On some problems of modelling the non-stationary heat conductivity process in an axisymmetric multilayer medium

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Abstract. Some possibilities of solving the non-stationary heat equation in an axisymmetric multilayer medium are considered. The problem is solved by combining Fourier method and the matrix method. The solutions of the first and third boundary value problems are considered. Examples of calculations by the specified method for three-layer and five-layer axisymmetric systems are given.

1. Introduction
Multilayer materials in the form of plates, shells, and screens are finding increasing practical application in technology. During operation, they are exposed to external thermal influences (as a result of heating by electromagnetic radiation and/or charged particles) and often operate under extreme conditions - for example, in space, in nuclear power, etc. The study of thermal regimes in a multilayer shell makes it possible to predict the behaviour of individual layers and to identify conditions conducive to possible deformation, melting, and other changes in the physical or chemical nature of individual layers. To predict the influence of external factors on materials and multilayer structures, it is necessary to have good calculation methods that allow predicting possible undesirable phenomena.

We give a short review of some recent research on this issue. In a paper [1], an example was considered when the coefficient of thermal conductivity changes exponentially depending on the coordinate. The resulting solution is expressed in terms of Bessel functions. A paper [2] includes conditions for the electromagnetic exchange of energy between layers, and the material layers, except for the extreme ones, are considered quite thin. In a paper [3], the most general formulation of the heat transfer problem in the fuel piping system is studied at a high theoretical level, including the possibility of solving it in various classes of functions.

In our research, we propose to use a combination of the matrix method and the method of generalized powers of Bers and the Fourier method in the case of a nonstationary process to solve the problem of heat and mass transfer in multilayer media.

The matrix method as applied to problems of conduction of heat in composite slabs is described in [4]. However, it was not widely used to solve heat- and mass-transfer problems in multilayer media, possibly due to the fact that the formulas of the analytical solution were extremely complex, computer algebra systems were just beginning to emerge at that time, and therefore numerical methods were preferred. At present, due to the development of symbolic computation systems, this method can be effectively used for calculations. Also, the idea of the matrix method is currently successfully applied in theoretical research [5, 6].
In our research, the analytical matrix method proposed in [4] was used in conjunction with the method of generalized powers of Bers [7–8], which allowed us to successfully describe in a single form the process of heat and mass transfer in multilayer medium with different geometries: planar, axisymmetric or layers with central symmetry. Earlier we used the matrix method to solve stationary problems [9, 10], as well as to construct a new computational algorithm [11, 12]. We also applied a similar idea to solve the non-stationary heat conduction problem in a multilayer medium by the Fourier method in combination with the use of complex functions [13].

This paper continues to research the possibility of combining the matrix method and the classical Fourier method for solving the non-stationary problem of heat conduction in an axisymmetric multilayer medium.

2. Statement of the problem

Consider the process of thermal conductivity in an axisymmetric multilayer medium (Figure 1) described by the equation

\[
\frac{1}{c(x)\rho(x)x} \frac{\partial}{\partial x} \left( \lambda(x)x \frac{\partial T(x,t)}{\partial x} \right) = \frac{\partial T(x,t)}{\partial t},
\]

where \( \lambda(x), c(x), \rho(x) \) are respectively, the coefficient of thermal conductivity, heat capacity and density of the medium. The flow is directed along the \( x \)-axis

\[
J(x,t) = -\lambda(x)x \frac{\partial T(x,t)}{\partial x}.
\]

![Figure 1. Scheme of an axisymmetric multilayer medium.](image)

If the physical parameters are constant on each layer of the multilayer medium, then the heat conduction process can be described by a system of differential equations

\[
\frac{1}{c^{(i)}\rho^{(i)}x} \frac{\partial}{\partial x} \left( \lambda^{(i)}x \frac{\partial T^{(i)}(x,t)}{\partial x} \right) = \frac{\partial T^{(i)}(x,t)}{\partial t}, \quad i = 1, n,
\]

where \( \lambda^{(i)}, c^{(i)}, \rho^{(i)} \) are the thermal conductivity coefficient, heat capacity and density on the \( i \)-th layer respectively. Then the flux on each layer is determined by the formula

\[
J^{(i)}(x,t) = -\lambda^{(i)}x \frac{\partial T^{(i)}(x,t)}{\partial x}, \quad i = 1, n.
\]

At the boundaries between the layers of the medium, we take the conditions of ideal contact

\[
T^{(i)}(x_{i+1}, t) = T^{(i+1)}(x_{i+1}, t), \quad J^{(i)}(x_{i+1}, t) = J^{(i+1)}(x_{i+1}, t), \quad i = 1, n-1.
\]

The initial temperature distribution is given

\[
T^{(i)}(x,0) = g(x), \quad x \in \left[x_i, x_{i+1}\right], \quad i = 1, n.
\]
where \( g(x) \) is a function that can be specified layer by layer, i.e. may have discontinuities of the first kind at the contact points of the layers.

In this paper, we will consider the solution of the first boundary value problem

\[
T^{(1)}(x_1,t) = 0, \quad T^{(n)}(x_{n+1},t) = 0,
\]

and the solution of the third boundary value problem

\[
T^{(1)}(x_1,t) - T_1 = -r^{(1)}J^{(1)}(x_1,t), \quad T_2 - T^{(n)}(x_{n+1},t) = -r^{(n)}J^{(n)}(x_{n+1},t),
\]

where \( r^{(1)} \) and \( r^{(n+1)} \) are thermal resistance coefficients at the boundaries of a multilayer medium, \( T_1 \) and \( T_2 \) are external temperatures at the boundaries of the medium.

3. The method of calculation

The solution of the posed problems (1 – 4) and (1, 2, 3, 5) will be sought by the combined application of the matrix method and the Fourier method.

We write a solution of equations (1) in the form

\[
T^{(i)}(x,t) = u^{(i)}(x)e^{-\mu t}, \quad i = 1, n.
\]

The amplitude function \( u^{(i)}(x) \) satisfies the equation

\[
\frac{1}{c^{(i)}\rho^{(i)}x} \frac{d}{dx} \left( \lambda^{(i)}x \frac{d}{dx}u^{(i)}(x) \right) + \mu^2 u^{(i)}(x) = 0,
\]

conditions of ideal contact of layers are

\[
u^{(i)}(x_{i+1}) = u^{(i+1)}(x_{i+1}), \quad j^{(i)}(x_{i+1}) = j^{(i+1)}(x_{i+1}), \quad i = 1, n,
\]

where \( j^{(i)}(x) = -\lambda^{(i)}xdu^{(i)}/dx \) and boundary conditions for the first boundary value problem (1 – 4) are

\[
u^{(i)}(x_1) = 0, \quad u^{(n)}(x_{n+1}) = 0,
\]

and for the third boundary value problem (1, 2, 3, 5) the conditions at zero external temperatures are

\[
u^{(i)}(x_1) = -r^{(i)}j^{(i)}(x_1), \quad u^{(n)}(x_{n+1}) = r^{(n+1)}j^{(n)}(x_{n+1}).
\]

If the external temperatures are not equal to zero, then the solution to problem (1, 2, 3, 5) is sought in the form of the sum of solutions of stationary and nonstationary subproblems

\[
T^{(i)}(x,t) = T^{(i)}_{st}(x,t) + T^{(i)}_{nonst}(x,t), \quad i = 1, n.
\]

Then the stationary subproblem has the form

\[
\frac{1}{c^{(i)}\rho^{(i)}x} \frac{d}{dx} \left( \lambda^{(i)}x \frac{dT^{(i)}_{st}(x,t)}{dx} \right) = 0, \quad i = 1, n,
\]

\[
T^{(i)}_{st}(x_{i+1}) = T^{(i+1)}_{st}(x_{i+1}), \quad J^{(i)}_{st}(x_{i+1}) = J^{(i+1)}_{st}(x_{i+1}), \quad i = 1, n-1.
\]

And nonstationary subproblem has the form

\[
T^{(i)}_{st}(x_1) - T_1 = -r^{(i)}J^{(i)}_{st}(x_1), \quad T_2 - T^{(n)}_{st}(x_{n+1}) = -r^{(n+1)}J^{(n)}_{st}(x_{n+1}).
\]
We solve the stationary subproblem (6) by the matrix method. On the segment $[x_i, x_{i+1}]$ we set the Cauchy problem

$$T_{\alpha}^{(i)}(x, t) = T_{\alpha}^{(i+1)}(x_{i+1}, t), \quad J_{\alpha}^{(i)}(x_i, t) = J_{\alpha}^{(i+1)}(x_{i+1}, t), \quad i = 1, n-1,$$

where

$$T_{\alpha}^{(i)}(x, 0) = g(x) - T_{\alpha}^{(i)}(x), \quad x \in [x_i, x_{i+1}], \quad i = 1, n.$$

The solution of the Cauchy problem for a segment $[x_i, x_{i+1}]$ in the Bers formalism $[7, 8]$ has the form

$$T_{\alpha}^{(i)}(x) = T_{\alpha}^{(i)}(x_i) - J_{\alpha}^{(i)}(x_i) X_{\alpha}(x_i, x), \quad J_{\alpha}^{(i)}(x) = J_{\alpha}^{(i)}(x_i).$$

where generalized power of Bers takes the form for axisymmetric layers

$$X_{\alpha}(x, x_i) = \int_{x_i}^{x} \frac{d\xi}{\alpha^{(i)}(\xi)} = \int_{x_i}^{x} \frac{d\xi}{\lambda^{(i)} x} = \frac{1}{\lambda^{(i)}} \ln \frac{x}{x_i}.$$

We write the solution (8) for the $i$-th layer in matrix form

$$V_{\alpha}^{(i)}(x) = K_{\alpha}^{(i)}(x, x_i) V_{\alpha}^{(i)}(x_i),$$

where

$$V_{\alpha}^{(i)}(x) = \begin{pmatrix} T_{\alpha}^{(i)}(x) \\ J_{\alpha}^{(i)}(x) \end{pmatrix}, \quad K_{\alpha}^{(i)}(x, x_i) = \begin{pmatrix} 1 & -X_{\alpha}(x, x_i) \\ 0 & 1 \end{pmatrix}, \quad V_{\alpha}^{(i)}(x_i) = \begin{pmatrix} T_{\alpha}^{(i)}(x_i) \\ J_{\alpha}^{(i)}(x_i) \end{pmatrix}.$$
\begin{align*}
V^{(i)}(x) = K^{(i)}(x, x_i) \cdots K^{(i)}(x, x_N) V^{(i)}(x_N) = K^{(i,1)}(x, x_i) V^{(i)}(x_i) \\
= \begin{cases}
1 & \sum_{k=1}^{i} X_k (x, x_i + 1) \\
0 & 1
\end{cases} T^{(i)}(x_i), \quad x_i \leq x \leq x_{i+1}
\end{align*}

(9)

Then, at the end point \(x_{i+1}\) of the layer system, we obtain

\[V^{(i)}(x_{i+1}) = K^{(i,1)}(x_{i+1}, x_i) V^{(i)}(x_i),\]

whence, using the boundary conditions, we find \(T^{(i)}(x_i)\) and \(J^{(i)}(x_i)\), substitute in (9) and obtain a solution to the stationary subproblem (6).

Next, we solve non-stationary problems (7). On the \(i\)-th layer, we define the Cauchy problem

\[u^{(i)}(x)\bigg|_{x=x_i} = u^{(i)}(x_i), \quad j^{(i)}(x)\bigg|_{x=x_i} = j^{(i)}(x_i),\]

then the solution to the Cauchy problem for each layer has the form

\begin{align*}
u^{(i)}(x) &= u^{(i)}(x_i) \cos \mu X_i (x, x_i) - \frac{1}{\mu} j^{(i)}(x_i) \sin \mu X_i (x, x_i), \\
j^{(i)}(x) &= u^{(i)}(x_i) \mu \sin \mu \tilde{X}_i (x, x_i) + j^{(i)}(x_i) \cos \mu \tilde{X}_i (x, x_i).
\end{align*}

(10)

Here the solution is written in the Bers formalism [7, 8]. We denote

\[V^{(i)}(x) = \begin{pmatrix} u^{(i)}(x) \\ j^{(i)}(x) \end{pmatrix}, \quad V^{(j)}(x_i) = \begin{pmatrix} u^{(j)}(x_i) \\ j^{(j)}(x_i) \end{pmatrix}, \quad K^{(i)}(x, x_i) = \begin{pmatrix} \cos \mu X_i (x, x_i) & -\frac{1}{\mu} \sin \mu X_i (x, x_i) \\ \mu \sin \mu \tilde{X}_i (x, x_i) & \cos \mu \tilde{X}_i (x, x_i) \end{pmatrix} \]

and write the solution (10) of the Cauchy problem on the \(i\)-th layer in matrix form

\[V^{(i)}(x) = K^{(i)}(x, x_i) V^{(i)}(x_i).\]

In the case of an axisymmetric medium with constant physical parameters on the layer, the elements of matrix \(K^{(i)}(x, x_i)\) can be expressed in terms of the Bessel functions of the first and second kind and have the form

\begin{align*}
k_{11}^{(i)} &= \frac{\pi \mu X_i}{2 \alpha^{(i)}}, \quad J_1 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) N_0 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) - N_1 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) J_0 \left( \frac{\mu X_i}{\alpha^{(i)}} \right), \\
k_{12}^{(i)} &= -\frac{\pi}{2 \alpha^{(i)} \beta^{(i)}} \left( N_0 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) J_0 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) - J_0 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) N_0 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) \right), \\
k_{21}^{(i)} &= \frac{\pi \beta^2}{2 \alpha^{(i)}} \left( J_1 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) N_1 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) - N_1 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) J_1 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) \right), \\
k_{22}^{(i)} &= \frac{\pi \mu X_i}{2 \alpha^{(i)}} \left( N_0 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) J_1 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) - J_0 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) N_1 \left( \frac{\mu X_i}{\alpha^{(i)}} \right) \right),
\end{align*}

where \(\alpha^{(i)} = \sqrt{\lambda^{(i)} (c^{(i)} \rho^{(i)})^2}, \quad \beta^{(i)} = \sqrt{\lambda^{(i)} c^{(i)} \rho^{(i)}}.\)

We write the ideal contact of the layers in matrix form
The solution for the first layer is

\[ V^{(1)}(x) = K^{(1)}(x, x_i)V^{(1)}(x_i), \quad x_i \leq x \leq x_2. \]

For the end point of the first layer, then we get

\[ V^{(1)}(x_2) = K^{(1)}(x_2, x_i)V^{(1)}(x_i) = V^{(2)}(x_2), \]

and, considering the conditions of ideal contact, the solution for the second layer can be written

\[ V^{(2)}(x) = K^{(2)}(x, x_2)V^{(2)}(x_2) = K^{(2)}(x, x_2)K^{(1)}(x_2, x_i)V^{(1)}(x_i). \]

Performing further sequential substitution by layers, we obtain a solution on the \( i \)-th layer

\[ V^{(i)}(x) = K^{(i)}(x, x_i)V^{(i)}(x_i), \quad x \in [x_i, x_{i+1}], \]

where \( K^{(i)}(x, x_i) = K^{(i)}(x, x_i)K^{(i-1)}(x_i, x_{i-1})...K^{(1)}(x_2, x_1). \)

Thus, at the end point of the layer system, we have

\[ V^{(n)}(x_{n+1}) = K^{(n+1)}(x_{n+1}, x_i)V^{(i)}(x_i). \tag{11} \]

Equation (11) connects the values of the function and the flow at the initial and end points, which makes it possible to obtain a condition for determining the eigenvalues \( \mu_k \) for various boundary value problems. The eigenvalues for the first boundary value problem (1 – 4) are found from the condition

\[ k^{(n,1)}_{12} = 0, \]

and the eigenvalues for the third boundary value problem (1 – 3, 5) at zero external temperatures are found from the condition

\[ \det \begin{pmatrix} k^{(n,1)}_{11} - r^{(2)} k^{(n,1)}_{21} & r^{(1)} \\ k^{(n,1)}_{12} - r^{(2)} k^{(n,1)}_{22} & r^{(1)} \end{pmatrix} = 0. \]

Next, we define the normalized eigenfunctions

\[ f^{(i)}_k(x) = \frac{u^{(i)}_k(x)}{N_k}, \]

where \( u^{(i)}_k(x) \) is the basis function corresponding to the eigenvalue \( \mu_k \).

The normalization condition has the form

\[ N_k^2 = \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} e^{(i)} \rho^{(i)} x \left( u^{(i)}_k(x) \right)^2 \, dx. \]

Then the coefficients in the Fourier expansion

\[ C_k = \frac{1}{N_k} \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left( g(x) \right) e^{(i)} \rho^{(i)} x u^{(i)}_k(x) \, dx. \]

Thus, the solution of the assigned tasks was obtained.

\[ T^{(i)}(x, t) = \sum_{k=1}^{n} C_k f^{(i)}_k(x) e^{-\mu^2_k t}. \]
4. Results of calculations and their discussion

Some model problems were solved by this method. In figure 2 shows the result of modelling the first boundary value problem for a three-layer axisymmetric system with the radii of the boundaries of layers \(x_1 = 0.1\) m, \(x_2 = 0.2\) m, \(x_3 = 0.3\) m and \(x_4 = 0.4\) m with thermophysical parameters characteristic of rubber (outer layers) and copper (inner layer): \(\lambda^{(2)} = 384\) W/(m·K), \(c^{(2)} = 390\) J/(kg·K), \(\rho^{(2)} = 8900\) kg/m\(^3\), \(\lambda^{(1)} = \lambda^{(3)} = 0.2\) W/(m·K), \(c^{(1)} = c^{(3)} = 1370\) J/(kg·K), \(\rho^{(1)} = \rho^{(3)} = 1160\) kg/m\(^3\).

At the initial moment of time, only the middle layer is heated to 20°C, that is, the initial conditions are symmetric. The graphs are based on 25 eigenvalues. Modelling shows that the cooling of a layer with a smaller radius occurs faster than a layer with a large radius, and since the outer layers have a significantly lower thermal conductivity than the inner layer, the temperature on it practically does not change along the coordinate, which corresponds to the physical course of the process.

![Figure 2](image)

**Figure 2.** Results of modelling the first boundary value problem. The temperature distribution as a function of time (a) and at time \(t = 10^5\) s (b) in a three-layer axisymmetric medium, constructed from 25 eigenvalues. At the initial moment of time, only the middle layer is heated to 20°C, the thickness of each layer is 0.1 m.

In figure 3 shows the modelling of the third boundary value problem in a three-layer axisymmetric medium with the radii of the layer boundaries \(x_1 = 0.01\) m, \(x_2 = 0.02\) m, \(x_3 = 0.03\) m and \(x_4 = 0.04\) m (a) and \(x_1 = 0.31\) m, \(x_2 = 0.32\) m, \(x_3 = 0.33\) m and \(x_4 = 0.34\) m (b).

Thermophysical parameters of the layers are typical for the of rubber (outer layers) and copper (inner layer). At the initial moment of time, the temperature is everywhere equal to zero. Outside temperatures \(T_1 = 10^\circ\)C, \(T_2 = 20^\circ\)C, thermal resistance coefficients at the boundaries of the medium \(r^{(1)} = r^{(4)} = 1\) K·m\(^2\)/W. The temperature distribution is based on 10 eigenvalues. Modelling shows that at smaller radii of the layer system, the transition to the stationary regime occurs faster, while the temperature drop at the boundary points of the multilayer cylinder is more pronounced. Calculations also show that at large radii of the cylindrical system, the temperature distribution approaches the temperature distribution in the corresponding system of plane layers of the same thickness, which corresponds to the physical meaning of the simulated phenomenon.
Figure 3. Results of modelling the third boundary value problem, constructed from 10 eigenvalues. The temperature distribution in three-layer axisymmetric system with an inner radius of 1 cm (a) and 31 cm (b). The thickness of each layers is 1 cm. The initial temperature is equal to zero everywhere. Outside temperatures \( T_1 = 10^\circ\text{C}, \) \( T_2 = 20^\circ\text{C} \), thermal resistance coefficients are \( r^{(1)} = r^{(4)} = 1 \text{ K} \cdot \text{m}^2/\text{W} \).

Figure 4. Results of modelling the third boundary value problem, constructed from 10 eigenvalues. The temperature distribution in five-layer medium with an inner radius of 1 cm (a) and 31 cm (b). The thickness of each layers is 1 cm. The initial temperature is equal to zero everywhere. Outside temperatures \( T_1 = 10^\circ\text{C}, \) \( T_2 = 20^\circ\text{C} \), thermal resistance coefficients are \( r^{(1)} = r^{(4)} = 1 \text{ K} \cdot \text{m}^2/\text{W} \).

Figure 4 shows the modelling results in a five-layer axisymmetric medium with parameters typical for the "plastic-glue-aluminum-glue-plastic" system: \( \lambda^{(1)} = \lambda^{(5)} = 0.4 \text{ W/(m} \cdot \text{K}), \)
\( \lambda^{(2)} = \lambda^{(4)} = 0.1 \text{ W/(m} \cdot \text{K}), \)
\( \lambda^{(3)} = 203.5 \text{ W/(m} \cdot \text{K}), \)
\( c^{(1)} = c^{(5)} = 2300 \text{ J/(kg} \cdot \text{K}), \)
\( c^{(2)} = c^{(4)} = 300 \text{ J/(kg} \cdot \text{K}), \)
\( c^{(3)} = 897 \text{ J/(kg} \cdot \text{K}), \)
\( \rho^{(1)} = \rho^{(3)} = 950 \text{ kg/m}^3, \)
\( \rho^{(2)} = \rho^{(4)} = 0.96 \text{ kg/m}^3, \)
\( \rho^{(3)} = 2712 \text{ kg/m}^3. \) Layer thicknesses in increasing radii are 0.8 mm, 0.1 mm, 0.2 mm, 0.1 mm, 0.8 mm. The initial temperature is equal to zero everywhere. Outside temperatures \( T_1 = 10^\circ\text{C}, \) \( T_2 = 60^\circ\text{C} \), thermal resistance coefficients at the boundaries of the medium are \( r^{(1)} = r^{(6)} = 1 \text{ K} \cdot \text{m}^2/\text{W} \). The temperature distribution is based on 10 eigenvalues.
All calculations were performed using the mathematical package Maple (Widows 10 Home, processor Intel® Core™ i5-8250U CPU® 1.60GHz 1.80GHz, 6.00GB). Most of the computational time according to the described algorithm is spent on finding eigenvalues. The computation time noticeably increases with an increase in the number of layers or the number of eigenvalues. Particularly, for a three-layer medium, modelling on 10 eigenvalues usually does not exceed 10 minutes.

Also note that, as before, the results of modelling using the matrix method described in this work correlate well with the results of numerical analysis of the considered heat and mass transfer processes.

5. Conclusion
The paper describes the combined use of the analytical matrix method and the classical Fourier method for solving the problem of heat conduction in an axisymmetric multilayer medium. Some possibilities of this approach for modelling first boundary value problem and third boundary value problem are considered. It is shown that the proposed matrix method makes it possible to calculate temperature distributions in a relatively short time with an accuracy sufficient for practical use.

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