CONFIGURATIONS OF POINTS ON A LINE UP TO SCALING OR TRANSLATION

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Abstract. We prove that the Losev–Manin compactification of the space of configurations of $n$ points on $\mathbb{P}^1\setminus\{0, \infty\}$ modulo scaling degenerates (isotrivially) to a compactification of the space of configurations of $n$ points on $\mathbb{A}^1$ modulo translation. The latter resembles the compactification constructed by Ziltener and Mau–Woodward, but allows the marked points to coincide, making it a $G_n^{n-1}$-variety, which mirrors the fact that the Losev–Manin space is toric. The degeneration is compatible with the actions of $G_n^{n-1}$ and $G_a^{n-1}$ in the sense that these actions fit together globally in the total space of the degeneration.

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1. Introduction

The goal of this note is to point out a connection between the following problems:

$(\oplus)$ compactifying the space of $n$-tuples of (not necessarily distinct) points on the affine line $x_1, \ldots, x_n \in \mathbb{C}$ modulo translation, that is,

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \text{ if } y_1 - x_1 = \cdots = y_n - x_n; \quad \text{and}$$

$(\times)$ compactifying the space of $n$-tuples of (not necessarily distinct) points on the punctured affine line $x_1, \ldots, x_n \in \mathbb{C}^*$ modulo scaling, that is,

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \text{ if } \frac{y_1}{x_1} = \cdots = \frac{y_n}{x_n}.$$ 

Both problems are open ended.

The best answer to $(\times)$ may be the Losev-Manin space, which we will denote by $L_n$, as in [LM00]. For $(\oplus)$, if we change the problem by instead insisting that the points are distinct, then a beautiful answer is given by the moduli space $\mathcal{Q}_n$ of ‘stable scaled marked curves’ constructed by Mau and Woodward as a projective variety [MW10], after Ziltener constructed it with symplectic methods [Zi06, Zi14]. The

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moduli space $\overline{Q}_n$ plays a central role in the context of gauged stable maps [Wo15, GSW17, GSW18]. The first goal of this paper is to construct a compactification $\overline{P}_n$ (related to $\overline{Q}_n$ but simpler) which answers (+) as stated above.

**Definition 1.1.** An $n$-marked $G_n$-rational tree is a reduced, connected, complex, projective curve $C$ of arithmetic genus 0 with at worst nodal singularities, that is, $\overline{C}_{p,p} \simeq \mathbb{C}[x,y]/(xy)$ or $\mathbb{C}[x]$ at all $p \in C(\mathbb{C})$, with a $G_n$-action and $n + 1$ nonsingular points $x_\infty, x_1, \ldots, x_n \in C(\mathbb{C})$, such that $x_\infty$ is fixed by the $G_n$-action, but $x_1, \ldots, x_n$ are not. The $n$-marked $G_n$-rational tree is stable if any irreducible component of $C$ which doesn’t contain any $x_i$, $1 \leq i \leq n$, either intersects at least 3 other irreducible components of $C$, or contains $x_\infty$ and intersects at least 2 other irreducible components of $C$.

We will construct a normal projective $\mathbb{G}_a^{n-1}$-variety (cf. [HT99, Definition 2.1]) $\overline{P}_n$ such that $\overline{P}_n(\mathbb{C})$ is the set of stable $n$-marked $\mathbb{G}_a$-rational trees. In fact, $\overline{P}_n$ is almost certainly the fine moduli space of stable $n$-marked $\mathbb{G}_a$-rational trees, but since $\overline{P}_n$ is ultimately a combinatorial object, we will prove the weaker ‘representability’ statement which only considers families over complex varieties rather than families over more general schemes cf. Corollary 5.6, to avoid some lengthy technical distractions. This weaker statement still determines $\overline{P}_n$ uniquely.

There is an obvious analogy between $\overline{T}_n$ and $\overline{P}_n$. If $(\overline{Y}_n, G)$ is either $(\overline{T}_n, G_m)$ or $(\overline{P}_n, G_a)$, then $\overline{Y}_n$ is a compactification of $\mathbb{G}_a^n/G$ (where $G \hookrightarrow \mathbb{G}_a^n$ diagonally), with the property that the usual $\mathbb{G}_a^n$-action on $Y_n = \mathbb{G}_a^n/G$ (or $\mathbb{G}_a^n/G = \mathbb{G}_a^{n-1}$-action, if we quotient out by the trivial diagonal action) extends to all of $\overline{Y}_n$. We call this $\mathbb{G}_a^{n-1}$-action on $\overline{Y}_n$ the natural $\mathbb{G}_a^{n-1}$-action on $\overline{Y}_n$. However, the relation between $\overline{T}_n$ and $\overline{P}_n$ goes beyond this analogy.

Our main result is that $\overline{P}_n$ deforms isotrivially to $\overline{T}_n$, in a manner compatible with the natural group actions. To state it, we first recall the elementary fact that $G_m$ degenerates isotrivially to $G_a$ as an algebraic group. (This seems to have been first recorded in [KM78, 3.1], although the Jacobians of nodal and cuspidal cubics were known much earlier.) Let $\gamma : G = \text{Spec } \mathbb{C}[t, x][tx+1] \to \text{Spec } \mathbb{C}[t] = \mathbb{A}^1$. The operation $G \times_{\mathbb{A}^1} G \to G$ given on $\mathbb{C}$-points by

$$(t,x) \ast (t,y) = (t, x + y + txy)$$

with the identity section $t \mapsto (t,0)$ makes $G$ an $\mathbb{A}^1$-group scheme. Note that

$$G_t \simeq \begin{cases} G_m & \text{if } t \neq 0, \\ G_a & \text{if } t = 0, \end{cases}$$

where $t \in \mathbb{C} = \mathbb{A}^1(\mathbb{C})$ and $G_t = \gamma^{-1}(t)$.

**Theorem 1.2.** For any positive integer $n$, there exist a complex variety $X$, a flat projective morphism $\xi : X \to \mathbb{A}^1$, and an action of

$$G_{\mathbb{A}^1}^{n-1} = G \times_{\mathbb{A}^1} \cdots \times_{\mathbb{A}^1} G$$

$n-1$ copies of $G$

on $X$ over $\mathbb{A}^1$ such that for all $t \in \mathbb{C}$, if $X_t = \xi^{-1}(t)$, then

- if $t \neq 0$, then $X_t$ is isomorphic to $\overline{T}_n$, and the action of $G_t^{n-1}$ on $X_t$ is isomorphic to the natural action of $G_m^{n-1}$ on $\overline{T}_n$;
- if $t = 0$, then $X_t$ is isomorphic to $\overline{P}_n$, and the action of $G_t^{n-1}$ on $X_t$ is isomorphic to the natural action of $G_a^{n-1}$ on $\overline{P}_n$. 
In fact, \( X \setminus X_0 \simeq (\mathbb{A}^1 \setminus \{0\}) \times T_n \) compatibly with the projections to \( \mathbb{A}^1 \setminus \{0\} \), and the restriction of the \( G_n^{-1} \)-action in Theorem 1.2 to \( X \setminus X_0 \) is the pullback of the \( G^{-1} \)-action on \( T_n \) along the projection \( X \setminus X_0 \to T_n \).

For instance, \( \mathcal{T}_1 \simeq \mathcal{P}_1 \simeq \mathbb{P}^1 \), \( \mathcal{T}_2 \simeq \mathcal{P}_2 \simeq \mathbb{P}^1 \), and \( \mathcal{T}_3 \) is the blowup of \( \mathbb{P}^2 \) at 3 general points, while \( \mathcal{P}_3 \) is the blowup of \( \mathbb{P}^2 \) at 3 collinear points. However, \( \mathcal{P}_n \) is mildly singular for \( n \geq 4 \) for the same reasons that \( \mathcal{Q}_n \) is \([MW10, Corollary 10.6 and Figure 19]\), whereas \( \mathcal{T}_n \) is nonsingular \([LM00, Theorem 2.2.a]\), and they are certainly not homeomorphic in general. Nevertheless, Theorem 1.2 still shows that \( \mathcal{T}_n \) and \( \mathcal{P}_n \) are related topologically. For instance, we have a trivial consequence:

**Corollary 1.3.** There exist polarizations on \( \mathcal{P}_n \) and \( \mathcal{T}_n \) for which their Hilbert polynomials coincide, that is, \( \chi(\mathcal{T}_n, \mathcal{O}_{\mathcal{T}_n}(m)) = \chi(\mathcal{P}_n, \mathcal{O}_{\mathcal{P}_n}(m)) \) for all \( m \in \mathbb{Z} \).

The values of \( \chi_{\text{top}}(\mathcal{P}_n) \) for small \( n \) can be found easily using a computer.

\[
\begin{array}{c|cccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 \chi_{\text{top}}(\mathcal{P}_n) & 1 & 2 & 6 & 27 & 170 & 1390 & 13979 \\
\end{array}
\]

Although this sequence doesn’t match any sequence in the Online Encyclopedia of Integer Sequences, at least it illustrates the fact that \( \mathcal{P}_n \) is singular for \( n \geq 4 \), since \( \mathcal{T}_n \) is nonsingular and \( \chi_{\text{top}}(\mathcal{T}_n) = n! \) \([LM00]\).

**Example 1.4.** The degeneration of \( \mathcal{T}_3 \) to \( \mathcal{P}_3 \) is illustrated in the figure below. The graphs in the top row represent strata of \( \mathcal{T}_3 \), while the graphs in the bottom row represent strata of \( \mathcal{P}_3 \). The vertical arrows \( A \to a + \cdots + z \) indicate that the fiber over 0 of the closure relative to \( X \) of \( A \times (\mathbb{A}^1 \setminus \{0\}) \) is set theoretically \( a \cup \cdots \cup z \).

Although 0 and \( \infty \) play symmetric roles in \( \mathcal{T}_n \), we see that this is no longer true in the context of the degeneration in Theorem 1.2.

To prove Theorem 1.2, we will interpret \( X \) as a moduli space of certain objects called stable \( n \)-marked field-decorated rooted rational trees, cf. Definition 4.1. The proof revolves around the observation that \( \mathcal{L}_{n+1} \) and \( \mathcal{P}_{n+1} \) are the universal curves over \( \mathcal{L}_n \) and \( \mathcal{P}_n \) respectively, and relies heavily on the techniques in Knudsen’s proof that \( \mathcal{M}_{g,n+1} \) is the universal curve over \( \mathcal{M}_{g,n} \) \([Kn83]\), with some additional input from \([Stacks, Tag 0E7B]\).

A side remark on the moduli of curves point of view. Consider the operations:

- glue 0 and \( \infty \) on a curve which corresponds to a \( \mathbb{C} \)-point in \( \mathcal{L}_n \);
- pinch \( \infty \) on a curve which corresponds to a \( \mathbb{C} \)-point in \( \mathcal{P}_n \).

It seems that it is possible to carry out these two operations globally on the universal curve over \( X \). In this way, we may think of \( X \) as the base of a family of curves of arithmetic genus 1. Clearly, these operations don’t compromise the group actions on the curves.
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2. Preliminaries on curves and differentials

Recall that a prestable curve is a proper flat morphism whose geometric fibers are connected curves with at worst nodal singularities, e.g. [BM96, Definition 2.1]. We say that it is of genus $g$ if all geometric fibers have arithmetic genus $g$. If $C \to S$ is a prestable curve and $x : S \to C$ is a section, we will often abuse notation by writing $x$ instead of $x(S)$ for the scheme-theoretic image of $x$ (recall that sections of separated morphisms are closed immersions [EGAI, 5.4.6]).

Let $\pi : C \to S$ be a projective prestable curve. We denote the sheaf of relative differentials and relative dualizing sheaf by $\Omega^1_{C/S}$ and $\omega_{C/S}$ respectively. Please see [Kn83, p. 163] for some of the fundamental properties of $\Omega^1_{C/S}$ and $\omega_{C/S}$. Their formation commutes with base change. There is a homomorphism $\psi : \Omega^1_{C/S} \to \omega_{C/S}$ whose formation also commutes with base change. Please see [ACG11, Ch. 10, (2.20)] for a discussion of the kernel and cokernel of $\psi$. The dual $\psi^\vee : \omega_{C/S}^\vee \to \Omega^1_{C/S}^\vee$ is always injective, so it possible to think of $\omega_{C/S}^\vee$ as a submodule of $\Omega^1_{C/S}$. This is proved on page 168 in [Kn83] for stable curves, and the same argument works equally well for prestable curves. If $S$ is a variety, we even have a completely explicit fiberwise description.

Lemma 2.1. If $S$ is a complex variety, then

$$\Gamma(U, \omega_{C/S}^\vee) = \left\{ \eta \in \Gamma(U, \Omega_{C/S}^\vee) : \eta_s \in \Gamma(C_s \cap U, \omega_{C_s}^\vee), \forall s \in S(\mathbb{C}) \right\},$$

where $\eta_s$ is the restriction of $\eta : \Omega_{C/S}|_U \to \mathcal{O}_C|_U$ to $C_s \cap U$, and the condition simply means that $\eta_s$ factors through $(\psi|_U)_s$.

Proof. Both $\psi$ and $\eta$ annihilate $\Omega_{C/S}^{tor}$. Using arguments similar to those in [ACG11, p. 98], we obtain the short exact sequence

$$0 \to \Omega_{C/S}/\Omega_{C/S}^{tor} \to \omega_{C/S} \to \omega_{C/S} \otimes \mathcal{O}_{C^{\text{sing}}} \to 0$$

which identifies $\Omega_{C/S}/\Omega_{C/S}^{tor}$ with $\omega_{C/S} \otimes \mathcal{I}_{C^{\text{sing}}}$. The scheme structure on $\pi^{\text{sing}}$ is the one implicit in loc. cit. Then the claim becomes showing that a homomorphism $\mathcal{I}_{\pi^{\text{sing}}}|_U \to \omega_{C/S}|_U$ which extends fiberwise to $\mathcal{O}_C|_U \to \omega_{C/S}|_U$ extends globally, which can be checked explicitly. The details are routine. \qed
We record for future use two elementary remarks on morphisms of schemes $f : X \to Y$ such that $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism. First, the canonical map $\mathcal{L} \to f_* f^* \mathcal{L}$ is an isomorphism for all invertible $\mathcal{L}$. Second, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, there exists a natural $\mathcal{O}_Y$-module homomorphism $f_* \mathcal{F}^\vee \to (f_* \mathcal{F})^\vee$. Indeed, an element $F_{f^{-1}(U)} \to \mathcal{O}_X|_{f^{-1}(U)}$ of $\Gamma(U, f_* \mathcal{F}^\vee)$ induces maps

$$\Gamma(V, f_* \mathcal{F}) = \Gamma(f^{-1}(V), \mathcal{F}) \to \Gamma(f^{-1}(V), \mathcal{O}_X) \simeq \Gamma(V, \mathcal{O}_Y)$$

for $V \subseteq U \subseteq Y$ which satisfy the obvious compatibilities and are thus an element of $\Gamma(U, (f_* \mathcal{F})^\vee)$. More generally, if $\mathcal{G}$ is an $\mathcal{O}_Y$-module, and $\phi : \mathcal{G} \to f_* \mathcal{F}$ is an $\mathcal{O}_Y$-module homomorphism, then we may define a map $f_* \mathcal{F}^\vee \to \mathcal{G}^\vee$ as the composition $f_* \mathcal{F}^\vee \to (f_* \mathcal{F})^\vee \to \mathcal{G}^\vee$.

**Definition 2.2.** Let $\pi_i : C_i \to S$ be prestable curves for $i = 1, 2$, and $f : C_1 \to C_2$ an $S$-morphism. We say that $f$ is contractive if $f^\#: \mathcal{O}_{C_2} \to f_* \mathcal{O}_{C_1}$ is an isomorphism and $R^1 f_* \mathcal{O}_{C_1} = 0$, and these continue to hold after any base change $S' \to S$.

Some remarkable properties of contractive morphism are stated and proved in [Stacks, Tag 0E7B]. These properties will be essential in §4.

**Proposition 2.3.** Assume that $S$ is a variety. If $f : C_1 \to C_2$ is contractive, then there exists a unique $\mathcal{O}_{C_2}$-module homomorphism $f_* : f_* \omega_{C_1/S}^\vee \to \omega_{C_2/S}^\vee$ which is the identity on every open subset $U \subseteq C_2$ on which $f$ restricts to an isomorphism $f^{-1}(U) \simeq U$. If $s \in S(\mathbb{C})$ and $v$ is a local section of $f_* \omega_{C_1/S}^\vee$, then $(f_* (v))_{s} = (f_s)_*(v_s)$, where $f_s = f|_{C_{1,s}}$, $v_s = v \otimes \kappa_s$, etc.

(We can rely on [Liu02, Definition 4.7 and Theorem 4.32] to make sense intrinsically of the dualizing sheaf on open subsets.)

**Proof.** To justify uniqueness, we note that $f$ is an isomorphism above $C_2 \setminus \pi_2^{\text{sing}}$, and invoke the defining property on this dense open subset. To prove existence, let $f_\Delta : f_1 \Omega_{C_1/S}^\vee \to \Omega_{C_2/S}^\vee$ be the map induced by the adjoint of $f^* \Omega_{C_2/S} \to \Omega_{C_1/S}$, as in the remark preceding Definition 2.2. We claim that $f_\Delta$ maps sections of $f_* \omega_{C_1}^\vee$ to sections of $\omega_{C_2}^\vee$, and thus induces a homomorphism $f_* : f_* \omega_{C_1/S}^\vee \to \omega_{C_2/S}^\vee$. If $S = \text{Spec} \mathbb{C}$, then $f$ simply contracts several $\mathbb{P}^1$ chains, and the claim can be checked directly using the explicit description of $\omega_{C_1}^\vee$ as a sheaf of meromorphic differentials. In general, for each $s \in S(\mathbb{C})$ and any open set $U \subseteq C_2$, we have a diagram

$$\begin{array}{ccc}
\Gamma(f^{-1}(U), \omega_{C_1/S}^\vee) & \longrightarrow & \Gamma(f^{-1}(U) \cap C_{1,s}, \omega_{C_{1,s}}^\vee) \\
\downarrow & & \downarrow \\
\Gamma(U \cap C_{2,s}, \omega_{C_{2,s}}^\vee) & \longrightarrow & \Gamma(U, \omega_{C_2/S}^\vee)
\end{array}$$

whose commutativity is left to the reader to check. Then the claim follows from this commutativity, the special case $S = \text{Spec} \mathbb{C}$, and Lemma 2.1. The construction
of \( f_\delta \) is complete and the fact that it satisfies the required property is clear. We also note that \( (f_\Delta(v))_s = (f_s)_\Delta(v_s) \) amounts to commutativity of the front face in the diagram above, and \( (f_\delta(v))_s = (f_s)_\delta(v_s) \) follows. \( \square \)

Proposition 2.3 is true over more general bases too, but we don’t know a simple direct proof and it will not be used in this greater generality.

Definition 2.4. The homomorphism \( f_\delta \) in Proposition 2.3 will be called the *skew logarithmic differential* of \( f \).

3. Stabilization

The operations described in §§3.1–3.2 will be iterated later to give an explicit inductive construction of the degeneration in Theorem 1.2.

Throughout §3, we work in one of the following situations.

**Situation 3.1.** Let \( S \) be a complex (irreducible) variety, \( \pi : C \to S \) a genus 0 projective prestable curve over \( S \), \( x, x_\infty : S \to C \) sections of \( \pi \), and an \( \mathcal{O}_C \)-module homomorphism \( \nu : \omega_{C/S} \to \mathcal{I}_{x_\infty} \). Consider the situations:

1. the above, and \( \pi \) is smooth at \( x_\infty(s) \), for all \( s \in S \);
2. same as (0), but also \( x(s) \neq x_\infty(s) \) and \( \pi \) is smooth at \( x(s) \), for all \( s \in S \);
3. same as (1), but also \( x^*\nu : x^*\omega_{C/S} \to \mathcal{O}_S \) is an isomorphism.

In all situations above, let \( v \in \Gamma(C, \omega^\vee_{C/S}(-x_\infty)) \) corresponding to \( \nu \).

Indeed, in Situation 3.1.(0), \( x_\infty \) is a relative effective Cartier divisor.

3.1. Knudsen stabilization. In §3.1, we are in Situation 3.1.(0). Here we review the stabilization procedure of [Kn83, §2]. The sole difference that in our setup there is a single given smooth section \( x_\infty \) (as opposed to \( n \) such sections in [Kn83, §2]) is unsubstantial, although we will need to keep track of the logarithmic vector field \( v \). In our situation, ‘stabilization’ may be a misnomer as the produced objects are not stable in general. In Situation 3.1.(0), let

\[
\mathcal{K} = \text{Coker} \left( \mathcal{O}_C \xrightarrow{\text{inclusion}} \mathcal{O}_C(x_\infty) \oplus \mathcal{I}_{x_\infty} \right),
\]

where \( \mathcal{I}_{x_\infty} \hookrightarrow \mathcal{O}_C \) is the inclusion, and \( \xi : C' = \mathbb{P}_C \mathcal{K} \to C \) and \( \pi' = \pi \xi \). (Here and thereafter, we use the notation \( \mathbb{P}_X \mathcal{F} = \text{Proj} \text{Sym}(\mathcal{F}) \).) It can be proved by arguments analogous to those of [Kn83, §2] that:

- the morphism \( \pi' : C' \to S \) is a genus 0 (projective) prestable curve;
- the sections \( x \) and \( x_\infty \) lift canonically to sections \( x', x'_\infty : S \to C' \);
- \( x'(s) \neq x'_\infty(s) \) and \( \pi' \) is smooth at \( x'(s) \), for all \( s \in S \);
- \( \xi \) is conractive, cf. Definition 2.2;
- the formation of all new data commutes with base change.

The curve \( \pi' : C' \to S \) together with the sections \( x'_\infty, x' : S \to C' \) will always be part of the data obtained by Knudsen stabilization. If

\[
C_\circ = \{ z \in C : z \neq x(\pi(z)) \},
\]
then \( C_\circ \) is open and \( \mathcal{K}|_{C_\circ} \) is invertible and hence \( \xi \) is an isomorphism above \( C_\circ \). Let \( C'_\circ = \xi^{-1}(C_\circ) \simeq C_\circ \). It remains to discuss \( v' \).

Definition 3.2. We say that \( v' \in \Gamma(C', \omega^\vee_{C'/S}(-x'_\infty)) \) is *compatible with \( v \) if \( v'|_{C_\circ} = v|_{C_\circ} \) under the obvious identification of the restrictions of the sheaves.
Proposition 3.3. There exists an isomorphism
\[ \xi_\ast \omega_\mathcal{C}'/\mathcal{S}(-x_\infty') \simeq \omega_\mathcal{C}/\mathcal{S}(-x_\infty) \]
which restricts to the identity on \( C_0 \).

Proof. Consider the skew logarithmic differential \( \xi_\delta : \xi_\ast \omega_\mathcal{C}'/\mathcal{S} \to \omega_\mathcal{C}/\mathcal{S} \), cf. Definition 2.4. We claim that \( \xi_\delta \) maps local sections of \( \omega_\mathcal{C}'/\mathcal{S}(-x_\infty') \) to local sections of \( \omega_\mathcal{C}/\mathcal{S}(-x_\infty) \). By the last part of Proposition 2.3, it suffices to check this on fibers, when it is elementary. We thus obtain an \( \mathcal{O}_\mathcal{C} \)-module homomorphism \( \xi_\ast \omega_\mathcal{C}'/\mathcal{S}(-x_\infty') \to \omega_\mathcal{C}/\mathcal{S}(-x_\infty) \) and it can be checked once more on fibers that it is actually an isomorphism. Indeed, any homomorphism from a coherent torsion-free \( \mathcal{O}_\mathcal{C} \)-module to an invertible \( \mathcal{O}_\mathcal{C} \)-module which is an isomorphism on fibers (of closed points) must be an isomorphism: this can be justified by first invoking Nakayama’s lemma to argue that the source must be invertible because its restrictions to all points are 1-dimensional, and then noting that all maps on stalks must be isomorphisms. \( \square \)

Note the similarity of Proposition 3.3 with [Kn83, Lemma 1.6.a]].

Corollary 3.4. There exists a unique \( \nu' \) compatible with \( \nu \).

Indeed, existence follows from Proposition 3.3 while uniqueness follows from the fact that the restriction map on sections of \( \omega_\mathcal{C}'/\mathcal{S}(-x_\infty') \) from \( \mathcal{C}' \) to \( C_0 \) is injective, which is a consequence of Proposition 3.3 once more and the density of \( C_0 \) in \( \mathcal{C} \).

Definition 3.5. In Situation 3.1.(0), with notation as above, the collection of data \((\mathcal{C}', S, \pi', x_\infty', x', \nu')\) is declared to be the Knudsen stabilization of the given data.

Remark 3.6. The output of this stabilization operation fits into Situation 3.1.(1). Note that the output of stabilization commutes with base changes \( T \to S \).

3.2. Stabilization relative to a vector field. In §3.2, we work in Situation 3.1.(1). Besides the stabilization à la Knudsen in §3.1, we will use another version of stabilization, which also creates a bubble when a certain degeneracy occurs at a newly inserted marked point; in this case, the bubbling up occurs when the given vector field vanishes at the marked point. In Situation 3.1.(1), let
\[ V = \operatorname{Coker} \left( \mathcal{O}_\mathcal{C} \xrightarrow{\omega_\mathcal{C}'/\mathcal{S}} \omega_\mathcal{C}/\mathcal{S}(-x_\infty) \right), \]
and \( \xi : \mathcal{C}' = \mathbb{P}_C V \to \mathcal{C} \) and \( \pi' = \pi \xi \). Let \( Z \subseteq S \) be the scheme-theoretic vanishing locus of \( \nu \nu \in \Gamma(S, x' \omega_\mathcal{C}'/\mathcal{S}) \), \( C_0 = \mathcal{C} \setminus x(Z) \), and \( C_0' = \xi^{-1}(C_0) \). It is clear that \( \mathcal{K}_{|C_0} \) is invertible, so \( \xi \) restricts to an isomorphism on \( C_0' \simeq C_0 \). Since \( x_\infty \) is contained in \( C_0 \) by the assumption \( x \cap x_\infty = \emptyset \), it trivially lifts uniquely to a section \( x_\infty' : S \to \mathcal{C}' \).

The canonical lift \( x' : S \to \mathcal{C}' \) of \( x \) corresponds to the quotient \( x^* V \to x^* \mathcal{O}_C(x) \to 0 \) in the context of the remark in [Kn83, §2, p. 173]. We also note that the formation of all data so far commutes with base change.

Proposition 3.7. In Situation 3.1.(1), with notation as above, we have:
1. The morphism \( \pi' : \mathcal{C}' \to S \) is a genus 0 projective prestable curve.
2. There exists an isomorphism \( \omega_\mathcal{C}'/\mathcal{S}(-x-x_\infty) \simeq \xi_\ast \omega_\mathcal{C}/\mathcal{S}(-x'-x'\infty) \) which restricts to the identity on \( C_0 \).
3. The sequence \( \xymatrix{ x^* \omega_\mathcal{C}'/\mathcal{S} \ar[r] & x^* \omega_\mathcal{C}/\mathcal{S} \ar[r] & x^* \omega_\mathcal{C}/\mathcal{S} \otimes \mathcal{O}_Z \ar[r] & 0 } \) in which the first map is the pullback along \( x' \) of the differential \( \Omega_\mathcal{C}'/\mathcal{S} \to \xi^* \Omega_\mathcal{C}/\mathcal{S} \) and the second map is the restriction, is exact.
Proof. Before proving Proposition 3.7, we make a general technical remark, stated with independent notation. Let $X$ be a scheme and $D \subset X$ an effective Cartier divisor. For $i = 1, 2$, let $\mathcal{L}_i$ be an invertible $\mathcal{O}_X$-module and $s_i \in \Gamma(X, \mathcal{L}_i)$. Assume that there exists an isomorphism $\phi : \mathcal{L}_1|_D \simeq \mathcal{L}_2|_D$ such that $\phi(s_1|_D) = s_2|_D$. Let

$$V_i = \text{Coker} \left( \mathcal{O}_X \xrightarrow{t \mapsto (t \cdot x_0)} \mathcal{L}_i \oplus \mathcal{O}_X(D) \right),$$

$p_i : X'_i = \mathbb{P}_X(V_i) \to X$, and $t_i : D \hookrightarrow X'_i$ the ‘proper transform’ of $D$, constructed using [Kn83, §2, p. 173]. Then there exists an open cover $\nu : U = \bigcup_{\alpha \in I} U_\alpha \to X$ and a $U$-isomorphism $\tau : U \times_X X'_i \simeq U \times_X X'_2$ such that $\tau \circ (\text{id}_U \times_X t_1) = \text{id}_U \times_X t_2$.

To prove the technicality above, we may assume without loss of generality that $X = \text{Spec}(R)$ for some ring $R$, $\mathcal{L}_i = \mathcal{O}_X$, and $D$ is cut out by some $g \in R$. Then $s_1, s_2 \in R$, and $s_1|_D = s_2|_D$ simply means that there exists $h \in R$ such that $s_2 - s_1 = hg$. Then $\text{id}_R$ and $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ give an isomorphism between the two complexes $[R \to R^2]$, which will induce $\tau$. The remaining details are straightforward.

Let us return to the proof of Proposition 3.7. It is straightforward that all geometric fibers of $\pi'$ have arithmetic genus 0, so to prove the first claim it remains to show that $\pi'$ is flat. Thus all that remains to prove (for all 3 parts) is Zariski local on $C$ or $C'$. Let

$$V_0 = \text{Coker} \left( \mathcal{O}_C \xrightarrow{\alpha \mapsto \alpha \pi^* x^\nu \otimes \eta} \pi^* x^\nu \omega_{C/S}^\vee \oplus \mathcal{O}_C(x) \right),$$

$\xi_0 : C'_0 = \mathbb{P}_C V_0 \to C$ the natural projection, and $C'_{0,0} = \xi_0^{-1}(C'_0)$. As above, we may construct lifts $x'_0$ and $x'_{0,\infty}$. Since $x^* \pi^* x^* \omega_{C/S}^\vee = x^* \omega_{C/S}^\vee$ and $x^* \pi^* x^* v = x^* v$, re general technicality at the beginning of this proof shows that $C'_0$ and $C'$ are locally isomorphic and the local isomorphisms are compatible with the projections to $C$ and $S$, and with the sections $x'$ and $x'_0$. Since everything left to check is Zariski local, we can check it for $C'_0$ instead of $C'$.

The advantage $C'_0$ has over $C'$ is that it is easy to construct a map to the local model. Let $h : Y = \text{Spec} \mathbb{C}[x,t] \to \text{Spec} \mathbb{C}[t] = Z$ be the projection map, $h : Y' \to Y$ the blowup at $(x,t)$, $r : Z \to Y$ the zero-section, and $r' : Z \to Y'$ its lift. To construct the map to the local model, we proceed as follows. Let $s \in S$. Since $\pi$ is smooth at $x(s)$, there exists an open subset $U \subset C$ such that $x(s) \in U$ and an étale morphism $\psi : U \to \mathbb{A}^1_S$ such that $\pi|_U$ is the composition of $\psi$ with the projection $\mathbb{A}^1_S \to S$. Composing with a suitable automorphism, we may assume that $x$ maps to the 0-section. By shrinking $S$ to $x^{-1}(U)$, we may assume that $x \subset U$, and that $x$ is the whole preimage of the zero section of $\mathbb{A}^1_S \to S$. Furthermore, we may shrink $S$ so that $x^* \omega_{C/S}^\vee$ is trivial, and choose a trivialization. Then $x^* v$ is simply a regular function on $S$, so it induces a morphism $S \to \mathbb{A}^1 = Z$ from which we may construct the desired map. Part 1 of the proposition follows from the fact that the composition $Y' \to Y \to Z$ is flat, part 2 follows from the existence of the isomorphism

$$h_* \omega_{Y'/Z}^\vee(-r'(Z)) \simeq \omega_{Y/Z}^\vee(-r(Z))$$

which restricts to the identity on $Y' \setminus \{0\}$, and part 3 follows from the fact that, on the proper transform of the zero-section, the differential $\Omega_{Y'/Z} \to h_* \Omega_{Y/Z}$ vanishes only at the point above 0. \hfill $\square$
Alternatively, part (2) could have been proved by an argument similar to that in Proposition 3.3.

**Definition 3.8.** We say that \( v' \in \Gamma(C', \omega^\vee_{C'/S}(-x'_\infty)) \) is compatible with \( v \) if \( x^*v' \) is nowhere vanishing and \( v'|_{C'_0} = v|_{C'_0} \).

**Remark 3.9.** Let \( \xi_{\delta} \) be the skew logarithmic differential of \( \xi \) (Definition 2.4). If \( v' \) is compatible with \( v \), then \( \xi_{\delta}(v') = v \). Indeed, \( \xi_{\delta}(v') - v \) restricts to 0 on the dense open \( C_\circ \subset C \), so it must be 0 since \( \omega^\vee_{C/S} \) is locally free.

Note that it is no longer true that such \( v' \) is unique, even when \( S = \text{Spec } \mathbb{C} \). However, we’ll see that this is true up to automorphisms of the rest of the data.

**Lemma 3.10.** Assume that \( S \) is a variety over \( \mathbb{C} \). Then the group of automorphisms \( \phi \) of \( C' \) over \( C \) such that \( \phi \circ x'_\infty = x'_\infty \) and \( \phi \circ x' = x' \) acts transitively on the set of \( v' \) compatible with \( v \).

**Proof.** Let \( v'_1 \) and \( v'_2 \) compatible with \( v \). In the first case, if \( v'_1 \neq v'_2 \) and \( S \) is a variety, then \( x^*v \equiv 0 \). Then there exists a section \( \rho : C \to C' \) of \( \xi \) which is a closed immersion, and in fact \( C' = \rho(C) \cup_{y(S)} P \), the gluing of \( \rho(C) \) with \( P \cong \mathbb{P}(S)[x(S),] \), which is now a \( \mathbb{P}^1 \)-bundle over \( x(S) \), along a section of \( \pi' \). Then we may choose \( \phi \) to be the identity on \( \rho(C) \) and to fix \( y \) and \( x' \) on \( P \).

Proving Lemma 3.10 over arbitrary bases is the main difficulty that persuaded us to focus on the case when \( S \) is a variety.

**Lemma 3.11.** There is a canonical surjection \( \xi_*(\mathcal{O}_{C'}(x')) \to \mathcal{I}_{x(Z),C}(x) \). Moreover, if \( Z \neq \emptyset \), then this map is an isomorphism.

**Proof.** Since \( x_* \) is right exact as \( x \) is finite, Proposition 3.7 part 3 shows that

\[
(3) \quad \xi_*x'_*x^*x^*\omega^\vee_{C'/S} = x_*x^*x^*\omega^\vee_{C'/S} \to x_*x^*\omega^\vee_{C'/S} \to \mathcal{O}_C(x) \otimes x_*\mathcal{O}_Z \to 0
\]

is exact, as \( x_*(x^*\omega^\vee_{C'/S} \otimes \mathcal{O}_Z) = x_*(N_{x(S),C} \otimes \mathcal{O}_Z) = x_*(x^*\mathcal{O}_C(x) \otimes \mathcal{O}_Z) = \mathcal{O}_C(x) \otimes x_*\mathcal{O}_Z \) by adjunction and the projection formula, where \( N \) stands for normal sheaf. On the other hand, there is a natural map \( \mathcal{O}_{C'}(x') \to \xi^*\mathcal{O}_C(x) \) obtained as the dual of the map \( \xi^*\mathcal{I}_{x,C} \to \mathcal{I}_{x',C'} \) adjoint to \( \mathcal{I}_{x,C} \to \xi_*,C' \), the restriction of the isomorphism \( \xi^* : \mathcal{O}_C \to \mathcal{O}_{C'} \). This map pushes forward to \( \xi_*\mathcal{O}_{C'}(x') \to \xi_*\xi^*\mathcal{O}_C(x) \cong \mathcal{O}_C(x) \), since the fact that \( \xi^* \) is an isomorphism implies that \( \xi_*\xi^*\mathcal{L} \cong \mathcal{L} \) for any invertible \( \mathcal{O}_C \)-module \( \mathcal{L} \). We have a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \xi_*\mathcal{O}_{C'} \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_C \\
\end{array}
\]

with exact rows. The exactness of the top row follows from \( R^1\xi_*\mathcal{O}_{C'} = 0 \), and the proof of commutativity is left to the reader. Then the snake lemma and (3) imply that the cokernel of \( \xi_*\mathcal{O}_{C'}(x') \to \mathcal{O}_C(x) \) is \( \mathcal{O}_C(x) \otimes x_*\mathcal{O}_Z = \mathcal{O}_C(x)|_{x(Z)} \).

The quotient map to the kernel \( \mathcal{O}_C(x) \to \mathcal{O}_C(x)|_{x(Z)} \) is the usual restriction, and the existence of the desired surjection follows. This map is an isomorphism if and only if \( \xi_*\mathcal{O}_{C'}(x') \to \mathcal{O}_C(x) \) is injective. The key point here is that the restriction maps \( \Gamma(U', \mathcal{O}_{C'}(x')) \to \Gamma(U' \setminus \xi^{-1}(x), \mathcal{O}_{C'}) \) are injective, which implies that so are the restriction maps \( \Gamma(U', \mathcal{O}_C(x')) \to \Gamma(U' \setminus \xi^{-1}(x), \mathcal{O}_C) \), since \( \mathcal{O}_{C'}(x') \) is invertible, from which the injectivity of \( \xi_*\mathcal{O}_{C'}(x') \to \mathcal{O}_C(x) \) follows easily. \( \square \)
Proposition 3.12. In Situation 3.1.(1), if \( Z \neq S \), then there exists \( v' \) compatible with \( v \).

Proof. We have \( \xi_1 \omega_{C/S}(-x_\infty) \simeq \mathcal{I}_{(Z),C} \otimes \omega_{C/S}(-x_\infty) \) by Proposition 3.7 part 2, Lemma 3.11, and the projection formula. If \( 0 \to \mathcal{I}_{(Z),C} \to \mathcal{O}_C \to x_* \mathcal{O}_Z \to 0 \) is twisted by \( \omega_{C/S}(-x_\infty) \) and we take global sections, we obtain an exact sequence

\[
0 \to \Gamma(C', \omega_{C'/S}(-x'_\infty)) \to \Gamma(C, \omega_{C/S}(-x_\infty)) \to \Gamma(C, \omega_{C/S}(-x_\infty) \otimes x_* \mathcal{O}_Z),
\]

and since \( v|_{x(Z)} = 0 \), there exists a unique \( v' \) which maps to \( v \). To show that \( x'^*v' \) is nowhere vanishing, note that the first map in Proposition 3.7 part 3 is injective on sections in this case, as it is an isomorphism above \( S \setminus Z \). Then Lemma 3.7 implies that \( x'^*\omega_{C'/S}(-x'_\infty) \simeq \mathcal{O}_Z \) given the definition of \( Z \). Combined with the fact that \( x'^*v' \) is nonzero on the dense open \( S \setminus Z \subset S \), it implies that \( x'^*v' \) is indeed nowhere vanishing.

The result of the stabilization relative to \( v \) is \( (C', S, \pi', x'_\infty, x', v') \), where all data except \( v' \) is the one constructed canonically above, and \( v' \) is compatible with \( v \). We have proved the existence of such a \( v' \) only with some additional assumptions cf. Proposition 3.12, and we have shown that if such \( v' \) exists then it is unique up to suitable automorphisms of the data cf. Lemma 3.10. Existence is true in general and will come ‘for free’ from an inductive hypothesis since we’ll see in §4 that the condition in Proposition 3.12 is satisfied in a universal case.

4. Marked field decorated rational trees

We will see very soon that the work done in §3 suffices to construct inductively the degeneration in Theorem 1.2. What we will do in §4 allows us to interpret this degeneration as a moduli space. The objects it parametrizes are the following. Again, we consider only the case when the base is a complex variety to simplify matters, although it is possible to allow arbitrary \( S \).

Definition 4.1. An \( n \)-marked field-decorated rooted rational tree consists of a variety \( S \), a prestable genus 0 curve \( \pi : C \to S \), sections \( x_\infty, x_1, \ldots, x_n : S \to C \) of \( \pi \), and an \( \mathcal{O}_C \)-module homomorphism \( \nu : \omega_{C/S} \to \mathcal{I}_{x_\infty(S),C} \), such that

1. \( x_i(s) \neq x_\infty(s) \), and \( \pi \) is smooth at \( x_\infty(s), x_1(s), \ldots, x_n(s) \), for all \( s \in S \);
2. \( x_i^* \nu : x_i^* \omega_{C/S} \to \mathcal{O}_S \) is an isomorphism for \( i = 1, \ldots, n \).

We say that the \( n \)-marked field-decorated rooted rational tree is stable if \( \omega_{C/S}(x_\infty + 2x_1 + \cdots + 2x_n) \) is relatively ample.

Let \( v \) be the image of \( \nu \) under \( \text{Hom}(\omega_{C/S}, \mathcal{I}_{x_\infty(S),C}) \simeq \Gamma(C, \omega_{C/S}(-x_\infty)) \). There is a natural isomorphism \( x_\infty^* \omega_{C/S}(x_\infty) \simeq x_\infty^* \mathcal{O}_{C/S}(x_\infty) \simeq \mathcal{O}_S \) by taking residues, or dually \( x_\infty^* \omega_{C/S}(-x_\infty) \simeq \mathcal{O}_S \). Then \( x_\infty^*v \) is simply a regular function on \( S \), so it induces a morphism \( S \to \mathbb{A}^1 \). We call this the coressidue morphism to \( \mathbb{A}^1 \).

Theorem 4.2. The moduli space \( \overline{F}_n \) of stable \( n \)-marked field-decorated rooted rational trees exists, in the sense that the functor \( \text{Var}_C \to \text{Set} \) which sends \( S \) to the set of stable \( n \)-marked field-decorated rooted rational tree over \( S \) modulo \( S \)-isomorphism is representable. Moreover, \( \overline{F}_n \) is normal and irreducible, and the coressidue morphism \( \eta_n : \overline{F}_n \to \mathbb{A}^1 \) is flat and projective.

To prove Theorem 4.2, we will need an extension of Definition 4.1.
Definition 4.3. Let \( n \) be a positive integer and \( \mathbf{a} = (a_1, \ldots, a_n) \in \{0, 1, 2\}^n \). An \( \mathbf{a} \)-stable field-decorated rooted rational tree (or \( \mathbf{a} \)-FDRT) consists of a complex variety \( S \), a prestable genus 0 curve \( \pi : C \to S \), sections \( x_{\ell,1}, \ldots, x_{\ell,n}, x_{\infty} : S \to C \) of \( \pi \), and an \( \mathcal{O}_C \)-module homomorphism \( \nu : \omega_{C/S} \to \mathcal{I}_{x_{\infty}}(S,C) \), such that

1. \( x_{\infty}(S) \) is contained in the open subset where \( \pi \) is smooth;
2. if \( a_i \geq 1 \), then \( x_i(S) \) is contained in the open subset where \( \pi \) is smooth;
3. if \( a_i = 2 \), then \( x_i^*\nu : x_i^*\omega_{C/S} \to \mathcal{O}_S \) is an isomorphism;
4. \( \omega_{C/S}(x_{\infty} + \mathbf{a} \cdot \mathbf{x}) \) is \( \mathbf{a} \)-ample, where \( \mathbf{a} \cdot \mathbf{x} = \sum_{j=1}^n a_j x_j(S) \).

(Note that \( \sum_{j=1}^n a_j x_j(S) \) is Cartier in (4), since \( x_i(S) \) is Cartier if \( a_i \neq 0 \).

If \( \mathbf{a} \) is some \( \mathbf{a} \)-stable field-decorated rooted rational tree, we will write \( S[\mathbf{a}], C[\mathbf{a}], \pi[\mathbf{a}], x_{\infty}[\mathbf{a}], x_i[\mathbf{a}], \nu[\mathbf{a}] \) for what was denoted \( S, C, \pi, x_{\infty}, x_i, \nu \) in Definition 4.3. We also write \( v[\mathbf{a}] \) for the global section of \( \omega_{\nu[\mathbf{a}]}(-x_{\infty}[\mathbf{a}]) \) corresponding to \( \nu[\mathbf{a}] \). For any \( \mathbf{a} \)-FDRT \( \mathbf{a} \) and any map \( \phi : S' \to S[\mathbf{a}] \), we define the pullback \( \phi^*\mathbf{a} \) of \( \mathbf{a} \) along \( \phi \) in the natural way: \( S'[\phi^*\mathbf{a}] = S' \times_S C[\mathbf{a}] \), etc. Then the class of all \( \mathbf{a} \)-FDRT becomes a category fibered over \( \text{Var}_C \).

For \( \ell \in \{0, 1, 2\} \) and any positive integer \( n \), let \( \mathbf{a}_{\ell,n} = (2, \ldots, 2, \ell) \in \{0, 1, 2\}^n \).

We construct inductively a sequence of objects \((t_{\ell,n})_{0 \leq \ell \leq n \in \mathbb{Z}^+}\), such that \( t_{\ell,n} \) is an \( \mathbf{a}_{\ell,n} \)-FDRT. Let \( t_{2,1} \) be defined by \( S[t_{2,1}] = \mathbb{A}^1 \), \( C[t_{2,1}] = \mathbb{A}^1 \times \mathbb{P}^1 \), \( \pi[t_{2,1}] \) the projection to \( \mathbb{A}^1 \):

\[
x_{\infty}[t_{2,1}](t) = (t, [1 : 0]) \quad x_1[t_{2,1}](t) = (t, [1 : 1]) \quad v[t_{2,1}] = (1 + tx) \frac{\partial}{\partial x}
\]

in the affine chart \( Y \neq 0 \), where \( x = X/Y \). It is clear that \( t_{2,1} \) is an \( \mathbf{a}_{2,1} \)-FDRT.

Note also that the corestriction morphism for \( t_{2,1} \) is the identity map on \( \mathbb{A}^1 \). Then we continue for \( n \geq 2 \) as follows:

- \( t_{0,n} \) is obtained from \( t_{2,n-1} \) by pulling back along \( \pi[t_{2,n-1}] \) and taking \( x_n \) to be the diagonal section \( C[t_{2,n-1}] \to C[t_{2,n-1}] \times_S t_{2,n-1} \times C[t_{2,n-1}] \);
- \( t_{1,n} \) is obtained from \( t_{0,n} \) applying the stabilization of §3.1;
- \( t_{2,n} \) is obtained from \( t_{1,n} \) applying the stabilization of §3.2.

It is straightforward to check that this construction is sound.

We can extend Theorem 4.2 to the form which will be proved inductively.

Theorem 4.4. Let \( n \geq 1 \) and \( 0 \leq \ell \leq 2 \) such that \( 3n + \ell \geq 5 \). Then \( t_{\ell,n} \) is terminal in the category of \( \mathbf{a}_{\ell,n} \)-FDRT.

Remark 4.5. Note that for all \( n \geq 1 \) and \( 0 \leq \ell \leq 2 \) such that \( 3n + \ell \geq 5 \), all \( \mathbf{a}_{\ell,n} \)-FDRT are free of automorphisms. Since we are assuming that bases are varieties, this boils down to the elementary case when the base is \( \text{Spec} \mathbb{C} \).

Definition 4.6. A contraction consists of the following data: \( \ell \in \{1, 2\} \), an \( \mathbf{a}_{\ell,n} \)-FDRT \( \mathbf{a} \), an \( \mathbf{a}_{\ell-1,n} \)-FDRT \( \mathbf{b} \), and a morphism \( f : C[\mathbf{a}] \to C[\mathbf{b}] \) such that

1. \( S[\mathbf{a}] = S[\mathbf{b}] \), \( \pi[\mathbf{b}] \circ f = \pi[\mathbf{a}] \), and \( f \circ x_\alpha[\mathbf{a}] = x_\alpha[\mathbf{b}] \), for \( \alpha = \infty, 1, \ldots, n \);
2. \( f \) is contractive, cf. Definition 2.2;
3. if \( \pi[\mathbf{b}] \circ f \) is the open subscheme where \( \pi[\mathbf{b}] \) is smooth, then the restriction of \( f \) induces an isomorphism \( f^{-1}(\pi[\mathbf{b}] \circ f) \to \pi[\mathbf{b}] \circ f \) under which \( \nu[\mathbf{a}] \circ f^{-1}(\pi[\mathbf{b}] \circ f) \) corresponds to \( \nu[\mathbf{b}] \circ \pi[\mathbf{b}] \circ f \) in the natural sense.

To prove the existence of contractions, we will rely on the powerful techniques of [Stacks, Tag 0E7B]. Although an elementary approach is also possible, the more
Lemma 4.7. For \( \ell = 1,2 \) and any \( n \), any \( a_{\ell,n} \)-FDRT admits a contraction to an \( a_{\ell-1,n} \)-FDRT, unique up to unique isomorphism.

Proof. If the base \( S \) is \( \text{Spec } \mathbb{C} \), this is elementary. For instance, we may take
\[
C[b] = \text{Proj} \bigoplus_{j \geq 0} \Gamma (C[a], \omega_{C[a]}(2x_1[a] + \cdots + 2x_{n-1}[a] + (\ell - 1)x_n[a])^{\otimes j}),
\]
and any \( a \)-FDRT, unique up to unique isomorphism.

Proof. If the base \( S \) is \( \text{Spec } \mathbb{C} \), this is elementary. For instance, we may take
\[
C[b] = \text{Proj} \bigoplus_{j \geq 0} \Gamma (C[a], \omega_{C[a]}(2x_1[a] + \cdots + 2x_{n-1}[a] + (\ell - 1)x_n[a])^{\otimes j}),
\]
and any

Contraction and stabilization are inverse operations, and now we partially prove this, ignoring the vector fields for now.

Lemma 4.8. Let \( a \) be an \( a_{1,n} \)-FDRT and \( b \) a contraction of \( a \). Then \( b \) satisfies Situation 3.1.(0) with \( x_n[b] \) in the role of \( x \). If \( C[\ell][b], x[\ell][b], x[\ell][b], v[\ell][b] \) is its Knudsen stabilization (§3.1), then there exists an isomorphism \( C[\ell][b] \simeq C[a] \) which is compatible with the projections to \( S = S[a] = S[b] \) and all the sections.

Note also that for \( i = 1, \ldots, n - 1 \), \( x_i[b] \) necessarily maps to the open subset of \( C[b] \) on which \( C[\ell][b] \to C[b] \) is an isomorphism.

Proof. This is similar to the proof of [Kn83, Lemma 2.5], so we will only sketch the argument. The claim that \( b \) satisfies Situation 3.1.(0) if \( x_n[b] \) plays the role of \( x \) boils down immediately to the special case \( S = \text{Spec } \mathbb{C} \), when it's elementary. Analogously to the proof of [Kn83, Lemma 2.5], we have an \( \mathcal{O}_{C[b]} \)-module homomorphism \( f_*\mathcal{O}_{C[a]}(x_n[a] - x_\infty[a]) \to T_{x_n[b],C[b]}(-x_\infty[b]) \), and it is straightforward to check on geometric fibers that this is an isomorphism. We claim that the adjoint
which restricts to the identity on any open subset where $f$ is surjective. Equivalently, $f^* \beta \omega \circ f_* \omega \circ f^* = f^* \omega$, which is surjective, which follows from [Kn83, Corollary 1.5]. Let $\mathcal{J} = f^* \mathcal{O}_C(x_n[b]) \otimes \mathcal{O}_C(-x_\infty[a])$. Consider the (solid arrow) commutative diagram with exact rows

\[
\begin{array}{ccc}
\mathcal{O}_C[a] - f^* \mathcal{O}_C[b] & \rightarrow & f^* \mathcal{O}_C[b](x_\infty[b]) \\
\downarrow & & \downarrow \begin{bmatrix} \text{id} & 0 \\ 0 & \alpha \otimes \text{id} \end{bmatrix} \\
0 & \rightarrow & \mathcal{J} \\
\end{array}
\]

Both rows are related to the exact sequence defining the cokernel $\mathcal{K}$ in (1). The top row is the $f$-pullback of this sequence for $C[b]$, the bottom row is this sequence for $C[a]$, twisted by $\mathcal{J}$. A basic diagram chase shows that there exists a unique dashed arrow which makes the diagram commute, and this map is surjective by the snake lemma. We’ve obtained a surjective map $f^* \mathcal{K} \rightarrow \mathcal{O}_C[a](x_n[a]) \otimes f^* \mathcal{O}_C[b](x_\infty[b])$. Of course, the target is invertible, so this induces a morphism $C[a] \rightarrow C[b]'$. The compatibilities as well as the fact that this is an isomorphism are straightforward because they can be checked on fibers.

\[\square\]

Lemma 4.9. Let $a$ be an $a_{2,n}$-FDRT and $b$ a contraction of $a$. Then $b$ satisfies Situation 3.1.(1) with $x_n[b]$ in the role of $x$. If $C[b]'$, $\pi[b]'$, $x_\infty[b]'$, $x[b]'$ is the data constructed canonically as part of its stabilization relative to $v[b]$ (last paragraph of §3.2), then there exists an isomorphism $C[b]' \simeq C[a]$ which is compatible with the projections to $S = S[a] = S[b]$ and all the sections.

A remark analogous to the one after Lemma 4.8 applies.

Proof. The claim that $b$ satisfies Situation 3.1.(1) if $x_n[b]$ plays the role of $x$ boils down immediately to the special case $S = \text{Spec} \ C$, when it’s elementary. We claim that there exists an isomorphism

\[
\beta : f^* \omega_{\pi[a]}^{-1}(x_n[b] - x_\infty[b]) \simeq \omega_{\pi[a]}^{-1}(x_n[a] - x_\infty[a])
\]

which restricts to the identity on any open subset where $f$ is an isomorphism. We have $\omega_{\pi[a]}^{-1}(x_n[b] - x_\infty[b]) \simeq f_* \omega_{\pi[a]}^{-1}(x_n[a] - x_\infty[a])$ by an argument ‘isomorphic’ to that used in the proof of Proposition 3.3. Indeed, only the details of the verifications on fibers differ slightly, otherwise the argument is the same. Then (4) follows by [Kn83, Corollary 1.5].

Let $\mathcal{U} = f^* \mathcal{O}_C[b](x_n[b])$. We have a commutative diagram with exact rows

\[
\begin{array}{ccc}
\mathcal{O}_C[a] = f^* \mathcal{O}_C[b] & \rightarrow & f^* \mathcal{O}_C[b](x_n[b]) \oplus f^* \omega_{\pi[a]}^{-1}(x_\infty[b]) \\
\downarrow & & \downarrow \begin{bmatrix} \text{id} & 0 \\ 0 & \beta \otimes \text{id} \end{bmatrix} \\
0 & \rightarrow & \mathcal{U}(x_n[a]) \\
\end{array}
\]

Both rows are related to the exact sequence defining the cokernel $\mathcal{V}$ in (2). The top row is the pullback of this sequence for $C[b]$ along $f$; the bottom row is this sequence for $C[a]$, twisted by $\mathcal{U}(x_n[a])$. Commutativity is left to the reader. A simple diagram chase shows that there exists a unique dashed arrow which makes the diagram commute, and by the snake lemma, this map is surjective. We’ve obtained a surjective $\mathcal{O}_C[a]$-module homomorphism $f^* \mathcal{V} \rightarrow \omega_{\pi[a]}^{-1}(x_\infty[a]) \otimes f^* \mathcal{O}_C[b](x_n[b])$. 
The target is invertible, so it induces an $S$-morphism $C[a] \to (C[b])'$. The fact that this is an isomorphism, as well as all the required compatibilities can be checked on fibers.

We may now prove Theorem 4.4, which then gives Theorem 4.2.

Proof of Theorem 4.4. We prove this by induction on $3n+\ell$. The base case $3n+\ell = 5$ is elementary and left to the reader. For the inductive step, note that the case $\ell = 0$ is straightforward. Assume that $\ell \in \{1, 2\}$ and let $a$ be an $a_{\ell,n}$-FDRT. Note that if a map $S[a] \to S[t_{\ell,n}]$ which exhibits $a$ as a pullback of $t_{\ell,n}$ exists, then such a map must be unique. To prove that such a map exists, we proceed as follows. Let $a^-$ be the contraction of $a$ to an $a_{\ell-1,n}$-FDRT, cf. Lemma 4.7. By the inductive hypothesis, there exists a morphism $\phi : S[a] = S[a^-] \to S[t_{\ell-1,n}] = S[t_{\ell,n}]$ which exhibits $a^-$ as a pullback of $t_{\ell-1,n}$, and let $\psi : C[a^-] \to C[t_{\ell,n-1}]$ be the corresponding map on the curves. Since the stabilization procedures of §3.1 and §3.2 are both functorial, by Lemma 4.8 and Lemma 4.9, there exists a morphism $\tilde{\psi} : C[a] \to C[t_{\ell,n}]$ which exhibits $C[a]$ as $S[a] \times_{S[t_{\ell,n}]} C[t_{\ell,n}]$, and is compatible with the marked sections. Then, by the uniqueness part of Corollary 3.4 if $\ell = 1$ respectively Lemma 3.10 if $\ell = 2$, we obtain that $a$ is isomorphic to $\phi^* t_{\ell,n}$. □

5. $G_m$ and $G_a$ Actions on Curves and Their Moduli

5.1. Preliminaries. What we will need from this section is the equivariance criterion in Proposition 5.2 below. The proposition concerns a reflexive sheaf associated to a Weil divisor. We refer the reader to [Sch10, Ha94] for the theory of reflexive sheaves associated to Weil divisors, and also state a very simple functoriality result which we’ll need very soon. If $X$ is a normal variety, then we write $\mathcal{O}_X(D)$ for the reflexive coherent sheaf associated to the divisor $D$; however, in other sections we will reserve this notation for the Cartier case.

Remark 5.1 (Functoriality). Let $X$ and $X'$ be normal quasi-projective varieties, $f : X' \to X$ a morphism, $D = \sum_{k=1}^m a_kW_k$ a Weil divisor on $X$ such that $W_i$ and $f^{-1}(W_i)$ are prime Weil divisors, and $D' = \sum_{k=1}^m a_kf^{-1}(W_k)$ on $X'$. Then $f^*\mathcal{O}_X(D) \cong \mathcal{O}_{X'}(D')$ holds provided that $f^{-1}(X^{\text{sing}})$ has codimension at least 2 in $X'$. Indeed, functoriality is well known for Cartier divisors, so the statement holds in codimension 2. Then the claim follows automatically, in light of [Sch10, Theorems 1.17 and 2.10], or equivalently [Ha94, Proposition 1.11 and Theorem 1.9].

Proposition 5.2. Let $\gamma : G \to T$ be a flat group scheme with reduced, connected and rationally connected fibers with trivial Picard groups, such that both $G$ and $T$ are complex quasi-projective varieties, and let $G_t = \gamma^{-1}(t)$. Let $\pi : X \to T$ be a geometrically integral flat projective morphism, and let $X_t = \pi^{-1}(t)$. Assume that $X$ is normal, that $T$ is smooth and $\text{Pic}_T = 0$, and that $G$ acts on $X$. Moreover, assume that for all $t \in T$ such that $X_t$ is singular, all of the following hold: $X_t$ is normal, any nonconstant regular function on $G_t$ vanishes at some point, and $id_{G_t}$ and the constant identity-element map are ‘rationally homotopic’ in the space of endomorphisms $\text{End}(G_t)$. Finally, let $D \subset X$ be a prime Weil divisor flat over $T$, and with integral fibers. Then $\mathcal{O}_X(D)$ admits a $G$-equivariant structure.

Note that $\mathcal{O}_X(D) = T'_{D,X}$ by [Sch10, Propositions 3.4 and 3.13.(b)]. In the statement of Proposition 5.2, by ‘rationally homotopic’ we mean that there exist a
rational curve $B \subset \mathbb{A}^1$ and a $B$-endomorphism of $B \times G_t$ whose fibers over 0 and 1 are the two endomorphisms of $G_t$ in question.

Proof. Let $\sigma : G \times_T X \to X$ be the given $G$-action, $Y = G \times_T X$, $D_1 = G \times_T D$, and $D_2 = \sigma^{-1}(D)$. We claim that $D_1 \sim D_2$. Let $E = D_2 - D_1$, and $\mathcal{F} = \mathcal{O}_Y(E)$. For each $t \in T$, let $E_t = D_{2,t} - D_{1,t}$.

We will show that $E_t \sim 0$. If $X_t$ is nonsingular, the fact that $G_t$ is rationally connected implies that $D_t \sim g \cdot D_t$ for all $g \in G_t$ as Cartier divisors on $X_t$, and then the see-saw lemma [Ha77, Exercise III.12.4] implies that $\mathcal{O}_{Y_t}(E_t)$ is the pullback of a line bundle on $G_t$ along the projection $Y_t \cong G_t \times X_t \to G_t$. However, $\text{Pic}_{G_t} = 0$, so $\mathcal{O}_{Y_t}(E_t) \cong \mathcal{O}_{Y_t}$ when $X_t$ is nonsingular. If $X_t$ is singular, the assumption that $\text{id}_{G_t}$ and the constant identity-element map are rationally homotopic as morphisms $G_t \to G_t$ implies that $D_{1,t}$ and $D_{2,t}$ are rationally equivalent, and hence linearly equivalent as Weil divisors on $X_t$ [Fu98, §1].

By Lemma 5.1, it follows that $\mathcal{F}_t \cong \mathcal{O}_{Y_t}$. In particular, this also implies that $\mathcal{F}$ is invertible because invertibility of torsion free coherent modules can be checked fiberwise (all fibers are vector spaces of dimension 1) by a standard corollary of Nakayama’s lemma. It then follows from another application of the see-saw lemma [Ha77, Exercise III.12.4] that $\mathcal{F}$ is the pullback of an invertible sheaf on $T$ along the projection $Y \to T$. However, $\text{Pic}_T = 0$, and hence $\mathcal{F} \cong \mathcal{O}_Y$, that is, $D_1 \sim D_2$.

We thus obtain an isomorphism $\mathcal{O}_Y(D_2) \cong \mathcal{O}_Y(D_1)$. By Remark 5.1 once more, $\mathcal{O}_Y(D_1) \cong \text{pr}_Y^* \mathcal{O}_X(D)$, and $\mathcal{O}_Y(D_2) \cong \sigma^* \mathcal{O}_X(D)$, so the isomorphism above is an isomorphism $\phi_0 : \sigma^* \mathcal{O}_X(D) \to \text{pr}_Y^* \mathcal{O}_X(D)$. We can modify $\phi_0$ to a ‘unitary’ isomorphism $\phi : \sigma^* \mathcal{O}_X(D) \to \text{pr}_Y^* \mathcal{O}_X(D)$, that is, an isomorphism which restricts to the identity on $\{e_G\} \times_T X \subset G \times_T X$. This can be achieved simply by composing $\phi_0$ with the pullback along $\text{pr}_X$ of the restriction of $\phi_0^{-1}$ to $\{e_G\} \times_T X$.

Finally, we claim that $\phi$ automatically satisfies the cocycle condition. It suffices to check it on fibers of $G \times_T G \times_T X \to T$. Let $t \in T$. If $X_t$ is non-singular, then $\mathcal{O}_{X_t}(D_t)$ is invertible, and the cocycle condition is well-known to be automatically satisfied, please see the proof of [MFK94, Proposition 1.5, §3, Ch. 1]. If $X_t$ is singular, our assumption is that $G_t$ admits no nowhere vanishing regular functions. Keeping Remark 5.1 in mind, the cocycle condition amounts to the agreement of two isomorphisms from two canonically identified sheaves on $G_t^2 \times X_t$ to $\mathcal{O}_{G_t^2 \times X_t}(G_t^2 \times D_t)$. The agreement above $e_{G_t^2}$ is automatic by the unitarity condition. Composing one isomorphism with the inverse of the other gives an automorphism $\alpha$ of $\mathcal{O}_{G_t^2 \times X_t}(G_t^2 \times D_t)$ which restricts to the identity automorphism on $\{e_{G_t^2}\} \times X_t$. By [Sch10, Proposition 3.13.(c)], $\alpha$ corresponds to a nowhere vanishing regular function on $G_t^2 \times X_t$ equal to 1 on $\{e_{G_t^2}\} \times X_t$. However, $\Gamma(\mathcal{O}_{X_t}) \cong \Gamma(\mathcal{O}_{G_t^2 \times X_t})$, hence $\alpha \equiv 1$.

Proposition 5.2 will be used in conjunction with Lemma 5.3, which will also be used by itself several times.

Lemma 5.3. Let $Y$ be an $S$-scheme, and $\sigma$ an action of $G$ on $Y$ relative to $S$. Let $\mathcal{F}$ be a $G$-equivariant quasi-coherent $\mathcal{O}_Y$-module, $X = \mathbb{P}_Y(\mathcal{F})$, and $f : X \to Y$ the natural projection map. Then there exists a $G$-action on $X$ over $S$, relative to which $f$ is $G$-equivariant. Moreover, the choice is compatible with any base change $S' \to S$.

5.2. Stabilization revisited. In Situation 3.1.(f), $\ell \in \{0,1,2\}$, assume that the base $S$ is a variety, and let $\eta : S \to \mathbb{A}^1$ be the corestriction morphism, as in §4. Let
now $\gamma : G \to \mathbb{A}^1$ as in \S 1. We say that a $G$-action $\sigma : G \times_{\mathbb{A}^1} C \to C$ on $C$ over $S$ and $\mathbb{A}^1$ (more accurately, an action of the $S$-group scheme $S \times_{\mathbb{A}^1} G$ on the $S$-scheme $C$) is generated by $v$ if its first order infinitesimal action corresponding to the vector field $\frac{\partial}{\partial v}$ along the identity section of $G$ is equal to $v$, and $x_\infty$ is fixed by $\sigma$. In this subsection, we will prove inductively that for $3n + \ell \geq 5$, $t_{\ell,n}$ admits a $G$-action generated by $v[t_{\ell,n}]$. In the base case $3n + \ell = 5$, with notation as in \S 4, we may take the action $(t, a) \cdot (t, [X : Y]) = (t, [X + aY + taX : Y])$, and it is an elementary calculation that it is generated by $v[t_{2,1}]$. To prove the induction step, we need to revisit stabilization. Specifically, the induction steps will be Lemma 5.4, Lemma 5.5 or the obvious fact that actions or prestable curves can be pulled back, depending on the value of $\ell$.

5.2.1. Knudsen stabilization revisited. In Situation 3.1.(0), assume in addition that $C$ and $S$ are complex normal quasi-projective varieties, that the fibers of $\pi$ are normal, and that $x \neq x_\infty$. It can be checked simply following through the construction in \S 4 that these hypotheses are satisfied for $t_{0,n}$ (for instance, regular in codimension 1 and lci, or more generally Cohen-Macaulay, implies normal [Stacks, Tag 0342]). Moreover, assume that an action $\sigma$ on $C$ generated by $v$ is given.

**Lemma 5.4.** In the situation above, there exists a $G$-action $\sigma'$ on $C'$ such that $\xi$ is $G$-equivariant relative to $\sigma$ and $\sigma'$, and $\sigma'$ generates $v'$.

**Proof.** It suffices to analyze separately two situations: $x(S)$ is contained in the open subset of $C$ where $\pi$ is smooth, respectively $x \cap x_\infty = \emptyset$. In the first case, $x$ is a Cartier divisor on $C$ and $T_{x,C}' = \mathcal{O}_C(x)$. Let $\mathcal{E} = \mathcal{O}_C(-x) \oplus \mathcal{O}_C(-x_\infty)$. Our hypotheses ensure that we have a short exact sequence

$$0 \to \wedge^2 \mathcal{E} \to \mathcal{E} \to \mathcal{I}_{x(S) \cap x_\infty(S),C} \to 0,$$

so $K \simeq \mathcal{I}_{x_\infty \cap x,C}(x_\infty + x)$, where $K$ is defined in (1), and $C' = \mathbb{P}_C \mathcal{I}_{x_\infty \cap x,C}$. However, $\mathcal{I}_{x_\infty \cap x,C}$ is $G$-equivariant, so we may apply Lemma 5.3 to construct a lift $\sigma'$. In the second case, $K = \mathcal{T}_{x,C}'(x_\infty)$, and hence $C' = \mathbb{P}_C \mathcal{T}_{x,C}'$. By Proposition 5.2, $\mathcal{T}_{x,C}'$ is $G$-equivariant, so again by Lemma 5.3, there exists a lift $\sigma'$ of $\sigma$ to $C'$.

5.2.2. Stabilization relative to a vector field revisited. In Situation 3.1.(1), assume in addition that $C$ and $S$ are complex normal quasi-projective varieties, and that $x^*v$ is not identically 0. These hypotheses are satisfied for $t_{1,n}$ and again this can be ascertained by following through the construction in \S 4. Moreover, assume that an action $\sigma$ on $C$ generated by $v$ is given.

**Lemma 5.5.** In the situation above, there exists a $G$-action $\sigma'$ on $C'$ such that $\xi$ is $G$-equivariant relative to $\sigma$ and $\sigma'$, and $\sigma'$ generates $v'$.

**Proof.** It is clear that $x(Z)$ is fixed (even scheme-theoretically, since it’s reduced) by $\sigma$, and in particular $\mathcal{I}_{x(Z),C}$ has a natural $G$-equivariant structure relative to $\sigma$. Let $\mathcal{E} = \mathcal{O}_C(-x) \oplus \omega_{C/S}$. We have a short exact sequence

$$0 \to \wedge^2 \mathcal{E} \to \mathcal{E} \to \mathcal{I}_{x(Z),C} \to 0,$$

hence $\mathcal{V} \simeq \mathcal{I}_{x(Z),C} \otimes \omega_{C/S}(x - x_\infty)$, where $\mathcal{V}$ is defined in (2), so $C' = \mathbb{P}_C \mathcal{I}_{x(Z),C}$. By Lemma 5.3, there exists a $G$-action $\sigma'$ on $C'$ for which $\xi$ is $G$-equivariant. In particular, the map $\Gamma(d\xi)$ sends the infinitesimal generator of $\sigma'$ to $v$, and it follows that the former must be precisely $v'$. In particular, $x'(s)$ is not fixed by $\sigma'$ by the second part of Proposition 3.12. It is easy to check that $x'_\infty$ is fixed by $\sigma$. \qed
5.3. The $G^n_{\mathbb{A}^1}$-action on the total space. We may now complete the proof of Theorem 1.2. In the previous subsection, we shown that $t_{2,n}$ admits a $G$-action generated by $v[t_{2,n}]$, which we will call $\sigma$. To review notation, we have $S[t_{2,n}] = \mathcal{F}_n$. Consider the $n$-marked field decorated rooted rational tree $y$ which coincides with the pullback of $t_{2,n}$ along the projection $\varpi_2 : G^n_{\mathbb{A}^1} \times \mathbb{A}^1 \mathcal{F}_n \to \mathcal{F}_n$, with the sole exception of the sections $x_i[y]$ which are instead defined as the composition

$$S[y] = G^n_{\mathbb{A}^1} \times \mathbb{A}^1 \mathcal{F}_n \xrightarrow{(\varpi_1, \sigma)} G^n_{\mathbb{A}^1} \times \mathbb{A}^1 \mathcal{F}_n = S[\varpi_2^* t_{2,n}] \xrightarrow{x_i[\varpi_2^* t_{2,n}]} C[\varpi_2^* t_{2,n}] = C[y],$$

where $\varpi_1 : G^n_{\mathbb{A}^1} \times \mathbb{A}^1 \mathcal{F}_n \to G^n_{\mathbb{A}^1}$ is the projection to the first factor. Since $S[y]$ is a variety, Theorem 4.2 shows that $y$ induces an $\mathbb{A}^1$-morphism

$$\rho : G^n_{\mathbb{A}^1} \times \mathbb{A}^1 \mathcal{F}_n \to \mathcal{F}_n.$$

The fact that $\rho$ is a group action can be checked purely formally using Theorem 4.2 again. It is also easy to check that the diagonal $G \to G^n_{\mathbb{A}^1}$ acts trivially, and we take the quotient action simply by restricting to

$$G^{n-1}_{\mathbb{A}^1} \simeq \{ (g_1, \ldots, g_n) : g_1 \ast g_2 \ast \cdots \ast g_n = 0 \} \subset G^n_{\mathbb{A}^1}.$$

It is straightforward to check that this restricted action restricts on $F_n \cong \mathcal{F}_n \subset \mathcal{F}_n$ parametrizing $n$-marked field decorated rooted rational trees with irreducible sources to the tautological action.

Finally, we have to argue that the fibers of the coresidue morphisms $\eta : \mathcal{F}_n \to \mathbb{A}^1$ are $\mathcal{L}_n$ if $t \neq 0$, respectively $\mathcal{L}_n$ if $t = 0$. For the Losev-Manin space, this can be checked by comparing the constructions. What the stabilization in [LM00] accomplishes in a single step, our stabilization accomplishes in two. We also note in passing that the 0-section in the universal curve over the Losev-Manin spaces extends on the universal curve over $\mathcal{F}_n$ not to a section, but to the vanishing locus of $v[t_{2,n}]$. Regarding $\mathcal{P}_n$, we simply define $\mathcal{P}_n$ to be $\eta^{-1}(0)$, but note that indeed stable $n$-marked field decorated rooted rational trees over $\text{Spec } \mathbb{C}$ with coresidue 0 at $x_\infty$ are precisely stable $n$-marked $\mathbb{G}_a$-rational trees, cf. Definition 1.1.

It is straightforward to extend Definition 1.1 to prestable curve over arbitrary bases $S$, which we will continue to consider complex varieties. On one hand, it is straightforward to check that any stable $n$-marked $\mathbb{G}_a$-rational tree is a stable $n$-marked field decorated rooted rational tree with an identically 0 coreside map, where the vector field corresponds to the infinitesimal first order automorphism induced by $\frac{d}{dx} \in \mathbb{g}_a$. On the other hand, the converse holds: all stable $n$-marked field decorated rooted rational tree with an identically 0 coresude map come from stable $n$-marked $\mathbb{G}_a$-rational trees, because Theorem 4.2 shows that the pullback of $t_{2,n}$ to $\eta^{-1}(0) = \mathcal{P}_n$ is terminal in the category of stable $n$-marked field decorated rooted rational tree with an identically 0 coresude map, and the discussion above shows that the curve over it is in fact a stable $n$-marked $\mathbb{G}_a$-rational tree.

**Corollary 5.6.** The space $\mathcal{P}_n$ constructed above represents the functor $\text{Var}_C \to \text{Set}$ which sends each $S$ to the set of stable $n$-marked $\mathbb{G}_a$-rational trees over $S$ up to isomorphism.

We conclude the paper with the following question.

**Question 5.7.** Is it possible to classify projective toric varieties which can degenerate isotrivially to $G^n_\mathbb{A}$-varieties in a manner compatible with the group actions?
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