Higher direct images of ideal sheaves

Charles Godfrey
Pacific Northwest National Laboratory
charles.godfrey@pnnl.gov

July 5, 2022

Abstract

We prove invariance results for the cohomology groups of ideal sheaves of simple normal crossing divisors under (a restricted class of) birational morphisms of pairs in arbitrary characteristic. As an application, we extend some foundational results in the theory of rational pairs that were previously known only in characteristic 0.

Contents

1 Introduction 1
  1.1 Overview .................................................. 4
  1.2 Acknowledgements ................................... 6

2 Regular sequences of divisors and descent spectral sequences 6
  2.1 Semi-simplicial schemes and their derived categories ............. 6
  2.2 Regular sequences of divisors ................................ 8
  2.3 Resolving sheaves of log-differentials .......................... 10
  2.4 Replacing the ideal sheaf with a filtered complex ................. 12

3 Simple normal crossing divisors and thriftness 13
  3.1 Definitions and basic properties ................................ 13
  3.2 The regular-to-regular case ................................ 16

4 Constructing semi-simplicial projective Macaulayfications 16
  4.1 Preliminaries .............................................. 16
  4.2 Gluing on simplices ....................................... 17
  4.3 Common admissible blowups .................................. 20
  4.4 Invariance results for cohomology of snc ideal sheaves ........... 22

5 Applications to rational pairs 24
  5.1 Rational resolutions of pairs are all-for-one .................... 25
  5.2 Semi-simplicial versus thrifty resolutions ....................... 25

A (Non-)examples of thrift 30

1 Introduction

A foundational problem in birational geometry, posed by Grothendieck in his 1958 ICM address [Gro60, Problem B], asked whether for every proper birational morphism of non-singular projective varieties $f : X \to Y,$

$$R^if_*\mathcal{O}_X = 0$$

for $i > 0.$
In characteristic 0 this was answered affirmatively by Hironaka as a corollary of resolution of singularities [Hir64, §7 Cor. 2]. In characteristic $p > 0$, where resolutions of singularities are not known to exist, answering Grothendieck’s question proved much harder, remaining open until 2011 when Chatzistamatiou and Rülling proved the following theorem.

**Theorem 1.1** ([CR11, Thm. 3.2.8], see also [CR15, Thm. 1.1] [Kov20, Thm. 1.6]). Let $k$ be a perfect field and let $S$ be a separated scheme of finite type over $k$. Suppose $X$ and $Y$ are two separated finite type $S$-schemes which are

(i) smooth over $k$ and

(ii) **properly birational** over $S$ in the sense that there is a commutative diagram

$$
\begin{array}{c}
Z \\
\downarrow r \\
X \\
\downarrow f \\
S \\
\uparrow g \\
Y \\
\uparrow s
\end{array}
$$

with $r$ and $s$ proper birational morphisms.

Let $n = \dim X = \dim Y = \dim Z$. Then, there are isomorphisms of sheaves

$$R^i f_* \mathcal{O}_X \sim\sim R^i g_* \mathcal{O}_Y$$

and

$$R^i f_* \omega_X \sim\sim R^i g_* \omega_Y$$

for all $i$.

One of the primary applications of Theorem 1.1 was to extend foundational results on rational singularities from characteristic 0 to arbitrary characteristic (for definitions of rational resolutions and rational singularities see **Definition 5.2**).

**Corollary 1.4** ([CR11, Cor. 3.2.10], [Kov20, Thm. 1.4]). If $S$ has a rational resolution, then every resolution of $S$ is rational.

In this article, we prove analogues of Theorem 1.1 for pairs.

**Definition 1.5** (slightly more general version of [Kol13, Def. 1.5]). In what follows a **pair** $(X, \Delta_X)$ will mean a reduced, equidimensional excellent scheme $X$ admitting a dualizing complex together with a Q-Weil divisor $\Delta_X = \sum_i a_i D_i$ on $X$ such that no irreducible component $D_i$ of $\Delta_X$ is contained in $\text{Sing}(X)$.

**Definition 1.6.** A **simple normal crossing pair** is an equidimensional, regular excellent scheme $X$ together with a reduced effective divisor $\Delta_X = \sum_i D_i$ such that for every subset $J \subseteq \{1, \ldots, N\}$ the scheme-theoretic intersection

$$D_J := \cap_{j \in J} D_j \subseteq X$$

is regular of codimension $|J|$.

**Remark 1.7.** If $X$ is regular as in **Definition 1.6** then it admits a dualizing complex. By an amazing result of Kawasaki [Kaw02, Cor. 1.4], a noetherian ring admits a dualizing complex if and only if it is a homomorphic image of a finite-dimensional Gorenstein ring.

As observed in [Kol13, §2.5], to generalize **Corollary 1.4** to pairs we must restrict attention to a special class of **thifty resolutions**.

**Definition 1.8.** A **stratum** of a simple normal crossing pair $(X, \Delta_X = \sum_i D_i)$ is a connected (equivalently, irreducible) component of an intersection $D_J = \cap_{j \in J} D_j$.

**Definition 1.9** (compare with [Kol13, Def. 2.79-2.80], [KX16, §1, discussion before Def. 10]). Let $(S, \Delta_S = \sum_i D_i)$ be a pair, and assume $\Delta_S$ is reduced and effective. A separated, finite type birational morphism $f : X \to S$ is **thifty with respect to $\Delta_S$** if and only if

(i) $f$ is an isomorphism over the generic point of every stratum of $\text{snc}(S, \Delta_S)$ and
We hope to show that this is a natural approach — for example, every pair $(X, \Delta_X)$ has a naturally associated semi-simplicial scheme categorifying its set of strata. We emphasize that all of the semi-simplicial schemes $X$, we consider have strong finite-dimensionality properties ($X_i = \emptyset$ for $i > 0$). Some of our machinery could be of interest even in characteristic 0; for example, we obtain a criterion for a pair $(S, \Delta_S)$ to have rational singularities in terms of a resolution of the associated semi-simplicial scheme $S_*$.

(ii) letting $D_i = f_s^{-1} D_i$ for $i = 1, \ldots, N$ be the strict transforms of the divisors $D_i$, and setting $\Delta_X := \sum_i D_i$, the map $f$ is an isomorphism at the generic point of every stratum of $\text{sn}c(X, \Delta_X)$.

Philosophically, a resolution $(X, \Delta_X) \to (S, \Delta_S)$ is thrifty if it induces a one-to-one birational correspondence between the strata of $(X, \Delta_X)$ and $(S, \Delta_S)$, thus preserving the combinatorial geometry of the pair $(S, \Delta_S)$.

**Theorem 1.10 (Theorem 4.21).** Let $S$ be an excellent noetherian scheme and let $(X, \Delta_X)$ and $(Y, \Delta_Y)$ be simple normal crossing pairs separated and of finite type over $S$. Suppose $(X, \Delta_X), (Y, \Delta_Y)$ are properly birational over $S$ in the sense that there is a commutative diagram

$$
\begin{array}{ccc}
(Z, \Delta_Z) & \xrightarrow{r} & (X, \Delta_X) \\
\downarrow & & \downarrow f \downarrow \uparrow \downarrow \uparrow s \\
(Y, \Delta_Y) & \xleftarrow{g} & (S, \Delta_S)
\end{array}
$$

(1.11)

where $r$, $s$ are proper and birational morphisms, and assume $\Delta_Z = r_*^{-1} \Delta_X = s_*^{-1} \Delta_Y$. If $r$ and $s$ are thrifty, then there are quasi-isomorphisms

$$
Rf_* \omega_X(-\Delta_X) \simeq Rg_* \omega_Y(-\Delta_Y) \quad \text{and} \quad Rf_* \omega_X(\Delta_X) \simeq Rg_* \omega_Y(\Delta_Y).
$$

(1.12)

In order to state an analogue of **Corollary 1.4**, we say what we mean by rational singularities of pairs.

**Definition 1.13** (compare with [Kol13, Def. 2.78]). Let $(S, \Delta_S)$ be a pair as in **Definition 1.5** and assume $\Delta_S$ is reduced and effective. A proper birational morphism $f : X \to S$ is a rational resolution if and only if

(i) $X$ is regular and the strict transform $\Delta_X := f_*^{-1} \Delta_S$ has simple normal crossings,

(ii) the natural morphism $\mathcal{O}_S(-\Delta_S) \to Rf_* \mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, and letting $\omega_X = h^{-\dim X} \omega_X^*$, where we use $\omega_X^* = f^! \omega_S^*$ as a normalized dualizing complex on $X$,

(iii) $R^if_* \omega_X(\Delta_X) = 0$ for $i > 0$.

The pair $(S, \Delta_S)$ is resolution-rational if and only if it has a thrifty rational resolution.

We wish to emphasize that this is not the only definition of rational pairs available in the literature: Schwede and Takagi adopted a different definition in [ST08]. Here we focus on the variant of rational pairs defined in [Kol13, §2.5], simply because it is the one to which our methods most directly apply, however we view identifying and studying a notion of rational pairs that simultaneously generalizes those of [ST08] and [Kol13] as an interesting question for future work. With **Definition 1.13** in hand, the precise statement of our result is:

**Theorem 1.14** ([Kol13, Cor. 2.86] in characteristic 0, **Lemma 5.6** in arbitrary characteristic). Let $(S, \Delta_S)$ be a pair, with $\Delta_S$ reduced and effective. If $(S, \Delta_S)$ has a thrifty rational resolution $f : (X, \Delta_X) \to (S, \Delta_S)$, then every thrifty resolution $g : (Y, \Delta_Y) \to (S, \Delta_S)$ is rational.

To prove **Theorem 1.10** and **Theorem 1.14** we take a small step outside of the category of schemes, to that of semi-simplicial schemes (for their definitions and basic properties we refer to **section 2**). We wish to show that this is a natural approach — for example, every pair $(X, \Delta_X)$ has a naturally associated semi-simplicial scheme categorifying its set of strata. We emphasize that all of the semi-simplicial schemes $X$, we consider have strong finite-dimensionality properties ($X_i = \emptyset$ for $i > 0$). Some of our machinery could be of interest even in characteristic 0; for example, we obtain a criterion for a pair $(S, \Delta_S)$ to have rational singularities in terms of a resolution of the associated semi-simplicial scheme $S_*$.
**Lemma 1.15 (Lemma 5.7).** Let \((S, \Delta_S)\) be a pair, with \(\Delta_S\) reduced and effective. Let \(\varepsilon^S : S_\ast \rightarrow S\) be the associated augmented semi-simplicial scheme and suppose

\[
\begin{array}{ccl}
X_\ast & \xrightarrow{f} & S_\ast \\
\downarrow \varphi^X & & \downarrow \varepsilon^S \\
X & \xrightarrow{f} & S
\end{array}
\]

is a resolution. Then, \((S, \Delta_S)\) is a rational pair if and only if the sheaf \(\Theta_S(-\Delta_S)\) is Cohen-Macaulay and the natural map \(\Theta_S(-\Delta_S) \rightarrow Rf_*\mathcal{K}\) is a quasi-isomorphism, where \(\mathcal{K}\) is naturally defined complex on \(X\).

A benefit of this lemma is that resolutions of the semi-simplicial scheme \(S_\ast\), are arguably easier to construct than thrifty resolutions of pairs: in [Kol13, Thm. 10.45], the existence of thrifty resolutions was proved using the refined log-resolution theorems of [BM97; Sza94]. In contrast, resolving semi-simplicial schemes requires only standard resolution of singularities together with elementary algorithms for inductively constructing semi-simplicial schemes (quite similar to the constructions of hyper-resolutions in the theory of Du Bois singularities [Del71; Del74; Du 81]). Of course, the question of which route is easier is subjective, but in light of Lemma 1.15 we think an overarching "slogan" of this work is:

**Slogan 1.17.** Rationality of a pair \((S, \Delta)\) can be tested using any resolution \(X_\ast \rightarrow S_\ast\) of a semi-simplicial stratification \(S_\ast\) of \(S\) coming from closures of strata of the snc locus snc\((S, \Delta)\), and thrifty resolutions simply provide one way to obtain such resolutions \(X_\ast \rightarrow S_\ast\).

### 1.1 Overview

We next provide some motivation for the appearance of semi-simplicial schemes in the context of Theorem 1.10, which will also serve to contextualize an overview of the following sections. To begin we may translate the condition that a birational morphism \(f : (X, \Delta_X) \rightarrow (S, \Delta_S)\) of simple normal crossing pairs\(^1\) with \(\Delta_X = f^{-1}_*\Delta_S\) is thrifty into the statement that the dual complexes \(\mathcal{D}(\Delta_X)\) and \(\mathcal{D}(\Delta_S)\) are isomorphic. The dual complex \(\mathcal{D}(\Delta_X)\) is usually described as the \(\Delta\)-complex (in the sense of [Hat02, §2.1]) with 0-cells the irreducible components \(D^X_i \backslash D^X_j\) of \(\Delta_X = \bigcup_i D^X_i\), 1-cells the components of intersections \(D^X_i \cap D^X_j\) for \(i < j\) with gluing maps corresponding to the inclusions \(D^X_i \cap D^X_j \subseteq D^X_j \cup D^X_i\), and so on — in terms of Definition 1.8, the cells of \(\mathcal{D}(\Delta_X)\) correspond to strata of \((X, \Delta_X)\), with gluing maps corresponding to inclusions of strata. The topological properties of \(\mathcal{D}(\Delta_X)\) have been extensively studied, for example in this non-exhaustive list of references: [ABW13; Dan75; FKKX17; Ste06]. Upon inspection we see that a \(\Delta\)-complex is precisely a semi-simplicial set, and that \(\mathcal{D}(\Delta_X)\) is the semi-simplicial set obtained from \(\pi_0(\text{connected components})\) of a semi-simplicial scheme \(X_\ast\), with

\[
X_1 = \prod_{|J| = i + 1} (\cap J \setminus D^X_j) \text{ for } i \geq 0
\]

The thriftyness hypotheses of Theorem 1.10 ensure that \((X, \Delta_X)\) and \((Y, \Delta_Y)\) have the same dual complex, which provides enough rigidity to attempt to prove Theorem 1.10 by induction on \(\dim X\) and the number of components of \(\Delta_X\), using Theorem 1.1 as a base case. For example, we have exact sequences

\[
0 \rightarrow \Theta_X(-\Delta_X) \rightarrow \Theta_X(-\Delta_X + D^X_1) \rightarrow \Theta_{D^X_1}(\Delta_X + D^X_1|D^X_2) \rightarrow 0
\]

---

\(^1\)What exactly “resolution” means will be made precise in Lemma 5.7

\(^2\)Again we defer to Lemma 5.7 for further details.

\(^3\)We could relax the condition that both pairs are snc, but it will make this motivational discussion simpler.
and similarly on $Y$. We can even assume by induction the existence of already-defined quasi-isomorphisms in a diagram

$$
\begin{align*}
Rf_*\mathcal{O}_X(-\Delta_X) & \longrightarrow Rf_*\mathcal{O}_X(-\Delta_X + D_1^X) \xrightarrow{\rho^X} Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{align*}
\tag{1.18}
Rg_*\mathcal{O}_Y(-\Delta_Y) \xrightarrow{\rho^Y} Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y|_{D_1^Y}) \longrightarrow \cdots
$$

If the square $(\ast)$ commutes, then using only the fact that $D^b_{coh}(S)$ is a triangulated category we get a quasi-isomorphism $\alpha$ on the dashed arrow. However, in this approach $\beta, \gamma$ are themselves defined by induction, and so to know $(\ast)$ commutes we must take one inductive step further, considering maps of distinguished triangles

$$
\begin{align*}
Rf_*\mathcal{O}_X(-\Delta_X + D_1^X) & \longrightarrow Rf_*\mathcal{O}_X(-\Delta_X + D_1^X + D_2^X) \longrightarrow Rf_*\mathcal{O}_{D_2^X}(-\Delta_X + D_1^X|_{D_2^X}) \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{align*}
\quad \quad \text{and}
\begin{align*}
Rg_*\mathcal{O}_Y(-\Delta_Y + D_1^Y) & \longrightarrow Rg_*\mathcal{O}_Y(-\Delta_Y + D_1^Y + D_2^Y) \longrightarrow Rg_*\mathcal{O}_{D_2^Y}(-\Delta_Y + D_1^Y|_{D_2^Y}) \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{align*}
\tag{1.19}
Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) \longrightarrow Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X + D_2^X|_{D_1^X}) \longrightarrow Rf_*\mathcal{O}_{D_1^X|_{D_2^X}}(-\Delta_X + D_1^X + D_2^X|_{D_1^X|_{D_2^X}}) \longrightarrow \cdots
\tag{1.20}
Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y|_{D_1^Y}) \longrightarrow Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y + D_2^Y|_{D_1^Y}) \longrightarrow Rg_*\mathcal{O}_{D_1^Y|_{D_2^Y}}(-\Delta_Y + D_1^Y + D_2^Y|_{D_1^Y|_{D_2^Y}}) \longrightarrow \cdots
$$

Together with a map from (1.19) to (1.20) including the square $(\ast)$, and so on. It is certainly possible that the correct induction hypothesis (building in not only quasi-isomorphisms like $\beta, \gamma$ in (1.18) but also commutativity hypotheses) and some careful analysis of diagrams in $D^b_{coh}(S)$ could make this strategy work, but the author had no such luck. A separate technical issue the above approach encounters is that at some point in the base case, we must analyze how the isomorphisms of Theorem 1.1 behave with respect to restrictions, i.e. diagrams of schemes like

$$
\begin{align*}
D^Y_1 & \longrightarrow D^Y_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
X & \longrightarrow Z & \longrightarrow Y
\end{align*}
$$

Delving into the methods of [CR11; CR15; Kov20], this analysis runs into subtle aspects of Grothendieck duality, especially since for this approach to work we do require morphisms in $D^b_{coh}(S)$, not simply of cohomology sheaves as in Theorem 1.1.

Despite the aforementioned technical issues, what is clear is that this attempted induction takes place on the semi-simplicial schemes $X_*$ and $Y_*$, underlying the dual complexes $\mathcal{D}(\Delta_X)$ and $\mathcal{D}(\Delta_Y)$. Under necessary thrustiness hypotheses, in the situation of Theorem 1.10 we find that there is also an auxiliary semi-simplicial scheme $Z_*$, together with morphisms $X_* \xrightarrow{\eta} Z_* \xrightarrow{\delta} Y_*$ which are birational in each simplicial degree. Using the refined forms of Chow’s lemma and resolution of indeterminacies from Conrad’s article on Deligne’s notes on Nagata compactifications [Con07], together with Kawasaki’s proof the existence of Macaulayficaitions [Kaw00] and its strengthening by Česnavičius [Čes21], we can prove the existence of such a $Z$, where each scheme $Z_i$ is Cohen-Macaulay and the morphisms $X_i \xrightarrow{\iota_i} Z_i \xrightarrow{\delta_i} Y_i$ are projective — this occupies sections 3 and 4. We then make essential use of recent work of Kovács [Kov20, Thm. 1.4] to conclude that there are natural maps $\mathcal{O}_{X_i} \rightarrow R\iota_{i!}\mathcal{O}_{Z_i}$ and $\mathcal{O}_{Y_i} \rightarrow R\delta_{i!}\mathcal{O}_{Z_i}$ are quasi-isomorphisms for all $i$. A more detailed overview of this construction is included at the beginning of section 4.
The remainder of our proof is pure homological algebra: in Section 2 we show that when \((X, \Delta_X)\) is a simple normal crossing pair (more generally, when the components \(D^i_X\) of \(\Delta_X\) form a regular sequence, see Definition 2.5) the ideal sheaf \(\mathcal{O}_X(-\Delta_X)\) admits a Čech-type resolution of the form

\[
\mathcal{O}_X(-\Delta_X) \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_1} \to \cdots,
\]

in other words we can recover \(\mathcal{O}_X(-\Delta_X)\) from an augmentation morphism \(X_0 \to X\). Moreover, we can recover the cohomology of \(\mathcal{O}_X(-\Delta_X)\) from a descent-type spectral sequence Corollary 2.18 — the last major technical ingredient is a comparison of the resulting spectral sequences associated to \(X, Y\) and \(Z\).

Section 5 deals with applications to rational pairs, in particular Theorem 1.14 and Lemma 1.15. Appendix A includes some new examples illustrating the subtleties of thrifty and rational resolutions of pairs, including an affirmative answer to a question of Erickson and Prelli on whether there exists a non-thrifty rational resolution of a pair \((S, \Delta)\) — our \((S, \Delta)\) is even a rational pair, and the resolution is related to the famous Atiyah flop.

1.2 Acknowledgements

I would like to thank Jarod Alper, Chi-yu Cheng, Kristin DeVleming, Gabriel Dorfsman-Hopkins, Max Lieblich, Takumi Murayama, Karl Schwede and Tuomas Tajakka for helpful conversations, and my advisor Sándor Kovács for proposing the problem of extending results Theorem 1.1 to pairs and for many valuable suggestions.

2 Regular sequences of divisors and descent spectral sequences

2.1 Semi-simplicial schemes and their derived categories

To any simple normal crossing pair we can naturally associate a semi-simplicial scheme. A primary reference for the theory of semi-simplicial schemes is [SGA4II, Vbis]; since many elementary facts about simplicial schemes carry over to semi-semi-simplicial schemes, [Con03], [Ols16, §2.4] and [Stacks, Tag 0162] are also relevant. What follows is a condensed summary of the machinery we need.

Let \(\Lambda\) denote the category with objects the sets \([i] := \{0, 1, 2, \ldots, i\}\) for \(i \in \mathbb{N}\) and with morphisms the strictly increasing functions \(j \rightarrow [i] \); in particular \(\text{Hom}_\Lambda([j],[i]) = \emptyset\) if \(j > i\).\footnote{In [SGA4II, Vbis] \(\Lambda\) is denoted by \(\Delta^+\) so this notation is non-standarad, but seemed necessary due to the number of divisors \(\Delta\) and pairs \((X, \Delta)\) considered below. My apologies.} A semi-simplicial object in a category \(\mathcal{C}\) is a functor \(\Lambda^{\text{op}} \to \mathcal{C}\); semi-simplicial \(\mathcal{C}\)-objects naturally form a category, the functor category \(\mathcal{C}^{\Lambda^{\text{op}}}\). Any morphism \(\varphi : [j] \to [i]\) can be written non-uniquely as a composition of the basic morphisms

\[
\delta^i_k : [i-1] \rightarrow [i] \text{ defined by } \delta^i_k(x) = \begin{cases} x & \text{if } x < k \\ x+1 & \text{otherwise} \end{cases}
\]

(so \(\delta^i_k\) skips \(k\)) [Stacks, Tag 0164], and hence a semi-simplicial object \(X : \Lambda^{\text{op}} \to \mathcal{C}\) is equivalent to a sequence of objects \(X_i := X([i])\) together with morphisms

\[
d^i_k := X(\delta^i_k) : X_i \to X_{i-1} \text{ subject to the relations } d^i_{k-1}d^i_k = d^i_{k-1}o_{k-1}d^i_k,
\]

and all semi-simplicial objects below will be obtained from such an explicit description. In what follows semi-simplicial objects will be denoted with a \(\ast\), e.g. “the semi-simplicial scheme \(X, \ast\)” (to distinguish them from plain schemes).

When \(\mathcal{C}\) is a category of schemes, a sheaf on a semi-simplicial scheme \(X\) is the data of a sheaf \(\mathcal{F}_\ast\) on each scheme \(X_i\) together with morphisms of sheaves \(\delta^i_k : \mathcal{F}_{i-1} \to d^i_{k-1} \ast \mathcal{F}_{i-1}\) on \(X_{i-1}\) satisfying...
compatibilities coming from (2.1). These sheaves form a topos \( \mathcal{X} \), such that morphisms of semi-simplicial schemes \( f : X \to Y \) induce functorial maps of topos \( \mathcal{X} \to \mathcal{Y} \), (see [SGA4II, Vbis, Prop. 1.2.15]) — the benefit of the topos-theoretic point of view is that it immediately implies the category of abelian sheaves \( \mathsf{Ab}(X) \) on \( X \) is an abelian category with enough injectives ([Stacks, Tag 01DL]), enables us to define pushforward functors \( Rf_* : D^+(\mathsf{Ab}(X)) \to D^+(\mathsf{Ab}(Y)) \) for morphisms of semi-simplicial schemes \( f : X \to Y \), and so on.

An augmented semi-simplicial scheme is a morphism of semi-simplicial schemes \( \varepsilon : X \to S \), where \( S \) is a constant semi-simplicial scheme (that is, \( S_i = S \) for all \( i \) for some fixed scheme \( S \), and all \( d_k^i = \text{id} \)). This is equivalent to the data of a semi-simplicial object of \( \mathsf{Sch}_S \). For such a constant semi-simplicial scheme \( S \), \( \mathsf{Ab}(S) \) is equivalent to the category \( \mathsf{Ab}(S)^{\Delta} \) of co-semi-simplicial sheaves of abelian groups on \( S \), that is, sequences of sheaves of abelian groups \( \mathcal{G}_i \) on \( S \) together with morphisms \( \delta^i_k : \mathcal{G}_{i-1} \to \mathcal{G}_i \) satisfying compatibilities forced by (2.1). As in the construction of the Čech complex setting \( d^i = \sum_k (-1)^k : \mathcal{G}_{i-1} \to \mathcal{G}_i \) gives a complex of abelian sheaves on \( S \) and hence in particular an abelian sheaf \( \mathcal{A}(\mathcal{G}_i) := \ker d^0 \). Writing \( \varepsilon_* := \alpha \varepsilon_* \), the composite derived functor

\[
\begin{array}{c}
R\varepsilon_* \\
\downarrow \\
D^+(\mathsf{Ab}(X)) \xrightarrow{R\varepsilon_*} D^+(\mathsf{Ab}(S)) \xrightarrow{Ra} D^+(\mathsf{Ab}(S))
\end{array}
\]

admits the following concrete description: given a sheaf \( \mathcal{F} \) on \( X \), one takes an injective resolution

\[ \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots \ \text{in} \ \mathsf{Ab}(X) \]

Here the \( \mathcal{F}^i \) are in particular sheaves on \( X \), with each \( \mathcal{F}^i \) an injective abelian sheaf on \( X \) — for further discussion of injective objects in \( \mathsf{Ab}(X) \) see [SGA4II, Vbis, Prop. 1.3.10] and [Con03, Lem. 6.4, comments on p. 42]. Then

\[ \varepsilon_* \mathcal{F}^0 \to \varepsilon_* \mathcal{F}^1 \to \varepsilon_* \mathcal{F}^2 \to \cdots \ \text{in} \ \mathsf{Ab}(S) \]

is a complex of co-semi-simplicial abelian sheaves which via the Čech construction becomes a complex of complexes. Applying the sign trick gives a double complex whose \( \text{Tot} \) computes \( R\varepsilon_* \mathcal{F} \). One of the spectral sequences of this double complex is displayed below. In our calculations it is crucial that this spectral sequence is (at least in a minimal sense) functorial.

**Lemma 2.2** (Descent spectral sequence, [SGA4II, Vbis §2.3], [Con03, Thms. 6.11-6.12]). If \( \mathcal{F} \) is an abelian sheaf on an augmented semi-simplicial scheme \( \varepsilon : X \to S \) then there is a spectral sequence

\[ E_1^{pq} = R^q \varepsilon_p \mathcal{F}_p \to R^{q+mp} \varepsilon_* \mathcal{F} \]

Moreover if \( \mathcal{G} \) is an abelian sheaf on another augmented semi-simplicial scheme \( \varepsilon' : Y \to T \) and

\[
\begin{array}{ccl}
Y & \xrightarrow{g} & X \\
\downarrow \varepsilon' & & \downarrow \varepsilon \\
\downarrow g' & & \\
S & & \end{array}
\]

is a map of augmented semi-simplicial schemes together with a map of abelian sheaves \( \varphi : \mathcal{F} \to g_* \mathcal{G} \) on \( X_* \), then \( \varphi \) induces a morphism of spectral sequences

\[ E_1^{pq}(\mathcal{F}_*) = R^q \varepsilon_p \mathcal{F}_p \to R^q (\varphi g)_p \mathcal{G}_p = E_1^{pq}(\mathcal{G}_*) \]

converging to the morphism \( R\varepsilon_* (\varphi) : R\varepsilon_* \mathcal{F} \to R\varepsilon_* Rg_* \mathcal{G} = Rg_* R\varepsilon_* \mathcal{G} \).

**Proof of the “Moreover ...”**. We work with the abelian categories of sheaves of abelian groups on \( Y_* \), \( X_* \). Let \( \mathcal{F}_* \) be an injective resolution of \( \mathcal{G}_* \) in \( \mathsf{Ab}(Y)_* \). Then \( f_* \mathcal{F}_* \) is a complex of injectives (this uses
the fact that $f_*$ has an exact left adjoint $f^{-1}$), $\mathcal{F}_* \to \mathcal{F}'$ is a quasi-isomorphism and we are given a map

$$\varphi : \mathcal{F}_* \to f_* \mathcal{G}_* \to f_* \mathcal{F}'$$

By [Stacks, Tag 013P] (see also [Wei94, Thm. 2.2.6]) there is a map of complexes of abelian sheaves on $X$, extending $\varphi$:

$$\hat{\varphi} : \mathcal{F}' \to f_* \mathcal{F}'$$

Applying $\varepsilon_*$ then gives a morphism of complexes of co-semi-simplicial abelian sheaves on $S$ consisting of morphisms

$$\varepsilon_p \mathcal{F}_p^q \to \varepsilon_p g_p \mathcal{F}_p^q$$

compatible with both the simplicial sheaf maps (in the $p$ direction) and the injective resolution maps (in the $q$ direction), to which we may apply the Čech construction and sign trick to obtain a map of double complexes. This reduces us to the claim that a map of double complexes (or more generally a filtered map of filtered complexes) induces a map of spectral sequences, which we take as well known. \qed

Remark 2.3. The above proof is at least suggested in the last sentence of [Con03, Thm. 6.11]. An alternative method would be to use Deligne’s trick of viewing $\varphi$ as an abelian sheaf on the $\Lambda \times I$ scheme associated to $f_*$, — for related discussion see [SGA4II, Vbis, §3.1].

Corollary 2.4. In the situation of Lemma 2.2 suppose in addition that the morphisms $\varphi_p : \mathcal{F}_p \to Rf_p \mathcal{G}_p$ are quasi-isomorphisms for all $p$. Then, the induced morphism

$$R\varepsilon_*(\varphi) : R\varepsilon_* \mathcal{F}_* \to R\varepsilon_* Rg_* \mathcal{G}_* = Rg_* R\varepsilon'_* \mathcal{G}'$$

is a quasi-isomorphism.

2.2 Regular sequences of divisors

Definition 2.5. Let $X$ be a locally noetherian scheme. A sequence of effective Cartier divisors $D_1, D_2, \ldots, D_N \subseteq X$ is called regular if and only if for each point $x \in X$, letting $f_1, \ldots, f_N \in \mathfrak{m}_x X$ be local generators for the divisors $D_i$ and letting $I(x) = \{i \mid x \in D_i\}$, the elements $(f_j \in \mathfrak{m}_x | j \in I(x))$ form a regular sequence.

This definition is designed to ensure that a permutation of a regular sequence of divisors is again a regular sequence (see [Mat80, §15, Thm. 27], [Stacks, Tag 00LJ]).

Let $X$ be a locally noetherian scheme together with a regular sequence of effective Cartier divisors $D_1, D_2, \ldots, D_N \subseteq X$. We define an augmented semi-simplicial scheme $X_*$ as follows: $X_{-1} = X$, $X_0 = \coprod D_i$ and for $k > 0$, $X_k = \coprod_{I \subseteq \{1, \ldots, N\} \mid |I| = k+1} D_I$, where $D_I = \bigcap_{j \in I} D_j$

The face maps are defined by the inclusions $d_i^j : D_I \to D_{I \setminus \{i\}}$ for $I = \{i_0, \ldots, i_k\}$ and $0 \leq j \leq i$, as in a Čech complex, and for each $k$ we have an augmentation map $\varepsilon_k : X_k \to X$ obtained from the inclusions $D_I \subseteq X$. In this situation the descent spectral sequence of Lemma 2.2 degenerates: since the $\varepsilon_p : X_p \to X$ are closed immersions and hence affine, $R^i \varepsilon_* \mathfrak{O}_{X_p} = 0$ for $q > 0$. It follows that $R^i \varepsilon_* \mathfrak{O}_X$, is the cohomology of the Čech type complex

$$\varepsilon_* \mathfrak{O}_{X_0} \xrightarrow{d^1} \varepsilon_* \mathfrak{O}_{X_1} \xrightarrow{d^2} \ldots \xrightarrow{d^N} \varepsilon_* \mathfrak{O}_{X_N} = \bigoplus_{I \subseteq \{1, \ldots, N\} \mid |I| = k+1} \mathfrak{O}_{D_I}$$

(2.6)
**Lemma 2.7.** The complex (2.6) is exact in degrees $i > 0$, with $\ker d^1 \cong \mathcal{O}_{\cup_i D_i}$. Equivalently, the extended complex

$$0 \to \mathcal{O}_X(- \sum_i D_i) \to \mathcal{O}_X \to \bigoplus_i \mathcal{O}_{D_i} \xrightarrow{d^1} \bigoplus_{i \neq j} \mathcal{O}_{D_i \cap D_j} \xrightarrow{d^2} \cdots \xrightarrow{d^N} \mathcal{O}_{\cap_i D_i} \to 0$$

where $\gamma : \mathcal{O}_X \to \bigoplus_i \mathcal{O}_{D_i}$ is restriction in each factor is exact, and hence there is a canonical quasi-isomorphism $\mathcal{O}_X(- \sum_i D_i) \cong \text{cone}(\mathcal{O}_X \to \mathcal{O}_{\cap_i D_i})[1]$.

**Proof.** We proceed by induction on the number $N$ of divisors. The base case $N = 0$ is vacuous ($X$ is empty). If that seems too weird, the case $N = 1$ simply says that the sequence $0 \to \mathcal{O}_X(-D_1) \to \mathcal{O}_X \to \mathcal{O}_{D_1} \to 0$ is exact, which is indeed the case as $D_1$ is an effective Cartier divisor.

Suppose now that $N > 1$. Then by the definition of a regular sequence, $D_1 \cap D_2, D_1 \cap D_3, \ldots, D_1 \cap D_N \subseteq D_1$ is a regular sequence of divisors, and by permutation invariance of regular sequences (for noetherian local rings [Mat80, §15, Thm. 27], [Stacks, Tag 00LJ] — this dictated Definition 2.5) $D_2, \ldots, D_N \subseteq X$ is a regular sequence. We form a short exact sequence of complexes (with cohomological degrees as indicated)

$$0 \to \mathcal{O}_{D_1} \xrightarrow{d'} \bigoplus_{1 < j} \mathcal{O}_{D_1 \cap D_j} \xrightarrow{d''} \bigoplus_{1 < j < k} \mathcal{O}_{D_1 \cap D_j \cap D_k} \to \cdots$$

(in fact by comparing ranges of indices we can see the columns are split short exact sequences). By inductive hypotheses,

$$h^i(C') = \begin{cases} \mathcal{O}_{D_1}(- \sum_{1 < j} D_1 \cap D_j) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h^i(C'') = \begin{cases} \mathcal{O}_X(- \sum_{1 < j} D_j) & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases}$$

showing that $h^i(C) = 0$ for $i > 0$, and that in low degrees there is an exact sequence

$$0 \to h^{-1}(C) \to \mathcal{O}_X(- \sum_{1 < j} D_j) \xrightarrow{\delta} h^0(C') = \mathcal{O}_{D_1}(- \sum_{1 < j} D_1 \cap D_j) \to 0 \quad (2.9)$$

To complete the proof, we must verify that the connecting map $\delta$ is indeed restriction of sections, so that (2.9) coincides with the usual exact sequence

$$0 \to \mathcal{O}_X(- \sum_j D_j) \to \mathcal{O}_X(- \sum_{1 < j} D_j) \to \mathcal{O}_{D_1}(- \sum_{1 < j} D_1 \cap D_j) \to 0$$

and indeed, by the snake lemma construction of the connecting map $\delta$ we lift a local section $\sigma \in \ker \gamma'' \subseteq \mathcal{O}_X$ along $\beta$, apply $\gamma : \mathcal{O}_X \to \bigoplus_i \mathcal{O}_{D_i}$ to obtain a local section $(\sigma|_{D_i}) \in \ker \beta \subseteq \bigoplus_i \mathcal{O}_{D_i}$, and then lift along $\alpha : \mathcal{O}_{D_1} \to \bigoplus_i \mathcal{O}_{D_i}$ — the net result is $\sigma|_{D_1}$ as claimed.

**Remark 2.10.** Here we sketch a different proof of Lemma 2.7, which could potentially shed more light on what happens if $D_1, \ldots, D_N \subseteq X$ deviates from being a regular sequence. For each $i$ let $\sigma_i : \mathcal{O}_X \to \mathcal{O}_X(D_i)$ be the canonical global section and let $\sigma_i^\vee : \mathcal{O}_X(-D_i) \to \mathcal{O}_X$ be its dual. For each subset
There's a map of chain complexes

\[
\begin{array}{cccccc}
0 = \mathcal{E}_0 & \rightarrow & \bigoplus_{|I|=1} \mathcal{E}_I & \rightarrow & \bigoplus_{|I|=2} \mathcal{E}_I & \rightarrow & \bigoplus_{|I|=3} \mathcal{E}_I & \rightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{O}_X & \rightarrow & \bigoplus_{|I|=1} \mathcal{O}_X & \rightarrow & \bigoplus_{|I|=2} \mathcal{O}_X & \rightarrow & \bigoplus_{|I|=3} \mathcal{O}_X & \rightarrow & \cdots 
\end{array}
\]

where the horizontal differentials are alternating sums of summand inclusions (in effect, they come from the singular co-chain complex of the \(N - 1\)-simplex \(\Delta^{N-1}\)) and the vertical maps are induced by the \(\sigma_i\). Applying the Koszul construction to the individual maps \(\sigma_j : \mathcal{O}_X \rightarrow \mathcal{E}_j\) (along with the usual sign trick) then results in a double complex \(C^{**}\) with \(C^{pq} = \bigoplus_{|I|=p} \wedge^q \mathcal{E}_j\).

I conjecture⁵ that the horizontal complexes

\[
C^q : 0 \rightarrow \cdots \rightarrow 0 \rightarrow \bigoplus_{|I|=0} \wedge^q \mathcal{E}_j^\vee \rightarrow \bigoplus_{|I|=1} \wedge^q \mathcal{E}_j^\vee \rightarrow \cdots \rightarrow \bigoplus_{|I|=N} \wedge^q \mathcal{E}_j^\vee = \wedge^q \left( \bigoplus_{i=1}^N \mathcal{O}_X(-D_i) \right)
\]

are exact for \(q > -N\), and hence \(\text{Tot}(C^{**})\) is quasi-isomorphic to \(\wedge^N \left( \bigoplus_{i=1}^N \mathcal{O}_X(-D_i) \right) = \mathcal{O}_X(-\sum_d D_i)\).

On the other hand, the vertical complexes

\[
C^{p*} : 0 \rightarrow \cdots \rightarrow 0 \rightarrow \bigoplus_{|I|=0} \wedge^p \mathcal{E}_j^\vee \rightarrow \bigoplus_{|I|=1} \wedge^p \mathcal{E}_j^\vee \rightarrow \cdots \rightarrow \bigoplus_{|I|=p} \mathcal{E}_j^\vee \rightarrow \bigoplus_{|I|=p} \mathcal{O}_X
\]

are direct sums of Koszul complexes by design, and so their cohomology is

\[
h^q(C^{p*}) = \bigoplus_{|I|=p} \text{Tor}^\mathcal{O}_X_{-q}(\mathcal{O}_D, \mathcal{O}_X),
\]

which reduces to

\[
h^q(C^{p*}) = \begin{cases} \bigoplus_{|I|=p} \mathcal{O}_D & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}
\]

precisely when the sequence \(D_1, \ldots, D_N\) is regular [Mat80, §18 Thm. 43], [Ful98, Lem. A.5.3]. As a technical aside, this approach might show that Lemma 2.7 holds under slightly weaker hypotheses of Koszul regularity (see e.g. [Stacks, Tag 062D]).

### 2.3 Resolving sheaves of log-differentials

In the case where \(X\) is smooth over a perfect field and \(\Delta_X := \bigcup_{i=1}^N D_i\) is snc in the strong sense that for each point \(x \in X\) there are regular parameters \(z_1 \cdots z_r\) so that \(\Delta_X = V(z_1 \cdots z_r)\) on a Zariski neighborhood of \(x\), we can say even more — however this additional information is not used in the sequel so the reader is welcome to proceed to Section 2.4.

Here the \(X_k\) are smooth, so in particular the sheaves of differential forms \(\Omega^1_{X_k}\) are locally free, and for each \(p\) the standard Čech construction applied to the co-semi-simplicial sheaf \(\Omega^p_{X_k}\) gives a cochain complex

\[
\varepsilon_0 \Omega^p_{X_k} : \varepsilon_0 \Omega^p_{X_k} \rightarrow \varepsilon_1 \Omega^p_{X_k} \rightarrow \varepsilon_2 \Omega^p_{X_k} \rightarrow \cdots
\]

on \(X\), together with a morphism \(\Omega^p_{X} \rightarrow \varepsilon \Omega^p_{X_k}\) induced by the augmentation. The shifted cone \(\Omega^p_{X, \Delta_X} := \text{cone}(\Omega^p_{X_k} \rightarrow \varepsilon \Omega^p_{X_k}(-1)\}) is then represented by the following complex, with derived

---

⁵It seems a proof by induction on \(N\) analogous to the argument in Lemma 2.7 works, although it is combinatorially more involved.
category degrees as indicated:\textsuperscript{6}

\[
\Omega_X^p \longrightarrow \varepsilon_0 \Omega_X^p \longrightarrow \varepsilon_1 \Omega_X^p \longrightarrow \varepsilon_2 \Omega_X^p \longrightarrow \cdots
\]

\[
= \Omega_X^p \rightarrow \prod_{\sigma \in D(\Delta_X)^0} \Omega_{D(\sigma)}^p \rightarrow \prod_{\sigma \in D(\Delta_X)^1} \Omega_{D(\sigma)}^p \rightarrow \prod_{\sigma \in D(\Delta_X)^2} \Omega_{D(\sigma)}^p \rightarrow \cdots \tag{2.11}
\]

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\end{array}
\]

**Lemma 2.12** ([Fri83, Prop. 1.5], [DI87, Rem. 4.2.2]). The complex

\[
0 \rightarrow \Omega_X^p(\log \Delta_X)(-\Delta_X) \rightarrow \Omega_X^p \rightarrow \prod_{\sigma \in D(\Delta_X)^0} \Omega_{D(\sigma)}^p \rightarrow \prod_{\sigma \in D(\Delta_X)^1} \Omega_{D(\sigma)}^p \rightarrow \cdots
\]

is exact. Equivalently, the complex (2.11) is a resolution of the sheaf \(\Omega_X^p(\log \Delta_X)(-\Delta_X)\). In particular (for \(p = 0\)) the complex

\[
\mathcal{O}_X \rightarrow \prod_{\sigma \in D(\Delta_X)^0} \mathcal{O}_{D(\sigma)} \rightarrow \prod_{\sigma \in D(\Delta_X)^1} \mathcal{O}_{D(\sigma)} \rightarrow \cdots
\]

is a resolution of \(\mathcal{O}_X(-\Delta_X)\).

We include a proof merely to make clear that the lemma is valid in arbitrary characteristic — the argument given follows [Fri83, Prop. 1.5] very closely.

**Proof.** We can check exactness on Zariski stalks over a point \(x \in X\). We may also check exactness after renumbering the divisors \(D_i\), and so we may assume that \(x \in D_1, \ldots, D_k\) and \(x \not\in D_i\) for \(i > k\).

By hypothesis, there are local coordinates \(z_1, \ldots, z_k \in \mathcal{O}_{X,x}\) such that in a Zariski neighborhood of \(x\), \(\Delta_x = V(\prod_{i=1}^k z_i)\) and \(D_i = V(z_i)\) for \(i = 1, \ldots, k\).

We now proceed by simultaneous induction on \(k\) and \(\dim X\). Letting \(\Delta_{D_i} = \sum_{i=2}^{k} D_i \cap D_1\), we have \(\dim D_1 < \dim X\) and \(k - 1 < k\), so denoting by \(\varepsilon' : D_1 \rightarrow D_1\) the semi-simplicial scheme associated to \((D_1, \Delta_{D_1})\), by inductive hypothesis the complex

\[
0 \rightarrow \Omega_{D_1}^p(\log \Delta_{D_1})(-\Delta_{D_1}) \rightarrow \Omega_{D_1}^p \rightarrow \Omega_{D_1}^p \rightarrow \cdots \tag{2.13}
\]

is exact. On the other hand, letting \(\Delta^{>1} = \sum_{i=2}^{k} D_i\) we obtain a divisor with \(k - 1 < k\) components, so denoting \(\varepsilon'' : X^{>1} \rightarrow X\) the semi-simplicial scheme associated to \((X, \Delta^{>1})\), by inductive hypothesis the complex

\[
0 \rightarrow \Omega_X^p(\log \Delta^{>1})(-\Delta^{>1}) \rightarrow \Omega_X^p \rightarrow \Omega_X^p \rightarrow \cdots
\]

is exact. Moreover, there is a sequence of complexes

\[
\begin{array}{cccc}
0 & \Omega_{D_1}^p & \Omega_{D_1}^p & \cdots \\
\downarrow & \varepsilon' & \varepsilon' & \cdots \\
\Omega_X^p & \varepsilon_0 \Omega_X^p & \varepsilon_1 \Omega_X^p & \cdots \\
\downarrow & \varepsilon'' & \varepsilon'' & \cdots \\
\Omega_X^p & \varepsilon_0'' \Omega_X^{>1} & \varepsilon_1'' \Omega_X^{>1} & \cdots \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\end{array}
\]

\textsuperscript{6}This notation is chosen to align with the fact that over \(\mathbb{C}\), the complex (2.11) represents the \(p\)th graded part of the Du Bois complex of the pair \((X, \Delta_X)\).
and since for each \( k \), \( X_k = X_{k-1}^\mathbb{Z} \prod D_{1,k-1} \) the columns are (split) exact. Using the long exact sequence of cohomology sheaves, the inductive hypotheses show that \( h^i(\Omega^p_{X_{\Delta X}}) = 0 \) for \( i > 1 \), and in low degrees we have an exact sequence

\[
0 \to \Omega^p_X(\log \Delta X)(-\Delta X) \to \Omega^p_X(\log \Delta X^\mathbb{Z})(-\Delta X^\mathbb{Z}) \to \Omega^p_{D_1}(\log \Delta D_1)(-\Delta D_1) \to h^1(\Omega^p_{X_{\Delta X}}) \to 0
\]

It remains to show \( h^1(\Omega^p_{X_{\Delta X}}) = 0 \). For this consider a local section

\[
(\varphi_i) = (\varphi_i | i = 1, \ldots, k) \in \ker d \subseteq \epsilon_{0*}\Omega^p_{X_{\Delta X}} = \prod_{i=1}^k \Omega^p_{D_i}
\]

As \( d''\beta(\varphi_i) = \beta d(\varphi_i) = 0 \), by inductive hypothesis there is a local section \( \omega \in \Omega^1_X \) such that \( \beta(\varphi_i) = e''d\omega \). Unravelling, \( \beta(\varphi_i) = (\varphi_2, \ldots, \varphi_k) \) and \( \omega | D_i = \varphi_i \) for \( i = 2, \ldots, k \). Since

\[
0 = d(\varphi_i) = (\varphi_i | D_i \cap D_j - \varphi_i | D_i \cap D_j | 1 \leq i < j \leq N), \text{ so in particular for } i = 1
\]

\[
0 = \varphi_1 | D_i \cap D_j - \varphi_1 | D_i \cap D_j = \varphi_1 | D_i \cap D_j - \omega | D_i \cap D_j \text{ for } j = 2, \ldots, k
\]

we find that \( \varphi_1 - \omega | D_i \) vanishes on \( \Delta D_1 \), and applying exactness of (2.13) once more we see \( \varphi_1 - \omega | D_i \in \Omega^p_{D_1}(\log \Delta D_1)(-\Delta D_1) \). At \( x \), \( \Omega^p_{D_1}(\log \Delta D_1)(-\Delta D_1) \) is generated by the forms

\[
\big( \prod_{i=2}^k \frac{dz_i}{z_i} \big) \cdot \frac{dz_i}{z_i} \wedge \cdots \wedge \frac{dz_i}{z_i} \wedge dz_{i+1} \wedge \cdots \wedge dz_p \text{ where } 1 < i_1 < \cdots < i_l \leq k < i_{l+1} < \cdots < i_p \leq N
\]

The key point is: each of these vanishes on \( D_i \) for \( i > 1 \) (since they each contain either a \( z_i \) or a \( dz_i \) for all \( 1 < i \leq k \)), and so we may find a local section \( \xi \in \Omega^1_X \) with

(i) \( \xi | D_1 = \varphi_1 - \omega | D_1 \);

(ii) \( \xi | D_i = 0 \) for \( i > 1 \).

Rearranging shows \( (\omega + \xi) | D_i = \varphi_i \) for all \( i \) — in other words \( (\varphi_i) = e''(\omega + \xi) \).

\[\Box\]

**Remark 2.15.** As a byproduct we obtain an exact sequence

\[
0 \to \Omega^p_X(\log \Delta X)(-\Delta X) \to \Omega^p_X(\log \Delta X^\mathbb{Z})(-\Delta X^\mathbb{Z}) \to \Omega^p_{D_1}(\log \Delta D_1)(-\Delta D_1) \to 0,
\]

and considering the snake-lemma definition of the connecting morphism shows this is, at least up to sign, restriction of log differential forms (see [EV92, §2])

### 2.4 Replacing the ideal sheaf with a filtered complex

Let \( X \) be a locally noetherian scheme and let \( D_1, \ldots, D_N \subseteq X \) be a regular sequence of effective Cartier divisors, with sum \( \Delta X := \sum_{i=1}^N D_i \). By **Lemma 2.7** the ideal sheaf \( \mathcal{O}_X(-\Delta X) \) is quasi-isomorphic to \( \text{cone}(\mathcal{O}_X \to R_{\epsilon^*} \mathcal{O}_X)[-1] \), which for convenience moving forward we give a name:7

**Definition 2.16.** \( \Omega^p_{X_{\Delta X}} := \text{cone}(\mathcal{O}_X \to R_{\epsilon^*} \mathcal{O}_X)[-1] \). By **Lemma 2.7** and its proof this complex has the explicit representation

\[
\mathcal{O}_X \to \bigoplus D_i \mathcal{O}_{D_i} \to \bigoplus_{i < j} \mathcal{O}_{D_i \cap D_j} \to \cdots \to \mathcal{O}_{\cap D_i} \to 0
\]

\[\begin{array}{cccccc}
0 & 1 & 2 & \cdots & N
\end{array}\]

7This notation is chosen to align with the fact that over \( \mathbb{C} \) and when \( (X, \Delta X) \) is a simple normal crossing pair, the complex (2.11) represents the 0th graded part of the Du Bois complex of the pair \( (X, \Delta X) \).
We can give $\Omega^0_{X,\Delta_X}$ a descending filtration by truncations

$$\Omega^0_{X,\Delta_X} = \sigma_0 \Omega^0_{X,\Delta_X} \supset \sigma_1 \Omega^0_{X,\Delta_X} \supset \sigma_2 \Omega^0_{X,\Delta_X} \supset \cdots$$

where

$$(\sigma_\geq \Omega^0_{X,\Delta_X}) / = \begin{cases} \Omega^0_{X,\Delta_X} / & \text{if } j < i \\ \prod_{j \subseteq \{1, \ldots, N\} \mid |j| = i} R^j f_* \sigma_j \end{cases}$$

Using this filtration we obtain a spectral sequence for higher direct images.

**Corollary 2.18.** Let $S$ be a locally noetherian scheme and let $f : X \to S$ be a finite type morphism. Let $D_1, \ldots, D_N \subseteq X$ be a regular sequence of effective Cartier divisors, with sum $\Delta_X$. Then there is a filtered complex $(Rf_* \Omega^0_{X,\Delta_X}, F)$ whose cohomology computes the higher direct images $R^{i+1}f_* \sigma_X(-\Delta_X)$.

For each $i$ there is a distinguished triangle

$$F^{i+1} Rf_* \Omega^0_{X,\Delta_X} \to F^i Rf_* \Omega^0_{X,\Delta_X} \to \prod_{j \subseteq \{1, \ldots, N\} \mid |j| = i} R^j f_* \sigma_j \to \cdots$$

In particular, there is a spectral sequence

$$E^{ij}_1 = \prod_{j \subseteq \{1, \ldots, N\} \mid |j| = i} R^j f_* \sigma_j \Rightarrow R^{i+j} f_* \sigma_X(-\Delta_X)$$

The filtration $F$ is defined as $F = Rf_* \sigma$, and the resulting spectral sequence is just the usual hypercohomology spectral sequence.

**Remark 2.19.** Viewing $\varepsilon : X \to X$ as a sort of resolution of the pair $(X, \Delta_X)$, we can consider the spectral sequence of **Corollary 2.18** as a sort of descent spectral sequence (see [SGA4II, Vbis], [Con03]).

## 3 Simple normal crossing divisors and thriftyness

### 3.1 Definitions and basic properties

**Definition 3.1 ([EGAIV], §7.8).** A scheme $X$ is excellent if and only if

- $X$ is locally noetherian,
- for every point $x \in X$ the fibers of the natural map $\text{Spec} \sigma_{X,x} \to \text{Spec} \sigma_{X,x}$ are regular,
- for every integral $X$-scheme $Z$ that is finite over an affine open of $X$, there is a non-empty regular open subscheme $U \subseteq Z$, and
d- every scheme $X'$ locally of finite type over $X$ is catenary (that is, if $x \in X'$ and $x \leadsto y$ is a specialization, then any 2 saturated chains of specializations $x = x_0 \leadsto x_1 \leadsto \cdots \leadsto x_n = y$ have the same length).

If $X$ is excellent, then the locus

$$\text{Reg}(X) = \{ x \in X \mid \sigma_{X,x} \text{ is regular} \}$$

is open [EGAIV, Prop. 7.8.6]; we will make repeated use of this fact.

We first relate the notion of a simple normal crossing pair to the regular sequences of effective Cartier divisors considered in the previous section.

**Lemma 3.2.** If $(X, \Delta_X = \sum_i D_i)$ is a simple normal crossing pair then $(D_i)$ is a regular sequence of effective Cartier divisors.
Proof. Let \( x \in X \) be a point and as above let \( I(x) = \{ i \mid x \in D_i \} \). Let \( f_j \in \mathfrak{m}_x \subseteq \mathcal{O}_{X,x} \) be local generators for the \( D_j \), for \( j \in I(x) \). By hypothesis for any subset \( J \subseteq I(x) \) the quotient \( A/(f_j \mid j \in J) \) is regular, and so by induction we reduce to the commutative algebra statement that if \( A \) is a regular local ring, \( f \in A \) and \( A/f \) is a regular local ring with dimension \( \dim A - 1 \) then \( f \) is a non-0-divisor (see for example [Stacks, Tag 0AGA]). \( \square \)

**Lemma 3.3.** Let \( X \) be an integral excellent scheme with an effective Weil divisor \( \Delta_X = \sum D_i \), and for each \( i \) let \( \mathcal{F}_i \subseteq \mathcal{O}_X \) be the ideal sheaf of \( D_i \). Then the locus

\[
\text{snc}(X, \Delta_X) := \{ x \in X \mid \sum_{i \in I(x)} \mathcal{F}^\wedge_i \subseteq \mathcal{O}^\wedge_X \text{ is a simple normal crossing pair} \} \subseteq X
\]

is open, and this is the largest open set \( U \subseteq X \) such that \( (U, \Delta_X|_U) \) is a simple normal crossing pair.

We could alternatively just declare \( \text{snc}(X, \Delta_X) \) to be the largest open set \( U \subseteq X \) such that \( (U, \Delta_X|_U) \) is a simple normal crossing pair; the content of the lemma is that in some sense the snc locus is “already open.”

**Proof.** Suppose \( J \subseteq \{1, \ldots, N\} \), and write \( \mathcal{F}_J = (f_j \mid \mathcal{O}_X \subseteq f_j \in \mathcal{O}_{X,x} \). Consider the co-cartesian diagram of noetherian local rings

\[
\begin{array}{ccc}
\mathcal{O}^\wedge_{X,x} & \longrightarrow & \mathcal{O}^\wedge_{X,x}/\mathcal{F}_J \mathcal{O}^\wedge_{X,x} \simeq (\mathcal{O}^\wedge_{X,x}/\mathcal{F}_J)^\wedge \\
\uparrow & & \uparrow \\
\mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{X,x}/\mathcal{F}_J
\end{array}
\]

The vertical homomorphisms are faithfully flat and by hypothesis \( \mathcal{O}^\wedge_{X,x}/\mathcal{F}_J \mathcal{O}^\wedge_{X,x} \) is regular — since regularity satisfies faithfully flat descent, \( \mathcal{O}_{X,x}/\mathcal{F}_J \) is also regular. Thus \( D_J \) is regular at the point \( x \in D_J \), and as \( X \) is excellent by hypothesis the regular locus of \( D_J \) is open. Letting \( x \in U_J \subseteq X \) be a neighborhood such that \( D_J \cap U_J \subseteq D_J \) is regular and then letting \( U = \cap_i D_J \) gives a neighborhood of \( x \) such that \( (U, (D_J \cap U)) \) is a simple normal crossing pair. \( \square \)

Note that for a simple normal crossing pair \((X, \Delta_X)\), since the intersections \( D_J = \cap_{j \in J} D_j \) are regular their connected components and irreducible components coincide. For convenience we recall the definitions of strata and thriftiness mentioned in the introduction.

**Definition 3.4.** A stratum of a simple normal crossing pair \((X, \Delta_X) = \sum D_i\) is a connected (equivalently, irreducible) component of an intersection \( D_J = \cap_{j \in J} D_j \).

**Definition 3.5** (compare with [Kol13, Def. 2.79-2.80], [KK16, §1, discussion before Def. 10]). Let \((S, \Delta_S = \sum D_i)\) be a pair in the sense of Definition 1.5, and assume \( \Delta_S \) is reduced and effective. A separated, finite type birational morphism \( f : X \to S \) is **thrift** with respect to \( \Delta_S \) if and only if

(i) \( f \) is an isomorphism over the generic point of every stratum of \( \text{snc}(S, \Delta_S) \) and

(ii) letting \( s_i = f^{-1}_* D_i \) for \( i = 1, \ldots, N \) be the strict transforms of the divisors \( D_i \), and setting \( \Delta_X := \sum s_i \), the map \( f \) is an isomorphism at the generic point of every stratum of \( \text{snc}(X, \Delta_X) \).

The restriction that \( D_i \cap \text{Reg}(S) \neq \emptyset \) for all \( i \) ensures that if \( \eta \in D_i \) is a generic point of a component, then \( \eta \in \text{Reg}(S) \). Since on a regular scheme every Weil divisor is Cartier, and as \( S \) is excellent and \( D_i \) is reduced by hypothesis, there is a neighborhood \( \eta \in U \subseteq S \) such that \( U, D_i \cap U \) is a simple normal crossing pair. In other words, \( \eta \in \text{snc}(S, \Delta_S) \) is the generic point of a stratum, so (i) implies \( f^{-1}(\eta) \) is a single (non-closed) point. For our purposes the strict transform \( D_i \) can be **defined** as

\[
D_i := \bigcup_{\eta \in D_i, \text{generic}} f^{-1}(\eta) \subseteq X.
\]

Since \( f \) is an isomorphism over \( \eta \), we also see \( f^{-1}(\eta) \subseteq \text{snc}(X, \Delta_X) \).
**Lemma 3.6.** Let $S$ be an integral excellent noetherian scheme with a sequence of reduced effective Weil divisors $D_1, \cdots, D_N \subseteq S$ such that no component of $\bigcup_i D_i$ is contained in $\text{Sing}(X)$, and let $f : X \to S$ be a separated, finite type birational morphism. Then, $f$ is thrifty if and only if there is a diagram of separated finite type $S$-schemes

$$S \leftarrow U \leftarrow X$$

with both morphisms (necessarily dense) open immersions, such that $U$ contains all generic points of strata of $\text{snc}(S, \Delta_S)$ and $\text{snc}(X, \Delta_X)$.

**Proof.** Since the existence of a common dense open $S \leftarrow U \leftarrow X$ as in the statement of the lemma certainly guarantees (i) and (ii), we focus on the “only if,” and in fact we show that one can take $U$ the maximal domain of definition of $f^{-1} : S \to X$. By (i) of Definition 3.5 this $U$ contains all generic points of strata of $\text{snc}(S, \Delta_S)$.

Suppose $\xi \in \text{snc}(X, \Delta_X)$ is a generic point of a stratum. By hypothesis there is a neighborhood $\xi \in V \subseteq X$ such that $f|_V : V \to \sim S$ is an isomorphism onto its image. Then $W := f(V)$ is a Zariski neighborhood of $f(\xi)$ and the inverse of $f|_V$ gives a section of the birational map $X_W = X \times_S W \to W$.

$$V \leftrightarrow X_W$$

$$f|_V$$

$$W$$

But then the inclusion $V \leftarrow X_W$ is a proper dense open immersion, hence an isomorphism. □

**Remark 3.7.** It seems that the above proof shows in addition that $f(\xi) \in S$ is the generic point of a stratum of $\text{snc}(S, \Delta_S)$.

We will make repeated use of a few blowup lemmas from the construction of Nagata compactifications in Section 4 — here, they are used to show that thrifty morphisms can be dominated by certain admissible blowups.

**Lemma 3.8 ([Con07, Lem. 2.4, Rmk. 2.5, Cor. 2.10]).** Let $S$ be a quasi-compact, quasi-separated scheme.

(i) If $X$ is a quasi-separated quasi-compact $S$-scheme and $Y$ is a proper $S$-scheme, and if $f : U \to Y$ is an $S$-morphism defined on a dense open $U \subseteq X$, then there exists a $U$-admissible blowup $\bar{X} \to X$ and an $S$-morphism $\bar{f} : \bar{X} \to Y$ extending $f$.

(ii) Let $j_i : U \to X_i$ be a finite collection of dense open immersions between finite type separated $S$-schemes. Then there exist $U$-admissible blowups $X'_i \to X_i$ and a separated finite type $S$-scheme $X$, together with open immersions $X'_i \leftarrow X$ over $S$, such that the $X'_i$ cover $X$ and the open immersions $U \leftarrow X'_i \leftarrow X$ are all the same.

**Corollary 3.9.** There exist $U$-admissible blowups

$$\bar{X} \leftarrow S$$

$$X \leftarrow S$$

In particular if $f$ is proper then $X$ and $S$ have a common $U$-admissible blowup.

**Proof.** By Lemma 3.8 there are a separated, finite type $S$-scheme $Y$, $U$-admissible blowups $\bar{S} \to S$ and $\bar{X} \to X$ and dense open immersions $\bar{S} \leftarrow Y \leftarrow \bar{S}$ over $S$ such that the diagram

$$U \leftarrow \bar{X}$$

$$S \leftarrow Y$$

is a proper dense open immersion, hence an isomorphism.
commutes. Since \( \hat{S} \) is proper over \( S \), the bottom arrow is necessarily an isomorphism, in other words \( Y = \hat{X} \). If \( f \) is proper then \( \hat{X} \) is proper over \( S \), so \( Y = \hat{X} \) as well.

\[ \text{Remark 3.10.} \] If \((S, \Delta_S)\) is a simple normal crossing pair and \( U \subseteq S \) is an open containing all strata, a \( U \)-admissible blowup \( f : X \to S \) need not be thrifty, see Example A.12.

### 3.2 The regular-to-regular case

Using Corollary 2.18 we can already obtain a restricted form of Theorem 1.10, the case of a thrifty proper birational morphism of simple normal crossing pairs.

**Theorem 3.11.** Let \((Y, \Delta_Y)\) be a simple normal crossing pair and let \( f : X \to Y \) be a thrifty proper birational morphism. Assume \((X, \Delta_X)\) is also a simple normal crossing pair. Then the natural map

\[ \Theta_Y(-\Delta_Y) \to Rf_* \Theta_X(-\Delta_X) \text{ is a quasi-isomorphism.} \]

**Proof.** Let \( X_i \) (resp. \( Y_i \)) be the semi-simplicial scheme associated to \((X, \Delta_X)\) (resp. \((Y, \Delta_Y)\)). For any \( J \subseteq \{1, \ldots, N\} \) \( f \) restricts to a morphism \( \cap_{j \in J} D_j \to \cap_{j \in J} D_j \), and in this way we obtain a morphism of semi-simplicial schemes

\[
\begin{array}{ccccccc}
\cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X \\
\downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\
\cdots & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \longrightarrow & Y
\end{array}
\]

The hypothesis that both pairs have simple normal crossings and \( f \) is thrifty implies that for each \( i \), \( f_i : X_i \to Y_i \) is a proper birational morphism of (possibly disconnected) regular schemes over \( k \). By [CR11, Cor. 3.2.10] (or [CR15, Thm. 1.1], [Kov20, Thm. 1.4])

\[ \Theta_{Y_i} \simeq Rf_{i*} \Theta_{X_i} \text{ is a quasi-isomorphism for all } i \] (3.13)

The diagram (3.12) induces a morphism of filtered complexes \( f^\# : \Omega^0_Y, \Delta_Y \to Rf_* \Omega^0_X, \Delta_X \), and by Lemma 2.7 and Corollary 2.18 it will suffice to show that the resulting map of descent spectral sequences

\[ E_1^{ij}(Y) = \left\{ \prod_{\sigma \in \Delta_Y} \Theta_D(\sigma) \right\}^{j = 0} \text{ otherwise } \rightarrow \prod_{\sigma \in \Delta_X} R^j f_* \Theta_D(\sigma) = E_1^{ij}(X) \]

is an isomorphism, and this last step is a consequence of (3.13).

### 4 Constructing semi-simplicial projective Macaulayfications

#### 4.1 Preliminaries

In the situation of Theorem 1.10, if \( Z \) is smooth and \( \Delta_Z \) is snc, then Theorem 3.11 applied to both \( r \) and \( s \) shows

\[ Rf_* \Theta_X(-\Delta_X) \simeq Rf_* Rr_* \Theta_Z(-\Delta_Z) = Rg_* Rs_* \Theta_Z(-\Delta_Z) \simeq Rg_* \Theta_Y(-\Delta_Y). \]

Of course, \( Z \) need not be smooth and in the absence of resolution of singularities away from characteristic \( 0 \),\(^8\) we cannot replace it by a resolution. In characteristic \( p > 0 \) we could replace \( Z \) with an alteration, but only at the cost of allowing \( r, s \) to be generically finite but not necessarily birational, and as such using alterations seems incompatible with the strategy of Theorem 3.11. Moreover, to

\[^8\text{At least at the time of this writing ...} \]
the best of our knowledge at the level of generality Theorem 1.10 is stated, even alterations are unavailable.\footnote{Ditto.}

Instead, we will replace $Z$ with a mildly singular (specifically Cohen-Macaulay) semi-simplicial scheme $Z$, together with morphisms $X, r_z: Z, s_z: Y$, over $S$ which are term-by-term proper birational equivalences over $S$. This construction is made possible by the existence of Macaulayfifications.

Theorem 4.1 ([Čes21, Thm. 1.6], see also [Kaw00, Thm. 1.1]). For every a CM-quasi-excellent noetherian scheme $X$ there exists a projective birational morphism $\pi: \tilde{X} \to X$ such that $\tilde{X}$ is Cohen-Macaulay and $\pi$ is an isomorphism over the Cohen-Macaulay locus $CM(X) \subset X$.

The notion of CM-quasi-excellence is a weakening of excellence introduced by Česnavičius, so in particular the theorem applies to excellent noetherian schemes. The usefulness of Macaulayfifications for the problem at hand stems from an extension of the results of Chatzistamatiou-Rülling due to Kovács.

Theorem 4.2 ([Kov20, Thm. 1.4]). Let $f: X \to Y$ be a locally projective birational morphism of excellent Cohen-Macaulay schemes. If $Y$ has pseudo-rational singularities then

$$\delta_Y = Rf_{*}\delta_X and Rf_{*}\omega_X = \omega_Y.$$  

By a result of Lipman-Teissier, if $Y$ is regular (so in particular if it is smooth over $k$) then $Y$ is pseudo-rational [LT81, §4], hence Theorem 4.2 applies when $Y$ is regular.

### 4.2 Gluing on simplices

In this section we describe an inductive method for constructing a sequence of truncated semi-simplicial schemes converging to $Z$. Here for any $i \in \mathbb{N}$ an $i$-truncated semi-simplicial object in a category $\mathcal{C}$ is a functor $\Lambda^{\text{op}}_{\leq i} \to \mathcal{C}$, where $\Lambda^{\text{op}}_{\leq i}$ is the full subcategory of $\Lambda^{\text{op}}$ generated by the objects $[j]$ with $j \leq i$. Given an $i-1$-truncated semi-simplicial object $X_*$ of $\mathcal{C}$, let

$$[i]_\leq := \{j, k \in [i] | j < k\}$$

and define two morphisms

$$\delta_+, \delta_- : X^{[i]}_{i-2} \to X^{[i]}_{i-1}$$

by $\delta_+(x_0, ..., x_i) = (d_i^{-1}(x_k) | j < k)$ and $\delta_-(x_0, ..., x_i) = (d_i^{-1}(x_j) | j < k)$. Assuming $\mathcal{C}$ has finite limits we may form the equalizer

$$E(X_*) := \text{Eq}(\delta_+, \delta_-) \xrightarrow{\delta_+} X^{[i]}_{i-1} \xrightarrow{\delta_-} X^{[i]}_{i-2} \quad (4.3)$$

one can check that this construction is functorial in $X_*$: indeed if $Y_*$ is another $i-1$-truncated semi-simplicial object then given a morphism $X_* \to Y_*$, we can form a commutative diagram

$$E(X_*) := \text{Eq}(\delta_+, \delta_-) \xrightarrow{\delta_+} X^{[i]}_{i-1} \xrightarrow{\delta_-} X^{[i]}_{i-2} \quad (4.4)$$

and obtain a unique morphism on the dashed arrow by functoriality of equalizers. Finally, let $I$ denote the category $0 \to 1$ (thought of as the “unit interval”). An object of $\mathcal{C}^I$ is a morphism $f: X \to Y$ in $\mathcal{C}$ and there are 2 functors $s : \mathcal{C}^I \to \mathcal{C}$ defined by $s(f) = X, t(f) = Y$ (source and target).
Lemma 4.5 (compare with [SGA4II, Vbis, Prop. 5.1.3], [Stacks, Tag 0AMA]). Let \( \mathcal{C} \) be a category containing finite limits. The functor

\[
\Phi_1 : \mathcal{C}^{\Lambda_{i1}^{op}} \to \mathcal{C}^{A_{i1}^{op}} \times_{\mathcal{C}} \mathcal{C}^I,
\]

where the right hand side is the 2-fiber product with respect to the functors \( E : \mathcal{C}^{\Lambda_{i1}^{op}} \to \mathcal{C} \) and \( t : \mathcal{C}^I \to \mathcal{C} \) that sends an \( i \)-truncated semi-simplicial object \( X \), to the pair \((sk_{i-1}X, X_1 \to E(sk_{i-1}X))\), is an equivalence of categories.

Proof. We first check that \( \Phi_1 \) is fully faithful. For faithfulness, note that for any 2 \( i \)-truncated semi-simplicial objects \( X, Y \), there is an injection

\[
\text{Hom}_{\mathcal{C}^{\Lambda_{i1}^{op}}}(X, Y) \hookrightarrow \prod_{j=0}^{I} \text{Hom}_{\mathcal{C}}(X_j, Y_j)
\]

(4.6)
since a morphism \( \alpha : X \to Y \) is equivalent to a sequence of morphisms \( \alpha_i : X_i \to Y_i \) commuting with differentials. By the definition of the 2-fiber product, the morphism \( \Phi_1(\alpha) : \Phi_1(X, Y) \to \Phi_1(Y, X) \) induced by \( \alpha \) consists of the morphism \( sk_{i-1} \alpha : sk_{i-1}X, \to sk_{i-1}Y, \) and the commutative diagram

\[
\begin{array}{ccc}
X_i & \longrightarrow & E(sk_{i-1}X) \\
\downarrow{\alpha_i} & & \downarrow{E(\alpha)} \\
Y_i & \longrightarrow & E(sk_{i-1}Y)
\end{array}
\]

This shows that (4.6) factors as

\[
\text{Hom}_{\mathcal{C}^{\Lambda_{i1}^{op}}}(X, Y) \xrightarrow{\Phi_1} \text{Hom}_{\mathcal{C}^{\Lambda_{i1}^{op}}}(\Phi_1(X), \Phi_1(Y)) \to \prod_{j=0}^{I} \text{Hom}_{\mathcal{C}}(X_j, Y_j)
\]

(4.7)
hence the first map is injective, or in other words \( \Phi_1 \) is faithful. On the other hand given an arbitrary morphism \( \Phi_1(X, Y) \to \Phi_1(X, Y) \) consisting of a map \( \beta : sk_{i-1}X, \to sk_{i-1}Y, \) a map \( \gamma : X_1 \to Y_1 \) and a commutative diagram

\[
\begin{array}{ccc}
X_i & \longrightarrow & E(sk_{i-1}X) \\
\downarrow{\gamma} & & \downarrow{E(\beta)} \\
Y_i & \longrightarrow & E(sk_{i-1}Y)
\end{array}
\]

(4.8)
we may verify commutativity of

\[
\begin{array}{c}
\begin{array}{c}
X_i \xrightarrow{d_k^i} E(sk_{i-1}X) \xrightarrow{\text{pr}_1} X_{i-1} \\
\downarrow{\gamma} \quad (1) \quad \downarrow{E(\beta)} \quad (2) \quad \downarrow{\beta_i} \\
Y_i \xrightarrow{d_k^i} E(sk_{i-1}Y) \xrightarrow{\text{pr}_1} Y_{i-1}
\end{array}
\end{array}
\]

as follows: commutativity of (1) is exactly (4.8), and commutativity of (2) can be deduced from that of the left square of (4.4). Hence \( \beta \) and \( \gamma \) define a map \( X \to Y \), and so \( \Phi_1 \) is full.

Next we show \( \Phi_1 \) is essentially surjective — this argument is inspired by and closely follows the proof of [Stacks, Tag 0186]. For this we consider an object of the 2-fiber product \( \mathcal{C}^{\Lambda_{i1}^{op}} \times_{\mathcal{C}} \mathcal{C}^I \) consisting of an \( i-1 \)-truncated semi-simplicial object \( X, \) and object \( Y \) and a morphism \( f : Y \to E(X, \),
and we must prove that there exists an $i$-truncated semi-simplicial object $Z$, and an isomorphism $\Phi(Z, \to X) \simeq (X, f)$. We first let $Z_j = X_j$ for $j < i$ and let $Z(\phi) = X(\phi)$ for any $\phi : [j'] \to [j]$ with $j' < j < i$. Then we set $Z_i = Y$, and we must define morphisms $Z(\phi) : Z_i = Y \to X_j = Z_j$ for increasing maps $[j] \to [i]$ which are functorial in $\phi$, in the sense that for any increasing $\psi : [j'] \to [j]$ the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{Z(\phi)} & X_j \\
\downarrow[Z(\psi)] & & \downarrow[X(\psi)] \\
X_{j'} & \xrightarrow{X(\phi)} & X_j
\end{array}
$$

commutes (note that the data of $X(\psi)$ is already included in $X$). We may assume $j < i$ (otherwise $\phi = id$ and we must set $Z(\phi) = id$), and so $\phi$ must factor as

$$[j] \xrightarrow{\psi} [i-1] \xrightarrow{\delta_k} [i]$$

for some $k$ and some $\psi$. We define $Z(\phi)$ to be the composition

$$Y \xrightarrow{f} E(X, \to X) \xrightarrow{pr_k} X_{i-1} \xrightarrow{X(\psi)} X_j$$

(so in particular we define $Z(\delta_k) = pr_k \circ f =: f_k$). To verify this definition is independent of $\psi$, suppose that there is another factorization

$$[j] \xrightarrow{\psi'} [i-1] \xrightarrow{\delta_k} [i]$$

Note that if $j = i - 1$ then $\psi = \psi' = id$ and $k = l$ for trivial reasons, so we may assume $j < i - 1$ and in that case $\phi$ misses both $k$ and $l$, so we may factor through $[i-2]$ as follows:

$$
\begin{array}{ccc}
\vdots & & \psi' \\
\vdots & & \downarrow \delta_k^{i-1} \\
[j] & \xrightarrow{\rho} & X_{i-1} \\
\downarrow \delta_k & & \downarrow \delta_k \\
[i] & & [i]
\end{array}
$$

By the defining property of the equalizer $E(X, \to X)$, we know $X(\delta_k^{i-1}) \circ f_k = X(\delta_k^{i-1}) \circ f_{i-1}$, and

$$X(\rho) \circ X(\delta_k^{i-1}) = X(\psi) \text{ and } X(\rho) \circ X(\delta_k^{i-1}) = X(\psi')$$

because $X_j$ is an $i-1$-truncated semi-simplicial object. It follows that $X(\psi) \circ f_k = X(\psi') \circ f_{i-1}$ as desired.

We now prove to prove the commutativity statement in (4.9). Again we may assume $j < i$, since otherwise $\phi = id$ and $\psi = \phi \circ \psi$ so commutativity is implied by the above proof that the $Z(\phi)$ are well defined. When $j < k$ the map $\phi$, and hence also $\phi \circ \psi$ must factor through some $\delta_{k}^{i} : [i - 1] \to [i]$ and we obtain the following situation:

$$
\begin{array}{ccc}
\vdots & & \phi \circ \psi \\
\vdots & & \downarrow \delta_k^{i-1} \\
[j] & \xrightarrow{\psi} & [j] \\
\downarrow \delta_k & & \downarrow \delta_k \\
[i] & & [i]
\end{array}
$$

Now by definition $Z(\phi) = X(\phi) \circ f_k$ and $Z(\phi \circ \psi) = X(\phi \circ \psi) \circ f_k$, and since $X_j$ is an $i-1$-truncated semi-simplicial object $X(\phi \circ \psi) = X(\phi) \circ X(\rho)$, so that

$$X(\psi) \circ Z(\phi) = X(\psi) \circ X(\rho) \circ f_k = X(\phi \circ \psi) \circ f_k = Z(\phi \circ \psi)$$

as claimed. □
4.3 Common admissible blowups

Using Lemma 4.5 to build the semi-simplicial scheme $Z$, inductively, at each step we encounter the situation of the lemma below.

Lemma 4.11. Suppose

$$
\begin{array}{c}
X \leftarrow U \xrightarrow{j} Y \\
\downarrow^\varphi \quad \downarrow^{\varphi^0} \\
F \leftarrow E \xrightarrow{g} G
\end{array}
$$

is a commutative diagram of schemes of finite type over a quasi-compact quasi-separated base scheme $S$, and assume that $f, g, \varphi$ and $\psi$ are proper and $i$ and $j$ are dense open immersions. Then, there is a commutative diagram

$$
\begin{array}{c}
X \leftarrow Z \xrightarrow{s} Y \\
\downarrow \varphi \quad \downarrow \varphi^0 \\
F \leftarrow E \xrightarrow{g} G
\end{array}
$$

where $r$ and $s$ are $U$-admissible blowups (hence in particular projective).

If in addition $S$ is a CM-quasi-excellent noetherian scheme and $U$ is Cohen-Macaulay, we may ensure that $Z$ is also Cohen-Macaulay.

Proof. First, $X$ and $E$ are proper over the scheme $F$, which is quasi-compact and quasi-separated since it is of finite type over $S$. By the first part of Lemma 3.8 applied to the map of $F$-schemes $\varphi^0 : U \to E$ defined on the dense open $U \subseteq X$, there is a $U$-admissible blowup $V_X \to X$ and an $F$-morphism $V_X \to E$ extending $\varphi^0$. A similar argument produces a $U$-admissible blowup $V_Y \to Y$ and a $G$-morphism $V_Y \to E$ extending $\varphi^0$. The current situation is summarized below:

$$
\begin{array}{c}
U \\
\downarrow^i \\
V_Y \\
\downarrow^j \\
Y \\
\downarrow^\varphi \\
X \\
\downarrow \varphi \\
F \\
\downarrow \varphi \\
E \\
\downarrow \varphi \\
E
\end{array}
$$

Since $V_X, V_Y$ are $U$-admissible blowups of $X, Y$ respectively, they still contain $U$ as a dense open ([Con07, comments before Lem. 1.1]). Note that since $V_X \to X$ is a blowup, $\varphi$ is proper and $f$ is proper the morphism $V_X \to E$ is also proper; similarly $V_Y$ is proper over $E$. Now applying the second part of Lemma 3.8 to $V_X$ and $V_Y$ over $E$ we obtain a separated finite type morphism $\varphi : Z \to E, U$ admissible blowups $V_X \to V_X$ and $V_Y \to V_Y$ and open immersions $V_X \leftarrow Z \leftarrow V_Y$ over $E$ such that the diagram

$$
\begin{array}{c}
U \\
\downarrow \\
\tilde{V}_Y \\
\downarrow \\
\tilde{V}_X \\
\downarrow \\
Z
\end{array}
$$

commutes and $E = \tilde{V}_X \cup \tilde{V}_Y$. Since $U$ is dense in both $\tilde{V}_X$ and $\tilde{V}_Y$, we see that $\tilde{V}_X$ and $\tilde{V}_Y$ are both dense in $Z$. Then as $\tilde{V}_X \to Z$ is a dense open immersion of separated finite type $E$-schemes where $\tilde{V}_X$ is proper over $E$, it must be that $\tilde{V}_X = Z$; similarly, $\tilde{V}_Y = Z$ (see also the comments following [Con07, 20].
Finally, we define $r$ and $s$ to be the compositions

\[
\begin{array}{ccc}
Z & \xrightarrow{r} & V_X \\
& & \xrightarrow{s} \downarrow \downarrow \downarrow \downarrow
\end{array}
\]

Finally if $S$ is CM-quasi-excellent, then since $Z$ is of finite type over $S$ it is also CM-quasi-excellent by [Čes21, Rmk.1.5]. By hypothesis $U \subseteq \text{CM}(Z)$, and by Theorem 4.1 there is a CM($X$)-admissible (hence also $U$-admissible) blowup $\tilde{Z} \to Z$ such that $\tilde{Z}$ is Cohen-Macaulay. In this case we replace $Z$ with $\tilde{Z}$.

**Lemma 4.12.** Let $S$ be a quasi-compact quasi-separated base scheme and let

\[
X_1 \xleftarrow{i_1} U_1 \xrightarrow{j_1} Y_1
\]

be morphisms of augmented semi-simplicial schemes of finite type over $S$. Assume that all differentials and augmentations of $X_1$ and $Y_1$ are proper,\(^{10}\) and that the morphisms $X_i \xleftarrow{i_i} U_i \xrightarrow{j_i} Y_i$ are dense open immersions for all $i$ (including $i = -1$). If there exists a finite-type $S$-scheme $Z_{-1}$ and $U_{-1}$-admissible blowups $X_{-1} \xleftarrow{r_{-1}} Z_{-1} \xrightarrow{s_{-1}} Y_{-1}$, then there exists an augmented semi-simplicial $S$-scheme $Z_i \to Z_{-1}$ together with morphisms

\[
X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i
\]

such that for all $i$ the morphisms $X_i \xleftarrow{i_i} Z_i \xrightarrow{s_i} Y_i$ are $U_i$-admissible blowups (hence in particular projective and birational).

Moreover if $S$ is a CM-quasi-excellent noetherian scheme, and each $U_i$ is Cohen-Macaulay, we may ensure that the $Z_i$ are also Cohen-Macaulay.

**Proof.** We construct a sequence of $i$-truncated semi-simplicial $S$-schemes $Z_i$, converging to $Z_\ast$, with the additional requirement that the morphisms $s_{k_{-1}}(U_i) \to s_{k_{-1}}(X_i)$ and $s_{k_{-1}}(U_i) \to s_{k_{-1}}(Y_i)$ factor through $Z_i$.\(^{11}\) The $i = -1$ case is included in the hypotheses. At the inductive step we may assume that there is an $i - 1$-truncated semi-simplicial $S$-scheme $Z_{i-1}$, together with a commutative diagram

\[
\begin{array}{ccc}
\text{sk}_{k_{-1}}(U_i) & & \text{sk}_{k_{-1}}(Y_i) \\
\downarrow & & \downarrow \\
\text{sk}_{k_{-1}}(X_i) & & \text{sk}_{k_{-1}}(Y_i) \\
\downarrow & & \downarrow \\
X_{i-1} & & Y_{i-1}
\end{array}
\]

such that for all $j < i$ the morphisms $X_j \xleftarrow{r_{j-1}^{i-j}} Z_{j-1} \xrightarrow{s_{j-1}^{i-j}} Y_j$ are $U_j$-admissible blowups. Letting $E$

---

\(^{10}\)This is equivalent to requiring that $X_i$ is a semi-semi-simplicial object in the category of proper $X_{-1}$-schemes (and similarly for $Y_i$).

\(^{11}\)I think that this isn’t actually an additional restriction, but including it makes the inductive step easier.
denote the equalizer functor of Lemma 4.5, we obtain a commutative diagram of the form

\[
\begin{array}{ccc}
X_i & \xleftarrow{i_i} & U_i & \xrightarrow{J_i} & Y_i \\
\downarrow{(X(\delta^j_k))} & & \downarrow{(U(\delta^j_k))} & & \downarrow{(Y(\delta^j_k))} \\
E(\text{sk}_{i-1}(X_i)) & \xleftarrow{E(k_i)} & E(\text{sk}_{i-1}(U_i)) & \xrightarrow{E(k_i)} & E(\text{sk}_{i-1}(Y_i)) \\
\end{array}
\]  

(4.16)

Next, we verify that (4.16) satisfies the hypotheses of Lemma 4.11, making repeated reference to the constructions in (4.3) and (4.4). Note that the bottom horizontal arrows are proper, since they are obtained as limits of the blowup maps \( \hat{r}_{i-1,j} : Z_{i-1,j} \rightarrow X_j \) and \( \hat{s}_{i-1,j} : Z_{i-1,j} \rightarrow Y_j \) for \( j = i-1, i-2 \). The vertical maps on the outside edges are proper since the differentials \( X(\delta^j_k) : X_i \rightarrow X_{i-1} \) and \( Y(\delta^j_k) : Y_i \rightarrow Y_{i-1} \) are proper by hypothesis. Hence applying Lemma 4.11 we obtain a commutative diagram

\[
\begin{array}{ccc}
U_i & \xleftarrow{i_i} & X_i & \xrightarrow{r_{i-1,j}} & Z_i & \xrightarrow{s_{i-1,j}} & Y_i \\
\downarrow{(U(\delta^j_k))} & & \downarrow{(X(\delta^j_k))} & & \downarrow{\rho} & & \downarrow{(Y(\delta^j_k))} \\
E(\text{sk}_{i-1}(U_i)) & \xrightarrow{E(k_i)} & E(\text{sk}_{i-1}(X_i)) & \xrightarrow{E(k_i)} & E(\text{sk}_{i-1}(Z_i)) & \xrightarrow{E(k_i)} & E(\text{sk}_{i-1}(Y_i)) \\
\end{array}
\]  

(4.17)

in which the maps \( r_{i-1,j} : Z_i \rightarrow X_j \) and \( s_{i-1,j} : Z_i \rightarrow Y_j \) are \( U_i \)-admissible blowups. In the case where \( S \) is CM-quasi-excellent we apply Lemma 4.11 to ensure that \( Z_i \) is Cohen-Macaulay.

Now Lemma 4.5 implies that there is an \( i \)-truncated semi-simplicial \( S \)-scheme \( Z_i \), such that \( \text{sk}_{i-1}(Z_i) = Z_{i-1} \), and \( Z_{i,i} = Z_i \), together with a commutative diagram

\[
\begin{array}{ccc}
sk_i(U_i), & \xleftarrow{k_i} & sk_i(U_i) \\
\downarrow{sk_i(U_i)} & & \downarrow{sk_i(U_i)} \\
sk_i(X_i), & \xleftarrow{r_i} & Z_i, & \xrightarrow{s_i} & sk_i(Y_i) \\
\downarrow{sk_i(X_i)} & & \downarrow{sk_i(Y_i)} & & \downarrow{sk_i(Y_i)} \\
X_{i-1} & \xleftarrow{r_{i-1,j}} & Z_{i-1} & \xrightarrow{s_{i-1,j}} & Y_{i-1} \\
\end{array}
\]  

(4.18)

such that for all \( j \leq i \) the morphisms \( X_j \xleftarrow{r_{i-1,j}} Z_{i-1,j} \xrightarrow{s_{i-1,j}} Y_j \) are \( U_j \)-admissible blowups.

\[\square\]

4.4 Invariance results for cohomology of snc ideal sheaves

Lemma 4.19. Let \( S \) be an excellent noetherian scheme and let \( (X, \Delta_X) \) and \( (Y, \Delta_Y) \) be simple normal crossing pairs separated and of finite type over \( S \), and let \( X \xleftarrow{r} Z \xrightarrow{s} Y \) be a thifty proper birational equivalence over \( S \). Then there exists a semi-simplicial separated finite type \( S \)-scheme \( Z \) and morphisms of semi-simplicial \( S \)-schemes \( X \xleftarrow{r_i} Z \xrightarrow{s_i} Y \) such that for all \( i \),

(i) \( Z_i \) is Cohen-Macaulay and

(ii) \( X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i \) is a projective birational equivalence over \( S \).
Proof. Set $\Delta_X = r_{s}^{-1}\Delta_X = s_{i}^{-1}\Delta_Y$. By Lemma 3.6 there is a dense open set $X \leftarrow U_X \leftarrow Z$ (resp $Z \leftarrow U_Y \leftarrow Y$) containing all generic points of strata of $\text{snc}(X, \Delta_X)$ and $\text{snc}(Z, \Delta_Z)$ (resp. $\text{snc}(Y, \Delta_Y)$ and $\text{snc}(Z, \Delta_Z)$). Then $U := U_X \cap U_Z$ is a dense open containing all generic points of strata of $\text{snc}(X, \Delta_X)$, $\text{snc}(Y, \Delta_Y)$ and $\text{snc}(Z, \Delta_Z)$. Set $\Delta_U := \Delta_U |_U$, so that $(U, \Delta_U)$ is simple normal crossing pair together with thrifty birational (but not necessarily projective) morphisms $(X, \Delta_X) \leftarrow (U, \Delta_U) \rightarrow (Y, \Delta_Y)$.

We now let $X$, $Y$, and $U$ be the augmented semi-simplicial schemes associated to $(X, \Delta_X), (Y, \Delta_Y)$ and $(U, \Delta_U)$ as in the discussion at the beginning of Section 2, and consider the resulting morphisms

\[
\begin{array}{ccc}
X & \leftarrow & U \\
\downarrow & & \downarrow \\
X_{i-1} = X & \leftarrow & U_{i-1} = U \\
\downarrow & & \downarrow \\
Y_{i-1} = Y & \rightarrow & Y
\end{array}
\]

(4.20)

Since $U$ contains the generic points of all strata of $\text{snc}(Z, \Delta_Z)$, the morphisms $X_i \leftarrow U_i \rightarrow Y_i$ are dense open immersions for all $i$, and the differentials and augmentations of $X_i$ and $Y_i$ are closed immersions, hence proper. Finally applying Lemma 3.8 to the collection of open immersions $U \subseteq X, Z$ over $X$, we obtain $U$-admissible blowups $\bar{X}, \bar{Y}$ of $X, Y$ respectively, as well as a separated finite type $X$-scheme $W$ with open immersions $\bar{X}, \bar{Z} \subseteq W$ covering $W$. Again properness of $\bar{X}, \bar{Y}$ over $X$ forces $\bar{X} = \bar{Z} = W$, hence replacing $Z$ with $\bar{Z}$ we can ensure $s : Z \rightarrow Y$ is an $U$-admissible blowup. Repeating this construction with $Y, Z$ in place of $X, Z$, we may ensure $s : Z \rightarrow Y$ is also a $U$-admissible blowup. Thus the hypotheses of Lemma 4.12 are satisfied.

Theorem 4.21. With the same hypotheses as Lemma 4.19, there exists a complex $\mathcal{K}$ on $Z$ together with quasi-isomorphisms $\theta_X(-\Delta_X) \simeq Rr_*\mathcal{K}$ and $\theta_Y(-\Delta_Y) \simeq Rs_*\mathcal{K}$. In particular there are quasi-isomorphisms $Rf_*\theta_X(-\Delta_X) \simeq Rf_*Rr_*\mathcal{K} = Rg_*Rs_*\mathcal{K} \simeq Rg_*\theta_Y(-\Delta_Y)$.

Proof. By Lemma 4.19 there is a commutative diagram of augmented semi-simplicial $S$-schemes

\[
\begin{array}{ccc}
X & \leftarrow & Z \\
\downarrow & & \downarrow \\
X & \leftarrow & Z \\
\downarrow & & \downarrow \\
Y & \rightarrow & Y
\end{array}
\]

such that for each $i$ the scheme $Z_i$ is Cohen-Macaulay and the maps $X_i \leftarrow Z_i \rightarrow Y_i$ define a projective birational equivalence over $S$. Defining $\mathcal{K} = \text{cone}(\theta_Z \rightarrow \text{Res}\theta_Z)[-1]$, from (4.22) we obtain a map of complexes $\Omega^0_{X, \Delta_X} \rightarrow Rr_*\mathcal{K}$ appearing in a map of distinguished triangles

\[
\begin{array}{ccc}
\Omega^0_{X, \Delta_X} & \rightarrow & \theta_X \\
\downarrow & & \downarrow \\
Rr_*\mathcal{K} & \rightarrow & Rr_*\theta_X
\end{array}
\]

By [Kov20, Thm. 1.4] $\beta$ is a quasi-isomorphism. Using commutativity of (4.22) we may identify $\gamma$ with the morphism

\[
\text{Res}\theta_X \rightarrow \text{Res}\theta_Z.
\]

The morphisms on cohomology induced by (4.23) are the abutment of a map of descent spectral sequences (see Lemma 2.2); the map of $E_1$ pages reads

\[
E_1^j(X) = \begin{cases} 
\text{Res}\theta_{X_i} & \text{if } j = 0 \\
0 & \text{else}
\end{cases} \rightarrow R^j\text{Res}\theta_{Z_i} = E_1^j(Z)
\]

(4.24)

By [Kov20, Thm. 1.4] again, for each $i$ the natural map $\theta_{X_i} \rightarrow Rr_*\theta_{Z_i}$ is a quasi-isomorphism. We conclude via Corollary 2.4 that (4.24) an isomorphism, and so $\gamma$ is a quasi-isomorphism.
By the 5-lemma, we conclude \( \alpha \) is a quasi-isomorphism. Applying \( Rf_* \) and using Lemma 2.7 then gives a quasi-isomorphism

\[
Rf_* \mathcal{O}_X(-\Delta_X) \approx Rf_* \Omega^0_{X,\Delta_X} \approx Rf_* Rr_* \mathcal{K}.
\]

A symmetric argument applied on the \( Y \) side gives the desired quasi-isomorphism \( Rg_* \mathcal{O}_Y(-\Delta_Y) \approx Rg_* Rr_* \mathcal{K}. \)

\[\square\]

5 Applications to rational pairs

In this section we make use of Grothendieck duality, as formulated in [Con00; R&D].

**Theorem 5.1** (Grothendieck duality, [R&D, Cor. VII.3.4], [Con00, Thm. 3.4.4].) Let \( f : X \to Y \) be a proper morphism of finite-dimensional noetherian schemes and assume \( Y \) admits a dualizing complex (for example \( X \) and \( Y \) could be schemes of finite type over \( k \)). Then for any pair of objects \( \mathcal{F} \in D^b_c(X) \) and \( \mathcal{G} \in D^b_c(Y) \) there is a natural isomorphism

\[
Rf_* R\text{Hom}_X(\mathcal{F}, \mathcal{G}) \cong R\text{Hom}_Y(Rf_* \mathcal{F}, \mathcal{G}) \text{ in } D^b_c(Y)
\]

If \( \omega_X^* \) is a dualizing complex on \( X \) then \( \omega_X := f^! \omega_X^* \) is a dualizing complex on \( X \) [R&D, §V.10, Cor. VI.3.5], and so in the case \( \mathcal{G} = \omega_Y \) we obtain a natural isomorphism

\[
Rf_* R\text{Hom}_X(\mathcal{F}, \omega_Y^*) \cong R\text{Hom}_Y(Rf_* \mathcal{F}, \omega_Y^*) \text{ in } D^b_c(Y)
\]

In the case where \( f \) is smooth of relative dimension \( n \), there is a quasi-isomorphism \( f^! \omega_Y \cong \omega_Y / [n] \).

**Definition 5.2** (compare with [Kol13, Def. 2.78].) Let \( (S, \Delta_S) \) be a pair as in Definition 1.5 and assume \( \Delta_S \) is reduced and effective. A proper birational morphism \( f : X \to S \) is a rational resolution if and only if

(i) \( X \) is regular and the strict transform \( \Delta_X := f^{-1}_\Delta \Delta_S \) has simple normal crossings,
(ii) the natural morphism \( \mathcal{O}_S(-\Delta_S) \to Rf_* \mathcal{O}_X(-\Delta_X) \) is a quasi-isomorphism, and

(iii) \( R^i f_* \omega_X(\Delta_X) = 0 \) for \( i > 0 \).

In the situation of Definition 5.2, the map \( \mathcal{O}_S(-\Delta_S) \to Rf_* \mathcal{O}_X(-\Delta_X) \) appearing in condition (ii) is Grothendieck dual to a morphism

\[
Rf_* \omega_X^*(\Delta_X) \xrightarrow{(1)} Rf_* R\text{Hom}_X(\mathcal{O}_X(-\Delta_X), \omega_X^*)
\]

\[
\xrightarrow{(2)} R\text{Hom}_S(Rf_* \mathcal{O}_X(-\Delta_X), \omega_X^*) \xrightarrow{(3)} R\text{Hom}_S(\mathcal{O}_S(-\Delta_S), \omega_X^*)
\]

where the equality (1) comes from the fact that \( \Delta_X \) is a Cartier divisor \((X, \Delta_X) \) is snc by hypothesis), the isomorphism (2) comes from Grothendieck duality and the map (3) is obtained from the morphism of (ii) by applying the derived functor \( R\text{Hom}_S(\mathcal{O}_S(-\Delta_S), \omega_X^*) \). As \( X \) is regular and the dualizing complex \( \omega_X^* \) is normalized to \( h^i \omega_X^* = 0 \) for \( i \neq -\dim X \); in other words, \( \omega_X^* \cong \omega_X [\dim X] \). Twisting this equation with the Cartier divisor \( \Delta_X \) gives \( \omega_X^*(\Delta_X) \cong \omega_X(\Delta_X) [\dim X] \). If \( \mathcal{O}_S(-\Delta_S) \to Rf_* \mathcal{O}_X(-\Delta_X) \) is a quasi-isomorphism, so is

\[
Rf_* \omega_X(\Delta_X) [\dim X] \cong Rf_* \omega_X^*(\Delta_X) \to R\text{Hom}_S(\mathcal{O}_S(-\Delta_S), \omega_X^*)
\]

and taking cohomology sheaves we see that \( R^{i - \dim X} f_* \omega_X(\Delta_X) \cong h^i R\text{Hom}_S(\mathcal{O}_S(-\Delta_S), \omega_X^*) \) for all \( i \).

Thus given conditions (i) and (ii) of Definition 5.2, condition (iii) is equivalent to Cohen-Macaulayness of the sheaf \( \mathcal{O}_S(\Delta_S) \). We record these observations as a lemma.

---

12 Which is to say we make explicit use of Grothendieck duality — that is, it has already been used implicitly via dependence on references quite a few times!
Lemma 5.4 (compare with [Kol13, Cor. 2.73, Props. 2.82-2.23], [Kov20, Def. 1.3]). With notation and setup as in Definition 5.2, the morphism \( f : X \to S \) is a rational resolution if and only if

(i) \( X \) is regular and the strict transform \( \Delta_X := f_*^{-1} \Delta_S \) has simple normal crossings,
(ii) the natural morphism \( \mathcal{O}_S(-\Delta_S) \to Rf_* \mathcal{O}_X(-\Delta_X) \) is a quasi-isomorphism, and
(iii) the sheaf \( \mathcal{O}_S(-\Delta_S) \) is Cohen-Macaulay.

As illustrated in the examples of Appendix A, even simple normal crossing pairs \((S, \Delta_S)\) may have non-rational resolutions in the absence of additional thriftiness restrictions, hence the following definition of rational singularities for pairs.

Definition 5.5. Let \((S, \Delta_S)\) be a pair such that \(\Delta_S\) is a reduced effective Weil divisor. Then, \((S, \Delta_S)\) is resolution-rational if and only if it has a thrifty rational resolution.

5.1 Rational resolutions of pairs are all-for-one

In the case where \(S\) is a normal variety over a field of characteristic 0, it is known that if \((S, \Delta_S)\) has a thrifty rational resolution then every thrifty resolution is rational [Kol13, Cor. 2.86]. The proof of this fact shows more generally that if \(f : X \to S\) and \(g : Y \to S\) are thrifty resolutions, then there are isomorphisms \(R^i f_* \mathcal{O}_X(-\Delta_X) \cong R^i g_* \mathcal{O}_Y(-\Delta_Y)\) for all \(i\). This remains true in arbitrary characteristic.

Lemma 5.6 ([Kol13, Cor. 2.86] in characteristic 0). Let \((S, \Delta_S)\) be a pair such that \(\Delta_S\) is a reduced effective Weil divisor, and let \(f : X \to S\) and \(g : Y \to S\) be thrifty resolutions. Then there is a quasi-isomorphism \(Rf_* \mathcal{O}_X(-\Delta_X) \cong Rg_* \mathcal{O}_Y(-\Delta_Y)\). In particular, \(f\) is a rational resolution if and only if \(g\) is.

Note that this includes Theorem 3.11 as a special case: indeed, if \((S, \Delta_S)\) is a simple normal crossing pair then given any thrifty resolution \(f : X \to S\) we may choose \(g\) to be the identity.

Proof. By Lemma 3.6, there are dense open immersions \(S \leftarrow U_X \leftarrow X\) and \(S \leftarrow U_Y \leftarrow Y\) such that \(U_X\) (resp. \(U_Y\)) contains all strata of \(\text{snc}(S, \Delta_S)\) and \((X, \Delta_X)\) (resp. \(\text{snc}(S, \Delta_S)\) and \((Y, \Delta_Y)\)). Then \(U := U_X \cap U_Y\) also contains all strata of \(\text{snc}(S, \Delta_S)\) -- moreover since \(f\) and \(g\) are thrifty, the strata of \((X, \Delta_X)\) and \((Y, \Delta_Y)\) are in one-to-one birational correspondence with those of \((S, \Delta_S)\), so it remains true that \(U\) contains all strata of \((X, \Delta_X)\) and \((Y, \Delta_Y)\). Replacing \(U\) with \(U \cap \text{snc}(X, \Delta_X)\), we may assume \((U, \Delta_U := \Delta_S \cap U)\) is an snc pair. We now have morphisms \(i : U \leftarrow X, j : U \leftarrow Y\) which are thrifty and birational, but not necessarily proper.

Now let \(X_i \rightarrow X_{i-1} := X, Y_i \rightarrow Y_{i-1} := Y\) and \(U_i \rightarrow U_{i-1} := U\) be the augmented semi-simplicial schemes associated to these simple normal crossing pairs. The inclusions \(i\) and \(j\) induce a diagram as in (4.13); we proceed to verify that the hypotheses of Lemma 4.12 are satisfied. All schemes in sight are defined over the noetherian and hence quasi-compact quasi-separated \(S\). The differentials and augmentations are all closed immersions and hence proper, and thriftiness of \(i\) and \(j\) implies that the morphisms \(X_i \leftarrow U_i \leftarrow Y_i\) are dense open immersions for all \(i\). Applying Lemma 3.8 to the collection of \(S\)-schemes \(X, Y\) and \(U\) with the common dense open \(U\) gives a common \(U\)-admissible blowup \(X \leftarrow Z \to Y\). Finally (for the moreover part of the lemma) \(S\) is excellent by hypothesis and the \(U_i\) are regular, hence Cohen-Macaulay.

The output of Lemma 4.12 is an augmented semi-simplicial scheme \(Z_i \rightarrow Z_{i-1} := Z\) such that each scheme \(Z_i\) is Cohen-Macaulay, together with morphisms \(X_i \leftarrow Z_i \leftarrow Y_i\), such that for each \(i\) the \(X_i \leftarrow Z_i \leftarrow Y_i\) are \(U_i\)-admissible blowups. For the remainder of the proof we argue exactly as in Theorem 4.21.

5.2 Semi-simplicial versus thrifty resolutions

We can also deduce the following lemma. While it's statement is quite verbose, it may be of interest as it opens the possibility of defining rational singularities of pairs without thrifty resolutions.

Lemma 5.7. Let \((S, \Delta_S)\) be a pair, with \(\Delta_S\) reduced and effective, and suppose \((S, \Delta_S)\) admits a thrifty resolution. Let \(e^S : S \to S\) be the associated semi-simplicial scheme and suppose there exists an
augmented semi-simplicial scheme $\xi^Y : Y, \to Y$ with proper differentials and augmentation, together with a map $g, : Y, \to S$, such that each $Y,,$ has rational singularities in the sense of [Kov20] and all of the morphisms $Y, \to S,$ are admissible blowups for $U_{Y,} \subseteq S,$ where $U_Y \subseteq S$ is a dense open containing the generic points of all strata of $\mathrm{sc}(S, \Delta_S)$ and $U_{Y,}$ is the semi-simplicial scheme associated to $(U_Y, U_Y \cap \Delta_S)$. Let $\mathcal{K} = \text{cone}(\mathcal{O}_Y \to \mathcal{R}_Y \mathcal{O}_Y, |(-1)|$ (this is a complex in the derived category of $Y$).

Then, $(S, \Delta_S)$ is a rational pair if and only if the sheaf $\mathcal{O}_S(-\Delta_S)$ is Cohen-Macaulay and the natural map $\mathcal{O}_S(-\Delta_S) \to \mathcal{R}g, \mathcal{K}$ is quasi-isomorphism.

**Proof.** Let $f : X \to S$ be a thrifty resolution. We observe that the first two paragraphs of the proof of Lemma 5.6 remain valid when $Y,$ is defined as in the lemma (as opposed to being obtained as the associated semi-simplicial scheme of an snc pair $(Y, \Delta_Y)$). Since the $Y,,$ are assumed to have rational singularities in the sense of [Kov20] the spectral sequence argument appearing in Lemma 5.6 and Theorem 4.21 remains valid and we conclude

$$Rf, \mathcal{O}_X(-\Delta_X) \approx \mathcal{R}g, \mathcal{K}. \tag{5.8}$$

Now, if $(S, \Delta_S)$ is a rational pair, by Lemma 5.6 $f$ is a thrifty resolution and hence by Definition 5.2 and Lemma 5.4 $\mathcal{O}_S(-\Delta_S) \approx Rf, \mathcal{O}_X(-\Delta_X)$ and $\mathcal{O}_S(-\Delta_S)$ is Cohen-Macaulay; from eq. (5.8) we conclude $\mathcal{O}_S(-\Delta_S) \approx \mathcal{R}g, \mathcal{K}$. On the other hand, if $\mathcal{O}_S(-\Delta_S)$ is Cohen-Macaulay and $\mathcal{O}_S(-\Delta_S) \approx \mathcal{R}g, \mathcal{K}$ eq. (5.8) shows $\mathcal{O}_S(-\Delta_S) \approx Rf, \mathcal{O}_X(-\Delta_X)$ and Definition 5.2 and Lemma 5.4 imply $f$ is a rational resolution. \qed

In characteristic 0 where we know resolutions exist, we can construct a $Y,$ as in the statement of the lemma with each $Y,$ smooth, using the methods of Section 4, but substituting Hironaka’s strong resolution of singularities for Theorem 4.1. Unlike an augmented semi-simplicial resolution obtained from a thrifty resolution $f : X \to S$, the differentials of $Y,$ need not be closed embeddings. To illustrate this distinction in the simplest possible situation, suppose $S$ is a smooth variety over $C$ and $D \subseteq S$ is a (not necessarily smooth) irreducible divisor. In this case a thrifty resolution is simply an embedded resolution of $D$. On the other hand, an augmented semi-simplicial resolution $g, Y, \to S,$ can be obtained as $\varepsilon : \tilde{D} \to S$ where $\pi : \tilde{D} \to D$ is any resolution of $D$ and $\varepsilon$ is the composition $\tilde{D} \to D \hookrightarrow S$. Unpacking the definition of $\mathcal{K}$ we see that it is the shifted cone of the morphism $\mathcal{O}_S \to \mathcal{R}\pi_* \mathcal{O}_D,$ and hence coincides with $\mathcal{O}_S(-D)$ precisely when $\mathcal{R}\pi_* \mathcal{O}_D = \mathcal{O}_D$. This recovers the fact that when $S$ is smooth and $D$ is irreducible, $(S, D)$ is a rational pair if and only if $D$ has rational singularities [Kol13, Rmk. 2.85].

**References**

[ABW13] Donu Arapura, Parsa Bakhtary, and Jarosław Włodarczyk. “Weights on cohomology, invariants of singularities, and dual complexes”. In: *Math. Ann.* 357.2 (2013), pp. 513–550. ISSN: 0025-5831. DOI: 10.1007/s00208-013-0912-7. URL: https://doi.org/10.1007/s00208-013-0912-7.

[BM97] Edward Bierstone and Pierre D. Milman. “Canonical desingularization by blowing up the maximum strata of a local invariant”. In: *Invent. Math.* 128.2 (1997), pp. 207–302. ISSN: 0020-9910. DOI: 10.1007/s002220050141. URL: https://doi.org/10.1007/s002220050141.

[Čes21] Kęstutis Česnavičius. “Macaulayfication of Noetherian schemes”. In: *Duke Mathematical Journal* 170.7 (2021), pp. 1419–1455. DOI: 10.1215/00127094-2020-0063. URL: https://doi.org/10.1215/00127094-2020-0063.

[Con00] Brian Conrad. *Grothendieck duality and base change*. Vol. 1750. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000, pp. vi+296. ISBN: 3-540-41134-8. DOI: 10.1007/b75857. URL: https://doi.org/10.1007/b75857.
[Con03] Brian Conrad. “Cohomological Descent”. In: (2003), p. 67. URL: https://math.stanford.edu/~conrad/papers/hypercover.pdf.

[Con07] Brian Conrad. “Deligne’s notes on Nagata compactifications”. In: J. Ramanujan Math. Soc. 22.3 (2007), pp. 205–257. ISSN: 0970-1249.

[CR11] Andre Chatzistamatiou and Kay Rülling. “Higher direct images of the structure sheaf in positive characteristic”. In: Algebra Number Theory 5.6 (2011), pp. 693–775. ISSN: 1937-0652. DOI: 10.2140/ant.2011.5.693. URL: https://doi.org/10.2140/ant.2011.5.693.

[CR15] Andre Chatzistamatiou and Kay Rülling. “Vanishing of the higher direct images of the structure sheaf”. In: Compos. Math. 151.11 (2015), pp. 2131–2144. ISSN: 0010-437X. DOI: 10.1112/S0010437X15007435. URL: https://doi.org/10.1112/S0010437X15007435.

[Dan75] V. I. Danilov. “Polyhedra of schemes and algebraic varieties”. In: Mat. Sb. (N.S.) 139.1 (1975), pp. 146–158, 160.

[Del71] Pierre Deligne. “Théorie de Hodge. II”. In: Inst. Hautes Études Sci. Publ. Math. 40 (1971), pp. 5–57. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1971__40__5_0.

[Del74] Pierre Deligne. “Théorie de Hodge. III”. In: Inst. Hautes Études Sci. Publ. Math. 44 (1974), pp. 5–77. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1974__44__5_0.

[DI87] Pierre Deligne and Luc Illusie. “Relèvements modulo $p^2$ et décomposition du complexe de de Rham”. In: Invent. Math. 89.2 (1987), pp. 247–270. ISSN: 0020-9910. DOI: 10.1007/BF01389078. URL: https://doi.org/10.1007/BF01389078.

[Du81] Philippe Du Bois. “Complexe de de Rham filtré d’une variété singulière”. In: Bull. Soc. Math. France 109.1 (1981), pp. 41–81. ISSN: 0037-9484. URL: http://www.numdam.org/item?id=BullSocMathFrance_1981__109__41_0.

[EGAIV2] A. Grothendieck. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II”. In: Inst. Hautes Études Sci. Publ. Math. 24 (1965), p. 231. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1965__24__231_0.

[Eri14a] Lindsay Erickson. Deformation invariance of rational pairs. Seattle: University of Washington, 2014.

[Eri14b] Lindsay Erickson. “Deformation invariance of rational pairs”. In: arXiv: Algebraic Geometry (2014).

[EV92] Hélène Esnault and Eckart Viehweg. Lectures on vanishing theorems. Vol. 20. DMV Seminar. Birkhäuser Verlag, Basel, 1992, pp. vi+164. ISBN: 3-7643-2822-3. DOI: 10.1007/978-3-0348-8600-0. URL: https://doi.org/10.1007/978-3-0348-8600-0.

[FKX17] Tommaso de Fernex, János Kollár, and Chenyang Xu. “The dual complex of singularities”. In: Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata’s sixtieth birthday. Vol. 74. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2017, pp. 103–129. DOI: 10.2969/aspm/07410103. URL: https://doi.org/10.2969/aspm/07410103.
[LT81] Joseph Lipman and Bernard Teissier. “Pseudorational local rings and a theorem of Briançon-Skoda about integral closures of ideals”. In: Michigan Math. J. 28.1 (1981), pp. 97–116. ISSN: 0026-2285. URL: http://projecteuclid.org/euclid.mmj/1029002461.

[Mat80] Hideyuki Matsumura. Commutative algebra. Second. Vol. 56. Mathematics Lecture Note Series. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980, pp. xv+313. ISBN: 0-8053-7026-9.

[Ols16] Martin Olsson. Algebraic spaces and stacks. Vol. 62. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2016, pp. xi+298. ISBN: 978-1-4704-2798-6. DOI: 10.1090/coll/062. URL: https://doi.org/10.1090/coll/062.

[Pre17] L. Prelli. “On rationalizing divisors”. In: Periodica Mathematica Hungarica 75 (2017), pp. 210–220.

[R&D] Robin Hartshorne. Residues and duality. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin-New York, 1966, pp. vii+423.

[RG71] Michel Raynaud and Laurent Gruson. “Critères de platitude et de projectivité. Techniques de “platification” d’un module”. In: Invent. Math. 13 (1971), pp. 1–89. ISSN: 0020-9910. DOI: 10.1007/BF01390094. URL: https://doi.org/10.1007/BF01390094.

[SAGAII] Théorie des topos et cohomologie étale des schémas. Tome 2. Lecture Notes in Mathematics, Vol. 270. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Springer-Verlag, Berlin-New York, 1972, pp. iv+418.

[ST08] Karl Schwede and Shunsuke Takagi. “Rational singularities associated to pairs”. In: Michigan Mathematical Journal 57. none (2008), pp. 625–658. DOI: 10.1307/mmj/1220879429. URL: https://doi.org/10.1307/mmj/1220879429.

[Stacks] The Stacks project authors. The Stacks project. 2021. URL: https://stacks.math.columbia.edu.

[Ste06] D. A. Stepanov. “A remark on the dual complex of a resolution of singularities”. In: Uspekhi Mat. Nauk 61.1(367) (2006), pp. 185–186. ISSN: 0042-1316. DOI: 10.1070/RM2006v061n01ABEH004309. URL: https://doi.org/10.1070/RM2006v061n01ABEH004309.

[Sza94] Endre Szabó. “Divisorial log terminal singularities”. In: J. Math. Sci. Univ. Tokyo 1.3 (1994), pp. 631–639. ISSN: 1340-5705.

[Wei94] Charles A. Weibel. An introduction to homological algebra. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1. DOI: 10.1017/CBO9781139644413. URL: https://doi.org/10.1017/CBO9781139644413.
A (Non-)examples of thrift

In this section we work over a field \( k \). Our first example is not new, and likely served as the original motivation for considering thrify morphisms.

**Example A.1.** Let \( S = \mathbb{A}^2 \) and \( \Delta = V(xy) \). Then \( f : X = \text{Bl}_0 S \to S \) is neither thrify nor rational. Indeed, letting \( D_1 = V(x), D_2 = V(y) \) we see that \( \Delta \) is the union of the 2 lines \( D_1, D_2 \) meeting at the origin. Let \( D_1 = f^{-1}_* D_1 \) be the strict transforms, \( E = f^{-1}(0) \) the exceptional divisor, and \( \tilde{\Delta} = \tilde{D}_1 + \tilde{D}_2 \) (see Figure 1). The map \( f : X \to S \) fails to be thrify since it is not an isomorphism over the stratum \( 0 = \tilde{D}_1 \cap \tilde{D}_2 \) of \((S, \Delta)\). We will calculate cohomology to show \( f \) isn’t rational either.

Since \( S = \mathbb{A}^2 \) is affine, we can identify the sheaves \( R^i f_* \mathcal{O}_X(-\tilde{\Delta}) \) as the sheaves associated to the \( k[x,y] \)-modules \( H^i(X, \mathcal{O}_X(-\tilde{\Delta})) \). Observe that \( X \) can be identified with the geometric line bundle \( \text{Spec}_{\mathbb{P}^1} \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \) associated to \( \mathcal{O}_{\mathbb{P}^1}(1) \). Under this identification, the projection \( \pi : \text{Spec}_{\mathbb{P}^1} \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{P}^1 \) corresponds to the composition \( \text{Bl}_0 S \subseteq \mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{P}^1 \), and the blowup map \( f : \text{Spec}_{\mathbb{P}^1} \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{A}^2 \) corresponds to the natural map

\[
\text{Spec}_{\mathbb{P}^1} \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \text{Spec}_k H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = \text{Spec}_k k[x,y] = \mathbb{A}^2
\]

Hence \( \tilde{\Delta} = \pi^*(0 + \infty) \). Now since \( \pi \) is affine its Leray spectral sequence degenerates to give

\[
H^i(X, \mathcal{O}_X(-\tilde{\Delta})) = H^i(\mathbb{P}^1, \pi_* \mathcal{O}_X(-\tilde{\Delta})) \text{ and via projection formula } \pi_* \mathcal{O}_X(-\tilde{\Delta}) = \pi_* \mathcal{O}_X(-\pi^*(0 + \infty)) = (\pi_* \mathcal{O}_X)(-0 - \infty)
\]

By the correspondence between affine schemes and sheaves of algebras,

\[
\pi_* \mathcal{O}_X = \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) = \bigoplus_{d \geq 0} \mathcal{O}_{\mathbb{P}^1}(d)
\]

Hence \( H^i(X, \mathcal{O}_X(-\tilde{\Delta})) = \bigoplus_{d \geq 0} H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 2)) \). In particular, when \( i = 1 \) and \( d = 0 \), we see \( H^1(X, \mathcal{O}_X(-\tilde{\Delta})) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \approx k \) by [Har77, Thm. III.5.1].

An elaboration of **Example A.1** shows in general that if \((S, \Delta)\) is a simple normal crossing pair and \( Z \subseteq S \) is a stratum, then \( f : X = \text{Bl}_Z S \to S \) fails to be thrify. Localizing at the generic point \( \eta \in Z \) we can reduce to the case where \( Z \) is replaced by a closed point \( \eta \in S \) and \( \Delta = V(x_1 \cdots x_n) \) where \( x_1, \cdots, x_n \in m_\eta \) is a regular system of parameters. Then the long exact sequence obtained by pushing forward \( \mathcal{O}_X(-\tilde{\Delta} - E) \to \mathcal{O}_X(-\tilde{\Delta}) \to \mathcal{O}_E(-\tilde{\Delta}|_E) \) ends in

\[
R^{n-1} f_* \mathcal{O}_X(-\tilde{\Delta} - E) \to R^{n-1} f_* \mathcal{O}_X(-\tilde{\Delta}) \to R^{n-1} f_* \mathcal{O}_E(-\tilde{\Delta}|_E) \to R^n f_* \mathcal{O}_X(-\tilde{\Delta} - E) = 0
\]

where the vanishing on the right holds since the maximal fiber dimension of \( f \) is \( n - 1 \) [Har77, Cor. III.11.2]. Thus \( R^{n-1} f_* \mathcal{O}_X(-\tilde{\Delta}) \to R^{n-1} f_* \mathcal{O}_E(-\tilde{\Delta}|_E) = H^{n-1}(E, \mathcal{O}_E(-\tilde{\Delta}|_E)) \) is surjective, and
identifying $E$ with the projectivized Zariski tangent space $\mathbb{P}(TS_n)$ with homogeneous coordinates $x_1, \ldots, x_n$ and $\Delta|_E$ with $V(\prod x_i)$ shows $H^{n-1}(E, \mathcal{O}_E(-\Delta|_E)) \cong H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-n)) \cong k$. For related discussion see [Kol13, p. 86].

The next example answers (in the affirmative!) a question of Erickson [Eri14b, p.2] and Prelli [Pre17, p.3] about whether there exists a resolution which is rational but not thrifty. In fact, we give such an example where the underlying pair $(S, \Delta)$ is rational.

**Example A.2.** Let $S = V(xy - zw) \subseteq \mathbb{A}^4_{xyzw}$, $D_0 = V(x, w)$ and $D_\infty = V(y, z)$; finally let $C_\infty = V(w, y)$.

We can identify $S = C(\mathbb{P}^1 \times \mathbb{P}^1)$ as the affine cone over the Segre embedding $\mathbb{P}^1_s \times \mathbb{P}^1_t \hookrightarrow \mathbb{P}^3_{xyzw}$ given by

$$
\begin{bmatrix}
    x & u \\
    z & v \\
\end{bmatrix} =
\begin{bmatrix}
    s_0 & t_0 \\
    s_1 & t_1 \\
\end{bmatrix}
\begin{bmatrix}
    s_0 t_0 & s_0 t_1 \\
    s_1 t_0 & s_1 t_1 \\
\end{bmatrix}
$$

Hence $D_0 = C([0] \times \mathbb{P}^1), D_\infty = C([\infty] \times \mathbb{P}^1)$ and $C_\infty = C(\mathbb{P}^1 \times [\infty])$.

Let $\Delta = D_0 + D_\infty + C_\infty$. Note that $\Delta$ is not Cartier, as it is not linearly equivalent to any multiple of $C([0] \times \mathbb{P}^1) + C(\mathbb{P}^1 \times [0])$ (here $[0] \times \mathbb{P}^1 + \mathbb{P}^1 \times [0]$ is a hyperplane section of the Segre embedding) — see e.g. [Har77, Ex. II.6.3],[Kol13, Prop. 3.14]. Since $K_S$ is $Q$-Cartier, it follows that the pair $(S, \Delta = D_0 + D_\infty)$ is not $Q$-Gorenstein — in particular it isn’t dlt, so we are not at risk of violating [Kol13, Thm. 2.87] which implies that a resolution of a dlt pair is thrifty if and only if it is rational.

To show that $S$ is not thrifty, let $f : X = Bl_{D_0} S \rightarrow S$ be the blowup at $D_0$, let $D_i = f^{-1}_* D_i$ for $i = 0, \infty$ and $C_\infty = f^{-1}_* C_\infty$, and let $\Delta = D_0 + D_\infty + C_\infty$. The map $f$ is a small resolution of $S$ (as mentioned in [KM98, Ex. 2.7]). This means we are not at risk of violating [Eri14b, Prop. 1.6] which states that if a log resolution of a pair is rational then it is thrifty. Indeed, the ambient blowup is described as

$$
Bl_{D_0} \mathbb{A}^4 \subseteq \{(x, y, z, w), [u, v] | (x, u) \propto (u, v)\} \subseteq \mathbb{A}^4 \times \mathbb{P}^1_{uv}
$$

so on the $D(u)$ patch $(x, u) = \lambda(1, u)$ and

$$
xy - zw = \lambda y - z\lambda v = \lambda(y - zv)
$$

Since $V(\lambda)$ is the exceptional divisor we see the strict transform $X \subseteq Bl_{D_0} \mathbb{A}^4$ of $S$ is $V(y - zw)$ on the $u = 1$ patch — this is smooth as it’s a graph. By symmetry in $x, w$, we conclude $X$ is smooth.

Even better, this allows us to parametrize $X \cap D(u)$ with coordinates $z, \lambda, v$:

$$
\mathbb{A}^3_{z, \lambda, v} \cong Bl_{D_0} S \cap D(u) \cong D(u) \cong \mathbb{A}^5_{xyzw}
$$

sending $(z, \lambda, v) \mapsto (\lambda, zv, z, \lambda v, v) = (x, y, z, w, v)$

So in particular the restriction of $f$ looks like $(z, \lambda, v) \mapsto (\lambda, zv, z, \lambda v)$ and we see that the exceptional locus is the $v$-axis. In this coordinate patch the strict transforms $\tilde{D}_0$ and $\tilde{D}_\infty$ are $V(\lambda)$ and $V(z)$ respectively, which intersect along the $v$-axis $V(\lambda, z)!$ Thus $\Delta$ has a stratum in Ex$(f)$ and $f$ isn’t thrifty. We also see that on this patch $C_\infty = V(u)$. As a philosophical aside, the blowup coordinates $[u, v]$ correspond to $[x, w] = [s_0 t_0, s_0 t_1] = [t_0, t_1]$ as long as $s_0 \neq 0$, so Ex $f$ can be viewed as a copy of the $\mathbb{P}^1_t$ appearing in $D_\infty = C([0] \times \mathbb{P}^1)$ — see Figure 2.

To show that $f$ is in fact a rational resolution we will use an alternative description of $X$. Starting with the ample invertible sheaf $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ we have natural morphisms of relative spectra

$$
Spec_{\mathbb{P}^1 \times \mathbb{P}^1} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \rightarrow Spec_{\mathbb{P}^1_t} \mathcal{O}_{\mathbb{P}^1_t}(1, 1) \xrightarrow{f'} Spec_{\mathbb{C}} H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))
$$

$$
(A.5)
$$

where $pr_i : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the projection. It is well known that the scheme on the left can be identified with the blowup $Bl_0 S$, and the scheme on the right is $S$.

**Claim A.6.** There is an isomorphism of $S \times \mathbb{P}^1$-schemes

$$
X = Bl_{D_0} S \cong Spec_{\mathbb{P}^1_t} \mathcal{O}_{\mathbb{P}^1_t}(1, 1)
$$

$$
(A.5)
$$

31
This can be proved via the universal property. On the other hand, at least when \( k \) is algebraically closed, a quick, dirty and more illuminating proof is possible: we have a morphism \((f^t, \pi) : \text{Spec}_{\mathbb{P}} \to \text{Spec} \times_{\mathbb{P}} (1, 1) \to S \times \mathbb{P}_t^1\); the first factor is the second map of (A.5), the second is the canonical projection

\[
\pi : \text{Spec}_{\mathbb{P}} \to \text{Spec} \times_{\mathbb{P}} (1, 1) \to \mathbb{P}_t^1
\]

from the relative Spec construction. \( X \subseteq S \times \mathbb{P}^1_t \) by construction, and we can check \( \varphi \) maps the \( k \)-points of \( \text{Spec}_{\mathbb{P}} \) bjectively onto those of \( X \). Indeed, the fiber of \( \text{Spec}_{\mathbb{P}} \) over \( t \in \mathbb{P}_t^1 \) can be described as follows: Note by projection formula

\[
\text{Sym} H^0(\mathbb{P}^1_t, \mathcal{O}_{\mathbb{P}^1_t}(1, 1)) \cong H^0(\mathbb{P}^1_t, \mathcal{O}_{\mathbb{P}^1_t}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^1_t}(1)
\]

Explicitly \( H^0(\mathbb{P}^1_t, \mathcal{O}_{\mathbb{P}^1_t}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^1_t}(1) \cong \mathcal{O} \mathbb{P}^1_t(1) \) where \( x \) denotes the product of graded rings of \([\text{Har77}, \text{Ex. II.5.11}]\) and hence for a \( k \)-point \( t \),

\[
\pi^{-1}(t) = \text{Spec} k[s_0, s_1], \quad \text{so that } f^t|_{\pi^{-1}(t)} : \pi^{-1}(t) \to S
\]

is a map \( \mathbb{P}^2_{s_0, s_1} \to \mathbb{P}^4_{x} \). Writing down the map of algebras corresponding to \( f^t \) shows that it is none other than the linear transformation of (A.3). Finally, referencing (A.4) we see that the fibers of \( X \to \mathbb{P}^1_t \) have the same description.\(^{13}\)

Using the claim, we proceed as in Example A.1 using degeneration of the Leray spectral sequence for the affine map \( \pi : X \to \mathbb{P}^1_t \) to calculate

\[
H^1(X, \mathcal{O}_X(-\Delta)) = H^1(\mathbb{P}^1_t, \pi_* \mathcal{O}_X(-\Delta))
\]

On \( \mathbb{P}^1_t \), noting that \( \mathcal{C}_\infty = \pi^*(\infty) \), the projection formula gives

\[
\pi_* \mathcal{O}_X(-\Delta) = (\pi_* \mathcal{O}_X(-D_0 - D_\infty))(\infty)
\]

and \( \pi_* \mathcal{O}_X(-D_0 - D_\infty) \subseteq \pi_* \mathcal{O}_X \) is the sheaf of ideals \( (s_0 \cdot s_1) \subseteq k[s_0, s_1] \times \text{Sym} \mathcal{O}_{\mathbb{P}^1_t}(1) \). Letting \( (s_0 \cdot s_1)_d \subseteq k[s_0, s_1] \) denote the \( d \)-th graded part, we see

\[
\pi_* \mathcal{O}_X(-\Delta) = \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k \mathcal{O}_{\mathbb{P}^1_t}(d - 1)
\]

\(^{13}\)In slogan form: \( X = \text{Bl}_{s_0} S \) is a pencil of 2-planes on \( S \) corresponding to the pencil of rulings \( \mathbb{P}^1_t \times \{t\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1_t \).
where the “-1” comes from the twist “\((-\infty)\)” in (A.8). This yields:
\[
H^i(P^1_1, \pi_* O_X(-\Delta)) = \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k H^i(P^1_1, O_{P^1_1}(d - 1))
\]
\[
= \begin{cases} 
\bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k (t_1)_d \subseteq k[s_0, s_1] \times k[t_0, t_1] = H^0(S, O_S) & \text{if } i = 0 \\
0 & \text{if } i = 1 
\end{cases}
\]
(A.9)

the key point being that \(H^i(P^1_1, O_{P^1_1}(d - 1)) = 0\) for \(d \geq 0\). This calculation shows \(f_* O_X(-\Delta) = O_S(-\Delta)\) (this holds for more general reasons, namely \(S\) is normal [Pre17, Lem. 2.1]) and \(R^1 f_* O_X(-\Delta) = 0\).

Finally, \((S, \Delta)\) is a rational pair, as a consequence of the theorem below — this was the main reason for including the additional divisor \(C_\infty\). If we had left it out, the above calculations would still show that \(f : X \to (S, D_0 + D_\infty)\) is a non-thrifty rational resolution, however the pair \((S, D_0 + D_\infty)\) isn’t rational (also by the theorem below).

**Theorem A.10** ([Pre17, Thm. 3.2]). Let \((Y, B)\) be a pair such that \(Y\) is a normal variety over \(k\) and \(B\) is a reduced effective Weil divisor on \(Y\) (for example a simple normal crossing pair) and let \(\mathcal{L}\) be an ample invertible sheaf on \(Y\). Let \((C_Y, CB)\) be the abstract affine cone over \((Y, B)\) with respect to \(\mathcal{L} : CY = \text{Spec}_k H^0(Y, \text{Sym} \mathcal{L})\) and \(CB\) is the image of \(\text{Spec}_k H^0(B, \text{Sym} \mathcal{L}|_B) \to \text{Spec}_k H^0(Y, \text{Sym} \mathcal{L}) = CY\) with its reduced subscheme structure. Then \((C_Y, CB)\) is a rational pair if and only if \((Y, B)\) is a rational pair and

\[
H^i(Y, \mathcal{L}^d(\mathcal{L}_\mathcal{L})) = 0 \quad \text{for } i > 0, d \geq 0.
\]

Applying the theorem to \(Y = P^1 \times P^1\) with the divisor \(B = \{0, \infty\} \times P^1 + P^1 \times \{\infty\}\) which has associated invertible sheaf \(O_Y(B) \simeq O_{P^1 \times P^1}(2, 1)\), together with the ample invertible sheaf \(\mathcal{L} = O_{P^1 \times P^1}(1, 1)\) we calculate (using Künneth)

\[
H^i(Y, \mathcal{L}^d(-B)) = H^i(P^1 \times P^1, O_{P^1 \times P^1}(d - 2, d - 1)) = \bigoplus_{j+k=i} H^j(P^1, O_{P^1}(d - 2)) \otimes_k H^k(P^1, O_{P^1}(d - 1))
\]

(A.11)

Noting that \(H^k(P^1, O_{P^1}(d - 1)) = 0\) for \(k > 0\) and \(d \geq 0\), we see that \(H^2(Y, \mathcal{L}^d(-B)) = 0\) for \(d \geq 0\), and

\[
H^1(Y, \mathcal{L}^d(-B)) = H^1(P^1, O_{P^1}(d - 2)) \otimes_k H^0(P^1, O_{P^1}(d - 1))
\]

Now \(H^1(P^1, O_{P^1}(d - 2)) = 0\) for \(d \neq 0\), but \(H^0(P^1, O_{P^1}(d - 1)) = 0\), so the tensor product is always 0.

The last example of this section shows that even when \((S, \Delta)\) is a simple normal crossing pair and \(f : X \to S\) is a \(U\)-admissible blowup for some \(U \subseteq S\) containing all strata, and \(\Delta = f_{\ast}^{-1}\Delta\) is snc, \(f\) may still fail to be thrifty. Unfortunately our presentation only makes sense in characteristic 0, but I would be shocked and appalled if this example doesn’t work in any characteristic \(p > 2\).

**Example A.12.** Let \(S = A^3_{xy}\), let \(\Delta = V((z - x)(z + x))\) and let \(Z = V(x, y); \) let \(U = S \setminus Z\). Then there is a \(U\)-admissible blowup \(f : X \to S\) such that \(f_{\ast}^{-1}\Delta\) is a simple normal crossing divisor but \(f\) is not thrifty.

We first blow up \(Z\) to obtain \(g : Bl_Z S \to S\), and claim that the strict transform of \(\Delta\) is no longer snc. Letting \(D_\pm = V(z \pm x)\) we can work in blowup coordinates described like

\[
\text{Bl}_Z S = \{(x, y, z), [u, v]) \in A^3 \mid (x, y) \propto (u, v)\}
\]

so that on the \(D(u)\) patch \((x, y) = \lambda(1, v)\) and

\[
z \pm x = z \pm \lambda, \text{ so in } (z, \lambda, u) \text{ coordinates } \tilde{D}_\pm \cap D(u) = V(z \pm \lambda)
\]
in other words \(\tilde{\Delta}\) is snc on the \(D(u)\) patch (as is expected since on \(D(x) \subseteq A^3, \Delta\) is smooth). But on the \(D(v)\) patch where \((x, y) = \lambda(u, 1),

\[
z \pm x = z \pm \lambda u, \text{ so in } (z, \lambda, u) \text{ coordinates } \tilde{D}_\pm \cap D(v) = V(z \pm \lambda u)
\]

(A.13)
and here we see the strict transforms intersect along \( V(\lambda u) \) and hence fail to be snc (Figure 3).

A global description of the situation: \( \text{Bl}_2 S \) is isomorphic to \( \mathbb{A}^1_z \times \text{Bl}_0 \mathbb{A}^2_{xy} \), and \( D_\pm \) are 2 copies of \( \text{Bl}_0 \mathbb{A}^2_{xy} \) embedded via the maps

\[
(\pm x, \text{id}) : \text{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^1_z \times \text{Bl}_0 \mathbb{A}^2_{xy}
\]

where the map \( \pm x : \text{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^1_z \) really means the composition \( \text{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^2_{xy} \xrightarrow{\pm x} \mathbb{A}^1_z \). From this perspective \( D_+ \cap D_- \) is the preimage of \( V(x) \) under the blowup map \( \text{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^2_{xy} \), the union \( \mathbb{P}^1_{xy} \cup \mathbb{A}^1_y \) glued along the points \([0,1] \in \mathbb{P}^1_{xy} \) and \( 0 \in \mathbb{A}^1_y \). Let \( p \) denote the point in \( \mathbb{P}^1_{xy} \cap \mathbb{A}^1_y \). Equivalently \( \text{Sing}(D_+ \cap D_-) \) consists of a single closed point which we call \( p \).

This discussion shows that the snc locus of \( (\text{Bl}_2 S, \Delta) \) is

\[
\text{snc}(\text{Bl}_2 S, \Delta) = \text{Bl}_2 S \setminus \{p\}
\]

By work of Szabó and Bierstone-Milman [BM97; Sza94] (this is where we use the characteristic 0 hypothesis) there exists a further blowup \( h : X \to \text{Bl}_2 S \) such that \( h^{-1}_-\Delta + \text{Ex} h \) is a simple normal crossing divisor and \( h \) is an isomorphism over snc(\( \text{Bl}_2 S, \Delta \)), that is, \( h \) must be a snc(\( \text{Bl}_2 S, \Delta \))-admissible blowup. Now by [Har77, Thm. II.7.17] we know that \( f := \gamma h : X \to S \) is a blowup at some closed subscheme \( W \subseteq S \) and since \( g(p) \in Z \) (equivalently) \( g^{-1}(U) \subseteq \text{snc}(\text{Bl}_2 S, \Delta) \), it must be that \( W \subseteq Z \) as closed sets (see also [RG71, Lem. 5.1.4]), hence \( f : X \to S \) is a \( U \)-admissible blowup.

On the other hand, by a proposition of Erickson [Eri14b, Prop. 1.4], since \( h^{-1}_-\Delta + \text{Ex} h \) is snc the map \( h \) is thrifty and so the strata of \( f^{-1}_{S} = h^{-1}_-\Delta \) are in 1-1 birational correspondence with those of \( \Delta \), in particular \( f^{-1}_{S} \Delta \) has a stratum in \( \text{Ex} f \).

While the application of [BM97; Sza94] is heavy-handed for this toy example, we point out that \( h \) is not simply the blowup at \( p \) as one might initially guess: starting from (A.13), blowing up the origin \( 0 \in \mathbb{A}^1_{z\lambda u} \) and introducing blowup coordinates

\[
\text{Bl}_0 \mathbb{A}^3_{z\lambda u} = \{(z, \lambda, u), (r, s, t) \in \mathbb{A}^3_{z\lambda u} \times \mathbb{P}^2_{rst} | (z, \lambda, u) \propto (r, s, t) \}
\]

we note that since \( V((z - \lambda u) \cdot (z + \lambda u)) \) is smooth on \( D(z) \) we can check that the strict transform remains smooth on the \( D(r) \) patch. We will investigate the \( D(s) \) patch — by symmetry of \( \lambda, u \) in the equation \( (z - \lambda u) \cdot (z + \lambda u) \) the situation is similar on the \( D(t) \) patch. On \( D(s) \) we have \( (z, \lambda, u) = \mu(r, 1, t) \) and so

\[
z \pm \lambda u = \mu r \pm \mu t = \mu(r \pm tu)
\]

Here \( V(\mu) \) is a copy of the exceptional divisor of \( \text{Bl}_0 \mathbb{A}^3_{z\lambda u} \to \mathbb{A}^3_{z\lambda u} \) but we are still left with strict transforms \( (r \pm \mu t) \) of exactly the same form as \( z \pm \lambda u \); in other words, blowing up \( 0 \in \mathbb{A}^3_{z\lambda u} \) does
not help! This is quite similar to the classical fact that blowing up the origin of the pinch point
\[ V(z^2 - \lambda u^2) \subseteq \mathbb{A}^3_{z,\lambda,u} \] gives another pinch point singularity. In fact, since \((z - \lambda u) \cdot (z + \lambda u) = z^2 - \lambda^2 u^2\)
our example is a double cover of the pinch point (that is, it is the preimage of the pinch point with
respect to \((z, \lambda, u) \mapsto (z, \lambda^2, u))\).