Polynomial and Chaplygin f(T)-gravity models

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Abstract. We reconstruct different f(T)-gravity models corresponding to a set of dark energy scenarios containing the polytropic, the standard Chaplygin and the generalized Chaplygin gas models. We also derive the equation of state parameter of the selected f(T)-gravity models and obtain the necessary conditions for crossing the phantom-divide line.

1. The f(T) theory of modified gravity

Recent observations indicate that our universe is currently expanding with an acceleration [1]. One explanation for the cosmic acceleration is the dark energy (DE), an exotic energy with negative pressure. Although the origin and nature of DE is still a mystery, a great variety of DE models like the polytropic gas [2], the standard Chaplygin gas (SCG) [3], the generalized Chaplygin gas (GCG) [4] and so forth have been proposed. Also one of interesting alternative proposals for DE is modified gravity. It can explain naturally the unification of earlier and later cosmological epochs. Recently, Ferraro and Fiorini [5] suggested a new model of modified gravity, named f(T) theory, by generalizing the action of teleparallel gravity as function of the torsion scalar T, and found that it can explain the observed acceleration of the universe. Here, our aim is to study how the f(T)-gravity can describe the polytropic, SCG and GCG models as effective theories of DE models. This motivated us to establish different models of f(T)-gravity according to the aforementioned scenarios.

In the framework of f(T) theory, the action of modified teleparallel gravity is given by [5]

\[ I = \frac{1}{2k^2} \int d^4x \; e \left[ f(T) + L_m \right], \]

where \( k^2 = 8\pi G \) and \( e = \det(e^i_{\mu}) = \sqrt{-g} \). Also T and \( L_m \) are the torsion scalar and the Lagrangian density of the matter inside the universe, respectively. Note that \( e^i_{\mu} \) is the vierbein field which uses as dynamical object in teleparallel gravity.

The modified Friedmann equations in the framework of f(T)-gravity in the flat spatial FRW universe are given by [6]

\[ \frac{3}{k^2} H^2 = \rho_m + \rho_T, \]

\[ \frac{1}{k^2} (2\dot{H} + 3H^2) = -(\rho_m + \rho_T), \]

where

\[ \rho_T = \frac{1}{2k^2}(2f_T - f - T), \]
\[ p_T = -\frac{1}{2k^2}[-8\dot{H}f_T + (2T - 4\dot{H})f_T - f + 4\dot{H} - T], \quad (5) \]
\[ T = -6H^2, \quad (6) \]
and \( H = \dot{a}/a \) is the Hubble parameter. Here \( \rho_m \) and \( p_m \) are the total energy density and pressure of the matter inside the universe, respectively. Also \( \rho_T \) and \( p_T \) are the energy density and pressure, respectively, due to the torsion contribution. The energy conservation laws are still given by
\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (7) \]
\[ \dot{\rho}_T + 3H(\rho_T + p_T) = 0. \quad (8) \]
Note that if \( f(T) = T \), from Eqs. (4) and (5) we have \( \rho_T = 0 \) and \( p_T = 0 \) then Eqs. (2) and (3) transform to the usual Friedmann equations in general relativity (GR).

The equation of state (EoS) parameter due to the torsion contribution is defined as
\[ \omega_T = \frac{p_T}{\rho_T} = -1 + \frac{8\dot{H}f_T + 4\dot{H}f_T - 4\dot{H}}{2Tf_T - f - T}. \quad (9) \]

For a given \( a = a(t) \), by the help of Eqs. (4) and (5) one can reconstruct the \( f(T) \)-gravity according to any DE model given by the EoS \( p_T = p_T(\rho_T) \) or \( \rho_T = \rho_T(a) \). Here we assume an ansatz for the scale factor as [7]
\[ a(t) = a_0(t_s - t)^{-h}, \quad t \leq t_s, \quad h > 0. \quad (10) \]
Using Eqs. (6) and (10) one can obtain
\[ H = \frac{h}{t_s - t}, \quad T = -\frac{6h^2}{(t_s - t)^2}, \quad \dot{H} = -\frac{T}{6h}. \quad (11) \]
From Eqs. (10) and (11) the scale factor \( a \) can be rewritten in terms of \( T \) as
\[ a = a_0 \left( -\frac{T}{6h^2} \right)^{\frac{1}{h}}. \quad (12) \]
In sections 2 to 4, we reconstruct different \( f(T) \)-gravities according to the polytropic, SCG and GCG models.

### 2. Polytropic \( f(T) \)-gravity model

Here we reconstruct the \( f(T) \)-gravity according to the polytropic gas DE model. Following [2], the EoS of the polytropic gas is given by
\[ p_\Lambda = K \rho_\Lambda^{1 + \frac{n}{m}}, \quad (13) \]
where \( K \) is a positive constant and \( n \) is the polytropic index. Using Eq. (8) the energy density evolves as
\[ \rho_\Lambda = \left( Ba_0^{\frac{n}{m}} - K \right)^{-n}, \quad (14) \]
where \( B \) is a positive integration constant [2]. Replacing Eq. (12) into (14) yields
\[ \rho_\Lambda = \left( \alpha T^\frac{m}{2n} - K \right)^{-n}, \quad (15) \]
where
\[ \alpha = Ba_0^{\frac{n}{m}} \left( -6h^2 \right)^{-\frac{3h}{2m}}. \quad (16) \]
Equating (4) with (15), i.e. $\rho_T = \rho_\Lambda$, we obtain the following differential equation

$$2T f_T - f - T - 2k^2 \left( \alpha T^{3h} - K \right)^{-n} = 0. \quad (17)$$

Solving Eq. (17) gives

$$f(T) = \beta T^{1/2} + T + (-1)^{1+n} \frac{2k^2}{Kn} 2F_1 \left( -\frac{n}{3h}, n; 1 - \frac{n}{3h}; \frac{\alpha T^{3h}}{K} \right), \quad (18)$$

where $2F_1$ denotes the first hypergeometric function. Replacing Eq. (18) into (9) one can obtain the EoS parameter of torsion contribution as

$$\omega_T = -1 - \frac{1}{\frac{K}{A} T^{3h} - 1} = -1 - \frac{1}{\frac{K}{A} \left[ a_0 \left( \frac{H}{K} \right)^h \right]^{\frac{1}{3h}} - 1}, \quad h > 0. \quad (19)$$

We see that for $\frac{K}{A} \left[ a_0 \left( \frac{H}{K} \right)^h \right]^{\frac{1}{3h}} > 1$, $\omega_T < -1$ which corresponds to a phantom accelerating universe. Recent observational data indicates that the EoS parameter $\omega$ at the present lies in a narrow strip around $\omega = -1$ and is quite consistent with being below this value [8].

3. Standard Chaplygin $f(T)$-gravity model

The EoS of the SCG model of DE is given by [3]

$$p_\Lambda = -\frac{A}{\rho_\Lambda}, \quad (20)$$

where $A$ is a positive constant. Inserting the above EoS into the energy conservation equation (8), leads to a density evolving as [3]

$$\rho_\Lambda = \sqrt{A + \frac{B}{a^6}}, \quad (21)$$

where $B$ is an integration constant. Inserting Eq. (12) into (21) one can get

$$\rho_\Lambda = \sqrt{A + \alpha T^{-3h}}, \quad (22)$$

where

$$\alpha = B a_0^{-6} (-6h^2)^{3h}. \quad (23)$$

Equating (22) with (4) one can obtain

$$2T f_T - f - T - 2k^2 \sqrt{A + \alpha T^{-3h}} = 0. \quad (24)$$

Solving the differential equation (24) yields

$$f(T) = \beta T^{1/2} + T - 2k^2 A^{\frac{1}{2}} \left( \frac{1}{6h}, -\frac{1}{2}; 1 + \frac{1}{6h}; -\frac{\alpha T^{-3h}}{A} \right). \quad (25)$$

Replacing Eq. (25) into (9) one can get

$$\omega_T = -1 + \frac{1}{\frac{A}{6h} T^{3h} + 1} = -1 + \frac{1}{\frac{A}{6h} \left[ a_0 \left( \frac{H}{K} \right)^h \right]^{6} + 1}, \quad h > 0, \quad (26)$$

which for $B < 0$ and $\frac{A}{6h} \left[ a_0 \left( \frac{H}{K} \right)^h \right]^{6} > 1$ then $\omega_T$ can cross the phantom-divide line.
4. Generalized Chaplygin $f(T)$-gravity model

The EoS of the GCG model is given by [4]

$$p_\Lambda = -\frac{A}{\rho_\Lambda^\alpha}, \quad (27)$$

where $\alpha$ is a constant in the range $0 \leq \alpha \leq 1$ (the SCG corresponds to the case $\alpha = 1$) and $A$ a positive constant. Using Eq. (8), the GCG energy density evolves as [4]

$$\rho_\Lambda = \left( A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}}, \quad (28)$$

where $B$ is an integration constant. Substituting Eq. (12) into (28) one can get

$$\rho_\Lambda = \left( A + \gamma T^{\frac{2}{3}h(1+\alpha)} \right)^{\frac{1}{1+\alpha}}, \quad (29)$$

where

$$\gamma = B a_0^{-3(1+\alpha)} (-6h^2)^{\frac{1}{3}h(1+\alpha)}.$$

Equating (29) with (4) gives

$$2Tf_T - f - T - 2k^2 \left( A + \gamma T^{\frac{2}{3}h(1+\alpha)} \right)^{\frac{1}{1+\alpha}} = 0. \quad (31)$$

Solving Eq. (31) yields

$$f(T) = \beta T^{1/2} + T - 2k^2 A^{1/3} \left( \frac{1}{3h(1+\alpha)} - \frac{1}{1+\alpha} - 1 + \frac{1}{3h(1+\alpha)} + \frac{\gamma}{A} T^{\frac{2}{3}h(1+\alpha)} \right), \quad (32)$$

Replacing Eq. (32) into (9) gives the EoS parameter as

$$\omega_T = -1 + \frac{1}{A^n T^{\frac{2}{3}h(1+\alpha)} + 1} = -1 + \left[ A \left( \frac{A}{n} \right)^h \right]^{3(1+\alpha)} + 1, \quad h > 0, \quad (33)$$

which for $B < 0$ and $A^n \left[ a_0 \left( \frac{A}{n} \right)^h \right]^{3(1+\alpha)} > 1$ then $\omega_T$ behaves like the EoS of phantom DE.

5. Conclusions

Here, we considered the polytropic gas, SCG and GCG models of DE. We reconstructed different theories of modified gravity based on the $f(T)$ action in the spatially-flat FRW universe and according to the selected DE models. We also obtained the EoS parameter of the polytropic, standard Chaplygin and generalized Chaplygin $f(T)$-gravity scenarios. We showed that crossing the phantom-divide line can occur when the parameters of the models to be chosen properly.

Note that the behavior of cosmological fluctuations basically depends on the squared speed of sound velocity $v_s^2 = dp/d\rho$ and the EoS parameter $\omega = p/\rho$. In the polytropic gas, SCG and GCG models the sound speed crucially depends on the model parameters. Also in different models of $f(T)$-gravity reconstructed according to the aforementioned scenarios, the model parameters appear in $\omega = p/\rho$. Thus, the allowed parameter space can be constrained by the data on the large scale structure (LSS) and cosmic microwave background (CMB). Since the LSSs in the universe have a fraction of baryons, the acoustic oscillations in the relativistic plasma would be imprinted onto the late-time power spectrum of the nonrelativistic matter. Therefore, the acoustic signatures in the large scale clustering of galaxies can be served as a test to constrain models of DE with detection of a baryon acoustic oscillation (BAO) peak in the correlation function of luminous red galaxies in the Sloan Digital Sky Survey (SDSS). Also the CMB peak from WMAP observations arises from acoustic oscillations of the primeval plasma just before the universe becomes transparent [9].
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