Synthesis over Regularly Approximable Data Domains

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Abstract. We study reactive synthesis of systems interacting with environments using infinite alphabets. Register automata and transducers are popular formalisms for specifying and modelling such systems. They extend finite-state automata by adding registers to store data values and to compare the incoming data values against stored ones. Synthesis from nondeterministic or universal register automata is undecidable in general. Its register-bounded variant, where additionally a bound on the number of registers in a sought transducer is given, is however known to be decidable for universal register automata which can compare data for equality.

In this paper, we generalise this result. We introduce the notion of $\omega$-regularly approximable data domains, and show that register-bounded synthesis from universal register automata on such domains is decidable. Importantly, the data domain $(\mathbb{N},=,<)$ with natural order is $\omega$-regularly approximable, and its closer examination reveals that the synthesis problem is decidable in time doubly exponential in the number of registers, matching the known complexity of the equality-only case $(\mathbb{N},=)$. We then introduce a notion of reducibility between data domains which we exploit to show decidability of synthesis over, e.g., the domains $(\mathbb{N}^d,=,<)$ of tuples of numbers equipped with the component-wise partial order and $(\Sigma^*,=,<)$ of finite strings with the prefix relation.

Keywords: Synthesis, Register Automata, Transducers, Ordered Data Domains

1 Introduction

Synthesis aims at the automatic construction of a system from its specification. A system is usually modelled as a transducer: in each step, it reads an input from the environment and produces an output. In this way, the transducer, reading an infinite sequence of inputs, produces an infinite sequence of outputs. Specifications are modelled as a language of desirable input-output sequences. Traditionally [26,3], the inputs and outputs have been modelled as letters from a finite alphabet. This however limits the application of synthesis, so recently researchers started investigating synthesis of systems working on data domains [11,21,14,22,1,13].
A well-studied formalism for specifying and modelling data systems are register automata and transducers [19,24,20,29]. Register automata extend classical finite-state automata to infinite alphabets $\mathcal{D}$ by introducing a finite number of registers. In each step, the automaton reads a data $d \in \mathcal{D}$, compares it with the values held in its registers, then depending on this comparison it decides to store $d$ into some of its registers, and finally moves to a successor state. This way it builds a sequence of configurations (pairs of state and register values) representing its run on reading a word from $\mathcal{D}^\omega$: it is accepted if the visited states satisfy a certain condition, e.g. parity. Transducers are similar except that in each step they also output the content of one register.

Unlike classical finite-state automata, the expressive power of register automata depends on whether they are deterministic, nondeterministic, or universal. Among these, universal register automata (URA) suit synthesis best. First, they can specify request-grant properties: every requested data shall be eventually output. This is a key property in reactive synthesis, and in the data setting it can be expressed by a universal register automaton but not by a nondeterministic one. Furthermore, universal register automata are closed, in linear time, under intersection. Hence they allow for succinct conjunction, which is desirable in synthesis as specifications are usually modular and consist of many independent properties. Finally, in the register-free setting universal automata are often used to obtain synthesis methods feasible in practice [23,27,16,3].

The second factor affecting expressivity of register automata is the comparison operator on data. Originally, register automata could compare data for equality only, i.e., operated on $(\mathbb{N}, =)$ [19]. To widen synthesis applicability, we aim at $(\mathbb{N}, =, <)$ and beyond. The following two examples illustrate these settings: the first one is a classical request-grant specification in the domain $(\mathbb{N}, =)$, and the second example extends it to $(\mathbb{N}, =, <)$.

**Example 1 (Arbiter in $(\mathbb{N}, =)$).** Consider the specification saying that every request must be eventually granted, expressed by a universal register automaton with a single register $r$ in Figure 1. The automaton reads words where each letter is a pair a finite-alphabet label and a data value from an infinite countable domain like $(\mathbb{N}, =)$. In this example, data words are infinite sequences in $[((\text{req, idle}) \times \mathbb{N}) \times ([\text{grt, idle}] \times \mathbb{N})]^\omega$, alternating between input and output letters, where the data component represents client IDs, and request req, grant grt, and idle are labels from a finite alphabet. A pair $(\text{req, }i)$ means that client $i \in \mathbb{N}$ has requested the resource, while $(\text{grt, }i)$ means that it has been granted to client $i$. The label idle means that no request has been made; the data value is then ignored. States are partitioned into input states (square) which read input pairs in $(\text{req, idle}) \times \mathbb{N}$ and are controlled by the environment, and output states (circle) reading pairs in $(\text{grt, idle}) \times \mathbb{N}$ and controlled by the system. In the figure, each transition is labelled, in addition to a label from $(\text{req, grt, idle})$, by a test and an assignment. A test is a literal $\star = r$ or $\star \neq r$, and the writing $\top$ denotes every test. An assignment dictates whether to save an incoming data into automaton registers (we have only $r$): $\downarrow r$ means that the incoming data is stored in $r$. The writing $\top$ can also denote every pair of letter-test. The double-circle
Fig. 1: Universal register automaton expressing the specification that every grant is eventually granted (Example 1).

state is rejecting and can be visited only finitely often. Thus, a run looping in wait states \( w_{\text{in}} \) and \( w_{\text{out}} \) is accepting. Branching is universal, hence some run always loops around \( w_{\text{in}} \) and \( w_{\text{out}} \). Whenever a request is received, a copy of the automaton moves from \( w_{\text{in}} \) to a pending state \( p_{\text{out}} \) while storing the client ID into \( r \); it stays in states \( p_{\text{out}} \) and \( p_{\text{in}} \) as long as the request is not granted, and an infinite such run is rejecting. If the request is eventually granted (transition \( p_{\text{out}} \) to a sink state \( s_{\text{in}} \)), the run dies, and the run is accepting.

Notice that such a specification is not expressible by a nondeterministic register automaton, because every client ID requesting the resource has to be stored in a register, in order to be able to check that eventually the request is granted\(^3\). This is not possible in a single run (even nondeterministically chosen), as the automaton has only finitely many registers. In contrast, in universal automata the values of the same register in different runs are independent, so we can devote one independent run to every client ID.

**Example 2 (Arbiter in \((\mathbb{N}, =, <)\)).** We now add priorities between clients: a client with a larger ID has a larger priority. The specification is: for all \( i \in \mathbb{N} \), whenever client \( i \) requests a resource, it must eventually be granted to it or to a client with a larger priority, for two consecutive steps. In a first-order like formalism where variables are positions in the word, and can be compared with an order \( \leq_p \) while their data can compared by an order \( \leq_D \), this would be written as:

\[
\forall x. \text{req}(x) \rightarrow \exists y >_p x. \: \text{grt}(y) \land \text{grt}(y+2) \land \delta(y) \geq_D \delta(x) \land \delta(y) =_D \delta(y+2).
\]

Here, the writing \( \delta(y) \) denotes the data value in position \( y \). Note that we compare data at moments \( y \) and \( y+2 \) (rather than \( y+1 \)) due to alternation of input-output letters in data words. A one-register URA defining this specification is given in Figure 2. In contrast to the previous example, this automaton compares data using predicates \( < \) and \( = \), so it operates in domain \((\mathbb{N}, =, <)\). This specification is realisable by a transducer with two registers \( r_1 \) and \( r_2 \) in Figure 3. It outputs \( \langle \text{grt}, r_1 \rangle \) twice, then outputs \( \langle \text{grt}, r_2 \rangle \) twice, ad infinitum. Since at every moment either \( r_1 \) or \( r_2 \) holds the maximal ID of any client requested the resource, this transducer realises the specification. The transducer realises the specification because \( r_1 \) and \( r_2 \) satisfy the following invariant: at any moment, either \( r_1 \) or \( r_2 \) hold the maximal ID of any client requested the resource.

\(^3\) This can also be seen as a consequence of the proof of [15, Proposition 4.7.].
Fig. 2: Universal register automaton expressing the specification that every request must eventually be granted to a client with a larger or equal ID, for two consecutive steps.

Fig. 3: Transducer with two registers $r_1, r_2$ realizing the specification of Figure 2 from Example 2. In red are the tests over the inputs received by the transducer. The test else is a shortcut for every test different from the first test on the transition. On the right from the vertical bar is the output action performed by the transducer. For example, from state $q_0$ to $q_1$, if the input is $(\text{req}, d)$ and the data $d$ is larger than the data stored in register $r_1$, then the transducer stores it into $r_1$, and outputs grt and the content of $r_1$. There is no acceptance condition, hence such transducers are an extension of Mealy machines by registers.

Finally, note that if we replace $\geq$ by $=\geq$ in the first-order formula above, i.e., if we ask that every client requesting a resource gets the resource granted eventually for two consecutive steps, then the specification becomes unrealisable. Indeed, in this case the clients could perform a DDoS attack, when at each step a new client requests the resource. This scenario makes the specification unrealisable, because the number of registers in transducers is finite, whereas the number of clients that requested but not yet obtained the resource is unbounded.

Already for $(\mathbb{N}, =)$, the synthesis problem of register transducers from universal register automata is undecidable [11,14]. Decidability is recovered in the deterministic case for $(\mathbb{N}, =)$ [14] and $(\mathbb{N}, =, <)$ [13], but, as argued above, universal automata are more desirable in synthesis. To circumvent the undecidability, the works [21,14,22] studied a decidable variant, called register-bounded synthesis: given a universal register automaton and a bound $k$ on the number of transducer registers, return a $k$-register transducer realising the automaton or ‘No’ if no such transducer exists. Register-bounded synthesis generalises classical
register-free synthesis from (data-free) \( \omega \)-regular specifications, for the number of transducer states is finite but unconstrained.

The key idea of existing approaches \([11,21,14,22]\) is to reduce the register-bounded synthesis problem in a data domain to a two-player game on a finite arena with an \( \omega \)-regular winning condition. Two players alternate play for an infinite number of rounds. Adam, modelling the environment, picks an input label and a test over the \( k \) registers, while Eve, modelling the system, picks a subset of the \( k \) registers (meant to store the data), an output label, and a register (whose value is meant to be output). No data are manipulated in the game and the arena is finite. Infinite plays in the game induce action words, which are infinite sequences of input and output labels, tests, and assignments of the \( k \) registers. Thus, action words use a finite alphabet. The reduction ensures that any finite-memory winning strategy in the game can be converted into a \( k \)-register transducer realising the specification, and vice versa. To this end, the game winning condition requires that: if a play induces a feasible action word, in the sense that there exists a data word \( w \) satisfying all its tests and assignments, then all the runs of the URA on all such data words are accepting. In the case of \((\mathbb{N},=)\), it is known that the winning condition is \( \omega \)-regular \([21,14,22]\). In \((\mathbb{N},=,<)\), however, the set of feasible action words is not \( \omega \)-regular \([13]\), and neither is the winning condition. Such winning conditions can be expressed by nondeterministic \( \omega \)-S automata \([4]\), but games with such objectives are not known to be decidable, to the best of our knowledge.

Contributions. To overcome the latter obstacle, we introduce the notion of \( \omega \)-regularly approximable (\( \text{regapprox} \)) data domains, for which it is required that there exists an \( \omega \)-regular over-approximation of the set of feasible action words, which is exact over lasso-shaped action words (of the form \( uv^\omega \)). In regapprox domains, the set of feasible lasso-shaped action words is therefore \( \omega \)-regular, which allows us to reduce synthesis to solving \( \omega \)-regular games. We prove that for regapprox domains, register-bounded synthesis from universal register automata is decidable, and the procedure is effective, in the sense that for realisable specifications, it outputs a transducer. As the data domains \((\mathbb{N},=),(\mathbb{Q},=,<)\), and other oligomorphic domains \([5]\) are regapprox (their sets of feasible action words are \( \omega \)-regular), our result subsumes works \([14,22,21]\). We then show that the data domain \((\mathbb{N},=,<)\) is regapprox (despite not being oligomorphic), and that the synthesis problem is decidable in time doubly exponential in the number of registers. If, additionally, the bound \( k \) and the number of registers in the specification automaton are fixed, the problem is \( \text{ExpTime-c} \) for \((\mathbb{N},=,<)\). These complexity results match those of \([14,22]\) for \((\mathbb{N},=)\), so we get the order for free.

Having the synthesis problem solved for regapprox domains and for \((\mathbb{N},<,=)\), we proceed to the question of reducibility between data domains. Intuitively, a data domain \( \mathcal{D} \) reduces to \( \mathcal{D}' \) if there is a rational transduction that relates action words in \( \mathcal{D} \) and \( \mathcal{D}' \) while preserving feasibility. We show that if \( \mathcal{D} \) reduces to \( \mathcal{D}' \), and \( \mathcal{D}' \) is regapprox, then \( \mathcal{D} \) is regapprox as well, so a synthesis procedure for \( \mathcal{D}' \) can be used to solve synthesis in \( \mathcal{D} \). Finally, we reduce to \((\mathbb{N},=,<)\) the domain \((\mathbb{N}^d,=^d,<^d)\) of tuples of numbers with the component-wise partial order and
the domain \((\Sigma^*, =, \prec)\) of finite strings with the prefix relation, which entails the decidability of register-bounded synthesis on these domains.

**Related works.** Our procedure generalises the results [21, 22, 14] on register-bounded synthesis. While we focus on universal register automata as most suitable for synthesis, the paper [14] additionally studies synthesis from nondeterministic register automata and shows it is undecidable for \((\mathbb{N}, =)\). Synthesis for data domain \((\mathbb{N}, =, \prec)\) is also studied in [13]; the authors focus on deterministic register automata which are less expressive than universal ones, but they consider general data strategies while we consider those expressible by transducers. Their proofs implicitly show that \((\mathbb{N}, =, \prec)\) is regapprox. See also [12] for a thorough study of synthesis on \((\mathbb{N}, =)\) and \((\mathbb{Q}, =, \prec)\).

Action words can be seen as infinite sequences of constraints between successive register contents. Infinite constraint sequences and their satisfiability have been studied in various papers, for example in [9, 28] for the domain \((\mathbb{N}, =, \prec)\) and in [10] for the domain \((\Sigma^*, =, \prec)\). The fact that \((\mathbb{N}, =, \prec)\) is regapprox was already shown implicitly in [13]. For the data domain \((\Sigma^*, =, \prec)\), we prove a reduction to \((\mathbb{N}, \prec, =)\) by using the same ideas as [10], although we cannot reuse their result as a blackbox.

## 2 Synthesis Problem

Let \(\mathbb{N} = \{0, 1, \ldots\}\) denote the set of natural numbers including 0.

**Data domain and data words.** A data domain is a tuple \(D = (\mathcal{D}, P, C, c_0)\) consisting of an infinite countable set \(\mathcal{D}\) of data, a finite set \(P\) of interpreted predicates (predicate names with arities and their interpretations) which must contain the usual equality predicate \(=\), a finite set \(C \subset \mathcal{D}\) of constants, and a distinguished initialiser constant \(c_0 \in C\). For example, \((\mathbb{N}, \{<, =\}, \{0\}, 0)\) is the data domain of natural numbers with the usual interpretation of \(<, =\), and 0. In the tuple notation, we often omit the brackets, as well as the mention of \(c_0\) when there is only one constant. E.g., we write \((\mathbb{N}, =, <, 0)\) for \((\mathbb{N}, \{<, =\}, \{0\}, 0)\). Another familiar example is \((\mathbb{Z}, =, <, 0)\), which is the data domain of integers with the usual \(<, =\), and 0. Throughout the paper we assume that the satisfiability problem of quantifier-free formulas built on the signature \((P, C)\) is decidable in \(D\), and whenever we state complexity results, the satisfiability problem is additionally assumed to be decidable in \(\text{PSPACE}\). This is the case for all data domains considered in this paper. Finally, data words are infinite sequences \(d_0d_1\ldots \in D^\omega\), and for two sets \(\Sigma\) and \(\Gamma\) and a language \(L \subseteq (\Sigma \cdot \Gamma)^\omega\), we call \(\Sigma\) and \(\Gamma\) its input and output alphabets respectively.

**Action words.** Fix a finite set \(R\) of elements called registers and a data domain \(D\). A register valuation (over \(D\)) is a mapping \(\nu : R \rightarrow D\). A test atom has the form \(p(x_1, \ldots, x_k)\) where \(p\) is a \(k\)-ary predicate from \(P\) and all \(x_i \in R \cup C \cup \{\ast\}\), where \(\ast\) is a fresh symbol used as a placeholder for incoming data. A literal is an atom or its negation. A register valuation \(\nu : R \rightarrow D\) and data \(d \in D\) satisfy an atom \(p(x_1, \ldots, x_k)\), written \((\nu, d) \models p(x_1, \ldots, x_k)\), if the predicate
holds, where $\nu' = \nu \cup \{ c \mapsto c | c \in C \} \cup \{ \ast \mapsto d \}$; define $(\nu, d) \models \neg p(x_1, \ldots, x_k)$ if $(\nu, d)$ does not satisfy $p(x_1, \ldots, x_k)$. A test $\text{tst}$ is a set of literals that contains every atom or its negation and such that there is $(\nu, d)$ satisfying all literals of $\text{tst}$ (i.e., maximally consistent, where consistency is wrt. the theory axioms of the domain). Let $\text{Tst}_R$ denote the set of all possible tests over registers $R$ in domain $(D, P; C, c_0)$. Note that the assumption on decidable satisfiability of the data domain is used here to be able to construct the set $\text{Tst}_R$.

**Example.** Consider domain $(\mathbb{N}, =, <, 0)$ and $R = \{ r \}$. Test atoms are: $r < \ast$, $\ast = r$, $r < 0$, $\ast = 0$, etc. We omit mentioning valid atoms like $0 = 0$, unsatisfiable atoms like $r < r$, and literals implied by theory axioms like $\neg (r = 0)$ when $r = 0$ is already known to hold. For readability, in describing tests we often use formula notation instead of set notation. Some examples of tests are $0 < r < \ast$ and $r = \ast = 0$, and some examples of non-tests are $r = \ast < 0$ (it is not satisfiable in $(\mathbb{N}, =, <, 0)$) and $r < \ast$ (it does not contain literals relating $\ast$ and $0$). The latter example highlights that tests describe in full the relations between the registers and input data.

An **assignment** is a set $\text{asgn} \subseteq R$; it describes registers that should store the current input data. Let $\text{Asgn}_R$ denote the set of all possible assignments. Given an assignment $\text{asgn}$, a data $d$, and a valuation $\nu$, define $\text{update}(\nu, d, \text{asgn})$ to be the valuation that maps $r$ to $d$ if $r \in \text{asgn}$ else to $\nu(r)$, for every $r \in R$.

An automaton action word, or simply action word, is a sequence of the form $\text{tst}_0(\text{asgn}_0)(\text{tst}_1, \text{asgn}_1) \ldots \in (\text{Tst}_R \times \text{Asgn}_R)^\omega$. It is feasible in $D$ (or simply feasible when $D$ is clear) by a sequence of valuation-data pairs $(\nu_0, d_0)(\nu_1, d_1) \ldots$ if for all $i$: $\nu_0 = c_0^R$ and $\nu_{i+1} = \text{update}(\nu_i, d_i, \text{asgn}_i)$, and $(\nu_i, d_i) \models \text{tst}_i$. An action word $\Sigma$ is feasible by $d_0d_1 \ldots$ if such $\nu_0\nu_1 \ldots$ exist, and it is simply feasible if such $d_0, d_1 \in R$ and $\nu_0, \nu_1 \in \text{asgn}_0$. Let $\text{FEAS}^\omega_R$ denote the set of action words over $R$ feasible in domain $D$. We may write either $\text{FEAS}_R$, or $\text{FEAS}^\omega_R$ or just $\text{FEAS}$ when $D$, $R$ or both are clear from the context.

**Example.** Consider domain $(\mathbb{N}, =, <, 0)$ and $R = \{ r, s \}$. For an assignment $\{ x \}$, where $x \in R$, we use the notation $\downarrow x$. An action word $(r = s = 0 < \ast, \downarrow r)(s < \ast < r, \downarrow r)^\omega$ is unfeasible, because it requires having an infinite chain of decreasing values for register $r$, which is not possible in $(\mathbb{N}, =, <, 0)$.

**Register automata.** To define a register automaton, we first fix a data domain $D$. A register automaton is a tuple $S = (Q, q_0, R, \delta, \alpha)$, where $Q$ is a finite set of states containing the initial state $q_0$, $R$ is a finite set of registers, $\delta \subseteq Q \times \text{Tst}_R \times \text{Asgn}_R \times Q$ is a transition relation, and $\alpha : Q \rightarrow \{ 1, \ldots, c \}$ is a priority function where $c$ is the priority index.

A configuration of $S$ is a pair $(q, \nu) \in Q \times D$, and $(q_0, c_0^R)$ is initial. A run of $S$ on a data word $d_0d_1 \ldots$ is a sequence of configurations $r = (q_0, \nu_0)(q_1, \nu_1) \ldots$ starting from the initial configuration and such that there exists an action word $\text{tst}_0(\text{asgn}_0)(\text{tst}_1, \text{asgn}_1) \ldots$ feasible by $(\nu_0, d_0)(\nu_1, d_1) \ldots$ and $(q_0, \text{tst}_0, \text{asgn}_0, q_1) \in \delta$ for all $i$. The run $r$ is said to be accepting if the maximal priority appearing infinitely often in $\alpha(q_0)\alpha(q_1) \ldots$ is even, otherwise it is rejecting. A word may induce several runs of $S$. For universal register automata, abbreviated URA, a word is accepted if all induced runs are accepting; for nondeterministic au-
tomata, there should be at least one accepting run. The set of all data words over $\mathcal{D}$ accepted by $S$ is called the \textit{language} of $S$ and denoted $L(S)$. We may write $L_\mathbb{D}(S)$ to emphasise that $L(S)$ is defined over $\mathbb{D}$. Figure 2 gives an example of URA over data of the form $(\sigma, \nu)$ where $\sigma$ belongs to some finite alphabet and $\nu$ is a natural number which can be tested wrt. $\leq$.

A \textit{finite-alphabet (parity) automaton} (without registers) is a tuple $\langle \Sigma, Q, q_0, \delta, \alpha \rangle$, where $\Sigma$ is a finite alphabet, $\delta \subseteq Q \times \Sigma \times Q$, and the definition of runs, accepted words, and language is standard. Such automata operate on words from $\Sigma^\omega$.

\textbf{Syntactical language of a register automaton.} Treating a register automaton $S = \langle Q, q_0, R, \delta, \alpha \rangle$ syntactically gives us a finite-alphabet automaton $S_{\text{synt}} = \langle \Sigma, Q, q_0, \delta, \alpha \rangle$ with $\Sigma = \text{Tst}_R \times \text{Asgn}_R$ and the original $Q, q_0, \delta, \alpha$. Note however that the language of $S_{\text{synt}}$ may contain action words which are not feasible, i.e., $L(S_{\text{synt}}) \not\subseteq \text{FEAS}_{\mathbb{D}}^\Sigma$. The set $L(S_{\text{synt}}) \cap \text{FEAS}_{\mathbb{D}}^\Sigma$ might not even be $\omega$-regular in general. It is the case of $\langle \mathbb{N}, <, 0 \rangle$: e.g., it easy to construct $S$ over $R = \{m, r\}$ such that $L(S_{\text{synt}})$ is the following set of action words:

$$(r = m = 0 < \ast, \downarrow m)(0 < r < \ast < m, \downarrow r)^* (0 = \ast < r < m, \downarrow r)^\omega$$

which correspond to initializing some maximal bound $d_m \in \mathbb{N}$ and then constructing increasing chains of non-negative integers bounded by $d_m$ and sometimes reset to 0. Clearly, the length of those increasing chains is bounded by $d_m$, which is the first data value. So, the subset of feasible action words is all words of the form $(r = m = 0 < \ast, \downarrow m) \prod_{i=1}^\omega (0 < r < \ast < m, \downarrow r)^{\omega_i} (0 = \ast < r < m, \downarrow r)$ such that $(n_i)$ is bounded. It is not $\omega$-regular but an $\omega$-$B$-language [4]. This motivates introducing a notion of regular approximation in this paper.

\textbf{Register transducers.} A \textit{transducer action word} is a sequence $\text{tst}_0(\text{asgn}_0, r_0) \ldots$ from $\langle \text{Tst} - \{\text{Asgn} \times R\} \rangle^*$; it is feasible by $(v_0, d_0, \delta_0^\omega)(v_1, d_1, \delta_1^\omega) \ldots$ if the automaton action word $(\text{tst}_0, \text{asgn}_0) \ldots$ is feasible by $(v_0, d_0^\omega)(v_1, d_1^\omega) \ldots$ and $d_i^\omega = \nu_{i+1}(r_i)$ for all $i$. It is feasible by $d_0^\omega d_1^\omega d_2^\omega \ldots$ when such $\nu_i$s exist, and simply feasible when such $(\nu_i, d_i, \delta_i^\omega)$s exist.

A $k$-\textit{register transducer} is a tuple $T = \langle Q, q_0, R, \delta \rangle$, where $Q, q, R (|R| = k)$ are as in automata but $\delta : Q \times \text{Tst} \rightarrow \text{Asgn} \times R \times Q$. Note that $\delta$ is a total function; moreover, since tests are maximally consistent, only one test holds per incoming data, so the transducers are deterministic. A \textit{run} of $T$ on an input data word $d_0^\omega d_1^\omega \ldots$ is a sequence $(q_0, v_0)(q_1, v_1) \ldots$ starting in the initial configuration such that there exists an action word $\text{tst}_0(\text{asgn}_0, r_0) \ldots$ feasible by $(v_0, d_0^\omega)(v_1, d_1^\omega) \ldots$, where $d_i^\omega = \nu_{i+1}(r_i)$, and $(\text{asgn}_i, r_i, q_{i+1}) = \delta(q_i, t_{\text{tst}_i})$ for all $i$. The sequence $d_0^\omega d_1^\omega \ldots$ is the \textit{output word} of $T$ on reading $d_0^\omega d_1^\omega \ldots$; since the transducers are deterministic and have a run on every input word, the output word is uniquely defined. The sequence $d_0^\omega d_1^\omega \ldots$ is the \textit{input-output word}. The \textit{language} $L(T)$ consists of all input-output words of $T$. Figure 3 gives an example (there, transducers read a finite-alphabet letter along data, which can also be modelled but are omitted for conciseness).

A \textit{finite-alphabet transducer} is a tuple $\langle \Sigma_I, \Sigma_O, Q, q_0, \delta \rangle$, where $\Sigma_I$ and $\Sigma_O$ are finite input and output alphabets, $\delta : Q \times \Sigma_I \rightarrow \Sigma_O \times Q$, and the definition of
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language is standard. Treating a register transducer \(T\) syntactically gives a finite-alphabet transducer \(T_{\text{synt}}\) of the same structure with \(\Sigma_T = \text{Tst}\) and \(\Sigma_O = \text{Asgn} \times R\).

**Synthesis problem.** Fix a data domain \((D, P, C, c_0)\). A register transducer \(T\) realises a register automaton \(S\) if \(L(T) \subseteq L(S)\). The **register-bounded synthesis problem** is:

- **input:** \(k \in \mathbb{N}\) and a URA \(S\);
- **output:** yes iff there exists a \(k\)-register transducer realising \(S\).

Moreover in this paper, when the latter problem is decidable, then we are able to synthesise, i.e., effectively construct, a transducer realising the specification. Notice the number of transducer states is finite but unconstrained, so register-bounded synthesis generalises classical register-free synthesis from (data-free) \(\omega\)-regular specifications. Finally, observe that transducers have to be complete for all possible tests, and therefore, any transducer \(T\) satisfies that the projection of \(L(T)\) on input data is exactly \(D^\omega\).

## 3 Sufficient Condition for Decidable Synthesis for URA

In this section, we first show a reduction from register-bounded synthesis to (register-free) finite-alphabet synthesis. More precisely, given a specification \(S\) (as a URA) and a bound \(k\), we show how to construct a finite-alphabet specification \(W_{S,k}^f\) on action words over \(k\) registers, which is realisable by a finite-alphabet transducer iff \(S\) is realisable by a \(k\)-register transducer (Lemma 2). The main idea is to see the actions of the URA and of the sought \(k\)-register transducer as finite-alphabet letters. In particular, the specification \(W_{S,k}^f\) accepts a transducer action word \(\pi_k\) iff every action word \(\pi_S\) of the specification \(S\), such that both \(\pi_k\) and \(\pi_S\) are feasible by the same data word, is accepted by \(S_{\text{synt}}\). The first technical step is to define a composition operator which combines both \(\pi_k\) and \(\pi_S\), in order to be able to talk about their joint feasibility (i.e. feasibility by the same action word) more easily (Lemma 1). Then, in general, \(W_{S,k}^f\) is not necessarily \(\omega\)-regular, and in a second step, we provide sufficient conditions on the data domain making synthesis wrt. \(W_{S,k}^f\) decidable, namely, that it can be under-approximated by an \(\omega\)-regular language which coincides with \(W_{S,k}^f\) over lasso words (Section 3.1). We obtain a general decidability result for data domains having this property (Theorem 1). We then instantiate this result for data domain \((\mathbb{N}, <, 0)\) (Section 3.2). Let us start with a definition of composition.

Given a transducer action word \(\pi_k = (\text{tst}_0^k(\text{asgn}_0^k, r_0^k) \ldots\) and an automaton action word \(\pi_S = (\text{tst}_0^S(\text{asgn}_0^S))(\text{tst}_0^S(\text{asgn}_0^S) \ldots\), a composition \(\pi_k \otimes \pi_S\) is any action word \((\text{tst}_0^k(\text{asgn}_0^k))(\text{tst}_0^S(\text{asgn}_0^S) \ldots\) over \(R_k \cup R_S\) satisfying for all \(j\): \(\text{tst}_j^k\) contains the atoms of \(\text{tst}_j^S \cup \text{tst}_j^k\), \(\text{asgn}_j^k = \text{asgn}_j^S \cup \text{asgn}_j^k\), \(\text{tst}_j^k\) contains the atoms of \(\text{tst}_j^S \cup \{* = r_j^k\}\), and \(\text{asgn}_j^S = \text{asgn}_j^S\). In other words, tests and assignments of a composition \(\pi_k \otimes \pi_S\) subsume those of \(\pi_k\) and \(\pi_S\), and additionally each output test contains the literal \(* = r_j^k\) ensuring that the current data equals the value stored in the output transducer register. Let \(\text{Comp}(\pi_k, \pi_S)\) denote the set of all compositions of given \(\pi_k\) and \(\pi_S\). Given \(\pi_k\) and \(\pi_S\), an extended composition
is a word from \(( (\text{Tst}_R \times \text{Asgn}_R) \cdot (\text{Tst}_R \times \text{Asgn}_R \times R_k))^\omega\) that satisfies the same conditions but also contains the elements \(r^j_k\) in the output letters.

**Example.** Let \(R_S = \{s\}, R_k = \{t\}, \overline{\pi}_S = (** > s, \emptyset)**(* > s, \downarrow s)\ldots, and \(\overline{\pi}_k = (** > t)(\downarrow t, t)\ldots\). A composition \(\overline{\pi}_k \otimes \overline{\pi}_S = (\text{tst}^k_0, \text{asgn}^k_0)\) can have:
- \(\text{tst}^k_0\) is either \(* > s > t\) or \(* > t > s\) or \(* > s = t,\)
- \(\text{asgn}^k_0\) performs \(\downarrow t\) (this is the storing operation performed on the input by \(\overline{\pi}_k\) while \(\overline{\pi}_S\) skips the storing).
- \(\text{tst}^k_0\) must contain \(* > s\) and \(* = t\), hence it is \(* = t > s,\)
- \(\text{asgn}^k_0\) performs \(\downarrow s\).

So, a composition \(\overline{\pi}_k \otimes \overline{\pi}_S\) can start with \((** > s = t, \downarrow t)(** > s > t, \downarrow s)\ldots\). Another composition is \(\overline{\pi}_k \otimes \overline{\pi}_S = (** > s > t, \downarrow t)\ldots,\) but this one is unfeasible in \((\mathbb{N}, =, <, 0)\) as all registers must start in \(0\), which falsifies the test \(* > s > t,\) \(\Box\)

It is possible that \(\overline{\pi}_k\) is feasible and \(\overline{\pi}_S\) is feasible yet they do not have a feasible composition \(\overline{\pi}_k \otimes \overline{\pi}_S\). For instance, in domain \((\mathbb{N}, =, <, 0)\), \(\overline{\pi}_k\) may require the input data to increase in each step while \(\overline{\pi}_S\) to stay constant: each such action word is feasible on its own, by different data words, but they do not have a feasible composition (that implies that \(\overline{\pi}_k\) and \(\overline{\pi}_S\) are feasible by the same data word). The following lemma describes a necessary and sufficient condition for having a feasible composition (the proof is in Appendix A.1).

**Lemma 1.** A transducer and automaton action words \(\overline{\pi}_k\) and \(\overline{\pi}_S\) have a feasible composition iff there is a data word such that \(\overline{\pi}_k\) and \(\overline{\pi}_S\) are feasible by it.

Having compositions defined, we now show how to abstract data specification, in the form of a URA \(S\) with registers \(R_S\), by a finite-alphabet specification over \(k\)-register transducer action words. Let \(\text{FEAS}_R\) be the set of feasible action words over \(R = R_S \sqcup R_k\) (defined on page 2), then we define
\[
W^f_{S,k} = \{\overline{\pi}_k \mid \forall \overline{\pi}_S : \text{Comp}(\overline{\pi}_k, \overline{\pi}_S) \cap \text{FEAS}_R \neq \emptyset \Rightarrow \overline{\pi}_S \in L(S_{\text{synt}})\}.
\]
Thus, \(W^f_{S,k}\) rejects a feasible transducer action word \(\overline{\pi}_k\) iff there is an automaton action word \(\overline{\pi}_S\) feasible by the same data word as \(\overline{\pi}_k\) and rejected by \(S\).

**Lemma 2.** These two are equivalent:
- a URA \(S\) is realisable by a \(k\)-register transducer,
- \(W^f_{S,k}\) is realisable (by a finite-alphabet transducer).

**Proof.** Direction \(\Rightarrow\). Suppose that \(S\) is realisable by a register transducer \(T\), thus \(L(T) \subseteq L(S)\), then we show that \(L(T_{\text{synt}}) \subseteq W^f_{S,k}\). By contradiction, assume that \(L(T_{\text{synt}}) \nsubseteq W^f_{S,k}\). By definition of \(W^f_{S,k}\), there is \(\overline{\pi}_k \in L(T_{\text{synt}})\) and \(\overline{\pi}_S \notin L(S_{\text{synt}})\), and there is a feasible composition \(\overline{\pi}_k \otimes \overline{\pi}_S\). By Lemma 1, there is a data word \(w = d_0d_1\ldots\) making \(\overline{\pi}_k\) and \(\overline{\pi}_S\) feasible. Let \(q_0q_1\ldots\) be a rejecting run of \(S_{\text{synt}}\) on \(\overline{\pi}_S\). Since \(\overline{\pi}_S\) is feasible by \(w\), there is \(\nu_0\nu_1\ldots\) such that \(\overline{\pi}_S\) is feasible by \((\nu_0, d_0)(\nu_1, d_1)\ldots\). The sequence \((q_0, \nu_0)(q_1, \nu_1)\ldots\) satisfies the definition of a run of \(S\) on \(w\) using \(\overline{\pi}_S\), and this run is rejecting, hence \(w \notin L(S)\). Since \(\overline{\pi}_k \in L(T_{\text{synt}})\) and \(\overline{\pi}_k\) is feasible by \(w, w \in L(T)\). Therefore \(L(T) \nsubseteq L(S)\), giving the sought contradiction.
Direction $\Leftarrow$. Suppose that $W_{S,k}^f$ is realisable by a finite-alphabet transducer $M$. The transducer $M$ uniquely defines register transducer $T$ through $T_{\text{syn}} = M$. We show that $L(T) \subseteq L(S)$. By contradiction, assume that $L(T) \not\subseteq L(S)$. Then there is a word $w = d_0d_1\ldots$ such that $w \in L(T)$ and $w \notin L(S)$. Let $((q_0,\nu_0)(q_1,\nu_1)\ldots$ be a rejecting run of $S$ on $w$. By definition of runs, there is an action word $\overline{a}_S$ feasible by $(\nu_0,d_0)(\nu_1,d_1)\ldots$, and the sequence $q_0q_1\ldots$ is a rejecting run of $S_{\text{syn}}$ on $\overline{a}_S$. Therefore $\overline{a}_S \notin L(S_{\text{syn}})$. Similarly we show there is $\overline{a}_k \in L(T_{\text{syn}})$ feasible by $w$. Both $\overline{a}_k$ and $\overline{a}_S$ are feasible by $w$, hence by Lemma 1, there is a feasible composition $\overline{a}_k \oplus \overline{a}_S$. As $\overline{a}_S \notin L(S_{\text{syn}})$, by definition of $W_{S,k}^f$ we have that $\overline{a}_k \notin W_{S,k}^f$, so $L(T_{\text{syn}}) \not\subseteq L(S_{\text{syn}})$, leading to contradiction. \hfill \Box

3.1 General Decidability Result

It is possible to show that $W_{S,k}^f$ in domain $(\mathbb{N},=,<,0)$ can be expressed by a nondeterministic $\omega$-S-automaton [4], which can recognise non-$\omega$-regular languages. However, games with such objectives are not known to be decidable, and even less is certain about the would-be complexity. To overcome this obstacle, we introduce the notion of $\omega$-regularly approximable data domains that allows us to construct $\omega$-regular and equi-realisable subset of $W_{S,k}^f$.

Let $\text{lasso}_R$ be the set of lasso-shaped action words over a given set of registers $R$; we write $\text{lasso}$ when $R$ is clear. A data domain $D$ is $\omega$-regularly approximable (regapprox) if for every set of registers $R$, there exists an $\omega$-regular language $QFEAS_R \subseteq (\text{Tst}_R \times \text{Asgn}_R)^\omega$, written as $QFEAS$ when $R$ is clear, satisfying

$$QFEAS \cap \text{lasso} \subseteq \text{FEAS} \subseteq QFEAS$$

and which is recognisable by a nondeterministic Büchi automaton that can be effectively constructed given $R$. Notice this implies that $\text{FEAS}$ and $QFEAS$ coincide on lasso words. Given a URA $S$ with registers $R_S$ and $k$, define:

$$W_{S,k}^{gf} = \{ \overline{a}_k \mid \forall \overline{a}_S : \text{Comp}(\overline{a}_k, \overline{a}_S) \cap QFEAS_R \neq \emptyset \Rightarrow \overline{a}_S \in L(S_{\text{syn}}) \},$$

where $R = R_S \uplus R_k$. The definition of $W_{S,k}^{gf}$ differs from $W_{S,k}^{f}$ only in using $QFEAS_R$ instead of $\text{FEAS}_R$. Thus, $W_{S,k}^{gf} \subseteq W_{S,k}^{f}$, since $\text{FEAS}_R \subseteq QFEAS_R$.

We now show that $W_{S,k}^{gf}$ is $\omega$-regular (which essentially follows from $\omega$-regularity of $QFEAS$ and $S_{\text{syn}}$), and estimate the size of an automaton recognising $W_{S,k}^{gf}$ and the time needed to construct it. For that we use the following terminology for functions of asymptotic growth: a function is $\text{poly}(t)$ if it is $O(t^\kappa)$, $\text{exp}(t)$ if it is $O(2^{\omega t})$, and $2\text{exp}(t)$ if it is $O(2^{\omega^2 t})$, for a constant $\kappa \in \mathbb{N}$. When $\text{poly}$, $\text{exp}$, and $2\text{exp}$ are used with several arguments, the maximal among them shall be taken for $t$.

\footnote{The works [7,8] about games with $\omega$B conditions, which are dual to $\omega$S, together with the Borel result for $\omega$B-$\omega$S [18, Theorem 4] and Martin’s determinacy theorem, can lead to solving register-bounded synthesis in domain $(\mathbb{N},=,<,0)$, but probably with high complexity, and this does not apply to all domains in general.}
Lemma 3. \( W_{S,k}^f \) is \( \omega \)-regular. Moreover, \( W_{S,k}^f \) is recognisable by a universal co-Büchi automaton with \( O(2^k N_{nc}) \) many states that can be constructed in time \( poly(N, n, 2^{\exp(r, k)}) \) where \( n, r, c \) are the number of states, registers, and priorities in \( S \), and \( N \) is the number of states in a nondet. Büchi automaton recogniseing QFEAS.

Proof. (The full proof is in Appendix A.2.) We prove the second claim, implying the first one. Let \( R_S \) be the registers of the URA \( S \), \( R_k \) – the \( k \) registers of the sought transducers, and \( R = R_S \cup R_k \). We assume that transition relations of all automata are complete (they can be made complete within required complexity).

1. We first define an automaton for the language of all extended compositions, which are words in \(( (Tst_R \times Asgn_R) \cdot (Tst_R \times Asgn_R \times R_k) )^\omega \) (see the definition on page 3). The definition requires letters to satisfy certain local conditions, which can be done by a deterministic Büchi automaton, called \( \text{Comp} \), with two states.

2. Let QFEAS’ be the nondet. Büchi automaton derived from that of QFEAS via replacing every edge labelled with an output letter \( (\text{tst, asgn}) \in Tst_R \times Asgn_R \) by edges labeled \( (\text{tst}, \text{asgn}, r^k) \), for every \( r^k \in R_k \). Thus, the projection of \( L(\text{QFEAS}') \) on \( Tst_R \times Asgn_R \) equals \( L(\text{QFEAS}) \).

3. Let \( \tilde{S}_{\text{synt}} \) be the nondet. parity automaton dual to \( S_{\text{synt}} \), thus \( L(\tilde{S}_{\text{synt}}) = \overline{S_{\text{synt}}} \). We construct \( \tilde{S}'_{\text{synt}} \) by extending the input and output alphabets of \( \tilde{S}_{\text{synt}} \) from \( Tst_S \times Asgn_S \) to \( Tst_R \times Asgn_R \) and to \( Tst_R \times Asgn_R \times R_k \), respectively, as follows. Every edge of \( \tilde{S}_{\text{synt}} \) labelled by an input letter \( (\text{tst}^i, \text{asgn}^i) \in Tst_S \times Asgn_S \) is replaced by the edges labelled \( (\text{tst}^{i'}, \text{asgn}^{i'}) \in Tst_R \times Asgn_R \) s.t. \( \text{tst}^i \subseteq \text{tst}^{i'} \) and \( \text{asgn}^i \subseteq \text{asgn}^{i'} \); and every edge labelled by an output letter \( (\text{tst}^o, \text{asgn}^o) \in Tst_S \times Asgn_S \) is replaced by the edges labelled \( (\text{tst}^{o'}, \text{asgn}^{o'}, r) \in Tst_R \times Asgn_R \times R_k \) s.t. \( \text{tst}^o \subseteq \text{tst}^{o'} \) and \( \text{asgn}^o \subseteq \text{asgn}^{o'} \).

4. We now construct \( \text{Comp} \land \text{QFEAS}' \land \tilde{S}'_{\text{synt}} \). First, translate the nondet. parity automaton \( \tilde{S}'_{\text{synt}} \) into a nondet. Büchi automaton with \( O(nc) \) states. Then project all automata and get a nondet. Büchi automaton with \( O(N_{nc}) \) states. Notice that every word accepted by \( \text{Comp} \land \text{QFEAS}' \land \tilde{S}'_{\text{synt}} \) is an extended composition for some transducer action word \( \pi_k \) and automaton action word \( \pi_S \) and such that \( \pi_S \) is rejected by \( \tilde{S}_{\text{synt}} \).

5. Project \( \text{Comp} \land \text{QFEAS}' \land \tilde{S}'_{\text{synt}} \) into the input alphabet \( Tst_S \times Asgn_k \) and output alphabet \( R_k \). Then shift the component \( Asgn_k \) from the input alphabet to the output alphabet, so the number of states becomes \( O(2^k N_{nc}) \). Call the result \( A' \). Note that \( L(A') = W_{S,k}^f \). Finally, we treat \( A' \) as universal co-Büchi: this is a sought automaton with \( O(2^k N_{nc}) \) many states.

Every step can be done in time \( poly(N, n, 2^{\exp(r, k)}) \) or less, and the correctness should be clear as the construction follows the definition of \( W_{S,k}^f \). \( \square \)

We now prove that \( W_{S,k}^f \) and \( W_{S,k}^2f \) are equi-realisable. It is well known that for \( \omega \)-regular specifications (like \( W_{S,k}^f \)) there is no distinction between realisability by finite- and infinite-state transducers. This is not known for \( W_{S,k}^2f \) specifications,
and the answer may depend on data domain. We leave this question for future work, and in this paper focus on realisability by finite-state transducers.

**Lemma 4.** $W_{S,k}^f$ is realisable by a finite-state transducer iff $W_{S,k}^{qf}$ is realisable.

**Proof.** Direction $\Leftarrow$ follows from the inclusion $W_{S,k}^{qf} \subseteq W_{S,k}^f$. Consider direction $\Rightarrow$. Suppose that a transducer $T$ does not realise $W_{S,k}^{qf}$, then we prove that $T$ does not realise $W_{S,k}^f$. We rely on the following claim $\dagger$ proven in Appendix A.3: there exists a lasso-shaped composition $\pi_k \otimes \pi_S \in QFEAS$ such that $\pi_k \in L(T)$ and $\pi_S \notin L(S_{syn})$. Then $\pi_k \otimes \pi_S \in FEAS$, as $QFEAS$ and $FEAS$ coincide over lasso words. Together with $\pi_k \in L(T)$ and $\pi_S \notin L(S_{syn})$, we conclude that $T$ does not realise $W_{S,k}^f$. We now sketch the proof of $\dagger$: it uses the automaton $A$ constructed in step 4 of Lemma 3, called there $Comp \land QFEAS \land S_{syn}$. We show that $T$ does not realise $W_{S,k}^{qf}$ iff the intersection of a slightly modified $T'$ with $A$ is nonempty. As this intersection is $\omega$-regular (and here we use the assumption that $T$ is finite state), it accepts a lasso word, which induces a lasso composition $\pi_k \otimes \pi_S$ with $\pi_k \in L(T)$ and $\pi_S \notin L(S_{syn})$. See Appendix A.3 for details. $\square$

We are now able to prove the main result of this paper.

**Theorem 1.** Let $\mathcal{D}$ be a regapprox data domain such that for every set of registers $R$, one can construct a nondeterministic Büchi automaton with $n_{qf}$ states recognizing $QFEAS_R$ in time $f(|R|)$ for some function $f$. Then:

1. $k$-register-bounded synthesis for URAs over $\mathcal{D}$ is decidable in time $f(k + r) + \exp(\exp(k,r),n_{qf},n,c)$, where $n$ is the number of states of the URA, $c$ its number of priorities, and $r$ its number of registers. Moreover, it is EXPTime-$c$ for fixed $r$ and $k$.

2. For every positive instance of the register-bounded synthesis problem, one can construct, within the same time complexities, a register transducer realizing the specification.

**Proof.** Lemmas 2, 3, 4 reduce register-bounded synthesis to (finite-alphabet) synthesis for the $\omega$-regular specification $W_{S,k}^{qf}$. Since synthesis wrt. to $\omega$-regular specifications is decidable, we get the decidability part of the theorem. Let us now study the complexity. Let $R_S$ be the set of $r$ registers of the URA and $R_k$ be a disjoint set of $k$ registers. First, one needs to construct an automaton recognizing $QFEAS_{R \cup R_k}$. This is done by assumption in time $f(k + r)$. Then, one can apply Lemma 3 and get that $W_{S,k}^{qf}$ can be recognised by universal co-Büchi automaton $A$ with $O(2^n_{qf} n c)$ states, which can be constructed in time $\text{poly}(n_{qf}, n, 2\exp(r,k))$. A universal co-Büchi automaton with $m$ states can be determined into a parity automaton with $\exp(m)$ states and $\text{poly}(m)$ priorities (see e.g. [25]). Remind that the alphabet of $A$ is $\text{Tst}_k \cup (\text{Asgn}_k \times R_k)$. Hence by determinising $A$, and seeing it as a two-player game arena, we get a parity game with $\exp(k)$ edges (corresponding to the actions of Adam and Eve), $\exp(\exp(k),n_{qf},n,c)$ states, and $\text{poly}(\exp(k),n_{qf},n,c)$ priorities. The latter
can be solved in polynomial time in the number of its states, as the number of priorities is logarithmic in the number of states (see e.g. [6]), giving the overall time complexity $\exp(\exp(k), n_{qf}, n, c)$ for solving the game. If we sum this to the complexity of constructing an automaton for $W_{S,k}^{qf}$ plus the complexity for construction an automaton for QFEAS, one gets $\exp(\exp(k), n_{qf}, n, c) + \text{poly}(n_{qf}, n, 2\exp(r, k)) + f(r + k)$, which is $\exp(\exp(k, r), n_{qf}, n, c) + f(r + k)$. If both $r$ and $k$ are fixed, then $\exp(k, r)$ and $f(r + k)$ are constants, so the complexity is exponential only. It is folklore that the hardness holds in the register-free setting (for $r = k = 0$). See for example [17, Proposition 6] for a proof in the finite word setting over a finite alphabet (which generalises trivially to infinite words). There, the proof is done for non-deterministic finite automata, but by determinacy, hardness also holds for its dual, which is a universal automaton.

Now, if a URA specification is realisable for some given $k$, then by Lemmas 2 and 4, $W_{S,k}^{qf}$ is realisable by a finite-alphabet transducer $M$. Since $W_{S,k}^{qf} \subseteq W_{S,k}^{f}$, $M$ also realises the specification $W_{S,k}^{f}$. The mapping $\cdot_{\text{synt}}$ which turns a register transducer into a finite-alphabet transducer is bijective, and hence there exists a register transducer $T$ such that $T_{\text{synt}} = M$. The proof of Lemma 2 exactly shows that $T$ realises $S$, concluding the proof. $\square$

3.2 Register-bounded Synthesis over Data Domain $(\mathbb{N}, =, <, 0)$

We instantiate Theorem 1 for the data domain $(\mathbb{N}, =, <, 0)$. In [13], though there was no general notion of $\omega$-regular approximability for data domains, this notion was introduced in the particular case of $(\mathbb{N}, =, <, 0)$:

**Fact 1 ([13, Thm.8])** *For all $R$, $(\mathbb{N}, =, <, 0)$ has a witness $\text{QFEAS}_R$ of $\omega$-regular approximability expressible by a deterministic parity automaton with $\exp(|R|)$ states and $\text{poly}(|R|)$ priorities, which can be constructed in time $\exp(|R|)$.*

A parity automaton can be translated to a nondeterministic Büchi automaton with a quadratic number of states, hence the above fact allows us to instantiate Theorem 1 to the data domain $(\mathbb{N}, =, <, 0)$:

**Theorem 2.** *For a URA in $(\mathbb{N}, =, <, 0)$ with $r$ registers, $n$ states, and $c$ priorities, $k$-register-bounded synthesis is solvable in time $\exp(\exp(r, k), n, c)$: it is singly exponential in $n$ and $c$, and doubly exponential in $r$ and $k$. It is EXPTIME-complete for fixed $k$ and $r$.*

4 Reducibility Between Data Domains

Theorem 2 relies on the study of feasibility of action words in $(\mathbb{N}, =, <, 0)$ of [13], which is quite involved. Such a study could in principle be generalised to domains such as $\mathbb{Z}$-tuples, as well as finite strings with the prefix relation, by leveraging the results of [10]. However, this would come at the price of a high level of technicality. We choose a different path, and introduce a notion of reducibility between domains, which allows to reuse the study of $(\mathbb{N}, =, <, 0)$ and yields
a compositional proof of the decidability of register-bounded synthesis for the quoted domains.

**Definition 1.** A data domain $\mathbb{D}$ reduces to a data domain $\mathbb{D}'$ if for every finite set of registers $R$, there exists a finite set of registers $R'$ and a rational relation $^5$ $K$ between $R$-automata action words in $\mathbb{D}$ and $R'$-automata action words in $\mathbb{D}'$ that preserves feasibility, in the sense that for every $R$-action word $\pi \in (\text{Tst}_R^R, \text{Asgn}_R)^\omega$, $\pi$ is feasible in $\mathbb{D}$ iff there exists an $R'$-action word in $K(\pi) \in (\text{Tst}_{R'}^{\mathbb{D}'}^R, \text{Asgn}_{R'})^\omega$ feasible in $\mathbb{D}'$.

**Remark 1.** Reducibility is a transitive relation, since rational relations are closed under composition [2, Theorem 4.4], and feasibility preservation is transitive.

**Lemma 5.** If $\mathbb{D}$ reduces to $\mathbb{D}'$ and $\mathbb{D}'$ is regapprox, then $\mathbb{D}$ is regapprox.

**Proof.** Let $R$ be a fixed set of registers, and let $R'$ be a set of registers satisfying the definition of reducibility. Let $\text{FEAS}$ (respectively, $\text{FEAS}'$) be the set of $R$-action words feasible in $\mathbb{D}$ (resp., feasible $R'$-action words in $\mathbb{D}'$).

Our goal is to define an $\omega$-regular set $\text{QFEAS}$ (for $R$) s.t. $\text{QFEAS} \cap \text{lasso} \subseteq \text{FEAS} \subseteq \text{QFEAS}$. Since $\mathbb{D}'$ is regapprox, there is an $\omega$-regular set $\text{QFEAS}'$ (for $R'$) s.t. $\text{QFEAS}' \cap \text{lasso} \subseteq \text{FEAS}' \subseteq \text{QFEAS}'$. Define $\text{QFEAS} = K^{-1}(\text{QFEAS}')$; as the preimage of an $\omega$-regular set by a rational relation, it is (effectively) $\omega$-regular, thus satisfying one of the conditions for $\mathbb{D}$ to be regapprox.

We now show that $\text{FEAS} \subseteq \text{QFEAS}$. Before proceeding, notice that $\text{FEAS} = K^{-1}(\text{FEAS}')$, since $K$ preserves feasibility. Since $\text{FEAS} \subseteq \text{QFEAS}'$, we have $K^{-1}(\text{FEAS}) \subseteq K^{-1}(\text{QFEAS}')$, hence $\text{FEAS} \subseteq \text{QFEAS}$.

It remains to show that $\text{QFEAS} \cap \text{lasso} \subseteq \text{FEAS}$. The inclusion $\text{QFEAS}' \cap \text{lasso} \subseteq \text{FEAS}'$ implies $K^{-1}(\text{QFEAS} \cap \text{lasso}) \subseteq K^{-1}(\text{FEAS}') = \text{FEAS}$ (the latter equality is because $\text{FEAS} = K^{-1}(\text{FEAS}')$). We prove that $\text{QFEAS} \cap \text{lasso} \subseteq K^{-1}(\text{QFEAS} \cap \text{lasso})$, which entails the desired result. Pick an arbitrary $\pi \in \text{QFEAS} \cap \text{lasso}$. Since $K$ is rational, $K(\pi)$ is $\omega$-regular. Moreover, $\text{QFEAS}'$ is $\omega$-regular, which entails that $K(\pi) \cap \text{QFEAS}'$ is as well. Since $\pi \in K^{-1}(\text{QFEAS}')$, the intersection $K(\pi) \cap \text{QFEAS}'$ is nonempty. Since $K(\pi) \cap \text{QFEAS}'$ is $\omega$-regular and nonempty, it contains a lasso word $\pi'$. Thus, $\pi' \in K(\pi) \cap \text{QFEAS}' \cap \text{lasso}$, hence $\pi \in K^{-1}(\text{QFEAS}' \cap \text{lasso})$.

As a direct consequence of Lemma 5 and Theorem 1, we get the following result:

**Theorem 3.** If a data domain $\mathbb{D}$ reduces to a regapprox data domain, then register-bounded synthesis is decidable for $\mathbb{D}$. Moreover, for any positive instance of the register-bounded synthesis problem over $\mathbb{D}$, one can effectively construct a register-transducer realizing the specification of that instance.

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$^5$ Given two finite alphabets $\Sigma$ and $\Gamma$, a relation $K \subseteq \Sigma^\omega \times \Gamma^\omega$ is rational if there exists an $\omega$-regular language $L \subseteq (\Sigma \cup \Gamma)^\omega$ such that $K = \{(\text{proj}_\Sigma(u), \text{proj}_\Gamma(u)) \mid u \in L\}$. This is equivalent to saying that it can be computed by a nondeterministic asynchronous finite-state transducer over input $\Sigma$ with output in $\Gamma^\omega$. See, e.g., [2, Section 3].

$^6$ Note that we do not forbid the existence of unfeasible action words in the image.
4.1 Adding Labels to Data Values

As a first application, we show that one can equip data values with labels from a finite alphabet while preserving regapproximability. By Theorem 3, this yields decidability of register-bounded synthesis for such domains.

Formally, given a data domain $\mathbb{D} = (\mathcal{D}, P, C, c_0)$ and a finite alphabet $\Sigma$, we define the domain of $\Sigma$-labeled data values over $\mathbb{D}$ as $\Sigma \times \mathbb{D} = (\mathcal{D} \times \mathcal{D}, P \cup \{ \text{lab}_\sigma \mid \sigma \in \Sigma \}, \Sigma \times C, (c_0, c_0))$, where $c_0 \in \Sigma$ is a fixed but arbitrary element of $\Sigma$ and, for each $\sigma \in \Sigma$, $\text{lab}_\sigma(\gamma, d)$ holds if and only if $\gamma = \sigma$.

**Lemma 6.** For all finite alphabet $\Sigma$ and data domain $\mathbb{D}$, $\Sigma \times \mathbb{D}$ reduces to $\mathbb{D}$.

**Proof.** Wlog we assume that the set of constants $C$ is empty (modulo adding new predicates to $P$). Let $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_n\}$, where $\sigma_0$ is such that $(\sigma_0, c_0)$ is the initialiser of $\Sigma \times \mathbb{D}$. We first define an encoding at the level of data words. Let $\mu : \Sigma \to \mathbb{D}$ be an injective mapping such that $\mu(\sigma_0) = c_0$. A data word $u$ over $\mathbb{D}$ is a $\mu$-encoding of $v = (\sigma_1, d_1)(\sigma_2, d_2)\ldots$ if it is equal to $\mu(\sigma_1)\ldots \mu(\sigma_n)\mu(\sigma_1)d_1\mu(\sigma_2)d_2\ldots$. The data word $u$ is a valid encoding of $v$ if it is a $\mu$-encoding of $v$ for some $\mu$.

Now, the idea is to define a rational relation $K$ from action words $\overline{\pi}$ over $\Sigma \times \mathbb{D}$ to actions words $\overline{b}$ over $\mathbb{D}$ such that $\overline{\pi}$ is feasible by some $u$ if there exists $\overline{b}$ such that $(\overline{\pi}, \overline{b}) \in K$ and $\overline{b}$ is feasible by a valid encoding of $u$. Let $R$ be a set of registers and assume $\overline{\pi}$ is built over $R$. Let $R^* = \{r_\sigma \mid \sigma \in \Sigma\} \cup R$. Then, any $\overline{b}$ such that $(\overline{\pi}, \overline{b}) \in K$ should make sure that the $n$ first data are different and store them in $r_{\sigma_1}, \ldots, r_{\sigma_n}$ respectively. So, we require that $\overline{b}$ is of the form $\overline{b} = b_\Sigma, b_\overline{\pi}$ where $b_\Sigma = \text{tst}_\pi, \{r_\sigma\}_1 \ldots \{r_\sigma\}_n$ such that for all $1 \leq i, j \leq n$, $\star \neq r_\sigma \in \text{tst}_\pi$. The second part $b_\overline{\pi}$ is an encoding of the tests and assignments of $\overline{\pi} = \text{tst}_1, \text{asgn}_1, \text{tst}_2, \text{asgn}_2 \ldots$. It is of the form

$$b_\overline{\pi} = \text{tst}_1, \text{asgn}_1, \text{tst}_2, \text{asgn}_2 \ldots$$

where for all $i \geq 1$, the following holds:

- for all predicate $p \in P$ of arity $n$, for all $x_1, \ldots, x_n \in R \cup \{\star\}$, if $(\neg)p(x_1, \ldots, x_n) \in \text{tst}_\pi$, then $(\neg)p(x_1, \ldots, x_n) \in \text{tst}_\pi$.

- for all $\sigma \in \Sigma$ and $x \in R \cup \{\star\}$, if $\text{lab}_\sigma(x) \in \text{tst}_\pi$, then $(\sigma = x) \in \text{tst}_\pi$.

Correctness follows from the construction; see Appendix B for details.

The latter result combined with Theorem 3 yields:

**Corollary 1.** Let $\mathbb{D}$ be a regapprox data domain and $\Sigma$ be a finite alphabet, then register-bounded synthesis is decidable for $\Sigma \times \mathbb{D}$.

4.2 Encoding Integers and Tuples

**Lemma 7.** $(\mathbb{Z}, =, <, 0)$ reduces to $(\mathbb{N}, =, <, 0)$.

**Proof.** We encode $\mathbb{Z}$ as two copies of $\mathbb{N}$. Fix a set of registers $R$, and let $R_+$ and $R_-$ be two copies of $R$. We define $K$ at the level of tests and assignments; it is then extended homomorphically. A test $\text{tst}$ generates tests $\text{tst}'$ that satisfy the following, for any pair of registers $r, s \in R$:
– If \( \text{tst} \) implies \( r \geq 0 \) and \( s \geq 0 \), then any atom \( r \triangleleft s \) translates to \( r^+ \triangleleft s^+ \) in \( \text{tst}' \); similarly \( \neg(r \triangleleft s) \) to \( \neg(r^+ \triangleleft s^+) \) for \( \triangleleft \in \{=,\prec\} \).

– If \( \text{tst} \) implies \( r < 0 \) and \( s < 0 \), then \( r = s \) translates to \( r^- = s^- \), \( \neg(r = s) \) to \( \neg(r^- = s^-) \) and \( \neg(
eg(r < s)) \) to \( \neg((r^- > s^-)) \).

Otherwise, we impose no conditions. It remains to translate assignments:

– If \( \text{tst} \) implies \( * > 0 \), then \( \text{asgn}^i = \{r^+ \mid r \in \text{asgn}\} \)

– If \( \text{tst} \) implies \( * = 0 \), then \( \text{asgn}^i = \{r^+ \mid r \in \text{asgn}\} \cup \{r^- \mid r \in \text{asgn}\} \)

– If \( \text{tst} \) implies \( * < 0 \), then \( \text{asgn}^i = \{r^- \mid r \in \text{asgn}\} \).

Since it is a morphism, \( K \) is in particular a rational relation; the fact that it preserves feasibility follows from the encoding. See Appendix B for details. \( \square \)

By Theorems 2 and 3, we get:

**Corollary 2.** Register-bounded synthesis is decidable for \( (\mathbb{Z},=,\prec,0) \).

We now generalise the result to \( d \)-uples of integers, by encoding a \( d \)-uple as a sequence of \( d \) data values. In the following, we fix \( d \geq 1 \).

For \( (n_1,...,n_d),(m_1,...,m_d) \in \mathbb{Z}^d \), define \( (n_1,...,n_d) <^d (m_1,...,m_d) \) if for all \( i \in \{1,...,d\} \), \( n_i \leq m_i \) and \( n_j < m_j \) for some \( j \in \{1,...,d\} \); it is a partial order on \( \mathbb{Z}^d \). The predicate \( =^d \) is defined as expected.

**Lemma 8.** \( (\mathbb{Z}^d,\geq^d ,<^d,0^d) \) reduces to \( (\mathbb{Z},=,\prec,0) \).

**Proof.** Fix a set of registers \( R \). Let \( R_i = \{r^i \mid r \in R\} \) be an \( i \)-th copy of \( R \), for every \( i \in \{1,...,d\} \). We construct a relation \( K \) between action words over \( R \) and action words over \( R' = \bigcup_i R_i \) satisfying the definition of reducibility. To this end, define a relation \( \kappa \) between actions over \( R \) and \( d \)-sequences of actions over \( R' \): \( \{(\text{tst},\text{asgn}), (\text{tst}^1,\text{asgn}^1),..., (\text{tst}^d,\text{asgn}^d)\} \in \kappa \) iff

\[- (r = s) \in \text{tst} \implies (r^i = s^i) \in \text{tst}^i, \text{ for every } i \in \{1,...,d\} \] and every \( r^i,s^i \in R'^i \cup \{*,0\} \);

\[- (r < s) \in \text{tst} \implies (r^i \triangleleft s^i) \in \text{tst}^i, \text{ for every } i \in \{1,...,d\} \] and every \( r^i,s^i \in R'^i \cup \{*,0\} \), where the \( \triangleleft \in \{<,=\} \) are such that at least one of them is \( < \);

\[- \neg(r = s) \in \text{tst} \text{ holds whenever there exists some } j \in \{1,...,d\} \text{ such that } r_j \neq s_j, \text{ i.e. } r_j < s_j \text{ or } r_j > s_j. \text{ Thus, we ask that } (r_j < s_j) \in \text{tst}_j \text{ or } (r_j > s_j) \in \text{tst}_j, \text{ for some } j \in \{1,...,d\} \];

– Finally, \( \neg(r < s) \) holds whenever \( r = s \) or there exists some \( j \in \{1,...,d\} \) such that \( r_j > s_j \). The first case corresponds to the previous item, and the second is encoded as expected.

\[- \text{asgn}^i = \{r^i \mid r \in \text{asgn}\}, \text{ for all } i \in \{1,...,d\}. \]

Then, define \( (a_0,a_1,...,a_d) \in K \) iff \( (a_j^1,a_j^2,...,a_j^d) \in \kappa \) for all \( j \), where every \( a_j^i \) is an action over \( R \) and every \( a_j^i \) is an action over \( R' \).

Again, \( K \) is a morphism, so it is in particular rational. Translating an action word over \( \mathbb{Z}^d \) to \( \mathbb{Z} \) is done by splitting the \( d \)-uples; conversely, one gets a \( d \)-uple by aggregating the \( d \) successive values. See Appendix B for more details. \( \square \)

Since \( (\mathbb{Z},=,\prec,0) \) reduces to \( (\mathbb{N},=,\prec,0) \), and reducibility is transitive, we get:

**Corollary 3.** Register-bounded synthesis is decidable for \( (\mathbb{Z}^d,\geq^d,\prec^d,0^d) \).

**Remark 2.** One can similarly show that \( \mathbb{N}^d \) reduces to \( \mathbb{N} \). Note that \( \mathbb{N}^d \) reduces to \( \mathbb{Z}^d \), by restricting \( \mathbb{Z}^d \) to nonnegative values.
4.3 Finite Words with the Prefix Relation

In this section, we show that synthesis is decidable over the data domain \((\Sigma^*, =, \prec, \epsilon)\), where \(\Sigma\) is a finite alphabet and \(\prec\) denotes the prefix relation. To that end, we use a result of [10], that encodes prefix constraints as integer ones. Showing that feasibility is preserved still requires some work, as registers induce dependencies between successive valuations. In the sequel, \(\Sigma\) is a fixed finite set of size \(l \geq 2\).

First, \((\Sigma^*, =, \prec, \epsilon)\) reduces to a slightly more general domain, namely \((\Sigma^*, =, \text{clen}_\omega, \text{clen}_\prec, \epsilon)\), where, given \(u, v \in \Sigma^*\), \(\text{clen}(u, v)\) denotes the length of the longest common prefix of \(u\) and \(v\), and, for \(a \in \{\prec, =\}\), \(\text{clen}_a(u, v, u', v')\) holds whenever \(\text{clen}(u, v) \prec \text{clen}(u', v')\). The reduction is direct, and follows the same lines as [10, Lemma 3]: \(u \prec v\) is encoded as \(\text{clen}(u, u) = \text{clen}(u, v)\), and \(K\) is actually a morphism on tests, and the identity over assignments.

**Lemma 9.** \((\Sigma^*, =, \prec, \epsilon)\) reduces to \((\Sigma^*, =, \text{clen}_\omega, \text{clen}_\prec, \epsilon)\).

Note also that satisfiability of tests over both domains is decidable, and NP-complete [10, Lemma 7]. It now remains to show that \((\Sigma^*, =, \text{clen}_\omega, \text{clen}_\prec, \epsilon)\) reduces to \((\mathbb{N}, =, \prec, 0)\). The proof draws on ideas similar to that of [10, Lemmas 8,9], which mainly relies on [10, Lemmas 5,6]. Here, it remains to lift them to our synthesis framework, and ensure that feasibility is preserved despite the dependencies induced by registers.

**Lemma 10.** \((\Sigma^*, =, \text{clen}_\omega, \text{clen}_\prec, \epsilon)\) reduces to \((\mathbb{N}, =, \prec, 0)\).

**Proof.** We describe the main ideas underlying the proof; a full proof can be found in Appendix B. From [10, Lemma 5,6], we know that a string valuation is characterised by the length of the longest common prefixes of all its pairs of values, when prefix constraints are concerned. This allows to encode \(\Sigma^*\) in \(\mathbb{N}\): given a set \(R\) of registers, we introduce a register \(\pi_{r,s}\) for each \((r, s) \in R^2\) = \((R \cup \{x\})^2\), where \(x\) is an additional register name that denotes the input data value \(*\) in \(\Sigma^*\). Along the execution, a register \(\pi_{r,s}\) is meant to contain \(\text{clen}(\nu(r), \nu(s))\). Note that in particular, \(\pi_{r,s}\) contains the length of the word stored in \(r\). At each step, we read a sequence of \(|R|\) integers that each corresponds to the value of \(\text{clen}(\ast, r)\) for some \(r \in R\), that we store in the corresponding register \(\pi_{\ast, r}\). We then check that they satisfy the \(\text{clen}\) constraints, as well as the properties of [10, Proposition 2]. The latter consist in logical formulas that can be encoded as tests in \((\mathbb{N}, =, \prec, 0)\), as they only use \(=\) and \(\prec\).

Using [10, Lemma 6], from a sequence of integer valuations (called counter valuations in [10]) that satisfy those properties, we can reconstruct a sequence of string valuations. As the integer valuations additionally satisfy the \(\text{clen}\) constraints, we know that the string valuations satisfy them as well. Thus, if an image \(R'\)-action word is feasible, the original action word is feasible. The converse direction is easier: given a sequence \(\nu_0 \nu_1 \ldots\) of string valuations that is compatible with the \(R\)-action word, at step \(i\) one fills each \(\pi_{r,s}\) with \(\text{clen}(\nu_i(r), \nu_i(s))\).

By Theorems 2 and 3, we get:

**Corollary 4.** Register-bounded synthesis is decidable for \((\Sigma^*, =, \prec, \epsilon)\).
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A Detailed Proofs of Section 3

A.1 Proof of Lemma 1

Proof. Direction $\Rightarrow$. Suppose $\pi_k$ and $\pi_S$ have a composition $\pi_k \otimes \pi_S$ feasible by some $(\nu_0, d_0)(\nu_1, d_1) \ldots$. Let $\nu_j^k$ be $\nu_j$ restricted to the domain $R_S$. The feasibility of $\pi_S$ by $(\nu_0^S, d_0) \ldots$ follows from two facts: (i) $\pi_k \otimes \pi_S$ subsumes the tests of $\pi_S$ and (ii) the assignments of $\pi_S$ on $(\nu_0^S, d_0) \ldots$ and the $R_S$-assignments of $\pi_k \otimes \pi_S$ on $(\nu_0, d_0) \ldots$ assign exactly the same values. Finally, let $\nu_0^k$ be the $R_k$-restriction of $\nu_0$, and for all $j \geq 1$ let $\nu_j^k$ be the $R_k$-restriction of $\nu_{2j-1}$. Then the sequence $\pi_k$ is feasible by $(\nu_0^k, d_0^k, d_0^k) \ldots$, where every $d_j^k = d_{2j}$ and $d_j^S = d_{2j+1}$, because $\pi_k \otimes \pi_S$ subsumes the tests and assignments of $\pi_k$ and $d_j^S = \nu_{j+1}^S(\nu_j^S)$ for all $j$. 


the latter holds because \( \text{tst}_{ij}^{\ast} \) contains the atom \( \ast = r_j^k \). Thus, both \( \text{tst}_{ij} \) and \( \text{tst}_{ik} \) are feasible by \( d_{o1}^j \) ....

Direction \( \Leftarrow \). Suppose \( \text{tst}_{ik} \) and \( \text{tst}_{ij} \) are feasible by \( d_{o1}^j \) .... meaning \( \text{tst}_{ik} \) is feasible by \( (\nu_j^k, d_{o1}^j) \) (see the definition on page 3). The definition requires each letter of an action word to satisfy certain local conditions, and there are no dependencies between letters in different positions, hence this language can be recognised by a deterministic Büchi automaton, called \( \text{Comp} \), with two states: the automaton loops in the initial state and the run is accepting if letters satisfy the conditions, otherwise it moves to a rejecting sink state. The automaton \( \text{Comp} \) can be constructed in time \( 2\exp(r, k) \) as follows: it has a self-loop in the initial state for every input letter in \( \text{Tst}_{R} \times \text{Asgn}_{R} \), as well as for every output letter \( (\text{tst}, \text{asgn}, r) \in \text{Tst}_{R} \times \text{Asgn}_{R} \times R_k \) such that \( (\ast = r) \in \text{tst} \) and \( \text{asgn} \subseteq \text{Asgn}_{S} \); for the other output letters, it goes to a rejecting sink state.

2. Let \( \text{QFEAS}' \) be the nondet. Büchi automaton derived from that of \( \text{QFEAS} \) via replacing every edge labelled with an output letter \( (\text{tst}, \text{asgn}) \in \text{Tst}_{R} \times \text{Asgn}_{R} \) by edges labeled \( (\text{tst}, \text{asgn}, r^k) \), for every \( r^k \in R_k \). Thus, the projection of \( L(\text{QFEAS}') \) on \( \text{Tst}_{R} \times \text{Asgn}_{R} \) equals \( L(\text{QFEAS}) \). We make the transition relation of \( \text{QFEAS}' \) complete if it is not the case. The automaton \( \text{QFEAS}' \) can be constructed in time \( \text{poly}(N, 2\exp(r, k)) \).

3. Let \( S_{\text{syst}} \) be the nondet. parity automaton dual to \( S_{\text{syst}} \), thus \( L(S_{\text{syst}}) = \overline{S_{\text{syst}}} \). We construct \( S_{\text{syst}}^{\prime} \) by extending the input and output alphabets of \( S_{\text{syst}} \) from \( \text{Tst}_{S} \times \text{Asgn}_{S} \) to \( \text{Tst}_{R} \times \text{Asgn}_{R} \) and to \( \text{Tst}_{R} \times \text{Asgn}_{R} \times R_k \), respectively, while preserving the literals of the original alphabets. Formally: every edge of \( S_{\text{syst}}^{\prime} \) labelled by an input letter \( (\text{tst}^i, \text{asgn}^i) \in \text{Tst}_{S} \times \text{Asgn}_{S} \) is replaced by the edges labelled \( (\text{tst}^i, \text{asgn}^i) \in \text{Tst}_{R} \times \text{Asgn}_{R} \) s.t. \( \text{tst}^i \leq \text{tst}^i \) and \( \text{asgn}^i \leq \text{asgn}^i \); and every edge labelled by an output letter \( (\text{tst}^i, \text{asgn}^i) \in \text{Tst}_{S} \times \text{Asgn}_{S} \) is replaced
by the edges labelled \((\text{tst}^\sigma, \text{asgn}^\sigma, r) \in \text{Tst}_R \times \text{Asgn}_R \times R_k\) s.t. \(\text{tst}^\sigma \subseteq \text{tst}^{\sigma'}\) and \(\text{asgn}^\sigma \subseteq \text{asgn}^{\sigma'}\). Since the size of the new alphabets is \(2^{\exp(r,k)}\) and the check \(\sigma \subseteq \sigma'\) takes time in \(\exp(r,k)\), we can construct \(\tilde{S}'_{\text{synt}}\) in time \(\text{poly}(n, 2^{\exp(r,k)})\).

4. We now construct \(\text{Comp} \land QFEAS' \land \tilde{S}'_{\text{synt}}\). First, translate the nondet. parity automaton \(\tilde{S}'_{\text{synt}}\) into a nondet. Büchi automaton with \(O(nc)\) states. Then produce all automata and get a nondet. Büchi automaton with \(O(Nnc)\) states. This can be done in time \(\text{poly}(N,n, 2^{\exp(r,k)})\). Notice that every word accepted by \(\text{Comp} \land QFEAS' \land \tilde{S}'_{\text{synt}}\) is an extended composition for some transducer action word \(\overline{a}_k\) and automaton action word \(\overline{a}_S\) and such that \(\overline{a}_S\) is rejected by \(S_{\text{synt}}\).

5. Project \(\text{Comp} \land QFEAS' \land \tilde{S}'_{\text{synt}}\) into the input alphabet \(\text{Tst}_k \times \text{Asgn}_k\) and output alphabet \(R_k\). This does not affect the number of states and can be done in time \(\text{poly}(N,n, 2^{\exp(r,k)})\).

6. Then we shift the component \(\text{Asgn}_k\) from the input alphabet to the output alphabet, which multiplies the number of states by \(|\text{Asgn}_k| = 2^k\), hence the number of states becomes \(O(2^k Nnc)\). This can be done in time \(\text{poly}(N,n, 2^{\exp(r,k)})\). Call the result \(A'\). Note that \(L(A') = \overline{W}_{\tilde{S},k}^d\).

7. Finally, we treat the nondeterministic Büchi automaton \(A'\) as universal co-Büchi, and this is a sought universal co-Büchi automaton with \(O(2^k Nnc)\) many states.

The correctness should be clear as the construction follows the definition of \(W_{\tilde{S},k}^d\). □

A.3 Proof of Lemma 4

Proof. Direction \(\Leftarrow\) follows from the inclusion \(\text{FEAS} \subseteq QFEAS\) that implies \(W_{\tilde{S},k}^d \subseteq W_{\tilde{S},k}^f\). Consider direction \(\Rightarrow\). Suppose that a transducer \(T\) does not realise \(W_{\tilde{S},k}^d\), then we prove that \(T\) does not realise \(W_{\tilde{S},k}^f\).

We rely on the following claim (\dagger) proven later: there exists a lasso-shaped composition \(\overline{a}_k \otimes \overline{a}_S \in QFEAS\) such that \(\overline{a}_k \in L(T)\) and \(\overline{a}_S \not\in L(S_{\text{synt}})\). Then \(\overline{a}_k \otimes \overline{a}_S \in \text{FEAS}\), as \(QFEAS\) and \(\text{FEAS}\) coincide over lasso words. Together with \(\overline{a}_k \in L(T)\) and \(\overline{a}_S \not\in L(S_{\text{synt}})\), we conclude that \(T\) does not realise \(W_{\tilde{S},k}^f\). We are left to prove (\dagger).

Firstly, from the proof of Lemma 3 (step 4), we use an automaton \(A\) (called there \(\text{Comp} \land QFEAS' \land \tilde{S}'_{\text{synt}}\)) over input alphabet \(\text{Tst}_R \times \text{Asgn}_R\) and output alphabet \(\text{Tst}_R \times \text{Asgn}_R \times R_k\) that accepts a word iff it is an extended composition for some \(\overline{a}_k\) and \(\overline{a}_S\) such that \(\overline{a}_S \not\in L(S_{\text{synt}})\). Note that the language of \(A\) is \(\omega\)-regular, and its every word \(\overline{a}\) uniquely maps to transducer action word \(\overline{a}_k\), automaton action word \(\overline{a}_S\), and composition \(\overline{a}_k \otimes \overline{a}_S\); moreover, if \(\overline{a}\) is a lasso then the mapped words are lassos as well.

Secondly, we define the language \(L'(T)\) from transducer \(T\) by extending its input alphabet from \(\text{Tst}_k\) to \(\text{Tst}_R \times \text{Asgn}_R\) and output alphabet from \(\text{Asgn}_k \times R_k\).
to \( Tst \times Asgn_R \times R_k \), to match those of \( A \), as follows:
\[
L'(T) = \{ (tst_j^0, asgn_j^0)(tst_j^1, asgn_j^1) \ldots : \forall j: tst_j \subseteq tst_j^1, asgn_j \subseteq asgn_j^1, tst_j^0 \in Tst, asgn_j^0 \in Asgn_R \text{ where } (tst_0)(asgn_0, r_0) \ldots \in L(T) \}.
\]
Since \( L(T) \) is \( \omega \)-regular, \( L'(T) \) is \( \omega \)-regular as well. Moreover, every word of \( L'(T) \) uniquely maps to a transducer action word in \( L(T) \), and every word of \( T \) induces a nonempty set of words in \( L'(T) \).

Finally, notice that \( T \) does not realise \( W_{S,k}^g \) if \( L'(T) \cap L(A) \) is nonempty. By the premise, \( T \) does not realise \( W_{S,k}^g \), hence \( L'(T) \cap L(A) \) is nonempty. Since \( L'(T) \cap L(A) \) is \( \omega \)-regular (this is where we need the assumption on finiteness of transducers), it has a lasso word \( \overline{\pi} \). By properties of \( L'(T) \) and \( L(A) \), the word \( \pi \) uniquely maps to \( \overline{\pi} \in L(T), \pi_S \notin L(S_{\text{sym}}) \), and \( \overline{\pi} \otimes \pi_S \in \text{QFEAS} \).

Since \( \overline{\pi} \) is lasso-shaped, they are all, and in particular \( \overline{\pi} \otimes \pi_S \), are lasso-shaped. This concludes the proof of (1) and of the lemma.

\[\Box\]

B Detailed Proofs of Section 4

B.1 Proof of Lemma 6

Proof. It remains to show that the construction is correct. First, \( K \) is rational: a finite transducer just has to first output \( \overline{\pi} \) (which is independent from \( \pi \)) and then, whenever it reads \( \text{tst, asgn} \) in \( \pi \), it outputs any \( \text{tst}_{\text{lab}} \otimes \text{tst}_{\text{data}} \text{asgn} \), which satisfies the two latter conditions (there are only finitely many). Let us show that \( K \) preserves feasibility. Suppose \( \overline{\pi} \) is feasible by \( u = (\sigma_1, d_1)(\sigma_2, d_2) \ldots \) then since \( \overline{\mathcal{D}} \) is assumed to be infinite, there exists an injective mapping \( \mu : \Sigma \to \mathcal{D} \) such that \( \mu(\sigma_0) = c_0 \). Consider the word \( v = \mu(\sigma_1) \ldots \mu(\sigma_n)\mu(\sigma_1)\mu(\sigma_2) \ldots \). It can be checked that there exists a unique action word \( \overline{b} \) feasible by \( v \) of the form \( \overline{\mu(\sigma_1) \ldots} \) and by construction of \( K \) we have \( (\pi, \overline{b}) \in K \).

Conversely, if \( \overline{b} \) is feasible by some \( v = e_1e_2 \ldots e_ne_1'd_1'd_2' \ldots \) such that \( (\overline{\pi}, \overline{b}) \in K \), then, by letting \( e_0 = c_0 \) we have, by construction of \( K \), that \( e_i \neq e_j \) for all \( 0 \leq i < j \leq n \), and for all \( i \geq 1 \), there exists \( 0 \leq j \leq n \) such that \( d_i' = e_j \). By letting \( \mu : \sigma_i \mapsto e_i \) for all \( 0 \leq i \leq n \), we get that \( v \) is a \( \mu \)-encoding of \( (\mu^{-1}(d_1'), d_1)(\mu^{-1}(d_2'), d_2) \ldots \) which by construction of \( K \) is a witness of feasibility of \( \pi \).

\[\Box\]

B.2 Proof of Lemma 7

Proof. We show that the construction is indeed correct. Fix an action word \( \pi_2 = (tst_0, asgn_0) \ldots \). Suppose that \( \pi_2 \) is feasible in \((\mathbb{Z}, \leq)\), by a sequence of valuation-data pairs \( \pi_2 = (v_0, d_0)(v_1, d_1) \ldots \). We construct \( \pi_\forall \in K(\pi_2) \) and a sequence of data-valuations \( \pi_\forall \in (v_0', d_0') \ldots \) that makes \( \pi_\forall \) feasible in \((\mathbb{N}, \leq, 0)\).

Define \( \nu_k(r) = |\nu_k(r)| \) and \( d_i' = d_i \) for all \( r \) and \( i \), where \(|.|\) is the absolute value.

Define \( \pi_\forall = (tst_0', asgn_0') \ldots \) as follows: for all \( i, tst_i' \) contains \( r \) if \( \nu_i'(r) \leq 0 \), where \( \nu_i' \) extends \( \nu_i \) with \( \nu_i'(s) = d_i' \) and \( \nu_i'(0) = 0 \), and \( r, s \in R \cup \{*, 0\} \) and
\( a \in \{ <, =, \} \). Note that \( \text{tst}' \) is uniquely defined by \( \nu'_a, d'_a \); moreover, \((\nu'_a, d'_a) \models \text{tst}' \) and \((\text{tst}, \text{tst}') \in \kappa \). Hence, \( \pi_N \in K(\pi_Z) \), and \( \pi_N \) makes \( \pi_N \) feasible in \((\mathbb{N}, \leq, 0)\).

Conversely, suppose there exists \( \pi_N \in K(\pi_Z) \) feasible in \((\mathbb{N}, \leq, 0)\) by some \( \pi_N = (\nu'_0, d'_0, \ldots) \). Let \( \pi_Z = (\nu_0, d_0, \ldots) \) be a sequence of valuation-data pairs in \((\mathbb{N}, \leq, 0)\) such that for all \( i \): \( \nu_i(r) = -\nu'_i(r) \) if \( r < 0 \in \text{tst}' \), otherwise \( \nu_i(r) = \nu'_i(r) \), for all \( r \in R \); and \( d_i = -d'_i \) if \( r < 0 \in \text{tst}' \), otherwise \( d_i = d'_i \). Since \( \pi_N \) makes \( \pi_N \) feasible in \((\mathbb{N}, \leq, 0)\), and by definition of \( \pi_Z \) and \( \kappa \), \( \pi_Z \) makes \( \pi_Z \) feasible in \((\mathbb{Z}, \leq, 0)\). This concludes the proof that \( \kappa \) satisfies property 1 of the definition of reducibility.

\[ \Box \]

### B.3 Proof of Lemma 8

**Proof.** We have described the construction of \( \kappa \) in the body of the paper. Since it is a morphism, it is in particular a rational relation. We now show that it preserves feasibility. Fix an action word \( \pi = (\text{tst}_0, \text{asgn}_0) \ldots \) over domain \((\mathbb{Z}, \leq, 0, \text{tst}) \) and registers \( Z \).

Suppose \( \pi \) is feasible in \((\mathbb{Z}, \leq, 0, \text{tst}) \) by some \( \pi = (\nu_0, d_0)(\nu_1, d_1) \ldots \). We construct \( \pi' = (\text{tst}'_0, \text{asgn}'_0)(\text{tst}'_1, \text{asgn}'_1) \ldots \) over \( R' \) that is feasible in \((\mathbb{Z}, \leq, 0, \text{tst}) \) by some \( \pi' = (\nu'_0, d'_0)(\nu'_1, d'_1) \ldots \) and such that \( \pi' \in K(\pi) \).

Conversely, assume that \( \pi' = a^0_1 \ldots a^d_1 \ldots d^d_1 \ldots \in K(\pi) \) is feasible by \( \pi = (\nu'_0, d'_0)(\nu'_1, d'_1) \ldots \) where \( \nu_j = \nu'_j \) and \( d_j = \nu'_j \) for all \( j \). Again, \( \nu_j, d_j \) satisfies every literal of every \( \text{tst}_j \) by definition of \( \kappa \), \( \nu_j, d_j \). Moreover, \( \nu_{j+1} = \text{update}(\nu_j, d_j, \text{asgn}_j) \). Therefore, \( \pi \) is feasible by \( \pi' \).

\[ \Box \]

### B.4 Proof of Lemma 10

**Proof.** Let \( R = \{ r_1, \ldots, r_k \} \) be a finite set of registers, and let \( \pi \) be an action word over \( R \). We let \( R' \) be the set of unordered pairs over \( \mathbb{R} \cup \{ x \} \), where \( x \) is an additional variable name that aims at denoting the input word at each step. For readability, a register \( \{ r, s \} \in R' \) is denoted \( \pi_{r,s} \). Note that we only need unordered pairs, since \( \text{clen} \) is a symmetric function. For clarity, in the following, we import the terminology of [10]: a register valuation \( \nu : R \to \Sigma^* \) is called a *string valuation*, while a register valuation \( \nu' : R' \to \mathbb{N} \) is a *counter valuation*.

Now, each action \( (\text{tst}, \text{asgn}) \) of \( \pi \) translates to a sequence of actions as follows:

- First, we read the values of \( \text{clen}(\ast, r) \) for each \( r \in \mathbb{R} \cup \{ x \} \), and store them. This corresponds to a sequence \( (\text{tst}_1, \{ \pi_{r_1,x} \}) \ldots (\text{tst}_k, \{ \pi_{r_k,x} \}) \)\( (\text{tst}_x, \{ \pi_{x,x} \}) \);
- at this point we put no constraints on the incoming data values so \( \text{tst}_1, \ldots, \text{tst}_k, \text{tst}_x \) can be any tests over \((\mathbb{N}, \leq, 0)\). Note that the last value corresponds to the length of the input data value (which is a word over \( \Sigma \)).
- Then, \( \pi' \) checks that the values that have been read indeed yield a string-compatible counter valuation, in the sense of [10, Proposition 2 and Section
3], i.e. a counter valuation from which one can reconstruct a string valuation. Any action \((\text{tst}_{\text{clen}}, \emptyset)\) such that \(\text{tst}_{\text{clen}}\) satisfies the following enforces that it is indeed the case:

- If \(\text{clen}(r, s, r', s') \in \text{tst}\), then \(\text{tst}_{\text{clen}}\) should contain \(\pi_{r,s} = \pi_{r',s'}\)
- If \(\text{clen}(r, s, r', s') \in \text{tst}\), then \(\text{tst}_{\text{clen}}\) should contain \(\pi_{r,s} < \pi_{r',s'}\)
- If \(r = s \in \text{tst}\), then \(\text{tst}_{\text{clen}}\) should contain \(\pi_{r,r} = \pi_{s,s}\)
- If \(r = \epsilon \in \text{tst}\), then \(\pi_{r,r} = 0 \in \text{tst}_{\text{clen}}\)
- If \(\neg \text{clen}(r, s, r', s') \in \text{tst}\), then \(\text{tst}_{\text{clen}}\) should contain \(\pi_{r,s} < \pi_{r',s'}\) or \(\pi_{r,s} > \pi_{r',s'}\)
- If \(\neg \text{clen}(r, s, r', s') \in \text{tst}\), then \(\text{tst}_{\text{clen}}\) should contain \(\pi_{r,s} = \pi_{r',s'}\) or \(\pi_{r,s} > \pi_{r',s'}\)
- If \(\neg \text{clen}(r, s, r', s') \in \text{tst}\), then \(\text{tst}_{\text{clen}}\) should contain \(\pi_{r,s} < \pi_{r',s'}\) (they mismatch at some point)
- If \(\neg \text{clen}(r, s, r', s') \in \text{tst}\), then \(\pi_{r,s} > 0 \in \text{tst}_{\text{clen}}\)

Additionally, we require that \(\pi_{r,r}\) is extended to action sequences homomorphically. Since the resulting \(\pi_{r,r}\) should contain \(\pi_{r',s'}\), and let \(n \in \text{clen}\) values. Formally, for all \(i \geq 0\), and for all \(0 \leq j \leq k\), we let \(n^r_i = \text{clen}(\nu_i(r), s_i)\). Additionally, \(n^r_i = \text{clen}(w_i, w_i) = |w_i|\), and \(n_i^{\text{clen}} \in \mathbb{N}\) is some value whose choice does not matter (recall that the action corresponding to \(\text{tst}_{\text{clen}}\) is dedicated to checking that the input values indeed satisfy the prefix constraints and the string-compatibility conditions). Let us show by induction on \(i\) that there exists an action word \(\vec{\pi}^r\) in \(K(\overline{\emptyset})\) such that \(n_0 \ldots n_0 n_0 n_0^{\text{clen}} n_0 \ldots n_1 n_1 n_1^{\text{clen}} \ldots\) is compatible with \(\vec{\pi}^r\), and that it yields the following sequence of valuations: for
all \( i \geq 0 \) and all \( j \in \{0, \ldots, k, *, \text{clen}\}, \nu^j_i : \{r, s\} \in \binom{R}{2} \mapsto \text{clen}(\nu_i(r), \nu_i(s)) \)
and, for all \( 0 \leq m \leq k \), \( \nu^j_i(\{r_m, x\}) = \begin{cases} 
\text{clen}(\nu_i(r_m), w_i) & \text{if } m \leq j \\
\nu^j_{i-1}(r_m, x) & \text{otherwise}
\end{cases} \). Finally, for all \( j \in \{0, \ldots, k, *, \text{clen}\}, \nu^j_i(\{x\}) = |w_{i-1}| \), and \( \nu^{\text{clen}}_i(\{x\}) = |w_i| \).

For simplicity, we initialise the induction at \(-1\) by picking \( w_{-1} = \epsilon \), which makes the properties true. Now, assume that we have built \( \pi \) up to \( i_0 \). Thus, the current valuation is \( \nu^\pi_{i_0} \) which is such that \( \nu^{\text{clen}}_{i_0} : \{r, s\} \mapsto \text{clen}(\nu_{i_0}(r), \nu_{i_0}(s)) \). Additionally, \( \nu^{\text{clen}}_{i_0}(r, x) \mapsto \text{clen}(r, x) \) for all \( r \in R \cup \{x\} \). Now, the \( R \)-action word reads \( n_{i_0+1}^0 \ldots n_{i_0+1}^k n_{i_0+1}^* \text{clen} n_{i_0+1}^0 \ldots n_{i_0+1}^k n_{i_0+1}^* \text{clen} \ldots \), as defined above. Since we set no conditions on \( \text{tst}_1, \ldots, \text{tst}_k, \text{tst}_* \), we can pick suitable tests so that \( n_{i_0+1}^0 \ldots n_{i_0+1}^k n_{i_0+1}^* \text{clen} \) satisfies them. Finally, \( \text{tst}_{\text{clen}} \) does not depend on \( * \), so any value of \( n_{i_0+1}^* \) is suitable. It remains to show that at this point, \( \nu^{\text{clen}}_{i_0+1} \) satisfies \( \text{tst}_{\text{clen}} \). By definition, it is a string-compatible counter valuation, since it is obtained from a string valuation. Thus, by [10, Proposition 2], it satisfies properties \( \psi_1, \psi_{II} \), and \( \psi_{III} \). Moreover, \( \nu_{i_0+1} \) along with \( w_{i_0+1} \), satisfies the \( \text{clen} \) constraints, so \( \nu^{\text{clen}}_{i_0+1} \) satisfies them as well. Thus, the property holds at step \( i_0 + 1 \).

Conversely, assume that some \( \pi \in K(\Sigma) \) is feasible by some data word \( n_{i_0+1}^0 \ldots n_{i_0+1}^k n_{i_0*}^0 n_{i_0+1}^* \text{clen} n_{i_0+1}^0 \ldots n_{i_0+1}^k n_{i_0+1}^* \text{clen} \ldots \in \mathbb{N}^\omega \), along with a sequence of valuations \( \nu^0 \ldots \nu^{i_0} \nu^{\text{clen}} \nu^1 \ldots \nu^{i_0} \nu^{\text{clen}} \ldots \). We build by induction on \( i \) a sequence of string valuations \( \nu_i : R \mapsto \Sigma^* \), along with a data word \( x = w_0 w_1 \ldots \) which are compatible with \( \pi \), and such that for all \( i \geq 0 \), \( \nu_i : (R, \{x\}) \mapsto \mathbb{N} \) is a counter valuation that is string-compatible with \( \nu_i[\{s \leftarrow w_{i-1}\}] \), with the convention that \( w_{-1} = \epsilon \).

Initially, \( \nu_0 : R \mapsto \epsilon \) and \( w_{-1} = \epsilon \), so \( \nu^0 : \{r, s\} \mapsto 0 \) for all \( r, s \in R \cup \{x\} \), is string-compatible with \( w_0 \).

Now, assume that we have built the \( \nu_i \) up to step \( i \geq 0 \). By construction of \( K(\pi) \), we know that \( \nu^{i+1}_i \) satisfies \( \psi_1 \wedge \psi_{II} \wedge \psi_{III} \), as we ask that \( \text{tst}_{\text{clen}} \) implies them, and is such that for all \( r, s \in R \), \( \nu^{\text{clen}}_i(\{r, s\}) = \text{clen}(\nu_i(r), \nu_i(s)) \).

Indeed, \( \nu^{\text{clen}}_i |_R = \nu^0_i |_R \), as the values of the \( \pi_{r,s} \) for \( r, s \in R \) are left untouched before updating the registers when transitioning to \( \nu^0_{i+1} \). In other words, the counters in \( \nu^{\text{clen}}_i \) indeed contain the length of the longest common prefixes of the strings of \( \nu_i \) (in the terminology of [10], this means that \( \nu^{\text{clen}}_i \preceq_R \nu_i \)). Then, since \( \nu^{\text{clen}}_i \) additionally satisfies conditions \( \psi_1, \psi_{II} \) and \( \psi_{III} \) with regards to \( x \), by [10, Lemma 6], we can construct a value for \( x \) that is consistent with \( \nu_i \). More precisely, by applying [10, Lemma 6] to \( X = \{x\} \), we know that there exists a string \( w_i \) such that for all \( \nu^{\text{clen}}_i \preceq_{R \cup \{x\}} \nu_i[\{s \leftarrow w_i\}] \), i.e. for all \( r, s \in R \cup \{x\} \) (note the addition of \( x \)), we have that \( \nu^{\text{clen}}_i(\{r, s\}) = \text{clen}(\nu_i(r), \nu_i(s)) \), where \( \nu_i(x) = w_i \). Moreover, we know that the \( \text{clen} \) constraints are satisfied, since we encoded them in \( \text{tst}_{\text{clen}} \). Thus, \( w_0 \ldots w_i w_{i+1} \) is compatible with \( \pi \) up to index \( i + 1 \), and the property holds at step \( i + 1 \). □