ASYMPTOTICS FOR THE SASA–SATSUMA EQUATION
IN TERMS OF A MODIFIED PAINLEVÉ II TRANSCENDENT

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Abstract. We consider the initial-value problem for the Sasa–Satsuma equation on
the line with decaying initial data. Using a Riemann–Hilbert formulation and steepest
descent arguments, we compute the long-time asymptotics of the solution in the sector
\(|x| \leq Mt^{1/3}\), \(M\) constant. It turns out that the asymptotics can be expressed in terms
of the solution of a modified Painlevé II equation. Whereas the standard Painlevé II
equation is related to a 2 \(\times\) 2 matrix Riemann–Hilbert problem, this modified Painlevé
II equation is related to a 3 \(\times\) 3 matrix Riemann–Hilbert problem.

AMS Subject Classification (2010): 35Q15, 37K15, 41A60.
Keywords: Sasa–Satsuma equation, Riemann–Hilbert problem, asymptotics, initial value
problem.

1. Introduction

In this paper, we consider the long-time behavior of the solution of the Sasa–Satsuma
equation \([11]\)

\[ u_t - u_{xxx} - 6|u|^2u_x - 3u(|u|^2)_x = 0, \tag{1.1} \]

with initial data \(u(x, 0) = u_0(x) \in S(\mathbb{R})\) in the Schwartz class. Our main result shows
that \(u(x, t)\) admits an expansion to all orders in the asymptotic sector \(|x| < Mt^{1/3}\) of
the form

\[ u(x, t) \sim \sum_{j=1}^{\infty} \frac{u_j(y)}{t^{j/3}}, \quad t \to \infty, \tag{1.2} \]

where \(\{u_j(y)\}_{j=1}^{\infty}\) are smooth functions of \(y \equiv x/(3t)^{1/3}\) and \(M > 0\) is a constant. It also
shows that the leading coefficient \(u_1(y)\) is given by

\[ u_1(y) = \frac{i u_P(y)}{3^{1/3} \sqrt{2}}, \]

where \(u_P(y)\) satisfies the following modified Painlevé II equation:

\[ u''_P(y) + yu_P(y) + 2u_P(y)|u_P(y)|^2 = 0. \tag{1.3} \]

Equation (1.3) coincides with the standard Painlevé II equation

\[ u''_P(y) - yu_P(y) - 2u_P(y)^3 = 0, \tag{1.4} \]

except for a sign difference and the presence of the absolute value squared in the last
term. We will show that (1.3) is related to a 3 \(\times\) 3 matrix RH problem much in the same
way that (1.4) is related to a 2 \(\times\) 2 matrix RH problem cf. [5]. In the case of a real-valued
solution, equation (1.1) reduces to a version of the mKdV equation, (1.3) reduces (up
to a sign) to (1.4), and the expansion (1.2) reduces to the analogous asymptotic formula
for the corresponding mKdV equation (see [4], and [2] for the higher order terms, in the
case of the standard mKdV equation).

It turns out that the leading coefficient \(u_1(y)\) in (1.2) has constant phase, that is,
\(u_1(y) = |u_1(y)|e^{i\alpha} \) where \(\alpha \in \mathbb{R}\) is independent of \(y\). It is somewhat remarkable that this
is the case for any choice of the complex-valued initial data \( u_0(x) = u(x, 0) \); however, we also recall that the Sasa–Satsuma has a class of one-soliton solutions of constant phase (see [11] or [1]):

\[
u_{1\text{-sol}}(x, t) = \frac{\sqrt{2}ae^{a(x+a^2t-x_0)}e^{i\phi}}{1+e^{2a(x+a^2t-x_0)}}, \quad a, \phi, x_0 \text{ real constants.}
\]

The starting point for our analysis is a Riemann–Hilbert (RH) representation for the solution of (1.1) obtained via the inverse scattering transform formalism. The asymptotic formula (1.2) is derived by performing a Deift–Zhou [4] steepest descent analysis of this RH problem. The main novelty compared with the analogous derivation for the mKdV equation is that the Lax pair of (1.1) involves \( 3 \times 3 \) instead of \( 2 \times 2 \) matrices.

The inverse scattering problem for (1.1) was studied already by Sasa and Satsuma [11]. The initial-boundary value problem for (1.1) on the half-line was considered in [12]. Asymptotic formulas for the long-time behavior in the sector \( 0 < c_1 < x < c_2 \) were obtained in [6, 10].

Our main results are presented in Section 2. They are stated in the form of three theorems (Theorem 1-3) whose proofs are given in Section 4, 5, and 6, respectively. Section 3 recalls the Lax pair formulation of (1.1). The RH problem associated with the modified Painlevé II equation (1.3) is discussed in Appendix A. Appendix B considers an extension of this RH problem which is needed to obtain the higher order terms in (1.2).

2. Main results

Our first theorem shows how solutions of (1.1) can be constructed starting from an appropriate spectral function \( \rho_1(k) \). We let \( \mathcal{S}(\mathbb{R}) \) denote the Schwartz class of smooth (complex-valued) rapidly decaying functions.

**Theorem 1** (Construction of solutions). Suppose \( \rho_1 \in \mathcal{S}(\mathbb{R}) \). Define the \( 3 \times 3 \)-matrix valued jump matrix \( v(x, t, k) \) by

\[
v(x, t, k) = \begin{pmatrix}
I_{2 \times 2} & \rho^\dagger(k) e^{-2ikx+8ik^3t} \\
\rho(k) e^{2ikx-8ik^3t} & 1 + \rho(k) \rho^\dagger(k)
\end{pmatrix},
\]

where

\[
\rho(k) \triangleq \begin{pmatrix} \rho_1(k) & \rho_2(k) \end{pmatrix}, \quad \rho^\dagger(k) \triangleq \begin{pmatrix} \rho_1(k) \\ \rho_2(k) \end{pmatrix}, \quad \rho_2(k) \triangleq \rho_1(-k).
\]

Then the \( 3 \times 3 \)-matrix RH problem

\begin{itemize}
  \item m(x, t, k) is analytic for \( k \in \mathbb{C} \setminus \mathbb{R} \) and extends continuously to \( \mathbb{R} \) from the upper and lower half-planes;
  \item the boundary values \( m_{\pm}(x, t, k) = m(x, t, k \pm i0) \) obey the jump condition \( m_+(x, t, k) = m_-(x, t, k) v(x, t, k) \) for \( k \in \mathbb{R} \);
  \item \( m(x, t, k) = I + O(k^{-1}) \) as \( k \to \infty \);
\end{itemize}

has a unique solution for each \( (x, t) \in \mathbb{R}^2 \) and the limit \( \lim_{k \to \infty} (km(x, t, k))_{13} \) exists for each \( (x, t) \in \mathbb{R}^2 \). Moreover, the function \( u(x, t) \) defined by

\[
u(x, t) = 2i \lim_{k \to \infty} (km(x, t, k))_{13}
\]

is a smooth function of \( (x, t) \in \mathbb{R}^2 \) with rapid decay as \( |x| \to \infty \) which satisfies the Sasa–Satsuma equation (1.1) for \( (x, t) \in \mathbb{R}^2 \).

**Proof.** See Section 4. \qed
Our second theorem gives the long-time asymptotics of the solutions constructed in Theorem 1 in the sector $|x| \leq M t^{1/3}$.

**Theorem 2** (Asymptotics of constructed solutions). Under the assumptions of Theorem 1, the solution $u(x,t)$ of (1.1) defined in (2.2) satisfies the following asymptotic formula as $t \to \infty$:

$$u(x,t) = \sum_{j=1}^{N} u_j(y) + O(t^{-N+1}), \quad |x| \leq M t^{1/3},$$  \hspace{1cm} (2.3)

where

- The formula holds uniformly with respect to $x$ in the given range for any fixed $M > 0$ and $N \geq 1$.
- The variable $y$ is defined by
  $$y = \frac{x}{(3t)^{1/3}},$$
- $\{u_j(y)\}_{1}^{\infty}$ are smooth functions of $y \in \mathbb{R}$.
- The function $u_1(y)$ is given by
  $$u_1(y) = \frac{i u_P(y; s)}{3^{1/3} \sqrt{2}},$$  \hspace{1cm} (2.4)

where $s = \rho_1(0)$ and $u_P(y; s)$ denotes the smooth solution of the modified Painlevé II equation (1.3) corresponding to $s$ according to Lemma A.1. In particular, $u_1(y)$ has a constant phase, that is, arg $u_1$ is independent of $y$.

**Proof.** See Section 5. \hfill $\square$

**Remark 2.1** (Hierarchy of differential equations). Substituting the expansion (2.3) into (1.1) and identifying coefficients of powers of $t^{-1/3}$, we infer that the coefficients $\{u_j(y)\}_{1}^{\infty}$ in (2.3) satisfy a hierarchy of linear ordinary differential equations. The first two equations in this hierarchy are

$$u_1'' + yu_1' + u_1 = -3^{5/3}(3|u_1|^2 u_1' + u_1^2 u_1'),$$  \hspace{1cm} (2.5a)

$$u_2'' + yu_2' + 2u_2 = -3^{5/3}(3|u_1|^2 u_2' + u_1^2 u_2' + 3u_1 u_1' u_2 + 2u_1 u_1' u_2' + 3u_1 u_1' u_2' + 3u_1 u_1' u_2').$$  \hspace{1cm} (2.5b)

As expected, the function $u_1(y)$ in (2.4) satisfies the first of these equations. Indeed, if $u_1(y)$ is given by (2.4) where $u_P(y; s) \equiv u_P(y; s)$ satisfies (1.3), then (2.5a) reduces to the equation $|u_P(y)|^{3}(\text{arg } u_P)'(y) = 0$, which is satisfied for solutions $u_P$ of constant phase.

By applying the above two theorems in the case when $\rho_1(k)$ is the “reflection coefficient” corresponding to some given initial data $u_0(x)$, we obtain our third theorem, which establishes the asymptotic behavior of the solution of the initial-value problem for (1.1) in the sector $|x| \leq M t^{1/3}$. Before stating the theorem, we introduce some notation.

Given $u_0 \in \mathcal{S}(\mathbb{R})$, define $U_0(x)$ and $\Lambda$ by

$$U_0(x) = \begin{pmatrix} 0 & 0 & u_0(x) \\ 0 & 0 & u_0(x) \\ -u_0(x) & -u_0(x) & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Define the $3 \times 3$-matrix valued function $X(x,k)$ as the unique solution of the Volterra integral equation

$$X(x,k) = I - \int_{x}^{\infty} e^{i k(x' - x) \Lambda} (U_0 X)(x',k) dx', \quad x \in \mathbb{R}, \; k \in \mathbb{R},$$
where $\hat{\Lambda}$ acts on a matrix $A$ by $\hat{\Lambda}A = [\Lambda, A]$, i.e., $e^{\hat{\Lambda}}A = e^{\Lambda}Ae^{-\Lambda}$. Define the scattering matrix $s(k)$ by

$$s(k) = I - \int_{\mathbb{R}} e^{ikx\hat{\Lambda}}(UX)(x,k)dx, \quad k \in \mathbb{R}. \quad (2.6)$$

Then the “reflection coefficient” $\rho_1(k)$ is defined by

$$\rho_1(k) = \frac{s_{13}(k)}{s_{33}(k)}, \quad k \in \mathbb{R}. \quad (2.7)$$

We will see in Section 6 that the $(33)$ entry $s_{33}(k)$ of $s(k)$ has an analytic continuation to the upper half-plane. Possible zeros of $s_{33}(k)$ give rise to poles in the RH problem, see (6.8). For simplicity, we assume that no such poles are present (solitonless case).

**Theorem 3** (Asymptotics for initial value problem). Suppose $u_0 \in \mathcal{S}(\mathbb{R})$ and define $s(k)$ and $\rho_1(k)$ by (2.6) and (2.7). Suppose the $(33)$-entry $s_{33}(k)$ is nonzero for $\text{Im} k \geq 0$.

Then $\rho_1 \in \mathcal{S}(\mathbb{R})$ and the solution $u(x,t)$ of (1.1) defined in terms of $\rho_1(k)$ by (2.2) is the unique solution of the initial value problem for (1.1) with initial data $u(x,0) = u_0(x)$ and rapid decay as $|x| \to \infty$. Moreover, $u(x,t)$ obeys the asymptotic formula (2.3) as $t \to 0$.

**Proof.** See Section 6. \qed

**Remark 2.2** (Scattering transform). Let $S$ denote the subset of $\mathcal{S}(\mathbb{R})$ consisting of all functions $u_0(x)$ such that the associated scattering matrix $s(k)$ defined in (2.6) satisfies $s_{33}(k) \neq 0$ for $\text{Im} k \geq 0$. Theorem 3 shows that the map which takes $u_0(x)$ to $\rho_1(k)$ (the scattering transform) is a bijection from $S$ onto its image in $\mathcal{S}(\mathbb{R})$. The inverse of this map (the inverse scattering transform) is given by the construction of Theorem 1 for $t = 0$.

### 3. Lax pair

An essential ingredient in the proofs of Theorem 1-3 is the fact that equation (1.1) is the compatibility condition of the Lax pair equations [11]

$$\begin{cases}
\psi_x(x,t,k) = L(x,t,k)\psi(x,t,k), \\
\psi_t(x,t,k) = Z(x,t,k)\psi(x,t,k),
\end{cases} \quad (3.1)$$

where $k \in \mathbb{C}$ is the spectral parameter, $\psi(x,t,k)$ is a $3 \times 3$-matrix valued eigenfunction, the $3 \times 3$-matrix valued functions $L$ and $Z$ are defined by

$$L(x,t,k) = \mathcal{L}(k) + U(x,t), \quad Z(x,t,k) = \mathcal{Z}(k) + V(x,t,k) \quad (3.2)$$

where $\mathcal{L}(k) = -ik\Lambda$, $\mathcal{Z}(k) = 4ik^3\Lambda$,

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & \bar{u} \\ -\bar{u} & -u & 0 \end{pmatrix}, \quad (3.3)$$

$$V = k^2V^{(2)} + kV^{(1)} + V^{(0)},$$

$$V^{(2)} = -4U, \quad V^{(1)} = -2i \begin{pmatrix} |u|^2 u^2 u_x \\ \bar{u}^2 |\bar{u}|^2 \bar{u}_x \\ \bar{u}_x u_x - 2|u|^2 \end{pmatrix},$$

$$V^{(0)} = 4|u|^2U + U_{xx} - (\bar{u}\bar{u}_x - u_x\bar{u}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.4)$$
Note that \( U \) and \( V \) are rapidly decaying as \( |x| \to \infty \) if \( u \) is, and that \( L, Z \) obey the symmetries
\[
L(x, t, k) = -L^\dagger(x, t, \bar{k}), \quad Z(x, t, k) = -Z^\dagger(x, t, \bar{k}),
\]
where \( A^\dagger \) denotes the complex conjugate transpose of a matrix \( A \) and
\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

4. Proof of Theorem 1

Suppose \( \rho_1 \in \mathcal{S}(\mathbb{R}) \). The associated jump matrix \( v(x, t, k) \) defined in (2.1) obeys the symmetries
\[
v(x, t, k) = v^\dagger(x, t, \bar{k}) = A v(x, t, -k) A, \quad k \in \mathbb{R}.
\]
In particular, \( v \) is Hermitian and positive definite for each \( k \in \mathbb{R} \). Hence the result of Zhou [13] implies that there exists a vanishing lemma for the RH problem for \( m(x, t, k) \), i.e., the associated homogeneous RH problem has only the zero solution.

Defining the nilpotent matrices \( w^\pm(x, t, k) \) by
\[
w^- = \begin{pmatrix} 0_{2 \times 2} & 0_{2 \times 1} \\ \rho(k) e^{2ikx-8ik^2 t} & 0 \end{pmatrix}, \quad w^+ = \begin{pmatrix} 0_{2 \times 2} & \rho^\dagger(k) e^{-2ikx+8ik^2 t} \\ 0_{1 \times 2} & 0 \end{pmatrix},
\]
we can write \( v(x, t, k) = (v^-)^{-1} v^+ \), where \( v^\pm = I \pm w^\pm \). For \( h \in L^2(\mathbb{R}) \), we define the Cauchy transform \( Ch \) by
\[
(Ch)(z) = \frac{1}{2\pi i} \int_\mathbb{R} \frac{h(s)}{s-z} ds, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
and denote the nontangential boundary values of \( Cf \) from the left and right sides of \( \mathbb{R} \) by \( C_+ f \) and \( C_- f \), respectively. Then \( C_+ \) and \( C_- \) are bounded operators on \( L^2(\mathbb{R}) \) and \( C_+ - C_- = I \). Given two functions \( w^\pm \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), we define the operator \( C_w : L^2(\mathbb{R}) + L^\infty(\mathbb{R}) \to L^2(\mathbb{R}) \) by
\[
C_w(f) = C_+(fw^-) + C_-(fw^+).
\]
For each \( (x, t) \in \mathbb{R} \times [0, \infty) \), we have \( v^\pm \in C(\mathbb{R}) \) and \( v^\pm, (v^\pm)^{-1} \in I + L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). In view of the vanishing lemma, this implies (see e.g. [9, Theorem 5.10]) that \( I - C_w \) is an invertible bounded linear operator on \( L^2(\mathbb{R}) \), and that the \( 3 \times 3 \) matrix \( L^2 \)-RH problem for \( m \) has a unique solution \( m(x, t, k) \) for each \( (x, t, k) \in \mathbb{R}^2 \) given by
\[
m = I + C(\mu(w^+ + w^-)),
\]
where
\[
\mu = I + (I - C_w)^{-1} C_w I \in I + L^2(\mathbb{R}).
\]
The smoothness and decay of \( w^\pm \) together with the smooth dependence on \( (x, t) \) implies that \( m \) is a classical solution of the RH problem and that \( m \) admits an expansion
\[
m(x, t, k) = I + \frac{m_1(x, t)}{k} + \frac{m_2(x, t)}{k^2} + O(k^{-3}), \quad k \to \infty,
\]
where the coefficients \( m_j(x, t) \) are smooth functions of \( (x, t) \in \mathbb{R}^2 \) (see e.g. [8, Section 4] for details in a similar situation). Since \( \rho_1 \in \mathcal{S}(\mathbb{R}) \), an application of the Deift-Zhou steepest descent method [4] implies that \( m \) and the coefficients \( m_j \) have rapid decay as \( |x| \to \infty \) for each \( t \). In particular, the limit in (2.2) exists for each \( (x, t) \in \mathbb{R}^2 \) and \( u(x, t) = 2i(m_1(x, t))_{13} \) is a smooth function of \( (x, t) \in \mathbb{R}^2 \) with rapid decay as \( |x| \to \infty \).
Lemma 4.1. Define \( u(x,t) \) by (2.2). Then
\[
\begin{align*}
\begin{cases}
  m_x + ik[A, m] &= Um, \\
  m_t - 4ik^3[A, m] &= Vm.
\end{cases}
\end{align*}
\]

where \( U \) and \( V \) are defined in terms of \( u(x,t) \) by (3.3) and (3.4), respectively.

Proof. The symmetries (4.1) of \( v \) together with the uniqueness of the solution of the RH problem imply the following symmetries for \( m \):
\[
m(x,t,k) = m^\dagger(x,t,\bar{k})^{-1} = \overline{Am(x,t,\bar{k})A}.
\]

In particular, the coefficient \( m_1 \) in (4.4) satisfies
\[
m_1(x,t) = -m_1^\dagger(x,t) = -\overline{Am_1(x,t)A}.
\]

It follows that the definition (2.2) of \( u(x,t) \) can be expressed as
\[
U(x,t) = i[A, m_1(x,t)].
\]

Define the operator \( \mathcal{L} \) by
\[
\mathcal{L}m = m_x + ik[A, m] - Um.
\]

Substituting the expansion (4.4) into (4.8), we find
\[
\mathcal{L}m = i[A, m_1] - U + O(k^{-1}), \quad k \to \infty.
\]

In view of (4.7), this implies that \( \mathcal{L}m \) satisfies the following homogeneous RH problem:
\begin{itemize}
  \item \( \mathcal{L}m \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \) with continuous boundary values on \( \mathbb{R} \);
  \item \( (\mathcal{L}m)_+ = (\mathcal{L}m)_- v \) for \( k \in \mathbb{R} \);
  \item \( \mathcal{L}m = O(k^{-1}) \) as \( k \to \infty \).
\end{itemize}

Thus, by the vanishing lemma, \( \mathcal{L}m = 0 \). This proves the first equation in (4.5).

In order to prove the second equation in (4.5), we define the operator \( \mathcal{Z} \) by
\[
\mathcal{Z}m = m_t - 4ik^3[A, m] - k^2A(x,t)m - kB(x,t)m - C(x,t)m,
\]

where the matrices \( A(x,t), B(x,t) \) and \( C(x,t) \) are yet to be determined. Substituting the asymptotic expansion (4.4) into (4.9), we find
\[
\mathcal{Z}m = (-4i[A, m_1] - A)k^2 + (-4i[A, m_2] - Am_1 - B)k
\]
\[
+ (-4i[A, m_3] - Am_2 - Bm_1 - C) + O(k^{-1}), \quad k \to \infty.
\]

Thus, we define \( A, B, C \) by the equations
\[
\begin{align*}
A &= -4i[A, m_1], \\
B &= -4i[A, m_2] - Am_1, \\
C &= -4i[A, m_3] - Am_2 - Bm_1.
\end{align*}
\]

If we can show that \( A = V^{(2)}, B = V^{(1)}, \) and \( C = V^{(0)} \), it will follow from the vanishing lemma that \( \mathcal{Z}m = 0 \), which will prove the second equation in (4.5).

Comparing (4.7) and (4.10a), we see that \( A = -4U = V^{(2)} \), and then (4.10b) becomes
\[
B = 4Um_1 - 4i[A, m_2].
\]

The terms of order \( O(k^{-1}) \) in the asymptotic expansion of the equation \( \mathcal{L}m = 0 \) yield
\[
m_{1,x} + i[A, m_2] = Um_1.
\]

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\[
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B &= -4i[A, m_2] - Am_1, \\
C &= -4i[A, m_3] - Am_2 - Bm_1.
\end{align*}
\]

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\]

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\[
m_{1,x} + i[A, m_2] = Um_1.
\]
Comparing (4.12) with (4.11), it follows that \( B = 4m_{1,x} \). Given a \( 3 \times 3 \) matrix
\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix},
\]
let us write \( A = A^{(o)} + A^{(d)} \), where
\[
A^{(o)} = \begin{pmatrix}
0 & 0 & a_{13} \\
0 & 0 & a_{23} \\
a_{31} & a_{32} & 0
\end{pmatrix}, \quad A^{(d)} = \begin{pmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & 0 \\
0 & 0 & a_{33}
\end{pmatrix}.
\]

Equation (4.7) can then be written as
\[
m^{(o)}_1 = -i\Lambda U_x,
\]
and hence
\[
m^{(o)}_{1,x} = -i\Lambda U_{xx}.
\]
(4.13)

According to (4.12), we have
\[
m^{(d)}_1 = U m^{(o)}_1 = -i\Lambda U.
\]
(4.14)

Equations (4.13) and (4.14) imply
\[
B = 4m_{1,x} = -2i(\Lambda U_x + U\Lambda U) = \mathcal{V}^{(1)}.\]

It only remains to prove that \( C = \mathcal{V}^{(0)} \). The terms of order \( O(k^{-2}) \) in the expansion of the equation \( \mathbb{L}m = 0 \) yield
\[
m_{2,x} + i[\Lambda, m_3] = Um_2.
\]
(4.15)

It follows that \( C = 4m_{2,x} - Bm_1 \). On the other hand, (4.12) and (4.15) imply
\[
m^{(o)}_2 = -\frac{i}{2}\Lambda(U m^{(d)}_1 - m^{(o)}_{1,x}), \quad m^{(d)}_{2,x} = Um^{(o)}_2.
\]
We conclude that
\[
C = 4m_{2,x} - Bm_1 = -Bm^{(o)}_1 - \frac{i}{2}\Lambda UB + 2i\Lambda m^{(o)}_{1,xx} = \mathcal{V}^{(0)},
\]
which proves the lemma. \( \square \)

The compatibility condition of (4.5) shows that \( u(x,t) \) satisfies (1.1). The proof of Theorem 1 is complete.

5. Proof of Theorem 2

Let \( \rho_1 \in \mathcal{S}(\mathbb{R}) \) and let \( u(x,t) \) be the associated solution of (1.1) defined by (2.2). Our goal is to find the asymptotics of \( u(x,t) \) in the sector \( \mathcal{P} \) defined by
\[
\mathcal{P} = \{(x,t) \in \mathbb{R}^2 \mid |x| \leq Mt^{1/3}, t \geq 1\},
\]
(5.1)
where \( M > 0 \) is a constant. Let
\[
\mathcal{P}_{\geq} = \mathcal{P} \cap \{x \geq 0\} \quad \text{and} \quad \mathcal{P}_{\leq} = \mathcal{P} \cap \{x \leq 0\}
\]
denote the right and left halves of \( \mathcal{P} \). For conciseness, we will give the proof of the asymptotic formula (2.3) for \( (x,t) \in \mathcal{P}_{\geq} \); the case when \( (x,t) \in \mathcal{P}_{\leq} \) can be handled in a similar way but requires some (minor) changes in the arguments (see [2] for the required changes in the case of the mKdV equation).

The jump matrix \( v(x,t,k) \) defined in (2.1) involves the exponentials \( e^{\pm t\Phi(\zeta,k)} \), where \( \Phi(\zeta,k) \) is defined by
\[
\Phi(\zeta,k) \doteq 2ik\zeta - 8ik^3 \quad \text{with} \quad \zeta \doteq x/t.
\]
(5.2)
Suppose \((x, t) \in \mathcal{P}_\geq\). Then there are two real critical points (i.e., solutions of \(\partial \Phi / \partial k = 0\)) located at the points \(\pm k_0\), where (see Figure 1)

\[
 k_0 = \sqrt{\frac{x}{12t}} > 0.
\]

As \(t \to \infty\), the critical points \(\pm k_0\) approach 0 at least as fast as \(t^{-1/3}\), i.e., \(0 \leq k_0 \leq Ct^{-1/3}\).

5.1. Analytic approximation. We first decompose \(\rho = (\rho_1, \rho_2)\) into an analytic part \(\rho_a\) and a small remainder \(\rho_r\). Let \(N \geq 1\) be an integer. Let \(\Gamma^{(1)} \subset \mathbb{C}\) denote the contour \(\Gamma^{(1)} = \mathbb{R} \cup \Gamma_1^{(1)} \cup \Gamma_2^{(1)}\), where

\[
 \Gamma_1^{(1)} = \{k_0 + re^{\frac{2\pi i}{6}} | r \geq 0\} \cup \{-k_0 + re^{\frac{2\pi i}{6}} | r \geq 0\},
\]

\[
 \Gamma_2^{(1)} = \{k_0 + re^{\frac{-2\pi i}{6}} | r \geq 0\} \cup \{-k_0 + re^{\frac{-2\pi i}{6}} | r \geq 0\}.
\]

We orient \(\Gamma^{(1)}\) to the right and let \(V\) (resp. \(V^*\)) denote the open subset between \(\Gamma_1^{(1)}\) (resp. \(\Gamma_2^{(1)}\)) and the real line, see Figure 2.

**Lemma 5.1** (Analytic approximation). There exists a decomposition

\[
 \rho_1(k) = \rho_{1,a}(x, t, k) + \rho_{1,r}(x, t, k), \quad k \in (-\infty, -k_0) \cup (k_0, \infty),
\]

where the functions \(\rho_{1,a}\) and \(\rho_{1,r}\) have the following properties:

(a) For each \((x, t) \in \mathcal{P}_\geq\), \(\rho_{1,a}(x, t, k)\) is defined and continuous for \(k \in \bar{V}\) and analytic for \(k \in V\).

(b) The function \(\rho_{1,a}\) obeys the following estimates uniformly for \((x, t) \in \mathcal{P}_\geq\):

\[
 |\rho_{1,a}(x, t, k)| \leq \frac{C}{1 + |k|} e^{\frac{4}{7}|\text{Re}\Phi(\zeta, k)|}, \quad k \in \bar{V},
\]

and

\[
 |\rho_{1,a}(x, t, k) - \sum_{j=0}^{N} \frac{\rho_{1}^{(j)}(k_0)}{j!} (k - k_0)^j| \leq C |k - k_0|^{N+1} e^{\frac{4}{7}|\text{Re}\Phi(\zeta, k)|}, \quad k \in \bar{V}.
\]
Figure 2. The sets $V$ and $V^*$ and the contour $\Gamma^{(1)}$.

(c) The $L^1$ and $L^\infty$ norms of $\rho_{1,r}(x,t,\cdot)$ on $(-\infty,-k_0) \cup (k_0,\infty)$ are $O(t^{-N})$ as $t \to \infty$ uniformly for $(x,t) \in \mathcal{P}_\geq$.

Proof. See [2, Lemma 5.1].

Letting $\rho_{2,a}(k) \doteq \rho_{1,a}(-k)$ and $\rho_{2,r}(k) \doteq \rho_{1,r}(-k)$, we obtain a decomposition $\rho = \rho_a + \rho_r$ of $\rho$ by setting
$$
\rho_a(k) \doteq (\rho_{1,a}(k) \quad \rho_{2,a}(k)), \quad \rho_r(k) \doteq (\rho_{1,r}(k) \quad \rho_{2,r}(k)).
$$

5.2. Opening of the lenses. The jump matrix $v$ enjoys the factorization
$$
v(x,t,k) = \begin{pmatrix} I_{2\times 2} & 0_{2\times 1} \\ \rho e^{t\Phi} & 1 \end{pmatrix} \begin{pmatrix} I_{2\times 2} & \rho^+ e^{-t\Phi} \\ 0_{1\times 2} & 1 \end{pmatrix}. \tag{5.3}
$$

It follows that $m$ satisfies the RH problem in Theorem 1 if and only if the function $m^{(1)}$ defined by
$$
m^{(1)}(x,t,k) = \begin{cases} m(x,t,k) \begin{pmatrix} I_{2\times 2} & -\rho_a^+(x,t,k)e^{-t\Phi} \\ 0_{1\times 2} & 1 \end{pmatrix}, & k \in V, \\ m(x,t,k) \begin{pmatrix} I_{2\times 2} & 0_{2\times 1} \\ \rho_a(x,t,k)e^{t\Phi} & 1 \end{pmatrix}, & k \in V^*, \\ m(x,t,k), & \text{elsewhere}, \end{cases} \tag{5.4}
$$
satisfies the RH problem

- $m^{(1)}(x,t,\cdot)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ with continuous boundary values on $\Gamma \setminus \{\pm k_0\}$;
- $m^{(1)}_+ = m^{(1)}_- v^{(i)}$ for $k \in \Gamma \setminus \{\pm k_0\}$;
- $m^{(1)} = I + O(k^{-1})$ as $k \to \infty$;
- $m^{(1)} = O(1)$ as $k \to \pm k_0$;
where the jump matrix \( v^{(1)}(x, t, k) \) is given by

\[
v^{(1)} = \begin{cases} 
   v_1^{(1)}(x, t, k) = \begin{pmatrix} I_{2 \times 2} & \rho^*_k(x, t, \bar{k}) e^{-t\Phi} \\ 0_{1 \times 2} & 1 \end{pmatrix}, & k \in \Gamma^{(1)}_1, \\
   v_2^{(1)}(x, t, k) = \begin{pmatrix} I_{2 \times 2} & 0_{2 \times 1} \\ \rho a(x, t, k) e^{t\Phi} & 1 \end{pmatrix}, & k \in \Gamma^{(1)}_2, \\
   v_3^{(1)}(x, t, k) = \begin{pmatrix} I_{2 \times 2} & 0_{2 \times 1} \\ \rho e^{t\Phi} & 1 \end{pmatrix}, & k \in (-k_0, k_0), \\
   v_4^{(1)}(x, t, k) = \begin{pmatrix} I_{2 \times 2} & 0_{2 \times 1} \\ \rho e^{t\Phi} & 1 \end{pmatrix}, & k \in \mathbb{R} \setminus [-k_0, k_0].
\]

(5.5)

Note that \( v^{(1)} \) and \( m^{(1)} \) obey the same symmetries (4.1) and (4.6) as \( v \) and \( m \).

5.3. Local model. Let us introduce new variables \( y \) and \( z \) by

\[
y = \frac{x}{(3t)^{1/3}}, \quad z = (3t)^{1/3} k,
\]

(5.6)

so that

\[
t\Phi(\zeta, k) = 2i \left( yz - \frac{4z^3}{3} \right). \quad (5.7)
\]

Fix \( \epsilon > 0 \) and let \( D_{\epsilon}(0) = \{ k \in \mathbb{C} \mid |k| < \epsilon \} \). Let \( Z^\epsilon = (\Gamma^{(1)} \cap D_{\epsilon}(0)) \setminus ((-\infty, -k_0) \cup (k_0, \infty)) \), see Figure 3. Let \( Z \) denote the contour defined in (B.1) with \( z_0 = (3t)^{1/3} k_0 = \sqrt{3}/2 \geq 0 \). The map \( k \mapsto z \) maps \( Z^\epsilon \) onto \( Z \cap \{ |z| < (3t)^{1/3} \epsilon \} \) and we have \( Z^\epsilon = \bigcup_{j=1}^5 Z_j^\epsilon \), where \( Z_j^\epsilon \) denotes the inverse image of \( Z_j \cap \{ |z| < (3t)^{1/3} \epsilon \} \) under this map.

Let \( p \) denote the \( N \)th order Taylor polynomial of \( \rho \) at \( k = 0 \), i.e.,

\[
p(t, z) = \sum_{j=0}^{N} \frac{(z/j)!}{j!} \frac{\rho^{(j)}(0)}{j!} \frac{z^j}{t^{j/3}}.
\]

(5.8)
For large $t$ and fixed $z$, the jump matrices $\{v_j^{(1)}\}_4$ can be approximated as follows:

\[
\begin{align*}
v_1^{(1)} &\approx \begin{pmatrix} I_{2\times2} & p^\dagger (t,\bar{z}) e^{-2i(yz-4z^3/3)} \\ 0_{1\times2} & 1 \end{pmatrix}, \\
v_2^{(1)} &\approx \begin{pmatrix} I_{2\times2} & 0_{2\times1} \\ p(t,z) e^{2i(yz-4z^3/3)} & 1 \end{pmatrix}, \\
v_3^{(1)} &\approx \begin{pmatrix} I_{2\times2} & p^\dagger (t,\bar{z}) e^{-2i(yz-4z^3/3)} \\ p(t,z) e^{2i(yz-4z^3/3)} & 1 + p(t,z) p^\dagger (t,\bar{z}) \end{pmatrix}, \\
v_4^{(1)} &\approx I.
\end{align*}
\]

Thus we expect that $m^{(1)}$ approaches the solution $m_0(x,t,k)$ defined by

\[m_0(x,t,k) = m_Z(y,t,z_0)\] (5.10)

for large $t$, where $m_Z(y,t,z_0)$ is the solution of the model RH problem of Lemma B.1 with $z_0 = \sqrt{y}/2$ and $p(t,z)$ given by (5.8). If $(x,t) \in \mathcal{P}_\geq$, then $(y,t,z_0) \in \mathcal{P}$, where $\mathcal{P}$ is the parameter subset defined in (B.4). Thus Lemma B.1 ensures that $m_0$ is well-defined by (5.10). By (B.6), $m_0$ obeys the same symmetries (4.6) as $m$.

5.4. The solution $\hat{m}$. Fix $\epsilon > 0$. Let $\hat{\Gamma} = \Gamma^{(1)} \cup \partial D_\epsilon(0)$ and assume that the boundary of $D_\epsilon(0)$ is oriented counterclockwise, see Figure 4. Define $\hat{m}(x,t,k)$ by

\[
\hat{m} = \begin{cases} 
  m^{(1)} m_0^{-1}, & k \in D_\epsilon(0), \\
  m^{(1)}, & k \in \mathbb{C} \setminus D_\epsilon(0).
\end{cases}
\]

Then $\hat{m}$ satisfies a small-norm RH problem with jump $\hat{m}_+ = \hat{m}_- \hat{v}$ across $\hat{\Gamma}$, where the jump matrix $\hat{v}$ is given by

\[
\hat{v} = \begin{cases} 
  m_0 - v^{(1)} m_0^{-1}, & k \in \hat{\Gamma} \cap D_\epsilon(0), \\
  m_0^{-1}, & k \in \partial D_\epsilon(0), \\
  v^{(1)}, & k \in \hat{\Gamma} \setminus D_\epsilon(0).
\end{cases}
\]

Using Lemma B.1, the rest of the proof proceeds as in the case of the mKdV equation (see [2]) and we only give a brief outline. Let $\hat{C}$ be the Cauchy operator associated with...
\[ \hat{\Gamma} \text{ and let } \hat{\mathcal{C}}_a f = \hat{\mathcal{C}}_a(f \hat{w}). \] Then
\[ \hat{m}(x, t, k) = I + \frac{1}{2\pi i} \int_{\hat{\Gamma}} (\hat{\mu}(\hat{w})(x, t, s) \frac{ds}{s - k}), \tag{5.12} \]
where \( \hat{w} = \nu - I \) and \( \hat{\mu}(x, t, k) \in I + L^2(\hat{\Gamma}) \) is defined by \( \hat{\mu} = I + (I - \hat{\mathcal{C}}_a)^{-1}\hat{\mathcal{C}}_a I \). The expansion (B.5) of \( m^Z \) translates into expansions of \( \hat{w} \) and \( \hat{\mu} \) in powers of \( t^{-1/3} \) with coefficients which are functions of \( y \). It follows that there are smooth functions \( h_j(y) \) such that
\[ \lim_{k \to \infty} k(m(x, t, k) - I) = \lim_{k \to \infty} k(\hat{m}(x, t, k) - I) = -\frac{1}{2\pi i} \int_{\hat{\Gamma}} \hat{\mu}(x, t, k) \hat{w}(x, t, k) dk \]
\[ = -\sum_{j=1}^{N} \frac{h_j(y)}{t^{j/3}} + O(t^{-N+\frac{1}{3}}), \quad t \to \infty, \]
uniformly for \( (x, t) \in P_{\geq} \), where \( h_1(y) \) is the coefficient of \( t^{-1/3} \) in the large \( t \) expansion of
\[ \frac{1}{2\pi i} \int_{\partial D_j(0)} \hat{w} dk = -\frac{1}{2\pi i} \int_{\partial D_j(0)} \frac{m^Z(y)}{(3t)^{1/3}k} dk + O(t^{-2/3}) = -\frac{m^Z_{10}(y)}{(3t)^{1/3}} + O(t^{-2/3}). \]
Hence, \( u(x, t) = 2i \lim_{k \to \infty} (km(x, t, k))_{13} \) has an expansion of the form (2.3) with leading coefficient given by
\[ u_1(y) = 2i \frac{m^Z_{10}(y)}{3^{1/3}} = i \frac{u_p(y; s)}{3^{1/3}\sqrt{2}}. \]
This completes the proof of Theorem 2.

6. Proof of Theorem 3

Let \( u_0 \in S(\mathbb{R}) \) and suppose \( u(x, t) \) is a smooth solution of (1.1) with initial data \( u(x, 0) = u_0(x) \) and with rapid decay as \( |x| \to \infty \). If \( \psi \) satisfies the Lax pair equations (3.1), then the eigenfunction \( \Psi \) defined by \( \psi = \Psi e^{-i(kx-4k^3t)^{1/3}} \) satisfies
\[ \begin{cases} 
\Psi_x + i k \sigma_3 [\Lambda, \Psi] = U \Psi, \\
\Psi_t - 4ik^3 \sigma_3 [\Lambda, \Psi] = V \Psi.
\end{cases} \tag{6.1} \]
We define two solutions \( \{\Psi_j\}^2_j \) of (6.1) as the unique solutions of the integral equations
\[ \Psi_1(x, t, k) = I + \int_{-\infty}^{x} e^{ik(x-x')} \Lambda(\U \Psi_1(x', t', k)) dx', \tag{6.2a} \]
\[ \Psi_2(x, t, k) = I - \int_{x}^{\infty} e^{ik(x-x')} \Lambda(\U \Psi_2(x', t', k)) dx'. \tag{6.2b} \]
Let \( C_+ = \{ \text{Im } k \geq 0 \} \). The third columns of the matrix equations (6.2) involves the exponential \( e^{2ik(x-x')} \). Since the equations in (6.2) are Volterra integral equations, it follows that the third column vectors of \( \Psi_1 \) and \( \Psi_2 \) are bounded and analytic for \( k \in C_- \) and \( k \in C_+ \), respectively, with smooth extensions to \( \mathbb{R} \). Similar considerations apply to the first and second columns; thus
\[ \Psi_1(x, t, k) \text{ is bounded and analytic for } k \in (C_+, C_+, C_-), \]
\[ \Psi_2(x, t, k) \text{ is bounded and analytic for } k \in (C_-, C-, C_+), \]
where \( k \in (C_+, C_+, C_-) \) indicates that the first, second, and third columns of the equation are valid for \( k \) in \( C_+, C_+ \) and \( C_- \), respectively. Moreover, for each \( t \) and each
\(j \geq 0\), there are bounded functions \(f_-(x)\) and \(f_+(x)\) of \(x \in \mathbb{R}\) with rapid decay as \(x \to -\infty\) and \(x \to +\infty\), respectively, such that
\[
\left| \frac{\partial^j}{\partial k^j} (\Psi_1(x, t, k) - I) \right| \leq f_-(x), \quad k \in (\bar{C}_+, \bar{C}_+), \hspace{1em} x \in \mathbb{R}, \tag{6.3a}
\]
\[
\left| \frac{\partial^j}{\partial k^j} (\Psi_2(x, t, k) - I) \right| \leq f_+(x), \quad k \in (\bar{C}_-, \bar{C}_-), \hspace{1em} x \in \mathbb{R}. \tag{6.3b}
\]
As \(k \to \infty\), \(\Psi_1\) and \(\Psi_2\) have asymptotic expansions of the form
\[
\Psi_j(x, t, k) \sim I + \sum_{n=1}^{\infty} \frac{\Psi_j^{(n)}(x, t)}{k^n}, \hspace{1em} j = 1, 2, \tag{6.4}
\]
where the coefficients \(\Psi_j^{(n)}(x, t)\) are smooth bounded functions of \(x\) for each \(t\) and the expansion is valid uniformly for \(k \in (\bar{C}_+, \bar{C}_+)\) if \(j = 1\) and for \(k \in (\bar{C}_-, \bar{C}_-)\) if \(j = 2\). The above properties follow from an analysis of the Volterra equations (6.2); see e.g. [3] or Theorem 3.1 in [7] for similar proofs.

The symmetries in (3.5) imply that (cf. (4.6))
\[
\Psi_j(x, t, k) = \Psi_j^\dagger(x, t, \bar{k})^{-1} = \mathcal{A}\Psi_j(x, t, -\bar{k})\mathcal{A}, \hspace{1em} j = 1, 2. \tag{6.5}
\]
Moreover, the tracelessness of \(\mathbf{U}\) and \(\mathbf{V}\) shows that \(\det \Psi_j \equiv 1\) for \(j = 1, 2\). Indeed, the solution \(\psi_j\) of (3.1) given by \(\psi_j = \psi_j e^{-i(kx-4kt)\hat{A}}\) satisfies
\[
\begin{aligned}
(\det \psi_j)_x &= \text{tr} (\psi_{jx}\psi_j^{-1}) \det \psi_j = -ik \det \psi_j, \\
(\det \psi_j)_t &= \text{tr} (\psi_{jt}\psi_j^{-1}) \det \psi_j = 4ik^3 \det \psi_j.
\end{aligned}
\]
Hence \(\det \psi_j = c_j e^{-i(kx-4kt)}\) for some constant \(c_j \in \mathbb{C}\). Thus, for each \(j\), \(\det \Psi_j(x, t, k)\) is independent of \((x, t)\); evaluation at \(x = \pm \infty\) shows that \(\det \Psi_j(x, t, k) \equiv 1\).

Define the \(3 \times 3\) matrix valued spectral function \(s(k)\) by
\[
\Psi_2(x, t, k) = \Psi_1(x, t, k)e^{-i(kx-4kt)\hat{A}}s(k), \hspace{1em} x \in \mathbb{R}, \hspace{1em} k \in \mathbb{R}. \tag{6.6}
\]
Letting \(X(x, k) \doteq \Psi_2(x, 0, k)\), we see that \(s(k)\) can be expressed as in (2.6). Since \(\det \Psi_j \equiv 1\), (6.6) yields \(\det s \equiv 1\). By (6.5), we have
\[
\det s(k) = s^\dagger(\bar{k})^{-1} = \mathcal{A}s(-\bar{k})\mathcal{A}. \tag{6.7}
\]

Define \(\rho_1(k)\) in terms of \(s(k)\) by (2.7). By assumption, \(s_{33}(k)\) is nonzero for \(\text{Im } k \geq 0\).

**Lemma 6.1.** The reflection coefficient \(\rho_1(k)\) belongs to the Schwartz class \(\mathcal{S}(\mathbb{R})\).

**Proof.** The expression in (2.6) for the \((ij)\)th entry of \(s(k)\) involves the exponential factor \(e^{ikx(\lambda_i-\lambda_j)}\), where \(\lambda_1 = \lambda_2 = -\lambda_3 = 1\). It follows from the properties of \(\Psi_2\) and \(\mathbf{U}\) that \(s(k)\) is a smooth function of \(k \in \mathbb{R}\) and that the \((33)\)-entry \(s_{33}\) admits an analytic continuation to the upper half-plane. It also follows (by replacing \(X\) in (2.6) by its large \(k\) expansion and integrating by parts repeatedly in the resulting expression) that \(s_{13}, s_{23}, s_{31}, s_{32}\) have rapid decay as \(|k| \to \infty\). For the diagonal element \(s_{33}(k)\), the exponential factor is absent from the integral in (2.6), and substituting in the large \(k\) expansion of \(X\) we instead obtain
\[
s_{33}(k) \sim 1 + \sum_{n=1}^{\infty} \frac{s_{33}^{(n)}(k)}{k^n}, \hspace{1em} k \to \infty,
\]
uniformly for \(k \in \bar{C}_+\) for some coefficients \(\{s_{33}^{(n)}\} \subset \mathbb{C}\). The lemma follows. \(\square\)
Let \( s^*_ij(k) = \overline{s_{ij}(k)} \) denote the Schwartz conjugate of \( s_{ij}(k), \ i,j = 1, 2, 3 \). Let \([A]_j\) denote the \( j\)th column of a matrix \( A \).

**Lemma 6.2.** The function \( m(x,t,k) \) defined by
\[
m = \begin{cases} 
\left( [\Psi_1]_1, [\Psi_1]_2, \frac{[\Psi_2]_3}{s_{33}}, s_{22}[\Psi_2]_2 - s_{21}[\Psi_2]_2, -s_{12}[\Psi_2]_2 + s_{11}[\Psi_2]_2, [\Psi_1]_3 \right), & \text{Im} \ k > 0, \\
\left( s_{33}, s_{33} \right), & \text{Im} \ k < 0.
\end{cases}
\] (6.8)

satisfies the RH problem of Theorem 3 with \( \rho_1(k) \) given by (2.7).

**Proof.** We saw in the proof of Lemma 6.1 that \( s_{33} \) admits an analytic continuation to the upper half-plane. A similar argument shows that \( s_{11}, s_{12}, s_{21}, s_{22} \) admit analytic continuations to the lower half-plane. Hence \( m \) is well-defined by (6.8) and the properties of \( \Psi_1, \Psi_2 \) together with the assumption that \( s_{33}(k) \neq 0 \) for \( \text{Im} \ k \geq 0 \) imply that \( m(x,t,k) \) is analytic for \( k \in \mathbb{C} \setminus \mathbb{R} \) with continuous boundary values on \( \mathbb{R} \) from above and below. The jump \( m_+ = m_- v \) across \( \mathbb{R} \) is a consequence of a long but straightforward computation which uses (6.6), the symmetries (6.7) of \( s \), and the fact that \( \det s = 1 \). Finally, the normalization condition \( m(x,t,k) = I + O(k^{-1}) \) follows from the large \( k \) behavior of \( \Psi_1, \Psi_2, \) and \( s \). \( \square \)

In view of Theorem 2, the next lemma completes the proof of Theorem 3.

**Lemma 6.3.** The solution \( u(x,t) \) is given by (2.2).

**Proof.** Substituting the expansions (6.4) into (6.1), we find that
\[
u(x,t) = 2i \lim_{k \to \infty} \left( k \Psi_j(x,t,k) \right)_{13}, \quad (x,t) \in \mathbb{R}^2, \ j = 1, 2.
\] (6.9)
The lemma then follows from the definition (6.8) of \( m \) and the fact that \( s_{33}(k) = 1 + O(k^{-1}) \) as \( k \to \infty \). \( \square \)

**Remark 6.4** (Motivation for (6.8)). The form of the expression (6.8) for \( m \) can be motivated as follows. Let \( D_1 = \mathbb{C}_+ \) and \( D_2 = \mathbb{C}_- \). Define a \( 3 \times 3 \)-matrix valued solution \( M_n(x,t,k), n = 1, 2, \) of (6.1) for \( k \in D_n \) by the Fredholm integral equations
\[
(M_n)_{ij}(x,t,k) = \delta_{ij} + \int_{\gamma^n_{ij}} \left( e^{(x-x')\mathcal{L}(k)}(UM_n)(x',t,k) \right)_{ij} \ dx', \quad i,j = 1, 2, 3,
\] (6.10)
where the contours \( \gamma^n_{ij}, n = 1, 2, i,j = 1, 2, 3 \), are defined by
\[
\gamma^n_{ij} = \begin{cases} 
(-\infty,x), & \text{Re} \ l_i(k) \leq \text{Re} \ l_j(k), \\
(x,\infty), & \text{Re} \ l_i(k) \geq \text{Re} \ l_j(k)
\end{cases}
\]
for \( k \in D_n \), with \( \mathcal{L} = -ikA = \text{diag} (l_1, l_2, l_3) \), i.e.,
\[
\gamma^1 = \begin{pmatrix} \gamma_1 & \gamma_1 & \gamma_2 \\
\gamma_1 & \gamma_1 & \gamma_2 \\
\gamma_1 & \gamma_1 & \gamma_1
\end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \gamma_1 & \gamma_1 & \gamma_1 \\
\gamma_1 & \gamma_1 & \gamma_1 \\
\gamma_2 & \gamma_2 & \gamma_1
\end{pmatrix}.
\]
Solving the matrix factorization problem
\[
s(k) = S_n(k)T_n^{-1}(k), \quad k \in \bar{D}_n,
\] (6.11)
combined with the relations
\[
(S_n(k))_{ij} = \delta_{ij} \quad \text{if} \quad \gamma^n_{ij} = (-\infty,x),
\]
\[
(T_n(k))_{ij} = \delta_{ij} \quad \text{if} \quad \gamma^n_{ij} = (\infty,x),
\]
we infer that
\[ M_n(x,t,k) = \Psi_1(x,t,k)e^{-i(kx-4k^3t)\hat{\Lambda}}S_n(k) \]
\[ = \Psi_2(x,t,k)e^{-i(kx-4k^3t)\hat{\Lambda}}T_n(k), \quad k \in \bar{D}_n, \quad n = 1, 2, \]
where the spectral functions \( S_n(k) \) and \( T_n(k) \) are given in terms of the entries of \( s(k) \) by
\[
S_1(k) = \begin{pmatrix} 1 & 0 & s_{14} \\ 0 & 1 & s_{34} \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{s_{14}}{s_{33}} & -\frac{s_{24}}{s_{33}} & 1 \end{pmatrix},
\]
and
\[
T_1(k) = \begin{pmatrix} s_{11} & s_{21} & 0 \\ s_{12} & s_{22} & 0 \\ s_{13} & s_{23} & \frac{1}{s_{33}} \end{pmatrix}, \quad T_2(k) = \begin{pmatrix} \frac{s_{22}}{s_{33}} & \frac{s_{12}}{s_{33}} & s_{31} \\ \frac{s_{21}}{s_{33}} & \frac{s_{11}}{s_{33}} & s_{32} \\ 0 & 0 & s_{33} \end{pmatrix}.
\]
The expression (6.8) for \( m \) is obtained by taking \( m = M_1 \) for \( k \in D_1 \) and \( m = M_2 \) for \( k \in D_2 \).

**Appendix A. Modified Painlevé II RH problem**

Let
\[ P_1 = \{ re^{\frac{\pi i}{6}} \mid r \geq 0 \} \cup \{ re^{\frac{5\pi i}{6}} \mid r \geq 0 \}, \quad P_2 = \{ re^{-\frac{\pi i}{6}} \mid r \geq 0 \} \cup \{ re^{-\frac{5\pi i}{6}} \mid r \geq 0 \}, \]
and let \( P \) denote the contour \( P = P_1 \cup P_2 \) oriented as in Figure 5.

**Lemma A.1** (modified Painlevé II RH problem). Let \( s \in \mathbb{C} \) be a complex number and define the matrices \( S_1 \) and \( S_2 \) by
\[
S_1 = \begin{pmatrix} 1 & 0 & \bar{s} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & \bar{s} & 1 \end{pmatrix}.
\]

Then the RH problem
\[
\begin{align*}
& \cdot \quad m^P(y,\cdot) \text{ is analytic in } \mathbb{C} \setminus P \text{ with continuous boundary values on } P \setminus \{0\}; \\
& \cdot \quad m^P_z = m^P v^P \text{ for } z \in P \setminus \{0\}; \\
& \cdot \quad m^P = I + O(z^{-1}) \text{ as } z \to \infty; \\
& \cdot \quad m^P = O(1) \text{ as } z \to 0;
\end{align*}
\]
where
\[
v^P(y,z) = e^{-i(uz - \frac{4\pi^3}{3})\hat{\Lambda}}S_n, \quad z \in P_n, \quad n = 1, 2,
\]
has a unique solution \( m^P(y, z) \) for each \( y \in \mathbb{R} \). Moreover, there are smooth functions \( \{m_j^P(y)\}_j \) of \( y \in \mathbb{R} \) with decay as \( y \to -\infty \) such that, for each integer \( N \geq 0 \),

\[
m^P(y, z) = I + \sum_{j=1}^{N} \frac{m_j^P(y)}{z^j} + O(z^{-N-1}), \quad z \to \infty, \tag{A.1}
\]

uniformly for \( y \) in compact subsets of \( \mathbb{R} \) and for \( \arg z \in [0, 2\pi] \). The \((13)\)-entry of the leading coefficient \( m_1^P \) is given by

\[
(m_1^P(y))_{13} = \frac{u_P(y)}{2\sqrt{2}},
\]

where \( u_P(y) \equiv u_P(y; s) \) satisfies the modified Painlevé II equation \((1.3)\) and has constant phase, that is, \( \arg u_P \) is independent of \( y \).

**Proof.** The jump matrix \( v^P \) obeys the symmetries

\[
v^P(y, z) = (v^P)\dagger(y, \bar{z}) = [Av^P(y, -\bar{z})]A. \tag{A.2}
\]

We infer from the first of these symmetries that the RH problem for \( m^P \) admits a vanishing lemma, see [13, Theorem 9.3]. As in Section 4, this implies that there exists a unique solution \( m^P \) which admits an expansion of the form \((A.1)\). A Deift-Zhou steepest descent analysis shows that the coefficients \( m_j^P \) (and their \( y \)-derivatives) have exponential decay as \( y \to -\infty \).

Let \( \phi(y, z) = m^P(y, z)e^{-i(yz-\frac{4\alpha}{3})}\Lambda \). Then the function \( U(y, z) \) defined by

\[
U = \phi y\phi^{-1} = (m_y^P - izm^P\Lambda)(m^P)^{-1} \tag{A.3}
\]

is an entire function of \( z \); hence \( U(y, z) = U_0(y) + U_1(y)z \). Equation \((A.3)\) then becomes

\[
m_y^P - izm^P\Lambda = (U_0 + U_1 z)m^P \tag{A.4}
\]

Substituting the expansion \((A.1)\) into \((A.4)\), we find

\[
U_1 = -i\Lambda, \quad U_0 = i[\Lambda, m_1^P].
\]

Similarly,

\[
V = \phi(\phi^{-1} = (m_z^P - i(y - 4z^2)m^P\Lambda)(m^P)^{-1} \tag{A.5}
\]

is entire, and hence \( V = V_0 + V_1 z + V_2 z^2 \). Substituting \((A.1)\) into \((A.5)\), we find

\[
V_2 = 4i\Lambda, \quad V_1 = -4i[\Lambda, m_1^P], \quad V_0 = -V_1 m_1^P - 4i[\Lambda, m_2^P] - iy\Lambda
\]

Substituting \((A.1)\) into \((A.4)\), it follows that

\[
m_{1y}^P + i[\Lambda, m_2^P] = U_0 m_1^P,
\]

which gives

\[
V_0 = -V_1 m_1^P - 4(U_0 m_1^P - m_{1y}^P) - iy\Lambda
\]

We have shown that \( \phi \) obeys the Lax pair equations

\[
\begin{cases}
\phi_y = U\phi, \\
\phi_z = V\phi,
\end{cases} \quad y \in \mathbb{R}^2, \quad z \in \mathbb{C} \setminus P, \tag{A.6}
\]

where \( U \) and \( V \) are expressed in terms of \( m_1^P(y) \).

As a consequence of \((A.2)\), \( m^P \) obeys the symmetries

\[
m^P(y, z) = (m^P)\dagger(y, \bar{z})^{-1} = Am^P(y, -\bar{z})A. \tag{A.7}
\]
In particular, the leading coefficient $m_1^P$ satisfies
\[ m_1^P(y) = -m_1^P(y) = -\mathcal{A}m_1^P(y)\mathcal{A}. \]

Hence we can write
\[ m_1^P(y) = \begin{pmatrix} \psi_1(y) & \psi_2(y) & \psi_3(y) \\ -\overline{\psi_2(y)} & \psi_1(y) & -\psi_3(y) \\ -\overline{\psi_3(y)} & \psi_3(y) & \psi_4(y) \end{pmatrix}, \]

where \( \{\psi_j(y)\}_3^4 \) are complex-valued functions such that \( \psi_1(y), \psi_4(y) \in i\mathbb{R} \). The compatibility condition
\[ \mathcal{U}_z - V_y + \mathcal{U}V - \mathcal{V}U = 0 \]
of the Lax pair (A.6) is then equivalent to the following four equations:
\begin{align*}
\psi_1'' + 2i(\overline{\psi_3}\psi_3)' = 0, & \quad (A.8a) \\
\psi_2'' - 4i\overline{\psi_3}\psi_3' = 0, & \quad (A.8b) \\
\psi_3'' - 2i\overline{\psi_3}\psi_2' + \psi_3(y + 2i\psi_1' - 2i\psi_4') = 0, & \quad (A.8c) \\
\psi_4'' - 4i(\overline{\psi_3}\psi_3)' = 0. & \quad (A.8d)
\end{align*}

Since \( m_1^P(y) \) and its derivatives decay as \( y \to -\infty \), equations (A.8a), (A.8b), and (A.8d) yield
\[ \psi_1' = -2i|\psi_3|^2, \quad \psi_2' = 2i\psi_3^2, \quad \psi_4' = 4i|\psi_3|^2. \quad (A.9) \]

Substituting (A.9) into (A.8c), we find
\[ \overline{\psi_3} + y\psi_3 + 16\psi_3|\psi_3|^2 = 0. \quad (A.10) \]

Writing \( \psi_3(y) = r(y)e^{i\alpha(y)} \) with \( r(y), \alpha(y) \in \mathbb{R} \), (A.10) reduces to the pair of equations
\begin{align*}
r'' + 16r^3 + yr - (\alpha')^2r = 0, & \quad (A.11a) \\
2r'\alpha' + r\alpha'' = 0. & \quad (A.11b)
\end{align*}

Equation (A.11b) yields \( r^2\alpha' = c_0 \), where \( c_0 \in \mathbb{R} \) is a constant. Using this relation to eliminate \( \alpha' \) from (A.11a), we obtain
\[ r'' + 16r^3 + yr - c_0^2r^{-3} = 0. \]

The decay of \( \psi_3 \) and its derivatives as \( y \to -\infty \) shows that we must have \( c_0 = 0 \). Hence \( \alpha(y) = \arg \psi_3(y) \) is independent of \( y \). The lemma follows by setting \( u_P(y) = 2\sqrt{2}\psi_3(y) \).

\[ \square \]

### Appendix B. Model problem for Sector \( \mathcal{P}_{\geq} \)

Given \( z_0 \geq 0 \), let
\begin{align*}
Z_1 &= \{ z_0 + re^{\frac{i\pi}{4}} \mid 0 \leq r < \infty \}, & Z_2 &= \{ -z_0 + re^{\frac{5i\pi}{4}} \mid 0 \leq r < \infty \}, \\
Z_3 &= \{ -z_0 + re^{-\frac{5i\pi}{8}} \mid 0 \leq r < \infty \}, & Z_4 &= \{ z_0 + re^{-\frac{i\pi}{8}} \mid 0 \leq r < \infty \}, \\
Z_5 &= \{ r \mid -z_0 \leq r \leq z_0 \},
\end{align*}

and let \( Z \equiv Z(z_0) \) denote the contour \( Z = \bigcup_{j=1}^{5} Z_j \) oriented as in Figure 6. Suppose
\[ p_1(t, z) = s + \sum_{j=1}^{n} \frac{p_{1,j}z^j}{t^{j/3}}, \quad (B.2a) \]
is a polynomial in \(zt^{-1/3}\) with coefficients \(s \in \mathbb{C}\) and \(\{p_{1,j}\}_1^n \subset \mathbb{C}\) for some integer \(n \geq 0\). Define the row-vector valued function \(p(t, z)\) by

\[
p(t, z) = \left( p_1(t, z) \quad p_2(t, z) \right), \quad p_2(t, z) = \overline{p_1(t, -z)}.
\]

The long-time asymptotics in \(\mathcal{P}_\geq\) is related to the solution \(m^Z\) of the following family of RH problems parametrized by \(y \geq 0\), \(t \geq 0\), and \(z_0 \geq 0\):

- \(m^Z(y, t, z_0, \cdot)\) is analytic in \(\mathbb{C} \setminus Z\) with continuous boundary values on \(Z \setminus \{\pm z_0\}\);
- \(m^Z = m^Z v^Z\) for \(z \in Z \setminus \{\pm z_0\}\);
- \(m^Z = I + O(z^{-1})\) as \(z \to \infty\);
- \(m^Z = O(1)\) as \(z \to \pm z_0\);

where the jump matrix \(v^Z(y, t, z_0, z)\) is defined by

\[
v^Z(y, t, z_0, z) = \begin{cases} 
\left( \begin{array}{cc} I_{2 \times 2} & p^I(t, \bar{z})e^{-2i(yz - 4z^3/3)} \\
0_{1 \times 2} & 1 \end{array} \right), & z \in Z_1 \cup Z_2, \\
\left( \begin{array}{cc} I_{2 \times 2} & 0_{2 \times 1} \\
p(t, z)e^{2i(yz - 4z^3/3)} & 1 \end{array} \right), & z \in Z_3 \cup Z_4, \quad (B.3) \\
\left( \begin{array}{cc} I_{2 \times 2} & p^I(t, \bar{z})e^{-2i(yz - 4z^3/3)} \\
0_{2 \times 1} & 1 + p(t, z)p^I(t, \bar{z}) \end{array} \right), & z \in Z_5,
\end{cases}
\]

with \(p(t, z)\) given by (B.2). Define the parameter subset \(\mathbb{P} \subset \mathbb{R}^3\) by

\[
\mathbb{P} = \{(y, t, z_0) \in \mathbb{R}^3 \mid 0 \leq y \leq C_1, t \geq 1, \sqrt[4]{y}/2 \leq z_0 \leq C_2\}, \quad (B.4)
\]

where \(C_1, C_2 > 0\) are constants.

**Lemma B.1** (Model problem for Sector \(\mathcal{P}_\geq\)). Let \(p(t, z)\) be of the form (B.2) for some \(s \in \mathbb{C}\) and \(\{p_{1,j}\}_1^n \subset \mathbb{C}\).

(a) The RH problem for \(m^Z\) with jump matrix \(v^Z\) given by (B.3) has a unique solution \(m^Z(y, t, z_0, z)\) whenever \((y, t, z_0) \in \mathbb{P}\).

(b) There are smooth functions \(\{m^Z_j(y)\}\) such that, for each integer \(N \geq 1\),

\[
m^Z(y, t, z_0, z) = I + \sum_{j=1}^{\infty} \sum_{l=0}^{N} m^Z_{j,l}(y) z^{-l/3} + O\left(\frac{t^{-(N+1)/3}}{|z|} + \frac{t^{-1/3}}{|z|^{N+1}}\right), \quad z \to \infty, \quad (B.5)
\]

uniformly with respect to \(\arg z \in [0, 2\pi]\) and \((y, t, z_0) \in \mathbb{P}\).

(c) \(m^Z(y, t, z_0, z)\) is uniformly bounded for \(z \in \mathbb{C} \setminus Z\) and \((y, t, z_0) \in \mathbb{P}\).

(d) \(m^Z\) obeys the symmetries

\[
m^Z(y, t, z_0, z) = (m^Z)^I(y, t, z_0, \bar{z})^{-1}, \quad m^Z(y, t, z_0, z) = \tilde{A} m^Z(y, t, z_0, -\bar{z}) A. \quad (B.6)
\]
(e) The (13)-entry of the leading coefficient $m^Z_{10}$ is given by

$$(m^Z_{10}(y))_{13} = \frac{u_P(y; s)}{2\sqrt{2}},$$

where $u_P(y; s)$ is the smooth solution of the modified Painlevé II equation (1.3) associated with $s$ according to Lemma A.1.

Proof. We have

$$\text{Re} \left( -2i \left( yz - \frac{4z^3}{3} \right) \right) = -\frac{8r^3}{3} - 4\sqrt{3}r^2 z_0 + r \left( y - 4z^2_0 \right) \leq -\frac{8r^3}{3} - 4\sqrt{3}r^2 z_0,$$

for all $z = z_0 + re^{\frac{\pi i}{3}} \in Z_1$ with $r \geq 0$, $z_0 \geq 0$, and $0 \leq y \leq 4z^2_0$. Consequently,

$$|e^{2i(yz+\frac{4z^3}{3})}| \leq Ce^{-|z-z_0|^2(z_0+|z-z_0|)}, \quad z \in Z_1,$$

uniformly for $(y, t, z_0) \in \mathbb{P}$. Analogous estimates hold for $z \in Z_j$, $j = 2, 3, 4$, and $|e^{\pm 2(yz-4z^3/3)}| = 1$ for $z \in Z_5$, showing that $v^Z - I$ has uniform decay for large $z$.

The jump matrix $v^Z$ obeys the same symmetries (A.2) as $v^P$. In particular, $v^Z$ is Hermitian and positive definite on $Z \cap \mathbb{R}$ and satisfies $v^Z(y, t, z_0, z) = (v^Z)^\dagger(y, t, z_0, z)$ on $Z \setminus \mathbb{R}$. This implies the existence of a vanishing lemma [13] from which we deduce the unique existence of the solution $m^Z$. The symmetries (B.6) follow from the symmetries of $v^Z$.

Let $m^P(y, z) \equiv m^P(y, z; s)$ solve the same RH problem as $m^Z$ except that the polynomial $p(t, z)$ in the jump matrix (B.3) is replaced with its leading term $s$. Then (up to a trivial contour deformation) $m^P$ is the solution of Lemma A.1 corresponding to $s$. The remainder of the proof is analogous to the corresponding proof for the mKdV equation (see [2]) and consists of considering the RH problem satisfied by the quotient $m^Y(m^P)^{-1}$. \hfill \Box

Acknowledgement The authors acknowledge support from the Göran Gustafsson Foundation, the European Research Council, Grant Agreement No. 682537, the Swedish Research Council, Grant No. 2015-05430, and the National Science Foundation of China, Grant No. 11671095.

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