A NEW LOOK AT THE HARDY-LITTLEWOOD-PÓLYA
INEQUALITY OF MAJORIZATION

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Abstract. The Hardy-Littlewood-Pólya inequality of majorization is extended
to the framework of ordered Banach spaces. Several applications illustrating
our main results are also included.

1. Introduction

In their celebrated book on Inequalities, G. H. Hardy, J. E. Littlewood and G.
Pólya [11] have proved an important characterization of convex functions in terms
of a preorder of vectors in \( \mathbb{R}^N \) called majorization. Given two vectors \( x \) and \( y \)
in \( \mathbb{R}^N \), we say that \( x \) is weakly majorized by \( y \) (denoted \( x \prec_{wHLP} y \)) if their
decreasing rearrangements, respectively \( x_1^\downarrow \geq \cdots \geq x_N^\downarrow \) and \( y_1^\downarrow \geq \cdots \geq y_N^\downarrow \)
verify the inequalities

\[
\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow \quad \text{for } k = 1, \ldots, N;
\]

we say that \( x \) is majorized by \( y \) (denoted \( x \prec_{HLP} y \)) if in addition

\[
\sum_{i=1}^{N} x_i^\downarrow = \sum_{i=1}^{N} y_i^\downarrow
\]

The basic result relating majorization to convexity is the Hardy-Littlewood-Pólya
inequality of majorization:

Theorem 1. (Hardy-Littlewood-Pólya [11]) If \( x \prec_{HLP} y \), then

\[
\sum_{k=1}^{N} f(x_k) \leq \sum_{k=1}^{N} f(y_k)
\]

for every real-valued continuous convex function \( f \) defined on an interval that con-
tains the components of \( x \) and \( y \). Conversely, if the inequality (1.3) holds for every
real-valued continuous convex function defined on an interval including the compo-
ents of \( x \) and \( y \), then \( x \prec_{HLP} y \).
The inequality (1.3) still works when $f$ is a nondecreasing convex function and $x \prec_{wHLP} y$. This important remark, due independently to Tomič and Weyl, can be derived directly from Theorem 1. See [15] and [19].

As we briefly noticed in [18], the Hardy-Littlewood-Pólya inequality can be extended to the framework of ordered Banach spaces alongside an argument that can be traced back to [14]. The aim of the present paper is to prove much more general results and to show that they are best possible (that is, no such theorems exist with less restrictions than ours).

The necessary background on ordered Banach spaces can be covered from [18]. Additional information is available in the classical books of Aliprantis and Tourky [1] and Meyer-Nieberg [16].

According to Choquet’s theory (see [19] and [22]), the right framework for developing the majorization theory is that of probability measures. In the case of the Hardy-Littlewood-Pólya preorder relation $\prec_{HLP}$ this can be done simply by identifying each vector $x = (x_1, ..., x_N)$ in $\mathbb{R}^N$ with the discrete probability measure $(1/N) \sum_{k=1}^N \delta_{x_k}$ acting on $\mathbb{R}$; as usually $\delta_{x_k}$ denotes the Dirac measure concentrated at $x_k$. We put

$$\frac{1}{N} \sum_{k=1}^N \delta_{x_k} \prec_{HLP} \frac{1}{N} \sum_{k=1}^N \delta_{y_k}$$

with the same understanding as $x \prec_{HLP} y$. Under these terms, the Hardy-Littlewood-Pólya inequality of majorization can be rephrased as

$$\mu \prec_{HLP} \nu \text{ if and only if } \int_I f d\mu \leq \int_I f d\nu$$

for every real-valued continuous and convex function $f$ whose domain of definition is an interval $I$ that includes the supports of the discrete probability measures $\mu$ and $\nu$.

Since in an ordered Banach space not every string of elements admits a decreasing rearrangement, in this paper we will concentrate to the case of pairs of discrete probability measures of which at least one of them is supported by a monotone string of points. The case where the support of the left measure consists of a decreasing string is defined as follows.

**Definition 1.** Suppose that $\sum_{k=1}^N \lambda_k \delta_{x_k}$ and $\sum_{k=1}^N \lambda_k \delta_{y_k}$ are two discrete Borel probability measures that act on the ordered Banach space $E$. We say that $\sum_{k=1}^N \lambda_k \delta_{x_k}$ is weakly $L^1$-majorized by $\sum_{k=1}^N \lambda_k \delta_{y_k}$ (denoted $\sum_{k=1}^N \lambda_k \delta_{x_k} \prec_{wL^1} \sum_{k=1}^N \lambda_k \delta_{y_k}$) if the left hand measure is supported by a decreasing string of points

\begin{equation}
(1.4) \quad x_1 \geq \cdots \geq x_N
\end{equation}

and

\begin{equation}
(1.5) \quad \sum_{k=1}^n \lambda_k x_k \leq \sum_{k=1}^n \lambda_k y_k \quad \text{for all } n \in \{1, \ldots, N\}.
\end{equation}
We say that \( \sum_{k=1}^N \lambda_k \delta_{x_k} \) is \( L^1 \)-majorized by \( \sum_{k=1}^N \lambda_k \delta_{y_k} \) (denoted \( \sum_{k=1}^N \lambda_k \delta_{x_k} \prec_{L^1} \sum_{k=1}^N \lambda_k \delta_{y_k} \)) if in addition

\[
\sum_{k=1}^N \lambda_k x_k = \sum_{k=1}^N \lambda_k y_k.
\]

Notice that the context of Definition 1 makes necessary that all weights \( \lambda_1, \ldots, \lambda_N \) belong to \((0, 1]\) and \( \sum_{k=1}^N \lambda_k = 1 \).

The three conditions (1.4), (1.5) and (1.6) imply \( y_N \leq x_N \leq x_1 \leq y_1 \) but not the ordering \( y_1 \geq \cdots \geq y_N \). For example, when \( N = 3 \), one may choose

\[
\lambda_1 = \lambda_2 = \lambda_3 = 1/3, \quad x_1 = x_2 = x_3 = x
\]

and

\[
y_1 = x, \quad y_2 = x + z, \quad y_3 = x - z
\]

where \( z \) is any positive element.

Under these circumstances it is natural to introduce the following companion to Definition 1, involving the ascending strings of elements as support for the right hand measure.

**Definition 2.** The relation of weak \( R^\uparrow \)-majorization,

\[
\sum_{k=1}^N \lambda_k \delta_{x_k} \prec_{wR^\uparrow} \sum_{k=1}^N \lambda_k \delta_{y_k},
\]

between two discrete Borel probability measures means the fulfillment of the condition (1.5) under the presence of the ordering \( y_1 \leq \cdots \leq y_N \);

assuming in addition the condition (1.6), we say that \( \sum_{k=1}^N \lambda_k \delta_{x_k} \) is \( R^\uparrow \)-majorized by \( \sum_{k=1}^N \lambda_k \delta_{y_k} \) (denoted \( \sum_{k=1}^N \lambda_k \delta_{x_k} \prec_{R^\uparrow} \sum_{k=1}^N \lambda_k \delta_{y_k} \)).

When every element of \( E \) is the difference of two positive elements, the weak majorization relations \( \prec_{wL^1} \) and \( \prec_{wR^\uparrow} \) can be augmented so to obtain majorization relations.

The corresponding extensions of the Hardy-Littlewood-Pólya inequality of majorization for \( \prec_{wL^1}, \prec_{L^1}, \prec_{wR^\uparrow} \) and \( \prec_{R^\uparrow} \) make the objective of two theorems in Section 4. The first one, Theorem 4, deals with Gâteaux differentiable convex functions whose differentials are isotone (that is, order preserving). The second one, Theorem 6, extends the conclusion of the preceding theorem to a nondifferentiable framework involving convex functions defined on open \( N \)-dimensional box of \( \mathbb{R}^N \) which verify a condition of monotonicity à la Popoviciu [23] (called by us 2-box monotonicity). This is done via the approximation Theorem 3, whose proof makes the objective of Section 3.

Unlike the case of functions of one real variable, when the isotonicity of the differential is automatic, for several variables, while this is not necessarily true in the case of a differentiable convex function of a vector variable. See Remark 3. Remarkably, the isotonicity of the differential is not only a sufficient condition for the validity of our differentiable generalization of the Hardy-Littlewood-Pólya theorem, but also a necessary one. See Remark 5.

For the convenience of the reader we review in Section 2 some very basic results concerning the various classes of convex or convex like functions and the gradient
inequalities they generate. This section also includes several significant examples of differentiable convex functions with isotonone differentials.

Not entirely surprising, the inequalities of majorization may occur outside the class of convex functions. This is illustrated by Theorem 6, that deals with the case of strongly smooth functions.

Applications of the above majorization theorems include the isotonicity of Jensen’s gap, a general form of the parallelogram law and also the extension of several classical inequalities to the setting of convex functions of a vector variable. They are all presented in Section 5.

2. Classes of Convex Functions

In what follows $E$ and $F$ are two ordered Banach spaces and $\Phi : C \to F$ is a function defined on a convex subset of $E$.

The function $\Phi$ is said to be a perturbed convex function with modulus $\omega : [0, \infty) \to F$ (abbreviated, $\omega$-convex function) if it verifies an estimate of the form

$$(2.1) \quad \Phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\Phi(x) + \lambda\Phi(y) - \lambda(1 - \lambda)\omega(\|x - y\|),$$

for all $x, y$ in $C$ and $\lambda \in (0, 1)$. The usual convex functions represent the particular case when $\omega$ is identically 0. Every $\omega$-convex function associated to a modulus $\omega \geq 0$ is necessarily convex. When $\omega$ is allowed to take negative values, there are $\omega$-convex functions which are not convex. See the case of semiconvex functions, described below.

The $\omega$-convex functions whose moduli $\omega$ are strictly positive except at the origin (where $\omega(0) = 0$) are usually called uniformly convex. In their case the inequality (2.1) is strict whenever $x \neq y$ and $\lambda \in (0, 1)$. More information on this important class of convex functions is available in [3], [5], [6], [26] and [27].

By changing $\Phi$ to $-\Phi$ one obtain the notions of $\omega$-concave function and uniformly concave function.

Remark 1. Much of the study of perturbed convex functions with vector values can be reduced to that of real-valued functions. Indeed, in any ordered Banach space $F$ with generating cone, any inequality of the form $u \leq v$ is equivalent to $w^*(u) \leq w^*(v)$ for all $w^* \in F^*_+$. See [18], Lemma 1 (c). As a consequence, a function $\Phi : C \to F$ is $\omega$-convex if and only if $w^* \circ \Phi$ is $(w^* \circ \omega)$-convex whenever $w^* \in F^*_+$.

There are several variants of convexity that play a prominent role in convex optimization, calculus of variations, isoperimetric inequalities, Monge–Kantorovich theory of transport etc. Some of them are mentioned in what follows.

A real-valued function $\Phi$ defined on a convex subset $C$ of $\mathbb{R}^N$ is called $\alpha$-strongly convex functions (that is, strongly convex with parameter $\alpha > 0$) if $\Phi - (\alpha/2)\|\cdot\|^2$ is convex. The function $\Phi$ is called $\beta$-semiconvex (that is, semiconvex with parameter $\beta > 0$) if it becomes convex after the addition of $(\beta/2)\|\cdot\|^2$. Equivalently, these are the functions that verify respectively estimates of the form

$$(2.2) \quad \Phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\Phi(x) + \lambda\Phi(y) - \frac{1}{2}\lambda(1 - \lambda)\alpha \|x - y\|^2$$

and

$$(2.3) \quad \Phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\Phi(x) + \lambda\Phi(y) + \frac{1}{2}\lambda(1 - \lambda)\beta \|x - y\|^2,$$
for all \( x, y \) in \( C \) and \( \lambda \in (0, 1) \). By changing \( \Phi \) to \(-\Phi\) one obtain the notions of \( \alpha\)-strong concavity and \( \beta\)-semiconcavity.

Under the presence of Gâteaux differentiability, each of the above classes of functions generates specific gradient inequalities that play a prominent role in our generalization of the Hardy-Littlewood-Pólya inequality of majorization.

**Lemma 1.** Suppose that \( C \) is an open convex subset of \( E \) and \( \Phi : C \to F \) is a function both Gâteaux differentiable and \( \omega\)-convex. Then
\[
\Phi(x) - \Phi(a) \geq \Phi'(a)(x - a) + \omega(\|x - a\|)
\]
for all points \( a \in C \) and \( x \in C \).

As is well known, if \( C \) is an open convex subset of \( \mathbb{R}^N \), then a twice continuously differentiable function \( \Phi : C \to \mathbb{R} \) is \( \alpha\)-strongly convex (respectively \( \beta\)-semiconvex) if and only if its Hessian matrix verifies the inequality \( \nabla^2 \Phi \geq \alpha I \) (respectively \( \nabla^2 \Phi \geq -\beta I \)). However, valuable characterizations are possible with less smoothness.

A continuously differentiable function \( \Phi : C \to \mathbb{R} \) defined on an open convex subset of \( \mathbb{R}^N \) is said to be \( \sigma\)-strongly smooth if its gradient is \( \sigma\)-Lipschitz, that is,
\[
\|\nabla \Phi(x) - \nabla \Phi(y)\| \leq \sigma \|x - y\| \quad \text{for all } x, y \in C.
\]
Notice that every \( \sigma\)-strongly smooth function \( \Phi \) verifies the following variant of the gradient inequality:
\[
\Phi(y) - \Phi(x) \leq \Phi'(x)(y - x) + \frac{1}{2\sigma} \|y - x\|^2
\]
for all \( x, y \) in \( C \). See [8], Lemma 3.4, p. 267.

**Lemma 2.** If \( \Phi \) is simultaneously convex and \( \sigma\)-strongly smooth, then
\[
\frac{1}{2\sigma} \|\Phi'(y) - \Phi'(x)\| \leq \Phi(y) - \Phi(x) - \Phi'(x)(y - x) \leq \frac{1}{2\sigma} \|y - x\|^2.
\]

**Proof.** The left-hand side inequality is just Lemma 3.5 in [8], p. 268, while the right-hand side inequality is a restatement of the inequality (2.5).

An important source of strongly smooth convex functions is offered by the following result:

**Lemma 3.** If \( \Phi \) is an \( \alpha\)-strongly convex function, then its Legendre-Fenchel conjugate
\[
\Phi^*(x^*) = \sup \{ x^*(x) - \Phi(x) : x \in C \}
\]
is an \((1/\alpha)\)-strongly smooth function and also a convex function. In particular, \( \Phi^* \) is defined and differentiable on the whole dual space \( E^* \).

For details, see [25], Lemma 15, p. 126. The converse also works. See [13], Theorem 6.

The connection between \( \sigma\)-strong smoothness and semiconvexity is outlined by the following theorem.

**Theorem 2.** (a) Suppose that \( C \) is an open convex subset of \( \mathbb{R}^N \). If \( \Phi : C \to \mathbb{R} \) is a \( \sigma\)-strongly smooth function, then \( \Phi + \frac{\sigma}{2} \|\cdot\|^2 \) is convex and \( \Phi - \frac{\sigma}{2} \|\cdot\|^2 \) is concave.

(b) Conversely, if \( \Phi : C \to \mathbb{R} \) is a function simultaneously semiconvex and semiconcave with parameter \( \sigma > 0 \), then \( \Phi \) is \( \sigma\)-strongly smooth.
The details are available in the book of Cannarsa and Sinestrari [9]; the assertion (a) follows from Proposition 2.1.2, p. 30, while the assertion (b) is motivated by Corollary 3.3.8, p. 61.

As was noticed by Amann [2], Proposition 3.2, p. 184, the Gâteaux differentiability offers a convenient way to recognize the property of isotonicity of functions acting on ordered Banach spaces: the positivity of the differential. In the case of convex functions his result can be stated as follows:

**Lemma 4.** Suppose that $E$ and $F$ are two ordered Banach space, $C$ is a convex subset of $E$ with nonempty interior $\text{int} C$ and $\Phi : C \to F$ is a convex function, continuous on $C$ and Gâteaux differentiable on $\text{int} C$. Then $\Phi$ is isotone on $C$ if and only if $\Phi'(a) \succeq 0$ for all $a \in \text{int} C$.

**Proof.** The "only if" part follows immediately from the definition of the Gâteaux derivative. For the other implication, notice that the gradient inequality mentioned by Lemma 2 shows that $\Phi$ is isotone on $\text{int} C$ if $\Phi'(a) \succeq 0$ for all $a \in \text{int} C$. As concerns the isotonicity on $C$, that follows by an approximation argument. Suppose that $x, y \in C$ and $x \leq y$. For $x_0 \in \text{int} C$ arbitrarily fixed and $t \in [0, 1)$, both elements $u_t = x_0 + t(x - x_0)$ and $v_t = x_0 + t(y - x_0)$ belong to $\text{int} C$ and $u_t \leq v_t$. Moreover, $u_t \to x$ and $v_t \to y$ as $t \to 1$. Passing to the limit in the inequality $\Phi(u_t) \leq \Phi(v_t)$ we conclude that $\Phi(x) \leq \Phi(y)$.

**Remark 2.** If the ordered Banach space $E$ has finite dimension, then the statement of Lemma 4 remains valid by replacing the interior of $C$ by the relative interior of $C$. See [9], Exercise 6, p. 81.

A key ingredient in our extension of the Hardy-Littlewood-Pólya inequality is the isotonicity of the differentials of the functions involved. Unlike the case of differentiable convex functions of one variable, the isotonicity of the differential is not mandatory for the differentiable convex functions of several variables.

**Remark 3.** (A difference between the differentiable convex functions of one real variable and those of several variables) The twice continuously differentiable function

$$
\Phi(x, y) = -2 \left( xy \right)^{1/2}, \quad (x, y) \in \mathbb{R}_{++}^2,
$$

is convex due to the fact that its Hessian,

$$
H = \frac{1}{2} \begin{pmatrix} x^{-3/2}y^{1/2} & -x^{-1/2}y^{-1/2} \\ -x^{-1/2}y^{-1/2} & x^{1/2}y^{-3/2} \end{pmatrix},
$$

is a positive semidefinite matrix. However, unlike the case of convex functions of one real variable, the differential of $\Phi$,

$$
d\Phi : \mathbb{R}_{++}^2 \to \mathbb{R}^2, \quad d\Phi(x, y) = -(x^{-1/2}y^{1/2}, x^{1/2}y^{-1/2}),
$$

is not isotone. Indeed, at the points $(1, 1) < (2, 1)$ in $\mathbb{R}_{++}^2$ we have

$$
d\Phi(1, 1) = -(1, 1) \quad \text{and} \quad d\Phi(2, 1) = -(1/\sqrt{2}, \sqrt{2})
$$

and these values are not comparable.

On the other hand, a simple example of nonconvex differentiable function whose differential is isotone is provided by the function

$$
H(x, y) = (2x - 1)(2y - 1), \quad (x, y) \in \mathbb{R}^2.
$$
Using the aforementioned result of Amann, one can easily prove the following criterion of isotonicity of the differentials.

**Lemma 5.** Suppose that \( C \) is an open convex subset of the Banach lattice \( \mathbb{R}^N \) and \( \Phi : C \to \mathbb{R} \) is a continuous function which is twice Gâteaux differentiable. Then \( \Phi' \) is isotone on \( C \) if (and only if) all partial derivatives of second order of \( \Phi \) are nonnegative.

When \( \Phi \) is also convex, the isotonicity of \( \Phi' \) is equivalent to the condition that all mixed derivatives \( \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \) are nonnegative.

In the light of Lemma 5, the example exhibited in Remark 3, shows that the property of positive definiteness of the Hessian matrix does not necessarily imply its positivity as a linear map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \).

Several examples of differentiable functions which are isotone and/or admit isotone differentials are presented in the Appendix of this paper.

### 3. An Approximation Result

One can characterize the isotonicity of the differential of a convex function by using the concept of 2-box monotonicity, first noticed by Popoviciu [23] in the case when \( N = 2 \). See also [10], where the 2-box monotonicity is described in its relationship with another concept due to Popoviciu, 2-box convexity. The natural domains of such functions are the open \( N \)-dimensional boxes, that is, the products \( \prod_{k=1}^{N} (a_k, b_k) \) of \( N \) open intervals.

**Definition 3.** A real-valued function \( \Phi \) defined on an open and solid subset \( C \) of the Banach lattice \( \mathbb{R}^N \) (\( N \geq 2 \)) is 2-box monotone if the increment of \( \Phi \) over every nondegenerate 2-dimensional box

\[
B_{ij} = \{ u_1 \} \times \cdots \times [v_i, w_i] \times \cdots \times \{ u_k \} \times \cdots \times [v_j, w_j] \times \cdots \times \{ u_N \}, \quad 1 \leq i < j \leq N,
\]

included in \( C \) and parallel to one of the planes of coordinates is nonnegative, that is,

\[
\Delta(\Phi; B_{ij}) = \Phi(u_1, ..., v_i, ..., v_j, ..., u_N) - \Phi(u_1, ..., v_i, ..., w_j, ..., u_N) - \Phi(u_1, ..., w_i, ..., v_j, ..., u_N) + \Phi(u_1, ..., w_i, ..., w_j, ..., u_N) \geq 0.
\]

The property of isotonicity of the differential of a convex function (of two or more variables) is equivalent to the property of 2-box monotonicity for the given function. When \( \Phi \) is twice continuously differentiable, this follows directly from Lemma 5. Indeed, for \( i < j \),

\[
\int_{v_i}^{w_i} \int_{v_j}^{w_j} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} (u_1, ..., x_i, ..., x_j, ..., u_N) dx_i dx_j = \Phi(u_1, ..., v_i, ..., v_j, ..., u_N) - \Phi(u_1, ..., v_i, ..., w_j, ..., u_N) - \Phi(u_1, ..., w_i, ..., v_j, ..., u_N) + \Phi(u_1, ..., w_i, ..., w_j, ..., u_N).
\]

Remarkably, the continuous differentiability of \( \Phi \) suffices as well.

**Lemma 6.** Suppose that \( C \) is an open box of the Banach lattice \( \mathbb{R}^N \) and \( \Phi : C \to \mathbb{R} \) is a continuously differentiable convex function. Then \( \Phi' \) is isotone on \( C \) if (and only if) \( \Phi \) is 2-box monotone.
Proof. The fact that \( \Phi' \) is isotone is equivalent to
\[
(3.1) \quad \frac{\partial \Phi}{\partial x_k}(u_1, \ldots, u_N) \leq \frac{\partial \Phi}{\partial x_k}(v_1, \ldots, v_N)
\]
for all indices \( k \in \{1, \ldots, N\} \) and all pairs of points \( u = (u_1, \ldots, u_N) \leq v = (v_1, \ldots, v_N) \) in \( C \).

Since \( \Phi \) is differentiable and convex in each variable, we have
\[
\frac{\partial \Phi}{\partial x_k}(u_1, \ldots, u_{k-1}, x_k, u_{k+1}, \ldots, u_N) \leq \frac{\partial \Phi}{\partial x_k}(u_1, \ldots, u_{k-1}, y_k, u_{k+1}, \ldots, u_N)
\]
whenever \((u_1, \ldots, u_{k-1}, x_k, u_{k+1}, \ldots, u_N) \leq (u_1, \ldots, u_{k-1}, y_k, u_{k+1}, \ldots, u_N)\) in \( C \).

Using the identity
\[
\int_{x_j}^{y_j} \left( \frac{\partial \Phi}{\partial x_j}(u_1, \ldots, y_i, \ldots, t, \ldots, u_N) - \frac{\partial \Phi}{\partial x_j}(u_1, \ldots, x_i, \ldots, t, \ldots, u_N) \right) dt
= \Phi(x_1, \ldots, y_i, \ldots, x_N) - \Phi(x_1, \ldots, x_i, \ldots, x_N)
- \Phi(x_1, \ldots, x_i, \ldots, y_j, \ldots, x_N) + \Phi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N),
\]
which works for every nondegenerate 2-dimensional box
\[
\{u_1\} \times \cdots \times [x_j, y_j] \times \cdots \times \{u_k\} \times \cdots \times [x_j, y_j] \times \cdots \times \{u_N\}
\]
included in \( C \), we can easily infer that the 2-box monotonicity is equivalent with the isotonicity of each partial derivative \( \frac{\partial \Phi}{\partial x_k} \) in each variable distinct from the \( k \)th, when the others are kept fixed. By mathematical induction one can put together all these facts to obtain the inequalities (3.1). \( \square \)

The reader can verify easily that the following two nondifferentiable functions,
\[
\min\{x_1, x_2\} \text{ and } \max\{x_1 + x_2 - 1, 0\}, \quad x_1, x_2 \in [0, 1],
\]
are 2-box monotone. They are known in the theory of copulas as the Fréchet-Hoeffding bounds. See [17]. The second function is convex but the first one is not convex. This fact combined with with Remark 3 shows that the notions of convexity and 2-box monotonicity are independent in dimension \( N \geq 2 \).

The analogue of 2-box monotonicity for functions \( f \) defined on an interval \([a, b]\) is the property of equal increasing increments (which is equivalent to convexity):
\[
f(x + z) - f(x) \leq f(y + z) - f(y)
\]
whenever \( x \leq y, z > 0 \) and \( x, y, y + z \in [a, b] \). See [19], Remark 1.4.1, p. 25 and Corollary 1.4.6, p. 29. An immediate consequence of this property is the fact that every function of the form
\[
\Phi(x) = f((x, v)), \quad x \in \mathbb{R}^N,
\]
associated to a convex function \( f : \mathbb{R} \rightarrow \mathbb{R} \) and a vector \( w \in \mathbb{R}_+^N \) is 2-box convex.

The increment of the log-sum-exp function over the box
\[
[0, 1] \times [0, 1] \times \{0\} \times \cdots \times \{0\}
\]
equals
\[
\log N - 2 \log(e + N - 1) + \log(2e + N - 2) < 0
\]
so that, this function is not 2-box monotone. According to Lemma 6, the differential of the log-sum-exp function is not isotone.

The usefulness of the concept of 2-box monotonicity is made clear by the following approximation result.
\textbf{Theorem 3.} Suppose that $\Phi$ is a 2-box monotone convex function defined on an open box $C$ included in $\mathbb{R}^N$. Then on every compact box $K \subset C$, $\Phi$ is the uniform limit of a sequence of infinitely differentiable strongly convex functions with isotone differentials.

When $C \subset \mathbb{R}^N_{++}$ and the function $\Phi$ is also isotone, then the approximants $\Phi_n$ can be chosen to be isotone.

\textit{Proof.} We use the usual convolution based smooth approximation. Let $\varepsilon > 0$ be arbitrarily fixed. Then the function $\Psi = \Phi + \varepsilon \| \cdot \|^2$ is 2-box monotone and $\varepsilon$-strongly convex. Besides,

$$\| \Psi(x) - \Phi(x) \| \leq \varepsilon \sup \left\{ \| x \|^2 : x \in K \right\}$$

for all $x \in K$.

According to the method of mollifiers, the convolution

$$(\Psi * \varphi)(x) = \int_{\mathbb{R}^N} \Psi(x-y) \varphi(y)dy,$$

of $\Psi$ with any infinitely differentiable function $\varphi : \mathbb{R}^N \to [0, \infty)$ such that $\varphi = 0$ on $\mathbb{R}^N \setminus K$ and $\int_{\mathbb{R}^N} \varphi(y)dy = 1$, is an infinitely differentiable function that provides a regularization of $\Psi$ since $\Psi * \varphi \to \Psi$ uniformly on $K$ as the support of $\varphi$ shrinks to $\{0\}$ . An easy computation shows that $\Psi * \varphi$ is also a 2-box monotone and $\varepsilon$-strongly convex function. Indeed, with the notation in Definition 3, we have

$$\Delta(\Psi * \varphi; B_{ij}) = \int_{\mathbb{R}^N} \Delta(\Psi(x-y); B_{ij})\varphi(y)dy \geq 0$$

and

$$(\Psi * \varphi)((1-\lambda)u + \lambda v) = \int_{\mathbb{R}^N} \Psi((1-\lambda)(u-y) + \lambda(v-y)) \varphi(y)dy$$

$$\leq \int_{\mathbb{R}^N} \left[ (1-\lambda) \Psi(u-y) + \lambda \Psi(v-y) \right] - \lambda(1-\lambda) \| u - v \|^2 \varphi(y)dy$$

$$= (1-\lambda) (\Psi * \varphi)(u) + \lambda (\Psi * \varphi)(v) - \varepsilon \lambda(1-\lambda) \| u - v \|^2.$$

Then the conclusion of Theorem 3 follows from Lemma 6. \hfill $\square$

4. \textbf{The majorization inequality in the context of ordered Banach spaces}

We start with the case of differentiable convex functions.

\textbf{Theorem 4.} Suppose that $\sum_{k=1}^{N} \lambda_k \delta_{x_k}$ and $\sum_{k=1}^{N} \lambda_k \delta_{y_k}$ are two discrete probability measures whose supports are included in an open convex subset $C$ of the ordered Banach space $E$. If $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{L^1} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, then

\begin{equation}
\sum_{k=1}^{N} \lambda_k \Phi(y_k) \geq \sum_{k=1}^{N} \lambda_k \Phi(x_k) + \sum_{k=1}^{N} \lambda_k \omega(\| x_k - y_k \|)
\end{equation}

for every Gâteaux differentiable $\omega$-convex function $\Phi : C \to F$ whose differential is isotone, while if $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{L^1} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, the inequality \ref{11} works in the reversed sense.

If $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{L^1} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, then

\begin{equation}
\sum_{k=1}^{n} \lambda_k \Phi(y_k) \geq \sum_{k=1}^{n} \lambda_k \Phi(x_k) + \sum_{k=1}^{n} \lambda_k \omega(\| x_k - y_k \|) \quad \text{for } n \in \{1, \ldots, N\}
\end{equation}
whenever $\Phi : C \to F$ is an isotope and Gâteaux differentiable $\omega$-convex function whose differential is isotope. Under the same hypotheses on $\Phi$, the inequality (1.2) works in the reverse way when $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{w^*} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$.

Proof. According to the gradient inequality (2.4),

$$
D = \sum_{k=1}^{N} \lambda_k \Phi(y_k) - \sum_{k=1}^{N} \lambda_k \Phi(x_k) = \sum_{k=1}^{N} \lambda_k (\Phi(y_k) - \Phi(x_k))
$$

$$
\geq \sum_{k=1}^{N} \Phi'(x_k)(\lambda_k y_k - \lambda_k x_k) + \sum_{k=1}^{N} \lambda_k \omega(\|x_k - y_k\|),
$$

whence, by using Abel’s trick of interchanging the order of summation (19), Theorem 1.9.5, p. 57), one obtains

$$
D = \sum_{k=1}^{N} \lambda_k \Phi(y_k) - \sum_{k=1}^{N} \lambda_k \Phi(x_k) - \sum_{k=1}^{N} \lambda_k \omega(\|x_k - y_k\|)
$$

$$
\geq \Phi'(x_1)(\lambda_1 y_1 - \lambda_1 x_1) + \sum_{m=2}^{N} \Phi'(x_m) \sum_{k=1}^{m} (\lambda_k y_k - \lambda_k x_k) - \sum_{k=1}^{m-1} (\lambda_k y_k - \lambda_k x_k)
$$

$$
= \sum_{m=1}^{N-1} \left[ (\Phi'(x_m) - \Phi'(x_{m+1})) \sum_{k=1}^{m} (\lambda_k y_k - \lambda_k x_k) \right] + \Phi'(x_N) \sum_{k=1}^{N} (\lambda_k y_k - \lambda_k x_k).
$$

When $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{L^*} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, the last term vanishes and the fact that $D \geq 0$ is a consequence of the isotonicity of $\Phi'$. When $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{w^*} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$ and $\Phi$ is isotope, one applies Lemma 4 (a) to infer that

$$
\Phi'(x_N) \left( \sum_{k=1}^{N} (\lambda_k y_k - \lambda_k x_k) \right) \geq 0.
$$

The other cases can be treated in a similar way.

The specific statement of Theorem 4 for the class of strongly convex functions, the class of semiconvex functions as well as its translation in the case of strongly concave functions and of semiconcave functions is left to the reader as an exercise. We will detail here only the case of $\sigma$-smooth functions, which in the light of Lemma 3 appears as a Legendre-Fenchel dual of the majorization inequality.

**Theorem 5.** Suppose that $\sum_{k=1}^{N} \lambda_k \delta_{x_k}$ and $\sum_{k=1}^{N} \lambda_k \delta_{y_k}$ are two discrete probability measures whose supports are included in an open convex subset $C$ of the ordered Banach space $E$. If $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{L^*} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, then

$$
\sum_{k=1}^{N} \lambda_k \Phi(y_k) \leq \sum_{k=1}^{N} \lambda_k \Phi(x_k) + \frac{\sigma}{2} \sum_{k=1}^{N} \lambda_k \|x_k - y_k\|^2
$$

for every Gâteaux differentiable and $\sigma$-smooth function $\Phi : C \to F$ whose differential is antidote on $C$.

If $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{w^*} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, then the conclusion (4.3) should be replaced by

$$
\sum_{k=1}^{N} \lambda_k \Phi(x_k) + \frac{\sigma}{2} \sum_{k=1}^{N} \lambda_k \omega(\|x_k - y_k\|) \geq \sum_{k=1}^{N} \lambda_k \Phi(y_k)
$$

(4.4)
Moreover, if the majorization relations \( \preceq_L \) and \( \preceq_R \) are replaced respectively by \( \preceq_{w_L} \) and \( \preceq_{w_R} \), then the inequalities (4.3) and (4.4) still work for those Gâteaux differentiable and \( \sigma \)-smooth functions \( \Phi : C \to F \) which are antitone and have antitone differentials.

One might wonder if the majorization relations \( \preceq_L \) and \( \preceq_R \) can be reformulated in terms of doubly stochastic matrices (as, for example, \( x \preceq_L y \) if and only if \( x = Py \) for some doubly stochastic matrix \( P \)). The answer is negative as shows the case of two pairs of elements

\[
x_1 \geq x_2 \text{ and } y_1 \geq y_2 = 0
\]

such that

\[
x_1 \leq y_1 \text{ and } (x_1 + x_2)/2 = (y_1 + y_2)/2.
\]

Clearly, no \( 2 \times 2 \)-dimensional real matrix \( A \) could exist such that

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = A \begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}.
\]

This would imply that \( x_1, x_2 \) belong necessarily to the segment \([y_2, y_1] \), which is not the case.

However, one implication is true.

**Remark 4.** If \( x_1 \geq \cdots \geq x_N \) and \( y_1 \geq \cdots \geq y_N \) are two families of points in the ordered Banach space \( E \) such that

\[
P \begin{pmatrix}
y_1 \\
\vdots \\
y_N
\end{pmatrix} = \begin{pmatrix}
x_1 \\
\vdots \\
x_N
\end{pmatrix}
\]

for a suitable doubly stochastic matrix \( P \), then

\[
\frac{1}{N} \sum_{k=1}^{N} \delta_{x_k} \preceq_L \frac{1}{N} \sum_{k=1}^{N} \delta_{y_k}.
\]

Indeed, the argument used by Ostrowski (see [15], Theorem A.4, p. 31) to settle the case \( E = \mathbb{R} \) extends verbatim to the case of ordered Banach spaces.

In the context of functions of several variables, one can take advantage of the approximation Theorem 3 to prove the following variant of Theorem 4, where the assumption on differentiability is discarded.

**Theorem 6.** Suppose that \( C \) is an open box included in \( \mathbb{R}_{++}^N \) and \( \sum_{k=1}^{N} \lambda_k \delta_x \) and \( \sum_{k=1}^{N} \rho_k \delta_y \) are two discrete probability measures supported at points in \( C \).

If \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \preceq_L \sum_{k=1}^{N} \lambda_k \delta_{y_k} \), then

\[
\sum_{k=1}^{N} \lambda_k \Phi(y_k) \geq \sum_{k=1}^{N} \lambda_k \Phi(x_k)
\]

for every \( 2 \)-box monotone convex function \( \Phi : C \to F \), while if

\[
\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_L \sum_{k=1}^{N} \lambda_k \delta_{y_k},
\]

the latter inequality works in the opposite direction.

**Proof.** Suppose that \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_L \sum_{k=1}^{N} \lambda_k \delta_{y_k} \) and choose a compact box \( K \subset C \) that contains all points \( x_k \) and \( y_k \). According to Theorem 3 for \( \varepsilon > 0 \) arbitrarily fixed, there is an infinitely differentiable, convex and isoto ne function
Ψ with isotone differential, such that sup_{x \in K} |Φ(x) − Ψ_ε(x)| < ε. Taking into account Theorem 4, we infer that

\[ \sum_{k=1}^{N} \lambda_k \Psi_\varepsilon(y_k) \geq \sum_{k=1}^{N} \lambda_k \Psi_\varepsilon(x_k). \]

Then

\[ \sum_{k=1}^{N} \lambda_k \Phi(y_k) \geq \sum_{k=1}^{N} \lambda_k \Phi(x_k) - 2\varepsilon. \]

As \( \varepsilon > 0 \) was arbitrarily fixed, we conclude that

\[ \sum_{k=1}^{N} \lambda_k \Phi(y_k) \geq \sum_{k=1}^{N} \lambda_k \Phi(x_k). \]

The case when \( \sum_{k=1}^{N} \lambda_k \delta x_k \sim_R \sum_{k=1}^{N} \lambda_k \delta y_k \) can be treated similarly. □

Remark 5. (The isotonicity of the differential is not only sufficient but also necessary for the validity of Theorem 4 and Theorem 6). As was already noticed in Remark 3, the infinitely differentiable function

\[ \Phi(x, y) = -2 (xy)^{1/2}, \quad x, y \in (0, \infty), \]

is convex and its differential

\[ d\Phi : \mathbb{R}_+^2 \to \mathbb{R}^2, \quad d\Phi(x, y) = -(x^{-1/2}y^{1/2}, x^{1/2}y^{-1/2}) \]

is not isotone. Therefore (see Lemma 6), the function \( \Phi \) is not 2-box monotone. Consider the points \( x_1 = (3/2, 1) > x_2 = (1/2, 1) \) and \( y_1 = (2, 2) > y_2 = (0, 0) \). Then

\[ x_1 \leq y_1 \text{ and } (x_1 + x_2)/2 = (y_1 + y_2)/2, \]

but

\[ \Phi(x_1) + \Phi(x_2) = -2 (3/2)^{1/2} - 2 (1/2)^{1/2} \approx -3.8637 > \Phi(y_1) + \Phi(y_2) = -4. \]

Therefore \( \Phi \) fails the conclusion of Theorem 4.

Remark 6. In the variant of weak majorization, the assertions of Theorem 6 remain valid for the 2-box monotone, isotone and convex functions defined on an open box included in \( \mathbb{R}^N \). Indeed, in this case the approximants \( \Psi_\varepsilon \) (that appear in the proof of Theorem 3) are not only 2-box monotone and strictly convex but also isotone.

5. Applications

The following consequence of Theorem 4 shows that the gap in Jensen’s inequality (the difference of the two sides of this inequality) decreases when the order interval under attention is shrinking.

Theorem 7. (The contractibility of Jensen’s gap) Suppose that \( E \) and \( F \) are ordered Banach spaces, \( C \) is an open convex subset of \( E \) and \( \Phi : C \to F \) is a differentiable convex function whose differential is isotone on \( C \). Then for every family of points \( x_1, x_2, y_1, y_2 \) in \( C \) and any \( \lambda \in (0, 1) \) such that

\[ y_2 \leq x_2 \leq (1 - \lambda)y_1 + \lambda y_2 \leq x_1 \leq y_1. \]
we have
\[
0 \leq (1 - \lambda)\Phi(x_1) + \lambda\Phi(x_2) - \Phi((1 - \lambda)x_1 + \lambda x_2) \\
\leq (1 - \lambda)\Phi(y_1) + \lambda\Phi(y_2) - \Phi((1 - \lambda)y_1 + \lambda y_2).
\]

**Proof.** Indeed, under the above hypotheses, we have \(x_1 \geq (1 - \lambda)y_1 + \lambda y_2 \geq x_2\) and also
\[
\frac{1 - \lambda}{2}x_1 \leq \frac{1 - \lambda}{2}y_1
\]
\[
\frac{1 - \lambda}{2}x_1 + \frac{1}{2}((1 - \lambda)y_1 + \lambda y_2) \leq \frac{1 - \lambda}{2}y_1 + \frac{1}{2}((1 - \lambda)x_1 + \lambda x_2)
\]
\[
\frac{1 - \lambda}{2}x_1 + \frac{1}{2}((1 - \lambda)y_1 + \lambda y_2) + \frac{\lambda}{2}x_2 = \frac{1 - \lambda}{2}y_1 + ((1 - \lambda)x_1 + \lambda x_2) + \lambda y_2.
\]
Therefore
\[
\frac{1 - \lambda}{2}\delta x_1 + \frac{1}{2}\delta(1 - \lambda)y_1 + \lambda \delta x_2 \preceq_L \frac{1 - \lambda}{2}\delta y_1 + \frac{1}{2}\delta(1 - \lambda)x_1 + \lambda \delta x_2 + \frac{\lambda}{2}\delta y_2
\]
so that, taking into account Theorem 7, we infer that
\[
(1 - \lambda)\Phi(x_1) + \Phi((1 - \lambda)y_1 + \lambda y_2) + \lambda\Phi(x_2)
\]
\[
\leq (1 - \lambda)\Phi(y_1) + \Phi((1 - \lambda)x_1 + \lambda x_2) + \lambda\Phi(y_2),
\]
an inequality that is equivalent to the conclusion of Theorem 7. \(\square\)

A particular case of Theorem 7 is as follows:

**Corollary 1.** *(The parallelogram rule)* Suppose that \(\Phi : C \to F\) is as in the statement of Theorem 7 and \(x_1, x_2, y_1\) and \(y_2\) are points in \(C\) such that \(y_2 \leq x_2 \leq x_1 \leq y_1\) and \((x_1 + x_2)/2 = (y_1 + y_2)/2\) then the following extension of the parallelogram law takes place:
\[
\Phi(x_1) + \Phi(x_2) \leq \Phi(y_1) + \Phi(y_2).
\]

**Remark 7.** *(A multiplicative version of the generalized parallelogram law)* Suppose that \(A_1, A_2, B_1, B_2\) are positively definite matrices from \(\text{Sym}(N, \mathbb{R})\) such that
\[
B_2 \preceq A_2 \preceq A_1 \preceq B_1, \quad A_1 A_2 = A_2 A_1, \quad B_1 B_2 = B_2 B_1,
\]
and \((A_1 A_2)^{1/2} = (B_1 B_2)^{1/2}\). Since the logarithm is an operator monotone function (see [12]), we have
\[
\log B_2 \leq \log A_2 \leq \log A_1 \leq \log B_1 \quad \text{and} \quad \log A_1 + \log A_2 = \log B_1 + \log B_2.
\]
From Example 15 (presented in Appendix) and Corollary 1 (applied to trace \(f(\exp(A))\)) we infer that
\[
\text{trace } f(A_1) + \text{trace } f(A_2) \leq \text{trace } f(B_1) + \text{trace } f(B_2),
\]
whenever \(f : (0, \infty) \to \mathbb{R}\) is a continuously differentiable and nondecreasing function such that \(f \circ \exp\) is convex.

**Remark 8.** *(Another variant of the generalized parallelogram law)* Suppose that \(E\) and \(F\) are ordered Banach spaces, \(C\) is an open convex subset of \(E\) and \(\Phi : C \to F\) is a differentiable, isotone and convex function whose differential is isotone on \(C\). Then for every family of points \(x_1, x_2, y_1, y_2\) in \(E_+\) such that
\[
x_2 \leq x_1 \leq y_1 \quad \text{and} \quad x_1 + x_2 \leq y_1 + y_2.
\]
we have
\[ \Phi(x_1) + \Phi(x_2) \leq \Phi(y_1) + \Phi(y_2). \]
Indeed, in this case \( x_1 \leq y_1 \) and \( x_1 + x_2 \leq y_1 + y_2 \). Though \( x_1 + x_2 = y_1 + y_2 \) could fail, Theorem 4 still applies because \( \Phi'(x_2) \geq 0 \) (see Lemma 4).

Numerous classical inequalities from real analysis can be extended to the context of ordered Banach spaces via Theorems 4-6. Here are three examples based on Theorem 4.

**Theorem 8.** (The extension of Szegő and Bellman inequalities) Suppose that \( E \) and \( F \) are two ordered Banach spaces, \( C \) is an open convex subset of \( E \) that contains the origin and \( \Phi : C \to F \) is a Gâteaux differentiable \( \omega \)-convex function whose differential is isotone. Then for every finite family \( x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \) of points in \( C \) we have

\[
(1 - \sum_{k=1}^{n} (-1)^{k+1}) \Phi(0) + \sum_{k=1}^{n} (-1)^{k+1} \Phi(x_k) \geq \Phi(\sum_{k=1}^{n} (-1)^{k+1} x_k) + \sum_{k=1}^{n} \omega(||x_k - x_{k+1}||) + \omega(||\sum_{k=1}^{n} (-1)^{k+1} x_k||).
\]

The proof is immediate, by considering separately the cases where \( n \) is odd or even. The weighted case of Theorem 8 can be easily deduced from it following the argument of Olkin [20] for the strings of real numbers.

**Theorem 9.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a nondecreasing, differentiable and convex function. If \( A_1, A_2, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) are two families of elements in \( \text{Sym}(N, \mathbb{R}) \) such that

\[
A_1 \geq A_2 \geq \cdots \geq A_n \geq 0 \quad \text{and} \quad \sum_{k=1}^{j} A_k \leq \sum_{k=1}^{j} B_k \quad \text{for} \quad j \in \{1, 2, \ldots, n\},
\]
then

\[
\sum_{k=1}^{n} \text{Trace } f(A_k) \leq \sum_{k=1}^{n} \text{Trace } f(B_k).
\]

This is a consequence of Theorem 4 when combined with Example 5 in the Appendix. The particular case where \( f(x) = x^2 \) is attributed by Petz [21] to K. L. Chung.

The third example concerns the case of Popoviciu’s inequality. In its simplest form this inequality asserts that every convex function \( \Phi \) defined on a real interval \( I \) verifies the inequality

\[
\frac{\Phi(x) + \Phi(y) + \Phi(z)}{3} - \Phi\left(\frac{x + y + z}{3}\right) \geq 2 \left[ \Phi\left(\frac{x+y}{2}\right) + \Phi\left(\frac{y+z}{2}\right) + \Phi\left(\frac{z+x}{2}\right) \right] - \Phi\left(\frac{x+y+z}{3}\right),
\]

whenever \( x, y, z \in I \) (which is an illustration of the contractibility of Jensen gap in the case of triplets of elements. See [24] and [19] for details. While Popoviciu’s inequality makes sense in any Banach space, it was shown in [4] that it actually works only for a special class of convex functions (including the norm of a Hilbert space). Based on Theorem 4, we will show that the class of useful functions can be enlarged at the cost of limiting the triplets of elements under consideration.
Theorem 10. Suppose that $E$ and $F$ are two ordered Banach spaces, $C$ is an open convex subset of $E$ and $x \geq y \geq z$ is a triplet of points in $C$. In addition, $\Phi : C \to F$ is a Gâteaux differentiable $\omega$-convex function whose differential is isotone.

(a) If $x \geq (x + y + z)/3 \geq y \geq z$, then

$$\frac{\Phi(x) + \Phi(y) + \Phi(z)}{3} + \Phi\left(\frac{x + y + z}{3}\right)$$

$$\geq \frac{2}{3} \left[ \Phi\left(\frac{x + y}{2}\right) + \Phi\left(\frac{y + z}{2}\right) + \Phi\left(\frac{z + x}{2}\right) \right] + \frac{1}{6}\omega (\|x - y\|)$$

$$+ \frac{1}{6}\omega \left(\frac{\|2z - x - y\|}{6}\right) + \frac{1}{3}\omega \left(\frac{\|2y - x - z\|}{6}\right) + \frac{1}{3}\omega \left(\frac{\|z - y\|}{2}\right).$$

(b) If $x \geq y \geq (x + y + z)/3 \geq z$, then

$$\frac{\Phi(x) + \Phi(y) + \Phi(z)}{3} + \Phi\left(\frac{x + y + z}{3}\right)$$

$$\geq \frac{2}{3} \left[ \Phi\left(\frac{x + y}{2}\right) + \Phi\left(\frac{y + z}{2}\right) + \Phi\left(\frac{z + x}{2}\right) \right] + \frac{1}{3}\omega (\|x - y\|)$$

$$+ \frac{1}{6}\omega \left(\frac{\|2x - y - z\|}{6}\right) + \frac{1}{3}\omega \left(\frac{\|2y - x - z\|}{6}\right) + \frac{1}{6}\omega \left(\frac{\|z - y\|}{2}\right).$$

Proof. (a) In this case

$$(x + y)/2 \geq (x + z)/2 \geq (y + z)/2$$

and $x + y \geq y \geq 2z/3 \geq z$ so the conclusion follows from Theorem 11 applied to the families of points

$$x_1 = x_2 = (x + y)/2 \geq x_3 = x_4 = (x + z)/2 \geq x_5 = x_6 = (y + z)/2$$

and

$$y_1 = x \geq y_2 = y_3 = y_4 = (x + y + z)/3 \geq y_5 = y \geq y_6 = z,$$

by noticing that

$$(x + y)/2 \leq x$$

$$(x + y)/2 + (x + y)/2 \leq x + (x + y + z)/3$$

$$(x + y)/2 + (x + y)/2 + (x + z)/2 \leq x + (x + y + z)/3 + (x + y + z)/3$$

and

$$(x + y)/2 + (x + y)/2 + (x + z)/2 + (x + z)/2 \leq x + (x + y + z)/3 + (x + y + z)/3$$

and

$$(x + y)/2 + (x + y)/2 + (x + z)/2 + (x + z)/2 + (y + z)/2 \leq x + (x + y + z)/3 + (x + y + z)/3 + (x + y + z)/3 + (x + y + z)/3$$

and

$$(x + y)/2 + (x + y)/2 + (x + z)/2 + (x + z)/2 + (y + z)/2 + (y + z)/2 \leq x + (x + y + z)/3 + (x + y + z)/3 + (x + y + z)/3 + (x + y + z)/3 + (y + z)/2$$

(b) The proof is similar, by considering the families

$$x_1 = x_2 = (x + y)/2 \geq x_3 = x_4 = (x + z)/2 \geq x_5 = x_6 = (y + z)/2$$

$$y_1 = x, \quad y_2 = y, \quad y_3 = y_4 = y_5 = (x + y + z)/3, \quad y_6 = z.$$
In the case where $E = \mathbb{R}$ and $C$ is an interval of $\mathbb{R}$, we have $[z, x] = [z, y] \cup [y, x]$, so $(x + y + z)/3$ lies automatically in one of the intervals $[z, y]$ and $[y, x]$. This allows us to recover the aforementioned result of Popoviciu.

6. Appendix: Examples of Differentiable Functions Which Are Isotone and/or Admit Isotone Differentials

Example 1. Let $I$ be one of the intervals $(-\infty, 0]$, $[0, \infty)$ or $(-\infty, \infty)$. The perspective function associated to a convex function $f : I \to \mathbb{R}$ is the convex function

$$\tilde{f} : I \times (0, \infty), \quad \tilde{f}(x, y) = yf(x/y).$$

See [19], Section 3.5. Assuming $f$ of class $C^2$, then

$$\frac{\partial \tilde{f}}{\partial x} = f'(\frac{x}{y}), \quad \frac{\partial \tilde{f}}{\partial y} = -\frac{x}{y}f'(\frac{x}{y}) + f(\frac{x}{y})$$

and

$$\frac{\partial^2 \tilde{f}}{\partial x \partial y} = -\frac{x}{y^2}f''(\frac{x}{y}).$$

As a consequence, if $I = (-\infty, 0)$, then $\tilde{f}$ is isotone; if in addition $f$ is nonnegative and increasing, then $\tilde{f}$ itself is isotone.

Example 2. Let $p \in (1, \infty)$. The function

$$\Phi : L^p(\mathbb{R}) \to \mathbb{R}, \quad \Phi(f) = \|f\|_p^p = \int_{\mathbb{R}}|f|^p \, dt$$

is convex and differentiable, its differential being defined by the formula

$$d\Phi(f)(h) = p \int_{\mathbb{R}} h |f|^{p-1} \text{sgn} f \, dt$$

for all $f, h \in L^p(\mathbb{R})$.

See [19], Proposition 3.7.8, p. 151. Clearly, $\Phi$ and its differential are isotone on the positive cone of $L^p(\mathbb{R})$. A variant of this example within the framework of Schatten classes is provided by Theorem 16 in [13].

Example 3. The negative entropy function, $E(x) = \sum_{k=1}^N x_k \log x_k$, is $C^\infty$-differentiable on

$$\mathbb{R}^N_+ = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1, \ldots, x_N > 0 \}$$

and strongly convex on any compact subset $K$ of $\mathbb{R}^N_+$. The differential of $\Phi$ is the map $d\Phi : \mathbb{R}^N_+ \to (\mathbb{R}^N)^*$ given by the formula

$$d\Phi(x)v = \sum_{k=1}^N (1 + \log x_k)v_k, \quad x \in \mathbb{R}^N_+ \text{ and } v \in \mathbb{R}^N,$$

so that $x \leq y$ in $\mathbb{R}^N_+$ implies $d\Phi(x) \leq d\Phi(y)$.

Example 4. The log-sum-exp function is defined on $\mathbb{R}^N$ by the formula

$$\text{LSE}(x) = \log(\sum_{k=1}^N e^{x_k}), \quad x \in \mathbb{R}^N.$$

This function is infinitely differentiable, isotone and convex, but it is not strongly convex. See [19], Example 3.8.9, pp. 157-158. A simple argument showing that the differential of LSE is not isotone is given in the comments after Lemma [8]. The log-sum-exp function is the Legendre-Fenchel conjugate of the restriction of the negative entropy function $E$ to the simplex $\Delta = \{ x : x \in \mathbb{R}^N, \sum_{k=1}^N x_k = 1 \}$. See [7], p. 93. Since $E$ is strongly convex, it follows from Lemma [8] that the log-sum-exp function is strongly smooth.
Example 5. (Trace functions of matrices) Denote by $\text{Sym}(N, \mathbb{R})$ the ordered Banach space of all $N \times N$-dimensional symmetric matrices with real coefficients endowed with the Frobenius norm and the Löwner ordering,

$$A \leq B \text{ if and only if } \langle Ax, x \rangle \leq \langle Bx, x \rangle \text{ for all } x \in \mathbb{R}^N.$$ 

If $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable (strongly) convex function, then the formula

$$\Phi(A) = \text{trace}(f(A))$$

defines a differentiable (strongly) convex function on $\text{Sym}(n, \mathbb{R})$. Since $d\Phi(A)X = \text{trace}(f'(A)X)$ and $f'$ is isotone, it follows that $d\Phi$ is isotone too. According to Weyl’s monotonicity principle (see [19], Corollary 4.4.3, p. 203), the function $\Phi$ is isotone if $f$ itself is isotone.

Two particular cases are of a special interest:

(a) The operator analogue of the negative entropy function presented in Example 4 is the negative von Neumann entropy, defined on the compact convex set $C = \{A \in \text{Sym}^+(N, \mathbb{R}) : \text{trace}(A) = 1\}$ via the formula

$$S(A) = \text{trace}(A \log A) = \sum_{k=1}^{N} \lambda_k(A) \log \lambda_k(A),$$

where $\lambda_1(A), ..., \lambda_N(A)$ are the eigenvalues of $A$ counted with their multiplicity. According to the preceding discussion, this function is convex and differentiable and its differential is isotone. One can prove (using Lemma 3) that the negative von Neumann entropy is $1/2$-strongly convex and its Legendre-Fenchel conjugate, the convex function $\log(\text{trace}(e^A))$, is $2$-smooth. See [13], Theorem 16.

(b) The function $\text{trace}(e^A)$ is log-convex and continuously differentiable, with isotone differential. However, the differential of the convex function $\log(\text{trace}(e^A))$ is not anymore isotone. See Example 5, which discusses the case of diagonal matrices.

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