EVERY CONTACT MANIFOLD CAN BE GIVEN A NON-FILLABLE CONTACT STRUCTURE

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Abstract. Recently Francisco Presas Mata constructed the first examples of closed contact manifolds of dimension larger than 3 that contain a plastikstufe, and hence are non-fillable. Using contact surgery on his examples we create on every sphere \( S^{2n-1} \), \( n \geq 2 \), an exotic contact structure \( \xi \) that also contains a plastikstufe. As a consequence, every closed contact manifold \( (M, \xi) \) (except \( S^1 \)) can be converted into a contact manifold that is not (semi-positively) fillable by taking the connected sum \( (M, \xi) \# (S^{2n-1}, \xi) \).

Most of the natural examples of contact manifolds can be realized as convex boundaries of symplectic manifolds. These manifolds are called symplectically fillable. An important class of contact manifolds that do not fall into this category are so-called overtwisted manifolds (\[Eli88\], [Gro85]). Unfortunately, the notion of overtwistedness is only defined for 3–manifolds. A manifold is overtwisted if one finds an embedded disk \( D^2 \subset \xi \), an overtwisted disk \( D_{\text{OT}} \). This topological definition gives an effective way to find many examples of contact 3–manifolds that are non-fillable.

Until recently no example of a non-fillable contact manifold in higher dimension was known, but Francisco Presas Mata recently discovered a construction that allowed him to build many non-fillable contact manifolds of arbitrary dimension ([Pre06]). He showed that after performing this construction on certain manifolds, they admit the embedding of a plastikstufe. Roughly speaking, a plastikstufe \( \mathcal{PS}(S) \) can be thought of as a disk-bundle over a closed \((n-2)\)-dimensional submanifold \( S \), where each fiber looks like an overtwisted disk. As shown in [Nie06b], the existence of such an object in a contact manifold \( (M^{2n-1}, \xi) \) excludes the existence of a symplectic filling.

In this paper we extend Presas’ results to a much larger class of contact manifolds. The idea is to start with one of his examples and to use contact surgery to simplify the topology and convert it into a contact sphere \( S_E \) that admits the embedding a plastikstufe. If \( (M, \xi) \) is any other contact manifold, then \( (M, \xi) \# S_E \cong M \) carries a contact structure that also has an embedded plastikstufe and is hence non-fillable.

Definition. Let \((M, \alpha)\) be a cooriented \(2n-1\)-dimensional contact manifold, and let \( S \) be a closed \((n-2)\)-dimensional manifold. A plastikstufe \( \mathcal{PS}(S) \) with singular set \( S \) in \( M \) is an embedding of the \( n \)-dimensional manifold

\[ \iota : \mathbb{D}^2 \times S \hookrightarrow M \]

that carries a (singular) Legendrian foliation given by the 1-form \( \beta := \iota^* \alpha \) satisfying:

- The boundary \( \partial \mathcal{PS}(S) \) of the plastikstufe is the only closed leaf.
- There is an elliptic singular set at \( \{0\} \times S \).
- The rest of the plastikstufe is foliated by an \( S^1 \)-family of stripes, each one diffeomorphic to \( (0,1) \times S \), which are spanned between the singular set on one end and approach \( \partial \mathcal{PS}(S) \) on the other side asymptotically.

The importance of the plastikstufe lies in the following theorem.

**Theorem 1.** Let \((M, \alpha)\) be a contact manifold containing an embedded plastikstufe. Then \( M \) does not have a semipositive strong symplectic filling. In particular, if \( \dim M \leq 5 \), then \( M \) does not have any strong symplectic filling at all.

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1In the scope of this article a contact sphere will be a smooth sphere carrying a contact structure.
Definition. A contact manifold is called \textit{PS–overtwisted} if it admits the embedding of a plastikstufe.

Whether \textit{PS–overtwisted} can be taken as the general definition of overtwisted in higher dimensions, has yet to be clarified in the future (see Remark \ref{Remark6}).

In dimension 3, the definition of \textit{PS–overtwisted} is identical to the standard definition of overtwistedness. Using the Lutz twist, it is easy to convert a tight contact structure on a 3–manifold \(M\) into an overtwisted one. Until recently, no example of a closed \textit{PS–overtwisted} contact manifold of dimension larger than 3 was known, but Francisco Presas Mata found a beautiful construction, which allowed him to create such examples in arbitrary dimension \cite{Pre06}.

Theorem 2 (F. Presas Mata). Let \((M, \alpha)\) be a contact manifold, which contains a \textit{PS–overtwisted} contact submanifold \(N\) of codimension 2. Assume that \(N\) has trivial normal bundle. Then we can glue \(N \times \mathbb{T}^2\) via a fiber sum to \(M\), and the resulting manifold

\[
\text{Glue}(M, N, \alpha) := M \cup_N N \times \mathbb{T}^2
\]

supports a \textit{PS–overtwisted} contact structure that coincides on \(M\) outside a small neighborhood of the gluing area with the original structure. More precisely, if \(N\) contains a plastikstufe \(\mathcal{P}S(S)\), then \(\text{Glue}(M, N, \alpha)\) contains \(\mathcal{P}S(S \times S^1)\).

With this construction, he was immediately able to find \textit{PS–overtwisted} contact manifolds in every odd dimension greater than 1.

Corollary 3. (1) There is a contact form \(\alpha_-\) on the 5–sphere \(S^5\) that is obtained by taking an open book with a single left-handed Dehn-twist (see Section \ref{Section1}). It restricts on the standard embedding of \(S^3\) to an overtwisted contact structure. Hence, one finds on the manifold

\[
M_0 = \text{Glue}(S^5, S^3, \alpha_-) = S^5 \cup_{S^3} S^3 \times \mathbb{T}^2
\]

a \textit{PS–overtwisted} contact structure.

(2) Let \(\Sigma_g\) denote the closed Riemann surface of genus \(g \geq 2\), and let \((M, \alpha)\) be a closed \textit{PS–overtwisted} contact manifold. Then the manifold \(M \times \Sigma_g\) also supports a \textit{PS–overtwisted} contact structure.

In this note, we apply contact surgery to examples that are similar to the manifold \(M_0\), and we obtain the following corollary.

Corollary 4. Every sphere \(S^{2n+1}\) with \(n \geq 1\) supports a \textit{PS–overtwisted} contact structure. More precisely, on \(S^{2n+1}\), with \(n \geq 1\), exists a contact structure which admits the embedding of a plastikstufe \(\mathcal{P}S(T^{n-1})\) (with \(T^0 := \{p\}\) and \(T^1 := \{S\}\)).

Corollary 5. It is possible to modify the contact structure \(\xi = \ker \alpha\) of a contact manifold \((M, \alpha)\) with \(\dim M \geq 3\) in an arbitrary small open set in such a way that the new contact structure is \textit{PS–overtwisted}.

Proof. Attach one of the contact spheres obtained in Corollary \ref{Corollary4} via connected contact sum to the manifold \(M\). \qed

Remark 6. The results above make it tempting to claim that \textit{PS–overtwisted} is the proper definition of overtwistedness in higher dimensions. The second author proved together with Frédéric Bourgeois that contact manifolds having an open book decomposition of a certain type have vanishing contact homology, which as \textit{folklore} tells also implies that they are non-fillable. In 3 dimensions, overtwisted can equivalently be defined via open book decompositions, and thus in higher dimensions it would also be interesting to find the precise relation between the definition using a plastikstufe or an open book decomposition.

In particular all of the spheres \((S^{2n-1}, \alpha_-)\) defined in Section \ref{Section1} have vanishing contact homology, and hence are non-fillable even before applying the Presas gluing.

Acknowledgments. This article was written at the \textit{Université Libre de Bruxelles}, where we are both being funded by the \textit{Fonds National de la Recherche Scientifique} (FNRS). We thank Hansjörg Geiges for fruitful discussions.
1. The contact spheres \((S^{2n-1}, \alpha_-)\)

In this section, we will describe an exotic contact structure on each sphere \(S^{2n-1}\) with \(n \geq 2\), with the interesting property that they can all be stacked into each other in a natural way.

Let \(S^{2n-1}\) be the unit sphere in \(\mathbb{C}^n\) with coordinates \(z = (z_1, \ldots, z_n) \in \mathbb{C}^n\), and let \(f\) be the polynomial

\[ f : \mathbb{C}^n \rightarrow \mathbb{C}, (z_1, \ldots, z_n) \mapsto z_1^2 + \cdots + z_n^2. \]

The 1–form

\[ \alpha_- := i \sum_{j=1}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j) - i \left( \frac{f}{\|z\|} \right) d\bar{z} - \frac{\bar{f}}{\|z\|} df \]

defines a contact structure on \(S^{2n-1}\), which is obtained by using an open book decomposition with left-handed Dehn-twist (see Remark 2). The first term of \(\alpha_-\) is just the standard contact form \(\alpha_{\text{std}}\) on the sphere, and it is the second term that is responsible for changing the properties of \(\alpha_-\).

**Proposition 7.** The sphere \((S^{2n-1}, \alpha_-)\) is a contact manifold.

**Proof.** Consider the 1–form

\[ \tilde{\alpha}_- := i \sum_{j=1}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j) - i \left( \frac{f}{\|z\|} \right) d\bar{z} - \frac{\bar{f}}{\|z\|} df \]

Its restriction to the unit sphere is equal to \(\alpha_-\). We will show that its exterior differential, the 2–form

\[ \omega_- := d\tilde{\alpha}_- = 2i \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j - 2i d \left( \frac{f}{\|z\|} \right) \wedge d \left( \frac{\bar{f}}{\|z\|} \right), \]

is symplectic on \(\mathbb{C}^n - \{0\}\). To compute the \(n\)-fold product \(\omega_-^n\), note that the last term can appear at most once in each term of the total product. Furthermore since the first terms always couple a \(dz_j\) with \(d\bar{z}_j\), this eliminates most mixed forms of the last term. Using this, one easily computes

\[ \omega_-^n = -\frac{(2i)^n n!}{\|z\|^{2n}} \left( 3 \|z\|^4 - 2 |f|^2 \right) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n. \]

This form does not vanish, because

\[ |f|^2 = z_1^2 + \cdots + z_n^2 = |(z\bar{z})|^2 \leq \|z\|^4. \]

Now, it follows that \(\tilde{\alpha}_-\) (and hence also \(\alpha_-\)) is a contact form by using that the Liouville field

\[ X_L = \frac{1}{2} \partial_r = \frac{1}{2} (z_1, \ldots, z_n). \]

for \(\omega_-\) satisfies \(\iota_{X_L} \omega_- = \tilde{\alpha}_-\). \qed

**Remark 8.**

1. We obtain sequences of contact embeddings

\( (S^3, \alpha_-) \hookrightarrow (S^5, \alpha_-) \hookrightarrow (S^7, \alpha_-) \hookrightarrow \cdots \),

where each map is just given by

\[ \iota_j : S^{2k-1} \rightarrow S^{2k+1}, (z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_{j-1}, 0, z_j, \ldots, z_k). \]

The normal bundle of every \((2k-1)\)-sphere in the following \((2k+1)\)-sphere is of course trivial.

2. Using similar computations as the ones in [KN03] or [Nie06a], it can be seen that \((S^{2n-1}, \alpha_-)\) is compatible with the open book \((B = f^{-1}(0), \vartheta = \bar{f}/|f|)\), which is equivalent to the abstract open book with page \((T^n S^{n-1}, d\lambda_{\text{can}})\) and monodromy map consisting of a single left-handed Dehn-twist. In particular, the 3–dimensional case \((S^3, \alpha_-)\) is overtwisted, and it is not difficult to localize an overtwisted disk: The intersection \(F\) between the 3–sphere...
\(\mathbb{S}^3\) and the hyperplane \(\{(x_1 + iy_1, x_2 + iy_2) \mid y_1 = x_2\}\) is diffeomorphic to a 2–sphere, which is foliated by \(\alpha_-\). Using stereographic projection

\[ \Phi : \mathbb{C} \to \mathbb{C}^2 \]

\[ x + iy \mapsto \frac{1}{\sqrt{2}(1 + x^2 + y^2)} \begin{pmatrix} (x + 1)^2 + y^2 - 2 \\ 2y \\ 2y \\ (x - 1)^2 + y^2 - 2 \end{pmatrix}, \]

we obtain for the pull-back of the contact form onto this sphere

\[ \Phi^* \alpha_- = \frac{4(3r^4 - 10r^2 + 3)}{(1 + r^2)^4} (y \, dx - x \, dy), \]

with \(r^2 = |z|^2 = x^2 + y^2\). This form does not vanish with exception of the points corresponding to the origin and the circles of radius \(1/\sqrt{3}\) and \(\sqrt{3}\). Hence we find an over-twisted disk in each of the hemispheres of \(F\) (the set \(\mathbb{D}^2_{z|z|^2 \leq 1/3} := \{z \in \mathbb{C} \mid |z|^2 \leq 1/3\}\) and \(\{z \in \mathbb{C} \mid |z|^2 \geq 3\} \cup \{\infty\}\).

2. \(\mathbb{S}^5\) supports a \(PS\)–overtwisted contact structure

In this section, we will prove the result stated in Corollary 4 for dimension 5. To achieve our goal, we will simply start with the manifold \(M_0\) (see Corollary 3), and then use contact surgery to kill the fundamental group. The general proof in Section 4 includes the 5-dimensional case, but the induction used there is relatively complicated, so that we preferred to work out this case explicitly. The following proposition is well known to topologists, but since it is key to our construction we include a proof.

**Proposition 9.** Let \(M\) be an orientable manifold of dimension \(n > 3\). Assume the fundamental group \(\pi_1(M)\) to be generated by the closed embedded paths \(\gamma_1, \ldots, \gamma_N\). Then by using surgery on these circles we obtain a simply connected manifold \(\bar{M}\).

**Proof.** We have to show that \(\pi_1(\bar{M})\) vanishes. First note that the statement in the proposition needs a little clarification. We want all generators \(\gamma_1, \ldots, \gamma_N\) to be disjoint, but this is strictly speaking not possible, because all elements in \(\pi_1(M, p_0)\) have at least the base point \(p_0 \in M\) in common. Instead move each of the circle with a small isotopy to make them disjoint from each other. Choose now new representatives \(\gamma'_1, \ldots, \gamma'_N \in \pi_1(M, p_0)\) such that \(\gamma'_j\) consists of a short segment connecting \(p_0\) with a point \((\gamma_j(0), x)\) on the boundary of the tubular neighborhood \(S^1 \times \mathbb{D}^{n-1}\) of \(\gamma_j\), the path \((\gamma_j, x) \subset \partial(S^1 \times \mathbb{D}^{n-1})\), and a copy of the first segment but with opposite orientation connecting \((\gamma_j(0), x) = (\gamma_j(1), x)\) back with \(p_0\). After the surgery, each of the \(\gamma'_j\) can be contracted to the point \(p_0\).

Let \(\gamma\) be a closed path that represents an element in \(\pi_1(\bar{M}, p_0)\), where we assume that the base point \(p_0\) lies outside the surgery area. With a homotopy, we can make it also everywhere disjoint from the surgery area which is essentially an \((n - 2)\)–sphere. This way we obtain a loop that lives not only in \(\bar{M}\) but also in \(M\), and represents an element in the fundamental group \(\pi_1(M, p_0)\). In \(M\), this circle is homotopic to a product of the \(\gamma'_1, \ldots, \gamma'_N\), and this homotopy can be made disjoint from the surgery regions \(S^1 \times \mathbb{D}^{n-1}\), because a homotopy of curves is a map \([0,1] \times S^1 \to M\), but in an \(n\)–dimensional manifold with \(n > 3\), it is always possible to make a the image of a 2–manifold by a perturbation disjoint from a 1–dimensional submanifold. This homotopy can thus be also realized in \(M\), and so \(\gamma\) is also in \(M\) homotopic to a product of the \(\gamma'_1, \ldots, \gamma'_N\), which are all contractible in \(\bar{M}\). If follows that \(\gamma\) represents the trivial element in \(\pi_1(M)\), and hence \(\pi_1(M) = \{0\}\). \(\square\)

We start by using the manifold \(M_0\) given in Corollary 3 part (1). The homology of this group is (as already stated by Presas) \(H_0(M_0) \cong H_5(M_0) \cong \mathbb{Z}, H_1(M_0) \cong H_4(M_0) \cong \mathbb{Z}^2\) and

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\(^{2}\)From now on, we will always assume integer coefficients for the homology groups.
Every loop in the surgered space $\widetilde{M}$ is homotopic to a concatenation of circles $\gamma_1, \ldots, \gamma_N$. The surgery makes each of these circles contractible so that $\widetilde{M}$ is simply connected.

$H_2(M_0) \cong H_3(M_0) = \{0\}$. The fundamental group $\pi_1(M_0)$ can be easily computed using the Seifert-van Kampen theorem. We obtain

$$\pi_1(M_0) = \langle a, b, c | aba^{-1}b^{-1} = c \rangle = \langle a, b \rangle = \mathbb{Z} \ast \mathbb{Z},$$

where $a, b$ are the generators of $\pi_1(\mathbb{S}^3 \times (\mathbb{T}^2 - \mathbb{D}^2)) \cong \pi_1(\mathbb{T}^2 - \mathbb{D}^2) \cong \mathbb{Z} \ast \mathbb{Z}$, and $c$ generates the fundamental group of $\mathbb{S}^5 - \mathbb{S}^1 \times \mathbb{D}^2 \cong \mathbb{D}^4 \times \mathbb{S}^1$. The relation follows from identifying elements in the intersection $\mathbb{S}^3 \times \mathbb{S}^1$.

Represent the two generators $a, b$ by smooth embedded paths $\gamma_1, \gamma_2 \subset \mathbb{S}^3 \times (\mathbb{T}^2 - \mathbb{D}^2)$ of the form

$$\gamma_1 : [0, 1] \to \mathbb{S}^3 \times (\mathbb{T}^2 - \mathbb{D}^2), \ t \mapsto (p_1; e^{2\pi it}, 1)$$

$$\gamma_2 : [0, 1] \to \mathbb{S}^3 \times (\mathbb{T}^2 - \mathbb{D}^2), \ t \mapsto (p_2; e^{2\pi it})$$

with fixed $p_1 \neq p_2 \in \mathbb{S}^3$ such that both curves are isotropic and do not intersect the plastikstufe $\mathcal{P}\mathcal{S}(\mathbb{S}^1)$ lying in $M_0$. This can be achieved by choosing both points $p_1, p_2$ in the 1-dimensional binding of $(\mathbb{S}^1, \alpha_-)$ (because then both functions $f_1, f_2$ used for the definition of $\alpha_-$ on $\mathbb{S}^3 \times \mathbb{T}^2$ vanish, cf. Section 3). One can find an overtwisted disk in $\mathbb{S}^3$ that intersects the binding at only one point. For the construction of $\mathcal{P}\mathcal{S}(\mathbb{S}^1)$, this disk is transported through $\mathbb{S}^3 \times \mathbb{T}^2$ along two paths which are parallel to $\gamma_1$ or $\gamma_2$. Hence there is sufficient space to choose $p_1, p_2$ in such a way that $\gamma_1$ and $\gamma_2$ do not intersect $\mathcal{P}\mathcal{S}(\mathbb{S}^1)$.

Now apply contact surgery on these generators of $\pi_1(M_0)$, i.e. cut out a tubular neighborhood of the two isotropic curves representing $a$ and $b$. This neighborhood is of the form $\mathbb{S}^1 \times \mathbb{D}^4 \cup \mathbb{S}^1 \times \mathbb{D}^4$ and it has boundary $\mathbb{S}^1 \times \mathbb{S}^3 \cup \mathbb{S}^1 \times \mathbb{S}^3$. Glue in two copies of $\mathbb{D}^4 \times \mathbb{S}^3$, which have the same boundary as the cavities. As explained in [We91], the manifold $\widetilde{M}_0$ we obtain this way, carries a contact structure, which coincides outside the surgery loci with the original structure, so in particular $\widetilde{M}_0$ still contains a plastikstufe. By Proposition 3 it follows that $\widetilde{M}_0$ is simply connected.

2.1. **The homology of $\widetilde{M}_0$.** We want to show that $H_* (\widetilde{M}_0) \cong H_* (\mathbb{S}^3)$, because this together with $\pi_1(\widetilde{M}_0) = \{0\}$, using the Poincaré conjecture proved by Smale and the non-existence of exotic 5-spheres shows that $\widetilde{M}_0$ is diffeomorphic to $\mathbb{S}^3$. 
All of the computations in this section are standard applications of the Mayer-Vietoris sequence. Use the following notation $B = S^1 \times \mathbb{D}^4 \cup S^1 \times \mathbb{D}^4$, $\hat{B} = \mathbb{D}^2 \times S^3 \cup \mathbb{D}^2 \times S^3$, $A = M_0 - B = M_0 - \hat{B}$, $A \cap B = A \cap \hat{B} = S^1 \times S^3 \cup S^1 \times S^3$. Then, because $H_2(A \cap B) = \{0\}$, both Mayer-Vietoris sequences for $M_0 = A \cup B$ and for $\hat{M}_0 = A \cup \hat{B}$ split at that homology group. Using that $H_2(B) \cong H_2(\hat{B}) \cong H_1(\hat{M}_0) = \{0\}$, the Mayer-Vietoris sequences for the pairs $(A, B)$ and $(A, \hat{B})$ reduce to

\[
\begin{align*}
0 \to &H_2(A) \oplus H_2(B) \to H_2(M_0) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(M_0) \to 0 \\
0 \to &H_2(A) \oplus H_2(\hat{B}) \to H_2(\hat{M}_0) \to H_1(A \cap \hat{B}) \to H_1(A) \oplus H_1(\hat{B}) \to 0.
\end{align*}
\]

Because $H_2(M_0) = \{0\}$, it follows that $H_2(A)$ also vanishes. The top sequence simplifies to

\[
0 \to \mathbb{Z}^2 \to H_1(A) \oplus \mathbb{Z}^2 \to \mathbb{Z}^2 \to 0
\]

As $H_1(A)$ cannot have torsion in such a short exact sequence, we obtain $H_1(A) \cong \mathbb{Z}^2$. With this, the second sequence simplifies to

\[
0 \to H_2(\hat{M}_0) \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to 0,
\]

so that $H_2(\hat{M}_0)$ vanishes.

By Poincaré duality and the universal coefficient theorem, we have $H_3(\hat{M}_0) \cong H_1(\hat{M}_0) = \{0\}$. This implies that $\hat{M}_0$ is homeomorphic to $S^5$ and in fact it is even diffeomorphic to $S^5$, because there are no exotic 5–spheres. This proves Corollary 4 for dimension 5.

3. Submanifolds and Presas gluing

In [Pre06], Presas starts out with a manifold $(M, \alpha)$ which contains a codimension 2 contact submanifold $N$ with trivial normal bundle. A neighborhood of $N$ in $M$ is contactomorphic to $(N \times \mathbb{D}^2, \alpha|_{TN} + r^2 d\varphi)$. The product manifold $N \times \mathbb{T}^2$ also supports a contact structure, namely $\alpha_e := \alpha|_{TN} + \varepsilon(f_1 d\varphi_1 + f_2 d\varphi_2)$, where $(e^{i\varphi_1}, e^{i\varphi_2})$ are the coordinates of the 2–torus and $f_1, f_2 : N \to \mathbb{R}$ are certain functions, which are obtained from an open book decomposition of $N$ (see [Bou02]). A fiber $N \times \{p_0\} \subset N \times \mathbb{T}^2$ has a neighborhood that is contactomorphic to $(N \times \mathbb{D}^2, \alpha|_{TN} - r^2 d\varphi)$, and one can perform the fiber connected sum of $N \times \mathbb{T}^2$ onto $M$ along $N \times \{p_0\}$ and $N \subset M$. Presas has shown that if $N$ is $PS$–overtwisted then so is $M \cup_N N \times \mathbb{T}^2$. This construction can be carried out simultaneously on several embedded contact manifolds. To this end we need a neighborhood theorem that is adapted to such a situation.

**Proposition 10.** Let $N, S_1, \ldots, S_r$ be codimension 2 contact submanifolds of $(M, \alpha)$ with trivial normal bundle. Assume that all of these contact submanifolds intersect $N$ and each other transversely, i.e. $T_p N + T_p S_j = T_p M$ at every $p \in S_j \cap N$ and $T_q S_i + T_q S_j = T_q M$ at every $q \in S_i \cap S_j$.

Then we find a neighborhood of $N$ in $M$ that can be represented as $(N \times \mathbb{D}^2, \alpha|_{TN} + r^2 d\varphi)$ such that $S_j$ is in this neighborhood of the form $(S_j \cap N) \times \mathbb{D}^2$.

**Proof.** Start by choosing a metric on $S_1 \cap \cdots \cap S_r$, and extend this metric first over all $S_1 \cap \cdots \cap \tilde{S}_j \cap \cdots \cap S_r$, then to all $S_1 \cap \cdots \cap \tilde{S}_j \cap \cdots \cap \tilde{S}_j \cap \cdots \cap S_r$, and so on until the metric is defined on all $S_1, \ldots, S_r$. Now define finally the metric on the rest of $M$. By considering the exponential map along $N$, we obtain a tubular neighborhood $N$ diffeomorphic to $N \times \mathbb{D}^2$ such that every $S_j$ is given by $(S_j \cap N) \times \mathbb{D}^2$.

The standard neighborhood theorem (see for example [Gei06]) guarantees that we have a contactomorphism from $(S_1 \cap \cdots \cap S_r \cap N) \times \mathbb{D}^2$ to itself, which deforms the contact form to $\alpha|_{T(S_1 \cap \cdots \cap S_r \cap N)} + r^2 d\varphi$. This map is generated by a vector field which vanishes on $(S_1 \cap \cdots \cap S_r \cap N) \times \{0\}$, and hence we can easily extend the contactomorphism to a diffeomorphism $\Phi$ from any $(S_1 \cap \cdots \cap \tilde{S}_j \cap \cdots \cap S_r \cap N) \times \mathbb{D}^2$ to itself such that $\Phi$ leaves $(S_1 \cap \cdots \cap \tilde{S}_j \cap \cdots \cap S_r \cap N) \times \{0\}$ fixed. By suitably extending the vector field successively over all of the submanifolds and then to the rest of the manifold, we finally obtain a diffeomorphism that converts the contact form on
Now use again the neighborhood theorem, this times for $(S_2 \cap \cdots \cap S_r \cap N) \times \mathbb{D}^2$ in $S_2 \cap \cdots \cap S_r \cap N \times \mathbb{D}^2$. Note that the Moser vector field vanishes on $(S_1 \cap \cdots \cap S_r \cap N)$, because there the contact form was already brought into the desired shape in the previous step. Extend the vector field not to destroy the form on $(S_1 \cap \cdots \cap S_r \cap N)$, but if the vector field is extended to vanish wherever possible, this submanifold $S_r$ is of codimension two. In this neighborhood. Then it follows that the Presas gluing $S' \cup S'' \cap S''' (N \cap S) \times \mathbb{T}^2$ contains the gluing $S' \cup S'' \cap S''' (N \cap S) \times \mathbb{T}^2$ as a contact submanifold. This construction works for several submanifolds $S_1, \ldots, S_r$ that satisfy the assumptions in Proposition \[11\].

### 4. The proof of Corollary \[11\]

The general construction to prove Corollary \[11\] is considerably more complicated than the 5–dimensional one. The proof works by induction. We start with a contact sphere that contains a $PS$–overtwisted contact submanifold of codimension $2k$. In each induction step, we raise the dimension of the submanifold by two until finally the sphere itself is $PS$–overtwisted.

Let $S := (S^{2n-1}, \alpha)$ be a contact sphere that contains codimension 2 contact submanifolds $S_1, \ldots, S_k$ (with $k \leq n - 2$) with the following properties (see also Figure 2):

1. Every $S_j$ is a sphere that is unknotted in $S$.
2. Every two spheres $S_i$ and $S_j (i \neq j)$ intersect transversely, and the intersection $S_{i_1, \ldots, i_r} := S_{i_1} \cap \cdots \cap S_{i_r}$ of any combination of these spheres is a contact $(2n - 2r - 1)$–sphere, which is unknotted in any of the spheres $S'_{i_1, \ldots, i_r}$.
3. Finally the lowest dimensional sphere $S_{1, \ldots, k}$ is $PS$–overtwisted.

\[
\begin{align*}
\dim &= 2n - 1 \\
S' &
\dim &= 2n - 3 \\
S'' &
\dim &= 2n - 5 \\
S''' &
\dim &= 2n - 7
\end{align*}
\]

**Figure 2.** The intersection of all codimension 2 spheres is $PS$–overtwisted. In each induction step we increase the dimension of the $PS$–overtwisted submanifold by two.

We are then able to construct a new contact sphere $S' = (S^{2n-1}, \alpha')$ which satisfies the conditions in the above list for $k - 1$ instead of $k$. More explicitly we mean that $S'$ contains $k - 1$
Corollary 11. The neighborhood of $M_0^{1,...,k-1}$ is contactomorphic to

$$\left( M_0^{1,...,k-1} \times \mathbb{C}^{k-1}, \alpha|_{TM_0^{1,...,k-1}} + \frac{i}{2} \sum_{j=1}^{k-1} (z_j \, d\bar{z}_j - \bar{z}_j \, dz_j) \right),$$

where $M_0^{j}$ is represented by $M_0^{1,...,k-1} \times \{(z_1, \ldots, z_{k-1}) \in \mathbb{C}^{k-1} | z_j = 0\}$.
Every contact manifold can be given a non-fillable contact structure

\[
S^{2n+1} \times \{p_0\}
\]

\[
S^{2n-1} \times T^2
\]

\[
\gamma_j
\]

Figure 3. Apply surgery on the curves $\gamma_1$ and $\gamma_2$. These two loops generate the fundamental group of all submanifolds considered.

**Proof.** Consider $M_0^1, \ldots, M_0^{k-1}$ in $M_0$. They satisfy the conditions of Proposition [10] and so there is a tubular contact neighborhood of $M_0^1$

\[
\left( M_0^1 \times \mathbb{C} = \{(p; z_1)\}, \alpha|_{T M_0^1} + \frac{i}{2} (z_1 \overline{d} - \overline{z}_1 \overline{d}) \right)
\]

such that $M_0^j$ is given by $M_0^{1,j} \times \mathbb{C}$. Repeat the step for $M_0^{1,2}, \ldots, M_0^{1,k-1}$ in $M_0^1$. We obtain a neighborhood of the form

\[
\left( M_0^{1,2} \times \mathbb{C}^2 = \{(p'; z_1, z_2)\}, \alpha|_{T M_0^{1,2}} + \frac{i}{2} \sum_{j=1}^2 (z_j \overline{d} - \overline{z}_j \overline{d}) \right).
\]

This step can be iterated until one arrives at the neighborhood described in the corollary we want to prove. \qed

With this neighborhood theorem, we will be able to apply contact 1–surgery on $M_0$ along the curves $\gamma_1$ and $\gamma_2$.

4.2.1. Surgery compatible with submanifolds. To describe the contact surgery, we briefly recall Weinstein’s picture for surgery [Wei91]. Consider $\mathbb{R}^{2n+4}$ with coordinates $(\vec{x}, \vec{y}, z_1, z_2, w_1, w_2)$ and symplectic form

\[
\omega = d\vec{x} \wedge d\vec{y} + \sum_{i=1}^2 dz_i \wedge dw_i.
\]

The vector field

\[
\frac{1}{2} (\vec{x} \frac{\partial}{\partial \vec{x}} + \vec{y} \frac{\partial}{\partial \vec{y}}) + \sum_{i=1}^2 \left( 2z_i \frac{\partial}{\partial z_i} - w_i \frac{\partial}{\partial w_i} \right)
\]

is Liouville, and it is transverse to the non-zero level sets of the function

\[
f = \frac{1}{4} (\vec{x}^2 + \vec{y}^2) + \sum_{i=1}^2 \left( z_i^2 - \frac{1}{2} w_i^2 \right).
\]
In particular, \( f^{-1}(-1) \cong \mathbb{R}^{2n+2} \times S^1 \) is a contact hypersurface, and the circle \( \gamma_{\text{model}} = \{ (0, 0, 0, 0, w_1, w_2) \mid w_1^2 + w_2^2 = 2 \} \) is isotropic. By a neighborhood theorem, the loops \( \gamma_1 \) and \( \gamma_2 \) considered above have a neighborhood that is contactomorphic to a neighborhood of \( \gamma_{\text{model}} \) in \( f^{-1}(-1) \).

The next step consists of gluing in the 1–handle lying between \( f^{-1}(-1) \) and \( f^{-1}(1) \). A piece of the set \( f^{-1}(-1) \) can be identified via the Liouville flow with a contact manifold that is close to \( f^{-1}(1) \) (see Figure 4). The precise construction that is necessary can be found in [Gei07]. Hence we can replace a neighborhood of the curve \( \gamma_{\text{model}} \) by the surgery \( D^2 \times S^{2n+1} \). As the Liouville field is transverse to the set \( f^{-1}(1) \), we see that the surgered manifold is contact.

We still need to show that we can choose the contact surgery on \( M_0 \) in such a way that it induces contact surgery on all the submanifolds \( M^J_0 \), where \( J \) is an index set. To see that the surgery can be made compatible, we need to specify the framing more precisely, which can be done by using an induction. Corollary [11] is of relevance here.

Let us denote the curve where we perform surgery by \( \gamma \). We start with the contact submanifold \( M_0^{1,...,k} \). This manifold also contains the curve \( \gamma \), which is still isotropic. Therefore we can identify a tubular neighborhood of \( \gamma \) with \( f^{-1}_J(-1) \) as in the above model. We have added the subscript \( J \) to indicate the different dimensions, i.e. the number of \( \vec{x} \) coordinates depends on the size of index set \( J \). At this stage there is still some freedom in choosing the framing. For clarity, we will also give \( \gamma \) a subindex to indicate in what submanifold we consider the curve, i.e. \( \gamma_J \) denotes the restriction of the curve \( \gamma \) to the submanifold \( M^J_0 \).

Next, suppose we have fixed the framing on \( M^J_0 \) such that contact surgery on \( \gamma_J \) induces the desired contact surgery on submanifolds \( M_0^{J,j,K} \) in \( M^J_0 \) indexed by \( K \). Let us now look at \( M_0^{J,j} \subset M^J_0 \). A neighborhood of \( M_0^{J,j} \) in \( M^J_0 \) looks like \( M_0^{J,j} \times D^2 \), so we may write for the contact form by Corollary [11]

\[
\alpha_J = \alpha_{J,j} + x \, dy - y \, dx ,
\]
by an embedded 2–sphere. On an abstract level this is clear by the Hurewicz theorem, but for so that $\mathbb{H}$ generator of $\gamma$ this almost proves property (2) of the list. There is, however, one missing part, namely we still need to show that the spheres are in fact diffeomorphic to the standard sphere. This shall be done so this almost proves property (2) of the list. There is, however, one missing part, namely we still need to show that the spheres are in fact diffeomorphic to the standard sphere. This shall be done next.

4.3. Every manifold $S'_{j_1,...,j_r}$ is diffeomorphic to the standard sphere. Our next aim is to see that all the manifolds $M_{0}^{j_1,...,j_r}$ are converted by the surgery into smooth spheres. For this, we will show that each of these surgered manifolds is the boundary of a ball.

Consider the manifold $H$ consisting of the thickened 2–torus $\mathbb{D}^{2n} \times \mathbb{T}^{2}$ where we attach a 2–handle along $\{p_1\} \times \mathbb{S}^{1} \times \{1\}$, another one along $\{p_2\} \times \{1\} \times \mathbb{S}^{1}$ and a $(2n)$–handle along $(\partial \mathbb{D}^{2n}) \times \{p_0\}$ (with $p_1 \neq p_2 \in \partial \mathbb{D}^{2n}$, $p_0 \in \mathbb{T}^{2} - (\mathbb{S}^{1} \times \{1\} \cup \{1\} \times \mathbb{S}^{1})$). If we have chosen above the right framing for attaching the handles, then the boundary of $H$ is diffeomorphic to a manifold $S'_{j_1,...,j_r}$ which was defined as $M_{0}^{j_1,...,j_r}$ surgered along $\gamma_1$ and $\gamma_2$. So $H$ provides a topological filling for $S'_{j_1,...,j_r}$. Using a further surgery on the interior of $H$, we will convert $H$ into a ball. This will show that every $S'_{j_1,...,j_r}$ is diffeomorphic to the standard sphere.

The fundamental group of the handle body $H$ constructed above is trivial, because $\pi_1(\mathbb{D}^{2n} \times \mathbb{T}^{2}) \cong \mathbb{Z}^2$, attaching the $(2n)$–handle does not change the fundamental group, and the final two 2–handles then kill $\pi_1(H)$. The homology of $H$ can be computed with a Mayer–Vietoris sequence, where we set $A = \mathbb{D}^{2n} \times \mathbb{T}^{2}$ and $B = \mathbb{D}^{2n} \cup \mathbb{D}^{2} \cup \mathbb{D}^{2}$ and $A \cap B = \mathbb{S}^{2n-1} \times \{p_0\} \cup \{p_1\} \times \mathbb{S}^{1} \times \{1\} \cup \{p_2\} \times \{1\} \times \mathbb{S}^{1}$, and so

$$0 \to H_2(A) \oplus H_2(B) \to H_2(H) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to 0,$$

which simplifies to

$$0 \to \mathbb{Z} \to H_2(H) \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to 0.$$

This means that $H_2(H) \cong \mathbb{Z}$. Hence the manifold $H$ is obviously not a $(2n+2)$–ball. For higher homology the sequence gives

$$0 \to H_{2n}(H) \to H_{2n-1}(A \cap B) \to 0,$$

so that $H_{2n}(H) \cong \mathbb{Z}$. All other homology groups $H_k(H)$ with $2 < k < 2n$ are trivial. The generator of $H_{2n}(H)$ can be represented by the cycle composed of the $(2n)$–handle of $H$ and the $(2n)$–disk $\mathbb{D}^{2n} \times \{p_0\}$.

To convert $H$ into the desired $(2n+2)$–ball, note that the generator of $H_{2n}(H)$ can be represented by an embedded 2–sphere. On an abstract level this is clear by the Hurewicz theorem, but for
later observations we will need to find the generator explicitly. The 2–torus \( \{0\} \times \mathbb{T}^2 \) at the core of the thickened torus with which the construction of \( H \) began, clearly generates \( H_2(H) \) as can be seen from the Mayer-Vietoris sequence. Of course this torus can be moved to \( \{p_1\} \times \mathbb{T}^2 \). One of the 2–handles has been attached to \( \{p_1\} \times \mathbb{S}^1 \times \{1\} \). We will cut out the annulus

\[
A = \{ \{p_1\} \times \mathbb{S}^1 \times \{e^t\} \mid t \in (-\varepsilon, \varepsilon) \},
\]

and attach two disks to \( \partial A \) which lie in the 2–handle. Depending on the framing we used to attach the 2–handle, the two boundary circles \( \partial A \pm \) of \( A \) are given by

\[
\mathbb{S}^1 \to \mathbb{D}^{2n} \times \partial \mathbb{D}^2, \quad e^{i\varphi} \mapsto (\pm z(\varphi), e^{i\varphi}).
\]

This curves can be easily extended to the interior of the 2–handle by writing

\[
\mathbb{D}^2 \to \mathbb{D}^{2n} \times \mathbb{D}^2, \quad re^{i\varphi} \mapsto (\pm r z(\varphi), re^{i\varphi}).
\]

By a general position argument, the two disks can also be made disjoint, because the lowest dimension for \( H \) we are considering is 6. The manifold obtained from the torus by cutting out \( A \), gluing the two disks and smoothing out the corners is an embedded 2–sphere, which represents the generator of \( H_2(H) \), because the annulus \( A \) together with the two disks represents a boundary in the chain complex of the 2–handle.

To perform surgery along this sphere, we have to show that its normal bundle in \( H \) can be made disjoint from the surgery region such that it represents a loop in \( \mathbb{S}^2 \). The manifold constructed this way is simply connected (a closed loop \( \gamma \) in \( H \) can be made disjoint from the surgery region such that it represents a loop in \( H \), which can be contracted without intersecting the surgery region.)

The Mayer-Vietoris sequence gives with the notation \( B = \mathbb{S}^2 \times \mathbb{D}^{2n} \), \( A = H - B \), \( A \cup B = H \), and \( A \cap B = \mathbb{S}^2 \times \mathbb{S}^{2n-1} \) that \( H_k(A) = \{0\} \) for \( 2 < k < 2n - 1 \), and with

\[
0 \to \mathbb{Z} \to H_2(A) \oplus \mathbb{Z} \to \mathbb{Z} \to H_1(A) \to 0,
\]

one sees that \( H_2(A) \cong \mathbb{Z} \). Below it will be important to understand the map \( \iota_A : H_2(A \cap B) \to H_2(A) \). In the sequence above, the generator of \( H_2(A \cap B) \) goes to \((a, 1) \in H_2(A) \oplus H_2(B) \) with \( a \in \mathbb{Z} \). The map \( H_2(B) \to H_2(H) \) sends generator to generator, so that \( a \) has to be a generator, because \((a, 1)\) lies in the kernel of the map \( \iota_A - \iota_B \) in the sequence above. It follows that \( \iota_A : H_2(A \cap B) \to H_2(A) \) is an isomorphism.

For the higher groups compute the rest of the Mayer-Vietoris sequence:

\[
0 \to \mathbb{Z} \to H_{2n+1}(A) \to 0 \to H_{2n}(A) \to \mathbb{Z} \to \mathbb{Z} \to H_{2n-1}(A) \to 0.
\]

It follows that \( H_{2n+1}(A) \cong \mathbb{Z} \), \( H_{2n}(A) \) is either trivial or isomorphic to \( \mathbb{Z} \), and \( H_{2n-1}(A) \) can be trivial or isomorphic to either \( \mathbb{Z}_p \) or \( \mathbb{Z} \). To recognize that these two groups are trivial, we will study the map \( H_{2n}(H) \to H_{2n-1}(\mathbb{S}^2 \times \mathbb{S}^{2n-1}) \) and prove that it is an isomorphism. The generator of \( H_{2n}(H) \) can be written as the core of the \((2n)\)–handle of \( H \) composed with the ball \( \mathbb{D}^{2n} \times \{p_0\} \subset \mathbb{D}^{2n} \times \mathbb{T}^2 \). To see the image under the connecting homomorphism we need to represent the generator as the sum of two chains, one which lies in \( A \) and one which lies in \( B \). The boundary of these chains lies in \( A \cap B \) (because they cancel each other), and the boundary of one of these chains gives a representative for the image of the generator under the connecting
homomorphism. In our situation $B$ is a neighborhood of the 2–sphere which generates $H_2(H)$, and as we showed above, this 2–sphere is obtained by taking the torus $\{0\} \times \mathbb{T}^2 \subset \mathbb{D}^{2n} \times \mathbb{T}^2$ and attaching two disks, which lie in one of the 2–handles. The set $B$ corresponds thus to a tubular neighborhood of $\mathbb{S}^2$, which coincides outside a small neighborhood of the 2–handle with a tubular neighborhood of $\{0\} \times \mathbb{T}^2$. The intersection of this tubular neighborhood with the generator of $H_{2n}(H)$ gives a small disk lying in a fiber of the normal bundle of $\mathbb{S}^2$. The boundary of the small disk represents the generator of $H_{2n-1}(B)$ as we wanted to show. Hence the connecting homomorphism is a bijection and both groups $H_{2n}(A)$ and $H_{2n-1}(A)$ are trivial.

Use now the notation $A = H - B$, $\tilde{B} = \mathbb{D}^3 \times \mathbb{S}^{2n-1}$, $A \cap \tilde{B} = \mathbb{S}^2 \times \mathbb{S}^{2n-1}$, and $\tilde{H} = A \cup \tilde{B}$. One sees immediately that $H_{2}(\tilde{H}) = \{0\}$ for all $2n - 1 > k > 3$, and

$$\cdots \to H_3(\tilde{H}) \to H_2(A \cap \tilde{B}) \to H_2(A) \to H_2(\tilde{H}) \to 0.$$  

We showed that the map $H_2(A \cap B) \cong \mathbb{Z} \to H_2(A) \cong \mathbb{Z}$ is an isomorphism, hence it follows that $H_2(\tilde{H}) \cong \{0\}$. The higher parts of the Mayer-Vietoris sequence give

$$0 \to \mathbb{Z} \to H_{2n+1}(A) \to H_{2n+1}(\tilde{H}) \to 0$$

and

$$0 \to H_{2n}(A) \to H_{2n}(\tilde{H}) \to \mathbb{Z} \to H_{2n-1}(A) \oplus \mathbb{Z} \to H_{2n-1}(\tilde{H}) \to 0.$$

In the first sequence, we can use that the first map is an isomorphism, as could be seen from the Mayer-Vietoris sequence when doing the first step of the surgery, and hence $H_{2n+1}(\tilde{H})$ is trivial. Since we know that $H_{2n}(A)$ and $H_{2n-1}(A)$ are trivial, the second sequence simplifies to

$$0 \to H_{2n}(\tilde{H}) \to \mathbb{Z} \to \mathbb{Z} \to H_{2n-1}(\tilde{H}) \to 0.$$  

The map in the middle is also an isomorphism, and we obtain that $H_{2n}(\tilde{H}) \cong H_{2n-1}(\tilde{H}) \cong \{0\}$. To compute $H_3(\tilde{H})$ analyze the sequence

$$\cdots \to H_4(\tilde{H}) \to H_3(A \cap \tilde{B}) \to H_3(A) \oplus H_3(\tilde{B}) \to H_3(\tilde{H}) \to 0.$$  

The case $n = 2$ is different from the case $n > 2$, but $H_3(\tilde{H})$ vanishes for both.

We just have shown that the homology of $\tilde{H}$ is equal to the one of a point, and it is also simply connected, hence $\tilde{H}$ is diffeomorphic to the ball $\mathbb{D}^{2n+2}$ ([Mil63]), and so the boundary $\partial \tilde{H} \cong S'_{j_1, \ldots, j_r}$, is a smooth standard sphere.

Together with our previous remarks, we have now established properties (2) and (3) from the list for the manifolds $S'_{j_i}$. In last section, we will prove the remaining property of the list to conclude the induction step.

4.4. The sphere $S'_{j_1, \ldots, j_r}$ is unknotted in $S'_{j_1, \ldots, j_r}$. Throughout this section, we shall use the index set $J = \{j_1, \ldots, j_r\}$ to abbreviate the notation. In order to show that the spheres are unknotted, we shall use the following characterization of unknotted due to Levine, [Lev65].

**Theorem 12** (Levine). A smoothly embedded sphere $\iota : \mathbb{S}^{k-2} \to \mathbb{S}^k$ is unknotted if and only if $\pi_1(\mathbb{S}^k - \iota(\mathbb{S}^{k-2})) \cong \mathbb{Z}$.

This theorem reduces the problem to a computation of the fundamental group of the complement of $S'_{j_i}$ in $S'_{j_i}$. It is helpful to first consider the fundamental group before the 1–surgery. In that case we have

$$M_0' = S_{j} \cup S_{j,k} \mathbb{S}^1 \times \mathbb{T}^2,$$

and we have a similar expression for $M_0^{j,j}$. Next, we note that $S_{j,k}$ and $S_{j,j,k}$ are, by induction, unknotted, so we can write

$$S_{j} = \mathbb{S}^1 \times \mathbb{D}^{2n} \cup \mathbb{D}^2 \times \mathbb{S}^{2n-1}$$

and

$$S_{j,k} = \mathbb{S}^1 \times \mathbb{D}^{2n-2} \cup \mathbb{D}^2 \times \mathbb{S}^{2n-3}.$$  

These decompositions are such that $S_{j,k} = \{0\} \times \mathbb{S}^{2n-1} \subset S_{j}$ and $S_{j,j,k} = \{0\} \times \mathbb{S}^{2n-3} \subset S_{j}$. Hence we have decompositions that are adapted to the fiber connected sum. In other words, we can write

$$M_0' = \mathbb{S}^1 \times \mathbb{D}^{2n} \cup_{A \times S_{j,k}} S_{j,k} \times (\mathbb{T}^2 - \{p\}),$$
where we glue along the annulus times fiber $A \times S_{J,k}$. We proceed by computing the fundamental group of the complement of $M^I_{0,J}$ in $M^0_I$. We have

$$M^I_0 - M^I_{0,J} = \mathbb{S}^1 \times (\mathbb{D}^{2n} - \mathbb{D}^{2n-2}) \cup_{A \times (S_{J,k} - S_{J,j,k})} (S_{J,k} - S_{J,j,k}) \times (\mathbb{T}^2 - \{p\}) .$$

Here we glue along an annulus times fiber $(S_{J,k} - S_{J,j,k}$ in this case). The Seifert-Van Kampen theorem gives us the fundamental group

$$\pi_1(M^I_0 - M^I_{0,J}) = \langle a, b, c, d, e \mid ab = ba, \ cd = dc, \ ce = ec, \ ded^{-1}e^{-1} = a, \ b = c \rangle$$

$$\cong \langle c, d, e \mid cd = dc, \ ce = ec \rangle \cong \mathbb{Z} \oplus (\mathbb{Z} \ast \mathbb{Z})$$

Here the generators $a, b$ denote the two commuting generators of the left-hand side, $\mathbb{S}^1 \times (\mathbb{D}^{2n} - \mathbb{D}^{2n-2})$. The generator $c$ can be represented by a curve in $S_{J,k} - S_{J,j,k}$ that generates $\pi_1(S_{J,k} - S_{J,j,k}) \cong \mathbb{Z}$. Finally, the elements $d$ and $e$ can be represented by a longitudinal curve and a meridian of $\mathbb{T}^2 - \{p\}$. Because of the product structure of the right-hand side, the two generators $d, e$ commute with $c$. The other two relations come from the amalgamation in the Seifert-Van Kampen theorem. We see in particular that the fundamental group of the complement of $M^I_{0,J}$ has already a simple structure before the surgery.

Now take any element $W$ in $\pi_1(S'_J - S'_{J,j}, q)$, where $q$ denotes the basepoint. By a homotopy, we can arrange $W$ to be represented by a product of curves that lie outside the surgery region, i.e. we assume that the curves lie in $M^I_0 - M^I_{0,J}$. When thought of as an element in $\pi_1(M^I_0 - M^I_{0,J}, q)$ our above computation shows that such a product of curves can be written as a word in $a, d$ and $e$. Note however that a curve $\Gamma$ representing $d$ or $e$ can be written in terms of $c$ after the surgery. Indeed, we can homotope such a curve $\Gamma$ close to the curves $\gamma_1$ or $\gamma_2$ from Equation (4.1). More precisely, for dimension reasons we can find a homotopy $H$ that is disjoint from the surgery locus such that

$$H(0, t) = \Gamma(t)$$

$$H(1, t) = \tilde{\gamma}_i(t) = (\tilde{p}; e^{2\pi i t}, 1) \in \mathbb{S}^n \times \mathbb{T}^2$$

in case $\Gamma$ is homotopic to $\gamma_1$ as a curve in $M^I_0$. We have a similar homotopy $H$ for the case that $\Gamma$ is homotopic to $\gamma_2$ as a curve in $M^I_0$. The homotopy $H$ takes place in $M^I_0 - M^I_{0,J}$ and only homotopes $\Gamma$ to a curve $\tilde{\gamma}_i$ that is close to $\gamma_i$ in the following sense.

We have chosen $\tilde{p}$ disjoint from a tubular neighborhood $N_{e_i}$ of $S_{1,...,k}$ that we use to perform surgery, but inside the larger tubular neighborhood $N_{2e_i}$ such that $\tilde{p}$ is close to either $p_1$ or $p_2$ as given by Equation (4.1). Now we see that $\Gamma$ can be simplified. Follow the homotopy $H$ and then push the curve $\tilde{\gamma}_i$ into the surgered region. Before the surgery, this region (with $M^I_{0,J}$ removed) looks like

$$(\mathbb{D}^{2n} - \mathbb{D}^{2n-2}) \times \mathbb{S}^1 .$$

When we homotope $\tilde{\gamma}_i$ into this region, $\tilde{\gamma}_i$ can arranged to have the form

$$t \mapsto (f(t); e^{2\pi i t}) .$$

The precise form of $f$ depends on the chosen framing of the neighborhood of $\mathbb{S}^1$. Now the surgery replaces the set $\mathbb{S}^n$ by

$$(\mathbb{S}^{n-1} \setminus \mathbb{S}^{n-3}) \times \mathbb{D}^2 .$$

Hence we can homotope $\tilde{\gamma}_i$ to a curve of the form

$$(f(t); 1) ,$$

which represents $c^k$ for some $k$. This establishes our claim.

We see therefore that $\pi_1(S'_J - S'_{J,j}, q)$ can be presented by a group with at most one generator. Since we know that $H_1(S'_J - S'_{J,j}) \cong \mathbb{Z}$, we have

$$\pi_1(S'_J - S'_{J,j}, q) \cong \mathbb{Z} .$$

As a result of Levine's criterion we obtain that the embedding $S'_{J,j} \hookrightarrow S'_J$ is unknotted. This proves property (1) of the list and finishes the induction step.
EVERY CONTACT MANIFOLD CAN BE GIVEN A NON-FILLABLE CONTACT STRUCTURE

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