Joint Design of Transmit Waveforms and Receive Filters for MIMO Radar via Manifold Optimization

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Abstract

The problem of joint design of transmit waveforms and receive filters is desirable in many application scenarios of multiple-input multiple-output (MIMO) radar systems. In this paper, the joint design problem is investigated under the signal-to-interference-plus-noise ratio (SINR) performance metric, in which case the problem is formulated to maximize the SINR at the receiver side subject to some practical transmit waveform constraints. A numerical algorithm is proposed for problem resolution based on the manifold optimization method, which has been shown to be powerful and flexible to address nonconvex constrained optimization problems in many engineering applications. The proposed algorithm is able to efficiently solve the SINR maximization problem with different waveform constraints under a unified framework. Numerical experiments show that the proposed algorithm outperforms the existing benchmarks in terms of computation efficiency and achieves comparable SINR performance.

Index Terms

MIMO system, SINR maximization, waveform constraints, manifold optimization, Riemannian gradient descent.

I. INTRODUCTION

Multiple-input multiple-output (MIMO) radar systems have attracted a lot of attentions due to its flexibility in transmitting different waveforms through multiple transmit antennas [15]. For different application scenarios, the waveforms in a MIMO radar system can be properly designed to achieve a desired target measured by a specific performance criterion, which may not be possible in the classical phased-array radar systems [14]. Hence, the intriguing property of waveform diversity has provided the MIMO radars many appealing features like higher resolution property and better parameter identifiability property [23].

The problem of joint design of transmit waveforms and receive filters is desirable in many application scenarios of the MIMO radar systems. In this paper, we study the joint design problem to maximize the signal-to-interference-plus-noise ratio (SINR) performance metric at the system receiver side subject to some practical transmit waveform constraints [20]. The problem is intrinsically non-convex due to the highly non-convex fractional objective and the non-convex waveform constraints. Since no analytical solution to the SINR maximization problem can be attained, many iterative algorithms have been applied to solve it. One of the classical methods is the sequential optimization algorithm which is based on the semidefinite relaxation (SDR) with randomization for rank-1 solution reconstruction [7]. Solving an SDR in each iteration has been argued to have high computational complexity [17], which is not applaudable and amenable to large-scale problems and real-time applications. In order to reduce the complexity, a widely used method is to resort to the majorization-minimization (MM) method [12]. The MM method converts the original non-convex problem to a series of relatively simpler problems to be solved in each iteration by choosing a proper upper-bound function. The MM-based algorithm has been shown to be efficient for the SINR maximization problem [20]. Besides, due to its flexibility in choosing the upper-bound function, the MM-based algorithm is able to handle various practical waveform constraints which are not feasible by SDR method.

Recently, the manifold optimization has shown its advantages in dealing these non-convex optimization problems for applications in many fields [11]. In manifold optimization, amounts of constrained optimization problems in the Euclidean space can be regarded as unconstrained optimization problems on the manifolds [11]. Therefore, unconstrained optimization methods (such as the gradient descent and conjugate gradient) can be implemented on the manifold. Similar to other engineering fields, manifold optimization methods have been exploited for problem solving in MIMO radar systems. In [2], a manifold optimization method called Riemannian gradient descent (RGD) has been applied for transmit beampattern synthesis under the unimodular constraint which is modeled as the complex circle manifold (CMM). However, besides unimodular constraint there are several other waveform constraints which have practical applicability with the consideration of hardware configuration. Besides that, there are few literatures studying the joint design of transmit waveforms and receive filters for SINR maximization problem. In this paper, the SINR maximization problem will be studied based on manifold optimization under multiple waveform constraints, where a unified projection operator and a unified retraction operator are defined to help to handle the waveform constraints. Numerical results depict that our algorithm outperforms the state-of-the-art methods in terms of computation efficiency and is able to achieve comparable SINR’s.

II. JOINT TX-RX DESIGN FOR SINR MAXIMIZATION

A MIMO radar system with $N_t$ transmit antennas and $N_r$ receive antennas is considered. Each transmit antenna can emit individual waveform and the $n$-th sample emitted from the $N_t$ transmitters is $s(n) = [s_1(n), \ldots, s_{N_t}(n)]^T \in \mathbb{C}^{N_t}$ with $n = 1, \ldots, N$, where $N$ denotes the total number of transmitted samples. The range-angle position of the target to be tracked is configured as $(r_0, \theta_0)$ and usually we set $r_0 = 0$. Additionally, $K$ signal-dependent interferers located at $(r_k, \theta_k)$ are also
taken into account with the range position \( r_k \in \{0, \ldots, N\} \) and the spatial angle \( \theta_k \in \{0, \ldots, L\} \times \frac{2\pi}{L+1} \) for \( k \neq 0 \) with \( k = 1, \ldots, K \) and \( L \) denoting the number of discrete azimuth sectors. Therefore, the signals at the receive antennas can be represented by

\[
x(n) = \alpha a_r(\theta_0) a_i(\theta_0)^T s(n) + d(n) + v(n),
\]

for \( n = 1, \ldots, N \). In (1), \( \alpha \) is the complex amplitude of the target with \( \mathbb{E} [|\alpha|^2] = \sigma_0^2 \), and \( a_r(\theta) \in \mathbb{C}^{N_r} \) and \( a_i(\theta) \in \mathbb{C}^{N_i} \) are the propagation vector and the steering vector, respectively, with \( a_i(\theta) = \frac{1}{\sqrt{N_r}} [e^{-j\pi \theta}, \ldots, e^{-j\pi (N_r-1) \theta}]^T \) and \( a_r(\theta) = \frac{1}{\sqrt{N_r}} [e^{-j\pi \theta}, \ldots, e^{-j\pi (N_i-1) \theta}]^T \) if the transmit and receive antennas are both assumed to be uniform linear arrays with half-wavelength separation. The term \( d(n) \) denotes \( K \) signal-dependent uncorrelated point-like interferers as \( d(n) = \sum_{k=1}^{K} \alpha_k a_r(\theta_k) a_i(\theta_k)^T s(n-r_k) \), where \( \alpha_k \) denotes a complex amplitude with \( \mathbb{E} [|\alpha_k|^2] = \sigma_k^2 \). The term \( v(n) \in \mathbb{C}^{N_i} \) is a noise term with covariance \( \sigma_n^2 \mathbf{I}_{N_i} \).

Let \( x = [x(1)^T, \ldots, x(N)^T]^T \), \( s = [s(1)^T, \ldots, s(N)^T]^T \), and \( v = [v(1)^T, \ldots, v(N)^T]^T \). We get the following compact form as

\[
x = \alpha A(r_0, \theta_0)s + \sum_{k=1}^{K} \alpha_k A(r_k, \theta_k)s + v,
\]

(2)

where \( A(r_k, \theta_k) = [I_N \otimes (a_r(\theta_k) a_i(\theta_k)^T)] J_{r_k} \) for \( k = 0, \ldots, K \) is a Hermitian matrix related to position \( r_k \) and angle \( \theta_k \) with a shift matrix \( J_{r_k} \in \mathbb{R}^{N \times N} \) given by

\[
J_{r_k}_{m,n} = \begin{cases} 1, & m - n = N_t r_k \\
0, & m - n \neq N_t r_k \\
[N_{r_k}^T]_{m,n}. & \end{cases}
\]

For notational simplicity, we denote \( A(r_k, \theta_k) = A_k \) hereafter.

Let \( w \in \mathbb{C}^{N_t, N} \) be the response receive filters, the SINR \([4]\) at the output side can be calculated as

\[
\text{SINR} = \frac{\sigma_0^2 |w^H A_0 s|^2}{w^H (\sum_{k=1}^{K} \sigma_k^2 A_k s s^H A_k^H) w + \sigma_n^2 w^H w}.
\]

(3)

Finally, the joint design of transmit waveforms and receive filters for SINR maximization (TxRx-SINR) problem is given as

\[
\text{maximize } s, w \quad \frac{|w^H A_0 s|^2}{w^H (\sum_{k=1}^{K} \theta_k A_k s s^H A_k^H) w + w^H w},
\]

(TxRx-SINR)

subject to \( s \in \mathcal{M} \).

where \( \theta_k = \sigma_k^2 / \sigma_n^2 > 0 \), and \( \mathcal{M} \) denotes the different considered waveform constraints to be detailed later.

### III. ALGORITHMIC FRAMEWORK

#### A. Optimization over a manifold

Consider a constrained optimization problem as follows:

\[
\text{minimize } f(x) \quad \text{subject to } x \in \mathcal{R},
\]

where \( \mathcal{R} \) is a constraint set treated as a Riemannian manifold embedded in an Euclidean space \( \mathcal{E} \subseteq \mathcal{R} \) equipping the Riemannian metric \([11]\). By doing this, optimizing \( f(x) \) can be viewed as an unconstrained optimization problem in the manifold \( \mathcal{M} \) rather than a constrained one with explicit constraint \( \mathcal{R} \). Hence, numerous unconstrained optimization algorithms like the gradient descent \([13]\) can be utilized to handle these manifold optimization problems.

In this paper, the classical unconstrained optimization method gradient descent will be implemented for optimization over the Riemannian manifold, which hence is named as Riemannian gradient descent (RGD) \([5]\). Given an initialization \( x^{(0)} \), a sequence \( \{x^{(i)}\} \) is generated by RGD through iteratively taking two steps until convergence. The first step is “descent with projection” where the gradient of any smooth extension of the objective function, i.e., \( \nabla f(x) \) with \( x \in \mathcal{E} \) is computed as \( \nabla f(x^{(i)}) \), i.e., the standard gradient in the Euclidean space, then the Riemannian (manifold) gradient is obtained by projecting \( \nabla f(x^{(i)}) \) onto the tangent space \( T_{x^{(i)}} \mathcal{M} \) by an orthogonal projection at \( x^{(i)} \) denoted by \( \text{Proj}_{x^{(i)}}(\cdot) \), and finally \( x^{(i)} \) is obtained by a descent step on \( T_{x^{(i)}} \mathcal{M} \) with the direction \( \nabla f(x^{(i)}) \) and a prespecified stepsize \( \alpha^{(i)} \). Due to the updated \( x^{(i)} \) is on \( T_{x^{(i)}} \mathcal{M} \) rather than the manifold \( \mathcal{M} \), a “retraction” at \( x^{(i)} \) denoted by the operator \( \text{Retr}(\cdot) \) is applied in the second step to map it back to \( \mathcal{M} \). To summarize, the update step of RGD at the \( i \)-th iteration is

\[
\begin{align*}
\bar{x}^{(i+1)} &= x^{(i)} - \alpha^{(i)} \text{Proj}_{x^{(i)}}(\nabla f(x^{(i)})) \quad \text{[descent with projection step]} \\
x^{(i+1)} &= \text{Retr}(\bar{x}^{(i+1)}) \quad \text{[retraction step]},
\end{align*}
\]

where stepsize \( \alpha^{(i)} \) can be chosen to be constant or according to a specific stepsize rule like the Armijo back-tracking line search \([3]\) for convergence guarantee, and the projection operator \( \text{Proj}_{x^{(i)}}(\cdot) \) and the retraction operator \( \text{Retr}(\cdot) \) vary from manifolds.

#### B. The projection and retraction in TxRx-SINR

In this section, we consider the projection operators and the retraction operators w.r.t. different constraints \( \mathcal{M} \)’s encountered in the TxRx-SINR problem \([\text{TxRx-SINR}]\). Three manifold constraints are considered which are highly non-convex in the
Euclidean space, namely the constant modulus (CM) constraint \( \mathcal{M}_c = \{ s \mid |s_n| = \frac{1}{\sqrt{\sum_{n=1}^{N}}} \} \) [10] (including the unimodular constraint, i.e., the CCM with \(|s_n| = 1\)), the \( \epsilon \)-uncertainty constant modulus (\( \epsilon \)-CM) constraint \( \mathcal{M}_c = \{ s \mid c_m - \epsilon_1 < |s_n| < c_m + \epsilon_2 \text{ with } 0 \leq \epsilon_1 \leq c_m \text{ and } 0 \leq \epsilon_2 \} \) [22], and the constant modulus and similarity (CM&S) constraint \( \mathcal{M}_c = \{ s \mid |s_n| = 1 \text{ and } \sum_{n=1}^{N} ||s - s_{ref}||_\infty \leq \epsilon, \text{ with } 0 \leq \epsilon \leq \frac{2}{\sqrt{\sum_{n=1}^{N}}} \} \) [2].

**Projection step.** The projection operator \( \text{Proj}_{\mathcal{M}_c} (\cdot) \) at an iterate \( s(i) \in \mathcal{M} \) with \( \mathcal{M}_c \)'s or \( \mathcal{M}_s \)'s is the same and has a closed-form solution. This result is classical in manifold optimization and can be easily proved by first showing the complex scalar case and then extending it to the vector case [5]. For any \( u \in \mathbb{C}^{N_M} \), the projection operator for these two constraints is given by

\[
\text{Proj}_{\mathcal{M}_c} (u) = u - \text{Re} \left\{ u^* \odot s(i) \right\} \odot s(i),
\]

(4)

where \( \odot \) denotes the Hadamard product. The \( \epsilon \)-CM constraint describes an annulus manifold, the projection operator of which is given by

\[
\text{Proj}_{\mathcal{M}_c} (u) = u,
\]

(5)

**Retraction step.** The retraction operators \( \text{Retr} (\cdot) \)'s w.r.t. different \( \mathcal{M} \)'s can be solved in closed-forms. For a given \( u \in \mathbb{C}^{N_M} \), a unified retraction function can be employed to handle all the manifold constraints, which is given by

\[
\text{Retr} (u) = \arg \min_{s \in \mathcal{M}} ||s - u||_2^2,
\]

(6)

where specifically the solution w.r.t. \( \mathcal{M}_c \) is given by \( \text{Retr} (u) = u \odot (\sqrt{\sum_{n=1}^{N} |u|^2})^{-1} \) with \(|\cdot|\) and \((\cdot)^{-1}\) applied element-wisely, w.r.t. \( \mathcal{M}_s \) is given in [8], and w.r.t. \( \mathcal{M}_c \) is given in [22].

**IV. SOLVING THE TXRX-SINR PROBLEM VIA RGD**

Now we are ready to derive the RGD algorithm for problem (TxRx-SINR). Noting that problem (TxRx-SINR) is invariant to a scaling in \( w \), for a fixed \( s \) it can be transformed to be a convex problem as

\[
\begin{align*}
\text{minimize} \quad & w^H \left[ \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I \right] w \\
\text{subject to} \quad & w^H A_0 s = 1,
\end{align*}
\]

(Rx Prob.)

to which a closed-form solution for \( w \) is obtained by [6]

\[
w^* = \frac{\sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I}{s^H A_0^H \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I} A_0 s.
\]

(Optim. Rx)

Substituting (Optim. Rx) into the original problem (TxRx-SINR) we get the subproblem for the transmit waveforms as

\[
\begin{align*}
\text{minimize} \quad & -s^H A_0^H \left[ \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I \right]^{-1} A_0 s \\
\text{subject to} \quad & s \in \mathcal{M},
\end{align*}
\]

(Tx Prob.)

where the waveform constraints \( \mathcal{M} \) can take different forms as discussed in Sec. III-B.

To solve the original problem (TxRx-SINR), it suffices to solve the problem (Tx Prob.) for \( s \) and then obtain \( w \) by (Optim. Rx). In this paper, we propose to solve (Tx Prob.) via the RGD method in Sec. III-A. For convergence concern, the objective function for problem (Tx Prob.) will be augmented with a constant term \( \gamma s^H s \) w.l.o.g. \( \gamma \) is a prescribed constant, the choice of which guarantees the monotonicity of “projection” step in RGD and is discussed in an online supplementary material [21] due to space limitation) to control the monotonicity of the retraction operator \( \text{Retr} (\cdot) \). Then we define the “augmented” objective function for problem (Tx Prob.) as

\[
g(s) = -s^H (A_0^H \left[ \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I \right]^{-1} A_0) s + \gamma s^H s.
\]

The gradient of a smooth extension of the objective function denoted by \( \tilde{g}(s) \) (extending \( g(s) \) to the Euclidean domain) is given by

\[
\nabla \tilde{g}(s) = -2 \left( A_0^H \left[ \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I \right]^{-1} A_0 \right) s + 2\gamma s - \left( s^H \frac{\partial}{\partial s} \left( A_0^H \left[ \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I \right]^{-1} A_0 \right) \right) s.
\]

\[
= -2 \left( A_0^H \left[ \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I \right]^{-1} A_0 \right) s + 2\gamma s - \left( A_0^H \left[ \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I \right]^{-1} \right) \cdot \left( s^H \left( \partial_{s_1} A_k^H \cdots \partial_{s_N} A_k^H \right) \left[ \sum_{k=1}^{K} \partial_k A_k s s^H A_k^H + I \right]^{-1} A_0 \right).
\]

(7)
In this paper, we have considered the SINR maximization problem in MIMO radar subject to multiple practical waveform constraints by jointly designing the transmit waveforms and receive filters. A manifold optimization algorithm called RGD is proposed for problem resolution. Numerical results validate the superiority of the proposed algorithms. It should be mentioned that a well-chosen stepsize $\alpha^{(t)}$ is to ensure the decrease of the objective function $g(\cdot)$ in the “descent with projection” step of RGD.
Fig. 1: Convergence rate comparisons.
TABLE I: Runtime comparisons under CM constraint.

| Algorithm    | \( (N_r, N_t, N) \)   |
|--------------|------------------------|
|              | (4,4,4)               | (10,10,4)       | (10,10,8)       | (15,15,8)       | (10,10,30)      |
| RGD-Armijo   | 0.0197sec.            | 0.7969sec.      | 0.7031sec.      | 1.875sec.       | 35.1021sec.     |
| MM-SQUAREM   | 0.3729sec.            | 1.2813sec.      | 3.7134sec.      | 4.0156sec.      | 31.4375sec.     |
| RGD          | 0.7536sec.            | 2.2971sec.      | 10.3421sec.     | 20.3203sec.     | 431.7156sec.    |
| MM           | 4.3909sec.            | 29.7969sec.     | 173.4153sec.    | 207.6719sec.    | 620.8147sec.    |
| SDR          | 7.43386sec.           | 518.7628sec.    | 1026.127sec.    | 1231.6143sec.   | 2029.9058sec.   |

Fig. 2: Range-angle ambiguity function for CM constraint.

that we note there is an independent work on manifold optimization for the SINR maximization problem available recently [16], which considers a different system setting (the airborne MIMO-STAP radar setting) and only focuses on the CM constraint. Besides that, the RGD algorithm proposed in our paper is leveraging the variable reduction technique in the alternating minimization which is different from the technique used in that paper.
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