DIMENSIONAL DUAL HYPEROVALS IN CLASSICAL POLAR SPACES

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Abstract. In this paper we show that \( n \)-dimensional dual hyperovals cannot exist in all but one classical polar space of rank \( n \) if \( n \) is even. This resolves a question posed by Yoshiara.

1. Definitions and preliminaries

An \( n \)-dimensional dual arc \( D \) in a vector space \( V(N, q) \) over a finite field \( F_q \) is a set of \( n \)-dimensional subspaces such that

1. each two intersect in exactly a one-dimensional space;
2. no three intersect non-trivially.

It is clear that \( |D| \leq \frac{q^{n-1}}{q-1} + 1 \). For let \( S \) be any element of \( D \). Then the other elements of \( D \) intersect \( S \) in distinct one-dimensional subspaces, of which there are \( \frac{q^{n-1}}{q-1} \). If \( D \) meets this bound, it is called an \( n \)-dimensional dual hyperoval. We will sometimes use the shorthand \( n \)-DA and \( n \)-DHO.

For background and a recent survey of known results and applications, we refer to [15]. Note that the definition therein are in terms of projective spaces, but here we use vector space terminology and notation, following [5]. In this paper we will mostly consider the case \( N = 2n \). In [5], this is required in the definition, but we will not impose this restriction here.

It is known [15] that \( n \)-dimensional dual hyperovals exist in \( V(2n, q) \) for all \( n \) and all \( q \) even, see for example [15]. It is an open problem whether any can exist when \( q \) is odd.

In this paper we will consider \( n \)-dimensional dual arcs in polar spaces, that is, where \( D \) consists of maximum totally isotropic subspaces with respect to some nondegenerate form on \( V(N, q) \). Necessarily then we have that \( N \in \{2n, 2n + 1, 2n + 2\} \).

It is known [15] that there exist \( n \)-dimensional dual hyperovals in the hyperbolic quadric \( Q^+(2n-1, q) \) for all \( n \) odd and \( q = 2 \) (see Section 2 for notation). Furthermore, there exists a 3-dimensional dual hyperoval in the hermitian variety \( H(5, 4) \), the Mathieu dual hyperoval.

In [15] Problem 4.7, the following (paraphrased) question is asked.

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Does the existence of an $n$-dimensional dual hyperoval in a polar space imply that
$n$ is odd?

Taniguchi [10] proved that $n$-dimensional “alternating doubly dual hyperovals” exist
in $V(2n, 2)$ if and only if $n$ is odd. Dempwolff [4], showed that $n$-dimensional
“symmetric doubly dual hyperovals” exist only if $n$ is odd. We will see in Section 4
that the existence of such implies the existence of an $n$-dimensional dual hyperovals
in the symplectic space $W(2n - 1, q)$.

We respond now to these questions with the following theorem.

**Theorem 1.** Suppose $\mathcal{D}$ is an $n$-dimensional dual hyperoval in a polar space $\mathcal{P}$ of
rank $n$. Then either $n$ is odd or $\mathcal{P}$ is an elliptic quadric.

The result is a simple application of a theorem of Vanhove.

2. Polar spaces

A classical polar space $\mathcal{P}$ is the geometry of totally singular subspaces with respect
to some non-degenerate quadratic form on $V(N, q)$, or totally isotropic with respect
to some non-degenerate symplectic or sesquilinear form on $V(N, q)$. The rank of
$\mathcal{P}$ is the maximum (vector space) dimension of a subspace of $\mathcal{P}$. If every $(n - 1)$-
dimensional space of a polar space of rank $n$ is contained in precisely $q^{e} + 1$ totally
isotropic $n$-dimensional spaces, then $\mathcal{P}$ is said to have parameters $(q, q^{e})$. See for
example [2] for background. We tabulate the relevant polar spaces of rank $n$ here.

| Name            | form       | Notation  | Ambient vector space | Parameters | $e$ |
|-----------------|------------|-----------|----------------------|------------|-----|
| Hyperbolic quadric | quadratic | $Q^{+}(2n - 1, q)$ | $V(2n, q)$ | $(q, 1)$ | 0   |
| Parabolic quadric | quadratic | $Q(2n, q)$ | $V(2n + 1, q)$ | $(q, q)$ | 1   |
| Elliptic quadric | quadratic | $Q^{-}(2n + 1, q)$ | $V(2n + 2, q)$ | $(q, q^{2})$ | 2   |
| Symplectic space | symplectic | $W(2n - 1, q)$ | $V(2n, q)$ | $(q, q)$ | 1   |
| Hermitian variety | sesquilinear | $H(2n - 1, q^{e})$ | $V(2n, q^{e})$ | $(q^{e}, q)$ | 1/2 |
| Hermitian variety | sesquilinear | $H(2n, q^{e})$ | $V(2n + 1, q^{e})$ | $(q^{e}, q^{e})$ | 3/2 |

Note that $n$-dimensional dual hyperovals in polar spaces defined by a quadratic
form are often referred to as being of orthogonal type.

If $q$ is even, $W(2n - 1, q)$ is isomorphic to $Q(2n, q)$, and contains $Q^{+}(2n - 1, q)$.

**Example 1.** Yoshiara defined in [14] the following $n$-dimensional dual hyperovals in
$V(2n, 2)$, and showed in [16] that they lie in $Q^{+}(2n - 1, 2)$ (and hence $W(2n - 1, 2)$)
if and only if $n$ is odd. Let $h$ be an integer coprime to $n$, and for each $t \in \mathbb{F}_{2^{n}}$ define

$$S_{t} = \{(x, x^{2^{2h}} + tx^{2^{h}}) : x \in \mathbb{F}_{2^{n}}\}.$$ 

Then $\mathcal{D} := \{S_{t} : t \in \mathbb{F}_{2^{n}}\}$ is an $n$-dimensional dual hyperoval in $Q^{+}(2n - 1, 2)$ (and
$W(2n - 1, 2)$), where the quadratic form on $V(2n, 2)$ is

$$(a, b) \mapsto \text{Tr}(ab^{2^{h}}),$$ 

and the associated symmetric (alternating) bilinear form on $V(2n, 2)$ is

$$(a, b, (c, d)) \mapsto \text{Tr}(ad^{2^{h}} - bc^{2^{h}}).$$
Dempwolff and Kantor [5] gave a geometric construction leading to many inequivalent examples in $Q^+(2n−1, 2)$. Dempwolff [3] gave further examples in $W(2n−1, 2)$ which cannot lie in $Q^+(2n−1, 2)$.

**Example 2.** There exists a 3-dimensional dual hyperoval in $V(6, 4)$ which lies in the polar space $H(5, 4)$ known as the Mathieu dual hyperoval, see e.g. [6].

To the author’s knowledge, no examples in other polar spaces are known. Del Fra [6] showed that the only 3-dimensional dual hyperovals in a polar space are the above examples.

Yoshiara [16] showed that $n$-dimensional dual hyperovals can exist in $Q^+(2n−1, q)$ only if $n$ is odd.

3. Dual polar graphs and Main result

Given a polar space $P$ of rank $n$, we define the dual polar graph $\Gamma_P$, whose vertices are the $n$-spaces of $P$, and where two vertices are adjacent if their intersection has dimension $n−1$. Many properties of this graph are known, see for example [1], [12].

For a set $D$ of $n$-spaces of $P$, the inner distribution is an $(n+1)$-tuple of integers $a = (a_0, a_1, \ldots, a_n)$, where

$$a_i = \frac{\{(S, T) : S, T \in D \mid \dim(S \cap T) = n−i\}}{|D|}.$$

Equivalently, if we view $D$ as a subset of $\Gamma_P$, and let $d(S, T)$ denote the distance function on $\Gamma$, then

$$a_i = \frac{\{(S, T) : S, T \in D \mid d(S, T) = i\}}{|D|}.$$

In [13, Lemma 3.2], the following was proved.

**Theorem 2 (Vanhove).** Let $P$ be a classical polar space of rank $n$ with parameters $(q, q^e)$, and let $D$ be a set of $n$-spaces in $P$ with inner distribution $(a_0, a_1, \ldots, a_n)$. Then

$$\sum_{i=0}^{n} \left(-\frac{1}{q^e}\right)^i a_i \geq 0.$$

Now suppose $D$ is a dimensional dual arc in $P$. Then it is clear that

$$a_0 = 1$$
$$a_{n−1} = |D| − 1$$
$$a_i = 0 \text{ otherwise}.$$

Hence we get that

$$1 + \left(-\frac{1}{q^e}\right)^{n−1} (|D| − 1) \geq 0,$$

and so if $n$ is even,

$$|D| \leq q^{(n−1)e} + 1.$$

Hence we get an upper bound for an $n$-dimensional dual arc in each classical polar space.
Theorem 3. Suppose \( \mathcal{D} \) is an \( n \)-dimensional dual arc in \( \mathcal{P} \), and suppose \( n \) is even. Then we have the following upper bounds on \( |\mathcal{D}| \).

| \( \mathcal{P} \) | Ambient vector space | Parameters | \( e \) | \( |\mathcal{D}| \leq \) | Size of DHO |
|---|---|---|---|---|---|
| \( Q^+(2n-1, q) \) | \( V(2n, q) \) | \( q, 1 \) | 0 | \(\frac{q^{n-1}}{2} + 1 \) | \( \frac{q^{n-1} + 1}{2} \) |
| \( Q(2n, q) \) | \( V(2n + 1, q) \) | \( q, q \) | 1 | \( q^{n-1} + 1 \) | \( \frac{q^{n-1}}{2} + 1 \) |
| \( Q^-(2n + 1, q) \) | \( V(2n + 2, q) \) | \( q, q^2 \) | 2 | \( q^{2n-2} + 1 \) | \( \frac{q^{2n-1}}{2} + 1 \) |
| \( W(2n - 1, q) \) | \( V(2n, q) \) | \( q, q \) | 1 | \( q^{n-1} + 1 \) | \( \frac{q^{n-1}}{2} + 1 \) |
| \( H(2n - 1, q^2) \) | \( V(2n, q^2) \) | \( q^2, q \) | \( 1/2 \) | \( q^{n-1} + 1 \) | \( \frac{q^{n-1}}{2} + 1 \) |
| \( H(2n, q^2) \) | \( V(2n + 1, q^2) \) | \( q^2, q^3 \) | \( 3/2 \) | \( q^{3(n-1)/2} + 1 \) | \( \frac{q^{3(n-1)/2}}{2} + 1 \) |

Proof of Theorem 1. This now now follows immediately by comparing the above upper bounds on \( n \)-dimensional dual arcs (fourth column) with the required size of an \( n \)-dimensional dual hyperoval (fifth column).

Remark 1. Note that an \( n \)-dimensional dual hyperoval is a special case of a constant-distance, constant-dimension subspace code, or equivalently, a clique in the graph \( \Gamma_{n-1} \), where \( \Gamma_1 \) is the graph whose vertices are the vertices of \( \Gamma \), and whose edges are between vertices at distance \( i \) in \( \Gamma \). Note however that not every clique of the correct size in \( \Gamma_{n-1} \) gives rise to an \( n \)-dimensional dual hyperoval. As the proof of Theorem 3 does not use the fact that no three spaces intersect nontrivially, the same bounds hold for the relevant constant-distance subspace codes in each polar spaces.

This is the same method used by Vanhove in [11] to prove that the maximum size of a partial spread in \( H(2n - 1, q) \) is \( q^n + 1 \) if \( n \) is odd.

Remark 2. This table does not imply any results for dimensional dual hyperovals in elliptic quadrics \( Q^-(2n + 1, q) \). This problem seems to require a different approach. Note that such objects do not satisfy the definition of a dimensional dual hyperoval in [5].

4. Alternating and Symmetric Doubly Dual Hyperovals

An \( n \)-dimensional dual hyperoval \( \mathcal{D} \) in \( V(2n, q) \) is said to be “doubly dual” if \( \mathcal{D}^+ := \{ S^+ : S \in \mathcal{D} \} \) is also an \( n \)-dimensional dual hyperoval, where \( \perp \) is some nondegenerate polarity. Note that if \( \mathcal{D} \) lies in some polar space \( \mathcal{P} \), it is doubly dual: we take \( \perp \) to be the polarity defined by the quadratic or sesquilinear form associated to \( \mathcal{P} \), whence \( S^+ = S \) for all maximum subspaces \( S \) in \( \mathcal{P} \). However, the converse is not necessarily true.

In [11] the concept of a (bilinear) symmetric doubly dual hyperoval was introduced, and it was proved that such objects can not exist in \( V(2n, q) \) for \( n \) even. We will now show that the existence of this implies the existence of an \( n \)-dimensional dual hyperoval in symplectic polar space.

Suppose there is some injective linear map \( \beta : V(n, q) \to \text{End}(V(n, q)) \). Let us represent elements of \( V(2n, q) \) with elements of \( V(n, q) \times V(n, q) \). For each \( y \in V(n, q) \), define an \( n \)-dimensional subspace \( S_y = \{ (x, \beta(y)(x)) : x \in V(n, q) \} \) of \( V(2n, q) \), and define \( \mathcal{D}_\beta = \{ S_y : y \in V(n, q) \} \). If \( \mathcal{D}_\beta \) is an \( n \)-dimensional dual
hyperoval, then it is called a bilinear dual hyperoval. Note that this can occur only if \( q = 2 \).

Define \( \beta^o : V(n,q) \to \text{End}(V(n,q)) \) by \( \beta^o(x)(y) = \beta(y)(x) \). If \( \beta = \beta^o \), that is if \( \beta(y)(x) = \beta(x)(y) \) for all \( x, y \in V(n,q) \), then \( \mathcal{D}_\beta \) is called a symmetric dual hyperoval. If furthermore \( \beta(x)(x) = 0 \) for all \( x \), then \( \mathcal{D}_{\beta} \) is called an alternating dual hyperoval.

Let \( \langle \cdot, \cdot \rangle : V(n,q) \times V(n,q) \to \mathbb{F}_q \) be a nondegenerate symmetric bilinear form on \( V(n,q) \). Let \( t \) denote the adjoint operator with respect to this form, i.e. \( \langle x, f(y) \rangle = \langle f^t(x), y \rangle \) for all \( x, y \in V(n,q) \), and define \( \beta^t : V(n,q) \to \text{End}(V(n,q)) \) by \( \beta^t(x) = \beta(x)^t \).

Taniguchi \cite{10} showed that alternating doubly dual hyperovals exist in \( V(2n,2) \) if and only if \( n \) is odd. Dempwolff \cite{4} improved this by showing that symmetric doubly dual hyperovals exist in \( V(2n,2) \) only if \( n \) is odd. We now show that this result follows also from Theorem 1 of this paper.

The following lemma follows from \cite{7}, and from \cite{3, Lemma 3.8}.

**Lemma 1.** If there exists a symmetric bilinear \( n \)-dimensional doubly dual hyperoval \( \mathcal{D} \) in \( V(2n,2) \), then there exists an \( n \)-dimensional dual hyperoval in \( W(2n-1,2) \).

Combining this with Theorem 1 immediately gives us the following corollary.

**Corollary 1.** There exists a symmetric bilinear \( n \)-dimensional doubly dual hyperoval \( \mathcal{D} \) in \( V(2n,2) \) only if \( n \) is odd.

Note that Theorem 1 applied to \( W(2n-1,q) \), does not require either bilinearity or that \( q = 2 \), and so this result is more general than the results of Taniguchi and Dempwolff.

Dempwolff further conjectured in \cite{3} that \( n \)-dimensional doubly dual hyperovals over \( \mathbb{F}_2 \) exist only if \( n \) is odd. This remains an open problem.

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