Research Article

Exponential Stability for an Opinion Formation Model with a Leader Associated with Fractional Differential Equations

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This paper studies the dynamics of an opinion formation model with a leader associated with a system of fractional differential equations. We applied the concept of \(\alpha\)-exponential stability and the uniqueness of equilibrium to show the consensus of the followers with the leader. A sufficient condition for the consensus is obtained for both fractional formation models with and without time-dependent external inputs. Moreover, numerical results are provided to illustrate the dynamical behavior.

1. Introduction

Humans and some animals are social animals. Living together as a society requires mutual assistance and generosity, which will lead to happiness. Whether it is a small or a large society, social animals, for instance, a flock of birds, bees, insects, and fish, have a complex thought structure and they tend to have a leader to lead their lives to peacefulness. Some research focuses on the behavior of these social animals, including an all-leader agent-based model for turning and flocking birds [1], the migration of honeybees to a new nest site [2], leadership in fish shoals [3], and the effective leadership and decision-making in animal groups on the move [4]. In these systems, individuals are mutually dependent and communicate some information to their peers in a certain region around them. Many researchers have studied and developed a group of opinion formation models, which have been developed into many approaches. Examples of these approaches are opinion formation models on a gradient [5], opinion formation models based on game theory [6], Boltzmann and Fokker–Planck equations for opinion formation models associated with strong leaders [7], opinion formation models based on kinetic equations [8], agent-based models for opinion formation [9], consensus and clustering in opinion formation on networks [10], and Boltzmann-type control of consensus of the opinion with leaders [11].

In recent years, fractional differential and integral operators have been utilized in various real-world problems and areas of study, such as physical and biological sciences [12], environmental science, signal and image processing, and engineering [13–19]. New generalizations and definitions of the fractional derivative are also of interest by choosing different kernels [20]. In particular, it motivates a generalization of the opinion formation model to involve fractional calculus.

Consider the fractional opinion formation model

\[
\begin{align*}
C_0^D_x (t) i = \sum_{j=1}^{N} a_{ij} \left( f_j (x_j (t)) - f_i (x_i (t)) \right),
\end{align*}
\]

with the initial condition \( x_i (0) = \xi_i \in \mathbb{R} \) for \( i = 0, 1, \ldots, N \). The coefficient \( a_{ij} \) reflects the weight for which agents influence each other. The fractional derivative of order \( \alpha \) is used to describe the memory effects of the interaction.
It is natural to introduce a leader to the above system in order to drive a consensus among all agents \(x_i\). The leader is regarded as a virtual agent whose opinion \(x_0\) is independent of all the remaining agents’ opinions. The fractional opinion formation model associated with a leader is described by

\[
C^{D_{0^+}}_{\alpha} [x_0] (t) = u(t),
\]

\[
C^{D_{0^+}}_{\alpha} [x_i] (t) = \sum_{j=1}^{N} a_{ij} \bigl(f_j(x_j(t))\bigr) + c_i(x_0(t) - x_i(t)),
\]

subject to the initial condition \(x_i(0) = \xi_i \in \mathbb{R}\) for \(i = 0, 1, \ldots, N\). The coefficient \(c_i\) describes the influence of the leader on another agent. In particular, \(c_i > 0\), when the leader has an impact on the \(i\)th agent’s opinion.

During the last few years, the opinion formation model with leadership based on the fractional differential system has attracted tremendous attention and has been extensively studied by many researchers. For example, Almeida et al. [21] discussed an optimal control strategy for two types of fractional opinion formation models with leadership to reach a consensus. The result was given by a numerical scheme to approximate the Caputo fractional derivative based on the Grunwald–Letnikov approximation. Next, in 2019, Almeida et al. [22] used Mittag-Leffler stability to discuss sufficient conditions of (2) to guarantee that all agents have a consensus opinion approaching the leader’s opinion. They also studied the opinion formation model with external inputs \(h_i(t)\) described by

\[
C^{D_{0^+}}_{\alpha} [x_0] (t) = u(t),
\]

\[
C^{D_{0^+}}_{\alpha} [x_i] (t) = \sum_{j=1}^{N} a_{ij} \bigl(f_j(x_j(t))\bigr) + c_i(x_0(t) - x_i(t)) + h_i(t),
\]

with the initial data \(x_i(0) = \xi_i \in \mathbb{R}\) for \(i = 0, 1, \ldots, N\). In the above system, optimal control strategies were designed for the leader to obtain a consensus opinion.

Motivated by [21, 22], this paper aims to study the dynamics of a nonautonomous nonlinear fractional leader-follow opinion formation model

\[
C^{D_{0^+}}_{\alpha} [x_0] (t) = u(t),
\]

\[
C^{D_{0^+}}_{\alpha} [x_i] (t) = \sum_{j=1}^{N} a_{ij} \bigl(f_j(x_j(t))\bigr) + c_i(x_0(t) - x_i(t)),
\]

with \(x_i(0) = \xi_i \in \mathbb{R}\) for \(i = 0, 1, \ldots, N\) as well as the model with time-dependent external inputs

\[
C^{D_{0^+}}_{\alpha} [x_0] (t) = u(t),
\]

\[
C^{D_{0^+}}_{\alpha} [x_i] (t) = \sum_{j=1}^{N} a_{ij} \bigl(f_j(x_j(t))\bigr) + c_i(x_0(t) - x_i(t)) + h_i(t),
\]

subject to \(x_i(0) = \xi_i \in \mathbb{R}\) for \(i = 0, 1, \ldots, N\). Here, we include the time-dependence in the source term \(f_j(t, x_j(t))\).

Rather than focusing on the design of an optimal control, we mainly focus on the exponential stability of the solutions and establish some sufficient conditions for the consensus opinion of all agents with the leader \(x_0\).

The rest of this paper is organized as follows. In Section 2, we provide some definitions and preliminary results in integral and differential fractional calculus and exponential stability of systems of fractional differential equations. The existence of a unique consensus equilibrium point of fractional opinion formation models associated with a leader will be investigated in Section 3 and Section 4 for the systems described by (4) and (5), respectively. In Section 5, we provide examples and numerical simulations to demonstrate our analytical results.

2. Fractional Calculus Framework

In this section, we briefly outline some notions from fractional calculus that will be used throughout the paper.

**Definition 1** (see [23]). For \(\alpha > 0\), the left Caputo fractional derivative of order \(\alpha\) is given by

\[
D^\alpha f (t) = \frac{d^{\alpha}}{dt^{\alpha}} f (t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-\alpha-1} D^n f (\tau) d\tau,
\]

where \(n\) is a natural number such that \(\alpha \in (n - 1, n)\) and \(D = dt/dt\).

**Definition 2** (see [23]). The Riemann–Liouville fractional integral of order \(\alpha > 0\) is given by

\[
I^\alpha f (t) = \frac{1}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds,
\]

provided the right-side integral exists point wisely, on \((0, \infty)\).

**Lemma 1** (see [23]). Let \(\alpha > 0\) be a real number. The general solution of the fractional Caputo type differential equation

\[
\frac{d^\alpha}{dt^\alpha} y(t) = 0,
\]

is a function satisfying

\[
y(t) = k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1},
\]

for some constants \(k_i\), \(i = 0, 1, \ldots, n - 1\) where \(n = [\alpha] + 1\) if \(\alpha \notin \mathbb{N}\), and

\[
y(t) = k t^{n-1},
\]

for some constant \(k\), if \(\alpha \in \mathbb{N}\).

**Lemma 2** (see [23]). For a positive constant \(\alpha > 0\) and \(y \in C^n[0, T]\), we have
\[
\frac{d^\alpha}{dt^\alpha} y(t) = y(t) + k_0 + k_1 t + \ldots + k_{n-1} t^{n-1}, \quad (11)
\]

for some constants \(k_i, \ i = 0, 1, \ldots, n-1\) where \(n = \lfloor \alpha \rfloor + 1\). Here, \(D^\alpha = d^\alpha/dt^\alpha(\cdot)\) is the Caputo fractional derivative.

Consider the differential equation with the Caputo fractional derivative defined by
\[
\begin{align*}
D^\alpha y(t) &= f(t, y(t)) \quad t \in [0, \infty), \\
y(0) &= y_0,
\end{align*}
\]

for some initial data \(y_0 \in \mathbb{R}^m\), where \(y = (y_1, y_2, \ldots, y_m)^T \in \mathbb{R}^m, 0 < \alpha < 1\), and the nonhomogeneous term \(f: [0, \infty) \times \mathbb{R}^m \rightarrow \mathbb{R}^m\) satisfies the continuity in \(t\) and locally Lipschitz continuity in \(y\).

**Definition 3** (see [24]). We say the \(y^* \in \mathbb{R}^m\) is an equilibrium of system (12) whenever \(f(t, y^*) = 0\) for all \(t \in [0, \infty)\).

**Definition 4** (see [24]). System (12) is \(\alpha\)-exponentially stable if there exist two positive constants \(M > 0\) and \(\lambda > 0\) such that for any solutions \(x(t)\) and \(y(t)\) of system (6) subject to the initial conditions \(x_0\) and \(y_0\), respectively, we have
\[
\|x(t) - y(t)\| \leq M\|x_0 - y_0\|\exp(-\lambda t^\alpha),
\]

for \(t \geq 0\) where \(\| \cdot \|\) denotes the Euclidean norm.

It should be noted that the notion of \(\alpha\)-exponential stability concerns the closeness of solutions \(x(t)\) and \(y(t)\) subject to different initial conditions \(x_0\) and \(y_0\), respectively, while the definition of the Mittag-Leffler stability in [12] concerns the convergence of a solution \(x(t)\) to an equilibrium point.

**Lemma 3** (see [17]). If \(d^\alpha x/dt^\alpha \leq d^\alpha y/dt^\alpha\) with \(\alpha \in (0, 1)\) and \(x(0) = y(0)\), then \(x(t) \leq y(t)\).

**Lemma 4** (see [17]). Let \(W : C[0, \infty)\) be a function such that \(d^\alpha y/dt^\alpha \leq 8W(t)\), where \(\alpha \in (0, 1)\) and \(\delta\) is an arbitrary constant. Then, we have
\[
W(t) \leq W(0)\exp\left[\frac{\delta}{\Gamma(\alpha + 1)} t^\alpha\right].
\]

We define the signum function as
\[
\text{sign}(x) = \begin{cases} 
1 & x \in (0, \infty), \\
-1 & x \in (-\infty, 0).
\end{cases}
\]

### 3. Stability and Consensus of Fractional Leader-Follower Opinion Formation Model

In this section, we present a sufficient condition for the consensus opinion in a leader-follower opinion formation model (4) based on the exponential stability concept and the uniqueness of the equilibrium. We first outline the following assumptions for the problem (4)

- (H1) The functions \(f_i, i = 1, 2, \ldots, N\) are Lipschitz continuous with respect to the second variable on \(\mathbb{R}\) with Lipschitz constants \(L_i > 0\), that is,
  \[
  |f_i(t, x) - f_i(t, y)| \leq L_i|x - y|,
  \]
  for all \(x, y \in \mathbb{R}\) uniformly with respect to \(t\),

- (H2) The constants \(c_i\) is positive for \(i = 1, 2, \ldots, N\) and satisfies
  \[
  k_i = c_i - \sum_{j=1}^{N} |a_{ij}| L_j > 0.
  \]

Under the Lipschitz condition of the source term (H1), we obtain the existence and uniqueness of the solution to the Cauchy problem (4) from the standard result [25]. Motivated by [24], we prove the following result.

**Theorem 1.** Under the assumptions (H1) and (H2), system (4) is exponentially stable.

**Proof.** Let \(x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T\) and \(y(t) = y_1(t), y_2(t), \ldots, y_N(t))^T\) be the solutions of system (4) subject to the different initial conditions \(x_i(0) = \xi_i\) and \(y_i(0) = \zeta_i, i = 1, 2, \ldots, N\).

Let \(w_i^*(t) = y_j(t) - x_i(t)\) for \(i = 1, 2, \ldots, N\), then \(w_i^*(0) \neq 0\). By (4) we get
\[
D^\alpha w_i^*(t) = -c_i w_i^*(t) + \sum_{j=1}^{N} a_{ij} (f_j(t, y_j(t)) - f_j(t, x_j(t))).
\]

If \(w_i^*(t)\) is positive, then
\[
D^\alpha|w_i^*(t)| = \frac{1}{\Gamma(1 - \alpha)} \int_{t_0}^{t} (t - s)^{-\alpha} w_i^*(s) ds = D^\alpha w_i^*(t),
\]

if \(w_i^*(t)\) is negative, then
\[
D^\alpha|w_i^*(t)| = -\frac{1}{\Gamma(1 - \alpha)} \int_{t_0}^{t} (t - s)^{-\alpha} w_i^*(s) ds = -D^\alpha w_i^*(t),
\]

hence,
\[
D^\alpha|w_i^*(t)| = \text{sign}(w_i^*(t)) D^\alpha w_i^*(t),
\]

next, we construct a function \(W\) by
\[
W(t) = \sum_{i=1}^{N} |w_i^*(t)|,
\]

we compute the Caputo fractional derivative of \(W(t)\) using the assumptions (H1) and (H2) to obtain
\[ D^p W(t) = \sum_{i=1}^{N} D^p |w_i^*(t)| \]
\[ = \sum_{i=1}^{N} \text{sign}(w_i^*(t)) \left( -c_i w_i^*(t) + \sum_{j=1}^{N} a_{ij} (f_j(t,y_j(t)) - f_j(t,x_j(t))) \right) \]
\[ \leq \sum_{i=1}^{N} \left( -c_i |w_i^*(t)| + \sum_{j=1}^{N} |a_{ij}| |f_j(t,y_j(t)) - f_j(t,x_j(t))| \right) \]
\[ = -\sum_{i=1}^{N} \left( c_i - \sum_{j=1}^{N} |a_{ij}| L_j \right) |w_i^*(t)| \]
\[ \leq -k W(t), \quad \text{(23)} \]

where \( k = \min_{1 \leq i \leq N} \xi_i \). Lemma 4 implies that
\[ W(t) \leq W(0) \exp \left( \frac{k}{\Gamma(\alpha + 1)} t^{\alpha} \right). \quad \text{(24)} \]

it follows that system (4) is exponentially stable.

Next, we assume that the leader’s opinion is kept constant as \( \xi_0 \) for all time, that is, \( u(t) = 0 \) in (4). Let \( y^* = (y_1, y_2, \ldots, y_N)^T \in \mathbb{R}^N \) be an equilibrium of (4). Hence, it satisfies the following system
\[ -c_i y_i + \sum_{j=1}^{N} a_{ij} f_j(t, y_j) + c_i \xi_0 = 0, \quad i = 1, 2, \ldots, N. \quad \text{(25)} \]

**Theorem 2.** Assume that (H1) and (H2) hold. Then, system (4) has a unique equilibrium.

**Proof.** Let \( c_i y_i = p_i \) for \( i = 1, 2, \ldots, N \). We get from (25) that
\[ p_i = \sum_{j=1}^{N} a_{ij} f_j(t, \frac{p_j}{c_j}) + c_i \xi_0, \quad i = 1, 2, \ldots, N, \quad \text{(26)} \]

consider the map \( \mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N \) defined by \( \mathcal{F}(p) = (\mathcal{F}_1(p), \mathcal{F}_2(p), \ldots, \mathcal{F}_N(p))^T \), where
\[ \mathcal{F}_i(p) = \sum_{j=1}^{N} a_{ij} f_j(t, \frac{p_j}{c_j}) + c_i \xi_0, \quad i = 1, 2, \ldots, N, \quad \text{(27)} \]

by the Lipschitz condition (H1), for any \( p, q \in \mathbb{R}^N \), we have
\[ |\mathcal{F}_i(p) - \mathcal{F}_i(q)| = \left| \sum_{j=1}^{N} a_{ij} \left( f_j(t, \frac{p_j}{c_j}) - f_j(t, \frac{q_j}{c_j}) \right) \right| \]
\[ \leq \sum_{j=1}^{N} |a_{ij}| \left| f_j(t, \frac{p_j}{c_j}) - f_j(t, \frac{q_j}{c_j}) \right| \]
\[ \leq \sum_{j=1}^{N} |a_{ij}| \frac{L_j}{c_j} |p_j - q_j| \quad \text{(28)} \]

We have from (H1) that \( \sum_{j=1}^{N} |a_{ij}| L_j / c_i < 1 \) for \( i = 1, 2, \ldots, N \). Hence,
\[ \| \mathcal{F}_i(p) - \mathcal{F}_i(q) \| \leq \sum_{j=1}^{N} |a_{ij}| L_j / c_i |p_j - q_j| \]
\[ \leq \sum_{i=1}^{N} \left( \sum_{j=1}^{N} \frac{|a_{ij}| L_j}{c_j} |p_j - q_j| \right) \]
\[ < \sum_{i=1}^{N} |p_i - q_i| \]
\[ = \| p - q \|. \quad \text{(29)} \]

Thus, the map \( \mathcal{F} \) is a contraction. Therefore, \( \mathcal{F} \) has a unique fixed point, showing that system (4) has a unique equilibrium.

The next result shows that all the agents’ opinions are in consensus with the leader.

**Theorem 3.** Assume that (H1) and (H2) hold. Suppose that \( f_i(t, \xi_0) = 0 \) for \( t \in [0, \infty) \) and for all \( i = 1, 2, \ldots, N \). Then, all agents’ opinions \( x_i(t) \) of (4) converge to the consensus opinion \( \xi_0 \).

**Proof.** The proof follows the same argument as in [22, Theorem 2] where the system is autonomous. We outline the proof when the nonautonomous source term is considered here. Let \( (x_0(t), x_1(t), \ldots, x_N(t)) \) be a solution of (4). Since we assume that \( u(t) = 0 \), we have that \( x_0(t) = \xi_0 \) is a constant function. Let \( y_i = x_i - \xi_0 \) and define \( g_i(t, y_i) = f_i(t, y_i + \xi_0) \), for \( i = 1, \ldots, N \). Then, system (4) can be written as
\[ D^p [y_i](t) = \sum_{j=1}^{N} a_{ij} (g_j(t, y_j(t))) - c_i y_i(t), \quad i = 1, 2, \ldots, N, \quad \text{(30)} \]
we see that for any \( y, \varphi \in \mathbb{R} \),
\[
|g_i(t, y) - g_i(t, \varphi)| \leq |f_i(t, y + \xi_0) - f_i(t, \varphi + \xi_0)| \leq L_i |y - \varphi|,
\]
(31)
for \( i = 1, 2, \ldots, N \). It follows from Theorem 2 that system (30) has a unique equilibrium \( y^* = (y_1^*, y_2^*, \ldots, y_N^*) \). Furthermore, the exponential stability in Theorem 1 implies that every solution \( y \) of (30) converges to the equilibrium, that is, \( \lim_{t \to \infty} \| y(t) - y^* \| = 0 \). Since \( g_i(t, 0) = f_i(t, \xi_0) = 0 \) for \( i = 1, 2, \ldots, N \), we see that \( y^* = (0, \ldots, 0)^T \). Consequently, we have \( \lim_{t \to \infty} x_i(t) = \xi_0 \) for every \( i = 1, \ldots, N \) showing the consensus of opinion.

4. Stability and Consensus of Leader-Follower Opinion Formation Model with Time-dependent External Inputs

In this section, we extend the study to consider a leader-follower opinion formation model with time-dependent external inputs.

Consider the leader-follower opinion formation model with time-dependent external inputs \( h_i(t) \), \( i = 1, 2, \ldots, N \) in (5).

We assume the following conditions for the external inputs.

(H3) The functions \( h_i \), \( i = 1, 2, \ldots, N \) are bounded that is, there exists \( O_i > 0 \) such that
\[
|h_i(t)| \leq O_i,
\]
(32)
for each \( i = 1, 2, \ldots, N \).

\[
D^\alpha W (t) = \sum_{i=1}^{N} D^\alpha |x_i(t)|
\]
\[
= \sum_{i=1}^{N} \text{sign} (x_i(t)) \left( -c_i x_i(t) + \sum_{j=1}^{N} a_{ij} f_j (t, x_j(t)) - f_j (t, 0) \right)
\]
\[
+ \sum_{i=1}^{N} \text{sign} (x_i(t)) \left( \sum_{j=1}^{N} a_{ij} f_j (t, 0) \right) + \sum_{i=1}^{N} \text{sign} (x_i(t)) \left( c_i \xi_0 + h_i(t) \right)
\]
\[
\leq \sum_{i=1}^{N} \left( -c_i |x_i(t)| + \sum_{j=1}^{N} L_{ij} |x_j(t)| + \sum_{j=1}^{N} |a_{ij}| |f_j (t, 0)| + c_i \xi_0 + O_i \right)
\]
\[
= - \sum_{i=1}^{N} \left( c_i - \sum_{j=1}^{N} |a_{ij}| L_{ij} \right) |x_i(t)| + \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}| |f_j (t, 0)| + c_i \xi_0 + O_i \right) \leq - k W (t) + w,
\]
where \( k = \min_{1 \leq i \leq N} k_i \).

Consider the fractional-order system
\[
D^\alpha V (t) = - k V (t) + w.
\]
(38)
It is easily seen that (38) is exponentially stable, so that any solution converges to the unique equilibrium \( V^* = w/k \), that is

\[
\text{Theorem 4. Assume that (H1), (H2), and (H3) hold. Then, system (5) is exponentially stable. Moreover, for any solution \( x(t) \), there exists } T \geq 0 \text{ such that}
\]
\[
\|x(t)\| \leq \frac{w}{k} + \varepsilon, \quad t \geq T,
\]
(33)
where \( k = \min_{1 \leq i \leq N} k_i \)
\[
w = \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}| |f_j (t, 0)| + c_i \xi_0 + O_i \right),
\]
(34)
and \( \varepsilon > 0 \) is arbitrary small.

Proof. Let \( x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T \) and \( y(t) = (y_1(t), y_2(t), \ldots, y_N(t))^T \) be the solutions of system (5) subject to the different initial conditions \( x_i(0) = \xi_i \) and \( y_i(0) = \xi_i \), \( i = 1, 2, \ldots, N \). By setting \( w^*(t) = y(t) - x(t) \) and using a similar argument as in the proof of Theorem 1, there exists \( M > 0 \) such that
\[
\lim_{t \to \infty} \| y(t) - x(t) \| \leq M |\xi_0 - \xi_0| \exp (-\lambda t^a) = 0, \quad t \geq 0,
\]
(35)
showing the exponential stability of (5). Next, we construct a function \( W \) by
\[
W (t) = \| x(t) \| = \sum_{i=1}^{N} |x_i(t)|.
\]
(36)
Using assumptions (H1) and (H2), we compute the Caputo fractional derivative of \( W(t) \) as
\[
\lim_{t \to \infty} |V (t) - V^*| = \lim_{t \to \infty} \| V (t) - \frac{w}{k} \| = 0.
\]
(39)
Hence, for any arbitrary \( \varepsilon > 0 \), there exists \( T > 0 \) such that
\[
V (t) \leq \frac{w}{k} + \varepsilon, \quad \text{whenever } t > T.
\]
(40)
By Lemma 3, we have \( W(t) \leq V(t) \) with \( W(0) = V(0) \). Thus, there exists \( T \geq 0 \) such that for any solution \( x(t) \),
Theorem 5. Assume that (H1), (H2), and (H3) hold. Suppose that \( f_i(t, \xi_0) = 0 \) for \( t \in [0, \infty) \) and \( \lim_{t \to \infty} h_i(t) = 0 \) for all \( i = 1, 2, \ldots, N \). Then, all agents’ opinions \( x_i(t) \) of (5) converge to the consensus opinion \( \xi_0 \).

Proof. Let \( (x_0(t), x_1(t), \ldots, x_N(t)) \) be a solution of (5). Since we assume that \( u(t) = 0 \), the leader’s opinion is a constant function \( x_0(t) = \xi_0 \). Let \( y_i = x_i - \xi_0 \) and define \( g_i(t, y_i) = f_i(t, y_i + \xi_0) \), for \( i = 1, \ldots, N \). Then, system (5) can be written as

\[
D^\alpha \mathbf{y}_i(t) = -\sum_{j=1}^{N} a_{ij} \left( g_j(t, y_j(t)) - c_i y_i(t) + h_i(t) \right). \tag{42}
\]

it can be seen that \( g_i \) satisfies Lipschitz continuity condition as

\[
\|g_i(t, y) - g_i(t, \varphi)\| \leq \|f_i(t, y + \xi_0) - f_i(t, \varphi + \xi_0)\| \leq L_i \|y - \varphi\|. \tag{43}
\]

for any \( y, \varphi \in \mathbb{R} \) and \( i = 1, 2, \ldots, N \). Hence, by applying Theorem 4 for (42), that is, considering \( g_i \) instead of \( f_i \) and taking \( \xi_0 = 0 \) in (5), and using the assumption \( g_i(t, 0) = f_i(t, \xi_0) = 0 \) for \( t \in [0, \infty) \), we see that the solution \( \mathbf{y}(t) = (y_1(t), y_2(t), \ldots, y_N(t)) \) of (42) satisfies

\[
\|\mathbf{y}(t)\| \leq \frac{w}{k} + \varepsilon, \quad t \geq T. \tag{44}
\]

for some \( T > 0 \) where

\[
w = \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}| \sup_{t \in [0, \infty)} |g_j(t, 0)| + c_i(0) + O_i \right) = \sum_{i=1}^{N} O_i,
\]

since \( \lim_{t \to \infty} h_i(t) = 0 \) for \( i = 1, 2, \ldots, N \), we have \( |h_i(t)| \leq \varepsilon \) whenever \( t \) is sufficiently large for all \( i = 1, 2, \ldots, N \). Hence, if we consider (37) for some sufficiently large time \( t \geq T \), then the bound of \( h_i \) in (H3) can be chosen as an arbitrary small \( \varepsilon \). Consequently, we obtain

\[
\|\mathbf{y}(t)\| \leq \frac{N \varepsilon}{k} + \varepsilon, \quad t \geq T. \tag{46}
\]

As \( \varepsilon > 0 \) was arbitrary, we have \( y_i(t) \to 0 \) as \( t \to \infty \). Consequently, we obtain the consensus opinion \( \lim_{t \to \infty} x_i(t) = \xi_0 \) for every \( i = 1, \ldots, N \). \( \qed \)

5. Numerical Examples

In this section, we give examples of the fractional opinion formation model of order \( 0 < \alpha < 1 \) to illustrate the dynamics of the system. In particular, we demonstrate through numerical examples that consensus opinion can be achieved when the assumptions of our results are satisfied.

Example 1. Consider the following system of opinion formation model with leader

\[
\begin{aligned}
C^\alpha_{D_0^\alpha}[x_0(t)] &= 0, \\
C^\alpha_{D_0^\alpha}[x_1(t)] &= a_{11} \left( 0.5 \tanh(x_1(t)) \right) + a_{12} \left( \frac{x_2}{x_2(t) + 1} \right) + a_{13} \left( \frac{x_3}{x_3(t) + 1} \right) + 9e^{-t} - 0.3x_1(t) + 0.3x_0(t), \\
C^\alpha_{D_0^\alpha}[x_2(t)] &= a_{21} \left( 0.5 \tanh(x_1(t)) \right) + a_{22} \left( \frac{x_2}{x_2(t) + 1} \right) + a_{23} \left( 0.03 \tanh(x_4(t)) \right) + 0.09e^{-t} - 0.3x_2(t) + 0.3x_0(t), \\
C^\alpha_{D_0^\alpha}[x_3(t)] &= a_{31} \left( 0.3 \tanh(x_1(t)) \right) + a_{32} \left( 0.3 \tanh(x_4(t)) \right) + 0.1e^{-t} - 0.2x_3(t) + 0.2x_1(t), \\
C^\alpha_{D_0^\alpha}[x_4(t)] &= a_{41} \left( 0.3 \tanh(x_1(t)) \right) + a_{42} \left( \frac{x_2}{x_2(t) + 1} \right) + a_{43} \left( 0.3 \tanh(x_5(t)) \right) - 0.4x_4(t) + 0.4x_0(t), \\
C^\alpha_{D_0^\alpha}[x_5(t)] &= a_{51} \left( \frac{x_3}{x_3(t) + 1} \right) + a_{52} \left( 0.3 \tanh(x_5(t)) \right) + a_{53} \left( \frac{x_4}{x_4(t) + 1} \right) + 0.02e^{-t} - 0.1x_5(t) + 0.1x_0(t), \\
C^\alpha_{D_0^\alpha}[x_6(t)] &= a_{61} \left( 0.5 \tanh(x_1(t)) \right) + a_{62} \left( \frac{x_2}{x_2(t) + 1} \right) + a_{63} \left( 0.3 \tanh(x_5(t)) \right) + a_{64} \left( \frac{x_4}{x_4(t) + 1} \right) - 0.4e^{-t} - 0.4x_6(t) + 0.4x_0(t),
\end{aligned}
\]

subject to the initial condition \( x_0(0) = 0, x_1(0) = 25, x_2(0) = 10, x_3(0) = 30, x_4(0) = 1.5, x_5(0) = -10, x_6(0) = -4.5 \).

In this system of opinion formation model (47) with leader \( x_0(t) = \xi_0 = 0 \), we set \( a_{11} = 0.06, a_{12} = -0.01, a_{16} = -0.02, a_{21} = 0.03, a_{23} = -0.02, a_{24} = -0.02, a_{33} = -0.01, a_{34} = 0.01, a_{44} = 0.03, a_{45} = -0.02, a_{62} = 0.02, a_{63} = 0.09, \) \( a_{53} = -0.09, a_{56} = 0.08, a_{61} = 0.03, a_{62} = 0.05, a_{63} = 0.09, \) \( a_{66} = 0.02, c_1 = 0.3, c_2 = 0.3, c_3 = 0.2, c_4 = 0.4, c_5 = 0.1, \) and \( c_6 = 0.4 \), respectively. We see that \( f_i(t, \xi_0) = 0 \) and \( \varepsilon_i = 1 \) for \( i = 1, 2, \ldots, 6 \) so that all the assumptions of Theorem 5 are satisfied. Hence, the leader-follower for opinion formation model (47) is exponentially
stable. The states of $x_0, x_1, x_2, x_3, x_4, x_5$ and $x_6$ of system (47) converges to a consensus state $(0, 0, 0, 0, 0, 0)$ as shown in Figure 1 for $\alpha = 0.5$.

Example 2. Consider the leader-follower opinion formation model

$$
\begin{align}
C_D^{\alpha_0}[x_0(t)] &= 0, \\
C_D^{\alpha_0}[x_1(t)] &= -0.2x_1(t) + a_{11} f(t, x_1(t)) + a_{12} f(t, x_2(t)) + a_{13} f(t, x_3(t)) + 0.2x_0(t), \\
C_D^{\alpha_0}[x_2(t)] &= -0.4x_2(t) + a_{21} f(t, x_1(t)) + a_{22} f(t, x_2(t)) + a_{23} f(t, x_3(t)) + \frac{1}{t^2} + 0.4x_0(t), \\
C_D^{\alpha_0}[x_3(t)] &= -x_1(t) + a_{31} f(t, x_1(t)) + a_{32} f(t, x_2(t)) + a_{33} f(t, x_3(t)) + 0.005 + x_0(t), \\
\end{align}
$$

where $f(t, x) = (x - 2/x^2 + e^t)$ subject to the initial condition $x_0(0) = 2, x_1(0) = 1.5, x_2(0) = -5, x_3(0) = 3$.

In this system of opinion formation model (48) with leader $x_0(t) = \xi_0 = 2$, we set $a_{11} = -0.06, a_{12} = 0.01, a_{13} = -0.04, a_{21} = 0.01, a_{22} = 0.01, a_{23} = -0.02, a_{31} = 0.06, a_{32} = -0.2, a_{33} = 0.01, c_1 = 0.2, c_2 = 0.4, c_3 = 1$, respectively. We see that $f(t, \xi_0) = 0$, $\lim_{t \to \infty} h_i(t) = 0$ and $L_i = 1$ for $i = 1, 2, 3$ so that all assumptions of Theorem 5 are satisfied.
Hence, the leader-follower for opinion formation model (48) is exponentially stable. The states of \( x_0, x_1, x_2, \) and \( x_3 \) of system (48) converges to a consensus state \((2,2,2,2)\) as shown in Figure 2 for \( \alpha = 0.8 \).

Example 3. Consider leader-follower for the opinion formation model

\[
CD^\alpha [x_0(t)] = 0,
\]

\[
CD^\alpha [x_1(t)] = a_{11}(0.5(|x_1 + 1| - |2x_1 - 4|)) + a_{12}(0.5(|x_2 + 1| - |2x_2 - 4|)) + \frac{\sin(t)}{t} + 0.2x_0(t) - 0.2x_1(t),
\]

\[
CD^\alpha [x_2(t)] = a_{21}(0.5(|x_1 + 1| - |2x_1 - 4|)) + a_{22}(0.5(|x_2 + 1| - |2x_2 - 4|)) - 2e^{-t} + 0.3x_0(t) - 0.3x_2(t).
\]

Subject to the initial condition \( x_0(0) = 5, x_1(0) = -1, x_2(0) = 15 \). In this system of opinion formation model (49) with leader \( x_0(t) = \xi_0 = 5 \), we set \( a_{11} = -0.06, a_{12} = 0.01, a_{21} = -0.03, a_{22} = -0.02, c_1 = 0.2, c_2 = 0.3 \), respectively. We see that \( f_i(t, \xi_0) = 0, \lim_{t \to +\infty} h_i(t) = 0 \) and \( L_i = 1 \) for \( i = 1,2 \) so that all assumptions of Theorem 5 are satisfied. Hence, the leader-follower for opinion formation model (49) is exponentially stable. The states of \( x_0, x_1, \) and \( x_2 \) of system (49) converges to a consensus state \( (5,5,5) \) as shown in Figure 3 for \( \alpha = 0.5 \).

6. Discussion and Conclusion

We study the exponential stability of a leader-follower opinion formation model given by the system of nonlinear fractional differential equations. We mainly focus on the sufficient conditions for a consensus opinion of the system when time-dependent inputs are involved. In particular, under the Lipschitz continuity of the source terms, the boundedness condition on the interaction of agents, and the boundedness of external inputs, we apply the concept of \( \alpha \)-exponential stability to prove the consensus of the followers with the leader. Our main theoretical result is illustrated by a numerical model to verify that consensus opinion can be achieved. The result is complementary to the optimal control problem for the leader-follower opinion formation model where external control is added to drive all agents’ opinions to consensus, for example, in [21, 22]. A general framework for \( \alpha \)-exponential stability can also be considered for fractional-order neural networks as in [24]. In many real-world phenomena, a group of agents can be described by a network where the information exchange between agents is represented by some weight. The results of this paper could be applied to obtain sufficient conditions for the system that ensures consensus with the leader.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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