Regular poles of local $L$-functions for $\text{GSp}(4)$ with respect to split Bessel models (the subregular cases)

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Piatetskii-Shapiro has defined local spinor $L$-factors for infinite-dimensional irreducible representations of $\text{GSp}(4, k)$ over a local non-archimedean field $k$, attached to a choice of a Bessel model. We classify the so-called subregular poles, these are the regular poles that do not come from the asymptotic of the Bessel functions. For anisotropic Bessel models there are no subregular poles.

1. Introduction

For infinite-dimensional irreducible smooth representations $\Pi$ of the symplectic group of similitudes $G = \text{GSp}(4, k)$, where $k$ is a local non-archimedean field, and a smooth character $\mu$ of $k^*$, Piatetskii-Shapiro [PS97] has constructed local $L$-factors

$$L^\text{PS}(s, \Pi, \mu, \Lambda)$$

attached to a choice of a Bessel model $\Lambda$ of $\Pi$. It was already expected by Piatetskii-Shapiro and Soudry [PSS81] that these $L$-factors do not depend on the choice of the Bessel model and they proved this in the special case of unitary fully Borel induced irreducible representations $\Pi$ and unitary characters $\Lambda$ [PSS81, Thm. 2.4]. The $L$-factor $L^\text{PS}(s, \Pi, \mu, \Lambda)$ factorizes into a regular and an exceptional part. Piatetskii-Shapiro stated that generically the regular poles should come from the asymptotic of the Bessel functions. However, the precise conditions of this expectation were left open in the formulation of [PS97, thm. 4.1]. Notice, more or less by definition, the poles coming from the asymptotic of the Bessel functions form the $L$-factor of the Bessel module. Danişman has shown that Piatetskii-Shapiro’s expectation holds true for the case of anisotropic Bessel models [D14, proposition 2.5]. Danişman

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[D14, D15a, D15b, D17] and the authors [RW18] explicitly determined the regular and exceptional $L$-factors.

In this paper we consider the case of split Bessel models, where in contrast to the anisotropic case certain regular poles are not determined by the asymptotic of the Bessel functions [RW17, 3.39]. We call such poles subregular poles. We determine them explicitly for every irreducible $\Pi$ and every split Bessel model, see table 2. This is related to the classification of certain $H_+ -$functionals on the representation $\Pi$, and this could be of independent interest. The results clarify the above mentioned expectations and supplement the results of [RW17] and [W17], where the other regular and exceptional poles were computed. In the following, we freely use notation from these two papers.

As a consequence we show that indeed the $L$-factor of Piatetskii-Shapiro is independent of the choice of the Bessel model $\Lambda$ and coincides with the $L$-factor of the representation of the Weil-Deligne group attached to $\Pi$ via the local Langlands correspondence, as computed by Roberts and Schmidt in table A.8 of [RS07]. In particular our results are in complete accordance with the local Langlands correspondence for $\text{GSp}(4)$ as established by Gan and Takeda [GT11]. Note that these authors used a different construction for the $L$-factors. It is interesting that Piateskii-Shapiro’s approach to $L$-series gives rise Euler systems for $\text{GSp}(4)$ as shown by Loeffler, Skinner and Zerbes [LSZ17]. See also the work of Lemma [L08, L17]. The authors are grateful to David Loeffler for comments and suggestions.

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2. Piatetskii-Shapiro’s local spinor $L$-factor for $GSp(4)$

The group of symplectic similitudes $G = GSp(4)$ is defined by the equation

$$gJg^t = \lambda_G(g)J$$

for $g \in \text{Gl}(4)$ and a scalar $\lambda_G(g) \in \text{Gl}(1)$ where we use Siegel’s notation $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Fix an irreducible smooth representation $\Pi$ of $G$ with central character $\omega$ and a smooth Bessel character $\Lambda = \rho\otimes \rho^*$ of the split torus $\tilde{T} = \{\text{diag}(t_1, t_2, t_2, t_1) \in G\} \cong k^\times \times k^\times$.

Attached to $v \in \Pi$ and Schwartz-Bruhat functions $\Phi \in C^\infty_c(k^4)$ is the zeta-function

$$Z_{PS}(s, v, \Lambda, \Phi, \mu) = \int_{\mathbb{N}\setminus H} W_v(h)\Phi((0, 0, 1, 1)h)\mu(\lambda_G(h))|\lambda_G(h)|^{s+\frac{1}{2}} \, dh.$$  

The integral converges for sufficiently large $\text{Re}(s) > 0$ and admits a unique meromorphic continuation to the complex plane. The local spinor $L$-factor

$$L_{PS}(s, \Pi, \mu, \Lambda)$$

attached to $\Pi$ and $\Lambda$ by Piatetskii-Shapiro [PSS81] is the regularization $L$-factor of the zeta functions $Z_{PS}(s, v, \Lambda, \Phi, \mu)$ varying over all $v \in \Pi$ and $\Phi \in C^\infty_c(k^4)$. The isomorphism $k^2 \times k^2 \rightarrow k^4$, $((x_1, y_1), (x_2, y_2)) \rightarrow (x_1, x_2, y_1, y_2)$ defines inclusions

$$0 \hookrightarrow C^\infty_c(k^2 \setminus \{0\} \times k^2 \setminus \{0\}) \hookrightarrow C^\infty_c(k^4 \setminus \{0\}) \hookrightarrow C^\infty_c(k^4)$$

and thus a filtration of $C^\infty_c(k^4)$ with quotients isomorphic to

$$C^\infty_c(k^2 \setminus \{0\}) \otimes C^\infty_c(k^2 \setminus \{0\}) \quad \text{and} \quad C^\infty_c(k^2 \setminus \{0\}) \oplus C^\infty_c(k^2 \setminus \{0\}) \quad \mathbb{C}.$$  

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The regularization $L$-factor of the family of zeta-functions $Z_{PS}(s,v,\Lambda,\Phi,\mu)$ varying over $v \in \Pi$ and $\Phi \in C_c^\infty(k^2 \setminus \{0\} \times k^2 \setminus \{0\})$ coincides with the regularization $L$-factor of the regular zeta-functions

$$Z_{\text{reg}}(s,W_v,\mu) = \int_T W_v(x_\lambda)\mu(\lambda)|\lambda|^{-\frac{3}{2}}\,dx_\lambda, \quad v \in \Pi,$$

again convergent for sufficiently large $\text{Re}(s) > 0$ with unique meromorphic continuation, see [RW17, 3.39] and compare Danisman [D15a] in the anisotropic case. This $L$-factor equals $L(s,\mu \otimes M)$ for $M = \nu^{-3/2} \otimes \beta_\mu(\Pi)/\beta_\mu(\Pi)^S$. It has been determined explicitly by the authors [RW17] for every $\Pi$ and $\Lambda$.

The subregular factor $L_{\text{reg}}^{PS}(s,\Pi,\mu,\Lambda)$ is the regularization $L$-factor of

$$\frac{Z^{PS}(s,v,\Lambda,\Phi,\mu)}{L(\mu \otimes M, s)},$$

varying over $v$ and $\Phi \in C_c^\infty(k^4 \setminus \{0\})$. In this work we determine the subregular factor explicitly as a product of Tate factors.

The exceptional factor $L_{\text{ex}}^{PS}(s,\Pi,\mu,\Lambda)$ is the regularization $L$-factor of

$$\frac{Z^{PS}(s,v,\Lambda,\Phi,\mu)}{L(\mu \otimes M, s)L_{\text{reg}}^{PS}(s,\Pi,\mu,\Lambda)},$$

varying over $v$ and $\Phi \in C_c^\infty(k^4)$. For the explicit values, see [W17]. Hence the $L$-factor of Piatetskii-Shapiro factorizes

$$L^{PS}(s,\Pi,\mu,\Lambda) = L_{\text{reg}}^{PS}(s,\Pi,\mu,\Lambda)L_{\text{ex}}^{PS}(s,\Pi,\mu,\Lambda)$$

where the regular factor is

$$L_{\text{reg}}^{PS}(s,\Pi,\mu,\Lambda) = L(\mu \otimes M, s)L_{\text{reg}}^{PS}(s,\Pi,\mu,\Lambda).$$

2.1. Notation

Fix a local non-archimedean number field $k$ of characteristic not two with ring of integers $\mathfrak{o}_k$ and its split quadratic extension $K = k \times k$. We also fix the additive Haar measure on $k$ with $\text{vol}(\mathfrak{o}) = 1$ and the multiplicative Haar measure on $k^\times$ with $\text{vol}(\mathfrak{o}^\times) = 1$.

The group $G = \text{GSp}(4)$ of symplectic similitudes $g \in \text{Gl}(4)$ is defined by the equation $gJg^t = \lambda_G(g)J$ with a scalar $\lambda_G(g) \in \text{Gl}(1)$ where we use Siegel’s notation $J = \left( \begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix} \right)$. The center $Z_G = \{ \lambda \cdot I_4 | \lambda \in \text{Gl}(1) \}$ is the subgroup of
scalar matrices. Fix the standard Siegel parabolic $P$ and the standard Klingen parabolic $Q$

$$P = \left\{ \begin{pmatrix}
* & * & * & *
* & * & * & *
0 & 0 & * & *
0 & 0 & * & *
\end{pmatrix} \right\} \cap G , \quad Q = \left\{ \begin{pmatrix}
* & * & * & *
0 & * & * & *
0 & 0 & * & 0
0 & 0 & * & *
\end{pmatrix} \right\} \cap G$$

and the standard Borel $B_G = P \cap Q$. The subgroups $T, \tilde{T}, \tilde{N} = S_A \times S_C, S$ of $G$ are defined in [RW17] and $N = \tilde{N}S$ is the unipotent radical of $P$. We denote by $Q_1$ the subgroup of $Q$ of matrices with third diagonal entry 1. The Weyl group of $G$ has order eight and is generated by

$$s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \end{pmatrix} , \quad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \end{pmatrix} .$$

The fiber product over the determinant $\text{Gl}(2) \times \text{Gl}(1) \text{Gl}(2)$ is identified with its image $H$ in $G$ under the embedding

$$\left( \begin{pmatrix} a_1 & b_1 \\
c_1 & d_1 \end{pmatrix} , \begin{pmatrix} a_2 & b_2 \\
c_2 & d_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\
0 & a_2 & 0 & b_2 \\
c_1 & 0 & d_1 & 0 \\
0 & c_2 & 0 & d_2 \end{pmatrix} \in G .$$

In the following $\text{Gl}(2)$ will always denote the image of $\text{Gl}(2)$ under the embedding $\text{Gl}(2) \hookrightarrow H$ via $g \mapsto (\text{diag}(\det(g), 1), g)$. The maximal parabolic subgroup $H_+ \subseteq H$ defined by $c_1 = 0$ factorizes as

$$H_+ = Z_G \cdot \text{Gl}(2) \cdot S_A .$$

The affine linear group is

$$\text{Gl}_a(n) = \left\{ [g \mid v] := \begin{pmatrix} g & v \\
0 & 1 \end{pmatrix} \mid g \in \text{Gl}(n), \ v \in k^n \right\} \subseteq \text{Gl}(n + 1) .$$

We always identify the above groups with their groups of $k$-valued points. That means we write $\text{Gl}(n)$ or $\text{GSp}(4)$ meaning $\text{Gl}(n, k)$ resp. $\text{GSp}(4, k)$. 

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3. Subregular poles

A subregular pole is a pole of the subregular $L$-factor $L_{sreg}^{PS}(s, \Pi, \mu, \Lambda)$.

**Proposition 3.1.** The subregular poles of $\Pi$ are the poles of the functions

1. Type 1:

$$\frac{Z_{sreg}^{PS}(s, W_v, \mu)}{L(\mu \otimes M, s)} L(s, v^{1/2} \rho \mu)$$

for $v \in \Pi$ invariant under the compact group $\text{id} \times \text{Sl}(2, o_k) \subseteq H$,

2. Type 2:

$$\frac{Z_{sreg}^{PS}(s, W_v, \mu)}{L(\mu \otimes M, s)} L(s, v^{1/2} \rho^* \mu)$$

for $v \in \Pi$ invariant under the compact group $\text{id} \times \text{Sl}(2, o_k) \subseteq H$.

**Proof.** We set $\mu = 1$ by a suitable twist. Recall that a subregular pole is a pole of $Z^{PS}(s, v, \Lambda, \Phi, \mu)/L(M, s)$ for some $\Phi$ with $\Phi(0, 0, 0, 0) = 0$. Without loss of generality, we can assume

1. $\Phi(x_1, x_2, y_1, y_2) = \Phi_1(x_1, y_1) \Phi_2(x_2, y_2)$ for $x, y \in K$, because these $\Phi$ span the tensor product $C^\infty_c(k \oplus k) \otimes C^\infty_c(k \oplus k)$,

2. $\Phi_2(0, 0) \neq 0$ but $\Phi_1(0, 0) = 0$, otherwise apply the Weyl involution $s_1$ that switches $\rho$ and $\rho^*$, compare [RW17, lemma 4.17],

3. $\Phi_2$ is the characteristic function of $o_k \oplus o_k$, otherwise add a suitable Schwartz function with support in $k^2 \backslash \{0\} \times k^2 \backslash \{0\}$.

By Iwasawa decomposition, $H = H_+ \cdot (\text{Sl}(2, o_k) \times \{\text{id}\})$, so for a suitable choice of Haar measure and sufficiently large $\text{Re}(s)$,

$$Z^{PS}(s, v, \Lambda, \Phi, 1) = \int_{\text{Sl}(2, o_k)} Z^{PS}_{sreg}(s, \Pi(k_1, 1)v, \Lambda, \Phi_1^{k_1} \otimes \Phi_2) \, dk_1$$

with $\Phi_1^{k_1}(x_1, y_1) = \Phi_1((x_1, y_1)k_1)$ and the partial zeta integral

$$Z^{PS}_{+}(s, v, \Lambda, \Phi, 1) = \int_{N \backslash H_+} W_v(h) \Phi((0, 0, 1, 1)h) |\lambda_G(h)|^{s + \frac{1}{2}} \delta_{H_+}^{-1}(h) \, d_R h,$$

with a right Haar measure $d_R h$ on $H_+$, see [B98 2.1.5(ii)] and [BZ76 1.21]. The modulus factor is defined by $\delta_{H_+}(g) = |\text{det}(g)|$ for $g \in \text{Gl}(2)$ embedded into $O_1$ as above, $\delta_{H_+}$ is trivial on the center $Z_G$ and on $N$. It is defined such that $\delta_{H_+}^{-1}(h) d_R h$ a left-invariant Haar measure.
The $k_1$-integral is a finite sum, so every pole of $Z^\text{PS}(s, v, \Lambda, \Phi, \mu)$ comes from a pole of the partial zeta integral. Conversely, $\Phi_1$ can be chosen arbitrarily in $C^\infty_c(k^2 \setminus \{0\})$, so every pole of the partial zeta integral gives rise to a pole of $Z^\text{PS}(s, v, \Lambda, \Phi, \mu)$ for some $v$ and $\Phi$. Hence the subregular poles are the poles of the partial zeta integrals varying over $\Phi_1$ and $v$.

By Iwasawa decomposition, every $h \in H_+$ factorizes as $h = n_x \lambda t(1, k_2) \in \tilde{N}T\tilde{T}(1 \times \text{Sl}(2, o_k))$ for $\tilde{t} = \text{diag}(t_1, t_2, t_2, t_1)$. The Bessel function transforms with $W_v(t(1, k_2)) = \rho(t_1) \rho^*(t_2)W_{\Pi(1, k_2)v}(t)$, so the partial zeta integral is

$$
\int_{\text{Sl}(2, o_k)} Z_{\text{reg}}^\text{PS}(s, W_{\Pi(1, k_2)v}) \, dk_2 \int_{\tilde{T}} \Phi_1((0, t_2)) \Phi_2((0, t_1)) \Lambda(\tilde{t}) |t_1 t_2|^{s+1/2} \, d\tilde{t} =
$$

$$
Z_{\text{reg}}^\text{PS}(s, W^\text{av}_v) \int_{k^2} \Phi_1(0, t_2) \rho^*(t_2)|t_2|^{s+1/2} \, dt_2 \int_{k^2} \Phi_2(0, t_1) \rho(t_1)|t_1|^{s+1/2} \, dt_1
$$

with the average $W^\text{av}_v(g) = \int_{\text{Sl}(2, o_k)} W_v(g(1, k_2)) \, dk_2$ over $\text{id} \times \text{Sl}(2, o_k) \subseteq H$.

The integral over $t_1$ in the last line equals $L(s, \nu^{1/2} \rho)$ for unramified $\rho$ and is zero otherwise. The integral over $t_2$ is holomorphic in $s$ because the integrand has compact support with respect to $t_2$. By choice of $\Phi_1$ we can arrange that the integral over $t_2$ is a non-zero constant.

**Corollary 3.2.** Every subregular pole attached to $\Pi$ and its split Bessel model with Bessel datum $\Lambda = \rho \boxtimes \rho^*$ is of the form $L(s, \nu^{1/2} \rho \mu)$ or $L(s, \nu^{1/2} \rho^* \mu)$.

In the following it is sufficient to discuss subregular poles of type 1. Every pole of type 2 corresponds to a pole of type 1 after switching $\rho$ and $\rho^*$. This does not change the $TS$-module $M$ by [RW17, 4.19]. Without loss of generality, we will set $\mu = 1$ from now on by a suitable twist of $\Pi$.

### 3.1. Smooth representations

For a locally compact totally disconnected group $X$ fix the modulus character $\delta_X : X \to \mathbb{R}_{>0}$ with

$$
\int_X f(x x_0) \, dx = \delta_X(x_0) \int_X f(x) \, dx , \quad x, x_0 \in X , \quad f \in C^\infty_c(X)
$$

with respect to a left-invariant Haar measure on $X$. The abelian category of smooth complex-valued linear representations $\rho$ of $X$ is denoted $C_X$. The full subcategory of representations with finite length is $C_X^{\text{fin}}$. The smooth dual or contragredient of $\rho$ is denoted $\rho^\vee$. For a closed subgroup $Y \subseteq X$, the
unnormalized compact induction of \( \sigma \in \mathcal{C}_Y \) is the representation \( \text{ind}_Y^X(\sigma) \in \mathcal{C}_X \), given by the right-regular action of \( X \) on the vector space of functions \( f: X \to \mathbb{C} \) that satisfy \( f(y x k) = \sigma(y) f(x) \) for every \( y \in Y, x \in X \) and \( k \) in some open compact subgroup of \( X \) and such that \( f \) has compact support modulo \( Y \). This defines an exact functor \( \text{ind}_Y^X \) from \( \mathcal{C}_Y \) to \( \mathcal{C}_X \).

**Lemma 3.3** (Frobenius reciprocity). For smooth \( \rho \in \mathcal{C}_X \) and \( \sigma \in \mathcal{C}_Y \), there is a canonical isomorphism

\[
\text{Hom}_X(\text{ind}_Y^X(\sigma), \rho) \cong \text{Hom}_Y(\sigma, (\rho|_Y) \otimes \delta_Y) .
\]

**Proof.** See [BZ76, 2.29]. \( \square \)

**Example 3.4.** Irreducible smooth \( \text{Gl}(1) \)-modules are characters \( \chi: k^* \to \mathbb{C}^* \). The valuation character \( \nu: \text{Gl}(1) \to \mathbb{C}^* \) is defined by \( \nu(\lambda) = |\lambda| \). Every smooth \( \text{Gl}(1) \)-module \( X \in \mathcal{C}_{\text{fin}}^{\text{Gl}(1)} \) of finite length is a direct sum of Jordan blocks \( \chi^{(n)} \) with respect to the monodromy operator \( \tau_X = (X(\varpi) - \chi(\varpi)) \text{id}_X \) with a uniformizer \( \varpi \in o_k \).

Let \( \mathcal{C}_n = \mathcal{C}_{\text{Gl}_a(n)} \) denote the category of smooth representations of \( \text{Gl}_a(n) \). As in [RW17] we write \( j!, j^!, i_*, i^* \) for the unnormalized functors \( \Phi^+, \Phi^-, \Psi^+, \Psi^- \) of [BZ76].

**Lemma 3.5.** There is a functorial short exact sequence of excision type

\[
0 \to j! j^! \to \text{id} \to i_* i^* \to 0 .
\]

The composition \( j^! i_* \) and \( i^* j_! \) are both zero.

For each smooth character \( \rho \) of \( \text{Gl}(1) \), the functors \( k_\rho, k^\rho : \mathcal{C}_{n+1} \to \mathcal{C}_n \) are constructed in [RW17] together with a functorial short exact sequence

\[
0 \to j! k_\rho \to k_\rho j_! \to i_* i^* \to 0
\]

and a natural equivalence \( j! k^\rho \cong k^\rho j_! \). The embedding

\[
\text{GL}_a(2) \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} ad - bc & sc - ta & * \\ 0 & a & s \\ 0 & 0 & 1 \\ 0 & c & t \\ d \end{bmatrix} \in Q_1/S_A
\]

defines a functor \( \eta: \mathcal{C}_G(\omega) \to \mathcal{C}_2 \) by taking \( S_A \)-coinvariant quotients. We write \( \Pi = \eta(\Pi) = \Pi_{S_A} \in \mathcal{C}_2 \) for \( \Pi \in \mathcal{C}_G(\omega) \). This defines the Bessel functor \( \beta_\rho = k_\rho \circ \eta \) from \( \mathcal{C}_G(\omega) \to \mathcal{C}_1 \). The Bessel module of an irreducible \( \Pi \in \mathcal{C}_G(\omega) \) is denoted \( \tilde{\Pi} = \beta_\rho(\Pi) \) if \( \rho \) is clear from the context. The Gelfand-Graev-module is \( \mathbb{S}_n = (j!)^n(\mathbb{C}) \in \mathcal{C}_n \).
3.2. Sufficient condition

This condition will be verified in section[5] in order to show the existence of the subregular poles listed in table[2]

Lemma 3.6. Fix an irreducible \( \Pi \in \mathcal{C}_G(\omega) \) and a smooth character \( \rho \) of \( \text{GL}(1) \) and let \( \tilde{\Pi} = k_\rho(\Pi) \) and \( K = \text{GL}(2, \mathfrak{o}_k) \). If the composition of the canonical morphisms

\[
\Pi^K \to \Pi \to \tilde{\Pi} \to \tilde{\Pi}^S \to k_{\nu^2 \rho}(\tilde{\Pi}/\tilde{\Pi}^S)
\]

is non-trivial, then \( \rho \) is unramified, the Bessel character \( \Lambda = \rho \boxtimes \rho^* \) yields a split Bessel model of \( \Pi \) and the associated subregular \( L \)-factor \( L_{\text{PS}}^{\text{reg}}(s, \Pi, \Lambda, 1) \) admits a pole of the form \( L(s, \nu^{1/2} \rho) \).

Proof. \( \rho \) is unramified since otherwise the composition cannot be non-trivial. Put \( M := \nu^{-3/2} \otimes (\tilde{\Pi}/\tilde{\Pi}^S) \) as before. If \( \Lambda \) does not yield a Bessel model, then \( M \) would be zero. Since \( M \) is perfect, there is an embedding of \( TS \)-modules (unique up to scalars)

\[
M \hookrightarrow C_b^\infty(k^\times)
\]

and we identify \( M \) with its image [RW17 lemma 3.17]. This defines the Bessel model of the representation \( \Pi \) with respect to the split Bessel datum \( \Lambda \). Lemma 3.31 of [RW17] implies that for each \( s_0 \in \mathbb{C} \), the functional

\[
M \ni f \mapsto \lim_{s \to s_0} \frac{Z(s, f; 1)}{L(s, M)} = \lim_{s \to s_0} \frac{Z(s, f; \nu^{s_0})}{L(s, \nu^{s_0} \otimes M)} =: I_{\nu^{s_0}}(f)
\]

spans the one-dimensional space of homomorphisms

\[
\text{Hom}_T(\nu^{s_0} \otimes M, \mathbb{C}) = \text{Hom}_T(M, \nu^{-s_0})
\]

Now let \( s_0 \) be a pole of the Tate \( L \)-factor \( L(\nu^{1/2} \rho, s) \), then \( \nu^{-s_0} = \nu^{1/2} \rho =: \chi_{\text{crit}} \). In other words, \( I_{\nu^{s_0}} \) spans the one-dimensional space

\[
\text{Hom}_T(M, \chi_{\text{crit}}) \cong \text{Hom}_T(\tilde{\Pi}/\tilde{\Pi}^S, \nu^{3/2} \chi_{\text{crit}}) \cong \text{Hom}_T(k_\chi(\tilde{\Pi}/\tilde{\Pi}^S), \mathbb{C})
\]

By assumption the image of \( \Pi^K \) in \( k_\chi(\tilde{\Pi}/\tilde{\Pi}^S) \) is non-trivial. The map \( \Pi^K \to \tilde{\Pi}^K \) is surjective, because \( K \) is compact. Hence there is \( v \in \Pi^K \) such that its image \( f_v \in M \) satisfies \( I_{\nu^{s_0}}(f_v) \neq 0 \), or in other words, such that \( \lim_{s \to s_0} Z_{\text{PS}}^{\text{reg}}(s, W_v, 1)/L(s, M) \) is non-zero. Proposition 3.1 implies the statement. \( \square \)
3.3. Necessary condition

This condition will be used in section 4 in order to show that there are not more subregular poles than listed in table 2. Fix an irreducible smooth representation \( \Pi \in \mathcal{C}_G(\omega) \) and a smooth character \( \rho \) of \( \text{Gl}(1, k) \). We write \( \tilde{\rho} = \nu \rho \) where \( \nu(x) = |x| \) for \( x \in k^\times \).

**Definition 3.7.** An \((H_+, \tilde{\rho})\)-functional attached to \((\Pi, \rho)\) is a nontrivial functional \( \ell : \Pi \to \mathbb{C} \), such that

\[
\ell(\Pi(s_A z g)v) = \omega(z)\tilde{\rho}(\det(g)) \cdot \ell(v)
\]

for \( z \in Z_G, \ s_A \in S_A \) and \( g \in \text{Gl}(2) \). Every \( H_+ \)-functional factorizes over the projection \( \Pi \to \Pi = \Pi S_A \) and this defines an isomorphism

\[
\{(H_+, \tilde{\rho})\text{-functionals}\} \cong \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \det).
\]

**Proposition 3.8.** Assume that \( \Lambda = \rho \boxtimes \rho^* \) yields a split Bessel model for \( \Pi \). If \( L(\rho \nu^{1/2}, s) \) is a subregular pole of \( \Pi \) with a unramified character \( \rho \), then \( \Pi \) admits a nontrivial \((H_+, \tilde{\rho})\)-functional \( \ell \). Especially, \( \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \det) \neq 0 \).

**Proof.** Assume \( L(\rho \nu^{1/2}, s) \) divides \( L_{\text{reg}}(s, \Pi, 1, \Lambda) \) and admits a subregular pole \( s_0 \in \mathbb{C} \). Without loss of generality we can assume that this pole belongs to type 1 in the sense of prop 3.1 since otherwise \( \rho = \rho^* \) holds and we can apply the Weyl involution \( s_1 \). Then this pole comes from a pole of a partial zeta integral \( Z_{s_0}^+(s, W_v, \Phi, \Lambda, 1) \) with \( \Phi = \Phi_1 \boxtimes \Phi_2 \) and \( \Phi_2(0, 0) \neq 0 \) as in the proof of proposition 3.1. The functional \( \ell \) is then defined as the residue functional

\[
f(v) = \text{res}_{s=s_0} Z_+(s, W_v, \Phi, \Lambda, 1)/L(s, M).
\]

It is non-trivial because the pole has order one. The functional \( \ell \) must factorize over the evaluation \( \Phi_2 \mapsto \Phi_2(0, 0) \). The partial zeta-integral is \( S_A \)-invariant because \( S_A \) is a normal in \( H_+ \). The center clearly acts with the central character \( \omega \) of \( \Pi \). The action of \( g \in \text{Gl}(2) \to H_+ \) on \( f(v) \) is by multiplication with

\[
\nu^{-s_0 - \frac{1}{2}}(\det g)\delta_{H_+}(g) = \nu^{-s_0 + \frac{1}{2}}(\det g).
\]

The pole at \( s = s_0 \) is a pole of \( L(\rho \nu^{1/2}, s) \), this means \( \rho \nu^{s_0 + \frac{1}{2}} \equiv 1 \). Hence \( g \in \text{Gl}(2) \) acts by multiplication with \( \nu \rho(\det g) = \tilde{\rho}(\det g) \). \( \square \)

Note that we did not discuss subregular poles of type 2. In fact, after switching \( \rho \) and \( \rho^* \) they become type 1 subregular poles. Hence we can apply our results also for type 2 subregular poles. The subregular \( L \)-factor \( L(\chi_{\text{crit}}, s) \) only depends on \( \Pi \) and the pair \((\rho, \rho^*)\), where the roles of type 1 and type 2 poles may switch.
4. Classification of $H_+^*$-functionals

In this section we classify the $(H_+, \tilde{\rho})$-functionals attached to irreducible smooth representations $\Pi$ of $G$ and smooth characters $\tilde{\rho}$ of $k^\times$. In other words, we determine $\dim \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \det)$. The final result is

**Theorem 4.1.** Suppose $\Lambda = \rho \boxtimes \rho^*$ defines a split Bessel model for $\Pi$.

1. If $\Pi$ is non-generic, then

$$\dim \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \det) = \begin{cases} 1 & \text{type IIb and } \tilde{\rho} \in \nu \Delta_+(\Pi) = \{ \nu \sigma, \nu \chi_1 \sigma \}, \\ 1 & \text{type IVc and } \tilde{\rho} = \nu^2 \sigma \in \nu \Delta_+(\Pi), \\ 1 & \text{type IVc and } \tilde{\rho} \in \nu \Delta_+(\Pi) = \{ \nu \sigma \}, \\ 1 & \text{type VId and } \tilde{\rho} \in \nu \Delta_+(\Pi) = \{ \nu \sigma \}, \\ 0 & \text{otherwise}. \end{cases}$$

These functionals all factorize over the projection $\Pi \to i^* (\Pi) = B$.

2. If $\Pi$ is generic, then

$$\dim \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \det) = \begin{cases} 1 & \text{type I, IIa, Va, Vla, X, XIa and } \tilde{\rho} \in \Delta_+(\Pi), \\ 0 & \text{otherwise}. \end{cases}$$

These functionals do not factorize over $B$.

For non-generic $\Pi$, see lemma 4.12. For generic $\Pi$, see theorem 4.14.

**Lemma 4.2.** For every irreducible $\Pi \in C_G(\omega)$ and every smooth character $\tilde{\rho}$ of $\text{Gl}(1)$, there is an embedding

$$\text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \det) \hookrightarrow \text{Hom}_T(k\tilde{\rho}(\Pi), \tilde{\rho}) \cong \text{Hom}_C(k\tilde{\rho}k\tilde{\rho}(\Pi), \mathbb{C}).$$

**Proof.** Every $\text{Gl}(2)$-equivariant map $\Pi \to (\mathbb{C}, \tilde{\rho} \circ \det)$ is obviously $B_{\text{Gl}(2)}$-equivariant for the standard Borel subgroup $B_{\text{Gl}(2)}$ of upper triangular matrices in $\text{Gl}(2)$. It thus factorizes over a unique $T$-linear map $k\tilde{\rho}(\Pi) \to (\mathbb{C}, \tilde{\rho})$ and hence a unique map $k\tilde{\rho}k\tilde{\rho}(\Pi) \cong k\tilde{\rho}k\tilde{\rho}(\Pi) \to \mathbb{C}$. \qed

In the following we tacitly assume that all $\Pi$ are normalized as [RW17, table 1]. Recall $\Delta_+ (\Pi) \cap \nu \Delta_+ (\Pi) \neq \emptyset$ if and only if $\Delta_- (\Pi) \cap \Delta_+ (\Pi) \neq \emptyset$ if and only if $\Pi$ is of type IIa with $\chi_1 = \nu^{\pm 1}$ by Lemma A.11 in [RW17]. Hence, for non-generic $\Pi$ the characters $\rho$ and $\tilde{\rho}$ both define a Bessel model only in these cases. For every other non-generic $\Pi$ with $\rho \in \Delta_+ (\Pi)$, the character $\tilde{\rho}$ is not in $\Delta_+ (\Pi)$.
Lemma 4.3. Suppose \( \rho \) is an arbitrary smooth character of \( k^\times \) that does not define a split Bessel model for a smooth irreducible representation \( \Pi \). Then \( \Pi \) is necessarily non-generic. The Bessel module \( \beta_\rho(\Pi) = k_\rho(\Pi) \) is given by

\[
\beta_\rho(\Pi) \cong \beta^\rho(\Pi) \oplus \bigoplus_{\chi \in \Delta_0(\Pi)} \nu^{3/2} \chi.
\]

Every \( T \)-character occurs at most once, so \( \beta_\rho(\Pi) \) is semisimple.

Proof. See lemma 5.25 and table 7 in [RW17]. For the list of constituents, see [RW17, table 3, prop. 6.3.3].

Lemma 4.4. \( \dim \beta_{\tilde{\rho}}(\Pi)_{T,\tilde{\rho}} \leq 1 \) holds for every irreducible \( \Pi \in C_G(\omega_\Pi) \) and smooth characters \( \rho \). If \( \rho \) provides a Bessel model for \( \Pi \), then \( \dim \beta_{\tilde{\rho}}(\Pi)_{T,\tilde{\rho}} = 1 \).

Proof. Suppose \( \tilde{\rho} \) defines a Bessel model for \( \Pi \), i.e. \( \deg(\beta_{\tilde{\rho}}(\Pi)) = 1 \). Then \( \dim \beta_{\tilde{\rho}}(\Pi)_{T,\chi} = 1 + \dim \beta_{\tilde{\rho}}(\Pi)_{T,\chi} \) holds for every smooth character \( \chi \) [RW17, lemma 3.12]. It has been shown that \( \dim \beta_{\tilde{\rho}}(\Pi)_{T,\chi} \) vanishes for every smooth character \( \chi \neq \nu^2 \rho \) [RW17, theorem 6.6]. If \( \tilde{\rho} \) does not define a Bessel model for \( \Pi \), then \( \dim \beta_{\tilde{\rho}}(\Pi)_{T,\tilde{\rho}} \leq 1 \) by lemma 4.3 because every \( T \)-character occurs at most once in \( \beta_{\tilde{\rho}}(\Pi) \).

Lemma 4.5. For every \( (\Pi, \rho) \),

\[
\dim \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \text{det}) \leq \dim \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \otimes \text{ind}_{\text{Gl}(2)}^B(1)) \leq 1.
\]

Proof. By dual Frobenius reciprocity, \( \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \text{det}) \) is contained in \( \text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \otimes \text{ind}_{\text{Gl}(2)}^B(1)) \cong \text{Hom}_{B_{\text{Gl}(2)}}(\Pi, \tilde{\rho} \circ \text{det}) \cong \text{Hom}_T(k_{\tilde{\rho}}(\Pi), \tilde{\rho}) \).

By lemma 4.4 that the right hand side has dimension \( \leq 1 \).

4.1. Gelfand-Kazhdan theory

For an irreducible \( \Pi \in C_G(\omega) \), fix the unnormalized Jacquet modules \( A = J_P(\Pi)_\psi \cong i^*(\Pi) \in C_{\text{Gl}(1)} \) and \( B = J_Q(\Pi) \cong i^*(\Pi) \in C_{\text{Gl}(2)} \). Let \( m_\Pi = 0, 1 \) be the dimension of Whittaker models of \( \Pi \).

Lemma 4.6. For every \( \Pi \in C_G(\omega) \) there is a short exact sequence of \( \text{Gl}_a(2) \)-modules

\[
0 \to j_*i_*(A) \to \Pi/\mathcal{S}_2^m \to i_*(B) \to 0.
\]
This is shown in [RW17, §4.2].

**Lemma 4.7.** For every \( \Pi \) in \( C_G(\omega) \) there is an isomorphism

\[
\text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \det) \cong \text{Hom}_{\text{Gl}(2)}(\Pi/S_2^{\text{un}}, \tilde{\rho} \circ \det).
\]

Further, there is an exact sequence

\[
0 \to \text{coker}(\delta) \to \text{Hom}_{\text{Gl}(2)}(\Pi/S_2^{\text{un}}, \tilde{\rho} \circ \det) \to \text{Hom}_{\text{Gl}(2)}(B, \tilde{\rho} \circ \det) \to 0
\]

with boundary map \( \delta : \text{Ext}^1_{\text{Gl}(2)}(B, \tilde{\rho} \circ \det) \to \text{Hom}_{\text{Gl}(2)}(j_!(i^*(A)), \tilde{\rho} \circ \det) \).

**Proof.** Lemma 2.2 in [W17] shows \( \text{Hom}_{\text{Gl}(2)}(S_2, \tilde{\rho} \circ \det) = 0 \). By lemma 4.6, the \((H_+, \tilde{\rho})\)-functionals of \( \Pi \) come from those of \( A \) or \( B \). Now both assertions follow from right-exactness of the functor \( \text{Hom}_{\text{Gl}(2)}(-, \tilde{\rho} \circ \det) \).

**Lemma 4.8.** For irreducible \( \Pi \in C_G(\omega) \) and every smooth character \( \tilde{\rho} \) of \( k^* \),

\[
\dim \text{Hom}_{\text{Gl}(2)}(B, \tilde{\rho} \circ \det) = \begin{cases} 1 & \text{type IIIb and } \tilde{\rho} \in \{ \nu, \chi_1 \nu \}, \\ 1 & \text{type IVc and } \tilde{\rho} = \nu^2, \\ 1 & \text{type IVd and } \tilde{\rho} = 1, \\ 1 & \text{type Vc and } \tilde{\rho} = \nu, \\ 0 & \text{otherwise}. \\ \end{cases}
\]

where we assume that \( \Pi \) is normalized as in [RW17], table 1.

**Proof.** The dimension cannot exceed one by lemma 4.5. By dual Frobenius reciprocity such a homomorphism exists if and only if \( \Pi \) is a submodule of the Klingen induced representation \( \text{ind}_{\text{Q}}^G(\omega_\Pi \boxtimes (\tilde{\rho} \circ \det)) \) for the character \( \omega_\Pi \boxtimes (\tilde{\rho} \circ \det) \) of the Levi subgroup \( Z_G \times [\text{Gl}(2)] \subseteq \text{Q} \). The submodules of Klingen induced representations are well-known [RS07, table A.4], [ST94].

**Lemma 4.9.** For every irreducible \( \Pi \in C_G(\omega) \), the \( T \)-module \( A = i^*j!(\Pi) \) is cyclic. In other words, every \( T \)-character occurs with at most a single Jordan block in \( A \).

**Proof.** If \( \Pi \) is generic, then the \( \text{Gl}_2(1) \)-module \( j!(\Pi) \) is perfect of degree one [RW17, 4.14], so \( A \) is cyclic by [RW17, 3.17]. If \( \Pi \) is non-generic, then the constituents of \( A \) are pairwise distinct, see [RW17, table A.3].
Lemma 4.10. For irreducible $\Pi \in C_G(\omega)$ and every smooth character $\tilde{\rho}$ of $k^\times$,

$$\dim \text{Hom}_{G(2)}(j_!(i_*(A)), \tilde{\rho} \circ \text{det}) = \begin{cases} 1 & \tilde{\rho} \in \Delta_+(\Pi) , \\ 0 & \text{otherwise} . \end{cases}$$

Proof. This follows from [W17], lemma 2.2.2. Let $G_a(1) = \{ (\ast, 0) \} \subseteq G(2)$ be the mirabolic subgroup of $G(2)$. For every Jordan block $\mu^{(n)}$ in the finite-dimensional $T$-module $A$, note that Frobenius reciprocity implies

$$\text{Hom}_{G(2)}(j_!(i_*(\mu^{(n)})), \tilde{\rho} \circ \text{det}) = \text{Hom}_{G_a(1)}(\nu^{-1} \mu^{(n)}, i_*(\tilde{\rho})) = \text{Hom}_{G(1)}(\mu^{(n)}, \tilde{\rho} \nu).$$

This space has dimension one for $\mu = \tilde{\rho} \nu$ and is zero otherwise. The assertion follows by lemma 4.9.

4.2. Non-generic cases

Lemma 4.11. Suppose $\Pi$ is non-generic and $\rho$ provides a split Bessel model for $\Pi$. Then $\text{Hom}_{G(2)}(j_! i_*(A), \tilde{\rho} \circ \text{det})$ vanishes, unless $\Pi$ is of type $\text{IIIb}$ with $\chi_1 = \nu^{\pm 1}$.

Proof. The classification of Bessel models implies $\rho \in \Delta_+(\Pi)$, see [RS16, thm. 6.2.2], [RW17, thm. 6.2]. If $\text{Hom}_{G(2)}(j_! i_*(A), \tilde{\rho} \circ \text{det})$ is non-zero, lemma 4.10 implies $\tilde{\rho} \in \Delta_+(\Pi)$. By lemma A.8 of [RW17], $\Pi$ belongs to type $\text{IIIb}$ with $\chi_1 = \nu^{\pm 1}$.

Lemma 4.12. If $\Pi$ is non-generic and $\rho$ provides a Bessel model for $\Pi$, then

$$\dim \text{Hom}_{G(2)}(\Pi, \tilde{\rho} \circ \text{det}) = \dim \text{Hom}_{G(2)}(B, \tilde{\rho} \circ \text{det}) = \begin{cases} 1 & \text{type } \text{IIIb} \text{ and } \rho \in \Delta_+(\Pi) = \{ 1, \chi_1 \} , \\ 1 & \text{type } \text{IVc} \text{ and } \rho = \nu \in \Delta_+(\Pi) , \\ 1 & \text{type } \text{VId} \text{ and } \rho = \nu \in \Delta_+(\Pi) = \{ 1 \} , \\ 0 & \text{otherwise} . \end{cases}$$

These functionals all factorize over $\Pi \twoheadrightarrow B \twoheadrightarrow (\tilde{\rho} \circ \text{det})$.

Proof. By lemma 4.11 and lemma 4.7 the first assertion holds for every case except possibly for $\Pi$ of type $\text{IIIb}$ with $\chi_1 = \nu^{\pm 1}$. In that case, $\Delta_+(\Pi) = \{ 1, \chi_1 \}$ and we can assume $\chi_1 = \nu$ by a Weyl reflection. Thus the only contribution for

$$\text{Hom}_{G(2)}(j_! i_*(A), \tilde{\rho} \circ \text{det}) \neq 0$$
comes from $\rho = 1$ and $\tilde{\rho} = \nu = \chi_1$. For this case we already have functionals that factorize over $B$ for $\tilde{\rho} = \nu$, and there are no others. Indeed, by lemma 4.5 the space of $H_+$-functionals of $\Pi$ has dimension $\leq 1$. Since $\rho$ provides a Bessel model and $\Pi$ is non-generic in these cases, $\tilde{\rho}$ is in $\nu \Delta_+(\Pi)$.

The second assertion is a restatement of lemma 4.8. The last assertion is clear by construction.

Remark 4.13. For type IVc the character $\tilde{\rho} = 1$ in $\nu \Delta_+(\Pi) = \{1, \nu^2\}$ does not give an $(H_+, \tilde{\rho})$-functional. Type VIc and VId are extended Saito-Kurokawa cases. Here the functionals have already been constructed in [W17], lemma 5.4. Both cases yield subregular poles of the form $L(s, \nu^{1/2})$.

4.3. Generic cases

Proposition 4.14. Fix a generic irreducible representation $\Pi \in \mathcal{C}_G(\omega)$ and a smooth character $\tilde{\rho}$ of $\text{Gl}(1)$. If $\tilde{\rho} \notin \nu \Delta_+(\Pi)$, then

$$\dim \text{Hom}_{\text{Gl}(2)}(\hat{\Pi}, \tilde{\rho} \circ \text{det}) = \begin{cases} 1 & \text{type I, IIa, Va, VIa, X, XIa and } \tilde{\rho} \in \Delta_+(\Pi), \\ 0 & \text{otherwise.} \end{cases}$$

These are exactly the generic exceptional cases discussed in [RW17].

Proof. By lemma A.1 it suffices to consider the central specialization $\hat{\Pi} = \zeta_\mu(\Pi) \in \mathcal{C}$ where $\mathcal{C} = \mathcal{C}_{\text{Gl}(2)}(\mu)$ is the full subcategory of $\mathcal{C}_{\text{Gl}(2)}$ with central character $\mu = \tilde{\rho}^2$. As in section A.3 consider

$$Y_\pm = (\tilde{\rho} \circ \text{det}) \otimes (\nu^{\pm 1/2} \times \nu^{\pm 1/2}) .$$

Suppose $\Pi$ is of type I, IIa, Va, Vla, X or Xla and $\tilde{\rho} \in \Delta_+(\Pi)$. The Bessel module $k_{\nu^{-1} \tilde{\rho}}(\Pi)$ is non-perfect [RW17, §6], so $\dim \text{Hom}_{\mathcal{C}}(\hat{\Pi}, Y_+) = 1$ and $\dim \text{Hom}_{\mathcal{C}}(\hat{\Pi}, Y_-) = 2$ by lemma A.10 and lemma 4.4. Now assume that $\text{Hom}_{\mathcal{C}}(\hat{\Pi}, \tilde{\rho} \circ \text{det}) = 0$ vanishes. By lemma A.18 $\text{Ext}_{\mathcal{C}}(\hat{\Pi}, \tilde{\rho} \circ \text{det}) = 0$ also vanishes. The long exact sequence in remark A.15 implies $\dim \text{Hom}_{\mathcal{C}}(\hat{\Pi}, \text{St}(\tilde{\rho})) = 1$ by lemma A.18. The long exact sequence for $Y_-$ implies $\text{Hom}_{\mathcal{C}}(\hat{\Pi}, \text{St}(\tilde{\rho})) = 2$. This is a contradiction, so $\text{Hom}_{\mathcal{C}}(\hat{\Pi}, \tilde{\rho} \circ \text{det})$ is non-zero and thus one by lemma 4.5.

For the cases IIIa and IVa the Bessel module $k_{\nu^{-1} \tilde{\rho}}(\Pi)$ is perfect, and this implies $\dim \text{Hom}_{\mathcal{C}}(\hat{\Pi}, Y_\pm) = 1$ by lemma A.10. By proposition A.13, $\text{Ext}_{\mathcal{C}}(\hat{\Pi}, Y_\pm) = 0$.
vanishes. By the long exact sequences in remark A.15 both $\text{Ext}_C(\hat{\Pi}, \tilde{\rho} \circ \text{det}) = 0$ and $\text{Ext}_C(\hat{\Pi}, \text{St}(\hat{\rho})) = 0$ vanish, so counting dimensions implies
\[
\dim \text{Hom}_C(\hat{\Pi}, \text{St}(\hat{\rho})) + \dim \text{Hom}_C(\hat{\Pi}, \hat{\rho} \circ \text{det}) = 1.
\]
For $\rho \in \Delta_+(\Pi)$ there are nontrivial $\text{Gl}(2)$-homomorphisms $\Pi \to B \to \text{St}(\hat{\rho})$ [RS07, table A.4], so $\text{Hom}_C(\hat{\Pi}, \hat{\rho} \circ \text{det}) = 0$ vanishes.
For $\rho \not\in \Delta_+(\Pi)$ the assertion follows from lemma 4.7, lemma 4.10 and lemma 4.8.

5. Construction of subregular poles

In this section we determine the subregular $L$-factors for the cases occurring in table 2 by verifying the sufficient condition of lemma 3.6.

5.1. Non-generic cases

Fix a non-generic irreducible $\Pi \in C_G(\omega)$, normalized as in [RW17, table 1], and a smooth character $\rho$ of $k^\times$ such that $\Lambda = \rho \boxtimes \rho^*$ defines a split Bessel model of $\Pi$, i.e. $\rho \in \Delta_+(\Pi)$. We assume that there is a $\text{Gl}(2)$-equivariant functional $\Pi \to (\mathbb{C}, \tilde{\rho} \circ \text{det})$ for $\tilde{\rho} = \nu \rho$, so $(\Pi, \rho)$ belongs to one of the cases of lemma 4.12 where $\Pi$ is of type IIIb, IVc, VIc, VId.

Lemma 5.1. The Bessel module $\tilde{\Pi} = k_\rho(\hat{\Pi})$ is perfect of degree one. Especially, $\tilde{\Pi}^2 = 0$ and $\dim k_\chi(\hat{\Pi}) = 1$ for every smooth character $\chi$ of $\text{Gl}(1)$.

Proof. See corollary 6.10 in [RW17].

Lemma 5.2. Consider an extension of smooth $\text{Gl}(2)$-modules
\[
0 \to j_! i_* (\hat{\rho})|_{\text{Gl}(2)} \to Q \to (\tilde{\rho} \circ \text{det}) \to 0.
\]
If $\overline{\zeta}(Q) = 0$ vanishes for $\mu = \tilde{\rho}^\vee$, then $\zeta_\mu(Q) = 0$ is isomorphic to the induced representation $\zeta_\mu \cong (\tilde{\rho} \circ \text{det}) \otimes (\nu^{1/2} \times \nu^{-1/2})$.

Proof. The proof is completely analogous to lemma 6.4 of [W17].

Lemma 5.3. Under the above assumptions, the $\text{Gl}(2)$-module $\Pi \in C_{\text{Gl}(2)}$ has a quotient $\hat{Q}$ isomorphic to
\[
\hat{Q} \cong (\tilde{\rho} \circ \text{det}) \otimes (\nu^{1/2} \times \nu^{-1/2}).
\]
Proof. Recall the exact sequence \(0 \to j_i^*(A) \to \Pi \to i_*\langle B \rangle \to 0\) of lemma 4.4.6. The constituents of \(B \in C_{Gl(2)}\) are isomorphic to \((\nu \chi' \circ \det) \otimes \tau\) for the \(Gl(1) \times Gl(2)\)-modules \(\chi' \otimes \tau\) that occur in table A.4 of [RS07]. The constituents of \(A\) are given by \(\nu^{3/2} \Delta_\emptyset(\Pi)\), see table 3 in [RW17]. Assume that \(\Pi\) is not of type \(\emptyset\emptyset\emptyset\) with \(\chi_1 = \nu^{\pm 1}\). Then \(A\) and \(B\) are semisimple because there are no non-trivial extensions between their constituents, see table 1. Note that lemma 6.3 in [W17] implies

\[
j_i^*(A)|_{Gl(2)} \cong \bigoplus_{X} \text{ind}^{Gl(2)}_{B_{Gl(2)}} (\chi \boxtimes S),
\]

where the sum runs over constituents \(\chi\) of \(A\). By Bernstein decomposition, \(\Pi\) splits as a direct sum of smooth \(Gl(2)\)-modules

\[
\Pi|_{Gl(2)} \cong Q \oplus Q',
\]

where \(Q\) is an extension of \(Gl(2)\)-modules \(0 \to j_i^*(\tilde{p}) \to Q \to i_*\langle \tilde{\rho} \circ \det \rangle \to 0\). Since \(\zeta^\mu(\Pi) = 0\) vanishes by [W17, lemma 6.1] for every smooth \(\mu\), so does its submodule \(\zeta^\mu(Q)\). Lemma 5.2 implies that \(\hat{Q} = \zeta^\mu(Q)\) is isomorphic to \((\tilde{\rho} \circ \det) \otimes (\nu^{1/2} \times \nu^{-1/2})\).

Consider the remaining case of type \(\emptyset\emptyset\emptyset\) with \(\chi_1 = \nu^{\pm 1}\). Without loss of generality we can assume \(\Pi = \nu \times (1 \circ \det) \in C_G(\omega)\) by a Weyl reflection and a suitable twist. By table A.4 in [RS07], the unnormalized Klingen-Jacquet-module \(J_Q(\Pi)\) has length four and contains the special representation \(\delta_Q^{1/2} \otimes (\nu^{-1} \boxtimes \text{St}(\nu))\) as a constituent. By Frobenius reciprocity, this constituent is neither a submodule nor a quotient. In other words, the \(Gl(2)\)-module \(B\) contains the constituent \(\text{St}(\nu)\), but neither as a submodule nor as a quotient. This means \(B\) is isomorphic to

\[
B \cong (\nu^2 \circ \det) \oplus ((\nu \circ \det) \otimes M_{(1;\text{St};1)}).
\]

Here \(M_{(1;\text{St};1)}\) is the \(Gl(2)\)-module of length three with constituents \((1 \circ \det)\) occurring twice and the Steinberg representation \(\text{St}\) such that the socle and top are both one-dimensional. This is an extension panachée in the sense of [SGA7, 9.3] and is unique up to isomorphism. For \(\tilde{\rho} = \nu\), the quotient \(\hat{Q} = B / \text{soc}(B)\) of \(B\) by its socle \(\text{soc}(B) = (\nu^2 \circ \det) \oplus (\nu \circ \det)\) is an indecomposable extension \(0 \to \text{St}(\nu) \to \hat{Q} \to (\nu \circ \det) \to 0\) and thus isomorphic to \(\hat{Q} = \tilde{\rho} \otimes (\nu^{1/2} \times \nu^{-1/2})\). For \(\tilde{\rho} = \nu^2\) one constructs \(\hat{Q}\) as in the previous cases. \(\square\)

The Waldspurger-Tunnell functor \(WT : C_{Gl(2)} \to C_T\) for \(T = \{\text{diag}(*, 1) \in \text{Gl}(2)\}\) sends a smooth \(Gl(2)\)-module \(X\) to its maximal quotient \(WT(X) = X_{\tilde{F}, \rho}\) of \(X\) on which \(\tilde{T} = \{\text{diag}(1, *) \in \text{Gl}(2)\}\) acts by the character \(\rho\). The \(TT\)-equivariant natural projection \(X \to WT(X)\) is also denoted \(WT\).
Table 1: $A$ and $B$-modules for the relevant non-generic cases.

| Type | $\Pi$ | $A \in \mathcal{C}_T$ | $B \in \mathcal{C}_{\text{Gl}(2)}$ | $\hat{\rho}$ |
|------|------|----------------|----------------|-------|
| IIIb | $\chi_1 \times (1 \circ \text{det})$, $\chi_1 \mp \nu^\pm 1$ | $\nu \oplus \nu \chi_1$ | $(\nu \chi_1 \circ \text{det}) \oplus (\nu \circ \text{det}) \oplus (\nu^{1/2} \chi_1 \times \nu^{1/2})$ | $\nu$, $\chi_1 \nu$ |
| IIIb | $\nu \times (1 \circ \text{det})$ | $\nu \oplus \nu^2$ | $(\nu^2 \circ \text{det}) \oplus \nu \otimes M_{[1:1:1]}$ | $\nu$, $\nu^2$ |
| IVc  | $L(\nu^{3/2}/t, \nu^{-3/2})$ | $1 \oplus \nu^2$ | $(\nu^2 \circ \text{det}) \oplus (\nu^{3/2} \times \nu^{-1/2})$ | $\nu^2$ |
| Vlc  | $L(\nu^{1/2}/t, \nu^{-1/2})$ | $\nu$ | $(\nu \circ \text{det})$ | $\nu$ |
| Vld  | $L(\nu, 1 \times \nu^{-1/2})$ | $\nu$ | $(\nu \circ \text{det}) \oplus (\nu^{1/2} \times \nu^{1/2})$ | $\nu$ |

**Lemma 5.4.** Under the above assumptions, there is a commutative diagram with the canonical projections

\[
\begin{array}{ccc}
\overline{\Pi} \in \mathcal{C}_{\text{Gl}(2)} & \overset{C_T S \ni k_\rho(\overline{\Pi})}{\longrightarrow} & \widehat{Q} \in \mathcal{C}_{\text{Gl}(2)} \\
\downarrow & & \downarrow_{WT} \\
C_T \ni k_{\nu^2 \rho} k_\rho(\overline{\Pi}) & \equiv & \text{WT}(\widehat{Q}) \in \mathcal{C}_T \\
& \equiv & \\
& & k_{\nu^2 \rho} k_\rho(\widehat{Q}).
\end{array}
\]

Both lower diagonal arrows are isomorphisms.

**Proof.** The center acts on $\widehat{Q}$ by the central character $\mu = \nu^2 \rho^2 = \hat{\rho}^2$, so the diagonal torus $TT$ of $\text{Gl}(2)$ acts on $WT(\widehat{Q})$ by multiplication with the character $\nu^2 \rho \boxtimes \rho$. It is well-known that $WT(\widehat{Q})$ is one-dimensional, compare [W17 lemma 5.3]. By construction, $k_{\nu^2 \rho} k_\rho(\widehat{Q})$ is one-dimensional, so the natural projection $WT(\widehat{Q}) \rightarrow k_{\nu^2 \rho} k_\rho(\widehat{Q})$ is an isomorphism. Perfectness of $\overline{\Pi}$ (lemma 5.1) implies $\dim k_{\nu^2 \rho} k_\rho(\overline{\Pi}) = 1$, see [RW17 lemma 3.13]. Therefore the natural projection $k_{\nu^2 \rho} k_\rho(\overline{\Pi}) \rightarrow k_{\nu^2 \rho} k_\rho(\widehat{Q})$ is also an isomorphism. By transitivity of coinvariant functors, both sides of the diagram yield the natural projection to the coinvariant quotient $k_{\nu^2 \rho} k_\rho(\widehat{Q})$ on which the Borel $B_{\text{Gl}(2)}$ acts with $\nu^2 \rho \boxtimes \rho$.

**Proposition 5.5.** Fix a non-generic irreducible representation $\Pi \in \mathcal{C}_G(\omega)$ and an unramified character $\rho$ such that the Bessel module $\overline{\Pi} = k_\rho(\overline{\Pi})$ has degree
one. In other words, assume that \( \Lambda = \rho \boxtimes \rho^* \) defines a split Bessel model for \( \Pi \). If \( \Pi \) admits a non-trivial \((H_+, \bar{\rho})\)-functional, then for \( K = \GL(2, \mathfrak{o}_k) \) the canonical morphism

\[
\Pi^K \rightarrow \Pi \rightarrow \tilde{\Pi} \rightarrow \kappa_{\nu, \rho}(\tilde{\Pi}/\tilde{\Pi}^S)
\]

is nontrivial.

Proof. The Bessel module is perfect of degree one by lemma \[5.1\], so \( \tilde{\Pi}^S = 0 \). Exactness of \( K \)-invariants implies that the natural projection \( \Pi^K \rightarrow \hat{Q}^K \) is surjective. The Waldspurger-Tunnell map \( \hat{Q} \rightarrow WT(\hat{Q}) \) is nonzero on \( \hat{Q}^K \) as shown in lemma 5.3 of [W17]. Since the diagram in lemma 5.4 is commutative, by going over the left side of the diagram we obtain the assertion.

\[\square\]

**Theorem 5.6.** For all non-generic irreducible \( \Pi \in \mathcal{C}_G(\omega) \) with a split Bessel model attached to \( \Lambda = \rho \boxtimes \rho^* \), the subregular factor \( L_{\text{ps}}^S(s, \Pi, 1, \Lambda) \) is given in table 2.

Proof. By proposition 3.8, proposition 5.5 and lemma 3.6, there is a one-to-one correspondence between \((H_+, \bar{\rho})\)-functionals and subregular poles. The functionals have been classified in lemma 4.12.

\[\square\]

### 5.2. Preparations for the generic cases

In the next section we will verify the criterion of lemma 3.6 using the Gelfand-Kazhdan filtration in the sense of lemma 4.6. We now discuss its constituents. Recall that \( B_{\GL_n(2)}S \) denotes \( \{[\begin{smallmatrix} * & \ast \\ \ast & 0 \end{smallmatrix}] \in \GL_n(2)\} \).

**Lemma 5.7.** For every smooth character \( \rho \) of \( \GL(1) \) there is an isomorphism of \( \GL_n(2) \)-modules \( I \cong j_!(\mathbb{E}[\nu^2 \rho]) \) where \( I = \text{ind}_{B_{\GL_2(2)}}^{\GL_n(2)}(\sigma) \) is compactly induced from \( \sigma = S_1 \boxtimes \rho \in \mathcal{C}_{TS \times T} \).

Proof. By Bruhat decomposition, the double coset space

\[
B_{\GL(2)}S \backslash \GL_n(2)/ \{[\begin{smallmatrix} * & \ast \\ \ast & \ast \end{smallmatrix}]\}
\]

is represented by \( \text{id} \) and \( \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \). The orbit filtration of Bernstein and Zelevinski [BZ77, 5.2] yields an exact sequence of \( TU \)-modules

\[
0 \rightarrow \text{ind}^{TU}_T(\mathbb{S}_{S, \psi} \otimes \nu \rho) \rightarrow j_!(I) \rightarrow \text{ind}^{TU}_{TU}(\mathbb{S}_S \otimes i_*(\nu \rho)) \rightarrow 0.
\]

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Since $S_S = 0$ and $S_{S, \psi} \cong \mathbb{C}$ we obtain an isomorphism
\[
j_! (I) \cong \text{ind}_{T}^{TU} (\nu \rho) .
\]
Finally, $\text{ind}_{T}^{TU} (\nu \rho)$ is isomorphic to the space $\nu^2 \rho \otimes E \cong E [\nu^2 \rho]$ in the sense of [RW17] example 3.4. An analogous argument shows $i^* (I) = 0$ where the double coset space $B_{G_l(2)} S \backslash G_{la} (2) / G_{la} (2)$ is obviously generated by $\text{id}$. The functorial short exact sequence of lemma 3.5 implies $I \cong j_! (E [\nu^2 \rho])$.  

**Proposition 5.8.** Fix unramified characters $\rho, \chi$ of $G_l(1)$. For $I = j_! (E [\nu^2 \rho]) \in C_2$ write $\tilde{I} = k_{\rho} (I)$ and let $K = G_l(2, \mathfrak{o}_k)$. Then the composition of the natural morphisms
\[
I^K \rightarrow I \rightarrow \tilde{I} \rightarrow (\tilde{I}/\tilde{I}^S)_{\chi}
\]
is non-zero.

**Proof.** Without loss of generality we can assume $\rho = 1$ by a suitable twist. By lemma 5.7 it is sufficient to show the assertion for the $G_{la} (2)$-module $I = \text{ind}_{B_{G_l(2)} S} (\sigma)$ for $\sigma = S_1 \otimes \rho \in C_{TS \times F}$. Evaluation at identity defines a $B_{G_{la}(2)} S$-equivariant projection
\[
eval_{\text{id}} : I \rightarrow \sigma , \quad f \mapsto f (\text{id}) .
\]
The action of $\tilde{T} U = \{ [1 \ast 0] \} \subseteq G_l(2)$ on $\sigma$ is trivial, so $\text{eval}_{\text{id}}$ factorizes over the projection to the Bessel module $\tilde{I}$. By [RW17] theorem 4.27 there is an isomorphism $\tilde{I} \cong S_1 \oplus i_* (\nu^2)$. The $TS$-module $\sigma$ is isomorphic to $S_1$, so the kernel of $\tilde{I} \rightarrow \sigma$ equals $\tilde{I}^S \cong i_* (\nu^2)$. This defines a commutative diagram of $TS$-modules with an exact horizontal sequence
\[
0 \rightarrow \tilde{I}^S \rightarrow \tilde{I} \rightarrow \sigma \rightarrow 0
\]
It remains to be shown that $I^K \rightarrow I \rightarrow \text{eval}_{\text{id}} \rightarrow \sigma$ is non-trivial. Indeed, for the characteristic function $\phi_0 \in C_{c}^\infty (k^\times) \cong S_1$ of $\mathfrak{o}_k^\times$, define $f^\circ \in I$ by
\[
f^\circ (g) = \begin{cases} 
\sigma (bs) \phi_0 & g = bsk \in B_{G_l(2)} S K , \\
0 & \text{otherwise} .
\end{cases}
\]
This is well-defined because $\phi_0$ is invariant under $B_{G_l(2)} S \cap K$ by construction, so $f^\circ$ is a $K$-invariant element in $I^K$. The evaluation $\text{eval}_{\text{id}}$ sends $f^\circ$ to $f^\circ (\text{id}) = \phi_0$. It is clear that $\phi_0$ has non-trivial image under the projection
\[
S_1 \rightarrow (S_1)_{\chi} , \quad \phi \mapsto \int_{k^\times} \phi (x) \chi^{-1} (x) \, d^\times x .
\]
Lemma 5.9. Fix an unramified character $\rho$ of $\text{Gl}(1)$ and a finite-dimensional $\text{Gl}(1)$-module $A$ that does not contain $\nu \rho$ as a constituent. For $I = j_! i_*(A) \in C_2$ and $K = \text{Gl}(2, \mathfrak{o}_k)$, the composition of canonical morphisms

$$I^K \longrightarrow I \longrightarrow k_\rho(I) \longrightarrow k_{\nu^2 \rho} k_\rho(I)$$

is surjective. It is non-zero if and only if $\nu^2 \rho$ occurs in $A$ as a constituent.

Proof. By compactness of $K$ and transitivity of coinvariant functors, it is sufficient to prove that the composition $\hat{I}^K \rightarrow \hat{I} \rightarrow k_\rho(\hat{I})$ is surjective for the central specialization $\hat{I} = \zeta_\rho \nu^2(I)$. We use induction over the length of $A$. For $A = 0$ the assertion is obvious. If $A = \mu$ is a character, then lemma A.3 implies $\hat{I} \cong \text{ind}^{\text{Gl}(2)}_{B_\text{Gl}(2)}(\mu \boxtimes \nu^2 \rho - 1)$. The well-known formulas for the Jacquet-module show that $k_\rho(\hat{I})$ vanishes for $\mu \neq \nu \rho, \nu^2 \rho$. If $\mu = \nu^2 \rho$, then $k_\rho(\hat{I}) \cong \hat{I}_{\text{F}, \rho}$ and lemma 5.3 in [W17] implies that $\hat{I}^K \rightarrow \hat{I} \rightarrow k_\rho(\hat{I})$ is surjective.

If $A$ has length greater than one, then there is an exact sequence of non-zero $\text{Gl}(1)$-modules $0 \rightarrow A_1 \rightarrow A \rightarrow A_2 \rightarrow 0$. Lemma A.4 yields an exact sequence $0 \rightarrow \hat{I}_1^K \rightarrow \hat{I}^K \rightarrow \hat{I}_2^K \rightarrow 0$ with $\hat{I}_i = \zeta_\rho \nu^2 j_! i_*(A_i)$. By compactness of $K$ and right-exactness of $k_\rho$, we obtain a commutative diagram with exact sequences

$$
\begin{array}{cccccc}
0 & \longrightarrow & \hat{I}_1^K & \longrightarrow & \hat{I}^K & \longrightarrow & \hat{I}_2^K & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& \cdots & k_\rho(\hat{I}_1) & \longrightarrow & k_\rho(\hat{I}) & \longrightarrow & k_\rho(\hat{I}_2) & \longrightarrow & 0
\end{array}
$$

The induction hypothesis states that the left and right vertical arrows are surjective. By the five lemma the middle vertical arrow is surjective. \qed

Lemma 5.10. Fix a generic irreducible $\Pi \in C_G(\omega)$ and a smooth character $\rho$ of $\text{Gl}(1)$ not in $\Delta_+^-(\Pi)$. For $B = i^*(\Pi) \in C_{\text{Gl}(2)}$ there are $C_1$-isomorphisms

$$k_\rho(i_*(B)) \cong k_\rho(i_*(B)) \cong \begin{cases} i_*(\nu^2 \rho) & \text{type IIa, IVa, VIa and } \rho \in \Delta_+^-(\Pi), \\
0 & \text{otherwise}. \end{cases}$$

Proof. $k_\rho(i_*(B)) \cong k_\rho(i_*(B))$ holds by lemma 4.3 in [RW17]. If $\Pi$ is cuspidal or of type X or Xa, there is nothing to show because $B = 0$. For type VII, VIIIa, IXa the assertion is also clear because $B$ is cuspidal. Suppose $\Pi$ is
of type I, IIa or Va. Every irreducible constituent of the normalized Siegel-Jacquet-module $\delta_p^{-1/2} \otimes J_P(\Pi)$ is generic and non-cuspidal [RS07] table A.3. Since $T = \{\text{diag}(*, *, 1, 1) \in G\}$ is in the center of the Levi-quotient of $P$, the following characters coincide:

1. characters in $\Delta_0(\Pi)$,
2. $T$-eigencharacters of the twisted Jacquet-module of $\delta_p^{-1/2} \otimes J_P(\Pi)$,
3. $T$-eigencharacters of the Siegel-Jacquet-module $\delta_p^{-1/2} \otimes J_P(\Pi)$,
4. $T$-eigencharacters of the Borel-Jacquet-module $\delta_{BG}^{-1/2} \otimes J_{BG}(\Pi)$.

In other words, $\Delta_0(\Pi)$ is the multiset of $\{\text{diag}(*, 1) \subseteq \text{GL}(2)\}$-eigencharacters on the normalized Borel-Jacquet-module of the $\text{GL}(2)$-module $B = J_0(\Pi)$. Every constituent of $B$ is fully induced as a $\text{GL}(2)$-module [RS07] table A.4, so the $T$-eigencharacters on $\delta_{BG}^{-1/2} \otimes B_U$ coincide with the $\widetilde{T}$-eigencharacters for $\widetilde{T} = \{\text{diag}(1, *) \subseteq \text{GL}(2)\}$. On $\widetilde{T}$ the modulus character $\delta_{BG}$ coincides with the valuation character $\nu$. That means $k_\rho(B) \neq 0$ implies that $\rho$ is a $\widetilde{T}$-eigencharacter of the unnormalized Borel-Jacquet-module of $B$, so $\nu^{-1/2} \rho \in \Delta_0(\Pi)$. Hence $k_\rho, i_*(B) \neq 0$ is only possible for $\rho \in \Delta_+(\Pi) = \nu^{1/2} \Delta_0(\Pi)$.

If $\Pi$ is of type Ila, then $B$ is isomorphic to $B \cong (\nu^{3/2} \sigma \circ \text{det}) \otimes (1 \times 1) \oplus \text{St}(\nu \sigma)$ [RS07] table A.4. Hence $k_\rho, i_*(B) \neq 0$ is only possible for $\rho = \nu \sigma$ in $\Delta_+(\Pi)$ and $\rho = \sigma$ in $\Delta_-(\Pi)$. It is straightforward to see $k_\rho, i_*(B) = i_*(\nu^2 \sigma)$ for $\rho = \sigma$.

For type IIa, Ila the argument is basically analogous, but for type IIIb the constituent $\nu^2 \rho \boxtimes \rho$ may occur more than once in the Jacquet module of $B$. It suffices to show that $\dim k^0(B) \leq 1$. Indeed, for type IIIa there is an exact sequence

$$0 = k_\rho(\Pi) \to k^0(B) \to k_\rho(J) \to k_\rho(\Pi) \to k_\rho(\Pi) \to k_\rho(B) \to 0$$

where $J = j_1 j_2(\Pi) \cong j_1(\mathbb{E}[A])$ by lemma 4.14 in [RW17]. The constituents of $A = i^* j_1(\Pi)$ are pairwise distinct, so lemma 4.27 in [RW17] ensures that the maximal finite-dimensional subspace of $k_\rho(J)$ is one-dimensional.

\section*{5.3. Generic cases}

Let $(\Pi, \rho)$ be a non-ordinary or extra-ordinary exceptional case in the sense of [RW17] cor. A.12. In other words, $\Pi$ is of type I, IIa, Va, Vla, X, Xla and $\nu^{1/2} \rho \in \Delta_0(\Pi)$, where $\Delta_0(\Pi)$ denotes the multiset of characters that occur as constituents of $i^* j_1(\Pi) \cong J_P(\Pi)$, see [RW17] table 3. We show that the subregular factor attached to $\Lambda = \rho \boxtimes \rho^*$ is $L_{\text{reg}}^{PS}(s, \Pi, 1, \Lambda) = L(s, \chi_{\text{crit}})$ for the critical character $\chi_{\text{crit}} = \nu^{1/2} \rho$ by verifying the sufficient criterion of lemma 3.6.
For technical reasons we distinguish the cases where the critical character occurs in $\Delta_0(\Pi)$ once and more than once. Recall that $\Delta_{\pm}(\Pi) = \nu^{\pm 1/2} \Delta_0(\Pi)$.

**Case 1:** The critical character occurs in $\Delta_0(\Pi)$ exactly once.

**Proposition 5.11.** For every generic exceptional case $(\Pi, \rho)$ where the critical character $\chi_{\text{crit}} = \nu^{1/2} \rho$ occurs in $\Delta_0(\Pi)$ exactly once, the composition of the canonical morphisms for $K = \text{Gl}(2, \mathfrak{o}_k)$ and $\tilde{\Pi} = k_\rho(\Pi)

\[ \Pi^K \rightarrow \Pi \rightarrow \tilde{\Pi} \rightarrow k_{\nu^2 \rho}(\tilde{\Pi}/\tilde{\Pi}^S) \]

is non-trivial.

**Proof.** By assumption $\Pi$ is of type I, IIa, Va, X or Xla. Further, $A = i^* j^!(\tilde{\Pi})$ splits as a direct sum $A \cong \nu^2 \rho \oplus A'$ for some $\text{Gl}(1)$-module $A'$ that does not contain $\nu^2 \rho$ as a constituent. By [RW17 4.14], $j^!(\tilde{\Pi})$ is perfect, so there is an embedding $E[\nu^2 \rho] \hookrightarrow j^!(\tilde{\Pi})$. Set $J = j_!(E[\nu^2 \rho])$, then exactness of $j_!$ and lemma 4.6 yield an exact sequence

\[ 0 \rightarrow J \rightarrow \tilde{\Pi} \rightarrow Y \rightarrow 0 \]

where the cokernel $Y$ is an extension $0 \rightarrow j_! i^*_s(A') \rightarrow Y \rightarrow i_! (B) \rightarrow 0$. By lemma 5.10 $k_\rho(B) = 0$ and $k^\rho(B) = 0$ both vanish, so $k_\rho(Y) \cong k_\rho j_! i^*_s(A')$ and $k^\rho(Y) \cong k^\rho j_! i^*_s(A')$. The intersection $\Delta_+(\Pi) \cap \Delta_-(\Pi)$ is empty by lemma A.8 in [RW17], so $\nu \rho$ is not a constituent of $A'$. Lemma 4.28 in [RW17] implies $k^\rho(Y) = 0$, $k_\rho(Y) \cong i^*_s(A')$.

Consider the following commutative diagram with the canonical arrows

\[
\begin{array}{cccccc}
J^K & \rightarrow & J & \rightarrow & \tilde{J} & \rightarrow & \tilde{J}/\tilde{J}^S & \rightarrow & k_{\nu^2 \rho}(\tilde{J}/\tilde{J}^S) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \cong \\
\Pi^K & \rightarrow & \Pi & \rightarrow & \tilde{\Pi} & \rightarrow & \tilde{\Pi}/\tilde{\Pi}^S & \rightarrow & k_{\nu^2 \rho}(\tilde{\Pi}/\tilde{\Pi}^S).
\end{array}
\]

We have shown above that the third vertical arrow is injective with cokernel $k_\rho(Y) \cong i^*_s(A')$. The fourth vertical arrow is injective with cokernel $i^*_s(A')$ because $\tilde{J}^S \cong \tilde{\Pi}^S$ is the unique submodule isomorphic to $i_*(\nu^2 \rho)$ as shown in [RW17 thm. 4.27, thm. 6.6]. Since $\nu^2 \rho$ does not occur in $A'$, both

\[ k_{\nu^2 \rho}(i^*_s(A')) = 0 \quad \text{and} \quad k^\rho(i^*_s(A')) = 0 \]

vanish, so the fifth vertical arrow is an isomorphism. By proposition 5.8 the composition of the upper horizontal arrows is non-trivial. By commutativity of the diagram the composition of the lower horizontal arrows is non-trivial. \qed
Case 2: The critical character occurs in $\Delta_0(\Pi)$ more than once.
Now we consider the remaining case and thus make the following

**Assumption:** Fix a generic irreducible representation $\Pi \in C_G(\omega)$ and a smooth character $\rho$ of $\text{Gl}(1)$ such that

1. $\text{Hom}_{\text{Gl}(2)}(\Pi, \tilde{\rho} \circ \text{det})$ is non-zero for $\tilde{\rho} = \nu \rho$,
2. $\chi_{\text{crit}} = \nu^{1/2} \rho$ occurs in the multiset $\Delta_0$ with multiplicity more than one.

Under these assumptions, the Bessel module $\tilde{\Pi} = k_{\rho}(\Pi) \in C_1$ is not perfect [RW17, 6.6]. It contains a unique one-dimensional submodule $\tilde{\Pi}^S \cong i_*(\nu^2 \rho)$.

The main idea now is to consider the commutative diagram of proposition 5.15. Its construction is explained in the following lemmas. For $B = i^*(\Pi)$ and $J = j_*j^!(\Pi)$, lemma 3.5 gives an exact sequence in $C_2$

$$0 \rightarrow J \xrightarrow{a} \Pi \rightarrow i_*(B) \rightarrow 0.$$

**Lemma 5.12.** $J \cong j_!(\mathbb{E}[A])$ is perfect.

*Proof.* Since $\Pi$ is generic, $j_!(\Pi)$ is perfect of degree one [RW17, 4.13]. It is uniquely determined by $A = i^*j!(\Pi)$ and thus isomorphic to $\mathbb{E}[A]$. The functor $j_!$ sends perfect modules to perfect ones [RW17, lemma 4.4]. \qed

**Lemma 5.13.** For the Bessel module $\tilde{J} = k_{\rho}(J)$, the composition $g = p \circ k_{\rho}(a)$ of the canonical $C_1$-morphisms

$$g : \tilde{J} \xrightarrow{k_{\rho}(a)} \tilde{\Pi} \xrightarrow{p} \tilde{\Pi}/\tilde{\Pi}^S$$

is surjective.

*Proof.* The functor $k_{\rho}$ applied to the exact sequence of lemma 5.12 yields a long exact sequence

$$\cdots \rightarrow k^\rho(\Pi) \rightarrow k^\rho i_*(B) \xrightarrow{\delta} k_{\rho}(J) \xrightarrow{k_{\rho}(a)} k_{\rho}(\Pi) \rightarrow 0.$$  

The left term $k^\rho(\Pi)$ vanishes because $\Pi$ is generic [RW17, lemma 5.10]. Lemma 5.10 implies

$$k^\rho(i_*(B)) \cong k_{\rho}(i_*(B)) \cong \begin{cases} i_*(\nu^2) & \text{type VIa}, \\ 0 & \text{otherwise}. \end{cases}$$

For every case not of type VIa this shows that $k_{\rho}(J) \rightarrow k_{\rho}(\Pi)$ is an isomorphism, which yields the assertion. For type VIa, lemma 4.29 in [RW17] implies that
$k_ρ(J)$ has a unique one-dimensional submodule isomorphic to $i_*(ν^2)$ and fits in an exact sequence in $C_1$

\[
0 \to i_*(ν^2) \to k_ρ(J) \to E[ν^2] \to 0.
\]

Especially, the cokernel of $δ$, or equivalently the image of $k_ρ(a)$, is isomorphic to $E[ν^2]$. Since $\tilde{Π} ∼ i_*(ν^2) ⊕ E[ν^2]$ by [RW17] theorem 6.6 and since $i_*(ν^2)$ is the kernel of $p$, the assertion follows.

**Lemma 5.14.** There is a unique $C_1$-morphism $h$ that makes the following diagram with the canonical surjective arrows in $C_1$ commute,

\[
\begin{array}{ccc}
\tilde{J} & \xrightarrow{g} & \tilde{J}/S_2 \\
\downarrow h & & \downarrow \exists h \\
\tilde{Π}/\tilde{Π}^S & \xrightarrow{\pi_0(\tilde{Π}/\tilde{Π}^S)} & \pi_0(\tilde{Π}/\tilde{Π}^S).
\end{array}
\]

Especially, $h$ is surjective.

**Proof.** It suffices to show the existence of $h$, then uniqueness and surjectivity follow from surjectivity of $g$ shown in lemma [5.13]. Consider the following commutative diagram in $C_1$

\[
\begin{array}{ccc}
\tilde{J}/S_2 & \xrightarrow{h} & \tilde{J} \\
\downarrow \pi_0(\tilde{J}/S_2) & & \downarrow \pi_0(\tilde{J}) \\
\pi_0(\tilde{Π}/\tilde{Π}^S) & \xrightarrow{\pi_0(\tilde{Π}/\tilde{Π}^S)} & \pi_0(\tilde{Π}/\tilde{Π}^S).
\end{array}
\]

where the tilde denotes for the functor $k_ρ$. Surjectivity of $g$ has been shown in lemma [5.13] and surjectivity of the other arrows follows from right-exactness of $k_ρ$. It only remains to be shown that the lower left horizontal arrow is an isomorphism. Indeed, $k_ρ$ applied to the exact sequence $0 \to S_2 \to J \to J/S_2 \to 0$ yields by [RW17] 4.7 a long exact sequence

\[
\cdots \to S_1 \xrightarrow{I_ν} \tilde{J} \to \tilde{J}/S_2 \to 0.
\]

The exact functor $\pi_0 = i_*i^*$ applied to this sequence shows that the natural morphism

\[
\pi_0(\tilde{J}) \to \pi_0(\tilde{J}/S_2)
\]

is an isomorphism because $\pi_0(S_1)$ vanishes. Now the diagonal arrow $h$ exists because the lower left horizontal arrow is an isomorphism. 

\[\square\]
Proposition 5.15. Fix a generic $\Pi \in \mathcal{C}_G(\omega)$ and a smooth character $\rho$ of $\text{GL}(1)$ as in the above assumption. Then the composition of the canonical morphisms

$$\Pi^K \rightarrow \Pi \rightarrow \tilde{\Pi} \rightarrow k_{\nu^2 \rho}(\Pi/\Pi^S)$$

is non-zero for $K = \text{GL}(2, \omega)$ and the Bessel module $\tilde{\Pi} = k_{\rho}(\Pi)$.

Proof. Let $J = j_!j^!(\Pi)$ and $\chi = \nu^2 \rho$. The functor of $K$-invariants and the coinvariant functors applied to the embedding $a : J \hookrightarrow \Pi$ yield a commutative diagram of complex vector spaces with the canonical arrows

The third vertical arrow $g$ is surjective by lemma 5.13. The fourth vertical arrow $h$ exists and is surjective by lemma 5.14. By right-exactness of $k_{\chi}$, the fifth vertical arrow $k_{\chi}(h)$ is also surjective. The composition of the upper horizontal arrows is surjective by lemma 5.9 and because $J^K \rightarrow (J/S_2)^K$ is surjective. Hence the composition of the lower horizontal arrows is surjective. By assumption, $\chi$ occurs at least once in $\pi_0(\Pi/\Pi^S)$, so $k_{\chi}\pi_0(\Pi/\Pi^S)$ is non-zero. The lower triangle is commutative by transitivity of coinvariant functors. Going over the diagonal arrows shows the assertion. \qed

Theorem 5.16. Fix a generic irreducible $\Pi \in \mathcal{C}_G(\omega)$ and a smooth character $\rho$ of $\text{GL}(1)$. If $(\Pi, \rho)$ is non-ordinary exceptional or extra-ordinary exceptional in the sense of [RW17], then the subregular $L$-factor with split Bessel model attached to the Bessel character $\Lambda = \rho \boxtimes \rho^*$ is

$$L_{\text{psreg}}^S(s, \Pi, 1, \Lambda) = L(s, \chi_{\text{crit}}), \quad \chi_{\text{crit}} = \nu^{1/2} \rho.$$

If neither $(\Pi, \rho)$ nor $(\Pi, \rho^*)$ are non-ordinary exceptional or extra-ordinary exceptional, then the subregular $L$-factor $L_{\text{psreg}}^S(s, \Pi, 1, \Lambda) = 1$ is trivial.

Proof. Proposition 5.11 and proposition 5.15 imply that the sufficient condition of lemma 3.6 is satisfied, so there is a subregular pole of the form $L(s, \nu^{1/2} \rho)$. Any additional subregular pole would be of the form $L(s, \nu^{1/2} \rho^*)$ where
\( \rho \neq \rho^* \) by lemma 3.1. This would give rise to an \((H_+, \nu \rho^*)\)-functional by proposition 3.8. Then proposition 4.14 implies that the two distinct characters \( \rho, \rho^* \) are both contained in \( \Delta_-(\Pi) \). This contradicts \([RW17, \text{lemma A.9}]\).

The last assertion follows from proposition 3.8 and proposition 4.14.

Theorem 5.6 and theorem 5.16 imply that the subregular poles are exactly the ones listed in table 2. This completes the results of \([RW17]\) and \([W17]\) and shows that the Piatetskii-Shapiro \(L\)-factors \(L^{PS}(s, \Pi, \mu, \Lambda)\) coincide with the \(L\)-factors formally defined in \([RS07]\), table A.8 using the Langlands classification (see \([RS07]\), section 2.4 of loc. cit. as well as Gan-Takeda \([GT11]\)). This proves our final result:

**Theorem 5.17.** For all irreducible smooth representation \( \Pi \) of \( \text{GSp}(4, k) \) and all split Bessel models \( \Lambda = \rho \boxtimes \rho^* \) of \( \Pi \) with Bessel module \( \tilde{\Pi} = k \rho(\Pi) \), the Piatetskii-Shapiro \(L\)-factor

\[
L^{PS}(s, \Pi, \mu, \Lambda) = L(s, \mu \nu^{-3/2} \otimes (\tilde{\Pi}/\tilde{\Pi}^S))L_{\text{reg}}^{PS}(s, \Pi, \mu, \Lambda)L_{\text{ex}}^{PS}(s, \Pi, \mu, \Lambda)
\]

is the expected \(L\)-factor defined for \((\Pi, \mu)\) given by the Langlands philosophy. It does not depend on the particular choice of the split Bessel model.

A. Appendix

A.1. Central specialization

Instead of the category \( \mathcal{C}_{\text{Gl}(n)} \) it is sometimes preferable to consider its full subcategory \( \mathcal{C} = \mathcal{C}_{\text{Gl}(n)}(\mu) \) of smooth representations with central character \( \mu \). For example, \( \mathcal{C}^{\text{fin}}_{\text{Gl}(n)}(\mu) \) has cohomological dimension \( \leq n \) \([SS97, \text{II.3.3}]\) and simplifies the consideration of the Euler characteristic. For the fixed smooth character \( \mu \) of the center \( Z \cong k^* \) of \( \text{Gl}(n) \), consider the functor

\[
\zeta_\mu : \mathcal{C}_{\text{Gl}(n)} \to \mathcal{C}_{\text{Gl}(n)}(\mu)
\]

sending \( \text{Gl}(n) \)-modules \( M \) to their \((Z, \mu)\)-coinvariant quotient \( M_{(Z, \mu)} \). The functor \( \zeta_\mu \) is right-exact and its left-derived functor is the functor of \((Z, \mu)\)-invariants \( \zeta^\mu : \mathcal{C}_{\text{Gl}(n)} \to \mathcal{C}_{\text{Gl}(n)}(\mu) \), \( M \mapsto M^{(Z, \mu)} \), compare \([RW17, \text{lemma A.2}]\).

By composition with the forgetful functor \( \mathcal{C}_n \to \mathcal{C}_{\text{Gl}(n)} \) we obtain functors \( \mathcal{C}_n \to \mathcal{C}_{\text{Gl}(n)}(\mu) \), also denoted \( \zeta_\mu, \zeta^\mu \). We use the analogous notation for representations of subgroups of \( \text{Gl}(n) \) containing \( Z \). The meaning will be clear from the context.
Lemma A.1. The central specialization functors satisfy:

1. $\zeta_\mu$ is left-adjoint and $\zeta^\mu$ is right-adjoint to the natural embedding $C_{\text{Gl}(n)}(\mu) \to C_{\text{Gl}(n)}$.

2. For every closed subgroup $M \subseteq \text{Gl}(n)$ with $Z \subseteq M$ there are natural equivalences
   $$\zeta_\mu \circ \text{ind}^M_{M} \cong \text{ind}_M^M \circ \zeta_\mu, \quad \zeta^\mu \circ \text{ind}^M_{M} \cong \text{ind}_M^M \circ \zeta^\mu.$$  

3. The functor $\zeta^\mu : C_n \to C_{\text{Gl}(n)}(\mu)$ factorizes over the functor $\kappa : C_n \to C_{\text{Gl}(n)}$ of invariants under $\ker(\text{Gl}_n(n) \to \text{Gl}(n))$.

4. The functor $\zeta_\mu \circ j_!$ is exact and $\zeta^\mu \circ j_! = 0$ is zero.

5. There is an isomorphism $\zeta_\mu(X) \cong \zeta^\mu(X)$ for irreducible $X \in C_{\text{Gl}(n)}$.

Proof. This is a straightforward consequence of results in [BW00, §X]. □

Remark A.2. The functor $\zeta^\mu : C^\text{fin}_n \to C^\text{fin}_{\text{Gl}(n)}$ preserves finite length. The functor $\zeta_\mu : C^\text{fin}_n \to C_{\text{Gl}(n)}(\mu)$ does not preserve finite length. Indeed, $\zeta_\mu(S_2) \in C_{\text{Gl}(2)}(\mu)$ does not have finite length because $k_\rho \zeta_\mu(S_2)$ is non-zero for every $\rho$ by corollary A.6. However, for $X \in C^\text{fin}_2$ with $(j^!)^2(X) = 0$ the central specialization $\zeta_\mu(X) \in C^\text{fin}_{\text{Gl}(2)}(\rho)$ does have finite length.

Lemma A.3. For a smooth character $\chi$ of $k^\times$ let $A = \chi^{(n)} \in C_{\text{Gl}(1)}$ be the attached Jordan block of length $n$. Then the $\text{Gl}(1)$-module $\zeta_\mu(j_! i_*(A))$ has a composition series of modules of length $n$ whose graded pieces are isomorphic to
   $$\zeta_\mu j_! i_*(\chi) \cong \chi^{-1/2} \times \chi^{-1} \nu^{1/2} \mu.$$  

Furthermore $\zeta^\mu(j_! i_*(A)) = 0$.

Proof. $\zeta^\mu \circ j_! = 0$ vanishes and the functor $\zeta_\mu \circ j_! i_*$ is exact by lemma A.1. Therefore $\zeta_\mu \circ j_! i_*(\chi^{(n)})$ has a composition series of length $n$ with graded pieces $\zeta_\mu j_! i_*(\chi)$. Lemma A.1 and transitivity of induction shows that there is an isomorphism
   $$\zeta_\mu j_! i_*(\chi) = \zeta_\mu(\text{ind}^G_{B_{\text{Gl}(2)}}(\chi \boxtimes S)) \cong \text{ind}^G_{B_{\text{Gl}(2)}}(\zeta_\mu(\chi \boxtimes S)).$$

Since $\zeta_\mu(\chi \boxtimes S) = \chi \boxtimes \mu \chi^{-1}$ this shows the statement. □

Lemma A.4. For smooth characters $\mu$ of $k^\times$, the central specializations of $S_n \in C_n$ are $\zeta^\mu(S_n) = 0$ and
   $$\zeta_\mu(S_n) \cong \text{ind}^G_{Z_{\text{Gl}(n)} \times \text{Gl}_n(n-1)}(\mu \boxtimes S_{n-1})$$

induced from the parabotic subgroup $Z_{\text{Gl}(n)} \times \text{Gl}_n(n-1) \subseteq \text{Gl}(n)$ where the mirabolic subgroup is $\text{Gl}_n(n-1) = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \subseteq \text{Gl}(n)$.
Proof. By definition, $S_n = j_!(S_{n-1})$, so the restriction to $\text{Gl}(n)$ is isomorphic to the compactly induced representation

$$S_n|_{\text{Gl}(n)} \cong \text{ind}^{\text{Gl}(n)}_{\text{Gl}(n-1)} S_{n-1} \cong \text{ind}^{\text{Gl}(n)}_{Z \times \text{Gl}(n-1)} (S \boxtimes S_{n-1})$$

where $S \equiv C_c(k^n) = \text{ind}_Z^1(1) \in C_Z$. The central specializations of $S \boxtimes S_{n-1}$ are $\zeta_\mu(S \boxtimes S_{n-1}) \cong \mu \boxtimes S_{n-1}$ and $\zeta^\mu(S \boxtimes S_{n-1}) = 0$. By lemma [A.1], central specialization commutes with $\text{ind}^{Z \times \text{Gl}(n-1)}$ and this implies the statement.

The functors $k_\rho, k^\rho : C_n \to C_{n-1}$ are defined in [RW17]. By restriction they define functors $C_{\text{Gl}(n)} \to C_{\text{Gl}(n-1)}$, also denoted $k_\rho, k^\rho$.

Lemma A.5. There is a commutative diagram

$$
\begin{array}{ccc}
C_n & \xrightarrow{\text{forget}} & C_{\text{Gl}(n)} \\
\downarrow_{k_\rho} & & \downarrow_{k_\rho} \\
C_{n-1} & \xrightarrow{\zeta_\mu} & C_{\text{Gl}(n-1)} \\
\downarrow_{\text{forget}} & & \downarrow_{\text{forget}} \\
C_{n-1} & \xrightarrow{\zeta_\mu/\rho} & C_{\text{Gl}(n-1)}(\mu/\rho)
\end{array}
$$

The functors $k_\rho : C_1 \to C_0$ and $\zeta_\rho : C_1 \to C_{k^n}(\rho) \cong C_0$ are naturally equivalent.

Proof. This follows by transitivity of coinvariant functors.

Corollary A.6. For $n \geq 2$ and smooth characters $\rho$ and $\mu$ of $k^\times$,

$$k_\rho \zeta_\mu(S_n) \cong \zeta_{\mu/\rho}(S_{n-1}), \quad k^\rho \zeta_\mu(S_n) = 0 \quad \text{in } C_{\text{Gl}(n)}(\mu/\rho).$$

Proof. Lemma [A.5] and [RW17, cor. 4.7] show $k_\rho \circ \zeta_\mu(S_n) \cong \zeta_{\mu/\rho} \circ k_\rho(S_n) \cong \zeta_{\mu/\rho}(S_{n-1})$ and thus the first assertion. For the second assertion we only give the proof for $n = 2$. The proof for $n > 2$ is similar using [BZ77, 5.2]. By Bruhat decomposition there is an exact sequence of $\text{B}_{\text{Gl}(2)} = \text{Gl}_1(1) \times Z$-modules

$$0 \to \text{ind}^B_{\text{Gl}(1) \times Z} (S \boxtimes \mu)^w \to \zeta_\mu(S_2) \to (S_1 \boxtimes \mu) \to 0.$$

with the Weyl group element $w = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ and the $\text{Gl}(1)$-module $S = S_1|_{\text{Gl}(1)}$. Let $U$ be the unipotent radical of $\text{B}_{\text{Gl}(2)}$. The $U$-coinvariants of the right hand side are zero because $(S_1)_U \cong i^*S_1$ vanishes. By integration over $U$, the $U$-coinvariant quotient of the left hand side is isomorphic to $S \boxtimes \mu$ as a representation of $\{ (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \} \times Z$. Finally, $S^p = 0$ vanishes.

Lemma A.7. There is a natural equivalence of functors $C_{\text{fin}}^{\text{Gl}(n)} \to C_{\text{fin}}^{\text{Gl}(n-1)}$,

$$k^\rho i_* \cong k_\rho i_*.$$

See [RW17, 4.3].
A.2. Ext-functor

For every connected reductive group $X$ over $k$, the categories $C_X$ and $C_X(\omega)$ have enough projectives and injectives, see [Vig96, §5.9f]. By definition, $\text{Ext}^n_{C_X}(\cdot, M)$ is the $n$-th right derived functor of the contravariant functor $\text{Hom}_{C_X}(\cdot, M)$ for fixed $M \in C_X$.

**Lemma A.8** (Dual Frobenius reciprocity). For a closed subgroup $Y \subseteq X$, there is a natural isomorphism

$$
\text{Ext}^n_{C_X}(V, \text{Ind}_Y^X(W)) \cong \text{Ext}^n_{C_Y}(V|_Y, W)
$$

for every $V \in C_X$ and $W \in C_Y$ and every $n \geq 0$.

**Proof.** See [Vig96, 5.10] and [BW00, X.1.7].

**Lemma A.9.** For every smooth character $\chi$ of $\text{Gl}(1)$ and $X \in C_{\text{Gl}(1)}$, there are natural isomorphisms

$$
\text{Hom}_{\text{Gl}(1)}(X, \chi) \cong \text{Hom}_C(X, \mathbb{C}) \quad , \quad \text{Ext}^1_{\text{Gl}(1)}(X, \chi) \cong \text{Hom}_C(X^\chi, \mathbb{C}) .
$$

**Proof.** The first assertion is clear. The functor $X \mapsto X^\chi$ is left-derived to $X \mapsto X$, see [RW17, A.2]. Dualization is exact and contravariant, so $X \mapsto \text{Hom}(X^\chi, \mathbb{C})$ is the right-derived functor of $X \mapsto \text{Hom}(X, \mathbb{C})$.

**Lemma A.10.** If $\omega = \rho \chi$ is a square, then for every $X \in C_{\text{Gl}(2)}(\omega)$ there are natural isomorphisms

$$
\text{Ext}^1_{C_{\text{Gl}(2)}(\omega)}(X, \text{Ind}_B^G(\chi \boxtimes \rho)) \cong \text{Hom}_C(k^\rho(X), \mathbb{C}) ,
$$

$$
\text{Hom}_{C_{\text{Gl}(2)}(\omega)}(X, \text{Ind}_B^G(\chi \boxtimes \rho)) \cong \text{Hom}_C(k^\rho(X), \mathbb{C}) .
$$

**Proof.** Since $\omega$ is a square, we may assume $\omega = 1$ by a twist. Then $C_{\text{Gl}(2)}(1) = C_{\text{PGl}(2)}$ and we can apply lemma A.8.

$$
\text{Ext}^1_{C_{\text{PGl}(2)}}(X, \text{Ind}_B^G(\chi \boxtimes \rho)) = \text{Ext}^1_{B/Z}(X, \chi \boxtimes \rho) = \text{Ext}^1_{T_{T/Z}}(X, \chi \boxtimes \rho) = \text{Hom}_C(k^\rho(X), \mathbb{C}) ,
$$

where the last step is lemma A.9. The second assertion is analogous.
A.3. Euler characteristic

Fix a smooth central character $\mu$ and write $\hat{X} = \zeta_\mu(X) \in \mathcal{C} := \mathcal{C}^\text{fin}_{\text{GL}(2)}(\mu)$ for the central specialization of $X \in \mathcal{C}^\text{fin}_2$. We consider the Euler characteristic of $X$ and $Y \in \mathcal{C}$ to be that of $\hat{X}$ and $Y$ in $\mathcal{C}$, i.e.

$$\chi(X, Y) = \sum_{i=0}^{1} (-1)^i \dim \text{Ext}^i_{\mathcal{C}}(\hat{X}, Y).$$

The higher $\text{Ext}^i_{\mathcal{C}}$-classes for $i \geq 2$ vanish [SS97, II.3.3]. In other words, the cohomological dimension of $\mathcal{C}$ is one.

**Lemma A.11.** The Euler characteristic $\chi(X, Y)$ is well-defined.

**Proof.** It has to be shown that $\text{Ext}^i_{\mathcal{C}}(\zeta_\mu(X), Y)$ is finite-dimensional for $i = 0, 1$, and irreducible $X$ and $Y$ as above. If $(j^1)^2(X) = 0$, then $\hat{X}$ has finite length as a $\text{GL}(2)$-module by remark [A.2] and the assertion follows. If $(j^1)^2(X) \neq 0$ then $X \cong S_2$ and the assertion follows from lemma [A.4].

**Lemma A.12.** For $Y = \text{ind}_{\text{Gl}(2)B}^\text{Gl}(2) (\chi \boxtimes \chi^{-1} \mu)$ and for every exact sequence $0 \to X_1 \to X \to X_2 \to 0$ in $\mathcal{C}^\text{fin}_2$,

$$\chi(X, Y) = \chi(X_1, Y) + \chi(X_2, Y).$$

Likewise, for every short exact sequence $0 \to Y_1 \to Y \to Y_2 \to 0$ in $\mathcal{C}$,

$$\chi(X, Y) = \chi(X, Y_1) + \chi(X, Y_2).$$

**Proof.** Since $\zeta_\mu(X)$ has finite length for every $X \in \mathcal{C}^\text{fin}_2$ by remark [A.2]

$$\sum_{i=0}^{1} (-1)^i \dim(\text{Ext}^i_{\mathcal{C}}(\zeta_\mu(X), Y)) = 0$$

vanishes by lemma [A.7] Central specialization yields a long exact sequence

$$0 \to \zeta_\mu(X_1) \to \zeta_\mu(X) \to \zeta_\mu(X_2) \to \zeta_\mu(X_1) \to \zeta_\mu(X) \to \zeta_\mu(X_2) \to 0$$

The well-known formulas for the Euler characteristic imply the statement.

**Proposition A.13.** For $X \in \mathcal{C}^\text{fin}_2$ and $Y = \text{ind}_{\text{Gl}(2)B}^\text{Gl}(2) (\mu \rho^{-1} \boxtimes \rho)$, the Euler characteristic is

$$\chi(X, Y) = \dim(j^1)^2(X)$$

with the functor $(j^1)^2 : \mathcal{C}_2 \to \mathcal{C}_0$. 

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Proof. By lemma A.12 it suffices to show the assertion for irreducible $X$. Note that $\dim \chi(X, Y) = \dim \text{Hom}(k^\rho \zeta_\mu, X) - \dim \text{Hom}(k^\rho \zeta_\mu, C)$ as shown in lemma A.10. For $X = i_* (\pi)$ with irreducible $\pi \in C_{\text{Gl}(2)}$ use lemma A.7. The argument for $X = j_! i_*(\chi)$ is similar using lemma A.3. Finally, for $X = \mathbb{S}_2$ the assertion follows from corollary A.6. \hfill \Box

Corollary A.14. For $Y = \text{ind}_{\text{Gl}(2)} (\mu \rho^{-1} \boxtimes \rho) \in C$ and $X \in C_{\text{fin}}$,\

$$\dim \text{Ext}^1 \rho(X, Y) = \dim k \chi k \rho(X) - \dim (j')^2(X).$$

Remark A.15. Fix a smooth character $\tilde{\rho}$ of $\text{Gl}(1)$. For $Y_+ = \text{ind}_{\text{Gl}(2)} (\tilde{\rho} \boxtimes \tilde{\rho})$ and $Y_- = \text{ind}_{\text{Gl}(2)} (\nu \tilde{\rho} \boxtimes \nu^{-1} \tilde{\rho})$ there are exact sequences\

$$0 \to (\tilde{\rho} \circ \text{det}) \to Y_+ \to \text{St} (\tilde{\rho}) \to 0,$$

$$0 \to \text{St} (\tilde{\rho}) \to Y_- \to (\tilde{\rho} \circ \text{det}) \to 0.$$

The functor $\text{Hom}_{\text{Gl}(2)} (\hat{X}, -)$ yields long exact sequences\

$$0 \to \text{Hom}_C (\hat{X}, \tilde{\rho} \circ \text{det}) \to \text{Hom}_C (\hat{X}, Y_+) \to \text{Hom}_C (\hat{X}, \text{St} (\tilde{\rho})) \to 0,$$

$$0 \to \text{Hom}_C (\hat{X}, \text{St} (\tilde{\rho})) \to \text{Hom}_C (\hat{X}, Y_-) \to \text{Hom}_C (\hat{X}, \tilde{\rho} \circ \text{det}) \to 0.$$

Recall that $\dim \text{Hom}_C (\hat{\Pi}, Y_+) = \dim \text{Hom}_C (k^\rho k^\mu \Pi, C) = 1$ for generic $\Pi \in C_G(\omega)$ by lemma 4.4.

Lemma A.16. $\chi(S_2, \tilde{\rho} \circ \text{det}) = 0$ and $\chi(S_2, \text{St} (\tilde{\rho})) = 1$.

Proof. $\text{Ext}^1 \rho(S_2, Y_+) = 0$ vanish by corollary A.14 and because $k \chi k \rho(S_2) \cong C$. The long exact sequences of remark A.15 imply that both\

$$\text{Ext}^1 \rho(S_2, \tilde{\rho} \circ \text{det}) = 0$$

vanish. $\text{Hom}_C (S_2, \tilde{\rho} \circ \text{det}) = 0$ vanishes [W17, lemma 2.2], so the first assertion follows. Proposition A.13 and counting dimensions in the long exact sequence yields the second assertion. \hfill \Box
Lemma A.17. For every finite-dimensional Gl(1)-module $A$ and for every $Y \in \mathcal{C}$, the Euler characteristic $\chi(j_!i_*(A),Y) = 0$ vanishes.

Proof. By lemma A.3 and lemma A.12 we can assume that $A$ is a character and $Y$ is irreducible. For $X = j_!i_*(A)$, lemma A.3 implies $\widehat{X} = \zeta_\mu(X) = \text{ind}_{B_{GL(2)}}^{\text{GL}(2)}(A \boxtimes \mu A^{-1})$. If $Y = \sigma \circ \det$ for a smooth character $\sigma$ with $\sigma^2 = \mu$, it is well-known that for both $i = 0, 1$

$$\dim \text{Ext}^i_C(\widehat{X}, \sigma \circ \det) = \begin{cases} 1 & A = \nu \sigma, \\ 0 & \text{otherwise}. \end{cases}$$

The proof for $Y = \text{St}(\tilde{\rho})$ is analogous where for both $i = 0, 1$

$$\dim \text{Ext}^i_C(\widehat{X}, \text{St}(\sigma)) = \begin{cases} 1 & A = \sigma, \\ 0 & \text{otherwise}. \end{cases}$$

Both $\text{Ext}^i$-terms vanish if $Y$ is cuspidal. For fully induced $Y$ the assertion has been shown in proposition A.13.

Lemma A.18. For irreducible $\Pi \in \mathcal{C}_G(\omega)$ with a split Bessel model and not of type VII, VIIIa, IXa and for non-cuspidal irreducible $Y \in \mathcal{C}$, the Euler characteristic is

$$\chi(\Pi, Y) = \chi(j_!j^!(\Pi), Y) = \chi(S^m_2, Y) = m_Y m_\Pi$$

where $m_Y \in \{0, 1\}$ is the dimension of Whittaker models for $Y$ and $m_\Pi \in \{0, 1\}$ is the dimension of Whittaker models for $\Pi$.

Proof. Recall that lemma 3.5 yields an exact sequence

$$0 \to J \to \Pi \to i_*(B) \to 0.$$

with $B = i^*(\Pi)$ and $J = j_!j^!(\Pi)$. By [W17, lemma 6.1], the associated long exact sequence of central specialization is

$$0 \to \zeta_\mu i_*(B) \to \zeta_\mu(J) \to \zeta_\mu(\Pi) \to \zeta_\mu j_*(B) \to 0.$$

Lemma A.1 implies $\zeta_\mu i_*(B) \cong \zeta_\mu j_*(B)$ and thus $\chi(\Pi, Y) = \chi(J, Y)$ holds by lemma A.12. The functor $\zeta_\mu \circ j_!$ is exact by lemma A.1 so there is a short exact sequence

$$0 \to \zeta_\mu(S^m_2) \to J \to \zeta_\mu \circ j_!i_*(A) \to 0$$

for $A = i^* j_!(\Pi)$. Since $\chi(j_!i_*(A), Y) = 0$ by lemma A.17, we obtain $\chi(J, Y) = m_\Pi \cdot \chi(S_2, Y)$. By lemma A.13 and lemma A.16 this equals $m_\Pi m_Y$. □
Table 2: The subregular $L$-factor for split Bessel models $\Lambda = \rho \boxtimes \rho^*$.

| Type | II | $\rho$ | $L_{\text{sreg}}(s, \Pi, 1, \Lambda)$ |
|------|----|--------|----------------------------------|
| I    | $\chi_1 \times \chi_2 \times \sigma$ | $\rho \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho)$ |
|      | | $\rho^* \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho^*)$ |
| Ila  | $\text{St}(\chi_1) \rtimes \sigma$ | $\rho \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho)$ |
|      | | $\rho^* \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho^*)$ |
| Va   | $\delta([\xi, \nu \xi], \nu^{-1/2} \sigma)$ | $\rho \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho)$ |
|      | | $\rho^* \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho^*)$ |
| Vla  | $\tau(S, \nu^{-1/2} \sigma)$ | $\rho \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho)$ |
|      | | $\rho^* \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho^*)$ |
| X    | $\pi \rtimes \sigma$ | $\rho \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho)$ |
|      | | $\rho^* \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho^*)$ |
| Xla  | $\delta(\nu^{1/2} \pi, \nu^{-1/2} \sigma)$ | $\rho \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho)$ |
|      | | $\rho^* \in \Delta_-(\Pi)$ | $L(s, \nu^{1/2} \rho^*)$ |
| IIIb | $\chi \rtimes \sigma \chi_{\text{sp}(2)}$ | $\rho \in \{\sigma, \chi \sigma\}$ | $L(s, \nu^{1/2} \chi \sigma)L(s, \nu^{1/2} \sigma)$ |
| IVc  | $L(\nu^{1/2} \pi, \nu^{-3/2} \sigma)$ | $\rho \in \{\nu^{1/2} \sigma\}$ | $L(s, \nu^{3/2} \sigma)$ |
| Vlc  | $L(\nu^{1/2} \pi, \nu^{-1/2} \sigma)$ | $= \sigma$ | $L(s, \nu^{1/2} \sigma)$ |
| Vld  | $L(\nu, 1 \rtimes \nu^{-1/2} \sigma)$ | $= \sigma$ | $L(s, \nu^{1/2} \sigma)$ |
|      | otherwise | | 1 |

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Table 3: Piatetskii-Shapiro’s spinor $L$-factor for split Bessel models

| Type | $\Pi$ | $\rho$ | $L^{PS}(s, \Pi, \Lambda, 1)$, $\Lambda = \rho \boxtimes \rho^*$ split |
|------|-------|--------|--------------------------------------------------------------|
| I    | $\chi_1 \times \chi_2 \times \sigma$ | all | $L(s, \sigma)L(s, \chi_1\chi_2\sigma)L(s, \chi_1\chi_2\sigma)$ |
| IIa  | $\chi_{St} \times \sigma$ | all | $L(s, \sigma)L(s, \chi_2\sigma)L(s, \nu^{1/2}\chi\sigma)$ |
| IIb  | $\chi_1 \times \sigma$ | $\chi\sigma$ | $L(s, \sigma)L(s, \chi_2\sigma)L(s, \nu^{-1/2}\chi\sigma)L(s, \nu^{1/2}\chi\sigma)$ |
| IIIa | $\chi \times \sigma_{St}$ | all | $L(s, \sigma)L(s, \nu^{1/2}\chi\sigma)L(s, \nu^{1/2}\sigma)$ |
| IIIb | $\chi \times \sigma 1$ | $\sigma, \chi\sigma$ | $L(s, \nu^{-1/2}\chi\sigma)L(s, \nu^{-1/2}\sigma)L(s, \nu^{1/2}\chi\sigma)L(s, \nu^{1/2}\sigma)$ |
| Va   | $\sigma_{St} G$ | none | — |
| IVa  | $\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$ | all | $L(s, \nu^{1/2}\sigma)L(s, \nu^{1/2}\xi\sigma)$ |
| IVb  | $L(\nu^{1/2}\sigma St, \nu^{-1/2}\sigma)$ | $\sigma$ | $L(s, \nu^{1/2}\sigma)L(s, \nu^{-1/2}\sigma)$ |
| IVc  | $L(\nu^{3/2}\sigma St, \nu^{-3/2}\sigma)$ | $\nu^{k_1}\sigma$ | $L(s, \nu^{1/2}\sigma)L(s, \nu^{-3/2}\sigma)L(s, \nu^{3/2}\sigma)$ |
| IVd  | $\sigma \chi G$ | none | — |
| Va   | $\tau(S, \nu^{-1/2}\sigma)$ | all | $L(s, \nu^{1/2}\sigma)^2$ |
| Vb   | $\tau(T, \nu^{-1/2}\sigma)$ | none | — |
| Vc   | $L(\nu^{1/2}\sigma St, \nu^{-1/2}\sigma)$ | $\sigma$ | $L(s, \nu^{-1/2}\sigma)L(s, \nu^{1/2}\sigma)^2$ |
| Vd   | $L(\nu\sigma, \chi\times \nu^{-1/2}\sigma)$ | $\xi\sigma$ | $L(s, \nu^{1/2}\sigma)L(s, \nu^{-1/2}\xi\sigma)L(s, \nu^{1/2}\xi\sigma)$ |
| VII  | $\chi \times \pi$ | all | 1 |
| VIIIa| $\tau(S, \pi)$ | all | 1 |
| VIIIb| $\tau(T, \pi)$ | none | — |
| IXa  | $\delta(\nu\sigma, \nu^{-1/2}\pi)$ | all | 1 |
| IXb  | $L(\nu\sigma, \nu^{-1/2}\pi)$ | none | — |
| X    | $\pi \times \sigma$ | all | $L(s, \sigma)L(s, \omega\sigma)$ |
| Xla  | $\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ | all | $L(s, \nu^{1/2}\sigma)$ |
| Xlb  | $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ | $\sigma$ | $L(s, \nu^{-1/2}\sigma)L(s, \nu^{1/2}\sigma)$ |

For every irreducible smooth representation $\Pi$ of $GSp(4, k)$ with central character $\omega$, the column $\rho$ lists the smooth characters such that the character $\Lambda = \rho \boxtimes \rho^*$ of $T \cong k^\times \times k^\times$ where $\rho^* = \omega \rho^{-1}$ yields a split Bessel model for $\Pi$. The right column lists Piatetskii-Shapiro’s spinor $L$-factors attached to this split Bessel model. For non-cuspidal $\Pi$ we use the classification symbols of [ST94] and [RS07].