D/M/1 Queue: Policies and Control

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Abstract. Equilibrium G/M/1-FIFO waiting times are exponentially distributed, as first proved by Smith (1953). For other client-sorting policies, such generality is not feasible. Assume that interarrival times are constant. Symbolics for the D/M/1-LIFO density are completely known; numerics for D/M/1-SIRO arise via an unpublished recursion due to Burke (1967). Consider a weighted sum of two costs, one from keeping clients waiting for treatment and the other from having the server idle. With this in mind, what is the optimal interarrival time and how does this depend on the choice of policy?

In an M/G/1 system, interarrival times are exponentially distributed with mean $1/\lambda$ and service lengths are arbitrary with mean $1/\mu$. In a G/M/1 system, it is service lengths that are exponentially distributed with mean $1/\mu$, while interarrival times are arbitrary with mean $1/\lambda$. All such intervals are taken to be independent; client waiting space is unlimited. The two systems display a kind of subtle symmetry, never overt. Let $\rho = \lambda/\mu$ denote the traffic intensity. Our focus in this paper is D/M/1, for which interarrival times are constant $a = 1/\lambda$.

We discussed first-in-first-out M/D/1 in [1]; both last-in-first-out M/D/1 and serve-in-random-order M/D/1 were covered in [2]. The two policies FIFO and LIFO are also known as FCFS (first-come-first-serve) and LCFS (last-come-first-serve). The policy SIRO is also variously known as ROS (random-order-of-service) and RSS (random-selection-for-service).

Given D/M/1, the functional equation

$$\zeta(s) = \exp \left[ -a (\mu + s - \mu \zeta(s)) \right]$$

is important. This is a consequence of $\exp(-as)$ being the Laplace transform of $\delta(x - a)$, where $\delta(x)$ is the Dirac delta. An analogous equation holds for non-constant interarrival times, depending on the probability density, but this would be more complicated. We have

$$\zeta(s) = -\frac{1}{a \mu} \omega [-a \mu \exp(-a \mu - a s)] = -\rho \omega \left[ -\frac{1}{\rho} \exp \left( -\frac{1}{\rho} - a s \right) \right]$$
where \( \omega(s) \) is the principal branch of the Lambert omega:

\[
\omega(s)e^{\omega(s)} = s, \quad -1 \leq \omega(x) \in \mathbb{R} \quad \forall \ x \geq -1/e, \quad \exists \text{ branch cut for } x < -1/e.
\]

As an example of aforementioned symmetry, the expression for \( \zeta(s) \) is identical to that for \( \Theta(s) \) corresponding to M/D/1 in [2] except \( \rho \) is everywhere replaced by \( 1/\rho \). A special value

\[
\zeta_0 = \zeta(0) = -\frac{1}{a \mu} \omega (-a \mu e^{-a \mu})
\]

will appear throughout. For instance, if \( \lambda = 2 \) and \( \mu = 3 \), then \( \zeta_0 = 0.41718835... \).

Upon differentiation, we have

\[
\omega'(s) = \frac{\omega(s)}{s (1 + \omega(s))}, \quad \zeta'(s) = -a \omega' [-a \mu \exp (-a \mu - a s)] \exp (-a \mu - a s).
\]

Another value

\[
\zeta'_0 = \zeta'(0) = -a \frac{\omega (-a \mu e^{-a \mu})}{(-a \mu e^{-a \mu}) [1 + \omega (-a \mu e^{-a \mu})]} e^{-a \mu}
\]

\[
= -a \frac{\mu \zeta_0}{(-a \mu) (1 - a \mu \zeta_0)} = -a \frac{\zeta_0}{1 - a \mu \zeta_0}
\]

is also needed; if \( \lambda = 2 \) and \( \mu = 3 \), then \( \zeta'_0 = -0.55741433... \).

1. FIFO

Let \( W_{\text{que}} \) denote the waiting time in the queue (prior to service). Under equilibrium (steady-state) conditions, the probability density function \( f(x) \) of \( W_{\text{que}} \) has Laplace transform [3, 4, 5, 6]

\[
F(s) = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\infty} \exp(-s x) f(x) dx = 1 - \zeta_0 + \zeta_0 \frac{\mu(1 - \zeta_0)}{s + \mu(1 - \zeta_0)} = F_{\text{alt}}(s) + 1 - \zeta_0
\]

and initial value [7]

\[
f(0^+) = \lim_{s \to 1^{-}} s F_{\text{alt}}(s) = \mu \zeta_0 (1 - \zeta_0).
\]

Consequently

\[
f(x) = (1 - \zeta_0) \delta(x) + \mu \zeta_0 (1 - \zeta_0) \exp (-\mu(1 - \zeta_0)x).
\]

In fact, exponentiality holds more generally for non-constant interarrival times, proved by Smith [3]. Moments are

\[
\text{mean} = -F'(0) = \frac{\zeta_0}{\mu(1 - \zeta_0)}, \quad \text{variance} = F''(0) - F'(0)^2 = \zeta_0 \frac{2 - \zeta_0}{\mu^2(1 - \zeta_0)^2}
\]
giving 0.23860673... and 0.21600433... respectively when \( \{ \lambda, \mu \} = \{2, 3\} \). If sampling is restricted only to \( W_{\text{que}} > 0 \), then \[8\]

\[
\text{mean}_{>0} = \frac{1}{\mu(1 - \zeta_0)}, \quad \text{variance}_{>0} = \frac{1}{\mu^2(1 - \zeta_0)^2}
\]
giving 0.57194007... and 0.32711544... respectively.

Let \( L_{\text{sys}} \) denote the number of patients in the system (both queue and service). Under equilibrium, with \( \{ \lambda, \mu \} = \{2, 3\} \), we have \[9\]

\[
\tilde{f}(\ell) = \mathbb{P}\{L_{\text{sys}} = \ell\} = (1 - \zeta_0)\zeta_0^\ell, \quad \ell = 0, 1, 2, 3, \ldots
\]

\[
\text{mean} = \frac{\zeta_0}{1 - \zeta_0}, \quad \text{variance} = \frac{\zeta_0}{(1 - \zeta_0)^2}
\]
giving 0.71582021... and 1.22821879... respectively.
Geometricity holds more generally for non-constant interarrival times. It is remarkable that classical distributions occur within G/M/1 universally but not within even M/D/1 specifically.

2. LIFO

The probability density function \( f(x) \) of \( W_{\text{que}} \) has Laplace transform

\[
F(s) = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\infty} \exp(-sx)f(x)dx = 1 - \zeta_0 + \zeta_0 \frac{\mu - \mu \zeta(s)}{s + \mu - \mu \zeta(s)} = F_{\text{alt}}(s) + 1 - \zeta_0
\]

and the inverse Laplace transform of \( \zeta(s) \) is

\[
\theta(x) = \sum_{k=1}^{\infty} e^{-k/\rho} \frac{(k/\rho)^{k-1}}{k!} \delta(x - a k).
\]

With regard to symmetry and \( F(s) \), we see that \( \{\zeta_0, \mu, \zeta\} \) play the roles of \( \{ \rho, \lambda, \Theta \} \) in \[2\], but an extra factor \( \zeta_0 \) is also present, i.e., the correspondence is not perfect. From

\[
(1 - \zeta_0)s + \mu [1 - \zeta(s)] (1 - \zeta_0 + \zeta_0) = s F(s) + \mu F(s) [1 - \zeta(s)]
\]

we have

\[
(1 - \zeta_0)s + \mu (1 - F(s)) [1 - \zeta(s)] = s F(s),
\]
i.e.,

\[
F(s) = 1 - \zeta_0 + \mu (1 - F(s)) \left[ \frac{1 - \zeta(s)}{s} \right]
\]
hence
\[
f(x) = (1 - \zeta_0)\delta(x) + \kappa + \mu \int_0^x (\delta(t) - f(t)) \left[ 1 - \int_0^{x-t} \theta(u) du \right] dt = (1 - \zeta_0)\delta(x) + \mu \zeta_0 + \mu \left[ 1 - \int_0^{x} \theta(u) du \right] - \mu \int_0^{x} f(t) \left[ 1 - \int_0^{x-t} \theta(u) du \right] dt.
\]

The indicated condition \( \kappa = \mu \zeta_0 \) is true by the initial value theorem \([7]\):
\[
\lim_{\varepsilon \to 0^+} f(\varepsilon) = \lim_{s \to 1^- \cdot \infty} s F_{\text{alt}}(s).
\]
Differentiating, we obtain
\[
f'(x) = \mu [0 - \theta(x)] - \mu f(x) [1 - 0] - \mu \int_0^x f(t) [0 - \theta(x - t)] dt
\]
\[
= -\mu \theta(x) - \mu f(x) + \mu \int_0^x f(t) \theta(x - t) dt
\]
\[
= -\mu \theta(x) - \mu f(x) + \mu \int_0^x f(t) \sum_{k=1}^{\infty} e^{-k/\rho} \frac{(k/\rho)^{k-1}}{k!} \delta(x - t - a k) dt
\]
\[
= -\mu \theta(x) - \mu f(x) + \mu \sum_{k=1}^{\infty} e^{-k/\rho} \frac{(k/\rho)^{k-1}}{k!} \int_0^x f(t) \delta(x - t - a k) dt
\]
\[
= -\mu \theta(x) - \mu f(x) + \mu \sum_{k=1}^{\infty} e^{-k/\rho} \frac{(k/\rho)^{k-1}}{k!} f(x - a k).
\]

For \(0 < x < a\),
\[
f'(x) = -\mu f(x), \quad f(0^+) = \mu \zeta_0
\]
implies
\[
f(x) = \mu \zeta_0 e^{-\mu x}.
\]
Note that \(\lim_{\varepsilon \to 0^+} f(a k + \varepsilon) = 0\) for each \(k \geq 1\) because, if a client arrives at the same moment the server becomes available, the client is taken immediately (by LIFO) and there is no waiting. Note also \(1/\rho = \mu/\lambda = \mu a\). For \(a < x < 2a\),
\[
f'(x) = -\mu f(x) + \mu e^{-1/\rho} \cdot \mu \zeta_0 e^{-\mu(x-a)}
\]
\[
= -\mu f(x) + \mu^2 \zeta_0 e^{-\mu x}.
coupled with $f(a^+) = 0$ implies

$$f(x) = \mu^2 \zeta_0 (x - a) e^{-\mu x}.$$ 

For $2a < x < 3a$,

$$f'(x) = -\mu f(x) + \mu e^{-1/\rho} \cdot \mu^2 \zeta_0 (x - 2a) e^{-\mu(x-a)} + \mu e^{-2/\rho} \frac{2}{2!} \cdot \mu \zeta_0 e^{-\mu(x-2a)}$$

$$= -\mu f(x) + \mu^3 \zeta_0 (x - 2a) e^{-\mu x} + (\mu^2 \zeta_0) (\mu a) e^{-\mu x}$$

$$= -\mu f(x) + \mu^3 \zeta_0 (x - a) e^{-\mu x}$$

coupled with $f(2a^+) = 0$ implies

$$f(x) = \frac{1}{2} \mu^3 \zeta_0 x(x - 2a) e^{-\mu x}.$$ 

More generally, for $k a < x < (k+1)a$, we obtain

$$f(x) = \frac{1}{k!} \mu^{k+1} \zeta_0 x^{k-1} (x - k a) e^{-\mu x}$$

and thus the waiting time density for LIFO is completely understood. Wishart [6] evidently holds priority in discovering this formula, building upon work by Conolly [10]. Stitching the fragments together gives the LIFO density function pictured in Figure 1, for parameter values $\lambda = 2$ and $\mu = 3$; hence $\rho = 2/3$ and $a = 1/2$.

Moments of $W_{\text{que}}$ for LIFO are [6]

mean $= -F'(0) = \frac{\zeta_0}{\mu(1 - \zeta_0)}$, variance $= F''(0) - F'(0)^2 = \zeta_0 \frac{2 - \zeta_0 - 2\mu \zeta_0}{\mu^2(1 - \zeta_0)^2}$

giving 0.23860673... and 0.67242217... respectively. The mean of $W_{\text{que}}$ for FIFO is the same as that for LIFO; the variance for FIFO is smaller. If sampling is restricted only to $W_{\text{que}} > 0$, then [8, 11, 12, 13]

mean$_{>0} = \frac{1}{\mu(1 - \zeta_0)}$, variance$_{>0} = \frac{1 - 2\mu \zeta_0}{\mu^2(1 - \zeta_0)^2}$

giving 0.57194007... and 1.42114846... respectively. The variance expression reported in [14] contains an apparent error.

3. SIRO

The probability density function $f(x)$ of $W_{\text{que}}$ has Laplace transform [8]

$$F(s) = 1 - \zeta_0 + \zeta_0 \Phi(s) = F_{\text{alt}}(s) + 1 - \zeta_0$$
where
\[ \Phi(s) = B(s, \zeta_0) - \int_{\zeta(s)}^{\zeta_0} \exp \left( - \int_{u}^{\zeta_0} \frac{dv}{v - e^{-a(u + \mu v)}} \right) \frac{\partial B}{\partial u}(s, u)du, \]
\[ B(s, z) = \frac{\mu(1 - \zeta_0)}{1 - z} \frac{1 - \exp[-a(s + \mu - \mu z)]}{s + \mu - \mu z}. \]

The integral underlying \( \Phi(s) \) is intractable; our symbolic approach for FIFO & LIFO seems inapplicable for SIRO.

We therefore turn to a numeric approach. An unpublished memorandum written in 1967 by Burke (the same author as of [15]) has regrettably been lost, although summaries are found in [16, 17]. Rosenlund [18] provided an especially clear algorithm for D/M/1 to follow. Since our interest is in densities, we differentiate his initial expression with respect to \( x \), i.e.,
\[ \frac{d}{dx}(x^{j+1-r}e^{-x}) = (j + 1 - r - x)x^{j-r}e^{-x}. \]

Define recursively
\[ h_{j,0}(x) = \sum_{r=1}^{j+1} \frac{r}{j+1} \frac{(j+1-r)x^{j-r}}{(j+1-r)!} e^{-x}, \quad j = 0, 1, 2, \ldots; \]
\[ h_{j,k}(x) = \sum_{r=1}^{j+1} \frac{r}{j+1} \frac{(1/\rho)^{j+1-r}}{(j+1-r)!} e^{-1/\rho} h_{r,k-1}(x), \quad j = 0, 1, 2, \ldots \text{ and } k = 1, 2, 3, \ldots. \]

We consequently have
\[ f(x) = (1 - \zeta_0)\delta(x) + \zeta_0 g(x) \]

where
\[ g(x) = -\mu(1 - \zeta_0) \sum_{j=0}^{\infty} \zeta_0^{j} h_{j,\lfloor \lambda x \rfloor} \left( \mu x - \frac{\lfloor \lambda x \rfloor}{\rho} \right), \quad x \geq 0. \]

For example, if \( 0 < x < a \), then
\[ f(x) = \mu(1 - \zeta_0) \int_{1-\zeta_0}^{1} \frac{e^{-\mu xt}}{t} dt = \mu(1 - \zeta_0) \left[ E(\mu(1 - \zeta_0)x) - E(\mu x) \right] \]

where \( E(x) = -\text{Ei}(-x) \) is the exponential integral. This corresponds to the leftmost curvilinear arc in Figure 2, surmounting the interval \([0, \frac{1}{2}]\). Verification that the Laplace transform of \( f(x) \) is equal to \( F_{alt}(s) \) remains open.
It is known (by other techniques) that the mean of $W_{\text{que}}$ for SIRO is the same as that for FIFO and LIFO; the corresponding variance is between the two extremes $[8, 18]$:

$$
\zeta_0 \frac{4 - 2\zeta_0 - 4\mu \zeta_0' + \mu \zeta_0 \zeta_0'}{\mu^2 (1 - \zeta_0)^2 (2 - \mu \zeta_0')}
$$
giving $0.34029290\ldots$ If sampling is restricted only to $W_{\text{que}} > 0$, then the variance is

$$
\frac{2 - 3\mu \zeta_0'}{\mu^2 (1 - \zeta_0)^2 (2 - \mu \zeta_0')}
$$
giving $0.62503500\ldots$.

## 4. Idle Period

We are concerned here with successive periods of server activity and inactivity. The left-hand subinterval of $k a \leq x < (k + 1) a$ is busy (since a new client has just arrived) and its right-hand complement is idle. It is possible that the idle period is empty. Jansson [19] proved that, under FIFO and equilibrium, the idle period length has probability density function

$$
\zeta_0 \delta(x) + \mu \zeta_0 (1 - \zeta_0) \exp(\mu(1 - \zeta_0)x), \quad 0 \leq x < a.
$$

Moments are

$$
\text{mean} = \frac{1}{\lambda} - \frac{1}{\mu} = (1 - \rho)a, \quad \text{variance} = \frac{1 + \zeta_0 - 2a \mu \zeta_0}{\mu^2 (1 - \zeta_0)}
$$
giving $1/6$ and $0.03157553\ldots$ respectively when $\{\lambda, \mu\} = \{2, 3\}$. The analysis of a busy period is more complicated, in part because it may span multiple adjacent intervals $[k a, (k + 1) a)$, but this issue is not pertinent for our study here.

Each client is associated with both a waiting time $t \in [0, \infty)$ and an idle period length $t \in [0, a)$. An expression for the bivariate density is available [19]. We report merely the cross-covariance

$$
(1 - \zeta_0)e^{-\alpha \mu} + \frac{\alpha \zeta_0}{\mu(1 - \zeta_0)} - \frac{1}{\mu^2 (1 - \zeta_0)}
$$
and cross-correlation $-0.44448913\ldots$ when $\{\lambda, \mu\} = \{2, 3\}$. Again, the proof is valid under FIFO and equilibrium. What is remarkable is that these results (marginal density and joint moments) appear via simulation to be the same under LIFO and SIRO as well. Likewise, the distribution of $L_{\text{sys}}$ (what we called $\tilde{f}$ in Section 1) seems to be invariant upon change in policy. Justification would be good to see someday.
5. Minimal Cost

The expression “queue control” may seem redundant because queues are themselves a method of control [20]. They exist to accommodate client demands on a service provider. A control, however, exists to ensure that costs remain sustainable. We wish to minimize cost as a function of $a = 1/\lambda$, for fixed $\mu$, where cost is a $c$-weighted sum of the mean idle period and the mean waiting time [19]:

$$C = (1 - c) \left( a - \frac{1}{\mu} \right) + c \frac{\zeta_0}{\mu(1 - \zeta_0)}.$$ 

The derivative of $\zeta_0$ with respect to $a$ will be written as $\zeta_0'$, which should not be confused with our earlier usage of the same symbol (the derivative of $\zeta$ with respect to $s$, evaluated at 0). From

$$\zeta_0 = -\frac{1}{a \mu} \omega (-a \mu e^{-a \mu})$$

we deduce

$$(\mu \zeta_0)' = \frac{1}{a^2} \omega (-a \mu e^{-a \mu}) - \frac{1}{a} \omega' (-a \mu e^{-a \mu}) (-\mu e^{-a \mu} + a \mu^2 e^{-a \mu})
= \frac{1}{a^2} \omega (-a \mu e^{-a \mu}) - \frac{1}{a} \frac{\omega (-a \mu e^{-a \mu})}{1 + \omega (-a \mu e^{-a \mu})} (-\mu) (1 - a \mu) e^{-a \mu}
= \frac{\omega (-a \mu e^{-a \mu})}{a^2} \left[ 1 - \frac{1}{a} \frac{1 - a \mu}{1 + \omega (-a \mu e^{-a \mu})} \right]
= \frac{-\mu \zeta_0 a \mu - a \mu \zeta_0}{a (1 - a \mu \zeta_0)} = -\frac{\mu^2 \zeta_0 (1 - \zeta_0)}{1 - a \mu \zeta_0}.$$ 

thus

$$\left( \frac{\zeta_0}{1 - \zeta_0} \right)' = \frac{\zeta_0'}{1 - \zeta_0} - \frac{\zeta_0}{(1 - \zeta_0)^2} (-\zeta_0')
= \left[ \frac{1}{1 - \zeta_0} + \frac{\zeta_0}{(1 - \zeta_0)^2} \right] \left( -\frac{\mu \zeta_0 (1 - \zeta_0)}{1 - a \mu \zeta_0} \right)
= \frac{1 - \zeta_0 + \zeta_0 \mu \zeta_0 (1 - \zeta_0)}{(1 - \zeta_0)^2} \frac{1}{1 - a \mu \zeta_0} = \frac{-\mu \zeta_0}{(1 - \zeta_0) (1 - a \mu \zeta_0)}.$$ 

thus

$$C' = (1 - c) - c \frac{\zeta_0}{(1 - \zeta_0) (1 - a \mu \zeta_0)} = 0$$ 

when

$$\frac{c}{1 - c} = \frac{(1 - \zeta_0) (1 - a \mu \zeta_0)}{\zeta_0} = \frac{1 - \zeta_0 - \zeta_0 a \mu (1 - \zeta_0)}{\zeta_0}.$$
It is additionally required \[19\] that 0 < \( \zeta_0 < 1 \). From

\[
\zeta_0 = \exp(-a \mu(1 - \zeta_0)), \quad \text{i.e.,} \quad a \mu(1 - \zeta_0) = -\ln(\zeta_0)
\]

we obtain

\[
\frac{c}{1-c} = \frac{1 - \zeta_0 + \zeta_0 \cdot \ln(\zeta_0)}{\zeta_0}
\]

hence

\[
\zeta_0 = -\frac{1}{\bar{\omega} \left[ -\exp\left(-\frac{1}{1-c}\right) \right]}
\]

where \( \bar{\omega}(s) \) is “the” secondary branch of the Lambert omega:

\[
\bar{\omega}(s)e^{\bar{\omega}(s)} = s, \quad -1 \geq \bar{\omega}(x) \in \mathbb{R} \ \forall \ x \in [-1/e, 0), \quad \exists \ \text{branch cut for} \ x \leq 0.
\]

For example, if \( \mu = 3 \) and \( c = 1/2 \), then \( \zeta_0 \approx 0.31784443 \) and \( a = 0.56008398 \). In words, if mean client waiting times are weighted the same as mean server idle periods, i.e., \( c = 1/2 \), then in terms of cost, the interarrival time \( a = 0.50 \) is far from optimal, but \( a = 0.56 \) is close.

If server idle periods are weighted more heavily than client waiting times, e.g., \( c = 1/5 \), then \( a = 0.44983251 \). If instead client waiting times are weighted more heavily than server idle periods, e.g., \( c = 4/5 \), then \( a = 0.75436304 \). This is consistent with intuition. Compressed interarrival times lead to less idleness but longer waits; expansive interarrival times lead to shorter waits but more idleness. Balancing these conflicting priorities makes life interesting.

To clarify: there exist countably infinite branches of the Lambert omega, but only two (\( \omega \) and \( \bar{\omega} \)) that assume real values on \([-1/e, 0)\), one increasing and the other decreasing. All other branches are complex-valued with nonzero imaginary parts. Our notation \( \bar{\omega} \) is unorthodox, as is referring to \( \bar{\omega} \) as “the” secondary branch. In Mathematica, the function \( \text{ProductLog}[k, x] \) gives \( \omega(x) \ & \bar{\omega}(x) \) for \( k = 0 \ & \ k = -1 \), respectively. Alternative notation \( \omega_+ \ & \omega_- \), proposed somewhat by \[21\], is intended to suggest “upper branch” and “lower branch”.

We have omitted discussion of the variance of \( C \). From the aforementioned joint distribution of idle period and waiting time \[19\], it would be possible to minimize cost as a function of \( a \), for fixed \( \mu \), where cost is the median of a \( c \)-weighted sum of idle period and waiting time. Solving this revised optimization problem could be advantageous because the median is more a robust estimator of centrality than the mean. We wonder too about the proper choice of \( 0 < c < 1 \) and whether a sum (rather than a product, say) is necessarily best. More recent work appears in \[22\ \[23\ \[24\ \[25\ \[26\]. Processes with constant interarrival times and exponential server queues are fundamental, as proved in \[27\].
Figure 1: Waiting time density plot for Deterministic\( \frac{1}{2} \) arrivals, last-in-first-out exponential service.
Figure 2: Waiting time density plot for Deterministic\(\left[\frac{1}{2}\right]\) arrivals, exponential serve-in-random order policy.
6. Addendum

With $\zeta_0$ as before (constant $\approx 0.417$ for $\lambda = 2 = 1/a$, $\mu = 3$), define \cite{5, 8, 28, 29}

\[
\Delta_n(x) = \left\{ \frac{1}{1 - \zeta_0} + \sum_{j=1}^{n} \binom{n}{j} \frac{Q_j(x)}{1 - e^{-a j x}} \right\}^{-1},
\]

\[
m_n = \frac{\Delta_n(\mu/n)}{\mu (1 - \zeta_0)^2}, \quad p_n = 1 - \frac{\Delta_n(\mu/n)}{1 - \zeta_0}
\]

where $n$ is a positive integer and

\[
Q_j(x) = \prod_{i=1}^{j} \frac{1 - e^{-a_i x}}{e^{-a_i x}}.
\]

For example, $\Delta_1 = (1 - \zeta_0) \zeta_0$, $m_1 = \zeta_0/ (\mu (1 - \zeta_0)) \approx 0.239$ and $p_1 = 1 - \zeta_0 \approx 0.583$. More generally, $m_n$ is the expected waiting time in a D/M/n queue with $n$ slow servers (more precisely, each server working with rate only $\mu/n$ when busy) and $p_n$ is the probability of zero wait. With $\{\lambda, \mu\} = \{2, 3\}$, we have

\[
m_2 = 0.16901950..., \quad p_2 = 0.70448039...;
\]
\[
m_3 = 0.12647170..., \quad p_3 = 0.77887245...;
\]
\[
m_4 = 0.09744181..., \quad p_4 = 0.82962932...;
\]
\[
m_5 = 0.07648770..., \quad p_5 = 0.89364299...;
\]
i.e., $n$ slow servers outperform one fast server, relative to average waiting time. The sum $S$ of idle periods over all servers would however be potentially significant; the mean of $S$ would be crucial in minimizing total cost as a function of $a$, for fixed $\mu$.

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