Chebyshev Polynomials and Continued Fractions Related

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Abstract
Let \( p, q \) be complex polynomials, \( \deg p > \deg q \geq 0 \). We consider the family of polynomials defined by the recurrence \( P_{n+1} = 2pP_n - qP_{n-1} \) for \( n = 1, 2, 3, \ldots \) with arbitrary \( P_1 \) and \( P_0 \) as well as the domain of the convergence of the infinite continued fraction

\[
f(z) = 2p(z) - \frac{q(z)}{2p(z) - \frac{q(z)}{2p(z) - \ldots}}
\]

Key words: Chebyshev polynomials, continued fractions, Binet formula, Cassini identity

1 Some polynomials of the Chebyshev type

Let \( P_0 \) and \( P_1 \) be polynomials of one complex variable, \( \deg P_1 > \deg P_0 \geq 0 \). Let \( p, q \) be polynomials of one complex variable, \( \deg p > \deg q \geq 0, q \neq 0 \). Define the family of polynomials \( P_n \) by the recurrence formula

\[
P_{n+1}(z) = 2p(z)P_n(z) - q(z)P_{n-1}(z), \quad n = 1, 2, 3, \ldots
\]

(1)

Note that (1) gives the Chebyshev polynomials of

- the first kind \( T_n \) for \( P_0(z) = 1, P_1(z) = z, p(z) = z \) and \( q(z) = 1 \)
- the second kind \( U_n \) for \( P_0(z) = 1, P_1(z) = 2z, p(z) = z \) and \( q(z) = 1 \)
- the third kind \( V_n \) for \( P_0(z) = 1, P_1(z) = 2z - 1, p(z) = z \) and \( q(z) = 1 \)
- the fourth kind \( W_n \) for \( P_0(z) = 1, P_1(z) = 2z + 1, p(z) = z \) and \( q(z) = 1 \).

(See [2], Appendix B, Table B.2).

We write the recurrence (1) in the matrix form

\[
\begin{bmatrix}
P_{n+1} \\
P_n \\
P_{n-1}
\end{bmatrix} =
\begin{bmatrix}
2p & -q \\
1 & 0 \\
0 & -q
\end{bmatrix}
\begin{bmatrix}
P_n \\
P_{n-1} \\
P_{n-2}
\end{bmatrix}
\]

(2)

proceeding as in [1], p.80, where the Fibonacci sequence was considered, defined by the similar recurrence

\[
\begin{bmatrix}
F_{n+1} \\
F_n \\
F_{n-1}
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
F_n \\
F_{n-1} \\
F_{n-2}
\end{bmatrix}
\]

with \( F_0 = 0 \) and \( F_1 = 1 \).

Note that the characteristic polynomial

\[
w(\lambda) = \det \begin{bmatrix} 2p - \lambda & -q \\ 1 & -\lambda \end{bmatrix} = (\lambda - p)^2 - p^2 + q
\]

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of the matrix
\[
\begin{bmatrix}
2p & -q \\
1 & 0
\end{bmatrix}
\quad (3)
\]
denotes two different roots
\[
\lambda_1 = p + \sqrt{p^2 - q} \text{ and } \lambda_2 = p - \sqrt{p^2 - q},
\quad (4)
\]
as the polynomial \( q \) is assumed to be nonzero.

**Theorem 1.1** For the polynomial \( P_n \) defined by (1) we get the following formula
\[
P_n = \frac{1}{\lambda_1 - \lambda_2} \left( (\lambda_1^n - \lambda_2^n) P_1 - \lambda_1 \lambda_2 (\lambda_1^{n-1} - \lambda_2^{n-1}) P_0 \right)
\quad (5)
\]
where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues (4) of the matrix (3).

**Proof.** By the Jordan decomposition of the matrix (3) we get
\[
\begin{bmatrix}
2p & -q \\
1 & 0
\end{bmatrix}
= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix}
\lambda_1 & \lambda_2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
1 & -\lambda_2 \\
-1 & \lambda_1
\end{bmatrix}.
\]
The \( n \)-th power of the matrix (3) equals
\[
\begin{bmatrix}
2p & -q \\
1 & 0
\end{bmatrix}^n
= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix}
\lambda_1 & \lambda_2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_1^n & 0 \\
0 & \lambda_2^n
\end{bmatrix}
\begin{bmatrix}
1 & -\lambda_2 \\
-1 & \lambda_1
\end{bmatrix}.
\]
Hence, by the recurrence
\[
\begin{bmatrix}
P_{n+1} & P_n \\
P_n & P_{n-1}
\end{bmatrix}
= \begin{bmatrix}
2p & -q \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
P_n & P_{n-1} \\
P_{n-1} & P_{n-2}
\end{bmatrix}
= \begin{bmatrix}
2p & -q \\
1 & 0
\end{bmatrix}^{n-1}
\begin{bmatrix}
P_2 & P_1 \\
P_1 & P_0
\end{bmatrix}
\]
we obtain
\[
\begin{bmatrix}
P_{n+1} & P_n \\
P_n & P_{n-1}
\end{bmatrix}
= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix}
\lambda_1 & \lambda_2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_1^{n-1} & 0 \\
0 & \lambda_2^{n-1}
\end{bmatrix}
\begin{bmatrix}
1 & -\lambda_2 \\
-1 & \lambda_1
\end{bmatrix}
\begin{bmatrix}
P_2 & P_1 \\
P_1 & P_0
\end{bmatrix}.
\]
Multiplying the above matrices we get
\[
P_n = \frac{1}{\lambda_1 - \lambda_2} \left( (\lambda_1^n - \lambda_2^n) P_1 - \lambda_1 \lambda_2 (\lambda_1^{n-1} - \lambda_2^{n-1}) P_0 \right)
\]
\[\square\]
Note that (5) corresponds to the well known Binet formula for the Fibonacci sequence
\[
F_n = \frac{\mu_1^n - \mu_2^n}{\mu_1 - \mu_2} = \frac{1}{\sqrt{5}} \left( \left( \frac{\sqrt{5} + 1}{2} \right)^n - \left( \frac{-\sqrt{5} + 1}{2} \right)^n \right)
\]
where $\mu_1 = \frac{\sqrt{5}+1}{2}$ and $\mu_2 = \frac{-\sqrt{5}+1}{2}$ are eigenvalues of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ that defines the Fibonacci sequence $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$ with $F_0 = 0$ and $F_1 = 1$.

**Remark 1.2** The formula (5) works well with two known formulae (see [2] 1.49 and 1.52) for the Chebyshev polynomials of the first kind $T_n(x)$ for the Chebyshev polynomials of the second kind $U_n(x)$ if we put $p(x) = x$, $q(x) = 1$, $\lambda_1(x) = x + \sqrt{x^2 - 1}$, $\lambda_2(x) = x - \sqrt{x^2 - 1}$, $P_0(x) = 1$ and $P_1(x) = x$:

$$T_n(x) = \frac{1}{\lambda_1 - \lambda_2} \left[ (\lambda_1^n - \lambda_2^n)x - (\lambda_1^{n-1} - \lambda_2^{n-1}) \right]$$

$$= \frac{1}{\lambda_1 - \lambda_2} \left[ \lambda_1^n \left( x - \frac{1}{\lambda_1} \right) - \lambda_2^n \left( x - \frac{1}{\lambda_2} \right) \right]$$

$$= \frac{1}{2} (\lambda_1^n + \lambda_2^n)$$

$$= \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right), \ |x| \geq 1,$$

and for the Chebyshev polynomials of the second kind $U_n(x)$ if we put $p(x) = x$, $q(x) = 1$, $\lambda_1(x) = x + \sqrt{x^2 - 1}$, $\lambda_2(x) = x - \sqrt{x^2 - 1}$, $P_0(x) = 1$ and $P_1(x) = 2x$:

$$U_n(x) = \frac{1}{\lambda_1 - \lambda_2} \left[ (\lambda_1^n - \lambda_2^n)2x - (\lambda_1^{n-1} - \lambda_2^{n-1}) \right]$$

$$= \frac{1}{\lambda_1 - \lambda_2} \left[ \lambda_1^n \left( 2x - \frac{1}{\lambda_1} \right) - \lambda_2^n \left( 2x - \frac{1}{\lambda_2} \right) \right]$$

$$= \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^{n+1} - \lambda_2^{n+1})$$

$$= \frac{1}{2\sqrt{x^2 - 1}} \left( (x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1} \right), \ |x| \geq 1.$$

Proceeding as above we get the next two formulae for the Chebyshev polynomials of the third and the fourth kind $V_n$, $W_n$, respectively:

$$V_n(x) = \frac{1}{\lambda_1 - \lambda_2} \left[ (\lambda_1^n - \lambda_2^n)(2x - 1) - (\lambda_1^{n-1} - \lambda_2^{n-1}) \right]$$

$$= \frac{1}{\lambda_1 - \lambda_2} \left[ \lambda_1^n \left( 2x - 1 - \frac{1}{\lambda_1} \right) - \lambda_2^n \left( 2x - 1 - \frac{1}{\lambda_2} \right) \right]$$

$$= \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^{n+1} - \lambda_2^{n+1}) - \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^n - \lambda_2^n)$$

$$= U_n(x) - U_{n-1}(x)$$
and

\[ W_n(x) = \frac{1}{\lambda_1 - \lambda_2} \left[ (\lambda_1^n - \lambda_2^n)(2x + 1) - (\lambda_1^{n-1} - \lambda_2^{n-1}) \right] \]

\[ = \frac{1}{\lambda_1 - \lambda_2} \left[ \lambda_1^n(2x + 1 - \frac{1}{\lambda_1}) - \lambda_2^n(2x + 1 - \frac{1}{\lambda_2}) \right] \]

\[ = \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^{n+1} - \lambda_2^{n+1}) + \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^n - \lambda_2^n) \]

\[ = U_n(x) + U_{n-1}(x) \]

**Remark 1.3** If \( \deg q = 0 \), i.e. \( q \) is a nonzero constant, one may continue defining polynomials \( P_n \) for negative integers putting initial polynomials \( P_0, P_1 \) arbitrarily and the recurrence formula \( P_{n-1} = -\frac{1}{q}P_{n+1} + \frac{2p}{q}P_n \) equivalent to the relation \( P_{n+1} = 2pP_n(z) - qP_{n-1} \). Same as before we have

\[
\begin{bmatrix}
P_n & P_{n-1}
\end{bmatrix}
= \left[ 2p - q \right]^{-1}
\begin{bmatrix}
P_{n+1} & P_n
\end{bmatrix}
\]

We obtain

\[
\begin{bmatrix}
P_0 & P_{-1}
\end{bmatrix}
= \left[ 2p - q \right]^{-1}
\begin{bmatrix}
P_1 & P_0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
P_{-n+1} & P_{-n}
\end{bmatrix}
= \left[ 2p - q \right]^{-n}
\begin{bmatrix}
P_1 & P_0
\end{bmatrix}
\]

\[ = \frac{1}{\lambda_1 - \lambda_2}
\begin{bmatrix}
\lambda_1 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
\lambda_1^{n-1} & 0
\end{bmatrix}
\begin{bmatrix}
1 & -\lambda_1
\end{bmatrix}
\begin{bmatrix}
P_1 & P_0
\end{bmatrix}
\]

Multiplying the above matrices we get an analogous formula as (5) in Theorem 1.1:

\[ P_{-n} = \frac{1}{\lambda_1 - \lambda_2} \left[ (\lambda_1^{n} - \lambda_2^{n})P_1 - \lambda_1\lambda_2(\lambda_1^{n-1} - \lambda_2^{n-1})P_0 \right] \]

Calculating the determinant of the matrix

\[
\begin{bmatrix}
P_{n+1} & P_n
\end{bmatrix}
= \frac{1}{\lambda_1 - \lambda_2}
\begin{bmatrix}
\lambda_1 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
\lambda_1^{n-1} & 0
\end{bmatrix}
\begin{bmatrix}
1 & -\lambda_1
\end{bmatrix}
\begin{bmatrix}
P_2 & P_1
\end{bmatrix}
\]

we get the Cassini type identity for the polynomials \( P_n \), corresponding to the Cassini identity for the Fibonacci sequence \( F_{n+1}F_{n-1} - F_n^2 = \frac{1}{1} \begin{bmatrix}
1 & 1
\end{bmatrix}^n \begin{bmatrix}
1 & 1
\end{bmatrix} = (-1)^n: \]

\[ (\lambda_1 - \lambda_2)^2 \det \begin{bmatrix}
P_{n+1} & P_n
\end{bmatrix} = \det \begin{bmatrix}
\lambda_1 & \lambda_2
\end{bmatrix} \det \begin{bmatrix}
\lambda_1^{n-1} & 0
\end{bmatrix} \det \begin{bmatrix}
1 & -\lambda_2
\end{bmatrix} \det \begin{bmatrix}
P_2 & P_1
\end{bmatrix} \]

\[ \det \begin{bmatrix}
P_{n+1} & P_n
\end{bmatrix} = \det \begin{bmatrix}
\lambda_1^{n-1} & 0
\end{bmatrix} \det \begin{bmatrix}
P_2 & P_1
\end{bmatrix} \]
Since $\lambda_1 \lambda_2 = q$ we get the following remark.

**Remark 1.4** The Cassini type identity for the polynomials $P_n$ defined by (1) holds:

$$P_{n+1}P_{n-1} - P_n^2 = q^{n-1}(P_2P_0 - P_1^2)$$

which implies the four known formulae for the Chebyshev polynomials of the first, second, third and fourth kind, respectively:

$$
\begin{align*}
T_{n+1}(x)T_{n-1}(x) - T_n^2(x) &= x^2 - 1 \\
U_{n+1}(x)U_{n-1}(x) - U_n^2(x) &= -1 \\
V_{n+1}(x)V_{n-1}(x) - V_n^2(x) &= -2x - 2 \\
W_{n+1}(x)W_{n-1}(x) - W_n^2(x) &= 2x - 2.
\end{align*}
$$

**Theorem 1.5** Let $P_n$ be the sequence of polynomials defined by (1). The quotient $P_{n+1}/P_n$ converges uniformly on compact subsets of the set

$$\{ z \in \mathbb{C} : \left| \frac{\lambda_1(z)}{\lambda_2(z)} \right| > 1 \}$$

to the limit $\lambda_1$. The limit does not depend on the initial polynomials $P_0$ and $P_1$.

**Proof.** By (5) the quotient of polynomials $P_{n+1}$ and $P_n$ equals

$$\frac{P_{n+1}}{P_n} = \frac{(\lambda_1^{n+1} - \lambda_2^{n+1})P_1 - \lambda_1\lambda_2(\lambda_1^n - \lambda_2^n)P_0}{(\lambda_1^n - \lambda_2^n)P_1 - \lambda_1\lambda_2(\lambda_1^{n-1} - \lambda_2^{n-1})P_0} = \frac{(\lambda_1 - \lambda_2(\lambda_2/\lambda_1)^n)P_1 - \lambda_1\lambda_2(1 - (\lambda_2/\lambda_1)^n)P_0}{(1 - (\lambda_2/\lambda_1)^n)P_1 - \lambda_2(1 - (\lambda_2/\lambda_1)^{n-1})P_0}$$

It converges on compact subsets the set $\{ z \in \mathbb{C} : \left| \frac{\lambda_1(z)}{\lambda_2(z)} \right| > 1 \}$ uniformly to the limit

$$\frac{\lambda_1P_1 - \lambda_1\lambda_2P_0}{P_1 - \lambda_2P_0} = \lambda_1$$

that is independent of $P_0$ and $P_1$. $\square$

### 2 Continued fractions related to polynomials $P_n$

Consider the infinite continued fraction

$$f(z) = 2p(z) - \frac{q(z)}{2p(z) - \frac{q(z)}{2p(z) - ...}}$$
and the rational functions \( r_k \) related to \( f \):

\[
\begin{align*}
    r_1 &= \frac{P_1}{P_0} \\
    r_2 &= \frac{P_2}{P_1} = \frac{2pP_1 - qP_0}{P_1} = 2p - \frac{q}{P_1/P_0} \\
    r_3 &= \frac{P_3}{P_2} = \frac{2pP_2 - qP_1}{P_2} = 2p - \frac{q}{P_2/P_1} \\
    \vdots & \quad \vdots \\
    r_{n+1} &= \frac{P_{n+1}}{P_n} = \frac{2pP_n - qP_{n-1}}{P_{n-1}} = 2p - \frac{q}{P_n/P_{n-1}} \\
    \vdots & \quad \vdots
\end{align*}
\]

It is easy to see that the function \( f \) is the limit of the sequence \( r_k \).

**Theorem 2.1** For arbitrary polynomials \( p \) and \( q \) such that \( \deg p > \deg q > 0, q \neq 0 \), the continued fraction

\[
f(z) = 2p(z) - \frac{q(z)}{2p(z) - \frac{q(z)}{2p(z) - \ldots}}
\]

is a holomorphic function on the set \( \{ z \in \mathbb{C} : \left| \frac{\lambda_1(z)}{\lambda_2(z)} \right| > 1 \} \) where \( \lambda_1 \) and \( \lambda_2 \) are eigenvalues (4) of the matrix (3).

**Proof.** The statement follows from Theorem 1.5. \( \square \)

**Remark 2.2.** In the simple case \( p(z) = 2\alpha z \) and \( q(z) = \beta^2 \) the set \( \{ z \in \mathbb{C} : \left| \frac{\lambda_1(z)}{\lambda_2(z)} \right| > 1 \} \) is the exterior of the interval connecting two points on the complex plane: \( \frac{\beta}{\alpha} \) and \( -\frac{\beta}{\alpha} \). See Figure 1 for the density plot of the absolute value of the function \( r_{60} \) for \( p(z) = 2z \) and \( q(z) = i = \left( \frac{1+i}{\sqrt{2}} \right)^2 \) with a visible scar connecting the points \( -\frac{1+i}{\sqrt{2}} \) and \( \frac{1+i}{\sqrt{2}} \). The following plots exhibit the density plot of \( |r_{60}| \) and more complex scars containing points where the sequence \( r_k \) is divergent if \( p(z) = z^2 + \frac{i}{10}, q(z) = i \) (Figure 2), \( p(z) = iz^3 + \frac{1}{z}, q(z) = -\frac{1}{3} \) (Figure 3), \( p(z) = iz^2 - \frac{1}{5}z + \frac{1}{10}, q(z) = (i - \frac{1}{3})z + \frac{1}{5} + \frac{1}{5} \) (Figure 4).

The plots were created using Mathematica Wolfram Research program.

**References**

[1] Donald E. Knuth, *The Art of Computer Programming*, Addison Wesley, 2nd edition, 1973.

[2] John C. Mason, David Handscomb, *Chebyshev polynomials*, Chapman & Hall, 2003.
Figure 1: Density plot of $|r_{60}|$ for $p(z) = 2z$ and $q(z) = i$.

Figure 2: Density plot of $|r_{60}|$ for $p(z) = z^2 + \frac{i}{10}$, $q(z) = i$. 
Figure 3: Density plot of $|r_{60}|$ for $p(z) = iz^3 + \frac{1}{2}$, $q(z) = -\frac{1}{3}$.

Figure 4: Density plot of $|r_{60}|$ for $p(z) = iz^2 - \frac{1}{5}z + \frac{1}{10}$, $q(z) = (i - \frac{1}{3})z + \frac{1}{5} + \frac{1}{5}$.