DOMAIN DECOMPOSITION WITH LOCAL IMPEDANCE CONDITIONS FOR THE HELMHOLTZ EQUATION

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Abstract. We consider one-level additive Schwarz preconditioners for the Helmholtz equation (with increasing wavenumber $k$), discretized using fixed-order nodal conforming finite elements on a family of simplicial fine meshes with diameter $h$, chosen to maintain accuracy as $k$ increases. The preconditioners combine independent local solves (with impedance boundary conditions) on overlapping subdomains of diameter $H$ and overlap $\delta$, and prolongation/restriction operators defined using a partition of unity; this formulation was previously proposed in [J.H. Kinn and M. Sarkis, Comp. Meth. Appl. Mech. Engrg. 196, 1507-1514, 2007]. In numerical experiments (with $\delta \sim H$) we observe robust (i.e. $k-$independent) GMRES convergence as $k$ increases, both with $H$ fixed, and with $H$ decreasing moderately as $k$ increases. This provides a highly-parallel, $k-$robust one-level domain-decomposition method. We provide supporting theory for this observation by studying the preconditioner applied to a range of absorptive problems, $k^2 \to k^2 + i\varepsilon$, with absorption parameter $\varepsilon$, including the “pure Helmholtz” case ($\varepsilon = 0$). Working in the Helmholtz “energy” inner product, we prove a robust upper bound on the norm of the preconditioned matrix, valid for all $\varepsilon, \delta$. Under additional conditions on $\varepsilon$ and $\delta$, we also prove a strictly-positive lower bound on the distance of the field of values of the preconditioned matrix from the origin. Using these results, combined with previous results of [M.J. Gander, I.G. Graham and E.A. Spence, Numer. Math. 131(3), 567-614, 2015] we obtain theoretical support for the observed robustness of the preconditioner for the pure Helmholtz problem with increasing wavenumber $k$.

Key words. Helmholtz equation, high frequency, preconditioning, GMRES, domain decomposition, robustness

AMS subject classifications. 65F08, 65F10, 65N55

1. Introduction. The efficient solution of the wave equation is of intense current interest because of its many applications (in, e.g., computational medicine, underwater acoustics, earthquake modelling, and seismic imaging). This paper concerns efficient iterative methods for computing conforming finite-element solutions of the Helmholtz equation (i.e. the wave equation in the frequency domain) in 2-d or 3-d. We formulate and analyse parallel preconditioners for use with the GMRES (generalised minimum residual) Krylov iterative method. We show (in a sense made precise below) that our preconditioners remain effective as the wavenumber $k$ increases. As $k$ increases, there are several difficulties that make the problem hard, both mathematically and numerically: (i) the solution becomes more oscillatory and, in general, meshes need to be increasingly refined, leading to huge linear systems with dimension growing at least with $O(k^d)$; (ii) the linear systems become more indefinite; (iii) many “standard” preconditioning techniques that are motivated by positive-definite problems become unusable in practice; (iv) there is relatively little rigorous theory for preconditioning such large, indefinite problems.

Our analysis is carried out for the model Helmholtz problem:

\begin{equation}
-\Delta u - (k^2 + i\varepsilon)u = f,
\end{equation}
on an open bounded polygonal (for $d=2$) or Lipschitz polyhedral (for $d=3$) domain $\Omega \subset \mathbb{R}^d$, with mixed boundary conditions

\begin{equation}
\frac{\partial u}{\partial n} - i\eta u = g \quad \text{on} \quad \Gamma_I, \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma_D,
\end{equation}

where the wavenumber $k > 0$, and $\Gamma = \Gamma_I \cup \Gamma_D$ is the boundary of $\Omega$, partitioned into $\Gamma_I$ and $\Gamma_D$, where $\Gamma_I$ has positive measure. In applications, $k = \omega/c$, with $\omega$ the angular frequency and $c$ the wave speed. Here we restrict to the case when $c$ is a positive constant. We allow the absorption parameter $\varepsilon$ to be negative, zero or positive (with $\varepsilon = 0$ corresponding to the “pure Helmholtz” case, which is our main interest); more details on $\varepsilon$ and $\eta$ are given in §2.

In practical wave scattering problems, the PDE (1.1) is commonly posed on the infinite domain exterior to a bounded scatterer, and the infinite domain is then truncated using an artificial boundary. The significance of the impedance boundary condition in (1.2) is that (with $\eta = \sqrt{k^2 + i\varepsilon}$) it is the simplest possible approximation to the Sommerfeld radiation condition. The problem (1.1),

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(1.2) can therefore model acoustic scattering by a sound-soft scatterer. Also included in (1.1), (1.2) is the interior impedance problem, where \( \Gamma_D = \emptyset \), and \( \Gamma_I \) is the boundary of \( \Omega \).

The standard variational formulation for (1.1), (1.2) is: Given \( f \in L^2(\Omega) \), \( g \in L^2(\Gamma_I) \), find \( u \in H^1_0(D) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \} \), such that

\[
(1.3) \quad a_\varepsilon(u,v) = F(v) \quad \text{for all } v \in H^1_0(D),
\]

where

\[
(1.4) \quad a_\varepsilon(u,v) := \int_\Omega \nabla u \cdot \nabla v - (k^2 + \varepsilon) \int_\Omega u \nabla v - \imath \eta \int_{\Gamma_I} u \partial_n v \quad \text{and} \quad F(v) := \int_\Omega f v + \int_{\Gamma_I} g v;
\]

when \( \varepsilon = 0 \) and \( \eta = k \) we write \( a \) instead of \( a_\varepsilon \).

We approximate the variational problem (1.3) in a conforming finite-element space \( V^h \subset H^1_0(D) \) (consisting of continuous piecewise polynomials of arbitrary fixed order), on a mesh \( T^h \) with mesh diameter \( h \), yielding the linear system

\[
(1.5) \quad A_\varepsilon U := (S - (k^2 + \varepsilon)M - \imath \eta N)U = F;
\]

where \( U \) is the vector of nodal values of the finite-element approximation \( u_h \approx u \), \( S \) is the stiffness matrix for the negative Laplace operator, \( M \) is the domain mass matrix and \( N \) is the boundary mass matrix (corresponding respectively to each of the terms in \( a_\varepsilon(u,v) \) in (1.4), and described in more detail in §2.2). \( A_\varepsilon \) is large, sparse, indefinite and generally highly non-normal. When \( \varepsilon = 0 \) we write \( A \) instead of \( A_\varepsilon \).

One way to understand the essential difficulty in preconditioning \( A \) (as \( k \) increases) is to recall that the fundamental solution of the operator in (1.1) with \( \varepsilon = 0 \) (in 3 dimensions) is \( G(x,y) = \exp(ikr)/r \), where \( r = |x-y| \), with \( | \cdot | \) denoting the Euclidean norm, and so a good preconditioner for (1.1) with \( \varepsilon = 0 \) should, roughly speaking, approximate the integral operator with kernel \( G \). When \( k = 0 \) this operator is “data-sparse”, since the \( j \)-th derivative of \( G \) decays with order \( \mathcal{O}(r^{-j+1}) \), when \( x \) and \( y \) are well-separated. Thus, a source in a given region is only felt weakly far away, a fact that underlies many successful preconditioners for Laplace-like problems (e.g. multigrid, domain decomposition, or \( H \)-matrices). However, when \( k \) is large, the \( j \)-th derivative of \( G \) decays with the much slower rate \( \mathcal{O}(k^j r^{-1}) \), and Laplace-like preconditioning strategies fail.

Introducing absorption, \( \varepsilon \neq 0 \), has the effect of improving the decay of the Green’s function. While absorptive problems do appear in applications (and our results here cover these), our deeper motivation for including \( \varepsilon \) is that it has proved useful for both constructing and providing the theory for preconditioners for the case \( \varepsilon = 0 \). In [21] it was proved (subject to certain natural conditions on \( \Omega \) and \( h \)), that there is a constant \( K \), independent of \( h \) and \( \varepsilon \), such that

\[
(1.6) \quad \|I - A_\varepsilon^{-1}A\|_2 \leq K \frac{|\varepsilon|}{k}.
\]

Thus the left-hand side of (1.6) can then be made small by choosing \( \varepsilon \) to be a small-enough multiple of \( k \). However \( A_\varepsilon^{-1} \) is not a practical preconditioner for \( A \), and we therefore seek to replace it by some practical approximation \( B_\varepsilon^{-1} \approx A_\varepsilon^{-1} \).

Using the classical results about GMRES in [11], we say that \( B_\varepsilon^{-1} \) is a good preconditioner for \( A \) if both (i) the matrix \( B_\varepsilon^{-1}A \) has Euclidean norm bounded above, and (ii) the field of values (in the Euclidean norm) of \( B_\varepsilon^{-1}A \) is bounded away from the origin, with both bounds independent of \( \varepsilon \) and \( k \). (If both (i) and (ii) are satisfied then, by [11], GMRES for \( B_\varepsilon^{-1}A \) will converge in a number of iterations independent of \( k \) and \( \varepsilon \).)

In order to find conditions that ensure \( B_\varepsilon^{-1} \) will be a good preconditioner for \( A \), we can proceed by writing

\[
(1.7) \quad B_\varepsilon^{-1}A = B_\varepsilon^{-1}A_\varepsilon - B_\varepsilon^{-1}A_\varepsilon(I - A_\varepsilon^{-1}A).
\]

Then (1.6) combined with (1.7) suggest that \( B_\varepsilon^{-1} \) will be a good preconditioner for \( A \) provided that

\[
(1.8) \quad B_\varepsilon^{-1} \text{ is a good preconditioner for } A_\varepsilon \text{ when } |\varepsilon| = ck, \text{ with } c \text{ sufficiently small.}
\]
(An argument making this statement rigorous, under some technical assumptions, is given in Appendix §A).

The main theoretical results of this paper show that (i) $B^{-1}_\varepsilon$ is a good preconditioner for $A_\varepsilon$ when $|\varepsilon| \sim k^{1+\beta}$, with $\beta$ arbitrarily close to $0$; and (ii) there exists a $C > 0$ such that $B^{-1}_\varepsilon$ is a good preconditioner for $A_\varepsilon$ when $|\varepsilon| = Ck$. Both results require $H$ and $\delta$ to be chosen appropriately; see Section 3.3 (in particular Corollary 3.11).

In a previous work [24] we analysed classical additive Schwarz methods (i.e. those originally designed for Laplace-type operators and using local Dirichlet conditions on subdomains) when applied to (1.5). For these we could only show that $B^{-1}_\varepsilon$ is a good preconditioner for $A_\varepsilon$ when $|\varepsilon| \sim k^2$, so there was a large gap from the requirement $|\varepsilon| \sim k$. Nevertheless, the methods in [24] still provided efficient solvers in practice, especially when implemented in their multilevel variants [25, 6], and this approach has since been extended both in theory and practice to the Maxwell equations [4, 5]. This approach, however, is limited since it applies “elliptic technology” (local Dirichlet solves) to a wave-like problem. An analogous situation occurs in the theory of shifted Laplace preconditioners; these precondition $A$ by applying geometric multigrid (an “elliptic technology”) to the absorptive problem $A_\varepsilon$, but are only robust as $k \to \infty$ when $|\varepsilon| \sim k^2$ [10].

In this paper we analyse the effect of introducing more “wave-friendly” subdomain problems (namely local impedance problems) into the preconditioner. Whilst the effect of “wave-friendly” subdomain problems has been analysed for domain-decomposition (DD) solvers for certain geometries (e.g. subdomains being infinite strips or half planes), this paper is the first to analyse preconditioners (rather than solvers) and also the first to treat general geometries and decompositions. The improvement is quite dramatic as we see in §§1.2 and 4.

1.1. The preconditioner. Our algorithm is a variation on one of the simplest domain-decomposition methods – the one-level additive Schwarz method – based on a set of polyhedral subdomains $\{\Omega_\ell\}_{\ell=1}^N$, forming an overlapping cover of $\Omega$, but otherwise having quite general geometries. We assume that each $\Omega_\ell$ is a union of elements of the mesh $T^h$ and we assume that the mesh resolves the interface $\Gamma_f \cap \Gamma_D$ (when this is non-empty). The key component of the preconditioner for (1.5) is the solution of discrete “local” versions of (1.1):

\begin{equation}
-\Delta u - (k^2 + i\varepsilon)u = f \quad \text{on } \Omega_\ell,
\end{equation}

subject to boundary conditions

\begin{equation}
\frac{\partial u}{\partial n} - i\varepsilon u = 0 \quad \text{on } \partial\Omega_\ell \setminus \Gamma_D, \quad u = 0 \quad \text{on } \partial\Omega_\ell \cap \Gamma_D.
\end{equation}

To knit these local problems together, we use a partition of unity $\{\chi_\ell\}_{\ell=1}^N$ (i.e. $\text{supp } \chi_\ell \subseteq \Omega_\ell$ for each $\ell$ and $\sum_\ell \chi_\ell \equiv 1$ on $\Omega$).

The finite-element space $V^h \subset H^1_D(\Omega)$ underlying (1.5) is assumed to have a nodal basis. By this we mean that each $v_h \in V^h$ is uniquely determined by its values $(V_p := v_h(x_p), p \in T^h)$, at nodes $\{x_p : p \in T^h\} \subseteq \Omega$ (where $T^h$ is a suitable index set). Nodes on the subdomain $\Omega_\ell$ are denoted $\{x_p : p \in T^h(\Omega_\ell)\}$. Using this notation, we can define a restriction matrix $R_\ell$ that uses $\chi_\ell$ to map a nodal vector defined on $\Omega$ to a nodal vector on $\Omega_\ell$:

\begin{equation}
(R_\ell V)_p = \chi_\ell(x_p)V_p, \quad p \in T^h(\Omega_\ell).
\end{equation}

We denote by $A_{\varepsilon,\ell}$ the matrix obtained by approximating (1.9) and (1.10) in $V^h$ (restricted to $\Omega_\ell$); this matrix is a local analogue of the matrix $A_\varepsilon$ in (1.5). Our preconditioner for $A_\varepsilon$ is then simply:

\begin{equation}
B^{-1}_\varepsilon := \sum_{\ell=1}^N R_\ell^T(A_{\varepsilon,\ell})^{-1}R_\ell,
\end{equation}

where $R_\ell^T$ is the transpose of $R_\ell$. Hence the action of $B^{-1}_\varepsilon$ consists of $N$ parallel “local impedance solves” added up with the aid of appropriate restrictions/prolongations.

The main results of the paper are estimates for the norm and field of values of the preconditioned matrix $B^{-1}_\varepsilon A_\varepsilon$, which in turn (following (1.8)) provide pointers to good preconditioners for $A$. To state the main results, we introduce the $k$-dependent inner product and norm:

\begin{equation}
\langle V, W \rangle_{D_k} := W^*D_kV, \quad \|V\|_{D_k} = \langle V, V \rangle_{D_k}^{1/2}, \quad \text{where } D_k = (S + k^2M).
\end{equation}
In fact, $D_k$ is stiffness matrix arising from approximating the Helmholtz energy

\[(1.14) \quad \|v\|_{1,k} := (v, v)_{1,k}^{1/2} \quad \text{where} \quad (v, w)_{1,k} := (\nabla v, \nabla w)_{L^2(\Omega)} + k^2(v, w)_{L^2(\Omega)}.
\]

using the Galerkin method in $V^h$. When $D$ is any subdomain of $\Omega$ we write $(\cdot, \cdot)_{1,k,D}$ and $\| \cdot \|_{1,k,D}$ for the corresponding inner product and norm on $D$. The preconditioner (1.12) with $R_\ell$ defined using a partition of unity coincides with the “OBDD-H” preconditioner of Kimm and Sarkis in [29]. However there is no existing theory for (1.12) applied to the Helmholtz equation with $k$ large. In this paper we present such a theory, justifying the robustness of (1.12) as $k$ increases.

1.2. The main result. Our main results are contained in Section 3.3 below; we give a particular case of them here as Theorem 1.1 to illustrate the principal features. The parameters in the results are $k$, $\varepsilon$, and $\eta$ (appearing in (1.1) and (1.2)), $h$ (the mesh diameter), $\Lambda$ (the maximum number of subdomains $\Omega_\ell$ that any point in $\Omega$ can belong to), $H$ (the upper bound on the subdomain characteristic diameter), and $\delta \leq H$ (the overlap of the subdomains).

**Theorem 1.1.** Suppose $k$ and $\varepsilon$ satisfy the natural assumptions in (2.1) below, and suppose that either $\eta = \text{sign}(\varepsilon) k$ or $\eta = \sqrt{k^2 + 2\varepsilon}$. Suppose the mesh diameter $h$ and overlap parameter $\delta$ are chosen to satisfy

\[(1.15) \quad (a) \quad kh \text{ bounded, and } (b) \quad k\delta \to \infty, \quad \text{as} \quad k \to \infty.
\]

Let $\sigma > 0$ be such that

\[\|A^{-1}_\varepsilon R_\ell A_\varepsilon - R_\ell\|_{D_k} \leq \sigma, \quad \ell = 1, \ldots, N.
\]

Then there exists a constant $C_1$ (independent of $h, k, \Lambda, \varepsilon$ and $\sigma$) such that

\[(1.16) \quad \|B_\varepsilon^{-1} A_\varepsilon\|_{D_k} \leq C_1 \Lambda \left(1 + \sigma + \frac{1}{k\delta}\right)^2.
\]

If in addition

\[(1.17) \quad \sigma < \frac{1}{\sqrt{2} \Lambda^2},
\]

then there exists a constant $C_2$ (independent of $h, k, \Lambda, \varepsilon, \sigma$) such that

\[(1.18) \quad \min_{\mathbf{V} \in \mathbb{C}^n} \frac{|(V, B_\varepsilon^{-1} A_\varepsilon V)_{D_k}|}{\|V\|_{D_k}^2} \geq \left(1 - \sqrt{2} \Lambda \right) - C_2 \Lambda \left(1 + \frac{1}{k\delta}\right).
\]

Using the “Elman estimate” [11], this result immediately implies a $k$-independent bound for the iteration count of GMRES applied to $B_\varepsilon^{-1} A_\varepsilon$ (working in the $D_k$ inner product), provided that $\sigma < 1/\sqrt{2} \Lambda^2$. This result about left preconditioning (working in the $D_k$ inner product) can be easily converted into a result about right preconditioning (working in the $D_k^{-1}$ inner product); see Remark 3.14 below. We therefore have the following corollary.

**Corollary 1.2.** Assume that (1.15) and (1.17) hold. If GMRES is applied to the linear system (1.5), with $B_\varepsilon^{-1}$ as a left (or right) preconditioner in the inner product induced by $D_k$ (or $D_k^{-1}$), then the number of iterations needed to achieve a prescribed accuracy remains bounded as $k \to \infty$.

We now discuss the implications of the assumptions (1.15) and (1.17). First, (1.15) is easy to satisfy: (a) is automatic if the finite-element mesh is chosen to maintain accuracy as $k \to \infty$ (in fact $h$ has to decrease faster than $O(k^{-1})$ if the pollution effect is to be avoided for fixed-order methods [28, 35]). The constraint (b) is satisfied by subdomains with overlap with $\delta \sim O(k^{-\alpha})$ for any $\alpha \in [0, 1)$, for example if the subdomains have diameter $H = O(k^{-\alpha})$, and we allow “generous overlap” $\delta \sim H$. However (b) is not satisfied if $\delta \sim h$ (“minimal overlap”), because of (a). Second, (1.17) is a stronger constraint and may lead to restrictions on $\varepsilon$ and $H$. Essentially it says that for each $\ell$, the “local impedance solve” $A_\varepsilon^{-1}$ should be a sufficiently good left inverse for $A_\varepsilon$ when it is restricted to $\Omega_\ell$. 

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In Corollary 3.11 below we give scenarios in which $\sigma$ can be controlled. We find that $\sigma$ is bounded above, independent of all parameters and for all $0 \leq |\varepsilon| \leq k^2$, if the subdomains are star-shaped with respect to a ball, $h$ is small enough, and there is generous overlap $\delta \sim H$. Therefore, by (1.16), in this scenario $\|B_k^{-1}A\|_{D_k}$ is bounded above, independent of $k$, $h$, and $H$, even in the pure Helmholtz case $\varepsilon = 0$. Furthermore, we show that the more restrictive bound (1.17) holds if $H$ is kept fixed and $\varepsilon = Ck$ for $C$ large enough. While it is not known if this constant $C$ can be chosen less than the constant $c$ appearing in (1.8), this analysis gives a strong indication that $B_k^{-1}$ should be a good preconditioner for $A$ when $H$ is fixed. The following initial experiment shows this is indeed the case.

**Experiment 1.** Table 1 gives results where (1.5) with $\varepsilon = 0$ and $h \sim k^{-3/2}$ is solved by GMRES with preconditioner given by (1.12) with $\varepsilon = k$. There are $(M + 1)^2$ uniform subdomains (fixed as $k$ increases) and chosen to be the supports of the piecewise bilinear basis functions on a square grid of elements of size $1/M \times 1/M$. These bilinear functions also provide the partition of unity. We observe in Table 1 that this method appears remarkably robust and in fact the number of iterations even appears to slightly decrease as $k$ increases. Full details, including the choice of source, boundary data, and starting guess, are given in §4. More variations on this experiment are given in Experiment 5 in §4. We also see in §4 that this preconditioner appears to remain robust even when $H$ decreases (moderately) as $k \to \infty$ and we give theoretical support for this observation in §3.

| $k \setminus M$ | 4 | 8 | 16 |
|------------------|---|---|----|
| 40               | 12| 27| 61 |
| 60               | 11| 25| 56 |
| 80               | 10| 22| 52 |
| 100              | 9 | 21| 48 |
| 120              | 9 | 20| 45 |
| 140              | 8 | 18| 41 |
| 160              | 7 | 19| 40 |

Table 1
GMRES iterations for $B_k^{-1}A$ with $M^2$ fixed subdomains of size $1/M \times 1/M$

Note that Corollary 1.2 concerns the behaviour of GMRES when applied using certain weighted inner products (e.g. $\langle \cdot, \cdot \rangle_{D_k}$). In [24, Experiment 1] we compared this “weighted GMRES” with standard GMRES (using the Euclidean inner product) for similar DD methods applied to Helmholtz problems and observed little difference in iteration counts. Thus, in this and all later experiments we used standard GMRES.

**Remark 1.3.** It is perhaps remarkable that a one-level additive Schwarz method (with no coarse grid) can be robust when the subdomain size $H \to 0$. This conflicts with standard intuition for one-level methods for self-adjoint coercive PDEs (e.g. Poisson’s equation); there, if $H \to 0$, the condition number of the preconditioned problem grows with $O((\delta H)^{-1})$. In the Helmholtz case, we are solving a family of problems parametrized by $k$. Even though the problem itself becomes “harder” as $k$ increases, the one level preconditioner can still remain robust; this is one of the significant contributions of the current work. Further discussion comparing the Helmholtz case with the self-adjoint coercive case is given in Appendix B.

1.3. Discussion of related literature. There have been two important recent ideas that have had a large effect on the field of iterative solvers for the Helmholtz equation.

The first is the “shifted Laplace” preconditioner, arising from initial ideas by in [1] and [32], and then developed and advocated in [18, 16, 43]. Since the fundamental solution of (1.1) enjoys “Laplace-like” decay when $\varepsilon$ is large enough, the “shifted Laplace” preconditioner uses a multigrid approximation of the absorptive problem to precondition the “pure Helmholtz” problem $\varepsilon = 0$.

The second idea concerns a class of multiplicative domain-decomposition methods that fall under the general heading of “sweeping”, e.g. [12, 13, 14, 41, 8, 38, 44, 22]. To describe these in a simple context, suppose (1.1) is discretized on a tensor product grid on a rectangular domain $\Omega$ and suppose the finite-element nodes all lie in one or other of two non-overlapping subdomains
\[ \Omega_1, \Omega_2 \] as shown in the figure below.

![Diagram of \( \Omega_1, \Omega_2 \)]

Writing the resulting finite-element equations as \( Au = f \), blocking this system according to the domain decomposition, and applying block Gaussian elimination we obtain:

\[
\begin{bmatrix}
  A_{1,1} & A_{1,2} \\
  A_{2,1} & A_{2,2}
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
= \begin{bmatrix}
  f_1 \\
  f_2
\end{bmatrix}
\implies
\begin{bmatrix}
  A_{1,1} & A_{1,2} \\
  0 & S
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
= \begin{bmatrix}
  f_2 - A_{2,1}A_{1,1}^{-1}f_1
\end{bmatrix}
\]

where \( S := A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2} \) is the Schur complement. Therefore, \( u_2 \) can be computed by first computing (via a solve on \( \Omega_1 \)) the “modified source” \( f_2 - A_{2,1}A_{1,1}^{-1}f_1 \) on \( \Omega_2 \), and then forming and inverting the Schur complement \( S \) on \( \Omega_2 \); subsequently, \( u_1 \) is found by back substitution. The bulk of the work involves the formation and inversion of the Schur complement \( S \). However, considering the same system again, but this time with \( f_1 = 0 \), we see that the action of \( S^{-1} \) is obtained by solving a Helmholtz problem on all of \( \Omega \) but having data confined to \( \Omega_2 \) and observing the solution only on \( \Omega_2 \). In [33] it is shown that (in 2-d) the action of \( S^{-1} \) can be expected to be data sparse, even when \( k \) is large, in the special case when \( \Omega_2 \) contains only a few lines of finite-element nodes. This result underlies the “sweeping method” in which block-elimination methods are implemented recursively by approximating the Schur complements, either by “moving perfectly-matched layer (PML)” or \( \mathcal{H} \)-matrix approximation.

Both these ideas have substantial limitations. For “sweeping”, the theory applies only to rectangular 2-d domains and tensor-product discretizations (since the low-rank result [33] does not hold for general domains and discretizations [15]), and to the elimination of nodes in blocks, each consisting of a small number of rows. Although the inner solves in each multiplicative sweeping step can be parallelized [38, 44], general parallelisation strategies are restricted by the inherently serial structure of sweeping methods. On the other hand the “shifted Laplace” algorithm has been applied to substantial industrial problems, but is not in general robust with respect to \( k \), since the choice \( |\epsilon| \sim k^2 \), which is needed to make multigrid work [10] turns out to be too large a perturbation of the pure Helmholtz problem to remain robust as \( k \to \infty \). However very efficient versions of the shifted Laplace preconditioner are available, especially those which employ deflation techniques [40, 39, 17]. A number of recent developments in the theory and application of shifted Laplace and related preconditioners is given in [31].

Unlike multigrid, domain-decomposition methods offer the attractive feature that their coarse grid and local problems can be adapted to allow for “wave-like” behaviour. There is a large literature on this (e.g. [3, 19, 20, 29, 30, 23]), but there is no rigorous theory when \( k \) is large, for methods with \( \textit{either} \) many subdomains of general shape \( \textit{or} \) coarse grids. The paper [24] provided the first such rigorous analysis for the problem with absorption, but the bounds for \( |\epsilon| \ll k^2 \) in [24] were very pessimistic. The current paper extends this line of research to the case when wave-like components are inserted into the domain-decomposition method. The results we obtain for the one-level method (i.e. with no coarse solver) with impedance boundary conditions on the subdomains give practical bounds for much lower levels of absorption than in [24].

**1.4. Structure of the paper.** In §2 we define the preconditioner and the underlying theoretical assumptions. As usual in domain-decomposition theory, the preconditioned matrix is identified with a projection onto local finite-element spaces, in this case corresponding to solutions of local impedance problems. In our analysis, a key rôle is played by estimates for the local impedance solution operator at the continuous (PDE) level; these are given in §2.1. In §3 we prove the main results, including Theorem 1.1 and Corollary 1.2. In §4 we give numerical results. In Appendix
A we give a rigorous basis for the discussion around (1.8), and in Appendix B we outline the key differences between the theory we have developed here and the standard projection-operator analysis for self-adjoint coercive elliptic problems.

2. Definition of the preconditioner and associated results. Throughout we write \( a \lesssim b \) when there exists a \( C > 0 \), independent of all parameters of interest (here \( \varepsilon, k, h, H, \delta, A, \) and \( \ell \) - with some of these defined later), such that \( a \leq C b \). We write \( a \sim b \) if \( a \lesssim b \) and \( b \lesssim a \). We make the following basic assumption on \( k, \varepsilon \) and \( \eta \) throughout the paper.

**Assumption 2.1.** The parameters \( k, \varepsilon \) and \( \eta \) satisfy

\[
k \gtrsim 1, \quad 0 \leq |\varepsilon| \leq k^2, \quad \text{and} \quad |\eta| \sim k.
\]

We recall the inequalities (valid for all \( a, b > 0 \) and \( \varepsilon > 0 \)),

\[
2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2, \quad \text{and} \quad \frac{1}{\sqrt{2}}(a + b) \leq \sqrt{a^2 + b^2} \leq a + b.
\]

2.1. A priori estimates. The basic well-posedness of (1.3) is classical:

**Proposition 2.2.** If either (i) \( \varepsilon > 0 \) and \( \Re(\eta) > 0 \), or (ii) \( \varepsilon < 0 \) and \( \Re(\eta) < 0 \), or (iii) \( \varepsilon = 0 \) and \( \Re(\eta) \neq 0 \), the problem (1.4) has a unique solution.

Uniqueness is established by taking the imaginary part of (1.4) and existence then follows via the Fredholm alternative, since \( a_\varepsilon \) satisfies a Gårding inequality.

In the domain-decomposition method below we will be interested in local impedance solves on subdomains that may shrink in diameter as \( k \to \infty \). For this reason we introduce the following notion.

**Definition 2.3 (Characteristic length scale).** A domain has characteristic length scale \( L \) if its diameter \( \sim L \), its surface area \( \sim L^{d-1} \), and its volume \( \sim L^d \).

**Lemma 2.4 (Continuity and coercivity of the sesquilinear form \( a_\varepsilon \)).**

(i) Assume that \( \Omega \) has characteristic length scale \( L \) and that \( \varepsilon \) and \( \eta \) satisfy (2.1). Then the sesquilinear form \( a_\varepsilon \) is continuous, i.e.

\[
|a_\varepsilon(u, v)| \leq C_{\text{cont}}\|u\|_{1,k}\|v\|_{1,k}, \quad \text{with} \quad C_{\text{cont}} \lesssim 1 + \frac{|\eta|}{k} \left( 1 + \frac{1}{kL} \right),
\]

for all \( u, v \in H^1(\Omega) \).

(ii) Let \( \sqrt{k^2 + i\varepsilon} \) be defined via the square root with the branch cut on the positive real axis. If \( \eta \) satisfies

\[
\Re \left( \eta \sqrt{k^2 + i\varepsilon} \right) \geq 0,
\]

then \( a_\varepsilon \) is coercive, i.e.

\[
|a_\varepsilon(v, v)| \gtrsim C_{\text{coer}}\|v\|_{1,k}^2, \quad \text{with} \quad C_{\text{coer}} \sim \frac{|\varepsilon|}{k^2},
\]

for all \( v \in H^1(\Omega) \).

**Proof.** The assertion (ii) is Lemma 2.4 in [24] (note that the omitted constants in that result do not depend on \( L \)). The assertion (i) follows from the Cauchy-Schwarz inequality and the multiplicative trace inequality,

\[
\|v\|^2_{L^2(\Gamma)} \lesssim \left( \frac{1}{L} \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \right)
\]

(see, e.g., [26, Last equation on p. 41]) and the inequalities (2.2).

**Remark 2.5 (Adjoint coercivity).** The definition of \( \sqrt{k^2 + i\varepsilon} \) implies that when \( \eta \) is chosen to satisfy (2.3), the coercivity constant for \( a_\varepsilon \) is exactly the same as the coercivity constant for the sesquilinear form for the adjoint problem obtained by replacing \( \varepsilon \) by \( -\varepsilon \) and \( \eta \) by \( -\eta \).
Recall that a Lipschitz open set \( D \) is called *starshaped with respect to a ball* if there exists \( a > 0 \) such that \((x - x_0) \cdot n(x) \geq a\) for all \( x \in \partial D \) for which the normal vector \( n(x) \) is defined; see, e.g., [36, Lemma 5.4.1].

**Theorem 2.6 (A priori bound on solution of (1.3)).** Let \( \Omega \) be starshaped with respect to a ball and have characteristic length scale \( L \), and recall that we have assumed that \( \Gamma_1 \) has positive measure. Let \( u \) be either the solution to (1.3) with \( f \in L^2(D) \) and \( g = 0 \), or the solution to the adjoint problem under the same assumptions on \( f \) and \( g \). Then, given \( k_0 > 0 \), there exists \( C_1, C_2 \) (independent of \( k, \varepsilon, \eta \), and \( L \)) such that, when \( k \geq k_0 \),

\[
\|u\|_{1,k} \leq C_1 L \|f\|_{L^2(\Omega)}
\]

provided that

\[
\frac{|\varepsilon|L}{k} \leq C_2.
\]

**Proof.** This result is essentially given by [21, Theorem 2.9 and Remark 2.5], except the dependence of the constants on \( L \) is not kept track of there. To see that the condition \( |\varepsilon|k \leq c \) in [21, Theorem 2.9] is really (2.5), one needs to examine the theorem near the end of the proof of [21, Theorem 2.9] (just before Remark 2.16) and observe that \( R := \sup_{x \in \Omega} |x| \sim L \). To see why the bound (2.4) has the factor of \( L \) on the right-hand side, observe that choosing \( \delta_1 = 1/(2R) \) and \( \delta_2 \sim k^2 \) in the proof of [21, Theorem 2.9] means that, in [21, (2.29)], the factor multiplying \( \|f\|_{L^2(\Omega)} \) is \( L^2 \). (The \( L \)-explicit bound (2.4) in the case \( \varepsilon = 0 \) is also obtained in [37, Remark 3.6].) \( \Box \)

For simplicity, in the rest of the paper we assume that either \( \eta = \text{sign}(\varepsilon) k \) or \( \eta = \sqrt{k^2 + i\varepsilon} \); observe that both these choices satisfy the requirements on \( \eta \) in (2.1), the conditions for uniqueness of the solution of (1.3) in Proposition 2.2, and the more-restrictive condition for coercivity (2.3) (see [24, Remark 2.5]).

### 2.2. Finite element method

Let \( T^h \) be a family of conforming simplicial meshes that are shape regular as the mesh diameter \( h \to 0 \). A typical element of \( T^h \) is written \( \tau \in T^h \) and is considered as a closed subset of \( \Omega \). Our approximation space \( V^h \) is then the space of all continuous functions on \( \Omega \) that are polynomial of (total) degree \( r - 1 \) with \( r \geq 2 \) (when restricted to any \( \tau \)) and vanish on \( \Gamma_D \). We assume we have a nodal basis for this space (for example the standard Lagrange basis), i.e. with nodes \( \mathcal{N}^h = \{ x_q : q \in T^h \} \), where \( T^h \) is a suitable index set and corresponding basis \( \{ \phi_p : p \in T^h \} \) with \( \phi_p(x_q) = \delta_{pq} \). For any continuous function \( g \) on \( \Pi \), we introduce the standard nodal interpolation operator

\[
\Pi^h g = \sum_{p \in T^h} g(x_p) \phi_p.
\]

We assume that \( V^h \) satisfies the standard error estimate (e.g. [9, §3.1]):

\[
\|(I - \Pi^h)v\|_{L^2(\tau)} + h|(I - \Pi^h)v|_{H^r(\tau)} \leq Ch^r|v|_{H^r(\tau)},
\]

for each \( \tau \in T^h \), provided \( v \in H^r(\tau) \). The Galerkin approximation of (1.3) in the space \( V^h \) is equivalent to the linear system (1.5) where \( F_\ell := \int_\Omega f \phi_\ell + \int_{\Gamma_1} g \phi_\ell \), and

\[
S_{\ell,m} = \int_\Omega \nabla \phi_\ell \cdot \nabla \phi_m, \quad M_{\ell,m} = \int_\Omega \phi_\ell \phi_m, \quad N_{\ell,m} = \int_{\Gamma} \phi_\ell \phi_m, \quad \ell, m \in T^h.
\]

### 2.3. Overlapping covering and local problems

We introduce a set of subdomains \( \{ \Omega_\ell : \ell = 1, \ldots, N \} \) that form an overlapping cover of \( \Omega \). Each \( \Omega_\ell \) is assumed to be non-empty and to consist of a union of elements of the mesh \( T^h \). We assume that the subdomains \( \Omega_\ell \) are Lipschitz polyhedra (polygons in 2-d) that are shape regular with parameter \( H_\ell \) in the sense that each \( \Omega_\ell \) has characteristic length scale \( H_\ell \), and we set \( H = \max_\ell H_\ell \). In our analysis we allow \( H \) to depend on \( k \) in such a way that \( H \) could approach 0 as \( k \to \infty \). Because each subdomain is a union of one or more fine grid elements we have \( h \leq H \).
Let $\Omega_{\ell,b} \subset \Omega_{\ell} \setminus \hat{\Omega}_{\ell} \subset \Omega_{\ell,\delta}$; the case $\delta \sim H$ is called the generous overlap.

We make the finite-overlap assumption that there exists a finite $\Lambda > 1$ independent of $N$ such that
\begin{equation}
\Lambda = \max \{ \# \Lambda(\ell) : \ell = 1, \ldots, N \}, \quad \text{where} \quad \Lambda(\ell) = \{ \ell' : \Omega_{\ell} \cap \Omega_{\ell'} \neq \emptyset \}.
\end{equation}

It follows immediately from (2.10) that, for all $v \in L^2(\Omega)$,
\begin{equation}
\sum_{\ell=1}^N \|v\|^2_{L^2(\Omega_{\ell})} \leq \Lambda \|v\|^2_{L^2(\Omega)}
\end{equation}
and thus also
\begin{equation}
\sum_{\ell=1}^N \|v\|^2_{1,k,\Omega_{\ell}} \leq \Lambda \|v\|^2_{1,k},
\end{equation}
when $v \in H^1(\Omega)$.

We describe this property by saying the overlapped $\Omega_{\ell}$ are starshaped with respect to a ball, uniformly in $\ell$.

Concerning the overlap, for each $\ell = 1, \ldots, N$, let $\Omega_{\ell}$ denote the part of $\Omega_{\ell}$ that is not overlapped by any other subdomains. (Note that $\Omega = \emptyset$ is possible.) For $\mu > 0$ let $\Omega_{\ell,\mu}$ denote the set of points in $\Omega_{\ell}$ that are a distance no more than $\mu$ from the interior boundary $\partial \Omega_{\ell} \setminus \Gamma$. Then we assume that there exist $h \leq \delta \lesssim H$ and $0 < b < 1$ such that, for each $\ell = 1, \ldots, N$,
\begin{equation}
\Omega_{\ell,\delta} \subset \Omega_{\ell} \setminus \hat{\Omega}_{\ell} \subset \Omega_{\ell,\delta};
\end{equation}
for each $\ell$, we introduce the space of finite-element functions on the finite-element mesh restricted to $\Omega_{\ell}$ and denote these spaces by $V^h_{\ell}$; i.e. $V^h_{\ell} := \{ \psi_h : \psi_h \in V^h \}$. Recall that functions in $V^h$ vanish on the (outer) Dirichlet boundary $\Gamma_D$. Thus functions in $V^h_{\ell}$ also vanish on $\partial \Omega_{\ell} \cap \Gamma_D$, but are otherwise unconstrained. The nodes for functions in $V^h_{\ell}$ are denoted by $N^h(\Omega_{\ell}) = \{ x_j : j \in I^h(\Omega_{\ell}) \}$, for some suitable index set $I^h(\Omega_{\ell})$. The local impedance sesquilinear form on $\Omega_{\ell}$ is
\begin{equation}
a_{\varepsilon,\ell}(v, w) := \int_{\Omega_{\ell}} (\nabla v \cdot \nabla \overline{w} - (k^2 + i\varepsilon)v\overline{w}) - i\eta \int_{\partial\Omega_{\ell} \setminus \Gamma_D} \overline{v}w,
\end{equation}
for $v, w \in H^1(\Omega_{\ell})$. Given $F_{\ell} \in (H^1(\Omega_{\ell}))'$, the continuous local impedance problem is: find $u_{\ell} \in H^1(\Omega_{\ell})$ such that
\begin{equation}
a_{\varepsilon,\ell}(u_{\ell}, v_{\ell}) = F_{\ell}(v_{\ell}), \quad \text{for all} \quad v_{\ell} \in H^1(\Omega_{\ell});
\end{equation}
for all $v_{\ell} \in L^2(\Omega_{\ell})$.

The finite-element approximation of (2.13) is: find $u_{h,\ell} \in V^h_{\ell}$ such that
\begin{equation}
a_{\varepsilon,\ell}(u_{h,\ell}, v_{h,\ell}) = F_{\ell}(v_{h,\ell}), \quad \text{for all} \quad v_{h,\ell} \in V^h_{\ell}.
\end{equation}

Let
\begin{equation}
(A_{\varepsilon,\ell})_{i,j} := a_{\varepsilon,\ell}(\phi_j, \phi_i) \quad \text{for} \quad i, j \in I^h(\Omega_{\ell}).
\end{equation}

The next theorem gives conditions under which (2.14) has a solution, or equivalently under which (2.15) is invertible.

**Theorem 2.7** (Bounds on the solutions of the local problems (2.14)). Assume that either
\begin{itemize}
\item[(i)] for all $|\varepsilon| > 0$, and for any mesh size $h$, (2.14) has a unique solution $u_{h,\ell}$ which satisfies
\begin{equation}
\|u_{h,\ell}\|_{1,k,\Omega_{\ell}} \lesssim \Theta(\varepsilon, H, k) \max_{v_{h} \in V^h_{\ell}} \left( \frac{||F(v_{h})||}{\|v_{h}\|_{1,k,\Omega_{\ell}}} \right),
\end{equation}
with
\begin{equation}
\Theta(\varepsilon, H, k) = \frac{k^2}{|\varepsilon|}.
\end{equation}
\end{itemize}
(ii) If each $\Omega_\ell$ is starshaped with respect to a ball uniformly in $\ell$, then for all $|\varepsilon| \geq 0$, there exists a mesh threshold function $\overline{h}(k, r)$ such that when $h \leq \overline{h}(k, r)$, (2.14) has a unique solution $u_{h, \ell}$ which satisfies (2.16) with

$$\Theta(\varepsilon, H, k) = \min \left\{ (1 + Hk), \frac{k^2}{|\varepsilon|} \right\}$$

where we adopt the convention that $\Theta(0, H, k) = 1 + Hk$.

Proof. The result (i) is a consequence of Lemma 2.4 and the Lax-Milgram lemma. The result (ii) follows from the fact (used in the case of Helmholtz problems by the authors of [34] and their associated work) that when a sesquilinear form satisfies a Gårding inequality and the solution of the variational problem is unique, a “Schatz-type” argument obtains quasi-optimality under conditions on the approximability of the adjoint problem, and then the Gårding inequality can be used to verify a discrete inf-sup condition. Indeed, following the proof of [34, Theorem 4.2] and using the bound (2.4) and the fact that $\Omega_\ell$ has characteristic length scale $h_\ell \leq H$, we find that, when $|\varepsilon|H/k \leq C_2$, (2.15),

$$\inf_{0 \neq v_h \in V_h^0} \sup_{0 \neq w_h \in V_h^0} \frac{|a_{\varepsilon, r}(v_h, w_h)|}{\|v_h\|_{1,k} \|w_h\|_{1,k}} \geq \frac{1}{2 + C_{cont}^{-1} + C_1 kH}.$$ 

Then, from (2.14),

$$\|u_{h, \ell}\|_{1,k, \Omega_\ell} \lesssim (1 + kH) \sup_{0 \neq v_h \in V_h^0} \frac{|F(v_h)|}{\|v_h\|_{1,k, \Omega_\ell}},$$

when $|\varepsilon|H/k \leq C_2$. If $|\varepsilon|H/k > C_2$, then $1 + Hk > C_2 k^2/|\varepsilon|$ and the estimate (2.18) follows from (2.16). 

Remark 2.8 (The mesh-threshold function $\overline{h}(k, r)$). Bounds on $\overline{h}(k, r)$ are discussed in detail in [35, §5.1.2 and 5.2]. For 2-d polygonal domains, $k(hk/(r-1))^{-1}$ is required to be sufficiently small [35, Equation 5.13], equivalently $h$ being a sufficiently small multiple of $(r-1)k^{-1/(r-1)}$. Therefore, when $r = 2$ we require $hk^2$ small, but the requirement relaxes as $r$ increases. In 1-d, numerical experiments indicate that the requirement $hk^2$ is necessary for quasioptimality [28, Figures 7-9], [27, §4.5.4 and Figure 4.12] but the relative error in both the $H^1$-semi-norm and the $L^2$-norm is bounded independently of $k$ if $hk^{3/2}$ is sufficiently small [28, Equation 3.25], [27, Equation 4.5.15], with numerical experiments indicating that this is sharp [28, Figure 11], [27, Figure 4.13]. The numerical experiments in [2, §3] indicate that, at least for certain 2-d problems, the relative error in the $L^2$-norm is bounded independently of $k$ if $hk^{3/2}$ is sufficiently small, although this has yet to be proven.

2.4. Partition of unity, restriction, and prolongation. Recall the partition of unity $\{\chi_\ell : \ell = 1, \ldots, N\}$ and the restriction and prolongation matrices $R_{\ell, r}, R_{\ell, r}^T$ defined in §1.1. Note that since the subdomains are assumed to be unions of fine grid elements, their boundaries (and the boundaries of their supports) are fine-grid dependent. This is standard for domain-decomposition methods (e.g. [42, p. 57]). The functions $\chi_\ell$ do not have to be smooth globally but need to be sufficiently smooth elementwise on the fine mesh; we assume

$$\|D^\beta \chi_\ell \|_{\infty, \tau} \lesssim \delta^{-|\beta|}, \quad \text{for all } \tau \in \mathcal{T}_h, \quad \text{and for all } |\beta| \leq r,$$

where the hidden constant is also required to be independent of $\tau$ and of the multi-index $\beta$.

With $\Pi^h$ denoting the nodal interpolant, we make frequent use of the operator $\Pi^h \circ \chi_\ell$, which provides a prolongation from $V_h^0$ to $V^h$. In fact if $w_{h, \ell} \in V_h^0$ with nodal values $W$, then

$$\Pi^h(\chi_\ell w_{h, \ell}) = \sum_{p \in Z^h} (R_{\ell, r}^T W)_p \phi_p.$$ 

2.5. Definition of the preconditioner and associated projections. With $A_{\varepsilon, \ell}$ defined by (2.15), the corresponding one-level additive Schwarz preconditioner is defined by (1.12). In
Consequently, for any \(u \in V^h\) given by

\[ (2.25) \quad (u_h, Q^h_{\varepsilon} v_h)_{1,k} = \langle U, B^{-1} A_h V \rangle_{D_k}. \]

Proof. With \(W\) as given in (2.24), we have \((A_{\varepsilon, \ell} W)_q = (R_{\ell} A V)_q\), for all \(q \in \mathcal{I}^{h}(\Omega)\), and so (recalling the definition of \(R_{\ell}\) in (1.11)),

\[
\sum_{p \in \mathcal{I}^{\varepsilon}(\Omega)} a_{\varepsilon, \ell}(\phi_q, \phi_q) W_p = \chi_{\ell}(x_q) \sum_{p \in \mathcal{I}^{\varepsilon}(\Omega)} a_{\varepsilon}(\phi_q, \phi_q) V_p, \quad \text{for each} \quad q \in \mathcal{I}^{h}(\Omega). 
\]

Then, letting \(w_h \in V^h\), \(v_h \in V^h\) be defined by the nodal values \(W, V\), we have

\[
(2.25) \quad (u_h, Q^h_{\varepsilon} v_h)_{1,k} = \langle U, B^{-1} A_h V \rangle_{D_k}. 
\]

By multiplying by \(v_h(x_q)\) and using the definition of \(\Pi^h\) and summing over \(q\), we then have that

\[
\sum_{q \in \mathcal{I}^{h}(\Omega)} a_{\varepsilon, \ell}(w_h, \phi_q) = a_{\varepsilon}(v_h, \chi_{\ell}(x_q) \phi_q) \quad \text{for each} \quad q \in \mathcal{I}^{h}(\Omega). 
\]

The definition of \(Q^h_{\varepsilon, \ell}\) (2.25) and uniqueness then imply that \(w_h = Q^h_{\varepsilon, \ell} v_h\) which proves (2.24). Recalling (1.13) and (2.23), we obtain as a consequence of (2.24) that

\[
(u_h, Q^h_{\varepsilon} v_h)_{1,k} = \sum_{\ell} (u_h, \Pi^h(\chi_{\ell} Q^h_{\varepsilon} v_h))_{1,k} 
= \sum_{\ell} \langle U, R_{\ell} A_{\varepsilon, \ell}^{-1} R_{\ell} A_h V \rangle_{D_k} = \langle U, B^{-1} A_h V \rangle_{D_k}. 
\]

3. The Main Results.

3.1. Estimates involving the overlapping decomposition.

Lemma 3.1 (Estimates on norms involving \(\chi_{\ell}\)).

\[
(3.1) \quad \| \chi_{\ell} v \|^2_{l, k, \Omega_{\ell}} \lesssim \left( 1 + \frac{1}{(k\delta)^2} \right) \| v \|^2_{l, k, \Omega_{\ell}} \quad \text{for all} \quad v \in H^{1}(\Omega). 
\]

\[
(3.2) \quad \sum_{l=1}^{N} \| \chi_{\ell} v \|^2_{l, k, \Omega_{\ell}} \lesssim A \left( 1 + \frac{1}{(k\delta)^2} \right) \| v \|^2_{l, k} \quad \text{for all} \quad v \in H^{1}(\Omega). 
\]
Using \( (3.7) \), \( (3.9) \) and \( (3.6) \), we find

\[
\ell \text{ applications of } (3.1), \text{ summing both sides of the resulting estimate over } \Omega
\]

which is the estimate \( (3.1) \).

For all \( x \), noting that \( (2.10) \) ensures

\[
(3.8)
\]

and, for all \( j \),

\[
(3.7)
\]

Then, for all \( j \), the Cauchy-Schwarz inequality yields

\[
(3.9)
\]

Using \( (3.7) \), \( (3.9) \) and \( (3.6) \), we find

\[
\sum_{\ell=1}^{N} \int_{\Omega} \chi_{\ell}^{2}(x)f^{2}(x)dx = \sum_{j=1}^{\Lambda} \sum_{\ell \in D(j)} \chi_{\ell}^{2}(x)f^{2}(x)dx = \sum_{j=1}^{\Lambda} \sum_{\ell \in D(j)} \int_{\Omega \cap D_{j}} \chi_{\ell}^{2}(x)f^{2}(x)dx \\
\geq \sum_{j=1}^{\Lambda} \int_{D_{j}} \left( \sum_{\ell \in D(j)} \chi_{\ell}^{2}(x) \right)f^{2}(x)dx \geq \frac{1}{\Lambda} \sum_{j=1}^{\Lambda} \int_{D_{j}} f^{2}(x)dx = \frac{1}{\Lambda} \int_{\Omega} f^{2}(x)dx,
\]

where \( C \) denotes a parameter-independent constant.

Proof. Using \( \nabla(\chi_{\ell}v) = (\nabla x_{\ell})v + \chi_{\ell}\nabla v \), the first inequality in \( (2.2) \), and the inequality \( (2.21) \), we have that

\[
|\nabla(\chi_{\ell}v)(x)|^{2} \lesssim \left( \frac{1}{\delta^{2}} |v(x)|^{2} + |\nabla v(x)|^{2} \right),
\]

for all \( x \in \Omega_{\ell} \). Then

\[
\|\chi_{\ell}v\|_{L^{2}(\Omega_{\ell})}^{2} \lesssim \frac{1}{\delta^{2}} \|v\|_{L^{2}(\Omega_{\ell})}^{2} + |\nabla v(\Omega_{\ell})|^{2} \lesssim \left( 1 + \frac{1}{(k\delta)^{2}} \right) \|v\|_{L^{2}(\Omega_{\ell})}^{2},
\]

which is the estimate \( (3.1) \).

From \( (2.11) \), we see that \( (3.2) \) follows from \( (3.1) \). The estimate \( (3.3) \) follows from two successive applications of \( (3.1) \), summing both sides of the resulting estimate over \( \ell \), and then using \( (2.11) \).

To prove \( (3.4) \), define, for each \( x \in \Omega \), a positive integer \( m = m(x) \in \mathbb{N} \) by

\[
m(x) = \# \{ \ell \in \{1, \ldots, N\} : x \in \text{supp}(\chi_{\ell}) \},
\]

noting that \( (2.10) \) ensures \( m(x) \) is finite, and in fact, \( 1 \leq m(x) \leq \Lambda \), for all \( x \in \Omega \). Then, for any integer \( j \in \{1, \ldots, \Lambda\} \), we define the subset of \( \Omega \) \( D_{j} = \{ x \in \Omega : m(x) = j \} \), so that \( x \in D_{j} \) if and only if \( x \) lies in the supports of exactly \( j \) of the partition of unity functions \( \{\chi_{\ell}\} \). Corresponding to these we also define the index sets:

\[
(3.6) \quad D(j) = \{ \ell \in \{1, \ldots, N\} : \text{supp}(\chi_{\ell}) \cap D_{j} \neq \emptyset \},
\]

that is \( D(j) \) contains all the indices of all subdomains which overlap with \( D_{j} \). Then, we have

\[
(3.7) \quad \Omega = \bigcup_{j=1}^{\Lambda} D_{j} \quad \text{and} \quad D_{i} \cap D_{j} = \emptyset \text{ if } i \neq j,
\]

and, for all \( j = 1, \ldots, \Lambda \),

\[
(3.8) \quad \sum_{\ell \in D(j)} \chi_{\ell}(x) = 1 \quad \text{when } x \in D_{j}.
\]

Then, for all \( x \in D_{j} \), the Cauchy-Schwarz inequality yields

\[
1 = \left( \sum_{\ell \in D(j)} \chi_{\ell}(x) \right)^{2} \leq j \sum_{\ell \in D(j)} \chi_{\ell}^{2}(x) \leq \Lambda \sum_{\ell \in D(j)} \chi_{\ell}^{2}(x).
\]
which is \((3.4)\). Finally, for \((3.5)\), we use \((2.21)\) and \((2.2)\) to obtain

\[
\|\chi_I f\|_{1,k,\Omega_t}^2 = k^2 \|\chi_I f\|_{L^2(\Omega_t)}^2 + \|\chi_I \nabla f\|_{L^2(\Omega_t)}^2 + 2\Re \int_{\Omega_t} \chi_I f \nabla \chi_I \cdot \nabla \chi_I + \|f \nabla \chi_I\|_{L^2(\Omega_t)}^2 \\
\geq k^2 \|\chi_I f\|_{L^2(\Omega_t)}^2 + \|\chi_I \nabla f\|_{L^2(\Omega_t)}^2 - \frac{C}{k^3} \|f\|_{L^2(\Omega_t)},
\]

and the result is obtained by summing, and using \((3.4)\) and \((2.11)\).

**Remark 3.2.** An interesting observation is that the estimate \((3.2)\) provides a “stable splitting”, i.e. any \(v \in H^1(\Omega)\) has a decomposition into components \(\chi_I v \in H^1(\Omega_t)\), with

\[
v = \sum_\delta \chi_I v,
\]

so that sum of the squares of the energies of the components is bounded in terms of the square of the energy of \(v\), with a constant that is independent of \(k, h, H\) and \(\delta\), provided only that \(k \delta \gtrsim 1\).

Corollary 3.5 below provides an analogous stable splitting for finite element functions. This result is perhaps a little surprising, since, for positive-definite elliptic problems, families of subdomains with decreasing diameter do not enjoy this property (and a coarse space is needed to restore it) \([42]\). Here the stable splitting holds without coarse space, as \(k \to \infty\) (i.e. for a family of Helmholtz problems of increasing difficulty), provided only that \(\delta \gtrsim k^{-1}\). This includes for example, subdomains of diameter \(H \sim k^{-\alpha}\) with \(\alpha \in [0,1]\) and overlap \(k^{-1} \lesssim \delta \lesssim H\).

**Lemma 3.3** (Error in interpolation of \(\chi_I w_h\)). For any \(\ell = 1, \ldots, N\), suppose \(w_h \in V^b_\ell\). Then

\[
(I - \Pi^h)(\chi_I w_h) \|_{1,k,\Omega_t} \lesssim (1 + kh) \frac{h}{\delta} \|w_h\|_{H^1(\Omega_t)}.
\]

**Proof.** For each element \(e \in T^h\), from \((2.7)\) we have

\[
(I - \Pi^h)(\chi_I w_h) \|_{L^2(\ell)} + h|I - \Pi^h)(\chi_I w_h)|_{H^s(\ell)} \lesssim h^s|\chi_I w_h|_{H^s(\ell)}.
\]

Now, for any multi-index \(\alpha\) of order \(r\), the Leibnitz formula yields

\[
D^\alpha(\chi_I w_h) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (D^\beta \chi_I) (D^{\alpha - \beta} w_h) = \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} (D^\beta \chi_I) (D^{\alpha - \beta} w_h).
\]

(Note \(D^\alpha w_h = 0\), because \(w_h|_r\) is polynomial of degree \(r - 1\).) Then using \((2.21)\), a standard inverse estimate (e.g. \([9, \text{Thm. 3.2.6}]\)), and the fact that \(\delta \gtrsim h\), we obtain

\[
D^\alpha(\chi_I w_h) \|_{L^2(\ell)} \lesssim \max_{0 < \beta \leq \alpha} \delta^{-\beta} |w_h|_{H^{s-\beta}(\ell)}
\]

\[
\lesssim \max_{0 < \beta \leq \alpha} \delta^{1-r} |w_h|_{H^s(\ell)} \leq \delta^{-1} h^{2-r} |w_h|_{H^s(\ell)}.
\]

Combining \((3.11)\) and \((3.12)\), and summing over all elements \(\ell \subseteq \Omega_t\), we get

\[
k |I - \Pi^h)(\chi_I w_h) \|_{L^2(\Omega_t)} \lesssim kh \frac{h}{\delta} \|w_h\|_{H^1(\Omega_t)}.
\]

and

\[
|I - \Pi^h)(\chi_I w_h) \|_{H^s(\Omega_t)} \lesssim \frac{h}{\delta} \|w_h\|_{H^s(\Omega_t)}.
\]

Combining \((3.13)\) and \((3.14)\), we obtain the estimate. \(\square\)

We remark that in Lemma 3.3 it is essential that \(w_h \in V^b_\ell\). If \(w_h\) is replaced by \(w \in H^1(\Omega_t)\), the proof would fail.

We now specify an assumption on \(h, k, \) and \(\delta\) that will considerably simplify the estimates below.
Assumption 3.4.

(3.15) \[ kh \lesssim 1, \quad \text{and} \quad \delta \gtrsim k^{-1}, \]

Note that from these it follows that \( h/\delta \lesssim hk \lesssim 1 \), i.e., the fine mesh resolves the oscillatory solution as \( k \) increases and the overlap of the subdomains is always big enough to see at least one oscillation.

From now on we shall assume that Assumptions 2.1 and 3.4 both hold.

Corollary 3.5. Let \( v_h \in V^h \). Then

\[
v_h = \sum_{\ell=1}^{N} \Pi^h(\chi_v h) \quad \text{and} \quad \sum_{\ell=1}^{N} \|\Pi^h(\chi_v h)\|_{1,k,\Omega_\ell}^2 \lesssim \Lambda \left(1 + \frac{1}{k\delta}\right)^2 \|v_h\|_{1,k,\Omega}^2 .
\]

Proof. Using the triangle inequality, (3.1), (3.15), and Lemma 3.3, we have

\[
\|\Pi^h(\chi_v h)\|_{1,k,\Omega_\ell} \lesssim \|\chi_v h\|_{1,k,\Omega_\ell} + \|I - \Pi^h\|_{1,k,\Omega_\ell} \sum_{\ell=1}^{N} \|v_h\|_{1,k,\Omega_\ell},
\]

and the result follows by squaring, summing, and applying (2.11).

The next result is a kind of converse to the stable splitting result discussed in Remark 3.2.

Lemma 3.6. For each \( \ell = 1, \ldots, N \), choose any function \( v_\ell \in H^1(\Omega) \), with \( supp v_\ell \subset \overline{\Omega_\ell} \). Then

\[
\left\| \sum_{\ell=1}^{N} v_\ell \right\|_{1,k}^2 \leq \Lambda \sum_{\ell=1}^{N} \|v_\ell\|_{1,k,\Omega_\ell}^2 .
\]

Proof. The proof follows almost verbatim that of [24, Lemma 4.2], with a little extra care needed to obtain the explicit constant \( \Lambda \) on the right-hand side.

3.2. Results about the projection operators. In this subsection, we study the projection operators \( Q^h_{\ell,\ell} \) defined in (2.22). Our goal is a bound on the operator \( Q^h_{\ell,\ell} - \Pi^h \chi_v \) with respect to the Helmholtz energy norm \( \cdot \|_{1,k} \) – see Lemma 3.8. This bound is a key ingredient of our main results – Theorem 3.10 (for projection operators) and Theorem 1.1 (for matrices).

Before beginning, we note that when \( w_{h,\ell} \in V^h_{\ell,\ell} \), \( \Pi^h(\chi_v w_{h,\ell}) \) is supported on \( \Omega_\ell \) and vanishes on \( \partial\Omega_\ell \cap \Gamma_D \). Thus, by (2.22), for all \( w_{h,\ell} \in V^h_{\ell,\ell} \) and \( v \in H^1(\Omega) \),

\[
a_{\varepsilon,\ell}(Q^h_{\varepsilon,\ell} v, w_{h,\ell}) = a_{\varepsilon,\ell}(v, \Pi^h(\chi_v w_{h,\ell}))
\]

and hence

\[
a_{\varepsilon,\ell}(Q^h_{\varepsilon,\ell} v - \Pi^h(\chi_v v), w_{h,\ell}) = a_{\varepsilon,\ell}(v, \Pi^h(\chi_v w_{h,\ell})) - a_{\varepsilon,\ell}(\Pi^h(\chi_v v), w_{h,\ell}).
\]

This shows that \( Q^h_{\varepsilon,\ell} v - \Pi^h(\chi_v v) \) satisfies a local impedance problem with “data” given by the “commutator” (appearing on the right-hand side of (3.18)). To estimate this commutator we write

\[
a_{\varepsilon,\ell}(v, \Pi^h(\chi_v w_{h,\ell})) - a_{\varepsilon,\ell}(\Pi^h(\chi_v v), w_{h,\ell}) = a_{\varepsilon,\ell}(I - \Pi^h)(\chi_v v), w_{h,\ell}) - a_{\varepsilon,\ell}(v, (I - \Pi^h)(\chi_v w_{h,\ell}))
\]

\[
+ b_{\ell}(v, w_{h,\ell}) ,
\]

where

\[
b_{\ell}(v, w) := a_{\varepsilon,\ell}(v, \chi_v w) - a_{\varepsilon,\ell}(\chi_v v, w) = (v, \chi_v w)_{1,k,\Omega_\ell} - (\chi_v v, w)_{1,k,\Omega_\ell}
\]

(3.20)

\[
= \int_{\Omega_\ell} \nabla \chi_v (\nabla v - v \nabla w) .
\]

The following lemma provides estimates for each of the terms on the right-hand side of (3.19).
LEMMA 3.7.

(i) For all \( v, w \in H^1(\Omega) \),
\[
|b_\varepsilon(v, w)| \leq \frac{1}{k\delta} \|v\|_{1,k,\Omega} \|w\|_{1,k,\Omega}.
\]

(ii) For all \( v_h, w_h \in V_h^b \),
\[
\max \left\{ |a_{\varepsilon,\ell}(v_h, (I - \Pi_h^b)(\chi_\varepsilon v_h)), |a_{\varepsilon,\ell}((I - \Pi_h^b)(\chi_\varepsilon v_h), w_h) \right\} \leq \left( 1 + \frac{1}{kH} \right) \frac{h}{\delta} \|v_h\|_{1,k,\Omega} \|w_h\|_{1,k,\Omega}.
\]

Proof. Applying the Cauchy-Schwarz inequality to (3.20) and using and (2.21), we obtain
\[
|b_\varepsilon(v, w)| \leq \frac{1}{k\delta} \left( k\|w\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} + k\|v\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)} \right),
\]
and the result follows after an application of the Cauchy-Schwarz inequality with respect to the Euclidean inner product in \( \mathbb{R}^2 \).

For (ii), recall Assumption 2.1 and use the continuity of \( a_\varepsilon \) (from Lemma 2.4) to obtain
\[
|a_{\varepsilon,\ell}(v, (I - \Pi_h^b)(\chi_\varepsilon v_h))| \leq (1 + (kH)^{-1}) \|v\|_{1,k,\Omega} \|\Pi_h^b(\chi_\varepsilon v_h)\|_{1,k,\Omega};
\]
the result follows on applying Lemma 3.3.

Combining Lemma 3.7 with Theorem 2.7, we obtain the following result.

LEMMA 3.8. Under the assumptions of Theorem 2.7, for all \( v_h \in V_h^b \),
\[
\|Q_{\varepsilon,\ell}^b v_h - \Pi_h^b(\chi_\varepsilon v_h)\|_{1,k,\Omega} \lesssim \frac{1}{k\delta} \Theta(\varepsilon, H, k) \|v_h\|_{1,k,\Omega},
\]
and
\[
\|Q_{\varepsilon,\ell}^b v_h\|_{1,k,\Omega} \lesssim \left[ 1 + \frac{1}{k\delta} \Theta(\varepsilon, H, k) \right] \|v_h\|_{1,k,\Omega}.
\]

Proof. Let \( v_h \in V_h^b \). By (3.19), we have
\[
\varepsilon_{\varepsilon,\ell}(Q_{\varepsilon,\ell}^b v_h - \Pi_h^b(\chi_\varepsilon v_h), w_h) = F(w_h),
\]
where
\[
F(w_h) := a_{\varepsilon,\ell}((I - \Pi_h^b)(\chi_\varepsilon v_h), w_h) - a_{\varepsilon,\ell}(v_h, (I - \Pi_h^b)(\chi_\varepsilon v_h)) + b_\varepsilon(v_h, w_h).
\]
Using Lemma 3.7 and (3.15), we have, for any \( w_h \in V_h^b \),
\[
|F(w_h)| \lesssim \left( \left( 1 + \frac{1}{kH} \right) \frac{h}{\delta} + \frac{1}{k\delta} \right) \|v_h\|_{1,k,\Omega} \|w_h\|_{1,k,\Omega},
\]
\[
= \frac{1}{k\delta} \left( 1 + hk + \frac{h}{H} \right) \|v_h\|_{1,k,\Omega} \|w_h\|_{1,k,\Omega} \lesssim \frac{1}{k\delta} \|v_h\|_{1,k,\Omega} \|w_h\|_{1,k,\Omega}.
\]
Then (3.22) follows from Theorem 2.7. To obtain (3.23), we write
\[
\|Q_{\varepsilon,\ell}^b v_h\|_{1,k,\Omega} \leq \|Q_{\varepsilon,\ell}^b v_h - \Pi_h^b(\chi_\varepsilon v_h)\|_{1,k,\Omega} + \|\Pi_h^b(\chi_\varepsilon v_h)\|_{1,k,\Omega},
\]
and then use (3.16) and (3.22), remembering that \( \Theta(\varepsilon, H, k) \geq 1 \).

Combining the previous lemma with the definition of \( \Theta \) in (2.17)/(2.18), we have the immediate corollary.

COROLLARY 3.9. Under the assumptions of Theorem 2.7,

(i) If \( |\varepsilon| > 0 \), then
\[
\|Q_{\varepsilon,\ell}^b v_h - \chi_\varepsilon v_h\|_{1,k,\Omega} \lesssim \frac{k}{|\varepsilon|} \|v_h\|_{1,k,\Omega};
\]

(ii) If \( |\varepsilon| \geq 0 \), \( h \leq \bar{k}(k,r) \) and each \( \Omega_\ell \) is starshaped with respect to a ball uniformly in \( \ell \), then
\[
\|Q_{\varepsilon,\ell}^b v_h - \chi_\varepsilon v_h\|_{1,k,\Omega} \lesssim \left( \frac{H}{\delta} + \frac{1}{k\delta} \right) \|v_h\|_{1,k,\Omega}.
\]
3.3. Bounds on the norm and field of values of $Q^h_{ε}$.

**Theorem 3.10.** Let the assumptions of Theorem 2.7 hold, and assume also that

\[
\begin{align*}
\tag{3.28} & kδ \to \infty \text{ as } k \to \infty. \\
\end{align*}
\]

Let $σ_ε > 0$, $ℓ = 1, \ldots, N$ be such that

\[
\begin{align*}
\tag{3.29} & \|Q^h_{ε,ℓ}v_h - Π^h(χℓv_h)\|_{1,k,Ω_ε} \leq σ_ℓ \|v_h\|_{1,k,Ω_ε} \\
\end{align*}
\]

for all $v_h \in V^h$ and set $σ := \max_ℓ σ_ℓ$. Then,

\[
\begin{align*}
\tag{3.30} & \max_{v_h \in V^h} \frac{\|Q^h_{ε,ℓ}v_h\|_{1,k}}{\|v_h\|_{1,k}} \lesssim Λ \left(1 + \sigma + \frac{1}{kδ}\right)^2. \\
\end{align*}
\]

Also, for all $v_h \in V^h$, and for $k$ sufficiently large,

\[
\begin{align*}
\tag{3.31} & \frac{|(v_h, Q^h_{ε,ℓ}v_h)_{1,k}|}{\|v_h\|_{1,k}^2} \geq \left(\frac{1}{Λ} - \sqrt{2σΛ}\right) + \text{H.O.T.} \\
\end{align*}
\]

where H.O.T. is a “higher-order term” that satisfies

\[
\begin{align*}
\tag{3.32} & |\text{H.O.T.}| \leq C \frac{Λ}{kδ} \left(1 + \sigma + \frac{1}{kδ}\right), \\
\end{align*}
\]

where $C$ is a constant independent of all parameters. Note that (3.31) is a genuine lower bound, since the unspecified constant $C$ appears only in H.O.T..

**Proof.** Throughout the proof, we use the notation

\[
\begin{align*}
\tag{3.33} & z_ℓ := Q^h_{ε,ℓ}v_h - Π^h(χℓv_h). \\
\end{align*}
\]

Then using the triangle inequality, (3.16) and (3.29), we have

\[
\begin{align*}
\tag{3.34} & \|Q^h_{ε,ℓ}v_h\|_{1,k,Ω_ε} \leq Π^h(χℓv_h)\|_{1,k,Ω_ε} + \|z_ℓ\|_{1,k,Ω_ε} \leq \left(1 + \sigma + \frac{1}{kδ}\right) \|v_h\|_{1,k,Ω_ε}. \\
\end{align*}
\]

Then, using Lemma 3.6, (3.16) and (3.34),

\[
\begin{align*}
\|Q^h_{ε,ℓ}v_h\|_{1,k,Ω_ε}^2 & = \left\| \sum_ℓ Π^h(χℓQ^h_{ε,ℓ}v_h) \right\|^2_{1,k,Ω_ε} \leq Λ \sum_ℓ \|Π^h(χℓQ^h_{ε,ℓ}v_h)\|^2_{1,k,Ω_ε} \\
& \lesssim Λ \left(1 + \frac{1}{kδ}\right)^2 \sum_ℓ \|Q^h_{ε,ℓ}v_h\|^2_{1,k,Ω_ε} \\
& \lesssim Λ \left(1 + \frac{1}{kδ}\right)^2 \left(1 + \sigma + \frac{1}{kδ}\right)^2 \sum_ℓ \|v_h\|^2_{1,k,Ω_ε}
\end{align*}
\]

and (3.30) then follows on employing (2.11).

To obtain (3.31), we first use Lemma 3.3, (3.15), (3.20) and Lemma 3.7 to obtain

\[
\begin{align*}
\tag{3.35} & (v_h, Π^h(χℓQ^h_{ε,ℓ}v_h))_{1,k,Ω_ε} = (v_h, χℓQ^h_{ε,ℓ}v_h)_{1,k,Ω_ε} + O\left(\frac{h}{δ}\right) \|v_h\|_{1,k,Ω_ε} \|Q^h_{ε,ℓ}v_h\|_{1,k,Ω_ε}, \\
\tag{3.36} & (v_h, χℓQ^h_{ε,ℓ}v_h)_{1,k,Ω_ε} = (χℓv_h, Q^h_{ε,ℓ}v_h)_{1,k,Ω_ε} + O\left(\frac{1}{δk}\right) \|v_h\|_{1,k,Ω_ε} \|Q^h_{ε,ℓ}v_h\|_{1,k,Ω_ε}.
\end{align*}
\]

Moreover, by the definition of $z_ℓ$ and Lemma 3.3,

\[
(χℓv_h, Q^h_{ε,ℓ}v_h)_{1,k,Ω_ε} = \|χℓv_h\|^2_{1,k,Ω_ε} + (χℓv_h, z_ℓ)_{1,k,Ω_ε} + (χℓv_h, Π^h(χℓv_h) - χℓv_h)_{1,k,Ω_ε}
\]

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Combining (3.35) - (3.37), and recalling (2.23) and (3.15), we obtain

\[
(v_h, Q^h_{\epsilon} v_h)_{1,k} = \sum_{\ell} (v_h, \Pi^h(\chi_{\ell} Q^h_{\epsilon,\ell} v_h))_{1,k,\Omega_{\ell}}
\]

\[
= \sum_{\ell} \left( \|\chi_{\ell} v_h\|^2_{1,k,\Omega_{\ell}} + \mathcal{O} \left( \frac{1}{k^\delta} \right) \|v_h\|_{1,k,\Omega_{\ell}} \|Q^h_{\epsilon,\ell} v_h\|_{1,k,\Omega_{\ell}} \right)
\]

\[
= \sum_{\ell} \left[ \|\chi_{\ell} v_h\|^2_{1,k,\Omega_{\ell}} + (\chi_{\ell} v_h, z_{\ell})_{1,k,\Omega_{\ell}} \right]
\]

\[
+ \sum_{\ell} \left[ \mathcal{O} \left( \frac{1}{k^\delta} \right) \|v_h\|_{1,k,\Omega_{\ell}} \|Q^h_{\epsilon,\ell} v_h\|_{1,k,\Omega_{\ell}} + \mathcal{O} \left( \frac{h}{\delta} \right) \|v_h\|_{1,k,\Omega_{\ell}} \|\chi_{\ell} v_h\|_{1,k,\Omega_{\ell}} \right]
\]

We now claim that the second sum in (3.38) can be estimated by

\[
\mathcal{O} \left( \frac{A}{k^\delta} \left( 1 + \sigma + \frac{1}{k^\delta} \right) \right) \|v_h\|^2_{1,k}.
\]

Indeed, this follows from using (3.34) and (2.11) in the first term and (3.1) and (3.15) in the second term.

Also, using (3.5), (3.1), (3.29) and then (2.11), the modulus of the first term in (3.38) can be estimated from below by

\[
\sum_{\ell} \|\chi_{\ell} v_h\|^2_{1,k,\Omega_{\ell}} - \sqrt{2} \sigma \left( 1 + \frac{1}{k^\delta} \right) \sum_{\ell} \|v_h\|^2_{1,k,\Omega_{\ell}}
\]

\[
\geq \left( \frac{1}{\Lambda} - \sqrt{2} \sigma \Lambda \right) \|v_h\|^2_{1,k} + \mathcal{O} \left( \frac{A}{k^\delta} \right) \|v_h\|^2_{1,k}.
\]

The result (3.31) then follows from (3.38) and (3.39).

Having proved this theorem we can now give the proof of Theorem 1.1 in the Introduction.

Proof of Theorem 1.1. First note that, from (1.13) and (1.14), if $v_h \in V^h$ is a finite element function with nodal vector $V$, then $\|v_h\|_{1,k} = \|V\|_{D_k}$. By Theorem 2.9, the nodal vectors of $Q^h_{\epsilon} v_h$ and $Q^h_{\epsilon,\ell} v_h$ are $B^{-1}_{\epsilon} A_{\epsilon} V$ and $A^{-1}_{\epsilon} R_{\ell} A_{\epsilon} V$ respectively. By (1.11), the nodal vector of $\Pi^h(\chi_{\ell} v_h)$ is $R_{\ell} V$. Thus

\[
\|Q^h_{\epsilon} v_h\|_{1,k} = \|B^{-1}_{\epsilon} A_{\epsilon} D_k\| \quad \text{and} \quad \|Q^h_{\epsilon,\ell} v_h - \Pi^h(\chi_{\ell} v_h)\|_{1,k} = \|A^{-1}_{\epsilon,\ell} R_{\ell} A_{\epsilon} V - R_{\ell} V\|_{D_k}.
\]

We use these relations and also (2.25) to translate the statements in Theorem 3.10 into statements about matrices, yielding Theorem 1.1.

Proof of Corollary 1.2. This follows directly from Theorem 1.1, the GMRES convergence theory in [11], and the correspondence between left- and right-preconditioning (see Remark 3.14).

The utility of the bounds (3.30) and (3.31) depends on the behaviour of $\sigma$; the following result is obtained immediately from Corollary 3.9 and Lemma 3.3.

Corollary 3.11. Let the assumptions of Theorem 3.10 hold.

(i) Assume that $h \leq \tilde{h}(k, r)$, and each $\Omega_{\ell}$ is starshaped with respect to a ball uniformly in $\ell$. Then, for all $\epsilon$ with $0 \leq |\epsilon| \leq k^2$, we have $\sigma \lesssim H/\delta$.

(ii) If $|\epsilon| > 0$, $\sigma \sim k^{1+\beta}$ for $0 < \beta < 1$, $\tilde{\sigma} \sim H \sim k^{-\alpha}$ for $0 < \alpha < 1$, then $\sigma \lesssim k^{\alpha-\beta}$.

(iii) If $|\epsilon| > 0$ and $\delta$ is fixed, then there exist constants $C$ and $k_0$ so that when $\epsilon = Ck$ and $k \geq k_0$,

\[
\sigma \leq \frac{1}{2\sqrt{2}A^2}.
\]

Using the bounds of Corollary 3.11 in Theorem 3.10, we obtain the following results about $Q^h_{\epsilon}$ (which then imply results about $B^{-1}_{\epsilon} A_{\epsilon}$ via (2.25) and (3.40)).
respectively. (Here \( H \sim k^{-3/2} \) (the level of refinement generally believed to keep the relative error of the finite-element solution bounded independently of \( k \) as \( k \to \infty \); see Remark 2.8). The preconditioner is characterised by the coarse grid diameter and the level of absorption used, denoted by

\[
H \quad \text{and} \quad \varepsilon_{\text{prec}}
\]

respectively. (Here \( H \sim 1/M \).)
In Experiments 2 and 3, we verify the theory by illustrating the performance of the preconditioner on some problems with \( \varepsilon_{\text{prob}} > 0 \). In Experiments 1 (given in the Introduction), 4, 5, and 6, we solve the “pure Helmholtz” problem, i.e. \( \varepsilon_{\text{prob}} = 0 \). Unless otherwise stated, the data \( f, g \) in (1.5) is chosen so that the exact solution of (1.3) - (1.4) is a plane wave \( u(x) = \exp(i k x \hat{d}) \) where \( \hat{d} = (1/\sqrt{2}, 1/\sqrt{2})^\top \). Note that oscillations in the solution are not resolved by the subdomains. We choose \( \Gamma_D = \emptyset \), so that \( \Gamma = \Gamma_I \). Except in Experiment 5, the initial guess for GMRES is chosen to be a random (uniformly distributed in \([0, 1]^n\)) vector in \( \mathbb{R}^n \). In all cases the GMRES stopping criterion is based on requiring the initial tolerance to be reduced by \( 10^{-6} \). Standard GMRES (with residual minimisation in the Euclidean norm) is used, even though the estimates in Theorem 1.1 are with respect to the norm induced by \( D_k \); the numerical experiments in [24] for a similar method found the iteration counts to be essentially identical when minimisation in the Euclidean norm is replaced by minimisation in the norm induced by \( D_k \).

Experiment 2 is a direct illustration of the result of Corollary 3.11 (i).

**Experiment 2.** We choose

\[
(4.1) \quad h_{\text{prob}} \sim k^{-3/2}, \quad \varepsilon_{\text{prob}} = \varepsilon_{\text{prec}} = k^{1+\beta}, \quad H_{\text{prec}} = k^{-\alpha}, \quad \text{where} \quad \beta = \alpha + 0.1.
\]

**Corollary 3.11 predicts a wavenumber-independent iteration count for GMRES and this behaviour is clearly visible in Table 2. Reading across the table, for fixed \( k \), larger \( \alpha \) corresponds to smaller subdomains (and thus the preconditioner becomes cheaper per iterate), but the number of iterations increases (albeit slightly).**

| \( k \) \( \alpha \) | 0.2 | 0.3 | 0.4 | 0.5 |
|-------------------|-----|-----|-----|-----|
| 10                | 6   | 6   | 9   | 8   |
| 20                | 5   | 5   | 7   | 8   |
| 40                | 4   | 6   | 7   | 9   |
| 60                | 4   | 5   | 7   | 10  |
| 80                | 3   | 6   | 8   | 9   |
| 100               | 5   | 6   | 7   | 9   |
| 120               | 4   | 5   | 7   | 9   |
| 140               | 4   | 5   | 7   | 9   |

**Table 2**

Number of GMRES iterations for the case (4.1).

Based on Experiment 2, and recalling the discussion in the Introduction (in particular (1.8)), we now investigate how well the preconditioner performs when we reduce the absorption in the problem being solved to \( \varepsilon_{\text{prob}} = k \).

**Experiment 3.** We choose

\[
(4.2) \quad h_{\text{prob}} \sim k^{-3/2}, \quad \varepsilon_{\text{prob}} = \varepsilon_{\text{prec}} = k, \quad \text{and} \quad H_{\text{prec}} = k^{-\alpha}.
\]

Comparing Tables 2 and 3, we see an increase in the iteration numbers (especially for larger \( \alpha \)) but growth with \( k \) appears to be avoided when \( \alpha \leq 0.4 \). In this case, \( B_k^{-1} \) is a good preconditioner for \( A_k \) and so by the heuristic argument centred on (1.8), we expect \( B_k^{-1} \) to be good preconditioner for \( A \). This appears to be true, as demonstrated by Experiment 4, where \( \varepsilon_{\text{prob}} \) is reduced from \( k \) to 0; we again see modest growth in iteration numbers, but apparent robustness for \( \alpha \leq 0.4 \).

**Experiment 4.** We choose

\[
(4.3) \quad h_{\text{prob}} \sim k^{-3/2}, \quad \varepsilon_{\text{prob}} = 0, \quad \text{and} \quad H_{\text{prec}} = k^{-\alpha}.
\]

We make two observations from the results of Experiments 2-4.

1. The one-level Schwarz method provides an optimal preconditioner for the pure Helmholtz problem in that the iteration numbers are bounded independently of \( k \) (and hence \( n \)) as \( k \)
increases, provided the subdomain diameter does not shrink too quickly. The experiments suggest that robustness is maintained when the subdomain diameters shrink no faster than $O(k^{-0.4})$.

2. The performance of the preconditioner is virtually the same whether it is built from the absorptive system $\epsilon_{\text{prec}} = k$ or from the pure Helmholtz system $\epsilon_{\text{prec}} = 0$. Whilst the results of the present paper give theoretical support for the observed robustness when $\epsilon = k$, (see the discussion in §1 and Appendix A); with existing theoretical tools it seems very difficult to prove results for the case $\epsilon_{\text{prec}} = 0$.

These experiments also support the observation in Experiment 1 in §1, which showed that a robust method could be formed by taking a fixed number of subdomains. Experiment 5 provides further evidence for this.

**Experiment 5.** This experiment contains two variations on Experiment 1. We choose

$$h \sim k^{-3/2}, \quad \epsilon_{\text{prob}} = 0, \quad H = 1/M.$$

The left-hand panel of Table 5 reproduces the results of Experiment 1 from Table 1. The middle panel gives the case when $\epsilon_{\text{prec}} = 0$ and the starting guess is chosen randomly. The right-hand panel gives the case when $\epsilon_{\text{prec}} = k$ and the starting guess is chosen as zero. Comparing the left and middle panels, we see that (at least in this particular situation) there is little effect in switching off the absorption in the preconditioner. Comparing the left and right panels, we see that a random starting guess leads to consistently lower iteration counts than a zero starting guess; we have no explanation for this surprising observation.

In our final experiment, we study the effect of changing the boundary condition on the subdomains from Impedance to Dirichlet. Note that we observed in Remark 3.15 that when we consider the theory for preconditioning the absorptive problem we can get similar estimates when using local Dirichlet conditions compared to those obtained with impedance conditions. However we see from this experiment that Dirichlet local solves give very poor preconditioners for the pure Helmholtz problem (compare Experiment 6 with Experiment 4). Similar observations are made in [24], where coarse grids were also employed to improve the robustness.

### Table 3

Number of GMRES iterations for the case (4.2).

| $k \backslash \alpha$ | 0.2 | 0.3 | 0.4 | 0.5 |
|-----------------------|-----|-----|-----|-----|
| 10                    | 6   | 6   | 10  | 10  |
| 20                    | 5   | 5   | 9   | 12  |
| 40                    | 4   | 7   | 10  | 17  |
| 60                    | 4   | 7   | 12  | 22  |
| 80                    | 4   | 9   | 13  | 21  |
| 100                   | 6   | 8   | 13  | 23  |
| 120                   | 5   | 8   | 15  | 24  |
| 140                   | 5   | 7   | 13  | 25  |

### Table 4

Number of GMRES iterations for the case (4.3).

| $k \backslash \alpha$ | $\epsilon_{\text{prec}} = k$ | $\epsilon_{\text{prec}} = 0$ |
|-----------------------|-------------------------------|------------------------------|
|                       | 0.2  | 0.3  | 0.4  | 0.5  | 0.2  | 0.3  | 0.4  | 0.5  |
| 10                    | 8    | 8    | 12   | 11   | 6    | 6    | 11   | 11   |
| 20                    | 7    | 6    | 10   | 14   | 6    | 6    | 10   | 14   |
| 40                    | 6    | 8    | 12   | 20   | 5    | 8    | 11   | 19   |
| 60                    | 5    | 8    | 14   | 25   | 5    | 7    | 14   | 25   |
| 80                    | 5    | 10   | 15   | 25   | 4    | 10   | 15   | 24   |
| 100                   | 7    | 9    | 15   | 27   | 7    | 9    | 15   | 27   |
| 120                   | 6    | 9    | 17   | 29   | 6    | 9    | 17   | 29   |
| 140                   | 6    | 9    | 17   | 31   | 6    | 8    | 16   | 31   |
Table 5
Number of GMRES iterations for the case (4.4)

| $k \backslash M$ | random starting guess | random starting guess | zero starting guess |
|-----------------|-----------------------|-----------------------|---------------------|
|                 | $\varepsilon_{\text{prec}} = k$ | $\varepsilon_{\text{prec}} = 0$ | $\varepsilon_{\text{prec}} = k$ |
| 40              | 12 27 61              | 11 27 61              | 16 36 82            |
| 60              | 11 25 56              | 10 25 56              | 15 36 81            |
| 80              | 10 22 52              | 10 22 52              | 15 33 75            |
| 100             | 9 21 48               | 9 21 48               | 15 33 71            |
| 120             | 9 20 45               | 9 20 45               | 15 31 69            |
| 140             | 8 18 41               | 8 18 41               | 14 31 70            |

Table 6
Number of GMRES iterations for the case (4.5) with homogeneous Dirichlet condition on subdomain boundaries

Experiment 6. We choose

\begin{equation}
    h_{\text{prob}} \sim k^{-3/2}, \quad \varepsilon_{\text{prob}} = 0, \quad \text{and} \quad H_{\text{prec}} = k^{-\alpha}.
\end{equation}

| $k \backslash \alpha$ | $\varepsilon_{\text{prec}} = k$ | $\varepsilon_{\text{prec}} = 0$ |
|-----------------------|-----------------------|-----------------------|
| 0.2                   | 7 7 12 12             | 6 6 15 15             |
| 0.3                   | 7 7 17 25             | 5 5 20 29             |
| 0.4                   | 6 16 34 86            | 5 22 43 110           |
| 0.5                   | 6 16 68 102           | 5 25 83 121           |
| 0.2                   | 5 46 127 239          | 5 78 173 256          |
| 0.3                   | 14 58 130 242         | 22 121 222 429        |
| 0.4                   |                      |                       |
| 0.5                   |                      |                       |

Acknowledgements
IGG thanks the Department of Mathematics at the Chinese University of Hong Kong for providing generous support and a stimulating research environment during his visits; he also thanks Paul Childs for first motivating him to study this problem. The authors thank Eero Vainikko (University of Tartu) for generously computing the numerical experiments given in the paper. We also thank Eric Chung (Chinese University of Hong Kong) for very useful discussions. EAS acknowledges support from EPSRC grant EP/R005591/1. and JZ acknowledges support from Hong Kong RGC General Research Fund (Project 14322516) and NSFC/Hong Kong RGC Joint Research Scheme 2016/17 (N_CUHK437/16).

Appendix A. A rigorous basis for the discussion around (1.8). 

Lemma A.1. Let $(\cdot, \cdot)$ be an inner product with associated norm $\|\cdot\|$. Assume that (1.6) holds with $\|\cdot\|_2$ replaced by $\|\cdot\|$ and with $K > 0$ independent of $\varepsilon$ and $k$. Assume also that for all $\varepsilon$ in some neighbourhood of the origin, there exist positive numbers $C_1(\varepsilon)$ and $C_2(\varepsilon)$ (which may depend on $\varepsilon$ but are independent of all other parameters), such that

\begin{equation}
    \|B^{-1}_\varepsilon A_\varepsilon\| \leq C_1(\varepsilon),
\end{equation}

and

\begin{equation}
    \frac{|(V, B^{-1}_\varepsilon A_\varepsilon V)|}{\|V\|^2} \geq C_2(\varepsilon) \quad \text{for all} \quad V \in \mathbb{C}^n.
\end{equation}

Then

\begin{equation}
    \|B^{-1}_\varepsilon A\| \leq C_1(\varepsilon) \left(1 + K\frac{|\varepsilon|}{k}\right).
\end{equation}
and

\[
\frac{|(V, B^{-1}_\varepsilon AV)|}{\|V\|^2} \geq C_2(\varepsilon) - K C_1(\varepsilon) \frac{|\varepsilon|}{k} \quad \text{for all} \quad V \in \mathbb{C}^n.
\]

**Remark A.2.** Observe that for (A.3) to remain bounded we simply need \(C_1(\varepsilon)\) to be bounded, while for the field of values (A.4) to be bounded away from the origin we need the stronger condition

\[
C_2(\varepsilon) > K C_1(\varepsilon) \frac{|\varepsilon|}{k}.
\]

**Proof of Lemma A.1.** The estimate (A.3) follows from (1.7), (1.6), and (A.1). To obtain (A.4) we again use (1.7), (A.1) and the inverse triangle inequality to obtain

\[
|(V, B^{-1}_\varepsilon AV)| \geq |(V, B^{-1}_\varepsilon A_\varepsilon V)| - K C_1(\varepsilon) \frac{|\varepsilon|}{k} \|V\|^2,
\]

and then use (A.2).

\[\Box\]

Appendix B. Comparison with the classical Schwarz theory.

In the classical Schwarz theory (e.g. [42, §2.3]), we start with an inner product \(c\) and a linear functional \(G\) on a Hilbert space \(V\). (Here we assume \(V = H^1_{0,D}(\Omega)\), although much greater generality is possible.) The variational problem to be solved (with solution \(u \in \mathcal{V}\)) and its finite element approximation (with solution \(u_h \in \mathcal{V}^h\)) are:

\[
c(u, v) = G(v), \quad \text{for all} \quad v \in \mathcal{V}, \quad \text{and} \quad c(u_h, v_h) = G(v_h) \quad \text{for all} \quad v_h \in \mathcal{V}_h.
\]

The finite element problem corresponds to a linear system with a symmetric positive-definite coefficient matrix. To formulate and analyse preconditioners for this, we choose subspaces \(\mathcal{V}^h \subset \mathcal{V}\), and define projections \(P^h : \mathcal{V} \to \mathcal{V}^h\) by

\[
c(P^h v, w_{h,\ell}) = c(v, w_{h,\ell}) \quad \text{for all} \quad w_{h,\ell} \in \mathcal{V}^h.
\]

Then (analogously to Theorem 2.9), the operator \(P^h := \sum_{\ell} P^h_{\ell}\) represents the finite element stiffness matrix, preconditioned using the classical additive Schwarz method. Because this problem is positive definite, the power of the preconditioner can be established by proving estimates for its spectral condition number, i.e. the ratio \(\lambda_{\text{max}}/\lambda_{\text{min}}\) of its maximum and minimum eigenvalues.

Since \(P^h_{\ell}\) is the orthogonal projection onto \(\mathcal{V}^h_{\ell}\) with respect to the inner product \(c\), we have \(\|P^h_{\ell}\| = 1\), for all \(\ell\), where \(\| \cdot \|_c\) is the norm induced by \(c\). In the case of the one-level Schwarz method (analogous to (1.12), with local subspaces \(\mathcal{V}^h_{\ell} \subset \mathcal{V}^h\)), an upper bound for \(\lambda_{\text{max}}\) is obtained by using the finite overlap assumption (2.10) to show that \(\|P^h\|_c\) can be bounded above in terms of the overlap parameter \(\Lambda\) (see [42, Lemma 3.11] and references therein). This yields a parameter-independent upper bound for \(\lambda_{\text{max}}\).

To bound \(\lambda_{\text{min}}\) below, one typically uses a “splitting” lemma, namely that any \(v_h \in \mathcal{V}^h\) can be written as \(v_h = \sum_{\ell} v_{h,\ell}\), where \(v_{h,\ell} \in \mathcal{V}^h_{\ell}\), with an energy estimate

\[
\sum_{\ell} \|v_{h,\ell}\|^2_c \leq C_1^2 \|v_h\|^2_c.
\]

Then, using several instances of the Cauchy-Schwarz inequality, we obtain, for any \(v_h \in \mathcal{V}^h\),

\[
\frac{1}{C_1^2} \|v_h\|^2_c \leq \sum_{\ell} c(v_{h,\ell}, P^h_{\ell} v_{h,\ell}) = c(v_h, P^h v_h) = c(v_h, P^h v_h),
\]

yielding

\[
\frac{1}{C_1^2} \|v_h\|^2_c \leq \sum_{\ell} c(v_{h,\ell}, P^h_{\ell} v_{h,\ell}) = c(v_h, P^h v_h) = c(v_h, P^h v_h),
\]

\[22\]
which (being a Rayleigh quotient estimate) tells us that \( \lambda_{\text{min}} \geq C_{\lambda}^{-2} \).

The analysis developed in this paper has some similarities to this classical argument, but there are many differences, leading to difficulties which had to be overcome. We finish the paper by highlighting a few of these differences.

Firstly, the sesquilinear form \( a_{\varepsilon} \) is neither Hermitian nor positive definite and so the matrices \( A_{\varepsilon} \) being preconditioned are not Hermitian (or positive definite) either. Hence, estimates for the spectrum of the preconditioned problem do not tell us anything rigorous about the convergence of iterative methods. This motivates the use of the GMRES convergence theory in [11], a technique introduced to the domain-decomposition community by the seminal paper [7].

Secondly, because the Helmholtz sesquilinear form is not coercive (when \( \varepsilon = 0 \), the local problems (2.22) are not necessarily well-posed in \( V^h \)). They either need to have absorption added (\( \varepsilon \neq 0 \), or else they need special boundary conditions (such as the impedance conditions used here) and may be subject to a mesh refinement threshold (as in Theorem 2.7 (ii)).

Thirdly, the spaces \( V^h \) used here are not subspaces of \( V^h \), because their values on \( \partial \Omega \setminus \Gamma_D \) do not necessarily vanish. Indeed, imposing a zero Dirichlet boundary condition on \( \partial \Omega \setminus \Gamma_D \) ensures that each element of \( V^h \) has a natural extension by zero to an element of \( V^h \). However, with the local impedance boundary conditions, elements in \( V^h \) are unconstrained on \( \partial \Omega \setminus \Gamma_D \) and thus have no natural extensions to members of \( V^h \). To make the local projections \( Q^h_{\varepsilon,\ell} \) well-defined, we have to multiply the test function \( w_{h,\ell} \) by \( \chi_{\ell} \) on the right-hand side of (2.22). The same definition has been used in [29].

Fourthly, the Helmholtz energy in which we estimate the norm and field of values of \( Q^h_{\varepsilon,\ell} \) is far from the sesquilinear form \( a_{\varepsilon,\ell} \) which is used to define the local problems (2.22). This is unlike the classical case where the energy norm comes directly from the inner product of the Laplacian, which meant they were able to do the analysis with respect to the \( \mathcal{L}^2 \) inner product.

Nevertheless, one striking artefact of the present analysis is the one-level Helmholtz stable splitting result (Corollary 3.5—see also Remark 3.2), which is stronger for Helmholtz than it is for Laplace problems.

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