We construct not only an induced gravity model with restricted diffeomorphisms, that is, transverse diffeomorphisms that preserve the curvature density, but also with full diffeomorphisms. By solving the equations of motion, it turns out that these models produce Einstein’s equations with a certain Newton constant in addition to the constraint for the curvature density. In the limit of the infinite Newton constant, the models give rise to induced gravity. Moreover, we discuss cosmological solutions on the basis of the gravitational models at hand.

Subject Index B27, E00, E03

1. Introduction We have recently seen the revival of a gravitational theory called unimodular gravity [1–11]. The original idea of unimodular gravity stems from an observation by Einstein that some equations of general relativity take simpler expressions when they are described in the unimodular coordinates where the determinant of the metric tensor takes \(-1\), that is, \(\det g_{\mu\nu} = -1\) [12].

It is considered that unimodular gravity is obtained from general relativity by choosing a gauge condition \(\sqrt{-g} = 1\) for diffeomorphisms. The resultant equations of motion are then given by the traceless part of Einstein’s equations in addition to the unimodular constraint \(\sqrt{-g} = 1\). One of the attractive points in unimodular gravity is that the cosmological constant appears as an integration constant that can take any value. This fact has previously been expected to give some insight into the well-known cosmological constant problem. However, it might be difficult to resolve the cosmological constant problem within the framework of unimodular gravity if we think that unimodular gravity is nothing but a gauge-fixed theory of general relativity as mentioned above.

Before describing our gravitational models in detail in the next section, let us begin with unimodular gravity since we find it to be useful to explain our basic idea. Let us start with the conventional Einstein–Hilbert action with the cosmological constant,\(^1\)

\(^1\) We follow the notation and conventions of [13]; for instance, the flat Minkowski metric \(\eta_{\mu\nu} = \text{diag}(-, +, +, +)\), the Riemann curvature tensor \(R^\alpha_{\nu\rho\sigma} = \partial_\rho \Gamma^\alpha_{\nu\sigma} - \partial_\sigma \Gamma^\alpha_{\nu\rho} + \Gamma^\alpha_{\rho\sigma} \Gamma^\rho_{\nu\mu} - \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\nu\sigma}\), and the Ricci tensor \(R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}\). The reduced Planck mass is defined as \(M_p = \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18}\) GeV, where \(G\) is the Newton constant. Moreover, as usual, we define the gravitational coupling constant \(\kappa\) in terms of \(\kappa^2 = 8\pi G = \frac{1}{M_p^2}\).
\[ S_{GR} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R - 2\Lambda \right) \]
\[ = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R - 2\Lambda' \int d^4x \sqrt{-g}, \]

where \( \Lambda \) is the cosmological constant and we have introduced the rescaled cosmological constant \( \Lambda' = \frac{1}{2\kappa^2} \Lambda \). Based on this action, unimodular gravity can be understood as follows: In unimodular gravity, we focus on the second term, i.e., the cosmological constant term, putting \( \Lambda = 0 \), and try to derive effectively the cosmological constant by taking a gauge condition \( \sqrt{-g} = 1 \) for one of the diffeomorphisms.

In unimodular gravity, therefore, the starting action takes the form

\[ S_{UM} = \int d^4x \left[ \frac{1}{2\kappa^2} \sqrt{-g} R - \lambda(x) \left( \sqrt{-g} - v_0 \right) \right] + S_m, \]

where \( \lambda(x) \) is the Lagrange multiplier field enforcing the unimodular constraint \( \sqrt{-g} = v_0 \), with \( v_0 \) being a constant introduced for generality instead of 1, and \( S_m \) means the matter action. Note that this action is not manifestly invariant under the full diffeomorphisms because of the presence of the term \( \lambda(x)v_0 \).

Variation of the metric tensor yields the Einstein equations:

\[ \frac{1}{\kappa^2} G_{\mu\nu} + \lambda(x)g_{\mu\nu} = T_{\mu\nu}, \]

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \) is the Einstein tensor and \( T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} S_m \) is the energy–momentum tensor of matter. Then, operating \( \nabla^\mu \) on the Einstein equations (3), with the help of the Bianchi identity \( \nabla^\mu G_{\mu\nu} = 0 \) and the conservation law \( \nabla^\mu T_{\mu\nu} = 0 \), one obtains the equations \( \partial_\mu \lambda = 0 \), which mean that the Lagrange multiplier field \( \lambda(x) \) is a constant, \( \lambda(x) = \lambda_0 \). As a result, the dynamics of unimodular gravity is classically equivalent to that of general relativity with a constant cosmological constant \( \lambda_0 \kappa^2 \).

Incidentally, sometimes in unimodular gravity the traceless Einstein equations are regarded as the fundamental equations,

\[ \frac{1}{\kappa^2} \left( R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R \right) = T_{\mu\nu} - \frac{1}{4}g_{\mu\nu} T, \]

which can be obtained as follows: First, taking the trace of the Einstein equations (3), we have

\[ \lambda = \frac{1}{4} \left( \frac{1}{\kappa^2} R + T \right). \]

Next, substituting this expression for the Lagrange multiplier field into the Einstein equations (3), it is easy to show that we can get (4).

One of the most attractive aspects in unimodular gravity is that the cosmological constant arises as an arbitrary constant of integration from field equations, so it is not a fixed constant from the beginning even if it has no preferred value, at least at the classical level. It would be splendid if quantum effects could pick up a small and positive value among arbitrary values via a still unknown dynamical mechanism.

We are now ready to present our idea. A natural question to ask ourselves is whether it is possible to pay attention to the first term in the action (1), i.e., the Einstein–Hilbert term, and attempt to derive
effectively the Newton constant as an integration constant from field equations by imposing a new constraint \( \sqrt{-g} R = 1 \).

Of course, there is a big difference between unimodular gravity and our model. The unimodular constraint \( \sqrt{-g} = 1 \) is a non-dynamical equation, while the new constraint \( \sqrt{-g} R = 1 \) contains the second derivative of the time variable so it is a dynamical one. In this sense, the constraint \( \sqrt{-g} R = 1 \) would control the dynamics of the model and provide us with a new perspective on the background geometry.

The structure of this article is the following: In Sect. 2, we present a simple model that accommodates the curvature density preserving condition, and derive the equations of motion. Moreover, we also construct a gravitational model respecting full diffeomorphisms. In Sect. 3, we examine the cosmological implications of our model. We present our conclusions in Sect. 4.

2. **Gravitational models with curvature density preserving diffeomorphisms**

Let us start with a gravitational model where the Einstein–Hilbert term with the cosmological constant is accompanied by the curvature density preserving constraint \( \sqrt{-g} R = \varepsilon_0 \):\(^2\)

\[
S = \int d^4x \left[ \frac{1}{2\kappa^2} \sqrt{-g} \left( R - 2\Lambda \right) - \lambda(x) \left( \sqrt{-g} R - \varepsilon_0 \right) \right] + S_m, \tag{6}
\]

where \( \lambda(x) \) is the Lagrange multiplier field enforcing the constraint \( \sqrt{-g} R = \varepsilon_0 \). Note that this action is not manifestly invariant under full diffeomorphisms because of the presence of the term \( \lambda(x) \varepsilon_0 \).

The Einstein equations, which come from variation with respect to the metric tensor \( g^{\mu\nu} \), read

\[
\left[ \frac{1}{\kappa^2} - 2\lambda(x) \right] G_{\mu\nu} + \frac{\Lambda}{\kappa^2} g_{\mu\nu} + 2 \left( \nabla_\mu \nabla_\nu \lambda - g_{\mu\nu} \nabla_\rho \nabla^\rho \lambda \right) = T_{\mu\nu}. \tag{7}
\]

Variation of the Lagrange multiplier field \( \lambda(x) \), of course, produces our constraint \( \sqrt{-g} R = \varepsilon_0 \).

Operating \( \nabla^\mu \) on the Einstein equations (7) leads to

\[
-2 \nabla^\mu \lambda \cdot G_{\mu\nu} + 2 \left( \nabla^2 \nabla_\nu \lambda - \nabla_\nu \nabla^2 \lambda \right) = 0, \tag{8}
\]

where the Bianchi identity and the conservation law of the energy–momentum tensor are used. Then, using the identity \( \nabla^2 \nabla_\nu \lambda = R_{\mu\nu} \nabla^\mu \lambda + \nabla_\nu \nabla^2 \lambda \), Eq. (8) can be cast to the form

\[
R \nabla_\nu \lambda = 0, \tag{9}
\]

from which, assuming \( R \neq 0 \), we have \( \lambda(x) = \lambda_0 \), where \( \lambda_0 \) is an integration constant as in unimodular gravity.

Inserting \( \lambda(x) = \lambda_0 \) in the Einstein equations (7), we arrive at the conventional Einstein equations except for a modified Newton constant \( \kappa' \) in front of the Einstein tensor, defined as \( \frac{1}{\kappa'^2} = \frac{1}{\kappa^2} - 2\lambda_0 \):

\[
\frac{1}{\kappa'^2} G_{\mu\nu} + \frac{\Lambda}{\kappa'^2} g_{\mu\nu} = T_{\mu\nu}. \tag{10}
\]

\(^2\) Here, for slight generality, we have introduced a constant \( \varepsilon_0 \) instead of \( \sqrt{-g} R = 1 \). Moreover, we have involved the Einstein–Hilbert plus the cosmological term for generality. In this setting, we can obtain induced gravity theory in the limit of the infinite Newton constant, \( \kappa^2 = 8\pi G \to \infty \).
At this stage, it is worth mentioning that in the limit of the infinite Newton constant, \( \kappa^2 \rightarrow \infty \), the gravitational model at hand reduces to an induced gravity [14–21]. Actually, in this case, the Einstein–Hilbert plus the cosmological constant terms drop out of the classical action (6), and if we set \(-2\lambda_0 = \frac{1}{\kappa^2}\), Eqs. (10) become the Einstein equations without the cosmological constant,

\[ G_{\mu\nu} = \kappa^2 T_{\mu\nu}. \] (11)

In other words, in this specific situation, we start with only the matter action without the gravitational action, but our constraint generates the gravitational dynamics in a natural way. The key point here is that the Lagrange multiplier field \( \lambda(x) \) takes any constant term as an integration constant, which is also a peculiar feature of unimodular gravity.

In order to understand the constraint \( \sqrt{-g}R = \varepsilon_0 \), let us recall that general relativity is invariant under diffeomorphisms whose infinitesimal forms are written as \( \delta g_{\mu\nu} = \nabla_{\mu} \epsilon_{\nu} + \nabla_{\nu} \epsilon_{\mu} \) where \( \epsilon_{\mu} \) are the infinitesimal local parameters. Under diffeomorphisms, the constraint \( \sqrt{-g}R = \varepsilon_0 \) is transformed as

\[ \delta \left( \sqrt{-g}R - \varepsilon_0 \right) = \sqrt{-g}R \nabla_{\mu} \epsilon^{\mu} = \varepsilon_0 \nabla_{\mu} \epsilon^{\mu}. \] (12)

Thus, as long as \( \varepsilon_0 \neq 0 \), the constraint \( \sqrt{-g}R = \varepsilon_0 \) can be interpreted as a gauge condition for one of the diffeomorphisms. To put it differently, as in unimodular gravity, this constraint breaks diffeomorphisms down to the transverse diffeomorphisms, which are defined as diffeomorphisms satisfying \( \nabla_{\mu} \epsilon^{\mu} = 0 \).

If one regards the above gravitational model as a gauge-fixed version of general relativity, together with the ghost term, the term involving the constraint in the action (6) can be combined into a BRST-exact form, by which the interesting dynamical result makes no sense at least physically. Furthermore, as another unsatisfactory problem, when \( R = 0 \), Eq. (9) does not give rise to the result that \( \lambda(x) \) is a constant. It would be desirable if we could find a model that holds even in the case of \( R = 0 \), since black holes satisfy \( R = 0 \).

On the basis of the model constructed so far, however, it is easy to overcome these two problems at the same time, as follows: The key idea is to apply the Henneaux–Teitelboim method [4] to our system, that is, the starting action is defined as

\[ S = \int d^4x \left[ \frac{1}{2\kappa^2} \sqrt{-g} \left( R - 2\Lambda \right) - \lambda(x) \left( \sqrt{-g}R - \varepsilon_0 \sqrt{-g} \nabla_{\mu} \tau^{\mu} \right) \right] + S_m, \] (13)

where \( \tau^{\mu} \) is a vector field. Let us note that this action is manifestly invariant under full diffeomorphisms.

Here we wish to clarify the physical meaning of \( \tau^{\mu} \). The action (13) is invariant under restricted topological transformations

\[ \delta \tau^{\mu} = \epsilon^{\mu}, \quad \nabla_{\mu} \epsilon^{\mu} = 0. \] (14)

Or equivalently, for a fixed background metric, the action (13) is invariant under new gauge transformations

\[ \delta \tau^0 = -\frac{1}{\sqrt{-g}} \partial_0 \epsilon^i, \quad \delta \tau^i = \frac{1}{\sqrt{-g}} \partial_i \epsilon^0, \] (15)

where we have used a formula \( \nabla_{\mu} (\sqrt{-g}A^{\mu}) = \partial_{\mu} (\sqrt{-g}A^{\mu}) \) for an arbitrary vector field \( A^{\mu} \), and \( \epsilon^i \) are three gauge parameters (\( i = 1, 2, 3 \)). Thus, using these gauge symmetries, we can take the
gauge conditions $\tau^i = 0$. Then, the remaining component $\tau^0$ in $\tau^\mu$ is the only dynamical degree of freedom. This extra degree of freedom supplies us with one additional degree of freedom to keep the full group of diffeomorphisms.

Taking variation of the metric tensor leads to the Einstein equations:

$$\frac{1}{\kappa^2} - 2\lambda(x) \right) G_{\mu\nu} + \frac{\Lambda}{\kappa^2} g_{\mu\nu} + 2 \left( \nabla_\mu \nabla_\nu \lambda - g_{\mu\nu} \nabla^2 \lambda \right) - 2\varepsilon_0 \left[ \tau_{(\mu} \nabla_{\nu)} \lambda - \frac{1}{2} g_{\mu\nu} \tau^\rho \nabla^\rho \lambda \right] = T_{\mu\nu},$$

where the round bracket indicates the symmetrization of indices of weight $\frac{1}{2}$. Variation of the Lagrange multiplier field $\lambda(x)$ gives us the equation of motion:

$$\sqrt{-g} R = \varepsilon_0 \sqrt{-g} \nabla^\mu \tau_\mu = \varepsilon_0 \partial_\mu \left( \sqrt{-g} \tau^\mu \right).$$

Finally, when one regards $\tau^\mu$ as a fundamental vector field, variation of $\tau^\mu$ leads to

$$\sqrt{-g} \varepsilon_0 \nabla^\mu \lambda = 0,$$

from which the Lagrange multiplier field takes any constant value, $\lambda(x) = \lambda_0$. Then, substituting $\lambda(x) = \lambda_0$ into Eqs. (16), we again have the standard Einstein equations (10) except for the Newton constant. As before, taking the limit of the infinite Newton constant and setting $-2\lambda_0 = \frac{1}{\kappa^2}$ yields the Einstein equations without the cosmological constant (11).

3. Cosmological solutions

In this section, first of all, we work with the final model with the full diffeomorphisms and consider the cosmological solutions to Eqs. (10) and (17) in the framework of the Friedmann–Robertson–Walker (FRW) universe with spatially flat metric since solutions in the other models can be obtained as special cases from this model.

Now, taking the trace of Eqs. (10), one obtains

$$-\frac{1}{\kappa^2} R + \frac{4\Lambda}{\kappa^2} = T.$$  

Next, eliminating $\Lambda$ in Eqs. (10) via this relation gives us

$$\frac{1}{\kappa^2} \left( R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) = T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T,$$

which are the traceless Einstein equations with the modified Newton constant. Moreover, Eqs. (17) and (19) determine the value of the covariant divergence of the vector $\tau^\mu$,

$$\nabla_\mu \tau^\mu = \frac{\kappa^2}{\varepsilon_0} \left( -T + \frac{4\Lambda}{\kappa^2} \right).$$

Since this general model (13) includes an arbitrary vector field $\tau^\mu$, it is possible to find various types of cosmological solutions by adjusting the vector field in an appropriate manner.

At this stage, in order to see the coordinate freedoms allowed by the equations of motion, the classical solutions, let us switch off the energy–momentum tensor, $T_{\mu\nu} = 0$. Then, Eqs. (20) reduce to the form

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = 0.$$  

The spaces of the solutions satisfying Eqs. (22) are called Einstein spaces [22]. The Bianchi identities in the absence of torsion then imply $\nabla_\mu R = 0$, that is, a constant curvature. Actually, the value of...
the constant curvature is fixed by $R = 4\Lambda \kappa'^2$ from Eq. (19). We are not aware of a similar statement in the case of a non-vanishing energy–momentum tensor.

Next, we therefore wish to turn our attention to a more specific model without the vector field. Here we take account of the first gravitational model where the equations of motion are given by the Einstein equations

$$\frac{1}{\kappa'^2} G_{\mu\nu} + \frac{\Lambda}{\kappa^2} g_{\mu\nu} = T_{\mu\nu},$$

and the constraint equation

$$\sqrt{-g} R = \varepsilon_0.$$ (24)

The cosmological constant term can always be interpreted as the contribution of vacuum energy to the Einstein equations, so let us henceforth include it in the energy–momentum tensor of matter and set $\Lambda = 0$ in Eqs. (23).

Since the constraint equation (24) includes second derivatives of time, it is a dynamical equation which should be contrasted with the unimodular constraint $\sqrt{-g} = v_0$, which is a non-dynamical one. In other words, our constraint (24) might restrict the whole class of classical equations satisfying the (modified) Einstein equations (23) to be its certain subgroup. Indeed, we will see shortly that only a universe filled with non-relativistic matter (“dust”) is allowed as a classical solution.

To do that, recalling that we have set $\Lambda = 0$, let us take the trace of Eqs. (23),

$$R = -\kappa'^2 T.$$ (25)

Taking this equation together with Eq. (24), the trace part of the energy–momentum tensor is described as

$$T = -\frac{\varepsilon_0}{\kappa'^2} \frac{1}{\sqrt{-g}} = -\frac{\varepsilon_0}{\kappa'^2} a(t)^{-\frac{3}{2}},$$ (26)

where we have worked with the FRW metric with spatially flat metric ($k = 0$)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$ (27)

with $a(t)$ being the scale factor. It is sufficient for our purposes to treat matter as a perfect fluid, $T_{\mu\nu} = diag(-\rho, p, p, p)$, and consider the equation of state $w = \frac{p}{\rho} = \text{const}$. Then, the covariant conservation law, $\nabla_{\mu} T_{\mu0} = 0$, is solved to be $\rho(t) = \rho_0 a(t)^{-3(1+w)}$, where $\rho_0$ is an integration constant. Using these facts, the trace part of the energy–momentum tensor takes the form

$$T = -\rho + 3p = \rho_0(-1 + 3w)a(t)^{-3(1+w)}.$$ (28)

By comparing this result with Eq. (26), we find that

$$w = 0, \quad \rho_0 = \frac{\varepsilon_0}{\kappa'^2}. $$ (29)

Finally, using Eq. (29), the Einstein equations (23) are solved and the scale factor is completely determined to be

$$a(t) = \left(\frac{3}{4} \varepsilon_0 \right)^{\frac{1}{3}} (t - t_0)^{\frac{2}{3}},$$ (30)

where $t_0$ is some constant. This solution describes a decelerating universe filled with non-relativistic matter whose equation of state is $w = 0$. 

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4. Conclusion In this article, along similar lines to the argument of unimodular gravity, we have constructed gravitational models which have either transverse diffeomorphisms or full diffeomorphisms. In the limit of the infinite Newton constant, these models reduce to those of induced gravity. Moreover, we have investigated classical solutions that satisfy the equations of motion.

One of the interesting features in unimodular gravity is that the cosmological constant appears as an integration constant unrelated to any parameters in the action. Usually, this feature has been utilized in order to solve the well-known cosmological constant problem. However, in this article, we have made use of this feature to show that the Newton constant appears as an integration constant from the induced gravity action. In modern times, we are tempted to consider that quantum gravity is not only an emergent phenomenon but also independent of the background metric. In such a viewpoint, the present work might shed some light on quantum gravity.

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References
[1] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989).
[2] J. van der Bij, H. van Dam, and Y. J. Ng, Physica A 116, 307 (1982).
[3] W. Buchmuller and N. Dragon, Phys. Lett. B 207, 292 (1988).
[4] M. Henneaux and C. Teitelboim, Phys. Lett. B 222, 195 (1989).
[5] W. Buchmuller and N. Dragon, Phys. Lett. B 223, 313 (1989).
[6] W. G. Unruh, Phys. Rev. D 40, 1048 (1989).
[7] Y. J. Ng and H. van Dam, J. Math. Phys. 32, 1337 (1991).
[8] L. Smolin, Phys. Rev. D 80, 084003 (2009).
[9] G. F. R. Ellis, H. van Elst, J. Murugan, and J.-P. Uzan, Class. Quant. Grav. 28, 225007 (2011).
[10] A. Padilla and I. D. Saltas, Eur. Phys. J. C 75, 561 (2015).
[11] I. D. Saltas, Phys. Rev. D 90, 124052 (2014).
[12] A. Einstein, H. A. Lorentz, H. Weyl, and H. Minkowski, The Principle of Relativity (Dover Publications, New York, 1952).
[13] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (W. H. Freeman and Co., New York, 1973).
[14] A. D. Sakharov, Sov. Phys. Dokl. 12, 1040 (1968).
[15] K. Akama, Y. Chikashige, T. Matsuki, and H. Terazawa, Prog. Theor. Phys. 60, 868 (1978).
[16] S. L. Adler, Phys. Lett. B 95, 241 (1980).
[17] A. Zee, Phys. Rev. D 23, 858 (1981).
[18] D. Amati and G. Veneziano, Nucl. Phys. B 204, 451 (1982).
[19] S. L. Adler, Rev. Mod. Phys. 54, 729 (1982).
[20] K. Akama and I. Oda, Phys. Lett. B 259, 431 (1991).
[21] K. Akama and I. Oda, Nucl. Phys. B 397, 727 (1993).
[22] A. L. Besse, Einstein Manifolds (Springer-Verlag, Berlin, 1987).