A SHARP INEQUALITY FOR HARMONIC
DIFFEOMORPHISMS OF THE UNIT DISK

DAVID KALAJ

Abstract. We extend the classical Schwarz-Pick inequality to the class of harmonic mappings between the unit disk and a Jordan domain with given perimeter. It is intriguing that the extremals in this case are certain harmonic diffeomorphisms between the unit disk and a convex domain that solve the Beltrami equation of second order.

1. Introduction

Let $U$ be the unit disk in the complex plane $\mathbb{C}$ and denote by $T$ its boundary. A harmonic mapping $f$ of the unit disk into the complex plane can be written by $f(z) = g(z) + h(z)$ where $g$ and $h$ are holomorphic functions defined on the unit disk. Two of essential properties of harmonic mappings are given by Lewy theorem, and Rado-Kneser-Choquet theorem. Lewy theorem states that a injective harmonic mapping is indeed a diffeomorphism. Rado-Kneser-Choquet theorem states that a Poisson extension of a homeomorphism of the unit circle $T$ onto a convex Jordan curve $\gamma$ is a diffeomorphism on the unit disk onto the inner part of $\gamma$. For those and many more important properties of harmonic mappings we refer to the book of Duren [2].

The standard Schwarz-Pick lemma for holomorphic mappings states that every holomorphic mapping $f$ of the unit disk onto itself satisfies the inequality

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}. \tag{1.1}$$

If the equality is attained in (1.1) for a fixed $z = a \in U$, then $f$ is a M"obius transformation of the unit disk.

It follows from (1.1) the weaker inequality

$$|f'(z)| \leq \frac{1}{1 - |z|^2} \tag{1.2}$$

with the equality in (1.2) for some fixed $z = a$ if and only if $f(z) = e^{it} \frac{z-a}{1-\bar{z}a}$. We will extend this result to harmonic mappings.

2000 Mathematics Subject Classification. Primary 31R05; Secondary 42B30.

Key words and phrases. Harmonic functions, Bloch functions, Hardy spaces.
2. Main result

**Theorem 2.1.** If \( f \) is a harmonic orientation preserving diffeomorphism of the unit disk \( U \) onto a Jordan domain \( \Omega \) with rectifiable boundary of length \( 2\pi R \), then the sharp inequality

\[
|\partial f(z)| \leq \frac{R}{1 - |z|^2}, \quad z \in U
\]  

holds. If the equality in (2.1) is attained for some \( a \), then \( \Omega \) is convex and there is a holomorphic function \( \mu : U \to U \) and a constant \( \theta \in [0, 2\pi] \) such that

\[
F(z) := e^{-i\theta} f \left( \frac{z + a}{1 + z\bar{a}} \right) = R \left( \int_0^z \frac{dt}{1 + t^2 \mu(t)} + \int_0^z \frac{\mu(t)dt}{1 + t^2 \mu(t)} \right).
\]

Moreover every function \( f \) defined by (2.2), is a harmonic diffeomorphism and maps the unit disk to a Jordan domain bounded by a convex curve of length \( 2\pi R \) and the inequality (2.1) is attained for \( z = a \).

**Corollary 2.2.** Under the conditions of Theorem 2.1, if \( |\mu|_\infty = k < 1 \), then the mapping \( F \) is \( K = 1 + k \) bi-Lipschitz, and \( K \)-quasiconformal.

**Proof.** We have that

\[
f_z(z) = \frac{1}{1 + z^2 \mu(z)}
\]

and

\[
\overline{F_z(z)} = \frac{\mu(z)}{1 + z^2 \mu(z)}.
\]

Thus

\[
\frac{1 - k}{1 + k} \leq |F_z| - |F_{\overline{z}}| = |dF| = |F_z| + |F_{\overline{z}}| \leq \frac{1 + k}{1 - k}.
\]

\( \square \)

**Corollary 2.3.** If \( \Omega = U \), then the equality is attained in (2.1) for some \( a \) if and only if \( f \) is a Möbius transformation of the unit disk onto itself.

**Proof of Corollary 2.3.** Under conditions of Theorem 2.1 the function (2.2) can be written as

\[
F(z) := e^{-i\theta} f \left( \frac{z + a}{1 + z\bar{a}} \right) = R \left( \int_0^z \frac{dt}{1 - t^2 h'(t)} \right) + h(z)
\]

where \( h(z) = \sum_{k=0}^\infty a_k z^k \) is defined on the unit disk and satisfies the condition

\[
\frac{|h'(z)|}{|1 - z^2 h'(z)|} < 1, \quad z \in U.
\]

Moreover

\[
|f(U)| = R^2 \pi \left( 1 - \sum_{k=0}^\infty \frac{|a_k|^2}{(n + 1)(n + 2)} \right).
\]

If \( R = 1 \), this implies that \( \Omega = U \) if and only if \( h \equiv 0 \). \( \square \)
By using the corresponding result in [1] and Theorem 2.1 we have

**Corollary 2.4.** If as in (2.3), $F(z) = g + \overline{h}$, then $F(z) = g(z) - h(z)$ is univalent and convex in direction of real axis.

By using Theorem 2.1 we obtain

**Corollary 2.5.** For every positive constant $R$ and every holomorphic function $\mu$ of the unit disk into itself, there is a unique convex Jordan domain $\Omega = \Omega_{\mu,R}$, with the perimeter $2\pi R$, such that the initial boundary problem

\[
\begin{cases}
    f(z) = \mu(z) f(z), \\
    f(0) = R, \\
    f(0) = 0.
\end{cases}
\]

admits a unique univalent harmonic solution $f = f_{\mu,R} : \Omega \to \Omega$.

**Remark 2.6.** If instead of boundary problem (2.5) we observe

\[
\begin{cases}
    g(z) = \mu(z) g(z), \\
    g(z) = \frac{R}{1-|a|^2}, \\
    g(a) = 0,
\end{cases}
\]

then the solution $g$ is given by

$g(z) = e^{i\theta} f\left(\frac{z-a}{1-|a|^2}\right)$

and thus $g(U) = e^{i\theta} \cdot \Omega_{\mu,R}$. Here $f$ is a solution of (2.5).

3. **Proof of the main result**

**Proof of Theorem 2.1.** Assume first that $f(z) = g(z)+\overline{h(z)}$ has $C^1$ extension to the boundary and assume without loss of generality that $R = 1$. Then we have

\[\partial_t (g(z)+\overline{h(z)}) = ig'(z)z + \overline{h'(z)z}\]

So for $z = e^{it}$,

\[|ig'(z)z + \overline{h'(z)z}| = |g'(z) - \bar{h}'(z)z^2|.

Thus

\[2\pi = \int_T |\partial_t (g(z)+\overline{h(z)})||dz| = \int_T |g'(z) - \bar{h}'(z)z^2||dz|.

As $|g'(z) - \bar{h}'(z)z^2|$ is subharmonic, it follows that

\[|g'(0)| \leq \frac{1}{2\pi} \int_T |g'(z) - \bar{h}'(z)z^2||dz|.

Thus we have that $|g'(0)| \leq 1$. Now if $m(z) = \frac{z+a}{1+za}$, then $m(0) = a$, and thus $F(z) = f(m(z))$ is a harmonic diffeomorphism of the unit disk onto itself. Further,

\[\partial F(0) = f'(a)m'(0) = \partial f(a)(1-|a|^2).\]
Therefore by applying the previous case to $F$ we obtain

$$|\partial f(a)| \leq \frac{1}{1 - |a|^2}.$$ 

Assume now that the equality is attained for $z = 0$. Then

$$|g'(0)| = \frac{1}{2\pi r} \int_{rT} |g'(z) - h'(z)z^2|dz|,$$

or what is the same

$$|g'(0)| = \frac{1}{2\pi} \int_{T} |g'(zr) - h'(zr)r^2z^2|dz| - |g'(0)| \equiv 0.$$ 

Thus for $0 \leq r \leq 1$ we have

(3.1) \hspace{1cm} \frac{1}{2\pi} \int_{T} |g'(zr) - h'(zr)r^2z^2|dz| - |g'(0)| = 0.

In order to continue recall the definition of the Riesz measure $\mu$ of a subharmonic function $u$. Namely there exists a unique positive Borel measure $\mu$ so that

$$\int_{U} \varphi(z)d\mu(z) = \int_{U} u\Delta \varphi(z)dm(z), \quad \varphi \in C_0^2(U).$$

Here $dm$ is the Lebesgue measure defined on the complex plane $\mathbb{C}$. If $u \in C^2$, then

$$d\mu = \Delta u dm.$$ 

**Proposition 3.1.** \cite[Theorem 4.5.1]{5} If $u$ is a subharmonic function defined on the unit disk then for $r < 1$ we have

$$\frac{1}{2\pi} \int_{T} u(rz)|dz| - u(0) = \frac{1}{2\pi} \log \frac{r}{|z|}d\mu(z)$$

where $\mu$ is the Riesz measure of $u$.

By applying Proposition 3.1 to the subharmonic function

$$u(z) = |g'(z) - h'(z)z^2|$$

in view of (3.1) we obtain that

$$\frac{1}{2\pi} \int_{|z|<r} \log \frac{r}{|z|}d\mu(z) \equiv 0.$$ 

Thus in particular we infer that $\mu = 0$, or what is the same $\Delta u = 0$. As $u = |w|$ where $w = |u|e^{i\theta}$ is harmonic, it follows that

$$\Delta u = u|\nabla \theta|^2 = 0.$$ 

Therefore $\nabla \theta \equiv 0$, and therefore $\theta = \text{const}$.

So

$$e^{-i\theta}(g'(z) - h'(z)z^2) = G(z) + \mathcal{H}(z),$$

is a real harmonic function. Here

$$G(z) = e^{-i\theta}g'(z)$$
and
\[ H(z) = -e^{i\theta} h'(z) z^2 \]
are analytic functions satisfying the condition \(|H(z)| < |G(z)|\) in view of Lewy theorem. Thus
\[ G(z) + \overline{H(z)} = \overline{G(z)} + H(z) \]
or what is the same
\[ G(z) - H(z) = \overline{G(z)} - \overline{H(z)}. \]
Thus \(G(z) - H(z)\) is a real holomorphic function and therefore it is a constant function. Further
\[ e^{-i\theta} g'(z) + e^{i\theta} h'(z) z^2 = G(z) - H(z) = G(0) - H(0) = e^{-i\theta} g'(0). \]
Hence
\[ G(z) + \overline{H(z)} = G(z) + \overline{G(z)} - e^{-i\theta} g'(0) = 2\Re \left[ e^{-i\theta} g'(z) \right] - e^{-i\theta} g'(0). \]
Assume without losing the generality that \(\theta = 0\) and \(g'(0) = 1\). Then
\[ g'(z) = 1 - h'(z) z^2. \]
Further for \(z = e^{it}\),
\[ \partial_t F(z) = i z (1 - 2 \Re(h'(z) z^2)) \]
and
\[ \left| \partial_t f(z) \right| = \left| g'(z) - \overline{h'(z) z^2} \right| = \left| 1 - 2 \Re(h'(z) z^2) \right| = 1 - 2 \Re(h'(z) z^2). \]
From (2.4), we infer that
\[ (1 - 2 \Re(h'(z) z^2)) > \left| h'(z) \right|^2 (1 - |z|^2). \]
In order to get the representation (2.2), by Lewy theorem, we have that the holomorphic mapping \(\mu(z) = \frac{h'(z)}{g'(z)}\) maps the unit disk into itself. By (3.2) we deduce that
\[ g(z) = R \int_0^z \frac{dt}{1 + t^2 \mu(t)} \]
and
\[ h(z) = R \int_0^z \frac{\mu(t) dt}{1 + t^2 \mu(t)}. \]
In order to prove that, every mapping \(f\) defined by (2.2) is a diffeomorphism we use Choquet-Kneser-Rado theorem. First of all
\[ \arg \partial_t F(z) = (\pi/2 + t). \]
Therefore
\[ \partial_t \arg \partial_t F(z) = 1 > 0 \]
which means that \(F(T)\) is a convex curve.
As
\[ \frac{\partial_t F(z)}{|\partial_t F(z)|} = i z, \]
if \( z_1, z_2 \in T \) with \( f(z_1) = f(z_2) \), then
\[
\frac{\partial_t F(z_1)}{|\partial_t F(z_1)|} = \frac{\partial_t F(z_2)}{|\partial_t F(z_2)|}
\]
and so \( z_1 = z_2 \). Thus by Choquet-Kneser-Rado theorem, \( F \) is a diffeomorphism.

If \( f \) is not \( C^1 \) up to the boundary, then we apply the approximating sequence. Let \( \Omega \) be a fixed Jordan domain and assume that \( \phi \) is a conformal mapping of the unit disk onto \( \Omega \), with \( \phi(0) = 0 \). For \( r_n = \frac{n}{n+1} \), let \( \Omega_n = \varphi(r_n U) \), and let \( U_n = f^{-1}(\Omega_n) \). Let \( \phi_n : U \to U_n \) be a conformal mapping satisfying the condition \( \phi_n(0) = 0 \). Then \( f_n = f \circ \phi_n \) is a conformal mapping of the unit disk onto the Jordan domain \( \Omega_n \). Further, by subharmonic property of \( |\phi'(z)| \) we conclude that
\[
R_n = |\partial \Omega_n| = \int_T |\phi'(r_n z)|dz < \int_T |\phi'(z)|dz = |\partial \Omega| = R.
\]

Then we have that
\[
|\partial f_n(z)| \leq \frac{R_n}{1 - |z|^2}, \quad z \in U.
\]

As \( \phi_n \) converges in compacts to the identity mapping, and thus \( \phi'_n \) converges in compacts to the constant 1, we conclude that the inequality (2.1) is true for non-smooth domains.

It remains to consider the equality statement in this case. But we know that \( \partial \Omega \) is rectifiable if and only if \( \partial f \in h_1(U) \). (See e.g. [4, Theorem 2.7]). Here \( h_1 \) stands for the Hardy class of harmonic mappings. Now the proof is just repetition of the previous approach, and we omit the details.

**Example 3.2.** If \( \mu(z) = z^n \), then \( F \) defined in (2.2), maps the unit disk to \( n + 2 \)-regular polygon of perimeter \( 2\pi R \) and centered at 0. Namely we have that
\[
\partial_z F(z) = \frac{R}{1 + z^{n+2}}, \quad \partial_{\bar{z}} F(z) = \frac{Rz^n}{1 + z^{n+2}}.
\]
The rest follows from the similar statement obtained by Duren in [2, p. 62].

**Remark 3.3.** If \( \mu \) is a holomorphic mapping of the unit disk onto itself and \( F \) is defined by (2.2), then \( F(0) = 0 \) and
\[
|DF|^2 := |F_z|^2 + |F_{\bar{z}}|^2 \geq \frac{R^2}{2}.
\]
Indeed we have that
\[
|DF|^2 = R^2 \frac{1 + |\mu|^2}{|1 + z^2 \mu|^2} \geq \frac{R^2}{2} = \frac{L^2}{8\pi^2} \geq \frac{\rho^2}{2}.
\]
Here \( \rho = \text{dist}(0, \partial \Omega) \). Thus we have the sharp inequality
\[
(3.4) \quad |DF|^2 \geq \frac{\rho^2}{2}.
\]
In [3] it is proved that we have the general inequality

\[ |Df|^2 \geq \frac{\rho^2}{16}, \]

for every harmonic diffeomorphism of the unit disk onto a convex domain \( \Omega \) with \( f(0) = 0 \). Some examples suggest that the best inequality in this context is

\[ |Df|^2 \geq \frac{\rho^2}{8}, \]

The last conjectured inequality is not proved. The gap between \( \frac{\rho^2}{2} \) and \( \frac{\rho^2}{8} \) in (3.4) and (3.6) appears as the mappings \( F \) are special extremal mappings which for the case of \( \Omega \) being the unit disk are just rotations.

References

[1] J. Clunie, T. Sheil-Small, Harmonic univalent functions. Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984), 325.
[2] P. Duren: Harmonic mappings in the plane. Cambridge Tracts in Mathematics, 156. Cambridge University Press, Cambridge, 2004. xii+212 pp.
[3] D. Kalaj: On harmonic diffeomorphisms of the unit disc onto a convex domain. Complex Variables, Theory Appl. 48, No.2, 175-187 (2003).
[4] D. Kalaj, M. Marković, M. Mateljević, Carathéodory and Smirnov type theorems for harmonic mappings of the unit disk onto surfaces. Ann. Acad. Sci. Fenn. Math. 38 (2013), no. 2, 565–580.
[5] M. Pavlović: Introduction to function spaces on the disk. Posebna Izdanja [Special Editions], 20. Matematiki Institut SANU, Belgrade, 2004. vi+184 pp.

University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 81000 Podgorica, Montenegro

E-mail address: davidk@ac.me