Potential analysis of $\mathcal{N} = 2$ SUSY gauge theory with the Fayet-Iliopoulos term

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We analyze the vacuum structure of spontaneously broken $\mathcal{N} = 2$ supersymmetric gauge theory with the Fayet-Iliopoulos term. Our theory is based on the gauge group $SU(2) \times U(1)$ with $N_f = 2$ massless quark hypermultiplets having the same $U(1)$ charges. In the classical potential, there are degenerate vacua even in the absence of supersymmetry. It is shown that this vacuum degeneracy is smoothed out, once quantum corrections are taken into account. While there is the runaway direction in the effective potential, we found the promising possibility that there appears the local minimum with broken supersymmetry at the degenerate dyon point.

1. Introduction

There has been much progress in understanding the dynamics of strongly coupled supersymmetric (SUSY) gauge theories. Seiberg and Witten derived the exact low energy Wilsonian effective action for $\mathcal{N} = 2$ SUSY $SU(2)$ Yang-Mills theory, and generalized their discussion to the case with up to four massive quark hypermultiplets. The key ingredients which allow us to derive the exact results are duality and holomorphy.

The results by Seiberg and Witten were extended to the case with the explicit soft SUSY breaking terms by using spurion technique. Unless these terms do not change the holomorphy and duality properties of the theory, we can derive the exact effective action for $\mathcal{N} = 1$ and $\mathcal{N} = 0$ (non-supersymmetric) SUSY gauge theories up to the leading order for the soft SUSY breaking terms. In Refs. [3], the exact superpotential and the phase structure in $\mathcal{N} = 1$ SQCD were discussed based on the $\mathcal{N} = 2$ SUSY gauge theory with some soft breaking terms. In Refs. [4, 5], the vacuum structure of non-SUSY gauge theory was investigated in which soft SUSY breaking terms directly break $\mathcal{N} = 2$ SUSY to $\mathcal{N} = 0$.

In this paper, we study a spontaneously broken $\mathcal{N} = 2$ SUSY gauge theory. It is well known that, in the frame work of $\mathcal{N} = 2$ SUSY theory, the only possibility to break SUSY spontaneously is to introduce the Fayet-Iliopoulos (FI) term. Therefore, in the following, we consider the gauge theory which includes $U(1)$ gauge interaction together with the FI term.

The simplest example of this type of theory is $\mathcal{N} = 2$ SUSY QED (SQED) with the FI term. At the classical level, although SUSY is spontaneously broken in Coulomb branch, there are degenerate vacua (moduli space) parameterized by the vacuum expectation value of the scalar field, $a$, in the $U(1)$ vectormultiplet. The direction of this vacuum degeneracy in the absence of SUSY is called “pseudo flat” direction. However, it is expected that this direction is lifted up, once quantum corrections are taken into account. By virtue of $\mathcal{N} = 2$ SUSY, the effective action is found to be one loop exact, and the effective gauge coupling is given by $\epsilon(a)^2 \sim 1/\log(\Lambda_L/a)$, where $\Lambda_L$ is the Landau pole. Note that there are two singular regions in moduli space, namely, the ultraviolet (UV) region such as $|a| \geq \Lambda_L$ and the massless singular point at the origin $a = 0$. Since the effective potential is described as $V \sim \epsilon(a)^2$, the potential minimum emerges at the origin, where SUSY is formally restored. However, since this
point is the singular point, we conclude that there is no well-defined vacuum in this theory.

In this paper, we investigate the vacuum structure of more interesting theory with spontaneous $\mathcal{N} = 2$ SUSY breaking. Our theory is based on the gauge group $SU(2) \times U(1)$ with $N_f = 2$ massless quark hypermultiplets having the same $U(1)$ charges. In the UV region, the behavior of the effective potential can be well understood based on the perturbative discussion, since the $SU(2)$ gauge interaction is weak there. On the other hand, it is expected that the behavior of the effective potential in the infrared region is drastically changed compared with SQED, because of the presence of the $SU(2)$ gauge dynamics.

2. Classical structure of $\mathcal{N} = 2 \; SU(2) \times U(1)$ gauge theory

In this section, we briefly discuss the classical structure of our theory. The analysis of the classical potential was originally addressed in Ref. [9]. We describe the classical Lagrangian in terms of more interesting theory with spontaneous $\mathcal{N} = 2$ SUSY breaking. Our theory is based on the gauge group $SU(2) \times U(1)$ with $N_f = 2$ massless quark hypermultiplets having the same $U(1)$ charges. In the UV region, the behavior of the effective potential can be well understood based on the perturbative discussion, since the $SU(2)$ gauge interaction is weak there. On the other hand, it is expected that the behavior of the effective potential in the infrared region is drastically changed compared with SQED, because of the presence of the $SU(2)$ gauge dynamics.

\[ L_{\text{HM}} + L_{\text{VM}} + L_{\text{FI}}, \]

\[ L_{\text{HM}} = \int d^4 \theta \left( Q_i^\dagger e^{2V_2 + 2V_1} Q_i \right. \]

\[ + \hat{Q}_i e^{-2V_2 - 2V_1} \hat{Q}_i \right) \]

\[ + \sqrt{2} \left( \int d^2 \theta \hat{Q}_i (A_2 + A_1) Q_i + h.c. \right), \]

\[ L_{\text{VM}} = \frac{1}{2\pi} \text{Im} \left[ \text{tr} \left\{ \tau_{22} \left( \int d^4 \theta A_1^\dagger e^{2V_2} A_2 e^{-2V_2} \right. \right. \]

\[ + \frac{1}{2} \int d^2 \theta W_2^2 \left) \right\} \right] \]

\[ + \frac{1}{4\pi} \text{Im} \left[ \tau_{11} \left( \int d^4 \theta A_1^\dagger A_1 \right. \right. \]

\[ + \frac{1}{2} \int d^2 \theta W_1^2 \right) \right], \]

\[ L_{\text{FI}} = \int d^4 \theta V_1 , \]

where $\tau_{22} = i \frac{4\pi}{g^2} + \frac{\theta}{2\pi}$ and $\tau_{11} = i \frac{4\pi}{g^2}$ are the gauge couplings of the $SU(2)$ and the $U(1)$ gauge interactions, respectively. Here we take the notation, $T(R)\delta^{ab} = \text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$. The same $U(1)$ charges of the hypermultiplets are normalized to be one. The last term in (6) is the FI term with the coefficient $\xi$ of mass dimension two.

From the above Lagrangian, the classical potential is read off as

\[ V = \frac{1}{g^2} \text{tr}[A_2, A_1]^2 + \frac{g^2}{2}(q_i^T T^a q_i - \tilde{q}_i T^a \tilde{q}_i)^2 \]

\[ + \frac{e^2}{2} \left( \xi + q_i^T T^a q_i + 2g^2 |\tilde{q}_i T^a q_i|^2 \right) \]

\[ + 2 \left( q_i^T A_2 + A_1^T q_i - \tilde{q}_i A_2 + A_1^T \tilde{q}_i \right), \]

\[ \left( 7 \right) \]

where $A_2$, $A_1$, $q_i$ and $\tilde{q}_i$ are scalar components of the corresponding chiral superfields, respectively. The potential minimum is obtained by solving the stationary conditions with respect to these scalar components. There are some solutions, and one example is given by

\[ |q_i|^2 = 0, \quad |\tilde{q}_i|^2 = \frac{e^2}{4g^2 + e^2} \xi, \quad |\tilde{q}_j|^2 = 0 \quad (j \neq 1), \]

\[ A_2 + A_1 = \left( \begin{array}{cc} \frac{a_1}{2} & 0 \\ 0 & -\frac{a_2}{2} \end{array} \right) + \left( \begin{array}{cc} a_1 & 0 \\ 0 & a_1 \end{array} \right) \]

\[ = \left( \begin{array}{cc} 0 & 0 \\ 0 & z \end{array} \right), \]

\[ \left( 8 \right) \]

where $a_1$ and $a_2$ are complex parameters, and $z$ is arbitrary constant. In this example, the gauge symmetry $SU(2) \times U(1)$ is broken to $U(1)$. The potential energy is given by

\[ V = \frac{\xi^2 g^2}{4e^2 + g^2}. \]

\[ \left( 9 \right) \]

Note that the classical potential has the pseudo flat direction parameterized by $a_1$ or $a_2$ with the condition $a_1 + \frac{1}{2} a_2 = 0$. We expect that this direction is lifted up, once quantum corrections are
3. Quantum structure of $\mathcal{N} = 2$ $SU(2) \times U(1)$ gauge theory

3.1. Effective Action

In this subsection, we describe the low energy Wilsonian effective Lagrangian of our theory. If we could completely integrate the action to zero momentum, the exact effective Lagrangian $\mathcal{L}_{\text{exact}}$ could be obtained, which is described by light fields, the dynamical scale and the coefficient of the FI term $\xi$. However, this is highly non-trivial and very difficult task. In the following discussion, suppose that the coefficient $\xi$, the order parameter of SUSY breaking, is much smaller than the dynamical scale of the $SU(2)$ gauge interaction. Then we consider the effective action up to the leading order of $\xi$. The exact effective Lagrangian, if it could be obtained, can be expanded with respect to the parameter $\xi$ as

$$\mathcal{L}_{\text{exact}} = \mathcal{L}_{\text{SUSY}} + \xi \mathcal{L}_1 + \mathcal{O}(\xi^2).$$  

(10)

Here, the first term $\mathcal{L}_{\text{SUSY}}$ is the exact effective Lagrangian containing full SUSY quantum corrections. The second term is the leading term of $\xi$, and nothing but the FI term at tree level. Analyzing the effective Lagrangian up to the leading order of $\xi$, we obtain the effective potential of the order of $\xi^2$. The coefficient of $\xi^2$ in the effective potential includes full SUSY quantum corrections. Therefore, what we need to analyze the effective potential is nothing but the effective Lagrangian $\mathcal{L}_{\text{SUSY}}$.

Except the FI term, the classical $SU(2) \times U(1)$ gauge theory has moduli space, which is parameterized by $a_2$ and $a_1$. On this moduli space except the origin, the gauge symmetry is broken to $U(1)_c \times U(1)$. Here $U(1)_c$ denotes the gauge symmetry in the Coulomb phase originated from the $SU(2)$ gauge symmetry. Before discussing the effective action of this theory, we should make it clear how to treat the $U(1)$ gauge interaction part. In the following analysis, this part is, as usual, discussed as a cut-off theory. Thus, the Landau pole $\Lambda_L$ is inevitably introduced in our effective theory, and the defined region of the moduli parameter $a_1$ is constrained within the region $|a_1| < \Lambda_L$. According to this constraint, the defined region for moduli parameter $a_2$ is found to be also constrained in the same region, since two moduli parameters are related with each other through the hypermultiplets. We take the scale of $\Lambda_L$ to be much larger than the dynamical scale of the $SU(2)$ gauge interaction $\Lambda_2$, so that the $U(1)$ gauge interaction is always weak in the defined region of moduli space. Note that, in our framework, we implicitly assume that the $U(1)$ gauge interaction has no effect on the $SU(2)$ gauge dynamics. This assumption is justified in the following discussion about the monodromy transformation (see Eq. (13)).

We first discuss the general formulae for the effective Lagrangian $\mathcal{L}_{\text{SUSY}}$, which consists of two parts described by light vectormultiplets and hypermultiplets, $\mathcal{L}_{\text{SUSY}} = \mathcal{L}_{VM} + \mathcal{L}_{HM}$. The vectormultiplet part $\mathcal{L}_{VM}$, which is consistent with $\mathcal{N} = 2$ SUSY and all the symmetries in our theory, is given by

$$\mathcal{L}_{VM} = \frac{1}{4\pi} \text{Im} \left\{ \sum_{i,j=1}^2 \left( \int d^4\theta \frac{\partial F}{\partial A_i} A_i^j \right) + \int d^4\theta \frac{1}{2} \tau_{ij} W_i W_j \right\},$$  

(11)

where $F(A_2, A_1, \Lambda_2, \Lambda_L)$ is the prepotential, which is the function of moduli parameters $a_2$, $a_1$, the dynamical scale $\Lambda_2$, and the Landau pole $\Lambda_L$. The effective gauge coupling $\tau_{ij}$ is defined as

$$\tau_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} \ (i = 1, 2),$$  

(12)

The part $\mathcal{L}_{HM}$ is described by a light hypermultiplet with appropriate quantum number $(n_m, n_e)_n$, where $n_m$ is magnetic charge, $n_e$ is electric charge, and $n$ is the $U(1)$ charge. This part should be added to the effective Lagrangian around a singular point on moduli space, since
the hypermultiplet is expected to be light there and enjoys the correct degrees of freedom in the effective theory. The explicit description is given by
\[
\begin{align*}
\mathcal{L}_{HM} = & \int d^4 \theta (M^\dagger e^{2n_a v_2 D + 2m_v v_2 + 2m_v \hat{M}} M \\
+ & \hat{M} e^{-2n_a v_2 D - 2m_v v_2 - 2m_v \hat{M}} \hat{M}^\dagger) \\
+ & \sqrt{3} \left( \int d^2 \theta \hat{M} (n_m A_{2D} + n_c A_2 \\
& + n A_1) (M + h.c.) \right),
\end{align*}
\]
where \( M \) and \( \hat{M} \) denote light quark or light dyon hypermultiplet (the BPS states), and \( V_{2D} \) is the dual gauge field of \( U(1)_c \).

In order to obtain an explicit description of the effective Lagrangian, let us consider the monodromy transformation of our theory. Suppose that moduli space is parameterized by the vector-multiplet scalars \( a_2, a_1 \) and their duals \( a_{2D}, a_{1D} \) which are defined as \( a_{iD} = \partial F/\partial a_i \ (i = 1, 2) \). These variables are transformed into their linear combinations by the monodromy transformation. In our case, the monodromy transformation is subgroup of \( Sp(4, \mathbb{R}) \), which leaves the effective Lagrangian invariant, and the general formula is found to be
\[
\begin{pmatrix}
a_{2D} \\
a_2 \\
a_{1D} \\
a_1
\end{pmatrix} \rightarrow \begin{pmatrix}
a a_{2D} + \beta a_2 + p a_1 \\
\gamma a_{2D} + \delta a_2 + q a_1 \\
a_{1D} + p a_{2p} - q a_{2q} - p q a_1 \\
a_1
\end{pmatrix},
\]
where \( a_{2p} = \gamma a_{2D} + \delta a_2, \ a_{2q} = \alpha a_{2D} + \beta a_2, \ \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \in SL(2, \mathbb{Z}), \) and \( p, q \in \mathbb{Q} \). Note that this monodromy transformation for the combination \((a_{2D}, a_2, a_1)\) is exactly the same as that for SQCD with massive quark hypermultiplets, if we regard \( a_1 \) as the same mass of the hypermultiplets such that \( m = \sqrt{2} a_1 \). This means that the \( U(1) \) gauge interaction part plays the only role as the mass term for the \( SU(2) \) gauge dynamics. This observation is consistent with our assumption. On the other hand, the \( SU(2) \) dynamics plays an important role for the \( U(1) \) gauge interaction part, as can be seen in the transformation law of \( a_{1D} \). This monodromy transformation is also used to derive dual variables associated with the BPS states. As a result, the prepotential of our theory turns out to be essentially the same as the result in \cite{2} with understanding the relation \( A_1 = m/\sqrt{2} \).

\[
F(A_2, A_1, A_2, A_L) = F_{SU(2)}(A_2, m, A_2) \bigg|_{A_1 = \frac{m}{\sqrt{2}}} + C A_1^2,
\]
where the first term is the prepotential of \( N = 2 \) SQCD with hypermultiplets having the same mass \( m \), and \( C \) is free parameter. The freedom of the parameter \( C \) is used to determine the scale of the Landau pole relative to the scale of the \( SU(2) \) dynamics.

3.2. Effective Potential
The effective potential can be read off from the above Lagrangian with the FI term. Eliminating the auxiliary fields by using the equations of motion, we obtain
\[
V = \frac{b_{22}}{2 \det b} \xi^2 + S(a_2, a_1) \left\{ (|M|^2 - |\hat{M}|^2)^2 \\
+ 4 |M \hat{M}|^2 \\
- U(a_2, a_1)(|M|^2 - |\hat{M}|^2) \right\},
\]
where \( S, T \) and \( U \) are defined as
\[
\begin{align*}
S(a_2, a_1) &= \frac{1}{2} - \frac{(b_{12} - nb_{22})^2}{2 \det b}, \\
T(a_2, a_1) &= |n_m a_{2D} + n_a a_2 + n a_1|^2, \\
U(a_2, a_1) &= \frac{b_{12} - nb_{22}}{\det b}. \xi.
\end{align*}
\]
Solving the stationary conditions with respect to the hypermultiplet, we obtain three solutions as follows:
\[
\begin{align*}
1. & \quad M = \hat{M} = 0; \quad V = \frac{b_{22}}{2 \det b} \xi^2, \\
2. & \quad |M|^2 = \frac{2T - U}{2S}, \hat{M} = 0; \quad V = \frac{b_{22}}{2 \det b} \xi^2 - S |M|^4, \\
3. & \quad M = 0, |\hat{M}|^2 = \frac{2T + U}{2S}; \quad V = \frac{b_{22}}{2 \det b} \xi^2 - S |\hat{M}|^4.
\end{align*}
\]
The solution (21) or (22), in which the light hypermultiplet acquires the vacuum expectation value, is energetically favored, because of det $b > 0$ and $S(a_2, a_1) > 0$. Since the hypermultiplet appears in the theory as the light BPS state around the singular point on moduli space, the potential minimum is expected to emerge there. On the other hand, the solution (20) describes the potential energy away from the singular points, which smoothly connects to the solution (21) or (22).

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3.3. Periods and Effective Couplings

It was shown that the effective potential is described by the periods $a_{2D}$, $a_2$ and the effective coupling $b_{ij}$. In this subsection, we derive the periods and the effective couplings in order to give an explicit description of the effective potential.

As already discussed, the periods $a_{2D}$ and $a_2$ are the same as that of SQCD. The periods are defined as the contour integrals $a_{2D} = \oint c_1 \lambda$, $a_2 = \oint c_2 \lambda$, where $\lambda$ is a meromorphic differential on the algebraic curve, and the cycles $c_1$ and $c_2$ are defined so as to encircle the roots of the algebraic curve $e_2$ and $e_3$, and $e_1$ and $e_3$, respectively. In our case, the roots are given by

$$e_1 = \frac{u}{24} - \frac{\Lambda^2}{64} - \frac{1}{8} \sqrt{u + \frac{\Lambda^2}{8} + \Lambda_2 m \sqrt{u + \frac{\Lambda^2}{8} - \Lambda_2 m}},$$

$$e_2 = \frac{u}{24} - \frac{\Lambda^2}{64} + \frac{1}{8} \sqrt{u + \frac{\Lambda^2}{8} + \Lambda_2 m \sqrt{u + \frac{\Lambda^2}{8} - \Lambda_2 m}},$$

$$e_3 = \frac{u}{12} + \frac{\Lambda^2}{32}. \quad (23)$$

Then, the periods are described as $\oint c_1 (a_{2D} \text{ and } a_2 \text{ are denoted by } a_{21} \text{ and } a_{22}, \text{ respectively})$

$$a_{2i} = -\frac{\sqrt{2}}{4\pi} \left(-\frac{4}{3} u I_1^{(i)} + 8 I_2^{(i)} + \frac{m^2 \Lambda^2}{2} I_3^{(i)} \left(-\frac{u}{12} - \frac{\Lambda^2}{32}\right) - \frac{m}{\sqrt{2}} \delta_{i2} \right), \quad (24)$$

with the integral $I_i^{(1)} (i = 1, 2, 3)$ given by

$$I_1^{(1)} = \int_{e_2}^{e_3} \frac{dX}{Y} = \frac{iK(k')}{\sqrt{e_2 - e_1}}, \quad (25)$$

$$I_2^{(1)} = \int_{e_2}^{e_3} \frac{XdX}{Y} = \frac{i e_1}{\sqrt{e_2 - e_1}} K(k') + i \sqrt{e_2 - e_1} E(k'), \quad (26)$$

$$I_3^{(1)} = \int_{e_2}^{e_3} \frac{dX}{Y(X-c)} = \frac{-i}{(e_2 - e_1)^{3/2}} \frac{K(k')}{k + c} + \frac{4k}{1 + k c^2} - k^2 \Pi_1 \left(\nu, \frac{1 - k}{1 + k}\right), \quad (27)$$

where $k^2 = \frac{e_1 - e_2}{e_2 - e_1}$, $k'^2 = 1 - k^2 = \frac{e_1 - e_3}{e_3 - e_1}$, $c = -\frac{\Lambda^2}{32}$ is the pole of the meromorphic differential, $\tilde{c} = \frac{e_1 - e_2}{e_2 - e_1}$, and $\nu = -\left(\frac{k + c}{k}ight)^2 \left(\frac{k^2 + k}{1 + k}\right)^2$. The formulae for $I_i^{(2)}$ are obtained from $I_i^{(1)}$ by exchanging the roots, $e_1$ and $e_2$. In Eqs. (23)-(27),

Figure 1. The flow of the singular points in $N_f = 2$ case with the same mass.


$K$, $E$, and $\Pi_1$ are the complete elliptic integrals \cite{11a}. Next we give the effective couplings defined as \cite{12}. After some calculations, we obtain

$$
\tau_{22} = \frac{\omega_1}{\omega_2}, \quad (28)
$$

$$
\tau_{12} = -\frac{2z_0}{\omega_2}, \quad (29)
$$

$$
\tau_{11} = -\frac{1}{\pi i} \left[ \log \sigma(2z_0) + \frac{4z_0^2}{\omega_2} \right] + C, \quad (30)
$$

where $\omega_i$ is the period of the Abelian differential, $\omega_i = \oint_{\gamma_i} d\phi = 2f^{(i)}_1$ $(i = 1, 2)$, $z_0 = -\frac{1}{\sqrt{\varepsilon_2 - \varepsilon_1}} F(\phi, k)$ ($F(\phi, k)$ is the incomplete elliptic integral of the first kind with $\sin^2 \phi = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}$), $\sigma$ is the Weierstrass sigma function, and $C$ is the constant in Eq. (13).

Note that, since the gauge coupling $b_{11}$ is found to be a monotonically decreasing function of large $|a_1|$ with fixed $u$, and vice versa, the scale of the Landau pole is defined as $|a_1| = \Lambda_L$ at which $b_{11} = 0$. The large $\Lambda_L$ required by our assumption is realized by taking appropriate value. In the following analysis, we fix $C = 4\pi i$, which corresponds to $\Lambda_L \sim 10^{18}$ for fixed $\Delta_2 = 2\sqrt{2}$

4. Potential Analysis

Based on the results given by the previous sections, now let us investigate the vacuum structure of our theory. Since the effective potential is the function of two complex moduli parameters $u$ and $a_1$, it is a very complicated problem to figure out behaviors of the effective potential in the whole parameter space. However, note that, for our aim, it is enough to evaluate the potential energy just around each singular points, because these points are energetically favored (see Eqs. (21)-(22)). The singular points on the moduli space parameterized by $u$ flow according to the variation of $a_1$. In the following discussion, we evaluate the effective potential along the flow of the singular points, and examine which point is energetically favored on the line of the flow.

Let us first consider the flow of the singular points. The discriminant of the algebraic curve can be easily solved and leads to the three singular points such that $u_1 = -\sqrt{2}a_1\Lambda_2 - \frac{\Delta_2^2}{8}$, $u_2 = \sqrt{2}a_1\Lambda_2 - \frac{\Delta_2^2}{8}$ and $u_3 = (\sqrt{2}a_1)^2 + \frac{\Delta_2^2}{8}$. We investigate the case Im$a_1 = 0$, for simplicity. The flow of the singular points is sketched in Fig. 1. For $a_1 = 0$, the singular points appear at $u_1 = u_2 = -1$ and $u_3 = 1$. Here, at $u = -1$, two singular points degenerate. For non-zero $a_1 > 0$, this singular point splits into two singular points $u_1$ and $u_2$, which corresponds to the BPS states with quantum numbers $(1, 1)_1$ and $(1, 1)_1$, respectively. As $a_1$ is increasing, these singular points, $u_1$ and $u_2$, are moving to the left and the right on real $u$-axis, respectively. Two singular points, $u_2$ and $u_3$, collide at $u = 3\Lambda_2^2/8$ for $a_1 = \Lambda_2/(2\sqrt{2})$. This collision point is called Argyres-Douglas (AD) point \cite{12}, at which the theory is believed to transform into $\mathcal{N} = 2$ superconformal theory. As $a_1$ is increasing further, We consider only the case $a_1 > 0$, since the result for $a_1 < 0$ can be obtained by exchanging $u_1 \leftrightarrow u_2$, as be seen from the solution of the discriminant.

Figure 2. The left figure shows the potential for $a_1 = 0.4$. The middle and right figures show the evolutions of the potential energies at the singular points $u_2$ and $u_3$, respectively.
there appear two singular points $u_2$ and $u_3$ again, and quantum numbers of the corresponding BPS states, $(1, 1)_1$ and $(0, 1)_0$, change into $(1, 0)_1$ and $(1, -1)_1$, respectively. The singular point $u_2$ is moving to the right faster than $u_3$.

We investigate the vacuum structure by varying the values of $a_1$. For $a_1 = 0.4$, the effective potential is plotted in Fig. 3 (left). While there appear the potential minima at two singular points $u_1$ and $u_2$, the monopole condensation is too small for the potential to have a minimum at the singular point $u_3$. The top figure with the cusps shows the effective potential without the dyon condensations, and the bottom figure shows one with the condensations. Note that the cusps are smoothed out in the bottom figure. This means that the dyons enjoy the correct degrees of freedom in our effective theory around the singular point. The evolutions of the values of one potential minimum, $4$, and we find that the effective potential is also bounded in this direction. We can check that the effective potential is bounded from below for all the values of small $|a_1|$. Therefore, there is a possibility that the effective potential has the local minimum at the points $u = -1$ and $a_1 = 0$.

However, note that our description is not applicable for small $|a_1|$, since the condensations of two dyon states are going to overlap with each other. Unfortunately, we have no knowledge about the correct description of the effective theory in this situation. Nevertheless, we conclude that there must appear the local minimum with broken SUSY in the limit $a_1 \to 0$ from the result in the following. For the limit $a_1 \to 0$, the effective potential without the dyon condensations is depicted in Fig. 4. We can find that there appears the minimum at $u = -1$, and the value of the effective potential on the cusp is non-zero, $V \sim 0.0061$. If we had the correct description of the effective theory for $a_1 = 0$, this cusp might be smoothed out. However, there is no reason that SUSY is restored at $u = -1$, because the correct effective theory must have no singularity for the Kahler metric. Therefore, there is the promising possibility of the appearance of the local minimum with broken SUSY at $u = 1$ and $a_1 = 0$.

Finally, let us get back to discussion of the case $\text{Im} a_1 = 0$. The effective potential for $a_1 > \frac{\Lambda}{2\sqrt{2}}$ of the values of one potential minimum,$^4$ and we find that the effective potential has the CP symmetry under the exchange $u \leftrightarrow u^\dagger$. 

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$^4$ Two potential minima are degenerate, since the effective potential has the CP symmetry under the exchange $u \leftrightarrow u^\dagger$. 

Figure 3. The effective potential (left) for $a_1 = i\frac{\sqrt{2}}{4}, \xi = 0.1$ along $u = -1$ and the evolution of the minimum (right).
has two minima only at two singular points $u_1$ and $u_2$. The monopole condensation is too small for the effective potential to have a minimum at $u_3$. While the evolution of the value of the potential minimum on $u_1$ is the same as for $0 < a_1 < \frac{\Lambda_2}{2\sqrt{2}}$, the value of the potential minimum on $u_2$ point is monotonically decreasing, as $a_1$ is increasing. Thus, there is a runaway directions along the flow of the quark singular point $u_2$. We can find that the runaway direction always appears along the quark singular point for general complex $a_1$ values.

![Figure 4. The effective potential without the contribution of the dyon condensations in the limit $a_1 \to 0$.](image)

5. Conclusion

We analyzed the vacuum structure of spontaneously broken $\mathcal{N} = 2$ SUSY gauge theory with the Fayet-Iliopoulos term. Our theory is based on the gauge group $SU(2) \times U(1)$ with $N_f = 2$ massless quark hypermultiplets having the same $U(1)$ charges.

We formulated the effective action up to the leading order of the SUSY breaking parameter. Then the effective potential is obtained as the function of the moduli parameters. Examining the minimum of the effective potential, we found that the singular points are energetically favored, because of the condensations of the light BPS states. The singular points flow according to the values of the moduli parameters. Thus, we analyzed the effective potential along the flows of the singular points, and examined which point is energetically favored on the line of the flow.

While there is the runaway directions along the flow of the quark singular point, we found the promising possibility that the local minimum with broken SUSY appears at the degenerate dyon point. Therefore, this point is the unique and promising candidate for the well-defined vacuum. Unfortunately, we have no knowledge about the correct description of the effective theory around the degenerate singular point, since the condensations of two BPS states well overlap there.

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