A Recursive Algorithm for Solving Simple Stochastic Games

Xavier Badin de Montjoye
Université Paris Saclay, UVSQ, DAVID
xavier.badin-de-montjoye2@uvsq.fr

Abstract
We present two recursive strategy improvement algorithms for solving simple stochastic games. First we present an algorithm for solving SSGs of degree $d$ that uses at most $O\left(\left\lfloor \left(\frac{d+1}{2}\right)^{n/2}\right\rfloor\right)$ iterations, with $n$ the number of MAX vertices. Then, we focus on binary SSG and propose an algorithm that has complexity $O(\varphi^{n\text{Poly}(N)})$ where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. To the best of our knowledge, this is the first deterministic strategy improvement algorithm that visits $2^c n$ strategies with $c < 1$.

2012 ACM Subject Classification Theory of computation → Algorithmic game theory

Keywords and phrases Simple Stochastic Games, Strategy Improvement, Parametrized Complexity, Recursif algorithm

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Acknowledgements We would like to thanks David Auger, Pierre Coucheney and Yann Strozecki for sharing their insights on SSGs.

1 Introduction

Simple stochastic games (SSG) are a restriction introduced by Condon [6, 7] of the notion of stochastic games defined by Shapley [12]. An SSG is a turn-based zero-sum game with perfect information played by two players named MAX and MIN. A token is placed on a directed graph and moves alongside the graph arcs. The set of vertices is partitioned in MAX, MIN, Random and Sink vertices. If the token is in a MAX or MIN vertices, the corresponding player chooses its next position in the outneighbourhood of the currently occupied vertex. In the case of random vertices, the token is moved randomly. Finally, when a sink vertex $s$ is reached, the game ends and player MIN must pay some penalty value $\text{Val}(s)$ to player MAX. The goal of MIN is to minimise the expected penalty, and the goal of MAX is to maximise it. One reason to study SSGs and their related complexity is that the stochastic versions of classical games such as parity games, mean and discounted payoff are all equivalent to SSG [1].

Simple stochastic games admit a pair of optimal strategies [7] whose expected value satisfies a Nash equilibrium. Our goal is to compute such a pair of optimal strategy. This problem is known to be in PPAD [10] a subset of the class FNP, but is not known to be in FP. SSG have several applications, such as modelling autonomous urban driving [11] or in the domain of model checking of modal $\mu$-calculus [13].

We note $n$ the number of MAX vertices, $N$ the number of total vertices and $r$ the number of random vertices. Every known algorithm in the literature for solving SSGs has exponential time complexity bounds. There exist several FPT algorithms that can be used for specific families of SSGs. For instance, for SSGs with random vertices of degree 2 and uniform probability distribution, Ibsen-Jensen and Milterson gives a value iteration algorithm in [9] with complexity $O(r^2(r \log(r)) + N))$.

One of the main families of algorithms for solving SSGs are strategy improvement algorithms. In the case of SSG with MAX vertices of outdegree exactly 2, Tripathi, Valkanova...
and Kumar present in [14] an algorithm with $O(2^n/n)$ iterations, which is the only current bounds that deterministically improve the trivial $2^n$ iterations of checking every possible strategies of player MAX. Ludwig offers a randomised algorithm in [11], which does $2^{O(\sqrt{n})}$ iterations on average. In this paper, we provide a deterministic algorithm for the same family of SSGs that has complexity $O(2^{cn\text{ Poly}(N)})$ for some $c<1$. Moreover, we will provide the first deterministic algorithm of this family with parametrised complexity in the degree $d$ of MAX vertices and number of MAX vertices, improving the trivial bound of $O(d^n \text{ Poly}(N))$ to $O\left(((d + 1)/\sqrt{2})^n \text{ Poly}(N)\right)$.

In the general case, Gimbert and Horn give in [8] a strategy improvement algorithm whose complexity is a function of the random vertices with an algorithm of complexity $O(r! \text{ Poly}(N))$. Moreover, in [3], Auger Coucheney and Strozecki present a stochastic algorithm that runs in $2^{O(r)}$.

Auger, Badin de Montjoye and Strozecki present in [2] a general formulation for strategy improvement algorithms that offers a general bound on the complexity of all such algorithms depending on the format of the probability value of the random nodes. It states that if there is some $q$ such that all probabilities are of the format $p/q$ and if each iteration of the algorithm is done in polynomial time, then any strategy improvement algorithm runs in $O(nq^r \text{ Poly}(N))$.

Contributions
In this paper, we focus on SSGs whose MAX vertices have outdegree $d$ with no constraints on the probability distribution of random vertices. We introduce two new recursive algorithms to solve SSGs. The first one, in Section 4, fixes the strategies on two vertices and recursively solves the rest of the SSG. This algorithm has bound $O\left(\left(\frac{(d + 1)^2}{2} - 1\right)^{n/2} \text{ Poly}(|G|)\right)$.

The second algorithm presented in Section 5 works only for SSG of degree 2. However, it achieves a better bound that the $O(\sqrt{3} \text{ Poly}(N))$ of the first algorithm by reaching $O(\varphi^n \text{ Poly}(N))$ with $\varphi = (1 + \sqrt{5})/2$ the golden ratio, which is, to the best of our knowledge, the best complexity for deterministic algorithm on this family of SSGs. Moreover, our algorithm does not require SSGs to be stopping, a common technical hypothesis, which may require a squarring of the number of vertices to be met.

2 An Overview of Simple Stochastic Games

Simple Stochastic Games where introduce by Anne Condon in [6]. We give a definition close to the one given in [14] [2].

Definition 1. A Simple Stochastic Game (SSG) is a directed graph $G = (V, E)$ with a partition of the vertex set $V$ in $V_{\text{MAX}}, V_{\text{MIN}}, V_{\text{R}}$ and $V_{\text{S}}$ respectively called, MAX, MIN, random and sink vertices such that:

- every vertex of $V_{\text{MAX}}$ and $V_{\text{MIN}}$ has outdegree at least two.
- every vertex $x$ of $V_{\text{R}}$ has outdegree at least one, and an associated rational probability distribution $p_x(\cdot)$ on the outneighbourhood of $x$.
- every vertex $x \in V_{\text{S}}$, there is an associated rational value $\text{Val}(x)$ in the closed interval $[0, 1]$.

Definition 2. A binary SSG is an SSG where every vertex of $V_{\text{MAX}}$ has outdegree 2. An SSG is of degree $d$ if its MAX vertices are of degree at most $d$. 
In this article we denote $|V_{\text{MAX}}|$ by $n$. We will write $|G|$ the size of the representations of the game $G$ in bits. We present an instance of an SSG in Figure 1.

The game is played by two players named MAX and MIN. The game starts by placing a token on some initial vertex $x_0$. Then, the token is moved according to the following rule. If the token is in a MAX or a MIN vertex $x$, then the corresponding player moves the token according to an outgoing arc from $x$. If the token is in a random vertex $x$, then the token is moved according to the probability distribution $p_x$. When the token reaches a sink $s$, then player MIN has to pay player MAX the value $\text{Val}(x)$. Informally, the goal of the game for MAX is to maximise the value of the final sink and, conversely, the goal of the game for MIN is to minimise it.

This game is turn-based, with perfect information. The strategies consider by both players should thus be deterministic and only relies on the current position of the token. This is a well-known result on simple stochastic games and a proof of this can be found in [6, 14]. We thus only consider positional strategies.

**Definition 3.** A positional MAX strategy is a function $\sigma$ from $V_{\text{MAX}}$ to $V$ such that for all $x$ in $V_{\text{MAX}}$, $(x, \sigma(x))$ is an arc of $G$.

A positional MIN strategy is a function $\tau$ from $V_{\text{MIN}}$ to $V$ such that for all $x$ in $V_{\text{MIN}}$, $(x, \tau(x))$ is an arc of $G$.

In this paper when we talk about a pair of strategies $(\sigma, \tau)$, $\sigma$ is a positional MAX strategy and $\tau$ is a positional MIN strategy. We can now define the value vector of a pair of strategy.

**Definition 4.** For $(\sigma, \tau)$ a pair of strategies, and $x_0$ a vertex of $V$, the value $v_{\sigma,\tau}(x_0)$ is the expected gain for MAX if both players play according to $\sigma$ and $\tau$. In other words:

$$v_{\sigma,\tau}(x_0) = \sum_{s \in V_S} P_{\sigma,\tau}(x_0 \rightarrow s) \text{Val}(s)$$

where $P_{\sigma,\tau}(x_0 \rightarrow s)$ is the probability that the game ends in $s$ while starting in $x_0$ and such that when the token is in a MAX vertex (resp. MIN vertex) $x$ it moves to $\sigma(x)$ (resp. $\tau(x)$).
The value vector $v_{\sigma, \tau}$ is the vector $(v_{\sigma, \tau}(x))_{x \in V}$. We compare value vectors according to the pointwise order. For two value vector $v$ and $v'$, $v > v'$ if, for all $x$, $v(x) \geq v'(x)$ and there is some $y \in V$ such that $v(y) > v'(y)$. As usual, $v \geq v'$ if $v > v'$ or if $v = v'$.

Computing the value vector of a pair of strategies is equivalent to solving a Markov chain which can be done in time polynomial in $|G|$.

For a MAX strategy $\sigma$ we say that a MIN strategy $\tau$ is a best response for $\sigma$ if and only if, for every MIN strategy $\tau'$, we have $v_{\sigma, \tau} \leq v_{\sigma, \tau'}$. It is also well known that best response exists.

**Proposition 5 ([6]).** For every positional MAX strategy $\sigma$, there exists a positional MIN strategy $\tau$ that is a best response for $\sigma$. Moreover, a best response can be computed in polynomial time by linear programming.

In the same way, we can define the best response for a MIN strategy.

**Definition 6.** For every MAX strategy $\sigma$, we write $v_{\sigma} = v_{\sigma, \tau(\sigma)}$ where $\tau(\sigma)$ is a best response to $\sigma$.

We say that $\sigma$ is better than $\sigma'$ or has greater value, or we note $\sigma > \sigma'$ if $v_{\sigma} > v_{\sigma'}$.

It is well known in the literature ([6] [14]) that there is a pair of positional strategies $(\sigma^*, \tau^*)$ that are called optimal strategies such that they are best response of each other. The value vector equilibrium $v_{\sigma^*, \tau^*}$ is unique. Our goal is to compute this value vector. A known characteristic of a pair of optimal strategy is that its associated value vector satisfies local optimality.

**Proposition 7 ([6]).** Let $G = (V, E)$ be an SSG. For $\sigma$ a MAX strategy, $(\sigma, \tau(\sigma))$ is a pair of optimal strategy if and only if $v_{\sigma}$ satisfies the following local optimality condition:

- for $x \in V_{\text{MAX}}$, $v_{\sigma}(x) = \max \{v_{\sigma, \tau}(y) \mid (x, y) \in E\}$
- for $x \in V_{\text{MIN}}$, $v_{\sigma}(x) = \min \{v_{\sigma, \tau}(y) \mid (x, y) \in E\}$

It is important to notice that the converse of the proposition is true only because we consider the best response to a MAX strategy. If we consider a MIN strategy and its best response, then it does not hold anymore because of possible loops.

### 3 Switch set and super-switch

#### 3.1 Switch set

In this section we present an important concept for strategy improvement algorithm: the switch set. We present several properties of the switch set for SSG that we use to prove the complexity of our algorithms.

**Definition 8.** Let $G = (V, E)$ be an SSG and $\sigma$ a MAX strategy. The switch set of $\sigma$, written $S_\sigma$, is the set of vertices $x$ such that there is $y$ with $(x, y) \in E$ and $v_{\sigma}(x) < v_{\sigma}(y)$.

**Definition 9.** Let $\sigma$ be a MAX strategy. For all $x \in S_\sigma$, the improvement set, written $IS_\sigma(x)$ is the set of neighbours $y$ of $x$ such that $v_{\sigma}(y) > v_{\sigma}(x)$ and the best improvement option is defined as $\text{bio}_\sigma(x) = \arg \max_{y \in IS_\sigma(x)} \{v_{\sigma}(y)\}$.

In other words, the switch set of a MAX strategy is the set of MAX vertices that do not satisfy the local optimality condition presented Proposition 7. This notion directly gives the concept of $\sigma$-switch.
Definition 10. Let $\sigma$ be a MAX strategy with a non-empty switch set $S_\sigma$. A MAX strategy $\sigma'$ is said to be a $\sigma$-switch if $\sigma' \neq \sigma$, for all $x \in V_{\text{max}} \setminus S_\sigma$, $\sigma(x) = \sigma'(x)$ and for all $x \in S_\sigma$ such that $\sigma'(x) \neq \sigma(x)$, $\sigma'(x) \in 1S_\sigma(x)$.

A $\sigma$-switch is a strategy, where the strategy on vertices that satisfy local optimality for $\sigma$ has been kept and it had been changed on some vertices that did not satisfy local optimality. Informally, as the name implies, it is a strategy where we "switch" the strategy on some vertices that can achieve immediate better value by selecting another child. A representation of a switch is given in Figure 3.

Definition 11. For $\sigma$ a MAX strategy, the total switch $\bar{\sigma}$ is the $\sigma$-switch where for all $x \in S_\sigma$, $\bar{\sigma}(x) = \text{bio}_\sigma(x)$.

Proposition 12 ([14]). For $\sigma$ a MAX strategy and $\sigma'$ a $\sigma$-switch, $v_{\sigma'} > v_\sigma$.

The demonstration of this proposition uses the condition that the SSG is stopping, which means that the SSG ends in a sink with probability 1 and this for any pair of strategy. However, it had been shown that this condition is unnecessary [4, 2].

Corollary 13. A positional MAX strategy $\sigma$ is optimal if and only if $S_\sigma$ is empty.

This property directly gives a family of algorithms called Hoffmann-Karp algorithms. Starting from some MAX strategy $\sigma$, compute $v_\sigma$ then, if $S_\sigma$ is not empty, choose a $\sigma$-switch and iterate. Since the number of strategies is bounded by $d^n$ where $d$ is the degree of $G$, this algorithm terminates and provides an optimal MAX strategy. Tripathi, Valkanova and Kumar shows in [13] that for binary SSG, if at each iteration, we consider the $\sigma$-switch $\bar{\sigma}$, then the algorithm needs $O(2^n/n)$ iterations. We will show in Theorem 19 an extended version of their main result that we use to find a bound of the Algorithm 1 presented in Section 4.

For $T$ a set of MAX vertices, and $\sigma$ a MAX strategy, we define the subgame $G_{|T|\sigma}$ as the game $G$ where some MAX vertices have been replaced with random vertices that go to some vertex with probability one according to the MAX strategy $\sigma$. In other words, $G_{|T|\sigma}$ is the game $G$ where MAX has to play as in $\sigma$ in $T$.

Definition 14. Let $G = (V, E)$ an SSG. For $\sigma$ a strategy and $T$ a subset of $V_{\text{max}}$ we write $G_{|T|\sigma} = (V, E')$ the SSG that is a copy of $G$ where $E' = E \setminus \{(x, y) \mid x \in T, y \neq \sigma(x)\}$ and all vertices $x$ of $T$ are random vertices, with associated probability distribution $p_x(\sigma(x)) = 1$.

We provide an example of this transformation in Figure 2.

There is a bijection between the strategy of $G_{|T|\sigma}$ and the strategy of $G$ that plays has in $\sigma$ in $T$. In the rest of the paper, we identify those two sets of strategy. Moreover, for a strategy $\sigma'$ that plays as in $\sigma$ in $T$ and any MIN strategy $\tau$, there is equality between the value vectors $v_{\sigma', \tau}$ in $G$ and $G_{|T|\sigma}$.

Lemma 15 ([14]). For any MAX strategy $\sigma$, $\sigma$ is optimal in $G_{|S_\sigma|\sigma}$.

Proof. The switch set of $\sigma$ in $G_{|T|\sigma}$ is empty, hence, by Corollary 13, $\sigma$ is optimal in $G_{|T|\sigma}$.

The following proposition also appears in [14] and it allows us to study strategies by just looking at their switch set.

Proposition 16. Let $G$ be a binary SSG. For $\sigma$ and $\sigma'$ two MAX strategies such that $S_\sigma \subsetneq S_{\sigma'}$, then $v_\sigma > v_{\sigma'}$.
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3.2 Super-Switch

In this section, we will extend the classic notion of switch in order to add a perfect resolution on some of its vertices.

A super-switch is obtained by switching some vertices of \( \sigma \), fixing the strategy on a set of \( \text{MAX} \) vertices that include the switched vertices and then considering the optimal strategy of the subgame.

Let \( \sigma \) be a \( \text{MAX} \) strategy with a non-empty switch set \( S_{\sigma} \) and \( T \subset V_{\text{MAX}} \) with \( T \cap S_{\sigma} \neq \emptyset \). A \((\sigma, T)\)-super-switch is a strategy \( \tilde{\sigma} \) obtained from an intermediate \( \sigma \)-switch \( \sigma' \) such that \( \forall x \notin T, \sigma(x) = \sigma'(x) \) by computing an optimal strategy of \( G_{|T|}[\sigma'] \).

We give a representation of a super-switch in Figure 3.

\[ \text{Figure 2} \] Transformation of the game \( G \) in the game \( G_{T[\sigma]} \) where \( T = \{x_1, x_3, x_4\} \) and \( \sigma(x_1) = 1 \), \( \sigma(x_3) = x_2 \) and \( \sigma(x_4) = r2 \). The probability distribution on the random vertices is the uniform distribution.

\[ \text{Lemma 17.} \] Let \( G \) be a binary SSG. For \( \sigma \) and \( \sigma' \) two \( \text{MAX} \) strategies such that \( S_{\sigma} \subseteq S_{\sigma'} \), then \( \sigma' \) is not optimal in \( G_{|S_{\sigma}|}[\sigma] \) by Corollary 13 and by Lemma 15, \( v_\sigma > v_{\sigma'} \). Otherwise, we consider \( \sigma'' \) that plays as in \( \sigma' \) in \( V_{\text{MAX}} \setminus S_{\sigma} \) and as in \( \sigma \) in \( S_{\sigma} \). \( \sigma'' \) is a \( \sigma' \)-switch, and a strategy of \( G_{|S_{\sigma}|}[\sigma] \), thus: \( v_{\sigma'} < v_{\sigma''} \leq v_\sigma \)

In the same way, we can also consider the case where both switch sets are equal.

\[ \text{Lemma 18.} \] For \( \sigma \) a non-optimal \( \text{MAX} \) strategy, \( T \) a set of \( \text{MAX} \) vertices and \( \sigma' \) a \((\sigma, T)\)-super-switch, \( v_{\sigma'} > v_\sigma \).
Figure 3 The strategy of max are represented by plain arcs and the probability distribution on the random vertices is the uniform distribution. The switch set of $\sigma$, $S_\sigma$ is $\{x_3, x_4, x_5\}$. The strategy $\sigma'$ is a $\sigma$-switch and $\sigma''$ is a $(\sigma, S_\sigma)$-super-switch.
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Proof. We consider the strategy \( \sigma'' \) that plays as \( \sigma' \) on \( T \) and as \( \sigma \) on \( V_{\text{max}} \setminus T \). The strategy \( \sigma'' \) is a \( \sigma \)-switch and by Proposition 12 \( v_{\sigma''} > v_{\sigma} \). Moreover, \( \sigma' \) is the optimal strategy of \( G_{|T|}^{(\sigma')} \) and \( \sigma'' \) is also a strategy of \( G_{|T|}^{(\sigma')} \). Hence, \( v_{\sigma''} \geq v_{\sigma'} \) and \( v_{\sigma'} > v_{\sigma} \). △

In [19], Tripathi, Valkanova and Kumar, show that for a strategy \( \sigma \), there is at least \( |S| - 1 \) \( \sigma \)-switch \( \sigma' \) different from \( \sigma \) such that \( v_{\sigma'} \leq v_{\sigma} \). In Theorem 19 we adapt their proof to show that this result can be extended to \( \sigma \)-switch.

For \( \sigma \) a \( \text{MAX} \) strategy, \( T \) a set of \( \text{MAX} \) vertices and \( \sigma' \) a \( (\sigma, T) \)-super-switch, we note \( D_{\sigma, \sigma'} \) the set \( \{x \mid \sigma(x) \neq \sigma'(x), x \in T\} \) the set of vertices that has been switched in the intermediate \( \sigma \)-switch.

- **Theorem 19.** For \( \sigma \) a non-optimal \( \text{MAX} \) strategy, \( T \) a set of \( \text{MAX} \) vertices intersecting \( S_\sigma \) and \( \sigma' \) a \( (\sigma, T) \)-super-switch, there is at least \( |D_{\sigma, \sigma'}| \) \( (\sigma, T) \)-super-switches \( \sigma'' \neq \sigma' \) such that \( v_{\sigma''} < v_{\sigma'} \leq v_{\sigma} \).

Proof. First, we show that Theorem 19 is true in the case \( D_{\sigma, \sigma'} = 2 \). Let \( D_{\sigma, \sigma'} = \{x; y\} \). We consider the game \( G' \) which is the game \( G_{|T^{\sigma}} \) where all edges \( (x', y') \) with \( x' \notin \{\sigma(x); \sigma'(x)\} \) and \( y' \notin \{\sigma(y); \sigma'(y)\} \) has been removed. Both \( \sigma \) and \( \sigma' \) are strategies in the game \( G' \). We write \( S_\sigma \) the set of a strategy \( \tilde{\sigma} \) in the game \( G' \) and \( S_\sigma' \) the switch set of a strategy \( \tilde{\sigma} \) in the game \( G' \). Hence, \( S_\sigma' = \{x; y\} \). Let call \( \sigma_x \) and \( \sigma_y \) the \( (\sigma, T) \)-super-switch where respectively only \( x \) and \( y \) has been switched in \( T \).

By Lemma 16 \( S_\sigma' \) is strictly included in \( \{x; y\} \) and thus is at most a singleton. If \( S_\sigma' \) is empty then, \( v_{\sigma} < v_{\sigma_x} \), \( v_{\sigma_y} \leq v_{\sigma'} \). We suppose that \( S_\sigma' = \{x\} \). Then, we notice that \( \sigma_x \) and \( \sigma' \) are both strategies of \( G_{|\{x\}|}^{(\sigma')} \) and \( \sigma' \) is optimal in \( G_{|\{x\}|}^{(\sigma')} \). Thus, \( v_{\sigma} < v_{\sigma_x} \leq v_{\sigma'} \) and \( v_{\sigma_y} > v_{\sigma'} \). This implies that if \( v_{\sigma_y} \neq v_{\sigma}, v_{\sigma} \geq v_{\sigma_x} \) and if we also have \( v_{\sigma_x} \neq v_{\sigma} \) then \( y \in S_\sigma' \).

Now we look at the general case. Let \( D_{\sigma, \sigma'} = \{x, \ldots, t\} \). For \( E \subset \{1, \ldots, t\} \), we write \( v_E \) the value of the \( (\sigma, T) \)-super-switch \( \sigma|_E \) where only the vertices \( x_i \) for \( i \in E \) has been switched. We assume that for all \( i < j \leq t \), \( v_{\{i\}} \neq v_{\{j\}} \). If for all \( j > 1 \), \( v_{\{i\}} \neq v_{\{j\}} \) then \( v_{\{i,j\}} \geq v_{\{1\}} \) and \( j \in S_{\sigma(\{1\})} \).

Otherwise, we suppose that there is \( k > 1 \) such that for all \( 2 \leq j \leq k \), \( v_{\{i\}} = v_{\{j\}} \) and for all \( j > k \), \( v_{\{i\}} \neq v_{\{j\}} \). We consider the game \( G' \) which is the game \( G_{|T^{\sigma}} \) where for all \( i \leq k \), all edges from \( x_i \) not towards \( \sigma(x_i) \) or \( \sigma'(x_i) \) are removed. We note \( S_\sigma' \) the switch sets in \( G' \) and \( \sigma' = \sigma(1: \ldots: k) \). By Lemma 16 \( S_\sigma' \) is strictly included in \( \{1; \ldots; k\} \). We suppose that \( S_\sigma' = \{1; \ldots; k'\} \) for some \( k' < k \). By induction hypothesis, we know that \( v_{\{1\}} \leq v_{\{1; \ldots; k'\}} \) and \( \sigma' \) and \( \sigma(1: \ldots: k') \) are both optimal strategies of \( G_{|\{1; \ldots; k'\}|}^{(\sigma')} \) and thus have same value. Thus, \( v_E \leq v_{\sigma'} \) for all \( E \subset \{1; \ldots; k\} \) and for all \( j > k \), \( x_j \in S_{\sigma'} \).

We conclude by induction on the \( \{x_j; \ldots; t\} \).

Thus, we have that \( v_{\sigma'} = v_{\{1,2; \ldots; t\}} \geq v_{\{1,2, \ldots; t-1\}} \geq \ldots \geq v_{\{1\}} \) and we have proven that there is at least \( |D_{\sigma, \sigma'}| \) \( (\sigma, T) \)-super-switches \( \sigma'' \neq \sigma' \) such that \( v_{\sigma} < v_{\sigma''} \leq v_{\sigma} \). △

We notice that a \( \sigma \)-switch is a \( (\sigma, V_{\text{max}}) \)-super-switch. Hence, Theorem 19 also proves the main theorem of [19].

4 A recursive algorithm with a pair of fixed vertices

In all this section, we only consider SSGs of degree \( d \).

Algorithm 1 works by fixing the strategy of the game on two vertices, then recursively solving the rest of the game. If this does not yield an optimal strategy, then it switches the
Lemma 20. \( \rightarrow \)

Which gives:

Algorithm 1: RecursivePair

\( \begin{align*}
\text{Data: } & G \text{ an SSG} \\
\text{Result: } & \sigma \text{ an optimal MAX strategy and } v \text{ the optimal value vector.} \\
\text{begin} \\
\text{if } |V_{\text{MAX}}| \leq 1 \text{ then} \\
& \text{Compute the optimal strategy } \sigma \text{ by testing all possibilities} \\
& \text{return } (\sigma, v_\sigma) \\
& \sigma \leftarrow \text{a MAX strategy} \\
& x, y \leftarrow \text{two vertices of } V_{\text{MAX}} \\
& (\sigma, v) \leftarrow \text{RecursivePair}(G_{\{x,y\}[\sigma]}) \\
\text{while } \sigma \text{ is not optimal do} \\
& \text{return } (\sigma, v) \\
\text{return } (\sigma, v)
\end{align*} \)

Let us first recall that computing \( v_\sigma \) can be done in polynomial time in \( |G| \) by solving a linear programming problem.

We write \( N \) the number of iterations of the loop. Let \( \sigma_i \) be the value of \( \sigma \) at the start of the \( i \)-th iteration of the loop line 8 and \( \sigma_{N+1} \) the value of \( \sigma \) after the last iteration of the loop. Algorithm 1 makes \( N+1 \) recursive calls to an instance with \( n-2 \) MAX vertices. We notice that for all \( i \), \( S_{\sigma_i} \subseteq \{x,y\} \).

**Lemma 20.** For all \( i < N + 1 \), \( |S_{\sigma_i}| \neq 0 \). Moreover, there is at most \( 2(d-1) \) indices \( k \) such that \( |S_{\sigma_k}| = 1 \).

**Proof.** If \( S_{\sigma_i} = \emptyset \), then the algorithm stops and \( i = N + 1 \). Thus, for all \( k < N + 1 \), \( S_{\sigma_k} \neq \emptyset \).

For all neighbours \( x' \) of \( x \) there is at most one \( k \) such that \( \sigma_k(x) = x' \) and \( S_{\sigma_k} = \{x\} \), since such strategies are all optimal in \( G_{\{x\}[\sigma_k]} \) and thus have the same value. Moreover, if there is an optimal strategy \( \sigma^* \) such that \( \sigma^*(x) = x' \), then all optimal strategies of \( G_{\{x\}[\sigma^*]} \) are optimal on \( G \) and there is no strategy \( \sigma \) such that \( \sigma(x) = x' \) and \( \sigma \subseteq \{x\} \). Hence, there is at most \( 2(d-1) \) visited strategies with switch set of size 1.

**Proposition 21.** Algorithm 1 runs in \( O\left( \left( \left\lfloor \frac{(d+1)^2}{2} \right\rfloor - 1 \right)^{n/2} \text{Poly}(|G|) \right) \).

**Proof.** If we write, \( n_0, n_1 \) and \( n_2 \) the number of indices \( i \) such that \( \sigma_i \) has a switch set of respectively size 0, 1 and 2, then \( N + 1 = n_0 + n_1 + n_2 \). By Theorem 19 if \( S_{\sigma_i} = \{x,y\} \), then there is a super-switch \( \sigma' \) such that \( v_{\sigma_i} < v_{\sigma'} \leq v_{\sigma_i+1} \). Thus, \( n_0 + n_1 + 2n_2 \leq d^2 \). We also know by Lemma 20 that \( n_0 + n_1 \leq 2d - 1 \). Then:

\[ 2(N + 1) = (n_0 + n_1) + (n_0 + n_1 + 2n_2) \leq d^2 + 2d - 1 \]

Which gives:

\[ N + 1 \leq \frac{(d+1)^2}{2} - 1 \]
Since $N + 1$ is an integer, we have $N + 1 \leq \left\lfloor \frac{(d + 1)^2}{2} - 1 \right\rfloor$. Hence, Algorithm 1 makes at most $\left\lfloor \frac{(d + 1)^2}{2} - 1 \right\rfloor$ recursive calls to an instance with $n - 2$ MAX vertices and Algorithm 1 runs in $O\left(\left\lfloor \frac{(d + 1)^2}{2} - 1 \right\rfloor^{n/2} \text{Poly}(|G|)\right)$. ▶

In the case of binary SSG, Algorithm 1 is similar to Ludwig’s Algorithm [11] which fixes the strategy on the vertices one at a time. The choice of which vertex to fix is random and provides an algorithm that runs in expected time $2^{O(\sqrt{n}) \text{Poly}(|V|)}$. However, despite the proximity of the two algorithms, we were yet not able to find a similar analysis as the one in [11] to the stochastic version of Algorithm 1.

On binary SSG, Algorithm 1 gives a complexity bound in $O\left(\sqrt{3^n} \text{Poly}(|G|)\right)$ which is better than the currently known one for binary SSG in [14]. However, it is still possible to improve this complexity, as we show in the next section.

5 A Recursive Algorithm for Binary SSGs

In all this section, we will only consider binary SSG.

The concept of Algorithm 1 is to fix a subset of vertices and recursively solve the rest of the game. Then, we switch the current strategy and fix a smaller subset of vertices and reiterate. We show that we never make a call to an instance with $n - 1$ MAX vertices. This is done by carefully selecting the set of fixed vertices.

Algorithm 2 DecreasingFixedSet

\begin{algorithm}
\begin{algorithmic}[1]
\Statex \textbf{Data:} $G$ an SSG
\Statex \textbf{Result:} $\sigma$ an optimal MAX strategy and $v$ the optimal value vector.
\Begin
\State $\sigma \leftarrow$ a MAX strategy
\State $S \leftarrow S_{\sigma}$
\State $\bar{\sigma} \leftarrow \sigma$
\State $v \leftarrow v_{\sigma}$
\State $T \leftarrow S \cup S_{\bar{\sigma}}$
\State $S \leftarrow S_{\bar{\sigma}}$
\While{$S \neq \emptyset$}
\State $\sigma \leftarrow \bar{\sigma}$
\State $(\sigma, v) \leftarrow$ DecreasingFixedSet($G_{[T \setminus \sigma]}$)
\State $T \leftarrow S \cup S_{\bar{\sigma}}$
\State $S \leftarrow S_{\bar{\sigma}}$
\EndWhile
\State \Return $(\sigma, v)$
\End
\end{algorithmic}
\end{algorithm}

As stated before, the goal of Algorithm 2 is to avoid the call to a game with $n - 1$ MAX vertices. In order to achieve this, the set of vertices that is fixed in the recursive call is the union of the previous and current switch set. Computing $S_{\sigma}$ at line 11 can be done in linear time with the value vector $v$ computed at line 10.

\begin{lemma}
Algorithm 2 terminates and computes an optimal MAX strategy and its value vector.
\end{lemma}
The golden ratio.

We denote by $T_i$ and $S_i$ the value of the variables $S$ and $\sigma$ after line 3. In addition, we call $\sigma_i$, $T_i$ and $S_i$ the value of the variables $\sigma$, $T$ and $S$ at the beginning of the $i$-th iteration of the while loop. Let $N$ be the number of iterations. We call $T_{N+1}$, $S_{N+1}$ and $\sigma_{N+1}$ the value of those variables at the end of the last iteration. By line 11 of Algorithm 2 for every $i$, $S_i \subset T_i$. We create a partition of $T_i$ by considering $S_i$ and $S_{i-1} \setminus S_i$.

If for all $i \geq 1$, $S_{i-1} \setminus S_i$ is not empty, then $|T_i| > |S_i|$ and $S_i$ not empty implies that $|T_i| \geq 2$. Then, all recursive calls to Algorithm 2 are made to a subgame with at most $n - 2$ MAX vertices.

**Proposition 23.** For every $1 \leq i \leq N$, $S_{i-1} \setminus S_i$ is not empty.

**Proof.** For every $1 \leq i \leq N$, by definition $T_i = S_{i-1} \cup S_i$. The strategy $\sigma_i$ is a $(\sigma_{i-1}, T_{i-1})$-super-switch, thus $v_{\sigma_i} > v_{\sigma_{i-1}}$ and $S_{i-1}$ is not a subset or equal to $S_i$ according to the contraposition of Proposition 16.

Now, we need to prove that each iteration of the loop strictly decreases the size of $T$.

**Proposition 24.** For $1 \leq i < N$, $T_{i+1} \subset T_i$.

**Proof.** Let $1 \leq i < N$. First of all, we notice that $S_i \subseteq T_{i-1}$ since $\sigma_i$ is optimal in $G_{T_{i-1}[\sigma_i]}$, and $T_i = S_{i-1} \cup S_i$. Thus, we have $T_i \subseteq T_{i-1}$.

We recall that strategy $\sigma_{i-1}$ is optimal in the game $G_{S_{i-1}[\sigma_{i-1}]}$. We notice that for every $x \in S_{i-1} \cap S_i$, $\sigma_{i+1}(x) = \sigma_{i-1}(x)$; the strategy of every vertex in $S_{i-1} \cap S_i$ has been changed twice, thus going back to its original value since we consider binary SSG. We recall that $S_{i-1} \setminus S_i$ is not empty by Proposition 23 and assume for the sake of contradiction that $S_{i-1} \setminus S_i \subset S_{i+1}$. Then we define the strategy $\sigma'$ as follows:

$$\forall x \in S_{i-1} \setminus S_i, \sigma'(x) \neq \sigma_{i+1}(x)$$
$$\forall x \notin S_{i-1} \setminus S_i, \sigma'(x) = \sigma_{i+1}(x)$$

Since $S_i \setminus S_{i-1}$ is a subset of $S_{i+1}$, $\sigma'$ is a $\sigma_{i+1}$-switch and by Proposition 12 $\sigma' > \sigma_{i+1} > \sigma_i$. However, for all $x \in S_{i-1}$, $\sigma'(x) = \sigma_{i-1}(x)$ and $\sigma'$ is a strategy of $G_{S_{i-1}[\sigma_{i-1}]}$ which contradicts the optimality of $\sigma_{i-1}$ on this game. This shows that there exists $x$ in $S_{i-1} \setminus S_i$ but not in $S_{i+1}$. In other words, there is $x$ in $S_{i-1}$ and thus in $T_i$ but not in $S_i \cup S_{i+1}$ and thus not in $T_{i+1}$. Therefore, we have proven that $T_{i+1} \subset T_i$. In order to better visualise this proof a representation of the successive switches is provided Figure 4.

We can now give a bound on the complexity of Algorithm 2.

**Theorem 25.** Algorithm 2 has time complexity $O(\varphi \cdot \text{Poly}(|G|))$, where $\varphi = \frac{1 + \sqrt{5}}{2}$ is the golden ratio.

**Proof.** We denote by $C(0,k)$, the complexity of solving an SSG with 0 MAX vertices and $k$ total vertices. This is the resolution of a one-player game and can be done in polynomial time in the size of the game by solving a linear programming problem. We define $C(n,k)$ as:

$$C(n,k) = C(n-2,k) + C(n-3,k) + \ldots + C(1,k) + 3C(0,k)$$
We show by induction that the complexity of solving an SSG using Algorithm 2 with \( n \) \text{max} vertices and \( k \) total vertices is bounded by \( C(n,k) \). According to Proposition 24, each call to DecreasingFixedSet is done on an SSG with a decreasing number of \text{max} vertices. Proposition 23 also stipulates that if \( S_i \) is not empty, then \( S_{i-1} \setminus S_i \) is also not empty and \(|T_i|\) is greater than one. Thus, each recursive call is made on an instance with at most \( n - 2 \) \text{max} vertices. Finally, \( v_\sigma \) is computed twice before the loop, costing \( C(0,k) \) operations. Therefore Algorithm 2 has time complexity \( O(C(n,k)) \). We notice that \( C(n,k) - C(n-1,k) = C(n-2,k) \). Thus, we have:

\[
C(n,k) = O(\varphi^n Poly(|G|))
\]

Thus, we have shown that Algorithm 2 has time complexity \( O(\varphi^n Poly(|G|)) \).

The polynomial factor in all our Algorithms corresponds to the complexity of computing \( v_\sigma \) from \( \sigma \). We recall that this is the complexity of solving a linear programming problem with \(|V|\) variables. It is the same polynomial factor as the one in Tripathi, Valkanova and Kumar’s algorithm [14] which runs in \( O(2/n \cdot Poly(|G|)) \).

However, the analysis of Algorithm 2 does not hold in the case of SSG with higher degree. Algorithm 1 can still be improved for some degree by changing the size of the fixed set according to \( d \). For instance, if we fix set of size 3 the complexity of solving SSG of degree 3 is \( O \left( 17^{n/3} \right) \) iterations instead of \( O \left( 7^{n/2} \right) \). For information, \( 17^{1/3} \approx 2.57 \) and \( 7^{1/2} \approx 2.65 \). However, increasing the size of the fixed set not always hold better complexity. For binary SSG, the number of iterations with set of size 2 is \( O \left( 3^{n/2} \right) \) and \( O \left( 6^{n/3} \right) \) for set of size 3 and we know that \( 6^{1/3} \approx 1.82 \) and \( 3^{1/2} \approx 1.73 \).
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