Resistivity as a function of temperature for models with hot spots on the Fermi surface.

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We calculate the resistivity $\rho$ as a function of temperature $T$ for two models currently discussed in connection with high temperature superconductivity: nearly antiferromagnetic Fermi liquids and models with van Hove singularities on the Fermi surface. The resistivity is calculated semi-classically by making use of a Boltzmann equation which is formulated as a variational problem. For the model of nearly antiferromagnetic Fermi liquids we construct a better variational solution compared to the standard one and we find a new energy scale for the crossover to the low-temperature behavior at low temperatures. This energy scale is finite even when the spin-fluctuations are assumed to be critical. The effect of additional impurity scattering is discussed. For the model with van Hove singularities a standard ansatz for the Boltzmann equation is sufficient to show that although the quasiparticle lifetime is anomalously short, the resistivity $\rho \propto T^2 \ln(1/T)$.

I. INTRODUCTION.

One of the most challenging questions motivated by the anomalous normal state of the high temperature superconductors (HTc’s) is to explain the linearity of the resistivity $\rho$ as a function of temperature $T$ down to the superconducting transition temperature $T_c$. In the case of the one-layer Bi-based material, $T_c \approx 10K$. This temperature is well below the Debye temperature and, for a three-dimensional material, one should expect that $\rho \propto T^5$ if the resistivity is dominated by electron-phonon scattering. Moreover, electron-electron scattering (if treatable in perturbation theory) is known to yield $\rho \propto T^2$ for temperatures much less than the Fermi energy $E_F$. Thus, conventional theories do not explain the observed behavior of the resistivity and one has to search for other mechanisms which could lead to a higher scattering rate for the electrons.

Since superconductivity appears in the HTc’s close to a metal-insulator transition driven by strong correlations, it is widely believed that some aspect of electron-electron interactions should be responsible for the anomalous normal state and, finally, also for the high superconducting transition temperatures in the copper oxides. Another feature specific to the HTc’s is their layered structure and the consequent two-dimensional nature of the electronic states. The resulting problem of interacting electrons in two dimensions has not been solved yet and so far, only phenomenological theories attempting to describe the low-energy behavior of such models are available. In the literature, two fundamentally different routes for explaining the anomalies of the normal state of the HTc’s were followed. In one class of theories it is assumed that in the HTc’s, Landau’s Fermi-liquid theory breaks down completely and an exotic metallic state with some features of one-dimensional solutions is realized. In the other class of theories, it is assumed that Landau’s concept of quasiparticles does apply. However, in order to explain the deviations from the usual metallic behavior, anomalous scattering mechanisms are assumed and treated in perturbation theory. In the present paper, we will study the temperature dependence of the resistivity in two models of the latter type: nearly antiferromagnetic Fermi liquids and models with van Hove singularities.

In the theory of nearly antiferromagnetic Fermi liquids, it is assumed that the effect of the strong local repulsion between electrons can be described in the low-energy sector by coupling the electrons to an overdamped low-lying paramagnon mode. The parameters of the spin-fluctuation spectrum can be determined by fitting the magnetic properties of the HTc’s. Standard weak-coupling calculations of the resistivity in the model of nearly antiferromagnetic Fermi liquids give $\rho \propto T$ for $T > T^*$ where $T^*$ measures the deviation from the antiferromagnetic critical point. More sophisticated strong-coupling calculations of the resistivity support the weak-coupling results in that there are only quantitative changes to the latter. In Section 2, we elaborate the following observation: since the spin-fluctuation spectrum is soft at the Brillouin-zone boundary, only electrons in the vicinity of special points on the Fermi line are strongly scattered. Along the rest of the Fermi line, the lifetime of an electron is $1/\tau \propto T^2$ even for $T > T^*$. Thus the contribution of the strongly scattering special points to the resistivity is short-circuited by the remaining electrons and the resistivity has the standard Fermi-liquid form $\rho \propto T^2$ up to a new energy scale described in Section 2. A similar idea was exploited by Fujimoto, Kohno, and Yamada in their discussion of the $T$-dependence of the resistivity for models where parts of the Fermi surface exhibit perfect nesting.

Another interesting proposal to explain the anomalies of the HTc’s is to assume that these are caused by the presence of van Hove singularities at or close to the Fermi line. For instance, the resistivity is enhanced according to these theories due to the increase of phase space available in the scattering process. Moreover, assuming a weak-coupling BCS formula for the superconducting...
transition temperature, $T_c$ becomes enhanced due to the large density of states at the Fermi level. It is considered as a success of the van Hove scenario that the anomalies of the normal state are strongest at that doping where $T_c$ is maximal. In Section 3 we show, however, that although the single-particle lifetime is anomalous, the resistivity is consistent with the standard result of Landau’s Fermi-liquid theory with a logarithmic correction, $\rho \propto T^2 \ln(1/T)$.

II. NEARLY ANTIFERROMAGNETIC FERMI LIQUIDS.

Following Monthoux and Pines [3], we consider electrons moving in a square lattice of Wannier orbitals with a simple tight-binding spectrum

$$\varepsilon_k = -2t \left[ \cos(k_x a) + \cos(k_y a) \right] + 4t' \cos(k_x a) \cos(k_y a),$$

where $a$ is the lattice constant. $t = 0.25 eV$ and $t' = 0.45t$ are nearest- and next-nearest-neighbor hoppings, respectively. The electrons are coupled to the spin-fluctuation operator $S$. The Hamiltonian reads

$$H = \sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + \frac{g}{2} \sum_{k,q,\alpha,\beta} c_{k+q,\alpha}^\dagger c_{k,\beta} \sigma_{\alpha,\beta} \cdot S_{-q},$$

(2.2)

where $g$ is a coupling constant. The spectrum of the spin-fluctuations is $\chi_{i,j}(q,\omega) = \delta_{i,j} \chi(q,\omega)$ with

$$\chi(q,\omega) = \frac{A}{\omega_q - i\omega},$$

(2.3)

where $A \approx 1$ (in what follows we take $A = 1$), $\omega_q = T^* + \alpha T + \omega_D \psi_q$ and $\psi_q = 2 + \cos(q_x a) + \cos(q_y a)$. $T^*$, $\alpha$, and $\omega_D$ are temperature-independent parameters. Note that the energy of the spin-fluctuations is minimal for $q = Q = (\pi/a, \pi/a)$ and $\omega_Q = T^* + \alpha T$. Thus the parameter $T^*$ measures the critical antiferromagnetic point.

A. Quasiparticle lifetime.

Before calculating the resistivity due to scattering on the spin-fluctuations, let us analyze first the single-particle lifetime. In the second order of perturbation theory in $g$, the lifetime of an electron with momentum $k$ at zero temperature is

$$\frac{1}{\tau_k} = 2g^2 \sum_{k'} \int_0^{\xi_{k'}} d\omega \Im \chi(k' - k,\omega) \delta(\xi_k - \xi_{k'} - \omega),$$

(2.4)

where we have introduced $q^2 = 3g^2/4$; $\Im \chi$ denotes the imaginary part of $\chi$. We can write $\int d^2 k = \int dk_1 dk_2$ where $k_1$ and $k_2$ are the directions perpendicular and parallel to the Fermi line. Since $\int dk_2 = \int d\varepsilon/|v|$ where $v = \nabla_k \varepsilon$ is the group velocity we have

$$\frac{1}{\tau_k} = 2 \left( \frac{ga}{2\pi} \right)^2 \int_0^{\xi_k} d\omega \int \frac{dk'}{|v|} \Im \chi(k' - k,\omega),$$

(2.5)

where we write $k'$ instead of $k'$. In evaluating $\Im \chi(k' - k,\omega)$ we will assume that both $k$ and $k'$ lie on the Fermi line, since corrections lead only to subleading terms in the denominator of Eq.2.3. After integrating over $\omega$ we have

$$\frac{1}{\tau_k} = \left( \frac{ga}{2\pi} \right)^2 \int \frac{dk'}{|v|} \ln \left( \frac{\xi_k^2 + \omega_k^2}{\omega_{k-k'}^2} \right).$$

We assume that along the whole Fermi line the group velocity is finite (the opposite case will be treated in Section 3). It is seen that $1/\tau_k \propto \xi_k^2$ for $\xi_k \ll T^*$. Thus $T^*$ defines an energy scale below which Landau’s Fermi-liquid theory applies. Let us investigate now the behavior of the lifetime for energies $T^* \ll \xi_k \ll \omega_D$. Strong scattering occurs only for those states $k$ (hot spots) on the Fermi line for which

$$\varepsilon_k = \varepsilon_{k+Q}.$$

For a wide range of fillings, there are only 8 points with anomalous scattering (see Fig.1). Note that for a commensurate $Q = (\pi/a, \pi/a)$ the hot spots are by no means special points of the Fermi line, since the group velocity in a point $k$ satisfying Eq.2.3 is not parallel to $Q$. The commensurate case when $Q$ is a locally extremal vector connecting 2 points on the Fermi line (like the $2k_F$-processes in continuum) will not be studied here.

Let $\delta k$ be the distance between the projection of a point $k$ to the Fermi line and a nearby hot spot. The lifetime of an electron in the state $k$ is

$$\frac{1}{\tau_k} \propto g^2 E_F \sqrt{\xi_k} \min \left[ 1, \frac{1}{\varphi} \left( \frac{\xi_k}{\omega_D} \right)^3 \right],$$

where $\varphi = \delta k a$ and $E_F \sim \nu F/a$; $\nu F$ is the Fermi velocity in the hot spot. Thus, in agreement with our expectations, the lifetime in the neighborhood of a hot spot becomes anomalously large: $1/\tau_k \propto \sqrt{\xi_k}$. Away from a hot spot, $1/\tau_k \propto \xi_k^2$ as in standard Landau’s Fermi-liquid theory. It is interesting to note that if one calculates the average of $1/\tau_k$ for states of fixed energy $\varepsilon > T^*$ along the Fermi line, one obtains

$$\left\langle \frac{1}{\tau_k} \right\rangle \propto \frac{g^2 E_F}{\omega_D} \frac{\varepsilon}{T^*},$$

i.e., the average lifetime of an electron is linear in energy! It is basically this feature which has led in previous calculations to the result $\rho \propto T$ for $T > T^*$. 

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B. Resistivity.

In calculating the temperature dependence of the resistivity due to spin-fluctuations, we will assume that a simple description in terms of a Boltzmann equation (BE) captures the essential physics. Let the stationary solution of the BE be \( \Phi_k = f_k - \Phi_k(\partial f_k^0/\partial \varepsilon) \), where \( f_k^0 \) is the equilibrium Fermi-Dirac distribution function and \( \Phi_k \) is a function to be determined. The linearized (in \( \Phi_k \)) collision term of the BE reads

\[
\left( \frac{\partial f_k}{\partial t} \right)_{\text{coll}} = \frac{2g^2}{T} \sum_{k'} \int_{-\infty}^{\infty} d\varepsilon \, n(\varepsilon) \, f_k^0(1 - f_k^0) \times \Im(\chi(k - k', \varepsilon)(\Phi_{k'} - \Phi_k) \delta(\varepsilon_k - \varepsilon_{k'} - \varepsilon_k)),
\]

where \( n(\varepsilon) \) is the Bose-Einstein distribution function. Note that we have assumed that the spin-fluctuations are in equilibrium. Following the standard arguments as described by Ziman [3], the resistivity \( \rho \) can be found as the minimum of a functional of \( \Phi_k \):

\[
\frac{\rho}{\rho_0} = \min \left[ \frac{\langle \Phi | W | \Phi \rangle}{\langle \Phi | \Phi \rangle} \right],
\]

(2.7)

where we have introduced \( \langle \Phi | W | \Phi \rangle = \sum_{k,k'} W_{k,k'} (\Phi_{k'} - \Phi_k)^2 \), \( \langle \Phi | \Phi \rangle = \sum_k \Phi_k \chi_k \) and \( \chi_k = (-\partial f_k^0/\partial \varepsilon) \frac{\varepsilon_k}{\hbar c^2} \). \( \rho_0 = h/e^2 \) is the quantum of resistivity and \( \mathbf{n} \) is a unit vector in the direction of the applied electric field. Assuming that \( \varepsilon_k = \varepsilon_{-k} \) and \( \Phi_k = -\Phi_{-k} \), we have

\[
W_{k,k'} = \frac{2(ga)^2}{T} f_k^0(1 - f_k^0) n(\varepsilon_k - \varepsilon_k) \Im(\chi(k' - k, \varepsilon_k - \varepsilon_k)).
\]

Similarly as in the discussion of the quasiparticle lifetime, we write \( \int d^2k = \int d\varepsilon \int d\varepsilon_k / v \) where the \( k \)-integration runs along the Fermi surface, and an analogous expression for \( \int d^2k' \). Defining \( \varepsilon' = \varepsilon + \omega \), we can perform the \( \varepsilon \)-integration and obtain

\[
\frac{\rho}{\rho_0} = \min \left[ \frac{\oint \frac{dE}{d\varepsilon} F_{k-k'} (\Phi_{k'} - \Phi_k)^2}{\langle \Phi_k | \varepsilon_k - \varepsilon_k | \Phi_k \rangle} \right],
\]

(2.8)

\[
F_{k-k'} = \frac{(ga)^2}{T} \int_{-\infty}^{\infty} d\varepsilon \, n(\varepsilon) [n(\varepsilon) + 1] \Im(\chi(k - k', \varepsilon)).
\]

Note that Eq.2.8 is a generalization of Eq.3.1 from Ref. [3] to the case of an arbitrary variational function \( \Phi_k \). Using Eq.2.3 for the spectrum of spin-fluctuations, we find \( F_{k-k'} = 2(ga)^2 I(\omega_{k-k'} / T) \), where

\[
I(x) = \int_0^{\infty} dt \, t^2 \, e^{-t^2 / x} \approx \frac{\pi^2 / \beta}{x(x + 2\pi/\beta)}.
\]

The last equality is an interpolation formula which becomes exact for \( x \to 0 \) and \( x \to \infty \). Summarizing, the sheet resistivity can be written in the following dimensionless form:

\[
\frac{\rho}{\rho_0} = \frac{\pi^2}{6} \left( \frac{g}{e^2} \right)^2 \Theta^3 \times \min \left[ \frac{\oint \frac{dE}{d\varepsilon} \frac{d\varepsilon'}{d\varepsilon} (\Phi_{k'} - \Phi_k)^2}{\langle \Phi_k | \varepsilon_k - \varepsilon_k | \Phi_k \rangle} \right]^{-\beta},
\]

(2.9)

where \( \mathbf{u}_k = \frac{v_k}{2T} \mathbf{n} \) is a dimensionless group velocity, \( \Theta = T/\omega_D \), \( \Theta^* = T^*/\omega_D \), and \( \beta = \alpha + 2\pi/3 \). The integrations run along the Fermi line. Note that for \( T \ll T^*/\beta \), the resistivity has a standard Landau-Fermi-liquid form \( \rho \propto T^2 \) in agreement with our results for the quasiparticle lifetime. In what follows, we will study the resistivity as given by Eq.2.9 for \( T^*/\alpha \ll T \ll \omega_D \). A standard ansatz for the variational function is

\[
\Phi_k = \mathbf{u}_k \cdot \mathbf{n}.
\]

(2.10)

For such \( \Phi_k \), there always exists a pair of hot spots \( k, k' \) such that \( \Phi_{k'} - \Phi_k \) is finite. In that case, the integral in the nominator of Eq.2.9 is dominated by \( k, k' \) close to the hot spots; \( \psi_{k-k'} \) can be approximated by a homogeneous quadratic polynomial in the deviations \( \delta k, \delta k' \) from the hot spots and by scaling, one obtains \( \rho \propto T \) in agreement with Moriya et al. [6].

We have seen that the quasiparticle lifetime is extremely anisotropic along the Fermi line. It is therefore natural to assume that the resistivity will be dominated by that part of the Fermi line where the scattering is weakest and the contribution from the hot spots will be short-circuited. To prove this, we take another ansatz for \( \Phi_k \) and show that it leads to a lower resistivity. Let us consider

\[
\Phi_k = \frac{\mathbf{u}_k \cdot \mathbf{n}}{e^{\beta |\Delta - \varphi| + 1}}.
\]

(2.11)

where \( \varphi = \delta k \mathbf{a} \) is the deviation from a hot spot and \( \Delta \) and \( \beta \) are variational parameters. The case \( \beta = 0 \) corresponds to the standard ansatz, while \( \beta \gg 1 \) and \( \Delta \neq 0 \) describe the situation when finite parts of the Fermi line around the hot spots do not contribute to the transport. If \( \beta \to \infty \) and \( \sqrt{T/\omega_D} \ll \Delta \ll 1 \), we obtain at low temperatures \( \rho \propto T^2 \) even for \( T^* = 0 \), since \( \langle \Phi | W | \Phi \rangle \) becomes temperature-independent. With increasing temperature, one is forced to choose larger \( \Delta \) in order to exclude the hot-spot regions. This leads, however, to a decrease of \( \langle \Phi | X \rangle \) and finally at high enough temperatures, the solution Eq.2.11 with a large \( \beta \) becomes unfavourable compared to the standard ansatz Eq.2.10. This happens if \( T/\omega_D > c \), where \( c \) is a numerical factor which depends on the details of the geometry of the Fermi line and of the hot spots. We were unable to make a reliable estimate of \( c \) analytically and therefore we calculated \( \rho \) as a function of \( T \) numerically.

In Fig.2, we show the results of a numerical calculation of the resistivity according to Eq.2.9 using both the standard and improved ansatz for \( \Phi_k \). For the spin-fluctuation spectrum we take \( T^* = 0, \alpha = 2.0 \), and \( \omega_D = 1760K \). The density of electrons is \( n = 0.75 \) and
the coupling constant $g = 0.64eV$. Note that with the standard ansatz Eq.2.10, we obtain for this critical system $\rho \propto T$ down to $T = 0$ in agreement with previous studies \cite{1} (see also \cite{2}). With the improved ansatz Eq.2.11, the resistivity is lower for all studied temperatures and it is proportional to $T^2$ up to $T \approx 70K$. For $T > 70K$, $\rho$ is a linear function of $T$ with a similar slope as for the standard ansatz. However, the extrapolation of the linear part down to $T = 0$ is negative.

Finally, let us consider the spin-fluctuation spectrum with $T^* = 110K$, $\alpha = 0.55$, and $\omega_p = 1760K$. We take again $n = 0.75$ and $g = 0.64eV$. These are the same parameters as those used in Ref. \cite{3} (see their Eq.37 and note that $\omega_D = 2\omega_{SF}(\xi/a)^2$; $\omega_{SF}$ and $\xi$ are the parameters for the spin-fluctuation spectrum used in Ref. \cite{3}).

The results of our numerical calculation are shown in Fig.3. The standard ansatz Eq.2.10 yields a resistivity-temperature curve qualitatively similar to that obtained by Monthoux and Pines \cite{3}, but our resistivity is approximately three times larger \cite{10}. $\rho$ is a linear function of $T$ down to $T \approx 100K$. Our improved solution Eq.2.11 yields smaller values of resistivity: for instance, $\rho(T_z)$ calculated using our ansatz is only $\approx 0.6$ of the value obtained with the standard ansatz. More importantly, the shape of the $\rho(T)$ curve changes: it is linear only above $T \approx 180K$. We believe that the latter feature will hold true also in a more sophisticated calculation than in our Boltzmann-equation approach. In order to proceed further in this direction it will be necessary to find a translation of the variational principle used here to the Green’s-function formulation of transport problems.

C. Influence of impurity scattering on the resistivity.

In the presence of impurities, one can expect that the anisotropy of the quasiparticle lifetime will be suppressed. Thus, a question arises what is the actual temperature dependence of the resistivity in such a case \cite{1}. We will address this question by assuming that the impurity scattering can be described by the Boltzmann equation (thus disregarding all fully quantum-mechanical effects like weak localization, etc.) In that case, the resistivity can be described by Eq.2.8 where $F_{k-k'}$ acquires an additional contribution $F_{k-k'}^{imp}$ from impurity scattering.

In the Born approximation, $F_{k-k'}^{imp} = \pi a^2 |H_{k-k'}'|^2$, where $H'$ describes the interaction of an electron with impurities. Since we are not interested here in a microscopic calculation of the resistivity due to impurities $\rho_{imp}$, we take $|H_{k,k'}| = V$ where $V$ is a free parameter to be chosen so as to give a realistic $\rho_{imp}$. Under these assumptions, we have

$$\rho_{imp} = \frac{\pi}{2} \frac{V^2}{T} \frac{\langle \Phi_k \rangle^2}{\langle \Phi_k u_k \cdot n \rangle^2},$$

where $\langle A \rangle = \langle \frac{\partial}{\partial t} A_k / \frac{\partial}{\partial t} A \rangle_{\tau}$ is an average of $A$ along the Fermi surface. It is easy to see that $\rho_{imp}$ is minimized by the standard ansatz Eq.2.10. Since the resistivity due to impurities is finite down to $T = 0$ while the contribution of spin-fluctuations vanishes in that limit, it is clear that the standard ansatz will be favourable for $T \to 0$. At higher temperatures, however, the decrease of the spin-fluctuation contribution to the resistivity for $\Phi_k$ given by Eq.2.11 may outweigh the increase of the contribution due to impurities. In order to test this possibility, we performed a calculation of the resistivity with the same parameters as those used in Fig.3; we assumed a residual resistivity $\rho_{imp}(T = 0) = 0.25\rho_0$. The result of this calculation is shown in Fig.4. It is seen that the presence of impurity scattering decreases the difference between the resistivity as calculated by the standard ansatz Eq.2.10 and our variational function Eq.2.11. However, even for the relatively large impurity scattering we have chosen, the resistivity is still not a linear function of temperature for $T > 100K$.

Summarizing the results of the present Section we can say that although the quasiparticle lifetime is anomalous around special points on the Fermi line (hot spots), the resistivity is proportional to $T^2$ at low enough temperatures. The energy scale where a crossover to $\rho \propto T$ occurs is determined not only by the parameter $T^*$ as found in previous studies, but also by some fraction of $\omega_D$.

III. MODELS WITH VAN HOVE SINGULARITIES ON THE FERMI LINE.

In the literature there appeared a number of attempts to explain the anomalies of the normal state of the HTc’s in the framework of a weak-coupling theory under the assumption of a special single-particle dispersion. The most promising among these are the models which assume that at the optimal doping, there is a van Hove singularity on the Fermi line. Such singularities always exist in a periodic energy band for topological reasons (see, e.g., Ref. \cite{12}). The special feature of the HTc’s according to these theories simply is that a crossing of the Fermi line with a van Hove singularity can occur. In the present Section, we will calculate the quasiparticle lifetime and the resistivity in the van Hove scenario. We consider electrons with the Hamiltonian

$$H = \sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + g \sum_i n_{i,\uparrow} n_{i,\downarrow}, \quad (3.1)$$

where $\varepsilon_k$ is the single-particle dispersion Eq.2.1 and $g$ is a weak screened interaction among the electrons. It is assumed that $t > 2t' > 0$. In that case, there exist two saddle-points $(\pi/a,0)$ and $(0,\pi/a)$ which lead to a van Hove singularity at energy $-4t'$. The Fermi line for a filling when it goes through a van Hove singularity is shown in Fig.5. The single-particle dispersion around the saddle points becomes anomalous: e.g., in the neighborhood of $(\pi/a,0)$, we have
\[ \epsilon_k \approx k_y^2/2m_y - (k_x - \pi/a)^2/2m_x, \quad (3.2) \]

where \( 1/m_x = (t - 2\ell')a^2 \) and \( 1/m_y = (t + 2\ell')a^2 \).

A. Quasiparticle lifetime.

Before considering the temperature dependence of the resistivity in the van Hove scenario, let us first calculate the quasiparticle lifetime. We rederive here the results of Gopalan et al.\,[13] in a simpler way which will enable us to calculate the resistivity in the next subsection. The quasiparticle lifetime is given by Eq.2.4 where \( \Im(\chi(q, \omega)) = \pi \sum \gamma_k (1 - f_k) \delta(\epsilon_k - \epsilon_q - \omega) \) is the imaginary part of the bare susceptibility. In what follows, we study the lifetime of an electron \( k \) which is scattered to \( k' \) by exciting a particle-hole pair \( K \rightarrow K' \).

Let us first consider the case when all involved states \( k, k', K, \) and \( K' \) are in the neighborhood of the saddle points. There are two types of such scatterings: intra- and inter-saddlepoint scatterings (with or without umklapp), respectively. If both \( K \) and \( K' \) lie close to the same saddle point, we can use the expression for the susceptibility found by Gopalan et al.\,[13] (for \( T = 0 \)):

\[ \Im(\chi(q, \omega)) \propto \min \left( 1, \frac{\omega}{|\epsilon_q|} \right), \quad (3.3) \]

where \( \epsilon_q \) is given by Eq.3.2. To simplify the analysis of the susceptibility in the case when \( K \) and \( K' \) are close to different saddle points, let us assume for the moment that the single-particle spectrum is

\[ \epsilon_k = \frac{1}{2m} \left( |k_x| - \frac{G}{2} \right) k_y, \quad (3.4) \]

The Fermi line for electrons with the dispersion Eq.3.4 is shown in Fig.6. The spectrum consists of two saddle points whose distance is \( G = (G, 0) \), where \( 2G \) is assumed to be a vector of the inverse lattice. The susceptibility at a wavevector \( G - q \) where \( q \) is small then is

\[ \Im(\chi(G - q, \omega)) \propto |\ln \frac{2\omega}{|\epsilon_q|} - \text{sgn}(\epsilon_q)|, \]

where \( \epsilon_q = q_x q_y / 2m \). The lifetime of the electron in the state \( k \) now reads

\[ \frac{1}{\tau_k} = 2g^2 \sum_{k'} \Im(\chi(k - k', \epsilon_k - \epsilon_{k'})), \]

where the prime on the sum means a restriction to those states \( k' \) which satisfy \( 0 < \epsilon_{k'} < \epsilon_k \). One finds from here that at zero temperature \( 1/\tau_k \propto \epsilon_k \) in agreement with the result of Gopalan et al.\,[13].

Let us calculate now the lifetime of an electron \( k \) away from the saddle points when two of the states \( k', K, \) and \( K' \) are close to a saddle point. We make use of Eq.2.3. Let us consider first the contribution of \( k' \) close to \( k \) (forward-scattering channel) for which the expression Eq.2.3 is valid. Let \( |k - k'| = q \). Since none of the asymptotes of the hyperbolae Eq.3.2 is parallel to the Fermi line (away from the saddle points), we have

\[ \frac{1}{\tau_k} \propto \int_0^{\epsilon_k} d\omega \int_0^\Lambda dq \min \left[ 1, |M\omega/q^2| \right], \]

where \( M \) is a constant and \( \Lambda \) a cut-off in momentum space. Taking the integral, we have \( 1/\tau_k \propto \epsilon_k^{3/2} \) in agreement with Ref.\,[13]. Let us consider now \( k' \) close to a saddle point. If we require that one of the points \( K \) and \( K' \) is close to a saddle point, we find that either \( K \approx k' \) and \( K' \approx k \) (exchange channel) or \( K \approx -k \) and \( K' \approx -k' \) (Cooper channel). The contribution to the lifetime of the exchange channel is analogous to that of the forward-scattering channel. In order to calculate the contribution of the Cooper channel we need to calculate the susceptibility \( \Im(\chi(K - K', \omega)) \) where \( K \) and \( K' \) are momenta away and close to a saddle point, respectively. Let \( P \) and \( Q \) be points on the Fermi line close to \( K \) and \( K' \), respectively such that \( Q - P = K' - K \). Let the spectrum in the vicinity of the saddle point and of \( P \) be \( \epsilon_k = k_x k_y / 2m \) and \( \epsilon_k = v \cdot (k - P) \), respectively and let \( Q = (Q, 0) \). Then the susceptibility is

\[ \Im(\chi(K' - K, \omega)) \approx \frac{1}{(2\pi)^2 v \cos \phi} \left( \sqrt{Q^2 + \frac{8m\omega}{\tan \phi} - Q} \right), \quad (3.5) \]

where \( \phi \) is the angle between the tangents to the Fermi line in the points \( P \) and \( Q \). Now we can calculate the lifetime in the Cooper channel according to Eq.2.4 by first integrating over \( \omega \) and introducing hyperbolic coordinates \( k' = (\epsilon', \phi') \) such that \( k'_x \propto \sqrt{\epsilon'/\tan \phi'} \) and \( k'_y \propto \sqrt{\epsilon'\tan \phi'} \). \( Q \) is dominated by the position of \( k' \) and we find \( Q \propto \sqrt{\epsilon'/\tan \phi'} \tan \phi' \). The resulting integral can be performed by scaling and we find \( 1/\tau_k \propto \epsilon_k^{3/2} \).

Finally, in case when \( k \) is away from saddle points and at most one of the points \( k', K \), and \( K' \) is close to a saddle point we obtain a lifetime analogous to the result for an isotropic spectrum \,[14]. As an example, let us consider the case when it is the momentum \( K' \) which is close to a saddle point. We calculate the lifetime according to Eq.2.3 where the susceptibility is given by Eq.3.3. Let \( k'_0 \) be that value of \( k' \) for which \( Q = 0 \). For a general \( k' \) we have \( Q = \alpha q \) where \( q = |k' - k'_0| \) and \( \alpha \) is a constant. The lifetime is

\[ \frac{1}{\tau_k} \propto \int_0^{\epsilon_k} d\omega \int_0^\Lambda dq \left( \sqrt{q^2 + \frac{8m\omega}{\alpha^2 \tan \phi} - q} \right). \]

where \( \Lambda \) is a cut-off in momentum space. The integration is straightforward and we find \( 1/\tau_k \propto \epsilon_k^2 \text{ln}(1/\epsilon_k) \). Thus the contribution to \( 1/\tau_k \) of the processes with one
of the scattering states close to a saddle point is subleading compared to the processes in the forward-scattering, exchange, and Cooper channels.

Summarizing, we have found that due to the presence of van Hove singularities on the Fermi line, the scattering rate is anomalously enhanced compared to the isotropic case; for electrons close to a saddle point $1/\tau \propto \varepsilon$, whereas for the remaining electrons $1/\tau \propto \varepsilon^{3/2}$.

B. Resistivity.

Let us calculate the resistivity in the van Hove scenario. We will work again in the quasiclassical formalism of the Boltzmann equation. Analogously to the discussion in Section 2, the resistivity can be found \[ \rho \propto \frac{1}{T} \sum_k \frac{f_k}{\tau_k^{TR}} \sim \int d\varepsilon \frac{1}{\tau_k^{TR}(k,T)}, \]
\[ \text{i.e., the resistivity can be found as an average over the Fermi line of the transport scattering rate at energy } \sim T. \]

For the transport lifetime we find an expression similar to that for the quasiparticle lifetime:
\[ \frac{1}{\tau_k^{TR}} \sim \int_0^{\varepsilon_k} d\omega \sum_{k'} \frac{f_{k'}}{\tau_k^{TR}(k',\omega)} \times \delta(\varepsilon_k - \varepsilon_{k'} - \omega), \]
\[ \sum\chi^{TR}(\omega;u) = \pi \sum_k f_k (1 - f_{k+q})(\Phi_k - \Phi_{k+q}), \]
\[ \times \delta(\varepsilon_k - \varepsilon_{k+q} - \omega), \]
where we have defined the ‘transport susceptibility’ $\chi^{TR}(\omega;u)$. Let us study first the transport lifetime for the processes when all electron states involved in the scattering are close to the saddle points. Note that assuming intra-saddle-point scatterings and a dispersion $\varepsilon_q = q^2 q_y/2m$, the conservation of momentum implies $\Phi_k + \Phi_K - \Phi_{k'} - \Phi_{K'} = 0$ similarly as in the case of an isotropic dispersion and the resistivity vanishes \[ [3]. \]

Thus we have to take into account inter-saddle-point scatterings. Unfortunately, the transport lifetime for the model dispersion Eq.\[1.4\] is different from the actual result for the spectrum Eq.\[2.1\] since the asymptotes of the hyperbolas of the two saddle points in the latter spectrum are not parallel to each other and we have to calculate more carefully. Let us assume without loss of generality that $k'$ lies in the vicinity of the point $(\pi/a,0)$ where the dispersion is described by Eq.\[3.2\]. The energy conservation together with the Pauli principle require $0 < \varepsilon_{k'} < \varepsilon_k$ and the allowed $k'$-points lie between two branches of a hyperbola centered at $(\pi/a,0)$. The dominant contribution to $1/\tau_k^{TR}$ comes from the $k'$-points in the tails of the hyperbolas, where the transport susceptibility $\chi^{TR}(k - k', \varepsilon_k - \varepsilon_{k'}) \approx \varepsilon_k$. Thus the transport lifetime reads
\[ \frac{1}{\tau_k^{TR}} \propto \varepsilon_k \int d\omega \int_0^{\Lambda} dqq^2 \min[1, |M\omega/q^2|] \approx \varepsilon_k^2. \]

Scattering in the exchange channel leads to a similar result. In the Cooper channel the relevant transport susceptibility has an additional factor $(\Phi_k + \Phi_K - \Phi_{k'} + \Phi_{K'})^2 \propto \varepsilon_k$ compared to Eq.\[3.2\] and hence $1/\tau_k^{TR} \propto \varepsilon_k^{5/2}$.

Finally, if only one of the states $k, k', K, K'$ is close to a saddle point there is in general no additional small factor distinguishing $1/\tau$ from $1/\tau^{TR}$ and thus the resistivity is $\rho \propto T^2 \ln(1/T)$. Note that although processes of this type give only a subdominant contribution to the quasiparticle lifetime, they provide a leading contribution to relaxation of momentum.

Summarizing, in Section 2 we calculated the quasiparticle lifetime and the resistivity in the van Hove scenario. We found that although the quasiparticle lifetime is anomalously short the resistivity exhibits the standard temperature dependence with a logarithmic correction $\rho \propto T^2 \ln(1/T)$ for $T \ll E_F$.

IV. CONCLUSIONS.

In this paper we have analyzed the resistivity as a function of temperature for two two-dimensional models with hot spots: nearly antiferromagnetic Fermi liquids and a model with van Hove singularities on the Fermi line.
To simplify the treatment, we decided to formulate the transport problem on the level of a Boltzmann equation.

In the case of nearly antiferromagnetic Fermi liquids, we have shown that the standard treatment which does not take into account the anisotropy of the electron lifetime along the Fermi line leads to $\rho \propto T$ for $T > T^*$. However, we constructed better variational solutions of the BE which exclude highly resistive points on the Fermi line and yield $\rho \propto T^2$ even above $T^*$. We have found a new energy scale for the crossover to the $\rho \propto T$ behavior at higher temperatures. This energy scale does not vanish even if the spectrum of the spin-fluctuations is critical. The presence of disorder was shown to decrease the difference between the standard solution and our ansatz; however, even for relatively strong disorder, the resistivity is not a linear function of temperature above 100K, if we use the parameters proposed by Monthoux and Pines [3].

More generally, our analysis suggests that if the electrons couple to a bosonic excitation which is soft at some long wavelengths. For example, consider electrons with the spectrum $\varepsilon = k^2/2m$ coupled to bosons described by the propagator

$$\chi(q, \omega) \propto \frac{q^\beta}{\omega_q - i\omega} \quad \text{or} \quad \chi(q, \omega) \propto \frac{q^\beta \omega_q}{\omega^2_q - \omega^2},$$

where $\omega_q = T^* + Aq^\alpha$; we take $\alpha \geq 1$, $\beta \geq 0$. A golden-rule calculation of the quasiparticle lifetime and resistivity for $T > T^*$ gives $1/\tau \propto T^{(D+\beta-1)/\alpha}$ and $\rho \propto T^{(D+\beta+1)/\alpha}$, respectively. $D$ is the spatial dimension (we assume that $D$ is the same for both electrons and bosons). E.g., for spinors of spinon on the gauge field we have $D = 2$, $\alpha = 3$, $\beta = 1$ and we obtain $1/\tau \propto T^{2/3}$ and $\rho \propto T^{4/3}$ in agreement with Ref. [2]. For $T \ll T^*$, we have $1/\tau \propto T^2$. Thus the concept of quasiparticles is valid and our use of a semiclassical approximation is supported. It is interesting to note that for $T^* = 0$, $\rho \propto T$ would require $1/\tau \propto T^\nu$ with $\nu < 1$; coupling the electrons to a bosonic excitation and requiring that Landau’s Fermi-liquid theory is applicable would lead to $\rho \propto T^\mu$ where $\mu > (2 + \alpha)/\alpha$.

In the second part of this paper, we considered a model with van Hove singularities close to the Fermi line and with weak screened electron-electron interactions. If the van Hove singularity is located at energy $E_F \pm T^*$, then the anomalous quasiparticle lifetime reported in Section 3 is valid for $\varepsilon \gg T^*$. At energies smaller than $T^*$ we obtain the standard results for interacting electrons in two dimensions [4] and the concept of quasiparticles should be applicable. Thus the existence of a nonvanishing $T^*$ provides, similarly as in the case of nearly antiferromagnetic Fermi liquids, support for our use of the semiclassical approach. At $\varepsilon \gg T^*$, small-angle scatterings or Cooper-channel processes in which electrons close to saddle points take part are responsible for the anomalous behavior of the quasiparticle lifetime. However, using the standard ansatz $\Phi_k = v_k \cdot n$ for the variational solution of the Boltzmann equation their contribution to the transport lifetime becomes regularized by the appearance of an additional small factor $(\Phi_k + \Phi_k - \Phi_k - \Phi_k)^2$ in Eq.3.4 and we obtain finally $\rho \propto T^2 \ln(1/T)$. Such a reduction of the transport scattering rate compared to the quasiparticle scattering is in fact quite common as can be seen, e.g., from our results for electrons interacting with a bosonic mode. Another example is a one-dimensional system away from half filling: the quasiparticle lifetime behaves as $1/\tau \propto T$, while the resistivity is exponentially small [6]. A similar tendency holds for electrons with the usual isotropic dispersion in two dimensions: Hodges et al. [4] found that $1/\tau \propto T^2 \ln(1/T)$, whereas it can be shown [8] that the resistivity $\rho \propto T^2$.

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in a straightforward way. Another difference is that we have used the bare electron dispersion in Eq. 2.9. It is seen from the Tables 1 and 2 in Ref. 1 that including the self-energy correction in the quasiparticle dispersion leads to a decrease of the density of states at the Fermi level and, consequently, to a decrease of resistivity Eq. 2.9.

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Fig. 1. Solid line: the Fermi line for the spectrum Eq. 2.1. We take $t' = 0.45t$ and $n = 0.75$. The hot spots satisfying the condition Eq. 2.6 are located in the cross-section points of the Fermi line with the dashed square.

Fig. 2. Resistivity due to spin-fluctuations as a function of temperature for the spin-fluctuation parameters $T^* = 0$, $\alpha = 2.0$, $\omega_D = 1760K$, the electron parameters $t = 0.25eV$, $t' = 0.45t$, $n = 0.75$, and the coupling constant $g = 0.64eV$. Solid line: calculation with the improved ansatz Eq. 2.11. Dashed line: calculation with the standard ansatz Eq. 2.10.

Fig. 3. Resistivity due to spin-fluctuations as a function of temperature for the spin-fluctuation parameters $T^* = 110K$, $\alpha = 0.55$, $\omega_D = 1760K$, the electron parameters $t = 0.25eV$, $t' = 0.45t$, $n = 0.75$, and the coupling constant $g = 0.64eV$. Solid line: calculation with the improved ansatz Eq. 2.11. Dashed line: calculation with the standard ansatz Eq. 2.10.

Fig. 4. Resistivity due to spin-fluctuations as a function of temperature for the same parameters as in Fig. 3. Additional impurity scattering $\rho_{imp}(T = 0) = 0.25\rho_0$ was assumed. Solid line: calculation with the improved ansatz Eq. 2.11. Dashed line: calculation with the standard ansatz Eq. 2.10.

Fig. 5. Fermi line for the spectrum Eq. 2.1. We take $t' = 0.28t$ and $n = 0.75$. The van Hove singularities are located in the points $(\pi/a, 0)$ and $(0, \pi/a)$.