Cobweb Posets and KoDAG Digraphs are Representing Natural Join of Relations, their Di-Bigraphs and the Corresponding Adjacency Matrices.

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Abstract: Natural join of di-bigraphs (directed bi-parted graphs) and their corresponding adjacency matrices is defined and then applied to investigate the so called cobweb posets and their Hasse digraphs called KoDAGs. KoDAGs are special orderable Directed Acyclic Graphs which are cover relation digraphs of cobweb posets introduced by the author a few years ago. KoDAGs appear to be distinguished family of Ferrers digraphs which are natural join of a corresponding ordering chain of one direction directed cliques called di-bicliques. These digraphs serve to represent faithfully corresponding relations of arbitrary arity so that all relations of arbitrary arity are their subrelations. Being this chain-way complete (compare with Kompletne , Kuratowski Kₙ,m bipartite graphs) their DAG denotation is accompanied with the letter K in front of descriptive abbreviation oDAG. The way to join bipartite digraphs of binary into multi-ary relations is the natural join operation either on relations or their digraph representatives. This natural join operation is denoted here by ⊕→ symbol deliberately referring - in a reminiscent manner - to the direct sum ⊕ of adjacency matrices as it becomes the case for disjoint di-bigraphs.

Key Words: posets, graded digraphs, Ferrers dimension, natural join
AMS Classification Numbers: 06A06, 05B20, 05C7
affiliated to The Internet Gian-Carlo Polish Seminar:

http://ii.uwb.edu.pl/akk/sem/sem_rota.htm

1 Introduction to cobweb posets

1.1 Notation

One may identify and interpret some classes of digraphs in terms of their associated posets. (see [1] Interpretations in terms of posets Section 9)

Definition 1 (see [1]) Let $D = (\Phi, \prec)$ be a digraph. $w, v \in \Phi$ are said to be equivalent iff there exists a directed path containing both $w$ and $v$ vertices. We then write: $v \sim w$ for such pairs and denote by $[v]$ the $\sim$ equivalence class of $v \in \Phi$.

Definition 2 (see [1]) The poset $P(D)$ associated to $D = (\Phi, \prec)$ is the poset $P(D) = (\Phi/\sim, \leq)$ where $[v] \leq [w]$ iff there exists a directed path from a vertex $x \in [v]$ to a vertex $y \in [w]$.

The graded digraphs case:
If $D = (\Phi, \prec)$ is graded digraph then $D = (\Phi, \prec)$ is necessarily acyclic. Then no two elements of $D = (\Phi, \prec)$ are $\sim$ equivalent and thereby $P(D) = (\Phi/\sim, \leq)$ associated to $D = (V, \prec)$ is equivalent to: $P(D) \equiv (\Phi, \leq)$ is transitive, reflexive closure of $D = (\Phi, \prec)$.

The cobweb posets where introduced in several paper (see [2]-[6] and references therein) in terms of their poset [Hasse] diagrams. Here we deliver their equivalent definition preceded by preliminary notation and nomenclature.

**Notation : nomenclature, di-bicliques and natural join**

In order to proceed proficiently we adopt the following.

**Definition 3** A digraph $D = (\Phi, \prec \cdot)$ is transitive irreducible iff transitive reduction $(D) = D$.

**Definition 4** A poset $P(D) = (\Phi, \leq)$ is associated to a graded digraph $D = (\Phi, \prec)$ iff $P(D)$ is the transitive, reflexive closure of $D = (\Phi, \prec)$.
If you imagine arrows → left to the right - you would see two examples of di-bicliques.

![Diagram of di-bicliques](image)

Figure 1: Examples of di-bicliques if edges are replaced by arrows of join direction.

if you imagine arrows ← right to the left, you would see another examples of di-bicliques.

**Convention 2 (recall)** The binary relation \( E \subseteq X \times Y \) is identified with its bipartite digraph \( B = (X \cup Y, E) \) unless otherwise denoted distinctively deliberately.

**The natural join.**

The natural join operation is a binary operation like \( \Theta \) operator in computer science denoted here by \( \oplus \rightarrow \) symbol deliberately referring - in a quite reminiscent manner - to direct sum \( \oplus \) of adjacency Boolean matrices and - as matter of fact and in effect - to direct the sum \( \oplus \) of corresponding biadjacency [reduced] matrices of digraphs under natural join.

\( \oplus \rightarrow \) is a natural operator for sequences construction. \( \oplus \rightarrow \) operates on multi-ary relations according to the scheme: \((n + k)_{ary} \oplus \rightarrow (k + m)_{ary} = (n + k + m)_{ary}\)

For example: \((1 + 1)_{ary} \oplus \rightarrow (1 + 1)_{ary} = (1 + 1 + 1)_{ary}\), binary \( \oplus \rightarrow \) binary = ternary.

Accordingly an action of \( \oplus \rightarrow \) on these multi-ary relations’ digraphs adjacency matrices is to be designed soon in what follows.

**Domain-Codomain F-sequence condition** \( \text{dom}(R_{k+1}) = \text{ran}(R_k), \ k = 0, 1, 2, ... \).

Consider any natural number valued sequence \( F = \{F_n\}_{n \geq 0} \). Consider then any chain of binary relations defined on pairwise disjoint finite sets with cardinalities appointed by \( F \)-sequence elements values. For that to start we specify at first a relations’ domain-co-domain \( F \)-sequence.

**Domain-Codomain F-sequence** \( (|\Phi_n| = F_n) \)

\( \Phi_0, \Phi_1, \ldots \Phi_1, \ldots \ \Phi_k \cap \Phi_n = \emptyset \ for \ k \neq n, |\Phi_n| = F_n; \ i, k, n = 0, 1, 2, ... \)

Let \( \Phi = \bigcup_{k=0}^n \Phi_k \) be the corresponding ordered partition \( \text{anticipating} - \Phi \) is the vertex set of \( D = (\Phi, \prec \cdot \cdot \cdot) \) and its transitive, reflexive closure \( (\Phi, \leq) \). Impose \( \text{dom}(R_{k+1}) = \text{ran}(R_k) \) condition, \( k \in N \cup \{\infty\} \). What we get is binary relations chain.
Definition 5 (Relation's chain) Let $\Phi = \bigcup_{k=0}^{n} \Phi_k$, $\Phi_k \cap \Phi_n = \emptyset$ for $k \neq n$ be the ordered partition of the set $\Phi$.

Let a sequence of binary relations be given such that $R_0, R_1, ..., R_i, ..., R_{i+n}, ..., R_k \subseteq \Phi_k \times \Phi_{k+1}$, $\text{dom}(R_{k+1}) = \text{ran}(R_k)$.

Then the sequence $(R_k)_{k \geq 0}$ is called natural join (binary) relation's chain. Extension to varying arity relations' natural join chains is straightforward.

As necessarily $\text{dom}(R_{k+1}) = \text{ran}(R_k)$ for relations' natural join chain any given binary relation's chain is not just a sequence therefore we use "link to link" notation for $k, i, n = 1, 2, 3, ...$ ready for relational data basis applications:

$R_0 \oplus \rightarrow R_1 \oplus \rightarrow ... \oplus \rightarrow R_i \oplus \rightarrow ... \oplus \rightarrow R_{i+n}, ...$ is an $F$-chain of binary relations

where $\oplus \rightarrow$ denotes natural join of relations as well as both natural join of their bipartite digraphs and the natural join of their representative adjacency matrices (see the Section 3.).

Relation’s $F$-chain naturally represented by [identified with] the chain of theirs bipartite digraphs

$$R_0 \oplus \rightarrow R_1 \oplus \rightarrow ... \oplus \rightarrow R_i \oplus \rightarrow ... \oplus \rightarrow R_{i+n}, ... \Leftrightarrow$$

$$\Leftrightarrow B_0 \oplus \rightarrow B_1 \oplus \rightarrow ... \oplus \rightarrow B_i \oplus \rightarrow ... \oplus \rightarrow B_{i+n}, ...$$

results in $F$-partial ordered set $(\Phi, \leq)$ with its Hasse digraph representation looking like specific "cobweb" image [see figures below].

1.3 Partial order $\leq$

The partial order relation $\leq$ in the set of all points-vertices is determined uniquely by the above equivalent $F$-chains. Let $x, y \in \Phi = \bigcup_{k=0}^{n} \Phi_k$ and let $k, i, n = 1, 2, 3, ...$. Then

$$x \leq y \Leftrightarrow \forall x \in \Phi : x \leq x \lor \Phi_1 \ni x < y \in \Phi_{i+k} \text{ iff } x(R_i \circ \ldots \circ R_{i+k-1})y$$

where "\( \circ \)" stays for [Boolean] composition of binary relations.

Relation $(\leq)$ defined equivalently:

$x \leq y$ in $(\Phi, \leq)$ iff either $x = y$ or there exist a directed path from $x$ to $y; x, y \in \Phi$.

Let now $R_k = \Phi_k \times \Phi_{k+1}, k \in N \cup \{0\}$. For "historical" reasons [2]-[6] we shall call such partial ordered set $\Pi = (\Phi, \leq)$ the cobweb poset as theirs Hasse digraph representation looks like specific "cobweb" image (imagine and/or draw also their transitive and reflexive cover digraph $(\Phi, \leq)$). Cobweb? Super-cobweb! ...with fog droplets loops?!

1.4 Cobweb posets $(\Pi = (\Phi, \leq))$

Convention 3 (recall) The binary relation $E \subseteq X \times Y$ is identified with its bipartite digraph $B = (X \cup Y, E) \equiv K_{m,n}$ where $|X| = m, |Y| = n$. 

4
Definition 6 (cobweb poset) Let $D = (\Phi, \prec \cdot)$ be a transitive irreducible digraph. Let $n \in N \cup \{\infty\}$. Let $D$ be a natural join $D = \oplus_{k=0}^{n-1} B_k$ of di-bicliques $B_k = (\Phi_k \cup \Phi_{k+1}, \Phi_k \times \Phi_{k+1}), n \in N \cup \{\infty\}$. Hence the digraph $D = (\Phi, \prec \cdot)$ is graded. The poset $\Pi(D)$ associated to this graded digraph $D = (\Phi, \prec \cdot)$ is called a cobweb poset.

Convention 4 In a case we want to underline that we deal with finite cobweb poset (a subposet of appropriate - for example infinite $F$-cobweb poset $\Pi(D)$) we shall use a subscript and write $P_n$.

See: [2]-[6], [10], [13], [18].

Comment 2.

Graded graph is a natural join of bipartite graphs that form a chain of consecutive levels [i.e. graded graphs' antichains]

Graded digraph is a natural join of bipartite digraphs that form a chain of consecutive levels [i.e. graded digraphs' antichains]

Comment 3. (Definition 6. Recapitulation in brief.)
Cobweb poset is the poset $\Pi = (\Phi, \leq)$, where $\Phi = \bigcup_{k=0}^{n} \Phi_k \times \Phi_{k+1}, n \in N \cup \{\infty\}$. Cobweb poset is the poset $\Pi = (\Phi, \leq)$, where $\Phi = \bigcup_{k=0}^{n} \Phi_k \times \Phi_{k+1}, n \in N \cup \{\infty\}$, where $\leq$ is the transitive, reflexive cover of $\prec \cdot$.

Comment 4. (F-partial ordered set)
Cobweb poset $\Pi = (\Phi, \leq)$ is naturally graded and sequence $F$ - denominated thereby we call it sometimes $F$-partial ordered set $(\Phi, \leq)$.

2 Dimension of cobweb posets-revisited.

2.1 oDAG [7]

Observation 1 (cobwebs are oDAGs) In [2] it was observed that cobweb posets' Hasse diagrams are the members of so called oDAGs family i.e. cobweb posets' Hasse diagrams are orderable Directed Acyclic Graphs which is equivalent to say that the associated poset $P(D) = (\Phi, \leq)$ of $D = (\Phi, \prec \cdot)$ is of is of dimension 2.

Recall: DAGs - hence graded digraphs with minimal elements always might be considered - up to digraphs isomorphism - as natural digraphs [5] i.e. digraphs with natural labeling (i.e. $x_i < x_j \Rightarrow i < j$).

Definition 7 (Plotnikov - see [7], [2] and then below) A digraph $D = (\Phi, \prec \cdot)$ is called the orderable digraph (oDAG) if there exists a dim 2 poset such that its Hasse diagram coincides with the digraph $G$.

The statement from [2] may be now restated as follows:
Observation 2 (oDAG) Cobweb $P(D) = (\Phi, \preceq)$ posets’ Hasse diagrams $D = (\Phi, \prec)$ are oDAGs.

Proof: Obvious. Cobweb posets are posets with minimal elements set $\Phi_0$. Cobweb posets Hasse diagrams are DAGs. Cobweb posets representing the natural join of are then dim 2 posets as their Hasse digraphs are intersection of a natural labeling linear order $L_1$ and its "dual" $L_2$ denominated correspondingly in a standard way by: $L_1 = \text{natural labeling: chose for the topological ordering} \ L_1$ the labeling of minimal elements set $\Phi_0$ with labels 1, 2,..., from the left to the right (see Fig2.) then proceed up to the next level $\Phi_1$ and continue the labeling "→" from the left to the right ($\Phi_1$ is now treated as the set of minimal elements if $\Phi_0$ is removed) and so on. Apply the procedure of subsequent removal of minimal elements i.e. removal of subsequent labeled levels $\Phi_k$ - labeling the vertices along the levels from the left to the right.

$L_2 = "\text{dual" natural labeling: chose for the topological ordering } L_2$ the labeling of minimal elements set $\Phi_0$ with labels 1, 2,..., from the right to the left (see Fig1.) then proceed up to the next level $\Phi_1$ and continue the labeling "←" from the right to the left ($\Phi_1$ is now treated as the set of minimal elements if $\Phi_0$ is removed) and so on. Apply the procedure of subsequent removal of minimal elements i.e. removal of subsequent labeled levels $\Phi_k$ - labeling now the vertices along the levels from the right to the left q.e.d.

2.2 Brief history of the short oDAG’s name life

On the history of oDAG nomenclature with David Halitsky and Others input one is expected to see more in [15]. See also the December 2008 subject of The Internet Gian Carlo Rota Polish Seminar ([http://ii.uwb.edu.pl/akk/sem/sem_rota.htm](http://ii.uwb.edu.pl/akk/sem/sem_rota.htm)). Here we present its sub-history leading the author to note that cobweb posets are oDAGs.

According to Anatoly Plotnikov the concept and the name of oDAG was introduced by David Halitsky from Cumulative Inquiry in 2004.

oDAG-2004 (Plotnikov)

Quote 1. "A digraph $G \in D_n$ will be called orderable (oDAG) if there exists are dim 2 poset such that its Hasse diagram coincide with the digraph $G$". The Quote 1 comes from [9] in [2] i.e. from A.D. Plotnikov A formal approach to the oDAG/POSET problem (2004) html : //www.cumulativeinquiry.com/Problems/solut2.pdf (submitted to publication - March 2005)

The quote of the Quote 1 is to be found in [9]

oDAG-2005 [2]

Quote 2 "A digraph $G$ is called the orderable digraph (oDAG) if there exists a dim 2 poset such that its Hasse diagram coincides with the digraph $G$". [2]

oDAG-2006 [7]

Quote 3 "A digraph $G$ is called the orderable if there exists a dim 2 poset such that its Hasse diagram coincides with the digraph $G$". [7]

For further use of oDAG nomenclature see [6], and references therein. For
further references and recent results on cobweb posets see [10] and [11].

**Definition 8 (KoDAG)** The transitive and reflexive reduction of cobweb poset $\Pi = \langle \Phi, \leq \rangle$ i.e. posets’ $\Pi$ cover relation digraph [Hasse diagram] $D = (\Phi, \prec \cdot)$ is called KoDAG.

See [11]-[14].

**Comment 5.** Apply Comment 1.

**Why do we stick** to call KoDAGs graded digraphs with associated poset $\Pi = \langle \Phi, \leq \rangle$ the orderable DAGs on their own independently of the nomenclature quoted ?

Let $D = (\Phi, \prec \cdot)$ denotes now any transitive irreducible DAG [for example any graded digraph including KoDAG digraph for example as above]. Let poset $P(D) = (\Phi, \leq)$ be associated to $D = (\Phi, \prec \cdot)$.

**Definition 9 (Ferrers dimension)** We say that the poset $P(D) = (\Phi, \leq)$ is of Ferrers dimension $k$ iff it is associated to $D = (\Phi, \prec \cdot)$ of Ferrers dimension $k$.

**Observation 3 (Ferrers dimension)** Cobweb posets are posets of Ferrers dimension equal to one.

**Proof.** Apply any of many characterizations of Ferrers digraphs to see that cobweb posets are posets’ cover relation digraphs [Hasse diagrams] are Ferrers digraphs. For example consult Section 3 and see that biadjacency matrix does not contain any of two $2 \times 2$ permutation matrices.

**Comment 6.** Any KoDAG digraph $D = (\Phi, \prec \cdot)$ is the digraph stable under the transitive and reflexive reduction i.e. ["irreducible"] Hasse portrait of Ferrers relation $\prec \cdot$. The positions of 1’s in biadjacency [reduced adjacency] matrix display the support of Ferrers relation $\prec \cdot$. The digraph $(\Phi, \leq)$ of the cobweb poset $P(D) = (\Phi, \leq)$ associated to KoDAG digraph $D = (\Phi, \leq)$ is the portrait of Ferrers relation $\leq$. The positions of 1’s in biadjacency [reduced adjacency] matrix display the support of Ferrers relation $\leq$. Note: for $F$-denominated cobweb posets the nomenclature identifies: biadjacency [reduced adjacency] matrix $\equiv$ zeta matrix i.e. the incidence matrix $\zeta_F$ of the $F$- poset (see: Fig.$\zeta_N$ and Fig.$\zeta_F$). Recall that this $F$-partial ordered set $(\Phi, \leq)$ is a natural join of $F$-chain of binary $K$-relations (complete or universal relations as called sometimes). These relations are represented by di-bicliques $K_{k,k+1}$ which are on their own the Ferrers dimension one digraphs. As for the other - not necessarily $K$-relations’ chains we may end up with Ferrers or not digraphs in corresponding di-bigraphs’ chain. See below, then Section 4 and more in [15].

### 3 The natural join $\oplus \to$ operation

We define here the adjacency matrices representation of the natural join $\oplus \to$ operation.
3.1 Recall

Let \( D(R) = (V(R) \cup W(R), E(R)) \equiv (V \cup W, E); V \cap W = \emptyset, E(R) \subseteq V \times W. \)
Let \( D(R) \) denotes here down the bipartite digraph of binary relation \( R \) with \( \text{dom}(R) = V \) and \( \text{rang}(R) = W. \) Colligate with the anticipated examples \( R = R_k \subseteq \Phi_k \times \Phi_{k+1} \equiv K_{k,k+1}, V(R) \cup W(R) = \Phi_k \cup \Phi_{k+1}. \)

3.2 The adjacency matrices and their natural join.

The adjacency matrix \( A \) of a bipartite graph with biadjacency (reduced adjacency) matrix \( B \) is given by

\[
A = \begin{pmatrix}
0 & B \\
B^T & 0
\end{pmatrix}.
\]

**Definition 10** The adjacency matrix \( A[D] \) of a bipartite digraph \( D(R) = (P \cup L, E \subseteq P \times L) \) with biadjacency matrix \( B \) is given by

\[
A[D] = \begin{pmatrix}
0_{k,k} & B(k \times m) \\
0_{m,k} & 0_{m,m}
\end{pmatrix},
\]

where \( k = |P|, m = |L|. \)

**Convention 5** \( S \odot R = \text{composition of binary relations } S \text{ and } R \Leftrightarrow B_R \odot B_S = B_R \odot B_S \text{ where } (|V| = k, |W| = m) \) \( B_R(k \times m) \equiv B_R \) is the \((k \times m)\) biadjacency [or another name: reduced adjacency] matrix of the bipartite relations’ \( R \) digraph \( B(R) \) and \( \odot \) apart from relations composition denotes also Boolean multiplication of these rectangular biadjacency Boolean matrices \( B_R, B_S. \) What is their form? The answer is in the block structure of the standard square \((n \times n)\) adjacency matrix \( A[D(R)]; n = k + m. \) The form of standard square adjacency matrix \( A[G(R)] \) of bipartite digraph \( D(R) \) has the following apparently recognizable block reduced structure: [ \( O_{s \times s} \) stays for \((k \times m)\) zero matrix]

\[
A[D(R)] = \begin{bmatrix}
O_{k \times k} & A_R(k \times m) \\
O_{m \times k} & O_{m \times m}
\end{bmatrix}
\]

Let \( D(S) = (W(S) \cup T(S), E(S)); W \cap T = \emptyset, E(S) \subseteq W \times T; (|W| = m, |T| = s) \); hence

\[
A[D(S)] = \begin{bmatrix}
O_{m \times m} & A_S(m \times s) \\
O_{s \times m} & O_{s \times s}
\end{bmatrix}
\]

**Definition 11 (natural join condition)** The ordered pair of matrices \((A_1, A_2)\) is said to satisfy the natural join condition iff they have the block structure of \( A[D(R)] \) and \( A[D(S)] \) as above i.e. iff they might be identified accordingly: \( A_1 = A[D(R)] \) and \( A_2 = A[D(S)]. \)

Correspondingly if two given digraphs \( G_1 \) and \( G_2 \) are such that their adjacency matrices \( A_1 = A[G_1] \) and \( A_2 = A[G_2] \) do satisfy the natural join condition we shall say that \( G_1 \) and \( G_2 \) satisfy the natural join condition. For matrices satisfying the natural join condition one may define what follows.
First we define the **Boolean reduced** or **natural join composition** $\circ\rightarrow$ and secondly the natural join $\oplus\rightarrow$ of adjacent matrices satisfying the natural join condition.

**Definition 12 (\(\circ\rightarrow\) composition)**

\[
A[D(R\circ S)] =: A[D(R)]\circ\rightarrow A[D(S)] = \begin{bmatrix} O_{k\times k} & A_{R\circ S}(k \times s) \\ O_{s\times k} & O_{s\times s} \end{bmatrix}
\]

where $A_{R\circ S}(k \times s) = A_{R}(k \times m)\circ A_{S}(m \times s)$.

according to the scheme:

\[
[(k + m) \times (k + m)]\circ\rightarrow [(m + s) \times (m + s)] = [(k + s) \times (k + s)]
\]

**Comment 7.** The adequate projection makes out the intermediate, joint in common $\text{dom}(S) = \text{rang}(R) = W$, $|W| = m$.

The above Boolean reduced composition $\circ\rightarrow$ of adjacent matrices technically reduces then to the calculation of just Boolean product of the **reduced** rectangular adjacency matrices of the bipartite relations’ graphs.

We are however now in need of the Boolean natural join product $\oplus\rightarrow$ of adjacent matrices already announced at the beginning of this presentation. Let us now define it.

As for the **natural join** notion we aim at the morphism correspondence:

\[
S \oplus\rightarrow R \Leftrightarrow M_{S\oplus\rightarrow R} = M_{R\oplus\rightarrow M_S}
\]

where $S \oplus\rightarrow R$ = natural join of binary relations $S$ and $R$ while $M_{S\oplus\rightarrow R} = M_{R\oplus\rightarrow M_S}$ = natural join of standard square adjacency matrices (with customary convention: $M(G(R)) = M_R$ adapted). Attention: recall here that the natural join of the above binary relations $R \oplus\rightarrow S$ is the ternary relation - and on one results in $k$-ary relations if with more factors undergo the $\oplus\rightarrow$ product.

As a matter of fact $\oplus\rightarrow$ **operates on multi-ary relations according to the scheme**:

\[
(n + k)_{\text{ary}} \oplus\rightarrow (k + m)_{\text{ary}} = (n + k + m)_{\text{ary}}.
\]

For example: $(1 + 1)_{\text{ary}} \oplus\rightarrow (1 + 1)_{\text{ary}} = (1 + 1 + 1)_{\text{ary}}, \text{binary} \oplus\rightarrow \text{binary} = \text{ternary}.$

Technically - the natural join of the $k$-ary and $n$-ary relations is defined accordingly the same way via $\oplus\rightarrow$ natural join product of adjacency matrices - the adjacency matrices of these relations’ Hasse digraphs.

With the notation established above we finally define the natural join $\oplus\rightarrow$ of two adjacency matrices as follows:

**Definition 13 (natural join $\oplus\rightarrow$ of biadjacency matrices)**.
\[
A[D(R \oplus \to S)] = A[D(R)] \oplus A[D(S)] = \\
= \begin{bmatrix}
O_{k \times k} & A_R(k \times m) \\
O_{m \times m} & O_{s \times s}
\end{bmatrix} \oplus \begin{bmatrix}
O_{m \times m} & A_S(m \times s) \\
O_{s \times s} & O_{s \times s}
\end{bmatrix} = \\
= \begin{bmatrix}
O_{k \times k} & A_R(k \times m) & O_{k \times s} \\
O_{m \times k} & O_{m \times m} & A_S(m \times s) \\
O_{s \times k} & O_{s \times m} & O_{s \times s}
\end{bmatrix}
\]

**Comment 8.** The adequate projection used in natural join operation leaves one copy of the joint in common “intermediate” submatrix \(O_{m \times m}\) and consequently leaves one copy of “intermediate” joint in common \(m\) according to the scheme:

\[
[(k + m) \times (k + m)] \oplus \to [(m + s) \times (m + s)] = [(k + m + s) \times (k + m + s)]
\]

### 3.3 The biadjacency matrices of the natural join of adjacency matrices.

Denote with \(B(A)\) the biadjacency matrix of the adjacency matrix \(A\).

Let \(A(G)\) denotes the adjacency matrix of the digraph \(G\), for example a di-biclique relation digraph. Let \(A(G_k), k = 0, 1, 2, \ldots\) be the sequence adjacency matrices of the sequence \(G_k, k = 0, 1, 2, \ldots\) of digraphs. Let us identify \(B(A) \equiv B(G)\) as a convention.

**Definition 14 (digraphs natural join)** Let digraphs \(G_1\) and \(G_2\) satisfy the natural join condition. Let us make then the identification \(A(G_1 \oplus \to G_2) \equiv A_1 \oplus \to A_2\) as definition. The digraph \(G_1 \oplus \to G_2\) is called the digraphs natural join of digraphs \(G_1\) and \(G_2\). Note that the order is essential.

We observe at once what follows.

**Observation 4**

\[
B(G_1 \oplus \to G_2) \equiv B(A_1 \oplus \to A_2) = B(A_1) \oplus B(A_2) \equiv B(G_1) \oplus B(G_2)
\]

**Comment 9.** The Observation 4 justifies the notation \(\oplus \to\) for the natural join of relations digraphs and equivalently for the natural join of their adjacency matrices and equivalently for the natural join of relations that these are faithful representatives of.

As a consequence we have.

**Observation 5**

\[
B(\oplus \to_{i=1}^n A(G_i)) = B[\oplus \to_{i=1}^n A(G_i)] = \oplus_{i=1}^n B[G_i] = \text{diag}(B_1, B_2, B_3, \ldots, B_n)
\]

\[
= \begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
\vdots \\
B_n
\end{bmatrix}
\]

\(n \in N \cup \{\infty\}\).
3.4 Applications

Once any natural number valued sequence \( F = \{F_n\}_{n \geq 1} \) is being chosen its KoDAG digraph is identified with Hasse cover relation digraph. Its adjacency matrix \( A_F \) is sometimes called Hasse matrix and is given in a plausible form and impressively straightforward way. Just use the fact that the Hasse digraph which is displaying cover relation \( \preceq \) is an \( F \)-chain of coined bipartite digraphs - coined each preceding with a subsequent one by natural join operator \( \oplus \rightarrow \) [resemblance of \( \oplus \rightarrow \) to direct matrix sum is not naive - compare ”natural join” of disjoint digraphs with no common set of marked nodes (“attributes”)].

Note: \( I(s \times k) \) stays for \( (s \times k) \) matrix of ones i.e. \([I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k\).

Let us start first with \( F = \{F_n\}_{n \geq 1} = N \). See Fig. 2. Then its associated \( F \)-partial ordered set \( \langle \Phi, \leq \rangle \) has the following Hasse digraph displaying cover relation of the \( \leq \) partial order

![Figure 2: Display of a finite subposet \( \Pi_6 \) of the \( N \) natural numbers cobweb poset](image)

The Hasse matrix \( A_N \) i.e. adjacency matrix of cover relation digraph i.e. adjacency matrix of the Hasse diagram of the \( N \)-denominated cobweb poset \( \langle \Phi, \leq \rangle \) is given by upper triangular matrix \( A_N \) of the form:

\[
A_N = \begin{bmatrix}
O_{1 \times 1} & I(1 \times 2) & O_{1 \times \infty} & & & \\
O_{2 \times 1} & O_{2 \times 2} & I(2 \times 3) & O_{2 \times \infty} & & \\
O_{3 \times 1} & O_{3 \times 2} & O_{3 \times 3} & I(3 \times 4) & O_{3 \times \infty} & \\
O_{4 \times 1} & O_{4 \times 2} & O_{4 \times 3} & O_{4 \times 4} & I(4 \times 5) & O_{4 \times \infty} \\
... & ... & ... & ... & ... & \\
end{bmatrix}
\]

One may see that the zeta function matrix of the \( F = N \) choice is geometrical series in \( A_N \) i.e. the geometrical series in the poset \( \langle \Phi, \leq \rangle \) Hasse matrix \( A_N \):

\[
\zeta = (1 - A_N)^{-1 \odot}
\]

Explicitly: \( \zeta = (1 - A_N)^{-1 \odot} = I_\infty + A_N + A_N^{\odot 2} + ... = \)
\[
A^k_{ij} = \begin{cases}
0 & \text{if } k \neq j - i \\
\frac{1}{k!} & \text{if } k = j - i
\end{cases}
\]

and the supports (nonzero matrices blocks) of \(A^k\) and \(A^m\) are disjoint for \(k \neq m\). Indeed: the entry in row \(i\) and column \(j\) of the inverse \((I - A)^{-1}\) gives the number of directed paths from vertex \(x_i\) to vertex \(x_j\). This can be seen from geometric series with adjacency matrix as an argument

\[
(I - A)^{-1} = I + A + A^2 + A^3 + ...
\]

taking care of the fact that the number of paths from \(i\) to \(j\) equals the number of paths of length 0 plus the number of paths of length 1 plus the number of paths of length 2, etc.

Therefore the entry in row \(i\) and column \(j\) of the inverse \((I - A)^{-1}\) gives the answer whether there exists a directed paths from vertex \(i\) to vertex \(j\) (Boolean value 1) or not (Boolean value 0) i.e. whether these vertices are comparable i.e. whether \(x_i < x_j\) or not.

**Remark:** In the cases - Boolean poset \(2^N\) and the "Ferrand-Zeckendorf" poset of finite subsets of \(N\) without two consecutive elements considered in \([\text{I}^2]\) one has

\[
\zeta = \exp[A] = (1 - A)^{-1} = I_{\infty \times \infty} + A + A^2 + \ldots
\]

because in those cases

\[
A^k_{ij} = \begin{cases}
0 & \text{if } k \neq j - i \\
\frac{1}{k!} & \text{if } k = j - i
\end{cases}
\]

How it goes in our \(F\)-case? Just see \(A_N^{\odot 2}\) and then add \(A_N^{\odot 0} \vee A_N^{\odot 1} \vee A_N^{\odot 2} \vee \ldots\) For example:

\[
A_N^{\odot 2} = \begin{bmatrix}
O_{1 \times 1} & O_{1 \times 2} & I(1 \times 3) & O_{1 \times \infty} \\
O_{2 \times 1} & O_{2 \times 2} & O_{2 \times 3} & I(2 \times 4) & O_{2 \times \infty} \\
O_{3 \times 1} & O_{3 \times 2} & O_{3 \times 3} & I(3 \times 5) & O_{3 \times \infty} \\
O_{4 \times 1} & O_{4 \times 2} & O_{4 \times 3} & I(4 \times 4) & O_{4 \times \infty} \\
\text{...etc} & \text{...} & \text{...} & \text{...} & \text{...}
\end{bmatrix}
\]

Consequently we arrive at the incidence matrix \(\zeta = \exp[A_N]\) for the natural numbers cobweb poset displayed by Fig 3. Note that incidence matrix \(\zeta\) representing uniquely its corresponding cobweb poset does exhibits (see below) a
staircase structure of zeros above the diagonal which is characteristic to Hasse diagrams of all cobweb posets.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$$

Figure $\zeta_N$. The incidence matrix $\zeta$ for the natural numbers i.e. N-cobweb poset

Comment 9. The given $F$-denominated staircase zeros structure above the diagonal of zeta matrix $\zeta$ is the unique characteristics of its corresponding $F$-KoDAG Hasse digraphs.

For example see Fig $\zeta_F$. below (from [4]).

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$$

Figure $\zeta_F$. The incidence matrix $\zeta$ for the Fibonacci cobweb poset associated to $F$-KoDAG Hasse digraph
The zeta matrix $\zeta_F$ i.e. the incidence matrix for the Fibonacci numbers cobweb poset $[F - KoDAG]$ determines completely its incidence algebra and corresponds to the poset with Hasse diagram displayed by the Fig. 3.

![Figure 3: Display of the $F$- Fibonacci numbers cobweb poset](image)

The explicit expression for zeta matrix $\zeta_F$ via known blocks of zeros and ones for arbitrary natural numbers valued $F$- sequence is readily found due to brilliant mnemonic efficiency of the authors up-side-down notation (see Appendix in [13]). With this notation inspired by Gauss and the reasoning just repeated with "$k_F$" numbers replacing $k$ - natural numbers one gets in the spirit of Knuth [18] the clean result:

$$A_F = \begin{pmatrix}
0_{1_F \times 1_F} & I(1_F \times 2_F) & 0_{1_F \times \infty} \\
0_{2_F \times 1_F} & 0_{2_F \times 2_F} & I(2_F \times 3_F) & 0_{2_F \times \infty} \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & 0_{3_F \times 3_F} & I(3_F \times 4_F) & 0_{3_F \times \infty} \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & 0_{4_F \times 4_F} & I(4_F \times 5_F) & 0_{4_F \times \infty} \\
\vdots & \text{etc} & \vdots & \text{and so on} & \vdots \\
\end{pmatrix}$$

and

$$\zeta_F = \exp\left(\frac{1}{2}\right)\left[A_F\right] \equiv (1 - A_F)^{-1} \equiv I_{\infty \times \infty} + A_F + A_F^2 + ... =$$

$$= \begin{pmatrix}
I_{1_F \times 1_F} & I(1_F \times \infty) \\
O_{2_F \times 1_F} & I(2_F \times \infty) \\
O_{3_F \times 1_F} & 0_{3_F \times 2_F} & I(3_F \times \infty) \\
O_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & I(4_F \times \infty) \\
\vdots & \text{etc} & \vdots & \text{and so on} & \vdots \\
\end{pmatrix}$$

**Comment 10.** (ad "upside down notation")

Concerning Gauss and Knuth - see remarks in [18] on Gaussian binomial coefficients.
Observation 6 Let us denote by \( \Phi_k \rightarrow \Phi_{k+1} \) (see the authors papers quoted) the di-bicliques denominated by subsequent levels \( \Phi_k, \Phi_{k+1} \) of the graded \( F \)-poset \( P(D) = (\Phi, \preceq) \) i.e. levels \( \Phi_k, \Phi_{k+1} \) of its cover relation graded digraph \( D = (\Phi, \prec) \) [Hasse diagram]. Then

\[
B(\oplus_{k=1}^{n} (\Phi_k \rightarrow \Phi_{k+1})) = \text{diag}(I_1, I_2, ..., I_n) = \\
\begin{bmatrix}
I(1_F \times 2_F) \\
I(2_F \times 3_F) \\
I(3_F \times 4_F) \\
\vdots \\
I(n_F \times (n + 1)_F)
\end{bmatrix}
\]

where \( I_k \equiv \text{diag}(I(k_F \times (k + 1)_F), k = 1, ..., n \) and where - recall - \( I(s \times k) \) stays for \( (s \times k) \) matrix of ones i.e. \( [I(s \times k)]_{ij} = 1, 1 \leq i \leq s, 1 \leq j \leq k \) and \( n \in N \cup \{\infty\} \).

Observation 7 Consider bigraphs’ chain obtained from the above di-bicliques’ chain via deleting or no arcs making thus [if deleting arcs] some or all of the di-bicliques \( (\Phi_k \rightarrow \Phi_{k+1}) \) not di-bicliques; denote them as \( G_k \). Let \( B_k = \text{B}(G_k) \) denotes their biadjacency matrices correspondingly. Then for any such \( F \)-denominated chain [hence any chain] of bipartite digraphs \( G_k \) the general formula is:

\[
\text{B}(\oplus_{i=1}^{n} G_i) \equiv \text{B}(\oplus_{i=1}^{n} A(G_i)) = \oplus_{i=1}^{n} B[A(G_i)] = \text{diag}(B_1, B_2, ..., B_n) = \\
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
\vdots \\
B_n
\end{bmatrix}
\]

\( n \in N \cup \{\infty\} \).

Observation 8 The \( F \)-poset \( P(G) = (\Phi, \preceq) \) i.e. its cover relation graded digraph \( G = (\Phi, \prec) = \oplus_{k=0}^{n} G_k \) is of Ferrers dimension one iff in the process of deleting arcs from the cobweb poset Hasse diagram \( D = (\Phi, \prec) = \oplus_{k=0}^{n} (\Phi_k \rightarrow \Phi_{k+1}) \) does not produces \( 2 \times 2 \) permutation submatrices in any bigraph \( G_k \) biadjacency matrix \( B_k = \text{B}(G_k) \).

Examples (finite subposets of cobweb posets)
Fig.4 and Fig.5 display a Hasse diagram portraits of finite subposets of cobweb posets. In view of the Observation 2 these subposets are naturally Ferrers digraphs i.e. of Ferrers dimension equal one.

4 Summary

4.1 Principal - natural identifications

Any KoDAG is a di-bicliques chain ⇔ Any KoDAG is a natural join of complete bipartite graphs [di-bicliques] =

\[
(\Phi_0 \cup \Phi_1 \cup ... \cup \Phi_n \cup ..., E_0 \cup E_1 \cup ... \cup E_n \cup ...) \equiv D(\bigcup_{k \geq 0} \Phi_k, \bigcup_{k \geq 0} E_k) \equiv D(\Phi, E)
\]
Figure 4: Display of the subposet $P_5$ of the $F$= Fibonacci sequence $F$-cobweb poset and $\sigma P_5$ subposet of the $\sigma$ permuted Fibonacci $F$-cobweb poset.

Figure 5: Display of the subposet $P_4$ of the $F$ = Gaussian integers sequence ($q = 2$) $F$-cobweb poset and $\sigma P_4$ subposet of the $\sigma$ permuted Gaussian ($q = 2$) $F$-cobweb poset.

where $E_k = \Phi_k \times \Phi_{k+1} \equiv K_{k,k+1}$ and $E = \bigcup_{k \geq 0} E_k$.

Naturally, as indicated earlier any graded posets’ Hasse diagram with finite width including KoDAGs is of the form

$$D(\Phi, E) \equiv D\left(\bigcup_{k \geq 0} \Phi_k, \bigcup_{k \geq 0} E_k\right) \iff \langle \Phi, \leq \rangle$$

where $E_k \subseteq \Phi_k \times \Phi_{k+1} \equiv K_{k,k+1}$ and the definition of $\leq$ from 1.3. is applied.

In front of all the above presentation the following is clear .

**Observation 9** "Many" graded digraphs with finite width including KoDAGs $D = (V, \prec \cdot)$ encode bijectively their correspondent $n$-ary relation ($n \in N \cup \{\infty\}$ as seen from its following definition: $E_k \subseteq \Phi_k \times \Phi_{k+1} \equiv K_{k,k+1}$ where (n-ary relation) $E = \bigoplus_{k=0}^{n-1} E_k \subseteq \times_{k=0}^{n} \Phi_k$

i.e. identified with graded poset $(V_n, E)$ natural join obtained $n + 1$-ary relation $E$ is a subset of Cartesian product obtained the universal $n + 1$-ary relation identified with cobweb poset digraph $(V_n, \cdot)$. $V_\infty \equiv V$.

Which are those "many"? The characterization is arrived at with au rebour point of view. Any $n$-ary relation ($n \in N \cup \{\infty\}$) determines uniquely [may be identified with] its correspondent graded digraph with minimal elements set $\Phi_0$ given by the (n-ary rel.) formula

$$E = \bigoplus_{k=0}^{n-1} E_k \subseteq \times_{k=0}^{n} \Phi_k,$$
where the sequence of binary relations $E_k \subseteq \Phi_k \times \Phi_{k+1} \equiv K_{k,k+1}$ is denominated by the source $n$-ary relation as the following example shows.

**Example** (ternary = $Binary_1 \oplus \rightarrow Binary_2$)

Let $T \subset X \times Z \times Y$ where $X = \{x_1, x_2, x_3\}$, $Z = \{z_1, z_2, z_3, z_4\}$, $Y = \{y_1, y_2\}$ and

$$T = \{(x_1, z_1, y_1), (x_1, z_2, y_1), (x_1, z_4, y_2), (x_2, z_3, y_2), (x_3, z_3, y_2)\}.$$

Comment 11. As a comment to the Observation 9 and the Observation 3 consider Fig.7 which was the source of inspiration for cobweb posets birth \[4, 3, 2, 5, 6\] and here serves as Hasse diagram $D_{Fib} \equiv (\Phi, \prec_{Fib})$ of the poset $P(D_{Fib}) = (\Phi, \leq_{Fib})$ associated to $D_{Fib}$. Obviously, $P(D_{Fib})$ is a subposet of the Fibonacci cobweb poset $P(D)$ and $D_{Fib}$ is a subgraph of the Fibonacci cobweb poset $P(D)$ Hasse diagram $D \equiv (\Phi, \prec)$.

The Ferrers dimension of $D_{Fib}$ is obviously not equal one.

... and so on up ...

Figure 7: Display of of Hasse diagram of the form of the Fibonacci tree.
Exercise. Find the Ferrers dimension of $D_{\text{Fib}}$. What is the dimension of the poset $P(D_{\text{Fib}}) = (\Phi, \leq_{\text{Fib}})$? (Compare with Observation 2). Find the chain $E_k \subset \Phi_k \times \Phi_{k+1}$, $k = 0, 1, 2, \ldots$ of binary relations such that $D_{\text{Fib},n} = \oplus_{k=0}^{n} E_k$, $n \in \mathbb{N} \cup \{\infty\}$. Find the Ferrers dimension of $D_{\text{Fib},n}$.

Ad Bibliography Remark
On the history of oDAG nomenclature with David Halitsky and Others input one is expected to see more in [15]. See also the December 2008 subject of The Internet Gian Carlo Rota Polish Seminar (http://ii.uwb.edu.pl/akk/sem/sem_rota.htm). Recommended readings on Ferrers digraphs of immediate use here are [19]-[25]. For example see pages 61 an 85 in [19], see page 2 in [21]. The J. Riguet paper [21] is the source paper including also equivalent characterizations of Ferrers digraphs as well as other [22, 23, 24]. The now classic reference on interval orders and interval graphs is [25].

Acknowledgments
Thank are expressed here to the Student of Gdańsk University Maciej Dziemiańczuk for applying his skillful TeX-nology with respect to the present work as well as for his general assistance and cooperation on KoDAGs investigation.

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