EQUIVARIANT BIRCH-SWINNERTON-DYER CONJECTURE FOR THE
BASE CHANGE OF ELLIPTIC CURVES: AN EXAMPLE

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Abstract. Let $E$ be an elliptic curve defined over $\mathbb{Q}$ and let $K/\mathbb{Q}$ be a finite Galois
extension with Galois group $G$. The equivariant Birch-Swinnerton-Dyer conjecture for
$h^1(E \times \mathbb{Q} K)(1)$ viewed as a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}[G]$ relates the twisted
$L$-values associated with $E$ with the arithmetic invariants of the same. In this paper we
prescribe an approach to verify this conjecture for a given data. Using this approach, we
verify the conjecture for an elliptic curve of conductor 11 and an $S_3$-extension of $\mathbb{Q}$.

1. Introduction

Let $E$ be an elliptic curve defined over $\mathbb{Q}$. Let $K/\mathbb{Q}$ be a finite Galois extension with Galois
group $G$ and let $E_K := E \times \text{Spec } \mathbb{Q} \text{Spec } K$. Our interest is in the motive $M := h^1(E_K)(1)$
which has a natural action of the semisimple algebra $\mathbb{Q}[G]$. We regard $M$ as a motive defined
over $\mathbb{Q}$ with coefficients in $\mathbb{Q}[G]$. For a ring $S$, we let $\zeta(S)$ denote its centre. Let
$\Xi = \text{Det}_{\mathbb{Q}[G]} E(K)_{\mathbb{Q}} \otimes \text{Det}_{\mathbb{Q}[G]}^{-1} E(K)_{\mathbb{Q}} \otimes \text{Det}_{\mathbb{Q}[G]}^{-1} H^1((E_K)(\mathbb{C}), \mathbb{Q})^+ \otimes \text{Det}_{\mathbb{Q}[G]} H^0(E_K, \Omega^1_{E_K})^*$,
where $\Omega^1_{E_K}$ is the sheaf of differentials and $\text{Det}_{\mathbb{Q}[G]}$ is a $((\mathbb{Q}[G])^\times$-module valued functor
introduced below. There is an isomorphism
$\vartheta_\infty : \zeta(\mathbb{R}[G]) \simeq \Xi \otimes \mathbb{R},$
given by the height pairing and the period isomorphism attached to the elliptic curve. The rationality
part of the equivariant conjecture says that the special value $L^*(M, 0)^{-1}$, which can be viewed as an element of $\zeta(\mathbb{R}[G])$, maps under $\vartheta_\infty$ into an element of $\Xi \otimes 1$.

Let $l$ be a rational prime and let $S_l$ be the finite set of primes in $\mathbb{Q}$ consisting of primes of
bad reduction, ramified primes, infinite prime and $l$. There exists a perfect complex
$R\Gamma_c(Z_{S_l}, H_1(M))$ of $\mathbb{Q}[G]$-modules along with an isomorphism
$\vartheta_l : \Xi(M) \otimes \mathbb{Q}_l \simeq \text{Det}_{\mathbb{Q}[G]} R\Gamma_c(Z_{S_l}, H_1(M))$
of $\mathbb{Q}_l[G]$-modules.

Let $T_l := \text{Ind}_{K}^{G} \left( \lim_{\rightarrow n} E(\mathbb{Q})[l^n] \right)$, a $\mathbb{Z}_l$-lattice in $V_l := H_1(M)$. Then, $R\Gamma_c(Z_{S_l}, T_l)$ is a
perfect complex of $\mathbb{Z}_l[G]$-modules with $R\Gamma_c(Z_{S_l}, T_l) \otimes_{\mathbb{Z}_l[G]} \mathbb{Q}_l[G] \simeq R\Gamma_c(Z_{S_l}, V_l)$. This along
with a trivialization $\tau_l : \Xi \otimes \mathbb{Q}_l \simeq \zeta(\mathbb{Q}_l[G])$ gives an element $\xi_l = (R\Gamma_c(Z_{S_l}, T_l); \tau_l) \in
K_0(\mathbb{Z}_l[G]; \mathbb{Q}_l)$. On the other hand, the leading coefficient $L^*(M, 0)$ of the motivic $L$-function
at $s = 0$ gives a class $\theta_l \in K_0(\mathbb{Z}[G]; \mathbb{Q}_l)$, obtained via the long exact sequence of $K$-theory.

Key words and phrases. Elliptic curves, Tamagawa numbers, equivariant conjecture, Birch-Swinnerton-
Dyer conjecture.
The equivariant conjecture formulated by Burns and Flach states that $\xi_l - \theta_l$ vanishes in $K_0(\mathbb{Z}[G]; \mathbb{Q}_l)$ for all primes $l$.

In this paper, we present a technique to verify the above conjecture for a given elliptic curve and a fixed extension under certain hypotheses. While the equality in the conjecture is exactly checked in the commutative case, we only give a numerical verification in the noncommutative case. We prove

**Theorem 1.1.** Let $E$ be the elliptic curve $y^2 + xy + y = x^3 + 11$ and let $K$ be the splitting field of $x^3 - 4x + 3$ over $\mathbb{Q}$. Let $G(\simeq S_3)$ denote the Galois group of $K$ over $\mathbb{Q}$. Let $l$ be an odd rational prime. If $|\text{III}(E/K)|_l = 1$ then the $l$-part of the equivariant conjecture holds numerically for the motive $h^1(E_K)\langle 1 \rangle$ and the $\mathbb{Z}$-order $\mathbb{Z}[G]$.

We analyse the complex $R\Gamma_c$ via the exact triangle

$$R\Gamma_c(\mathbb{Z}_{S_l},T_l) \to R\Gamma_f(\mathbb{Q},T_l) \to \oplus_{p \in S_l} R\Gamma_f(\mathbb{Q}_p,T_l),$$

where the cohomology of the complexes $R\Gamma_f(\mathbb{Q},T_l)$ can be described in terms of the group of points and the Tate-Shafarevich group of $E$. However, to ensure that all the terms in the triangle are perfect complexes of $\mathbb{Z}[G]$-modules we have to make certain noncanonical choices (see Section 3 below).

On the analytic side, we use the theory of modular symbols to write down the special value $L(E \otimes \chi, 1)$ for a Dirichlet character $\chi$. However, no such theory exists for elliptic curves defined over a general number field, or for twists by nonabelian characters. We therefore numerically compute the special values using methods of Totiis as explained in [10].

The paper is organized as follows. In Section 2, we give a brief description of the conjecture specialized to the case we are interested in. We then indicate methods to compute the arithmetic and the analytic sides of the conjecture in Sections 3 and 4 respectively. We prove the main result in the last section.

2. **Equivariant conjecture**

2.1. **Algebraic $K$-groups.** Let $R$ and $S$ be rings, and let $\phi : R \to S$ be a ring homomorphism. Let $K_i(\ast)$ denote the associated $K$-groups. We have the following long exact sequence

$$K_1(R) \longrightarrow K_1(S) \xrightarrow{\delta} K_0(R;S) \longrightarrow K_0(R) \longrightarrow K_0(S)$$

where $K_0(R;S)$ is the relative $K_0$-group.

In terms of generators and relations, $K_0(R;S)$ is generated by triples $(M,N;\lambda)$ where $M,N$ are finitely generated projective $R$-modules and $\lambda : M \otimes S \to N \otimes S$ is an $S$-isomorphism, and with relations given by short exact sequences (see [8]).

Similarly, for any ring $R$, the group $K_1(R)$ is generated by pairs $(M;\phi)$ where $M$ is a free $R$-module of finite rank and $\phi$ is an $R$-automorphism of $M$, and with relations given by the inclusions of free $R$-modules. We use these descriptions to denote elements of $K_1$ and the relative $K_0$-group.

For a $\mathbb{Q}$-algebra $A$ and a $\mathbb{Z}$-order $\mathfrak{A}$ in $A$, we denote by $Cl(\mathfrak{A})$ the associated class group. That is,

$$Cl(\mathfrak{A}) := \ker (K_0(\mathfrak{A}) \to K_0(A)).$$

We use analogous notation in the local case as well.
2.2. Virtual objects. Let $R$ be a ring and let $\text{PMod}(R)$ denote the category of finitely generated projective $R$-modules. In [9], Deligne has constructed a Picard category $V(R)$ of virtual objects and a universal determinant functor

$$[\,] : \text{PMod}(R) \to V(R)$$

satisfying certain conditions. This functor naturally extends to a functor

$$[\,] : D^b(R) \to V(R)$$

where $D^b(R)$ is the category of perfect complexes of $R$-modules (see [5] [14] for details).

It follows from the proof of the existence of $V(R)$ in [9] that there are isomorphisms

$$(2) \quad K_i(R) \overset{\sim}{\to} \pi_i(V(R))$$

for $i = 0, 1$ where $K_i(R)$ denotes the algebraic $K$-group associated to $R$, and $\pi_0(V(R))$ is the group of isomorphism classes of objects of $V(R)$ and $\pi_1(V(R)) = \text{Aut}_{V(R)}(1_{V(R)})$.

Given a finitely generated subring $S$ of $\mathbb{Q}$, an $S$-order $\mathfrak{A}$ of a finite dimensional $\mathbb{Q}$-algebra $A$ is a finitely generated $S$-module such that $\mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Q} = A$. For any such $S$-order, and any field extension $F$ of $\mathbb{Q}$, one has a notion of relative virtual objects $V(\mathfrak{A}; F)$ and $V(\mathfrak{A}_p; \mathbb{Q}_p)$, where $\mathfrak{A}_p = \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (see [5] for details). There are isomorphisms

$$(3) \quad \pi_0(V(\mathfrak{A}; F)) \overset{\sim}{\to} K_0(\mathfrak{A}; F)$$

and

$$(4) \quad \pi_0(V(\mathfrak{A}_p; \mathbb{Q}_p)) \overset{\sim}{\to} K_0(\mathfrak{A}_p; \mathbb{Q}_p),$$

which are compatible with the Mayer-Vietoris sequences (cf. Prop. 2.5 in [5]). Given a perfect complex $C_\bullet$ of $\mathfrak{A}$-modules and an isomorphisms $\tau : [C_\bullet \otimes_{\mathfrak{A}} F] \simeq 1_{V(F)}$, we denote by $([C_\bullet]; \tau)$ the corresponding element of $K_0(\mathfrak{A}; F)$ under the above isomorphism. We use analogous notation in the local case as well.

Remarks.

(1) Let $R$ be a commutative ring. Then an $R$-module $P$ is projective if and only if it is locally free at all the prime ideals of $R$. In this case, the determinant functor can be defined locally by

$$\text{Det}_{R_p}(P_p) = \left( \bigwedge^{\text{rank}_{R_p}(P_p)} P_p, \text{rank}_{R_p}(P_p) \right)$$

for every prime $p \in \text{Spec}(R)$. Note that thus defined $\text{Det}_R(P)$ is a graded line bundle. Let $\mathcal{P}(R)$ denote the category of graded line bundles over $R$. Then the universal property of $V(R)$ defines a tensor functor $f_R : V(R) \to \mathcal{P}(R)$. The functor $f_R$ is an equivalence if $R$ is either semisimple or a finite flat $\mathbb{Z}$-algebra. For such rings, we assume that the determinant functor is constructed as above.

(2) Even if $R$ is not commutative, but is semisimple, one can construct the determinant functor in a similar fashion by looking at the indecomposable idempotents. We give this construction below since it is key to some of our computations.

By Wedderburn’s decomposition we can assume that $R$ is a central simple algebra over a field $F$. So $R \simeq M_n(D)$ for some division ring $D$ with centre $F$. Further, by fixing an exact Morita equivalence $\text{PMod}(M_n(D)) \to \text{PMod}(D)$, we may assume that $R = D$. Fix a field extension $F'/F$ such that $D \otimes_F F' \simeq M_d(F')$. Let $e$ be an indecomposable idempotent of $M_d(F')$ and let $e_1, \ldots, e_d$ be an ordered $F'$-basis of
eM_d(F'). Let V be a finitely generated projective (and hence free) D-module. Let \( \{v_1, \ldots, v_r\} \) be a D-basis of V. Set \( b := \wedge_{i \neq j} v_j. \) This is an \( F' \)-basis of \( \text{Det}_F(e(V \otimes_F F')) \). Since any change in the basis \( \{v_i\}_i \) multiplies \( b \) by an element of \( F^* \), the \( F \)-space spanned by \( b \) yields a well-defined graded \( F \)-line bundle. This defines the determinant functor. Thus, for a semisimple ring, the determinant functor can be constructed as taking values in graded line bundles over the centre. We use this concrete construction when referring to the functor over \( \mathbb{R}[G] \), \( \mathbb{Q}[G] \) and \( \mathbb{Q}_l[G] \).

2.3. The motive \( h^1(E_K)(1) \). Let \( K/\mathbb{Q} \) be a finite Galois extension with Galois group \( G \). Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \). Denote by \( E_K \) the base change \( E \times_{\text{Spec} \mathbb{Q}} \text{Spec} K \). The Galois group \( G \) acts on the motive \( M = h^1(E_K)(1) \) and thus the \( \mathbb{Q} \)-algebra \( \mathbb{Q}[G] \) acts on the realizations of \( M \). We regard \( M \) as a motive defined over \( \mathbb{Q} \) with coefficients in \( \mathbb{Q}[G] \). Note that the algebra \( \mathbb{Q}[G] \) acts on

\[
H_{dR} := H^1_{dR}(E_K)/\text{Fil}^1 \simeq H^0(E_K, \Omega^1_{E_K})^*,
\]

and on

\[
H^1_{\text{wt}} := \bigoplus_{v \in S_{\infty}(K)} H^1(vE(\mathbb{C}), 2\pi i \mathbb{Q})^{G_v},
\]

where * denotes the dual, and \( S_{\infty}(K) \) denotes the set of infinite places of \( K \). Here \( H^1(vE(\mathbb{C}), *) = H^1(\sigma E(\mathbb{C}), *) \) for \( \sigma \in G \) corresponding to \( v \). Denoting by \( c \in G \) the complex conjugation, we identify \( H^1(\sigma E(\mathbb{C}), *) \) and \( H^1(c \circ \sigma E(\mathbb{C}), *) \) via the isomorphism \( \sigma E(\mathbb{C}) \simeq c \circ \sigma E(\mathbb{C}) \).

After identifying \( H^1(vE(\mathbb{C}), *) \) with the dual of the homology group, we define period isomorphism

\[
(5) \quad \pi : H^1_{\text{wt}} \otimes_{\mathbb{Q}} \mathbb{R} \to H_{dR} \otimes_{\mathbb{Q}} \mathbb{R}
\]

by

\[
\gamma \mapsto (\omega \mapsto \int_{\gamma} \omega)
\]

for \( \gamma \in \bigoplus_{v \in S_{\infty}(K)} H^1(vE(\mathbb{C}), \mathbb{Q})^{G_v} \) and \( \omega \in H^0(E_K, \Omega^1_{E_K}) \).

Let

\[
\Xi := \text{Det}(E(K))_Q \otimes \text{Det}^{-1}(E_K)^* \otimes \text{Det}^{-1} H^1(E_K(\mathbb{C}), \mathbb{Q})^+ \otimes \text{Det} H^0(E_K, \Omega^1_{E_K})^*,
\]

where all the determinants are taken over the ring \( \mathbb{Q}[G] \). Thanks to the above defined period isomorphism and the canonical height pairing associated with the rational points on the elliptic curve, we get an isomorphism

\[
\vartheta_{\infty} : \Xi \otimes_{\mathbb{Q}} \mathbb{R} \simeq \zeta(\mathbb{R}[G])
\]

of \( \mathbb{R}[G] \)-modules.

Let \( l \) be a rational prime and let \( S \) be the finite set of primes in \( \mathbb{Q} \) consisting of primes of bad reduction of \( E \), ramified primes in \( K/\mathbb{Q} \) and the infinite prime. Let \( S_l := S \cup \{l\} \). Also, let \( S_{\infty} = \{\infty\} \), \( S_f = S \setminus \{\infty\} \) and \( S_{fl} = S_l \setminus \{\infty\} \). For any set \( S_0 \) of primes in \( \mathbb{Q} \), we let \( S_0(K) \) denote the primes in \( K \) lying above those in \( S_0 \).

Let \( T_l := \text{Ind}_K^Q \left( \lim_{\longrightarrow} E(\mathbb{Q})[l^n] \right) \) be the Tate module associated with \( E_K \). Then we have \( V_l := H_l(M) \simeq T_l \otimes_{\mathbb{Q}_l} \mathbb{Q}_l \). There exists a perfect complex \( R\Gamma_e(\mathbb{Z}_{S_l}, V_l) \) of \( \mathbb{Q}_l[G] \)-modules along with an isomorphism

\[
\vartheta_l : \Xi \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq \text{Det}_{\mathbb{Q}_l[G]} R\Gamma_e(\mathbb{Z}_{S_l}, V_l)
\]
of $\mathbb{Q}_l[G]$-modules obtained via comparison isomorphisms (see [3, 14] for details).

Analogously, one can define a perfect complex $R\Gamma_c(\mathbb{Z}_{S_l}, T_l)$ of $\mathbb{Z}_l[G]$-modules with a mapping to $R\Gamma_c(\mathbb{Z}_{S_l}, V_l)$ such that

$$R\Gamma_c(\mathbb{Z}_{S_l}, T_l) \otimes_{\mathbb{Z}_l[G]} \mathbb{Q}_l[G] \simeq R\Gamma_c(\mathbb{Z}_{S_l}, V_l).$$

2.4. Special values of $L$-functions. In this section we state the equivariant conjecture as formulated by Burns and Flach. See [5] or [14] for more details.

Let $\hat{G}$ denote the set of irreducible complex characters of $G$. The $\mathbb{Q}[G]$-equivariant $L$-function associated with the motive $h^1(E_K)(1)$ is the tuple

$$L(h^1(E_K)(1), s) := (L(E \otimes \eta, s + 1))_{\eta \in \hat{G}},$$

where $L(E \otimes \eta, s)$ is the twist of the Hasse-Weil $L$-function attached to $E$. For $\eta \in \hat{G}$ let $L^*(E \otimes \eta, 1)$ denote the leading nonzero coefficient in the Taylor expansion of $L(E \otimes \eta, s)$ at $s = 1$. Then we let

$$L^*(h^1(E_K)(1)) := (L^*(E \otimes \eta, 1))_{\eta \in \hat{G}}.$$

Note that $L(h^1(E_K)(1), s) \in \prod_{\eta \in \hat{G}} \mathbb{C} \simeq \zeta(\mathbb{C}[G])$ where $\zeta$ denotes the centre. Hence, we can identify $L^*(h^1(E_K)(1))$ with an element of $\zeta(\mathbb{R}[G])^\times$. The rationality part of the equivariant conjecture states that

**Conjecture 2.1.** With the notations as above, one has

$$\vartheta_\infty^{-1}(L^*(h^1(E_K)(1))^{-1}) \in \Xi \otimes 1 \subset \Xi \otimes \mathbb{R}.$$

Now consider the reduced norm map

$$nr : K_1(\mathbb{R}[G]) \to \zeta(\mathbb{R}[G])^\times.$$

Note that the special value $L^*(h^1(E_K)(1)) \in \zeta(\mathbb{R}[G])^\times$. By the weak approximation theorem, there exists $\lambda \in \zeta(\mathbb{Q}[G])^\times$ such that $\lambda L^*(h^1(E_K)(1)) \in nr(K_1(\mathbb{R}[G]))$. By the above conjecture, we get an isomorphism

$$\tau : \Xi \simeq \zeta(\mathbb{Q}[G])$$

which when tensored with $\mathbb{R}$ gives $-nr^{-1}(\lambda L^*(h^1(E_K)(1)) \cdot \vartheta_\infty$. We therefore get

$$R\Gamma_c(\mathbb{Z}_{S_l}, V_l) \simeq \Xi \otimes \mathbb{Q}_l \simeq \zeta(\mathbb{Q}_l[G]).$$

Denoting the composition by $\tau_l$ we have an element

$$\xi_l := ([R\Gamma_c(\mathbb{Z}_{S_l}, T_l)]; \tau_l) \in K_0(\mathbb{Z}_l[G]; \mathbb{Q}_l).$$

On the other hand, the local reduced norm map

$$nr_l : K_1(\mathbb{Q}_l[G]) \to \zeta(\mathbb{Q}_l[G])^\times$$

is an isomorphism. Thus, via the connecting morphism $\delta_l : K_1(\mathbb{Q}_l[G]) \to K_0(\mathbb{Z}_l[G]; \mathbb{Q}_l)$ we get an element

$$\theta_l := \delta_l(nr_l^{-1}(\lambda)) \in K_0(\mathbb{Z}_l[G]; \mathbb{Q}_l).$$

The integrality part of the equivariant conjecture says that

**Conjecture 2.2.** Assuming Conjecture 2.1, one has that $\xi_l - \theta_l$ vanishes in $K_0(\mathbb{Z}_l[G]; \mathbb{Q}_l)$.

Our aim is to prescribe a method to verify the above conjecture for a given elliptic curve and a finite Galois extension.
3. Arithmetic calculations

Henceforth we work under the following hypotheses.
- $K/Q$ is a finite Galois extension with Galois group $G$.
- $\mathcal{O}_K$ is a free $\mathbb{Z}[G]$-module of rank 1. By a theorem of Taylor, this condition holds if and only if $K/Q$ is tamely ramified and $G$ has no symplectic character (cf. [19]).
- $E(K)$ and $\mathcal{H}(E/K)$ are finite.
- $(\text{cond}(E), \text{disc}(K)) = 1$. This ensures that the deRham cohomology with integral coefficients is a free $\mathbb{Z}[G]$-module.

3.1. Period isomorphism. Since $E(K)$ is torsion, we have
\begin{equation}
\Xi = \text{Det}^{-1} H^1(E_K(\mathbb{C}), \mathbb{Q})^* \otimes \text{Det} H^0(E_K, \Omega^1_{\mathbb{E}_K})^*.
\end{equation}

Let $\mathcal{E}$ be a Néron model for $E$ over $\mathbb{Z}$. Let $\omega_0$ be a generator of $H^0(\mathcal{E}, \Omega^1_{\mathcal{E}})$. Then the image of the map
\[ H_1(E(\mathbb{C}), \mathbb{Z}) \to \mathbb{C} \]
\[ \gamma \mapsto \int_\gamma \omega_0 \]

is a $\mathbb{Z}$-lattice in $\mathbb{C}$. This lattice is generated by $\Omega^+$ and $\Omega^-$, the real and purely imaginary periods associated with $E$ respectively (cf. [18]).

Let $\alpha_0$ be a $\mathbb{Z}[G]$-generator for $\mathcal{O}_K$. Fix an element $c$ of $G$ corresponding to complex conjugation (so $c$ is trivial if $K/Q$ is real). For a complex representation $\rho : G \to GL(V)$, we let $d(\rho)_+$ and $d(\rho)_-$ denote the dimension of the eigenspaces of $V$, with eigenvalues 1 and $-1$ respectively, for the action of $c$.

**Proposition 3.1.** In the above setting, the image of $\Xi$ in $\Xi \otimes \mathbb{R} \simeq \zeta(\mathbb{R}[G])$ under the isomorphism $\vartheta_\infty$ is given by
\[ \Omega_+^{d(\rho)_+} \cdot \Omega_-^{d(\rho)_-} \cdot \left( \det \left( \sum_{g \in G} g(\alpha_0) \rho_\eta(g^{-1}) \right) \right)_{\eta \in \tilde{G}} \cdot \zeta(\mathbb{Q}[G]), \]

where $\rho_\eta$ is the representation corresponding to the character $\eta$.

**Proof.** Recall that the isomorphism $\vartheta_\infty : \Xi \otimes \mathbb{R} \simeq \zeta(\mathbb{R}[G])$ is constructed using the period map $\pi$ as in [5]. Therefore, we shall first write down the Betti and deRham realizations, and the corresponding period map between them.

Recall that $H_B^* = \bigoplus_{v \in S_{\infty}} H^1(vE_K(\mathbb{C}), 2\pi iv)^G_v$. We shall identify each summand on the right hand side with the dual homology via the isomorphism
\[ H^1(vE_K(\mathbb{C}), 2\pi iv) \simeq \text{Hom}(H_1(vE_K(\mathbb{C}), \mathbb{Q}), 2\pi iv). \]

Therefore, we have
\[ H_B^* \simeq \text{Hom} \left( \bigoplus_{v \in S_{\infty}} H_1(vE_K(\mathbb{C}), \mathbb{Q}), (2\pi iv)^G_v \right). \]

For a path $\gamma \in H_1(vE_K(\mathbb{C}), \mathbb{Q})$, we let $\gamma^+$ denote the corresponding element in the dual that maps $\gamma$ to 1 and the orthogonal complement of $\gamma$ to 0.

Let $\gamma_+$ and $\gamma_-$ be $\mathbb{Z}$-generators of $H_1(E_K(\mathbb{C}), \mathbb{Z})$ that are eigenvectors, with eigenvalues 1 and $-1$ respectively, for the action of the complex conjugation. We also assume that these
generators satisfy \( \int_{\gamma_v} \omega_0 = \Omega_+ \) and \( \int_{\gamma_v} \omega_0 = \Omega_- \). Note that, \( \sigma_{\gamma_+} \) and \( \sigma_{\gamma_-} \) are eigenvectors generating \( H_1(vE_K(\mathbb{C}), \mathbb{Z}) \), for \( v = v(\sigma) \). Now define 
\[
\tilde{\gamma} : \oplus_{v \in S_{\infty}} H_1(vE_K(\mathbb{C}), \mathbb{Q}) \to (2\pi i)\mathbb{Q}
\]
by setting 
\[
\tilde{\gamma}(\sigma_{\gamma_*}) = \begin{cases} 
2\pi i & \text{if } \sigma = \text{id}, * = \pm \\
0 & \text{otherwise}.
\end{cases}
\]
It is easy to see that the restriction of \( \tilde{\gamma} \) to \( H^+_B \), which we shall denote by the same symbol, generates \( H^+_B \) as a \( \mathbb{Q}[G] \)-module.

Now, we consider the deRham realization. Recall that by Serre duality one has 
\[
H_{dR}/E^0 = H^1(E_K, \mathcal{O}_{E_K}) \cong H^0(E_K, \Omega^1_{E_K})^*
\]
where \( \Omega^1_{E_K} \) is the sheaf of differentials, and \( * \) denotes the dual.

**Lemma 3.2.** Let \( \mathcal{E} \) and \( \mathcal{E}_{\mathcal{O}_K} \) be the Néron models for \( E \) over \( \mathbb{Z} \) and for \( E_K \) over \( \mathcal{O}_K \), respectively. Suppose that the conductor of \( E \) and the discriminant of \( K \) are relatively prime to each other. Then 
\[
\mathcal{E}_{\mathcal{O}_K} \cong \mathcal{E} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathcal{O}_K).
\]

**Proof.** Note that the primes of bad reduction of \( E \) and the primes that ramify in \( K/\mathbb{Q} \) do not intersect. Therefore, \( \mathcal{E} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathcal{O}_K) \) is an abelian scheme over \( \text{Spec}(\mathcal{O}_K) \). The Lemma now follows from Corollary 1.4 in [2].

From the above Lemma we have 
\[
H^0(E_K, \Omega^1_{E_K}) \cong H^0(\mathcal{E}_{\mathcal{O}_K}, \Omega^1_{\mathcal{E}_{\mathcal{O}_K}}) \otimes_{\mathcal{O}_K} K \cong (H^0(\mathcal{E}, \Omega^1_{\mathcal{E}}) \otimes_{\mathbb{Q}} \mathcal{O}_K) \otimes_{\mathcal{O}_K} K.
\]
Recall that \( \omega_0 \) is a generator for \( H^0(\mathcal{E}, \Omega^1_{\mathcal{E}}) \). Therefore, \( ((\omega_0 \otimes \alpha_0) \otimes 1)^* \) is a \( \mathbb{Q}[G] \)-generator for \( H_{dR}/\text{Fil}^0 \).

After identifying the Betti and deRham realizations as above, the period map 
\[
\pi : H^+_B, \mathbb{R} \to H_{dR, \mathbb{R}}/\text{Fil}^0
\]
is given by 
\[
\gamma^* \mapsto \left( \omega \otimes \alpha \mapsto \left( \int_{\gamma} \omega \right)^{-1} \alpha \right),
\]
for \( \gamma \in \bigoplus_{v \in S_{\infty}} H_1(vE(\mathbb{C}), \mathbb{Q})^{G_v}, \omega \in H^0(\mathcal{E}, \Omega^1_{\mathcal{E}}) \) and \( \alpha \in K \).

Since \( \mathbb{Q}[G] \) is semisimple, the determinant functor is given by the map Det defined in Section 2.2. Let \( \rho_\eta : G \to M_d(F) \) be an irreducible complex representation of \( G \) of dimension \( d \), with character \( \eta \). Here \( F \) is a finite extension of \( \mathbb{Q} \). By choosing a suitable basis, we assume that \( \rho(c) \) is diagonal with the first \( d(\rho)_+ \) diagonal entries being 1 and the rest of the diagonal entries being \(-1\). We shall now compute the \( \eta \)-component of the generator 
\( \tilde{\gamma} \otimes ((\omega_0 \otimes \alpha_0) \otimes 1)^* \) of \( \Xi \otimes \mathbb{R} \).

Let \( e_{i1} \) be the indecomposable idempotent of \( \mathbb{Q}[G] \) such that \( \rho(e_{i1}) = (b_{ij}) \) with \( b_{ij} = 1 \) for \( (i, j) = (1, 1) \) and \( b_{ij} = 0 \) otherwise. Let \( e_{11}, e_{12}, ..., e_{1d} \) be an \( F \)-basis for \( e_{11} \mathbb{Q}[G] \). We
choose this basis such that for $1 \leq r \leq d$, one has that $\rho_\eta(e_{1r})$ is the matrix $(b_{ij})$ with $b_{ij} = 1$ of $(i, j) = (1, r)$ and $b_{ij} = 0$ otherwise.

For $1 \leq k \leq d$ and $g \in G$, we have

$$\left( \pi(e_{1k}) \right) (\omega_0 \otimes g(\alpha_0)) = \Omega^{-1}_{s(k)} e_{1k} g(\alpha_0),$$

where $s(k) = +$ if $1 \leq k \leq \rho^+(\rho) + \rho^+(\rho)$ and $s(k) = -$ otherwise. Therefore it follows that under the period isomorphism $\pi$, the image of $e_{1k} \tilde{\gamma}$ is

$$\left( \Omega^{-1}_{s(k)} e_{1k} \sum_{g \in G} g(\alpha_0) g^{-1} \right) (\omega_0 \otimes \alpha_0)^*.$$

Now, for an element $\theta \in \mathbb{Q}[G]$, one has that

$$e_{1k} \theta = \sum_{r=1}^{d} a_{kr} e_{1r}$$

where $\rho_\eta(\theta) = (a_{ij})$. Thus it follows that the image of $\land_{k=1}^d e_{1k} \tilde{\gamma}$ under the period isomorphism is

$$\Omega^{-d(\rho)}_+ \cdot \Omega^{-d(\rho)}_- \cdot \det \left( \sum_{g \in G} g(\alpha_0) \rho_\eta(g^{-1}) \right) \cdot \land_{k=1}^d e_{1k} (\omega_0 \otimes \alpha_0)^*.$$

The proposition now follows. \hfill \Box

3.2. The cohomology of finite complexes. We shall now define the complexes $RG_f$ and the exact triangle $\Gamma$. Let $T_i(E) := \lim_{\to F} E(\bar{\mathbb{Q}})[1/\nu]$ and $V_i(E) := T_i(E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Recall that $T_i := \text{Ind}_{K}^\mathbb{Q} (T_i(E))$ and $V_i := T_i \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is the $l$-adic realization of $M = h^1(E_K)(1)$. Note that $T_i$ is a Galois stable $\mathbb{Z}_l$-lattice in $V_i$.

Let $M_K := h^1(E_K)(1)$, regarded as a motive defined over $K$. Then $H_i(M_K) = V_i(E)$ and by Proposition 4.1 of $[\mathbb{H}]$ we have

$$RG_c(\mathbb{Z}_{S_i}, V_i) \simeq RG_c(O_{K,S_i(K)}, V_i(E)).$$

We will therefore consider the following exact triangle instead of $\Gamma$:

$$(7) \quad RG_c(O_{K,S_i(K)}, T_i(E)) \to RG_f(K,T_i(E)) \to \oplus_{v \in S_i(K)} RG_f(K,v,T_i(E)).$$

We can then set $RG(\mathbb{Q}, T_i) := RG(K,T_i(E))$ and $RG_f(\mathbb{Q}_p, T_i) := \oplus_{v|p} RG_f(K,v,T_i(E))$ for any $p \in S_i$. Our aim is to define complexes $RG_f(K,v,T_i(E))$ for $v \in S_i(K)$ such that they satisfy the following:

- there is a map $RG_f(K,v,T_i(E)) \to RG_f(K,v,T_i(E))$ of complexes which upon tensoring with $\mathbb{Q}_l$ becomes quasi-isomorphic to $RG_f(K,v,V_i(E)) \subseteq RG(K,v,V_i(E));$
- $RG_f(K,v,T_i(E)) := \oplus_{v|p} RG_f(K,v,T_i(E))$ is a perfect complex of $\mathbb{Z}_l[G]$-modules.

We shall denote by $RG_f(K,v,T_i)_{BK}$ the finite integral complexes defined by Bloch-Kato (cf. $[\mathbb{K}]$). Note that these finite complexes in general satisfy only the first condition and therefore doesn’t fit our framework.
3.2.1. **Case** $v \in S_\infty(K)$. In this case, one has that

$$R\Gamma_f(K_v, V_l(E)) = R\Gamma(K_v, V_l(E)),$$

which is the standard complex of continuous cochains. We therefore define $R\Gamma_f(K_v, T_l(E))$ to be the complex $H^0(K_v, T_l(E))$ (concentrated in degree zero). Note that $R\Gamma_f(K_\infty, T_l(E))$ is free of rank 1 over $\mathbb{Z}[G]$. Further, under Grothendieck’s comparison isomorphism, $\oplus_{v \mid \infty} H^0(K_v, T_l(E))$ is mapped onto $\oplus_{v \in S_\infty(K)} H^1(vE(C), \mathbb{Z}) \otimes \mathbb{Z}/l$.

3.2.2. **Case** $v \in S_f(K)$. Note that for $v \in S_f(K)$, the complex $R\Gamma_f(K_v, V_l(E))$ is

$$(8) \quad V_l(E)^{T_v} \xrightarrow{1-\text{Fr}_v} V_l(E)^{I_v},$$

where $I_v$ is the inertia group, $\text{Fr}_v$ is the Frobenius at $v$ and the modules are placed in degrees 0 and 1. If $l$ does not divide the order of the image of $I_v$ in $G$ then

$$T_l(E)^{I_v} \xrightarrow{1-\text{Fr}_v} T_l(E)^{I_v}$$

is a good choice for $R\Gamma_f(K_v, T_l(E))$. The complex $(8)$ has trivial cohomology and therefore we can $R\Gamma_f(K_v, T_l(E))$ to be the zero complex whenever $l$ divides the order of the image of $I_v$ in $G$.

3.2.3. **Case** $v \mid l$. In this case, one has

$$H^1_f(K_v, V_l(E)) \simeq (\lim_{n} E(K_v)/l^n) \otimes \mathbb{Q}_l.$$

We first note that to define $R\Gamma_f(K_v, T_l(E))$ it is enough to define a Galois stable $H^1_f(K_v, T_l(E)) \subset H^1_f(K_v, V_l(E))$ (cf. [12]). The classical definition due to Bloch-Kato (cf. [3]) is to take this cohomology to be

$$\lim_{n} E(K_v)/l^n.$$

However, this doesn’t work for our purpose since this group might not have finite projective dimension over $\mathbb{Z}[G]$.

So we consider the short exact sequence

$$0 \to E_1(K_v) \to E_0(K_v) \to E(k_v) \to 0,$$

where $E_0(K_v)$ is the group of nonsingular points, $E_1(K_v)$ is the subgroup of points that are trivial modulo the maximal ideal $m_v$ and $k_v$ is the residue field. One has an isomorphism between $E_1(K_v)$ and the formal group $\hat{E}(m_v)$. We let $n_v$ to be the least positive integer such that $\hat{E}(m_v^{n_v}) \simeq m_v^{n_v}$ (see [18] for details) and such that the ramification index $e(K_v/\mathbb{Q}_l)$ divides $n_v$.

We now define for $v \mid l$

$$H^1_f(K_v, T_l(E)) := \text{im}(\hat{E}(m_v^{n_v})), $$

where the right hand side is the image of the $l$-adic completion of the formal group in $\lim_{n} E(K_v)/l^n$.

**Remark.** Note that for a tamely ramified extension $K_v/\mathbb{Q}_l$ with ramification index $e := e(K_v/\mathbb{Q}_l)$, and for a positive integer $r$, one has that $m_v^{e^r} = l^r \mathcal{O}_{K_v} \simeq \mathcal{O}_{K_v} \simeq \mathbb{Z}[\text{Gal}(K_v/\mathbb{Q}_l)]$, as $\mathbb{Z}[\text{Gal}(K_v/\mathbb{Q}_l)]$-modules. Therefore, the above definition of $R\Gamma_f$ ensures that $R\Gamma_f(K_v, T_l(E))$ is a free $\mathbb{Z}[G]$-module of rank 1.
Lemma 3.3. The below definitions of $R\Gamma_f(K_v, T_l(E))$ for $v \in S_l(K)$ satisfy the conditions that

- there exists a map $R\Gamma_f(K_v, T_l(E)) \to R\Gamma(K_v, V_l(E))$ of complexes which upon tensoring with $\mathbb{Q}_l$ becomes quasi-isomorphic to $R\Gamma_f(K_v, V_l(E)) \to R\Gamma(K_v, V_l(E))$;

- $R\Gamma_f(K_v, T_l(E))$ is a perfect $\mathbb{Z}_l[G]$-complex.

(i) If $v \in S_{\infty}(K)$ then $R\Gamma_f(K_v, T_l(E)) := H^0(K_v, T_l(E))$ (placed in degree 0).

(ii) If $v \in S_f(K)$ and $l$ divides the order of the image of $I_v$ in $G$ then define $R\Gamma_f(K_v, T_l(E))$ to be the complex

$$T_l(E)^I_v \xrightarrow{1-Fr^{-1}} T_l(E)^I_v.$$ 

(iii) If $v \in S_f(K)$ and $l$ divides the order of the image of $I_v$ in $G$ then define $R\Gamma_f(K_v, T_l(E))$ to be the zero complex.

(iv) If $v \mid l$ then $R\Gamma_f(K_v, T_l(E)) := \text{im}(\tilde{E}(m_v^{n_v}))$, where $m_v$ is the maximal ideal in $\mathcal{O}_{K_v}$ and $n_v$ is smallest positive integer divisible by the ramification index $e(K_v/\mathbb{Q}_l)$ and such that $\tilde{E}(m_v^{n_v}) \simeq m_v^{n_v}$.

Remarks.

1. For $v \in S_f(K)$, the above definition of $R\Gamma_f(K_v, T_l(E))$ differs from the classical definition by a factor of $|c_v L_v(E, 1)^{-1}l|$ or $|c_v| l$ depending on whether or not $l$ divides the order of the image of $I_v$ in $G$. Here $c_v$ is the order of the group of components.

2. For $v \nmid l$ the above definition differs by the classical definition by

$$\lim_{n} E(K_v)/l^n : H^1_f(K_v, T_l) = |c_v| E(k_v)|l \cdot |k_v|^{n_v-1} = |c_v L_v(E, 1)^{-1}l \cdot |k_v|^{n_v},$$

where $k_v$ is the residue field at $v$.

3. For $v \mid l$, note that the following diagram is commutative:

$$\begin{array}{ccc}
E(K_v) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l & \xrightarrow{\exp} & \text{Tan}(E_{K_v}) \\
\uparrow & & \uparrow \omega_0 \\
\tilde{E}(m_v^{n_v}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l & \xrightarrow{\log} & K_v
\end{array}$$

We shall therefore compute the image of $\oplus_{v|l} H^1_f(K_v, T_l(E))$ under the comparison isomorphism via the logarithm map.

We can now define the global finite complex as in [4]. To be precise, we let

$$R\Gamma_f(K_v, T_l(E)) := \text{Cone}(R\Gamma_f(K_v, T_l(E)) \to R\Gamma(K_v, T_l(E)))$$

and

$$R\Gamma_f(K, T_l(E)) := \text{Cone}(R\Gamma(\mathcal{O}_{K,S_l}, T_l(E)) \to \oplus_{v \in S} R\Gamma_f(K_v, T_l(E)))[−1].$$

The cohomology of $R\Gamma_f(K, T_l(E))$ is given by the following lemma.

Lemma 3.4. If $\Pi(E(K))$ is finite then

$$H^0_f(K, T_l(E)) = 0, \quad H^3(K, T_l(E)) \simeq \text{Hom}_{\mathbb{Z}_l}(E(K)|_{K}, \mathbb{Q}_l/\mathbb{Z}_l),$$
and there are exact sequences of $\mathbb{Z}_p[G]$-modules
\[
0 \to H^1_f(K, T_l(E)) \to E(K) \otimes \mathbb{Z}_p \to \oplus_{v \in S_{l,f}(K)} \Phi_v \\
\to H^2_f(K, T_l(E)) \to H^2_f(K, T_l(E))_{BK} \to 0,
\]
where,
\[
\Phi_v := \left( \lim_{n \to \infty} \frac{E(K_v)}{n} \right) / H^1_f(K_v, T_l(E)).
\]

Proof. By the above definitions of the finite complexes one has a commutative diagram

\[
\begin{array}{ccc}
R\Gamma_c(\mathcal{O}_{K,S_{l,f}(K)}, T_l(E)) & \longrightarrow & R\Gamma_f(K, T_l(E)) \\
\downarrow & & \downarrow \\
R\Gamma_c(\mathcal{O}_{K,S_{l,f}(K)}, T_l(E))_{BK} & \longrightarrow & R\Gamma_f(K, T_l(E))_{BK} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \oplus_{v \in S_{l,f}(K)} W_v \\
\end{array}
\]

with rows being distinguished triangles. It follows from the octahedral axiom that there exists complexes $W_v$ such that the columns in the above diagram are also distinguished. We note that $\Phi_v = H^1(W_v)$. The lemma now follows from the results of [3] on the classical finite cohomology of $R\Gamma_f(K, T_l(E))_{BK}$ and from the long exact sequences arising from the distinguished columns. \qed

**Corollary 3.5.** If $E(K)$ is torsion and $\text{III}(E(K))$ is finite, then one has
\[
H^0_f(K, T_l(E)) = H^1_f(K, T_l(E)) = 0, H^2_f(K, T_l(E)) \cong \text{Hom}_\mathbb{Z}(E(K)_{l\infty}, \mathbb{Q}_l/\mathbb{Z}_l),
\]
and the following sequence is exact:
\[
0 \to E(K)_{l\infty} \to \oplus_{v \in S_{l,f}(K)} \Phi_v \to H^2_f(K, T_l(E)) \to \text{III}(E(K))_{l\infty} \to 0.
\]

**4. Special values of L-functions**

**4.1. Modular symbols.** Let $\mathcal{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}$ denote the upper half plane and $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{ \infty \}$ denote the extended upper half plane. Let $\Gamma$ be a congruence subgroup and let $X_\Gamma := \Gamma \backslash \mathcal{H}^*$ be the corresponding modular curve. For cusps $\alpha, \beta \in \mathbb{Q} \cup \{ \infty \}$ consider a smooth path in $\mathcal{H}^*$ from $\alpha$ to $\beta$. Let $\{ \alpha, \beta \}_\Gamma$ denote the image of the path in $X_\Gamma$. The Manin-Drinfeld theorem says that for any cusps $\alpha, \beta \in \mathbb{Q} \cup \{ \infty \}$ one has $\{ \alpha, \beta \}_\Gamma \in H_1(X_\Gamma, \mathbb{Q})$. For a cusp form $f \in S_2(\Gamma)$ we let
\[
\langle \{ \alpha, \beta \}_\Gamma, f \rangle := 2\pi i \int_{\alpha}^{\beta} f(z)dz.
\]
Note that the element $\{ \alpha, \beta \}_\Gamma \in H_1(X_\Gamma, \mathbb{Q})$ and the above integral are both independent of the path chosen.

Using Mellin transformation one can deduce the following:
Proposition 4.1. Let $N$ be a prime, $\Gamma = \Gamma_0(N)$, $f$ be a new form and $\eta$ be a Dirichlet character of prime conductor $l$ with $(l, N) = 1$. Then

$$L(f, 1) = -\langle \{0, \infty\} \Gamma, f \rangle,$$

and

$$L(f \otimes \eta, 1) = \frac{g(\eta)}{l} \sum_{a=1}^{l} \bar{\eta}(a) \langle \{0, a/l\} \Gamma, f \rangle,$$

where $g(\eta) = \sum_{n=1}^{l} \eta(n) e^{2\pi in/l}$ is the Gauss sum.

See [7] for a proof of the above proposition.

Now consider the well-known pairing $H_1(X_\Gamma, \mathbb{C})^+ \times S_2(\Gamma) \rightarrow \mathbb{C}$ given by

$$\langle \gamma, f \rangle = \int_{\gamma} f(z) dz.$$

This gives an isomorphism $S_2(\Gamma) \simeq H_1(X_\Gamma, \mathbb{C})^{++}$ of $\mathbb{C}$-vectorspaces. Given a rational new form $f \in S_2(\Gamma)$, it corresponds to an element $\{\alpha, \beta\} \in H_1(X_\Gamma, \mathbb{Q})^+$ via this isomorphism. Note that this element is unique up to sign if we further restrict it to be an element of $H_1(X_\Gamma, \mathbb{Z})^+$. We shall denote this 1-cycle by $\gamma_f$. Since the pairing $\langle \cdot, \cdot \rangle$ is compatible with the action of the Hecke operators, it follows that $\gamma_f$ is a common eigen vector for all the Hecke operators with eigenvalues same as that of the new form. Thus, by looking at the Hecke action on $H_1(X_\Gamma, \mathbb{Z})^+$ we can compute $\gamma_f$. Let $V$ be the $\mathbb{Q}$-vector space generated by $\gamma_f$. Note that we can construct the complementary subspace $V'$ on which the pairing $\langle \cdot, f \rangle$ is trivial. Then, for any $\gamma \in H_1(X_\Gamma, \mathbb{Q})^+$ we have

$$\int_{\gamma} f = \int_{\gamma|_V} f$$

where $\gamma|_V$ is the projection of $\gamma$ onto the subspace $V$. In fact, $\gamma|_V$ is a rational multiple of $\gamma_f$. So, if we let $\Omega_f := \int_{\gamma_f} f$ then we see that $\int_{\gamma} f$ is a rational multiple of $\Omega_f$. The following well-known proposition gives the equality between the periods of an elliptic curve and the corresponding modular form.

Proposition 4.2. Let $E$ be a strong Weil curve defined over $\mathbb{Q}$ and let $f$ be the normalized rational new form corresponding to $E$. Then, the real period associated to $E$ equals $c_E d \Omega_f$, where $d$ is the number of components in the real locus of $E$ (that is, $d = 2$ if the corresponding lattice is rectangular, $d = 1$ otherwise) and $c_E$ is the Manin constant.

Note that, $\{0, a/l\} + \{0, (l - a)/l\} \in H_1(X_\Gamma, \mathbb{Z})^+$ and $\{0, \infty\} \in H_1(X_\Gamma, \mathbb{Z})^+$. Hence we can compute the $L$-values of a rational new form and its twists using the above computation. Implementation of such a computation is studied in detail by Cremona (see [7]).

Remarks.

1. The Manin constant $c_E$ is known to be trivial for elliptic curves $E$ of prime conductor (see [1]).

2. An analogous result can be obtained for the imaginary period of the elliptic curve by looking at $H_1(X_\Gamma, *)^-$, the eigenspace for the action of the complex conjugation with eigenvalue $-1$. 
4.2. **Nonabelian twists.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$. Let $\tau$ be an (irreducible) self-dual representation of $G = \text{Gal}(K/\mathbb{Q})$ of dimension $d$ and let $N(E, \tau)$ be the conductor of $E \otimes \tau$. The $L$-function $L(E \otimes \tau, s)$ has a meromorphic continuation to the whole $s$-plane and it satisfies a functional equation

\[
\widehat{L}(E \otimes \tau, s) = \pm \widehat{L}(E \otimes \tau, 2 - s)
\]

where

\[
\widehat{L}(E \otimes \tau, s) = \left( \frac{\sqrt{N(E, \tau)}}{\pi^d} \right)^s \Gamma \left( \frac{s}{2} \right)^d \Gamma \left( \frac{s + 1}{2} \right)^d L(E \otimes \tau, s).
\]

Let $A = A(E, \tau) := \frac{\sqrt{N(E, \tau)}}{\pi^d}$ and $\gamma(s) = \Gamma(\frac{s}{2})^d \Gamma(\frac{s+1}{2})^d$.

Let $\phi(s)$ be the inverse Mellin transform of $\gamma(s)$, that is,

\[
\gamma(s) = \int_{0}^{\infty} \phi(t) t^{s-1} \frac{dt}{t}.
\]

Let

\[
G_s(t) = t^{-s} \int_{t}^{\infty} \phi(x) x^{s-1} \frac{dx}{x}
\]

be the incomplete Mellin transform of $\phi(t)$. Then one has

**Proposition 4.3.**

\[
\widehat{L}(E \otimes \tau, s) = \sum_{n=1}^{\infty} a_n G_s \left( \frac{n}{A} \right) \pm \sum_{n=1}^{\infty} a_n G_{2-s} \left( \frac{n}{A} \right).
\]

**Proof.** See [10] or [20].

For fixed $s$, the series [10] converges exponentially with $t$ and therefore we can use this series to get numerical approximations of the value $L(E \otimes \tau, 1)$. The rate of convergence of the series depends on the conductor $N(E \otimes \tau)$. We roughly need to sum $\sqrt{N(E \otimes \tau)}$ terms in the series to obtain an approximation. Note that if the bad primes of $E$ and $\tau$ do not intersect, then the conductor of $E \otimes \tau$ is $N(E, \tau) = N(E)^d N(\tau)^2$. Therefore, obtaining numerical approximations to the value $L(E \otimes \tau, 1)$ is computationally infeasible for large field extensions.

In [10] Dokchitser has explained how to compute $G_s(t)$ efficiently and this has been implemented in [6]. Our numerical approximations use this particular implementation.

**Remark.** The Proposition 4.3 is in fact a special case of a more general result that holds for any $L$-series having a meromorphic continuation and satisfying a functional equation of type (9).

5. **An example**

5.1. **The group $S_3$.** Let $r$ and $s$ denote elements of $S_3$ of order 2 and 3 respectively. Let $C_3$ denote the subgroup of $S_3$ of order 3. We denote by $\chi_0$, $\chi$ and $\psi$ the trivial, nontrivial abelian and the nonabelian characters of $S_3$ respectively. We define $\rho : S_3 \to GL_2(\mathbb{C})$ by

\[
\rho(r) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \rho(s) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.
\]
This is an irreducible representation of $S_3$ whose character is $\psi$.

As before, for a character $\eta \in \hat{S}_3$, we let $e_{\eta} = \sum_{g \in S_3} \eta(g) g^{-1}$. Further, we fix indecomposable idempotents $e_{\psi,1}$ corresponding to the character $\psi$. Let $\{e_{\psi,1}, e_{\psi,2}\}$ be a basis of $e_{\psi,1}\mathbb{Q}_l[S_3]$. To be precise, we have

$$
e_{\chi_0} = (1 + s + s^2 + r + rs + rs^2)/6,$$

$$e_{\chi} = (1 + s + s^2 - r - rs - rs^2)/6,$$

$$e_{\psi} = (2 - s - s^2)/3,$$

$$e_{\psi,1} = (1 + rs^2 - s^2 - rs)/3,$$

$$e_{\psi,2} = (s + rs - s - r)/3.$$

For any finite $\mathbb{Z}_l[S_3]$-module $N$ of finite projective dimension, there exists a short exact sequence

$$0 \to P_1 \to P_0 \to N \to 0,$$

of $\mathbb{Z}_l[S_3]$-modules, where $P_i$’s are projective over $\mathbb{Z}_l[S_3]$. Let $\tau : P_0 \otimes \mathbb{Q}_l \simeq P_1 \otimes \mathbb{Q}_l$ be the induced map. Since the class group $\text{Cl}(\mathbb{Z}_l[S_3])$ is trivial (cf. [9], 49.11), it follows that $P_0 \simeq P_1$ as $\mathbb{Z}_l[S_3]$-modules. Picking such an isomorphism, $\tau$ gives an automorphism of $P_0 \otimes \mathbb{Q}_l$. This defines an element of $\zeta(\mathbb{Q}_l[S_3])^\times$ whose image under

$${\tilde{\delta}} : \zeta(\mathbb{Q}_l[S_3])^\times \to K_0(\mathbb{Z}_l[S_3]; \mathbb{Q}_l)$$

is $(P_1, P_0; \tau) \in K_0(\mathbb{Z}_l[S_3]; \mathbb{Q}_l)$. We shall denote this element by $e(N)$. Note that by choosing a different presentation for $N$ and by choosing a different isomorphism between the modules in the presentation, $e(N)$ changes by an element of $K_1(\mathbb{Z}_l[S_3])$. Thus, $e(N)$ is well-defined up to an element of $K_1(\mathbb{Z}_l[S_3])$. For $a, b \in \zeta(\mathbb{Q}_l[S_3])^\times$ we shall write $a \sim b$ if $a/b$ is in the image of $K_1(\mathbb{Z}_l[S_3])$. If $N$ is the trivial module, then we have $e(N) \sim 1$. If $C_\bullet$ is a perfect complex of $\mathbb{Z}_l[S_3]$-modules with finite cohomology then, by Remark 2.2 of [11], it corresponds to an element $\sigma(N)$ of $K_0(\mathbb{Z}_l[S_3]; \mathbb{Q}_l)$. We let $e(C_\bullet)$ to be an element of $\zeta(\mathbb{Q}_l[S_3])^\times$ whose image in $K_0(\mathbb{Z}_l[S_3]; \mathbb{Q}_l)$ equals $\sigma(N)$. Note that if $C_\bullet = N[i]$, a complex concentrated in a single degree, then $e(C_\bullet) \sim e(N)^{(-1)^i}$.

For $q \in S_f$, if $\Phi_q$ is of finite projective dimension, then we fix a $\mathbb{Z}_l[S_3]$-presentation for $\Phi_q$ and also fix isomorphisms between the modules appearing in the presentation. This fixes $e(\Phi_q) \in \zeta(\mathbb{Q}_l[S_3])^\times$.

5.2. **Setting.** Let $K$ be the splitting field of $p(x) = x^3 - 4x + 1$. Let $E$ be the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$. This is the curve 11A1 in the sense of Cremona. Then we have the following.

1. $K/\mathbb{Q}$ is a real $S_3$-extension since $p(x)$ is an irreducible polynomial with discriminant 229.

2. The conductor of $E$ is 11, and the discriminant of $K$ is $229^3$. Since the conductor of $E$ is a prime, the corresponding Manin constant $c_E = 1$. Also, the real locus of $E$ has only one component and therefore the real period associated to $E$ equals $\Omega_f$ where $f$ is the normalized rational new form corresponding to $E$.
(3) The set $S = \{11, 229, \infty\}$. The residual indices of 11 and 229 are 3 and 1 respectively. We note that $|E_{\text{tor}}(k_v)| = 1330$, for $v \mid 11$, and $|E_{\text{tor}}(k_v)| = 215$, for $v \mid 229$.

(4) The above shows that $|E(K)_{\text{tors}}| \leq 5$ (see [13] for details). But, from the tables in [7] we know that $|E(\mathbb{Q})| = 5$. Therefore we have $|E(K)_{\text{tors}}| = 5$.

(5) $L(E/K, 1) \neq 0$. This follows from the value computed below. Thus, by a theorem of Zhang (cf. [21]), we have that $E(K)$ is finite.

(6) $E$ has split multiplicative reduction at $p = 11$. The Tamagawa factor $c_{11}$ is 5 and therefore, $c_v = 125$ for all $v \mid 11$.

(7) Suppose that $|\mathfrak{III}(E/K)_{2\infty}| = |\mathfrak{III}(E/K)_{3\infty}| = 1$. Then, for any $l$, we have that $\Phi_l$ is of finite projective dimension for all $q \in S_f$.

5.3. Arithmetic values.

5.3.1. Global finite cohomologies. We henceforth assume that $\mathfrak{III}(E/K)$ is trivial. So, by Corollary 3.5, the global finite cohomologies are concentrated in degree two and three, and are given by

$$H^3_l(K, T_l(E)) \simeq \text{Hom}(E(K)_l^{\infty}, \mathbb{Q}_l/\mathbb{Z}_l),$$

and,

$$0 \to E(K)_l^{\infty} \to \oplus_{v \in S_{l,f}(K)} \Phi_v \to H^3_l(K, T_l(E)) \to 0.$$

Note that, in our setting, for $l \neq 5$, $H^3_l$ vanishes and $H^2_l$ is isomorphic to $\oplus_{v \in S_{l,f}(K)} \Phi_v$. And for $l = 5$, we have

$$E(K)_l^{\infty} \simeq \text{Hom}(E(K)_l^{\infty}, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \mathbb{Z}/l\mathbb{Z},$$

as $\mathbb{Z}_l[S_3]$-modules. So, defining $u_l := \epsilon(\mathbb{Z}/l\mathbb{Z})^2$ for $l = 5$ and $u_l := 1$ for $l \neq 5$, we have

$$\epsilon(R\Gamma_f(K, T_l(E))) \sim \epsilon(\oplus_{v \in S_{l,f}(K)} \Phi_v) \cdot u_l^{-1}. \tag{11}$$

5.3.2. Local finite cohomologies. For $q \in S_f$, the complex $R\Gamma_f(K_q, T_l(E))$ has finite cohomologies. Therefore, it’s contribution to $\Xi \otimes \mathbb{Q}_l$ can be measured by $\epsilon(R\Gamma_f(K_q, T_l(E)))$. This is trivial if the complex is the zero complex. Otherwise, it equals $\left(\sum_{\eta \in \hat{G}} |L_q(E \otimes \eta, 1)|_l \right)^{-1}$.

For $v \mid l$, we have that $H^1_l(K_v, T_l(E)) = \hat{E}(m_v^n)$, where $n_l$ is trivial for $l > 3$, $n_l = 2$ for $l = 3$, and $n_l = 3$ for $l = 2$. Define $\bar{\Phi}_v$ by the exact sequence

$$0 \to m_v/m_v^{n_l} \to \Phi_v \to \bar{\Phi}_v \to 0.$$ 

So, $\bar{\Phi}_v$ is trivial if $l = 2, 3$, and $\bar{\Phi}_v = \Phi_v$ otherwise. Next, we note that via the exponential map, we have an exact sequence

$$0 \to \oplus_{v \mid l} H^1_l(K_v, T_l(E)) \to \oplus_{v \mid l} \mathcal{O}_{K_v} \cdot \omega_0 \to \oplus_{v \mid l} \mathcal{O}_{K_v}/m_v^{n_l} \to 0.$$ 

Also, we have an exact sequence

$$0 \to \oplus_{v \mid l} m_v/m_v^{n_l} \to \oplus_{v \mid l} \mathcal{O}_{K_v}/m_v^{n_l} \to \oplus_{v \mid l} k_v \to 0.$$ 

From the above three exact sequences we get the following identity in $V(\mathbb{Z}_l[G])$:

$$\left[\oplus_{v \mid l} H^1_l(K_v, T_l(E))\right] \oplus \left[\oplus_{v \mid l} \Phi_v\right] = \left[\oplus_{v \mid l} \mathcal{O}_{K_v} \cdot \omega_0\right] \oplus \left[\oplus_{v \mid l} k_v\right]^{-1} \oplus \left[\oplus_{v \mid l} \bar{\Phi}_v\right]. \tag{12}$$

Recall also that for $v \mid \infty$, we have $H^1_l(K_v, T_l(E)) \simeq T^{S_3}_l[0]$. As before, we choose $\mathbb{Z}_l[S_3]$-generators $\alpha_0 \cdot \omega_0$ and $\tilde{\gamma}_1$ for $\mathcal{O}_{K_v} \cdot \omega_0$ and $T^{S_3}_l[0]$ respectively.
5.3.3. Generator $\mu_l$. The above choice of generators for $O_{K_v} \otimes \omega_0$ and $T_l^3[0]$ define a $\zeta(\mathbb{Q}_l[S_3])$-generator $\mu_l$ of $\text{Det}_Q(S_3)$. From (11)-(14), the inverse image $\nu$ is given via the quasi-isomorphism $L^{5.4}$. Given this, we get $\xi_l \cdot \gamma_1 \otimes (\omega_0 \otimes \alpha_0)^*$ where

$$\xi_l := \epsilon(\oplus_{v \mid l} k_v)^{-1} \cdot u_l^{-1} \cdot \epsilon(\Phi_l) \cdot \prod_{q \in S_f} \epsilon(\Phi_q) \prod_{q \in S_f} \epsilon(\Gamma_f(K_v, T_l(E)))^{-1} \cdot \prod_{\eta \in S_3} L_q(E \otimes \eta, 1) e_\eta^{-1}.$$ 

The first term is the contribution of $\oplus_{v \mid l} k_v$ from (12). This term equals $(l e_\chi + le_\psi + l^2 e_\psi)$ for $l \neq 229$, and equals $(l e_\chi + e_\chi + l^2 e_\psi)$ for $l = 229$.

5.4. L-values and equivariant conjecture. We now prove Theorem 1.1.

Proof of Theorem 1.1. Let $(E, K)$ be as above. Using the methods prescribed in Sections 4.1 and 4.2 we compute the following special values of the twisted $L$-functions.

$$L(E, 1) = \Omega/5,$$
$$L(E \otimes \chi, 1) = 5\Omega/\sqrt{229},$$
$$L(E \otimes \psi, 1) \approx 25\Omega^2/\sqrt{229}.$$

On the other hand, fixing an $\alpha_0$ we get

$$\det \left( \sum_{q \in S_3} \Omega^{-1} g(\alpha_0) \rho_\eta(g^{-1}) \right) \begin{cases} \Omega^{-1} & \text{if } \eta = \chi_0, \\ \Omega^{-1} \cdot \sqrt{229} & \text{if } \eta = \chi, \\ \Omega^{-2} \cdot \sqrt{229} & \text{if } \eta = \psi. \end{cases}$$

These computations were carried out using PARI/GP (cf. [15]). Thus, from Proposition 3.1 we get

$$\vartheta_\infty^{-1}(L^*(h^1(E_K)(1))^{-1}) = (e_{\chi_0}/5 + 5e_\chi + 25e_\psi)(\tilde{\gamma} \otimes (\omega_0 \otimes \alpha_0)^*).$$

This verifies the rationality conjecture. The local $L$-values $L_p(E \otimes \eta, 1)^{-1}$ are given by the following table.
Table 1. Local $L$-values

| $p$ | $L_p(\chi_0, 1)$ | $L_p(\chi, 1)$ | $L_p(\psi, 1)$ |
|-----|-----------------|----------------|----------------|
| 2   | 5/2             | 1/2            | 5/4            |
| 3   | 5/3             | 5/3            | 4/9            |
| 5   | 1               | 1              | 28/25          |
| 11  | 10/11           | 10/11          | 133^2/11^4     |
| 229 | 215/229         | 1              | 215/229        |

Recall that there is an isomorphism $\tau : \Xi \to \zeta(Q[S_3])$ which when tensored with $\mathbb{R}$ gives $-nr^{-1}(L^*(h^1(E_R)(1)))\cdot \vartheta_\infty$. This gives us an isomorphisms $\eta : \Xi \otimes Q_l \to \zeta(Q_l[S_3])$. Combining this with $\vartheta_l^{-1}$, we get

$$\beta_l : \text{Det}_{Q_l[S_3]} R\Gamma_c(O_{K,S}, V_l(E)) \to \zeta(Q_l[S_3]).$$

From (15) and (17), we have

$$\beta_l(\mu_s) = (e_{\chi_0}/5 + 5e_{\chi} + 25e_{\psi}) \cdot \xi_l.$$  

Our aim is to show that this is an element of $K_1(Z_2[S_3])$ and this verifies the integrality part of the conjecture. We split the verification into various cases.

5.4.1. Case $l = 2$. In this case, we first note that $\Phi_v$ is trivial for all $v \in S_f(K)$. Also, $\Phi_2$ is trivial and $u_2 = 1$. So, from (15) we have

$$\xi_l = (2e_{\chi_0} + 2e_{\chi} + 4e_{\psi}) \cdot \left( \sum_{\eta \in S_3} |L_{11}(E \otimes \eta, 1)|_2 e_{\eta} \right) \cdot \prod_{q \in \{2,11,229\}} \left( \sum_{\eta \in S_3} L_q(E \otimes \eta, 1)e_{\eta} \right).$$

So from (18) we have

$$\beta_2(\mu_2) = (a_{\chi_0}e_{\chi_0} + a_{\chi}e_{\chi} + a_{\psi}e_{\psi}),$$

where $a_{\eta}$’s are units in $Z_2$. Therefore it follows that $\beta_2(\mu_2), \beta_2(\mu_2)^{-1} \in Z_2[S_3]$. Hence, multiplication by $\beta_2(\mu_2)$ defines an isomorphism from between free rank-1 $Z_2[S_3]$-modules. This shows that $\beta_2(\mu_2)$ is in the image of $K_1(Z_2[S_3])$. This verifies the equivariant conjecture for $l = 2$.

5.4.2. Case $l = 3$. In this case, the modules $\Phi_v$ are all trivial. Also, $\Phi_3$ is trivial and $u_3 = 1$. Further, the cohomologies of $R\Gamma_f(K_q, T_l(E))$ are also trivial for $q = 11, 229$. Thus (18) reduces to

$$\beta_3(\mu_3) = (a_{\chi_0}e_{\chi_0} + a_{\chi}e_{\chi} + a_{\psi}e_{\psi}),$$

where $a_{\chi_0} \equiv -a_{\chi} \equiv -a_{\psi} \equiv 1$ (mod $Z_3$).

Now, multiplication by $r$ defines an isomorphism from between free rank-1 $Z_3[S_3]$-modules. Thus, $r$ defines an element $\theta \in K_1(Z_3[S_3])$. The image of $\theta$ in $\zeta(Q_3[S_3])^\times$ is $\overline{\theta} := (e_{\chi_0} - e_{\chi} - e_{\psi})$. Consider the element $\overline{\theta} \cdot \beta_3(\mu_3) \in \zeta(Q_3[S_3])^\times$. All the coefficients of $e_{\eta}$ in $\overline{\theta} \cdot \beta_3(\mu_3)$ are in $1 + 3Z_3$. It therefore follows that $\overline{\theta} \cdot \beta_3(\mu_3)$ and its inverse are both elements of $Z_3[S_3]$, and thus they are in the image of $K_1(Z_3[S_3])$. Since $\overline{\theta}$ is in the image of $K_1(Z_3[S_3])$, so is $\beta_3(\mu_3)$. This verifies the equivariant conjecture for $l = 3$. 


5.4.3. Case $l = 5$. First, consider the sequence
\[ 0 \rightarrow \mathbb{Z}/5[3] \xrightarrow{\nu} \mathbb{Z}/5[3] \rightarrow \mathbb{Z}/5 \rightarrow 0, \]
where $\nu$ is multiplication by $\theta = 5e_{\chi_0} + e_\chi + e_\psi$. Note that for any $g \in S_3$, $(1 - g) \cdot \theta = (1 - g)$ and $(e_{\chi_0} + 5e_\chi + 5e_\psi) \cdot \theta = 5$, and therefore it follows that the above sequence is exact. Thus, we have $u_5 = 25e_{\chi_0} + e_\chi + e_\psi$.

The contributions of $\Phi_q$'s are given by the following.

**Lemma 5.1.** For $l = 5$ we have
\[ \epsilon(\Phi_q) \sim \begin{cases} 5e_{\chi_0} + 5e_\chi + e_\psi & \text{if } q = 5, \\ 25e_{\chi_0} + 25e_\chi + 25e_\psi & \text{if } q = 11, \\ 1 & \text{if } q = 229. \end{cases} \]

**Proof.** We have that $\Phi_5 \simeq \text{Ind}_{\mathbb{Z}/5[S_3]}^{\mathbb{Z}/5[3]} \mathbb{Z}/5\mathbb{Z}$ with trivial action of $C_3$ on $\mathbb{Z}/5\mathbb{Z}$. Consider the sequence
\[ 0 \rightarrow \mathbb{Z}/5[S_3] \xrightarrow{\nu} \mathbb{Z}/5[S_3] \rightarrow \Phi_5 \rightarrow 0, \]
where $\nu$ is multiplication by $\theta = 5e_{\chi_0} + 5e_\chi + e_\psi$. Note that $(1 - s) \cdot \theta = (1 - s)$ and $(e_{\chi_0} + 5e_\chi + 5e_\psi) \cdot \theta = 5$. Therefore it follows that the above sequence is exact. Thus, we have $\epsilon(\Phi_5) = 5e_{\chi_0} + 5e_\chi + e_\psi$.

For $q = 11$ and $v \mid q$, we have the following exact sequence:
\[ 0 \rightarrow \lim_{n} E(K_v)/(E_0(K_v), 5^n) \rightarrow \Phi_v \rightarrow \lim_{n} E(k_v)/5^n \rightarrow 0. \]

Since $E$ has split multiplicative reduction at $11$ and the residual index of $v$ is $3$, we have
\[ \lim_{n} E(K_v)/(E_0(K_v), 5^n) \simeq \mathbb{F}_{125} \simeq \mathbb{Z}/5\mathbb{Z}[C_3] \]
as $\mathbb{Z}/5[C_3]$-modules. Also, it is easy to see that $\lim_{n} E(k_v)/5^n \simeq \mathbb{Z}/5\mathbb{Z}$, with trivial action of $C_3$. Thus we get
\[ \epsilon(\Phi_{11}) = \epsilon(\mathbb{Z}/5\mathbb{Z}[S_3]) \cdot \epsilon(\text{Ind}_{\mathbb{Z}/5[S_3]}^{\mathbb{Z}/5[3]} \mathbb{Z}/5\mathbb{Z}). \]

And thus, we get $\epsilon(\Phi_{11}) = 25(e_{\chi_0} + e_\chi + e_\psi)$.

Finally, $\epsilon(\Phi_{229}) \sim 1$ since $\Phi_{229}$ is trivial. This completes the proof the lemma. \hfill $\Box$

Plugging the above values into (15), we get
\[ \xi_l = (5e_{\chi_0} + e_\chi/5 + e_\psi/25) \cdot u \]
for some unit $u$ in $\mathbb{Z}/5[S_3]$. But then $\beta_l(\mu_l) = u$ is a unit in $\mathbb{Z}/5[S_3]$ and is therefore in the image of $K_1(\mathbb{Z}/5[S_3])$. This verifies the equivariant conjecture for $l = 5$.

5.4.4. Case $l > 5$. Fix an $l > 5$. In this case $\Phi_q$ is trivial for all $q \in S_f$. The contribution of $\Phi_l = \Phi_l$ is given by the following:

**Lemma 5.2.** Let $I$ be the inertia group corresponding to $l$. Then one has
\[ \epsilon(\Phi_l) \sim \sum_{\eta \in G} |l^{\dim(V_\eta)} L_l(E \otimes \eta, 1)^{-1} |v e_\eta, \]
where $V_\eta$ is a representation of $G$ corresponding to $\eta$. 
Proof. For \( l = 229 \) the module \( \Phi_l \) is trivial. In this case the lemma easily follows from the table of \( L \)-values above. Now, assume that \( l \neq 229 \), so the inertia group \( I \) is trivial.

Let \( \nu \) be a prime in \( K \) lying above \( l \). Then \( \Phi_l = \text{Ind}_{z_l[D]}^{Z_l[G]} E(k_v), \) where \( D \) is the corresponding decomposition group. Let \( f = |D| \). The possible values of \( f \) are 1, 2 and 3.

Suppose that \( f = 1 \). Since \( E(\mathbb{Q}) = 5 \) and \( l > 5 \), we have \( E(\mathbb{F}_l) \neq l \). Thus, \( \Phi_l \) is trivial and \( \epsilon(\Phi_l) \sim 1 \). On the other hand, the Frobenius element is trivial and therefore \( L_l(E \otimes \eta, s) = (1 - l^{-s})^{-\dim \eta} \). This proves the lemma when \( f = 1 \).

If \( f = 2 \) then \( k_v = \mathbb{F}_2 \). The \( l \)-component of the local \( L \)-values are given by

\[
|lL_l(E, 1)^{-1}|_l = |l^{-1} - a_l + l|_l = |E(\mathbb{F}_l)|_l,

|lL_l(E \otimes \chi, 1)^{-1}|_l = |l^{-1} + a_l + l|_l = |E(\mathbb{F}_2)/E(\mathbb{F}_l)|_l,

|l^2 L_l(E \otimes \psi, 1)^{-1}|_l = |l^{-2} + a_l + l(1 - a_l + l)|_l = |E(\mathbb{F}_2)|_l.
\]

Therefore, if \( E(\mathbb{F}_2)^{l}\mathbb{F}_l \) is trivial then the lemma holds. If \( E(\mathbb{F}_2)^{l}\mathbb{F}_l \neq l \), it follows that \( E(\mathbb{F}_2)^{l}\mathbb{F}_l \simeq \mathbb{Z}/l\mathbb{Z} \), with nontrivial action of \( D \). Since \( |D| = 2 \) we will suppose that \( D = \{ 1, r \} \). Consider the sequence

\[
0 \longrightarrow \mathbb{Z}_l[S_3] \overset{\nu}{\longrightarrow} \mathbb{Z}_l[S_3] \longrightarrow \Phi_l \longrightarrow 0,
\]

where \( \nu \) is multiplication by \( \theta = \frac{l+1}{2} e - \frac{l-1}{2} r \). Since \( (1-r)\theta = (1-r) \) and \( (\frac{l+1}{2} e + \frac{l-1}{2} r)\theta = l \), it follows that the above sequence is exact. Further, we have

\[
eq e_{x_0}, \quad e_{x} = le_x,

eq e_{\psi, 1} + e_{\psi, 2},
\]

Thus, we get that \( \epsilon(\Phi_l) \sim e_{x_0} + le_x + le_\psi \). This proves the lemma when \( f = 2 \).

Assume now that \( f = 3 \), so \( k_v = \mathbb{F}_3 \). The \( l \)-component of the local \( L \)-values are given by

\[
|lL_l(E, 1)|_l = |l^{-1} - a_l + l|_l = |E(\mathbb{F}_l)|_l,

|lL_l(E \otimes \chi, 1)|_l = |l^{-1} + a_l + l|_l = |E(\mathbb{F}_2)/E(\mathbb{F}_l)|_l,

|l^2 L_l(E \otimes \psi, 1)|_l = |l^{-2} + a_l + l(1 - a_l + l)|_l = |E(F_{\mathbb{F}_l})/l|_l^2.
\]

Therefore, the lemma holds if \( E(F_{\mathbb{F}_l})^{l}\mathbb{F}_l \) is trivial. If \( E(F_{\mathbb{F}_l})^{l}\mathbb{F}_l \neq l \) it follows that \( E(F_{\mathbb{F}_l})^{l}\mathbb{F}_l \simeq \mathbb{Z}/l\mathbb{Z} \), with nontrivial action of \( D \). Consider the sequence

\[
0 \longrightarrow \mathbb{Z}_l[S_3] \overset{\nu}{\longrightarrow} \mathbb{Z}_l[S_3] \longrightarrow \Phi_l \longrightarrow 0,
\]
where $\nu$ is multiplication by
\[\theta = \frac{(l+1)}{3}e - \frac{(l-1)}{3}s - \frac{(l-1)}{3}s^2.\]
Since $(1-s)\theta = (1-s)$ and $\left(\frac{(l+2)}{3}e + \frac{(l-1)}{3}s + \frac{(l-1)}{3}s^2\right)\theta = l$, it follows that the above sequence is exact. Further, we have
\[
\begin{align*}
  e_{\chi_0}\theta &= e_{\chi_0}, \\
  e_{\chi}\theta &= e_{\chi}, \\
  e_{\psi,1}\theta &= le_{\psi,1}, \\
  e_{\psi,2}\theta &= le_{\psi,2}.
\end{align*}
\]
Thus, we get that $\epsilon(\Phi_l) \sim e_{\chi_0} + e_{\chi} + le_{\psi}$. Thus the lemma holds when $f = 3$. This completes the proof of the lemma. □

Thus, the $l$-parts of the local $L$-factors appearing in $\xi_l$ get cancelled with $\epsilon(\Phi_l), \epsilon(\oplus_v k_v)$ and $\epsilon(\text{RT}_f(K_q, T_l(E)))$’s. Therefore, $\xi_l = (a_{\chi_0}e_{\chi_0} + a_{\chi}e_{\chi} + a_{\psi}e_{\psi})$ for some $a_{\chi_0}, a_{\chi}, a_{\psi} \in \mathbb{Z}^\times_l$. This implies that $\beta_l(\mu_l)$ and its inverse are elements of $\mathbb{Z}_l[S_3]$. Hence $\beta_l(\mu_l)$ is in the image of $K_1(\mathbb{Z}[S_3])$. This verifies the equivariant conjecture for $l > 5$.

The proof of Theorem 1.1 is now complete. □

Remarks.

(1) Theorem 1.1 provides the first evidence for the (noncommutative) equivariant conjecture for motives arising from elliptic curves.

(2) The above argument prescribes a method to verify the equivariant conjecture (not just numerically) in the commutative case. However, no methods are known to compute the $L$-values of the nonabelian twists of an elliptic curve, and therefore the above method will only give a numerical verification in the noncommutative case.

(3) For the abelian twists, one can write down the $L$-values in terms of an Euler system constructed by Kato (cf. [13]). One can possibly use this to get a description of the $L$-values in the fundamental line defined via a module over an Iwasawa algebra. This should relate the equivariant conjecture and Kato’s main conjecture as formulated in [13].

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