STABLE CATEGORIES OF SPHERICAL MODULES AND TORSIONFREE MODULES

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Abstract. Auslander and Bridger introduced the notions of \( n \)-spherical modules and \( n \)-torsionfree modules. In this paper, we construct an equivalence between the stable category of \( n \)-spherical modules and the category of modules of grade at least \( n \), and provide its Gorenstein analogue. As an application, we prove that if \( R \) is a Gorenstein local ring of Krull dimension \( d > 0 \), then there exists a stable equivalence between the category of \((d−1)\)-torsionfree \( R \)-modules and the category of \( d \)-spherical modules relative to the local cohomology functor.

1. Introduction

Throughout this paper, let \( R \) be a two-sided noetherian ring. All subcategories are assumed to be strictly full. Denote by \( \mod R \) the category of finitely generated (right) \( R \)-modules.

Auslander and Bridger \cite{1} introduced the notion of \( n \)-spherical modules for each positive integer \( n \): a finitely generated \( R \)-module \( M \) is called \( n \)-spherical if \( \text{Ext}^i_R(M, R) = 0 \) for all \( 1 \leq i \leq n−1 \) and \( M \) has projective dimension at most \( n \). Note that when this is the case, \( \text{Ext}^i_R(M, R) = 0 \) for all \( i \neq 0, n \). Auslander and Bridger found various important properties related to \( n \)-spherical modules. For example, the spherical filtration theorem they proved asserts that any finitely generated module \( M \) satisfying a certain grade condition has a filtration

\[
M_m \subset M_{m-1} \subset \cdots \subset M_1 \subset M_0 = M \oplus P,
\]

where \( P \) is projective, such that \( M_{j-1}/M_j \) is \( j \)-spherical for all \( 1 \leq j \leq m \). Recently, Huang \cite{6} proved the dual version of the spherical filtration theorem.

In this paper, we study the stable category of \( n \)-spherical modules. Moreover, we introduce the notion of \( n \)-G-spherical modules by replacing projective dimension in the definition of \( n \)-spherical modules with Gorenstein dimension, and give similar results for the stable category of \( n \)-G-spherical modules. These are related to the category of modules with high grade and the category of totally reflexive modules. To be precise, the following theorem holds.

**Theorem 1.1.** Let \( n \) be a positive integer. Consider the following subcategories of \( \mod R \).

\[
\begin{align*}
\text{Sph}_n(R) &= \{ M \in \mod R \mid M \text{ is } n\text{-spherical} \}, \\
\text{Grd}_n(R) &= \{ M \in \mod R \mid \text{grade}_R M \geq n \}, \\
\text{Sph}^G_n(R) &= \{ M \in \mod R \mid M \text{ is } n\text{-G-spherical} \}, \\
\text{Ref}^T(R) &= \{ M \in \mod R \mid M \text{ is totally reflexive} \}.
\end{align*}
\]

One then has the equivalences

\[
\begin{align*}
\text{Sph}_n(R) &\overset{\text{Ext}^n(\cdot, R)}{\cong} \text{Grd}_n(R^{op}), \\
\text{Sph}^G_n(R) &\overset{\text{Tr} \Omega^{n-1}}{\cong} \text{Grd}_n(R^{op}) \ast \text{Ref}^T(R^{op}).
\end{align*}
\]

Here, \( \Omega(\cdot) \) and \( \text{Tr}(\cdot) \) respectively stand for the syzygy and the (Auslander) transpose, while the stable category of a subcategory \( \mathcal{X} \) of \( \mod R \) is denoted by \( \mathcal{X}^{s} \). For two subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of \( \mod R \), we
denote by $\mathcal{X} = \mathcal{Y}$ the subcategory of mod $R$ consisting of modules $M$ such that there is an exact sequence $0 \to L \to M \to N \to 0$ with $L \in \mathcal{X}$ and $N \in \mathcal{Y}$.

The notion of $n$-torsionfree modules was also introduced by Auslander and Bridger \cite{1}, and played a central role in the stable module theory they developed. The structure of $n$-torsionfree modules has been well-studied; see \cite{1, 4, 5, 8}. In the following, applying Theorem \ref{thm:1.1} to the case where $(R, \mathfrak{m})$ is a commutative local ring, we describe the structure of $n$-torsionfree modules. Denote by FL$(R)$ the subcategory of mod $R$ consisting of modules of finite length, and by $H^n_m(\mathfrak{r})$ the $i$-th local cohomology functor with respect to $\mathfrak{m}$. Also, we say that a finitely generated $R$-module $M$ is $n$-H-spherical if $H^n_m(M) = 0$ for all $i \neq 0, n$.

**Corollary 1.2.** Suppose that $R$ is commutative, local and with Krull dimension $d > 0$. Let $\mathfrak{m}$ be the maximal ideal of $R$. For a nonnegative integer $n$, consider the following subcategories of mod $R$.

$$\text{TF}_n(R) = \{ M \in \text{mod } R \mid M \text{ is } n\text{-torsionfree } \},$$
$$\text{Sph}_n^R(R) = \{ M \in \text{mod } R \mid M \text{ is } n\text{-H-spherical } \}.$$

1. If $R$ is regular, then one has the equivalence
   $$\text{TF}_{d-1}(R) \cong \frac{\text{Ext}^d(\text{Tr}(-), R)}{\Omega^{d-1}} = \text{FL}(R).$$
2. If $R$ is Gorenstein, then one has the equivalence
   $$\text{TF}_{d-1}(R) \cong \frac{\text{Tr} \Omega^{d-1} \text{Tr}}{\Omega^{d-1}} = \text{Sph}_n^R(R).$$

It may be well-known to experts that when $R$ is a three dimensional regular local ring, there is an equivalence
$$\text{Ref}(R) \cong \frac{\text{Ext}^1((-)^*, R)}{\Omega^2} = \text{FL}(R),$$
where $\text{Ref}(R)$ stands for the subcategory of mod $R$ consisting of reflexive $R$-modules, while $(-)^*$ denotes the $R$-dual. The above corollary gives a higher dimensional version of this result.

2. **Our results and proofs**

In this section we give several definitions, and state and prove our results. We also give proofs of Theorem \ref{thm:1.1} and Corollary \ref{cor:1.2} which are displayed in the previous section. We begin with recalling fundamental notions.

**Definition 2.1.** (1) Let $\mathcal{X}$ be a subcategory of mod $R$. The **stable category** $\mathcal{X}$ of $\mathcal{X}$ is defined as follows:

The objects of $\mathcal{X}$ are the same as those of $\mathcal{X}$. The morphism set of objects $X, Y$ of $\mathcal{X}$ is the quotient of the additive group $\text{Hom}_R(X, Y)$ by the subgroup consisting of $R$-homomorphisms factoring through some finitely generated projective $R$-modules. Note that $\mathcal{X}$ is none other than the subcategory of mod $R$ consisting of objects $M$ such that $M \in \mathcal{X}$, that is,

$$\mathcal{X} = \{ M \in \text{mod } R \mid M \in \mathcal{X} \}.$$

2. Let $M$ be a finitely generated $R$-module and $P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} M \to 0$ a finite projective presentation of $M$. The **syzygy** $\Omega M$ of $M$ is defined as $\text{Im} d_1$. Note that $\Omega M$ is uniquely determined by $M$ up to projective summands. Taking the syzygy induces an additive functor $\Omega : \text{mod } R \to \text{mod } R$. Inductively, we define $\Omega^n = \Omega \circ \Omega^{n-1}$ for an integer $n > 0$. The (Auslander) transpose $\text{Tr} M$ of $M$ is defined as $\text{Coker} d_1^*$. Note that $\text{Tr} M$ is uniquely determined by $M$ up to projective summands. Taking the transpose induces an additive functor $\text{Tr} : \text{mod } R \to \text{mod } R^{\text{op}}$.

3. Let $m, n \in \mathbb{Z}_{\geq 0} \cup \{ \infty \}$. We denote by $\mathcal{G}_{m,n}(R)$, or simply $\mathcal{G}_{m,n}$, the subcategory of mod $R$ consisting of $R$-modules $M$ such that $\text{Ext}^i_m(R, M) = 0$ for all $1 \leq i \leq m$ and $\text{Ext}^j_{\text{proj}}(\text{Tr} M, R) = 0$ for all $1 \leq j \leq n$. We denote by $\text{proj}(R)$ (resp. $\text{GP}(R)$) the subcategory of mod $R$ consisting of finitely generated projective (resp. Gorenstein projective) $R$-modules. Note that $\text{GP}(R) = \mathcal{G}_{\infty, \infty}$. A finitely generated $R$-module $M$ is called $n$-torsionfree if $M$ belongs to $\mathcal{G}_{0,n}$. We denote by $\text{TF}_n(R)$ the subcategory of mod $R$ consisting of $n$-torsionfree modules, that is, we set $\text{TF}_n(R) = \mathcal{G}_{0,n}$. 


(4) The **projective dimension** (resp. **Gorenstein dimension**) of a finitely generated $R$-module $M$ is defined to be the infimum of integers $n$ such that there exists an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

of finitely generated $R$-modules with $X_i$ projective (resp. Gorenstein projective).

(5) The **grade** of a finitely generated $R$-module $M$ is defined to be the infimum of integers $i$ such that $\text{Ext}_R^i(M, R) = 0$, and denoted by $\text{grade}_R M$. For an integer $n$, we denote by $\text{Grd}_n(R)$ the subcategory of $\text{mod } R$ consisting of $R$-modules $M$ satisfying that $\text{grade}_R M \geq n$.

(6) Let $M, N$ be finitely generated $R$-modules. We say that $M$ and $N$ are **stably isomorphic** if there are finitely generated projective modules $P, Q$ such that $M \oplus P \cong N \oplus Q$, and then write $M \approx N$. Note that $M \approx N$ if and only if $M$ and $N$ are isomorphic as objects of $\text{mod } R$.

(7) Let $\mathcal{X}$ be a subcategory of $\text{mod } R$. We say that $\mathcal{X}$ is **closed under stable isomorphism** if for finitely generated modules $M, N$ with $M \in \mathcal{X}$ and $M \approx N$, it holds that $N \in \mathcal{X}$.

We extend the definition of $n$-spherical modules due to Auslander and Bridger. Namely, for any subcategory $\mathcal{X}$ of $\text{mod } R$ closed under stable isomorphism, we introduce the concept of $n$-$\mathcal{X}$-spherical modules as follows.

**Definition 2.2.** Let $\mathcal{X}$ be a subcategory of $\text{mod } R$. Suppose that $\mathcal{X}$ is closed under stable isomorphism. Let $n \geq 1$ be an integer and $M$ a finitely generated $R$-module. We say that $M$ is $n$-$\mathcal{X}$-spherical if $\text{Ext}_R^n(M, R) = 0$ for all $1 \leq i \leq n-1$ and $\Omega^i M \in \mathcal{X}$. We denote by $\text{Sph}_n^\mathcal{X}(R)$ the subcategory of $\text{mod } R$ consisting of $n$-$\mathcal{X}$-spherical $R$-modules. We call $n\text{-proj}(R)$-spherical (resp. $n\text{-GP}(R)$-spherical) simply $n$-spherical (resp. $n$-$\mathcal{G}$-spherical). We denote by $\text{Sph}_n(R)$ (resp. $\text{Sph}_n^\mathcal{G}(R)$) the subcategory of $\text{mod } R$ consisting of $n$-spherical (resp. $n$-$\mathcal{G}$-spherical) modules.

**Remark 2.3.** Let $n \geq 1$ be an integer and $M$ a finitely generated $R$-module. Then $M$ is $n$-spherical if and only if $\text{Ext}_R^n(M, R) = 0$ for all $1 \leq i \leq n-1$ and $M$ has projective dimension at most $n$. Hence our convention is consistent with the original definition by Auslander and Bridger. Similarly, $M$ is $n$-$\mathcal{G}$-spherical if and only if $\text{Ext}_R^n(M, R) = 0$ for all $1 \leq i \leq n-1$ and $M$ has Gorenstein dimension at most $n$.

For a finitely generated $R$-module $M$, we denote by $D(M)$ the image of the canonical map $\sigma_M : M \to M^{**}$ given by $\sigma_M(m)(f) = f(m)$ for $m \in M$ and $f \in M^*$. This correspondence induces an additive functor $D : \text{mod } R \to \text{mod } R$. Note that $D(M) \approx \Omega \text{Tr} \Omega \text{Tr} M$; see [1, Appendix]. The following proposition is an essential part of the proof of the main theorem.

**Proposition 2.4.** Let $n \geq 1$ be an integer and $M$ a finitely generated $R$-module. Let $\mathcal{X}$ be a subcategory of $\text{mod } R$. Assume that $\mathcal{X}$ satisfies the following conditions.

(i) The subcategory $\mathcal{X}$ is closed under stable isomorphism.

(ii) One has $D(\mathcal{X}) \subseteq \mathcal{X}$.

(iii) One has $\mathcal{X} \subseteq \text{Grd}_{n,0}$.

Then the following are equivalent.

1. There exists an exact sequence $0 \to L \to M \to N \to 0$ with $\text{grade}_R L \geq n$ and $N \in \mathcal{X}$.

2. The module $M$ belongs to $\mathcal{G}_{n-1,0}$ and the module $D(M)$ belongs to $\mathcal{X}$.

In other words, one has $\text{Grd}_n(R) * \mathcal{X} = D^{-1}(\mathcal{X}) \cap \mathcal{G}_{n-1,0}$.

**Proof.** Suppose that there is an exact sequence $0 \to L \to M \to N \to 0$ with $L \in \text{Grd}_n(R)$ and $N \in \mathcal{X}$. Then $M$ is in $\mathcal{G}_{n-1,0}$ as $\text{Grd}_n(R)$ and $\mathcal{X}$ are contained in $\mathcal{G}_{n-1,0}$. Since $L^* = 0$, by [1, Lemma 3.9], there is an exact sequence $0 \to A \to B \to C \to 0$ such that $A \approx \text{Tr} N$, $B \approx \text{Tr} M$ and $C \approx \text{Tr} L$. As $\Omega \text{Tr} L$ is projective, we have $\Omega \text{Tr} M \approx \Omega \text{Tr} N$. Hence $D(M) \approx D(N)$. We get $D(N) \in \mathcal{X}$ by the assumption (ii), and we obtain $D(M) \in \mathcal{X}$ by the assumption (i). The implication $(1) \Rightarrow (2)$ holds.

Conversely, assume that $M \in \mathcal{G}_{n-1,0}$ and $D(M) \in \mathcal{X}$. We consider the exact sequence $0 \to \text{Ext}^1(\text{Tr} M, R) \to M \to \text{ext}^1(\text{Tr} M, R) \to D(M) \to 0$; see [1, Proposition 2.6]. Set $L = \text{Ext}^1(\text{Tr} M, R)$ and $N = D(M)$. As $\sigma_M^*$ is surjective, so is $\sigma_M^*$. We obtain a long exact sequence

$$0 \to L^* \to \text{Ext}^1(N, R) \to \text{Ext}^1(M, R) \to \text{Ext}^1(L, R) \to \cdots$$

Now $M$ belongs to $\mathcal{G}_{n-1,0}$, and $N$ belongs to $\mathcal{G}_{n,0}$ by the assumption (iii). Thus $L$ has grade at least $n$, and the implication $(2) \Rightarrow (1)$ holds. \(\blacksquare\)
Here are some comments about Proposition 2.4.

Remark 2.5. Let \( m \geq n \) be integers.

1. The subcategory \( \text{proj}(R) \) satisfies the three assumptions (i), (ii) and (iii) of Proposition 2.4 and so do the subcategories \( \text{GP}(R) \) and \( \mathcal{G}_{m,m+1}(R) \); see [7, Proposition 1.1.1].
2. Let \( \mathcal{X} \) be a subcategory of \( \text{mod} R \) satisfying the three assumptions (i), (ii) and (iii) of Proposition 2.4. By Proposition 2.4, if \( \mathcal{X} \) is closed under direct summands, then so is \( \text{Grd}_n(R) * \mathcal{X} \). Hence \( \text{Grd}_n(R) * \text{GP}(R) \) and \( \mathcal{G}_{m,m+1}(R) \) are closed under direct summands.

The following lemma connects Proposition 2.4 to the notion of \( n \times \)-spherical modules.

Lemma 2.6. Let \( n \geq 1 \) be an integer.

1. Let \( \mathcal{X} \) be a subcategory of \( \text{mod} R \). Assume that \( \mathcal{X} \) is closed under stable isomorphism. One then has the duality
   \[
   [D^{-1}(\mathcal{X}) \cap \mathcal{G}_{n-1,0}(R) \xrightarrow{\text{Tr} \Omega^{n-1}} \{ M' \in \mathcal{G}_{n-1,0}(R^{\text{op}}) | \text{Tr} \Omega \Omega^n M' \in \mathcal{X} \}].
   \]
2. Let \( M' \) be a finitely generated \( R^{\text{op}} \)-module and \( m \geq 0 \) an integer. Let \( \mathcal{X}(R) \) be any of the subcategories \( \text{proj}(R), \text{GP}(R) \) and \( \mathcal{G}_{m,m+1}(R) \). Then \( \Omega \text{Tr} \Omega^n M' \) belongs to \( \mathcal{X}(R) \) if and only if \( \Omega^n M' \) belongs to \( \mathcal{X}(R^{\text{op}}) \).

Proof. (1) By [7, Proposition 1.1.1], the functor \( \text{Tr} \Omega^{n-1} : \text{mod} R \leftrightarrow \text{mod} R^{\text{op}} \) gives a duality \( \mathcal{G}_{n-1,0}(R) \leftrightarrow \mathcal{G}_{n-1,0}(R^{\text{op}}) \). Hence it is enough to show that both of the above restricted correspondences are well-defined.

Let \( M \) be a finitely generated \( R \)-module satisfying that \( M \in \mathcal{G}_{n-1,0}(R) \) and \( D(M) \in \mathcal{X} \). Set \( M' = \text{Tr} \Omega^{n-1} M \).

Thus the functor from the left hand side is well-defined. The converse is proved similarly.

(2) Since \( \Omega^n M' \) is in \( \mathcal{G}_{n-1,0}(R^{\text{op}}) \), one has \( \Omega \text{Tr} \Omega^n M' \approx \Omega^n M' \). Moreover, the inclusion \( \Omega \text{Tr}(\mathcal{X}(R)) \subset \mathcal{X}(R^{\text{op}}) \) holds now. Hence the assertion follows.

The following theorem is the main result of this paper.

Theorem 2.7. Let \( n \geq 1 \) be an integer.

1. One has the equivalence
   \[
   \text{Sph}_n(R) \xrightarrow{\text{Ext}^n(-,R)} \mathcal{G}_{m,m+1}(R^{\text{op}}).
   \]
2. Let \( m \in \mathbb{Z}_{\geq n} \cup \{ \infty \} \). One then has the equivalence
   \[
   \text{Sph}_n^{\mathcal{G}_{m,m+1}}(R) \xrightarrow{\text{Tr} \Omega^{n-1}} \mathcal{G}_{m,m+1}(R^{\text{op}}) \ast \mathcal{G}_{m,m+1}(R^{\text{op}}).
   \]

Proof. The assertion (2) follows from Proposition 2.4 Remark 2.5(1) and Lemma 2.6. It is easily seen that the category \( \mathcal{G}_{m,m+1}(R) \ast \text{proj}(R) \) is naturally equivalent to the category \( \mathcal{G}_{m,m+1}(R) \ast \text{Grd}_n(R) \). Moreover, since \( n \)-spherical modules have projective dimension at most \( n \), taking \( \text{Tr} \Omega^{n-1} \) on \( \text{Sph}_n(R) \) is the same as taking \( \text{Ext}^n(-,R) \). Hence the assertion (1) is seen similarly.

Proof of Theorem 1.1. The first assertion of Theorem 1.1 is the same as Theorem 2.7(1). The second assertion follows by letting \( m = \infty \) in Theorem 2.7(2).

Remark 2.8. In general, the grade of a non-zero finitely generated module is less than or equal to its projective dimension. A finitely generated \( R \)-module \( M \) is said to be \( \text{perfect} \) if \( M = 0 \) or the grade of \( M \) equals its projective dimension. For a positive integer \( n \), we denote by \( \text{Perf}_n(R) \) the subcategory of \( \text{mod} R \) consisting of \( n \)-perfect \( R \)-modules with grade \( n \) or infinity. Then the equality \( \mathcal{G}_{m,m+1}(R) \cap \text{Sph}_n(R) = \text{Perf}_n(R) \) holds. By restricting the correspondence of Theorem 2.7(1), we obtain the duality \( \text{Ext}^n(-,R) : \text{Perf}_n(R) \to \text{Perf}_n(R^{\text{op}}) \). Theorem 2.7(1) can be regarded as a generalization of this classical result.
In the rest of this paper, we assume that \( R \) is commutative and local, and denote by \( m \) the maximal ideal of \( R \). Let \( \Gamma_m(\cdot) \) be the \( m \)-torsion functor; recall that \( \Gamma_m(M) = \{ x \in M \mid m^r x = 0 \text{ for some } r > 0 \} \) for an \( R \)-module \( M \). Let \( H^i_m(\cdot) \) be the \( i \)-th local cohomology functor, that is, the \( i \)-th right derived functor of \( \Gamma_m(\cdot) \).

A finitely generated \( R \)-module \( M \) is said to be maximal Cohen–Macaulay if the depth of \( M \) is greater than or equal to the (Krull) dimension of the ring \( R \); see [3] Chapter 2 for details. We denote by \( \text{CM}(R) \) the subcategory of \( \text{mod} \ R \) consisting of maximal Cohen–Macaulay \( R \)-modules. By Grothendieck’s vanishing theorem [3] Theorem 3.5.7, a finitely generated \( R \)-module \( M \) is maximal Cohen–Macaulay if and only if \( H^i_m(M) = 0 \) for all \( i < d \), where \( d \) is the dimension of \( R \). The following proposition provides various equivalent conditions for a finitely generated module \( M \) to be spherical relative to the local cohomology functor \( H_m \). Here, we say that a finitely generated \( R \)-module \( M \) is \( n \)-H-spherical if \( H^i_m(M) = 0 \) for all \( i \neq n \).

**Proposition 2.9.** Suppose that \( R \) is Cohen–Macaulay and with dimension \( d > 0 \). Let \( M \) be a finitely generated \( R \)-module. Consider the following four conditions.

(a) There exists an exact sequence \( 0 \to L \to M \to N \to 0 \) of \( R \)-modules such that \( L \) has finite length and \( N \) is maximal Cohen–Macaulay.

(b) The module \( M/\Gamma_m(M) \) is maximal Cohen–Macaulay.

(c) The module \( M \) is \( d \)-H-spherical.

(d) The module \( M \) is locally maximal Cohen–Macaulay on the punctured spectrum of \( R \).

Then the following hold.

1. The implications (a) \( \iff \) (b) \( \iff \) (c) \( \Rightarrow \) (d) hold.
2. If \( d = 1 \), then the implication (d) \( \Rightarrow \) (c) holds.
3. If \( d > 2 \), then the implication (d) \( \Rightarrow \) (c) never holds.

**Proof.** (1) Suppose that (a) holds. Since \( \Gamma_m(L) = L \) and \( \Gamma_m(N) = 0 \), the short exact sequence \( 0 \to L \to M \to N \to 0 \) yields an isomorphism \( \Gamma_m(f) : L \xrightarrow{\sim} \Gamma_m(M) \). Hence an isomorphism \( N \xrightarrow{\sim} M/\Gamma_m(M) \) is induced, and (b) holds. Conversely, since the module \( \Gamma_m(M) \) has finite length, if (b) holds, then the short exact sequence \( 0 \to \Gamma_m(M) \to M \to M/\Gamma_m(M) \to 0 \) satisfies the required condition in (a). We get the equivalence (a) \( \iff \) (b). Next, we note that \( \Gamma_m(M/\Gamma_m(M)) = 0 \) and \( H^i_m(M) \cong H^i_m(M/\Gamma_m(M)) \) for all \( i > 0 \); see [2] Chapter 2 for instance. The equivalence (b) \( \iff \) (c) follows from these and Grothendieck’s vanishing theorem. As \( M \) and \( M/\Gamma_m(M) \) are locally isomorphic on the punctured spectrum of \( R \), the implication (b) \( \Rightarrow \) (d) clearly holds.

(2) When \( d = 1 \), the module \( M/\Gamma_m(M) \) is maximal Cohen–Macaulay for all finitely generated \( R \)-modules \( M \) as \( \Gamma_m(M/\Gamma_m(M)) = 0 \), and therefore the assertion follows.

(3) The maximal ideal \( m \) of \( R \) is locally isomorphic to \( R \) on the punctured spectrum of \( R \). However, one has \( \Gamma_m(R) = H^1_m(R) = 0 \) by the assumption. The long exact sequence

\[
0 \to \Gamma_m(m) \to \Gamma_m(R) \to \Gamma_m(R/m) \to H^1_m(m) \to H^1_m(R) \to \cdots
\]

indicates that \( H^1_m(m) \cong \Gamma_m(R/m) = R/m \neq 0 \). Hence the implication (d) \( \Rightarrow \) (c) never holds. \( \blacksquare \)

Applying Theorem 2.7 to the stable category \( \text{TF}_n(R) \) gives rise to the following theorem. Here, \( \text{FL}(R) \) and \( \text{Sph}^H_d(R) \) respectively stand for the subcategory of \( \text{mod} \ R \) consisting of \( R \)-modules of finite length, and the subcategory of \( \text{mod} \ R \) consisting of \( n \)-H-spherical \( R \)-modules.

**Theorem 2.10.** Suppose that \( R \) is Cohen–Macaulay and with dimension \( d > 0 \).

1. If \( R \) is regular, then one has the equivalence

\[
\text{TF}_{d-1}(R) \xrightarrow{\text{Ext}^d_{T(R)}(\cdot,R)} \text{FL}(R).
\]

2. If \( R \) is Gorenstein, then the following equalities hold.

\[
\text{Sph}^H_d(R) = \text{FL}(R) \ast \text{CM}(R) = \{ M \in \text{mod} R \mid M/\Gamma_m(M) \text{ is maximal Cohen–Macaulay} \}.
\]
Moreover, one then has the equivalence
\[
\frac{\text{TF}_{d-1}(R)}{\Omega^{d-1}} \xrightarrow{\text{Tr} \Omega^{d-1}} \frac{\text{Sph}_{d}^{H}(R)}{\Omega^{d-1}}.
\]

Proof. Note that the equality $\text{Grd}_{d}(R) = \text{FL}(R)$ holds; see [3, Proposition 1.2.10]. By [3, Theorem 2.2.7] and [1, Theorem 4.20], if $R$ is regular (resp. Gorenstein), then any finitely generated $R$-module has projective (resp. Gorenstein) dimension at most $d$. When this is the case, $\text{Sph}_{d}(R)$ (resp. $\text{Sph}_{G}^{d}(R)$) is equal to $G_{d-1,0}$. Moreover, if $R$ is Gorenstein, then maximal Cohen–Macaulay modules are totally reflexive, and the converse is also true; see [3, Theorem 3.3.10] for instance. Since one has the duality $G_{d-1,0} \xrightarrow{\text{Tr}} \text{TF}_{d-1}(R)$ by [7, Proposition 1.1.1], the assertion follows from Theorem 2.7 and Proposition 2.9.

Proof of Corollary 1.2. The assertion of Corollary 1.2 is included in Theorem 2.10.

We denote by $\text{Ref}(R)$ the subcategory of $\text{mod} R$ consisting of reflexive $R$-modules. The exact sequence
\[
0 \to \text{Ext}^{1}_{R^{op}}(\text{Tr} M, R) \to M \xrightarrow{\text{Tr} M^{\ast}} M^{\ast} \to \text{Ext}^{2}_{R^{op}}(\text{Tr} M, R) \to 0
\]
for each finitely generated $R$-module $M$ shows that $M$ is reflexive if and only if it is 2-torsionfree. In other words, the equality $\text{Ref}(R) = \text{TF}_{2}(R)$ holds. The following corollary is a special case of Theorem 2.10.

Corollary 2.11. Let $R$ be a three dimensional regular local ring. One then has the equivalence
\[
\text{Ref}(R) \xrightarrow{\text{Ext}^{1}((-)^{\ast}, R)} \frac{\text{Ext}^{1}(\cdot, R)}{\Omega^{2}} \xrightarrow{\text{FL}(R)}.
\]

Proof. Since $(-)^{\ast} \approx \Omega^{2} \text{Tr}(-)$, the assertion is none other than the case $d = 3$ of Theorem 2.10(1).

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