EULER CHARACTERISTICS OF \( p \)-SUBGROUP CATEGORIES

MARTIN WEDEL JACOBSEN AND JESPER M. MÖLLER

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Abstract. Let \( G \) be a finite group and \( p \) a prime number. We compute the Euler characteristic in the sense of Leinster for some categories of nonidentity \( p \)-subgroups of \( G \). The \( p \)-subgroup categories considered include the poset \( S^*_G \), the transporter category \( T^*_G \), the linking category \( L^*_G \), the Frobenius, or fusion, category \( F^*_G \), and the orbit category \( O^*_G \) of all nonidentity \( p \)-subgroups of \( G \).

1. Introduction

In this note we apply Tom Leinster’s theory of Euler characteristics of (some) finite categories \([17]\) to \( p \)-subgroup categories associated to finite groups.

For a finite group \( G \) and a fixed prime number \( p \), let \( S_G \) denote the poset of consisting of all \( p \)-subgroups of \( G \) ordered by inclusion. Also, let \( T_G \) (the transporter category), \( L_G \) (the linking category \([6]\)), \( F_G \) (the Frobenius category \([19, 6]\)), \( O_G \) (the orbit category), and \( \tilde{F}_G \) (the exterior quotient of the Frobenius category \([19, 1.3, 4.8]\)) be the categories whose objects are the \( p \)-subgroups of \( G \) and whose morphism sets are

\[
T_G(H, K) = N_G(H, K) \\
L_G(H, K) = O^p C_G(H) \backslash N_G(H, K) \\
F_G(H, K) = C_G(H) \backslash N_G(H, K) \\
O_G(H, K) = N_G(H, K) / K \\
\tilde{F}_G(H, K) = C_G(H) \backslash N_G(H, K) / K
\]

for any two \( p \)-subgroups, \( H \) and \( K \), of \( G \). Here \( N_G(H, K) = \{ g \in G \mid H^g \leq K \} \) denotes the transporter set. Composition in any of these categories is induced from group multiplication in \( G \). The morphisms in \( F_G(H, K) \) are restrictions to \( H \) of inner automorphisms of \( G \), morphisms in \( O_G(H, K) \) are right \( G \)-maps \( H \backslash G \to K \backslash G \), and morphisms in \( \tilde{F}_G(H, K) \) are \( K \)-conjugacy classes of restrictions to \( H \) of inner automorphisms of \( G \). The endomorphism groups in these categories of the \( p \)-subgroup \( H \) of \( G \) are \( S^*_G(H) = \)
The five categories \( \mathcal{T}_G, \mathcal{L}_G, \mathcal{F}_G, \mathcal{O}_G \), and \( \tilde{\mathcal{F}}_G \) are related by functors
\[
\begin{array}{c c c c}
\mathcal{T}_G & \longrightarrow & \mathcal{L}_G & \longrightarrow & \mathcal{F}_G & \longrightarrow & \tilde{\mathcal{F}}_G \\
\downarrow & & & & & & \\
\mathcal{O}_G & & & & & & 
\end{array}
\]
If \( \mathcal{C} \) is any of these categories
- \( \mathcal{C}^* \) is the full subcategory of \( \mathcal{C} \) generated by all nonidentity \( p \)-subgroups
- \( \mathcal{C}^a \) is the full subcategory of \( \mathcal{C}^* \) generated by all elementary abelian \( p \)-subgroups
- \( \mathcal{C}^c \) is the full subcategory of \( \mathcal{C}^* \) generated by all \( p \)-selfcentralizing \( p \)-subgroups

Here is a summary of our main results appearing in Table 1, Table 2, and Theorem 4.1.

**Theorem 1.1** (Euler characteristics of \( p \)-subgroup categories). The Euler characteristics are
\[
\chi(\mathcal{C}^*) = \sum_{[H]} -\frac{\mu(H)}{|\mathcal{C}^*(H)|}, \quad \mathcal{C} = \mathcal{T}_G, \mathcal{L}_G, \mathcal{F}_G
\]
where the sum runs over the set of conjugacy classes of nonidentity \( p \)-subgroups of \( G \). Also, \( \chi(\mathcal{S}^*_G) = |G|\chi(\mathcal{T}^*_G), \chi(\tilde{\mathcal{F}}^*_G) = \chi(\mathcal{F}^*_G)\), and
\[
\chi(\mathcal{O}^*_G) = \chi(\mathcal{T}^*_G) + \frac{p-1}{p} \sum_{[C]} 1_{|\mathcal{O}^*_G(C)|}
\]
where the sum runs over the set of conjugacy classes of nonidentity cyclic \( p \)-subgroups of \( G \).

Here, \( \mu(K) = \mu_K(1, K) \) is the Möbius function of finite groups [13] [22, Chp 3.7]. (The formula for the Euler characteristic of \( \mathcal{S}^*_G \) is already known, of course.) Theorem 1.1 implies that
\[
\chi(\mathcal{C}^*) = \chi(\mathcal{C}^a)
\]
because these categories have coweightings supported (precisely) on the elementary abelian subgroups of \( G \). Dually, the weightings for \( \mathcal{S}^*_G \), \( \mathcal{O}^*_G \), and \( \mathcal{F}^*_G \) are supported on the nonidentity \( p \)-radical subgroups of \( G \) (Definition 3.18, Corollary 3.19), and, if \( G \) has a normal Sylow \( p \)-subgroup, \( P \), the weighting for \( \mathcal{F}^*_G \) is supported (precisely) on the subgroups of \( P \) of the form \( C_{p^x}(x), x \in G \) (Corollary 5.4). This shows that these \( p \)-subgroup categories carry information, retrieved by the weighting or the coweighting, about which objects are elementary abelian, \( p \)-radical, or centralizers of group elements. The Euler characteristics of Theorem 1.1 are rational numbers. However, \( |G|^{p^x}\chi(\mathcal{O}^*_G) \) and \( |G|^{p^x}\chi(\mathcal{F}^*_G) \) are integers (Corollaries 4.6 and 5.2).

As a spin-off of our investigations of (co)weightings we establish three combinatorial identities in (3.16), (3.17), and (4.5).

**Corollary 1.2.** For any finite group \( G \) and any prime \( p \),
\[
\sum_{[H]} (1 - \chi(\mathcal{S}^*_C(H)) + \mu(H)) = 0
\]
\[
\sum_{[H]} \sum_{x \in C_G(H)} (1 - \chi(\mathcal{S}^*_{C_{H}(x)}(H)) + \mu(H)) = 0
\]
\[
\sum_{[H]} (|H| - \chi(\mathcal{O}^*_C(H))|H| + \mu(H)) = \frac{p-1}{p} \sum_C |C|
\]
where \( H \) runs over the set of nonidentity \( p \)-subgroups of \( G \) and \( C \) over the set of nonidentity cyclic \( p \)-subgroups of \( G \).

Theorems 6.1 and 6.2 establish formulas for Euler characteristics of posets and Frobenius categories of nonidentity subgroups stating that
\[
1 - \chi(\mathcal{S}^*_n) = \prod_{i=1}^{n} (1 - \chi(\mathcal{S}^*_G)), \quad 1 - \chi(\mathcal{F}^*_n) = \prod_{i=1}^{n} (1 - \chi(\mathcal{F}^*_G))
\]
where \( G_1, \ldots, G_n \) are finite groups.

For the sake of quick reference we list here the notation that we are using throughout this paper:
• $p$ is a fixed prime number
• $n_p$ is $p$-part of the integer $n$, the highest power of $p$ dividing $n$, and $n_{p'} = n/n_p$ is the $p'$-part of $n$
• $G$ is a finite group
• $H \leq K$ means that $H$ is a subgroup of $K$
• $\Phi(K)$ is the Frattini subgroup of $K$ [10, Definition 3.14]
• $C$ is a finite category, $C(a, b)$ is the set of morphisms from object $a$ to object $b$, and $C(a) = C(a, a)$ is the monoid of endomorphisms of $a$
• $\text{Ob}(C)$ is the set of objects of $C$
• $[C]$ the set of isomorphism classes of objects of $C$ and $[a] \in [C]$ the isomorphism class of $a \in \text{Ob}(C)$

2. Euler characteristics

In this section we review the relevant parts of Tom Leinster’s concept of Euler characteristic of a finite category $C$ [17].

2.1. The Euler characteristic of a square matrix. Let $S$ be a finite set and $\zeta : S \times S \to \mathbb{Q}$ a rational function on $S \times S$. Equivalently, $\zeta = (\zeta(a, b))_{a, b \in S}$ is a square matrix with rows and columns indexed by the finite set $S$ and with rational entries $\zeta(a, b) \in \mathbb{Q}$, $a, b \in S$.

Definition 2.1. [17, Definition 1.10] A weighting for $\zeta$ is a column vector $(k^\bullet)$ and a coweighting for $\zeta$ is a row vector $(k_{\bullet})$ solving the linear equations

$$
(\zeta(a, b))(k^b) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad (k_a)(\zeta(a, b)) = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}
$$

If $\zeta$ admits both a weighting $k^\bullet$ and a coweighting $k_{\bullet}$, then the sum of the values of the weighting

$$
\sum_b k^b = \sum_b (\sum_a k_a \zeta(a, b)) k^b = \sum_a k_a (\sum_b \zeta(a, b) k^b) = \sum_a k_a
$$

equals the sum of the values of the coweighting (and then this sum is independent of the choice of weighting or coweighting).

Definition 2.3. [17, Definition 2.2] The square matrix $\zeta$ has Euler characteristic if it admits both a weighting and a coweighting and its Euler characteristic is then the sum

$$
\chi(\zeta) = \sum_b k^b = \sum_a k_a
$$

of all the values of any weighting $k^\bullet$ or any coweighting $k_{\bullet}$.

In case the square matrix $\zeta$ is invertible, if we let $\mu = (\mu(a, b))_{a, b \in S}$ denote the inverse of $\zeta$, the Möbius inversion formula

$$
\forall a, c \in S: \sum_b \zeta(a, b) \mu(b, c) = \delta(a, c) = \sum_b \mu(a, b) \zeta(b, c)
$$

simply expresses that $\zeta$ and $\mu$ are inverse matrices. When $\zeta$ is invertible the vectors

$$
(k^\bullet) = (\mu(a, b)) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = (\sum_{b \in S} \mu(a, b)), \quad (k_b) = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} (\mu(a, b)) = (\sum_{a \in S} \mu(a, b))
$$

are the unique weighting and coweighting for $\zeta$ and the Euler characteristic of $\zeta$

$$
\chi(\zeta) = \sum_{a, b \in S} \mu(a, b)
$$

is the sum of all the entries in the inverse matrix.
2.2. The Euler characteristic of a finite category. Define the \( \zeta \)-matrix for the finite category \( C \) to be the square matrix

\[
\zeta(C) = ([C(a, b)])_{a,b \in \text{Ob}(C)}
\]

that counts the number of morphisms between pairs of objects of \( C \). We say that the category \( C \) admits a weighting, admits a coweighting, or has Euler characteristic if its \( \zeta \)-matrix \( \zeta(C) \) does. This means that a weighting for \( C \) is a rational function \( k^\bullet : \text{Ob}(C) \to \mathbb{Q} \) and a coweighting for \( C \) is a rational function \( k_\bullet : \text{Ob}(C) \to \mathbb{Q} \) such that

\[
(2.5) \quad \forall a \in \text{Ob}(C): \sum_{b \in \text{Ob}(C)} \zeta(a, b)k^b = 1, \quad \forall b \in \text{Ob}(C): \sum_{a \in \text{Ob}(C)} k_\alpha \zeta(a, b) = 1
\]

and the Euler characteristic of \( C \) is

\[
\chi(C) = \sum_{b \in \text{Ob}(C)} k^b = \sum_{a \in \text{Ob}(C)} k_\alpha
\]

provided that \( C \) admits both a weighting and a coweighting. We say that \( C \) has Möbius inversion if \( \zeta(C) \) is invertible and then

\[
k^a = \sum_{b \in \text{Ob}(C)} \mu(a, b), \quad \zeta(a, b) = \sum_{a \in \text{Ob}(C)} \mu(a, b), \quad \chi(C) = \sum_{a,b \in \text{Ob}(C)} \mu(a, b)
\]

is the unique weighting, the unique coweighting, and the Euler characteristic of \( C \) where \( \mu = \zeta(C)^{-1} \) denotes the inverse of the \( \zeta \)-matrix.

Example 2.6. [17, Examples 1.1.c] Suppose that \( C \) has Euler characteristic. If \( C \) has an initial element 0 then the Kronecker function \( k_\bullet = \delta(0, \bullet) \) is a coweighting concentrated at the initial element because

\[
\sum_a \delta(0, a)\zeta(a, b) = \zeta(0, b) = 1
\]

and the Euler characteristic of \( C \) is \( \chi(C) = \sum_a \delta(0, a) = \delta(0, 0) = 1 \). Dually, if \( C \) has a terminal element 1, then \( k^\bullet = \delta(\bullet, 1) \) is a weighting concentrated at the terminal element, and \( \chi(C) = 1 \).

As usual, \( \delta \) stands for Kronecker’s \( \delta \)-function

\[
\delta(a, b) = \begin{cases} 
1 & a = b \\
0 & a \neq b 
\end{cases}, \quad a, b \in \text{Ob}(C),
\]

Lemma 2.7. [17, Proposition 2.4] Let \( C \) and \( D \) be finite categories.

1. \( C \) has Euler characteristic if and only if its opposite category \( C^{\text{op}} \) has and then \( \chi(C) = \chi(C^{\text{op}}) \).
2. If both \( C \) and \( D \) have Euler characteristics and there is an adjunction \( C \rightleftarrows D \) then \( \chi(C) = \chi(D) \).
3. If \( C \) and \( D \) are equivalent then \( C \) has Euler characteristic if and only if \( D \) has Euler characteristic and then \( \chi(C) = \chi(D) \).

Lemma 2.8. Let \( C \) be a full subcategory of \( D \) and suppose that both categories have Euler characteristics.

1. If \( \text{Ob}(C) \) contains the support of some weighting \( k^\bullet \) on \( D \), then the restriction \( k^\bullet|\text{Ob}(C) \) is a weighting for \( C \) and \( \chi(C) = \chi(D) \).
2. If \( \text{Ob}(C) \) contains the support of some coweighting \( k_\bullet \) on \( D \), then the restriction \( k_\bullet|\text{Ob}(C) \) is a coweighting for \( C \) and \( \chi(C) = \chi(D) \).

Proof. (2) The assumption is that \( \forall a \in \text{Ob}(D): k_\alpha \neq 0 \implies a \in \text{Ob}(C) \). For any \( b \in \text{Ob}(C) \)

\[
\sum_{a \in \text{Ob}(C)} k_\alpha \zeta(a, b) = \sum_{a \in \text{Ob}(D)} k_\alpha \zeta(a, b) = 1
\]

so that the restriction to \( \text{Ob}(C) \) of \( k_\bullet \) is indeed a coweighting for \( C \). The Euler characteristic of \( C \) is \( \chi(C) = \sum_{a \in \text{Ob}(C)} k_\alpha = \sum_{a \in \text{Ob}(D)} k_\alpha = \chi(D) \). \( \square \)
2.3. The Euler characteristic of a finite poset. In particular, any finite poset, $S$, has Möbius inversion [22]. The value of the Möbius function

$$\mu(a, b) = \chi((a, b)) - 1, \quad a < b,$$

depends only on the open interval $(a, b)$ from $a$ to $b$ and not on the whole poset [22, Proposition 3.8.5] [17, Corollary 1.5].

Example 2.9. Let $S$ be a finite poset with a least element, 0. Then $\chi(S) = 1$ by Example 2.6. For any element $b$ of $S$,

$$\sum_{a: 0 \leq a \leq b} \mu(a, b) = \sum_{a: 0 \leq a \leq b} |S(0, a)| \mu(a, b) = \delta(0, b)$$

and for any element $b > 0$ of $S$,

$$(2.10) \quad \sum_{0 \leq a \leq b} \mu(a, b) = -\mu(0, b) + \sum_{0 \leq a \leq b} \mu(a, b) = -\mu(0, b)$$

The Euler characteristic of the subposet $S^*$ of all elements $\neq 0$ is

$$\chi(S^*) = \sum_{a,b > 0} \mu(a, b) = \sum_{b > 0} \sum_{a > 0} \mu(a, b) = -\mu(0, b)$$

Alternatively,

$$(2.11) \quad 1 - \chi(S^*) = \mu(0, 0) + \sum_{b > 0} \mu(0, b) = \sum_{b \in \text{Ob}(S)} \mu(0, b)$$

with summation over all elements of the poset $S$.

2.4. The Euler characteristic of $[C]$. Let $[\zeta(C)] : [C] \times [C] \to \mathbb{Q}$ be the function induced by the $\zeta$-function $\zeta(C)$: $\text{Ob}(C) \times \text{Ob}(C) \to \mathbb{Q}$ for $C$. We say that the set $[C]$ of isomorphism classes of $C$-objects admits a weighting, a coweighting, or has Euler characteristic if its $\zeta$-matrix $[\zeta(C)]$ does. This means that a weighting for $[C]$ is a rational function $k^\bullet : [C] \to \mathbb{Q}$ and a coweighting for $[C]$ is a rational function $k_* : [C] \to \mathbb{Q}$ such that

$$(2.12) \quad \forall [a] \in [C] : \sum_{[b] \in [C]} [\zeta(C)]([a], [b]) k^{|b|} = 1, \quad \forall [b] \in [C] : \sum_{[a] \in [C]} k_{{[a]}} [\zeta(C)]([a], [b]) = 1$$

and the Euler characteristic of $[C]$ is

$$\chi([C]) = \sum_{[b] \in [C]} k^{|b|} = \sum_{[a] \in [C]} k_{[a]}$$

provided that $[C]$ admits both a weighting and a coweighting. Clearly, if $C$ has a weighting $k^\bullet$ and a coweighting $k_*$, then $[C]$ has weighting $k^{|a|} = \sum_{c \in [a]} k^{|c|}$ and coweighting $k_{{[b]}} = \sum_{c \in [b]} k^{|c|}$ and $\chi(C) = \chi([C])$.

We say that $[C]$ has Möbius inversion if its $\zeta$-matrix $[\zeta(C)]$ is invertible and then

$$k^{|a|} = \sum_{[b] \in [C]} [\mu]([a], [b]), \quad k_{{[b]}} = \sum_{[a] \in [C]} [\mu]([a], [b]), \quad \chi([C]) = \sum_{[a], [b] \in [C]} [\mu]([a], [b])$$

is the unique weighting, the unique coweighting, and the Euler characteristic of $[C]$ where $([\mu]([a], [b])_{[a], [b] \in [C]}$ denotes the inverse of $[\zeta(C)]$.

Theorem 2.13. Suppose that $[C]$ has Möbius inversion. Then the functions

$$k^a = |[a]|^{-1} k^{|a|}, \quad k_{{|b|}} = |[b]|^{-1} k_{|[b]|}$$

are a weighting and a coweighting for $C$, and $\chi([C]) = \chi(C)$.

Proof. Let $a$ be any object of $C$. Since the function $k^\bullet$ is constant on the isomorphism class $[a]$ we find that

$$\sum_{b} \zeta(a, b) k^b = \sum_{b} [\zeta]([a], [b]) k^b = \sum_{[b]} [\zeta([a], [b]) k^{|b|}] = \sum_{[b]} [\zeta]([a], [b]) k^{|b|} = 1$$

according to (2.12) and this shows that $k^\bullet$ is a weighting on $C$ according to (2.5). A symmetric argument shows that $k_*$ is a coweighting. Thus $C$ has Euler characteristic $\chi(C) = \sum_a k^a = \sum_{[a]} |[a]| k^{|a|} = \sum_{[a]} k^{|a|} = \chi([C])$. \qed
Observe for instance that the transporter category \(\mathcal{T}_G\) (in general) does not have Möbius inversion as its \(\zeta\)-matrix \(\zeta(\mathcal{T}_G)\) has two identical rows as soon as there are two nonidentical nonidentity subgroups of \(G\) that are conjugate in \(G\). However, \([\mathcal{T}_G]\), the set of conjugacy classes of subgroups of \(G\), always has Möbius inversion: Extend the partial subconjugation ordering, \([H] \leq_G [K] \iff \mathcal{T}_G(H,K) \neq \emptyset\), to a total ordering of \([\mathcal{T}_G]\). If \([H] > [K]\) in this total order, then \(\mathcal{T}_G(H,K) = \emptyset\). This means that the \(\zeta\)-matrix \(\langle \mathcal{T}_G(H,K) \rangle_{[H], [K]\in[\mathcal{T}_G]}\) is upper-triangular in this total ordering and, as the diagonal entries are nonzero, it is invertible. We shall determine the Möbius function \([\mu]\) for \([\mathcal{T}_G]\) in Proposition 3.9.

2.5. The Euler characteristic of a homotopy orbit category. Let \(S\) be a finite category with a \(G\)-action. (This means that there is a functor from \(G\) to the category of finite categories taking the single object of \(G\) to \(S\).) The homotopy orbit category, \(\mathcal{S}_h G\), is the Grothendieck construction on the \(G\)-action on \(S\): The category with the same set of objects as \(S\) and with morphism sets

\[
\mathcal{S}_h G(a,b) = \coprod_{g\in G} S(a g, b) = \coprod_{g\in G} S(a, b g^{-1}) \quad \text{and} \quad |\mathcal{S}_h G(a,b)| = \sum_{g\in G} |S(a, b g^{-1})|
\]

Theorem 2.15. Let \(F\) be a finite category with the same objects as \(S\) such that \(d_a|F(a,b)|t^b = |\mathcal{S}_h G(a,b)|\) for all \(a,b\in \text{Ob}(S)\).

1. If \(m^* : \text{Ob}(S) \to \mathbb{Q}\) is a rational function so that \(\sum_b |S(a,b)|m^b = d_a\) for all \(a\in \text{Ob}(S)\) and \(d_*\) is \(G\)-invariant, then \(G^{-1} t^* m^*\) is a weighting for \(F\).
2. If \(m_* : \text{Ob}(S) \to \mathbb{Q}\) is a rational function so that \(\sum_a m_a |S(a,b)| = t^b\) for all \(b\in \text{Ob}(S)\) and \(t^*\) is \(G\)-invariant, then \(G^{-1} m_* d_*\) is a coweighting for \(F\).
3. Suppose that \(S\) has Möbius inversion and \(\mu\) is the Möbius function. If \(d_*\) is \(G\)-invariant then \(k^a = |G|^{-1} \sum_{g\in G} t^a \mu(a,b) db\) is a weighting for \(F\), and if \(t^*\) is \(G\)-invariant then \(k_b = |G|^{-1} \sum_{g\in G} t^b \mu(a,b) db\) a coweighting for \(F\).

Proof. \(1\) The proofs of \((1)\) and \((2)\) are dual to each other.

\(2\) For every \(b\in \text{Ob}(S)\),

\[
\sum_a m_a d_a |F(a,b)| = \sum_a m_a |\mathcal{S}_h G(a,b)| (t^b)^{-1} = \sum_{g\in G} \sum_a m_a |S(a, b g^{-1})| (t^{g^{-1}})^{-1} = \sum_{g\in G} t^{bg^{-1}} (t^b)^{-1} = |G|
\]

as \(t^*\) is \(G\)-invariant so that \(t^{bg^{-1}} = t^b\) for all \(g\in G\).

\(3\) If \(m^a = \sum_b \mu(a,b) db\) then \(\sum_b |S(a,b)|m^b = d_a\) by the Möbius inversion formula (2.4). By \((1)\), \(k^a\) is a weighting for \(F\) if \(d_*\) is \(G\)-invariant. Dually, if \(m_b = \sum_a t^a \mu(a,b)\) then \(\sum_a m_a |S(a,b)| = t^b\) by the Möbius inversion formula (2.4). By \((2)\), \(k_b\) is a coweighting for \(F\) if \(t^*\) is \(G\)-invariant.

3. The Möbius function of a finite group

The Möbius function for the finite group \(G\) is the Möbius function \(\mu\) for the poset \(\mathcal{S}_G\) of all subgroups of \(G\). Note that \(\mu\) restricts to Möbius functions for the convex subposets \(\mathcal{S}_G\) and \(\mathcal{S}_G^\ast\) of \(p\)-subgroups. For any subgroup \(K \leq G\), \(\mu(1,K)\) only depends on \(K\) and not on the whole group \(G\) and it is customary to write \(\mu(K)\) for \(\mu(1,K)\) [13].

Lemma 3.1. [12][13, Corollary 3.5] Let \(H\) and \(K\) be \(p\)-subgroups of \(G\). Then

\[
\mu(H,K) = \begin{cases} (-1)^n p^\binom{2n}{n} & \Phi(K) \leq H, \ k^p = |K: H| \\ 0 & \text{otherwise} \end{cases}
\]

In particular, \(\mu(K) = \mu(1,K) = 0\) unless \(K\) is elementary abelian where

\[
\mu(K) = (-1)^n p^\binom{2n}{n}, \quad k^n = |K|
\]

Proof. If \(\mu(H,K) \neq 0\) then \(H \lhd K\) with \(H\setminus K\) elementary abelian and \(\mu(H,K) = \mu(H\setminus K)\) [15, Proposition 2.4] [16, Lemme 4.1]. Burnside’s basis theorem [21, 5.3.2] [10, Lemma 3.15], \(\Phi(K) = [K,K][K:p]\), shows that \(H \lhd K\) with \(H\setminus K\) elementary abelian if and only if \(\Phi(K) \leq H\).

Theorem 3.2. Weightings \(k^*\), coweighings \(k_*\), and Euler characteristics for the \(p\)-subgroup categories \(\mathcal{S}_G^\ast\), \(\mathcal{T}_G^\ast\), \(\mathcal{L}_G^\ast\), \(\mathcal{F}_G^\ast\), and \(\mathcal{O}_G^\ast\) are as in Table 1.
| $\mathcal{C}$ | $k^H$ | $k_K$ | $\chi(\mathcal{C})$ |
|---|---|---|---|
| $\mathcal{S}_G^*$ | $\sum_K \mu(H, K)$ | $-\mu(K)$ | $|G| \chi(T_G^*)$ |
| $\mathcal{T}_G^*$ | $|G|^{-1} \sum_K \mu(H, K)$ | $-|G|^{-1} \mu(K)$ | $\sum [K] -\mu(K)$ |
| $\mathcal{L}_G^*$ | $|G|^{-1} \sum_K \mu(H, K) |O^p C_G(K)|$ | $-|G|^{-1} \mu(K) |O^p C_G(K)|$ | $\sum [K] -\mu(K)$ |
| $\mathcal{F}_G^*$ | $|G|^{-1} \sum_K \mu(H, K) |C_G(K)|$ | $-|G|^{-1} \mu(K) |C_G(K)|$ | $\sum [K] -\mu(K)$ |
| $\mathcal{O}_G^*$ | $|G|^{-1} |H| \sum_K \mu(H, K)$ | $|G|^{-1} \sum [H] |H| \mu(H, K)$ | $|G|^{-1} \sum [K] |H| \mu(H, K)$ |

Table 1. Categories of nonidentity $p$-subgroups

**Proof.** This follows almost immediately from Theorem 2.15 because the transporter category $T_G^* = (\mathcal{S}_G^*)_{hG}$ is the homotopy orbit category for the conjugation action of $G$ on the poset $\mathcal{S}_G^*$ and

$$|T_G^*(H, K)| = |O^p C_G(H)| |O^p C_G(K)| = |C_G(H)| |C_G(K)| = |O_G^*(H, K)| [K]$$

Since the poset $\mathcal{S}_G^*$ has Möbius inversion, $k^H = |G|^{-1} \sum_K \mu(H, K)$ is a weighting, $k_K = |G|^{-1} \sum_H \mu(H, K) = -|G|^{-1} \mu(K)$ (Example 2.9) a coweighting for $T_G^*$ by Theorem 2.15.(3). In case of $\mathcal{F}_G^*$, Theorem 2.15.(3) provides the weighting and the coweighting

$$k^H = |G|^{-1} \sum_{H \subseteq K} \mu(H, K) |C_G(K)|, \quad k_K = |G|^{-1} \sum_{H \subseteq K} \mu(H, K) |C_G(K)| = -|G|^{-1} \mu(K) |C_G(K)|$$

where the expression for the coweighting simplifies when using $\sum_{H \in \mathcal{O}(\mathcal{S}_G^*)} \mu(H, K) = -\mu(K)$ from identity (2.10). The Euler characteristic of $\mathcal{F}_G^*$, calculated as the sum of the values of the coweighting, is

$$\chi(\mathcal{F}_G^*) = \sum_K k_K = -|G|^{-1} \sum_K \mu(K) |C_G(K)| = -|G|^{-1} \sum_K \mu(K) |C_G(K)| |G: N_G(K)| = \sum_K -\frac{\mu(K)}{|\mathcal{F}_G^*(K)|}$$

because the coweighting $k_K$ is constant over the conjugacy class $[K]$ of $K$ and $|[K]| = |G: N_G(K)|$.

The quotient category $\mathcal{F}_G^*$ is missing from Table 1 because Theorem 2.15 does not directly apply. We shall later see that $\mathcal{F}_G^*$ and $\mathcal{F}_G^*$ have identical Euler characteristics (Corollary 3.6).

Lemma 2.8 implies that $\chi(\mathcal{S}^*) = \chi(\mathcal{S}^a)$ for $\mathcal{C} = \mathcal{S}, \mathcal{T}, \mathcal{L}, \mathcal{F}$ because the coweighings for these categories are concentrated on the elementary abelian $p$-subgroups of $G$ (Lemma 3.1). Quillen shows in [20, Proposition 2.1] the much stronger result that the posets $\mathcal{S}_G^*$ and $\mathcal{S}_G^a$ are homotopy equivalent.

If $P$ is a nonidentity $p$-group we immediately have that

$$\chi(\mathcal{S}_P^a) = 1, \quad \chi(\mathcal{T}_P^a) = |P|^{-1}, \quad \chi(\mathcal{L}_P^a) = |P|^{-1}, \quad \chi(\mathcal{F}_P^a) = 1, \quad \chi(\mathcal{O}_P^a) = 1, \quad \chi(\mathcal{F}_P^a) = 1$$

because $P$ is terminal in $\mathcal{S}_P^a$ and $\mathcal{O}_P^a$, $\mathcal{T}_P^a = \mathcal{L}_P^a$ and $\chi(\mathcal{T}_P^a) = |P|^{-1} \chi(\mathcal{S}_P^a) = |P|^{-1}$ by Theorem 2.15, Proposition 5.1 applies to $\mathcal{F}_P^a$, and Corollary 3.6 to $\mathcal{F}_P^a$. More generally, if $G$ has a normal $p$-complement, then $\chi(\mathcal{F}_G^a) = 1$ because $\mathcal{F}_G^a = \mathcal{F}_P^a$ according to the Frobenius normal $p$-complement theorem [11, Proposition 16.10][21, 10.3.2]. For some more examples, let $D_{pn}$ be the dihedral group of order $2pn$, $n \geq 1$, $A_p$ the alternating group of order $p > 2$, and $\operatorname{SL}_n(\mathbb{F}_q)$ the special linear group where $q$ is a power of $p$ and $n \geq 2$. Then

$$\chi(\mathcal{S}_{D_{pn}}) = 1, \quad \chi(\mathcal{S}_{A_p}) = (p - 2)!, \quad \chi(\mathcal{S}_{\operatorname{SL}_n(\mathbb{F}_q)}) = 1 + (-1)^n q^{(2)}$$

See [20, Example 2.7] for the Euler characteristic of $\mathcal{S}_{A_p}$. Let $V_n(q)$ be an $n$-dimensional vector space over $\mathbb{F}_q$ and $L_n(q)$ the poset of $\mathbb{F}_q$-subspaces of $V_n(q)$. As $\mathcal{S}_{\operatorname{SL}_n(\mathbb{F}_q)}$ and the open interval $(0, V_n(q))$, the building for $\operatorname{SL}_n(\mathbb{F}_q)$ [1, Example 6.5], are homotopy equivalent posets [20, Theorem 3.1],

$$\chi(\mathcal{S}_{\operatorname{SL}_n(\mathbb{F}_q)}) = \chi((0, V_n(q))) = 1 + \mu_{L_n(q)}(0, V_n(q)) = 1 + (-1)^n q^{(2)}$$

by the computation of the Möbius function $\mu_{L_n(q)}$ in $L_n(q)$ [22, Example 3.10.2] [16, Proposition 3.6]. In this example we may replace $\operatorname{SL}_n(\mathbb{F}_q)$ by any of the groups $\operatorname{GL}_n(\mathbb{F}_q)$, $\operatorname{PSL}_n(\mathbb{F}_q)$, or $\operatorname{PGL}_n(\mathbb{F}_q)$ since they
all have identical \( p \)-subgroup posets. The computer-generated Table 3 displays Euler characteristics of poset categories at \( p = 2 \) of small alternating groups.

The Euler characteristics of the subgroup categories generated by all \( p \)-subgroups of \( G \) (including the identity subgroup) are

\[
\chi(S_G) = 1, \quad \chi(T_G) = |G|^{-1}, \quad \chi(L_G) = |G: O^p G|^{-1}, \quad \chi(F_G) = 1, \quad \chi(O_G) = |G|^{-1} + \frac{p - 1}{p} \sum_{\kappa \in \{1\}} \frac{1}{|\mathcal{O}_G(\kappa)|}, \quad \chi(\bar{F}_G) = 1
\]

Observe that \( S_G, F_G, \) and \( \bar{F}_G \) have initial objects and that \( T_G \) deformation retracts onto \( T_G(1) = G \) and \( L_G \) deformation retracts onto \( L_G(1) = O^p G \backslash G \). See Remark 4.7 for \( \chi(O_G) \).

3.1. **Euler characteristic of the exterior quotient of the Frobenius category.** The equation

\[
(3.4) \quad |C_G(H)||\bar{F}_G(H, K)||K| = \sum_{n \in T_G(H, K)} |C_K(H^n)|
\]

follows from Burnside’s counting lemma (Lemma 3.7) applied to the action of \( C_G(H) \times K \) on the transporter set \( N_G(H, K), C_G(H) \times N_G(H, K) \times K \to N_G(H, K); (h, n, k) \to h n k \), with isotropy subgroup \( C_K(H^n) \) at \( n \in N_G(H, K) \). In particular, \( |\bar{F}_G(H)||H| = |F_G(H)||Z(H)| \).

Define \( \bar{S}_G^* \) to be the \( G \)-category with objects the nonidentity \( p \)-subgroups of \( G \) and morphisms

\[
\bar{S}_G^*(H, K) = \begin{cases} C_K(H) & H \leq K \\ \emptyset & H \not\leq K \end{cases}
\]

with composition in \( \bar{S}_G^* \) induced from composition in the group \( G \). Using that \( G \) acts on \( \bar{S}_G^* \) by conjugation, we may rewrite equation (3.4) as

\[
(3.5) \quad |C_G(H)||\bar{F}_G^*(H, K)||K| = |(\bar{S}_G^*)h_G(H, K)|
\]

according to (2.14).

**Corollary 3.6.** \( \chi(\bar{F}_G^*) = \chi(F_G^*) \)

**Proof.** For any nonidentity \( p \)-subgroup \( K \) of \( G \), Theorem 3.2 says that

\[
\sum_{1 < H \leq K} -\mu(H)|C_K(H)| = |K|\chi(F_G^*K) = |K|
\]

because \( \chi(F_G^*K) = 1 \) by Proposition 5.1. By Theorem 2.15.(2) and equation (3.5), \( k_K = -|G|^{-1}|C_G(K)|\mu(K) \) is a coweighting for \( \bar{F}_G^* \). But this function is also a coweighting for \( F_G^* \) by Theorem 3.2 and Table 1. \( \square \)

**Lemma 3.7** (Burnside’s counting lemma). [18] If \( X \) is a finite right \( G \)-set then

\[
\sum_{g \in G} |X^g| = |X/G||G| = \sum_{x \in X} |xG|
\]

where \( X^g \subset X \) is the fixed set for \( g \in G \) and \( xG \leq G \) is the isotropy subgroup for \( x \in X \).

3.2. **Alternative weightings and coweightings.** We shall first reformulate the expressions for the weightings \( \mu^* \) from Table 1 using the Möbius function for \( [T_G^*] \) (Theorem 2.13).

The rational number

\[
(3.8) \quad \lambda(H, K) = \frac{1}{|N_G(H)|} \sum_{L \in [K]} \mu(H, L)
\]

only depends on the conjugacy classes of \( H \) and \( K \). In particular, \( \nu(K) = |N_G(K)|^{-1}\mu(K) \), where \( \nu(K) \) is short for \( \nu(1, K) \).

**Proposition 3.9.** The function \( \mu([H], [K]) \) defined by equation (3.8) is the Möbius function for \( [T_G^*] \).

\( ^1 \)The function \( \nu(H, K) \) is not the same as \( |N_G(H)|^{-1}\lambda(H, K) \) where \( \lambda \) is the Möbius function for the poset of \( p \)-subgroup classes ordered by subconjugation.
The third column of Table 2 is simply the sum $\sum_{[K]} k_{[K]}$ of the coweightings for $[C]$ as in Theorem 2.13. □

Remark 3.11. Define the $\mu$-transporter from $H$ to $K$ to be the set

$$N^p_G(H, K) = \{ g \in G \mid \Phi(K) \leq H^p \leq K \}, \quad H, K \in \text{Ob}(S_G^p)$$

of group elements $g$ that conjugate $H$ into $K$ such that $\mu(H^p, K) \neq 0$.

The map $g \to K^{g^{-1}}$ is a bijection between $N^p_G(H, K)/N_G(K)$ and the set $\{L \in [K] \mid H \leq L, \mu(H, L) \neq 0\}$ of subgroups $L$ of $G$ conjugate to $K$ and containing $H$ with $\mu(H, L) \neq 0$ and therefore

$$[\mu]|([H], [K]) = (-1)^{n} p(n)|\frac{|N^p_G(H, K)|}{|N_G(H)||N_G(K)|}, \quad H, K \in \text{Ob}(F_G^p), \quad |K| = p^n |H|$$

can be computed from these transporter sets.

Next we note that the values of the weightings for the $p$-subgroup categories of Table 1 can be computed locally.

Fix $H$, a nonidentity $p$-subgroup of $G$, and consider the projection $T_G^p(H) = N_G(H) \to N_G(H) = N_G(H)/H = O_G^p(H)$ of the $p$-local subgroup $N_G(H)$ onto its quotient $N_G(H)/H$. The functor

$$(3.12) \quad C_G : (S_{O_G^p(H)})^{op} \to C_{G}(H)$$

takes the nonidentity $p$-subgroup $K$ of $O_G^p(H)$ to the subgroup $C_G(K)$ of $C_G(H)$ where $K \leq N_G(H)$ is the preimage of $\overline{K} \leq N_G(H)/H$. For every $x \in C_G(H),

(3.13) \quad C_G/\langle x \rangle = \{ \overline{K} \in \text{Ob}(S_{O_G^p(H)}) \mid C_G(K) \ni x \} = S_{C_{N_G(H)}(x)}$$

is the preimage under $C_G$ of the subposet $\{ Y : \langle x \rangle \leq Y \leq C_G(H) \}$. Following the bar convention of [9, p 18], we write $\overline{C_{N_G(H)}(x)}$ for the image in $O_G^p(H)$ of the centralizer in $N_G(H)$ of $x \in C_G(H)$. 

Proposition 3.14. The weightings for $S^*_G$, $T^*_G$, and $O^*_G$ are

$$k^H_{S^*} = 1 - \chi(S^*_{O^*_G(H)})$$
$$k^H_{T^*} = \frac{1 - \chi(S^*_{O^*_G(H)})}{|G|},$$
$$k^H_{O^*} = \frac{1 - \chi(S^*_{O^*_G(H)})}{|G: H|}$$

and the weighting for $F^*_G$ is

$$k^H_{F^*} = |G|^{-1} \sum_{x \in C_G(H)} (1 - \chi(C_G/Ax))$$

Proof. Equation (2.11) shows that

$$k^H_{S^*} = \sum_K \mu(H, K) = \sum_{K \in [H, N_G(H)]} \mu(H, K) = 1 - \chi((H, K)) = 1 - \chi((H, N_G(H))) = 1 - \chi(S^*_{O^*_G(H)})$$

as $\mu(H, K) = 0$ unless $H$ is normalized by $K$ (Lemma 3.1). (Indeed, the subposets $(H, K)$ and $(H, N_G(H))$ of $S^*_G$ are homotopy equivalent [20, Proposition 6.1].) Similarly,

$$|G|k^H_{F^*} = \sum_{H \leq K \leq N_G(H)} \mu(H, K)|C_G(K)| = \sum_{H \leq K \leq N_G(H)} \mu(H, K)|C_{N_G(H)}(K)| = \sum_{K \leq O_G(H)} \mu(K)|C_{N_G(H)}(K)|$$

because $C_G(K) = C_{N_G(H)}(K)$ as $C_G(K) \leq C_G(H) \leq N_G(H)$ when $H \leq K \leq N_G(H)$. The sum that occurs in this formula for $|G|k^H_{F^*}$ is the Euler characteristic [17, Proposition 2.8] of the Grothendieck construction for the presheaf $C_G$ (3.12). Since the opposite of this Grothendieck construction is the direct sum [22, Chp 3.2] over $x \in C_G(H)$ of the subposets (3.13) we arrive at the formula that we wanted to prove. \qed

Using the expressions from Proposition 3.14, the Euler characteristics of $S^*_G$, $T^*_G$, and $O^*_G$ are

$$\chi(S^*_G) = \sum_H (1 - \chi(S^*_{O^*_G(H)})),$$
$$\chi(T^*_G) = \sum_{[H]} \frac{1 - \chi(S^*_{O^*_G(H)})}{|T^*_{O^*_G(H)}|},$$
$$\chi(O^*_G) = \sum_{[H]} \frac{1 - \chi(S^*_{O^*_G(H)})}{|O^*_{O^*_G(H)}|}$$

The first of these equation can also be written

$$\sum_H (1 - \chi(S^*_{O^*_G(H)}) + \mu(H)) = 0$$

as $\chi(S^*_G) = \sum_H -\mu(H)$ (Table 1). Similarly, we obtain the alternative formula

$$\chi(F^*_G) = |G|^{-1} \sum_{x \in C_G(H)} (1 - \chi(C_G/Ax))$$

for the Euler characteristic of $F^*_G$. Comparing this new formula with the one from Table 1 we arrive at the combinatorial identity

$$\sum_H \sum_{x \in C_G(H)} (1 - \chi(C_G/Ax) + \mu(H)) = 0$$

where $H$ runs over the set of nonidentity $p$-subgroups of $G$.

Finally, we observe that only $p$-radical $p$-subgroups contribute to the weightings for $S^*_G$, $T^*_G$, and $O^*_G$.

Definition 3.18. The $p$-subgroup $H$ of $G$ is

- $p$-radical if $O_p^*O^*_G(H) = 1$ [4, Proposition 4]
- $F^*_G$-radical if $O_p^*F^*_G(H) = 1$ [6, Definition A.9]

Corollary 3.19. The weightings for $S^*_G$, $T^*_G$, and $O^*_G$ are supported on the nonidentity $p$-radical subgroups of $G$.

Proof. If $O_p^*O^*_G(H) > 1$ then $\chi(S^*_{O^*_G(H)}) = 1$ [20, Proposition 2.4] and the weightings $k^H_H = 0$ for the categories $S^*_G$, $T^*_G$, and $O^*_G$ (Proposition (3.14)). \qed
Consequently, 
\[ \chi(C') = \chi(C^*), \quad C = S_G, T_G, O_G \]
where \( C' \) is the subposet of \( C^* \) of nonidentity \( p \)-radical \( p \)-subgroups. Bouc [3, Corollaire] shows the stronger result that \( S_G^* \) and \( S_C^* \) are homotopy equivalent posets. Thévenaz and Webb [23, Theorem 2.3] describe \( S_G^* \) when \( G \) is simple group of Lie type in defining characteristic \( p \).

We suspect that the weightings for \( S_G^*, T_G^*, \) and \( O_G^* \) are supported precisely on the nonidentity \( p \)-radical \( p \)-subgroups, ie that
\[ (3.20) \]
\[ \chi(S_{O_G^*(H)}) \neq 1 \iff O_p O_G^*(H) = 1 \]
This would be true if the strong Quillen conjecture
\[ (3.21) \]
\[ \chi(S_G^*) \neq 1 \iff O_p G = 1 \]
turns out to be true for all finite groups \( G \). (It is true, as used above, that \( O_p(G) \neq 1 \implies \chi(S_G^*) = 1 \) but the problem is that \( \chi(S_G^*) = 1 \implies O_p(G) \neq 1 \) is only known to hold for \( p \)-solvable groups with abelian Sylow \( p \)-subgroups [14, Lemma 1.1, Theorem A]. The original Quillen conjecture [20, Conjecture 2.9], that \( S_G^* \simeq \ast \implies O_p(G) \neq 1 \), is true when \( G \) is solvable. Also, it is known that \( |G|_p \) divides \( 1 - \chi(S_G^*) \) [7, 20, 24, 13].)

Explicit computations with Magma [Lemma 4.3, Theorem 2.3] describe \( S_G^* \) and \( O_G^* \). Bouc [3, Corollaire] shows the stronger
\[ (3.20) \]
\[ \chi(S_{O_G^*(H)}) \neq 1 \iff O_p O_G^*(H) = 1 \]
when \( G \) is simple group of Lie type in defining characteristic \( p \).

We are not aware of any similar characterization of the support of the weighting for \( F_G^* \).

The two concepts of radical subgroups introduced in Definition 3.18 are unrelated in general [6, Appendix A]. If \( P \) is an abelian nonidentity \( p \)-group, then all subgroups of \( P \) are \( F_G \)-radical but only \( P \) itself is \( p \)-radical. However, if \( H \) is a \( p \)-selfcentralizing \( p \)-subgroup of \( G \) (Definition 8.1) then \( O_p C_G(H) \) is a \( p' \)-group (Lemma 8.2.(1)) and the short exact sequence
\[ 1 \to O_p C_G(H) \to O_G^*(H) \to \bar{F}_G(H) \to 1 \]
can be used to verify the implication
\[ H \text{ is } p \text{-selfcentralizing and } F_G \text{-radical } \iff H \text{ is } p \text{-selfcentralizing and } p \text{-radical} \]
The converse implication does not hold in general: Let \( p = 2 \). The normal cyclic subgroup \( H = O_p G \) of order 4 in the dihedral group \( G = D_{24} \) of order 24 is a \( p \)-selfcentralizing subgroup with \( O_G^*(H) = \Sigma_3 \) and \( \bar{F}_G(H) = C_2 \). Thus \( H \) is \( p \)-radical but not \( F_G \)-radical.

4. Euler characteristics of orbit categories

We shall now derive a more concise expression than the ones given in Table 1 or Table 2 for the Euler characteristic of \( O_G^* \).

**Theorem 4.1.** The Euler characteristic of the orbit category \( O_G^* \) is
\[ \chi(O_G^*) = \chi(T_G^*) + \frac{p - 1}{p} \sum_{C \in \text{Ob}(O_G^*) \text{ cyclic}} |C| \]

**Proof.** The coweighting \( k_G^* \) from Table 1 for \( O_G^* \) multiplied by \( |G| \) is
\[ |G| k_G^* = \sum_{1 < H} |H| \mu(H, K) = -\mu(K) + \sum_{1 < H} |H| \mu(H, K) = \begin{cases} -\mu(K) + \frac{p - 1}{p} |K| & K \text{ is cyclic} \\ -\mu(K) & K \text{ is not cyclic} \end{cases} \]
by Theorem 3.2 and Corollary 4.3. Thus the Euler characteristic of \( O_G^* \) is
\[ \chi(O_G^*) = \frac{1}{|G|} \sum_{K \in \text{Ob}(O_G^*)} -\mu(K) + \frac{p - 1}{p} \sum_{C \in \text{Ob}(O_G^*) \text{ cyclic}} |C| \]
where the first term is the Euler characteristic of the transporter category \( T_G^* \). \( \square \)

Equivalently, the Euler characteristic of the orbit category \( O_G^* \) is
\[ (4.2) \]
\[ \chi(O_G^*) = \chi(T_G^*) + \frac{p - 1}{p} \sum_{C \in \text{Ob}(O_G^*) \text{ cyclic}} \frac{1}{|O_G^*(C)|} \]
where the sum is taken over the set of conjugacy classes of nonidentity cyclic \( p \)-subgroups of \( G \).
Corollary 4.3. For any $K \in \text{Ob}(\mathcal{S}_p)$

\[ \frac{1}{|\Phi(K)|} \sum_{1 \leq H \leq K} |H| \mu(H, K) = \begin{cases} p - 1 & K \text{ is cyclic} \\ 0 & K \text{ is not cyclic} \end{cases} \]

Proof. Suppose that the Frattini quotient $K/\Phi(K)$ is elementary abelian of order $p^n$ for some $n > 0$. Recall that $n = 1$, $K/\Phi(K)$ is cyclic, if and only if $K$ is cyclic [8, Chp 5, Corollary 1.2]. The sum of this corollary,

\[ \sum_{H : \Phi(K) \leq H \leq K} |H| \mu(K/H) = \sum_{d=0}^{n} (-1)^{n-d} \binom{n}{d} p^{(n-d)} = \sum_{d=0}^{n} (-1)^{d} \binom{n}{d} p^{(2)} p^{n-d}, \]

is evaluated in Lemma 4.4. It is nontrivial only if $n = 1$ where it has value $p - 1$. \hfill \Box

The Gaussian $p$-binomial coefficient

\[ \binom{n}{d} = \frac{\prod_{j=1}^{d} (p^n - p^{j-1})}{\prod_{j=1}^{d} (p^{d-p^{j-1}})} = \frac{\prod_{j=1}^{d} (p^{n+1-j} - 1)}{\prod_{j=1}^{d} (p^j - 1)} \]

counts the number of $d$-dimensional subspaces of the $n$-dimensional $\mathbb{F}_p$-vector space $\mathbb{F}_p^n$ [22, 1.3.18].

Lemma 4.4. For any $n \geq 1$,

\[ \sum_{d=0}^{n} (-1)^d \binom{n}{d} p^{(2)} p^{n-d} = \begin{cases} p - 1 & n = 1 \\ 0 & n > 1 \end{cases} \]

Proof. Note first the formulas [22, p 26]

\[ \binom{n}{d} + p^{n-d} \binom{n-1}{d} = 1, \quad \binom{n}{d} = \binom{n}{0}, \quad \binom{2}{1} = 1 + p, \]

for the Gaussian $p$-binomial coefficients.

For $n = 1$ and $n = 2$, the sums we are evaluating are the polynomials

\[ \binom{1}{0} p - \binom{1}{1} = p - 1, \quad \binom{2}{0} p^2 - \binom{2}{1} p = p^2 - (1 + p) p = p = 0 \]

For $n > 2$ the sum has the value

\begin{align*}
\sum_{d=0}^{n} (-1)^d \binom{n}{d} p^{(2)} p^{n-d} &= p^n + \sum_{d=1}^{n-1} (-1)^d \binom{n}{d} p^{(2)} p^{n-d} + (-1)^n p^{(2)} \\
&= p^n + \sum_{d=1}^{n-1} (-1)^d \left( \binom{n-1}{d} + p^{n-d} \binom{n-1}{d-1} \right) p^{(2)} p^{n-d} + (-1)^n p^{(2)} \\
&= \left( p^n + \sum_{d=1}^{n-1} (-1)^d \binom{n-1}{d} p^{(2)} p^{n-d} \right) + \sum_{d=1}^{n-1} (-1)^d \binom{n-1}{d-1} p^{(2)} p^{2(n-d)} + (-1)^n p^{(2)} \\
&= \sum_{d=0}^{n-1} (-1)^d \binom{n-1}{d} p^{(2)} p^{n-d} + \sum_{d=1}^{n-1} (-1)^d \binom{n-1}{d-1} p^{(2)} p^{2(n-d)} \\
&= p^{(2)} p^{n-1-d} + \sum_{d=1}^{n-1} (-1)^d \binom{n-1}{d-1} p^{(2)} p^{2(n-d)} \\
&= \sum_{d=0}^{n-1} (-1)^d \binom{n-1}{d} p^{(2)} p^{2(n-d)} \]

The first term is

\[ \sum_{d=0}^{n-1} (-1)^d \binom{n-1}{d} p^{(2)} p^{2(n-d)} = \sum_{d=0}^{n-1} (-1)^d \binom{n-1}{d} p^{(2)} p^{2(n-d)} \]

and the second term is

\[ \sum_{d=1}^{n} (-1)^d \binom{n-1}{d-1} p^{(2)} p^{2(n-d-2)} = \sum_{d=0}^{n-1} (-1)^d \binom{n-1}{d} p^{(2)} p^{2(n-d-1)} \]
We have now proved the recursive relation
\[
\sum_{d=0}^{n} (-1)^d \binom{n}{d} p(d) p^{n-d} = p(1 - p^{n-2}) \sum_{d=0}^{n-1} (-1)^d \binom{n-1}{d} p(d) p^{n-1-d}
\]
for \( n > 2 \). Since the sum equals 0 for \( n = 2 \), it equals 0 for all \( n \geq 2 \).

Equating the two expressions for \( \chi(O_G^\ast) \) from Theorem 4.1 and (3.15) we arrive at the combinatorial identity
\[
(4.5) \quad \sum_H (\lvert H \rvert - \chi(S_{O_G^\ast(H)}^\ast) \lvert H \rvert + \mu(H)) = \frac{p-1}{p} \sum_C \lvert C \rvert
\]
where \( H \) runs over the set of nonidentity \( p \)-subgroups of \( G \) and \( C \) over the set of nonidentity cyclic \( p \)-subgroups of \( G \).

**Corollary 4.6.** \( |G|_{p'} \chi(O_G^\ast) \) is an integer.

**Proof.** In fact, all values of
\[
|G|_{p'} k^H_G = |G|_{p'} \frac{1 - \chi(S_{N_G(H)}^\ast(H))}{|N_G(H) : H|} = \frac{|G|_{p'}}{|N_G(H) : H|} \frac{1 - \chi(S_{N_G(H)}^\ast(H))}{|N_G(H) : H|} p
\]
are integers because \( |G|_p \), and hence also \( |N_G(H) : H|_p \), divides \( 1 - \chi(S_{N_G(H)}^\ast(H)) \) [7, Corollary 2].

**Remark 4.7** (Euler characteristic of \( O_G \)). The function
\[
k^H_G = \begin{cases} |G|^{-1} - \chi(T_G^\ast) & H = 1 \\ \lvert H \rvert \sum_{K > 1} \mu(H, K) & H > 1 \end{cases}
\]
is a weighting for \( O_G \), and
\[
\chi(O_G) = |G|^{-1} - \chi(T_G^\ast) + \chi(O_G^\ast) = |G|^{-1} + \frac{p-1}{p} \sum_{\lvert C \rvert \in \{F_G^\ast \text{ cyclic}\}} \frac{1}{\lvert O_G^\ast(C) \rvert}
\]
is the Euler characteristic of the orbit category \( O_G \) of \( G \).

5. The range of \( \chi(F_G^\ast) \)

We shall first identify a class of finite groups \( G \) for which \( \chi(F_G^\ast) = 1 \).

**Proposition 5.1.** If \( G \) contains a central nonidentity \( p \)-subgroup then \( \chi(F_G^\ast) = 1 \) and \( \chi(L_G^\ast) = |G:O^pG|^{-1} \).

**Proof.** We concentrate on \( F_G^\ast \) and leave the similar case of \( L_G^\ast \) to the reader. Let \( Z \) be a nonidentity central \( p \)-subgroup of \( G \) and \( Z^+ \) the full subcategory of \( F_G^\ast \) generated by \( p \)-subgroups containing \( Z^+ \). \( Z^+ \) is a deformation retract of \( F_G^\ast \) in the sense that there are functors
\[
Z^+ \xrightarrow{R} L \rightarrow F_G^\ast, \quad 1_{Z^+} = LR, \quad 1_{F_G^\ast} \Rightarrow RL,
\]
where \( R \) is the inclusion functor and \( L \) is the functor that takes \( Q \leq G \) to \( LQ = QZ \) and the \( F_G^\ast \)-morphism \( c_g: P \rightarrow Q \) to \( c_g: LP \rightarrow LQ \) (where \( c_g: x \mapsto x^g \) is conjugation by \( g \in G \)). If \( P \) and \( Q \geq Z \) are nonidentity \( p \)-subgroups of \( G \) then
\[
Z^+(LP, Q) = F_G^\ast(PZ, Q) = C_G(PZ) \lvert N_G(PZ, Q) = C_G(P) \lvert N_G(P, Q) = F_G^\ast(P, Q) = F_G^\ast(P, RQ)
\]
showing that \( L \) and \( R \) are adjoint functors with \( L \dashv R \). By Lemma 2.7, the EI-categories \( Z^+ \) and \( F_G^\ast \) have the same Euler characteristics and, by Example 2.6, \( \chi(Z^+) = 1 \) as \( Z^+ \) has initial object \( Z \).

The converse of Proposition 5.1 is not true as \( \chi(F_G^\ast) = 1 \) and \( Z(G) = 1 \) for \( G = \Sigma_3 \) and \( p = 2 \).

The Euler characteristic \( \chi(F_G^\ast) = |G|^{-1} \sum_K k_K = -|G|^{-1} \sum_K \mu(K)|C_G(K)| \) is a rational number such that \( |G| \chi(F_G^\ast) \) is an integer. We now improve this observation.

**Corollary 5.2.** \( |G|_{p'} \chi(F_G^\ast) \) is an integer.
Proof. In fact, all values of
\[ |G|_{p'} k_{[K]}^P = |G|_p \frac{\mu(K)}{|\mathcal{F}_G^*(K)|} = \frac{\mu(K)}{|\mathcal{F}_G^*(K)|_p'}, \quad K \in \text{Ob}(S_G^*), \quad |K| = p^n, \]
are integers because \(|\mathcal{F}_G^*(K)|_p\) divides \(|\text{Aut}(K)|_p = p^{\mu(K)}\) as \(|\mathcal{F}_G^*(K)|\) divides \(|\text{Aut}(K)|\), and \(|\mathcal{F}_G^*(K)|_p'\) divides \(|G|_{p'}\) as \(|\mathcal{F}_G^*(K)| = |\mathcal{N}_G(K): C_G(K)|\) divides \(|G|\). (Remember that \(\mu(K) = 0\) unless \(K \in \text{Ob}(S_G^*)\) is elementary abelian of order \(|K| = p^n, n > 0\).) \( \square \)

We now show that the computation of the Euler characteristic of the Frobenius category can be reduced to the computation of Euler characteristics of posets.

**Proposition 5.3.** \(\chi(\mathcal{F}_G^*) = |G|^{-1} \sum_{x \in G} \chi(S_{G(x)}^*)\)

**Proof.** Recall that \(S_G\) denotes the poset of all subgroups of \(G\). Note that for any subgroup \(K\) of \(G\)
\[ |C_G(K)| = |\{x \in G \mid x \in C_G(K)\}| = |\{x \in G \mid C_G(x) \geq K\}| = \sum_{x \in G} S_G(K, C_G(x)) \]
and therefore
\[ |G| \chi(\mathcal{F}_G^*) = \sum_{K \in \text{Ob}(S_G^*)} -\mu(K) |C_G(K)| = \sum_{K \in \text{Ob}(S_G^*)} \sum_{x \in G} -\mu(K) S_G(K, C_G(x)) \]
\[ = \sum_{x \in G} \sum_{K \in \text{Ob}(S_G^*)} -\mu(K) S_G(K, C_G(x)) = \sum_{x \in G} \sum_{K \in \text{Ob}(S_G^*)} -\mu(K) = \sum_{x \in G} \chi(S_{G(x)}^*) \]
where the final equality uses the formula from Table 1 for \(\chi(S_G^*)\). \( \square \)

**Corollary 5.4.** Suppose that \(G\) has a normal Sylow \(p\)-subgroup, \(P\), (so that \(G = P \rtimes G/P\)) and let \(k_P^H, H \in \text{Ob}(S_G^*)\), be the weighting for \(\mathcal{F}_G^*\) from Table 1.

1. \(|G| k_P^H = |\{x \in G \mid C_P(x) = H\}|\)
2. \(k_P^H \geq 0\) and \(k_P^H > 0\) if and only if \(H = C_P(x)\) for some \(x \in G\)
3. \(|G| \chi(\mathcal{F}_G^*) = |\{x \in G \mid C_P(x) > 1\}|\)

**Proof.** For any nonidentity \(p\)-subgroup \(K\) and any element \(x \in G\), since \(K \leq P\),
\[ x \in C_G(K) \iff K \leq C_G(K) \iff K \leq P \cap C_G(x) = C_P(x) \]
so that
\[ |G| k_P^H = \sum_K \mu(H, K)|C_G(K)| = \sum_{x \in G} \sum_K \mu(H, K) S_G(K, C_P(x)) = \sum_{x \in G} \delta(H, C_P(x)) \]
by the Möbius inversion formula (2.4). This proves (1) which immediately implies (2) and (3). \( \square \)

**Example 5.5.** Let \(p = 2\) and \(G = P \rtimes C_3\) where the cyclic group \(C_3\) cyclically permutes the three factors of \(P = C_2^3\). Then \(\chi(\mathcal{F}_G^*) = 1\) by Proposition 5.1; indeed, \(k_{\mathcal{F}_G^*}^{Z(G)} = 2/3, k_{\mathcal{F}_G^*}^{P} = 1/3, \) and \(k_{\mathcal{F}_G^*}^{H} = 0\) for all other nonidentity \(2\)-subgroups \(H \leq G\) by Corollary 5.4.

**Corollary 5.6.** Suppose that \(G\) has an abelian Sylow \(p\)-subgroup, \(P\). Then
\[ \chi(\mathcal{F}_G^*) = \frac{|\{\varphi \in \mathcal{F}_G(P) \mid C_P(\varphi) > 1\}|}{|\mathcal{F}_G(P)|} \]

**Proof.** When \(P\) is abelian, \(\mathcal{F}_G(P)\) has order prime to \(p\) and
\[ \mathcal{F}_G = \mathcal{F}_{N_G(P)} = \mathcal{F}_{P \rtimes \mathcal{F}_G(P)} \]
where the first identity is Burnside’s Fusion Theorem [10, Lemma 16.9] which says that \(N_G(P)\) controls \(p\)-fusion in \(G\). For the second equality, observe that all morphisms in the Frobenius category of \(N_G(P)\) extend to automorphisms of \(P\). Now apply Corollary 5.4.(3) to \(P \rtimes \mathcal{F}_G(P)\). \( \square \)
For instance, let $D_{pn}$ be the dihedral group of order $2pn$, $n \geq 1$, $A_p$ the alternating group of order $p > 2$, and $SL_2(F_q)$ the special linear group where $q$ is a power of $p$. Then

$$
\chi(F_{D_{pn}}) = \frac{1}{(2, p-1)}, \quad \chi(F_{A_p}) = \frac{2}{p-1}, \quad \chi(F_{SL_2(F_q)}) = \frac{2, q-1}{q-1}
$$

The computer-generated Table 3 displays Euler characteristics of Frobenius categories at $p = 2$ of small alternating groups. (The Frobenius categories for $A_{2n}$ and $A_{2n+1}$ at $p = 2$ are equivalent.) We do not know if the sequence $\chi(F_{A_p})$ converges.

Example 5.7. The group $H = (C_3 \times C_3) \times C_2$, where $C_2$ swaps the two copies of $C_3$, has an irreducible 4-dimensional representation over $F_2$. Let $G = C_2^4 \times H$ be the associated semi-direct product. Then $|G| = 288$ and $\chi(F_G) = 10/9$ at $p = 2$.

In all the examples we have checked (and also if $G$ has a normal or abelian Sylow $p$-subgroup as in Corollary 5.4 and Corollary 5.6) $\chi(F_G)$ is positive when $p$ divides the order of $G$. Example 5.7 shows that $\chi(F_G)$ can be greater than 1. Prompted by these observations, we would like to pose two questions:

- Is $\chi(F_G)$ always positive when $p$ divides the order of $G$?
- Can $\chi(F_G)$ get arbitrarily large?

6. PRODUCT FORMULAS

We present product formulas for the Euler characteristics of the subgroup poset $S_{G_1 \times G_2}$ and the Frobenius category $F_{G_1 \times G_2}$ for the product of two finite groups $G_1$ and $G_2$.

The formula of Theorem 3.2 for $\chi(S_G^*)$ may be written in the alternative form of (2.11) as

$$
1 - \chi(S_G^*) = \sum_{H \in \text{Ob}(S_G)} \mu(H)
$$

with summation over all $p$-subgroups of $G$. We shall use this expression to derive a formula for the Euler characteristic of the subgroup poset of a direct product of groups.

Theorem 6.1. Let $G_1, \ldots, G_n$ be finite groups. Then

$$
1 - \chi(S_{G_1 \times \cdots \times G_n}^*) = \prod_{i=1}^{n} 1 - \chi(S_{G_i}^*)
$$

Proof. By induction over $n$ it is enough to prove the formula for a product of two groups, $G_1$ and $G_2$. It is then equivalent to

$$
\sum_{H \in \text{Ob}(S_{G_1 \times G_2})} \mu(H) = \sum_{H_1 \in \text{Ob}(S_{G_1})} \mu(H_1) \cdot \sum_{H_2 \in \text{Ob}(S_{G_2})} \mu(H_2)
$$

Let $\pi_1 : G_1 \times G_2 \to G_1$ and $\pi_2 : G_1 \times G_2 \to G_2$ be the projections. The product poset $S_{G_1} \times S_{G_2}$ [22, Chp 3.2] is a deformation retract of $S_{G_1 \times G_2}$ in the sense that there are poset morphisms

$$
S_{G_1} \times S_{G_2} \overset{L}{\longrightarrow} S_{G_1} \times S_{G_2}, \quad 1_{S_{G_1} \times S_{G_2}} = LR, \quad 1_{S_{G_1} \times S_{G_2}} \Rightarrow RL,
$$

where $LH = (\pi_1(H), \pi_2(H))$, $H \leq G_1 \times G_2$, $R(H_1, H_2) = H_1 \times H_2$, $H_1 \leq G_1$, $H_2 \leq G_2$, and $H \leq R(H_1, H_2) \iff LH \leq (H_1, H_2)$.
In this situation
\[ \sum_{H \in \text{Ob}(S_{G_1 \times G_2}) : L_H = (H_1, H_2)} \mu(H) = \mu(H_1)\mu(H_2) \]
for all \( p \)-groups \( H_1 \leq G_1, H_2 \leq G_2 \), and \( H \leq G_1 \times G_2 \) by [17, Proposition 4.4] and the formula [22, Proposition 3.8.2]
\[ \mu_{S_{G_1 \times G_2}}((1, 1), (H_1, H_2)) = \mu(H_1)\mu(H_2) \]
for the Möbius function \( \mu_{S_{G_1 \times G_2}} \) of the product poset \( S_{G_1} \times S_{G_2} \). The theorem now easily follows. \( \square \)

At \( p = 2 \), \( 1 - \chi(S^*_G) = 16 \) and \( 1 - \chi(S^*_G/A_6) = 1 - \chi(F_{C_2}^*) = 0 \), where \( \Sigma_6 = A_6 \rtimes C_2 \) is the permutation group and \( A_6 \) the alternating group. This example shows that Theorem 6.1 does not generalize to semi-direct products.

Theorem 6.1 also follows from work of Quillen. According to [20, Proposition 2.6], \( S_{G_1 \times G_2}^0 \) is homotopy equivalent to the join \( S_{G_1}^0 \star S_{G_2}^0 \) and therefore
\[ 1 - \chi(S^*_G) = 1 - \chi(S^*_{G_1 \times G_2}) = (1 - \chi(S^*_{G_1}))(1 - \chi(S^*_{G_2})) \]
as \( 1 - \chi(X \ast Y) = (1 - \chi(X))(1 - \chi(Y)) \) for any two finite abstract simplicial complexes, \( X \) and \( Y \).

The formula of Theorem 3.2 for \( \chi(F_G^*) \) may be rewritten as
\[ 1 - \chi(F_G^*) = \frac{1}{|G|} \sum_{H \in \text{Ob}(S_G)} \mu(H)|C_G(H)| \]
with summation over all \( p \)-subgroups \( H \) of \( G \). We shall use this expression to derive a formula for the Euler characteristic of the Frobenius category of a direct product of groups.

**Theorem 6.2.** Let \( G_1, \ldots, G_n \) be finite groups. Then
\[ 1 - \chi(F_{\prod_{i=1}^n G_i}^*) = \prod_{i=1}^n 1 - \chi(F_{G_i}^*) \]

**Proof.** By induction over \( n \) it is enough to prove the formula for a product of two groups, \( G_1 \) and \( G_2 \). It is then equivalent to
\[ \sum_{H \in \text{Ob}(S_{G_1 \times G_2})} \mu(H)|C_{G_1 \times G_2}(H)| = \sum_{H_1 \in \text{Ob}(S_{G_1})} \mu(H_1)|C_{G_1}(H_1)| \cdot \sum_{H_2 \in \text{Ob}(S_{G_2})} \mu(H_2)|C_{G_2}(H_2)| \]
But this follows as in the proof of Theorem 6.1 because \( C_{G_1 \times G_2}(H) = C_{G_1}(H_1) \times C_{G_2}(H_2) \) when \( H \leq G_1 \times G_2 \) and \( H_1 = \pi_1(H), H_2 = \pi_2(H) \) are the projections of \( H \). \( \square \)

At \( p = 2 \), \( 1 - \chi(F_{\Sigma_4}^*) = 1/3 \) and \( 1 - \chi(F_{\Sigma_4/A_4}^*) = 1 - \chi(F_{C_2}^*) = 0 \), where \( \Sigma_4 \) is the permutation group and \( A_4 \) the alternating group. This example shows that Theorem 6.2 does not generalize to semi-direct products. (Note also that \( O_2(\Sigma_4) > 1 \) and \( \chi(F_{\Sigma_4}^*) \neq 1 \) in contrast to [14, Lemma 1.1] according to which \( \chi(S^*_G) = 1 \) whenever \( O_p(G) > 1 \).)

7. Variations on Frobenius categories

Recall that \( \overline{S}_G \) is the poset of all subgroups of \( G \). For any two subgroups, \( H \) and \( K \), of \( G \),

\[ \overline{S}_G(H, K) = \begin{cases} 1 & \text{if } H \leq K \\ 0 & \text{if } H \nleq K \end{cases} \]

Writing \( [H] \) for the \( G \)-conjugacy class of \( H \), let
\[ \overline{S}_G([H], K) = \sum_{H \in [H]} \overline{S}_G(H, K) \]
denote the number of subgroups of \( K \) that are \( G \)-conjugate to \( H \). In particular, \( \overline{S}_G([H], G) = |[H]| = |G : N_G(H)| \) is the number of conjugates of \( H \) in \( G \).

We next formulate an alternative expression for the Euler characteristic of a Frobenius category. Let \( P \) be a subgroup of \( G \) of index prime to \( p \), for instance, a Sylow \( p \)-subgroup of \( G \). Write \( P \cap F_G^* \) for the full subcategory of \( F_G^* \) generated by all nonidentity \( p \)-subgroups of \( P \). Then \( P \cap F_G^* \) and \( F_G^* \) are equivalent so they have identical Euler characteristics (Lemma 2.7).
Corollary 7.2. The function
\[ k_K = \frac{-\mu(K)}{|F^*_G(K, P)|}, \quad K \in \text{Ob}(P \cap F^*_G), \]
is a coweighting for \( P \cap F^*_G \) and the Euler characteristic of \( P \cap F^*_G \) is
\[ \chi(P \cap F^*_G) = \sum_{K \in \text{Ob}(P \cap F^*_G)} -\frac{\mu(K)}{|F^*_G(K, P)|} = \chi(F^*_G) \]
with summation over the nonidentity elementary abelian \( p \)-subgroups \( K \) of \( P \).

Proof. As \( k_K = -\mu(K)/|G: C_G(K)| \) is a coweighting for \( F^*_G \) (Table 1), the function
\[ k_K \frac{S_G([K], G)}{S_G([K], P)} = -\frac{\mu(K)}{|F^*_G(K, P)|} = -\frac{1}{||K||} \frac{\mu(K)}{|F^*_G(K)|} \]
is a coweighting for \( P \cap F^*_G \). The identities
\[ S_G([K], G) = |G: N_G(K)| \text{ and } S_G([K], P) = |F_G(K, P)|/|F_G(K)| \]
go into the above equality sign.

Let \( P \) be a finite \( p \)-group and \( F \) an abstract Frobenius category over \( P \) [19, Chp 2] [6]. The objects of \( F \) are the subgroups of \( P \). Define \( F^* \) to be the full subcategory of \( F \) generated by all nonidentity subgroups of \( P \).

Theorem 7.3. The function
\[ k_K = \frac{-\mu(K)}{|F^*(H, K)|}, \quad K \in \text{Ob}(F^*), \]
is a coweighting for \( F^* \) and the Euler characteristic of \( F^* \) is
\[ \chi(F^*) = \sum_{K \in [F^*]} -\frac{\mu(K)}{|F^*(K)|} \]
The Euler characteristic \( \chi(F^*) \in \mathbb{Z}_{(p)} \) is a \( p \)-local integer.

Proof. The Divisibility Axiom [19, 2.3.1] implies that
\[ |F^*(H, K)| = |F^*(H)|S^p([H], K) \]
where \([H] \subset \text{Ob}(F) = \text{Ob}(F_P)\) is the set of \( F \)-objects \( F \)-isomorphic to \( H \) and \( S^p([H], K) = \sum_{H \in [H]} S^p(H, K) \) is the number of \( F \)-objects \( F \)-isomorphic to \( H \) and contained in \( K \). In particular, \( S^p([H], P) = ||H|| \). For any object \( K \) of \( \text{Ob}(F^*) \)
\[ \sum_{H \in \text{Ob}(F^*)} -\frac{\mu(H)}{|F^*(H, P)|} |F^*(H, K)| = \sum_{H \in \text{Ob}(F^*)} -\mu(H)S^p([H], K) = \sum_{|H| \in [F^*]} \sum_{H \in [H]} -\mu(H)S^p([H], K) = \sum_{|H| \in [F^*]} -\mu(H) = \chi(S^p_h) = 1 \]
This shows that
\[ k_K = \frac{-\mu(K)}{|F^*(K, P)|} = -\frac{1}{||K||} \frac{\mu(K)}{|F^*(K)|}, \quad K \in \text{Ob}(F^*), \]
is a coweighting for \( F^* \). The formula for the Euler characteristic follows.

Since \(|F^*(K)|_p \) divides \(|\text{Aut}(K)|_p = \mu(K)\) when \( K \) is elementary abelian (see proof of Corollary 5.2), the Euler characteristic of \( F^* \) is a \( p \)-local integer. \( \Box \)

8. Self-centralizing subgroups

This section deals with the \( p \)-subgroup categories generated by the \( p \)-selfcentralizing subgroups. We mention here some facts to justify our interest in these subcategories of \( p \)-selfcentralizing subgroups:
- The centric linking category \( L^*_G \) is a complete algebraic invariant of the \( p \)-completed classifying space of \( G \) [5, Theorem A]
- The Frobenius category \( F^*_G \) is completely determined by its centric subcategory \( F^*_G \) [19, Chp 4–5]
- All morphisms in the category \( F^*_G \) are epimorphisms [19, Corollary 4.9]
\[ k_{[H]} = |G|^{-1} |H| \sum_{K \in \mathcal{T}_{G}^{C}} |\mu(H, K)|C_{G}(K)|_{p'}^{\mathcal{C}} \quad \text{and} \quad k_{K} = |G|^{-1} |C_{G}(K)|_{p'} \sum_{H \in \mathcal{T}_{G}^{C}} |H| \mu(H, K). \]

\[ k^{[H]} = \frac{|H|}{|G| : N_{G}(H)} \sum_{[K] \in \mathcal{T}_{G}^{C}} |\mu([H], [K])|C_{G}(K)|_{p'}^{\mathcal{C}} \quad \text{and} \quad k^{[K]} = |H| \sum_{[K]} |\mu([H], [K])|C_{G}(K)|_{p'}^{\mathcal{C}}. \]

Table 4. Categories of p-selfcentralizing p-subgroups.

- All morphisms in the category \( \mathcal{F}_{G}^{C} \) have unique maximal extensions [19]

Now follow the definition and a few standard properties of p-selfcentralizing p-subgroups.

**Definition 8.1.** [19, 4.8.1] [6, Definition A.3] The p-subgroup \( H \) of \( G \) is p-selfcentralizing if \( C_{H}(H) \to C_{G}(H) \) is a p-Sylow inclusion.

**Lemma 8.2.** [19, Chp 4] [6, Appendix A] Let \( H \) be a p-subgroup of \( G \) and let \( P \) be a Sylow p-subgroup of \( G \).

1. If \( H \) is p-selfcentralizing, then \( H \) is a p-subgroup of \( G \).

2. If \( H \) is p-selfcentralizing, then \( H \) is a p-subgroup of \( G \).

3. If \( H \) is p-selfcentralizing and \( H^{q} \leq K \) for some \( q \in G \) and some p-subgroup \( K \) of \( G \), then \( K \) is p-selfcentralizing.

4. If \( Q \leq P \) and \( C_{P}(Q) \) is a Sylow p-subgroup of \( C_{G}(Q) \), then \( QC_{P}(Q) \) is p-selfcentralizing.

**Proof.** (1) If \( Z \) is a central Sylow p-subgroup of \( C \) then \( Z = Z \times C/Z \) where \( C/Z = O_{p}C = O^{p}C \).

(2) Assume that \( H \) is p-selfcentralizing. Then \( H^{q} \) is also p-selfcentralizing. The p-subgroup \( C_{P}(H^{q}) \) is contained in the unique Sylow p-subgroup \( Z(H^{q}) \) which is contained in \( H^{q} \). Conversely, assume that \( H \) has property (2). Choose \( g \in N_{G}(H, P) \) so that \( C_{P}(H^{q}) \) is a Sylow p-subgroup of \( C_{G}(H^{q}) \). By assumption, \( C_{P}(H^{q}) = C_{P}(H^{q}) \cap H^{q} = Z(H^{q}) \). This shows that \( Z(H^{q}) \) is a Sylow p-subgroup of \( C_{G}(H^{q}) \).

(3) Let \( h \in N_{G}(K, P) \). Then \( gh \in N_{G}(H, P) \) and \( C_{P}(K^{h}) \leq C_{P}(H^{q}) \) whenever \( h^{q} \leq K \). According to (2), \( K \) is p-selfcentralizing. Since \( Z(H) \) is central in \( C_{H}(H) \) it is the unique Sylow p-subgroup of \( C_{H}(H) \). The p-subgroup \( C_{K}(H^{q}) \) of \( C_{G}(H^{q}) \) is a subgroup of \( Z(H^{q}) \).

Now the chain of inclusions

\[ Z(H^{q}) = C_{H^{q}}(H^{q}) \leq C_{K}(H^{q}) \leq Z(H^{q}) \]

shows that \( Z(H^{q}) = C_{K}(H^{q}) \). Obviously, \( Z(H^{q}) = C_{K}(H^{q}) \geq C_{K}(K) = Z(K) \).

(4) According to [19, Proposition 2.11], \( C_{P}(Q) \leq C_{P}(Q^{q}) \) and \( C_{P}((QC_{P}(Q))^{q}) \leq C_{P}(Q^{q}) \leq C_{P}(Q^{q}) \leq (QC_{P}(Q))^{q} \).

Now apply item (2).

**Theorem 8.3.** Weightings \( k^{*} \), coweighings \( k^{*} \), and Euler characteristics for the finite categories \( \mathcal{T}_{G}^{C} \), \( \mathcal{L}_{G}^{C} \), \( \mathcal{F}_{G}^{C} \), \( \mathcal{O}_{G}^{C} \), and \( \mathcal{F}_{G}^{C} \) of p-selfcentralizing p-subgroups of \( G \) are as in Table 4.

**Proof.** This follows almost immediately from Theorem 2.15. We comment on the most interesting case, \( \mathcal{F}_{G}^{C} \). If \( H \) is p-selfcentralizing and \( K \geq H \), then \( K \) is p-selfcentralizing and \( C_{K}(H) \) is isomorphic to \( Z(H) \). (Lemma 8.2.3). Equality (3.4) simplifies to

\[ |C_{G}(H)|_{p'} |\mathcal{F}_{G}^{C}(H, K)| |K| = |\mathcal{T}_{G}^{C}(H, K)| \]

so that the functions

\[ k^{[H]} = |G|^{-1} |H| \sum_{K \in \mathcal{T}_{G}^{C}} |\mu(H, K)|C_{G}(K)|_{p'}^{\mathcal{C}} \quad \text{and} \quad k^{[K]} = |G|^{-1} |C_{G}(K)|_{p'} \sum_{H \in \mathcal{T}_{G}^{C}} |H| \mu(H, K) \]

are a weighting and a coweighting for \( \mathcal{F}_{G}^{C} \) by Theorem 2.15. Rewriting the weighting as

\[ k^{[H]} = \frac{|H|}{|G| : N_{G}(H)} \sum_{[K] \in \mathcal{T}_{G}^{C}} |\mu([H], [K])|C_{G}(K)|_{p'}^{\mathcal{C}} \quad \text{and} \quad k^{[K]} = |H| \sum_{[K]} |\mu([H], [K])|C_{G}(K)|_{p'}^{\mathcal{C}} \]
we calculate the Euler characteristic

\[ \chi(\tilde{F}_G) = \sum_{[H]} k[H] = \sum_{[H]} |H| \mu([H], [K]) |C_G(K)|_{p'} \]

as the sum of the values of the weighting.

The function \(|\mu|\) of Table 2 is the Möbius function of \(|T_G^c|\) (Proposition 3.9). Because the \(p\)-selfcentralizing property is upward closed (Lemma 8.2.3) the Möbius for \(|T_G^c|\) is simply the restriction of the Möbius function for \(|T_G|\). Also, the weightings for \(C^c_G\) from Table 4 are the restrictions of the weightings for \(C^c_G\) for \(\mathcal{C} = \mathcal{T}, \mathcal{L}, \mathcal{F}, \mathcal{O}\) from Table 2.

We next note that the weightings for \(L^c_G\) and \(F^c_G\) can be computed locally (cf Proposition 3.14).

Fix \(H\), a \(p\)-selfcentralizing \(p\)-subgroup of \(G\), and consider the projection \(\tilde{T}_G(H) = N_G(H) \rightarrow \tilde{N}_G(H) = N_G(H)/H = \mathcal{O}^c_G(H)\) of the \(p\)-local subgroup \(N_G(H)\) onto its quotient \(N_G(H)/H\). The functor

\[ O^pC_G: (\mathcal{S}_{C^c_G(H)})^{op} \rightarrow \mathfrak{S}_{O^pC_G(H)} \]

takes the nonidentity \(p\)-subgroup \(K\) of \(\mathcal{O}^c_G(H)\) to the subgroup \(O^pC_G(K)\) of \(O^pC_G(H)\) where \(K \leq N_G(H)\) is the preimage of \(K\) under \(C_G\). For every \(x \in O^pC_G(H)\),

\[ O^pC_G/\langle x \rangle = \{ K \in \text{Ob}(\mathcal{S}_{O^c_G(H)}) \mid O^pC_G(K) \ni x \} \]

is the preimage of the subposet \(\{Y \mid \langle x \rangle \leq Y \leq O^pC_G(H)\}\) under the functor \(O^pC_G\).

**Proposition 8.6.** The value of the weighting for \(F^c_G\) at the \(p\)-selfcentralizing \(p\)-subgroup \(H \leq G\) is

\[ k^H_{F_G} = |G|^{-1} \sum_{x \in O^pC_G(H)} (1 - \chi(O^pC_G/\langle x \rangle)) \]

**Proof.** By Table 4 and (3.8) the weighting for \(F^c_G\) at the \(p\)-selfcentralizing \(p\)-subgroup \(H \leq G\) is given by

\[ |G:H|k^H_{F_G} = \sum_{K \in \mathcal{H}(H)} \mu(H, K) |C_G(K)|_{p'} \]

For any nonidentity \(p\)-subgroup \(H \leq G\), there is a commutative diagram

\[
\begin{array}{c}
Z(H) \longrightarrow H \longrightarrow H/Z(H) \\
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \\
C_G(H) \longrightarrow \tilde{T}_G(H) \longrightarrow \tilde{F}_G(H) \\
\downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow \\
C_G(H)/Z(H) \longrightarrow \mathcal{O}^c_G(H) \longrightarrow \tilde{F}_G(H)
\end{array}
\]

with exact rows and columns. Let \(K\) be a \(p\)-subgroup such that \(H \leq K \leq N_G(H)\). Then \(C_G(K) \leq C_G(H) \leq N_G(H)\) so that \(C_G(K) = C_{N_G(H)}(K)\). In case \(H\) is a \(p\)-selfcentralizing subgroup of \(G\), the chain of inequalities, obtained using Lemma 8.2.3,

\[ Z(K) \leq Z(H) \cap C_G(K) \leq H \cap C_G(K) \leq K \cap C_G(K) = Z(K) \]
is, in fact, a chain of identities so that \( Z(K) = Z(H) \cap C_G(K) = H \cap C_G(K) \). The projection \( T_G(H) = N_G(H) \to \mathcal{O}_G(H) = N_G(H)/H \) takes \( C_G(K) \) to \( C_G(K)/H \) with kernel \( H \cap C_G(K) = Z(K) \), the Sylow \( p \)-subgroup of \( C_G(K) \). Thus \( \mathcal{O}_G(K) = C_G(K)/Z(K) = O^p C_G(K) \) and \( |C_G(K)| = |C_G(K)|_{p'} \).

This means that
\[
|G: H|k^H_{f^G} = \sum_{R \in \mathcal{O}_G(H)} \mu(R)|C_G(K)| = |C_G(H)| - \sum_{1 \leq R \leq \mathcal{O}_G(H)} -\mu(R)|C_G(K)|
\]

We now proceed as in the proof of Proposition 3.14. The sum in the above formula is the Euler characteristic of the Grothendieck construction on the presheaf \( O^p C_G \). The opposite of this Grothendieck construction is the direct sum over \( x \in O^p C_G(H) \) of the of the posets \((8.5)\).

Based on explicit computations we suspect, first, that \( \chi(\mathcal{F}_G) = \chi(\mathcal{F}_G^x) \) (cf Corollary 3.6) and, second, that the weighting for \( \mathcal{F}_G^x \) is supported precisely on the \( p \)-selfcentralizing \( \mathcal{F}_G \)-radical subgroups (Definition 3.18), i.e.

\[
\sum_{x \in O^p C_G(H)} \chi(O^p C_G(x)) \neq |O^p C_G(H)| \iff O^p \mathcal{F}_G^x(H) = 1
\]

holds for any \( p \)-selfcentralizing subgroup \( H \) of any group \( G \), cf (3.20). Explicit computations with Magma [2] reveal that \((8.7)\) true at \( p = 2 \) for all groups of order \( \leq 760 \) and for the alternating groups \( A_n \), \( 4 \leq n \leq 13 \), of Table 5.

**Lemma 8.8.** Let \( H \) and \( K \) be two nonidentity \( p \)-subgroups of \( G \). Then \( H \) and \( K \) are isomorphic in \( \mathcal{F}_G^x \) if and only if they are isomorphic in \( \mathcal{F}_G \).

**Proof.** Suppose that \( H \) and \( K \) are isomorphic in \( \mathcal{F}_G^x \). Then there exist \( x \in N_G(H,K) \), \( y \in N_G(K,H) \) so that conjugation by \( xy \) is an inner automorphism of \( H \) and conjugation by \( yx \) is an inner automorphism of \( K \). By replacing \( y \) by another element of \( yH \), if necessary, we obtain that \( yx \in C_G(H) \). Then \( yx = yx^x \in C_G(H)^x = C_G(K) \). This means that \( xy \) represents the identity of \( \mathcal{F}_G^x(H) \) and \( yx \) represents the identity of \( \mathcal{F}_G^x(K) \).

If \( P \) is a nonidentity \( p \)-group then
\[
\chi(S_P^x) = 1, \quad \chi(T_P^x) = |P|^{-1}, \quad \chi(L_P^x) = |P|^{-1}, \quad \chi(F_P^x) = 1, \quad \chi(O_P^x) = 1, \quad \chi(\mathcal{F}_G^x) = 1
\]

because \( S_P^x, O_P^x \), and \( \mathcal{F}_G^x \) have \( P \) as terminal object and \( L_P^x = T_P^x \) is the Grothendieck construction for the \( P \)-action on \( S_P^x \). Corollary 5.4.(2) shows that the weighting for \( \mathcal{F}_P^x \) and is supported on the subgroups of the form \( C_P(x), x \in P \), so that \( \chi(\mathcal{F}_P^x) = \chi(F_P^x) = 1 \) as these subgroups are \( p \)-selfcentralizing by Lemma 8.2.(4). (By Example 5.5 it is not true for general groups \( G \) that the weighting for \( \mathcal{F}_G^x \) is supported on the \( p \)-selfcentralizing subgroups and Tables 3 and 5 contain several examples of alternating groups where the nonidentity and the centric Frobenius categories have different Euler characteristics.)

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Institut for Matematiske Fag, Universitetsparken 5, DK–2100 København E-mail address: tau.wedel@gmail.com

Institut for Matematiske Fag, Universitetsparken 5, DK–2100 København E-mail address: moller@math.ku.dk

URL: http://www.math.ku.dk/~moller