Closed-formed solutions of geometric albedos and phase curves of exoplanets for any reflection law

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Abstract

The albedo of a celestial body is the fraction of light reflected by it. Studying the albedos of the planets and moons of the Solar System dates back at least a century [1, 2, 3, 4, 5]. Of particular interest is the relationship between the albedo measured at superior conjunction (full phase), known as the “geometric albedo”, and the albedo considered over all phase angles, known as the “spherical albedo” [2, 6, 7]. Modern astronomical facilities enable the measurement of geometric albedos from visible/optical secondary eclipses [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the inference of the Bond albedo (spherical albedo measured over all wavelengths) from infrared phase curves [21, 22, 23, 24, 25] of transiting exoplanets. Determining the relationship between the geometric and spherical albedo or Bond albedo usually involves complex numerical calculations [26, 27, 28, 29, 30, 31, 32] and closed-form solutions are restricted to simple reflection laws [33, 34]. Here we report the discovery of closed-form solutions for the geometric albedo and integral phase function that apply to any law of reflection. The integral phase function is used to obtain the phase integral, which is the ratio of the spherical to the geometric albedos. The generality of the solutions stems from a judicious choice of the coordinate system in which to perform different parts of the derivation. The closed-formed solutions have profound implications for interpreting observations. The shape of a reflected light phase curve and the secondary eclipse depth may now be self-consistently inverted to retrieve fundamental physical parameters (single-scattering albedo, scattering asymmetry factor, cloud cover). Fully-Bayesian phase curve inversions for reflectance maps and simultaneous light curve detrending may now be performed, without the need for binning in time, due to the efficiency of computation. We demonstrate these innovations for the hot Jupiter Kepler-7b, inferring a revised geometric albedo of 0.12 ± 0.02, a Bond albedo of 0.18 ± 0.03 and a phase integral of 1.5 ± 0.1, which is consistent with isotropic scattering. The scattering asymmetry factor is 0.04±0.15, implying that the aerosols are small compared to the wavelengths probed by the Kepler space telescope. In the near future, one may use the closed-form solutions discovered here to extract fundamental parameters, across wavelength, from multi-wavelength phase curves of both gas-giant and terrestrial exoplanets measured by the James Webb Space Telescope.

Keywords: exoplanetary atmospheres, albedos, reflected light phase curves, phase integral, analytical formulae, data analysis and interpretation

Main Text

Named after the astronomer George Bond [1], the Bond albedo $A_B$ is the fraction of starlight that is reflected by a celestial body—either by its atmosphere or surface—over all viewing angles and wavelengths of light. For example, the Bond albedo of the Earth is about 0.3. The spherical albedo $A_s$
is considered over all angles, but only at one wavelength of light. The geometric albedo \( A_g \) is the albedo measured when an exoplanet is at superior conjunction or full phase—when the star is between the observer on Earth and the exoplanet. This occurs at an orbital phase angle \( \alpha \) of zero. The phase integral \( q \) relates the spherical and geometric albedos [2, 6],

\[
A_s = q A_g, 
\]

(1)

The phase integral is notoriously difficult to evaluate [7] and requires knowledge of the integral phase function \( \Psi \) [2, 6],

\[
q = 2 \int_0^\pi \Psi \sin \alpha \, d\alpha. 
\]

(2)

An exact, closed-form solution for \( \Psi \) exists for a Lambertian sphere [2, 6]—a perfectly reflective sphere that is equally bright regardless of viewing direction.

Insight is gained by considering the geometry of the problem [3, 6, 26], which may be formulated in two different coordinate systems (Figure 1). The observer-centric coordinate system defines a latitude \( \Theta \) and longitude \( \Phi \), such that the location on the exoplanet closest to Earth (i.e., the sub-Earth point) always sits at \( \Phi = 0 \). The observer on Earth views an exoplanet with observer-centric coordinate system 

\[
\text{observer} \rightarrow (\mu, \phi, \alpha) \rightarrow \text{exoplanet}.
\]

Figure 1: Schematic describing the geometry of the system. An observer on Earth views an exoplanet with observer-centric latitude \( \Theta \) and longitude \( \Phi \). At each point in its orbit, the exoplanet resides at a phase angle \( \alpha \). Starlight impinges upon the exoplanet at an angle \( \theta \) from its zenith. An infinitesimal area element on the exoplanet may be described by the exoplanet-centric coordinate system \( (\theta, \phi) \). Trios of these angles form triangles on the surface of the sphere, and may be related by the spherical law of cosines.

\[
\mu \equiv \cos \Theta, \\
\phi \equiv \cos \phi, \\
\alpha \equiv \cos \alpha.
\]

(3)

Starlight impinges upon each point at a zenith angle \( \theta \). The observer- and exoplanet-centric coordinate systems are generally not aligned in the same plane (Figure 1). In principle, calculations may be performed in either coordinate system and all six angles are mathematically related (see Methods).

All of the quantities of interest \( (A_g, A_s, \Psi, q) \) fundamentally involve the reflection coefficient [6, 7, 27, 28, 33],

\[
\rho = \frac{I_0}{I_s \mu_*}, 
\]

(3)

which relates the intensities of incident starlight \( I_s \) and reflected starlight \( I_0 \). We have defined \( \mu_* = \cos \theta_* \). By definition, a Lambertian sphere has \( \rho = 1 \) [6]. The geometric albedo may be evaluated in the exoplanet-centric coordinate system [6, 27],

\[
A_g = \frac{\int_0^\pi \rho_0 I_* \mu^2 \, d\mu}{\int_0^\pi I_* \mu \, d\mu} = 2 \int_0^1 \rho_0 \mu^2 \, d\mu, 
\]

(4)

where we have defined \( \mu = \cos \theta \). The reflection coefficient evaluated at zero phase angle \( \alpha = 0 \) is represented by \( \rho_0 \). The integral phase function may be written as \( \Psi = F/F_0 \) and is evaluated in the observer-centric coordinate system using [6]

\[
F = \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} \rho \mu \mu_* \cos \Theta \, d\Theta \, d\Phi, 
\]

(5)

where \( F_0 \) is \( F \) evaluated at \( \alpha = 0 \). Alternatively, the geometric albedo may be evaluated in the exoplanet-centric coordinate system as \( A_g = 2F_0/\pi \) [6].

The reflection coefficient is derived from the radiative transfer equation (see Methods). If the approximation is made that single scattering may be described by any reflection law, but multiple scattering of light occurs isotropically [35], then the reflection coefficient of a semi-infinite atmosphere is

\[
\rho = \frac{\omega}{4 \mu_* + \mu} \left( P_* - 1 + HH_* \right), 
\]

(6)

where \( H(\mu) \) is Chandrasekhar’s H-function [36] and \( H_* = H(\mu_*) \) (see Methods). The single-scattering albedo \( \omega \) is the fraction of light reflected during a single scattering event. The reflection law or scattering phase function \( P_* \) relates the incident direction of collimated starlight to the emergent direct-
Figure 2: (a) Geometric and Bond albedos and (b) phase integral for the isotropic, Lambertian, Rayleigh and Henyey-Greenstein reflection laws. Thick curves include multiple scattering, while the thin lines exclude it. For illustration, the scattering asymmetry factor of the Henyey-Greenstein reflection law is chosen to be $g = 0.508$ such that the phase integral (single scattering only) has a value of 3.5, which matches the ensemble averaged value of a population of hot Jupiters [24]. The Lambertian sphere has geometric and spherical albedos of $2/3$ and 1, respectively.

The integral phase function $\Psi$ may be derived using the stated reflection coefficient. In the observer-centric coordinate system, the two-dimensional integral over $\Theta$ and $\Phi$ may be evaluated as two independent, one-dimensional integrals. In the exoplanet-centric coordinate system, $\alpha$ is instead the third independent angle, which allows $P_\star(\alpha)$ to be taken out of the integrals involving $\Theta$ and $\Phi$. These insights render the derivation of $\Psi$ analytically tractable (see Methods).

Physically, $\Psi$ consists of three components. The first component $\Psi_S$ exhibits single-scattering behaviour. The second component $\Psi_L$ behaves like a Lambertian sphere [35]. The third component $\Psi_C$ is a correction term that is significant only when $\omega \sim \epsilon \sim 1$ (strong reflection). Our derivation yields

$$\Psi = \frac{12\rho_S \Psi_S + 16\rho_L \Psi_L + 9\rho_C \Psi_C}{12\rho_{S0} + 16\rho_L + 6\rho_C},$$

where the various coefficients are

$$\rho_S = P_\star - 1 + \frac{1}{4} (1 + \epsilon)^2 (2 - \epsilon)^2,$$

$$\rho_{S0} = P_0 - 1 + \frac{1}{4} (1 + \epsilon)^2 (2 - \epsilon)^2,$$

$$\rho_L = \frac{\epsilon}{2} (1 + \epsilon)^2 (2 - \epsilon),$$

$$\rho_C = \epsilon^2 (1 + \epsilon)^2.$$

The geometric albedo $A_g$ is defined as $A_g = \frac{\int_{0}^{\pi} \sin^2 \theta d \theta}{\int_{0}^{\pi} \sin \theta d \theta}$, which is equal to $A_g = \frac{1}{2} (1 + \epsilon)$ for the pure reflection limit ($\omega = 1$).

For the geometric albedo $A_g$, the Term involving $P_0$ accounts for the single scattering of light, while the other terms account for isotropic multiple scattering. For $\omega = 1$ and isotropic single scattering ($P_0 = 1$), one obtains $A_g = 17/24$, which is slightly higher than the $A_g = 2/3$ value of the Lambertian sphere by 1/24.
The three components of the integral phase function are

\[ \Psi_S = 1 - \frac{1}{2} \left[ \cos \left( \frac{\alpha}{2} \right) - \sec \left( \frac{\alpha}{2} \right) \right] \Psi_0, \]

\[ \Psi_L = \frac{1}{\pi} \left[ \sin \alpha + (\pi - \alpha) \cos \alpha \right], \]

\[ \Psi_C = -1 + \frac{5}{3} \cos^2 \left( \frac{\alpha}{2} \right) - \frac{1}{2} \tan \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\alpha}{2} \right) \Psi_0, \]

\[ \Psi_0 = \ln \left[ \frac{(1 + \alpha_-)(\alpha_+ - 1)}{(1 + \alpha_+)(1 - \alpha_-)} \right], \]

and we have defined \( \alpha_\pm = \sin(\alpha/2) \pm \cos(\alpha/2) \). For \(-\pi \leq \alpha \leq 0\), one needs to replace \( \alpha \) by \( |\alpha| \).

While these formulae may appear long and unwieldy, they constitute a closed-form solution of the integral phase function for any reflection law \( P_r \) that includes both single and (isotropic) multiple scattering. The phase integral \( q \) may be straightforwardly obtained by a numerical integration of \( \Psi \) over the phase angle. The spherical albedo is \( A_s = q A_g \) (see Methods for an alternative way of computing \( A_s \)). Figure 2 shows calculations of \( A_g \), \( A_s \) and \( q \) for the isotropic, Rayleigh and Henyey-Greenstein reflection laws. Multiple scattering is dominant in determining the values of the geometric and spherical albedos. For example, if only isotropic single scattering is considered, the maximum values of the geometric and spherical albedos are \( 1/8 \) and \( 2(1 - \ln 2)/3 \approx 0.2 \), respectively. If only single scattering is considered, the phase integral is independent of the single-scattering albedo \( \omega \). With multiple scattering included, \( q \) has a nontrivial dependence on \( \omega \), especially for the Henyey-Greenstein reflection law. We validated our calculations of \( A_g \) and \( A_s \) against those from previous studies [27, 33, 34] for various reflection laws (see Methods and Figure 5).

There are profound implications of the closed-form solutions of \( A_g \) and \( \Psi \) for fitting and interpreting the measured reflected light phase curves of exoplanets. Let the fluxes of the exoplanet and star observed at Earth be \( F_p \) and \( F_\star \), respectively. The reflected light phase curve is [7]

\[ \frac{F_p}{F_\star} = \left( \frac{R}{a} \right)^2 A_g \Psi, \]

where \( R \) is the radius of the exoplanet and \( a \) is its orbital semi-major axis. Figure 3 shows several examples of \( \Psi \). At low and intermediate values of the single-scattering albedo \( \omega \), the shapes of reflected light phase curves encode information on the reflection laws. As \( \omega \) approaches unity, the phase curve shape follows that of a Lambertian sphere [34]. A severe test of the theory is to measure reflected light phase curves of cloudfree exoplanetary atmospheres, where Rayleigh scattering of light is mediated by atoms and molecules and the phase curve is predicted to have a distinct shape if \( \omega < 1 \) (Figure 3).

We use the closed-form solutions to fit the phase curve of the hot Jupiter Kepler-7b measured by the
Kepler space telescope. Since the reflected light component of the phase curve has a peak offset [38], a model of an inhomogeneous atmosphere [39, 40] is needed to fit the data (see Methods). For tidally-locked hot Jupiters, an inhomogeneous atmosphere is a natural consequence of aerosols having finite condensation temperatures [41, 42]. It is assumed that the atmosphere has a single-scattering albedo $\omega_0$, the total single-scattering albedo $\omega$, the scattering asymmetry factor $g$, the local longitudes $x_1$, $x_2$ between which the single-scattering albedo is $\omega_0$, the power in the spherical harmonic mode describing the temperature map for thermal emission $C_{11}$; and posterior distributions for the derived parameters (blue labels), which can be computed from the fundamental parameters: the geometric albedo $A_g$, the Bond albedo $A_B$, the phase integral $q$, and the ratio of flux in reflected light at secondary eclipse $F_0$ to the maximum reflected light $F_{\text{max}}$.

Standard phase curve analyses perform detrending of the raw telescope measurements and fitting to a phase curve model as separate steps [15, 16, 20, 38]. Often after detrending, the photometry is binned in time to reduce data volume and increase the signal-to-noise of the weak phase curve signal. The shape of the phase curve, geometric albedo, and secondary eclipse depth are usually treated as independent fitting parameters. A key improvement from previous analyses is the ability to jointly and self-consistently fit the shape of the phase curve and secondary eclipse depth in terms of fundamental physical parameters ($\omega_0$, $\omega$, $g$, $x_1$, and $x_2$). A major implication is the ability to simultaneously detrend the phase curve in the time domain and fit for physical parameters at the native temporal resolution of the photometry—without the need to bin in time (see Methods). Performing both steps simultaneously and without binning [43] ensures that uncertainties in the detrending process are accurately propagated into the physical parameters.

Figure 4 shows the model fit to the Kepler-7b phase curve, which indicate that the atmosphere has a dark region with $\omega_0 = 0.04^{+0.05}_{-0.03}$ between the local longitudes of $x_1 = -12^{+8}_{-9}$ and $x_2 = 80^{+15}_{-11}$ degrees. The reflective regions of the atmosphere have a single-scattering albedo of $\omega = 0.20^{+0.07}_{-0.09}$ and a scattering asymmetry factor of $g = 0.04^{+0.15}_{-0.13}$ implying roughly isotropic scattering of starlight by aerosols with sizes smaller than the wavelengths probed by the Kepler bandpass (0.42–0.90 μm). It is important to note that the inference of $g \approx 0$ is not a consequence of assuming isotropic multi-
ple scattering as \( \omega \) is considerably less than unity. The geometric albedo is \( A_g = 0.12 \pm 0.02 \), which significantly revises the previously reported value of \( 0.35 \pm 0.02 \) [13, 38]. Using the inferred values of \( \omega_0 \), \( \omega \), \( q \), \( x_1 \) and \( x_2 \), the Bond albedo and phase integral are predicted to be \( A_B = 0.18 \pm 0.03 \) and \( q = 1.46^{+0.10}_{-0.06} \), respectively. The transition between the dark and reflective regions of the atmosphere occurs at \( \sim 2000 \) K, which we interpret as the condensation temperature of the aerosols. A broad class of aerosol compositions, including the olivine group, are ruled out for solar metallicity [44].

In the upcoming era of the James Webb Space Telescope, phase curves of exoplanets from 0.6 to 24 \( \mu \)m will be procured. The closed-form solutions presented here enable fundamental physical parameters to be extracted from multi-wavelength phase curves. Alternatively, one may construct phase curve models where the single-scattering albedo and scattering asymmetry factor are inserted, from first principles (using Mie theory), as functions of wavelength. Both approaches will allow phase curve models to be rigorously and efficiently confronted by data and open up new avenues of cloud/haze research in the atmospheres of exoplanets.

KH combined insights from the historical literature, formulated the problem, derived the equations, made all of the figures except Figure 4 and led the writing of the manuscript. BMM designed and engineered open-source software that implemented the derived equations, performed the analysis of Kepler-7b data, made Figure 4 and co-wrote the manuscript. DK participated in decisive discussions of the problem with KH. We acknowledge partial financial support from the Center for Space and Habitability (CSH), the PlanetS National Center of Competence in Research (NCCR) and an European Research Council (ERC) Consolidator Grant awarded to KH (number 771620). KH acknowledges a honorary professorship from the Department of Physics of the University of Warwick.

Methods

Spherical trigonometry

The six angles of the exoplanet- and observer-centric coordinate systems (Figure 1) are related by spherical trigonometry [6],

\[
\begin{align*}
\mu &= \cos \Theta \cos \Phi, \\
\mu_* &= \cos \Theta \cos (\alpha - \Phi), \\
\cos \alpha &= \mu \mu_* - \sqrt{(1 - \mu^2)(1 - \mu_*^2)} \cos \phi.
\end{align*}
\]

At zero phase angle (\( \alpha = 0 \)), one obtains \( \mu = \mu_* \) and \( \phi = \pi \).

Radiative transfer equation

Let the scattering phase function be represented by \( P \), which describes the mathematical relationship between the incident and emergent angles of radiation. It is chosen to have no physical units; an alternative choice is for it to have units of per unit solid angle. The fraction of light reflected during a single scattering event is given by the single-scattering albedo \( \omega \). With these definitions, the radiative transfer equation reads

\[
\frac{\partial I}{\partial \tau} = I - \frac{\omega}{4\pi} \int_0^{4\pi} I P \, d\Omega' - \frac{\omega I_*}{4} P_* e^{-\tau/\mu_*}. \quad (14)
\]

The optical depth \( \tau \) is the generalisation of length in radiative transfer and takes into account not just the spatial extent of the atmosphere, but how dense it is and the ability of its constituent atoms, molecules, ions and aerosols to absorb and scatter radiation. Since we are interested in calculating the intensity of reflected light, equation (14) ignores the contribution of thermal (blackbody) emission. The term involving the integral accounts for the multiple scattering of radiation, while the last term describes the collimated beam of incident starlight.

The integral is generally difficult to evaluate, because the integration is performed over all incident angles of radiation (with \( d\Omega' \) denoting the incident solid angle).

The scattering phase function is normalised such that [37, 47]

\[
\frac{1}{4\pi} \int_0^{4\pi} P \, d\Omega = \frac{1}{4\pi} \int_0^{4\pi} P \, d\Omega' = 1, \quad (15)
\]

with \( d\Omega \) denoting the emergent solid angle. It depends on both the incident and emergent angles: \( P(\mu', \mu, \phi) \). For the stellar beam, we have \( P_* = P(-\mu*, \mu, \phi) \); the minus sign accounts for the convention chosen that radiation travelling into the
atmospheric column, towards the center of the exoplanet, corresponds to \( \mu' < 0 \). At zero phase angle, the scattering phase function becomes \( P_0 = P(-\mu_*, \mu_*, \pi) \). The scattering asymmetry factor is the first moment of the scattering phase function [37, 47],

\[
g = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi P(\cos \beta \sin \beta \, d\beta \, d\phi).
\]

(16)

When \( P = 1 \) (isotropic scattering), one naturally obtains \( g = 0 \).

**Solutions of radiative transfer equation**

If one ignores the integral in equation (14), then only single scattering is considered. For a semi-infinite atmosphere, the intensity of reflected light is [35]

\[
I_{0,5} = \frac{\omega I_*}{4} \frac{\mu_*}{\mu_* + \mu} P_*.
\]

(17)

In the limit of isotropic single scattering \( (P_* = 1) \), one obtains the Lommel-Seeliger law of reflection [49, 50]. For isotropic multiple scattering, Chandrasekhar’s exact solution ignoring the stellar beam is [3, 36]

\[
I_{0,M} = \frac{\omega I_*}{4} \frac{\mu_*}{\mu_* + \mu} H_*
\]

(18)

where \( H_* = H(\mu_*) \) and \( H(\mu) \) is the Chandrasekhar H-function [36],

\[
H = 1 + \frac{1}{2} \omega H \int_0^1 \frac{H_*}{\mu_* + \mu} \, d\mu_*.
\]

(19)

An identical solution for \( I_{0,M} \) may be found by evaluating the integral in equation (14) using the two-stream solutions, but with the H-function taking on an approximate form (Hapke’s solution) [35, 51],

\[
H = \frac{1 + 2\mu}{1 + 2\gamma \mu}.
\]

(20)

where we have \( \gamma = \sqrt{1 - \omega} \). To combine the solutions, one needs to account for isotropic multiple scattering of the stellar beam, which yields [35]

\[
I_0 = \frac{\omega I_*}{4} \frac{\mu_*}{\mu_* + \mu} (P_* - 1 + HH_*)
\]

(21)
The derivation may be performed in either the exoplanet- or observer-centric coordinate systems. In the local longitude inhogeneous atmosphere in terms of the longitude $\Phi$ within the observer-centric coordinate system. In the local longitude $\Phi$ within the observer-centric coordinate system, the atmosphere has a single-scattering albedo of $\omega_0$ across $x_1 \leq x \leq x_2$. When this region is within view of the observer, it is highlighted with a thick blue line. Regions of enhanced reflectivity (with a single-scattering albedo of $\omega = \omega_0 + \omega'$) that are within the observer’s view are highlighted with thick red lines.

To evaluate the geometric albedo for isotropic multiple scattering, one uses Hapke’s linear approximation of the Chandrasekhar H-function [35],

$$H = (1 + \epsilon) \left( 1 - \frac{\epsilon}{2} + \epsilon \mu \right).$$

The geometric albedo associated with isotropic multiple scattering only is

$$A_g,M = \frac{\omega}{4} \int_0^1 \mu H^2 \, d\mu - \frac{\omega}{8}$$

The total geometric albedo is $A_g = A_{g,S} + A_{g,M}$.

There is an alternative way of deriving the spherical albedo. In Figure 1, the flux reflected by the infinitesimal area element with projected solid angle $d\mu d\Omega$ is $\int I_0 \mu \, d\Omega = \int P_s \mu \, d\Omega$. In the exoplanet-centric coordinate system, one has $d\Omega = d\mu \, d\phi$.

The incident flux of starlight is $\pi \mu_s I_*$. Therefore, the plane albedo associated with each atmospheric column is [6]

$$A = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \rho \mu \, d\mu \, d\phi,$$

where $0 \leq \mu \leq 1$ represents only the emergent (and not downwelling) radiation from the exoplanet. The fraction of incident starlight reflected by the entire exoplanet needs to consider all of the atmospheric columns. The reflected flux is $2\pi \int \mu_s \pi I_* A \, d\mu_s$, while the incident flux is $\pi^2 I_*$. The ratio of these two quantities yields the spherical albedo [6],

$$A_s = 2 \int_0^1 A_{\mu_s} \, d\mu_s.$$

The isotropic multiple scattering component of the spherical albedo is [35]

$$A_{s,M} = \frac{5\epsilon}{6} + \frac{\epsilon^2}{6} - \frac{2}{3} \omega (1 - \ln 2),$$

where Hapke’s linear approximation of the Chandrasekhar H-function has again been invoked. Finally, the spherical albedo including both single and isotropic multiple scattering is

$$A_s = \frac{q_S}{P_0} + A_{s,M},$$

where the single scattering phase integral $q_S$ is obtained from

$$q_S = \frac{2}{P_0} \int_0^\pi P_s \Psi_S \sin \alpha \, d\alpha.$$

For isotropic single scattering ($P_s = P_0 = 1$), we verified numerically that $q_S = 16(1 - \ln 2)/3 \approx 1.63655$ [2, 28]. For Rayleigh scattering, the single-scattering component of the spherical albedo may be derived exactly,

$$A_{s,R} = \frac{3\omega}{8} \int_0^1 \int_0^1 \frac{\mu \mu_s}{\mu_s + \mu} \left( 3 + 3 \mu_s^2 \mu^2 - \mu_s^2 - \mu^2 \right) \, d\mu \, d\mu_s = \frac{\omega}{35} (26 - 27 \ln 2),$$

which has a value of about 0.208144$\omega$. We verified numerically that $q_S \approx 0.208144\omega$.
In the limit of conservative ($\omega = 1$) Rayleigh single scattering and isotropic multiple scattering, we obtain

$$\frac{A_g}{A_s} = \frac{1295}{1808 - 176 \ln 2} \approx 0.768,$$

(31)

which is not inconsistent with the estimate of 0.751 for conservative Rayleigh scattering [27] that is the basis for the commonly quoted factor of 3/4 [29, 34]. The slight discrepancy of about 2.3% stems from the assumption of isotropic multiple scattering.

**Dimensionless Sobolev Fluxes**

Using Hapke’s linear approximation of the Chandrasekhar H-function [35], the reflection coefficient may be written as

$$\rho = \frac{\omega}{4} \left( \frac{\rho_S}{\mu + \rho_L + \rho_C} \right),$$

(32)

where the various coefficients have already been defined in equation (10). The dimensionless flux of Sobolev may be written as three terms,

$$F = F_S + F_L + F_C.$$

(33)

For a homogeneous atmosphere, the term that exhibits single scattering behaviour is

$$F_S = \frac{\omega \rho_S}{4} \int_0^{\pi/2} \cos^2 \Theta \, d\Theta$$

(34)

$$\times \int_{\alpha - \pi/2}^{\pi/2} \cos \Phi \cos (\alpha - \Phi) \, d\Phi.$$  

The Lambertian-sphere-like term is

$$F_L = \frac{\omega \rho_L}{4} \int_0^{\pi/2} \cos^3 \Theta \, d\Theta$$

(35)

$$\times \int_{\alpha - \pi/2}^{\pi/2} \cos \Phi \cos (\alpha - \Phi) \, d\Phi.$$  

The correction term is

$$F_C = \frac{\omega \rho_C}{4} \int_0^{\pi/2} \cos^4 \Theta \, d\Theta$$

(36)

$$\times \int_{\alpha - \pi/2}^{\pi/2} \cos^2 \Phi \cos^2 (\alpha - \Phi) \, d\Phi.$$  

Two key properties of all three terms are worth belabouring: the integrals over $\Theta$ and $\Phi$ may be evaluated independently, and closed-form solutions exist for them. By defining $F_0 = F(\alpha = 0)$, the integral phase function of a homogeneous atmosphere may be obtained using $\Psi = F/F_0$.

**Henyey-Greenstein Reflection Law**

The widely used Henyey-Greenstein scattering phase function is [52]

$$P_* = 1 - g^2 \left( 1 + g^2 - 2g \cos \beta \right)^{3/2},$$

(37)

where the scattering angle $\beta$ is [37, 47, 48]

$$\cos \beta = -\mu \mu_* + \sqrt{(1 - \mu^2)(1 - \mu_*^2)} \cos \phi.$$  

(38)

The scattering asymmetry factor $g$ quantifies the asymmetry of scattering: $g = -1$, 0 and 1 correspond to purely reverse, isotropic and purely forward scattering, respectively. The scattering phase function at zero phase angle ($\cos \beta = -\cos \alpha = -1$) is

$$P_0 = \frac{1 - g^2}{(1 + g)\sqrt{1 - \mu^2}}.$$  

(39)

**Validation**

To validate our calculations of $A_g$ and $A_s$, we compare them against the numerical calculations of Dlugach & Yanovitskij [27], who tabulated the geometric and spherical albedos from $\omega = 0.7$ to 1 (Figure 5). For isotropic and Rayleigh scattering, the discrepancies associated with $A_g$ and $A_s$ are typically $\sim 0.1$–1%. For the Henyey-Greenstein law of reflection, the discrepancies associated with $A_g$ and $A_s$ are $\sim 10\%$ for $g = 0.5$. The larger discrepancies indicate that the assumption of isotropic multiple scattering starts to break down when scattering becomes strongly asymmetric. For isotropic scattering and non-conservative ($\omega \neq 1$) Rayleigh scattering, we perform further comparisons of the spherical albedo to the empirical fitting functions of van de Hulst [33] and Madhusudhan & Burrows [34], respectively. The differences between these fitting functions and our calculations are negligible ($< 1\%$), as shown in Figure 5.
Inhomogeneous atmosphere

Let the local longitude of the exoplanet be \( x \). Since it lies in the same plane as \( \alpha \), it is trivially related to the phase angle by \( x = \Phi - \alpha \) [39]. It is assumed that the region from \( x_1 \leq x \leq x_2 \) has a single-scattering albedo of \( \omega_0 \), while all of the other regions have an enhanced single-scattering albedo of \( \omega_0 + \omega' \) [39]. The dimensionless Sobolev fluxes are generalised to

\[
F_S = \frac{\pi}{16} (\omega_0 \rho_S \Psi_S + \omega' \rho_S' \Psi_S'),
\]

\[
F_L = \frac{\pi}{12} (\omega_0 \rho_L \Psi_L + \omega' \rho_L' \Psi_L),
\]

\[
F_C = \frac{3\pi}{64} (\omega_0 \rho_C \Psi_C + \omega' \rho_C' \Psi_C'),
\]

where \( \rho_S = \rho_S(\omega_0), \rho_L = \rho_L(\omega_0), \rho_C = \rho_C(\omega_0), \rho_S' = \rho_S(\omega'), \rho_L' = \rho_L(\omega'), \rho_C' = \rho_C(\omega') \). The coefficients \( \rho_S, \rho_L, \rho_C \) are given by equation (10), while \( \Psi_S, \Psi_L, \Psi_C \) are given by equation (11). The total dimensionless Sobolev flux is \( F = F_S + F_L + F_C \), while \( F_0 = F(\alpha = 0) \) and \( A_g = 2F_0/\pi \). The integral phase function, \( \Psi = F/F_0 \), is generally not symmetric about \( \alpha = 0 \) and the generalised expression for the phase integral does not assume this symmetry [39],

\[
q = \int_{-\pi}^{\pi} \Psi \sin |\alpha| \ d\alpha. \tag{41}
\]

To compute \( \Psi_S', \Psi_L, \) and \( \Psi_C' \), one needs to determine the regions of the inhomogeneous atmosphere that are within view of the observer [39] in the observer-centric coordinate system (Figure 6). If we write \( \Psi'_i \) for short (with the index \( i = S, L, \) or \( C \)), then we can express \( \Psi'_i \) in terms of the following functions,

\[
I_S = -\frac{1}{2} \sec \left( \frac{\alpha}{2} \right) \left[ \sin \left( \frac{\alpha}{2} - \Phi \right) + (\cos \alpha - 1) \Psi'_0 \right],
\]

\[
I_L = \frac{1}{\pi} \left[ \Phi \cos \alpha - \frac{1}{2} \sin(\alpha - 2\Phi) \right],
\]

\[
I_C = -\frac{1}{24} \sec \left( \frac{\alpha}{2} \right) \left[ -3 \sin \left( \frac{\alpha}{2} - \Phi \right) + \sin \left( \frac{3\alpha}{2} - 3\Phi \right) + 6 \sin \left( \frac{3\alpha}{2} - \Phi \right) - 6 \sin \left( \frac{\alpha}{2} + \Phi \right) + 24 \sin \left( \frac{\alpha}{2} \right) \Psi'_0 \right],
\]

\[
\Psi'_0 = \tanh^{-1} \left[ \sec \left( \frac{\Phi}{2} \right) \sin \left( \frac{\alpha}{2} - \Phi \right) \right]. \tag{42}
\]

Table 1 states explicit formulae for \( \Psi'_i \) in terms of \( I_i \). Corresponding to Figure 6, one has to construct \( \Psi'_i \) according to the correct sextant, which is expressed as a set of inequalities of \( \Phi \). In comparison to a homogeneous atmosphere, an inhomogeneous atmosphere has \( \omega', x_1 \) and \( x_2 \) as additional parameters.

For an inhomogeneous atmosphere, the reflected light phase curve is still given by equation (12). The expression relating the secondary eclipse depth \( D \) to the geometric albedo \( A_g \) is [7]

\[
D = A_g \left( \frac{R}{a} \right)^2. \tag{43}
\]

Another way to understand the preceding expression is to realise that the flux of reflected starlight at secondary eclipse is \( 2F_0 I_* \) (the factor of 2 accounts for the fact that the expression for \( F \) assumes latitudinal symmetry), which is divided by the flux from a Lambertian disk \( \pi I_* \) to obtain \( A_g \).

Fit to data

We simultaneously fit the long-cadence fluxes of the hot Jupiter-host Kepler-7 observed by the Kepler mission, spanning the times corresponding to the phase curve and the secondary eclipse. We retrieve the Pre-search Data Conditioning Simple Aperture Photometry (PDCSAP) flux light curve from MAST with lightkurve [53], and mask 3σ outliers in flux. For the purposes of this analysis, we have masked out transits. There are 60,934 remaining fluxes at 30 minute cadence from Quarters 0-17.

The light curve model consists of four components: reflected light, thermal emission, the secondary eclipse and a Gaussian process. All components are fit simultaneously to the entire out-of-transit light curve.

The reflected light component is modelled with an inhomogeneous atmosphere and the Henyey-Greenstein law of reflection. The planet has two hemispheres defined by the single-scattering albedos \( 0 \leq \omega_0 \) and \( \omega = \omega_0 + \omega' \leq 1 \), with one global scattering asymmetry factor \( g \) that follows a Gaussian prior \( N(0, 0.25) \), where the darker surface is bounded by the local longitudes \( -\pi/2 < x_1 < x_2 < \pi/2 \). We note that the parameter pair \( (\omega_0, \omega) \) shares similar constraints to the quadratic limb-darkening parameters [54] and thus adapt the
The medical harmonic power in the temperature map in sion model requires two free parameters: the spherical harmonic basis \[55\]. The thermal emitterising the temperature map with a generalised efficient, uninformative sampling.

The mid-transit time, period, impact parameter, planetary radius and stellar radius \[16\], as well as the stellar density \[60\], are fixed to their previously reported values. The ratio of the exoplanet flux to the stellar flux is assumed to drop to zero during secondary eclipse. We account for the 30 minute exposure times by super-sampling the eclipse curve.

The reflected light and thermal emission phase curve model components are implemented in \texttt{exoplanet ecoss-}

| Sextant | Integration Angles | \(\Psi_i'\) |
|---------|--------------------|-------------|
| \(-\frac{\pi}{2} \leq \alpha - \frac{\pi}{4} \leq \frac{\pi}{2} \leq \alpha + x_1 \leq \alpha + x_2\) | \(\Phi_2 = \frac{\pi}{2}, \Phi_1 = \alpha - \frac{\pi}{2}\) | \(I_i(\Phi_2) - I_i(\Phi_1)\) |
| \(-\frac{\pi}{2} \leq \alpha - \frac{\pi}{4} \leq \alpha + x_1 \leq \frac{\pi}{2} \leq \alpha + x_2\) | \(\Phi_2 = \alpha + x_1, \Phi_1 = \alpha - \frac{\pi}{2}\) | \(I_i(\Phi_2) - I_i(\Phi_1)\) |
| \(\alpha + x_1 \leq \alpha + x_2 \leq \frac{\pi}{2}\) | \(\Phi_2 = \frac{\pi}{2}, \Phi_1 = \alpha + x_2\) | \(I_i(\Phi_2) - I_i(\Phi_1)\) |
| \(\alpha + x_1 \leq \alpha + x_2 \leq \frac{\pi}{2}\) | \(\Phi_2 = \alpha + \frac{\pi}{2}, \Phi_1 = -\frac{\pi}{2}\) | \(I_i(\Phi_2) - I_i(\Phi_1)\) |
| \(-\frac{\pi}{2} \leq \alpha + x_1 \leq \alpha + x_2 \leq \frac{\pi}{2}\) | \(\Phi_2 = \alpha + x_1, \Phi_1 = \alpha - \frac{\pi}{2}\) | \(I_i(\Phi_2) - I_i(\Phi_1)\) |

Note: The index \(i\) refers to the components S, L and C.

The thermal emission is computed by parameterising the temperature map with a generalised spherical harmonic basis \[55\]. The thermal emission model requires two free parameters: the spherical harmonic power in the temperature map in the \(m = \ell = 1\) mode \(C_{\ell1}\), and \(f = f_0(1 - A_B)\). The heat redistribution parameter \(f_0\) is bounded between 1/4 (full redistribution) and 2/3 (no redistribution) \[22\]. The Bond albedo is given by \(A_B = qA_g\), where \(A_g\) is the geometric albedo integrated over the Kepler bandpass. Thus, \(A_B\) is self-consistently derived at each step in the chains from the other phase curve parameters. Since infrared phase curves have not been measured for Kepler-7 b, an empirical relationship between the peak offset of the thermal component and the irradiation temperature is used \[56, 57\], which yields an expected peak offset consistent with zero; we fix the orbital phase of maximum thermal emission to \(\alpha = 0\).

The secondary eclipse is modelled with the \texttt{exoplanet} package with no limb-darkening \[58, 59\]. The reflected light and thermal emission phase curve model components are implemented in \texttt{exoplanet ecoss-}.
Re-parameterisation

For the Henyey-Greenstein reflection law, one may treat $\omega_0$, $\omega'$, $x_1$, $x_2$ and $A_g$ as the independent parameters and express $g$ as a dependent parameter,

$$g = -\frac{(2C_3 + 1) + \sqrt{1 + 8C_3^2}}{2C_3},$$

where the various quantities are

$$C_3 = \frac{16\pi A_g - 32C_1 - 2\pi\omega_0 C + \pi\omega'C_2 C_0}{2\pi\omega_0 + \pi\omega'C_2},$$

$$C_2 = 2 + \sin x_1 - \sin x_2,$$

$$C_1 = \frac{\omega_0 \rho L_C \pi}{12} + \frac{\rho C_0}{12} \left[ x_1 - x_2 + \pi \right] + \frac{\pi \omega_0 \rho C_0}{32}$$

$$+ \frac{3\pi \rho C_0}{64} \left[ \frac{2}{3} + \frac{3}{8} \left( \sin x_1 - \sin x_2 \right) \right]$$

$$+ \frac{1}{24} \left( \sin 3x_1 - \sin 3x_2 \right),$$

$$C = -1 + \frac{1}{4} \left( 1 + \epsilon' \right)^2 \left( 2 - \epsilon \right)^2,$$

$$C' = -1 + \frac{1}{4} \left( 1 + \epsilon' \right)^2 \left( 2 - \epsilon' \right)^2,$$

where $\epsilon' = (1 - \gamma')/(1 + \gamma')$ and $\gamma' = \sqrt{1 - \omega'}$.

### Table 2: Maximum-likelihood parameters inferred from the Kepler light curve of Kepler-7 b fit with a model consisting of the inhomogeneous atmosphere in reflected light, thermal emission, a secondary eclipse and a Gaussian process.

| Parameter                                      | Value                     |
|------------------------------------------------|---------------------------|
| Geometric albedo, $A_g$                        | 0.123$^{+0.020}_{-0.019}$ |
| Bond albedo, $A_B$                             | 0.178$^{+0.033}_{-0.027}$ |
| Single-scattering albedo, $\omega_0$           | 0.044$^{+0.046}_{-0.030}$ |
| Single-scattering albedo, $\omega'$            | 0.200$^{+0.072}_{-0.085}$ |
| Western limit, $x_1$ (degree)                  | $-11.8^{+7.9}_{-8.8}$     |
| Eastern limit, $x_2$ (degree)                  | $80.3^{+6.7}_{-10.6}$     |
| Scattering asymmetry factor, $g$               | 0.04$^{+0.15}_{-0.13}$    |
| Phase integral, $q$                            | 1.461$^{+0.108}_{-0.082}$ |
| Heat redistribution parameter, $f_0$           | 0.650$^{+0.012}_{-0.022}$ |
| Spherical harmonic power, $C_{11}$             | 0.0155$^{+0.0076}_{-0.0079}$ |
| Reflected flux at eclipse, $F_0/F_{\text{max}}$| 0.846$^{+0.047}_{-0.054}$ |

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