Supplementary Information:

Ultra-coherent nanomechanical resonators via soft clamping and
dissipation dilution

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Appendix A: Simulated displacement

FIG. S1. Simulated displacement fields. Mode shapes of the localised vibrational modes of the defect, showing excellent agreement with the measurements in Fig. 3.

FIG. S2. Projections of the displacement fields. Projections of the simulated displacement fields along the x- (orange) and y-directions (red) for modes A – E.

Appendix B: Quality factors outside the bandgap

Here we present a comparison of measured quality factors inside and outside the phononic bandgap of a perforated membrane resonator of thickness \( h = 121 \text{ nm} \) and lattice constant \( a = 364 \mu \text{m} \). The modes outside the bandgap are delocalised vibrational modes of the entire membrane structure and have been chosen randomly. As shown in Fig. S3, the defect modes have on average an order of magnitude higher quality factors than the modes outside the bandgap. The outer dimensions of the membrane structure are \( 6755 \mu \text{m} \times 6582 \mu \text{m} \) and according to the standard theory of dissipation dilution [15], one would expect appreciably higher Q-factors for the delocalised (outside-bandgap) modes than seen in Fig. S3, approximating the perforated membrane by a plain one. Since the wafer with the perforated membrane is clamped, we suspect the discrepancy to be predominantly due to radiation losses, against which these modes are not protected.
Appendix C: Comparison with higher-order modes of square membranes

It has previously been demonstrated that high $Qf$-products can also be obtained with higher-order modes of square membranes. For example, for a $5 \times 5 \text{mm}^2$ membrane, $Qf \approx 100 \text{THz}$ has been achieved at $f \approx 2.7 \text{MHz}$ [17]. This is fully consistent with the standard model of dissipation dilution in such structures, which suggests [9, 15]

$$Q^{-1}_{\square} = (2\lambda + (i^2 + j^2) \pi^2 \lambda^2) Q^{-1}_{\text{int}}$$

(S1)

for modes with $i$ and $j$ antinodes in $x$- and $y$-direction, respectively. Equation (4) in the main manuscript is the special case $i = j = 1$ of the fundamental mode. While the $Qf$-products achieved in this manner are nearly as high as the ones we demonstrate, we emphasise (see Introduction) that working with such modes in practice is mostly hindered by their large mass and intrinsically dense mode spectrum.

To illustrate the latter, Fig. S4 shows the eigenfrequencies of all $(i, j)$-mode pairs and a histogram accumulating the number of modes. At $f \approx 2.7 \text{MHz}$, where $Qf \approx 100 \text{THz}$ has been reached [17], the mode density is very high, about 1 mode per kHz. This is consistent with the theoretically expected value of

$$\frac{dN}{df} = 2\pi f L^2 \rho / \sigma,$$

(S2)

which is straightforward to calculate for this 2-dimensional system with linear dispersion. From the decoherence rate of such modes, $\gamma = k_B T_R / hQ \approx 2\pi 130 \text{kHz}$ at room temperature, it is clear that quantum optomechanical experiments are impossible with such modes. For example, ground-state cooling would require optical damping, and thus broadening the resonance, by $\gtrsim \gamma$. A large number of modes would start to overlap, making it impossible to work with single vibrational modes of the membrane. Secondly, the large mass $m_{\text{eff}} \sim \mathcal{O}(1 \text{\mu g})$ would make it very difficult to achieve the required cooling rates in the first place.

This is in stark contrast to the soft clamped modes we introduce here, in both aspects—mass and mode density. This is rooted in the fact that the modes we discuss are localised to the defect. Both their oscillating mass, and the mode density, are thus determined by the size of the defect $\sim a < L$, much smaller than the size of the embedding membrane, and of a plain square membrane that achieves similar $Qf$-product.

To better illustrate this comparison, we present here the measured broadband displacement noise spectra of a $3 \times 3 \text{mm}^2$ plain square membrane similar to that of ref. [17] (Fig. S5), and a patterned membrane with $a = 160 \mu\text{m}$ (Fig. S6), whose outer dimensions are the same as those of the plain one. The dramatically reduced mode spectral density in the phononic bandgap of the patterned membrane is clearly evident.

At the same time, the effective mass of the higher-order plain membrane mode is much larger, since it involves motion of the entire membrane. In this concrete example, the mass of any mode of the plain square membrane is calculated as $m_{\text{eff}} = \rho L^2 h / 4 \approx 480 \text{ng}$ for thickness $h = 66 \text{nm}$. In contrast, the A-mode in the defect of a phononic membrane with $a = 160 \mu\text{m}$ and $h = 66 \text{nm}$ has been measured to be $m_{\text{eff}} \sim 4 \text{ng}$ (see p. 6 of main manuscript), two orders of magnitude lower.

However, the $Q$-factors and $Qf$-products for these two scenarios are comparable. For example, for a higher order mode $(i, j) = (12, 12)$ of the square membrane, $f \sim 1.6 \text{MHz}$, and $Q \sim 16 \cdot 10^6$ according to standard theory [15], which yields $Qf \sim 25 \text{THz}$. On the other hand, the in-bandgap A-mode of the patterned membrane is expected to have $Q \sim 25 \cdot 10^6$ according to our model. With $f \sim 1.46 \text{MHz}$ this yields $Qf \sim 37 \text{THz}$.
FIG. S4. **Calculated mode density** of a $5 \times 5.25 \text{ mm}^2$ plain (unpatterned) SiN membrane under 1 GPa tensile stress. The 5% sidelength difference lifts the $(i, j)$-$(j, i)$ mode degeneracies for better illustration. Blue lines indicate calculated eigenfrequencies, yellow histogram is the mode density (number of modes/100 kHz). Red curve is the analytically calculated mode density.
FIG. S5. Measured thermal noise spectrum of a plain square membrane sized $3 \times 3\text{mm}^2$ (5 averages). A high mode density throughout the spectrum is apparent, especially for larger mode numbers where $Q_f$-products are appreciable ($>10\text{THz}$).
FIG. S6. Measured thermal noise spectrum of a patterned membrane with $a = 160 \mu m$ (average over 5 different locations on the defect). The dramatically reduced mode density in the bandgap (shaded yellow) is apparent. The outer dimensions of the patterned membrane resonator are identical to the unpatterned resonator from Fig. S5.