RADII OF COVERING DISKS FOR LOCALLY UNIVALENT HARMONIC MAPPINGS

SERGEY YU. GRAF, SAMINATHAN PONNUSAMY, AND VICTOR V. STARKOV

Abstract. For a locally univalent sense-preserving harmonic mapping $f = h + g$ defined on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, let $d_f(z)$ be the radius of the largest (univalent) disk on the manifold $f(D)$ centered at $f(z_0)$ ($|z_0| < 1$).

One of the aims of the present investigation is to obtain sharp upper and lower bounds for the quotient $d_f(z_0)/d_h(z_0)$, especially, for a family of locally univalent $Q$-quasiconformal harmonic mappings $f = h + g$ on $D$. In addition to several other consequences of our investigation, the disk of convexity of functions belonging to certain linear invariant families of locally univalent $Q$-quasiconformal harmonic mappings of order $\alpha$ is also established.

1. Introduction and Main Results

For a smooth univalent mapping $h$ of the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ onto two-dimensional manifold $M$, we define $d_h(z)$ to be the radius of the largest (univalent) disk centered at $h(z)$ on the manifold $M$. If $LU$ denotes the family of functions $h$ analytic and locally univalent ($h'(z) \neq 0$) in $D$, then the classical Schwarz lemma for analytic functions gives the following well-known sharp upper estimate of the radius $d_h(z)$:

$$d_h(z) \leq |h'(z)|(1 - |z|^2).$$

The sharp and nontrivial lower estimate of the value $d_h(z)$ was obtained by Pommerenke in a detailed analysis of what is called linear invariant families of locally univalent analytic functions in $D$. Throughout we denote by $\text{Aut}(D)$, the set of all conformal automorphisms (Möbius self-mappings) $\omega(z) = e^{i\theta}z + z_0 + \sum_{k=2}^{\infty}a_k(h)z^k$, where $|z_0| < 1$ and $\theta \in \mathbb{R}$, of the unit disk $D$.

Definition 1. (cf. [9]) A non-empty collection $\mathcal{M}$ of functions from $LU$ is called a linear invariant family (LIF) if for each $h \in \mathcal{M}$, normalized such that $h(z) = z + \sum_{k=2}^{\infty}a_k(h)z^k$, the functions $H_\omega(z)$ defined by

$$H_\omega(z) = \frac{h(\omega(z)) - h(\omega(0))}{h'(\omega(0))\omega'(0)} = z + \cdots,$$

belong to $\mathcal{M}$ for each $\omega \in \text{Aut}(D)$.

The order of the family $\mathcal{M}$ is defined to be $\alpha := \text{ord}\mathcal{M} = \sup_{h \in \mathcal{M}}|a_2(h)|$. The universal LIF, denoted by $\mathcal{U}_\alpha$, is defined to be the collection of all linear invariant...
families $\mathcal{M}$ with order less than or equal to $\alpha$ (see [9]). An interesting fact about the order of a LIF family is that many properties of it depend only on the order of the family. It is well-known [9] that $U_\alpha \neq \emptyset \iff \alpha \geq 1$, and that $U_1$ is precisely the family $\mathcal{K}$ of all normalized convex univalent (analytic) functions whereas $S \subset U_2$.

Here $S$ denotes the classical family of all normalized univalent (analytic) functions in $\mathbb{D}$ investigated by a number of researchers (see [3, 5, 10]).

Note that the universal LIF is the largest LIF such that for each $h \in U_\alpha$ the following inequality holds:

$$|h'(z)| \leq \frac{(1 + |z|)^{\alpha - 1}}{(1 - |z|)^{\alpha + 1}}.$$  

In [9], Pommerenke has proved that for each $h \in U_\alpha$ the following sharp lower estimate of $d_h(z)$ holds:

$$d_h(z) \geq \frac{1}{2\alpha} |h'(z)|(1 - |z|^2).$$

In the present paper the question about the alteration of the estimate of the functional $d_f(z)$ is explored in the case when instead of analytic functions $h(z)$ we consider harmonic locally univalent mappings of the form

$$f(z) = h(z) + g(z) = \sum_{k=1}^{\infty} \left( a_k z^k + a_{-k} z^{-k} \right),$$

i.e. when the co-analytic part is added to the function $h$. We say that $f = h + \overline{g}$ is sense-preserving if the Jacobian of $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ of $f$ is positive. Lewy’s theorem [7] (see also for example [4, Chapter 2, p. 20] and [11]) implies that every harmonic function $f$ on $\mathbb{D}$ is locally one-to-one and sense-preserving on $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$. The condition $J_f(z) > 0$ is equivalent to $h'(z) \neq 0$ and the existence of an analytic function $\mu_f$ in $\mathbb{D}$ such that

$$|\mu_f(z)| < 1 \text{ for } z \in \mathbb{D},$$

where $\mu_f(z) = g'(z)/h'(z)$ and $\mu_f$ is referred to as the (complex) dilatation of the harmonic mapping $f = h + \overline{g}$. When it is convenient, we simply use the notation $\mu$ instead of $\mu_f$.

There are different generalizations of the notion of the linear invariant family to the case of harmonic mappings. For example, the question about a lower estimate of the radius $d_f(0)$ of the univalent disk centered at the origin was examined by Sheil-Small [13] in the linear and affine invariant families of univalent harmonic functions $f$. The concept of linear and affine invariance was also discussed by Schaubroeck [12] for the case of locally univalent harmonic mappings.

**Definition 2.** The family $\mathcal{LU}_H$ of locally univalent sense-preserving harmonic functions $f$ in the disk $\mathbb{D}$ of the form (1.1) is called linear invariant (LIF) if for each $f = h + \overline{g} \in \mathcal{LU}_H$ the following conditions are fulfilled: $a_1 = 1$ and

$$\frac{f(\omega(z)) - f(\omega(0))}{h'(\omega(0))\omega'(0)} \in \mathcal{LU}_H.$$
for each \( \omega \in \text{Aut}(\mathbb{D}) \). A family \( \mathcal{AL}_H \) is called linear and affine invariant (LAIF) if it is LIF and in addition each \( f \in \mathcal{AL}_H \) satisfies the condition that

\[
\frac{f(z) + \varepsilon f(z)}{1 + \varepsilon f'(0)} \in \mathcal{AL}_H \quad \text{for every} \quad \varepsilon \in \mathbb{D}.
\]

The number \( \text{ord} \mathcal{AL}_H = \sup_{f \in \mathcal{AL}_H} |a_2| \) is known as the order of the LAIF \( \mathcal{AL}_H \).

The order of LIF \( \mathcal{LU}_H \) without the assumption of affine invariance property is defined in the same way: \( \text{ord} \mathcal{LU}_H = \sup_{f \in \mathcal{LU}_H} |a_2| \).

Throughout the discussion, we suppose that the orders of these families, namely, \( \text{ord} \mathcal{AL}_H \) and \( \text{ord} \mathcal{LU}_H \) are finite. The universal linear and affine invariant family, denoted by \( \mathcal{AL}_H(\alpha) \), is the largest LAIF \( \mathcal{AL}_H \) of order \( \alpha = \text{ord} \mathcal{AL}_H \). Thus, the subfamily \( \mathcal{AL}_H^\alpha \) of LAIF \( \mathcal{AL}_H \) consists of all functions \( f \in \mathcal{AL}_H \) such that \( f(0) = 0 \). If \( f \in \mathcal{AL}_H^0 \) is univalent in \( \mathbb{D} \), then according to the result of Sheil-Small [13] one has the following sharp lower estimate:

\[
d_f(0) \geq \frac{1}{2\alpha}.
\]

For \( \alpha > 0 \) and \( Q \geq 1 \), denote by \( \mathcal{H}(\alpha, Q) \) the set of all locally univalent \( Q \)-quasiconformal harmonic mappings \( f = h + \overline{g} \) in \( \mathbb{D} \) of the form (1.1) with the normalization \( a_1 + a_{-1} = 1 \) such that

\[
h(z)/h'(0) \in \mathcal{U}_\alpha, \quad |g'(z)/h'(z)| \leq k, \quad k = (Q - 1)/(Q + 1) \in [0, 1).
\]

The family \( \mathcal{H}(\alpha, Q) \) was introduced and investigated in details by Starkov [15, 16]. In particular, he established double-sided estimates of the value \( d_f(z) \) for functions belonging to the family \( \mathcal{H}(\alpha, Q) \).

We shall restrict ourselves to the case of finite \( Q \). In [15, 16], it was also shown that the family \( \mathcal{H}(\alpha, Q) \) possess the property of linear invariance in the following sense: for each \( f = h + \overline{g} \in \mathcal{H}(\alpha, Q) \) and for every \( \omega(z) = e^{i\theta} \frac{z + a}{1 + \overline{a}z} \in \text{Aut}(\mathbb{D}) \), the transformation

\[
f(\omega(z)) - f(\omega(0)) \overline{\partial_\theta f(\omega(0))|\omega'(0)|} \in \mathcal{H}(\alpha, Q),
\]

where \( \partial_\theta f(z) = h'(z)e^{i\theta} + g'(z)e^{i\theta} \) denotes the directional derivative of the complex-valued function \( f \) in the direction of the unit vector \( e^{i\theta} \).

In [17], it was also proved that for each \( f \in \mathcal{H}(\alpha, Q) \) and \( z \in \mathbb{D} \),

\[
\frac{1 - |z|^2}{2\alpha Q} \max_\theta |\partial_\theta f(z)| \leq d_f(z) \leq Q(1 - |z|^2) \min_\theta |\partial_\theta f(z)|
\]

which is equivalent to

\[
\frac{1 - |z|^2}{2\alpha Q} (|h'(z)| + |g'(z)|) \leq d_f(z) \leq Q(1 - |z|^2) (|h'(z)| - |g'(z)|),
\]

and the lower estimate is sharp in contrast to the upper one.

One of the main aims of this article is to establish the sharp estimations of the ratio \( d_f(z)/d_h(z) \) for \( Q \)-quasiconformal harmonic mappings \( f = h + \overline{g} \) and, in particular, the sharp upper estimate in (1.5) is also obtained. We now state our first result.
Theorem 1. Let \( f = h + \bar{g} \in \mathcal{H}(\alpha, Q) \) for some \( Q \in [1, \infty) \), and \( \mu(z) = g'(z)/h'(z) \) be the complex dilatation of the mapping \( f \). Then for \( z \in \mathbb{D} \),

\[
1 - k \leq m \left( \left| \frac{\mu(z)}{k} \right|, Q \right) \leq d \left( \left| \frac{\mu(z)}{k} \right|, k \right) \leq 1 + k,
\]

where \( k = (Q - 1)/(Q + 1) \in [0, 1] \). Here the functions \( M(., k) \) and \( m(., Q) \) are defined as follows:

\[
M(x, k) = \begin{cases} 1 + \frac{k}{x} \left( 1 - \frac{1}{x} - x \right) \log (1 + x) & \text{when } x \in (0, 1], \\ \lim_{x \to 0^+} M(x, k) = 1 + \frac{k}{2} & \text{when } x = 0, \end{cases}
\]

and

\[
\frac{1}{m(x, Q)} = \begin{cases} \int_0^1 \frac{1 + \varphi^{-1}(Q^{-1}\varphi(t)x)}{1 - kx + \varphi^{-1}(Q^{-1}\varphi(t))(x - k)} \, dt & \text{when } Q < \infty, \\ 0 & \text{when } Q = \infty, \end{cases}
\]

with

\[
\varphi(t) = \frac{\pi K'(t)}{2 \mathcal{K}(t)} \quad (t \in (0, 1))
\]

where \( \mathcal{K} \) denotes the (Legendre) complete elliptic integral of the first kind given by

\[
\mathcal{K}(t) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - t^2 \sin^2 x}} = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - t^2 x^2)}}
\]

and \( \mathcal{K}'(t) = \mathcal{K}(\sqrt{1 - t^2}) \). The argument \( t \) is sometimes called the modulus of the elliptic integral \( \mathcal{K}(t) \).

Estimations in (1.6) are sharp for the family \( \mathcal{H}(\alpha, Q) \) for \( Q < \infty \) and for each \( \alpha \geq 1 \). When \( Q = \infty \), estimations in (1.6) are sharp in the sense that for each \( z \in \mathbb{D} \),

\[
\inf_{f \in \mathcal{H}(\alpha, \infty)} \frac{d_f(z)}{d_h(z)} = m(x, \infty) = 0 \quad \text{and} \quad \sup_{f \in \mathcal{H}(\alpha, \infty)} \frac{d_f(z)}{d_h(z)} = M(1, 1) = 2.
\]

Remark 1. In some neighbourhood of the origin, it is also possible to obtain a simple lower estimate in the inequality (1.6) without the involvement of elliptic integrals. For example, the well-known theorem of Mori [8] reveals that for \( Q \)-quasiconformal automorphism \( F \) of the disk \( \mathbb{D} \), such that \( F(0) = 0 \), one has

\[
|F(z)| \leq 16|z|^{1/Q}.
\]

Using this result in the estimation of the value of \( |\ell_0(t)| \) in Part 3 of the proof of Theorem 1, one can easily obtain that

\[
|\ell_0(t)| = |f^{-1}(At)| = \begin{cases} 16t^{1/Q} \text{ when } 0 \leq t < 1/16^{-Q} \\ 1 \text{ when } 16^{-Q} \leq t < 1. \end{cases}
\]
The last relation provides an opportunity to estimate the ratio \( \frac{dh(z)}{df(z)} \) by means of an integral of an elementary function, namely,

\[
\frac{dh(z)}{df(z)} \leq \frac{1}{m(x, Q)} \\
\leq \frac{1}{1-k} \left( 1 - 16^{-Q} + (1-k) \int_0^{16^{-Q}} \frac{1+yx}{1-kx+y(x-k)} \, dt \right) \\
\leq \frac{1}{1-k},
\]

where \( y = 16 t^{1/Q} \leq 1 \) for \( t \in [0, 16^{-Q}] \). Here \( x = |\mu(z)|/k, z \in \mathbb{D} \).

**Remark 2.** For fixed \( \zeta \in \mathbb{D} \), the least value of the upper estimation in (1.6) is attained when \( x = 0 \); that is when \( \mu(\zeta) = 0 \). In this case the estimation in (1.6) takes the form

\[
\frac{df(\zeta)}{dh(\zeta)} \leq 1 + \frac{k}{2}.
\]

Suppose that \( f = h+g \in \mathcal{H}(\alpha, Q), \alpha \in [1, \infty], \) and \( f_1(z) = C \cdot f(z) = h_1(z) + g_1(z), \) where \( C \) is a complex constant. Then the following relations hold:

\[
df_1(z) = |C| \, df(z) \quad \text{and} \quad dh_1(z) = |C| \, dh(z), \quad z \in \mathbb{D}.
\]

Moreover, after appropriate normalization, every \( Q \)-quasiconformal harmonic mapping in \( \mathbb{D} \) hits the family \( \mathcal{H}(\alpha, Q) \) for some \( \alpha \). Therefore an equivalent formulation of Theorem 1 may now be stated.

**Theorem 2.** Let \( f = h+g \in \mathcal{H}(\alpha, Q) \) and \( h(z) = h'(0)H(z) \) from \( \mathcal{U}_\alpha \). Then we have (see [15, 16])

\[
\frac{1}{1+k} \leq |h'(0)| \leq \frac{1}{1-k}
\]

and thus,

\[
\frac{dh(z)}{1+k} \leq \frac{dh(z) = |h'(0)| \cdot dH(z)}{1-k} \leq \frac{dH(z)}{1-k}.
\]

These inequalities and (1.6) give the following.

**Corollary 1.** Let \( f = h+g \in \mathcal{H}(\alpha, Q) \) and \( h(z) = h'(0)H(z) \). Then we have

\[
\frac{dH(z)}{Q} \leq df(z) \leq Q \cdot dH(z) \quad \text{for} \; z \in \mathbb{D}.
\]

The sharpness of the last double-sided inequalities at the point \( z = 0 \) follows from the proof of Theorem 1.

We now state the remaining results of the article.
Theorem 3. Let $f = h + \overline{g}$ be a locally quasiconformal harmonic mapping belonging to the family $\mathcal{AL}_H$ with $\text{ord}(\mathcal{AL}_H) = \alpha < \infty$, $\mu(z) = g'(z)/h'(z)$ and $|\mu(z)| < 1$. Then

\begin{equation}
(1.9)
d_f(z) \geq \frac{1 - |\mu(z)|}{2\alpha} \left(\frac{1 - |z|}{1 + |z|}\right)^{\alpha} \text{ for } z \in \mathbb{D}.
\end{equation}

The estimation $d_f(0)$ is sharp for example in the universal $\text{LAIF} \mathcal{AL}_H(\alpha)$.

Recall that a locally univalent function $f$ is said to be convex in the disk $D(z_0, r) := \{z : |z - z_0| < r\}$ if $f$ maps $D(z_0, r)$ univalently onto a convex domain. The radius of convexity of the family $\mathcal{F}$ of functions defined on the disk $D$ is the largest number $r_0$ such that every function $f \in \mathcal{F}$ is convex in the disk $D(0, r_0)$.

Theorem 4. If $f \in \mathcal{H}(\alpha, Q)$, then for every $z \in \mathbb{D}$, the function $f$ is convex in the disk $D(z, R(z))$, where

\begin{equation}
(1.10)
R(z) = \frac{1}{2} \left( R_0 + R_0^{-1} - \sqrt{(R_0 - R_0^{-1})^2 + 4|z|^2} \right),
\end{equation}

and

\begin{equation}
(1.11)
R_0 = \alpha + k^{-1} - \sqrt{k^{-2} - 1} - \sqrt{(\alpha + k^{-1} - \sqrt{k^{-2} - 1})^2 - 1}.
\end{equation}

In particular, the radius of convexity of the family $\mathcal{H}(\alpha, Q)$ is no less than $R_0$.

The proofs of Theorems 1, 3 and 4 will be presented in Section 2.

2. Proofs of the Main results

2.1. Proof of Theorem 1. The proof of the theorem is divided into three parts.

Part 1: Let $f = h + \overline{g}$ satisfy the assumptions of Theorem 1. In compliance with the definition of the value $d_f(0)$, there exists a boundary point $A$ of the manifold $f(\mathbb{D})$ such that $A \in \{w : |w| = d_f(0)\}$. Consider the smooth curve $\ell_0 = f^{-1}([0, A])$, namely, the preimage of the half-interval $[0, A)$ with the starting point 0 in the disk $\mathbb{D}$. Then

\[ d_f(0) = |A| = \left| \int_{\ell_0} df(z) \right| = \min_{\gamma} \left| \int_{\gamma} df(z) \right|, \]

where the minimum is taken over all smooth paths $\gamma(t), t \in [0, 1)$, such that $\gamma(0) = 0$, $|\gamma(t)| < 1$ and $\lim_{t \to 1^-} |\gamma(t)| = 1$.

Similarly we define the value

\[ d_h(0) = |B| = \left| \int_{\ell} dh(z) \right| = \min_{\gamma} \left| \int_{\gamma} dh(z) \right|, \]

where the simple smooth curve $\ell = h^{-1}([0, B])$ is emerging from the origin, the preimage of the half-interval $[0, B)$ under the mapping $h$. Consider the following
The parametrization of the curve $\ell: \ell(t) = h^{-1}(Bt), t \in [0, 1)$. Then $h'(\ell(t))\ell'(t) = B$ and

\[
\begin{align*}
\left. df(0) \right| = \left| \int_0^1 df(\ell_0(t)) \right| & \leq \left| \int_0^1 df(\ell(t)) \right| \\
& = \left| \int_0^1 \left\{ h'(\ell(t))\ell'(t) + g'(\ell(t))\ell'(t) \right\} dt \right| \\
& = \left| B \right| \int_0^1 \left\{ 1 + \frac{g'(\ell(t))\ell'(t)}{h'(\ell(t))\ell'(t)} \right\} dt \\
& \leq d_h(0) \left\{ 1 + \int_0^1 |\mu(\ell(t))| dt \right\}. 
\end{align*}
\]

(2.1)

At first we consider the case $k = \sup_{z \in \mathbb{D}} |\mu(z)| < 1$. Since $|\mu(z)| \leq k$ for $z \in \mathbb{D}$, we have

\[
\mu(0)/k = \frac{\partial}{\partial t} (k a_t) =: u \in \mathbb{D}.
\]

If $|u| = 1$ for $k < 1$, then we have the inequality

\[
d_f(0) \leq d_h(0)(1 + k) = d_h(0)M(1, k)
\]

which proves the upper estimate in the inequality (1.6) for $z = 0$.

Let us now assume that $|u| < 1$ for some $k < 1$. Then, from a generalized version of the classical Schwarz lemma (see for example [5, Chapter VIII, §1]), it follows that

\[
\frac{|\mu(z)|}{k} \leq \frac{|z| + |u|}{1 + |u||z|}.
\]

(2.2)

Consequently, by (2.1), one has

\[
\begin{align*}
\left. df(0) \right| & \leq d_h(0) \left\{ 1 + k \int_0^1 \left| \frac{\ell(t)}{1 + |u|\ell(t)} \right| dt \right\}. 
\end{align*}
\]

(2.3)

Also, the function $h^{-1}(B\zeta)$ maps biholomorphically $\mathbb{D}$ onto some sub domain of the disk $\mathbb{D}$. Applying the classical Schwarz lemma, we obtain the inequality $|h^{-1}(B\zeta)| \leq |\zeta|$ and hence, $|\ell(t)| \leq t$ holds. Using the last estimate and the inequality (2.3), one can obtain, after evaluating the integral, the inequality

\[
\left. df(0) \right| \leq d_h(0) \left\{ 1 + k \int_0^1 \left| t + \frac{|u|}{1 + |u|\ell(t)} \right| dt \right\} = d_h(0)M(|u|, k),
\]

where $M(x, k)$ is defined by (1.7). The function $M(x, k)$ is strictly increasing on $[0, 1]$ with respect to the variable $x$ and for each fixed $k \in [0, 1]$. This follows from the observation that (see (1.7))

\[
\frac{\partial M(x, k)}{\partial x} = -\frac{k}{x^2} + 2k \frac{x}{x^3} \log(1 + x) - k \left( \frac{1 - x}{x^2} \right),
\]

which is positive, since $\log(1 + x) > x - x^2/2$. Hence

\[
d_f(0) \leq d_h(0)M(|u|, k) \leq d_h(0)M(1, k) = (1 + k)d_h(0).
\]

(2.4)
Let us now set \( k = 1 \). According to Lewy's theorem \cite{7} for locally univalent harmonic mapping \( f \), we obtain that \(|\mu(z)| \neq 1\) for all \( z \). Next we obtain the inequality (2.3) in the case \( k = 1 \) by repeating the argument of the case \( k < 1 \).

Let us now begin to prove that the upper estimate in (1.6) is true for all \( \zeta \in \mathbb{D} \). As mentioned above, the family \( \mathcal{H}(\alpha, Q) \) is linear invariant in the sense of \cite{15, 16} (see (1.4) above). Hence, for each fixed \( \zeta = re^{i\theta} \in \mathbb{D} \), where \( H \) and \( G \) are analytic in \( \mathbb{D} \) such that \( H(0) = G(0) = 0 \). Therefore, in view of (2.3) for \( k \in [0, 1] \), we have

\[
F(z) = \frac{f\left(e^{i\theta} z + r e^{i\theta}/1 + rz\right) - f(re^{i\theta})}{\partial f(re^{i\theta})(1 - r^2)} = H(z) + G(z)
\]

belongs to the family \( \mathcal{H}(\alpha, Q) \), where \( H \) and \( G \) are analytic in \( \mathbb{D} \). We complete the proof of the upper estimate in (1.6).

Part 2: We now deal with the sharpness of the upper estimate in (1.6). Consider the case \( k \in [0, 1] \). For every \( \alpha \in \mathbb{N} \) and every \( \zeta \in \mathbb{D} \), we shall indicate functions from the families \( \mathcal{H}(\alpha, Q) \) such that \( d_f(\zeta)/d_h(\zeta) = M(x) = 1 + k \), where \( x = |\mu(\zeta)|/k \).

Since the families \( \mathcal{H}(\alpha, Q) \) are enlarging with increasing values of \( \alpha \), the sharpness of the upper estimate in (1.6) will be shown for every \( \zeta \in \mathbb{D} \) and each \( \alpha \in [1, \infty) \).

Consider the sequence of \( \{k_n\}_{n=1}^{\infty} \) functions from \( \mathcal{U}_n \) defined by

\[
k_n(z) = \frac{i}{2n} \left[ \left(\frac{1 - iz}{1 + iz}\right)^n - 1 \right].
\]

Then we have \( d_{k_n}(0) = 1/2n \) (see \cite{9}) and observe that \( k_n \) maps the unit disk \( \mathbb{D} \) univalently onto the Riemann surface \( k_n(\mathbb{D}) \) whose boundary

\[
\partial k_n(\mathbb{D}) = \left\{ \frac{i}{2n} [(i\lambda)^n - 1] : \lambda \in \mathbb{R} \right\} = \left\{ \frac{i}{2n} [s e^{\pm in/2} - 1] : s \geq 0 \right\}
\]

consists of two rays. Then the univalent image of the disk \( \mathbb{D} \) under the mapping

\[
f_n(z) = h_n(z) + g_n(z) = \frac{1}{1 - k}[k_n(z) - k\overline{k_n(z)}] \in \mathcal{H}(n, Q), \quad k \in [0, 1),
\]

defines the family of the upper estimate in (1.6).
\(a_1 = \frac{1}{1-k}, \ a_{-1} = -\frac{k}{1-k}\) represents the manifold with the boundary

\[
\partial f_n(D) = \left\{ \frac{i}{2n(1-k)}[s(e^{\pm i\pi/2} + k e^{\mp i\pi/2}) - 1 - k] : s \geq 0 \right\},
\]

which consists of two rays parallel to the coordinate axes and arising from the point \(-\frac{i}{2n}Q\). Note that the function \(f_n\) maps the half-interval \([0, -i]\) bijectively onto \([0, -\frac{i}{2n}Q]\) and thus, we conclude that \(df_n(0) = \frac{Q}{2n}\); that is,

\[
df_n(0) = dk_n(0)Q = d\mu_n(0)(1 + k),
\]

where \(h_n(z) = k_n(z)/(1 - k)\). The sharpness of the upper estimate in (1.6) is proved for \(\zeta = 0\) and \(k < 1\).

Next we let \(0 \neq \zeta \in \mathbb{D}, \ k < 1\) and consider a conformal automorphism \(\omega(z) = (z + \zeta)/(1 + \zeta z)\) of the unit disk \(\mathbb{D}\). Then the inverse mapping is given by \(\omega^{-1}(z) = (z - \zeta)/(1 - \zeta z)\). From the condition (1.4) of the linear invariance property of the family \(\mathcal{H}(\alpha, Q)\), it follows that the function \(f\) defined by

\[
f(z) = \frac{f_n(\omega^{-1}(z)) - f_n(-\zeta)}{\partial \omega f_n(-\zeta)(1 - \zeta^2)} = h(z) + g(z)
\]

belongs to \(\mathcal{H}(\alpha, Q)\), where \(h\) and \(g\) have the same meaning as above. Taking into account of the normalization condition for functions in the family \(\mathcal{H}(\alpha, Q)\), we deduce that

\[
\frac{f(\omega(z)) - f(\zeta)}{\partial \omega(\zeta)(1 - |\zeta|^2)} = f_n(z) = h_n(z) + g_n(z).
\]

Therefore,

\[
df_n(0) = \frac{df(\zeta)}{\partial \omega f(\zeta)(1 - |\zeta|^2)} = d\mu_n(0)(1 + k).
\]

On the other hand, a direct computation gives

\[
h_n(z) = \frac{h(\omega(z)) - h(\zeta)}{\partial \omega h(\zeta)(1 - |\zeta|^2)} \quad \text{and} \quad d\mu_n(0) = \frac{dh(\zeta)}{\partial \omega f(\zeta)(1 - |\zeta|^2)}
\]

showing that

\[
df_n(0)|\partial \omega f(\zeta)(1 - |\zeta|^2) = df(\zeta) = dh(\zeta)(1 + k),
\]

which completes the proof of the upper estimation in Theorem 1 for \(k \in [0, 1)\).

If \(k = 1\) then for \(j \in \mathbb{N}\), we consider the sequence \(\{f_{n,j}\}\) of functions

\[
f_{n,j}(z) = h_{n,j}(z) + g_{n,j}(z) = j k_n(z) - (j - 1) k_n(z).
\]

We see that \(f_{n,j} \in \mathcal{H}(n, 2j - 1) \subset \mathcal{H}(n, \infty)\) for each \(j \in \mathbb{N}\). Therefore,

\[
df_{n,j}(0) = d\mu_{n,j}(0)M(1, 1 - 1/j).
\]

Hence

\[
\sup_{j \in \mathbb{N}} \frac{df_{n,j}(0)}{d\mu_{n,j}(0)} = M(1, 1) = 2.
\]

The sharpness of the upper estimation in (1.6) for \(k = 1, \ \zeta \neq 0\), can be proved analogously. So, we omit the details.
**Part 3:** Finally, we deal with the correctness of the lower estimation of \( df(z) \). If \( k = 1 \), then the lower estimation in \((1.6)\) is trivial because \( m(x, \infty) = 0 \). So, we may assume that \( k \in [0, 1) \). As in Part 1, we define the boundary points \( A \) and \( B \) of the manifolds \( f(\mathbb{D}) \) and \( h(\mathbb{D}) \), respectively, and smooth curves \( \ell_0 = f^{-1}([0, A]) \) and \( \ell \) in the same manner as in Part 1. Consider the parametrization of the curve \( \ell_0 \):

\[
\ell_0(t) = f^{-1}(At), \ t \in [0, 1).
\]

Then \( df(\ell_0(t)) = Adt \) and thus,

\[
dh(0) = \left| \int_0^1 dh(\ell(t)) \right| \\
\leq \left| \int_0^1 h'(\ell_0(t)) \ell_0'(t) \ dt \right| \\
= \left| \int_0^1 \left\{ h'(\ell_0(t)) \ell_0'(t) + g'(\ell_0(t)) \ell_0'(t) \right\} \ dt \right| \\
\times \left( 1 - \frac{g'(\ell_0(t)) \ell_0'(t)}{h'(\ell_0(t)) \ell_0'(t) + g'(\ell_0(t)) \ell_0'(t)} \right) \\
= \left| \int_0^1 h'(\ell_0(t)) \ell_0'(t) \ dt \right| \\
\leq |A| \int_0^1 \frac{dt}{1 - |\mu(\ell_0(t))|}.
\]

(2.5)

In view of the inequality \((2.2)\), we find that

\[
|\mu(\ell_0(t))| \leq k \frac{|\ell_0(t)| + x}{1 + x|\ell_0(t)|},
\]

where \( x = |\mu(0)|/k \).

It is possible to obtain an estimation of the value \( |\ell_0(t)| = |f^{-1}(At)|, \ t \in [0, 1) \), with the help of the analog of the Schwarz lemma for \( Q \)-quasiconformal automorphisms of the disk. Let \( F \) be a \( Q \)-quasiconformal automorphism of \( \mathbb{D} \), and \( F(0) = 0 \). It is known (see for example [1] Chapter 10, equality (10.1)) that the sharp estimation

\[
|F(z)| \leq \varphi^{-1}(Q^{-1}\varphi(|z|))
\]

holds, where \( \varphi \) and \( Q \) are as in the statement. The function \( f^{-1}(Aw) \) defined on the unit disk \( \{ w : |w| < 1 \} \) satisfies the conditions \( f^{-1}(0) = 0 \) and \( |f^{-1}(Aw)| < 1 \). Let \( \Phi \) be the univalent conformal mapping of the domain \( f^{-1}(A\mathbb{D}) \) onto the unit disk \( \mathbb{D} \) and \( \Phi(0) = 0 \). Then the composition \( \Phi \circ f^{-1}(Az) \) is a \( Q \)-quasiconformal automorphism of \( \mathbb{D} \) and \( \Phi^{-1} \) satisfies the conditions of the classical Schwarz lemma for analytic functions. Hence, we have

\[
|\ell_0(t)| = |\Phi^{-1}(\Phi \circ f^{-1}(At))| \leq |\Phi \circ f^{-1}(At)| \leq \varphi^{-1}(Q^{-1}\varphi(t)).
\]

As a result of it and taking into account of the last estimation, inequalities \((2.5)\) and \((2.6)\), and the fact that the function \((1 + yx)/(1 - kx + y(x - k))\) is strictly
increasing with respect to $y$ on $(0,1)$, we conclude that
\[ d_h(0) \leq d_f(0) \int_0^1 \frac{1 + yx}{1 - kx + y(x-k)} \, dt \leq \frac{d_f(0)}{1 - k}, \]
where $y = \varphi^{-1}(Q^{-1}\varphi(t)) \leq 1$ for $t \in (0,1)$. Therefore the lower estimate in (1.6) is sharp at the origin.

The proof of the lower estimation in (1.6) for $0 \neq \zeta \in \mathbb{D}$ follows easily if we proceed with the same manner as in Part 1 and use the linear invariance property of the family $\mathcal{H}(\alpha, Q)$.

For the sharpness of the left side of the inequality in (1.6) for $k \in [0,1)$, we consider the functions (see [15, 16])

\[ h_{\alpha}(z) = \frac{1}{2i\alpha} \left[ \left( \frac{1 + iz}{1 - iz} \right)^{\alpha} - 1 \right] \in \mathcal{U}_{\alpha} \]
and
\[ f(z) = h(z) + g(z) := \frac{h_{\alpha}(z)}{1 + k} + \frac{k h_{\alpha}(z)}{1 + k}. \]
Then it is a simple exercise to see that
\[ d_f(0) = \frac{1}{2\alpha Q} \quad \text{and} \quad d_h(0) = \frac{1}{2\alpha(1 + k)}. \]
If $k \to 1^-$ then from the last equality we obtain
\[ \lim_{k \to 1^-} d_f(0) = 0 \quad \text{and} \quad \lim_{k \to 1^-} d_h(0) = \frac{1}{4\alpha}, \]
so that
\[ \inf_{f \in \mathcal{H}(\alpha, \infty)} \frac{d_f(0)}{d_h(0)} = 0. \]
Thus the last equality is sharp not only at the origin but also at points $z \in \mathbb{D}$, in view of the degeneration of functions
\[ f(z) = \frac{h_{\alpha}(z) + k h_{\alpha}(z)}{1 + k} \]
when $k \to 1^-$. The proof of the theorem is complete. □

2.2. **Proof of Theorem 3.** Let us first prove the inequality (1.9) for $z = 0$. As in the proof of Theorem 1 consider on the circle $\{ w : |w| = d_f(0) \}$ the boundary point $A$ of the manifold $f(\mathbb{D})$ and define a curve $\ell_0 = f^{-1}([0, A])$ with the starting point 0 in $\mathbb{D}$. Then

\[ d_f(0) = |A| = \left| \int_{\ell_0} df(\zeta) \right| = \int_{\ell_0} |df(\zeta)| \geq \int_{\ell_0} (|h'(\zeta)| - |g'(\zeta)|) \, |d\zeta|. \]

In view of the affine invariance property of the family $\mathcal{AL}_H$, the function $F$ defined by
\[ F(\zeta) = H(\zeta) + G(\zeta) = \frac{f(\zeta) - \varepsilon f(\zeta)}{1 - \varepsilon f(\zeta)} \]
belongs $\mathcal{AL}_H$ for every $\varepsilon$ with $|\varepsilon| < 1.$
For a fixed $\zeta$, we introduce $\theta(z) = \arg h'(z) - \arg g'(z)$ when $g'(z) \neq 0$, and $\theta(z) = \arg h'(z)$ otherwise. Consider then $\epsilon = se^{i\theta(z)}$ for $s \in [0, 1)$. Therefore, taking into account of the relation $f_z(0) = \mu(0)$, we obtain that

$$H'(\zeta) = \frac{h'(\zeta) - sg'(\zeta)e^{i\theta(z)}}{1 - \epsilon \mu(0)}$$

and thus,

$$|H'(\zeta)| \leq \frac{|h'(\zeta)| - s|g'(\zeta)|}{1 - s|\mu(0)|}.$$  

(2.9)

For the other side of the inequality for functions in the family $\mathcal{AL}_H$, the inequality

$$|H'(\zeta)| \geq \frac{(1 - |\zeta|)^{\alpha - 1}}{(1 + |\zeta|)^{\alpha + 1}}$$

(2.10)

holds, where $\alpha = \text{ord}(\mathcal{AL}_H)$ is defined as in the sense of Definition 2. The inequality (2.10) was obtained in [13] for LAIF of univalent harmonic mappings, but the proof is still valid without a change for any LAIF $\mathcal{AL}_H$ of finite order $\alpha$. Using inequalities (2.9) and (2.10), we obtain the inequality

$$|h'(\zeta) - s|g'(\zeta)| \geq (1 - s|\mu(0)|)\frac{(1 - |\zeta|)^{\alpha - 1}}{(1 + |\zeta|)^{\alpha + 1}}$$

for every $s \in (0, 1)$. Allowing in the last inequality $s \to 1^-$ and substituting the resulting estimate into (2.8), we easily obtain that

$$d_f(0) \geq (1 - |\mu(0)|)\int_0^1 \frac{(1 - |\zeta|)^{\alpha - 1}}{(1 + |\zeta|)^{\alpha + 1}}|d\zeta|
\geq (1 - |\mu(0)|)\int_0^1 \frac{(1 - t)^{\alpha - 1}}{(1 + t)^{\alpha + 1}}dt = \frac{1 - |\mu(0)|}{2\alpha}.$$  

If $0 < |z| < 1$, then as in the proof of Theorem 1, we may use the linear invariance property of the family $\mathcal{AL}_H$ in accordance with the function $F_1 \in \mathcal{AL}_H$, where

$$F_1(\zeta) = h'\left(\frac{\zeta + z}{h'(z)}\right) - h(z).$$

In this way, applying the estimation of $d_f(0)$ to the function $F_1$, we see that

$$d_{F_1}(0) \geq \frac{1 - |\mu_f(0)|}{2\alpha}.$$  

Also, we have

$$d_{F_1}(0) = \frac{d_f(z)}{|h'(z)|(1 - |z|^2)}.$$  

It remains to note that $|\mu_{F_1}(0)| = |\mu_f(z)|$ and apply the inequality (2.10) to the function $h'(z)$.

In order to prove the sharpness of the estimate of $d_f(0)$, we first note that the functions $p(z) = h_\alpha(z) + kh_\alpha(z)$, where each $h_\alpha$ has the form (2.7), belong to
belongs to $C$ for every $k = |\mu(0)| \in [0, 1)$. Indeed, for each $\alpha$, the function $p$ is locally univalent and meet the normalization condition of the family $AL_H(\alpha)$, and $|p_{zz}(0)/2| = |h_{\alpha}''(0)/2| = \alpha$. Affiliation of the functions

$$q(z) = \frac{p(\omega(z)) - p(\omega(0))}{h_{\alpha}'(\omega(0))\omega'(0)} = \frac{h_{\alpha}'(\omega(z)) - h_{\alpha}'(\omega(0))}{h_{\alpha}'(\omega(0))\omega'(0)} + k \frac{h_{\alpha}'(\omega(z)) - h_{\alpha}'(\omega(0))}{h_{\alpha}'(\omega(0))\omega'(0)},$$

and

$$w(z) = \frac{q(z) + \varepsilon q(z)}{1 + \varepsilon q(z)} = \frac{h_{\alpha}(\omega(z)) - h_{\alpha}(\omega(0))}{h_{\alpha}'(\omega(0))\omega'(0)} + \frac{h_{\alpha}'(\omega(z)) - h_{\alpha}'(\omega(0))}{h_{\alpha}'(\omega(0))\omega'(0)} \left( \frac{k + \varepsilon h_{\alpha}'(\omega(0))\omega'(0)/(h_{\alpha}'(\omega(0))\omega'(0))}{1 + \varepsilon k h_{\alpha}'(\omega(0))\omega'(0)/(h_{\alpha}'(\omega(0))\omega'(0))} \right)$$

[to the family $AL_H(\alpha)$ for every conformal automorphism $\omega$ of the disk $D$ and every $\varepsilon \in \mathbb{D}$, follow from the membership of the function $h_{\alpha}$ to the universal LIF $U_{\alpha}$. The analogous reasoning is true after the change of order of the linear and affine transforms of the function $p$.

Therefore, $p = h_{\alpha} + kh_{\alpha} \in AL_H(\alpha)$ for each $k \in [0, 1)$ and at the same time

$$d_{p}(0) = \frac{1 - |\mu(0)|}{2\alpha},$$

which proves the sharpness of the established estimate in the universal LIF $AL_H(\alpha)$. The proof of the theorem is complete. \hfill \Box

**Remark 3.** (a) Recall that a domain $D \subset \mathbb{C}$ is called close-to-convex if its complement $\mathbb{C} \setminus D$ can be written as an union of disjoint rays or lines. The family $C_H$ of all univalent sense-preserving harmonic mappings $f$ of the form (11) such that $a_1 = 1$ and $f(D)$ is close-to-convex, is LAIF (cf. [13]). Also, the inequality in Theorem 3 is sharp in the LAIF $C_H$. The order of the family $C_H$ is proved to be 3 ([2]). The harmonic analog of the analytic Koebe function $K(z) = z/(1 - z)^2$ (see for example [4 Chapter 5, p. 82]) is given by

$$F(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3} + \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3},$$

where $F \in C_H$ and $F(D) = \mathbb{C} \setminus (-\infty, -1/6]$ which is indeed a domain starlike with respect to the origin. From the affine invariance property of the family $C_H$, we deduce that for every $b \in [0, 1)$, the affine mapping

$$f(z) = F(z) - b F(z)$$

belongs to $C_H$ such that $\mu(0) = f_{zz}(0)/f_z(0) = -b$. The function $f$ is a composition of the univalent harmonic mapping $F$ of the disk $D$ onto $\mathbb{C} \setminus (-\infty, -1/6]$ and affine transformation $\psi(w) = w - b \overline{w}$. The plane $\mathbb{C}$ with a slit $(-\infty, -1/6]$ under the transformation $\psi$ is the plane with a slit along the ray emanating from the point $\psi(-1/6) = -(b-1)/6$ through the point $\psi(1) = b - 1 < (b-1)/6$, since $b \in [0, 1)$. 

Linear invariant families of locally univalent harmonic mappings 13
Therefore, \( f(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -(1 - b)/6] \) and thus, \( d_f(0) = (1 - b)/6 \) and the lower estimate of \( d_f(0) \) is sharp in the LAIF \( \mathcal{C}_H \).

(b) In the first part of the present paper, we are concerned with the question about the covering of the manifold \( f(\mathbb{D}) \) by the disks. Now we turn our attention on the problem related with the covering of \( f(\mathbb{D}) \) by convex domains.

Sheil-Small [13] proved that the radius of convexity of the univalent subfamily of the linear and affine invariant family \( \mathcal{AL}_H \) of harmonic mappings is equal to

\[
(2.11) \quad r_0 = \alpha - \sqrt{\alpha^2 - 1},
\]

where \( \alpha = \text{ord}(\mathcal{AL}_H) \). Later this result was generalized to the families of locally univalent harmonic mappings [6]. Now we will show the radius of convexity will be altered under the assumption of \( Q \)-quasiconformality of functions \( f \).

**Lemma 1.** Let \( \mathcal{LU}_H(\alpha, Q) \) denote the LIF of locally univalent \( Q \)-quasiconformal harmonic mappings of the order \( \alpha < \infty \), where \( Q \leq \infty \). Then the affine hull

\[
\mathcal{AL}_H = \left\{ F(z) = \frac{f(z) + \varepsilon f(z)}{1 + \varepsilon a_{-1}} : f \in \mathcal{LU}_H(\alpha, Q), \varepsilon \in \mathbb{D} \right\}
\]

of the family \( \mathcal{LU}_H(\alpha, Q) \) is linear and affine invariant of the order no greater than \( \alpha + 1 - \sqrt{\frac{1}{1 - k^2}} \), where \( k = (Q - 1)/(Q + 1) \).

**Proof.** In [14], it was shown that the affine hull of the linear invariant in the sense of Definition 2 of the family of the locally univalent harmonic mappings is the LAIF \( \mathcal{AL}_H \). Thus, it remains to determine the estimate of the order of the family \( \mathcal{AL}_H \).

We begin with \( F = H + \overline{G} \in \mathcal{AL}_H \). Then there exists an \( f = h + \overline{\mu} \in \mathcal{LU}_H(\alpha, Q) \) of the form (1.1) with the normalization \( f(0) = 0, h'(0) = 1, \) and \( \varepsilon \in \mathbb{D} \) such that

\[
F(z) = \frac{f(z) + \varepsilon f(z)}{1 + \varepsilon g'(0)} = H(z) + \overline{G(z)}.
\]

It is easy to compute that

\[
A_2 = \frac{H''(0)}{2} = \frac{a_2 + \varepsilon a_{-2}}{1 + \varepsilon g'(0)},
\]

where \( a_2 = h''(0)/2 \) and \( a_{-2} = g''(0)/2 \). Taking into account of the relation \( g'(z) = \mu(z)h'(z) \), where \( \mu \) is the complex dilatation of \( f \) with \( |\mu(z)| < k \), we see that

\[
g'(0) = \mu(0) \quad \text{and} \quad g''(0) = h''(0)\mu(0) + h'(0)\mu'(0),
\]

so that

\[
a_{-2} = a_2\mu(0) + \mu'(0)/2.
\]

If we apply the Schwarz-Pick lemma (see for example [5, Chapter VIII, §1]) to the function \( \mu(z)/k \), then the inequality (1.3) in this case leads to

\[
\frac{|\mu'(0)|}{k} \leq 1 - \frac{|\mu(0)|^2}{k^2}.
\]
Using the expression for $a_2$, we deduce that

$$|A_2| = \left| \frac{a_2(1 + \varepsilon \mu(0)) + \varepsilon \mu'(0)}{1 + \varepsilon \mu(0)} \right| \leq |a_2| + \frac{k^2 - |\mu(0)|^2}{2k(1 - |\mu(0)|)}$$

(since $|\varepsilon| < 1$). Calculating the maximum of the function $u(t) = (k^2 - t^2)/(1 - t)$ over the interval $[0, k]$, we obtain the estimate

$$|A_2| \leq |a_2| + \frac{1 - \sqrt{1 - k^2}}{k} \leq \alpha + 1 - \sqrt{1 - k^2} < \alpha + 1.$$ 

The proof of the lemma is complete. \hfill \Box

Using Lemma 1 and the equality (2.11), one obtains the estimate of the radius of convexity of functions in the family $\mathcal{H}(\alpha, Q)$.

2.3. Proof of Theorem 3. Let $f_0 = h_0 + g_0 \in \mathcal{H}(\alpha, Q)$. It is easy to see that function $f_0$ is convex in the same disks as the normalized function

$$f(z) = f_0(z)/h_0'(0) = h(z) + g(z)$$

that belongs to some LIF $\mathcal{LU}_H(\alpha, Q)$. So it is enough to prove the statement of the theorem for such functions $f$. We first show that function $f$ is convex in the disk centered at the origin with radius $R_0$ defined by (1.11).

Clearly, the function $f$ belongs to the affine hull $\mathcal{AL}_H$ of the family $\mathcal{LU}_H(\alpha, Q)$. In view of Lemma 1, the family $\mathcal{AL}_H$ has the order $\alpha_1 \leq \alpha + 1 - \sqrt{1 - k^2}$. Taking into consideration of the equality (2.11), we conclude that the function $f$ is convex in the disk of radius $R_0 = \alpha_1 - \sqrt{\alpha_1^2 - 1}$ centered at the origin.

We now let $0 \neq z_0 \in \mathbb{D}$. Consider a conformal automorphism $\Phi$ of the unit disk $\mathbb{D}$ given by

$$\Phi(\zeta) = e^{i \arg z_0} \left( \frac{\zeta + |z_0|}{1 + |z_0|} \right).$$

We see that $\Phi$ maps the disk $\mathbb{D}(0, R_0)$ onto the disk $\mathbb{D}(z_0, R(z_0))$, where $R(z_0)$ is defined in (1.10). In view of the linear invariance property of the family $\mathcal{LU}_H(\alpha, Q)$, the function $F$ defined by

$$F(\zeta) = \frac{f(\Phi(\zeta)) - f(z_0)}{h'(z_0)\Phi'(0)}$$

belongs to $\mathcal{LU}_H(\alpha, Q)$ and as remarked above, the function $F$ maps the disk $\mathbb{D}(0, R_0)$ onto a convex domain. Therefore, the function

$$f(z) = F(\Phi^{-1}(z)) \cdot h'(z_0)\Phi'(0) + f(z_0)$$

is convex and univalent in the disk $\mathbb{D}(z_0, R(z_0))$. The proof of the theorem is complete. \hfill \Box

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S. Yu. Graf, Tver State University, ul. Zhelyabova 33, Tver, 170000 Russia.
E-mail address: sergey.graf@tversu.ru

S. Ponnusamy, Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and Security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India.
E-mail address: samy@isichennai.res.in, samy@iitm.ac.in

V. V. Starkov, Department of Mathematics, University of Petrozavodsk, ul. Lenina 33, 185910 Petrozavodsk, Russia
E-mail address: vstarv@list.ru