Noncommutative Geometry of Lattice and Staggered Fermions

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Abstract

Differential structure of a d-dimensional lattice, which is essentially a noncommutative exterior algebra, is defined using reductions in first order and second order of universal differential calculus in the context of noncommutative geometry (NCG) developed by Dimakis et al. This differential structure can be realized adopting a Dirac-Connes operator proposed by us recently within Connes' NCG. With matrix representations being specified, our Dirac-Connes operator corresponds to staggered Dirac operator, in the case that dimension of the lattice equals to 1, 2 and 4.

Key words: Dirac operator, noncommutative geometry, reduction, staggered fermion

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I Introduction

In our recent work [9], we formulated a Dirac operator on discrete abelian group to bridge the noncommutative geometry devised by Dimakis et al based on reduced differential calculus [1] and spectral noncommutative geometry (sNCG) [2][3], and we referred this operator as Dirac-Connes operator. It has no longer been a new intuition to consider lattice Dirac operator within the framework of noncommutative geometry (NCG). Feng et al employed the intuition of “half-spacing” lattice in [4], while Vaz generalized Clifford algebra to be non-diagonal in spacetime [5]; Balachandran et al studied another type of Dirac operator in the context of discrete field theories upon fuzzy sphere [6] and its Cartesian products [7]. It is worthy to be remarked that all these ideas that we mention above were more or less with the aim to resolve the species doubling puzzle of massless fermion on lattices [8] which has been explored for more than two decades by lattice field theorists.

In this contribution, we will show that under a specific matrix representation, our Dirac-Connes operator which is rooted in pure geometry possesses an interpretation of staggered Dirac operator emerged from lattice field theory (LFT) [10]. Below we give an outline of this article. A canonical differential structure can be implemented onto a lattice as a quotient algebra of the universal differential calculus on this lattice by a collection of first order and second order reductions. This differential structure is essentially a noncommutative exterior algebra. Our Dirac-Connes operator provides a natural representation for this reduced calculus on a spinor Hilbert space and this representation naturally has the same dimension as that of staggered fermions (Section II). After matrix representation being assigned, the correspondence between our Dirac-Connes operator and staggered Dirac operator can be computed explicitly in cases that the dimension of the underlying lattice equals to one, two and four (Section III). We will also discuss the relation of our formalism and that of Takami et al [11] (Section IV).
II Noncommutative Geometry on Lattice: Two Approaches

A noncommutative space can be described in either quantum algebraic way or operator algebraic way. NCG of a discrete point set, with or without a group structure being endowed, has been formulated well along the first approach by Dimakis et al \[1\]. A d-dimensional lattice, being a specific object in this category, can be parametrized by a direct-product group $\mathbb{Z}^d$ where $\mathbb{Z}$ is the integer addition group, namely each element in $\mathbb{Z}^d$ can be labeled by one $d$-tuple vector $x$ whose components $x^i, i = 1, 2, ..., d$ are integers. Let $\mathcal{A}$ be the algebra of complex functions on $\mathbb{Z}^d$. The group translations on $\mathbb{Z}^d$ being pulled back onto $\mathcal{A}$ are defined by $(T_x f)(y) = f(x + y), \forall x, y \in \mathbb{Z}^d, \forall f \in \mathcal{A}$. A natural linear basis of $\mathcal{A}$ is a complete class of delta-functions on this lattice \{\(e^x, x \in \mathbb{Z}^d: e^x(y) = \Pi_i \delta^{x_i y_i}, \forall y \in \mathbb{Z}^d\}\}. One can easily check that $T_x e^y = e^{x-y}$.

We will use $a = 1$ as the convention for lattice constant all through this work.

**Definition 1** Universal Differential Calculus $(\Omega_u(\mathcal{A}), d)$ over $\mathbb{Z}^d$:

i) $\Omega_u(\mathcal{A}) = \bigoplus_{k=0}^{\infty} \Omega_u^k(\mathcal{A})$ is a bimodule over $\mathcal{A}$ with $\Omega_u^0(\mathcal{A}) = \mathcal{A}$ and the elements in $\Omega_u^p(\mathcal{A})$ are referred as $p$-order forms;

ii) $\Omega_u(\mathcal{A})$ is a $\mathbb{Z}$-graded algebra: $\Omega_u^p(\mathcal{A}) \cdot \Omega_u^q(\mathcal{A}) \subset \Omega_u^{p+q}(\mathcal{A})$;

iii) $d: \Omega_u^k(\mathcal{A}) \to \Omega_u^{k+1}(\mathcal{A}), k = 0, 1, ...$ is a linear homomorphism satisfying graded Leibnitz rule

$$d(\omega \omega') = d(\omega)\omega' + (-)^p\omega d(\omega'), \forall \omega, \omega' \in \Omega_u^p(\mathcal{A}), \omega' \in \Omega_u^q(\mathcal{A})$$

and nilpotent rule $d \cdot d = 0$;

iv) If $1$ is the unit of $\mathcal{A}$, then $1$ is the unit of $\Omega_u(\mathcal{A})$.

Accordingly, one can check that

**Lemma 1** i) $e^x de^y$ for all $x \neq y$ form a linear basis of $\Omega_u^1(\mathcal{A})$.

ii) $\chi^x = \sum_{y \in \mathbb{Z}^d} e^y de^{y-x}$ form a module basis of $\Omega_u^1(\mathcal{A})$ which is translation-invariant;

iii) (Fundamental noncommutative relation of lattice differential):

$$\chi^x f = (T_x f) \chi^x$$

(1)

iv) Define a formal partial derivative $\partial_x f(y) = f(y + x) - f(y) = ((T_x - 1)f)(y)$, then there is

$$df = \sum_x \partial_x f \chi^x$$

(2)
for all \( f \in A \).

The universal differential of function defined in Eq.(2) is highly non-local in the sense that lattice is treated as a spacetime model in physics. So we need a reduction procedure, namely introducing a set of equivalent relation on \( \Omega_u(A) \) and considering the quotient as differential structure of this lattice. We will use \((\Omega(A), d)\) to denote the quotient differential algebra.

Here begins what we hope to deliver in this paper.

**Definition 2 (Symmetric Nearest First Order Reduction)**

\[
\text{df} \cong \sum_{\mu=1}^{d} (\partial_\mu f \chi^\mu + \partial_{-\mu} f \chi^{-\mu}) \quad (3)
\]

in which \( \mu \) is the unit vector along the \( \mu \)th axis of \( \mathbb{Z}^d \)

To be compatible with constructive axioms in Definition 1, esp. nilpotent rule \( d^2 = 0 \), relations in order two are inferred.

**Lemma 2** For all \( \mu, \nu = 1, 2, \ldots, d \),

i) Exterior product:

\[
\{ \chi^{\pm \mu}, \chi^{\pm \nu} \} = 0 \quad (4)
\]

ii) Maurer-Cartan Equation:

\[
\{ \chi^{\pm \mu}, \chi^{\mp \nu} \} = \delta^{\mu \nu} d \chi^\mu = \delta^{\mu \nu} d \chi^{-\nu} \quad (5)
\]

Additional to equivalent relation Eq.(3) in first order, a set of second order reduction \( d \chi^\mu \cong 0 \cong d \chi^{-\nu} \) is put into Eq.(5), for all \( \mu, \nu = 1, 2, \ldots, d \), while the consistency is obvious. Consequently, we reach a 2d-dimensional exterior algebra generated by 2d translation-invariant 1-forms \( \chi^{\pm \mu} \), together with noncommutative relation Eq.(1) as a canonical differential structure on \( \mathbb{Z}^d \).

The differential structure \((\Omega(A), d)\) which we introduce onto a lattice is able to be represented as a quantized calculus \[2\].
Lemma 3  Let $\mathcal{H} = A \otimes \mathbb{C}^{2d}$ be a complex $l^2$-space defined in the standard way; $A$ acts on $\mathcal{H}$ by multiplication and the action is written as $\pi$. Let

$$D = \sum_{\mu}(\Gamma^\mu T^-_\mu + \Gamma^-^{\mu} T^\mu)$$

in which $\Gamma^{\pm \mu}$ are gamma-matrices in 2d-Euclidean space satisfying generating relations of Clifford algebra $\text{Cl}(E^{2d})$:

$$\{\Gamma^{\pm \mu}, \Gamma^{\pm \nu}\} = 0, \{\Gamma^{\pm \mu}, \Gamma^{\mp \nu}\} = \delta^{\mu \nu}, (\Gamma^{\mu})^\dagger = \Gamma^{-\mu}$$

and being represented on $\mathbb{C}^{2d}$ irreducibly. Then $(\mathcal{H}, D)$ forms a Fredholm Module over $A$.

In fact, one can verify geometric square-root condition $D^2 = d1$, hence $D$ is a Fredholm operator up to a scalar normalization. The first step to implement differential representation of $(\Omega(A), d)$ is the introduction of a quantized differential

$$\hat{df} = [D, \pi(f)]$$

and the extension of $\pi$ to be a linear homomorphism from $\Omega(A)$ into $\text{End}_\mathbb{C}(\mathcal{H})$ by

$$\pi(f_0 df_1 df_2 ... df_p) = \pi(f_0) \cdot \hat{df}_1 \cdot \hat{df}_2 \cdot ... \cdot \hat{df}_p$$

Note that we omit an “$i$” in RHS of Eq.(7) which appears in usual literature due to the reason that we do not concern the involutive property of differential algebra in this work. Then, one can check that

Proposition 1  (Representation of First Order Reductions)

$$\pi(\chi^{\pm \mu}) = \Gamma^{\pm \mu} T^{\pm \mu}_{\pm \mu}\ (\text{no summation to } \mu)$$

$$\pi(d\chi^{\mu}) = \pi(d\chi^{-\nu}) = 1, \forall \mu, \nu = 1, 2, ..., d$$

Eq.(8) depicts the common feature for constructing calculus in sNCG that differential forms in different orders are mixed. However, this drawback can be cured in our specified model by implement the second order reductions, namely define the product of two adjunct gamma matrices to be a wedge product. Note importantly that this definition is consistent with Eq.(4).
because of the abelian nature of $\mathcal{Z}^d$. Inner product of two forms in $\Omega(A)$ is pulled back from the trace of operators on $\mathcal{H}$

$$(\omega, \omega') = Tr(\pi(\omega)\pi(\omega'))), \forall \omega, \omega' \in \Omega(A)$$

in the conventional way of sNCG. One can check that the perpendicularity between forms in different order is an outcome instead of a prerequisite, thanks to our wedge product definition.

Remarks:
1) It is important to realize that Eqs.(4)(5) are not assumptions, but inferences.
2) Second order reductions, though at the first sight appearing to be ad hoc and not so intuitive as the first order ones, are necessary for the requirement $\Omega^p(A) \subset \Omega^q(A)^\perp, p \neq q$ when inner product of forms is defined.
3) If there be no geometric square root condition, the choice of $\mathcal{D}$ is not unique, due to that only $[\mathcal{D}, \pi(f)]$ is concerned to implement $\Omega(A)$ onto $\mathcal{H}$. In fact, $\mathcal{D}' = \mathcal{D} + \mathcal{O}$ will do the same work if $\mathcal{O} \in \pi(A)'$ where $\pi(A)'$ is the commutants of $\pi(A)$ on $\mathcal{H}$. Nevertheless, one has to consider “Junk-idea” by using $\mathcal{D}'$ to realize differential forms in $End_c(\mathcal{H})$, if $\mathcal{D}'$ is not Fredholm operator.
4) The distinction between Fredholm operator and Dirac operator which has essential implication in operator algebraic approach to NCG is not relevant to our stage. In fact, we can make a compactification $\mathcal{Z} \rightarrow \mathcal{Z}_N$ with a large enough $N$, then we would just handle a finite dimensional NCG.

### III Staggered Fermions

Now we make the transition

$$\mathcal{D} = \sum_{\mu} (\Gamma^\mu T_\mu + \Gamma^{-\mu} T_{-\mu}) \rightarrow \mathcal{D}_{dyn} = \sum_{\mu} (\Gamma^\mu \partial_\mu + \Gamma^{-\mu} \partial_{-\mu})$$

in which $\mathcal{D}_{dyn}$ satisfies that

**Proposition 2 (Physical Square-Root Condition):**

$$\mathcal{D}^2 = \Delta$$

where $\Delta = \sum_\mu \partial_\mu \partial_{-\mu}$ is lattice Laplacian.
We will try to show the nontrivial correspondence between \(D_{\text{dyn}}\) and staggered Dirac operator under a specific matrix representation for \(\Gamma^{\pm \mu}\) when \(d = 1, 2, 4\) in this section.

Massless staggered Dirac operator on \(d\)-dimensional lattice, acting on \(\mathcal{H}_S = \mathcal{A}\) directly, can be written as

\[
D_S = \sum_{\mu} \eta^{\mu} \nabla_{\mu}
\]  

(10)

in which \(\nabla_{\mu} = \frac{1}{2}(T_{\mu} + T_{-\mu})\) and \(\eta^{\mu}(x) = i(-)^{\sum_{l<\mu} x_l}\) is referred as staggered phase \([10]\). Note that our definition of staggered phase has an additional “\(i\)” to insure \(D_S\) to be hermitian instead of to be anti-hermitian. Chirality operator is defined to be \(\epsilon(x) = (\sum_i x_i) \)\([12]\). The main advantage of staggered formalism in LFT is the remnant \(U(1)\)-chiral symmetry generated by \(\epsilon(x)\), compared with Wilson-Dirac formalism. However, flavor interpretation is a problem for staggered fermion \([13]\). When a “double spacing” transformation being performed, staggered Dirac operator in Susskind form as in Eq.(10) could converted into a bi-module form to which the right module is interpreted as flavor space; this equivalent is broken when a gauge potential presents on lattice. We will adopt this “double spacing” tech also below. Dynamics of staggered fermions and gauge fields has been well studied in \([14]\).

In the present work, staggered fermion field is denoted as \(\phi\) whose classical action functional is \(A[\phi] = (\phi, D_S \phi)_{\mathcal{H}_S} = \sum_x \phi(x) (D_S \phi)(x)\). As for our formalism, fermion fields are elements in \(\mathcal{H}\), being written as \(\psi\) with \(2^d\)-components; classical action is \(A[\psi] = (\psi, D_{\text{dyn}} \psi)_{\mathcal{H}} = \sum_x \psi^\dagger(x) (D_{\text{dyn}} \psi)(x)\). The subtlety concerning anti-commutativity for Euclidean spinor is not relevant in this work, so we do not use the notation like \(\bar{\phi}, \bar{\psi}\).

**Proposition 3** \(A[\phi] = A[\psi] \) when \(d = 1, 2, 4\).

**Proof:**

\(d=1:\) We modify the representation used by Dimakis and Müller-Hoissen in \([13]\) to be that

\[
\Gamma^1 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad \Gamma^{-1} = (\Gamma^1)^\dagger,
\]

and introduce a “double-spacing” lattice by defining the map

\[
\psi_1(x) = \phi(2x)/\sqrt{2}, \quad \psi_2(x) = \phi(2x + 1)/\sqrt{2},
\]

then one can check \(A[\psi] = A[\phi]\).
\(d=2\): Let

\[
\Gamma^{(1,0)} = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & i & 0
\end{pmatrix}, \quad \Gamma^{(0,1)} = \begin{pmatrix}
0 & -i & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & i & 0
\end{pmatrix}
\]

\(\Gamma^{(-1,0)} = (\Gamma^{(1,0)})^\dagger, \Gamma^{(0,-1)} = (\Gamma^{(0,1)})^\dagger\), and let “double-spacing” map be \(\psi_3(x^1, x^2) = \phi(2x^1, 2x^2)/\sqrt{2}, \psi_2(x^1, x^2) = \phi(2x^1 + 1, 2x^2)/\sqrt{2}, \psi_4(x^1, x^2) = \phi(2x^1, 2x^2 + 1)/\sqrt{2}, \psi_1(x^1, x^2) = \phi(2x^1 + 1, 2x^2 + 1)/\sqrt{2}\). Then \(A[\psi] = A[\phi]\) still holds. Note that \(\epsilon\) being mapped onto \(\mathcal{H}\) equals to \(\text{diag}(1, -1, 1, -1)\).

\(d=4\): Label spinor components of \(\psi\) by \(\psi_3\) in which \(\hat{\delta} = (\delta^1, \delta^2, \delta^3, \delta^4), \delta^i \in \{0, 1\}, i = 1, ..., 4, \) and order \(\hat{\delta}\) as \((0, 0, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), (1, 0, 0, 1), (0, 1, 1, 0), (1, 0, 1, 0), (1, 1, 0, 0), (1, 1, 1, 1), (0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1), (1, 1, 1, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0)\). Under this ordering, define the representation of \(\Gamma^{(1,0,0,0)}, \Gamma^{(0,1,0,0)}, \Gamma^{(0,0,1,0)}, \Gamma^{(0,0,0,1)}\) to be
respectively, and $\Gamma^{-\mu} = (\Gamma^{\mu})^\dagger$; let “double-spacing” map to be $\psi_\delta(x) = \phi(2x + \hat{\delta})/\sqrt{2}$. After a tedious algebra, one will reach still $A[\psi] = A[\phi]$; while $\epsilon = diag(1_8, -1_8)$.

We would like to conjecture that there exists a representation for $\Gamma^\mu, \Gamma^{-\mu}, \mu = 1, 2, ..., d$ and a “double-spacing” map from $H_S$ to $H$, such that $A[\psi] = A[\phi]$ holds for any $d$.

Remarks:
1) Square root property Eq. (9) which is emphasized by Vaz in [5] has been ignored by previous authors [13]. In our understanding, this property characterizes staggered Dirac operator in an abstract sense.
2) Our Dirac-Connes operator can be understood also as an abstract definition of staggered operator where “abstract” refers to representation independent.

IV Discussions

Some similarity in formalism can be found in the work of Takami et al [11]. In fact, they were considering a discretized Weyl-equation on lattice in their papers. Combine a discrete time axis
\( \mathcal{T} \) to the above \( Z^d \), and define forward action of translation along \( \mathcal{T} \) to be \( (T^+_0 f)(t, x) = f(t+1, x) \) and \( \partial_t f = T^+_0 f - f \). Then their Dirac operator \( \Lambda \) can be essentially written as

\[
\Lambda = -\partial_t + T^+_0 D
\]

where \( D \) is just \( D_{\text{dyn}} \), Dirac-Connes operator discussed in the last section. \( \Lambda \) is not hermitian, though these author showed that this lost would not do harm to physics.

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