SIMPLY CONNECTED PROJECTIVE MANIFOLDS IN
CHARACTERISTIC $p > 0$ HAVE NO NONTRIVIAL
STRATIFIED BUNDLES

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ABSTRACT. We show that simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles. This gives a positive answer to a conjecture by D. Gieseker (1975). The proof uses Hrushovski’s theorem on periodic points.

1. Introduction

Let $X$ be a smooth complex variety. The category of bundles with integrable connections on $X$ is the full subcategory of the category of coherent $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent as well. It is a $\mathbb{C}$-linear abelian rigid category. If $X$ is projective or if we restrict to connections which are regular singular at infinity, then it is equivalent by the Riemann-Hilbert correspondence to the $\mathbb{C}$-linear abelian rigid category of local systems of complex vector spaces ([2]). Upon neutralizing those categories by the choice of a point $x \in X(\mathbb{C})$, the Riemann-Hilbert correspondence translates via the Tannaka formalism into an isomorphism between the proalgebraic completion of the topological fundamental group $\pi^\text{top}_1(X, x)$ and the Tannaka pro-algebraic group $\pi^\text{strat}_1(X, x)$ of flat bundles. Malcev ([14]) and Grothendieck ([7]) showed that if the étale fundamental group $\pi^\text{ét}_1(X, x)$ is trivial, that is if $X$ does not have any nontrivial connected finite étale covering, then $\pi^\text{strat}_1(X, x)$ is trivial as well, thus there are no nontrivial flat bundles. The proof has nothing to do with flat bundles, but with the fact that $\pi^\text{top}_1(X, x)$ is an abstract group of finite type, and that, as a consequence of the Riemann existence theorem, $\pi^\text{ét}_1(X, x)$ is its profinite completion. The theorem says that if the profinite completion of a group of finite type is trivial, so is its proalgebraic completion.

Let $X$ now be a smooth variety defined over a perfect field $k$ of characteristic $p > 0$. The full subcategory of the category of coherent $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent is again a $k$-linear rigid tensor category. Katz shows [5, Theorem 1.3] that it is equivalent to the category of objects $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ where $E_n$ is a vector bundle, $\sigma_n : F^*E_{n+1} \simeq E_n$ is a $\mathcal{O}_X$-linear isomorphism, and where the
morphisms respect all the structures. Let us call it the category $\text{Strat}(X)$ of stratified bundles. One neutralizes the category via the choice of a rational point $i_x : x \to X(k)$, defining the functor $\omega_x : \text{Strat}(X) \to \text{Vec}_k$, $\omega_x(E) = i_x^* E_0$. This defines the pro-algebraic group $\pi_1^{\text{strat}}(X, x) = \text{Aut}^\oplus(\omega_x)$.

For a rational point $x \in X(k)$, we denote by $\bar{x}$ a geometric point above it. The purpose of this article is to show, when $X$ is projective, the analog in characteristic $p > 0$ of Malcev-Grothendieck theorem (see Theorem 3.15): \[\text{Theorem 1.1.} \text{ Let } X \text{ be a smooth connected projective variety defined over a perfect field } k \text{ of characteristic } p > 0. \text{ Let } \bar{x} \in X \text{ be a geometric point. If } \pi_1^{\text{et}}(X \otimes_k \bar{k}, \bar{x}) = \{1\}, \text{ there are no nontrivial stratified bundles.}\]

This gives a (complete) positive answer to Gieseker’s conjecture [5, p.8]. If $X$ has a rational point $x \in X(k)$, one can rephrase by saying that under the assumptions of the theorem, $\pi_1^{\text{strat}}(X, x) = \{1\}$. The theorem implies that if $\pi_1^{\text{et}}(X \otimes_k \bar{k}, \bar{x})$ is a finite group, then irreducible stratified bundles come from representations of $\pi_1^{\text{et}}(X \otimes_k \bar{k}, \bar{x})$ (see Corollary 3.16).

Dos Santos [3] studied the $k$-pro-group $\pi_1^{\text{strat}}(X, x)$ when $k = \bar{k}$. He showed that all quotients in $GL(\omega_x(E))$ are smooth algebraic groups (3, Corollary 12), see also [15]). In fact, the proof is written only for the finite part, but it applies more generally. This is an important fact pleading in favor of the conjecture.

We now describe the philosophy of the proof of Theorem 1.1. Since there is no known group of finite type which controls $\pi_1^{\text{strat}}(X, x)$ and $\pi_1^{\text{et}}(X, x)$, it is impossible to adapt Grothendieck’s proof. Instead, one can first think of the full subcategory spanned by rank one objects. Over $\bar{k}$, the maximal abelian quotient $\pi_1^{\text{ab}}(X, x)$ of $\pi_1^{\text{et}}(X, x)$ nearly controls the Picard variety: if $\pi_1^{\text{ab}}(X, x) = \{1\}$, then, for any prime $\ell \neq p$, the $\ell$-adic Tate module $\varprojlim \text{Pic}^0(X)(\bar{k})[\ell^n]$ is trivial, thus $\text{Pic}^\tau(X)$ is finite. So a rank one stratified bundle $L = (L_n, \sigma_n)_{n \in \mathbb{N}}$ on $X \otimes \bar{k}$ must have the property that for a strictly increasing sequence $n_i$, $i \geq 0$, the line bundles $L_{n_i}$ are all isomorphic, thus all $L_{n_i}$ are fixed by some power of the Frobenius, so define a Kummer covering of $X$, which then has to be trivial by our assumption. This implies that all the $L_n$, $n \geq 0$ are trivial. Since on $X$ proper, a stratified bundle $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ is uniquely recognized by the bundles $E_n$ ([5, Proposition 1.7]), this shows the statement.

Thus the rank one case does not rely directly on the Tannaka property of the category. It rather uses the representability of the Picard functor together with the fact that those bundles with are fixed by a power of the Frobenius defined étale coverings on one hand, and are dense in $\text{Pic}^\tau(X \otimes \bar{k})$ on the other. We try to follow the same idea in the higher rank case.

We now describe the main steps of the proof over $k = \bar{k}$. In order to be able to use moduli of vector bundles, we first reduce the problem to the case where all the underlying bundles $E_n$ of the stratified bundles $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ are stable
with vanishing Chern classes. Here we use \( \mu^- \) (or slope) stability with respect to a fix polarization, which is defined by Mumford by the growth of the degree of subsheaves. Brenner-Kaid ([11 Lemma 2.2]) show that if \( E \) is a stratified bundle, then \( E_n \) is semistable of slope 0 for \( n \) large. Using Langer’s boundedness [12 Theorem 4.2], the authors apply the same argument as for the classical one in rank 1 sketched above, to conclude that if all the \( E_n \) are defined over the same finite field \( \mathbb{F}_q \) and \( \pi_1^{\text{et}}(X_{\mathbb{F}_q}) = 0 \), then all stratified bundles are trivial ([11 Lemma 2.4]).

We show in general that the stratified bundle \( E(n_0) = (E_{n_0+m}, \sigma_{n_0+m})_{m \in \mathbb{N}} \), for \( n_0 \) large enough, is always a successive extension of stratified bundles \( (U_m, \tau_m)_{m \in \mathbb{N}} \) such that all the \( U_m \) are \( \mu^- \)-stable. This, together with dos Santos’ theorem [3 (9)], describing with the projective system \( \lim_{\leftarrow m} H^1(X, \mathcal{O}_X) \) over the Frobenius, the full subcategory of \( \text{Strat}(X) \) spanned by successive extensions of the trivial object by itself, is enough to perform the reduction (see Proposition [2.4]).

The moduli scheme \( M \) of \( \chi \)-stable torsionfree sheaves of Hilbert polynomial \( p_{\mathcal{O}_X} \) and of rank \( r > \) pissionfree sheaves with Hilbert polynomial \( M \) is well known [4, Theorem 1], even if \( E \) over \( \mathbb{F}_q \) and \( \pi_1^{\text{et}}(X_{\mathbb{F}_q}) = 0 \), then all stratified bundles are trivial ([11 Lemma 2.4]).

While on \( \text{Pic}(X) \), the Verschiebung morphism, which, to a line bundle \( L \), assigns its Frobenius pullback \( V(L) := F^*(L) \simeq L^p \), is well defined, on \( M \) it is not. As is well known [4 Theorem 1], even if \( E \) is stable, the bundle \( F^*(E) \) need not be stable. We define the sublocus \( M^s \) of \( \mu^- \)-stable points \([E] \) for which there is a stratified bundle \((E_n, \sigma_n)_{n \in \mathbb{N}} \) with \( E_0 \cong E \). The bundles \( E_n \) are necessarily \( \mu^- \)-stable, thus define points \([E_n] \in M^s \). In particular, \([E] = V([E_1]) \) and \( M^s \) lies in the image of the sublocus of \( M^s \) on which \( V \) is well defined. If we assume that there are nontrivial stratified bundles with \([E_n] \in M \), this sublocus is not empty. We define \( N \subset M \) to be the Zariski closure of \( M^s \). We show that the Verschiebung is a rational dominant morphism \( V : N \longrightarrow N \) (see Lemma [3.8]).

On the other hand, there is a smooth affine variety \( S \), defined over \( \mathbb{F}_p \), such that \( X \) has a smooth model \( X_S \rightarrow S \), the Frobenius \( F_k : \text{Spec } k \rightarrow \text{Spec } k \) descends to the absolute Frobenius \( F_S : S \rightarrow S \), and the absolute Frobenius \( F : X \rightarrow X \) over \( F_k \) descends to the absolute Frobenius \( F_{X_S} : X_S \rightarrow X_S \) over \( F_S \). By the representability theorem [13 Theorem 4.1], \( M \) has a model \( M_S \) with the property that for all morphisms \( T \rightarrow S \) of \( \mathbb{F}_p \)-varieties, \( M_S \times_S T = M_T \), where \( M_T \) is the moduli of stable vector bundles of the same Hilbert polynomial and the same rank \( r > 1 \) on \( X_T \rightarrow T \), where \( X_T = X_S \times_S T \). We take \( T \) smooth so that \( N \subset M \) has a model \( N_T \subset M_T \), and its connected components are also defined over \( T \). We further require that \( V \) has a model \( V_T : N_T \longrightarrow N_T \). There is a dense open subvariety \( T^0 \subset T \) such that for all closed points \( t \in T^0 \), the reduction
\( V_t : N_t \longrightarrow N_t \) is still defined as a rational map, is dominant, and some power fixes the irreducible components of \( N_t \). We show that \( V_t \) is the Verschiebung from \( M_t \) restricted to \( N_t \) (see Lemma 3.10).

Replacing \( N_t \) by an irreducible component \( Y \) say, we denote by \( \Gamma \subset Y \times_t Y \) the Zariski closure of the graph, where it is defined, of \( V_t \) respecting \( Y \). One can apply Hrushovski’s fundamental theorem [8, Corollary 1.2] to find a dense subset of closed points of the shape \((u, \Phi^{m_q}(u)) \in \Gamma\), where \( \Phi^{m_q}(u) \) is the geometric Frobenius of \( N_t \) raising coordinates to the \( q \)-th power, and where \( F_q(t) \) is the residue field of \( t \). From this and the representability theorem loc. cit., one deduces that if \( N \) is not empty, \( N_t \otimes_{F_q} \bar{F}_q \) contains closed points which are fixed under the Verschiebung (see Theorem 3.14). Since by Grothendieck’s specialisation theorem [17, Théorème 3.8], \( \pi_1^{et}(X_t \otimes_{F_q} \bar{F}_q) = \{1\} \), we conclude that \( N_t \) is empty, so thus is \( N \).

The reduction to the stable case is written in section 2, the proof is performed in section 3. In fact in section 3, we do a bit more. We show that torsion points (see Definition 3.12) are dense in good models of stratified schemes (see Definition 3.3 and Theorem 3.14). In section 4, we make a few remarks and raise some questions. In particular, if \( X \) is quasi-projective, in view to Grothendieck’s theorem in characteristic 0, one would expect a relation between the fundamental group \( \pi_1^{et}(X \otimes_k \bar{k}, \bar{x}) \), and tame stratified bundles. Without resolution of singularities and without theory of canonical extension like Deligne’s one [2] in characteristic 0, our method of proof can’t be extended to this case.

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geometrically simply connected (see Theorem 3.14). Furthermore, he directly contributed an illustration of Theorem 3.14 (see Proposition 3.17). We also thank Burt Totaro for pointing out [14] to us.

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2. Reduction to the case where the bundles are stable

Let $X$ be a smooth connected projective variety defined over a perfect field $k$ of characteristic $p > 0$. By Katz’ equivalence of category between coherent $D_X$-modules which are $\mathcal{O}_X$-coherent and stratified bundles [5, Theorem 1.3], any stratified bundle $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$, $\sigma_n : F^*E_{n+1} \cong E_n$ of rank $r > 0$ is isomorphic as a stratified bundle to one for which for all $n \geq 0$ and all $a \geq 0$, $(F^n)^{-1}E_{n+a} \subset E_n$ is a subsheaf of abelian groups, which is a $(F^n)^{-1}\mathcal{O}_X$-locally free module of rank $r$.

From now on, we will always take a representative of a stratified bundle which has this property and we will drop the isomorphisms $\sigma_n$ from the notation. Let $\mathcal{O}_X(1)$ be an ample line bundle. For any bundle $E$ of rank $r > 0$, one defines the Hilbert polynomial of $E$ relative to $\mathcal{O}_X(1)$ by $p_E(m) = \frac{1}{r} \chi(X, E(m)) \in \mathbb{Q}[m]$ for $m >> 0$ large enough so that it is a polynomial, and where $E(m) = E \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$. Recall ([6, section 0]) that $E$ is said to be $\chi$-stable ($\chi$-semistable) if for all subsheaves $U \subset E$ one has $p_U < p_E$ ($p_U \leq p_E$). Here the order is defined by the values of the polynomials for $n$ large, and the rank of the (necessarily torsionfree) subsheaf $U$ is its generic rank. Recall that $E$ is said to be $\mu$-stable ($\mu$-semistable) if for all subsheaves $U \subset E$ on has $\mu(U) < \mu(E)$ ($\mu(U) \leq \mu(E)$).

We use the notation $CH^i(X)$ for the Chow group of codimension $i \geq 0$ cycles and denote by $\cdot$ the cup-product $CH^i(X) \times CH^j(X) \to CH^{i+j}(X)$. If $X$ is connected and has dimension $d$, we denote by $\deg : CH^d(X) \to \mathbb{Z}$ the degree homomorphism.

With this notation, the slope $\mu(E)$ of a torsionfree sheaf $E$ is defined by the formula $\mu(E) = \frac{1}{\text{rank}(E)} \deg(c_1(E) \cdot \mathcal{O}_X(1)^{d-1})$, where $d = \dim_k X$.

**Lemma 2.1.** Let $X$ be a smooth connected projective variety of dimension $d \geq 1$ defined over a perfect field $k$ of characteristic $p > 0$. Let $E = (E_n)_{n \in \mathbb{N}}$ be a stratified bundle. For any class $\xi \in CH^i(X)$, for all $n \geq 0$, and all $0 \leq i \leq d - 1$, one has $\deg(\xi \cdot \gamma_{d-i}(E_n)) = 0$, where $\gamma_{d-i}$ is any homogeneous polynomial of degree $d - i$ with rational coefficients in the Chern classes.

**Proof.** One has $\deg(\xi \cdot c_{d-i}(E_n)) = \deg(\xi \cdot c_{d-i}(F^a E_{n+a}))$ for all $a \geq 0$. On the other hand, for any bundle $E$, one has $\gamma_{d-i}(F^*E) = p^{d-i-1} \gamma_{d-i}(E)$. Thus $\deg(\xi \cdot c_{d-i}(E_n)) \in \frac{1}{p} \mathbb{Z}$ is infinitely $p$-divisible, where $D$ is the bounded denominator $\in \mathbb{N} \setminus \{0\}$. Thus this is 0. \qed
Corollary 2.2. Let $X$ be a smooth connected projective variety of dimension $d \geq 1$ defined over a perfect field $k$ of characteristic $p > 0$. Let $E = (E_n)_{n \in \mathbb{N}}$ be a stratified bundle. Then $p_{E_n} = p_{\mathcal{O}_X}$ and $\mu(E_n) = 0$ for any $n \geq 0$.

Proposition 2.3. Let $X$ be a smooth connected projective variety defined over a perfect field $k$ of characteristic $p > 0$. For any stratified bundle $E = (E_n)_{n \in \mathbb{N}}$, there is a $n_0 \in \mathbb{N}$ such that the stratified bundle $E(n_0) = (E_n)_{n \geq n_0, n \in \mathbb{N}}$ is a successive extension of stratified bundles $U = (U_n)_{n \in \mathbb{N}}$ with the property that all $U_n$ for $n \in \mathbb{N}$ are $\mu$-stable of slope 0. In particular, all $U_n$ for $n \in \mathbb{N}$ are $\chi$-stable bundles of Hilbert polynomial $p_{\mathcal{O}_X}$.

Proof. Since $\mu$-stability implies $\chi$-stability (\cite{B1} Lemma 1.2.13), it is enough to prove the proposition with $\mu$-stability. We first show that for $n_0$ large enough, $E_n, n \geq n_0$ is $\mu$-semistable (see \cite{B2} Lemma 2.2]). Let $U_n$ be the a nontrivial subsheaf of $E_n$. Assume $\mu(U_n) > 0 = \mu(E_n)$ (Corollary 2.2). Since $(F^n)^*(U_n) \subset E_n$, $\mu((F^n)^*(U_n)) = p^n\mu(U_n)$ is bounded by $\mu_{\max}(E_0)$, the slope of the maximal destabilizing subsheaf of $E_0$. One concludes that there is a $n_0 \geq 0$ such that for all $n \geq n_0$, one has $\mu_n(U_n) \leq 0$. Thus $E_n$ is $\mu$-semistable for $n \geq n_0$.

To show the proposition, we may now assume that $n_0 = 0$, that is all $E_n$ are $\mu$-semistable of slope 0. Let $U_n \subset E_n$ be the socle of $E_n$, that is the maximal nontrivial subsheaf which is $\mu$-polystable of slope 0. Then $(F^n)^*(U_n) \subset E_n$ has still slope 0, thus has to lie in $U_0$. This yields a decreasing sequence $\ldots \subset (F^{n-1})^*(U_{n+1}) \subset (F^n)^*(U_n) \subset \ldots \subset E_0$, which has to be stationary for $n$ large. Thus there is a $n_1 \geq 0$ such that for all $n \geq n_1$, $F^nU_{n+1} = U_n$. So $(U_{n1}, U_{n1+1}, \ldots) \subset E(n_1)$ is a substratified sheaf. Thus it is a substratified bundle, as $\mathcal{O}_X$-coherent $\mathcal{D}_X$-coherent modules are locally free (\cite{B3} Lemma 6)).

So we may assume $n_1 = 0$. Write $U_n = \oplus_{b=1}^{a(n)} S^b_n$ where $S^b_n$ is $\mu$-stable of slope 0. Then one has an exact sequence $0 \to F^*S^b_{n+1} \cap S^c_n \to F^*S^b_{n+1} \oplus S^c_n \to F^*S^b_{n+1} + S^c_n \to 0$ where the sum on the right is taken in $U_n$. Thus $\mu(F^*S^b_{n+1} + S^c_n) \leq 0$ and $\mu(F^*S^b_{n+1} \cap S^c_n) \leq 0$, and we conclude that $F^*S^b_{n+1} \cap S^c_n$ is either $S^c_n$ or else is equal to 0. We conclude that $U = (U_n)_{n \in \mathbb{N}} \subset E = (E_n)_{n \in \mathbb{N}}$ is a direct sum $U = \oplus_{k=1}^{a} U^b_k$ where $U^b = (U^b_n)_{n \in \mathbb{N}}$ is a stratified bundle which has the property that $U^b$ is $\mu$-stable of slope 0.

We now finish the proof using that the category of stratified bundles is abelian: we replace $E$ by $E/U$, which has a strictly lower rank, and we redo the argument.

□

Proposition 2.4. Let $X$ be a smooth projective variety defined over a perfect field $k$ of characteristic $p > 0$. Let $x \to X$ be a geometric point. If $\pi_1^e(X \otimes_k k, x)$ has no quotient isomorphic to $\mathbb{Z}/p$, then stratified bundles which are successive extensions of the trivial stratified bundle by itself are trivial.

Proof. We assume $k = \bar{k}$. We have to show that a stratified bundle $E = (E_n)_{n \in \mathbb{N}}$ which is a successive extension of $\mathbb{I} := (\mathcal{O}_X, \mathcal{O}_X, \ldots)$ by itself is trivial. By
dos Santos' theorem \[3\] (9), the isomorphism class of \(E\) lies in the projective system \(\lim_{\leftarrow n} H^1(X, \mathcal{O}_X)\) where the transition maps are the pullback maps \(F^*: H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)\) via the absolute Frobenius. On the other hand, since \(k\) is perfect, one has the decomposition \(H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)_{\text{ss}} \oplus H^1(X, \mathcal{O}_X)_{\text{nilp}}\) where \(H^1(X, \mathcal{O}_X)_{\text{ss}} = H^1(X, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}_p} k\) and \(H^1(X, \mathcal{O}_X)_{\text{nilp}} \subset H^1(X, \mathcal{O}_X)\) is defined as the \(k\)-subvector space of classes on which \(F^*\) is nilpotent. Since \(H^1(X, \mathbb{Z}/p\mathbb{Z}) = \text{Hom}(\pi_1(X), \mathbb{Z}/p\mathbb{Z})\), this group is 0 by the assumption. Since \(H^1(X, \mathcal{O}_X)\) is a finite dimensional vector space, and \(F\) is semilinear, there is a \(N \in \mathbb{N} \setminus \{0\}\) such that \(F^N\) annihilates \(H^1(X, \mathcal{O}_X)_{\text{nilp}}\). Thus we conclude \(\lim_{\leftarrow n} H^1(X, \mathcal{O}_X) = \lim_{\leftarrow n} H^1(X, \mathcal{O}_X)_{\text{nilp}} = 0\). Thus the stratified bundle \(E\) is trivial. This finishes the proof over \(k = \bar{k}\). In general, this shows that if \(E\) is a stratified bundle on \(X\) which is a successive extension of the trivial stratified bundle by itself, then \(E \otimes_k \bar{k}\) is a trivial stratified bundle on \(X \otimes_k \bar{k}\). Thus for any \(n \geq 0\), \(H^0(X, E_n) \otimes_k \mathcal{O}_X \to E_n\) is an isomorphism after tensoring with \(\bar{k}\) over \(k\), thus it is an isomorphism. Thus \(E_n\) is trivial for all \(n \geq 0\), so \(E\) is trivial. This finishes the proof in general. \(\square\)

3. The Proof of the Main Theorem and its Corollary

Let \(X\) be a smooth projective connected variety defined over a perfect field \(k\) of characteristic \(p > 0\). We fix an ample line bundle \(\mathcal{O}_X(1)\), a rank \(r > 1\) and consider the quasi-projective moduli scheme \(M\) of \(\chi\)-stable torsionfree sheaves with Hilbert polynomial \(p_{\mathcal{O}_X}\) and rank \(r\), as defined by Gieseker \[6\] Theorem 0.2] in dimension 2 and and Langer \[13\] Theorem 4.1] in any dimension.

Even if one can reduce the statement of Theorem \[17\] to the dimension 2 case by a Lefschetz type argument on stratified bundles, we will need the strength of Langer’s theorem. Let us recall from loc. cit. that if \(S\) is a smooth absolutely connected affine variety over a finite field \(\mathbb{F}_q\) such that \(\mathbb{F}_q(S) \subset k\), and if \(X_S \to S\) is a smooth projective model of \(X\), then there is a quasiprojective scheme \(M_S \to S\) which universally corepresents the functor of families of \(\chi\)-stable torsionfree sheaves of rank \(r\) and Hilbert polynomial \(p_{\mathcal{O}_X}\) on geometric fibers. This concept is due to Simpson \[16\] p. 60]. The scheme \(M_S\) has several properties. It represents the étale sheaf associated to the moduli functor. If \(s \in S\) is any closed point, then \(M_S \times_S s\) is \(M_s\). If \(u \in M_S\) is any closed point above the closed point \(s \in S\), there is a \(\chi\)-stable torsionfree sheaf \(E\) on \(X_S \times_S s\) with moduli point \(u\).

We will use the notation \([E] \in M\) to indicate the closed point in \(M\) which represents the stable bundle \(E\). As is well known \[4\] Theorem 1], even if \(E\) is stable, the bundle \(F^*(E)\) need not be stable. On the other hand, stability is an open condition. Thus there is an open subscheme \(M^0 \subset M\) such that \(F^*(E)\) is stable for all \([E] \in M^0\).

**Definition 3.1.** The morphism \(V: M^0 \to M\) is called the Verschiebung.
**Definition 3.2.** We define the locus $M^s \subset M$ consisting of points $[E] \in M$ which are $\mu$-stable of slope 0 such that there is a stratified bundle $(E_n)_{n \in \mathbb{N}}$ with $E = E_0$. (The upper script $s$ here stands for stratified.)

This is the key definition of the article due to the following remark.

**Remark 3.3.** All the bundles $E_n, n \geq 0$ in Definition 3.2 have the property that they are $\mu$-stable of slope 0 by the computation of Proposition 2.3 $\mu((F^n)U_n) = p^n\mu(U_n)$, hence $\mu(U_n) < 0$. Thus a point $[E] \in M^s$ defines a sequence of points $[E_n]_{n \in \mathbb{N}, n \geq 0} \in M^s$ such that $(E_n)_{n \in \mathbb{N}}$ is a stratified bundle.

We now define various closed subschemes of $M$ using $M^s$.

**Definition 3.4.** Let $E = (E_n)_{n \in \mathbb{N}}$ be a stratified bundle with $E_0 \in M^s$ (thus $E_n \in M^s$ for all $n \geq 0$).

1) We define $A(E)$ to be the Zariski closure of the locus $\{E_n, n \geq 0\}$ in $M$.

2) We define $N(E) = \cap_{n \geq 0} A(E(n))$ where $E(n) = (E_{n+m}, m \geq 0)$.

Closed subschemes of $M$ of type $N(E)$ as in 2) are called subschemes of $M$ spanned by stratifications.

**Remarks 3.5.**

1) As $M^s$ is defined by its $k$-points, any subscheme of $M$ spanned by stratifications lies in $M_{\text{red}}$.

2) Let $E = (E_n)_{n \in \mathbb{N}}$ be a stratified bundle with $E_n \in M^s$ for all $n \geq 0$. As $A(E(n+1)) \subset A(E(n))$ are closed subschemes of $M_{\text{red}}$, by the noetherian property there is a $n_0 \geq 0$ such that $A(E(n)) = A(E(n_0)) = N(E)$ for all $n \geq n_0$. Thus there is a hierarchy in the definition: subschemes of type $N(E)$ are of type $A(E)$, and to show that $M^s$ is empty is equivalent to showing that subschemes of $M$ spanned by stratifications are empty.

**Definition 3.6.** A closed subscheme $N \subset M_{\text{red}}$ is called Verschiebung divisible if $V|_N : N \to N$ is a rational map which is dominant on all the components of $N$. We denote by $N^1 \subset N$ the dense locus on which $V|_N$ is defined.

**Lemma 3.7.** Let $N$ be a Verschiebung divisible subscheme of $M$. For all natural numbers $a \in \mathbb{N} \setminus \{0\}$, the composite $V|_N^a := V|_N \circ \ldots \circ V|_N$ (a-times) is a dominant rational map $V|_N^a : N \to N$ and there is a natural number $a \in \mathbb{N} \setminus \{0\}$ such that $V|_N^a$ stabilizes all the irreducible components of $N$.

**Proof.** Since $V|_N$ is a rational dominant self-map, the iterate maps $V|_N^a$ are rational dominant self-maps as well. On the other hand, the image of an irreducible component by a rational map lies in an irreducible component. So by a counting argument, a rational dominant self-map permutes the irreducible components. Thus an iterate of the map stabilizes the components. 

**Lemma 3.8.** Subschemes of $M$ which are spanned by stratifications are Verschiebung divisible.
Proof. If \( N \) is empty, there is nothing to prove. Else \( N \) is constructed as the Zariski closure of family of points \( S = \{ e_n, n \geq n_0 \} \) in \( M \) which have the property that for all \( n \geq n_0 \), there is a point \( e_{n+1} \in N \) with \( V(e_{n+1}) = e_n \), and which in addition have the property that \( N \) is also the Zariski closure of the family of points \( S[m] = \{ e_n, n \geq m + n_0 \} \) for any \( m \geq 0 \). Taking \( m = 1 \), one has \( V(S) = S[1] \).

Let \( N^1 \subset N \) be the locus on which \( V \) is defined and \( V(N^1) \subset N \). Then \( N^1 \) contains \( S[1] \). Thus \( V|_N : N^1 \to N \) is dominant on \( N \) and is dense in \( N \). This finishes the proof.

\( \square \)

Recall that if \( Z \) is any scheme of finite type defined over \( k \), and \( S \) is a smooth affine variety, which is defined over a finite field \( \mathbb{F}_q \), is geometrically irreducible over \( \mathbb{F}_q \), then a model \( Z_S/S \) is a \( S \)-flat scheme such that \( Z_S \otimes_S k = Z \).

Let \( S_0 \) be a smooth affine variety, which is defined over a finite field \( \mathbb{F}_{q_0} \), is geometrically irreducible over \( \mathbb{F}_{q_0} \), such that

1. \( X/k \) has a smooth projective model \( X_{S_0} \to S_0 \),

2. the absolute Frobenius \( F = F_X : X \to X \) over the Frobenius \( F_k : Spec \, k \to Spec \, k \) has a model \( F_{X_{S_0}} : X_{S_0} \to X_{S_0} \) over the absolute Frobenius \( F_{S_0} : S_0 \to S_0 \).

By Langer’s theorem [12, Theorem 4.1], there is then a quasiprojective model \( M_{S_0} \to S_0 \) of \( M/k \) which universally corepresents the functor of \( \chi \)-stable torsion-free sheaves of rank \( r \) and Hilbert polynomial \( p_{\mathcal{O}_X} \).

Definition 3.9. Let \( S \to S_0 \) by a morphism, such that \( S \) is a smooth affine variety, geometrically irreducible over a finite extension \( \mathbb{F}_q \) of \( \mathbb{F}_{q_0} \), with \( \mathbb{F}_{q_0}(S_0) \subset \mathbb{F}_q(S) \subset k \). Let \( X_S, M_S \) the base changed varieties \( X_{S_0}, M_{S_0} \). Let \( N \) be a Verschiebung divisible subscheme of \( M_{\text{red}} \). Then a model \( N_S \) of \( N \) is called a good model if

1. all irreducible components \( N_i \subset N \subset M_{\text{red}} \) of \( N \), for \( i = 1, \ldots, \rho \), have a model \( N_{iS} \subset M_S \) over \( S \),
2. \( V|_N : N^1 \to N \) has a model \( V_S : N^1_S \to N_S \), where \( N_S = \bigcup_{i=1}^\rho N_{iS} \subset M_S \) and \( N_S^1 \subset N_S \) is dense.

Note, for some closed point \( s \to S \), \( N_{iS} \times_S s \) might be reducible.

For a closed point \( s \to S_0 \), we denote by \( M_{S_0} \times_{S_0} s \), \( X_{S_0} \times_{S_0} s = X_s \), \( F_{S_0} \times_{S_0} s \) the reductions to \( s \). By the universal corepresentability, one has \( M_{S_0} \times_{S_0} s = M_s \), Langer’s moduli scheme of \( \chi \)-stable torsionfree sheaves of rank \( r \) with Hilbert polynomial \( p_{\mathcal{O}_X} = p_{\mathcal{O}_{X_s}} \). By definition, \( F_{S_0} \times_{S_0} s \) is the absolute Frobenius \( F_s \).

We denote by \( V_s \) the Verschiebung on \( M_s \) and by \( V_{S_0} \times_S s \) the reduction of \( V_S \) to \( s \) on the locus where it is defined. Let \( M_S^0 \subset M_S \) be a model of \( M_0 \), with \( S \to S_0 \), with \( S \) smooth affine over a finite extension \( \mathbb{F}_q \) of \( \mathbb{F}_{q_0} \). Then by definition, \( V_S \times_S s \) is defined on \( M_S^0 \times_S s \).

Lemma 3.10. One has \( V_s = V_S \times_S s \) on \( M_S^0 \times_S s \).
Proof. Let \( t \to M^0_S \times_S s \) be a closed point. Since \( \mathbb{F}_q \) is perfect, \( t \to s \) is étale. There is a morphism of schemes \( T \to M^0_S \), such that \( T \) is irreducible over \( \mathbb{F}_q \), the composite \( T \to M^0_S \to S \) is finite and étale onto its image which is a neighbourhood containing \( s \), \( t \in M^0_S \) is the image of \( t_0 \in T \), \( T \to M^0_S \) is a closed embedding in a Zariski open containing \( t_0 \). In particular, one has a factorization \( \mathbb{F}_q(S) \subset \mathbb{F}_q(T) \subset k \). By the universal corepresentability, there is a torsion-free sheaf \( E_T \) over \( X_T \), such that \( E_T \times_T k \) is \( \chi \)-stable of rank \( r \) and with Hilbert polynomial \( p_{\mathcal{O}_X} \). By definition of \( M^0_S \), \( [E_T \times_T k] \in M^0 \), thus \( F^*(E_T \times_T k) \) is \( \chi \)-stable of Hilbert polynomial \( p_{\mathcal{O}_X} \) and rank \( r \). The sheaf \( F^*(E_T \times_T k) \) has a model which we explain now. The pullback of the absolute Frobenius \( F_S \to T \) over \( X_T \), such that \( E_T \times_T k \) is \( \chi \)-stable of rank \( r \) and with Hilbert polynomial \( p_{\mathcal{O}_X} \). By definition of \( M^0_S \), \( [E_T \times_T k] \in M^0 \), thus \( F^*(E_T \times_T k) \) is \( \chi \)-stable of Hilbert polynomial \( p_{\mathcal{O}_X} \) and rank \( r \). The sheaf \( F^*(E_T \times_T k) \) has a model which we explain now. The pullback of the absolute Frobenius \( F_S : S \to S \) by \( T \to S \) is \( F_T : T \to T \), as \( T \to S \) is étale. Thus the pullback of the absolute Frobenius \( F_{X_S} : X_S \to X_S \) over \( F_S \) by \( T \to S \) is the absolute Frobenius \( F_{X_T} : X_T \to X_T \). This defines the model \( F_{X_T}^* E_T \) of \( F^*(E_T \times_T k) \). We conclude that \( F_{X_T}^* (E_T) \times_T t_0 = F_{X_T}^* E_{t_0} \). By definition of \( t_0 \), \( F_{X_T}^* E_{t_0} = F_{X_T}^* E_t \). By definition, \( [F_{X_T}^* E_t] \in M_t \). We conclude \( [F_{X_T}^* E_t] = V_t([E_t]) \). \( \square \)

Corollary 3.11. Let \( N \) be a Verschiebung divisible subscheme of \( M_{\text{red}} \), and let \( N_S \) be a good model. Let \( a \in \mathbb{N} \setminus \{0\} \) such that \( V|_N \) stabilizes the irreducible components of \( N \) as in Lemma 3.10. Then

i) there is a nontrivial open subscheme \( T \subset S \) such that for all closed points \( s \in T \), and for all \( i \in \{1, \ldots, \rho \} \), \( V^i_{T|N_T} \times_T s \) is a dominant rational self-map of \( N \times_T s \);

ii) for any closed point \( s \in T \), there is a dense open subscheme of \( N \times_T s \) on which \( V^a_{T|N_T} \times_T s \) is well defined and is equal to \( V^a_s = V_s \circ \ldots \circ V_s \) (a-times).

Proof. By definition, the restriction of \( V \) on \( N \) is a rational dominant self-map. As \( V_S \) is a good model, \( V_S : N_S^1 \to N_S \) is a well defined dominant map, and \( N_S^1 \subset N_S \) is dense. Thus there is a nontrivial open \( S_a \subset S \) and a dense subscheme \( N^1_S \subset N_S \), such that the composite \( V^a_S = V_{S_a} \circ \ldots \circ V_{S_a} \) (a-times) is well defined and dominant. So there is a nontrivial open \( T \subset S_a \) such that for all closed points \( t \in T \), the restriction \( V^a_T \times_T t : N^a_T \times_T t \to N_T \times_T t = N_t \) is well defined and is dominant. By the choice of \( a \), \( V^a_T \) respects the components \( V_T \). This shows i). As for ii), this is then a direct consequence of Lemma 3.10. This finishes the proof. \( \square \)

Definition 3.12. Let \( X \) be a smooth projective variety defined over a perfect field \( k \) of characteristic \( p > 0 \). Let \( M \) be the moduli of \( \chi \)-stable sheaves of rank \( r \) and Hilbert polynomial \( p_{\mathcal{O}_X} \).

1) A closed point \( [E] \in M \) is called a torsion point if there is a \( m \in \mathbb{N} \setminus \{0\} \) such that \( (F^m)^* E \cong E \).

2) Let \( S \) be a smooth affine variety defined over \( \mathbb{F}_q \) such that \( X \) has a smooth projective model \( X_S \), and let \( M_S \) Langer's quasi-projective moduli \( S \)-scheme. Let \( u \in M_S \) be a closed point, mapping to the closed point
s ∈ S. Then u is said to be a torsion point if u is a torsion point viewed as a closed point in Ms.

Remark 3.13. If u ∈ Ms is a torsion point mapping to s ∈ S, then the torsionfree sheaf E on Xs it corresponds to has a stratification E = E0, E1 = (F^{N-1})^*E, . . . , EN = E, E_{N+1} = E1, . . . It follows by Katz’ theorem [5, Theorem 1.3] that E = (En)_{n∈N} is a stratified bundle (that is, all the En are locally free). Furthermore, all En are stable of slope 0, so all En define modular points in Ms.

Theorem 3.14. Let X be a smooth projective variety defined over a perfect field k of characteristic p > 0. Then M is the moduli of χ-stable torsionfree sheaves of rank r. Let N ⊂ Ms be a Verschiebung divisible subscheme, and let NS be a good model of N. Then the torsion points of NS are dense in NS.

Proof. Our goal is to show that if Z ⊂ NS is any strict closed subscheme, then NS \ Z contains torsion points. So it is enough to show that NT \ Z contains torsion points, where T is defined in Corollary 3.11. Let s0 ∈ T be a closed point and let q(s0) be the cardinality of the residue field κ(s0). We want to show that there are torsion points in NT ×T s0 \ (Z ∩ NT ×T s0).

We apply Hrushovski’s theorem [3, Corollary 1.2]: let i ∈ {1, . . . , ρ}. This fixes the component NT we consider. We define κ to be a finite extension of κ(s0) of cardinality q, so that the irreducible components of Ni := NT ⊗T κ are defined over κ. Here s = s0 ⊗κ(s0) κ. We pick one such irreducible component of maximal dimension, say Y′. Since Vs stabilizes Ni, the same argument as in Lemma 3.7 implies that Vsab stabilizes Y′ for some b ∈ N \ {0}. We choose an open dense affine subvariety Y ⊂ Y′. Then Y is irreducible over Fq. Let Γ ⊂ Y ×s Y be the Zariski closure of the graph of Ψ := Vsab where it is well defined. Then the first projection Γ → Y is birational (and therefore dominant) and the second projection Γ → Y is dominant as Ψ is dominant. Let Φq : Y ⊗κ Fq → Y ⊗κ Fq be the Frobenius raising coordinates to the q-th power. Hrushovski’s theorem loc. cit. asserts that for any proper closed subvariety W ⊂ Y, there is a closed point u ∈ Y \ W, thus that for a suited m ∈ N \ {0} large enough, (u, Φq^m(u)) is a closed point of Γ. Taking W to contain both the locus on which Ψ is not defined, and Z ∩ Y, we obtain that (u, v) ∈ Γ if and only if v = Ψ(u), thus Φq^m(u) = Ψ(u), and u /∈ Z ∩ Y. On the other hand, one has Φq ◦ V_s(u) = V_s ◦ Φq(u) as V_s is defined over κ. Thus from Φq^m(u) = V_s^{ab}(u), we deduce that Φq^m(u) = V_s^{abc}(u) for all c ∈ N \ {0}. As u ∈ T is a closed point, there is a c such that Φq^m(u) = u. We deduce that for all proper closed subvarieties W ⊂ Y containing the indeterminacy locus of Ψ and Z ∩ Y, there is a closed point u ∈ Y \ W and a natural number d ∈ N \ {0}, such that u = V_s^d(u). Thus u is a torsion point. This finishes the proof.

□
Theorem 3.15. Let $X$ be a smooth connected projective variety defined over a perfect field $k$ of characteristic $p > 0$. Let $\bar{x} \to X$ be a geometric point. If $\pi_1^{et}(X \otimes_k \bar{k}, \bar{x}) = \{1\}$, there are no nontrivial stratified bundles.

Proof. We consider rank $r \geq 2$ stratified bundles, as for rank 1, we gave the proof in the introduction. We want to show that $M^s$ is empty. This is equivalent to saying that the Zariski closure of $M^s$ in $M$ is empty, and is also equivalent to saying that any subscheme $N$ of $M_{\text{red}}$ spanned by stratifications is empty (see Remarks 3.5)). Let $N_S$ be a good model of $N$. By Lemma 3.8 together with Theorem 3.14, torsion points are dense in $N_S$. By Remark 3.13, a torsion point $u \in N_T$ mapping to $s \in S$ represents in particular a vector bundle $E$ on $X_s$. By the theorem of Lange-Stuhler [11, Satz 1.4], there is a (noncommutative) geometrically connected étale finite covering $h : Z \to X_s$, such that $h^*E$ is trivial (beware that the other statement of loc.cit, asserting that for any vector bundle $E$, an étale finite covering $h$ which trivializes $h^*E$ exists if and only if $(F^m)^*E \cong E$ for some $m \in \mathbb{N} \setminus \{0\}$ is not correct, although there are many ways to correct it). On the other hand, since $X_S \to S$ has good reduction and is proper, the specialization map $\pi_1(X_k) \to \pi_1(X_s \times_s \text{Spec} \bar{\mathbb{F}}_q)$ is surjective ([17, Exposé X, Théorème 3.8]). The assumption implies then that $\pi_1(X_s \times_s \text{Spec} \bar{\mathbb{F}}_q) = \{1\}$, thus $h$ is the identity, and $E \cong \oplus^r_1 \mathcal{O}_{X_s}$. But $E$ has to be stable of rank $r \geq 2$. This is impossible. So we conclude that $N_S$ is empty, thus $N$ is empty. So there are no stratified bundles $E = (E_n)_{n \in \mathbb{N}}$ with $E_n$ $\mu$-stable of rank $r$ and slope zero, except $E = 1$. By Proposition 2.3 for any stratified bundle, $E(n_0)$ is a successive extension of 1 by itself for $n_0$ large enough. We apply Proposition 2.4 to finish the proof over $k = \bar{k}$. If $k$ is perfect and not algebraically closed, as already noticed in the proof of Proposition 2.3, a stratified bundle $E$ is trivial if and only if $E \otimes_k \bar{k}$ is trivial. This finishes the proof in general. \qed

In the remaining part of this section, we illustrate the theorem with two examples. The second one is due to M. Raynaud.

Let $X$ be a smooth projective variety defined over an algebraically closed field $k = \bar{k}$ of characteristic $p > 0$. We know that if $k \neq \bar{\mathbb{F}}_p$, and if $H^1(X, \mathcal{O}_X)_{ss}$ has $k$-dimension at least $\geq 2$, one easily constructs an infinite family of extensions of 1 by itself in $\text{Strat}(X)$ (see [11, Proposition 2.9]). Furthermore, there are stratified bundles which are not semistable, so they can’t be trivialized after a finite étale covering ([15]). Thus the assumption on the finiteness of $\pi_1^{et}(X \otimes_k \bar{k}, \bar{x})$ in the next corollary is really necessary.

Corollary 3.16. Let $X$ be a smooth connected projective variety defined over an algebraically closed field $k$ of characteristic $p > 0$. Let $x \in X(k)$ be a rational point. Assume $\pi_1^{et}(X, x)$ is finite. Then

i) the surjective homomorphism $\pi^{strat}(X, x) \to \pi_1^{et}(X, x)$ induces an isomorphism on irreducible representations;
ii) if \( \pi_1^{\text{ét}}(X, x) \) has order prime to \( p \), then the surjective homomorphism 
\[ \pi^{\text{strat}}_1(X, x) \to \pi_1^{\text{ét}}(X, x) \]
is an isomorphism; in particular, every stratified bundle is a direct sum of irreducible ones.

**Proof.** Let \( h : Y \to X \) be the universal cover based at \( x \), so it is a Galois cover under \( \pi_1^{\text{ét}}(X, x) \). Let \( \mathbb{I}_Y \) be the trivial stratified bundle on \( Y \), and \( E = h_*\mathbb{I}_Y \) be its direct image. It is an object of \( \text{Strat}(X) \) which comes from the regular representation \( k[\pi_1^{\text{ét}}(X, x)] \) of \( \pi_1^{\text{ét}}(X, x) \). If \( E = (E_n)_{n \in \mathbb{N}} \) is a stratified object, then by Theorem 3.15, \( h^*E \) is trivial, thus \( E \subset H^0(Y, h^*E_0) \otimes \mathbb{E} \). We first show i). If \( E \) is irreducible as a stratified bundle, there is a projection of \( H^0(Y, h^*E_0) \) to a \( k \)-line \( \ell \) such that \( E \) is still injective in \( \ell \otimes_k \mathbb{E} \). Thus \( E \subset \ell \otimes_k \mathbb{E} \) is a subrepresentation of \( \pi^{\text{strat}}_1(X, x) \). Since \( \pi^{\text{strat}}_1(X, x) \to \pi_1^{\text{ét}}(X, x) \) is surjective, \( E \subset \ell \otimes_k \mathbb{E} \) is a subrepresentation of \( \pi_1^{\text{ét}}(X, x) \). This shows i). As for ii), \( h \) then has degree prime to \( p \), thus \( \mathbb{E} \) is a direct sum of irreducible representations of \( \pi_1^{\text{ét}}(X, x) \), thus \( E \subset H^0(Y, h^*E_0) \otimes_k \mathbb{E} \) as well, and is in particular a subrepresentation of \( \pi_1^{\text{ét}}(X, x) \). \( \square \)

We now illustrate Theorem 3.14 in rank 2 over \( k = \mathbb{F}_p \). To this aim, recall that if \( E \) is a stratified bundle over \( X \) projective smooth over \( k = \mathbb{F}_p \), and if \( x \in X(k) \), the Tannaka \( k \)-group \( \text{Aut}^\otimes(\langle E \rangle, x) \subset GL(E_0|_x) \) is also called the **monodromy group**. Recall further that if \( [E] \) is a torsion point (see Definition 3.12) in \( M \), then its monodromy group is in fact a finite quotient of \( \pi_1^{\text{ét}}(X, x) \), as \( E \) is trivialized on the Lange-Stuhler finite étale covering \( h \) (loc. cit.), so the monodromy group is a quotient of the Galois group of \( h \).

**Proposition 3.17** (M. Raynaud). Let \( X \) be a smooth projective variety defined over \( k = \mathbb{F}_p \). Let \( E = (E_n)_{n \in \mathbb{N}} \) be a rank 2 stratified bundle with \( E_n \in M \), the moduli of \( X \)-stable rank 2 torsionfree sheaves, and with \( \det(E) = I \) (i.e. \( \det(E_n) = O_X \) for all \( n \geq 0 \)). Let \( N(E) \) be defined in Definition 3.4. We assume \( N(E) \) irreducible of dimension > 0. Then

i) either there is a dense subset of torsion points \( a_i \in N(E) \) with monodromy group \( SL(2, k_i) \subset GL(E_0|_x) \) where \( k_i \) are finite subfields of \( k \) of increasing order;

ii) or there is a dense subset of torsion points \( a_i \in N(E) \) with monodromy group a dihedral group \( D_{n_i} \) of order \( 2n_i \) with increasing \( n_i \).

One can characterize geometrically the second case: it happens precisely when there is an étale degree 2 covering \( h : Y \to X \) and a rank one stratified bundle \( L \) on \( Y \) such that \( h_*L = E \), or, equivalently, such that \( h^*E \) becomes reducible: \( h^*E \cong L \oplus L^\sigma \), where \( \langle \sigma \rangle \) is the Galois group of \( h \) and \( L^\sigma \) is the Galois translate of \( L \).

**Proof.** Assume there is a degree 2 étale covering \( h : Y \to X \) and a rank 1 stratified bundle \( L \) on \( Y \) such that \( L \subset h^*E \). Then \( L^\sigma \subset h^*E \) and thus \( h^*E \cong L \oplus L^\sigma \). Since \( \det(E) = I \), one has \( L^\sigma \cong L^{-1} \). One has in particular \( h_*L_n = E_n \) for all \( n \geq 0 \). All \( L_n \) lie in \( \text{Pic}^\sigma(Y) \). We define \( N(L) \subset \text{Pic}^\sigma(Y) \) as in Definition 3.4. Then the
morphism \( \text{Pic}^\ell(Y) \to M, L \mapsto h_*L \) sends \( N(L) \) to a closed subscheme of \( M_{\text{red}} \), as \( \text{Pic}^\ell(Y) \) is proper, which is contained in \( N(E) \) by construction, but which contains all the \( E_n = h_*L_n \). Thus \( h_*N(L) = N(E) \). The subscheme \( N(L) \) is the closure of an infinite family of torsion points \( t_m \). Since \( h_*t_m \) lies in \( N(E) \), and since the locus of \( M \) on which the determinant is \( O_X \) is closed, all points of \( N(E) \) have determinant equal to \( O_X \). In particular, one has \( \det(h_*(t_m)) = O_X \). This implies that \( t_m \simeq t_m^{-1} \) as \( \det(h^*h_*t_m) \simeq \det(t_m \oplus t_m^* \sigma) = O_Y \). Let \( y \in Y(k) \) mapping to \( x \in X(k) \). The exact sequence \( 1 \to \pi^\ell_1(Y, y) \to \pi^\ell_1(X, x) \to \mathbb{Z}/2 \to 0 \) induces by pushout an exact sequence \( 1 \to H \to D \to \mathbb{Z}/2 \to 0 \) for any finite quotient \( \pi^\ell_1(Y, y) \to H \). Taking for \( H \) the cyclic quotients \( \mathbb{Z}/n(m)\mathbb{Z} \) corresponding to the \( t_m \) yields a stratified line bundle \( L_m \) on \( Y \), such that the generator \( \sigma \in \pi^\ell_1(X, x) \) of the automorphism group of \( Y \) over \( X \) acts via \( x \mapsto -x \) on \( \mathbb{Z}/n(m)\mathbb{Z} \). Thus \( D \) is the dihedral group \( D_{n(m)} \). Summarizing: if there is a \( h : Y \to X \) such that \( h^*E \) becomes strictly semistable, then points with monodromy group a dihedral group \( D_{n(m)} \) of increasing order \( n(m) \) are dense in \( N(E) \). (Note we do not need the irreducibility of \( N(E) \) for this point).

Vice-versa, assume the points \( \delta_m \) with dihedral monodromy \( D_{n(m)} \) are dense in \( N(E) \). Since \( \pi^\ell_1(X, x) \) has only finitely many \( \mathbb{Z}/2\mathbb{Z} \) quotients, there is an infinite sequence of such points \( \delta_n \) such that the induced quotient \( \pi^\ell_1(X, x) \to \mathbb{Z}/2\mathbb{Z} \) is fixed. Let \( h : Y \to X \) be the corresponding covering and \( y \in Y(k) \) mapping to \( x \). The representation \( \pi^\ell_1(X, x) \to D_{n(m)} \) induces a representation \( \pi^\ell_1(Y, y) \to \mathbb{Z}/n(m)\mathbb{Z} \), defining a stratified rank 1 bundle \( L_m \) on \( Y \), such that \( L_m \simeq L_m^{-1} \). Then the points \( h_*L_m \) are dense in \( N(E) \). Thus the Zariski closure \( Z \) of the \( L_m \) in \( \text{Pic}^\ell(Y) \) has the property that \( h_*Z = N(E) \) by the argument we had before. Thus there is a \( L \in Z \) such that \( h_*L = E \). This finishes the characterization of the second case.

To show that one has either case i) or ii), one has to appeal to the classification of finite groups of \( SL(2, \overline{\mathbb{F}}_p) \): one deduces from Dickson’s Theorem \cite{Dickson}, Hauptsatz 8.27] that there are only 2 infinite families of finite subgroups of \( SL(2, \overline{\mathbb{F}}_p) \) such that the natural rank 2 representation is geometrically irreducible, the dihedral subgroups \( D_m \) and the finite subgroups \( SL(2, \mathbb{F}_{q_i}) \). Arguing again that fixing a finite group \( H \), there are finitely many surjections \( \pi^\ell_1(X, x) \to H \), one shows the dichotomy.

\[ \square \]

4. Some remarks and questions

4.1. Using the effective Lefschetz properties \cite{Deligne} Corollaire 3.4, the property that stratified bundles uniquely lift to the formal neighbourhood of a smooth divisor \cite{Illusie}, Proposition 1.5, and the fact that coherent \( \mathcal{D}_X \)-modules which are \( \mathcal{O}_X \)-coherent are locally free (\cite{Illusie} Lemma 6), one shows that if \( Y \subset X \) is a smooth ample divisor on a smooth projective variety defined over a perfect field
of characteristic \( p > 0 \), and if \( y \in Y(k) \), then the homomorphism \( \pi^\text{strat}_1(Y, y) \to \pi^\text{strat}_1(X, x) \) induced by the restriction of bundles is an isomorphism, if \( \dim_k(Y) \geq 2 \). But in order to reduce the proof of the main theorem of this article to surfaces, one would need the full strength of Langer’s theorem on corepresentatibility, which is hard to extract from Gieseker’s article. So we did not present the argument via this reduction.

4.2. Simpson constructed in [16] quasi-projective moduli schemes of semi-simple bundles with flat connection on a complex smooth projective variety. It is not unlikely that his ideas, combined with Langer’s methods in characteristic \( p > 0 \) ([12]) and Proposition 2.3 yield the existence of quasi-projective moduli schemes of stratified bundles. Those moduli, aside of their own interest, could then be used directly to prove Theorem 1.1.

4.3. As mentioned in the introduction, Grothendieck’s theorem over \( \mathbb{C} \) has nothing to do with stability questions, and applies to the algebraic completion of the topological fundamental group of any smooth complex quasi-projective manifold. If we translate his theorem via the Riemann-Hilbert correspondence on \( X \) smooth complex but not necessarily proper, then he shows that if the profinite completion topological fundamental group is trivial, there is no nontrivial regular singular bundles with flat connection. The analog in characteristic \( p > 0 \) over a quasi-projective smooth variety \( X \) would relate Grothendieck’s étale fundamental group over \( \bar{k} \) and tame stratified bundles. This would require a completely different proof. At any rate, we do not even have at disposal a theory of good lattices in \( j_*E \), where \( E \) is stratified on \( X \) and \( j : X \to \bar{X} \) is a normal compactification. Unfortunately, we can not say anything on this subject.

4.4. The proof of the main theorem may be seen as an application of Hrushovski’s theorem. On one hand, it is very nice to see how his profound theorem works concretely for some natural question coming from algebraic geometry. On the other hand, to have another proof anchored in algebraic geometry would perhaps shed more light on the original problem.

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