BOSON-FERMION CORRESPONDENCE FOR HALL-LITTLEWOOD POLYNOMIALS REVISITED

G. NECOECEHA AND N. ROZHKOVSKAYA

Abstract. We connect twisted vertex operator presentation of Hall-Littlewood polynomials with the action of charged free fermions, describe a boson-fermion correspondence that relates twisted vertex operators with classical Heisenberg algebra. We also show that the elements of the orthogonal to Schur functions basis of the one-parameter deformation of the ring of symmetric functions $\Lambda[t]$ are $\tau$-functions of KP hierarchy.

1. INTRODUCTION

This note was inspired by the vertex operator realization of Hall-Littlewood polynomials and the twisted version of boson-fermion correspondence developed in [8, 9] and further applied by many authors. We observe that the algebraic structures of these constructions are closely related to classical (non-twisted) algebra of charged free fermions. This allows us to simplify significantly the calculations of vertex operators, to give new insight into the algebraic structures, and to propose another version of boson-fermion correspondence directly relating the twisted vertex operators with the classical (non-twisted) Heisenberg algebra.

In [11], twisted vertex operators of [8, 9] were applied in interpretation of deformations of generating functions for random plane partitions. One of the results of [11] describes a connection between the KP hierarchy and an orthogonality condition in $\Lambda[t]$ that involves classical Schur functions $s_\lambda$ and the dual basis $S_\lambda$. In this note we show explicitly that $S_\lambda$’s are $\tau$-functions of KP hierarchy and find the generating function and two different vertex operator presentations of these symmetric functions.

In Sections 2, 3 we review necessary facts about symmetric functions and the action of charged free fermions on the ring of symmetric functions. In Section 4 we connect twisted fermions with classical charged free fermions. In Section 5 we propose a version of boson-fermion correspondence for twisted fermions and compare it with the construction of [9]. In Section 6 we prove that the elements of the orthogonal to Schur functions basis are $\tau$-functions of KP hierarchy and provide two different versions of their vertex operator realizations.

2. Preliminaries on symmetric functions

Statements reviewed in this section can be found in e.g. [3, 7].

Let $\Lambda = \Lambda[x_1, x_2, \ldots]$ be the ring of symmetric functions in variables $(x_1, x_2, \ldots)$. Schur symmetric functions $s_\lambda$ labeled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ form a linear basis of $\Lambda$. They are defined by

$$s_\lambda(x_1, x_2, \ldots) = \frac{\det[x_i^{\lambda_j + n-j}]}{\det[x_i^{n-j}]}.$$
Complete symmetric functions $h_r = s(r)$ and elementary symmetric functions $e_r = s(1^r)$ are Schur functions that correspond to one-row and one-column partitions:

$$h_r(x_1, x_2 \ldots) = \sum_{1 \leq i_1 \leq \ldots \leq i_r < \infty} x_{i_1} \ldots x_{i_r},$$
$$e_r(x_1, x_2 \ldots) = \sum_{1 < i_1 < \ldots < i_r < \infty} x_{i_1} \ldots x_{i_r}.$$  

One also introduces power sums

$$p_k(x_1, x_2, \ldots) = \sum_i x_i^k.$$  

Then the ring of symmetric functions $\Lambda$ is a polynomial ring in algebraically independent variables $(h_1, h_2, \ldots), (e_1, e_2, \ldots)$ or $(p_1, p_2, \ldots)$:

$$\Lambda = \mathbb{C}[h_1, h_2, \ldots] = \mathbb{C}[e_1, e_2, \ldots] = \mathbb{C}[p_1, p_2, \ldots].$$  

Based on the interpretation of a polynomial ring as the ring of symmetric functions, one defines boson Fock space $\mathcal{B}$. Let $\mathcal{B} = \mathbb{C}[z, z^{-1}, p_1, p_2, \ldots]$ be the graded space of polynomials

$$\mathcal{B} = \oplus_{m \in \mathbb{Z}} \mathcal{B}^{(m)}, \quad \mathcal{B}^{(m)} = z^m \cdot \mathbb{C}[p_1, p_2, \ldots] = z^m \cdot \Lambda.$$  

We write generating functions for complete and elementary symmetric functions:

$$H(u) = \frac{1}{u} \sum_{k \geq 0} \frac{h_k}{u^k}, \quad E(u) = \frac{1}{u} \sum_{k \geq 0} \frac{e_k}{u^k} = \prod_{i \geq 1} \frac{1}{1-x_i/u}.$$  

Then

$$H(u)E(-u) = 1,$$  

and

$$H(u) = \exp \left( \sum_{n \geq 1} \frac{p_n}{n} \frac{1}{u^n} \right), \quad E(u) = \exp \left( - \sum_{n \geq 1} \frac{(-1)^n p_n}{n} \frac{1}{u^n} \right).$$  

Heisenberg algebra is a complex Lie algebra generated by elements $\{\alpha_m\}_{m \in \mathbb{Z}}$ and central element $1$ with commutation relations

$$[\alpha_k, \alpha_n] = k \delta_{k,-n} \cdot 1.$$  

Combining generators into formal distribution $\alpha(u) = \sum_k \alpha_k u^{-k-1}$, we can rewrite this relation as

$$[\alpha(u), \alpha(v)] = \partial_v \delta(u, v),$$  

where $\delta(u, v) = \sum_{k \in \mathbb{Z}} u^k v^{-k-1}$ is formal delta distribution. There is a natural action of Heisenberg algebra on the graded components $\alpha_m : \mathcal{B}^{(m)} \to \mathcal{B}^{(m)}$ defined by

$$\begin{cases} 
\alpha_{-n} = p_n, & n > 0, \\
\alpha_n = n \partial_{\alpha_m}, & n > 0, \\
\alpha_0 = m.
\end{cases}$$  

The ring of symmetric functions $\Lambda$ possesses a natural scalar product, where classical Schur functions $s_\lambda$ constitute an orthonormal basis: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$. Then for the operator of multiplication by a symmetric function $f$ one can define the adjoint operator $f^\perp$ acting on the ring of symmetric functions by the standard rule: $\langle f^\perp g, w \rangle = \langle g, f w \rangle$, where $g, f, w \in \Lambda$ (3, 1.5.). We set

$$E^\perp(u) = \sum_{k \geq 0} e_k^\perp u^k, \quad H^\perp(u) = \sum_{k \geq 0} h_k^\perp u^k.$$
One can prove ([3] I.5) that

\[ E^\perp(u) = \exp \left( -\sum_{k \geq 1} (-1)^k \frac{\partial}{\partial p_k} u^k \right), \quad H^\perp(u) = \exp \left( \sum_{k \geq 1} \frac{\partial}{\partial p_k} u^k \right). \] (2.7)

The following commutation relations (cf. Exercise 29 I.5 in [3]) serve as the foundation of most of calculations in this note.

**Proposition 2.1.**

\[ \left( 1 - \frac{u}{v} \right) E^\perp(u) E(v) = E(v) E^\perp(u), \]
\[ \left( 1 - \frac{u}{v} \right) H^\perp(u) H(v) = H(v) H^\perp(u), \]
\[ H^\perp(u) E(v) = \left( 1 + \frac{u}{v} \right) E(v) H^\perp(u), \]
\[ E^\perp(u) H(v) = \left( 1 + \frac{u}{v} \right) H(v) E^\perp(u). \]

**Remark 2.1.** Statements of Proposition 2.1 should be understood as equalities of series expansions in powers of \( u^k v^{-m} \) for \( k, m \geq 0 \).

### 3. Fermions and Schur symmetric functions

Following [1], [2], [4] define the action of algebra of charged free fermions on the boson Fock space. Let \( R(u) : \mathcal{B} \to \mathcal{B} \) act on elements \( z^m f, f \in \Lambda \) by the rule

\[ R(u)(z^m f) = \left( \frac{z}{u} \right)^{m+1} f. \]

(see e.g. [12] Lecture 5). Then \( R^{-1}(u) \) acts as

\[ R^{-1}(u)(z^m f) = (zu)^{m-1} f. \]

One should think of \( R^\pm(u) \) as operators that transport the action of other operators along the grading of the boson Fock space: \( R^\pm(u) : \mathcal{B}^{(m)} \to \mathcal{B}^{(m \pm 1)} \). More precisely, we set

\[ \Phi^+(u) = R(u) H(u) E^\perp(-u), \quad \Phi^-(u) = R^{-1}(u) E(-u) H^\perp(u). \] (3.1) (3.2)

Observe that

\[ \Phi^+(u)|_{\mathcal{B}^{(m)}} = zu^{-m-1} H(u) E^\perp(-u), \]
\[ \Phi^-(u)|_{\mathcal{B}^{(m)}} = z^{-1} u^{m-1} E(-u) H^\perp(u). \]

**Proposition 3.1.** Vertex operators \( \Phi^\pm(u) \) satisfy the relations of the algebra of fermions:

\[ \Phi^\pm(u) \Phi^\pm(v) + \Phi^\pm(v) \Phi^\pm(u) = 0, \]
\[ \Phi^+(u) \Phi^-(v) + \Phi^-(v) \Phi^+(u) = \delta(u, v). \] (3.3) (3.4)

Here \( \delta(u, v) = \sum_{k \in \mathbb{Z}} \frac{u^k v^{-k}}{k!} \) is formal delta distribution.

**Proof.** We use Proposition 2.1 to prove this classical result, thus illustrating the simplicity of this approach. Relations between other vertex operators further in this text are proved along the same lines.
From Proposition 2.1 we compare the products of given operators with their corresponding “normal form”, where all differentiations are applied before multiplication operators. Namely, observe that for any “normal form”, where all differentiations are applied before multiplication operators. Namely, observe that for any \( f \in \Lambda \),

\[
\Phi^+(u)\Phi^+(v)(z^m f) = z^{m+2}u^{m-2}v^{-m-1}H(u)E^\perp(-u)H(v)E^\perp(-v)(f)
\]

\[
= z^{m+2}u^{m-2}v^{-m-1}
\left(1 - \frac{u}{v}\right)H(u)H(v)E^\perp(-u)E^\perp(-v)(f)
\]

\[
= z^{m+2}(uv)^{-m-2}(v-u)H(u)H(v)E^\perp(-u)E^\perp(-v)(f).
\]

\[
\Phi^-(u)\Phi^-(v)(z^m f) = z^{m}u^{m-2}v^{-m-1}E(-u)H^\perp(u)E(-v)H^\perp(v)(f)
\]

\[
= z^{m}u^{m-2}v^{-m-1}
\left(1 - \frac{u}{v}\right)E(-u)E(-v)H^\perp(u)H^\perp(v)(f)
\]

\[
= z^{m}(uv)^{-m-2}(v-u)E(-u)E(-v)H^\perp(u)H^\perp(v)(f).
\]  

Changing the roles of \( u \) and \( v \) in these calculations one gets the first two relations.

For the last equality note that

\[
\left(1 - \frac{u}{v}\right)\Phi^+(u)\Phi^-(v)(z^m f) = z^{m}u^{-m}v^{m-1}H(u)E(-v)E^\perp(-u)H^\perp(v)(f),
\]

\[
\left(1 - \frac{v}{u}\right)\Phi^-(v)\Phi^+(u)(z^m f) = z^{m}u^{-m-1}v^m H(u)E(-v)E^\perp(-u)H^\perp(v)(f).
\]

Denote as \( i_{u/v}(F) \) the expansion of rational function \( F \) as a series in powers of \( u/v \). Then one can write

\[
(\Phi^+(u)\Phi^-(v) + \Phi^-(v)\Phi^+(u))(z^m f)
\]

\[
= z^{m}v^{m}
\left(\frac{1}{u}i_{u/v}\left(1 - \frac{u}{v}\right)^{-1} + \frac{1}{v}i_{v/u}\left(1 - \frac{v}{u}\right)^{-1}\right)H(u)E(-v)E^\perp(-u)H^\perp(v)(f)
\]

\[
= z^{m}v^{m}
\left(\sum_{k \geq 0}u^{-k} + \sum_{k \geq 0}\frac{v^k}{u^{k+1}}\right)H(u)E(-v)E^\perp(-u)H^\perp(v)(f)
\]

\[
= z^{m}v^{m}\delta(u,v)H(u)E(-v)E^\perp(-u)H^\perp(v)(f) = \delta(u,v) \cdot z^m f.
\]

We used (2.3) along with the property of formal delta distribution \( \delta(u,v)A(v) = \delta(u,v)A(u) \) (see e.g. [12]).

Let \( 1 \in \mathcal{B}^{(0)} \) be the constant function (often called vacuum vector and denoted as \( |0\rangle \)).

**Proposition 3.2.**

\[
\Phi^+(u_1)\ldots\Phi^+(u_l)(1) = z^l u_1^{-l} \ldots u_l^{-l} Q(u_1,\ldots,u_l),
\]

where

\[
Q(u_1,\ldots,u_l) = \prod_{1 \leq i < j \leq l} \left(1 - \frac{u_i}{u_j}\right) \prod_{i=1}^l H(u_i).
\]

**Proof.** Using Proposition 2.1 and that \( E^\perp(-u)(1) = 1 \), rewrite the product in “normal form”:

\[
\Phi^+(u_1)\ldots\Phi^+(u_l)(1) = z^l u_1^{-l} \ldots u_l^{-l} \prod_{1 \leq i < j \leq l} \left(1 - \frac{u_i}{u_j}\right) \prod_{i=1}^l H(u_i)E^\perp(-u_i)(1)
\]

\[
= z^l u_1^{-l} \ldots u_l^{-l} Q(u_1,\ldots,u_l).
\]
It is known [3, 10] that \( Q(u_1, \ldots, u_l) \) is the generating function for Schur symmetric functions. More precisely, consider the series expansion of the rational function

\[
Q(u_1, \ldots, u_l) = \sum_{(\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^l} Q_{\lambda_1} u_1^{-\lambda_1} \cdots u_l^{-\lambda_l}
\]

in the region \(|u_1| < \cdots < |u_l|\). Then for any partition \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_l \geq 0) \) the coefficient of \( u_1^{-\lambda_1} \cdots u_l^{-\lambda_l} \) is exactly Schur symmetric function: \( Q_{\lambda} = s_{\lambda} \). Thus, Proposition 3.2 describes vertex operator presentation of Schur symmetric functions.

4. Twisted fermions and Hall-Littlewood symmetric functions.

Let \( \lambda \) be a partition of length at most \( n \). Hall-Littlewood polynomials are defined by

\[
P_{\lambda}(x_1, \ldots, x_n; t) = \left( \prod_{i \geq 0} \prod_{j=1}^{m(i)} \frac{1-t}{1-v^i} \right) \sum_{\sigma \in S_n} \sigma \left( \prod_{i<j}^{x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j}^{x_i - x_j} \right),
\]

where \( m(i) \) is the number of parts of the partition \( \lambda \) that are equal to \( i \), and \( S_n \) is the symmetric group of \( n \) letters (Chapter III.2, [3]).

In this section we construct vertex operators presentation of Hall-Littlewood polynomials from vertex operators of Schur symmetric functions. This approach and application of Proposition 2.1 simplifies the proofs of [8, 9] and gives new insight into the original results of these papers.

Consider the deformed boson Fock space \( \mathcal{B}[t] = \mathbb{C}[t, z, z^{-1}, p_1, p_2, \ldots] = \oplus \mathcal{B}^{(m)}[t] \). We extend the action of the operators in Section 2 to \( z^m \mathcal{A}[t] \) by \( t \)-linearity. Define

\[
\Psi^+ (u) = E(-u/t) \Phi^+ (u) = R(u) E(-u/t) H(u) E^\perp (-u), \tag{4.1}
\]

\[
\Psi^- (u) = H(u/t) \Phi^- (u) = R(u) H(u/t) E(-u) H^\perp (u). \tag{4.2}
\]

**Proposition 4.1.** Coefficients of formal distributions \( \Psi^\pm (u) \) create and algebra of twisted fermions. More precisely, these formal distributions satisfy relations

\[
\left( 1 - \frac{ut}{v} \right) \Psi^+ (u) \Psi^+ (v) + \left( 1 - \frac{vt}{u} \right) \Psi^+ (v) \Psi^+ (u) = 0,
\]

\[
\left( 1 - \frac{vt}{u} \right) \Psi^+ (u) \Psi^- (v) + \left( 1 - \frac{ut}{v} \right) \Psi^- (v) \Psi^+ (u) = \delta(u, v)(1-t)^2.
\]

**Proof.** The proof follows the same lines as the proof of Proposition 3.2. \( \Box \)

It turns out that the operators \( \Psi^\pm (u) \) give vertex operators realization [8] of Hall-Littlewood polynomials.

**Proposition 4.2.**

\[
\Psi^+ (u_1) \cdots \Psi^+ (u_l) (1) = z^l u_1^{-l} \cdots u_l^{-1} F(u_1, \ldots, u_l; t),
\]

where

\[
F(u_1, \ldots, u_l; t) = \prod_{1 \leq i < j \leq l} \frac{u_j - u_i}{u_j - u_i t} \prod_{i=1}^l H(u_i) E(-u_i/t),
\]

and \( \prod_{1 \leq i < j \leq l} \frac{u_j - u_i}{u_j - u_i t} \) understood as the series expansion of this rational function in the region \( |u_1| < \cdots < |u_l| \).
Proof. This statement immediately follows from rewriting the product of vertex operators in "normal" form

\[ \Psi^+(u_1) \cdots \Psi^+(u_l) = z^l \prod_{1 \leq i < j \leq l} \left( 1 - \frac{u_i}{u_j} \right) \left( 1 - \frac{u_i}{u_j} \right)^{-1} \prod_{i=1}^l u_i^{-1} H(u_i) E(-u_i/t) \prod_{i=1}^l E(-u_i). \]

\[ \square \]

It is known [3], [8] that \( F(u_1, \ldots, u_l; t) \) is the generating function for Hall-Littlewood symmetric functions. Namely, in the expansion in the region \(|u_1| < \cdots < |u_l|\) of the rational function

\[ F(u_1, \ldots, u_l; t) = \sum_{\lambda_1, \ldots, \lambda_l \in \mathbb{Z}} F_{\lambda_1} \cdots F_{\lambda_l}, \]

for any partition \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_l \geq 0) \) the coefficient of \( u_1^{-\lambda_1} \cdots u_l^{-\lambda_l} \) is exactly Hall-Littlewood symmetric function: \( F_{\lambda} = P_{\lambda}(x_1, x_2, \ldots; t) \). Thus, Proposition 4.2 gives vertex operator presentation of Hall-Littlewood symmetric functions.

We also have the following simple observation.

**Corollary 4.1.** Generating functions \( F(u_1, \ldots, u_l; t) \) for Hall-Littlewood polynomials and \( Q(u_1, \ldots, u_l) \) for Schur symmetric functions are related by the formula

\[ F(u_1, \ldots, u_l; t) = \prod_{1 \leq i < j \leq l} \left( 1 - \frac{tu_i}{u_j} \right)^{-1} \prod_{i=1}^l E(-u_i/t) Q(u_1, \ldots, u_l). \]

5. **Boson-fermion correspondence for Hall-Littlewood polynomials revisited**

In its most general form, the boson-fermion correspondence establishes relations between actions of different algebraic structures on a certain space. Let us review the classical version of this phenomena related to classical charged free fermions, since we will literally transport the construction to the twisted case. We will also compare our version of the boson-fermion correspondence with the one constructed in [9].

In the classical case, one may start with Heisenberg algebra action (5.6) on the boson space \( \mathcal{B} \) through operators \( p_n, \partial/\partial p_n \). Then one can express the action of fermion algebra through Heisenberg operators using (2.4), (2.7). More precisely, the vertex operators \( \Phi^\pm(u) \) defined by (3.1), (3.2) can be written as

\[ \Phi^+(u) = R(u) \exp \left( \sum_{n \geq 1} \frac{p_n}{n} \frac{1}{u^n} \right) \exp \left( -\sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n \right), \]

\[ \Phi^-(u) = R^{-1}(u) \exp \left( -\sum_{n \geq 1} \frac{p_n}{n} \frac{1}{u^n} \right) \exp \left( \sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n \right). \]
One can easily see that the twisted version of these formulas provides expression of $\Psi^\pm(u)$ through classical Heisenberg operators in a similar manner:

$$\Psi^+(u) = R(u) \exp \left( \sum_{n \geq 1} \frac{1 - t^n}{n} p_n \frac{1}{u^n} \right) \exp \left( - \sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n \right),$$

$$\Psi^-(u) = R^{-1}(u) \exp \left( - \sum_{n \geq 1} \frac{1 - t^n}{n} p_n \frac{1}{u^n} \right) \exp \left( \sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n \right).$$

(5.3) (5.4)

Remark 5.1. Formulas (5.3), (5.4) were used as definitions of $\Psi^\pm(u)$ in [8], [9].

In the other direction of classical boson-fermion correspondence, one can start with the algebra of charged free fermions defined through formal distributions $\Phi^\pm(u)$ satisfying relations (3.3), (3.4). Define $\{\Phi^\pm_{k+1/2}\}_{k \in \mathbb{Z}}$ as coefficients of expansions

$$\Phi^\pm(u) = \sum_{k \in \mathbb{Z}} \Phi^\pm_{k+1/2} u^{\pm k}.$$  

(5.5)

Introduce the parts $\Phi^+(u) \pm$ of distribution $\Phi^+(u)$

$$\Phi^+(u) + = \sum_{k \geq 1} \Phi^+_{k+1/2} u^k, \quad \Phi^+(u) - = \sum_{k \leq 0} \Phi^+_{k+1/2} u^k.$$

It is known (see e.g. 16.3 in [12]) that coefficients of formal distribution $\alpha(u) = \sum \alpha_k u^{-k-1}$ defined by

$$\alpha(u) =: \Phi^+(u) \cdot \Phi^-(u) := \Phi^+(u) + \Phi^-(u) - \Phi^+(u) \Phi^-(u) -$$

satisfy relations of generators of Heisenberg algebra (2.5). Moreover, for any $\beta \in \mathbb{C}$, the formal distribution

$$L^{(\beta)}(u) = \sum_{k \in \mathbb{Z}} L_k u^{-k-2}$$

defined by the formula

$$L^{(\beta)}(u) = \beta : \partial \Phi^+(u) \Phi^-(u) : + (1 - \beta) : \Phi^+(u) \partial \Phi^-(u) :$$

is a Virasoro formal distribution with central charge $c_\beta = -12 \beta^2 + 12 \beta - 2$:

$$[L^{(\beta)}(u), L^{(\beta)}(v)] = \partial_v L(v) \delta(u, v) + 2 L(v) \partial_v \delta(u, v) + \frac{c_\beta}{12} \partial^3_v \delta(u, v).$$

(see e.g. 15.4 in [12]).

Given relations (4.11), (4.12) between twisted fermions $\Psi^\pm(u)$ and classical fermions $\Phi^\pm(u)$ we can use these standard constructions to relate $\Psi^\pm(u)$ with formal distributions $\alpha(u)$ of classical (untwisted) Heisenberg algebra and $L^{(\beta)}(u)$ of Virasoro algebra. Substitution

$$\Phi^+(u) = H(u/t) \Psi^+(u), \quad \Phi^-(u) = E(-u/t) \Psi^-(u)$$

provides expression of Heisenberg algebra formal distribution through $\Psi^\pm(u)$:

$$\alpha(u) =: H(u/t) \Psi^+(u) \cdot E(-u/t) \Psi^-(u) :,$$

and the property (1.2 (2.10) in [3])

$$P(u) = \partial H(u)/H(u), \quad P(-u) = \partial E(u)/E(u),$$

allows one to write, for example, the following expression of $L^{(\beta)}(u)$:

$$L^{(\beta)}(u) = \beta : H(u/t) \left( t^{-1} P(-u/t) \Psi^+(u) + \partial \Psi^+(u) \right) \cdot E(-u/t) \Psi^-(u) : + (1 - \beta) : H(u/t) \Psi^+(u) \cdot E(-u/t) \left( -t^{-1} P(-u/t) \Psi^-(u) + \partial \Psi^-(u) \right) :.$$
Thus, we described the transitions between the actions on $B[t]$:

(I) Heisenberg algebra $\rightarrow$ twisted fermions $\rightarrow$ Heisenberg algebra.

In [8],[9] the boson-fermion correspondence is based on the transitions

(II) twisted Heisenberg algebra $\rightarrow$ twisted fermions $\rightarrow$ twisted Heisenberg algebra.

The twisted Heisenberg algebra is defined as an algebra with generators $\{h_k\}$ and central element $c$ with relations

$$[h_n, h_m] = \frac{m\delta_{m,-n}}{1 - t^{|m|}} \cdot c.$$ 

One can obtain the twisted Heisenberg algebra from the classical one identifying

$$\begin{cases} h_{-n} = \alpha_{-n}, & n \geq 0, \\ h_n = \frac{1}{1 - \alpha_n} & n > 0. \end{cases} \quad (5.6)$$

For the second arrow in (II) a deformed version of normal ordered product in [9] was introduced.

We believe that the advantage of the version of the boson-fermion correspondence described here is that the regular Heisenberg algebra and the Virasoro algebra are naturally present in the picture, and we do not need to adjust the definition of normal ordered product to obtain the bosonisation of twisted fermions.

6. Basis dual to Schur symmetric functions as tau-functions of KP hierarchy

The ring $\Lambda[t]$ possesses a scalar product $\langle \cdot, \cdot \rangle_t$, which is a deformation of the previously defined natural scalar product $\langle \cdot, \cdot \rangle$ on $\Lambda$ (III.4 in [3]). In this scalar product, the basis dual to classical Schur functions $s_\lambda = s_\lambda(x_1, x_2, \ldots)$ consists of elements $S_\lambda = S_\lambda(x_1, x_2, \ldots; t)$ with the defining duality property

$$\langle S_\lambda, s_\mu \rangle_t = \delta_{\lambda,\mu}.$$ 

**Proposition 6.1.** Let

$$S(u_1, \ldots, u_l) = \prod_{i<j} \left(1 - \frac{u_i}{u_j}\right) \prod_{i=1}^l H(u_i)E(-u_i/t).$$

Then for any partition $\lambda$, the coefficient of $u_1^{-\lambda_1} \ldots u_l^{-\lambda_l}$ is $S_\lambda$.

**Proof.** By III.4 formula (4.3) and III.2 formula (2.10) in [3], $S_\lambda$ can be expressed through Hall-Littlewood polynomials $P_{(k)}$ by an analogue of Jacobi-Trudi formula:

$$S_\lambda = \det[(1 - t)P_{(\lambda_i-i+j)}], \quad (6.1)$$

and $P_{(k)} = P_{(k)}(x_1, x_2, \ldots; t)$ are coefficients of the expansion

$$S(u; t) = H(u)E(-u/t) = (1 - t) \sum_{k=0}^{\infty} P_{(k)} \frac{1}{u^k}.$$
Then

\[ S(u_1, \ldots, u_l) = \prod_{i<j} \left(1 - \frac{u_i}{u_j}\right) \prod_{i=1}^l S(u_i; t) \]

\[ = \det[u_i^{i+j}] \prod_i S(u_i; t) = \det[u_i^{i+j}S(u_i; t)] \]

\[ = (1 - t)^l \sum_{\sigma \in S_l} \text{sgn}(\sigma) \sum_{a_1, \ldots, a_l} P_{(a_1)} u_1^{-a_1 - 1 + \sigma(1)} \ldots P_{(a_l)} u_l^{-a_l - l + \sigma(l)} \]

\[ = (1 - t)^l \sum_{\lambda_1, \ldots, \lambda_l} \sum_{\sigma \in S_l} \text{sgn}(\sigma) P_{(\lambda_1 - l + \sigma(1))} \ldots P_{(\lambda_l - l + \sigma(l))} u_1^{-\lambda_1} \ldots u_l^{-\lambda_l} \]

\[ = \sum_{\lambda_1, \ldots, \lambda_l} \det[(1 - t)P_{(\lambda_1 - i + j)}] u_1^{-\lambda_1} \ldots u_l^{-\lambda_l}. \]

\[ \square \]

Corollary 6.1. Generating functions \( S(u_1, \ldots, u_l) \) for \( S_\lambda \) and \( Q(u_1, \ldots, u_l) \) for \( s_\lambda \) are related by the formula

\[ S(u_1, \ldots, u_l) = E(-u_l/t) \ldots E(-u_1/t)Q(u_1, \ldots, u_l). \]

The vertex operator presentation of \( S(u_1, \ldots, u_l) \) can be written as

\[ z_i^{-l} u_1^{-l} \ldots u_l^{-l} S(u_1, \ldots, u_l) = E(-u_l/t) \ldots E(-u_1/t)\Phi^+(u_1) \ldots \Phi^+(u_l) (1). \]

In [11] the relation of \( S_\lambda \) to \( \tau \)-functions of KP hierarchy is discussed. From the duality of the bases \( s_\lambda \) and \( S_\lambda \) and pl"ucker coordinates-type property of \( S_\lambda \) it is proved that the expression \( \sum_\lambda s_\lambda S_\lambda \) is a solution of KP hierarchy.

Here we prove that the symmetric functions \( S_\lambda \)'s themselves are solutions of KP hierarchy and give explicit formula for the corresponding classical charged free fermion algebra action.

Reviewing the bilinear form of the KP hierarchy, consider

\[ \Omega = \text{Res}_{u=0} \left( \frac{1}{u} \Phi^+(u) \otimes \Phi^-(u) \right) = \sum_{k \in \mathbb{Z}} \Phi_k^+ \otimes \Phi_k^-. \]

Then the system of differential equation of KP hierarchy is equivalent to the bilinear identity on the functions \( \tau \in \mathcal{B}^{(0)} = \Lambda = \mathbb{C}[p_1, p_2, \ldots] \)

\[ \Omega(\tau \otimes \tau) = 0. \]

It is known (4, 5, 6) that Schur symmetric functions \( s_\lambda \) are solutions of the KP hierarchy.

Recall that \( \Phi^\pm(u) \) are represented in terms of operators \( p_i \)'s and \( \partial/\partial p_i \)'s by the formulas \[ \text{[1.1]} \]

\[ \text{[5.2]} \]

while Schur symmetric functions are expressed through complete symmetric functions by Jacobi-Trudi formula \( s_\lambda = \det[h_{\lambda_i - i + j}], \) and complete symmetric functions are defined as functions of power sums \( h_k = h_k(p_1, p_2, \ldots) \) through the relation

\[ \sum_{k=0}^\infty h_k \frac{u^k}{k!} = H(u) = \exp \left( \sum_{n \geq 1} \frac{p_n}{n} \frac{1}{u^n} \right). \]

At the same time, from [6, 11] the elements of the dual basis are given by \( S_\lambda = \det[q_{\lambda_i - i + j}], \) where symmetric functions \( q_k = (1 - t)P_k \) are defined as functions \( q_k = q_k(p_1, p_2, \ldots) \) of power sums
through the relation
\[
\sum_{k=0}^{\infty} \frac{q_k}{u_k} = H(u)E(-u/t) = \exp \left( \sum_{n \geq 1} \left( 1 - t^n \right) \frac{p_n}{n} \frac{1}{u^n} \right).
\]
Thus, \( S_\lambda \), as a function of \( (p_1, p_2, \ldots) \), can be obtained from \( s_\lambda \) by the substitution of variables \( p_n \rightarrow (1 - t^n)p_n \). Moreover, it also shows that \( S_\lambda \) satisfies the bilinear identity
\[
\Omega_t(S_\lambda \otimes S_\lambda) = 0,
\]
where \( \Omega_t = \text{Res}_{u=0} \left( \frac{1}{u} \Phi_t^+(u) \otimes \Phi_t^-(u) \right) \),
\[
\Phi_t^\pm(u) = R(u)^\pm \exp \left( \mp \sum_{n \geq 1} \frac{1}{n} \frac{1}{(1 - t^n)} \frac{\partial}{\partial p_n} u^n \right).
\]
Using the geometric series expansion \( 1/(1 - t^n) = \sum_i (t^i)^n \) in the region \( |t| < 1 \) we can write the second exponential factor as
\[
\exp \left( \pm \sum_{n \geq 1} \frac{1}{(1 - t^n)} \frac{\partial}{\partial p_n} u^n \right) = \exp \left( \pm \sum_{i=0}^{\infty} \sum_{n \geq 1} \frac{\partial}{\partial p_n} (t^i u)^n \right) = \prod_{i=0}^{\infty} \exp \left( \pm \sum_{n \geq 1} \frac{\partial}{\partial p_n} (t^i u)^n \right).
\]
Comparing these expressions with \( (2.4), (2.7) \) we get
\[
\Phi_t^+(u) = R(u)H(u)E(-u/t) \prod_{i=0}^{\infty} E_+(-u/t^i),
\]
\[
\Phi_t^-(u) = R(u)^{-1}E(-u)H(u/t) \prod_{i=0}^{\infty} H_+(u/t^i).
\]
We summarize the statements in the following proposition.

**Proposition 6.2.** Let \( \Phi_t^\pm(u) \) be defined by \( (6.4), (6.5) \) with \( |t| < 1 \).

(a) Symmetric functions \( S_\lambda \) satisfy bilinear identity \( (6.2) \) with \( \Omega_t \) defined by \( (6.3) \).

(b) The operators \( \Phi_t^\pm(u) \) satisfy exactly the same relations as classical fermions \( \Phi_t^\pm(u) \) in Proposition 3.1. Thus, operators \( \Phi_t^\pm(u) \) provide the action of classical algebra of charged free fermions on \( B[t] \).

(c) Consequently, bilinear identity \( (6.4) \) is equivalent to the bilinear identity of KP hierarchy, and symmetric functions \( S_\lambda \) are \( \tau \)-functions of KP hierarchy.

(d) Generating function \( S(u_1, \ldots, u_l) \) of \( S_\lambda \)'s has one more vertex operator presentation:
\[
\Phi_t^+(u_1) \ldots \Phi_t^+(u_l)(1) = z^l u_1^{-l} \ldots u_l^{-1} S(u_1, \ldots, u_l).
\]
Remark 6.1. Parts (b) and (d) follow from the fact that $\Phi_t^\pm(u)$ are obtained from classical charged free fermions $\Phi^\pm(u)$ just by a substitution of variables, but one can also prove it directly along the same lines as the proof of Proposition 3.1 using the definition (6.4), (6.5) and commutation relations of Proposition 2.1.

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