Topologies Induced by Relations with Applications

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Abstract: Some methods for inducing topological structures by relations were initiated and their importance in applications were indicated. Topologies generated by equivalence relations were all quasi-discrete spaces. We induced the topologies generated using similarity relations and pre-order relations. Also, the topologies generated using general binary relations on the universe of discourse were initiated. Finally, rheumatic fever data reduction using topologies induced by relations were studied.

Key words: Topological spaces, binary relations, information system, rough sets, data reduction

INTRODUCTION

Topology is an important and interesting area of mathematics, the study of which will not only introduce you to new concepts and theorems but also put into context old ones like continuous functions. It is so fundamental that its influence is evident in almost every other branch of mathematics. This makes the study of topology relevant to all who aspire to be mathematicians whether their first love is algebra, analysis, category theory, chaos, continuum mechanics, dynamics, geometry, industrial mathematics, mathematical biology, mathematical economics, mathematical finance, mathematical modeling, mathematical physics, mathematics of communication, number theory, numerical mathematics, operation research or statistics. Topological notions like compactness, connectedness and denseness are as basic to mathematicians of today as sets and functions were to those of last century\cite{3,8,9}.

For a long time, many individuals believed that abstract topological structures have limited application in the generalization of real line and complex plane or some connections to Algebra and other branches of mathematics. And it seems that there is a big gap between these structures and real life applications. We noticed that in some situations, the concept of relation is used to get topologies that are used in important applications such as computing topologies\cite{15}, recombination spaces\cite{2,7,17} and information granulation\cite{21} which are used in biological sciences and some other fields of applications.

The aim of rough set theory is to give a description of the set of objects entails as well relationships and functional or near to functional dependencies among various similarity relations generated by various sets of the set of objects.

Rough sets were first introduced by\cite{10,11} and are based on approximation spaces. An approximation space is a pair A = (Ob, R). Here, R is an equivalence relation, also called indiscernibility relation, imposing a granularity on the universe Ob such that R \subseteq Ob\times Ob. Furthermore, we assume Ob to be finite. For x\in Ob, let [x]_R be the equivalence class containing x, i.e.,

\[ [x]_R = \{ y: y R x \} \]

Given an arbitrary set X \subseteq Ob, we wish to describe X in terms of elements or granules of Ob/R. Pawlak proposed the use of lower and upper approximations of a set X, denoted \( \underline{R}(X) \) and \( \overline{R}(X) \), respectively. Lower and upper approximations are defined as:

\[\underline{R}(X) = \{ x\in Ob:[x]_R \subseteq X \} \]
\[\overline{R}(X) = \{ x\in Ob: [x]_R \cap X \neq \emptyset \} \]

The semantics of the approximations of sets may be defined as follows:

- Elements of the universe that belong to \( \underline{R}(X) \) are those elements that surely belong to the set X.
- Elements that belong to \( \overline{R}(X) \) possibly belong to the set X.
- Elements that belong to Ob/\( \overline{R}(X) \) are elements of the universe that surely do not belong to the set X. Hence, the uncertainty lies in \( \overline{R}(X)/\underline{R}(X) \) which is also called area of uncertainty. Elements of the area of uncertainty may, or may not, belong to X.
The approximation operators can also be considered using membership functions. It is possible to define a rough membership function as presented in [12].

**MATERIALS AND METHODS**

**Topologies induced by relations:** Let \( A = (\text{Ob}, R) \) be an approximation space. The equivalence classes \( \text{Ob} \cap R \) of the relation \( R \) will be called elementary sets (atoms) in \( A \). Every finite union of elementary sets in \( A \) will be called a composed set in \( A \). The family of all composed sets in \( A \) will be denoted by \( \text{com}(A) \).

The family \( \text{com}(A) \) in the approximation space \( A = (\text{Ob}, R) \) is a topology on the set \( \text{Ob} \). Since the approximation space \( A = (\text{Ob}, R) \) is a topology on the set \( \text{Ob} \), the family \( \text{com}(A) \) is both the set of open sets in \( \text{com}(A) \) and \( \text{com}(A) \) is the family of all open sets in \( \text{com}(A) \).

**Example:** Consider the partitions \( B_1 = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}\} \) and \( B_2 = \{\{x_1, x_2\}, \{x_3, x_4\}\} \) of the set \( \text{Ob} = \{x_1, x_2, x_3, x_4\} \). Then \( B_1 \cap B_2 \leq \tau_1 \) and \( \tau_2 \leq \tau_1 \) where \( \tau_1 = \{\text{Ob}, \phi, \{x_3\}, \{x_2, x_3, x_4\}\} \). \( \tau_2 = \{\text{Ob}, \phi, \{x_1, x_3\}, \{x_3, x_4\}\} \) are the topologies generated by \( B_1 \) and \( B_2 \) respectively.

For any topological space \((\text{Ob}, \tau)\), we define the equivalence relation \( E(\tau) \) on the set \( \text{Ob} \) by \( (x, y) \in E(\tau) \) iff \( \text{cl}_1([x]) = \text{cl}_1([y]) \), \( \forall x, y \in \text{Ob} \). The set of all equivalence classes of \( E(\tau) \) is denoted by \( \text{Ob}/E(\tau) \).

**Proposition 3:** Let \( A = (\text{Ob}, R) \) be an approximation space and let \( \tau_R \) be the topology generated by the base \( B_R = \text{Ob}/R \). If \((\text{Ob}, \tau)\) is the quasi-discrete topological space has \( \text{Ob}/E(\tau) \) as a base. Then \( \tau_R = \tau \) iff for all \( x \in B_R \) there exists \( B \in \text{Ob}/E(\tau) \) such that \( x \in B \).

**Lemma 4 [15]:** For any topology \( \tau \) on a set \( \text{Ob} \) and for all \( x, y \in \text{Ob} \), if \( y \in \text{cl}_1([x]) \) and \( x \in \text{cl}_1([y]) \) then \( \text{cl}_1([x]) = \text{cl}_1([y]) \).

**Lemma 5 [15]:** If \( \tau \) is a quasi-discrete topology on a set \( \text{Ob} \), then \( y \in \text{cl}_1([x]) \) implies \( x \in \text{cl}_1([y]) \) for all \( x, y \in \text{Ob} \).

**Lemma 6 [15]:** If \( \tau \) is a quasi-discrete topology on a set \( \text{Ob} \), then the family \( \{\text{cl}_i([x]): x \in \text{Ob}\} \) is a partition of \( \text{Ob} \).

**Proposition 7:** Let \( \tau \) be the topology induced by the partition \( B_R = \text{Ob}/R \). Then \( B_R = \text{Ob}/E(\tau) \).

**Proof:** \( x \in B, B \in B_R \):
1. iff \( x \in \text{cl}_1(B) = \bigcup_{y \in B} \text{cl}_1([y]) \)
2. iff \( y_o \in B \) and \( x \in \text{cl}_1([y_o]) \) iff \( \text{cl}_1([x]) = \text{cl}_1([y_o]) \) (Lemma 2.2)
3. iff \( (x, y_o) \in E(\tau) \)
4. iff \( A \in \text{Ob}/E(\tau) \) such that \( x \in A \)
5. iff \( B_R = \text{Ob}/E(\tau) \)

For any \( n \) approximation spaces \( A_1 = (\text{Ob}, R_1), A_2 = (\text{Ob}, R_2), \ldots, A_n = (\text{Ob}, R_n) \) we define the partition \( \text{Ob}/E(\tau_{\text{ind}}) = \bigcap_{i=1}^{n} \text{Ob}/E(\tau_i) \).

**Theorem 8:** \( \tau \leq \tau_{\text{ind}}, i = 1, 2, \ldots, n \) where \( \tau_i \) and \( \tau_{\text{ind}} \) are the topologies generated by the partitions \( \text{Ob}/(\tau_i) \) and \( \text{Ob}/E(\tau_{\text{ind}}) \) respectively.

**Proof:** Since \( \text{Ob}/E(\tau_{\text{ind}}) \subseteq \text{Ob}/E(\tau_i) \) for all \( i = 1, 2, \ldots, n \) then \( \tau_i \leq \tau_{\text{ind}} \).
Example: Consider the topological space \((\text{Ob}, \tau)\) where \(\text{Ob} = \{x_1, x_2, x_3, x_4\}\) and \(b = \{\{x_1\}, \{x_2, x_3\}, \{x_4\}\}\) is the base of \(\tau\), then \(\tau\) is a quasi-discrete topology and:
\[
\text{cl}_\tau(\{x_1\}) = \{x_1\}, \text{cl}_\tau(\{x_2\}) = \{x_2, x_3\}, \text{cl}_\tau(\{x_4\}) = \{x_2, x_3\}, \text{cl}_\tau(\{x_1, x_2\}) = \{x_1, x_2, x_3\}, \text{cl}_\tau(\{x_2, x_3\}) = \{x_2, x_3, x_4\}.
\]
Then \(\text{Ob}/\text{E}(\tau) = \{\{x_1\}, \{x_2, x_3\}, \{x_4\}\} = \beta\).

Example: Consider the approximation spaces \(A_1 = (\text{Ob}, R_1)\), \(A_2 = (\text{Ob}, R_2)\) and \(A_3 = (\text{Ob}, R_3)\) where \(\text{Ob} = \{x_1, x_2, x_3, x_4\}\) and \(\text{Ob}/\text{E}(\tau_1) = \{\{x_1\}, \{x_2, x_3\}, \{x_4\}\}\) and \(\text{Ob}/\text{E}(\tau_2) = \{\{x_1, x_2\}, \{x_3, x_4\}\}\) and \(\text{Ob}/\text{E}(\tau_3) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}\) are the bases of \(\tau_1, \tau_2\) and \(\tau_3\), respectively, then \(\text{Ob}/\text{E}(\tau_{ind}) = (\text{Ob}/\text{E}(\tau_1)) \cap (\text{Ob}/\text{E}(\tau_2)) \cap (\text{Ob}/\text{E}(\tau_3))\) is the partition induced by \(\text{E}(\tau_{ind})\). Then \(\tau_{ind} \subset \tau_i\) for \(i = 1, 2, 3\).

Topologies generated using similarity relations: A similar relation \(R\) on \(\text{Ob}\) is any relation satisfies:

- For any \(x \in \text{Ob}\), \(xRx\) (reflexive)
- For any \(x, y \in \text{Ob}\), if \(xRx\) then \(yRx\) (Symmetric)

For \(x \in \text{Ob}\), we define the similar class containing \(x\) by \(R(x) = \{y \in \text{Ob}: xRy\}\).

The relation \(R\) on \(\text{Ob}\) defined by \(xRy\) iff \(d(x,y) < n\) where \((\text{Ob}, d)\) is a metric space with a metric function \(d\) defined as: \(d(x,y) = |x - y|\) and \(n = \text{card}(\text{Ob})\), is a similar relation.

**Proposition 9:** For any similar relation \(R\) defined on \(\text{Ob}\) we have:

- \(x \in R(x)\)
- \(y \in R(x)\) iff \(x \in R(y)\)
- \(xRy\) iff \(x \in R(y)\) and \(y \in R(x)\)

The class \(\beta = \{B(x): x \in X\}\) is called a symmetric covering of a set \(X\) if \(x \in B(y)\) iff \(y \in B(x)\). Then the class \(\beta = \{R(x): x \in \text{Ob}\}\) is a symmetric covering of the set of objects \(\text{Ob}\).

Let \(\beta\) be the symmetric covering of \(\text{Ob}\) by the similar relation \(R\). Then we define a relation \(R_\beta\) induced by \(\beta\) by \(xR_\beta y\) iff there exist \(B \in \beta\) and \(x, y \in B\).

**Proposition 10:** The relation \(R_\beta\) is a similar relation on the set of objects \(\text{Ob}\).

Since \(\beta\) is a covering of \(\text{Ob}\), then for any \(x \in \text{Ob}\) there exists \(B \in \beta\) such that \(x \in B\) hence \(x \in B\) then \(xR_\beta x\). Let \(xR_\beta y\) then there exists \(B \in \beta\) such that \(x, y \in B\) then \(y \in B\) hence \(yR_\beta x\).

**Proposition 10** For every \(x \in \text{Ob}\) we have:

\[R_\beta(x) = \bigcup_{B \in \beta(x)} B\,
\]

where \(\beta(x) = \{B \in \beta: x \in B\}\)

**Proof:**

\[y \in R_\beta(x) \iff \exists B \in \beta \text{ and } x, y \in B\]
\[\iff \exists B \in \beta \text{ and } x \in B \text{ and } y \in B\]
\[\iff \exists B \in \beta \text{ and } y \in B\]
\[\iff \bigcup_{B \in \beta(x)} B\]

Let \(\beta\) be the covering of \(\text{Ob}\). Then we define the class \(\beta^* = \{R_\beta(x): x \in \text{Ob}\}\).

**Proposition 11:** The class \(\beta^*\) is a symmetric covering of the set of objects \(\text{Ob}\) and \(R_\beta \subseteq R_\beta^*\).

**Proof:**

- \(xR_\beta(y) \iff \exists B \in \beta(y) \text{ and } x \in B \iff \exists B \in \beta^*(y) \text{ and } y \in B\)
- \(\text{Let } (x,y) \in R_\beta \implies \exists B \in \beta \text{ and } x, y \in B\)
- \(\implies B \in \beta(x) \text{ and } B \in \beta(y)\)
- \(\implies B \in \beta^*(x) \text{ and } B \in \beta^*(y)\)
- \(\implies x, y \in \beta^*\)
- \(\implies x, y \in R_\beta^*\)

Let \(A \subseteq \text{Ob}\) be any nonempty subset of the set of objects. Then \(A\) is called a similar pre-class of \(R\) if for any \(x, y \in A\) \(\implies (x, y) \in R\).

**Proposition 12:** Every similar class \(R(x)\) is a maximal similar pre-class.

For an element \(x \in \text{Ob}\) we define a class called the pre-similar class of \(x\) as follows:

\[L_\beta(x) = \{A \subseteq \text{Ob}: x \in A \text{ and } A \text{ is similar pre-class of } R\}\]

Let \(L_\beta = \{L_\beta(x): x \in \text{Ob}\}\) be the family of all pre-similar classes. Then we define a relation \(R^*\) on \(L_\beta\) by for any \(L_\beta(x), L_\beta(y) \in L_\beta\), \(L_\beta(x)R^*L_\beta(y)\) iff there exist \(A \in L_\beta(x)\) and \(B \in L_\beta(y)\) and \(A \cap B \neq \emptyset\).

**Proposition 13:**

- The relation \(R^*\) on \(L_\beta\) is a similar relation
- \(xR^*y\) iff \(L_\beta(x)R^*L_\beta(y)\) for any \(x, y \in \text{Ob}\).
Proof:

- Since for any \( L_R(x) \in L_R \) and \( A \subseteq L_R(x) \), \( A \cap \emptyset \neq \emptyset \) then \( L_R(x) \cap L_R(y) \) hence \( R^* \) is reflexive. Also if \( L_R(x) \cap L_R(y) \) then there exist \( A \subseteq L_R(x) \) and \( B \subseteq L_R(y) \) such that \( A \cap B \neq \emptyset \), hence \( L_R(x) \cap L_R(y) \). then \( R^* \) is symmetric.

- Firstly, we will prove that \( x \in R(y) \implies L_R(x) \cap L_R(y) \)

Let \( (x, y) \in R \Rightarrow \{x, y\} \) is a similar pre-class of \( R \).

\( \Rightarrow \) there exist a similar class \( R(x) \subseteq L_R(x) \) and \( R(x) \subseteq L_R(y) \) but \( R \) is symmetric then \( R(x) \subseteq L_R(y) \), then there exist \( A = R(x) \subseteq L_R(x) \) and \( B = R(x) \subseteq L_R(y) \) and \( A \cap B \neq \emptyset \), hence \( L_R(x) \cap L_R(y) \).

Conversely, let for some \( x, y \in Ob \), \( L_R(x) \cap L_R(y) \)

then there exist \( R(z) \subseteq L_R(x) \) and \( R(z) \subseteq L_R(y) \) a similar class of \( R \), hence \( x \in R(z) \) and \( y \in R(z) \) then \( x, y \in R(z) \) hence \( x \in R(y) \).

Let \( L_R(x) \) be the pre-similar class of \( x \in Ob \). Then we define a set \( L_R(x) = \bigcup_{A \in L_R(x)} A \) is called the R-link of \( x \), where \( A \subseteq L_R(x) \) and \( A \neq R(x) \).

- \( L_R(x) = Ob \) then it is called open \( R \)-link of \( x \) and if \( L_R(x) \subseteq Ob \) then it is called closed \( R \)-link of \( x \).

The class \( M = \{L_R(x) : x \in Ob\} \) of all \( R \)-links of \( x \in Ob \) is a subbase of a topology on \( Ob \) called the linked topology and denoted \( \tau_{L_R} \).

Proposition 14:

- For any \( x \in Ob \), \( L_R(x) \subseteq R(x) \)
- The class \( M \) is a symmetric covering of \( Ob \)

Proof:

- Let \( y \in L_R(x) \implies y \subseteq \bigcup_{A \in L_R(x)} A \) there exists \( A \subseteq L_R(x) \) and \( y \in A \) then \( y \in L_R(x) \implies L_R(x) \subseteq R(x) \)
- For any \( x \in Ob \), \( x \subseteq L_R(y) \) that \( M \) is covering of \( Ob \)

Now let \( x \in L_R(y) \implies x \subseteq \bigcup_{A \in L_R(x)} A : \)

- there exist \( A \subseteq R(x) \subseteq L_R(y) \) and \( x \in R(x) \)
- \( x, y \in R(x) \)
- \( y \in L_R(x) \)

then \( M \) is a symmetric covering of \( Ob \).

Proposition 15:

- The linked topology \( \tau_{L_R} \) is finer than the similar topology \( \tau_R \), where \( \tau_R \) is the topology generated by the subbase \( \{R(x) : x \in Ob\} \)
- \( x \in R(y) \implies \exists \) open set \( u \) \( \subseteq \tau_{L_R} \) and \( x, y \in u \)

Example: Let \( Ob = \{c_1, c_2, \ldots, c_7\} \) be the set of objects which is seven computers in a local network in a certain company. Let \( \tau \) be the irregular topology on the set of objects which induced by a general relation on \( Ob \) which makes the following graph. We define a similar relation \( R \) on the set of objects by: Two computers \( x \) and \( y \) are in relation by \( R \) iff the computer \( x \) has a copy of a certain program in the computer \( y \).

Then we can define the similar classes of \( R \) as follows:

- \( R(c_1) = \{c_1, c_2, c_4\}, R(c_2) = \{c_1, c_2, c_3, c_4, c_6, c_7\}, R(c_3) = \{c_2, c_3, c_4, c_6, c_7\}, R(c_4) = \{c_1, c_2, c_3, c_4, c_6\} \)
- \( R(c_5) = \{c_5, c_6, c_7\}, R(c_6) = \{c_5, c_6, c_7\} \)
- \( L_R(c_1) = \{\{c_1\}, \{c_1, c_2\}, \{c_1, c_3\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_1, c_6\}, \{c_1, c_7\}\} \)
- \( L_R(c_2) = \{\{c_2\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_2, c_5\}, \{c_2, c_6\}, \{c_2, c_7\}\} \)
- \( L_R(c_3) = \{\{c_3\}, \{c_3, c_4\}, \{c_3, c_5\}, \{c_3, c_6\}, \{c_3, c_7\}\} \)
- \( L_R(c_4) = \{\{c_4\}, \{c_4, c_5\}, \{c_4, c_6\}, \{c_4, c_7\}\} \)
- \( L_R(c_5) = \{\{c_5\}, \{c_5, c_6\}, \{c_5, c_7\}\} \)
- \( L_R(c_6) = \{\{c_6\}, \{c_6, c_7\}\} \)
- \( L_R(c_7) = \{\{c_7\}\} \)

Then the linked topology \( \tau_{L_R} \) is finer than the similar topology, such that \( L_R(c_i) \subseteq R(c_i) \) for all \( i = 1, 2, \ldots, 7 \).

For any subset \( A \) of the set of objects, we define two sets \( R(A) \) and \( \overline{R(A)} \), they are called the lower and upper similar classes of \( A \) by:

\[
R(A) = \bigcup \{R(x) : R(x) \subseteq A\}
\]

and

\[
\overline{R(A)} = \bigcup \{R(x) : R(x) \cap A \neq \emptyset\}
\]
Let \( \tau_\# \) be the topology induced by the subbase \( \{ R(A) : A \subseteq \text{Ob} \} \) of \( \text{Ob} \) this topology is called the lower similar topology. Also we define the topology \( \tau_\# \) which is called the upper similar topology and generated by the subbase \( \{ \overline{R}(A) : A \subseteq \text{Ob} \} \).

**Proposition 16:** Let \( \tau_\# \) and \( \tau_\# \) be the lower and upper similar topologies then:

- \( \tau_\# \subseteq \tau_\# \) if \( R \) is an equivalence relation
- \( \tau_\# \subseteq \tau_\# \) if \( R \) is a similar relation
- \( \tau_\# \) and \( \tau_\# \) are in general not comparable if \( R \) is a general relation

The following proposition presents another way to generate topologies from similarity relations.

**Proposition 17:** \( \tau^**_R = \{ A \subseteq \text{Ob} : \forall x \in A, R(x) \subseteq A \} \) is a topology on \( \text{Ob} \).

**Proof:**

- \( \text{Ob}, \varphi \in \tau^**_R \) is clearly
- If \( A_i, A_2, \ldots \in \tau^**_R \) and \( x \in \bigcup_i A_i \) for some \( i \), then \( R(x) \subseteq A_i \), then \( R(x) \subseteq \bigcup_i A_i \) hence \( \bigcup_i A_i \in \tau^**_R \)
- Let \( A_1, A_2 \in \tau^**_R \), then \( \forall x \in A_1 \cap A_2 \) we have \( R(x) \subseteq A_1 \) and \( R(x) \subseteq A_2 \) hence \( R(x) \subseteq A_1 \cap A_2 \) then \( A_1 \cap A_2 \in \tau^**_R \)

**Example:** Consider \( \text{Ob} = \{ a, b, c, d \} \) be the set of objects with a similar relation \( R \) its similar classes are:

- \( R(a) = \{ a, c \}, R(b) = \{ b, d \}, R(c) = \{ a, c, d \} \) and \( R(d) = \{ b, c, d \} \). Then: \( \tau_\# = \{ \text{Ob}, \varphi, \{ c \}, \{ d \}, \{ c, d \}, \{ a, c \}, \{ b, d \}, \{ a, c, d \}, \{ b, c, d \} \} \) and \( \tau^**_R = \{ \text{Ob}, \varphi, \{ d, c \}, \{ a, c, d \}, \{ b, c, d \} \} \) and \( \tau^**_R = \{ \text{Ob}, \varphi \} \) then \( \tau^**_R \subseteq \tau_\# \subseteq \tau^**_R \)
- The conjugate relation \( \overline{R} \) of \( R \) is defined by \( (x, y) \in \overline{R} \) iff \( (x, y) \notin R \) or \( x = y \)

**Proposition 18:**

- \( R \cap \overline{R} = I \) is the identity relation
- \( \overline{R} \) is a similar relation
- \( \overline{R} \) is a similar relation
- \( \overline{R} \) is a similar relation

**Proof:**

- \( (x, y) \in R \cap \overline{R} \) iff \( x = y \Rightarrow R \cap \overline{R} = I \)
- \( (x, y) \in \overline{R} \) such that \( x = x \) and if \( (x, y) \in \overline{R} \) then \( (y, x) \notin \overline{R} \) or \( x = y \) hence \( (y, x) \notin R \)
- \( \overline{R} \) is a similar relation
- \( \overline{R} \) is a similar relation
- \( \overline{R} \) is a similar relation
- \( \overline{R} \) is a similar relation

**Example:** Let \( \text{Ob} = \{ a, b, c, d \} \) be the set of objects with the similar relation \( R = \{ (a, a), (b, b), (c, c), (d, d), (c, b), (b, c), (b, a), (a, b) \} \). Then \( \overline{R} = \{ (a, a), (b, b), (c, c), (d, d), (c, a), (a, c), (d, a), (a, d) \} \).

**Example:** Let \( \text{Ob} = \{ 1, 2, 3, 4, 5, 6 \} \) and \( (x, y) \in R \) if and only if \( x \neq y \) and \( x, y \in \text{Ob} \).

- \( R = \{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (4, 4), (5, 5), (6, 6), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6) \} \). The relation \( R \) is a dominance relation.

In a finite space \( \langle \text{Ob}, \tau \rangle \), it is clear that \( \tau^* = \{ \text{Ob} - G : G \in \tau \} \) is also a topology.

A subset \( F \subseteq X \) is a closed set iff \( F = \bigcup_{y \in F} F_y \) such that \( F_y = \{ x : x \notin F \} \) (\( F_y \) is the smallest closed set about \( x \)). This is the dual of our representation of open sets.

If \( R \) is a dominance relation on a set \( \text{Ob} \), then its dual \( R_D \) is defined by the requirement \( y \in F_{x} \) if and only if \( x \in F_y \).

A point \( x \) in a subset \( U \) of \( \text{Ob} \) is insulated from \( \text{Ob} - U \) if and only if there is no point \( y \) in \( \text{Ob} - U \) such that \( y \) dominates \( x \).

**Lemma 19:** Let \( R \) be a dominance relation on a set \( \text{Ob} \), \( U \subseteq \text{Ob} \), \( P \subseteq U \) the following are equivalent:

- A point \( x \) in a subset \( U \) of \( \text{Ob} \) is insulated from \( \text{Ob} - U \)
- \( P \in U \), \( (y, P) \in R \), then \( y \in U \)
Table 1: Application for Openness algorithm

| P(Ob)         | Insulated points | Open sets |
|---------------|------------------|-----------|
| {1}           | √                | √         |
| {2}           |                   |           |
| {3}           |                   |           |
| {4}           |                   |           |
| {1, 2}        | √                |           |
| {1, 3}        |                   |           |
| {1, 4}        |                   |           |
| {2, 3}        |                   |           |
| {2, 4}        |                   |           |
| {3, 4}        |                   |           |
| {1, 2, 3}     | √                |           |
| {1, 2, 4}     |                   |           |
| {1, 3, 4}     |                   |           |
| Ob            |                   |           |
| ∅             |                   |           |

**Proof:** First, consider p ∈ U is insulated from Ob−U, (y, p) ∈ R. Let y ∈ U, then y ∈ Ob−U, So (y, p) ∈ R, but (y, p) ∈ R, a contradiction, then y ∈ U.

Second, consider p ∈ U, x ∈ Ob−U, suppose (x, p) ∈ R. Then x ∈ U contradicts that x ∈ U, then U = {y ∈ X | (y, x) ∈ R}.

**Proposition 20:** If R is a dominance relation on a set Ob, then is topology on Ob.

**Proof:** Clearly Ob and ∅ are elements in tR. Let U, U ∈ tR, for every i ∈ I. For any x ∈ i ∈ U, and (y, x) ∈ R, there is i ∈ I such that x ∈ U. By openness of U, we have y ∈ i ∈ U. Therefore U = {y ∈ X | (y, x) ∈ R}.

According to the above proposition we give the following algorithm to check the openness of a subset U ⊂ Ob with respect to a dominance relation R.

**Openness algorithm:**

1. Find Ob−U
2. Investigates the existence of any pair (a, b) ∈ R, a ∈ Ob−U, b ∈ U, we have two cases:
   - If there exists such pair, then U is not open
   - If there is not such pair (a, b), then U is open

The following example (Table 1) is an application for the above algorithm.

**Example 4:** Consider Ob = {1, 2, 3, 4}, (x, y) ∈ R whenever x ≤ y, ∀ x, y ∈ Ob, then R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}.

Table 2: Closure space generated by a general relation

| A               | ClR(A)  | intR(A) |
|-----------------|---------|---------|
| {[a]            | {[a]}   | ∅       |
| {[b]            | {[b]}   |         |
| {[c]            | {[a, c]}| ∅       |
| {[a, b]         | Ob      | {[b]}   |
| {[a, c]         | {[a, c]}| ∅       |
| {[b, c]         | Ob      | {[b, c]}|
| Ob              | Ob      | Ob      |
| ∅               | ∅       | ∅       |

Then the induced topology is τR = {Ob, ∅, {1, 2}, {1, 2, 3}}.

Let Ob be the set of objects and let R be any binary relation on Ob. The relation R gives rise to a closure operator clR as follows:

clR(Ob) = A ∪ [y ∈ X | ∃ x ∈ A : (y, x) ∈ R] for every A ⊆ Ob

**Lemma 21:** The interior operator corresponding to clR is given by:

intR(A) = {y ∈ A : ∀ x ∈ A, ¬Rx}

**Proof:** intR(A) = [clR(A)]c
= {Ac ∪ {y ∈ X : ∃ x ∈ A, (y, x) ∈ R}}c
= A ∩ {y ∈ X : ∃ x ∈ A, (y, x) ∈ R}c
= A ∩ {y ∈ X : ∀ x ∈ A, ¬Rx}

Thus the interior operator of A consist of those elements of A which are not R-related to any elements outside A.

**Lemma 22:** For any relation R on Ob , (Ob, clR) is closure space.

In the following we will give an example (Table 2) for closure space generated by a general relation.

**Example:** Consider Ob = {a, b, c} and R is a binary relation on Ob, R = {(a, b), (c, b), (a, c)}. Then we have the Table 2 for closures and interiors of the subsets of Ob: We note from Table 2 that:

- clR(∅) = ∅
- A ⊆ clR(A)
- clR(A ∪ B) = clR(A) ∪ clR(B) for all A, B ⊆ Ob

**Lemma 23** If Ob be non-empty set and R is transitve relation, then (x, clR) is topological space.

**Example 24** Consider the relation R = {(1, 1), (2, 3), (3, 2), (2, 2)} on Ob = {1, 2, 3}. Table 3 shows closures and interiors of the subsets of Ob.
Table 3: Closures and interiors of the subsets of Ob

| A    | Cl_R(A) | int_R(A) |
|------|---------|----------|
| {1}  | {1}     | {1}      |
| {2}  | {2, 3}  | φ        |
| {3}  | {2, 3}  | φ        |
| {1, 2}| Ob      | Ob       |
| {1, 3}| Ob      | {2, 3}   |
| {2, 3}| {2, 3}  | {2, 3}   |
| Ob   | Ob      | Ob       |
| φ    | φ       | φ        |

From Table 3, we have:

- $\text{cl}_R(\phi) = \phi$
- $A \subseteq \text{cl}_R(A)$
- $\text{cl}_R(A \cup B) = \text{cl}_R(A) \cup \text{cl}_R(B)$ for all $A, B \subseteq \text{Ob}$
- $\text{cl}_R(\text{int}_R(A)) = \text{int}_R(\text{cl}_R(A))$ for all $A \subseteq \text{Ob}$

**Topologies generated using general binary relations:**

The basic aim of this section is to generate topological structures using the lower and the upper approximations of any binary relation. Given general approximation space $A = (\text{Ob}, R)$ where $R$ here is any general binary relation on $\text{Ob}$. For any subset $X$ of $\text{Ob}$ we define lower and upper approximations as follows:

$$R(X) = \{x \in \text{Ob} : \forall y((x, y) \in R \Rightarrow y \in X)\}$$

and

$$\bar{R}(X) = \{x \in \text{Ob} : \exists y((x, y) \in R \land y \in X)\}$$

Then the following structures are topologies on $\text{Ob}$:

- $\tau_1^1 = [G \subseteq \text{Ob} : R(G) = G]$  
- $\tau_1^2 = [G \subseteq \text{Ob} : R^2(G) = R(R(G)) = G]$  
- $\tau_3^3 = [G \subseteq \text{Ob} : R^3(G) = R(R(R(G))) = G]$ 
- $\tau_4^n = [G \subseteq \text{Ob} : R^{n+4}(G) = G, n = |\text{Ob}|]$ 

These topologies have the property that $\tau_1^1 \subseteq \tau_{n-1}^n \subseteq \ldots \subseteq \tau_{n+1}^n$.

Also, if we deal with the upper approximation instead of the lower approximation we can construct the following topologies:

- $\tau_1^1 = [G \subseteq \text{Ob} : \bar{R}(G) = \bar{R} \circ R(G) = \phi]$  
- $\tau_2^2 = [G \subseteq \text{Ob} : \bar{R}^2(G) = \bar{R}(R(G)) = \phi]$  
- $\tau_3^3 = [G \subseteq \text{Ob} : \bar{R}^3(G) = \bar{R}(R(R(G))) = \phi]$ 
- $\tau_4^n = [G \subseteq \text{Ob} : \bar{R}^{n+4}(G) = \phi, n = |\text{Ob}|]$ 

These topologies have the property that $\tau_1^1 \subseteq \tau_{n-1}^n \subseteq \ldots \subseteq \tau_{n+1}^n$.

In the following we will give some illustrative examples and remarks.

**Example:** Let $\text{Ob} = \{a, b, c, d\}$ be the universe and let $R = \{(a, b), (c, d), (b, d), (d, a), (c, b)\}$ be a general binary relation on $\text{Ob}$. Then we have the following topologies on $\text{Ob}$ using the lower approximation:

- $\tau_1 = [\text{Ob}, \phi, \{a, b, d\}]$
- $\tau_2 = [\text{Ob}, \phi, \{a, b, d\}]$
- $\tau_3 = [\text{Ob}, \phi, \{a, b, c, b, d, \{a, b, \{a, c, \{b, c, \{a, c, d, \{b, c, d\}\}\}\}\}\}$

If we made more iteration to introduce more topologies using the lower approximation we will obtain that $\tau_1 = \tau_2 = \tau_3$ and $\tau_4 = \tau_5$, and so on.

Also we have the following topologies on $\text{Ob}$ using the upper approximation:

- $\tau_1 = [\text{Ob}, \phi, \{c\}]$
- $\tau_2 = [\text{Ob}, \phi, \{c\}]$
- $\tau_3 = [\text{Ob}, \phi, \{d, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d, \{b, c, d\}\}\}$

If we made more iteration to introduce more topologies using the upper approximation we will obtain that $\tau_1 = \tau_2 = \tau_3$ and $\tau_4 = \tau_5$, and so on.

**Remark:** If the relation $R$ on the universe $\text{Ob}$ is constant, then all topologies induced by the lower or the upper approximations are indiscrete.

If the relation $R$ on the universe $\text{Ob}$ is identity or contain the identity relation, then all topologies induced by the lower or the upper approximations are discrete.

If we made more iteration to introduce more topologies using the lower approximation or the upper approximation, then all new iterations will introduce the same topologies we before obtained.

Another method for constructing topologies using the lower and the upper approximations is presented bellow:

All the following are topologies on $\text{Ob}$:

- $\tau^1 = [R(G) : R(G) = R(G)]$
- $\tau^2 = [\bar{R}(G) : \bar{R}(G) = \bar{R}(G)]$

885
\[ \tau^3 = (R^3(G) : R^3(G) = R(G)), \]

\[ \ldots, \tau^s = (R^{s+1}(G) : R^{s-1}(G) = R(G)) \]

Also, all the following structures are topologies on \( Ob \).
\[ \tau^1 = (R(R(G)) : R(R(G)) = R(R(G))), \tau^2 = (R(R(G)) : R(R(G)) = R(R(G))) \]

Example: According to Example 4.1 we have:
\[ \tau^1 = \{Ob, \phi, \{d\}, \{a, b, c\} \} \]
\[ \tau^2 = \{Ob, \phi, \{d\} \} \]
\[ \tau^3 = \{Ob, \phi, \{d\} \} \]

RESULTS AND DISCUSSION

Here we will give the main conventions that we will apply in this work. These conventions will be indicated by examples.

We briefly describe the rheumatic fever datasets used in our example. No doubt that, the rheumatic fever is a very common disease. It has many symptoms differs from patient to another but though the diagnosis it is the same. So, we obtained the following data on seven rheumatic fever patients from Banha fever hospital, Egypt. All patients are between 9-12 years old with history of Arthurian began from age 3-5 years. This disease has many symptoms and it is usually started in young age and still with the patient along his life. Table 4 introduced the seven patients characterized by 8 symptoms (Attributes) using them to decide the diagnosis for each patient (Decision Attribute). Table 5 shows the rheumatic fever information system.

Let us consider the topological space \( \tau_a \) generated using binary relation defined on the attribute \( a \). Also, using the same terminology the topological space \( \tau_b \) is the topology generated using general relation defined on a subset of attributes \( B \) of all condition attributes \( At \). The decision attribute generates the topology \( \tau_0 \).

Now, we will use the following suggestion:

- The set of attributes \( B \subseteq At \) is called a reduct if \( \tau_b \leq \tau_0 \) and \( B \) is minimal, where:

\[ \text{red}(B) = \{ \phi, \{d\}, \{a, b, c\} \} \]

When the classical technique of rough set theory (ROSETTA software) used to obtain reducts and core of our data we found that we have 8 reducts of Table 5 with out any intersections among them. So, we do not have any core of Table 5. The set of obtained reducts is as follows:

| Attribute name | Attribute values | Attribute refers to |
|----------------|------------------|---------------------|
| Sex (S)        | S_1              | Male                |
|                | S_2              | Female              |
| Pharyngitis (P)| yes              | Yes                 |
|                | no               | No                  |
| Arthritis (A)  | a_0              | No arthritis        |
|                | a_1              | Began in the knee   |
|                | a_2              | Began in the ankle  |
| Carditis (C)   | r_1              | Affected            |
|                | r_2              | Not affected        |
| Chorea (Ch)    | yes              | Yes                 |
|                | no               | No                  |
| ESR            | e_1              | Normal              |
|                | e_2              | High                |
| Abdominal pain (Ap) | p_1 | Absent |
|                | p_2              | Present             |
| Headache (H)   | yes              | Yes                 |
|                | no               | No                  |
| Diagnosis (D)  | d_1              | Rheumatic arthritis |
|                | d_2              | Rheumatic carditis  |
|                | d_3              | Rheumatic arthritis and carditis |

| Patients | S | P | A | C | Ch | ESR | Ap | H | D |
|----------|---|---|---|---|----|-----|----|---|---|
| p1       | S_1 | yes | a_1 | r_1 | yes | e_1 | p_1 | no | d_3 |
| p2       | S_1 | yes | a_1 | r_1 | yes | e_2 | p_1 | yes | d_3 |
| p3       | S_2 | yes | a_2 | r_1 | no  | e_1 | p_1 | no  | d_3 |
| p4       | S_1 | yes | a_1 | r_2 | no  | e_1 | p_1 | no  | d_1 |
| p5       | S_1 | no  | a_0 | r_1 | no  | e_1 | p_2 | no  | d_2 |
| p6       | S_1 | yes | a_1 | r_1 | no  | e_2 | p_1 | no  | d_3 |
| p7       | S_1 | yes | a_2 | r_1 | no  | e_1 | p_1 | yes | d_3 |

Now, after getting the reducts of Table 5 using the ROSETTA software. We will convert Table 4-7 using Table 6.

Now we will apply the above contributions on Table 6 where \( Ob = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \) is the set of objects, the set of condition attributes is \( At = \{\alpha, \beta, \delta\} \) and the decision attribute is the diagnosis \( D \).
For any subset B of At, the B-neighborhood of x is defined by:

\[ N_B(x, r) = \{ y \in Ob : |f_x(y) - f_y(x)| \leq r \} \]

For any subset X of Ob, we define two mappings \( \text{Int,Cl}: P(Ob) \rightarrow P(Ob) \) as follows:

\[ \text{Int}_B(X) = \{ x \in Ob : N_x(r, x) \subseteq X, \forall a \in B \} , s \]

\[ \text{Cl}_B(X) = \{ x \in Ob : N_x(x, r) \cap X \neq \emptyset, \forall a \in B \} . \]

Thus, the set of attributes of equal highest degree of dependency is the reduct of our system. So we conclude that \( \{ \alpha \} \) is the reduct of our data using the topological method also, \( \{ \alpha \} \) is the core of our system.

Now, we observe that the reduction that we got by using the GMIS is contained in the reduction that we got using the discernibility matrix and this clears for us that our method for getting the reduction is more precise than using the ROSETTA method. Because, the ROSETTA method cannot apply on general binary relations.

**Topological reduction of single valued datasets:** By reduction we mean if we can remove some data from the data table given in our information system preserving its basic properties. To express this idea more precisely, let \( S = \{ \text{Ob, At, } \{ V_{\alpha} : \alpha \in \text{At} \}, f \} \) be an information system (numerical system). Let \( r \) be a positive real, for each object \( x \in \text{Ob} \) and for \( a \in \text{At}, N_{\alpha}(x, r) \) is the \( a \)-neighborhood of \( x \) and defined by:

\[ N_{\alpha}(x, r) = \{ y \in \text{Ob} : |f_x(x) - f_y(y)| \leq r \} \]

According to the binary relation \( R_\beta = \{(x, y), f_\beta(x) \subseteq f_\beta(y), B \subseteq A \} \) we can construct the following topologies:

\( \tau_\alpha = \{ \text{Ob, } \phi, \{ x_2 \}, \{ x_1 \}, \{ x_2, x_1 \} \} \), \( \tau_\beta = \{ \text{Ob, } \phi \} \), \( \tau_\delta = \{ \text{Ob, } \phi \} \), \( \tau_\alpha \subseteq \{ \text{Ob, } \phi \} \), \( \tau_\beta \subseteq \{ \text{Ob, } \phi \} \), \( \tau_\delta \subseteq \{ \text{Ob, } \phi \} \)

Now we will apply the relation \( R_\delta = \{(x, y), f(x) \subseteq f(y)\} \) to deal with the decision attribute \( D \) and we can construct the following topology:

\[ \tau_\delta = \{ \text{Ob, } \phi, \{ x_1, x_2, x_3, x_4, x_5 \} , \{ x_1, x_2, x_3, x_4, x_5 \} \} \]

We observe that, \( \tau_\delta \) is the core of our system. Then we can get the degree of dependency for each attribute as follows:

\[ \gamma(\alpha, D) = \gamma(\alpha, \delta, D) = \gamma(\beta, \delta, D) = \gamma(\delta, D) = 0 \]

\[ \gamma(\alpha, D) = \gamma(\alpha, \delta, D) = \gamma(\beta, \delta, D) = \gamma(\delta, D) = 0 \]

\[ \gamma(\alpha, \beta, D) = \gamma(\alpha, \delta, D) = \gamma(\beta, \delta, D) = \gamma(\delta, D) = 0 \]

Now let \( \text{At} = \{ a_1, a_2, ..., a_n \} \) and let \( \tau_{a_1}, \tau_{a_2}, ..., \tau_{a_n} \) be the topologies induced by the subbases \( \text{Int}_{\alpha_1}(X) \subseteq \text{Ob}, \{ \text{Int}_{\alpha_2}(X) : \subseteq \text{Ob} \} \), \( \{ \text{Int}_{\alpha_n}(X) : \subseteq \text{Ob} \} \), \( \{ \text{Cl}_{\beta_1}(X) : \subseteq \text{Ob} \} \), \( \{ \text{Cl}_{\beta_2}(X) : \subseteq \text{Ob} \} \), \( \{ \text{Cl}_{\alpha_n}(X) : \subseteq \text{Ob} \} \), \( \{ \text{Cl}_{\beta_n}(X) : \subseteq \text{Ob} \} \), respectively. These topologies called interior, closure and neighborhood topologies respectively.

\[ \gamma(\alpha, \beta, D) = \gamma(\alpha, \delta, D) = \gamma(\beta, \delta, D) = \gamma(\delta, D) = 0 \]
One of the two attributes $a_i, a_j$ , $i \neq j$ is called interior-dispensable in $At$ if $\tau_{i,j} = \tau_{i}$, otherwise, $a_i$ or $a_j$ is indispensable in $At$. Let $\tau_{1,2}, \tau_{1,3}, \ldots, \tau_{n-1,n}$ be the topologies induced by $\tau_{i,j} \cup \tau_{i,j}$, $\tau_{i,j} \cup \tau_{i,j}$, $\tau_{i,j} \cup \tau_{i,j}$ if interior topologies are used (the same terminology used if closure topologies or neighborhood topologies is replaced).

Now if $\tau_{i,j}$ is the topology induced by $(\text{Int}_A(X) : X \subseteq \text{Ob})$ ($\tau_{c,i}$ or $\tau_{c,i}$ can be used alternately), then when $\tau_{i,j} = \tau_{i,j}$, the set $\{a_i, a_j\}$ is a second order reducet of $At$ in $S$. On the other hand, if $\tau_{i,j} \neq \tau_{i,j}$ for all $i, j = 1, 2, \ldots, n$ we must calculate the highest topologies $\tau_{2,3}, \ldots, \tau_{n-2,n}$ and the subset $\{a_i, a_j\}$ is another order reducet of $At$ in $S$ when $\tau_{i,j} \neq \tau_{i,j}$. By the same manner, we can define a highly order reducets of $At$ in $S$.

In each case, the topological core of $At$ in $S$ is the intersection of all reducets (intersection of all the same order reducets). This core called the interior core and denoted $Core_{cl}(At)$. By the same terminology, we can define the closure core ($Core_{cl}(At)$) and the neighborhood core ($Core_{cl}(At)$).

Illustrated Example Consider the information system given by Table 8 and if we choose $r = 2$, then $N_{a}(x, r) = \{y \in \text{Ob}: |f_{a}(x) - f_{a}(y)| \leq 2\}$, hence we have the following subbases:

$$
\begin{align*}
\zeta_1 &= \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\} \\
\zeta_2 &= \{x_1, x_2, x_3, x_4\}, \{x_1, x_3, x_5\} \\
\zeta_3 &= \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_5\} \\
\zeta_4 &= \{x_2, x_3, x_4, x_5\}, \{x_1, x_5\}, \text{Ob}
\end{align*}
$$

The corresponding bases are:

$$
\begin{align*}
\beta_1 &= \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\} \\
\beta_2 &= \{x_1, x_2, x_3\}, \{x_3, x_5\} \\
\beta_3 &= \{x_1, x_2, x_3, x_4\}, \{x_3, x_5\}, \{x_2, x_3, x_5\} \\
\beta_4 &= \{x_2, x_3, x_4, x_5\}, \{x_1, x_5\}, \text{Ob}
\end{align*}
$$

The corresponding topologies are:

$$
\begin{align*}
\tau_1 &= \{\text{Ob}, \phi, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\}, \{x_4, x_5\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}\} \\
\tau_2 &= \{\text{Ob}, \phi, \{x_1, x_2, x_3, x_4\}, \{x_3, x_5\}\} \\
\tau_3 &= \{\text{Ob}, \phi, \{x_1, x_2, x_3, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}, \{x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\}\} \\
\tau_4 &= \{\text{Ob}, \phi, \{x_2, x_3, x_4, x_5\}, \{x_1, x_5\}\}
\end{align*}
$$

If we considered the set of all attributes then $\tau_{n,a}$ is the discrete topology, but the second order topologies are given such that $\tau_{1,2} \neq \tau_{n,a}$, $\tau_{1,3} = \tau_{n,a}$, $\tau_{1,4} \neq \tau_{n,a}$, $\tau_{2,3} = \tau_{n,a}$, $\tau_{1,4} \neq \tau_{n,a}$. Then $\{a_1, a_3\}$ and $\{a_2, a_4\}$ are second order reducets of $At$ and the second order core is given by $Core_{cl}(At) = \{a_1\}$.

**CONCLUSION**

There are many approaches for obtaining topologies by relations and we used some of them in data reduction. These approaches were generalizations to Pawlak approaches namely, we ignored the notion of equivalence relations. Also, these approaches open the way for other approximations if we use the general topological recent concepts such as pre-open sets or semi-open sets. Make use of this terminology to obtain the missing values in incomplete datasets will be a good future work. Implementing software for large data sets reduction using advanced programming languages will be also a good future work.

**REFERENCES**

1. Abd El-Monsef, M.E., E.F. Lashien and A.A. Nasef, 1992. Some topological operators via ideals. Kyungpook Math. J., 32: 273-284.
2. Chachuri, M., 1993. On rough sets in topological Boolean algebra. RSKD, 9: 157-160. http://portal.acm.org/citation.cfm?id=646470.692048
3. Davies, P.C.W. and J. Brown, 2000, Superstring, a Theory of Every Thing. Cambridge University Press, Canto.
4. Krysikiewicz, M., 1999. Rules in incomplete information systems. Inform. Sci., 113: 271-292. http://portal.acm.org/citation.cfm?id=309463
5. Krysikiewicz, M., 1998. Rough set approach to incomplete information systems. Inform. Sci., 112: 39-49. DOI: 10.1016/S0020-0255(98)10019-1
6. Yee, L. and Deyu Li, 2003. Maximal consistent block technique for rule acquisition in incomplete information systems. Inform. Sci., 153: 85-106. DOI: 10.1016/S0020-0255(03)00061-6
7. Maritz, P., 1996. Pawlak and topological rough sets in terms of multifunction. Glasnik Math., 31: 77-99. http://web.math.hr/glasnik/vol_31/no1_17.ps
8. Mashhour, A.S. and A.A. Allam, 1983. On supratopological spaces. Indian J. Pure. Appl. Math., 14: 502-510. http://www.new.dli.ernet.in/rawdataupload/upload/insa/INSA_2/20005a7c_502.pdf
9. Munkres, J.R., 1974. Topology, a First Course. Prentice-Hall. New Jersey, ISBN-10: 0139254951, pp: 448.
10. Pawlak, Z., 1981. Information systems-theoretical foundations. Inform. Syst., 6: 205-218.
11. Pawlak, Z., 1982. Rough sets. Int. J. Comput. Inform. Sci., 11: 341-356.
12. Pawlak, Z. and Skowron A., 2007. Rough sets: Some extensions. Inform. Sci., 177: 28-40. doi:10.1016/j.ins.2006.06.006
13. Pawlak, Z. and Skowron A., 2007. Rough sets and boolean reasoning. Inform. Sci., 177: 41-73. doi:10.1016/j.ins.2006.06.007
14. Pawlak, Z. and A. Skowron, 2007. Rudiments of rough sets. Inform. Sci., 177: 3-27. doi:10.1016/j.ins.2006.06.003
15. Pawlak, Z., 1984. Rough classification. Int. J. Man-Mach. Stud., 20: 469-483. DOI: 10.1016/S0020-7373(84)80022-X
16. Stefanowski, J. and A. Tsoukias, 2001. Incomplete information tables and rough classification. Comput. Intel., 17: 545-566. DOI: 10.1111/0824-7935.00162
17. Wiweger, A., 1989. On topological rough sets. Bull. Pol. Ac. Mat., 37:89-93. http://cat.inist.fr/?aModele=afficheN&cpsidt=4615952
18. William Zhu, 2007. Topological approaches to covering rough sets. Inform. Sci., 177: 1499-1508. doi:10.1016/j.ins.2006.06.009
19. Greco, S., Matarazzo B. and Slowinski R., 2001. Rough set theory for multicriteria decision analysis. Eur. J. Operat. Res., 129: 1-47. http://www.elsevier.com/author_subject_section/S03/Anniversary/EJOR_free29.pdf
20. Orłowska, E., 1987. Reasoning about vague concepts. Bull. Polish Acad. Sci., Math., 35: 643-652.
21. Yao, Y.Y., 2001. Information granulation and rough set approximation. Int. J. Intel. Syst., 16: 87-104. http://cat.inist.fr/?aModele=afficheN&cpsidt=859147
22. Yao, Y.Y., 2003. Probabilistic approaches to rough sets. Expert Syst., 20: 287-297. DOI: 10.1111/1468-0394.00253