Generalized LKF Transformations for N-Point Fermion Correlators in QED

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Abstract—Within the worldline approach to quantum electrodynamics (QED), a change of the photon’s covariant gauge parameter \( \xi \) is investigated to analyse the non-perturbative gauge dependence of the configuration space fermion correlation functions, deriving a generalization of the Landau–Khalatnikov transformations (LKFt). These transformations reveal how the non-perturbative gauge dependence of position space amplitudes can be extracted into a multiplicative exponential factor.

Keywords: quantum field theory, LKF transformations, gauge symmetry, worldline formalism

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1. INTRODUCTION

The LKFt, derived by Landau and Khalatnikov [1], and independently by Fradkin [2], are non-perturbative transformations which relate field theory propagators (or vertices) calculated in different covariant gauges, originally phrased in position space. The transformation was derived first in QED and then generalized to correlation functions in quantum chromodynamics (QCD) [3, 4]. In particular cases, as in scalar QED in 3-dimensions, it is possible to obtain exact LKFt in momentum space [5].

Their applications range from non-perturbative analyses, implying constraints for the construction of a non-perturbative ansatz for the fermion-boson vertex [6] in the context of the Schwinger–Dyson equations, to perturbative ones, such as to obtain information on Feynman diagrams at higher loop orders, starting from an amplitude at a fixed loop level [4, 7]. The extra information at higher loops is always gauge dependent, and is such that the propagator at fixed loop order calculated in some gauge (e.g., Landau gauge, with gauge parameter \( \xi = 0 \)), determined to order \( \mathcal{O}(\alpha^n) \) in the fine structure constant, fixes—through the LKFt—the coefficients of the contributions at order \( \alpha^{i+j}\xi^i \), with \( i = 0, 1, ..., j = 0, 1, ..., n \) [8].

In 2016, a generalized LKFt was derived for the case of \( N (= 2n) \)-point correlator functions [9] in scalar QED, applying the worldline formalism (for reviews see [10, 11]), an alternative formulation of quantum field theory based on first quantized particle path integrals [12]. In this contribution we recap more recent work extending and improving this to spinor QED first presented in [13].

2. THE DRESSED PROPAGATOR

In Section II of [14], also in these proceedings, the worldline representation of the fermion propagator, \( S^{\pi\pi} = \langle x | [m - i\not{D}]^{-1} | x \rangle \), in an electromagnetic field, \( A_\mu \) is discussed. In the covariant derivative, \( D_\mu = \partial_\mu + ieA_\mu \) the field is specialized to plane waves,

\[
A^\pi_\mu(x) = \sum_i \varepsilon_{i\mu} e^{ik_i \cdot x},
\]

to describe external photons scattering off the fermion line, with definite momenta, \( k_i \), and polarizations, \( \varepsilon_{i\mu} \).

2.1. Supersymmetric Invariance on the Wordline

Noting that the worldline action \( S[x, \psi, A^\gamma] \) described in (3) of [14] is invariant under the supersymmetric transformations on the worldline (for Grassmann variable \( \zeta \))

\[
\delta x^\mu = -2\zeta \psi^\mu, \quad \delta \psi^\mu = \zeta \dot{x}^\mu,
\]
we can rewrite that action in a more compact form in terms of the superfield and superderivative:
\[ X^\mu(\tau, \theta) = x^\mu(\tau) + \sqrt{2} \theta \psi^\mu(\tau) + \eta^\mu, \]
\[ \mathcal{D} = \partial_\theta - \theta \partial_x. \quad (3) \]
Here, \( \theta \) is a Grassmann parameter which extends the parameter domain, \( \tau \rightarrow \tau | \theta \). Then the action is written:
\[ S[x, \psi, A^\gamma] = \int_0^T d\tau \int d\theta \left[ -\frac{1}{4} \mathcal{D}^2 X - ie A[X] \cdot \mathcal{D} X \right]. \quad (4) \]

2.2. Interaction with Virtual Photons

The interaction with a quantum field, \( A_\mu \), can be included by splitting the field \( A \) as \( A^\gamma + \bar{A} \), and computing the path integral over the field \( A \) according to the background method [15]
\[ Z_A = \int \mathcal{D} \bar{A} e^{-\int d^D x (-\bar{F}_{\mu\nu} F^{\mu\nu} - S_{\bar{g}}(\xi))}, \quad (5) \]
with the field \( A^\gamma \) an external electromagnetic field and where \( S_{\bar{g}}(\xi) \) is the gauge-fixing action depending on gauge parameter \( \xi \)
\[ S_{\bar{g}} = -\int d^4x \frac{(\bar{\partial} \cdot \bar{A})^2}{2\xi} \quad (6) \]

The inclusion of the quantum field makes the propagator in the covariant gauge \( \xi \) become
\[ S^{x_1 x}(\xi) = \left[ m + i \slashed{D} + i \frac{\delta}{\delta J(x')} \right] \]
\[ \times \left( e^{ie \int d^D x J(x) A[x]} K^{x_1 x} \right) \bigg|_{A, \xi, J=0} \]
\[ = \left[ m + i \slashed{D} + i \frac{\delta}{\delta J(x')} \right] 2^{-D/2} \operatorname{symb}^{-1} \]
\[ \times \int_0^\infty dT e^{-m^2 T} \int_{x(0)=x} \mathcal{D}x(\tau) \]
\[ \times \int_{\operatorname{APC}} \mathcal{D}\psi(\tau) e^{-\sum_{x, \psi, A^\gamma} S[\psi, A^\gamma] - S_{\bar{g}}} \bigg|_{J=0}, \quad (7) \]
where the insertion of \( \bar{A} \) in the pre-factor of the kernel, in (1) of [14], is now generated by the functional derivative of the source, \( J[x] \), in \( e^{ie \int d^D x J(x) A[x]} \). After evaluating the expression on \( J = 0 \), the term \( S_i \) represents the interaction with virtual photons, and is given by
\[ S_i = \frac{e^2}{2} \int d^D y d^D y' \mathcal{J}(y) \times G(y - y'; \xi) \cdot \mathcal{J}(y'), \quad (8) \]
\[ \mathcal{J} = J^\mu(y) + \int_0^T d\tau \int d\theta \delta^D (y - X) \mathcal{D}X, \quad (9) \]
where \( G_{\mu\nu}(y; \xi) \) is the configuration space photon propagator in \( D \) dimensions in covariant gauge \( \xi \).

3. N-POINT FERMION CORRELATOR

The \( N(=2n) \)-point correlation function determined with covariant gauge parameter \( \xi \), defined by
\[ S(x_1 \ldots x_n; x'_1 \ldots x'_n|\xi) = \langle \psi(x_1) \ldots \psi(x_n) \psi(x'_1) \ldots \psi(x'_n) \rangle \]
\[ = \sum_{\pi \in S_n} S_{\pi}(x_1 \ldots x_n; x'_1 \ldots x'_n|\xi), \quad (10) \]
where the fermion lines in \( S_{\pi} \) start at \( x_i \) and end at \( x'_i \). The partial \( N \)-point function in the presence of an external electromagnetic field, \( A_\mu^\gamma \), and which interacts with virtual photons, is
\[ S_{\pi}(x_1 \ldots x_n; x'_1 \ldots x'_n|\xi) = \left\langle \prod_{i=1}^n S^{x_{\pi(i)} x_i} \right\rangle_{\xi}, \quad (11) \]
with the fermion propagator, \( S^{x_{\pi(i)} x_i} \), as in (1) of [14], but now with \( A_\mu^\gamma + \bar{A}_\mu \) instead of just \( A_\mu^\gamma \). The world-line representation of the \( N \)-point partial amplitude is
\[ S_{\pi}(x_1 \ldots x_n; x'_1 \ldots x'_n|\xi) = \prod_{j=1}^n \left( \left[ m + i \slashed{D}_j' + i \frac{\delta}{\delta J_j(x'_j)} \right] \right) \]
\[ \times \left( e^{ie \int d^D x J_j(x) A_j(x)} \prod_{j=1}^n \left( K^{x_j x_{\pi(j)}} \right) \right) \bigg|_{J_j=0} \]
\[ = \prod_{j=1}^n \left( \left[ m + i \slashed{D}_j' + i \frac{\delta}{\delta J_j(x'_j)} \right] \right) \]
\[ \times \prod_{j=1}^n 2^{-D/2} \operatorname{symb}^{-1} \int_0^\infty dT_j e^{-m^2 T_j} \]
\[ \mathcal{D}x_j(T_j) \int_{x_j(0)=x_j} \mathcal{D}x_j(T_j), \quad (12) \]
with

\[ S^{(l)}[\xi] \]

where \( S^{(l)}[\xi] \) is the action (4) of the fermion line \( l \) and \( S_{i,n} \) is as expressed in (8), but with a sum over lines

\[
\mathcal{J}^\mu(y) = \mathcal{J}^\mu(y) \\
+ \sum_{l=1}^n \int d\tau_l \int d\theta_l \delta^D(y - \xi_l) \mathcal{D}_{\tau_l} \mathcal{X}^\mu_l, \tag{14}
\]

where we used (2) of [14] for the kernel \( K_{\varphi n}^{(l)}[k_1, \epsilon_1; \ldots; k_N, \epsilon_N | A^\gamma + \bar{A}] \).

4. THE GENERALIZED LKF \((F)\) IN SPINOR QED

Interactions with external photons can be expressed in terms of the vertex operator in (5) of [14]. Then, a gauge transformation can be done by making the replacement \( \varphi_{\mu} \rightarrow \varphi_{\mu} + \xi k_{\mu}, \) in the vertex operator and the on-shell invariance of the amplitude is well understood by the Ward-Takahashi identity [16]. So the non-trivial transformation properties of the \( N \)-point correlation functions only require analysis of the gauge transformation of virtual photons under a variation in gauge parameter \( \xi. \)

A gauge transformation of virtual photons can be realised by sending \( \xi \rightarrow \xi + \Delta \xi \) in the photon propagator \( G^\mu_{\nu}(y; \xi). \) The transformation properties of the \( N \)-point correlation functions are then determined by how \( S_{i,n} \) transforms under this change. The derivation of this generalized LKF for spinor QED can be found in [13]. Essentially, the idea is to analyse the expectation value of the product of kernels, \( \langle \prod_{i=1}^n K_{\varphi n}^{(l_i)}^{x_i} \rangle_{A, \xi} \) given by

\[
\prod_{j=1}^n 2^{-D/2} \text{ symb}^{-1} \int_0^\infty dT_j e^{-m^2 T_j} \int_{x_j(0)=x_j}^{x_j(T)=x_j} \mathcal{D}_{x_j}(\tau_j) \\
\times \int_{\text{APC}} \mathcal{D}_{\psi_j}(\tau_j) e^{-\sum_{i=1}^{n} S^{(l_i)}[x_i, \psi_i, A_i^\gamma]} - \sum_{k,l=1}^{n} S^{(k,l)}_{\pi}, \tag{15}
\]

with \( S_{\pi}^{(k,l)} \) as defined in (8), but with \( J = 0. \) The change of \( S_{\pi}^{(k,l)} \) due to a gauge transformation of virtual photons is \( S_{\pi}^{(k,l)} \rightarrow S_{\pi}^{(k,l)} + \Delta \xi S_{\pi}^{(k,l)}, \) with

\[
\Delta \xi S_{\pi}^{(k,l)} = \Delta \xi \frac{e^2}{32\pi^2} \Gamma \left( \frac{D}{2} - 2 \right) \left\{ [(x_k - x_l)^2]^{2-D/2} - [(x'_k - x'_l)^2]^{2-D/2} \right. \\
\left. + [(x'_k - x'_{\pi l})^2]^{2-D/2} \right\}, \tag{16}
\]

which does not depend on the spinor degrees of freedom, since the spin interaction is already gauge invariant. Hence this is the same as in the scalar case [9] and we can conclude that for the derivation of the transformation properties of the \( N \)-point correlation function the quenched approximation of the QED is sufficient, since \( \Delta \xi S_{\pi}^{(k,l)} \) is zero when the virtual photons are attached, at least to one end, to closed fermion lines \( (x_l = x'_{\pi l}). \) Then, the transformation rule for the product of kernels is

\[
\langle K_{\varphi n}^{(l_1)} \cdots K_{\varphi n}^{(l_n)} \rangle_{A, \xi + \Delta \xi} = \langle K_{\varphi n}^{(l_1)} \cdots K_{\varphi n}^{(l_n)} \rangle_{A, \xi} e^{- \sum_{k,l=1}^{n} \Delta \xi S_{\pi}^{(k,l)}}, \tag{17}
\]

where the exponential can be factorised since it only depends on endpoints of the lines. This transformation shares the multiplicative form of the original LKF.

The next step is to check how the result changes when the pre-factors \( [m + i P^1 - e A_i] \) are included. In [13] it was found that the partial derivatives of the exponential factor in (17) which arises after the gauge transformation of the kernels cancel with other terms coming from the expected values involving insertions of \( \bar{A}. \) So at the end, the transformation rule which arises from this analysis is

\[
S_{\pi}^{LKF}(x_1 \cdots x_n; x'_{\pi (1)} \cdots x'_{\pi (n)}) | \xi + \Delta \xi \rangle = e^{- \sum_{k,l=1}^{n} \Delta \xi S_{\pi}^{(k,l)}} \\
\times S_{\pi}(x_1 \cdots x_n; x'_{\pi (1)} \cdots x'_{\pi (n)}) | \xi \rangle, \tag{18}
\]

where the label “LKF” indicates that the LHS involves the \( N \)-point amplitude in the gauge \( \xi + \Delta \xi \) plus extra gauge dependent parts of higher order diagrams generated by the multiplicative exponential factor acting upon the original amplitude.

5. CONCLUSIONS

Since the factor \( \sum_{k,l=1}^{n} \Delta \xi S_{\pi}^{(k,l)} \) is independent of the permutation, i.e., the same for each partial amplitude, the same transformation rule is valid for the total amplitude, which defines the complete, generalized LKF transformation for spinor QED, which turns out the same as in the scalar case (the original LKF are recovered for \( N = 2 \)). In ongoing work these transformations are being analysed in the context of the Schwinger model, extended to the case of propagation in external electromagnetic fields and applied to derive the analogous transformations of the interaction vertex.
CONFLICT OF INTEREST
The author declares that they have no conflicts of interest.

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