ON HARMONIC FUNCTIONS AND THE HYPERBOLIC METRIC

MARIJAN MARKOVIĆ

Abstract. Motivated by some recent results of Kalaj and Vuorinen (Proc. Amer. Math. Soc., 2012), we prove that positive harmonic functions defined in the upper half–plane are contractions w.r.t. hyperbolic metrics of half–plane and positive part of the real line, respectively.

1. Introduction and the main result

Denote by \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) the unit disc of the complex plane \( \mathbb{C} \) and by \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \} \) the upper half–plane. \( \mathbb{R} \) is the whole real axis, and the positive real axis is denoted by \( \mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \} \).

Let \( d_h \) stands for the hyperbolic distance on the disc \( U \). With the same letter we denote the hyperbolic distance on \( \mathbb{H} \) and \( \mathbb{R}^+ \), since we believe that misunderstanding will not occur. We have

\[
d_h(z, w) = \inf_{\gamma} \int_{\gamma} \frac{|d\omega|}{1 - |\omega|^2} = 2 \tanh^{-1} \left| \frac{z - w}{1 - \overline{z}w} \right|,
\]

where \( \gamma \subseteq U \) is any regular curve connecting \( z \in U \) and \( w \in U \). On the other hand, the hyperbolic distance between \( z \in \mathbb{H} \) and \( w \in \mathbb{H} \) is

\[
d_h(z, w) = \inf_{\gamma} \int_{\gamma} \frac{|d\omega|}{\text{Im} \, \omega} = 2 \tanh^{-1} \left| \frac{z - w}{z - \overline{w}} \right|,
\]

where now \( \gamma \subseteq \mathbb{H} \). In particular, the hyperbolic distance between \( x \in \mathbb{R}^+ \) and \( y \in \mathbb{R}^+ \), where \( x \leq y \) is

\[
d_h(x, y) = d_h(ix, iy) = \int_{x}^{y} \frac{dt}{t} = \log \frac{y}{x}.
\]

Recall the classical Schwarz–Pick lemma. An analytic function \( f \) of the unit disk into itself satisfies

\[
\left| \frac{f(z) - f(w)}{1 - f(z)f(w)} \right| \leq \left| \frac{z - w}{1 - \overline{z}w} \right|
\]

for all \( z, w \in U \). The equality sign occurs if and only if \( f \) is a Möbius transform of \( U \) onto itself.

The previous result has a counterpart for analytic functions \( f : \mathbb{H} \to \mathbb{H} \). Using the Cayley transform one easily finds that the Schwarz–Pick inequality in this settings says

(1) \[
\left| \frac{f(z) - f(w)}{f(z) - f(w)} \right| \leq \left| \frac{z - w}{z - w} \right|
\]

2010 Mathematics Subject Classification. Primary 31A05.
Key words and phrases. Positive harmonic functions.
for every $z, w \in \mathbb{H}$. Letting $z \to w$ in (1), we obtain

$$\frac{|f'(z)|}{\text{Im } f(z)} \leq \frac{1}{\text{Im } z}. \tag{2}$$

Regarding the expression for the hyperbolic distance in the upper half-plane, (1) may be rewritten as

$$d_h(f(z), f(w)) \leq d_h(z, w), \tag{3}$$

which means that $f$ is a contraction in the hyperbolic metric of $\mathbb{H}$. It is well known that the equality sign attains in (1), (2), and (3) (for some $z$ or for some distinct $z$ and $w$, and therefore for all such points) if and only if

$$f = \text{a M"obius transform of } \mathbb{H} \text{ onto itself.}$$

During the past decade, harmonic mappings and functions have been extensively studied and many results from the theory of analytic functions have been extended for them.

Quite recently Kalaj and Vuorinen [3] proved that a harmonic function $f : \mathbb{U} \to (-1, 1)$ is a Lipschitz function in the hyperbolic metric, i.e., for every $z, w \in \mathbb{U}$ they obtained that

$$d_h(u(z), u(w)) \leq \frac{4}{\pi} d_h(z, w). \tag{4}$$

Actually, using the classical results they firstly established

$$|\nabla u(z)| \leq \frac{4}{\pi} \frac{1 - |u(z)|^2}{1 - |z|^2} \tag{5}$$

for $z \in \mathbb{U}$. Both inequalities are sharp.

We refer to [1] for a related result.

We are interested here in the positive harmonic functions defined in $\mathbb{H}$. As we have said in the abstract, our main aim is to prove

**Theorem 1.1.** Let $u : \mathbb{H} \to \mathbb{R}^+$ be harmonic. Then

$$d_h(u(z), u(w)) \leq d_h(z, w) \tag{6}$$

for all $z, w \in \mathbb{H}$. In other words, a positive harmonic function is a contractible function in the hyperbolic metric.

Moreover, if the equality sign holds in (6) for some pair of distinct points $z$ and $w$, then the function $u$ must be of the following form

$$u(z) = \text{Im}(a \text{ M"obius transform of } \mathbb{H} \text{ onto } \mathbb{H}).$$

**Remark 1.2.** The group of all conformal mappings of $\mathbb{H}$ onto itself is given by

$$\left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R}, ad - bc > 0 \right\}.$$ 

Thus, a positive harmonic function $u(z)$ is extremal for the inequality (6) if and only if it has the form

$$u(z) = k \cdot \text{Im } z \quad \text{or} \quad u(z) = k \cdot P(z, t),$$

where $k > 0$ and $t \in \mathbb{R}$; here

$$P(z, t) = \frac{1}{\pi} \frac{y}{(x - t)^2 + y^2},$$

$z = x + iy \in \mathbb{H}, t \in \mathbb{R}$ is the Poisson kernel for the upper half-plane.
2. Proof of the result

Let $\Omega$ be any domain in $\mathbb{C}$ (or in $\mathbb{R}$). A metric density $\rho$ is any continuous function in $\Omega$ with nonnegative values everywhere in $\Omega$. The $\rho$-length (or just a length) of a curve $\gamma$ in $\Omega$ is given by

$$\int_{\gamma} \rho(z) |dz|.$$  \hfill (7)

The $\rho$-distance (or just a distance) between $z \in \Omega$ and $w \in \Omega$ is

$$d_{\rho}(z, w) = \inf_{\gamma} \int_{\gamma} \rho(z) |dz|,$$

where $\gamma$ is a regular curve in $\Omega$ connecting point $z$ and $w$. Of course, if $\Omega$ is an interval in $\mathbb{R}$, then we need not the infimum sign in the preceding expression for the distance function.

For a regular curve $\gamma$ we denote by $t_\gamma(\omega)$ the unit tangent vector at a point $\omega \in \gamma$ consistent with the orientation of $\gamma$.

We will prove the following lemma of somewhat general character.

**Lemma 2.1.** Let $\Omega \subseteq \mathbb{C}$ be a domain, $I \subseteq \mathbb{R}$ an open interval, and $u : \Omega \to I$ a smooth function, i.e., $u \in C^1(\Omega)$. Let $\rho$ be a metric density in $\Omega$ and let $\tilde{\rho}$ be a metric density in the interval $I$. If

$$\tilde{\rho}(u(\omega)) |\nabla u(\omega)| \leq \rho(\omega)$$

for every $\omega \in \Omega$, then we have

$$d_{\tilde{\rho}}(u(z), u(w)) \leq d_{\rho}(z, w)$$

for $z, w \in \Omega$.

If the equality sign is attained in $\text{(10)}$ for some pair of distinct points $z$ and $w$, then equality holds in $\text{(9)}$ for some $\omega \in \Omega$.

**Proof.** Let $z$ and $w$ be distinct fixed points in $\Omega$. Without lost of generality, we may assume that $u(z) \leq u(w)$. Let $\gamma$ be a regular curve connecting $z$ and $w$. Orient it from $z$ to $w$. Denote

$$I_0 = \{ \omega \in \gamma : \text{there exist } \lambda(\omega) > 0 \text{ such that } \nabla u(\omega) = \lambda(\omega) t_\gamma(\omega) \}.$$  

This set may be decomposed as

$$I_0 = \bigcup_{n=1}^{\infty} I_n,$$

where $I_n$ are intervals in $\gamma$ (if the union if finite, we assume that $I_n = \emptyset$, starting from an integer $n_0$). Let $J_1 = I_1$,

$$J_2 = I_2 \setminus \{ \omega \in \gamma : u(\omega) \in u(J_1) \},$$

and by induction for $n > 1$, let

$$J_{n+1} = \left\{ \omega \in I_n : u(\omega) \notin u \left( \bigcup_{k=1}^{n} J_k \right) \right\}.$$  

Denote

$$J = \bigcup_{k=1}^{\infty} J_k.$$
Then \([u(z), u(w)] \subseteq u(\gamma)\) and \(|[u(z), u(w)]| = |u(J)|\). Moreover, \(u\) is injective in \(J\).

Therefore, by using inequality (9) we obtain
\[
d_{\tilde{\rho}}(u(z), u(w)) = \int_{u(z)}^{u(w)} \tilde{\rho}(\tilde{\omega}) \, |d\tilde{\omega}| = \int_{J} \rho(u(\omega)) \, |\nabla u(\omega)| \, |d\omega|
\]
\[
\leq \int_{J} \rho(\omega) \, |d\omega| \leq \int_{\gamma} \rho(\omega) \, |d\omega|.
\]

Since \(\gamma\) is any curve, we have
\[
d_{\tilde{\rho}}(u(z), u(w)) \leq d_{\rho}(z, w),
\]
what we have to prove.

The second part of this lemma follows immediately. \(\square\)

As an application of the preceding lemma and the Schwarz–Pick lemma we prove our main result here.

Proof of Theorem 1.1. Let \(u : \mathbb{H} \to \mathbb{R}^+\) be a harmonic function. Denote by \(f\) an analytic mapping in the upper half–plane such that
\[
\text{Im } f(z) = u(z)
\]
for \(z \in \mathbb{H}\). Then \(f\) maps \(\mathbb{H}\) into \(\mathbb{H}\). Since \(|f'(z)| = |\nabla u(z)|, z \in \mathbb{H}\), applying the version of Schwarz’s lemma for analytic function in the upper half–plane, i.e. (2), we obtain
\[
(11) \quad \frac{|\nabla u(z)|}{u(z)} = \frac{|f'(z)|}{\text{Im } f(z)} \leq \frac{1}{\text{Im } z}
\]
for \(z \in \mathbb{H}\). The equality sign appears if and only if \(f\) is a Möbius transform of \(\mathbb{H}\) onto \(\mathbb{H}\). This is a counterpart of the sharp estimate (5) for harmonic functions in the unit disc.

Thus \(u\) satisfies the condition of Lemma 2.1 with the hyperbolic metric density on each sides. Thus, according to this lemma we obtain
\[
(12) \quad d_{h}(u(z), u(w)) \leq d_{h}(z, w)
\]
for all \(z, w \in \mathbb{H}\).

If the equality sign holds for a pair of distinct points \(z, w \in \mathbb{H}\), then we must have the equality sign in (11) for some \(z\). As we know, this means that \(u\) must be of the form
\[
u = \text{Im}(a \text{ Möbius transform of } \mathbb{H} \text{ onto itself}),
\]
what proves the second part of our theorem. \(\square\)

Remark 2.2. In order to prove that the inequality (5) is sharp the authors of [3] have found a function for which the inequality (5) reduces to the equality. However, following the proof in [3], one can deduce that the equality sign holds in (5) (for some \(z\), and therefore for all \(z\)) if and only if \(u(z) : U \to (-1, 1)\) has the form
\[
u(z) = \text{Re} \left\{ \frac{2i}{\pi} \log \frac{1 + b(z)}{1 - b(z)} \right\} = -\frac{2}{\pi} \arg \frac{1 + b(z)}{1 - b(z)},
\]
where \(b(z)\) is a Möbius transform of the unit disc onto itself.

Using Lemma 2.1 and the sharp estimate (5) for harmonic functions \(u : U \to (-1, 1)\) one can derive (2) (what is the main result of [3]), along with all extremal functions.
References

[1] H. Chen, *The Schwarz–Pick lemma and Julia lemma for real planar harmonic mappings*, Science China Mathematics 56 (2013), 2327–2334.

[2] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[3] D. Kalaj and M. Vuorinen, *On harmonic functions and the Schwarz lemma*, Proc. Amer. Math. Soc. 140 (2012), 161–165.

Faculty of Natural Sciences and Mathematics
University of Montenegro
Cetinjski put b.b. 81000 Podgorica
Montenegro
E-mail address: marijanmarkovic@gmail.com