Are Fractional Brownian Motions Predictable?

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Abstract. We provide a device, called the local predictor, which extends the idea of the predictable compensator. It is shown that a fBm with the Hurst index greater than 1/2 coincides with its local predictor while fBm with the Hurst index smaller than 1/2 does not admit any local predictor.

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1. Introduction

The question in the title is provocative, of course. Everybody familiar with the theory of stochastic processes knows that a continuous adapted process on the stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) is predictable, in the sense it is measurable with respect to the \(\sigma\)-algebra of predictable subsets of \(\Omega \times \mathbb{R}^+\). And fractional Brownian motions are continuous.

The point is that the predictability has a clear meaning in the discrete time, while in continuous time it looses its intuitive character. Brownian motion serves in many models as a source of unpredictable behavior, but it is predictable in the sense of the general theory of processes.

We are not going to suggest any change in the established terminology, although the old alternative of “well-measurable” sounds more reasonable. Our aim is to provide a device for verifying whether some fractional Brownian motions are “more predictable” than others.

2. The local predictor and its existence for fBms

We develop the idea of a predictable compensator in somewhat unusual direction. Let, as before, \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)\) be a stochastic basis, satisfying the “usual”
conditions, i.e., the filtration \(\{\mathcal{F}_t\}\) is right-continuous and \(\mathcal{F}_0\) contains all \(P\)-null sets of \(\mathcal{F}_T\). By convention, we set \(\mathcal{F}_\infty = \mathcal{F}\).

Let \(\{X_t\}_{t \in [0,T]}\) be a stochastic process on \((\Omega, \mathcal{F}, P)\), adapted to \(\{\mathcal{F}_t\}_{t \in [0,T]}\) (i.e., for each \(t \in [0,T]\), \(X_t\) is \(\mathcal{F}_t\) measurable) and with càdlàg (or regular) trajectories (i.e., its \(P\)-almost all trajectories are right-continuous and possess limits from the left on \((0,T]\)).

Suppose we are sampling the process \(\{X_t\}\) at points \(0 = t_0^\theta < t_1^\theta < t_2^\theta < \ldots < t_{k^\theta} = T\) of a partition \(\theta\) of the interval \([0,T]\). By the discretization of \(X\) on \(\theta\) we mean the process

\[
X^\theta(t) = X_{t_k^\theta} \quad \text{if} \quad t_k^\theta \leq t < t_{k+1}^\theta, \quad X_T^\theta = X_T.
\]

If random variables \(\{X_t^\theta\}_{t \in [0,T]}\) are integrable, we can associate with any discretization \(X^\theta\) its “predictable compensator”

\[
A_t^\theta = 0 \quad \text{if} \quad 0 \leq t < t_1^\theta,
\]

\[
A_t^\theta = \sum_{j=1}^k E\left(X_{t_j}^\theta - X_{t_{j-1}}^\theta \mid \mathcal{F}_{t_{j-1}}^\theta\right) \quad \text{if} \quad t_k^\theta \leq t < t_{k+1}^\theta, \quad k = 1, 2, \ldots, k^\theta - 1,
\]

\[
A_T^\theta = \sum_{j=1}^{k^\theta} E\left(X_{t_j}^\theta - X_{t_{j-1}}^\theta \mid \mathcal{F}_{t_{j-1}}^\theta\right).
\]

Notice that \(A_t^\theta\) is \(\mathcal{F}_{t_{k-1}}^\theta\)-measurable for \(t_k^\theta \leq t < t_{k+1}^\theta\), and so the processes \(A^\theta\) are predictable in a very intuitive manner, both in the discrete and in the continuous case. It is also clear, that the discrete-time process \(\{M_t^\theta\}_{t \in \theta}\) given by

\[
M_t^\theta = X_t^\theta - A_t, \quad t \in \theta,
\]

is a martingale with respect to the discrete filtration \(\{\mathcal{F}_t\}_{t \in \theta}\).

If we have square integrability of \(\{X_t\}_{t \in [0,T]}\), then the predictable compensator \(\{A_t^\theta\}_{t \in \theta}\) possesses also a clear variational interpretation. Fix \(\theta\) and let \(A^\theta\) be the set of discrete-time stochastic processes \(\{A_t\}_{t \in \theta}\) which are \(\mathcal{F}_t\)-predictable, i.e., for each \(t = t_k^\theta \in \theta\), \(A_{t_k}^\theta\) is \(\mathcal{F}_{t_{k-1}}^\theta\)-measurable. Then the predictable compensator \(\{A_t^\theta\}_{t \in \theta}\) minimizes the functional

\[
\mathcal{A}^\theta \ni A \mapsto E[X - A]_T,
\]

where the discrete quadratic variation \([\cdot]_T\) is defined as usual by

\[
[Y]_T = \sum_{t \in \theta} (\Delta Y_t)^2 = \sum_{k=1}^{k^\theta} (Y_{t_k}^\theta - Y_{t_{k-1}}^\theta)^2.
\]

Now consider a sequence \(\Theta = \{\theta_n\}\) of normally condensing partitions of \([0,T]\). This means we assume \(\theta_n \subset \theta_{n+1}\) and the mesh

\[
|\theta_n| = \max_{1 \leq k \leq k^\theta_n} t_k^\theta_n - t_{k-1}^\theta_n \to 0, \quad \text{as} \quad n \to \infty.
\]