ON THE RAJCHMAN PROPERTY FOR SELF-SIMILAR MEASURES ON $\mathbb{R}^d$

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Abstract. We establish a complete algebraic characterization of self-similar iterated function systems $\Phi$ on $\mathbb{R}^d$, for which there exists a positive probability vector $p$ so that the Fourier transform of the self-similar measure corresponding to $\Phi$ and $p$ does not tend to 0 at infinity.

1. Introduction and the main result

1.1. Introduction. Let $d \geq 1$ be an integer. Given a Borel probability measure $\nu$ on $\mathbb{R}^d$ its Fourier transform is denoted by $\hat{\nu}$. That is,

$$\hat{\nu}(\xi) := \int e^{i\langle \xi, x \rangle} \, d\nu(x) \text{ for } \xi \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^d$. It is said that $\nu$ is a Rajchman measure if $|\hat{\nu}(\xi)| \to 0$ as $|\xi| \to \infty$. The Riemann–Lebesgue lemma says that $\nu$ is Rajchman whenever it is absolutely continuous with respect to the Lebesgue measure. For singular measures determining which ones are Rajchman is a subtle question with a long history (see [21]). In this paper we study the Rajchman property in the context of self-similar measures on $\mathbb{R}^d$.

Denote the orthogonal group of $\mathbb{R}^d$ by $O(d)$. A similarity of $\mathbb{R}^d$ is a map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\varphi(x) = rUx + a$, where $r > 0$, $U \in O(d)$ and $a \in \mathbb{R}^d$. When $0 < r < 1$, the map $\varphi$ is said to be a contracting similarity. A finite collection $\Phi = \{\varphi_i\}_{i=1}^\ell$ of contracting similarities is called a self-similar iterated function system (IFS) on $\mathbb{R}^d$. It is well known (see [15]) that there exists a unique nonempty compact $K \subset \mathbb{R}^d$ which satisfies the relation

$$K = \bigcup_{i=1}^\ell \varphi_i(K).$$

It is called the self-similar set, or attractor, corresponding to $\Phi$.

Following [14] we make the following definition.

**Definition 1.1.** We say that $\Phi$ is affinely irreducible if there does not exist a proper affine subspace $V$ of $\mathbb{R}^d$ so that $\varphi_i(V) = V$ for all $1 \leq i \leq \ell$.

It is easy to see that $\Phi$ is not affinely irreducible if and only if its attractor $K$ is contained in a proper affine subspace $V$ of $\mathbb{R}^d$. Observe that when $d = 1$, the IFS $\Phi$ is affinely irreducible if and only if the maps in $\Phi$ do not all have the same fixed point.

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It is also well known (again, see [15]) that given a probability vector \( p = (p_i)_{i=1}^\ell \) there exists a unique Borel probability measure \( \mu \) on \( \mathbb{R}^d \) which satisfies the relation,

\[
\mu = \sum_{i=1}^\ell p_i \cdot \varphi_i \mu,
\]

where \( \varphi_i \mu := \mu \circ \varphi_i^{-1} \) is the pushforward of \( \mu \) via \( \varphi_i \). The measure \( \mu \) is called the self-similar measure corresponding to \( \Phi \) and \( p \), and it is supported on the attractor \( K \). If \( p_i > 0 \) for all \( 1 \leq i \leq \ell \) we say that \( p \) is positive and write \( p > 0 \). When \( p > 0 \) the support of \( \mu \) is equal to \( K \). If \( \Phi \) is not affinely irreducible then \( \mu(\mathbb{V}) = 1 \) for some proper affine subspace \( \mathbb{V} \subset \mathbb{R}^d \), in which case it is easy to see that \( \mu \) is not Rajchman. For this reason, we shall always assume that our function systems are affinely irreducible.

Before stating our main theorem we mention some relevant previous results, mainly regarding the Fourier decay of self-similar measures on \( \mathbb{R} \). We start with the basic case of Bernoulli convolutions. Given \( \lambda \in (0, 1) \), write \( \nu_\lambda \) for the distribution of the random sum \( \sum_{n \geq 0} \pm \lambda^n \), where the \( \pm \) are independent unbiased random variables. This measure is called the Bernoulli convolution with parameter \( \lambda \). It can also be realised as the self-similar measure corresponding to the IFS \( \{ t \mapsto t \pm 1 \} \) and the probability vector \( (\frac{1}{2}, \frac{1}{2}) \).

Erdős [10] proved that \( \nu_\lambda \) is not Rajchman whenever \( \lambda^{-1} \) is a Pisot number different from 2. Recall that a Pisot number, also called a Pisot–Vijayaraghavan number or a P.V. number, is an algebraic integer greater than one whose algebraic (Galois) conjugates are all less than one in modulus. Note that \( \nu_{1/2} \) is absolutely continuous and in particular Rajchman. Later Salem [26] showed that if \( \lambda^{-1} \) is not a Pisot number then \( \nu_\lambda \) is Rajchman, thus providing a characterization of Rajchman Bernoulli convolution measures. Erdős [11] proved that \( \hat{\nu}_\lambda \) has power decay for a.e. \( \lambda \in (0, 1) \). That is, there exist \( s > 0 \) and \( C > 1 \) so that \( |\hat{\nu}_\lambda(\xi)| \leq C|\xi|^{-s} \) for \( \xi \in \mathbb{R} \). Kahane [16] later observed that this actually holds for all \( \lambda \in (0, 1) \) outside a set of zero Hausdorff dimension.

We turn to discuss the Fourier decay of general orientation preserving self-similar measures on the real line, in which case a lot of recent progress has been made. Let \( \Phi = \{ \varphi_i(t) = r_it + a_i \}_{i=1}^\ell \) be a self-similar IFS on \( \mathbb{R} \), with \( r_i > 0 \) for \( 1 \leq i \leq \ell \). Set

\[
\Delta := \{ (p_i)_{i=1}^\ell \in (0, 1]^\ell : p_1 + \ldots + p_\ell = 1 \},
\]

and for \( p \in \Delta \) write \( \mu_p \) for the self-similar measure corresponding to \( \Phi \) and \( p \). Let \( \mathcal{H} \subset \mathbb{R}_{>0} \) be the group generated by the contractions \( \{ r_i \}_{i=1}^\ell \), where \( \mathbb{R}_{>0} \) is the multiplicative group of positive real numbers. It is desirable to characterize the systems \( \Phi \) for which there exists \( p \in \Delta \) so that \( \mu_p \) is non-Rajchman. The following result, due to Li and Sahlsten, reduces this problem to the case in which \( \mathcal{H} \) is cyclic.

**Theorem 1.2** (Li–Sahlsten, [19]). Suppose that \( \Phi \) is affinely irreducible and that \( \mathcal{H} \) is not cyclic. Then \( \mu_p \) is Rajchman for every \( p \in \Delta \).

A related result has recently been obtained by Algom, Rodriguez Hertz and Wang [11 Corollary 1.2], which verifies the Rajchman property for self-conformal measures under mild assumptions. In [25], Sahlsten and Stevens have established power Fourier decay for self-conformal measures under certain conditions.

The proof of Theorem 1.2 is based on the classical renewal theorem for transient random walks on \( \mathbb{R} \). This approach was initiated by Li [18], who established the
Rajchman property for the Furstenberg measure on $\mathbb{R}^d$ under mild assumptions. Renewal theory also plays a major role in the proof of the main result of this paper.

The situation in which $\Phi$ is cyclic has been considered by Brémont (see also the paper by Varjú and Yu [29]). We continue to consider the orientation preserving system $\Phi = \{\varphi_i(t) = r_i t + a_i\}_{i=1}^\ell$ on $\mathbb{R}$.

**Theorem 1.3** (Brémont, [5].) Suppose that $\Phi$ is affinely irreducible and that $\Phi$ is cyclic. Let $r \in (0,1)$ be with $H = \{r^n\}_{n \in \mathbb{Z}}$. Then there exists $p \in \Delta$ so that $\mu_p$ is non-Rajchman if and only if, $r^{-1}$ is a Pisot number and $\Phi$ can be conjugated by a suitable similarity to a form such that $a_i \in \mathbb{Q}(r)$ for $1 \leq i \leq \ell$.

Brémont also proved that when $H = \{r^n\}_{n \in \mathbb{Z}}$ for a Pisot number $r$ and $a_i \in \mathbb{Q}(r)$ for $1 \leq i \leq \ell$, then in fact $\mu_p$ is non-Rajchman for every $p \in \Delta$ outside a finite union of proper submanifolds of $\Delta$. Moreover, in this case he also showed that $\mu_p$ is absolutely continuous whenever it is Rajchman.

Theorems 1.2 and 1.3 provide a complete algebraic characterization of the systems $\Phi$ on $\mathbb{R}$ for which there exists $p \in \Delta$ so that $\mu_p$ is non-Rajchman. The purpose of this paper is to extend this characterization to arbitrary self-similar IFSs on $\mathbb{R}^d$.

We point out that Solomyak [27] has recently shown that there exists $E \subset (0,1)^\ell$ of zero Hausdorff dimension so that when $\Phi$ is affinely irreducible and $(r_i)_{i=1}^\ell \notin E$, it holds that $\hat{\mu}_p$ has power decay for all $p \in \Delta$. For explicit parameters, and under additional diophantine assumptions, logarithmic decay rate has recently been obtained in [19], [29] and [1]. In the present paper we are only interested in the complete characterization of self-similar IFSs generating non-Rajchman measures, and do not consider the Fourier rate of decay.

Finally, we mention that in the context of self-affine measures on $\mathbb{R}^d$, the Rajchman property has recently been considered by Li and Sahlsten [20]. Assuming the group generated by the linear parts of the affine maps is proximal and totally irreducible and that the attractor is not a singleton, they have established that all self-affine measures, corresponding to positive probability vectors, are Rajchman. When $d = 2, 3$, or under additional assumptions on the group generated by the linear parts, they have also obtained power Fourier decay. The proximality assumption makes the situation studied in that paper very different compared to the self-similar setup studied here.

### 1.2. The main result

Following [8] and [2, Section 9.2] we make the following definition, which is necessary in order to state our main result.

**Definition 1.4.** Given $k \geq 1$, a finite collection $\{\theta_1, ..., \theta_k\}$ of distinct algebraic integers is said to be a P.V. $k$-tuple if the following conditions are satisfied.

1. $|\theta_j| > 1$ for $1 \leq j \leq k$;
2. there exists a monic polynomial $P \in \mathbb{Z}[X]$ so that $P(\theta_j) = 0$ for $1 \leq j \leq k$, and $|z| < 1$ for $z \in \mathbb{C} \setminus \{\theta_1, ..., \theta_k\}$ with $P(z) = 0$.

We make some remarks regarding this definition. In what follows, let $\{\theta_1, ..., \theta_k\}$ be a P.V. $k$-tuple.

- For each $1 \leq j_0 \leq k$ there exists $1 \leq j_1 \leq k$ so that $\theta_{j_1} = \overline{\theta_{j_0}}$. Additionally, writing $J$ for the set of $1 \leq j \leq k$ so that $\theta_j$ is conjugate to $\theta_{j_0}$ over $\mathbb{Q}$, it holds that $\{\theta_j\}_{j \in J}$ is also a P.V. tuple.
Note that a positive real number $\theta$ is a Pisot number precisely when $\{\theta\}$ is a P.V. 1-tuple. A nonreal complex number $\theta$ so that $\{\theta, \bar{\theta}\}$ is a P.V. 2-tuple is commonly called a complex Pisot number.

We shall be interested in P.V. tuples whose elements have the same modulus. Obviously every P.V. 1-tuple and every P.V. 2-tuple of the form $\{\theta, \bar{\theta}\}$ has this property. Assuming $|\theta_1| = \ldots = |\theta_k|$, for every $m \geq 1$ the collection

$$\{z \in \mathbb{C} : z^m = \theta_j \text{ for some } 1 \leq j \leq k\}$$

is a P.V. $mk$-tuple with this property. Further examples can be obtained by considering the products of real or complex Pisot numbers with certain primitive roots of unity. For instance, as pointed out in [4], if $\theta$ and $\bar{\theta}$ are the complex Pisot numbers whose minimal polynomial is $X^3 + X^2 - 1$ and $u$ and $\pi$ are the primitive 6th roots of unity, then $\{\theta u, \bar{\theta u}, \theta \pi, \bar{\theta \pi}\}$ is a P.V. 4-tuple whose elements are all conjugates over $\mathbb{Q}$ and have the same modulus.

Suppose that $|\theta_1| = \ldots = |\theta_k|$ and that $\theta_1, \ldots, \theta_k$ are conjugates over $\mathbb{Q}$. It is natural to ask whether we can say more about the structure of the P.V. $k$-tuple $\{\theta_1, \ldots, \theta_k\}$ under these additional assumptions. From a result of Boyd [4] and Ferguson [12] it follows that if $\theta_j$ is real for some $1 \leq j \leq k$ then $\{\theta_1, \ldots, \theta_k\} = \{e^{2\pi i/j} \theta_1\}^k_{j=1}$. Considering this result and the previous remark, one might think that, under the additional assumptions, for every $1 \leq j \leq k$ at least one of the numbers $\theta_j/\theta_1$ and $\theta_j/\theta_k$ is a root of unity. In Example 1.9 below we show that this is not the case.

As noted above, we shall always assume that our function systems are affinely irreducible. On the other hand, the situation in which there exists a nontrivial linearly invariant subspace should be taken into account. Let $\Phi = \{\varphi_i(x) = r_iU_i x + a_i\}^{\ell}_{i=1}$ be a self-similar IFS on $\mathbb{R}^d$. Given a linear subspace $\mathbb{V}$ of $\mathbb{R}^d$ we write $\pi_{\mathbb{V}}$ for the orthogonal projection onto $\mathbb{V}$. Observe that if $d' := \dim \mathbb{V} > 0$, $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \leq i \leq \ell$, and $S: \mathbb{V} \to \mathbb{R}^d$ is an isometry (which is necessarily an affine map), then $\{S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}\}^{\ell}_{i=1}$ is a self-similar IFS on $\mathbb{R}^{d'}$. Moreover, in this situation for every $1 \leq i \leq \ell$ there exists $U'_i \in O(d')$ and $a'_i \in \mathbb{R}^{d'}$ so that

$$S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_iU'_i x + a'_i \text{ for } x \in \mathbb{R}^{d'}.$$

We are now ready to state the main result of this paper. In what follows we consider $\mathbb{R}^d$ as a subset of $\mathbb{C}^d$. We denote the standard inner product of $\mathbb{C}^d$ by $\langle z, w \rangle$, that is $\langle z, w \rangle = \sum_{j=1}^{d} z_j w_j$ for $z, w \in \mathbb{C}^d$. Given a linear operator $A$ on $\mathbb{R}^d$ we consider it also as a linear operator on $\mathbb{C}^d$ in the natural way, that is by setting $A(x + iy) := Ax + iAy$ for $x, y \in \mathbb{R}^d$.

**Theorem 1.5.** Let $\Phi = \{\varphi_i(x) = r_iU_i x + a_i\}^{\ell}_{i=1}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$, with $0 < r_i < 1$, $U_i \in O(d)$ and $a_i \in \mathbb{R}^d$ for $1 \leq i \leq \ell$. Then there exists a probability vector $p = (p_i)_{i=1}^{\ell} > 0$ such that the self-similar measure corresponding to $\Phi$ and $p$ is non-Rajchman if and only if there exists a linear subspace $\mathbb{V} \subset \mathbb{R}^d$, with $d' := \dim \mathbb{V} > 0$ and $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \leq i \leq \ell$, and an isometry $S: \mathbb{V} \to \mathbb{R}^{d'}$ so that the following conditions are satisfied.

1. For $1 \leq i \leq \ell$ let $U'_i \in O(d')$ and $a'_i \in \mathbb{R}^{d'}$ be with $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_iU'_i x + a'_i$. Let $H \subset GL_d(\mathbb{R})$ be the group generated by $\{r_iU'_i\}^{\ell}_{i=1}$, and set $N := H \cap O(d')$. Then $N$ is finite, $\mathbf{N} \triangleleft H$ and $H/\mathbf{N}$ is cyclic.
(2) For every contracting $A \in \mathcal{H}$ with $\{A^nN\}_{n \in \mathbb{Z}} = \mathcal{H}/N$, there exist $k \geq 1$, $\theta_1, \ldots, \theta_k \in \mathbb{C}$ and $\zeta_1, \ldots, \zeta_k \in \mathbb{C}^d \setminus \{0\}$, so that
(a) $\{\theta_1, \ldots, \theta_k\}$ is a P.V. $k$-tuple;
(b) $A^{-1}\zeta_j = \theta_j \zeta_j$ for $1 \leq j \leq k$;
(c) for every $1 \leq i \leq \ell$ and $V \in \mathcal{N}$ there exists $P_{i,V} \in \mathbb{Q}[X]$ so that $\langle Vd_i, \zeta_j \rangle = P_{i,V}(\theta_j)$ for $1 \leq j \leq k$.

We make some remarks regarding the theorem.

- When $d = 1$ and the system $\Phi$ is orientation preserving, Theorem 1.5 is easily seen to be equivalent to Theorems 1.2 and 1.3.
- As the proof will show, if condition (1) holds and condition (2) is satisfied for some contracting $A \in \mathcal{H}$ with $\{A^nN\}_{n \in \mathbb{Z}} = \mathcal{H}/N$, then there exists a probability vector $p = (p_i)_{i=1}^\ell > 0$ so that the corresponding self-similar measure is non-Rajchman.
- In condition (1), since $N$ is the kernel of the homomorphism sending $rU \in \mathcal{H}$ with $r > 0$ and $U \in O(d')$ to $r$, it is obvious that $\mathcal{N} \triangleleft \mathcal{H}$. The statements regarding the finiteness of $\mathcal{N}$ and $\mathcal{H}/\mathcal{N}$ being cyclic are the interesting part of this condition.
- In condition (2), note that by restricting to a suitable nonempty subset of $\{\theta_1, \ldots, \theta_k\}$ we may assume that $\theta_1, \ldots, \theta_k$ are conjugates over $\mathbb{Q}$.
- As we show in Example 1.8 below, the parameters $k$ and $\theta_1, \ldots, \theta_k$ in condition (2) may depend on the choice of $A$. On the other hand, it is not hard to show that if conditions (1) and (2) are satisfied and $A_1, A_2 \in \mathcal{H}$ are contractions with $\{A^nN\}_{n \in \mathbb{Z}} = \mathcal{H}/N$ for $i = 1, 2$, then for every eigenvalue $\theta$ of $A_1$ there exists a root of unity $u$ so that $u\theta$ is an eigenvalue of $A_2$.
- In condition (2), since $A^{-1}$ is a member of $\mathcal{H}$ all of its eigenvalues have the same modulus. In particular $|\theta_1| = \ldots = |\theta_k|$.
- Theorem 1.5 provides many explicit examples of affinely irreducible self-similar function systems for which there exists a positive probability vector so that the corresponding self-similar measure is non-Rajchman. In fact, for every $k \geq 1$ and $\theta_1, \ldots, \theta_k \in \mathbb{C}$ such that $\{\theta_1, \ldots, \theta_k\}$ is a P.V. $k$-tuple and $|\theta_1| = \ldots = |\theta_k|$, we can construct a corresponding self-similar IFS on $\mathbb{R}^k$ with these properties (see Example 1.7 below).
- For a system $\Phi$ satisfying conditions (1) and (2), it could be interesting to study the exceptional set of positive probability vectors for which the corresponding self-similar measure is Rajchman. As noted after the statement of Theorem 1.3, this has been carried out by Brémond [5] in the case of orientation preserving systems on $\mathbb{R}$.

Theorem 1.5 can be used to verify the Rajchman property in many situations. For instance we have the following simple corollary.

**Corollary 1.6.** Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^\ell$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose that there exists a probability vector $p = (p_i)_{i=1}^\ell > 0$ such that the self-similar measure corresponding to $\Phi$ and $p$ is non-Rajchman. Then there exists an algebraic integer $\theta > 1$ so that for every $1 \leq i \leq \ell$ there exists a rational integer $n_i \geq 1$ with $r_i = \theta^{-n_i}$.

**Proof.** By Theorem 1.5 there exist $V$ and $S$ as in the statement of the theorem. Let $A$ and $\theta_1$ be as in condition (2), and note that $A = rU$ for some $0 < r < 1$ and $U \in O(d')$. Since $\theta_1^{-1}$ is an eigenvalue of $A$ we have $\|A\| = |\theta_1|^{-1}$, where $\|\cdot\|$ is the
operator norm. From \( \{ A^n N \}_{n \in \mathbb{Z}} = H/N \) and \( \| A \| < 1 \), it follows that for \( 1 \leq i \leq \ell \) there exists \( n_i \geq 1 \) and \( V_i \in N \) so that \( r_i U_i' = A^{n_i} V_i \). Thus,
\[
r_i = \| r_i U_i' \| = \| A^{n_i} V_i \| = \| A \|^{|n_i|} = |\theta_1|^{-n_i}.
\]

Since \( \theta_1 \) is a member of a P.V. \( k \)-tuple it is an algebraic integer. Since the modulus of an algebraic integer is still an algebraic integer, this completes the proof of the corollary.

\[ \square \]

1.3. Examples.

**Example 1.7.** Let \( k \geq 1 \) and \( \theta_1, ..., \theta_k \in \mathbb{C} \) be such that \( \{ \theta_1, ..., \theta_k \} \) is a P.V. \( k \)-tuple and \( |\theta_1| = ... = |\theta_k| \). In this example we show that it is possible to construct an affinely irreducible self-similar IFS \( \Phi \) on \( \mathbb{R}^k \) so that conditions (1) and (2) in Theorem 1.5 are satisfied with the parameters \( k \) and \( \theta_1, ..., \theta_k \), where we take \( V = \mathbb{R}^k \) and \( S = Id \).

Since \( \{ \theta_1, ..., \theta_k \} \) is a P.V. \( k \)-tuple, for every \( 1 \leq j_1 \leq k \) there exists \( 1 \leq j_2 \leq k \) so that \( \theta_{j_2} = \theta_{j_1} \). Thus, there exists \( A \in GL_k(\mathbb{R}) \) so that \( \theta_1, ..., \theta_k \) are the eigenvalues of \( A^{-1} \) and \( A = rU \) for some \( 0 < r < 1 \) and \( U \in O(k) \). Let \( \zeta_1, ..., \zeta_k \in \mathbb{C}^k \) be such that \( \{ \zeta_1, ..., \zeta_k \} \) is an orthonormal basis for \( \mathbb{C}^k \). \( A^{-1} \zeta_j = \theta_j \zeta_j \) for \( 1 \leq j \leq k \), and \( \zeta_j = \zeta_j \) for \( 1 \leq j_1, j_2 \leq k \) with \( \theta_{j_2} = \theta_{j_1} \). Set \( \xi := \sum_{j=1}^k \xi_j \), so that \( \xi \in \mathbb{R}^k \). Let \( \Phi := \{ \varphi_i \}_{i=0}^k \) be the self-similar IFS on \( \mathbb{R}^k \) with \( \varphi_0(x) = A x \) and \( \varphi_i(x) = A x + A^{-1} i \xi \) for \( 1 \leq i \leq k \).

Let us show that \( \Phi \) is affinely irreducible. Denote the attractor of \( \Phi \) by \( K \). Let \( y_0 \) be the zero vector of \( \mathbb{R}^k \), and for \( 0 \leq i \leq k \) write \( y_i := (I - A)^{-1} A^{-1} i \xi \). For \( 0 \leq i \leq k \) we have \( \varphi_i(y_i) = y_i \), and so \( y_0, ..., y_k \in K \). The matrix \( \{ (A^{-1} \xi, \xi_j) \}_{i,j=1}^k \) is equal to the Vandermonde matrix \( \{ \theta_j^{-1} \}_{i,j=1}^k \), and so its determinant is nonzero. It follows that \( \{ A^{-1} \xi \}_{i=1}^k \) are linearly independent, and so \( \{ y_i \}_{i=1}^k \) are also linearly independent. This shows that the affine span of \( K \) is equal to \( \mathbb{R}^k \), which implies that \( \Phi \) is affinely irreducible.

It is obvious that condition (1) in Theorem 1.5 is satisfied with \( N = \{ Id \} \). Moreover, for \( 1 \leq i, j \leq k \) we have \( \langle \varphi_i(0), \xi_j \rangle = \theta_j^{-1} \). From this, and since \( \langle \varphi_0(0), \xi_j \rangle = 0 \) for \( 1 \leq j \leq k \), it follows that condition (2) is also satisfied. From Theorem 1.5 we now get that there exists a probability vector \( p = (p_i)_{i=1}^k > 0 \) so that the self-similar measure corresponding to \( \Phi \) and \( p \) is non-Rajchman.

**Example 1.8.** The purpose of this example is to show that the parameters \( k \) and \( \theta_1, ..., \theta_k \), appearing in condition (2) in Theorem 1.5 may depend on the choice of \( A \). Set \( r_1 = r_2 = 1/2 \), let \( U_1 \) be the identity map of \( \mathbb{R}^2 \), let \( U_2 \in O(2) \) be a planar rotation of angle \( \pi/2 \) (i.e. \( U_2(x_1, x_2) = (-x_2, x_1) \)), set \( a_1 = (1, 0) \) and \( a_2 = 0 \), and set \( \varphi_i(x) = r_i U_i x + a_i \) for \( i = 1, 2 \) and \( x \in \mathbb{R}^2 \). It is easy to verify that the IFS \( \Phi := \{ \varphi_1, \varphi_2 \} \) is affinely irreducible.

Let \( H \subset GL_2(\mathbb{R}) \) be the group generated by \( r_1 U_1 \) and \( r_2 U_2 \), and set \( N := H \cap O(2) \). We have \( N = \{ U_2^m \}_{m=1}^k \), and for \( A_1 := r_1 U_1 \) and \( A_2 := r_2 U_2 \) it holds that \( \{ A_1^n N \}_{n \in \mathbb{Z}} = \{ A_2^n N \}_{n \in \mathbb{Z}} = H/N \). Thus, condition (1) of Theorem 1.5 is satisfied (with \( V = \mathbb{R}^2 \) and \( S = Id \)). It is also easy to verify that if we take \( k = 1 \), \( \theta_1 = 2 \) and \( \zeta_1 = (1, 0) \) then condition (2) holds with respect to \( A_1 \), and if we take \( k = 2 \), \( \theta_1 = 2i \), \( \theta_2 = -2i \), \( \zeta_1 = (1, i) \) and \( \zeta_2 = (1, -i) \) then condition (2) holds with respect to \( A_2 \). Moreover, since \( 2i \) and \( -2i \) are conjugates over \( \mathbb{Q} \), with \( k = 1 \) condition (2) cannot hold with respect to \( A_2 \). This shows that the parameters \( k \) and \( \theta_1, ..., \theta_k \) depend on the choice of \( A \).
Example 1.9. The purpose of this example is to construct a P.V $k$-tuple $\{\theta_1, \ldots, \theta_k\}$ such that $k \geq 3$, $\theta_1, \ldots, \theta_k$ are all conjugates over $\mathbb{Q}$, $|\theta_1| = \cdots = |\theta_k|$, and for every $1 \leq j_1 < j_2 \leq k$ the number $\theta_{j_1} \theta_{j_2}^{-1}$ is not a root of unity.

A polynomial $P \in \mathbb{Z}[X]$ of degree $n$ is said to be reciprocal if $P(X) = X^n P(X^{-1})$. In this case the roots of $P$ fall into reciprocal pairs, that is $z^{-1}$ is a root of $P$ whenever $z \in \mathbb{C}$ is a root of $P$. We say that $P \in \mathbb{Z}[X]$ is a Salem polynomial if it is the minimal polynomial of a Salem number. This means that $P$ is irreducible, monic, reciprocal, it has degree at least 4, there exists $s > 1$ with $P(s) = 0$, and $|z| = 1$ for every $z \in \mathbb{C} \setminus \{s, s^{-1}\}$ with $P(z) = 0$. The number $s$ is called a Salem number.

Let $m \geq 4$ be even, and let $P$ be a Salem polynomial of degree $2m$. For example, we can take $P(X)$ to be $X^8 - X^5 - X^4 - X^3 + 1$. Let $z_1 > 1$ be the Salem number corresponding to $P$, and let $z_2, \ldots, z_m$ be the roots of $P$ located on the upper half of the unit circle in $\mathbb{C}$. Set $I := \{1, \ldots, m\}$, and for $J \subset I$ write $\theta_J := \Pi_{j \in J} z_j \cdot \Pi_{j \in I \setminus J} z_j^{-1}$. The Galois group of $P$ is analysed in [9, Theorem 1.1]. From that result it follows that $\{\theta_J\}_{J \subset I}$ is a complete set of algebraic conjugates over $\mathbb{Q}$.

Let $F \subset \mathbb{C}$ be the splitting field of $P$ over $\mathbb{Q}$. Note that given $j_0 \in I$, $J_1, J_2 \subset I$ with $J_0 \subset J_1 \setminus J_2$, and an automorphism $\sigma : F \to F$ with $\sigma(z_{j_0}) = z_1$, we have $|\sigma(\theta_{J_1})| = z_1 > 1$ and $|\sigma(\theta_{J_2})| = z_1^{-1} < 1$. Since $\sigma(u)$ is a root of unity whenever $u \in F$ is a root of unity, it follows that $\theta_{J_1} \neq \theta_{J_2} e^{2\pi i q}$ for all distinct $J_1, J_2 \subset I$ and $q \in \mathbb{Q}$. This shows that $\{\theta_J : 1 \leq J \subset I\}$ is a P.V. $2^{m-1}$-tuple, and that it satisfies the required properties.

1.4. About the proof. Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^\ell$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Most of the proof of Theorem 1.5 deals with the direction in which $\Phi$ is assumed to generate a non-Rajchman measure. We present a general outline of the argument for this direction. Everything will be repeated in a rigorous manner in later parts of the paper.

Let $p = (p_i)_{i=1}^\ell$ be a positive probability vector, let $\mu$ be the self-similar measure corresponding to $\Phi$ and $p$, and suppose that $\mu$ is non-Rajchman. Write $G \subset \mathbb{R} \times O(d)$ for the closed subgroup generated by $\{\log r_i^{-1}, U_i\}_{i=1}^\ell$. For $(t, U) = g \in G$ set $\psi g = t$. Since $\psi$ is a proper continuous map, $\psi(G)$ is a closed subgroup of $\mathbb{R}$. Let $\gamma : \psi(G) \to G$ be a continuous homomorphism with $\psi \circ \gamma = Id$. We define a right action of $G$ on $\mathbb{R}^d$ by setting $x.(t, U) := 2^{-t} U^{-1} x$ for $(t, U) \in G$ and $x \in \mathbb{R}^d$.

Let $X_1, X_2, \ldots$ be i.i.d. $G$-valued random elements with $\mathbb{P}\{X_1 = (\log r_i^{-1}, U_i)\} = p_i$ for $1 \leq i \leq \ell$, and for $n \geq 1$ set $Y_n := X_1 \cdot \ldots \cdot X_n$. For $t > 0$ denote by $\tau_t$ the stopping time which is equal to the smallest $n \geq 1$ for which $\psi Y_n \geq t$. Using a result obtained in [7], which extends the classical renewal theorem, we show that as $t \to \infty$ the random elements $\gamma_{-t} Y_{\tau_t}$ converge in distribution to a probability measure $\nu$ on $G$ which is absolutely continuous with respect to the Haar measure of $G$. This key fact will be used several times during the paper. In particular, we use it to prove the following lemma.

Lemma 1.10. For every $\epsilon > 0$ there exists $T > 1$ such that the following holds. Let $t \geq T$ be with $t \in \psi(G)$ and let $\xi \in \mathbb{R}^d$ be with $|\xi| \leq \epsilon^{-1}$, then

$$
|\hat{\mu}(\xi - \ell)|^2 \leq \epsilon + \int \int |e^{i\xi \cdot \ell} u \cdot v| \, d\nu(g) \, d\mu(x) \, d\mu(y).
$$

This lemma is inspired by the argument in [19] used in the proof of Theorem 1.2. The lemma is only useful when the group $G$ is nondiscrete.
As we show below, from the affine irreducibility of $\Phi$ it follows that $\mu(V) = 0$ for every proper affine subspace $V$ of $\mathbb{R}^d$. Using this fact we prove the following lemma.

**Lemma 1.11.** For every $\epsilon > 0$ there exists $S > 1$ so that the following holds. Let $s \geq S$ and let $c : [0, 1] \rightarrow \mathbb{R}^d$ be a smooth curve with $|c'(t)| \geq \epsilon$ and $|c''(t)| \leq \epsilon^{-1}$ for all $0 \leq t \leq 1$, then

$$\int \int \left| \int_0^1 e^{is(c(t),x-y)} \, dt \right| \, d\mu(x) \, d\mu(y) < \epsilon.$$  

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Indeed, assuming this is not the case we can represent $\nu$ as an average of smooth 1-dimensional probability measures, each of which is supported on a single coset of the subgroup $\gamma(\mathbb{R})$. By using this decomposition together with Lemmata 1.10 and 1.11 we show that $\mu$ must be Rajchman which contradicts our assumption.

Next we want to make a reduction from the case in which $G$ is nondiscrete with $\psi(G) \neq \mathbb{R}$, to the case in which $G$ is discrete. For this we need to choose appropriately the subspace $V \subset \mathbb{R}^d$ appearing in the statement of Theorem 1.5. Denote by $G_0$ the connected component of $G$ containing the identity element. We choose $V$ to be the linear subspace consisting of all $x \in \mathbb{R}^d$ so that $x.g = x$ for all $g \in G_0$. From $G_0 \subset G$ it follows that $x.g \in V$ for $x \in V$ and $g \in G$, which implies that $U_i(V) = V$ for $1 \leq i \leq l$.

Note that from $\psi(G) \neq \mathbb{R}$ it follows that the connected Lie group $G_0$ is contained in the compact group $\{0\} \times O(d)$. By using this fact, by representing $\nu$ as an average of certain smooth 1-dimensional measures, and by applying Lemmata 1.10 and 1.11 once more, we prove the following proposition. It will enable us to perform the aforementioned reduction.

**Proposition 1.12.** For every $\epsilon > 0$ there exists $R > 1$ so that $|\hat{\mu}(\xi)| < \epsilon$ for every $\xi \in \mathbb{R}^d$ with $|\pi_V \xi| \geq \max\{R, \epsilon|\pi_V \xi|\}$.

Next we consider the case in which $G$ is discrete. Clearly $\psi(G) \neq \mathbb{R}$ in this case, and so $\psi(G) = \beta\mathbb{Z}$ for some $\beta > 0$. Let $U \in O(d)$ be with $(\beta, U) \in G$, and set $A = 2^{-\beta}U$. Under the additional technical assumption $a_1 = 0$, we show that condition (2) in the statement of Theorem 1.5 holds for the matrix $A$. The proof is a nontrivial extension of the argument used in [29] for the direction of Theorem 1.3 in which the IFS is assumed to generate a non-Rajchman measure. One of the main ingredients of that argument is a classical theorem of Pisot. This theorem says that if $\theta > 1$ and $0 \neq \lambda \in \mathbb{R}$ satisfy $\sum_{n \geq 0} \|\lambda\theta^n\|^2 < \infty$, where $\| \cdot \|$ is the distance to the nearest integer, then $\theta$ is a Pisot number and $\lambda \in \mathbb{Q}(\theta)$. In our proof we shall need to use a generalisation of this result for P.V. $k$-tuples, which is basically contained in Pisot’s original paper [23].

Observe that from proposition 1.12 and since $\mu$ is not Rajchman, it follows that $d' := \text{dim} \mathbb{V} > 0$. Let $S : \mathbb{V} \rightarrow \mathbb{R}^{d'}$ be an isometry. By using Proposition 1.12 the fact that $\mu$ is not Rajchman, and the self-similarity of $\mu$, we can show that $S\pi_V \mu$ is also not Rajchman. The measure $S\pi_V \mu$ is the self-similar measure corresponding to the self-similar IFS $\Psi' := \{S \circ \pi_V \circ \varphi_i \circ S^{-1}\}_{i=1}^l$ on $\mathbb{R}^{d'}$ and the probability vector $p$. Let $H$ be the closed group generated by the linear parts of $\Psi'$, and set $N := H \cap O(d')$. By using our choice of $\mathbb{V}$, it is not hard to show that $H$ is discrete, $N$ is finite and $H/N$ is cyclic. Moreover, we can choose the isometry $S$ so that the
technical assumption $a_1 = 0$ is satisfied for the IFS $\Phi'$. At this point we complete the proof by applying on $\Phi'$ our result for the case in which $G$ is discrete.

**Organisation of the paper.** In Section 2 we develop notations and establish some basic properties of the group $G$. Assuming irreducibility, we also prove that self-similar measures vanish on proper affine subspaces. In Section 3 we state the version of the renewal theorem for $G$, and derive the statement regarding the limit distribution of $\gamma - t\tau_t$. Section 4 deals with the parts of the argument in which $G$ is assumed to be nondiscrete. In Section 5 we consider the case in which $G$ is discrete. In particular, in this section we construct non-Rajchman self-similar measures when $G$ is discrete and the IFS satisfies assumptions similar to condition (2) in Theorem 1.5. In Section 6 we connect all the pieces, and complete the proof of Theorem 1.5.

2. Preliminaries

2.1. General notations. For an integer $m$ we write $\mathbb{Z}_{\geq m} := \{m, m+1, \ldots\}$. We use the notations $\mathbb{Z}_{>m}$, $\mathbb{Z}_{\leq m}$ and $\mathbb{Z}_{<m}$ in a similar way.

Let $d \in \mathbb{Z}_{\geq 1}$ be fixed. We denote the standard inner product of $\mathbb{R}^d$ or $\mathbb{C}^d$ by $\langle \cdot, \cdot \rangle$, that is

$$\langle z, w \rangle = \sum_{j=1}^{d} z_j \overline{w_j} \text{ for } z, w \in \mathbb{C}^d.$$ 

For a linear subspace $V$ of $\mathbb{R}^d$ or $\mathbb{C}^d$, the orthogonal projection onto $V$ is denoted by $\pi_V$. We write $V^\perp$ for the orthogonal complement of $V$. The orthogonal group of $\mathbb{R}^d$ is denoted by $O(d)$. Given a Borel probability measure $\sigma$ on $\mathbb{R}^d$ its Fourier transform $\hat{\sigma}$ is defined by

$$\hat{\sigma}(\xi) := \int e^{i\langle \xi, x \rangle} \, d\sigma(x) \text{ for } \xi \in \mathbb{R}^d.$$ 

For a locally compact Hausdorff space $X$, we write $C_c(X)$ for the space of continuous functions $f : X \to \mathbb{R}$ with compact support. We denote by $\mathcal{M}(X)$ the collection of all compactly supported Borel probability measures on $X$. If $Y$ is another topological space, $\sigma$ is a Borel measure on $X$, and $F : X \to Y$ is Borel measurable, then we write $F\sigma$ for the pushforward of $\sigma$ via $F$. That is, $F\sigma := \sigma \circ F^{-1}$.

Throughout the paper $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ is an affinely irreducible self-similar IFS on $\mathbb{R}^d$, so that $0 < r_i < 1$, $U_i \in O(d)$ and $a_i \in \mathbb{R}^d$ for $1 \leq i \leq \ell$. We consider $\Phi$ as fixed, and so usually the dependence of various parameters on $\Phi$ will not be indicated. We denote by $K$ the attractor of $\Phi$, that is $K$ is the unique nonempty compact subset of $\mathbb{R}^d$ with

$$K = \bigcup_{i=1}^{\ell} \varphi_i(K).$$ 

Given a probability vector $p = (p_i)_{i=1}^{\ell}$ there exists a unique $\mu \in \mathcal{M}(K)$ which satisfies the relation

$$(2.1) \quad \mu = \sum_{i=1}^{\ell} p_i \cdot \varphi_i \mu.$$ 

It is called the self-similar measure corresponding to $\Phi$ and $p$. We usually assume that $p_i > 0$ for $1 \leq i \leq \ell$, in which case we say that $p$ is positive and write $p > 0$. 9
We sometimes write \( \Lambda \) for the index set \( \{1, \ldots, \ell\} \), and denote the set of finite words over \( \Lambda \) by \( \Lambda^* \). Following [3], we say that a finite set of words \( W \subset \Lambda^* \) is a minimal cut-set for \( \Lambda^* \) if every infinite sequence in \( \Lambda^N \) has a unique prefix in \( W \).

Given a group \( Y \), indexed elements \( \{y_i\}_{i=1}^\ell \subset Y \), and a word \( i_1 \ldots i_n = w \in \Lambda^* \), we often write \( y_w \) in place of \( y_{i_1} \ldots y_{i_n} \). For the empty word \( \emptyset \) we write \( y_\emptyset \) in place of \( 1_Y \), where \( 1_Y \) is the identity of \( Y \). Note that if \( W \) is a minimal cut-set for \( \Lambda^* \), then by the self-similarity relation \( \mu = \sum_{w \in W} \phi_w \cdot \varphi_w \cdot \mu \).

For \( 1 \leq i \leq \ell \) set
\[
g_i := (\log r_i^{-1}, U_i) \in \mathbb{R} \times O(d),
\]
where throughout the paper the base of the log function is always 2. Let \( G \) be the smallest closed subgroup of \( \mathbb{R} \times O(d) \) containing the elements \( \{g_i\}_{i=1}^\ell \). We always equip \( G \) with the subspace topology inherited from \( \mathbb{R} \times O(d) \). Since \( G \) is a closed subgroup of the Lie group \( \mathbb{R} \times O(d) \), it is itself a Lie group. We denote by \( G_0 \) the connected component of \( G \) containing the identity element. We write,
\[
x, (t, U) := 2^{-t} U^{-1} x \text{ for } (t, U) \in G \text{ and } x \in \mathbb{R}^d,
\]
which defined a right action of \( G \) on \( \mathbb{R}^d \).

Let \( \psi : G \to \mathbb{R} \) be the projection onto the first coordinate, that is \( \psi(t, U) = t \) for \( (t, U) \in G \), and write \( N \) for the kernel of \( \psi \). Since the homomorphism \( \psi \) is continuous and proper, it is also a closed map. In particular \( \psi(G) \) is a closed subgroup of \( \mathbb{R} \). Given \( T \in \mathbb{R} \) we write \( G_{\psi \leq T} \) for the set \( \psi^{-1}(-\infty, T] \), and use the notations \( G_{< T}, G_{\geq T} \) and \( G_{\leq T} \) in a similar way.

Let \( m_\mathbb{R} \) be the Lebesgue measure of \( \mathbb{R} \). Let \( m_{\psi(G)} \) be the Haar measure of \( \psi(G) \), normalized so that \( m_{\psi(t, U)} = m_\mathbb{R} \) if \( \psi(g) = \mathbb{R} \) and \( m_{\psi(t, U)} \{t\} = \beta \) for all \( t \in \psi(G) \) if \( \psi(g) = \beta \mathbb{Z} \). We show in Corollary [2.4] below that \( G \) is unimodular. It is easy to see that if \( m \) is a Haar measure for \( G \) then \( \psi m \) is a Haar measure for \( \psi(G) \). We denote by \( m_G \psi \) the Haar measure of \( G \), normalized so that \( \psi m_G = m \psi(G) \).

Our choice of normalization for \( m_G \) can be explained by the version of the renewal theorem for \( G \) stated below (see Section [3.1]).

### 2.2. Basic properties of \( G \).

**Lemma 2.1.** There exists a continuous and proper homomorphism \( \gamma : \psi(G) \to G \) such that \( \psi \circ \gamma = 1d \). If \( \psi(G) = \mathbb{R} \), then \( \gamma \) is smooth. If \( \psi(G) = \beta \mathbb{Z} \) for \( \beta > 0 \), then for any \( g \in G \) with \( \psi(g) = \beta \) it is possible to define \( \gamma \) so that \( \gamma(\beta) = g \).

**Proof.** If \( \psi(G) = \beta \mathbb{Z} \) for \( \beta > 0 \) then the lemma is trivial. Given \( g \in G \) with \( \psi(g) = \beta \), we simply set \( \gamma(\beta n) = g^n \) for \( n \in \mathbb{Z} \). Clearly \( \gamma \) satisfies the required properties.

Suppose next that \( \psi(G) = \mathbb{R} \). Let \( \mathfrak{g} \) and \( o(d) \) be the Lie algebras of \( G \) and \( O(d) \) respectively. We may identify \( \mathfrak{g} \) as a Lie subalgebra of \( \mathbb{R} \times o(d) \). Since \( \psi(G) = \mathbb{R} \), there exists \( X \in \mathfrak{g} \) so that its projection onto the first coordinate of \( \mathbb{R} \times o(d) \) is equal to 1. Set \( \gamma(t) = \exp(tX) \) for every \( t \in \mathbb{R} \), where \( \exp : \mathfrak{g} \to G \) is the exponential map of \( G \). It is easy to check that \( \gamma \) satisfies the required properties. \( \square \)

We consider the homomorphism \( \gamma \) from the previous lemma as fixed. In later sections we shall often write \( \gamma_t \) in place of \( \gamma(t) \). Recall that we write \( N \) for the kernel of \( \psi \), and let \( H \) denote the image of \( \gamma \). Since \( \gamma \) is continuous and proper, \( H \)
is a closed subgroup of $G$. Since $N \triangleleft G$, the subgroup $H$ acts on $N$ by conjugation. For $h \in H$ and $n \in N$ we write $n^h$ in place of $hnh^{-1}$. Let $N \rtimes H$ be the semidirect product of $N$ by $H$. That is, $N \rtimes H$ is the group whose underlying set is $N \times H$ with the following group operation,

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1n_2^{h_1}, h_1h_2) \text{ for } (n_1, h_1), (n_2, h_2) \in N \times H.$$ We equip $N$ and $H$ with the subspace topologies inherited from $G$, and $N \times H$ with the product topology. It is easy to verify that this makes $N \times H$ into a locally compact group. Let $F : N \rtimes H \to G$ be with $F(n, h) = nh$ for $(n, h) \in N \times H$.

**Lemma 2.2.** $G$ is a split extension of $N$ by $H$, that is $HN = G$ and $H \cap N = \{1\}$. Consequently, the map $F$ is an isomorphism of topological groups.

**Proof.** For $g \in G$ we have $\psi(\gamma(\psi g^{-1})g) = 0$. Hence,

$$g = \gamma(\psi g) \cdot \gamma(\psi g)^{-1}g \in HN,$$

which shows that $HN = G$. Next let $g \in H \cap N$, then $\psi g = 0$ and there exists $t \in \mathbb{R}$ with $\gamma(t) = g$. Thus,

$$t = \psi(\gamma(t)) = \psi g = 0,$$

and so $1_G = \gamma(0) = \gamma(t) = g$, which shows that $H \cap N = \{1\}$.

It is easy to verify that $F$ is a homomorphism. From $HN = G$ and $H \cap N = \{1\}$ it follows that $F$ is a group isomorphism. It is obvious that $F$ is continuous. It is also easy to see that $F$ is a proper map, and so it is a closed map. This shows that $F$ is an isomorphism of topological groups, and completes the proof of the lemma.

Since $N$ is a closed subgroup of $\{0\} \times O(d)$ it is compact. Let $m_N$ be the Haar measure of $N$, normalized so that $m_N(N) = 1$. By Lemma 2.1 the map $\gamma : \psi(G) \to H$ is an isomorphism of topological groups. Write $m_H$ for $\gamma m_{\psi(G)}$, so that $m_H$ is a Haar measure for $H$.

**Lemma 2.3.** $m_N \times m_H$ is a left and right Haar measure for $N \rtimes H$.

**Proof.** Since $N$ is compact and $H$ is abelian, $m_N$ and $m_H$ are both left and right Haar measures. Let $(n_0, h_0) \in N \times H$ and $f \in C_c(N \times H)$ be given. Since $N$ is compact, the automorphism $n \to n^{h_0}$ preserves $m_N$ (see e.g. Section 1.1]). Thus,

$$\int f((n_0, h_0) \cdot (n, h)) \, dm_N \times m_H(n, h) = \int \int f(n_0n^{h_0}, h_0h) \, dm_N(n) \, dm_H(h)$$

$$\quad = \int \int f(n_0n, h_0h) \, dm_N(n) \, dm_H(h)$$

$$\quad = \int f(n, h) \, dm_N \times m_H(n, h),$$

which shows that $m_N \times m_H$ is a left Haar measure for $N \rtimes H$. The proof that it is also a right Haar measure is even simpler, and is therefore omitted.

**Corollary 2.4.** $G$ is unimodular, and $m_G = F(m_N \times m_H)$. 

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Proof. From Lemmata 2.2 and 2.3 it follows that \( G \) is unimodular and that \( F(m_N \times m_H) \) is a Haar measure for \( G \). For every \((n, h) \in N \times H\) we have \( \psi F(n, h) = \psi h \). Hence,

\[
\psi F(m_N \times m_H) = \psi m_H = \psi \gamma m_{\psi(G)} = m_{\psi(G)}.
\]

Since \( m_G \) is the unique Haar measure for \( G \) whose image under \( \psi \) is equal to \( m_{\psi(G)} \) the corollary follows. \( \square \)

2.3. Self-similar measures vanish on proper affine subspaces. The following lemma is a consequence of the affine irreducibility of \( \Phi \). It is well known, but since we could not find a proof in the existing literature we provide one for completeness. The lemma will be used in Section 4 when we consider the case in which the group \( G \) is nondiscrete.

Lemma 2.5. Let \( p = (p_i)_{i=1}^\ell > 0 \) be a probability vector, and let \( \mu \) be the self-similar measure corresponding to \( \Phi \) and \( p \). Then \( \mu(\mathbb{V}) = 0 \) for every proper affine subspace \( \mathbb{V} \) of \( \mathbb{R}^d \).

Proof. For \( 0 \leq k \leq d \) denote by \( A_k \) the collection of all \( k \)-dimensional affine subspaces of \( \mathbb{R}^d \). Let \( m \) be the smallest nonnegative integer for which there exists \( \mathbb{V} \in A_m \) with \( \mu(\mathbb{V}) > 0 \). Assume by contradiction that \( m < d \).

Set

\[
\kappa := \sup\{ \mu(\mathbb{V}) : \mathbb{V} \in A_m \},
\]

and

\[
M = \{ \mathbb{V} \in A_m : \mu(\mathbb{V}) = \kappa \}.
\]

By the definition of \( m \) we have \( \kappa > 0 \). From this, since \( \mu \) is a finite measure, and since \( \mu(\mathbb{V}) = 0 \) for all \( \mathbb{V} \in \cup_{k=0}^{m-1} A_k, \) it follows that \( M \) is nonempty and finite. Let \( \{V_j\}_{j=1}^\ell \) be an enumeration of the elements in \( M \).

Set \( \mathbb{Y} := \cup_{j=1}^s \mathbb{V}_j \). For \( 1 \leq j \leq s \),

\[
\kappa = \mu(\mathbb{V}_j) = \sum_{i=1}^\ell p_i \cdot \mu(\varphi_i^{-1}(\mathbb{V}_j)).
\]

By the definition of \( \kappa \) we have \( \mu(\varphi_i^{-1}(\mathbb{V}_j)) \leq \kappa \) for \( 1 \leq i \leq \ell \), and so \( \varphi_i^{-1}(\mathbb{V}_j) \in M \) for \( 1 \leq i \leq \ell \). This implies that \( \mathbb{Y} \) is invariant with respect to the maps in \( \Phi \). From this and since \( \mathbb{Y} \) is closed, it follows that the attractor \( K \) is contained in \( \mathbb{Y} \).

For every \( 1 \leq j_1 < j_2 \leq s \) it holds that \( \mathbb{V}_{j_1} \cap \mathbb{V}_{j_2} \) is either empty, or it is an affine subspace of dimension strictly less than \( m \). Thus, by the definition of \( m \) we have \( \mu(\mathbb{V}_{j_1} \cap \mathbb{V}_{j_2}) = 0 \) for \( 1 \leq j_1 < j_2 \leq s \). Since \( \mu \) is supported on \( K \), this implies that \( K \) is not contained in \( \cup_{1 \leq j_1 < j_2 \leq s} \mathbb{V}_{j_1} \cap \mathbb{V}_{j_2} \).

Hence, by reordering \( \{\mathbb{V}_j\}_{j=1}^s \) if necessary, we may assume that there exists \( x \in K \cap \mathbb{V}_1 \) with \( x \notin \mathbb{V}_j \) for \( 2 \leq j \leq s \). From

\[
\inf\{ \text{dist}(x, \mathbb{V}_j) : 2 \leq j \leq s \} > 0
\]

and

\[
\inf\{ \text{diam}(\varphi_w(K)) : w \in \Lambda^* \text{ and } x \in \varphi_w(K) \} = 0,
\]

it follows that there exists \( w \in \Lambda^* \) such that \( \varphi_w(K) \cap \mathbb{V}_j = \emptyset \) for \( 2 \leq j \leq s \). From this and \( \varphi_w(K) \subset K \subset \mathbb{Y} \), we obtain that \( K \subset \varphi_w^{-1}(\mathbb{V}_1) \). Since \( \dim(\varphi_w^{-1}(\mathbb{V}_1)) = m < d \), this contradicts the affine irreducibility of \( \Phi \) and completes the proof of the lemma. \( \square \)
3. Renewal theory and first hitting distribution

**Definition 3.1.** Given \( q \in \mathcal{M}(G) \) we say that \( q \) is adapted if the subgroup generated by the support of \( q \) is dense in \( G \).

Throughout this section we fix an adapted finitely supported probability measure \( q \) on \( G \) with \( q(G_{>0}) = 1 \), where recall that \( G_{>0} := \psi^{-1}(0, \infty) \). We write \( \lambda \) in place of \( \int \psi \, dq \), so that \( \lambda > 0 \).

**3.1. A version of the renewal theorem for \( G \).** Set \( Q := \sum_{n \geq 0} q^n \), where \( q^n \) is the \( n \)-fold convolution of \( q \) with itself for \( n \geq 1 \) and \( q^0 \) is the Dirac mass as \( 1_G \).

Since \( q(G_{>0}) = 1 \), it is obvious that \( Q \) is a Radon measure on \( G \). For \( g \in G \) let \( L_g : G \to G \) be with \( L_g g' = gg' \) for \( g' \in G \), and note that \( L_g Q := Q \circ L^{-1}_g \) is also a Radon measure on \( G \).

The following theorem follows directly from [7, Theorem A.1]. It extends the classical renewal theorem for closed subgroups of \( \mathbb{R} \) (see e.g. [24, Chapter 5]) to the group \( G \).

**Theorem 3.2.** Let \( h_1, h_2, \ldots \in G \) be with \( \psi h_n \xrightarrow{n} -\infty \). Then for every \( f : G \to \mathbb{R} \) which is Borel measurable, bounded, compactly supported and satisfies

\[
\mathbf{m}_G \{ g \in G : f \text{ is not continuous at } g \} = 0,
\]

we have

\[
\lim_{n \to \infty} \int f \, dL_{h_n}Q = \lambda^{-1} \int f \, d\mathbf{m}_G.
\]

**3.2. First hitting distribution.** Let \( X_1, X_2, \ldots \) be i.i.d. \( G \)-valued random elements with distribution \( q \). Set \( Y_0 := 1_G \), and for \( n \geq 1 \) let \( Y_n := X_1 \cdot \cdots \cdot X_n \). For \( t > 0 \) write,

\[
\tau_t := \inf \{ n \geq 1 : \psi Y_n \geq t \}.
\]

For \( g \in G \) set,

\[
\rho(g) := \lambda^{-1} \mathbb{P} \{ \psi X_1 > \psi g \geq 0 \}.
\]

Note that since \( \lambda = \mathbb{E}[\psi X_1] \), and by our choice of \( \mathbf{m}_G \), it follows that \( \int \rho \, d\mathbf{m}_G = 1 \). Write \( \nu \) in place of \( \rho \, d\mathbf{m}_G \), so that \( \nu \in \mathcal{M}(G) \).

Recall the homomorphism \( \gamma : \psi(G) \to G \) from Lemma [2,1] and that for \( t \in \psi(G) \) we often write \( \gamma_t \) in place of \( \gamma(t) \). The idea of the proof of the following proposition is based on [28, Section 4, Proof of Theorem 3].

**Proposition 3.3.** The random elements \( \{ \gamma_t Y_{\tau_t} \}_{t \in \psi(G_{>0})} \) converge in distribution to \( \nu \) as \( t \to \infty \). That is, for every continuous and bounded \( f : G \to \mathbb{C} \),

\[
\lim_{t \to \infty} \mathbb{E} [f(\gamma_t Y_{\tau_t})] = \int f \, d\nu.
\]

For the proof we need the following lemma.

**Lemma 3.4.** For \( f \in C_c(G) \) we have,

\[
\lim_{t \to \infty} \int 1_{G_{>0}}(g) \int f(gh) \, dq(h) \, dL_{-t}Q(g) = \frac{1}{\lambda} \int 1_{G_{>0}}(g) \int f(gh) \, dq(h) \, d\mathbf{m}_G(g).
\]

**Proof.** Let \( f \in C_c(G) \), and for \( g \in G \) set

\[
\tilde{f}(g) = 1_{G_{>0}}(g) \int f(gh) \, dq(h).
\]

For the rest of this section, we write \( \mathcal{C}(G) \) for the space of all \( \lambda \)-finite Radon measures on \( G \), i.e., the dual space of \( \mathcal{M}(G) \).

Recall the homomorphism \( \gamma : \psi(G) \to G \) from Lemma [2,1] and that for \( t \in \psi(G) \) we often write \( \gamma_t \) in place of \( \gamma(t) \). The idea of the proof of the following proposition is based on [28, Section 4, Proof of Theorem 3].
It is clear that \( \tilde{f} \) is Borel measurable and bounded. Since \( q \) and \( f \) are compactly supported so does \( \tilde{f} \). The set of points at which \( \tilde{f} \) is discontinuous is contained in the boundary of \( G_{<0} \), which we denoted by \( \partial G_{<0} \). If \( \psi(G) = \mathbb{R} \) then,
\[
m_G(\partial G_{<0}) = m_G(\psi^{-1}\{0\}) = m_{\mathbb{R}}(0) = 0.
\]
If \( \psi(G) \neq \mathbb{R} \) then \( \partial G_{<0} = \emptyset \). From Theorem 3.2 and since \( \psi \circ \gamma = Id \) we now get,
\[
\lim_{t \to \infty} \int \tilde{f} \, dL_{\gamma^{-1}}Q = \lambda^{-1} \int \tilde{f} \, dm_G,
\]
which completes the proof of the lemma.

Proof of Proposition 3.3 Let \( f \in C_c(G) \) be nonnegative and with \( f(g) = 0 \) for \( g \in G_{<0} \). Since \( \nu(G_{<0}) = 0 \), it suffices to show
\[
\lim_{t \to \infty} \mathbb{E} [f(\gamma_t Y_{\tau_t})] = \int f \, d\nu.
\]
Note that for \( n \geq 1 \) the distribution of \( Y_n \) is equal to \( g^n \). Hence, for \( t \in \psi(G_{>0}) \)
\[
\int 1_{G_{<0}}(g) \int f(gh) \, dq(h) \, dL_{\gamma^{-1}}Q(g) = \sum_{n \geq 0} \int 1_{G_{<0}}(g) \int 1_{G_{>0}}(gh) f(gh) \, dq(h) \, dL_{\gamma^{-1}}g^n(g)
\]
\[
= \sum_{n \geq 0} \mathbb{E} [1_{G_{<0}}(\gamma^{-1}Y_n)1_{G_{>0}}(\gamma^{-1}Y_{n+1})f(\gamma^{-1}Y_{n+1})]
\]
\[
= \sum_{n \geq 0} \mathbb{E} [1_{\psi Y_n < t}1_{\psi Y_{n+1} \geq t}f(\gamma^{-1}Y_{n+1})]
\]
\[
= \sum_{n \geq 0} \mathbb{E} [1_{\{\tau_t = n+1\}} f(\gamma^{-1}Y_{n+1})] = \mathbb{E} [f(\gamma^{-1}Y_{\tau_t})].
\]
Moreover, by the right-invariance of \( m_G \)
\[
\lambda^{-1} \int 1_{G_{<0}}(g) \int f(gh) \, dq(h) \, dm_G(g) = \lambda^{-1} \int f(g) \int 1_{G_{>0}}(g)1_{G_{<0}}(gh^{-1}) \, dq(h) \, dm_G(g)
\]
\[
= \lambda^{-1} \int f(g)P\{\psi X_1 > \psi g \geq 0\} \, dm_G(g)
\]
\[
= \int f(g)\rho(g) \, dm_G(g) = \int f \, d\nu.
\]
Thus, the equality (3.1) \( \square \) follows from Lemma 3.4 which completes the proof of the proposition.

We shall need a uniform version of Proposition 3.3 Define a metric \( d_{op} \) on \( G \) by setting
\[
d_{op}((r,U),(s,V)) = \|2^{-r}U^{-1} - 2^{-s}V^{-1}\| \text{ for } (r,U),(s,V) \in G,
\]
where \( \|\cdot\| \) is the operator norm. It is clear that the topology induced by \( d_{op} \) is equal to the subspace topology inherited from \( \mathbb{R} \times O(d) \). Given \( C > 0 \), we say that \( f : G \to \mathbb{C} \) is \( C \)-Lipschitz with respect to \( d_{op} \) if
\[
|f(g) - f(g')| \leq C d_{op}(g,g') \text{ for } g,g' \in G.
\]
Observe that there exists a compact subset \( B \) of \( G \) so that \( \nu(B) = 1 \) and \( P\{\gamma_t Y_{\tau_t} \in B\} = 1 \) for all \( t \in \psi(G_{>0}) \). The following corollary follows directly from this, from Proposition 3.3 and from [3] Lemma A.3.3.
Corollary 3.5. For every $\epsilon > 0$ there exists $T = T(q, \epsilon) > 1$ so that the following holds. Let $f : G \to \mathbb{C}$ be $\epsilon^{-1}$-Lipschitz with respect to $d_{up}$, then

$$|E[f(\gamma_i Y_{t})] - \int f \, d\nu| < \epsilon \text{ for all } t \in \psi(G_{\geq T}).$$

4. The nondiscrete case

In this section we consider the situation in which the group $G$ is nondiscrete. In Section 4.3 we show that self-similar measures corresponding to positive probability vectors are always Rajchman whenever $\psi(G) = \mathbb{R}$. In Section 4.4 we assume $\psi(G) \neq \mathbb{R}$, and prove a result which will enable us to make a reduction to the case in which $G$ is discrete.

Recall that $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^\ell$ is an affinely irreducible self-similar IFS on $\mathbb{R}^d$, and that $g_i = (\log r_i^{-1}, U_i)$ for $1 \leq i \leq \ell$. Throughout this section let $p = (\rho_i)_{i=1}^\ell$ be a fixed positive probability vector. Since we consider $p$ as fixed for this section, usually the dependence of various parameters on $p$ will not be indicated. Let $\mu$ be the self-similar measure corresponding to $\Phi$ and $p$. Let $\sigma \in \mathcal{M}(\mathbb{R}^d)$ be defined by,

$$\sigma(f) = \int \int f(x-y) \, d\mu(x) \, d\mu(y) \text{ for } f \in C_c(\mathbb{R}^d).$$

Set

$$q := \sum_{i=1}^\ell \rho_i \delta_{g_i},$$

where $\delta_{g_i}$ is the Dirac mass at $g_i$. By definition $G$ is the closed subgroup generated by the support of $q$, and so $q$ is adapted. As before, we write $\lambda$ in place of $\int q \, dq$.

4.1. An initial upper bound. The purpose of this subsection is to prove Lemma 4.1. Recall that for $(t, U) = g \in G$ and $x \in \mathbb{R}^d$ we write $x.g := 2^{-i}U^{-1}x$.

Lemma 4.1. For every $\epsilon > 0$ there exists $T > 1$ such that the following holds. Let $t \in \psi(G_{\geq T})$ and let $\xi \in \mathbb{R}^d$ be with $|\xi| \leq \epsilon^{-1}$, then

$$||\hat{\mu}(\xi, \gamma_{\cdot-1})||^2 \leq \epsilon + \int \left| \int e^{i(\xi \cdot g.x)} \, d\nu(g) \right| \, d\sigma(x).$$

We first need the following lemma, whose proof is similar to the proof of [19, Lemma 3.1].

Lemma 4.2. For every $\xi \in \mathbb{R}^d$ and $t \in \psi(G_{>0})$,

$$||\hat{\mu}(\xi)||^2 \leq \mathbb{E}|e^{i(\xi \cdot Y_{t}x)}| \, d\sigma(x).$$

Proof. Recall that $\Lambda := \{1, \ldots, \ell\}$, and that for a group $Z$, elements $\{z_i\}_{i=1}^\ell \subset Z$ and a word $i_1 \ldots i_n = w \in \Lambda^*$, we write $z_w$ in place of $z_{i_1} \cdot \ldots \cdot z_{i_n}$. Let

$$W = \{i_1 \ldots i_n \in \Lambda^* : \psi(g_{i_1} \ldots i_n) \geq t > \psi(g_{i_1} \ldots i_{n-1})\}.$$

Since $W$ is a minimal cut-set (see Section 2.1),

$$\mu = \sum_{w \in W} p_w \cdot \varphi_w \mu.$$
This implies,
\[ \hat{\mu}(\xi) = \sum_{w \in \mathcal{W}} p_w \int e^{i\langle \xi, \varphi_w(x) \rangle} \, d\mu(x). \]

Thus be Jensen’s inequality,
\[ |\hat{\mu}(\xi)|^2 \leq \sum_{w \in \mathcal{W}} p_w \left| \int e^{i\langle \xi, \varphi_w(x) \rangle} \, d\mu(x) \right|^2 \]
\[ = \sum_{w \in \mathcal{W}} p_w \int e^{i\langle \xi, \varphi_w(x) \rangle} \, d\mu(x) \cdot \int e^{-i\langle \xi, \varphi_w(y) \rangle} \, d\mu(y) \]
\[ = \int \int \sum_{w \in \mathcal{W}} p_w e^{i\langle \xi, \varphi_w(x-y) \rangle} \, d\mu(x) \, d\mu(y) \]
\[ = \int \sum_{w \in \mathcal{W}} p_w e^{i\langle \xi, g_w \cdot x \rangle} \, d\sigma(x). \]

The lemma now follows since the distribution of \( Y_{\tau} \) is equal to \( \sum_{w \in \mathcal{W}} p_w \delta_{y_w}. \) \[\Box\]

**Proof of Lemma 4.1.** Let \( \epsilon > 0 \) and let \( t \in \psi(G_{\varepsilon}) \) be large with respect to \( \epsilon, p \) and \( \Phi. \) Fix \( \xi \in \mathbb{R}^d \) with \( |\xi| \leq \epsilon^{-1}. \) By Lemma 4.2,
\[ |\hat{\mu}(\xi, \gamma_{-t})|^2 \leq \int \mathbb{E} \left[ e^{i\langle \xi, (\gamma_{-t} Y_t, x) \rangle} \right] \, d\sigma(x). \]

Recall the metric \( d_{op} \) on \( G \) defined in (3.2). Observe that \( \sigma \) is supported on the compact set \( K - K, \) where \( K \) is the attractor of \( \Phi. \) From this and since \( |\xi| \leq \epsilon^{-1}, \) there exists a constant \( C > 1, \) which depends only on \( \epsilon \) and \( \Phi, \) so that for every \( x \in \text{supp}(\sigma) \) the map which takes \( g \in G \) to \( e^{i\langle \xi, g \cdot x \rangle} \) is \( C \)-Lipschitz with respect to \( d_{op}. \) Thus, by assuming that \( t \) is large enough and by Corollary 3.5,
\[ |\hat{\mu}(\xi, \gamma_{-t})|^2 \leq \epsilon + \int \left| \int e^{i\langle \xi, g \cdot x \rangle} \, d\nu(g) \right| \, d\sigma(x), \]
which completes the proof of the lemma. \[\Box\]

**4.2. Average Fourier decay of measures on curves.** The purpose of this subsection is to prove Lemma 4.5. It will enable us to make use of the upper bound obtained in the previous section.

For \( y \in \mathbb{R}^d \) and \( \delta > 0 \) we write \( B(y, \delta) \) for the closed ball in \( \mathbb{R}^d \) with centre \( y \) and radius \( \delta. \) Let \( \mathbb{R}^{d-1} \) be the projective space of \( \mathbb{R}^d, \) and for \( 0 \neq x \in \mathbb{R}^d \) write \( \mathbf{p} \in \mathbb{R}^{d-1} \) for the line spanned by \( x. \) Recall that for a linear subspace \( V \subset \mathbb{R}^d \) we denote its orthogonal projection by \( \pi_V. \)

**Lemma 4.3.** For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that,
\[ \pi_{\mathbf{p}} \sigma(B(y, \delta)) < \epsilon \text{ for all } \mathbf{p} \in \mathbb{R}^{d-1} \text{ and } y \in \mathbf{p}. \]

**Proof.** Assume by contradiction that the lemma fails for some \( \epsilon > 0. \) Then for every \( n \geq 1 \) there exist \( \mathbf{p}_n \in \mathbb{R}^{d-1} \) and \( y_n \in \mathbf{p}_n \) so that \( \pi_{\mathbf{p}_n} \sigma(B(y_n, 1/n)) \geq \epsilon. \) For every \( n \geq 1 \) we have
\[ B(y_n, 1/n) \cap \pi_{\mathbf{p}_n} \text{supp}(\sigma) \neq \emptyset. \]
From this, and since \( \sigma \) is compactly supported, it follows that there exists \( M > 1 \) so that \( y_1, y_2, \ldots \in B(0, M). \) Thus, there exist \( \mathbf{p} \in \mathbb{R}^{d-1}, y \in \mathbf{p} \) and an increasing sequence \( \{n_k\}_{k \geq 1} \subset \mathbb{Z}_{\geq 1}, \) so that \( \mathbf{p}_{n_k} \to \mathbf{p} \) and \( y_{n_k} \to y. \)
For \( \eta > 0 \) and sufficiently large \( k \geq 1 \),
\[
\pi^{-1}_n \left( B\left( y_{n_k}, \frac{1}{n_k} \right) \right) \cap \text{supp}(\sigma) \subset \pi^{-1}_n \left( B\left( y, \eta \right) \right).
\]
Since \( \pi^{-1}_n \sigma \left( B\left( y_{n_k}, \frac{1}{n_k} \right) \right) \geq \epsilon \), this implies \( \pi^{-1}_n \sigma\left( B\left( y, \eta \right) \right) \geq \epsilon \). Since this holds for all \( \eta > 0 \) we have \( \pi^{-1}_n \left\{ y \right\} \geq \epsilon \). Hence by the definition of \( \sigma \),
\[
\epsilon \leq \int \int 1_{\pi^{-1}_n \left( y \right)} \left( z - \xi \right) \mu(\xi) \, d\mu(\xi) = \int \mu\left( \pi^{-1}_n \left\{ y + \pi^{-1}_n \xi \right\} \right) \, d\mu(\xi).
\]
Thus, there exists a proper affine subspace \( V \) of \( \mathbb{R}^d \) such that \( \mu(V) > 0 \). This contradicts Lemma 2.5 which completes the proof. \( \square \)

**Lemma 4.4.** For every \( \epsilon > 0 \) there exists \( S > 1 \) so that the following holds. Let \( s \geq S \) and let \( u \in \mathbb{R}^d \) be with \( |u| \geq 1 \), then
\[
\int \left| \int_0^1 e^{i(tu, sx)} \, dt \right| \, d\sigma(x) < \epsilon.
\]

**Proof.** Let \( \epsilon > 0 \), let \( \delta > 0 \) be small with respect to \( \epsilon \), and let \( s > 4/(\delta \epsilon) \). Fix \( u \in \mathbb{R}^d \) with \( |u| \geq 1 \). By Lemma 4.3 we may assume that \( \pi^{-1}_n \sigma\left( B(0, \delta) \right) < \epsilon/2 \). For every \( x \in \mathbb{R}^d \) with \( |\pi^{-1}_n x| \geq \delta \),
\[
\left| \int_0^1 e^{i(tu, sx)} \, dt \right| \, d\sigma(x) = \left| \frac{1}{s \langle u, x \rangle} \left( e^{i\langle u, x \rangle} - 1 \right) \right| \leq 2s^{-1} \delta^{-1} < \epsilon/2.
\]
Hence,
\[
\int \left| \int_0^1 e^{i(tu, sx)} \, dt \right| \, d\sigma(x) < \int 1_{\{|\pi^{-1}_n x| \geq \delta\}} \left\| e^{i(tu, sx)} \right\| \, d\sigma(x) + \frac{\epsilon}{2} < \epsilon,
\]
which completes the proof of the lemma. \( \square \)

**Lemma 4.5.** For every \( \epsilon > 0 \) there exists \( S > 1 \) so that the following holds. Let \( s \geq S \) and let \( c : [0, 1] \to \mathbb{R}^d \) be a smooth curve with \( |c'(t)| \geq \epsilon \) and \( |c''(t)| \leq \epsilon^{-1} \) for all \( 0 \leq t \leq 1 \), then
\[
\int \left| \int_0^1 e^{i(c(t), sx)} \, dt \right| \, d\sigma(x) < \epsilon.
\]

**Proof.** Let \( \epsilon > 0 \) and let \( s > 1 \) be large with respect to \( \epsilon \) and \( \text{supp}(\sigma) \). Fix a smooth curve \( c : [0, 1] \to \mathbb{R}^d \) with \( |c'(t)| \geq \epsilon \) and \( |c''(t)| \leq \epsilon^{-1} \) for all \( 0 \leq t \leq 1 \). Let \( n \geq 1 \) be such that \( (n-1)^{3/2} < s \leq n^{3/2} \). By assuming that \( s \) is large enough,
\[
(4.1) \quad \frac{s}{n} > \frac{1}{2n-1} > \frac{1}{2} (n-1)^{1/2} > \frac{1}{4} n^{1/2} > \frac{1}{4} \frac{s^{1/3}}{n}.
\]
Firstly, we have
\[
(4.2) \quad \int \left| \int_0^1 e^{i(c(t), sx)} \, dt \right| \, d\sigma(x) \leq \sum_{k=0}^{n-1} \int \left| \int_0^{1/n} e^{i(c(t+k/n), sx)} \, dt \right| \, d\sigma(x).
\]
For \( 0 \leq k < n \) set \( v_k = c(k/n) \) and \( u_k = c'(k/n) \), and note that \( |u_k| \geq \epsilon \). By Taylor’s theorem it follows that for \( 0 \leq t \leq 1/n \),
\[
|c(t+k/n) - v_k - tu_k| \leq d \frac{\|c''\|_\infty}{2} n^{-2} \leq \frac{d}{2cn^2}.
\]
Hence for \( x \in \text{supp}(\sigma) \),

\[
\left| e^{i(x(t + \frac{k}{n}), sx)} - e^{i(v_k + tu_k, sx)} \right| \leq \left| c(t + \frac{k}{n}) - v_k - tu_k, sx \right| \\
\leq n^{3/2}|x| \cdot |c(t + \frac{k}{n}) - v_k - tu_k| \\
\leq \frac{d|x|}{2\epsilon n^{3/2}} \leq \frac{d|x|}{2\epsilon s^{3/2}}.
\]

By taking \( s \) to be large enough, we may assume that the last expression is at most \( \epsilon \). Thus,

\[
\sum_{k=0}^{n-1} \left| \int_0^{1/n} e^{i(x(t + \frac{k}{n}), sx)} \, dt \right| \, d\sigma(x) \leq \epsilon + \sum_{k=0}^{n-1} \left| \int_0^{1/n} e^{i(v_k + tu_k, sx)} \, dt \right| \, d\sigma(x).
\]

(4.3)

Additionally, for every \( 0 \leq k < n \),

\[
\int_0^{1/n} \left| \int_0^{1/n} e^{i(tu_k, sx)} \, dt \right| \, d\sigma(x) = \frac{1}{n} \int_0^1 \left| \int_0^{1/n} e^{i(tu_k, esn^{-1}x)} \, dt \right| \, d\sigma(x).
\]

From this, since \( |e^{-1}u_k| \geq 1 \), by Lemma 4.4 by (4.1), and by assuming that \( s \) is large enough,

\[
\int_0^{1/n} \left| \int_0^{1/n} e^{i(tu_k, sx)} \, dt \right| \, d\sigma(x) \leq \epsilon/n.
\]

From this, (4.2) and (4.3),

\[
\int_0^{1} \left| \int_0^{1} e^{i(x(t), sx)} \, dt \right| \, d\sigma(x) \leq 2\epsilon,
\]

which completes the proof of the lemma. \( \square \)

4.3. **The case \( \psi(G) = \mathbb{R} \).** Recall that \( \psi(t, U) = t \) for \((t, U) \in G\), that \( N \) is the kernel of \( \psi \), and that \( m_N \) is the Haar measure of \( N \) normalized so that \( m_N(N) = 1 \).

By reordering the maps \( \{\varphi_i\}_{i=1}^{j} \) if necessary, we may assume that \( \log r_i^{-1} \leq \log r_j^{-1} \) for \( 1 \leq i < j \leq \ell \). For \( 1 \leq i \leq \ell \) set \( b_i = \log r_i^{-1} \) and \( \alpha_i = \sum_{k=i}^{j} p_k \), and write \( b_0 = 0 \). Let \( \rho_0 : \mathbb{R} \to [0, \infty) \) be such that,

\[
\rho_0(t) = \begin{cases} 
0 & \text{for } t < 0 \text{ and } t \geq b\ell \\
\frac{\alpha_i}{\lambda} & \text{for } 1 \leq i \leq \ell \text{ and } b_{i-1} \leq t < b_i.
\end{cases}
\]

**Lemma 4.6.** Suppose that \( \psi(G) = \mathbb{R} \). Then \( \int \rho_0(t) \, dt = 1 \), and for every bounded and continuous \( f : G \to \mathbb{C} \)

\[
\nu(f) = \int \int f(\gamma_i(n))\rho_0(t) \, dt \, d\text{m}_N(n).
\]

**Proof.** By the definition of \( \rho \) (see Section 3.2) we have \( \rho(g) = \rho_0(\psi g) \) for \( g \in G \).

Since \( \psi(G) = \mathbb{R} \), and by the way we defined \( m_G \) (see Section 2.1), it follows that \( \psi m_G \) is equal to the Lebesgue measure \( m_{\mathbb{R}} \). Thus,

\[
1 = \int \rho \, dm_G = \int \rho_0 \, d\psi m_G = \int \rho_0(t) \, dt.
\]
Let \( f : G \to \mathbb{C} \) be bounded and continuous. By Corollary 2.1 and since \( \mu_H = \gamma \mu_{\mathbb{R}} \),

\[
\nu(f) = \int f \rho \, d\mu_G = \int \int f(nh) \rho(nh) \, d\mu_N(n) \, d\gamma \mu_{\mathbb{R}}(h) = \int \int f(n\gamma_t) \rho_0(\psi(n\gamma_t)) \, d\mu_N(n) \, dt.
\]

Since \( \psi \circ \gamma = Id \) (see Lemma 2.1) we have \( \psi(n\gamma_t) = t \) for \( n \in N \) and \( t \in \mathbb{R} \). Thus,

\[
\nu(f) = \int \int f(\gamma(\gamma_t^{-1}n\gamma_t)) \rho_0(t) \, d\mu_N(n) \, dt.
\]

Fort every \( t \in \mathbb{R} \) the map \( n \to \gamma_t^{-1}n\gamma_t \) is a continuous automorphism of \( N \). Since \( N \) is compact this automorphism preserves \( \mu_N \). The lemma now follows from the last equality. \( \square \)

**Proposition 4.7.** Recall that \( \mu \) is the self-similar measure corresponding to \( \Phi \) and the positive probability vector \( p \). Suppose that \( \psi(G) = \mathbb{R} \), then \( \mu \) is a Rajchman measure. That is, \( \hat{\mu}(\xi) \to 0 \) as \( \xi \to \infty \).

**Proof.** Let \( \epsilon > 0 \), let \( r > 1 \) be large with respect to \( \epsilon \), \( \Phi \), \( p \) and \( \gamma \), and let \( T > 1 \) be large with respect to \( r \). Fix \( \xi \in \mathbb{R}^d \) with \( |\xi| \geq 2^T r \). We prove the proposition by showing that \( |\hat{\mu}(\xi)|^2 \leq \epsilon \).

Write \( S^{d-1} \) for the unit sphere in \( \mathbb{R}^d \). Let \( u \in S^{d-1} \) and \( t \geq T \) be such that \( \xi = 2^T ru \). Note that since \( \psi(G) = \mathbb{R} \), the domain of \( \gamma \) is \( \mathbb{R} \). Let \( U \in O(d) \) be such that \( \gamma_{-t} = (-t, U) \). By Lemma 4.1 and by assuming that \( T \) is large enough with respect to \( r \) and \( \epsilon \),

\[
|\hat{\mu}(\xi)|^2 = |\hat{\mu}(ruU, \gamma_{-t})|^2 \leq \epsilon/2 + \int \left| \int e^{i((ruU)g, x)} \, d\nu(g) \right| \, d\sigma(x).
\]

From this, Lemma 4.6 and the definition of \( \rho_0 \),

\[
|\hat{\mu}(\xi)|^2 \leq \epsilon/2 + \int \left| \int e^{i((ruU)g, x)} \rho_0(s) \, ds \right| \, d\mu_N(n) \, d\sigma(x)
\]

(4.4)

\[
\leq \epsilon/2 + \int \sum_{j=1}^\ell \frac{\alpha_j}{\lambda} \int_{b_{j-1}}^{b_j} \left| \int e^{i((ruU)g, x)} \, ds \right| \, d\sigma(x) \, d\mu_N(n).
\]

For \( 1 \leq j \leq \ell, n \in N \) and \( v \in S^{d-1} \), let \( c_{j,n}^v : [0, 1] \to \mathbb{R}^d \) be such that

\[
c_{j,n}^v(s) := v(\gamma_{s(b_j - b_{j-1}) + b_{j-1}}) \quad \text{for} \quad s \in [0, 1].
\]

From (4.4) we get,

(4.5) \[ |\hat{\mu}(\xi)|^2 \leq \epsilon/2 + \sum_{j=1}^\ell \frac{\alpha_j}{\lambda} \int_{b_{j-1}}^{b_j} \left| \int_0^1 e^{i(c_{j,n}^v(s), x)} \, ds \right| \, d\sigma(x) \, d\mu_N(n).
\]

By Lemma 2.1 and since \( \psi(G) = \mathbb{R} \), it follows that \( \gamma \) is smooth. Hence, the curves \( c_{j,n}^v \) are also smooth. Let \( C > 1 \) be large with respect to \( \{b_j\}_{j=0}^\ell \) and the curve \( \gamma \). For \( x \in \mathbb{R}^d \) set \( f(x) = |x|^2 \). From \( \psi \circ \gamma = Id \) and \( N \subset \{0\} \times O(d) \), we get that for every \( 1 \leq j \leq \ell, n \in N, v \in S^{d-1} \) and \( s \in [0, 1] \),

\[
f(c_{j,n}^v(s)) = 2^{-2s(b_j - b_{j-1}) + b_{j-1}}.
\]
By differentiating the last equality with respect to $s$ and by assuming that $C$ is sufficiently large, it follows that for every $1 \leq j \leq \ell$ with $b_j > b_{j-1}$,

$$\left| \frac{d}{ds} v_{j,n}^v(s) \right| \geq C^{-1} \text{ for } n \in N, \ v \in S^{d-1} \text{ and } s \in [0, 1].$$

Additionally, by assuming that $C$ is sufficiently large,

$$\left| \frac{d^2}{ds^2} v_{j,n}^v(s) \right| \leq C \text{ for } 1 \leq j \leq \ell, \ n \in N, \ v \in S^{d-1} \text{ and } s \in [0, 1].$$

Hence, from Lemma 4.5 from (4.5), and by assuming as we may that $r$ is large enough with respect to $\epsilon$, $C$, $\{\alpha_j/\lambda_j\}_{j=1}^\ell$ and $\{b_j\}_{j=0}^\ell$, we get $|\hat{\mu}(\xi)|^2 \leq \epsilon$. This completes the proof of the proposition. \hfill \Box

4.4. Reduction to the discrete case. Throughout this subsection we assume that $\psi(G) \neq \mathbb{R}$. Recall that we write $G_0$ for the connected component of $G$ containing the identity element. Since $G$ is a Lie group it is locally path connected, and so $G_0$ is an open and closed normal subgroup of $G$. From $\psi(G) \neq \mathbb{R}$ it follows that $\psi(G)$ is a discrete subgroup of $\mathbb{R}$. This implies that $N$ is also open and close in $G$, and so $G_0 \subset N$. In particular $G_0$ is compact.

Let $\mathbb{V}$ be the linear subspace of $\mathbb{R}^d$ consisting of all $x \in \mathbb{R}^d$ so that $x.g = x$ for all $g \in G_0$. Recall that $\mathbb{V}^\perp$ denotes the orthogonal complement of $\mathbb{V}$.

**Lemma 4.8.** The subspaces $\mathbb{V}$ and $\mathbb{V}^\perp$ are $G$-invariant. That is, $v.g \in \mathbb{V}$ and $w.g \in \mathbb{V}^\perp$ for all $v \in \mathbb{V}$, $w \in \mathbb{V}^\perp$ and $g \in G$.

**Proof.** Let $g_0 \in G_0$, $g \in G$ and $v \in \mathbb{V}$ be given. Since $G_0 \triangleleft G$, there exists $g'_0 \in G_0$ with $g.g_0 = g'_0.g$. Thus,

$$(v.g).g_0 = v.(g.g_0) = v.(g'_0.g) = (v.g'_0).g = v.g.$$

This shows that $v.g \in \mathbb{V}$, and so $\mathbb{V}$ is $G$-invariant. It is now obvious that $\mathbb{V}^\perp$ is also $G$-invariant. \hfill \Box

The purpose of this subsection is to prove the following proposition. In Section 4 when we complete the proof of our main result, it will enable us to make a reduction to the case in which $G$ is discrete.

**Proposition 4.9.** Recall that $\mu$ is the self-similar measure corresponding to $\Phi$ and the positive probability vector $p$. Suppose that $\psi(G) \neq \mathbb{R}$. Then for every $\epsilon > 0$ there exists $R = R(\epsilon, p) > 1$ so that $|\hat{\mu}(\xi)| < \epsilon$ for every $\xi \in \mathbb{R}^d$ with $|\pi_{\mathbb{V}^\perp}\xi| \geq \max\{R, \epsilon|\pi_{\mathbb{V}\xi}|\}$.

We start working towards the proof of the proposition. If $\mathbb{V} = \mathbb{R}^d$ then the proposition holds trivially, and so we may assume that $\dim \mathbb{V}^\perp > 0$. Write $S_{\mathbb{V}^\perp}$ for the unit sphere of $\mathbb{V}^\perp$. That is $S_{\mathbb{V}^\perp} := \mathbb{V}^\perp \cap S^{d-1}$, where $S^{d-1}$ is the unit sphere of $\mathbb{R}^d$.

Let $g_0$ be the Lie algebra of $G_0$. For the rest of this section, fix some compact neighbourhood $B_{g_0}$ of 0 in $g_0$. That is, $B_{g_0}$ is a compact subset of $g_0$ and 0 $\in\text{Int}(B_{g_0})$. Since $B_{g_0}$ is fixed, usually the dependence of various parameters on $B_{g_0}$ will not be indicated. For $X \in g_0$ and $y \in \mathbb{R}^d$ let $c_{X,y} : \mathbb{R} \to \mathbb{R}^d$ be such that

$$c_{X,y}(t) = y.\exp(tX) \text{ for } t \in \mathbb{R},$$

where $\exp : g_0 \to G_0$ is the exponential map of $G_0$.\hfill 20
Lemma 4.10. There exists $\delta > 0$ so that for every $y \in S_{V^\perp}$ there exists $X \in B_{g_0}$, such that $|c'_{X,y}(t)| \geq \delta$ for all $t \in \mathbb{R}$.

Proof. For $y \in S_{V^\perp}$, $X \in B_{g_0}$ and $t, s \in \mathbb{R}$,

$$c_{X,y}(t + s) = (y, \exp(sX)). \exp(tX) = c_{X,y}(s). \exp(tX).$$

Differentiating with respect to $s$ at $s = 0$ we get

$$(4.6) c'_{X,y}(t) = c'_{X,y}(0). \exp(tX).$$

Since $\exp(tX) \in G_0 \subset N$ and $N \subset \{0\} \times O(d)$, it follows that $|c'_{X,y}(t)| = |c'_{X,y}(0)|$. Thus, it suffices to show that there exists $\delta > 0$ so that

$$(4.7) \quad \text{for every } y \in S_{V^\perp} \text{ there exists } X \in B_{g_0} \text{ with } |c'_{X,y}(0)| \geq \delta.$$

Assume by contradiction that such a $\delta$ does not exist. Let $M : S_{V^\perp} \to [0, \infty)$ be with

$$M(y) = \max_{X \in B_{g_0}} |c'_{X,y}(0)| \quad \text{for } y \in S_{V^\perp}.$$ 

Since there does not exist $\delta > 0$ which satisfies (4.7), we have

$$(4.8) \quad \inf_{y \in S_{V^\perp}} M(y) = 0.$$

Since $B_{g_0}$ is compact and since the map which takes $(X, y) \in g_0 \times S_{V^\perp}$ to $|c'_{X,y}(0)|$ is continuous, it is easy to check that $M$ is also continuous. From this, from (4.8) and since $S_{V^\perp}$ is compact, it follows that there exists $y_0 \in S_{V^\perp}$ so that $M(y_0) = 0$. That is, $c'_{X,y_0}(0) = 0$ for all $X \in B_{g_0}$. Hence, by (4.6) we have for all $t \in \mathbb{R}$

$$c'_{X,y_0}(t) = c'_{X,y_0}(0). \exp(tX) = 0.$$

This shows that for all $X \in B_{g_0}$ and $t \in \mathbb{R}$,

$$y_0 = c_{X,y_0}(0) = c_{X,y_0}(t) = y_0. \exp(tX).$$

Since $0 \in \text{Int}(B_{g_0})$ we have $g_0 = \cup_{t>0} tB_{g_0}$, and so $y_0 = y_0. \exp(X)$ for all $X \in g_0$. Additionally, since $G_0$ is a compact and connected Lie group its exponential map is surjective (see e.g. [13, Corollary 11.10]), that is $\exp(g_0) = G_0$. Thus $y_0 = y_0.g$ for all $g \in G_0$, which gives $y_0 \in V$. This contradicts $y_0 \in S_{V^\perp}$ and completes the proof of the lemma. □

Given a compact Hausdorff topological group $Y$, let $m_Y$ be its Haar measure normalized so that $m_Y(Y) = 1$. The following equidistribution lemma is standard. We provide the simple proof for completeness.

Lemma 4.11. Let $X \in g_0$, and let $G_1$ be the smallest closed subgroup of $G_0$ containing $\{\exp(tX)\}_{t \in \mathbb{R}}$. Then for every $f \in C_c(G_1)$,

$$m_{G_1}(f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\exp(tX)) \, dt.$$

Proof. Since $G_1$ is compact and separable, $M(G_1)$ is compact and metrizable with respect to the weak-* topology. For $T > 0$ let $\alpha_T \in M(G_1)$ be such that

$$\alpha_T(f) = \frac{1}{T} \int_0^T f(\exp(tX)) \, dt \quad \text{for } f \in C_c(G_1).$$

Let $\{T_k\}_{k \geq 1} \subset (0, \infty)$ and $\alpha \in M(G_1)$ be with $T_k \xrightarrow{k \to \infty} \infty$ and $\alpha T_k \xrightarrow{k \to \infty} \alpha$ in the weak-* topology. In order to prove the lemma it suffices to show that $\alpha = m_{G_1}$. 21
Let $s \in \mathbb{R}$, and write $g = \exp(sX)$. Recall that we write $L_gh = gh$ for $h \in G$. For $f \in C_c(G_1)$,
\[
|\alpha(f) - L_\alpha \alpha(f)| = \lim_{k \to \infty} |\alpha_{T_k}(f) - \alpha_{T_k}(f \circ L_\alpha)| = \lim_{k \to \infty} \frac{1}{T_k} \left| \int_{T_0}^{T_k} f(\exp(tX)) dt - \int_{s}^{s+T_k} f(\exp(tX)) dt \right| = 0.
\]

Thus $\alpha = L_\alpha \alpha$ for all $g \in \{\exp(tX)\}_{t \in \mathbb{R}}$. Since $G_1$ is the closure of $\{\exp(tX)\}_{t \in \mathbb{R}}$, it follows that $\alpha = L_\alpha \alpha$ for all $g \in G_1$. Hence $\alpha$ is a Haar measure for $G_1$. Since it is also a probability measure we get $\alpha = m_{G_1}$, which completes the proof of the lemma.

**Lemma 4.12.** For every $\epsilon > 0$ there exists $R > 1$ so that the following holds. Let $y \in S_{G_1}$, then there exists a closed subgroup $G_1$ of $G_0$ so that for all $t \geq R$ and $g_0 \in G_0$,
\[
\int \left| \int e^{i(y \cdot g_0, rz)} \, dm_{G_1}(g) \right| \, d\sigma(x) < \epsilon.
\]

**Proof.** It follows from \textbf{[4.6]} that for $X \in B_{g_0}$, $y \in S_{G_1}$ and $s, t \in \mathbb{R}$,
\[
c'_{X,y}(s + t) = (c'_{X,y}(0), \exp(sX)), \exp(tX) = c'_{X,y}(s) \exp(tX).
\]
Differentiating with respect to $s$ at $s = 0$ and using $\exp(tX) \in G_0 \subset \{0\} \times O(d)$, we get $|c'_{X,y}(t)| = |c'_{X,y}(0)|$. From this and since $B_{g_0}$ and $S_{G_1}$ are compact, it follows that there exists a constant $C > 1$, depending only on $B_{g_0}$, so that $|c'_{X,y}(t)| \leq C$ for all $X \in B_{g_0}$, $y \in S_{G_1}$ and $t \in \mathbb{R}$.

Let $\delta > 0$ be as obtained in Lemma \textbf{4.10}. Let $\epsilon > 0$, and let $r > 1$ be large with respect to $\epsilon$, $\delta$ and $C$. Let $y \in S_{G_1}$, then there exists $X \in B_{g_0}$ so that $|c'_{X,y}(t)| \geq \delta$ for all $t \in \mathbb{R}$. Let $G_1$ be the smallest closed subgroup of $G_0$ containing $\{\exp(tX)\}_{t \in \mathbb{R}}$. For $g_0 \in G_0$ and $k \in \mathbb{Z}_{\geq 0}$, let $c_{X,y}^{g_0,k} : [0,1] \to \mathbb{R}^d$ be with
\[
c_{X,y}^{g_0,k}(t) = c_{X,y}(k + t) \cdot g_0 \text{ for } t \in [0,1].
\]

Let $g_0 \in G_0$, then by Lemma \textbf{4.11},
\[
\int \left| \int e^{i(y \cdot g_0, rz)} \, dm_{G_1}(g) \right| \, d\sigma(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \int_0^T e^{i(c_{X,y}(t), g_0, rz)} \, dt \right| \, d\sigma(x)
\]
\[
\leq \lim_{M \to \infty} \frac{1}{M} \sum_{k=0}^{M-1} \int_0^1 \left| \int_0^1 e^{i(c_{X,y}^{g_0,k}(t), rz)} \, dt \right| \, d\sigma(x).
\]

Since $g_0 \in \{0\} \times O(d)$, we have for $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq t \leq 1$
\[
\left| \frac{d}{dt} c_{X,y}^{g_0,k}(t) \right| = |c'_{X,y}(k + t)| \geq \delta \quad \text{and} \quad \left| \frac{d^2}{dt^2} c_{X,y}^{g_0,k}(t) \right| = |c''_{X,y}(k + t)| \leq C.
\]
From this, from Lemma \textbf{4.5} by assuming that $r$ is large enough, and by \textbf{(4.9)}
\[
\int \left| \int e^{i(y \cdot g_0, rz)} \, dm_{G_1}(g) \right| \, d\sigma(x) < \epsilon,
\]
which completes the proof of the lemma. \hfill \square
Lemma 4.13. For every $\epsilon > 0$ there exists $R > 1$ so that for all $y \in \mathbb{S}_{\mathbb{V}}$ and $r \geq R$,

$$
\int \left| \int e^{i(y, g_0, rx)} \, dm_{G_0}(g_0) \right| \, d\sigma(x) < \epsilon.
$$

**Proof.** Let $\epsilon > 0$, let $r > 1$ be large with respect to $\epsilon$, and let $y \in \mathbb{S}_{\mathbb{V}}$. By Lemma 4.12 and by assuming that $r$ is large enough, it follows that there exists a closed subgroup $G_1$ of $G_0$ so that for all $g_0 \in G_0$

(4.10) \[ \int \left| \int e^{i(y, g_0, rx)} \, dm_{G_1}(g) \right| \, d\sigma(x) < \epsilon. \]

Additionally, for all $f \in C_c(G_0)$

$$
\int f \, dm_{G_0} = \int \int f(g_0) \, dm_{G_1}(g) \, dm_{G_0}(g_0).
$$

Hence,

$$
\int \left| \int e^{i(y, g_0, rx)} \, dm_{G_0}(g_0) \right| \, d\sigma(x) = \int \int \left| \int e^{i(y, g_0, rx)} \, dm_{G_1}(g) \, dm_{G_0}(g_0) \right| \, d\sigma(x) 
\leq \int \int \left| \int e^{i(y, g_0, rx)} \, dm_{G_1}(g) \right| \, d\sigma(x) \, dm_{G_0}(g_0).
$$

This together with (4.10) completes the proof of the lemma. □

**Proof of Proposition 4.9.** Let $\epsilon > 0$, let $r > 1$ be large with respect to $\epsilon$, $p$ and $\Phi$, and let $T > 1$ be large with respect to $r$. Fix $\xi \in \mathbb{R}^d$ with

$$
|\pi_{\mathbb{V}} \xi| \geq \max\{2^T r, \epsilon|\pi_{\mathbb{V}} \xi|\}.
$$

Since $\psi(G) \neq \mathbb{R}$, there exists $\beta > 0$ so that $\psi(G) = \beta \mathbb{Z}$. Let $t \in \psi(G_{\geq T})$ be such that

(4.11) \[ 2^{t-\beta} r < |\pi_{\mathbb{V}} \xi| \leq 2^t r. \]

Since $\psi_\gamma_\mathbb{T} = t$,

$$
|\xi, \gamma_\mathbb{T}| = 2^{-t} |\xi| \leq 2^{-t} (|\pi_{\mathbb{V}} \xi| + |\pi_{\mathbb{V}} \xi|) \leq 2^{-t} |\pi_{\mathbb{V}} \xi| (1 + \epsilon^{-1}) \leq r (1 + \epsilon^{-1}).
$$

By Lemma 4.1 from $|\xi, \gamma_\mathbb{T}| \leq r (1 + \epsilon^{-1})$, and by assuming that $T$ is large enough with respect to $r$ and $\epsilon$,

$$
|\tilde{\mu} (\xi)|^2 = |\tilde{\mu}((\xi, \gamma_\mathbb{T})).| \leq \epsilon + \int \left| \int e^{i(\xi, (\gamma_\mathbb{T}) g).x} \, d\nu(g) \right| \, d\sigma(x).
$$

From this, since $\nu = \rho \, dm_G$, and since

$$
m_G(f) = \int \int f(g_0) \, dm_{G_0}(g_0) \, dm_{G}(g) \text{ for } f \in C_c(G),
$$

we get,

$$
|\tilde{\mu} (\xi)|^2 \leq \epsilon + \int \left| \int e^{i(\xi, (\gamma_\mathbb{T}) g).x} \, \rho(g) \, dm_G(g) \right| \, d\sigma(x)
\leq \epsilon + \int \rho(g) \left| \int e^{i(\xi, (\gamma_\mathbb{T} g_0).x)} \, dm_{G_0}(g_0) \right| \, d\sigma(x) \, dm_G(g).
$$

(4.12)

We have also used here the fact that $\rho(g_0) = \rho(g)$ for $g \in G$ and $g_0 \in G_0$, which holds since $G_0 \subset N$. 

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Let $g \in G$ and write $w_g = (\pi_{V^\perp}\xi).\gamma_t g$ and $y_g = \psi_g/\|\psi_g\|$. It follows from Lemma 4.8 that the subspaces $V$ and $V^\perp$ are $\gamma_t g$-invariant. Hence,

$$\pi_{V^\perp}(\xi.\gamma_t g) = \pi_{V^\perp}((\pi_{V^\perp}\xi).\gamma_t g + (\pi_{V^\perp}\xi).\gamma_t g) = w_g.$$ 

Additionally, given $g_0 \in G_0$ it follows by the definition of $V$ that $v.g_0 = v$ for $v \in V$. Thus,

$$\xi.(\gamma_t g_0) = (\pi_{V}(\xi.\gamma_t g) + \pi_{V^\perp}(\xi.\gamma_t g)).g_0 = \pi_{V}(\xi.\gamma_t g) + w_g g_0.$$ 

From this and (4.12),

$$(4.13) \quad |\hat{\mu}(\xi)|^2 \leq \epsilon + \int \rho(g) \int \left| \int e^{ix(y_g.g_0,|w_g|x)}\,d\mu_{G_0}(g_0) \right| \,d\sigma(x) \,d\mu_{C}(g).$$

By the definition of $\rho$ there exists a constant $C > 1$, which depends only on $\Phi$, so that $\psi g \leq C$ for all $g \in G$ with $\rho(g) \neq 0$. For such a $g$ it follows by (4.11) that,

$$|w_g| = |(\pi_{V^\perp}\xi).\gamma_t g| = 2^{-\psi g}2^{-t}|\pi_{V^\perp}\xi| > 2^{-C-\beta}r.$$ 

Thus, from Lemma 4.13 since $y_g \in S_{V^\perp}$, and by assuming as we may that $r$ is sufficiently large with respect to $\epsilon, \beta$ and $C$, we get

$$\int \left| \int e^{ix(y_g.g_0,|w_g|x)}\,d\mu_{G_0}(g_0) \right| \,d\sigma(x) \leq \epsilon .$$

This together with (4.13) shows that $|\hat{\mu}(\xi)|^2 \leq 2\epsilon$, which completes the proof of the Proposition. \qed

5. The discrete case

Throughout this section we always assume that the group $G$ is discrete. That is, we assume that the subspace topology on $G$, inherited from $\mathbb{R} \times O(d)$, is equal to the discrete topology. Since $N$ is a compact subset of a discrete space, it holds that $N$ is finite. From the discreteness of $G$ it also follows that there exists $\beta > 0$ with $\psi(G) = \beta \mathbb{Z}$. Hence, $l_i := \psi(g_i)/\beta$ is a positive integer for all $1 \leq i \leq \ell$.

Fix some $h \in G$ with $\psi(h) = \beta$, and recall that $H := \gamma \circ \psi(G)$. By Lemma 2.1 we may assume that $\gamma h = h$, which gives $H = \{h^j\}_{j \in \mathbb{Z}}$. By Lemma 2.2 and since $N \triangleleft G$, it follows that for every $g \in G$

$$(5.1) \quad g = nh^l = h^l n' \text{ for some } l \in \mathbb{Z} \text{ and } n, n' \in N.$$ 

Let $U \in O(d)$ be such that $h = (\beta, U)$. Set $A := 2^{-\beta}U$ and $B := A^*$, where $A^* = 2^{-\beta}U^{-1}$ is the transpose of $A$. Write,

$$N_0 := \{V \in O(d) : (0, V) \in N\},$$ 

so that $N_0$ is a finite subgroup of $O(d)$. Since $N \triangleleft G$, we also have $A^{-1}N_0A = N_0$. From (5.1) it follows that for each $1 \leq i \leq \ell$,

$$(5.2) \quad r_i U_i = VA^i = A^i V' \text{ for some } V, V' \in N_0 .$$

The following proposition is the main result of this section. Its proof is a non-trivial extension of an argument used in [29] to prove one of the directions of Theorem 1.3 stated in the introduction. Recall that $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^\ell$ is an affinely irreducible self-similar IFS on $\mathbb{R}^d$, and recall from Section 1.2 the definition of a P.V. $k$-tuple. We shall consider $A$ as a linear operator on $\mathbb{C}^d$ in the natural way, that is by setting $A(x + iy) := Ax + iAy$ for $x, y \in \mathbb{R}^d$. 24
Proposition 5.1. Suppose that $G$ is discrete and that $a_1 = 0$. Moreover, assume that there exists a probability vector $p = (p_i)_{i=1}^\ell > 0$ so that the self-similar measure corresponding to $\Phi$ and $p$ is non-Rajchman. Let $A$ and $N_0$ be as defined above. Then there exist $k \geq 1$, $\theta_1, \ldots, \theta_k \in \mathbb{C}$ and $\zeta_1, \ldots, \zeta_k \in \mathbb{C}^d \setminus \{0\}$, so that

1. $\{\theta_1, \ldots, \theta_k\}$ is a $P.V.$ k-tuple;
2. $A^{-1}\zeta_j = \theta_j\zeta_j$ for $1 \leq j \leq k$;
3. for every $1 \leq i \leq \ell$ and $V \in N_0$ there exists $P_{i,V} \in \mathbb{Q}[X]$ so that $(Va_i, \zeta_j) = P_{i,V}(\theta_j)$ for all $1 \leq j \leq k$;

The assumption $a_1 = 0$ might seem somewhat arbitrary. It simplifies the statement of condition (3), and some of the arguments that follow.

The proof of the proposition is carried out in Sections 5.1 and 5.2. In Section 5.3 we state and prove a converse to it.

5.1. A preliminary proposition. Throughout this subsection let $p = (p_i)_{i=1}^\ell$ be a fixed positive probability vector. Let $\mu$ be the self-similar measure corresponding to $\Phi$ and $p$. Recall that $G$ is assumed to be discrete, and that we write $B$ in place of $A^*$. For a real number $x$ let $\|x\|$ be the distance from $x$ to its nearest integer, that is

$$\|x\| := \inf\{|x - k| : k \in \mathbb{Z}\}.$$ 

Recall that $A := \{1, \ldots, \ell\}$, and that a finite set of words $W \subset A^*$ is said to be a minimal cut-set for $A^*$ if every infinite sequence in $A^\mathbb{N}$ has a unique prefix in $W$. The purpose of this subsection is to prove the following proposition.

Proposition 5.2. Let $W$ be a minimal cut-set for $A^*$, and let $u, u' \in W$. Suppose that $G$ is generated by $\{g_w\}_{w \in W}$, and that $g_u = g_{u'}$. Then for every $\epsilon > 0$ there exists $C = C(\epsilon, W, p) > 1$ so that for all $V \in N_0$,

$$\sum_{j \geq 0} \|\langle V(\varphi_u(0) - \varphi_{u'}(0)), B^j \xi \rangle \|^2 \leq C \text{ for } \xi \in \mathbb{R}^d \text{ with } |\hat{\mu}(2\pi \xi)| \geq \epsilon.$$ 

For the rest of this subsection fix $W \subset A^*$ and $u, u' \in W$ as in the statement of the proposition. Note that since $W$ is a minimal cut-set, $(p_w)_{w \in W}$ is a probability vector. Let $I_1, I_2, \ldots$ be i.i.d. $W$-valued random words with $\mathbb{P}\{I_1 = w\} = p_w$ for $w \in W$. Set $Y_0 = 1_G$, and for $k \geq 1$ let $X_k := g_{I_k}$, $Y_k := X_1 \cdot \ldots \cdot X_k$, and

$$\tau_\beta(k) := \inf\{m \geq 1 : \psi Y_m \geq k\beta\}.$$ 

For $\xi \in \mathbb{R}^d$ and $w \in \{u, u'\}$ set,

$$\alpha_w(\xi) := \frac{1}{p_u + p_{u'}} \left| p_u e^{i\xi, \varphi_u(0)} + p_{u'} e^{i\xi, \varphi_{u'}(0)} \right|,$$

and for $w \in W \setminus \{u, u'\}$ write $\alpha_w(\xi) := 1$. Set $Z_{\xi,0} := 1$, and for $n \geq 1$ let

$$Z_{\xi,n} := \prod_{k=1}^n \alpha_{I_k}(\xi, Y_{k-1}).$$

Lemma 5.3. For $k \geq 1$ and $\xi \in \mathbb{R}^d$ we have $|\hat{\mu}(\xi)| \leq \mathbb{E}[Z_{\xi,\tau_\beta(k)}].$
Proof. Since $\mathcal{W}$ is a minimal cut-set and since $g_w = g_w'$, it follows that for $y \in \mathbb{R}^d$

$$|\hat{\mu}(y)| = \left| \sum_{w \in \mathcal{W}} p_w \int e^{i(y, \hat{\varphi}_w(x))} \, d\mu(x) \right|$$

$$= \left| \sum_{w \in \mathcal{W}} p_w e^{i(y, \hat{\varphi}_w(0))} \hat{\mu}(y,g_w) \right| \leq \sum_{w \in \mathcal{W}} p_w |\hat{\mu}(y,g_w)| \cdot \alpha_w(y).$$

Let $n \geq 0$, then by applying the last inequality with $y = \xi Y_n$,

$$Z_{\xi,n}|\hat{\mu}(\xi Y_n)| \leq Z_{\xi,n} \sum_{w \in \mathcal{W}} p_w |\hat{\mu}((\xi Y_n),g_w)| \cdot \alpha_w(\xi Y_n)$$

$$= \mathbb{E} \left[ Z_{\xi,n} |\hat{\mu}((\xi Y_n),g_{I_{n+1}})| \cdot \alpha_{I_{n+1}}(\xi Y_n) \mid I_1, \ldots, I_n \right]$$

$$= \mathbb{E} \left[ Z_{\xi,n+1}|\hat{\mu}(\xi Y_{n+1})| \mid I_1, \ldots, I_n \right].$$

This shows that $\{Z_{\xi,n}|\hat{\mu}(\xi Y_n)|\}_{n \geq 0}$ is a submartingale with respect to the filtration $\{\mathcal{F}_n\}_{n \geq 0}$, where $\mathcal{F}_n$ is the $\sigma$-algebra generated by $I_1, \ldots, I_n$. Thus, since $\tau_{\beta}(k)$ is a bounded stopping time with respect to the filtration $\{\mathcal{F}_n\}_{n \geq 0}$, and by Doob’s optional stopping theorem, we get

$$|\hat{\mu}(\xi)| = \mathbb{E}[Z_{\xi,0}|\hat{\mu}(\xi Y_0)|] \leq \mathbb{E} \left[ Z_{\xi,\tau_{\beta}(k)}|\hat{\mu}(\xi Y_{\tau_{\beta}(k)})| \right] \leq \mathbb{E} \left[ Z_{\xi,\tau_{\beta}(k)} \right],$$

which completes the proof of the lemma.

\[\square\]

Lemma 5.4. There exists a constant $C = C(\mathcal{W}, p) > 1$ so that,

$$\mathbb{P} \left\{ \gamma_{-\beta k} Y_{\tau_{\beta}(k)} = g \right\} > C^{-1} \text{ for every integer } k \geq C \text{ and } g \in \mathcal{N}.$$  

Proof. Set $q := \sum_{w \in \mathcal{W}} p_w g_w$ and $\lambda := \int \psi \, dq$. For $g \in \mathcal{N}$ let,

$$\rho(g) := \lambda^{-1} \mathbb{P} \{ \psi X_1 > \psi g \geq 0 \},$$

and write $\nu$ in place of $\rho |\mu_G|$. Since $\psi(G) = \beta \mathbb{Z}$ and $|N| < \infty$, it follows by our choice of $\mathbf{m}_G$ (see Section 2.4) that $\mathbf{m}_G \{ g \} = \beta |N|$ for $g \in G$. For $g \in \mathcal{N}$ we have $\psi(g) = 0$, and so $\rho(g) = \lambda^{-1}$. Since $G$ is generated by $\{g_w\}_{w \in \mathcal{W}}$ it holds that $q$ is adapted, and so we can apply Proposition 3.3. It follows that for $g \in \mathcal{N}$,

$$\lim_{k \to \infty} \mathbb{P} \left\{ \gamma_{-\beta k} Y_{\tau_{\beta}(k)} = g \right\} = \nu \{ g \} = \rho(g) \mathbf{m}_G \{ g \} = \beta / (\lambda |N|).$$

Since $N$ is finite this completes the proof of the lemma.

\[\square\]

Recall that $\psi(g_i)/\beta = l_i \in \mathbb{Z}_{>0}$ for $1 \leq i \leq \ell$. Given a word $i_1 \ldots i_n = w \in \Lambda^*$ we write $l_w$ in place of $l_{i_1} + \ldots + l_{i_n}$.

Lemma 5.5. There exists an integer $C = C(\mathcal{W}, p) > 1$ so that for every $k \in \mathbb{Z}_{>0}, V \in \mathcal{N}_0$ and $\xi \in \mathbb{R}^d$,

$$\mathbb{E} \left[ Z_{\xi,\tau_{\beta}(k)} \right] \leq \mathbb{E} \left[ Z_{\xi,\tau_{\beta}(k-C)} \right] (1 - C^{-1}(1 - \alpha_n(V B^{k-l}(\xi)))) .$$

Proof. Let $C \in \mathbb{Z}_{>1}$ be large with respect to $\Phi$, $p$ and $\mathcal{W}$. Set $l_{\text{max}} = \max_{w \in \mathcal{W}} l_w$, and suppose that $C > 2l_{\text{max}}$. Fix $k \in \mathbb{Z}_{>C}, V \in \mathcal{N}_0$ and $\xi \in \mathbb{R}^d$. Let $n_V \in \mathbb{N}$ be with $n_V = (0, V^{-1})$. Denote by $\mathcal{W}^*$ the set of finite words over $\mathcal{W}$. For $w_1 \ldots w_m = w \in \mathcal{W}^*$ we write,

$$g_w := g_{w_1} \cdot \ldots \cdot g_{w_m} \text{ and } l_w := l_{w_1} + \ldots + l_{w_m} .$$
Let,
\[ \mathcal{Y} := \{ w_1 \ldots w_m \in \mathcal{W}^* : \psi(g_{w_1 \ldots w_m}) \geq \beta(k - C) > \psi(g_{w_1 \ldots w_{m-1}}) \} \]

For \( y \in \mathcal{Y} \) set,
\[ \eta_y := \mathbb{P} \{ Y_{\tau_\beta(k-l_u)} = h^{k-l_u}n \mid I_1 \ldots I_{\tau_\beta(k-C)} = y \} \]

For \( m \in \mathbb{Z}_{\geq 0} \) and \( b \in \mathbb{Z}_{\geq 1} \) write,
\[ \tau_{\beta,m}(b) := \inf \{ j > m : \psi(X_{m+1} \ldots X_j) \geq b\beta \} \]

Fix \( w_1 \ldots w_m = y \in \mathcal{Y} \) for the moment. From (5.1) and since \( \psi(h) = \beta \), it follows that there exists \( n_y \in N \) with \( g_y = h^{\beta}n_y \). Additionally, by the definition of \( \mathcal{Y} \),
\[ (5.3) \quad k - C \leq l_y < k - C + l_{\text{max}} \]

Note that,
\[ \mathbb{P} \{ Y_m = g_y \mid I_1 \ldots I_{\tau_\beta(k-C)} = y \} = 1 \]

From this and since \( \psi(g_y) = \beta l_y < \beta(k - l_u) \),
\[ \mathbb{P} \{ \tau_\beta(k-l_u) = \tau_{\beta,m}(k-l_u - l_y) \mid I_1 \ldots I_{\tau_\beta(k-C)} = y \} = 1 \]

Hence, by multiplying from the left both sides of the equation \( Y_{\tau_\beta(k-l_u)} = h^{k-l_u}n \) by \( g_y^{-1} = n_y^{-1}h^{-l_y} \), we get
\[ \eta_y = \mathbb{P} \{ X_{m+1} \ldots X_{\tau_\beta(m-k-l_u+1)} = n_y^{-1}h^{k-l_u-l_y}n \mid I_1 \ldots I_{\tau_\beta(k-C)} = y \} = \mathbb{P} \{ Y_{\tau_\beta(k-l_u-l_y)} = n_y^{-1}h^{k-l_u-l_y}n \}, \]

where in last equality we have used the stationarity of the process \( \{ X_j \}_{j \geq 1} \). Set,
\[ z_y := \gamma - \beta(k-l_u-l_y)n_y^{-1}h^{k-l_u-l_y}n \]

then
\[ (5.4) \quad \eta_y = \mathbb{P} \{ \gamma - \beta(k-l_u-l_y)Y_{\tau_\beta(k-l_u-l_y)} = z_y \} \]

From \( \psi \circ \gamma = \text{Id} \), \( \psi(h) = \beta \) and \( n_y, n \) \( \forall \) \( y \in N \) it follows that \( z_y \in N \). Also, by (5.3) we have \( k - l_u - l_y > C - 2l_{\text{max}} \). Hence, by Lemma 5.4 and by assuming that \( C \) is sufficiently large, it follows that \( \eta_y > p^{-1}_uC^{-1} \). This holds for all \( y \in \mathcal{Y} \), which implies that almost surely
\[ \mathbb{P} \{ Y_{\tau_\beta(k-l_u)} = h^{k-l_u}n \mid I_1 \ldots I_{\tau_\beta(k-C)} \} \geq p^{-1}_uC^{-1} \]

From the last inequality we get,
\[ (5.5) \quad \mathbb{P} \{ Y_{\tau_\beta(k-1)} = h^{k-l_u}n \} \text{ and } I_{\tau_\beta(1)} = u \mid I_1 \ldots I_{\tau_\beta(k-C)} \}
\[ = \mathbb{P} \{ Y_{\tau_\beta(k-l_u)} = h^{k-l_u}n \} \text{ and } I_{\tau_\beta(1)} = u \mid I_1 \ldots I_{\tau_\beta(k-C)} \}
\[ = \mathbb{P} \{ Y_{\tau_\beta(k-l_u)} = h^{k-l_u}n \} \mathbb{P} \{ I_1 = u \} \geq C^{-1}. \]

Since \( C > l_{\text{max}} \) we have \( \tau_\beta(k - C) \leq \tau_\beta(k - 1) \). Thus, since \( \alpha_w(x) \leq 1 \) for all \( w \in \mathcal{W} \) and \( x \in \mathbb{R}^d \), it follows that \( Z_{\xi,\tau_\beta(k-C)} \geq Z_{\xi,\tau_\beta(k-1)} \). Hence,
\[ Z_{\xi,\tau_\beta(k)} = Z_{\xi,\tau_\beta(k-1)} \cdot \alpha_{I_{\tau_\beta(1)}(\xi,Y_{\tau_\beta(k-1)})} \leq Z_{\xi,\tau_\beta(k-C)} \cdot \alpha_{I_{\tau_\beta(1)}(\xi,Y_{\tau_\beta(k-1)})}. \]
Hence, by Lemma 5.3 we get,
\[
\begin{align*}
\mathbb{E}[Z_{\xi,T_\beta(k)} | I_1, \ldots, I_{T_\beta(k-C)}] & \leq \mathbb{E}[Z_{\xi,T_\beta(k-C)} | I_1, \ldots, I_{T_\beta(k-C-1)}] \\
& \leq Z_{\xi,T_\beta(k-C)} (1 - C^{-1} + C^{-1} \alpha_u (\xi, h^{k-l_u} \nu_V)) \\
& = Z_{\xi,T_\beta(k-C)} (1 - C^{-1} + C^{-1} \alpha_u (VB^{k-l_u} \xi)) .
\end{align*}
\]

This gives,
\[
\begin{align*}
\mathbb{E}[Z_{\xi,T_\beta(k)}] &= \mathbb{E}[\mathbb{E}[Z_{\xi,T_\beta(k)} | I_1, \ldots, I_{T_\beta(k-C-1)}]] \\
& \leq \mathbb{E}[Z_{\xi,T_\beta(k-C)}] (1 - C^{-1} + C^{-1} \alpha_u (VB^{k-l_u} \xi)),
\end{align*}
\]

which completes the proof of the lemma.

Proof of Proposition 5.3. Let \(0 < \epsilon < 1\), let \(C \in \mathbb{Z}_{>1}\) be large with respect to \(\epsilon\), \(\mathcal{W}, p\) and \(\Phi\), let \(V \in \mathbb{N}_0\), let \(\xi_0 \in \mathbb{R}^d\) be with \(|\hat{\mu}(2\pi \xi_0)| \geq \epsilon\), and write \(\xi = 2\pi \xi_0\). For \(y \in \mathbb{R}^d\) set \(\Psi(y) = 1 - \alpha_u(y)\), and note that \(0 \leq \Psi(y) \leq 1\). By Lemma 5.5 it follows that for \(k \in \mathbb{Z}_{>C}\),
\[
\mathbb{E}[Z_{\xi,T_\beta(k)}] \leq \mathbb{E}[Z_{\xi,T_\beta(k-C)}] (1 - C^{-1} \Psi(VB^{k-l_u} \xi)) .
\]

Iterating this and using the fact that \(0 \leq Z_{\xi,n} \leq 1\) for all \(n \in \mathbb{Z}_{\geq1}\), we get
\[
\mathbb{E}[Z_{\xi,T_\beta(k)}] \leq \prod_{j=0}^{[k/C]-2} (1 - C^{-1} \Psi(VB^{k-jC-l_u} \xi)),
\]

where \([k/C]\) is the smallest integer which is at least as large as \(k/C\). Let \(n \in \mathbb{Z}_{\geq1}\), then by applying the last inequality for \(nC < k \leq nC + C\) we get,
\[
\prod_{k=nC+1}^{nC+C} \mathbb{E}[Z_{\xi,T_\beta(k)}] \leq \prod_{k=nC+1}^{nC+C} \prod_{j=0}^{[k/C]-2} (1 - C^{-1} \Psi(VB^{k-jC-l_u} \xi)) \\
= \prod_{k=1}^{C} \prod_{j=0}^{n-1} (1 - C^{-1} \Psi(VB^{nC+k-jC-l_u} \xi)) \\
= \prod_{j=C+1}^{nC+C} (1 - C^{-1} \Psi(VB^{j-l_u} \xi)) .
\]

Hence, by Lemma 5.3
\[
e^C \leq |\hat{\mu}(\xi)|^C \leq \prod_{j=C+1}^{nC+C} (1 - C^{-1} \Psi(VB^{j-l_u} \xi)) .
\]

From this and the inequality \(1 + t \leq e^t\),
\[
e^C \leq \exp \left( -C^{-1} \sum_{j=C+1}^{nC+C} \Psi(VB^{j-l_u} \xi) \right) .
\]

Since this holds for all \(n \in \mathbb{Z}_{\geq1}\),
\[
C^2 \ln \epsilon^{-1} \geq \sum_{j=C+1}^{\infty} \Psi(VB^{j} \xi) .
\]  

(5.6)
Set $\delta := p_a/(p_a + p_{a'})$ and $\delta' := p_{a'}/(p_a + p_{a'})$. By Taylor’s theorem, given $0 \leq s \leq 1/8$ there exists $0 \leq t \leq 2\pi s$ so that
\[
\cos(2\pi s) - 1 = -\frac{\cos(t)}{2}(2\pi s)^2 \leq -\frac{\cos(\pi/4)}{2}(2\pi s)^2 \leq -s^2.
\]
Hence,
\[
|\delta e^{2\pi is} + \delta'|^2 = (\delta \cos(2\pi s) + \delta')^2 + \delta^2 \sin^2(2\pi s) = 1 + 2\delta'(\cos(2\pi s) - 1) \leq 1 - 2\delta's^2,
\]
and so,
\[
1 - |\delta e^{2\pi is} + \delta'| \geq 1 - (1 - 2\delta's^2)^{1/2} \geq \delta's^2.
\]
It follows that if $y \in \mathbb{R}^d$ satisfies $\|\langle y, \varphi_u(0) - \varphi_{u'}(0)\rangle\| \leq 1/8$, then
\[
\Psi(2\pi y) = 1 - |\delta \exp (2\pi i \|\langle y, \varphi_u(0) - \varphi_{u'}(0)\rangle\|) + \delta'|
\]
and so,
\[
1 - |\delta e^{2\pi is} + \delta'| \geq 1 - (1 - 2\delta's^2)^{1/2} \geq \delta's^2.
\]
It follows that if $y \in \mathbb{R}^d$ satisfies $\|\langle y, \varphi_u(0) - \varphi_{u'}(0)\rangle\| > 1/8$, then
\[
\Psi(2\pi y) \geq 1 - \frac{\delta'}{4} \geq 1 - \frac{\delta'}{2}
\]
and so,
\[
1 - |\delta e^{2\pi is} + \delta'| \geq 1 - (1 - 2\delta's^2)^{1/2} \geq \delta's^2.
\]
It follows that if $y \in \mathbb{R}^d$ satisfies $\|\langle y, \varphi_u(0) - \varphi_{u'}(0)\rangle\| > 1/8$, then
\[
\Psi(2\pi y) \geq 1 - |\delta \exp (2\pi i \|\langle y, \varphi_u(0) - \varphi_{u'}(0)\rangle\|) + \delta'| \geq \delta's^2.
\]
Now recall that $\xi = 2\pi \xi_0$, then from (5.6), (5.7) and (5.8)
\[
C^2 \ln^{-1} \geq \sum_{j=C+1}^{\infty} \Psi(2\pi V B^j \xi_0) \geq \frac{1}{4} \delta's \sum_{j=C+1}^{\infty} \|\langle V B^j \xi_0, \varphi_u(0) - \varphi_{u'}(0)\rangle\|^2,
\]
which completes the proof of the proposition. \qed

5.2. Proof of Proposition 5.1. We continue to assume that $G$ is discrete. In order to apply Proposition 5.2 we need the following lemma.

**Lemma 5.6.** Suppose that $a_1 = 0$. Then there exists $\mathcal{W} \subset \Lambda^*$, $L \in \mathbb{Z}_{\geq 1}$ and $\{u_j\}_{i=1}^L, \{u'_j\}_{i=1}^L \subset \mathcal{W}$ so that,

- (1) $\mathcal{W}$ is a minimal cut-set for $\Lambda^*$;
- (2) $G$ is generated by $\{g_\omega\}_{\omega \in \mathcal{W}}$;
- (3) $g_\omega = g_{\omega'}$ for $1 \leq j \leq \ell$;
- (4) $\varphi_{u_j}(0) - \varphi_{u_j}(0) = a_j - A^L a_j$ for $1 \leq j \leq \ell$;
- (5) $A^L V = \Lambda^L V$, $V \in N_0$.

**Proof.** For every $k \geq 1$ we have $h^{k_1} g_1^{-k} \in N$. Since $N$ is finite there exist $k_1 > k_2 \geq 1$ with $h^{k_1} h_1^{-k_1} = h^{k_2} g_1^{-k_2}$, and so $g_1^{k_1-k_2} = h^{k_1-k_2 j_1}$. For every $g, g' \in G$ it holds that $[g, g'] \in N$, where $[g, g']$ is the commutator of $g$ and $g'$. Since $N$ is finite there exist $m_1 > m_2 \geq 1$ such that $g^{m_1}, g' = g^{m_2}$, and so $g^{m_1-m_2} g' = g' g^{m_1-m_2}$. It follows that there exist $b \in \mathbb{Z}_{\geq 1}$ such that $g_{b_1} = h^{b_1}$, $g_{b_1}^b g_j = g_j g_{b_1}^b$ for $1 \leq j \leq \ell$, $g_{b_2} g_1 = g_1 g_{b_2}$, and $h^b n = h^b n$ for $n \in N$. We set $L := b_1$. 29
Recall that $h = (\beta, U)$. For $V \in N_0$ we have $(0, V) \in N$, thus

$$(b\beta, VU^b) = (0, V)h^b = h^b(0, V) = (b\beta, U^b V),$$

and so $VU^b = U^b V$. Since $A = 2^{-\beta} U$ this implies that $VA^k = A^k V$, and so the fifth condition in the statement of the lemma is satisfied.

For $m \geq 1$ denote the set of $m$-words over $A$ by $\Lambda^m$. For $1 \leq j \leq \ell$ we write $j^m$ for the word $i_1 \ldots i_m \in \Lambda^m$ with $i_k = j$ for $1 \leq k \leq m$. Given $m_1, m_2 \geq 1$, $w_1 \in \Lambda^{m_1}$ and $w_2 \in \Lambda^{m_2}$, we write $w_1 w_2 \in \Lambda^{m_1 + m_2}$ for the concatenation of $w_1$ with $w_2$.

Set

$$W := (\Lambda^{b+1} \setminus \{ \ell^b \}) \cup \{ \ell^b 1 i : 1 \leq i \leq \ell \}.$$ 

It is clear that $W$ is a minimal cut-set for $\Lambda^\ell$. For $1 \leq j \leq \ell$ set $u_j := j^1 b$ and $u_j' := 1^b j$. Note that since $\Phi$ is affinely irreducible we must have $\ell > 1$. From this and $b > 1$, it follows that $u_j, u_j' \in W$.

From $g_i^b g_j = g_j g_i^b$ it follows that the third condition is satisfied. From $g_1 = (\log r_1^{-1}, U_1)$, $h = (\beta, U)$ and $g_i^b = h^{b_{1i}}$ it follows that,

$$r_1^b U_1^b = 2^{-\beta b_{1i}} U^{b_{1i}} = A^L.$$ 

Thus, since $a_1 = 0$

$$\varphi_{u_j}(0) - \varphi_{u_j'}(0) = a_j - r_1^b U_1^b a_j = a_j - A^L a_j,$$

which shows that the fourth condition is satisfied.

It remains to show that $G$ is generated by $\{ g_w \}_{w \in W}$. By definition $G$ is the closed subgroup of $\mathbb{R} \times O(d)$ generated by $\{ g_i \}_{i=1}^\ell$. From this and since $G$ is discrete, it follows that $G$ is generated by $\{ g_i \}_{i=1}^\ell$. Write $G_1$ for the group generated by $\{ g_w \}_{w \in W}$. For every $1 \leq i \leq \ell$ we have $1^b, \ell^b i, 1^b i \in W$. Hence from $g_i^{b_i} g_{1^b i} = g_{1^b i}^{b_{1i}}$

$$g_i = (g_{1^b i})^{-1} g_{1^b i} \in G_1.$$ 

This shows that $G_1 = G$, which completes the proof of the lemma.

The treatment of the 1-dimensional case, carried out in [6] and [29], relies on a classical theorem of Pisot (see [6] Theorem 2.1)). In the proof of Proposition 5.1 we shall need the following extension of this result. It follows directly from [23, Chapter III, Theorem III] together with [17, Theorem 1]. A result similar to [17, Theorem 1] was obtained in [22, Lemma 2].

**Theorem 5.7.** Let $k \geq 1$ and $\theta_1, \ldots, \theta_k, \lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be with $|\theta_j| > 1$ and $\lambda_j \neq 0$ for $1 \leq j \leq k$, and $\theta_j \neq \theta_i$ for $1 \leq j < i \leq k$. For $n \geq 0$ set $\eta_n = \sum_{j=1}^k \lambda_j \theta_j^n$, and suppose that $\eta_n \in \mathbb{R}$ for all $n \geq 0$. Moreover assume that $\sum_{n \geq 0} \| \eta_n \|^2 < \infty$. Then,

1. $\{ \theta_1, \ldots, \theta_k \}$ is a P.V. $k$-tuple;
2. $\lambda_j \in \mathbb{Q}(\theta_j)$ for each $1 \leq j \leq k$;
3. if $1 \leq j, i \leq k$ are such that $\theta_j$ and $\theta_i$ are conjugates over $\mathbb{Q}$ and $\sigma : \mathbb{Q}(\theta_j) \rightarrow \mathbb{Q}(\theta_i)$ is an isomorphism with $\sigma(\theta_j) = \theta_i$, then $\sigma(\lambda_j) = \lambda_i$.

**Proof of Proposition 5.1.** Recall that $A = 2^{-\beta} U$ and $B = A^\ell$, where $U \in O(d)$. Let $\theta_1, \ldots, \theta_s \in \mathbb{C}$ be the distinct eigenvalues of $A^{-1}$. For $1 \leq j \leq s$ let $V_j \subset \mathbb{C}^d$ be the eigenspace of $A^{-1}$ corresponding to $\theta_j$. Since $B^{-1} = (A^{-1})^\ell$, the numbers $\theta_1, \ldots, \theta_s$ are also the distinct eigenvalues of $B^{-1}$, and $V_j$ is the eigenspace of $B^{-1}$ corresponding to $\theta_j$ for each $1 \leq j \leq s$.

Assume that there exists a probability vector $p = (p_i)_{i=1}^\ell > 0$ so that the self-similar measure $\mu$ corresponding to $\Phi$ and $p$ is non-Rajchman. There exist $\epsilon > 0$
and \(\xi_1, \xi_2, \ldots \in \mathbb{R}^d\) so that \(|\xi_k| \geq 1\) and \(|\tilde{\mu}(2\pi \xi_k)| \geq \epsilon\) for all \(k \geq 1\), and also \(|\xi_k| \to \infty\). For \(k \geq 1\) set
\[
n_k := \min\{n \geq 1 : |B^n \xi_k| \leq 1\},
\]
then \(2^{-\beta} \leq |B^n \xi_k| \leq 1\). Thus, by moving to a subsequence without changing the notation, we may assume that there exists \(0 \neq \xi \in \mathbb{R}^d\) so that \(B^n \xi_k \to \xi\).

Recall that we assume \(a_1 = 0\), and let \(W, \ell, \{u_\ell\}_{\ell=1}^\ell\) and \(\{u'_\ell\}_{\ell=1}^\ell\) be as obtained in Lemma \[5.6\]. Let \(C > 1\) be large with respect to \(\epsilon, W, \Phi\) and \(p\). For \(1 \leq i \leq \ell\) and \(V \in N_0\) we have \(g_{ai} = g_{ai'}\) and,
\[
V(\varphi_{ai}(0) - \varphi_{ai'}(0)) = V(a_i - A^L a_i) = (I - A^L)V a_i,
\]
where \(I\) is the identity operator here. Set \(b_{i,V} := (I - A^L)V a_i\), then by Proposition \[5.2\] it follows that for all \(k \geq 1\),
\[
C \geq \sum_{n \geq 0} \| V(\varphi_{ai}(0) - \varphi_{ai'}(0)) B^n \xi_k \|^2 = \sum_{n \geq -n_k} \| b_{i,V} B^n \xi_k \|^2 .
\]
From \(|\xi_k| \to \infty\) it follows that \(n_k \to \infty\). Thus, for every fixed \(T \geq 1\) and \(k \geq 1\) large enough with respect to \(T\),
\[
\sum_{n=0}^T \| b_{i,V} B^{-n} B^n \xi_k \|^2 \leq C .
\]
From this and since \(B^n \xi_k \to \xi\),
\[
\sum_{n=0}^\infty \| b_{i,V} B^{-n} \xi \|^2 \leq C .
\]
Hence, since this holds for every \(T \geq 1\),
\[
(5.9) \quad \sum_{n=0}^\infty \| b_{i,V} B^{-n} \xi \|^2 < \infty \text{ for all } 1 \leq i \leq \ell \text{ and } V \in N_0 .
\]

Recall that for a linear subspace \(\mathbb{V}\) of \(\mathbb{C}^d\) we denote by \(\pi_\mathbb{V}\) the orthogonal projection onto \(\mathbb{V}\). For \(1 \leq j \leq s\) set
\[
\zeta_j := (1 - \frac{1}{\theta_j^L}) \pi_\mathbb{V}_j \xi ,
\]
where we consider \(\xi\) as a vector in \(\mathbb{C}^d\) here. Let \(\lambda_j : \mathbb{R}^d \to \mathbb{C}\) be with \(\lambda_j(x) = \langle x, \zeta_j \rangle\) for \(x \in \mathbb{R}^d\). Regarding \(\mathbb{C}\) as a 2-dimensional vector space over \(\mathbb{R}\), the maps \(\lambda_1, \ldots, \lambda_s\) are \(\mathbb{R}\)-linear. Additionally, for \(1 \leq i \leq \ell, V \in N_0\) and \(n \geq 0\)
\[
(5.10) \quad \langle b_{i,V} B^{-n} \xi \rangle = \left( I - A^L \right) V a_i , \sum_{j=1}^s \theta_j^n \pi_\mathbb{V}_j \xi ,
\]
which in particular implies that \(\sum_{j=1}^s \theta_j^n \lambda_j(V a_i) \in \mathbb{R}\). From \[5.9\] and \[5.10\],
\[
(5.11) \quad \sum_{n=0}^\infty \| \sum_{j=1}^s \theta_j^n \lambda_j(V a_i) \|^2 < \infty \text{ for all } 1 \leq i \leq \ell \text{ and } V \in N_0 .
\]
For every $1 \leq j \leq s$ we have $|\theta_j| = 2^j > 1$. From this and since $\xi \neq 0$, we get that there exists $1 \leq j_0 \leq s$ so that $\zeta_{j_0} \neq 0$. Hence $\lambda_{j_0}$ is not identically 0, and so $\ker \lambda_{j_0}$ is a proper subspace of $\mathbb{R}^d$. Let us show that,

\begin{equation}
\lambda_{j_0}(V a_i) \neq 0 \text{ for some } 1 \leq i \leq \ell \text{ and } V \in N_0.
\end{equation}

Assume by contradiction that this is false. Then,

$$\{a_i\}_{i=1}^s \subseteq \bigcap_{V \in N_0} V(\ker \lambda_{j_0}) =: W.$$ 

For $x \in \mathbb{R}^d$,

$$\lambda_{j_0}(A^{-1} x) = \langle A^{-1} x, \zeta_{j_0} \rangle = \langle x, B^{-1} \zeta_{j_0} \rangle = \theta_j \lambda_{j_0}(x),$$

from which it follows that $A(\ker \lambda_{j_0}) = \ker \lambda_{j_0}$. Moreover, from $N \triangleleft G$ it follows that $AN_0 = N_0 A$, which implies

$$A(W) = \bigcap_{V \in N_0} V A(\ker \lambda_{j_0}) = W.$$ 

Since $N_0$ is a group, we also have $V(W) = W$ for all $V \in N_0$. By (5.1), for every $1 \leq i \leq \ell$ there exists $V_i \in N_0$ so that $\varphi_i(x) = V_i A_i x + a_i$. From all of this it follows that $\varphi_i(W) = W$ for all $1 \leq i \leq \ell$. Since $W \subseteq \ker \lambda_{j_0}$ and since $\ker \lambda_{j_0}$ is a proper subspace of $\mathbb{R}^d$, this contradicts the affine irreducibility of $\Phi$, which shows that (5.12) must hold. For $1 \leq i \leq \ell$ and $V \in N_0$ set,

$$J_{i, V} := \{ 1 \leq j \leq s : \lambda_j(V a_i) \neq 0 \}.$$ 

From (5.12) it follows that $J_{i, V} \neq \emptyset$ for some $1 \leq i \leq \ell$ and $V \in N_0$.

For $1 \leq i \leq \ell$ and $V \in N_0$ it follows from (5.11) and Theorem 5.7 that,

1. $\{ \theta_j \}_{j \in J_{i, V}}$ is a P.V., $|J_{i, V}|$-tuple or $J_{i, V} = \emptyset$;
2. $\lambda_j(V a_i) \in \mathbb{Q}(\theta_j)$ for $1 \leq j \leq s$;
3. $\sigma(\lambda_j(V a_i)) = \lambda_{j_2}(V a_i)$ for every $j_1, j_2 \in J_{i, V}$ and isomorphism $\sigma : \mathbb{Q}(\theta_j) \rightarrow \mathbb{Q}(\theta_{j_2})$ with $\sigma(\theta_{j_1}) = \theta_{j_2}$ (if such a $\sigma$ exists).

Let $1 \leq i_0 \leq \ell$ and $V_0 \in N_0$ be with $J_{i_0, V_0} \neq \emptyset$, so that $\{ \theta_j \}_{j \in J_{i_0, V_0}}$ is a P.V. $|J_{i_0, V_0}|$-tuple. By the definition of a P.V. tuple, there exists a nonempty subset $J$ of $J_{i_0, V_0}$ so that $\{ \theta_j \}_{j \in J}$ is a P.V. $|J|$-tuple and $\theta_{j_1}, \theta_{j_2}$ are conjugates over $\mathbb{Q}$ for all $j_1, j_2 \in J$. For $j \in J$ we have,

$$\langle V_0 a_{i_0}, \zeta_j \rangle = \lambda_j(V_0 a_{i_0}) \neq 0,$$

and so $\zeta_j \neq 0$. Recall that $\zeta_j := (1 - \theta_j - \bar{\theta}_j - \theta_j) \pi_V \xi$, which implies $A^{-1} \zeta_j = \theta_j \zeta_j$ for $j \in J$. It remains to construct to polynomials $P_{i, V}$. Let $1 \leq i \leq \ell$ and $V \in N_0$ be given. Since $\{ \theta_j \}_{j \in J}$ are algebraic conjugates, since $|\theta_j| > 1$ for $j \in J$, and since $\{ \theta_j \}_{j \in J_{i, V}}$ is either empty or a P.V. tuple, it follows that $J \cap J_{i, V} = \emptyset$ or $J \subseteq J_{i, V}$. If $J \cap J_{i, V} = \emptyset$ we set $P_{i, V}(X) := 0$. For $j \in J$ we have $j \notin J_{i, V}$, and so

$$(V a_i, \zeta_j) = \lambda_j(V a_i) = 0 = P_{i, V}(\theta_j).$$

Next suppose that $J \subseteq J_{i, V}$, and let $j_1 \in J$. Since $\theta_{j_1}$ is algebraic and from $\lambda_{j_1}(V a_i) \in \mathbb{Q}(\theta_{j_1})$, it follows that there exist $P_{i, V}(X) \in \mathbb{Q}[X]$ so that $\lambda_{j_1}(V a_i) = P_{i, V}(\theta_{j_1})$. Let $j \in J$, then $\theta_j$ and $\theta_{j_1}$ are conjugates over $\mathbb{Q}$, and so there exists an isomorphism $\sigma : \mathbb{Q}(\theta_{j_1}) \rightarrow \mathbb{Q}(\theta_j)$ with $\sigma(\theta_{j_1}) = \theta_j$. From this and $j_1, j \in J \subseteq J_{i, V}$ we get,

$$\langle V a_i, \zeta_j \rangle = \lambda_j(V a_i) = \sigma(\lambda_{j_1}(V a_i)) = \sigma(P_{i, V}(\theta_{j_1})) = P_{i, V}(\sigma(\theta_{j_1})) = P_{i, V}(\theta_j).$$
and so $P_{i,V}$ satisfies the required property. This completes the proof of the proposition.

5.3. **Construction of non-Rajchman self-similar measures.** The purpose of this subsection is to prove the following converse to Proposition 5.1.

**Proposition 5.8.** Suppose that $G$ is discrete, and let $A$ and $N_0$ be as defined before the statement of Proposition 5.7. Assume that there exist $k \geq 1$, $\theta_1, \ldots, \theta_k \in \mathbb{C}$ and $\zeta_1, \ldots, \zeta_k \in \mathbb{C}^d \setminus \{0\}$, so that

1. $\{\theta_1, \ldots, \theta_k\}$ is a P.V. $k$-tuple;
2. $A^{-1}\zeta_j = \theta_j\zeta_j$ for $1 \leq j \leq k$;
3. for every $1 \leq i \leq \ell$ and $V \in N_0$ there exists $P_{i,V} \in \mathbb{Q}[X]$ so that $\langle Va_i, \zeta_j \rangle = P_{i,V}(\theta_j)$ for all $1 \leq j \leq k$;

Then there exists a positive probability vector $p = (p_i)_{i=1}^s$ so that the self-similar measure corresponding to $\Phi$ and $p$ is non-Rajchman.

The proof of the proposition relies on the following lemma. A version of it can be found in [8, Theorem 3.5], but we provide the short proof for the reader’s convenience.

**Lemma 5.9.** Let $\{\theta_1, \ldots, \theta_k\}$ be a P.V. $k$-tuple and let $P \in \mathbb{Z}[X]$. Then there exist $C > 1$ and $0 < \delta < 1$ such that,

\[
\|P(\theta_1)^{n_1} + \ldots + P(\theta_k)^{n_k}\| \leq C\delta^n \text{ for all } n \geq 0.
\]

**Proof.** Let $Q \in \mathbb{Z}[X]$ be the monic polynomial of smallest degree with $Q(\theta_j) = 0$ for $1 \leq j \leq k$. Let $\theta_{k+1}, \ldots, \theta_s$ be the remaining roots of $Q$. Set

\[
\delta := \max_{1 \leq j \leq s} |\theta_j| \text{ and } C := \sum_{j=k+1}^s |P(\theta_j)|,
\]

then $0 < \delta < 1$ since $\{\theta_1, \ldots, \theta_k\}$ is a P.V. $k$-tuple. Since $\theta_1, \ldots, \theta_s$ are all the roots of $Q$, and by the fundamental theorem of symmetric polynomials, it follows that for all $n \geq 0$

\[
P(\theta_1)^{n_1} + \ldots + P(\theta_s)^{n_s} \in \mathbb{Z}.
\]

Hence,

\[
\|\sum_{j=1}^k P(\theta_j)^{n_j}\| \leq \sum_{j=k+1}^s |P(\theta_j)| \leq C\delta^n,
\]

which completes the proof of the lemma. \qed

The following lemma is a consequence of the affine irreducibility of $\Phi$. For $(z_1, \ldots, z_d) = z \in \mathbb{C}^d$ we write $\bar{z}$ in place of $(\overline{z_1}, \ldots, \overline{z_d})$.

**Lemma 5.10.** Assume the conditions of Proposition 5.8 are satisfied. Let $1 \leq j_1, j_2 \leq k$ be with $\theta_{j_2} = \overline{\theta_{j_1}}$, then $\zeta_{j_2} = \overline{\zeta_{j_1}}$.

**Proof.** The proof is similar to the argument used in the proof of Proposition 5.1 to establish (5.12). Set,

\[
\mathbb{V} := \{x \in \mathbb{R}^d : \langle x, \zeta_{j_2} - \overline{\zeta_{j_1}} \rangle = 0\} \text{ and } \mathbb{W} := \cap_{V \in N_0} V(\mathbb{V}) .
\]

For $1 \leq i \leq \ell$ and $V \in N_0$,

\[
\langle Va_i, \zeta_{j_2} - \overline{\zeta_{j_1}} \rangle = P_{i,V}(\theta_{j_2}) - \overline{P_{i,V}(\theta_{j_1})} = 0,
\]

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and so \(a_i \in \mathbb{W}\). For \(x \in \mathbb{V}\),
\[
\langle A^{-1}x, \zeta_{j_2} - \zeta_{j_1} \rangle = \langle x, B^{-1}\zeta_{j_2} - B^{-1}\zeta_{j_1} \rangle = \theta_{j_2} \langle x, \zeta_{j_2} - \zeta_{j_1} \rangle = 0,
\]
and so \(A(\mathbb{V}) = \mathbb{V}\). Moreover, since \(AN_0 = N_0A\),
\[
A(\mathbb{W}) = \cap_{V \in N_0} VA(\mathbb{V}) = \mathbb{W}.
\]
Since \(N_0\) is a group, we also have \(V(\mathbb{W}) = \mathbb{W}\) for all \(V \in N_0\). By \(5.1\), for every \(1 \leq i \leq \ell\) there exists \(V_i \in N_0\) so that \(\varphi_i(x) = V_iA^tx + a_i\). From all of this it follows that \(\varphi_i(\mathbb{W}) = \mathbb{W}\) for all \(1 \leq i \leq \ell\). Since \(\Phi\) is affinely irreducible and \(\mathbb{W} \subset \mathbb{V}\), we must have \(\mathbb{V} = \mathbb{R}^d\). This implies that \(\zeta_{j_2} = \zeta_{j_1}\), which completes the proof of the lemma.

The following lemma will enable us to assume that \(a_1 = 0\), which will be useful in the proof of Proposition \(5.8\).

**Lemma 5.11.** Assume the conditions of Proposition \(5.8\) are satisfied. Suppose also that \(\theta_1, \ldots, \theta_k\) are all conjugates over \(\mathbb{Q}\). For \(x \in \mathbb{R}^d\) set \(T_x = x - (I - r_1U_1)^{-1}a_1\), where \(I\) is the identity operator. Then \(T \circ \varphi_1 \circ T^{-1}(0) = 0\), and for every \(1 \leq i \leq \ell\) and \(V \in N_0\) there exists \(Q_i,V \in \mathbb{Q}[X]\) so that
\[
\langle V T \circ \varphi_1 \circ T^{-1}(0), \zeta_j \rangle = Q_i,V(\theta_j) \quad \text{for all } 1 \leq j \leq k.
\]

**Proof.** For \(1 \leq i \leq \ell\) we have,
\[
(T \circ \varphi_i \circ T^{-1}(0)) = a_i - (I - r_1U_i)(I - r_1U_1)^{-1}a_1,
\]
which shows that \(T \circ \varphi_1 \circ T^{-1}(0) = 0\).

Let \(V \in N_0\). By \(5.2\), since \(BN_0B^{-1} = N_0\) and since \(N_0\) is finite, there exists \(m \geq 1\) so that \(r_1^mU_1^{-m} = B^m, V = \overline{B}^m\). Additionally, for every \(b \in \mathbb{Z}_{\geq 0}\) there exists \(V_b \in N_0\) so that \(r_1^bU_1^{-b}V = V_b\). Set \(S := \sum_{b=0}^{m-1} r_1^bU_1^{-b}\), then for \(1 \leq j \leq k\),
\[
\langle a_1, SV_1 \zeta_j \rangle = \sum_{b=0}^{m-1} \langle a_1, V_bB^{bl_1} \zeta_j \rangle = \sum_{b=0}^{m-1} \theta_j^{-bl_1} P_{1,V_b^{-1}}(\theta_j).
\]
On the other hand, since \((I - r_1U_1^{-1})SV_1 \zeta_j = (I - r_1^mU_1^{-m})V_1 \zeta_j = V(I - B^{ml_1}) \zeta_j = (1 - \theta_j^{-ml_1})V \zeta_j\), we have
\[
\langle a_1, SV_1 \zeta_j \rangle = \langle (I - r_1U_1)^{-1}a_1, (I - r_1U_1^{-1})SV_1 \zeta_j \rangle = (1 - \theta_j^{-ml_1}) \langle V^{-1}(I - r_1U_1)^{-1}a_1, \zeta_j \rangle.
\]
From this and \(5.14\), we get,
\[
\langle V^{-1}(I - r_1U_1)^{-1}a_1, \zeta_j \rangle = (1 - \theta_j^{-ml_1}) \sum_{b=0}^{m-1} \theta_j^{-bl_1} P_{1,V_b^{-1}}(\theta_j) \in \mathbb{Q}(\theta_j).
\]
Since \(\theta_1, \ldots, \theta_k\) are algebraic conjugates, it follows that for every \(V \in N_0\) there exists \(Q_V \in \mathbb{Q}[X]\) so that
\[
\langle V(I - r_1U_1)^{-1}a_1, \zeta_j \rangle = Q_V(\theta_j) \quad \text{for } 1 \leq j \leq k.
\]
Fix $1 \leq i \leq \ell$ and $V \in N_0$. There exists $V' \in N_0$ so that $r_i VU_i = A^i V'$. Hence for $1 \leq j \leq k$,
\[
\langle r_i VU_i (I - r_i U_1)^{-1} a_1, \zeta_j \rangle = \langle V' (I - r_i U_1)^{-1} a_1, B^i \zeta_j \rangle = \theta_j^{-1} Q_{V'}(\theta_j).
\]
It follows that there exists $R_{i,V} \in \mathbb{Q}[X]$ so that,
\[
\langle r_i VU_i (I - r_i U_1)^{-1} a_1, \zeta_j \rangle = R_{i,V}(\theta_j) \quad \text{for} \quad 1 \leq j \leq k.
\]
From this, (5.15) and (5.13), we get that for \(1 \leq j \leq k\)
\[
\langle VT \circ \varphi_i \circ T^{-1}(0), \zeta_j \rangle = P_{t,V}(\theta_j) - Q_{V}(\theta_j) + R_{i,V}(\theta_j),
\]
which completes the proof of the lemma.

**Proof of Proposition 5.8.** There exists $\emptyset \neq J \subset \{1,...,k\}$ so that \(\{\theta_j\}_{j \in J}\) is a P.V. $|J|$-tuple, and such that $\theta_{j_1}$ and $\theta_{j_2}$ are conjugates over $\mathbb{Q}$ for all $j_1, j_2 \in J$. Thus, be replacing $\{\theta_j\}_{j=1}^k$ with $\{\theta_j\}_{j \in J}$ and $\{\zeta_j\}_{j=1}^k$ with $\{\zeta_j\}_{j \in J}$, without changing the notation, we may assume that $\theta_1, ..., \theta_k$ are all conjugates over $\mathbb{Q}$.

Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be as in Lemma 5.11. By that lemma $T \circ \varphi_1 \circ T^{-1}(0) = 0$, and there exists $M \in \mathbb{Z}_g$ so that for every $1 \leq i \leq \ell$ and $V \in N_0$ there exists $Q_{i,V} \in \mathbb{Z}[X]$ such that,
\[
\langle VT \circ \varphi_i \circ T^{-1}(0), M_{\zeta} \rangle = Q_{i,V}(\theta_j) \quad \text{for} \quad 1 \leq j \leq k.
\]
Set $\Phi' = \{T \circ \varphi_i \circ T^{-1}\}_{i=1}^k$, and note that $\Phi'$ is affinely irreducible (since $\Phi$ is), and that the linear parts of the maps in $\Phi'$ are equal to the linear parts of the maps is $\Phi$. Additionally, observe that if $p = (p_i)_{i=1}^k$ is a probability vector and $\mu$ is the self-similar measure corresponding to $\Phi$ and $p$, then $T \mu$ is the self-similar measure corresponding to $\Phi'$ and $p$. Moreover, it is clear that $\mu$ is Rajchman if and only if $T \mu$ is Rajchman. From all of this it follows that by replacing $\Phi$ with $\Phi'$, $\{\zeta_j\}_{j=1}^k$ with $\{M_{\zeta_j}\}_{j=1}^k$ and $\{P_{i,V}\}$ with $\{Q_{i,V}\}$, without changing the notation, we may assume that $a_1 = 0$ and $P_{i,V} \in \mathbb{Z}[X]$ for all $1 \leq i \leq \ell$ and $V \in N_0$.

By Lemma 5.9, since $\{P_{i,V}\} \subset \mathbb{Z}[X]$ and since $N_0$ is finite, there exists $C > 1$ and $0 < \delta < 1$ so that for all $1 \leq i \leq \ell$, $V \in N_0$ and $b \in \mathbb{Z}_{\geq 0}$,
\[
(5.16) \quad \| \sum_{j=1}^k \theta_j^b \langle V a_i, \zeta_j \rangle \| = \| \sum_{j=1}^k \theta_j^b P_{i,V}(\theta_j) \| \leq C \delta^b.
\]
Set $\xi = \sum_{j=1}^k \zeta_j$. Since $\zeta_1, ..., \zeta_k$ are eigenvectors of $A^{-1}$ corresponding to distinct eigenvalues, they are independent. In particular $\xi \neq 0$, and $\zeta_j, \neq \zeta_{j_2}$ for $1 \leq j_1 < j_2 \leq k$. Since $\{\theta_1, ..., \theta_k\}$ is a P.V. $k$-tuple, for every $1 \leq j_1 < j_2 \leq k$ there exists $1 \leq j_2 \leq k$ with $\theta_{j_2} = \overline{\theta}_{j_1}$. By Lemma 5.10 this implies $\zeta_{j_2} = \overline{\zeta}_{j_1}$, which shows that $\xi \in \mathbb{R}^d$. From (5.16) it follows that for all $1 \leq i \leq \ell$, $V \in N_0$ and $b \in \mathbb{Z}_{\geq 0}$,
\[
(5.17) \quad \| \langle V a_i, B^{-b} \xi \rangle \| = \| \sum_{j=1}^k \theta_j^b \langle V a_i, \zeta_j \rangle \| \leq C \delta^b.
\]
Set,
\[
\Delta := \{(p_1, ..., p_\ell) \in [0,1]^\ell : p_1 + ... + p_\ell = 1\}.
\]
For $(p_i)_{i=1}^\ell = p \in \Delta$ let $\mu_p$ be the self-similar measure corresponding to $\Phi$ and $p$. Additionally, set
\[
q_p := \sum_{i=1}^\ell p_i \delta_{g_i} \in \mathcal{M}(G),
\]
let $X_{p,1}, X_{p,2}, \ldots$ be i.i.d. $G$-valued random elements with distribution $q_p$, and write $\lambda_p := \mathbb{E} [\psi X_{p,1}]$. For $g \in G$ set

$$\rho_p(g) = \lambda_p^{-1} \mathbb{P} \{ \psi X_{p,1} > \psi g \geq 0 \},$$

and write $\nu_p$ in place of $\rho_p \, d\nu_G$. Note that $\nu_p \in \mathcal{M}(G)$.

Let $m \geq 1$ be large with respect to $\delta$ and $C$. Let $f : \Delta \to \mathbb{C}$ be such that,

$$f(p) = \int \widetilde{\mu}_p(2\pi(B^{-m} \xi).g) \, d\nu_p(g) \quad \text{for } p \in \Delta.$$ 

Let $(1,0,...,0) =: e_1 \in \Delta$, then $\mu_{e_1}$ is unique member of $\mathcal{M}([\mathbb{R}^d]$ which satisfies $\mu_{e_1} = \varphi_1 \mu_{e_1}$. Since $a_1 = 0$, this relation is also satisfied by $\delta_0$, where $\delta_0$ is the Dirac mass centred at 0. This implies that $\mu_{e_1} = \delta_0$, and so $f(e_1) = 1$. It is easy to see that $f$ is continuous, and so there exists $(p_1,\ldots,p_\ell) = p \in \Delta$ with $|f(p)| \geq 1/2$ and $p_i > 0$ for $1 \leq i \leq \ell$. Fix this $p$ until the end of the proof. We shall show that $\mu_p$ is non-Rajchman. Since $p$ is positive, this will complete the proof of the proposition.

Let $n \geq 1$ be large with respect to $m$ and $p$. Set,

$$W_n := \{ i_1,\ldots,i_s \in \Lambda^* : \psi(g_{i_1}\cdots g_{i_s}) \geq \beta n > \psi(g_{i_1}\cdots g_{i_{s-1}}) \}.$$ 

As noted in the beginning of the present section, by Lemma 2.1 we may assume that $\gamma_\beta = h$. Thus, for $y \in \mathbb{R}^d$

$$B^{-1} y = 2^\delta U y = y. h^{-1} = y. \gamma_\beta.$$ 

Additionally,

$$r_w U_w^{-1} y = y.g_w \quad \text{for } w \in \Lambda^* \text{ and } y \in \mathbb{R}^d.$$ 

Hence, since $W_n$ is a minimal cut-set for $\Lambda^*$,

$$\widetilde{\mu}_p(2\pi B^{-m-n} \xi) = \sum_{w \in W_n} p_w \int e^{2\pi i \langle B^{-m-n} \xi, \varphi_w(x) \rangle} \, d\mu_p(x)$$

$$= \sum_{w \in W_n} p_w e^{2\pi i \langle B^{-m-n} \xi, \varphi_w(0) \rangle} \widetilde{\mu}_p(2\pi(B^{-m} \xi). (\gamma \beta g_w)).$$

Let,

$$f_n(p) := \sum_{w \in W_n} p_w \mu_p(2\pi(B^{-m} \xi). (\gamma \beta g_w)).$$

For $s \in \mathbb{Z}_{\geq 1}$ set $Y_s := X_{p,1} \cdots X_{p,s}$, and let

$$\tau_\beta(n) := \inf \{ s \in \mathbb{Z}_{\geq 1} : \psi Y_s \geq \beta n \}.$$ 

Observe that,

$$f_n(p) = \mathbb{E} \left[ \widetilde{\mu}_p(2\pi(B^{-m} \xi). (\gamma \beta Y_{\tau_\beta(n)}) \right].$$

Additionally, since $\operatorname{supp}(g_p) = \{ g_1 \}_{i=1}^\ell$ and since $G$ is generated by $\{ g_1 \}_{i=1}^\ell$, the measure $q_p$ is adapted. Thus, by Proposition 3.3 and by taking $n$ to be large enough with respect to $m$ and $p$, we may assume that $|f_n(p)| \geq |f(p)| - \frac{1}{4} \geq \frac{1}{2}$.

By (5.2) it follows that for every $1 \leq i \leq \ell$ there exists $V_i \in N_0$ so that $r_i U_i = A^i V_i$. Hence for $i_1\cdots i_s = w \in W_n$,

$$\varphi_w(0) = \sum_{j=1}^s r_{i_1}\cdots r_{i_{j-1}} U_{i_1}\cdots U_{i_{j-1}} a_{i_j} = \sum_{j=1}^s A_{i_j} V_{i_1} \cdots A_{i_{j-1}} V_{i_{j-1}} a_{i_j}.$$
From $N_0 A = N_0 A$ it follows that there exist $V_{w,1}, \ldots, V_{w,s} \in N_0$ so that
\[
\varphi_w(0) = \sum_{j=1}^s A^\sigma_{w,j} V_{w,j} a_i,
\]
where $\sigma_{w,j} := l_{i_1} + \ldots + l_{i_j}$ for $1 \leq j \leq s$. From (5.17) we now get that for all $b \in \mathbb{Z}_{\geq \sigma_{w,s}}$
\[
\| \langle B^{-b} \xi, \varphi_w(0) \rangle \| \leq \sum_{j=1}^s \| \langle B^\sigma_{w,j} - b \xi, V_{w,j} a_i \rangle \| \leq C \sum_{j=b - \sigma_{w,s}}^\infty \delta^j = \frac{C}{1 - \delta} \delta^{b - \sigma_{w,s}}.
\]
Additionally, since $w \in W_n$
\[
\beta n > \psi(g_{i_1 \ldots i_{s-1}}) = \sum_{j=1}^{s-1} \psi(g_{i_j}) = \beta \sigma_{w,s},
\]
which implies,
\[
\| \langle B^{-m - n} \xi, \varphi_w(0) \rangle \| \leq \frac{C \delta^m}{1 - \delta}.
\]
Hence,
\[
\left| 1 - e^{2\pi i \langle B^{-m - n} \xi, \varphi_w(0) \rangle} \right| \leq \frac{2 \pi C \delta^m}{1 - \delta} \text{ for } w \in W_n.
\]
Now from this, from (5.18) and by assuming that $m$ is large enough with respect to $\delta$ and $C$,
\[
|\hat{\mu}_p(2\pi B^{-m - n} \xi) - f_n(p)| \leq \sum_{w \in W_n} p_w \left| 1 - e^{2\pi i \langle B^{-m - n} \xi, \varphi_w(0) \rangle} \right| \leq \frac{1}{8}.
\]
Since $|f_n(p)| \geq 1/4$, it follows that $|\hat{\mu}_p(2\pi B^{-m - n} \xi)| \geq 1/8$. Note that this inequality holds for all sufficiently large $n \geq 1$. Since $\xi \neq 0$, this shows that $\mu_p$ is not a Rajchman measure, which completes the proof of the proposition. \hfill \square

6. Proof of the main result

In this section we prove Theorem 1.5 which we now restate. As always, recall that $\Phi = \{ \varphi_i(x) = r_i U_i(x) + a_i \}_{i=1}^\ell$ is an affinely irreducible self-similar IFS on $\mathbb{R}^d$.

**Theorem.** There exists a probability vector $p = (p_i)_{i=1}^\ell > 0$ such that the self-similar measure corresponding to $\Phi$ and $p$ is non-Rajchman if and only if there exists a linear subspace $\mathcal{V} \subset \mathbb{R}^d$, with $\dim \mathcal{V} > 0$ and $U_i(\mathcal{V}) = \mathcal{V}$ for $1 \leq i \leq \ell$, and an isometry $S : \mathcal{V} \to \mathbb{R}^d$ so that the following conditions are satisfied.

1. For $1 \leq i \leq \ell$ let $U_i' \in O(d')$ and $a_i' \in \mathbb{R}^{d'}$ be with $S \circ \pi_{\mathcal{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i'(x) + a_i'$. Let $H \subset GL_{d'}(\mathbb{R})$ be the group generated by $\{ r_i U_i' \}_{i=1}^\ell$, and set $N := H \cap O(d')$. Then $N$ is finite, $N \subset H$ and $H/N$ is cyclic.
2. For every contracting $A \in H$ with $\{ A^n \} \in \mathbb{Z}$, there exist $k \geq 1$, $\theta_1, \ldots, \theta_k \in \mathbb{C}$ and $\xi_1, \ldots, \xi_k \in \mathbb{C}^d \setminus \{ 0 \}$, so that
   a. $\{ \theta_1, \ldots, \theta_k \}$ is a P.V. $k$-tuple;
   b. $A^{-1} \xi_j = \theta_j \xi_j$ for $1 \leq j \leq k$;
   c. for every $1 \leq i \leq \ell$ and $V \in N$ there exists $P_{i,V} \in \mathbb{Q}[X]$ so that $\langle V a_i', \xi_j \rangle = P_{i,V}(\theta_j)$ for all $1 \leq j \leq k$.  

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We shall need following lemma.

Lemma 6.1. Let \( V \subset \mathbb{R}^d \) be a linear subspace with \( d_i := \dim V > 0 \) and \( U_i(V) = V \) for \( 1 \leq i \leq \ell \), and let \( S : V \to \mathbb{R}^d \) be an isometry. For \( 1 \leq i \leq \ell \) set \( \varphi_i' := S \circ \pi_V \circ \varphi_i \circ S^{-1} \), and write \( \Phi' \) for the self-similar IFS \( \{ \varphi_i' \}_{i=1}^\ell \). Then \( \Phi' \) is affinely irreducible.

Proof. Let \( W \) be an affine subspace of \( \mathbb{R}^d \) so that \( \varphi_i'(W) = W \) for \( 1 \leq i \leq \ell \). Set \( W_0 := S^{-1}(W) \) and let \( 1 \leq i \leq \ell \), then \( W_0 \subset V \) and \( \pi_V \circ \varphi_i(W_0) = W_0 \). From this, \( U_i(V) = V \) and \( U_i(V^\perp) = V^\perp \), it follows that for \( x \in W_0 \),

\[
\varphi_i(x + y) = r_i U_i x + \pi_V a_i + r_i U_i y + \pi_V \varphi_i(y),
\]

and so \( \varphi_i(W_0 + V^\perp) = W_0 + V^\perp \). Since this holds for every \( 1 \leq i \leq \ell \) and \( \Phi \) is affinely irreducible, it follows that \( W_0 + V^\perp = \mathbb{R}^d \). Since \( W_0 \subset V \), we must have \( W_0 = V \). Hence \( W = \mathbb{R}^d \), which shows that \( \Phi' \) is affinely irreducible.

Proof of Theorem 1.3 Suppose first that there exist a linear subspace \( V \) and an isometry \( S \) as in the statement of the theorem. For \( 1 \leq i \leq \ell \) set \( \varphi_i' := S \circ \pi_V \circ \varphi_i \circ S^{-1} \), and let \( \Phi' := \{ \varphi_i' \}_{i=1}^\ell \). For every \( 1 \leq i \leq \ell \) and \( x \in \mathbb{R}^d \) we have \( \varphi_i'(x) = r_i U_i x + a_i \). By Lemma 6.1 it follows that \( \Phi' \) is affinely irreducible.

Let \( G' \subset \mathbb{R} \times O(d') \) be the group generated by \( \{ (\log r_i^{-1}, U'_i) \}_{i=1}^\ell \). Let \( A \in \mathcal{H} \) be contracting and with \( \{ A^n N \}_{n \in \mathbb{Z}} = \mathcal{H}/\mathbb{N} \), and let \( \beta > 0 \) and \( U \in O(d') \) be with \( A = 2^{-\beta} U \). From \( \{ A^n N \}_{n \in \mathbb{Z}} = \mathcal{H}/\mathbb{N} \) it follows that,

\[
G' = \{ (n\beta, U V) : n \in \mathbb{Z} \text{ and } V \in \mathbb{N} \}.
\]

Since \( \mathbb{N} \) is finite, \( G' \) is easily seen to be discrete and closed in \( \mathbb{R} \times O(d') \). By Proposition 5.3 and by condition (2) in the statement of the theorem, it now follows that there exists a probability vector \( p = (p_i)_{i=1}^\ell > 0 \) so that the self-similar measure \( \mu' \in \mathcal{M}(\mathbb{R}^d) \) corresponding to \( \Phi' \) and \( p \) is non-Rajchman.

Let \( \mu \in \mathcal{M}(\mathbb{R}^d) \) be the self-similar measure corresponding to \( \Phi \) and \( p \). Since for \( 1 \leq i \leq \ell \) we have \( U_i(V^\perp) = V^\perp \), it follows that for \( x \in \mathbb{R}^d \),

\[
\pi_V \varphi_i(x) = \pi_V (r_i U_i \pi_V x + r_i U_i \pi_V x) = \pi_V a_i = \pi_V \varphi_i \pi_V x.
\]

From this and by the self-similarity of \( \mu \),

\[
S \pi_V \mu = \sum_{i=1}^\ell p_i \cdot S \circ \pi_V \circ \varphi_i \circ \pi_V \mu = \sum_{i=1}^\ell p_i \cdot \varphi_i' \circ S \circ \pi_V \mu.
\]

Since \( \mu' \) is the unique member of \( \mathcal{M}(\mathbb{R}^d) \) which satisfies the relation

\[
\mu' = \sum_{i=1}^\ell p_i \cdot \varphi_i' \mu',
\]

it follows that \( \mu' = S \pi_V \mu \). From this and since \( \mu' \) is non-Rajchman, we get that there exist \( \epsilon > 0 \) and \( \xi_1, \xi_2, ... \in \mathbb{V} \) so that \( |\xi_n| \to \infty \) and \( |\pi_V \mu(\xi_n)| > \epsilon \). Since \( \pi_V \mu(\xi) = \tilde{\mu}(\xi) \) for \( \xi \in \mathbb{V} \), this shows that \( \mu \) is also non-Rajchman, which completes the proof of the first direction of the theorem.

Suppose next that there exists a probability vector \( p = (p_i)_{i=1}^\ell > 0 \) so that the self-similar measure \( \mu \) corresponding to \( \Phi \) and \( p \) is non-Rajchman. By Proposition 4.7 it follows that \( \psi(G) \neq \mathbb{R} \). Recall that \( G_0 \) denotes the connected component of \( G \).
containing the identity. Let $V$ be the linear subspace of $\mathbb{R}^d$ consisting of all $x \in \mathbb{R}^d$ so that $x.g = x$ for all $g \in G_0$. By Proposition 4.9 and since $\mu$ is non-Rajchman, we have $d^d := \dim V > 0$. By Lemma 4.8
\begin{equation}
U_i(V) = V \quad \text{and} \quad U_i(V^\perp) = V^\perp \quad \text{for all} \quad 1 \leq i \leq \ell .
\end{equation}

The map $\pi \varphi_1|_V$ is a strict contraction of $V$, and so there exists $y \in V$ with $\pi \varphi_1(y) = y$. Let $S : V \to \mathbb{R}^d$ be an isometry with $Sy = 0$. For $1 \leq i \leq \ell$ set $\varphi'_i := S \circ \varphi_i \circ S^{-1}$, and let $U'_i \in O(d')$ and $a'_i \in \mathbb{R}^d$ be with $\varphi'_i(x) = r_i U'_i x + a'_i$ for $x \in \mathbb{R}^d$. Let $H$ be the smallest closed subgroup of $GL_d(\mathbb{R})$ containing $\{r_i U'_i\}_{i=1}^\ell$.

Since $S$ is also an affine map, there exists a linear isometry $L : V \to \mathbb{R}^d$ so that $Sx = Lx - Ly$ for $x \in V$. From (6.1) it follows that $L \circ U \circ L^{-1} \in O(d')$ for every $(t, U) \in G$. Note that $U'_i = L \circ U_i \circ L^{-1}$ for $1 \leq i \leq \ell$. For $(t, U) \in G$ set $F(t, U) = 2^{-i}L \circ U \circ L^{-1}$, so that $F : G \to GL_d(\mathbb{R})$ is a continuous homomorphism. It is easy to verify that $F$ is a proper map, which implies that $F$ is a closed map. From this and since the group generated by $\{r_i U'_i\}_{i=1}^\ell$ is dense in $F(G)$, it follows that $H = F(G)$ and that $F$ descends to an isomorphism of topological groups from $G/\ker F$ onto $H$.

Since $\psi(G) \neq \mathbb{R}$, we have $G_0 \subset \{0\} \times O(d)$. From this and by the definition of $V$, it follows that $G_0 \subset \ker F$. Since $G_0$ is an open subgroup of $G$, it follows that $\ker F$ is also an open subgroup of $G$. This implies that $G/\ker F$ is discrete, and so that $H$ is also discrete. From this and by the definition of $H$, it follows that $H$ is equal to the group generated by $\{r_i U'_i\}_{i=1}^\ell$ (and not just to the closed subgroup generated by these elements).

Set $N := H \cap O(d')$. Since $N$ is the kernel of the homomorphism taking $rU \in H$ to $r$, where $U \in O(d')$ and $r > 0$, we have $N \vartriangleleft H$. Since $H$ is closed in $GL_d(\mathbb{R})$ and $O(d')$ is compact, it follows that $N$ is compact. From this and since $H$ is discrete, it follows that $N$ is finite. Since $\psi(G) \neq \mathbb{R}$ and $H = F(G)$, there exists $A \in H$ so that $|A| < 1$ and $|A| \geq |B|$ for all $B \in H$ with $|B| < 1$, where $|| \cdot ||$ is the operator norm here. It is now obvious that $\{A^n N\}_{n \in \mathbb{Z}} = H/N$, which shows that condition (1) in the statement of the theorem is satisfied.

We turn to prove that condition (2) is also satisfied. First we show that
\begin{equation}
\lim_{M \to \infty} \sup_{\xi \in V} \{ |\tilde{\mu}(\xi)| : \xi \in V \text{ and } |\xi| \geq M \} > 0 .
\end{equation}
Since $\mu$ is non-Rajchman, there exists $\epsilon_0 > 0$ so that
\[ \lim_{|\xi| \to \infty} \sup |\tilde{\mu}(\xi)| > \epsilon_0 . \]
Let $0 < \epsilon < 1$ be small with respect to $\Phi$, $p$ and $\epsilon_0$, let $R > 1$ be large with respect to $\epsilon$, and let $\xi \in \mathbb{R}^d$ be with $|\xi| > R$ and $|\tilde{\mu}(\xi)| > \epsilon_0$. By Proposition 4.9 we may assume that,
\[ |\pi \varphi_\perp \xi| < \epsilon R/2, \epsilon |\pi \varphi \xi| \]
If $|\pi \varphi_\perp \xi| \geq \epsilon |\pi \varphi \xi|$ then $|\pi \varphi_\perp \xi| < \epsilon R/2$, and so
\[ R < |\pi \varphi \xi| + |\pi \varphi_\perp \xi| \leq (\epsilon^{-1} + 1)|\pi \varphi_\perp \xi| < 2\epsilon^{-1}(\epsilon R/2) = R, \]
which is not possible. Hence we must have $|\pi \varphi_\perp \xi| < \epsilon |\pi \varphi \xi| \leq |\xi|$. We may assume that $R > \epsilon^{-1/2}$, which gives $|\xi|^{-1}\epsilon^{-1/2} < 1$. Set,
\[ W = \{i_1 \ldots i_n \in \Lambda^* : |r_{i_1 \ldots i_n}| \leq |\xi|^{-1}\epsilon^{-1/2} < r_{i_1 \ldots i_{n-1}} \} . \]
Since $W$ is a minimal cut-set,
\[
\epsilon_0 < |\hat{\mu}(\xi)| = \left| \sum_{w \in W} p_w \int e^{i(x,\varphi_w(x))} d\mu(x) \right| \leq \sum_{w \in W} p_w \left| \int e^{i(r_wU_w^{-1}\xi,x)} d\mu(x) \right|,
\]
and so there exists $w \in W$ with $|\hat{\mu}(r_wU_w^{-1}\xi)| > \epsilon_0$. By (6.1) and the definition of $W$,
\[
|r_wU_w^{-1}\xi - \pi_\varphi(r_wU_w^{-1}\xi)| = |\pi_{\varphi_1}(r_wU_w^{-1}\xi)| = r_w|\pi_{\varphi_1}\xi| \leq r_w|\xi| \leq \epsilon^{1/2}.
\]
Since $\mu$ is compactly supported, the map which takes $\eta \in \mathbb{R}^d$ to $\hat{\mu}(\eta)$ is uniformly continuous. Hence, by assuming that $\epsilon$ is sufficiently small with respect to $\Phi$, $p$ and $\epsilon_0$, we get
\[
|\pi_{\varphi_1}\mu(r_w\pi_\varphi U_w^{-1}\xi)| = |\hat{\mu}(r_w\pi_\varphi U_w^{-1}\xi)| \geq |\hat{\mu}(r_wU_w^{-1}\xi)| - \epsilon_0/2 > \epsilon_0/2.
\]
Set $r_{\min} := \min_{1 \leq i \leq \ell} r_i$, then by the definition of $W$ we have $r_w > r_{\min}|\xi|^{-1}\epsilon^{-1/2}$. Thus,
\[
|r_w\pi_\varphi U_w^{-1}\xi| \geq r_w|\xi| - |\pi_{\varphi_1}(r_wU_w^{-1}\xi)| > r_{\min}\epsilon^{-1/2} - \epsilon^{1/2}.
\]
Since $\epsilon$ can be chosen to be arbitrarily small, the last expression can be made arbitrarily large (while keeping $\epsilon_0$ fixed). This together with (6.3) gives (6.2).

Set $\Phi' := \{\varphi'_i\}_{i=1}^\ell$, where recall that $\varphi'_i := S \circ \pi_\varphi \circ \varphi_i \circ S^{-1}$ for $1 \leq i \leq \ell$. By Lemma 6.1 it follows that $\Phi'$ is affinely irreducible. Since $\pi_\varphi \varphi_1(y) = y$ and $Sy = 0$, we have $a'_1 = \varphi'_1(0) = 0$. As in the proof of the first direction of the theorem, it holds that $S\pi_\varphi \mu$ is the self-similar measure corresponding to $\Phi'$ and $p$. From (6.2) it clearly follows that $S\pi_\varphi \mu$ is non-Rajchman. Since the closed subgroup generated by $\{r_iU_i\}_{i=1}^\ell$ is discrete, it follows that all of the assumptions in Proposition 5.1 are satisfied for the IFS $\Phi'$. This implies that condition (2) in the statement of the theorem holds, which completes the proof.  

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References

[1] A. Algom, F. Rodriguez Hertz, and Z. Wang. Pointwise normality and Fourier decay for self-conformal measures, 2021. arXiv:2012.06529.

[2] M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J.-P. Schreiber. Pisot and Salem numbers. Birkhäuser Verlag, Basel, 1992. With a preface by D. Boyd.

[3] C. J. Bishop and Y. Peres. Fractals in probability and analysis, volume 162. Cambridge University Press, 2012.

[4] D. W. Boyd. Irreducible polynomials with many roots of maximal modulus. Acta Arith., 68(1):85–88, 1994.

[5] J. Brémont. Self-similar measures and the Rajchman property. To appear in Ann. H. Lebesgue, 2020. arXiv:1910.03463.

[6] Y. Bugeaud. Distribution modulo one and Diophantine approximation, volume 193. Cambridge University Press, 2012.

[7] D. Buraczewski, E. Damek, Y. Guivarc’h, A. Hulanicki, and R. Urban. Tail-homogeneity of stationary measures for some multidimensional stochastic recursions. Probab. Theory Related Fields, 145(3-4):385, 2009.

[8] D. G. Cantor. On sets of algebraic integers whose remaining conjugates lie in the unit circle. Trans. Amer. Math. Soc., 105(3):391–406, 1962.
[9] C. Christopoulos and J. McKee. Galois theory of salem polynomials. In *Math. Proc. Cambridge Philos. Soc.*, volume 148, page 47. Cambridge University Press, 2010.

[10] P. Erdős. On a family of symmetric Bernoulli convolutions. *Amer. J. Math.*, 61(4):974–976, 1939.

[11] P. Erdős. On the smoothness properties of a family of bernoulli convolutions. *Amer. J. Math.*, 62(1):180–186, 1940.

[12] R. Ferguson. Irreducible polynomials with many roots of equal modulus. *Acta Arith.*, 78(3):221–225, 1997.

[13] B. Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer, second edition, 2015.

[14] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy in $\mathbb{R}^d$. To appear in *Mem. Amer. Math. Soc.*, 2015. arXiv:1503.09043.

[15] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[16] J.-P. Kahane. Sur la distribution de certaines séries aléatoires. In *Colloque de Théorie des Nombres (Univ. Bordeaux, Bordeaux, 1969)*, pages 119–122. Soc. Math. France, Paris, 1971.

[17] I. Környei. On a theorem of Pisot. *Publ. Math. Debrecen*, 34(3-4):169–179, 1987.

[18] J. Li. Decrease of fourier coefficients of stationary measures. *Math. Ann.*, 372(3):1189–1238, 2018.

[19] J. Li and T. Sahlsten. Trigonometric series and self-similar sets. To appear in *J. Eur. Math. Soc.*, 2019. arXiv:1902.00426.

[20] J. Li and T. Sahlsten. Fourier transform of self-affine measures. *Adv. Math.*, 374:107349, 2020.

[21] R. Lyons. Seventy years of rajchman measures. *J. Fourier Anal. Appl.*, 1:363–378, 1995.

[22] C. Mauduit. Caractérisation des ensembles normaux substitutifs. *Invent. Math.*, 95(1):133–147, 1989.

[23] C. Pisot. La répartition modulo 1 et les nombres algébriques. *Ann. Scuola Norm. Super. Pisa Cl. Sci.*, 7(3-4):205–248, 1938.

[24] D. Revuz. *Markov chains*, volume 11 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1984.

[25] T. Sahlsten and C. Stevens. Fourier transform and expanding maps on Cantor sets, 2020. arXiv:2009.01703.

[26] R. Salem. Sets of uniqueness and sets of multiplicity. *Trans. Amer. Math. Soc.*, 54(2):218–228, 1943.

[27] B. Solomyak. Fourier decay for self-similar measures, 2019. arXiv:1906.12164.

[28] C. J. Stone. Infinite particle systems and multi-dimensional renewal theory. *J. Math. Mech.*, 18(3):201–227, 1968.

[29] P. Varjú and H. Yu. Fourier decay of self-similar measures and self-similar sets of uniqueness. To appear in *Anal. PDE*, 2020. arXiv:2004.09358.

[30] P. Walters. *An introduction to ergodic theory*, volume 79. Springer Science & Business Media, 2000.

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