Invisible non-Hermitian potentials in discrete-time photonic quantum walks

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Discrete-time photonic quantum walks on a synthetic lattice, where both spatial and temporal evolution of light is discretized, have provided recently a fascinating platform for the observation of a wealth of non-Hermitian physical phenomena and for the control of light scattering in complex media. A rather open question is whether invisible potentials, analogous to the ones known for continuous optical media, do exist in such discretized systems. Here it is shown that, under certain conditions, slowly-drifting Kramers-Kronig potentials behave as invisible potentials in discrete-time photonic quantum walks. © 2022 Optical Society of America

Introduction. The scattering of waves through inhomogeneous or disordered systems is ubiquitous in different areas of classical and quantum physics. In optics, light scattering arises in any inhomogeneous medium where the refractive index rapidly changes on a spatial scale of the order of the optical wavelength. However, it is known since long time that some inhomogeneous distributions of the refractive index do not reflect light [1]. Recently, there has been a surge of interest in controlling the scattering of waves through inhomogeneous or disordered media based on the special engineering of both the real and imaginary parts of the refractive index [2–29].

Refractive index engineering enables to realize new kinds of reflectionless and even invisible potentials, such as those based on parity-time (PT) symmetry and transformation optics [2–7], reverse engineering [8–10] and the spatial Kramers-Kronig relations [11–22]. Complex potential engineering offers as well a systematic method to construct a complex medium where a desired waveform can freely propagate, free of scattering, even in highly-disordered media [23–29]. However, the experimental feasibility to engineer at sub-wavelength scale both real and imaginary parts of the refractive index in continuous media is challenging, and there are few experiments demonstrating such new classes of synthetic materials either at microwaves [20] or for acoustic waves [25].

Discrete-time photonic quantum walks on a lattice [30] provide a different and important class of optical systems, where the evolution of light is discretized both in space and time. As compared to continuous optical media, they offer the rather unique possibility of a simple implementation of complex (non-Hermitian) potentials with rather arbitrary profiles, providing a fascinating platform to experimentally access a wealth of novel non-Hermitian phenomena, such as PT symmetry breaking [31, 32], non-Hermitian topological physics [33–37], non-Hermitian Anderson localization [38] and non-Hermitian phase transitions [39]. Recently, the observation of constant-intensity waves and induced transparency has been reported using discrete-time photonic quantum walks with complex tailored potentials [40], thus overcoming the limitations of refractive index engineering required in continuous media. However, an open question is whether in such discrete-time systems there exist potentials that are invisible for any incident wave, i.e. not only for a given target wave shape. This question is specially demanding since the discrete nature of the dynamics can deeply modify the scattering properties of non-Hermitian reflectionless potentials known in continuous media [19].

In this Letter we consider wave scattering in a discrete-time photonic quantum walk setting from non-Hermitian complex potentials that drift on the lattice at a constant speed, and show that the class of Kramers-Kronig potentials are invisible in the slowly-drifting regime.

Model. We consider a discrete-time photonic quantum walk realized using optical pulses in a synthetic mesh lattice [30–32, 35, 38–40]. The system consists of two fiber loops of slightly different lengths $L \pm \Delta L$ (short and long paths) that are connected by a fiber coupler with a coupling angle $\beta$ [Fig.1(a)]. In the short loop, we place an amplitude (AM) and a phase (FM) modulator, which provide a desired control of both phase and amplitude of the traveling pulses. Light dynamics in the fiber loops mimic a one-dimensional discrete-time quantum walk of a single quantum particle [30] with time steps $m$ and spatial positions $n$ [Fig.1(b)]. Neglecting pulse broadening and cross-talk effects, the coupled equations for the amplitudes $u_n^{(m)}$ and $v_n^{(m)}$ of the optical pulses in the short and long fiber loops read [31, 32, 38, 40, 41]

\[
\begin{align*}
    u_{n+1}^{(m+1)} &= \cos \beta u_{n+1}^{(m)} + i \sin \beta v_{n+1}^{(m)} \exp(-i V_{n,m+1}) \\
    v_{n+1}^{(m+1)} &= \cos \beta v_{n}^{(m)} + i \sin \beta u_{n}^{(m)}
\end{align*}
\]

where $V_{n,m}$ is the discrete complex potential that is generated by the combination of AM and PM modulators. The ratio $L/\Delta L = T/\Delta T$ determines the number of
Fig. 1. Discrete-time photonic quantum walk on a synthetic lattice with a non-Hermitian scattering potential. (a) Schematic of two coupled fiber loops with slightly length mismatch \(L \pm \Delta L\). A fiber coupler with coupling angle \(\beta\) mixes the light waves between the two fiber loops. An amplitude (AM) and a phase (PM) modulator are placed in the short loop, providing an effective complex potential \(V_{n,m}\) for the traveling pulses in the loop. (b) Schematic of the synthetic mesh lattice. The physical time \(t\) is mapped at the discretized times \(t_n^m = n\Delta T + m\tau\), where \(T = L/c\) is the mean travel time and \(\Delta T = \Delta L/c \ll T\) is the travel time mismatch between the two fiber loops. The index \(n\) corresponds to the site index in a synthetic one-dimensional spatial lattice, while the integer \(m\) corresponds to a discrete time along which the system evolves. The scattering potential is assumed of the form \(V_{n,m} = \varphi(n + mv)\), i.e. drifting along the lattice in the backward direction at a speed \(v\).

sites in the lattice, whereas the travel time mismatch \(\Delta T = \Delta L/c\) gives the upper limit to the duration \(\Delta \tau\) of pulses that can be injected in the loops [41]. Typical values of mean travel time \(T = L/c\) and time mismatch \(\Delta T\) are \(T \sim 27 \mu s\) and \(\Delta T \sim 50 ns\) [39], so that the lattice can accommodate more than 200 sites. For a typical pulse duration \(\Delta \tau \sim 5 ns\), pulse broadening due to fiber dispersion is negligible over several thousands of time steps \(m\), so that the mesh lattice equations (2) describe with excellent accuracy the pulse dynamics in the two rings (see e.g. [32, 35, 39]).

In the absence of the potential, i.e. for \(V_{n,m} = 0\), the system shows discrete translational invariance both in space and time, and the eigenfunctions are of the Bloch-Floquet form, i.e. \((u_n^{(m)}, v_n^{(m)})^T = (\tilde{U}_\pm, \tilde{V}_\pm)^T \exp[iq\alpha - iE_{\pm}(q)m],\) where \(q\) is the (spatial) Bloch wave number,

\[
E_{\pm}(q) = \pm \frac{1}{2} \alpha \cos(\beta \cos q) \tag{3}
\]

are the quasi-energies of the two bands of the binary lattice, and \(\tilde{U}_\pm = i\sin \beta \exp(iq), \tilde{V}_\pm = \exp(-iE_{\pm}) - \cos \beta \exp(iq)\) are the amplitudes of Bloch waves. Note that a wave packet with carrier Bloch wave number \(q\) in either one of the two bands travels in the lattice at the speeds (group velocities) \(v_{g\pm} = (dE_{\pm}/dq)\), which take the largest absolute value \(v_{g}(max) = \cos \beta\) at \(q = \pm \pi/2\). Therefore, any excitation in the discrete lattice cannot propagate faster than \(v_{g}(max)\).

Wave scattering analysis. Let us consider the discrete-time photonic quantum walk in the presence of a space-time complex potential \(V_{n,m}\), localized in space and vanishing fast enough as \(n \to \pm \infty\). We aim at establishing whether one can find a class of potentials that are fully invisible for any arbitrary wave packet propagating in the system, i.e. such that after the scattering event the wave packet evolves exactly as if the potential were not present. We note that such a kind of transparency has been recently demonstrated in Ref. [40], however in that case the invisibility holds only for a target incident waveform. Additionally, we mention that the families of static reflectionless potentials known in continuous media, such as the class of Kramers-Kronig potentials [14], may lose their reflectionless property owing to space discretization [19]. To search for a class of invisible potentials, let us assume, for the sake of definiteness, that the wave packet comes from \(n = -\infty\) and propagates in the forward direction of the lattice. Since in the clean system there is an upper bound \(v_{g}(max)\) to the propagation speed of excitation, it readily follows that any potential of the form \(V_{n,m} = \varphi(n + mv)\), i.e. drifting on the lattice at a speed \(v\) in the backward direction [Fig.1(b)], is reflectionless provided that \(v > v_{g}(max) = \cos \beta\) [42]: in fact, any scattered wave cannot appear as a reflected wave in the reference frame where the potential is at rest. Here \(\varphi(x)\) is a continuous function of the variable \(x\), that defines the shape of the scattering potential. The absence of reflected waves, however, does not ensure that the potential is invisible. To study the scattering dynamics, it is worth considering the moving reference frame

\[
x = n + vm, \quad t = m \tag{4}
\]

where the potential is at rest. Note that in the moving reference frame the variable \(t\) remains discrete, while \(x\) should be considered as a continuous variable. After let-
Note that, in the absence of the scattering potential, the vanishing of the amplitudes $\phi^\pm(x)$ in the lower band of the lattice, so that its quasi energy is conserved apart from integer multiples than $2\pi$, the asymptotic solution of the scattered wave as $x \to \infty$ must be of the form

$$\left( \frac{f(x,t)}{g(x,t)} \right)_x \approx \sum_{\alpha, \pm} t^{\pm}_{\alpha} \left( \frac{F_{\pm}(q_{n0}^{\alpha})}{G_{\pm}(q_{n0}^{\alpha})} \right) \exp(iq^{\pm}_{\alpha}x - i\epsilon_{\alpha}t)$$

(8)

with some amplitudes $t^{\pm}_{\alpha}$, where $\alpha = 0, \pm 1, \pm 2, \ldots$ is the order of scattering channel and $q_{n0}^{\alpha}$ are the roots of the equation $\epsilon_{\pm}(q_{n0}^{\alpha}) = 0 + 2\pi\alpha$ (see Fig. 2). Basically, the amplitudes $t^{\pm}_{\alpha}$ are the transmission coefficients of various scattering channels in the two bands, labelled by the index $\alpha$ and arising from the discrete nature of time evolution. Clearly, the scattering potential is invisible provided that all amplitudes $t^{\pm}_{\alpha}$ vanish, apart from $t^{+}_{0}$ which should be $t^{+}_{0} = 1$. The analytic form of $t^{+}_{0}$ can be derived in the weak potential limit $|\phi(x)| \ll 1$ using a first-order (Born) approximation, and turns out to be proportional to $f_{\alpha}(q_{n0}^{\alpha} - q_{00}^{\alpha})$, where $f_{\alpha}(q) = \int dxq(x) \exp(-iqx)$ is the Fourier spectrum of the potential (Sec. I of the Supplemental document). Therefore, it is not possible rather generally to have an invisible potential, owing to the infinite number of scattering channels. However, in the limit of a slowly-drifting potential $v \to 0$, all the wave numbers $q_{n0}^{\alpha}$ of scattered waves diverge like $\sim 1/v$, apart from $q_{00}^{\alpha}$ which does not depend on $v$. Note that the slowly-drifting regime $v \to 0$ necessarily implies $\beta \to \pi/2^\pm$, with $\cos \beta < v$. Since the Fourier spectrum $\tilde{\phi}(q)$ of the scattering potential vanishes at high spatial frequencies, in the $v \to 0$ regime the amplitudes $t^{\pm}_{\alpha}$ of scattered waves are vanishing, with the exception of $t^{+}_{0}$. In the Born approximation and for a slowly-driving potential, invisibility is thus attained provided that $\tilde{\phi}(q) = 0$. As shown in Sec. II of the Supplemental document, using the method of complex spatial displacement [15] it can be shown that invisibility is found for any slowly-drifting potential $\varphi(x)$ of the Kramers-Kronig type, i.e.
In Fig. 3(a), the scattering potential drifts slowly on the lattice and it appears invisible, whereas in Fig. 3(b) the potential drifts fast and ceases to be invisible, according to the theoretical analysis. Clearly, the invisibility of the potential is lost even in the slowly-drifting regime when $\varphi(x)$ is not of the Kramers-Kronig type. This is shown, as an example, in Fig.3(c), where the scattering potential $\varphi(x)$ is real (Hermitian scattering) and given by the real part solely of the potential $A/(x-x_0)^2$. Finally, we note that, as mentioned above, the shape of the Kramers-Kronig potential can be made arbitrarily irregular by considering the multi-pole potential (9) with the sum extended over a suitable large number of terms: in spite of the irregular shape of the potential, it is invisible for any incident wave in the slowly-drifting regime. An example of an irregular multi-pole invisible potential is shown in Fig.4.

**Conclusion.** We predicted the existence of invisible potentials of Kramers-Kronig type in discrete-time photonic quantum walks, extending to time-discrete systems the fascinating property of such a class of non-Hermitian potentials, unraveling some limitations arising from the discrete nature of the temporal dynamics. Our results could be readily extended to discrete-time multi-band lattice systems, providing a deeper understanding of non-Hermitian wave scattering and invisibility in discrete-time systems and opening up new possibilities for non-Hermitian wave control.

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