Self-delayed feedback car-following control with the velocity uncertainty of preceding vehicles on gradient roads

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Abstract  Uphill and downhill roads are prevalent in mountainous areas and freeways. Despite the advancements of vehicle-to-vehicle (V2V) communication technology, the driving field of vision could be still largely limited under such a complex road environment, which hinders the sensors from accurately perceiving the speed of front vehicles. As such, a fundamental question for autonomous traffic management is how to control traffic flow associated with the velocity uncertainty of preceding vehicles? This paper aims to answer this question by devising a cooperative control method for autonomous traffic to stabilize the traffic flow under such a complex road environment. To this end, this paper first develops a traffic flow model accounting for the uncertainty of preceding vehicle’s velocity on gradient roads and further devises a new self-delayed feedback controller based on the velocity and headway differences between the current time step and historical time step. The sufficient condition where traffic jams do not occur is derived from the perspective of the frequency domain via Hurwitz criteria and $H_\infty$ norm of transfer functions. The Bode diagram reveals that the robustness of the closed-loop traffic flow model can be significantly enhanced. Simulation results show that the key parameters (control gain coefficient and delay time) of the designed controller contribute to the stability of traffic flow, which is consistent with the theoretical analysis conclusion.

Keywords  Car-following model · Self-delayed feedback controller · Gradient road · Frequency domain · Stability

1 Introduction

Over the past decades, the rapid development of the urban economy has spurred the growth of car ownership and traffic congestion. Along with this, a variety of approaches and countermeasures have been proposed to alleviate traffic congestion by making use of emerging technologies. On the one hand, numerous traffic management and planning approaches have emerged, such as bus route optimization tactics [1, 2] and online electric vehicle charge scheduling [3]. On the other hand, another stream of research direction concentrated on the development of traffic flow models to reproduce traffic congestion and explore the underlying mechanism. With the advancements in
the V2V communication technology, connected and automatic vehicles (CAVs) are envisioned to become mainstream in future transportation, while the topic of autonomous traffic flow has attracted increased interest recently to support traffic control and management.

Efficient traffic flow control is crucial for the large-scale adoption of CAVs, while the complex traffic environment is a fundamental issue for managing autonomous traffic. As a typical road pattern, gradient roads are not uncommon in economically underdeveloped areas or mountainous freeways, as shown in Fig. 1. The road slope may exert great influence on road traffic jams [4, 5]. Despite the development of the Internet of Vehicles, the driving field of vision could be still limited in a traffic environment with gradient roads, which hinders the sensors from perceiving the speed of the front vehicle accurately and raises “velocity uncertainty,” posing significant challenges to traffic control and safety. Therefore, the velocity uncertainty of preceding vehicles should not be neglected in such a complex road environment.

The control scheme for traffic flow plays an important role in controlling the CAV system. In the provision of gradient roads and velocity uncertainty, it is essential to design an effective control scheme that can adapt to such a traffic environment to improve traffic efficiency and safety. Although many control theories have been applied in the CAV control, most

![Fig. 1 Common gradient roads in an urban and mountainous area. a the 66th highway in Hebei province, the longitude and latitude information is approximately (36°01′N, 113°04′E); b the 318th highway in Guizhou province, the longitude and latitude information is approximately (26°21′N, 107°79′E); c continuous downhill sign on Hexi section of Xiaorong highway in Fujian province, the longitude and latitude information is approximately (24°35′N, 118°05′E); d continuous uphill sign on Yizhao highway in Xinjiang province, the longitude and latitude information is approximately (43°77′N, 87°68′E)](image-url)
controllers require accurate road traffic information as
the control input. Based on the above considerations,
the research scope of this paper is to devise a self-
delayed feedback controller for autonomous traffic
considering the uncertain velocity of preceding vehi-
cles on gradient roads. The autonomous traffic is
modeled as a car-following model under a designed
controller. This research is expected to provide an
insight into microscopic traffic flow simulation and
traffic flow control in the ear of CAVs.

The remainder of this paper is as follows: In Sect. 2,
we review the related work and highlight our contri-
butions. In Sect. 3, a modified car-following model
taking into account the velocity uncertainty of pre-
ceding vehicles on gradient roads is presented. In
Sect. 4, the stability condition of the closed-loop car-
following model is obtained via Hurwitz criterion and
\( H_\infty \) norm of transfer functions. In Sect. 5, simulations
are carried out to verify the findings of the theoretical
analysis. Finally, the conclusions and future works are
given in Sect. 6.

2 Related work

According to the research subjects, current traffic flow
theoretical models can be generally divided into the
following two branches: macroscopic and microscopic
traffic flow models. The former mainly includes the
continuous models [6, 7] and lattice hydrodynamic
models [8, 9], while the latter includes the car-
following models [10, 11] and the cellular automata
models [12, 13].

In the former category, several en-route vehicles
are modeled as compressible continuous fluid with the
time- and location-dependent vehicle density and
average speed. By constructing the partial differential
equations of local density and local velocity, the
dynamic behavior of traffic flow can be revealed by
solving the equations. The development of the
macroscopic traffic flow model began with the well-
known LWR model proposed by Lighthill, Whitham,
and Richards, also termed as the kinematic wave
model [14, 15]. Based on this model, the propagation
characteristics of the nonlinear density waves can be
described, and the evolution characteristics of shock
waves and rarefaction waves can be captured via the
characteristic-line method or numerical simulation.
Later, researchers found that the vehicle speeds are
always equilibrated in the LWR model, such that the
non-equilibrium states, i.e., the stop-and-go phe-
nomenon, cannot be reproduced. To tackle this issue,
some high-order derivative models have been pro-
posed [16]. Although these models are less computa-
tionally expensive, they cannot analyze the
microscopic driving behavior of vehicles.

In contrast, the microscopic traffic flow models take
individual vehicles as the research subject and
describe the vehicle motion behavior based on differen-
tial equations as a function of the speed, headway,
and acceleration information of vehicles. In 1995,
Bando [17] proposed a seminal work of a car-
following model termed the optimal velocity (OV)
model. The OV function has an inflexion point and
elegant mathematical and physical properties, which
facilitate linear and nonlinear theoretical derivation.
Helbing and Tilch [18] used the real traffic data to
identify the parameters of the OV model and found
that the model presented excessive acceleration,
unreasonable deceleration, and crash phenomena.
Then, they presented a new generalized force (GF)
model. Jiang et al. [19] applied the GF model to
simulate the start-up process of a stationary traffic flow
and found that the starting wave speed was too small.
To address this issue, they proposed a full velocity
difference (FVD) model. Since then, the FVD model
has attracted wide attention, yielding a number of
extended and more realistic traffic flow models. To
exemplify, Natagani [20] developed an OV model
considering the headway information of the multiple
preceding vehicles. Nakayama [21] incorporated the
headway information of the following vehicles, called
the backwards-looking effect, into the OV model. In a
more general setting, Hasebe [22] proposed a car-
following model with both multiple headway infor-
mation of preceding vehicles and the backwards-
looking effect of following vehicles. The representa-
tive works can be referred to Table 1.

There are a large number of uncertainties in the real
traffic, while any small disturbance would gradually
be expanded over time and induce traffic jams. Given
the unstable nature of road traffic, how to maintain
traffic flow stability and suppress traffic jams under
uncertainties remains a prominent research topic due
to its inherent complexity. This raises a number of
traffic flow control methods, including the feedback
controller based on speed (headway) differences
[55, 59] or multiple vehicle information [60], delay
feedback controller [61], and sliding mode controller [62]. To exemplify, Li et al. [55] devised a feedback controller taking the velocity and headway differences of successive vehicles as the control input, where the solution procedure of the controller gain is given via the Lyapunov function analysis method. Providing that the velocity information of multiple preceding vehicles is difficult to be detected in the real traffic environment, Yu et al. [60] developed a feedback controller taking into account the velocity difference information between the current vehicle and two successive vehicles. Li et al. [62] proposed a sliding mode controller (SMC) concerning the errors between the desired headway and the actual headway, where the stability of the controller is guaranteed using the Lyapunov technique. Li et al. [63] devised a self-delayed feedback controller considering the velocity difference between the current velocity and historical velocity. Different from feedback controllers, such a self-delayed feedback controller relies only on its vehicle traffic information without acquiring additional vehicle information, which is suitable for the complex traffic environment with the provision of certain errors between the perceived data and the real data associated with vehicle information.

Although many traffic flow models and associated feedback control methods have been developed, how to effectively control traffic flow in the provision of gradient roads and velocity uncertainty is still an open question. This paper aims to fill in this gap. The main contributions of this paper are twofold: First, this paper proposes an extended traffic flow model accounting for the effect of velocity uncertainty of preceding vehicles on gradient roads. Our model is a generalized form of previous models, which can better reflect the real traffic environment. Second, a new self-delayed feedback controller is designed, which is prone to stabilize traffic flow in such a complex traffic environment. Compared to the traditional feedback controller, the designed controller can adjust the vehicle status during the delay time relying only on the velocity and headway information of the current vehicle without additional vehicle information. The stability criterion of the closed-loop system is also derived.

### 3 Model

We now revisit the traditional model and introduce the rationale behind our proposed model. The primary notations used in this paper are listed in Table 2. In 1995, Bando [17] proposed an OV model to study the car-following characteristics of vehicles on a single
lane where overtaking is forbidden, and the dynamic equation is expressed as:

\[
\frac{dv_n(t)}{dt} = a[V(y_n(t)) - v_n(t)]
\]  

(1)

\[
\frac{dy_n(t)}{dt} = v_{n+1}(t) - v_n(t)
\]  

(2)

where the function \(V(\cdot)\) is uniquely determined by the parameter \(y_n(t)\), specifically

\[
V(y_n(t)) = \frac{v_{\text{max}}}{2} \left[ \tanh(y_n(t) - y_c) + \tanh(y_c) \right]
\]  

(3)

To eliminate the phenomenon of excessive acceleration and unreasonable deceleration in the traditional OV model, Helbing and Tilch [18] presented an improved GF model expressed as follows:

\[
\frac{dv_n(t)}{dt} = a[V(y_n(t)) - v_n(t)] + \lambda H(-\Delta v_n(t)) \Delta v_n(t)
\]  

(4)

where \(\Delta v_n(t) = v_{n+1}(t) - v_n(t)\). \(H(\cdot)\) represents the Heaviside function, and

\[
H(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
1, & \text{if } x > 0 
\end{cases}
\]  

(5)

Later, to overcome the low starting wave speed in the aforementioned GF model, Jiang et al. [19] presented a FVD model, which is written as follows:

\[
\frac{dv_n(t)}{dt} = a[V(y_n(t)) - v_n(t)] + \lambda \Delta v_n(t)
\]  

(6)

Since the FVD model, extensive derivative models are developed. Given that uphill and downhill roads are prevalent in mountainous areas and freeways, the literature [4] proposed a traffic flow model considering road slope information to explore the impact of road

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**Table 2** Primary notations used in the proposed model

| Symbols | Definition |
|---------|------------|
| \(n\)  | The subscript of a vehicle |
| \(a\)  | Driver’s sensitivity |
| \(y_n(t)\) | Instantaneous headway information |
| \(v_n(t)\) | Instantaneous velocity information |
| \(V(\cdot)\) | Optimal speed function |
| \(v_{\text{max}}\) | Maximum speed |
| \(y_c\) | Safety distance between vehicles on the general road without any slope |
| \(\Delta v_n\) | Velocity difference information |
| \(\dot{\lambda}\) | Weight coefficient of velocity difference term |
| \(V^{\text{opt}}(\cdot)\) | Modified optimal velocity function constrained by the slope information |
| \(v_{g,\text{max}}(\theta)\) | Speed adjustment item affected by the acceleration of gravity |
| \(y_c(\theta)\) | Safety distance item affected by the acceleration of gravity |
| \(v_{f,\text{max}}\) | Maximum velocity without any slope |
| \(m\) | The mass of the vehicle |
| \(g\) | The acceleration of gravity, which is set \(g = 9.8 \text{ m/s}^2\) |
| \(\mu\) | The friction coefficient, which is set as \(\mu = mg\) |
| \(\eta\) | Adjustment coefficient, which is taken as \(\eta = 1\) |
| \(u_n(t)\) | Designed control input |
| \(\varepsilon\) | Error coefficient representing the difference between the perceived data and real data |
| \(\tau_1\) | Delay time |
| \(k_1\) | The weight coefficients corresponding to the headway difference |
| \(k_2\) | The weight coefficients corresponding to the speed difference |
slope on traffic jams, and the kinetic equation is as follows:

\[
d\frac{v_n(t)}{dt} = a[V^{\text{op}}(y_n(t)) - v_n(t)] + \lambda \Delta v_n(t)
\]

where the function \( V^{\text{op}}(\cdot) \) is taken as follows:

\[
V^{\text{op}}(y) = \frac{v_{\text{max}}(y) - v_{g,\text{max}}(\theta)}{2} \left[ \tanh(y - y_c(\theta)) + \tanh(y_c(\theta)) \right]
\]

Figure 2 shows a schematic diagram of the force of the vehicle on the uphill and downhill scenarios. Suppose that the angle information of the sloping road is \( \theta \). We now decompose the gravity of the vehicle on the road into two forces. One is parallel to the road, and the other is perpendicular to the road. As a result, the magnitudes of the two forces are \( mg \cos \theta \) and \( mg \sin \theta \), respectively, where the direction of the arrow is the direction of the force. The specific value of \( v_{\text{max}}(\theta) \) can be determined by

\[
v_{g,\text{max}}(\theta) = \frac{mg \sin \theta}{\mu}
\]

Since the safety distance differs in the uphill (\( \theta > 0 \)) and downhill (\( \theta < 0 \)) scenarios [4], we have that

\[
y_c(\theta) = y_c(1 - \eta \sin \theta)
\]

As a result, the corresponding OV function can be rewritten as

\[
V^{\text{op}}(y) = \frac{v_{g,\text{max}} - \sin \theta}{2} V_0(y)
\]

where

\[
V_0(y) = \tanh(y - y_c(\theta)) + \tanh(y_c(\theta))
\]

To date, the existing studies on traffic flow assume that the current vehicle can accurately receive the traffic information of preceding vehicles. However, as discussed previously, the driving field of vision will be still limited in a traffic environment with gradient roads even though the V2V technology is in use, such that the sensors are difficult to accurately perceive the speed of the front vehicle. In other words, certain errors of perceived speed may exist between the information of the vehicle ahead and the real data. As a result, there is an imminent need to incorporate the velocity uncertainty of preceding vehicles in the design of car-following control under such a complex road environment. Given this fact, we introduce the perception error term of the preceding vehicle into Eq. (7), yielding a new traffic flow model as follows:

\[
d\frac{v_n(t)}{dt} = a[V^{\text{op}}(y_n(t)) - v_n(t)] + \lambda[(1 + \epsilon)v_{n+1}(t) - v_n(t)] + u_n(t)
\]

where the parameter \( \epsilon \) represents the uncertainty coefficient. Here, \( \epsilon > 0 \) indicates a positive error. To avoid the reversing movement of the preceding vehicle, the term of \( (1 + \epsilon) \) should be larger than 0, that is, \( |\epsilon| < 1 \). When \( \epsilon = 0 \), the proposed model is consistent with the literature [4]. Therefore, the new traffic flow model can be regarded as a generalized form of previous research.

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**Fig. 2** Schematic diagram of vehicle forces on different scenarios: **a** uphill; **b** downhill

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4 The control scheme for the car-following model

In this section, the stability of the above car-following model will be explored. It proceeds to devise a control scheme for the model. The stability of traffic flow is to investigate the impact of disturbances on the traffic state along the road. Specifically, if the traffic flow model is unstable, the small disturbances will propagate upstream along with the vehicle flows, gradually inducing traffic jams; conversely, if the traffic flow model is stable, the disturbance will gradually be inhibited and eventually disappear during the propagation process, forcing the traffic flow to return to the equilibrium state. To improve road traffic efficiency, we propose a new self-delayed feedback control for the car-following model. This control scheme relies only on the information of velocity and headway difference between the current time and historical time, specifically

\[ u_j(t) = k_1(y_j(t) - y_j(t - \tau_1)) + k_2(v_j(t) - v_j(t - \tau_1)) \]  

(14)

Remark 1: The traditional linear feedback controller, with the vehicle speed difference or vehicle spacing difference of successive vehicles being the control input, requires the accurate road traffic information of the remaining vehicles. In contrast, the proposed self-feedback controller Eq. (14) relies only on the speed and headway difference between the current time and the previous historical time, without any additional external traffic information. In the current underdeveloped traffic perception environment, the accuracy of preceding vehicles’ speeds may be limited. Given that the proposed designed controller only utilizes its vehicle information, the impact of velocity uncertainty on the control performance is negligible. As such, it can handle the issue of the velocity uncertainty effect. In other words, the proposed controller Eq. (14) is particularly suitable for the current underdeveloped Internet of Vehicles environment.

To obtain the range of control gain coefficients \( k_1 \) and \( k_2 \), we perform a stability analysis on the closed-loop traffic flow model via the \( H_\infty \) optimal control theory and transfer function method. The transfer function represents the relationship between the input and output of a linear time-invariant traffic flow system described by the system parameters. When a perturbation is injected into the system parameters, the system can be described by a set of transfer functions, and the largest gain can be taken as the function set gain. The principle of \( H_\infty \) optimal control theory, which is a typical robust control theory, is that the gain of the transfer function from the interference to the output is suppressed to achieve the purpose of anti-interference. Based on this principle, the steady state of the model is first given as follows:

![Fig. 3 Delay-independent stable regions of characteristic polynomial \( d(s) \) on phase diagram \((k_1, k_2)\) with \( a = 2, \theta = -5^\circ, \lambda = 0.1\); a condition (22); b condition (23)](image)
\[ y_j, v_j \] = \begin{bmatrix} y^*, v^* \end{bmatrix}^T \quad (15) \]

where \( y^* = h \), \( v^* = \frac{\nu_{\text{op}}(h)}{1-\xi} \).

Let \( \delta v_j = v_j - v^* \) and \( \delta y_j = y_j - y^* \), substituting the above items into Eq. (13) and linearizing this equation yields the following equations:

\[
\begin{align*}
\frac{d\delta v_n(t)}{dt} &= a\left[\Lambda \delta y_n(t) - \delta v_n(t)\right] + \lambda \left[\left(1 + e\right)\delta v_{n+1}(t) - \delta v_n(t)\right] + k_1 \left[\delta y_j(t) - \delta y_j(t - \tau)\right] + k_2 \left[\delta v_j(t) - \delta v_j(t - \tau)\right] \\
\frac{d\delta y_n(t)}{dt} &= \delta v_{n+1}(t) - \delta v_n(t)
\end{align*}
\]

(16)

where \( \Lambda = \left(\frac{\nu_{\text{op}} - \sin \theta}{2}\right)V'_0 \), \( V'_0 = \frac{dV_0}{d\rho}\bigg|_{\rho=\rho_0} \),

\( \delta y_j(t - \tau) = y_j(t - \tau) - y_j^* \),

\( \delta v_j(t - \tau) = v_j(t - \tau) - v_j^* \).

Fig. 4 Delay-independent stable regions of characteristic polynomial \( d(s) \) on phase diagram \((k_1, k_2)\) with \( a = 2, \theta = 5^\circ, \lambda = -0.1; \ a \) condition (22); \( b \) condition (23)

Applying the Laplace transform to the above formula, we have that where \( P_j(s) = L(p_j) \) and \( Q_j(s) = L(q_j) \). \( L(\cdot) \) represents the Laplace transform. \( s \) is the complex variable.
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\[
\begin{align*}
&\{ sV_n(s) - V_n(0) = a[AY_n(s) - V_n(s)] + \dot{\lambda}(1 + \varepsilon)V_{n+1}(s) - V_n(s) + k_1(1 - e^{-\tau_1})V_n(s) + k_2(1 - e^{-\tau_2})V_n(s) \\
&\{ sY_n(s) - Y_n(0) = V_{n+1}(s) - V_n(s)
\end{align*}
\]  
(17)

By transforming Eq. (17) to eliminate the intermediate variable \( Y_n(s) \), then the transition function is

\[
G(s) = \frac{a\Lambda + \dot{\lambda}(1 + \varepsilon)s + k_1(1 - e^{-\tau_1})}{d(s)}
\]  
(18)

where \( d(s) \) represents the characteristic polynomial, and

\[ d(s) = s^2 + (a + \dot{\lambda} - k_2)s + a\Lambda + k_1, \]

where \( P(s) = k_2s - k_1 \).

Furthermore, Eq. (19) is written as follows:

\[
F(w) = [\text{ReP}(iw)]^2 + [\text{ImP}(iw)]^2 - [\text{ReQ}(iw)]^2 - [\text{ImQ}(iw)]^2
\]

\[ = (-w^2 + a\Lambda + k_1)^2 + (a + \dot{\lambda} - k_2)^2w^2 - k_1^2 - k_2^2w^2
\]

\[ = w^4 + m_1w^2 + m_2
\]  
(20)

where \( m_1 = (a + \dot{\lambda})^2 - 2(a + \dot{\lambda})k_2 - 2(a\Lambda + k_1), \)

\[ m_2 = a^2\Lambda^2 + 2a\Lambda k_1. \]

If the following conditions are met, then the critical function \( F(w) \) has no real roots. We now use the Routh–Hurwitz criterion to identify the stability of the above polynomials, and then we have

\[
m_1 \geq 0, m_2 \geq 0 \text{ or } m_1 < 0, m_1^2 - 4m_2 < 0
\]  
(21)

**Remark 2** (Routh–Hurwitz stability criterion): In the classical control theory, the Hurwitz criterion can be applied to analyze the root distribution of the characteristic equation on the s-plane to identify the stability of traffic flow. To minimize the influence of the disturbance on the controllable output, the above problem can be transformed into the minimum norm of the transfer function matrix via the optimal control theory. The necessary and sufficient conditions when the polynomial is stable are: (1) The coefficients of the characteristic equation are all positive; (2) The subdeterminants of each order are greater than zero.

Substituting \( m_1 \) and \( m_2 \) into Eq. (21), we have that

\[
(a + \dot{\lambda})k_2 + k_1 \leq \frac{(a + \dot{\lambda})^2}{2} - a\left(\frac{v_{\text{max}} - \sin \theta}{2}\right) V_0, k_1 \geq -a\left(\frac{v_{\text{max}} - \sin \theta}{2}\right) V_0
\]  
(22)

or

\[
(a + \dot{\lambda})k_2 + k_1 > \frac{(a + \dot{\lambda})^2}{2} - a\left(\frac{v_{\text{max}} - \sin \theta}{2}\right) V_0, k_1 < \frac{(a + \dot{\lambda})^2 - 2(a + \dot{\lambda})k_2}{2} + 4k_2^2 < 4\left[(a + \dot{\lambda})^2 - 2(a + \dot{\lambda})k_2\right]
\]

\[ k_1 > \left[a\left(\frac{v_{\text{max}} - \sin \theta}{2}\right) V_0 + k_1\right]
\]  
(23)

In summary, if the condition (22) or (23) holds, the characteristic polynomial \( d(s) \) satisfies the stability condition. Figures 3 and 4 show the delay-independent stable regions of \( d(s) \) in phase diagram \((k_1, k_2)\) for uphill and downhill scenarios, respectively. The shaded region represents the set of parameter combinations where the characteristic polynomial stability is guaranteed.

Moreover, to ensure that the designed control algorithm meets the condition (II) in Lemma 1, similar to the literature [64–67], we use the \( H_{\infty} \) norm in the subsequent theoretical derivation, that is,
\[
\|G(s)\|_\infty = \sup_{w \in [0, \infty)} |G(iw)| \leq 1 \\
|G(iw)| = \sqrt{G(iw)G(-iw)} \\
= \sqrt{\frac{[a\Lambda + k_1(1 - \cos(w\tau_1)) + k_2w\sin(w\tau_1)]^2 + [\lambda(1 + \epsilon)w + k_1\sin(w\tau_1) - k_2w(1 - \cos(w\tau_1))]^2}{[-w^2 + a\Lambda + k_1(1 - \cos(w\tau_1)) + k_2w\sin(w\tau_1)]^2 + [a\lambda + k_1(1 - \cos(w\tau_1))]^2 + [\lambda(1 + \epsilon)w + k_1\sin(w\tau_1) - k_2w(1 - \cos(w\tau_1))]^2}} \leq 1
\]

Therefore, if the inequality \( |G(iw)| \leq 1 \) holds for any \( w \in [0, +\infty) \), the following condition is specifically met:

\[
[-w^2 + a\Lambda + k_1(1 - \cos(w\tau_1)) + k_2w\sin(w\tau_1)]^2 + [(a + \lambda)w + k_1\sin(w\tau_1) - k_2w(1 - \cos(w\tau_1))]^2 + [a\lambda + k_1(1 - \cos(w\tau_1))]^2 + [\lambda(1 + \epsilon)w + k_1\sin(w\tau_1) - k_2w(1 - \cos(w\tau_1))]^2 \geq 0
\]

(25)

**Theorem 1:** Based on the traffic flow model (13), for given parameters \( \lambda, k_1, k_2 \) and \( \tau_1 \), if the conditions (22) (or (23)) and (25) are met, then traffic congestion can be effectively suppressed under the delay feedback controller Eq. (14).

Figure 5 shows the amplitude change of the transfer function under different values of parameters \( \theta \) and \( \epsilon \) without the designed controller. As shown in Fig. 5a, when the value of the parameter \( \theta \) is higher, the amplitude of the transfer function decreases, such that the traffic flow stability is stronger. The stability of traffic flow in the uphill scenario is stronger than that in the downhill scenario, which is consistent with the literature [4]. As shown in Fig. 5b, with the increase of parameter \( \epsilon \), the amplitude of the transfer function increases, such that the stability of the traffic flow weakens. This indicates that a small disturbance may be enlarged over time and eventually induce the traffic jam phenomenon.

Figures 6 and 7 present the amplitude change of the transfer function under different values of key parameters \( k_1, k_2 \) and \( \tau_1 \) for the designed controller Eq. (14), where Figs. 6 and 7 correspond to downhill and uphill scenario, respectively. As shown in Fig. 6a, the amplitude of the transfer function \( |G| \) is effectively reduced as the parameters \( k_1 \) and \( k_2 \) increase; when \( k_1 = 2.5 \) and \( k_2 = 2.5 \), Theorem 1 is satisfied, the amplitude vertex of the transition function is less than 1. Therefore, the designed controller Eq. (14) is conducive to enhancing the robustness of traffic flow. Figure 6b compares the amplitude of the transfer function \( |G| \) under different values of delay time \( \tau_1 \). When \( \tau_1 = 0.5 \), the amplitude of the Bode curve is greater than 0, and the traffic disturbance will exacerbate over time and induce traffic jam phenomenon. As the parameter \( \tau_1 \) continues to increase, the amplitude of the Bode curve keeps decreasing. When \( \tau_1 = 2.5 \), the amplitude of the Bode curve is approximately less than 1, which means that the traffic flow is stable. Therefore, it can be concluded that a higher value of the parameter \( \tau_1 \) is conducive to improving the robustness of traffic flow.

5 Simulation example

In this section, numerical simulations are conducted to verify the effectiveness of the designed self-delay feedback controller on suppressing traffic congestion and reducing energy consumption. Technically speaking, the differential equations of the model are solved by the ordinary differential equation (ODE) function tool in MATLAB 2020b software. This tool uses the Runge–Kutta algorithm to solve the above partial differential equations given the initial conditions. To better reproduce the evolution of traffic congestion, the simulation environment is set as periodic boundary conditions, where the starting point is also the ending point on the road, forming a closed-loop system as shown in Fig. 8. On the initial conditions, it is assumed that all vehicles are running with a uniform speed \( v^* \) and a uniform headway \( y^* \). Additionally, to explore the propagation and evolution mechanism of traffic jams
over time in the closed-loop traffic environment, an initial disturbance is injected into the vehicles in the middle of the road under the aforementioned uniform state, and the initial position information is as follows:

\[
x_n(0) = \begin{cases} 
  4, & \text{if } n \neq \frac{N}{2}, \frac{N}{2} + 1 \\
  4 - \delta, & \text{if } n = \frac{N}{2} \\
  4 + \delta, & \text{if } n = \frac{N}{2} + 1
\end{cases}
\]  

(26)

where the number of vehicles \( N = 100 \) and the disturbance term \( \delta = 0.1 \). The other default parameters are selected as follows: \( v_{\max} = 2, y_c = 4, a = 1.2, \lambda = 0.1, \varepsilon = 0.1 \). These benchmark data are widely adopted in the literature (e.g., [68, 69]).

Figure 9 shows the spatiotemporal evolution of headways with different controller gain coefficients \( k_1 \) and \( k_2 \) under the downhill scenario. Figure 9a corresponds to the traffic flow without a designed controller (\( k_1 = 0 \) and \( k_2 = 0 \)). Since the above stability conditions (22) (or (23)) and (25) are not satisfied, the initial disturbance gradually expands over time, and the interactions between vehicles become more frequent and eventually the stop-and-go phenomenon appears.

![Fig. 5 Bode plot for different values of parameter \( \theta, \epsilon \)](image)

(a)  
(b)

![Fig. 6 Bode plot for different values of parameter \( k_1, k_2, \tau_1 (\theta = -6^\circ) \)](image)

(a)  
(b)
which indicates that the traffic jam phenomenon is prone to occur. In Fig. 9b, when the parameter $k_2$ increases to 0.4 and the parameter $k_1$ remains unchanged, the controller relies only on the information of velocity difference between the current time and the previous historical time, which is equivalent to the delayed feedback controller in the literature [63]. Although the stability conditions are still not satisfied, the speed difference between successive vehicles can be applied to the current vehicle to achieve the consistency of vehicle platoon, such that the amplitude and frequency of the disturbance have been alleviated to a certain extent, which verifies the effectiveness of the delayed feedback controller [63]. When the value of parameter $k_1$ increases from 0 (Fig. 9b) to 0.4 (Fig. 9c), the fluctuating amplitude and frequency are further reduced based on the designed controller Eq. (14), as compared with the delay feedback controller in the literature [63]. Therefore, the parameters $k_1$ and $k_2$ contribute to the stability of traffic flow, and it is feasible to suppress the influence of external disturbance on the stability of traffic flow via the information of headway difference and velocity difference. When $k_1 = 0.8$ and $k_2 = 0.8$, the stability conditions (22) (or (23)) and (25) are met, and the initial disturbance gradually decreases and disappears over time, forcing the traffic flow to return to the equilibrium state. In summary, the designed control scheme Eq. (14) is effective in eliminating traffic disturbance and improving the anti-interference ability of traffic flow, while the new controller outperforms the traditional one.
Figure 10 shows the instantaneous headway distribution of all vehicles corresponding to Fig. 9 at $t = 10300$ s. One can see that the fluctuating amplitude of headways without control in Fig. 9a is the largest. The fluctuating amplitude of the headway has been reduced to some extent under the delay feedback controller in the literature \[63\]. The upper boundary of oscillation decreases from 4.71 to 4.63, and the lower boundary of oscillation increases from 2.45 to 2.53. The fluctuating amplitude is approximately equal to 0 in Fig. 10d. This indicates that the traffic flow is stable.

Figure 11 describes the evolution of the headway between vehicles on the road under different values of the parameter $s_1$, where Fig. 11a–d, respectively, corresponds to $s_1 = 0, k_2 = 0$; b $k_1 = 0, k_2 = 0.4$; c $k_1 = 0.4, k_2 = 0.4$; d $k_1 = 0.8, k_2 = 0.8$. \( \tau_1 = 0.1, \theta = -6^\circ \)

Figure 10 shows the instantaneous headway distribution of all vehicles corresponding to Fig. 9 at $t = 10300$ s. One can see that the fluctuating amplitude of headways without control in Fig. 9a is the largest. The fluctuating amplitude of the headway has been reduced to some extent under the delay feedback controller in the literature \[63\]. The upper boundary of oscillation decreases from 4.71 to 4.63, and the lower boundary of oscillation increases from 2.45 to 2.53. The fluctuating amplitude is approximately equal to 0 in Fig. 10d. This indicates that the traffic flow is stable.

Figure 11 describes the evolution of the headway between vehicles on the road under different values of the parameter $\tau_1$, where Fig. 11a–d, respectively, corresponds to $\tau_1 = 0.1 \sim 0.4$. When $\tau_1 = 0.1 \sim 0.3$, the stability conditions (22) (or (23)) and (25) are not satisfied, and the stop-and-go waves and traffic jams appear. Nevertheless, compared to the other three sub-graphs, the frequency and amplitude of headway fluctuation between vehicles have been alleviated to a certain extent under the same parameters $k_1$ and $k_2$. Furthermore, when the parameter $\tau_1$ increases to 0.4, the stability conditions are met, such that the stop-and-go wave disappears and the traffic flow returns to the equilibrium state. To sum up, a higher value of the parameter $\tau_1$ is conducive to improving the anti-interference ability of traffic flow.

Figure 12 shows the instantaneous headway distribution of all vehicles corresponding to Fig. 11 at
$t = 10300 \text{ s}$. With the increase of the parameter $\tau_1$, the fluctuating amplitude of the headway has been reduced from $2.26 (= 4.71 - 2.45)$ as in Fig. 12a to a negligible value as in Fig. 12d. In other words, the initial disturbance has been completely diluted when the designed controller (14) is employed.

Figures 13, 14, 15, 16 analyze the relationship between control gain coefficients $k_1$ and $k_2$, delay time $\tau_1$ and traffic flow stability for the uphill scenario. With the increase of the gain coefficients $k_1$ and $k_2$ or delay time $\tau_1$, the robustness of the traffic flow model can be enhanced. Therefore, the designed self-delayed feedback controller is effective in suppressing traffic congestion and enhancing the anti-interference ability for both the uphill or downhill scenarios.

6 Concluding remarks

This paper proposes a new self-delayed feedback controller for the car-following model with velocity uncertainty of preceding vehicles on gradient roads.
The stability condition of the closed-loop traffic flow model is obtained via Hurwitz criteria and $H_\infty$ norm of transfer functions. The influence of parameters $k_1$ and $k_2$ and delay time $s_1$ on traffic jams from the uphill and downhill scenarios is investigated. Key findings and their implications are summarized as follows:

a. As the values of parameters $k_1$ and $k_2$ increase, the traffic disturbance can be effectively suppressed, which indicates that the designed controller is effective in improving the robustness of road traffic flow. Compared to the traditional delayed feedback controller [63], the proposed controller presents better control performance, owing to the comprehensive consideration of velocity and headway difference.

b. A higher value of delay time $s_1$ will decrease the fluctuating amplitude of headways, which indicates that the robustness of the traffic flow model is enhanced. Therefore, the delay time plays an important role in the control performance, and the whole traffic flow can be stabilized with a higher delay time step.

c. Under the gradient road environment, the driving field of vision would be largely limited, which results in errors between the perceived data and the real traffic data. The control performance of traditional controllers would be degraded in the
provision of perceived errors. However, under the proposed controller, the vehicle state is optimized only based on its accurate information, which effectively avoids the above shortcomings and adapts for such a complex road environment. In addition, subject to limited bandwidth constraints in the underdeveloped Internet of Vehicles environment, communication delays between vehicles and data packet loss phenomenon are not uncommon. Therefore, when the real-time and accurate data cannot be guaranteed, the designed self-delayed feedback controller outperforms the traditional feedback controllers.

Despite the newly developed autonomous traffic flow control method, there are a few strong assumptions in the current research and thus possible future research directions. First, all drivers are assumed to be...
homogeneous; thus, the heterogeneous traffic flow with different types of drivers can be considered. Second, for the design of the control algorithm, the delay time of drivers is set as a constant, whereas the delay time could fluctuate to some extent over time. Hence, a more realistic time-varying delay function can be constructed to replace the constant delay time.

Finally, the feasibility of the designed control scheme is only verified based on simulation test data. Therefore, future research can verify the model with the open data set, such as the well-known Next Generation Simulation (NGSIM) data set.

Fig. 13 The spatiotemporal evolutions of headway under different values of parameter $k_1$ and $k_2$ after $t = 10^4$ s, where a $k_1 = 0, k_2 = 0$; $b k_1 = 0, k_2 = 0.15$; c $k_1 = 0.15, k_2 = 0.15$; d $k_1 = 0.3, k_2 = 0.3$. ($\tau_1 = 0.1, \theta = 6^\circ$)
Fig. 14  The snapshots of headway distribution of all vehicles at $t = 10,300$ s corresponding to Fig. 13
Fig. 15 The spatiotemporal evolutions of headway under different values of parameter $\tau_1$ after $t = 10^4$ s, where a $\tau_1 = 0.1$; b $\tau_1 = 0.2$; c $\tau_1 = 0.3$; d $\tau_1 = 0.4$. ($k_1 = 0.6$, $k_2 = 0.5$, $\theta = 6^\circ$)
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Data availability Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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