Temperature Dependence of the Magnetic Susceptibility for Triangular-Lattice Antiferromagnets with spatially anisotropic exchange constants

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We present the temperature dependence of the uniform susceptibility of spin-half quantum antiferromagnets on spatially anisotropic triangular-lattices, using high temperature series expansions. We consider a model with two exchange constants, $J_1$ and $J_2$ on a lattice that interpolates between the limits of a square-lattice ($J_1 = 0$), a triangular-lattice ($J_2 = J_1$), and decoupled linear chains ($J_2 = 0$). In all cases, the susceptibility which has a Curie-Weiss behavior at high temperatures, rolls over and begins to decrease below a peak temperature, $T_p$. Scaling the exchange constants to get the same peak temperature, shows that the susceptibilities for the square-lattice and linear chain limits have similar magnitudes near the peak. Maximum deviation arises near the triangular-lattice limit, where frustration leads to much smaller susceptibility and with a flatter temperature dependence. We compare our results to the inorganic materials $\text{Cs}_2\text{CuCl}_4$ and $\text{Cs}_2\text{CuBr}_4$ and to a number of organic molecular crystals. We find that the former ($\text{Cs}_2\text{CuCl}_4$ and $\text{Cs}_2\text{CuBr}_4$) are weakly frustrated and their exchange parameters determined through the temperature dependence of the susceptibility are in agreement with neutron-scattering measurements. In contrast, the organic materials considered are strongly frustrated with exchange parameters near the isotropic triangular-lattice limit.

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I. INTRODUCTION

Understanding the interplay of quantum and thermal fluctuations and geometrical frustration in low-dimensional quantum antiferromagnets is a considerable theoretical challenge\textsuperscript{1,2,3,4,5,6} Research in frustrated quantum antiferromagnets was greatly stimulated by Anderson’s “resonating valence bond” (RVB) paper\textsuperscript{7} in which he suggested that the parent insulators of the cuprate superconductors might have spin liquid ground states and excitations with fractional quantum numbers, motivated by his earlier suggestion of such a ground state for the Heisenberg antiferromagnet on the triangular lattice.\textsuperscript{8} The Ising model on a triangular lattice illustrates the rich physics that can arise due to frustration: it is known to have a macroscopic number of degenerate ground states.\textsuperscript{9} The antiferromagnetic Heisenberg model with spatially anisotropic exchange interactions on the triangular lattice is of interest both theoretically and experimentally. It describes the spin excitations in $\text{Cs}_2\text{CuCl}_4$ and $\text{Cs}_2\text{CuBr}_4$ and the Mott insulating phase of several classes of superconducting organic molecular crystals.\textsuperscript{10} Other materials for which this model is relevant include $\text{NaTiO}_2$, $\text{CuCl}_2$ graphite intercalation compounds,\textsuperscript{11} and the anhydrous alum, $\text{KTi(SO}_4)_2$.\textsuperscript{12} Theoretically, this Heisenberg model is a candidate for a system with spin liquid ground states and possibly excitations with fractional quantum numbers.\textsuperscript{8,16,17} For the triangular-lattice model with spatially isotropic interactions, the preponderence of numerical evidence\textsuperscript{18,19,20,21} suggests that the ground state has long-range magnetic order. However, making the interactions spatially anisotropic can lead to a very rich ground-state phase diagram.\textsuperscript{22}

The spatially anisotropic model, defined by a nearest-neighbor exchange constant $J_1$ along one axis and $J_2$ along all other axes [see Fig. 1], interpolates between the limits of square-lattice ($J_1 = 0$), triangular-lattice ($J_2 = J_1$) and decoupled linear chain ($J_2 = 0$) limits.\textsuperscript{23,24} It has been studied by spin wave theory,\textsuperscript{25} series expansions,\textsuperscript{26} large-N techniques,\textsuperscript{27} slave fermions,\textsuperscript{28} Schwinger bosons with gaussian fluctuations and variational quantum Monte Carlo. Quantum fluctuations are largest for $J_1 \approx 0.8J_2$ and for $J_1 > 4J_2$\textsuperscript{22,25} and so for these parameter regions one is most likely to observe quantum disordered phases.

From an experimental point of view, it is highly desirable to have a definitive way to determine the values of the exchange parameters for individual material systems. Recently, it has been shown how for materials with relatively small values for the Heisenberg exchange constants $J$ this can be done in high magnetic fields, using inelastic neutron scattering to measure the spin wave dispersion in the field induced ferromagnetic phase.\textsuperscript{29}}
The temperature dependence of the magnetic susceptibility is one of the most common experimental measurements and it would be very useful if that can be used to determine the extent of frustration and the various exchange constants directly. It is particularly important to have a scheme for materials, where the very high temperature behavior of the system ($T \gg J$) is not accessible to experiments. Previously, Castilla, Chakravarty, and Emery pointed out how the temperature dependence of the magnetic susceptibility of the antiferromagnetic spin chain compound CuGeO$_3$ implied significant magnetic frustration. In that case, it constrains the ratio of the nearest and next-nearest neighbour exchanges along the chain. Similarly, it is reasonable to expect that the temperature dependence of the magnetic susceptibility should depend on frustration in two-dimensional models also and hence constrain the ratio $J_1/J_2$.

The Mott insulating phase of the organic molecular crystals is of particular interest because under pressure the materials considered become superconducting. A possible RVB theory of superconductivity in such materials, emphasising the role of frustration, have recently been proposed. These materials have exchange constants in the range of several hundred Kelvin, and their behaviour has led to several puzzles. Tamura and Kato measured the temperature dependence of the magnetic susceptibility for five organic molecular crystals in the family, $\beta$-[Pd(dmit)$_2$]X (where dmit is the electron acceptor molecule thiol-2-thione-4, 5-dithiolate, C$_5$S$_2$) and the cation X = Me$_4$As, Me$_4$P, Me$_4$Sb, Et$_2$Me$_2$P, and Et$_2$Me$_2$Sb, where Me = CH$_3$ and Et = C$_2$H$_5$ (denote methyl and ethyl groups, respectively). They compared their results with the predictions for the square and triangular lattices and found that for all the materials the results could be fitted by the high temperature series expansion for the triangular lattice. However, some and not all of them undergo a transition to a magnetically ordered state at low temperatures.

Recently, Shimizu et al. showed using $^1$H nuclear magnetic resonance that $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$ did not undergo magnetic ordering and that the temperature dependence of the uniform magnetic susceptibility could still be fit by that for the triangular lattice. However, it should be stressed that for these molecular crystals the underlying triangular lattice of molecular dimers (to which each spin is associated) is not isotropic and so it is important to know the extent of the spatial anisotropy because this has a significant effect on the possible ground state. The isotropic triangular lattice is believed to be ordered, but for $J_1/J_2 = 0.7-0.9$ the anisotropic lattice could be quantum disordered. Hence, determination of the actual ratio is important for understanding these materials.

Here, we use high temperature series expansions to calculate the temperature dependent uniform susceptibility of the spatially anisotropic triangular-lattice models. Such calculations have been done previously for the pure square and triangular-lattice cases, but not for the spatially anisotropic triangular-lattice model. This method is particularly useful here, as it allows one to cover the full range of $J_2/J_1$ ratios at once. Our main finding is that the susceptibility, for these antiferromagnets, shows a broad maximum at a temperature (which we call the peak temperature $T_p$) of order the Curie-Weiss temperature. If the exchange constants are scaled to give the same peak location, the magnitude of the peak susceptibility varies with frustration. The unfrustrated models, represented by the square-lattice and the linear-chain limits have similar peak susceptibilities. The triangular-lattice deviates the most from them, having a much smaller peak value, and a much flatter temperature dependence. The parameter regimes, where the ground states could be spin-disordered do not stand out in these calculations and are similar to the triangular-lattice limit. The reason for this is probably that at the temperature scales considered the susceptibility is largely determined by short-range frustration, rather than long length scale physics such as the existence of spin liquid states at zero temperature.

Comparison with the measured susceptibility of Cs$_2$CuCl$_4$ and Cs$_2$CuBr$_4$ leads to exchange parameters in agreement with previous neutron measurements. For the organic materials, it shows that they are all close to the isotropic triangular-lattice limit. But, some of them could be weakly anisotropic, leading to a quantum-disordered ground state. Since, the organic materials are close to a Mott metal-insulator transition, we consider the possible role of multiple-spin exchange. Such interactions can be necessary for a quantitative description of such materials.

II. THE FRUSTRATED MODEL

The spatially anisotropic triangular-lattice is shown in Fig. 1. The antiferromagnetic Heisenberg model is described by the Hamiltonian

$$H = J_1 \sum_a S_a \cdot S_j + J_2 \sum_{b,j} S_b \cdot S_j$$

where the first sum runs over all nearest neighbor pairs along the x-axis and the second sum runs over all other nearest neighbor pairs. The vectors $S$ represent spin-half operators. It is evident that, for $J_1 = 0$, the model is equivalent to the square-lattice Heisenberg model, for $J_2 = J_1$ it is equivalent to the isotropic triangular lattice model, and in the limit $J_2 \rightarrow 0$, it is equivalent to a model of decoupled linear chains.

We now discuss how we might quantify how the amount of frustration in the model varies with $J_2/J_1$. Possible measures of frustration which have been discussed before include:

1. The number of degenerate ground states.
2. How the competing interactions prevent the pairwise collinear alignment of spins that would give neighboring spins the lowest interaction energy.
In order to quantify 2., Lacorre considered classical spins and introduced a “constraint” function \( F_c = -E_0/E_b \), which is the ratio of the ground state energy \( E_0 \) of the system to the “base energy” \( E_b \), which is the sum of all bond energies if they are independently fully satisfied, i.e.

\[
E_b = - \sum_{ij} |J_{ij}|(S_i \cdot S_j)_{\text{max}} \quad (2)
\]

Lacorre suggested that \( F_c \) has values ranging from -1 (no frustration) to +1 (complete frustration). However, for spin models that have a traceless Hamiltonian the ground state energy cannot become positive. So, \( F_c \) must lie between -1 (unfrustrated) and 0 (fully frustrated)- the largest possible value of \( F_c \) is zero. Considering a single isosceles triangular plaquette taken from the lattice in Fig. 1. Lacorre found that for classical (large-S) spins as a function of \( J_2/J_1 \), \( F_c \) had its maximum value \((-1/2)\) for the isotropic triangle \((J_1 = J_2)\). The same result holds for the infinite lattice.

Kahn recently stressed that for Heisenberg spins the degeneracy of the ground state depends on the value of the spin quantum number, \( S \), as well as the geometry of the plaquette. For example, on an isotropic triangle, the ground state is four-fold degenerate for \( S = 1/2 \) but non-degenerate for \( S = 1 \). On a single isosceles triangle, for \( S = 1/2 \), the ground state has total spin \( S_T = 1/2 \) and is 2-fold degenerate for \( J_1 \neq J_2 \) and 4-fold degenerate at the isotropic point \( J_1 = J_2 \). We find that both \( F_c \) is maximal \((-1/3)\) and the ground state has the highest degeneracy for \( J_1 = J_2 \). On the other hand, for spin \( S = 1 \) the ground state is a non-degenerate singlet \((S_T = 0)\) for a wide region near the isotropic limit \((0.5 < J_1/J_2 < 2)\), is three-fold degenerate \((S_T = 1)\) outside this range \((J_1/J_2 < 0.5 \text{ or } J_1/J_2 > 2)\) and has accidental 4-fold degeneracy at the special points \( J_1/J_2 = 0.5 \). The function \( F_c \) has no singular maximum, but a plateau at \(-0.5\) for the whole range \(0.5 < J_1/J_2 < 2\), so the spin-1 case is much less frustrated than the extreme spin-1/2 case.

The above properties of the degeneracy and constraint function are not unique to quantum spins but also hold for the Ising model on the same lattice. For a single isosceles triangle and for \( S = 1/2 \) the ground state energy changes at \( J_1 = J_2 \) from \(-J_1/4\) for \( J_1 > J_2 \), to \((-2J_2 + J_1)/4\) for \( J_1 < J_2 \). The “base energy” is \( E_b = -(J_1 + 2J_2)/4 \) and hence \( F_c \) has its maximum value \((-1/3)\) when \( J_1 = 2J_2 \). The degeneracy of the ground state is 2 (only up-down symmetry) for \( J_1 < J_2 \), 6 (only up and all down are not ground states) for \( J_1 = J_2 \), and 4 (either one of the \( J_2 \) bonds can be dissatisfied) for \( J_1 > J_2 \). So indeed by both measures for \( J_1 = J_2 \) the model on a triangle is most frustrated. Extending this analysis for a single triangle to a large lattice of \( N \) sites the difference is even more dramatic as the degeneracy is \( \exp(cN) \) for \( J_1 = J_2 \), and is easily seen to be only 2 for \( J_1 < J_2 \) and \( \exp(c'N^{1/2}) \) for \( J_1 > J_2 \), where \( c, c' \) are numbers of order one. So the model has the largest ground-state degeneracy at the isotropic point.

Although, this paper is concerned with the quantum spin model, the reason we mention the above properties of classical models is because an important question is whether our results concerning the connection between the amount of frustration and the temperature dependence of the susceptibility are also exhibited by the corresponding classical Heisenberg and Ising models. This may be the case if the temperature dependence of the susceptibility down to the peak is largely determined by the frustration and correlations associated with a single plaquette.

With regard to measures of frustration we also note that from an experimental point of view two measures that have been proposed previously\(^{\text{10}}\) (i) The ratio of the Curie-Weiss temperature to the magnetic ordering temperature. This increases with increasing frustration. (ii) The amount of entropy at temperature scales much less than the exchange energy.

\[
\chi = \frac{N_A g^2 \mu_B^2}{kT} \chi
\]

where \( N_A \) is Avogadro’s number, \( g \) the g-factor, \( \mu_B \) a Bohr-magneton, \( k \) the Boltzmann constant and \( T \) the absolute temperature. The dimensionless quantity \( \chi \) can be expressed in a high-temperature expansion in \( J_2/T \) and \( y = J_1/J_2 \), as

\[
\chi = \sum_{n=0}^{\infty} (J_2/T)^n \sum_{m=0}^{n} c_{m,n} y^m / (4^{n+1} n!)
\]
The integer coefficients $c_{m,n}$ complete to order $n = 10$ are presented in Table I.

**IV. CURIE-WEISS BEHAVIOR AND BEYOND: SERIES EXTRAPOLATIONS**

As is well known, the high temperature behavior of the susceptibility, per mole, is given by a Curie-Weiss law

$$\chi = \frac{C}{T + T_{cw}}$$

For our model, the Curie constant

$$C = N_A g^2 \mu_B^2 / 4k = A g^2,$$

with $A = 0.0938$ in cgs units. The Curie-Weiss temperature is

$$T_{cw} = J_2 + J_1 / 2.$$  \hspace{1cm} (7)

From an experimental point of view, an important question is: How low in temperature is the Curie-Weiss law valid? To investigate this, we plot in Fig. 2a, the normalized inverse susceptibility as a function of $T/T_{cw}$ for several parameters, together with the Curie-Weiss law. It is clear that below $T < 10T_{cw}$, the Curie-Weiss fit is no longer accurate. Deviations from the Curie-Weiss behavior are the smallest near the triangular-lattice limit, and largest for linear chains. If one were to fit the inverse susceptibility below some temperature to a Curie-Weiss behavior, one would get a systematically larger Curie-Weiss parameter ratios. It shows that attempts to fit to a Curie-Weiss behavior below four times the Curie-Weiss temperature can result in an overestimate in the Curie-Weiss constant. The effective temperature dependent Curie-Weiss constant $T_{cw}^{eff}$ as

$$T_{cw}^{eff} = -T - \frac{\chi}{d\chi/dT}$$  \hspace{1cm} (8)

If one was to fit $\chi^{-1}$ to a linear curve in the vicinity of some temperature ($T$) and use the intercept to estimate the Curie-Weiss constant, one would get $T_{cw}^{eff}$. Fig. 2b, shows how $T_{cw}^{eff}$ varies with temperature for several parameter ratios. It shows that attempts to fit to a Curie-Weiss behavior below four times the Curie-Weiss temperature can result in an overestimate in the Curie-Weiss constant by less than 20 percent for the isotropic triangular lattice, whereas for the square lattice an error of 40 percent is possible. Similar observations were made previously for the classical Heisenberg antiferromagnet on a kagome lattice.  \hspace{1cm} (3)

To obtain the susceptibility for $T \leq T_{cw}$, we need to develop a series extrapolation scheme. We have used d-log Padé and the integral differential approximants to extrapolate the series. For the linear chain model we use very long series given by Takahashi and for the square and triangular-lattice cases we have also used longer series. In the former case, the calculated susceptibility agrees well with the exact results obtained from Thermodynamic Bethe Ansatz calculations. For the square and triangular-lattice cases it also agrees well with previous numerical calculations. In all cases, several integral/dlog-Padé approximants are calculated, and in the plots below two outer approximants are shown, i.e. a large number of approximants lie between those shown. Based on our general experience with series extrapolations, we feel confident that as long as the upper and lower curves are not too far from each other, they bracket the true value of the thermodynamic susceptibility. In general, we find that the extrapolations work well down to the peak temperature and begin to deviate from each other below the peak. It is not possible to address the zero and very low-temperature behavior of the susceptibility from these calculations.  

In Fig. 3, we show the uniform susceptibility, for different temperatures, with previous numerical calculations. In all cases, several integral/dlog-Padé approximants are calculated, and in the plots below two outer approximants are shown, i.e. a large number of approximants lie between those shown. Based on our general experience with series extrapolations, we feel confident that as long as the upper and lower curves are not too far from each other, they bracket the true value of the thermodynamic susceptibility. In general, we find that the extrapolations work well down to the peak temperature and begin to deviate from each other below the peak. It is not possible to address the zero and very low-temperature behavior of the susceptibility from these calculations.

In Fig. 3, we show the uniform susceptibility, for different temperatures, with previous numerical calculations.

![Graph showing the Curie-Weiss Law](image)
different $y = J_1/J_2$, as a function of temperature. For all $J_1/J_2$ ratios, there are two plots showing the upper and lower limits of extrapolated values as discussed in the previous paragraph. The susceptibility is scaled to have a peak value of unity, and the temperature axis is scaled by the peak susceptibility to a dimensionless relative temperature. One finds that the susceptibility peaks at a comparable relative temperature for the unfrustrated square-lattice and linear chains. The primary difference between these two models lies in the behavior of the susceptibility below the peak. It decreases much more slowly for the linear chains than it does for the square-lattice. We believe, this is related to the fact that longer-range antiferromagnetic interactions grow faster for the square-lattice than they do for linear-chains. Thus the shift of the spectral weight away from zero wavevector occurs more gradually for linear chains. For the triangular-lattice, the peak is shifted to much lower relative temperatures. Note that the triangular-lattice has a peak at a temperature even lower than for $J_1/J_2 = 0.8$, where $T = 0$ calculations show an absence of long range order.

From Fig. 3 it is clear that frustration leads to a reduction in the magnitude of the product $\chi_p T_p$ as well as a reduction in the peak temperature $T_p$ with respect the Curie-Weiss temperature $T_{cw}$. These parameters are plotted in appropriate dimensionless units in Fig. 4 as a function of the frustration ratio $J_1/(J_1 + J_2)$, and both have a minimum around the triangular lattice limit $J_1 = J_2$. To connect with experiments we also show the ratio of the peak temperature $T_p$ and the Zeeman energy required to fully polarize the spins $g\mu_B B_{sat}$, related to the couplings strength by

$$g\mu_B B_{sat} = \begin{cases} 2J_1 + 2J_2 + \frac{J_2^2}{J_1} & \text{for } J_2 \leq 2J_1 \\ 4J_2 & \text{for } J_2 \geq 2J_1 \end{cases}$$

Fig. 4 also shows the ratio $\chi(4T_p)/\chi_p$, which is a measure of the flatness of the curves on the high-temperature side of the peak. A larger value of this ratio implies a slower decay of the susceptibility with temperature. These quantities clearly show that the triangular-lattice is the most frustrated, with the lowest peak temperature relative to the scale of the exchange interactions, $T_p/T_{cw}$ or $kT_p/g\mu_B B_{sat}$, the smallest dimensionless ratio $T_p\chi_p$ and the flattest peak denoted by the largest $\chi(4T_p)/\chi_p$. The plots look very symmetrical around the triangular-lattice limit, and there is nothing anomalous about the case of $J_1/J_2 = 0.8$, where zero-temperature studies give a disordered and gapped dimerized ground state. We note that all of the extracted parameters in Fig. 4 are from the susceptibility curve at temperatures above the peak and in order to see evidence for the presence of a gap for $J_1/J_2 \sim 0.8$ as opposed to no gap in the isotropic
triangular-lattice case one would be required to analyze the susceptibility curve at temperatures much below the estimated gap $\Delta \sim 0.25J_2 \sim 0.5 T_p$ in the dimerised state and such low temperatures are not accessible by the present series calculations.

V. COMPARISON WITH EXPERIMENTAL SYSTEMS

In this section, we compare our theoretical results with experimental data on Cs$_2$CuCl$_4$, Cs$_2$CuBr$_4$ and various organic materials. In Fig. 5, we show the susceptibility as a function of temperature for different $J_1/J_2$ ratio, where the temperature is scaled by the peak temperature ($T_p$) and the susceptibility itself is scaled by the peak temperature to give a dimensionless reduced susceptibility. This plot is very instructive as it allows one to clearly read out the $J_1/J_2$ ratios. Also shown are the susceptibilities for the materials Cs$_2$CuCl$_4$ and Cs$_2$CuBr$_4$, with their g-values taken from ESR experiments. In this plot with no free parameters, it is apparent that the $J_1/J_2$ ratio is near 3.0 for Cs$_2$CuCl$_4$ and near 2.0 for Cs$_2$CuBr$_4$. Some of these results can also be seen from Fig. 4 where key dimensionless ratios of the temperature dependent susceptibility are shown.

A more detailed comparison of the susceptibility for the materials, Cs$_2$CuCl$_4$ and Cs$_2$CuBr$_4$, allowing $g$ to vary freely is shown in Fig. 6. Once $g$ is allowed to vary, the material Cs$_2$CuCl$_4$ can be fit above the peak not too badly even with the pure square-lattice model (not shown). However, a much improved fit happens with $J_1/J_2 = 3$ and $J_2 = 1.49$ K in excellent agreement with the exchange values extracted directly from neutron scattering measurements. Also shown are fits to linear chain and triangular-lattice limits, which bracket $J_1/J_2 = 3$. One can see significant deviation in both limits. The large deviation from the isotropic triangular-lattice case shows that frustration is relatively weak in this material.

For Cs$_2$CuBr$_4$, the best fit for $J_1/J_2 = 2$ arises with $J_2 = 6.99$ K. However, when $g$ is allowed to vary, a range of $J_1/J_2$ values from 1.8 to 2.8 give comparable fits, several of which are shown in figure. In general, the high temperature data is better fit by a larger $J_1/J_2$ value, whereas the data at and around the peak is better fit by a smaller $J_1/J_2$ value. No choice of parameters can fit the very low temperature data (below half the peak temperature). These values are also consistent with previous estimates. Using the value of the incommensurate ordering wavevector $Q = 0.575(1)b^*$ observed by neutron scattering classical spin-wave theory gives $J_1/J_2 = 2.14$ whereas including quantum renormalization corrections as predicted by large $N$ Sp($N$) theory gives $J_1/J_2 \sim 1.8$, and series expansions gives $J_1/J_2 \sim 1.4$. This calls into question the rather large renormalization of the ordering wavevector found in the series expansion study.

Now we turn to the organic materials. In Fig. 7 we
show a corresponding comparison for the material $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$. Only the theoretical data for the isotropic triangular-lattice are shown. One can see an important difficulty in using the $T_p$-scaled plots near the triangular-lattice limit to determine $J_1/J_2$. For the organic material, $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$, the measured susceptibility is very flat and it is difficult to determine the peak temperature $T_p$. From the data, the peak temperature appears to be between 65 K and 95 K. Using the values for $T_p$ of 65 K and 95 K, one can either get the data to fall above or below the triangular-lattice values. A suitably chosen peak temperature allows one to get very close agreement with the triangular-lattice limit. This peak temperature can also be used to determine the exchange constant. However, for the triangular-lattice, there is theoretical uncertainty in the peak location. Hence, it is more accurate to directly fit the experimental data to theory to obtain the exchange constants. For $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$, fixing $g = 2.006$ and $J_1 = J_2$, the best fit leads to $J_1 = 256$ K, a value close to that obtained by Shimizu et al.\textsuperscript{35}

The ability to fit flat susceptibilities to the isotropic triangular-lattice model is further illustrated in Fig. 8, where the susceptibility data are shown from five different molecular crystals in the family $\beta'\beta''$-[Pd(dmit)$_2$]$_2$X (where dmit is the electron acceptor molecule thiol-2,5-dithiolate, C$_3$S$_2$) and the cation X = Me$_4$As, Me$_4$Se, Me$_2$Sb, Et$_2$Me$_2$P, and Et$_2$Me$_2$Sb, where Me = CH$_3$ and Et = C$_2$H$_5$, denote methyl and ethyl groups, respectively). We have taken the $g$-value to be 2.04. By adjusting the peak temperature, they can all be brought to rough agreement with the triangular-lattice model. Assuming isotropic interactions, and $g = 2.04$, we estimate the exchange constants to be 283 K, 289 K, 279 K, 279 K and 247 K respectively. It is clear that none of these organic materials are far from the isotropic triangular-lattice limit. But, we emphasize, that by this method it is difficult to discriminate between $J_1/J_2$ ratios in the range $0.85 < J_1/J_2 < 1.15$. Note that the latter regions also include quantum disordered phases.

To avoid the problem of determining the peak temperature, we go back to Fig. 4 and scale the data by the peak susceptibility. These can be inferred accurately from the data, even when the peak temperature cannot. In Fig. 4 we show such a comparison of experimental data with theory. The data for $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$ lies extremely close to the isotropic triangular-lattice case. The other materials deviate from the $J_1 = J_2$ limit, but still lie in the range $0.85 < J_1/J_2 < 1.15$. If we assume that the systems are described by the isotropic triangular-lattice, the exchange constant can be read from the peak susceptibility by using the relation $J = 0.0035g^2/\chi_p$. This leads to exchange constants of 250 K for $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$ and 280 K, 289 K, 260 K, 273 K and 236 K for the other materials. These values are close to those obtained from the best fits.

It should be noted here that in the experimental data, a Curie term from magnetic impurities and a diamagnetic term has been subtracted and these can also influence the determination of exchange parameters. However, it is unlikely that any of these materials are very far from the isotropic triangular lattice limit.

From the fits the Heisenberg couplings are comparable for all materials and around 250 K. We now consider how these compare with quantum chemistry calculations. The exchange constants can be related to parameters in an underlying Hubbard model\textsuperscript{12,34,54}, where $J = 2U/\lambda$ and $t$ is the intersite (i.e., inter-dimer) hopping and $U$ is the cost of double occupancy for two electrons or holes on a dimer. If the Coulomb repulsion $U_0$ on a single molecule within the dimer is much larger than the inter-molecular hopping $t_0$ within a dimer then $U \approx 2t_0$. For $\beta'\beta''$-[Pd(dmit)$_2$]$_2$X electronic structure calculations based on the local-density approximation (LDA)\textsuperscript{34,51,55} give $t \sim 30$ meV and $t_0 \sim 500$ meV, and so $J \sim 50$ K. For $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$ Hückel electronic structure calculations give $t \sim 50$ meV and $t_0 \sim 200$ meV\textsuperscript{56,57}. The resulting $U \sim 400$ meV is comparable to that deduced from measurements of the frequency dependent optical conductivity of similar $\kappa$ materials\textsuperscript{12,58}. This value of $U$ is smaller than values deduced from quantum chemistry calculations on isolated dimers, which do not take into account screening\textsuperscript{59}. Using the above values of $t$ and $U$ gives $J \sim 100$ K.

FIG. 7: (Color online) Comparison of the temperature dependence of the magnetic susceptibility of $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$ with series expansions calculations for isotropic triangular-lattice. The experimental data is from Ref. [40]. A value of $g=2.006$ was used based on electronic spin resonance measurements\textsuperscript{36}. We see that this material is well described by a Heisenberg model on the isotropic triangular lattice, with peak temperature $T_p = 85$ K. Note also that the agreement is quite sensitive to changes in the value of $T_p$, a quantity that is difficult to pin-point in a flat curve.

Note that the quality of fit is best for the material $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$, where it really fits well with the isotropic triangular-lattice model. However, it is also a system that does not order down to very low
interactions do not need to be taken into account. The isotropic triangular lattice, assuming that ring-exchange terms in the Hamiltonian.

For the triangular lattice triple exchange is also possible. However, for spin-1/2 this just corresponds to a renormalisation of the nearest-neighbour two-particle exchange. The frustrating effects of multiple-spin exchange on the isotropic triangular lattice lead to rich physics and have an experimental realisation in monolayers of solid $^3$He on graphite. Let $J$ denote the nearest-neighbour exchange and $J_4$ the multiple spin-exchange, involving the four spins comprising a pair of triangular plaquettes. This model has been studied extensively and exact diagonalisation calculations suggest that the 120 degree Neel state, which is the ground state for $J_4 = 0$, is destroyed when $J_4 > 0.1J$. It is appealing to think that this could be the explanation for why $\kappa$-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$ does not magnetically order, whereas it should if it is really described by the isotropic triangular lattice nearest neighbour model. This material is close to a Mott-Hubbard metal-insulator transition since the insulating state is destroyed under pressure or uniaxial stress. However, it is not clear that $J_4$ will be large enough in the actual material. The expressions derived from a $t/U$ expansion give $J_4/J = 10(t/U)^2$. This means one must have $U < 8t$ to obtain a spin liquid. However, exact diagonalisation of the Hubbard model on the isotropic triangular lattice at half filling, shows that the insulating state only occurs for $U > 12t$. Hence, it is not clear that multiple-spin exchange could account for the fact that this material appears to be close to the isotropic triangle but does not magnetically order. However, to definitively resolve this issue would require a detailed study of the spatially anisotropic model with four-spin exchange.

VI. CONCLUSIONS

In this paper, we have developed high temperature expansions for the uniform susceptibility of the spatially anisotropic triangular lattice Heisenberg model. We find that the temperature dependence of the susceptibility at temperatures of order the exchange constants, is sensitive to frustration, that is, the ability of spins to align antiparallel to all their neighbors. The square-lattice and linear chain limits have similar reduced susceptibilities at and above the peak, while the triangular-lattice limit appears most frustrated, with the smallest and flattest susceptibilities. Comparison with various experimental systems shows that a variety of organic materials are close to the isotropic triangular-lattice limit, whereas the inorganic materials Cs$_2$CuCl$_4$ and Cs$_2$CuBr$_4$ are much less frustrated.

It would be nice to have a simple formalism which

![Graph showing the temperature dependence of the uniform susceptibility of the spatially anisotropic triangular lattice Heisenberg model.](image)

**FIG. 8:** (Color online) Comparison of the temperature dependence of the uniform susceptibility of the spatially anisotropic triangular lattice Heisenberg model. The metallic phase of the organics are in the regime $U/t \sim 5 - 10$ [Ref. 12] and so one might expect multiple-exchange terms to be relevant in the insulating phase. Then these terms may be important. Recent neutron scattering studies showed the effect of such interactions on the dispersion of spin excitations in La$_2$CuO$_4$. The metallic phase of the organics are in the regime $U/t \sim 5 - 10$ [Ref. 12] and so one might expect multiple-exchange terms to be relevant in the insulating phase. Then these terms may be important. Recent neutron scattering studies showed the effect of such interactions on the dispersion of spin excitations in La$_2$CuO$_4$.
could provide an analytic relation between the peak susceptibility and exchange parameters. Qualitatively, our arguments show that short-range frustration, or the inability to align parallel with respect to neighbors as quantified by the parameter $F_1$ in Section II, is maximum near the isotropic limit and this is what pushes the peak in the susceptibility down to lower temperatures. For a wide range of frustrated antiferromagnets it has been previously pointed out that the Curie-Weiss law holds to relatively low temperatures. Several theoretical models, mostly for classical spins, have been developed to explain this. Basically, frustration leads to individual plaquettes or spin clusters behaving essentially independently. However, our models are less frustrated than that and hence always develop substantial correlations. This means that any simplistic explanation is unlikely.

In organic molecular crystals a weak temperature dependence of the magnetic susceptibility is often interpreted as being evidence for metallic behavior, since for a Fermi liquid the susceptibility is weakly temperature dependent. However, this is inconsistent with the fact that in most of these materials above temperatures of about 50 K there is no Drude peak in the optical conductivity and the resistivity has a non-monotonic temperature dependence and values of order the Mott limit. This work shows that due to the substantial magnetic frustration the susceptibility can actually be due to local magnetic moments, even though in the range up to 300 K one does not see a clear Curie temperature dependence.

In a future study we will consider the temperature dependence of the specific heat capacity for this model. A previous study of the square lattice, single chain, and triangular lattice Heisenberg model found that the peak in the specific heat versus temperature curve occurred around $J$ for all models but was much broader for the triangular lattice. A related issue was that as the temperature decreases the entropy decreases much more slowly for the triangular lattice than the others.

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TABLE I: Series coefficients for the high-temperature expansions of the uniform susceptibility $\chi$ in Eq. (4). Nonzero coefficients $c_{m,n}$ up to order $n = 10$ are listed.

| $(m, n)$ | $c_{m,n}$ | $(m, n)$ | $c_{m,n}$ | $(m, n)$ | $c_{m,n}$ | $(m, n)$ | $c_{m,n}$ |
|----------|-----------|----------|-----------|----------|-----------|----------|-----------|
| (0, 0)   | 1         | (3, 5)   | -7680     | (7, 7)   | 20480     | (7, 9)   | -129328128 |
| (0, 1)   | -4        | (4, 5)   | 1920      | (0, 8)   | 4205056   | (8, 9)   | -159694848 |
| (1, 1)   | -2        | (5, 5)   | -672      | (1, 8)   | -58877952 | (9, 9)   | 19133440   |
| (0, 2)   | 16        | (0, 6)   | 23488     | (2, 8)   | 110985216 | (0, 10)  | -2574439424 |
| (1, 2)   | 32        | (1, 6)   | 293376    | (3, 8)   | -501760   | (1, 10)  | 52032471040 |
| (0, 3)   | -64       | (2, 6)   | 111552    | (4, 8)   | 101972480 | (2, 10)  | -735774720  |
| (1, 3)   | -264      | (3, 6)   | 411392    | (5, 8)   | -84013056 | (3, 10)  | -29924454400 |
| (2, 3)   | -96       | (4, 6)   | -115968   | (6, 8)   | 29817856  | (4, 10)  | 15318384640 |
| (3, 3)   | 16        | (5, 6)   | 70656     | (7, 8)   | -15618048 | (5, 10)  | 38033190912 |
| (0, 4)   | 416       | (6, 6)   | -12768    | (8, 8)   | 2932776   | (6, 10)  | -40192143360 |
| (1, 4)   | 1216      | (0, 7)   | 207616    | (0, 9)   | -198295552| (7, 10)  | 48646737920 |
| (2, 4)   | 2400      | (1, 7)   | -1766016  | (1, 9)   | -571327488| (8, 10)  | -13533921280 |
| (3, 4)   | 512       | (2, 7)   | -7739648  | (2, 9)   | 3934844928| (9, 10)  | 4594278400 |
| (4, 4)   | 80        | (3, 7)   | -1804992  | (3, 9)   | -4115195904| (10, 10) | -869608960 |
| (0, 5)   | -4544     | (4, 7)   | 3373440   | (4, 9)   | 3772164096 |
| (1, 5)   | -10880    | (5, 7)   | 689920    | (5, 9)   | -1888413696 |
| (2, 5)   | -20480    | (6, 7)   | 120064    | (6, 9)   | 1134317568 |