On Kaluzhnin-Krasner’s embedding of groups

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Abstract. In this note, we consider a 'thrifty' version of Kaluzhnin-Krasner’s embedding in wreath products and apply it to extensions by finite groups and to metabelian groups.

1. Introduction

This note goes back to the pioneer paper of L. Kaluzhnin and M. Krasner [6], where wreath products of groups were introduced and studied. Later many other group theorists applied wreath products to construct various counter-examples and to prove embedding theorems, and now wreath products are among the main tools of Group Theory. Here I pay attention to a feature of Kaluzhnin-Krasner’s works, that probably has not been used in subsequent research papers. Herewith I consider only standard wreath products of abstract groups (i.e., in terms of [6], of permutation groups with regular actions).

Let $A$ and $B$ be groups and $F$ a group of all functions $f : B \to A$ with multiplication $(f_1f_2)(x) = f_1(x)f_2(x)$ for $x \in B$. The group $B$ acts on $F$ from the right by shift automorphisms: $(f \circ b)(x) = f(xb^{-1})$ for all $f \in F$, $b, x \in B$, and the associated with this action semidirect product $B \ltimes F$ is called the (complete) wreath product of the groups $A$ and $B$, denoted by $A \text{Wr} B$. Thus, every element of $A \text{Wr} B$ has a unique presentation as $bf$ ($b \in B, f \in F$) and the multiplication rule follows from the conjugation formula

$$ (b^{-1}fb)(x) = f(xb^{-1}) $$

in $A \text{Wr} B$ for any $b, x \in B$ and $f \in F$.

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Observe that any homomorphism $A \to \bar{A}$ induces the homomorphism $A \text{Wr} B \to \bar{A} \text{Wr} B$ by the rule $bf \mapsto b\bar{f}$, where $\bar{f} \in \bar{F}$ is obtained by replacing the values of $f$ by their images in $\bar{A}$.

Given an arbitrary group $G$ with a normal subgroup $A$, one has a canonical homomorphism $\pi$ of $G$ onto the factor group $G/A = B$. Let $b \mapsto b^s$ be any transversal $B \to G$, i.e. $\pi(b^s) = b$. Then the Kaluzhnin-Krasner monomorphism $\phi$ of the (abstract) group $G$ into $A \text{Wr} B$ is given by the formula (see [5], [10])

$$
\phi(g) = \pi(g)f_g, \quad \text{where } f_g(x) = (x\pi(g)^{-1})^s g (x^s)^{-1}
$$

Applying $\pi$ to $(x\pi(g)^{-1})^s g (x^s)^{-1}$, one obtains 1, and so $f_g \in F$. To check that $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$, one just exploits the formulas (1,2). Finally, $\phi$ is injective since obviously we have $\ker \phi \subseteq A$, and by (2), $f_g(x) = x^s g(x^s)^{-1} \neq 1$ if $g \in A\backslash 1$.

The above-defined form of the Kaluzhnin-Krasner embedding $\phi$ is well known, and, up to conjugation, the image $\phi(G)$ does not depend on the transversal $s$. However, the original paper [6] suggested a stronger form of such an embedding. Namely, assume now that a subgroup $C$ is normal in the normal subgroup $A$ of $G$, but $C$ does not contain nontrivial normal subgroups of $G$. Then consider the product $\kappa$ of $\phi$ and the homomorphism $A \text{Wr} B \to \bar{A} \text{Wr} B$ induces by the canonical homomorphism $A \to \bar{A} = A/C$ with $a \mapsto \bar{a}$ for $a \in A$. Thus

$$
\kappa(g) = \pi(g)\bar{f}_g, \quad \text{where } \bar{f}_g(x) = (x\pi(g)^{-1})^s g (x^s)^{-1}
$$

If $g \in \ker \kappa$, then $\pi(g) = 1$ and so $g \in A$. Then formula (3) shows that the values $x^s g(x^s)^{-1}$ are trivial in $A/C$ for any $x \in B$, i.e. $x^s g(x^s)^{-1} \in C$. Taking $x = 1$, we have $g \in C$ since $C$ is normal in $A$. Thus, $\ker \kappa \subseteq C$, which implies $\ker \kappa = 1$ because $C$ contains no nontrivial normal in $G$ subgroups.

Therefore $\kappa$ is also an embedding of $G$, and the following is a slightly modified version of the old Kaluzhnin-Krasner statement:

**Proposition 1.1.** Let $G \triangleright A \triangleright C$ be a subnormal series of a group $G$ with factors $\bar{A} = A/C$ and $B = G/A$, and let $C$ contain no non-trivial normal subgroups of $G$. Then there is an isomorphic embedding $\kappa$ of the group $G$ in the wreath product $\bar{A} \text{Wr} B = B \ltimes \bar{F}$, the embedding $\kappa$ can be defined by the formula (3), so $\kappa(A) \trianglelefteq \bar{F}$ and $\kappa(G)\bar{F} = \bar{A} \text{Wr} B$.

The group $\bar{A}$ can be much smaller than $A$, and making use of this below, we
apply the embedding $\kappa$ to wreath products with finite active group $B$, 
• embed finitely generated metabelian groups into $\bar{A}\Wr B$ with 'small' abelian $\bar{A}$ and $B$, and
• observe that $(\mathbb{Z}/p\mathbb{Z})\Wr \mathbb{Z}$ and $\mathbb{Z}\Wr \mathbb{Z}$ contain $2^{\aleph_0}$ non-isomorphic locally polycyclic subgroups.

2. Splittings of some group extensions

The first application gives a characterization of wreath products with finite active group. If the normal subgroup $A$ is abelian, then the statement of Theorem 2.1 follows from Proposition I.8.3 [7].

Theorem 2.1. Assume that a normal subgroup $A$ of a group $G$ is a direct product $\times_{i=1}^{n}H_i$, where $\{H_i\}_{1\leq i\leq n}$ is the set of all conjugate to $H = H_1$ subgroups of $G$. Also assume that the normalizer $N_G(H)$ is equal to $A$. Then $A$ has a semidirect compliment $B$ in $G$ and $G = H\Wr B$.

Proof. Note that $n$ is the index of the normalizer of $H$ in $G$, therefore the group $B = G/A$ has order $n$.

Define $C = C_1 = \prod_{i\neq 1} H_i$. The normal in $A$ subgroup $C$ is conjugate in $G$ to any $C_j = \prod_{i \neq j} H_i$, as it follows from the assumptions. Therefore $\cap_{j=1}^{n} C_j = 1$, and $C$ does not contain nontrivial normal in $G$ subgroups.

By Proposition 1.1, we have the embedding $\kappa$ of $G$ in the wreath product $(A/C)\Wr B$. If $g \in A$, then by formula (3), $\kappa(g) = \bar{f}_g$, where $\bar{f}_g(x) = x^sg(x^s)^{-1}$. We may further assume that $x^s = 1$ for $x = 1$. Then for $g \in H$, every value $x^sg(x^s)^{-1}$ of $f_g$ belongs to some $H_i \leq C$ if $x \neq 1$ and $f_g(1) = g$. Therefore all the values of $\bar{f}_g$, except for one, are trivial, and $\kappa(H) = H(1)$, where $H(1)$ is the subgroup of functions $\bar{f}$ supported by $\{1\}$ only. Since $H(1) \simeq A/C \simeq H_1 = H$, the subgroup $H(1)$ can be identified with $H$.

The conjugacy class of the subgroup $H(1)$ in the wreath product $H\Wr B$ consists of $n$ subgroups $H(b) = b^{-1}H(1)b$, where $b \in B$. Therefore the set $\{H(b)\}_{b \in B}$ is the $\kappa$-image of the set $\{H_i\}_{i=1}^{n}$. Since $\bar{F} = \times_{b \in B} H(b)$, we have $\kappa(A) = \bar{F}$ and, by Proposition 1.1, $A\Wr B = \kappa(G)\bar{F} = \kappa(G)$. Hence $\kappa$ is an isomorphism, and the theorem is proved.

The following example shows that one cannot remove the finiteness of the quotient $B = G/A$ from the assumption of Theorem 2.1.

Example. Let $G$ be a free metabelian group with two generators $a$ and $b$. The commutator subgroup $A = [G, G]$ is the normal closure of
one commutator \([a, b]\). Since the subgroup \(A\) is abelian, every subgroup conjugate to \(H = \langle [a, b] \rangle\) is of the form \(H_{ij} = c^{-1}_{ij} [a, b] c_{ij}\), where \(c_{ij} = a^i b^j, i, j \in \mathbb{Z}\). We explain below the known fact that \(A\) is the direct product \(\times_{i,j} H_{i,j}\). It follows that \(N_G(H) = A\). However \(G\) does not split over \(A\) because by A. Shmelkin’s [12] result (also see [10], Theorem 42.56) two independent modulo \(A\) elements must generate free metabelian subgroup of \(G\).

To check that the elements \(d_{ij} = c^{-1}_{ij} [a, b] c_{ij}\) are linearly independent over \(\mathbb{Z}\), one can apply the homomorphism \(\mu\) of \(G\) (in fact, a version of Magnus’s embedding [8, 8]) into the metabelian group of upper triangle \(2 \times 2\) matrices over the group ring \(\mathbb{Z}(x, y)\) of a free abelian group of rank two given by the rule

\[
\mu(a) = \text{diag}(x, 1), \quad \mu(b) = \text{diag}(y, 1) + E_{12},
\]

where \(E_{12}\) is the matrix with 1 at position \((1, 2)\) and zeros everywhere else. The matrix multiplication shows that \(\mu(d_{ij}) = I + x^{-i} y^{-j} - 1 (1 - x^{-1}) E_{12}\), where the elements \(x^{-i} y^{-j} - 1 (1 - x^{-1})\) of \(\mathbb{Z}(x, y)\) \((i, j \in \mathbb{Z})\) are linearly independent over \(\mathbb{Z}\).

3. Embeddings of metabelian groups

There are finitely generated torsion free metabelian groups which are not embeddable in \(W = \mathbb{Z} \Wr B\) with finitely generated abelian \(B\). For example, the derived subgroup \([G, G]\) of the Baumslag - Solitar group \(G = \langle a, b \mid b^{-1} a b = a^n \rangle\) is isomorphic to the additive group of rationals whose denominators divide some powers of \(n\), denote it by \(D_n\); but for \(|n| \geq 2\), \([W, W]\) has no nontrivial elements divisible by all powers of \(n\). (Moreover, it is easy to construct 2-generated metabelian groups \(G\) with \([G, G]\) containing an infinite direct power of the group \(D_n\) [9].) However the following is true.

**Theorem 3.1.** Let \(G\) be a finitely generated metabelian group with infinite abelianization \(B = G/[G, G]\). Then for some \(n = n(G) \geq 1\),

(a) \(G\) embeds in the wreath product \((D_n \times \mathbb{Z}/n\mathbb{Z}) \Wr B\).

(b) \(G\) is isomorphic to a subgroup of \(W = D_n \Wr B\) if the derived subgroup \([G, G]\) is torsion free;

(c) \(G\) is isomorphic to a subgroup of \(W = (\mathbb{Z}/n\mathbb{Z}) \Wr B\), provided the derived subgroup \([G, G]\) is a torsion group.
The proof is based on the following

**Lemma 3.2.** Let $G$ be a finitely generated metabelian group and $A$ an abelian normal subgroup of $G$. Then there is a subgroup $C$ in $A$ containing no non-trivial normal in $G$ subgroups, such that for some $n \geq 1$,

(a) the factor group $A/C$ is isomorphic to a subgroup of a finite direct power of the group $D_n \times (\mathbb{Z}/n\mathbb{Z})$;

(b) $A/C$ is isomorphic to a subgroup of a finite direct power of $D_n$ if $A$ is torsion free;

(c) $A/C$ is isomorphic to a subgroup of a finite direct power of $\mathbb{Z}/n\mathbb{Z}$ if $A$ is a torsion group.

**Proof.** (b) We denote by $R$ the maximal normal in $G$ subgroup of finite (torsion-free) rank contained in $A$. Since $G$ satisfies the maximum condition for normal subgroups [3], this ‘radical’ $R$ exists and contains all normal in $G$ subgroups of $A$ having finite rank. Moreover, the maximal torsion subgroup $T/R$ of $L = A/R$ is trivial since the subgroup $T$ is normal in $G$ and its rank is equal to the rank of $R$.

By Ph.Hall’s theorem on normal subgroups in metabelian groups (see [4], lemmas 8 and 5.2), there is $n \geq 1$, and a basis $(a_1, a_2, \ldots)$ of a free abelian subgroup $M \leq A$ such that $A/M$ is a torsion group having non-trivial $p$-subgroups for $p | n$ only, and so the group $A$ embeds in a direct product $D_1^n \times D_2^n \times \ldots$ of the copies of $D_n$ (such that $a_i \mapsto D_i^n$).

We will use additive notation for $A$. If $R = A$, i.e. if the basis of $M$ is finite, the statement (b) holds for $C = 0$. So we assume further that $M$ has infinite rank.

Since $R$ has finite rank, it follows from the above embedding of $A$ into the countable direct power of $D_n$ that the group $A$ has a subgroup $K$ such that the intersection $R \cap K$ is trivial and the factor group $A/K$ is isomorphic to a subgroup of a finite direct power of $D_n$.

Let us enumerate non-trivial normal subgroups $N$ of $G$ of infinite rank, which are contained in $A$ and have torsion free factor groups $A/N$: $N_1, N_2, \ldots$. (This set is countable since $G$ satisfies the maximum condition for normal subgroups.) Using this enumeration, we will transform the basis of $M$ as follows.

Let $a_{01} = a_1, a_{02} = a_2, \ldots$, and assume that the basis $(a_{i-1,1}, a_{i-1,2}, \ldots)$ of $M$ is defined for some $i \geq 1$, and $a_{i-1,k} = a_k$ if $k$ is greater than $m(i-1)$, where $m(0) = 0$. 
Since the subgroup $N_i$ has infinite rank, $N_i \cap M$ has a non-zero element $g_i = \sum_{j>m(i-1)} \lambda_j a_j$. Let $m(i)$ be the maximal subscript at non-zero coefficients $\lambda_j$ of this sum. We may assume that the greatest common divisor of the coefficients $\lambda_{m(i-1)+1}, \ldots, \lambda_{m(i)}$ is 1 since the group $A/N_i$ is torsion free. Therefore there exists a basis $\langle a_{i,m(i-1)+1}, \ldots, a_{i,m(i)} \rangle$ of the free abelian subgroup $\langle a_{m(i-1)+1}, \ldots, a_{m(i)} \rangle$ with $a_{i,m(i-1)+1} = g_i$.

The other elements of the $(i-1)$-th basis of $M$ are left unchanged, i.e. $a_{i,k} = a_{i-1,k}$ if $k \leq m(i-1)$ or $k > m(i)$.

Now we define $e_k = a_{i,k}$ if $m(i-1) < k \leq (m(i))$ for some $i$ and obtain a new basis of $(e_1, e_2, \ldots)$ of $M$ because we see from the construction that $\langle e_1, \ldots, e_{m(i)} \rangle = \langle a_1, \ldots, a_{m(i)} \rangle$ for every $m(i) > 0$. Thus every subgroup $N_i$ contains an element $g_i = a_{i,m(i-1)+1} = e_{m(i-1)+1}$ from this new basis.

Since the group $A$ is torsion free, every element of $A$ has a unique finite presentation of the form $\sum_i \lambda_i e_i$ with rational coefficients (although not every such a rational combination belongs to $A$). Hence the subgroup $H$ of $A$ given by the equation $\sum_i \lambda_i = 0$ is well defined, and the factor group $A/H$ is torsion free and has rank 1. Note that $(M+H)/H \simeq M/(H \cap M)$ is infinite cyclic and $A/(M+H)$ is the factor group of the torsion group $A/M$. Therefore the group $A/H$ is isomorphic to a subgroup of $D_n$.

It follows from the definition of $H$ that for every $i$, the subgroup $H$ does not contain any non-zero multiple $mg_i$ of the element $g_i = e_{m(i-1)+1}$.

Finally we set $C = H \cap K$. Then $A/C$ is embeddable in a finite direct power of $D_n$ since both $A/H$ and $A/K$ are. Assume now that $C$ contains a nontrivial normal in $G$ subgroup $L$. Then $L$ has to have infinite rank since $C \cap R \leq K \cap R = 0$. Let $T/L$ be the torsion part of $A/L$. Then $T$ is a normal in $G$ subgroup with torsion free factor group $A/T$. Hence $T = N_i$ for some $i$. Therefore $L$ and $C$ must contain a non-zero multiple of the element $g_i \in N_i$. But $H$ does not contain such elements, and the statement (b) is proved by contradiction.

(c) The subgroup $A$ is a direct sum of its Sylow $p$-subgroups $A_p$. For every prime $p$, the elements $x$ of $A_p$ with $px = 0$ form a normal in $G$ subgroup $A(p)$. It follows from the maximum condition for normal subgroups of $G$ that there are only finitely many primes $p$ with nonzero $A(p)$. For the same reason, $A$ has a finite exponent $n$. Arguing as in the proof of (b), but replacing $A$ by $A(p)$ and taking the maximal set $(a_1, a_2, \ldots)$ in $A(p)$ as there (but linearly independent over $\mathbb{Z}/p\mathbb{Z}$), we obtain a subgroup (and subspace) $C_p$ of finite codimension in $A(p)$, which does not contain any nonzero normal in $G$ subgroup.

Now consider a maximal in $A_p$ subgroup $E_p$ such that $E_p \cap A(p) = C_p$. Then every element $x + E_p$ of order $p$ from $A_p/E_p$ must belong to the
canonical image \((A(p) + E_p)/E_p\) of the subgroup \(A(p)\) in \(A_p/E_p\) since otherwise \((\langle x + E_p \rangle/E_p) \cap ((A(p) + E_p)/E_p)\) is trivial, and so \((x + E_p) \cap A(p) \leq E_p\), contrary to the maximality of \(E_p\). Since the subgroup \((A(p) + E_p)/E_p \cong A(p)/C_p\) is finite, we have finitely many elements of order \(p\) in the \(p\)-group \(A_p/E_p\) of finite exponent dividing \(n\). Hence \(A_p/E_p\) is a finite \(p\)-group.

If \(N\) is a non-trivial \(p\)-subgroup normal in \(G\) and \(N \leq E_p\), then \(N \cap A(p) \leq E_p \cap A(p) = C_p\), where \(N \cap A(p)\) is non-trivial and normal in \(G\), contrary to the choice of \(C_p\). Thus \(E_p\) contains no such subgroups \(N\).

Since \(A_p\) is a direct summand of \(A\), for every \(E_p\), one can find a subgroup \(F_p \leq A\) with \(A_p \cap F_p = E_p\) and \(A/F_p \cong A_p/E_p\). If a normal subgroup \(N\) in \(G\) with nontrivial \(p\)-torsion were contained in \(F_p\), then \(A_p \cap N \leq A_p \cap F_p = E_p\), which would provide a contradiction.

Now the intersection \(C = \cap_{p^n} F_p\) contains no nonzero subgroups of \(A\), which are normal in \(G\). Since every \(A/F_p\) is a finite group of exponent dividing \(n\), the group \(A/C\) is embeddable in a finite direct power of the group \(\mathbb{Z}/n\mathbb{Z}\), as desired.

(a) Let \(T\) be the torsion subgroup of \(A\). By the statement (b) applied to \(G/T\), we have a subgroup \(C'\) containing \(T\) but containing no bigger normal in \(G\) subgroups, and \(A/C'\) is embeddable in a finite direct power of some \(D_n\). Note that \(C'\) contains no nontrivial torsion free subgroup \(N\) normal in \(G\) since \(N + T > T\).

Since \(T\) has a finite exponent \(m\), it has a torsion free direct compliment \(K\) in \(A\) by Kulikov’s theorem [11,14]. The intersection \(S\) of the subgroups conjugated to \(K\) in \(G\) is torsion free, normal in \(G\), and the exponent of \(A/S\) is equal to the exponent of \(A/K \cong T\), i.e. it is \(m\).

The statement (c) applied to \(G/S\) provides us with a subgroup \(C''\) containing \(S\) but containing no bigger normal in \(G\) subgroups, with \(A/C''\) embeddable in a finite direct power of \(\mathbb{Z}/m\mathbb{Z}\). Note that \(C''\) contains no nontrivial torsion subgroup \(N\) normal in \(G\) since \(N + S > S\).

It follows that \(C = C' \cap C''\) contains no nontrivial normal in \(G\) subgroups and \(A/C\) is embeddable in a finite direct power of \(D_n \times \mathbb{Z}/m\mathbb{Z}\). Since both \(m\) and \(n\) can be replaced by their common multiple, the lemma is proved.

\[ \square \]

**Remark 3.3.** One cannot generalize Lemma 3.2 to the slightly larger class of central-by-metabelian groups. Indeed, every countable abelian group \(A\) embeds as a central subgroup in some finitely generated central-by-metabelian group \(G\) [3], and so every subgroup \(C\) of \(A\) becomes normal in \(G\).
Proof of Theorem 3.1. (b) By Lemma 3.2 (b) and Proposition 1.1, for some \( n, m \geq 1 \), the group \( G \) embeds in a wreath product \( W = D \text{ Wr } B \), where \( D \) is a direct product of \( m \) copies of the group \( D_n \). Since \( B \) is infinite, it follows from the Fundamental Theorem of finitely generated abelian groups, that \( B \) contains a subgroup of index \( m \) isomorphic to \( B \). In other words, \( B \) is a subgroup of index \( m \) in a group \( B_0 \) isomorphic to \( B \). The group \( W_0 = D_n \text{ Wr } B_0 = B_0 \ltimes F \), where \( F \) is the subgroup of functions \( B_0 \to D_n \), contains the subgroup \( W_1 = B \ltimes F \), and it remains to show that \( W_1 \) is isomorphic with \( W \). This isomorphism is identical on \( B \) and maps every function \( f_0 : B_0 \to D_n \) to the function \( f : B \to D \) given by the rule \( f(b) = (f(t_1 b), \ldots, f(t_m b)) \in D \), where \( \{t_1, \ldots, t_m\} \) is a transversal to the subgroup \( B \) in \( B_0 \).

(c,a) One should argue as in the proof of (b) but with reference to the items (c) and (a) of Lemma 3.2 and with the group \( D_n \) replaced by \( \mathbb{Z}/n\mathbb{Z} \) and \( D_n \times \mathbb{Z}/n\mathbb{Z} \), respectively.

\[ \square \]

4. Subgroups of \((\mathbb{Z}/p\mathbb{Z}) \text{ Wr } \mathbb{Z}\)

Let us fix a prime number \( p \). For every prime \( q \neq p \), there is a finite-dimensional faithful, irreducible presentation of \( \mathbb{Z}/q\mathbb{Z} \) over the Galois field \( \mathbb{F}_p \). Let \( V_q \) be the corresponding representation module. Each of these representations lifts to a representation of an infinite cyclic group \( \langle b \rangle \), so the direct sum \( V = \bigoplus_{q \neq p} V_q \) is a \( \mathbb{F}_p \langle b \rangle \)-module. The action of \( \langle b \rangle \) defines the semidirect product \( G = \langle b \rangle \ltimes V \).

Lemma 4.1.

(1) Every finitely generated subgroup of \( G \) is finite-by-cyclic, i.e. \( G \) is locally finite-by-cyclic.

(2) \( G \) contains \( 2^{\aleph_0} \) non-isomorphic subgroups.

(3) There is a subgroup \( C \) in \( V \) such that \( V/C \) has order \( p \) and \( C \) contains no nontrivial normal in \( G \) subgroups.

Proof. (1) It is easy to see that any finite subset of \( G \) is contained in a subgroup \( \langle b \rangle \ltimes U \), where \( U \) is the direct sum of the subgroups \( V_{q_i} \) over a finite subset of prime numbers \( q_i \). Since \( U \) is finite, the statement (1) follows.

(2) Denote by \( H_S \) the subgroup \( \langle b \rangle \ltimes V_S \), where \( S \) is a set of prime numbers \( q \neq p \) and \( V_S = \bigoplus_{q \in S} V_q \). Since different \( V_q \)-s are irreducible and non-isomorphic \( \mathbb{F}_p \langle b \rangle \)-modules, every normal in \( H_S \) finite subgroup of \( V_S \)
is $V_T = \bigoplus_{q \in T} V_q$ for a finite subset $T \subset S$. The centralizer of $V_T$ in $H_S$ has index $\prod_{q \in T} q$. It follows that the groups $H_{S_1}$ and $H_{S_2}$ are not isomorphic for $S_1 \neq S_2$, which implies the statement (2).

(3) Let form an $\mathbb{F}_p$-basis $(e_1, e_2, \ldots)$ as the union of the bases of the subspaces $V_q$ and define the subspace $C$ by the equation $\sum_i x_i = 0$ in the coordinates. Then clearly $V/C$ has order $p$ and $C$ contains none of the summands $V_q$. Since every normal in $G$ subgroup of $V$ is a submodule of the direct sum of some non-isomorphic irreducible $\mathbb{F}_p\langle b \rangle$-modules $V_q$, it coincides with some $V_S$, and the statement is proved. □

The next theorem demonstrates that the structure of subgroups of wreath products of cyclic groups is quite rich.

**Theorem 4.2.** The wreath product $\mathbb{Z}_p \wr \mathbb{Z}$ contains $2^{{\aleph}_0}$ non-isomorphic subgroups $H$, where each $H$ is locally finite-by-cyclic.

**Proof.** The group $G$ from Lemma 4.1 is embeddable in $\mathbb{Z}_p \wr \mathbb{Z}$ by the property (3) of Lemma 4.1 and Proposition 1.1. Therefore Theorem 4.2 follows from the properties (1) and (2) of Lemma 4.1. □

**Remark 4.3.** Similar approach shows that the wreath product $\mathbb{Z} \wr \mathbb{Z}$ contains $2^{{\aleph}_0}$ countable locally polycyclic, non-isomorphic subgroups. But now, instead of $V_q$, one should start with $\mathbb{Z}\langle b \rangle$-modules $V_i$, which are non-isomorphic, finite-dimensional and irreducible over $\mathbb{Q}$.

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