Influence of the Rashba effect on the Josephson current through a Superconductor/Luttinger Liquid/Superconductor tunnel junction

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Abstract

The Josephson current through a 1D quantum wire with Rashba spin-orbit and electron-electron interactions is calculated. We show that the interplay of Rashba and Zeeman interactions gives rise to a supercurrent through the 1D conductor that is anomalous in the sense that it persists in the absence of any phase difference between the two superconducting leads to which it is attached. The electron dispersion asymmetry induced by the Rashba interaction in a Luttinger-liquid wire plays a significant role for poorly transmitting junctions. It is shown that for a weak or moderate electron-electron interaction the spectrum of plasmonic modes confined to the normal part of the junction becomes quasi-random in the presence of dispersion asymmetry.
I. INTRODUCTION

In recent years the concept of a Luttinger liquid (LL) as a realistic model of interacting electrons in one-dimensional (1D) metallic structures has received experimental support (see e.g. Refs. [1,2,3,4,5]). Quantum wires (QW) in laterally constrained 2D electron gases (2DEG) and single wall carbon nanotubes (SWNT) - are the two best known structures where LL behavior has been established both theoretically and experimentally.

In SWNTs, where the interaction effects have been shown to be strong[3,4,5], interactions influence the charge and spin transport through the nanotube. When a repulsively interacting LL is coupled to leads (M) of noninteracting electrons, two qualitatively different regimes of charge transport may be realized depending on the quality of the LL/M electrical contacts. For tunnel contacts charge transport through the system is strongly suppressed at low temperatures and bias voltages[6] by the repulsive electron-electron (e-e) interaction. In contrast, for adiabatic contacts when electron backscattering is negligibly small, the conductance is not renormalized by the interaction[7,8,9].

These two types of charge transport behavior also characterize the superconducting properties of a LL wire coupled to superconductors. For adiabatic contacts only Andreev scattering of electrons occurs at the boundaries between LL and bulk superconductors (LL/S boundaries). This process does not lead to a redistribution of charge density along the wire and therefore the Coulomb interaction does not influence the supercurrent through a perfect LL. The above result was proved in Ref. [10] by a direct calculation of the Josephson current through a long S/LL/S junction, \( L \gg \xi_0 \), \( (L \) is the junction length, \( \xi_0 = \hbar v_F/|\Delta| \) is the superconducting coherence length, \( \Delta \) is the superconducting order parameter). In the case of tunnel S/I/LL/I/S junction — where "I" denotes the insulating "layer" — the repulsive e-e interaction results in a renormalization of the junction transparency and the critical Josephson current is strongly suppressed[11].

Here we consider the influence of spin-orbit (s-o) interaction on the Josephson current through a long S/I/LL/I/S junction. It has been known for a long time that the s-o interaction is strong in a 2DEG formed in a GaAS/AlGaAs inversion layer (the Rashba effect[12]) and that it can be controlled by a gate voltage[13,14,15]. In what follows we will consider a quantum wire in a laterally confined 2DEG coupled to superconducting electrodes via tunnel barriers.
FIG. 1: Schematic energy spectrum of 1D spin-1/2 electrons with dispersion asymmetry. The subbands "1" and "2" are characterized by their Fermi velocities $v_{1F} \neq v_{2F}$. In the case of weak Rashba interaction spin projections in the subbands for each given momentum are opposite. For strong Rashba interaction spins in subbands are parallel and they are opposite for right- and left-moving particles.

The influence of the Rashba effect on the electron spectrum and on the transport properties of quasi-1D quantum wires has been studied theoretically in Refs. [16,17], where it was shown that the s-o interaction not only splits the electron spectrum into spin-"up" and spin-"down" subbands, but additionally breaks the chiral symmetry. This implies that left- and right-moving electrons with the same spin projection have different Fermi velocities. Since the time invariance (T-symmetry) of the spin-orbit Hamiltonian implies that $v_{R\uparrow}^{(F)} = v_{L\downarrow}^{(F)} = v_{1F}$ and $v_{R\downarrow}^{(F)} = v_{L\uparrow}^{(F)} = v_{2F}$ the strength of the Rashba effect in a single-channel QW can be characterized by a dispersion asymmetry parameter $\lambda_a = (v_{1F} - v_{2F})/(v_{1F} + v_{2F})$. In Refs. [16,17] it was assumed that in the presence of Rashba interactions the electron spins in a quasi-1D wire are aligned as in the 2D case (see Fig. 1 solid lines for spin projections). Although this assumption is not valid for a strong Rashba coupling [18], the model considered by the authors of these references is interesting in itself. It allows one to study the effects of dispersion asymmetry ($\lambda_a \neq 0$) on the electron dynamics and in the limit $\lambda_a \to 0$ it reproduces the standard results for spin-1/2 electrons without s-o interaction.

Since the electron spin is not conserved in the presence of s-o interactions the classification of spin states assumed in Refs. [16,17] is not evidently correct. Actually, as was shown in Ref. [18], it can be justified only for a weak Rashba interaction. In the most interesting case of
strong Rashba interaction, when the characteristic energy scale introduced by s-o coupling is compared with the energy spacing of 1D subbands, the average spin projections for electrons with large (Fermi) momentum are different. The total energy is minimized when all right-moving (R) electrons have parallel spins and they are in the opposite direction to the spins of left-moving (L) electrons. In what follows we choose the sign of the Rashba interaction so that R-electrons have ”down”-spin and L-electrons are ”up”-spin particles (see Fig.1, dashed lines for spin projections). Notice that under conditions when the Rashba effect is active the electron spin lies in a (2D) plane and orthogonal to the electron momentum (in the 1D case this direction is fixed and ”up” and ”down” spin projections are well defined).

At first we consider the influence of electron dispersion asymmetry in the model elaborated in Refs.[16,17] on the superconducting properties of S/I/LL/I/S junction. We calculate Josephson current perturbatively on the junction transparency \( D = |t_l t_r|^2 \), (\(|t_l t_r|^2 \ll 1\) are the transparencies of tunnel barriers at left and right LL/S interfaces) and for arbitrary strength of electron-electron interaction, dispersion asymmetry \( \lambda_a \) and Zeeman splitting \( \Delta_Z = g \mu_B B \) (\( g \) is g-factor, \( \mu_B \) is Bohr magneton and \( B \) is the magnetic field). Two different geometries of S/LL/S junction are considered. In the first case an effectively infinite LL is connected by the side electrodes to the bulk superconductors (Fig.2). In this geometry one can use periodic boundary conditions for plasmons and all calculations can be done analytically even in the presence of s-o interaction. When dispersion asymmetry is negligibly small \( (\lambda_a \to 0) \) we reproduce the formula for Josephson current derived in Ref.[11]. As was shown in the cited paper the influence of Coulomb interaction on a supercurrent through a tunnel junction \( J = J_c \sin \varphi \) results in suppression of the critical current, which in the presence of Zeeman splitting takes the form \( J_c = J_c^{(0)} R_i(g_c) \cos(\Delta_Z/\Delta_L) \), where \( J_c^{(0)} \sim D\Delta_L \) (\( \Delta_L = \hbar v_F/L \) is the critical current for noninteracting electrons and the interaction-induced renormalization factor \( R_i \) (the subindex ”i” labels the case of effectively infinite LL wire) is small for repulsively interacting electrons \( R_i(g_c \ll 1) \ll 1 \) (here \( g_c \) is the LL correlation parameter in the charge sector). The Zeeman interaction in the absence of Rashba effect results only in additional sign alternating harmonic factor in the critical current (see also Refs.[19]).

We assume that the magnetic field is local and influencing only the normal (nonsuperconducting) part of the junction. (This can be realized in an experiment for instance with the help of a magnetic tip in a scanning tunnelling microscope). The interplay of dispersion
asymmetry and the Zeeman interaction leads to beatings in the supercurrent considered as a function of the local magnetic field $B$. A more unusual prediction is the appearance of supercurrent $J_\alpha$ even at $\phi = 0$. The existence of this anomalous Josephson current is related to the breaking of chiral invariance in quasi-1D quantum wires and the effect manifests itself already for noninteracting particles (see Ref. [21]).

A more realistic geometry for an S/LL/S junction is a finite LL wire (of the length $L$) coupled via tunnel barriers to bulk superconductors (Fig. 3). We will assume that the barrier transparencies are unequal and small (nonsymmetric tunnel junction $|t_L| \neq |t_R| \ll 1$) and evaluate the $\phi$-dependent part of the ground state energy in perturbation theory using the junction transparency $D = |t_L t_R|^2$ as expansion parameter. Normal and Andreev scattering at the interfaces can be taken into account by the boundary terms in the Hamiltonian of the S/I/LL/I/S junction. To first order in the junction transparency the problem is reduced to the evaluation of four-fermion correlation functions for a two channel LL Hamiltonian with the boundary conditions implying the absence of particle current through the S/LL interfaces at $x = 0, L$. In the absence of spin-orbit interaction the problem of quantization of plasmon modes in a finite LL with open ends was solved in Ref. [22]. Here, we generalize the quantization procedure proposed in the cited paper to the case of spin-1/2 fermions with dispersion asymmetry.

The two-channel LL Hamiltonian describing our system is diagonalized exactly by the canonical transformation suggested in Ref. [23]. We show that the spectrum of plasmons in a LL with open ends in the presence of dispersion asymmetry is determined by a transcendental equation. In the general case the spectrum forms a set of quasi-random energy levels. For noninteracting electrons, or when the energy dispersions are symmetric ($v_{1F} = v_{2F} = v_F$), the spectrum is reduced to a set of equidistant energy levels. In the limit of strongly
interacting particles the plasmon spectrum also becomes regular. We calculate the Josephson current in the cases when the spectral equation can be solved analytically.

We find that for noninteracting electrons the critical Josephson current through a tunnel $S/QW/S$ junction is enhanced by the presence of dispersion asymmetry. This behavior is specific for 1D electrons and the effect disappears in 2D junctions.$^{25}$

When the Rashba effect is not pronounced ($\lambda_a \to 0$), the Josephson current through a tunnel $S/LL/S$ junction is described by an expression analogous to the formula derived for an effectively infinite LL. However, the interaction-induced renormalization (suppression) of the critical supercurrent is much stronger for a finite LL than for an infinite one $R_f(g_c \ll 1) \ll R_i(g_c \ll 1)$ provided the electron-electron interaction is short-ranged.

As already mentioned in the Introduction, the electron spin states in a 1D quantum wire in the regime of strong Rashba effect are fully determined by s-o interaction and the electrons with large (Fermi) momenta behave as truly chiral particles. That is the electron spin polarizations and the direction of their motion (right/left) are strongly correlated and all right(left)-moving particles, irrespective of their Fermi velocities, have parallel spins$^{18}$ which are opposite to the spin polarizations of left(right)-moving electrons. So it is reasonable to expect that in this case the magnetic field via the Zeeman interaction will induce an anomalous supercurrent (at $\varphi = 0$) even in the absence of dispersion asymmetry. We calculate the Josephson current in a SILLIS junction in a model when the Rashba s-o interaction is smoothly switched on in a 1D QW and spin-flips are not accompanied by electron backscattering. An anomalous influence of Zeeman splitting on the critical supercurrent is predicted.
II. PROXIMITY-INDUCED SUPERCONDUCTIVITY IN A LUTTINGER LIQUID WIRE WITH CHIRAL SYMMETRY BREAKING

It is physically evident that the Coulomb interaction in a long S/I/LL/I/S junction suppresses the critical supercurrent due to a strong Kane-Fisher renormalization of the bare tunneling matrix elements. The Josephson current through a Luttinger liquid coupled to bulk superconductors via tunnel contacts was first calculated by Fazio et al. who showed that the critical supercurrent is multiplicatively renormalized (suppressed) by a repulsive electron-electron interaction. The calculations were performed in linear (Fig. 2) and ring-like geometries. In both cases periodic boundary conditions for the plasmonic modes can be imposed. Although from an experimental point of view the considered geometries of an SNS junction look rather artificial, they do allow one to simplify the calculations.

For noninteracting electrons the critical supercurrents in an SNS junction formed by a long (effectively infinite) quantum wire connected to superconductors by side tunnel contacts (separated by a distance $L$) and in an SNS junction where a finite length QW bridges the gap (of the same length L) between two superconductors differ only by a numerical factor. If the QW is treated as a Luttinger liquid this factor becomes a function of the interaction strength and can be evaluated analytically (see below). When both electron-electron interactions and dispersion asymmetry are present the calculations are more cumbersome. We start with the case of a side-contacted QW where we are able to analytically evaluate the supercurrent for arbitrary interaction strength and dispersion asymmetry parameter.

The Hamiltonian $H = H_{LL} + H_b$ of a S/I/LL/I/S junction is a sum of the LL Hamiltonian $H_{LL}$ and the boundary Hamiltonian $H_b$. The latter describes the effective boundary pairing and scattering interactions produced by the superconducting and normal scattering potentials at the points $x = 0$ and $x = L$ (see Ref. [26]). When chiral symmetry is broken the corresponding spin-1/2 LL Hamiltonian expressed in terms of charge densities of chiral fields takes the form

$$H_{LL} = \pi\hbar \int dx \left\{ u_1 (\rho_{R\uparrow}^2 + \rho_{L\downarrow}^2) + u_2 (\rho_{R\uparrow}^2 + \rho_{L\downarrow}^2) \right\} + \frac{V_0}{\pi\hbar} \left( \rho_{R\uparrow}\rho_{R\downarrow} + \rho_{L\uparrow}\rho_{L\downarrow} + \rho_{R\uparrow}\rho_{L\downarrow} + \rho_{R\downarrow}\rho_{L\uparrow} \right) + \rho_{R\uparrow}\rho_{L\downarrow} + \rho_{R\downarrow}\rho_{L\uparrow} \right\},$$

where $\rho_{R/L,\uparrow/\downarrow}$ are the charge density operators of right/left-moving electrons with up/down-
spin projection, \( V_0 \) is the strength of electron-electron interaction \( (V_0 \sim e^2) \) and \( u_{1(2)} = v_{1(2)F} + \frac{V_0}{2\pi \hbar} \). The Fermi velocities \( v_{1F} \neq v_{2F} \) are different due to an assumed electron dispersion asymmetry (see Fig. 1). We have neglected the magnetic field-induced corrections to the Fermi velocities and assumed that the effective electron-electron interaction has no significant magnetic field dependence. Both the neglected effects are of "1/\( \varepsilon_F \)"-order (see e.g. Ref. [27]) and they are irrelevant for Zeeman splittings \( \Delta_Z \ll |\Delta| \ll \varepsilon_F \).

The Hamiltonian (1) is equivalent to a two-channel LL Hamiltonian and can be diagonalized by the canonical transformation suggested in Ref. [23] (see Appendix I). The diagonalized Hamiltonian is

\[
H_d = \pi \hbar \int dx \{ s_1(\rho_{R1}^2 + \rho_{L1}^2) + s_2(\rho_{R2}^2 + \rho_{L2}^2) \},
\]

where \( s_{1,2} \) are the velocities of noninteracting bosonic modes (see Appendix I).

We assume strong normal backscattering at the S/N-boundaries (tunnel junction). In this limit the pairing Hamiltonian contains a small factor - the amplitude of Andreev backscattering

\[
r_A^{r,l} \simeq D_{r,l} \exp \left[ i \left( \frac{\pi}{2} + \varphi_{r,l} \right) \right],
\]

where \( D_{r,l} \ll 1 \) is the transparency of the barrier at the right(left) interface, \( \varphi_{r,l} \) is the phase of the superconducting order parameter on the right(left) bank of the junction. The boundary Hamiltonian for our two-channel system can be expressed in terms of the Andreev scattering amplitudes Eq. (3) up to an overall numerical factor \( C \), which will be specified later

\[
H_b/C = \hbar v_{1F} \left[ r_A^{s(r)}(0)\Psi_{R\uparrow}(0)\Psi_{L\downarrow}(L) - r_A^{s(r)}(L)\Psi_{R\uparrow}(L)\Psi_{L\downarrow}(0) \right] + \hbar v_{2F} \left[ r_A^{s(r)}(L)\Psi_{R\downarrow}(L)\Psi_{L\uparrow}(L) - r_A^{s(r)}(0)\Psi_{R\downarrow}(0)\Psi_{L\uparrow}(0) \right] + h.c.
\]

To second order in the Andreev scattering amplitude the phase dependent part of the ground state energy takes the form

\[
\delta E^{(2)}(\varphi) = \sum_j \frac{|\langle j|H_b|0\rangle|^2}{E_{0} - E_{j}} = \frac{1}{\hbar} \int_0^\infty d\tau \langle 0|H^\dagger_b(\tau)H_b(0)|0 \rangle ,
\]

where \( H_b(\tau) \) is the boundary Hamiltonian (4) in the imaginary time Heisenberg representation. After substituting Eq. (4) into Eq. (5) we get the following expression for \( \delta E^{(2)}(\varphi) \) expressed in terms of electron correlation functions

\[
\delta E^{(2)}(\varphi) = -4\hbar \Re \left\{ r_A^{s(r)}r_A^{s(r)} \int_0^\infty d\tau [v_{1F}^2(\Psi_{R\uparrow}(\tau,0)\Psi_{L\downarrow}(\tau,0)\Psi_{L\downarrow}^\dagger(0,L)\Psi_{R\uparrow}^\dagger(0,L)) + v_{2F}^2(\uparrow \leftrightarrow \downarrow)] \right\}.
\]
We will calculate the electron correlation functions in Eq. (6) by making use of the bosonization technique. Notice that the Zeeman splitting introduces an extra $x$-dependent phase factor in the chiral components of the fermion fields. This interaction can be taken into account (see e.g. Ref. [30]) by replacing the fermion operators in Eq. (6) by $\Psi_{\mu,\sigma}^{(Z)}$, where

$$\Psi_{\mu,\sigma}^{(Z)} = \exp(iK_Z x)\Psi_{\mu,\sigma}, \quad K_Z = \frac{\Delta Z}{4\hbar v_F} \frac{\mu \sigma - \lambda a}{1 - \lambda a^2},$$

(7)

Here $v_F = (v_{1F} + v_{2F})/2$, $\mu = (R, L) \equiv (1, -1)$, $\sigma = (\uparrow, \downarrow) \equiv (1, -1)$, $\Delta Z$ is the Zeeman splitting, and $\lambda a = (v_{1F} - v_{2F})/(v_{1F} + v_{2F})$ is the parameter which characterizes the strength of chiral symmetry breaking.

The standard bosonization formulae now read

$$\Psi_{R(L),\uparrow}(x, t) = \exp\left\{ \pm i \sqrt{\frac{4\pi}{2\pi a_1(2)}} \Phi_{R(L),\uparrow}(x, t) \right\}, \quad \Psi_{R(L),\downarrow}(x, t) = \exp\left\{ \pm i \sqrt{\frac{4\pi}{2\pi a_2(1)}} \Phi_{R(L),\downarrow}(x, t) \right\},$$

(8)

where $a_{1,2}$ are the cutoff parameters of the two-channel LL. The chiral bosonic fields in Eq. (8) for a finite length LL are represented as follows (see e.g. Ref. [31])

$$\Phi_{R(L),\uparrow}(x, t) = \frac{1}{2} \hat{\phi}_{R(L),\uparrow} + \hat{\Pi}_{\uparrow} \frac{x \mp v_{1(2)} t}{L_{1(2)}} + \varphi_{R(L),\uparrow}(x, t),$$

$$\Phi_{R(L),\downarrow}(x, t) = \frac{1}{2} \hat{\phi}_{R(L),\downarrow} + \hat{\Pi}_{\downarrow} \frac{x \mp v_{2(1)} t}{L_{2(1)}} + \varphi_{R(L),\downarrow}(x, t).$$

(9)

(10)

Here the zero mode operators ($\hat{\phi}_{R(L),\sigma}, \hat{\Pi}_{\sigma}$) obey the commutation relations $[\hat{\phi}_{R(L),\sigma}, \hat{\Pi}_{\sigma'}] = \mp i \delta_{\sigma,\sigma'}$ and the nontopological (harmonic) components $\varphi_{R(L),j}(x, t)$ are

$$\varphi_{R(L),j}(x, t) = \sum_q \sqrt{\frac{1}{2qL_j}} \left\{ e^{i q(x - v_j t)} \hat{b}_q + h.c. \right\},$$

(11)

where $b_q (b^d_q)$ are the standard bosonic annihilation(creation) operators. The effective quantization lengths $L_j (j = 1, 2)$ depend on the boundary conditions and will be specified in the next section.

As is well known (see e.g. Ref. [32]), the topological excitations for an effectively infinite LL play no role and can be omitted in Eqs. (9) and (10). After straightforward transformations Eq. (6) is reduced to the following expression

$$\delta E^{(2)}(\varphi) = 4\hbar D \left\{ v_{1F}^2 \cos(\varphi - \frac{\Delta Z}{\Delta_{1L}}) \int_0^\infty d\tau \Pi_1(\tau) + v_{2F}^2 \cos(\varphi + \frac{\Delta Z}{\Delta_{2L}}) \int_0^\infty d\tau \Pi_2(\tau) \right\},$$

(12)
where \( D = D_i D_r \) is the junction transparency, \( \Delta_{1(2)\ell} = \hbar v_{1(2)\ell}/L \) and

\[
\Pi_{1(2)}(\tau) = \frac{1}{(2\pi a_{1(2)})^2} \exp \left\{ \frac{2\pi}{2} \left[ \langle \varphi_\sigma(\tau, -L) \varphi_\sigma \rangle + \langle \Theta_\rho(\tau, -L) \Theta_\rho \rangle \pm \langle \Theta_\rho(\tau, -L) \varphi_\sigma \rangle \pm \langle \varphi_\sigma(\tau, -L) \Theta_\rho \rangle \right] \right\} \quad .
\]

Here \( \varphi_\sigma \equiv \varphi_\sigma(0, 0), \Theta_\rho \equiv \Theta_\rho(0, 0) \) and double brackets denote the subtraction of the corresponding vacuum average at the points \( \tau, x = 0 \). The charge \( (\rho) \) and spin \( (\sigma) \) bosonic fields in Eq. \( (13) \) are related to the chiral fields \( \varphi_{R(L),\uparrow(\downarrow)} \) introduced above by the simple linear equations

\[
\varphi_\sigma(\Theta_\rho) = \frac{1}{\sqrt{2}} (\varphi_{R,\uparrow} \pm \varphi_{L,\uparrow} \mp \varphi_{R,\downarrow} - \varphi_{L,\downarrow})
\]

(the upper sign corresponds to \( \varphi_\sigma \) and the lower sign denotes \( \Theta_\rho \)). With the help of the canonical transformation Eq. \( (55) \) the chiral bosonic fields in Eq. \( (14) \) can be expressed in terms of noninteracting plasmonic modes \( \varphi_{R/L,j} \) \( (j = 1, 2) \). For an infinitely long LL the propagators of these fields are (see e.g. Ref. \[32\])

\[
\langle \langle \varphi_{R/L,j}(t, x) \varphi_{R/L,k} \rangle \rangle = -\frac{\delta_{jk}}{4\pi} \ln \frac{a_k + x + is_{k} t}{a_k} \quad .
\]

where the velocities \( s_j \) are defined in Eqs. \( (58) \) and \( (59) \). Finally, the expression for the Josephson current through a side-coupled LL wire (Fig. \[1\]) takes the form

\[
J^{(i)}(V_0, \lambda_\alpha, \Delta_Z; \varphi) = \frac{ev_{F}}{L} D \frac{C}{2\pi^2} \left\{ \left( \frac{a_1}{L} \right)^{2(\gamma_1 - 1)} \frac{v_{1F}^2}{s_{1} v_{F}} B(1/2, \gamma_1 - 1/2) F(1/2, \gamma_{1s}; \gamma_1; 1 - (s_2/s_1)^2) \sin \left( \varphi - \frac{\Delta_Z}{\Delta_{1L}} \right) \right. \\
+ \left. \left( \frac{a_2}{L} \right)^{2(\gamma_2 - 1)} \frac{v_{2F}^2}{s_{1} v_{F}} B(1/2, \gamma_2 - 1/2) F(1/2, \gamma_{2s}; \gamma_2; 1 - (s_2/s_1)^2) \sin \left( \varphi + \frac{\Delta_Z}{\Delta_{2L}} \right) \right\} \quad ,
\]

where \( B(x, y) = \Gamma(x) \Gamma(y)/\Gamma(x + y) \) is the beta function, \( F(\alpha, \beta; \gamma; z) \) is the hypergeometric function (see e.g. Ref. \[24\]), \( v_F = (v_{1F} + v_{2F})/2 \), \( \gamma_{j} = \gamma_{js} + \gamma_{jc} \) \( (j = 1, 2) \) and

\[
\gamma_{1s} = \frac{v_{2F}}{v_{1F}} \frac{\sin^2 \psi}{g_2}, \quad \gamma_{1c} = \frac{\cos^2 \psi}{g_1}, \quad \gamma_{2s} = \gamma_{1s}(1 \leftrightarrow 2), \quad \gamma_{2c} = \gamma_{1c}(1 \rightarrow 2)
\]

Here \( g_j = s_j/v_{jF} \) are the correlation parameters of a two-channel LL (see Appendix I) and angle parameter \( \psi \) is defined by Eq. \( (54) \).

By using the properties of the hypergeometric function it is easy to show that for a given strength of the electron-electron interaction the Josephson current \( J^{(i)} \) satisfies the equations

\[
J^{(i)}(-\lambda_\alpha, \Delta_Z; \varphi) = J^{(i)}(\lambda_\alpha, -\Delta_Z; \varphi) = -J^{(i)}(\lambda_\alpha, \Delta_Z; -\varphi)
\]
which describe the symmetries of electric current with respect to space and time reflections. In particular one can infer from Eq. (18) that when both the chiral symmetry breaking ($\lambda_a \neq 0$) and the Zeeman ($\Delta_Z \neq 0$) interaction are present the supercurrent can persist even at $\varphi = 0$. This anomalous supercurrent exists already for noninteracting electrons ($V_0 = 0$) and at first we analyze Eq. (16) in the limit of weak e-e interaction.

For noninteracting electrons ($V_0 = 0, g_1 = g_2 = 1$) Eq. (16) is much simplified to

$$J_0^{(i)}(\varphi) = J_c^{(0)} \frac{1}{2} \left\{ \frac{v_{1F}}{v_F} \sin \left( \varphi - \frac{\Delta_Z}{\Delta_{1L}} \right) + \frac{v_{2F}}{v_F} \sin \left( \varphi + \frac{\Delta_Z}{\Delta_{2L}} \right) \right\} \tag{19}$$

where $J_c^{(0)} = (D ev_F/4L)(C/\pi)$ is the critical Josephson current. We see that in the absence of magnetic interaction ($\Delta_Z = 0$) the Rashba interaction in the considered geometry of SNS junction does not affect Josephson current at all (see also Ref. [25] where an analogous result was derived for a short 2D SNS junction in the presence of Rashba spin-orbit interactions).

The interplay of the Zeeman interaction and the dispersion asymmetry in quantum wires results in the appearance of an anomalous (at $\varphi = 0$) Josephson current $J_a^{(i)} \equiv J_0^{(i)}(\varphi = 0)$ which it is convenient to express in terms of the asymmetry parameter $\lambda_a$ and the magnetic phase $\chi_B = \Delta_z/\Delta_L$ ($\Delta_L = \bar{h} v_F/L$) as

$$J_a^{(i)}(\lambda_a, \chi_B) = J_c^{(0)} \frac{1}{2} \left\{ (1 - \lambda_a) \sin \left( \frac{\chi_B}{1 - \lambda_a} \right) - (1 + \lambda_a) \sin \left( \frac{\chi_B}{1 + \lambda_a} \right) \right\}. \tag{20}$$

As is evident from the above equation, the anomalous supercurrent $J_a$ appears only when both the dispersion asymmetry and the Zeeman interaction are present $J_a(\lambda_a = 0, \Delta_Z) = J_a(\lambda_a, \Delta_Z = 0) = 0$. In the limit of weak dispersion asymmetry $\lambda_a \ll 1$ (a realistic case for quantum wires formed in 2DEG) the Josephson current as a function of Zeeman splitting demonstrates a simple harmonic behavior with a slow periodically varying amplitude (beats)

$$J_a^{(i)} \approx J_c^{(0)} \sin \left[ \varphi + \lambda_a \left( \frac{\Delta_Z}{\Delta_L} - \tan \frac{\Delta_Z}{\Delta_L} \right) \right] \cos \left( \frac{\Delta_Z}{\Delta_L} \right). \tag{21}$$

Now we analyze Eq. (16) in the limit when the Rashba interaction is negligibly small ($\lambda_a = 0$). In this case the Josephson current through the LL wire takes the form

$$J_g^{(i)} = J_c^{(g)} \cos \chi_B \sin \varphi, \quad J_c^{(g)} = R(g_c) J_c^{(0)}, \tag{22}$$

where the interaction-induced renormalization factor $R(g_c)$ (here $g_c^{-1} = \sqrt{1 + 2V_0/\pi \bar{h} v_F}$ is the LL correlation parameter in the charge sector) is equivalent to the one evaluated in Ref. [11]

$$R(g_c) = g_c \frac{\Gamma(1/2g_c)}{\sqrt{\pi} \Gamma(1/2 + 1/2g_c)} F \left( \frac{1}{2}, \frac{1}{2}; \frac{1}{2g_c} + \frac{1}{2}; 1 - g_c^2 \right) \left( \frac{a}{L} \right)^{g_c^{-1} - 1}. \tag{23}$$
In the limit of strong interaction $V_0/\hbar v_F \gg 1$ the renormalization factor is small

$$R(g_c \ll 1) \simeq \frac{\pi}{2} \left( \frac{\hbar v_F}{V_0} \right)^{3/2} \left( \frac{a}{L} \right)^2 \sqrt{2V_0/\pi \hbar v_F} \ll 1 \quad (24)$$

and the Josephson current through a S/I/LL/I/S junction is strongly suppressed.

When both the electron-electron interaction and the dispersion asymmetry are strong, only one of the two terms in Eq. (16) dominates. The corresponding critical current (for definiteness we assume that $v_{1F} \simeq v_F/2 \gg v_{2F}$)

$$J_c^{(i)} = J_c^{(0)} \pi \left( \frac{\hbar v_{1F}}{V_0} \right)^{3/2} \left( \frac{a}{L} \right)^2 \sqrt{2V_0/\pi \hbar v_{1F}} \quad (25)$$

is much smaller than the critical current $J_c$ in the absence of dispersion asymmetry ($v_{1F} = v_{2F}$). It means that chiral symmetry breaking in quantum wires enhances the interaction-induced suppression of the Josephson current.

III. DISPERSION ASYMMETRY AND QUASI-RANDOM ENERGY SPECTRUM OF PLASMONS

In this section we evaluate the spectrum of topological excitations and plasmonic modes in a LL wire of the length $L$ end-coupled to bulk superconductors (see Fig.3). The electron normal backscattering at the N/S interfaces is assumed to be strong. The Josephson current can be calculated to the first order on junction transparency using Eq.(6) for the $\phi$-dependent part of the ground state energy. For a finite length LL the zero modes in Eqs. (9) and (10) contribute to the energy and after some algebra we get for $\delta E^{(2)}(\phi)$ an expression analogous to Eq. (12) where now $\Pi_{1(2)}(\tau)$ is replaced by the product $\Pi_{1(2)}(\tau)Q_{1(2)}(\tau)$. The zero mode contributions $Q_{1(2)}(\tau)$ are ($j = 1, 2$)

$$Q_j(\tau) = \exp \left\{-2\pi \left\{ \frac{L}{L_j} \left( \hat{\Pi}_\uparrow - \hat{\Pi}_\downarrow \right) + \frac{iv_j \tau}{L_j} \left( \hat{\Pi}_\uparrow + \hat{\Pi}_\downarrow \right) \right\} \right\} \exp \left( \frac{2\pi v_j \tau}{L_j} \right). \quad (26)$$

To zeroth order of perturbation theory in the barrier transparencies the electrons are confined to the normal region. Therefore the correlation functions in Eq. (12) have to be calculated with the appropriated boundary conditions. The natural boundary condition for our problem is the requirement that the particle current through the interfaces at $x = 0, L$ is zero

$$J_\sigma \sim \Re \{ i \Psi^\dagger_{\sigma} \partial_x \Psi_{\sigma} \} |_{x=0,L} = 0 \quad , \sigma = \uparrow, \downarrow. \quad (27)$$
Here the wave function $\Psi_{\sigma}$ for the nonsymmetric electron dispersion is represented as

$$\Psi_{\uparrow(\downarrow)} \simeq e^{ik_{1(2)}F x}\Psi_{R(1)(2)}(x) + e^{-ik_{2(1)}F x}\Psi_{L(1)(2)}(x). \quad (28)$$

Notice that Eqs. (27) and (28) determine more general boundary conditions than $\Psi_{\sigma}(x = 0, L) = 0$ usually assumed in the literature (see e.g. Ref. [20]). The last b.c. is the particular case of so called "hard wall" b.c.’s $\Psi^{(j)}(x_b) = 0$ $j = 1, ..., 2N$ for a multichannel ($N$) spin-1/2 LL. They do not mix the channels and allows one to reduce the multichannel problem to calculations for a single channel situation with an additional summation of channel dependent quantities over channel quantum numbers. In our case scattering at the boundaries changes the channel "index" ($1 \leftrightarrow 2$) and the correct b.c. for "slow" fields $\Psi_{R(L)}$ has to take this fact into account. The decomposition Eq. (28) holds at distances much larger then $\lambda_F$. In a general case, the wave function at the boundary is of a more complicated and unknown form and one may not put $\Psi_{\sigma} = 0$ in order to find the relations between the two terms in Eq. (28). In contrast, the requirement that the particle current through the boundary is zero is robust and its consequences hold at any distance from the boundary due to current conservation.

For the bare electron spectrum without dispersion asymmetry ($k_{1F} = k_{2F} = k_F$) the formulated requirement is equivalent to the following boundary conditions for the chiral (R,L) fermionic fields (see also Ref. [22])

$$\Psi^\dagger_{R\sigma}(x)\Psi_{R\sigma}(x)|_{x=0,L} = \Psi^\dagger_{L\sigma}(x)\Psi_{L\sigma}|_{x=0,L}. \quad (29)$$

The boundary conditions Eq. (29) correspond to a LL with open ends and result in zero eigenvalues of the momentum-like zero-mode operator $\hat{\Pi}_\sigma$ and in quantization of harmonic modes (plasmons) on a ring with circumference $2L$ (see Ref. [22]). In this case the spectrum of plasmons is equidistant and the propagators take the form ($j, k = 1, 2$)

$$\langle \langle \phi_{R(L)j}(t, x)\phi_{R(L)k}(t, x) \rangle \rangle = -\frac{\delta_{jk}}{4\pi} \ln \frac{1 - \exp[i\pi(\pm x - s_{1(2)}t + ia)/L]}{\pi a/L}.$$  

Here $a$ is the cutoff length and $s_{1(2)}$ are the velocities of charge and spin excitations (for noninteracting fermions $s_1 = s_2 = v_F$).

Now we generalize the quantization procedure elaborated in Ref. [22] to an electron spectrum with dispersion asymmetry. We will assume that electron normal backscattering at the boundaries is not accompany by spin-flip processes. Therefore each backscattering
for our spectrum (Fig.1) leads to the change of the channel index ("1" ↔ "2") and the corresponding Fermi velocity.

It is worthwhile at first to consider the general case of boundary scattering in a two-channel system of noninteracting electrons confined to the interval [0,L]. The electron backscattering at the boundaries is described by 2 × 2 unitary symmetric matrix which is convenient to parameterize as follows

\[ S = e^{i\delta} \left( \begin{array}{cc} r & i|t| \\ i|t| & r^* \end{array} \right), \quad (31) \]

where \( r = |r| e^{i\delta r} \) is the intrachannel backscattering amplitude (1 ↔ 1, 2 ↔ 2) and \( t \) is the interchannel backscattering (1 ↔ 2) amplitude \(|r|^2 + |t|^2 = 1\). By matching the electron wave functions at the boundaries \( x = 0 \) and \( x = L \) with the help of the S-matrix Eq. (31) one easily finds the spectrum equation

\[ \cos^2 \left[ \frac{\varepsilon L}{2} \left( \frac{1}{v_1} + \frac{1}{v_2} \right) + \delta \right] = |r|^2 \cos^2 \left[ \frac{\varepsilon L}{2} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) + \delta \right]. \quad (32) \]

For purely intrachannel reflection, \( t = 0 \), we get from Eq. (32) two independent sets \((j = 1, 2)\) of equidistant levels with spacing \( \Delta \varepsilon_j = \pi \hbar v_j F / L \). In the opposite case of purely interchannel backscattering \((r = 0)\) the spectrum is also equidistant

\[ \varepsilon_n = \frac{2\pi \hbar}{L} \frac{v_1 v_2}{v_1 + v_2} \left( n + \frac{1}{2} - \frac{\delta}{\pi} \right), \quad n = 1, 2, ... \quad (33) \]

In a general case the spectral equation (32) yields a set of quasi-random energy levels.

The bozonization technique is compatible only with the two considered limiting cases: \(|r| = 1\) (this was demonstrated in Ref. [22]), and \( r = 0 \) as we will show now. Let us start at first with the case on noninteracting fermions. The boundary condition Eq. (27) for \( v_1 \neq v_2 \) results in the equations

\[ v_1 \Psi_{R1}^\dagger(x) \Psi_{R1}(x)|_{x=0,L} = v_2 \Psi_{L2}^\dagger(x) \Psi_{L2}(x)|_{x=0,L}, \quad (34) \]

\[ \Re \left[ \Psi_{L2}^\dagger(x) \Psi_{R1}(x) e^{i(k_1 + k_2)x} \right]|_{x=0,L} = 0. \quad (35) \]

These equations are satisfied if

\[ \frac{a_1}{a_2} = \frac{L_1}{L_2} = \frac{v_1}{v_2}, \quad \frac{1}{L_1} + \frac{1}{L_2} = \frac{1}{L} \quad (36) \]

\[ \varepsilon_n^F = \frac{2\pi}{L} \frac{v_1 v_2}{v_1 + v_2} n, \quad n = 1, 2, ... \quad (37) \]
and
\[
[\Phi_{L\sigma}(x, t) + \Phi_{R\sigma}(x, t)]|_{x=0,L} = \sqrt{\pi} n_\sigma, \quad \sigma = \uparrow, \downarrow, \quad (38)
\]
where \(n_\uparrow\) and \(n_\downarrow\) are integers. Eq. (38) in its turn is satisfied for topological sector with quantum numbers \((\hat{\varphi}_{R\sigma} + \varphi_{L\sigma})/\sqrt{\pi} = n_\sigma\), \(\hat{\Pi}_\sigma = 0\) and the harmonic modes \(\varphi_{R(L)\sigma}(x, t)\) which obey the relations
\[
\varphi_{R\sigma}(x, t)|_{x=0,L} = -\varphi_{L\sigma}(x, t)|_{x=0,L}. \quad (39)
\]
From Eqs. (11), (36) and (39) one easily gets the plasmon spectrum
\[
\varepsilon_n = \frac{2\pi}{L} \frac{s_1 s_2}{s_1 + s_2} n \quad (40)
\]
\((s_1, 2)\) are the plasmon velocities, which coincide with the Fermi velocities for noninteracting fermions) and the desired correlation functions \((j, k = 1, 2)\)
\[
\langle\langle \varphi_{R(L)j}(x, t) \varphi_{R(L)k} \rangle\rangle = \frac{\delta_{jk}}{4\pi} \ln \frac{1 - \exp[i2\pi(\pm x - s_k t + i a_k)/L_k]}{2\pi a_k/L_k}, \quad (41)
\]
where the effective quantization lengths \(L_j\) according to Eq. (36) are
\[
L_{1(2)} = \frac{v_{1F} + v_{2F}}{v_{2(1)F}} L \quad (42)
\]
In the limit \(v_{1F} = v_{2F}\) Eqs. (10)-(12) reproduce the plasmon spectrum and the correlation functions of a single channel LL with open ends\(^{22}\).

Now we are ready to consider the effects of interaction. For a single-mode LL (or for a multichannel LL, provided the backscattering is allowed only to its own channel) the boundary condition Eq. (39) for harmonic modes holds also for interacting fermions as one can check using a Bogoliubov-like transformation which diagonalizes the LL Hamiltonian. Hence in the presence of interaction one can still use the same correlation functions as for noninteracting fermions with the only difference that the velocities are renormalized by interaction.

This is not the case for our problem. With the help of exact transformations (see Eq. (55)) which diagonalize the 2-channel LL Hamiltonian\(^{23}\) one can show that if the chiral bosonic fields satisfy Eq. (39), the diagonalized ones \(\tilde{\varphi}_{R(L)j}\) are connected at the boundaries by the effective ”scattering matrix” \(\hat{S}^e\)
\[
\tilde{\varphi}_{Rj}(x = 0, L) = \sum_{k=1}^{k=2} S^e_{ijk} \tilde{\varphi}_{Lk}(x = 0, L), \quad \hat{S}^e = \frac{1}{B} \begin{pmatrix} -A & 1 \\ 1 & A \end{pmatrix}, \quad (43)
\]
where the coefficients $A, B$ are

$$A = -\left[\cos 2\psi - \sinh(\vartheta_1 - \vartheta_3) \sin 2\psi\right]^{-1} \left[\sinh(\vartheta_1 - \vartheta_2) \cosh(\vartheta_1 - \vartheta_3)\right],$$

$$B = -\left[\cos 2\psi - \sinh(\vartheta_1 - \vartheta_3) \sin 2\psi\right]^{-1} \left[\cosh(\vartheta_1 - \vartheta_2) \cosh(\vartheta_1 - \vartheta_3)\right].$$

and the ”rotation angles” $\vartheta_l$ ($l = 1, ..., 4$) and $\psi$ are defined in Appendix I (see Eqs. (56) and (57)). One can check after some algebra that the coefficients $A$ and $B$ satisfy the simple relation $B^2 - A^2 = 1$, which makes the S-matrix in Eq. (43) unitary. This observation allows us to use the scattering matrix formalism when evaluating the energy spectrum of plasmons. Notice that in the parametrization Eq. (31) we have $r = iA/B$, $|t| = 1/B$, $\delta = \pi/2$.

For monochromatic bosonic fields with amplitudes $b_{R(L)}$ the scattering at the boundaries $x = 0$ and $x = L$ are determined by the equations

$$x = 0: \quad b_{Rj} = \sum_{k=1}^{2} S_{jk}^e b_{Lk}, \quad x = L: \quad e^{-i\alpha_j} b_{Lj} = \sum_{k=1}^{2} S_{jk}^e e^{i\alpha_k} b_{Rk},$$

where the phases $\alpha_j = \varepsilon L/s_j$ and $s_j$ are the plasmon velocities (see Eqs. (58),(59)). From the above set of linear equations one easily finds the spectrum equation for plasmons

$$\sin^2 \left[\frac{\varepsilon L}{2} \left(\frac{1}{s_1} + \frac{1}{s_2}\right)\right] = R \sin^2 \left[\frac{\varepsilon L}{2} \left(\frac{1}{s_1} - \frac{1}{s_2}\right)\right],$$

where $R \equiv (A/B)^2 \leq 1$ is the effective backscattering coefficient for plasmons. It depends both on the dimensionless interaction strength $\kappa = V_0/\pi\hbar(v_{1F} + v_{2F})$ and on the dispersion asymmetry parameter $\lambda_a$. Notice that the spectral equation (47) is the special case of Eq. (32) for $\delta = \delta_r = \pi/2$.

The derived spectral equation has simple exact analytic solutions in two limiting cases: (i) noninteracting fermions and, (ii) when dispersion asymmetry is absent, $v_{1F} = v_{2F} = v_F$. For noninteracting particles ($V_0 = 0$ in Eq. (11)) the ”rotation angles” are $\psi = 0, \vartheta_1 = \vartheta_3 = 0$ (see Appendix I) and the velocities $s_1 = v_{1F}, s_2 = v_{2F}$. Then $A = 0, B = -1$ and the effective backscattering coefficient $R = 0$. Eq. (47) in this limit reproduces the equidistant spectrum of ”noninteracting” plasmons, Eq. (33). For interacting fermions in the absence of dispersion asymmetry the ”rotation angles” are $\vartheta_1 = \vartheta_3, \vartheta_2 = \vartheta_4, \cos 2\psi = 0$. In this
limit $s_1 = s, s_2 = v_F$ and $R = 1$. The corresponding plasmon energies $\varepsilon_n^{(1)} = \pi s n/L$, $\varepsilon_n^{(2)} = \pi v_F n/L$, $(n = 1, 2, \ldots)$ represent the standard excitations in the charge and spin sector of a finite length ($L$) Luttinger liquid with open ends.\(^{22}\)

For a general case Eq. (47) has to be solved numerically and the plasmon spectrum represents a set of quasi-random energy levels. The plasmonic energies can not be separated into two independent sets of levels - one for charge density excitations, another for spin density excitations. It means that the considered boundary conditions strongly mix the charge and spin excitations and the phenomena of charge-spin separation, well known in a LL, generally speaking, can disappear when both spin-orbit interaction and finite size effects are present.

**IV. JOSEPHSON CURRENT THROUGH A FINITE-LENGTH LL WIRE WITH DISPERSION ASYMMETRY**

It is clear from a physical point of view that the effects of a dispersion asymmetry in the bare electron spectrum have to be most significant in the quantum dynamics of non-interacting electrons. In this case the mismatch in Fermi velocities when an electron is backscattered at the boundaries leads to an intricate interference pattern. The more strongly particles interact, the less important are the effects of dispersion asymmetry. For instance, in the limiting case of a 1D Wigner crystal (strong repulsive long-range interactions) it is hard to imagine any interference produced by the quantum dynamics of plasmons in two Wigner crystals pinned by structural imperfections at the boundaries. So in our problem it is reasonable to expect the restoration of the regular plasmon spectrum and the spin-charge separation in the limit of strong interaction.

For strongly interacting electrons $\kappa = V_0/\pi \hbar (v_{1F} + v_{2F}) \gg 1$ and $v_{1F} \sim v_{2F}$ (i.e. for weak or moderate dispersion asymmetry) the coefficient $R$ (intra-channel backscattering probability) in Eq. (47) takes the form

$$R \approx 1 - \frac{\lambda^2 \left(4 - 3\lambda^2\right)}{\kappa^{3/2} 2 \sqrt{1 - \lambda^2}}.$$  \hspace{1cm} (48)

We see that when $\kappa \gg 1$ the backscattering is always an intrachannel process ($R \approx 1$ with a high accuracy) and the spin-charge separation and the equidistant character of plasmon spectra are indeed restored. This observation (see also Ref. 23) justifies for strongly inter-
acting multichannel \((j = 1, \ldots N)\) LL the boundary conditions \((\Psi_j(x=0,L)=0)\) usually postulated in the literature (see e.g. Ref. 20) for arbitrary interaction.

It is straightforward to evaluate the Josephson current using the exact plasmon spectrum for \(R=1\) (i.e. when \(\lambda_a=0\)) and the propagators Eq. (30). The result for zero Zeeman splitting \((\Delta_Z=0)\) is

\[
J^{(f)}(\Delta Z; \varphi) = J_c^{(0)} R_f(g_c) \sin \varphi,
\]

where \(J_c^{(0)} = (Dev_F/4L)(C/\pi)\) is the critical current through a 1D SNS junction and the interaction-induced renormalization factor \(R_f(g_c)\) is

\[
R_f(g_c) = \frac{2g_c^2}{2-g_c^2} F(2g_c^{-1},2g_c^{-1}-g_c;2g_c^{-1}-1;-1) \left( \frac{a}{L} \right)^{2(g_c^{-1}-1)}.
\]

Here \(F(\alpha,\beta;\gamma;z)\) is the hypergeometric function and \(g_c\) is the LL correlation parameter (see Eq. (22)). For noninteracting electrons \(R_f(g_c=1)=1\) and our formula has to reproduce the known expression for the Josephson current through a 1D SNS junction (see e.g. Ref. 33). From this comparison one finds \(C = \pi\).

As we have already mentioned in this section, \(R \to 1\) in the limit of strong interaction \(\kappa \gg 1\). This observation allows us to evaluate the correlation functions and the Josephson current for strongly interacting electrons with dispersion asymmetry. The Josephson current is described by Eqs. (49) and (50) after the replacement \(g_c \to \kappa^{-1/2}\) and in the limit \(\kappa \gg 1\). The renormalization factor now takes the form

\[
R_f(\kappa \gg 1) \approx \frac{1}{\kappa} \left( \frac{\pi a}{L} \right)^{2\sqrt{\kappa}} \ll 1.
\]

The formulae (49) and (51) show that in the considered limit the Josephson current does not depend on the parameter \(\lambda_a\) of dispersion asymmetry. By comparing Eq. (51) and Eq. (25) we see that the interaction suppresses supercurrent more strongly in a long end-coupled quantum wire than in a side-coupled one, which is in agreement with intuition.

Dispersion asymmetry affects the supercurrent of weakly interacting electrons. The influence, however, numerically is not strong even for the most favourable case of noninteracting particles. With the help of the correlation functions (41) it is easy to calculate the Josephson current of noninteracting electrons with dispersion asymmetry

\[
J^{(f)}(\lambda_a,\Delta Z; \varphi) = J_c^{(0)} R(\lambda_a) \cos \left[ \frac{\Delta Z}{2} \left( \frac{1}{\Delta L_1} + \frac{1}{\Delta L_2} \right) \right].
\]
Here \( J_c^{(0)} \) is the critical current in the absence of dispersion asymmetry (see Eq. (49)) and \( R(\lambda_a) \) is the renormalization factor induced by the asymmetry of electron dispersion

\[
R(\lambda_a) = \frac{\pi \lambda_a (1 - \lambda_a^2)}{\sin(\pi \lambda_a)} \simeq \begin{cases} 
1 + \left(\frac{\pi^2}{6} - 1\right) \lambda_a^2 & \lambda_a \ll 1 \\
2 & \lambda_a \to 1 
\end{cases}
\]  

(53)

We see from Eq. (53) that the dispersion asymmetry always slightly enhances the critical current. The analysis of the Josephson current in a 1D SQWS junction in the presence of dispersion asymmetry was performed in Ref. [21] using the Andreev level approach. It was shown that the observed enhancement of the Josephson current is due to less perfect cancellations (different Fermi velocities) of the partial supercurrents carried by adjacent Andreev levels.

V. THE RASHBA EFFECT, CHIRAL ELECTRONS IN 1D QUANTUM WIRES AND THE JOSEPHSON CURRENT IN SLLS JUNCTION

Now we consider the limit of strong Rashba interaction. In this case the electrons in a quasi-1D quantum wire behave like truly chiral particles, that is the spin polarization of an electron irrespective of its subband index is determined by the direction of electron motion along the wire — right-moving and left-moving electrons have opposite spin projections. We will assume for definiteness (it depends on the sign of Rashba coupling) that "R"-electrons are "down"-polarized and "L"-electrons are "up"-polarized (see Fig.3, dashed lines for spin polarizations). We have already seen in section II, that the left/right symmetry breaking in the presence of the Zeeman interaction results in the appearance of an anomalous Josephson current. Physically it means that when the spin projection is correlated with the direction of motion (left, right), the magnetic field, via the Zeeman interaction, induces partial Josephson currents (for each subband "1" and "2") even if the superconducting phase difference in the SNS junction is zero. For the spin alignments assumed in Refs. [16,17] the subbands contribute to the Josephson current with opposite signs and therefore the anomalous supercurrent vanishes for symmetrical spectrum \( v_1 F = v_2 F \). As we see, the electron dispersion asymmetry is indispensible property to get anomalous Josephson current for a weak Rashba interaction. In the limit of strong s-o interaction when all right(left) moving particles have parallel spins, the contributions of subbands have the same
sign and the existence of electron dispersion asymmetry ceases to be crucial in appearance of anomalous (at $\varphi = 0$) Josephson current.

What is more important are the spin-flip processes which may take place in the transition regions between the 2D or 3D electron reservoirs (superconducting leads in our case) and the 1D quantum wire with a pronounced Rashba effect. The electrons in the reservoirs have two possible spin states, while deep inside the wire, where the s-o interaction is strong, the electron spins have to be aligned according to the above discussed prescription. So particles with the ”wrong” spin projection should turn their spins toward the ”right” direction.

One can imagine two different types of transition regions. In the case when s-o interaction is changed abruptly at the lead/wire interfaces, the spin-flips induced by the Rashba interaction will be accompanied by normal electron backscattering. Such non-adiabatic contacts were studied in Ref. [18] when evaluating normal electron transport through a 1D quantum wire with strong Rashba interaction attached to leads where the s-o interaction was assumed to be negligibly small. In this model the transparency of the junction strongly depends on the spin-orbit coupling and normal electron transport is suppressed even for perfect contacts.

Here we will assume that the Rashba interaction in the QW is switched on smoothly and that therefore the backscattering of the electron at the boundary and the rotation of its spin induced by the Rashba interaction the in quasi-1D quantum wire are spatially separated. For instance, the left-moving electron (spin-”up”) at first is normally backscattered at the left interface keeping the spin-”up” and then after travelling some length $\lambda_F \ll l \ll L$ its spin becomes ”down”-polarized. In this model, the Rashba interaction does not lead to additional electron backscattering at the interfaces and does not suppress the supercurrent.

It is also convenient for calculations to assume that the electron-electron interaction in the 1D QW is strong. As was shown in section IV, the interchannel electron backscattering at the I/LL interfaces is suppressed in the limit of strong repulsive interaction and one can use a simple quantization procedure proposed in Ref. [22] to evaluate the plasmon spectrum and the correlation functions. After straightforward calculations the desired expression for the Josephson current takes the form

$$J^{(R)}(\varphi) \simeq J_{c}^{(0)} R_f \sin \left[ \varphi + \frac{\Delta Z}{2} \left( \frac{1}{\Delta L_1} + \frac{1}{\Delta L_2} \right) \right] \cos \left[ \frac{\Delta Z}{2} \left( \frac{1}{\Delta L_1} - \frac{1}{\Delta L_2} \right) \right],$$  \hspace{1cm} (54)

where the interaction-induced renormalization coefficient $R_f$ is determined by Eq. [51]. As was already evident from physical considerations, the anomalous supercurrent $J^{(R)}(\varphi = 0)$
in the limit of strong Rashba interaction is induced by a magnetic field ($\Delta Z \neq 0$) even in the absence of electron dispersion asymmetry. We see from Eq. (5.4) that the dependence of the supercurrent on magnetic field is absolutely different for chiral and normal electrons. In particular the critical current for symmetric electron spectrum ($v_{1F} = v_{2F}$) in the case of chiral electrons does not at all depend on the Zeeman splitting, while in the ordinary situation one gets a periodic dependence.

VI. CONCLUSION

The problem we have studied allows one to consider the interplay of proximity-induced superconductivity, the Rashba, Zeeman and Coulomb interactions on the transport properties of quasi-1D quantum wires. We have shown that the Rashba and Zeeman effects strongly influence the supercurrent. In particular, the Rashba effect in quantum wires results in a strong correlation between electron spin polarization and the direction of electron motion. In other words a strong Rashba interaction creates chiral particles in the 1D electron system. The influence of a magnetic field via the Zeeman interaction on chiral particles leads to the appearance of a net electric current in the wire. When the leads that the quantum wire is attached to are superconducting this current persists even at zero phase difference across the junction. The effect exists already for noninteracting particles and it is strongly sensitive to the electron dispersion asymmetry for weak Rashba coupling and ceases to depend on the asymmetry parameter in the regime of strong Rashba interactions.

It is well known\textsuperscript{10,26} that the Josephson current in a perfectly transmitting junction (i.e. without normal electron backscattering) is not influenced by the Coulomb interaction. In contrast, any potential barrier inside the normal region which induces electron backscattering is renormalized (upwards) by the repulsive interaction (the Kane-Fisher effect\textsuperscript{6}) and therefore strongly suppresses the supercurrent through a (poorly transmitting) SIL-LIS junction\textsuperscript{10,11,20,26}. We have shown that the electron dispersion asymmetry, which is induced by the Rashba interaction in quasi-1D quantum wires\textsuperscript{16,17} is significant for the superconducting properties of a LL junction only for weak or moderate Coulomb interaction. In this case the interplay of interaction and dispersion asymmetry leads to an intricate interference pattern in the plasmon quantum dynamics of a finite length two-channel LL and makes the plasmon spectrum quasi-random. Strong Coulomb interactions suppress this kind of
quantum behavior and restores a regular (equidistant) plasmon spectrum. The tendency of strong Coulomb interactions to suppress quantum interference can be traced in different 1D electronic systems, for instance, in a LL double barrier (absence of resonant tunnelling for a strong repulsive interaction\cite{34}) or in mesoscopic coupled rings (ordering effect of Coulomb interaction on persistent current oscillations\cite{35}).

Acknowledgments

This research is supported by the Royal Swedish Academy of Sciences (KVA) and by the Swedish Research Council (LIG,RIS). The authors thanks E. Bezuglyi, Yu. Galperin, L. Gorelik and V. Shumeiko for numerous fruitful discussions. IVK and AK acknowledge the hospitality of the Department of Applied Physics at Chalmers University of Technology and Göteborg University, Sweden. AK also acknowledges the hospitality of the Theoretische Physik III Institut, Ruhr-Universität Bochum, Germany. IVK gratefully acknowledges discussions with B. Altshuler, L. Glazman, V. Kravtsov and A. Nersesyan, and the hospitality and the financial support from the Abdus Salam ICTP (Trieste, Italy), where this work was completed.
Appendix I

The canonical pseudoorthogonal transformations, which diagonalize the Luttinger liquid Hamiltonian (1) are

\[
\begin{pmatrix}
\rho_{R\uparrow} \\
\rho_{L\downarrow} \\
\rho_{R\downarrow} \\
\rho_{L\uparrow}
\end{pmatrix} =
\begin{pmatrix}
\cosh \vartheta_1 \cos \psi & \sinh \vartheta_1 \cos \psi & -\cosh \vartheta_2 \sin \psi & -\sinh \vartheta_2 \sin \psi \\
\sinh \vartheta_1 \cos \psi & \cosh \vartheta_1 \cos \psi & -\sinh \vartheta_2 \sin \psi & -\cosh \vartheta_2 \sin \psi \\
\cosh \vartheta_3 \sin \psi & \sinh \vartheta_3 \sin \psi & \cosh \vartheta_4 \cos \psi & \sinh \vartheta_4 \cos \psi \\
\sinh \vartheta_3 \sin \psi & \cosh \vartheta_3 \sin \psi & \sinh \vartheta_4 \cos \psi & \cosh \vartheta_4 \cos \psi
\end{pmatrix}
\begin{pmatrix}
\rho_{R\uparrow} \\
\rho_{L\downarrow} \\
\rho_{R\downarrow} \\
\rho_{L\uparrow}
\end{pmatrix},
\]

where the "rotation angles" \( \vartheta_j \) and \( \psi \) are expressed in terms of the Fermi velocities \( v_{1F}, v_{2F} \) and the interaction strength \( V_0 \) by the following equations

\[
\begin{align*}
\vartheta_1 &= \frac{1}{2} \ln g_1, \\
\vartheta_2 &= \frac{1}{2} \ln \left( \frac{v_{1F}}{v_{2F}} \right), \\
\vartheta_3 &= \frac{1}{2} \ln \left( \frac{v_{2F}}{v_{1F}} g_1 \right), \\
\vartheta_4 &= \frac{1}{2} \ln g_2, \\
\tan 2\psi &= \frac{2 V_0 \sqrt{v_{1F} v_{2F}}}{(v_{1F} - v_{2F}) \left[ V_0 + \pi \hbar (v_{1F} + v_{2F}) \right]}.
\end{align*}
\]

Here \( g_j = v_{jF}/s_j \) \((j = 1, 2)\) are the correlation parameters of a 2-channel LL and the plasmon velocities are

\[
\begin{align*}
s_1 &= v_{1F} \left\{ \cos^2 \psi + \left( \frac{v_{2F}}{v_{1F}} \right)^2 \sin^2 \psi + \frac{V_0}{\pi \hbar v_{1F}} \left[ \cos \psi + \sqrt{\frac{v_{2F}}{v_{1F}}} \sin \psi \right]^2 \right\}^{1/2}, \\
s_2 &= s_1(\psi \leftrightarrow -\psi, v_{1F} \leftrightarrow v_{2F}).
\end{align*}
\]

For noninteracting electrons, \( V_0 = 0, \) the correlation parameters are \( g_1 = g_2 = 1 \) and, according to Eqs. (56) and (57) \( \vartheta_1 = \vartheta_4 = 0, \) \( \psi = 0. \) In the limit \( v_{1F} = v_{2F} = v_F \) Eqs. (56)-(59) reproduce the well-known expressions for the correlation parameters of a spin-1/2 LL

\[
\begin{align*}
\vartheta_1 &= \vartheta_3 = \frac{1}{2} \ln g_c, \\
\vartheta_2 &= \vartheta_4 = 0, \\
g_c &= \left( 1 + \frac{2 V_0}{\pi \hbar v_F} \right)^{-1/2}.
\end{align*}
\]

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