Critical behavior of a generalized Bak-Sneppen model

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Abstract. We propose an alternative to the Bak-Sneppen model for species co-evolution. In our model the closest neighbors of the least adapted species are replaced by new species with a certain probability $\alpha$. The probability $\alpha$ can be regarded as the interaction strength between nearest species. We show that the system can always self-organizes to a critical state when the interaction strength $\alpha$ is larger than zero. The critical exponents of our model are estimated through the finite size scaling analysis, and found to be the same as those of the original Bak-Sneppen model. We also find that the critical dimension of our model is 4, below which avalanches are compact and above which the critical behavior is mean-field like. Moreover, a strong interaction strength leads to a lower critical fitness.

1. Introduction

Self-organized criticality (SOC) [1] has become one of the most studied concepts of non-equilibrium statistical mechanics. It can be used to explain the origin of the ubiquitous occurrence of fractal structures in nature [2], noise with $1/f$ power spectrum [1] and punctuated equilibrium [3].

Among many SOC models, the Bak-Sneppen (BS) model [4], which mimics co-evolution between interacting species, is the simplest one. In this model, an ecosystem is characterized by $L^d$ species on a $d$-dimensional lattice of linear size $L$. A random number called fitness value $f_i$, drawn from a uniform distribution between 0 and 1, is assigned to each site of the lattice as the initial state of the system. At each time step, the site with the smallest fitness and its $2d$ nearest neighbors are replaced by new random numbers drawn from the same distribution. After some transient time, which depends on the system size, the model can reach a statistically stationary state where the density of fitness is uniformly distributed between $f_c$ and 1, and vanishes under this critical fitness. In the stationary state, the model exhibits punctuated equilibrium and the complexity of this regime can be revealed by the existence of spatio-temporal power-law distribution of avalanches.

The BS model has received much attention [3, 5] in the statistical physics community and has been studied through various approaches, including numerical simulation [6, 7], theoretical analysis [8], and mean-field theory [9]. Many variants of the Bak-Sneppen model have been proposed, such as (1) the discrete Bak-Sneppen model, in which fitnesses are only take the values 0 and 1; (2) the anisotropic Bak-Sneppen (aBS) model [10], in which only the least fit species and its right-hand nearest neighbor are mutated; (3) the random neighbors BS model
[11], in which the least fit species and $K - 1$ randomly chosen other ones are replaced by new species. In the BS model, the least fit species and its $2d$-nearest neighbors have full interaction, and are replaced by new species at the same time. This is not true in most real situations. In Ref. [12], the authors distinguish the strength of interactions of the BS model in one dimensional lattice. They found that as long as the interaction strength is non-zero, the model can always evolve to a critical state where the avalanche sizes subject to a power-law distribution. The critical exponent $\tau$ for the avalanche life distribution slightly relies on $\alpha$. The critical exponent $\gamma$ for average avalanche size distribution and the avalanche fractal dimension $D$ increase slowly as $\alpha$ increases. Here, we focus on the BS model with variable interaction strength in high dimensions. We implemented extensive simulations, estimated critical exponents and critical fitness accurately, and verified the scaling relations which are derived from the BS model are still valid in our model.

2. The model description
Our model is defined as follows:

(1) Initially, on a $d$-dimensional lattice with linear length $L$, a random number called fitness drawn from distribution $P(f)$, is assigned to each site.

(2) At each time step, the global minimum fitness is located, and replaced by a new random number drawn from the same distribution $P(f)$. Its $2d$-nearest neighbors are replaced by new random numbers also taken from $P(f)$ with probability $\alpha$. Here, $P(f)$ is a uniform distribution between 0 and 1.

(3) Repeat Step (2)

As mentioned above, $\alpha$ is interpreted as interaction strength, it is fixed during the system evolution and is independent of species’ fitness. The value of $\alpha$ ranges from 0 to 1. If $\alpha$ is set to 0, there is no interaction and all fitness values will eventually become 1 where no SOC can be observed. If $\alpha$ is set to 1, the original BS model is restored.

3. Simulation results
We use a generalized branching process [3] to describe an $f_0$ avalanche, which provides an efficient way to study the $f_0$ avalanches and is completely free from system size corrections. In simulations of this branching process, to describe an individual avalanche, we can assume that it started at the origin, and all times we need only keep track of those numbers below $f_0$ with explicit lattice reference. We have first used this simulated branching process to extrapolate threshold values $f_c$ for $1 \leq d \leq 8$ with $\alpha = 0.1, 0.2, \ldots, 1$. Then run up to $4 \times 10^6$ critical avalanches to determine their statistical properties.

3.1. Critical exponent of average avalanche size $\gamma$ distribution and critical fitness $f_c$

Normally, we need to observe the avalanches based on $f_c$. However, the critical fitness cannot be precisely identified. An alternative way is to choose an auxiliary parameter $f_0$ very close to $f_c$ and observe the so-called $f_0$-avalanche near the critical point. From the hierarchical structure of the $f_0$-avalanche, the following exact differential equation for the average size of $f_0$ avalanches can be derived [3]

$$\frac{d \ln(S)}{df_0} = \frac{\langle n_{cov}\rangle f_0}{1 - f_0}$$

where $S$ is the avalanche size, $\langle n_{cov}\rangle f_0$ is the average number of sites covered by $f_0$-avalanches, $f_0$ is the auxiliary parameter to define an $f_0$-avalanche. With the assumption that when
$f_0$ approaches the critical value $f_c$, the average size of avalanches diverges as $(f_c - f_0)^{-\gamma}$. Substituting this scaling ansatz into Eq. (1) gives the so-called $\gamma$ equation [3]

$$\gamma = \lim_{f_0 \to f_c} \frac{\langle n_{\text{cov}} \rangle f_0 (f_c - f_0)}{1 - f_0},$$

where $\gamma$ is the critical exponent of average avalanche size distribution, and $f_c$ is the critical fitness. Note that $\gamma$ appears as a constant for a certain model [3]. So we can plot $(1 - f_0) / \langle n_{\text{cov}} \rangle f_0$ for different values of $f_0$, as shown in Figure 1, the intersection with the vertical axis gives $f_c$ very accurately, and the asymptotic slope gives $\gamma$.

![Figure 1](image1)

**Figure 1.** $f_0$ as a function of $(1 - f_0) / \langle n_{\text{cov}} \rangle$ of our model in two-dimensional case with $\alpha = 0.9$. The solid line is a least square fit of the data. The intersection with the vertical axis gives $f_c = 0.348474(3)$, and the slope gives $\gamma = 1.69(1)$.

![Figure 2](image2)

**Figure 2.** $\ln(1 - f_c(\alpha))$ as a function of $\ln(\alpha)$ for our model in dimensions 1 to 8. The solid lines are least square fits of the data. The intersection with the vertical axis gives $\ln(1 - f_c(1))$. Clearly, the critical fitness depends heavily on the underlying interaction details and have a simple relation with $\alpha$: $f_c(\alpha) = 1 - C\alpha^2$, where $C = 1 - f_c(1)$.

We have estimated $\gamma$ and $f_c$ for our model with $\alpha = 0.1, 0.2, \cdots, 1.0$ in dimensions 1 to 8. We found that $f_c$ depends heavily on $\alpha$, but the critical exponent $\gamma$ only depend on dimensionality...
and are entirely insensitive to the changes in $\alpha$. Fig. 2 shows $\ln(1 - f_c(\alpha))$ as a function of $\ln(\alpha)$ for our model in different dimensions. We found that the critical fitness and the interaction strength has a simple relation: $f_c(\alpha) = 1 - C\alpha^\beta$, with different values of $\beta$ in different dimensions, where $C = 1 - f_c(1)$. In a certain dimension, a stronger interaction strength leads to a lower critical fitness. The values of critical exponent $\gamma$ for our model in dimensions 1 to 8 are listed in Table 1. As can be seen, $\gamma$ does not change when $d > 4$, which is in line with the result in Ref. [13].

3.2. Scaling function and data collapse

The statistics of the avalanches clearly depends on the value of $f_0$. The larger it is, the larger the expected size of the avalanche is. We use the following scaling ansatz for the probability distribution $P(s, f_0)$ and complementary cumulative probability distribution $P(S \geq s, f_0)$ of $f_0$

$\text{avalanches of size } s,$

\begin{align}
 P(s, f_0) &= s^{-\tau} g \left( s(f_c - f_0)^{1/\sigma} \right), \quad (3a) \\
 P(S \geq s, f_0) &= s^{-\tau+1} h \left( s(f_c - f_0)^{1/\sigma} \right). \quad (3b)
\end{align}

Here $\tau$ and $\sigma$ are model dependent exponents. $g(x)$ and $h(x)$ are scaling functions, which decays rapidly to zero when $x \gg 1$ and goes to a constant when $x \to 0$. This ansatz has been confirmed by numerical simulations in Refs. [4, 14–21]. When $f_0$ is lowered below $f_c$, the avalanche distribution acquire a cutoff $S_{co} = (f_c - f_0)^{-1/\sigma}$.

From Eq (3b) we can see that two completely independent exponents combined in a nontrivial way to form a single one leads to an enormous simplification in the description of the model. The values of $P(S \geq s, f_0)$ for various $s$ and $f_0$ can be made to collapse on a single curve if $s^\tau P(S \geq s, f_0)$ is plotted against $(f_c - f_0)^{1/\sigma}$. In order to remove the subjectiveness of the data collapse we adopt a method from [22], in which a measure is proposed to quantify the ‘goodness of collapse’. The measure $P_b$ in our scenario is defined as,

\begin{equation}
 P_b = \left[ \frac{1}{N_{\text{over}} \sum_p \sum_{j \neq p} \sum_{i, \text{over}}} s_{i,j}^{\tau-1} P_{i,j}^{-1} - \varepsilon_p \left( s_{i,j} (f_c - f_{0j})^{1/\sigma} \right)^q \right]^{-1/q}, \quad (4)
\end{equation}

where $\varepsilon_p(x)$ is the interpolating function based on the values of set $p$ bracketing the argument in question of set $j$. Different $f_0$ leads to different data sets. $s_{i,j}$ is the $i$th avalanche size for the $j$th set of $f_0$. $P_{i,j}$ is the $i$th complementary cumulative avalanche distribution probability for $j$th set (i.e. $P_{i,j} = P(S \geq s_{i,j}, f_{0j})$). Here we use the complementary cumulative avalanche distribution $P(S \geq s, f_0)$ instead of the avalanche distribution $P(s, f_0)$ because the statistics of the former is much better than latter (binned data will lose a lot of information). The innermost sum over $i$ is done only over the overlapping regions (denoted by the ‘$i, \text{over}$’), i.e. only those values of set $j$ are considered for which $x_{i,j} = s_{i,j} (f_c - f_{0j})^{1/\sigma}$ belongs to the interval spanned by the corresponding $x$-values of set $p$. $N_{\text{over}}$ is the total number of such pairs. Here, we use $q = 1$ and a cubic B-spline interpolation for $\varepsilon_p(x)$. A minimization of $P_b$ over $(\tau, \sigma)$ can be used to extract the optimal values of the exponents. Estimates of errors can be obtained from the width of the minimum. From an expansion of $\ln P_b$ around the minimum at $(\tau_0, \sigma_0)$, the width is estimated as:

\begin{align}
 \Delta \tau &= \eta \tau_0 \left[ 2 \ln \frac{P_b(\tau_0 \pm \eta \tau_0, \sigma_0)}{P_b(\tau_0, \sigma_0)} \right]^{-1/2} \quad (5a) \\
 \Delta \sigma &= \eta \sigma_0 \left[ 2 \ln \frac{P_b(\tau_0, \sigma_0 \pm \eta \sigma_0)}{P_b(\tau_0, \sigma_0)} \right]^{-1/2} \quad (5b)
\end{align}
Figure 3. The measure $P_b$ with $q = 1$ for our model in two-dimensional case is shown over the $(\tau, \sigma)$ plane. The $z$-axis is in log-scale. A few contours of constant $\ln(P_b)$ are shown by projecting the surface on the $(\tau, \sigma)$ plane. Here, $\alpha = 0.5$, $f_c - f_0 = 0.001, 0.002, \cdots, 0.01$.

As can be seen, this method not only can be used to measure the goodness of data collapse but also can estimate the exponents $\tau$ and $\sigma$ all at once. We also estimate $\tau$ by fitting the avalanche data using methods in Ref. [23], and the results are consistent with those of the data collapse method.

We counted $2 \times 10^6$ avalanches for various values of $f_0$ in each dimension with $\alpha = 0.1, 0.2, \cdots, 1.0$. Fig. 3 shows the measure of data collapse $P_b$ of our model in two-dimensional case with $\alpha = 0.5$. A minimization of $P_b$ gives us the exponents $\tau = 1.244$, $\sigma = 0.443$, with $P_b = 0.18$. We also estimate $\tau$ and $\sigma$ for different values of $\alpha$ in dimensions 1 to 8, and find that the exponents depend only on the dimensionality (see Table 1). Fig. 4 shows the data collapse of our model in two-dimensional case with $\alpha = 0.5$ and 1. So increasing the interaction strength $\alpha$ modifies the critical fitness $f_c$ but not the critical exponents, which is consistent with results in Ref. [24]. The critical fitness $f_c$ depends heavily on the underlying lattice details, while the critical exponents only depend on the dimensionality and are entirely insensitive to the underlying lattice details.

Table 1. List of the critical exponents of our model, for $1 \leq d \leq 8$. The errors in the exponent are less than 1%.

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $\tau$ | 1.08 | 1.25 | 1.37 | 1.46 | 1.50 | 1.50 | 1.50 | 1.50 |
| $\sigma$ | 0.34 | 0.44 | 0.47 | 0.46 | 0.50 | 0.50 | 0.50 | 0.50 |
| $\gamma$ | 2.69 | 1.69 | 1.33 | 1.17 | 1.00 | 1.00 | 1.00 | 1.00 |
| $D$ | 2.43 | 2.92 | 3.35 | 3.92 | 4.00 | 4.00 | 4.00 | 4.00 |
| $\mu$ | 0.41 | 0.68 | 0.91 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 |
| $d_f$ | 1 | 2 | 3 | 4 | 4 | 4 | 4 | 4 |
3.3. Fractal of avalanches
We recorded the instances of having activity at a distance \( r \) relative to the origin at update step \( s \). Similar to a random walk, the moments of the distribution define the avalanche dimension exponent \( D \) via \( \langle r \rangle_s \sim s^{1/D} \). We estimated \( D \) by least square fit \( \ln(s) \) and \( \ln(\langle r \rangle_s) \) in each dimension with different \( \alpha \), which shows that \( D \) is insensitive to changes in \( \alpha \). Fig. 5 shows \( s \) vs \( \langle r \rangle \) of our model in two-dimensional case with \( \alpha = 0.5 \). We used the value of our best fit for the exponent \( D \) as given for each dimension in Table 1. For all dimensions \( s/\langle r \rangle^D \) tends to be stable for increasing \( \langle r \rangle \). For \( d \geq 4 \), the fractal exponent \( D \) corresponds to the mean-field result [25, 26]

![Figure 5](image)

**Figure 5.** \( \ln(s) \) as a function of \( \ln(\langle r \rangle) \) of our model in two-dimensional case with \( \alpha = 0.5 \). The solid line is a least square fit of the data and the slope gives the fractal exponent \( D = 2.92 \).

We also measured the scaling of the coverage with the lifetime of an avalanche \( \langle n_{\text{cov}} \rangle \sim s^\mu \) and listed our best fit for the exponent \( \mu \) in Table 1. Fig. 6 shows \( \langle n_{\text{cov}} \rangle \) as a function of \( s \) of our model in two-dimensional case with \( \alpha = 0.5 \).

To test whether the avalanche is compact or not in a certain dimension, we estimated exponent \( d_f \) defined by \( \langle n_{\text{cov}} \rangle \sim R^{d_f} \), where \( R \) is the radius of gyration of a avalanche covered domain. Avalanches are fractal if \( d_f < d \) and compact if \( d_f = d \). Fig. 7 shows the plot of \( \langle n_{\text{cov}} \rangle \) as a function of \( R \) of our model in two-dimensional case with \( \alpha = 0.5 \). Since there should be only one characteristic length for a compact avalanche, there should have \( \langle r \rangle \sim R \), and thus, \( \mu = d/D \)
Figure 6. $\ln(\langle n_{\text{cov}} \rangle)$ as a function of $\ln(s)$ of our model in two-dimensional case with $\alpha = 0.5$. The solid line is a least square fit of the data and the slope gives exponents $\mu = 0.68$.

for a compact avalanche, and $\mu = 1$ in the mean-field limit. Clearly, the exponents in Table 1 for $d \leq 4$ in line with $\mu = d/D$, and for $d = 5, 6, 7, 8$ are in perfect agreement with mean-field values. That is to say, avalanches of our model are compact until $d = 4$, which is consistent with [13]. Because of the coincidence of the exponents of our model with those of the BS model, the scaling relations in [3] are preserved. We can easily verify the scaling relations of exponents in Table 1.

Figure 7. $\ln(\langle n_{\text{cov}} \rangle)$ as a function of $\ln(R)$ of our model in two-dimensional case with $\alpha = 0.5$. The solid line is a least square fit of the data and the slope gives exponents $d_f = 2.0$.

4. Conclusions
To summarize, we have studied a stochastic Bak-Sneppen model in dimensions 1 to 8 by extensive numerical simulations. The avalanches are regarded as a branching process, and the branching factor $r_b = 1 + 2d\alpha$, as long as $\alpha > 0$, the branching factor $r_b > 1$ in all case, and the nontrivial SOC states are observed in every dimension. The critical fitness changes with $\alpha$ according to $f_c(\alpha) = 1 - C\alpha^{-\beta}$, with different $\beta$ in different dimensions, where $C = 1 - f_c(1)$. The critical
exponents of our model do not change with $\alpha$, and are consistent with those of the origin BS model.

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