THOMAE FORMULA FOR GENERAL CYCLIC COVERS OF $\mathbb{CP}^1$

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Abstract. Let $X$ be a general cyclic cover of $\mathbb{CP}^1$ ramified at $m$ points, $\lambda_1, \lambda_m$, we define a class of non positive divisors on $X$ of degree $g - 1$ supported in the pre images of the branch points on $X$, such that the Riemann theta function doesn’t vanish on their image in $J(X)$. We generalize the results of [BR],[Na] and [EG] and prove that up to a certain determinant of the non standard periods of $X$, the value of the Riemann theta function at these divisors raised to a high enough power is a polynomial in the branch point of the curve $X$. Our approach is based on a refinement of Accola’s results for 3 cyclic sheeted cover [Ac1] and a generalization of Nakayashiki’s approach explained in [Na] for general cyclic covers.

1. Introduction

Let $X$ be an algebraic curve given by the equation:

$$y^N = \prod_{i=1}^{m} (x - \lambda_i)^{R_i},$$

such that $\sum_{i=1}^{m} R_i = 0 \mod N$ and $(R_i, N) = 1$. Let $\phi : X \rightarrow \mathbb{CP}^1$ be a map defined by: $\phi(x, y) = x$. Choose a base point $z_0$, a normalized homology basis $a_1, b_1, ..., a_g, b_g$ and a normalized holomorphic differentials $v_1, ..., v_g$ to define the Jacobian of $J(X)$ and a standard map $u : X \rightarrow J(X)$. Let $K_{z_0}$ be the Riemann constant and $\tau$ is the period matrix associated with the homology basis and the differentials selected above. We prove the following theorem:

Theorem 1.1. Let $r$ be a total ramification of $\phi : X \rightarrow \mathbb{CP}^1$. Select an integer vector $\beta = (\beta_1, ..., \beta_m)$ such that:

1) $0 \leq \beta_i \leq N - 1$

2) for $0 \leq k \leq N - 1$, $\sum_{i=1}^{m} (\beta_i + kR_i) = \frac{r}{2}$

and $j$ denotes the smallest positive integer $j_0$ such that $j \mod N = j_0 \mod N$.

Let $K_{z_0}$ be the Riemann’s constant and let $\theta(z, \tau)$ be the Riemann theta function. Then

$$\theta\left[u\left(\sum_{i=1}^{m} \beta_i P_i\right) + K_{z_0} - u(\sum_{j=1}^{N} \infty_j)\right](0, \tau) \neq 0$$

and there exists a complex number $\alpha$ not depending on $\tau$ such that:

$$\theta\left[u(\beta_i P_i) + K_{z_0} - u(\sum_{i=1}^{N} \infty_i)\right] = \alpha \sqrt{\det C} \times \prod_{i,j=1 \ldots m, i \neq j} (\lambda_i - \lambda_j)^{\beta_{ij} + \gamma_{ij}}$$

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where $\det C$ is a certain determinant of the $g \times g$ matrix of non normalized holomorphic differentials evaluated at $a_i, 1 \leq i \leq g$ and

$$
\gamma_{ij} = \sum_{w=0}^{N-1} \left\{ wR_i/N \right\} \left\{ wR_j/N \right\},
$$

$$
\beta_{ij} = \sum_{k=0}^{N-1} \left\{ (\beta_i + kR_i)/N \right\} \times \left\{ (\beta_j + kR_j)/N \right\}
$$

and $\{ \alpha \}$ is the fractional part of $\alpha$.

the theorem is a generalization of the work started by [BR],[Na] and [EG]. Using methods from String and Quantum field theory Bershadsky and Radul generalized the Thomae formula for hyper-elliptic covers for a non singular covers of the sphere i.e. when $R_i = 1$ and the number of branch points $m$ is a multiple of $N$. [Na] gave a more rigorous proof for the formula suggested by [BR] while [EG] modified Nakayashiki’s method and treated a special singular case. In this note we follow the approach of [Na] to prove the formula we stated above. In the first part of the paper we show that the Riemann theta function doesn’t vanish on the images of the divisors we defined above. In the second part of the paper we modify [Na] to the case when $(R_i, N) = 1$ and $\sum_{i=1}^{m} R_i = 0 \bmod N$. As far as we know it provides Thomae formula for the most wide class of cyclic covers of the Sphere.

The main idea of [Na] is to produce an integrable differential equation that describes the variation of the logarithm of the theta function with respect to the branch points. This is accomplished by constructing certain analytic quantities of the Riemann surfaces locally (as algebraic expressions supported by local coordinates around the branch points) and comparing them to the global expression as derived in [Fa]. Equating the expansions of the global and the local constructions produces the result. We carry this program below. It turns out that the general case of cyclic covers we handle doesn’t differ much from the case considered by [Na] and [BR].

It is interesting to look for Thomae formulas since they should be useful in mathematical physics, representation and number theory. [Na1] applied the formulas he found in a non singular case to investigate solutions of KZ equations. [EG] relied on the formula the Thomae formula they found to solve the Riemann Hilbert problem for special class of cyclic covers. On the other hand these formulas might be of interest in representation theory since Symmetric groups (or products of Symmetric groups) act on polynomials on the RHS of the formulas. Consequently powers of the theta functions realize representations of Symmetric groups (or their products). One can then use the machinery of representation theory to derive a basis for the theta functions and enhance our understanding of their modular properties and perhaps characterize the periods coming from cyclic of covers of $\mathbb{C}P^1$. For an example of this approach, when the cyclic cover is of degree 3 see [Ko]. In number theory Thomae formulas can probably be used to improve existing algorithms for counting points on cyclic covers of the projective line above finite fields. Mestre applied Thomae formula and duplication formulas of the theta functions count points of these curves above finite fields. It is plausible that his approach can be generalized to the cyclic cover case as well using the generalized Thomae formula. Finally one can’t ignore the inherent mystery of the formulas where the LHS is a highly transcendental object while a RHS is a product of differences of points.
The approach presented here isn’t the only one to look for these formulas. In a series of papers Hershel Farkas and his collaborators ([EF],[EfF]) reproved Thomae’s original result and used classical approach of Riemann to write the branch points as of cyclic covers as ratios of theta functions. Consequently they were able to get the \( \beta \) part of the formula in certain cases. In a book currently written with his student Zemel, Farkas [FZ] generalizes his work avoiding the variational approach used in this note. Lastly [KT] derived a similar result for the case \( p = 3 \) by assuming Nakayashiki’s result and analyzing the degenerations of the homology basis when ramification are coming together. Their approach enables them to calculate the constant \( \alpha \) explicitly. In subsequent we plan to analyze their method and attempt to reprove results obtained in this note.

2. Non positive divisors on Riemann surface

Let \( X \) be a Riemann surface and assume that \( D = \sum d_i z_i \) is a divisor (not necessarily positive) on it.

**Definition 2.1.** \( H^0(X,O(D)) \) is the collection of functions \( f : X \mapsto \mathbb{CP}^1 \) on \( X \) such that \( \text{div}(f) \geq D \). In the [FK] notation this is the space \( \mathfrak{K}(D) \).

Let \( r(D) = \dim H^0(X,O(D)) \).

We seek conditions when \( \exists E \) a divisor on \( X \) such that \( E = \sum e_i x_i, e_i \geq 0 \) and \( D \equiv E \). Assume that \( D \) is not a positive divisor (otherwise you can set \( E = D \)). Then if \( E \) is positive and equivalent to \( D \) there exists a non constant function \( f \) such that \( \text{div}(f) = E/D \). Therefore \( f \in H^0(X,O(-D)) \). (That is \( f \) is a function such that \( \text{div}(f) \geq D \)). Conclude that \( r(-D) > 0 \). We showed the following:

**Lemma 2.2.** Let \( D \) be a non positive divisor. Then if there is \( E \) a positive divisor such that \( E \equiv D \) then \( r(-D) > 0 \).

Note that because of Jacobi’s inversion theorem if \( D \) is a divisor such that \( r(-D) > 0 \) there is always a positive divisor \( E \) of degree \( g(X) - 1 \) and \( D \equiv E \).

Now assume that \( \deg E = g(X) - 1 \). Apply Riemann Roch and conclude that:

\[
\text{r}(-D) = i(-D).
\]

Choose a base point \( z_0 \) on \( X \) and let \( u : X \mapsto \text{Jac}(X) \) the standard mapping from \( X \) into its Jacobian. Let \( K_{z_0} \) be the Riemann constant. Then Using Riemann vanishing theorem for theta functions we have the following non vanishing criteria for theta functions:

**Lemma 2.3.** Let \( D, \deg D = g - 1 \) be a non positive divisor such that \( r(-D) = 0 \) then \( \theta(u(D) + K_{z_0}) \neq 0 \).

3. Cyclic covers

Let \( \phi : X \mapsto \mathbb{CP}^1 \) be a cyclic cover of the sphere of order \( N \) prime number ramified above \( m \) points \( \lambda_1...\lambda_m \). Assume that \( \lambda_i \neq \infty \) and let \( P_i \) be the ramification point above \( \lambda_i \). Riemann Hurwitz formula applies \( g(X) = \frac{(n-1)(m-2)}{2} \). It is easy to see that \( X \) satisfies the equation:

\[
y^N = \prod_{i=1}^{m} (x - \lambda_i)^{R_i}.
\]

and \( R_i \in \{1,2,...N - 1\} \). \( X \) has a cyclic group of automorphisms of order \( N \). An explicit generator of this group is: \( T(y,x) = (\omega y, x), \omega^N = 1 \). since \( \lambda_i \neq \infty \) conclude: \( \sum_{i=1}^{m} R_i = 0 \mod N \). For each \( j \in \{1,...,N - 1\} \) define \( t_j \) to be the number of
Theorem 3.4. Jacobian. Let $R_i = j$. Then $\sum_{i=1}^{N-1} t_j = m$. Associate to $X$ a vector $\alpha \in \mathbb{Z}^m$, such that: $\alpha = (1, 1, 2, 2, 3, 3, \ldots, N-1, N-1)$. The index $j$ appears $t_j$ times. Let $\alpha_i$ be the $i - \text{th}$ of $\alpha$. Since $\sum_{i=1}^{m} R_i = 0 \mod N$, $\infty$ has precisely $N$ pre images in $X$. Let these images be: $(\infty_1, \ldots, \infty_N) = \phi^{-1}(\infty)$.

**Definition 3.1.** For $\xi$ is a divisor of degree 1 on the sphere which is not a ramification point define $a\xi = \sum_{i=1}^{N} \phi^{-1}\xi$. Extend the map to a divisor of any degree on the sphere.

On $X$ select a normalized homology basis $a_i, b_j$ and the set of the normalized canonical differentials $\omega_i$. Choosing a base point $z_0 \in X$ define the mapping: $u : \mathbb{C} \rightarrow J(X)$.

**Definition 3.2.** For the base point $z_0$ define the divisor: $g_{N}^{-1} = \sum_{i=0}^{N-1} T(i)(z_0)$.

**Definition 3.3.** Let $G_0 \in \text{Jac}(X)$ be a point such that $NG_0 = u(g_{N}^{-1})$.

Since $NP_i - g_{N}^{-1} = 0$ in the Jacobian conclude that: $u(P_k) = C_k + G_0$ and $NC_k = 0$. Let $\Delta$ be the canonical class and let $K_{z_0}$ be the Riemann constant i.e. $-2K_{z_0} = u(\Delta)$. Then $u(\Delta) = (N-1)\sum_{i=1}^{m} u(P_i) - 2u(g_{N}^{-1})$. Using the definition of $C_k$ rewrite the last expression as: $(N-1)\sum_{i=1}^{m} C_i + 2(g(X) - 1)G_0$. Therefore Riemann’s constant $K_{z_0}$ equals to:

$$K_{z_0} = -\frac{N-1}{2} \sum_{i=1}^{m} C_i - (g(X) - 1)G_0 + E_2$$

and $E_2$ is a point of order 2, i.e. $2E_2 = 0$. Let $E_1 = E_2 + \frac{N-1}{2} \sum_{i=1}^{m} C_i$ then $E_1 = -K_{z_0} - (g(X) - 1)G_0$, if $N$ is an odd number conclude that $2NE_1 = 0$ in $J(X)$. We like to formulate the main theorem which is the adaptation of $[\text{Ac2}]$ p.26. This describes the vanishing order of theta functions at certain points of the Jacobian. Let $r$ be a total ramification of $f$ Note that: $r = m(N-1)$.

**Theorem 3.4.** Let $\beta = (\beta_1, \ldots, \beta_m) \in (0, \ldots, N-1)$ such that $\sum \beta_j - \frac{r}{2} = 0 \mod N$ ($r$ is always even.) Define a sequence of $N$ numbers $\tau_0, \ldots, \tau_{N-1}$ satisfying the equations:

$$\sum_{i=1}^{m} \beta_i = \frac{r}{2} - N\tau_k, 0 \leq k \leq N - 1$$

Then the order of the theta function vanishing on the point $\sum_{i=1}^{m} \beta_i C_i - E_1$ in $\text{Jac}(X)$ is $\sum_{i=0}^{N-1} \text{Max}(0, \tau_i)$.

**Remark 3.5.** Since $\beta = \beta$, notice:

$$\sum_{i=1}^{m} \beta_i = \frac{r}{2} - N\tau_0$$

**Proof:**

The theta function vanishes on the point $\sum_{i=1}^{m} \beta_i C_i - E_1$ if and only if there is a positive divisor of degree $g - 1, \psi$ such that: $\sum_{i=1}^{m} \beta_i C_i - E_1 = u(\psi) + K_{z_0}$. Use the definition of $E_1$ and $C_i$ and the formula: $g(X) - 1 = r/2 - N$ to write the last equality as:

$$u(\psi) = \sum_{i=1}^{m} \beta_i C_i - E_1 - K_{z_0} = \sum_{j=1}^{m} \beta_j u(P_j) + (\tau_0 - 1)g_{N}^{-1}$$
Where $\sum_{i=1}^{m} \beta_i = \frac{2}{3} - \tau_0 N$ by [3.3] Let:

$$D_1 = \sum_{i=1}^{m} \beta_i P_i + (\tau_0 - 1)g_N^{-1},$$

$D_1$ is a divisor (not necessarily positive). Its degree given by the next proposition:

**Proposition 3.6.** $D_1$ has degree $g(X) - 1$.

**Proof:**

$\deg(D_1) = \sum_{i=1}^{m} \beta_i + N \times (\tau_0 - 1)$ By definition of $\tau_0$:

$$\deg(D_1) = r/2 - N \times \tau_0 + N \times \tau_0 - N$$

Recall that: $2g(X) - 2 = r - 2N$, hence $g - 1 = r/2 - N$ and the proposition follows.

since $D_1$ is invariant under $T$, $\forall f \in H^0(X, \mathcal{O}(-D_1))$. $T$ is cyclic and unitary therefore $H^0(X, \mathcal{O}(-D_1))$ has a decomposition: $H^0(X, \mathcal{O}(-D_1)) = \bigoplus L_\lambda$ and $L_\lambda$ is the vector space of $T$ eigenvectors with a character: $\chi: \mathbb{Z}_n \rightarrow \mathbb{C}$. If $N_\chi = \dim L_\chi$ then Riemann Theorem applies that the order of vanishing of the theta function is: $N = \sum N_\chi$. We attempt to find $N_\chi$. $T$ is cyclic hence its characters are of the form $\omega^k$ for some $k$, and $\omega^N = 1$. Now $Ty = \omega y$, and $Ty^k = \omega^k y^k$. Conclude that if $f \in N_\chi$ then $f/y^k$ for some $k$ is a pullback of a function $g$ on $\mathbb{CP}^1$. But $f/y^k \in H^0(X, \mathcal{O}(-D_1 - \text{div}(y^k)))$. Further $f/y^k$ corresponds to the functions that are pullbacks from the functions on the $\mathbb{CP}^1$ in the space: $H^0(X, \mathcal{O}(-D_1 - \text{div}(y^k)))$. Let $V_k$ be the space of $f \in H^0(X, \mathcal{O}(-D_1 - \text{div}(y^k)))$ that are pullbacks from functions on the sphere.

**Lemma 3.7.** There is a divisor $\sigma_0$ with support on $\mathbb{CP}^1$ such that: $H^0(\mathbb{CP}^1, \mathcal{O}(-\sigma_0))$ is isomorphic to $V_k^0$.

**Proof:**

For a ramification point $\lambda_j$ let $\gamma_j = \left[\frac{kR_j + \beta_j}{N}\right] \times N$(i.e. $\gamma_j$ is the maximal number such that $\gamma_j \leq kR_j + \beta_j$ and $\gamma_j = 0 \mod N$), Let $Q_{\lambda_j}$ be the point on $\mathbb{CP}^1$ that corresponds to $\lambda_j$. Define

$$\sigma_0 = \sum_{j=1}^{m} \frac{\gamma_j}{N} Q_{\lambda_j} + (\tau_0 - 1)\phi(z_0) - k \frac{\sum_{j=1}^{m} R_j}{N} \infty.$$

We show that $\sigma_0$ is the desired divisor. Let $h: \mathbb{CP}^1 \mapsto \mathbb{CP}^1$ be a function such that $\text{div}h \geq -\sigma_0$, assume that $h$ is a lift of $h$ to $X$. Then

$$\text{div}h \geq -\gamma_j P_j - (\tau_0 - 1)g_N^{-1} + k \frac{\sum_{j=1}^{m} R_j}{N} \sum_{i=1}^{N} \infty_i.$$ 

but $\gamma_j \leq kR_j + \beta_j$, and from the definition of $V_k^0$ conclude, $h \in V_k^0$. Now assume that $\hat{f}$, a lift of a function $f$ on the sphere and $\hat{f} \in V_k^0$. By definition at a point $P_j$ the $\text{Ord}_{P_j}(\hat{f}) \geq -\beta_j - kR_j$, $\hat{f}$ is a lift of $f$ hence, $\text{Ord}_{P_j}(\hat{f}) = 0 \mod N$ conclude that $\text{Ord}_{P_j}(\hat{f}) \geq \gamma_j$ Or $\text{Ord}_{Q_{\lambda_j}}(\hat{f}) \geq \frac{\gamma_j}{N} \infty$. In other points of its support The order of $\text{div}(-D_1 - y^k)$ is divisible by $N$. Thus: $f \in H^0(\mathbb{CP}^1, \mathcal{O}(-\sigma_0))$. ■
The immediate conclusion from the lemma is that: \( \dim V_k^0 = \dim H^0(\mathbb{CP}^1, O(-\sigma_0)) \).
Let us compute the degree of \( \sigma_0 \). By definition of \( \sigma_0 \) we have:
\[
\deg \sigma_0 = \frac{1}{N} \times \left( \sum_{i=1}^{m} \gamma_i + (\tau_0 - 1)N - \sum_{j=1}^{m} kR_i \right)
\]
But \( \gamma_i = \beta_i + kR_i - \overline{\beta_i + kR_i} \) by the definition of \( \gamma_i \). Substituting this expression into \( \gamma_i \) rewrite the last expression as:
\[
\frac{1}{N} \times \left( \sum_{i=1}^{m} \beta_i + kR_i - \overline{\beta_i + kR_i} + (\tau_0 - 1)N - \sum_{j=1}^{m} kR_i \right)
\]
Cancel \( kR_i \) and apply the definition of \( \tau_0 \) to simplify further:
\[
\frac{1}{N} \times \left( \frac{r}{2} - \sum_{i=1}^{m} (\beta_i + kR_i) \right) - 1.
\]
By definition of \( \tau_i \) this equals to: \( \tau_k - 1 \). Apply Riemann Roch on \( \mathbb{CP}^1 \) to conclude that \( \dim H^0(\mathbb{CP}^1, O(-\sigma_0)) = \text{Max}(\tau_i, 0) \).

The discussion in section 2 produces the following corollary:

**Corollary 3.8.** Choose \( \beta_i \) as in theorem but \( \tau_i = 0 \) then under the conditions of last theorem \( \theta(\beta_i, C_i - E_1) \) is not vanishing.

Recall that \( -E_1 - K_{z_0} = -(g(X) - 1)G_0 \), Or \( -E_1 = K_{z_0} + (g(X) - 1)G_0 \). Rewrite the divisor from corollary as:
\[
\sum_{i=1}^{m} \beta_i (P_i - G_0) + K_{z_0} + (g(X) - 1)G_0 = \sum_{i=1}^{m} \beta_i P_i - \beta_i G_0 + K_{z_0} + (g(X) - 1)G_0 = \sum_{i=1}^{m} \beta_i P_i - \frac{r}{2} G_0 + \left( \frac{r}{2} - N \right) G_0 + K_{z_0}
\]
and the last expression is readily seen to be equal to: \( \sum_{i=1}^{m} \beta_i P_i + K_{z_0} - \sum_{i=1}^{N} \infty_i \).

**Corollary 3.9.** Let \( \beta_i \) be selected such that \( \tau_i = 0, 0 \leq i \leq N - 1 \) then
\[
\theta \left[ u \left( \sum_{i=1}^{m} \beta_i P_i \right) + K_{z_0} - u(\sum_{j=1}^{N} \infty_j) \right] (0, \tau) \neq 0.
\]

**Remark 3.10.** Gabino in [GG] obtained similar results but the theorem stated here seems to be stronger and was independently obtained. See also [EF] for an alternative proof where non positive divisors of degree \( g - 1 \) are replaced with the more traditional special divisors of degree \( g \).

### 4. N=3 Example

We work out the general \( N = 3 \) example following [Ac1]. Let us represent the curve as: \( y^3 = \prod_{i=1}^{s} (x - p_i) \prod_{j=1}^{t} (x - q_j) \). and \( s + 2t = 0 \mod 3 \). Then \( G_0 \) be a point satisfying \( 3G_0 = u(z_1 + z_2 + z_3) \) in the Jacobian. The ramification points lying above \( p_i \) and \( q_j \) will be respectively: \( a_k = A_k + G_0 \) and \( b_k = B_k + G_0 \). Consider the sum \( \sum_{i=1}^{s} \epsilon_i A_i + \sum_{j=1}^{t} \delta_j B_j - E_1 \). In the vector \( \alpha = (1, 1, 2, 2) \) the ones appearing \( s \) times and 2 appearing \( t \) times. Then we have the following conditions on \( \epsilon_i, \delta_j \):
• \( \sum \epsilon_i + \frac{\delta_j}{2} = s + t - 3\tau_0 \)
• \( \frac{1}{2}(\epsilon_i + 1) + \frac{\delta_j + 2}{2} = s + t - 3\tau_1 \)
• \( \frac{1}{2}(\epsilon_i + 2) + \frac{\delta_j + 1}{2} = s + t - 3\tau_2 \)

Now rewrite the period as: \( E_1 - \sum_{S_1} A - 2\sum_{S_2} A - \sum_{T_1} B - \sum_{T_2} 2B \) where \( S_1, S_2 \) are subsets where appearing 1 and 2 in the A part of the sum and \( T_1 \) and \( T_2 \) appearing in the B part of the sum. Accordingly \( |S_1| = s_1 \) and similarly \( |T_i| = t_i \).

Finally \( S_0, T_0 \) be subsets of indices such that \( \epsilon_i = \delta_i = 0 \). Now define \( \mu_0 = s_0 - t_2, \mu_1 = s_1 - t_1 \) and \( \mu_2 = s_2 - t_0 \). Then we can write the condition on \( \tau_i \) as follows:

\[ 3\tau_0 = \mu_0 - \mu_2, 3\tau_1 = \mu_2 - \mu_1, 3\tau_2 = \mu_1 - \mu_0. \]

Thus we obtain: \( \mu_0 = \mu_1 = \mu_2 = \frac{(t-s)}{3} \)
guarantees non vanishing. We showed:

**Theorem 4.1.** Let \( \{S_0, S_1, S_2\}, \{T_0, T_1, T_2\} \) be a partition of \( a_i, 1 \leq i \leq s \) and \( b_j, 1 \leq j \leq t \) respectively. if \( t_2 = s_0 - (t-s)/3, t_1 = s_1 - (t-s)/3, \tau_0 = s_2 - (t-s)/3 \). Then \( \theta \) does not vanish on the following divisor:

\[
(\sum_{S_1} A + 2\sum_{S_2} B + \sum_{T_1} B + 2\sum_{T_2} B) - \infty_1 - \infty_2 - \infty_3 + K_{z_0}.
\]

5. THE NON SINGULAR CASE

As a second example of applying the result assume \( R_i = 1 \). Then \( m = pN \).

Then we can identify the coefficients \( \beta_i \) with a vector \( v \) in the integral lattice: \( Z^m \) such that the \( i \)-th coordinate of \( v \) is \( \beta_i \). Now the vector \( \alpha = (1, 1, 1...1,1) \) and consequently the \( \tau_k \) are defined as:

\[
\sum_{i=1}^{m} \beta_i + k = \frac{r}{2} - N\tau_k.
\]

since \( r = m(N-1) \) we can rewrite the last expression as :

\[
\sum_{i=1}^{m} \beta_i + k = m\frac{N-1}{2} - N\tau_k
\]

Let \( t_i \) be the number of times that \( \beta_i = l, 0 < l < N - 1 \), w.l.o.g we can assume that \( \max(t_1...t_{N-1}) = t_1 \) then:

\[
\sum_{i=1}^{m} \beta_i + k = N\frac{N-1}{2} t_1 + 2(t_2 - t_1) + 3(t_3 - t_1) + ... (N-1)(t_{N-1} - t_1)
\]

Because \( t_i - t_1 \leq 0 \) The condition for non vanishing turns out to be: \( t_i = t_1 \forall i \) and \( t_1 = p \). We obtained the following theorem:

**Theorem 5.1.** Assume that the \( R_i = 1, \forall i \) Then the divisor \( \sum_{i=1}^{N} \beta_i P_i + K_{z_0} - \sum_{i=1}^{N} \infty_i \) is non vanishing if and only if the number of \( \beta_i \) such that \( \beta_i = l, 0 \leq l \leq N - 1 \) is \( k \).

6. PROPERTIES OF DIVISORS

Having established **Corollary 3.9** we show various properties of these divisors useful in the sequel:

**Proposition 6.1.** Let \( D = \sum_{i=1}^{m} \beta_i P_i \) and let \( E = \sum_{i=1}^{m} (\beta_i + k\alpha_i) P_i \). Then \( D \equiv E \)
Proof:
We show that there exists a function on \( X \) such that its divisor is: \( D/E \). Define:

\[
f = \frac{\prod^{m}_{i=1} (x - \lambda_i)^{\beta_i/N + k\alpha_i/N}}{\prod^{m}_{i=1} (x - \lambda_i)^{\beta_i/N + k\alpha_i/N}}
\]

Clearly the divisor of \( f \) is precisely \( D/E \). We verify that this is indeed a function in \( X \). Using the definition of \( y \) we rewrite \( f \) as:

\[
y^{-k} \times \frac{\prod^{m}_{i=1} (x - \lambda_i)^{\beta_i/N + k\alpha_i/N}}{\prod^{m}_{i=1} (x - \lambda_i)^{\beta_i/N + k\alpha_i/N}}
\]

and by definition \( \beta_i + k\alpha_i - \beta_i + k\alpha_i = h_i N, h_i \in \mathbb{Z} \). Hence \( f = y^{-k} \times \prod^{m}_{i=1} (x - \lambda_i)^{h_i} \)

\[\blacksquare\]

Proposition 6.2. Let \( D = u(\sum^{m}_{i=1} \beta_i P_i) + K_{\lambda_0} - u(\sum^{N}_{i=1} \infty_i) \) as in (3.9). Then

\[D = u(\sum^{m}_{i=1} (N - 1 - \beta_i)) + K_{\lambda_0} - u(\sum^{N}_{i=1} \infty_i)\]

Proof:
We can write:

\[-D = -u(\sum^{m}_{i=1} \beta_i P_i) - K_{\lambda_0} + u(\sum^{N}_{i=1} \infty_i) =
\]

\[-u(\sum^{m}_{i=1} \beta_i P_i) - 2K_{\lambda_0} + K_{\lambda_0} + 2 \sum^{N}_{i=1} \infty_i - \sum^{N}_{j=1} \infty_i\]

Because of the definition of \( K_{\lambda_0} \) we obtain that \(-2K_{\lambda_0} = u\left(\sum^{m}_{i=1} (N - 1) P_i - 2 \sum^{m}_{j=1} \infty_i\right)\).

Hence:

\[-\sum^{m}_{i=1} \beta_i P_i - 2K_{\lambda_0} + K_{\lambda_0} + 2u(\sum^{N}_{i=1} \infty_i) - u(\sum^{N}_{i=1} \infty_i) =
\]

\[\sum^{m}_{i=1} (N - 1 - \beta_i) P_i + K_{\lambda_0} - \sum^{N}_{j=1} \infty_i\]

\[\blacksquare\]

Lemma 6.3. For \( 0 \leq k \leq N - 1 \) Let

\[f_k(z) = \prod^{m}_{i=1} (z - \lambda_i)^{\frac{\beta_i + k\alpha_i - (N - 1)}{N}} \sqrt{dz}\]

Then \( f_k(z) \) is a meromorphic whose divisor is equivalent to: \( \sum^{m}_{i=1} \beta_i P_i - \sum^{N}_{i=1} \infty_i \)

Proof:
The order of \( z - \lambda_i \) at \( P_i \) is \( N \). Hence the order of the of \( (z - \lambda_i)^{\frac{\beta_i + k\alpha_i - (N - 1)}{N}} \sqrt{dz} \)

is exactly \( \beta_i + k\alpha_i \). at \( \infty_i \) the order is: \( \sum^{m}_{j=1} \beta_j + k\alpha_j - \frac{N}{2} - 1 = -1 \) and hence the divisor is:

\[\sum^{m}_{j=1} \beta_j + k\alpha_j P_j - \sum^{N}_{i=1} \infty_i\]

which is equivalent to: \( \sum^{m}_{j=1} \beta_j P_j - \sum^{N}_{i=1} \infty_i \). \[\blacksquare\]
7. Algebraic construction of the Szego Kernel

Let us recall the definition of the Szego Kernel.

**Definition 7.1.** For $e \in \mathbb{C}^n$, if $\theta[e] \neq 0$ define the Szego kernel by the following equation:

$$S(P, Q|e) = \frac{\theta[e](P - Q)}{\theta[e](1, \tau)E(P, Q)}, P, Q \in C.$$ 

$E(P, Q)$ is the prime form. $e$ depends only on its in the Jacobian, $J(X)$. $R(P, Q|e)$ has the following properties that are well known [F] (p.19,p.123), [EG2] (Proof of Theorem 4.7):

- $S[e](P, Q)$ is a $(\frac{1}{2}, \frac{1}{2})$ form with a simple pole along the diagonal
- $S[e](P, Q)$ has divisor $[e - K_{\omega}]$ with respect to variable $Q$
- $S[e](P, Q)$ has a divisor $[-e - K_{\omega}]$ with respect to variable $P$.
- $S[e](P, Q)$ is a unique up to a constant $(\frac{1}{2}, \frac{1}{2})$ form that satisfies the previous properties

We generalize the approach of Nakayashiki to give the following expression to the Szego kernel:

**Theorem 7.2.** Let $P = (x_1, y_1), Q = (x_2, y_2) \in C$. Choose $\mathbf{\beta} = (\beta_1, ..., \beta_m) \in \mathbb{Z}^m$ be as in 3.7 if $e = \sum_{i=1}^{m} \alpha_i \beta_i + K_{\omega} - u(\sum_{i=1}^{N} \lambda_i)$ Let

$$F_{\beta}(P, Q) = \frac{1}{N} \sum_{k=0}^{N-1} \prod_{i=1}^{m} (x_1 - \lambda_i)^{\beta_k + \kappa_i - \frac{N-1}{n}} \times \prod_{i=1}^{m} (x_2 - \lambda_i)^{\beta_k + \kappa_i - \frac{N-1}{n}} \sqrt{dx_1 \sqrt{dx_2}}$$

Then

$$(2) \quad S[e](P, Q) = F_{\beta}(P, Q)$$

**Proof:**

We verify that $F_{\beta}(P, Q)$ satisfies the properties characterizing $S[e](P, Q).$ the RHS of the equation (1). $F_{\beta}(P, Q)$ is regular outside $P = Q$. If $P \neq Q$ we need to check the case when $x_1 = x_2$ and $y_1 \neq y_2$. This means that $y_2 = \omega^ky_1$ and $\omega^N = 1$. Now, $\beta_i + \kappa_i = \beta_i + \kappa_i - h_{ki}N$. Rewrite $F_{\beta}(P, Q)$ as:

$$\frac{1}{N} \sum_{k=0}^{N-1} \prod_{i=1}^{m} \left( \frac{y_1}{y_2} \right)^k \times \left( \frac{x_1 - \lambda_i}{x_2 - \lambda_i} \right)^{\beta_k + \kappa_i - \frac{N-1}{n}} \sqrt{dx_1 \sqrt{dx_2}}$$

if $y_2 = \omega^ky_1$. In the limit when $x_1 \rightarrow x_2$, $y_2 \rightarrow \omega^k y_1$ and $\sum_{k=0}^{N-1} \left( \frac{y_1}{y_2} \right)^k \rightarrow 0$. Therefore $F_{\beta}(P, Q)$ is regular when $x_1 = x_2$ but $y_1 \neq y_2$.

Let us calculate the expansion of

$$\frac{1}{N} \sum_{k=0}^{N-1} \prod_{i=1}^{m} \left( x_1 - \lambda_i \right)^{\beta_k + \kappa_i - \frac{N-1}{n}} \times \prod_{i=1}^{m} \left( x_2 - \lambda_i \right)^{-\beta_k - \kappa_i + \frac{N-1}{n}}$$

as a function of $x_3$ when the expansion is around $x_1$. Assuming $x_1 = x_2$ we get that the leading coefficient is $\frac{1}{N} \sum_{i=0}^{N-1} 1 = 1$ Now for the coefficient in $x_2 - x_1$ we
obtain using the derivative product rule that the coefficient is:

\[
\frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=1}^{m} \frac{-\beta_j - k\alpha_j/N + (N-1)/2N}{x_2 - \lambda_j} = \frac{1}{N} \sum_{j=1}^{m} \sum_{i=0}^{N-1} \frac{-\beta_j - k\alpha_j/N + \frac{r}{2}}{x_2 - \lambda_j}
\]

But \( \sum_{j=1}^{m} \frac{-\beta_j - k\alpha_j/N + \frac{r}{2}}{x_2 - \lambda_j} = 0 \) and hence

\[
\frac{1}{N} \sum_{j=1}^{m} \sum_{i=0}^{N-1} \left( -\beta_j - k\alpha_j/N \right) + \frac{r}{2} \times \frac{1}{x_2 - \lambda_j} = 0
\]

as well. Taking the second derivative according to \( x_1 \), to calculate the coefficient of \( (x_1 - x_2) \) we arrive to the following result:

**Proposition 7.3.** The expansion of \( F_{\beta} (P, Q) \) around \( P \) a non branch point is:

\[
F_{\beta} (P, Q) = \sqrt{dx_1} \sqrt{dx_2} \left[ 1 + \frac{1}{2N} \sum_{i,j=1}^{i,j=m} \frac{q(\beta_i, \beta_j)}{(x_2 - \lambda_i)(x_2 - \lambda_j)} \times (x_1 - x_2)^2 + \ldots \right]
\]

where

\[
q(\beta_i, \beta_j) = \sum_{k=0}^{N-1} \left\{ (\beta_i + kR_i)/N \times (\beta_j + kR_j)/N \right\}
\]

Where \( x_1, x_2 \) are the local coordinate around \( P, Q \) respectively.

To complete the proof of theorem (7.2) note that \( L(P, Q) = F_{\beta}(P, Q) - S[\varepsilon](P, Q) \) are a section of a line bundle \( L[e^{-K_{\varepsilon}}] \otimes L[e^{-K_{\varepsilon}}] \). Because of the expansion of \( F_{\beta}(P, Q) \) conclude that \( L(P, Q) \) is a holomorphic section of the line bundle. But

\[
H^0 \left( L[e^{-K_{\varepsilon}}] \otimes L[e^{-K_{\varepsilon}}] \right) = H^0 \left( L[e^{-K_{\varepsilon}}] \right) \otimes H^0 \left( L[e^{-K_{\varepsilon}}] \right) = 0
\]

and thus \( F_{\beta}(P, Q) = S[\varepsilon](P, Q) \) as required. \( \blacksquare \)

**Remark 7.4.** The above argument is exactly the method adopted in [Na] to show this. See [EG] for a slightly different approach.

Based on the the formula given at the beginning of the section [Na] shows the following expansion for \( S[\varepsilon](P, Q) \) in terms of theta functions:

**Corollary 7.5.** The expansion of the Szego kernel can be given in terms of theta functions as follows:

\[
S[\varepsilon](P, Q) = \frac{\sqrt{dx_1} \sqrt{dx_2}}{x_1 - x_2} \times \left[ 1 + \sum_{i=1}^{g} \frac{\partial \log \theta[\varepsilon]}{\partial z_i}(0) u_i(x_1)(x_1 - x_2) + \ldots \right]
\]

\( u_i(x) \) is the coefficient of \( dx_1 \) in the expansion of the holomorphic \( v_i(x) \).

Comparing the expansions conclude the following result:

**Corollary 7.6.**

\[
\frac{\partial \theta[\varepsilon]}{\partial z_i}(0) = 0
\]

The following is obtained by multiplying the expansions:
Lemma 7.7.

\[ S[e](P, Q)S[-e](P, Q) = \frac{dx_1 dx_2}{(x_1 - x_2)} \left[ 1 + \frac{1}{N} \sum_{i,j=1}^{m} \frac{q(\beta_i, \beta_j)}{(x_2 - \lambda_i)(x_2 - \lambda_j)} (x_1 - x_2)^2 + \ldots \right] \]

8. Algebraic Construction for the Canonical Differential

We construct the canonical differential algebraically for cyclic curves.

Definition 8.1. The canonical symmetric differential is a \( \omega(x, y) \) is a meromorphic one differential with respect to \( x, y \in C \), having a unique pole of second order when \( z \) tends to \( w \) with a leading expansion coefficient of 1. Further for a canonical homology basis \( a_i, b_j, 1 \leq i, j \leq g \) we have:

\[ \int_{a_i} \omega(x, y) = 0 \]

for fixed \( y \).

First we remind the reader of a possible basis for the holomorphic differentials on \( C \).

Lemma 8.2. Let \( s_l(z) = \prod_{i=1}^{m} (x - \lambda_i)^{\frac{lR_i}{N}} \), \( 0 \leq l \leq N - 1 \) Then a basis for holomorphic differentials is given by:

\[ \frac{z^{j-1} dz}{s_l(z)} \]

where \( j = 1 \ldots d(l), d(l) = \text{Max} \left( \sum_{i=1}^{m} \frac{lR_i}{N} - 1, 0 \right) \).

Proof:

The order of \( z - \lambda_i \) at \( \lambda_i \) is \( N \). Hence the order of \( (z - \lambda_i)^{\frac{lR_i}{N}} \) is \( \frac{lR_i}{N} < N - 1 \). Hence we have non trivial 0 at \( \lambda_i \). For \( \infty_i, i = 1 \ldots n \). The order of \( y_i(z) \) is \( \sum_{i=1}^{N} \frac{lR_i}{N} \). Thus if \( j < \sum_{i=1}^{N} \frac{lR_i}{N} \) we will not have a pole at \( \infty_i \).

Let,

\[ P_l^{(l)}(z, w) = \sum_{n=0}^{d(l)+1} A_n^{(l)}(w)(z-w)^n \]

such that:

1. \( A_0^{(l)}(w) = \prod_{i=1}^{m} (w - \lambda_i) \)
2. \( A_1^{(l)} = \sum_{i=1}^{m} \frac{lR_i}{N} \times A_n^{(l)}(w) \)
3. \( \deg_w P_l^{(l)} \leq d(N - l) + 1 \)

Set:

\[ \xi_0(x, y) = \frac{dz(x)dz(y)}{(z(x) - z(y))^2} \]

\[ \xi_l(x, y) = \frac{P_l^{(l)}(z(x), z(y))dz(x)dz(y)}{s_l(x)s_{N-l}(y)(z(x) - z(y))^2} \]

\[ \xi(x, y) = \frac{1}{N} \sum_{i=0}^{N-1} \xi_i(x, y) \]
Proposition 8.3. 

1. \( \xi(x, y) \) is holomorphic outside the diagonal set \( \{ x = y \} \).
2. For a non branch point \( P \in X \) take \( z \) to be a local coordinate around \( P \). Then the expansion in \( z(x) \) at \( z(y) \) is:

\[
\xi(x, y) = \frac{dz(x)dz(y)}{(z(x) - z(y))^2} + O((z(x) - z(y))^0)
\]

Proof:
To show the proposition we need first need to show that if \( P \neq Q \) on \( X \), then \( \xi(P, Q) \) is still non singular. Assume that \( P = (p_1, q_1) \) and \( Q = (p_1, \omega^q q_1) \). Let us examine the leading term of the expansion of \( \xi_l(x, y) \) around \( Q \).

By definition of \( \xi_l(x, y) \) it is:

\[
\begin{align*}
A^{(l)}_0(z(Q)) & = \omega^{rt} r_1 \prod_{i=1}^{m} (z - \lambda_i)^{\frac{1}{(Nq)_1}} (z(P) - z(Q))^{-2} \\
& = \omega^{rt} (z(P) - z(Q))^{-2},
\end{align*}
\]

( by definition of \( A^{(l)}_0 \) ) Then if \( r = 0 \) (i.e. \( P \to Q \) the leading coefficient of \( \xi_l(x, y) \) is:

\[
\frac{1}{(z(P) - z(Q))}
\]

if \( r > 0 \), summing we obtain that the coefficient is 0. Next we compute the coefficient of \( \frac{1}{z(P) - z(Q)} \) in the expansion of \( \xi_l(x, y) \), \( l > 0 \). Apply the product rule for derivatives to obtain that the coefficient of \( \frac{1}{z(P) - z(Q)} \) is expanding around \( Q \) is:

\[
\sum_{i=1}^{m} (lR_i/N) A^{(l)}_0(z(Q)) s_l(Q) - \sum_{i=1}^{m} (lR_i/N) s_l(z(Q)) z - \lambda_i A^{(l)}_0(z(Q)) = 0
\]

therefore the coefficient of \( \frac{1}{z(P) - z(Q)} \) is 0 and the proposition is proved.

As an immediate corollary of the proposition we have that:

Corollary 8.4.

\( \omega(x, y) - \xi(x, y) \)

is holomorphic on \( X \times X \).

Thus by the corollary there exist polynomials \( P^{(l)}_{k} \) such that:

\[
\omega(x, y) - \xi(x, y) = \sum_{l=1}^{N-1} \sum_{k=1, k \neq l}^{N-1} \frac{P^{(l)}_k(z(x), z(y)) dz(x)dz(y)}{s_k(z(x))s_{N-1}(z(y))}
\]

Where by modifying the definition of \( P^{(l)}_k \) we can exclude the terms \( k = l \). as before we can write

\[
P^{(l)}_k(z, w) = \sum_{j=0}^{d(k)} A^{(l)}_{kj}(w)(z - w)^j.
\]

Note that \( \text{deg}_w P^{(l)}_k(z, w) \leq d(N - l) \). Our aim is to show the following proposition:

Proposition 8.5.

\[
\sum_{l=1}^{N-1} A^{(l)}_{12} (\lambda_i) = - \prod_{j=1, i \neq j}^{m} (\lambda_i - \lambda_j) \frac{\partial}{\partial \lambda_i} \log det C
\]
and \(C\) is a \(g(X) \times g(X)\) period matrix of non normalized form:

\[
\left(\int_{a_i} z^{j-1}dz/s_l(z)\right).
\]

**Proof:**

Let us take a local \(t = (z - \lambda_i)^{+}\) coordinate around \(Q_1\). Then we have that the condition that \(\int \omega(x,y) = 0\) is equivalent to the coefficient of the expansion around \(Q_i\) in \(dt,tdt,...t^{N-2}dt\) is vanishing. A short calculation shows that this is equivalent to

\[
\omega^{(l)}(x) = \frac{1}{N} P_{l1}^{(l)}(z(x),\lambda_i)dz(x) + \sum_{k=1,k\neq l}^{N-1} \frac{P_{k}^{(l)}(z(x),\lambda_i)dz(x)}{s_k(x)}
\]

vanishing when we integrate around \(a_j\). Let us write this explicitly: Note,

\[
\frac{\partial}{\partial \lambda_i} \frac{dz}{s_l(z - \lambda_i)} = \frac{dz}{s_l(z - \lambda_i)}
\]

and

\[
P_{l1}^{(l)}(z,\lambda_i) = \{LR_i/N\} \prod_{j=1}^m (\lambda_i - \lambda_j)(z - \lambda_i) + \sum_{j=0}^{d(l)-2} A_{l,j+2}^{(l)}(z - \lambda_i)^{j+2}.
\]

hence:

\[
\prod_{i=1}^m \frac{(\lambda_i - \lambda_j)}{N} \frac{\partial}{\partial \lambda_i} \int_{a_h} \frac{dz}{s_l} + \frac{1}{N} \sum_{j=0}^{d(l)-1} A_{l,j+2}^{(l)}(\lambda_i) \int_{a_h} (z - \lambda_i)^j dz/s_l
\]

\[
+ \sum_{k=1,k\neq l}^{d(l)-1} \sum_{j=0}^{d(k)-1} A_{k,j}^{(l)}(\lambda_i) \int_{a_h} (z - \lambda_i)^j dz/s_k = 0
\]

Following [BR],[Na](see also [EG] for a slightly different approach.) For a fixed \(l\) regard the equations above as \(g(X)\) equations in \(g(X)\) variables, \(A_{k,r}^{(l)}\). The matrix of these equations is the \(g(X) \times g(X)\) matrix 

\[
B = \int_{a_h} (z - \lambda_i)^j dz/s_l(z), l = 1...N - 1, j = 1...d(l).
\]

For each \(l\) define matrices \(B_l\) obtain from \(B\) by replacing the column \(\int_{a_h} \frac{dz}{s_l(x)}, 1 \leq g(X)\) with the column: \(\frac{\partial}{\partial \lambda_i} \int_{a_h} \frac{dz}{s_l(z)}\). Then by Cramer’s rule:

\[
A_{l2}^{(l)} = \frac{\det B_l}{\det B}.
\]

Expand \((z - \lambda_i)^k, k \geq 0\) using the binomial formula and perform elementary operations on columns to obtain that \(\det B = \det C\). We now prove a similar proposition for the matrix \(\det B_l\).

**Proposition 8.6.**

\[
\det B_l = \sum_{i=0}^{d(l)} \det C_i
\]

where \(C_i\) is the matrix \(C\) where the \(i\) - th column is replaced with \(\frac{\partial}{\partial \lambda_i} \frac{z^i dz}{s_l(z)}\).
Proof: Assume inductively that the proposition is true for any minor of $B_l$ that contains the first $v$ columns. The polynomials $1, (z - \lambda_i), \ldots, (z - \lambda_i)^{m-1}$ are basis for the polynomials $p(z), deg_x p(z) \leq h - 1$. Write:

$$(z - \lambda_i)^h = (z - \lambda_i)^h - z^h + z^h - \lambda_i^h = \sum_{i=0}^{v=h-1} c_v(z - \lambda_i)^v + z^h - \lambda_i^h$$

Thus the det $B_l$ is equal to the determinant of a matrix where the last column is replaced by $\int_{a_i} \left( z^{d(l)} - \lambda_i^{d(l)} \right) \times \frac{dz}{s_i}$. Hence $\det B_l = \det D_l - \det E_l$ where the last columns of $D_l, E_l$ are replaced by: $z^{d(l)}$ and $\lambda_i^{d(l)}$ respectively. By induction assumption we see that $\det D_l = \sum_{v=1}^{d(l)-1} \det C_v$. So to finish the proof we need to show that $\det E_l = - \det C_{di}$. On the matrix $E_l$ perform the elementary operations by swapping the first and last column. dividing the first column by $\lambda_i^h$ and multiplying the last column by $\lambda_i^h$ respectively results in a matrix $E'_l$ where the first $d(l) - 1$ columns are $\int_{a_i} (z - \lambda_i)^v dz / s_1(z)$ and the last column is $\lambda_i^{d(l)} \frac{dz}{s_1(z)}$. Perform elementary operations on the first $d(l) - 1$ to replace the columns with $\int_{a_i} z^d dz$. Now we can write:

$$\frac{z^j}{z - \lambda_i} = \sum_{h=1}^{j-1} \lambda_i^{h-1} z^{j-h} + \lambda_i^j \frac{z-j}{z - \lambda_i}.$$

But:

$$\frac{\partial}{\partial \lambda_i} \frac{z^j dz}{s_i} \left( \int_{a_k} \frac{dz}{s_i} \right) = \left\{ I R_e / N \right\} \frac{z^j dz}{s_i (z - \lambda_i)}$$

Hence:

$$\frac{\partial}{\partial \lambda_i} \int_{a_k} \frac{z^j dz}{s_i} = \lambda_i^j \times \frac{\partial}{\partial \lambda_i} \int_{a_k} \frac{dz}{s_i} + \left\{ I R_e / N \right\} \sum_{h=1}^{j-1} \lambda_i^{h-1} \int_{a_k} z^{j-h} dz / s_i$$

Set $j = d(l)$ and using the fact (yet again!) that elementary operations don’t alter the determinant to conclude the result. To finish the proof of 8.5 observe that

$$\frac{\sum_{b=1}^{N-1} \sum_{j=0}^{d(h)} \det C_i}{\det C} = \frac{\partial}{\partial \lambda_i} \log(\det C).$$

Let us define the following object we will work closely when showing Thomae:

Definition 8.7. Let $P = (x, y) \in X$ be a non branch point with a local coordinate $z$. Define:

$$G_z(z) = \lim_{y \to z} \left[ \omega(x, y) - \frac{dz(x)dz(y)}{(z(y) - z(x))} \right]$$

Now taking the local coordinate $t = (z - \lambda_i)\frac{dz}{dz}$ around the branch point $\lambda_i$ we have the following corollary:

Corollary 8.8. The coefficient of $t^{N-2} dt^2$ in the expansion of $G_z(z)$ in $t = (z - \lambda_i)\frac{dz}{dz}$ is:

$$-N \sum_{j=1, j \neq i}^{m} \frac{\gamma_{ij}}{\lambda_i - \lambda_j} - N \log(\det C),$$

where $\gamma_{ij}$ are coefficients of $G_z(z)$.
where
\[
\gamma_{ij} = \sum_{h=0}^{N-1} \{hR_i/N\} \{hR_j/N\}
\]

To proceed further we learned the following from [Na] see ([F] Corollary 2.12) that if \( c \) belongs to the Jacobian such that \( \theta[c](0, \tau) \neq 0 \) then:
\[
S[c](x, y) = \omega(x, y) + \sum_{i,j=1}^{g} \frac{\partial^2 \log \theta[e](0)}{\partial z_i \partial z_j} v_i(x)v_j(y),
\]

\( v_i(x), v_j(x) \) are holomorphic differentials on the surface. Using this we obtain the following expression for \( G_z(z) \)

**Proposition 8.9.**
\[
G_z(z) = \frac{1}{N} \sum_{i,j=1}^{m} \frac{q(\beta_i, \beta_j)}{(z(x) - \lambda_i)(z(x) - \lambda_j)} - \sum_{i,j=1}^{g} \frac{\partial^2 \log \theta[e](0)}{\partial z_i \partial z_j} v_i(x)v_j(y),
\]

Passing to the local coordinate \( t = (z - \lambda_i)^{\frac{1}{n}} \) we obtain the following corollary:

**Corollary 8.10.** The coefficient of \( t^{N-2}dt^2 \) in the Laurent expansion of \( G_z(z) \) is:
\[
2N \sum_{j=1}^{m} \frac{q(\beta_i, \beta_j)}{(\lambda_i - \lambda_j)} - \frac{1}{(N-2)!} \sum_{r,s=1}^{N} \frac{(N-2)!}{N-\alpha} \sum_{i,j=1}^{g} \frac{\partial^2 \log \theta[e](0)}{\partial z_i \partial z_j} v_r^\alpha(P_i)v_s^{N-2-\alpha}(P_i),
\]

\( v_r^\alpha(P_i) \) is the coefficient of \( dt \) in the expansion of \( v_r(x) \) in the local coordinate \( t \).

9. Variational formula for the period matrix and Thomae for general cyclic covers

[Na] shows the following formula that can be generalized to any cyclic cover:

**Theorem 9.1.** If \( \tau \) is a period matrix with respect to the fixed homology basis \( a_i, b_j, 1 \leq i, j \leq g \) then
\[
\frac{d\tau_{jk}(0)}{dt} = \frac{1}{N(N-2)!} \sum_{\alpha=0}^{N-2} \left( \begin{array}{c} N-2 \\ \alpha \end{array} \right) v_j^\alpha(Q_i)v_k^{N-2-\alpha}(Q_i)
\]

Now let us show Thomae formula. As in [Na] we write the logarithmic derivative of the theta function on the divisor: \( e_\beta = u(\beta_iP_i) + K_{x_0} - u(\sum_{i=1}^{N} \infty_i) \)
\[
\frac{\partial \log \theta[e_\beta]}{\partial \lambda_i}(0, \tau) = \frac{d}{dt} \log \theta_i \left( e_\beta \right)(0)|_{t=0}(0, \tau)
\]

By the chain rule the last expression is:
\[
\frac{d}{dt} \log \theta_i \left( e_\beta \right)(0)|_{t=0}(0, \tau) = \sum_{1 \leq k, r \leq g} \frac{\partial \log \theta[e_\beta]}{\partial \tau_{kr}}(0) \frac{d\tau_{kr}}{dt}
\]

Now use the heat equation to rewrite the last expression as:
\[
\frac{1}{2} \sum_{1 \leq k, r \leq g} \frac{1}{\theta \left( e_\beta \right)(0, \tau)} \frac{\partial \theta[e_\beta]}{\partial z_k \partial z_r}(0) \frac{d\tau_{kr}}{dt}
\]
We showed in (7.6)\[
\frac{\partial \theta}{\partial z_i}(0) = 0
\]
The last sum equals:
\[
\frac{1}{2} \sum_{1 \leq k, r \leq g} \partial \log \theta \left[ \frac{e^\beta}{\partial z_i} \right](0) d\tau_{kr}.
\]
Use the theorem 9.1 and corollaries 8.10 8.8 to conclude that:
\[
\frac{\partial \log \theta \left[ e^\beta \right]}{\partial \lambda_i}(0, \tau) = \frac{1}{2} \frac{\partial}{\partial \lambda_i} \log \det C + \sum_{j=1, j \neq i}^m \frac{q(\beta_i, \beta_j)}{\lambda_i - \lambda_j} + \frac{1}{2} \sum_{j=1, j \neq i}^m \frac{\gamma_{ij}}{\lambda_i - \lambda_j}
\]
Integrate the system of first order differential equations to get the following theorem:

**Theorem 9.2.** Let $\beta_i$ be integer numbers and $0 \leq \beta_i \leq N - 1$. such that
\[
\sum_{i=0}^m \beta_i + kR_i = \frac{r}{2}.
\]
Then there is a complex number $\alpha$ such that:
\[
\theta \left[ u(\beta_i P_i) + K_{z_0} - u(\sum_{i=1}^N \infty_i) \right] = \alpha \sqrt{\det C} \times \prod_{i,j=1, i \neq j}^{N-1} (\lambda_i - \lambda_j)^{\beta_{ij} + 2\gamma_{ij}}
\]
where $\beta_{ij} = \sum_{k=0}^{N-1} \{\beta_i + kR_i\} \{\beta_j + kR_j\}$ and $\gamma_{ij} = \sum_{w=0}^{N-1} \{wR_i\} \{wR_j\}$.

**References**

[Ac1] R.Accola *On Cyclic Trigonal Riemann Surfaces* Transactions of The American Mathematical Society Vol.283 pp.423-449

[Ac2] R.Accola *Riemann Surfaces, Theta Functions, and Abelian Automorphism Groups* Lecture Notes in Mathematics vol.483 Springer Verlag

[BR] M.Bershadsky and A.Radul *Fermionic Fields on $\mathbb{Z}_n$ curves* Communications in Mathematical Physics vol.116 pp 689-700 (1988)

[BR1] M.Bershadsky and A.Radul *Conformal Field Theories with additional $\mathbb{Z}_N$ symmetry* International Journal of Modern Physics A vol.2 pp 165-178 (1987)

[EF] D.Ebin, H.Farkas *Thomae Formula for $\mathbb{Z}_N$ curves* to appear in Journal D’Analyse

[EiF] M.Eizenmann, H.Farkas *An elementary proof of Thomae formula* Online Journal of Combinatorics No.3 (2008), Art.2 14 pages [FZ] H.Farkas,S.Zemel *On Thomae formula and its generalizations* In preparation

[EG] V.Enolskii, T.Grava *Thomae type formulae for singular $\mathbb{Z}_N$ curves* Lett. Math. Phys. 76 (2006), no. 2-3, 187-214

[EG2] V.Enolskii, T.Grava *Singular $\mathbb{Z}_N$ curves and the Riemann Hilbert problem* Int. Math Research Notices No.32(2004) pp. 1619-1683 [Fa] J. Fay *Theta functions on Riemann Surfaces* Lecture Notes in Mathematics Vol. 352, 1973

[FK] H.Farkas, I.Kra *Riemann Surfaces* Graduate Text in Mathematics 1992, Springer Verlag

[FM] H.Farkas, S.Meisel *Thomae formula for cyclic covers* In preparation
[Fr] M. Fried Private communications that took place in March 2009

[GG] Gabino Gonzalez Dies. *Non Special divisors supported on the branch set of p-gonal Riemann Surfaces* London Mathematical Society Lecture Note Series vol.368, 2009

[Ko] Kopeliovich Yaacov. *Theta constant identities at periods of covers of degree 3* International Journal of Number Theory vol.4, (5) pp. 1-9 (2008)

[MT] Matsumoto Keiji, Terasoma Tomohide. *Degenerations of triple covering and Thomae’s formula* Bibliographic code: 2010arXiv1001.4950M

[Na] A. Nakayashiki, *On the Thomae formula for $Z_N$ curves* Publ. Res. Inst. Math Sci. vol.33 (6)(1997) 987-1015

[Na1] A. Nakayashiki, *Trace Construction of a Basis for the Solution Space of $sl_N$ qKZ Equation* Communications in Mathematical Physics, Volume 212, Issue 1, pp. 29-61 (2000).

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