Laplacian eigenmodes for spherical spaces

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Abstract

The possibility that our space is multi–rather than singly–connected has gained renewed interest after the discovery of the low power for the first multipoles of the CMB by WMAP. To test the possibility that our space is a multi-connected spherical space, it is necessary to know the eigenmodes of such spaces. Except for lens and prism space, and to some extent for dodecahedral space, this remains an open problem. Here we derive the eigenmodes of all spherical spaces. For dodecahedral space, the demonstration is much shorter, and the calculation method much simpler than before. We also apply our method to tetrahedric, octahedric and icosahedric spaces. This completes the knowledge of eigenmodes for spherical spaces, and opens the door to new observational tests of the cosmic topology. The vector space \( V_k \) of the eigenfunctions of the Laplacian on the 3-sphere \( S^3 \), corresponding to the same eigenvalue \( \lambda_k = -k(k + 2) \), has dimension \( (k + 1)^2 \). We show that the Wigner functions provide a basis for such a space. Using the properties of the latter, we express the behaviour of a general function of \( V_k \) under an arbitrary rotation \( G \) of \( SO(4) \). This offers the possibility of selecting those functions of \( V_k \) which remain invariant under \( G \). Specifying \( G \) to be a generator of the holonomy group of a spherical space \( X \), we give the expression of the vector space \( V^k_X \) of the eigenfunctions of \( X \). We provide a method to calculate the eigenmodes up to an arbitrary order. As an illustration, we give the first modes for the spherical spaces mentioned.

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1. Introduction

1.1. Cosmic topology

The possibility that spatial sections of our universe have a multi-connected topology offers interesting alternatives to standard big bang models [10]. The recent suggestion [13] that a
peculiar variant, namely the Poincaré dodecahedral space, may explain the deficiency in the
first modes of the angular power spectrum of the CMB, as observed by WMAP, has renewed
the interest in this question. It is presently an important task of cosmology to check this
possibility.

The best hope to test the possible multi-connectedness of space comes from the CMB
fluctuations. For this task, the most important characteristic of a multi-connected space \( X \)
is the set of its eigenmodes, a subspace of the modes of its universal covering [15]. The main
consequences on the CMB are (i) a suppression of power at the scales comparable to, or larger
than, the circumference of \( X \) and (ii) a violation of isotropy at these scales. Thus, observational
signatures may occur as a reduction of power in the angular CMB spectrum (as reported by
[13]), and as the imprint of correlations between different modes. The predictions of these
observable effects require knowledge of the eigenmodes (of the Laplacian) of \( X \). For instance,
[13] use a limited number of eigenmodes of the Poincaré space, calculated numerically. More
precise checks require knowledge of a larger number of modes of this space. The examination
of other models requires knowledge of the eigenmodes of other spherical spaces.

This paper is devoted to the calculation of the eigenmodes of all variants of the multi-
connected spherical spaces. They are defined as quotients \( X \equiv S^3/\Gamma \), where the 3-sphere
\( S^3 \) is the universal covering space of \( X \), and \( \Gamma \subset SO(4) \) is the holonomy group of \( X \). The
transformations of \( \Gamma \) leave \( X \) invariant. The compactness of \( S^3 \) implies the finiteness of \( \Gamma \).
The latter is finitely generated, i.e. by a finite number of transformations.

1.2. Spherical spaces

In the family of spherical spaces, the eigenmodes of lens and prism spaces have been calculated
by [11, 12] (see also [8]) in an analytic form. For dodecahedral space, only the very first modes
had been calculated numerically, by [13]. Later, [9] reduced the calculation to an eigenvalue
problem of a reduced dimensionality, allowing their estimations up to an arbitrary order. For
the other spherical spaces, knowledge of the eigenmodes remains an open problem. Here, we
present a general method to calculate the eigenmodes of any of the spherical spaces \( X = S^3/\Gamma \).

For a given eigenvalue \( \lambda_k \), the eigenmodes of \( X \) describe an eigenspace \( V_k^X \). The dimension
of this vector space has been calculated, very generally, by Ikeda [6]. We use his results as a
cross-check of the calculations below.

Our calculation is based on the fact that each \( \Gamma \) has two generators \( G_\pm \), which are single
left action rotations of \( SO(4) \). We obtain the straightforward decomposition

\[ V_k^X = V_k^G_+ \cap V_k^G_- , \]

and we give explicit analytic expressions of the \( V_k^G_\pm \), which are the vector spaces of the
eigenfunctions of \( S^3 \) invariant under \( G_\pm \).

Although the expressions of the two \( V_k^G_\pm \) are very simple, there is no general analytic
expression of their intersection. Thus, to obtain an explicit expression of the modes, we
propose a general method, which applies to any spherical space. It is based on the fact that
a basis of \( V_k \) is provided by the Wigner functions. From the rotation properties of the latter,
we derive a simple invariance condition of the modes of \( S^3 \) under \( G_\pm \), and thus under \( \Gamma \).
This provides the eigenmodes of \( X \), which are the eigenmodes of \( S^3 \) which remain invariant
under \( \Gamma \).

In section 2, we analyse the eigenfunctions of \( S^3 \). We prove that the Wigner functions
provide a convenient basis \( (B_2) \), different from the widely used basis \( (B_1) \) provided by the
usual hyperspherical harmonics. In section 3, we derive the rotation properties of this basis
under \( SO(4) \), from which we deduce those of all eigenfunctions of \( S^3 \). This allows us to give a
criterion for the invariance of such a function under an arbitrary rotation of SO(4). In section 4, we give an explicit analytic expression for the invariant subspaces \( V_{G_{\alpha}}^{k} \), and we provide a general procedure to calculate the eigenmodes of any spherical space. The first modes are explicitly given in the appendix.

2. Eigenmodes of \( S^3 \)

The eigenmodes of \( S^3 \) are the eigenfunctions of the Laplacian \( \Delta_{S^3} \) of \( S^3 \). The eigenvalues are known to be of the form \( \lambda_k = -k(k+2) \), where \( k \in \mathbb{N} \). The eigenmodes corresponding to a given value of \( k \) span the eigenspace \( V_k \), a vector space of dimension \((k+1)^2\). They are the solutions of the Helmholtz equation

\[
\Delta_{S^3} f = \lambda_k f = -k(k+2) f, \quad k \in \mathbb{N}.
\]

Each eigenspace \( V_k \) realizes the \((k+1)^2\) dimensional irreducible representation of SO(4), the isometry group of \( S^3 \).

The corresponding eigenspace \( V_k^X \) of \( X = S^3 / \Gamma \), the goal of our search, is the vector space of functions of \( V_k \) which are \( \Gamma \)-invariant. Each \( V_k^X \) is a subvector space of \( V_k \) whose dimension (possibly zero) is the degeneracy of \( \lambda_k \) on \( X \).

2.1. Basis \((B_1)\): hyperspherical harmonics

Let us call \((B_1)\) the most widely used (orthonormal) basis for \( V_k \) provided by the hyperspherical harmonics

\[
(B_1) \equiv \{ Y_{k\ell m}, 0 \leq \ell \leq k, -\ell \leq m \leq \ell \}.
\]

It generalizes the usual spherical harmonics \( Y_{\ell m} \) on the sphere (note that \( Y_{k\ell m} \propto Y_{\ell m} \)). In fact, it can be shown ([1, 3 p 240, 4]) that a basis of this type exists on any sphere \( S^n \). Moreover, [3, 4] show that the \((B_1)\) basis for \( S^n \) is ‘naturally generated’ by the \((B_1)\) basis for \( S^{n-1} \). In this sense, the \((B_1)\) basis for \( S^3 \) is generated by the usual spherical harmonics \( Y_{\ell m} \) on the 2-sphere \( S^2 \).

The generation process involves harmonic polynomials constructed from null complex vectors. The basis \((B_1)\) is in fact based on the reduction of the representation of SO(4) to representations of SO(3): each \( Y_{k\ell m} \) is an eigenfunction of an SO(3) subgroup of SO(4) which leaves a selected point of \( S^3 \) invariant. This makes these functions useful when one considers the action of that particular SO(3) subgroup. But they show no simple behaviour under a general rotation.

This basis is adapted to the usual polar spherical coordinates \((\psi, \theta, \phi)\).

2.2. The parabolic basis \((B_2)\)

The second orthonormal basis of \( V_k \) is specific to \( S^3 \):

\[
(B_2) \equiv \{ T_{k,m_1,m_2}, -k/2 \leq m_1, m_2 \leq k/2 \},
\]

where \( m_1 \) and \( m_2 \) vary independently by integers (and, thus, take integral or semi-integral values according to the parity of \( k \)). It is, for instance, introduced in [1] by group theoretical arguments, it appears naturally adapted to the systems of toroidal coordinates to describe \( S^3 \):

\[
\begin{align*}
x^0 &= r \cos \chi \cos \theta \\
x^1 &= r \sin \chi \cos \phi \\
x^2 &= r \sin \chi \sin \phi \\
x^3 &= r \cos \chi \sin \theta
\end{align*}
\]
spanning the range $0 \leq \chi \leq \pi/2$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. Here, the $(x^\mu)$ represent the Cartesian coordinates of the embedding space $\mathbb{R}^4$ (see [8, 11]), and as we consider only spherical spaces, $r$ is thus the radius of $S^3$, which is a constant assumed hereafter to be 1.

The explicit expression of the $T_{k;m_1,m_2}$ can be found in [1, 11] (see [8]). It is proportional to a Jacobi polynomial

$$T_{k;m_1,m_2}(X) = C_{k;m_1,m_2}(\cos \chi e^{i\theta})^\ell (\sin \chi e^{i\phi})^d P_d^{m_1,m_2}(\cos(2\chi)),$$

where we have written

$$\ell = m_1 + m_2, \quad m = m_2 - m_1, \quad d = k/2 - m_2.$$

The factor $C_{k;m_1,m_2}$ can be computed from normalization requirements (there is a factor $\sqrt{2}$ lacking in [8]) as

$$C_{k;m_1,m_2} = \frac{\sqrt{k + 1}}{2\pi^{3/2}} \frac{(k/2 + m_2)!(k/2 - m_2)!}{(k/2 + m_1)!(k/2 - m_1)!}.\tag{5}$$

Since $(B_2)$ is a basis, we have $V^k = \text{span}(T_{k;m_1,m_2})_{-k/2 \leq m_1,m_2 \leq k/2}$. It appears convenient to define the subvector spaces

$$V^k_{m_1} = \text{span}(T_{k;m_1,m_2})_{-k/2 \leq m_2 \leq k/2}.\tag{6}$$

We have

$$V^k = \bigoplus_{m_1 = -k/2}^{k/2} V^k_{m_1}.$$

We show now that the $T_{k;m_1,m_2}$ identify, up to a constant, the Wigner D-functions.

### 2.3. The Wigner D-functions

The Wigner D-functions are defined as functions on the Lie group $SO(3)$:

$$SO(3) \rightarrow \mathbb{C},
\quad g \mapsto D^j_{m'm}(g) \equiv \langle jm'|R_g|jm \rangle.\tag{7}$$

Here, $g$ is the rotation of $SO(3)$, and $R_g$ expresses its natural left action $f \mapsto R_g f$ on functions: $R_g f(x) \equiv f(g^{-1}x)$. The vectors $|jm\rangle, m \in [-j, j]$ form an orthonormal basis for $\mathcal{H}^j$, the $(2j + 1)$-dimensional irreducible representation of $SU(2)$. When $j$ is integer, this is an irreducible unitary representation of $SO(3)$, and the $|jm\rangle$ can be taken as the usual spherical harmonics $Y_{jm}$. This implies

$$R_g Y_{jm} = \sum_{m'} D^j_{m'm}(g^{-1}) Y_{jm'}.\tag{8}$$

(Care must be taken that one finds in the literature other definitions with indices exchanged; we adopt here the notation of [2].)

As manifolds, the 3-sphere $S^3$ and $SU(2)$ are identical. This allows us to identify any point of $S^3$ with an element of $SU(2)$ by the following relation:

$$S^3 \rightarrow SU(2),
\quad x = (\chi, \theta, \phi) \mapsto u_x \equiv \begin{pmatrix} \cos \chi e^{i\theta} & i \sin \chi e^{i\phi} \\ i \sin \chi e^{-i\phi} & \cos \chi e^{-i\theta} \end{pmatrix}.\tag{9}$$

(Note that one finds other phase choices for this identification in the literature.)

On the other hand, there is a group isomorphism between $SU(2)$ and $SO(3)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ refers to the multiplicative group $\{-1, 1\}$. Thus, each element $u$ of $SU(2)$ defines an $SO(3)$
rotation \( g_u \). In practice, a rotation of \( SO(3) \) is parametrized by its Euler angles \( \alpha, \beta, \gamma \). Taking into account the identification above, the correspondence takes the form

\[
SU(2) \quad \mapsto \quad SO(3) \quad \quad \quad (10)
\]

\[
u = (\chi, \theta, \phi) \quad \mapsto \quad g_u = (\alpha, \beta, \gamma). \quad (11)
\]

where

\[
\chi = \frac{\beta}{2} \quad \theta = \frac{\alpha + \gamma}{2} \quad \phi = \frac{\alpha - \gamma}{2}. \quad (12)
\]

This allows us to consider the Wigner D-functions as functions on \( SU(2) \) and, thus, on \( S^3 \). Their explicit expression (see for instance [2]) shows their identification with the previous functions \( T_{k,m_1,m_2} \):

\[
D^{k/2}_{m_1,m_2}(u) = \frac{2^{k+1}}{k+1} T_{k,m_1,m_2}(u), \quad (13)
\]

with \(-k/2 \leq m_1, m_2 \leq k/2\). This will allow us to use their very convenient properties with respect to rotations.

### 2.4. Change of basis

We have two bases for \( \mathcal{V}^k \). Calculations may be more convenient in one or the other. Most numerical codes use a development in the basis \((B_1)\). Here we will calculate the eigenmodes \( X \) in the basis \((B_2)\). A conversion requires the formulae for the change of basis, which result from the properties of the Wigner functions.

The expansion may be found in [5] (see also [1], which however uses different conventions):

\[
\mathcal{Y}_{k,l,m}(u) = i^{k+l-m} \sum_{m_1,m_2} (-1)^{m_2} \left( \begin{array}{c}
\frac{k}{2}, -m_1; \frac{k}{2}, m_2 \mid l, m \end{array} \right) T_{k,m_1,m_2}(u), \quad (14)
\]

where the \((j, m_1; j, m_1 \mid l, m)\) are the Clebsch–Gordan coefficients. We can invert this formula by multiplying it by another Clebsch–Gordan coefficient \((j, m_2; j, m_2 \mid l, m)\) and by summing on the two indices \(l\) and \(m\). The orthonormality of the Clebsch–Gordan coefficients then gives the relation (after renaming indices):

\[
T_{k,m_1,m_2}(u) = (-1)^{m_2} \sum_{l,m} i^{-k+l} \left( \begin{array}{c}
\frac{k}{2}, -m_1; \frac{k}{2}, m_2 \mid l, m \end{array} \right) \mathcal{Y}_{k,l,m}(u). \quad (15)
\]

The properties of the Clebsch–Gordan coefficients (non-zero only when \(-m_1 + m_2 = m\) with the previous notation) reduce these formulae, respectively, to

\[
\mathcal{Y}_{k,l,m}(u) = \sum_{m_1} i^{k+l+m_1+m_2} \left( \begin{array}{c}
\frac{k}{2}, -m_1; \frac{k}{2}, m_1 + m \mid l, m \end{array} \right) T_{k,m_1,m_1+m}(u) \quad (16)
\]

and

\[
T_{k,m_1,m_2}(u) = \sum_{l} \left( \begin{array}{c}
\frac{k}{2}, -m_1; \frac{k}{2}, m_2 \mid l, m \end{array} \right) \mathcal{Y}_{k|l,m_2-m_2}(u) \frac{i^{k+l+m_1+m_2}}{i^{k+l+m_1+m_2}}. \quad (17)
\]
3. SO(4) rotations

3.1. General case

The isometries of $S^3$ form the group SO(4), the rotation group of the embedding space $\mathbb{R}^4$. We denote such an isometry as $G : x \mapsto Gx$, and define its action on a function $f$ of $S^3$ as

$$R_G : f \mapsto R_G f ; \quad R_G f(x) \equiv f(G^{-1}x).$$

Using the previous identification between $S^3$ and $SU(2)$, it is very convenient to use the $SU(2) \times SU(2)$ representation of SO(4). Any rotation $G$ of SO(4) is represented (modulo a sign) as a pair $(u_l, u_r)$ of two $SU(2)$ matrices, respectively acting on the left and on the right on $u_x$, an element of $SU(2)$:

$$(u_l, u_r) : SU(2) \longrightarrow SU(2)
\quad u_x \longmapsto u_l \cdot u_x \cdot u_r = u_{Gx},$$

where all matrices belong to $SU(2)$. (The SO(4) matrices of the form $(u_l, u_r) = (u_l \cdot 1, u_r)$ form a specific SO(3) subgroup.)

The resolution of identity in $\mathcal{H}$ (or equivalently, the composition of two successive SO(3) rotations, with $R_{g_1} = R_{g_2} \circ R_{g_1}$), implies the well-known addition formula

$$D^j_{mn} (uv) = \sum_{m'} D^j_{m'n'} (u) D^j_{m'm} (v).$$

Using (13), this implies

$$Tk_{k,m}(xuv) = \sum_{m'} D^{k/2}_{m'm}(u)Tk_{k,m'}(xv).$$

Here, $xu$ is the point of $S^3$ corresponding to $u \in SU(2)$. Now we will use the same notation for a function on $S^3$ and on $SU(2)$: $f(xu) = f(u)$.

In these formulae, $uv$ refers to the product of two $SU(2)$ matrices. We are now able to compute the action of a rotation $G$ of SO(4) on the basis functions $Tk_{k,m}$ by using the decomposition (18) of $G$ and by using twice the addition formula (20) for $Tk_{k,m}$. Thus, we obtain the rotation properties of the basis $(B_2)$ of eigenfunctions under the rotation $G \approx (u_l, u_r)$:

$$R_GT_{k,m}(v) = \sum_{m'} D^{k/2}_{m'm}(u_l)D^{k/2}_{m'm}(u_r^{-1})Tk_{k,m'}(v).$$

This completely specifies the rotation properties of the basis $(B_2)$ under SO(4).

3.2. Single action rotations

A special case is a single left action rotation (hereafter SLAR), $G \equiv (u_l, u_r = 1_{SU(2)})$, where $u_l \in SU(2)$ defines an SO(3) rotation $g$. The previous formula reduces to

$$R_GT_{k,m}(v) = Tk_{k,m}(u_l^{-1}v)$$

$$= \sum_mD^{k/2}_{m'm}(u_l^{-1})Tk_{k,m'}(v).$$

We note that $G$ preserves the first index $M = m_1$. This means that each $\chi^{k,M}$ (see (6)) is invariant under $G$. This implies that the search for eigenfunctions invariant under a SLAR $G$
may be performed in each $\mathbb{V}^{k,M}$ separately. Moreover, since the invariance equation does not depend on $M = m_1$, the solution in one of them will give the solution for all of them.

From (22), the invariance condition reads

$$T_{k;M,m_2}(v) = \sum_m D_{m_1m}^{k/2}(u^{-1}_I) T_{k;Mm}(v).$$  \hspace{1cm} (23)

3.3. Invariant functions

The eigenfunctions of $X \equiv S^3/\Gamma$ are the eigenfunctions of $S^3$ which remain invariant under $\Gamma$. A necessary and sufficient condition is that they remain invariant under the generators of $\Gamma$, which are SLARs.

Thus, we search for the eigenfunctions invariant under an arbitrary SLAR $G$. For a given value $k$, let us call their space $\mathbb{V}^k_G \subset \mathbb{V}^k$. Since $G$ preserves the vector space $\mathbb{V}^{k,M}$, we have the decomposition

$$\mathbb{V}^k_G = \bigoplus_{M = -k/2}^{k/2} \mathbb{V}^{k,M}_G; \quad \mathbb{V}^{k,M} \equiv \mathbb{V}^{k,M}_G \cap \mathbb{V}^k_G.$$

The search for the $G$-invariant eigenmodes is equivalent to the search for the vector spaces $\mathbb{V}^{k,M}_G$, for the different values of $k, M$, in $\mathbb{V}^{k,M}$.

In $\mathbb{V}^{k,M}$, the expansion of a function $f$ on the canonical basis,

$$f = \sum_{m = -k/2}^{k/2} f_{k;Mm} T_{k;M,m},$$

describes it as the vector $F$ with the $k + 1$ components $F_m \equiv f_{k;Mm}$. Similarly, we write for simplicity the matrix $M_{mn} \equiv D_{m_1m}^{k/2}(u^{-1}_I)$. Then, equation (23) takes the very simple form

$$F_m = \sum_n M_{mn} F_n. $$  \hspace{1cm} (24)

This is an eigenvalue equation $MF = F$. Its solutions are the eigenvectors of the matrix $M$, with the eigenvalue 1.

We recall (see equation (7)) that the Wigner functions are the rotation matrix elements in the $(k + 1)$-dimensional representation $k/2$. Assuming $k = 2\ell$ even, the vector $F$ also represents a harmonic function on the 2-sphere $\psi_{V} \equiv \sum_m F_m Y_{\ell m} \in \mathbb{V}^{\ell}$. Thus, (24) becomes

$$\psi_{V} = (R_g)^{t}\psi_{V}$$

(the superscript $t$ holds for matrix transposition). This is the condition for the function $f_V$ to be invariant under the $SO(3)$ rotation $g^{-1}$. The solution of this problem is well known. Let us call $n$ and $2\pi/N$ the oriented (unit) axis and angle (assumed an integer divisor of $2\pi$) of the rotation $g$.

- **Diagonal case.** When $n$ is along Oz, the rotation is diagonalized: the $SU(2)$ matrix,

$u = \text{diag}(e^{i\pi/N}, e^{-i\pi/N})$, $[R_g]_{mn} = \delta_{mn} e^{-2i\pi m/N}$ and, using (4) and (13),

$$D_{m_1m}^{k/2}(u^{-1}) = \delta_{m_1m} (e^{-i\pi N}\text{e}^m + \text{e}^{-m} \text{P}_{k/2-m}^{m-m_2+m_2})$$

$$= e^{-2i\pi m/N} \delta_{m_1m_2}. $$  \hspace{1cm} (25)

The eigenspace corresponding to the eigenvalue 1 is thus span($e_{-J}$) $-J_N \leq m \leq J_N$, after defining $J_N \equiv [k/2N]$ ($[\cdot]$ means integer part), and where $e_j$ is the $i$th basis vector. In
other words, the non-zero components of an invariant vector have as index an integer multiple of \( N \). The dimension of this space is \( 2J_N + 1 \).

Coming back to \( SO(4) \), this result implies immediately that

\[
V_{k,M}^G = \text{span}(T_{k,M,N\mu}), \quad -J_N \leq \mu \leq J_N.
\]

- When \( n \) is not along \( Oz \), we may diagonalize \( R_g \) as \( g = \rho^{-1} D\rho \). The eigenvalue equation becomes

\[
fV = \rho^{-1} D\rho fV.
\]

This is the same eigenvalue equation as above, with the same solution, although not for \( V \) but for \( \rho V \). The eigenspace corresponding to the eigenvalue 1 is thus \( \text{span}(\rho^{-1} e_{N\mu}) \), \( -J_N \leq \mu \leq J_N \).

Coming back to \( SO(4) \), this result implies immediately that

\[
V_{k,M}^G = \text{span}(\rho^{-1} T_{k,M,N\mu}), \quad -J_N \leq \mu \leq J_N.
\]

## 4. Eigenmodes of spherical spaces

When we have two generators \( G_{\pm} \), the eigenspace is simply

\[
V_X^{k,M} = V_{G_+}^{k,M} \cap V_{G_-}^{k,M}.
\]

In any spherical space, we chose a frame where one of the generators of \( \Gamma \) is diagonalized, so that the invariance condition relative to this generator takes the simple form (28). Thus, the problem is reduced to the search for the invariance condition relative to the other generators.

### 4.1. Diagonal matrices

Applying equation (22) to a diagonal matrix as above, we obtain

\[
R_g T_{k,m_1,m_2}(v) = e^{-2\pi i m_2/N} T_{k,m_1,m_2}(v).
\]  

(26)

Let us assume that a generator of the holonomy group of a spherical space \( X \) is represented by a diagonal left action \( SU(2) \) matrix \( g \approx (u_1, 1) \) with \( u_1 \) of the diagonal form as given above. The invariance condition, under \( g \), of the basis functions for this space takes the form

\[
T_{k,m_1,m_2}(u_1) = e^{-2\pi i m_2/N} T_{k,m_1,m_2}(u_1).
\]

(27)

Its solution is

\[
m_2 = 0 \quad (\text{mod } N),
\]

(28)

or \( m_2 = N\mu \), where \( \mu \) varies in the range \([ -J_N, \ldots, J_N ]\).

Hereafter, all sums involving \( \mu, \nu \) are assumed to go in this range \((2J_N + 1 \text{ values})\). This is the selection rule for the eigenmodes invariant under \( g \).

### 4.2. A general procedure

The spherical spaces considered below all have two generators which are SLARs, \( G_{\pm} \equiv (\gamma_{\pm}, 1) \). Let us choose a frame where \( G_+ \) is diagonal. Then, form above, we can write

\[
V_{G}^{k,M} = \text{span}(T_{k,M,N\mu}), \quad -J_N \leq \mu \leq J_N.
\]

In other words, all eigenfunctions \( T_{k,M,m} \) with \( m = N\mu \) are \( G_+ \)-invariant.
Since there is no simple way to calculate the intersection
\[ \mathcal{V}^k, M_X = \mathcal{V}^k, M_{G+} \cap \mathcal{V}^k, M_{G-}, \]
we propose a practical procedure.

Let us expand a \( G_+ \)-invariant function in the basis \((B_2)\) as
\[
f = \sum_m \sum_{\mu} f_{k:m,N\mu} T_{k:m,N\mu}, \quad \mu = -J_N, \ldots, J_N. \tag{29}
\]
The invariance condition under the rotation \( \gamma_+ \), for such a function, takes the form
\[
R_{G_+} f = f : \sum_m \sum_{\mu} f_{k:m,N\mu} R_{G_+} T_{k:m,N\mu}(x) = \sum_m \sum_{\mu} f_{k:m,N\mu} T_{k:m,N\mu}(x). \tag{29}
\]
With the transformation law \((22)\), this becomes
\[
f_{k:m,N\mu} = \sum_{\nu} f_{k:m,N\nu} D^{k/2}_{N\nu,N\mu} ((\gamma_-)^{-1}). \tag{30}
\]
This equation is independent of index \( m \).

This implies that the eigenspace for \( k \) splits as the direct sum
\[
\mathcal{V}^k_X = \bigoplus_m \mathcal{V}^k, m_X.
\]
The degeneracy of the eigenvalue \( \lambda_k \) is the dimension of \( \mathcal{V}^k_X \). This implies that
\[
\text{dim}(\mathcal{V}^k_X) = (k + 1) \text{dim}(\mathcal{V}^k, m_X).
\]
Thus, the solution for \( m \) does not depend on \( m \), and we can write \( f_{k:m,n} = f_{k:m_0,n} \), where \( m_0 \) is some index in \([-k/2; k/2]\]. Equation \((30)\) is still an eigenvalue equation, but in a vector space of the restricted dimension \( 2J_N + 1 \): let us define the matrix \( \mathcal{M}_k \), of order \( 2J_N + 1 \), through its coefficients
\[
[\mathcal{M}_k]_{\mu,\nu} \equiv D^{k/2}_{N\nu,N\mu} ((\gamma_-)^{-1}), \quad \mu, \nu = -J_N, \ldots, J_N. \tag{31}
\]
Note that this matrix does not depend on the index \( M \). Let us also define the \((2J_N + 1)\)-vector \( \mathcal{F} \) through its components
\[
[\mathcal{F}]_{\mu} \equiv f_{k:M,N\mu}.
\]
Equation \((30)\) could thus be written in matrix form as
\[
\mathcal{M}_k \mathcal{F} = \mathcal{F}. \tag{31}
\]
This eigenvalue equation may be easily solved, for instance with Mathematica. The results are given in the appendix.

Having found the solution under the form of the vectors \( \mathcal{F}_s \), with \( 1 \leq s \leq \sigma \), with \( \sigma = 1/(2J_N + 1) \) times the degeneracy of the eigenvalue (\( \sigma \) is found as a result of the eigenvalue problem; the value coincides with that given by \([6]\)), we have finally a basis
\(\mathcal{Y}_{k,M,s} = \sum_{\mu=-J_N}^{J_N} (\mathcal{F}_\mu)_{\mu} T_{k,M,N}\mu\).

(32)

5. Application to the spherical spaces

For each space, the steps are the following:

- We write the two generators in the SU(2) \(\times\) SU(2) forms. Since both are SLARs, this corresponds to two SU(2) matrices \(\gamma_{\pm}\). (We thank J Weeks for providing generators in suitable form.)
- We diagonalize the matrix \(\gamma_+\).
- We deduce the value \(J_N\).
- We solve the eigenvalue equation (31) in the vector space of dimension \(2J_N + 1\). This gives the vector(s) \(\mathcal{F}\) by their components.
- Then equation (32) gives the eigenmodes of \(X\).

5.1. Dodecahedral space

The two SU(2) generators,

\[
\begin{pmatrix}
c + iC \\
\pm i/2 \\
c - iC
\end{pmatrix}; \quad c \equiv \cos(\pi/5), \quad C \equiv \cos(2\pi/5),
\]

become, after diagonalization,

\[
\gamma_+ = \begin{pmatrix}
e^{i\pi/5} & 0 \\
0 & e^{-i\pi/5}
\end{pmatrix}, \\
\gamma_- = \begin{pmatrix}
\cos(\pi/5) + i\sin(\pi/5)/\sqrt{5} & 2i\sin(\pi/5)/\sqrt{5} \\
2i\sin(\pi/5)/\sqrt{5} & \cos(\pi/5) - i\sin(\pi/5)/\sqrt{5}
\end{pmatrix}.
\]

(34)

We have \(N = 5, J_N = [k/10]\).

Equation (32) gives the eigenmodes. We have checked that they correspond to those found by [9].

As we mentioned, it may be more convenient to express these modes in the more widely used basis \((B_{11})\). The solution of our problem in this basis is, for the dodecahedral space,

\[
f_k(x) = \sum_{l,m,m'} 2^{m-k-l} (-1)^m f_{k,m'} \left(\frac{k}{2}, m' - m; \frac{k}{2}, m' | l, m\right) \mathcal{Y}_{klm},
\]

(35)

where the index \(m'\) satisfies the condition (28) for the dodecahedral space, \(l\) vary in \([0, k]\) and \(m\) vary in \([-l, l]\).

5.2. Other multi-connected spherical spaces

Besides lens and prism spaces, three other multi-connected spherical spaces remain. In each space, holonomy groups are generated by two left-action SO(4) rotations. We follow the same procedure.
We give in the table below a list of these spaces with their two generators, under their $SU(2)$ form.

| Tetrahedron | Diagonal form | Octahedron | Icosahedron |
|-------------|--------------|------------|-------------|
| \( \frac{1}{2} \begin{pmatrix} 1-i & -1-i \\ 1+i & 1+i \end{pmatrix} \) | \( e^{\pi/3} \begin{pmatrix} 0 & 0 \\ -1-i & 1-i \end{pmatrix} \) | \( e^{-i\pi/2} \begin{pmatrix} 0 & 0 \\ e^{-i\pi/2} & 0 \end{pmatrix} \) | \( e^{i\pi/5} \begin{pmatrix} 0 & 0 \\ e^{i\pi/5} & 0 \end{pmatrix} \) |
| | \( \frac{1}{2} \begin{pmatrix} \sqrt{3}+i/2 & 1-i/2 \\ \sqrt{3}-i/2 & 1+i/2 \end{pmatrix} \) | \( e^{i\pi/2} \begin{pmatrix} 0 & 0 \\ e^{-i\pi/2} & 0 \end{pmatrix} \) | \( \begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix} \) |
| Diagonal form | \( \frac{1}{2} \begin{pmatrix} \sqrt{3}+i & 1-i/2 \\ \sqrt{3}-i & 1+i/2 \end{pmatrix} \) | \( e^{-i\pi/2} \begin{pmatrix} 0 & 0 \\ e^{i\pi/2} & 0 \end{pmatrix} \) | \( e^{i\pi/5} \begin{pmatrix} 0 & 0 \\ e^{i\pi/5} & 0 \end{pmatrix} \) |
| Icosahedron | \( e^{i\pi/3} \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \) | \( e^{i\pi/2} \begin{pmatrix} 0 & 0 \\ e^{-i\pi/2} & 0 \end{pmatrix} \) | \( \begin{pmatrix} c+i & 1/2-i/s \\ -1/2-ic & c-ic \end{pmatrix} \) |
| | \( \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \) | \( e^{i\pi/2} \begin{pmatrix} 0 & 0 \\ e^{-i\pi/2} & 0 \end{pmatrix} \) | \( \begin{pmatrix} c+i & 1/2-i/s \\ -1/2-ic & c-ic \end{pmatrix} \) |

with \( c := \cos(\pi/5) \) and \( s := \sin(\pi/5)/\sqrt{5} \)

Computations give the tables in the appendix, where we present the first modes for the different spherical spaces. More modes can be obtained by sending us an email at cailleri@discovery.saclay.cea.fr. Note that the modes for an icosahedron are the same as those of the tetrahedron, as expected; they are however given here in a different frame.

### 6. Conclusion

The introduction of the basis \((B_2)\) of Wigner functions provided the explicit behaviour of the modes of \( S^3 \), under the rotations of \( SO(4) \). This allowed us to write an invariance condition of these modes under any rotation, and under the holonomy group \( \Gamma \) of a spherical space \( S^3/\Gamma \). The solutions of the corresponding equations give the vector spaces of the eigenmodes of \( S^3/\Gamma \). These modes are calculated explicitly, for the first values of \( k \), for the spherical spaces, in the basis \((B_2)\). We give the transformation relations to express them in the more usual basis \((B_1)\) of hyperspherical harmonics.

This allows us to calculate the power angular spectrum of the CMB fluctuations, as predicted by the cosmological model with the Poincaré dodecahedral space, up to arbitrary multipoles \( k \). This work is in progress. It will allow us to check if the predictions of [13] hold up to higher orders. In any case, the confrontation of the predicted power spectrum with observations may not offer enough discriminating power. Thus, we intend to calculate the cross correlations predicted from this model. They are zero for Gaussian isotropic fluctuations, and thus allow more specific tests of the cosmic topology. This work is also in progress.

The present results allow us to extend these calculations to the remaining spherical spaces, a work also in progress. Hopefully, this will provide a definitive test of the case of a multi-connected space with a positive curvature, at least for a range of scales comparable with that of the last scattering surface.

### Appendix

#### Table of eigenfunctions for the dodecahedral space

Eigenmodes of dodecahedral space correspond to the basis of functions \((f_k(x))_{m \in \mathbb{Z}/k/2} \equiv \mathbb{Z}/k/2\) where \( f_k = \sum_{m,m'} f_{k,m} T_{k,m,m'} \). In this table are indicated the allowed \( k \) (those where \( D_k \) admits 1 as an eigenvalue) and the corresponding \( f_{k,m'} \) for \( m' = 0 \pmod{5} \) (for other values of \( m' \), \( f_{k,m'} = 0 \)).
Table of eigenfunctions for the tetrahedral space

For this space, we present modes in the form: $f_{k,m}$, with the notation used above, where $m = 0 \pmod{3}$. For $k = 12$, we can observe that modes are degenerate and we thus have two series of $f_{k,m}$.

$k = 12$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 12   | 0.529 150 262i   | 0.663 324 958i   | 0.529 150 262i   |

$k = 20$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 20   | -0.315 805 8475i | 0.578 273 291i   | 0.362 950 869i   |
|      | -0.315 805 8475i |                  | 0.578 273 291i   |

$k = 24$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 24   | -0.486 949 689   | 0.302 522 7264i  | 0.585 422 924i   |
|      | -0.486 949 689   |                  | 0.302 522 7264i  |

$k = 30$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 30   | -0.282 984 0984  | 0.391 305 507i   | -0.516 526 862i  |
|      |                  | -0.391 305 507i  | 0.282 984 0984   |

$k = 32$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 32   | -0.329 930 901i  | 0.425 449 096i   | -0.331 083 071i  |
|      |                  |                  | 0.448 380 790i   |

Table of eigenfunctions for the octahedral space

For this space, we present modes in the form: $f_{k,m}$, with the notation used above, where $m = 0 \pmod{2}$.

$k = 2$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 2    | 0.333 333 333    | -0.333 333 333i  | 0.745 355 992    |
|      |                  |                  |                  |

$k = 8$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 8    | 0.608 580 619    |                  | -0.360 041 149   |
|      |                  |                  | -0.360 041 149i  |

$k = 12$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 12   | -0.003 233 321 62 + 0.003 233 321 62i | 0.738 332 205 | 0.266 691 7602 + 0.266 691 7602i |
|      |                  |                  | 0.282 869 3156i  |
|      | -0.341 013 640 + 0.341 013 640i  | 0.583 280 119 + 0.354 280 119i | -0.086 144 655i   |
|      |                  |                  | -0.353 087 976i  |
|      | -0.092 925 9330  - 0.092 925 9330i |                  | 0.689 157 164   |

$k = 14$

| $k$  | $f_1$            | $f_2$            | $f_3$            |
|------|------------------|------------------|------------------|
| 14   | 0.491 351 820i   |                  |                  |
|      |                  | -0.188 852 5745 + 0.188 852 5745i |
|      | 0.611 952 283    |                  | 0.188 852 5745 + 0.188 852 5745i |
|      | -0.491 351 820i  |                  |                  |
Table of eigenfunctions for the icosahedral space

For this space, we present modes in the form: $f_{k,m}$, with the notation used above, where $m = 0 \pmod{5}$.

| $k$  | $f_1$                                             | $f_2$                                             |
|------|---------------------------------------------------|---------------------------------------------------|
| 4    | $-0.612372435$                                    | $0.500000000$                                     |
|      | $-0.612372435$                                    |                                                   |
| 8    | $-0.379649579$                                    | $-0.1447862189$                                  |
|      | $-0.379649579$                                    | $-0.818416944$                                   |
|      | $-0.530330085$                                    | $-0.493006648$                                   |
|      | $-0.530330085$                                    | $0.3118047822$                                   |
|      |                                                   | $0.3118047822$                                   |
| 10   | $0.353553390$                                      | $-0.612372435$                                   |
|      | $0.353553390$                                      |                                                   |
| 12   | $0.507752400$                                      | $-0.125000000$                                  |
|      | $0.342326598$                                      | $0.342326598$                                   |
|      | $0.1230304012$                                    | $0.612372435$                                   |
|      | $0.1230304012$                                    |                                                   |
| 20   | $-0.2115446757$                                   | $0.3004753102$                                  |
|      | $0.1486544684$                                    | $-0.3633777752i$                                |
|      | $0.1798126817$                                    | $0.655649414$                                   |
| 24   | $-0.0622550053$                                   | $-0.2352778614$                                  |
|      | $-0.3529784282$                                   | $-0.2352778614$                                  |
|      | $0.6406372434$                                    | $-0.1437372075$                                 |
| 30   | $0.02978235622 + 0.1557458776i$                   | $0.2506842548 + 0.0642750991i$                  |
|      | $-0.2564612869 + 0.3101497151i$                   | $0.0002877519 + 0.0006113259i$                  |
|      | $0.560360227$                                      | $0.2128816958$                                  |
|      | $0.2718985718 + 0.3288186907i$                    | $-0.4522652865i$                                |

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