U-Equivalence Spaces

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ABSTRACT

In this paper the notion of $\mathcal{U}$-equivalence space is introduced. It is proved that the topology induced by a $\mathcal{U}$-equivalence space is regular. $\mathcal{U}$-equivalent continuous functions and $\mathcal{U}$-equivalent open functions are studied. Finally, the quotient $\mathcal{U}$-equivalence spaces are introduced and discussed.

KEYWORDS: U-Equivalence; space; topology; function

INTRODUCTION

Uniform spaces are somewhere the midway points between metric spaces on one hand and abstract topological spaces on the other hand.

There are however a few aspects of metric spaces that are lost in general topological spaces. For example, since the notion of nearness is not defined for a general topological space, we cannot define the notion of uniform continuity in abstract topological spaces. The same can be said about the other notions such as total boundedness. A uniform space, which is due to A. Weil [7] is a mathematical construction in which such ‘uniform’ concepts are still available.

In this paper we introduce a new construction, namely, $\mathcal{U}$-equivalence space that is almost like a uniform space [4, 5]. We will show that the topological space induced by a $\mathcal{U}$-equivalence space, is a regular topological space. In the theory of $\mathcal{U}$-equivalence spaces, the structure-preserving functions, in the inverse image sense, are $\mathcal{U}$-equivalent continuous functions which are considered in section 3. Also, there is another way of forming a category where the objects are $\mathcal{U}$-equivalence spaces and the morphisms are structure-preserving functions in the direct image sense. We refer to these functions as the $\mathcal{U}$-equivalently open functions (see [4, 5]).

The notion of quotient uniform space was introduced by I.M. James [4]. We introduce and discuss a suitable notion for the quotient $\mathcal{U}$-equivalence space in section 4. In particular we explore several properties of such spaces.

BASIC NOTIONS

Let us begin this section with the definition of the $\mathcal{U}$-equivalence class on a set.

Definition. A $\mathcal{U}$-equivalence class on a set $X$ is a non-empty collection $\mathcal{U}_e$ of equivalence relations on $X$ such that $\mathcal{U}_e$ is closed under finite intersections.

A simple example of a $\mathcal{U}$-equivalence class on a set $X$, is the collection of all equivalence relations on $X$ which is called discrete $\mathcal{U}$-equivalence class.

Theorem. The collection $\gamma_\mathcal{U} = \{ U(\alpha) \mid \alpha \in X, U \in \mathcal{U}_e \}$, where $U(\alpha) = \{ x \in X \mid (\alpha, x) \in U \}$ forms a base for a topology on $X$.

The topology generated by this base, is called $\mathcal{U}$-equivalence topology and denoted by $\tau_\mathcal{U}$.

Corollary. Let $G \in \tau_\mathcal{U}$ and $x \in G$. Then there exists $U \in \mathcal{U}_e$ such that $x \in U(x) \subseteq G$. Hence the collection $\{ U(\alpha) \mid U \in \mathcal{U}_e \}$ forms a local base [1,3] at $x$.

Proof. By theorem 2.2, there exists $U(\alpha)$ such that $x \in U(\alpha) \subseteq G$.

Since $x \in U(\alpha)$ and $U$ is an equivalence relation on $X$, then $U(x) = U(\alpha)$. Hence $x \in U(x) \subseteq G$ as asserted. ■

Proposition. Let $(X, \mathcal{U}_e)$ be a $\mathcal{U}$-equivalence space. Then the following statements are equivalent:

1. The topological space $(X, \tau_\mathcal{U})$ is a Hausdorff topological space.
2. The intersection of all members of $\mathcal{U}_e$ coincides with $\Delta$.

Proof. Suppose (1) holds. Since $\Delta$ is contained in any member of $\mathcal{U}_e$, then $\Delta \subseteq \cap U$ as $U$ ranges over all members of $\mathcal{U}_e$.

For the other way inclusion, assume $(x, y)$ belongs to each $U$, we will show that

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x \neq y. If this is not so, then since X is Hausdorff, there exists U \in \mathcal{U}_e and V \in \mathcal{U}_e such that U(x) \cap V(y) = \emptyset. If W = U \cap V, then W \in \mathcal{U}_e and W(x) \cap W(y) = \emptyset, whence (x, y) \notin W that is a contradiction with assumption.

(2)\Rightarrow(1). Assume x, y are distinct members of X. Then by (2), there exists U \in \mathcal{U}_e such that (x, y) \notin U. Hence U(x) \cap U(y) = \emptyset. So the topological space (X, \tau_e) is a Hausdorff topological space. ■

**Definition.** Let A, B be subsets of a \mathcal{U}-equivalence space (X, \mathcal{U}_e) We say that A and B are \mathcal{U}-adjacent if for each U \in \mathcal{U}_e there exists a \in A and b \in B such that \langle a, b \rangle \in U. In particular if x_o \in X and A \subseteq X, x_o is adjacent to A if and only if for each U \in \mathcal{U}_e, there exists a \in A such that \langle x_o, a \rangle \in U.

**Proposition.** Let (X, \mathcal{U}_e) be a \mathcal{U}-equivalence space, x_o \in X and let A \subseteq X. Then x_o is \mathcal{U}-adjacent to A if and only if x_o \in \bar{A} where \bar{A} is the closure of A with respect to \tau_e.

**Proof.** Suppose x_o is adjacent to A and G is a neighbourhood of x_o. By corollary 2.3, there exists U \in \mathcal{U}_e such that U(x_o) \subseteq G.

Since x_o is adjacent to A, then there exists a \in A such that \langle x_o, a \rangle \in U.

Hence U(x_o) \cap A \neq \emptyset. This implies G \cap A \neq \emptyset. So x_o \in \bar{A}.

Conversely let x_o \in \bar{A} and let U \in \mathcal{U}_e. Since U(x_o) is a neighbourhood of x_o, then U(x_o) \cap A \neq \emptyset. Let a \in U(x_o) \cap A.

Then a \in A and \langle x_o, a \rangle \in U as required. ■

**Theorem.** Every \mathcal{U}-equivalence space is a regular topological space.

**Proof.** We first show that the set U(A) = \{ x \in X \mid (a, x) \in U for some a \in A \} is open in \mathcal{A} \subseteq U(A), where U \in \mathcal{U}_e and A \subseteq X.

Let x \in U(A). Then there exists a \in A such that \langle a, x \rangle \in U. We claim U(x) \subseteq U(A). If z \in U(x), then \langle x, z \rangle \in U. Since U is transitive, then \langle a, z \rangle \in U and it follows that z \in U(A).

So U(A) is open. Obviously, \mathcal{A} \subseteq U(A).

Now suppose x_o \in X and A is a closed subset of X not containing x_o. Then there exists G \in \tau_e such that x_o \in G and G \cap A \neq \emptyset. By proposition 2.3, there exists U \in \tau_e such that U(x_o) \subseteq G. Hence, U(x_o) \cap A \neq \emptyset. We claim U(x_o) \cap U(A) \neq \emptyset.

If this is not so, then U(x_o) \cap U(A) = \emptyset. If z \in U(x_o) \cap U(A), then \langle x_o, z \rangle \in U and there exists a \in A such that \langle a, z \rangle \in U.

Hence \langle x_o, a \rangle \in U or \langle a, x_o \rangle \in U, which contradicts that U(x_o) \cap A \neq \emptyset. This implies the result. ■

The following corollary follows from theorem 2.7 and proposition 2.4.

**Corollary.** Let (X, \mathcal{U}_e) be a \mathcal{U}-equivalence space so that \cap \{ U : U \in \mathcal{U}_e \} = \Delta_X. Then (X, \tau_e) is a T_3-space [1].

**Theorem.** Let (X, \mathcal{U}_e) be a \mathcal{U}-equivalence space. Then the topological space (X, \tau_e) is connected if and only if it admits the trivial \mathcal{U}-equivalence class \{X^2\}.

**Proof.** First suppose X admits the trivial equivalence class, i.e. \mathcal{U}_e = \{X^2\}. We show that X is connected. To see this, let G be open and closed in X (with respect to \tau_e) and let G \neq \emptyset. We have to show that G \subseteq X. By proposition 2.3, there exists U \in \mathcal{U}_e such that U(x) \subseteq G. Since \mathcal{U}_e is trivial, then U(x) = X. So G = X.

So the empty set and the whole set are the only sets in X which are both open and closed and hence X is connected.

Conversely, assume (X, \tau_e) is connected and U \in \mathcal{U}_e. We have to show that U \subseteq X^2. Let x_o \in X. From the definition of \tau_e, we see immediately that U(x_o) is open. Also we show that U(x_o) is closed. To do this, it is sufficient to show that \bar{U}(x_o) \subseteq U(x_o), where \bar{U}(x_o) is the closure of U(x_o) with respect to \tau_e [1].

If z \in \bar{U}(x_o), then U(z) \cap U(x_o) \neq \emptyset. Consequently, \langle x_o, z \rangle \in U or z \in U(x_o). So U(x_o) is also closed. So U(x_o) = \emptyset or U(x_o) = X. Since x_o \in U(x_o), then U(x_o) = X. So U(x) = X for all x \in X. Consequently, U \subseteq X^2. Hence \mathcal{U}_e = \{X^2\}. ■
**Proposition.** Let \((X, \mathcal{U}_e)\) be a \(\mathcal{U}\)-equivalence space and let \(A \subseteq X\). Then:

a) \(\bar{A} \cap \{U : U \in \mathcal{U}_e\}\), where \(\bar{A}\) is the closure of \(A\) with respect to \(\tau_e\).

b) \(U(A)\) is closed and open in \(X\).

**Proof.** a) Let \(x \in \bar{A}\) and let \(U \in \mathcal{U}_e\). Then \(A \cap U(x) \neq \emptyset\).

If \(a \in A \cap U(x)\), then \((a, x) \in U\) and hence, \(x \in U(a)\). So \(\bar{A} \subseteq \{U : U \in \mathcal{U}_e\}\).

For the other way inclusion, let \(x \in \{U : U \in \mathcal{U}_e\}\) and let \(G\) be a neighbourhood of \(x_0\). By corollary 2.3, there exists \(U \in \mathcal{U}_e\) such that \(U(x) \subseteq G\).

Since \(x \in U(A)\), then there exists \(a \in A\) such that \((a, x) \in U\). Thus \(G \cap A \neq \emptyset\). Hence \(x \in \bar{A}\). This shows that \(\{U : U \in \mathcal{U}_e\}\) is closed.

b) Evidently, \(U(A)\) is open.

On the other hand, by using (a), \(U(A) \cap V(U(A)) \subseteq U(V(A)) \subseteq U(A)\). The last statement is true, because \(U\) is an equivalence relation on \(X\). So, \(U(A)\) is closed.

The following corollary is easily obtained from part (a) of proposition 2.10.

**Corollary.** Let \((X, \mathcal{U}_e)\) be a \(\mathcal{U}\)-equivalence space. A subset \(A\) of \(X\) is dense in \(X\) (w.r.t \(\tau_e\)) if and only if \(U(A) \supseteq X\) for every \(U \in \mathcal{U}_e\).

**CONTINUITY**

In the theory of \(\mathcal{U}\)-equivalence spaces the structure-preserving functions, in the inverse-image sense, are the \(\mathcal{U}\)-equivalently continuous functions, defined as follows.

**Definition.** Let \((X, \mathcal{U}_e), (Y, \mathcal{V}_g)\) be \(\mathcal{U}\)-equivalence spaces, and let \(f : X \rightarrow Y\) be a function. \(f\) is said to be \(\mathcal{U}\)-equivalently continuous if \(f_2^{-1}(V) \in \mathcal{U}_e\) for each \(V \in \mathcal{V}_g\), where \(f_2^{-1}(V) = \{(x, y) \in X \times Y | (f(x), f(y)) \in V\}\).

Clearly the identity function on any \(\mathcal{U}\)-equivalence space \((X, \mathcal{U}_e)\) is \(\mathcal{U}\)-equivalently continuous.

**Definition.** A \(\mathcal{U}\)-equivalence class \(\mathcal{U}_e\) is said to be saturated if \(U \in \mathcal{U}_e\) and \(U \subseteq V\), where \(V\) is an equivalence relation on \(X\), then \(V \in \mathcal{U}_e\). Also, \(\mathcal{U}_e\) is said to be rich if \(X^2 \in \mathcal{U}_e\).

**Proposition.** Let \((X, \mathcal{U}_e)\) and \((Y, \mathcal{V}_g)\) be two \(\mathcal{U}\)-spaces and let \(\mathcal{U}_e\) be saturated.

Then a function \(f : X \rightarrow Y\) is \(\mathcal{U}\)-equivalently continuous, if for each \(V \in \mathcal{V}_g\) there exists \(U \in \mathcal{U}_e\) such that \(f_2(U) \subseteq V\).

**Proof.** The ‘only if’ part of the proposition is a simple consequence of definition 3.1. To prove the ‘if’ part, let \(V \in \mathcal{V}_g\). We will show that \(f_2^{-1}(V) \in \mathcal{U}_e\). If \(U \in \mathcal{U}_e\) and \(f_2(U) \subseteq V\), then \(U \subseteq f_2^{-1}(V)\). Since \(V\) is an equivalence relation on \(Y\), then \(f_2^{-1}(V)\) is an equivalence relation on \(X\). Now since \(\mathcal{U}_e\) is saturated, \(f_2^{-1}(V) \in \mathcal{U}_e\) as asserted.

**Proposition.** Let \((X, \mathcal{U}_e), (Y, \mathcal{V}_g)\) be \(\mathcal{U}\)-equivalence spaces and let \(f : X \rightarrow Y\) be \(\mathcal{U}\)-equivalently continuous function. Then \(f\) is continuous when regarded as a function from topological space \(X\) in to topological space \(Y\).

**Definition.** The \(\mathcal{U}\)-equivalence space \((X, \mathcal{U}_e)\) is said to be \(\mathcal{U}\)-connected if for each \(U \in \mathcal{U}_e\), \(X = \bigcup_{i=1}^{n} U(x_i)\) where \(U^n = U \cup U \cup \ldots \cup U\) (n-times).

For example, the discrete \(\mathcal{U}\)-equivalence space \(X\) is never \(\mathcal{U}\)-connected provided that the underlying set has at least two points. On the other hand, the trivial \(\mathcal{U}\)-equivalence space is always \(\mathcal{U}\)-connected.

**Definition.** The \(\mathcal{U}\)-equivalence space \((X, \mathcal{U}_e)\) is totally bounded if for each \(U \in \mathcal{U}_e\), there exist \(x_1, x_2, \ldots, x_n \in X\) such that \(X = \bigcup_{i=1}^{n} U(x_i)\). For example, the trivial \(\mathcal{U}\)-equivalence space is always totally bounded.

**Definition.** Let \((X, \mathcal{U}_e), (Y, \mathcal{V}_g)\) be \(\mathcal{U}\)-equivalently spaces and \(f : X \rightarrow Y\) be a function. \(f\) is said to be \(\mathcal{U}\)-equivalently open if for each \(U \in \mathcal{U}_e\), there exists \(V \in \mathcal{V}_g\) such that \(V(f(x)) \subseteq f(U(x))\) for all \(x \in X\).

**Proposition.** Let \(f : X \rightarrow Y\) be a \(\mathcal{U}\)-equivalently continuous surjection, where \((X, \mathcal{U}_e)\) and \((Y, \mathcal{V}_g)\) are \(\mathcal{U}\)-equivalence spaces. Moreover let \(X\) be totally bounded. Then so is \(Y\).
Proof. Let \( V \in \vartheta_e \). We claim that there exist \( y_1, y_2, \ldots, y_n \in Y \) so that \( Y = \bigcup_{i=1}^n V(y_i) \).

Suppose \( U = f_2^{-1}(V) \), then \( U \in \vartheta \), because \( f \) is \( \mathcal{U} \)-equivalently continuous.

Since \( X \) is totally bounded, then there exist \( x_1, x_2, \ldots, x_n \in X \) such that \( X = \bigcup_{i=1}^n U(x_i) \).

If \( y_i = f(x_i) \), then we will show that \( Y = \bigcup_{i=1}^n V(y_i) \). Let \( y \in Y \). Since \( f \) is surjective, then \( y = f(x) \) for some \( x \in X \).

For \( i = 1, 2, \ldots, n \), let \( (x, x) \in U = f_2^{-1}(V) \), then \((f(x_i), f(x)) \in V \). Hence \( Y = \bigcup_{i=1}^n V(y_i) \), as asserted. \( \square \)

**Proposition.** Let \( f : X \to Y \) be a \( \mathcal{U} \)-equivalently continuous surjection, where \( (X, \vartheta) \) and \( (Y, \theta) \) are \( \mathcal{U} \)-spaces. If \( X \in \mathcal{U} \)-connected, then \( Y \in \mathcal{U} \)-connected.

**Proof.** Let \( V \in \vartheta \). Since \( f \) is surjection, then so is \( f_2 \). Since \( f \) is \( \mathcal{U} \)-equivalently continuous, then \( U = f_2^{-1}(V) \in \vartheta \). So, \( Y = f_2(X^2) = f_2(\bigcup_{i=1}^n U^n) = \bigcup_{i=1}^n f_2(U^n) = \bigcup_{i=1}^n V^n \).

Hence, \( Y \in \mathcal{U} \)-connected and the proof is now complete. \( \square \)

**Proposition 3.10.** Let \( (X, \mathcal{U}_e) \), \( (Y, \vartheta_e) \) and \( (Z, \mathcal{W}_e) \) be \( \mathcal{U} \)-equivalence spaces and \( f : X \to Y \) be a \( \mathcal{U} \)-equivalently continuous surjection and let \( g : Y \to Z \) be a function.

If \( g \circ f : X \to Z \) is \( \mathcal{U} \)-equivalently open then so is \( g \).

**Proof.** Let \( V \in \vartheta_e \) and \( U = f_2^{-1}(V) \). Since \( f \) is \( \mathcal{U} \)-equivalently continuous, then \( U \in \mathcal{U}_e \). Moreover, since \( f \) is \( \mathcal{U} \)-equivalently open, then there exists \( W \in \mathcal{W} \) such that \( W(g \circ f)(x) \subseteq (g \circ f)(U(x)) \) for all \( x \in X \). We claim that \( W(g)(y) \subseteq g(V(y)) \) for all \( y \in Y \). To see this, let \( y \in Y \) and \( z \in W(g)(y) \). Since \( f \) is surjection, then \( y = f(x) \) for some \( x \in X \). So \( W(g)(y) \subseteq (g \circ f)(U(x)) \) (I).

Hence by (I), there exists \( x_i \in X \) such that \((x, x_i) \in U, z = g(f(x_i)) \). Let \( t = f(x_i) \). Then \( z = g(t), (y, t) \in V \) i.e. \( z \in g(V(y)) \) as required. \( \square \)

Let us present another classification of saturated \( \mathcal{U} \)-connected spaces as follows.

**Theorem.** In a saturated \( \mathcal{U} \)-equivalence space \((X, \vartheta)\) the following statements are equivalent:

1) \( X \) is \( \mathcal{U} \)-connected

2) for each discrete space \( D \), every \( \mathcal{U} \)-equivalently continuous function \( \lambda : X \to D \) is constant.

**Proof.** (1) \( \Rightarrow \) (2). Given a \( \mathcal{U} \)-equivalently continuous function \( \lambda : X \to D \)

\( \lambda(x) = 0 \) when \((x, x) \in D \) for some \( i \) and \( \lambda(x) = 1 \) otherwise. Hence \( \lambda(x) \) is constant. \( \lambda \) is \( \mathcal{U} \)-equivalently continuous.

We first show that \( U \subseteq \lambda_2^{-1}(\Delta_D) \). If this is not so, then there exists \((x, x) \in U \), \( \lambda(x) \neq \lambda(x) \). Assume that \( \lambda(x) = 0, \lambda(x) = 1 \). Hence there exists \( i \in N \), \((x, x) \in U^i \). Consequently, \((x, x) \in U^{n+1} \) contradicting that \( \lambda(x) = 1 \). Hence \( U \subseteq \lambda_2^{-1}(\Delta_D) \) So for each \( V \in \mathcal{U}_D \), \( \lambda_2^{-1}(V) \supseteq \lambda_2^{-1}(\Delta_D) \) \( \subseteq \mathcal{U} \). Hence \( \mathcal{U}_e \) is saturated.
Hence \( \lambda \) is \( \mathcal{U} \)-equivalently continuous function while it is not constant, that is a contradiction. This proves that \( X \) is \( \mathcal{U} \)-connected.

We omit the straightforward proof of the following proposition.

**Proposition.** Let \((X, \mathcal{U}_e), (Y, \vartheta_y)\) be \( \mathcal{U} \)-equivalence spaces where \( \vartheta_y \) is saturated. Then a bijection \( f : X \to Y \) is \( \mathcal{U} \)-equivalently open if and only if its inverse is \( \mathcal{U} \)-equivalently continuous.

**Proof.** Let \( U \in \mathcal{U}_e \). Then there exists \( W \in \vartheta_y \), \( W(h(x)) \subseteq h(U(x)) \) for all \( x \in X \) where \( h = g \circ f \). Since \( g \) is \( \mathcal{U} \)-equivalently continuous, then the pre-image \( V = g^{-1}_2(W) \) is a member of \( \vartheta_y \). Now it is easy to see \( V(f(x)) \subseteq f(U(x)) \) for all \( x \in X \). It follows that \( f \) is \( \mathcal{U} \)-equivalently open as asserted.

**Proposition.** Let \( f : X \to Y \) be a \( \mathcal{U} \)-equivalently open function, where \( X \) is non-empty, \((X, \mathcal{U}_e)\) is rich and \((Y, \vartheta_y)\) is \( \mathcal{U} \)-connected. Then \( f \) is surjection.

**Proof.** Let \( U \subseteq X^2 \). Then there exists \( V \in \vartheta_y \) such that \( V(f(x)) \subseteq f(U(x)) \) for all \( x \in X \). Consequently, \( V(f(x)) \subseteq f(X) \) for all \( x \in X \). Hence for each \( n \) and each \( x \in X \), \( V^n(f(x)) \subseteq f(X) \). Let \( x_0 \in X \) and let \( y_0 = f(x_0) \). We claim \( Y = f(X) \).

To see this, let \( y \in Y \), then \((y_0, y) \in Y^2 = \bigcup_{n=1}^{\infty} V^n \). Hence, \( y \in V^n(f(x_0)) \) for some \( n \). Since \( V^n(f(x_0)) \subseteq f(X) \), then \( y \in f(X) \). This proves \( Y = f(X) \).

**Definition.** Let \( f : X \to Y \) be a map where \((X, \mathcal{U}_e)\) is a \( \mathcal{U} \)-equivalence space and \( Y \) is a set. We say that \( f \) is transverse to \( X \) if there exists \( U \in \mathcal{U}_e \) such that \( U \cap f^{-1}_2(D_X) = D_Y \). By a local \( \mathcal{U} \)-equivalence we mean, a \( \mathcal{U} \)-equivalently continuous and \( \mathcal{U} \)-equivalently open function \( f : X \to Y \), where \((X, \mathcal{U}_e)\) and \((Y, \vartheta_y)\) are \( \mathcal{U} \)-equivalence spaces such that \( f \) is transverse to \( X \).

**Proposition.** Let \( f : X \to Y \) be a \( \mathcal{U} \)-equivalently continuous function. Suppose \( f \) admits a left inverse \( g \) which is local \( \mathcal{U} \)-equivalence. Then \( f \) is \( \mathcal{U} \)-equivalently open.

**Proof.** Let \( U \in \mathcal{U}_e \). Then \( V_1 \subseteq g^{-1}_2(U) \in \vartheta_y \) because \( g \) is \( \mathcal{U} \)-equivalently continuous. Since \( g \) is transverse to \( Y \), then there exists \( V_2 \in \vartheta_y \) such that \( V_2 \cap g^{-1}_2(D_X) = D_Y \). Let \( V_2 = (f \circ g)_2^{-1}(V_0) \). Then since \( f \circ g \) is \( \mathcal{U} \)-equivalently continuous, \( V_2 \in \vartheta_y \). Finally let \( V = V_0 \cap V_1 \cap V_2 \). We claim \( V(f(x)) \subseteq f(U(x)) \) for all \( x \in X \). Suppose \( y \in V(f(x)) \). Then \( (x, g(y)) \in U \). Finally, we have to show that \( f(g(y)) = y \).

Since \( (g(y), g(y)) \in D_Y \), then \( f(g(y), y) \in g^{-1}_2(D_X) \).

Also, \( (y, f(x)) \in V_0 \) and \((f(x), f(g(y))) \in V_0 \). Hence, \((f(g(y)), y) \in V_0 \). Consequently, \((f(g(y)), y) \in D_Y \) that means, \( f(g(y)) = y \).

**Proposition.** Let \( f : X \to Y \) and \( g : Y \to Z \) be \( \mathcal{U} \)-equivalently continuous functions, where \((X, \mathcal{U}_e), (Y, \vartheta_y)\) and \((Z, \vartheta_z)\) are \( \mathcal{U} \)-equivalence spaces \( g \circ f \) is \( \mathcal{U} \)-equivalently open, \( f \) is injective and \( g \) is transverse to \( Y \). The \( g \circ f \) is a local \( \mathcal{U} \)-equivalence.

**Proof.** Since \( g \) is transverse to \( Y \), there exists \( V \in \vartheta_y \) such that \( V \subseteq g^{-1}_2(D_X) = D_Y \). Let \( U = f^{-1}_2(V) \). Then \( U \in \mathcal{U}_e \).

Now we have to show that \( U \cap ((g \circ f)^{-1}_2(D_Z)) = D_X \).

Clearly, \( D_X \subseteq U \cap ((g \circ f)^{-1}_2(D_Z) \). For the other way inclusion, let \((x_1, x_2) \in U \) and \( g(f(x_1)) = g(f(x_2)) \). Then \((f(x_1), f(x_2)) \in V \). Since \( f(x_1) = f(x_2) \) and since \( f \) is injective, \( x_1 = x_2 \). Hence \( U \cap ((g \circ f)^{-1}_2(D_Z) = D_X \).

**QUOTIENT \( \mathcal{U} \)-EQUIVALENCE SPACES**

Let \((X, \mathcal{U}_e)\) be a \( \mathcal{U} \)-equivalence space and let \( \mathcal{R} \) be an equivalence relation on \( X \).
Also, let \( \pi : X \to X/\mathcal{R} \) is the function defined by \( \pi(x) = \mathcal{R}[x] \), where \( \mathcal{R}[x] = \{ y \in X \mid (x, y) \in \mathcal{R} \} \). The function \( \pi \) is called the natural projection.

Now we ask whether \( X/\mathcal{R} \) can inherit a \( U \)-equivalence class from \( X \) such that makes the natural projection \( \pi \) \( U \)-equivalently continuous, and if the answer is yes, then we discuss the relationships between these two spaces.

**Definition.** An equivalence relation \( \mathcal{R} \) on a \( U \)-equivalence space \( (X, \mathcal{U}_e) \) is compatible with \( \mathcal{U}_e \) if for each \( U \in \mathcal{U}_e \), \( \mathcal{R} \circ U = U \)

For example, let \( X \) be a non-empty set and \( \mathcal{R} = \Delta_X \) Then \( \mathcal{R} \) is compatible with \( \{ X^2 \} \).

The following lemma is often useful.

**Lemma.** Let \( \mathcal{R} \) be an equivalence relation on a \( U \)-equivalence space \( (X, \mathcal{U}_e) \). Then the following statements are equivalent:

i) \( \mathcal{R} \) is compatible with \( \mathcal{U}_e \).

ii) For each \( U \in \mathcal{U}_e \), \( U \circ \mathcal{R} = U \).

iii) For each \( U \in \mathcal{U}_e \), \( \mathcal{R} \circ U \circ \mathcal{R} = U \).

iv) For each \( U \in \mathcal{U}_e \), \( U \circ \mathcal{R} \circ U = U \).

**Proof.** The equivalence of (i) with (ii) is trivial.

Assume (ii) holds and suppose \( U \in \mathcal{U}_e \). Then \( U \circ \mathcal{R} = U \) and hence \( \mathcal{R} \circ U \circ \mathcal{R} = \mathcal{R} \circ U \). Since \( \mathcal{U} \circ \mathcal{R} = U \), then the equivalence of (i) with (ii) implies \( \mathcal{R} \circ U = U \). Hence \( \mathcal{R} \circ U \circ \mathcal{R} = U \). The other parts result by straightforward calculations.

**Theorem.** Let \( \mathcal{R} \) be a compatible equivalence relation on a \( U \)-equivalence space \( (X, \mathcal{U}_e) \). Then the images of the members of \( \mathcal{U}_e \) under \( \pi_2 \), form a \( U \)-equivalence class on \( X/\mathcal{R} \). We refer to this class as the quotient \( U \)-equivalence class and to \( X/\mathcal{R} \) with this structure, as the quotient \( U \)-equivalence space.

We recall that \( X/\mathcal{R} \) is the collection of all equivalence classes \( \mathcal{R}[X] \), and \( \pi_2(x, y) = \pi([x], [y]) \)

**Proof.** Let \( U_e^\mathcal{R} \) denotes this collection i.e. \( U_e^\mathcal{R} = \{ \pi_2(U) \mid U \in \mathcal{U}_e \} \).

We first show each member of \( U_e \) is an equivalence relation on \( X/\mathcal{R} \). Let \( V \in \pi_2(U) \), where \( U \in \mathcal{U}_e \) and let \( x \in X \).

Then \( (\mathcal{R}[x], \mathcal{R}[x]) \in \pi_2(x, x) \) and \( (x, x) \in \Delta_X \subseteq U \). Hence \( \Delta_X \subseteq \mathcal{R} \subseteq V \) and so \( V \) is reflexive. Clearly \( V \) is symmetric. Now we show that \( V \) is transitive.

Let \( (\mathcal{R}[x], \mathcal{R}[y]) \in V \) and let \( (\mathcal{R}[y], \mathcal{R}[z]) \in V \). Then \( (\mathcal{R}[x], \mathcal{R}[y]) \in (\mathcal{R}[t_1], \mathcal{R}[t_2]) \), \( (t_1, t_2) \in U \). Also \( (\mathcal{R}[y], \mathcal{R}[z]) \in (\mathcal{R}[u_1], \mathcal{R}[u_2]) \), \( (u_1, u_2) \in U \).

Hence \( \mathcal{R}[x], \mathcal{R}[z] \in (\mathcal{R}[t_1], \mathcal{R}[u_2]) \). Since \( (t_1, t_2) \in U \), \( (t_2, u_1) \in \mathcal{R} \) and \( (u_1, u_2) \in U \), then \( (t_1, u_2) \in \mathcal{R} \). Now compatibility of \( \mathcal{R} \) with \( \mathcal{U}_e \), implies \( (t_1, u_2) \in U \). Hence \( \mathcal{R}[x], \mathcal{R}[z] \in \pi_2(U) \), \( (t_1, u_2) \in U \) and \( \mathcal{R}[x], \mathcal{R}[z] \in \pi_2(U) \).

So \( (\mathcal{R}[x], \mathcal{R}[z]) \in \pi_2(U) \) and \( V \) is transitive.

Finally, We show that the intersection of two members of \( U_e^\mathcal{R} \) is a member of \( U_e \). Let \( V_1 = \pi_2(U_1) \) and \( V_2 = \pi_2(U_2) \), where \( U_1, U_2 \in \mathcal{U}_e \), be two members of \( U_e^\mathcal{R} \).

We contend that \( V_1 \cap V_2 = \pi_2(U_1 \cap U_2) \) which shows that \( V_1 \cap V_2 = \pi_2(U_1 \cap U_2) \).

Clearly \( \pi_2(U_1 \cap U_2) \subseteq \pi_2(U_1) \cap \pi_2(U_2) \). Now let \( (\mathcal{R}[x], \mathcal{R}[y]) \in \pi_2(U_1) \cap \pi_2(U_2) \).

Then \( (\mathcal{R}[x], \mathcal{R}[y]) \in (\mathcal{R}[t_1], \mathcal{R}[t_2]) \), \( (t_1, t_2) \in U_1 \)

\( (\mathcal{R}[u_1], \mathcal{R}[u_2]) \), \( (u_1, u_2) \in U_2 \).

Consequently, \( (\mathcal{R}[x], \mathcal{R}[y]) \in (\mathcal{R}[t_1], \mathcal{R}[u_2]) \). But \( (t_1, u_2) \in \mathcal{R} \) when \( \pi_2(U_2) \) and \( (t_1, u_2) \in U_1 \). Hence \( \mathcal{R}[x], \mathcal{R}[y] \in (\mathcal{R}[t_1], \mathcal{R}[u_2]) \), \( (t_1, u_2) \in U_1 \) and \( U_2 \). So \( (\mathcal{R}[x], \mathcal{R}[y]) \in \pi_2(U_1 \cap U_2) \). Hence, \( \pi_2(U_1 \cap U_2) \subseteq \pi_2(U_1) \cap \pi_2(U_2) \).

**Theorem.** Let \( \mathcal{R} \) be an equivalence relation on \( X \), compatible with \( \mathcal{U}_e \) where \( (X, \mathcal{U}_e) \) is a \( U \)-equivalence space. Then \( \pi \) is \( U \)-equivalently continuous and \( U \)-equivalently open.
Proof. We first show that $\pi$ is $\mathcal{U}$-equivalently open. Let $U \in \mathcal{U}_e$ and $V = \pi_2(U)$. Then $V \in \mathcal{U}_e$. We claim that $V(\pi(x)) \subseteq \pi(U(x))$ for all $x \in X$.

Let $x \in X$ and let $\mathcal{R}[t] \subseteq V(\pi(x))$. We will show there exists $u \in X$ such that $\mathcal{R}[t] \subseteq \mathcal{R}[u]$ and $(x, u) \in U$. Since $\mathcal{R}[t] \subseteq V(\pi(x))$, then there exists $(t_1, t_2) \in U$ such that $\mathcal{R}[x] \mathcal{R}[t] (\mathcal{R}[t_1], \mathcal{R}[t_2])$. Hence $\mathcal{R}[t] \subseteq \mathcal{R}[t_2]$ and $(x, t_2) \in \mathcal{R} \circ U \circ \mathcal{R} = U$. Let $u = t_2$. Then $\mathcal{R}[t] = \mathcal{R}[u]$ and $(x, u) \in U$ as required.

Now we prove that $\pi$ is $\mathcal{U}$-equivalently continuous. Let $V \in \mathcal{U}_e$. We show that $\pi_2^{-1}(V) \in \mathcal{U}_e$. There exists $U \in \mathcal{U}_e$ such that $V = \pi_2(U)$. On one hand we have $\pi_2^{-1}(V) \subseteq \pi_2^{-1}(\pi_2(U)) \supseteq U$. On the other hand, if $(x_1, x_2) \in \pi_2^{-1}(V)$, then $\mathcal{R}[x_1] \mathcal{R}[x_2] (\mathcal{R}[t_1], \mathcal{R}[t_2])$, $(t_1, t_2) \in U$. Hence $(x_1, x_2) \in \mathcal{R} \circ U \circ \mathcal{R} = U$.

So $\pi_2^{-1}(V) \subseteq U$. And hence $\pi_2^{-1}(V) = U \in \mathcal{U}_e$.

Acknowledgment
The authors declare that they have no conflicts of interest in the research.

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