I. INTRODUCTION

The chimera state, a novel collective phenomenon observed in coupled oscillator systems, that spontaneously emerges as a spatio-temporal pattern of co-existing synchronous and asynchronous groups of oscillators, has attracted a great deal of attention in recent years [1, 2]. First observed and identified by Battogtokh and Kuramoto [3] for a model system of identical phase oscillators that are non-locally coupled, such states have now been shown to occur in a wide variety of systems [2] and under less restrictive conditions than previously thought of [4, 5]. Theoretical and numerical studies have established the existence of chimera states in neuronal models [6], in systems with non-identical oscillators [7, 8], time delay coupled systems [9] and globally coupled systems that retain the amplitude dynamics of the oscillators [4, 5]. Chimera states have also been observed experimentally in chemical [10, 11], optical [12], mechanical [13], electronic [14] and electro-chemical [15] oscillator systems. Furthermore, chimera states have been associated with some natural phenomena such as unihemispheric sleep [16] in certain birds and mammals during which one half of the brain is synchronized while the other half is in a de-synchronized state [17]. In fact, the electrical activity of the brain resulting in the collective dynamics of the cortical neurons provides a rich canvas for the application of chimera states and is the subject of many present studies [18–20]. It is believed that a prominent feature of the dynamics of the brain is the generation of a multitude of meta-stable chimera states that it keeps switching between. Such a process, presumably, is at the heart of our ability to respond to different stimuli and to much of our learning behavior. Chimera states are thus vital to the functioning of the brain and it is important therefore to investigate their existence conditions and robustness to changes in the system environment. An interesting question to ask is how the formation of chimeras can be influenced by the loss of functionality of some of the constituent components of the system. In the case of the brain it could be the damage suffered by some of the neuronal components due to aging or disease [21]. For a model system of coupled oscillators this can be the loss of oscillatory behavior of some of the oscillators. In this paper we address this question by studying the existence and characteristics of the amplitude mediated chimera (AMC) states in an ensemble of globally coupled Ginzburg-Landau oscillators some of which are in a non-oscillatory (inactive) state. The coupled set of Ginzburg-Landau oscillators display a much richer dynamics than the coupled phase oscillator systems on which many past studies on chimeras have been carried out. The conditions governing the emergence of the AMCs are also free of the topological and coupling constraints of the classical phase oscillator chimeras and may therefore have wider practical applications. Past studies on the robustness of the collective states of a population of coupled oscillators that are a mix of oscillatory and non-oscillatory (inactive) elements have been restricted to the synchronous state [22, 23]. Our work extends such an analysis to the emergent dynamics of the amplitude mediated chimera state. We find that the presence of inactive elements in the system can significantly impact the existence domain and the nature of the AMCs. The existence domain of the AMC states is found to shrink and shift in parametric space as the fraction of inactive oscillators is increased in the system. Under the influence of the coupling the inactive oscillators experience a revival and turn oscillatory to form another separate coherent region. They also inject a sort of ‘inertia’ in the system that decreases the collective frequency of the coherent regions. These results are established through extensive numerical simulations of the coupled system and by a systematic comparison with past results obtained in the absence of the inactive oscillators. To provide a deeper understanding of the numerical results and to trace the dynamical origins of these changes we also present a detailed bifurcation analysis of a reduced model system that has the form of a single driven oscillator equation with a forcing term obtained from a mean field approximation.

The paper is organized as follows. In the next section
(section II) we describe the model equations and summarize the past results on the AMC states obtained from these equations in the absence of any inactive oscillators. Section III details our numerical simulation results obtained for various fractions of inactive oscillators and discusses the consequent modifications in the existence regions and characteristics of the AMC states. In section IV we provide some analytic results based on a mean field theory to explain the findings of section III. Section V gives a brief summary and some concluding remarks on our results.

II. MODEL EQUATIONS

We consider a large population of globally coupled identical complex Ginzburg-Landau type oscillators whose time evolution is governed by the following set of equations

\[
\dot{W}_j = \alpha_j W_j - (1 + iC_2)|W_j|^2 W_j + K(1 + iC_1)(\bar{W} - W_j) \tag{1}
\]

\[
\bar{W} = \frac{1}{N} \sum_{i=1}^{N} W_i \tag{2}
\]

where, \( j = 1, \ldots, N, \) \( N \) being the number of oscillators and \( \bar{W} \) is the mean field. In Equation (1) \( W_j \) is the complex amplitude of the \( j^{th} \) oscillator, overdot represents a differentiation w.r.t. time (t), \( C_1, C_2, K \) are real constants and \( \alpha_j \) is a parameter specifying the distance from the Hopf bifurcation point. In the absence of coupling, the \( j^{th} \) oscillator exhibits a periodic oscillation if \( \alpha_j > 0 \) while if \( \alpha_j < 0 \) then, the \( j^{th} \) oscillator settles down to the fixed point \( W_j = 0 \) and does not oscillate. Oscillators with \( \alpha_j > 0 \) are termed as active and those with \( \alpha_j < 0 \) are termed as inactive or dead. Since the oscillators are identical and globally coupled, for the sake of convenience, and without any loss of generality, we set the group of inactive elements to be \( j = \{1, \ldots, \lfloor Np \rfloor \} \) and the active elements to be \( j = \{ \lfloor Np + 1 \rfloor, \ldots, N \} \), where, \( 0 < p < 1 \). For simplicity we also assume, \( \alpha_j = \alpha > 0, \forall j \in \{1, \ldots, \lfloor Np \rfloor \} \) and \( \alpha_j = -b < 0, \forall j \in \{ \lfloor Np + 1 \rfloor, \ldots, N \} \), where both \( a \) and \( b \) are parameters. Eq. (1), in the absence of the inactive oscillators (\( b = 0 \)) has been extensively studied in the past \cite{20} and shown to possess a variety of collective states including synchronous states, splay states, single and multi-cluster states, chaotic states and more recently, also, amplitude mediated chimera states \cite{5}. Fig. (1), reproduced from \cite{5} shows a phase diagram in the \( C_1 - K \) space for \( C_2 = 2 \) that summarizes these past results. Our objective in this paper is to study the effect of inactive elements on the nature of the AMCs and of modifications if any of their existence region (the shaded region of Fig.1). Accordingly we carry out an extensive numerical exploration of Eq. (1) in the relevant parametric domain of \( C_1 - K \) space and compare them to previous results obtained in the absence of inactive oscillators.

III. AMPLITUDE MEDIATED CHIMERA STATES

In Fig. 2 we display a snapshot of the time evolution of a typical AMC state obtained by a numerical solution of Eq. (1) for \( N = 201, K = 0.7, C_1 = -1, C_2 = 2, a = b = 1 \) and \( p = 0.1 \). The figure shows the distribution of the 201 oscillators in the complex plane of \( (\text{Re}(W_j) \text{ vs } \text{Im}(W_j)) \). We observe the typical signature of an AMC in the form of a string like object representing the incoherent oscillators and a cluster (represented by a black filled circle) marking the coherent oscillators which move together at a fixed frequency. The new feature compared to the standard AMC state \cite{5} is the presence of another cluster (marked by a red triangle) that represents the dynamics of the dead oscillators which are no longer inactive now but have acquired a frequency and form another coherent region in the AMC. These oscillators have a lower amplitude than the originally active oscillators but oscillate at the same common collective frequency as them. Thus the inactive components of the system experience a revival due to the global coupling and modulate the coherent portion of the AMC profile.

This can be seen more clearly in the snapshots of the profiles of \( |W_j| \) and phase \( \phi_j \) shown in Figs. 3 and 4 respectively for \( N = 201, K = 0.6, C_1 = -0.7, C_2 = 2, a = b = 1 \) and \( p = 0.2 \). The red solid line in Fig. 3 marks the coherent region arising from the initially inactive oscillators. They have a smaller amplitude of oscillations than the initially active ones that form the coherent region shown by the blue solid line. The scattered dots show the incoherent region whose oscillators drift at different frequencies. The oscillator phases of the three
regions are shown in Fig. 4 where one observes that the phases of the oscillators in each coherent region remain the same and there is a finite phase difference between the two regions. The incoherent regions have a random distribution of phases among the oscillators.

We have next carried out extensive numerical explorations to determine the extent and location of the parametric domain where such AMCs can occur in order to determine the changes if any of the shaded existence region shown in Fig. 1. Our results are shown in Fig. 5 for several different values of $p$. We find that, for a fixed value of $b = -1$, as the value of $p$ is increased the parametric domain of the existence region of AMC shrinks and also shifts upward and away from the $p = 0$ region towards a higher value of $K$. This can be understood qualitatively by the argument that as the number of inactive oscillators increases one needs a higher amount of coupling to pull them in and create an AMC state. The area of this region asymptotically goes to zero as $p \to 1$. Note that there is a finite parametric domain of existence even for a $p$ value that is as large as 0.9 indicating that AMCs can occur even in the presence of a very large number of inactive oscillators and thus are very robust to aging related changes in the system environment.

IV. MEAN FIELD THEORY

In order to gain a deeper understanding of the dynamical origin of our numerical results we now analyse a re-
duced model that has been employed in the past to study the chaotic and AMC states of Eqn. 1 in the absence of inactive oscillators [3, 20]. We divide the whole population (W) into two sub-populations, namely, (W_A) for the active oscillators with α = a and (W_I) for the inactive oscillators (to start with) having α = -b. Along with the general mean field parameter, R, we next construct two other mean field parameters, one for the active oscillators and the other for the inactive ones, as follows,

\[
W(t) = \frac{1}{N} \sum_{i=1}^{N} W_i = Re^{i\omega t} \\
W_A(t) = \frac{1}{N(1-p)} \sum_{Active} W_i = R_Ae^{i\omega t} \\
W_I(t) = \frac{1}{Np} \sum_{Inactive} W_i = R_Ie^{i\omega t}
\]

where \( R \) is the amplitude of the mean field for all the oscillators, \( R_A \) is the amplitude of the mean field for the active oscillators and \( R_I \) is the same for the inactive oscillators. The nature of \( R, R_A \) and \( R_I \) can be seen from our numerical simulation results shown in Fig. 6, where we have plotted the time evolution of \( |W_{A,I}| \) and \( Re(W_{A,I}) \). We observe that the time variation of \( Re(W) \) is nearly periodic and identical for the active and inactive oscillators. A power spectrum plot of the numerical solution (4) further shows that the periodicity is primarily around a single frequency, \( \omega \), as denoted by the sharp peak in the power spectrum. The amplitudes \( R_A = |W_A| \) and \( R_I = |W_I| \) evolve on a slower time scale and are nearly constant with small fluctuations around a mean value. We can thus treat them as constants with \( R_I < R_A \).

\[
B(t) = (1 - K)^{-1/2}W(t)e^{-i(\omega t + \phi)}
\]

and re-scaling time as

\[
\tau = (1 - K)t
\]

equation (1) can be reduced to

\[
\frac{\partial B}{\partial \tau} = (1 + i\Omega)B - (1 + iC_2)|B|^2B + F
\]

with

\[
\Omega = -\omega + KC_1 \\
F = \frac{K\sqrt{1+C_1^2}}{(1-K)^{3/2}}R \\
C_1 = \tan(\phi), \quad \frac{\pi}{2} < \phi < \frac{\pi}{2}
\]

In the above relations, \( \Omega \) and \( F \) are functions of \( p \) since \( R \) and \( \omega \) now vary with \( p \) as seen from Figs. 6 and 7. From our numerical simulation results the variation of \( R \) and \( \omega \) can be represented by the following approximate relations,

\[
R = R_0\sqrt{a - K[1 - (\alpha^*_0 + \gamma b)p]} \\
\omega = \omega_0[1 - (\beta^*_0 + \gamma b)p]
\]

where for \( K = 0.7, C_1 = -1, C_2 = 2 \) and \( a = 1 \) we get, \( R_0 = 1.291, \omega_0 = 1.57, \alpha^*_0 = 0.63, \beta^*_0 = 0.48 \) and \( \gamma = 0.141 \). We can now try to understand the dynamical origin of the behaviour of the existence region of the AMC as a function of \( p \) by making use of the bifurcation diagram of eqn. 8 and the relations 12 and 13. The bifurcation diagram of 8 in the \( F - \Omega \) space for \( p = 0 \), is shown in Fig. 7, where the red (solid) line is the Hopf bifurcation line and the blue (dotted) line represents the saddle-node bifurcation line. As discussed in
the AMC states arise from the coexistence of a stable node and a limit cycle or a spiral attractor close to the saddle-node curve. The fluctuations in the amplitude of the mean field then drive the oscillators towards these equilibrium points with those that go to the node constituting the coherent part of the AMC while those that populate the limit cycle or the stable spiral forming the incoherent part of the AMC. The distribution of the oscillators among these two sub-populations depends on the initial conditions and the kicks in phase space that they receive from the amplitude fluctuations. The location of these $p = 0$ AMCs are marked by the open and filled circles in the figure. With the increase of $p$, $\Omega$ decreases, causing the determinant of the Jacobian (det $J$) of Equation (8) to monotonically decrease while keeping the trace of the Jacobian ($\text{Tr} \ J$) to remain unchanged. This results in the loss of a spiral into a node leading the AMC to collapse into two or three coherent cluster states. Also, as $p$ increases, $R$ decreases and $[\text{Tr} \ J]^2 - 4 \text{det} \ J$ decreases. This leads to the loss of a node and the generation of a spiral causing the AMC states near the Saddle-Node bifurcation line, for $p = 0$, to evolve into chaotic states. The above two reasons account for the shrinkage of the existence region of the AMC with the increase of $p$. Furthermore due to the re-scaling of $\Omega$ and $F$ for a given value of $p$, the bifurcation plot also shifts in the $F - \Omega$ space leading to a relocation of the position of the new AMC states. The direction and amount of these shifts are indicated by the direction and length of arrows marked in Fig (8) for different values of $p$.

\[
\begin{align*}
\delta C_1 &= \frac{1}{K_0} (\Omega_0 \delta K - \delta \omega - \delta K C_{10}) \\
&\quad + \left[3F_0^2(1 - K_0)^2 - 2K_0R_0^2(1 + C_{10}\Omega_0)\right] \delta K \\
&\quad - 2K_0R_0\left[\Omega_0 \delta R + K_0C_{10}^2 \delta R - C_{10}R_0 \delta \omega\right]
\end{align*}
\]  

(14)  

The above equations can be solved for $\delta K$ and $\delta C_1$ using the numerically obtained values for $\delta \omega$, $\delta R$ for a chosen small value of $p$. For $p = 0.1$ we have $\delta \omega = -0.12$ and $\delta R = -0.06$. Using the above values at $C_{10} = -0.7$ and $K_0 = 0.7$ we get $\delta K = 0.004$ and $\delta C_1 = 0.18$. These shifts in the values of $K$ and $C_1$ are depicted by an arrow in Fig. (3) and indicate the shift in the location of an AMC in the $K - C_1$ space. The direction of the shift agrees quite well with the observed shift in the domain of the $p = 0.1$ AMC compared to the domain with $p = 0$.

\section{Summary and Discussion}

To summarize, we have studied the influence of a population of inactive oscillators on the formation and dynamical features of amplitude mediated chimera states in an ensemble of globally coupled Ginzburg-Landau oscillators. From our numerical investigations we find that the inactive oscillators influence the AMCs in several distinct ways. The coupling with the rest of the active oscillators revives their oscillatory properties and they become a part of the AMC as a separate coherent cluster thereby modulating the structure of the AMC. Their presence also introduces an additional inertia in the system that reduces the overall frequency of the coherent group of oscillators. Finally they shrink the existence region of the AMCs in the parametric space of the coupling strength $K$ and the constant $C_1$ (where $KC_1$ is the imaginary component of the coupling constant). This region continuously shrinks and shifts in this parametric domain as a function of $p$. Remarkably, the AMCs continue to exist (albeit in a very small parametric domain) even for $p$ as large as 0.9 which is indicative of their robustness to aging effects in the system. Our numerical results can be well understood from an analytic study of a reduced model oscillator that is derived from a mean field theory and consists of a single driven nonlinear oscillator. The driving term is in fact the mean field arising from the
global coupling and that effectively captures the dynamics of both the active and the inactive oscillators. Our numerical results provide some simple scaling relations between the amplitude of the driver and its primary frequency component as a function of $p$. A comparison of the bifurcation diagrams of this equation obtained for $p = 0$ and finite $p$ values then provides a good qualitative understanding of the changes taking place in the existence domain of the AMCs due to the presence of the inactive oscillators. AMCs are a generalized class of chimera states in which both amplitude and phase variations of the oscillators are retained and their existence is not constrained by the need to have non-local forms of coupling. Our findings can therefore have a wider applicability and be practically relevant for such states in biological or physical systems where aging can diminish the functional abilities of component parts.

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