SOME IDENTITIES INVOLVING DEGENERATE STIRLING NUMBERS ASSOCIATED WITH SEVERAL DEGENERATE POLYNOMIALS AND NUMBERS

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ABSTRACT. The aim of this paper is to investigate some properties, recurrence relations and identities involving degenerate Stirling numbers of both kinds associated with degenerate hyperharmonic numbers and also with degenerate Bernoulli, degenerate Euler, degenerate Bell and degenerate Fubini polynomials.

1. INTRODUCTION

It is noteworthy that various degenerate versions of quite a few special polynomials and numbers have been investigated recently, which began from the pioneering work of Carlitz in [2,3]. These explorations for degenerate versions are not only limited to special polynomials and numbers but also extended to some transcendental functions, like gamma functions. In the course of this quest, many different tools are used, which include generating functions, combinatorial methods, \( p \)-adic analysis, umbral calculus, operator theory, differential equations, special functions, probability theory and analytic number theory (see [8-14,16] and the references therein).

The aim of this paper is to investigate some properties, recurrence relations and identities involving degenerate Stirling numbers (see (10), (11)) associated with degenerate hyperharmonic numbers (see (5), (19)) and also with degenerate Bernoulli, degenerate Euler, degenerate Bell and degenerate Fubini polynomials (see (15), (16), (13), (17)). The novelty of the present paper is that a degenerate version of the hyperharmonic numbers, namely the degenerate hyperharmonic numbers, is firstly introduced.

The outline of this paper is as follows. In Section 1, we recall degenerate exponentials, degenerate logarithms, degenerate Stirling numbers of both kinds and hyperharmonic numbers. Also, we remind the reader of degenerate Bell polynomials, degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Fubini polynomials and degenerate polylogarithms, which are, respectively, degenerate versions of Bell polynomials, Bernoulli polynomials, Euler polynomials, Fubini polynomials and polylogarithms. Section 2 is the main result of this paper. We introduce degenerate hyperharmonic numbers, which are a degenerate version of the hyperharmonic numbers, and derive the generating function of them in Theorem 1. We obtain an identity involving the degenerate hyperharmonic numbers and the degenerate Stirling numbers of the second kind in Theorem 2, and its inversion identity in Theorem 3. Obtained are identities involving the degenerate Euler polynomials and the degenerate Stirling numbers of both kinds in Theorem 4. Derived are identities relating the degenerate Bernoulli polynomials and the degenerate Stirling numbers of both kinds in Theorems 5, 6, 7 and 10. In Theorems 8 and 9, we get identities connecting the degenerate hyperharmonic numbers, the degenerate Bernoulli polynomials and the degenerate Stirling numbers of both kinds. A recurrence relation for the degenerate Bell polynomials are deduced in Theorem 11. An identity connecting the degenerate Stirling numbers of the second kind, the

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The Fubini polynomials are defined by
\[ H_n(x; \lambda) = \frac{1}{n!} \frac{d^n}{dx^n} \left[ e^{\lambda x} \left( e^{\lambda x - 1} - 1 \right) \right] \]
where
\[ H_n(0; \lambda) = \lambda^n \]

An explicit expression and a recurrence relation for the degenerate Fubini polynomials are obtained respectively in Theorem 13 and Theorem 14. Finally, an interesting identity on a finite sum of the degenerate Stirling numbers of the second kind is derived in Theorem 15.

For any \( \lambda \in \mathbb{R} \), the degenerate exponential is defined by
\[ e^\lambda(t) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}, \quad ( \text{see [9, 10, 13]} ) \]

where
\[ (x)_{0, \lambda} = 1, \quad (x)_{k, \lambda} = x(x - \lambda)(x - 2\lambda) \cdots (x - (k - 1)\lambda), \quad (k \geq 1) \]

When \( x = 1 \), we write \( e^\lambda = e^\lambda_1(t) \).

Let \( \log_{\lambda}(t) \), called the degenerate logarithm, be the compositional inverse of \( e^\lambda(t) \) such that \( \log_{\lambda}(e^\lambda(t)) = e^\lambda(\log_{\lambda}(t)) = t \). Then we have
\[ \log_{\lambda}(1 + t) = \sum_{k=1}^{\infty} \lambda^{k-1}(1)_{k, \lambda} \frac{t^k}{k!}, \quad ( \text{see [8, 14]} ) \]

Note that \( \lim_{\lambda \to 0} e^\lambda(t) = e^t \), \( \lim_{\lambda \to 0} \log_{\lambda} t = \log t \).

It is well known that the harmonic numbers are defined by
\[ H_0 = 0, \quad H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad (n \in \mathbb{N}), \quad ( \text{see [4, 5, 15]} ) \]

The generating function of the harmonic numbers is given by
\[ -\frac{1}{1-t} \log(1-t) = \sum_{n=0}^{\infty} H_n t^n, \quad ( \text{see [4, 5, 15]} ) \]

Recently, the degenerate harmonic numbers are introduced by Kim–Kim as
\[ H_{0, \lambda} = 0, \quad H_{n, \lambda} = \frac{1}{\lambda} \sum_{k=1}^{n} \binom{\lambda}{k} (-1)^{k-1}, \quad (n \in \mathbb{N}), \quad ( \text{see [9]} ) \]

Note that \( \lim_{\lambda \to 0} H_{n, \lambda} = H_n \).

In [5], Conway and Guy introduced the hyperharmonic numbers given by
\[ H_{n}^{(1)} = H_n, \quad H_{n}^{(r)} = \sum_{k=1}^{n} H_{k}^{(r-1)}, \quad (r \geq 2) \]

From (6), we see that
\[ H_{n}^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}), \quad (r \geq 2), \quad H_{0}^{(r)} = 0, \quad ( \text{see [5]} ) \]

The Fubini polynomials are defined by
\[ \frac{1}{1-x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}, \quad ( \text{see [4, 6, 12, 16]} ) \]

By (8), we easily get
\[ F_n(x) = \sum_{k=0}^{n} S_2(n, k)x^{(k)} = \sum_{k=0}^{n} S_2(n, k)x^k, \quad (n \geq 0), \quad ( \text{see [12]} ), \]
where \( S_2(n,k) \) are the Stirling numbers of the second kind given by
\[
\frac{1}{k!} (e^t - 1)^k = \sum_{n=0}^{\infty} S_2(n,k) \frac{t^n}{n!}, \quad (k \geq 0), \quad \text{ (see [1, 11, 17, 18]).}
\]

Recently, Kim-Kim introduced the degenerate Stirling numbers of the first kind defined by
\[
(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k) (x)_k^\lambda, \quad (n \geq 0), \quad \text{ (see [8]),}
\]

where
\[
(x)_0 = 1, \quad (x)_n = x(x-1)(x-2) \cdots (x-n+1), \quad (n \geq 1), \quad \text{ (see [15]).}
\]

As the inversion formula of (10), the degenerate Stirling numbers of the second kind are given by
\[
(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k) (x)_k, \quad (n \geq 0), \quad \text{ (see [8]).}
\]

From (11), we note that
\[
S_{2,\lambda}(n,k) = S_{2,\lambda}(n-1,k-1) + (k - (n-1)\lambda) S_{2,\lambda}(n-1,k), \quad \text{ (see [8]),}
\]

where \( n,k \in \mathbb{N} \) and with \( n \geq k \).

In [10], the degenerate Bell polynomials are defined by
\[
e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{ (see [10, 14]).}
\]

When \( x = 1 \), \( \phi_{n,\lambda} = \phi_{n,\lambda}(1) \) are called the degenerate Bell numbers.

From (13), we note that
\[
\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k) x^k, \quad (n \geq 0), \quad \text{ (see [10, 14]).}
\]

Note that \( \lim_{\lambda \to 0} \phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_2(n,k) x^k \) are the ordinary Bell polynomials.

For \( k \in \mathbb{Z} \), the degenerate polylogarithm is defined by Kim-Kim as
\[
\text{Li}_{k,\lambda}(t) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1} (1^n/n_1^{\lambda})(-1)^{n-1} t^n}{(n-1)! n^k}, \quad \text{ (see [8, 13]).}
\]

Note that \( \text{Li}_{1,\lambda}(t) = -\log_\lambda(1-t) \).

Carlitz considered the degenerate Bernoulli polynomials given by
\[
\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{ (see [2, 3]).}
\]

When \( x = 0 \), \( \beta_{n,\lambda} = \beta_{n,\lambda}(0) \) are called the degenerate Bernoulli numbers.

He also defined the degenerate Euler polynomials given by
\[
\frac{2}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \epsilon_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{ (see [2]).}
\]

Recently, Kim-Kim considered the degenerate Fubini polynomials defined by
\[
\frac{1}{1-x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{ (see [12]).}
\]
Continuing this process, we have
\[ F_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)k!x^{k}, \quad (n \geq 0), \quad (\text{see [12]}). \]

2. Some identities involving degenerate Stirling numbers associated with several degenerate polynomials and numbers

In view of (6), we consider the degenerate hyperharmonic numbers given by
\[ H^{(1)}_{n,\lambda} = H_{n,\lambda}, \quad H^{(r)}_{n,\lambda} = \sum_{k=1}^{n} H^{(r-1)}_{k,\lambda}, \quad (r \geq 2). \]

By (5), we easily get
\[ -\frac{\log_{\lambda}(1-t)}{1-t} = \sum_{n=1}^{\infty} H_{n,\lambda}t^{n}, \quad (\text{see [9]}). \]

In the view of (20), we try to derive the generating function of the degenerate hyperharmonic numbers.

From (20), we note that
\[
\begin{align*}
-\frac{\log_{\lambda}(1-t)}{(1-t)^r} &= \frac{1}{(1-t)^{r-1}} \left( -\frac{\log_{\lambda}(1-t)}{1-t} \right) \\
&= \frac{1}{(1-t)^{r-2}} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} H_{k,\lambda} \right) t^{n} \\
&= \frac{1}{(1-t)^{r-3}} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} H_{k,\lambda}^{(2)} \right) t^{n}.
\end{align*}
\]

Continuing this process, we have
\[ -\frac{\log_{\lambda}(1-t)}{(1-t)^r} = \sum_{n=1}^{\infty} H^{(r)}_{n,\lambda}t^{n}, \quad (r \geq 1), \quad H^{(r)}_{0,\lambda} = 0. \]

Therefore, by (22), we obtain the following theorem.

**Theorem 1.** The generating function of the degenerate hyperharmonic numbers $H^{(r)}_{n,\lambda}$ is given by
\[ -\frac{\log_{\lambda}(1-t)}{(1-t)^r} = \sum_{n=1}^{\infty} H^{(r)}_{n,\lambda}t^{n}, \quad H^{(r)}_{0,\lambda} = 0, \quad (r \geq 1). \]

Replacing $t$ by $1 - e_{\lambda}(t)$ in (22), we get
\[
-\frac{\log_{\lambda}(1-t)}{(1-t)^r} = \sum_{k=1}^{\infty} H^{(r)}_{k,\lambda}(-1)^k \frac{1}{k!} \left( e_{\lambda}(t) - 1 \right)^k
\]
\[ = \sum_{k=1}^{\infty} H^{(r)}_{k,\lambda}(-1)^k \sum_{n=0}^{\infty} S_{2,\lambda}(n,k) \frac{t^{n}}{n!}
\]
\[ = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} H^{(r)}_{k,\lambda}(-1)^k S_{2,\lambda}(n,k) \right) \frac{t^{n}}{n!}. \]
On the other hand, by (11), we get

\[-te_{\lambda}^{r}(t) = -t \sum_{n=0}^{\infty} \frac{(-r)_{n,\lambda}t^{n}}{n!} = t \sum_{n=0}^{\infty} (-1)^{n-1} \frac{\langle r \rangle_{n,\lambda}t^{n}}{n!}\]

(24)

\[= \sum_{n=1}^{\infty} (-1)^{n} \langle r \rangle_{n-1,\lambda} \frac{t^{n}}{(n-1)!} = \sum_{n=1}^{\infty} (-1)^{n} \langle r \rangle_{n-1,\lambda} n! t^{n},\]

where \(\langle x \rangle_{0,\lambda} = 1, \langle x \rangle_{n,\lambda} = x(x + \lambda)(x + 2\lambda) \cdots (x + (n-1)\lambda), \) \((n \geq 1).\)

Therefore, by (23) and (24), we obtain the following theorem.

**Theorem 2.** For \(n, r \in \mathbb{N},\) we have

\[\sum_{k=1}^{n} (-1)^{k} H_{k,\lambda}^{(r)} k! S_{2,\lambda}(n,k) = (-1)^{n} \langle r \rangle_{n-1,\lambda} n.\]

In particular, for \(r = 1,\) we get

\[\frac{1}{(1)_{n-1,\lambda}} \sum_{k=1}^{n} (-1)^{n-k} H_{k,\lambda} k! S_{2,\lambda}(n,k) = n.\]

From (10), we note that

(25)

\[\frac{1}{k!}(\log \lambda (1+t))^{k} = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^{n}}{n!}, \quad (k \geq 0).\]

By (22) and (25), we get

(26)

\[\sum_{n=1}^{\infty} H_{n,\lambda}^{(r)} t^{n} = -\frac{\log \lambda (1-t)}{(1-t)^{r}} = -\log \lambda (1-t) e_{\lambda}^{-t} (\log \lambda (1-t))\]

\[= -\log \lambda (1-t) \sum_{k=0}^{\infty} (-1)^{k} \frac{1}{k!} (\log \lambda (1-t))^{k}\]

\[= \sum_{k=0}^{\infty} (-1)^{k} \frac{1}{(k-1)!} (\log \lambda (1-t))^{k}\]

\[= \sum_{k=1}^{\infty} (-1)^{k} \frac{1}{k-1,\lambda} k \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)(-1)^{n} \frac{t^{n}}{n!}\]

\[= \sum_{n=1}^{\infty} \left( k \sum_{k=1}^{n} (-1)^{n-k} \frac{1}{k-1,\lambda} S_{1,\lambda}(n,k) \right) \frac{t^{n}}{n!}.\]

Therefore, by comparing the coefficients on both sides of (26), we obtain the following theorem.

**Theorem 3.** For \(n, r \in \mathbb{N},\) we have

\[H_{n,\lambda}^{(r)} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \langle r \rangle_{k-1,\lambda} k! S_{1,\lambda}(n,k).\]

In particular, for \(r = 1,\) we have

\[H_{n,\lambda} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} k S_{1,\lambda}(n,k).\]
Replacing $t$ by $\log_\lambda(1-t)$ in (16), we get

$$
\frac{2}{2-t}(1-t)^x = \sum_{k=0}^{\infty} \mathcal{E}_{k,\lambda}(x) \frac{1}{k!} (\log_\lambda(1-t))^k
$$

By binomial expansion, we get

$$
\frac{2}{2-t}(1-t)^x = \frac{1}{1 - e^{-t\lambda}} (1-t)^x = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{t^j}{k!} (1-e^{-t\lambda})^k
$$

Thus, by (27) and (28), we get

$$
\sum_{n=0}^{\infty} \binom{x}{k} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{2} \right)^{n-k} (-1)^k t^k \right) \frac{t^n}{n!}.
$$

Replacing $t$ by $1 - e^{-t\lambda}$ in (28), we have

$$
\frac{2}{e^{-t\lambda} + 1} e^{-t\lambda}(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \left( \frac{1}{2} \right)^{k-j} (-1)^j (1-e^{-t\lambda})^k.
$$

By (16) and (30), we get

$$
\mathcal{E}_{n,\lambda}(x) = \sum_{k=0}^{n} k! S_{2,\lambda}(n,k) \sum_{j=0}^{k} \binom{x}{j} \left( -\frac{1}{2} \right)^{k-j} , \quad (n \geq 0).
$$

In particular, for $x = \frac{1}{2}$, we obtain

$$
\mathcal{E}_{n,\lambda}\left(\frac{1}{2}\right) = \sum_{k=0}^{n} k! S_{2,\lambda}(n,k) \sum_{j=0}^{k} \binom{1}{j} \left( -\frac{1}{2} \right)^{k-j} = \sum_{k=0}^{n} k! S_{2,\lambda}(n,k) (-1)^k \sum_{j=0}^{k} \binom{2j}{j} \frac{1}{(1-2j)^{2k+j}}.
$$

Therefore, by (29) and (31), we obtain the following theorem.

**Theorem 4.** For $n \geq 0$, we have

$$
\sum_{k=0}^{n} S_{1,\lambda}(n,k) \mathcal{E}_{k,\lambda}(x) = n! \sum_{k=0}^{n} \binom{x}{k} \left( -\frac{1}{2} \right)^{n-k} ,
$$
and
\[ \mathcal{E}_{n,\lambda}(x) = \sum_{k=0}^{n} k! S_{2,\lambda}(n,k) \sum_{j=0}^{k} \binom{x}{j} \left( -\frac{1}{2} \right)^{k-j}. \]

From (35), we note that
\[ \frac{1}{t} \log_{\lambda} (1 + t) = \frac{1}{t} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}(1)_{n+1} t^n}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^n(1)_{n+1} t^n}{(n+1)!}. \]

Replacing \( t \) by \( e_{\lambda}(t) - 1 \) in (33), we get
\[ \sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{t^n}{n!} = \frac{t}{e_{\lambda}(t) - 1} = \sum_{k=0}^{\infty} \frac{\lambda^k(1)_{k+1/\lambda} t^k}{(k+1)!} (e_{\lambda}(t) - 1)^k = \sum_{k=0}^{\infty} \frac{\lambda^k(1)_{k+1/\lambda}}{k+1} \sum_{n=0}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{\lambda^k(1)_{k+1/\lambda}}{k+1} S_{2,\lambda}(n,k) \right) \frac{t^n}{n!}. \]

By comparing the coefficients on both sides of (34), we get
\[ \beta_{n,\lambda} = \sum_{k=0}^{n} \frac{\lambda^k(1)_{k+1/\lambda}}{k+1} S_{2,\lambda}(n,k), \quad (n \geq 0). \]

From (15), we note that
\[ \beta_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}(x)_{n-k,\lambda} = \sum_{k=0}^{n} \binom{n}{k} (x)_{n-k,\lambda} \beta_{k,\lambda}, \quad (n \geq 0). \]

By (35) and (36), we see that
\[ \beta_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} (x)_{n-k,\lambda} \beta_{k,\lambda} = \sum_{k=0}^{n} \binom{n}{k} (x)_{n-k,\lambda} \sum_{j=0}^{k} \frac{\lambda^j(1)_{j+1/\lambda}}{j+1} S_{2,\lambda}(k,j) = \sum_{j=0}^{n} \frac{\lambda^j(1)_{j+1/\lambda}}{j+1} \sum_{k=j}^{n} \binom{n}{k} S_{2,\lambda}(k,j)(x)_{n-k,\lambda}. \]

Therefore, by (37), we obtain the following theorem.

**Theorem 5.** For \( n \geq 0 \), we have
\[ \beta_{n,\lambda}(x) = \sum_{j=0}^{n} \frac{\lambda^j(1)_{j+1/\lambda}}{j+1} \sum_{k=j}^{n} \binom{n}{k} S_{2,\lambda}(k,j)(x)_{n-k,\lambda}. \]
Replacing $t$ by $\log_\lambda(1 + t)$ in (15), we have

$$
\frac{\log_\lambda(1 + t)}{t}(1 + t)^x = \sum_{k=0}^{\infty} \beta_{k, \lambda}(x) \frac{1}{k!} (\log_\lambda(1 + t))^k
= \sum_{k=0}^{\infty} \beta_{k, \lambda}(x) \sum_{n=0}^{\infty} S_{1, \lambda}(n, k) \frac{t^n}{n!}
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \beta_{k, \lambda}(x) S_{1, \lambda}(n, k) \right) \frac{t^n}{n!}.
$$

On the other hand, by (3), we get

$$
\log_\lambda(1 + t) = \sum_{k=0}^{\infty} \lambda^k \frac{1}{k!} (1 + t)^k
= \sum_{k=0}^{\infty} \lambda^k (1 + t)^k = \sum_{k=0}^{\infty} \lambda^k \frac{1}{(k+1)!} \sum_{k=0}^{\infty} \frac{x^j}{j!} \frac{t^k}{k!}
= \sum_{n=0}^{\infty} \left( n! \sum_{k=0}^{n} \frac{x^j}{j!} \lambda^{n-j} \frac{1}{(n-j)!} \frac{t^k}{k!} \right) \frac{t^n}{n!}.
$$

Therefore, by (38) and (39), we obtain the following theorem.

**Theorem 6.** For $n \geq 0$, we have

$$
\sum_{k=0}^{n} \beta_{k, \lambda}(x) S_{1, \lambda}(n, k) = n! \sum_{k=0}^{n} \frac{x^j}{j!} \lambda^{n-j} \frac{1}{(n-j)!} \frac{t^k}{k!} \frac{t^n}{n!}.
$$

In (39), by replacing $t$ by $e_\lambda(t) - 1$, we get

$$
\sum_{n=0}^{\infty} \beta_{n, \lambda}(x) t^n = \frac{t}{e_\lambda(t) - 1} e_\lambda^x(t)
= \sum_{k=0}^{\infty} \frac{k!}{k!} \frac{1}{(k+1)!} \sum_{j=0}^{k} \frac{x^j}{j!} \frac{\lambda^{k-j}}{(k-j+1)!} \frac{1}{k!} (e_\lambda(t) - 1)^k
= \sum_{k=0}^{\infty} \frac{k!}{k!} \frac{1}{(k+1)!} \sum_{j=0}^{k} \frac{x^j}{j!} \frac{\lambda^{k-j}}{(k-j+1)!} \frac{1}{k!} \sum_{n=0}^{\infty} S_{2, \lambda}(n, k) \frac{t^n}{n!}
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} k! S_{2, \lambda}(n, k) \sum_{j=0}^{k} \frac{x^j}{j!} \frac{\lambda^{k-j}}{(k-j+1)!} \frac{1}{(k-j+1)!} \frac{t^n}{n!} \right) \frac{t^n}{n!}.
$$

Therefore, by comparing the coefficients on both sides of (40), we obtain the following theorem.

**Theorem 7.** For $n \geq 0$, we have

$$
\beta_{n, \lambda}(x) = \sum_{k=0}^{n} k! S_{2, \lambda}(n, k) \sum_{j=0}^{k} \frac{x^j}{j!} \frac{\lambda^{k-j}}{(k-j+1)!} \frac{1}{(k-j+1)!} \frac{t^n}{n!}.
$$

In particular, for $x = 0$, we have

$$
\beta_{n, \lambda} = \sum_{k=0}^{n} k! S_{2, \lambda}(n, k) \frac{\lambda^k}{(k+1)!} \frac{1}{(k+1)!} \frac{t^n}{n!}.
$$
Let us take \( k = 2 \) in (44). Then

\[
\text{Li}_{2,\lambda}(t) = \sum_{n=1}^{\infty} \frac{\lambda^{-1} (1)_{n+1}^{-1}}{(n-1)n!} (-1)^{n-1} t^n, \quad \text{(see [13])}.
\]

Note that

\[
-\int_{0}^{t} \frac{1}{x} \log \lambda (1-x)dx = \int_{0}^{t} \sum_{n=1}^{\infty} \frac{\lambda^{-1}}{n!} (1)_{n+1}^{-1} (-1)^{n-1} x^{n-1}dx
\]

\[
= \sum_{n=1}^{\infty} \frac{\lambda^{-1}}{(n-1)n!} (1)_{n+1}^{-1} (-1)^{n-1} t^n = \text{Li}_{2,\lambda}(t).
\]

Thus, by (42), we get

\[
\text{Li}_2(1-e_\lambda(-t)) = -\int_{0}^{1-e_\lambda(-t)} \frac{1}{x} \log \lambda (1-x)dx
\]

\[
= \int_{0}^{1-e_\lambda(-t)} \frac{-t}{e_\lambda(-t)-1} e_\lambda^{-1}(-t)dt
\]

\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \beta_{n-1,\lambda}(1-\lambda) \frac{t^n}{n!}.
\]

From (42), we note that

\[
\text{Li}_{2,\lambda} \left( -\frac{t}{1-t} \right) = -\int_{0}^{1-t} \frac{1}{x} \log \lambda (1-x)dx
\]

\[
= -\int_{0}^{t} \left( -\frac{1-t}{t} \right) \left( \log \lambda \left( -\frac{1-t}{1-t} \right) \right) \frac{-1}{(1-t)^2}dt
\]

\[
= -\int_{0}^{t} \frac{1}{t} \left( -\frac{\log \lambda (1-t)}{1-t} \right) dt = -\int_{0}^{t} \frac{1}{t} \sum_{n=1}^{\infty} H_{n-\lambda} t^n dt
\]

\[
= -\sum_{n=1}^{\infty} (n-1)! H_{n-\lambda} \frac{t^n}{n!}.
\]

Replacing \( t \) by \(-\log \lambda \frac{1}{1-t}\) in (43), we have

\[
\text{Li}_{2,\lambda} \left( -\frac{-t}{1-(-t)} \right) = \sum_{k=1}^{\infty} (-1)^{k-1} \beta_{k-1,\lambda}(1-\lambda) \frac{1}{k!} (-\log \lambda \left( -\frac{1}{1-t} \right) )^k
\]

\[
= \sum_{k=1}^{\infty} (-1)^{k-1} \beta_{k-1,\lambda}(1-\lambda) \frac{1}{k!} (\log \lambda (1-t))^k
\]

\[
= \sum_{k=1}^{\infty} (-1)^{k-1} \beta_{k-1,\lambda}(1-\lambda) \sum_{n=k}^{\infty} S_{1-\lambda}(n,k) \frac{(-t)^n}{n!}
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{n-k-1} \beta_{k-1,\lambda}(1-\lambda) S_{1-\lambda}(n,k) \right) \frac{t^n}{n!}
\]

Therefore, by (44) and (45), we obtain the following theorem.

**Theorem 8.** For \( n \in \mathbb{N} \), we have

\[
(n-1)! H_{n-\lambda} = \sum_{k=1}^{n} (-1)^{n-k} \beta_{k-1,\lambda}(1-\lambda) S_{1-\lambda}(n,k).
\]
In (44), by replacing $t$ by $1 - e^{\lambda}(t)$, we get

\begin{equation}
\text{Li}_{2,\lambda}\left(\frac{e^{\lambda}(t) - 1}{e^{\lambda}(t)}\right) = -\sum_{k=1}^{\infty} \frac{1}{k^2} (1 - e^{\lambda}(t))^k = -\sum_{k=1}^{\infty} (k - 1)! H_{k-\lambda} \left(1 - e^{\lambda}(t)^k\right) = \sum_{n=0}^{\infty} \frac{S_{2,\lambda}(n,k)}{n!} t^n.
\end{equation}

On the other hand, by (42), we get

\begin{equation}
\text{Li}_{2,\lambda}\left(1 - e^{\lambda}(t)\right) = \text{Li}_{2,\lambda}\left(1 - e^{\lambda}(t)\right) = \int_{0}^{t} \frac{-1}{e^{\lambda}(t) - x} \log(1 - x) dx = \int_{0}^{t} \frac{-t}{e^{\lambda}(t) - x} e^{2\lambda + 1}(1 - t) dt = \int_{0}^{t} \sum_{n=0}^{\infty} \frac{1}{k^2} (1 - e^{\lambda}(t))^k = \sum_{n=0}^{\infty} \frac{S_{2,\lambda}(n,k)}{n!} t^n.
\end{equation}

Therefore, by (46) and (47), we obtain the following theorem.

**Theorem 9.** For $n \in \mathbb{N}$, we have

$$
\beta_{n-1,\lambda}(2\lambda + 1) = \sum_{k=1}^{n} (k - 1)! (-1)^{n-k} H_{k-\lambda} S_{2,\lambda}(n,k).
$$

Replacing $t$ by $-\log(1 - t)$ in (43), we have

\begin{equation}
\sum_{n=0}^{\infty} \frac{\lambda^{n-1}(1)}{(n-1)! n^2} (-1)^{n-1} t^n = \text{Li}_{2,\lambda}(t) = \sum_{k=1}^{\infty} (-1)^{k-1} \beta_{k-1,\lambda}(1 - \lambda) \frac{(-\log(1 - t))^k}{k!}
\end{equation}

\begin{equation}
= -\sum_{k=1}^{\infty} \beta_{k-1,\lambda}(1 - \lambda) \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)(-t)^n = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{n-k-1} \beta_{k-1,\lambda}(1 - \lambda) S_{1,\lambda}(n,k) \right) \frac{t^n}{n!}.
\end{equation}

Therefore, by comparing the coefficients on both sides of (48), we obtain the following theorem.

**Theorem 10.** For $n \in \mathbb{N}$, we have

$$
\frac{\lambda^{n-1}}{n}(1)_{n,\lambda} = \sum_{k=1}^{n} \beta_{k-1,\lambda}(1 - \lambda) S_{1,\lambda}(n,k).
$$

From (13), we note that

$$
\phi_{n,\lambda}(x) = \frac{1}{e^x} \sum_{k=0}^{\infty} \frac{(x)^k}{k!} = \frac{1}{e^x} \left( x \frac{d}{dx} \right) e^x, \quad (n \geq 1).
$$

Thus, by (49), we get

$$
\phi_{n,\lambda}(x) = e^{-x} \left( x \frac{d}{dx} \right) e^x = \sum_{k=1}^{n} S_{2,\lambda}(n,k)x^k, \quad (n \geq 1).
$$

From (13), we have

$$
S_{2,\lambda}(n+1,k) = S_{2,\lambda}(n,k-1) + (k - n\lambda) S_{2,\lambda}(n,k),
$$

where $n,k$ are nonnegative integers with $n \geq k$. 


By (50) and (51), we get

\begin{equation}
\phi_{n+1, \lambda}(x) = \sum_{k=0}^{n+1} S_{2, \lambda}(n+1, k)x^k = \sum_{k=0}^{n+1} \{S_{2, \lambda}(n, k-1) + (k-n\lambda)S_{2, \lambda}(n, k)\}x^k
\end{equation}

= \sum_{k=1}^{n+1} S_{2, \lambda}(n, k-1)x^k + \sum_{k=0}^{n+1} S_{2, \lambda}(n, k)(k-n\lambda)x^k

= x\sum_{k=0}^{n} S_{2, \lambda}(n, k)x^k + \left(\frac{d}{dx} - n\lambda\right)\sum_{k=0}^{n} S_{2, \lambda}(n, k)x^k

= x\phi_{n, \lambda}(x) + \left(\frac{d}{dx} - n\lambda\right)\phi_{n, \lambda}(x).

By (13), we easily get

\begin{equation}
\sum_{n=0}^{\infty} \phi_{n+1, \lambda}(x)\frac{t^n}{n!} = \frac{d}{dt}\sum_{n=0}^{\infty} \phi_{n, \lambda}(x)\frac{t^n}{n!}
\end{equation}

= \frac{d}{dt}e^{t(\lambda(1-t))} = \lambda e^{1-\lambda}(1-t)e^{t(\lambda(1-t))}

= \sum_{n=0}^{\infty} \left(x\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)_{n-k, \lambda} \phi_{k, \lambda}(x)\right)\frac{t^n}{n!}.

Therefore, by (52) and (53), we obtain the following theorem.

Theorem 11. For \(n \in \mathbb{N}\), we have

\[\phi_{n+1, \lambda}(x) = x\phi_{n, \lambda}(x) + \left(\frac{d}{dx} - n\lambda\right)\phi_{n, \lambda}(x)\]

= \left(x\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)_{n-k, \lambda} \phi_{k, \lambda}(x)\right)\frac{t^n}{n!}.

For \(p \geq 0\), we observe that

\begin{equation}
\sum_{k=1}^{p+1} S_{2, \lambda}(p+1, k)x^k = \phi_{p+1, \lambda}(x) = e^{-x} \sum_{n=1}^{\infty} (n+1-\lambda)_{p, \lambda} \frac{x^n}{n!}
\end{equation}

= e^{-x} \sum_{n=0}^{\infty} \left((n+1-\lambda)_{p, \lambda} + (n+1-\lambda)_{p, \lambda} + \ldots + (n+1-\lambda)_{p, \lambda} - ((n-1-\lambda)_{p, \lambda} + \ldots + (n-1-\lambda)_{p, \lambda})\right)\frac{x^n}{n!}

= e^{-x} \sum_{n=1}^{\infty} \left((1-\lambda)_{p, \lambda} + (2-\lambda)_{p, \lambda} + \ldots + (n-1-\lambda)_{p, \lambda}\right)\frac{x^n}{(n-1)!}

- e^{-x} \sum_{n=0}^{\infty} \left((1-\lambda)_{p, \lambda} + (2-\lambda)_{p, \lambda} + \ldots + (n-1-\lambda)_{p, \lambda}\right)\frac{x^n}{n!}

= \frac{d}{dx} \left(e^{-x} \sum_{n=0}^{\infty} \left((1-\lambda)_{p, \lambda} + (2-\lambda)_{p, \lambda} + \ldots + (n-1-\lambda)_{p, \lambda}\right)\frac{x^n}{n!}\right).
From (55), we note that

\[
\frac{d}{dx} \sum_{k=1}^{p+1} S_{2,\lambda}(p+1,k) \frac{x^k}{k!} = \frac{d}{dx} \left( e^{-x} \sum_{n=0}^{\infty} \left( (1-\lambda)_{p,\lambda} + (2-\lambda)_{p,\lambda} + \cdots + (n-\lambda)_{p,\lambda} \right) \frac{x^n}{n!} \right).
\]

Thus, by (56), we get

\[
\sum_{k=1}^{p+1} S_{2,\lambda}(p+1,k) \frac{x^k}{k!} = e^{-x} \sum_{n=0}^{\infty} \left( (1-\lambda)_{p,\lambda} + (2-\lambda)_{p,\lambda} + \cdots + (n-\lambda)_{p,\lambda} \right) \frac{x^n}{n!}.
\]

Now, we observe that

\[
\sum_{p=0}^{n-1} (k-\lambda)_{p,\lambda} \frac{t^p}{p!} = \sum_{k=0}^{n-1} e^{k-\lambda} (t) = e^{1-\lambda} (t) \frac{e^{t} - 1}{e^{\lambda} - 1} = \sum_{p=0}^{\infty} \left( \beta_{p+1}(n+1-\lambda) - \beta_{p+1,\lambda}(1-\lambda) \right) \frac{t^p}{p!}.
\]

Comparing the coefficients on both sides of (58), we have

\[
\sum_{k=1}^{n} (k-\lambda)_{p,\lambda} = \frac{1}{p+1} \left( \beta_{p+1,\lambda}(n+1-\lambda) - \beta_{p+1,\lambda}(1-\lambda) \right)
\]

\[
= \frac{1}{p+1} \left\{ \sum_{j=0}^{p+1} \left( \frac{p+1}{j} \beta_{j,\lambda}(1-\lambda)(n)_{p+1-j,\lambda} - \beta_{p+1,\lambda}(1-\lambda) \right) \right\}
\]

\[
= \frac{1}{p+1} \sum_{j=0}^{p} \left( \frac{p+1}{j} \beta_{j,\lambda}(1-\lambda)(n)_{p+1-j,\lambda} \right).
\]

So, by (57) and (59), we get

\[
\sum_{k=1}^{p+1} S_{2,\lambda}(p+1,k) \frac{x^k}{k!} = e^{-x} \sum_{n=0}^{\infty} \left( (1-\lambda)_{p,\lambda} + \cdots + (n-\lambda)_{p,\lambda} \right) \frac{x^n}{n!}
\]

\[
= e^{-x} \sum_{n=0}^{\infty} \left( \frac{1}{p+1} \sum_{j=0}^{p} \left( \frac{p+1}{j} \beta_{j,\lambda}(1-\lambda)(n)_{p+1-j,\lambda} \right) \right) \frac{x^n}{n!}
\]

\[
= \frac{1}{p+1} \sum_{j=0}^{p} \left( \frac{p+1}{j} \beta_{j,\lambda}(1-\lambda) \left( e^{-x} \sum_{n=0}^{\infty} (n)_{p+1-j,\lambda} \frac{x^n}{n!} \right) \right)
\]

\[
= \frac{1}{p+1} \sum_{j=0}^{p} \left( \frac{p+1}{j} \beta_{j,\lambda}(1-\lambda) \phi_{p+1-j,\lambda}(x) \right), \quad (p \geq 0).
\]

Therefore, by (60), we obtain the following theorem.

**Theorem 12.** For \( n \in \mathbb{Z} \) with \( n \geq 1 \), we have

\[
\sum_{k=1}^{n} S_{2,\lambda}(n,k) \frac{x^k}{k!} = \frac{1}{n} \sum_{j=0}^{n-1} \left( \frac{n}{j} \beta_{j,\lambda}(1-\lambda) \phi_{n-j,\lambda}(x) \right).
\]

In particular, for \( x = 1 \), we get

\[
\sum_{k=1}^{n} S_{2,\lambda}(n,k) \frac{x^k}{k!} = \frac{1}{n} \sum_{j=0}^{n-1} \left( \frac{n}{j} \beta_{j,\lambda}(1-\lambda) \phi_{n-j,\lambda} \right).
\]
From Theorem 12, we note that
\[
(61) \quad \sum_{k=1}^{n} S_{2,\lambda}(n, k) \frac{x^k}{k!} = \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j} \beta_{j,\lambda}(1 - \lambda) \int_{0}^{x} \frac{\phi_{n-j,\lambda}(t)}{t} \, dt \\
= \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j} \beta_{j,\lambda}(1 - \lambda) \sum_{l=1}^{n-j} S_{2,\lambda}(n-j, l) \frac{x^l}{l!} \\
= \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j} \beta_{j,\lambda}(1 - \lambda) \frac{1}{n-j} \sum_{m=1}^{n-j} \binom{n-j}{m} \beta_{n-j-m,\lambda}(1 - \lambda) \phi_{m,\lambda}(x).
\]
Thus, by (61), we get
\[
\sum_{k=1}^{n} S_{2,\lambda}(n, k) \frac{x^k}{k!} = \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j} \beta_{j,\lambda}(1 - \lambda) \frac{1}{n-j} \sum_{m=1}^{n-j} \binom{n-j}{m} \beta_{n-j-m,\lambda}(1 - \lambda) \phi_{m,\lambda}(x),
\]
for \( n \geq 1 \).

From (17) and (18), we have
\[
(62) \quad \int_{0}^{x} \frac{F_{n,\lambda}(t)}{t} \, dt = \sum_{k=1}^{n} S_{2,\lambda}(n, k)(k-1)!x^k, \quad (n \geq 1).
\]
By (51), we get
\[
\sum_{k=1}^{n+1} S_{2,\lambda}(n+1, k)(k-1)!x^k = \sum_{k=1}^{n+1} \{ S_{2,\lambda}(n, k-1) + (k-n\lambda)S_{2,\lambda}(n, k) \}(k-1)!x^k \\
= \sum_{k=1}^{n} k!S_{2,\lambda}(n, k)x^k + x \sum_{k=1}^{n} S_{2,\lambda}(n, k)k!x^k - n\lambda \sum_{k=1}^{n} S_{2,\lambda}(n, k)(k-1)!x^k \\
= (1+x)F_{n,\lambda}(x) - n\lambda \int_{0}^{x} \frac{F_{n,\lambda}(t)}{t} \, dt, \quad (n \geq 1).
\]
Therefore, by (62), we obtain the following theorem.

**Theorem 13.** For \( n \in \mathbb{N} \), we have
\[
(1+x)F_{n,\lambda}(x) = \sum_{k=1}^{n+1} S_{2,\lambda}(n+1, k)(k-1)!x^k + n\lambda \sum_{k=1}^{n} S_{2,\lambda}(n, k)(k-1)!x^k.
\]
Moreover, we have
\[
\int_{0}^{x} \frac{F_{n+1,\lambda}(t)}{t} \, dt + n\lambda \int_{0}^{x} \frac{F_{n,\lambda}(t)}{t} \, dt = (1+x)F_{n,\lambda}(x).
\]
From Theorem 13, we have
\[
(63) \quad \sum_{k=1}^{n+1} S_{2,\lambda}(n+1, k)k!x^{k-1} = \frac{d}{dx} \sum_{k=1}^{n+1} S_{2,\lambda}(n+1, k)(k-1)!x^k \\
= -n\lambda \sum_{k=1}^{n} S_{2,\lambda}(n, k)k!x^k + F_{n,\lambda}(x) + (1+x)F'_{n,\lambda}(x),
\]
where \( F_{n,\lambda}'(x) = \frac{d}{dx}F_{n,\lambda}(x) \).

Thus, by (63), we get
\[
(64) \quad \sum_{k=1}^{n+1} S_{2,\lambda}(n+1, k)k!x^k = -n\lambda F_{n,\lambda}(x) + xF_{n,\lambda}(x) + (x + x^2)F_{n,\lambda}'(x).
\]
Therefore, by (18) and (64), we obtain the following theorem.
Theorem 14. For $n \in \mathbb{N}$, we have

$$F_{n+1,\lambda}(x) = (x - n\lambda)F_{n,\lambda}(x) + (x + x^2)F'_{n,\lambda}(x).$$

Remark. By Theorem 13, we easily get

$$\sum_{k=1}^{n+1} S_{2,\lambda}(n+1,k)(k-1)! + n\lambda \sum_{k=1}^{n} S_{2,\lambda}(n,k)(k-1)! = \begin{cases} 2F_{n,\lambda}(1), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

We observe that

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} x^k(n,k,\lambda) \right) \frac{t^n}{n!} = \sum_{k=0}^{\infty} x^k(\lambda)(t) = \frac{1}{1 - xe^{t}}(t)$$

Comparing the coefficients on both sides of (65), we have

$$\frac{1}{1 - x}F_{n,\lambda}(\frac{x}{1 - x}) = \sum_{k=0}^{\infty} x^k(n,k,\lambda), \quad (n \geq 0).$$

Let us take $x = \frac{1}{2}$ in (66). Then we get

$$F_{n,\lambda}(1) = \sum_{k=0}^{\infty} (k)n_{,\lambda} \left( \frac{1}{2} \right)^{k+1}, \quad (n \geq 0).$$

Let

$$f(t) = (1 + t) \log \lambda (1 + t) - t.$$

Then, by (33), we get

$$f(t) = t \log \lambda (1 + t) + \log \lambda (1 + t) - t$$

$$= t \sum_{n=1}^{\infty} \frac{\lambda^{n-1}(1,n+1,\lambda)}{n!} t^n + \sum_{n=2}^{\infty} \frac{\lambda^{n-1}(1,n+1,\lambda)}{n!} t^n$$

$$= t^2 \sum_{n=0}^{\infty} \frac{\lambda^{n}(1,n+1,\lambda)}{(n+1)!} t^n + t^2 \sum_{n=0}^{\infty} \frac{\lambda^{n+1}(1,n+1,\lambda)}{(n+2)!} t^n$$

$$= t^2 \sum_{n=0}^{\infty} \frac{\lambda^{n}(1)n+1,\lambda}{n+1} t^n - t^2 \sum_{n=0}^{\infty} \frac{\lambda^{n}(1)n+1,\lambda}{n+2} t^n + t^2 \sum_{n=0}^{\infty} \frac{\lambda^{n+1}(1)n+1,\lambda}{(n+2)!} t^n$$

$$= t^2 \sum_{n=1}^{\infty} \frac{\lambda^{n}(1)n+1,\lambda}{n+1} t^n + t^2 \sum_{n=0}^{\infty} \frac{\lambda^{n}(1)n+1,\lambda}{(n+2)!} t^n$$

$$= \sum_{n=2}^{\infty} \lambda^{n-2}(1)n-1,\lambda t^n + \lambda \sum_{n=2}^{\infty} \lambda^{n-2}(1)n-1,\lambda t^n$$

$$= \sum_{n=2}^{\infty} (1 + \lambda)\lambda^{n-2}(1)n-1,\lambda t^n.$$
In (69), by replacing \( t \) by \( e_\lambda(t) - 1 \), we get

\[
(70) \quad t e_\lambda(t) - e_\lambda(t) + 1 = f(e_\lambda(t) - 1) = (1 + \lambda) \sum_{k=2}^{\infty} (1)_{k-1} \frac{\lambda^{k-2}}{k!} \left( e_\lambda(t) - 1 \right)^k
\]

\[
= (1 + \lambda) \sum_{k=2}^{\infty} (1)_{k-1} \frac{\lambda^{k-2} \sum_{n=k}^{\infty} S_{2,\lambda}(n, k)}{n!} t^n
\]

\[
= \sum_{n=2}^{\infty} \left( (1 + \lambda) \sum_{k=2}^{n} (1)_{k-1} \frac{\lambda^{k-2} S_{2,\lambda}(n, k)}{n!} \right) t^n.
\]

On the other hand, by (1), we get

\[
(71) \quad t e_\lambda(t) - e_\lambda(t) + 1 = t \sum_{n=0}^{\infty} \frac{(1)_{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{(1)_{n+1}}{n!} t^n + 1
\]

\[
= \sum_{n=0}^{\infty} \frac{(1)_{n+1}}{n!} - \sum_{n=1}^{\infty} \frac{(1)_{n+1}}{n!} t^n
\]

\[
= \sum_{n=1}^{\infty} \frac{(1)_{n+1}}{n!} \frac{t^n}{n} - \sum_{n=1}^{\infty} \frac{(1)_{n+1}}{n!} \frac{(1 - (n - 1)\lambda)}{n!} t^n
\]

\[
= \sum_{n=1}^{\infty} \frac{(1)_{n+1}}{n!} \lambda (n-1) \frac{t^n}{n} - \sum_{n=1}^{\infty} \frac{(1)_{n+1}}{n!} \frac{(1 - (n - 1)\lambda)}{n!} t^n
\]

\[
= \sum_{n=1}^{\infty} \frac{(1)_{n+1}}{n!} \lambda (n-1) \frac{t^n}{n} + \sum_{n=1}^{\infty} \frac{(1)_{n+1}}{n!} \frac{(1 - (n - 1)\lambda)}{n!} t^n
\]

\[
= \sum_{n=2}^{\infty} \frac{(1 + \lambda)(n-1)(1)_{n-1,\lambda}}{n!} t^n.
\]

Therefore, by (70) and (71), we obtain the following theorem.

**Theorem 15.** For \( n \geq 2 \), we have

\[
\sum_{k=2}^{n} (1)_{k-1} \frac{\lambda^{k-2} S_{2,\lambda}(n, k)}{n!} = (n - 1)(1)_{n-1,\lambda}.
\]

We remark that, by letting \( \lambda \to 0 \), we get the following corollary.

**Corollary 16.** For \( n \geq 2 \), we have

\[
\sum_{k=2}^{n} (-1)^{k-2} (k-2)! S_{2}(n, k) = n - 1.
\]

3. Conclusion

In this paper, by exploiting generating functions we investigated some properties, recurrence relations and identities involving degenerate Stirling numbers associated with degenerate hyperharmonic numbers and also with degenerate Bernoulli, degenerate Euler, degenerate Bell and degenerate Fubini polynomials. In particular, the degenerate hyperharmonic numbers were introduced as a degenerate version of the hyperharmonic numbers.

It is one of our future projects to continue to study various degenerate versions of special polynomials and numbers by using such diverse methods as it is mentioned in Introduction.
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