Gauge dependence of spontaneous radiation spectrum in a time-dependent relativistic non-perturbative Coulomb field

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Abstract. Lamb triggered a continuous debate on the gauge choice for atomic interactions with electromagnetic fields, particularly with plane waves and the vacuum field. Modern technologies of Rydberg atoms and relativistic atomic beams make it possible to explore interactions with a more intriguing non-perturbative, adiabatic Coulomb field. In such cases, one would face the well-known tricky issue about the physical significance of the scalar gauge potential when it is time-dependent. We start attacking this issue by studying a simplest system: a one-dimensional oscillator interacting adiabatically with a relativistic charge. We reveal that a gauge dependence much severer than the one Lamb observed is encountered when calculating the transient radiation spectrum of this oscillator by the external-field method, which is currently the only available tool. The obtained peak frequency can differ by 10 MHz or larger for the commonly used Coulomb, Lorentz, and Multipolar gauges. Contrary to the popular view, we explain that such a gauge dependence is not really a disaster, but actually an advantage here: The relativistic bound-state problem is so complicated that a full quantum-field method is still lacking; thus, the external-field approximation cannot be derived and hence not guaranteed. However, by fitting to experimental data, one may always define an effective external field, which may likely be parameterized with the gauge potential in a particular gauge. This effective external field would not only be of phenomenological use, but also shed light on the physical significance of the gauge potential. We thereby encourage further investigations of this fundamental problem with more realistic systems involving Rydberg atoms and relativistic atomic beams, both theoretically and experimentally.

1 Introduction

Gauge symmetry is trivial in classical electrodynamics, which is essentially a theory of the electric field \( \vec{E} \) and magnetic field \( \vec{B} \). A local quantum formulation of electromagnetic interactions has to utilize the gauge potential \( A^\mu \) and therefore, does not share the absolutely safe gauge invariance of the classical case. It is known that for scattering problems of elementary particles, gauge symmetry is still well preserved in a quantum theory. However, the issue seems to be tricky as bound states are involved. It was first noticed by Lamb about 70 years ago [1] when studying the renowned Lamb-shift transition \( 2S_{1/2} \rightarrow 2P_{1/2} \), that a straightforward calculation can correctly give the observed line-shape in just one particular gauge, namely the so-called length gauge with the \(-q\vec{E} \cdot \vec{r}\) interaction, here, the “straightforward calculation” means that the atomic state is defined as the eigen-state of the conventional energy operator

\[
\mathcal{E}_n = \frac{\vec{p}^2}{2m} + qV(\vec{x}), \quad \mathcal{E}_n|\psi^0_n\rangle = E_n|\psi^0_n\rangle, \quad (1)
\]

where \( n \) labels the state, \( \vec{p} = -i\vec{\nabla} \) (we set \( \hbar = c = 1 \)) is the canonical momentum operator, \( q = -e \) is the electron charge, and \( V(\vec{x}) \) is the nuclear Coulomb potential.

Lamb’s observation is somehow surprising, and also puzzling, because the quantum-mechanical formulation of electromagnetic interactions, for example, in the Schrödinger equation

\[
i\partial_t \psi(\vec{x}, t) = H(t)\psi = \left\{ \frac{1}{2m}(\vec{p} - q\vec{A}(\vec{x}, t))^2 + q\phi(\vec{x}, t) + qV(\vec{x}) \right\} \psi(\vec{x}, t), \quad (2)
\]

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is constructed purposely to be invariant under the joint gauge transformation

\[
\psi = U \psi' = e^{-iqA(\vec{x}, t)} \psi',
\]

\[
A_\mu = A'_\mu - \frac{i}{q} U \partial_\mu U^{-1} = A'_\mu + \partial_\mu \Lambda(\vec{x}, t),
\]

where \( U = e^{-iqA(\vec{x}, t)} \), with an arbitrary function \( \Lambda(\vec{x}, t) \).

At the root of Lamb’s observation is that although the Schrödinger equation is gauge invariant, the energy operator \( \mathcal{E}_0 \) in Eq. (1) has a gauge-dependent expectation value. A way out of this difficulty is therefore quite natural, as many authors argued [2–6], that in the presence of an electromagnetic field the atomic states should be defined instead as the eigen-states of the gauge invariant energy operator

\[
\mathcal{E}_A(t) = \frac{\vec{\pi}^2}{2m} + qV(\vec{x}), \quad \mathcal{E}_A(t) \big| \psi_n^A(t) \big> = E_n \big| \psi_n^A(t) \big>,
\]

where \( \vec{\pi} = \vec{p} - q\vec{A}(\vec{x}, t) \) is the mechanical momentum operator. Accidentally, in the length gauge \( \vec{A} = 0 \), \( \mathcal{E}_A \) coincides with \( \mathcal{E}_0 \), thus the “straightforward calculation” can indeed work. As Lamb et. al. elaborated in Ref. [5], a gauge with nonzero \( \vec{A} \) leads to complications, but careful and consistent calculations can give the same result as in the length gauge. The major cause of complication is that the operator \( \mathcal{E}_A(t) \) is time-dependent and normally does not commute with itself at different times, therefore Eq. (4) defines instantaneous rather than stationary eigen-states \( \psi_n^A(t) \), which are also time-dependent. (The eigenvalues \( E_n \), nevertheless, are gauge invariant and time-independent. In some cases, \( E_n \) do not even depend on the explicit form of \( \vec{A} \). We will come back to this simple but delicate point in the next section.)

So far so good. The gauge-choice problem raised by Lamb was thus claimed to have been solved, especially by Lamb himself and co-workers in a “concluding paper” in 1987 [5]. However, we would like to remind that Eq. (4) cannot really be taken for granted and is rather a conjecture. Especially, what is the justification for Eq. (4) to include \( qV(\vec{x}) \) but discard \( q\phi(\vec{x}, t) \)? Why not just take the “total” \( H(t) \) in Eq. (2) to define the instantaneous atomic eigen-states in the presence of electromagnetic interactions? 1 Certainly, the “total” \( H(t) \) is gauge dependent, and if Eq. (4) is replaced by

\[
H(t) \big| \psi_n(t) \big> = E_n(t) \big| \psi_n(t) \big>,
\]

we will show that this would make both the eigenvalues and the transition rates gauge dependent. But quite interestingly, for the problem as Lamb considered, by Eq. (5) the correct result can still be obtained in one particular gauge, namely the Coulomb gauge with \( \phi = 0 \), instead of the length gauge with \( \phi = -\vec{E} \cdot \vec{x} \). This time, it is the Coulomb gauge that stands out.

We call attention to the fact that the above concern is not merely of academic nature. Equation (4) relies crucially on the separation of the nuclear Coulomb potential \( V(\vec{x}) \) from the external scalar potential \( \phi(\vec{x}, t) \). However, with the advent of modern technologies, one may encounter cases where the time-dependent scalar potential \( \phi(\vec{x}, t) \) can be adiabatic and comparable to \( V(\vec{x}) \) in effect and therefore, must be treated at the same footing as \( V(\vec{x}) \). Then, Eq. (4) could not possibly apply, and Eq. (5) is a more reasonable option; thus, one must face sharply the well-known long-standing puzzle about the physical significance of the scalar gauge potential when it is time-dependent. One such a case is the Rydberg atom, which is very sensitive to the external electric field, including a varying Coulomb field associated with a time-dependent \( \phi(\vec{x}, t) \). Another case is the relativistic atomic beam [7,8], which feels a non-uniform field as a time-dependent one in the beam frame. In both cases, a time-dependent, non-perturbative, and adiabatic scalar potential \( \phi(\vec{x}, t) \) can appear; hence, the computation of radiation spectra has to rely on the definition of instantaneous atomic eigenstates involving \( \phi(\vec{x}, t) \). We will show that a significant gauge dependence severer than what Lamb observed would appear, and cannot be overcome by existing methods. To see unambiguously the origin of this severer gauge dependence in the computed observable spectra, we will employ in this paper one of the simplest physical systems in this context: a one-dimensional quantum oscillator influenced by a moving charge. In the forthcoming studies, we will consider more realistic systems with feasible parameters for possible actual experimental tests.

To check the correctness and consistency of our calculation, we follow an approach in the literature by repeating our calculation with an adiabatic “switch-on” and “switch-off” of the interactions [9–11]. This is achieved with an factor \( \exp(-\epsilon |t|) \) [12]. Under these conditions, the gauge transformation of the wave function amounts to the unity transformation in the infinite past and future. We indeed find that gauge invariance of the transition probabilities is recovered as \( \epsilon \) gets big enough. With this important check, we believe that our findings are reliable.

The paper is organized as follows: In Sect. 2, we introduce our physical system: a cluster of relativistic charged particles passing by a one-dimensional oscillator. Parameters can be adjusted to make \( \phi(\vec{x}, t) \) of the moving charge cluster to be adiabatic and non-perturbative for the oscillator. In Sect. 3, we discuss the only available tool for computing the radiation spectrum of such a system, namely the external-field approximation. We explain in detail why this method is inherently gauge dependent for bound-state problems.

1 Note that including the total \( H(t) \) to define the instantaneous eigen-states, thus the interaction term is apparently absent, does not mean that the system will not make quantum transition. The reason is that \( H(t) \) may not commute at different times, therefore the eigen-state at one moment is not always the eigen-state later. As is known in the discussion of adiabatic approximation, quantum transitions are avoided only if \( H(t) \) varies slowly enough in time.
The one-dimensional system is particularly advantageous for revealing such a gauge dependence. This method is adopted in Sec. 4 to solve the instantaneous eigen-states of our system. The solutions indeed differ significantly for the commonly used Coulomb, Lorentz, and multipolar gauges. Then, in Sect. 5, we compute the transient spontaneous radiation spectrum explicitly and find again a significant gauge dependence. In Sect. 6, we check calculation with an adiabatic “switch-on” and “switch-off” of the interactions. In the last section, we summarize our results and discuss their physical implications.

2 A time-dependent relativistic non-perturbative Coulomb field acting adiabatically on a quantum oscillator

Figure 1 shows a schematic design of our physical system. The electromagnetic field is produced by a cluster of relativistic protons of a huge number $N$, like a bunch from an accelerator. (Certainly, electrons and heavy ions may also be utilized.) This charge cluster passes by a one-dimensional oscillator, formed of an electron moving in a nanowire or carbon nanotube along the $x$-axis, and constrained by two other nearby negative charges, placed at the coordinates $(x, y) = (l, 0)$ and $(-l, 0)$, respectively. The charge cluster moves in the same direction, with an impact distance of $Y$. If excited, the oscillator can emit a photon by spontaneous radiation. It can be expected that by enlarging the parameters $N, l,$ and $Y$, the scalar potential $\phi(\vec{x}, t)$ of the charge cluster can be non-perturbative and adiabatic for the oscillator for a duration which is sufficiently long for the excited oscillator to emit a transient photon. This emission must then be computed by including both $\phi(x, t)$ and $V(\vec{x})$ when defining the instantaneous eigen-states of the oscillator. Namely, we have to employ Eqs. (5) instead of (4), as we remarked above in Sect. 1.

For simplicity, we treat the proton cluster as a point charge and leave the task of considering the actual spatial distribution and possible dispersion of the cluster to a future study. This would influence some quantitative detail but not the main concern of this paper about gauge dependence. The cluster velocity $\beta = v/c$ should be chosen large enough to produce a considerable difference for $\phi(\vec{x}, t)$ among various gauges, and at the same time small enough to allow for a rough external-field approximation. As we will comment further in the last section, to compute the radiation spectrum of our system involving a relativistic source, the only available tool at present is the external-field method, by which the moving charge just provides a classical field acting on the quantum oscillator. In the next section, we give the details of this method and explain how it gives rise to a serious gauge dependence for the system we consider.

3 The external-field method and its gauge dependence

To compute the radiation spectrum of a bound state, one must first define its eigen-states. In our case, this is done by solving Eq. (5). The Hamiltonian takes the form in Eq. (2), in which $V(\vec{x})$ is provided by the two constraining stationary charges, and $\phi(\vec{x}, t)$ is provided by the moving cluster. Upon solving Eq. (5) for our one-dimensional oscillator, we first note that it shares the same feature as Eq. (4) that the eigenvalues do not depend explicitly on $\vec{A}$. This is a simple fact, but may often cause confusion, so we elaborate a little bit here.

We may always apply to Eq. (5) the following unitary transformation:

$$|\psi_n(t)\rangle = U|\tilde{\psi}_n(t)\rangle, H(t) = U\tilde{H}(t)U^{-1},$$  \hspace{1cm} (6)

then Eq. (5) becomes

$$\tilde{H}(t)|\tilde{\psi}_n(t)\rangle = E_n(t)|\tilde{\psi}_n(t)\rangle.$$  \hspace{1cm} (7)

Note that although the factor $U$ in Eq. (6) can be the same as in Eqs. (3), (6) is not a gauge transformation. It is merely a mathematical technique, after fixing a gauge, for the convenience of solving the eigen-equation. Especially, the new operator,

$$\tilde{H}(t) = U^{-1}H(t)U = \frac{1}{2m}(\vec{p} - q\vec{A}(\vec{x}, t) - q\vec{V}(\vec{x}))^2 + q\phi(\vec{x}, t) + qV(\vec{x}),$$  \hspace{1cm} (8)

is also merely a mathematical tool without much physical meaning. It is not the gauge-transformed Hamiltonian $H'(t)$. The latter should be obtained by applying $\psi = U\psi'$ to the time-evolution Eq. (2) instead of the eigen-equation (5). The result is familiar and reads.

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**Fig. 1** Schematic design of the physical system. The big solid circle represents the proton cluster. Two small solid circles represent stationary charges. The small hollow circle represents the electron. See the text for a description of the parameters.
where both the vector and scalar potentials are the gauge-transformed ones. In contrast, $\hat{H}(t)$ looks like a hybrid-gauge expression in which the vector potential is transformed, while the scalar potential remains the same.

It is exactly the hybrid feature of $\hat{H}$ in Eq. (8) that makes the unitary transformation (6) advantageous when solving Eq. (5). For a one-dimensional system located on the $x$-axis, all three components of $\vec{A}$ may be transformed to be zero along the $x$-axis, namely, $\vec{A}'(x,0,0,t)=0^2$ and $\hat{H}$ is thereby simplified for solving $\psi_n$. After obtaining the simpler solution $\tilde{\psi}_n$, the original $\psi_n$ is easily obtained by multiplying the factor $U$, and the eigenvalues $E_n(t)$ remain unchanged. Hence, when solving Eq. (5) for our one-dimensional oscillator, the vector potential $\vec{A}$ is trivial, and only the scalar potential $\phi(\vec{x},t)$ is essential. (Certainly, $\vec{A}$ still leaves an imprint on the quantum state through the unitary factor $U$, which must be included when computing quantum transitions.)

It is worth repeating this simple but extremely important fact: in the external-field method, the eigenvalues $E_n(t)$ of a one-dimensional charged oscillator depend only on the scalar potential $\phi(\vec{x},t)$, but not on the vector potential $\vec{A}(\vec{x},t)$; and consequently, the $E_n(t)$ are strongly gauge dependent, just as the scalar potential $\phi(\vec{x},t)$.

One may wonder why the above simple but important property was rarely emphasized, and even largely eluded people’s attention. The point is that in most cases there exists an apparently “natural gauge choice.” For example, for a stationary system one always chooses $A^0$ to be stationary as well. However, there is no rationale that forbids one to do otherwise. Should one choose a gauge such that the scalar potential $\phi$ cancels any binding potential $V$ in Eq. (2), then there would be no bound state at all for a one-dimensional system.

Certainly, such a gauge dependence is not limited to one-dimensional problems. It persists in higher dimensions. Especially, when the magnetic field is absent, $\vec{A}$ would merely contribute a unitary factor in whatever gauge, then according to Eqs. (6, 7, 8), bound-state solutions can again disappear by tuning $\phi$. When a magnetic field is present, $\vec{A}$ can no longer be transformed to be zero all over the space, so the contrast is not as sharp and vivid as in one dimension.

We call special attention to the fact that the above observation is by no means a declaration that gauge symmetry is broken, or that in the quantum domain electromagnetic effects are determined by the gauge potential $A^\mu$ instead of the field strength $F^{\mu\nu}$, and $A^\mu$ is more essential than $F^{\mu\nu}$. The encountered gauge dependence here is just an inherent defect of the external-field method, especially when applied to bound-state problems. On the other hand, the external-field method is always the most convenient one, and very often the only one that is solvable, so it is important to explore the applicability and limitations of this method. For example, when solving the stationary hydrogen spectrum, the common practice of treating the proton as an external source which just provides a Coulomb potential can actually be justified as a good leading-order approximation, by analyzing the total Hamiltonian including the proton, the electron, and also the electromagnetic field [13].

For the relativistically time-dependent, non-perturbative, and adiabatic system as we discuss here, there is no more “natural gauge choice,” so it is especially interesting to analyze the possible gauge dependence, and see whether any particular gauge might work.

4 The instantaneous eigen-states

Having explained our method clearly, we now proceed to calculate the instantaneous eigen-states of our system. In doing so, we must face the bitter issue about the physical meaning of the time-dependent scalar potential, which is counted directly as part of the Hamiltonian in the external-field method. Since in our system the moving cluster and the fixed charges have comparable effects on the electron, we include all their contributions into $\phi(\vec{x},t)$. The expression is easy to calculate, and details are given in the Appendix.

To simplify our calculation, we approximate the electron motion as a harmonic oscillator, with the equilibrium position influenced by the charge cluster. To this end, we Taylor-expand $\phi$ at a particular point $\vec{x}_0=(x_0,0,0)$ to the second order:

$$\phi(x,t) \approx \phi(x_0,t) + \partial_x \phi(x,t)|_{x=x_0}(x-x_0) + \frac{1}{2} \partial_x^2 \phi(x,t)|_{x=x_0}(x-x_0)^2.$$  (10)

Here, $\phi(x,t)$ is an abbreviation for $\phi(x,0,0,t)$. By solving $\partial_x \phi(x,t)|_{x=x_0}=0$, $x_0$ is identified as the instantaneous equilibrium position of the harmonic oscillator. Note again that, as we elaborated in the previous section, for a one-dimensional system the vector potential $\vec{A}$ only contributes a unitary factor $U$, without affecting the absolute value $|\psi|$ and the equilibrium position.
The above practice is standard, but here one may immediately find two alerting points: (1) the so-obtained equilibrium position $x_0$ is gauge dependent; (2) $x_0$ does not agree with the classical equilibrium position at $\vec{E} = 0$, except in the multipolar gauge.

Our comments on these two points are the following:

1. For a quantum system influenced by a relativistic agent, it is not certain that equilibrium position would necessarily be at $\vec{E} = 0$. After all, the quantum formulation of the electromagnetic interaction is via the minimal coupling of $A^\mu$, and not $F^{\mu\nu}$.

2. If gauge dependence is encountered, it is an indication that the adopted method might be questionable, and there is no rationale that the particular multipolar-gauge result must be correct.

3. In principle, the equilibrium position is measurable, though less convenient than the radiation spectrum as an experimental observable. Certainly, for the simple oscillator as we study here, there should be only one actual equilibrium position for a particular set of physical parameters. Therefore, the gauge dependence of $x_0$ is a strong indication that the gauge dependence of the external-field method as we explained in the previous section is indeed vital. However, we will explain in the last section that such a gauge dependence can actually be utilizable in practice. So, we proceed with the calculations and record any emerged gauge dependence as it is.

We add a gauge label $G$ and write approximately the Hamiltonian of the oscillator as the unified expression

$$H_{0G}(t) \approx \frac{1}{2m}(p - iU_G^{-1}\partial_t U_G)^2 + \frac{1}{2}k_G(t)(x - x_{0G}(t))^2.$$

(11)

Here, $k_G(t) = q\partial_\xi^2\phi(x,t)|_{x=x_0}$ is the strength of the harmonic potential and varies with the equilibrium position. $U_G$ is the PZW-transformation factor [14,15]

$$U_G = \exp \left[-i\frac{1}{m} \int_0^t d\lambda \vec{A}_G(\lambda \vec{x} + (1 - \lambda)\vec{x}_0, t) \cdot (\vec{x} - \vec{x}_0) \right].$$

(12)

For the multipolar gauge, $U_G$ is simply unity. With this Hamiltonian (11), we can solve the instantaneous eigenvalues and eigen-states as

$$E_{n,G}(t) = (n + \frac{1}{2})\omega_G(t),$$

$$|n_G(t)\rangle = U_G^{-1} N_{nG}(t) \exp \left[ -\frac{1}{2} \gamma_G(t)^2 (x - x_{0G}(t))^2 \right] |H_n[\gamma_G(t)(x - x_{0G}(t))]\rangle,$$

(13)

where, $\omega_G(t) = \sqrt{k_G(t)/m}$, $\gamma_G(t) = \sqrt{m\omega_G(t)}$, $N_{nG}(t) = [\gamma_G(t)/\sqrt{2^m m!}]^{1/2}$, and $H_n$ denote Hermite polynomials.

In the following, we set up the parameters and compute the numerical results. We take the physical mass of the electron and keep in mind that its effective mass might be different in an actual system. After some rough estimation, we find that the following values suffice our study: $N = 10^{12}$, $\beta = 0.1$, $l = 6.33$ nm, and $Y/l = 10^6$. The cluster moves from $x = -100Y$ to $x = 100Y$.

Figure 2 depicts the multipolar-gauge results of the instantaneous equilibrium position and the frequency of the oscillator. As we commented above, this is the gauge that essentially sets the equilibrium position at $\vec{E} = 0$. Figure 3 shows the corresponding results in the Lorentz and Coulomb gauges, expressed as the deviation from the values in the multipolar gauge.

Before solving the eigen-equations, we must check that with our chosen parameters, $\phi(\vec{x},t)$ is indeed an adiabatic potential for the oscillator, in all three gauges we use. The criterion for the adiabatic approximation is the parameter

$$r_{nm} = \left| \frac{\langle n_G(t) | \hat{H}_G(t) | m_G(t) \rangle}{(E_{nG}(t) - E_{mG}(t))^2} \right|,$$

(14)

where $n,m$ label two different states. Figure 4 gives the results of $r_{nm}$ for the lowest two states, which show that for the Lorentz, Coulomb, and multipolar gauges, we all have $r_{01} = r_{10} \ll 1$. Thus, if prepared in the first excited state $|1\rangle$ before the cluster moves in, the oscillator will largely stay in the state $|1_G(t)\rangle$ during the whole process as the cluster moves by.

With the adiabatic conditions verified, the gauge dependence of the instantaneous energy levels, as we
just aforementioned for our designed quantum oscillator, may in principle be tested experimentally. A possible experimental observable, which is the cleanest theoretically, is the spectrum of spontaneous radiation, as we will compute it in the next section.

5 The transient spontaneous radiation spectrum and its gauge dependence

We now add into our system the coupling to the background vacuum electromagnetic field \( A_B^\mu = (A_B, \phi_B) \). For the convenience of imbedding our one-dimensional oscillator into a three-dimensional formulation, we introduce an artificial harmonic potential \( V(y,z) = \frac{m}{2} (\omega_y^2 y^2 + \omega_z^2 z^2) \), with \( \omega_y, \omega_z \to \infty \). The entire Hamiltonian is now expressed as

\[
H(t) = \frac{1}{2m} [\vec{p} - q(\vec{A}(\vec{x}, t) + \vec{A}_B(\vec{x}, t))]^2 + q(\vec{\phi}(\vec{x}, t) + \phi_B(\vec{x}, t)) + V(y,z) + H_B. \tag{15}
\]

Here, \( H_B \) is the Hamiltonian of the vacuum photon:

\[
H_B = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \omega_k \epsilon_{k,\lambda} a_{k,\lambda} a_{k,\lambda}^\dagger. \tag{16}
\]

The expression of \( A_B^\mu \) also depends on the gauge. For the Coulomb gauge, it is

\[
\vec{A}_B = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \frac{1}{2\omega_k} \left( \epsilon_{k,\lambda} a_{k,\lambda} e^{i\vec{k} \cdot \vec{x} - i\omega_k t} + \epsilon_{k,\lambda} a_{k,\lambda}^\dagger e^{-i\vec{k} \cdot \vec{x} + i\omega_k t} \right), (\epsilon_{k,\lambda} \cdot \vec{k} = 0)
\]

\[
\phi_B = 0, \tag{17}
\]

whereas for the Lorentz gauge it has the form

\[
A_B^\mu = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \frac{1}{2\omega_k} \left( \epsilon_{\lambda,\mu} a_{k,\lambda} e^{i\vec{k} \cdot \vec{x} - i\omega_k t} + \epsilon_{\lambda,\mu} a_{k,\lambda}^\dagger e^{-i\vec{k} \cdot \vec{x} + i\omega_k t} \right). \tag{18}
\]
Since we shall only consider the emission of a physical photon and only compute this effect to leading order, in the Lorentz gauge the part of the Hamiltonian (15) becomes

\[ H_G(t) \approx H_{0G}(t) + e \widetilde{E}_B(\vec{x}_{0G}(t), t) \cdot (\vec{p} + e \vec{A}(\vec{x}, t)) + H_B. \]  

We would like to remark that for our system the gauge dependence from the vacuum field is in fact not serious, as compared with that from the external field. Even though we do not apply the transformation in Eq. (21) and go ahead in any gauge with \( |n_G(t) \rangle \) and the minimal-coupling Hamiltonian

\[ H_G(t) \approx H_{0G}(t) + \frac{e}{m} \vec{A}_g(\vec{x}_{0G}(t), t) \cdot (\vec{p} + e \vec{A}(\vec{x}, t)) + H_B, \]  

the result would only differ from that using Eq. (23) by an undetectable amount, as we will check at the end of this section.

Note that our quantum oscillator is actually one-dimensional, so we only need to consider the physical photon with \( e_x \) polarization, and the vacuum photon Hamiltonian (16) reduces to

\[ H_B = \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}} a_{\vec{k},x}^\dagger a_{\vec{k},x}. \]  

Moreover, due to the constraint \( \vec{e}_x \cdot \vec{k} = 0 \), the photon momentum-space integration is reduced as well:

\[ \int d^3k \Rightarrow \int_0^{\infty} \frac{\omega_k^2 d\omega_k}{k} \int_0^\pi \sin \theta_k \theta_k^2 d\theta_k \int_0^{2\pi} d\phi_k = 2\pi \]

where \( \theta_k \) is the angle between the photon momentum \( \vec{k} \) and the x-axis, and \( \phi_k \) is the azimuthal angle of \( \vec{k} \) in the y-z plane.

To facilitate our discussion, we write collectively in all gauges the interaction term between the vacuum field and the time-dependent quantum system as

\[ \int \frac{d^4k}{(2\pi)^4} \gamma_1(\vec{a}_{k,x}^\dagger, \vec{a}_{k,x}, \vec{x}, t). \]  

Using energy-level raising and lowering operators \( \sigma_+ \) and \( \sigma_- \) to represent the position operator \( x - x_{0G}(t) \) and the mechanical-momentum operator \( p_x + eA_x \):

\[ x - x_{0G}(t) = \frac{1}{\sqrt{2\gamma_G(t)}}(\sigma_+ + \sigma_-), \]

\[ p_x + eA_x = i\frac{\gamma_G(t)}{\sqrt{2}}(\sigma_+ - \sigma_-), \]

where \( \sigma_+ |n_G(t) \rangle = \sqrt{n+1} |(n+1)G(t) \rangle, \sigma_- |n_G(t) \rangle = \sqrt{n} |(n-1)G(t) \rangle \). Applying a standard rotating-wave
approximation, the interaction Hamiltonian can be expressed as

\[ H_I(a_{\kappa,x}, a^\dagger_{\kappa,x}, t) \approx H_I(\sigma_+, a_{\kappa,x}, t) + H_I(\sigma_-, a^\dagger_{\kappa,x}, t). \] (28)

The explicit expressions, if the multipolar gauge is employed for the vacuum field, are

\[ H_I(\sigma_+, a_{\kappa,x}, t) = \frac{ie}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{\varepsilon_k}}{\gamma_0(t)} \sigma_+ a_{\kappa,x} e^{i\varepsilon_{\kappa_0}(t) + i\omega_k t} \]

\[ H_I(\sigma_-, a^\dagger_{\kappa,x}, t) = -\frac{ie}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{\varepsilon_k}}{\gamma_0(t)} \sigma_- a^\dagger_{\kappa,x} e^{-i\varepsilon_{\kappa_0}(t) + i\omega_k t} \quad (G = M, L, C). \] (29)

Note that the gauge imprint from the external field still persists here. If the Lorentz or Coulomb gauge is employed for the vacuum field, then

\[ H_I(\sigma_+, a_{\kappa,x}, t) = \frac{ie}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\gamma_0(t)}{\sqrt{\varepsilon_k}} \sigma_+ a_{\kappa,x} e^{i\varepsilon_{\kappa_0}(t) - i\omega_k t} \]

\[ H_I(\sigma_-, a^\dagger_{\kappa,x}, t) = -\frac{ie}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\gamma_0(t)}{\sqrt{\varepsilon_k}} \sigma_- a^\dagger_{\kappa,x} e^{-i\varepsilon_{\kappa_0}(t) + i\omega_k t} \quad (G = M, L, C). \] (30)

The full quantum state of the time-dependent oscillator plus the possibly emitted photon can be written as

\[ |\psi(t)\rangle = \sum_n b_n(t)|n_G(t), 0\rangle 
+ \sum_n \int \frac{d^3k}{(2\pi)^3} b_{n,\kappa}(t)|n_G(t), \gamma_{\kappa,x}\rangle \] (31)

Here, \(|n_G(t), 0\rangle \equiv |n_G(t)\rangle \otimes |0\rangle, |n_G(t), \gamma_{\kappa,x}\rangle = |n_G(t)\rangle \otimes |\gamma_{\kappa,x}\rangle, |n_G(t)\rangle\rangle is the instantaneous eigen-state of the oscillator as we constructed in the previous section, and \(|0\rangle, |\gamma_{\kappa,x}\rangle\) are the Fock states of the photon. We only consider the single-photon process and the lowest two levels of the oscillator. Then, the whole state can be approximated as

\[ |\psi(t)\rangle \approx \sum_{n=0,1} b_n(t)|n_G(t), 0\rangle 
+ \sum_{n=0,1} \int \frac{d^3k}{(2\pi)^3} b_{n,\kappa}(t)|n_G(t), \gamma_{\kappa,x}\rangle. \] (32)

In order to derive the coefficients \(b_n\) and \(b_{n,\kappa}\), we introduce the dynamic phase \(\theta_n = -\int_t^t E_n(s)ds\) and adiabatic phase \(\gamma_n = i\int_t^t \langle n(s)|\dot{n}(s)\rangle\) and write

\[ b_n(t) = \exp[i(\gamma_n(t) + \theta_n(t))]|c_n(t), \]

\[ b_{n,\kappa}(t) = \exp[i(\gamma_n(t) + \theta_n(t))]|c_{n,\kappa}(t). \] (33)

Using the state-evolution equation \(i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle\), we obtain differential equations for the new coefficients \(c_n\) and \(c_{n,\kappa}\) given by

\[ \dot{c}_0(t) = -\langle 0_G(t), 0|\partial_t|m_G(t)\rangle \otimes |0\rangle \]

\[ -\langle n_G(t)|\partial_t|m_G(t)\rangle \otimes |n_G(t)\rangle = \langle n_G(t)|\partial_t|m_G(t)\rangle \otimes |n_G(t)\rangle \]

\[ \langle \gamma_{\kappa,x}\rangle \rangle = (2\pi)^3\delta(\vec{k} - \vec{k}'). \] (34)

In general, the calculation of the complete evolution process is rather involved. Fortunately, we demonstrated in Sect. 2 that the moving charge cluster acts adiabatically; therefore, the transition terms which conserve the photon number can be neglected, and we only need to consider the emission and absorption of photons caused by the coupling with the vacuum field. In this case, Eqs. (34) simplify greatly to

\[ \dot{c}_0(t) = -i\langle 0_G(t), \gamma_{\kappa,x}\rangle |H_1(\sigma_+, a_{\kappa,x}, t)|1_G(t), 0\rangle |c_1(t) \exp \[i(\gamma_1(t) + \theta_1(t) - i(\gamma_0(t) + \theta_0(t))], \]

\[ \dot{c}_1(t) = - i\langle 1_G(t), 0|H_1(\sigma_+, a_{\kappa,x}, t)|0_G(t), \gamma_{\kappa,x}\rangle |c_0, \kappa(t) \exp \[i(\gamma_1(t) + \theta_1(t) + i(\gamma_0(t) + \theta_0(t))]. \] (35)

In order to handle the decay of the excited state, we follow a method similar to the Weisskopf-Wigner approximation [16] viz.

\[ \dot{c}_1(t) = - i\int \frac{d^3k}{(2\pi)^3} \langle 1_G(t), 0|H_1(\sigma_+, a_{\kappa,x}, t)|0_G(t), \gamma_{\kappa,x}\rangle \exp \[i(\gamma_1(t) + \theta_1(t) + i(\gamma_0(t) + \theta_0(t))], \]

\[ \times \int_{-\infty}^t dt' \langle 0_G(t'), \gamma_{\kappa,x}\rangle |H_1(\sigma_+, a_{\kappa,x}, t')|1_G(t'), 0\rangle |c_1(t') \exp \[i(\gamma_1(t') + \theta_1(t') - i(\gamma_0(t') + \theta_0(t'))]. \] (36)
To proceed with the computation in a clearer form, we introduce $\Delta G(t) = E_G(t) - E_G(t) - i(1_G(t)|G(t)) + i(0_G(t)|0_G(t))$. Since the radiation spectrum typically has a peak frequency, we can effectively perform the frequency integration in the range $(-\infty, +\infty)$, yielding

$$\int_{-\infty}^{\infty} d\omega_k e^{i(\omega_\bf{k})(t-t')} = 2\pi \delta(t-t').$$ (37)

Using Eqs. (26), (36) and (37), we thereby obtain

$$\dot{c}_1(t) = -\Gamma_G(t)c_1(t),$$ (38)

where the time-dependent decay rate is

$$\Gamma_G(t) = \frac{\Delta_G^2(t)}{4\pi} \langle 1_G(t), 0|H_I(\sigma_+, a_{\Delta_G(t), x}, t) e^{i\Delta_G(t)|t} |0_G(t), \gamma_{\Delta_G(t), x} \rangle \times \langle 0_G(t), \gamma_{\Delta_G(t), x}|H_I(\sigma_-, a_{\Delta_G(t), x}^\dagger, t) e^{-i\Delta_G(t)|t} |1_G(t), 0 \rangle.$$ (39)

Then, to obtain the solutions for $c_1(t)$ and $c_{0,\bf{k}}(t)$, we combine Eq. (38) with the first line of Eqs. (35), compute the relevant matrix elements, and perform the numerical integration step by step. Ultimately, the to-be-observed radiation spectrum, accumulated until a time $t_f$, is computed via

$$\int d\omega_\bf{k} S(\omega_\bf{k}, t_f) = \int \frac{d^3k}{(2\pi)^3} |c_{0,\bf{k}}(t_f)|^2 \delta(\theta_\bf{k} - \frac{\pi}{2})$$

$$= \int d\omega_\bf{k} \frac{\omega^2_\bf{k}}{(2\pi)^3} \int d\theta_\bf{k} d\phi_\bf{k} \sin \theta_\bf{k} |c_{0,\bf{k}}(t_f)|^2$$

$$\delta(\theta_\bf{k} - \frac{\pi}{2}),$$ (40)

which explicitly yields,

$$S(\omega_\bf{k}, t_f) = \frac{\omega^2_\bf{k}}{(2\pi)^2} |c_{0,\bf{k}}(t_f)|^2, \quad (\theta_\bf{k} = \frac{\pi}{2}).$$ (41)

Figure 5 shows our calculated results for the spontaneous radiation spectrum $S(\omega_\bf{k}, t_f)$ around the peak frequency. These results show clearly that the computed spontaneous radiation spectrum depends considerably on the gauge, in both the peak frequency and the radiation intensity. Even with a relatively mild velocity of $\beta = 0.1$ of the charge cluster, the computed peak frequency in the Lorentz gauge is higher than that in the multipolar gauge by a detectable amount $\sim 10$ MHz, and higher than that in the Coulomb gauge by $\sim 60$ MHz.

For another comparison, we also plot in Fig. 5 the unperturbed spectrum in the absence of the moving cluster, with a peak frequency around $6.3369 \times 10^{13}$ Hz, which is higher than that of the perturbed spectrum in the Lorentz gauge by about 120 MHz. This is a cross-check that the moving cluster indeed acts non-perturbatively on the quantum oscillator.

In this paper, we displayed gauge dependence for the accumulated radiation spectrum without time-resolution. Registering accurately the arrival time of the emitted photons would reveal more detailed information. If time-resolution is applied, one may also consider induced transitions, either absorption or emission.

To close this section, we would like to comment that the significant gauge dependence we revealed above comes mainly from the non-perturbative, adiabatic scalar potential produced by the relativistic charge cluster. In comparison, the gauge dependence similar to that as Lamb originally noticed, Lamb [1] is quantitatively negligible in our case, as we show in Fig. 6.

### 6 Gauge-invariance under an adiabatic “switch-on” and “switch-off” of the interactions

To confirm that the gauge dependence which we found is reliable, we follow an approach in the literature by introducing a factor $\exp(-\epsilon |t|)$ to adiabatically switch on and off [9, 11, 12]. Then, in the infinite past and future, the gauge transformation of the wave function would amount to the unity transformation, and we expect to recover gauge invariance for the transition probabilities as $\epsilon$ gets big enough.

Explicitly, we add the factor $\exp(-\epsilon |t|)$ to the electric field $\vec{E}$ generated by the moving charge cluster and repeat all our calculations to obtain the radiation spectrum. We choose several values of $\epsilon$. In the previous section, we showed that the relevant gauge dependence for our system comes mainly from the external field, not from the vacuum field. So, in this section we fix the multipolar gauge for the vacuum field and focus on the gauge choice for the external field.

Figure 7 shows the new results of the spontaneous radiation spectrum with different values of $\epsilon$. For each $\epsilon$, the results of different gauges are displayed. These results are the counterparts of Fig. 5, which is essentially the result with $\epsilon = 0$. From Fig. 7, we see clearly that the gauge dependence is gradually suppressed as $\epsilon$ becomes larger. Eventually, the gauge dependence disappears at a large enough $\epsilon$, and we indeed recover the expected gauge invariance for the radiation spectrum. This is an important check of the consistency and correctness for the relatively tricky and complex calculations performed in our paper.

### 7 Summary and discussion

In this paper, we proposed a physical system in which a cluster of relativistic charges produces a non-perturbative Coulomb field and acts adiabatically on a quantum system. This is a case where we have to face sharply the long-standing concerns about: (1) what is actually the physical significance of the scalar potential
Fig. 5  The transient spontaneous radiation spectrum of our physical system. In the legend, the subscript “0” represents the gauge choice for the field produced by the moving cluster, the subscript “B” represents the gauge choice for the vacuum field; and the last two lines refer to the unperturbed spectrum without the moving cluster.

Fig. 6  The “Lamb-type” gauge dependence, displayed as the spectrum deviation from the multipolar-gauge result when it is time-dependent; (2) how the electromagnetic field actually interacts with a quantum system, especially, how to quantize a bound state in the presence of a time-dependent, non-perturbative, and adiabatic scalar potential. We calculated the spontaneous radiation spectrum of such a quantum system and find a significant gauge dependence which cannot be cured by existing methods. As an important check, we also re-calculated the radiation spectrum under an adiabatic “switch-on” and “switch-off” of the interactions and confirmed the expected recovering of gauge invariance.

We should compare our physical system with related phenomena. The key elements of our design include: a relativistic charge cluster of a huge number, a quantum oscillator of relatively large spatial extension and low frequency, and adiabatic conditions for the oscillator to complete transient spontaneous radiation during the interacting period. These elements have been encountered individually in some studies. For example, Coulomb excitation [17,18] may also utilize a relativistic projectile, but in a perturbative and/or non-adiabatic way. The term “strong-field,” as in the study of the AC-stark effect [19], typically refers to an intensive electromagnetic wave or the laser field, not a relativistic Coulomb field. It is our study that combines all these key elements and brings out sharply a “Coulomb-type” gauge dependence. We invent this terminology to refer to the gauge-choice problem for the Coulomb field of charged particles, especially relativistic ones. Naturally, the gauge-choice problem for an electromagnetic wave or the vacuum field may be termed “Lamb-type.”

Since our system has very low energy, the decay rate is rather small. This also reveals clearly that the gauge dependence we encounter comes mainly from defining the quantum basis and is not the Lamb-type. To further address, this new type of gauge dependence, other quantum systems of large spatial extension may also be employed, especially Rydberg...
atoms, for which the Coulomb field of a relativistic charged cluster may become non-perturbative relatively easily.

The design we proposed also makes a tunable system to study the relativistic bound-state problems. The usual relativistic bound states often go to two extremes: either all particles are relativistic and the problem becomes over complicated, or, the acted-upon particle is relativistic while the acting particles are non-relativistic, which can be easily handled by the Dirac equation. In our design, the acting cluster is relativistic, while the acted-upon particle is non-relativistic. (One may feel like to turn the situation around by going to the rest frame of the acting cluster. But this idea would not work, since we may easily arrange two or more acting clusters with different velocities.) Moreover, the design makes both the relativity parameter $\beta$ and the interaction strength continuously adjustable.

At the end of this paper, we discuss the far-reaching implications of the “Coulomb-type” gauge dependence.
Whenever some gauge dependence stands in the way, there is often an intent to discard \( (\vec{A}, \phi) \) and stick to \( (\vec{E}, \vec{B}) \), especially in the early practices and debates. However, a consensus was gradually reached that this is not possible, at least in a local formulation. For example, the multipolar-gauge expressions involve only \( (\vec{E}, \vec{B}) \), but in terms of path-integrals. It should be especially reminded that such integrals are in general path-dependent. Namely, the multipolar gauge itself is not unique. Another intent of long history is to distinguish between energy and Hamiltonian for a non-conservative system, and to refuse the assignment of any physical significance to the time-dependent \( \phi(\vec{x}, t) \). We do not claim against such an intent and merely present a case where the existing methods have to utilize \( \phi(\vec{x}, t) \), with a gauge-dependent result.

There also exists an intent in the opposite way, especially after the work of Aharanov and Bohm [20] to think of \( (\vec{A}, \phi) \) as a physical reality. Another support for this idea is that the Coulomb field always accompanies a charge and can never be stripped away. We do not claim either that our findings add to that intent. In fact, we must indeed be clear that our starting position, namely the semi-classical Hamiltonian (5) or (15) under the external-field approximation, is known to be unsafe if the external source is relativistic [21]. While an appropriate and full quantum-field method is still lacking [22, 23]. In such a circumstance, the gauge dependence we just found is not a disaster, but an advantage: Since the Hamiltonian (5) or (15) is not justified anyway, there is no sense in sticking to the actual \( (\vec{A}, \phi) \) generated by the external source. Instead, one may always try to define an effective external field which, when plugged into the Hamiltonian (5) or (15), may fit well with experimental data. This is indeed quite probable, as we saw in Sect. 5 that by just tuning the gauge, we may fit the peak frequency of the radiation spectrum; and another overall factor may be multiplied to fit the radiation intensity. Namely, the effective external field may likely be parameterized with the gauge potential in a particular gauge. This gauge is not necessarily among the most common ones and might possibly be an individualized condition for the field of each charge, instead of the usual conditions for the overall field.

Such an effective external field is not merely of important phenomenological use, but might also shed light on the physical significance of the gauge potential. Should it turn out to be closer to the multipolar-gauge expression, it might suggest that \( \vec{E} \) is indeed more essential than \( \phi \). Should it turn out to be closer to the expression of the Lorentz or Coulomb gauge, and deny that of the multipolar gauge, however, we would find an extremely interesting phenomenon which may be termed “kinematic scalar Aharanov-Bohm effect”: Due to Lorentz contraction, a charged particle close to the speed of light produces negligible electric field along its trajectory. However, its effective scalar potential could possibly be felt by a quantum system. We may even put this in a science-fiction style: The classical bodies can never feel a projectile flying directly to them at nearly the speed of light, no matter how immensely energetic, until the moment it crashes in; while a quantum system may act as an alerting whistle. Pitifully, we human beings may belong to the “classical bodies.” Such a projectile might be a particle, or even a planet.

It would be also very interesting to explore how a quantum system is influenced by the gravitational field of a relativistic object, and whether a similar “A-B” effect would occur. So far as gravity is involved, a full quantum-field approach is not even conceivable, and the external-field method is the only option. However, even if one day the possible “kinematic scalar Aharanov-Bohm effect” could eventually be derived via a rigorous quantum-field approach, (hopefully in QED first.), it is in all senses a novel quantum-field effect, just like the Lamb shift [24] and the asymptotic freedom [25, 26].

The ultimate goal of this study is to thoroughly examine how the gauge field actually interacts with a quantum system and explore more closely the physical significance of the gauge potential. In closing, we should mention that the most easily accessible facility for this study is the relativistic atomic beam. It only needs to build a strong and large-scale Coulomb field into the facility so that in the beam frame, the atom can equivalently feel a non-perturbative and adiabatic field from a relativistic charge.

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Author contributions X-NC and Y-HL contributed equally to this work under the tutoring of X-SC. The results in the work were independently calculated and cross-checked by X-NC and Y-HL, respectively. All authors participated in the presentation and discussion of this work.

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Appendix A: The scalar potential in different gauges

In this Appendix, we give the details of the scalar potential \( \phi(\vec{x}, t) \) for our system in the commonly used
Lorentz, Coulomb, and multipolar gauges. For the Lorentz gauge,
\[
\phi_L = \frac{1}{4\pi} \left( \frac{-e}{l+x} + \frac{-e}{l-x} + \frac{N e}{\sqrt{(x-L(t))^2 + (1-\beta^2)Y^2}} \right),
\]
and for Coulomb gauge,
\[
\phi_C = \frac{1}{4\pi} \left( \frac{-e}{l+x} + \frac{-e}{l-x} + \frac{N e}{\sqrt{(x-L(t))^2 + Y^2}} \right).
\]
Here, \(x\) is the coordinate of the electron, \(L(t)\) is the coordinate of the charge cluster at time \(t\). The subscripts \(L, C\) refer to expressions in the Lorentz and Coulomb gauges, respectively, and a subscript \(G\) will denote a general gauge.

By applying the PZW transformation (12), one obtains \(\phi_M\) in the multipolar gauge,
\[
\phi_M = \left\{ \phi_G(x_0, t) + \int_0^t d\lambda \vec{E}(\lambda x + (1-\lambda)x_0, t) \cdot (\vec{x} - \vec{x}_0) \right\}.
\]

As we explained in the text, we Taylor-expand \(\phi(\vec{x}, t)\).

For the Lorentz gauge,
\[
\phi_L \approx \frac{1}{4\pi} \left\{ \frac{-e}{(x_0+l)} - \frac{-e}{x_0-l} + \frac{N e}{\sqrt{(x_0-L(t))^2 + (1-\beta^2)Y^2}} \right. \\
- \frac{-e}{(x_0+l)^3} - \frac{-e}{(x_0-l)^3} \\
+ \frac{N e (x_0-L(t))}{\sqrt{(x_0-L(t))^2 + (1-\beta^2)Y^2}} (x-x_0) \\
+ \frac{-e}{(x_0+l)^3} - \frac{-e}{(x_0-l)^3} \\
- \frac{1}{2} \left( \frac{N e}{\sqrt{(x_0-L(t))^2 + (1-\beta^2)Y^2}} \right) (x-x_0)^2 \\
\left. - \frac{3N e (x_0-L(t))^2}{\sqrt{(x_0-L(t))^2 + (1-\beta^2)Y^2}} \right\}.
\]

For the Coulomb gauge,
\[
\phi_C \approx \frac{1}{4\pi} \left\{ \frac{-e}{x_0+l} - \frac{-e}{x_0-l} + \frac{N e}{\sqrt{(x_0-L(t))^2 + Y^2}} \right. \\
- \frac{-e}{(x_0+l)^3} - \frac{-e}{(x_0-l)^3} \\
+ \frac{N e (x_0-L(t))}{\sqrt{(x_0-L(t))^2 + Y^2}} (x-x_0) \\
- \frac{1}{2} \left( \frac{N e}{\sqrt{(x_0-L(t))^2 + Y^2}} \right) (x-x_0)^2 \\
\left. - \frac{3N e (x_0-L(t))^2}{\sqrt{(x_0-L(t))^2 + Y^2}} \right\}.
\]

And for the multipolar gauge,
\[
\phi_M \approx \phi_G(x_0, t) + \frac{1}{4\pi} \left\{ \frac{-e}{(x_0+l)^3} - \frac{-e}{(x_0-l)^3} \right. \\
+ \frac{N e (1-\beta^2)(x_0-L(t))}{\sqrt{(x_0-L(t))^2 + (1-\beta^2)Y^2}} (x-x_0) \\
+ \frac{-e}{(x_0+l)^3} - \frac{-e}{(x_0-l)^3} \\
- \frac{1}{2} \left( \frac{N e (1-\beta^2)}{\sqrt{(x_0-L(t))^2 + (1-\beta^2)Y^2}} \right) (x-x_0)^2 \\
\left. - \frac{3N e (1-\beta^2)(x_0-L(t))^2}{\sqrt{(x_0-L(t))^2 + (1-\beta^2)Y^2}} \right\}.
\]

The value of \(x_0\) in each gauge is found by solving \(\partial_x \phi_G(x, t)|_{x=x_0} = 0\).

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