Fluid-plate interaction under periodic forcing

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The motion of a thin elastic plate interacting with a viscous fluid is investigated. A periodic force acting on the plate is considered, which in a setting without damping could lead to a resonant response. The interaction with the viscous fluid provides a damping mechanism due to the energy dissipation in the fluid. Moreover, an internal damping mechanism in the plate is introduced. In this setting, we show that the periodic forcing leads to a time-periodic (non-resonant) solution. We employ the Navier-Stokes and the Kirchhoff-Love plate equation in a periodic cell structure to model the motion of the viscous fluid and the elastic plate, respectively. Maximal $L^q$ regularity for the linearized system is established in a framework of time-periodic function spaces. Existence of a solution to the fully nonlinear system is subsequently shown with a fixed-point argument.

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1 Introduction

The interaction of an incompressible viscous fluid with a thin elastic structure is investigated. We consider a three-dimensional fluid filled container with an elastic plate as part of its boundary. Our aim is to obtain a good understanding of the damping effects on the elastic structure in such a setting. To this end, we introduce a periodic force, which in a setting without damping could lead to onset of resonance in the structure. We develop a framework in which the damping effects under periodic forcing can be quantified, and employ the framework to show how damping prevents resonance when the periodic force is sufficiently restricted in magnitude.

In order to simplify the mathematical analysis, we consider a simple geometry in which the container’s stress free configuration is a cuboid with its bottom face an elastic plate. We assume the motion of the viscous fluid is governed by the Navier-Stokes equations, and the motion of the plate by the Kirchoff-Love plate equations. To further simplify the analysis, we impose periodic boundary conditions on the lateral faces of the cube. In this setting we can study the linearized equations of motion directly in Fourier space, where we are able to quantify the damping effects in a precise manner. From this characterization we are able to establish time-periodic a priori estimates, which we utilize to show that a solution to the nonlinear coupled fluid-structure system is necessarily time-periodic, that is, non-resonant, when the time-periodic force is sufficiently small.

In order to show the full mathematical potential of our technique, we have chosen to include an additional damping term in the structure equations that regularizes the system and enables us to establish comprehensive $L^q$ estimates of maximal regularity type. Without this internal damping, our approach only yields a priori estimates in an $L^2$ framework. We do not go further into this case in the current article. Further research is also required to extend our results to include more physically reasonable boundary conditions and geometries.

We denote by

$$\omega := (0, L) \times (0, L) \subset \mathbb{R}^2$$

a square that represents a flat elastic plate in its stress free configuration, and consider the cuboid

$$\Omega := \omega \times (0, 1) \subset \mathbb{R}^3$$

as container for the viscous fluid. More precisely, when no outer forces act on the system, the fluid filled cuboid container $\Omega$ with an elastic bottom face represents the stress free configuration of the fluid-structure system. As customary in fluid-structure problems, we utilize the stress free configuration $\Omega$ as the reference configuration and will refer to it as such.

Two $T$-time-periodic outer body forces are introduced. A force

$$f : \mathbb{R} \times \Omega \to \mathbb{R}^3, \quad f(t + T, \cdot) = f(t, \cdot) \quad (1.1)$$

is defined as

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that acts on the fluid, and a force

\[ h : \mathbb{R} \times \omega \to \mathbb{R}, \quad h(t + \mathcal{T}, \cdot) = f(t, \cdot) \tag{1.2} \]

that acts in normal direction on the plate (tangential forces are neglected in the Kirchhoff-Love plate model). Here, \( \mathbb{R} \) denotes the time axis. We consider generic forces formulated as functions on the reference configuration.

With outer forces acting on the system, the dynamics of the elastic structure is described by the displacement

\[ \eta : \mathbb{R} \times \omega \to \mathbb{R} \]

of the plate in normal direction \(-e_3\). The current configuration of the container at time \( t \) is then

\[ \Omega_\eta(t) := \{ x = (x', x_3) \in \mathbb{R}^3 \mid x' \in \omega, -\eta(t, x') < x_3 < 1 \} \].

A canonical mapping that takes the reference configuration into the current configuration at time \( t \) is given by

\[ \varphi_\eta(t) : \Omega \to \Omega_\eta(t), \quad x \mapsto (x', x_3 - (1 - x_3)\eta(t, x')) \tag{1.3} \]

Provided \( \eta \) is sufficiently small, \( \varphi_\eta(t) \) is a bijection with inverse given by

\[ \varphi_\eta^{-1}(t) : \Omega_\eta(t) \to \Omega, \quad x \mapsto \left( x', x_3 + \eta(t, x') \frac{1}{1 + \eta(t, x')} \right) \].

The dynamics of the incompressible viscous fluid flow is described in terms of its Eulerian velocity field and pressure term

\[ u : \bigcup_{t \in \mathbb{R}} \{ t \} \times \Omega_\eta(t) \to \mathbb{R}^3, \quad p : \bigcup_{t \in \mathbb{R}} \{ t \} \times \Omega_\eta(t) \to \mathbb{R}, \]

respectively. We assume the fluid is Newtonian with Cauchy stress tensor given by

\[ T(u, p) := \mu f(\nabla u + \nabla u^\top) - pI. \]

Letting \( \nu_t \) denote the outer normal of the container’s current configuration \( \Omega_\eta(t) \), we can thus express the surface force exerted by the fluid in normal direction \(-e_3\) on the structure in reference configuration coordinates as

\[ e_3 \cdot (T(u, p)\nu_t) \circ \varphi_\eta \big|_{x_3 = 0}. \]

The Kirchoff-Love plate equation (with damping) govern the motion of the elastic structure:

\[ \partial_t^2 \eta + \Delta^2 \eta - \mu_s \Delta \partial_t \eta = h + e_3 \cdot (T(u, p)\nu_t) \circ \varphi_\eta \big|_{x_3 = 0} \quad \text{in } \mathbb{R} \times \omega. \tag{1.4} \]
The term $\mu_s \Delta' \partial_t \eta$ with $\mu_s > 0$ introduces an internal Kelvin-Voigt type damping in the structure. We use $\Delta'$ to denote the Laplacian with respect to the coordinates of the plate $\omega$.

The Navier-Stokes equations govern the motion of the incompressible viscous fluid:

$$
\begin{align*}
\partial_t u - \mu_f \Delta u + (u \cdot \nabla)u + \nabla p &= f_\eta \quad \text{in} \quad \bigcup_{t \in \mathbb{R}} \{t\} \times \Omega_\eta(t), \\
\text{div} \ u &= 0 \quad \text{in} \quad \bigcup_{t \in \mathbb{R}} \{t\} \times \Omega_\eta(t).
\end{align*}
$$

Here, $f_\eta := f \circ \varphi_{\eta}^{-1}$ denotes the outer force $f$ expressed in current configuration coordinates, and $\mu_f > 0$ the kinematic viscosity constant.

We assume a no-slip boundary condition for the fluid velocity on both the top and bottom (elastic) face of the container:

$$
\begin{align*}
u &= 0 \quad \text{on} \quad \mathbb{R} \times \omega \times \{1\}, \\
u \circ \varphi_\eta &= -\partial_t \eta e_3 \quad \text{on} \quad \mathbb{R} \times \omega \times \{0\}.
\end{align*}
$$

We assume periodic boundary conditions on the lateral faces of the container, that is,

$$
\begin{align*}
\eta(:, x_1 + L, \cdot) &= \eta(:, x_1, \cdot), \\
\eta(\cdot, x_2 + L) &= \eta(\cdot, x_2), \\
u(:, x_1 + L, \cdot) &= \nu(:, x_1, \cdot), \\
u(\cdot, x_2 + L, \cdot) &= \nu(\cdot, x_2, \cdot).
\end{align*}
$$

Moreover, we augment the system with the volume constraint

$$
\int_\omega \eta(t, x') \, dx' = 0.
$$

We investigate the conditions under which the system (1.4)–(1.8) admits a $T$-time-periodic (non-resonant) solution under $T$-time-periodic forcing (1.1)–(1.2). For this purpose, it is convenient to reformulate the system in a setting where the time-axis $\mathbb{R}$ is replaced with the torus $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$. In such a setting all function are generically $T$-time-periodic. By the same token, we introduce the torus $\mathbb{T}_0^2 := (\mathbb{R}/L\mathbb{Z})^2$ and incorporate the periodic boundary conditions (1.7) into the setting by replacing the elastic square $\omega$ and the cuboid container $\Omega$ with (we keep the notation $\omega$ and $\Omega$)

$$
\omega := \mathbb{T}_0^2 \quad \text{and} \quad \Omega := \mathbb{T}_0^2 \times (0, 1),
$$

respectively. We denote the corresponding time-space current configuration by

$$
\Omega^T_\eta := \bigcup_{t \in \mathbb{T}} \{t\} \times \Omega_\eta(t).
$$

Summarizing, we identify $T$-time-periodic solutions $(u, p, \eta)$ to (1.4)–(1.8) as solutions
As the main result of the article we establish existence of a solution to (1.9) under a smallness condition on the data:

**Theorem 1.1.** Let $q \in (2, \infty)$. There is an $\varepsilon > 0$ such that for all $(f, h) \in L^q(T \times \Omega)^3 \times L^q(T; W^{1-\frac{1}{q}}(T_0))$,

there is a solution $(\eta, u, p)$ to (1.9) satisfying

$$
\eta \in W^{2,q}(T; W^{1-\frac{1}{q}}(T_0)^3) \cap L^q(T; W^{5-\frac{1}{q}}(T_0)^3),
$$

$$
u \circ \varphi_\eta \in W^{1,q}(T; L^q(\Omega))^3 \cap L^q(T; W^{2,q}(\Omega))^3,$n

$$
p \circ \varphi_\eta \in L^q(T; W^{1,q}(\Omega)).$$

The main novelty of the article, however, concerns the technique we introduce to establish a priori estimates of the corresponding linearization:

$$
\begin{aligned}
\partial_t^2 \eta + \Delta^2 \eta - \mu_s \Delta' \partial_t \eta &= h + e_3 \cdot (T(u, p) e_3)_{|x_3=0} & \text{in } T \times T_0^2, \\
\partial_t u - \mu_f \Delta u + (u \cdot \nabla)u + \nabla p &= f & \text{in } T \times \Omega, \\
\text{div } u &= g & \text{in } T \times \Omega, \\
u(t, x', 0) &= -\partial_t \eta(t, x') e_3 & \text{on } T \times T_0^2, \\
u(t, x', 1) &= 0 & \text{on } T \times T_0^2, \\
\int_{T_0^3} \eta(t, x') \, dx' &= 0.
\end{aligned}
$$

More specifically, (1.10) is obtained as the linearization of (1.9) by reformulating the fluid equations (1.9) in the reference configuration and subsequently neglecting all higher order terms in $(\eta, u, p)$. Moreover, an inhomogeneous right-hand side $g$ is introduced in (1.10), which is critical in the utilization of (1.10) towards the resolution of (1.9). We establish the following a priori $T$-time-periodic estimates of (1.10):
Theorem 1.2. Let $q \in (1, \infty)$. For all $(f, g, h) \in \mathcal{Y}^q(\mathbb{T} \times \Omega)$ with

\[
\mathcal{Y}^q(\mathbb{T} \times \Omega) := L^q(\mathbb{T} \times \Omega)^3 \\
\times L^q(T; W^{1,q}(\Omega)) \cap W^{1,q}(\mathbb{T}; W^{-1,q}(\Omega)) \times L^q(T; W^{1,\frac{3}{2}q}(\mathbb{T}^2_0))
\] (1.11)

satisfying

\[
\int_{\Omega} g \, dx = 0,
\]

there is a unique solution $(u, p, \eta) \in \mathcal{X}^q(\mathbb{T} \times \Omega)$ to (1.10) with

\[
\mathcal{X}^q(\mathbb{T} \times \Omega) := W^{1,q}(T; L^q(\Omega))^3 \cap L^q(T; W^{2,q}(\Omega))^3 \\
\times L^q(T; W^{1,q}(\Omega)) \times W^{2,q}(T; W^{1,\frac{3}{2}q}(\mathbb{T}^2_0)) \cap L^q(T; W^{5,\frac{3}{2}q}(\mathbb{T}^2_0)).
\] (1.12)

Moreover,

\[
\|(u, p, \eta)\|_{\mathcal{X}^q(\mathbb{T} \times \Omega)} \leq C_1 \|(f, g, h)\|_{\mathcal{Y}^q(\mathbb{T} \times \Omega)}
\] (1.13)

with $C_1 = C_1(q, \mathbb{T}) > 0$.

We solve (1.10) via a representation formula in which the damping effect in the structure of both the fluid force $e_3 \cdot \nabla(T(u, p)e_3)$ and the internal damping $\mu \Delta' \partial_t \eta$ are quantified in the Fourier space with respect to the Fourier transform $\mathcal{F}_{\mathbb{T} \times \mathbb{T}^2_0}$; see Remark 3.6. From this characterization we obtain the a priori estimate (1.13) via a transference principle for Fourier multipliers. We first establish the estimate in a half-space setting, and subsequently employ a (non-trivial) localization.

The coupling of an incompressible viscous fluid with an elastic plate has previously been investigated in [8, 16, 9, 11, 3, 5, 7, 14, 5, 3, 4, 15]. Most of these articles cover the corresponding initial-value problem. To our knowledge, the investigation of time-periodic solutions in an $L^q$ setting to the fully non-linear problem (1.9) is new.

2 Preliminaries

The periodic time-space domain $\mathbb{T} \times \mathbb{T}^2_0 \times \mathbb{R}$ with $\mathbb{T} := \mathbb{R}/\mathbb{T} \mathbb{Z}$ and $\mathbb{T}^2_0 := (\mathbb{R}/L\mathbb{Z})^2$ inherits its differentiable structure as a manifold from the physical time-space domain $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ via the quotient mappings $\pi : \mathbb{R} \to \mathbb{T}$ and $\pi_0 : \mathbb{R}^2 \to \mathbb{T}^2_0$. It is easy to verify that the reformulation (1.9) in the subdomain $\mathbb{T} \times \mathbb{T}^2_0 \times (0, 1)$ of $\mathbb{T} \times \mathbb{T}^2_0 \times \mathbb{R}$ is equivalent to the original system (1.4)–(1.8) defined in the physical time-space domain. Sobolev spaces defined on subdomains of $\mathbb{T} \times \mathbb{T}^2_0 \times \mathbb{R}$, such as those appearing in Theorem 1.1 and 1.2, can be defined analogously to classical Sobolev spaces. We refer to [2] for a systematic approach.
We take advantage of the structure of $\mathbb{T} \times \mathbb{T}_0^2$ as a compact abelian group (with normalized Haar measure) and utilize the corresponding Fourier transform $\mathcal{F}_{\mathbb{T} \times \mathbb{T}_0^2}$. We identify the dual group of $\mathbb{T} \times \mathbb{T}_0^2$ as $\mathbb{T} \times (\mathbb{T}_0^2 \times \mathbb{T}_0^2)$, and use $(k, \xi) \in \mathbb{T} \times (\mathbb{T}_0^2 \times \mathbb{T}_0^2)$ as canonical notation for its elements. Formally, the Fourier transform takes the following form on functions $u : \mathbb{T} \times \mathbb{T}_0^2 \to \mathbb{R}$:

$$\hat{u}(k, \xi) := \mathcal{F}_{\mathbb{T} \times \mathbb{T}_0^2}[u](k, \xi) := \int_{\mathbb{T} \times \mathbb{T}_0^2} u(t, x) e^{-ix \cdot \xi - ikt} \, dx \, dt.$$  

The Fourier transform $\mathcal{F}_{\mathbb{T} \times \mathbb{T}_0^2}$ can be expressed as the composition of the Fourier transforms $\mathcal{F}_\mathbb{T}$ and $\mathcal{F}_{\mathbb{T}_0^2}$, which takes functions defined on $\mathbb{T}$ and $\mathbb{T}_0^2$ into their Fourier coefficients with respect to indices $k \in \mathbb{Z}$ and $\xi \in (\mathbb{T}_0^2 \times \mathbb{T}_0^2)$. At one point in the following, namely when we establish the $L^q$ estimates in Theorem 1.2, it is critical that a single Fourier transform $\mathcal{F}_{\mathbb{T} \times \mathbb{T}_0^2}$ covering the whole time-space domain is employed instead of a sequential utilization of $\mathcal{F}_\mathbb{T}$ and $\mathcal{F}_{\mathbb{T}_0^2}$.

For functions $f$ defined on $\mathbb{T} \times \mathbb{T}_0^2 \times \mathbb{R}$ we define

$$f_s := \mathcal{P} f(t, \cdot) := \int_{\mathbb{T}} f(s, \cdot) \, ds, \quad f_{tp} := \mathcal{P}_\perp f(t, \cdot) := f(t, \cdot) - \mathcal{P} f(t, \cdot)$$  

whenever the integral is well defined. Since $f_s$ is independent of time $t$, we shall implicitly treat $f_s$ as a function in the spatial variable $x$ only and refer to it as the steady-state part of $f$. The function $f_{tp}$ is referred to as the purely oscillatory part of $f$. When using the projections $\mathcal{P}$ and $\mathcal{P}_\perp$ to decompose a function space, we use the symbol $\perp$ as subscript to denote the purely oscillatory part, for example $L^q_\perp(\mathbb{T} \times \Omega) := \mathcal{P}_\perp L^q(\mathbb{T} \times \Omega)$.

Finally, we observe that the divergence problem

$$\begin{cases} 
\text{div } u = f & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega
\end{cases} \tag{2.2}$$

set in the torus domain $\Omega := \mathbb{T}_0^2 \times (0, 1)$ possesses the same properties as the corresponding divergence problem set in classical subdomains of $\mathbb{R}^3$. As in [12, Section III.3], a so-called Bogovskiĭ operator $\mathcal{B} : C_0^\infty(\Omega) \to C_0^\infty(\Omega)^3$ can be constructed such that $u := \mathcal{B}(f)$ satisfies (2.2) whenever $f$ satisfies

$$\int_{\Omega} f \, dx = 0. \tag{2.3}$$

**Theorem 2.1 (Bogovskiĭ Operator).** The Bogovskiĭ operator $\mathcal{B} : C_0^\infty(\Omega) \to C_0^\infty(\Omega)^3$ has a continuous (linear) extension $\mathcal{B} : W_0^{m,q}(\Omega) \to W_0^{m+1,q}(\Omega)^3$ for $q \in (1, \infty)$ and $m \in \mathbb{N}_0$. If $f \in W_0^{m,q}(\Omega)$ satisfies (2.3), then $u := \mathcal{B}(f)$ is a solution to (2.2) satisfying

$$\|\nabla u\|_{l,q} \leq C_2 \|f\|_{l,q}, \tag{2.4}$$

for all $l = 0, \ldots, m$. Moreover, there exists a constant $C_3 = C_3(n, q, \Omega) > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C_3 \|f\|^*_1 \quad \text{for all } f \in W_0^{m,q}(\Omega). \tag{2.5}$$
where
\[ |f|_{-1,q}^* = \sup_{\varphi \in W^{1,q}(\Omega): \|\varphi\|_{1,q'} = 1} |(f, \varphi)| \quad (2.6) \]
for all \( f \in L^q(\Omega) \).

**Proof.** The classical construction of the Bogovskii operator can be adapted to the domain \( \Omega \) without significant modifications; see for example [12, Section III.3]. The first part of the theorem can be shown as in [12, Theorem III.3.3]. Estimate (2.5) follows as in the proof of [12, Theorem III.3.5]. \( \square \)

## 3 Linearized system

We employ the projections \( \mathcal{P} \) and \( \mathcal{P}_\perp \) introduced in (2.1) to decompose (1.10) into a steady-state part and a purely oscillatory part. These two problems are different by nature, and we therefore study them separately. The purely oscillatory part is investigated in Section 3.1–3.2, and the steady-state problem in Section 3.3. A proof of Theorem 1.2 is presented in Section 3.4. To simplify the notation, we set, without loss of generality, \( \mu_s = \mu_f = 1 \).

### 3.1 Resolvent problem

Applying the Fourier transform \( \mathcal{F}_T \) to the linear system (1.10), we obtain for each \( k \in \frac{2\pi}{T}\mathbb{Z} \) the following resolvent type system for the (complex valued) Fourier coefficients \( (\eta_k, u_k, p_k) \):

\[
\begin{aligned}
-k^2 \eta_k + \Delta'^2 \eta_k - ik \Delta' \eta_k &= h_k - e_3 \cdot (T(u_k, p_k)e_3) \big|_{x_3=0} \quad \text{in} \ T^2_0, \\
-iku_k - \Delta u_k + \nabla p_k &= f_k \quad \text{in} \ \Omega, \\
\text{div } u_k &= g_k \quad \text{in} \ \Omega, \\
-u_k|_{x_3=0} &= -ik \eta_k e_3 \quad \text{on} \ T^2_0, \\
u_k|_{x_3=1} &= 0 \quad \text{on} \ T^2_0, \\
\int_{T^2_0} \eta_k(x') \, dx' &= 0.
\end{aligned}
(3.1)
\]

In the homogeneous case \( g_k = 0 \), we can solve this system with an application of Lax-Milgram’s theorem:

**Lemma 3.1 (Existence).** Let \( k \in \frac{2\pi}{T}\mathbb{Z} \) with \( k \neq 0 \). For every \( (f_k, h_k) \in L^2(\Omega; \mathbb{C})^3 \times L^2(T^2_0; \mathbb{C}) \) there is a weak solution \((u_k, \eta_k)\) to (3.1) with \( g_k = 0 \), that is, \((\eta_k, v_k) \in \mathcal{V}_k\) with

\[
\mathcal{V}_k := \{(u, \eta) \in W^{1,2}(\Omega; \mathbb{C})^3 \times W^{2,2}(T^2_0; \mathbb{C}) \mid \int_{T^2_0} \eta(x') \, dx' = 0, \quad u|_{x_3=0} = -ik \eta e_3 \text{ on } T^2_0, \quad u|_{x_3=1} = 0 \text{ on } T^2_0 \}.
(3.2)
\]
and satisfies
\[ \forall (w, \zeta) \in V_k : \quad B((u_k, \eta_k), (w, \zeta)) = -ik \int_{T^2_0} h_k \bar{\zeta} \, dx' + \int_{\Omega} f_k \cdot \bar{w} \, dx \] \tag{3.3}

where
\[ B((u, \eta), (w, \zeta)) := \int_{T^2_0} ik^3 \eta \bar{\zeta} - i k \Delta' \eta \Delta' \bar{\zeta} + k^2 \nabla' \eta \cdot \nabla' \bar{\zeta} \, dx' + \int_{\Omega} \nabla u : \nabla \bar{w} + i k u \cdot \bar{w} \, dx. \] \tag{3.4}

Moreover,
\[ \|u_k\|_{W^{r+2,2}(\Omega)} + \|\eta_k\|_{W^{r+1,2}(\Omega)} + \|k \eta_k\|_{W^{r+4,2}(T^2_0)} \leq C_4 (\|f_k\|_{W^{r,2}(\Omega)} + \|h_k\|_{W^{r,2}(T^2_0)}) \] \tag{3.5}

with constant \( C_4 = C_4(T, L) > 0 \) independent of \( k \).

**Proof.** The sesquilinear form \( B \) is clearly bounded in the Hilbert space \( V_k \). By computing the real and imaginary part of \( B((u, \eta), (u, \eta)) \), one readily verifies that \( B \) is also coercive. Existence of a solution to (3.3) therefore follows from the theorem of Lax-Milgram. Setting \((w, \zeta) = (u_k, \eta_k)\) in (3.3) and taking real and imaginary parts in the equation, one obtains (3.5) by an application of Young’s inequality. \( \square \)

A pressure field corresponding to the weak solution obtained in Lemma 3.1 can be constructed and higher order regularity subsequently established.

**Lemma 3.2 (Pressure and regularity).** Let \( k \in \frac{2\pi}{T} \mathbb{Z} \setminus \{0\} \) and \( r \in \mathbb{N}_0 \). Moreover, let \((f_k, h_k) \in W^{r,2}(\Omega; \mathbb{C})^3 \times W^{r,2}(T^2_0; \mathbb{C})\) and \((u_k, \eta_k) \in V_k \) be the corresponding weak solution to (3.1) with \( g_k = 0 \) constructed in Lemma 3.1. Then there exists a pressure field \( p_k \) such that
\[ (u_k, p_k, \eta_k) \in W^{r+2,2}(\Omega)^3 \times W^{r+1,2}(\Omega) \times W^{r+4,2}(T^2_0) \] \tag{3.6}
solves (3.1) and satisfies
\[ \|u_k\|_{W^{r+2,2}(\Omega)} + \|p_k\|_{W^{r+1,2}(\Omega)} + \|\eta_k\|_{W^{r+4,2}(T^2_0)} \leq C_5 \left( \|f_k\|_{W^{r,2}(\Omega)} + \|h_k\|_{W^{r,2}(T^2_0)} \right) \] \tag{3.7}

with \( C_5 = C_5(T, L) > 0 \) independent of \( k \).

**Proof.** First consider only the Stokes part (3.1)2-5 of the system. By well known methods (see for example Theorem 1.2. in [10]) a pressure \( \tilde{p}_k \in L^2(\Omega) \) can be constructed such that \((u_k, \tilde{p}_k)\) solves this resolvent type Stokes problem. Since the weak formulation (3.3) is obtained by multiplying (3.1) with a pair of test functions and subsequent integration by parts, one readily verifies that \((u_k, p_k, \eta_k)\) with
\[ p_k := \tilde{p}_k - \int_{T^2_0} \tilde{p}_k \, dx' - \int_{T^2_0} h_k \, dx' \]
solves the full system (3.1) in a distributional sense.

At the outset $\eta_k \in W^{2,2}(T_0^*)$. Standard elliptic regularity theory for the Stokes system (3.1)\textsubscript{2,5} (see for example [12, Chapter 4]) therefore yields $(u_k, p_k) \in W^{2,2}(\Omega)^3 \times W^{1,2}(\Omega)$. Consequently, $T(u_k, p_k) \in W^{1,2}(T_0^*)$. Applying the Fourier transform $\mathcal{F}_{T_0^*}$ to the plate equation (3.1), we obtain the representation

$$\eta_k = \mathcal{F}_{T_0^*}^{-1}\left[\frac{1}{|\xi|^4 - k^2 + i k|\xi|^2} \mathcal{F}_{T_0^*}[c_3 \cdot (T(u_k, p_k)c_3)_{|x_3=0} + h_k]\right]. \quad (3.8)$$

Due to the regularizing damping term $i k|\xi|^2$ in the Fourier multiplier, a simple application of Parseval’s theorem implies that $\eta_k$ admits a regularity gain of 4 derivatives over the right-hand side of the (damped) plate equation (3.1). Consequently we deduce $\eta_k \in W^{5,2}(T_0^*)$. Iterating this procedure, we find that $(u_k, p_k, \eta_k)$ is as regular as the data $(f, h)$ allows for and thus conclude (3.6). In this process, the elliptic regularity theory of the Stokes system and the representation (3.8) also yields (3.7).

3.2 Purely oscillatory problem

By expressing a solution to (1.10) in terms of its Fourier series, we can utilize the existence and regularity of the Fourier coefficients in Lemma 3.1 and Lemma 3.2 to construct a solution to (1.10):

**Lemma 3.3** (Existence). Let $q \in (1, \infty)$. For any $(f, h) \in C^\infty_{0,1}(T \times \Omega)^3 \times C^\infty_{0,1}(T \times T_0^*)$ the system (1.10) with $g = 0$ admits a solution $(u, p, \eta) \in \mathcal{A}^q_0(T \times \Omega)$.

**Proof.** We expand the data into Fourier series with respect to $\mathbb{T}$. Since $\mathcal{P}f = \mathcal{P}h = 0$, the zeroth order Fourier coefficients are 0, that is,

$$f = \sum_{k \in \mathbb{Z} \setminus \{0\}} f_k e^{ikt} \quad \text{and} \quad h = \sum_{k \in \mathbb{Z} \setminus \{0\}} h_k e^{ikt}$$

with Fourier coefficients $f_k = \mathcal{F}_{\mathbb{T}}[f](k)$ and $h_k = \mathcal{F}_{\mathbb{T}}[h](k)$. By Parseval’s theorem, these identities are valid in $W^{r,2}(T; W^{2,r}(\Omega))$ and $W^{r,2}(\mathbb{T}; W^{r,2}(T_0^*))$, respectively, for any $r \in \mathbb{N}_0$. From Lemma 3.1 and Lemma 3.2 we obtain for each pair $(f_k, h_k)$ of Fourier coefficients a solution

$$(u_k, p_k, \eta_k) \in W^{r+2,2}(\Omega)^3 \times W^{r+1,2}(\Omega) \times W^{r+4,2}(T_0^*)$$

to (3.1). By (3.7) and Parseval’s theorem, the corresponding Fourier series

$$u := \sum_{k \in \mathbb{Z} \setminus \{0\}} u_k e^{ikt}, \quad p := \sum_{k \in \mathbb{Z} \setminus \{0\}} p_k e^{ikt}, \quad \eta := \sum_{k \in \mathbb{Z} \setminus \{0\}} \eta_k e^{ikt}$$

are well-defined in the Hilbert spaces $W^{r,2}(\mathbb{T}; W^{r+2,2}(\Omega))^3$, $W^{r,2}(\mathbb{T}; W^{r+1,2}(\Omega))$ and $W^{r,2}(\mathbb{T}; W^{r+4,2}(T_0^*))$, respectively. Clearly, $(u, p, \eta)$ solves (1.10). Choosing $r$ sufficiently large, we obtain $(u, p, \eta) \in \mathcal{A}^q_0(T \times \Omega)$ by Sobolev embedding. \qed
The Fourier analysis carried out above is rather crude, and it is restricted to the Hilbert space setting due to the application of Parseval’s theorem. It is, however, only a step towards a more refined Fourier analysis that leads to maximal regularity $L^q$ estimates. To this end, we need the following uniqueness property:

**Lemma 3.4 (Uniqueness).** Let $q \in (1, \infty)$. A solution to (1.10) is unique in the class $X^q(T \times \Omega)$.

**Proof.** It suffices to consider a solution $(u, p, \eta) \in X^q(T \times \Omega)$ to (1.10) with homogeneous right-hand side $(f, g, h) = (0, 0, 0)$ and show that necessarily $(u, p, \eta) = (0, 0, 0)$. We employ a duality argument. Let $(\varphi, \psi) \in C^\infty_0(T \times \Omega)^3 \times C^\infty_0(T \times T_0^2)$. By the same argument that leads to Lemma 3.3, existence of a solution $(w, \pi, \zeta) \in X^q_\perp(T \times \Omega)$ to the dual of (1.10) follows, that is,

$$
\begin{align*}
\begin{cases}
\partial_t^2 \zeta + \Delta^2 \zeta + \Delta^\prime \partial_t \zeta = \psi - e_3 \cdot (T(w, \pi)e_3)_{|x_3=0} & \text{in } T \times T_0^2, \\
-\partial_t w - \Delta w - \nabla \pi = \varphi & \text{in } T \times \Omega, \\
div w = 0 & \text{in } T \times \Omega, \\
w(t, x', 0) = -\partial_t \zeta(t, x')e_3 & \text{on } T \times T_0^2, \\
w(t, x', 1) = 0 & \text{on } T \times T_0^2, \\
\int_{T_0^2} \zeta(t, x') \, dx' = 0.
\end{cases}
\end{align*}
$$

An straightforward integration by parts shows that

$$
\int_{T \times \Omega} u \cdot \varphi \, dx \, dt = \int_{T_0^2} \partial_t \eta \psi \, dx'.
$$

Since $(\varphi, \psi)$ can be chosen arbitrarily, $u = 0$ and $\partial_t \eta = 0$ follows. Since $\int_{T_0^2} \eta \, dx' = 0$ we deduce $\eta = 0$ and in turn $p = 0$.

Finally we can move towards $L^q$ estimates. We first consider the problem (1.10) in the periodic half space $\Omega_+ = T_0^2 \times \mathbb{R}_+$. In this geometry we can compute a formula that represents the solution $\eta$ in terms of the data.

**Lemma 3.5 (L^q-Estimate in $\Omega_+$).** Let $q \in (1, \infty)$. For any $(f, g, h) \in Y^q_\perp(T \times \Omega_+)$ a solution $(v, p, \eta) \in X^q_\perp(T \times \Omega_+)$ to

$$
\begin{align*}
\begin{cases}
\partial_t^2 \eta + \Delta^2 \eta - \Delta^\prime \partial_t \eta = h - e_3 \cdot (T(v, p)e_3)_{|x_3=0} & \text{in } T \times T_0^2, \\
\partial_t v - \Delta v + \nabla p = f & \text{in } T \times \Omega_+, \\
div v = g & \text{in } T \times \Omega_+, \\
v_{|x_3=0} = -\partial_t \eta e_3 & \text{on } T \times T_0^2, \\
\int_{T_0^2} \eta(t, x') \, dx' = 0
\end{cases}
\end{align*}
$$

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obeys the $L^q$ estimate

\[
\|v\|_{W^{1,2,q}(T \times \Omega_+)} + \|\nabla p\|_{L^q(T \times \Omega_+)} + \|\eta\|_{W^{2,1,q}(T;W^{1,2,q}(T \times \Omega_+))} \\
\leq C_6(\|f\|_{L^q(T \times \Omega_+)} + \|g\|_{L^q(T;W^{1,2,q}(\Omega_+))} + \|h\|_{L^q(T;W^{1,2,q}(T \times \Omega_+))}).
\]  

(3.11)

**Proof.** From [2] (see also [1, Theorem 4.4.7]) we obtain a solution $(w, \pi) \in W^{1,2,q}(T \times \Omega_+) \times L^q_\perp(T; W^{1,q}(\Omega_+))$ to the half-space Stokes problem

\[
\begin{align*}
\partial_t w - \Delta w + \nabla \pi &= f, \\
\text{div } w &= g, \\
\big|w\big|_{T \times \partial \Omega_+} &= 0
\end{align*}
\]  

(3.12)

satisfying

\[
\|w\|_{W^{1,2,q}(T \times \Omega_+)} + \|\nabla \pi\|_{L^q(T \times \Omega_+)} \\
\leq c_0(\|f\|_{L^q(T \times \Omega_+)} + \|g\|_{L^q(T;W^{1,2,q}(\Omega_+))} + \|h\|_{L^q(T;W^{1,2,q}(T \times \Omega_+))}).
\]  

(3.13)

Letting

\[ u := v - w, \quad p := \pi + \int_\Omega \pi \, dy, \]

we find that $(u, p, \eta)$ solves

\[
\begin{align*}
\partial_t^2 \eta + \Delta' \eta - \Delta' \partial_t \eta &= F - e_3 \cdot (T(u, p)e_3) \big|_{x_3 = 0} \quad \text{in } T \times T_0^2, \\
\partial_t u - \Delta u + \nabla p &= 0 \quad \text{in } T \times \Omega_+, \\
\text{div } u &= 0 \quad \text{in } T \times \Omega_+, \\
u|_{x_3 = 0} &= -\partial_t \eta \, e_3 \quad \text{on } T \times T_0^2,
\end{align*}
\]  

(3.14)

with

\[
F := h - e_3 \cdot T(w, \tilde{\pi})e_3, \quad \tilde{\pi} := \pi - \int_\Omega \pi \, dx.
\]

At this point we mimic the arguments in [2, Proof of Proposition 3.1] and utilize the Fourier transform $\mathcal{F}_{T \times T_0^2}$ in (3.14). The result is a system of ODEs we can solve explicitly obtaining

\[
u = (U, V), \quad p = \mathcal{F}_{T \times T_0^2}^{-1}[q_0(k, \xi') e^{-|\xi'|x_3}]
\]  

(3.15)
Similarly, we let \( \delta \)

\[
U := \mathcal{F}^{-1}_{T \times T_0} \left[ -\frac{\xi' q_0}{k} e^{-i|\xi'| x_3} + \frac{\xi' q_0}{k} e^{-\sqrt{|\xi'|^2 + ik} x_3} \right],
\]

\[
V := \mathcal{F}^{-1}_{T \times T_0} \left[ \frac{\xi' q_0}{ik} e^{-i|\xi'| x_3} - \left( ik\hat{\eta} + \frac{\xi' q_0}{ik} \right) e^{-\sqrt{|\xi'|^2 + ik} x_3} \right],
\]

\[
q_0(k, \xi') := \left[ -ik \left( |\xi'| + \sqrt{|\xi'|^2 + ik} \right) + k^2 \right] \hat{\eta}.
\]

(3.16)

Observing that \(-e_3 \cdot T(u, p)e_3 = p|x_3=0\) for \(u\) satisfying (3.14)3-4, we deduce from (3.14)1 that

\[
\left[ |\xi'|^4 - k^2 + ik|\xi'|^2 - \frac{k^2}{|\xi'|} + ik \left( |\xi'| + \sqrt{|\xi'|^2 + ik} \right) \right] \hat{\eta} = \hat{F}.
\]

(3.17)

We let \(\delta_{\mathbb{T}^Z}\) denote the Dirac measure on the group \(\mathbb{T}^Z\), that is, the function that takes the value 1 for \(k = 0\) and otherwise 0. By assumption \(P\eta = 0\), which means that the zeroth order Fourier coefficient of \(\eta\) with respect to \(\mathcal{F}_T\) vanishes and consequently

\[
(1 - \delta_{\mathbb{T}^Z}(k)) \hat{\eta} = \hat{\eta}.
\]

(3.18)

Similarly, we let \(\delta_{(\mathbb{T}^Z)^2}\) denote the Dirac measure on the group \((\mathbb{T}^Z)^2\), and conclude from (3.14)5 that

\[
(1 - \delta_{(\mathbb{T}^Z)^2}(\xi')) \hat{\eta} = \hat{\eta}.
\]

(3.19)

We derive from (3.17)–(3.19) that

\[
\eta = \mathcal{F}^{-1}_{T \times T_0} \left[ \frac{(1 - \delta_{\mathbb{T}^Z}(k)) (1 - \delta_{(\mathbb{T}^Z)^2}(\xi'))}{|\xi'|^4 - k^2 + ik|\xi'|^2 - \frac{k^2}{|\xi'|} + ik \left( |\xi'| + \sqrt{|\xi'|^2 + ik} \right)} \hat{F} \right].
\]

(3.20)

Following the exact same steps as in [2, Proof of Proposition 3.1], a Fourier multiplier argument based on the representation formulas in (3.16) yields

\[
\|u\|_{W^{1, 2,q}(T \times \Omega_+)} + \|\nabla p\|_{L^q(T \times \Omega_+)} \leq c_1 \left( \|\partial_\eta\|_{W^{1, \frac{1}{2} - \frac{1}{q}, q}(T_0)} + \|\partial_\eta\|_{W^{1, q}(T; W^{1, \frac{1}{2} - \frac{1}{q}, q}(T_0))} \right)
\]

\[
\leq c_2 \|\eta\|_{W^{2,q}(T; W^{1, \frac{1}{2} - \frac{1}{q}, q}(T_0)) \cap L^q(T; W^{1, \frac{1}{2} - \frac{1}{q}, q}(T_0))}.
\]

A similar argument further yields an estimate of \(\eta\) based on the representation formula (3.20) and an analysis of the Fourier multiplier \(M: \mathbb{T}^Z \times (\mathbb{T}^Z)^2 \to \mathbb{C}\) given by

\[
M(k, \xi') := \frac{(1 - \delta_{\mathbb{T}^Z}(k)) (1 - \delta_{(\mathbb{T}^Z)^2}(\xi'))}{|\xi'|^4 - k^2 + ik|\xi'|^2 - \frac{k^2}{|\xi'|} + ik \left( |\xi'| + \sqrt{|\xi'|^2 + ik} \right)}.
\]
Observe that
\[
(k, \xi') \mapsto (1 + |k|^2 + |\xi'|^4) M(k, \xi')
\] (3.21)
is bounded. In fact, one can verify that a canonical extension of the multiplier in (3.21) from the domain \(\frac{\mathbb{Z}}{\Delta} \times (\frac{\mathbb{Z}}{\Delta})^2\) to the domain \(\mathbb{R} \times \mathbb{R}^2\) satisfies the conditions of Marcinkiewicz’s multiplier theorem; see [1, Lemma A.2.3 and Lemma A.2.4]. It follows from de De Leeuw’s transference principle [6] in combination with Marcinkiewicz’s multiplier theorem (see [17] for a comprehensive explanation of the argument) that
\[
\|\eta\|_{W^2,q(T;W^{1-\frac{1}{2}}q(T_0^3))} = \|\mathfrak{F}^{-1}_{T \times T_0} \left[ (1 + |k|^2 + |\xi'|^4) \hat{\eta} \right] \|_{L^q(T;W^{1-\frac{1}{2}}q(T_0^3))} \\
\leq c_3 \|F\|_{L^q(T;W^{1-\frac{1}{2}}q(T_0^3))},
\]
Since \(\hat{\pi}\) has a vanishing mean value, we obtain by utilizing (3.13) and the properties of the trace operator
\[
\text{Tr}_0 : L^q(T;W^{1,q}(\Omega)) \to L^q(T;W^{1-\frac{1}{2}}q(T_0^3)), \quad \varphi \mapsto \varphi|_{x_3=0}, \quad (3.22)
\]
stated in [12, Theorem II.4.3] that
\[
\|F\|_{L^q(T;W^{1-\frac{1}{2}}q(T_0^3))} \leq c_4 \left( \|h\|_{L^q(T;W^{1-\frac{1}{2}}q(T_0^3))} + \|\nabla w\|_{L^q(T;W^{1,q}(\Omega_+))} + \|\hat{\pi}\|_{L^q(T;W^{1,q}(\Omega_+))} \right) \\
\leq c_5 \left( \|h\|_{L^q(T;W^{1-\frac{1}{2}}q(T_0^3))} + \|w\|_{W^{1,2,q}(\Omega_+)} + \|\nabla \pi\|_{L^q(T \times \Omega_+)} \right),
\]
which together with (3.13) completes the proof. \(\square\)

Remark 3.6. As a key observation in the proof above, we recognize, quantified in the formula (3.20), the damping effect both the viscous fluid and the internal damping has on the displacement \(\eta\) of the plate. Indeed, without the coupling of the viscous fluid and the introduction of internal damping the representation formula would read
\[
\eta = \mathfrak{F}^{-1}_{T \times T_0} \left[ \frac{(1 - \delta_{\mathbb{Z},T}(k))(1 - \delta_{\mathbb{Z},T}(\xi'))}{|\xi'|^4 - k^2} \hat{F} \right].
\] (3.23)
The additional terms
\[
-\frac{k^2}{|\xi'|} + ik \left( |\xi'| + \sqrt{|\xi'|^2 + ik} \right) \quad \text{and} \quad ik|\xi'|^2
\] (3.24)
in the denominator in (3.20) manifest the damping effect of the viscous fluid and internal damping, respectively. Observe that the multiplier in (3.23) is unbounded, whereas the damping terms in (3.20) lead to a bounded multiplier that decays to 0 as \(|(k, \xi')| \to \infty\). The decay rate of the multiplier in (3.20) as \(|(k, \xi')| \to \infty\) dictates the order of the a priori estimates that can be established for \(\eta\) in terms of the data, and can thus be interpreted as a quantification of the damping effects.
Next we seek to utilize the $L^q$ estimate established in the half-space case in Lemma 3.5 to the original problem (1.10) set in the cuboid $\Omega$. To this end, we employ a standard localization argument. Since (1.10) contains a non-homogeneous Stokes problem, the lower order pressure term that appears naturally in such a localization argument poses a non-trivial challenge.

**Lemma 3.7** (Pressure Field Estimates). Let $s \in (1, \infty)$,

$$(f, h) \in L^s_{\perp}(\mathbb{T} \times \Omega)^3 \times L^s_{\perp}(\mathbb{T}; W^{1-\frac{1}{s}, s}(\mathbb{T}_0^2))$$

and $(u, p, \eta) \in X^s_{\perp}(\mathbb{T} \times \Omega)$ be a solution to (1.10) with $g = 0$. Then there exists a constant $C_7 = C_7(\Omega, s) > 0$ such that

$$
\|p(t, \cdot)\|_{L^s_{\perp}(\Omega)} \leq C_7(\|\nabla u(t, \cdot)\|_{L^s(\Omega)} + \|\nabla'\Delta'\eta(t, \cdot)\|_{L^s(\mathbb{T}_0^2)} + \|\nabla'\partial_t\eta(t, \cdot)\|_{L^s(\mathbb{T}_0^2)})
\quad + \|f(t, \cdot)\|_{L^s(\Omega)} + \|h(t, \cdot)\|_{W^{1-\frac{1}{s}, s}(\mathbb{T}_0^2)} + \|\nabla u(t, \cdot)\|_{L^s(\Omega)} \|\nabla u(t, \cdot)\|_{L^s(\Omega)}^{\frac{1}{2}}
$$

for a.e. $t \in \mathbb{T}$. Moreover, there is a constant $C_8 = C_8(\Omega, s) > 0$ such that for a.e. $t \in \mathbb{T}$ the additional estimate

$$
\|\nabla p(t, \cdot)\|_{L^s(\mathbb{T}_0^2 \times (\frac{1}{2}, \frac{3}{2}))} \leq C_8(\|f(t, \cdot)\|_{L^s(\Omega)} + \|p(t, \cdot)\|_{L^s(\Omega)})
$$

holds.

**Proof.** We use the approach from [13, Proof of Lemma 5.4]. For simplicity, we do not explicitly denote the $t$-dependency of functions in the following.

First consider an arbitrary $\varphi \in C^\infty(\overline{\Omega})$, and observe that due to the periodicity of $u$ and $\varphi$, as well as the boundary condition $u_{|x_3 = 0} = -\partial_t \eta e_3$,

$$
\int_{\Omega} \partial_t u \cdot \nabla \varphi \, dx = \int_{\mathbb{T}^2_0} \partial_t^2 \eta \varphi \, dx'
$$

holds for a.e. $t \in \mathbb{T}$. Hence, by multiplication of (1.10)$_2$ with $\nabla \varphi$ we identify $p$ as a solution to the weak Laplace problem with homogeneous Robin and Neumann boundary conditions on the bottom and top face of $\Omega$, respectively, i.e., for any $\varphi \in C^\infty(\overline{\Omega})$

$$
\int_{\Omega} \nabla p \cdot \nabla \varphi \, dx + \int_{\mathbb{T}^2_0} p \varphi \, dx' = \int_{\Omega} f \cdot \nabla \varphi \, dx + \int_{\Omega} \Delta u \cdot \nabla \varphi \, dx
\quad - \int_{\mathbb{T}^2_0} h \varphi \, dx' + \int_{\mathbb{T}^2_0} \Delta' \eta \varphi \, dx'
\quad - \int_{\mathbb{T}^2_0} \partial_t \eta \varphi \, dx'.
$$

For $g \in C_0^\infty(\Omega)$, existence of a solution $\Phi$ to

$$
\begin{cases}
-\Delta \Phi = g & \text{in } \Omega, \\
\partial_x \Phi_{|x_3 = 0} + \Phi_{|x_3 = 0} = 0 & \text{on } \mathbb{T}^2_0, \\
\partial_x \Phi_{|x_3 = 1} = 0 & \text{on } \mathbb{T}^2_0
\end{cases}
$$

(3.29)
obeying for any $q \in (1, \infty)$ the $L^q$ estimate

$$\|\Phi\|_{W^{2,q}(\Omega)} \leq c_0\|g\|_{L^q(\Omega)} \quad (3.30)$$

follows by classical methods; one may mimic the proof for the pure Neumann problem in [18]. Then $\Phi$ obeys the weak formulation

$$\int_\Omega \nabla \Phi \cdot \nabla \psi \, dx + \int_{\partial \Omega} \Phi \psi' \, d\nu = \int_\Omega g \psi \, dx$$

for any $\psi \in C^\infty(\overline{\Omega})$. In view of (3.28), we can thus compute

$$\int_\Omega pg \, dx = -\int_\Omega p\Delta \Phi \, dx = \int_\Omega \nabla p \cdot \nabla \Phi \, dx + \int_{\partial \Omega} p \Phi \, d\nu = I_1 + \ldots + I_5,$$

with

$$I_1 := \int_\Omega f \cdot \nabla \Phi \, dx, \quad I_2 := \int_\Omega \Delta u \cdot \nabla \Phi \, dx, \quad I_3 := -\int_{\partial \Omega} h \Phi \, d\nu,$$

$$I_4 := \int_{\partial \Omega} \Delta^2 \psi \Phi \, d\nu, \quad I_5 := -\int_{\partial \Omega} \Delta \partial_\nu \Phi \, d\nu.$$

By Sobolev embedding we find that

$$\|\Phi\|_{W^{1,s'}(\Omega)} \leq c_1\|\Phi\|_{W^{2,s}(\Omega)} \leq c_2\|g\|_{L^{\frac{2s}{s+2}}(\Omega)}.$$

Using Hölder’s inequality, we deduce

$$|I_1| \leq \|f\|_{L^s(\Omega)} \|\nabla \Phi\|_{L^{s'}(\Omega)} \leq c_3\|f\|_{L^s(\Omega)} \|g\|_{L^{\frac{2s}{s+2}}(\Omega)}.$$

To find a similar estimate for $I_2$, we twice integrate by parts to deduce

$$I_2 = \int_{\partial \Omega} \partial_\nu u_i \partial_x^j \Phi \nu_j \, d\nu - \int_\Omega \partial_\nu u_i \partial_\nu \Phi \, dx = \int_{\partial \Omega} (\partial_\nu u_i \partial_x^j \Phi \nu_j - \partial_\nu u_i \partial_x^j \Phi \nu_i) \, d\nu,$$

where we utilized the Einstein summation convention and the fact that $\text{div} \, u = 0$. Hence, applying Hölder’s inequality, Sobolev embeddings and a trace inequality (see [12, Theorem II.4.1]), we obtain

$$|I_2| \leq c_4\|\nabla u\|_{L^s(\partial \Omega)} \|\nabla \Phi\|_{L^{s'}(\partial \Omega)} \leq c_5\|\nabla u\|_{L^s(\partial \Omega)} \|\nabla \Phi\|_{W^{1,\frac{2s}{s+2}}(\Omega)}$$

$$\leq c_6 \left( \|\nabla u\|_{L^s(\Omega)} + \|\nabla u\|_{L^{s'}(\Omega)} \|\nabla u\|_{W^{1,s}(\Omega)} \right) \|g\|_{L^{\frac{2s}{s+2}}(\Omega)}.$$

The estimates for the final three integrals will be established similarly to the estimates of $I_2$ by an application of the same trace inequality as above. It follows that

$$|I_3| \leq \|h\|_{L^s(\partial \Omega)} \|\Phi\|_{L^{s'}(\partial \Omega)} \leq c_7 \|h\|_{L^s(\partial \Omega)} \|\Phi\|_{W^{1,s'}(\Omega)} \leq c_8 \|h\|_{L^s(\partial \Omega)} \|g\|_{L^{\frac{2s}{s+2}}(\Omega)}.$$
In order to utilize the same arguments as for \( I_2 \), we integrate by parts in \( I_4 \) and \( I_5 \) to find

\[
|I_4| = \left| \int_{T_0^3} \nabla' \Delta' \eta \cdot \nabla \Phi \, dx \right| \leq c_9 \| \nabla' \Delta' \eta \|_{L^r(T_0^3)} \| g \|_{L^{\frac{2r}{r-2}}(\Omega)},
\]

\[
|I_5| = \left| \int_{T_0^3} \nabla' \partial_t \eta \cdot \nabla \Phi \, dx \right| \leq c_{10} \| \nabla' \partial_t \eta \|_{L^r(T_0^3)} \| g \|_{L^{\frac{2r}{r-2}}(\Omega)}.
\]

It follows from the estimates of \( I_1 - I_5 \) that

\[
\int_{\Omega} pg \, dx \leq c_{11} \left( \| f \|_{L^r(\Omega)} + \| h \|_{L^r(T_0^3)} + \| \nabla u \|_{L^{r}(\Omega)} \| \nabla \|_{W^{1,r}(\Omega)} \right) + \| \nabla u \|_{L^r(\Omega)} + \| \nabla' \Delta' \eta \|_{L^r(T_0^3)} + \| \nabla' \partial_t \eta \|_{L^r(T_0^3)} \| g \|_{L^{\frac{2r}{r-2}}(\Omega)}.\]

By using the duality \((L^{\frac{2r}{r-2}}(\Omega))' = L^\frac{2r}{r-2}(\Omega)\), we obtain (3.25).

To show (3.26), we let \( \chi \in C_0^\infty(\mathbb{R}) \) be a cut-off function such that

\[
\chi(x_3) = 1 \quad \text{for} \quad x_3 \in \left( \frac{1}{3}, \frac{2}{3} \right), \quad \text{and} \quad \chi(x_3) = 0 \quad \text{for} \quad x_3 \in \mathbb{R} \setminus \left[ \frac{1}{6}, \frac{5}{6} \right].
\]

We then set \( \pi(x) := \chi(x_3) p(x) \), \( f_1 := 2p \nabla \chi + f \chi \) and \( f_2 := p \Delta \chi + f \cdot \nabla \chi \), and observe that \( \pi \) is a solution to the weak Neumann problem

\[
\int_{\Omega} \nabla \pi \cdot \nabla \varphi \, dx = \int_{\Omega} f_1 \cdot \nabla \varphi \, dx + \int_{\Omega} f_2 \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega). \tag{3.31}
\]

Since \( \text{supp} \, f_1, \text{supp} \, f_2 \subset T_0^3 \times (\frac{1}{6}, \frac{5}{6}) \), we clearly have

\[
\sup_{\| \nabla \|_{L^r(\Omega)}} \left| \int_{\Omega} f_1 \cdot \nabla \varphi \, dx \right| \leq c_{12} (\| f \|_{L^r(\Omega)} + \| p \|_{L^r(\Omega)}),
\]

\[
\sup_{\| \nabla \|_{L^r(\Omega)}} \left| \int_{\Omega} f_2 \cdot \varphi \, dx \right| \leq c_{13} (\| f \|_{L^r(\Omega)} + \| p \|_{L^r(\Omega)}).
\]

Finally, (3.26) follows via a standard \textit{a priori} estimate for the weak Neumann problem (3.31) and \( p|_{T_0^2 \times (\frac{1}{6}, \frac{5}{6})} = \pi|_{T_0^2 \times (\frac{1}{6}, \frac{5}{6})} \). \qed

We can now establish \( L^q \) estimates for the purely periodic part of problem (1.10).

**Lemma 3.8.** Let \( q \in (1, \infty) \). For every \((f, h) \in L^q_1(\mathbb{T} \times \Omega)^3 \times L^q_1(\mathbb{T} \times W^{1-\frac{1}{q}, q}(T_0^3))\) and \( g = 0 \) there is a unique solution

\[
(u, p, \eta) \in \mathcal{X}^q_1(\mathbb{T} \times \Omega)
\]

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Proof of Theorem 5.1] and employ Lemma 3.5. Let follow from Lemma 3.3 and Lemma 3.4. To show (3.32), we mimic the steps in [13, general case then follows via (3.32) by a density argument. Existence and uniqueness it suffices to show the statement for (\(u, p, \eta\)) with \(\phi\), \(\chi\) to (1.10) with \(\nu\).

Letting \(u, p, \eta\) we have extended (\(f, g, h\)) to a solution \((U, P, \eta)\) to (3.10) in the half space with right-hand side \((f, g, h) = (\tilde{f}, \tilde{g}, h)\) with

\[
\begin{align*}
\tilde{f}_j(t, x) &:= \begin{cases} f_j(t, x) & \text{if } x_3 < 1, \\
\varepsilon_j ((1 - \chi(2 - x_3)) f_j(t, x', 2 - x_3) - H_j(t, x)) & \text{if } x_3 > 1,
\end{cases} \\
\tilde{g}(t, x', x_3) &:= \begin{cases} 0 & \text{if } x_3 < 1, \\
-\chi'(2 - x_3) u_3(t, x', 2 - x_3) & \text{if } x_3 > 1,
\end{cases}
\end{align*}
\]

where \(\varepsilon_j := 1\) for \(j = 1, 2\), \(\varepsilon_3 := -1\), and

\[
H(t, x) := \chi''(2 - x_3) u(t, x', 2 - x_3) + 2 \partial_{x_3} u(t, x', 2 - x_3) \chi'(2 - x_3) + \chi'(2 - x_3) p(t, x', 2 - x_3) \varepsilon_3.
\]

Observer that \(U(t, x, 0) = u(t, x', 0)\) and \(P(t, x', 0) = p(t, x', 0)\). Due to the identity

\[
\chi' \Delta u_3 = \text{div} [\chi' \nabla u_3] - \partial_{x_3} u_3 \chi''
\]

we deduce by [12, Theorem III.3.4] and (2.4) that

\[
\|B(\chi' \Delta u_3)\|_{L^q(T \times \Omega)} \leq \|B(\text{div}[\chi' \nabla u_3])\|_{L^q(T \times \Omega)} + \|B(\partial_{x_3} u_3 \chi'')\|_{L^q(T \times \Omega)} \leq c_0(\chi) \|u\|_{L^q(T; W^{1,q}(\Omega))}.
\]
Letting $V := B(\chi' u_3)$, we have supp $V \subset Q_T := T \times T_0^2 \times (\frac{1}{3}, \frac{2}{3})$ and in view of (3.10) and the identity $\partial_t V = B(\chi' \partial_t u_3)$ that

$$
\|\partial_t V\|_{L^q(\mathcal{T} \times \Omega)} \leq c_1 (\|B(\chi' f_3)\|_{L^q(\mathcal{T} \times \Omega)} + \|B(\chi' \Delta u_3)\|_{L^q(\mathcal{T} \times \Omega)} + \|B(\chi' \partial_t u_3)\|_{L^q(\mathcal{T} \times \Omega)})
$$

$$
\leq c_2(\chi)(\int_{Q_T} \|f\|_{L^q(\mathcal{T} \times \Omega)} + \int_{Q_T} \|u\|_{L^r(\mathcal{T}; W^{1,q}(\Omega))} + \|\nabla p\|_{L^q(\mathcal{T})}).
$$

Now utilizing (3.26) and (2.4), we obtain

$$
\|V\|_{W^{1,2,q}(T \times \omega_+)} \leq c_3(\|f\|_{L^q(\mathcal{T} \times \Omega)} + \|u\|_{L^r(\mathcal{T}; W^{1,q}(\Omega))} + \|p\|_{L^q(\mathcal{T})}).
$$

By setting

$$
w: T \times \omega_+ \to \mathbb{R}^3, \quad w = U - V, \quad \text{and} \quad \pi: T \times \omega_+ \to \mathbb{R}, \quad \pi = P,
$$

we obtain a solution $(w, \pi, \eta) \in \mathcal{X}^r_\omega(T \times \omega_+)$ to the half space problem (3.10) with $(f, g, h) = (\hat{f} - [\partial_t - \Delta] V, 0, h)$. Moreover, due to Lemma 3.5 and (3.36) $(w, \pi, \eta)$ obeys

$$
\|w\|_{W^{1,2,q}(T \times \omega_+)} + \|\nabla \pi\|_{L^q(T \times \omega_+)} + \|\eta\|_{W^{2,q}(T; W^{1,q}(\omega_+))} \leq C_{10}(\|f\|_{L^q(T \times \omega_+)} + \|f\|_{L^q(T \times \omega_+)} + \|h\|_{L^r(T; W^{1,q}(\omega_+))} + \|u\|_{L^r(T; W^{1,q}(\Omega))} + \|p\|_{L^q(T)}).
$$

In view of (3.33) and (3.34), we deduce

$$
\|\hat{f}\|_{L^q(T \times \omega_+)} \leq (\|f\|_{L^q(T \times \omega_+)} + \|f\|_{L^q(T \times \omega_+)} + \|h\|_{L^r(T \times \omega_+ \times (1, \infty))})
$$

$$
\leq c_4(\chi)(\|f\|_{L^q(T \times \omega_+)} + \|u\|_{L^r(T; W^{1,q}(\Omega))} + \|p\|_{L^q(T \times \omega_+)})
$$

with

$$
f(t, x) := (1 - \chi(2 - x_3)) f(t, x', 2 - x_3),
$$

where the second inequality above follows by utilizing Hölder’s inequality and shifting the coordinates in the second and third norm above. Therefore, we conclude from $(u, p, \eta) = (w|_{T \times \omega_+}, \pi|_{T \times \omega_+}) + (V|_{T \times \omega}, 0, 0)$ as well as the estimates (3.36) and (3.37) that

$$
\|u\|_{W^{1,2,q}(T \times \omega_+)} + \|\nabla \pi\|_{L^q(T \times \omega_+)} + \|\eta\|_{W^{2,q}(T; W^{1,q}(\omega_+))} \leq c_5(\|f\|_{L^q(T \times \omega_+)} + \|h\|_{L^r(T; W^{1,q}(\omega_+))}) + \|u\|_{L^r(T; W^{1,q}(\Omega))} + \|p\|_{L^q(T \times \omega_+)}).
$$

In order to complete the proof, it remains to show that the final two terms on the right-hand side in (3.38) can be omitted. For this purpose, we utilize Young’s inequality and Ehrling’s Lemma to deduce

$$
\int_{T} \left(\|\Delta u\|_{L^q(\Omega)}^{\frac{q}{2} - 1} \|\nabla u(t, \cdot)\|_{W^{1,2,q}(\Omega)}^{\frac{1}{2}} \right)^q dt \leq c_6(q) \varepsilon^{-\frac{1}{q-1}} \|\nabla u\|_{L^q(\Omega)}^{q} + \varepsilon \|\nabla u\|_{L^q(\Omega)}^{q},
$$

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for any $\varepsilon > 0$, as well as
\[
\|u\|_{L^q(T; W^{1,q}(\Omega))} \leq c_7(\delta)\|u\|_{L^q(T \times \Omega)} + \delta\|u\|_{L^q(T; W^{2,\eta}(\Omega))}
\]
with $\delta > 0$. Consequently, (3.25) and (3.38) yield
\[
\|u\|_{W^{1,2,q}(T \times \Omega)} + \|\nabla p\|_{L^q(T \times \Omega)} + \|\eta\|_{W^{2,q}(T; W^{1,\frac{1}{2},q}(\Omega_0^2))} + \|\eta\|_{L^q(T; W^{5,\frac{1}{2},q}(\Omega_0^2))} \\
\leq c_8(\|f\|_{L^q(T \times \Omega)} + \|h\|_{L^q(T \times T_0^2)} + \|u\|_{L^q(T \times \Omega)} \\
+ \|\nabla' \Delta' \eta\|_{L^q(T \times T_0^2)} + \|\nabla' \partial_t \eta\|_{L^q(T \times T_0^2)}) + c_9\|u\|_{L^q(T; W^{2,\eta}(\Omega))},
\]
with $\delta = \delta(\varepsilon) > 0$, $c_8 = c_8(q, \delta, \varepsilon) > 0$ and $c_9 = c_9(q, \delta, \varepsilon) > 0$. Observe that we have utilized Ehrling’s Lemma to estimate the third norm on the right-hand side in (3.38). Choosing $\varepsilon$ and $\delta$ sufficiently small, we finally deduce
\[
\|u\|_{W^{1,2,q}(T \times \Omega)} + \|\nabla p\|_{L^q(T \times \Omega)} + \|\eta\|_{W^{2,q}(T; W^{1,\frac{1}{2},q}(\Omega_0^2))} + \|\eta\|_{L^q(T; W^{5,\frac{1}{2},q}(\Omega_0^2))} \\
\leq c_{10}(\|f\|_{L^q(T \times \Omega)} + \|h\|_{L^q(T \times T_0^2)} + \|u\|_{L^q(T \times \Omega)} \\
+ \|\nabla' \Delta' \eta\|_{L^q(T \times T_0^2)} + \|\nabla' \partial_t \eta\|_{L^q(T \times T_0^2)}). \tag{3.39}
\]

Since the embeddings
\[
W^{1,2,q}(T \times \Omega) \hookrightarrow L^q(T \times \Omega), \\
W^{2,q}(T; W^{1,\frac{1}{2},q}(\Omega_0^2)) \cap L^q(T; W^{5,\frac{1}{2},q}(\Omega_0^2)) \hookrightarrow W^{1,q}(T; W^{1,q}(\Omega)), \\
W^{2,q}(T; W^{1,\frac{1}{2},q}(\Omega_0^2)) \cap L^q(T; W^{5,\frac{1}{2},q}(\Omega_0^2)) \hookrightarrow L^q(T; W^{3,q}(\Omega)),
\]
are compact and the solution to (1.10) with homogeneous right-hand side is zero by Lemma 3.4, the $L^q$ estimate (3.32) follows by a standard contradiction argument. \hfill \Box

### 3.3 Steady-state problem

Applying the projection $P$ to the linear system (1.10), we obtain
\[
\begin{align*}
\Delta^2 \eta_s &= h_s - e_3 \cdot (T(u_s, p_s)e_3)_{|x_3=0} \quad \text{in } T_0^2, \\
-\Delta u_s + \nabla p_s &= f_s \quad \text{in } \Omega, \\
\text{div } u_s &= g_s \quad \text{in } \Omega, \\
u_s &= 0 \quad \text{on } \partial \Omega, \tag{3.40}
\end{align*}
\]

This weakly coupled system of elliptic equations (Stokes problem coupled with the biharmonic equation) is relatively simple to analyze.
Lemma 3.9. Let $q \in (1, \infty)$. For $(f_s, h_s) \in L^q(\Omega)^3 \times W^{1, \frac{1}{2}}(T_0^2)$ and $g_s = 0$ there is a unique solution $(u_s, p_s, \eta_s) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega) \times W^{5, \frac{5}{4}}(T_0^2)$ to (3.40) satisfying
\[
\|u_s\|_{W^{2,q}(\Omega)} + \|p_s\|_{W^{1,q}(\Omega)} + \|\eta_s\|_{W^{5, \frac{5}{4}}(T_0^2)} \leq C_{11}(\|f_s\|_{L^q(\Omega)} + \|h_s\|_{W^{1, \frac{1}{2}}(T_0^2)}) \tag{3.41}
\]
with $C_{11} = C_{11}(q) > 0$.

Proof. Existence of a solution $(u_s, \tilde{p}_s)$ to the steady-state Stokes system (3.40)$_{2,4}$ that is unique up-to addition of a constant to $\tilde{p}_s$ and satisfying
\[
\|u_s\|_{W^{2,q}(\Omega)} + \|\nabla \tilde{p}_s\|_{L^q(\Omega)} \leq c_0\|f_s\|_{L^q(\Omega)} \tag{3.42}
\]
follows by the same arguments typically employed in the analysis of the steady-state Stokes problem set in classical bounded subdomains of $\mathbb{R}^n$; see for example [12, Theorem IV.6.1]. Letting
\[
p_s := \tilde{p} - \frac{1}{|T_0^2|} \int_{T_0^2} h_s \, dx' - \frac{1}{|T_0^2|} \int_{T_0^2} \tilde{p} \, dx \tag{3.43}
\]
we obtain a solution to
\[
\begin{cases}
\Delta' \eta_s = h_s - e_3 \cdot (T(u_s, p_s)e_3) |_{x_3 = 0} & \text{in } T_0^2, \\
\int_{T_0^2} \eta_s(x') \, dx' = 0
\end{cases} \tag{3.44}
\]
via the formula
\[
\eta_s := \mathcal{F}_{T_0^2}^{-1} \left[ \frac{1 - \delta_{(\frac{2\pi}{T}, \mathbb{Z})}^2(\xi')}{|\xi'|^4} \mathcal{F}_{T_0^2}[h_s - e_3 \cdot (T(u_s, p_s)e_3) |_{x_3 = 0}] \right], \tag{3.45}
\]
where $\delta_{(\frac{2\pi}{T}, \mathbb{Z})}$ denotes the Dirac measure on the group $(\frac{2\pi}{T}, \mathbb{Z})^2$. The term $1 - \delta_{(\frac{2\pi}{T}, \mathbb{Z})}(\xi')$ appears in the numerator above due to (3.43), which implies
\[
\mathcal{F}_{T_0^2}[h_s - e_3 \cdot (T(u_s, p_s)e_3) |_{x_3 = 0}](0) = 0
\]
since $e_3 \cdot (T(u_s, p_s)e_3) |_{x_3 = 0} = p_s |_{x_3 = 0}$ on $T_0^2$. The term $1 - \delta_{(\frac{2\pi}{T}, \mathbb{Z})}(\xi')$ in (3.45) implies
\[
\int_{T_0^2} \eta_s(x') \, dx' = 0.
\]
Utilizing DE LEEUW's Transference Principle [6] on the Fourier multiplier in (3.45), we obtain
\[
\|\eta_s\|_{W^{5, \frac{5}{4}}(T_0^2)} \leq c_1 \|h_s - e_3 \cdot (T(u_s, p_s)e_3) |_{x_3 = 0}\|_{W^{1, \frac{1}{2}}(T_0^2)}
\leq c_2 (\|h_s\|_{W^{1, \frac{1}{2}}(T_0^2)} + \|f_s\|_{L^q(\Omega)}).
\]
By an application of Poincaré's inequality to $p_s$, we conclude (3.41). \hfill \Box

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3.4 Fully inhomogeneous linearized problem

Finally, we consider the fully inhomogeneous system (1.10) and establish Theorem 1.2.

Proof of Theorem 1.2. Recall Theorem 2.1 and set \( w := B(g) \). Since \( g \) has a vanishing mean over \( \Omega \), \( w \) solves

\[
\begin{align*}
\text{div} w &= g \quad \text{in} \ T \times \Omega, \\
 w &= 0 \quad \text{on} \ T \times \partial \Omega.
\end{align*}
\]

Moreover,

\[
\|w(t,\cdot)\|_{W^{2,q}(\Omega)} \leq c_0 \|g(t,\cdot)\|_{W^{1,q}(\Omega)},
\]

\[
\|w(t,\cdot)\|_{L^q(\Omega)} \leq c_1 |g(t,\cdot)|^{-q}.\]

Introducing the vector-field \( v := u - w \), we reduce the investigation of (1.10) with fully inhomogeneous right-hand side to that of (1.10) with right-hand side \( \tilde{f} := f - \partial_t w + \Delta w, \tilde{g} := 0, \tilde{h} := h - g|_{x_3=0} \).

Specifically, letting \((u_s,p_s,\eta_s) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega) \times W^{5-\frac{1}{q},q}(T_0^2)\) be the solution from Lemma 3.9 corresponding to data \((\mathcal{P} \tilde{f},0,\mathcal{P} \tilde{h})\), and \((u_{tp},p_{tp},\eta_{tp}) \in X_q(T,\Omega)\) the solution from Lemma 3.8 corresponding to data \((\mathcal{P}_\perp \tilde{f},0,\mathcal{P}_\perp \tilde{h})\), then

\[
(u,p,\eta) := (u_s + u_{tp} + w,p_s + p_{tp},\eta_s + \eta_{tp})
\]
solves (1.10). From (3.32) and (3.41) it follows that

\[
\|(u,p,\eta)\|_{X_q(T,\Omega)} \leq c_2 \|(\tilde{f},0,\tilde{h})\|_{Y_q(T,\Omega)}
\]

\[
\leq c_3 (\|f\|_{L^q(T,\Omega)} + \|h\|_{L^q(T,W^{1-\frac{1}{q},q_0}(T_0^2))} + \|B(\partial_t g)\|_{L^q(T,\Omega)} + \|w\|_{L^q(T,W^{2,q}(\Omega))})
\]

\[
\leq c_4 (\|f\|_{L^q(T,\Omega)} + \|h\|_{L^q(T,W^{1-\frac{1}{q},q_0}(T_0^2))} + \|g\|_{L^q(T,W^{5,q}(\Omega))} \cap W^{1,q}(T,\Omega) \cap W^{1,q}(T,\Omega))
\]

\[
\leq c_5 \|(f,g,h)\|_{Y_q(T,\Omega)}.
\]

\(\square\)

4 Proof of the main theorem

Employing Theorem 1.2, we shall now prove Theorem 1.1 using a fixed point argument.

We need the following embedding properties to estimate the nonlinear terms in this approach:

Theorem 4.1 (Sobolev embedding). Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be the whole space \( \mathbb{R}^n \), the half space \( \mathbb{R}^n_+ \), or a bounded domain with a Lipschitz boundary. Let \( q \in (1,\infty) \). Moreover, let \( m \in \mathbb{N}, M_t \in \mathbb{N}_0 \) and \( m_x \in \mathbb{N}_0 \) such that

\[
0 \leq M_x + 2M_t \leq 2m,
\]

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with $M_x := |m_x|$, and $\alpha, \beta \in [0, 2(m - M_t) - M_x]$ such that $\beta := 2(m - M_t) - M_x - \alpha$. Assume that $p, r \in [q, \infty]$ satisfy
\[
\begin{cases}
  r \leq \frac{2q}{2 - \alpha q} & \text{if } \alpha q < 2, \\
  r < \infty & \text{if } \alpha q = 2, \\
  r \leq \infty & \text{if } \alpha q > 2,
\end{cases}
\begin{cases}
  p \leq \frac{n q}{n - \beta q} & \text{if } \beta q < n, \\
  p < \infty & \text{if } \beta q = n, \\
  p \leq \infty & \text{if } \beta q > n.
\end{cases}
\]

Then there exists a constant $C_{12} = C_{12}(T, \Omega, p, q, r) > 0$ such that
\[
\|\partial^m_x \partial^M_t u\|_{L^r(T; L^p(\Omega))} \leq C_{12} \|u\|_{W^{m,2n,q}(T \times \Omega)}
\]
holds for all $u \in W^{m,2n,q}(T \times \Omega) := W^{m,q}(T; L^q(\Omega)) \cap L^q(T; W^{2m,q}(\Omega))$.

**Proof.** See [1, Theorem 2.6.3].

To simplify notation, we again set $\mu_s = \mu_f = 1$. We start by reformulating (1.9) in the reference configuration. To this end, we recall the transformation $\varphi_\eta$ from (1.3) and set
\[
\tilde{u} := u \circ \varphi_\eta, \quad \tilde{p} := u \circ \varphi_\eta.
\]

We thus obtain the following reformulating of (1.9) in the reference configuration:
\[
\begin{cases}
  \partial_t^2 \eta + \Delta' \eta - \Delta' \partial_t \eta = h - e_3 \cdot (T(\tilde{u}, \tilde{p}) e_3) |_{x_3 = 0} + R_\eta \quad \text{in } T \times T^2_0, \\
  \partial_t \tilde{u} - \Delta \tilde{u} + (\tilde{u} \cdot \nabla) \tilde{u} + \nabla \tilde{p} = \tilde{f} + \tilde{R}_f \quad \text{in } T \times \Omega, \\
  \text{div } \tilde{u} = \tilde{R}_d \quad \text{in } T \times \Omega, \\
  \tilde{u}(t, x', 0) = -\partial_t \eta(t, x') e_3 \quad \text{on } T \times T^2_0, \\
  \tilde{u}(t, x', 1) = 0 \quad \text{on } T \times T^2_0, \\
  \int_\omega \eta(t, x') \, dx' = 0,
\end{cases}
\]

where $\tilde{f} := f \circ \varphi_\eta$ and the nonlinear terms $R_\eta$, $R_f$ and $\tilde{R}_d$ are given by
\[
\begin{align*}
\tilde{R}_f(\tilde{u}, \tilde{p}, \eta) & := R_f - (\tilde{u} \cdot \nabla) \tilde{u}, \\
\tilde{R}_d(\tilde{u}, \eta) & := \text{div } R_d, \\
R_\eta(\tilde{u}, \tilde{p}, \eta) & := -e_3 \cdot \left[ (T(\tilde{u}, \tilde{p}) |_{x_3=0}(\tilde{\nu}_t - e_3)) + S_\eta \tilde{\nu}_t \right],
\end{align*}
\]

the normal vectors given by
\[
\tilde{\nu}_t := \nu_t \circ \varphi_\eta \quad \text{and} \quad \nu_t = (1 + |\nabla' \eta|^2)^{-\frac{1}{2}} \begin{pmatrix} \nabla' \eta \\ -1 \end{pmatrix},
\]

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and the terms $R_f$, $R_d$ and $S_\eta$ given by

$$R_f(\tilde{u}, \tilde{p}, \eta) := \partial_{x_3} \tilde{u} \frac{\rho \partial_x \eta}{1 + \eta} + \sum_{j,k,l=1}^3 \partial_{x_l} \partial_{x_k} \tilde{u} (E_{k,l} + E_{l,k} + E_{j,k} E_{j,l}) \circ \varphi_\eta$$

$$+ \sum_{j,k=1}^3 \partial_{x_k} \tilde{u} (\partial_{x_j} E_{j,k}) \circ \varphi_\eta - \partial_{x_3} \tilde{p} \left( \frac{\rho \nabla' \eta}{1 + \eta} \right) + \nabla \tilde{u} (E \circ \varphi_\eta) \tilde{u},$$

$$R_d(\tilde{u}) := - \begin{pmatrix} \eta \tilde{u}' \\ \rho \nabla' \eta \cdot \tilde{u}' \end{pmatrix},$$

$$S_\eta(\tilde{u}, \eta) := ( (\nabla \tilde{u} (E \circ \varphi_\eta) + (\nabla \tilde{u} (E \circ \varphi_\eta)^T)) |_{x_3=0},$$

where $\rho$ and the matrix $E \circ \varphi_\eta$ are defined as

$$\rho(x_3) := 1 - x_3, \quad \text{and} \quad E_{i,j} \circ \varphi_\eta := \begin{cases} 0 & \text{if } i \neq 3, \\ \frac{\rho \partial_x \eta}{1 + \eta} & \text{if } i = 3 \text{ and } j \neq 3, \\ -\eta & \text{if } i = 3 = j. \end{cases}$$

To simplify notation in the following, we set

$$S^q(T \times T_0^2) := W^{2,q}(T; W^{1,\frac{1}{2}q}(T_0^2)) \cap L^q(T; W^{5-\frac{1}{2}q}(T_0^2)).$$

**Lemma 4.2.** Let $q \in (1, \infty)$. Then there exists an $\varepsilon_0 > 0$ and a constant $C_{13} > 0$ such that if $\eta$ satisfies

$$\|\eta\|_{S^q(T \times T_0^2)} \leq \varepsilon_0, \quad (4.3)$$

then $E \circ \varphi_\eta \in L^q(T \times \Omega)$ and

$$\|E \circ \varphi_\eta\|_{L^q(T \times \Omega)} \leq C_{13} \|\eta\|_{L^q(T; W^{1,q}(T_0^2))}. \quad (4.4)$$

**Proof.** Due to the definition of $E \circ \varphi_\eta$, we have

$$\|E \circ \varphi_\eta\|_{L^q(T \times \Omega)} \leq \left\| \frac{\rho \nabla' \eta}{1 + \eta} \right\|_{L^q(T \times \Omega)} + \left\| \frac{\eta}{1 + \eta} \right\|_{L^q(T \times \Omega)}. \quad (4.5)$$

Utilizing the Sobolev embedding theorem (Theorem 4.1) with $m = 2$, $m_x = 0 = M_f$ and $\alpha = 2$, we find for any $q > 1$ that

$$\|\eta\|_{L^\infty(T \times T_0^2)} \leq c_0 \|\eta\|_{W^{2,4,q}(T \times T_0^2)} \leq c_0 \|\eta\|_{S^q(T \times T_0^2)} \leq c_0 \varepsilon_0,$$

and consequently by choosing $\varepsilon_0 = \frac{1}{2c_0}$ that

$$\left\| \frac{1}{1 + \eta} \right\|_{L^\infty(T \times T_0^2)} \leq \frac{1}{1 - \|\eta\|_{L^\infty(T \times T_0^2)}} \leq \frac{1}{1 - c_0 \varepsilon_0} \leq 2 < \infty.$$

In view of (4.5), this estimate implies (4.4) since $x_3 \in (0, 1).$ \hfill \Box
Lemma 4.3 (Estimates of nonlinear terms). Let $\varepsilon_0 > 0$ and $q \in (2, \infty)$. Then for any $(u, p, \eta) \in \mathcal{X}^q((T \times \Omega)$ satisfying (4.3) the nonlinear terms obey

\[
\|\tilde{R}_f\|_{L^q(T \times \Omega)} \leq C_{14}(1 + \varepsilon_0)\|u\|_{W^{1,2,q}(T \times \Omega)} + \|\nabla p\|_{L^2(T \times \Omega)} \\
+ \|u\|_{W^{1,2,q}(T \times \Omega)}^2 \|\eta\|_{S^q(T \times T^2_\alpha)},
\]

(4.6)

\[
\|\tilde{R}_d\|_{L^q(T; W^{1, q}(\Omega)) / W^{1, q}(T; W^{-1, q}(\Omega))} \leq C_{15}\|u\|_{W^{1,2,q}(T \times \Omega)} \|\eta\|_{S^q(T \times T^2_\alpha)},
\]

(4.7)

\[
\|R_d\|_{L^q(T; W^{1, q}(T^2_\alpha))} \leq C_{16}(1 + \varepsilon_0) \|\eta\|_{S^q(T \times T^2_\alpha)} + \|u\|_{W^{1,2,q}(T \times \Omega)} + \|p\|_{L^q(T; W^{1, q}(\Omega))}.
\]

(4.8)

Proof. We begin by proving estimate (4.7). Observe that Theorem 4.1 with parameters $m = 2, M_t = 0, M_x = 2$ and $\alpha = 1$ (or $m = 2, M_t = 1, M_x = 0$ and $\alpha = 1$) yields

\[
\|\eta\|_{W^{1,2,\infty}(T \times \Omega)} \leq c_0 \|\eta\|_{W^{2,4,\infty}(T \times T^2_\alpha)} \leq c_0 \|\eta\|_{S^q(T \times T^2_\alpha)}
\]

for $q > 2$. Further observer that for a.e. $t \in T$

\[
\|\text{div } R_d(t, \cdot)\|_{W^{-1, q}(\Omega)} = \sup_{v \in W^{1, q'}(\Omega), \|\nabla v\|_{L^q(\Omega)} = 1} |\langle \text{div } R_d(t, \cdot), v(t, \cdot) \rangle|
\]

\[
= \sup_{v \in W^{1, q'}(\Omega), \|\nabla v\|_{L^q(\Omega)} = 1} |\langle R_d(t, \cdot), \nabla v(t, \cdot) \rangle| \leq \|R_d\|_{L^q(\Omega)}
\]

holds. Hence, $\tilde{R}_d = \text{div } R_d$ fulfills

\[
\|\tilde{R}_d\|_{L^q(T; W^{1, q}(\Omega)) / W^{1, q}(T; W^{-1, q}(\Omega))} \leq \|R_d\|_{W^{1,2,q}(T \times \Omega)} \\
\leq c_1 \|u\|_{W^{1,2,q}(T \times \Omega)} \|\eta\|_{W^{1,2,\infty}(T \times T^2_\alpha)} \leq c_2\|u\|_{W^{1,2,q}(T \times \Omega)} \|\eta\|_{S^q(T \times T^2_\alpha)}
\]

(4.9)

and thus (4.7). The $L^q$-estimates (4.6) and (4.8) follow similarly by an application of Hölder’s inequality and Theorem 4.1. \qed

Proof of Theorem 1.1. We employ the contraction mapping principle based on the $L^q$ estimates deduced for the linear system (1.10). To this end, let

\[
\mathcal{S}: \mathcal{Y}^q_{(0)}(T \times \Omega) \to \mathcal{X}^q(T \times \Omega)
\]

be the solution operator corresponding to (1.10) with

\[
\mathcal{Y}^q_{(0)}(T \times \Omega) := \mathcal{L}^q(T \times \Omega)^3 \times \mathcal{L}^q(T; W^{1, q}(\Omega)) \cap W^{1, q}(\Omega) \times \mathcal{L}^q(T; W^{1, -\frac{1}{q}}(T^2_\alpha))
\]

and

\[
W^{1, q}(\Omega) := \{g \in W^{1, q}(\Omega) \mid \int_{\Omega} g \, dx = 0\}.
\]

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By Theorem 1.2, $S$ is a well-defined bounded operator. We seek a solution to (4.2) as a fixed point of the mapping

$$F: \mathcal{A}(\mathbb{T} \times \Omega) \to \mathcal{A}(\mathbb{T} \times \Omega), \quad F(u, p, \eta) := S(f + \tilde{R}_f, \tilde{R}_d, h + R_\eta).$$

Let $r > 0$ and consider some $(u, p, \eta) \in \mathcal{A}(\mathbb{T} \times \Omega) \cap B_r$, where $B_r$ denotes the closed ball in $\mathcal{A}(\mathbb{T} \times \Omega)$ with radius $r$. By Lemma 4.3 we obtain

$$\|\tilde{R}_d\|_{L^\alpha(\mathbb{T} \times \Omega)} + \|\tilde{R}_f\|_{L^\alpha(\mathbb{T} \times \Omega)} + \|R_\eta\|_{L^\alpha(\mathbb{T} \times \mathbb{T}_0^2)} \leq c_0(1 - \varepsilon)(r^2 + r^3)$$

and therefore

$$\|F\|_{\mathcal{A}} \leq \|S\| \left( \|f\|_{L^\alpha(\mathbb{T} \times \Omega)} + \|\tilde{R}_f\|_{L^\alpha(\mathbb{T} \times \Omega)} + \|R_\eta\|_{L^\alpha(\mathbb{T} \times \mathbb{T}_0^2)} \right) + \|\tilde{R}_d\|_{L^\alpha(\mathbb{T} \times \mathbb{T}_0^2)} + \|R_\eta\|_{L^\alpha(\mathbb{T} \times \mathbb{T}_0^2)} \leq c_1(\varepsilon + r^2 + r^3).$$

Choosing $r = \sqrt{\varepsilon}$ and $\varepsilon$ sufficiently small, we have $c_1(\varepsilon + r^2 + r^3) \leq r$, in which case $F$ becomes a self-mapping on $B_r$. To complete the proof, it remains to show that $F$ is a contraction. For this purpose we utilize Theorem 4.1 with $m = 2, M_t = 1, M_x = 0, \alpha = 1$ and subsequently $m = 2, M_t = 0, M_x = 0, \alpha = 2$ to deduce

$$\|\eta\|_{L^\infty(\mathbb{T} \times \mathbb{T}_0^2)} + \|\partial_t \eta\|_{L^\infty(\mathbb{T} \times \mathbb{T}_0^2)} \leq c_2 \|\eta\|_{W^{2,4}(\mathbb{T} \times \mathbb{T}_0^2)} \leq c_2 \|\eta\|_{S^{(\Theta)}} \ell^{\infty}(\mathbb{T} \times \mathbb{T}_0^2).$$

We thus conclude from (4.3) that

$$\left\| \nabla u \partial_t \eta \frac{1}{1 + \eta} - \nabla v \partial_t \zeta \frac{1}{1 + \zeta} \right\|_{L^\alpha(\mathbb{T} \times \Omega)} \leq c_3\varepsilon \|(u, p, \eta) - (v, p, \zeta)\|_{\mathcal{A}},$$

and similarly

$$\left\| \partial_{\varepsilon t} \rho \nabla \eta \frac{1}{1 + \eta} - \partial_{\varepsilon t} \rho \left( \nabla \zeta \frac{1}{1 + \zeta} \right) \right\|_{L^\alpha(\mathbb{T} \times \Omega)} \leq c_4\varepsilon \|(u, p, \eta) - (v, p, \zeta)\|_{\mathcal{A}}.$$

The additional terms in $\tilde{R}_f$ can be estimated analogously to deduce

$$\|\tilde{R}_f(u, p, \eta) - \tilde{R}_f(v, p, \zeta)\|_{L^\alpha(\mathbb{T} \times \Omega)} \leq c_5\varepsilon \|(u, p, \eta) - (v, p, \zeta)\|_{\mathcal{A}}.$$

A straightforward calculation yields

$$\|\tilde{R}_d(u, p, \eta) - \tilde{R}_d(v, p, \zeta)\|_{L^\alpha(\mathbb{T} \times \mathbb{T}_0^2)} \leq c_6\varepsilon \|(u, p, \eta) - (v, p, \zeta)\|_{\mathcal{A}}.$$

In the case of $R_\eta$, we further have to employ the properties of the trace operator $\text{Tr}$ defined in (3.22) to deduce that

$$\|R_\eta(u, p, \eta) - R_\eta(v, p, \zeta)\|_{L^\alpha(\mathbb{T} \times \mathbb{T}_0^2)} \leq c_7\varepsilon \|(u, p, \eta) - (v, p, \zeta)\|_{\mathcal{A}}.$$

Collecting the estimates deduced for $R_f$, $\tilde{R}_d$ and $R_\eta$, we obtain that

$$\|F(u, p, \eta) - F(v, p, \zeta)\|_{\mathcal{A}} \leq c_8\varepsilon \|(u, p, \eta) - (v, p, \zeta)\|_{\mathcal{A}}.$$

Choosing $\varepsilon$ sufficiently small, we deduce that $F$ is a contracting self-mapping. By the contraction mapping principle, existence of a fixed point for $F$ follows and completes the proof. \qed
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