Characterizations of centrality by local convexity of certain functions on $C^*$-algebras

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Abstract. We provide a quite large function class which is useful to distinguish central and non-central elements of a $C^*$-algebra in the following sense: for each element $f$ of this function class, a self-adjoint element $a$ of a $C^*$-algebra is central if and only if $f$ is locally convex at $a$.

Mathematics Subject Classification (2010). Primary: 46L05.

Keywords. $C^*$-algebra, centrality, convexity.

1. Introduction

1.1. Motivation

Connections between algebraic properties of $C^*$-algebras and some essential properties of functions defined on them by functional calculus have been investigated widely.

The first results concern the relation between the commutativity of a $C^*$-algebra and the monotonicity (with respect to the order induced by positivity) of certain functions defined on the positive cone of it. It was shown by Ogasawara in 1955 that a $C^*$-algebra is commutative if and only if the map $a \mapsto a^2$ is monotone increasing on its positive cone [9]. Later on, Pedersen provided a generalization of Ogasawara’s result for any power function $a \mapsto a^p$ with $p > 1$ [10]. More recently, Wu proved that the exponential function is also useful to distinguish commutative and non-commutative $C^*$-algebras in the above sense [13], and in 2003, Ji and Tomiyama described the class of all functions that can be used to decide whether a $C^*$-algebra is commutative or not [5].

Some "local" results were also obtained in this topic. First, Molnár showed that a self-adjoint element $a$ of a $C^*$-algebra is central if and only if the exponential function is locally monotone at $a$ [8]. Later on, we managed...
to provide a quite large class of functions (containing all the power functions with exponent greater than 1 and also the exponential function) which has the property that each element of this function class can distinguish central and non-central elements via local monotonicity \[12\].

Investigating the connections between the commutativity of a \(C^*\)-algebra (or locally, the centrality of an element) and the global (or local) convexity property of some functions is of a particular interest, as well.

In 2010, Silvestrov, Osaka and Tomiyama showed that a \(C^*\)-algebra \(A\) is commutative if and only if there exists a convex function \(f\) defined on the positive axis which is not convex of order 2 (that is, it is not convex on the \(C^*\)-algebra of the \(2 \times 2\) matrices) but convex on \(A\)[11, Thm. 4].

Motivated by the above mentioned result in \[11\], the main aim of this paper is to provide a large class of functions which have the property that they are locally convex only at central elements, that is, they characterize central elements by local convexity.

1.2. Basic notions, notation

Throughout this paper, \(C^*\)-algebras are always assumed to be unital. The spectrum of an element \(a\) of the \(C^*\)-algebra \(A\) is denoted by \(\sigma(a)\). The symbol \(A_s\) stands for the set of all self-adjoint elements of \(A\). A self-adjoint element of a \(C^*\)-algebra is called positive if its spectrum is contained in \([0, \infty)\). The order induced by positivity on the self-adjoint elements is defined as follows: \(a \leq b\) if \(b - a\) is positive. In the sequel, the symbol \(\mathcal{H}\) stands for a complex Hilbert space and \(B(\mathcal{H})\) denotes the algebra of all bounded linear operators on \(\mathcal{H}\). The inner product on a Hilbert space is denoted by \(\langle \cdot, \cdot \rangle\) and the induced norm is denoted by \(\|\cdot\|\). If \(u\) and \(v\) are elements of a Hilbert space, the symbol \(u \otimes v\) stands for the linear map \(z \mapsto \langle z, v \rangle u\).

2. The main theorem

In this section we provide the main result of this paper. In order to do so, we first need a definition.

**Definition 1 (Local convexity).** Let \(A\) be a \(C^*\)-algebra and let \(f\) be a continuous function defined on some open interval \(I \subset \mathbb{R}\). Let \(a \in A_s\) with \(\sigma(a) \subset I\). We say that \(f\) is locally convex at the point \(a\) if for every \(b \in A_s\) such that \(\sigma(a + b) \cup \sigma(a - b) \subset I\) we have

\[
  f(a) \leq \frac{1}{2} \left( f(a + b) + f(a - b) \right).
\]

**Remark 2.** Note that in fact the above definition is the definition of the midpoint convexity. However, in this paper every function is assumed to be continuous, so there is no difference between midpoint convexity and convexity.

Now we are in the position to present the main result of the paper.
Theorem 3. Let $I \subset \mathbb{R}$ be an open interval and let $f$ be a convex function in $C^2(I)$ such that the second derivative $f''$ is strictly concave on $I$. Let $A$ be a $C^*$-algebra and let $a \in A$ be such that $\sigma(a) \subset I$. Then the followings are equivalent.

(1) The element $a$ is central, that is, $ab = ba$ for every $b \in A$.
(2) The function $f$ is locally convex at $a$.

Example 1. On the interval $I = (0, \infty)$ the functions $f(x) = x^p$ ($2 < p < 3$) satisfy the conditions given in Theorem 3. That is, these functions are useful to distinguish central and non-central elements via local convexity.

3. Proof of the main theorem

This section is devoted to the proof of Theorem 3. We believe that some of the main ideas of the proof can be better understood if we provide the proof first only for the special case of the $C^*$-algebra of all $2 \times 2$ matrices and then turn to the proof of the general case.

3.1. The case of the algebra of $2 \times 2$ matrices

Let $I \subset \mathbb{R}$ be an open interval and $f$ be a function defined on $I$ that satisfies the conditions given in Theorem 3. Let $A$ be the $C^*$-algebra of all $2 \times 2$ complex matrices (which is denoted by $M_2(\mathbb{C})$). Let $A \in M_2(\mathbb{C})$ be a self-adjoint matrix with $\sigma(A) \subset I$.

The proof of the direction $[1] \Rightarrow [2]$ is clear. If $A$ is central, that is, $A = \lambda I_2$ (where $I_2$ denotes the identity element of $M_2(\mathbb{C})$) for some $\lambda \in I$, then $f(A) \leq \frac{1}{2} (f(A + B) + f(A - B))$ holds for every self-adjoint $B \in M_2(\mathbb{C})$ (such that $\sigma(A + B) \cup \sigma(A - B) \subset I$) because of the convexity of $f$ as a scalar function.

The interesting part is the proof of the direction $[2] \Rightarrow [1]$. We will prove it by contraposition, that is, we show that if $A$ is not central, then $f$ is not locally convex at the point $A$. So assume that the self-adjoint matrix $A$ is not central, which means that it has two different eigenvalues, say, $x$ and $y$ in $I$.

Let us use the formula for the (higher order) Fréchet derivatives of matrix valued functions defined by the functional calculus given by Hiai and Petz [4, Thm. 3.33]. This formula is essentially based on the prior works of Daleckii and Krein [2], Bhatia [1], and Hiai [3].

This formula gives us that if $A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then the second order Fréchet derivative of the function $f$ (defined by the functional calculus) at the point $A$ with arguments $(B, B)$ is

$$\partial^2 f(A)(B, B) = 2 \begin{bmatrix} f^{[2]}[x, x, x] + f^{[2]}[x, x, y] & f^{[2]}[x, x, x] + f^{[2]}[x, y, y] \\ f^{[2]}[x, x, y] + f^{[2]}[x, y, y] & f^{[2]}[x, y, y] + f^{[2]}[y, y, y] \end{bmatrix},$$

where $f^{[2]}[\cdot, \cdot, \cdot]$ denotes the second divided difference with respect to $f$. (For the Fréchet derivatives, we use the notation of Hiai and Petz [4].)
It is well-known that
\[ \frac{\partial^2 f(A)(B,B)}{\partial t^2} = \lim_{t \to 0} \frac{1}{t^2} \left( f(A + tB) - 2f(A) + f(A - tB) \right). \]  
(1)

Now we show that \( \partial^2 f(A)(B,B) \) is not positive semidefinite. Indeed, if \( w = [1 - 1]^T \), then
\[
\langle \partial^2 f(A)(B,B)w, w \rangle = 2 \left( f\left([x, x, x]\right) - f\left([x, x, y]\right) - f\left([y, y, y]\right) \right),
\]
(2)

where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( \mathbb{C}^2 \). Using the basic properties of the divided differences (which can be found e.g. in Section 3.4 of the book [4]) on can compute that the above expression (2) is equal to
\[
f''(x) - 2f'(x) \frac{f(x) - f(y)}{x - y} - f''(y) \frac{f(x) - f(y)}{x - y} + f''(y)
\]
= \( f''(x) - 2f'(x) \frac{f(x) - f'(y)}{x - y} + f''(y) \).  
(3)

And the expression (3) is negative by the strict concavity of the function \( f'' \) as one can see for example by the following calculation:
\[
2 \left( \frac{1}{2} \left( f''(x) + f''(y) \right) - \frac{f'(x) - f'(y)}{x - y} \right)
\]
= \( 2 \left( \int_0^1 t f''(x) + (1 - t) f''(y) \, dt - \int_0^1 f''(tx + (1 - t)y) \, dt \right) \)
= \( 2 \int_0^1 t f''(x) + (1 - t) f''(y) - f''(tx + (1 - t)y) \, dt \).  
(4)

The integrand in (4) is continuous in \( t \) and is negative for every \( 0 < t < 1 \) because \( x \neq y \) and \( f'' \) is strictly concave, hence the above integral (4) is negative. So we deduced that \( \langle \partial^2 f(A)(B,B)w, w \rangle < 0 \). (It is fair to remark that the above computation is essentially a possible proof of the well-known Hermite-Hadamard inequality.)

So, by (1), we have
\[
\left\langle \lim_{t \to 0} \frac{1}{t^2} \left( f(A + tB) - 2f(A) + f(A - tB) \right), w, w \right\rangle < 0.
\]

This means that
\[
\lim_{t \to 0} \frac{1}{t^2} \langle (f(A + tB) - 2f(A) + f(A - tB))w, w \rangle < 0,
\]
so there exists some \( t_0 > 0 \) such that
\[
\langle (f(A + t_0B) - 2f(A) + f(A - t_0B)) w, w \rangle < 0. \tag{5}
\]
(For further use, let us denote the negative number in (5) by \(-\delta\).) So, we obtained that \( f(A + t_0B) - 2f(A) + f(A - t_0B) \) is not positive semidefinite, i.e.,
\[
0 \not\leq f(A + t_0B) - 2f(A) + f(A - t_0B),
\]
in other words,
\[
f(A) \not\leq \frac{1}{2} (f(A + t_0B) + f(A - t_0B)).
\]
This means that \( f \) is not locally convex at the point \( A \). The proof is done.

### 3.2. The general case

The proof of Theorem 3 in the case of a general \( C^* \)-algebra is heavily based on our arguments given in [12]. For the convenience of the reader, we repeat some of the arguments of [12] here in this subsection instead of referring to [12] all the time.

Also in this general case, the proof of the direction \( 1 \implies 2 \) is easy. As \( f \) is continuous and convex as a function of one real variable, the map \( a \mapsto f(a) \) is also convex on any set of commuting self-adjoint elements of a \( C^* \)-algebra (provided that the expression \( f(a) \) makes sense). So, centrality automatically implies local convexity.

To prove the direction \( 2 \implies 1 \), we use contraposition again. Assume that \( a \in \mathcal{A} \), \( \sigma(a) \subset I \) and \( a \) is not central, that is, \( aa' - a'a \neq 0 \) for some \( a' \in \mathcal{A} \). Then, by [7, 10.2.4. Corollary], there exists an irreducible representation \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) such that \( \pi( aa' - a'a) \neq 0 \), that is, \( \pi(a) \pi(a') \neq \pi(a') \pi(a) \). Let us fix this irreducible representation \( \pi \). So, \( \pi(a) \) is a non-central self-adjoint (and hence normal) element of \( \mathcal{B}(\mathcal{H}) \) with \( \sigma(\pi(a)) \subset I \) (as a representation does not increase the spectrum). By the non-centrality, \( \sigma(\pi(a)) \) has at least two elements, and by the normality, every element of \( \sigma(\pi(a)) \) is an approximate eigenvalue [6, 3.2.13. Lemma]. Let \( x \) and \( y \) be two different elements of \( \sigma(\pi(a)) \), and let \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \) and \( \{v_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \) satisfy
\[
\lim_{n \to \infty} (\pi(a)u_n - xu_n) = 0, \quad \lim_{n \to \infty} (\pi(a)v_n - yv_n) = 0,
\]
\[
\langle u_m, v_n \rangle = 0 \quad \text{for all } m, n \in \mathbb{N}.
\]
(As \( x \neq y \), the approximate eigenvetors can be chosen to be orthogonal.)

Set \( \mathcal{K}_n := \text{span}\{u_n, v_n\} \) and let \( E_n \) be the orthoprojection onto the closed subspace \( \mathcal{K}_n^\perp \subset \mathcal{H} \). Let
\[
\psi_n(a) := xu_n \otimes u_n + yv_n \otimes v_n + E_n \pi(a) E_n.
\]
We intend to show that
\[
\lim_{n \to \infty} \psi_n(a) = \pi(a)
\]
in the operator norm topology. Let \( h \) be an arbitrary non-zero element of \( \mathcal{H} \) and consider the orthogonal decompositions \( h = h_1^{(n)} + h_2^{(n)} \), where \( h_1^{(n)} \in \mathcal{K}_n \) and \( h_2^{(n)} \in \mathcal{K}_n^\perp \) for any \( n \in \mathbb{N} \). Let us introduce the symbols \( \varepsilon_{u,n} := \)
\(\pi(a)u_n - xu_n\) and \(\varepsilon_{v,n} := \pi(a)v_n - yv_n\) and recall that \(\lim_{n \to \infty} \varepsilon_{u,n} = 0\) and \(\lim_{n \to \infty} \varepsilon_{v,n} = 0\) in the standard topology of the Hilbert space \(\mathcal{H}\). Now,

\[
\frac{1}{||h||} \left|\left| (\pi(a) - \psi_n(a)) h \right|\right| \\
\leq \frac{1}{||h||} \left|\left| (\pi(a) - \psi_n(a)) h_1^{(n)} \right|\right| + \frac{1}{||h||} \left|\left| (\pi(a) - \psi_n(a)) h_2^{(n)} \right|\right|.
\]

Both the first and the second term of the right hand side of the above inequality are bounded by the term \(||\varepsilon_{u,n}|| + ||\varepsilon_{v,n}||\) because

\[
\frac{1}{||h||} \left|\left| (\pi(a) - \psi_n(a)) h_1^{(n)} \right|\right| = \frac{1}{||h||} \left|\left| (\pi(a) - \psi_n(a)) (\alpha_n u_n + \beta_n v_n) \right|\right| \\
= \frac{1}{||h||} \left|\left| \alpha_n xu_n + \alpha_n \varepsilon_{u,n} - xu_n + \beta_n yv_n + \beta_n \varepsilon_{v,n} - y\beta_n v_n \right|\right| \\
\leq \frac{||\alpha_n||}{||h||} ||\varepsilon_{u,n}|| + \frac{||\beta_n||}{||h||} ||\varepsilon_{v,n}|| \leq ||\varepsilon_{u,n}|| + ||\varepsilon_{v,n}||
\]

as the sequences \(\{\alpha_n\}\) and \(\{\beta_n\}\) are obviously bounded by \(||h||\), and

\[
\frac{1}{||h||} \left|\left| (\pi(a) - \psi_n(a)) h_2^{(n)} \right|\right| = \frac{1}{||h||} \left|\left| (I_{\mathcal{H}} - E_n) \pi(a) h_2^{(n)} \right|\right| \\
= \frac{1}{||h||} \left|\left| (u_n \otimes u_n + v_n \otimes v_n) \pi(a) h_2^{(n)} \right|\right| \\
= \frac{1}{||h||} \left|\left| \langle \pi(a) h_2^{(n)} , u_n \rangle u_n + \langle \pi(a) h_2^{(n)} , v_n \rangle v_n \right|\right| \\
= \frac{1}{||h||} \left|\left| h_2^{(n)} , \pi(a) u_n \rangle u_n + \langle h_2^{(n)} , \pi(a) v_n \rangle v_n \right|\right| \\
\leq \frac{1}{||h||} \left|\left| \langle h_2^{(n)} , xu_n + \varepsilon_{u,n} \rangle \right|\right| + \frac{1}{||h||} \left|\left| \langle h_2^{(n)} , yv_n + \varepsilon_{v,n} \rangle \right|\right| \\
= \frac{1}{||h||} \left|\left| h_2^{(n)} , \varepsilon_{u,n} \rangle \right|\right| + \frac{1}{||h||} \left|\left| h_2^{(n)} , \varepsilon_{v,n} \rangle \right|\right| \\
\leq \frac{|h_2^{(n)}|}{||h||} ||\varepsilon_{u,n}|| + \frac{|h_2^{(n)}|}{||h||} ||\varepsilon_{v,n}|| \leq ||\varepsilon_{u,n}|| + ||\varepsilon_{v,n}||.
\]

We used that \(a\) is self-adjoint and that hence so is \(\pi(a)\). So, we found that

\[
\sup \left\{ \frac{1}{||h||} \left|\left| (\pi(a) - \psi_n(a)) h \right|\right| \middle| h \in \mathcal{H} \setminus \{0\} \right\} \leq 2 (||\varepsilon_{u,n}|| + ||\varepsilon_{v,n}||) \to 0,
\]

which means that \(\psi_n(a)\) tends to \(\pi(a)\) in the operator norm topology.

Let us use the notation \(B_n := (u_n + v_n) \otimes (u_n + v_n)\) and \(w_n := u_n - v_n\).

By the result of Subsection 3.1 (the proof for the case of \(A = M_2(\mathbb{C})\)) we have

\[
\langle (f(\psi_n(a) + t_0B_n) - 2f(\psi_n(a)) + f(\psi_n(a) - t_0B_n)) w_n, w_n \rangle \\
= -\delta < 0,
\]

(6)
where \( t_0 \) is the same as in \((5)\) and \(-\delta\) is the left hand side of \((5)\), for any \( n \in \mathbb{N} \). That is, the left hand side of \((6)\) is independent of \( n \).

The operator \( B_n \) is a self-adjoint element of \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{K}_n \) is a finite dimensional subspace of \( \mathcal{H} \), hence by Kadison’s transitivity theorem \([7, 10.2.1. \text{ Theorem}]\), there exists a self-adjoint \( b_n \in \mathcal{A} \) such that

\[
\pi(b_n)|_{\mathcal{K}_n} = B_n|_{\mathcal{K}_n}.
\]

So, we can rewrite \((6)\) as

\[
\langle (f(\psi_n(a) + t_0\pi(b_n)) - 2f(\psi_n(a)) + f(\psi_n(a) - t_0\pi(b_n)))w_n, w_n\rangle = -\delta < 0, \tag{7}
\]

A standard continuity argument which is based on the fact that \( \psi_n(a) \) tends to \( \pi(a) \) in the operator norm topology shows that

\[
\lim_{n \to \infty} ||f(\psi_n(a)) - f(\pi(a))|| = 0. \tag{8}
\]

Moreover, by Kadison’s transitivity theorem, the sequence \( \pi(b_n) \) is bounded (for details, the reader should consult the proof of \([6, 5.4.3. \text{ Theorem}]\)), and hence

\[
\lim_{n \to \infty} ||f(\psi_n(a) \pm t_0\pi(b_n)) - f(\pi(a) \pm t_0\pi(b_n))|| = 0 \tag{9}
\]

also holds. By \((8)\) and \((9)\), for any \( \delta > 0 \) one can find \( n_0 \in \mathbb{N} \) such that for \( n > n_0 \) we have

\[
||f(\psi_n(a)) - f(\pi(a))|| < \frac{1}{16}\delta
\]

and

\[
||f(\psi_n(a) \pm t_0\pi(b_n)) - f(\pi(a) \pm t_0b_n)|| < \frac{1}{16}\delta.
\]

Therefore, by \((7)\), for \( n > n_0 \), the inequality

\[
\langle (f(\pi(a + t_0b_n)) - 2f(\pi(a)) + f(\pi(a - t_0b_n)))w_n, w_n\rangle < -\delta/2 < 0,
\]

holds. In other words,

\[
f(\pi(a)) \not\leq \frac{1}{2}(f(\pi(a + t_0b_n)) + f(\pi(a - t_0b_n))
\]

or equivalently (as functional calculus commutes with every representation of a \( C^* \)-algebra),

\[
\pi(f(a)) \not\leq \pi\left(\frac{1}{2}(f(a + t_0b_n) + f(a - t_0b_n))\right).
\]

Any representation of a \( C^* \)-algebra preserves the semidefinite order, hence this means that

\[
f(a) \not\leq \frac{1}{2}(f(a + t_0b_n) + f(a - t_0b_n)),
\]

which means that \( f \) is not locally convex at \( a \). The proof is done.
Acknowledgement

The author is grateful to Lajos Molnár for proposing the problem discussed in this paper and for great conversations about this topic and about earlier versions of this paper. The author is grateful to Albrecht Böttcher for suggestions that helped to improve the presentation of this paper.

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