Stochastic user equilibrium model with a tradable credit scheme and application in maximizing network reserve capacity

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ABSTRACT
The tradable credit scheme (TCS) outperforms congestion pricing in terms of social equity and revenue neutrality, apart from the same perfect performance on congestion mitigation. This article investigates the effectiveness and efficiency of TCS on enhancing transportation network capacity in a stochastic user equilibrium (SUE) modelling framework. First, the SUE and credit market equilibrium conditions are presented; then an equivalent general SUE model with TCS is established by virtue of two constructed functions, which can be further simplified under a specific probability distribution. To enhance the network capacity by utilizing TCS, a bi-level mathematical programming model is established for the optimal TCS design problem, with the upper level optimization objective maximizing network reserve capacity and lower level being the proposed SUE model. The heuristic sensitivity analysis-based algorithm is developed to solve the bi-level model. Three numerical examples are provided to illustrate the improvement effect of TCS on the network in different scenarios.

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1. Introduction
Traffic congestion has become one of the most severe social problems in almost all large cities worldwide. For years, the main solution to mitigate traffic congestion has been adding new capacity to the transportation network. However, such an approach is greatly restricted by the limited urban spatial and financial constraints, and many empirical lessons have also proved that this approach is usually self-defeating (Goodwin 1996). Since Pigou (1920) found that congestion pricing (CP) can completely eliminate the queue delay of bottleneck sections, CP has been extensively investigated by transportation researchers. The essential idea of CP is to charge travellers the marginal external costs that their trips impose on society to reduce traffic congestion or increase social welfare. Although CP is perfect in theory, it has not yet been well accepted in practice because of the intractable issues of social equity and taxation neutrality.

Given the general political resistance to CP, some transportation researchers and planners have turned to the tradable credit scheme (TCS). Verhoef, Nijkamp, and Rietveld (1997) explored the feasibility of tradable road-pricing smart cards to regulate road transportation externalities and their potential application in transportation on both the demand and supply sides. Viegas (2001) suggested allocating ‘mobility credits’ to all taxpayers monthly, and providing the flexibility to pay either transit fares or road tolls for private cars, as well as permitting the free trading of credits. Other variants of the TCS in transportation areas are investigated in Raux (2004) and Kockelman and Kalmanje
(2005), with different credits endowed with different legitimate rights. Apart from these conceptual studies, Akamatsu (2007) conducted some mathematical analyses on TCSs. He proposed a tradable bottleneck permits scheme (TBPS) under which the queuing delay can be eliminated. In his article, he established an equilibrium model under TBPS and discussed the theoretical relationships between TBPS and CP. However, such a TBPS has some problematic issues, such as the complicated permit trading market and the controversial market selling scheme.

To develop a simple but forceful TCS, Yang and Wang (2011) proposed a new tradable credit distribution and charging scheme where credits were universal for all links but link specific in the amount of credit charging, and then they quantitatively analysed how to manage network mobility with TCS in the deterministic user equilibrium (UE) context. In this vein, numerous efforts have been made to expand Yang and Wang’s landmark work on TCSs. Wang et al. (2012) and Zhu et al. (2015) extended the work by considering heterogeneous users with discrete or continuous value of time (VOT) models. Nie (2012) investigated the impacts of transaction costs on the existence of system optimal (SO) TCS in both auction markets and negotiated markets. Shirmohammadi et al. (2013) explored the deviation in managing network mobility between CP and TCS under uncertain conditions. He et al. (2013) analysed TCS on the network with Cournot–Nash (CN) players and Wardrop-equilibrium (WE) players. Bao et al. (2014) extended the studies on managing network mobility with TCS by considering travellers’ loss aversion behaviours and the transaction costs. Besides, there are also many investigations on managing bottleneck congestion with TCS, such as Nie and Yin (2013), Tian, Yang, and Huang (2013) and Xiao, Qian, and Zhang (2013). Tables 1 provides a selective summary of the modelling methodologies on TCSs managing network mobility. For comprehensive reviews, readers can see Fan and Jiang (2013) and Grant-Muller and Xu (2014).

Note that although the studies mentioned above have investigated TCSs from different angles, they share the same mathematical modelling framework, i.e. using a UE model framework to describe the traveller’s path choice behaviours. It is well known that in a UE context travellers are assumed to have perfect knowledge of the generalized path travel costs and have the ability to recognize the shortest path at all times, which is obviously a rather strict assumption for the complex transportation system. Therefore, this article attempts to relax such an assumption and consider the traveller’s perception errors of generalized path travel costs. Under such a relaxed assumption, all the available routes are likely to be chosen according to a certain probability distribution, which can be exactly captured by the stochastic user equilibrium (SUE) model (Sheffi 1985). Furthermore, it is believed that the SUE principle plays the same role as the UE principle in describing a traveller’s route choice behaviour. However, the SUE traffic assignment problem with TCS has received very little attention even though its counterpart, i.e. the UE assignment problem with TCS, has been investigated extensively. One possible reason for this imbalance may be that establishing an equivalent mathematical programming model for the network SUE and market equilibrium (ME) condition is more difficult and more challenging than doing so for the UE case. Owing to the speciality of the SUE model, adding the linear constraint of total credit amount into Sheffi’s unconstrained SUE model (Sheffi 1985) cannot yield an equivalent minimization model for the SUE and credit ME conditions. Thus, it remains an open question as to how to establish the general SUE traffic assignment with TCS.

| Literature                      | Perception errors | VOT      | Equilibrium model | Modelling forms |
|---------------------------------|-------------------|----------|-------------------|-----------------|
| Akamatsu (2007); Yang and Wang (2011); Nie (2012); Shirmohammadi et al. (2013); He et al. (2013); Bao et al. (2014) | No                 | Homogeneous | UE model          | MP/VI           |
| Wang et al. (2012); Zhu, Yang, and Li (2015) | No, Yes           | Homogeneous | UE model          | VI              |
| This article                    |                   |           | SUE model         | MP              |

Note: VOT = value of time; UE = user equilibrium; MP = mathematical programming; VI = variational inequality.
The most commonly used optimization objectives in the previous TCS design problems are either minimizing total travel time (TTT) or maximizing total social benefit (TSB). However, under the SUE principle, such an SO or approximately SO network flow pattern cannot necessarily be decentralized with a meaningful and finite TCS (Smith, Eriksson, and Lindberg 1995). Although minimizing the perceived TTT [i.e. stochastic system optimal (SSO)] can always be taken as the optimization objective in the SUE context, it is trivial work to obtain the solution set of SSO-TCS, considering that the solution set of SSO-CP has been established in Maher, Stewart, and Rosa (2005). In this research, maximizing the network reserve capacity is taken as the optimization objective for the optimal TCS design problem. Besides, a heuristic sensitivity analysis-based (SAB) algorithm is also developed to solve the optimal TCS design problem, with analytical expressions of gradient information of the SUE link flow and ME credit price provided.

The remainder of the article is organized as follows. Section 2 describes the SUE and credit ME conditions, and then establishes an equivalent general SUE model with a TCS. In Section 3, a bi-level mathematical programming model for the optimal TCS design problem is formulated based on the proposed SUE model, and the SAB algorithm is used to solve the bi-level model. Section 4 presents some numerical examples. In Section 5, some concluding comments are provided.

2. Equivalent stochastic user equilibrium model with a tradable credit scheme

2.1. Stochastic user equilibrium and market equilibrium conditions

Consider a general strongly connected transportation network \( G = (N, A) \), with a set \( N \) of nodes and a set \( A \) of directed links. Let the set of origin-destination (O-D) pairs be denoted by \( W \), and the fixed traffic demand for O-D pair \( w \in W \) be denoted by \( q_w \); let \( R_w \) denote the set of all paths between O-D pair \( w \in W \), let \( f^w_r \) denote the flow on path \( r \in R_w \), and let \( v_a \) denote the traffic flow on link \( a \in A \). With these notations, the set \( \Phi \) of all feasible network flow patterns \( (f, v) \) can be described as follows:

\[
\Phi = \left\{ (f, v) \mid \sum_{r \in R_w} f^w_r = q_w, v_a = \sum_{w \in W} \sum_{r \in R_w} f^w_r \delta_{a,r} f^w_r \geq 0, \forall a \in A, \forall r \in R_w, w \in W \right\}
\]  

(1)

where \( \delta_{a,r} \), which is an element of the link/path incidence matrix, is equal to 1 if path \( r \in R_w \), \( w \in W \), uses link \( a \in A \) and 0 otherwise.

As advocated in Yang and Wang (2011), in a TCS the government initially distributes equal amounts of credits free to all eligible travellers, and travellers have to use credits to pay for all the link credit charges in the route. If a traveller’s initial credits are less than the total charges on a specific route, he or she cannot travel on that route unless additional credits are purchased from other travellers who have leftover credits. The credits can be freely traded in the competitive credit market and thus the credit price is determined by the market. Let \( K \) denote the total amount of credits, and \( \bar{k} \) denote the initial credit amount distributed to each traveller, thus \( K = \bar{k} \sum_{w \in W} q_w \). Let \( k_a \) denote the credit charges on link \( a \in A \), thus \( k = \{k_a, a \in A\} \) denotes a link credit charge scheme. Hence, notation \((K,k)\) can represent an entire TCS. Since not all TCSs can guarantee the existence of feasible network flow patterns \((f,v)\), the corresponding feasible TCS set \( \Psi \) is defined as follows:

\[
\Psi = \left\{ (K,k) \mid \exists (f, v) \in \Phi \text{ such that } \sum_{a \in A} k_a v_a \leq K, k_a \geq 0 \right\}
\]  

(2)

where \( \Phi \) has been defined in constraint (1). Hereafter, assume that \( \Psi \) is non-empty.

For simplicity, consider a separable and monotonically increasing link travel time function, denoted by \( t_a(v_a), \forall a \in A \), and thus the path travel time \( t^w_r \) on route \( r \in R_w \) can be obtained:

\[
t^w_r = \sum_{a \in A} t_a(v_a) \delta_{a,r}, \quad r \in R_w, w \in W
\]  

(3)
It should be emphasized that under a given TCS, the generalized path travel cost \( c_r^w \) on the route \( r \in R_w \) is the sum of path travel time \( t_r^w \) and total credit costs on that route. Let \( p \) denote the unit credit price in the credit market, then:

\[
c_r^w = t_r^w + p \sum_{a \in A} \kappa_a \delta_{a,r}^w, \quad r \in R_w, w \in W
\]

(4)

Note that the credit price has been converted to time unit for the brevity of the mathematical expressions. The homogeneous VOT is assumed in this article and, without loss of generality, the VOT is assumed to be unity.

It is further assumed that each traveller’s perceived path travel cost \( C_r^w \) is equal to the actual generalized path travel cost \( c_r^w \) plus a random perception error \( \xi_r^w \):

\[
C_r^w = c_r^w + \xi_r^w, \quad r \in R_w, w \in W
\]

(5)

where \( \xi_r^w, r \in R_w, w \in W \) are assumed to be random variables with the expected values \( E[\xi_r^w] = 0 \) and a constant variance \( \text{Var}(\xi_r^w) \).

Given a feasible TCS \((K, k)\), the probability \( P_r^w(c^w) \) of choosing the specific route \( r \in R_w \) can be expressed as

\[
P_r^w(c^w) = \Pr(C_r^w \leq C_l^w, \forall l \in R_w \text{ and } l \neq r|c^w), r \in R_w, w \in W
\]

(6)

where \( c^w = [c_1^w, c_2^w, \cdots] \). Specially, if the random perception error \( \xi_r^w \) follows the independent and identical Gumbel distribution, the analytical expression of the path choice probability \( P_r^w(c^w) \) can be obtained (Sheffi 1985):

\[
P_r^w(c_r^w) = \frac{\exp[-\theta c_r^w]}{\sum_{r' \in R_w} \exp[-\theta c_{r'}^w]} = \frac{\exp[-\theta (t_r^w + p \sum_{a \in A} \kappa_a \delta_{a,r}^w)]}{\sum_{r' \in R_w} \exp[-\theta (t_{r'}^w + p \sum_{a \in A} \kappa_a \delta_{a,r'}^w)]}, r \in R_w, w \in W
\]

(7)

where \( \theta \) is the travellers’ perception dispersion parameter. Thus, the SUE and ME conditions under a feasible TCS \((K, k)\) can be written as:

\[
f_r^w = q_w P_r^w(c^w), r \in R_w, w \in W
\]

(8)

\[
p \left( K - \sum_{a \in A} \kappa_a v_a \right) = 0, p \geq 0, \sum_{a \in A} \kappa_a v_a \leq K
\]

(9)

where \((f, v) \in \Phi\). Equation (9) represents the credit market clearing conditions, which stipulate that the equilibrium credit price \( p \) is positive only when all the initially issued credits are consumed. Note that the TCS is ineffective when \( p^* = 0 \), and thus the resultant SUE network flow pattern is the same as the SUE one with no TCS. Since neither \( p^* = 0 \) nor the SUE network flow pattern with no TCS is a desirable solution, hereafter the article focuses on the following effective TCS:

\[
\Psi = \left\{ (K, k) \left| \sum_{a \in A} k_a v_a^0 > K, \text{ and } (K, k) \in \Psi \right. \right\}
\]

(10)

where \( v_a^0 \) is the SUE link flow pattern without the intervention of the TCS.

### 2.2. An equivalent general stochastic user equilibrium model with tradable credit scheme

Since such a modelling conjecture, that the equivalent SUE model with TCS may be established just like the UE model case by adding a linear constraint into the conventional unconstrained SUE model,
is not true in the SUE context, it is very difficult to establish a SUE model with TCS directly based on the conventional SUE model. To find such an equivalent model, the following constructed function corresponding to the original link travel time function \( t_a(x) \), \( a \in A \), is defined:

\[
\tilde{t}_a(v_a) = \begin{cases} 
\int_0^{v_a} t_a(x) \, dx, & v_a > 0 \\
t_a^0, & v_a = 0
\end{cases}
\]  

(11)

where \( t_a^0 \) is the free flow travel time of original link travel time function \( t_a(x) \), and thus the new function \( \tilde{t}_a(x) \) possesses the same desirable mathematical properties (i.e. positive, strictly increasing and continuously differentiable) as the original link travel time function \( t_a(x) \).

According to Maher, Stewart, and Rosa (2005), it is easy to understand that for any feasible positive path flow pattern \( f > 0 \in \Phi \), there exists the path-specific continuously differentiable vector function \( d^w(f^w) = (\cdots, d^w_r, \cdots) \in R|\mathbf{R}_w|, w \in \mathbf{W} \), such that the path flow pattern \( f \) can be supported as SUE by the latter, where the notation \(|\mathbf{R}_w|\) is the dimension of route set \( \mathbf{R}_w \) between O-D pair \( w \in \mathbf{W} \):

\[
P_r^w(t^w(f)+d^w(f^w)) = \frac{f_r^w}{q^w_w}, \text{where } t^w(f) = [\cdots, t^w_r, \cdots], w \in \mathbf{W}, r \in \mathbf{R}_w
\]  

(12)

It is noteworthy that Equation (12) also holds if the original link travel time function \( t_a(x) \) and the path-specific vector function \( d^w(f^w) \) are simultaneously replaced by the newly constructed function \( \tilde{t}_a(x) \) and the corresponding path-specific vector function \( \tilde{d}^w(f^w) \) (Meng, Lam, and Yang 2008):

\[
P_r^w(t^w(f)+\tilde{d}^w(f^w)) = \frac{\tilde{f}_r^w}{q^w_w}, \text{where } \tilde{t}^w(f) = [\cdots, \tilde{t}_r^w, \cdots], w \in \mathbf{W}, r \in \mathbf{R}_w
\]  

(13)

Now, a linearly constrained equivalent minimization model can be established for the SUE and ME conditions (8)–(9) as follows:

\[
\begin{align*}
\min \quad & Z(f) = \sum_{w \in \mathbf{W}} q_w S^w(\tilde{t}^w(f)+\tilde{d}^w(f^w)) - \sum_{w \in \mathbf{W}} \sum_{r \in \mathbf{R}_w} \tilde{d}_r^w(f^w)f_r^w \\
\text{s.t.} \quad & \sum_{a \in A} k_a v_a \leq K \\
& (f, v) \in \Phi
\end{align*}
\]  

(14)

where \( S^w(\bullet) \) is the satisfaction function defined in Sheffi (1985, 269), and the first linear constraint is the total credit amount constraint.

**Proposition 1:** Any local minimum solution \( f^* \) of the model (14) satisfies the SUE and ME conditions (8)–(9), and the Lagrangian multiplier associated with the linear TCS constraint is the ME credit price.

**Proof:** According to the minimization model (14), the corresponding Lagrangian function can be obtained as below:

\[
L(f, \mu^w, \rho) = Z(f) + \sum_{w \in \mathbf{W}} \mu^w \left( q^w - \sum_{r \in \mathbf{R}_w} f_r^w \right) - \rho \left( K - \sum_{a \in A} k_a v_a \right)
\]  

(15)

where \( \mu^w, \rho \) are the Lagrangian multipliers associated with the corresponding linear constraints. Note that the following two equations always hold (Sheffi 1985, 269):

\[
\frac{\partial S^w(\bullet)}{\partial f_r^w} = \sum_{r \in \mathbf{R}_w} P_r^w \ast \left( \frac{\partial \tilde{t}_r^w}{\partial f_r^w} + \frac{\partial \tilde{d}_r^w}{\partial f_r^w} Y_r^w \right)
\]  

(16)

\[
t_a(v_a) = \tilde{t}_a(v_a) + v_a \tilde{\gamma}_a(v_a)
\]  

(17)
where the coefficient $\gamma_w^w = 1$ if $w = w'$, or otherwise $\gamma_w^w = 0$. From the Karush–Kuhn–Tucker (KKT) conditions of Lagrangian function (15), the following equations can be obtained:

$$\frac{\partial L(\bar{f}, \mu, \rho)}{\partial f_w} = -\bar{d}_w + \sum_{a \in A} v_a \bar{t}_a \delta_{a,w} + \rho \sum_{a \in A} k_a \delta_{a,w} - \mu = 0$$  \hspace{1cm} (18)

$$\rho \left( K - \sum_{a \in A} k_a v_a \right) = 0, K - \sum_{a \in A} k_a v_a \geq 0, \rho \geq 0$$  \hspace{1cm} (19)

Then, Equation (13) can be rewritten by substituting the corresponding parts with Equations (17) and (18), as follows:

$$P_w^x (\bar{t}_w (f) + \bar{d}_w^w (f^w)) = P_r (\ldots, \sum_{a \in A} \bar{t}_a (v_a) \delta_{a,r} + \bar{d}_r^w, \ldots)$$

$$= P_r (\ldots, \sum_{a \in A} [t_a (v_a) + \rho \kappa_a] \delta_{a,r}^w - \mu, \ldots)$$  \hspace{1cm} (20)

$$= P_r^w (\ldots, \sum_{a \in A} [t_a (v_a) + \rho \kappa_a] \delta_{a,r}^w, \ldots) = \frac{f_r}{q_r}$$

Obviously, Equations (19) and (20) are just the SUE and ME conditions, and the Lagrangian multiplier $\rho$ is the ME credit price. This completes the proof. \hfill \blacksquare

It should be noted that model (14) is a general SUE model with a TCS, and different model specifications can be further derived with different probability distributed perception errors. In particular, the following two equations can be obtained with Gumbel distributed perception errors (Maher, Stewart, and Rosa 2005),

$$\bar{d}_r^w (f^w) = -\bar{t}_r^w + \bar{t}_{|R_w|}^w - \frac{1}{\theta} \ln \frac{f_r^w}{f_{R_w}^w}, r \in R_w$$  \hspace{1cm} (21)

$$S^w (\bar{t}^w + \bar{d}^w) = -\frac{1}{\theta} \ln \sum_{r \in R_w} \exp \left( -\theta \left( \bar{t}_{|R_w|}^w - \frac{1}{\theta} \ln \frac{f_r^w}{f_{|R_w|}^w} \right) \right), w \in W$$  \hspace{1cm} (22)

where $f_{|R_w|}^w$ and $\bar{t}_{|R_w|}^w$ denote, respectively, the path flow and ‘nominal’ travel time on the $|R_w|$th route. After substituting $\bar{d}_r^w$ and $S^w (\cdot)$ in model (14) with Equations (21) and (22), model (14) can be simplified as the following expressions:

$$\min \ Z(\bar{f}) = \sum_{a \in A} \int_0^{v_a} t_a (\omega) d\omega + \frac{1}{\theta} \sum_{w \in W} \sum_{r \in R_w} f_r^w \ln f_r^w - \frac{1}{\theta} \sum_{w \in W} q_w \ln q_w$$

$$\text{s.t.} \sum_{a \in A} k_a v_a \leq K$$

$$\bar{f}, v \in \Phi$$  \hspace{1cm} (23)

The above simplified minimization model (23) looks like Fisk’s logit-type SUE model, with the only distinction of an additional linear credit amount constraint, as well as a constant term $-\frac{1}{\theta} \sum_{w \in W} q_w \ln q_w$ in the objective function. It is worth mentioning that the method of successive averaging (MSA) cannot solve the proposed model (14) owing to the additional linear constraint; thus, the Lagrangian dual method (LDM) proposed in Meng, Lam, and Yang (2008) for solving the capacitated SUE model is adopted to solve the proposed model (14) after some modifications (Han and Cheng 2016).
3. Optimal tradable credit scheme design for maximizing network reserve capacity

3.1. Bi-level model for the optimal tradable credit scheme design problem

In the existing literature, the optimization objectives of TCS design are mainly minimizing TTT or maximizing TSB. However, with the assumption of SUE route choice behaviours, the existence of a meaningful SO TCS cannot be always guaranteed. For example, if the SO path flow pattern always contains nil flow, then the SO state can never be supported as the SUE under finite TCS (Han and Cheng 2015; Smith, Eriksson, and Lindberg 1995). Meanwhile, it is just trivial work to take the SSO (i.e. minimizing the perceived TTT) as the optimization objective owing to the work of Maher, Stewart, and Rosa (2005). In this research, maximizing the network reserve capacity is taken as the optimization objective of the TCS design problem. A bi-level programming model of the optimal TCS design problem is established, of which the upper level model is the TCS optimization design problem for maximizing network reserve capacity, and the lower level model is the SUE assignment problem under a certain TCS. The network reserve capacity is defined by the maximum multiplier such that no link capacity constraint is violated when the multiplier is multiplied to all of the O-D demand (Wong and Yang 1997; Yang and Wang 2002; Zhang and Wee 2012). Thus, the bi-level model can be formulated as follows:

\[
\begin{align*}
\text{max} & \quad \sigma, \kappa \\
\text{s.t.} & \quad v_a(\sigma q, \kappa) \leq C_a, \quad \forall a \in A \\
& \quad \sigma > 0, \quad \kappa_a \geq 0, \quad \forall a \in A \tag{24c}
\end{align*}
\]

where \( v_a(\sigma q, \kappa) \) is obtained by solving the following SUE assignment problem under a certain TCS \((K, k)\):

\[
\begin{align*}
\min & \quad Z(f) = \sum_{w \in W} q_w S^w(i^w(f) + d^w(f^w)) - \sum_{w \in W} \sum_{r \in R_w} d^w_r(f^w)f^w_r \\
\text{s.t.} & \quad \sum_{r \in R_w} f^w_r = \sigma q_w, \quad \forall w \in W \tag{24e} \\
& \quad v_a = \sum_{w \in W} \sum_{r \in R_w} f^w_r \delta_w a, \quad \forall a \in A \tag{24f} \\
& \quad f^w_r \geq 0, \quad \forall r \in R_w, w \in W \tag{24g} \\
& \quad \sum_{a \in A} \kappa_a v_a \leq \sigma K \tag{24h}
\end{align*}
\]

It should be pointed out that the existing total amount of credits \(K\) is also multiplied by the multiplier \(\sigma\) in the constraint (24h), in addition to the existing O-D demands \(q_w, \forall w \in W\) in the constraint (24e). There are three main considerations for this modelling form. First, if the multiplier \(\sigma\) is not added into the constraint (24h) and the multiplier \(\sigma\) is continuously increasing, such an undesirable problem may happen just before the link capacity constraint (24b) is violated, so that the initially issued credit amount \(K\) is not large enough to ensure that all travellers complete their trips. Secondly, since the credits are uniformly distributed to all eligible travellers, to ensure fairness the initially issued credit amount must also be enlarged by the same scaling factor when the O-D demands increase to \(\sigma(\geq 1)\) times as large as the existing demands \(q_w, \forall w \in W\). Furthermore, it can also ensure that under the new TCS \((\sigma K, k)\), there always exists at least one feasible network flow pattern satisfying the set \(\Phi(\sigma)\), where \(\Phi(\sigma)\) is the set of feasible network flow patterns with O-D demands \(\sigma \cdot q_w, \forall w \in W\).

**Proposition 2:** If a feasible network flow pattern \((f, v) \in \Phi\) exists under a given TCS \((K, k) \in \tilde{\Psi}\), then there always exists at least one feasible network flow pattern satisfying the set \(\Phi(\sigma)\) under the TCS
(\(\sigma K, k\)), or in short, the set \(\Psi(\sigma)\) is always non-empty if \(\Psi\) is non-empty, where \(\Psi(\sigma)\) is the set of effective TCS with O-D demands \(\sigma \bullet q_w, \forall w \in W\).

**Proof:** According to the sets \(\Phi\) and \(\Psi\) defined, respectively, in constraints (1) and (23), the feasible network flow pattern \((f, v) \in \Phi\) must satisfy the following equations:

\[
\sum_{r \in R_w} f^w_r = q_w, \forall w \in W
\]

\[
v_a = \sum_{w \in W} \sum_{r \in R_w} f^w_r \delta^{w}_{a,r}, \forall a \in A
\]

\[
f^w_r \geq 0, \forall r \in R_w, w \in W
\]

\[
\sum_{a \in A} \kappa_a v_a \leq K
\]

Then, by multiplying the above four expressions by the multiplier \(\sigma\), the following new expressions can be obtained:

\[
\sum_{r \in R_w} \sigma f^w_r = \sigma q_w, \forall w \in W
\]

\[
\sigma v_a = \sum_{w \in W} \sum_{r \in R_w} \sigma f^w_r \delta^{w}_{a,r}, \forall a \in A
\]

\[
\sigma f^w_r \geq 0, \forall r \in R_w, w \in W
\]

\[
\sum_{a \in A} \kappa_a (\sigma v_a) \leq \sigma K
\]

Obviously, the network flow pattern \((\sigma f, \sigma v)\) is always a feasible network flow pattern satisfying the set \(\Phi(\sigma)\) under the TCS \((\sigma K, k)\), that is, the set \(\Psi(\sigma)\) is always non-empty. This completes the proof. 

### 3.2. Sensitivity analysis-based algorithm

Since the bi-level model (24) is NP hard and non-convex, the exact solution algorithms become powerless. Consequently, the heuristics SAB algorithm is adopted to solve the problem. The flowchart of the SAB algorithm is as shown in Figure 1. For a more detailed solution procedure, see Yang, Yagar, and Iida (1994).

From Figure 1, it can be seen that once the feasible current solution \((\sigma^*, k^*)\) and the corresponding SUE solution \(v(\sigma^*, k^*)\) are obtained, then a new improved solution \((\sigma, k)\) satisfying all the link capacity constraints and credit amount constraint can be obtained by solving a linear approximation subproblem. Clearly, the linear approximation of SUE solution \(v(\sigma, k)\) is the key process to find such an improved solution. The linear approximation function is as follows:

\[
v_a(\sigma, k) \approx v_a(\sigma^*, k^*) + \frac{\partial v_a(\sigma^*, k^*)}{\partial \sigma} (\sigma - \sigma^*) + \sum_{a' \in A} \frac{\partial v_a(\sigma^*, k^*)}{\partial \kappa_{a'}} (\kappa_{a'} - \kappa_{a'}^*)
\]

The gradient information needed in Equation (27) can be obtained based on the sensitivity analysis (SA) of the proposed model (14). Note that the SA of SUE model itself is a difficult problem, and the computation technologies of SA for different types of SUE model (e.g. logit-based, probit-based) are also different (Clark and Watling 2000; Ying and Miyagi 2001). Furthermore, the proposed model (14) is even more complicated than the conventional SUE model, and thus the SA of proposed model (14) is very challenging work, which can be a topic for future research. For completeness, the SA for
the lower level model in bi-level model (24) is given briefly as follows, with the logit network loading adopted.

From the analysis in Section 2.2, the KKT conditions for the perturbed SUE assignment model in the lower level of bi-level model (24) can be obtained as:

\[
\frac{\partial L(\bullet)}{\partial f^w_r} = \frac{1}{\theta} \ln f^w_r + \sum_{a \in A} t_a(v_a) \delta^w_a,r + p \sum_{a \in A} \kappa_a(\epsilon) \delta^w_{a,r} - \mu'_w = 0, \forall w \in W, r \in R_w
\]

\[
p \left( \sigma(\epsilon) K - \sum_{a \in A} \kappa_a(\epsilon) v_a \right) = 0
\]

\[
\sigma(\epsilon) q_w - \sum_{r \in R_w} f^w_r = 0, \forall w \in W
\]

where \( \mu'_w = \mu_w + \tilde{n}^w_{|R_w|} + (1/\theta) \ln f^w_{|R_w|} \) and \( \mu_w, p \) is the Lagrangian multiplier associated with the corresponding linear constraints.

Let \( M(\epsilon), N(\epsilon) \) be the Jacobian matrices of the KKT conditions (28) with respect to vector variables \( (f, p, \mu') \) and \( (k, \sigma) \), respectively:

\[
M(\epsilon) = \begin{bmatrix}
\nabla^2 f L & \left( \sum_{a \in A} \kappa_a \delta^w_{a,r} \right) \ast I' - \Gamma' \\
-(p \sum_{a \in A} \kappa_a \delta^w_{a,r}) \ast I' & \sigma K - \sum_{a \in A} \kappa_a v_a & 0 \\
-\Gamma & 0 & 0
\end{bmatrix}
\]

\[
N(\epsilon) = \begin{bmatrix}
\nabla_k (\nabla f L) & 0 \\
\nabla_k \left[ p(\sigma K - \sum_{a \in A} \kappa_a v_a) \right] & \sigma K \\
0 & q
\end{bmatrix}
\]

where \( \Gamma \) is the O-D/path incidence matrix, \( I \) represents a column vector full of one with the dimension \( \sum_{w \in W} |R_w| \), and \( I \) is an identity matrix with the dimension \( |W| \).
According to Fiacco’s Corollary 3.2.4 (Fiacco 1983), the gradient information of \((v, p, \mu')\) with respect to the perturbed parameters \(\varepsilon = (\varepsilon_k, \varepsilon_{\sigma})\) can be obtained from the following expression:

\[
\begin{bmatrix}
\nabla_{\varepsilon} v \\
\nabla_{\varepsilon} p \\
\nabla_{\varepsilon} \mu'
\end{bmatrix} = \left[ \Delta_{|A| \times |R|} \right] \begin{bmatrix}
\nabla_{\varepsilon} f \\
\nabla_{\varepsilon} p \\
\nabla_{\varepsilon} \mu'
\end{bmatrix} = -\left[ \Delta_{|A| \times |R|} \right] M(\varepsilon)^{-1} N(\varepsilon)
\]

where \(\Delta_{|A| \times |R|}\) is the link/path incidence matrix, and \(I\) is an identity matrix with the proper dimension.

4. Numerical examples

4.1. A toy network

Example 1: A toy network with three nodes and four links is considered in Figure 2. There are three O-D pairs \((1 \rightarrow 2, 2 \rightarrow 3 \text{ and } 1 \rightarrow 3)\), with the current O-D demands, respectively, 300, 300 and 200 units of flow. The link travel time functions are standard Bureau of Public Roads (BPR) functions \(t_a(v_a) = t^0_a[1 + 0.15(v_a/C_a)^4], \forall a \in A\), with the parameters presented in Figure 2. Assume that a total amount of 1750 credits is uniformly distributed to all current travellers. To compare the efficiency of different TCSs, an original TCS is taken for reference, with the credit charging scheme \(k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 1\). The logit network loading is adopted and the dispersion parameter \(\theta = 0.05\).

From Figure 3, it can be seen that the proposed SAB algorithm is convergent at different dispersion parameters. The stopping criterion of the SAB algorithm is \(||(\mathbf{k}, \sigma)^{(n+1)} - (\mathbf{k}, \sigma)^{(n)}||_2 / ||(\mathbf{k}, \sigma)^{(n)}||_2\) and the convergence accuracy is \(10^{-4}\).

From Tables 2, it can be seen that when there is no TCS imposed on the network, the maximum demand multiplier \(\sigma\) is 1.651 and the corresponding reserve capacity is 1320.8, which are both smaller than those under TCS. Specifically, under the optimal TCS, the maximum demand multiplier \(\sigma\) is 2.4 and the reserve capacity of the network becomes 1920, which are, respectively, 1.45 times those under no TCS, and 1.16 times of those under the original TCS. Therefore, a properly designed TCS can effectively improve the transportation network capacity, which is a valuable feature in practical engineering problems. Furthermore, by virtue of modern electronic toll collection technologies, TCS can improve network reserve capacity and mitigate traffic congestion in less time and with less financial cost than adding or broadening roads.

Besides, it can also be seen from Tables 2 that there are more links reaching saturation capacity under the optimal TCS than in the other two scenarios. In other words, the appropriate TCS can improve the usage rate of the links in the network, i.e. make the trip flows more evenly distributed over the network, by affecting the travellers’ path choice behaviour. In addition, the ME credit prices under both TCSs are very close to each other, which is a desirable and important feature for transportation policy makers. Since the volatility of credit price is usually undesirable in the realistic credit market, it is of great practical significance to design an optimal TCS with a steady (or almost steady) credit price.

![Figure 2. The toy network.](image)
Figure 3. Convergence of the sensitivity analysis-based (SAB) algorithm at different dispersion parameters. TCS = tradable credit scheme.

Table 2. Numerical results for the optimal tradable credits scheme (TCS) design problem.

| Link  | TCS Linkflow | V/C | Original TCS Linkflow | V/C | No TCS Linkflow | V/C |
|-------|--------------|-----|------------------------|-----|----------------|-----|
| Link 1| 0.56         | 600.00| 1.00                   | 600.00| 1.00          | 468.65| 0.78|
| Link 2| 1.52         | 200.00| 0.40                   | 2.00    | 194.65| 0.39  | 331.35| 0.66|
| Link 3| 4.45         | 800.00| 1.00                   | 3.00    | 794.66| 0.99  | 800.00| 1.00|
| Link 4| 0.00         | 400.00| 1.00                   | 1.00    | 236.79| 0.59  | 25.29 | 0.06|

Max. multiplier \( \sigma = 2.40 \) \( \sigma = 2.063 \) \( \sigma = 1.651 \)

Total demands \( \sigma Q = 1920 \) \( \sigma Q = 1650.4 \) \( \sigma Q = 1320.8 \)

Total credits \( \sigma K = 4200 \) \( \sigma K = 3610 \)

Credit price \( p^* = 16.99 \) \( p^* = 16.96 \)

Example 2: To comprehensively explore the improving effect of TCS on the network reserve capacity, the maximum demand multipliers at different dispersion parameters are compared, as well as the corresponding congestion distribution. Note that all the cases are considered in two scenarios, ‘With TCS’ and ‘No TCS’, and only the optimal TCS is considered in the ‘With TCS’ scenario. The network in Figure 2 is again used as the example here. The results are as shown in Figures 4–6.

Figure 4 shows that the maximum demand multipliers \( \sigma \) in the ‘With TCS’ scenario are all greater than those in the ‘No TCS’ scenario, regardless of the magnitude of the dispersion parameter. This result implies that the TCS can effectively improve the network reserve capacity, no matter whether the travellers are familiar with the traffic network state or not. Besides, the maximum demand multipliers in the ‘With TCS’ scenario are all 2.4 for different dispersion parameters, which illustrates that the corresponding optimal TCS can always ensure the best system performance of the transportation network, even if the travellers are not familiar with the traffic network state (e.g. \( \theta < 0.05 \)). In the ‘No TCS’ scenario, the maximum demand multiplier \( \sigma \) depends on the dispersion parameters \( \theta \) and critical bottleneck links. Furthermore, Figure 4 shows that the maximum demand multiplier \( \sigma \) in the ‘No TCS’ scenario becomes smaller with the increasing dispersion parameter \( \theta \). Specially, when \( \theta \) is small enough (e.g. \( \theta < 0.0001 \)), the traveller would choose every path with approximately equal probability, and thus the binding link capacity would be \( C_3 = 800 \), then the maximum demand multiplier \( \sigma = 1.846 \) can be obtained without any difficulty. By contrast, when \( \theta \) is great enough (e.g. \( \theta > 1 \)), the SUE model would reduce to the UE model, and thus the maximum demand multiplier

\[ p^* = \frac{\sigma K}{\sigma Q} \]

where \( \sigma K = 4200 \) and \( \sigma Q = 1920 \).

\[ p^* = \frac{3610}{1320.8} \approx 2.73 \]

\[ p^* = \frac{4200}{1920} = 2.16 \]

\[ p^* = \frac{3610}{1320.8} \approx 2.73 \]

\[ p^* = \frac{4200}{1920} = 2.16 \]

\[ p^* = \frac{3610}{1320.8} \approx 2.73 \]
\[ \sigma = 1.2, \] with the binding link capacity, would be \( C_1 = 600. \) It can also be seen from Figure 4 that the ME credit price \( p^* \) in the ‘With TCS’ scenario is almost keeping steady for any dispersion parameter \( \theta \geq 0.05, \) while for the cases with \( \theta < 0.05 \) the ME credit price \( p^* \) would increase substantially with \( \theta \) becoming smaller. The most probable reason is that the traveller’s perception of the generalized path costs is so insensitive with \( \theta < 0.05 \) that the traveller’s original path choice probability will not change unless the credit cost \( p^* \sum_{a \in A} K_a \delta w_a \) levied on that path is great enough. Therefore, it is only when ME credit price \( p^* \) becomes very high that the desirable link flows satisfying \( V/C \leq 1 \) and \( \sigma = 2.4 \) could be possibly decentralized as SUE under TCS. Note that both the total credit amount \( \sigma K \) and link credit charges \( k_a \) are limited.

Figures 5 and 6 show the congestion distribution across the network in two scenarios. It can be known from the two figures which links are congested bottlenecks when the O-D travel demands reach the network reserve capacity. Specifically, in the ‘With TCS’ scenario, links 3 and 4 are always the bottleneck links regardless of the magnitude of the dispersion parameter, while in the ‘No TCS’ scenario link 3 is the bottleneck link at small \( \theta \) and link 1 is the bottleneck at relatively large \( \theta. \) In fact, it is exactly these bottleneck links that would limit the network reserve capacity, just like the ‘cask
Figure 6. Distribution of bottleneck links at different theta in ‘No TCS’ scenario.

effect’. Thus, once additional capacity is added on to the bottleneck link, the reserve capacity of the entire transportation network could become even greater.

4.2. Medium-scale Sioux Falls network

Example 3: The classical Sioux Falls network (Clark and Watling 2000) is considered in Example 3. The network contains 24 nodes, 76 links and 528 O-D pairs with positive demand, and the link travel time functions are standard BPR functions. The network topology and the test data are provided in the supplemental online material owing to space limitations. Assume that the current total credit amount is 362,720 units, and the current link credit charges are 0.5 units on links 1–20, 1.0 unit on links 21–40, 1.5 units on links 41–60 and 1.0 unit on links 61–76.

The SUE assignment problem under the given TCS is solved by the LDM combined with logit network loading. The dispersion parameter \( \theta \) is 0.1. The LDM converges after 726 iterations and the ME credit price is \( p^* = 5.1876 \), with the initial solution \( p^{(0)} = 0.5 \) and the step size sequences \( \{0.618/n, n = 1, 2, \ldots \} \). Also, it is found that the current O-D demands have exceeded the network reserve capacity. There are 16 oversaturated links in the transportation network and the three most congested links are links 29, 48 and 40, with V/C rates of 1.53, 1.49 and 1.33.

To explore the extent to which a properly designed TCS can enhance the network reserve capacity in a medium-scale network, as well as the influence of dispersion parameters, the optimal TCS design problem is further investigated. The numerical results are summarized in Tables 3 and Figure 7.

Table 3 compares the numerical results of the SAB algorithm under different dispersion parameters for the two scenarios, i.e. the ‘With TCS’ scenario and the ‘No TCS’ scenario. It can be seen that

Table 3. Comparison of numerical results of the sensitivity analysis-based (SAB) algorithm in different cases for the two scenarios.

| Scenarios | Maximum multiplier | Bottleneck links | Total demand | Total travel time | Iterations of SAB | No. of LDM | No. of MSA |
|-----------|--------------------|------------------|--------------|------------------|-------------------|------------|-----------|
| \( \theta = 0.01 \) | No TCS | 0.5822 | 29,48 | 83,977 | 9.03E + 05 | 4 | – | 4 |
| With TCS | 0.7174 | 40,70 | 103,478 | 1.06E + 06 | 6 | 6 | 86 |
| \( \theta = 0.1 \) | No TCS | 0.5545 | 29,48 | 79,981 | 8.24E + 05 | 2 | – | 2 |
| With TCS | 0.6813 | 29 | 98,271 | 1.00E + 06 | 4 | 4 | 1374 |
| \( \theta = 1 \) | No TCS | 0.471 | 29,48 | 67,937 | 6.26E + 05 | 4 | – | 4 |
| With TCS | 0.6193 | 30,51 | 89,328 | 9.14E + 05 | 3 | 3 | 2367 |
| \( \theta = 10 \) | No TCS | 0.4374 | 29,48 | 63,091 | 5.74E + 05 | 2 | – | 2 |
| With TCS | 0.5004 | 29,48 | 72,178 | 6.62E + 05 | 4 | 4 | 4309 |

Note: The initial solution is \( k^{(0)} = [0.5, \ldots , 0.5, 1, \ldots , 1, 0.5, \ldots , 0.5, 1, \ldots , 1] \); \( \sigma^{(0)} = 0.2 \).

TCS = tradable credit scheme; LDM = Lagrangian dual method; MSA = method of successive averaging.
the network reserve capacity becomes smaller with increasing dispersion parameter in the 'No TCS' scenario, which is exactly the same as in Example 2. However, this changing trend also occurs in the 'With TCS' scenario. The reasons for this result are multiple and complex, but one of the most possible causes is the network scale and topology. Specifically, the Sioux Falls network has more paths and more complicated topology than the toy network in Example 2, i.e. more equations need be balanced for the SUE and ME conditions, thus with $\theta$ increasing there may no longer exist a TCS that can

Figure 7. Comparison of network bottleneck distribution in different cases for the two scenarios. TCS = tradable credit scheme.
decentralize the link flow pattern corresponding to the larger $\sigma$ as the SUE one.\(^1\) Besides, it can also be found that the total times of invoking the MSA (i.e. inner iteration) greatly increase as the dispersion parameter becomes larger, even though the times of the LDM (i.e. outer iteration) become less.

Figure 7 compares the distribution of bottleneck links over the network at full reserve capacity under different dispersion parameters for the two scenarios. It can be seen that for the ‘No TCS’ scenario links 29 (from node 10 to 16) and 48 (from node 16 to 10) are always the bottleneck links in all cases, while for the ‘With TCS’ scenario the bottleneck links in different cases are generally different, which shows the great effectiveness and efficiency of the TCS in regulating traffic demand.

5. Conclusions

This article has mainly investigated the SUE model with a TCS and the optimal TCS design problem. First, the SUE and ME conditions are presented to describe the equilibrium state of the transportation network under a given TCS, and then the SUE and ME conditions are formulated as an equivalent minimization model. Note that the minimization model is a general SUE model with TCS, and thus it can be simplified into the logit-based SUE model with TCS under Gumbel distributed perception errors. To investigate the effectiveness and efficiency of the TCS on enhancing transportation network capacity, a bi-level mathematical programming model for TCS design problem was established based on the proposed SUE model, with the upper level model being maximizing the network reserve capacity. Besides, the heuristic SAB algorithm is developed to solve the bi-level model, and the explicit analytical expressions of the gradient information are also derived. In future work, the heterogeneity of a traveller’s value of time, multi-mode transportation network, and other probability distributions of perception errors can be further considered.

Note

1. Note that the credit charging scheme is link specific in this article. It can be inferred from Equation (12) that the maximum multiplier $\sigma$ at different $\theta$ would be identical if the credit charging scheme is path specific.

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