Linear-Programming Approximations of AC Power Flows

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Outline

• Motivation
• The LPAC Models
• Experimental Results
  • LDC versus LPAC versus AC solutions
  • LPAC Variants
• Capacitor Placement Problem
• Power Restoration
Motivation

Collaboration with LANL
Motivation
Power Restoration

• One challenge (PSCC’11)
  • Schedule a fleet of repair crews to repair the grid and minimize the overall size of the blackout after a disaster

• Two fundamental aspects
  • Scheduling the repairs
  • Scheduling the power restoration
  • Both are challenging in their own right

• Assumptions for Last-Mile Restoration
  • Steady state behavior of the power grid
  • Ability to shed load and generation continuously
  • Transient/configuration aspects in a second step
Power Restoration

Restoration Timeline

Minimize

Power Flow

Time

Increase in served demand
Component repair
Power Restoration

- **Optimal Activation Problem**
  - Generalized optimal line switching [Fisher et al, 98]

- **Approximate the power flows equations**
  - Linear DC Model

- **Discrete optimization over the LDC model**
  - MIP solver

- **Solutions to large benchmarks [CPAIOR’12]**
  - 4000 components, a third of which were damaged
  - Using hybrid optimization (MIP + CP + LNS)
Optimal Activation

- find which items to activate
- find how much power to produce and consume
- find the phase angles at buses
- to maximize the served load

Generalized optimal line switching [Fisher et al, 98]

Figure 1: A MIP Model for the Unserved Load.
Power Restoration
A fundamental open question

- Is this “optimal” restoration plan “feasible” operationally?

- These are not normal operating conditions
  - “Maddeningly difficult” to find an AC solution in cold start contexts [Overbye et al, 2004]

- The network is stressed
  - Does the LDC model “overfit”?

- How accurate is the LDC model?
  - Can the LDC solution be turned into an AC solution?
N-3 Contingencies (IEEE-30)

Line Apparent Power Correlation (MVA)

- Small Line Phase Angle
- Large Line Phase Angle

LDC Power Flow

AC Power Flow

IEEE PES’12
N-3 Contingencies
N-3 Contingencies
Power Restoration

AC Restoration Timeline

AC Power Flow (MW)

Restoration Action

HELP WANTED

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From imagination to impact
• Find an approximation of AC power flows that
  • is more accurate than the LDC model
  • is useful outside normal operating conditions
  • reasons about voltage magnitudes and reactive power
  • can be embedded in discrete optimization solvers
    • mixed integer programming solvers

• Applications
  • Power restoration, vulnerability analysis, capacitor placement, expansion planning, …
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\[ p_n = \sum_{m \neq n} p_{nm} \]

\[ q_n = \sum_{m \neq n} q_{nm} \]

\[ p_{nm} = |\tilde{V}_n|^2 g_{nm} - |\tilde{V}_n||\tilde{V}_m|g_{nm} \cos(\theta_n^\circ - \theta_m^\circ) - |\tilde{V}_n||\tilde{V}_m|b_{nm} \sin(\theta_n^\circ - \theta_m^\circ) \]

\[ q_{nm} = -|\tilde{V}_n|^2 b_{nm} + |\tilde{V}_n||\tilde{V}_m|b_{nm} \cos(\theta_n^\circ - \theta_m^\circ) - |\tilde{V}_n||\tilde{V}_m|g_{nm} \sin(\theta_n^\circ - \theta_m^\circ) \]
Linear Programming Approximations

• Hot-Start Context
  • An AC base-point solution is available

• Warm-Start Context
  • Target voltage magnitudes are available and “useful”
  • E.g., from normal operating conditions

• Cold-Start Context
  • No useful information is available on voltage magnitudes
Hot-Start LP Approximation

\[ \hat{p}_{nm}^h = |\tilde{V}_n^h|^2 g_{nm} - |\tilde{V}_n^h||\tilde{V}_m^h|g_{nm}\cos(\theta_n^o - \theta_m^o) - |\tilde{V}_n^h||\tilde{V}_m^h|b_{nm}(\theta_n^o - \theta_m^o) \]

\[ \hat{q}_{nm}^h = -|\tilde{V}_n^h|^2 b_{nm} + |\tilde{V}_n^h||\tilde{V}_m^h|b_{nm}\cos(\theta_n^o - \theta_m^o) - |\tilde{V}_n^h||\tilde{V}_m^h|g_{nm}(\theta_n^o - \theta_m^o) \]

- Two approximations
  - \( \sin(x) \) is approximated by \( x \)
  - piecewise approximation of \( \cos(x) \)
Fig. 1. A Piecewise-Linear Approximation of Cosine using 7 Inequalities.
Warm-Start LP Approximation

• Understanding power flows [Grainger, 94]
  • Phase angle differences determine active power
  • Voltage magnitude differences determine reactive power

• Experiments
  • Per unit system
  • Look at how the equations behave when
    • $g = 0.2$ and $b = 1.0$
    $$|\tilde{V}_n| = 1.0, |\tilde{V}_m| \in (1.2, 0.8), \theta_n^\circ - \theta_m^\circ \in (-\pi/6, \pi/6)$$
Warm-Start LP Approximation

Active Power Field

Reactive Power Field

Voltage Difference

Angle Difference (rad)

0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4

0.8 0.9 1.0 1.1 1.2

0.8 0.9 1.0 1.1 1.2

0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4

0.1 0.05 0
-0.05
-0.1
-0.2
-0.25
-0.3

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Warm-Start LP Approximation

• Assumptions
  • We have target voltages

• Basic approach
  • Active power as in the hot-start model
  • Reactive power should capture voltage magnitudes and phase angles

• Key idea
  • Substitute $|\tilde{V}| = |\tilde{V}^t| + \phi$ into the power flow equations
Warm-Start LP Approximation

- Reactive power

\[ q_{nm} = q_{nm}^t + q_{nm}^\Delta \]

- Target part

\[ q_{nm}^t = -|\tilde{V}_n^t|^2 b_{nm} + |\tilde{V}_n^t| |\tilde{V}_m^t| b_{nm} \cos(\theta_n^\circ - \theta_m^\circ) - |\tilde{V}_n^t| |\tilde{V}_m^t| g_{nm} \sin(\theta_n^\circ - \theta_m^\circ) \]

- Delta part

\[ q_{nm}^\Delta = -(2|\tilde{V}_n^t|\phi_n + \phi_n^2) b_{nm} - (|\tilde{V}_n^t|\phi_m + |\tilde{V}_m^t|\phi_n + \phi_n\phi_m)(g_{nm} \sin(\theta_n^\circ - \theta_m^\circ) - b_{nm} \cos(\theta_n^\circ - \theta_m^\circ)) \]
Warm-Start LP Approximation

- Target part approximation

\[ \hat{q}^t_{nm} = -|\tilde{V}_n^t|^2 b_{nm} + |\tilde{V}_n^t||\tilde{V}_m^t|b_{nm}\cos(\theta^o_n - \theta^o_m) - |\tilde{V}_n^t||\tilde{V}_m^t|g_{nm}(\theta^o_n - \theta^o_m) \]

- Delta part approximation

\[ \hat{q}^\Delta_{nm} = -|\tilde{V}_n^t|b_{nm}(\phi_n - \phi_m) - (|\tilde{V}_n^t| - |\tilde{V}_m^t|)b_{nm}\phi_n \]
Model 1 The Warm LPAC Model.

Inputs:
\[ \mathcal{PN} = \langle N, L, G, s \rangle \]
- the power network

\[ |\tilde{V}^t| \]
- target voltage magnitudes

\[ c_s \]
- cosine approximation segment count

Variables:
\[ \theta_n^o \in (-\infty, \infty) \]
- phase angle on bus \( n \) (radians)

\[ \phi_n \in (-|V^t|, \infty) \]
- voltage change on bus \( n \) (Volts p.u.)

\[ \hat{\cos}\delta_{nm} \in (0, 1) \]
- Approximation of \( \cos(\theta_n^o - \theta_m^o) \)

Maximize:
\[ \sum_{(n,m) \in L} \hat{\cos}\delta_{nm} \]  
(M1.1)

Subject to:
\[ \theta_s^o = 0, \phi_s = 0 \]  
(M1.2)

\[ \phi_i = 0 \quad \forall i \in G \]  
(M1.3)

\[ p_n = \sum_{\substack{m \in N \\cap L \\cap L}} p_{n,m} \quad \forall n \in N \quad n \neq s \]  
(M1.4)

\[ q_n = \sum_{\substack{m \in N \\cap L \\cap L}} q_{n,m} + q_{n,m}^\Delta \quad \forall n \in N \quad n \neq s \quad n \neq G \]  
(M1.5)

\[ \forall (n,m), (m,n) \in L \]

\[ \hat{p}_{n,m} = |\tilde{V}_n^t|^2 g_{nm} - |\tilde{V}_m^t|^2 |\tilde{V}_m^t| (g_{nm} \hat{\cos}\delta_{nm} + b_{nm}(\theta_n^o - \theta_m^o)) \]  
(M1.6)

\[ \hat{q}_{n,m} = -|\tilde{V}_n^t|^2 b_{nm} - |\tilde{V}_m^t|^2 |\tilde{V}_m^t| (g_{nm}(\theta_n^o - \theta_m^o) - b_{nm} \hat{\cos}\delta_{nm}) \]  
(M1.7)

\[ \text{PWL} \langle \cos\rangle(\hat{\cos}\delta_{nm}, (\theta_n^o - \theta_m^o)，-\pi/3, \pi/3, c_s) \]  
(M1.8)

\[ \hat{q}_{n,m}^\Delta = -|\tilde{V}_n^t| b_{nm}(\phi_n - \phi_m) - (|\tilde{V}_n^t| - |\tilde{V}_m^t|) b_{nm} \phi_n \]  
(M1.9)
Cold-Start LP Approximation

- Simply use the warm-start model with
  - Target voltages at 1.0
  - Use an appropriate $\phi$ for voltage-controlled generators

- Note that the delta part of reactive power becomes

$$\hat{q}_{nm}^\Delta = -b_{nm}(\phi_n - \phi_m)$$
Extensions of the LPAC Model

- **Range for generators**
  - Simply include a decision variable

- **Removing the slack bus**
  - No need for a slack bus in the LPAC model

- **Shedding load**
  - Simply use decision variables for loads

- **Additional constraints**
  - **Voltages:** \[ |V| \leq |V_n^t| + \phi_n \quad \forall n \in N \]
  - **Apparent power:** \[ (\hat{P}_{nm}^t)^2 + (\hat{q}_{nm}^t + \hat{q}_{nm}^\Delta)^2 \leq |S_{nm}|^2 \]
  - **Reactive power** \[ \sum_{m \in N} \hat{q}_{nm}^t + \hat{q}_{nm}^\Delta \leq q_n \quad \forall n \in G \]
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Experimental Results

- Wide variety of IEEE and MATPOWER Benchmarks
  - ieee14, mp24, ieee30, mp30, mp39, ieee57, ieee118, ieedd17, mp300
  - Small benchmarks are easy in general
  - IEEE 118 is also easy
  - All LPAC models solved almost instantly (LPs)
- This talk
  - MP300 for scalability and brevity
- Comparison with an AC Solver
  - LDC and LPAC solutions versus an AC solution
- Comparison with alternative linearizations
  - Evaluating the importance of all components
Line Active Power

LDC Model

Cold-start LPAC Model
# Line Active Power

Table 1: Active Power Flow Accuracy Comparison

| Benchmark | Corr  | $\mu(\Delta)$ | max(\Delta) | $\delta(\text{arg max}(\Delta))$ | $\mu(\delta)$ | max(\delta) | $\Delta(\text{arg max}(\delta))$ | approx(%) |
|-----------|-------|----------------|-------------|---------------------------------|----------------|-------------|----------------------------------|-----------|
| **The LDC Model** |       |                |             |                                 |                |             |                                  |           |
| ieee14    | 0.9994 | 1.392          | 10.64       | 6.783                           | 6.052          | 24.33       | 0.3927                           | 65        |
| mp24      | 0.9989 | 5.659          | 19.7        | 23.65                           | 6.447          | 29.89       | 6.656                            | 47.06     |
| ieee30    | 0.9993 | 1.046          | 13.1        | 7.562                           | 6.406          | 31.23       | 0.5646                           | 80.49     |
| mp30      | 0.9993 | 0.2964         | 2.108       | 19.36                           | 3.086          | 19.36       | 2.108                            | 82.93     |
| mp39      | 0.9995 | 7.341          | 43.64       | 6.527                           | 9.566          | 52.18       | 12.86                            | 76.09     |
| iee57     | 0.9989 | 1.494          | 8.216       | 8.055                           | 105.8          | 4193        | 0.9607                           | 52.56     |
| iee118    | 0.9963 | 3.984          | 56.3        | 44.74                           | 29.2           | 445.9       | 6.526                            | 51.96     |
| ieeedd17  | 0.9972 | 4.933          | 201.3       | 13.84                           | 15.2           | 215         | 0.5265                           | 50.71     |
| ieeedd17m | 0.9975 | 4.779          | 191.1       | 13.23                           | 14.56          | 231.3       | 3.066                            | 51.43     |
| mp300     | 0.991  | 11.09          | 418.5       | 90.02                           | 29.35          | 2859        | 46.14                            | 67.73     |
| **The LPAC-Cold Model** |       |                |             |                                 |                |             |                                  |           |
| ieee14    | 0.9989 | 1.636          | 5.787       | 13.13                           | 11.52          | 35.67       | 2.623                            | 40        |
| mp24      | 0.9999 | 1.884          | 6.159       | 2.933                           | 3.871          | 17.23       | 3.837                            | 41.18     |
| ieee30    | 0.9998 | 0.5475         | 2.213       | 2.523                           | 5.751          | 31.33       | 0.5666                           | 75.61     |
| mp30      | 0.9995 | 0.2396         | 1.641       | 15.07                           | 2.402          | 15.07       | 1.641                            | 78.05     |
| mp39      | 1      | 2.142          | 8.043       | 3.288                           | 4.357          | 24.78       | 6.106                            | 43.48     |
| iee57     | 0.9995 | 0.9235         | 4.674       | 9.728                           | 110.1          | 4500        | 1.031                            | 46.15     |
| iee118    | 1      | 0.622          | 3.708       | 2.038                           | 5.318          | 99.61       | 0.5519                           | 55.31     |
| ieeedd17  | 0.9999 | 1.827          | 30.38       | 2.088                           | 10.92          | 420.2       | 1.029                            | 55.36     |
| ieeedd17m | 0.9999 | 1.475          | 20.21       | 1.399                           | 7.766          | 144.5       | 2.547                            | 56.79     |
| mp300     | 0.9998 | 2.455          | 18          | 8.675                           | 7.104          | 337.2       | 6.95                             | 57.21     |
Bus Angles

LDC Model

Cold-Start LPAC Model
Line Reactive Power

Cold-Start LPAC Model

Warm-Start LPAC Model
Bus Voltages

Cold-Start LPAC Model

Warm-Start LPAC Model
Cosine Approximation

Quality of a Piece-wise Linear Cosine Approximation

\[ \cos(x) \]

\[ \text{pwl-\cos}(x) \]

Radians

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Importance of $g$: Reactive Power

Cold-Start LPAC Model (g=0) Cold-Start LPAC Model
Importance of $g$: Bus Voltages

Cold-Start LPAC Model ($g=0$)

Cold-Start LPAC Model
Importance of cos: Reactive Power

Cold-Start LPAC Model (cos=1)
Importance of cos: Bus Voltages

Cold-Start LPAC Model (cos=1)
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  - Build on top of the cold LPAC model
- Power Restoration
Capacitor Placement

- The Problem
  - place capacitors in a power network to improve voltage stability

- Minimize the number of capacitors subject to
  - lower bounds on the voltages
  - upper bounds on reactive capacitor injection
  - upper bounds on reactive generation injection
Capacitor Placement

Inputs:
\[
\begin{align*}
\overline{q}_n & \quad \text{- injection bound for generator } n \\
\overline{q}_c & \quad \text{- capacitor injection bound} \\
|\overline{V}| & \quad \text{- minimum desired voltage magnitude}
\end{align*}
\]
Inputs from The Cold LPAC Model

Variables:
\[
\begin{align*}
q_c & \in (0, \overline{q}_c) \quad \text{- capacitor reactive injection} \\
c_n & \in \{0, 1\} \quad \text{- capacitor placement indicator}
\end{align*}
\]
Variables from The Cold LPAC Model

Minimize:
\[
\sum_{n \in N} c_n
\]
Subject to:
\[
\begin{align*}
|\overline{V}| & \leq 1.0 + \phi_n \leq 1.05 \quad \forall n \in N \\
q_c & \leq M c_n \\
q_n & \leq \overline{q}_n \quad \forall n \in G \\
q_n & = \sum_{m \in N, n \neq m} \hat{q}_{nm} + \hat{q}_{nm}^\Delta \quad \forall n \in G \\
q_n - q_c & = \sum_{m \in N, n \neq m} \hat{q}_{nm} + \hat{q}_{nm}^\Delta \quad \forall n \in N : n \neq s \land n \notin G
\end{align*}
\]
Constraints from The Cold LPAC Model
Experimental Results

- Modified IEEE 57 Benchmark
  - Remove the transformers
  - Remove the synchronous condensers
  - This induces severe voltage problems
  - Impose increasingly tighter voltage lower bounds
- The capacitor placement model
  - Meets all voltage requirements but is an approximation

How well does it do compared to the AC model?
IEEE57: 0 Capacitor
Table 1: Capacitor Placement: Effects of $|\tilde{V}|$ on IEEE57-C, $q^c = 30$ MVar

| $|\tilde{V}|$ | $\min(|\tilde{V}|)$ | $\max(|\tilde{V}|)$ | $\max(q_n)$ | $\sum c_n$ | Time (sec.) |
|------------|-----------------|-----------------|-------------|----------|-------------|
| 0.8850     | 0.000000        | 0.0             | 0.0         | 1        | 1           |
| 0.9350     | 0.000000        | 0.0             | 0.0         | 3        | 8           |
| 0.9600     | 0.000000        | 0.0             | 0.0         | 5        | 156         |
| 0.9750     | -0.000000       | 0.0             | 0.0         | 6        | 177         |
| 0.9775     | -0.000000       | 0.0             | 0.0         | 6        | 139         |
| 0.9800     | -0.000000       | 0.0             | 0.0         | 6        | 75          |
| 0.9840     | -0.000802       | 0.0             | 0.0         | 7        | 340         |
Bus Voltage Correlation (Volts p.u.)

LL-LDC Power Flow

AC Power Flow
Bus Voltage Correlation (Volts p.u.)

LL-LDC Power Flow vs. AC Power Flow
Bus Voltage Correlation (Volts p.u.)

- AC Power Flow
- LL-LDC Power Flow

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Bus Voltage Correlation (Volts p.u.)
Bus Voltage Correlation (Volts p.u.)

LL-LDC Power Flow

AC Power Flow
Bus Voltage Correlation (Volts p.u.)

AC Power Flow vs. LL-LDC Power Flow

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Bus Voltage Correlation (Volts p.u.)
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  • Build on top of the warm LPAC model
Demand Maximization

Inputs:
\[
\begin{align*}
\bar{p}_n^g & \quad - \text{maximum active injection for bus } n \\
\underline{p}_n^l & \quad - \text{desired active load at bus } n \\
q_n^l & \quad - \text{desired reactive load at bus } n
\end{align*}
\]
Inputs from the Warm LPAC Model

Variables:
\[
\begin{align*}
p_n^g & \in (0, \bar{p}_n^g) \quad - \text{active generation at bus } n \\
q_n^g & \in (-\infty, \infty) \quad - \text{reactive generation at bus } n \\
l_n & \in (0, 1) \quad - \text{percentage of load served at bus } n
\end{align*}
\]
Variables from the Warm LPAC Model

Maximize:
\[
\sum_{n \in N} l_n
\]

Subject to:
\[
\begin{align*}
p_n & = -\underline{p}_n^l l_n + \bar{p}_n^g \quad \forall n \in N \\
q_n & = -q_n^l l_n + q_n^g \quad \forall n \in N \\
q_n^g & = 0 \quad \forall n \in N \setminus G \\
q_n & = \sum_{m \in N} \hat{q}_{nm}^t + \hat{q}_{nm}^\Delta \quad \forall n \in G
\end{align*}
\]
Constraints from the Warm LPAC Model
### IEEE-30 Contingencies

|     | N-9  | N-11 | N-12 | N-13  | N-15  | N-16  | N-17  | %    |
|-----|------|------|------|-------|-------|-------|-------|------|
| LDC | 7436 | 6511 | 5344 | 6805  | 5931  | 7236  | 6877  | 66%  |
| LPAC| 9998 | 10000| 9996 | 9981  | 9998  | 10000 | 9911  | 99.8%|

|     | N-9 | N-11 | N-12 | N-13 | N-15 | N-16 | N-17 |
|-----|-----|------|------|------|------|------|------|
| LDC | 14.2| 20.14| 57.77| 73.67| 44.54| 64.58| 67.88|
| LPAC| 35.96| 30.38| 57.74| 62.83| 57.49| 66.69| 64.73|
Power Restoration

AC Restoration Timeline

AC Power Flow (MW)

Restoration Action

LDC–ROP
Power Restoration

AC Restoration Timeline

AC Power Flow (MW)

Restoration Action

- LDC–ROP
- LPAC–ROP
Power Restoration

DC Restoration Timeline

DC Power Flow (MW)

Restoration Action

LDC–ROP
LPAC–ROP
Power Restoration

AC Line Overloads

Cumulative Overload (MVA)

Restoration Action

- LDC
- LPAC+R
- LPAC+R+V

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AC Voltage Stability

Restoration Action

Cumulative Instability (Volts p.u.)

LDC
LPAC+R
LPAC+R+V
Conclusion

- **LPAC Models: Linear-Programming approximations**
  - Much more accurate than the LDC model
  - useful outside normal operating conditions
  - reason about voltage magnitudes and reactive power
  - can be embedded in MIP solvers

- **Experimental results**
  - Very high accuracy when compared to AC solutions

- **Case studies**
  - Capacitor placement problem
  - Power Restoration