AMALGAMATION CLASSES WITH ∃-RESOLUTIONS

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Abstract. Let \( K_d \) denote the class of all finite graphs and, for graphs \( A \subseteq B \), say \( A \leq_d B \) if distances in \( A \) are preserved in \( B \); i.e. for \( a, a' \in A \) the length of the shortest path in \( A \) from \( a \) to \( a' \) is the same as the length of the shortest path in \( B \) from \( a \) to \( a' \). In this situation \( (K_d, \leq_d) \) forms an amalgamation class and one can perform a Hrushovski construction to obtain a generic of the class.

One particular feature of the class \( (K_d, \leq_d) \) is that a closed superset of a finite set need not include all minimal pairs obtained iteratively over that set but only enough such pairs to resolve distances; we will say that such classes have ∃-resolutions.

In [8], Larry Moss conjectured the existence of graph \( M \) which was \( (K_d, \leq_d) \)-injective (for \( A \leq_d B \) any isometric embedding of \( A \) into \( M \) extends to an isometric embedding of \( B \) into \( M \)) but without finite closures. We examine Moss’s conjecture in the more general context of amalgamation classes. In particular, we will show that the question is in some sense more interesting in classes with ∃-resolutions and will give some conditions under which the possibility of such structures is limited.

1. Introduction

This paper arises from an investigation of the class of distanced graphs first explored by Larry Moss in [8]. In that paper, Moss produced a universal distanced graph; this is a countable graph \( U \) into which every finite graph \( A \) embeds isometrically. That is, for any finite graph \( A \) there is a map \( f : A \mapsto U \) such that for every \( a, a' \in A \), the length of the shortest path in \( A \) from \( a \) to \( a' \) is the same as the length of the shortest path in \( U \) from \( a \) to \( a' \).

Moss’s construction proceeded by expanding the language of graphs with predicates \( d_n(x,y) \) which were interpreted to indicate that the (path-length) distance from \( x \) to \( y \) was precisely \( n \). While Moss did not directly use a Fraïssé limit to obtain his structure, a number of his results can be recovered by doing so (see [6]).

It is also possible to reinterpret Moss’s work in the framework of the Hrushovski construction. Here we work in the language of graphs, let \( K_d \) be the class of all finite graphs and specify that (for \( A, B \in K_d \) \( A \leq_d B \) just in case the inclusion map \( A \hookrightarrow B \) is an isometry. This is an amalgamation class (see Section 3 or [8] for details) and can thus be associated with a \((K_d, \leq_d)\)-generic. The latter is the unique (up to isomorphism) countable graph \( G_d \) satisfying the following conditions

(Age): Every finite subgraph of \( G_d \) is in \( K_d \)

(Injectivity): If \( f : A \mapsto G_d \) so that \( f(A) \) is isometric in \( G_d \), then for any \( B \) with \( A \leq_d B \), \( f \) extends to an embedding \( g : B \mapsto G_d \) such that \( g \) is an isometry.

(Finite Closures): For any finite subset \( X \subseteq G_d \), there is a finite \( Y \) with \( X \subseteq Y \subseteq G_d \) and \( Y \) is isometric in \( G_d \)
It is easy to show that in this situation $G_d$ is also a countable universal distanced graph\(^1\). Moss conjectured that there was a graph $M$ which satisfied the first two of the three properties above but not the third. For any amalgamation class $(K, \leq)$ we will call $M$ a Moss structure if it satisfies appropriate analogues of the first two conditions but not the third (see Definition 6.1).

There are three main results of this paper. The first is that the existence of a $(K, \leq)$ Moss structure depends on a geometric property of $(K, \leq)$. In particular, we will define what it means for $(K, \leq)$ to have $\forall$-closures (as most studied classes do) versus $\exists$-resolutions (as the class of distanced graphs $(K_d, \leq_d)$ does). We will show the following

**Theorem 1.1.**

1. If $(K, \leq)$ is an amalgamation class with $\forall$-closures, then a $(K, \leq)$ Moss structure exists exactly when the $(K, \leq)$-generic is not $\omega$-saturated. Further, if a $(K, \leq)$ Moss structure exists, it can be taken to be a model of the theory of the $(K, \leq)$-generic.

2. If $(K, \leq)$ is an amalgamation class with finitary $\exists$-closures, then no $(K, \leq)$ Moss structure exists which is a model of the theory of the $(K, \leq)$-generic.

We do not resolve the general question of the existence of Moss structures for classes with $\exists$-closures, nor do we answer the specific question of whether or not a $(K_d, \leq_d)$ Moss structure exists (although the second part of the above theorem implies that if it does it cannot be a model of the theory of the generic).

We will also show that classes with $\exists$-resolutions and full amalgamation have superstable theories and prove a transfer theorem for classes with $\exists$-resolutions and related classes (which we will call $\forall$-companions).

## 2. Amalgamation

We will work throughout with amalgamation classes $(K, \leq)$ in a finite relational language $L$. These are classes $K$ of finite $L$-structures partially ordered by the relation $A \leq B$ (read $A$ is strong or closed in $B$) with which one can produce a canonical structure (the generic of the class) via an imitation of the Fraïssé construction. We will specifically require our classes to be closed under substructure and isomorphism, that $\leq$ be isomorphism invariant (if $A \leq B$ and $f : B \cong B'$, then $f(A) \leq B'$) and to satisfy the following axioms of Baldwin and Shi ([3]):

- **A1** For $M \in K$, $M \leq M$
- **A2** For $M, N \in K$, if $M \leq N$, $M \subseteq N$
- **A3** For $A, B, C \in K$, if $A \leq B \leq C$, then $A \leq C$
- **A4** For $A, B, C \in K$, if $A \leq C$ and $A \subseteq B \subseteq C$, then $A \leq B$
- **A5** We have $\emptyset \in K$ and $\emptyset \leq A$ for every $A \in K$.

In most examples of the Hrushovski construction in the current literature, the following additional axiom is also satisfied.

- **A6** If $A \leq B$ and $X$ is embedded in a common superstructure of $B$, $A \cap X \leq B \cap X$

\(^1\)It is in fact elementarily equivalent to Moss’s $U$ (see Theorem 4 of [6]) and will consist of countably many copies of $U$.\n
This axiom guarantees that in the \((K, \leq)\)-generic the smallest closed superset of a finite set is uniquely defined. Our primary interest in this paper is in examining amalgamation classes which do not satisfy \(A6\).

In order to produce a generic of \((K, \leq)\), we also require that the class have the \textit{amalgamation property}, defined below. We quickly summarize the main ideas of the construction; see (e.g.) [3, 5, 9] for more details.

\textbf{Notation 2.1.}

1. We will write \(A \subseteq \omega B\) to indicate that \(A\) is a finite substructure of \(B\).
2. We will write \(XY\) to denote \(X \cup Y\).

\textbf{Definition 2.2.}

Let \((K, \leq)\) be a class of \(L\)-structures partially ordered by \(\leq\) satisfying \(A1-A5\) as above.

1. For any \(L\)-structure \(M\), the set of \(M\)’s finite substructure is referred to as the age of \(M\).
2. If \(M\) is an \(L\)-structure whose age in contained in \(K\), we say that \(K\) is \textit{cofinal} in \(M\).
3. If \(K\) is cofinal in \(M\) and \(A \subseteq \omega M\), we say that \(A \leq M\) if for every \(X\) with \(A \subseteq X \subseteq \omega M\), \(A \leq X\).
4. If \(M\) has \(K\) cofinal in it and \(f : A \rightarrow M\) is an embedding, we say that \(f\) is a \textit{strong} or \textit{closed} embedding if \(f(A) \leq M\). We write this as \(f : A \overset{\leq}{\hookrightarrow} M\).
5. We say that \((K, \leq)\) has the \textit{amalgamation property} if for \(f : A \overset{\leq}{\hookrightarrow} B\), \(g : A \overset{\leq}{\hookrightarrow} C\), there is a structure \(D \in K\) and strong embeddings \(f' : B \overset{\leq}{\hookrightarrow} D\), \(g' : C \overset{\leq}{\hookrightarrow} D\) such that \(f' \circ f = g' \circ g\). In other words, given the solid part of the following diagram it can be completed to commute

\[
\begin{array}{ccc}
  B & \leq f \leq & D \\
  \downarrow f' & \downarrow & \downarrow g' \\
  C & \leq g \leq & A \\
\end{array}
\]

We will call \(D\) an \textit{amalgam} of \(B\) and \(C\) over \(A\).

We will call \((K, \leq)\) an \textit{amalgamation class} if it satisfies \(A1-A5\) and has the amalgamation property.

In fact, we will often want to work with especially nice forms of amalgamation.

\textbf{Definition 2.3.}

Suppose \(A, B, C\) are elements of \(K\) with \(A = B \cap C\), and let \(D\) be the structure whose universe is \(BC\) and whose relations are precisely those of \(B\) and those of \(C\). Then we will denote \(D\) by \(B \oplus_A C\).

1. If \((K, \leq)\) is an amalgamation class in which \(B \oplus_A C\) is an amalgam of \(B\) and \(C\) over \(A\), then we will call \(B \oplus_A C\) the \textit{free amalgam} of \(B\) and \(C\) over \(A\) and say that \((K, \leq)\) is a \textit{free amalgamation class}.
2. A free amalgamation class is full if for \(A, B, C \in K\) and \(A \leq B, A \subseteq C\), then \(C \leq D\), where \(D = B \oplus_A C\).

Given an amalgamation class \((K, \leq)\), one can imitate the Fraïssé construction to produce a \textit{generic} of the class. This is the unique (up to isomorphism) countable \(L\)-structure \(G\) which satisfies:
(Age): \( K \) is cofinal in \( G \).

(Injectivity): For \( A, B \in K \), if \( f : A \hookrightarrow G \) and \( A \leq B \), then \( f \) extends to some \( \hat{f} : B \hookrightarrow G \).

(Finite closures): For every \( A \subseteq \omega \), there is a finite \( B \) with \( A \subseteq B \leq G \).

We will say that any structure which satisfies the first two of these conditions is \((K, \leq)\)-injective.

It turns out that the model theory of the generic is largely determined by the complexity of the closure operation (that is, by the complexity of finding a minimal finite superset \( B \) of a given \( A \subseteq \omega \) for which \( B \leq G \)). We introduce some notions for analyzing such supersets. The fundamental one is that of a minimal pair, which is a minimal example of an extension which is not strong.

Definition 2.4. Let \((K, \leq)\) be an amalgamation class.

1. For \( X, Y \in K \) with \( X \subseteq Y \), we say that \((X, Y)\) is a minimal pair if \( X \not\leq Y \) but \( X \leq Y \) for \( X \subseteq Y \not\subseteq X \). We will also call \( Y \) a minimal extension of \( X \).

2. If \((X, Y)\) is a minimal pair, we say that it is a biminimal pair if whenever \( X_0 \subseteq X, Y_0 \subseteq Y \setminus X \) and \((X_0, Y_0)\) is a minimal pair, we must have \( X = X_0 \) and \( Y = Y_0 \). We will also say that \( Y \) is a biminimal extension of \( X \).

In [3], Baldwin and Shi noted that in classes satisfying A6, for \( A, B \in K \) we have \( A \leq B \) exactly when for \( X \subseteq A, Y \subseteq B \) and \((X, Y)\) a biminimal pair, we must have \( Y \subseteq B \). This will not hold as biconditional without A6, but does give a sufficient condition.

Lemma 2.5. If \((K, \leq)\) is an amalgamation class (and in particular satisfies A1-A5) and \( A \subseteq B \) are structures from \( K \) such that for every biminimal pair \((X, Y)\) with \( X \subseteq A, Y \subseteq B \) we have \( Y \subseteq A \), then \( A \leq B \).

Proof. Suppose \( A \not\leq B \). If \((A, B)\) is not a biminimal pair then by definition there must be some \( X \subseteq A \) and \( Y \subseteq B \) such that \((X, Y)\) is a biminimal pair. \(\square\)

3. Distanced Graphs

In this section we will formally explore that class of distanced graphs mentioned in the introduction. This class was studied by Moss in [8] and will form our canonical example of a class that does not satisfy A6 but does have what we will call \( \exists \)-resolutions. The remainder of the paper will examine this more general property, with a special eye toward the question of the existence of Moss structures.

Definition 3.1. Let \( G \) be a graph. For \( a, b \in G \), a path from \( a \) to \( b \) in \( G \) is a set of vertices \( a = p_0, \ldots, p_n = b \) where:

1. \( p_i \in G \)
2. There is an edge from \( p_i \) to \( p_{i+1} \) for \( i < n \)
3. For \( i, j < n \), \( p_i = p_j \) only if \( i = j \) (thus we consider all paths to be simple)

The length of a path is the number of edges in the path.

As before, let \( K_d \) be the class of all finite graphs and say \( A \leq_d B \) when for every \( a, a' \in A \), the minimal path length from \( a \) to \( a' \) in \( A \) is the same as the minimal path length from \( a \) to \( a' \) in \( B \). It is easy then to verify that A1 – A5 hold, and the
“strong amalgamation lemma” of [8] shows that \((K_d, \leq_d)\) has free amalgamation. We show that in fact it has full amalgamation.

To that end, fix \(A, B, C \in K_d\) with \(A \leq_d B, A \subseteq C\) and let \(D = B \oplus_A C\). We have to show that \(C \leq_d D\). Let \(p\) be a minimal length path from \(x\) to \(y\) in \(C\) and suppose by way of contradiction that it is shorter than any path from \(x\) to \(y\) in \(C\). List the vertices of \(p\) as \(x = x_0, \ldots, x_n = y\) where there is an edge between each \(x_i\) and \(x_{i+1}\). Since the amalgam is free, we can choose pairs of indices \((s_j, e_j)\) such that:

- \(s_j < e_j\) for all \(j\)
- \(p_{s_j}, p_{e_j} \in A\)
- For \(s_j < k < e_j\), we have \(p_k \in B \setminus A\)

Since \(p\) shorter than any path from \(x\) to \(y\) in \(C\), we must have that for some \(j\), \(p_{s_j}, p_{s_{j+1}}, \ldots, p_{e_j}\) is shorter than any path from \(p_{s_j}\) to \(p_{e_j}\) in \(A\). This contradicts that \(A \leq_d B\).

Let \(M\) be the \((K_d, \leq_d)\)-generic. Then we can define a closed superset of \(A \subseteq M\) by recursively adding minimal length paths over subsets of \(A\). In particular, for \(x, y \in M\) we define \(\chi(x, y)\) to be any minimal length path from \(x\) to \(y\). Then let \(J_0(A) = \bigcup \{\chi(x, y) : x, y \in A\}\) and having defined \(J_n\) let \(J_{n+1}(A) = \{\chi(x, y) : x, y \in J_n(A)\}\). Letting \(B = \bigcup_{n \in \omega} J_n\) it is easy to see that \(A \subseteq B \leq_d M\). Thus we can define a minimal closed superset of \(A\) by inductively adding to \(A\) a single minimal pair \((xy, \chi(xy))\) for each \(x, y \in A\). This idea will form the basis for our definition of a class with \(\exists\)-resolutions.

**Lemma 3.2.** For any finite graph \(X\), the pair \((X, Y)\) is a biminimal pair in \((K_d, \leq_d)\) if and only if \(X\) is a pair of points not joined by an edge and \(Y\) is a path between them.

**Proof.** Clear. \(\square\)

### 4. \(\exists\)-Resolutions

In this section we formally define the notion of having \(\exists\)-resolutions and examine its consequences. The notion will correspond to being able to form a closed superset of a finite base by iterative adding some minimal pair extensions which occur over the base. This is in contrast to Baldwin and Shi’s building of the intrinsic closure by iteratively adding all minimal pair extensions which occur over a given base.

**Definition 4.1.** Let \((K, \leq)\) be an amalgamation class satisfying the axioms A1 through A5 above. Following Baldwin and Shi [3], for any \(M\) cofinal with \(K\) and \(A \subseteq \omega M\), we define the **maximal** closure (mcl) as follows

- \(I_0(A) := A \cup \bigcup \{B \subseteq M : \exists A_0 \subseteq \omega A \text{ with } (A_0, B) \text{ a minimal pair }\}\)
- \(I_{n+1}(A) := \bigcup \{B \subseteq M : \exists A_0 \subseteq \omega I_n(A) \text{ with } (A_0, B) \text{ a minimal pair }\}\)
- \(\text{mcl}_M(A) := \bigcup_{n \in \omega} I_n(A)\)

It is worth noting that under axiom A6 \(\text{mcl}_M(A)\) is precisely the closure of \(A\) in \(M\). In general it is clear that \(\text{mcl}_M(A) \subseteq M\) (by Lemma 2.5). If \(\text{mcl}_M(A)\) is always the smallest closed superset of \(A\) which is closed in \(M\), then we say that \((K, \leq)\) has \(\forall\)-closures or \(\forall\)-resolutions. Intuitively, in these classes minimal pairs
can be thought of as obstructions to a set’s being closed, and all such obstructions must be included in the closure of a finite set.

**Lemma 4.2.** For any amalgamation class \((K, \leq)\) satisfying A1 – A5, Axiom A6 holds if and only if \((K, \leq)\) has \(\forall\)-closures.

*Proof.* Suppose \((K, \leq)\) has \(\forall\)-closures. Let \(A \subseteq B\) and let \(C\) be any element of \(K\) which is in a common superstructure of \(A, B\) (so that the intersections \(A \land C, B \land C\) make sense). We have to show that \(A \land C \leq B \land C\). Let \((X, Y)\) be a minimal pair with \(X \subseteq A \land C\) and \(Y \subseteq B \land C\). Then \((X, Y)\) is also a minimal pair with \(X \subseteq A\), so by \(\forall\)-closures \(Y \subseteq A\). Since \(Y\) was assumed to be contained in \(Y \land C\), \(Y\) is contained in \(A \land C\) as needed.

Conversely, if \((K, \leq)\) does not have \(\forall\)-closures, then there is a finite \(A\) and some \(M\) in which \(K\) is cofinal such that \(A \subseteq M\) but for some minimal pair \((X, Y)\) with \(X \subseteq A\) we have \(Y \not\subseteq A\). Let \(C = \text{mcl}_M(A)\); then \(A \leq C \leq M\). However \(A \land Y \not\subseteq C \land Y\) since \(A \land Y \supseteq X\) and \(C \land Y = Y\).

\[\Box\]

For any \(M\) as above, let us call \(B\) a **resolution** of \(A\) in \(M\) if \(A \subseteq B \subseteq M\). Such a resolution will be called **minimal** if there no resolution \(B'\) of \(A\) with \(B' \subsetneq B\). Minimal resolutions will also be called **closures**; note that in the absence of axiom A6, minimal closures need not be unique while in classes with \(\forall\)-closures the notion is precisely that of the \(M\)-closure of \(A\).

**Definition 4.3.** Let us say that a reflexive binary relation \(R\) on a set \(Z\) induces a **partial order** if, letting \([a]_R := \{ b \in Z : a R b \land b R a \}\), and defining \([a]_R R' [b]_R\) whenever \(a_0 R b_0\) for some \(a_0 \in [a]_R, b_0 \in [b]_R\), we have that \(R'\) is well defined and is a partial order on the equivalence classes of \(Z\).

Note that \(R\) induces a partial order on \(Z\) exactly when there is a map \(f : Z \to (P, \leq)\) where \((P, \leq)\) is partial order and for \(z, z' \in Z\), \(z R z'\) exactly when \(f(z) \leq f(z')\).

**Definition 4.4.** We say that a class \((K, \leq)\) satisfying A1-A5 has \(\exists\)-**resolutions** if for any \(X\), there is a relation \(\leq_X\) which induces a partial order on the class \(\{Y : (X, Y)\text{ is a biminimal pair}\}\) such that for \(A \subseteq B \subseteq K\), we have \(A \leq B\) if and only if for every \(X \subseteq A\), if \((X, Y')\) is a biminimal pair with \(Y' \subseteq B\) then there is a biminimal pair \((X, Y')\) with \(Y \subseteq A\) and \(Y \leq_X Y'\).

A class \((K, \leq)\) with \(\exists\)-closures will further be said to be **coherent** if for any \((K, \leq)\)-biminimal pair \((X, Y)\) if

1. There is some \(Z\) with \(X \subseteq Z \subseteq Y\) such that there is a biminimal pair \((U, V)\) with \(U \subseteq Z\) and \(V = Y \setminus Z\); and
2. there is some \(V'\) with \((U, V')\) a biminimal pair; and
3. \(V' \leq_U V\)

Then \((X, ZV')\) is a biminimal pair and \(ZV' \leq_X ZV\); i.e. \(ZV' \leq_X Y\).

Intuitively, the coherence of a class indicates that if we swap a part of a biminimal extension with something smaller with respect to a subextension, then the resulting biminimal extension will be smaller as well. In many of our examples the partial order will be induced by a number-valued function and coherence will come from a kind of monotonicity of that function.\(^2\)

\(^2\)In fact, the notion of coherence can be seen as a generalization of the principle of optimality in computer science. The latter states that optimal solutions to a global problem are comprised of
As for classes with $\prec$-closures, we want to extend the definition of $\leq$ to arbitrary structures $M, N$ in which $K$ is cofinal.

**Definition 4.5.** If $(K, \leq)$ is an amalgamation class with $\exists$-resolutions and $K$ is cofinal in $M, N$ with $M \subseteq N$, then we say that $M \leq N$ when for any biminimal pair $(X, Y)$ with $X \subseteq M, Y \subseteq N$, there is some $Y' \preceq_X Y$ with $Y' \subseteq M$.

In analogy with Baldwin and Shi’s characterization of the intrinsic closure in classes which satisfy $A_6$, we offer the following

**Lemma 4.6.** If $(K, \leq)$ is a free amalgamation class which satisfies $A_1$-$A_5$, then it has $\exists$-closures if and only if for any $M$ in which $K$ is cofinal, there are functions $\pi_M : [M]^{<\omega} \to 2^{[M]^{<\omega}}$ (a “potential function”) and $\chi_M : [M]^{<\omega} \to [M]^{<\omega}$ (a “choice function”) such that

1. For $X \subseteq M$, $\pi_M(X)$ is a subset of $\{Y \subseteq M : (X, Y) \text{ is a biminimal pair}\}$
2. For $A \subseteq M$, $\chi_M(A) = \pi(A)$ such that $A_X \leq M$, where $A_X$ is defined inductively as follows
   - $J_0(A) := A \cup \{\chi_M(A_0) \text{ for } A_0 \subseteq A\}$
   - $J_{n+1}(A) := J_n(A) \cup \{\chi_M(A_0) \text{ for } A_0 \subseteq J_n(A)\}$
   - $A_X := \bigcup_{n \in \omega} J_n(A)$

**Proof.** Suppose $(K, \leq)$ has $\exists$-resolutions. Fix $M$ as above and for $X \subseteq M$, let $\pi(X)$ be the set of $\preceq_X$-minimal extensions that occur over $X$ (or $\emptyset$ if there are no biminimal extensions), and choose $\chi(X)$ arbitrarily from $\pi(X)$. Then it is clear that $A_X \leq M$ as desired.

Conversely, suppose that our condition is satisfied for any $M$ with age $K$. Let $M$ be the $(K, \leq)$-generic. For $X \in K$ and biminimal extensions $Y, Y'$, let $D = Y \oplus_X Y'$ and choose a strong embedding of $D$ into $M$; abusing notation we will identify $D$ with its image in $M$. Let us say $Y \preceq_X Y'$ exactly when $Y \leq M$. Note that in such a case we will have $Y \leq D$ and that the isomorphism invariance of $\leq$ guarantees that this definition does not depend on the particular embedding of $D$ into $M$.

We must show that this induces a partial order. Reflexivity and antisymmetry come from the definition of the equivalence classes induced by $\preceq_X$, noting that we must have $\chi_D(X) \in \{Y, Y'\}$. For transitivity, we have to show that if $Y \preceq_X Y'$ and $Y' \preceq_X Y''$, then $Y \preceq_X Y''$. Let $D = (Y \oplus_X Y') \oplus_X Y''$ and assume $D$ is a strong substructure of $M$. We assume that each of $Y, Y', Y''$ are in different $\preceq_X$-equivalence classes and we will show that $Y \leq M$. First note that $\chi_D(X)$ must be exactly one of $Y, Y'$ or $Y''$; if not then letting $\chi_D(X) = Z$ we may assume that $X \leq (Z \cap Y), X \leq (Z \cap Y')$ and $X \leq (Z \cap Y'')$ so by free amalgamation $X \leq Z$, a contradiction. Further, we must have that $X\chi_D(X) \leq D$, since by bimimality and free amalgamation we cannot have $\chi_D(\chi_D(X)) \in D$. Thus $X\chi_D(X) \leq M$. If $\chi_D(X) = Y'$, then we have $XY' \preceq XYY'$ contradicting that $Y, Y'$ are in different $\preceq_X$ classes. Similarly if $\chi_D(X) = Y''$. So we must have $\chi_D(X) = Y$ and $XY \preceq XYY'$ as desired.

Intuitively, in classes with $\exists$-resolutions minimal pairs can be thought of as resolving some undetermined property of $A$. Once that property is resolved other potential resolutions add nothing new; thus one only needs at most one biminimal extension of any finite subset in a resolution of a finite set $A$. 

optimal solutions to local problems. Here global optimality would correspond to $\preceq_X$-minimality while local optimality would correspond to $\preceq_Y$-minimality
Note that in such a situation the definition of $\chi$ may not be preserved by substructures: that is, one might have a class with $\exists$-closures and models $M \subseteq N$ with $A_1$ in $M$ defined differently from $A_1$ in $N$. In fact, it is easy to see that if $\chi_M(A) = \chi_N(A)$ for every $A \subseteq M$, then $M \subseteq N$ (this implication does not reverse since a strong substructure could allow arbitrary choices of $\chi_M$; the possible choices $\pi_M$ should be preserved however). Also note that in general $\chi$ need not be unique. For example, in the class of distanced graphs discussed above, $\chi$ can be chosen so that for $a, b \in A$, $\chi(\{a, b\})$ is any minimal length path from $a$ to $b$ in $M$.

4.1. Companions. We now turn our attention to classes with $\exists$-resolutions which are derived from a given class.

**Definition 4.7.** For a given class $(K, \leq)$, a class $(K_3, \leq_3)$ is an $\exists$-companion of $(K, \leq)$ if:

1. $K_3 = K$
2. $(X, Y)$ is a biminimal pair in $(K, \leq)$ exactly if $(X, Y)$ is a biminimal pair in $(K_3, \leq_3)$
3. $(K_3, \leq_3)$ has $\exists$-closures.

It is natural to wonder when $\exists$-companions exist. For example, the class of Shelah-Spencer graphs are discussed in [3, 2, 7]. For a fixed irrational $\alpha \in (0, 1)$, the class $(K_\alpha, \leq_\alpha)$ is defined by saying that for graphs $A \subseteq B$, $A \leq_\alpha B$ exactly when, letting $e(Y/X)$ denote the number of edges in $Y$ but not in $X$, we have $|B_0 \setminus A| - \alpha e(B_0/A) > 0$ for every $A \subseteq B_0 \subseteq B$. We then let $K_\alpha := \{ A : \emptyset \leq_\alpha A \}$. A biminimal pair $(X, Y)$ will have $|Y \setminus X| - \alpha e(Y/X) < 0$; if we had an $\exists$-companion with free amalgamation we would be able to form, for $n \in \omega$, $D_n$ as the free amalgam of $X$ with $n$ copies of $Y$ and have $D_n \in K_\alpha$ (one copy would form the resolution of $X$ and the others would be then be strong extensions of the pair). But for sufficiently large values of $n$, $|D_n| - \alpha e(D_n/\emptyset)$ would be negative, contradicting the definition of $K_\alpha$. On the other hand, this is the main obstruction as the following lemma shows.

**Lemma 4.8.** Let $(K, \leq)$ be an amalgamation class satisfying A1–A6, and let $(K', \leq')$ be formed from $(K, \leq)$ by imposing an arbitrary partial ordering on the biminimal extensions of a fixed $X \in K$ and defining $\leq_3$ in accordance with the definition of a class with $\exists$-closures. Then if $(K', \leq')$ is an amalgamation class it is an $\exists$-companion for $(K, \leq)$.

**Proof.** One need only check that axioms A1–A5 hold, but this is straightforward.

For example, it is straightforward to show that the class of distanced graphs $(K_\alpha, \leq_\alpha)$ is an $\exists$-companion to the class $(K_C, \leq_C)$ of all graphs with $A \leq_C B$ exactly when every path from $a$ to $a'$ with length at least 2 in $B$ is contained in $A$. For a fixed pair $X = \{ a, a' \} \in K_C$ with no edge $(a, a')$, the biminimal pairs over $X$ will be the set of all paths between $a$ and $a'$, with the path length inducing the relevant partial ordering. A consequence of the previous lemma is that we could change the partial ordering to achieve a different $\exists$-companion, provided that the resulting class is an amalgamation class, as we will see with example 4.16.

**Lemma 4.9.** Let $(K, \leq)$ be a full amalgamation class with $\forall$-closures. If $(K_3, \leq_3)$ is an $\exists$-companion for $(K, \leq)$ with free and coherent amalgamation, then $(K_3, \leq_3)$ is also a full amalgamation class.
Proof. We will show that for \( A \subseteq B \) and \( A \subseteq C \) we have \( C \subseteq D = B \oplus A \). Let \((X, Y)\) be any \((K_3, \leq_3)\) biminimal pair with \( X \subseteq C \) and \( Y \subseteq D \). We have to show that there is some \( Y' \) with \( Y' \subseteq X \) and \( Y' \subseteq C \). We can assume without loss that \( X \cap C \) and \( Y \cap C \) are not both empty; our proof will be a generalization of the proof of full amalgamation for distanced graphs.

Let \( Z = (Y \cap C) \). Since \((X, Y)\) is a minimal pair, we must have that \((Z, Y)\) is also minimal and in particular that there is a biminimal pair \((U, UV)\) with \( U \subseteq Z \) and \( V \subseteq Y \). In fact, I claim that \( U \subseteq A \). If not, then \( U \cap (C \setminus A) \neq \emptyset \). Thus \( U \cap A \subseteq U \), so that by full amalgamation we have \( U \subseteq (V \cap B) \oplus_{UV} U = V \), a contradiction. Thus we have \( U \subseteq A \) as desired.

Given this, we have a biminimal pair \((U, UV)\) with \( U \subseteq A \) and \( V \subseteq (B \setminus A) \). Since \((X, Y)\) is biminimal, we must have \( V \setminus X = Y \setminus X \) (if \( V \setminus X = Y \setminus X \), then \((Z, Y)\) a minimal pair implies \( Z \subseteq Y_0 \) so \( Z \cap (UV Y_0) \subseteq Y_0 \cap (UV Y_0) \) so \( U \subseteq V \) a contradiction). Thus \((U, V)\) is a biminimal pair with \( U \subseteq A \).

Since \( A \subseteq B \), we have some \( V' \subseteq U \) with \( V' \subseteq A \). By coherence we must have \( ZV' \subseteq X \).

We can also pass from classes with \( \exists\)-resolutions to classes with \( \forall\)-closures.

**Definition 4.10.** Let \((K, \leq)\) be any amalgamation class satifying \( A_1 - A_5 \). We say that an amalgamation class \((K_\forall, \leq_\forall)\) is a \( \forall\)-companion of \((K, \leq)\) if

1. \( K_\forall = K \)
2. \((X, Y)\) is a biminimal pair in \((K, \leq)\) exactly when \((X, Y)\) is a biminimal pair in \((K_\forall, \leq_\forall)\)
3. \((K_\forall, \leq_\forall)\) has \( \forall\)-closures.

The \( \forall\)-companion always exists and is easily characterized.

**Lemma 4.11.** For any class \((K, \leq)\), a class \((K', \leq')\) is an \( \forall\)-companion of \((K, \leq)\) if and only if \( K' = K \) and \( A \leq' B \) exactly when for \((X, Y)\) a \((K, \leq)\) minimal-pair with \( X \subseteq A \) and \( Y \subseteq B \) we have \( Y \subseteq A \) as well.

**Proof.** Suppose \((K', \leq')\) is a \( \forall\)-companion of \((K, \leq)\). Then it is clear that \( K' = K \), we have to show that the condition on \( A \leq' B \) holds. But this is clear since the notion of biminimal pairs stays the same in an \( \forall\)-companion and the condition on \( \leq' \) is then equivalent to having \( \forall\)-closures by definition.

For the converse, suppose that \((K', \leq')\) is as described. The first and third conditions of being a \( \forall\)-companion are clear; we need to show that the notions of biminimal pair coincide. We first show that \((K, \leq)\) biminimal pairs are \((K', \leq')\) minimal pairs and vice versa. We then show that they are in fact biminimal pairs.

Suppose \((X, Y)\) is a \((K, \leq)\) biminimal pair. We want to show that \( X \leq' Y \) and for \( X \subseteq Y_0 \subseteq Y \) \( X \leq' Y_0 \). That \( X \leq' Y \) is clear from the definition of \( \leq' \). If \( X \leq' Y_0 \), then there is a \((K, \leq)\) minimal pair \((U, V)\) with \( U \subseteq X \) and \( V \subseteq Y_0 \) and \( V \subseteq X \). By biminimality, we must have \( U = X = V = Y \), contradicting that \( Y_0 \subseteq Y \). Thus any \((K, \leq)\) biminimal pair is a \((K', \leq')\) minimal pair.

Suppose now that \((X, Y)\) is a \((K', \leq')\) biminimal pair. We have to show that \((X, Y)\) is a \((K, \leq)\) minimal pair \((U, V)\) with \( U \subseteq X \) and \( V \subseteq Y \). If \( X \leq' Y \) then for every \((K, \leq)\) minimal pair \((U, V)\) with \( U \subseteq X \) and \( V \subseteq Y \) we have \( V \subseteq X \) as well. But by lemma ?? this implies that \( X \leq Y \) as well. The same lemma implies that \( X \leq Y_0 \) as well (since \( X \leq' Y_0 \)). Thus any \((K', \leq')\) biminimal pair is also a \((K, \leq)\) minimal pair.

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Now suppose that \((X, Y)\) is a \((K, \leq)\) biminimal pair. By the above, it is also a \((K', \leq')\) minimal pair. If there is a \((K', \leq')\) minimal pair \((U, V)\) with \(U \subseteq X, V \subseteq Y\), then there must be a \((K', \leq')\) biminimal pair \((U_0, V_0)\) with \(U_0 \subseteq U, V_0 \subseteq V\). Since \((U_0, V_0)\) is also \((K, \leq)\) minimal, we must have \(U_0 = X = U\) and \(V_0 = Y = V\) and \((X, Y)\) is \((K', \leq')\) biminimal as required. The same argument, mutatis mutandis, shows that \((K', \leq')\) biminimal pairs are also \((K, \leq)\) biminimal.

**Corollary 4.12.** For any \((K, \leq)\) as above, the \(\forall\)-companion \((K_\forall, \leq_\forall)\) exists and is unique

**Proof.** The previous lemma shows us that if we define \((K_\forall, \leq_\forall)\) to have \(K_\forall = K\) and \(A \leq_\forall B\) whenever for \((X, Y)\) a \((K, \leq)\) minimal pair with \(X \subseteq A, Y \subseteq B\) we have \(X \subseteq A\) as well, then \((K_\forall, \leq_\forall)\) will be an \(\forall\)-companion as long as it is an amalgamation class. It is easily checked that \((K_\forall, \leq_\forall)\) so defined will satisfy A1–A6. Uniqueness also follows from the previous lemma.

For amalgamation classes \((K, \leq)\) and \((L, \subseteq)\), we will say that \((K, \leq)\) is isomorphic to \((L, \subseteq)\) if there is a bijection \(F : K \rightarrow L\) such that for \(A, B \in K\), \(F(A) \subseteq F(B)\) exactly when \(A \leq B\). Note that it sufficient to have \(K = L\) and for \(\leq, \subseteq\) to induce the same notion of biminimal pair.

**Lemma 4.13.** For any class \((K, \leq)\) with \(\forall\)-closures, the \(\forall\)-companion of any \(\exists\)-companion \((K, \leq)\) is isomorphic to \((K, \leq)\).

**Proof.** Let \((K', \leq')\) denote the \(\forall\)-companion of some \(\exists\)-companion \((K, \leq_3)\) of \((K, \leq)\). It suffices to show that \((X, Y)\) is a biminimal pair in \((K, \leq)\) if and only if \((X, Y)\) is a biminimal pair in \((K', \leq')\). But this is clear from the definition.

As the examples which follow will show, a single \(\forall\)-companion can be associated with multiple (non-isomorphic) \(\exists\)-companions.

### 4.2. Examples

**Example 4.14.** Let \(L\) be the language whose non-logical symbols are \(E(x, y)\) and \(<\). Let \(K_p\) be the class of all finite \(L\)-structures in which \(E\) and \(<\) are interpreted so that for \(A \in K_p\)

- \(E^A\) is symmetric and antireflexive (thus \((A, E)\) is a simple graph).
- \((A, <)\) is a finite linear order.
- For \(x, y \in A\), if \(A \models E(x, y)\) then either \(y\) is the successor of \(x\) in \((A, <)\) or vice-versa.

Thus we can view \(A\) as a finite linear order with possible edges between pairs of successive vertices. For \(x \in A\), we will write \(x \pm 1\) for the successor or predecessor of \(x\) in \((A, <)\) (if there is one).

For \(A \subseteq B \in K_p\), let us say that \(A \leq_p B\) when the only connected components of \(B\) with vertices in \(A\) are entirely contained in \(A\). It is clear that \((K_p, \leq_p)\) has free amalgamation and \(\forall\)-closures. In the generic, the closure of a finite set will be a minimal set of connected components containing that set; the generic thus consists of finite paths separated by dense linear orders of vertices.

\((K_p, \leq_p)\) does not have \(\exists\)-resolutions: for \(A \in K_p\) and \(x \in A\), if there are edges in \(A\) from \(x\) to \(x + 1\) and \(x - 1\), then \((x, x + 1)\) and \((x, x - 1)\) will both be biminimal pairs that must be in the closure of \(\{x\}\). In fact all biminimal pairs will be of the
form \((x, xy)\) where \(y = \pm 1\) and there is an edge from \(x\) to \(y\). Thus we have three possibilities for an \(\exists\)-companion of \((K_p, \leq_p)\): we can choose \((x - 1) \leq_x (x + 1)\), or \((x + 1) \leq_x (x - 1)\), or the two possibilities could be equivalent. In the first case, we will have \(A \leq_B \exists B\) when for every \(x \in A\):

- If \((x + 1) \in B\) and \(B \models E(x, x + 1)\), then either \((x + 1) \in A\) or \((x - 1) \in A\).
- If \((x - 1) \in B\) and \(B \models E(x, x - 1)\), then \((x - 1) \in A\).

Thus if \(A \subseteq B\) with \(|A| > 1\) and both are connected, a resolution of \(A\) in \(B\) can be formed by adding all the vertices in \(B\) which are less than those in \(A\) (in the case that \(A\) is a singleton \(x\) and \(B = \{x, x + 1\}\) we would have \(B\) as the only resolution).

The \((K_{\exists}, \leq_{\exists})\)-generic will thus consist of countably many infinite paths of order type \(\omega, <\) separated by dense linear orders of vertices. If we had chosen to have \((x + 1) \leq_x (x - 1)\), we would get paths of order type \(\omega^*\) and by having them equivalent we would get countably many paths of order type \((Z, <)\).

It is clear that these are all the \(\exists\)-companions for \((K, \leq)\) and that \((K, \leq)\) is the \(\forall\)-companion of each of these (one need only notice that the biminimal pairs are pairs \((x, x \pm 1)\) with an edge between them in each case).

**Example 4.15.** Let \((K_{d, \leq d})\) be the class of distanced graphs as above. Then biminimal pairs in \((K_{d, \leq d})\) consist of pairs \(\{x_0, x_1\}, p\) where \(p\) is a simple path between \(x_0\) and \(x_1\) and \(x_0, x_1\) are not joined by an edge. We can form an \(\forall\)-companion \((K_{C, \leq C})\) by saying \(A \leq_C B\) if for \(a_0, a_1 \in A\) every simple path from \(a_0\) to \(a_1\) of length at least 2 in \(B\) is also in \(A\).

We have that \((K_{d, \leq d})\) is an \(\exists\)-companion of \((K_{C, \leq C})\). We could attempt to form another \(\exists\)-companion \((K_{m, \leq m})\) by saying \(A \leq_m B\) if for every \(a_0, a_1 \in A\) the longest simple path from \(a_0\) to \(a_1\) in \(B\) is no longer than the longest simple path from \(a_0\) to \(a_1\) in \(A\). However, the example illustrated below shows that this would not be a free amalgamation class. Let \(A\) consist of the five vertices on the solid ellipse, let \(B = A \cup \{b\}\) and let \(C = A \cup \{c\}\). Then by inspection we see that \(A \leq_m B, A \leq_m C\) but \(C \leq_m B \oplus_A C\) since there is a simple path of length 5 from \(c\) to \(a\) in the latter structure.

![Diagram](image.png)

**Example 4.16.** Let \(K_H\) be the class of all finite graphs; and for \(A \in K_H\) define a “(non)-Hamiltonicity” function \(H\) by \(H(A) = |A| - \lambda(A)\) where \(\lambda(A)\) is the number of vertices in the longest simple path in \(A\) and for \(\forall \subseteq B\) let \(H(B/A) = H(B) - H(A)\). Let us say that \(A \leq_H B\) if for every \(A_0 \subseteq A\) and \(B_0 \subseteq B\setminus A\) we have \(H(A_0B_0/A_0) \geq 0\). Then \((K_{H, \leq H})\) clearly satisfies A1 through A6.

It is not difficult to see that a biminimal pair in this class will be a pair \(\{x, y\}, p\) where there is no edge from \(x\) to \(y\) and \(p\) is a simple path between \(x\) and \(y\). Since this is the same notion of a biminimal pair that is used in the last example and since \((K_{H, \leq H})\) has \(\forall\)-closures, we see that this class is isomorphic to \((K_{C, \leq C})\).

While \((K_{C, \leq C})\) and \((K_{H, \leq H})\) are isomorphic classes, their different definitions allows for some insights into the generic \(G_{CH}\) of these classes. For example, it is easy to see from the definition of \((K_{H, \leq H})\) that every finite graph \(A\) without a Hamiltonian path embeds into \(G_{HC}\) in such a way that no finite extension of
A contains a Hamiltonian path either. The definition of \((K_C, \leq_C)\) makes it clear that any path between vertices \(\{x, y\}\) which are joined by an edge must be in the closure of \(\{x, y\}\).

The definition of \((K_H, \leq_H)\) also suggests another \(\exists\)-companion for these classes besides \((K_d, \leq_d)\): we simply treat all paths from \(x\) to \(y\) as equal in the induced partial order.

5. Properties of Classes with \(\exists\)-Resolutions

In a class with \(\exists\)-closures, for any \(M\) in which \(K\) is cofinal and \(A \subseteq \omega \ M\), there are two notions of closure in play: (non-unique) resolutions of \(A\) and the unique maximal closure \(\text{mcl}_M(A)\). We will examine the model-theoretic information given by each of these closures.

It will be useful in this analysis to make use of some definitions and results from [4]. First we define intrinsic formulae which will give an approximate description of the maximal closure of a finite set.

**Definition 5.1.** For a fixed amalgamation class \((K, \leq)\) a 0-intrinsic formula over a finite tuple \(\bar{a}\) is a formula of the form

\[
\phi(\bar{x}; \bar{a}) := \Delta_B(\bar{a}\bar{x})
\]

where \((\bar{a}_0, B)\) is a minimal pair for \(\bar{a}_0 \subseteq \bar{a}\) and \(\Delta_B(\bar{a}\bar{x})\) asserts that \(\bar{a}_0\bar{x}\) is isomorphic to \(B\).

Having defined \(k\)-intrinsic formulae, we define a \(k + 1\)-intrinsic formula to be of the form

\[
\phi(\bar{x}; \bar{a}) := \Delta_B(\bar{a}\bar{x}) \land \bigwedge_{i < m} \exists \bar{w}_i \phi_i(\bar{w}_i; \bar{a}\bar{x}) \land \bigwedge_{j < n} \neg \exists \bar{z}_j \psi_j(\bar{z}_j; \bar{a}\bar{x})
\]

where again \((\bar{a}_0, B)\) is a minimal pair for some \(a_0 \subseteq A\) and the \(\phi_i, \psi_j\) are \(k\)-minimal formulae.

We will call a formula \(\phi(\bar{x}; \bar{a})\) intrinsic over \(\bar{a}\) when it is \(k\)-intrinsic over \(\bar{a}\) for some \(k \in \omega\).

We get a full elementary description of a finite set’s maximal closure by passing to the closure type.

**Definition 5.2.** Let \((K, \leq)\) be an amalgamation class and let \(\mathfrak{C}\) be a monster model for the theory of the \((K, \leq)\)-generic (that is, a “sufficiently” saturated model of the theory of the \((K, \leq)\) generic). For any fixed tuple \(\bar{a} \subseteq \omega \mathfrak{C}\), the closure-type of \(\bar{a}\), denoted cltp(\(\bar{a}\)), is defined by

\[
\text{cltp}(\bar{a}) = \{ \exists \bar{x} \phi(\bar{x}; \bar{w}) : \phi \text{ is intrinsic over } \bar{w}, \mathfrak{C} \models \exists \bar{x} \phi(\bar{x}; \bar{a}) \} \cup \\
\{ \neg \exists \bar{y} \psi(\bar{y}; \bar{w}) : \psi \text{ is intrinsic over } \bar{w}, \mathfrak{C} \models \neg \exists \bar{y} \psi(\bar{y}; \bar{a}) \}
\]

For fixed \(\bar{a}\) and \(M \subseteq \mathfrak{C}\) any set, the closure type of \(\bar{a}\) over \(M\) is defined by cltp(\(\bar{a}/M\)) = \(\bigcup_{\bar{m} \subseteq \omega \ M} \text{cltp}(\bar{a}\bar{m})\).

**Lemma 5.3.** Suppose \((K, \leq)\) is a full amalgamation class and \(\mathfrak{C}\) is an associated monster model. Suppose \(\bar{a}, \bar{b} \subseteq \omega \mathfrak{C}\) are finite tuples with cltp(\(\bar{a}\)) = cltp(\(\bar{b}\)). Fix \(a_0 \in \mathfrak{C} \setminus \text{cl}(\bar{a})\), and let \(\Sigma\) be a finite fragment of cltp(\(\bar{a}a_0\)). Then there is a finite \(D \subseteq \text{cl}(\bar{a}a_0)\) such that if \(f : D \to \mathfrak{C}\) is an embedding of \(D\) with \(f : \bar{a} \to \bar{b}\) and clp(\(f(D)\)) = clp(\(\bar{b}\)) \(\oplus_b f(D)\), then there is some \(b_0 \in f(D)\) so that \(\mathfrak{C} \models \Sigma (\bar{b}_0)\).
Proposition 5.4. Let $\mathcal{C}$ denote a sufficiently saturated model of the theory of the $(K, \leq)$-generic, where $(K, \leq)$ is a full amalgamation class. Let $M \subseteq \mathcal{C}$ with $|M| < |\mathcal{C}|$. Let $a, b \in \mathcal{C}$ and suppose $cltp(a/M) = cltp(b/M)$. Then $tp_\mathcal{C}(a/M) = tp_\mathcal{C}(b/M)$.

Proof. These are proved in [4] under the assumptions of A1–A6. It is an easy check that A6 is not used in the proofs. □

One natural question one might ask is what a resolution of $A$ tells us about how $A$ sits inside of $M$ versus what $mcl_M(A)$ tells us. A resolution gives the possibilities for $mcl_M(A)$. Somewhat surprisingly, it turns out that in the generic, they provide the same information in that a single resolution of $A$ will determine $mcl_M(A)$. Under full amalgamation and $\exists$-closures we make this precise in the following.

Proposition 5.5. Let $(K, \leq)$ be any full amalgamation class, let $\mathcal{C}$ be a sufficiently saturated model of the theory of the $(K, \leq)$-generic, and let $M \subseteq \mathcal{C}$ be the $(K, \leq)$-generic.

1. Let $A, A'$ be finite substructures of $\mathcal{C}$ with an isomorphism $f : A \to A'$ which extends to an isomorphism $\hat{f} : mcl_\mathcal{C}(AM) \cong mcl_\mathcal{C}(A'M)$. Then $tp_\mathcal{C}(A/M) = tp_\mathcal{C}(A'/M)$.

Further, it suffices that the structure $(mcl_\mathcal{C}(AM), A)$ (this is the structure with universe $mcl_\mathcal{C}(AM)$ and parameter set $A$) be elementarily equivalent to $(mcl_\mathcal{C}(A'M), A')$.

2. Let $A, A' \subseteq_\omega M$. If there are finite resolutions $B, B'$ of $A, A'$ respectively and an isomorphism $f : B \cong B'$ with $f(A) = A'$, then $tp_M(A) = tp_M(A')$.

Proof. The first follows from Proposition 5.4. For the second, note that $B \leq mcl_M(A)$ and $B' \leq mcl_M(A')$. Thus by injectivity and finite closures in $M$ we can easily construct a winning strategy for the player $\exists$ in any finite length Ehrenfeucht-Fraïssé game. By the Fraïssé-Hintikka theorem, we have $tp_M(A) = tp_M(A')$. □

It is also fairly easy to see that in classes with $\exists$-resolutions, finite sets have countable resolutions. As a result, the generic of such a class will have a superstable theory.

Theorem 5.6. Let $(K, \leq)$ satisfy A1–A5 and have $\exists$-resolutions. Then

1. If $M$ is any structure with $K$ cofinal in $M$ and $A \subseteq M$, there is a countable resolution $B$ of $A$ in $M$.

2. If $(K, \leq)$ also has full amalgamation, then the theory of the $(K, \leq)$-generic is superstable.

Proof. For the first statement, for each $X \subseteq A$, choose $Y_X \subseteq_\omega M$ so that $(X, Y_X)$ is a biminimal pair and $Y_X$ is minimal in the $\preceq_X$ ordering. Let $A_1 := \bigcup_{X \subseteq A} Y_X$ and note that $A_1$ is finite. Thus we can iterate this procedure and define $A_{n+1} := \bigcup_{X \subseteq A_n} Y_X$ where $Y_X \subseteq_\omega M$ is such that $(X, Y_X)$ is an $\preceq_X$-minimal biminimal pair. It is clear that letting $B = \bigcup_{n \in \omega} A_n$ we have $B \leq M$.

For the second statement, let $M$ be a model of the theory of the generic with $|M| \geq 2^{|K|}$. Let $p \in S_1(M)$, let $\mathcal{C}$ be a $|M|^+$-saturated elementary extension of $M$, and let $a \in \mathcal{C}$ be a realization of $p$. Note that for $\vec{a} \subseteq M$ there are at most $2^{|K|} \cdot |M| = |M|$ possibilities for $cltp(a/M)$. Since the latter determines $tp(a/M)$, we have $|S_1(M)| \leq |M|$. □
From [8] we have that the theory of the generic of the class \((K_d, \leq_d)\) of distanced graphs is not \(\omega\)-stable; it is thus strictly superstable\(^3\).

We have the following lemma about full amalgamation

**Lemma 5.7.** Let \((K, \leq)\) be a full amalgamation class with \(\exists\) closures. Then for any \(M\) which is \((K, \leq)\)-injective, if \(A \subseteq C\) and \(A \leq B\) (with \(B \in K\)) then \(C \leq B \oplus_A C\).

**Proof.** Let \((X, Y)\) be a biminimal pair with \(X \subseteq C, Y \subseteq B \oplus_A C\). Choose a finite \(C_0\) so that \(Y \subseteq B \oplus_A C_0\). Then \(C_0 \leq C_0 \oplus_A B\) by full amalgamation, so we must have \(Y' \leq_X Y\) with \(Y' \subseteq C_0 \subseteq C\). \(\square\)

We also show that under fairly modest assumptions, the class of \((K, \leq)\)-injective structures is elementary.

**Definition 5.8.** A class \((K, \leq)\) with \(\exists\)-resolutions has finitary \(\exists\)-resolutions if for every \(X \in K\), the partial order induced by \(\leq_X\) is a well-ordering and every equivalence class is finite.

**Lemma 5.9.** If \((K, \leq)\) has finitary \(\exists\)-resolutions, then for any \(B \in K\) there is a universal formula \(k_B(x)\) such that for \(M\) in which \(K\) is cofinal, \(M \models k_B(b)\) exactly when \(b\) is isomorphic to \(B\) (under some fixed ordering of the ordering of the elements of \(B\)) and is closed in \(M\).

**Proof.** Let \(k_B\) assert that \(x\) has the quantifier free type of \(B\) and also, for every \(x_0 \leq_X x\), that there does not exist \(y\) with \((x_0, y)\) a biminimal pair and \(y\) smaller under \(\leq_{x_0}\) then any \(\leq_x\)-minimal biminimal extension which occurs in \(x\). The set off all such extensions is finite by finitariness of the \(\exists\)-resolutions. \(\square\)

We note that the class of distanced graphs has finitary \(\exists\)-resolutions as does the class of Example 4.16. The class from Example 4.14, however, does not.

**Lemma 5.10.** Let \((K, \leq)\) have finitary \(\exists\)-resolutions. Let \(\Sigma_I\) be the following sentences:

- \(\forall x \Delta_B(x)\) for finite \(B \not\in K\) and \(\Delta_B(x)\) the quantifier-free type of \(B\).
- \(\forall x k_A(x) \rightarrow \exists y k_B(x, y)\) for \(A \leq B\) and \(k_A, k_B\) as guaranteed by the previous lemma.

The \(\Sigma_I\) axiomatizes the class of \((K, \leq)\)-injective structures.

**Proof.** Clear. \(\square\)

### 5.1. Transfer

We note that for a given class with \(\forall\)-closures, any corresponding \(\exists\)-companion will have a generic that is no more complex then that of the original class. This allows us to prove various transfer theorems.

Throughout, fix a full amalgamation class \((K, \leq)\) with \(\forall\)-closures and let \((K, \leq_\exists)\) be an \(\exists\)-companion. Let \(G_\forall, G_\exists\) denote the corresponding generics.

Note that for \(A, B \in K\), if \(A \leq_\forall B\) then \textit{a fortiori} \(A \leq_\exists B\). This has the following consequences

**Lemma 5.11.** Let \(G_\forall, G_\exists\) respectively denote the \((K, \leq_\forall), (K, \leq_\exists)\) generics. Then

- (1) \(G_\forall\) embeds as a \(\leq_\exists\)-strong substructure of \(G_\exists\).
- (2) If \(M_3\) is any \((K, \leq_\exists)\) injective structure, then \(G_\exists\) embeds as a \(\leq_\exists\)-strong substructure of \(M_3\)

\(^3\)It is not small, which is of interest since Baldwin has conjectured that no strictly superstable generic structure exists which is also small (assuming A1–A6); see [1]
(3) If $M_3$ is any $(\mathbf{K}, \leq_3)$ injective structure, then $G_\forall$ embeds as a $\leq_3$-strong substructure of $M_3$.

Proof. For the first item, write $G_\forall = \bigcup_{i<\omega} A_i$ where $A_i \subseteq A_{i+1}$. Then by induction $A_i$ embeds as a $\leq_3$-strong substructure of $G_3$. Thus $G_\forall$ embeds into $G_3$, say with image $G'_\forall$; if $(X, Y)$ is a biminimal pair in $G_3$ with $X \subseteq G'_\forall$, then for some $i$ $X \subseteq A'_i$ (the image of $A_i$). Since $A'_i \leq_3 G_3$, we must have some $Y' \subseteq X Y$ with $Y' \subseteq Y$.

The same argument modified to use a decomposition $G_3 = \bigcup_{i<\omega} B_i (B_i \leq_3 B_{i+1})$ establishes the second item, while the third is an immediate consequence of the first two. \qed

Theorem 5.12. Assume that $(\mathbf{K}, \leq_\forall)$ and $(\mathbf{K}, \leq_3)$ both have full amalgamation. Let $\text{Th}(G_\forall)$ denote the theory of the $(\mathbf{K}, \leq_\forall)$-generic and similarly for $\text{Th}(G_3)$. Let $M_3 \models \text{Th}(G_3)$. Then there is some $M_\forall \models \text{Th}(G_\forall)$ with $|M_\forall| = |M_3|$ so that $|S_1(M_\forall)| \leq |S_1(M_3)|$, where $S_1(X)$ denotes the set of complete 1-types over parameter set $X$. In particular, if $\text{Th}(G_\forall)$ is stable ($\omega$-stable, superstable, etc), then so is $\text{Th}(G_3)$.

Proof. Let $C \cup D$ be a set of $|M_3|$ new constants, and let $S$ be a set of sentences asserting that $C$ is isomorphic to $M_3$ along with sentences asserting that for $A \in \mathbf{K}$, some elements of $D$ form an $\forall$-closed copy of $A$ in any model of $S$. Since $(\mathbf{K}, \leq_\forall)$ has the same age as $(\mathbf{K}, \leq_3)$, by compactness there is some $M_\forall \models \text{Th}(G_\forall) \cup S$. Let $h : M_3 \to M_\forall$ be an embedding. We define an injection $f : S_1(M_3) \to S_1(M_\forall)$. Fix $p \in S_1(M_3)$ and let $a_p$ be a realization in some $|M_3|$-saturated extension $N$ of $M_3$. Let $C_p = \text{cltp}(a_p/M_3)$, and for $\phi(x; \bar{m}_3) \in C_p$ let $h(\bar{m}) = \phi(x; h(\bar{m}_3))$ and let $P(x) = \bigcup \{ h(\bar{m}) \}$. By Lemma 5.3, each finite fragment $\Sigma$ of $P(x)$ is associated with a structure $D$ such that any $\leq_3$-strong embedding of $D$ into $M_\forall$ will contain a realization of $\Sigma$. Since the age of $M_\forall$ is the same as that of $M_3$ and since $M_\forall$ embeds every element of $\mathbf{K}$ as closed substructure, there will be such an embedding. Thus by compactness, $P(x)$ extends to a completion in $S_1(M_\forall)$. Letting $f(p)$ be any such completion, we note that since different types in $S_1(M_3)$ differ in their closure-types, and since this difference must be witnessed by a finite subset of $M_\forall$, we have that $f$ is an injection as required. \qed

6. Moss Structures

In this section we discuss the existence of Moss structures for amalgamation classes. We will see that the question of the existence of such structures is quite straightforward in classes with $\forall$-closures but more delicate for classes with $\exists$-resolutions.

Definition 6.1. Let $(\mathbf{K}, \leq)$ be any amalgamation class. A $(\mathbf{K}, \leq)$ Moss structure is a structure $M$ which is $(\mathbf{K}, \leq)$-injective but which does not have finite closures. That is, there is some $A \subseteq M$ such that there is no $B$ with $A \subseteq B \subseteq M$ and $B \leq M$.

We first note that the existence of Moss structures is easily established in classes $(\mathbf{K}, \leq)$ with $\forall$-closures, unless every structure in which $\mathbf{K}$ is cofinal has finite closures (in which case, following [3], we say that $(\mathbf{K}, \leq)$ has finite closures).
Proposition 6.2. If \((K, \leq)\) is a full amalgamation class with \(\forall\)-closures and without finite closures, then there is a Moss structure which satisfies the theory of the \((K, \leq)\)-generic.

Proof. By our supposition, there is an infinite chain of minimal pairs \((X_i, X_{i+1})\) with \(X_i \neq X_{i+1}\). Let \(p\) be the type which asserts that a copy of \(\bigcup_{i<\omega} X_i\) exists in any realization of \(p\). Then \(p\) is finitely satisfied in the generic, so that some elementary extension of the generic realizes \(p\). \(\square\)

We will not give a general account of the existence of such structures in classes with \(\exists\)-resolutions. We do note that under full amalgamation such structures are obstructed by a countably infinite version of injectivity.

Theorem 6.3. Suppose \((K, \leq)\) has \(\exists\)-resolutions. Let us say that \(M\) is \((K, \leq)\) countably injective when for any countable structures \(C, D\), if \(C \leq D\) and \(f : C \hookrightarrow M\) is a strong embedding, then \(f\) extends to a strong embedding \(\hat{f} : D \hookrightarrow M\). Then if

1. \(M\) is countably injective; OR
2. \(M\) is a model of the theory of the \((K, \leq)\)-generic, and \((K, \leq)\) has finitary \(\exists\)-resolutions; OR
3. \(M\) is an \(\omega\)-saturated model of the \((K, \leq)\)-generic

then \(M\) is not a Moss structure.

Proof. For all cases let \(A \subseteq M\) and for \(X \subseteq A\), choose \(Y_X \subseteq M\) so that \((X, Y_X)\) is a bimiminal pair and \(Y_X\) is minimal in the \(\leq_X\) ordering. Let \(B := A \cup \bigcup_{X \subseteq A} Y_X\)

1. Let \(C\) be a countable resolution of \(A\) note that \(C \leq CB\) since \(C\) is closed. Thus by countable injectivity \(B\) embeds strongly into \(M\) over \(C\). Letting \(B'\) be the image of \(B\) under such an embedding, we have that \(B'\) is a finite resolution of \(A\) in \(M\).
2. Let \(h : B \hookrightarrow G\) be any embedding of \(B\) into the \((K, \leq)\)-generic and let \(C_h\) be any finite resolution of \(h(A)\). Note that \(C_h \leq C_b h(B)\) so that \(h\) extends to a strong embedding \(B \hookrightarrow G\). Thus every embedding of \(A\) into the generic which extends to a copy of \(B\) also extends to a closed copy of \(B\). Since the class is finitary, this is a first-order statement and thus holds in \(M\) as well.
3. The argument is similar to the previous one. Now, for \(n \in \omega\) let \(\sigma_n\) be a sentence which says of its models \(N\) that for any \(h : B \hookrightarrow N\), \(h\) can be extended to some \(\bar{h} : B \hookrightarrow N\) such that \(\bar{h}(B)\) is closed in every cardinality \(|B| + n\) extension of \(\hat{h}(B)\). As before, we must have \(M \models \sigma_n\) for all \(n\) (every \(h : B \hookrightarrow G\) gives rise to a strong \(\bar{h} : B \hookrightarrow G\) over \(h(A)\) so that \(G \models \sigma_n\), hence \(M\) does as well.) Thus there is a type \(p(\bar{x})\) which asserts that \(A\bar{x}\) is isomorphic to \(B\) and is closed. By saturation, \(p\) is realized in \(M\). \(\square\)

Let \((K_3, \leq_3)\) be the \(\exists\)-companion to \((K_p, \leq_p)\) from Example 4.14 in which \((x - 1) \leq_3 (x + 1)\) and let \(M\) consist of countably many infinite paths with order types \((\mathbb{Z}, <)\) separated by dense linear orders of vertices; then \(M\) will be a Moss structure. The only finite closed sets are sets of vertices of degree 0 so we have injectivity. On the other hand, no finite subset of one of the paths of order-type \((\mathbb{Z}, <)\) will have a finite closure. In fact, noting that an \(\omega\)-saturated model of the \((K_3, \leq_3)\)-generic
will have a path with order type \((\mathbb{Z}, <)\), this example essentially shows that a Moss structure exists which is elementarily equivalent to the generic. This illustrates the necessity of requiring finitary closures in the above theorem, since it is a quick check that \((K_\exists, \leq_\exists)\) has full amalgamation.

We conclude with a restatement and proof of the main theorem.

**Theorem 1.1.**

1. If \((K_\forall, \leq_\forall)\) is an amalgamation class with \('\forall'-closures, then a \((K_\forall, \leq_\forall)\) Moss structure exists exactly when the \((K_\forall, \leq_\forall)\)-generic is not \(\omega\)-saturated. Further, if a \((K_\forall, \leq_\forall)\) Moss structure exists, it can be taken to be a model of the theory of the \((K_\forall, \leq_\forall)\)-generic.

2. If \((K_\exists, \leq_\exists)\) is an amalgamation class with finitary \('\exists'-closures, then no \((K_\exists, \leq_\exists)\) Moss structure exists which is a model of the theory of the \((K_\exists, \leq_\exists)\)-generic.

**Proof.** For the first item, we need only note that \(\omega\)-saturation of the \((K, \leq)\)-generic corresponds to the class not having finite closures (this is proved in [3]) and cite Proposition 6.2. The second item is precisely Theorem 6.3.2.

\(\square\)

**References**

[1] John T Baldwin. Problems on pathological structures. In Proceedings of 10th Easter conference in model theory, Wendisches Rietz, pages 1–9. Citeseer, 1993.
[2] John T. Baldwin and Saharon Shelah. Randomness and semigenericity. *Trans. Amer. Math. Soc.*, 349(4):1359–1376, 1997.
[3] John T. Baldwin and Niandong Shi. Stable generic structures. *Ann. Pure Appl. Logic*, 79(1):1–35, 1996.
[4] Justin Brody. Full amalgamation classes with instrinsic transcendentals. arXiv:math.lo/1512.03888.
[5] D. W. Kueker and M. C. Laskowski. On generic structures. *Notre Dame J. Formal Logic*, 33(2):175–183, 1992.
[6] David Kueker. Homogeneous-universal graphs with respect to isometric maps.
[7] Michael C. Laskowski. A simpler axiomatization of the Shelah-Spencer almost sure theories. *Israel J. Math.*, 161:157–186, 2007.
[8] Lawrence S Moss. Distanced graphs. *Discrete mathematics*, 102(3):287–305, 1992.
[9] Frank O. Wagner, Relational structures and dimensions. In *Automorphisms of first-order structures*, Oxford Sci. Publ., pages 153–180. Oxford Univ. Press, New York, 1994.

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