The Perturbative Onset of Multiparticle Production in Weak Interactions*

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Abstract

We use perturbation theory to estimate the energy scale beyond which multiparticle final states become a dominant feature of high energy weak interactions. Using estimates from a weak parton model and comparing two, three and four body final states we deduce that multiparticle states become important at energy scales in the range $10^7 - 10^9$ GeV.

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1 Introduction

There have recently been numerous conjectures regarding spectacular multi-TeV electroweak phenomena, characterized by the production of many electroweak gauge and Higgs bosons, which may be accessible to the next generation of colliders. Though such phenomena were initially studied in the context of topologically induced baryon plus lepton number violating interactions in the Standard Model [1], it was subsequently realized that they are a general feature of large order tree-level processes, where dramatic enhancements in production cross sections may be attributed to phase space and combinatorial factors [2].

Yet another proposal for multiple weak gauge boson production was put forward recently by Ringwald et al. [3, 4] under the name of geometrical flavour interactions. They propose an analogy between multiparticle production in Quantum Chromodynamics (QCD), which occurs above energy scales of a few GeV, and possible multiparticle production in weak interactions above the TeV scale, where weak gauge boson masses may be neglected. More precisely, they relate QCD to a prototypical confined phase of weak SU(2) and, using scaling arguments, extrapolate the analogy to the Higgs phase. As a consequence they allege that above parton-parton center of mass energies of 2 to 20 TeV, the non-perturbative production of \(O(\alpha^{-1}_w) \approx 30\) electroweak bosons could occur with cross sections of \(O(1 \text{ nb} - 10 \mu \text{b})\). Irrespective of the appropriateness of their scaling arguments, Ringwald et al. qualitatively suggest that, as far as weak interactions are concerned, colliding quarks and leptons may be thought of as being surrounded by clouds of virtual weak bosons which eventually result in “black disc” cross sections, and hence the name of geometrical flavour interactions.

We remark that Ringwald et al.’s qualitative view of high energy weak interactions is reminiscent of the QCD parton model in which an energetic proton may be viewed as a collection of quarks and gluons which form a
cloud of transverse size $O(\Lambda_{\text{QCD}}^{-1})$. In contrast to the approach of Refs. \[3, 4\], we investigate in this paper the possibility of high energy weak multiparticle production in the context of perturbation theory using techniques similar to those used for multiparticle production in QCD. We take two successively refined viewpoints. First, we adopt a weak parton model in which weak gauge bosons assume the role played by gluons in QCD and look for the failure of the lowest order approximations to the corresponding weak parton distribution functions which we interpret as signalling the increased importance of states with many gauge bosons. Second, we perform exact calculations of selected $2 \to 2$, $2 \to 3$, and $2 \to 4$ weak processes to investigate particle production at mass scales inaccessible to the parton model. For illustrative purposes we concentrate on contributions to the $\nu_e \overline{\nu}_\mu$ total cross section, $\sigma^{\nu_e \overline{\nu}_\mu}_{\text{total}}$, chosen because it is naively free from QCD and QED complications.

Our goal is to quantitatively explore the energy range in which multiparticle final states become a dominant feature of weak interactions. Since we limit ourselves to the first few orders of perturbation theory, we will only consider final states with up to three or four particles (as opposed to the production of $O(\alpha_w^{-1})$ particles discussed in Refs. \[3, 4\]). Furthermore, we are interested in the perturbative decomposition of $\sigma^{\nu_e \overline{\nu}_\mu}_{\text{total}}$ into its various multiparticle channels. The more global question of the perturbative growth of total (inclusive) weak cross sections with $\sqrt{s}$ has been considered recently by Pindor, Rączka and Wetterich in Ref. \[5\] in which they solve an integral equation suggested by Fadin, Kuraev and Lipatov\[6, 7\]. They find slowly growing weak total cross sections (i.e., $O(100 \text{ pb})$ for $\sqrt{s} \lesssim 10^{15}$ GeV) but, due to the nature of their methods, they cannot decompose their results into contributions with definite numbers of particles in the final state.

Using a variety of techniques we will argue that weak multiparticle production becomes a dominant characteristic of weak interactions at energy scales in the range $10^7 - 10^9$ GeV which is firmly beyond the reach of any planned accelerator. Kane and Scanio \[8\] have pointed out that cross sections
for exclusive processes such as $e^+e^- \rightarrow \mu^+\mu^-$ can be surpassed by higher order processes (such as $e^+e^- \rightarrow \bar{\nu}_e\nu_e\mu^+\mu^-$ through $W^+W^-$ fusion) at center of mass energies of $O(10^3 \text{ GeV})$. However this dominance is only relative to a rapidly falling $s-$channel Born contribution. In contrast, lowest order $\nu_e\bar{\nu}_\mu$ scattering proceeds through $t-$channel exchanges which are analogous to dominant $t-$channel processes in QCD; at higher orders multiparticle production arises from very forward (i.e., small angle) inelastic reactions. We shall not consider the possibility of a strongly interacting Higgs sector which could, by itself, dominate the characteristics of high energy weak interactions.

The paper is organized as follows. In section 2 we review the QCD parton model approach for describing perturbative contributions to the antiproton-proton total cross section $\sigma_{\bar{p}p}^{\text{total}}$. In section 3 we consider the lowest order distribution functions of the weak parton model. We use them to determine contributions to $\sigma_{\text{total}}^{\nu_e\bar{\nu}_\mu}$ through gauge boson fusion and investigate the energy scale at they become unreliable. In section 4 we present results of calculations beyond the weak parton model and argue that the dominance of multiparticle states at energy scales of $10^7 - 10^9 \text{ GeV}$ is more than an artifact of large logarithms in finite order calculations.

2 QCD Parton Model and $\sigma_{\text{total}}^{\bar{p}p}$

Let us abstract those aspects of the perturbative QCD parton model of relevance to high energy weak interactions. Underlying this ambition is the operating premise, which is not without controversy, that multiparticle production and the growth of $\sigma_{\text{total}}^{\bar{p}p}$ with center of mass energy may be deduced from the QCD parton model. We briefly review the support and criticisms of the QCD parton model view of $\sigma_{\text{total}}^{\bar{p}p}$ to set the stage for the weak parton model in the next section.
Though an old idea\textsuperscript{10}, perturbative contributions to rising hadronic cross sections gained prominence with the UA1 observation of minijets\textsuperscript{11} and have subsequently led to many theoretical studies of the perturbative component of $\sigma_{\text{total}}$\textsuperscript{12,13}. However, realistically, the detailed successes of a perturbative parton model at momentum transfers at $O(\Lambda_{\text{QCD}})$ must be viewed with a degree of skepticism since there is little \textit{a priori} justification for a perturbative treatment in that regime. Indeed, descriptions of $\sigma_{\text{total}}$ in terms of the QCD pomeron have also been shown to accommodate existing data\textsuperscript{14}.

As an approximation to $\sigma_{\text{total}}$ in the perturbative QCD parton model one may write\textsuperscript{12}

$$
\sigma_{\text{total}}(s) \simeq \sum_{i,j} \int_{p_{T}^{2} > 1 \text{ GeV}^2} d^2 p_T \int dx_1 dx_2 f_{i/p}(x_1, Q^2) f_{j/\bar{p}}(x_2, Q^2) \frac{d\hat{\sigma}_{ij}}{dp_{T}^2}(Q^2)
$$

(1)

where $Q^2 = x_1 x_2 s$, $f_{i/p}(x)$ is the number density of parton species $i$ carrying a proton momentum fraction $x$ (i.e., a parton distribution function) and $\hat{\sigma}_{ij}(Q^2)$ is the subprocess cross section for colliding partons of species $i$ and $j$. (It is actually more appropriate to evaluate the distribution functions at a scale $O(p_{T}^2)$ but such details will not concern us.) The sum extends over all possible quark and gluon species. The \textit{ad hoc} infrared cutoff $p_{T}^2 \gtrsim 1 \text{ GeV}^2$ avoids the non-perturbative growth of $\alpha_s(p_{T}^2 \to \Lambda_{\text{QCD}}^2)$.

In the parton framework the perturbative behaviour of $\sigma_{\text{total}}(s)$ is accommodated as follows. As the center of mass energy $\sqrt{s}$ increases, successively smaller $x$ values become kinematically accessible. For example, the small-$x$ behavior of the gluon distribution function $f_{g/p}(x, Q^2)$ is thought to behave as $\simeq x^\delta$ (where $\delta < 0$) so that gluon-gluon collisions give large contributions to the total cross section\textsuperscript{15}. The cooperation of these kinematic and dynamic effects may be illuminated by a toy calculation in which one considers only gluon-gluon collisions in Eq. 1 and takes $f_{g/p}(x) \simeq 1/x$ and $\hat{\sigma}_{gg} \simeq$ constant, which gives $\sigma_{\text{tot}} \simeq \ln^2 s$. The enhanced small-$x$ behavior of parton densities
inside hadrons suggests the picture of a high energy proton as a cloud of small-$x$ partons. Indeed, using more realistic distribution functions and hard cross sections one can reproduce the Tevatron result for $\sigma_{\text{total}}^{pp} \simeq 70 - 80$ mb at $\sqrt{s} = 1.8$ TeV\[16]. The ability to mimic this result using perturbative techniques is based upon the assumption that the total cross section is dominated by semi-hard collisions; the infrared cut-off is then interpreted as the scale at which semi-hard processes become relevant. In the following section we exploit this apparent success of perturbative QCD and apply similar techniques to describe high energy weak interactions where weak gauge bosons assume the role played by gluons in QCD.

3 The Weak Parton Model

3.1 Factorization and Distribution Functions

In QED there are already examples of rising cross sections which may be understood in the context of a parton picture. Among them is the two-photon production of a fermion pair ($f\bar{f}$) through $e^+e^- \rightarrow e^+e^-f\bar{f}$ which has a lowest order cross section\[17]\

$$\sigma(e^+e^- \rightarrow e^+e^-f\bar{f}) = \frac{28\alpha^4}{27\pi m_f^2} \ln^2 \left( \frac{s}{m_e^2} \right) \ln \left( \frac{s}{m_f^2} \right),$$  \hspace{1cm} (2)

which may be obtained by convoluting the probability of finding quasi-real photons inside electrons with the subprocess cross section for fermion pair production from two on-shell photons. This is the Weiszacker-Williams effective photon approximation \[18], which is in good agreement with exact calculations\[19]. The parton model approach to QED has also been successfully applied to describe other higher order electromagnetic processes such as corrections to the Z boson line shape\[20]

The enhanced small-$x$ behaviour of the gluon distribution function in QCD and the large QED logarithms in Eq. \[2] have counterparts which we
wish to exploit by considering a parton model for the weak interactions. By analogy with QCD, we consider writing contributions to the $\sigma^{\nu_e \bar{\nu}_\mu}_{\text{total}}$ in the factorized form

$$\sigma^{\nu_e \bar{\nu}_\mu}_{\text{total}}(s) = \sum_{i,j} \int dx_1 \, dx_2 \, f_{i/\nu_e}(x_1, Q^2) \, f_{j/\bar{\nu}_\mu}(x_2, Q^2) \, \hat{\sigma}_{ij}(Q^2)$$

where $Q^2 = x_1 x_2 s$. In principle, the sum in Eq. 3 extends over all weak parton species found “inside” neutrinos (e.g., $W$, $Z$, $\nu$, $\cdots$). The weak parton distribution functions and the on-shell parton-parton subprocess cross sections have interpretations similar to those in QCD. At the level of this discussion we emphasize that the factorization of $\sigma^{\nu_e \bar{\nu}_\mu}_{\text{total}}$ in Eq. 3 is performed strictly by analogy with QCD and as such is an assumption without rigorous justification[22]. As in QCD, the scale at which the distribution functions are evaluated is not actually $x_1 x_2 s$ but rather is determined by the subprocess under consideration.

In contrast to the QCD factorization, the integral in Eq. 3 does not have an arbitrary infrared cut-off imposed by the dynamics of the theory. Instead, there are implicit kinematic constraints of the form

$$x_1 x_2 \geq \frac{(m_i + m_j)^2}{s},$$

as required by the assumption that the weak partons participating in the hard interaction are on-shell with masses $m_i$ and $m_j$. In principle, similar constraints are present in QCD but are usually irrelevant since they are superseded by the more restrictive constraint on $p_T^2$ associated the onset of non-perturbative phenomena.

The constraints of Eq. 4 concern us for the following reasons. Drawing from our experience with QCD and QED, we anticipate that the small-$x$ enhancement of the distribution functions of weak gauge bosons is a promising place to look for cross section enhancements. However due to Eq. 4 the weak parton model can only address gauge boson subprocesses above energy
scales of $O(M_W)$. It is uncertain beforehand whether the bulk of the multiparticle production will occur above or below this scale. While multiple gauge boson production certainly requires energies of $O(M_W)$, the production of single gauge bosons or light quarks and leptons can occur below this scale. For weak interactions we can, unlike QCD, always perform an exact perturbative calculation (and we do) to check the sensibility of our assumptions. However we initially follow the more intuitive route of the weak parton model to elucidate the operative mechanisms.

The weak parton distribution functions in Eq. 3 obey coupled evolution equations analogous to the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi\[23\] equations in QCD and the Lipatov equations in QED\[24\]. Due to the relatively small coupling constants involved, the evolution equations for QED and weak interactions can be solved iteratively in terms of parton splitting functions (see Appendix A) starting from well defined input distributions (e.g., $f_{\frac{\nu}{\nu}}(x, Q^2 = 0) = \delta(1 - x)$ ). In contrast, the strong QCD coupling constant forbids such an expansion in most regions of interest and one must solve the evolution equations numerically starting from input distributions measured, for example, in deep inelastic scattering experiments\[15\].

Suppose we wish to find the distribution function of $W^+$ bosons with helicity $\lambda$, carrying a momentum fraction $x$, inside an electron neutrino, $f_{W^+_{\lambda}/\nu_e}(x, Q^2)$. The lowest order expansion for this distribution function is

$$f_{W^+_{\lambda}/\nu_e}(x, Q^2) = P_{W^+_{\lambda}/\nu_e}(x, Q^2)$$

where $P_{W^+_{\lambda}/\nu_e}(x, Q^2)$ is the splitting function which embodies the details of an elementary $W^+\nu_e\nu_e^{-}$ vertex (see Appendix A). Keeping only this first term of the expansion is known as the effective vector boson approximation (EVBA)$^{[25]}$ which is the weak interaction analog of the effective photon approximation. Weak splitting functions have been derived in many different contexts under various assumptions$^{[26, 27, 28, 29]}$. Different authors agree for terms proportional to $\ln(Q^2/M_W^2)$, the so-called “leading-log” contribu-
tions, but usually not further. Additional terms in the splitting functions can
depend on the context of the derivation which can lead to process dependence
and gauge dependence. If the EVBA is used in the context of, for example,
Higgs production where \( Q^2 \gg M_W^2 \), the “leading-log” term in the splitting
function is, in fact, the most important. However, our calculations require
the use of more refined splitting functions, listed in Appendix A, which are
slight variations of those of Johnson et al.\(^{[28]}\).

3.2 Naive Onset of Multiple Gauge Boson Emission

The effective number of small-\( x \) partons associated with a high energy
proton is closely related to the strength of the QCD coupling constant and the
inability to solve the QCD evolution equations in a iterative manner starting
from first principles. Likewise, the number of weak partons associated with
a high energy fermion can suggest the suitability of keeping only the lowest
order approximations to the weak parton distribution functions. Turning
this argument around, the failure of the lowest order approximation to the
weak parton distribution functions suggests an energy scale above which
multipartile processes become important. Neglecting, for the moment, the
separate issue of restrictions which govern the feasibility of factorizing weak
cross sections, it is amusing to consider two qualitative arguments leading to
an estimate of such an energy scale.

Consider a neutrino entering a reaction characterized by a center of mass
energy \( \sqrt{s} \). If, for example, the hard subprocess in the reaction involves
the emission of a \( Z \) boson from the neutrino, the integral of the \( Z \) boson
distribution function over all momentum fractions roughly corresponds to
the number of \( Z \) bosons, \( N_{Z/\nu}(s) \), “inside” the neutrino

\[
N_{Z/\nu}(s) = \int^{1}_{M_{Z}/s} dx P_{Z/\nu}(x, s).
\] (6)
The lower limit of integration accounts for the Z boson mass and it is assumed that $\sqrt{s}$ is large enough to accommodate real Z boson emission. The precise kinematic limits are not of critical importance for our argument. In Fig. 1 we plot $N_{Z/\nu}(s)$ which is summed over all boson polarizations although the transverse polarizations dominate the sum. It is relatively straightforward to show (see Appendix A) that as $s \to \infty$,

$$N_{Z/\nu}(s) \simeq \frac{\alpha_w}{8\pi \cos^2 \theta_w} \ln^2(s/M_Z^2),$$

which becomes of order unity at $\sqrt{s} \simeq 10^8$ GeV. We interpret this simple estimate as a scale above which the assumption of single Z boson emission is inappropriate; higher order corrections representing the possibility of multiple emissions should then be considered.

Had we used the “leading-log” approximation to the Z boson distribution function, we would have found

$$N_{Z/\nu}(s) \simeq \frac{\alpha_w}{4\pi \cos^2 \theta_w} \ln^2(s/M_Z^2),$$

which is of order unity at $\sqrt{s} \simeq 10^6$ GeV. This lower scale reflects the well known shortcoming of the “leading-log” approximation for describing small-$x$ fractions within the EVBA\[28\]. For most applications discussed previously in the literature this small-$x$ region is irrelevant but for us it is not.

A second technique to estimate the onset of multiple gauge boson emission involves a consideration of the iterative solution to the evolution equations governing the parton distribution functions. In QCD this approach forces one to iterate to all orders because of the large coupling constant and the singular nature of splitting functions near the endpoints $x = 0$ and $x = 1$. In the weak parton model the gauge boson masses avoid splitting function singularities and the small coupling suggests the validity of a series expansion. At asymptotically high energies the weak parton model should behave more like QCD in the sense that the gauge boson masses can be neglected. A crude
measure of when this occurs is to ask when the second order term in the weak parton model expansion of the distribution function becomes comparable to the first order term.

The second order contribution to the $Z$ distribution function in terms of first order splitting functions, depicted in Fig. 2, may be written as

$$
\int_{x}^{1-M_{Z}^{2}/s} dy \frac{y}{P_{\nu/\bar{\nu}}(y, s)} P_{Z/\nu}(x/y, ys)
$$

where $P_{\nu/\bar{\nu}}(y, s) = P_{Z/\nu}(1 - y, s)$ corresponds to the emission of a $Z$ boson by a neutrino. It is convenient to consider the ratio of the second order contribution to the first order (splitting function) contribution

$$
R(x, s) = \frac{\int_{x}^{1-M_{Z}^{2}/s} dy \frac{y}{P_{\nu/\bar{\nu}}(y, s)} P_{Z/\nu}(x/y, ys)}{P_{Z/\nu}(x, s)}.
$$

The quantity $R(x, s)$ is insensitive to $x$ for small $x$, where the distribution functions are largest. The plot of $R(x = 10M_{Z}^{2}/s, s)$ in Fig. 3 suggests that the second order correction becomes comparable to the first order term at $\sqrt{s} \approx 10^{8}$ GeV.

It is pleasing, though perhaps not surprising, to find agreement between the above two techniques which suggest the onset of multiple gauge boson emission. Nevertheless, these arguments must not be taken too literally since we have ignored the subtleties of whether weak factorization sanctions such calculations. Indeed, in our estimation of $\sigma_{\nu e \bar{\nu} \mu}^{\nu e \bar{\nu} \mu}$ in the next section, we will encounter a context in which we must be considerate of factorization concerns.

### 3.3 Parton Model Contributions to $\sigma_{\nu e \bar{\nu} \mu}^{\nu e \bar{\nu} \mu}$

The elastic and inelastic Born cross sections for $\nu_{e} \bar{\nu}_{\mu}$ scattering at center
of mass energy $\sqrt{s}$ are given by
\[ \sigma_{\text{elas}}^{\nu_e \bar{\nu}_\mu}(s) = \frac{\pi \alpha_w^2}{4 \cos^4 \theta_w M_Z^2} \left[ 1 + 2z - 2z(1 + z) \log \left( \frac{1 + z}{z} \right) \right] \] (11)
\[ \sigma_{\text{inel}}^{\nu_e \bar{\nu}_\mu}(s) = \frac{\pi \alpha_w^2}{M_W^2} \left[ 1 + 2w - 2w(1 + w) \log \left( \frac{1 + w}{w} \right) \right] \] (12)
where $w = M_W^2 / s$ and $z = M_Z^2 / s$. For $\alpha_w^{-1} \simeq 30$, $M_Z = 91.17$ GeV and $M_W = 80$ GeV these expressions for $\sigma_{\text{elas}}^{\nu_e \bar{\nu}_\mu}(s)$ and $\sigma_{\text{inel}}^{\nu_e \bar{\nu}_\mu}(s)$ respectively asymptote to $\simeq 70$ pb and $\simeq 210$ pb for $\sqrt{s} \simeq 1$ TeV.

In order to estimate the energy at which $\sigma^{\nu_e \bar{\nu}_\mu}_{\text{total}}$ may begin to receive significant contributions from multiparticle production, consider the gauge invariant set of diagrams of Fig. 4 corresponding to the exclusive production of an additional fermion pair in the reaction $\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ f \bar{f}$. We shall focus on this process as being representative of a class of reactions at this order of perturbation theory, though many other final states are certainly possible. Let us consider how these diagrams fit into a distribution function framework.

In analogy with QCD, the electroweak factorization ansatz of Eq. 3 associates the diagrams of Figs. 4a,b with a $W^+ W^-$ fusion subprocess. Included in Fig. 4c are final state bremsstrahlung processes corresponding to fragmentation functions of the final leptons and initial state bremsstrahlung processes related to redefinitions of the neutrino structure functions.

We can accommodate the $2 \rightarrow 2$ Born processes and the processes of Figs. 4 in the parton framework by summing only over $W^+ W^-$ and $\nu_e \bar{\nu}_\mu$ subprocesses in the factorized expression for $\sigma^{\nu_e \bar{\nu}_\mu}_{\text{total}}$ in Eq. 3. In addition, we only keep terms up to $O(\alpha_w)$ in $f_{W^+ / \nu_e}(x, Q^2)$ and $f_{W^- / \nu_\mu}(x, Q^2)$ as in Eq. 5, and in
\[ f_{\nu_e / \nu_e}(x, Q^2) = \delta(1 - x) + P_{Z / \nu_e}(1 - x, Q^2), \] (13)
\[ f_{\bar{\nu}_\mu / \nu_\mu}(x, Q^2) = \delta(1 - x) + P_{Z / \bar{\nu}_\mu}(1 - x, Q^2), \] (14)
where the delta functions reflect the initial conditions of the neutrinos. Consequently, in the factorization approximation, we use the above expansions
to obtain

$$
\sigma^{\nu_\mu}_{\text{total}}(s) = \sigma^{\text{Born}}_{\text{elas}}(s) + \sigma^{\text{Born}}_{\text{inel}}(s) + \sigma^{\text{fusion}}_{W^+W^-}(s) + \sigma^{\text{brem}}(s) + \cdots \tag{15}
$$

where the explicitly listed terms correspond to particular subprocesses which we will discuss in turn.

The cross sections $\sigma^{\text{Born}}_{\text{elas}}(s)$ and $\sigma^{\text{Born}}_{\text{inel}}(s)$ are given by Eqs. (11,12). The $W^+W^-$ fusion cross section is given by

$$
\sigma^{\text{fusion}}_{W^+W^-}(s) = \sum_{\lambda,\lambda'} \int_0^1 dx_1 \int_0^1 dx_2 P_{W^+\lambda/\nu_\mu}(x_1, s) P_{W^-\lambda'/\bar{\nu}_\mu}(x_2, x_1 s) \hat{\sigma}_{W^+\lambda W^-\lambda' \to f\bar{f}}(x_1 x_2 s) \\
\times \Theta(x_1 x_2 s - 4M_W^2) \tag{16}
$$

where the sum extends over the polarization states of the on-shell $W$ bosons. The subprocess cross sections for $W^+\lambda W^-\lambda' \to f\bar{f}$ are straightforward to calculate and may be found in the literature[31]. The kinematic constraints of the parton model strictly limit the applicability of this particular contribution to fermion pair masses $m_{f\bar{f}} \geq 2M_W$. The bremsstrahlung contributions $\sigma^{\text{brem}}$ correspond to the emission of a near-mass-shell photon or $Z$ which subsequently decays into a $f\bar{f}$ pair; since these only contribute to $m_{f\bar{f}} < 2M_W$ (which falls outside the constraints of Eq. (16) we neglect them in this approximation.

In Fig. 5b we show $\sigma^{\text{fusion}}_{W^+W^-}$ summed over all lepton and quark species (except for $t$ quarks) such that $m_{f\bar{f}} \geq 2M_W$. The $\ln^2 s$ growth of this contribution to $\sigma^{\nu_\mu}_{\text{total}}$ originates from the logarithms in the splitting functions corresponding to finding transverse gauge bosons inside neutrinos, in complete analogy with the QED expression of Eq. (2). In the next section we compare this result with an exact calculation.
4 Multiparticle Production Beyond the Parton Model

Fig. 5c gives the result of an exact calculation for fermion pair production for $m_{ff} \geq 2M_W$ which we performed using helicity techniques\[32\]. The helicity calculation incorporates interference effects between the diagrams of Fig. 4 and grows as $\ln s$. As discussed in more detail in Refs. [29, 30] the fact that the parton model behaviour disagrees with the exact calculation only by a power of $\ln s$ and not by some power of $s$ (which would be characteristic of a violation of gauge invariance in the EVBA) is somewhat of a success for the EVBA which neglects the effects of bremsstrahlung diagrams. Furthermore, closer inspection reveals that in the weak parton model the dominant contribution to $f \bar{f}$ production originates from the threshold region ($m_{W^+W^-} = m_{ff} \gtrsim 2M_W$) whereas the validity of the EVBA to $W^+W^-$ fusion requires that $m_{W^+W^-} \gg 2M_W$. It is at this level of agreement (or disagreement) that the weak parton model is useful for calculating low order contributions to $\sigma^{\nu\bar{\nu}\mu}$ total. In any case, the growing contribution to $\sigma^{\nu\bar{\nu}\mu}_{\text{total}}$ from fermion pair masses above $2M_W$ is not significant when compared with the much larger Born terms.

Fortunately, we are not limited to using the weak parton model exclusively. We next consider particle production processes below the $2M_W$ threshold of the weak parton model. By integrating our exact $2 \to 4$ calculation over all fermion pair masses we obtain a contribution to $\sigma^{\nu\bar{\nu}\mu}_{\text{total}}$ of $O(\alpha_w/M_W^2 \ln(s/M_W^2))$, as shown in Fig. 5a, which is dominated by the $Z$ boson pole. The same behavior is more easily deduced from the $2 \to 3$ production of an on-shell $Z$ boson (see Fig. 6) which agrees with the $2 \to 4$ calculation (except that the latter has an additional $O(\alpha_w^4)$ component corresponding to the region $|m_{ff} - M_Z| > \Gamma_Z$).

In Fig. 7 we plot additional contributions to $\sigma^{\nu\bar{\nu}\mu}_{\text{total}}$ from the production of on-shell $W^\pm$ and Higgs bosons (treating them as stable particles). At
\[ \sqrt{s} \simeq 10^7 \text{ GeV} \] the sum of the $2 \rightarrow 3$ contributions is approximately half that of $\sigma_{\text{el}}^{\text{Born}} + \sigma_{\text{inel}}^{\text{Born}} \simeq 290 \text{ pb}$. To be consistent, we should more appropriately compare the $2 \rightarrow 3$ contributions with the radiatively corrected $2 \rightarrow 2$ final states. To do this we consider the limit of the standard electroweak model without QED (i.e., $\sin \theta_w = 0$) and take $M_W = M_Z = 91.17 \text{ GeV}$. In Fig. 8 we show the sum of the analogous $2 \rightarrow 3$ contributions as well as an estimate of the of the radiatively corrected $2 \rightarrow 2$ contributions deduced from the work of Frankfurt and Sherman\cite{33}.

For the $2 \rightarrow 2$ contribution to $\sigma^{\nu \bar{\nu} \mu}_{\text{total}}$ in Fig. 8 we take

\[ \sigma_{2 \rightarrow 2} \simeq \sigma_{\text{el}}^{\nu \bar{\nu} \mu}(s) + \sigma_{\text{inel}}^{\nu \bar{\nu} \mu}(s) - \Theta(s - s_0) \frac{15\pi\alpha_w^3}{16M_Z^2} \ln \left( \frac{s}{s_0} \right), \]  

(17)

where $\sigma_{\text{el}}^{\nu \bar{\nu} \mu}(s)$ and $\sigma_{\text{inel}}^{\nu \bar{\nu} \mu}(s)$ are given by Eqs. 11,12 and the coefficient of the logarithm may be extracted, with the help of the optical theorem, from the leading log calculations of Ref.\cite{33}. As an additional approximation which fixes the nonleading logarithmic contribution to the radiative correction, we include the theta function and adjust the argument of the logarithm so that the radiative corrections only contribute above a center of mass energy $\sqrt{s_0}$.

In this approximation the $2 \rightarrow 3$ contributions surpass the $2 \rightarrow 2$ contributions to $\sigma^{\nu \bar{\nu} \mu}_{\text{total}}$ in the neighbourhood of $10^7 \text{ GeV}$ for $\sqrt{s_0}$ in the range from $M_Z$ to $10 \text{ TeV}$.

The appearance of large logarithms in both the lowest order $2 \rightarrow 3$ processes and the next to leading order $2 \rightarrow 2$ processes raises concerns over the contributions of higher orders of perturbation theory. As a step towards estimating the effects of higher orders we adopt a conjecture of Fadin, Kuraev and Lipatov of Ref.\cite{6} whereby we modify our $2 \rightarrow 2$ and $2 \rightarrow 3$ amplitudes in an attempt to sum leading logarithms to all orders. For the $2 \rightarrow 3$ amplitudes, this corresponds to recognizing the dominance of the multiperipheral diagrams (e.g., Fig. 6a) and replacing the the vector meson propagators with those of reggeons. We should emphasize that unlike Ref.\cite{6}, it is not clear
that such replacements are sensible in our context. With this caveat in mind, we will show how our results are modified before discussing their significance.

Consider the contribution to the $2 \rightarrow 3$ amplitude from the diagram in Fig. 6a. Due to vector boson propagators this diagram is proportional to $(q_1^2 - M^2)^{-1}(q_2^2 - M^2)^{-1}$ with momentum transfers $q_1 = p_{\nu e} - p_{e^-}$ and $q_2 = p_{\bar{\nu}_{\mu}} - p_{\mu^+}$. In the ansatz of Fadin, Kuraev and Lipatov these propagator factors are replaced by

$$\frac{1}{q_1^2 - M^2} \frac{1}{q_2^2 - M^2} \rightarrow \frac{(s_1/M^2)^{\alpha(q_1^2)}}{q_1^2 - M^2} \frac{(s_2/M^2)^{\alpha(q_2^2)}}{q_2^2 - M^2},$$

where

$$s_1 = (p_{e^-} + p_Z)^2, \quad s_2 = (p_{\mu^+} + p_Z)^2,$$

and

$$\alpha(q^2) = \frac{g^2}{(2\pi)^3} \int dk_{\perp}^2 \frac{q_{\perp}^2 - M^2}{(k_{\perp}^2 - M^2)((q - k_{\perp})^2 - M^2)}.$$

Likewise, the $2 \rightarrow 2$ amplitudes are modified by an overall factor of $(s/M^2)^{\alpha(q^2)}$. Incorporating these modifications in our helicity calculations leads to the cross sections shown in Fig. 9. The $2 \rightarrow 3$ curves in Fig. 9 are extrapolated beyond $10^7$ GeV since our unoptimized numerical integration becomes unreliable beyond that energy. Nevertheless, in this approximation we see that the $2 \rightarrow 2$ and $2 \rightarrow 3$ contributions to $\sigma_{\mu e^0}^\nu_{\bar{\nu}_{\mu}}$ become comparable in the vicinity of $10^{10}$ GeV. Again, these results are appropriate for $\sin \theta_w = 0$ and $M = M_W = M_Z = 91.17$ GeV.

As mentioned above, the conjecture of Fadin, Kuraev and Lipatov deserves scrutiny in our context. The implied modification of the $2 \rightarrow 2$ amplitude most plausibly accounts only for the exchange of weak isospin $T = 1$ in the $t$–channel without manifestly accounting for the more important $T = 0$ (pomeron) exchange. On the other hand, our estimate of the radiative corrections to $2 \rightarrow 2$ in Fig. 8 contains both $T = 0$ and $T = 1$ contributions (but is only of first order in $\alpha_w \ln s$). Hence it is perhaps safer to focus on the qualitative nature of the higher order corrections suggested in Fig. 9. Namely,
that while higher order corrections tend to suppress the contributions of individual channels, the energy scale at which \(2 \to 3\) reactions surpass \(2 \to 2\) reactions is not extensively affected (to logarithmic accuracy).

Though we have only explicitly dealt with \(2 \to 2\), \(2 \to 3\), and, to some extent, \(2 \to 4\) contributions to \(\sigma_{\text{total}}^{\nu_\tau \bar{\nu}_\tau}\), it is tempting to speculate beyond few body final states. Fig. 9 suggests that radiative corrections to \(2 \to 3\) processes subdue the growth of those channels and may eventually lead to overall decreasing contributions. Meanwhile Pindor et al.\cite{5} suggest a slowly growing \(\sigma_{\text{total}}^{\nu_\tau \bar{\nu}_\tau}\) which implies that \(2 \to n \ (n > 3)\) channels must be opening up to compensate for the decreasing contributions from \(2 \to 2\) and \(2 \to 3\) channels. This picture of multiparticle production is analogous to what is observed experimentally in QCD, albeit at a much different energy scale.

In summary, we have shown that by considering the lowest order weak parton model and \(2 \to 2\) and \(2 \to 3\) contributions to \(\sigma_{\text{total}}^{\nu_\tau \bar{\nu}_\tau}\) that multiparticle production may become an important feature of weak interactions at energy scales of \(10^7 - 10^9\) GeV. In each case the appearance of large logarithms plays an important role yet we suspect that the relevant energy scale is not an artifact of a finite order of perturbation theory.

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Appendix A

For completeness, we include in this appendix explicit expressions for the vector boson splitting functions used in the text. The following formulae are essentially those of Johnson et al. from Ref. [28] with some minor but notable modifications.

Consider a fermion entering a reaction with a body $A$ at center of mass energy $\sqrt{s}$ (see Fig. A1). The splitting function formalism addresses processes which correspond to the emission of a vector boson by the fermion so the vector boson is considered as an initial state particle in a subprocess with the body $A$. If the lagrangian coupling between vector bosons and fermions is $\overline{\Psi} \Gamma_\mu \Psi V^\mu$, where

$$\Gamma_\mu = g_R \gamma_\mu \frac{1 + \gamma_5}{2} + g_L \gamma_\mu \frac{1 - \gamma_5}{2}, \tag{21}$$

then the longitudinal vector boson splitting function is

$$P_{V_L/f}(x, s) = F_s \left( \frac{g_L^2 + g_R^2}{16\pi^2} \right) \left[ \frac{2(1-x)(r+x)}{x(x-r)} - \frac{2r(2+r-x)}{(x-r)^2} \ln \left( \frac{(1+r-x)x}{r} \right) \right] \tag{22}$$

where $r = M_V^2/s$ accounts for the vector boson mass $M_V$. The spin factor $F_s = 2$ for neutrinos (and $F_s = 1$ otherwise) compensates for the fact that the formulae of Ref. [28] are spin averaged assuming the initial fermion has two helicity states.

The splitting functions correspond to the emission of a positive or negative helicity boson are, respectively,

$$P_{V_+/f}(x, s) = F_s \left[ g_L^2 h_1(x, r) + g_R^2 h_2(x, r) \right] \tag{23}$$

$$P_{V_-/f}(x, s) = F_s \left[ g_R^2 h_1(x, r) + g_L^2 h_2(x, r) \right] \tag{24}$$
where

\[ h_1(x, r) = \frac{1}{16\pi^2} \left[ -\frac{(1-x)(2+r-x)}{(x-r)} - (x-r)\ln(x) \right. \]
\[ \left. + (1+r-x)\left(\frac{x+r}{(x-r)^2} - 1\right)\ln\left(\frac{(1+r-x)x}{r}\right) \right] \]

\[ h_2(x, r) = \frac{1}{16\pi^2} \left[ -\frac{(1-x)(2+r-x)}{(1+r-x)(x-r)} + \frac{x+r}{(x-r)^2}\ln\left(\frac{(1+r-x)x}{r}\right) \right] \]  

(26)

Eqs. 22, 25, 26 are slightly different from the corresponding expressions of Ref. 28. Our use of an on-shell flux factor has the effect of replacing an overall factor of \(x\) in the equations of Ref. 28 with an overall factor of \(x-r\). A consequence of this modification is that the distribution functions vanish at \(x = r\).

The quantity \(N_{V/f}(s)\) very roughly corresponds to the probability of finding a vector boson (of any polarization) “inside” a fermion. It is defined as the integral over \(x\) of the vector boson distribution functions, which to lowest order, is given by

\[ N_{V/f}(s) = \int_r^1 dx \left[ P_{V_L/f}(x, s) + P_{V_+/f}(x, s) + P_{V_-/f}(x, s) \right]. \]  

(27)

Using the above expressions for the splitting functions results in

\[ N_{V/f}(s) = F_s \left( \frac{g_L^2 + g_R^2}{16\pi^2} \right) \left[ \ln^2 r + \frac{7}{2} \ln r + \frac{r^2}{4} - 5r + \frac{19}{4} \right]. \]  

(28)
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Figure Captions

Figure 1. Integral of the lowest order $Z$ boson distribution function for a neutrino.

Figure 2. Kinematics relevant to second order contribution to the $Z$ boson distribution function for a neutrino. $s$ is the invariant mass squared of the original $\nu A$ system. One $Z$ boson carries away a momentum fraction $1 - y$ whereas the $Z$ boson entering the hard subprocess carries a fraction $x$ of the original neutrino momentum.

Figure 3. Ratio of second order contribution to lowest order $Z$ boson distribution function for a neutrino.

Figure 4. Gauge invariant set of diagrams for the production of an additional fermion pair (where $f \neq \mu, e$).

Figure 5. (a): $\sigma(\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ Z)$ calculated exactly at tree level; (b) $\sigma(\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ f \bar{f})$ in effective vector boson approximation ($m_{ff} \geq 2M_W$); (c) $\sigma(\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ f \bar{f})$ calculated exactly at tree level ($m_{ff} \geq 2M_W$);

Figure 6. Diagrams for the production of an on-shell $Z$ boson. (a) Dominant multiperipheral contribution (b) Subdominant bremsstrahlung diagrams.

Figure 7. $2 \rightarrow 3$ contributions to $\sigma^{\nu\bar{\nu}}_{\text{total}}$ in the standard model with $M_Z = 91.17$ GeV, $M_W = 80$ GeV and $M_H = 500$ GeV. Cross sections for final states containing a Higgs boson ($\sigma(\nu_e \bar{\nu}_\mu H) + \sigma(e^-\mu^+H)$), W boson ($\sigma(\nu_e \mu^+ W^-) + \sigma(e^-\bar{\nu}_\mu W^+)$), Z boson ($\sigma(e^-\mu^+Z) + \sigma(\nu_e \bar{\nu}_\mu Z)$) and their sum.

Figure 8. Contributions to $\sigma^{\nu\bar{\nu}}_{\text{total}}$ in the limit $\sin \theta_w = 0$, $M_W = M_Z = M_H = 91.17$ GeV. Sum of $2 \rightarrow 2$ contributions contains estimated
lowest order radiative corrections which are assumed to vanish below below $\sqrt{s} = 1$ TeV. $2 \rightarrow 3$ curve is sum of exact tree level calculation for states containing final state $Z$, $W$ or $H$ bosons.

**Figure 9.** Contributions to $\sigma_{\text{total}}^{\nu_e\nu_\mu}$ in the limit $\sin \theta_w = 0$, $M_W = M_Z = M_H = 91.17$ GeV with amplitudes modified by the Fadin, Kuraev and Lipatov (FKL) conjecture described in the text. (a) Sum of $2 \rightarrow 2$ Born contributions, (b) Sum of FKL modified $2 \rightarrow 2$ contributions, (c) Sum of $2 \rightarrow 3$ Born contributions, (d) Sum of FKL modified $2 \rightarrow 3$ contributions. The dashed portions of (c) and (d) are extrapolations.

**Figure A1.** Diagram corresponding the process $fA \rightarrow f'X$ in which the initial fermion $f$ emits a gauge boson $V$ which, in the effective vector boson approximation, is considered to be on-shell in the subprocess $AV \rightarrow X$. 
