Gain-Scheduled $H_\infty$ Control for Discrete-Time Polynomial LPV Systems Using Homogeneous Polynomial Path-Dependent Lyapunov Functions

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Abstract: This paper provides linear matrix inequality (LMI) analysis and synthesis conditions for the design of $H_\infty$ robust and gain-scheduled static output feedback controllers, for discrete-time linear parameter-varying systems. It is assumed that the system matrices have a homogeneous polynomial dependence of arbitrary degree on the time-varying scheduling parameters, which are assumed to vary inside a polytope and to have known bounds on their rates of variation. The geometric properties of the polytopic domain are exploited in order to derive a finite set of LMIs that takes into account bounds on the variation rate of the scheduling parameters. LMI conditions are obtained using a quadratic Lyapunov function with a homogeneous polynomial dependence on the scheduling parameters at successive instants of time. Numerical results show the benefits of the proposed approach.

Keywords: Discrete-time, Time-varying systems, $H_\infty$ performance, robust control, gain-scheduling, static output feedback, LMIs.

1. INTRODUCTION

System and control theory, in terms of linear matrix inequalities (LMIs), has been intensively investigated. It has been shown that a wide range of control problems can be described in terms of LMIs (Apkarian and Tuah 2000, Boyd et al. 1994). Thus, with the advance of computational tools, LMIs have become a powerful and efficient framework to solve analysis and synthesis problems for linear systems (Gahinet et al. 1995, Löfberg 2004).

In particular, for linear parameter varying (LPV) systems, LMI conditions obtained using a parameter-dependent Lyapunov function must be satisfied for the entire parameter space. Such conditions, though convex, involve an infinite set of LMIs, which is not numerically tractable. However, a finite set of LMIs can be derived by imposing a particular structure on the Lyapunov function.

Several results have appeared in literature that propose different parameterizations for the Lyapunov matrix. In Bernussou et al. (1989), Geromel et al. (1991), Kaminer et al. (1993), analysis and synthesis conditions are based on the notion of quadratic stability. In Daafouz and Bernussou (2001), Montagner et al. (2005), LMI conditions are obtained by imposing an affine parameter-dependent structure on the Lyapunov matrix. In Lee (2006), Lee and Dullerud (2006), an affine path-dependent Lyapunov function is used. However, these approaches are conservative for practical applications, since they allow for arbitrarily fast parameter variation.

To reduce this conservatism, some works have included information about the bounds on the rate of variation of the scheduling parameters. For instance, in Amato et al. (2005), piecewise constant Lyapunov functions are used to provide gain-scheduled full-state feedback controllers. In Oliveira and Peres (2008), gain-scheduled full-state feedback controllers are provided for polytopic time-varying systems. The static output feedback case is found in De Caigny et al. (2008a). LMI conditions for robust stability and control design using a path-dependent Lyapunov function are found in Oliveira and Peres (2009). Assuming the plant and the controller can have a homogeneous polynomial dependence on the scheduling parameters, synthesis conditions for gain-scheduled dynamic output feedback controllers are provided in De Caigny et al. (2012a). Techniques that provide polynomial LPV models can be found in De Caigny et al. (2008b; 2012b).

The paper is organized as follows. Section 2 presents the system matrices, which can have a homogeneous polynomial dependence of arbitrary degree on the scheduling parameters. This section also describes the modeling of the uncertainty domain considering bounds on the rate of variation of the scheduling parameters. Section 3 presents LMIs conditions for $H_\infty$ performance analysis, based on the existence of a quadratic Lyapunov function that has a homogeneous polynomial path-dependence on the scheduling parameters. Section 4 provides the design of both robust and gain-scheduled static output feedback controllers. Section 5 presents the numerical results.

Notation

The symbols $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{N}^*$ denote, respectively, the set of real numbers, nonnegative integers, and positive integers. The identity matrix of size $n$ is denoted as $I_n$. The notation $0_{n,m}$ indicates an $n \times m$ matrix of zeros. The symbol $(\cdot)'$ indicates transpose. The symbol $(>0)$ means positive definite. The
symbol (*) represents the symmetric block. For \( N, n, m, g \in \mathbb{N}^* \), \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \), and \( \ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n \), the operations \( v \geq \ell, v + \ell \), and \( v - \ell \) are defined element-wise. The scalar \( v^\top \) is defined as \( v^\top = [v_1^\top, \ldots, v_n^\top] = \prod_{i=1}^n v_i^\top \). The scalar \( \pi(v) \) is defined as \( \pi(v) = \prod_{i=1}^n v_i! \), where \( ! \) denotes factorial. \( \mathcal{X}(N, g) \) denotes the set of \( N \)-tuple obtained from all possible combinations of \( N \) nonnegative integers whose sum is \( g \).

2. SYSTEM DESCRIPTION

Consider the discrete-time linear parameter-varying system
\[
x[k+1] = A(\alpha[k])x[k] + B_u(\alpha[k])w[k] + B_u(\alpha[k])u[k],
\]
\[
z[k] = C(\alpha[k])x[k] + D_u(\alpha[k])w[k] + D_u(\alpha[k])u[k],
\]
where \( x[k] \in \mathbb{R}^{n_0} \) is the state, \( w[k] \in \mathbb{R}^{n_w} \) is the exogenous input, \( u[k] \in \mathbb{R}^{n_u} \) is the control input, \( z[k] \in \mathbb{R}^{n_z} \) is the system output, and \( \alpha[k] \in \mathbb{R}^N \) is the vector of time-varying scheduling parameters.

It is assumed that the first \( n_s \) states of the system can be measured in real time for feedback, that is
\[
y[k] = C_y x[k], \quad C_y = [I_{n_s} \ 0_{n_s, n_n - n_s}].
\]
If this is not the case, one can use a similarity transformation as proposed in Geromel et al. (1996) whenever output matrix \( C_y \) is not affected by the time-varying parameter.

Considering the static output feedback control law
\[
u[k] = K(\alpha[k])y[k]
\]
the closed-loop system is given by
\[
x[k+1] = A_C(\alpha[k])x[k] + B_u(\alpha[k])w[k],
\]
\[
z[k] = C_C(\alpha[k])x[k] + D_u(\alpha[k])w[k],
\]
with
\[
A_C(\alpha[k]) = A(\alpha[k]) + B_u(\alpha[k])K(\alpha[k])C_y, \quad C_C(\alpha[k]) = C(\alpha[k]) + D_u(\alpha[k])K(\alpha[k])C_y.
\]
The matrices \( A(\alpha[k]) \in \mathbb{R}^{n_0 \times n_0}, B_u(\alpha[k]) \in \mathbb{R}^{n_0 \times n_w}, B_u(\alpha[k]) \in \mathbb{R}^{n_0 \times n_u}, C(\alpha[k]) \in \mathbb{R}^{n_z \times n_0}, D_u(\alpha[k]) \in \mathbb{R}^{n_z \times n_w}, D_u(\alpha[k]) \in \mathbb{R}^{n_z \times n_u}, \) and \( K(\alpha[k]) \in \mathbb{R}^{n_u \times n_0}, \) have a homogeneous polynomial dependence of degree \( p \) on parameter \( \alpha[k] \), given by
\[
(A, B_u, B_u, C_u, D_u, D_u)(\alpha[k]) = \sum_{\ell \in \mathcal{X}(N, p)} \alpha[k]^\ell (A_{\ell}, B_{\ell}, B_{\ell}, C_{\ell}, D_{\ell}, D_{\ell}),
\]
with \( A_{\ell}, B_{\ell}, B_{\ell}, C_{\ell}, D_{\ell}, D_{\ell} \) constant matrices for all \( \ell \in \mathcal{X}(N, p) \). Scheduling parameter \( \alpha[k] \) takes values in the unit-simplex \( \Lambda_N \), given by
\[
\Lambda_N = \left\{ \zeta \in \mathbb{R}^N : \sum_{i=1}^N \zeta_i = 1, \zeta_i \geq 0 \right\}.
\]
The rate of variation of the scheduling parameters, given by
\[
\Delta \alpha[k,i] = \alpha[k+1,i] - \alpha[k,i], \quad i = 1, \ldots, N,
\]
is assumed to be bounded by an \textit{a priori} known bound \( b \in [0, 1] \) such that
\[
-b \leq \Delta \alpha[k,i] \leq b, \quad i = 1, \ldots, N,
\]

2.1 Modeling the uncertainty domain

The region where \( \Delta \alpha[k,i] \) can take values as a function of \( \alpha[k,i] \), considering (4), (5), and (6), is presented in Figure 1. This region can be modeled as a polytope with six vertices (De Caigray et al. 2010, Oliveira and Peres 2009), given by
\[
\Gamma_i = \left\{ \xi \in \mathbb{R}^2 : \xi = \sum_{j=1}^6 \gamma_j h^j, \gamma \in \Lambda_6, \right. \]
\[
\left. [h^1 \ h^6] = \begin{bmatrix} 0 & 0 & 1 & b & 1 & b \\ 1 & 1 & b & 0 & 0 & b \end{bmatrix} \right\}
\]
Due to the fact that \( \alpha[k] \in \Lambda_N \) and to (5), \( \Delta \alpha[k] \) satisfies
\[
\sum_{i=1}^N \Delta \alpha[k,i] = 0, \quad \forall k \geq 0.
\]
Thus, using the Cartesian product of \( \Gamma_i \), for \( i = 1, \ldots, N \), reordering the vertices \( h^j \), and discarding vertices that do not satisfy (4) and (7), it is possible to model the region in the \((\alpha[k], \Delta \alpha[k])\)-plane where vector \( \Delta \alpha[k] \) assumes values as a function of \( \alpha[k] \), taking into account bound \( b \). As an illustration, if \( N = 2 \), then \( \alpha \in \Lambda_2 \), and \( \Gamma \), constructed using the Cartesian product of \( \Gamma_1 \) and \( \Gamma_2 \), is given by
\[
\Gamma = \left\{ \xi \in \mathbb{R}^4 : \xi = \sum_{j=1}^6 \gamma_j h^j, \gamma \in \Lambda_6, \right. \]
\[
\left. [h^1 \ h^6] = \begin{bmatrix} 0 & 0 & 1 & b & 1 & b \\ 1 & 1 & b & 0 & 0 & b \end{bmatrix} \right\}
\]
Note that vectors \( f_l^1 = (f_{l1}^1, f_{l2}^1) \in \mathbb{R}^2 \) belong to \( \Lambda_2 \), and vectors \( f_l^2 = (f_{l1}^2, f_{l2}^2) \in \mathbb{R}^2 \) satisfy \( \sum_{i=1}^2 f_{li} = 0 \). These vectors were obtained after an appropriate reordering of the entries of \( h^j \), such that \( (\alpha[k], \Delta \alpha[k]) \in \Gamma \).

In a similar way, the uncertainty domain where vector \( (\Delta \alpha[k])' \), \( \Delta \alpha[k]' \), \ldots, \( \Delta \alpha[k] + L - 1)' \) \in \( \mathbb{R}^{(L+1)N} \), with \( L \in \mathbb{N}^* \), assumes values can be modeled by the set
\[
\Gamma = \left\{ \xi \in \mathbb{R}^{(L+1)N} : \xi = \sum_{j=1}^M \gamma_j h^j, \gamma \in \Lambda_M, \right. \]
\[
\left. [h^1 \ h^M] = \begin{bmatrix} f_1^1 \\ \vdots \\ f_L^1 \end{bmatrix}, f_k^j = (f_{k1}^j, \ldots, f_{kM}^j) \in \mathbb{R}^N, \right. \]
\[
\left. \sum_{j=1}^N f_{kj} = 1, \text{ with } f_{kj} \geq 0, j = 1, \ldots, M, \right. \]
\[
\left. \sum_{j=1}^N f_{kj} = 0, j = 1, \ldots, M, k = 2, \ldots, L + 1 \right\}.
\]
The vectors \( h^j \) above can be constructed in a systematic way as a function of \( N, b \) and \( L \), using the Cartesian product of \( \Gamma_i \), for \( i = 1, \ldots, N \), where each \( \Gamma_i \) can be constructed searching for all possible solutions of the equalities \( \sum_{i=1}^N \alpha[k,i] = 1 \).
\[ \sum_{i=1}^{N} \Delta \alpha[k]_i = 0, \ldots, \sum_{i=1}^{N} \Delta \alpha[k+L-1]_i = 0. \] From the definition of \( \Gamma \), it is immediate that
\[ \begin{bmatrix} \alpha[k+1] \\ \Delta \alpha[k] \\ \vdots \\ \Delta \alpha[k+L-1] \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{L+1} \end{bmatrix} \gamma[k] \in \Gamma, \]
with \( F_k = [f^1_k, \ldots, f^M_k] \in \mathbb{R}^{N \times M} \), for \( k = 1, \ldots, L+1 \), and \( \gamma[k] \in \mathcal{A}_M \). Using this linear transformation, and defining \( F_k = \sum_{k=1}^{L+1} F_k \), one obtains the following change of variables
\[ \alpha[k + \bar{k}] = F_k \gamma[k], \quad \bar{k} = 0, \ldots, L, \quad (8) \]
that is used in the next section to derive a finite set of LMI.

3. \( \mathcal{H}_\infty \) PERFORMANCE ANALYSIS

This section provides sufficient conditions, in terms of a finite set of LMIs, to guarantee an upper bound on the \( \mathcal{H}_\infty \) performance of LPV systems. First, an LMI characterization of the \( \mathcal{H}_\infty \) performance is presented and, afterwards, a finite set of LMI conditions is derived using the modeling of the uncertainty domain previously introduced.

Based on the bounded-real lemma, an upper bound on the \( \mathcal{H}_\infty \) performance can be characterized using a parameter-dependent LMI, as described in next lemma (see De Caigny et al. (2010), De Souza et al. (2006)).

**Lemma 1.** If there exist a scalar \( \eta > 0 \) and bounded parameter-dependent matrices \( P(\alpha[k]) = P(\alpha[k])^T > 0 \) and \( G(\alpha[k]) \), for all \( \alpha[k] \in \Lambda_M, k \geq 0 \), such that
\[ \begin{bmatrix} P(\alpha[k+1]) \\ G(\alpha[k])^T A(\alpha[k])^T \\ G_{22} \\ 0 \\ 0 \\ C(\alpha[k]) D(\alpha[k]) D(\alpha[k])^T \end{bmatrix} \begin{bmatrix} \eta \\ \eta \\ \eta \\ \eta \\ \eta \\ \eta \end{bmatrix} > 0 \quad (9) \]
with \( G_{22} = G(\alpha[k]) + G(\alpha[k])^T - P(\alpha[k]) \), then system (1) is asymptotically stable with a guaranteed upper bound \( \eta \) on its \( \mathcal{H}_\infty \) performance.

Notice that LMI (9), from Lemma 1, depends on the values of the time-varying scheduling parameter \( \alpha \in \Lambda_M \). This leads to an infinite-dimensional problem. Consequently, Lemma 1 is not numerically tractable. To derive a finite set of LMIs, two steps are required. First, a particular structure is imposed on Lyapunov matrix \( P(\alpha[k]) \) and on slack variable \( G(\alpha[k]) \). Second, the region where \( \alpha[k+1] \) can take values is modeled as a convex polytope.

In order to deal with the first step, the following structure for the Lyapunov matrix is chosen:
\[ P(\alpha[k]) = \sum_{\lambda_1 \in \mathcal{X}(N,g)} \cdots \sum_{\lambda_L \in \mathcal{X}(N,g)} \alpha[k]^\lambda_1 \alpha[k+1]^\lambda_2 \cdots \alpha[k+L-1]^\lambda_L P(\lambda_1, \ldots, \lambda_L), \quad (10) \]
which is a homogeneous polynomial of arbitrary degree \( g \) on the scheduling parameter \( \alpha[k] \), at \( L \) successive instants of time. This structure is also used for slack variable \( G(\alpha[k]) \). The parameterization given by (10) encompasses other structures, such as: the path-dependent structure, with \( g = 1 \), used in Lee (2006), Lee and Dullerud (2006), Oliveira and Peres (2009); the homogeneous polynomial structure, with \( L = 1 \), used in De Caigny et al. (2012a), Oliveira and Peres (2009); and the affine structure, with \( L = g = 1 \), used in Montagner et al. (2005).

Note that, by using the structure (10), LMI (9) depends on \( \alpha[k] \), at \( L \) future instants of time. Thus, it is also necessary to consider the region where \( \alpha[k+1], \ldots, \alpha[k+L] \) can take values.

### 3.1 Finite set of LMIs

Using the change of variables (8), all parameter-dependent matrices in the \( \alpha \)-space can be rewritten in the \( \gamma \)-space as
\[ (A, B, C, D) \rightarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) \]
for \( \gamma[k] \in \mathcal{A}_M \). This structure is also used for slack variable \( G(\alpha[k]) \) equivalent to the LMI
\[ \begin{bmatrix} P(\gamma[k]) \hat{A}(\gamma[k]) \hat{C}(\gamma[k]) \hat{D}(\gamma[k]) \\ 0 \\ \eta \\ \eta \end{bmatrix} \begin{bmatrix} \gamma[k] \\ \gamma[k]^T \\ \gamma[k] \\ \gamma[k]^T \end{bmatrix} \geq 0, \quad (11) \]
with \( G_{22} = \hat{G}(\gamma[k]) + \hat{G}(\gamma[k])^T - \hat{P}(\gamma[k]) \).

Using LMI (11), a finite set of LMIs can be derived, as presented in the next theorem.

**Theorem 1.** Assume that matrices \( \hat{F}_k \), for \( k = 0, \ldots, L \), in (8) are given. If there exist a scalar \( \eta > 0 \), a path length \( L \in \mathbb{N} \), a degree \( g \in \mathbb{N} \), a relaxation \( \lambda \in \mathbb{N} \), a level \( d \in \mathbb{N} \), symmetric matrices \( P(\lambda_1, \ldots, \lambda_L) \in \mathbb{R}^{n_\lambda \times n_\lambda} \), matrices \( G(\lambda_1, \ldots, \lambda_L) \in \mathbb{R}^{n_\lambda \times n_\lambda} \), such that
\[ \begin{bmatrix} \Psi_{22} \\ \Psi_{12} \\ \Psi_{11} \end{bmatrix} \begin{bmatrix} (L_g)_{\lambda_1, \lambda_1} \cdots (L_g)_{\lambda_L, \lambda_L} \\ (L_g)_{\lambda_1, \lambda_1} \cdots (L_g)_{\lambda_L, \lambda_L} \\ (L_g)_{\lambda_1, \lambda_1} \cdots (L_g)_{\lambda_L, \lambda_L} \end{bmatrix} \geq 0, \quad (12) \]
where matrices \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \) can be constructed as presented in the Appendix. Consequently, LMI (9) with the structure (10) for Lyapunov matrix \( P(\alpha[k]) \) and slack variable \( G(\alpha[k]) \) equivalent to the LMI
\[ \begin{bmatrix} \Psi_{22} \hat{A}(\gamma[k]) \hat{C}(\gamma[k]) \hat{D}(\gamma[k]) \\ 0 \\ \eta \\ \eta \end{bmatrix} \begin{bmatrix} \gamma[k] \\ \gamma[k]^T \\ \gamma[k] \\ \gamma[k]^T \end{bmatrix} \geq 0, \quad (11) \]
with \( G_{22} = \hat{G}(\gamma[k]) + \hat{G}(\gamma[k])^T - \hat{P}(\gamma[k]) \).

Using LMI (11), a finite set of LMIs can be derived, as presented in the next theorem.

**Proof:** Take any \( \gamma[k] \in \mathcal{A}_M \). Multiplying (12) by \( \gamma[k]^T \) and then summing, for \( i \in \mathcal{X}(M, L_g + d + p) \), yields LMI (11), since \( \gamma[k] \in \mathcal{A}_M \) for all \( k \geq 0 \). The set of LMIs (12) guarantees that (11) is positive definite. Due to (8), LMI (11) is equivalent to LMI (9) under the imposed parameterization (10) for Lyapunov.
matrix $P(\alpha[k])$ and slack variable $G(\alpha[k])$. Therefore, feasibility of (12) implies feasibility of (9), consequently, system (1) is asymptotically stable.

4. $\mathcal{H}_\infty$ STATIC OUTPUT FEEDBACK CONTROL DESIGN

In this section, Theorem 1 is extended to provide a finite set of LMI conditions for the synthesis of robust and gain-scheduled $\mathcal{H}_\infty$ static output feedback controllers for system (1).

4.1 Robust $\mathcal{H}_\infty$ static output feedback

The goal is to design a robust gain $K(\alpha[k]) = K$, that does not depend on the scheduling parameter $\alpha[k]$, such that the closed-loop system (3) is asymptotically stable, with guaranteed $\mathcal{H}_\infty$ performance.

Theorem 2. Assume that matrices $\hat{F}_k$, for $k = 0, \ldots, L$, in (8) are given. If there exist a scalar $\eta > 0$, a path length $L \in \mathbb{N}^*$, a degree $g \in \mathbb{N}^*$, a relaxation level $d \in \mathbb{N}$, symmetric matrices $P_{\lambda_1, \ldots, \lambda_d} \in \mathbb{R}^{n_a \times n_a}$, with $\lambda_d \in \mathcal{K}(N, g)$, for $\tau = 1, \ldots, L$, matrices $G_1 \in \mathbb{R}^{n \times p}$, $G_2 \in \mathbb{R}^{(n-n_\gamma) \times n}$, $G_3 \in \mathbb{R}^{(n-n_\gamma) \times (n-n_\gamma)}$, and $Z_1 \in \mathbb{R}^{p_a \times n_a}$, such that

$$
\begin{aligned}
j \sum_{j=1}^N \sum_{i=1} \sum_{\ell=1} \sum_{j=1}^p d_{\ell(\ell-j)}^j \\
[\Psi_{1,2}, \Psi_{2,2}] &> 0,
\end{aligned}
$$

with

$$
\Psi_{1,2} = \frac{(L_g)^p}{\pi(\pi-j)^{\tau}} (\hat{A}_G + \hat{B}_w(X_G)),
$$

$$
\Psi_{2,2} = \frac{(L_g)^p}{\pi(\pi-j)^{\tau}} (G + \gamma - P(\gamma))^T - \frac{p}{\pi(\pi-j)^{\tau}} (\hat{P}(\gamma)^T)
$$

for all $i \in \mathcal{X}(M, L_g + d + p)$, where

$$
G = [G_1, 0_{n - n_\gamma}, G_2], \quad \tau = [Z_1, 0_{n_a - n_a}],
$$

and matrices $\hat{A}_G$, $\hat{B}_w$, $\hat{B}_u$, $\hat{C}_z$, $\hat{D}_u$, $\hat{D}_a$, $\hat{P}_i$, and $\hat{P}_i$ are, respectively, the coefficients of matrices $\hat{A}(\gamma[k])$, $\hat{B}_w(\gamma[k])$, $\hat{B}_u(\gamma[k])$, $\hat{C}_z(\gamma[k])$, $\hat{D}_u(\gamma[k])$, $\hat{D}_a(\gamma[k])$, $\hat{P}(\gamma[k])$, and $\hat{P}(\gamma[k + 1])$, obtained from $A(\alpha[k])$, $B_w(\alpha[k])$, $B_u(\alpha[k])$, $C_z(\alpha[k])$, $D_u(\alpha[k])$, $D_a(\alpha[k])$, $P(\alpha[k])$, and $P(\alpha[k + 1])$ using the change of variables given by (8), then the static output feedback gain

$$
K = Z_1 G_1^{-1}
$$

ensures that closed-loop system (3) is asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance bounded by $\eta$.

Proof: Take any $\gamma[k] \in \mathcal{X}$, Multiplying (14) by $\gamma[k]^T$, and summing, for $i \in \mathcal{X}(M, L_g + d + p)$, gives

$$
\begin{aligned}
\sum_{j=1}^M \sum_{i=1} \sum_{\ell=1} \sum_{j=1}^p d_{\ell(\ell-j)}^j \\
[\Psi_{1,2}, \Psi_{2,2}] &> 0,
\end{aligned}
$$

with $\Psi_{1,2} = \hat{A}(\gamma[k]) G + \hat{B}_w(\gamma[k]) Z$ and $\Psi_{2,2} = G' C_z(\gamma[k]) + Z' \hat{B}_a(\gamma[k])$.

Using (2), (15) and (16), one has

$$
\hat{A}(\gamma[k]) G + \hat{B}_w(\gamma[k]) Z
= \hat{A}(\gamma[k]) G + \hat{B}_u(\gamma[k]) Z_1 G_1^{-1} G_1^{-1},
$$

and, analogously,

$$
\Psi_{2,2} = G' C_z(\gamma[k]) Z + Z' \hat{B}_a(\gamma[k])
$$

Finally, one obtains

$$
[\hat{P}(\gamma[k]), \hat{A}_G, \hat{B}_w(X_G), \hat{B}_w(X_G)] > 0.
$$

The set of LMIs (14) guarantees that (17) is positive definite. Due to (8), LMI (17) is equivalent to LMI (9) under the imposed parameterization (10) for Lyapunov matrix $P(\alpha[k])$ and the following parameterization for slack variable $G(\alpha[k]) = G$. Therefore, feasibility of (14) implies feasibility of (9), consequently, the closed-loop system (3) is asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance bounded by $\eta$.

4.2 Gain-scheduled $\mathcal{H}_\infty$ static output feedback

The goal is to design a parameter-dependent gain $K(\alpha[k])$ such that the closed-loop system (3) is asymptotically stable with guaranteed $\mathcal{H}_\infty$ performance.

Theorem 3. Assume that matrices $\hat{F}_k$, for $k = 0, \ldots, L$, in (8) are given. If there exist a scalar $\eta > 0$, a path length $L \in \mathbb{N}^*$, a degree $g \in \mathbb{N}^*$, a relaxation level $d \in \mathbb{N}$, symmetric matrices $P_{\lambda_1, \ldots, \lambda_d} \in \mathbb{R}^{n_a \times n_a}$, matrices $G_1(\alpha[k]), G_2(\alpha[k]), G_3(\alpha[k]), Z_1(\alpha[k]), Z_2(\alpha[k]), G(\alpha[k])$, and $Z(\alpha[k])$, for $i = 1, \ldots, L$, such that

$$
\begin{aligned}
j \sum_{j=1}^N \sum_{i=1} \sum_{\ell=1} \sum_{j=1}^p d_{\ell(\ell-j)}^j \\
[\Psi_{1,2}, \Psi_{2,2}] &> 0,
\end{aligned}
$$

with $\Psi_{1,2} = \hat{A}(\gamma[k]) G + \hat{B}_w(\gamma[k]) Z$ and $\Psi_{2,2} = G' C_z(\gamma[k]) Z + Z' \hat{B}_a(\gamma[k])$.

Using (2), (15) and (16), one has

$$
\hat{A}(\gamma[k]) G + \hat{B}_w(\gamma[k]) Z
= \hat{A}(\gamma[k]) G + \hat{B}_u(\gamma[k]) Z_1 G_1^{-1} G_1^{-1},
$$

and, analogously,

$$
\Psi_{2,2} = G' C_z(\gamma[k]) Z + Z' \hat{B}_a(\gamma[k])
$$

Finally, one obtains

$$
[\hat{P}(\gamma[k]), \hat{A}_G, \hat{B}_w(X_G), \hat{B}_w(X_G)] > 0.
$$

The set of LMIs (14) guarantees that (17) is positive definite. Due to (8), LMI (17) is equivalent to LMI (9) under the imposed parameterization (10) for Lyapunov matrix $P(\alpha[k])$ and the following parameterization for slack variable $G(\alpha[k]) = G$. Therefore, feasibility of (14) implies feasibility of (9), consequently, the closed-loop system (3) is asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance bounded by $\eta$.
then the parameter-dependent static output feedback gain

\[ K(\alpha[k]) = Z_1(\alpha[k])G_1(\alpha[k])^{-1} \]

ensures that closed-loop system (3) is asymptotically stable with a guaranteed \( H_{\infty} \) performance bounded by \( \eta \).

The proof of Theorem 3 is omitted since it is similar to the proof of Theorem 2.

5. NUMERICAL RESULTS

This section presents numerical examples to show the benefits of the proposed analysis and synthesis techniques.

5.1 Example 1 - \( H_{\infty} \) performance analysis

Consider the time-varying discrete-time system

\[
x[k+1] = \begin{bmatrix} 0.9979 & 0.008 \omega[k] - 0.01 \\ 0.01 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 0.2 \omega[k] \\ 0 \end{bmatrix} w[k],
\]

\[ z[k] = [1, 0]^T x[k], \quad |\omega[k]| \leq 0.3, \]

(19)

taken from Amato et al. (1998).

The goal is to determine, using Theorem 1, the minimum guaranteed upper bound \( \eta \) on the \( H_{\infty} \) performance as a function of bound \( b \) on parameter variation rate \( \Delta \omega[k] \).

The affine LPV system (19) can be represented by the following polytopic model

\[
(A, B_w, C_z) = \sum_{i=1}^{2} \alpha[k]_i (A, B_w, C_z)_i, \quad \alpha[k] \in \Lambda_2,
\]

where the vertices of the polytope \((A, B_w, C_z)_1\) and \((A, B_w, C_z)_2\) are computed using the minimum \( \omega \) and maximum \( \bar{\omega} \) values of \( \omega[k] \), respectively. The scheduling parameter \( \alpha[k] = (\alpha[k]_1, \alpha[k]_2) \) is given by

\[
\alpha[k]_1 = \frac{\omega[k] - \omega}{\bar{\omega} - \omega}, \quad \alpha[k]_2 = 1 - \frac{\omega[k] - \omega}{\bar{\omega} - \omega}.
\]

The polytopic model above is equivalent to a homogeneous polynomial system, as given by (1), of degree \( p = 1 \), with \( n_x = 2, n_w = 1, n_z = 1, N = 2, \) and

\[
(A, B_w, C_z) = \sum_{j \in \mathcal{X}(2,1)} \alpha[k]_j (A, B_w, C_z)_j,
\]

where

\[
(A, B_w, C_z)(1,0) = (A, B_w, C_z)_1
\]

and

\[
(A, B_w, C_z)(0,1) = (A, B_w, C_z)_2.
\]

Figure 2 shows the minimum guaranteed upper bound \( \eta \) on the \( H_{\infty} \) performance of this system, as a function of bound \( b \) on parameter variation rate \( \Delta \omega[k] \), for the following four cases: (i) \( L = 1, g = 1 \); (ii) \( L = 1, g = 2 \); (iii) \( L = 2, g = 1 \); (iv) \( L = 2, g = 2 \). For all the simulations, \( d = 0 \). The maximum allowed bound \( b_{\text{max}} \) is computed using Theorem 1. For the cases (i), (iii), and (iv), bound \( b_{\text{max}} \) is, respectively, given by \( b_{\text{max}} = 0.0145, b_{\text{max}} = 0.0383, \) and \( b_{\text{max}} = 0.0683 \). Cases (i) and (ii) provides similar results. Clearly, the results become less conservative as \( L \) and \( g \) increases.

5.2 Example 2 - \( H_{\infty} \) control design

Consider the following polynomial LPV system

\[
\begin{bmatrix}
A(2,0) \\
A(1,1)
\end{bmatrix} = 0.4 \times 
\begin{bmatrix}
1 & 2 & -1 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\
0 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & -2
\end{bmatrix},
\]

\[
\begin{bmatrix}
B(2,0) \\
B(1,1)
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix},
\]

\[
C_1 = \begin{bmatrix}
1 & 1 \end{bmatrix}, \quad D_{ui} = 0,
\]

with \( i = \{(2,0), (1,1), (0,2)\} \), \( n_i = 3, n_w = 1, n_u = 1, n_z = 1, N = 2, p = 2 \), and bound \( b \) on the parameter variation rate given by \( b = 0.525 \).

The goal is to design robust and gain-scheduled static output feedback controllers in order to minimize an upper bound on the \( H_{\infty} \) performance of the closed-loop system. The robust and gain-scheduled controllers are obtained using Theorem 2 and Theorem 3, respectively, for different combinations, using: \( L = 1, 2, 3; g = 1, 2, 3; n_y = 2, 3; \) and \( d = 0 \). The results are shown in Table 1.

Table 1. Minimum guaranteed upper bound \( \eta \).

| Parameters | Theorem 1 | Theorem 2 | Theorem 3 |
|------------|-----------|-----------|-----------|
| \( L = 1, g = 1 \) | \( 5.897 \) | \( 2.165 \) | \( 3.003 \) |
| \( L = 1, g = 2 \) | \( 3.810 \) | \( 2.115 \) | \( 2.164 \) |
| \( L = 2, g = 1 \) | \( 4.113 \) | \( 2.274 \) | \( 2.171 \) |
| \( L = 2, g = 2 \) | \( 3.744 \) | \( 2.077 \) | \( 2.069 \) |
| \( L = 3, g = 1 \) | \( 3.942 \) | \( 2.662 \) | \( 1.566 \) |

As shown in Table 1, the results obtained with the robust \( H_{\infty} \) static output feedback controllers, given by Theorem 2, are more conservative than those obtained with the gain-scheduled version, given by Theorem 3. The results are less conservative whenever the number of measured states \( n_u \), the path length \( L \), and the polynomial degree \( g \) increase, for both Theorem 2 and Theorem 3, as was expected.

CONCLUSION

The paper provided sufficient LMI conditions for \( H_{\infty} \) performance analysis and for the design of both robust and gain-scheduled \( H_{\infty} \) static output feedback controllers for discrete-time LPV systems whose parameters vary inside a polytope with bounded rate of variation. The new parameterization with homogeneous polynomial path dependence is used for the Lyapunov matrix and for the parameter-dependent controller. The LMI conditions become less conservative, yielding lower \( H_{\infty} \) guaranteed costs, whenever the path length and the polynomial degree of the parameter dependency increase, as was shown by the numerical examples. Furthermore, gain-scheduled controllers always provide better results than the robust ones.
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APPENDIX

This section presents the necessary manipulations for writing Lyapunov matrix $P(\alpha[k])$ in the $\gamma$-space. The manipulations for all the other matrices follow similar steps and, thus, are omitted. Recall that Lyapunov matrix $P(\alpha[k])$ has the following homogeneous polynomial path-dependent parameterization:

$$P(\alpha[k]) = \sum_{\lambda_1, \lambda_2, \ldots, \lambda_N} \alpha[k] \sum_{\gamma_1} \alpha[k+1] \sum_{\gamma_2} \cdots \sum_{\gamma_N} \sum_{\gamma_T} \hat{P}_k = \hat{P}(\gamma).$$

with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. Using (8), one has

$$\alpha[k] = \frac{1}{\gamma_1} F_0 \gamma_2 \cdots F_N \gamma_T = \frac{1}{\gamma_1} F_0 \gamma_2 \cdots F_N \gamma_T,$$

for $k = 0, \ldots, L$. Thus,$^2$

$$P(\alpha) = \sum_{\lambda_1, \lambda_2, \ldots, \lambda_N} \sum_{\lambda_N} \sum_{\gamma_1} \sum_{\gamma_2} \cdots \sum_{\gamma_N} \sum_{\gamma_T} \frac{1}{\gamma_1} F_0 \gamma_2 \cdots F_N \gamma_T \hat{P}_k.$$

Rearranging the terms, one has

$$P(\alpha) = \sum_{\lambda_1, \lambda_2, \ldots, \lambda_N} \sum_{\lambda_N} \sum_{\gamma_1} \sum_{\gamma_2} \cdots \sum_{\gamma_N} \sum_{\gamma_T} \frac{1}{\gamma_1} F_0 \gamma_2 \cdots F_N \gamma_T \hat{P}_k.$$

Defining $g_L = (g_1, g_2, \ldots, g_L) \in \mathbb{N}^L$, $k = (k_1, \ldots, k_L) \in \mathbb{N}^{L,k}$, with $k = (k_1, \ldots, k_L) \in \mathbb{N}^{L,k}$, for $i = 1, \ldots, L$, $M_{	ext{Lin}} = (M, \ldots, M) \in \mathbb{N}^{L,M}$, and $N_{	ext{Lin}} = (N, \ldots, N) \in \mathbb{N}^{L,N}$, one has

$$P(\alpha) = \sum_{\lambda_1, \lambda_2, \ldots, \lambda_N} \sum_{\gamma_1} \sum_{\gamma_2} \cdots \sum_{\gamma_N} \frac{1}{\gamma_1} F_0 \gamma_2 \cdots F_N \gamma_T \hat{P}_k,$$

where $X(N_{	ext{Lin}}, g_L)$ is defined as the Cartesian product of the $L$ terms $X(N, g)$, and $X(M_{	ext{Lin}}, \lambda)$ is defined as the Cartesian product of all the terms $X(M, \lambda)$. Defining $\sum_{k=1}^L \sum_{i=1}^N k_h \lambda_i = t$, one has $t \in X(M, \sum_{i=1}^N k_h \lambda_i) = X(M, Lg)$, thus,

$$P(\alpha) = \sum_{t \in X(M, Lg)} \sum_{\lambda_1, \lambda_2, \ldots, \lambda_N} \sum_{\lambda_N} \sum_{\gamma_1} \sum_{\gamma_2} \cdots \sum_{\gamma_N} \sum_{\gamma_T} \frac{1}{\gamma_1} F_0 \gamma_2 \cdots F_N \gamma_T \hat{P}_k.$$

Finally, defining

$$\hat{P}_t = \sum_{\lambda_1, \lambda_2, \ldots, \lambda_N} \sum_{\lambda_N} \sum_{\gamma_1} \sum_{\gamma_2} \cdots \sum_{\gamma_N} \sum_{\gamma_T} \frac{1}{\gamma_1} F_0 \gamma_2 \cdots F_N \gamma_T \hat{P}_k,$$

one has

$$P(\alpha) = \sum_{t \in X(M, Lg)} \frac{1}{\gamma_1} F_0 \gamma_2 \cdots F_N \gamma_T \hat{P}_t.$$