Scattering Matrices for Close Singular Selfadjoint Perturbations of Unbounded Selfadjoint Operators

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Abstract. In this paper, we consider an unbounded selfadjoint operator $A$ and its selfadjoint perturbations in the same Hilbert space $H$. As in [5], we call a selfadjoint operator $A_1$ the singular perturbation of $A$ if $A_1$ and $A$ have different domains $\mathcal{D}(A), \mathcal{D}(A_1)$ but $A = A_1$ on $\mathcal{D}(A) \cap \mathcal{D}(A_1)$. Assuming that $A$ has absolutely continuous spectrum and the difference of resolvents $R_z(A_1) - R_z(A)$ of $A_1$ and $A$ for non-real $z$ is a trace class operator we find the explicit expression for the scattering matrix for the pair $A, A_1$ through the constituent elements of the Krein formula for the resolvents of this pair. As an illustration, we find the scattering matrix for the standardly defined Laplace operator in $L_2(\mathbb{R}^3)$ and its singular perturbation in the form of an infinite sum of zero-range potentials.

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1. Introduction

Let $A, A_1$ be unbounded selfadjoint operators in Hilbert space $\mathcal{H}$, $\mathcal{D}, \mathcal{D}_1$ and $R(z), R_1(z)$, $\text{Im} z \neq 0$, are the domains and resolvents of $A, A_1$, respectively. Following [5] we call $A_1$ a singular perturbation of $A$ if

1) $\mathcal{D} \cap \mathcal{D}_1$ is dense in $\mathcal{H}$; 2) $A_1 = A$ on $\mathcal{D} \cap \mathcal{D}_1$.

In accordance with this definition $A, A_1$ are selfadjoint extensions of the same densely defined symmetric operator

$$B = A|_{\mathcal{D} \cap \mathcal{D}_1} = A_1|_{\mathcal{D} \cap \mathcal{D}_1}.$$
Therefore, the specific properties of the operator $A_1$ as a perturbation of the given operator $A$ can be investigated in the framework of the extension theory of symmetric operators. However, such a way can be long and involved, since in this case, it is not possible to directly operate with the difference of the singular perturbation $A_1$ of $A$ and $A$. Starting from the known properties of the operator $A$, the spectral analysis of the perturbation $A_1$ and the scattering theory for the pair $A, A_1$ can be developed bypassing the constructions of extension theory by operating directly and exclusively with the resolvent of $A$ and the constituent elements of Krein’s formula \[15\] for the difference of resolvents $R(z) - R_1(z)$. One can do this for a fairly wide class of unbounded selfadjoint operators of mathematical physics and their singular perturbations based on the following version of Krein’s formula \[1\], which combines all the characteristic properties of the resolvents of such perturbations.

**Theorem 1.1.** Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $A$ be an unbounded selfadjoint operator in $\mathcal{H}$ and $R(z), \Im z \neq 0$, is the resolvent of $A$, $G(z)$ is a bounded holomorphic in the open upper and lower half-planes operator function from $\mathcal{K}$ to $\mathcal{H}$ satisfying the conditions

- for any non-real $z, z_0$

$$G(z) = G(z_0) + (z - z_0)R(z)G(z_0), \quad (1.1)$$

- zero is not an eigenvalue of the operator $G(z)^*G(z)$ at least for one and hence for all non-real $z$ and the intersection of the domain $\mathcal{D}(A)$ of $A$ and the subspace $\mathcal{N} = \overline{G(z_0)\mathcal{K}} \subset \mathcal{H}$ consists only of the zero-vector.

Let $Q(z)$ be a holomorphic in the open upper and lower half-planes operator function in $\mathcal{K}$ such that

- $Q(z)^* = Q(\bar{z}), \ z \neq 0$;
- for any non-real $z, z_0$

$$Q(z) - Q(z_0) = (z - z_0)G(\bar{z}_0)^*G(z). \quad (1.2)$$

Then for any selfadjoint operator $L$ in $\mathcal{K}$ such that the operator $Q(z) + L, \ \Im z \neq 0$, is boundedly invertible the operator function

$$R_L(z) = R(z) - G(z) [Q(z) + L]^{-1} G(\bar{z})^* \quad (1.3)$$

is the resolvent of some singular selfadjoint perturbation $A_1$ of $A$.

A singular perturbation $A_1$ of a given selfadjoint operator $A$ is hereinafter referred to as close (to $A$) if the difference of the resolvents $R_1(z) - R(z), \ \Im z \neq 0$, is a trace class operator. In accordance with this definition, the absolutely continuous components of the close perturbation and the operator $A$ are similar and absolutely continuous components of $A_1$ can be described using the constructions of scattering theory \[13\].

The aim of this study is

- to derive an explicit expression relating the scattering matrix for an unbounded selfadjoint operator $A$ with absolutely continuous spectrum and its close singular perturbation $A_1$ with operator $L$ and the function
Q(z) in the Krein’s formula for the same pair A, A₁, slightly modifying the approach developed in [2];
• to illustrate the corresponding results by the example of singular self-adjoint perturbations of the Laplace operator

$$A = -\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$$

in $L_2(\mathbb{R}_3)$ defined on the Sobolev subspaces $H_2^2(\mathbb{R}_3)$.

In Section 2, for completeness, we give a sketch of the proof of Theorem 1.1 and indicate additional conditions under which the difference of resolvents $R_1(z) - R(z)$ of a singular perturbation $A_1$ of an unbounded self-adjoint operator $A$ and $A$ itself is a trace class operator, that is the conditions under which the singular perturbation $A_1$ and the unperturbed operator $A$ are close. Further, these conditions are refined for singular perturbations of the Laplace operator in in $L_2(\mathbb{R}_3)$.

In Section 3, which is a modified version of the paper [2], we calculate the scattering operator for a self-adjoint operator $A$ with an absolutely continuous spectrum and its close singular perturbation $A_1$ and find an explicit expression of the scattering matrix for the pair $A_1, A$ through the entries in the Krein formula (1.3).

Section 4 is devoted to the scattering theory for self-adjoint singular perturbations of the standardly defined Laplace operator $A$ in $L_2(\mathbb{R}_3)$ formally given as infinite sum of zero-range potentials. If such a perturbation $A_1$ and $A$ are close, then using the results of previous Section we find an explicit expression for the scattering matrix for this pair.

2. Resolvents of singular selfadjoint perturbations of Laplace operator

To make the paper self-contained we outline the proof of Theorem 1.1.

Recall the following well-known criterium (see, for example [1]): a holomorphic function $R(z)$ on the open upper and lower half-planes of the complex plane whose values are bounded linear operators in Hilbert space $\mathcal{H}$ is the resolvent of a selfadjoint operator $A$ in $\mathcal{H}$ if and only if

- $\ker R(z) = \{0\}$; \hspace{1cm} (2.1)
- $R(z)^* = R(\overline{z})$; \hspace{1cm} (2.2)
- for any non-real $z_1, z_2$ the Hilbert identity

$$R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2)$$ \hspace{1cm} (2.3)

holds.
So to verify the validity of Theorem 1.1, it is only necessary to check up that its assumptions about $R_L(z)$ guarantee the fulfillment of the conditions (2.1) - (2.3). But relation (2.2) and identity (2.3) for $R_L(z)$ directly follow from the fact that $R(z)$ is the resolvent of the self-adjoint operator and the properties that the operator functions $G(z)$ and $Q(z)$ possesses according to the assumptions of Theorem 1.1.

To see that (2.1) is true for $R_L(z)$ suppose that there is a vector $h \in \mathcal{H}$ such that $R_L(z)h = 0$ for some non-real $z$. By (1.3) this means that

$$R(z)h = G(z)[Q(z) + L]^{-1}G(\bar{z})^*h. \quad (2.4)$$

But by assumption, $R(z)\mathcal{H} \cap G(z)\mathcal{K} = \{0\}$ for non-real $z$. Hence $R(z)h = 0$. Given that $R(z)$ is the resolvent of a self-adjoint operator, we conclude from this that $h = 0$. Thus, the statement of Theorem 1.1 is true.

Turning to the description of singular perturbations of the Laplace operator $A = -\Delta$ in $L_2(\mathbb{R}_3)$ remind that the resolvent $R(z)$ of is the integral operator

$$\left(R(z)f\right)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}|x-x'|}}{|x-x'|} f(x') \, dx', \quad \text{Im} \sqrt{z} > 0, \quad x = (x_1, x_2, x_3), f(\cdot) \in L_2(\mathbb{R}_3). \quad (2.5)$$

In the simplest case, when the support of the singular perturbation of the Laplace operator is a finite set of points $x_1, \ldots, x_N$ of $\mathbb{R}_3$ the $N$-dimensional space $\mathbb{C}_N$ may be taken as an auxiliary Hilbert space $\mathcal{K}$ in Krein’s formula (1.3) for the perturbed resolvent $R_L(z)$. In this case, the linear mappings $G(z)$ of $\mathbb{C}_N$ into $L_2(\mathbb{R}_3)$, which transforms vectors of the canonical orthonormal basis in $\mathbb{C}_N$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

respectively, into the functions

$$(G(z)e_n)(x) = g_n(z; x) = R(z)\delta(\cdot - x_n)(x) = \frac{1}{4\pi} \frac{e^{i\sqrt{z}|x-x_n|}}{|x-x_n|}, \quad 1 \leq n \leq N, \quad (2.6)$$

can be substituted as $G(z)$ into (1.3), while, the matrix function

$$Q(z) = \left(q_{mn}(z)\right)_{m,n=1}^N = \begin{cases} q_{mn}(z) = g_n(z; x_m - x_n), & m \neq n, \\ q_{mm}(z) = \frac{i\sqrt{z}}{4\pi} \end{cases} \quad (2.7)$$

may acts as $Q$-function in (1.3). As a result, for any invertible Hermitian matrix $L = \left(w_{mn}\right)_{m,n=1}^N$ the arising operator function $R_L(z)$ satisfies all the conditions of Theorem 1.1 and hence appears to be the resolvent of a singular selfadjoint perturbation of the Laplace operator $A_L$.

Setting $\mathcal{N} = G(z_0)\mathbb{C}_N$, $\text{Im} z_0 \neq 0$, one can easily deduce from the Krein formula for $R_L(z)$ that in this case $A_L$ is nothing else than the Laplace
differential operator $-\Delta$ with the domain $[1]$

$$\mathcal{D}_L := \{ f : f = f_0 + g, \; f_0 \in H^2_2(\mathbb{R}_3), g \in \mathcal{N}, \lim_{\rho_n \to 0} \left[ \frac{\partial}{\partial \rho_n} (\rho_m f(x)) \right] + \sum_{n=1}^{N} 4\pi w_{mn} \lim_{\rho_n \to 0} [\rho_n f(x)] = 0, \rho_n = |x - x_n|, \; 1 \leq n \leq N \}. \tag{2.8}$$

**Remark 2.1.** The boundary conditions (2.8) correspond to singular perturbation $\hat{V}_L$, which is formally defined as an operator acting on any continuous function $f(\cdot) \in L^2(\mathbb{R}_3)$ by the formula

$$\left( \hat{V}_L f \right)(x) = \sum_{m,n=1}^{N} v_{mn} \delta(x - x_m) f(x_n), \left( v_{mn} \right)_{m,n=1}^{N} = L^{-1}. \tag{2.9}$$

If the matrix $L$ is diagonal, then by (2.9) we are dealing with a perturbation in the form of a finite sum of zero-range potentials [6, 8] and the corresponding boundary conditions (2.8) legalize the formal expression

$$A_L = -\Delta + \sum_{n=1}^{N} w_n^{-1} \delta(x - x_n).$$

In what follows, it is assumed that the elements of canonical basis $\{e_n = (\delta_{nm})_{m \in \mathbb{Z}}\}$ of the Hilbert space $l^2$ belong to the domains of all mentioned operators in this space, and no distinction is made between those operators and the matrices that represent them in the canonical basis of $l^2$.

For perturbations of the Laplace operator $A$ in the form of an infinite sum of zero-range potentials located at points $\{x_n\}_{m,n \in \mathbb{Z}}$ the set of boundary conditions of the form (2.11) also generates a singular self-adjoint perturbation $A_L$ of $A$ if

$$\inf_{m,n \in \mathbb{Z}} |x_m - x_n| = d > 0. \tag{2.10}$$

**Theorem 2.2 ([11]).** Let $Q(z)$ be the operator function in $l^2$ generated by the infinite matrix

$$Q(z) = (q_{mn}(z))_{m,n \in \mathbb{Z}}, \; q_{mn}(z) = \begin{cases} q_{mn}(z) = g(z; x_m - x_n), & m \neq n, \\ q_{mm}(z) = \frac{i\sqrt{z}}{4\pi} & \end{cases} \tag{2.11}$$

and $L$ be a selfadjoint operator in $l^2$ defined by an infinite Hermitian matrix $(w_{mn})_{m,n \in \mathbb{Z}}$. If the condition (2.11) holds, then the operator function

$$R_L(z) = R(z) - \sum_{m,n \in \mathbb{Z}} \left( [Q(z) + 4\pi L]^{-1} \right)_{mn} (., g_n(z; .)) g_m(z; .), \tag{2.12}$$

is the resolvent of the selfadjoint operator $A_L$ in $L^2(\mathbb{R}_3)$, which is the Laplace operator with the domain

$$\mathcal{D}_L := \{ f : f = f_0 + g, \; f_0 \in H^2_2(\mathbb{R}_3), \; g \in \mathcal{N}, \}$$
\[
\lim_{\rho_n \to 0} \left[ \frac{\partial}{\partial \rho_m} (\rho_m f(x)) \right] + \sum_{n \in \mathbb{Z}} w_{mn} \lim_{\rho_n \to 0} [\rho_n f(x)] = 0,
\]
\[
\rho_n = |x - x_n|, \quad n \in \mathbb{Z}.
\]

The above results concerning perturbations of the Laplace operator in the form of a finite or infinite sum of zero-range potentials do not contain anything new compared to those that were previously collected in the books [4, 5]. However, they can be obtained without the use of concepts and labor-consuming constructions of the theory of selfadjoint extensions of symmetric operators (see [1]).

Of course, the sparseness (2.10) of the set \( \{x_n\}_{n \in \mathbb{Z}} \) is not necessary for the expression (2.12) in Theorem 2.2 to be the resolvent of a self-adjoint perturbation of the Laplace operator in the form of an infinite sum of zero-range potentials. If (2.10) is not satisfied, then the operator functions \( R_L(z) \), formally defined as in Theorem 2.2, can nevertheless be resolvents of close singular selfadjoint perturbations \( A_L \) of \( A \) for account of special choice of the parameter \( L \).

**Theorem 2.3.** Let \( \{x_n\}_{n \in \mathbb{Z}} \) be a sequence of different points of \( \mathbb{R}_3 \) such that each compact domain of \( \mathbb{R}_3 \) contains a finite number of accumulation points of this set. Set

\[
\delta_0 = |x_0|; \quad \delta_n = \min_{-n \leq j \neq k \leq n} |x_j - x_k|, \quad n = 1, 2, ...
\] (2.13)

Let \( L \) be a selfadjoint operator in \( L_2 \) such that the point \( z = 0 \) is in the resolvent set of \( L \) and the matrix \((b_{nm})_{m,n \in \mathbb{Z}}\) of operator \( |L|^{-\frac{1}{2}} \) in the canonical basis \( \{e_n = (\delta_{nm})_{m \in \mathbb{Z}}\}_{n \in \mathbb{Z}} \) of \( L_2 \) satisfies the conditions:

\[
\sum_{nm \in \mathbb{Z}} |b_{nm}| < \infty, \quad \sum_{nm \in \mathbb{Z}} |b_{nm}| \frac{1}{\delta_m} < \infty.
\] (2.14)

Then the operator function

\[
R_L(z) = R(z) - \sum_{m,n \in \mathbb{Z}} \left( |Q(z) + 4\pi L|^{-1} \right)_{mn} (\cdot, g_n(\bar{z}; \cdot)) g_m(z; \cdot)
\]

\[
= R(z) - \bar{G}(z) \left( |Q(z) + 4\pi J_L|^{-1} \right) \bar{G}(z)^*,
\]

\[
g_n(z; x) = R(z) \delta(\cdot - x_n)(x),
\]

\[
\bar{G}(z) h = \sum_n \left( |L|^{-\frac{1}{2}} h, e_n \right) g_n(z; \cdot), \quad h \in L_2,
\]

\[
\bar{Q}(z) = |L|^{-\frac{1}{2}} Q(z)|L|^{-\frac{1}{2}}, \quad J_L = L \cdot |L|^{-1},
\] (2.15)

is the resolvent of the close singular perturbation \( A_L \) of the Laplace operator \( A \) with the domain

\[
\mathcal{D}_L := \{ f : f = f_0 + g, \quad f_0 \in H_2^2(\mathbb{R}_3), \quad g \in \mathcal{N} \},
\]

\[
\lim_{\rho_m \to 0} \left[ \frac{\partial}{\partial \rho_m} (\rho_m f(x)) \right] + 4\pi \sum_{n \in \mathbb{Z}} (Le_n, e_m) \lim_{\rho_n \to 0} [\rho_n f(x)] = 0,
\] (2.16)

\[
\rho_n = |x - x_n|, \quad n \in \mathbb{Z}.
\]
Proof. First of all, note that the mapping \( \tilde{G}(z) \) from \( L_2(\mathbb{R}_3) \) belongs to the Hilbert-Schmidt class. Indeed, for the canonical basis \( \{ \mathbf{e}_n \}_{n \in \mathbb{Z}} \) in \( L_2 \) by virtue of (2.14) we have
\[
\sum_{n \in \mathbb{Z}} \left| \tilde{G}(z) \mathbf{e}_n \right|^2 = \sum_{n \in \mathbb{Z}} \left| L^{-\frac{1}{2}} G(z)^* G(z) L^{-\frac{1}{2}} \mathbf{e}_n, \mathbf{e}_n \right| = \sum_{n,m,m' \in \mathbb{Z}} b_{nm'} \cdot \frac{1}{8\pi} e^{-|m' - m| |x_m - x_{m'}|} \sin \frac{\sin \text{Re} \sqrt{2 |x_m - x_{m'}|}}{\text{Re} \sqrt{2 |x_m - x_{m'}|}} b_{mn}' \quad (2.17)
\]
\[
\leq \frac{1}{8\pi} \sum_{n,m,m' \in \mathbb{Z}} |b_{nm}| \cdot |b_{m'n}| \leq \frac{1}{8\pi} \left( \sum_{n,m \in \mathbb{Z}} |b_{mn}| \right) < \infty.
\]

Turning to the operator function \( \tilde{Q}(z) \), we represent the corresponding matrix function as the sum \( D(z) + M(z) \) of matrices
\[
D(z) = \frac{i \sqrt{z}}{4\pi} \left( \sum_{j \in \mathbb{Z}} b_{mj} b_{jn} \right)_{m,n \in \mathbb{Z}} \quad (2.18)
\]
and
\[
M(z) = \left( u_{mn}(z) = \sum_{n',m' \in \mathbb{Z}} b_{mn'} g(z; x_{n'} - x_{m'}) b_{m'n}, \; m' \neq n' \right)_{m,n \in \mathbb{Z}} \quad (2.19)
\]

Note that, according to the first assumption in (2.14), \( L^{-\frac{1}{2}} \) is a Hilbert-Schmidt operator, since
\[
\sum_{n,j \in \mathbb{Z}} b_{nj} b_{jn} = \sum_{n,j \in \mathbb{Z}} |b_{nj}|^2 \leq \left( \sum_{n,j \in \mathbb{Z}} |b_{nj}| \right)^2 < \infty.
\]

As follows, \( D(z) = \frac{i \sqrt{z}}{4\pi} |L|^{-\frac{1}{2}} \) is a trace class operator.

Taking into account, further, that
\[
\left| \sum_{|m'| \leq |n'|} b_{m'm} \right| \leq \frac{1}{4\pi \delta_{m'}} \sum_{|m'| \leq |n'|} |b_{m'm}| + \frac{1}{4\pi \delta_{m'}} \sum_{|m'| > |n'|} \left| b_{m'm} \right| \leq \frac{1}{\delta_{m'}} C^{(0)}(m) + C^{(1)}(m),
\]
\[
C^{(0)}(m) = \frac{1}{4\pi} \sum_{j \in \mathbb{Z}} |b_{mj}| = \frac{1}{4\pi} \sum_{j \in \mathbb{Z}} |b_{mj}|, \quad C^{(1)}(m) = \frac{1}{4\pi} \sum_{j \in \mathbb{Z}} |b_{mj}| \frac{1}{\sigma_j} = \frac{1}{4\pi} \sum_{j \in \mathbb{Z}} |b_{mj}| \frac{1}{\sigma_j},
\]
and that by (2.14)
\[
\sum_{m \in \mathbb{Z}} C^{(0)}(m) < \infty, \quad \sum_{m \in \mathbb{Z}} C^{(1)}(m) < \infty, \quad (2.20)
\]
we conclude that
\[
|u_{mn}(z)| \leq 4\pi \left[ C^{(1)}(n)C^{(0)}(m) + C^{(1)}(m)C^{(0)}(n) \right] \quad (2.21)
\]

\[
|u_{nm}(z)| \leq 4\pi \left[ C^{(1)}(n)C^{(0)}(m) + C^{(1)}(m)C^{(0)}(n) \right]. \quad (2.22)
\]
Therefore
\[
\sum_{nm \in \mathbb{Z}} |u_{nm}(z)|^2 \leq \left( \sum_{nm \in \mathbb{Z}} |u_{nm}(z)| \right)^2 < \infty. \tag{2.23}
\]
We see that \(M(z)\) is a Hilbert-Schmidt operator and hence the sum \(\tilde{Q}(z) = D(z) + M(z)\) is at least a Hilbert-Schmidt operator.

Remember now that \(\tilde{Q}(z) + J_L\) is an operator function of the Nevanlinna class and that for any \(h \in L_2\) and any non-real \(z\) we have
\[
\frac{1}{\text{Im} z} \cdot \text{Im} \left[ (\tilde{Q}(z)h, h) + (J_Lh, h) \right] = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \sum_{n \in \mathbb{Z}} \left( |L| \frac{1}{2} h, e_n \right) e^{-k \cdot x_n} dx \geq 0 \tag{2.24}
\]

The function
\[
\hat{h}(k) = \sum_{n \in \mathbb{Z}} \left( |L| \frac{1}{2} h, e_n \right) e^{-k \cdot x_n}, \; h \in L_2, \tag{2.25}
\]
that appears in this case in (2.24) is bounded and continuous, since
\[
\sum_{n \in \mathbb{Z}} \left( |L| \frac{1}{2} h, e_n \right) \leq \sum_{n \in \mathbb{Z}} \left( |L| \frac{1}{2} e_n \right) \cdot \|h\| \leq \left( \sum_{m,n \in \mathbb{Z}} |b_{mn}|^2 \right)^{\frac{1}{2}} \cdot \|h\| < \infty.
\]
Obviously, the equality sign in (2.24) is possible if and only if \(\hat{h}(k) \equiv 0\). But for an expanding family of domains
\[
\Omega_K = \{k : -K < k_1, k_2, k_3 < K\} \subset \mathbb{R}_3
\]
and any \(n \in \mathbb{Z}\) in this case it turns out that
\[
\lim_{K \to \infty} \frac{1}{8K^3} \int_{\Omega_K} e^{i k \cdot x_n} \hat{h}(k) dk = \left( |L| \frac{1}{2} h, e_n \right) = 0.
\]
This means that \(|L| \frac{1}{2} h = 0\) and taking into account our assumptions, it turns out that \(h = 0\). Therefore "0" cannot be an eigenvalue of the operator \(\tilde{Q}(z) + J_L\), \(\text{Im} z \neq 0\) and, evidently, the same is true for the operator \((J_L \cdot \tilde{Q}(z) + I)\). In other words, "-1" is not an eigenvalue of the operator \(J_L \cdot \tilde{Q}(z)\). But \(J_L \cdot \tilde{Q}(z)\) is a compact operator. As follows, \(J_L \cdot \tilde{Q}(z) + I\) boundedly invertible and so is the operator \(\tilde{Q}(z) + J_L\),
\[
\left[ \tilde{Q}(z) + J_L \right]^{-1} = J_L \cdot \left[ J_L \cdot \tilde{Q}(z) + I \right]^{-1}.
\]
We proved that for non-real \(z\) the values of operator function \(\left[ \tilde{Q}(z) + J_L \right]^{-1}\) are bounded operators.

Suppose further that for some non-real \(z\) there is a vector \(f \in L_2(\mathbb{R}_3)\) from the linear set \(\tilde{G}(z)l_2\) that belongs to the domain \(D(A)\) of the Laplace operator \(A\). This vector as any vector from \(D(A)\) can be represented in the
form \( f = R(z)w \) with some \( w \in L_2(\mathbb{R}_3) \) while by our assumption there is a vector \( h \in l_2 \) such that

\[
R(z)w - \tilde{G}(z)h = 0. \tag{2.26}
\]

Now recall that for each \( w \in L_2(\mathbb{R}_3) \) and any infinitesimal \( \varepsilon > 0 \) it is possible to find an infinitely smooth compact function \( \phi(x) \) which is also equal to zero at some \( \eta \)-neighborhood of isolated points and all the accumulation points of the set \( \{x_n\} \) on the support of \( \phi(x) \) to satisfy the condition

\[
\left| (w, \phi)_{L_2(\mathbb{R}_3)} \right| \geq (1 - \varepsilon) \|w\|_{L_2(\mathbb{R}_3)}^2. \tag{2.27}
\]

Taking into account further that for \( \phi(r) \), as well as for any smooth compact function,

\[
\phi(x) = \frac{1}{4\pi} \int_{\mathbb{R}_3} e^{i\sqrt{|x-x'|}} \frac{[-\Delta \phi(x') - z\phi(x')]}{|x-x'|} dx', \tag{2.28}
\]

we notice that

\[
(G(z)w, [-\Delta \phi - \bar{z}\phi])_{L_2(\mathbb{R}_3)} = (w, \phi)_{L_2(\mathbb{R}_3)},
\]

\[
\left( \tilde{G}(z)h, [-\Delta \phi - \bar{z}\phi] \right)_{L_2(\mathbb{R}_3)} = 0, \quad h \in l_2.
\]

Hence by virtue of (2.27) we conclude that

\[
\|R(z)w - \tilde{G}(z)h\|_{L_2(\mathbb{R}_3)} \cdot \|\Delta \phi - \bar{z}\phi\|_{L_2(\mathbb{R}_3)} \geq \left| \left( R(z)w - \tilde{G}(z)h \right), \left[ -\Delta \phi - \bar{z}\phi \right]_{L_2(\mathbb{R}_3)} \right| = \left| (w, \phi)_{L_2(\mathbb{R}_3)} \right| \geq (1 - \varepsilon) \|w\|_{L_2(\mathbb{R}_3)}^2. \tag{2.29}
\]

But in view of (2.26), the last inequality in (2.29) must necessarily be violated unless \( w = 0 \). Therefore

\[
\tilde{G}(z)l_2 \cap \mathfrak{D}(A) = \{0\}. \tag{2.30}
\]

From (2.30) and the established properties of the operator function \( R_L(z) \) it follows, in particular, that \( \ker R_L(z) = \{0\}, \ \text{Im} z \neq 0 \). An obvious consequence of (2.30) and the established properties of the operator function \( R_L(z) \) is the fulfillment of condition \( \ker R_L(z) = \{0\}, \ \text{Im} z \neq 0 \). Indeed by virtue of (2.30), the equality

\[
R(z)w - \tilde{G}(z) \left( [\tilde{Q}(z) + 4\pi J_L]^{-1} \right) \tilde{G}(\bar{z})^*w = 0, \ \text{Im} z \neq 0,
\]

for some \( w \in L_2(\mathbb{R}_3) \) implies \( w = 0 \).

The verification of the remaining conditions of Theorem 1.1 for \( R_L(z) \) is not difficult and is left to the reader.
3. Scattering matrices

From now on we will assume that the spectrum of selfadjoint operator $A$ is absolutely continuous and as long as no mention of another that the operator functions $G(z)$ and $Q(z)$ and the operator $L$ in the expression (1.3) are such that all the conditions of Theorem 1.1 hold and that the difference of the resolvent of selfadjoint perturbation $A_L$ of $A$ and the resolvent of $A$ is a trace class operator. Hence for the pair of operators $A, A_L$, the following assertions of abstract scattering theory are true (for proofs and details, see [13], [18]).

The wave operators $W_\pm (A_L, A)$ defined as strong limits

$$ W_\pm (A_L, A) = s - \lim_{t \to \pm \infty} e^{iA_L t} e^{-iAt} 
$$

exist and are isometric mappings of $\mathcal{H}$ onto the absolutely continuous subspace of $A_L$. The wave operators $W_\pm (A_L, A)$ are intertwining for the spectral functions $E_\lambda, E^{(L)}_\lambda$, $-\infty < \lambda < \infty$, of the operators $A, A_L$ in the sense that

$$ W_\pm (A_L, A) E_\lambda = E^{(L)}_\lambda W_\pm (A_L, A). $$

The scattering operator, which is defined as the product of wave operators

$$ \mathcal{S} (A_L, A) = W_+ (A_L, A)^* W_- (A_L, A) 
$$

is an isometric operator in $\mathcal{H}$ and

$$ E_\lambda \mathcal{S} (A_L, A) = \mathcal{S} (A_L, A) E_\lambda, \quad -\infty < \lambda < \infty. 
$$

Therefore for the spectral representation of $A$ in $\mathcal{H}$ as the multiplication operator by $\lambda$ in the direct integral of Hilbert spaces $\mathfrak{h}(\lambda)$,

$$ \mathcal{H} \Rightarrow \int_{-\infty}^{\infty} \oplus \mathfrak{h}(\lambda) d\lambda, $$

the scattering operator $\mathcal{S} (A_L, A)$ acts as the multiplication operator by a contractive operator function $S(A_L, A)(\lambda)$, which will be below referred to as the scattering matrix.

These rather general assertions are specified below for close singular perturbations.

Let $\sigma(A)$ denote the spectrum of $A$ and $\Delta$ is some interval that $\subseteq \sigma(A)$. We will assume that the part of $A$ on $E(\Delta) \mathcal{H}$ has the Lebesgue spectrum of multiplicity $n \leq \infty$. Then there exists an isometric operator $\mathfrak{F}$, which maps $E(\Delta) \mathcal{H}$ onto the space $L^2(\Delta; \mathcal{N})$ of vector function on $\Delta$ with values in the auxiliary Hilbert space $\mathcal{N}$, $\dim \mathcal{N} = n$, and such that $\mathfrak{F} A |_{E(\Delta) \mathcal{H}} \mathfrak{F}^{-1}$ is the multiplication operator by independent variable in $L^2(\Delta; \mathcal{N})$. Then using the notation

$$ \mathfrak{F} (E(\Delta) f)(\lambda) = f(\lambda), \quad f \in \mathcal{H}, \ f(\cdot) \in L^2(\Delta; \mathcal{N}), $$

for any $g$ from the domain of $A$ we can write

$$ \mathfrak{F} (E(\Delta) A f)(\lambda) = \lambda \cdot f(\lambda) $$
and by (3.3) for any \( f, g \in E(\Delta)H \) we have
\[
(f, g)_H = \int_\Delta (f(\lambda), g(\lambda))_{\mathcal{N}'} d\lambda,
\]
\[
(\mathcal{G}(A_L, A)f, g)_H = \int_\Delta (S(A_L, A)(\lambda)f(\lambda), g(\lambda))_{\mathcal{N}'} d\lambda.
\]

**Theorem 3.1.** Let \( A \) be a selfadjoint operator in Hilbert space \( H \) with absolutely continuous spectrum, \( A_L \) be its close selfadjoint perturbation and the resolvent \( R_L(z) \) of \( A_L \) admits the representation (1.3) with selfadjoint operator \( L \) in Hilbert space \( K \), functions \( G(z) \) and \( Q(z) \) with values being bounded operators from \( K \) to \( H \) and in \( K \), respectively satisfy all the conditions of Theorem 1.1.

Suppose additionally that

• for fixed \( \gamma > 0 \), some linearly independent system \( \{g_k\}_{k \in \mathbb{Z}} \subset H \) and orthonormal system \( \{h_j\}_{j \in \mathbb{Z}} \subset K \) and a set of numbers \( \{b_{jk}\}_{jk \in \mathbb{Z}} \) such that
\[
\sum_{j,k \in \mathbb{Z}} |b_{jk}| < \infty
\]
the operator \( G(-i\gamma) \) admits the representation
\[
G(-i\gamma) = \sum_{j,k \in \mathbb{Z}} b_{jk} \langle \cdot, h_j \rangle_K g_k;
\] (3.4)

• on some interval \( \Delta \subset \sigma(A) \) the operator \( A \) has the Lebesgue spectrum of multiplicity \( n \leq \infty \) and for the operator function
\[
\Gamma(z) = \left[ |L|^{-\frac{1}{2}} Q(z)|L|^{-\frac{1}{2}} + J_L \right]^{-1}, \quad J_L = L \cdot |L|^{-1}, \quad \text{Im} z \neq 0,
\]
almost everywhere on \( \Delta \) there is a weak limit
\[
\Gamma(\lambda + i0) = \lim_{\epsilon \downarrow 0} \Gamma(\lambda + i\epsilon), \quad \lambda \in \Delta,
\]
such that
\[
\text{esssup}_{\lambda \in \Delta} \|\Gamma(\lambda + i0)\| < \infty.
\] (3.5)

Then the image \( \mathfrak{S} \mathcal{S}(A_L, A)\mathfrak{S}^{-1} \) of scattering operator acts in \( L^2(\Delta; N) \) as the multiplication by operator function
\[
S(A_L, A)(\lambda) = I - 4\pi i \sum_{j,k \in \mathbb{Z}} (\Gamma(\lambda + i0)h_j, h_k)_{\mathcal{K}} \langle \cdot, \mathfrak{g}_j(\lambda) \rangle_{\mathcal{N}'} \mathfrak{g}_k(\lambda),
\] (3.6)

\[
\mathfrak{g}_j(\lambda) = \sqrt{2\pi}(\lambda + i\gamma)^{-1} \sum_{l \in \mathbb{Z}} b_{jl} g_l(\lambda).
\]

**Proof.** Notice, that the existence of strong limits in (3.1) ensures the validity of relations
\[
\mathfrak{M}_\pm (A_1, A) = s - \lim_{\epsilon \downarrow 0} \int_0^\infty e^{-\epsilon t} e^{\pm iA_L t} e^{\mp iAt} dt
\]
\[
= s - \lim_{\epsilon \downarrow 0} \int_0^\infty e^{-\epsilon t} e^{\pm iA_L t} \int_{-\infty}^{\infty} e^{\mp i\lambda t} dE_\lambda
\]
\[
= s - \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} R_L(\lambda \pm i\epsilon) dE_\lambda.
\] (3.7)
By (3.7) the quadratic form of \( \mathcal{S}(A_L, A) \) for any \( f_1, f_2 \in \mathcal{H} \) can be written as follows
\[
(\mathcal{S}(A_L, A)f_1, f_2) = (\mathfrak{M}_-(A_L, A)f_1, \mathfrak{M}_+(A_L, A)f_2) = \lim_{\epsilon, \eta \downarrow 0} -\eta \epsilon \int_{-\infty}^{\infty} \left( R_L(\lambda + i\epsilon)dE_\lambda f_1, R_L(\mu + i\eta)dE_\mu f_2 \right) = \lim_{\epsilon, \eta \downarrow 0} -\eta \epsilon \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu + i(\epsilon - \eta)} \left( [R_L(\lambda + i\epsilon) - R_L(\mu + i\eta)]dE_\lambda f_1, dE_\mu f_2 \right).
\]
(3.8)

To proceed to the limits in (3.8) first we note that
\[
\mp i\epsilon R(\lambda \pm i\epsilon)dE_\lambda f = f, \quad \epsilon > 0, \ f \in \mathcal{H},
\]
\[
-\eta \epsilon \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu + i(\epsilon - \eta)}dE_\mu \left[ R(\lambda + i\epsilon) - R(\mu + i\eta) \right] dE_\lambda f = -\eta \epsilon \int_{-\infty}^{\infty} dE_\mu R(\mu + i\eta)R(\lambda + i\epsilon)dE_\lambda f = f.
\]
(3.9)

Then taking into account that by (1.1) for the fixed \( \gamma > 0 \) the equality
\[
G(z) = G(-i\gamma) + (z + i\gamma)R(z)G(-i\gamma)
\]
(3.10)
is true let us use Krein’s formula (1.3). Then for any \( f_1, f_2 \) from the domain \( D_A \) we obtain the expression
\[
(\mathcal{S}(A_L, A)f_1, f_2) = (f_1, f_2) - \lim_{\epsilon, \eta \downarrow 0} \int_{-\infty}^{\infty} \frac{[\mu + i(\gamma + \eta)][\lambda - i(\gamma - \epsilon)]}{\lambda - \mu + i(\epsilon - \eta)} \times \left( \left[ \frac{\gamma - \mu}{\lambda - \mu + i(\epsilon - \eta)} \Gamma(\lambda + i\epsilon) - \frac{\gamma - \mu}{\lambda - \mu + i(\eta)} \Gamma(\mu + i\eta) \right] G(-i\gamma)^*dE_\lambda f_1, G(-i\gamma)^*dE_\mu f_2 \right).
\]
(3.11)

But in accordance with (3.4) for any \( f \in \mathcal{H} \) we have
\[
G(-i\gamma)^* f = \sum_{j \in \mathbb{Z}} (f, \bar{g}_j)_{\mathcal{H}} h_j, \quad \bar{g}_j = \sum_{j \in \mathbb{Z}} b_{jk}g_k.
\]
(3.12)

Substituting (3.12) into (3.8) and assuming that \( f_1, f_2 \in E(\Delta)\mathcal{H} \cap D_A \) yields
\[
(\mathcal{S}(A_L, A)f_1, f_2) = \int_{\Delta} (f_1(\lambda), f_1(\lambda))_{\mathcal{N}} d\lambda - \lim_{\epsilon, \eta \downarrow 0} \int_{\Delta \times \Delta} d\lambda d\mu \times \left[ \frac{\gamma - i(\gamma - \epsilon)}{\lambda - \mu + i(\epsilon - \eta)} \right] \sum_{k, j \in \mathbb{Z}} \left[ \frac{\gamma - \mu}{\lambda - \mu + i(\epsilon - \eta)} \Gamma_{kj}(\lambda + i\epsilon) - \frac{\gamma - \mu}{\lambda - \mu + i(\eta)} \Gamma_{kj}(\mu + i\eta) \right] \times (f_1(\lambda), g_j(\lambda))_{\mathcal{N}} \cdot (g_k(\mu), f_2(\mu))_{\mathcal{N}}, \quad \Gamma_{kj}(z) = (\Gamma(z)h_j, h_k).
\]
(3.13)

To pass to the limit in (3.13), recall that in the Hilbert space \( L^2(\mathbb{R}; \mathcal{M}) \) of vector functions \( f(\lambda) \) on \( \mathbb{R} \) with values in some Hilbert space \( \mathcal{M} \) for any \( \epsilon > 0 \) operators
\[
\left( \pi^{(\epsilon)}_\pm f \right)(\lambda) = \pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{\mu - \lambda \mp i\epsilon} f(\mu) d\mu
\]
are contractions and strong limits \( \pi_\pm \) of this operators as \( \epsilon \downarrow 0 \) are orthogonal projectors onto the Hardy subspace \( H^2_+(\mathcal{M}) \) in the upper half-plane and its orthogonal complement \( H^2_-\mathcal{M} \), respectively. In particular, for any \( f \in
Let us assume for a moment that the scalar functions \((f_s(\lambda), \tilde{g}_j(\lambda))_{\mathcal{N}'}\), \(s = 1, 2\), in (3.13) satisfy the condition
\[
\int_{\Delta} \sum_{j \in \mathbb{Z}} \left| (f_s(\lambda), \tilde{g}_j(\lambda))_{\mathcal{N}'} \right|^2 d\lambda < \infty. 
\]
(3.15)

In other words, let us assume that vector functions
\[
f_s(\lambda) = \sum_{j \in \mathbb{Z}} (f_s(\lambda), \tilde{g}_j(\lambda))_{\mathcal{N}'} h_j
\]
belong to the space \(L^2(\Delta; \mathcal{K})\). In this case, using relations (3.14), one can directly carry out the passage to the limit in (3.13) and, as a result, obtain the expression
\[
\mathcal{S}(A_L, A) f_1, f_2 \rangle = \int_{\Delta} (f_1(\lambda), f_2(\lambda))_{\mathcal{N}'} d\lambda - \int_{\Delta} \sum_{k,j \in \mathbb{Z}} 2i \left( h_j, h_k \right)_\mathcal{K} \left( \bar{f}_1(\lambda), \bar{g}_j(\lambda) \right)_{\mathcal{N}'} \cdot \left( \bar{g}_k(\lambda), f_2(\lambda) \right)_{\mathcal{N}'} d\lambda.
\]
(3.16)

It remains to verify that the set of vector functions from \(L^2(\Delta; \mathcal{N})\) satisfying condition (3.15) is dense in \(L^2(\Delta; \mathcal{N})\). To this end let us take the scalar function
\[
\psi(\lambda) = \sum_{j \in \mathbb{Z}} \| \tilde{g}_j(\lambda) \|^2_{\mathcal{N}'}.
\]
According to the conditions of the theorem, this function is integrable since
\[
\sum_{j \in \mathbb{Z}} \int_{\Delta} \| \tilde{g}_j(\lambda) \|^2_{\mathcal{N}'} d\lambda = \sum_{j \in \mathbb{Z}} \int_{\Delta} \sum_{k', k \in \mathbb{Z}} \bar{b}_{jk'} b_{jk} (g_k(\lambda), g_{k'}(\lambda)) d\lambda \\
= \sum_{j, k \in \mathbb{Z}} |b_{jk}|^2 \leq \left( \sum_{j, k \in \mathbb{Z}} |b_{jk}| \right)^2 < \infty.
\]
For any \(f \in L^2(\Delta; \mathcal{N})\) and any \(\delta > 0\) put
\[
f_\delta = \frac{1}{\sqrt{1 + \delta \psi(\lambda)}} f(\lambda).
\]
Then, on the one hand,
\[
\int_{\Delta} \sum_{j \in \mathbb{Z}} \left| (f_\delta(\lambda), \tilde{g}_j(\lambda))_{\mathcal{N}'} \right|^2 d\lambda \leq \int_{\Delta} \frac{\psi(\lambda)}{1 + \delta \psi(\lambda)} \| f(\lambda) \|^2_{\mathcal{N}'} d\lambda \leq \frac{1}{\delta} \| f \|^2 < \infty,
\]
and on the other, it is obvious that
\[
\lim_{\delta \downarrow 0} \| f - f_\delta \| = 0.
\]
Remark 3.2 ([3]). By our assumption the scattering operator $S(A_1, A)$ is unitary. Therefore the scattering matrix $S(A_1, A)(\lambda)$ is to be unitary almost everywhere on $\Delta$. Notice, that the fact that $S(A_1, A)(\lambda)\ast S(A_1, A)(\lambda) = I$

follows directly from the relation

$$\frac{1}{2\pi} \left( \Gamma(\lambda + i0)^{-1} - [\Gamma(\lambda + i0)^{\ast}]^{-1} \right)_{j,k \in \mathbb{Z}} = \frac{1}{2\pi} \left( |L|^{-\frac{1}{2}} Q(\lambda + i0) |L|^{-\frac{1}{2}} - |L|^{-\frac{1}{2}} Q(\lambda + i0)^{\ast} |L|^{-\frac{1}{2}} \right)_{j,k \in \mathbb{Z}} \quad (3.17)$$

and general

Proposition 3.3. Let $g_1, ..., g_N$, $N < \infty$, be a set of linearly independent vectors of the Hilbert space $\mathcal{N}$, $\Upsilon = (\gamma_{nm})_1^N$ is the corresponding Gram-Schmidt matrix and $\Lambda$ is any Hermitian $N \times N$ - matrix such that the matrix $\Lambda + i\Upsilon$ is invertible. Then the matrix

$$\Omega = I - 2i \sum_{n,m=1}^{N} ([\Lambda + i\Upsilon])^{-1})_{jk} \langle \cdot, g_j \rangle g_k \quad (3.18)$$

is unitary.

The unitarity of the matrix $\Omega$ is verified by direct calculation. The extension of the statement of Proposition 3.3 to the case of infinite sequences of linearly independent vectors $\{g_j\}_{j \in \mathbb{Z}}$ from $\mathcal{N}$ under the additional conditions of boundedness and bounded invertibility of the operator in $l_2$ generated by the infinite matrix $\Lambda + i\Upsilon$ is left to the reader.

4. Scattering matrices for perturbations of the Laplace operator by infinite sums of zero-range potentials

The standardly defined Laplace operator $A = -\Delta$ in $L_2(\mathbb{R}^3)$ has uniform Lebesgue spectrum of infinite multiplicity, which coincides with the half-axis $[0, \infty)$ of the complex plane.

Let $S_2$ be the unit sphere in $\mathbb{R}^3$. The unitary mapping

$$(\tilde{g}f)(\lambda, n) = \frac{2^{\frac{3}{2}}}{\sqrt{2(2\pi)^2}} \int_{\mathbb{R}^3} f(x) e^{-i\sqrt{\lambda}(n \cdot x)} dx, \quad f \in L_2(\mathbb{R}^3), \quad n \in S_2, \quad (4.1)$$

of $L_2(\mathbb{R}^3)$ onto the space of vector function $L_2(\mathbb{R}^3; L_2(S_2))$ with values from the Hilbert space $L_2(S_2)$ transforms the Laplace operator $A$ into the selfadjoint operator of multiplication by the independent variable $\lambda$.

Let us consider a sequence $\{x_m\}_{m \in \mathbb{Z}_+}$ of different points of $\mathbb{R}^3$ that may have only finite number of accumulation points in compact domains of $\mathbb{R}^3$ and a corresponding sequence $(\delta_m)_{m \in \mathbb{Z}_+} > 0$ of positive numbers

$$\delta_0 = |x_0|; \quad \delta_m = \min_{0 \leq j \neq k \leq m} |x_j - x_k|, \quad m = 1, 2, \ldots.$$
Let \((w_m)_{m \in \mathbb{Z}^+}\) be a sequence of non-zero real numbers such that
\[
\sum_{m \in \mathbb{Z}^+} \frac{1}{\sqrt{|w_m|}} < \infty, \quad \sum_{m \in \mathbb{Z}^+} \frac{1}{|w_m| \delta_m} < \infty.
\]
By virtue of Theorem 2.3 substituting the diagonal matrix \(L = (w_m \delta_{mn})_{mn \in \mathbb{Z}^+}\) generated by this sequence into Krein’s formula (2.15) for the Laplace operator yields the resolvent \(R_L\) of close singular perturbation \(A_L\) of the Laplace operator.

To obtain an explicit expression of the scattering matrix \(S(A_L, A)(\lambda)\) for the pair \(A_L, A\) we note that in this case \(L_2(S_2)\) and the mentioned above space \(I_2\) may successfully act as the Hilbert space \(H\) and the auxiliary space \(K\), respectively in Krein’s formula (2.15) and the canonical basis \(e_j = (\delta_{jk})_{j,k \in \mathbb{Z}}\) in \(I_2\) may be engaged there as the orthonormal basis \(\{h_j\}\).

If the set of carriers \((x_j)_{1 \leq j \leq N}\) of the singular perturbation is finite, i.e. \(0 \leq N < \infty\), \((w_j)_{1 \leq j \leq N}\) are any real numbers and \(L_N = (w_m \delta_{mn})_{m,n = 1}^{N}\) is , then the absolutely continuous spectrum of the corresponding singular perturbation \(A_{L_N}\) of \(A\) fills the half-axis \((0, \infty)\) and the expression (3.6) for the scattering matrix \(S_N(\lambda), \lambda > 0\), for the pair \((A_{L_N})\) has form (4)
\[
S_N(\lambda; n, n') = \delta(n, n') - \frac{\sqrt{\lambda}}{4\pi} \sum_{j,j'=0}^{N} \Gamma_{j,j'}(\lambda + i0) 
\times e^{i\sqrt{\lambda} n \cdot x_j} \cdot e^{-i\sqrt{\lambda} n' \cdot x_{j'}}.
\]
Setting
\[
q_j(n) = \frac{\sqrt{\lambda}}{4\pi} e^{i\sqrt{\lambda} n \cdot x_j}
\]
we see that that the scattering matrix \(S_N(\lambda)\) in this case is an operator in \(L_2(S_2)\) of the form
\[
S_N(\lambda) = I - 2i \sum_{j,j'=1}^{N} \left( [\Omega_N(\lambda) + iG_N(\lambda)]^{-1} \right)_{j,j'}(\cdot, q_j(\cdot)) q_j'(n),
\]
where \(I\) is the unity operator in \(L_2(S_2)\), \(\Omega_N(\lambda)\) is a Hermitian \(N \times N\)-matrix function and \(G_N(\lambda)\) is the Gramm-Schmidt matrix for the set of vectors \(\{q_j(\cdot)\}_{j=1}^{N} \subset L_2(S_2)\), i.e.
\[
G_N(\lambda) = \left( (q_j, q_{j'})_{L_2(S_2)} \right)_{j,j'=1}^{N}.
\]
Note that according to (4.4)
\[
S_N(\lambda) q_s = \sum_{j=0}^{N} \left( [\Omega_N(\lambda) - iG_N(\lambda)] \cdot [\Omega_N(\lambda) + iG_N(\lambda)]^{-1} \right)_{s,j} q_j.
\]
Let us potentiate the perturbation \(A_{L_N}\) by adding at extra point \(x_{N+1}\) one more point potential with some real parameter \(w_{N+1}\). To compare the arising scattering matrix \(S_{N+1}(\lambda)\) with the preceded \(S_N(\lambda)\) remember the following proposition.
Let $\mathcal{W}_N$ and $\mathcal{W}_{N+1}$ be invertible $N \times N$ and $(N+1) \times (N+1)$ matrices respectively and the left upper $N \times N$ block of $\mathcal{W}_{N+1}$ coincides with $\mathcal{W}_N$. Let $\{\mathcal{W}\}_{N}^{-1}$ be the $(N+1) \times (N+1)$ matrix the upper left $N \times N$ block of which is the matrix $\mathcal{W}_N^{-1}$ and other its entries are zeros and $e_{N+1}$ be the $(N+1) \times 1$-matrix (column-vector) all the entries of which are zeros except the lowest one that equals 1. Then

$$\mathcal{W}_N^{-1} = \mathcal{W}_{N+1}^{-1} - [e_{N+1}^T \mathcal{W}_{N+1}^{-1} e_{N+1}]^{-1} \mathcal{W}_{N+1}^{-1} e_{N+1} e_{N+1}^T \mathcal{W}_N^{-1}$$

(4.6)

As follows in our case for the corresponding matrices in (4.4) we have

$$[\Omega_{N+1}(\lambda) + i\mathbf{G}_{N+1}(\lambda)]^{-1} = [\Omega_N(\lambda) + i\mathbf{G}_N(\lambda)]^{-1} + [e_{N+1}^T [\Omega_{N+1}(\lambda) + i\mathbf{G}_{N+1}(\lambda)]^{-1} e_{N+1}]^{-1}$$

$$\times [\Omega_{N+1}(\lambda) + i\mathbf{G}_{N+1}(\lambda)]^{-1} e_{N+1} e_{N+1}^T [\Omega_{N+1}(\lambda) + i\mathbf{G}_{N+1}(\lambda)]^{-1}.$$ \hspace{1cm} (4.7)



and setting further

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{N+1} \end{pmatrix} = [\Omega_{N+1}(\lambda) + i\mathbf{G}_{N+1}(\lambda)]^{-1} e_{N+1},$$

$$f_{N+1}(n) = \sum_{j=1}^{N+1} \xi_j q_j(n),$$

$$\mathbf{d}_{N+1}(\lambda) = \frac{\det[\Omega_{N+1}(\lambda) + i\mathbf{G}_{N+1}(\lambda)]}{\det[\Omega_N(\lambda) + i\mathbf{G}_N(\lambda)]}$$

we conclude that

$$S_{N+1}(\lambda; n, n') = S_N(\lambda; n, n') - 2i\mathbf{d}_{N+1}(\lambda) f_{N+1}(n) \overline{f_{N+1}(n')}.$$ \hspace{1cm} (4.9)

Note that by (4.8)

$$\xi_j = -\mathbf{d}_{N+1}(\lambda)^{-1} \sum_{k=1}^{N} \left([\Omega_N(\lambda) + i\mathbf{G}_N(\lambda)]^{-1}\right)_{jk} g(z; x_k - x_{N+1}), \quad j = 1, ..., N;$$

$$\xi_{N+1} = \mathbf{d}_{N+1}(\lambda)^{-1}. $$

(4.10)

Note also that by virtue of (4.8), (4.10)

$$\sum_{k=1}^{N} \xi_j (S^*_N(\lambda) q_j)(n) = -\mathbf{d}_{N+1}(\lambda)^{-1}$$

$$\times \sum_{j, k=1}^{N} \left([\Omega_N(\lambda) - i\mathbf{G}_N(\lambda)]^{-1}\right)_{jk} g(z; x_k - x_{N+1}) q_j(n)$$

(4.11)

and by (4.4)

$$S^*_N(\lambda) q_{N+1} = q_{N+1}$$

$$+ 2i \sum_{j, k=1}^{N} \left([\Omega_N(\lambda) - i\mathbf{G}_N(\lambda)]^{-1}\right)_{jk} \text{Im}g(z; x_k - x_{N+1}) q_j(n).$$

(4.12)

It follows from (4.10)-(4.12) that
\[(S_N^*(\lambda)f_{N+1})(n) = d_{N+1}(\lambda)^{-1}q_{N+1}(n)
- \sum_{j,k=1}^{N} \left(\Omega_N(\lambda) - i\mathfrak{G}_N(\lambda)\right)^{-1}_{jk} g(z;x_k - x_{N+1})q_j(n). \]  

(4.13)

and by virtue of above relations we conclude that

\[S_{N+1}(\lambda) = S_N(\lambda) \left[ I - 2i d_{N+1}(\lambda)^{-1} (\cdot, w_{\lambda,N+1}) J w_{\lambda,N+1} \right],\]

\[w_{\lambda,N+1}(n) = q_{N+1}(n) - \sum_{j,k=1}^{N} \left(\Omega_N(\lambda) - i\mathfrak{G}_N(\lambda)\right)^{-1}_{jk} g(z;x_k - x_{N+1})q_j(n),\]

\[J w(n) = \overline{w(-n)} \]  

(4.14)

and hence

\[
\det S_{N+1}(\lambda) = \det S_N(\lambda) \left[ 1 - 2i d_{N+1}(\lambda) (J w_{\lambda,N+1}, w_{\lambda,N+1}) \right].
\]  

(4.15)

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