A new Riemann fit for circular tracks

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Abstract. We present in this contribution a new Riemann track fit which operates on centered and scaled measurements. With these transformations, the fit becomes invariant under translations and similarity transforms of the measurements. We show in a simulation study in a generic, cylindrical detector that the modified Riemann fit is more precise than the standard Riemann fit, in particular if the hit resolution is large.

1. Introduction
The fitting of circular arcs to a set of measurements is of high importance in high-energy physics experiments due to the fact that tracking detectors often are embedded in homogeneous magnetic fields, resulting in a circular track in the bending plane of the particle trajectories. Even if the homogeneity of the magnetic field is imperfect, fast fits of circular tracks are important, e. g. for quality checks in track finding, in a track trigger, in defining a fast reference track for a Kalman filter [1], or for creating a preliminary track for a broken line fit [2]. Other types of detectors such as RICH detectors create circular patterns without the presence of any magnetic field, but with the same need of applying circle fit algorithms for the analysis of the data.

The problem of fitting a set of measurements to a circular arc is non-linear. Non-linear approaches to solving this problem are in general iterative and quite time-consuming. If the radius of curvature of the circle is not too small, the track model can be linearized around a reference track, making possible the application of linear methods such as a global least-squares fit or the Kalman filter. Alternatively, direct approaches such as the conformal mapping method [3], the Karimäki method [4] or the Riemann track fit [5] can be applied.

We present in this contribution a new version of the Riemann track fit method, which is based on centering and scaling of the data in order to achieve invariance under translations and similarity transforms. In a simulation experiment of a generic tracking detector system we show that this new method in some cases is significantly more precise than the standard Riemann track fit method [5].

2. Track fitting on the Riemann sphere
The Riemann sphere sits on the $(x, y)$-plane. Its south pole is the origin, and its north pole is $(0, 0, 1)$. Points in the plane are mapped to the sphere by a stereographic projection. If the track measurements in the $(x, y)$-plane are given in polar coordinates,

$$(R_i, \Phi_i), \quad i = 1, \ldots, N$$
Figure 1. An illustration of the Riemann sphere and the lines defining the transformed points. The imaginary axis corresponds to the $x$-axis, the real axis corresponds to the $y$-axis of the bending plane of the track. By courtesy of Encyclopaedia Britannica, Inc., copyright 2002; used with permission.

The mapping to the Riemann sphere is defined as:

$$
x_i = R_i \cos \Phi_i / (1 + R_i^2)
$$

$$
y_i = R_i \sin \Phi_i / (1 + R_i^2)
$$

$$
z_i = R_i^2 / (1 + R_i^2)
$$

The transformed point $(x_i, y_i, z_i)$ is the point of intersection of the straight line from the point $(R_i, \Phi_i)$ in the plane to the north pole of the sphere with the sphere itself, as indicated in Figure 1. The mapping is conformal and maps circles in the plane to circles on the sphere, illustrated in Figure 2. Since a circle on the sphere is the intersection of a unique plane with the sphere itself, the mapped points lie in a plane, or close to a plane if the measurement errors are taken into account. Due to this correspondence of a circle in the plane and a plane in space, the circle fit is transformed into fitting a plane to the mapped points.

All points with a position vector $r$ satisfying $n^T r + c = 0$ are lying in the same plane, where $n$ is a unit vector normal to the plane and $c$ is the signed distance from the plane to the origin. The plane is fitted to the transformed set of measurements by minimizing the following objective function:

$$
S = \sum_{i=1}^{N} (1 + R_i^2) d_i^2 = \sum_{i=1}^{N} p_i d_i^2
$$

where $d_i$ is the distance from the point $r_i = (x_i, y_i, z_i)^T$ to the plane. The solution to this minimization problem is that the normal vector $n$ is the unit eigenvector corresponding to the smallest eigenvalue of the sample covariance matrix:

$$
A = \frac{1}{N} \sum_{i=1}^{N} p_i (r_i - r_0) (r_i - r_0)^T
$$
Figure 2. An illustration of the Riemann sphere $K$ (with north pole $N$ and south pole $S$) and the important feature that a circle $L'$ in the plane $E$ maps onto a circle $L$ on the sphere.

with $r_0 = (x_0, y_0, z_0)^T$ and

$$x_0 = \sum_{i=1}^{N} p_i x_i / \sum_{i=1}^{N} p_i, \quad y_0 = \sum_{i=1}^{N} p_i y_i / \sum_{i=1}^{N} p_i, \quad z_0 = \sum_{i=1}^{N} p_i z_i / \sum_{i=1}^{N} p_i$$

Given $n$, $c$ is computed by:

$$c = -n^T r_0$$

The parameters $n$ and $c$ of the plane can then be mapped to a set of parameters of the corresponding circle in the $(x, y)$-plane [5].

3. The modified Riemann fit

Chernov [6] recommends centering and scaling of the measurements before mapping to the Riemann sphere in order to achieve invariance of the fit under translations and similarity transformations. For this purpose, it is convenient to transform the measured points to Cartesian coordinates:

$$X_i = R_i \cos \Phi_i, \quad Y_i = R_i \sin \Phi_i, \quad i = 1, \ldots, N$$

Centering is then achieved by:

$$X_{c,i} = X_i - \overline{X}, \quad Y_{c,i} = Y_i - \overline{Y}, \quad i = 1, \ldots, N$$

with

$$\overline{X} = \frac{1}{N} \sum_{i=1}^{N} X_i, \quad \overline{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i$$

After collecting all centered measurements $X_{c,i}$ and $Y_{c,i}$ in vectors $X_c$ and $Y_c$, the scaling is done by defining a scaling factor $s$:

$$s = b / \sqrt{(X_c^T X_c + Y_c^T Y_c) / N}$$

$$X_{cs} = s \cdot X_c, \quad Y_{cs} = s \cdot Y_c$$
with $b$ being an arbitrary, preselected constant. We have chosen the value $b = 0.5$, which has turned out to work well in our simulation studies. After centering and scaling, the measurements are transformed back to polar coordinates before performing the Riemann fit.

4. Simulation study in a generic cylindrical detector
In order to focus on the basic performance of the circle fit, we have simulated a generic type of a cylindrical detector system embedded in a perfectly homogeneous magnetic field. In the bending plane, the track model is in this case a circle. In order to test out the performance of the algorithms under a variety of conditions, we have generated a sample of 10000 tracks coming from the origin with radii of curvature in a range from about 1.5 m to about 750 m. This corresponds to arcs between less than 0.1 degrees and about 20 degrees, following a reasonably flat distribution in this range. There are about 10–12 hits per track, and the hit resolution, which is the standard deviation of the position error, varies between 0 mm, i.e. infinite precision, and 1.6 mm. We assume no background and thereby implicitly a perfect pattern recognition. An example of a simulated track is shown in Figure 3.

We have compared the modified fit with the standard Riemann fit by considering the generalized variances of the residuals, which are the difference of the estimated and the true track parameters in both the standard and the modified fit. The generalized variance is the determinant of the sample covariance matrix of the residuals. Figure 4 shows the ratio of the generalized variance of the modified fit with respect to that of the standard fit as a function of the hit resolution. The figure shows that the modified Riemann fit performs better at all values of the measurement uncertainties, and the improvement grows larger with increasing resolution.
Figure 4. The generalized variance of the modified Riemann fit divided by the generalized variance of the standard Riemann fit as a function of the hit resolution, i.e. the standard deviation of the position error.

5. Conclusions and outlook
We have in this contribution presented a new Riemann track fit which operates on centered and scaled measurements [6], making the fit invariant under translations and similarity transforms of the measurements. With these transformations, the fit becomes more precise than the traditional Riemann fit, in particular for large measurement uncertainties.

For the sake of completeness, a track fit also should be able to provide the covariance matrix of the track parameters. For the standard Riemann fit, the necessary Jacobians for this purpose have been derived earlier [7]. We intend to derive similar Jacobians for the new Riemann track fit in a future study.

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