Passive tracer in a flow corresponding to two-dimensional stochastic Navier–Stokes equations

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Abstract

In this paper we prove the law of large numbers and central limit theorem for trajectories of a particle carried by a two-dimensional Eulerian velocity field. The field is given by a solution of a stochastic Navier–Stokes system with non-degenerate noise. The spectral gap property, with respect to the Wasserstein metric, for such a system was shown in Hairer and Mattingly (2008 Ann. Probab. 36 2050–91). In this paper we show that a similar property holds for the environment process corresponding to the Lagrangian observations of the velocity. Consequently we conclude the law of large numbers and the central limit theorem for the tracer. The proof of the central limit theorem relies on the martingale approximation of the trajectory process.

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1. Introduction

Consider the Navier–Stokes equations (NSE) on a two-dimensional torus $\mathbb{T}^2$,

$$
\begin{align*}
\partial_t \vec{u}(t, x) + \vec{u}(t, x) \cdot \nabla x \vec{u}(t, x) &= \Delta_x \vec{u}(t, x) - \nabla_x p(t, x) + \vec{F}(t, x), \\
\nabla \cdot \vec{u}(t, x) &= 0, \\
\vec{u}(0, x) &= \vec{u}_0(x).
\end{align*}
$$

(1.1)

The two-dimensional vector field $\vec{u}(t, x)$ and scalar field $p(t, x)$ over $[0, +\infty) \times \mathbb{T}^2$, are called the Eulerian velocity and pressure, respectively. The forcing $\vec{F}(t, x)$ is assumed to be a Gaussian white noise in $t$, homogeneous and sufficiently regular in $x$ defined over a certain
The trajectory of a tracer particle is defined as the solution of the ordinary differential equation (ODE)
\[
\frac{dx(t)}{dt} = \vec{u}(t, x(t)), \quad x(0) = x_0,
\]
where \(x_0 \in \mathbb{R}^2\). Thanks to the well known regularity properties of the solutions of NSE, see e.g. [23], \(\vec{u}(t, x)\) possesses continuous modification in \(x\) for any \(t > 0\). However, since \(\vec{u}(t, x)\) need not be Lipschitz in \(x\), the equation might not define \(x(t), t \geq 0\), as a stochastic process over \((\Omega, \mathcal{F}, \mathbb{P})\), due to the possible non-uniqueness of the solutions. In our first result we construct a solution process (see proposition 4.6) and show (see corollary 4.4) that the law of any process satisfying (1.2) and adapted to the natural filtration of \(\vec{u}\) is uniquely determined.

The main objective of this paper is to study the ergodic properties of the trajectory process. We prove, see part 1 of theorem 3.5, that the Stokes drift vanishes, i.e.
\[
\lim_{t \to +\infty} \frac{x(t)}{t} = 0,
\]
where the limit above is understood in probability. A similar result for a Markovian and Gaussian velocity field \(\vec{u}\) (that need not be a solution of a NSE) that decorrelates sufficiently fast in time has been considered in [15]. Next, we investigate the size of a ‘typical fluctuation’ of the trajectory around its mean. We prove, see part 3 of the theorem, that
\[
Z(t) := \frac{x(t)}{\sqrt{t}} \Rightarrow Z,
\]
as \(t \to +\infty\) (1.4) where \(Z\) is a random vector with normal distribution \(N(0, D)\) and the convergence is understood in law. Moreover, we show that the asymptotic variance of \(Z(t)\), as \(t \to +\infty\), exists and coincides with the covariance matrix \(D\). The question of the law of the iterated logarithm is addressed in our companion paper, see [16].

In our approach a crucial role is played by the Lagrangian process
\[
\vec{\eta}(t, x) := \vec{u}(t, x(t) + x), \quad t \geq 0, \quad x \in \mathbb{T}^2
\]
that describes the environment from the vantage point of the moving particle. It turns out that its rotation in \(x\),
\[
\omega(t, x) = \text{rot} \vec{\eta}(t, x) := \partial_2 \eta_1(t, x) - \partial_1 \eta_2(t, x), \quad t \geq 0, \quad x \in \mathbb{T}^2,
\]
satisfies a stochastic partial differential equation (SPDE) (4.1) that is similar to the stochastic NSE in the vorticity formulation, see (3.1). The position \(x(t)\) of the particle at time \(t\) can be represented as an additive functional of the Lagrangian process, i.e.
\[
x(t) = \int_0^t \psi_s(\omega(s)) \, ds,
\]
see the beginning of section 6 for the definition of \(\psi_s\). Then, (1.3) and (1.4) become the statements about the law of large numbers and central limit theorem for an additive functional of the process \(\eta(\cdot)\).

Following the ideas of Hairer and Mattingly [8, 9], we are able to prove (see theorem 5.1) that the transition semigroup of \(\omega(\cdot)\) satisfies the spectral gap property in a Wasserstein metric defined over the Hilbert space \(H\) of square integrable mean zero functions. If \(\psi_s(\cdot)\) were Lipschitz, the proof of the law of large numbers and central limit theorem would be routine, in view of [27] (see also [17, 20]). However, in our case the observable \(\psi_s\) is not Lipschitz. In fact, it is not even defined on the state space \(H\) of the process. Nevertheless, it is a bounded linear functional over another Hilbert space \(V\) that is compactly embedded in \(H\). Adopting the approach of Mattingly and Pardoux from [23], see theorem 5.2, we are able to prove that...
the equation for \( \omega \) has regularization properties similar to the NSE and that \( \omega(t) \) belongs to \( V \) for any \( t > 0 \). In consequence, one can show that the corresponding transition semigroup can be defined on \( \psi_\omega \) and has the same contractive properties as the semigroup defined on Lipschitz functions on \( H \). The law of large numbers can then be shown (section 6.4) by a modification of the argument of Shirikyan from [27] (see also [17]). To prove the central limit theorem we construct a corrector field \( \chi \) (see section 6.1) over the ‘larger’ space \( H \). Then, we proceed with the classical martingale proof of the central limit theorem, see section 6.4. A similar argument has been used in [27] to demonstrate an analogous theorem for a Lipschitz observable of the solution of a NSE. The proof of the existence of the asymptotic variance is presented in section 6.3.

Equation (1.2), which describes one of the most fundamental model of transport of particles in a fluid flow, is sometimes referred to as the equation of a passive tracer, see e.g. chapter 5 of [31]. The \( d \)-dimensional vector field \( \vec{u} \) on the right hand side of (1.2) is usually assumed to be random and stationary, and in principle it may have nothing in common with the solution of the NSE. Since the fluid flow is incompressible, equation (1.2) is complemented by the condition \( \nabla \cdot \vec{u}(t, x) \equiv 0 \). This model was introduced by G Taylor in the 1920s (see [29] and also [19]) and plays an important role in describing transport phenomena in fluids, e.g. in the investigation of ocean currents (see [28]). Extensive literature exists concerning the passive tracer both from the mathematical and physical points of view, see, e.g. [21] and references therein. In particular, it can be shown (see [26]) that the incompressibility assumption implies that the Lagrangian process \( \vec{u}(t, x(t)) \), \( t \geq 0 \), is stationary and if one can prove its ergodicity, the Stokes drift coincides with the mean of the field \( v_\ast = E\vec{u}(0, 0) \). The weak convergence of \( (x(t) - v_\ast t)/\sqrt{t} \) towards a normal law has been shown for flows possessing good relaxation properties either in time, or both in time and space, see [1, 5, 12, 18] for the Markovian case, or [13] for the case of non-Markovian Gaussian fields with a finite decorrelation time. According to our knowledge this is the first result when the central limit theorem has been shown for the tracer in a flow that is given by the actual solution of the two-dimensional NSE.

2. Preliminaries

2.1. Some function spaces and operators

Denote by \( \mathbb{T}^2 \) the two-dimensional torus understood as the product of two segments \([-1/2, 1/2]\) with identified endpoints. The trigonometric monomials \( e_k(x) = e^{2\pi ik \cdot x} \), \( k = (k_1, k_2) \in \mathbb{Z}^2 \), form the orthonormal base in the space of all complex-valued, square integrable functions \( L^2(\mathbb{T}^2) \) with the standard scalar product \( \langle \cdot, \cdot \rangle \) and norm \( |\cdot| \). For a given \( w \in L^2(\mathbb{T}^2) \) let \( \hat{w}_k = \langle w, e_k \rangle \). Let \( H \) be the linear subspace of \( L^2(\mathbb{T}^2) \) over the field of reals consisting of those real-valued functions \( w \), for which \( \hat{w}_0 = 0 \). For any \( r \in \mathbb{R} \) let

\[
(-\Delta)^{r/2} w := \sum_{k \in \mathbb{Z}^2} |k|^r \hat{w}_k e_k, \quad w \in H^r,
\]

where \( H^r \) consists of such \( w \), for which \( \sum_{k \in \mathbb{Z}^2} |k|^{2r} |\hat{w}_k|^2 < +\infty \) and \( \mathbb{Z}_2^2 := \mathbb{Z}^2 \setminus \{(0, 0)\} \). We equip \( H^r \) with the graph Hilbert norm \( |\cdot|_r := |(-\Delta)^{r/2} \cdot| \). Let \( V := H^1 \) and let \( V' \) be the dual to \( V \). Then \( H \) can be identified with a subspace of \( V' \) and \( V \hookrightarrow H \hookrightarrow V' \). We will also denote by \( \| \cdot \| \) the respective norm \( |\cdot|_1 \). It is well known (see e.g. corollary 7.11 of [7]) that \( H^{1+s} \) is continuously embedded in \( C(\mathbb{T}^2) \) for any \( s > 0 \). Moreover, there exists a constant \( C > 0 \) such that

\[
\|w\|_\infty \leq C |w|_{1+s}, \quad \forall \, w \in C^s(\mathbb{T}^2).
\]
Here $\|w\|_{\infty} := \sup_{x \in \mathbb{T}^2} |w(x)|$. In addition, the following estimate, sometimes referred to as the Gagliardo–Nirenberg inequality, holds, see e.g. p 27 of [10]. For any $s > 0, \beta \in [0, 1]$ there exists $C > 0$ such that
\[
|w|_{\beta,s} \leq C |w|^{1-\beta} |w|_p^\beta, \quad \forall w \in C^\infty(\mathbb{T}^2).
\] (2.2)

Define $K: H' \to H^{r+1} \times H^{r+1}$ by
\[
K(w) = (K_1(w), K_2(w)) := i \sum_{k \in \mathbb{Z}_*^2} |k|^{-2} k \parallel \hat{w}_k e_k. \tag{2.3}
\]

We have
\[
|K_i(w)|_{r+1} \leq |w|r, \quad w \in H_r.
\] (2.4)

For a given $x \in \mathbb{R}^2$ and $w \in H'$ we let $\tau x w \in H'$ be defined by
\[
\tau x w := w(\cdot + x) = \sum_{k \in \mathbb{Z}_*^2} e^{-2\pi i k \cdot x} \hat{w}_k e_k.
\]

Define also the reflection of $w$ by letting $s w(x) := w(-x)$.

2.2. The homogeneous Wiener process

Write
\[
Z_*^2 := \{(k_1, k_2) \in \mathbb{Z}_*^2: k_2 > 0\} \cup \{(k_1, k_2) \in \mathbb{Z}_*^2: k_1 > 0, k_2 = 0\}
\]
and let $Z_*^2 := -Z_*^2$. Let $(B_k(t))_{t \geq 0}, k \in Z_*^2$, be independent, standard, one-dimensional complex Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $B_k(t) := B_k^*(t)$ for $k \in Z_*^2$. Assume that the function $k \mapsto q_k$ is complex even, i.e. $q_{-k} = q_k^*$, $k \in Z_*^2$. A cylindrical Wiener process in $H$, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, can be written as
\[
W(t) := \sum_{k \in \mathbb{Z}_*^2} B_k(t) e_k, \quad t \geq 0.
\]

Let $Q: H \to H'$ be a bounded linear operator given by
\[
Q \hat{w}_k := q_k \hat{w}_k, \quad k \in Z_*^2. \tag{2.5}
\]

The Hilbert–Schmidt norm of the operator, see appendix C of [3], can be computed from the formula
\[
\|Q\|_{L(H,H')}^2 := \sum_{k \in \mathbb{Z}_*^2} \|Q e_k\|_{H'}^2 = \sum_{k \in \mathbb{Z}_*^2} |k|^{2r} |q_k|^2. \tag{2.6}
\]

**Proposition 2.1.** If $\|Q\|_{L(H,H')}^2 < +\infty$ then the process $(QW(t))_{t \geq 0}$ has realizations in $H'$, $\mathbb{P}$-a.s. Its law is invariant under the reflection and translations. The above means that the law of $(QW(t))_{t \geq 0}$ and that of $(s QW(t))_{t \geq 0}$ are the same, and the laws of $(\tau x QW(t))_{t \geq 0}$ are independent of $x \in \mathbb{R}^2$.

**Proof.** The first part of the proposition follows directly from proposition 4.2, p 88 of [3]. The second part is a simple consequence of the fact that the processes in question have the same covariance operator as $(QW(t))_{t \geq 0}$. \(\Box\)
3. Formulation of the main results

In this section we rigorously define the notion of a solution of (1.2) with the vector field $\vec{u}$ given by the solution of the NSE (1.1) and formulate the main results of the paper dealing with the long time, large scale behaviour of the trajectory.

Since, as it turns out, the components of the solution of the NSE belong to $V$, see [24], if the initial condition $\vec{u}_0 \in V$, we cannot use equation (1.2) for a direct definition of the solution because the point evaluation for the field is not well defined (not to mention the question of the existence and uniqueness of solutions to the ODE in question).

3.1. Vorticity formulation of the NSE

Note that the rotation

$$\xi(t) := \text{rot} \vec{u}(t) = \partial_2 u_1(t) - \partial_1 u_2(t)$$

of $\vec{u}(t, x) = (u_1(t, x), u_2(t, x))$, given by (1.1), satisfies

$$\partial_t \xi(t) = \Delta \xi(t) - B_0(\xi(t)) + \text{rot} \vec{F}, \quad \xi(0) = w \in H,$$

where $B_0(\xi) := B_0(\xi, \xi)$, $\xi \in V$, with $B_0(h, \xi) := \vec{v} \cdot \nabla \xi$, and $\vec{v} := \mathcal{K}(h)$. Since $\vec{F}(t, x)$ is homogeneous in space we may assume that $\vec{F}(t, x) = Q dW(t, x)$, where $Q$ is a Hilbert–Schmidt diagonal operator of the form (2.5) and $W$ is a cylindrical Wiener process on $H$. Thus, we suppose that $\xi(t)$ satisfies

$$d\xi(t) = [\Delta \xi(t) - B_0(\xi(t))] dt + Q dW(t), \quad \xi(0) = w \in H. \quad (3.1)$$

Let $\mathcal{E}_T := C([0, T]; H) \cap L^2([0, T]; V)$ and let $W(t)$, $t \geq 0$ be non-anticipative with respect to a filtration $\{\mathcal{F}_t, t \geq 0\}$.

Definition 3.1. A measurable and $(\mathcal{F}_t)$-adapted, $H$-valued process $\xi = [\xi(t), t \geq 0]$ is a solution to (3.1) if for any $T \in (0, +\infty)$, $\xi \in L^2(\Omega, \mathcal{E}_T, P)$ and

$$\xi(t) = e^{\Delta t} w - \int_0^t e^{\Delta (t-s)} B_0(\xi(s)) ds + \int_0^t e^{\Delta (t-s)} Q dW(s) \quad (3.2)$$

for all $t \geq 0$.

The following estimate comes from [23], see lemma A.3, p 39.

Proposition 3.2. For any $T, N > 0$ there exists $C > 0$ such that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \left( |\xi(t)|^2 + t \|\xi\|_2^2 \right)^N \right] \leq C(1 + |w|^{4N}), \quad \forall w \in H. \quad (3.3)$$

Let $\vec{u}(t) := \mathcal{K}(\xi(t))$. Using the above proposition and (2.1) we conclude that

Corollary 3.3. For any $t > 0$, $\vec{u}(t) \in C([0, T]; H)$ and

$$\int_0^t \|\vec{u}(s)\|_\infty ds < +\infty, \quad \mathbb{P}\text{-a.s.} \quad (3.4)$$

Proof. The continuity of $\vec{u}(t, x)$ with respect to $x$, follows from Sobolev embedding. From (2.4) we conclude that there exists $C > 0$ such that

$$\|\vec{u}(s)\|_\infty \leq C \|\xi(s)\|, \quad \forall s \geq 0. \quad (3.5)$$

On the other hand, from (3.3) we conclude that for any $t > 0$ there exists a random variable $\tilde{C}$ that is almost surely finite and such that $\|\xi(s)\| \leq \tilde{C}s^{-1/2}$ for all $s \in (0, t]$. Combining this with (3.5) we conclude (3.4). □
3.2. Definition of the trajectory process and its ergodic properties

**Definition 3.4.** Let $x_0 \in \mathbb{R}^2$. By a solution to (1.2) we mean any $(\mathcal{F}_t)$-adapted process $x(t)$, $t \geq 0$, with continuous trajectories, such that

$$x(t) = x_0 + \int_0^t \tilde{u}(s, x(s)) \, ds, \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.} \quad (3.6)$$

For a given $\nu > 0$ denote $e_\nu(w) := \exp\{\nu|w|^2\}$, $w \in H$.

**Theorem 3.5.** Assume that $Q$ in (4.1) belongs to $L(H_S) (H, V)$ and has a trivial null space, i.e. $Qw = 0$ implies $w = 0$. Suppose that the initial vorticity is random, distributed on $H$ according to the law $\mu_0$ for which

$$\int_H e_\nu_0(w) \mu_0(dw) < +\infty$$

with a certain $\nu_0 > 0$. Finally, assume that $\{x(t; x_0), t \geq 0\}$ is a solution of (1.2) corresponding to the initial data $x_0 \in \mathbb{R}^2$. Then, the following are true:

1. (Weak law of large numbers) for any $x_0 \in \mathbb{R}^2$ we have

$$\lim_{T \to +\infty} \frac{x(T; x_0)}{T} = 0 \quad (3.8)$$

in probability.

2. (Existence of the asymptotic variance) there exists $D_{ij} \in [0, +\infty)$ such that

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E}\left[x_i(T; x_0)x_j(T; x_0)\right] = D_{ij}, \quad i, j = 1, 2. \quad (3.9)$$

3. (Central limit theorem) Random vectors $x(T; x_0)/\sqrt{T}$ converge in law, as $T \to +\infty$, to a zero mean normal law whose covariance matrix equals $D = [D_{ij}]$.

**Remark.** In our companion paper [16] it is shown that under our assumption about non-degeneracy of the noise, i.e. that $\ker(Q) = \{0\}$, we have $\det D \neq 0$.

4. Lagrangian and tracer trajectory processes

4.1. Uniqueness in law of the trajectory process

Define the Lagrangian velocity process as

$$\tilde{\eta}(t, x) = (\eta_1(t, x), \eta_2(t, x)) := \tilde{u}(t, x(t) + x), \quad t \geq 0, \quad x \in \mathbb{R}^2,$$

Using Ito's formula we obtain that its vorticity, given by,

$$\omega(t, x) := \text{rot} \tilde{\eta}(t, x) = \xi(t, x(t) + x)$$

satisfies $\omega(0) = \tau_{x_0}w \in H$ and

$$d\omega(t) = [\Delta \omega(t) - B_0(\omega(t)) + B_1(\omega(t))] \, dt + Q \, d\tilde{W}(t), \quad (4.1)$$

where $\tilde{W}$ is some $(\mathcal{F}_t)$-adapted cylindrical Wiener process on $H$ (different from the original $W$ in (3.1)) and

$$B_1(\omega) := B_1(\omega, \omega) \quad \text{and} \quad B_1(h, \omega) := K(h)(0) \cdot \nabla \omega, \quad \omega \in V,$$

for more details see [6, 14]. Since we have assumed that $\omega \in V$ and, by Sobolev embedding, $K(V)$ is embedded into the space $C(\mathbb{T}^2; \mathbb{R}^2)$ of the two-dimensional, continuous trajectory vector fields on $\mathbb{T}^2$, we see that the evaluation of $\tilde{\eta}$ is well defined, and therefore there is no ambiguity in the definition of $B_1(\omega)$ for $\omega \in V$. In what follows we will omit writing tilde over the cylindrical Wiener process.
Definition 4.1. A measurable, \((\mathcal{F}_t)\)-adapted, \(H\)-valued process \(\omega = \{\omega(t), t \geq 0\}\) is a solution to \((4.1)\), with the initial condition \(\omega(0) = w\), if for any \(T > 0\), \(\omega \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})\) and
\[
\omega(t) = e^{\Delta t} w - \int_0^t e^{\Delta(t-s)} B_0(\omega(s)) \, ds + \int_0^t e^{\Delta(t-s)} B_1(\omega(s)) \, ds + \int_0^t e^{\Delta(t-s)} Q \, dW(s),
\]
\(\mathbb{P}\)-a.s. for all \(t \geq 0\).

Sometimes, when we wish to highlight the dependence on the initial condition and the Wiener process, we will write \(\omega(t; w, W)\). We will omit writing one, or both of these parameters when they are obvious from the context.

Using a Galerkin approximation argument, as in section 3 of [24] (see also appendix A below for an outline of the argument), we conclude the following:

Theorem 4.2. Given an initial condition \(w \in H\) and an \((\mathcal{F}_t)\)-adapted cylindrical Wiener process \((W(t))_{t \geq 0}\), there exists a unique solution to \((4.1)\) in the sense of definition 4.1. Moreover, processes \(\{\omega(t; w), t \geq 0\}\) form a Markov family with the corresponding transition probability semigroup \(\{P_t, t \geq 0\}\) defined on the space \(C_b(H)\) of continuous and bounded functions on \(H\).

Using the Yamada–Watanabe result, see e.g. [32] (corollary after theorem 4.1.1), or [11], from the above theorem we can conclude the following result, see [14].

Corollary 4.3. Solutions of \((4.1)\) have uniqueness in law property, i.e. the laws over \(C([0, +\infty); H)\) of any two solutions of \((4.1)\) starting with the same initial data (but possibly based on different cylindrical Wiener processes) coincide.

This immediately implies the uniqueness in law property for solutions of \((1.2)\).

Corollary 4.4. Suppose that \(\xi\) and \(\xi'\) are two solutions of \((3.1)\) with identical initial data but possibly based on two cylindrical Wiener processes with the respective filtrations \((\mathcal{F}_t)\) and \((\mathcal{F}_t')\). Assume also that \(x(\cdot)\) and \(x'(\cdot)\) are the solutions of \((1.2)\) corresponding to \(\tilde{u}(t) = K(\xi(t))\) and \(\tilde{u}'(t) = K(\xi'(t))\), respectively. Then, the laws of the pairs \((x(\cdot), \xi(\cdot))\) and \((x'(\cdot), \xi'(\cdot))\) over \(C([0, +\infty), \mathbb{R}^2) \times C([0, +\infty), H)\) coincide.

Proof. Both \(\omega(t, \cdot) = \xi(t, x(t) + \cdot)\) and \(\omega'(t, \cdot) = \xi'(t, x'(t) + \cdot)\) satisfy \((4.1)\). According to corollary 4.3 they have identical laws on \(C([0, +\infty), H)\) with the initial condition \(\tau_{\mathbb{P}} w\). In fact, due to an analogue of proposition 3.2 that holds for the process \(\omega(\cdot)\) (see part 1 of theorem 5.2), this law is actually supported in \(L^1_{\text{loc}}([0, +\infty), V)\). We can write therefore that \((x(\cdot), \xi(\cdot)) = \Psi(\omega(\cdot))\) and \((x'(\cdot), \xi'(\cdot)) = \Psi(\omega'(\cdot))\), where the mapping
\[
\Psi = (\Psi_1, \Psi_2) : L^1_{\text{loc}}([0, +\infty), V) \to C([0, +\infty), \mathbb{R}^2) \times C([0, +\infty), H)
\]
is defined as
\[
\Psi_1(X)(t) := x_0 + \int_0^t K(X(s))(0) \, ds,
\Psi_2(X)(t, x) := X(t, x - \Psi_1(X)(t)), \quad \forall X \in L^1_{\text{loc}}([0, +\infty), V),
\]
and the uniqueness claim made in the corollary follows. \(\square\)
4.2. Existence of solution of (1.2)

**Definition 4.5.** Suppose that \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) is a filtered probability space. Let \(x_0 \in \mathbb{R}^2\). By a weak solution to (1.2) we mean a pair consisting of a continuous trajectory \((\mathcal{F}_t)\)-adapted process \(x(t), t \geq 0\), and an \((\mathcal{F}_t)\)-adapted solution \(\xi(t), t \geq 0\) to (3.1) such that (3.6) holds.

Suppose now that we are given a filtration \((\mathcal{F}_t)\) and an \((\mathcal{F}_t)\)-adapted solution \(\omega\) of (4.1) with the initial condition \(\omega(0) = \tau_{x_0}w\). Define \((x(\cdot), \xi(\cdot)) := \Psi(\omega(\cdot))\). One can easily check, using Itô’s formula, that \((x(\cdot), \xi(\cdot))\) is a weak solution in the sense of definition 4.5. Therefore we conclude the following.

**Proposition 4.6.** Given a filtered probability space there exists a weak solution of (1.2).

Since the reflected vorticity \(\tau_{x_0}w\) satisfies (4.1) with the reflected noise \(sQW(t)\) we conclude the following.

**Proposition 4.7.** If the law of \(\omega(0)\) is invariant under \(s\), then the laws of \((\omega(t))\) and that of \((\tau_{s0}w)\) over \(C[0, +\infty)\) are identical.

5. Spectral gap and regularity properties of the transition semigroup

Here we present the basic results that will be instrumental in the proof of theorem 3.5 formulated in the previous section. In the case of the Navier–Stokes dynamics on a two-dimensional torus, corresponding results have been shown in [9], see theorem 5.10, proposition 5.12 and parts 2 and 3 of lemma A.1 from [9]. The proofs of analogous results for the Lagrangian dynamics are not much different, some additional care is needed due to the presence of function \(B_1(\cdot, \cdot)\), but it usually does not create any difficulty.

Let us introduce the space \(C_0^n(H)\) consisting of all functionals \(\phi\), for which there exist \(n \geq 1\), a function \(F\) from \(C_0^n(\mathbb{R}^d)\) and vectors \(v_1, \ldots, v_n \in H\) such that
\[
\phi(v) = F(\langle v, v_1 \rangle, \ldots, \langle v, v_n \rangle), \quad \forall v \in H.
\]

Given \(n > 0\) define \(B_n\) as the completion of \(C_0^n(H)\) under the norm
\[
\|\phi\|_n := \sup_{w \in H} e_{-n}(w) (|\phi(w)| + \|D\phi(w)\|),
\]
where, as we recall, \(e_{-n}(v) = \exp(|v||w|^2\}.\) Here \(\|D\phi(w)\| = \sup_{|\xi| \leq 1} |D\phi(w)[\xi]|\), where \(D\phi(w)[\xi]\) denotes the Fréchet derivative of a function \(\phi : H \to \mathbb{R}\) at \(w\) in the direction \(\xi \in H\). By \(B_n\) we understand the Banach space of all Fréchet differentiable functions \(\phi\) such that \(\|\phi\|_n < +\infty\). Let \(\mathcal{P}(H)\) be the space of all Borel probability measures on \(H\). Recall also that \(\mu_\ast \in \mathcal{P}(H)\) is called an invariant measure for \((P_t)_{t \geq 0}\) if
\[
\langle \mu_\ast, P_t\phi \rangle = \langle \mu_\ast, \phi \rangle, \quad \forall \phi \in C_b(H), \quad t \geq 0.
\]

Here \(\langle \mu, \phi \rangle := \int_H \phi \, d\mu\) for any \(\mu \in \mathcal{P}(H)\) and \(\phi\) that is integrable. Our first result can be stated as follows.

**Theorem 5.1.** Under the assumptions of theorem 3.5, the following are true:

1. there exist \(v_0, C > 0\) such that for any \(v \in (0, v_0]\) we have
\[
\mathbb{E}e_{-n}(\omega(t; w)) \leq Ce_v(w), \quad \forall t \geq 0, \quad w \in H, \quad (5.1)
\]

2. there exists a unique Borel probability measure \(\mu_\ast\) that is invariant for \((P_t)\), and such that
\[
\int_H e_{-n}(w) \mu_\ast(dw) < +\infty, \quad \forall v \in (0, v_0]. \quad (5.2)
\]

This measure is invariant under \(s\), i.e. \(\mu_\ast \circ s^{-1} = \mu_\ast\).
The constant $\nu_0$ can be further adjusted in such a way that for any $\nu \in (0, \nu_0]$ the semigroup $(P_t)$ extends to $\tilde{B}_\nu$, and
\[ P_t(B_\nu) \subset B_\nu, \quad \forall t \geq 0. \]

In addition, for any $\nu$ as above there exist $C, \gamma > 0$ such that for any $\nu \in (0, \nu_0]$ the semigroup $(P_t)$ extends to $\tilde{B}_\nu$ and
\[ P_t(B_\nu), \quad \forall t \geq 0, \quad \phi \in \tilde{B}_\nu. \quad (5.3) \]

The property described in (5.3) is referred to as the spectral gap of the transition semigroup. Since we will use an extension of this property to functions defined on a smaller space than $H$, we introduce the following definition. For $N > 0$ and $\phi \in C^1(V)$ define
\[ \|\phi\|_N := \sup_{w \in V} |\phi(w)| + \|D\phi(w)\| \left(1 + \|w\|\right)^N \]
and denote by $C^1_N(V)$ the space made of functions, for which $\|\phi\|_N < +\infty$.

**Theorem 5.2.** Under the assumptions of theorem 3.5 the following are true:

1. for any $t, N > 0$ there exists $C_{t,N}$ such that
\[ E\|\phi(t; w)\|_N^N \leq C_{t,N} \left(|w|^{2N} + 1\right), \quad \forall w \in H, \quad (5.4) \]

2. the definition of the transition semigroup can be extended to an arbitrary $\phi \in C^1_N(V)$ by letting $P_t\phi(w) := E\phi(\omega(t; w))$, where $\phi$ is an arbitrary, measurable extension of $\phi$ from $V$ to $H$. Moreover, for any $t, N > 0$ there exists $C_{t,N}$ such that for any $\nu > 0$,
\[ \|P_t\phi\|_\nu \leq C_{t,N}\|\phi\|_N, \quad \forall \phi \in C^1_N(V). \quad (5.5) \]

Define
\[ p(w) := \begin{cases} \|w\|^2 & \text{for } w \in V, \\ +\infty & \text{for } w \in H \setminus V. \end{cases} \]

**Corollary 5.3.** For any $N > 0$ we have $\langle \mu_*, p^N \rangle < +\infty$. Thus, in particular $\mu_*(V) = 1$.

**Proof.** Suppose that $\phi_R : [0, +\infty) \rightarrow [0, R + 1]$ is a continuous function such that $\phi_R(u) = u$ if $u \in [0, R]$ and it vanishes on $u \geq R + 1$. For a fixed $K > 0$ we denote
\[ p_K(w) := \sum_{0 < |k| \leq K} |k|^2 |\hat{w}(k)|^2. \]

Thanks to part 2 of theorem 5.1 we have $P_t p^N \in B_\nu$ for any $t > 0$ and therefore from (5.4) and (5.2) we get
\[ \langle \mu_*, P_t p^N \rangle \leq \langle \mu_*, P_t \phi^N \rangle < +\infty. \quad (5.6) \]

We have therefore
\[ \langle \mu_*, P_t \phi_R \circ p^N_K \rangle = \langle \mu_*, \phi_R \circ p^N_K \rangle < \langle \mu_*, P_t \phi^N \rangle. \quad (5.7) \]

The first equality follows from the fact that $\mu_*$ is invariant. Letting first $K \rightarrow +\infty$ and then subsequently $R \rightarrow +\infty$ we conclude the corollary. \( \square \)

Combining the results of theorem 5.2 with part 2 of theorem 5.1 we conclude the following.

**Corollary 5.4.** For any $N > 0$ there exist $C, \nu_0, \gamma > 0$ such that for any $\nu \in (0, \nu_0]$ we have
\[ \|P_t\phi - \langle \mu_*, \phi \rangle\|_\nu \leq C e^{-\gamma t}\|\phi\|_N, \quad \forall t \geq 0, \quad \phi \in C^1_N(V). \quad (5.8) \]
6. Proof of theorem 3.5

For the sake of brevity we assume that \( x_0 = 0 \) and we drop it from our notation. Let \( \psi_\ast = (\psi_\ast^{(1)}, \psi_\ast^{(2)}): V \to \mathbb{R}^2 \) be defined as \( \psi_\ast(\omega) := K(\omega)(0) \). Since, for any \( s > 0 \), \( H_{1+s} \) is embedded into \( C(\mathbb{T}^2) \), for any \( s > 0 \) there exists \( C > 0 \) such that
\[
|\psi_\ast^{(i)}(w)| \leq C|K_\ast(w)|_{1+s} \leq C|w|_1, \quad \forall \ w \in H_s, \quad i = 1, 2. \tag{6.1}
\]

It is clear therefore that the components of \( \psi_\ast \) are bounded linear functional on \( V \) and \( \psi_\ast \in C^1(V) \). It follows from corollary 5.3 that the components of \( \psi_\ast \) are integrable with respect to \( \mu_\ast \). In addition, since \( K(s\omega) = -sK(\omega) \) and measure \( \mu_\ast \) is invariant under \( s \) we obtain
\[
\int \psi_\ast d\mu_\ast = \int \psi_\ast s d\mu_\ast = -\int s\psi_\ast d\mu_\ast = -\int \psi_\ast d\mu_\ast.
\]

Thus,
\[
\int \psi_\ast d\mu_\ast = 0. \tag{6.2}
\]

Suppose also that \( \omega(t) \) is the solution of (7.14) with the initial data distributed according to \( \mu_0 \).

6.1. Proof of part 1

To prove the weak law of large numbers it suffices to only show that for \( i = 1, 2 \),
\[
\lim_{T \to +\infty} \frac{1}{T} \mathbb{E}_{x_i}(T) = 0 \quad \text{and} \quad \lim_{T \to +\infty} \frac{1}{T^2} \mathbb{E}_{x_i^2}(T) = 0. \tag{6.3}
\]

Using the Markov property we can write that
\[
\frac{1}{T} \mathbb{E}_{x_i}(T) = \frac{1}{T} \int_0^T \langle \mu_0, P_t \psi_\ast^{(i)} \rangle \ ds, \quad i = 1, 2. \tag{6.4}
\]

Suppose that \( v_0 \) is chosen in such a way that the conclusions of theorem 5.1 and corollary 5.4 hold. Assume also that \( v \in (0, v_0] \). We will adjust its value later on. By virtue of (5.8) we conclude that there exists a constant \( C > 0 \) such that
\[
|P_t \psi_\ast(w)| \leq Ce^{-\gamma t} v_\ast \|\psi_\ast\|_1. \tag{6.5}
\]

Hence, the right-hand side of (6.4) converges to 0, by estimate (3.7) and the Lebesgue dominated convergence theorem. On the other hand,
\[
\frac{1}{T^2} \mathbb{E}_{x_i^2}(T) = \frac{1}{T^2} \mathbb{E} \left( \int_0^T \psi_\ast^{(i)}(\omega(t)) \ dr \int_0^T \psi_\ast^{(i)}(\omega(s)) \ ds \right)
\]
\[
= \frac{2}{T^2} \int_0^T \int_0^T \mathbb{E} [\psi_\ast^{(i)}(\omega(t))\psi_\ast^{(i)}(\omega(s))] \ dr \ ds. \tag{6.6}
\]

The utmost right-hand side of (6.6) equals
\[
\frac{2}{T^2} \int_0^T \int_0^T \mathbb{E} [\psi_\ast^{(i)}(\omega(s))P_{t-s}\psi_\ast^{(i)}(\omega(s))] \ dt \ ds = \frac{2}{T^2} \int_0^T \int_0^T \langle \mu_0 P_{t-s}\psi_\ast^{(i)} \psi_\ast^{(i)} \rangle \ dr \ ds. \tag{6.7}
\]

Using (6.5) we can estimate the right hand side of (6.7) by
\[
\frac{C}{T^2} \int_0^T \int_0^t e^{-\gamma(t-s)} \langle \mu_0 P_s, |\psi_\ast^{(i)}| e_\ast \rangle \ dt \ ds = \frac{C(1 - e^{-\gamma T})}{\gamma T^2} \int_0^T \langle \mu_0 P_s, |\psi_\ast^{(i)}| e_\ast \rangle \ ds. \tag{6.8}
\]
Applying Hölder’s inequality with $q \in (1, v_0/v)$ and an even integer $p$ such that $p^{-1} := 1 - q^{-1}$, we conclude that the right-hand side is smaller than
\[
\frac{C}{\gamma T^2} \int_0^T (\mu_0, P_s |\psi|^p) ^{1/p} (\mu_0 P_s, e_q) ^{1/q} \, ds \leq \frac{C_1}{\gamma T^2} \int_0^T (\mu_0, P_s |\psi|^p) ^{1/p} \, ds
\]
for some constants $C, C_1$ independent of $T$. The last inequality follows from (5.1) and (5.2).

Since $|\psi|^p$ belongs to $C_p(V)$ we conclude from corollaries 5.4, 5.3 and condition (3.7) that the right hand side of the above expression can be estimated by $C_2 T/(\gamma T^2)$, with $C_2$ a constant independent of $T$, which tends to 0, as $T \rightarrow +\infty$. Thus, part 1 follows.

6.2. Definition and basic properties of the corrector

We start with the following.

**Proposition 6.1.** Functions
\[
\chi_t(w) = (\chi_t^{(1)}(w), \chi_t^{(2)}(w)) := \int_0^t P_s \psi_s(w) \, ds, \quad w \in H,
\]
converge uniformly on bounded sets, as $t \rightarrow \infty$. For any $\nu \in (0, v_0)$ there is $C > 0$ such that
\[
|\chi_t^{(i)}| \leq C e_\nu, \quad \forall t \geq 1, \quad i = 1, 2.
\]
The limit
\[
\chi = (\chi^{(1)}, \chi^{(2)}) := \lim_{t \rightarrow +\infty} \chi_t = \int_0^{+\infty} P_s \psi_s \, ds,
\]
called a corrector, satisfies
\[
|\chi^{(i)}| \leq C e_\nu, \quad i = 1, 2,
\]
with the same constant as in (6.11).

**Proof.** As a consequence of corollary 5.4 we conclude that the functions
\[
\int_0^t P_s \psi_s^{(i)}(w) \, ds, \quad t \geq 1, \quad i = 1, 2,
\]
are well defined on $H$ and converge uniformly on bounded sets. The convergence part of the proposition follows from the fact that there exists a constant $C > 0$ such that for $\nu \in (0, v_0]$,
\[
\int_0^1 \mathbb{E} \|\omega(s, w)\|^2 \, ds \leq C e_\nu(w), \quad \forall w \in H,
\]
see (7.10) below. This estimate together with (6.5) imply both (6.11) and (6.13). □

**Proposition 6.2.** One can choose $v_0 > 0$ in such a way that $\chi^{(i)} \in B_v$ for any $v \in (0, v_0]$, $i = 1, 2$.

**Proof.** Since $\psi_s^{(i)} \in C^1(V)$, $i = 1, 2$, from corollary 5.4 we conclude that $P_t \psi_s^{(i)} \in B_v$ for $t \geq 1$ and there exists $v_0 > 0$ such that for any $v \in (0, v_0]$ one can find $C, \gamma > 0$, for which
\[
\|P_t \psi_s^{(i)}\| \leq C e^{-\gamma t} \|\psi_s^{(i)}\|_1, \quad \forall t \geq 1, \quad i = 1, 2.
\]
This guarantees that $\int_1^{+\infty} P_t \psi_s^{(i)} \, dt$ belongs to $B_v$. Thanks to estimate (6.13) it suffices only to show that
\[
\left| \int_0^1 D P_t \psi_s^{(i)}(w)[\xi] \, dt \right| \leq C e_\nu(w), \quad \forall w, \xi \in H, \quad |\xi| \leq 1.
\]
To prove the above estimate note that
\[ \int_0^1 D P_t \psi_\ast^{(i)}(w)[\xi] \, dt := \mathbb{E} [K(\Xi(1))(0)], \]
where \( \Xi(w) := \int_0^1 \xi(t; w) \, dt \) and \( \xi(t) := D\omega(t; w)[\xi] \). From (6.1) for \( s = 1 \) there exists \( C > 0 \) such that
\[ |K(\Xi(w))(0)| \leq C \| \Xi(w) \|, \quad \forall w \in H. \]
Hence, from (7.9), we conclude that for any \( \nu > 0 \) there exists \( C > 0 \) such that
\[ \left\| \int_0^1 D P_t \psi_\ast^{(i)}(w)[\xi] \, dt \right\|^2 \leq E \left\| \xi \right\|^2 \exp \left\{ \nu |\omega(1)|^2 + \frac{\nu}{2e} \int_0^1 \|\omega(s)\|^2 \, ds \right\} \]
and (6.15) follows from estimate (7.10) formulated below. \( \square \)

6.3. Proof of part 2

After a simple calculation we get
\[ D_{ij}(T) := \frac{1}{T} \mathbb{E} \left[ x_i(T)x_j(T) \right] = D_{ij}^1(T) + D_{ij}^2(T), \]
with
\[ D_{ij}^1(T) := \frac{1}{T} \int_0^T \left\{ \mu_0 P_s, \psi_\ast^{(i)} \int_0^{T-s} P_t \psi_\ast^{(j)} \, dt \right\} \, ds, \]
\[ D_{ij}^2(T) := \frac{1}{T} \int_0^T \left\{ \mu_0 P_s, \psi_\ast^{(j)} \int_0^{T-s} P_t \psi_\ast^{(i)} \, dt \right\} \, ds. \]

It suffices to only deal with the limit of \( D_{ij}^1(T) \), the other term can be handled in a similar way. We can write
\[ \left| D_{ij}^1(T) - \frac{1}{T} \int_0^T \left\{ \mu_0 P_s, \psi_\ast^{(i)} \chi^{(j)} \right\} \, ds \right| = \frac{1}{T} \left| \int_0^T \left\{ \mu_0 P_s, \psi_\ast^{(i)} \left( \chi^{(j)} - \chi^{(j)}_{T-s} \right) \right\} \, ds \right| = R_{ij}(T), \]
where
\[ R_{ij}(T) := \left\| \int_0^T \left\{ \mu_0 P_s T, \psi_\ast^{(i)} \left( \chi^{(j)} - \chi^{(j)}_{T(1-s)} \right) \right\} \, ds \right\|. \] (6.16)

**Lemma 6.3.** We have
\[ \lim_{T \to +\infty} R_{ij}(T) = 0. \] (6.17)

**Proof.** Suppose that \( p \) is a positive even integer and \( q \) is sufficiently close to 1 so that \( q \nu < \nu_0 \) and \( 1/q = 1 - 1/p \), where \( \nu \) is as in (6.11) and (6.13), while \( \nu_0 \) is such that (3.7) is in force. Then, we can find a constant \( C > 0 \) such that
\[ |\chi^{(j)}(w) - \chi^{(j)}_{T(1-s)}(w)|^q \leq C e_{\nu_0}(w), \quad \forall w \in H, \quad \forall s \in [0, 1], \quad T > 0. \] (6.18)

Using proposition 6.1 and (3.7) we conclude that
\[ \lim_{T \to +\infty} \left\| \mu_0 P_s T, \chi^{(j)} - \chi^{(j)}_{T(1-s)} \right\|^q = 0, \quad \forall s \in [0, 1]. \]

Equality (6.17) can be concluded, provided we can substantiate passage to the limit with \( T \) under the integral appearing on the right-hand side of (6.16). Suppose first that the argument \( s \) appearing in the integral satisfies \( sT \geq 1 \). Using Hölder’s inequality, in the same way as
As a result the left-hand side of (6.19) is bounded for all
\( r \geq 1 \), and estimates (6.11) and (6.13) the expression under the integral can be estimated by
\[
\langle \mu_0, P_T |\psi_*^{(i)}|p \rangle^{1/p} \langle \mu_0 P_T, |\psi_*^{(i)}|q \rangle^{1/q} \leq \sup(\mu_0, P_T |\psi_*^{(i)}|p \rangle^{1/p} \langle \mu_0 P_T, |\psi_*^{(i)}|q \rangle^{1/q}. \tag{6.19}
\]
Since \( |\psi_*|^p \in C^1_p(V) \) we have \( \sup_{r \geq 1} \langle \mu_0, P_T |\psi_*|^p \rangle < +\infty \), thanks to part 2 of theorem 5.2. As a result the left-hand side of (6.19) is bounded for all \( s \in [1/T, 1] \). From the Lebesgue dominated convergence theorem we conclude therefore that
\[
\lim_{T \to +\infty} \int_{1/T}^1 \langle \mu_0 P_T, \psi_*^{(i)}(\chi^{(i)} - \chi_T^{(i)}) \rangle \, ds = 0. \tag{6.20}
\]
Next we will prove that there exists \( C > 0 \) such that
\[
\left| \int_{0}^{1/T} \langle \mu_0 P_T, \psi_*^{(i)}(\chi^{(i)} - \chi_T^{(i)}) \rangle \, ds \right| \leq \frac{C}{T}. \tag{6.21}
\]
provided that \( T \geq 1 \). Indeed, using first the Cauchy–Schwartz inequality and then (6.11), and (6.13) we get that the left-hand side can be estimated by
\[
C \mathbb{E} \left\{ \left\{ \int_{0}^{1/T} |\psi_*^{(i)}(\omega(sT))|^2 \, ds \right\}^{1/2} \left\{ \int_{0}^{1/T} e^{2v_0(\omega(sT))} \, ds \right\}^{1/2} \right\}.
\]
Applying Hölder’s inequality with \( q \in (1, 2) \) and \( 1/p = 1 - 1/q \) we get that this expression can be estimated by
\[
C \mathbb{E} \left\{ \left\{ \int_{0}^{1/T} |\psi_*^{(i)}(\omega(sT))|^2 \, ds \right\}^{p/2} \right\}^{1/p} \left\{ \mathbb{E} \left\{ \int_{0}^{1/T} e^{2v_0(\omega(sT))} \, ds \right\}^{q/2} \right\}^{1/q} \leq C_1 \left\{ \mathbb{E} \left\{ \frac{1}{T} \int_{0}^{1/T} |\omega(s)|^2 \, ds \right\}^{p/2} \right\}^{1/p} \left\{ \mathbb{E} \left\{ \int_{0}^{1/T} e^{2v_0(\omega(sT))} \, ds \right\} \right\}^{1/2} \leq \frac{C_2}{T} \mathbb{E} \{ \psi_0 \} \left\{ \int_{0}^{1/T} |\omega(s)|^2 \, ds \right\}^{p/2} \leq \frac{C_3}{T},
\]
provided \( 2v < v_0 \). The penultimate inequality follows from (5.1) and assumption (3.7), while the last estimate is a consequence of (7.11) stated below. Thus, (6.21) follows.

We are left therefore with the problem of finding the limit of
\[
S_{ij}(T) = \frac{1}{T} \int_0^T \langle \mu_0 P_s, \psi_*^{(i)} \chi^{(j)} \rangle \, ds \tag{6.22}
\]
as \( T \to +\infty \). Let \( R \geq 1 \) be fixed and \( \varphi_R : \mathbb{R} \to \mathbb{R} \) be a smooth mapping such that \( \varphi_R(x) = 1 \)
for \( |x| \leq R \) and \( \varphi_R(x) = 0 \) for \( |x| \geq R + 1 \). Observe that
\[
\hat{\chi}^{(R)}(w) := \chi^{(j)}(w) \varphi_R(|w|^2)
\]
belongs to \( C^1_p(H) \), and thus also to \( C^1_p(V) \). Therefore, \( \psi_*^{(i)} \hat{\chi}^{(R)} \in C^1_p(V) \). Denote by \( S^{(R)}(T) \) the expression in (6.22) with \( \chi^{(j)} \) replaced by \( \hat{\chi}^{(R)} \).

Let \( \varepsilon > 0 \) be arbitrary. Using the same argument as in the proof of lemma 6.3 one can show that for any \( \varepsilon > 0 \) there exists a sufficiently large \( R \geq 1 \) and \( T_0 > 0 \) so that
\[
\left| \frac{1}{T} \int_0^T \langle \mu_0 P_s, \psi_*^{(i)} \chi^{(j)} - \hat{\chi}^{(R)} \rangle \, ds \right| < \frac{\varepsilon}{2}.
\]
Likewise, we can choose $R \geq 1$ and $T_0 > 0$ so large that
\[ \| \langle \mu_+, \psi^{(i)} \chi^{(j)} - \hat{\chi}^{(R)} \rangle \| \leq \varepsilon. \]
By corollary 5.4 we have
\[ \| P_t (\psi^{(i)} \hat{\chi}^{(R)}) - \langle \mu_+, \psi^{(i)} \hat{\chi}^{(R)} \rangle \| \leq C e^{-\gamma t} \| \psi^{(i)} \hat{\chi}^{(R)} \|^2, \quad \forall t \geq 0. \]
In consequence we conclude that
\[ \lim_{T \to +\infty} S(R)(T) = \langle \mu_+, \psi^{(i)} \hat{\chi}^{(R)} \rangle. \]
Hence,
\[ \limsup_{T \to +\infty} | S_{ij}(T) - \langle \mu_+, \psi^{(i)} \chi^{(j)} \rangle | \leq \limsup_{T \to +\infty} | S_{ij}(T) - S(R)(T) | + | \langle \mu_+, \psi^{(i)} \hat{\chi}^{(R)} \rangle - \langle \mu_+, \psi^{(i)} \chi^{(j)} \rangle | < \varepsilon. \]
This proves that
\[ \lim_{T \to +\infty} S_{ij}(T) = \langle \mu_+, \psi^{(i)} \chi^{(j)} \rangle. \]
We have therefore shown part 2 of the theorem with
\[ \lim_{T \to +\infty} D_{ij}(T) := \langle \mu_+, \psi^{(i)} \chi^{(j)} \rangle + \langle \mu_+, \psi^{(j)} \chi^{(i)} \rangle. \] (6.23)
\[ \square \]

6.4. Proof of part 3

6.4.1. Reduction to the central limit theorem for martingales. Note that
\[ \frac{1}{\sqrt{T}} \int_0^T \psi_*(\omega(s)) \, ds = \frac{1}{\sqrt{T}} M_T + R_T, \] (6.24)
where
\[ M_T := \chi(\omega(T)) - \chi(\omega(0)) + \int_0^T \psi_*(\omega(s)) \, ds \] (6.25)
and
\[ R_T := \frac{1}{\sqrt{T}} [\chi(\omega(0)) - \chi(\omega(T))]. \]

Proposition 6.4. The process $\{ M_T, T \geq 0 \}$ is a square integrable, two-dimensional vector martingale with respect to the filtration $\{ \mathcal{F}_T, T \geq 0 \}$. Moreover, random vectors $R_T$ converge to 0, as $T \to +\infty$, in the $L^1$-sense.

The proof of this result is quite standard and can be found in [17], see proposition 5.2 and lemma 5.3.

6.4.2. Central limit theorem for martingales. Assume that $\{ M_n, n \geq 0 \}$ is a zero mean martingale subordinated to a filtration $\{ \mathcal{F}_n, n \geq 0 \}$ and $Z_n := M_n - M_{n-1}$ for $n \geq 1$, is the respective sequence of martingale differences. Recall that the quadratic variation of the martingale is defined as
\[ \langle M \rangle_n = \sum_{j=1}^n \mathbb{E} [ Z_j^2 | \mathcal{F}_{j-1} ], \quad n \geq 1. \]
The following theorem has been shown in [17], see theorem 4.1.
Theorem 6.5. Suppose also that
\[(M1) \quad \sup_{n \geq 1} \mathbb{E} Z_n^2 < +\infty, \quad (6.26)\]
\[(M2) \text{ for every } \varepsilon > 0, \quad \lim_{N \to +\infty} \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[ Z_{j+1}, |Z_{j+1}| \geq \varepsilon \sqrt{N} \right] = 0. \quad (6.27)\]
\[(M3) \text{ there exists } \sigma \geq 0 \text{ such that } \lim_{K \to \infty} \limsup_{\ell \to \infty} \frac{1}{\ell} \sum_{m=1}^{\ell} \mathbb{E} \left[ \left| \langle M \rangle_{mK} - \langle M \rangle_{(m-1)K} \right| - \sigma^2 \right] = 0. \quad (6.28)\]
\[(M4) \text{ for every } \varepsilon > 0 \quad \lim_{K \to \infty} \limsup_{\ell \to \infty} \frac{1}{\ell K} \sum_{m=1}^{\ell} \mathbb{E} \left[ 1 + Z_{j+1}, |M_j - M_{(m-1)K}| \geq \varepsilon \sqrt{\ell K} \right] = 0. \quad (6.29)\]
Then,
\[\lim_{N \to +\infty} \frac{\mathbb{E} \langle M \rangle N}{N} = \sigma^2 \quad (6.29)\]
and
\[\lim_{N \to +\infty} \mathbb{E} e^{i\theta M_N \sqrt{N}} = e^{-\sigma^2 \theta^2/2}, \quad \forall \theta \in \mathbb{R}. \quad (6.30)\]

6.4.3. Proof of the central limit theorem for \(M_T / \sqrt{T}\). We prove that \(M_n / \sqrt{n}\), where \(n \geq 1\) is an integer, converge in law to a Gaussian random vector, as \(n \to +\infty\). This suffices to conclude that in fact \(M_T / \sqrt{T}\) satisfy the central limit theorem. Indeed, let \(Z_n := M_n - M_{n-1}\) for \(n \geq 1\). Note that for any \(\varepsilon > 0\)
\[\lim_{N \to +\infty} \sup_{T \in [N, N+1)} |M_T / \sqrt{T} - M_N / \sqrt{N}| = 0, \quad \mathbb{P}\text{-a.s.}, \quad (6.31)\]
For a given \(\varepsilon_N > 0\) we let
\[A_N := \left\{ \sup_{T \in [N, N+1)} |M_T / \sqrt{T} - M_N / \sqrt{N}| \geq \varepsilon_N \right\}.
\]
We have
\[\mathbb{P}[A_N] \leq \mathbb{P}\left[ \sup_{T \in [N, N+1)} |M_T - M_N| \geq \varepsilon_N \sqrt{N}/2 \right] + \mathbb{P}[|M_N|[N^{-1/2} - (N + 1)^{-1/2}] \geq \varepsilon_N/2] \leq \frac{C}{N^{1/4}} \mathbb{E}|Z_{N+1}|^4 + \frac{C}{N^{3/2}} \sum_{j=1}^{N} \mathbb{E}|Z_j|^2.
\]
The last inequality follows from the Doob and Chebyshev estimates and the elementary inequality \(N^{-1/2} - (N + 1)^{-1/2} \leq C N^{-3/2}\) that holds for all \(N \geq 1\) and some constant \(C > 0\). We denote the first and second terms on the right hand side by \(I_N\) and \(H_N\), respectively. We claim that there exists \(C > 0\) such that
\[\mathbb{E}|Z_{N+1}|^4 \leq C, \quad \forall N \geq 0. \quad (6.32)\]
Indeed, we have
\[\mathbb{E}|Z_{N+1}|^4 \leq C \left\{ \mathbb{E}|\chi(\omega(N+1))|^4 + \mathbb{E}|\chi(\omega(N))|^4 + \mathbb{E} \left[ \int_N^{N+1} \psi_\varepsilon(\omega(s)) \, ds \right]^4 \right\}.
\]
To estimate the first two terms appearing on the right hand side we use (6.13) and then subsequently (5.2). We conclude that all these terms can be estimated by a constant independent of \( N \). The last expectation can be estimated using (6.1) by

\[
C E \left[ \int_N^{N+1} \| \omega(s) \|^2 \, ds \right]^2 = C \left( \mu_0 P_N, E \left[ \int_0^1 \| \omega(s; \cdot) \|^2 \, ds \right]^2 \right).
\]

Applying (7.10) and then again (5.2) we obtain that also this term can be estimated independently of \( N \). Hence

\[
I_N \leq C N^2 \varepsilon_N^4.
\]

On the other hand, from (6.32) we conclude also that for some constants \( C, C_1 > 0 \) independent of \( N \) we have

\[
I_{1N} = C \sum_{k=1}^N \mathbb{E} |Z_k|^2 \leq C_1 \frac{N}{N^2 \varepsilon_N^2}.
\]

Choosing \( \varepsilon_N \) tending to 0 sufficiently slowly we can guarantee that

\[
\sum_{N \geq 1} \mathbb{P}[A_N] < +\infty,
\]

and (6.31) follows from an application of the Borel–Cantelli lemma.

Choose \( a \in \mathbb{R}^2 \) and let \( M_n := M_n \cdot a \). Condition M1 obviously holds in light of (6.32). Condition M2 also easily follows from (6.32) and the Chebyshev inequality. Before verifying hypothesis M3 let us introduce some additional notation. For a given probability measure \( \mu \) on \( H \) and a Borel event \( A \) write

\[
\mathbb{P}_\mu[A] := \int_H \mathbb{P}[A|\omega(0) = w] \mu(dw).
\]

The respective expectation will be denoted by \( \mathbb{E}_\mu \). We write \( \mathbb{P}_w \) and \( \mathbb{E}_w \) in case of \( \mu = \delta_w \). We can write that

\[
\frac{1}{K} \mathbb{E} \left[ \langle M \rangle_{mK} - \langle M \rangle_{(m-1)K} \right] = \frac{1}{K} \sum_{j=0}^{K-1} P_j \Psi(\omega((m - 1)K))
\]

with \( \Psi(w) := \mathbb{E}_w M_1^2 \). Suppose that \( \sigma^2 = \langle \mu_*, \Psi \rangle \). Let also \( \tilde{\Psi}(w) := \Psi(w) - \sigma^2 \),

\[
S_K(w) := \frac{1}{K} \sum_{j=0}^{K-1} P_j \Psi(w)
\]

and

\[
\tilde{S}_K(w) := |S_K(w)| - \langle \mu_*, |S_K| \rangle, \quad w \in H.
\]

We can rewrite the expression under the limit in (6.27) as being equal to

\[
\frac{1}{\ell} \sum_{m=1}^{\ell} \mathbb{E} \left[ \frac{1}{K} \sum_{j=0}^{K-1} P_j \tilde{\Psi}(\omega((m - 1)K)) \right] = \langle \mu_0 Q^K_\ell, |S_K| \rangle, \quad (6.33)
\]

where

\[
Q^K_\ell := \frac{1}{\ell} \sum_{m=1}^{\ell} P_{(m-1)K}.
\]
It is obvious that the second term on the right-hand side of (6.33) does not contribute to the limit in hypothesis M3. We prove that

$$\lim_{\ell \to +\infty} \sum_{m=1}^{\ell} (\mu_0 Q^K_{\ell}, \tilde{S}_K) = 0.$$  \hspace{1cm} (6.34)

Then M3 will follow upon subsequent applications of (6.34), as $\ell \to +\infty$, and Birkhoff’s individual ergodic theorem, as $K \to +\infty$. To prove (6.34) it suffices only to show that the function $S_K(\cdot)$ is continuous on $H$ and for any $K$ fixed there exists a constant $C > 0$ such that

$$|S_K(w)| \leq C e_v(w), \quad \forall \ w \in H.$$  \hspace{1cm} (6.35)

Equality (6.34) is then a consequence of the fact that measures $\mu_0 Q^K_{\ell}$ converge weakly to $\mu^*$ as $\ell \to +\infty$, and estimate (5.1). The continuity of $S_K(\cdot)$ follows from the fact that $\tilde{\Psi} \in B_v$. On the other hand estimate (6.35) follows from the fact that for any $j \geq 1$ fixed there exists a constant $C > 0$ such that

$$P_j \tilde{\Psi} \leq C e_v(w), \quad w \in H.$$  \hspace{1cm} (6.36)

The last estimate can be seen as follows:

$$P_j \tilde{\Psi} \leq C e_v(w), \quad w \in H.$$  \hspace{1cm} (6.38)

Finally we verify condition M4. For that purpose it suffices to only prove that

$$\lim_{K \to +\infty} \limsup_{\ell \to +\infty} \frac{1}{K} \sum_{j=0}^{K-1} (\mu_0 Q^K_{\ell}, G_{\ell,j}) = 0,$$

where

$$G_{\ell,j}(w) := E_w \left[ 1 + |Z_{j+1}|, |M_j| \geq \varepsilon \sqrt{\ell K} \right].$$

The latter follows if we show that

$$\limsup_{\ell \to +\infty} (\mu_0 Q^K_{\ell}, G_{\ell,j}) = 0, \quad \forall \ j = 0, \ldots, K - 1.$$  \hspace{1cm} (6.39)

From the Markov inequality we obtain

$$\mathbb{P}_w \left[ |M_j| \geq \varepsilon \sqrt{\ell K} \right] \leq \frac{E_w |M_j|}{\varepsilon \sqrt{\ell K}} \leq I_1 + I_2,$$

where

$$I_1 := \frac{1}{\varepsilon \sqrt{\ell K}} \sum_{i=1}^{2} E_w |\chi^{(i)}(\omega(j)) - \chi^{(i)}(w)|$$
and
\[ I_2 := \frac{1}{\epsilon \sqrt{\ell K}} \sum_{i=1}^{2} E_w \left| \int_0^j \tilde{\psi}_w^{(i)}(\omega(s)) \, ds \right|. \]

Using (6.13) we conclude that
\[ I_1 \leq \frac{C_1 e(\omega)}{\epsilon \sqrt{\ell K}}. \]

On the other hand, we have
\[ I_2 \leq \frac{C_2}{\epsilon \sqrt{\ell K}} \sum_{i=1}^{2} E_w \int_0^j \|\omega(s)\| \, ds \]
and from (7.11) we get that
\[ I_2 \leq \frac{C_3 e(\omega)}{\epsilon \sqrt{\ell K}}. \]

Summarizing, we have shown that for any \( R > 0 \),
\[ \sup_{|\omega| \leq R} \sum_{j=1}^{M_j} \left[ |M_j| \geq \epsilon \sqrt{\ell K} \right] \leq C \sqrt{\ell K}. \] (6.39)

Furthermore,
\[ \sup_{|\omega| \leq R} \sum_{j=1}^{M_j} \left[ |Z_{j+1}|^2, |M_j| \geq \epsilon \sqrt{\ell K} \right] \]
\[ \leq 2 \sum_{i=1}^{2} \left\{ \sup_{|\omega| \leq R} \left[ \left( \chi^{(i)}(\omega(j + 1)) - \chi^{(i)}(\omega(j)) \right)^2, |M_j| \geq \epsilon \sqrt{\ell K} \right] \right. \]
\[ + \sup_{|\omega| \leq R} \left[ \left( \int_j^{j+1} \tilde{\psi}_w^{(i)}(\omega(s)) \, ds \right)^2, |M_j| \geq \epsilon \sqrt{\ell K} \right] \}
\[ \leq C \sup_{|\omega| \leq R} \sup_{i,j} E_w \left[ e(\omega(t)), |M_j| \geq \epsilon \sqrt{\ell K} \right] \] (6.40)

for some constant \( C \) independent of \( \ell \). The above argument shows that
\[ \lim_{\ell \to +\infty} \sup_{|\omega| \leq R} |G_{\ell,j}(\omega)| = 0. \]

To obtain (6.38) it suffices only to prove that for \( \delta > 0 \) as in H3 we have
\[ \lim_{\ell \to +\infty} \sup_{|\omega| \leq R} |G_{\ell,j}(\omega)| = +\infty, \quad \forall K \geq 1, \quad 0 \leq j \leq K - 1. \] (6.41)

Note that
\[ \langle \mu_0 Q^K, G^4_{\ell,j} \rangle \leq \mathbb{E}_{\mu_0} (1 + |Z_{j+1}|^2)^{1+4/3}. \] (6.42)

This however is a consequence of (5.1). Thus condition M4 follows.

Summarizing, we have shown that
\[ \lim_{n \to +\infty} \exp \left\{ i a \cdot M_N / \sqrt{N} \right\} = \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^{2} D_{ij} a_i a_j \right\}, \]
where
\[ D_{ij} := \left\langle \mu_*, \mathbb{E} \left\{ \prod p_{\omega} \left[ \chi^{(p)}(\omega(1; w)) - \chi^{(p)}(w) + \int_0^1 \tilde{\psi}_w^{(p)}(\omega(s; w)) \, ds \right] \right\} \right\rangle. \]
After a somewhat lengthy, but straightforward calculation, using the stationarity of $\mu_+$ and the fact that
\[
\mu_+ \left[ P_s \chi^{(i)} - \chi^{(i)} + \int_0^s P_s \tilde{\psi}_e^{(i)} \, ds \right] = 0, \quad \forall s \geq 0
\]
we conclude that $D_{ij}$ coincides with the expression on the right hand side of (6.23). \hfill \Box

7. Proof of the results from section 5

7.1. Proof of theorem 5.1

Part 3 is a direct consequence of parts 1 and 2. The invariance of $\mu_+$ under $s_*$ follows from proposition 4.7.

7.1.1. Proof of part 1. Suppose that $\omega(t) := \omega(t; w)$. Let $q = e^{-1/2}$. From (7.10) one can conclude that for $v \in (0, v_0]$, where $v_0 = 1/(4\|Q\|)$, there exists a constant $C > 0$ such that
\[
\mathbb{E} \exp \left\{ v |\omega(n + 1)|^2 \right\} \leq C \mathbb{E} \exp \left\{ q |\omega(n)|^2 \right\}, \quad \forall n \geq 0. \tag{7.1}
\]
The right hand side can be further estimated using Jensen’s inequality
\[
C \mathbb{E} \exp \left\{ q |\omega(n)|^2 \right\} \leq C (\mathbb{E} \exp \left\{ v |\omega(n)|^2 \right\})^{q^2} \leq C^{1+q} \left( \mathbb{E} \exp \left\{ q |\omega(n - 1)|^2 \right\} \right)^q.
\]
Iterating this procedure we conclude that for any $n \geq 0$
\[
\mathbb{E} \exp \left\{ v |\omega(n + 1)|^2 \right\} \leq C^{1+q^n-1} \left( \exp \left\{ q^{2^n} |\omega(0)|^2 \right\} \right)^{1/2^{n-1}} \leq C^{1/(1-q)} \mathbb{E} \exp \left\{ v |w|^2 \right\}. \tag{7.2}
\]
Therefore (see part 3 of lemma A.1 of [9]) we have the following.

Lemma 7.1. There exists a constant $C > 0$ such that
\[
\mathbb{E} \exp \left\{ v |\omega(t; w)|^2 \right\} \leq C \mathbb{E} \exp \left\{ v |w|^2 \right\}, \quad \forall t \geq 0, \quad v \in (0, v_0], \quad w \in H. \tag{7.3}
\]
The above lemma obviously implies (5.1).

7.1.2. A stability result of Hairer and Mattingly. In our proof we use theorems 3.4 and 3.6 of [9], which we recall below. Suppose that $(H, \| \cdot \|)$ is a separable Hilbert space with a stochastic flow $\Phi; \mathcal{H} \times \Omega \to \mathcal{H}, t \geq 0$, i.e. a family of $C^1$-class random mappings of $\mathcal{H}$ defined over a probability space $(\Omega, F, \mathbb{P})$ that satisfies $\Phi_t(\Phi_s(x; \omega); \omega)) = \Phi_{t+s}(x; \omega)$ for all $t, s \geq 0, x \in \mathcal{H}$ and $\mathbb{P}$ a.s. $\omega \in \Omega$. We assume that $P_t$ and $P_t(x, \cdot)$, $x \in \mathcal{H}$, are a transition semigroup and a family of transition probabilities corresponding to the flow, i.e.
\[
P_t \phi(x) = \int \phi(y) P_t(x, dy) = \mathbb{E} \phi(\Phi_t(x)), \quad \forall \phi \in B(\mathcal{H}), \quad x \in \mathcal{H}.
\]
Here $B(\mathcal{H})$ is the space of Borel and bounded functions on $\mathcal{H}$. The dual semigroup acting on a Borel probability measure $\mu$ will be denoted by $\mu P_t$. We adopt the following hypotheses on the flow.

Assumption 1. There exists a measurable function $V: \mathcal{H} \to [1, +\infty)$ and two increasing continuous functions $V_+, V_*: [0, +\infty) \to [1, +\infty)$ that satisfy
\begin{enumerate}
\item $V_+(|x|) \leq V(x) \leq V^* (|x|), \quad \forall x \in \mathcal{H},$
\item $\lim_{a \to +\infty} V_+(a) = +\infty$
\item there exist $C > 0$ and $k_1 > 1$ such that $a V^* (a) \leq C V_+^k (a), \quad \forall a > 0$.
\end{enumerate}
(3) there exist \( \kappa_0 < 1 \), \( C > 0 \) and a decreasing function \( \alpha: [0, 1] \to [0, 1] \) with \( \alpha(1) < 1 \) such that
\[
\mathbb{E}
\left[
V^x(\Phi_t(x)) \left( 1 + |D\Phi_t(x)(h)| \right)
\right]
\leq
C V^{\alpha(t)}(x),
\forall x, \ h \in \mathcal{H}, \ |h| = 1,
\]
and \( t \in [0, 1], \ \kappa \in [\kappa_0, \kappa_1] \). Here \( D\Phi_t(x)(h) \) denotes the Fréchet derivative at \( x \) in the direction \( h \).

**Assumption 2.** There exist \( C > 0 \) and \( \kappa_2 \in [0, 1] \) such that for any \( \epsilon \in (0, 1) \) one can find \( C(\epsilon), \ T(\epsilon) > 0 \), for which
\[
|DP_t\phi(x)| \leq C V^{\alpha(x)} \left\{ C(\epsilon) \left| P_t(\phi^2)(x) \right|^{1/2} + \epsilon \left| P_t(|D\phi|^2)(x) \right|^{1/2} \right\},
\tag{7.4}
\]
for all \( x \in \mathcal{H}, \ t \geq T(\epsilon) \).

We now introduce the following family of metrics on \( \mathcal{H} \). For \( \kappa \geq 0 \) and \( x, y \in \mathcal{H} \) let
\[
d_x(x, y) := \inf_{c \in \Pi(x, y)} \int_0^1 V^x(c(t)) |\dot{c}(t)| \, dt,
\]
where the infimum extends over the set \( \Pi(x, y) \) consisting of all \( C^1 \) regular paths \( c: [0, 1] \to \mathcal{H} \) such that \( c(0) = x, c(1) = y \). In the special case of \( \kappa = 1 \) we set \( d = d_1 \). For two Borel probability measures \( \mu_1, \mu_2 \) on \( \mathcal{H} \) denote by \( C(\mu_1, \mu_2) \) the family of all Borel measures on \( \mathcal{H} \times \mathcal{H} \) whose marginals on the first and second coordinate coincide with \( \mu_1, \mu_2 \) respectively. We denote also by
\[
d(\mu_1, \mu_2) := \sup \left\{ \langle |\mu_1 - \mu_2|, \phi \rangle : \text{Lip}(\phi) \leq 1 \right\}.
\]
Here \( \text{Lip}(\phi) \) is the Lipschitz constant of \( \phi: \mathcal{H} \to \mathbb{R} \) in the metric \( d(\cdot, \cdot) \). By \( P_1(\mathcal{H}, d) \) we denote the space of all Borel probability measures \( \mu \) on \( \mathcal{H} \) satisfying \( \int_{\mathcal{H}} d(x, 0) \mu(dx) < +\infty \).

Let \( A \subset \mathcal{H} \times \mathcal{H} \) be Borel measurable. For a given \( t \geq 0 \) and \( x, y \in \mathcal{H} \) denote
\[
P_t(x, y; A) = \sup \{ \mu[A] : \mu \in C(P_t(x, \cdot), P_t(y, \cdot)) \}.
\]

**Assumption 3.** Given any \( \kappa \in (0, 1) \) and \( \delta, R > 0 \) there exists \( T_0 > 0 \) such that for any \( T \geq T_0 \) there exists \( a > 0 \) for which
\[
\inf_{|x|, |y| \leq R} P_T(x, y; \Delta_{\delta, a}) \geq a.
\]
Here,
\[
\Delta_{\delta, a} := \{(x, y) \in \mathcal{H} \times \mathcal{H} : d_\delta(x, y) < \delta \}, \quad \forall \kappa, \delta > 0.
\]

**Theorem 7.2.** Suppose that assumptions 1, 2, 3 stated above are in force. Then the following are true:

1. there exist \( C, \gamma > 0 \) such that
\[
d(\mu_1 P_t, \mu_2 P_t) \leq Ce^{-\gamma t} d(\mu_1, \mu_2), \quad \forall \mu_1, \mu_2 \in P_1(\mathcal{H}, d), \tag{7.5}
\]
2. there exists a unique probability measure \( \mu_* \in P_1(\mathcal{H}, d) \) invariant under \( \{P_t, \ t \geq 0\} \), i.e. \( \mu_* = \mu_* P_t \) for all \( t \geq 0 \),
3. we have
\[
\|P_t\phi - \langle \mu_*, \phi \rangle\|_{\text{Lip}} \leq Ce^{-\gamma t} \|\phi - \langle \mu_*, \phi \rangle\|_{\text{Lip}}, \quad \forall \phi \in C^1(\mathcal{H}), \ t \geq 0.
\tag{7.6}
\]
Here
\[
\|\phi\|_{\text{Lip}} := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{d(x, y)} + |\phi|_*.
\]
7.1.3. Proof of part 2.

Verification of assumption 1

Write $\Phi_1(w; W) := \omega(t; w, W)$, where $W$ is the cylindrical Wiener process appearing in (4.1).

Let

\[ \xi(t; w, \xi) := D\Phi_1(w)[\xi], \quad \xi \in H. \]  

(7.7)

In what follows we suppress $w$ and $\xi$ in our notation when their values are obvious from the context. The following result holds.

**Proposition 7.3.** For any $\nu > 0$ there exists $C > 0$ such that for any $w, \xi \in H$ we have

\[ |\xi(t)| \leq |\xi| \exp \left\{ \nu \int_0^t \|\omega(s)\|^2 \, ds + Ct \right\} \]  

(7.8)

and

\[ \left\{ \int_0^t \|\xi(s)\|^2 \, ds \right\}^{1/2} \leq |\xi| \exp \left\{ \nu \int_0^t \|\omega(s)\|^2 \, ds + Ct \right\}, \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.} \]  

(7.9)

In addition, there exist $v_0, C_1 > 0$ such that

\[ \mathbb{E} \exp \left\{ v|\omega(t)|^2 + \frac{v}{2} \int_0^t \|\omega(s)\|^2 \, ds \right\} \leq C_1 \exp \left\{ v|w|^2e^{-t/2} \right\}, \quad \forall t \in [0, 1], \quad v \in [0, v_0] \]  

(7.10)

and

\[ \mathbb{E} \exp \left\{ v \sup_{t \geq 0} \left[ |\omega(t)|^2 + \int_0^t \|\omega(s)\|^2 \, ds - t \text{tr} Q^2 \right] \right\} \leq Ce_v(w), \quad \forall v \in [0, v_0]. \]  

(7.11)

**Proof.** Note that $\xi(t)$ satisfies a (non-stochastic) equation

\[ \partial_t \xi(t) = \Delta \xi(t) - \eta(t) \cdot \nabla \xi(t) + K(\xi(t)) \cdot \nabla \omega(t) \]  

(7.12)

\[ + \eta(t, 0) \cdot \nabla \xi(t) + K(\xi(0)) \cdot \nabla \omega(t), \quad \xi(0) = \xi \in H. \]

Hence,

\[ \partial_t |\xi(t)|^2 = -2 \|\xi(t)\|^2 - 2 \langle K(\xi(t)) \cdot \nabla \omega(t), \xi(t) \rangle + 2 \langle K(\xi(t)) \cdot \nabla \omega(t), \xi(t) \rangle. \]

Using (A.5) and (A.6) for $r = 1/2$ we conclude that for some deterministic $C > 0$,

\[ \partial_t |\xi(t)|^2 \leq -2 \|\xi(t)\|^2 + C \|\xi(t)\|_1 \|\omega(t)\|\|\xi(t)\| \]  

\[ \leq -2 \|\xi(t)\|^2 + v \|\omega(t)\|^2 |\xi(t)|^2 + \frac{C^2}{4v} |\xi(t)|^3. \]

An application of the Gagliardo–Nirenberg inequality (2.2) with $s = 1, \beta = 1/2$ yields

\[ |\xi(t)|_1 \leq C \|\xi(t)\|^{1/2} |\xi(t)|^{1/2} \]

for some constant $C > 0$. In consequence, there exist $C, C_1 > 0$ such that

\[ \partial_t |\xi(t)|^2 \leq -\|\xi(t)\|^2 + v \|\omega(t)\|^2 |\xi(t)|^2 + \frac{C^2}{2 \cdot 4v} |\xi(t)|^2 \]  

(7.13)

\[ \leq -\|\xi(t)\|^2 + (v \|\omega(t)\|^2 + C_1) |\xi(t)|^2. \]
Estimate (7.8) follows upon the application of Gronwall’s inequality. In addition, from (7.8) and (7.9) we conclude that there exists $C > 0$ such that
\[
\int_0^t \| \xi(s) \|^2 \, ds \leq \| \xi \|^2 + \int_0^t (v\|\omega(s)\|^2 + C_1)|\xi(s)|^2 \, ds
\]
\[
\leq \| \xi \|^2 + \| \xi \|^2 \int_0^t (v\|\omega(s)\|^2 + C) \exp \left\{ v \int_0^t \|\omega(u)\|^2 \, du + Cs \right\} \, ds
\]
\[
\leq \| \xi \|^2 \exp \left\{ v \int_0^t \|\omega(s)\|^2 \, ds + Ct \right\}.
\]
This ends the proof of (7.9).

Estimates (7.10) and (7.11) can be found in [9], see (5.2) for the first one, while the second one is contained in part 1 of lemma 4.10 of the same reference. A minor modification of the argument is required, due to the fact that equation (4.2) also contains the expression corresponding to the bilinear form $B_1(\cdot)$.

Define $V(w) := V_\ast(|w|) = e^{\|w\|^2}$. Assumption 1 of theorem 7.2 is a consequence of proposition 7.3.

7.2. Verification of assumption 2

Suppose that $\Psi: H \to \mathcal{H}$ is a Borel measurable function. Given an $(\mathcal{F}_t)$-adapted process $g: [0, \infty) \times \Omega \to H$ satisfying $\mathbb{E} \int_0^t |g_\nu|^2 \, ds < +\infty$ for each $t \geq 0$ we denote by $D_g \Psi(\omega(t))$ the Malliavin derivative of $\Psi(\omega(t))$ in the direction of $g$; that is
\[
D_g \Psi(\omega(t; w)) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left[ \Psi(\omega(t; w, W + \varepsilon g)) - \Psi(\omega(t; w, W)) \right],
\]
where the limit is understood in the $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$ sense. Recall that $\omega_\nu(t; w) := \omega(t; w, W + g)$ solves the equation
\[
d\omega_\nu(t; w) = [\Delta \omega_\nu(t) - B_0(\omega_\nu(t; w)) + B_1(\omega_\nu(t; w))] \, dt + Q \, dW(t) + Qg(t) \, dt,
\]
$\omega(0; w) = w \in H$. \hfill (7.14)

Directly from the definition of the Malliavin derivative we conclude the chain rule: suppose that $\Psi \in C^1_b(H; \mathcal{H})$ then
\[
D_g \Psi(\omega(t; w)) = D \Psi(\omega(t; w))[D(t)],
\]
with $D(t; w, g) := D_g \omega(t; w), t \geq 0$. In addition, the integration by parts formula holds, see lemma 1.2.1, p 25 of [25]. Suppose that $\Psi \in C^1_b(H)$ then
\[
\mathbb{E}[D_g \Psi(\omega(t; w))] = \mathbb{E} \left[ \Psi(\omega(t; w)) \int_0^t (g(s), dW(s)) \right].
\]
\hfill (7.16)

In particular, one can easily show that when $H = \mathcal{H}$ and $\Psi = I$, where $I$ is the identity operator, the Malliavin derivative of $\omega(t; w)$ exists and the process $D(t; w, g)$ (we omit writing $w$ and $g$ when they are obvious from the context) solves the linear equation
\[
\frac{dD}{dt}(t) = \Delta D(t) - \eta(t) \cdot \nabla D(t) - \delta k(t) \cdot \nabla \omega(t)
\]
\[
+ \eta(t, 0) \cdot \nabla D(t) + \delta k(t, 0) \cdot \nabla \omega(t) + Qg(t),
\]
$D(0) = 0$. \hfill (7.17)

Here $\delta k(t) := K(D(t))$. Write $\rho(t; w, \xi) := \xi(t) - D_g \omega(t; w)$. We have the following.
**Proposition 7.4.** For any \( \nu, \gamma > 0 \) there exists a constant \( C > 0 \) such that for any given \( w, \xi \in H \) one can find an \((\mathcal{F}_t)\)-adapted \( H \)-valued process \( g(t) = g(t; w, \xi) \) that satisfies
\[
\sup_{|\xi| \leq 1} \mathbb{E} |\xi|^2 \leq Ce_v(w)e^{-\gamma t}, \quad \forall t \geq 0,
\] (7.18)
and
\[
\sup_{|\xi| \leq 1} \int_0^\infty \mathbb{E} |g(s; w, \xi)|^2 ds \leq Ce_v(w), \quad \forall w \in H.
\] (7.19)

**Proof.** The argument can be adapted directly from the proof of proposition 4.11 from [8]. Estimate (7.18) follows from (4.13) of [8] (modulo minor modification, due to the presence of the form \( B_1(\cdot) \) in equation (4.2)). Estimate (7.19) follows from the estimate appearing in the displayed equation following (4.13) in [8] and lemma A.1 therein.

We use the above result to verify assumption 2. We have
\[
DP_t\phi(w)[\xi] = \mathbb{E} \{ D\phi(\omega(t; w))[\xi] \} + \mathbb{E} \{ D\phi(\omega(t; w))|\rho(t; w, \xi)| \}.
\]
Using the chain rule, see (7.15), the right-hand side can be rewritten as
\[
\mathbb{E} \{ D\phi(\omega(t; w)) \} + \mathbb{E} \{ D\phi(\omega(t; w))|\rho(t; w, \xi)| \}
\]
\[
= \mathbb{E} \left\{ \phi(\omega(t; w)) \int_0^t \langle g(s), dW(s) \rangle + \mathbb{E} \{ D\phi(\omega(t; w))|\rho(t; w, \xi)| \} \right\}.
\]
The last equality follows from integration by parts formula (7.16). We have
\[
\left| \mathbb{E} \left\{ \phi(\omega(t; w)) \int_0^t \langle g(s), dW(s) \rangle \right\} \right| \leq (P_t|\phi|^2(w))^{1/2} \left( \mathbb{E} \int_0^\infty |g(s)|^2 ds \right)^{1/2}
\]
and
\[
|\mathbb{E} \{ D\phi(\omega(t; w))|\rho(t; w, \xi)| \}| \leq (P_t|D\phi|^2(w))^{1/2} \left( \mathbb{E} |\rho(t; w, \xi)|^2 \right)^{1/2}.
\]
Hence, by (7.19) and (7.18), given \( \kappa \in (0, 1) \), \( \nu > 0 \), the corresponding \( V(w) = e_v(w) \) and \( \varepsilon \in (0, 1) \), we conclude estimate (7.4) with \( T_0, C(\varepsilon) \), such that
\[
\left( \mathbb{E} \int_0^\infty |g(s)|^2 ds \right)^{1/2} \leq C(\varepsilon)V^{\kappa}(w)
\]
and
\[
\sup_{|\xi| \leq 1} \mathbb{E} |\rho(t; w, \xi)|^2 \leq \varepsilon V^{\kappa}(w).
\]

7.3. **Assumption 3**

To verify this assumption consider the solution \( y(t; w), t \geq 0 \), to the deterministic equation
\[
\frac{dy(t)}{dt} = \Delta y(t) + B(y(t)), \quad t \geq 0,
\]
with the initial condition \( y(0) = w \). Then
\[
\lim_{t \to +\infty} \sup_{|w| \leq R} |y(t; w)| = 0, \quad \forall R > 0.
\]

Fix \( \delta > 0 \) and \( R > 0 \). Let \( T_0 > 0 \) be such that
\[
\sup_{|w| \leq R} d_v(y(T; w), 0) \leq \delta/4, \quad \forall T \geq T_0.
\]
Since
\[ W_{Δ, Q}(t) := \int_0^t e^{Δ(t-s)} Q \, dW(s) \]
is a centred Gaussian random element in the Banach space \( C([0, T]; V) \) with the uniform norm
\[ \|f\|_{\infty,T} := \sup_{t \in [0,T]} \|f(t)\|, \quad f \in C([0, T]; V), \]
itst topological support is a closed linear subspace (see e.g. [30]). Thus, in particular, 0 belongs to the support of its law and for any \( \varrho > 0 \), \( \mathbb{P}(F_0) > 0 \), where
\[ F_0 = \{ \pi \in Ω: \|W_{Δ, Q}(\pi)\|_{\infty,T} < \varrho \}. \]
Choose \( \varrho_0 > 0 \) such that
\[ d_κ(ω(T; w_i)(π), ν(T; w_i))) |≤ δ/4 \quad \text{for all} \quad π \in F_0, \quad i = 1, 2 \quad \text{and} \quad |w| ≤ R, \]
and set \( a := \mathbb{P}(F_0) > 0 \). For any \( |w_1|, |w_2| ≤ R \) we have
\[ \mathbb{P}(T_1, T_2; Δ, \varrho) > \mathbb{P}[π \in Ω: d_κ(ω(T; w_i)(π), ν(T; w_i))) |≤ δ/4, \quad i = 1, 2] > \mathbb{P}(F_0) = a, \]
and thus we have finished the verification of assumption 3.

7.4. Proof of theorem 5.2

7.4.1. Proof of part 1). Let us fix an arbitrary \( T > 0 \) and define \( ζ(t) := |ω(t)|^2 + t\|ω(t)\|^2 \) and \( tr\ Q_1 := \sum_{k\in \mathbb{Z}^2} |k|^2 q_k^2 \). By Itô’s formula we have
\[ dζ(t) = \left[ tr\ Q_2 + tr\ Q_1 - 2t\|ω(t)\|^2 - 2t\langle B(ω(t)), Δω(t) \rangle \right] dt + dM_t. \] (7.20)
and
\[ dM_t := 2\langle Q \, dW(t), (I + tΔ)ω(t) \rangle. \]

According to (A.5) there exist \( C, C_1 > 0 \) such that
\[ |⟨B(ω), Δω⟩| ≤ C|ω|_{1/2}||ω||_2 ≤ \frac{1}{2}|ω|^2 + C_1|ω|^4, \quad \forall \omega ∈ H_2. \]

To estimate the respective bilinear form corresponding to \( B_1(\cdot) \) we use the following estimates, see proposition 6.1 of [2]. Suppose that \( s_1, s_2, s_3 ≥ 0 \) such that \( s_1 + s_2 + s_3 > 1 \) and \( s_1 > 1 \). Then, there exists \( C > 0 \) such that
\[ |⟨B_{1}(h, ω_1), ω_2⟩| ≤ C|h|_{s_1-1} |ω_1|_{1+s_2} |ω_2|_{s_3}, \quad \forall (h, ω_1, ω_2) ∈ H_{s_1-1} × H_{1+s_2} × H_{s_3}. \] (7.21)

From (7.21) with \( s_1 = 3/2, s_2 = s_3 = 0 \) it follows that
\[ |⟨B_{1}(ω), Δω⟩| ≤ C|ω|_{1/2}||ω||_2, \quad \forall ω ∈ H_2. \]

With these inequalities we conclude that
\[ |⟨B(ω), Δω⟩| ≤ \frac{1}{2}|ω|^2 + C_1|ω|^4, \quad \forall ω ∈ H_2. \]

From this point on we proceed as in the proof of lemma A.3 of [23] and conclude from (7.20) that
\[ ζ(t) ≤ |w|^2 + tr\ Q_2^2 + \frac{t^2 tr\ Q_1^2}{2} + C \int_0^t |s| \|ω(s)\|^4 ds + U(t), \] (7.22)
where \( U(0) = 0 \) and
\[ dU(t) = -(t|ω(t)|^2 + \|ω(t)\|^2) \, dt + dM_t. \]
Since
\[ U(t) \leq M_t - (\alpha/2)(M), \]
for some sufficiently small \( \alpha > 0 \) we conclude from the exponential martingale inequality that
\[
P\left[ \sup_{t \in [0,T]} U(t) \geq K \right] \leq e^{-\alpha K}, \quad \forall \ K > 0.
\]
This, of course, implies that \( \mathbb{E} \exp[\alpha' \sup_{t \in [0,T]} U(t)] < +\infty \) for any \( \alpha' \in (0, \alpha) \). From (7.11) we get
\[
\mathbb{E} \exp \left\{ \nu \sup_{t \in [0,T]} |\omega(t)|^2 \right\} \leq C e^{\nu w},
\]
which in turn implies that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |\omega(t)|^{4N} \right] \leq C |w|^{4N}.
\]
Summarizing the above consideration, we obtain from (7.22) that for any \( T > 0 \) and \( N \geq 0 \) there exists a constant \( C > 0 \) such that
\[
\mathbb{E} \left[ \sup_{s \in [0,T]} \zeta_{2N}(s) \right] \leq C |w|^{4N} + 1.
\] (7.23)
We conclude therefore the proof of part 1 of theorem 5.2.

7.4.2. Proof of part (2). Suppose that \( \phi \in C^1_N(V) \). Then, \( P_t \phi(w) \) is well defined thanks to the already established estimate (5.4). In addition, we have
\[
e^{-\nu(w)} P_t \phi(w) \leq \|\phi\|_N e^{-\nu(w)(1 + \mathbb{E}\|\omega(t; w)\|^N)} \leq C \|\phi\|_N, \quad \forall \ w \in H. \] (7.24)
To deal with \( DP_t \phi(w) \xi \) we use the following:

Lemma 7.5. Suppose that \( [\xi(t), \ t \geq 0] \) is defined by (7.7). Then, for any \( t, \nu > 0 \) there exists a constant \( C > 0 \) such that
\[
\|\xi(t)\|^2 \leq C \|\xi\|^2 \exp \left\{ \nu \int_0^t \|\omega(s; w)\|^2 \, ds + C t \right\}, \quad \forall \ t \geq 0, \ w \in H, \ \xi \in V,
\]
P-\text{a.s.} \] (7.25)

Proof. This estimate can be established analogously to the corresponding bound obtained in lemma B.1 of [23] (with \( \alpha = 0 \). Minor modifications needed to account for the term corresponding to \( B_2(\cdot) \) present no difficulty and we leave them to the reader. \( \square \)

Concerning the estimates of \( |DP_t \phi(w)\xi| \) we can write that
\[
e^{-\nu(w)} |DP_t \phi(w)\xi| = e^{-\nu(w)} \mathbb{E}[(D\phi)(\omega(t; w))\xi(t)]
\leq \|\phi\|_N e^{-\nu(w)} \mathbb{E} \left[ (1 + \|\omega(t; w)\|^N) \|\xi(t)\| \right]
\leq C \|\phi\|_N e^{-\nu(w)} \mathbb{E}(1 + \|\omega(t; w)\|)^{2N} \left\{ \mathbb{E}\|\xi(t)\|^2 \right\}^{1/2} \] \quad \forall \ w \in H. \]
(7.27)
By the already proved part 1 of the theorem and lemma 7.5, we obtain that the utmost right hand side is less than, or equal to
\[
C_1 \|\xi\| \|\phi\|_N e^{-\nu(w)}(1 + |w|^{4N}) \mathbb{E} \exp \left\{ \nu \int_0^t \|\omega(s; w)\|^2 \, ds + C_1 t \right\} \leq C_2 \|\xi\| \|\phi\|_N.
\]
Hence
\[ e^{-\nu(w)} \| DP_t \phi(w) \| \leq C_2 \| \phi \|_N \]
and thus we finish the proof of part 2 of theorem 5.2.

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Appendix A. Existence of the Markov, Feller family

Proof of theorem 4.2. Given \(N \in \mathbb{N}\), denote by \(\Pi_N\) the orthogonal projection of \(H\) into \(H_N := \text{span}\{e_k, 0 < |k| \leq N\}\). Consider the following finite dimensional Itô stochastic differential equation
\[
d\omega^{(N)}(t) = \left[ \Delta \omega^{(N)}(t) - B_0^{(N)}(\omega^{(N)}(t)) - B_1^{(N)}(\omega^{(N)}(t)) \right] dt + Q^{(N)} dW(t),
\omega^{(N)}(0) = w^{(N)} \in H,
\]
(A.1)
with \(W^{(N)}(t) := \Pi_N W(t), Q^{(N)} := \Pi_N Q\), and
\[
B_0^{(N)}(\omega) := \Pi_N B_0(\omega), \quad B_1^{(N)}(\omega) := \Pi_N B_1(\omega), \quad \omega \in H_N.
\]
The local existence and uniqueness of the solution to (A.1) follows from the respective result for finite dimensional stochastic differential equations. By Itô’s formula we get the estimate
\[
E \left[ |\omega^{(N)}(T)|^2 + \frac{1}{2} \int_0^T \|\omega^{(N)}(t)\|^2 dt \right] \leq |w^{(N)}|^2 + \|Q^{(N)}\|^2_{L^2(\Omega, E, T)} \quad (A.2)
\]
From this we conclude that the sequence \(\{\omega^{(N)}(t), t \in [0, T]\}, N \geq 1\) is compact in \(L^2(\Omega, E_T)\).

In addition,
\[
\omega^{(N)}(t) = e^{\Delta t} w^{(N)} - \int_0^t e^{\Delta(t-s)} B_0^{(N)}(\omega^{(N)}(s)) ds + \int_0^t e^{\Delta(t-s)} B_1^{(N)}(\omega^{(N)}(s)) ds + \int_0^t e^{\Delta(t-s)} Q^{(N)} dW(s).
\]
Any weak limiting point therefore satisfies (4.2). To show uniqueness we need the following.

Lemma A.1. There exists a constant \(C > 0\) such that for all \(w_0, w_1 \in H\), \(t \geq 0\),
\[
|\omega(t; w_0) - \omega(t; w_1)| \leq |w_0 - w_1| \exp \left\{ C \int_0^t \|\omega(s; w_0)\|^2 ds \right\}, \quad \mathbb{P}\text{-a.s.} \quad (A.3)
\]
Proof. Let \(\rho(t) := \omega(t; w_1) - \omega(t; w_0)\) and \(r(t) := K(\rho(t))\). From (7.14) we conclude
\[
\frac{d}{dt} |\rho(t)|^2 = -2\|\rho(t)\|^2 - 2((r(t) \cdot \nabla)\omega(t; w_0), \rho(t)) + 2((r(t, 0) \cdot \nabla)\omega(t; w_0), \rho(t)).
\]
To deal with the second term on the right-hand side we use the following estimate. Suppose that \(v = K(h)\). Then, for any \(r > 0\) there exists a constant \(C > 0\) such that
\[
|(v \cdot \nabla) f, g) | \leq C \|f\| \|g\| \|h\|, \quad \forall f \in V, \quad g \in H_r, \quad h \in H \quad (A.5)
\]
and

\[ |\langle (v \cdot \nabla) f, g \rangle| \leq C \|f\|_V \|g\|_H, \quad \forall \ g \in H, \ f \in V, \ h \in H_r, \quad (A.6) \]

see e.g. (6.10) of [2]. With these two inequalities in mind we conclude from (A.4) that

\[
\frac{d}{dt} \|\rho(t)\|^2 \leq -2\|\rho(t)\|^2 + C\|\omega(t; w_0)\|_1 \|\rho(t)\|_{1/2} \|\rho(t)\| \\
\leq -2\|\rho(t)\|^2 + C_1 \|\omega(t; w_0)\|^2 \|\rho(t)\|^2 + 2\|\rho(t)\|^2.
\]

By Gronwall’s inequality we conclude then (A.3). \qed

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