Pressure - velocity projection method with mixed type approximation for Oseen discrete operator

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Abstract. We are considering the problem of divergence-free projection in rectangular domains for the Oseen (stationary Navier-Stokes) operator. It uses nodal Finite Element method to reconstruct pressure and Finite Difference method for scalar functions and vector functions for other parts of the equations. This allows us to use simple finite difference schemes for linear part and WENO-type schemes for the nonlinear part in the space of discrete velocities. The boundary conditions are formulated in Finite Volume conservative approach for velocity and in Finite Element approach for pressure. We illustrate that the convergence rate depends on the smoothness of the projected function and that the projection method allows one to obtain divergence-free solutions with desired tolerance. We numerically analyze the projection operator that acts in the space of divergence operator image. It is shown that all eigenvalues of the operator are less than unity for Oseledets-type wall boundary conditions for velocity and zero Neumann conditions for the Poisson equation. We show that for smooth functions we obtain forth order convergence in divergence. Finally we demonstrate the method by solving the stationary lid driven cavity problem for the 2D Oseen operator.

1. Introduction

We are considering 2D or 3D incompressible Oseen (Navier–Stokes) operator \( F(u, p) \):

\[
F(u, p) := \begin{cases} 
(u, \nabla)u + \nabla p - \nu \Delta u - g = 0, \\
\nabla \cdot u = 0.
\end{cases}
\]

Here \( u \) is the vector–function of velocity, \( g \) is the vector–function of external forcing, \( p \) is the scalar function of pressure and the parameter \( \nu \) is the kinematic viscosity. The latter mass conservation equation reduces to forcing solenoidal field on velocity vector–function. These equations are considered in a piecewise-continuous domain \( \Omega \). The boundary conditions can be given as defining velocity components, its derivatives or given stress vector components. The pressure is determined up to a constant, but can be uniquely determined by setting the value at a given spatial point or setting pressure gauge, e.g. \( \int_\Omega p d\mathbf{x} = 0 \). We assume that for a correct boundary value problem (BVP) there exists a smooth solution of (1).

There are many methods dedicated to the numerical solution of (1). Most of these methods are either developed for the Finite Element Method discretization type. Their drawback is the application of finite elements for the advection term that results in oscillatory behaviour for high velocity flows and requires stabilization. This problem is solved by the application of stabilization, see [1, 2, 3] for examples. The pressure–velocity coupling can be achieved via either
segregated or coupled mechanism, for more details see [4, 5, 6, 7]. Coupled mechanism usually applies Uzawa–type iterations that have poor convergence speed and very poorly conditioned matrices with complex preconditioners [8, 9, 10]. Segregated methods, such as SIMPLE, may have poor convergence properties and usually require increase of iteration count with the decrease of the $\nu$ parameter value. This is usually tolerable for the solution of the problem (1) itself but poses difficulty for such problems as continuation in parameter space or detection of leading eigenvalues for the linearized operator of (1). Both problems arise in the bifurcation analysis of the Navier–Stokes equations.

Let us now consider some spatial discretization of the problem (1) with parameter $h$ (say, maximum mesh spacing size). Most approaches to the numerical solution of this discrete problem require solution of the linear system with the following matrix form as their internal step:

$$
\begin{pmatrix}
A & G \\
D & 0 \\
\end{pmatrix}
\begin{pmatrix}
\hat{u} \\
\hat{p} \\
\end{pmatrix}
= 
\begin{pmatrix}
g \\
0 \\
\end{pmatrix},
$$

(2)

where operator $A$ includes all velocity terms, operator $G$ is the gradient, $D$ is the divergence, vector $\hat{p}$ is the vector of discrete pressure and vector $\hat{u}$ is the vector of discrete velocity. One must solve discrete saddle point problem (2) in order to find the solution. The system is solved in two steps. Observe, that the velocity can be extracted by inverting $A$:

$$
\hat{u} = A^{-1}(g - G\hat{p}).
$$

(3)

Then, in the first step, the pressure from the continuity equation is found:

$$
D A^{-1} \hat{p} = D A^{-1} g.
$$

(4)

On the second step the velocity is adjusted by (3), now respecting continuity equation. We now observe two problems. First, matrix $A$ should be non–singular. This is usually true for all correct approximations. Next, we observe that in order to find $\hat{p}$, the pressure Schur complement matrix $S := DA^{-1}G$ must be non–singular. Even more, as $h \to 0$ the operator norms of these inverse matrices must be bounded, i.e. $||A^{-1}||_A \leq M_0 < +\infty, ||S^{-1}||_S \leq M_1 < +\infty$. This is related to the famous inf–sup condition by Ladyzhenskaya–Babuska–Brezzi (LBB). In order to satisfy the LBB condition one must choose proper discretization for spatial operators. The easiest choice is the staggered finite difference approximation where pressure is found in the center of a cell and velocities are found in a cell’s edges, introduced in the MAC method [16] and used in SIMPLE–type methods. Different appropriate finite element approximations are discussed in [11]. Then one finds the solution of the problem in coupled way (equations (3) and (4) simultaneously) or in segregated way. We refer reader to [11, 12] and other publications. But it is very difficult to formulate in parallel computational architecture which is needed for DNS computations since resulting block matrix (2) is badly conditioned and requires complex preconditioners for iterative solvers. The segregated strategy is less accurate in terms of divergence field correction process but is more easily implemented in modern parallel computers. We refer the reader to [6, 15] for overviews. Another way to avoid LBB constrain is to use divergence–free basis functions in finite elements.

We use segregated method to solve system (2) without explicitly using LBB condition due to mixed discretization. We adopt high order finite difference methods for velocity approximation and use finite element approximation to reconstruct pressure filed that can be efficiently solved by geometric multigrid method (GMG). The application of this method for the evolutional Navier–Stokes equations is discussed in [13].

The paper is laid out as follows. First we formulate the discretization variants of the method and lay out the general pressure correction algorithm. Next we formulate iterative matrices and investigate the spectrum of these matrices in terms of divergence annihilation. Next we formulate two approaches that solve the problem either by simple iterations or by the application of the Newton’s method. Finally, we solve the stationary 2D lid driven cavity flow as a benchmark.
2. Discretization and pressure correction method

We start with discretization of the Oseen problem mostly following [14] in terms of pressure coupling with slight modifications, reported in [12]. For the sake of brevity we are considering the 2D spacial setup, mostly. We use some discretization methods (given bellow) to translate (1) into the following discrete system (we use the same designation for continuous and discrete variable if it is clear from the context):

\[
\begin{align*}
B(u, u) + Gp &= Cu + g, \\
Du &= 0.
\end{align*}
\]

(5)

Here \(B\) is a discrete nonlinear advection operator and \(C\) is a diffusion matrix multiplied by the parameter \(\nu\). In order to solve (5) one needs to linearise the system and use either Newton’s method or simple iterations. One step in these iteration methods requires to solve the following linear system which is the main computational difficulty provided that solenoidal field \(u_0\) is the linearisation point:

\[
\begin{align*}
N(u_0)u + cN(u_0)u_0 + Gp &= Cu + g + c(Cu_0 - B(u_0, u_0) - Gp_0), \\
Du &= 0.
\end{align*}
\]

(6)

Here \(N(u)\) is the matrix of advection by the field \(u\) and parameter \(c = 1\) for the Newton’s method and \(c = 0\) for the simple iterations method. One can rewrite this system by introducing matrix \(A := N(u_0) + cN(\cdot)u_0 - C\) and vector \(b := g + c(Cu_0 - B(u_0, u_0) - Gp_0)\) as:

\[
\begin{align*}
Au &= -Gp + b, \\
Du &= 0.
\end{align*}
\]

(7)

We can formulate the Schur complement matrix of the pressure: \(S := DA^{-1}G\) which is later used for the verification of the LBB condition. Application of \(D\) matrix to the first equation yields:

\[
DGp = Db - DAu,
\]

(8)

and by back substitution we arrive at the following equation:

\[
P Au = Pb,
\]

(9)

where the divergence correction matrix is found as: \(P = E - G(DG)^{-1}D\) with the continuity equation \(Du = 0\) as additional constraint on the solution.

For the solution of the original discrete system (5) the Newton’s method with \(c = 1\) can be applied with the following iterative scheme:

\[
\begin{align*}
P A(u)v &= P (-B(u, u) + Cu + g), \quad \text{with } Dv = 0, \\
u &\leftarrow u + \alpha v.
\end{align*}
\]

(10)

The scheme is applied either until maximum number of iterations is reached or when \(\|v\| \leq \varepsilon\). The parameter \(0 < \alpha \leq 1\) can be adjusted to ensure convergence.

Another approach is to solve the system iteratively by applying the following SIMPLE–like method (we set \(c = 0\) here). We can always decompose the main matrix into diagonal (\(O\)) and off–diagonal (\(H\) parts as \(A = O + H\) and introduce iterations. The solution of the original system (5) can be presented as:

\[
\begin{align*}
\epsilon^* &= (\epsilon E + A)^{-1} (-Gp + g + \alpha u), \\
D (O^{-1}G\varphi) &= -D(O^{-1}(H\epsilon^* - g - \alpha u)), \\
p &\leftarrow p + \varphi, \\
\epsilon^* &\leftarrow \epsilon^* - G\varphi.
\end{align*}
\]

(11)
The parameter $0 < \alpha < 1$ is called a relaxation parameter and is adjusted to ensure convergence (actually serving as pseudo timestep parameter).

Our main simplification in the following schemes is the usage of Laplace operator $L$ instead of composition operator $DG$. This Laplace operator for pressure is formulated in terms of finite elements. Now we will introduce the discretization of the operators. In discretization we use nodal finite element method (space $P$) to reconstruct pressure and finite difference methods (space $V$ for scalar functions and $V^d$ for vector functions with $d = 2, 3$) for other parts of the equations. Matrices are having the following mapping: identity $E_d : V^d \rightarrow V^d$, $E_p : P \rightarrow P$, gradient $G : P \rightarrow V^d$, Laplace operator in finite element space $L : P \rightarrow P$, interpolation matrix from finite difference to finite element spaces $I : V \rightarrow P$, divergence operator $D : V^d \rightarrow V$ and general advection – diffusion operator matrix $A : V^d \rightarrow V^d$.

The divergence operator is given by the 4-th order central difference approximation $\nabla u_{jkl} = \sum_{(m)=(x,y,z)} (-1/12u(m)SW(j+2,k,l) + 2/3u(m)SW(j+1,k,l) - 2/3u(m)SW(j-2,k,l) + 1/12u(m)SW(j-2,k,l)/h(m)$, where $SW()$ is the operator of increment index interchange, or by the 2-n order central finite differences or by compact differences [17]. In this paper the advection operator is approximated by the upwind scheme (in the 2D case):

$$N(u_0)u_{jk} = \left( \frac{u_{0,j,k} (ux - uwx)/hx + u_{0,j,k} (uxs - uxn)/hyn}{ux_{0,j,k} (uye - uyw)/hx + uy_{0,j,k} (uys - uyn)/hyn} \right),$$

where fluxes $fe, fw, fs, fn$ with $f = \{ux, uy\}$ depend on the vector field $u_0$. We use either 1-st order fluxes:

$$fe = \begin{cases} f_{j,k}; u_{0,j+1,k} + u_{0,j,k} \geq 0, \\ f_{j+1,k}; u_{0,j+1,k} + u_{0,j,k} < 0, \end{cases}$$

$$fw = \begin{cases} f_{j-1,k}; u_{0,j-1,k} + u_{0,j,k} \geq 0, \\ f_{j,k}; u_{0,j-1,k} + u_{0,j,k} < 0, \end{cases}$$

$$fs = \begin{cases} f_{j,k}; u_{0,j,k+1} + u_{0,j,k} \geq 0, \\ f_{j,k+1}; u_{0,j,k+1} + u_{0,j,k} < 0, \end{cases}$$

$$fn = \begin{cases} f_{j,k-1}; u_{0,j,k-1} + u_{0,j,k} \geq 0, \\ f_{j,k}; u_{0,j,k-1} + u_{0,j,k} < 0, \end{cases}$$

or the QUICK scheme:

$$fe = \begin{cases} 0.75f_{j,k} + 0.375f_{j-1,k} - 0.125f_{j+1,k}; u_{0,j+1,k} + u_{0,j,k} \geq 0, \\ 0.75f_{j+1,k} + 0.375f_{j+2,k} - 0.125f_{j,k}; u_{0,j+1,k} + u_{0,j,k} < 0, \end{cases}$$

$$fw = \begin{cases} 0.75f_{j-1,k} + 0.375f_{j-2,k} - 0.125f_{j,k}; u_{0,j-1,k} + u_{0,j,k} \geq 0, \\ 0.75f_{j,k} + 0.375f_{j+1,k} - 0.125f_{j-1,k}; u_{0,j-1,k} + u_{0,j,k} < 0, \end{cases}$$

$$fs = \begin{cases} 0.75f_{j,k} + 0.375f_{j,k-1} - 0.125f_{j,k+1}; u_{0,j,k+1} + u_{0,j,k} \geq 0, \\ 0.75f_{j,k+1} + 0.375f_{j,k+2} - 0.125f_{j,k}; u_{0,j,k+1} + u_{0,j,k} < 0, \end{cases}$$

$$fn = \begin{cases} 0.75f_{j,k-1} + 0.375f_{j,k-2} - 0.125f_{j,k}; u_{0,j,k-1} + u_{0,j,k} \geq 0, \\ 0.75f_{j,k} + 0.375f_{j,k+1} - 0.125f_{j,k-1}; u_{0,j,k-1} + u_{0,j,k} < 0, \end{cases}$$

The linearisation adjoint matrix $N()u$ is given by the 2-nd or 4-th order central differences. The diffusion matrix $C$ is also given either by 2-nd or 4-th order finite difference or by compact differences. The boundary conditions for velocity discretization are set in the finite volume framework [13].

The pressure term is given by the finite element method discretization. Let us introduce rectangular hexahedron $W_{jk}$ that form a 3D (or 2D in case of two dimensions) tessellation of a rectangular domain $\Omega = \bigcup W_j$, such that $W_j \cap W_k = 0, j \neq k$, where $j$ is a multi index with $j$ being a center of $W_j$. We introduce another set of tessellation $U_k$ that is constructed from swapping central nodes and vertices, thus each vertex of $W_j$ becomes a center for $U_k$ and vice versa.
We define basis functions in an element $Q_m$, where $Q$ can be $W$ or $U$, as follows:

$$
\psi_m(x) = \begin{cases} 
  c_{j,k,l}(x), & j \in Q_m, \\
  0, & j \notin Q_m,
\end{cases}
$$

(13)

where $c_{j,k,l}$ is the trilinear function of a rectangular element. Let coordinates of an element be transformed as $\xi_m = x_l / h_m$, where $m = 1, 2, 3$ and $h_m$ is the length of an element in the direction $m$. Then the reference function for a canonical element $\tilde{c}$ is defined as:

$$
\tilde{c}_{v_1,v_2,v_3}(\xi_m) = \prod_{n=1}^{3} (v_n \xi_n + (1 - v_n)(1 - \xi_n)),
$$

where $v_m = \{0, 1\}$ are functions of an element vertices, that map global indexes $j, k, l$ into local indexes.

The following expansion in the element space for a scalar function $p(x)$ is used:

$$
p(x) = \sum_{j \in \{Q\}} p(j) \psi_j(x),
$$

(14)

The approximation of $L$ and $G$ is done using Bubnov – Galerkin projection (choosing test function from the same space of finite elements). We consider the correction equation for the pressure (8) in the form $DGp = f$. The composition operator is replaced with the Laplace operator. Then we obtain the following system:

$$
\int_{\Omega} \Delta p(x) \psi_j(x) dx = \int_{\Omega} f(x) \psi_j(x) dx.
$$

(15)

Inserting (14) into (15) and doing integration by parts:

$$
p_j \int_{\Omega} \nabla \psi_j(x) \nabla \psi_k(x) dx = g_{j,0} \int_{\partial \Omega_{r,0}} \psi_j \psi_k dS + g_{j,1} \int_{\partial \Omega_{r,1}} \psi_j (\nabla \psi_k, n_r) dS + f_j \int_{\partial \Omega} \psi_j \psi_k dS,
$$

(16)

where $g_{j,0}, g_{j,1}$ are coefficients of expansion for Dirichlet and Neumann boundary conditions. For the pressure correction equation both coefficients are zero. This system can be written in matrix form as:

$$
Lp = If.
$$

(17)

The gradient operator $G\phi$ for a scalar function $\phi$ at a central node $k_c$ that has coordinates $x_c$ is defined as:

$$
G\phi_{k_c} := \sum_{j \in \{Q\}} P(j) \nabla \psi_j(x_c).
$$

(18)

The fulfilment of the LBB condition for such approximation is given in [13]. The divergence correction operator is applied iteratively in order to obtain divergence free field. The divergence itself is annulled up to some constant error field which is proportional to the discretization size, which is also shown in [13].

3. Correction matrix properties

Let us consider the following correction operation in term of solenoid decomposition. Let the vector field $u$ be given. We apply divergence and solve the equation $\nabla \cdot \nabla p = -\nabla \cdot u$ to obtain the scalar potential. The vector function is later corrected as $u \leftarrow u - \nabla p$. We can repeat this process in order to obtain the divergence free vector field $u$. This method can be rearranged
in the matrix form by the substitution of the scalar function \( p \) into the correction equation: 
\[ u \leftarrow u - \nabla \left( -\left( \nabla \cdot \nabla \right)^{-1} \nabla \cdot \right) u. \]
Using discrete operators introduced above, one obtains:
\[ u \leftarrow \left( E_d + GL^{-1}ID \right) u. \tag{19} \]

It is more convenient to analyse the divergence evolution matrix. We apply divergence operator again, i.e. 
\[ \mathbf{D} \left( E_d + GL^{-1}ID \right) u \] and obtain the following divergence evolution matrix:
\[ \mathbf{Du} \leftarrow \left( E + DGL^{-1}I \right) \mathbf{Du}. \tag{20} \]

This matrix shows how divergence evolves from one iteration to another. If all eigenvalues of \( \mathbf{M} \) lie in the unit circle then the divergence decreases with iterations. In order to obtain inverse Laplace operator explicitly we set to one the eigenvalue of \( L \) that corresponds to the constant. The eigenvalues of \( \mathbf{M} \) are shown in figure 1. This matrix corresponds to the 2D discretization sized \( 38 \times 38 \). All zero eigenvalues are limited by \( 1 \cdot 10^{-12} \) to be shown in log-scale. The eigenvectors that correspond to the maximum valued eigenvalues are shown in figure 2.

**Figure 1.** Eigenvalues of the divergence evolution matrix \( \mathbf{M} \).

**Figure 2.** Three eigenvectors corresponding to the three largest eigenvalues.
One can observe checkerboard patterns. It indicates that the suggested method handles checkerboard instabilities poorly. Nevertheless, maximum eigenvalue is lesser than unity. We analized the mesh size dependence on the value of the maximum eigenvalue, results are shown in figure 3 that were found using ARPACK with Lanczos method. One can see that the maximum eigenvalue is decreasing with the increase of the spacial resolution.

![Figure 3. The maximum eigenvalue of the divergence evolution matrix $M$ as function of the problem size.](image)

This explains the behaviour of the convergence results obtained in [13] where divergence decreased proportionally to the problem discretization.

Now we are considering the application of correction matrix (19) to the given velocity field, defined by two rectangular blocks in a square domain with zero boundary conditions. We are considering two variants — the first one is pre–smoothed with the Laplace kernel weighted 0.0625 and the second one with no smoothing. The initial velocity in two variants, final velocity, potential function distribution and residual divergence and gradient are shown in figure 4.

Notice, that the residual divergence obtained is similar in nature with the eigenvalues represented in figure 2.

We show that the smoothness of the function plays important role, as can be seeing in figures 5 and 6, where convergence history is shown for different initial smoothing factors. We obtain 3-rd order convergence in divergence with the grid refinement, at worst.

4. Verification

We are considering the lid driven cavity flow problem which is a very good and complicated test case. It is formulated as follows: domain is $[0, 1]^2$ with no–slip boundary conditions imposed. The upper section of the boundary segment $0 \leq x \leq 1$ at $y = 1$ (the "lid") is moving with velocity $(u_x = 1, u_y = 0)$, thus a tangential flow on the boundary is imposed with zero discharge through computational domain. Such setup has two singular points on top with discontinuous boundary conditions. This causes problems with our divergence correction method, so we can analize the behaviour of the suggested method in a stress test conditions. Canonical way to solve the problem is to use ”vorticity–stream function” formulation. This problem was solved many times and very good and exact results are available, e.g. [18, 19, 20, 21] and many others. As the reference solution we take a very thorough work [20, 21], where 4-th order finite difference methods are applied to the problem for very fine grids and results are benchmarked with many common sources, including canonical work by Ghia et.al. [18]. Most importantly, that the author in [20] provides a link to his website where raw data can be found and checked. It is known, that this problem exhibits stationary solutions for super fine resolutions at least for $R \leq 21000$, see [20].
Observe, that we don’t discuss preconditioning technique here that is used in order to solve the Newton’s methods. Here we use geometric multigrid method with the Gauss–Seidel solver for 1-st order upwind method with one iteration of the same divergence corrector method applied on each multigrid level.

We shall compare two approaches, namely, the Newton’s method with projection (10) and the SIMPLE-like iteration method (11). We used maximum of 10 of the divergence correction with the tolerance of $1.0 \cdot 10^{-8}$ in the all operators of the Newton’s method (10) and a single iteration of the correction in the SIMPLE method (11). We used the grid of $110 \times 110$ points and performed all calculations using single thread of Intel(R) Core(TM) i7-5500U CPU @ 2.40GHz.
We solved this problem for three parameter values: \( \nu = \{1/500, 1/1000, 1/5000\} \). Obtained velocity vector fields are presented in figure 7.

First, we observe that the Newton’s method convergence speed decreases with the approach to the solution, see figure 8. This can be caused by the divergence error induced by the inexact projection. We can also see that this drawback increases with the decrease of viscosity. This may pose problems for high Reynolds flows. We also used variable value of the convergence parameter \( \alpha \) in (10) in order to ensure convergence for \( \nu = 1/5000 \). We present comparison of different methods with the reference data from [21] in figure 9. Observe, that we obtain an overall good
agreement with the Newton’s method results, having minor differences for $\nu = 1/5000$. These results are impressive taking into account that we used a coarse grid compared to the reference one in [21].

The SIMPLE method was able to converge to the solution for $\nu = \{1/500, 1/1000\}$ and failed to converge for $\nu = 1/5000$. The values of regularization parameter in (11) for different viscosity are $\alpha = \{0.85, 0.9, [0.85; 0.95; 0.97; 0.99; 0.95]\}$. The last values for the lowest viscosity were tested with no positive result. The convergence history is shown in figure 8 where for the lowest viscosity value we used $\alpha = 0.97$. We may assume that the observed osculatory behaviour can be treated by the grid refinement, but it would take substantial amount of wall time. The reference data comparison is shown in figure 9. It is shown that the method converges to the solution with lower accuracy than the Newton’s method.

We can compare these two methods in terms of wall time that is presented in table 1. Single iteration of the Newton’s method take substantial time compared to the SIMPLE method. However the Newton’s method is beneficial in terms of wall time. This is possible only if a good preconditioner is used for the solution of the linear system.
5. Conclusion
In the paper we use previously described pressure correction method [13] for different solution strategies of the Oseen problem. We reformulate the method in order to be used for stationary solutions and derive matrix operators that can be used to correct divergence of the vector field. Two solution strategies are formulated: Newton’s method and SIMPLE-type iterations method. We numerically analyze the correction operator that acts in the space of divergence operator image. It is shown that all eigenvalues of the operator are less than unity for Osledec's-type wall boundary conditions for velocity and zero Neumann conditions for the Poisson equation. We analized the evolution of the largest eigenvalue and observed that its value decreases with the increase of the grid resolution. We then performed numerical analysis of iterations needed to achieve solenoid vector field depending on its smoothness. It is observed that the method is very efficient for smooth solutions, while it may converge slowly for velocity fields with discontinuities. We also demonstrate the solution of the 2D lid driven cavity problem for moderate Reynolds numbers. It is observed that the Newton’s method is more efficient than the SIMPLE-type iterations. We may suggest to use the method for low or moderate Reynolds numbers flows. Such divergence correction method can be used in Krylov–type solvers provided a good preconditioner is available.

Acknowledgments
This work was supported by the Russian Foundation for Basic Research (grant No. 18-29-10008 mk).

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