Energy Conditions Constraints and Stability of Power Law Solutions in $f(R, T)$ Gravity

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Abstract

The energy conditions are derived in the context of $f(R, T)$ gravity, where $R$ is the Ricci scalar and $T$ is the trace of the energy-momentum tensor, which can reduce to the well-known conditions in $f(R)$ gravity and general relativity. We present the general inequalities set by the energy conditions in terms of Hubble, deceleration, jerk and snap parameters. In this study, we concentrate on two particular models of $f(R, T)$ gravity namely, $f(R) + \lambda T$ and $R + 2f(T)$. The exact power-law solutions are obtained for these two cases in homogeneous and isotropic $f(R, T)$ cosmology. Finally, we find certain constraints which have to be satisfied to ensure that power law solutions may be stable and match the bounds prescribed by the energy conditions.

Keywords: $f(R, T)$ gravity; Raychaudhuri equation; Energy conditions; Power law.

PACS: 04.50.-h; 04.50.Kd; 98.80.Jk; 98.80.Cq.

1 Introduction

Recent astrophysical observations form supernova type Ia\textsuperscript{1)}, cosmic microwave background anisotropies\textsuperscript{2)}, large scale structure\textsuperscript{3)}, baryon acoustic oscillations\textsuperscript{4)}

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and weak lensing\textsuperscript{5}) indicate that the universe is accelerating in the current epoch. The most promising feature of the universe is the dominance of exotic energy component with large negative pressure, known as \textit{dark energy} (DE). A number of alternative models have been proposed in the framework of general relativity (GR) to explain the role of DE in the present cosmic acceleration. Unfortunately, up to now, no suitable candidate is found, which boosts our interest in modified theories of gravity. Firstly, the Einstein-Hilbert action has been modified by replacing scalar curvature $R$ by an arbitrary function of $R$, this theory is known as $f(R)$ gravity\textsuperscript{6}). The other alternative theories of gravity include $f(T)$ gravity\textsuperscript{7}), where “$T$” is the torsion scalar in teleparallel gravity, Gauss-Bonnet gravity\textsuperscript{8}) and $f(R, T)$ gravity\textsuperscript{9}).

The $f(R, T)$ gravity is the generalization of $f(R)$ gravity involving the dependence of the trace of energy-momentum tensor $T$. The dependence of $T$ may be induced by exotic imperfect fluids or quantum effects. The cosmological reconstruction of $f(R, T)$ gravity has been studied in recent literature\textsuperscript{9−13}). In a paper\textsuperscript{9)}, the reconstruction of FRW cosmology is presented for $f(R, T) = R + 2f(T)$ model. Houndjo and Piattella\textsuperscript{10)} constructed $f(R, T)$ models describing the unification as well as transition of matter dominated phase to late accelerating phase. The chaplygin gas $f(R, T)$ models are investigated in\textsuperscript{11−12}) and it is shown that dust fluid reproduces $Λ$CDM, Einstein static universe and phantom cosmology\textsuperscript{12}). In our previous work\textsuperscript{13)}, we have reconstructed some explicit models of $f(R, T)$ gravity for anisotropic universe and explored the phantom era of dark energy. We have also discussed the validity of first and second laws of thermodynamics in this modified gravity\textsuperscript{14}). The existence of exact power law solutions for FRW spacetime has been investigated in modified theories of gravity\textsuperscript{15−16}). Here, we shall show that FRW power law solutions exist for a particular class of $f(R, T)$ gravity.

The classical energy conditions of GR are profound to the Hawking-Penrose singularity theorems and classical black hole laws of thermodynamics\textsuperscript{17}). These conditions have been used to address several important issues in GR and cosmology\textsuperscript{18}). Many authors have investigated the energy conditions in the context of modified theories including $f(R)$ gravity\textsuperscript{19−20}), $f(R)$ gravity with nonminimal coupling to matter\textsuperscript{21}), modified Gauss-Bonnet gravity\textsuperscript{22}), modified $f(G)$ gravity with curvature-matter coupling\textsuperscript{23}), Brans-Dicke theory\textsuperscript{24}), $f(T)$ gravity\textsuperscript{25}) and $f(R, T)$ gravity\textsuperscript{26}). In recent work\textsuperscript{26)}, the authors study the energy conditions for a special form of $f(R, T)$ gravity, $f(R, T) = R + 2f(T)$ and discussed the stability of two $f(T)$ models.
In this work, we are interested to set energy conditions bounds on exact power law solutions in \( f(R, T) \) gravity. The FRW power law solutions are obtained for \( f(R, T) = f(R) + \lambda T \) and \( f(R, T) = R + 2f(T) \) gravity. We derive the energy conditions for more general as well as particular class of \( f(R, T) \) gravity. The standard form of energy conditions in GR and \( f(R) \) gravity can be recovered in the limit of \( f(R, T) = f(R) \) and \( f(R, T) = R \). We show that for \( f(R, T) = f(R) + \lambda T \), the null energy condition (NEC) and strong energy condition (SEC) can be derived by using the Raychaudhuri equation with the requirement that gravity is attractive. The resulting inequalities for NEC and SEC are equivalent to the energy conditions obtained in terms of effective energy-momentum tensor.

The paper is organized as follows: In next section, we present the general formulation of the field equations of \( f(R, T) \) gravity in FRW cosmology. In section 3, the energy conditions are derived and hence presented in terms of deceleration \((q)\), jerk \((j)\) and snap \((s)\) parameters. Section 4 is devoted to obtain exact power law solutions for two specific forms of \( f(R, T) \) gravity. We also analyze the constraints of energy conditions for these models. In section 5, we investigate the perturbation and stability of power law solutions. Finally, section 6 summarizes the obtained results.

2 \( f(R, T) \) Gravity

The \( f(R, T) \) theory of gravity is an interesting modification to the Einstein gravity by introducing an arbitrary function of scalar curvature \( R \) and trace of the energy-momentum tensor \( T \). The action for this theory coupled with matter Lagrangian \( \mathcal{L}_{\text{matter}} \) is given by \(^9\)

\[
\mathcal{A} = \int dx^4 \sqrt{-g} \left[ f(R, T) + \mathcal{L}_{\text{matter}} \right],
\]

(1)

where \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \), we use the units \( 8\pi G = c = 1 \). The energy-momentum tensor of matter is defined as \(^{27}\)

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_{m})}{\delta g^{\mu\nu}}.
\]

(2)
Varying this action with respect to the metric tensor, we obtain the field equations of \( f(R, T) \) gravity as

\[
R_{\mu\nu} f_R(R, T) - \frac{1}{2} g_{\mu\nu} f(R, T) + (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) f_R(R, T) = T_{\mu\nu} - f_T(R, T) T_{\mu\nu} - f_T(R, T) \Theta_{\mu\nu},
\]

where \( f_R(R, T) \) and \( f_T(R, T) \) denote derivatives of \( f(R, T) \) with respect to \( R \) and \( T \) respectively; \( \Box = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \) is the d’Alembert operator, \( \nabla_{\mu} \) is the covariant derivative associated with the Levi-Civita connection of the metric tensor and \( \Theta_{\mu\nu} \) is defined by

\[
\Theta_{\mu\nu} = g^{\alpha\beta} \delta T_{\alpha\beta} = -2T_{\mu\nu} + g_{\mu\nu} L_m - 2g^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}.
\]

The contribution to the energy momentum tensor of matter is defined as

\[
T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - pg_{\mu\nu},
\]

where \( u_{\mu} \) is the four velocity of the fluid, \( \rho \) and \( p \) denote the energy density and pressure, respectively. We can take \( L_{\text{(matter)}} = -p \), then \( \Theta_{\mu\nu} \) becomes

\[
\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}.
\]

Consequently, the field equations \([3]\) can be expressed as effective Einstein field equations of the form

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T^{\text{eff}}_{\mu\nu},
\]

where \( T^{\text{eff}}_{\mu\nu} \) is the effective energy-momentum tensor in \( f(R, T) \) gravity which is defined as

\[
T^{\text{eff}}_{\mu\nu} = \frac{1}{f_R(R, T)} \left[ (1 + f_T(R, T)) T_{\mu\nu} + p g_{\mu\nu} f_T(R, T) + \frac{1}{2} f(R, T) - R f_R(R, T) g_{\mu\nu} + (\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box) f_R(R, T) \right].
\]

We consider the homogeneous and isotropic flat FRW spacetime as

\[
ds^2 = dt^2 - a^2(t) dx^2,
\]
where $a(t)$ is the scale factor and $d\mathbf{x}^2$ contains the spatial part of the metric. In the FRW background, the field equations may be rewritten as

$$3H^2 = \rho_{\text{eff}}, \quad -(2\dot{H} + 3H^2) = p_{\text{eff}}, \quad (9)$$

where $\rho_{\text{eff}}$ and $p_{\text{eff}}$ are the energy density and pressure respectively, defined as

$$\rho_{\text{eff}} = \frac{1}{f_R} \left[ \rho + (\rho + p)f_T + \frac{1}{2}(f - Rf_R - 3H(\dot{R}f_{RR} + \ddot{T}f_{RT})) \right], \quad (10)$$

$$p_{\text{eff}} = \frac{1}{f_R} \left[ p - \frac{1}{2}(f - Rf_R) + (\dot{R} + 2\dot{H})f_{RR} + \dot{R}^2 f_{RRR} ight. \right. \left. \left. + 2\dot{R}\dot{T}f_{RRT} + (\ddot{R} + 2\ddot{T}H)f_{RT} + \ddot{T}^2 f_{RTT} \right] \right), \quad (11)$$

the Hubble parameter $H$ is defined by $H = \dot{a}/a$ and dot denotes derivative with respect to cosmic time $t$.

## 3 Energy Conditions

Raychaudhuri equation is the key to SEC and NEC together with the requirement that gravity is attractive for a spacetime manifold endowed with a metric $g_{\mu\nu}$. For the congruence of timelike geodesics defined by vector field $u^\mu$, the Raychaudhuri equation reads,

$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}u^\mu u^\nu, \quad (12)$$

where $R_{\mu\nu}$ is the Ricci tensor, and $\theta$, $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are the expansion parameter, the shear and the rotation associated with the congruence respectively. The evolution equation for $\theta$, the expansion scalar of a congruence of null geodesics defined by the null vector field $\kappa^\mu$, is given by

$$\frac{d\theta}{d\tau} = -\frac{1}{2} \theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}\kappa^\mu \kappa^\nu. \quad (13)$$

Raychaudhuri equation is known to be purely geometric and hence, develops no reference to any theory of gravity. As the shear tensor is purely spatial $\sigma^{\mu\nu}\sigma_{\mu\nu} \geq 0$, thus, for any hypersurface of orthogonal congruence ($\omega_{\mu\nu} = 0$), the conditions for attractive gravity become

SEC : $R_{\mu\nu}u^\mu u^\nu \geq 0$, \quad NEC : $R_{\mu\nu}\kappa^\mu \kappa^\nu \geq 0. \quad (14)$
One can use the field equations of any gravity to relate $R_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$. Thus, the combination of the field equations and Raychaudhuri equations can set the physical conditions for the energy-momentum tensor. In the framework of GR, the conditions (14) can be written as

$$R_{\mu\nu}u^\mu u^\nu = (T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}) u^\mu u^\nu \geq 0, \quad R_{\mu\nu} \kappa^\mu \kappa^\nu = T_{\mu\nu} \kappa^\mu \kappa^\nu \geq 0. \quad (15)$$

For perfect fluid, this equation is reduced to the well-known form of SEC and NEC in GR,

$$\rho + 3p \geq 0, \quad \rho + p \geq 0. \quad (16)$$

In modified theories of gravity including $f(R)$ and $f(T)$ gravity, $R_{\mu\nu}$ can be obtained in terms of the energy-momentum tensor by using the corresponding field equations. However, this does not seem apparent in $f(R,T)$ gravity.

We consider the effective energy-momentum tensor $T_{\mu\nu}^{\text{eff}}$, so that the conditions in Raychaudhuri equations are represented as

$$(T_{\mu\nu}^{\text{eff}} - \frac{T^{\text{eff}}}{2} g_{\mu\nu}) u^\mu u^\nu \geq 0 \quad \& \quad T_{\mu\nu}^{\text{eff}} \kappa^\mu \kappa^\nu \geq 0.$$ 

Hence, the energy conditions in GR can be applied by replacing energy density $\rho$ and pressure $p$ by $\rho_{\text{eff}}$ and $p_{\text{eff}}$, respectively. Since the Raychaudhuri equation holds for any geometrical theory of gravity, we will keep the physical motivation of focusing of geodesic congruences along with attractive property of gravity to develop the energy conditions in $f(R,T)$ gravity. We also assume that standard matter obey the four energy conditions. Using the effective modified field equations (10), the energy conditions for $f(R,T)$ gravity are given by

**NEC** :

$$\rho_{\text{eff}} + p_{\text{eff}} = \frac{1}{f_R} \left[ (\rho + p)(1 + f_T) + (\ddot{R} - \dot{R}H)f_{RR} + \dot{R}^2 f_{RRR} 
+ 2\dot{R}\dot{T} f_{RT} + (\ddot{T} - \dot{T}H)f_{RT} + \dot{T}^2 f_{RTT} \right] \geq 0, \quad (17)$$

**WEC** :

$$\rho_{\text{eff}} = \frac{1}{f_R} \left[ \rho + (\rho + p)f_T + \frac{1}{2}(f - Rf_R) - 3H(\dot{R}f_{RR} 
+ \dot{T} f_{RT}) \right] \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \quad (18)$$
\textbf{SEC}:
\[ \rho_{\text{eff}} + 3p_{\text{eff}} = \frac{1}{f_R} \left[ (\rho + 3p) + (\rho + p)f_T - f + R f_R + 3\dddot{R}^2 f_{RRR} \right. \]
\[ + 3(\dddot{R} + \dot{R}H) f_{RR} + 6\dddot{R}\dot{T} f_{RRT} + 3(\dddot{T} + \dot{T}H) f_{RT} \]
\[ + 3\dddot{T}^2 f_{RTT} \right] \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \tag{19} \]

\textbf{DEC}:
\[ \rho_{\text{eff}} - p_{\text{eff}} = \frac{1}{f_R} \left[ (\rho - p) + (\rho + p)f_T + f - R f_R - \dddot{R}^2 f_{RRR} \right. \]
\[ - (\dddot{R} + 5\dot{R}H) f_{RR} - 2\dddot{R}\dot{T} f_{RRT} - (\dddot{T} + 5\dot{T}H) f_{RT} \]
\[ - \dddot{T}^2 f_{RTT} \right] \geq 0, \quad \rho_{\text{eff}} - p_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} \geq 0. \tag{20} \]

The inequalities (17)-(20) represent the null, weak, strong and dominant energy conditions in the context of \( f(R, T) \) gravity for FRW spacetime.

We define the Ricci scalar and its derivatives in terms of deceleration, jerk and snap parameters as
\[
R = -6H^2(1 - q), \quad \dot{R} = -6H^3(j - q - 2),
\]
\[ \dddot{R} = -6H^4(s + q^2 + 8q + 6), \tag{21} \]
where
\[
q = -\frac{1}{H^2}, \quad j = \frac{1}{H^3}, \quad \text{and} \quad s = \frac{1}{H^4}.
\]

Using the above definitions, the energy conditions (17)-(20) can be rewritten as
\[
\text{NEC} : \quad (\rho + p)(1 + f_T) - 6H^4(s - j + (q + 1)(q - 8)) f_{RR} + H^4[6H(j - q - 2)]^2 f_{RRR} - 12H^3(j - q - 2)\dddot{T} f_{RRT} + (\dddot{T} - \dot{T}H) f_{RT} + \dddot{T}^2 \times f_{RTT} \geq 0,
\]
\[
\text{WEC} : \quad \rho + (\rho + p)f_T + \frac{1}{2} f + 3H^2(1 - q)f_R + 18H^4(j - q - 2)f_{RR}
\]
\[ - 3H\dot{T} f_{RT} \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0,
\]
\[
\text{SEC} : \quad (\rho + 3p) + (\rho + p)f_T - f - 6H^2(1 - q)f_R + 3[6H^3(j - q - 2)]^3 \times f_{RRR} - 18(s + j + q^2 + 7q + 4)f_{RR} - 36H^3(j - q - 2)\dddot{T} f_{RRT}
\]
\[ + 3(\dddot{T} + \dot{T}H) f_{RT} + 3\dddot{T}^2 f_{RTT} \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0,
\]
DEC : 

\[
\begin{align*}
& (\rho - p) + (\rho + p)f_T + f + 6H^2(1 - q)f_R - [6H^3(j - q - 2)]^2 \\
& \times f_{RRR} - 6H^4(s + 5j + (q - 1)(q + 4))f_{RRR} + 12H^3(j - q - 2)\dot{T} \\
& \times f_{RRT} - (\dot{T} + 5\ddot{T}H)f_{RT} - \dot{T}^2 f_{RTT} \geq 0, \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \\
& \rho_{\text{eff}} \geq 0.
\end{align*}
\]

The energy conditions in \( f(R) \) gravity\(^{19} \) can be recovered for \( f(R, T) = f(R) \) and also in case of GR for particular choice \( f(R, T) = R \). To illustrate how above conditions can be used to place bounds on \( f(R, T) \) gravity, we consider two particular forms of \( f(R, T) \) gravity, (i) \( f(R) + \lambda T \), (ii) \( R + 2f(T) \).

We shall obtain the power law solutions for each case and hence the constraints set by the respective energy conditions.

## 4 Power Law Solutions for \( f(R, T) \) Gravity

It is important to study the existence of exact power solutions corresponding to different phases of cosmic evolution. Such solutions are particularly relevant because in FRW background they represent all possible cosmological evolutions such as radiation dominated, matter dominated or dark energy eras. We discuss power law solutions for two particular models of \( f(R, T) \) gravity.

### 4.1 \( f(R, T) = f(R) + \lambda T \)

For the particular case \( f(R, T) = f(R) + \lambda T^{11,13} \), the effective Einstein field equations are given by Eq.(6) with

\[
T^\text{eff}_{\mu\nu} = \frac{1}{f_R} \left[ (1 + \lambda)T_{\mu\nu} + (\lambda p + \frac{1}{2}\lambda T)g_{\mu\nu} + \frac{1}{2}(f - Rf_R)g_{\mu\nu} + (\nabla_{\mu}\nabla_{\nu} - g_{\mu\nu}\Box)f_R \right],
\]

where \( f_R \) is the derivative of \( f(R) \) with respect to scalar curvature \( R \). The Friedmann equation and the trace of the field equations are given by

\[
\Theta^2 = \frac{3}{f_R} \left[ \rho + \lambda(\rho + p) + \frac{\lambda T}{2} + \frac{1}{2}(f - Rf_R) - \Theta \dot{R}f_{RR} \right],
\]

\[
Rf_R + 3\Box f_R(R, T) - 2f = (1 + 3\lambda)T + 4\lambda p,
\]
where $\Theta = 3\dot{a}/a$ is the expansion scalar.

The standard matter satisfies the following energy conservation equation
\[ \dot{\rho} = -\Theta(\rho + p). \] (25)

For the homogeneous and isotropic spacetime, the field equations can be represented by Raychaudhuri equation
\[ \dot{\Theta} + \frac{1}{3}\Theta^2 = -\frac{1}{2f_R}[\rho + 3p + 4\lambda p - f + Rf_R + (3\dot{R} + \Theta\dot{R})f_{RR} + 3\dot{R}f_{RRR}]. \] (26)

Combination of Raychaudhuri and Friedmann equations yields
\[ R = -2(\dot{\Theta} + \frac{2}{3}\Theta^2). \] (27)

We assume that there exists an exact power law solution to the modified field equations
\[ a(t) = a_0 t^m, \] (28)
where $m$ is a positive real number. If $0 < m < 1$, then the required power law solution is decelerating while for $m > 1$ it exhibits accelerating behavior. For the equation of state $p = \omega\rho$, the energy conservation equation leads to
\[ \rho(t) = \rho_0 t^{-3m(1+\omega)}. \] (29)

Using Eq.(28) in Eq.(27), the scalar curvature becomes
\[ R = -6m(2m - 1)t^{-2} = -\eta_m t^{-2}, \] (30)
where $\eta_m = 6m(2m - 1)$. We see that sign of $R$ depends on the value of $m$, $R > 0$ if $0 < m < \frac{1}{2}$ and $R < 0$ for $m > \frac{1}{2}$. Since $m = \frac{1}{2}$ leads to vanishing of $R$, so we exclude this value of $m$ in our discussion.

Using Eqs.(29) and (30), Friedmann equation can be written in terms of Ricci scalar $R$, $f$ and $f_R$ as
\[ f_{RR}R^2 + \frac{m - 1}{2}Rf_R + \frac{1 - 2m}{2}f - (2m - 1)A\rho_0 \left(\frac{-R}{\eta_m}\right)^{3m(1+\omega)\over 2} = 0, \] (31)
where $A = 1 + \frac{1}{2}(3 - \omega)$. This represents second order differential equation for $f(R)$ whose general solution is

$$f(R) = X_{m\omega} \left(\frac{-R}{\eta_m}\right)^{\frac{3m(1+\omega)}{2}} + C_1 R^{\frac{1}{2}(3-m-\sqrt{\delta_m})} + C_2 R^{\frac{1}{2}(3-m+\sqrt{\delta_m})},$$  \hspace{1cm} (32)

where

$$X_{m\omega} = \frac{4A(2m-1)\rho_0}{3m^2(3\omega + 4)(\omega + 1) - m(9\omega + 13) + 2}, \quad \delta_m = m^2 + 10m + 1,$$

and $C_1$, $C_2$ are arbitrary constants of integration. Since $m > 0$, so $\delta_m > 0$ for cosmologically viable solutions. $X_{m\omega}$ is found to be real valued but it diverges for $3m^2(3\omega + 4)(\omega + 1) - m(9\omega + 13) + 2 = 0$, i.e., $m$ and $\omega$ satisfy any of the relations $\omega = \frac{3-7m+\sqrt{6m}}{6m}$ or $m = \frac{13+9\omega+\sqrt{9\omega^2+66\omega+73}}{6(\omega+1)(3\omega+4)}$. Since $R < 0$, so $(-R/\eta_m) > 0$ for all $R$, thus we have real valued solution $f(R,T) = f(R) + \lambda T$ showing that the power law solution exists for this model.

For $\lambda = 0$, we obtain solution as in $f(R)$ gravity$^{16}$. To check whether the $f(R,T)$ gravity reduces to GR, we need to put $C_1 = C_2 = \lambda = 0$. For $m = \frac{2}{3(1+\omega)}$ and $\rho_0 = \frac{4}{3(1+\omega)^2}$, this theory reduces to GR. We are interested to construct the $f(R,T)$ model of the form $\alpha R^n + \lambda T$. If we put $m = \frac{2n}{3(1+\omega)}$, then $f(R)$ is given by

$$f(R) = \alpha_{n\omega}(-R)^n,$$ \hspace{1cm} (33)

where

$$\alpha_{n\omega} = \frac{2^{3-2n}3^{n-1}nA(n(4n - 3(1 + \omega))^{1-n} (1 + \omega)^{2n-2}}{n^2(6\omega + 8) - n(9\omega + 13) + 3(\omega + 1)}},$$

and hence $f(R,T) = \alpha_{n\omega}(-R)^n + \lambda T$. This model represents the exact Friedmann-like power law solution $a \propto t^{\frac{2n}{3(1+\omega)}}$ and the limit $n \to 1$ with $\lambda = 0$ leads to GR. For $n = 1$, our solutions represents $\Lambda$CDM model of the form $f(R,T) = R + \lambda T$.

### 4.1.1 Phantom Phase Power Law Solution

We construct the phantom phase power law solution which lead to big rip singularity. For this case, the scale factor and Hubble parameter are expressed as

$$a(t) = a_0(t_s - t)^{-m}, \quad H(t) = \frac{m}{t_s - t}.$$
The scale factor diverges within finite time \((t \to t_s)\) leading to big rip singularity for \(m \geq 1^{29}\). The results for this case can be recovered by just replacing \(m\) by \(-m\) in the previous section. Hence, the phantom phase power law solution exist for \(f(R) + \lambda T\) gravity.

### 4.1.2 Constraining \(f(R, T) = f(R) + \lambda T\) Gravity

In section 2, we have found that Raychaudhuri equations with attractive behavior of gravitational interaction give rise to SEC and NEC which hold for any theory of gravitation. In this form of \(f(R, T)\) gravity, one can employ an approach similar to that in GR to develop the energy conditions.

Equations (6) and (22) can be written as

\[
R_{\mu \nu} = T_{\mu \nu} - \frac{T}{2} g_{\mu \nu},
\]

where

\[
T_{\mu \nu} = \frac{1}{f_R} \left[ (1 + \lambda)T_{\mu \nu} + \frac{\lambda}{2} (\rho - p) g_{\mu \nu} + (\nabla_\mu \nabla_\nu - g_{\mu \nu} \Box) f_R \right],
\]

\[
T = \frac{1}{f_R} \left[ (1 + 3\lambda)T + 4\lambda p - R f_R - 3 \Box f_R \right].
\]

Equations (14) and (34) lead to following inequalities

\[
\text{NEC} : \quad \frac{1}{f_R} \left[ (\rho + p)(1 + \lambda) + (\ddot{R} - \dot{R} H) f_{RR} + \dot{R}^2 f_{RRR} \right] \geq 0, \quad (35)
\]

\[
\text{SEC} : \quad \frac{1}{2f_R} \left[ (\rho + 3p) + 4p\lambda - f + R f_R + 3\dot{R}^2 f_{RRR} + 3(\ddot{R} + \dddot{R}) f_{RR} \right] \geq 0.
\]

For \(\lambda = 0\), one can obtain the NEC and SEC in \(f(R)\) gravity. Furthermore, the more familiar forms of NEC and SEC in GR, i.e., \(\rho + p \geq 0\) and \(\rho + 3p \geq 0\), can be recovered if \(\lambda = 0\) and \(f(R) = R\).

To derive the WEC and DEC, we can extend the GR approach by introducing an effective energy-momentum tensor. The above inequalities of the SEC and NEC are obtained directly from Raychaudhuri equations, however equivalent results can be derived by using the conditions \(\rho_{eff} + p_{eff} \geq 0\) and
\( \rho_{\text{eff}} + 3p_{\text{eff}} \geq 0 \). From Eq. (22), the effective energy density and effective pressure are given by

\[
\rho_{\text{eff}} = \frac{1}{f_R} \left[ \rho + \frac{\lambda}{2} (3\rho - p) + \frac{1}{2} (f - Rf_R) - 3\dot{R} f_{RR} \right], \quad (37)
\]

\[
p_{\text{eff}} = \frac{1}{f_R} \left[ (p - \frac{\lambda}{2} (\rho - 3p) - \frac{1}{2} (f - Rf_R) + (\ddot{R} + 2\dot{R}H)f_{RR} + \dot{R}^2 f_{RRR} \right]. \quad (38)
\]

The WEC and DEC in \( f(R) + \lambda T \) gravity can be obtained by following the effective energy-momentum tensor approach. The WEC is obtained by satisfying inequality (35) and the constraint

\[
\frac{1}{f_R} \left[ \rho + \frac{\lambda}{2} (3\rho - p) + \frac{1}{2} (f - Rf_R) - 3\dot{R} f_{RR} \right] \geq 0, \quad (39)
\]

and DEC is obtained by satisfying inequalities (35), (39) and the constraint

\[
\frac{1}{f_R} \left[ (\rho - p) + 2\lambda (\rho + p) + f - Rf_R - (\ddot{R} + 5\dot{R}H)f_{RR} - \dot{R}^2 f_{RRR} \right] \geq 0. \quad (40)
\]

We find that by setting \( \lambda = 0 \), the WEC and DEC in \( f(R) \) gravity can be recovered. Moreover, for \( f(R) = R \) and \( \lambda = 0 \), the well-known form of weak and dominant energy conditions in GR can be reproduced.

The above energy conditions can be used to put constraints on a given \( f(R) \) model in the context of \( f(R, T) \) gravity. We assume that \( f_R > 0 \) to keep the effective gravitational constant positive. Using the relations (21), the energy conditions for \( f(R) + \lambda T \) gravity in terms of present day values of \( H, q, j \) and \( s \) are given by

- **NEC** :
  \[
  (1 + \lambda)(\rho_0 + p_0) - 6H^4(s_0 - j_0 + (q_0 + 1)(q_0 + 8))f_{0RR} + H_0[6 \times H_0(j_0 - q_0 - 2)]^2 f_{0} f_{RRR} \geq 0,
  \]

- **WEC** :
  \[
  \rho_0 + \frac{\lambda}{2} (3\rho_0 - p_0) + \frac{1}{2} f_0 + 3H_0^2(1 - q_0) f_{0R} + 18H_0^4[j_0 - q_0 - 2] \times f_{0RR} \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0,
  \]

- **SEC** :
  \[
  (\rho_0 + 3p_0) + 4p_0\lambda - f_0 - 6H_0^2(1 - q_0) f_{0R} + 3[6H_0^3(j_0 - q_0 - 2)]^2 \times f_{0RR} - 18H_0^4[s_0 + j_0 + q_0^2 + 7q_0 + 4] f_{0} f_{RRR} \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0,
  \]

- **DEC** :
  \[
  (\rho_0 - p_0) + 2\lambda(\rho_0 - p_0) + f_0 + 6H_0^2(1 - q_0) f_{0R} - [6H_0^3(j_0 - q_0 - 2)]^2 f_{0} f_{RRR} - 6H_0^4[s_0 + 5j_0 + (q_0 - 1)(q_0 + 4)] f_{0} f_{RRR} \geq 0,
  \]

\[
\rho_{\text{eff}} + p_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} \geq 0.
\]
In order to present the concrete application of the above energy conditions in \( f(R, T) \) gravity, we employ the exact power law solution of \( f(R) + \lambda T \) gravity. We consider the present day values of deceleration, jerk and snap parameters as \( q_0 = -0.81 \pm 0.14, \; j_0 = 2.16_{-0.75}^{+0.81} \) and \( s_0 = -0.22_{-0.19}^{+0.2128} \). We shall discuss the WEC requirement to illustrate how the above conditions can be used to place constraints on \( f(R, T) \) gravity. We note that all the above conditions depend on the present value of pressure \( p_0 \), so for simplicity we assume \( p = 0 \).

Now, we take the power law solution as an objective model which is given by

\[
f(R, T) = \alpha_n(-R)^n + \lambda T, \tag{41}\]

where \( n \) is an integer and \( \alpha_n = \frac{2^{5-2n^3-1} n A (4n^2-3n)^{1-n}}{(8n^2-13n+3)} \). The constraints to fulfill the WEC, i.e., \( \rho_{eff} \geq 0, \rho_{eff} + p_{eff} \geq 0 \), are respectively obtained as

\[
(2 + 3\lambda)\rho_0 + \alpha_n[6H_0^2(1 - q_0)]^n[B_1(n^2 - n) - n + 1] \geq 0, \tag{42}
\]

\[
(1 + \lambda)\rho_0 + \alpha_n(n - 1)6H_0^4[6H_0^2(1 - q_0)]^{n-2}[-(s_0 - j_0 + (q_0 + 1)
\times (q_0 + 8)) - B_2(n - 2)] \geq 0, \tag{43}
\]

where \( B_1 = (j_0 - q_0 - 2)/(1 - q_0)^2 \) and \( B_2 = (j_0 - q_0 - 2)^2/(1 - q_0) \). As the standard matter is assumed to satisfy the necessary energy conditions and \( \lambda > 0 \), so \((2 + 3\lambda)\rho_0 > 0 \) and \((1 + \lambda)\rho_0 > 0 \). Hence, the inequality (42) is reduced to

\[
\alpha_n(3.3H_0)^2n\beta_n \geq 0, \quad \text{where} \quad \beta_n = B_1(n^2 - n) - n + 1.
\]

It is clear from above expression, the result is trivial for \( n = 0, 1 \). We consider the following two cases:

(i) \( \alpha_n > 0 \), the allowed values for \( n \) are \( n = \{2, 3, 4, \ldots\} \). Now \( \beta_n > 0 \) in the range \( n = \{4, 5, 6, \ldots\} \) and \( \beta_n < 0 \) for \( n = 2, 3 \).

(ii) \( \alpha_n < 0 \), the acceptable values of \( n \) are \( n = \{-1, -2, \ldots\} \) and in this particular range we have \( \beta_n < 0 \). Thus, the inequality \( \rho_{eff} \geq 0 \) is satisfied for \( n = \{-2, -1, 4, 5, \ldots\} \).

Now we check the validity of Eq. (43) except \( n = 0, 1 \) as the result is trivial for this choice. The inequality is transformed to the following form

\[
\alpha_n(3.3H_0)^2n\mu_n \geq 0, \quad \text{where} \quad \mu_n = (n^2 - n)(2.054 - 0.52n).
\]

The results of the above inequality can be interpreted as:

(i) \( \mu_n > 0 \), if \( n = \{2, 3, -1, -2, \ldots\} \) and for \( \mu_n < 0 \), the acceptable values of
\( n \) are \( n = \{4, 5, 6, \ldots\} \).

(ii) \( \alpha_n > 0 \), with acceptable range \( n = \{2, 3, 4, \ldots\} \) and \( \alpha_n < 0 \), when \( n = \{-1, -2, -3, \ldots\} \). Hence, the condition \( \rho_{\text{eff}} + p_{\text{eff}} \geq 0 \) is satisfied for \( n = 2, 3 \).

### 4.2 \( f(R, T) = R + 2f(T) \)

Now, we construct the power law solutions for \( R + 2f(T) \) gravity, where \( f(T) \) is an arbitrary function of the trace of energy-momentum tensor. The effective Einstein field equations are given by Eq. (6) with

\[
T^\text{eff}_{\mu\nu} = (1 + 2f_T)T_{\mu\nu} + (2pf_T + f)g_{\mu\nu},
\]

where \( f_T \) is the derivative of \( f \) with respect to the trace of energy-momentum tensor \( T \). The Friedmann equation and the trace equation can be obtained as

\[
\Theta^2 = 3[\rho + 2(\rho + p)f_T + f], \quad R = -(\rho - 3p) - 2(\rho + p)f_T - 4f. \quad (44)
\]

It follows that the field equations can be represented as the Raychaudhuri equation

\[
\dot{\Theta} + \frac{1}{3}\Theta^2 = -\frac{1}{2}[(\rho + 3p) + 2(\rho + p)f_T - 2f]. \quad (45)
\]

Combining Eqs. (44) and (45), we can get the Ricci scalar \( R \) given in Eq. (27). Using Eq. (29), the Friedmann equation can be written in terms of trace of the energy-momentum tensor \( T \), \( f(T) \) and its derivative with respect to \( T \) as

\[
Tf_T + \frac{T}{2(1 + \omega)} + \frac{(1 - 3\omega)f}{2(1 + \omega)} - \frac{K(1 - 3\omega)T^{\frac{2}{3m(1+\omega)}}}{2(1 + \omega)} = 0, \quad (46)
\]

where \( K = 3m^2(\rho_0(1 - 3\omega))^{\frac{2}{3m(1+\omega)}} \). This is the first order differential equation in \( f(T) \) whose solution is

\[
f(T) = \frac{T}{\omega - 3} + L_{m\omega}T^{\frac{2}{3m(1+\omega)}} + C_1 T^{-\frac{(1-3\omega)}{2(1+\omega)}}, \quad (47)
\]

where \( L_{m\omega} = \frac{9m^3(1-3\omega)(\rho_0(1-3\omega))^{\frac{-2}{4+3m(1-3\omega)}}}{4 + 3m(1-3\omega)} \) and \( C_1 \) is arbitrary constant of integration, \( L_{m\omega} \) is finite and real valued unless \( 4 + 3m(1-3\omega) = 0 \). In general, the
function $f(T)$ is real valued if $m$ and $\omega$ do not satisfy the relation $m = \frac{4}{3(1-3\omega)}$ and if $\omega > 3$. Therefore, the power law solutions exist for $R + 2f(T)$ gravity.

For $m = 0$, we have $a = a_0$, so that $H = R = 0$, it represents the Einstein static universe. The solution for this case is

$$f(T) = \frac{T}{\omega - 3} + C_1 T^{-\frac{(1-3\omega)}{2(1+\omega)}}.$$  \hspace{1cm} (48)

The standard Einstein gravity can be recovered for the choice $C_1 = 0$, $m = \frac{2}{3(1+\omega)}$, and $\rho_0 = \frac{4}{3(1+\omega)^2}$. In order to develop a more general form of function $f(T)$, we put $m = \frac{2}{3(1+\omega)}$, so that

$$f(T) = \frac{T}{\omega - 3} + a_{n\omega} T^n,$$  \hspace{1cm} (49)

where $a_{n\omega} = \frac{2^{3-2n}3^{n-1}n^{1-2n}(1-3\omega)^{1-n}(1+\omega)^{2n-2}}{4(1+\omega)^2 + 2n(1-3\omega)}$. We can ensure that this theory reduces to GR for $n = 1$. It is remarked that phantom power law solutions exist for this form of $f(R, T)$ gravity, which can be obtained in a similar fashion as in section 4.1.1.

### 4.2.1 Constraining $f(R, T) = R + 2f(T)$ Gravity

The effective energy density $\rho_{\text{eff}}$ and effective pressure $p_{\text{eff}}$ for this particular $f(R, T)$ gravity are defined as

$$\rho_{\text{eff}} = \rho + 2(\rho + p)f_T + f, \quad p_{\text{eff}} = p - f.$$  \hspace{1cm} (50)

Using Eq.(50) in energy conditions (17)-(20), the following form is obtained\textsuperscript{26}

- **NEC**: $(\rho + p)[1 + 2f_T] \geq 0$
- **WEC**: $\rho + 2(\rho + p)f_T + f \geq 0$, $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$
- **SEC**: $\rho + 3p + 2(\rho + p)f_T - 2f \geq 0$, $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$
- **DEC**: $\rho - p + 2(\rho + p)f_T + 2f \geq 0$, $\rho_{\text{eff}} \geq 0$, $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$

To check how these conditions place bounds on power law solution (49) in $R + \lambda T$ gravity we put $p = 0$, so that $T = \rho$. Hence, the function $f(\rho)$ is of the form

$$f(\rho) = -\frac{\rho}{3} + a_n \rho^n,$$  \hspace{1cm} (51)
where $a_n = \frac{2^{(1-n)3^n-1}n^{3-n}}{n+2}$. The constraints to accomplish the above energy conditions are obtained as follows:

\[
\begin{align*}
\text{NEC} : & \quad \frac{\rho}{3} + 2na_n\rho^n \geq 0, \\
\text{WEC} : & \quad \frac{\rho}{3} + 5na_n\rho^n \geq 0, \\
\text{SEC} : & \quad \frac{4\rho}{3} + 2(2n + 1)a_n\rho^n \geq 0, \\
\text{DEC} : & \quad \rho + (7n + 2)a_n\rho^n \geq 0.
\end{align*}
\]

Above conditions are trivially satisfied for $n = 0, 1$. The quantities $na_n$, $(2n+1)a_n$ and $(7n+2)a_n$ are negative when $n = \{-3, -4, -5, \ldots\}$ and positive for $n = \{-1, 2, 3, \ldots\}$. Since $\rho$ is assumed to be positive, so it is obvious that these conditions are satisfied within the range of $n = \{-1, 2, 3, \ldots\}$.

## 5 Stability of Power Law Solutions

In this section, we are interested to study the stability of power law solutions against linear perturbations in $f(R, T)$ gravity. First, we assume a general solution $H(t) = H_h(t)$ for the cosmological background of FRW universe that satisfies Eqs.(23) and (44). The matter fluid is assumed to be dust and evolution of the matter energy density can be expressed in terms of $H_h(t)$ as

\[\rho_h(t) = \rho_0 e^{-\int H_h(t) dt},\]

where $\rho_0$ is an integration constant. Since the matter perturbations also contribute to the stability, so we introduce perturbations in Hubble parameter and energy density to study the perturbation around the arbitrary solution $H_h(t)$ as follows\(^{30}\)

\[H(t) = H_h(t)(1 + \delta(t)), \quad \rho(t) = \rho_h(1 + \delta_m(t)).\]  

In the following, we develop perturbation equations for two specific cases $f(R, T) = f(R) + \lambda T$ and $f(R, T) = R + 2f(T)$.

### 5.1 $f(R, T) = f(R) + \lambda T$

To study the linear perturbations, we expand function $f(R)$ in powers of $R_h$ evaluated at $H(t) = H_h(t)$ as

\[f(R) = f^h + f^R_h(R - R_h) + O^2,\]
where function $f(R)$ and its derivative are evaluated at $R_h$. The term $O^2$ includes all the terms proportional to the square or higher powers of $R$. The Ricci scalar $R$ at $H(t) = H_h(t)$ is given by

$$R_h = -6(\dot{H}_h + 2H^2_h). \quad (55)$$

By introducing the expressions (53) and (54) in the FRW equation (23), the equation for the perturbation $\delta(t)$ becomes

$$\dot{\delta}(t) + c(t)\delta(t) = \frac{A\rho_h}{3H_hR_hf_{RR}^h}\delta_m, \quad (56)$$

where

$$c(t) = \frac{d}{dt}\left[\ln\left(H_h^{-1}R_h^2f_{RR}^h\right)\right] + H^2_h[2(\frac{d}{dt}[\ln(f^h_R)])^{-1} - 1].$$

The conservation equation (25) implies the second perturbation equation as

$$\dot{\delta}_m(t) + 3H_h(t)\delta(t) = 0. \quad (57)$$

We can eliminate $\delta(t)$ from Eqs. (56) and (57) and arrive at the following second-order perturbation equation

$$\ddot{\delta}_m(t) + c_1(t)\dot{\delta}_m(t) + \frac{A\rho_h}{3H_hR_hf_{RR}^h}\delta_m = 0, \quad (58)$$

where

$$c_1(t) = \frac{d}{dt}\left[\ln\left(H_h^{-2}R_h^2f_{RR}^h\right)\right] + H^2_h[2(\frac{d}{dt}[\ln(f^h_R)])^{-1} - 1].$$

Here, we consider the $f(R, T)$ model proposed in section 4.1 for the dust case which is defined as $f(R) = \alpha_n(-R)^n + \lambda T$. We evaluate $f(R)$ and its derivatives at $H(t) = H_h(t)$ and hence the perturbation $\delta_m(t)$ is given by

$$\delta_m(t) = C_+t^{\mu_+} + C_-t^{\mu_-}, \quad (59)$$

where $C_\pm$ are arbitrary constants and

$$\mu_\pm = \frac{8n^2 - 15n + 13}{6(n - 1)} \pm \frac{\sqrt{n^2(8n^2 - 15n + 3)^2 + 18(8n^3 - 21n^2 + 16n - 3)\rho_0}}{6n(n - 1)}.$$
In order to study the stability of perturbation given by Eq. (59), one needs to check the signs of exponents $\mu _\pm$. The exponents are found to be negative provided that $n \leq -2$, otherwise $\mu _\pm$ would be positive and the perturbation is unstable. The perturbation $\delta (t)$ is found to be

$$
\delta (t) = \frac{-1}{3H_h} (C_+ \mu_+ t^{\nu_+} + C_- \mu_- t^{\nu_-}), \quad (60)
$$

where $\nu _\pm = \mu _\pm - 1$. It can be seen that exponent $\nu _+$ is negative for $n \leq -2$ and $\nu _-$ is always negative. Hence, as the time evolves the condition $n \leq -2$ ensures the decay of perturbations $\delta (t)$ and $\delta_m (t)$ which implies the stability of power law solution for this $f(R, T)$ gravity.

### 5.2 $f(R, T) = R + 2f(T)$

We explore the behavior of perturbations (53) for this $f(R, T)$ model and expand the function in powers of $T_h (= \rho_h)$ as (60)

$$
f(T) = f^h + f^h_T (T - T_h) + \mathcal{O}^2, \quad (61)
$$

where $\mathcal{O}$ term includes all the terms proportional to the squares or higher powers of $T$. The function $f(T)$ and its derivatives are evaluated at $T = T_h$. Using Eqs. (53) and (61) in FRW equation (44), it follows that

$$
(T_h + 3T_h f^h_T + 2T_h^2 f^h_{TT}) \delta_m (t) = 6H_h^2 \delta (t). \quad (62)
$$

Combining Eqs. (57) and (62), the first order matter perturbation equation is

$$
\dot{\delta}_m (t) + \frac{1}{2H_h} (T_h + 3T_h f^h_T + 2T_h^2 f^h_{TT}) \delta_m (t) = 0. \quad (63)
$$

which leads to

$$
\delta_m (t) = C_4 \exp \left\{ \frac{-1}{2} \int C_T dt \right\}, \quad C_T = \frac{T_h}{H_h} (1 + 3f^h_T + 2T_h f^h_{TT}). \quad (64)
$$

The behavior of perturbation $\delta (t)$ can be seen from the relation

$$
\delta (t) = \frac{C_4 C_T}{6H_h^3} \exp \left\{ \frac{-1}{2} \int C_T dt \right\}. \quad (65)
$$
We explore the stability of power law model (proposed in section 4.2) of the form
\[ f(T) = a_1 T + a_2 T^n, \] (66)
where \(a_1\) and \(a_2\) are parameters. One can evaluate the expression \(C_T\) and integral \(-\frac{1}{2} \int C_T dt\) for the model (64) as
\[ C_T = \frac{3}{2n} \left[ \rho_0 (3a_1 + 1) t^{-2n+1} + a_2 \rho_0^n n (2n + 1) t^{-2n^2+1} \right], \] (67)
\[ -\frac{1}{2} \int C_T dt = \frac{3}{8n(n-1)} \left[ \rho_0 (3a_1 + 1) t^{-(2n-1)} + a_2 \rho_0^n n (2n + 1) t^{-2(n^2-1)} \right]. \] (68)
As the time evolves, we need to set the conditions for decay of perturbations. It is obvious that expression (67) and (68) decay as time increases for the choice \(n > 1\) which results in decay of \(\delta(t)\) and \(\delta_m(t)\). Hence, for large values of \(t\) perturbation decays, this corresponds to the stability of power law solutions for \(R + 2f(T)\) gravity. We find that the conditions developed for stability are compatible with some constraints to fulfil the energy conditions. Hence, we may remark that power law solutions are acceptable regarding to the stability, energy conditions and late time acceleration of the universe.

6 Conclusions

The issue of accelerated expansion of the universe can be explained by taking into account the modified theories of gravity such as \(f(R, T)\) gravity. The \(f(R, T)\) gravity provides an alternative way to explain the current cosmic acceleration with no need of introducing either the existence of extra spatial dimension or an exotic component of DE. In this modified gravity, cosmic acceleration may result not only due to geometrical contribution to the total cosmic energy density but it also depends on matter contents. This theory depends upon matter source term, so each choice of matter Lagrangian \(\mathcal{L}_m\) would generate a specific set of field equations. The various forms of Lagrangian in this gravity give rise to question how to constrain the \(f(R, T)\) gravity theories on physical grounds. We have made an attempt to address this issue and classify the particular \(f(R, T)\) models. In this respect, we have developed the energy conditions on general as well as particular forms of this gravity. The energy conditions in modified theories of gravity have a well defined physical motivation, i.e., Raychaudhuri’s equation along with attractive nature of gravity.
The energy conditions and exact power law solutions are studied in the framework of \( f(R, T) \) gravity. We have derived the energy conditions directly from effective energy-momentum tensor approach under the transformation \( \rho \rightarrow \rho_{\text{eff}} \) and \( p \rightarrow p_{\text{eff}} \). The general inequalities imposed by these conditions are presented in terms of deceleration \((q)\), jerk \((j)\) and snap \((s)\) parameters. In order to get some insights on the application of these conditions, we consider two particular forms of \( f(R, T) \) gravity, i.e., \( f(R, T) = f(R) + \lambda T \) and \( f(R, T) = R + 2f(T) \). In standard paradigm, the expansion history of the universe underwent a power law decelerating phase followed by late time acceleration. Therefore, power law solutions are important in cosmology to represent the matter dominated phase that later connects to an accelerating phase. We have shown that exact power law solutions exist for a special class of \( f(R, T) \) models. These solutions mimic the \( \Lambda \)CDM model as particular case. We have obtained the necessary constraints to fulfil the energy conditions for this particular class of \( f(R, T) \) gravity. We summarize the results of these two models as follows:

- \( f(R) + \lambda T \)

It is shown that exact power law solution exists for this form of \( f(R, T) \) gravity given in Eq. (32). In the limit of \( \lambda = 0 \), the corresponding result can be recovered in \( f(R) \) gravity. To ensure that this theory reduces to GR, we need to set \( C_1 = C_2 = \lambda = 0 \), with \( m = \frac{2}{3(1+w)} \) and \( \rho_0 = \frac{4}{3(1+w)^2} \). We have constructed the general form of \( f(R) + \lambda T \) model which corresponds to \( R^n \) gravity. For this particular model of \( f(R, T) \) gravity, the NEC and SEC are derived from the Raychaudhuri equation together with the condition that gravity is attractive. It is shown that these conditions differ from those derived in the context of GR and \( f(R) \) gravity\(^{19}\). The general expression of weak and dominant energy conditions are obtained by introducing the effective energy-momentum tensor in the context of GR. We have examined the WEC bounds on \( \alpha_n (R)^n + \lambda T \) model in terms of present day observational values \( H_0 \), \( q_0 \), \( j_0 \) and \( s_0 \).

- \( R + 2f(T) \)

The model \( f(R, T) = R + 2f(T) \) corresponds to gravitational Lagrangian with time dependent cosmological constant being function of trace of the energy-momentum tensor\(^{31}\). This model appears to be interesting and has widely been studied in literature\(^{8-12}\). A general form of \( f(T) \) model \(^{49}\)
is obtained which corresponds to GR in the limit $n = 1$. We have applied the energy conditions to set the possible constraints on this $f(R,T)$ model. It is found that energy conditions are globally satisfied within the range of $n = \{-1, 2, 3, \ldots\}$. It is worth mentioning here that results of power law solutions and energy conditions obtained in this paper are quite general which correspond to GR and $f(R)$ gravity.

We have also analyzed the stability of power law solutions under linear homogeneous perturbations in the FRW background for $f(R,T)$ gravity. In particular, perturbations for energy density and Hubble parameter are introduced which produce linearized perturbed field equations. It is shown that stability/instability can be studied for particular $f(R,T)$ models under some restrictions. The stability conditions are found to be compatible with energy conditions bounds to some extent. Hence, power law solutions in $f(R,T)$ gravity can be considered as viable models to explain the cosmic history of the universe. It is also interesting to note that for these $f(R,T)$ models, the gravitational coupling becomes an effective and time dependent coupling which modifies the gravitational interaction between matter and curvature. Our analysis shows that stable power law solutions are contained in class of $f(R,T)$ models, at least considering a given background evolution of the universe.

Acknowledgment

The authors would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the Indigenous Ph.D. 5000 Fellowship Program Batch-VII.

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