CYCLIC LORENTZIAN LIE GROUPS

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Abstract. We consider Lie groups equipped with a left-invariant cyclic Lorentzian metric. As in the Riemannian case, in terms of homogeneous structures, such metrics can be considered as different as possible from bi-invariant metrics. We show that several results concerning cyclic Riemannian metrics do not extend to their Lorentzian analogues, and obtain a full classification of three- and four-dimensional cyclic Lorentzian metrics.

1. Introduction

Homogeneous Riemannian manifolds were characterized in terms of homogeneous structures by Ambrose and Singer [1] (see also [16]). Gadea and Oubiña [9] introduced homogeneous pseudo-Riemannian structures, to give a corresponding characterization of reductive homogeneous pseudo-Riemannian manifolds. Let $G$ denote a (connected) Lie group and $\mathfrak{g}$ its Lie algebra. It is well known that left-invariant pseudo-Riemannian metrics $g$ on $G$ are in a one-to-one correspondence with nondegenerate inner products on $\mathfrak{g}$, which we shall denote again by $g$. If $g$ is such an inner product on $\mathfrak{g}$ and $\nabla$ denotes its Levi-Civita connection, then tensor $S_{x,y} = \nabla_{x}y, x, y \in \mathfrak{g}$, is a homogeneous pseudo-Riemannian structure. Conversely, among homogeneous pseudo-Riemannian manifolds, pseudo-Riemannian Lie groups are characterized by the fact that they admit a global pseudo-orthonormal frame field $\{e_i\}$, such that $S_{e_i,e_j} = \nabla_{e_i}e_j$ defines a homogeneous pseudo-Riemannian structure (see for example [4]).

A systematic study of left-invariant Riemannian cyclic metrics started in [11], with particular regard to the semi-simple and solvable cases and a complete classification of the examples of dimension up to five. Following [11], a left-invariant pseudo-Riemannian metric $g$ is said to be cyclic if the homogeneous pseudo-Riemannian structure $S$ described above falls within $S_1 \oplus S_2$ in Tricerri-Vanhecke’s classification of homogeneous structures. Explicitly, this means that

$$\mathcal{G}_{x,y,z}g([x,y],z) = 0 \quad \text{for all } x, y, z \in \mathfrak{g},$$

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where $\mathcal{S}$ stands for the cyclic sum. Note that, as bi-invariant metrics are characterized by condition $S \in \mathcal{S}_3$, cyclic metric can be considered as different as possible from the bi-invariant ones.

In this paper, we undertake the investigation of left-invariant cyclic pseudo-Riemannian metrics, starting from the Lorentzian ones. Although four-dimensional connected, simply connected Lorentzian Lie groups coincide with the Riemannian ones, their geometry proves to be richer, also with regard to cyclic metrics. We shall classify cyclic Lorentzian Lie groups of dimension up to four and show that several rigidity results valid for Riemannian cyclic metrics do not extend to pseudo-Riemannian settings. In particular, differently from the Riemannian case, we show the existence of compact or nilpotent non-abelian cyclic Lorentzian Lie groups.

The paper is organized in the following way. In Section 2 we shall report some basic information concerning homogeneous structures and cyclic metrics. In Sections 3 and 4 we shall give the complete classification of left-invariant cyclic Lorentzian metrics in dimension three and four, respectively. In particular, Theorems 3.2, 4.3, 4.4 and 4.5 below show that contrarily to the Riemannian case, all possible connected and simply connected three- and four-dimensional Lie groups admit an appropriately chosen left-invariant Lorentzian cyclic metric. We conclude in Section 5 with the classification of cotorsionless Lorentzian three-manifolds, and some observations, concerning in particular the link between three- and four-dimensional cyclic Lie groups, and the obstruction to the construction of non-symmetric solvmanifolds from solvable cyclic groups.

2. Preliminaries

Let $M$ be a connected manifold and $g$ a pseudo-Riemannian metric on $M$. We denote by $\nabla$ the Levi-Civita connection of $(M, g)$ and by $R$ its curvature tensor. The following definition was introduced by Gadea and Oubiña:

**Definition 2.1.** [9] A homogeneous pseudo-Riemannian structure on $(M, g)$ is a tensor field $S$ of type $(1, 2)$ on $M$, such that the connection $\tilde{\nabla} = \nabla - S$ satisfies

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0.$$

The geometric meaning of the existence of a homogeneous pseudo-Riemannian structure is explained by the following result.

**Theorem 2.2.** [9] Let $(M, g)$ be a connected, simply connected and complete pseudo-Riemannian manifold. Then, $(M, g)$ admits a pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.

Observe that if any of the hypotheses of connectedness, simple connectedness or completeness is missing, the existence of a homogeneous structure characterizes local homogeneity of the manifold. We remark that, while any
homogeneous Riemannian manifold is reductive, a homogeneous pseudo-Riemannian manifold needs not be reductive. This restriction also happens when considering local homogeneity, although a precise definition of local reductivity is required in this context (see [12]). Definition 2.1 and Theorem 2.2 above extend the characterization of homogeneous Riemannian manifolds by means of homogeneous structures [1] to reductive homogeneous pseudo-Riemannian manifolds.

We explicitly recall that for the reductive homogeneous pseudo-Riemannian manifold \((M = G/H, g)\), with reductive decomposition \(g = \mathfrak{h} \oplus \mathfrak{m}\), the linear connection \(\tilde{\nabla} = \nabla - T\) is the canonical connection associated to the reductive decomposition [16].

Let \(V\) denote an \(n\)-dimensional real vector space, equipped with a non-degenerate inner product \(\langle \cdot, \cdot \rangle\) of signature \((k, n-k)\). It is the model space for the tangent space at each point of a homogeneous pseudo-Riemannian manifold \((M, g)\). Let \(S(V)\) denote the vector space of \((0,3)\)-tensors \(S\) on \(V\), satisfying the same condition as the first equation \(\tilde{\nabla} g = 0\) of a homogeneous structure, that is,

\[
S(V) = \left\{ S \in \bigotimes^3 V^*: S_{xyz} = -S_{xzy}, \ x, y, z \in V \right\},
\]

where \(S_{xyz} := \langle x, y, z \rangle\). Then, \(\langle \cdot, \cdot \rangle\) induces an inner product on \(S(V)\), given by

\[
\langle S, S' \rangle = \sum_{i,j,k=1}^{n} \epsilon_i \epsilon_j \epsilon_k S_{e_i e_j e_k} S'_{e_i e_j e_k},
\]

where \(\{e_i\}\) denotes a pseudo-orthonormal basis of \(V\) and \(\epsilon_i = \langle e_i, e_i \rangle\) for all indices \(i\). The following result was proved [10].

**Theorem 2.3.** [10] If \(\dim V \geq 3\), then \(S(V)\) decomposes into the orthogonal direct sum

\[
S(V) = S_1(V) \oplus S_2(V) \oplus S_3(V),
\]

where

\[
S_1(V) = \{ S \in S(V) : S_{xyz} = \langle x, y \rangle \omega(z) - \langle x, z \rangle \omega(y), \ \omega \in V^* \},
\]

\[
S_2(V) = \{ S \in S(V) : \mathfrak{S}_{xyz} S_{xyz} = 0, \ c_{12}(S) := \sum_{i=1}^{n} \epsilon_i S_{e_i e_i} = 0 \},
\]

\[
S_3(V) = \{ S \in S(V) : S_{xyz} + S_{yxz} = 0 \}
\]

are invariant and irreducible under the action of \(O(k, n-k)\). If \(\dim V = 2\), then \(S(V) = S_1(V)\). Furthermore,

\[
S_1(V) \oplus S_2(V) = \{ S \in S(V) : \mathfrak{S}_{xyz} S_{xyz} = 0 \},
\]

\[
S_2(V) \oplus S_3(V) = \{ S \in S(V) : c_{12}(S) = 0 \},
\]

\[
S_1(V) \oplus S_3(V) = \left\{ S \in S(V) : \frac{S_{xyz} + S_{yxz}}{2} = \frac{2 \langle x, y \rangle \omega(z)}{-\langle x, z \rangle \omega(y) - \langle y, z \rangle \omega(x)}, \ \omega \in V^* \right\}.
\]
As proved in [10], naturally reductive homogeneous pseudo-Riemannian manifolds are all and the ones admitting a homogeneous structure $S \in S_3(V)$, while cotorsionless manifolds are characterized by the existence of homogeneous structures $S \in S_1(V) \oplus S_2(V)$.

Among homogeneous pseudo-Riemannian manifolds, pseudo-Riemannian Lie groups are characterized by the existence of a special homogeneous pseudo-Riemannian structure (see also [4]). In fact, when $(M = G, g)$ is a Lie group equipped with a left-invariant Lorentzian metric $g$, uniquely determined at the algebraic level by a non-degenerate inner product $g$ on the Lie algebra $\mathfrak{g}$, tensor $S_{xy} = \nabla_x y$, $x, y \in \mathfrak{g}$, defines a homogeneous pseudo-Riemannian structure. In this case $\tilde{\nabla}$, which vanishes when evaluated on left invariant vector fields, is the so-called $(-)$-connection of Cartan-Schouten, whose curvature and torsion are respectively given by $\tilde{R} = 0$ and $\tilde{T}(X, Y) = -[X, Y]$.

It is well known that the left-invariant pseudo-Riemannian metric corresponding to $g$ is bi-invariant if and only if the above special homogeneous structure $S$ belongs to $S_3(V)$. On the other hand, $g$ is called cyclic when $S \in S_1(V) \oplus S_2(V)$. Thus, taking into account the orthogonal decomposition of $S(V)$, left-invariant cyclic metrics can be considered “as far away as possible” from the bi-invariant ones.

We report below several strong rigidity results obtained in [11] for Riemannian cyclic metrics. As a consequence of the classifications given in the next sections, we shall see that most of these result do not hold any more for Lorentzian cyclic metrics.

**Proposition 2.4.** [11] A connected cyclic Riemannian Lie group is flat if and only if it is abelian. Moreover, let $G$ be a non-abelian cyclic Riemannian Lie group.

(i) If $G$ is solvable, then it has strictly negative scalar curvature.

(ii) If $G$ is unimodular, then it has positive sectional curvatures. If moreover it is solvable, then it has both positive and negative curvatures.

(iii) If $G$ is not unimodular there exist negative sectional curvatures.

**Theorem 2.5.** [11] Every non-abelian cyclic Riemannian Lie group is not compact.

**Theorem 2.6.** [11] The universal covering $\tilde{SL}(2, \mathbb{R})$ of $SL(2, \mathbb{R})$ is the only connected, simply connected simple real Riemannian Lie group.

**Proposition 2.7.** [11] Non-abelian nilpotent Lie groups do not admit left-invariant Riemannian cyclic metrics.

We end this section clarifying the relationship between Riemannian and Lorentzian Lie groups. Let $G$ be an $n$-dimensional connected Lie group and $\mathfrak{g}$ its Lie algebra. Left-invariant Lorentzian metrics on $G$ are in a one-to-one correspondence with inner products on $\mathfrak{g}$ of signature $(n - 1, 1)$. If $g$ is such a Lorentzian inner product, then it exists a pseudo-orthonormal basis
\{e_1, \ldots, e_n\} of \frak{g}, with \(e_n\) time-like. But then, \(G\) also admits a corresponding left-invariant Riemannian metric, completely determined at the Lie algebra level by having \(\{e_1, \ldots, e_n\}\) as an orthonormal basis of \(\frak{g}\).

Conversely, given a positive definite inner product \(\bar{g}\) over \(\frak{g}\), and a \(\bar{g}\)-orthonormal basis \(\{e_1, \ldots, e_n\}\) of \(\frak{g}\), it suffices to change the causal character of one of vectors in the basis, choosing it to be time-like, to determine a left-invariant Lorentzian metric on \(G\). Therefore, the following result holds (see also [7]).

**Proposition 2.8.** The class of \(n\)-dimensional connected, simply connected Lorentzian Lie groups (respectively, Lorentzian Lie algebras) coincides with the class of the Riemannian ones.

We explicitly observe that, although connected, simply connected Lorentzian Lie groups coincide with the Riemannian ones (Proposition 2.8), the geometry of left-invariant Lorentzian metrics is much richer than the one of their Riemannian counterpart. The fundamental reason for such a difference is the existence in Lorentzian settings of vectors with different causal characters. Some consequences of this fact are:

- that (contrarily to the Riemannian case) a self-adjoint operator with respect to a Lorentzian metric needs not be diagonalizable. For example, this yields four standard forms of three-dimensional unimodular Lorentzian Lie groups [15], while just one form occurs in Riemannian settings [13];
- that every subspace of a vector space endowed with a positive definite inner product, inherits a positive inner product, while a subspace of a Lorentzian vector space inherits an inner product that can be either positive definite, Lorentzian, or even degenerate. In particular, this fact yields the differences in the classifications of three-dimensional non-unimodular Lorentzian [8] and Riemannian [13] Lie groups, and of left-invariant Lorentzian [7] and Riemannian metrics [2] on four-dimensional Lie groups.

### 3. Three-dimensional cyclic Lorentzian Lie groups

As proved in [10] and reported in the above Theorem 2.3, for a two-dimensional vector space \(V\), one has \(\mathcal{S}(V) = \mathcal{S}_1(V)\). Consequently, any two-dimensional pseudo-Riemannian Lie group is cyclic. Next, homogeneous Lorentzian three-manifolds were classified in [4], taking into account previous results of Rahmani [15] and Cordero and Parker [8]. The classification result is the following.

**Theorem 3.1.** [4] A three-dimensional connected, simply connected complete homogeneous Lorentzian manifold \((M, g)\) is either symmetric, or \(M = G\) is a three-dimensional Lie group and \(g\) is left-invariant. Precisely, one of the following cases occurs:
I) If $G$ is unimodular, then there exists a pseudo-orthonormal frame field 
\{e_1, e_2, e_3\}, with $e_3$ time-like, such that the Lie algebra of $G$ is one of the
following:

\[
\begin{align*}
[e_1, e_2] &= \alpha e_1 - \beta e_3, \\
[e_1, e_3] &= -\alpha e_1 - \beta e_2, \\
[e_2, e_3] &= \beta e_1 + \alpha e_2 + \alpha e_3 & \alpha \neq 0.
\end{align*}
\]

(3.1) $\mathfrak{g}_1$ :

If $\beta \neq 0$, then $G$ is $\tilde{SL}(2, \mathbb{R})$, while for $\beta = 0$, $G = E(1, 1)$ is the group
of rigid motions of the Minkowski two-space.

\[
\begin{align*}
[e_1, e_2] &= -\gamma e_2 - \beta e_3, \\
[e_1, e_3] &= -\beta e_2 + \gamma e_3, & \gamma \neq 0, \\
[e_2, e_3] &= \alpha e_1.
\end{align*}
\]

(3.2) $\mathfrak{g}_2$ :

In this case, $G = \tilde{SL}(2, \mathbb{R})$ if $\alpha \neq 0$, while $G = E(1, 1)$ if $\alpha = 0$.

(3.3) ($\mathfrak{g}_3$) :

\[
\begin{align*}
[e_1, e_2] &= -\gamma e_3, & [e_1, e_3] &= -\beta e_2, & [e_2, e_3] &= \alpha e_1.
\end{align*}
\]

The following Table I (where $\tilde{E}(2)$ and $H_3$ respectively denote the universal
covering of the group of rigid motions in the Euclidean two-space and the
Heisenberg group) lists all the Lie groups $G$ which admit a Lie algebra $\mathfrak{g}_3$,
according to the different possibilities for $\alpha$, $\beta$ and $\gamma$:

| Lie group | $\alpha$ | $\beta$ | $\gamma$ |
|-----------|----------|----------|----------|
| $\tilde{SL}(2, \mathbb{R})$ | $+$ | $+$ | $+$ |
| $\tilde{SL}(2, \mathbb{R})$ | $+$ | $-$ | $-$ |
| $SU(2)$ | $+$ | $+$ | $-$ |
| $\tilde{E}(2)$ | $+$ | $+$ | $0$ |
| $\tilde{E}(2)$ | $+$ | $0$ | $-$ |
| $E(1, 1)$ | $+$ | $-$ | $0$ |
| $E(1, 1)$ | $+$ | $0$ | $+$ |
| $H_3$ | $+$ | $0$ | $0$ |
| $H_3$ | $0$ | $0$ | $-$ |
| $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ | $0$ | $0$ | $0$ |

Table I: 3D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_3$

\[
\begin{align*}
[e_1, e_2] &= -\varepsilon e_2 + (2\varepsilon - \beta)e_3, & \varepsilon = \pm 1, \\
[e_1, e_3] &= -\beta e_2 + e_3, \\
[e_2, e_3] &= \alpha e_1.
\end{align*}
\]

(3.4) $\mathfrak{g}_4$ :
Table II below describes all Lie groups $G$ admitting a Lie algebra $\mathfrak{g}_4$:

| Lie group         | $\varepsilon = 1$ | $\alpha$ | $\beta$ | Lie group         | $\varepsilon = -1$ | $\alpha$ | $\beta$ |
|-------------------|-------------------|----------|----------|-------------------|-------------------|----------|----------|
| $\tilde{SL}(2, \mathbb{R})$ | $\neq 0$ | $\neq 1$ |          | $\tilde{SL}(2, \mathbb{R})$ | $\neq 0$ | $\neq -1$ |          |
| $E(1, 1)$         | $0$               | $\neq 1$ |          | $E(1, 1)$         | $0$               | $\neq -1$ |          |
| $E(1, 1)$         | $< 0$             | $1$      |          | $E(1, 1)$         | $> 0$             | $-1$     |          |
| $E(2)$            | $> 0$             | $1$      |          | $E(2)$            | $< 0$             | $-1$     |          |
| $H_3$             | $0$               | $1$      |          | $H_3$             | $0$               | $-1$     |          |

Table II: 3D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_4$

II) If $G$ is non-unimodular, then there exists a pseudo-orthonormal frame field \( \{e_1, e_2, e_3\} \), with $e_3$ time-like, such that the Lie algebra of $G$ is one of the following:

\[
\begin{align*}
\mathfrak{g}_5 & : \quad [e_1, e_2] = 0, \\
& \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \\
& \quad [e_2, e_3] = \gamma e_1 + \delta e_2, \quad \alpha + \delta \neq 0, \alpha \gamma + \beta \delta = 0. \\
\mathfrak{g}_6 & : \quad [e_1, e_2] = \alpha e_2 + \beta e_3, \\
& \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad \alpha + \delta \neq 0, \alpha \gamma - \beta \delta = 0. \\
\mathfrak{g}_7 & : \quad [e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \\
& \quad [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \\
& \quad [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \quad \alpha + \delta \neq 0, \alpha \gamma = 0.
\end{align*}
\]

With the obvious exception of $\mathbb{S}^2 \times \mathbb{R}$, every three-dimensional Lorentzian symmetric space can also be realized in terms of a suitable Lorentzian Lie group \[5, \text{Theorem 4.2}\]. Hence, apart from $\mathbb{S}^2 \times \mathbb{R}$, the classification of three-dimensional Lorentzian cotorsionless manifolds reduces to the one of three-dimensional Lorentzian Lie groups.

In order to have a cyclic metric $g$, it suffices to check condition \[1.1\] on the vectors of a basis $\{e_i\}$ of $\mathfrak{g}$, that is,

\[
\mathcal{S}_{i,j,k=1}^3 g([e_i, e_j], e_k) = 0 \quad \text{for all indices } i, j, k.
\]

Note that if two of indices $i, j, k$ coincide, then equation \[3.8\] is trivially satisfied. Hence, in the three-dimensional case, $g$ is cyclic if and only if

\[
g([e_1, e_2], e_3) + g([e_2, e_3], e_1) + g([e_3, e_1], e_2) = 0.
\]

For each three-dimensional Lorentzian Lie group, the above Theorem \[3.1\] provides an explicit description of the corresponding Lie algebra in terms of a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$, with $e_3$ time-like. We now check equation \[3.9\] for these examples and we get the following cases:

1) $\mathfrak{g}_1$ is cyclic if and only if $\beta = 0$;
2) \( g_2 \) is cyclic if and only if \( \alpha = -2\beta \);
3) \( g_3 \) is cyclic if and only if \( \alpha + \beta + \gamma = 0 \);
4) \( g_4 \) is cyclic if and only if \( \alpha = 2(\varepsilon - \beta) \);
5) \( g_5 \) is cyclic if and only if \( \beta - \gamma = 0 \);
6) \( g_6 \) is cyclic if and only if \( \beta + \gamma = 0 \);
7) \( g_7 \) is cyclic if and only if \( \gamma = 0 \).

Therefore, taking into account the above Theorem 3.1, we proved the following result.

**Theorem 3.2.** A three-dimensional connected, simply connected non-abelian cyclic Lorentzian Lie group is isometrically isomorphic to one of the following Lie groups:

I) In the unimodular case:

(a) \( E(1,1) \), with Lie algebra described by one of the following cases:
   \( g_1 \) with \( \beta = 0 \); \( g_2 \) with \( \alpha = \beta = 0 \); \( g_3 \) with \( \alpha + \beta = \gamma = 0 \);
(b) \( \tilde{SL}(2,\mathbb{R}) \), with Lie algebra described by one of the following cases:
   \( g_2 \) with \( \alpha = -2\beta \neq 0 \); \( g_3 \) with \( \alpha = -(\beta + \gamma) > 0 \) and \( \beta, \gamma < 0 \); \( g_4 \) with \( \alpha = 2(\varepsilon - \beta) \neq 0 \);
(c) \( SU(2) \), with Lie algebra described by \( g_3 \) with \( \alpha = -(\beta + \gamma) \) and \( \gamma < 0 < \beta \);
(d) \( E(2) \), with Lie algebra described by \( g_3 \) with \( \beta = \alpha + \gamma = 0 \) and \( \gamma < 0 \);
(e) \( H_3 \), with Lie algebra described by \( g_4 \) with \( \alpha = \varepsilon - \beta = 0 \).

II) In the non-unimodular case: the connected, simply connected Lie group \( G \), whose Lie algebra is either \( g_5 \) with \( \beta = \gamma \), \( g_6 \) with \( \beta = -\gamma \), or \( g_7 \) with \( \gamma = 0 \).

Note that in general, each of the cases listed in the above Theorem 3.2 gives rise to a family of left-invariant cyclic Lorentzian metrics, depending on one or more parameters.

Curvature properties of three-dimensional Lorentzian Lie groups have been determined in [5]. Together with the examples classified in Theorem 3.2, the results of [5] already permit to emphasize some deep differences among Lorentzian and Riemannian cyclic metrics. In fact:

1) \( SU(2) \) is a connected, simply connected Lie group, both compact and simple. Hence, case (c) of Theorem 3.2 yields a Lorentzian counterexample to both Theorem 2.5 and Theorem 2.6.

2) The Heisenberg group \( H_3 \) is non-abelian and nilpotent. Hence, case (e) of Theorem 3.2 yields a Lorentzian counterexample to both Proposition 2.4 and Proposition 2.7.

3) Non-unimodular Lie group \( G \), with Lie algebra \( g_7 \) satisfying either \( \alpha = \gamma = 0 \) or \( \gamma = 0 \neq \alpha = \delta \), is equipped with a flat cyclic Lorentzian metric, giving a Lorentzian counterexample to Proposition 2.4(iii).
4. Four-dimensional cyclic Lorentzian Lie groups

As we observed in Section 2 (Proposition 2.8), in any dimension $n$, connected, simply connected Lorentzian Lie groups coincide with the Riemannian ones. Taking into account the classification of four-dimensional Riemannian Lie groups given by Bérard-Bérgery in [3], we then have the following.

**Proposition 4.1.** The connected and simply connected four-dimensional Lorentzian Lie groups are:

(i) the (unsolvable) direct products $SU(2) \times \mathbb{R}$ and $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$;

(ii) one of the following solvable Lie groups:

(ii1) the non-trivial semi-direct products $\widetilde{E}(2) \rtimes \mathbb{R}$ and $E(1,1) \rtimes \mathbb{R}$;

(ii2) the non-nilpotent semi-direct products $H_3 \rtimes \mathbb{R}$ ($H_3$ denoting the Heisenberg group);

(ii3) the semi-direct products $\mathbb{R}^3 \rtimes \mathbb{R}$.

We observe that all the examples classified in the above Proposition share the same fundamental structure, in the sense that all their Lie algebras $\mathfrak{g}$ are of the form $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, where $\mathfrak{r}$ is a one-dimensional Lie algebra, spanned by a vector acting (possibly in a trivial way) as a derivation on a three-dimensional unimodular Lie algebra $\mathfrak{h}$.

Semi-direct products involving a three-dimensional non-unimodular Lie algebra do not appear in the above classification. Indeed, it is easy to check that a semi-direct product $\mathfrak{h} \rtimes \mathfrak{r}$, with $\mathfrak{h}$ non-unimodular, is also isomorphic to a semi-direct product $\mathfrak{h} \rtimes \widetilde{\mathfrak{r}}$, with $\mathfrak{h}$ unimodular.

To make the Lorentzian case more interesting than its Riemannian counterpart, we have the following fundamental difference: if $\mathfrak{g}$ is a positive definite inner product on $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, the same is true for its restriction $\mathfrak{g}|_{\mathfrak{h}}$ over $\mathfrak{h}$. However, if $\mathfrak{g}$ is Lorentzian, then three different cases can occur, as $\mathfrak{g}|_{\mathfrak{h}}$ is either

(a) positive definite, (b) Lorentzian, or (c) degenerate.

We now give the following key result.

**Proposition 4.2.** [7] Let $(\mathfrak{g}, g)$ be an arbitrary four-dimensional Lorentzian Lie algebra. Then, there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$, such that

- $\mathfrak{h} = \text{span}(e_1, e_2, e_3)$ is a three-dimensional Lie algebra and $e_4$ acts as a derivation on $\mathfrak{h}$ (that is, $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, where $\mathfrak{r} = \text{span}(e_4)$), and

- with respect to $\{e_1, e_2, e_3, e_4\}$, the Lorentzian inner product takes one of the following forms:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

**Proof.** The following argument partially corrects and replaces the proof of Proposition 2.3 in [7]. Consider a semi-direct product $\mathfrak{g} = \mathfrak{k} \rtimes \mathfrak{r}$ of two Lie
algebras $\mathfrak{r}$ and $\mathfrak{t}$, with $\mathfrak{r} = \text{span}(v)$ one-dimensional. Note that for any vector $w \in \mathfrak{t}$ we have again $\mathfrak{g} = \mathfrak{t} \rtimes \mathfrak{r}$, where $\tilde{\mathfrak{r}} = \text{span}(v + w)$. In fact, since $\mathfrak{r}$ is one-dimensional, $\mathfrak{g} = \mathfrak{t} \rtimes \mathfrak{r}$ means that $[\mathfrak{r}, \mathfrak{r}] = 0$, $[\mathfrak{t}, \mathfrak{r}] \subset \mathfrak{t}$ and $[\mathfrak{r}, \mathfrak{t}] \subset \mathfrak{t}$. From these equations and the definition of $\tilde{\mathfrak{r}}$ it then follows at once that the same conditions hold replacing $\mathfrak{r}$ by $\tilde{\mathfrak{r}}$, that is, $\mathfrak{g} = \mathfrak{t} \rtimes \tilde{\mathfrak{r}}$.

Let $g$ denote a Lorentzian inner product on a four-dimensional Lie algebra $\mathfrak{g}$. Then, by the above Proposition 4.2, we know that $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, where $\mathfrak{r} = \text{span}(v)$ is one-dimensional. We now study separately three cases, according on whether the restriction of $g$ on $\mathfrak{h}$ is respectively (a) positive definite, (b) Lorentzian, or (c) degenerate.

**Case (a).** Since $g|_{\mathfrak{h}}$ is positive definite, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ for $g|_{\mathfrak{h}}$.

If $\mathfrak{r} = \text{span}(v)$, we now consider the orthogonal projection $w$ of $v$ on $\mathfrak{h}$, that is, $w := \sum_{i=1}^3 g(v, e_i)e_i$. Next, we put $\tilde{v} := v - w$ and $\tilde{\mathfrak{r}} := \text{span}(\tilde{v})$. By the above remark, we still have $\mathfrak{g} = \mathfrak{h} \rtimes \tilde{\mathfrak{r}}$.

Moreover, $\tilde{v}$ is orthogonal to $e_1, e_2, e_3$ and so, $\tilde{\mathfrak{r}} = \mathfrak{h}^\perp$. Since $g|_{\mathfrak{h}}$ is non-degenerate, so is $\tilde{\mathfrak{r}} = \mathfrak{h}^\perp$, and the index of $g$ is the sum of the indices of $g|_{\mathfrak{h}}$ and $g|_{\mathfrak{h}^\perp}$ [14]. Hence, $\tilde{v}$ is necessarily time-like, and $g$ takes the form (a) with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$, where we put $e_4 = \tilde{v}/\sqrt{-g(\tilde{v}, \tilde{v})}$.

**Case (b).** We proceed like in Case (a), with the following slight differences: in $\mathfrak{h}$ we now fix a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with $e_3$ time-like, and the orthogonal projection $w$ of $v$ on $\mathfrak{h}$ is given by $w := \sum_{i=1}^3 g(v, e_i)e_i$, where $e_i = g(e_i, e_i)$. Then, $\mathfrak{g} = \mathfrak{h} \rtimes \tilde{\mathfrak{r}}$, where $\tilde{\mathfrak{r}} := \text{span}(\tilde{v} = v - w) = \mathfrak{h}^\perp$ (and so, $\tilde{v}$ is necessarily space-like), and $g$ takes the form (b) with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$, where $e_4 = \tilde{v}/\sqrt{g(\tilde{v}, \tilde{v})}$.

**Case (c).** Since $g$ is Lorentzian, a subspace of $\mathfrak{g}$ (and so, of $\mathfrak{h}$) on which $g$ vanishes has dimension at most one [14]. Thus, being $g|_{\mathfrak{h}}$ degenerate, its signature is necessarily $(2,0,1)$, since all the other possibilities would give a subspace of $\mathfrak{h}$ dimension $\geq 2$ on which $g$ vanishes, which cannot occur. Hence, $\mathfrak{h}$ admits an orthogonal basis $\{e_1, e_2, e_3\}$, with $e_1, e_2$ unit space-like vectors and $e_3$ a light-like vector.

If $\mathfrak{r} = \text{span}(v)$, we consider $\tilde{v} := v - \sum_{i=1}^2 g(v, e_i)e_i$ and obtain $\mathfrak{g} = \mathfrak{h} \rtimes \tilde{\mathfrak{r}}$, with $\tilde{\mathfrak{r}} := \text{span}(\tilde{v})$ and $\tilde{v}$ orthogonal to $e_1, e_2$. Moreover, because of the non-degeneracy of $g$, necessarily $g(\tilde{v}, e_3) \neq 0$.

Next, there exists a unique $\lambda_0 \in \mathbb{R}$, such that $\tilde{v} + \lambda_0 e_3$ is light-like: explicitly, $\lambda_0 = -g(\tilde{v}, \tilde{v})/2g(\tilde{v}, e_3)$. Putting $k = g(\tilde{v} + \lambda_0 e_3, e_3) = g(\tilde{v}, e_3) \neq 0$ and $e_4 = \frac{1}{k}(\tilde{v} + \lambda_0 e_3)$, we get that $e_4$ acts as a derivation on $\mathfrak{h}$, and $g$ takes the form (b) with respect to the basis $\{e_1, e_2, e_3, e_4\}$. \(\square\)

In the following subsections we shall classify four-dimensional cyclic Lorentzian Lie groups, treating separately the three cases occurring in the above Proposition 4.2.
4.1. **First case: \( h \) Riemannian.** Following [13], there exists an orthonormal basis \( \{e_1, e_2, e_3\} \) of \( h \), such that

\[
[e_1, e_2] = a_3 e_3, \quad [e_2, e_3] = a_1 e_1, \quad [e_3, e_1] = a_2 e_2,
\]

providing the cases listed in the following Table III, depending on the signs of \( a_1, a_2 \) and \( a_3 \).

| Lie group | \( a_1 \) | \( a_2 \) | \( a_3 \) |
|-----------|---------|---------|---------|
| \( SU(2) \) | +       | +       | +       |
| \( SL(2, \mathbb{R}) \) | +       | +       | -       |
| \( \tilde{E}(2) \) | +       | +       | 0       |
| \( E(1, 1) \) | +       | -       | 0       |
| \( H_3 \) | +       | 0       | 0       |
| \( \mathbb{R}^3 \) | 0       | 0       | 0       |

Table III: Simply connected Riemannian 3D Lie groups

Since \( e_4 \) acts as a derivation on \( h_3 \), we also have

\[
\begin{align*}
[e_1, e_4] &= c_1 e_1 + c_2 e_2 + c_3 e_3, \\
[e_2, e_4] &= p_1 e_1 + p_2 e_2 + p_3 e_3, \\
[e_3, e_4] &= q_1 e_1 + q_2 e_2 + q_3 e_3,
\end{align*}
\]

for some constants \( c_i, p_i, q_i \), which in addition must satisfy the Jacobi identity

\[
[[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j] = 0.
\]

Applying the cyclic condition (3.8) to the pseudo-orthonormal basis satisfying (4.1) and (4.2), we easily get conditions

\[
a_3 + a_1 + a_2 = 0, \quad p_1 = c_2 \quad q_1 = c_3, \quad q_2 = p_3.
\]

Requiring that the Jacobi identity (4.3) holds, and after some computations, we get the following possible solutions:

1. \( \{ a_2 = a_3 = 0 \} \). In this case, taking into account (4.4), we have that

\[
\begin{align*}
[e_1, e_4] &= c_1 e_1 + p_1 e_2 + q_1 e_3, \\
[e_2, e_4] &= p_1 e_1 + p_2 e_2 + q_2 e_3, \\
[e_3, e_4] &= q_1 e_1 + q_2 e_2 + q_3 e_3.
\end{align*}
\]

2. \( \{ a_3 = -a_2, c_1 = p_1 = q_1 = 0, q_3 = p_2 \} \). In this case, by (4.4) and the above Table III, we conclude that \( h_3 = \mathfrak{e}(1, 1) \), with the action of \( \mathbb{R} = \text{span}\{e_4\} \) on it defined as

\[
\begin{align*}
[e_1, e_4] &= 0, \quad [e_2, e_4] = p_2 e_2 + q_2 e_3, \quad [e_3, e_4] = q_2 e_2 + p_2 e_3.
\end{align*}
\]
Imposing the Jacobi identity, we only have the solution
\begin{align}
(3) \quad \{c_1 = p_2, a_3 = q_1 = q_2 = q_3 = 0\}. \text{ In this case, } h_3 &= c(1, 1), \text{ with the action of } R = \text{span}\{e_4\} \text{ on it defined as}
\end{align}
\begin{align}
[1] \quad &e_1, e_4 = p_2 e_1 + c_2 e_2, \quad [e_2, e_4] = c_2 e_1 + p_2 e_2, \quad [e_3, e_4] = 0.
\end{align}
\begin{align}
(4) \quad \{c_1 = q_3, a_2 = a_3 = p_1 = p_2 = q_2 = 0\}. \text{ This corresponds to } h_3 &= c(1, 1), \text{ with the action of } R = \text{span}\{e_4\} \text{ on it defined as}
\end{align}
\begin{align}
[1] \quad &e_1, e_4 = q_3 e_1 + q_1 e_3, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = q_1 e_1 + q_3 e_3.
\end{align}
\begin{align}
(5) \quad \{c_1 = p_1 = p_2 = q_1 = q_2 = q_3 = 0\}. \text{ In this case, by the above Table III, } h_3 &= sl(2) \text{ with the trivial action of } R = \text{span}\{e_4\} \text{ on it.}
\end{align}

It is clear that the above cases (2), (3) and (4) coincide, up to a renumeration of $e_1, e_2, e_3$. Thus, we proved the following result.

**Theorem 4.3.** Let $G = H \rtimes R$ be a connected and simply connected four-dimensional Lie group, equipped with a left-invariant Lorentzian metric $g$, such that $g|_H$ is Riemannian. If $g$ is cyclic, then the Riemannian Lie algebra $h$ of $H$ admits an orthonormal basis $\{e_1, e_2, e_3\}$, such that (4.1) holds with $a_1 + a_2 + a_3 = 0$, and one of the following cases occurs:

1) \( G = \mathbb{R}^3 \rtimes \mathbb{R} \) and the action of $\mathbb{R} = \text{span}\{e_4\}$ (time-like) on $h = \mathbb{R}^3$ is described by (4.5) and (4.2), for arbitrary real constants $c_1, p_1, p_2, q_1, q_2, q_3$.

2) \( G = E(1, 1) \rtimes \mathbb{R} \) and the action of $\mathbb{R} = \text{span}\{e_4\}$ (time-like) on $h = c(1, 1)$ is described by (4.6), for arbitrary real constants $p_2, q_2$.

3) \( G = SL(2, \mathbb{R}) \times \mathbb{R} \).

**4.2. Second case: $h$ Lorentzian.** In this case, $h$ is one of the unimodular Lorentzian Lie algebras \( g_1 - g_4 \) classified in Theorem 3.1. We treat these cases separately.

1) \( h = g_1 \). The brackets of $g = h \rtimes \mathfrak{r}$ are then completely described by (3.1) and (4.2), and the cyclic condition (3.8) gives
\begin{align}
\beta &= 0, \quad e_2 = p_1, \quad c_3 = -q_1, \quad p_3 = -q_2.
\end{align}

Imposing the Jacobi identity, we only have the solution
\begin{align}
p_1 &= 0, \quad p_2 = -q_3, \quad q_1 = 0, \quad q_2 = q_3,
\end{align}
so that taking into account Theorem 3.1 we have $g = h \rtimes \mathfrak{r}$ with $h = c(1, 1) = \text{span}\{e_1, e_2, e_3\}$, $\mathfrak{r} = \text{span}\{e_4\}$ and the action given by
\begin{align}
[1] \quad &e_1, e_4 = c_1 e_1, \quad [e_2, e_4] = -q_3 (e_2 + e_3), \quad [e_3, e_4] = q_3 (e_2 + e_3).
\end{align}

2) \( h = g_2 \). The brackets of $g = h \rtimes \mathfrak{r}$ are now described by (3.2) and (4.2). The cyclic condition (3.5) yields
\begin{align}
\alpha &= -2 \beta, \quad e_2 = p_1, \quad c_3 = -q_1, \quad p_3 = -q_2.
\end{align}

Finally, the Jacobi identity (4.3) admits the following two solutions:

1) \( \{\beta = 0, c_1 = p_1 = q_1 = q_2 = 0\} \). Then, $g = h \rtimes \mathfrak{r}$ with $h = c(1, 1) = \text{span}\{e_1, e_2, e_3\}$, $\mathfrak{r} = \text{span}\{e_4\}$ and the action defined as
\begin{align}
[1] \quad &e_1, e_4 = 0, \quad [e_2, e_4] = p_2 e_2, \quad [e_3, e_4] = q_3 e_3.
\end{align}
(2) \( \{ c_1 = p_1 = p_2 = q_1 = q_2 = q_3 = 0 \} \). So, \( \mathbb{R} = \text{span}\{e_4\} \) acts trivially. Taking into account Proposition 4.1, we have \( g = \mathfrak{h} \times \mathbb{R} \) with \( \mathfrak{h} = \text{span}\{e_1, e_2, e_3\} = \mathfrak{sl}(2) \).

3) \( \mathfrak{h} = \mathfrak{g}_3 \). Starting from (3.3) and (4.2), the cyclic condition (3.8) now gives
\[
\alpha + \beta + \gamma = 0, \quad c_2 = p_1, \quad c_3 = -q_1, \quad p_3 = -q_2.
\]

Imposing the Jacobi identity and taking into account Proposition 4.1, we have the following sets of solutions:

1) \( \{ \beta = \gamma = 0 \} \). This case correspond to \( g = \mathfrak{h} \times \mathbb{R} \) with \( \mathfrak{h} = \mathbb{R}^3 \) and
\[
[e_1, e_4] = c_1 e_1 + p_1 e_2 - q_1 e_3, \quad [e_2, e_4] = p_1 e_1 + p_2 e_2 - q_2 e_3, \quad [e_3, e_4] = q_1 e_1 + q_2 e_2 + q_3 e_3.
\]

2) \( \{ \beta = 0, c_1 = q_3, p_1 = p_2 = q_2 = 0 \} \). Since \( \alpha + \beta + \gamma = 0 \), we get \( \alpha + \gamma = \beta = 0 \). If \( \alpha = 0 \), we then have a special case of the previous one. For \( \alpha \neq 0 \), taking into account Table I, we have \( g = \mathfrak{h} \times \mathbb{R} \), where \( \mathfrak{h} = \text{span}\{e_1, e_2, e_3\} = \mathfrak{e}(2) \), \( \mathbb{R} = \text{span}\{e_4\} \) and the action is defined as
\[
[e_1, e_4] = q_3 e_1 - q_1 e_3, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = q_1 e_1 + q_3 e_3.
\]

3) \( \{ \beta = -\gamma, c_1 = p_1 = q_1 = 0, q_3 = p_2 \} \), which is isometric to the above case, interchanging the space-like vectors \( e_1 \) and \( e_2 \).

4) \( \{ \gamma = 0, c_1 = p_2, q_1 = q_2 = q_3 = 0 \} \). If \( \alpha = 0 \) we obtain a special case of case (1). When \( \alpha \neq 0 \), we get \( g = \mathfrak{h} \times \mathbb{R} \), where \( \mathfrak{h} = \text{span}\{e_1, e_2, e_3\} = \mathfrak{e}(1,1) \), \( \mathbb{R} = \text{span}\{e_4\} \) and the action is defined as
\[
[e_1, e_4] = c_1 e_1 + p_1 e_2, \quad [e_2, e_4] = c_2 e_1 + c_1 e_2, \quad [e_3, e_4] = 0.
\]

5) \( \{ c_1 = p_1 = p_2 = q_1 = q_2 = q_3 = 0 \} \). In this case, the action of \( e_4 \) on \( \mathfrak{h} \) is trivial. Hence, by Proposition 4.1 and Table I, we find that \( g = \mathfrak{h} \times \mathbb{R} \), where \( \mathfrak{h} \) is either \( \mathfrak{su}(2) \) or \( \mathfrak{sl}(2) \).

4): \( \mathfrak{h} = \mathfrak{g}_4 \). By (3.3) and (4.2), the cyclic condition holds if and only if
\[
\alpha = 2(\varepsilon - \beta), \quad c_2 = p_1, \quad c_3 = -q_1, \quad p_3 = -q_2.
\]

Then, imposing the Jacobi identity and taking into account Proposition 4.1, we have the following two non-isometric cases:

1) \( \{ \beta = \varepsilon, c_1 = 0, p_1 = \varepsilon q_1, q_2 = \frac{\varepsilon}{2} (p_2 - q_3) \} \). In this case, \( g = \mathfrak{h} \times \mathbb{R} \), where \( \mathfrak{h} = \mathfrak{n}_3 = \text{span}\{e_1, e_2, e_3\} \) is the Heisenberg Lie algebra, \( \mathbb{R} = \text{span}\{e_4\} \) and the action is defined as
\[
[e_1, e_4] = q_1 (e_2 - e_3), \quad [e_2, e_4] = q_1 e_1 + p_2 e_2 - q_2 e_3, \quad [e_3, e_4] = q_1 e_1 + q_2 e_2 + q_3 e_3,
\]

with \( q_2 = \frac{\varepsilon}{2} (p_2 - q_3) \).
(2) \( \{ c_1 = p_1 = p_2 = q_1 = q_2 = q_3 = 0 \} \), so that \( g = h \times \mathbb{R} \) trivially, and, taking into account Proposition 4.1, \( h = \mathfrak{sl}(2) \).

Collecting all the above cases, we obtain the following.

**Theorem 4.4.** Let \( G = H \rtimes \mathbb{R} \) be a connected and simply connected four-dimensional Lie group, equipped with a left-invariant Lorentzian metric \( g \), such that \( g|_H \) is Lorentzian. If \( g \) is cyclic, then the Lorentzian Lie algebra \( \mathfrak{h} \) of \( H \) admits a pseudo-orthonormal basis \( \{ e_1, e_2, e_3 \} \), with \( e_3 \) time-like, such that one of the following cases occurs:

**I:** \( G = E(1, 1) \times \mathbb{R} \) and one of the following holds:

(a) \( \mathfrak{e}(1, 1) = \text{span}\{ e_1, e_2, e_3 \} \) is of the form \( \mathfrak{g}_1 \) with \( \beta = 0 \), and the action of \( \mathbb{R} = \text{span}\{ e_4 \} \) on \( \mathfrak{e}(1, 1) \) is described by (4.8).

(b) \( \mathfrak{e}(1, 1) = \text{span}\{ e_1, e_2, e_3 \} \) is of the form \( \mathfrak{g}_2 \) with \( \alpha = \beta = 0 \), and the action of \( \mathbb{R} = \text{span}\{ e_4 \} \) on \( \mathfrak{e}(1, 1) \) is described by (4.9).

(c) \( \mathfrak{e}(1, 1) = \text{span}\{ e_1, e_2, e_3 \} \) is of the form \( \mathfrak{g}_3 \) with \( \gamma = 0 \), and the action of \( \mathbb{R} = \text{span}\{ e_4 \} \) on \( \mathfrak{e}(1, 1) \) is described by (4.12).

**II:** \( G = \tilde{S}L(2, \mathbb{R}) \times \mathbb{R} \), with \( \mathbb{R} = \text{span}\{ e_4 \} \) acting trivially on \( \mathfrak{sl}(2) = \text{span}\{ e_1, e_2, e_3 \} \), and one of the following holds:

(a) \( \mathfrak{sl}(2) \) is of the form \( \mathfrak{g}_2 \) with \( \alpha = -2\beta \neq 0 \).

(b) \( \mathfrak{sl}(2) \) is of the form \( \mathfrak{g}_3 \) with \( \alpha + \beta + \gamma = 0 \).

(c) \( \mathfrak{sl}(2) \) is of the form \( \mathfrak{g}_4 \) with \( \alpha = 2(\varepsilon - \beta) \neq 0 \).

**III:** \( G = \tilde{E}(2) \times \mathbb{R} \), where \( \mathfrak{e}(2) = \text{span}\{ e_1, e_2, e_3 \} \) is of the form \( \mathfrak{g}_3 \) with \( \alpha + \gamma = \beta = 0 \), and the action of \( \mathbb{R} = \text{span}\{ e_4 \} \) on \( \mathfrak{e}(2) \) is described by (4.11).

**IV:** \( G = \mathbb{R}^3 \times \mathbb{R} \), where \( \mathbb{R}^3 = \text{span}\{ e_1, e_2, e_3 \} \) and the action of \( \mathbb{R} = \text{span}\{ e_4 \} \) on \( \mathbb{R}^3 \) is described by (4.10).

**V:** \( G = SU(2) \times \mathbb{R} \), where \( \mathfrak{su}(2) = \text{span}\{ e_1, e_2, e_3 \} \) is of the form \( \mathfrak{g}_3 \) with \( \alpha + \beta + \gamma = 0 \), and the action of \( \mathbb{R} = \text{span}\{ e_4 \} \) on \( \mathfrak{su}(2) \) is trivial.

**VI:** \( G = H_3 \times \mathbb{R} \), where \( \mathfrak{n}_3 = \text{span}\{ e_1, e_2, e_3 \} \) is of the form \( \mathfrak{g}_4 \) with \( \alpha = \beta - \varepsilon = 0 \), and the action of \( \mathbb{R} = \text{span}\{ e_4 \} \) on \( \mathfrak{n}_3 \) is described by (4.13).

### 4.3. Third case: \( \mathfrak{h} \) degenerate.

We now assume that the restriction of the metric \( g \) on \( \mathfrak{h} \) is degenerate. It is enough to restrict to the case when the derived algebra is the full subalgebra \( \mathfrak{h} \), that is,

\[
g' = [g, g] = \mathfrak{h}.
\]

In fact, if \( \dim g < 3 \), then there are at least two linearly independent vectors acting as derivations in \( g \). Since \( g \) is Lorentzian, the subspace spanned by these two vectors cannot be completely null [14] and so, we can pick a derivation that is either space-like or time-like. Henceforth, we are in one of the non-degenerate situations already studied in the previous subsections.
We shall now investigate the different possibilities, compatible with condition $g' = \mathfrak{h}$, determined by the dimension of the derived algebra $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ of $\mathfrak{h}$.

\textbf{dim} $\mathfrak{h}' = 0$. In this case, $\mathfrak{h} = \mathbb{R}^3$ is abelian. As the only non-vanishing Lie brackets are given by (4.2) and $\mathfrak{h} = g'$ is abelian, the Jacobi identity holds trivially. Moreover, the metric $g$ is cyclic if and only if $c_2 = p_1, q_1 = q_2 = 0$. Therefore, the Lie algebra is completely described by

\begin{equation}
+c_1 e_1 + p_1 e_2 + c_3 e_3, [e_2, e_4] = p_1 e_1 + p_2 e_2 + p_3 e_3, [e_3, e_4] = q_3 e_3.
\end{equation}

\textbf{dim} $\mathfrak{h}' = 1$. Then, $\mathfrak{h} = \mathfrak{n}_3$ is the three-dimensional Heisenberg Lie algebra and so, $\mathfrak{h}' = \text{span}(X)$.

As it follows from case (c) in Proposition 4.1, $g|_\mathfrak{h}$ has signature $(2, 0, 1)$. Thus, we can write $X = V + \lambda e_3$, where $V$ is spacelike and $e_3 \perp V$ is null. We have the following two possibilities.

\textbf{(a):} $V \neq 0$.

We consider $e_1 = X/\|X\|$ (space-like) and complete the basis of $\mathfrak{h}$ with another space-like unit vector $e_2$ and the null vector $e_3$, so that

$$g|_{\mathfrak{n}_3} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{cases}
[e_1, e_2] = \alpha e_1, \\
[e_1, e_3] = \beta e_1, \\
[e_2, e_3] = \mu e_1.
\end{cases}$$

Imposing the cyclic condition, we get

\begin{equation}
\mu = 0, \quad c_2 = p_1, \quad q_1 = 0, \quad q_2 = 0.
\end{equation}

Next, we apply the Jacobi identity (4.3) and find the following four possible solutions:

- $\{c_3 = p_1 = q_3 = 0, p_2 \alpha = -p_3 \beta\}$. Taking into account (4.15), We have

\begin{equation}
[e_1, e_2] = \alpha e_1, \quad [e_1, e_3] = \beta e_1, \quad [e_2, e_3] = 0, \\
[e_1, e_4] = c_1 e_1, \quad [e_3, e_4] = 0, \quad [e_2, e_4] = p_2 e_2 + p_3 e_3, \quad p_2 \alpha + p_3 \beta = 0.
\end{equation}

- $\{\alpha = \beta = 0\}$. But since $\mu = 0$ by (4.15), this case would contradict \textbf{dim} $\mathfrak{h}' = 1$ and so, it does not occur.

- $\{c_3 = p_1 = p_2 = p_3 = q_3 = 0\}$. Then, by (4.15), we would conclude that \text{dim}$(\mathfrak{g}, \mathfrak{g}) < 3$, against our assumption.

- $\{\beta = c_3 = p_1 = p_2 = 0\}$, which, taking into account (4.15), contradicts again \text{dim}$(\mathfrak{g}, \mathfrak{g}) = 3$.

\textbf{(b):} $V = 0$.

We can then choose an orthogonal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{h}$, such that

$$g|_{\mathfrak{n}_3} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{cases}
[e_1, e_2] = \alpha e_3, \\
[e_1, e_3] = \beta e_3, \\
[e_2, e_3] = \mu e_3.
\end{cases}$$
Imposing the cyclic condition we get
\[ \mu = 0, \quad e_2 = p_1 + \alpha, \quad q_2 = \mu = 0, \quad q_1 = -\beta \]
and applying the Jacobi identity we have the following possible solutions:

- \{\beta = 0, c_1 = -p_2 + q_3\}. Then, we have

\[
\begin{align*}
[e_1, e_2] &= \alpha e_3, \\
[e_1, e_4] &= (q_3 - p_2)e_1 + (p_1 + \alpha)e_2 + c_3e_3, \\
[e_2, e_3] &= 0, \\
[e_3, e_4] &= q_3e_3.
\end{align*}
\]
(4.17)

- \{\alpha = \beta = 0\}. But since \( \mu = 0 \), this contradicts \( \dim h' = 1 \) and so, it cannot occur.

\( \dim h' = 2 \). Thus, either \( h = e(1,1) \) or \( h = e(2) \).

Taking into account the signature of \( g|_{\mathfrak{h}} \) as in the previous case, we now have \( h' = \text{span}\{X_1, X_2\} \), where \( X_i = V_i + \lambda_ie_3 \), with \( V_i \) space-like and \( e_3 \) null and orthogonal to \( V_1, V_2 \). We consider the following subcases.

(a): \( V_1 \) and \( V_2 \) are linearly independent.
Since \( V_1, V_2 \) are space-like, there exist orthonormal vectors \( e_1 \) and \( e_2 \), such that \( h' = \text{span}\{X_1, X_2\} = \text{span}\{e_1, e_2\} \). With respect to the basis \( \{e_1, e_2, e_3\} \) of \( h \), we then have

\[
g|_{\mathfrak{g}_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{cases} [e_1, e_2] = a_1e_1 + a_2e_2, \\
[e_1, e_3] = b_1e_1 + b_2e_2, \\
[e_2, e_3] = t_1e_1 + t_2e_2. \end{cases}
\]

Imposing the cyclic condition, we find
\[ b_2 = t_1, \quad c_2 = p_1, \quad q_1 = 0, \quad q_2 = 0. \]
However, when we apply the Jacobi identity, all the solutions we get turn out to be incompatible with either \( \dim h' = 3 \) or \( \dim h' = 2 \). For example, one of such solutions is given by

\[ \{b_1 = 0, c_1a_2^2 = p_2a_1^2, p_1a_2 = p_2a_1, p_3a_2 = -a_1c_3, t_1 = 0, t_2 = 0\}. \]
But then, \( [e_1, e_3] = [e_2, e_3] = 0 \), contradicting the fact that \( \dim h' = 2 \). So, this case does not occur.

(b): \( V_1 \) and \( V_2 \) are linearly dependent.
Then, we can choose \( \{V_1, e_3\} \) as a basis for \( h' \). We consider \( e_1 = V_1/\|V_1\| \), and a space-like vector \( e_2 \), orthogonal to both \( e_1 \) and \( e_3 \), so that we have

\[
g|_{\mathfrak{g}_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{cases} [e_1, e_2] = a_1e_1 + a_3e_3, \\
[e_1, e_3] = b_1e_1 + b_3e_3, \\
[e_2, e_3] = t_1e_1 + t_3e_3. \end{cases}
\]

Imposing the cyclic condition, we get
\[ t_1 = 0, \quad c_2 = a_3 + p_1, \quad q_1 = -b_3, \quad q_2 = -t_3. \]
Also in this case, the Jacobi identity does not provide any solutions compatible with \( \dim h' = 3 \) and \( \dim h' = 2 \). Therefore, this case cannot occur.
dim h' = 3.

As above, we consider that e3 ∈ h is orthogonal to h itself. Since h' = h, we have either h = sl(2) or h = su(2). In order to distinguish these two cases, we consider ad_{e_3} : h → h, which, since h' = h, is necessarily of rank 2. Besides 0, ad_{e_3} has either two real eigenvalues or two conjugate complex eigenvalues. In addition, if we write e_3 = [X_1, X_2], we have

$$\text{ad}_{e_3} = \text{ad}_{X_1} \circ \text{ad}_{X_2} - \text{ad}_{X_2} \circ \text{ad}_{X_1}$$

so that tr(ad_{e_3}) = 0. We thus have the following possible cases.

(a): Eigenvalues of ad_{e_3} are 0, λ ≠ 0 and −λ.

We choose e_1 and e_2 (unitary) eigenvectors, that is, [e_3, e_1] = λ e_1, [e_3, e_2] = −λ e_2. The Jacobi identity (rescaling e_3 if needed) gives [e_2, e_1] = e_3. With respect to \{e_1, e_2, e_3\}, the metric is given by

$$g|_h = \begin{pmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Imposing the cyclic condition, we then find

\[
\begin{align*}
2k\lambda &= 0, \\
q_1 + kq_2 &= 0, \\
kq_1 + q_2 &= 0, \\
1 + kc_1 - p_1 + c_2 - kp_2 &= 0,
\end{align*}
\]

which, since λ ≠ 0, easily reduces to k = q_1 = q_2 = 0, p_1 = 1 + c_2.

Imposing the Jacobi identity to g, we get c_1 = −p_2 + c_3 and λ = 0, which is a contradiction. Hence, this case cannot occur.

(b): Eigenvalues of ad_{e_3} are 0, iβ and −iβ, with β ≠ 0.

We choose e_1 and e_2 (unitary) Jordan vectors, that is, [e_3, e_1] = β e_2, [e_3, e_2] = −β e_1. The Jacobi identity (rescaling e_3 if needed) then gives [e_1, e_2] = β e_3, and the metric is described by

$$g|_h = \begin{pmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Imposing the cyclic condition for e_1, e_2 and e_3, we have

$$0 = g([e_1, e_2], e_3) + g([e_2, e_3], e_1) + g([e_3, e_1], e_2) = 2\beta$$

which is not admissible. Therefore, this case does not occur.

Collecting all the above cases, we obtain the following.

**Theorem 4.5.** Let $G = H \times \mathbb{R}$ be a connected, simply connected four-dimensional Lie group, equipped with a left-invariant Lorentzian metric $g$, such that $g|_H$ is degenerate. If $g$ is cyclic, then we can choose a basis \{e_1, e_2, e_3, e_4\} of the Lie algebra $\mathfrak{g} = h \times \mathbb{R}$, such $h = \text{span}(e_1, e_2, e_3)$, with respect to \{e_1\} the the metric is described as in case (c) of Proposition 4.2 and one of the following cases occurs:
I): $G = \mathbb{R}^3 \times \mathbb{R}$, with brackets as in (4.14).

II): $G = H_3 \times \mathbb{R}$ with brackets either as in (4.16) or as in (4.17).

5. **Final remarks**

5.1. **Homogeneous manifolds with homogeneous structures in $S_3$ and in $S_1 \oplus S_2$.** We consider the question whether a homogeneous manifold can admit homogeneous structures both in $S_3$ and in $S_1 \oplus S_2$.

If we require that the same homogeneous structure $S$ belongs to both $S_3$ and $S_1 \oplus S_2$, then it means that $S = 0$, that is, the manifold is symmetric, and conversely.

Observe that for a metric Lie group $G$, equipped with a left-invariant pseudo-Riemannian metric $g$, we are considering a specific homogeneous structure $\tilde{S}$, namely, the one giving to it the Lie group structure ($G$ acting transitively on itself by isometries). Thus, the fact that such a structure belongs to both $S_3$ and $S_1 \oplus S_2$ is equivalent to require that $(G, g)$ is a symmetric Lie group.

On the other hand, for example, it follows from Theorem 4.4 that the homogeneous structure $\tilde{S}$ of $SU(2) \times \mathbb{R}$ belongs to $S_1 \oplus S_2$, since the left-invariant metric $g$ is cyclic. At the same time, $SU(2)$ is a non-symmetric naturally reductive homogeneous Lorentzian manifold [6, Theorem 4.3]. Consequently, being the (non-symmetric) direct product of naturally reductive manifolds, four-dimensional Lorentzian Lie group $SU(2) \times \mathbb{R}$ also admits a (non-trivial) homogeneous structure $S \in S_3$.

5.2. **Three-dimensional cotorsionless Lorentzian manifolds.** We already recalled in Section 3 that all connected, simply connected homogeneous Lorentzian three-manifolds can be realized as Lorentzian Lie groups, with the only exception of $\mathbb{S}^2 \times \mathbb{R}$ with the product metric $g = g_{32} - dt^2$. It is obvious that as a product of symmetric spaces, $\mathbb{S}^2 \times \mathbb{R}$ is again symmetric and so, it is (trivially) a cotorsionless manifold. With regard to all homogeneous structures on $\mathbb{S}^2 \times \mathbb{R}$, it is possible to check by direct calculation that they are parametrized by one parameter, and the only tensor belonging to $S_1 + S_2$ is $S = 0$.

The next result then follows from the above observations about $\mathbb{S}^2 \times \mathbb{R}$ and the classification of three-dimensional cyclic Lorentzian Lie groups given in Theorem 3.2.

**Theorem 5.1.** A three-dimensional connected, simply connected cotorsionless homogeneous Lorentzian manifold is either isometric to $\mathbb{S}^2 \times \mathbb{R}$, or to one of the cyclic Lorentzian Lie groups classified in Theorem 3.2.

5.3. **Relating three- and four-dimensional cyclic Lie groups.** As proved in [11], a three-dimensional Riemannian Lie group $H$ is cyclic if and only if its Lie algebra is of the form (4.1) with $a_1 + a_2 + a_3$. Moreover, by direct calculation (see also the proof of Theorem 6.2 in [11]), we see that
if \((G = H \times \mathbb{R}, g)\) is a four-dimensional cyclic Riemannian Lie group, then \((H, gh)\) is again cyclic.

With regard to cyclic Lorentzian metrics, by Theorem 4.3 we see that if \((G = H \times \mathbb{R}, g)\) is a four-dimensional cyclic Lorentzian Lie group, with \(H\) Riemannian, then \((H, gh)\) is cyclic. Similarly, Theorems 3.2 and 4.4 show that if \((G = H \times \mathbb{R}, g)\) is a four-dimensional cyclic Lorentzian Lie group and \(H\) is Lorentzian, then \((H, gh)\) is cyclic.

Hence, when \(gh\) is either Riemannian or Lorentzian, left-invariant cyclic Lorentzian metrics on four-dimensional Lie groups can be interpreted as semi-direct product extensions of corresponding cyclic metrics on three-dimensional Lie algebras. But clearly, the examples listed in Theorem 4.5 do not show such a correspondence, because for them \(gh\) is degenerate.

So, we see once more that geometric behaviours occurring in Lorentzian settings are richer than their Riemannian analogues: four-dimensional Riemannian cyclic metrics are semi-direct product extensions of three-dimensional Riemannian cyclic metrics, while not all four-dimensional Lorentzian cyclic metrics arise from a corresponding construction.

5.4. **Compact homogeneous solvmanifolds from cyclic Lie groups.**

From the classification results obtained in Section 4, all four-dimensional simply connected Lorentzian cyclic Lie groups \(G\) are non compact. One could ask about the existence compact Lorentzian cotorsionless manifolds by considering quotients \(G/\Gamma\) by an appropriate lattice subgroup \(\Gamma \subset G\). This is precisely the way compact homogeneous solvmanifolds or nilmanifolds are constructed. However, the following results holds (for arbitrary dimension of \(G\)).

**Proposition 5.2.** Let \(M = G/\Gamma\) be a compact pseudo-Riemannian homogeneous solvmanifold (in particular, a nilmanifold) given by the quotient of the right action of a lattice \(\Gamma\) in a solvable (in particular, nilpotent) Lie group \(G\). We assume that \(G\) is equipped with a left-invariant metric \(g\) such that the projection \(\pi : G \to M\) is a local isometry. Then, the metric \(g\) is also right-invariant, the group is naturally reductive and the homogeneous structure associated to \(g\) belongs to of class \(S_3\).

Consequently, the only possible cyclic homogeneous structure for \(G\) is the trivial one and occurs when \(M\) is locally symmetric.

**Proof.** The bi-invariance follows from the classification of homogeneous compact Lorentzian spaces obtained in [17]. From here, the only cyclic homogeneous structure is the trivial one and hence \(G\) (and \(M\)) is locally symmetric.

\[\square\]

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