-\beta N_n a_n} = \prod_{n=0}^{\infty} \frac{1}{1 - g_n e^{-\beta a_n}} \quad (7)$$

Note that we also sum over different link numbers, the links being quantum variables that can be created or annihilated.

For a macroscopic black hole the area and degeneracy of states grow large and thus the above partition function must diverge. As we increase the ‘temperature’, that is, decrease $\beta$ from infinity, a macroscopic black hole will form at some critical $\beta_c$ for which $Z$ diverges. Generically, this happens when one of the denominators in (7), corresponding to a specific spin $n_c$, diverges; that is,

$$\beta_c = max\{\beta_n\} \equiv \beta_{n_c}, \quad \text{where} \quad \beta_n = \frac{\ln g_n}{a_n} = \frac{\ln(n + 1)}{a_n} \quad (8)$$

At this point a condensate of spins $j_c = n_c/2$ forms, while the number of other spins remains finite and of order one. The area and entropy are dominated by the spins of the condensate and we have

$$A = N a_{n_c}, \quad S = \ln \Omega = \ln(g_{n_c}^N) \quad (9)$$

where $N$ is the total number of spins in the condensate. The above lead to the entropy law

$$S = \frac{\ln g_{n_c}}{a_{n_c}} A \quad (10)$$

Conformity with the Bekenstein-Hawking (BH) result $S = \frac{1}{4} A$ requires

$$a_{n_c} = 4 \ln g_{n_c} \quad (11)$$

Transitions of this black hole to one with different area will be statistically dominated by processes increasing or decreasing the number of spins in the condensate. That is, the emission spectrum of the black hole will exhibit the area quantum

$$\Delta A = a_{n_c} \quad (12)$$

Conformity with the ringing mode calculation $\Delta A = 4 \ln 3$, then, implies

$$g_{n_c} = 3, \quad a_{n_c} = 4 \ln 3 \quad (13)$$

corresponding to $n = 2$ (for $g_n = n + 1$). This picks spin-1 links as the ones dominating the composition of the black hole.

With the standard area spectrum (14), $a_n = \gamma \sqrt{n(n+2)}$, the critical point is at $\beta_c = \beta_1 = \ln 2/\gamma \sqrt{3}$. (Note that spin-0 links, which do not contribute to the
area, are considered as part of the vacuum and omitted from (7)). This picks spin-$\frac{1}{2}$ components for the condensate and $g_{nc} = 2$, thus leading to a disagreement with the ringing mode calculation. The remedy proposed for this problem is to accept only integer spins in the spectrum; that is, consider representations of $SO(3)$ rather than $SU(2)$ [7]. As stated before, this has the drawback that the discarded half-integer spins are desirable in order to have the black hole couple to fermion fields.

The first main observation of this paper is that the modified, equidistant spectrum (5), $a_n = \gamma(n+1)$, remedies this problem without discarding half-integer spins. Indeed, with this spectrum we have

$$\beta_0 = 0, \quad \beta_1 = \beta_3 = \frac{\ln 2}{2\gamma} = 0.3466$$
$$\beta_2 = \frac{\ln 3}{3\gamma} = 0.3662 \gamma \ldots \quad (14)$$

We see that the critical parameter is now shifted to $\beta_2$, which picks spin-1 components for the condensate. Note, further, that although spin-0 links now contribute an amount $a_1 = \gamma$ to the area and thus cannot be considered as part of the vacuum, they are thermodynamically suppressed. Thus, we obtain agreement with the ringing mode calculation while retaining the full $SU(2)$ spectrum.

5 Fully distinguishable components

The state-counting formula (6) corresponds to a peculiar, partially distinguishable statistics for the components. To make this point clear, consider two links with spins $j_1$ and $j_2$ and internal states (eigenvalues of the third component) $m_1$ and $m_2$ respectively. Formula (6) counts this as a single state, irrespective of which link carries which spin. So the quantum state is invariant under permutation of these two spins:

$$\langle (j_1, m_1), (j_2, m_2) \rangle = \langle (j_2, m_2), (j_1, m_1) \rangle \quad (15)$$

On the other hand, for two equal spins $j$, formula (6) counts all $(2j + 1) \times (2j + 1)$ states as distinct; that is

$$\langle (j, m_1), (j, m_2) \rangle \neq \langle (j, m_2), (j, m_1) \rangle \quad (16)$$

We see no justification for such a counting. If, indeed, nothing else distinguishes the links except their spin, then quantum mechanics implies that they should be treated as indistinguishable objects, in which case (16) is incorrect and the entropy-area law cannot be recovered. In [14], however, we have argued that the links are fully distinct because of their connectivity to the lattice representing spacetime outside of the black hole. In this case, they should be treated as completely distinguishable objects. (A similar point of view was put forth by Krasnov in [27], and also by Rovelli [28], but was apparently abandoned in subsequent work.)

This leads to different thermodynamics. The partition function of $N$ distinguishable links is

$$Z_N = Z_1^N, \quad \text{where} \quad Z_1 = \sum_{n=0}^{\infty} g_n e^{-\beta a_n} \quad (17)$$
The full partition function for all possible numbers of links is, thus,

\[ Z = \sum_{N=0}^{\infty} Z_N = \frac{1}{1 - Z_1} \]  

(18)

A macroscopic black hole will, again, form when \( Z \) diverges. This will happen for the critical value \( \beta_c \) for which

\[ Z_1(\beta_c) = 1 \]  

(19)

Standard thermodynamic arguments lead to the expression for the entropy

\[ S = \beta A + \ln Z \]  

(20)

At the critical point, \( Z \) is of order \( A \); thus, \( \ln Z \) is a sub-leading logarithmic correction. Such corrections will be present, but cannot be reliably calculated in the ensemble formulation since this implies fluctuations for the total black hole area which would not be there in a true area eigenstate. The leading (thermodynamic) contribution, however, is accurately given by the first term in (20) as

\[ S = \beta_c A \]  

(21)

Conformity with the standard BH result implies \( \beta_c = \frac{1}{4} \).

It should be stressed that, now, the black hole state is \textit{not} dominated by any single spin value; rather, there is a Boltzmann distribution of spins for all possible values. In fact, the probability that any given link carries spin \( j = n/2 \) is given by

\[ P_n = g_n e^{-\beta_c a_n} = (n + 1)e^{-\beta_c a_n} \]  

(22)

Transition to a black hole of different (lesser) area would involve transitions between different values of spin, as well as the disappearance of links. Therefore, black hole transitions involve quanta of area \( \Delta A = a_n - a_n' \), corresponding to transitions, as well as \( \Delta A = a_n \), corresponding to annihilation of links. Note, further, that spin-0 links \textit{cannot} contribute zero area in the fully distinguishable picture; this would lead to a thermodynamic collapse of the black hole into a state with an arbitrarily large number of spin-0 links and arbitrarily large entropy.

With the above conditions, it is easy to see that the standard spectrum \( a_n = \gamma \sqrt{n(n + 2)} \) is inconsistent with the quantization of area implied by the ringing mode calculation, since it leads to an essentially continuous area emission spectrum with no apparent quantum. The equidistant spectrum \( a_n = \gamma(n + 1) \), however, is consistent, predicting a quantum of area \( \Delta A = \gamma \). This fixes \( \gamma = 4 \ln 3 \). An explicit calculation of \( Z_1 \), on the other hand, leads to

\[ \beta_c = \gamma^{-1} \ln \frac{3 + \sqrt{5}}{2} = \frac{0.876}{4} \]  

(23)

This differs (although not by much) from the value \( \beta_c = \frac{1}{4} \) required by the BH area-entropy law.
The only possibility, therefore, is that the spin content is modified. We shall, again, only consider changing the group from \( SU(2) \) to \( SO(3) \), seeing no justification for any other modification. Thus, we now label the states in terms of the integer spin \( j = n \). The degeneracy is \( g_n = 2n + 1 \). For the spectrum, we must consider a renormalization of the standard result consistent with the requirement for a quantum of area, as explained two paragraphs above. The result \( a_n = 2\gamma(j + \frac{1}{2}) = \gamma(2n + 1) \), is not satisfactory, since it predicts an area quantum \( 2\gamma \) for transitions but a different area quantum \( \gamma \) for creation or annihilation of links (for spin-0 links). We thus propose the modified equidistant spectrum

\[
a_n = \gamma(n + 1) \quad g_n = 2n + 1 \quad \gamma = 4 \ln 3
\]

This corresponds to a quantum \( \gamma = 4 \ln 3 \) for transitions between different integer spins, and the same quantum for the creation or annihilation of links.

We should stress that the above spectrum is determined only by the requirement of a fixed area quantum. It is still not guaranteed that it will reproduce the correct black holes thermodynamics. To that end, we calculate the single-link partition function:

\[
Z_1 = \sum_{n=0}^{\infty} (2n + 1)e^{-\beta \gamma(n+1)} = \frac{e^{\beta \gamma} + 1}{(e^{\beta \gamma} - 1)^2}
\]

The critical point \( Z_1 = 1 \) is now at \( e^{\beta \gamma} = 3 \), or

\[
\beta_c = \frac{\ln 3}{\gamma} = \frac{1}{4}
\]

This leads to the correct BH area-entropy relation. We therefore conclude that distinguishable components with an equidistant spectrum of integer spins correctly reproduce both the thermodynamics and the ringing mode properties of black holes. No single spin value dominates. The probabilities \( P_n \) of appearance of the value \( j = n \) for the spin of a link are

\[
P_0 = \frac{1}{3} \quad P_1 = \frac{1}{3} \quad P_2 = \frac{5}{27} \ldots
\]

6 Conclusions and discussion

To summarize, we have made two main points:

1. If the standard counting formula for states \( 6 \) is assumed, then the equidistant area spectrum as proposed in \( 14 \) naturally explains the domination of spin-1 links and reproduces the ringing mode properties of black holes, without the need to eliminate half-integer spins.

2. If totally distinguishable link statistics are assumed, then an equidistant area spectrum of components, carrying integer spins only, correctly reproduces ringing mode properties, without domination of spin-1 components.
Which, if any, of the above cases is realized must be the subject of further investigation. The successful model should account in a natural way for the properties of black holes carrying charge and angular momentum.

A criticism for an equidistant area spectrum, as advocated in this paper, is that it leads to substantial deviations from the black-body spectrum of Hawking radiation. Indeed, an area and energy quantum implies a discrete emission spectrum with the spacing of spectral lines being of the same order of magnitude as the black body thermal frequency.

In fact, the same criticism would apply to the standard loop quantum gravity spectrum and degeneracy (4) and (6). It has been argued that the irrational character of the area eigenvalues (4) creates a spread of possible area quanta under transitions which effectively reproduce a continuum. However, the state of the black hole is dominated by a condensate of spin-$\frac{1}{2}$ links, and the number of links with different spin values is of order one. All transitions of the black hole are dominated by processes increasing or decreasing the number of links in the condensate and thus reproduce a discrete emission spectrum. It is unreasonable to expect that the remaining few links account for the bulk of the (continuous) Hawking radiation.

Distinguishable statistics actually cure that, since now the black hole state contains all spins. This still leaves us with a disagreement with the ringing mode result, as stated in the previous section. It seems that to take the ringing mode connection seriously, we must admit an equidistant area spectrum and thus accept the fact that the Hawking radiation spectrum is discrete.

We do not feel that this is damning. The high-frequency exponential part of the spectrum is accurately reproduced, the discreteness there being inconsequential. This is the energy range in which the photons (or other emitted particles) behave essentially like classical particles, whose scattering properties are expected to be accurately reproduced by the classical black hole metric. For frequencies close to the thermal frequency, however, the wavelength of the photons becomes comparable to the size of the black hole and they sense global properties of its geometry. Back-reaction due to geometry change at emission and absorption of such photons is expected to be important, the energy of these photons being of the same order as the energy spacing of the black hole. A deviation from ideal black-body spectrum, which assumes a fixed metric and ignores back-reaction, would seem reasonable.

In summary, the essence of the Hawking radiation argument is that a black hole would be in thermal equilibrium (although an unstable one) with a heat bath at temperature $T_{\text{BH}} = \frac{8}{3}pM$. Removal of the heat bath would deprive the black hole of all absorption, leaving only the emitted Hawking radiation. This does not necessarily mean that radiation is black-body. A large atom can be in thermal equilibrium with a heat bath of some temperature, but in the absence of the heat bath it would radiate in its own line spectrum. Perhaps we are facing a similar situation.
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