Wave Superposition Method Applied To Calculate the Radiation Sound Power of a Stiffened Plate

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Abstract. An indirect boundary element method using discrete sound field of virtual sound source is used to calculate the acoustic and vibration coupling response of the stiffened plate in water. In this paper, the relationship between the structural vibration mode coefficient and the virtual sound source intensity is established by using the superposition principle through Fourier transform using structural dynamics equations and boundary compatibility condition satisfying the function of virtual sound source intensity. Once the virtual sound source intensity is obtained, the radiation power can be determined. Taking the simply supported rectangular thin plate as an example, the sound radiation power is calculated on the condition that surface velocity is unknown, and the results are compared with the results of the analytical method, which shows that the method has high accuracy.

1. Introduction
Determining the radiation power of the structure has always been one of the main concerns in acoustic research. Structures with simple shape, such as spherical shell, simple rectangular plate and so on, can be used to solve the acoustic radiation characteristics with analytical methods. For more complex structures, the parameters such as vibration velocity and sound pressure can be determined by finite element (FEM), boundary element (BEM), finite element / boundary element (FEM / BEM) 1-6. However, there are some shortcomings in the above methods, such as the existence of surface singularity of the boundary element, the non-uniqueness of the solution at the characteristic frequency. In order to overcome the shortcomings of the above methods, Koopmann proposed the wave superposition method7-8, by arranging the virtual sound source inside the structures to simulate the external radiation field equivalently. The use of the normal surface velocity of the radiator can obtain the strength of the virtual sound source, and then determine sound power and other characteristic parameters. As the plane where virtual sources are located and the structural surface do not coincide, thus it can avoid the singularity integral problem, and the calculation is simple to implement. Based on the experimental results, Xiangyang et al. 9-10 calculated the surface acoustic pressure of the piston source and the pulsating sphere source on the rigid spherical surface by using the method. The effect of the number of the element, the number of nodes and the shape on the efficiency and accuracy of the algorithm is discussed. Xiayyu and Huang11-12 used this idea to calculate the acoustic radiation problem in the medium, and further constructed the fast convergence algorithm using IFFT.
This paper studies the radiation characteristics of rectangular stiffened plate in water. By using the principle of wave superposition and virtual sound sources to make acoustic field discrete, the strengths of sound sources satisfying the structural dynamic equation and the boundary compatibility condition are obtained by Fourier transform under the condition that the surficial vibration velocity is unknown, and the acoustic radiation power can be further determined. Through the comparison with the analytical solution, it verifies that the method has good accuracy. The sensitivity of the method to the number of virtual sound sources is also discussed in this paper.

2. The basic principle of sound field dispersion

In this paper, the calculation process of the virtual sound source method is illustrated by taking the simply supported rectangular thin plate on the infinite large baffle as an example. The length of the plate is \( a \), \( b \) in the direction of the axis \( x \), \( y \), and the thickness of the plate is \( h \). The ribs are arranged in the direction \( x \) in the condition that the spaces are \( \Delta x_1, \Delta x_2, \ldots, \Delta x_n \). One half space up the plate is full of heavy medium, and the other space is vacuum state. Choose thin plate theory to calculate plate vibration response and Euler-Bernoulli's beam theory to analysis ribs vibration. Simulate the role of the ribs on the substrate as line excitation, and give full consideration to the lateral force and bending moment force between the ribs and the substrate. When calculating the sound radiation power, it is assumed that the ribs do not produce acoustic radiation. The structural model is shown in Figure 1.

![Figure 1. The coordinates of a flat plate](image)

By the wave superposition principle, in the half space, virtual sound sources are collocated, with the sound source intensity \( \alpha_v \) (subscript indicates the \( v \)th virtual sound source) to be determined. The sound pressure at the field point \( r = (x, y, z) \) can be represented by the sum of the sound pressure generated by all the virtual sound sources [3]

\[
p(r) = \sum_{v=1}^{n} \alpha_v G(r, r_v)
\]

Where \( r_v = (x_v, y_v, z_v) \) is the location of the \( v \)th virtual sound source shown in Fig. 1. \( G(r, r_v) \) is Green function of free space

\[
G(r, r_v) = e^{ik|r-r_v|/c}
\]

Where \( |r-r_v| = \sqrt{(x-x_v)^2 + (y-y_v)^2 + (z-z_v)^2} \) indicates the distance between the field and virtual source. \( k = \omega/c \) is the wave number, \( \omega \) is angular frequency, \( c \) is wave speed.
The sound pressure acting on the board surface can be obtained by (1)

\[ p(r) \mid_{r=0} = \sum_{\nu=1}^{N} \alpha_{\nu} \cdot G(r, r') \mid_{r=0} \]  

(3)

3. Calculation of virtual sound source strength

In the range of linear elasticity, the transverse vibration equation of the plate under the external force \( F_{\text{out}} \) perpendicular to the plate surface is

\[ BV^4w + \bar{m} \frac{\partial^2 w}{\partial t^2} = F_{\text{out}} - \sum_{s=1}^{n} q_s(x, y) \delta(x-x_s) - \sum_{s=1}^{n} \kappa_s(x, y) \delta(x-x_s) - p(r) \mid_{r=0} \]  

(4)

Where \( w \) is plate deflection. \( B = E\bar{h}^3 / (12(1-v^2)) \) is the bending stiffness, \( E \) is Young’s modulus, \( h \) is the plate thickness and \( v \) is Poisson’s ratio. \( \bar{m} = \rho \bar{h} \) indicates the mass in unit area. And \( q_s \) and \( \kappa_s \) are the transverse shear force and moment of the interaction between the \( s \)th rib and the plate of the unit length respectively. \( x_s \) indicates the abscissa of the ribs. \( p(r) \mid_{r=0} \) indicates the effect of the medium on the plate structure.

Noting that the unknown quantity \( q_s \), \( \kappa_s \), reflect the interaction between the substrate and the ribs. The bending vibration and torsional vibration of the ribs in the forced vibration occur, shown as

\[ E_s I_s \frac{\partial^2 U_s(x, y)}{\partial y^2} - \rho_s A_s \partial^2 U_s(x, y) = q_s(x, y) \]  

(5)

\[ G_s J_s \frac{\partial^2 \theta_s(x, y)}{\partial y^2} - \rho_s I_{ps} \partial^2 \theta_s(x, y) = \kappa_s(x, y) \]  

(6)

Where \( U_s(x, y) \) is the lateral displacement of the ribs, \( \theta_s(x, y) \) is the corner of the ribs, \( E_s I_s \) and \( G_s J_s \) represent the flexural rigidity and torsional stiffness of the ribs respectively. \( \rho_s \) and \( A_s \) indicate the density and cross-sectional area of the ribs respectively. \( I_{ps} \) is the polar moment of inertia.

Since the ribs and the substrate vibrate as a whole, the ribs and the substrate at the junction meet the displacement and mechanical continuity conditions

\[ U_s(x, y) = w(x, y), \quad \theta_s(x, y) = \frac{\partial w(x, y)}{\partial x} \]  

(7)

For a simple rectangular plate, it can be seen from the literature [9] that the deflection of the plate can be expressed as a linear superposition of the vibration mode of the plate

\[ w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \psi_{mn}(x, y) \]  

(8)

Where \( A_{mn} \) is the amplitude of the vibration mode. \( \psi_{mn}(x, y) = \sin(m \pi x / a) \sin(n \pi y / b) \). Substituting the above formula to (7), \( U_s(x, y), \theta_s(x, y) \) can be expressed as

\[ U_s(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \psi_{mn}(x, y), \quad \theta_s(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{\partial \psi_{mn}(x, y)}{\partial x} \]  

(9)
Substitute (9) into the formula (5) and (6) and merge the same type

\[\sum_{m=-n}^{n} \sum_{n=1}^{N} \left( E_{i} I_{i} + \rho_{s} A_{s} \omega^2 \right) A_{mn}(x, y) = q_{s}(x, y) \]

(10)

\[\sum_{m=-n}^{n} \sum_{n=1}^{N} \left( G_{i} J_{i} + \omega^2 \rho_{s} I_{s} \right) A_{mn} \left( \frac{\partial \psi_{mn}(x, y)}{\partial x} \right) = \kappa_{s}(x, y) \]

(11)

The deflection in the vibration equation is expressed in the form of superposition of vibration modes, one obtains

\[\sum_{m=-n}^{n} \sum_{n=1}^{N} A_{mn} \left[ B \left( L_{mn} + 2 L_{mn}^2 + L_{mn}^3 \right) - \omega^2 m \right] \psi_{mn}(x, y) = F_{out} - \sum_{m=-n}^{n} \sum_{n=1}^{N} \left( E_{i} I_{i} + \rho_{s} A_{s} \omega^2 \right) A_{mn} \psi_{mn}(x, y) \]

\[\delta(x - x_{s}) - \sum_{m=-n}^{n} \sum_{n=1}^{N} \left( G_{i} J_{i} + \omega^2 \rho_{s} I_{s} \right) A_{mn} \left( \frac{\partial \psi_{mn}(x, y)}{\partial x} \right) \delta(x - x_{s}) - p(r) \mid_{r=0} \]

(12)

Where \( L_{mn} = m\pi \backslash a \), \( L_{n} = n\pi \backslash b \).

Express the sound pressure in the form of virtual source function and take Fourier transform to the above formulation

\[abA_{mn} \left( L_{mn}^4 + \omega^2 m \right) = 4 \int_{0}^{L} \int_{0}^{L} F_{out} \psi_{mn}(x, y) \, dx \, dy - 2b \sum_{m=-n}^{n} \sum_{n=1}^{N} \left( A_{mn} \left( E_{i} I_{i} + \rho_{s} A_{s} \omega^2 \right) \sin L_{mn} x \sin L_{mn} y \right) \]

\[-2b \sum_{m=-n}^{n} \sum_{n=1}^{N} \left( A_{mn} \left( G_{i} J_{i} + \omega^2 \rho_{s} I_{s} \right) L_{mn} \cos L_{mn} x \sin L_{mn} y \right) \]

\[4 \int_{0}^{L} \int_{0}^{L} \sum_{m=-n}^{n} \alpha_{r} G(r, r) \psi_{mn}(x, y) \, dx \, dy \]

(13)

Where \( L_{mn}^2 = L_{m}^2 + L_{n}^2 \), \( L_{m} = r\pi \backslash a \).

The above equation reflects the relationship between the intensity \( \alpha_{r} \) of the virtual source and the vibration mode coefficient \( A_{mn} \) of the plate from the vibration equations. The above formula is expressed in matrix form

\[MA = C - \sum_{i=1}^{N} D \cdot N^r \cdot A - \sum_{i=1}^{N} T \cdot P^r \cdot A - \alpha \]

(14)

Where \( M \) is matrix of \( mn \times mn \) order, \( M_{p_{i}, q_{j}} = ab(bL_{p_{i}, q_{j}} + \omega^2 m) \), \( C_{i} = 4 \int_{0}^{L} \int_{0}^{L} F_{out} \psi_{i}(x, y) \, dx \, dy \), \( D_{r_{i}, r_{j}} = 2b \sin L_{r_{m}} x \left( E_{i} I_{i} + \rho_{s} A_{s} \omega^2 \right) \), \( T_{r_{i}, r_{j}} = 2b \sin L_{r_{m}} x \left( G_{i} J_{i} + \omega^2 \rho_{s} I_{s} \right) \), \( Q_{mn, r} = 4 \int_{0}^{L} \int_{0}^{L} G(r, r) \psi_{mn}(x, y) \, dx \, dy \), \( N^r \) is matrix of \( mn \times mn \) order expressed in the below form

\[N^r = \begin{bmatrix} R^r \\ R^r \\ \vdots \end{bmatrix}, \]

\[R_{r_{i}, m} = \sin L_{r_{m}} x \]

\( P^r \) is matrix of \( mn \times mn \) order expressed in the below form
Since the matrix $N'$ is not a diagonal matrix, calculating vibration modes needs to consider the coupling between the vibration modes, which brings difficulty to calculate the modes. In the interface between the plate and the medium, the compatibility condition is satisfied

$$\frac{\partial p(r)}{\partial z} \bigg|_{z=0} = i\omega\rho \dot{w}(x, y)$$

Where $\dot{w}(x, y)$ is the velocity of the location $(x, y)$ on the plate. The sound pressure can be obtained from the equation (8), and the deflection of the plate can be obtained from the equation (1). Substitute (1), (8) into the above equation

$$\sum_{i=1}^{N} \alpha_i \frac{\partial G(r_i, r)}{\partial z} \bigg|_{z=0} = i\omega\rho \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \psi_{mn}(x, y)$$

Take Fourier transform to the above equation

$$4 \sum_{i=1}^{N} \alpha_i \int_0^\infty \int_0^\infty \frac{\partial G(r_i, r)}{\partial z} \bigg|_{z=0} \psi_{mn}(x, y) dxdy \bigg|_{z=0} = \omega^2 \rho \omega A_{mn}$$

The above equation reflects the relationship between the intensity $\alpha_i$ of virtual sound sources and vibration mode coefficient $A_{mn}$ of the plate from the kinematic boundary condition. $A_{mn}$ can be expressed as from (10) equation

$$A_{mn} = \frac{4}{\omega^2 \rho ab} \sum_{i=1}^{N} \alpha_i \int_0^\infty \frac{\partial G(r_i, r)}{\partial z} \bigg|_{z=0} \psi_{mn}(x, y) dxdy$$

The formula (18) is expressed in the form of a matrix

$$A = \Omega \alpha$$

The matrix $A$ in (14) is represented by the matrix of (19) containing the intensity of the virtual sound sources

$$M + \sum_{x=1}^{N} D\Omega' \alpha + \sum_{x=1}^{N} T\Omega' \alpha + Q\alpha = C$$

And further through the merger of the same type of simplification, the below formulation can be obtained

$$\left( \left( M + \sum_{x=1}^{N} D\Omega' + \sum_{x=1}^{N} T\Omega' \right) \Omega + Q \right) \alpha = C$$
\[ a = \left( \begin{array}{c} M + \sum_{i=1}^{N} D^i N^i + \sum_{i=1}^{N} T^i P^i \end{array} \right) \Omega + Q \right)^{-1} C \]  

(22)

After calculating the matrix \( a \), the average acoustic radiation power of the plate and the mean square value of the normal velocity of the plate surface can be calculated in comparison with the results obtained by the analytical method.

4. Calculation average acoustic radiation power

The sound pressure on the plate surface can be expressed as the form of sum function of virtual sound sources. Substitute (1) into the (15) and express the surface vibration velocity as a virtual source and function

\[ \dot{w}(x,y) = \frac{1}{io\rho_f} \sum_{i=1}^{N} a_i \frac{\partial G(r,r_i)}{\partial z} \bigg|_{r=0} \]  

(23)

Substitute the above equation into (19)

\[ \Pi_{av} = \frac{1}{2} a^H \Lambda a \]  

(24)

Where \( \Lambda_{i,j} = \frac{1}{io\rho_f} \int_{S} \int_{S} \left( \frac{\partial G(r,r_i)}{\partial z} \right)^* G(r,r_j) \psi_{ij} dS \). \( H \) represents a conjugate transpose of a matrix.

5. Case analysis

Select a ribbed simply supported rectangular steel plate as an example, the length of the substrate is \( a = 1m \), the width is \( b = 0.75m \), the thickness is \( h = 0.003m \). The density is \( \rho_f = 7800kg/m^3 \), Poisson's ratio is \( \nu = 0.3 \), the elastic modulus is \( E = 2.16 \times 10^7 N/m^2 \), the density of the medium is \( \rho_f = 1000kg/m^3 \), the speed of sound is \( c = 1500m/s \), and the reference sound power is \( W_r = 10^{-12} W \) regardless of the damping coefficient of the plate. The plate is subjected to a concentrated load acting at the geometric center, and the magnitude of the load is 1N. The analytical results can be obtained by using the method in [13-14].

\[ MA + i\omega ZA + \sum_{i=1}^{N} D^i N^i A + \sum_{i=1}^{N} T^i P^i A = C \]

Where \( Z_{p_{r,s}} = \frac{2\omega \rho_f}{\pi} \int_0^{\infty} \int_0^{\infty} \psi_{p_{r,s}} (x,y) \psi_{r,s} (x,y) e^{-\omega \sqrt{x^2+y^2}} dx dy \)

Discuss the influence of the number of virtual sound sources on the sound radiation power of the plate through using three kinds of virtual sources evenly arranged on a virtual plane. The first arrangement is \( 15 \times 12 \) (the previous number indicates the number of rows in the direction of \( x \), the latter number indicates the number of rows in the direction of \( y \)), the second arrangement is \( 10 \times 8 \), and the third arrangement is \( 5 \times 4 \). The sound power is obtained with three kinds of the layouts of virtual sound sources described above and compared with the analytical results. Figure 2, 3, 4 are the figs respectively for the first, second and third kinds of virtual sound source cases. The maximum relative error of the two computational methods is listed in Tab.1.
Figure 2. The result of $15 \times 12$ virtual sources compared with analytical result

Figure 3. The result of $10 \times 8$ virtual sources compared with analytical result

Figure 4. The result of $5 \times 4$ virtual sources compared with analytical result
Tab. 1 the Maximum relative error of different numbers of virtual sound sources

| number    | 5×4  | 10×8 | 15×12 |
|-----------|------|------|--------|
| Maximum relative error | 12.3% | 7.5% | 6.6% |

It can be shown that when the frequency is low, the results obtained using three different numbers of virtual sound sources are almost the same. As the frequency increases, the error with fewer number of virtual sound sources gradually increases and is higher than the results with more number of virtual sources. However, with the further increase in the number of virtual sources, the relative error slows down. It can be seen that the method has a small error in the distance away from the resonant frequency, and there is a large error near the resonant frequency.

6. Conclusion

The results of the above research show that the virtual sound source method can estimate the radiated sound power of the radiator surface well by solving the intensity of the virtual sources in the case that the surface velocity is unknown.

As can be seen from the example of a rectangular plate, the virtual sound source method has the following advantages over other algorithms. Compared with the boundary element method, it does not need to deal with the singularity problem, and the calculation is simplified. It needs the number of virtual sound sources fewer than the number of elements required for the finite element method and can estimate the radiation power well. The whole process to calculating sound power does not need to solve the surface acoustic pressure and radiation impedance of the structure showing that the method is simple and the computational efficiency is improved.

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