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Majnu John
Hofstra University, Hempstead, NY 11549, USA, majnu.john@hofstra.edu

Yihren Wu
Hofstra University, Hempstead, NY 11549, USA, yihren.wu@hofstra.edu

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DYNAMIC CORRELATION ESTIMATORS FOR BIVARIATE BROWNIAN AND GEOMETRIC BROWNIAN MOTIONS

MAJNU JOHN* AND YIHREN WU

ABSTRACT. Estimating dynamic correlation between a pair of time series is of importance in many applications. We present new estimators for the dynamic correlation between a pair of correlated Brownian motions and separately for dynamic correlation between a pair of correlated Geometric Brownian motions. We show that, as the sample size increases, all estimators presented in this paper converge in probability to the underlying true dynamic correlation.

1. Introduction

Dynamic correlation between a pair of time series is of long-standing interest in many fields such as economics [1, 7] and neuroscience [5]. In neuroimaging fMRI studies, for example, the connectivity (not anatomical connectivity but functional connectivity-) of different brain regions are often hypothesized to change over time. Such dynamic functional connectivity reveals underlying biological mechanisms and are often studied. In neurophysiological studies of rodent brains, correlations between local field potential time series from a pair of regions may change over time. Dynamic changes of correlation in such settings reveal time segments during which the corresponding brain regions work in tandem (e.g. during certain specific tasks or in response to a stimulus) and other time segments when they do not.

Recently, a nonparametric estimator of dynamic correlation, robust in the presence of extreme values, was introduced in [4]. The new estimator may be conceptualized based on a particular algorithm for conversion of time series into a weighted graph and considering the edge-weights of the graph. In this paper we consider extensions of the estimators presented in [4]. Specifically, we present estimators for a pair of correlated bivariate Brownian motions and estimators for a pair of correlated Geometric Brownian motions. The main focus of the paper is in showing weak consistency of the estimators.

The paper is structured as follows. The new estimators are presented in section 2. Next, in section 3, the weak consistency results are presented. Section 4 summarizes the conclusions and includes a brief discussion.

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* Corresponding author.
2. New Estimators

First we focus on a correlated (bivariate) Brownian motion pair denoted by 
\((X_t, Y_t), t \in [0, T]\), and present a weakly-consistent estimator for the dynamic 
correlation, \(\rho_t = \gamma_t / \sigma_t \sigma_t^\gamma = \gamma_t / t\) between \(X_t\) and \(Y_t\). Assume \(n\) points are sampled 
from each time series at time points \(\{t_1, \ldots, t_n = T\} \equiv I\). We assume without 
loss of generality, \(E(X_t) = 0 = E(Y_t), \forall t\) and for convenience, we take \(t_i = i, \forall i;\) however, our results will hold in general, even without these simplifications. The 
estimators that we consider are \(\hat{\rho}^{\gamma,p}_u = \hat{\gamma}^{\rho,p}_u / (\hat{\sigma}^{\gamma,p}_u \hat{\sigma}^{\gamma,p}_u), u \in I, q, p \geq 0,\) where

\[
\hat{\gamma}^{\rho,p}_u = \frac{1}{T-1} \sum_{v \in I, v \neq u} \frac{(v^\rho X_u - v^p X_u)(v^\rho Y_u - v^p Y_u)}{(u - v)^2},
\]

\[
(\hat{\sigma}^{\gamma,p}_u)^2 = \frac{1}{T-1} \sum_{v \in I, v \neq u} \frac{(v^\rho X_u - v^p X_u)^2}{(u - v)^2},
\]

\[
(\hat{\sigma}^{\gamma,p}_u)^2 = \frac{1}{T-1} \sum_{v \in I, v \neq u} \frac{(v^\rho Y_u - v^p Y_u)^2}{(u - v)^2}.
\]

The estimators that we originally considered in [4] were special cases of the 
estimator presented in (2.1), specifically with \(p = q = 0\). The main reason that we 
consider the more general version is that, while studying the weak consistency of 
the original estimators theoretically, we realized that the asymptotic bias is non-
zero for those estimators. In other words, weak consistency does not hold in the 
special case \(p = q = 0\). In the general case, weak consistency do hold for certain 
specific range of values for \(p\) and \(q\), as shown in section 3 below.

Next we consider a correlated (bivariate) Geometric Brownian motion pair 
\((R_t, S_t), t \in [0, T]\). The most general form that we should consider is

\[R_t = R_0 \exp \left\{ \left( \mu_R - \frac{\sigma_R^2}{2} \right) t + \sigma_R W_t \right\}\] 
\[S_t = S_0 \exp \left\{ \left( \mu_S - \frac{\sigma_S^2}{2} \right) t + \sigma_S U_t \right\},\]

where \((W_t, U_t)\) is a correlated Brownian motion pair. For ease of exposition, we 
restrict our attention to the case with \(R_0 = S_0 = 1\) and \(\mu_R = \mu_S = \sigma_R^2/2 = \sigma_S^2/2\) 
and \(\sigma_R = \sigma_S = \sigma\) so that

\[R_t = e^{\sigma W_t} \text{ and } S_t = e^{\sigma U_t}\]

is the Geometric Brownian motion pair that we consider. Note that in this case 
we have \(E(R_t) = E(S_t) = e^{\sigma^2 t/2}\) and \(\text{Var}(R_t) = \text{Var}(S_t) = e^{\sigma^2 t}(e^{\sigma^2 t} - 1)\). We 
also mention here that we use the same notation \((\rho_t \text{ and } \gamma_t, \text{ respectively})\) 
for the correlation and covariance between \(R_t\) and \(S_t\) as well. That is, whether \(\rho_t\) 
(similarly \(\gamma_t\)) stands for the correlation (covariance) between the Brownian motion 
pair \((X_t, Y_t)\) or for the Geometric Brownian motion pair \((R_t, S_t)\) will hopefully 
be understood from the context. We consider two estimators for the dynamic
correlation \( \rho_t = \frac{\gamma_t}{\sigma_t R_t \sigma_t S_t} = \frac{\gamma_t}{(\sigma_{t,W} \sigma_{t,U})} \) between \( R_t \) and \( S_t \). In the first case

\[
\hat{\gamma}_t = \frac{1}{e^{c \sigma^2 T}} \sum_{k=1}^{T} \left\{ e^{-a \sigma^2 k} \left( e^{\sigma W_k} - e^{\sigma^2 t} \right) - e^{-a \sigma^2 k} \left( e^{\sigma W_t} - e^{\sigma^2 t} \right) \right\} \times \left\{ e^{-b \sigma^2 k} \left( e^{\sigma U_k} - e^{\sigma^2 t/2} \right) - e^{-b \sigma^2 k} \left( e^{\sigma U_t} - e^{\sigma^2 t/2} \right) \right\}.
\]

In the second case, we have the estimator for \( \gamma_t \) as

\[
\hat{\gamma}_t = e^{-c \sigma^2 T} \sum_{k=1}^{T} \left\{ e^{a \sigma^2 k} \left( e^{\sigma W_k} - e^{\sigma^2 t/2} \right) - e^{-a \sigma^2 k} \left( e^{\sigma W_t} - e^{\sigma^2 t/2} \right) \right\} \times \left\{ e^{b \sigma^2 k} \left( e^{\sigma U_k} - e^{\sigma^2 k/2} \right) - e^{-b \sigma^2 k} \left( e^{\sigma U_k} - e^{\sigma^2 k/2} \right) \right\}.
\]

By considering \( W_k = U_k \) for all \( k \) (and \( W_t = U_t \)) in equations (2.2) and (2.3) we get the corresponding estimates for \( \sigma_{t,W}^2 \) and \( \sigma_{t,U}^2 \). Here \( a, b \) and \( c \) are real-valued constants with appropriate ranges given in section 3.

If the correlation between \( W_t \) and \( U_t \) is denoted by \( r_t \), then it is easy to verify that

\[
r_t = \frac{1}{\sigma^2 t} \log \left[ 1 + \rho_t (e^{\sigma^2 t} - 1) \right] \quad \text{or} \quad \rho_t = \frac{e^{\sigma^2 t} - 1}{e^{\sigma^2 t} - 1}.
\]

Based on this, we may also consider an estimate for \( \rho_t \) by plugging in an estimate for \( r_t \) (for example, the one given in (2.1)). However we do not pursue such estimators in this paper as our focus is on estimators that are generalizations of the original estimators that we considered in [4].

3. Results

3.1. Weak consistency of dynamic correlation estimators in the Brownian Motion case. Our main result in this subsection is the following theorem.

**Theorem 3.1.** \( \hat{\rho}_t^{q,p} \rightarrow \rho_u \) in probability for each \( u \) and \( p > q = 1/2 \), as \( T \rightarrow \infty \).

We first make a few remarks which will be used later.

**Remark 3.2.** A random variable \( V \) follows a variance-gamma distribution with parameters \( r > 0, \theta \in \mathbb{R}, \sigma > 0, \mu \in \mathbb{R} \) (denoted VG(\( r, \theta, \sigma, \mu \)) if it has probability density function given by

\[
p_{VG}(x; r, \theta, \sigma, \mu) = \frac{1}{\sigma \sqrt{\pi T} (\theta/2)^{r+1/2}} \exp \left( \frac{\theta}{\sigma^2} (x - \mu) \right) \left( \frac{|x - \mu|}{2\sqrt{\theta^2 + \sigma^2}} \right)^{r+1} \times K_{\frac{r+1}{2}} \left( \frac{\sqrt{\theta^2 + \sigma^2}}{\sigma} |x - \mu| \right),
\]

where \( x \in \mathbb{R} \) and \( K_{\nu}(\cdot) \) is a modified Bessel function of the second kind [3, 6]. We have \( \mathbb{E}(V) = \mu + r \theta, \Var(V) = r(\sigma^2 + 2\theta^2) \) [2].

**Remark 3.3.** If \( (X, Y) \) denote a bivariate normal random vector with zero means, variances \( (\sigma^2_x, \sigma^2_y) \) and correlation coefficient \( \rho \), then \( Z = XY \) follows \( \text{VG}(1, \rho \sigma_x \sigma_y, \sigma_x \sigma_y \sqrt{1 - \rho^2}, 0) \) [3]. In particular, based on Remark 3.2, \( \mathbb{E}(Z) = \)
Remark 3.4. Based on Remark 3.3, for the bivariate Brownian motion \((X_t, Y_t)\) mentioned in the first paragraph, we have \(\mathbb{E}(X_t Y_t) = t\rho_t\), \(\text{Var}(X_t Y_t) = (1 + \rho_t^2)/t^2 \leq 2/t^2\), \(\mathbb{E}(X_t Y_t)^2/t^4 = [(1 + \rho_t^2)t^2 + \rho_t^2 t^4]/t^4 \leq 3t^{-2}\) and for \(s < t\),

\[
\mathbb{E}\left(\frac{X_t Y_t}{t^2 s^2}\right) = \frac{1}{t^2 s^2} \left[\mathbb{E}\left((X_t - X_s)(Y_t - Y_s) X_s Y_s\right) + \mathbb{E}(Y_t - Y_s) X_s^2 Y_s\right]
\]

\[
+ \mathbb{E}\left([X_t - X_s] X_s Y_s^2\right) + \mathbb{E}(X_s^2 Y_s^2)\]

\[
\leq C_1 (t - s)s + C_2 (t - s)^{1/2} s^{3/2} + C_3 s^2
\]

\[
= \frac{C}{t s} \left[\frac{1}{t} + \frac{\sqrt{1 - (s/t)}}{t^{3/2} s^{1/2}} + \frac{1}{t^2}\right] \leq C s^2,
\]

(3.1)

Here \(C_1, C_2, C_3\) and \(C\) are generic finite positive constants.

In order to get the inequality in (3.1), we first note that \(X_t\) and \(Y_t\) may be written as

\[
X_t = \rho_t Y_t + \sqrt{1 - \rho_t^2} X_t^{(1)}, \quad Y_t = \rho_t X_t + \sqrt{1 - \rho_t^2} Y_t^{(1)},
\]

(3.2)

where \(\{X_t^{(1)}\}\) and \(\{Y_t^{(1)}\}\) are mean zero Brownian motions with \(\{X_t^{(1)}\}\) independent of \(\{Y_t\}\) and \(\{Y_t^{(1)}\}\) independent of \(\{X_t\}\). Using (3.2) we may write

\[
(X_t - X_s) X_s = (\rho_t X_t)(Y_t - Y_s) Y_s + (\rho_t Y_t + \sqrt{1 - \rho_t^2}) X_t^{(1)} Y_s \]

(3.3)

so that

\[
(X_t - X_s)(Y_t - Y_s) X_s Y_s \leq (Y_t - Y_s)^2 Y_s^2 + 2|Y_t - Y_s||Y_s|^3
\]

\[
+ 2|X_t^{(1)} - X_s^{(1)}||Y_t - Y_s||Y_s|^2 + 2|X_s^{(1)}||Y_t - Y_s|Y_s^2.
\]

Hence, using the fact that for a Brownian motion \(\{B_u\}\), \(\mathbb{E}|B_u|^a \leq K u^{a/2}\),

\[
\mathbb{E}|(X_t - X_s)(Y_t - Y_s) X_s Y_s| \leq C_1 (t - s)s + C_2 \sqrt{t - s} s^3.
\]

Using (3.3) again, we will get

\[
\mathbb{E}|(X_t - X_s) X_s Y_s^2| \leq C_2 \sqrt{t - s} s^3
\]

and similarly

\[
\mathbb{E}|(Y_t - Y_s) X_s^2 Y_s| \leq C_2 \sqrt{t - s} s^3.
\]

We also have \(\mathbb{E}(X_s^2 Y_s^2) \leq 4s^2\). Putting all these together we get the inequality in (3.1). This completes Remark 3.4.

We denote \(S_n = \sum_{i=1}^n n^{-a}\) for \(a > 0\); \(S_n < \infty\) for \(a > 1\). Although the primary focus of this subsection and the main result is for the case \(p > q = 1/2\), we have the following result for \(p = q = 0\).

Lemma 3.5. \(\text{Var}(\gamma_{0,0}^t) \rightarrow 0\), \(\text{Var}([\hat{\sigma}_1^{x,0,0}]^2) \rightarrow 0\) and \(\text{Var}([\hat{\sigma}_1^{y,0,0}]^2) \rightarrow 0\), for each \(i\), as \(T \rightarrow \infty\).
Proof.

\[ \text{Var}(\hat{\gamma}_i^{0,0}) \leq \mathbb{E}(\hat{\gamma}_i^{0,0})^2 \]

\[ = \frac{1}{(T - 1)^2} \sum_{j=1}^{T} \frac{\mathbb{E}[(X_i - X_j)(Y_i - Y_j)]^2}{(i - j)^4} \]

\[ + \frac{1}{(T - 1)^2} \sum_{j=1}^{T} \sum_{k=1}^{T} \frac{\mathbb{E}[(X_i - X_j)(Y_i - Y_j)(X_i - X_k)(Y_i - Y_k)]}{(i - j)^2(i - k)^2} \]

Using one of the facts mentioned in Remark 3.4 in the inequality below, there are \( L \) values for which \( (i - j)^2 < (i - k)^2 \), for some fixed \( L \) in \( \{1, \ldots, T\} \), and for the remaining \( T - L - 2 \) values \( (i - j)^2 > (i - k)^2 \). So, applying another fact mentioned in Remark 3.4, we get,

\[ 2^{nd} \text{ term} \leq \frac{1}{(T - 1)^2} \sum_{j=1}^{T} \left\{ \frac{L}{(i - j)^2} + C_1 S_2 \right\} \leq \frac{(C_2 + C_1 T) S_2}{(T - 1)^2} = O(T^{-1}), \]

where \( C_1 \) and \( C_2 \) are finite constants. Putting this together, we get \( \text{Var}(\hat{\gamma}_i^{0,0}) \to 0 \) as \( T \to \infty \). Using similar arguments it is easy to prove the remaining statements in the lemma. \( \square \)

**Lemma 3.6.** \( \text{Var}(\hat{\gamma}_i^{q,p}) \to 0, \text{Var}([\hat{\sigma}_i^{q,p}]^2) \to 0 \) and \( \text{Var}([\hat{\sigma}_i^{q,p}]^2) \to 0 \), for each \( i \) and for each \( q \) and \( p \) with \( 0 < q \leq 1/2, p > 1/2, \) as \( T \to \infty \).

**Proof.**

\[ \text{Var}(\hat{\gamma}_i^{q,p}) \leq \frac{1}{(T - 1)^2} \sum_{j=1}^{T} \frac{\mathbb{E}[(j^q X_i - X_j^{q-p})(j^q Y_i - Y_j^{q-p})]^2}{(i - j)^4} \]

\[ + \frac{1}{(T - 1)^2} \sum_{j=1}^{T} \sum_{k=1}^{T} \frac{\mathbb{E}[(j^q X_i - X_j^{q-p})(j^q Y_i - Y_j^{q-p})(k^q X_i - X_k^{q-p})(k^q Y_i - Y_k^{q-p})]}{(i - j)^2(i - k)^2} \]

\[ \times \frac{1}{(i - k)^2} \]
and
\[
(j^qX_i - X_j,j^{-p})^2(j^qY_i - Y_j,j^{-p})] ^2 \nonumber \\
= [j^{2q}(X_i - X_j)^2 + 2X_j(X_i - X_j)j^q(j^q - j^{-p}) + X_j^2(j^q - j^{-p})^2] \nonumber \\
\times [j^{2q}(Y_i - Y_j)^2 + 2Y_j(Y_i - Y_j)j^q(j^q - j^{-p}) + Y_j^2(j^q - j^{-p})^2].
\]

Expanding and taking expectations, many terms become zero by applying properties of Brownian motion. With the remaining non-zero terms, the first term in (3.5) becomes
\[
\frac{1}{(T-1)^2} \sum_{j=1}^{T} \sum_{j\neq 1} (j-i)^{-4} \{ [j^{2q}(X_i - X_j)^2(Y_j - Y_i)^2] \nonumber \\
+ \mathbb{E}[j^{2q}(X_i - X_j)^2]\mathbb{E}[Y_j^2(j^q - j^{-p})^2] \nonumber \\
+ 4\mathbb{E}[X_j Y_j]\mathbb{E}[(X_j - X_i)(Y_j - Y_i)j^{2q}(j^q - j^{-p})^2] \nonumber \\
+ \mathbb{E}[X_j^2]\mathbb{E}[Y_j - Y_i]^2j^{2q}(j^q - j^{-p})^2] \nonumber \\
+ \mathbb{E}[X_j^2 Y_j^2(j^q - j^{-p})^4] \}. \tag{3.6}
\]

We will now show that the product of $(j-i)^{-4}$ and each expectation term in (3.6) to (3.10) can be written as $C_1 + C_2 T$, for some finite constants $C_1$ and $C_2$ so that this product when multiplied by $(T-1)^{-2}$ is $O(T^{-1})$. Using Remark 3.4, for $j > i$,
\[
\frac{j^{4q} \mathbb{E}[(X_j - X_i)^2(Y_j - Y_i)^2]}{(j-i)^4} \leq \frac{3j^{4q}}{(j-i)^2} < \frac{3j^2}{(j-i)^2} \text{ if } 4q \leq 2; \text{ i.e. if } q \leq 1/2 \nonumber \\
\leq 3 \left(1 + \frac{2i}{j-i} + \frac{i^2}{(j-i)^2}\right) < 12, \text{ if } j > 2i. \tag{3.11}
\]

Since the first $2i$ terms in the sum in (3.6) is a constant, it is easy to see based on (3.11) that the whole sum can be bounded by $C_1 + C_2 T$, for some finite constants $C_1$ and $C_2$.
\[
\frac{j^{2q}(j^q - j^{-p})^2 \mathbb{E}[X_j - X_i]^2 \mathbb{E}[Y_j^2]}{(j-i)^4} < \frac{j^{2q+1}(j^{2q} + 1)}{(j-i)^4} \nonumber 
\]
can be bounded by a constant if $4q + 1 \leq 3$; i.e. if $q \leq 1/2$. Hence the sum corresponding to (3.7) can be expressed as $C_1 + C_2 T$. Similar reasoning can be applied to sums in (3.8), (3.9) and (3.10) so that the first term in (3.5) is $O(T^{-1})$.

Expanding the second term in (3.5) gives 16 terms. The double-sum ranging over $j$ and $k$ for each of these 16 terms can be expressed as $[C_1 S_a + C_2] S_b T$ for some $a > 1$ and $b > 1$, so that after multiplication with $(T-1)^{-2}$ each term becomes $O(T^{-1})$. For example, the last term will be
\[
\frac{\mathbb{E}[X_j Y_j j^{-2p} X_k Y_k k^{-2p}]}{(j-i)^2(k-i)^2} \leq \frac{j^{-2p+2k-2p+2}}{(j-i)^2(k-i)^2} \times \frac{3}{j^2}, \text{ for } k > j > i. \nonumber 
\]

The sum over $k$ of the above terms $< \frac{j^{-2p}}{(j-i)^2}(C + S_{2p})$,

and further sum over $j < [C_2 + C_1 S_{2p}] S_{2p+2} T$. 

Proceeding similarly with other terms, the 2nd term in (3.5) is also seen to be $O(T^{-1})$ and hence $\text{Var}(\hat{\gamma}_t^{q,p}) \to 0$ as $T \to \infty$. Using similar arguments the remaining statements in the lemma can also be proved.

**Lemma 3.7.** For each $t \in \{1, \ldots, n(=T)\}$,

$$Q_t = \mathbb{E}(\hat{\gamma}_t^{q,p})/\sqrt{\mathbb{E}((\hat{\sigma}_t^{x,q,p})^2)\mathbb{E}((\hat{\sigma}_t^{y,q,p})^2)} \to \rho_t$$

as $T \to \infty$, if $p > q \geq 1/2$.

**Proof.**

$$(T - 1)\mathbb{E}(\hat{\gamma}_t^{q,p}) = tp_t \sum_{s=1 \atop s \neq t}^T \frac{s^{2q}}{(s-t)^2} + \sum_{s=1 \atop s \neq t}^T \rho_s s^{-2p+1}$$

$$- 2 \left[ \sum_{s=1}^{t-1} \rho_s s^{-p+q+1} + \sum_{s=t+1}^T t \rho_s s^{-p+q} \right]$$

$$= tp_t \sum_{s=1 \atop s \neq t}^T \frac{s^{2q}}{(s-t)^2} + \sum_{s=1 \atop s \neq t}^T \rho_s s^{-2p+1}$$

$$- 2 \sum_{s=1 \atop s \neq t}^T \frac{s^{-p+q+1}}{(s-t)^2} + 2 \sum_{s=t+1}^T \frac{s^{-p+q}(s - t)}{(s-t)^2}. $$

Similarly,

$$(T - 1)\mathbb{E}((\hat{\sigma}_t^{x,q,p})^2) = (T - 1)\mathbb{E}((\hat{\sigma}_t^{y,q,p})^2) = t \sum_{s=1 \atop s \neq t}^T \frac{s^{2q}}{(s-t)^2} + \sum_{s=1 \atop s \neq t}^T s^{-2p+1}$$

$$- 2 \sum_{s=1 \atop s \neq t}^T \frac{s^{-p+q+1}}{(s-t)^2} + 2 \sum_{s=t+1}^T \frac{s^{-p+q}(s - t)}{(s-t)^2}. $$

Hence $Q_t$ can be written as a ratio

$$Q_t = \frac{tp_t + K_T^{-1}A_{1,T} - 2K_T^{-1}A_{2,T} + 2K_T^{-1}A_{3,T}}{t + K_T^{-1}B_{1,T} - 2K_T^{-1}B_{2,T} + 2K_T^{-1}B_{3,T}}, \quad (3.12)$$

where

$$K_T = \sum_{s=1 \atop s \neq t}^T \frac{s^{2q}}{(s-t)^2}, \quad A_{1,T} = \sum_{s=1 \atop s \neq t}^T \frac{\rho_s s^{-2p+1}}{(s-t)^2},$$

$$A_{2,T} = \sum_{s=1 \atop s \neq t}^T \frac{\rho_s s^{-p+q+1}}{(s-t)^2}, \quad A_{3,T} = \sum_{s=t+1}^T \frac{s^{-p+q}(s - t)}{(s-t)^2},$$

$$B_{1,T} = \sum_{s=1 \atop s \neq t}^T \frac{s^{-2p+1}}{(s-t)^2}, \quad B_{2,T} = \sum_{s=1 \atop s \neq t}^T \frac{s^{-p+q+1}}{(s-t)^2}, \quad B_{3,T} = \sum_{s=t+1}^T \frac{s^{-p+q}(s - t)}{(s-t)^2}. $$
Theorem 3.8. \( \hat{\rho}_{t}^{a,b,c} \rightarrow \rho_{t} \) in probability for each \( t, c > a > 0 \) and \( b > a + 10 \), as \( T \to \infty \), where \( \hat{\rho}_{t}^{a,b,c} = \gamma_{t}/(\hat{\sigma}_{t,W}^{2}\hat{\sigma}_{t,U}^{2}) \) with \( \hat{\gamma}_{t} \) given in (2.2) and \( \hat{\sigma}_{t,W}^{2}, \hat{\sigma}_{t,U}^{2} \) the corresponding estimates of the variances.

Theorem 3.9. \( \hat{\rho}_{t}^{a,b,c} \rightarrow \rho_{t} \) in probability for each \( t, b > 15 \) and \( c > a > 0 \), as \( T \to \infty \), where \( \hat{\rho}_{t}^{a,b,c} = \gamma_{t}/(\hat{\sigma}_{t,W}^{2}\hat{\sigma}_{t,U}^{2}) \) with \( \hat{\gamma}_{t} \) given in (2.3) and \( \hat{\sigma}_{t,W}^{2}, \hat{\sigma}_{t,U}^{2} \) the corresponding estimates of the variances.

We first consider Theorem 3.8 and begin with a lemma similar to Lemma 3.6.

Lemma 3.10. For \( \hat{\gamma}_{t} \) given in (2.2), and the corresponding estimates of the variances \( \hat{\sigma}_{t,W}^{2} \) and \( \hat{\sigma}_{t,U}^{2} \), \( \text{Var}(\hat{\gamma}_{t}) \to 0 \), \( \text{Var}(\hat{\sigma}_{t,W}^{2}) \to 0 \) and \( \text{Var}(\hat{\sigma}_{t,U}^{2}) \to 0 \), for each \( t \) and for \( c > a > 0 \) and \( b > a + 10 \), as \( T \to \infty \).

Based on (2.2)

\[
\hat{\gamma}_{t} = \frac{1}{e^{c\sigma^{2}T}} \sum_{k=1}^{T} \left[ e^{-\frac{k\sigma^{2}b}{2}} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 b}{2}} \right) - e^{-\frac{k\sigma^{2}a}{2}} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 a}{2}} \right) \right] \\
\times \left[ e^{-\frac{k\sigma^{2}c}{2}} \left( e^{\sigma U_k} - e^{\frac{\sigma^2 c}{2}} \right) - e^{-\frac{k\sigma^{2}a}{2}} \left( e^{\sigma U_k} - e^{\frac{\sigma^2 a}{2}} \right) \right]
\]

where

\[
G_1 = e^{-c\sigma^{2}T} \sum_{k=1}^{T} \left[ e^{-\frac{k\sigma^{2}b}{2}} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 b}{2}} \right) - e^{-\frac{k\sigma^{2}a}{2}} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 a}{2}} \right) \right],
\]

\[
G_2 = e^{-c\sigma^{2}T} \sum_{k=1}^{T} \left[ e^{(a-b)\sigma^{2}k/2} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 b}{2}} \right) - e^{(a-b)\sigma^{2}k/2} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 a}{2}} \right) \right],
\]

\[
G_3 = e^{-c\sigma^{2}T} \sum_{k=1}^{T} \left[ e^{(a-b)\sigma^{2}k/2} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 b}{2}} \right) - e^{(a-b)\sigma^{2}k/2} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 a}{2}} \right) \right],
\]

\[
G_4 = e^{-c\sigma^{2}T} \sum_{k=1}^{T} \left[ e^{a\sigma^{2}k} \left( e^{\sigma W_k} - e^{\sigma^2 k} \right) - e^{a\sigma^{2}k} \left( e^{\sigma W_k} - e^{\sigma^2 k} \right) \right].
\]
In order to show that $\text{Var}(\hat{\gamma}_t) \to 0$, it suffices to show that $E(\hat{\gamma}_t^2) \to 0$. Now

$$\hat{\gamma}_t^2 = G_1^2 + G_2^2 + G_3^2 + G_4^2 - 2G_1G_2 - 2G_1G_3 + 2G_2G_3 - 2G_2G_4 - 2G_3G_4. \quad (3.13)$$

We will show that the expectation of each of the 10 terms in the right-hand side of (3.13) converges to zero as $T \to \infty$ for the range of $a, b$ and $c$ assumed in the statement of Lemma 3.10, thereby showing that $E(\hat{\gamma}_t^2) \to 0$, as $T \to \infty$. The proof is lengthy but the calculations involved are very routine and repetitious. Hence we relegate the proof of Lemma 3.10 to the appendix. Next we state and prove a lemma similar to Lemma 3.7.

**Lemma 3.11.** For $\hat{\gamma}_t$ given in (2.2), and the corresponding estimates of the variances $\hat{\sigma}^2_{t,W}$ and $\hat{\sigma}^2_{t,U}$, $E(\hat{\gamma}_t)/\sqrt{E(\hat{\sigma}^2_{t,W})E(\hat{\sigma}^2_{t,U})} \to \rho$, as $T \to \infty$, if $b > a > 0$.

**Proof.** $e^{\alpha^2 T} \hat{\gamma}_t = \sum_{k=1}^{T-1} \{\cdot\} + \sum_{k=t}^{T} \{\cdot\}$ where

$$\{\cdot\} = \left[ e^{-b\sigma^2 k/2}(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma U_k} - e^{\sigma^2 t/2}) \right]$$

$$\times \left[ e^{-a\sigma^2 k/2}(e^{\sigma U_k} - e^{\sigma^2 k/2})(e^{\sigma U_k} - e^{\sigma^2 t/2}) \right]$$

$$= e^{-b\sigma^2 k} \left[ e^{\sigma(W_k+U_k)} - e^{\sigma^2 k/2}e^{\sigma W_k} - e^{\sigma^2 k/2}e^{\sigma U_k} + e^{\sigma^2 k} \right]$$

$$- e^{(a-b)\sigma^2 k/2} \left[ e^{\sigma(W_k+U_k)} - e^{\sigma^2 k/2}e^{\sigma U_k} - e^{\sigma^2 t/2}e^{\sigma W_k} + e^{\sigma^2 (k+t)/2} \right]$$

$$- e^{(a-b)\sigma^2 k/2} \left[ e^{\sigma(U_k+W_t)} - e^{\sigma^2 t/2}e^{\sigma U_k} - e^{\sigma^2 k/2}e^{\sigma W_k} + e^{\sigma^2 (k+t)/2} \right]$$

$$+ e^{a\sigma^2 k} \left[ e^{\sigma(W_t+U_k)} - e^{\sigma^2 t/2}e^{\sigma W_t} - e^{\sigma^2 t/2}e^{\sigma U_k} + e^{\sigma^2 t} \right].$$

We recall

$$E(e^{\sigma(W_k+W_t)}) = e^{\sigma^2 k[1 + \rho_k(e^{\sigma^2 k} - 1)],}$$

$$E(e^{\sigma(W_k+U_t)}) = e^{\sigma^2 t[1 + \rho_t(e^{\sigma^2 t} - 1)],}$$

$$W_k = \rho_k(U_k + \sqrt{1 - \rho_k^2}M^{(1)}_k)$$

$$E(e^{\sigma(W_k+U_t)}) = E\left( \exp\{\sigma[\rho_k U_k + U_t + \sqrt{1 - \rho_k^2}M^{(1)}_k]\} \right)$$

so that when $k \geq t$,

$$E(e^{\sigma(W_k+U_t)}) = E(e^{\sigma\rho_k(U_k-U_t)})E(e^{\sigma(1+\rho_k)U_t})E(e^{\sigma\sqrt{1-\rho_k^2}M^{(1)}_k})$$

$$= e^{\frac{\sigma^2 k}{2}[\rho_k^2(t-k) + (1+\rho_k)^2t + (1-\rho_k^2)k]} = e^{\sigma^2(k+t)/2}e^{\sigma^2 t}e^{\rho_k^2 t} \leq e^{\sigma^2 k/2}e^{3\sigma^2 t/2}.$$

The same relationship as above holds for $E(e^{\sigma(U_k+W_t)})$ also. When $k \leq t$,

$$E(e^{\sigma(W_k+U_t)}) = E(e^{\sigma\rho_k(U_k-U_t)})E(e^{\sigma(1+\rho_k)U_t})E(e^{\sigma\sqrt{1-\rho_k^2}M^{(1)}_k})$$

$$= e^{\frac{\sigma^2 k}{2}[\rho_k^2(t-k) + (1+\rho_k)^2t + (1-\rho_k^2)k]} = e^{\sigma^2(k+t)/2}e^{\rho_k^2 k} \leq e^{\sigma^2 t/2}e^{3\sigma^2 k/2};$$
again the same relationship holding for \( \mathbb{E}(e^{\sigma(U_k + W_i)}) \) also. Hence

\[
e^{\sigma T \mathbb{E}(\hat{\gamma}_t)} = \sum_{k=1}^{T} e^{-\delta \sigma^2 k} \{ e^{\sigma^2 k [1 + \rho_k (e^{\sigma^2 k} - 1)]} - e^{\sigma^2 k} \}
+ \sum_{k=1}^{T} e^{\sigma^2} \{ e^{\sigma^2 t [1 + \rho_t (e^{\sigma^2 t} - 1)]} - e^{\sigma^2 t} \}
- 2 \sum_{k=1}^{t-1} e^{(a-b)\sigma^2 k/2} \{ e^{\sigma^2 (k+t/2) [e^{\sigma^2 \rho_k} - 1]} \}
- 2 \sum_{k=1}^{T} e^{(a-b)\sigma^2 k/2} \{ e^{\sigma^2 (k+t/2) [e^{\sigma^2 \rho_k} - 1]} \}
= \left\{ \sum_{k=1}^{t-1} \rho_k \{ e^{(2-b)\sigma^2 k} - e^{(1-b)\sigma^2 k} \} \right\} + \rho_t (e^{\sigma^2 t} - 1) e^{(a+t)\sigma^2} \left[ \frac{e^{\sigma^2 T} - 1}{e^{\sigma^2} - 1} \right]
- 2 \sum_{k=1}^{T} e^{(a-b)\sigma^2 k/2} \{ e^{\sigma^2 (k+t/2) [e^{\sigma^2 \rho_k} - 1]} \}
- 2 \sum_{k=1}^{T} e^{(a-b)\sigma^2 k/2} \{ e^{\sigma^2 (k+t/2) [e^{\sigma^2 \rho_k} - 1]} \}
= A_T + D_T \rho_t - 2B_T - 2C_T,
\]

where \( B_t \) is a finite sum of \((t-1)\) terms that does not depend on \( T \), \( A_T \) and \( C_T \) converge to finite constants as \( T \to \infty \) since we assume \( b > a \), and

\[
D_T = e^{(a+t)\sigma^2} (e^{\sigma^2 t} - 1) \left[ \frac{e^{\sigma^2 T} - 1}{e^{\sigma^2} - 1} \right] \text{ diverges as } T \to \infty.
\]

Similarly, \( e^{\sigma^2 T \mathbb{E}(\hat{\gamma}_t)} = \sum_{k=1}^{t-1} \{ \} + \sum_{k=t}^{T} \{ \} \) where

\[
\{ \} = \left[ e^{-b \sigma^2 k/2} (e^{\sigma W_k} - e^{\sigma^2 k/2}) - e^{a \sigma^2 k/2} (e^{\sigma W_t} - e^{\sigma^2 t/2}) \right]^2
= e^{-b \sigma^2 k} (e^{\sigma W_k} - e^{\sigma^2 k/2})^2 + e^{a \sigma^2 k} (e^{\sigma W_t} - e^{\sigma^2 t/2})^2
- 2 e^{(a-b)\sigma^2 k/2} (e^{\sigma W_k} - e^{\sigma^2 k/2}) (e^{\sigma W_t} - e^{\sigma^2 t/2}).
\]

We have \( \mathbb{E}(e^{\sigma W_k} - e^{\sigma^2 k/2})^2 = \mathbb{E}[2 \sigma W_k - 2 e^{\sigma W_k} e^{\sigma^2 k/2}] = e^{2 \sigma^2 k} - e^{\sigma^2 k} = e^{\sigma^2 k} (e^{\sigma^2 k} - 1) \) and \( \mathbb{E}(e^{\sigma W_t} - e^{\sigma^2 t/2})^2 = e^{\sigma^2 (e^{\sigma^2 t} - 1)} \).

When \( k \geq t \),

\[
\mathbb{E}(e^{\sigma (W_k + W_i)}) = \mathbb{E}(e^{\sigma (W_k - W_i)}) \mathbb{E}(e^{2 \sigma W_i}) = \exp \{ \frac{\sigma^2}{2} (k - t) + 2 \sigma^2 t \} = e^{\sigma^2 (k+t)/2} e^{\sigma^2 t},
\]

and when \( k \leq t \),

\[
\mathbb{E}(e^{\sigma (W_k + W_i)}) = e^{\sigma^2 (k+t)/2} e^{\sigma^2 k}.
\]
Putting it all together,
\[
e^{-\sigma^2 T}E(\hat{\sigma}_{t,W}^2) = \sum_{k=1}^{T} e^{(2-b)\sigma^2 k} - e^{(1-b)\sigma^2 k} + e^{\sigma^2 t}(e^{\sigma^2 t} - 1)e^{\sigma^2} \left[ \frac{e^{\sigma^2 T} - 1}{e^{\sigma^2} - 1} \right] - 2 \sum_{k=1}^{T-1} e^{(a-b)\sigma^2 k/2}e^{\sigma^2(k+t)/2}[e^{\sigma^2 k} - e^{\sigma^2 t}] - 2e^{\sigma^2 t/2}(e^{\sigma^2 t} - 1) \sum_{k=1}^{T} e^{\sigma^2 k(a-b+1)/2}.
\]

Hence \( e^{-\sigma^2 T}E(\hat{\sigma}_{t,W}^2) \) can be written as \( E_T + D_T - 2F_T - 2G_T \), where \( D_T \) is same as the one given further above, \( F_t \) is a finite sum that does not depend on \( T \) and \( E_T \) and \( G_T \) converges to finite constants as \( T \to \infty \). The same result holds for \( E(\hat{\sigma}_{t,U}^2) \) also. Hence

\[
E(\gamma_t)/\sqrt{E(\hat{\sigma}_{t,W}^2)E(\hat{\sigma}_{t,U}^2)} = \frac{\rho_t + A_T/D_T - 2B_t/D_T - 2C_T/D_T}{1 + E_T/D_T - 2F_T/D_T - 2G_T/D_T}
\]

\( \to \rho_t \), as \( T \to \infty \),

since \( D_T \) diverges as \( T \to \infty \), \( B_t \) and \( F_t \) are finite terms that do not depend on \( T \) and \( A_T, C_T, E_T \) and \( G_T \) are all finite constants in the passage to the limit. This proves the lemma. \( \square \)

Lemmas 3.10 and 3.11 suffice to prove Theorem 3.8. Similarly lemmas 3.12 and 3.13 given below will suffice to prove Theorem 3.9.

**Lemma 3.12.** For \( \gamma_t \) given in (2.3), and the corresponding estimates of the variances \( \hat{\sigma}_{t,W}^2 \) and \( \hat{\sigma}_{t,U}^2 \), \( Var(\gamma_t) \to 0, Var([\hat{\sigma}_{t,W}]^2) \to 0 \) and \( Var([\hat{\sigma}_{t,U}]^2) \to 0 \), for each \( t \) and for \( c > a > 0 \) and \( b > 1 \), as \( T \to \infty \).

**Proof.**

\[
\gamma_t^2 = e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{k} \left\{ e^{\sigma^2 (k+j)}(e^{\sigma W_i} - e^{\sigma^2 t/2})(e^{\sigma U_i} - e^{\sigma^2 t/2}) - e^{\sigma^2 k}e^{-\sigma^2 j}(e^{\sigma W_i} - e^{\sigma^2 t/2})(e^{\sigma U_i} - e^{\sigma^2 j/2})(e^{\sigma W_i} - e^{\sigma^2 j/2})(e^{\sigma U_i} - e^{\sigma^2 j/2}) - e^{-\sigma^2 k}e^{\sigma^2 j}(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma W_i} - e^{\sigma^2 t/2})(e^{\sigma U_i} - e^{\sigma^2 t/2}) + e^{-\sigma^2 (k+j)}(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma U_k} - e^{\sigma^2 k/2})(e^{\sigma W_i} - e^{\sigma^2 t/2})(e^{\sigma U_i} - e^{\sigma^2 t/2}) \right\}
\]

It is easy to see that when taking expectations, the first term converges to zero as \( T \to \infty \) if \( c > a \). The expectations of the second and third terms are the same. We exhibit the calculation for the third term. Within the third term if we expand

\[
(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma W_i} - e^{\sigma^2 t/2})(e^{\sigma U_i} - e^{\sigma^2 t/2})
\]

there will be 16 terms. We exhibit only the calculation for the term containing \( e^{\alpha(W_k + U_k + W_i + U_i)} \) since similar calculations apply to the remaining 15 terms.
Recalling again $W_k = \mu_k U_k + \sqrt{1 - \mu_k^2} M_k^{(1)}$ we have
\[
W_k + U_k + W_t + U_t \leq (1 + \rho_k) U_k + \sqrt{1 - \rho_k^2} M_k^{(1)} + (1 + \rho_t) U_t + \sqrt{1 - \rho_t^2} M_t^{(1)} \\
\leq 2|U_k| + |M_k^{(1)}| + 2|U_t| + |M_t^{(1)}|
\]
so that when $k \geq t$,
\[
W_k + U_k + W_t + U_t \leq 2|U_k - U_t| + 4|U_t| + |M_k^{(1)} - M_t^{(1)}| + 2|M_t^{(1)}| \\
\mathbb{E}(e^{\sigma(W_k + U_k + W_t + U_t)}) \leq C \exp\left(\frac{\sigma^2}{2}(4(k - t) + 16t + (k - t) + 4t)\right) \leq C e^{5\sigma^2 k/2}
\]
and when $k \leq t$,
\[
W_k + U_k + W_t + U_t \leq 2|U_t - U_k| + 4|U_k| + |M_k^{(1)} - M_t^{(1)}| + 2|M_t^{(1)}| \\
\mathbb{E}(e^{\sigma(W_k + U_k + W_t + U_t)}) \leq C \exp\left(\frac{\sigma^2}{2}(4(t - k) + 16k + (t - k) + 4k)\right) \leq C e^{15\sigma^2 k/2}.
\]
Hence the expectation of the term containing $e^{\sigma(W_k + U_k + W_t + U_t)}$ in the third term is
\[
\leq C e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{a \sigma^2 j} e^{-b \sigma^2 k} e^{15\sigma^2 k/2} \\
= C e^{-2\sigma^2 T} e^{a \sigma^2} \left[\frac{e^{a \sigma^2 T} - 1}{e^{a \sigma^2} - 1}\right] e^{(7.5-b)\sigma^2} \left[\frac{1 - e^{(7.5-b)\sigma^2 T}}{1 - e^{(7.5-b)\sigma^2}}\right] \\
\leq C [e^{(a-2c)\sigma^2 T} - e^{-2c\sigma^2 T}] [1 - e^{(7.5-b)\sigma^2 T}] \\
\rightarrow 0,
\]
as $T \rightarrow \infty$, if $c > a/2 > 0$ and $b > 15/2$. Finally we exhibit the calculation for the first term in the expansion of the fourth term of $\gamma_t^2$; the corresponding expectation is
\[
= e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b \sigma^2 (k+j)} \mathbb{E}(e^{\sigma(W_k + U_k + W_j + U_j)}) \\
\leq C e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(15-b)\sigma^2 (k+j)} \\
\rightarrow 0,
\]
as $T \rightarrow \infty$, if $c > 0$ and $b > 15$. The calculations for the remaining terms within the expansion of the fourth term are similar. \hfill \Box

Lemma 3.13. For $\hat{\gamma}_t$ given in (2.3), and the corresponding estimates of the variances $\hat{\sigma}_t^2 W$ and $\hat{\sigma}_t^2 U$, $\mathbb{E}(\hat{\gamma}_t)/\sqrt{\mathbb{E}(\hat{\sigma}_t^2 W) \mathbb{E}(\hat{\sigma}_t^2 U)} \rightarrow \rho_1$, as $T \rightarrow \infty$, if $b > 2$ and $a > 0$. 

Proof.

\[ e^{c \sigma^2 T} \hat{\gamma}_t = \sum_{k=1}^{T} \left\{ e^{a \sigma^2 k} [(e^{\sigma^2 W_k} - e^{\sigma^2 t/2})(e^{\sigma^2 U_k} - e^{\sigma^2 t/2})] \right. \\
- e^{-b \sigma^2 k} [(e^{\sigma^2 W_k} - e^{\sigma^2 k/2})(e^{\sigma^2 U_k} - e^{\sigma^2 k/2})] \right\} \]

\[ e^{c \sigma^2 T} E(\hat{\gamma}_t) = \sum_{k=1}^{T} \left\{ e^{a \sigma^2 k} \left( e^{\sigma^2 t} [1 + \rho t (e^{\sigma^2 t} - 1)] - e^{\sigma^2 t} \right) \\
- e^{-b \sigma^2 k} \left( e^{\sigma^2 k} [1 + \rho_k (e^{\sigma^2 k} - 1)] - e^{\sigma^2 k} \right) \right\} \]

\[ = \rho t e^{\sigma^2 t} (e^{\sigma^2 t} - 1) e^{a \sigma^2} - \sum_{k=1}^{T} \left\{ e^{(1-b) \sigma^2 k} \rho_k (e^{\sigma^2 k} - 1) \right\} \]

\[ = DT \rho_t - AT, \]

where \( DT \) diverges and \( AT \) converges if \( b > 2 \) as \( T \to \infty \).

\[ e^{c \sigma^2 T} \hat{\sigma}^2_{t,W} = \sum_{k=1}^{T} \left\{ e^{a \sigma^2 k} [(e^{\sigma^2 W_k} - e^{\sigma^2 t/2})^2 - e^{-b \sigma^2 k} [(e^{\sigma^2 W_k} - e^{\sigma^2 k/2})^2] \right\} \]

\[ e^{c \sigma^2 T} E(\hat{\sigma}^2_{t,W}) = \sum_{k=1}^{T} \left\{ e^{a \sigma^2 k} e^{\sigma^2 t} (e^{\sigma^2 t} - 1) - e^{(1-b) \sigma^2 k} (e^{\sigma^2 k} - 1) \right\} \]

\[ = DT \rho_t - BT, \]

where \( BT \) converges if \( b > 2 \). The same result holds for \( e^{c \sigma^2 T} E(\hat{\sigma}^2_{t,U}) \) also. Hence we have

\[ \sqrt{E(\hat{\gamma}_t)^2 / E(\hat{\sigma}^2_{t,W}) E(\hat{\sigma}^2_{t,U})} = \frac{DT \rho_t - AT}{DT - BT} \]

\[ = \frac{\rho_t - (AT/DT)}{1 - (BT/DT)} \]

\[ \to \rho_t, \]

as \( T \to \infty \), if \( a > 0 \) and \( b > 2 \). This proves the lemma.

\[ 4. \text{ Conclusions and Discussion} \]

In this paper we presented estimators for dynamic correlation between a pair of correlated Brownian motions and separately estimators for dynamic correlation between a pair of correlated Geometric Brownian motions. The main thrust of this paper was in showing the weak consistency of all the estimators presented.

The type of estimators that we presented here are generalizations of estimators that we introduced in an earlier work [4]. The special case that we considered in [4] had shown good empirical performance in a few simulation scenarios. One
reason for the good empirical performance of the special case could be additional steps such as smoothing based on median filtering and the arctan function that we applied to the estimators. We conjecture that such extra processing steps may lead to reduction of bias empirically even for estimators which exhibit asymptotic bias theoretically. We also note here that the empirical study conducted in [4] did not include Brownian motions or Geometric Brownian motions.

The generalizations that we considered in this paper are based on inserting certain “weights” within the estimator: \( v^{-p} \) and \( v^q \) in the Brownian motion case, and \( e^{a \sigma^2 k/2} \) and \( e^{-b \sigma^2 k/2} \) in the Geometric Brownian motion case. When considering the estimator for correlation \( \rho_t \) at a time point \( t \), such terms up-weight the contribution of the time series value at \( t \) and down-weight the values corresponding to other time points. Such down-weighting gets intensified for time series values further away from the currently considered time point \( t \). In other words, the contribution of values further away, towards estimating the current correlation, is smaller compared to the contribution of values at nearby time points. In this sense, although the generalized estimator is global in nature, the emphasis is placed on local information. The special case (i.e. \( p = q = 0 \) and \( a = b = 0 \)) estimators give equal emphasis to values even much further away. Such equal emphasis is not intuitively very appealing and could also be the reason for theoretical asymptotic bias for the special case estimators.

The asymptotics considered in this paper is based on the sample size \( T \) going to infinity. Another type of asymptotics that could be of interest (e.g. in high-frequency time series framework) is when the “mesh-size” goes to zero. Because of the scaling properties of Brownian motion, reducing the mesh-size is effectively same as increasing the sample size, roughly speaking. Hence, at least in the Brownian motion case, it may be possible to obtain consistency results theoretically, as the mesh-size goes to zero.

Assessment of empirical performance of generalized versions introduced in this paper will be of interest for future work. Currently, we are working on a program to see whether these types of estimators may be obtained (and weak consistency proved for) in other models such as pairs of fractional Brownian motions, autoregressive (e.g. AR(1)) or moving average (e.g. MA(1)) processes. We plan to conduct the empirical assessment after the estimators in the above-mentioned program are hopefully worked out theoretically. The ‘hyperparameters’ \( p, q, a, b \) and \( c \) occurring in the estimators presented in this paper may be considered as tuning parameters, similar to considering the ‘window-size’ as a tuning parameter in the dynamic correlation estimation approach based on sliding windows. Computational approaches such as cross-validation may be utilized in obtaining the optimal hyperparameters. The empirical study that we will conduct in future will help to provide a better understanding on those topics as well.

Appendix (Proof of Lemma 3.10)

We provide the proof of Lemma 3.10 for the convenience of the reader.
Proof. Based on equation (2.2)

\[ \hat{\gamma}_t = \frac{1}{e^{\sigma^2 T}} \sum_{k=1}^{T} \left\{ \left[ e^{-\frac{\sigma^2 k}{2}} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \right) - e^{-\frac{\sigma^2 k}{2}} \left( e^{\sigma W_t} - e^{\frac{\sigma^2 k}{2}} \right) \right] \times \left[ e^{-\frac{\sigma^2 j}{2}} \left( e^{\sigma U_k} - e^{\frac{\sigma^2 j}{2}} \right) - e^{-\frac{\sigma^2 j}{2}} \left( e^{\sigma U_t} - e^{\frac{\sigma^2 j}{2}} \right) \right] \right\} \]

where

\[ G_1 = e^{-\sigma^2 T} \sum_{k=1}^{T} \left\{ e^{-b\sigma^2 k} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \right) \left( e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \right) \right\}, \]

\[ G_2 = e^{-\sigma^2 T} \sum_{k=1}^{T} \left\{ e^{(a-b)\sigma^2 k/2} \left( e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \right) \left( e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \right) \right\}, \]

\[ G_3 = e^{-\sigma^2 T} \sum_{k=1}^{T} \left\{ e^{(a-b)\sigma^2 k/2} \left( e^{\sigma W_t} - e^{\frac{\sigma^2 k}{2}} \right) \left( e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \right) \right\}, \]

\[ G_4 = e^{-\sigma^2 T} \sum_{k=1}^{T} \left\{ e^{a\sigma^2 k} \left( e^{\sigma W_t} - e^{\frac{\sigma^2 k}{2}} \right) \left( e^{\sigma U_t} - e^{\frac{\sigma^2 k}{2}} \right) \right\}. \]

In order to show that Var(\hat{\gamma}_t) → 0, it suffices to show that \( E(\hat{\gamma}_t^2) \) → 0.

\[ \hat{\gamma}_t^2 = [G_1^2 + G_2^2 + G_3^2 + G_4^2 - 2G_1G_2 - 2G_1G_3 + 2G_1G_4 + 2G_2G_3 - 2G_2G_4 - 2G_3G_4]. \]  

(4.1)

We will show that the expectation of each of the 10 terms in the right-hand side of (4.1) converges to zero as \( T \to \infty \) for the range of \( a, b \) and \( c \) assumed in the statement of Lemma 3.10, thereby showing that \( E(\hat{\gamma}_t^2) \) → 0, as \( T \to \infty \).
We label the 16 terms in the right-hand side of the equation for $G_1^2$ as $G_1^2_{\text{term}1}$, $G_1^2_{\text{term}2}$, ..., $G_1^2_{\text{term}16}$, in the same order as they appear above, and we will show the expectation of each of these terms to converge to zero as $T \to \infty$. We start with the last term since it is the easiest.

\[
E(G_1^2_{\text{term}16}) = e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\sigma^2 (U_k+U_j)}
\]

\[
= e^{-2c\sigma^2 T} e^{2(1-b)\sigma^2} \left[ \frac{1 - e^{(1-b)\sigma^2 T}}{1 - e^{(1-b)\sigma^2}} \right]^2
\]

\[
\to 0 \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > 1.
\]

Next we show that the expectation of $G_1^2_{\text{term}8}$ converges to zero. The same proof holds for the terms $G_1^2_{\text{term}12}$, $G_1^2_{\text{term}14}$ and $G_1^2_{\text{term}15}$ also.

\[
E(G_1^2_{\text{term}8}) = e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\sigma^2 (U_k+W_j)} E(e^{\sigma W_j})
\]

\[
= e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(1-b)\sigma^2 (k+j)}
\]

\[
= e^{-2c\sigma^2 T} e^{2(1-b)\sigma^2} \left[ \frac{1 - e^{2(1-b)\sigma^2 T}}{1 - e^{2(1-b)\sigma^2}} \right]^2
\]

\[
\to 0 \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > 1.
\]

We will use the following inequality frequently in the remaining parts of the proof of Lemma 3.10.

\[
E(e^{\sigma |W_i|}) \leq 2e^{\sigma^2 t/2}. \tag{4.2}
\]
The inequality in (4.2) can be obtained by direct calculation as follows.

\[
\mathbb{E}(e^{\sigma|W_t|}) = \frac{1}{\sqrt{2\pi t}} \left\{ \int_{0}^{\infty} e^{(\sigma x - \frac{x^2}{2t})} \, dx + \int_{-\infty}^{0} e^{-\frac{(\sigma x + x^2/2t)}{2t}} \, dx \right\}
= 2e^{\sigma^2 t/2} \left( \frac{1}{\sqrt{2\pi t}} \int_{0}^{\infty} e^{-(x - \sigma^2 t/2)} \, dx \right)
= 2e^{\sigma^2 t/2} \left( 1 - \Phi(-\sigma\sqrt{t}) \right) \leq 2e^{\sigma^2 t/2},
\]

where \( \Phi \) denotes the standard normal distribution function.

Recall that we may write \( W_k = \rho_k U_k + \sqrt{1 - \rho_k^2} M_k^{(1)} \), where \( M_k^{(1)} \) is a Brownian motion process independent of \( U_k \). With this, we have

\[
W_k + U_k = (1 + \rho_k) U_k + \sqrt{1 - \rho_k^2} M_k^{(1)} \leq 2|U_k| + |M_k^{(1)}|
\leq 2\mathbb{E}(e^{\sigma^2 U_k}) = C e^{2\sigma^2 k} e^{\sigma^2 k/2} = C e^{\sigma^2 k/2}
\leq e^{2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{2j}{T}} e^{\frac{2j^2}{T}}
= C e^{-2\sigma^2 T} e^{(3.5-b)\sigma^2} \left[ \frac{1 - e^{(1-b)\sigma^2 T}}{1 - e^{(1-b)\sigma^2}} \right] \left[ \frac{1 - e^{(2.5-b)\sigma^2 T}}{1 - e^{(2.5-b)\sigma^2}} \right]
\rightarrow 0\text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > 2.5.
\]

Here and below \( C \) denotes a generic positive finite constant (\( 0 < C < \infty \)). The same type of calculation for \( \mathbb{E}(G_{2,T}^{(1)}) \) also.

When \( k \geq j \),
\[
\mathbb{E}(e^{W_k+U_j}) = \mathbb{E}(e^{\sigma|\rho_k U_k + U_j + \sqrt{1 - \rho_k^2} M_k^{(1)}|}) \leq \mathbb{E}(e^{\sigma|U_k + U_j + |M_k^{(1)}|})
= \mathbb{E}(e^{\sigma|U_k + U_j|}) \mathbb{E}(e^{\sigma|\sqrt{1 - \rho_k^2} M_k^{(1)}|}) \leq \mathbb{E}(e^{\sigma|U_k - U_j|}) \mathbb{E}(e^{\sigma|\sqrt{1 - \rho_k^2} M_k^{(1)}|})
\leq C e^{\frac{2\sigma^2}{T}(k-j)} e^{2\sigma^2 j} e^{\frac{2j^2}{T}} \leq C e^{\frac{2\sigma^2}{T}(k+j)}.
\]

Because of the symmetry in the upper bound, the above inequality holds for \( k \leq j \) as well. Hence
\[
\mathbb{E}(G_{2,T}^{(7)}) \leq e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} + \frac{2\sigma^2}{T} \sigma^2 (k+j) + \frac{2\sigma^2}{T} \sigma^2 (k+j)
= e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(2-b)\sigma^2 (k+j)}
= e^{-2\sigma^2 T} e^{(2-b)\sigma^2} \left[ \frac{1 - e^{(2-b)\sigma^2 T}}{1 - e^{(2-b)\sigma^2}} \right]^2
\rightarrow 0\text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > 2.
\]

A similar calculation holds for \( \mathbb{E}(G_{2,T}^{(10)}) \) also.
\[
\mathbb{E}(e^{\sigma(U_k + U_j)}) = \mathbb{E}(e^{\sigma(U_k - U_j)}) \mathbb{E}(e^{2\sigma U_j}) = e^{\frac{2\sigma^2}{T}(k-j)} e^{2\sigma^2 j} \leq e^{\frac{2\sigma^2}{T}(k+j)},
\]
when \( k \geq j \), but the same holds true for \( k \leq j \) also because of symmetry. Hence

\[
\mathbb{E}(G_{1, \text{term11}}^2) = e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b \sigma^2 (k+j)} e^{2 \sigma^2 (k+j)} \mathbb{E}(e^{\sigma(U_k + U_j)}) \\
\leq e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b \sigma^2 (k+j)} e^{2 \sigma^2 (k+j)} \\
= e^{-2\sigma^2 T} e^{2(2-b)\sigma^2} \left[ \frac{1 - e^{(2-b)\sigma^2 T}}{1 - e^{(2-b)\sigma^2}} \right]^2 \\
\rightarrow 0 \text{ as } T \rightarrow \infty, \text{ if } c > 0 \text{ and } b > 2.
\]

Same calculation works for \( G_{1, \text{term6}}^2 \) also. So far, we have shown convergence to zero of the expectation the terms 4, 6, 7, 8, 10, 11, 12, 13, 14, and 15 of \( G_{1}^2 \). It remains to show the convergence for terms 1, 2, 3, 5, and 9. The calculations for terms 2, 3, and 9 are similar. So, we will exhibit just the calculation for \( G_{1, \text{term5}}^2 \) and then for \( G_{1, \text{term1}}^2 \).

\[
W_k + W_j = \rho_k U_k + \sqrt{1 - \rho_k^2} M_{1,k}^{(1)} + [(1 + \rho_j) U_j + \sqrt{1 - \rho_j^2} M_{1,j}^{(1)}] \\
\leq |U_k| + |M_{1,k}^{(1)}| + 2|U_j| + |M_{1,j}^{(1)}| \\
\leq |U_k - U_j| + 3|U_j| + |M_{1,k}^{(1)} - M_{1,j}^{(1)}| + 2|M_{1,j}^{(1)}| \text{ when } k \geq j.
\]

\[
\mathbb{E}(e^{\sigma(W_k + W_j + U_j)}) \leq \mathbb{E}(e^{\sigma(U_k - U_j)}) \mathbb{E}(e^{3\sigma|U_j|}) \mathbb{E}(e^{\sigma|M_{1,k}^{(1)} - M_{1,j}^{(1)}|}) \mathbb{E}(2\sigma|M_{1,j}^{(1)}|) \\
\leq C e^{\frac{\sigma^2}{2}(k-j)} e^{\frac{4\sigma^2}{2}(k-j)} e^{2\sigma^2} = Ce^{\frac{\sigma^2}{2}(k-j)} e^{\frac{4\sigma^2}{2}(k-j)} = Ce^{\frac{\sigma^2}{2}(k-j)} e^{\frac{4\sigma^2}{2}(k-j)} \leq Ce^{\frac{\sigma^2}{2}(k-j)}.
\]

We will get the same upper bound for \( k \leq j \) also because of symmetry. Hence,

\[
\mathbb{E}(G_{1, \text{term5}}^2) \leq Ce^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b \sigma^2 (k+j)} e^{\frac{\sigma^2}{2} + e^{\frac{4\sigma^2}{2}} (k+j)} \\
= Ce^{-2\sigma^2 T} e^{(1.5 - 2b)\sigma^2} \left[ \frac{1 - e^{(5.5 - b)\sigma^2 T}}{1 - e^{(5.5 - b)\sigma^2}} \right] \left[ \frac{1 - e^{(6.5 - b)\sigma^2 T}}{1 - e^{(6.5 - b)\sigma^2}} \right] \\
\rightarrow 0 \text{ as } T \rightarrow \infty, \text{ if } c > 0 \text{ and } b > 6.5.
\]

As mentioned above, the calculations for \( G_{1}^2 \) terms 2, 3, and 9 are similar to that of term 5. Thus, it remains to show the calculation for \( G_{1, \text{term1}}^2 \) to complete the proof for \( G_{1}^2 \).

\[
W_k + U_k + W_j + U_j = [(1 + \rho_k) U_k + \sqrt{1 - \rho_k^2} M_{1,k}^{(1)}] \\
+ [(1 + \rho_j) U_j + \sqrt{1 - \rho_j^2} M_{1,j}^{(1)}] \\
\leq 2|U_k| + |M_{1,k}^{(1)}| + 2|U_j| + |M_{1,j}^{(1)}| \\
\leq 2|U_k - U_j| + 4|U_j| + |M_{1,k}^{(1)} - M_{1,j}^{(1)}| + 2|M_{1,j}^{(1)}|.
\]
The above inequality was obtained assuming $k \geq j$ but the same upper bounds hold for $k \leq j$ as well. Hence

$$E(G^2_{1, \text{term}1}) \leq e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(7.5-b)\sigma^2 (k+j)}$$

$$= e^{-2c\sigma^2 T} e^{(7.5-b)\sigma^2 \frac{1}{(1-e^{2c\sigma^2 \frac{1}{T}})^2}}$$

$$\rightarrow 0 \text{ as } T \rightarrow \infty, \text{ if } c > 0 \text{ and } b > 7.5.$$
Thus for any $t, k \geq 20$ MAJNU JOHN AND YIHREN WU

Same type of calculations as above works for $G_k^2$. term3, $G_k^2$. term6 and $G_k^2$. term9.

For $G_k^2$. term4, we start by noting that

$$W_k + 2U_t \leq |U_k| + |M_k^{(1)}| + 2|U_t| \leq |U_k - U_t| + 3|U_t| + |M_k^{(1)}|.$$  

$$E(e^{\sigma(W_k+2U_t)}) \leq Ce^{\frac{\sigma^2}{2}(k-t)} e^{\frac{\tau^2}{2}k} \leq Ce^{\sigma^2k}, \text{ when } k \geq t,$$

$$W_k + 2U_t \leq 2|U_t - U_k| + 3|U_k| + |M_k^{(1)}|,$$

$$E(e^{\sigma(W_k+2U_t)}) \leq Ce^{2\sigma^2(t-k)} e^{\frac{\tau^2}{2}k} \leq Ce^{3\sigma^2k}, \text{ when } k \leq t.$$ 

Combining both cases, $k \geq t$ and $k \leq t$, we get $E(e^{\sigma(W_k+2U_t)}) \leq Ce^{3\sigma^2k}$. Hence

$$E(G_k^2.\text{term}4) \leq e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)} e^{\frac{\tau^2}{2}j} e^{3\sigma^2k}$$

$$\leq e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(6+a-b)\frac{\sigma^2}{2}(k+j)}$$

$$\leq e^{-2\sigma^2 T} e^{(6+a-b)\sigma^2 T} \frac{1 - e^{(6+a-b)\frac{\sigma^2}{2}T}}{1 - e^{(6+a-b)\frac{\sigma^2}{2}}}$$

$\rightarrow 0$, as $T \rightarrow \infty$, if $c > 0$ and $b > a + 6$.

Same type of calculations as above works for $E(G_k^2.\text{term}7)$ also.

For $G_k^2.\text{term}5$, when $k \geq t$, $W_k + U_t \leq |U_k - U_t| + 2|U_t| + |M_k^{(1)}|$, so that

$$E(e^{\sigma(W_k+U_t)}) \leq C \exp \left\{ \frac{\sigma^2}{2}[(k-t) + 4t + k] \right\} \leq Ce^{\sigma^2k}$$

and when $k \leq t$, $W_k + U_t \leq |U_t - U_k| + 2|U_k| + |M_k^{(1)}|$, so that

$$E(e^{\sigma(W_k+U_t)}) \leq C \exp \left\{ \frac{\sigma^2}{2}[(t-k) + 4k + k] \right\} \leq Ce^{\sigma^2k}.$$

Thus for any $t, k$, $E(e^{\sigma(W_k+U_t)}) \leq Ce^{\sigma^2k}$. Hence

$$E(G_k^2.\text{term}5) \leq e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)} e^{\frac{\tau^2}{2}j} e^{3\sigma^2k}$$

$$\leq Ce^{-2\sigma^2 T} e^{(4+a-b)\sigma^2 T/2} \frac{1 - e^{(4+a-b)\frac{\sigma^2}{2}T}}{1 - e^{(4+a-b)\frac{\sigma^2}{2}}}$$

$\rightarrow 0$, as $T \rightarrow \infty$, if $c > 0$ and $b > a + 6$.

Same type of calculations as above works for $E(G_k^2.\text{term}8)$ also.

For $G_k^2.\text{term}2$, we consider four cases: $\{k \geq t, j \geq t\}, \{k \geq t, j \leq t\}, \{k \leq t, j \geq t\}$ and $\{k \leq t, j \leq t\}$. 
Case 1a: \( \{k \geq j \geq t\} \).

\[
W_k + W_j + U_t \leq |U_k| + |M_k^{(1)}| + |U_j| + |M_j^{(1)}| + |U_t|
\]

\[
\leq |U_k - U_j| + 2|U_j| + |U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j|
\]

\[
\leq |U_k - U_j| + 2|U_j - U_t| + 3|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j|,
\]

\[
\mathbb{E}(e^{\sigma(W_k + W_j + U_t)}) \leq C \exp\left\{ \frac{\sigma^2}{2} [(k - j) + 4(j - t) + 9t + (k - j) + 4j]\right\} \leq C e^{\sigma^2(k+3t)}.
\]

Case 1b: \( \{j \geq k \geq t\} \). The result for this case can be obtained by interchanging \( k \) and \( j \) in the result for Case 1a.

\[
\mathbb{E}(e^{\sigma(W_k + W_j + U_t)}) \leq C e^{\sigma^2(3k+j)}.
\]

Case 2: \( \{k \geq t \geq j\} \).

\[
W_k + W_j + U_t \leq |U_k - U_t| + 2|U_t| + |U_j| + |M_k^{(1)} - M_j^{(1)}| + |M_t^{(1)}| + |M_j^{(1)}|
\]

\[
\leq |U_k - U_t| + 2|U_t - U_j| + 3|U_j| + |M_k^{(1)} - M_j^{(1)}|
\]

\[
\quad + |M_t^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|,
\]

\[
\mathbb{E}(e^{\sigma(W_k + W_j + U_t)}) \leq C \exp\left\{ \frac{\sigma^2}{2} [(k - t) + 4(t - j) + 9j + (k - t) + (t - j) + 4j]\right\}
\]

\[
\leq C e^{\sigma^2(k+4j)} \quad (4.3)
\]

Case 3: \( \{j \geq t \geq k\} \). The result for this case can be obtained by interchanging \( k \) and \( j \) in the result for Case 2.

\[
\mathbb{E}(e^{\sigma(W_k + W_j + U_t)}) \leq C e^{\sigma^2(4k+j)}.
\]

Case 4a: \( \{k \leq j \leq t\} \).

\[
W_k + W_j + U_t \leq 3|U_k| + 2|M_k^{(1)}| + 2|U_j - U_k| + |M_j^{(1)} - M_k^{(1)}| + |U_t - U_j|.
\]

\[
\mathbb{E}(e^{\sigma(W_k + W_j + U_t)}) \leq C \exp\left\{ \frac{\sigma^2}{2} [9k + 4k + 4(j - k) + (j - k) + (t - j)]\right\} \leq C e^{\sigma^2(4k+2j)}.
\]

Case 4b: \( \{j \leq k \leq t\} \). By interchanging \( k \) and \( j \) in the result for Case 4a, we get

\[
\mathbb{E}(e^{\sigma(W_k + W_j + U_t)}) \leq C e^{\sigma^2(2k+4j)}.
\]

For all cases combined we see that

\[
\mathbb{E}(e^{\sigma(W_k + W_j + U_t)}) \leq C e^{4\sigma^2(k+j)}.
\]
Hence

$$E(G_2^2, \text{term} 2) \leq e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(a-b)\frac{k^2}{2}(k+j)} e^{4\sigma^2(k+j)}$$

$$\leq Ce^{-2c\sigma^2 T}e^{(8+a-b)\sigma^2 T/2} \left[ \frac{1 - e^{(8+a-b)\sigma^2 T/2}}{1 - e^{(8+a-b)\sigma^2 /2}} \right]^2,$$

$$\to 0, \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > a + 8.$$  

For $G_2^2, \text{term} 1$ also we again deal with four cases.

**Case 1a:** $\{k \geq j \geq t\}$.

$$W_k + W_j + 2U_t \leq |U_k| + |M_k^{(1)}| + |U_j| + |M_j^{(1)}| + 2|U_t|$$

$$\leq |U_k - U_j| + 2|U_j| + 2|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j|$$

$$\leq |U_k - U_j| + 2|U_j - U_t| + 4|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j|,$$

$$E(e^{\sigma(W_k+W_j+2U_t)}) \leq C \exp \left( \frac{g^2}{2} ([k-j] + 4(j-t) + 16t + (k-j) + 4j) \right) \leq Ce^{\frac{a^2}{T}(2k+6j)}.$$  

**Case 1b:** $\{j \geq k \geq t\}$. By interchanging $k$ and $j$ in the result for Case 1a, we get

$$E(e^{\sigma(W_k+W_j+2U_t)}) \leq Ce^{\frac{a^2}{T}(6k+2j)}.$$  

**Case 2:** $\{k \geq t, j \leq t\}$.

$$W_k + W_j + 2U_t \leq |U_k - U_t| + 3|U_t - U_j| + 4|U_t| + |M_k^{(1)} - M_J^{(1)}| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|$$

$$E(e^{\sigma(W_k+W_j+2U_t)}) \leq Ce^{\frac{a^2}{T}(2k+10j)}.$$  

**Case 3:** $\{k \leq t, j \geq t\}$. By interchanging $k$ and $j$ in Case 2, we get,

$$E(e^{\sigma(W_k+W_j+2U_t)}) \leq Ce^{\frac{a^2}{T}(10k+2j)}.$$  

**Case 4a:** $\{k \leq j \leq t\}$.

$$W_k + W_j + 2U_t \leq 2|U_k - U_j| + 3|U_j - U_k| + 4|U_k| + |M_j^{(1)} - M_k^{(1)}| + 2|M_k^{(1)}|$$

$$E(e^{\sigma(W_k+W_j+2U_t)}) \leq Ce^{\frac{a^2}{T}(10k+6j)}.$$  

**Case 4b:**

$$E(e^{\sigma(W_k+W_j+2U_t)}) \leq Ce^{\frac{a^2}{T}(6k+10j)}.$$  

Combining all cases,

$$E(G_2^2, \text{term} 1) \leq e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(a-b)\frac{k^2}{2}(k+j)} e^{5\sigma^2(k+j)}$$

$$\leq Ce^{-2c\sigma^2 T}e^{(10+a-b)\sigma^2 T/2} \left[ \frac{1 - e^{(10+a-b)\sigma^2 T/2}}{1 - e^{(10+a-b)\sigma^2 /2}} \right]^2,$$

$$\to 0, \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > a + 10.$$
For the remaining terms $G_1G_2, G_1G_3, \ldots, G_3G_4$ we work out the calculations for only the first term in each of them. The conditions on $a$, $b$ and $c$ required for convergence of the first term in each product, will ensure the convergence for the remaining terms in each as well. We start with $G_1G_2$.

$$G_1G_2 = e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 k e(a-b)\frac{a^2}{2}} \left[ (e^{\sigma W_k} - e^{\frac{a^2}{2}})(e^{\sigma U_k} - e^{\frac{a^2}{2}}) \right] \times \left[ (e^{\sigma W_j} - e^{\frac{a^2}{2}})(e^{\sigma U_j} - e^{\frac{a^2}{2}}) \right]$$

$$G_1G_2.\text{term}1 = e^{-2\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 k e(a-b)\frac{a^2}{2}} \left[ e^{\sigma (W_k+U_k+W_j+U_t)} \right].$$

$$W_k + U_k + W_j + U_t = (1 + \rho_k)U_k + \sqrt{1 - \rho_k^2} M_k^{(1)} + \rho_j U_j + \sqrt{1 - \rho_j^2} M_j^{(1)} + U_t$$

$$\leq 2|U_k| + |M_k^{(1)}| + |U_j| + |M_j^{(1)}| + |U_t|$$

**Case 1a:** $\{k \geq j \geq t\}$.

$$W_k + U_k + W_j + U_t \leq 2|U_k - U_j| + 3|U_j - U_t| + 4|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$  

$$\mathbb{E}(e^{\sigma (W_k+U_k+W_j+U_t)}) \leq C e^{\frac{1}{2} (5k+8j)}.$$

**Case 1b:** $\{j \geq k \geq t\}$.

$$W_k + U_k + W_j + U_t \leq 2|U_j - U_k| + 3|U_k - U_t| + 4|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$  

$$\mathbb{E}(e^{\sigma (W_k+U_k+W_j+U_t)}) \leq C e^{\frac{1}{2} (11k+2j)}.$$

**Case 2:** $\{k \geq t \geq j\}$.

$$W_k + U_k + W_j + U_t \leq 2|U_k - U_t| + 3|U_t - U_j| + 4|U_j| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$  

$$\mathbb{E}(e^{\sigma (W_k+U_k+W_j+U_t)}) \leq C e^{\frac{1}{2} (5k+10j)}.$$

**Case 3:** $\{j \geq t \geq k\}$.

$$W_k + U_k + W_j + U_t \leq |U_j - U_t| + 2|U_t - U_k| + 4|U_k| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$  

$$\mathbb{E}(e^{\sigma (W_k+U_k+W_j+U_t)}) \leq C e^{\frac{1}{2} (15k+2j)}.$$

**Case 4a:** $\{t \geq k \geq j\}$.

$$W_k + U_k + W_j + U_t \leq |U_t - U_k| + 3|U_k - U_j| + 4|U_j| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$  

$$\mathbb{E}(e^{\sigma (W_k+U_k+W_j+U_t)}) \leq C e^{\frac{1}{2} (9k+10j)}.$$

**Case 4b:** $\{t \geq j \geq k\}$.

$$W_k + U_k + W_j + U_t \leq |U_t - U_j| + 2|U_j - U_k| + 4|U_k| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$  

$$\mathbb{E}(e^{\sigma (W_k+U_k+W_j+U_t)}) \leq C e^{\frac{1}{2} (15k+4j)}.$$
Combining all cases, we get \( \mathbb{E}(e^{\sigma(W_k+U_k+W_i+U_t)}) \leq Ce^{\frac{\sigma^2}{2}(15k+10j)}. \)

\[
\mathbb{E}(G_{1G_2}\text{term}1) \leq Ce^{-2\sigma^2T}\sum_{k=1}^{T}\sum_{j=1}^{T}e^{-b\sigma^2k}e^{15k}\frac{\sigma^2}{2} \left\{ \sum_{j=1}^{T}e^{(a-b)\frac{\sigma^2}{2}}e^{(10j)\frac{\sigma^2}{2}} \right\}
\]

\[
\leq Ce^{-2\sigma^2T}e^{(25+a-3b)\frac{\sigma^2}{2}}
\times \begin{bmatrix} 1 - e^{(10+a-b)\sigma^2T/2} \\
1 - e^{(10+a-b)\sigma^2T/2} \end{bmatrix}
\begin{bmatrix} 1 - e^{(15-2b)\sigma^2T/2} \\
1 - e^{(15-2b)\sigma^2T/2} \end{bmatrix}
\to 0, \text{ as } T \to \infty, \text{ if } c > 0
\]

and \( b > \max\{a+10, 15/2\} = a+10, \text{ since } a > 0. \)

We skip the proof for \( G_{1G_3}, \) since it is very similar to that of \( G_{1G_2} \) (and the results are the same). We move onto \( G_{1G_4}. \)

\[
G_{1G_4} = e^{-2\sigma^2T}\sum_{k=1}^{T}\sum_{j=1}^{T}e^{-b\sigma^2k}e^{a\sigma^2j} \left[ (e^{\sigma W_k} - e^{\frac{\sigma^2k}{2}})(e^{\sigma U_k} - e^{\frac{\sigma^2k}{2}}) \right]
\]

\[
G_{1G_4}\text{term}1 = e^{-2\sigma^2T}\sum_{k=1}^{T}\sum_{j=1}^{T}e^{-b\sigma^2k}e^{a\sigma^2j} \left[ e^{\sigma(W_k+U_k+W_i+U_t)} \right].
\]

When \( k \geq t, \)

\[
W_k + U_k + W_t + U_t \leq 2|U_k| + |M^{(1)}_k| + 2|U_t| + |M^{(1)}_t|.
\]

and when \( k \leq t, \)

\[
W_k + U_k + W_t + U_t \leq 2|U_k| - U_t| + 4|U_t| + |M^{(1)}_k| - M^{(1)}_t| + 2|M^{(1)}_t|.
\]

In the first case, we will get \( \mathbb{E}(e^{\sigma(W_k+U_k+W_i+U_t)}) \leq Ce^{\frac{\sigma^2}{2}} \) and in the second case we will get \( \mathbb{E}(e^{\sigma(W_k+U_k+W_i+U_t)}) \leq Ce^{\frac{15\sigma^2}{2}} \) and so combining both cases we see that \( \mathbb{E}(e^{\sigma(W_k+U_k+W_i+U_t)}) \leq Ce^{\frac{15\sigma^2}{2}} \). Hence,

\[
\mathbb{E}(G_{1G_4}\text{term}1) \leq Ce^{-2\sigma^2T}e^{a\sigma^2} \left( \frac{1 - e^{a\sigma^2T}}{1 - e^{a\sigma^2}} \right) e^{(15-2b)\frac{\sigma^2}{2}} \left( \frac{1 - e^{(15-2b)\sigma^2T/2}}{1 - e^{(15-2b)\sigma^2T/2}} \right)
\]

\[
= Ce^{(15-2b)\frac{\sigma^2}{2}} \left( \frac{e^{a\sigma^2}}{1 - e^{a\sigma^2}} \right)
\times \begin{bmatrix} 1 - e^{(15-2b)\sigma^2T/2} \\
1 - e^{(15-2b)\sigma^2T/2} \end{bmatrix}
\begin{bmatrix} e^{(15-2b)\tilde{\sigma}^2T} - e^{(a-2c)\sigma^2T} \\
1 - e^{(15-2b)\sigma^2T/2} \end{bmatrix}
\to 0, \text{ as } T \to \infty, \text{ if } b > 15/2
\]

and \( c > \max\{0, a/2\} = a/2, \text{ since } a > 0. \)
Overall, the following upper bound is satisfied in all the cases:

\[
E(e^{\sigma(W_k + U_j + W_t + U_t)}) \leq C e^{\frac{2}{T}(10k+j)}.
\]
Hence we have,
\[ \mathbb{E}(G_2G_3.\text{term} 1) \leq Ce^{-2a^2T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(10+a-b)\frac{a^2}{T}} \left[ \frac{1 - e^{(10+a-b)\frac{a^2}{T}}}{1 - e^{(10+a-b)\frac{a^2}{T}}} \right]^2 \]
\[ \to 0, \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > a + 10. \]

Next we consider \( G_2G_4 \).
\[ G_2G_4 = e^{-2a^2T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(a-b)\frac{a^2}{T}} k e^{a\frac{a^2}{T}} \left[ (e^{\sigma W_k} - e^{a\frac{a^2}{T}}) \right] \times \left[ (e^{\sigma U} - e^{a\frac{a^2}{T}}) (e^{\sigma W_t} - e^{a\frac{a^2}{T}}) \right] \]
\[ G_2G_4.\text{term} 1 = e^{-2a^2T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(a-b)\frac{a^2}{T}} k e^{a\frac{a^2}{T}} \left[ e^{\sigma (W_k+2U_t+W_t)} \right]. \]

For \( k \geq t \):
\[ W_k + 2U_t + W_t \leq |U_k| + |M_k^{(1)}| + 3|U_t| + |M_t^{(1)}| \]
\[ \mathbb{E}(e^{\sigma (W_k+2U_t+W_t)}) \leq C \exp \left[ \frac{\sigma^2}{2} [(k-t) + 16t + (k-t) + 4t] \right] \leq Ce^{\sigma^2k}. \]

For \( t \geq k \):
\[ W_k + 2U_t + W_t \leq 3|U_t - U_k| + 4|U_t| + |M_t^{(1)} - M_k^{(1)}| + 2|M_t^{(1)}| \]
\[ \mathbb{E}(e^{\sigma (W_k+2U_t+W_t)}) \leq C \exp \left[ \frac{\sigma^2}{2} [9(t-k) + 16k + (t-k) + 4k] \right] \leq Ce^{5\sigma^2k}. \]

Combining the the two cases \( k \geq t \) and \( k \leq t \), we get
\[ \mathbb{E}(e^{\sigma (W_k+2U_t+W_t)}) \leq C e^{5\sigma^2k} \]
so that
\[ \mathbb{E}(G_2G_4.\text{term} 1) \leq Ce^{-2a^2T} \sum_{j=1}^{T} e^{a\sigma^2} \sum_{k=1}^{T} e^{(10+a-b)\frac{a^2}{T}} k \]
\[ \leq Ce^{-2a^2T} e^{(10+a-b)\frac{a^2}{T}} \left[ \frac{1 - e^{(10+a-b)\frac{a^2}{T}}}{1 - e^{(10+a-b)\frac{a^2}{T}}} \right] e^{a\sigma^2} \left[ \frac{1 - e^{a\sigma^2}}{1 - e^{a\sigma^2}} \right] \]
\[ \to 0, \text{ as } T \to \infty, \text{ if } b > a + 10 \]
and \( c > \max \{a/2, 0\} = a/2, \text{ since } a > 0 \).

Proof and results for \( G_3G_4 \) are very similar to that of \( G_2G_4 \) and so we skip it. Combing through all the terms we see that if \( c > a > 0 \) and \( b > a + 10 \) each of the expectations converges to zero (and hence \( \mathbb{E}(\hat{\gamma}_k^2) \to 0 \)) as \( T \to \infty \), thereby proving the lemma. \( \square \)
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Majnu John: Departments of Mathematics and of Psychiatry, Hofstra University, Hempstead, NY 11549, USA
E-mail address: Majnu.John@Hofstra.edu

Yihren Wu: Department of Mathematics, Hofstra University, Hempstead, NY 11549, USA
E-mail address: Yihren.Wu@Hofstra.edu