ALGEBRAIC AND HAMILTONIAN APPROACHES TO ISOSTOKES DEFORMATIONS

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Abstract. We study a generalization of the isomonodromic deformation to the case of connections with irregular singularities. We call this generalization Isostokes Deformation. A new deformation parameter arises: one can deform the formal normal forms of connections at irregular points. We study this part of the deformation, giving an algebraic description. Then we show how to use loop groups and hypercohomology to write explicit hamiltonians. We work on an arbitrary complete algebraic curve, the structure group is an arbitrary semisimple group.

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The isomonodromic deformation is a classical subject pertaining to many areas of mathematics (see [BBT, I] for example). In [JMU] it was generalized to the case of connections with arbitrary order poles. In this case one requires the monodromy data and the Stokes multipliers to remain constant. Thus we suggest the term *Isostokes Deformation*. In irregular case a new direction of deformation arises: one can deform the irregular types of connections at irregular singular points. Thus one can deform the curve, the divisor, and the irregular types. The deformation of the curve and the divisor was further studied in [K] and [BF]. We study the deformation of the irregular types, our approach is close to that of [BF].

The deformation of the irregular types was also studied in [B1]. In that paper the algebraic curve is $\mathbb{C}P^1$, the structure group is $GL(n)$. Our interest in this subject was evoked by the papers [B2] and [BF]. In the former the deformation for the divisor $2(0) + (\infty)$ is studied for an arbitrary complex reductive group $G$. Its monodromy turns out to coincide with the action of a generalized braid group on the dual Poisson group $G^*$. The hamiltonian approach in this case is obtained in [H]. It turns out that in this case the isostokes connection is the quasi-classical limit of the De Concini–Milson–Toledano Laredo (or DMT) connection (see [T, T2]). A conjecture of De Concini and Toledano Laredo says that the monodromy of the DMT connection coincides with the action of the braid group on a quantum group (this conjecture has been proved recently by Toledano Laredo, see [T3]). Thus the result of [B2] can be thought as a geometrization of the quasi-classical limit of this conjecture.

We give an algebraic description of the isomonodromic deformation. Then we give a hamiltonian description with explicit hamiltonians. As by-products we obtain a description of algebraic and Poisson structures on the moduli spaces of connections.

The structure of the paper is as follows. In the first section we present main definitions and results. We generalize the notion of analytic isostokes deformation to arbitrary complete smooth complex algebraic curves and arbitrary complex semisimple groups (the precise definition is given in §2, see Proposition 3). To give an algebraic description of the deformation we put a structure of algebraic stack on the moduli space of connections. The result of Proposition 1 looks classical but we could not find any reference. Similar constructions are discussed in [A]. Then we give an algebraic description (Theorem 1) of the isostokes deformation. To obtain a hamiltonian description we define the moduli stack of connections with unipotent structures (see §1.5 and Proposition 2). Finally, we give a hamiltonian description (Theorem 2).

In the rest of the paper we prove these theorems. In section 5 we give an explanation of Theorem 2 via loop groups — this is how the theorem was invented. This is an explanation in the spirit of [BF] we were looking for. Another proof is given in [B]. In the last section we count the dimensions and explain how to write explicit formulae.

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1. Main results

1.1. Bundles with connections.

Non-resonant connections. Let us fix a smooth complete algebraic curve $X$ over $\mathbb{C}$, an effective divisor $D = \sum n_i x_i$ ($n_i > 0$) on $X$, and a connected semisimple group $G$ over $\mathbb{C}$. We shall call $x_i$ an irregular point if $n_i \geq 2$. We assume that there is at least one irregular point.

Fix an analytic coordinate $z_i$ at $x_i$ for $i = 1, \ldots, l$. These coordinates will be fixed throughout the paper. We could have avoided fixing coordinates, then we would have to work with jets. However, it would make things more complicated.

Consider a pair $(E, \nabla)$, where $E$ is a principal $G$-bundle on $X$ with a left action of $G$, $\nabla$ is a singular connection on $E$ such that the polar divisor of this connection is bounded by $D$. Choose any trivialization of $E$ in the formal neighborhood of $x_i$. This trivialization allows us to identify connections on the restriction of $E$ to this neighborhood with formal $g$-valued 1-forms, where $g$ is the Lie algebra of $G$. Thus we can write:

$$\nabla = d + A_{n_i} \frac{dz_i}{z_i^{n_i}} + O \left( z_i^{1-n_i} \right),$$

where $A_{n_i} \in g$. Denote by $g^{rs}$ the set of regular semisimple elements of $g$. The connection $\nabla$ is called non-resonant at an irregular point $x_i$ if $A_{n_i} \in g^{rs}$; the connection $\nabla$ is called non-resonant if it is non-resonant at all the irregular points.

The conjugacy class of $A_{n_i}$ does not depend on the choice of trivialization of $E$ in the neighborhood of $x_i$. Thus the notion of non-resonant connection does not depend on the choice of trivialization. We shall call pairs $(E, \nabla)$ connections for brevity.

We shall denote by $\text{Conn}$ the moduli space of pairs $(E, \nabla)$, where $E$ is a principal $G$-bundle, $\nabla$ is a connection on $E$ with the polar divisor bounded by $D$. Let $\text{Conn}$ be the subspace of $\text{Conn}$ corresponding to the non-resonant connections. We shall see below that $\text{Conn}$ has a natural structure of an algebraic stack, clearly, $\text{Conn}$ is its open substack (see Proposition 1).

Remark 1. Notice that in [B2] and elsewhere it is customary to write a connection as $d - A(z)$, since one thinks about a connection as about a differential equation. We always write connections in the form $d + A(z)$. Notice also that in some papers on the subject (in particular, in [B2]) $G$-bundles with right actions are considered.

Compatible framings. Let $x_i$ be an irregular point. A framing of $E$ at $x_i$ is a choice of an element $s_i$ in the fiber of $E$ over $x_i$.

We shall fix a maximal torus $T$, a maximal unipotent subgroup $U$ and a Borel subgroup $B$ in $G$ such that $T \subset B$, $U \subset B$. Let $\mathfrak{h}$, $\mathfrak{u}$ and $\mathfrak{b}$ be the corresponding Cartan, maximal nilpotent, and Borel subalgebras respectively.

After a framing at $x_i$ is chosen, the coefficient $A_{n_i}$ at the leading term in (1) is well-defined (not up to a conjugation). The framing is called compatible with $\nabla$ if $A_{n_i} \in \mathfrak{h}$. Compatible framings at $x_i$ for a non-resonant connection $\nabla$ form an $N(T)$-torsor, where $N(T)$ is the normalizer of $T$ in $G$. Of course, not every resonant connection has a compatible framing.
Denote by $\overline{\text{Conn}}_{fr}$ the moduli space of triples $(E, \nabla, s)$, where $(E, \nabla) \in \overline{\text{Conn}}$, $s = \{s_i\}$ is a collection of compatible framings at all the irregular points. There is a natural forgetful map $\overline{\text{Conn}}_{fr} \to \overline{\text{Conn}}$, the preimage of $\overline{\text{Conn}}$ will be denoted by $\overline{\text{Conn}}_{fr}$. Clearly, $\overline{\text{Conn}}_{fr}$ is an open subspace of $\overline{\text{Conn}}_{fr}$.

**Proposition 1.** The moduli space $\overline{\text{Conn}}$ is an algebraic stack, while $\overline{\text{Conn}}_{fr}$ is an algebraic space.

We shall prove it in §4.

### 1.2. Formal normal forms of connections at irregular singular points.

Set $h^r = h \cap g^r \ast$. Let $\nabla$ be a non-resonant connection, $x_i$ be an irregular point, $s_i$ be a compatible framing at $x_i$. Then we can choose a trivialization of $E$ in the formal neighborhood of $x_i$ such that $\nabla$ takes its formal normal form (see [B2], Lemma 1):

$$\nabla = d + h_{n_i} \frac{dz_i}{z_i} + h_{n_i-1} \frac{dz_i}{z_i^{n_i-1}} + \ldots + h_1 \frac{dz_i}{z_i}, \quad n_i \geq 2,$$

where $h_{n_i} \in h^r$, $h_j \in h$ for $j = n_i - 1, \ldots, 1$. Notice that without the framing this normal form would be defined up to the diagonal action of the Weyl group of $g$.

Taking the formal normal form at every irregular point $x_i$ we get a map:

$$\text{Conn}_{fr} \to (h^r)^{l_{irr}} \times (h)^{\deg D - l},$$

where $l_{irr}$ is the number of irregular points, $l$ is the total number of singular points of $D$, $\deg D$ is the sum of multiplicities of singular points.

Consider the map obtained from the previous map by forgetting the formal residue $h_1$ at every irregular point $x_i$ (in other words, this map assigns to a connection its **irregular type**):

$$\text{IT} : \text{Conn}_{fr} \to (h^r)^{l_{irr}} \times (h)^{\deg D - l - l_{irr}}.$$

This map will be of primary interest for us.

### 1.3. Analytic isostokes deformations.

**Convention.** Let $\Delta$ be an algebraic scheme or a smooth analytic manifold. By a **non-resonant family of connections** over $\Delta$ we mean a triple $(E(t), \nabla(t), s(t))$, where $t \in \Delta$ is the deformation parameter, $E(t)$ is a principal $G$-bundle on $X \times \Delta$, $\nabla(t)$ is a connection on $E(t)$ along $X$ with the polar divisor bounded by $D \times \Delta$ such that $\nabla(t)$ is non-resonant on the fiber over any point of $\Delta$, $s(t)$ is a set of compatible framings at all the irregular points. A **framing** for a family $E(t)$ at $x_i \in X$ is a section of $E(t)|_{\{x_i\} \times \Delta}$.

In §2 we shall construct a natural connection on the map $\text{IT}$ in the following sense. Given $(E, \nabla, s) \in \text{Conn}_{fr}$ and an analytic map $f$ from a polydisk $\Delta$ to the target space of $\text{IT}$ such that

$$\text{IT}(E, \nabla, s) = f(0),$$

we produce a canonical way to extend $(E, \nabla, s)$ to a non-resonant family of connections over $\Delta$. This will be called **Isostokes Deformation**. Heuristically, we deform the connection in such a way that both monodromy data and Stokes data remain constant.
Remark 2. We think about the fibers of $IT$ as about generalized topological data associated to connections. Thus, loosely speaking, the isostokes deformation is the deformation of non-topological data (i.e. of the irregular type), while preserving the topological data. More generally, one can consider a deformation preserving generalized topological data but changing irregular types, the curve $X$, and the divisor $\mathcal{D}$. An approach based on loop groups to the deformation of the curve and the divisor is given in [BF].

1.4. Algebraic approach to isostokes deformations. Consider deformations that do not change a given connection but change framings. It will be convenient for us to regard such deformations as isostokes.

Let $\Delta \ni t_0$ be a smooth manifold, $v \in T_{t_0}\Delta$, $(E(t), \nabla(t), s(t))$ be a non-resonant family, parameterized by $\Delta$ ($T_{t_0}\Delta$ is the tangent space to $\Delta$ at $t_0$). To give an algebraic description of the isostokes deformations we use a notion of a family in-finitesimally isostokes in the direction of $v$. Intuitively, it means that the restriction of this family to $I = \text{Spec } \mathbb{C}[e]/e^2$ is isostokes (where we view $v$ as a map $I \to \Delta$) and we shall give a precise definition in [2.2].

Let $X'$ be a subset of $X$. We say that the restriction of $(E(t), \nabla(t), s(t))$ onto $X'$ is algebraically (analytically) constant if this restriction is algebraically (analytically) isomorphic to the pullback of a triple $(E_0, \nabla_0, s_0)$ along the projection $X' \times \Delta \to X'$. In our applications $X'$ will be either open or closed in $X$.

Definition 1. Consider the open algebraic curve $X = X \setminus \mathcal{D}$ and let $v$ be a tangent vector to $\text{Conn}_{fr}$ at $(E, \nabla, s)$. As a tangent vector to the moduli space it induces an algebraic family of connections over $I$. We call $v$ isostokes if the restriction of this family to $X \times I$ is algebraically constant.

For any map $f$ we denote the corresponding tangent map by $f_*$.

Theorem 1. Consider an algebraic non-resonant family $(E(t), \nabla(t), s(t))$ parameterized by a smooth variety $\Delta \ni t_0$, $v \in T_{t_0}\Delta$. Let $f : \Delta \to \text{Conn}_{fr}$ be the induced map to the moduli space. The family is infinitesimally isostokes at $t_0$ in the direction of $v$ iff $f_*v$ is an isostokes vector.

Remark 3. Notice that since we work on the open curve, an analytically constant family is not always algebraically constant. Moreover, roughly speaking, the theorem above states that the Stokes data does not change in a family, whose restriction to $X$ is algebraically constant. Of course, it is not true for a family with the analytically constant restriction.

1.5. Hamiltonian approach. Now we want to give a hamiltonian description of the isostokes deformation. Actually, $\text{Conn}_{fr}$ is a Poisson space. It is not hard to see that connections that are in the same symplectic leaf of $\text{Conn}_{fr}$ have the same irregular type. Hence, our deformation is transversal to the leaves. Thus to make a hamiltonian description we need to extend $\text{Conn}_{fr}$. This is also known as time dependent hamiltonians.

A symplectic extension of $\text{Conn}_{fr}$. A level-$\mathcal{D}$ unipotent structure on a principal $G$-bundle $E$ is a reduction of $E\mid_\mathcal{D}$ to $U$, where we view $\mathcal{D}$ as a closed subscheme of $X$. Such a structure $\eta$ gives rise to a Borel structure (i.e. to a reduction of $E\mid_\mathcal{D}$ to $B$), which we denote by $\eta_b$.

Let $\overline{\text{Conn}}_{fr}$ be the moduli space of triples $(E, \nabla, \eta)$, where $(E, \nabla) \in \overline{\text{Conn}}$, $\eta$ is a unipotent level-$\mathcal{D}$ structure such that $\eta_b$ is compatible with $\nabla$. In other words,
this structure is a trivialization of $E$ at $x_i$ up to the order $n_i - 1$ (for all $i$) with the requirement that the coefficients of the polar part of $\nabla$ are in $b$. Two trivializations are considered the same if they differ by an element of $U(O_D)$.

Denote by $\mathcal{Conn}_U$ the open subspace of $\overline{\mathcal{Conn}_U}$ parameterizing triples $(E, \nabla, \eta)$ with an additional condition that $\nabla$ is non-resonant.

We shall construct a natural map $\nu$ from $\overline{\mathcal{Conn}_U}$ to $\mathcal{Conn}_{fr}$. Take any triple $(E, \nabla, \eta) \in \overline{\mathcal{Conn}_U}$, let $x_i$ be an irregular point. Let $\tilde{\eta}$ be any trivialization of $E$ at $x_i$ up to the order $n_i - 1$ such that $\tilde{\eta}$ extends $\eta$. Then $\tilde{\eta}$ gives rise to a framing $\eta_0$ of $E$ at $x_i$. Let $A$ be the coefficient of $\nabla$ at $z_i^{-n_i} dz_i$ relative to the framing $\eta_0$. Then $A \in b \cap g^\infty$, thus there is a unique $u \in U$ such that $Ad_u A \in \mathfrak{h}$. Then $u \eta_0$ is a unique framing at $x_i$ compatible with both $\nabla$ and $\eta$, this gives the desired map:

$$(E, \nabla, \eta) \mapsto (E, \nabla, u \eta_0).$$

If follows from Proposition 1 that $\overline{\mathcal{Conn}_U}$ is an algebraic stack, while $\mathcal{Conn}_U$ is an algebraic space.

**Proposition 2.** $\mathcal{Conn}_U$ is a smooth algebraic space with a natural symplectic structure.

This proposition will be proved in §6.

**Isostokes Hamiltonians.** Let $\mathcal{Conn}_n$ be the scheme (of infinite type) of non-resonant connections with a pole of order $n$ on the trivial $G$-bundle on formal disk. We define $\mathcal{Conn}^B_n$ as the subscheme of $\mathcal{Conn}_n$, consisting of connections with polar part in $B$. We define $G^U_n$ as the group of loops of the form

$$\exp(u_0 + u_1 z + \ldots + u_{n-1} z^{n-1} + g_n z^n + \ldots),$$

where $u_i \in u$ for $i = 0, \ldots, n - 1$, $g_i \in \mathfrak{g}$ for $i \geq n$.

Take $(E, \nabla, \eta) \in \mathcal{Conn}_U$. Restricting $\nabla$ to the formal neighborhood of $x_i$, we get a singular connection on the trivialized punctured disk. Since $E$ is reduced to $U$ up to the order $n_i - 1$ at $x_i$ and $\nabla$ is compatible with this reduction, we get an element of $\mathcal{Conn}^B_{n_i} / G^U_{n_i}$. Thus we get a map:

$$IT_U : \mathcal{Conn}_U \to \prod_{i : n_i \geq 2} \mathcal{Conn}^B_{n_i} / G^U_{n_i}.$$  

The target of this map is a Poisson variety, indeed, the $i$-th multiple is an open subset in the hamiltonian reduction at 0 of the space of all connections on the formal punctured disk with respect to $G^U_{n_i}$ (we shall see in §7 that the target of $IT_U$ is a smooth affine variety).

We come to the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{Conn}_U & \xrightarrow{IT_U} & \prod_{i : n_i \geq 2} \left( \mathcal{Conn}^B_{n_i} / G^U_{n_i} \right) \\
\nu \downarrow & & \downarrow \\
\mathcal{Conn}_{fr} & \xrightarrow{IT} & \left( \mathfrak{h}^r \right)^{l_{irr}} \times \left( \mathfrak{h} \right)^{\deg D - l_{irr}}
\end{array}$$

(5)

We call a tangent vector $v$ to $\mathcal{Conn}_U$ isostokes if $\nu_* v$ is isostokes (see Definition 1).

**Theorem 2.** (a) The map $IT_U$ is a Poisson map. If a hamiltonian on $\mathcal{Conn}_U$ factors through $IT_U$, then the corresponding hamiltonian vector field on $\mathcal{Conn}_U$ is isostokes.
This construction gives the whole isostokes deformation in the following sense: if \( v \) is a tangent vector to the target of \( IT \) (see (2)) at the point \( IT(\nu(E, \nabla, \eta)) \), where \( (E, \nabla, \eta) \in \text{Conn}_U \), then there is a hamiltonian vector field \( v_H \), whose hamiltonian factors through \( IT_U \), such that \( v_H(E, \nabla, \eta) \) projects to \( v \).

We shall give an heuristic proof of the part (a) in § 5 and a rigorous proof in § 6. The part (b) will be proved in § 7. We shall also see in § 7 that the number of linearly independent (at a generic point) hamiltonian vector fields produced by the part (a) of the theorem is equal to the dimension of the isostokes distribution on \( \text{Conn}_U \) given by (8).

Remark 4. If the residues of \( \nabla \) at regular singular points are in \( g_r^{*} \), we can think about unipotent level-\( D \) structures as follows: reduce \( E|_D \) to a \( B \)-bundle. Under the condition that \( \nabla \) preserves this \( B \)-structure, this reduction is unique up to the action of \( l \) copies of the Weyl group. This \( B \)-bundle gives rise to a \( B/U \)-bundle. The unipotent level-\( D \) structure is a discrete choice of a \( B \)-reduction plus a trivialization of the \( B/U \)-bundle over \( D \). Thus the dimension of a generic fiber of \( \nu \) is

\[
(\deg D - l_{irr}) \text{ rk } g
\]

(recall that we do not have framings at regular points).

Another approach is to trivialize \( E \) up to the order \( n_i \) at every singular point (this is called level-\( D \) structure). In that way we also obtain a smooth symplectic extension of \( \text{Conn} \). We decided on using unipotent level-\( D \) structures because the dimension of a generic fiber of \( \nu \) is equal to the codimension of the symplectic leaf of \( \text{Conn}_{fr} \). Thus \( \text{Conn}_U \) is a minimal symplectic extension of \( \text{Conn}_{fr} \).

Dimensions. The dimension of the analytic isostokes deformation is given by the dimension of the target of \( IT \), this is equal to

\[
(\deg D - l) \text{ rk } g.
\]

To calculate the dimension of the isostokes distribution on \( \text{Conn}_U \) we need to add up (6), (7), and \( l_{irr} \text{ rk } g \), where the last term comes from the deformations changing frames. The answer is

\[
(2 \deg D - l) \text{ rk } g.
\]

2. Analytic isostokes deformations

In this section we shall give precise definitions of analytic isostokes deformations and infinitesimal analytic isostokes deformations. Our primary reference is [22].

2.1. Stokes solutions and multipliers. Consider \( (E, \nabla, s) \in \text{Conn}_{fr} \). Let \( x_i \) be an irregular point, recall that \( z_i \) is an analytic coordinate at \( x_i \). We can assume that \( z_i(x_i) = 0 \). Let \( U_i \) be the neighbourhood of \( x_i \) given by \( |z_i| < \rho_i \) and \( V_i \) be given by \( |z_i| < 2\rho_i \) for some \( \rho_i > 0 \). The disks \( U_i \) and \( V_i \) will be fixed throughout \( [22] \) and \( [3] \). We can assume that \( V_i \) are disjoint.

For every irregular point \( x_i \) we shall define the Stokes solutions and the Stokes multipliers of \( (E, \nabla, s) \) at \( x_i \). Let us emphasize that the discs \( U_i \) and \( V_i \) are defined only for irregular points \( x_i \). We fix an irregular point \( x_i \) for the whole of § 2.1. Set \( n = n_i, z = z_i \) for brevity.
Stokes solutions. Consider the coefficient \( \lambda = \lambda_n \in \mathfrak{g}^+ \) at the leading term of the formal normal form (2) of \( \nabla \). Let \( \alpha \) be a root of \( \mathfrak{g} \) relative to \( \mathfrak{h} \). An anti-Stokes direction corresponding to \( \alpha \) is a ray in \( \mathbb{C} \), emerging from the origin, on which \( \alpha(h)z^{1-n} \) is real and negative.

Let \( r_1 \) and \( r_2 \) be consecutive anti-Stokes directions. A Stokes sector is a sector with the vertex at \( x_i \) bounded by the directions \( r_1 - \frac{2\pi}{2n-2} \) and \( r_2 + \frac{2\pi}{2n-2} \). We choose some Stokes sector \( S_1 \) and then enumerate all the other Stokes sectors counterclockwise: \( S_2, S_3, \ldots, S_m \).

Note that the Stokes sectors cover \( V_i \) and that angular size of each sector is greater than \( \frac{2\pi}{n-1} \). Notice also, that a single anti-Stokes direction can correspond to more than one root, thus the number of Stokes sectors can be different for irregular points of the same order.

We can write (2) in the following form:

\[
\nabla = d + dQ(z) + h_1 \frac{dz}{z},
\]

where \( Q(z) \) is an \( \mathfrak{h} \)-valued polynomial in \( \frac{1}{z} \). Then \( \Upsilon(z) = \exp(-h_1 \ln z - Q(z)) \) is a formal solution of \( \nabla \), it is called the canonical formal solution of \( \nabla \), \( \Upsilon(z) \) will be thought as a multi-valued section of the trivial \( G \)-bundle over \( V_i \setminus x_i \). We shall make it single-valued on every sector in the following way: choose a branch of \( \Upsilon(z) \) over \( S_1 \) and subsequently extend it to \( S_2, S_3, \ldots, S_m \). Notice that these choices do not agree on the intersection of \( S_1 \) and \( S_m \).

Choose any trivialization of \( E \) over \( V_i \), compatible with \( s_i \). Let \( F(z) \) be a unique formal gauge transformation such that \( F(z) = 1 + O(z) \) and \( F(z) \) takes \( \nabla \) into the formal normal form (3). Consider the Stokes sector \( S_j \). The Stokes solution \( \Phi_j(z) \) in \( S_j \) of \( \nabla \) is a unique solution of \( \nabla \) in \( S_j \) determined by the requirement that the asymptotic expansion of \( \Upsilon(z)^{-1}\Phi_j(z) \) at the origin coincides with that of \( F(z) \) (see [12], Theorem 1). This solution of \( \nabla \) does not depend on the trivialization chosen. However, it depends on a choice of a branch of \( \Upsilon(z) \) and a choice of a numeration of Stokes sectors (i.e. a choice of a first Stokes sector).

Stokes multipliers and the analytic classification of connections. Let \( S_j \) and \( S_{j+1} \) be a pair of consecutive Stokes sectors \( (j \neq m) \). The Stokes multiplier for this pair of sectors is \( \Phi_j(z)\Phi_{j+1}(z)^{-1} \in G \). The Stokes multiplier, corresponding to \( S_m \) and \( S_1 \), is \( e^{-2\pi i h_1} \Phi_m(z)\Phi_1(z)^{-1} \in G \). Notice that Stokes multipliers are constant functions of \( z \), since any two solutions of a connection differ by a constant element of \( G \). It is well known (at least for \( G = GL \)) that the formal normal form together with the Stokes multipliers constitute a complete set of invariants of the local analytic classification of connections, see Theorem 2 of [12]. The Stokes multipliers belong to certain subgroups of \( G \) that are Weyl conjugate to \( U \) (see [12], Lemma 6). This explains a somewhat peculiar formula for the Stokes multiplier corresponding to \( S_m \) and \( S_1 \).

Stokes solutions for families of connections. Recall that Stokes solutions depend on a choice of a branch of \( \Upsilon(z) \) and of a first Stokes sector. Let \( (E(t), \nabla(t), s(t)) \) be a non-resonant family of connections over a polydisk \( \Delta \ni 0 \) (see Convention). Let us make the choices above for \( (E(0), \nabla(0), s(0)) \). Let \( \tilde{S} \) be a sector whose closure is contained in \( S_j \) (it is assumed that the sectors \( S_j \) and \( \tilde{S} \) have the same vertex). Then, shrinking \( \Delta \) if necessary and making a continuous choice of a first Stokes sector for \( (E(t), \nabla(t), s(t)) \), we can assume that the closure of \( \tilde{S} \) is in the
j-th Stokes sector for all \( t \in \Delta \). We can also make a continuous choice of a branch of \( \Upsilon(z) \) for all \( t \in \Delta \). Now we get a family of Stokes solutions \( \Phi_j(z, t) \) defined on \( S \times \Delta \). It depends analytically on \( z \) and \( t \), see Lemma 7 of [122]. We shall often omit \( z \) in the notation below thus denoting this family by \( \Phi_j(t) \).

2.2. Analytic isostokes deformations. Let \((E(t), \nabla(t), s(t))\) be a non-resonant family. Denote the \( j \)-th Stokes sector at \( x_1 \) by \( S_j^1(t) \). We assume that \( S_j^1(t) \) depends continuously on \( t \) (see above). Set \( \tilde{X} = X \setminus (\cup \mathcal{U}_i) \).

**Proposition 3.** Given an analytic map \( f \) from a polydisk \( \Delta \) to the target space of IT and \((E, \nabla, s) \in \text{Conn}_r\), satisfying [4], there is a unique up to isomorphism extension of \((E, \nabla, s)\) to a non-resonant family \((E(t), \nabla(t), s(t))\) over \( \Delta \) such that

(a) \( IT(E(t), \nabla(t), s(t)) = f(t) \) for all \( t \in \Delta \).

(b) The restriction of \((E(t), \nabla(t), s(t))\) onto \( \tilde{X} \) is analytically constant.

(c) Let the closure of a sector \( \hat{S} \) be contained in \( S_j^1(t) \) for all \( t \). Let \( \Delta' \subset \Delta \) be small enough to define Stokes solution \( \Phi_j^1(t) \) on \( \hat{S} \times \Delta' \), then the restriction of \( \Phi_j^1(t) \) to \((\partial \mathcal{U}_i \cap \hat{S}) \times \Delta' \) does not depend on \( t \) as a section of \( E(t)|_{(\partial \mathcal{U}_i \cap \hat{S}) \times \Delta'} \). (This condition makes sense due to the condition [6]).

Notice that (c) is stronger than the requirement that Stokes multipliers do not change in the family.

**Proof.** To simplify notation we restrict to the case of a single irregular point \( x_1 \). The general case is completely similar.

Denote by \( \mathcal{M} \) the moduli space of triples \((\hat{E}, \hat{\nabla}, \hat{s})\), where \( \hat{E} \) is a principal \( G \)-bundle over \( V_1 \), \( \hat{\nabla} \) is a non-resonant connection on \( E \) with the only pole at \( x_1 \) of the order \( n_1 \), \( \hat{s} \) is a compatible framing at \( x_1 \). Taking irregular type gives a map \( \mathcal{M} \to \mathfrak{h}^r \times (\mathfrak{h})^{n_1-2} \). This is a fibred bundle with a canonical flat connection, obtained by deforming the irregular type, while preserving the Stokes data. This is explained in [122], where this is called the isomonodromic connection (some details are given for \( n_1 = 2 \) only but the general case is completely similar). We prefer to call this connection the local isostokes connection.

With this at hand we can finish the proof of the proposition. Let \((\hat{E}, \hat{\nabla}, \hat{s})\) be the restriction of \((E, \nabla, s)\) to \( V_1 \). Then we use the local isostokes connection to extend \((\hat{E}, \hat{\nabla}, \hat{s})\) to a family \((E(t), \nabla(t), s(t))\) of connections on \( V_1 \) such that the irregular type of \((\hat{E}(t), \hat{\nabla}(t), \hat{s}(t))\) is \( f(t) \). Let \((\hat{E}, \hat{\nabla})\) be the restriction of \((E, \nabla)\) to \( X \setminus \mathcal{U}_1 \). It remains to patch \((\hat{E}(t), \hat{\nabla}(t))\) and \((\hat{E}, \hat{\nabla})\) together on \((V_1 \setminus \mathcal{U}_1) \times \Delta \).

The condition (c) of the proposition gives a unique way to make such a patch.

In more detail, let \( S_j^1(t), \ldots, S_m^j(t) \) be all the Stokes sectors for \( x_1 \). Shrinking \( \Delta \) if necessary we can choose a system of sectors \( \hat{S}_j \) such that (a) the closure of \( \hat{S}_j \) is contained in \( S_j^1(t) \) for all \( t \) and (b) \( V_1 \subset \cup \hat{S}_j \). We have a natural identification of \( \hat{E} \) and \( E \) over \((V_1 \setminus \mathcal{U}_1) \times \{0\}\) and we use the condition (c) to extend it to an identification of \( \hat{E} \) and \( E \) over \((V_1 \setminus \mathcal{U}_1) \cap \hat{S}_j \) for every \( j \). These identifications agree on the intersections, since the Stokes multipliers do not change in \((\hat{E}(t), \hat{\nabla}(t), \hat{s}(t))\).

The identifications respect the connection because \( \Phi_j^1(t) \) is a solution of \( \nabla \).

It is clear that the way we patched \( \hat{E} \) and \( E \) is the only way that satisfies (c), thus the uniqueness.
2.3. Infinitesimal isostokes deformations. Unfortunately, we do not know whether $\text{Conn}_G$ is an algebraic scheme. Therefore we shall use somewhat oblique way to define the algebraic isostokes deformation. The problem is that $\text{Conn}_G$ parameterizes algebraic families of connections, while there are no algebraic isostokes families of connections parameterized by smooth varieties. Thus we shall introduce the notion of infinitesimally isostokes family of connections.

Let $(E(t), \nabla(t), s(t))$ be an isostokes family of connections, parameterized by a smooth manifold $\Delta$. The restriction of $E(t)$ onto $\hat{X} \times \Delta$ can be trivialized locally over $\Delta$. Indeed, if $\Delta' \subset \Delta$ is an analytic disk, then $\hat{X} \times \Delta'$ is a Stein manifold and the claim follows from the Oka–Grauert principle, see 3.

Fix such a trivialization. Then the restriction of $\nabla(t)$ onto $\hat{X} \times \Delta'$ becomes a family of $g$-valued 1-forms on $\hat{X}$. Thus the condition 4 of Proposition 5 reads as follows: there is a $g$-valued 1-form $\nabla$ on $\hat{X}$ and a family of $G$-valued functions $R(t)$ (where $t \in \Delta'$) on $\hat{X}$ such that $\nabla(t) = \text{Ad}_{R(t)} \nabla$. Here ‘Ad’ is the natural action of $G$-valued functions on connections by gauge transformations. Below we shall also use the infinitesimal action of $g$-valued functions on connections, which we denote by $\text{ad}$.

The restrictions of the Stokes solutions to $\partial \mathcal{U}_t$ can be also viewed as $G$-valued functions in this trivialization. Then the condition 5 of the definition becomes: the restriction of $\Phi_j^i(t) R(t)$ to $(\partial \mathcal{U}_t \cap \tilde{S}) \times \Delta'$ does not depend on $t$.

As was mentioned in 4, it will be convenient for us to work with an extended version of the isostokes deformation. We want to add the deformations that do not change connections but change framings at irregular points. If we change a framing $s_i$ to $Cs_i$, where $C \in G$, then the Stokes solution $\Phi_j^i$ transforms into $C^{-1}\Phi_j^i$. Thus we get a weaker version of 4:

4. For all $i$ there exists a family $C_i(t)$ of elements of $T$ such that the restriction of $C_i(t)^{-1}\Phi_j^i(t) R(t)$ to $(\partial \mathcal{U}_t \cap \tilde{S}) \times \Delta'$, where $\tilde{S}$ as in the part 4 of Proposition 5 does not depend on $t$.

It is easy to write the infinitesimal version of these conditions.

Definition 2. A non-resonant family $(E(t), \nabla(t), s(t))$ over a smooth manifold $\Delta$ is called infinitesimally isostokes at $t_0 \in \Delta$ in the direction of $v \in T_{t_0}\Delta$ if after trivializing the restriction of $E(t)$ onto $\hat{X} \times \Delta'$ (for some small neighbourhood $\Delta'$ of $t_0$) we can find a $g$-valued function $R$ on $\hat{X}$ and for each irregular $x_i$ an element $c_i \in \mathfrak{h}$ such that

(a) $L_v \nabla(t) = \text{ad}_R \nabla(t_0)$ on $\hat{X}$;
(b) $L_{v_i} \Phi_j^i(t) = -\Phi_j^i(t_0) R + c_i \Phi_j^i(t_0)$ on $\partial \mathcal{U}_t \cap S_j^i$ for all $i$ and $j$. Here $L_v$ is the directional derivative in the direction of $v$ at $t_0$, $\Phi_j^i(t_0) R$ is the usual left shift of $R$ on the tangent bundle of $G$, $c_i \Phi_j^i(t_0)$ is the right shift of $c_i$.

It is clear that an isostokes family is infinitesimally isostokes at every point and in every direction.

3. Proof of Theorem 1

Consider an algebraic non-resonant family $(E(t), \nabla(t), s(t))$ over a smooth variety $\Delta \ni t_0$. It gives rise to a map $f : \Delta \to \text{Conn}_G$. Choose $p \in X$, $p \notin \cup V_i$. 

3.1. Proof of the part ‘If’ of the theorem. Suppose that \( v \in T_{x_0} \Delta \) is such that \( f_* v \) is an isostokes vector. We need to show that \( (E(t), \nabla(t), s(t)) \) is an infinitesimally isostokes family in the direction of \( v \).

There is an étale neighbourhood \( \iota : \Delta' \to \Delta \) of \( t_0 \) such that \( E(t)|_{\hat{X} \times \Delta'} \) is trivial and \( E(t)|_{(X \setminus \mu) \times \Delta'} \) is trivial, since every \( G \)-bundle over a family of affine curves is trivial locally over the base in the étale topology, see [DS]. It is enough to show that the restriction of \( (E(t), \nabla(t), s(t)) \) to \( \Delta' \) is infinitesimally isostokes in the direction of \( (\iota_*)^{-1} v \), since every étale morphism is a local analytic diffeomorphism. Thus we can assume from the beginning that the restrictions of \( E(t) \) to \( \hat{X} \times \Delta \) and to \( (X \setminus \mu) \times \Delta \) are trivial.

Let us trivialize the restriction of \( E(t) \) onto \( \hat{X} \times \Delta \). In this trivialization the restriction of \( \nabla(t) \) becomes a family of \( \mathfrak{g} \)-valued 1-forms, denote it by \( \nabla(t) \). Then we can re-write the definition of \( f_* v \) being isostokes in the following form: there is a \( \mathfrak{g} \)-valued (algebraic) function \( R \) on \( \hat{X} \) such that

\[
L_v \hat{\nabla}(t) = \text{ad}_R \hat{\nabla}(t_0).
\]

We shall have to work in a neighbourhood of \( \mathfrak{D} \), thus we need a trivialization of \( E(t) \) in this neighbourhood. To this end we trivialize \( E(t) \) on \( (X \setminus \mu) \times \Delta \). In this trivialization \( \nabla(t) \) is again a \( \mathfrak{g} \)-valued 1-form, denote it by \( \nabla(t) \). These two trivializations (over \( \hat{X} \times \Delta \) and over \( (X \setminus \mu) \times \Delta \)) are related by a transition function \( Z(t) : (\hat{X} \setminus \mu) \times \Delta \to G \), \( t \in \Delta \).

We identify \( G \) with some subgroup of \( GL \) via any exact representation, this will simplify calculations. The condition (10) in Definition 2 is obvious: indeed, we just restrict \( R \) in any trivialization of \( E(t) \) from \( \hat{X} \) to \( X \). Thus it suffices to check the condition (10). In the trivialization of \( E(t) \) on \( (X \setminus \mu) \times \Delta \) we get

\[
L_v \hat{\nabla}(t) = \text{ad}_P \hat{\nabla}(t_0),
\]

where

\[
P = (L_v Z(t))Z(t_0)^{-1} + Z(t_0)RZ(t_0)^{-1}.
\]

Fix an irregular point \( x_i \). Recall that \( z_i \) is an analytic coordinate on \( V_i \) such that \( z(x_i) = 0 \). We restrict \( P \) and \( \nabla(t) \) onto \( V_i \) and \( V_i \times \Delta \) respectively. From now on we shall be working on \( V_i \), since the statement we need to prove depends solely on the restrictions of our objects to this disc. We can also assume that \( t_0 = 0 \). We shall work in the analytic setup, thus we view \( \Delta \) as an analytic manifold. Moreover, we can assume that \( \Delta \) is a disk in \( \mathbb{C} \) (indeed, first we reduce to the case when \( \Delta \) is a polydisk, then we take the appropriate 1-dimensional section of this polydisk). We emphasize that our objects depend on \( z_i \), which we omit in the notation. Finally, we set \( z = z_i \), \( n = n_i \), \( U = U_i \), and \( V = V_i \) for brevity.

Write

\[
\hat{\nabla}(t) = d + \sum_{j=-n}^{\infty} A_j(t)z^j dz.
\]

Lemma 1. Changing the trivialization of \( E(t) \) on \( V \times \Delta \) by an analytic gauge transformation we can assume that

(a) this trivialization is compatible with the framing \( s_i \);
(b) \( A_j(t) \in \mathfrak{h} \) for \(-n \leq j \leq n \) and all \( t \);
(c) \( P \) is a polynomial in \( \frac{1}{t} \);
(d) all the coefficients of \( P \) are in \( \mathfrak{h} \).
Proof. Clearly, we can assume that (11) is satisfied. Every connection can be brought to its formal normal form up to any power of \( z \) by an analytic (even a polynomial) gauge action. It is easy to see that this can be done in a family. Thus (12) is clear. Note that since we have a compatibly framed connection, this gauge change can be taken of the form \( 1 + O(z) \). It follows from (11) that the coefficients of \( P \) are in \( \mathfrak{h} \) up to the coefficient at \( z^{2n} \). Indeed, otherwise RHS of (11) would not be in \( \mathfrak{h} \) up to the order \( n \), since \( \tilde{\nabla}(0) \) is non-resonant. Write

\[
P = P_- + P_+,
\]

where \( P_- \) is a polynomial in \( \frac{1}{z} \), \( P_+ \) is a polynomial in \( z \) without the constant term. We can assume that \( t \) is a coordinate on \( \Delta \) such that \( v = \partial / \partial t \). Change the trivialization of \( E(t) \) on \( V \times \Delta \) by means of \( \exp(-tP_+) \). Then \( P \) changes to (see (12)):

\[
P - P_+ = P_-.
\]

Thus we get (c). The condition (a) of the lemma is not corrupted by this trivialization change since \( P_+ \) has no constant term and the condition (b) is not corrupted, since the coefficients of \( -tP_+ \) are in \( \mathfrak{h} \) up to the order \( 2n \). □

The infinitesimal change of the canonical formal solution. Write

\[
\tilde{\nabla}(t) = d + d_z Q(t) + \Lambda(t) \frac{dz}{z} + O(1),
\]

where \( d_z \) is the differential with respect to \( z \), \( Q(t) = \sum_{j=1}^{-n} A_{j-1}(t) z^j \). By (10) of Lemma 1 the formal normal form of \( \tilde{\nabla}(t) \) is just its polar part, denote it by \( \tilde{\nabla}_0(t) \).

Let

\[
\Upsilon(t) = \exp(-\Lambda(t) \ln z - Q(t))
\]

be the canonical formal solution of \( \tilde{\nabla}(t) \). We want to study how \( \Upsilon(t) \) changes in the direction of \( v \). Set

\[
\Theta = \Upsilon(0)^{-1}(L_v \Upsilon(t)) = -L_v Q(t)
\]

(here we use that \( L_v \Lambda(t) = 0 \), which follows from (11)). Let \( P_0 \) be the constant term of \( P \), we claim that

\[
(13) \quad \Theta = P_0 - P.
\]

Indeed, the definition of \( \Theta \) shows that \( -d_z \Theta \) is the polar part of

\[
L_v \tilde{\nabla}(t) = \text{ad}_P \tilde{\nabla} = d_z P + [P, A(0)],
\]

where we have written \( \tilde{\nabla}(t) = d + A(t)dz \). This polar part is equal to \( d_z P \) by Lemma 1. Thus \(-\Theta = P \) and \( P \) differ by a term which does not depend on \( z \) but \( \Theta \) is a polynomial in \( \frac{1}{z} \) without the constant term, and the claim follows.

The infinitesimal change of the Stokes solutions. Recall the notion of Stokes solutions. The disk \( V \) is covered by \( m \) Stokes sectors. Let \( F(t) = 1 + O(z) \) be the formal series in \( z \) taking \( \tilde{\nabla}(t) \) into its formal normal form. According to Lemma 1 we can assume that \( F(t) = 1 + O(z^{n+1}) \). Fix a Stokes sector \( S \) for \( \tilde{\nabla}(0) \). Take a sector \( \hat{S} \) of angular size greater than \( \frac{\pi}{m-1} \) whose closure is in \( S \). As before, we may assume that the corresponding Stokes solution \( \Phi(t) \) is defined on \( \hat{S} \times \Delta \). This solution of \( \tilde{\nabla}(t) \) is uniquely determined by the requirement that the asymptotic expansion of \( \Upsilon(t)^{-1} \Phi(t) \) in \( z \) at the origin coincides with one of \( F(t) \) for all \( t \).
Lemma 2.

\[(14) \quad L_v \Phi = -\Phi(0)P + c\Phi(0),\]

where \(c \in \mathfrak{h}\) is a constant matrix.

Proof. We shall prove this by a direct computation. Set \(\Psi = L_v \Phi\). We have
\[d_z \Phi(t) = -\Phi(t)A(t)dz.\]

Applying \(L_v\) to both sides of this equation, we get a variation equation for \(\Psi\):
\[d_z \Psi = -\Psi A(0)dz + \Phi(0)(d_z P + [P, A(0)] dz).\]

It is easy to verify that \(-\Phi(0)P\) satisfies the same differential equation. Thus \(\Psi + \Phi(0)P\) is a solution of \(\tilde{\nabla}(0)\), which gives
\[(15) \quad \Psi = -\Phi(0)P + c\Phi(0),\]

where \(c\) is a constant matrix. It remains to show that \(c \in \mathfrak{h}\). We have
\[(16) \quad \Upsilon(t)^{-1} \Phi(t) \sim F(t)\]

for every \(t\). Moreover, it follows from the proof of Lemma 7 of \[B2\] that this asymptotic expansion is uniform in some neighbourhood of the origin. Thus it follows from the Cauchy integral formula that we can apply \(L_v\) to both sides of this equation: we get
\[(17) \quad \Upsilon(0)^{-1} \Psi - \Theta \Upsilon(0)^{-1} \Phi(0) \sim L_v F(t).\]

Substituting \(\Psi\) from \[(14)\] and using \[(13)\] and \[(10)\], we get
\[\sim F(0)P + \Upsilon(0)^{-1}c \Phi(0) + PF(0) - P_0 F(0) \sim L_v F(t).\]

Since \(F(0) = 1 + O(z^{n+1})\) and \(L_v F(t) = O(z^{n+1})\), we get that
\[\Upsilon(0)^{-1}c \Upsilon(0) = P_0 + O(z).\]

Now, let \(\bigoplus_\alpha \mathfrak{g}_\alpha \oplus \mathfrak{h}\) be the root decomposition of \(\mathfrak{g}\). Suppose that the projection of \(c\) to \(\mathfrak{g}_\alpha\) is not zero for some \(\alpha\). Denoting the corresponding character of \(T\) by \(\exp(\alpha)\), we see that \(\exp(\alpha)(\Upsilon(0))\) must be bounded in \(\tilde{S}\). However, this function is of the form \(z^\lambda e^{f(z)}\), where \(f\) is a polynomial in \(z\) of degree \(n - 1\). Such a function cannot be bounded in \(\tilde{S}\), since the angular size of \(\tilde{S}\) is greater than \(\frac{\pi}{n-1}\). This contradiction shows that \(c \in \mathfrak{h}\). \(\square\)

In the trivialization of \(E(t)\) over \(\tilde{X} \times \Delta\) the Stokes solution \(\Phi(t)\) transforms into \(\hat{\Phi}(t) = Z(t)^{-1} \Phi(t)\), and \[(14)\] becomes
\[L_v \hat{\Phi}(t) = -\hat{\Phi}(0)R + c\hat{\Phi}(0),\]

and we see that \((E(t), \nabla(t), s(t))\) is infinitesimally isostokes in the direction of \(v\).
3.2. Proof of the part ‘Only if’ of the theorem. This is, in some sense, a rearrangement of the previous proof. Let us assume that \((E(t), \nabla(t), s(t))\) is infinitesimally isostokes in the direction of \(v\). Again, we can, after passing to an étale neighbourhood of \(t_0 \in \Delta\) assume that the restrictions of \(E(t)\) to \(\tilde{X} \times \Delta\) and \((X \setminus P) \times \Delta\) are trivial.

Then we have to show that there is an algebraic \(g\)-valued function \(R\) on \(\tilde{X}\) such that (10) holds on \(\tilde{X}\). The condition (10) of Definition 2 gives an analytic function \(R\) such that (10) holds on \(\tilde{X}\). Shrinking \(\Delta\) if necessary, we can choose a system of sectors \(\tilde{S}_j^i\) such that (a) the closure of \(\tilde{S}_j^i\) is contained in \(S_j^i(t)\) for all \(t\) and (b) \(V_i\) is contained in \(\cup_j \tilde{S}_j^i\).

We extend \(R\) to \(\tilde{X}\) solving the equation in the Definition 2, i.e. setting

\[ R = \Phi_j^i(0)^{-1}(c\Phi_j^i(0) - L_0 \Phi_j^i(t)) \]

on \(V_i \cap \tilde{S}_j^i\). The condition (1) of Definition 2 together with the analytic continuation principle show that these definitions of \(R\) agree on \(\tilde{X} \cap \tilde{X} \cap \tilde{S}_j^i\). The compatibility on \(\tilde{S}_j^i \cap \tilde{S}_j^{i+1}\) follows from the compatibility on \(V_i \cap \tilde{X} \cap \tilde{S}_j^i \cap \tilde{S}_j^{i+1}\) by analytic continuation. It remains to show that \(R\) does not have essential singularities.

This is easy to check, employing parts of the proof above. First, it is enough to check that \(P\) given by (12) has no essential singularities, since \(Z(t)\) is meromorphic. In \(U_t\) we have (14) with some \(c \in \mathfrak{g}\). We also have (17). Substituting the former into the latter we see that

\[ P \sim F(0)^{-1}cF(0) - F(0)^{-1}\Theta F(0) - F(0)^{-1}L_0 R F(t) \]

and therefore \(P\) has no essential singularity at \(x_i\). Theorem 1 is proved.

4. Algebraic structures on moduli spaces of connections

In this section we shall prove Proposition 1.

Let \(\text{Bun}_G X\) be the moduli stack of principal \(G\)-bundles on \(X\). This is an algebraic stack, locally of finite type over \(\mathbb{C}\), see [S], Corollary 3.6.6, see also [LM], 4.14.2.1 for the case \(G = GL\). Thus we can restrict ourselves to families over locally noetherian schemes below.

Let \(\mathcal{F}\) be a divisor on \(X\), denote by \(\text{Conn}_\mathcal{F}\) the lax functor (or 2-functor) from affine schemes to groupoids defined by:

\[ S \mapsto \{ \text{pairs } (E(t), \nabla(t)) \} + \{ \text{isomorphisms} \}. \]

Here \(E(t)\) is a \(G\)-bundle on \(X \times S\), \(\nabla(t)\) is a connection along \(X\) with the pole divisor bounded by \(\mathcal{F} \times S\). This is a stack in the étale topology, the proof is essentially the same as in the case of \(\text{Bun}_G X\) (see [Dr], Theorem 4.5). The only additional ingredient is that connections can be glued in the étale topology, this is obvious.

Now we shall prove that \(\text{Conn} = \text{Conn}_\mathcal{D}\) is an algebraic stack. Choose an ample divisor \(\mathcal{E}\) on \(X\). Let \(\text{Bun}_G^{(k)} X\) be the stack of \(G\)-bundles \(E\) such that

\[ H^1(X, \text{ad } E \otimes \Omega^1(\mathcal{D} + k\mathcal{E})) = 0. \]

Precisely, this stack parameterizes bundles \(E(t)\) over \(X \times S \ (t \in S)\) with

\[ R^1 p_* (\text{ad } E(t) \otimes \Omega((\mathcal{D} + k\mathcal{E}) \times S)) = 0, \]

(18)
where \( p : X \times S \to S \) is the natural projection, \( \Omega = \Omega_{X \times S/S}^1 \) is the sheaf of relative differentials. Clearly, this is an open (and hence algebraic) substack in \( \text{Bun}_G X \), it is also clear that \( \text{Bun}_G X = \bigcup_k \text{Bun}_G^{(k)} X \) (see [LM], 4.14.2.1 for more details).

Let \( \text{Conn}^{(k)}_{\mathcal{D}+k\mathcal{E}} \) be the substack of \( \text{Conn}^{(k)}_{\mathcal{D}+k\mathcal{E}} \) parameterizing pairs \((E, \nabla)\) such that \( E \in \text{Bun}_G^{(k)} X \). Consider the forgetful 1-morphism of stacks

\[ \lambda : \text{Conn}^{(k)}_{\mathcal{D}+k\mathcal{E}} \to \text{Bun}_G^{(k)} X. \]

We claim that it is representable. Indeed, let \( S \to \text{Bun}_G^{(k)} X \) be any morphism. It corresponds to a \( G \)-bundle \( E(t) \) over \( X \times S \). This bundle has a connection with poles on \( (\mathcal{D}+k\mathcal{E}) \times S \) étale locally over \( S \), since the local obstruction to the existence of such a connection is in the vanishing sheaf \( \mathcal{I}_S \).

Thus (locally over \( S \)) the set of connections on \( E(t) \) is identified with the total space of \( R^0 p_*(\text{ad}(E(t)) \times \Omega((\mathcal{D}+k\mathcal{E}) \times S)) \).

By \([\mathcal{I}_S]\) and the Riemann–Roch theorem, this sheaf is locally free. It follows that the fiber

\[ S \times_{\text{Bun}_G^{(k)} X} \text{Conn}^{(k)}_{\mathcal{D}+k\mathcal{E}} \]

is an affine bundle over \( S \), hence, a scheme.

Thus \( \lambda \) is representable, therefore \( \text{Conn}^{(k)}_{\mathcal{D}+k\mathcal{E}} \) is an algebraic stack. It follows that the substack of \( \text{Conn}_{\mathcal{D}} \) corresponding to the bundles satisfying \([\mathcal{I}_S]\) is algebraic as well, since it is a closed substack of \( \text{Conn}^{(k)}_{\mathcal{D}+k\mathcal{E}} \). Thus \( \text{Conn} \) is a union of an increasing sequence of open algebraic substacks, hence algebraic.

It remains to show that \( \text{Conn}_{fr} \) is an algebraic space. The stack of connections with arbitrary (not necessarily compatible) framings is algebraic since the forgetful 1-morphism to \( \text{Conn} \) is representable. Thus \( \text{Conn}_{fr} \) is an algebraic stack, since it is a closed substack of the latter stack. However, framed connections do not possess automorphisms, thus \( \text{Conn}_{fr} \) is an algebraic space.

This completes the proof of Proposition 1.

5. THE DOUBLE QUOTIENT CONSTRUCTION AND ISOSTOKES HAMILTONIANS

In this section we give an heuristic proof of the part \([\mathfrak{a}]\) of Theorem 2. It is based on an infinite-dimensional symplectic reduction. Unfortunately, there are some technical difficulties in such an approach, therefore we give another proof in the next section. The current proof explains how the theorem was invented making clear the connection with the paper \([\mathfrak{B}]\).

5.1. The double quotient construction. Let \( G((z)) \) be the group of \( G \)-valued functions on the punctured formal disk (the loop group). Denote by \( LG \) the group \( \prod_i G((z)) \). We can identify \( LG \) with the group of \( G \)-valued functions on the formal neighbourhood of \( \mathcal{D} \). Let

\[ L_+ G = \prod_{i=1}^d G^U_{n_i} \subset LG \]

be the subgroup of “positive loops”. Let \( L_X G = G(\hat{X}) \) be the group of \( G \)-valued functions on \( \hat{X} \). Such a function can be restricted to the formal neighbourhood
of $\mathfrak{D}$, which gives an embedding $L_XG \hookrightarrow LG$. Then the stack of $G$-bundles with level-$\mathfrak{D}$ unipotent structures is isomorphic to the double quotient 

$$L_XG \backslash LG/L_+G.$$  

The similar statement is well known for the stack of $G$-bundles without additional structures, see [S, Theorem 5.1.1] and [BR, Theorem 4.1.1]. In our case the proof is completely similar. Morally, an element of $LG$ is viewed as a $G$-bundle, trivialized over both $X$ and the formal neighbourhood of $\mathfrak{D}$, while the factoring by $L_XG$ and $L_+G$ amounts to forgetting these trivializations.

Notice that $LG$ and $L_XG$ are ind-groups, $L_+G$ is an affine group of infinite type, see [S], §8.

5.2. Generalities on hamiltonian quotients. Notice that we use the expressions “hamiltonian quotient” and “hamiltonian reduction” as synonyms.

Let $Y$ be a Poisson ind-scheme, equipped with a hamiltonian action of an ind-group $K$. Let $O$ be a coadjoint orbit in Lie($K$). Denote by $Y//OK$ the hamiltonian quotient of $Y$ by $K$ at $O$. This is the quotient $\mu^{-1}(O)/K$, where $\mu$ is the moment map.

Let $H : Y//OK \rightarrow \mathbb{C}$ be a function. We can lift it to a function

$$\tilde{H} : \mu^{-1}(O) \rightarrow \mathbb{C}.$$  

By a lift of $H$ to $Y$ we mean a function $\tilde{H} : Y \rightarrow \mathbb{C}$ such that its restriction to $\mu^{-1}(O)$ coincides with $\tilde{H}$. Clearly, such a lift is not unique. Actually, we can start with a function $H$, defined on an open subset of $Y//OK$, then its lift is a function on an open subset of $Y$. Note that in our case “open” means open in étale topology, this is where the difficulties come from.

Denote by $v_H$ the hamiltonian vector field corresponding to $H$, by $\{\cdot,\cdot\}$ the Poisson bracket.

**Lemma 3.** Let $H$ and $H_1$ be functions on an open subset of $Y//OK$, $\tilde{H}$ and $\tilde{H}_1$ be their lifts to $Y$. Then

(a) $\{H, H_1\}$ is a lift of $\{H, H_1\}$.

(b) A vector field $v_{\tilde{H}}$ is tangent to $\mu^{-1}(O)$, the restriction of $v_H$ to $\mu^{-1}(O)$ is $K$-equivariant and descends to $v_H$.

These are standard hamiltonian reduction facts.

5.3. Double quotient presentation of $\underline{Conn}$. Let $T^*_1LG$ be a twisted cotangent bundle to $LG$, parameterizing pairs $(g, \nabla)$, where $g \in LG$ and $\nabla$ is a connection on the formal punctured neighbourhood of $\mathfrak{D}$. Let $\underline{Conn} = \varinjlim \underline{Conn}_n$ (see [EL]) be the ind-scheme of connections on the trivial formal punctured disk. We may view $T^*_1LG$ as the space parameterizing $G$-bundles on $X$, trivialized both on $X$ and the formal neighbourhood of $\mathfrak{D}$ with a singular connection $\nabla$ on this formal neighbourhood (compare with [5.1]). We may view $\nabla$ as an element of $(\underline{Conn})^\ell$, using any of two trivializations of $T^*_1LG$, which gives two isomorphisms

$$T^*_1LG \cong LG \times (\underline{Conn})^\ell.$$  

Denote the corresponding projections $T^*_1LG \rightarrow (\underline{Conn})^\ell$ by $p^R$ and $p^L$. Precisely, $p^R$ corresponds to the trivialization of the bundle over the open curve $X$, while $p^L$ corresponds to the trivialization over the formal neighbourhood of $\mathfrak{D}$. Clearly, the adjoint action of $LG$ on $(\underline{Conn})^\ell$ intertwines these projections.
This twist can be also explained by symplectic reduction. Let \( \hat{\mathfrak{g}} \) be an affine Kac-Moody algebra that is the canonical central extension of the loop algebra \( \mathfrak{g}(\mathbb{C}) \). Then \( \hat{\mathfrak{g}} = \prod_{i=1}^{l} \hat{\mathfrak{g}} \) is a central extension of \( \text{Lie}(\mathbb{C}^{\mathfrak{g}}) \). Let \( \hat{\mathbb{C}} \) be the corresponding central extension of \( \mathbb{C} \), then \( \hat{\mathbb{C}} \) is a central subgroup of \( \hat{\mathbb{C}} \), isomorphic to \( (\mathbb{C}^{\hat{\mathfrak{g}}})^{l} \). This gives a \( (\mathbb{C}^{\hat{\mathfrak{g}}})^{l} \) action on \( T^{\ast} \hat{G} \). We have

\[
T^{\ast}_{\mathfrak{g}} \mathbb{C} = T^{\ast} \hat{G} / \mathfrak{g}(\mathbb{C})^{l},
\]

where \( /_{1} \) states for the symplectic reduction at \( 1 = (1, \ldots, 1) \in (\mathbb{C}^{\mathfrak{g}})^{l} \).

The proof of Theorem 2 is based on the following presentation:

\[
\mathbb{C} \text{on}_{\mathcal{U}} = L X G \setminus T^{\ast}_{\mathfrak{g}} \mathbb{C} / /_{0} \tilde{\mathfrak{g}} + \mathbb{C} L G.
\]

This formula also gives the desired symplectic structure on \( \mathbb{C} \text{on}_{\mathcal{U}} \). Notice that this double quotient can be thought as a single symplectic quotient with respect to the group \( L X G \times \mathbb{C} L G \).

We call \( v \in T^{\ast}_{\mathfrak{g}} \mathbb{C} \) an isostokes vector, if \( p^{R} v = 0 \). This definition, clearly, agrees with (20) and Definition 1 in the following sense: suppose \( v \) is a vector, tangent to the zero-level of the moment map and such that its projection to \( \mathbb{C} \text{on}_{\mathcal{U}} \) is tangent to \( \mathbb{C} \text{on}_{\mathcal{U}} \). Then it is isostokes iff its projection to \( \mathbb{C} \text{on}_{\mathcal{U}} \) is.

Remark 5. We have isomorphisms of ind-schemes:

\[
T^{\ast}_{\mathfrak{g}} \mathbb{C} \cong T^{\ast} \mathbb{C} \cong \mathbb{C} \times \text{Lie}(\mathbb{C}^{\mathfrak{g}})^{l}.\]

However, each of these spaces is equipped with both left and right actions of \( G \). The isomorphisms can be chosen either left or right equivariant but not bi-equivariant. Thus these spaces are different as \( G \)-bimodules. In addition, first two spaces carry different symplectic structures.

5.4. Completion of the “proof” of the first part of Theorem 2. The target of \( IT_{\mathcal{U}} \) can be identified with an open subset of

\[
(\text{Conn})^{l}_{\text{irr}} / /_{0} \left( \bigcap_{i: n_{i} \geq 2} G_{n_{i}}^{U} \right).
\]

Let \( H \) be a hamiltonian on \( \mathbb{C} \text{on}_{\mathcal{U}} \) that factors through \( IT_{\mathcal{U}} \): \( H = f \circ IT_{\mathcal{U}} \). Let \( \tilde{f} \) be a lift of \( f \) to \( (\text{Conn})^{l}_{\text{irr}} \) (recall, that this is a function on an open subset of \( (\text{Conn})^{l}_{\text{irr}} \)). Using the natural projection \( (\text{Conn})^{l} \to (\text{Conn})^{l}_{\text{irr}} \) we can lift \( \tilde{f} \) to a function \( \tilde{f} \) on an open subset of \( (\text{Conn})^{l} \). Then it is easy to see that \( \tilde{H} = \tilde{f} \circ p^{L} \) is a lift of \( H \) with respect to (20).

Taking into account Lemma 3 and the discussion in the end of §5.3, we see that our theorem reduces to the following statement:

\[
p^{L} \text{ is a Poisson map, } p^{R}_{\tilde{H}} v_{\tilde{H}} = 0.
\]

This is not hard to verify by a direct calculation. However, we can reduce it to some general nonsense, using (19). Indeed, let \( \tilde{p}^{L} \) and \( \tilde{p}^{R} \) be the projections \( T^{\ast} \hat{G} \to \hat{\mathfrak{g}}^{*} \), corresponding to the left and right trivializations of the cotangent bundle.

We can identify \( (\text{Conn})^{l} \) with a subspace in \( \hat{\mathfrak{g}}^{*} \) as usual. Now, let us lift \( \tilde{f} \) to \( \tilde{f} : \hat{\mathfrak{g}}^{*} \to \mathbb{C} \) and set \( \tilde{H} = \tilde{f} \circ \tilde{p}^{L} \). Applying Lemma 3 again we reduce the theorem to the following statement:

\[
\tilde{p}^{L} \text{ is a Poisson map, } \tilde{p}^{R}_{\tilde{H}} v_{\tilde{H}} = 0.
\]
This is true for any Lie group $K$: the left projection $T^*K \to \text{Lie}(K)^*$ is Poisson; hamiltonians that factor through this projection preserve the leaves of the right trivialization of $T^*K$.

6. The symplectic structure on $\text{Conn}_U$ via hypercohomology

In this section we give a rigorous proof of the part (a) of Theorem 2.

6.1. Tangent Space to $\text{Conn}_U$. The following presentation of the tangent space to the stack $\text{Conn}$ is well known:

$$T_{(E,\nabla)} \text{Conn} = \mathbb{H}^1(X, \text{ad} E \xrightarrow{\text{ad} \nabla} \text{ad} E \otimes \Omega(-\mathcal{D})).$$

Here $\omega = \Omega^1(X)$ is the canonical bundle on $X$. We are going to use not the formula above but its version for the tangent space to $\text{Conn}_U$ at $(E,\nabla,\eta)$. Denote by $\text{ad}(E,\eta) \subset \text{ad} E$ the sheaf of infinitesimal automorphisms of $E$ preserving $\eta$. Its stalk at $x \notin \text{Supp} \mathcal{D}$ coincides with the one of $\text{ad} E$, while its stalk at $x_i \in \text{Supp} \mathcal{D}$ is (non-canonically) isomorphic to the set of loops of the form

$$g_0 + g_1 z + g_2 z^2 + \ldots,$$

where $g_j \in \mathfrak{u}$ for $0 \leq j < n_i$. Denote by $\text{Higgs}(E,\eta)$ the sheaf of $\text{ad} E$-valued 1-forms with polar part bounded by $\mathcal{D}$ that are compatible with $\eta$.

Proposition 4. There is a canonical isomorphism:

$$T_{(E,\nabla,\eta)} \text{Conn}_U \cong \mathbb{H}^1(X, \text{ad}(E,\eta) \xrightarrow{\text{ad} \nabla} \text{Higgs}(E,\eta)).$$

Proof. Consider an affine open cover $\mathcal{U}_\alpha$ of $X$ such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \text{Supp} \mathcal{D} = \emptyset$ for all $\alpha \neq \beta$. Consider a section $s_\alpha$ of $E$ over each of $\mathcal{U}_\alpha$ such that $s_\alpha$ agrees with $\eta$. The transition functions between $s_\alpha$ form a $G$-valued 1-cocycle $\phi_{\alpha\beta}$. In the trivialization $s_\alpha$ the connection $\nabla$ becomes a $\mathfrak{g}$-valued 1-form on $\mathcal{U}_\alpha$, denote it by $\theta_\alpha$. The pair $(\phi_{\alpha\beta},\theta_\alpha)$ determines the triple $(E,\nabla,\eta)$ up to an isomorphism.

Denote the complex in (21) by $K^*$ and consider its Czech resolution with respect to $\mathcal{U}_\alpha$:

$$\begin{array}{c}
0 \rightarrow \text{ad}(E,\eta) \xrightarrow{\text{ad} \nabla} \text{Higgs}(E,\eta) \rightarrow 0 \rightarrow \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow C^0(\text{ad}(E,\eta)) \rightarrow C^1(\text{ad}(E,\eta)) \oplus C^0(\text{Higgs}(E,\eta)) \rightarrow C^2(\text{ad}(E,\eta)) \oplus C^1(\text{Higgs}(E,\eta)) \rightarrow 
\end{array}$$

The complex at the bottom is the cone of the morphism

$$\text{ad} \nabla : C^*(\text{ad}(E,\eta)) \rightarrow C^*(\text{Higgs}(E,\eta)).$$

Suppose that we have an infinitesimal deformation

$$\phi_{\alpha\beta} \mapsto \phi_{\alpha\beta} \exp(\varepsilon \psi_{\alpha\beta}), \quad \theta_\alpha \mapsto \theta_\alpha + \varepsilon \nu_\alpha.$$

Changing the trivializations $s_\alpha$ we can check by a direct computation that the pair $(\psi_{\alpha\beta},\nu_\alpha)$ is naturally identified with an element of

$$C^1(\text{ad}(E,\eta)) \oplus C^0(\text{Higgs}(E,\eta)).$$

It is also easy to see that the compatibility condition on intersections is equivalent to $(\psi_{\alpha\beta},\nu_\alpha)$ being a cocycle of $K^*$. To conclude the proof of the proposition it
remains to show that two cocycles give isomorphic deformations iff they differ by a coboundary. This can be also done by a direct computation. □

6.2. Smoothness of $\text{Conn}_U$. It is a standard fact, that the obstruction to smoothness of $\text{Conn}_U$ is in $\mathbb{H}^2(X, \mathcal{K})$. Note that $\mathcal{K}^*$ is a self-dual complex, thus by the Grothendieck duality $\mathbb{H}^2(X, \mathcal{K}^*)$ is dual to $\mathbb{H}^0(X, \mathcal{K}^*)$. The latter space vanishes. Indeed, framed connections have no automorphisms and there is an algebraic map $\nu : \text{Conn}_U \to \text{Conn}_H$. Thus $\text{Conn}_U$ is a smooth algebraic space.

6.3. The symplectic structure on $\text{Conn}_U$. Since $\mathcal{K}^*$ is a self-dual complex, the Grothendieck duality gives a non-degenerate 2-form $\varpi$ on $\text{Conn}_U$. We need to check that this 2-form is closed. Let us write the explicit formulae first. Suppose that we have two tangent vectors represented by cocycles

\[ (\psi^i_{\alpha \beta}, \nu^i_{\alpha}) \in C^1(\text{ad}(E, \eta)) \oplus C^0(\text{Higgs}(E, \eta)), \quad i = 1, 2. \]

Then the value of the symplectic form on these vectors is

\[ \text{Res}(\psi^1_{\alpha \beta} \nu^1_{\alpha} - \psi^2_{\alpha \beta} \nu^1_{\alpha}), \]

where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form, $\text{Res} : H^1(X, \omega) \to \mathbb{C}$ is the natural isomorphism.

Let $v_x, v_y, v_z$ be in $T_{(x,y,z)}\text{Conn}_U$, we need to check that $d\varpi(v_x, v_y, v_z) = 0$. Since $\text{Conn}_U$ is smooth, we can find a map $F : \text{Spf}[[x, y, z]] \to \text{Conn}_U$ such that $F_*(\frac{d}{dx}) = v_x, F_*(\frac{d}{dy}) = v_y, F_*(\frac{d}{dz}) = v_z$ (Spf states for formal spectrum). It is enough to show that

\[ d(F^*\varpi)(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}) = 0 \]

at the unique closed point of $\text{Spf}[[x, y, z]]$. Let us calculate $F^*\varpi$ in the coordinates $x, y, z$. The map $F$ corresponds to a family of connections parameterized by $\text{Spf}[[x, y, z]]$. We can write it by a cocycle as before:

\[ C = (\psi^\alpha_\beta + \psi^\alpha_\beta x + \psi^\alpha_\beta y + \psi^\alpha_\beta z + \psi^\alpha_\beta xy + \psi^\alpha_\beta yz + \psi^\alpha_\beta xz + \ldots, \]

\[ \nu^\alpha + \nu^\alpha y x + \nu^\alpha y z + \nu^\alpha x y + \nu^\alpha x y + \nu^\alpha x z + \nu^\alpha y z + \nu^\alpha x y + \nu^\alpha y z + \nu^\alpha x z + \ldots \]

(we omit irrelevant terms). Consider the vector field $\frac{d}{dx}$ on $\text{Spf} \mathbb{C}[[x, y, z]]$; it gives rise to a vector field on $\text{Conn}_U$ along $\text{Spf} \mathbb{C}[[x, y, z]]$, i.e. a map $\text{Spf} \mathbb{C}[[x, y, z]] \times \mathbb{I} \to \text{Conn}_U$ given by the following cocycle:

\[ C + \varepsilon(\psi^\alpha_\beta + \psi^\alpha_\beta xy + \psi^\alpha_\beta xz + \ldots, \nu^\alpha + \nu^\alpha xy y + \nu^\alpha x z + \ldots). \]

We have similar vector fields for $\frac{d}{dy}$ and $\frac{d}{dz}$. Thus, up to irrelevant terms we have

\[ F^*\varpi\left(\frac{d}{dy}, \frac{d}{dz}\right) = \varpi((\psi^\alpha_\beta + \psi^\alpha_\beta y + \psi^\alpha_\beta z, \nu^\alpha + \nu^\alpha y + \nu^\alpha z), (\psi^\alpha_\beta + \psi^\alpha_\beta y + \psi^\alpha_\beta z, \nu^\alpha y + \nu^\alpha x y + \nu^\alpha y z)). \]

We see that

\[ \left. \frac{d}{dz}\right|_{x=y=z} F^*\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right) = \text{Res}(\psi^\alpha_\beta - \psi^\alpha_\beta, \nu^\alpha - \nu^\alpha). \]

It remains to write out two similar expressions obtained from this by a cyclic permutation of $x, y$ and $z$ and add them up. They add up to zero, this concludes the proof of Proposition 2.
6.4. Isostokes vectors. Using the description of the tangent space above, it is easy to see that \( v \in T(E, \nabla, \eta) \) is an isostokes vector iff it is in the kernel of the natural map

\[
H^1(X, \mathcal{K}^\bullet) \rightarrow H^1(\hat{X}, \mathcal{K}^\bullet) = H^1(X, j_* j^* \mathcal{K}^\bullet),
\]

where \( j : \hat{X} \hookrightarrow X \) is the natural embedding. One may think about this as about the tangent map to the restriction map from connections on \( X \) to connections on \( \hat{X} \).

Clearly,

\[
j_* j^* \mathcal{K}^\bullet = \lim_{N \rightarrow +\infty} \mathcal{K}^\bullet \otimes O(N\mathcal{D}).
\]

Thus the space of isostokes vectors can be written as

\[
(22) \lim_{N \rightarrow +\infty} \ker(H^1(X, \mathcal{K}^\bullet) \rightarrow H^1(X, \mathcal{K}^\bullet \otimes O(N\mathcal{D}))).
\]

6.5. Hamiltonians that factor through \( IT_U \). Let \( H : \text{Conn}_U \rightarrow \mathbb{C} \) be a function that factors through \( IT_U \). Then \( dH |_{E, \nabla, \eta} \) vanishes on the corresponding vertical subspace \( T_0 \subset T(E, \nabla, \eta) \text{Conn}_U \). We want to describe this subspace \( T_0 \) in terms of hypercohomology. Let

\[
\mathcal{D}' = \sum_{i : n_i \geq 2} n_i x_i
\]

be the irregular part of \( \mathcal{D} \). We claim that

\[
(23) T_0 = \lim_{N \rightarrow +\infty} \text{Image}(H^1(X, \mathcal{K}^\bullet \otimes O(-N\mathcal{D}')) \rightarrow H^1(X, \mathcal{K}^\bullet)).
\]

Indeed, suppose that \( (\psi_{\alpha\beta}, \nu_\alpha) \in T_0 \). Look at the Launt expansion of \( \theta_\alpha + \varepsilon \eta \) at an irregular point \( x_i \in U_\alpha \). This expansion can be viewed as a tangent vector to \( \text{Conn}_U^{\mathcal{D}} \), and this vector is tangent to an orbit of \( G_{\mathcal{D}} \), by the definition of \( T_0 \).

It follows that for all \( N \) there is an infinitesimal gauge transformation \( Z \) such that \( (1) \) \( Z \) is defined on \( U_\alpha \), \( (2) \) \( Z \) preserves \( \eta \) and \( (3) \) \( \text{ad}_Z \nu_\alpha \) vanishes up to the order \( N \) at \( x_i \). In other words, we can make \( \nu_\alpha \) vanish up to any order at irregular points by adding a coboundary to \( (\psi_{\alpha\beta}, \nu_\alpha) \). And the claim follows.

6.6. End of the proof of Theorem 2. It remains to prove that the space \( 22 \) contains the symplectic complement to the space \( 22 \). Since \( \mathcal{D} > \mathcal{D}' \), it is enough to show that

\[
\ker(H^1(X, \mathcal{K}^\bullet) \rightarrow H^1(X, \mathcal{K}^\bullet \otimes O(N\mathcal{D})))
\]

is the symplectic complement to

\[
\text{Image}(H^1(X, \mathcal{K}^\bullet \otimes O(-N\mathcal{D})) \rightarrow H^1(X, \mathcal{K}^\bullet)).
\]

for all \( N > 0 \). Let \( i : \mathcal{K}^\bullet \hookrightarrow \mathcal{K}^\bullet \otimes O(N\mathcal{D}) \) be the natural inclusion. Our statement follows from the functoriality of the Grothendieck duality and the following commutative diagram

\[
(K^\bullet \otimes O(N\mathcal{D}))^{op} \otimes \omega \xrightarrow{\text{op} \otimes \text{Id}} (K^\bullet)^{op} \otimes \omega \\
\downarrow \simeq \downarrow \simeq \\
K^\bullet \otimes O(-N\mathcal{D}) \longrightarrow K^\bullet
\]

The bottom map is the natural inclusion. This concludes the proof of the first part of the theorem.
7. Dimensions and an explicit construction of hamiltonians

In this section we shall complete the proof of Theorem 2.

**Proposition 5.** Every element of $\text{Conn}_B^B/G^U_n$ has a unique representative in $\text{Conn}_B^B$ of the form

$$d + \alpha_n \frac{dz}{z^n} + \ldots + \alpha_1 \frac{dz}{z} + \beta_0 dz + \ldots + \beta_{n-2} z^{n-2} dz,$$

where $\alpha_i, \beta_i \in \mathfrak{h}$ for all $i$.

**Proof.** Take an element of $\text{Conn}_B^B/G^U_n$, let

$$d + \alpha_n \frac{dz}{z^n} + \ldots + \alpha_1 \frac{dz}{z} + \beta_0 dz + \beta_1 z dz + \ldots$$

be any of its representatives. We already have $\alpha_i \in \mathfrak{b}$ for all $i$. We can put $\alpha_n$ into $\mathfrak{h}$ by the gauge action of a constant loop $g \in U$.

Further, $[\alpha_n, u] = u$. It allows to put all the polar part of the connection into $\mathfrak{h}$ by the gauge action of an appropriate loop of the form

$$\exp(u_1 z + u_2 z^2 + \ldots + u_{n-1} z^{n-1}),$$

where $u_i \in \mathfrak{u}$ for all $i$. Since $[\alpha_n, \mathfrak{g}] + \mathfrak{h} = \mathfrak{g}$, we can put the terms of positive order into $\mathfrak{h}$, using the gauge action of a loop of the form

$$\exp(g_n z^n + g_{n+1} z^{n+1} + \ldots).$$

To kill the terms of order higher than $n-2$, we use the appropriate loop of the form

$$\exp(h_n z^n + h_{n+1} z^{n+1} + \ldots),$$

where all $h$’s are in $\mathfrak{h}$. This proves the existence part of the proposition. We leave the uniqueness to the reader. \qed

**Corollary.** (a) For $n > 1$ we have an isomorphism of varieties:

$$\text{Conn}_B^B/G^U_n \approx \mathfrak{h}^r \times (\mathfrak{h})^{2n-2}.$$

(b) $\dim \text{Conn}_B^B/G^U_n = (2n - 1) \text{rk} \mathfrak{g}$.

**Remark 6.** We see that the target space of $IT_U$ is affine. Thus there are a lot of global isostokes hamiltonians. These hamiltonians do not commute, since this space has a non-trivial Poisson structure. It would be interesting to construct commuting hamiltonians.

**Proposition 6.** The symplectic leaves of $\text{Conn}_B^B/G^U_n$ are given by $\alpha_1 = \text{const}$, see (24).

**Proof.** Let $A$ be an element of $\text{Conn}_B^B/G^U_n$ whose representative $\tilde{A}$ is given by (24) (we use Proposition 5). We shall calculate the tangent space to the symplectic leaf containing (24) in $\text{Conn}$. The Poisson structure on $\text{Conn}$ comes from the immersion $\text{Conn} \hookrightarrow \mathfrak{g}^*$. Thus the tangent space to the symplectic leaf at $\tilde{A}$ is given by $\text{ad}_\mathfrak{g} A$. It is easy to see that it consists of exactly those $v \in T_{\tilde{A}} \text{Conn}$ whose residue has no diagonal part (notice that $T_{\tilde{A}} \text{Conn} = \mathfrak{g}(\langle z \rangle)^*)$).

Now, unwinding the definition of the hamiltonian reduction we obtain the required statement. \qed
Now we can prove the part (b) of Theorem 2. Consider the map

\[ \varphi_n : \text{Conn}_n^B / G_n^U \to \mathfrak{h}^r \times (\mathfrak{h})^{n-2} \]

that assigns the \((n-1)\)-tuple \((\alpha_n, \ldots, \alpha_2)\) to \((24)\). Set

\[ \varphi = \prod_{i:n_i \geq 2} \varphi_{n_i} \]

(this is the right vertical arrow in \([\mathbf{1}]\)). We see that for every tangent vector \(v\) to the target of \(IT\) at \(\varphi(A)\) there is a hamiltonian \(f\) on the target space of \(IT_U\) such that \(\varphi_*(vf|A) = v\), where \(vf\) is the hamiltonian vector field corresponding to \(f\).

Taking \(H = f \circ IT_U\) one completes the proof of Theorem 2.

7.1. ‘Stupid’ hamiltonians. Some of isostokes hamiltonians, produced by Theorem 2, are ‘stupid’: they do not change irregular types of connections but the unipotent structures only (see the diagram \([\mathbf{5}]\)). Here we shall describe these hamiltonians. According to Proposition 5, we can consider \(\alpha_i\) for \(i = 1, \ldots, n\) and \(\beta_i\) for \(i = 0, \ldots, n-2\) as the coordinates on \(\text{Conn}_n^B / G_n^U\). Define a coordinate \(\alpha_j^i\) on the target of \(IT_U\) as the composition of \(\alpha_i\) and the projection to the \(j\)-th multiple. Similarly we define \(\beta_j^i\).

**Proposition 7.** \(f \circ IT_U\) is a stupid hamiltonian iff \(f\) does not depend on \(\beta_j^i\).

**Proof.** Clearly, it suffices to prove that a hamiltonian \(f : \text{Conn}_n^B / G_n^U \to \mathbb{C}\) satisfies \(\varphi, vf = 0\) iff it does not depend on \(\beta_j^i\)’s.

Notice first, that

\[ \{\alpha_i, \alpha_j\} = 0 \]

for all \(i\) and \(j\), this easily follows from the presentation of \(\text{Conn}_n^B / G_n^U\) as a hamiltonian reduction of \(\text{Conn}\) (compare with the proof of Proposition \([\mathbf{3}]\)). Thus \(\varphi\) is a lagrangian fibration and the claim follows. \(\square\)

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