Sea-ice dynamics on triangular grids

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Abstract

We present a stable discretization of sea-ice dynamics on triangular grids that can straightforwardly be coupled to an ocean model on a triangular grid with Arakawa C-type staggering. The approach is based on a nonconforming finite element framework, namely the Crouzeix-Raviart finite element. As the discretization of the viscous-plastic and elastic-viscous-plastic stress tensor with the Crouzeix-Raviart finite element produces oscillations in the velocity field, we introduce an edge-based stabilization. Based on an energy estimate for the viscous-plastic sea-ice model we define an energy for the elastic-viscous-plastic and viscous-plastic sea-ice model. In a numerical analysis we show that the stabilization is fundamental to achieve stable approximations of the sea-ice velocity field and a bounded energy of the sea-ice system.

1 Introduction

Sea-ice, located at high-latitudes and at the boundary between ocean and atmosphere, plays an important role in the climate system. Modelling the complex mechanical and thermodynamical behaviour of sea-ice at a broad range of spatio-temporal scales poses a manifold of challenges. Freezing sea water forms a composite of pure ice, liquid brine, air pockets and solid salt. The details of this formation depend on the laminar or turbulent environmental conditions. This composite responds differently to heating, pressure or mechanical forces than for example the (salt-free) glacial ice of the ice sheets. Climate models need to describe the dynamics of sea-ice on large scales and couple the large-scale sea-ice models to ocean general circulation models. This is also the perspective we pursue in this work.

This paper treats the problem of formulating the discrete sea-ice dynamics in a way such that the internal sea-ice dynamics are captured well while at the same time the external coupling to the ocean is accomplished in a natural way. The modelling problem we aim to solve consists in choosing approximation spaces that capture (compressible) sea-ice dynamics as well as the (incompressible) ocean dynamics and that allow a minimal-invasive coupling between the two models that avoids interpolations or projections. In global ocean modelling we observe a trend towards a Arakawa C-type staggering of variables, where scalar
variables are described as piecewise constant functions that are located at the center of the grid cell and velocity fields are represented by normal components of the velocity vector along the cell boundary (see e.g. [6], [25], [29], [18]). This trend reflects the increase in computational power towards high-resolution simulation where the efficient C-type staggering has advantageous discrete wave propagation properties once the Rossby radius is resolved over large part of the global domain. For sea-ice dynamics the small stencil of C-type staggering has the advantage that transport in narrow straits, which are only one cell wide is possible as well as an accurate representation of internal waves [2]. The modelling challenge stems from the fact that sea-ice dynamics requires the full strain rate tensor \( \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \) which is difficult to discretize if only partial information about the velocity field is available.

We consider this problem on triangular grids in the context of the ocean general circulation model ICON-O [18]. ICON-O uses a refined triangular mesh of an icosahedron that is approximating the surface of the sphere as described in [18]. On triangular grids the C-type-staggering is equivalent to the lowest order Raviart-Thomas finite element (RT-0). The space of the Raviart-Thomas finite element is not rich enough to approximate the full strain rate tensor [1].

We propose to enlarge the approximation space by including the tangential velocity at the mid point of an edge. This variable arrangement allows the desired natural coupling to the underlying ocean variables on the same grid. This enrichment of the Raviart-Thomas element results in the specification of the complete velocity vector at edge midpoints and equals to the first order nonconforming Crouzeix-Raviart finite element (CR). The Crouzeix-Raviart element is a classical finite element that has been applied for approximations of the Poisson problem and the Navier-Stokes equations. Lietear et al. [23] used the CR element to discretize the Canadian Arctic Archipelago in an uncoupled sea-ice model.

A direct application of the Crouzeix-Raviart element to the sea-ice equations leads to an unstable discretization that has its origin in the discretization of the symmetric strain rate tensor in the rheology. The reason of this instability is the non-trivial kernel of the strain rate tensor discretized with the Crouzeix-Raviart element. Thus, the element does not satisfy the first Korn inequality \( \| \nabla \mathbf{v} \|^2 \leq c \| \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \|_2^2 \) [8]. This inequality bounds the Jacobian matrix of partial derivatives by the symmetric part of the Jacobian matrix in terms of the \( L^2 \)-norm. The symmetric part of the Jacobian matrix is part of the stress tensor of the viscous-plastic sea-ice rheology.

In order to circumvent this instability we introduce in this paper a stabilization of the Crouzeix-Raviart element and demonstrate through numerical experiments that the stabilized Crouzeix-Raviart element is capable to discretize the viscous-plastic and elastic-viscous-plastic sea-ice model. Adding the suggested stabilization the Crouzeix-Raviart element fulfills a generalized version of Korn’s inequality [28]. Given the parallels between the stress tensor in the viscous-plastic rheology and models of linear elasticity [13], the stabilization is inspired from the stabilization of the Crouzeix-Raviart element for a linear elastic problem, introduced by Hansbo and Larson [1]. We furthermore show numerically...
that a stabilization of the sea-ice velocity is necessary in the viscous-plastic and elastic-viscous-plastic model. The numerical experiments are carried out in the framework of the ocean general circulation model ICON-O [18] that operate on a triangular C-grid. To analyze the effect of the stabilization, we derive an energy estimate for the viscous-plastic sea-ice model, where Korn’s inequality is essential to limit the viscous-plastic stress tensor. Based on this estimate we introduce an energy for the viscous-plastic and elastic-viscous-plastic rheology and investigate the limitation of the energy using the new discretization in ICON-O.

The paper is structured as follows. In Section 2 we introduce the sea-ice model in a strong an variational form and derive the energy estimate for the viscous-plastic sea-ice momentum equation. In Section 3 we introduce the realization of Crouzeix-Raviart element in ICON-O and describe the stabilization for the viscous-plastic and elastic-viscous-plastic model. In Section 4 we numerically analyze and validate the stabilized Crouzeix-Raviart element. We conclude in Section 5.

2 Model description

The motion of sea-ice is prescribed in a two-dimensional framework [22]. The momentum of sea-ice is modeled as

$$\rho h \partial_t \mathbf{v} = \text{div}(\mathbf{\sigma}) + F,$$  \hspace{1cm} (1)

where \( \rho \) is the sea-ice density and \( h \) the mean ice thickness. All external forces are collected in \( F \).

$$F = \tau - \rho h g \nabla h_d - \rho h f_c \mathbf{e}_r \times \mathbf{v},$$

where \( f_c \) is the Coriolis parameter, \( g \) is the gravity, \( \mathbf{e}_r \) unit normal vector to the surface and \( h_d \) is the ocean surface height. In order to express the surface height with the Coriolis term, we follow Coon [4] and use

$$\rho h g \nabla h_d \approx -\rho h f_c \mathbf{e}_r \times \mathbf{v}_w,$$

where \( \mathbf{v}_w \) is the ocean current. The stresses due wind and ocean are expressed as

$$\tau = \rho_a C_a \| \mathbf{v}_a \| \mathbf{v}_a + \rho_w C_w \| \mathbf{v}_w \| (\mathbf{v}_w - \mathbf{v}),$$

where \( \rho_a, \rho_w \) are the air and water densities, \( C_a, C_w \) are the wind and water drag coefficients and \( \mathbf{v}_a \) is the geostrophic wind.

The viscous-plastic sea-ice rheology (VP) The internal stresses \( \mathbf{\sigma} \) are related to the strain rate \( \dot{\mathbf{e}} = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \) by the viscous-plastic rheology

$$\mathbf{\sigma} = 2\eta \dot{\mathbf{e}} + (\zeta - \eta) \text{tr}(\dot{\mathbf{e}})I - \frac{P}{2}I,$$  \hspace{1cm} (2)
with the viscosities $\eta, \zeta$, and the ice strength $P$. Following Hibler the ice strength is modeled as

$$P = P^* h \exp(-C(1 - A)),$$

with $P^*$ the ice strength parameter, $A$ the ice concentration and $C$ the ice concentration parameter [10]. In the viscous-plastic sea-ice model introduced by Hibler [10], the viscosities are derived from an elliptic yield curve with eccentricity $e=2$ and a normal flow rule. They are modeled as

$$\zeta = \frac{P}{2\Delta}, \quad \eta = \zeta e^{-2}, \quad \Delta = \sqrt{2e^{-2} \dot{\varepsilon}' : \dot{\varepsilon}' + \text{tr}(\dot{\varepsilon})^2},$$

where we apply the decomposition of the strain rate tensor $\dot{\varepsilon} = \dot{\varepsilon}' + \frac{1}{2} \text{tr}(\dot{\varepsilon}) I$ into the deviatoric part $\dot{\varepsilon}'$ and into its trace $\text{tr}(\dot{\varepsilon}) I$. As described by Hibler [10] the plastic viscosity are limited by a viscous regime given as

$$\zeta_{\text{min}} \leq \zeta \leq \zeta_{\text{max}}, \quad \zeta_{\text{min}} = 4 \cdot 10^8 \text{kg/s}, \quad \zeta_{\text{max}} = \frac{P}{2\Delta_{\text{min}}}, \quad \Delta_{\text{min}} = 2 \cdot 10^{-9} \text{s}.$$

The limitation avoids that $\zeta, \eta \to \infty$ for $\Delta \to 0$. To regularize the transition from the viscous to the plastic regime we follow Harder [19] and use

$$\Delta = \sqrt{\Delta_{\text{min}}^2 + 2e^{-2} \dot{\varepsilon}' : \dot{\varepsilon}' + \text{tr}(\dot{\varepsilon})^2}. $$

For the numerical analysis done in the paper we do not use a replacement pressure [19] and minimal values for the viscosities.

**The elastic-viscous-plastic sea-ice rheology (EVP)** The elastic-viscous-plastic model was introduced to regularize the VP rheology, such that the VP model results from the EVP model in the steady state $\partial_t \sigma = 0$. We reformulate the viscous-plastic model to

$$\frac{1}{2\eta} \sigma + \frac{\eta - \zeta}{4\eta \zeta} \text{tr}(\sigma) I + \frac{P}{4\zeta} I = \frac{\zeta}{T} \dot{\varepsilon}',$$

and add an artificial elastic strain behaviour with a parameter $E$

$$\frac{1}{E} \partial_t \sigma + \frac{1}{2\eta} \sigma + \frac{\eta - \zeta}{4\eta \zeta} \text{tr}(\sigma) I + \frac{P}{4\zeta} I = \dot{\varepsilon}.$$

Hunke [11] introduced $T = \frac{\zeta}{E}$ such that the elastic-viscous-plastic model results as

$$\partial_t \sigma + \frac{e^2}{2T} \sigma + \frac{1 - e^2}{4T} \text{tr}(\sigma) I + \frac{P}{4T} I = \frac{\zeta}{T}.$$

The EVP model allows a fully explicit discretization in time with relatively large time steps such that it gains in numerical efficiency compared to the VP model [11]. In Section 3 we introduce a modified EVP approach (mEVP), a pseudo-time solver for the VP sea-ice model based on the EVP formulation.

All constants used in the momentum equation are defined in Table 1.
| Parameter | Definition          | Value       |
|-----------|---------------------|-------------|
| $\rho$    | sea-ice density     | $900 \text{ kg/m}^3$ |
| $\rho_a$  | air density         | $1.3 \text{ kg/m}^3$ |
| $\rho_w$  | water density       | $1026 \text{ kg/m}^3$ |
| $C_a$     | air drag coefficient | $1.2 \cdot 10^{-3}$ |
| $C_w$     | water drag coefficient | $5.5 \cdot 10^{-3}$ |
| $f_c$     | Coriolis parameter  | $1.46 \cdot 10^{-4} \text{s}^{-1}$ |
| $P^*$     | ice strength parameter | $27.5 \cdot 10^3 \text{ N/m}^2$ |
| $C$       | ice concentration parameter | 20 |
| $e$       | ellipse ratio       | 2           |

Table 1: Physical parameters of the momentum equation.

**Transport equations**  The mean ice thickness and ice concentration are advected in time by

$$\partial_t h + \text{div}(vh) = Q_h, \quad \partial_t A + \text{div}(vh) = Q_A,$$

(5)

where $A$ is limited from above by 1.0. Here we skip thermodynamic source terms on the right hand side of the transport equations as our analysis focuses on the sea-ice dynamics and set $Q_h = 0$ and $Q_A = 0$.

**Weak formulation**  We multiply the momentum equation with a test function $\phi \in H^1_0(\Omega)^2$ and integrate over space and time. The transport equations are multiplied with test functions $\phi \in L^2(\Omega)$

$$\int_I (\rho \partial_t v, \phi) - (F, \phi) + (\sigma, \nabla \phi) dt = 0$$

(6)

$$\left(\partial_t h + \text{div}(vh), \phi\right) = 0,$$

(7)

$$\left(\partial_t A + \text{div}(vh), \phi\right) = 0,$$

(8)

where $(\cdot, \cdot)$ denotes the $L^2$-inner product on a two-dimensional domain $\Omega \subseteq \mathbb{R}^2$. The $L^2$-norm is denoted by $\| \cdot \|$.

2.1 Energy estimate

In this section we derive an energy estimate for the viscous-plastic momentum equation. Based on the derivation we define an energy for the VP and EVP model.

**Theorem 1 (Energy estimate viscous-plastic sea-ice momentum equation)**

*Let $h > h_{\text{min}} > 0$ and fixed, then the solution of the sea-ice momentum equation...*
equation (1) $v \in H^1_0(\Omega)$ fulfills the following energy estimate

$$
\|\rho_{h_{\min}} v(T)\|^2 + \int_0^T 2\rho_w C_{w1}\|v\|^2 dt
+ c_k \kappa_{\min} \|\nabla v\|^2 dt
\leq \int_0^T \frac{2c_p}{c_k \kappa_{\min}} \|R\|^2 + \frac{2}{3\kappa_{\min}} \|P\|^2 dt + \|\rho_{h_{\min}} v(0)\|^2,
$$

where $c_k$ and $c_p$ are positive constants that depend on the domain and where $R := \rho_a C_a \|v_a\| v_a + \rho_w C_{w1} v_w - \rho h_{\text{fe}} r \times v_w$.

**Proof.** We consider the momentum equation given in (1), multiply it with $v$ and integrated it over a domain $\Omega$.

$$(\rho h \partial_t v, v) - (F v + \text{div}(\sigma), v) = 0.$$  

Then, we apply the chain rule to the time dependent integral and get

$$(\rho h \partial_t v, v) = \frac{1}{2} \int_\Omega \rho h \partial_t |v|^2 dxdy \geq \frac{1}{2} \int_\Omega \rho h_{\min} \partial_t |v|^2 dxdy = \frac{1}{2} \partial_t \|\rho h_{\min} v\|^2. \quad (9)$$

We proceed with analyzing the external forces

$$(F, v) = (\tau + \rho h f_{\text{fe}} \times v_w - \rho h f_{\text{fe}} \times v, v). \quad (10)$$

The integral over the Coriolis force $(f_{\text{fe}} \times v, v)$ vanishes as the product $(\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{a} \times \vec{c}) \cdot \vec{b}$ is anti-commutative. The wind and ocean drag simplifies to

$$(\tau, v) = (\rho_a C_a \|v_a\| v_a + \rho_w C_{w1} \|v - v_w\| (v_w - v), v)
= (\rho_a C_a \|v_a\| v_a + \rho_w C_{w1} (v_w - v), v)
= (\rho_a C_a \|v_a\| v_a + \rho_w C_{w1} v_w, v) - \rho_w C_{w1} \|v\|^2,$$

where we assume a linearized wind drag. Following Leppäranta [22] we use fixed scaling speeds, indicated with the subscript 0, and estimate the wind drag as

$$C_{w1} = C_w [v_w - v]_0. \quad (11)$$

We collect all $v$ independent terms of equation (10)

$$R = (\rho_a C_a \|v_a\| v_a + \rho_w C_{w1} v_w, v) + \rho h f_{\text{fe}} \times v_w,$$

and move $\rho_w C_{w1} \|v\|^2$ to the left hand side of the equation. Next we reformulate
the stress tensor given in equation (2).

\[
- \text{div}(\sigma) = (\sigma, \nabla v) = \frac{1}{2} (\sigma + \sigma^T, \nabla v) = \frac{1}{2} (\sigma, \nabla v + \nabla v^T) = (\sigma, \dot{\epsilon})
\]

\[
= (2\eta \dot{\epsilon} + (\zeta - \eta) \text{tr}(\dot{\epsilon}) I - \frac{P}{2} I, \dot{\epsilon})
\]

\[
= \left( \frac{1}{2} \dot{\epsilon} + \frac{3}{4} \zeta \text{tr}(\dot{\epsilon}) I - \frac{P}{2} I, \dot{\epsilon} \right)
\]

\[
= \left( \frac{1}{2} \zeta \dot{\epsilon}, \dot{\epsilon} \right) + \left( \frac{3}{4} \zeta \text{tr}(\dot{\epsilon}), \text{tr}(\dot{\epsilon}) \right) - \left( \frac{P}{2}, \text{tr}(\dot{\epsilon}) \right)
\]

\[
\geq \| \sqrt{2}^{-1} \sqrt{\zeta} \dot{\epsilon} \| ^2 + \| \frac{\sqrt{3}}{2} \sqrt{\zeta} \text{tr}(\dot{\epsilon}) \| ^2
\]

\[
- \| \frac{1}{\sqrt{2c_p^2}} P \| ^2 - \| \frac{\sqrt{c_p}}{2} \text{tr}(\dot{\epsilon}) \| ^2,
\]

where we applied Young’s inequality in the last term with \( c_p = \frac{3\zeta}{4} \). We move the \( P \) dependent term to the right hand side and apply Korn’s first inequality for homogeneous Dirichlet boundary values [3],

\[
\| \dot{\epsilon} \| ^2 \geq c_k \| \nabla v \| ^2,
\]

where \( c_k \) is the positive constant of Korn’s inequality. We get

\[
\| \sqrt{2}^{-1} \sqrt{\zeta} \dot{\epsilon} \| ^2 + \| \frac{\sqrt{3}}{2} \sqrt{\zeta} \text{tr}(\dot{\epsilon}) \| ^2 \geq \| \frac{\zeta_{\min}}{2} \| \dot{\epsilon} \| \geq c_k \frac{\zeta_{\min}}{2} \| \nabla v \| ^2.
\]

Combining the estimate for the water drag and the stress tensor gives

\[
\left( \rho_u C_w v - \text{div}(\sigma), v \right) \geq \rho_u C_w \| v \| ^2 + c_k \frac{\zeta_{\min}}{2} \| \nabla v \| ^2 - \| \frac{1}{\sqrt{2c_p}} P \| ^2. \quad (12)
\]

Finally, we reformulate the right hand side \( \mathcal{R} \). We apply Young’s and Poincare’s inequality with homogeneous Dirichlet boundaries [7],

\[
\| u \| \leq c_p \| \nabla u \|,
\]

where \( c_p \) is a positive constant of Poincare’s inequality. Thus, we get

\[
\left( \mathcal{R}, v \right) \leq \frac{1}{2\epsilon} \| \mathcal{R} \| ^2 + \frac{\epsilon}{2} c_p \| \nabla v \| ^2,
\]

where we choose

\[
\epsilon = c_k c_p^{-1} \frac{\zeta_{\min}}{2},
\]

to move \( \zeta c_p \| \nabla v \| ^2 \) to the left hand side of the equation. Taken into account the estimates for the temporal integral [9], the spatial operator [12] as well as the right hand side [13] we get

\[
\partial_t \| \rho h_{\min} v \| ^2 + 2 \rho_u C_w \| v \| ^2 + c_k \frac{\zeta_{\min}}{2} \| \nabla v \| ^2 \leq \frac{1}{\epsilon} \| \mathcal{R} \| ^2 + \| \frac{1}{\sqrt{2c_p}} P \| ^2.
\]
We integrate over a time interval $I = [0, T]$ and get the final estimate

$$\|\rho h_{\min} v(T)\|^2 - \|\rho h_{\min} v(0)\|^2 + \int_0^T 2\rho_w C_{w1} \|v\|^2 + \frac{c_k}{2} \|\nabla v\|^2 dt \leq \int_0^T \frac{1}{\epsilon} \|R\|^2 + \|\frac{1}{\sqrt{2\epsilon_p}} \frac{P}{2}\|^2 dt.$$  

q.e.d

The viscosities can change in space locally over several orders of magnitudes \[27\]. Thus limiting the viscosities globally with a $\zeta_{\min}$ in order to apply Korn’s inequality is a very rough estimate. Motivated by the energy estimate derived in Theorem 1, we define an energy which is more sensitive to the plastic model behavior. As the contribution from the ocean stress is comparable small to the gradient weighted with the viscosity we define the energy as

$$E(v) = \|\frac{\zeta}{2} \nabla v\|^2. \quad (14)$$

The limitation of the energy is numerically analyzed in Section 4.

3 Discretization

In this section we describe the spatial and temporal discretization of the VP and EVP model. We introduce an edge-based stabilization for both models discretized with the Crouzeix-Raviart finite element.

Time discretization  To solve the coupled sea-ice system \[6\] it is standard to use a splitting approach in time. As described by Lemieux et al. \[20\] we first compute the solution of the sea-ice momentum equation (1), followed by the solution of the transport equations (5). Ip et al. \[14\] pointed out that a fully explicit time stepping scheme for the momentum equation with a VP rheology would require a small time step of less than a second - even on a grid resolution as coarse as 100 km by 100 km. Therefore the authors recommended an implicit treatment. An implicit discretization asks for implicit solution methods such as a Picard solver \[30\] or Newton like methods \[21\], \[27\]. So far the applied solvers are difficult to parallelize as efficient linear solver are missing \[24\]. To avoid an implicit discretization Hunke and Dukowicz \[12, 11\] introduced the EVP model, where they add an artificial elastic term to the VP rheology, to allow an explicit discretization of the momentum equation with relatively large time steps. However, the EVP model produces large differences compared to approximations of the VP model. Thus, Kimmritz et al. \[15\] and Boullion et al. \[2\] developed, based on the EVP model formulation, explicit pseudo-time stepping methods that converge against the solution of the VP model. As spatial discretization errors dominate the temporal discretization errors, first order time stepping methods are sufficient to discretize the sea-ice momentum.
equation [26]. In [20] Lemieux and coauthors describe a second-order time stepping scheme to solve the VP model. For the temporal discretization of the momentum equation (1) we apply a first order semi-implicit time stepping scheme. We introduce the time partitioning \( 0 = t_0 < ... < t_N = T \) and the time step size \( k := t_n - t_{n-1} \). Let \( \mathbf{v}^n := \mathbf{v}(t_n) \), \( h^n := h(t_n) \), \( A^n := A(t_n) \).

**VP model** The discretized viscous-plastic momentum equation reads as

\[
\left( \rho h^{n-1} \frac{\mathbf{v}^n - \mathbf{v}^{n-1}}{k}, \phi \right) = \left( \rho h^{n-1} f_e e \times (\mathbf{v}_w^{n-1} - \mathbf{v}_w^{n-1}), \phi \right) - (\sigma^n, \nabla \phi) + \tau^n_{VP},
\]

with

\[
\tau^n_{VP} := \left( \rho_a C_a \|\mathbf{v}_w^{n-1}\| \mathbf{v}_w^{n-1}, \phi \right) + \left( \rho_a C_w \mathbf{v}_w^{n-1} - \mathbf{v}_w^{n-1} \right)(\mathbf{v}_w^{n-1} - \mathbf{v}_w^n), \phi \).
\]

(15)

**EVP model** To solve the elastic-viscous-plastic model we sub-cycle the momentum equation. Let \( t^{n-1} \leq t^{s-1} < t^s \leq t^n \) and \( k_s := t^s - t^{s-1} \). Then the subsycled momentum equation reads as

\[
\left( \rho h^{n-1} \frac{\mathbf{v}^s - \mathbf{v}^{s-1}}{k_s}, \phi \right) = \left( \rho h^{n-1} f_e e \times (\mathbf{v}_w^{n-1} - \mathbf{v}_w^{n-1}), \phi \right) - (\sigma^s, \nabla \phi) + \tau^s_{EVP},
\]

with

\[
\tau^s_{EVP} := \left( \rho_a C_a \|\mathbf{v}_w^{n-1}\| \mathbf{v}_w^{n-1}, \phi \right) + \left( \rho_a C_w \mathbf{v}_w^{n-1} - \mathbf{v}_w^{n-1} \right)(\mathbf{v}_w^{n-1} - \mathbf{v}_w^s), \phi \).
\]

(16)

The elastic-viscous-plastic stress is calculated via

\[
\frac{\sigma_1^s - \sigma_1^{s-1}}{k_s} + \frac{\sigma_1^s}{2T} = \frac{\zeta^{s-1} (\dot{\epsilon}_{11}^{s-1} + \dot{\epsilon}_{22}^{s-1})}{T} - \frac{P}{2T},
\]

\[
\frac{\sigma_2^s - \sigma_2^{s-1}}{k_s} + \frac{4\sigma_2^s}{2T} = \frac{\zeta^{s-1} (\dot{\epsilon}_{11}^{s-1} - \dot{\epsilon}_{22}^{s-1})}{T},
\]

\[
\frac{\sigma_{12}^s - \sigma_{12}^{s-1}}{k_s} + \frac{4\sigma_{12}^s}{2T} = \frac{\zeta^{s-1} \dot{\epsilon}_{12}^{s-1}}{T},
\]

with \( \sigma_1 = \sigma_{11} + \sigma_{22}, \sigma_2 = \sigma_{11} - \sigma_{22}, \dot{\epsilon}_1 = \dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_2 = \dot{\epsilon}_{11} - \dot{\epsilon}_{22} \) and \( \zeta^{s-1} := \zeta(v^{s-1}), \dot{\epsilon}^{s-1} := \dot{\epsilon}(v^{s-1}). \) For all computations done in this paper we use the tuning parameter \( T = 100. \) The number of sub-cycles \( s = 1, ..., N_{evp} \) is usually a large number around 100 or more [12] and \( k_s = \frac{k}{N_{evp}}, \) where \( k \) is the larger time step of the advection.

**mEVP solver** We approximate the viscous-plastic stress tensor with an elastic-viscous-plastic formulation. The elastic-viscous-plastic model is sub-cycled in
Figure 1: Left: C-grid staggering in the ICON earth system model, where the normal of horizontal velocities $v \cdot n$ are placed at edge mid points and the scalars are saved at cell centers. Right: The Crouzeix-Raviart finite element, where the horizontal velocity is represented by its normal and tangential component.

time, such that the formulation converges in time against the viscous-plastic formulation. Each sub-iteration $s$ of the momentum equation reads as

$$\left( \beta \frac{\rho h^{n-1}}{k_s} (v^s - v^{s-1}), \phi \right) = \left( \rho h^{n-1} \frac{v^n + v^n}{k_s}, \phi \right) - \left( \sigma^s, \nabla \phi \right)$$

$$- \left( \rho h^{n-1} f_c \varepsilon \times (v^{s-1} - v^{n-1}), \phi \right) + \tau_{mEVp}^n$$

with

$$\tau_{mEVp}^n := \left( \rho a C_a \| v_{atm}^{n-1} \| v_{atm}^{n-1}, \phi \right) + \left( \rho w C_w \| v_w^{n-1} - v^{s-1} \| (v_w^{n-1} - v_s), \phi \right).$$

(17)

where we time step the stress tensor $\sigma^s$ as

$$\alpha(\sigma_1^s - \sigma_1^{s-1}) = \sigma_1^{s-1} + 2\zeta^{s-1}(\varepsilon_1^{s-1} - P),$$

$$\alpha(\sigma_2^s - \sigma_2^{s-1}) = \sigma_2^{s-1} \frac{\zeta^{s-1}}{2} \varepsilon_2^{s-1},$$

$$\alpha(\sigma_{12}^s - \sigma_{12}^{s-1}) = \sigma_{12}^{s-1} \frac{\zeta^{s-1}}{2} \varepsilon_{12}^{s-1}.$$

A more detailed description of the mEVP solver can be found in the work of Boullion et al. [2]. Here $\alpha$ and $\beta$ are large constants. As analyzed by Boullion et al. [2] and Kimmritz et al. [15], the product $\alpha \beta$ should be sufficient large to fulfill the CFL-criterium. If not further specified we follow Koldunov et al. [17] and use $\alpha = \beta = 500$ in this paper.

**Spatial discretization** In the following we describe the realization of the Crouzeix-Raviart finite element in ICON-O. Figure shows the Raviart-Thomas
Figure 2: **Left**: Basis function $\phi_i$ of the Crouzeix-Raviart finite element with $\phi_i(x_i) = 1$, $x \in e_i$ and $\phi_i(E_j) = 0$, for $j \neq i$. **Right**: Support of a basis function at edge $e_i$.

...element that corresponds to the C-type staggering in ICON-O and the Crouzeix-Raviart finite element (CR). Both elements are coupled to a $P^0$ element for the scalar variables.

By $\Omega_h$ we introduce the triangulation of a domain $\Omega$ into triangles $T_i$ that fulfill the usual assumption of structure and shape regularity. Let $V_h = \text{span} \{ \phi_i, i = 1, \ldots, N \} \subset L^2(\Omega)$ be the space of the Crouzeix-Raviart element. The edge mid points are denoted by $E_i \in \Omega_h$ and the edge itself by $e_i$. In each node $E_i$ a finite element basic function is given as

\[ \phi_i(E_j) = \delta_{ij}, \quad \forall i, j = 1, \ldots, N, \quad \partial_n \phi_i(E_j) = \frac{2}{h_i}, \quad \partial_t \phi_i(E_j) = 0, \]

where $N$ represents the number of edges in $\Omega_h$ and $h_i$ is the height orthogonal to edge $e_i$ of a triangle. To simplify the notation we define $\phi_i = \phi_i(E_i)$. The outward normal vector and the tangential vector to edge $e_i$ at node $E_i$ is denoted by $n_i$ and $\tau_i$. The transposed is indicated by $T$. In Figure 2 we motivate that $\partial_t \phi_i = 0$ as the solid line through 1 is constant. Further, the differential quotient along the dashed blue lines in Figure 2 gives $\partial_n \phi_i = \frac{1}{2^{-1}h_i} = \frac{2}{h_i}$. Using the discrete space $V_h$ we can express the velocity vector $v_h$ as

\[
v_h = \sum_{i=1}^{N} (v_i n_i + u_i \tau_i) \phi_i. \tag{18}\]

The gradient of $v_h$ is given as

\[
\nabla v_h = \sum_{i=1}^{N} (v_i n_i + u_i \tau_i) \nabla^T \phi_i = \sum_{i=1}^{N} (v_i n_i + u_i \tau_i) n_i^T \partial_n \phi_i,
\]
and the transposed gradient $\nabla v_h^T$ reads

$$\nabla v_h^T = \sum_{i=1}^N (v_i n_i^T + u_i \tau_i^T) \nabla \phi_i = \sum_{i=1}^N (v_i n_i^T + u_i \tau_i^T) n_i \partial_n \phi_i.$$

We also decompose the basis functions $\phi$ and $\nabla \phi$ into the normal and tangential component.

$$\phi = \sum_{j=1}^N n_j \phi_j^n + \tau_j \phi_j^\tau, \quad \nabla \phi = \sum_{j=1}^N n_j n_j^T \partial_n \phi_j^n + \tau_j n_j^T \partial_n \phi_j^\tau.$$

Using these decomposition of the domain $\Omega_h$ in triangles, we can formulate the discrete momentum equation \([6]\) over a time interval $I = [0, T]$ as

$$\int_I \left\{ \sum_{i,j=1}^3 \left( \rho h \partial_t (v_i n_i + u_i \tau_i) \phi_i - F(v_i n_i + u_i \tau_i) \phi_i, n_j n_j^T \partial_n \phi_j^n + \tau_j n_j^T \partial_n \phi_j^\tau \right) \right\}_T + \left( \sigma_{i,h}, \sum_{j=1}^N n_j n_j^T \partial_n \phi_j^n + \tau_j n_j^T \partial_n \phi_j^\tau \right)_T \right\} dt = 0,$$

where the subscript $T$ denotes the $L^2$-integral over a triangle $T$. All integrals except of the one with stress tensor $(\sigma_{i,h}, \sum_{j=1}^N n_j n_j^T \partial_n \phi_j^n + \tau_j n_j^T \partial_n \phi_j^\tau)_T$ are zero order expressions and require an approximation of the mass matrix $M$, which is defined as

$$M = \begin{pmatrix} M_{nn} & M_{nr} \\ M_{rn} & M_{rr} \end{pmatrix}, \quad M_{i,j} = \begin{pmatrix} (n_i \phi_i^n, n_j \phi_j^n) & (\tau_i \phi_i^\tau, n_j \phi_j^n) \\ (n_i \phi_i^n, \tau_j \phi_j^\tau) & (\tau_i \phi_i^\tau, \tau_j \phi_j^\tau) \end{pmatrix}.$$

We approximate this integral by applying the trapezoid rule and derive the lumped mass matrix

$$M_L = \text{diag} \left( \begin{pmatrix} (n_i \phi_i^n, n_i \phi_i^n)_{tr} & (\tau_i \phi_i^\tau, n_i \phi_i^n)_{tr} \\ (n_i \phi_i^n, \tau_i \phi_i^\tau)_{tr} & (\tau_i \phi_i^\tau, \tau_i \phi_i^\tau)_{tr} \end{pmatrix} \right),$$

where the subscript $tr$ indicates the evaluation with the trapezoid rule. As the scalar product of $\langle n_i, \tau_i \rangle = 0$, the lumped mass matrix simplifies to

$$M_L = \text{diag} \left( \begin{pmatrix} (n_i \phi_i^n, n_i \phi_i^n)_{tr} & 0 \\ 0 & (\tau_i \phi_i^\tau, \tau_i \phi_i^\tau)_{tr} \end{pmatrix} \right).$$

The support of $(\phi_i, \phi_i)_tr$ consists only of the two triangles $T_i^1, T_i^2$ that share the edge $e_i$, as shown in Figure\([2]\). In the framework of equation \([19]\) the evaluation of the integral with the trapezoid rule on a triangle $T$ yields

$$\langle \phi_i, \phi_i \rangle_{tr|T} = \frac{|T|}{3},$$

12
where \( E_i(T) \) refers to the edge midpoints of triangle \( T \). Next we consider the integral over the discretized stress tensor \( \sigma_h = (\sigma_{11,h}, \sigma_{12,h}, \sigma_{22,h}) \), which can be written as

\[
(\sigma_h, \nabla \phi)_T = \sum_{E_i(T), E_j(T)} (\sigma_{i,h} n_j n_i^T \partial_n \phi_j^m + \tau_j n_j^T \partial_n \phi_j^m) T
\]

\[
= \sum_{E_i(T), E_j(T)} (\sigma_{i,h} n_j n_i^T \frac{2}{|h_j|} + \tau_j n_j^T \frac{2}{|h_j|}) T.
\]

We evaluate the integral with the midpoint rule and get

\[
(\sigma_{i,h} n_j n_i^T \partial_n \phi_j^m) T = \sum_{E_i(T), E_j(T)} |T| \frac{2}{|h_j|} (\sigma_{i,h}^{11} n_j^1 n_j^1 + 2\sigma_{i,h}^{12} n_j^1 n_j^2 + \sigma_{i,h}^{22} n_j^2 n_j^2),
\]

where we calculate the entries \( \sigma_{i,h} \) based on the strain rate tensor

\[
\frac{1}{2} (\nabla v_h + \nabla v_h^T) T = \sum_{E_i(T)} (v_i n_i + u_i \tau_i) n_i^T \frac{2}{|h_i|} + (v_i n_i^T + u_i \tau_i^T) n_i \frac{2}{|h_i|}.
\]

The transport equations are discretized with an upwind scheme.

**Stabilization** As detailed by Falk [8] discretizing the strain rate tensor with the Crouzeix-Raviart element, \( \nabla v_h + \nabla v_h^T \), has a non trivial kernel. Thus, the element violates the discrete version of Korn inequality [8, 16]. However by adding weighted velocity jumps to the triangle edges, the element fulfills a generalized discrete version of Korn’s inequality [28],

\[
\|\nabla v_h\|^2 \leq c \left( \|\nabla v_h + \nabla v_h^T\|^2 + \sum_{e_i} \int_{e_i} \frac{1}{|e_i|} [v_{e_i}] [v_{e_i}] \; dx dy, \right)
\]

(20)

Here \([v_{e_i}] = v_{e_i}^+ - v_{e_i}^-\) is the jump of the function \( v_h \in V_h \) at an edge \( e_i \) with \( v^\pm = \lim_{x \to 0} v(x \pm e_i) \) and \( x \in e_i \).

To stabilize the Crouzeix-Raviart element in the VP model we follow the analysis of Hansbo and Larson [9], where they developed a stable Crouzeix-Raviart discretization for a linear elastic problem. Based on this idea we add to viscous-plastic momentum equation (15) at each edge \( e_i \) the term

\[
S_{V,P,i,j}^c := 2\zeta_{e_i} \alpha_{VP} \frac{\alpha_{VP}}{|e_i|} \int_{e_i} [v_{e_i}] [\phi_{e_i}] \; dx dy
\]

(21)

with \( i,j = 1, .. N \). Here, the positive constant \( \alpha_{VP} \) was found by experimental tuning and is chosen as \( \alpha_{VP} = \frac{1}{2} \times 10^{-5} \). The weakly consistent stabilization is derived from a discontinuous Galerkin formulation. More details can be found in [9].
Adding the stabilization to spatially discretized momentum equation (19) yields

\[
\int_I \left\{ \sum_{i,j=1}^3 \left( \rho \partial_t (v_i n_i + u_i \tau_i) \phi_i - (F(v_i n_i + u_i \tau_i) \phi_i, n_j \phi_j^T + \tau_j^T \phi_j^T) \right)_T 
+ \left( \sigma_{i,k} n_j^T \partial_n \phi_j^T + \tau_j n_j^T \partial_n \phi_j^T \right)_T + \frac{1}{2} S_{VP,i,j}^{ST} \right\} dt = 0,
\]

where \( \partial T \) is the boundary of a triangle \( T \). We add \( \frac{1}{2} \) to the stabilization as we loop over all cells and each interior edge of the domain is called twice.

In numerical test we found that the stabilization \( S_{VP,i,j}^{ST} \) (21), which we add to the discretization of the VP model leads unstable numerical solutions if the model is solved with the mEVP solver. We observe the same behavior for the EVP model stabilized with \( S_{VP,i,j}^{ST} \). In both cases, these instabilities stem from the spatial variation of \( \Delta \), which is part of the viscosity \( \zeta = \frac{P}{\Delta} \) (see equation (3)).

Inspired by the generalized discrete version of Korn’s inequality (20), we found that a spatial and temporal constant \( \Delta_s \) produces stable results, if the mEVP solver or the EVP model is used. In this cases, we replace the stabilization \( S_{VP,i,j}^{ST} \) in the momentum equation (22) with

\[
S_{EV,i,j}^{V,i,j} := \alpha \frac{P}{\Delta_s} \int_e [v_{e,i}] [\phi_{e,j}] \, dx dy,
\]

where we choose a fixed \( \Delta_s \). In this paper we set \( \Delta_s = 1 \) s\(^{-1}\) and \( \alpha = \alpha_{EV,i,j} = \alpha_{mEVP} = 0.01 \) when we apply the mEVP solver.

It is left to outline the calculation of integrals \( \int_e [v_{e,i}] [\phi_{e,j}] \) in (21) and (23). The support of the integral over an edge \( e_i \) consists of the two triangles that share the edge \( e_i \) (see Figure 2). In order to evaluate

\[
\int_{e_i} [v_{e,i}] [\phi_{e,i}]
\]

we have to take into account the coupling of the test and ansatz functions along the five neighboring edges shown in Figure 2. Since \( \int_{e_i(T)} [\phi_i(T)] = 0 \, dx \) for \( T = T_{i1}, T_{i2} \), the stencil reduces to the test and ansatz functions defined at the four surrounding edges \( e_{ij}, j = 1, \ldots, 4 \). We define \( l_i := |e_i| \). For \( j, k = 1, \ldots, 4 \) the integral over an edge \( e_i \) is given as

\[
\int_{e_i} \phi_{ij} \phi_{ik} = \int_{\frac{l_i}{2}}^{\frac{3l_i}{2}} 2x \frac{2x}{l_i} \, dx = \begin{cases} \frac{1}{3} l_i, & \text{if } j = k, \\ -\frac{1}{3} l_i, & \text{else}. \end{cases}
\]

Next, we reformulate expression (24) and get for \( j, k = 1, \ldots, 4 \)
To calculate the stabilization in each edge $e_{ij}$, we again loop over all triangles and evaluate in each edge the contribution of the velocity jump. In a second step algorithm, in which we first loop over all triangles and calculate for each edge of the triangle the contribution of the velocity jump. In a second step, we again loop over all triangles and evaluate in each edge the contribution of the weighted integral.

**Algorithm 1 (Computation of weighted cross element jumps)** Let $e_{ik}$, $k = 1, \ldots, 3$ be the counter-clockwise numbered edges of a triangle $T_i$. By $S^x_i$, $S^y_i$ and $A^n_i$, $A^t_i$ we denote edge based vectors, where $S_x = [S^x_i, S^y_i]$ and $A_x = [A^n_i, A^t_i]$. To calculate the stabilization in each edge $e_{ik}$:

1. **Velocity jumps** $[v_{e_{ij}}]$ : we loop over all triangles $T_i$ and calculate for each edge of the triangle the entry $S^x_{e_{ij}} = \sum E_{ij} [v_{ij} n_{ij} \phi^0_{ij} + u_{ij} \tau_{ij} \phi^T_{ij}]$ of $S^x_{e_{ij}}$, with $j = 1, \ldots, 3$, such that

\[
S^x_{e_{11}} = S_{e_{11}} + v_{i2} n_{12} - v_{i3} n_{13} + u_{i2} \tau^1_{12} - u_{i3} \tau^1_{13},
\]

\[
S^x_{e_{12}} = S^x_{e_{12}} + v_{i3} n_{13} - v_{i1} n_{11} + u_{i3} \tau^1_{13} - u_{i1} \tau^1_{11},
\]

\[
S^x_{e_{13}} = S^x_{e_{13}} + v_{i1} n_{11} - v_{i2} n_{12} + u_{i1} \tau^1_{11} - u_{i2} \tau^1_{12}.
\]

The entries of $S^y_{e_{ij}}$ are calculated analogously where we use the second component of the normal and tangential vectors $n_{ij}^2$ and $\tau^2_{ij}$.

2. **Weighted velocity jumps** $[v_{e_{ij}}][\phi_{e_{ij}}]$. We loop over all $T_i$ and compute $A_{e_{ij}} = \sum E_{ij}, E_{ik} \int [v_{ij} n_{ij} \phi^0_{ij} + u_{ij} \tau_{ij} \phi^T_{ij}] [n_{ik} \phi^0_{ik} + \tau_{ik} \phi^T_{ik}], \text{ with } j, k = 1, \ldots, 3$

\[
A_{e_{11}} = A_{e_{11}} + \left( S_{e_{11}} n_{e_{11}} - S_{e_{12}} n_{e_{12}} + S_{e_{13}} \tau_{e_{11}} - S_{e_{12}} \tau_{e_{12}} \right) \frac{1}{3} l_{e_{11}},
\]

\[
A_{e_{12}} = A_{e_{12}} + \left( S_{e_{11}} n_{e_{11}} - S_{e_{12}} n_{e_{12}} + S_{e_{13}} \tau_{e_{11}} - S_{e_{12}} \tau_{e_{12}} \right) \frac{1}{3} l_{e_{12}},
\]

\[
A_{e_{13}} = A_{e_{13}} + \left( S_{e_{12}} n_{e_{12}} - S_{e_{13}} n_{e_{13}} + (S_{e_{12}} \tau_{e_{12}} - S_{e_{13}} \tau_{e_{13}} \right) \frac{1}{3} l_{e_{13}}.
\]

If we sort by normal and tangential vectors we get $A^n_i$ and $A^t_i$ respectively.
\[ \sigma = \frac{\zeta}{2} \left( \nabla v + \nabla v^T \right) \]

\[ \sigma_1 = \frac{\zeta}{2} \nabla v \]

\[ \sigma \text{ stabilized} \]

Figure 3: Evaluation of a simplified form of the momentum equation in the viscous regime with \( \zeta = \zeta_{\text{min}} \).

Figure 4: Test case with a stationary solution. We evaluate the approximation of the EVP and VP model with and without stabilization of the momentum equation. The VP model is solved either purely explicit or with the mEVP solver. The EVP model is solved semi-implicit.
4 Numerical evaluation

In this section we numerically analyze the discretization of the sea-ice momentum equation with the Crouzeix-Raviart element. In Section 4.1 we start with analyzing the strain rate tensor in the viscous regime. In Section 4.2 we move to the full viscous-plastic and elastic-viscous-plastic rheology and investigate the effect of the stabilization on the velocity field and the energy of the sea-ice system, defined in (14). Finally in Section 4.3, we evaluate the full system describing the sea-ice dynamics including the advection of the ice thickness and sea-ice concentration. We analyze a box test, which is a slight modified version of the test case introduced by Danilov et al. [5].

4.1 Strain rate tensor

We start our analysis with a simplified version of the momentum equation (1).

\[ \partial_t v - \text{div}(\sigma(v)) = R, \]  

and consider the viscous-plastic stress tensor in the viscous regime.

\[ \sigma = \frac{\zeta}{2} (\nabla v + v^T), \quad \zeta = \frac{P}{2\Delta_{\text{min}}}, \quad h = 1, A = 1. \]

We switch off the advection of the sea-ice thickness and the concentration and use \( h = 1, A = 1 \).

The domain is a planar quadrilateral with length of \( L_x = L_y = 500 \) [km] in \( x \) and \( y \) direction. The domain is tesselated by a mesh of equilateral triangles. We start the simulation with zero initial velocities. As boundary conditions we use homogeneous Dirichlet boundary conditions. Given an analytic solution \( v_1 = v_2 = -\sin(\pi x) \sin(\pi y) \), with \( \pi_x := \frac{\pi}{L_x} \) and \( \pi_y := \frac{\pi}{L_y} \), the right hand side of (25) is

\[ R = \frac{\zeta}{2} \left( \pi_x^2 \sin(\pi_x x) \sin(\pi_y y) + \frac{\pi_y^2}{2} \sin(\pi_x x) \sin(\pi_y y) \right) - \frac{1}{2} \pi_x \pi_y \cos(\pi_x x) \cos(\pi_y y). \]

We observe instabilities in the velocity field shown in left plot in Figure 3. This behavior is consistent with fact that the Crouzeix-Raviart element does not fulfill the discrete version of Korn’s inequality

\[ c_k \| \nabla v_h \|^2 \leq \| \nabla v_h + \nabla v_h^T \|, \]

where \( c_k \) is a positive constant depending on the size of the domain. This instabilities vanish if we consider \( \sigma_1 = \frac{\zeta}{2} \nabla v \) instead of the symmetric stress tensor \( \sigma = \frac{\zeta}{2} (\nabla v + v^T) \). The instabilities also vanish by adding a stabilization term \( S_{\text{VP},i,j} \) to the velocity. Here, we set \( \alpha_{VP} = \frac{1}{6} \cdot 10^{-8} \). We observe that
Figure 5: We zoom into the velocity calculated with the mEVP solver with and without stabilization. We show the lower left corner of the velocity \( v^1 \) shown in Figure (4).

Figure 6: We show the advected mean ice thickness \( h \) with and without applying a stabilization to the mEVP solver.

the stabilization slightly damps the solution. Without stabilization the velocity components reach their maximum at \( v^1_h = 1.052 \) and \( v^2_h = 1.28 \). Replacing \( \sigma \) by \( \sigma_1 \) reduces the maximal velocity to \( v^1_h = v^2_h = 1.027 \). Adding the edge-stabilization for \( \sigma \) to equation (25) the velocities further decrease the maxima to \( v^1_h = 1.023 \) and \( v^2_h = 1.005 \). For the runs that include \( \sigma_1 \) we adjusted the right hand side of equation (25) to \( R = \zeta \left( \pi_x^2 + \pi_y^2 \right) \sin \left( \pi_x x \right) \sin \left( \pi_y y \right) \) to converge against the same analytic solution \( v^1 = v^2 = - \sin \left( \pi_x x \right) \sin \left( \pi_y y \right) \). All simulations presented in Figure [3] are computed with an explicit Euler method using a time step \( k = 1 \cdot 10^{-6} [s] \) on a triangular mesh with 3833 edges in ICON.
4.2 Viscous-plastic and elastic-viscous-plastic rheology

We numerically analyze the effect of the Crouzeix-Raviart discretization on the approximations of the EVP and VP model. For our investigation we neglect the wind stress and the Coriolis force and consider

\[ \rho h \partial_t \mathbf{v} = \text{div}(\mathbf{\sigma}) - \rho_w C_{uw} \| \mathbf{v} - \mathbf{v}_w \| (\mathbf{v} - \mathbf{v}_w). \]

We follow Hunke [11] and Danilov et al. [5] and choose the ocean velocity as

\[ v_{1w} = 0.1(2y - L_y)/L_y, \quad v_{2w} = -0.1(L_x - 2x)/L_x. \]

The viscous-plastic stress tensor and the elastic-viscous-plastic stress tensor is given in (2) and in (4). We consider the same quadrilateral domain as in Section 4.1 with a triangular grid and use homogeneous Dirichlet boundary conditions. The initial ice velocities are

\[ \mathbf{v}(t_0) = 0. \]

In a first test we switch off the advection of the ice thickness and ice concentration and use \( h = 1 \) and \( A = \frac{1}{x} \). For this configuration the velocities converge against a stationary solution, such that we can compare the solutions of the EVP model to those of the VP model. The approximation of the momentum equation with the viscous-plastic rheology is solved in two ways. First fully explicit, using the forward Euler time-stepping method and a time step of \( k = 0.1 \) [s], second with the mEVP solver described in Section 3 and a time step of \( k = 600 \) [s] and \( N_{\text{mvp}} = 2000 \) sub-iterations. For both cases we observe instabilities in the velocity field. The same holds for the EVP model. Here we advect the ice concentration and the mean ice thickness with \( k = 600 \) [s], whereas the momentum equation is subcycled with \( k_s = 10 \) [s].

As shown in Figure 4, we observe instabilities in the velocity in regions with high ice concentration. With the corresponding stabilization these instabilities vanish and all three approximations produce similar results. In Figure 5 we zoom into the approximation of the VP model computed with the mEVP solver and show the lower left corner of the domain. One can see that using the edge-stabilization the oscillation in the Crouzeix-Raviart element completely disappears.

In a second test we switch on the advection and observe that the instabilities of the velocities propergate into the tracers fields. This can be seen in Figure 6 which shows the mean ice thickness after one simulated day. Here, the mean ice thickness is computed with the mEVP solver with and without stabilization. With this configuration we also analyse the energy of the sea-ice system

\[ E(\mathbf{v}) = \| \sqrt{\zeta} \mathbf{v} \|^2 \]

defined in Section 2.1. We evaluate the energy for the EVP model and the VP model with and without stabilization. In case of the VP model we use either a fully explicit discretization or the mEVP solver. We plot \( E(\mathbf{v}) \) in Figure 7 and observe that the energy deteriorates without stabilization, whereas the energy is bounded in the stabilized case.
Figure 7: Energy $E(\mathbf{v})$ evaluated for the EVP and VP model with and without stabilization. The approximation of the VP model is calculated fully explicit (VP) or with the mEVP solver.

### 4.3 Box test

We investigate the full system of sea-ice equations. The test case is a slightly modified version of the box test described by Danilov et al. [5]. We simulate the sea-ice dynamics for one month on a squared domain of length $L_x = L_y = 1000$ [km]. The domain is discretized with a triangular mesh of equilateral triangles with a side length of approximately 15 km and 15190 edges. We use the ocean current described in Section 4.2. The wind velocity is prescribed by

\[
\begin{align*}
\mathbf{v}_{1\text{ atm}}^x &= 5 + (\sin(2\pi t/T) - 3)(\sin(2\pi x/L_x) \sin(2\pi y/L_y)), \\
\mathbf{v}_{2\text{ atm}}^x &= 5 + (\sin(2\pi t/T) - 3)(\sin(2\pi y/L_y) \sin(2\pi x/L_x)).
\end{align*}
\]

We assume homogeneous Dirichlet boundary conditions and initial data given by $\mathbf{v}(t_0) = 0$, $h(t_0) = 1$ and $A(t_0) = \frac{\mathbf{v}}{L_x}$. We use a time step of $k = 600$ [s] and $N_{\text{evp}} = 500$ sub-cycles. In a first configuration, we solve the momentum equation with the mEVP solver and keep the ice concentration and the ice thickness constant with $A = \frac{\mathbf{v}}{L_x}$ and $h = 1.0$.

After one month we observe instabilities in the velocities in regions with large gradients in $\Delta$ (see (3)) and high ice concentrations. As shown in Figure 8 the oscillations vanish if the stabilization is applied.
With active advection we note also instabilities in the velocity field in regions with high ice concentration. Now the oscillation are not visible in the ice thickness and ice concentration field. As shown in Figure 9 stabilizing the momentum equation results into a smooth representation of the velocities. The approximation of the velocity with the stabilized Crouzeix-Raviart element leads to a slightly different distribution of the sea-ice thickness in the upper right corner of the domain. Here, the stabilization is chosen as described in Section 3.

5 Conclusion

In this paper we introduced a new sea-ice discretization on the triangular grid in ICON. The discretization is based on a stabilized nonconforming Crouzeix-Raviart finite element, which consists of normal and tangential velocity components staggered at the edge midpoints of a triangle. The velocities are coupled to cell wise constant representations of the tracers. This staggering allows a straightforward coupling to C-grid ocean and atmosphere discretizations. We numerically showed that a direct discretization with the Crouzeix-Raviart element leads to an unstable approximation of the velocities, as the element does
not fulfill a discrete version of Korn’s inequality. To overcome this issue we introduced an edge-based stabilization. We showed numerically that stabilizing the sea-ice velocity is necessary for both the viscous-plastic and elastic-viscous-plastic model. To analyze the effect of the stabilization in the VP and EVP model, we defined an energy of the sea-ice system. The definition is based on an energy estimate, which we derived for the viscous-plastic sea-ice momentum equation. Considering a time interval of interest the energy estimate limits the velocity in space and time by the right hand side of the momentum equation, the initial and final sea-ice velocity fields. We numerically evaluated the defined energy for an approximation of the viscous-plastic and elastic-viscous-plastic model and found that with the stabilization of the Crouzeix-Raviart element the energy stays bounded as in the continuous estimate. Without stabilization the energy of the viscous-plastic and elastic-viscous-plastic model blows up in time. This underlines the importance of stabilizing the Crouzeix-Raviart element when discretizing the sea-ice momentum equation.

References

[1] G. Acosta, T. Apel, R. Duran, and A. Lombardi. Error estimates for Raviart-Thomas interpolation of any order on anisotropic tetrahedra. *Math. Comput.*, 80:141–163, 2011.
[2] S. Bouillon, T. Fichefet, V. Legat, and G. Madec. The elastic-viscous-plastic method revisited. *Ocean Modelling*, 71:2–12, 2013.

[3] P.G. Ciarlet. On Korn’s inequality. *Chin. Ann. Math.*, 31:607–618, 2010.

[4] M.D. Coon. A review of AIDJEX modeling. In *Sea Ice Processes and Models: Symposium Proceedings*, pages 12–27. Univ. of Wash. Press, Seattle., 1980.

[5] S. Danilov, Q. Wang, R. Timmermann, N. Iakovlev, D. Sidorenko, M. Kimmritz, T. Jung, and J. Schröter. Finite-Element Sea Ice Model (FESIM), version 2. *Geosci. Model Dev.*, 8:1747–1761, 2015.

[6] A. Adcroft et al. The GFDL Global Ocean and Sea Ice Model OM4.0: Model Description and Simulation Features. *Journal of Advances in Modelling the Earth System*, 11:3167–3211, 2019.

[7] L. C. Evans. *Partial differential equations*. American Mathematical Society, 2010.

[8] R. Falk. Nonconforming Finite Element Methods for the Equations of Linear Elasticity. *Mathematics of Computation*, 57:529–529, 1991.

[9] P. Hansbo and M. Larson. Discontinuous Galerkin and the Crouzeix–Raviart element: Application to elasticity. *ESAIM*, 37:63–72, 2003.

[10] W.D. Hibler. A dynamic thermodynamic sea ice model. *J. Phys. Oceanogr*, 9:815–846, 1979.

[11] E.C. Hunke. Viscous-plastic sea ice dynamics with the EVP model: Linearization issues. *J. Comp. Phys.*, 170:18–38, 2001.

[12] E.C. Hunke and J.K. Dukowicz. An elastic-viscous-plastic model for sea ice dynamics. *J. Phys. Oceanogr.*, 27:1849–1867, 1997.

[13] J.K. Hutchings, H. Jasak, and S.W. Laxon. A strength implicit correction scheme for the viscous-plastic sea ice model. *Ocean Modelling*, 7:111–133, 2004.

[14] C.F. Ip, W.D. Hibler, and G.M. Flato. On the effect of rheology on seasonal sea-ice simulations. *Annals of Glaciology*, 15:17–25, 1991.

[15] M. Kimmritz, S. Danilov, and M. Losch. On the convergence of the modified elastic-viscous-plastic method for solving the sea ice momentum equation. *J. Comp. Phys.*, 296:90–100, 2015.

[16] P. Knobloch. On korn’s inequality for nonconforming finite elements. *Technische Mechanik* 20, 375:205 – 214, 2000.
[17] N. Koldunov, S. Danilov, D. Sidorenko, N. Hutter, M. Losch, H. Goessling, N. Rakowsky, P. Scholz, D. Sein, Q. Wang, and T. Jung. Fast evp solutions in a high-resolution sea ice model. *Journal of Advances in Modeling Earth Systems*, 11(5):1269–1284, 2019.

[18] P. Korn. Formulation of an unstructured grid model for global ocean dynamics. *J. Comp. Phys.*, 339:525–552, 2017.

[19] M. Kreyscher, M. Harder, P. Lemke, G. Flato, and M. Gregory. Results of the Sea Ice Model Intercomparison Project: Evaluation of sea ice rheology schemes for use in climate simulations. *J. Geophys. Res.*, 105:11299–11320, 2000.

[20] J.F. Lemieux, D. Knoll, M. Losch, and C. Girard. A second-order accurate in time IMplicit–EXplicit (IMEX) integration scheme for sea ice dynamics. *J. Comp. Phys.*, 263:375–392, 2014.

[21] J.F. Lemieux and B. Tremblay. Numerical convergence of viscous-plastic sea ice models. *J. Geophys. Res.*, 114(C5), 2009.

[22] M. Leppärinta. *The Drift of Sea Ice*. Springer-Verlag Berlin Heidelberg, 2011.

[23] O. Lietaer, T. Fichefet, and V. Legat. The effects of resolving the Canadian Arctic Archipelago in a finite element sea ice model. *Ocean Modelling*, 24:140–152, 2008.

[24] M. Losch, A Fuchs, J.F. Lemieux, and A. Vanselow. A parallel Jacobian-free Newton-Krylov solver for a coupled sea ice-ocean model. *J. Comp. Phys.*, 257:901–911, 2014.

[25] G. Madec. *NEMO ocean engine, Institut Pierre-Simon Laplace (IPSL), France*, 2012.

[26] C. Mehlmann. *Efficient numerical methods to solve the viscous-plastic sea ice model at high spatial resolutions*. PhD thesis, Otto-von-Guericke Universität Magdeburg, 2019.

[27] C. Mehlmann and T. Richter. A modified global Newton solver for viscous-plastic sea ice models. *Ocean Modeling*, 116:96–107, 2017.

[28] A. Ouazzi. *Finite Element Simulation of Nonlinear Fluids. Application to Granular Material and Powder*. Shaker, 2006. ISBN 3-8322-5201-0.

[29] T. Ringler, M. Petersen, R.L. Higdon, D. Jacobsen, P. W. Jones, and M. Maltrud. A multi-resolution approach to global ocean modeling. *Ocean Modeling*, 69:211–232, 2013.

[30] J. Zhang and W.D. Hibler. On an efficient numerical method for modeling sea ice dynamics. *J. Geophys. Res.*, 102:8691–8702, 1991.