Rapid Mixing for the Lorenz Attractor and Statistical Limit Laws for Their Time-1 Maps

V. Araújo¹, I. Melbourne², P. Varandas¹

¹ Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, Salvador 40170-110, Brazil. E-mail: vdaraujo99@gmail.com; vitor.d.araujo@ufba.br. http://www.sd.mat.ufba.br/~vitor.d.araujo.
E-mail: paulo.varandas@ufba.br. http://www.pgmat.ufba.br/varandas/

² Institute of Mathematics, University of Warwick, Coventry CV4 7AL, UK.
E-mail: i.melbourne@warwick.ac.uk

Received: 21 November 2013 / Accepted: 26 June 2015
Published online: 22 September 2015 – © Springer-Verlag Berlin Heidelberg 2015

Abstract: We prove that every geometric Lorenz attractor satisfying a strong dissipativity condition has superpolynomial decay of correlations with respect to the unique Sinai–Ruelle–Bowen measure. Moreover, we prove the central limit theorem and almost sure invariance principle for the time-1 map of the flow of such attractors. In particular, our results apply to the classical Lorenz attractor.

1. Introduction

The statistical point of view on dynamical systems is one of the most useful tools available for the study of the asymptotic behavior of transformations or flows. Statistical properties are often easier to study than pointwise behavior, since the future behavior of an initial data point can be unpredictable, but statistical properties are often regular and with simpler description.

One of the main concepts introduced is the notion of physical [or Sinai–Ruelle–Bowen (SRB)] measure for a flow (or transformation). An invariant probability measure $\mu$ for a flow $Z_t$ is a physical probability measure if the subset of points $z$ satisfying for all continuous functions $w$

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t w(Z_s(z)) \, ds = \int w \, d\mu,$$

I.M. was partially supported by a Santander Staff Mobility Award at the University of Surrey, by a European Advanced Grant StochExtHomog (ERC AdG 320977) and by CNPq (Brazil) through PVE Grant No. 313759/2014-6. V.A. and P.V. were partially supported by CNPq, PRONEX-Dyn.Syst. and FAPESB (Brazil). This research has been supported in part by EU Marie-Curie IRSES Brazilian–European partnership in Dynamical Systems (FP7-PEOPLE-2012-IRSES 318999 BREUDS). We are grateful to Oliver Butterley for very helpful discussions regarding the regularity of the strong stable foliation and to the anonymous referees for pointing out several details in our argument that had to be addressed, greatly improving the final text of this work.
has positive volume in the ambient space. These time averages are in principle physically observable if the flow models a real world phenomenon admitting some measurable features.

In 1963, the meteorologist Edward Lorenz published in the *Journal of Atmospheric Sciences* [20] an example of a polynomial system of differential equations

\[
\begin{align*}
\dot{x} &= 10(y - x) \\
\dot{y} &= 28x - y - xz \\
\dot{z} &= xy - \frac{8}{3}z
\end{align*}
\] (1.1)

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast.

Numerical simulations performed by Lorenz for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a chaotic attractor, whose well known picture can be easily found in the literature.

The mathematical study of these equations began with the geometric Lorenz flows, introduced independently by Afraimović et al. [1] and Guckenheimer and Williams [16, 35] as an abstraction of the numerically observed features of solutions to (1.1). The geometric flows were shown to possess a “strange” attractor with sensitive dependence on initial conditions. It is well known, see e.g. [6], that geometric Lorenz attractors have a unique SRB (or physical) measure. Tucker [31] showed that the attractor of the classical Lorenz equations (1.1) is in fact a geometric Lorenz attractor (see Remark 2.3 below). For more on the rich history of the study of this system of equations, the reader can consult [5,33].

An invariant probability measure \( \mu \) for a flow is mixing if

\[
\mu(Z_t(A) \cap B) \to \mu(A)\mu(B)
\]

as \( t \to \infty \) for all measurable sets \( A, B \). Mixing for the SRB measure of geometric Lorenz attractors was proved in [21] and, by [31], this includes the classical Lorenz attractor [20].

Results on the speed of convergence in the limit above, that is, of rates of mixing for the Lorenz attractor were obtained only recently: a first result on robust exponential decay of correlations was proved in [7] for a nonempty open subset of geometric Lorenz attractors. However, this open set does not contain the classical Lorenz attractor. Also, it follows straightforwardly from [23] that a \( C^2 \)-open and \( C^\infty \)-dense set of geometric Lorenz flows has superpolynomial decay of correlations (in the sense of [14]). It is likely, but unproven, that this open and dense set includes the classical Lorenz attractor.

### 1.1. Statement of results

In this paper, we introduce an additional open assumption, strong dissipativity, which is satisfied by the classical Lorenz attractor, under which we can prove superpolynomial decay of correlations.

We consider \( C^\infty \) vector fields \( G \) on \( \mathbb{R}^3 \) possessing an equilibrium \( p \), which is Lorenz-like: the eigenvalues of \( DG_p \) are real and satisfy

\[
\lambda_{ss} < \lambda_s < 0 < -\lambda_s < \lambda_u.
\] (1.2)

We say that \( G \) is strongly dissipative if the divergence of the vector field \( G \) is strictly negative: there exists a constant \( \delta > 0 \) such that \( (\text{div} \ G)(x) \leq -\delta \) for all \( x \in U \),