Shadowing, Finite Order Shifts and Ultrametric Spaces

Udayan B. Darji Daniel Gonçalves *
Marcelo Sobottka †

Abstract

Inspired by recent novel work of Good and Meddaugh, we establish fundamental connections between shadowing, finite order shifts and ultrametric Polish spaces. We develop a theory of shifts of finite type for infinite alphabets. We call them shifts of finite order. We develop the basic theory of the shadowing property in general Polish spaces, exhibiting similarities and differences with the theory in compact spaces. We connect these two theories in the setting of zero dimensional Polish spaces, showing that a uniformly continuous map of an ultrametric Polish space has the finite shadowing property if, and only if, it is an inverse limit of shifts of finite order. Furthermore, in this context, we show that the shadowing property is equivalent to the finite shadowing property and the fulfillment of the Mittag-Leffler Condition in the inverse limit description of the system. As corollaries we obtain that a variety of maps in ultrametric Polish spaces have the shadowing property, such as similarities and, more generally, maps which themselves, or their inverses, have Lipschitz constant 1. Finally, we apply our results to the dynamics of $p$-adic integers and $p$-adic rationals.

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1 Introduction

Very recently Good and Meddaugh [17] made a fundamental structural connection between the notions of shifts of finite type and the shadowing property. A classical result of Walters [34] states that a shift has the shadowing property if, and only if, it is a shift of finite type. The fundamental result in [17] shows that a continuous function \( f : X \to X \), where \( X \) is a compact totally disconnected space, has the shadowing property if, and only if, the system \((X, f)\) is conjugate to the inverse limit of a directed system consisting of shifts of finite type and satisfying the Mittag-Leffler Condition. As tempting as it is to reiterate the history and the importance of shifts of finite type and shadowing, we refrain from it here and let the reader browse the beautiful exposition in [17].

Our goal in this paper is to connect the shadowing property of zero dimensional Polish dynamical systems \((X, f)\) with simple objects such as shifts. Our general approach is inspired by [17], however, it departs from it in several important ways.

First, we must find an appropriate concept of "shift of finite type" in the noncompact setting. There are many approaches to shifts over infinite alphabets, e.g., [14, 28]. We will focus on the approach which takes the full shift space over an alphabet to be the product space, with product topology, and the usual shift map. These shifts have been studied extensively, e.g., [21]. In Section 2, we introduce the notion of shift of finite order, which is an analogue of a shift of finite type in the setting of countable alphabets.

Another important difference is the notion of shadowing in the noncompact framework. It is well known that, for compact metric spaces, finite shadowing and infinite shadowing are equivalent. In fact, in [17] it is shown that in the setting of compact metric spaces, shadowing is a topological property. Using this fact the authors define a meaningful notion of shadowing in an arbitrary compact space. In the noncompact setting things are very different, as finite shadowing and shadowing are not equivalent properties, the former being more general. Moreover, two dynamical systems, one with the shadowing property and the other without, may be conjugate to each other, see Example 2.3.4. We show, in Proposition 2.2.5, that uniform conjugacy remedy this problem, i.e., two uniformly conjugate Polish dynamical systems either both have the shadowing property or neither has it. Of course, in the compact metric space setting uniform conjugacy and conjugacy are equivalent.

Since the shadowing property in Polish spaces is necessarily metric dependent, the metric on the space space plays a significant role in our work. We therefore introduce the notions of "defining sequence" and "tame defin-
A defining sequence \( \{U_n\} \) can be thought of as a sequence of clopen partitions of a zero dimensional Polish space \( X \), where the clopen partitions get finer as \( n \) increases. A tame defining sequence is a defining sequence where the mesh of the diameter of the partition goes to zero as \( n \) increases and, at the same time, given a partition in the sequence any two elements of it are a fixed distance apart. These definitions may seem contrive, however, we show that any ultrametric Polish space admits a tame defining sequence. Our main theorem which connects shadowing and shifts of finite order in the general zero dimensional Polish space setting is the following.

**Theorem 4.2.10.** Let \( X \) be a metric space with a tame defining sequence and \((X, f)\) be a dynamical system with the finite shadowing property. Then, \( f \) is conjugate to the inverse limit of a sequence of 1-step shifts on a countable alphabet. More precisely, \((X, f)\) is conjugate to \((\lim_{\leftarrow} \{\rightarrow, \mathcal{P}O(U_n)\}, \sigma^*)\). Moreover, if \( f \) is uniformly continuous, then the conjugacy can be made uniform.

As it happens, finite shadowing and shadowing also coincide in some noncompact spaces such as uniformly locally compact spaces, see Proposition 2.3.2. Moreover, extending the classical result of Walters mentioned before, we show that for shifts on a countable alphabet, the finite shadowing property, the shadowing property and being a shift of finite order are equivalent, see Proposition 2.3.5. (This was also proved independently by Meddaugh and Raines in [26].) In the context of uniformly continuous Polish dynamical systems with tame defining sequences, we show that equivalence between the shadowing property and the finite shadowing property depends on the fulfillment of the Mittag-Leffler Condition in the inverse limit description of a natural system, see Proposition 4.2.11.

We also show that the inverse limit of Polish spaces with the shadowing property also has the shadowing property, provided that the inverse limit satisfies the Mittag-Leffler Condition, see Theorem 4.1.11. Putting this together with the above theorem, we have the following corollary.

**Corollary 4.2.12.** Let \((X, d)\) be a metric space with a tame defining sequence and \( f : X \to X \) be a uniformly continuous map. Then, \( f \) has the shadowing property if, and only if, \( f \) is uniformly conjugate to an inverse limit, satisfying Mittag-Leffler Condition, of a sequence of 1-step shifts on a countable alphabet, with bonding maps of the inverse limit uniformly continuous.

We apply our results above and the techniques developed in the article to ultrametric Polish spaces. An ultrametric space is a space where the triangle inequality is strengthened to \( d(x, z) = \max\{d(x, y), d(y, z)\} \). Ultrametric
spaces have topological dimension zero. They naturally appear in a variety of places, including general topology, mathematical logic, theoretical computer science, $p$-adic dynamics and theoretical biology. For example, results regarding Lipschitz and uniformly continuous and Borel reductions are studied in [3, 33], generic elements in isometry groups are described in [23], Polish ultrametric spaces on which each Baire one function is first return recoverable are characterized in [7] and locally contracting maps on perfect Polish ultrametric spaces are studied in [11].

As ultrametric Polish spaces are zero dimensional, they have a basis of clopen sets. Standard notions such as being Lipschitz or being an isometry are often localized. For example, if we consider the balls of radius $\varepsilon > 0$ for a fixed $\varepsilon$, we obtain a clopen partition of an ultrametric space. Often maps in question are Lipschitz or an isometry at this local level. We call this local behavior eventual, see Definition 5.1.1. The following corollaries establish the shadowing property of maps of these types.

**Corollary 5.1.3.** Let $X$ be a Polish ultrametric space. If $f : X \to X$ is an eventually Lipschitz 1 function, then $f$ has the shadowing property.

**Corollary 5.1.4.** Suppose that $X$ is an ultrametric Polish space with the additional property that for some $\varepsilon > 0$, all balls of radius less than $\varepsilon$ are compact. Let $f : X \to X$ be a uniformly continuous map.

(i) If $f^{-1}$ is an eventually Lipschitz 1 map, then $f$ has the shadowing property.

(ii) If $f^{-1}$ is uniformly continuous and $f$ is an eventual similarity, then $f$ has the shadowing property.

In particular, all similarities in compact ultrametric Polish spaces have the shadowing property. Of course, for contractions and dilations, this is well known in general Polish spaces. What may be somewhat surprising is that the identity map has the shadowing property. Of course, such is never the case in a connected metric space with more than one element. In Example 5.1.6, we point out that shifts which are not of finite order are Lipschitz, but do not have the shadowing property. Hence, our results are, in some sense, sharp.

An important class of ultrametric dynamical systems consists of the $p$-adic dynamics, which has both practical and theoretical applications (we refer the reader to the Aims and Scope section of the journal "$p$-Adic Numbers, Ultrametric Analysis and Applications" for a comprehensive description of fields which intersect with $p$-adic dynamics). Among developments in the area we mention that minimal polynomial dynamics on the set of 3-adic integers is studied in [8], strict ergodicity of affine $p$-adic dynamical systems
on $\mathbb{Z}_p$ is described in [10], minimal decomposition of $p$-adic homographic dynamical systems is studied in [9], and shadowing and stability in $p$-adic dynamics are studied in the recent paper [2]. We are particularly interested in the results of [2]. As consequences of our results on ultrametric Polish spaces, we recover many of the shadowing results in [2], in addition to answering a question left open. More precisely, we prove the following.

**Corollary 5.2.1** The three statements below hold.

(i) [2, Theorem 1] If $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a $(p^{-k}, p^m)$ locally scaling function, where $1 \leq m \leq k$ are integers, then $f$ has the shadowing property.

(ii) [2, Proposition 18] If $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a Lipschitz 1 map, then $f$ has the shadowing property.

(iii) [2, Remark 19] If $f : \mathbb{Q}_p \to \mathbb{Q}_p$ is a Lipschitz 1 map, then $f$ has the shadowing property.

In the final remarks of this introduction, we would like to point out that dynamics outside the realm of compact metric space is a thriving area. For instance, strong orbit equivalence and topological equivalence of locally compact Cantor minimal systems are studied in [5, 25] and topological full groups as invariants for groupoids associated to locally compact spaces (and with connections to graph and ultragraph C*-algebras) are studied in [12, 27]. The theory of chaos (including Li-Yorke Chaos, Devaney chaos, distributional chaos) is also explored in the context of locally compact dynamics, see for example [4, 15, 16, 31]. Our result provides a strong connection, via inverse limits, between the theory of shadowing of Polish dynamical systems and the theory of shift spaces with the product topology. The inverse limit description of a dynamical system has various advantages, as it is often applied in the computation of invariants for the system (for example, Čech cohomology of a tiling dynamical system is computed from its inverse limit description (see [1, 13]). We hope our results and techniques apply to shadowing in various noncompact settings mentioned above.

The paper is organized as follows. In Section 2, we define and develop symbolic dynamics on countable alphabets pertinent for our results. We also develop the basic theory of the finite shadowing property and the shadowing property in the setting of general Polish spaces. Notions of defining sequence, ultrametric spaces and $p$-adics, the settings in which our main results reside, are developed in Section 3. Section 4 concerns shadowing, inverse limits and proofs of our main results. In the final Section 5, we give applications of our results.
2 Symbolic Dynamics and Shadowing

Throughout our work \( \mathbb{N} \) and \( \mathbb{N}^* \) denote the set of non-negative integers, and positive integers, respectively. For the sake of ease of readability, we denote points of a space \( X \) in bold style (e.g., \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in X \)). A Polish space a complete, separable metric space and a Polish dynamical system \((X, f)\) is a Polish space \( X \) equipped with a continuous map \( f : X \to X \).

2.1 Symbolic Dynamics of Countable Alphabets

We start the section recalling the definition of shift spaces.

**Definition 2.1.1.** Let \( A \) be a non-empty countable set endowed with the discrete topology. The (one-sided) full shift on the alphabet \( A \) is the set

\[
A^\mathbb{N} := \{ \mathbf{x} = (x_i)_{i \in \mathbb{N}} : x_i \in A \forall i \in \mathbb{N} \},
\]

with the associated prodiscrete topology, i.e., the product of discrete topology. Moreover, we will use the metric \( d(x, y) = 2^{-i} \) on \( A^\mathbb{N} \), where \( i \) is the least integer where \( x(i) \neq y(i) \).

**Definition 2.1.2.** The shift map is the map \( \sigma : A^\mathbb{N} \to A^\mathbb{N} \) defined, for all \( \mathbf{x} = (x_i)_{i \in \mathbb{N}} \in A^\mathbb{N} \), by

\[
\sigma(\mathbf{x}) := (x_{i+1})_{i \in \mathbb{N}}.
\]

**Definition 2.1.3.** We say that \( X \subseteq A^\mathbb{N} \) is a (one-sided) shift (or subshift) space if it is a closed set and it is shift invariant (that is, \( \sigma(X) \subseteq X \)).

In a general space \( X \), we will denote a sequence indexed by \( I \subseteq \mathbb{N} \) as \((x_n)_{n \in I} \). In the particular case that each \( x_n \) is itself a sequence, we will use the notation \( x_{n,i} \) to denote the \( i^{th} \) entry of the sequence \( x_n \).

Given a shift space \( X \subseteq A^\mathbb{N} \) and \( n \in \mathbb{N}^* \), we define \( \mathcal{L}_n(X) \) as the set of all words of length \( n \) that appear in some sequence of \( X \), that is,

\[
\mathcal{L}_n(X) := \{ (a_0 \ldots a_{n-1}) \in A^n : \exists \mathbf{x} \in X \text{ s.t. } (x_0 \ldots x_{n-1}) = (a_0 \ldots a_{n-1}) \}.
\]

Clearly \( \mathcal{L}_n(A^\mathbb{N}) = A^n \). The language of \( X \) is the set \( \mathcal{L}(X) \) which consist of all finite words that appear in some sequence of \( X \) (plus the empty word \( \varepsilon \) which has length zero):

\[
\mathcal{L}(X) := \bigcup_{n \in \mathbb{N}} \mathcal{L}_n(X).
\]

A well-known equivalent way to define shift spaces is given in terms of forbidden words. Given \( F \subset \mathcal{L}(A^\mathbb{N}) \), we can define \( X_F \subseteq A^\mathbb{N} \) as the set of all
sequences in $A^N$ which do not contain any word of $F$. In particular, one can check that given a shift space $X \subseteq A^N$, setting $F := \mathcal{L}(A^N) \setminus \mathcal{L}(X)$, we have that $X = X_F$.

**Definition 2.1.4.** A shift space $X$ is said to have order $p$, for some $p \in \mathbb{N}$, if there is a set of words $F$ such that $X = X_F$ and every word in $F$ has length $p$. When $X$ is a shift of order $p$, we refer to it simply as a shift of finite order.

The next proposition corresponds to [23, Theorem 2.1.8] and gives an alternative characterization of shift spaces of finite order. Although in [23] it is stated for shift spaces over finite alphabets, the proof given there also works for shift spaces over infinite alphabets. Such a characterization will be used in Proposition 2.3.5 to characterize shift spaces with the shadowing property.

**Proposition 2.1.5.** $X$ is shift of order $p$ if, and only if, for all $u, v, w \in \mathcal{L}(X)$, with $|v| = p - 1$ and $uv, vw \in \mathcal{L}(X)$, it follows that $uvw \in \mathcal{L}(X)$.

**Remark 2.1.6.** Notice that if $A$ is finite, then the class of shift spaces with finite order always coincides with the class of shift spaces of finite type (SFT), i.e., those subshifts that can be obtained from finite sets of forbidden words. On the other hand, if $A$ is infinite, then the class of shift spaces of finite type is strictly contained in the class of shift spaces of finite order.

### 2.2 Shadowing in Polish Spaces

In this section we recall the definition of shadowing and develop some of its properties in Polish spaces.

**Definition 2.2.1.** Let $X$ be a metric space and $I$ be an initial segment of $\mathbb{N}$. We say that,

(i) The sequence $(f^n(x))_{n \in \mathbb{N}}$ is the orbit, or trajectory, of the point $x \in X$.

(ii) A (finite or infinite) sequence $(x_n)_{n \in I} \in X$ is a (finite or infinite) $\delta$-chain, or $\delta$-pseudo-orbit (trajectory), if $d(f(x_n), x_{n+1}) < \delta$ for all $n < \sup(I)$.

(iii) A point $x \in X$ $\varepsilon$-shadows a (finite or infinite) sequence $(x_n)_{n \in I}$ if $d(f^n(x), x_n) < \varepsilon$, for all $n \in I$. 

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Definition 2.2.2. We say that a dynamical system \((X, f)\) has the finite shadowing property if, for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that any finite \(\delta\)-pseudo-orbit is \(\varepsilon\)-shadowed by some point. We say that \((X, f)\) has the shadowing property if, for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that any infinite \(\delta\)-pseudo-orbit is \(\varepsilon\)-shadowed by some point.

There are other concepts of shadowing in topological dynamics, such as \(\ell_p\) shadowing, asymptotic shadowing, \(s\)-limit shadowing, etc. For a study of the relations between these notions the reader is referred to [18]. We also point out that a different notion of shadowing was studied in [6] and a spectral decomposition theorem was proved. For general references on shadowing, we suggest the classical texts [29, 30].

It was shown in [19, Lemma 6] that for compact metric spaces shadowing is a topological concept, i.e., shadowing is independent of the metric as long as the topology is the same. The following simple example shows such is not the case in Polish spaces.

Example 2.2.3. There exists a locally compact space \(X\), a continuous function \(f : X \rightarrow X\), and two metrics on \(X\) which generate the same topology, one yielding that \(f\) has the shadowing property and the other not.

Construction: Our space \(X\) will be a countable subset of \(\mathbb{R}\). Let \(\{x_n\}_{n \in \mathbb{Z}}\) be a sequence of distinct elements of \((0, 1]\] with the following properties.

- \(x_0 = 1\),
- \(\lim_{|n| \to \infty} x_n = 0\).

Let \(y_n = -x_n\) for all \(n \in \mathbb{Z}\). We let \(X\) be the union of \(\{x_n\}_{n \in \mathbb{Z}}\) and \(\{y_n\}_{n \in \mathbb{Z}}\). Let \(f : X \rightarrow \mathbb{R}\) be defined by \(f(x_n) = x_{n+1}\) and \(f(y_n) = y_{n+1}\). It is clear that \(f\) is a homeomorphism of \(X\). It is also clear that if one uses the discrete metric on \(X\), in which the distance between any two distinct points is 1, then \(f\) has the shadowing property with respect to this metric. It is also clear that this metric generates the same topology as the topology induced by the metric on \(\mathbb{R}\).

We will show that with respect to the usual metric of \(\mathbb{R}\), \(f\) does not have the shadowing property. To this end, let \(\varepsilon = 1/2\) and \(\delta > 0\). We will construct a \(\delta\)-pseudo-trajectory that cannot be \(\varepsilon\) shadowed by a trajectory. Let \(N\) be a positive integer such that \(|x_{N+1}| + |y_{-N}| < \delta\). We define a \(\delta\)-pseudo-trajectory \(\{z_n\}_{n=1}^{\infty}\) as follows: for \(0 \leq i \leq N\), let \(z_i = x_i\) and for \(i \geq N + 1\), let \(z_i = y_{i-2N-1}\). Note each of \(-1, 1\) belongs to the \(\delta\)-pseudo-trajectory \(\{z_n\}_{n=1}^{\infty}\). However, any trajectory must be a subset of \((0, 1]\) or a subset of \([-1, 0)\]. Hence, \(\{z_n\}_{n=1}^{\infty}\) cannot be \(\varepsilon\) shadowed by a trajectory. 

\(\square\)
As the above example shows, conjugacy is not enough to preserve shadowing among Polish dynamical systems. We need a stronger version of conjugacy, namely uniform conjugacy.

**Definition 2.2.4.** Suppose \((X, f)\) and \((Y, g)\) are Polish dynamical systems. We say that \((X, f)\) and \((Y, g)\) are uniformly conjugate if there is a surjective homeomorphism \(h : X \to Y\), with \(h\) and \(h^{-1}\) uniformly continuous, such that \(h \circ f = g \circ h\).

**Proposition 2.2.5.** Suppose \((X, f)\) and \((Y, g)\) are uniformly conjugate Polish dynamical systems. If \((X, f)\) has the shadowing property, then so does \((Y, g)\).

**Proof.** Let \(h : X \to Y\) be a uniform conjugacy between \((X, f)\) and \((Y, g)\). Let \(d_X\) and \(d_Y\) be metrics on \(X, Y\), respectively. Let \(\varepsilon > 0\). Let \(\varepsilon' > 0\) witness the uniform continuity of \(h\) with respect to \(\varepsilon\). Let \(\delta' > 0\) witness the shadowing property of \(f\) with respect to \(\varepsilon'\). Let \(\delta > 0\) witness the uniform continuity of \(h^{-1}\) with respect to \(\delta'\). We will show that \(\delta\) witnesses the shadowing property of \(g\) with respect to \(\varepsilon\).

Indeed, let \(\{y_n\}\) be \(\delta\)-pseudo-trajectory in \(Y\). Let \(x_n = h^{-1}(y_n)\). By our choice of \(\delta\),

\[
d_Y(g(y_n), y_{n+1}) < \delta \implies d_X(h^{-1}(g(y_n)), h^{-1}(y_{n+1})) < \delta'.
\]

However, as \(f \circ h^{-1} = h^{-1} \circ g\) we have that

\[
d_X(h^{-1}(g(y_n)), h^{-1}(y_{n+1})) = d_X(f(h^{-1}(y_n)), h^{-1}(y_{n+1}))
\]

\[
= d_X(f(x_n), x_{n+1})
\]

\[
< \delta',
\]

implying that \(\{x_n\}\) is a \(\delta'\)-pseudo-trajectory. Now by the fact that \(\delta'\) witnesses the shadowing property of \(f\) with respect to \(\varepsilon'\), we may choose \(x \in X\) such that for all \(n \in \mathbb{N}\) we have that

\[
d_X(f^n(x), x_n) < \varepsilon'.
\]

Let \(y = h(x)\). We will show that \(y\) \(\varepsilon\)-shadows \(\{y_n\}\). By the uniform continuity of \(h\), and our choice of \(\varepsilon'\), we have that

\[
d_Y(h(f^n(x)), h(x_n)) < \varepsilon.
\]

However, as \(h \circ f = g \circ f\), we have that

\[
d_Y(g^n(y), y_n) = d_Y(g^n(h(x)), h(x_n))
\]

\[
= d_Y(h(f^n(x)), h(x_n))
\]

\[
< \varepsilon,
\]

verifying that \(y\) \(\varepsilon\)-shadows \(\{y_n\}\) and completing the proof. \(\square\)
We note that having uniform continuity of $h$ in just one direction is not enough as Example 2.2.3 shows. In particular, the identity map on $X$ of Example 2.2.3, where the domain has the discrete metric and the range has the metric of $\mathbb{R}$, is a conjugacy, which is uniformly continuous, but does not preserve the shadowing property.

2.3 Finite Shadowing vs Shadowing

As we are dealing with possibly locally compact sets it is not straightforward that finite and infinite shadowing agree. We therefore provide a sufficient condition on general dynamical systems for finite and infinite shadowing to agree and also show that a shift space has finite order if, and only if, it has the (finite) shadowing property. In Section 4 in the context of Polish dynamical systems, we will describe precisely the relation between finite shadowing and shadowing, see Proposition 4.2.11 and Corollary 4.2.15.

It is well known that, for compact spaces, shadowing and finite shadowing are equivalent, see [30, Lemma 1.1.1] for example. Following the general idea from there, below we show this fact for uniformly locally compact spaces.

Definition 2.3.1. We say that a metric space is uniformly locally compact, if there exists $\varepsilon > 0$ such that for any $x \in X$ the open ball centred at $x$ with radius $\varepsilon$ is contained in a compact set.

Proposition 2.3.2. Let $(X, f)$ be a dynamical system where $X$ is uniformly locally compact. Then $(X, f)$ has the shadowing property if, and only if, it has the finite shadowing property.

Proof. It is straightforward that if $(X, f)$ has the shadowing property then it has the finite shadowing property.

To prove the converse, given $\varepsilon > 0$, take $\delta > 0$ such that any finite $\delta$-pseudo-orbit is $\varepsilon/2$-shadowed by some point. Since $X$ is uniformly locally compact, we can assume, without loss of generality, that $\varepsilon$ is sufficiently small so that the closure of any open ball $B_\varepsilon(x)$ is compact.

Given $(x_i)_{i \in \mathbb{N}} \in X$ an infinite $\delta$-pseudo-orbit we can consider, for each $k \geq 1$, the finite $\delta$-pseudo-orbit $(x_i)_{0 \leq i \leq k}$. Let $z_k \in X$ be the point that $\varepsilon/2$-shadows $(x_i)_{0 \leq i \leq k}$. Note that for each $k$ the point $z_k$ belongs to $B_{\varepsilon/2}(x_0)$ and, since $\overline{B_{\varepsilon/2}(x_0)}$ is compact, there exists $z \in B_\varepsilon(x_0)$ which is an accumulation point of $(z_k)_{k \geq 1}$. Since $f^i(z_k)_{k \geq 1} \in B_{\varepsilon/2}(x_i)$ for all $k \geq 1$ and $0 \leq i \leq k$, $f^i$ is continuous for all $i \geq 1$, and $z$ is accumulation point of $(z_k)_{k \geq 1}$, it follows that $f^i(z) \in \overline{B_{\varepsilon/2}(x_i)} \subset B_\varepsilon(x_i)$ for all $i \geq 0$, that is, $z$ $\varepsilon$-shadows $(x_i)_{i \in \mathbb{N}}$. 

\qed
Corollary 2.3.3. The notions of finite shadowing and shadowing coincide in \( \mathbb{R}^n \) with the usual metric.

Proof. This simply follows from applying Proposition 2.3.2 to the uniformly locally compact metric space \( \mathbb{R}^n \).

The following example shows that Proposition 2.3.2 is sharp.

Example 2.3.4. There exists a locally compact space \( X \subseteq \mathbb{R} \) that has the finite shadowing property but not the shadowing property. Moreover, this space admits a tame defining sequence (see Definition 3.1.2).

Construction: Our space \( X \) will be a subset of

\[
Y = \left\{ n + \frac{1}{k} : k, n \in \mathbb{N}, k \geq 2 \right\}.
\]

It is clear that endowed with the metric of \( \mathbb{R} \), any \( X \subseteq Y \) is locally compact as each point of \( X \) is an isolated point of \( X \). Moreover, any \( X \subseteq Y \) admits a tame defining sequence. Indeed, for \( n \in \mathbb{N} \), let \( U_n \) consist of nonempty sets of the form \( (k, k + \frac{1}{(n+1)\sqrt{2}}) \cap X \) and singletons in \( \bigcup_{k \in \mathbb{Z}} (k + \frac{1}{(n+1)\sqrt{2}}, k + 1) \cap X \). It is easy to verify that \( \{U_n\} \) is a tame defining sequence.

Let us now proceed with the construction of our space \( X \).

We first enumerate the set of all finite sequences of positive integers as \( \{s_k\}_{k \in \mathbb{N}} \), i.e., \( s_k = (n_{k,0}, \ldots, n_{k,l(k)}) \) where \( l(k) \in \mathbb{N} \) and \( n_{k,i} \) is a positive integer. Moreover, we require that every finite sequence of positive integers occurs infinitely often in \( \{s_k\} \), i.e., given \( (m_0, \ldots, m_j) \), there are infinitely many \( k \)'s such that \( s_k = (m_0, \ldots, m_j) \).

We now construct \( A_k \), a finite subset of \( Y \), based on \( s_k \). More specifically, let \( \{A_k\}_{k \in \mathbb{N}} \) be a sequence of subsets of \( Y \) such that the following properties hold.

(i) \( A_k \subseteq \bigcup_{i=0}^{l(k)} (i, i + \frac{1}{k+2}) \).

(ii) For each \( k \) and \( 0 \leq i \leq l(k) \) we have that the cardinality of \( A_k \cap (i, i + \frac{1}{k+2}) \) is \( n_{k,i} \).

(iii) If \( k \neq k' \), then \( A_k \cap A_{k'} = \emptyset \).

Finally, we define our space \( X = \bigcup_{k \in \mathbb{N}} A_k \).

We next define a map \( f : X \to X \). It will have the property that \( f(A_k) \subseteq A_k \) for all \( k \). If \( x \in A_k \) is not the largest element of \( A_k \), then \( f(x) \) is the smallest element of \( A_k \) greater than \( x \). If \( x \) is the largest element of
$A_k$, then $f(x) = x$. Clearly, $f$ is a well-defined continuous function on $X$. Moreover, the orbit of every $x \in X$ is bounded under $f$.

We now observe that $f$ does not have the shadowing property. For this it suffices to construct, for all $\delta > 0$, an unbounded $\delta$-pseudo-orbit. Indeed, let $\delta > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. For each $i \in \mathbb{N}$ choose $k_i > N$ so that $l(k_i) > i$. Let $x_i$ be the largest element of $A_k \cap (i, i + \frac{1}{k_i+2})$. Note that for all $i \in \mathbb{N}$, $x_i \in (i, i + \frac{1}{k_i+2}) \subseteq (i, i + \frac{1}{N})$. Moreover, as $l(k_i) > i$, we have that

$$f(x_i) \in (i + 1, i + 1 + \frac{1}{k_i+2}) \subseteq (i + 1, i + 1 + \frac{1}{N}).$$

Hence, we have that $x_{i+1}$ and $f(x_i)$ are in $(i + 1, i + 1 + \frac{1}{N})$, implying that $\{x_i\}_{i \in \mathbb{N}}$ is an unbounded $\delta$-pseudo-orbit.

We next show that $f$ has the finite shadowing property. Let $\varepsilon > 0$. Let $N > 2$ be a positive integer such that $\frac{1}{N} < \varepsilon$. As each point of the finite set $\bigcup_{i=0}^{N} A_i$ is an isolated point of $X$, we may choose $0 < \delta < \frac{1}{N}$ sufficiently small so that a $\delta$ interval around any point of $\bigcup_{i=0}^{N} A_i$ is a singleton set. Let $\{x_i\}_{i=0}^{j}$ be a finite $\delta$-pseudo-trajectory. By our choice of $\delta$, we have that the entire pseudo-trajectory $\{x_i\}_{i=0}^{j}$ is a subset of $\bigcup_{i=0}^{N} A_i$, or the entire pseudo-trajectory $\{x_i\}_{i=0}^{j}$ is a subset of $\bigcup_{i=N+1}^{\infty} A_i$. In the former case, the pseudo-trajectory $\{x_i\}_{i=0}^{j}$ is actually a trajectory and we are done. In the latter case, we proceed as follows. We first observe that by our construction of space $X$, map $f$ and the fact that $\delta < 1/2$, we have that if $x_i \in (u, u + 1)$ (for some $u \in \mathbb{N}$) then $x_{i+1} \in (u, u + 1)$ or $x_{i+1} \in (u + 1, u + 2)$. Hence, the pseudo-trajectory $\{x_i\}_{i=0}^{j}$ starts in some interval of the form $(u, u + 1)$, then eventually proceeds to the interval $(u + 1, u + 2)$ and so on, until it terminates. Keeping this structure in mind, we proceed to approximate $\{x_i\}_{i=0}^{j}$ by a real trajectory. Let $u \in \mathbb{N}$ be such that $x_0 \in (u, u + 1)$ and let $v \in \mathbb{N}$ be such that $x_j \in (v, v + 1)$. For each $u \leq t \leq v$, let $n_t$ be the cardinality of elements of $\{x_i\}_{i=0}^{j}$ in the interval $(t, t+1)$. Hence the pseudo-trajectory $\{x_i\}_{i=0}^{j}$ starts in $(u, u + 1)$, and stay there $n_u$ times. Then it moves to $(u + 1, u + 2)$ and spends $n_{u+1}$ times there, etc. Now choose $k > N$ such that $s_k = (n_{k,0}, \ldots, n_{k,k(k)})$ has the property that $n_{k,t} = n_t$ for all $u \leq t \leq v$. Let $y$ be the smallest element of $A_k$ in $(u, u + 1)$. We claim that $y \varepsilon$-shadows $\{x_i\}_{i=0}^{j}$. Indeed, as $\{x_i\}_{i=0}^{j}$ and $\{f^i(y)\}_{i \in \mathbb{N}}$ are subsets of $\bigcup_{i=0}^{\infty} A_i$, we have that the intersection of either one with $(t, t + 1)$, $u \leq t \leq v$, is actually contained in $(t, t + 1/N)$. Moreover, both sequences start in $(u, u + 1/N)$ and spend the same amount of time in each interval before moving to the next one. As $\frac{1}{N} < \varepsilon$, we have that $y \varepsilon$-shadows $\{x_i\}_{i=0}^{j}$, completing the proof. \hfill \Box

In the case of a finite alphabet, it is a classical result of Walters that a
subshift has the shadowing property if, and only if, it is a SFT, [22, Theorem 3.33]. Below we prove an analogous result for infinite alphabets, with the appropriate modifications.

**Proposition 2.3.5.** Let $A$ be a countable alphabet and $X \subseteq A^\mathbb{N}$ be a shift space. Then the following statements are equivalent:

(i) $X$ is a shift of finite order;

(ii) $(X, \sigma)$ has the shadowing property;

(iii) $(X, \sigma)$ has the finite shadowing property;

Proof. Let $X$ be a shift space over a countable alphabet.

(i) $\implies$ (ii) Suppose that $X$ is a shift of order $p$. Given $\varepsilon > 0$, take $\delta := 2^{-k}$, where $k \geq p - 1$ is an integer such that $2^{-k} < \varepsilon$. Now, let $(x_i)_{i \in \mathbb{N}}$ be any infinite $\delta$-pseudo-orbit. Since $d(\sigma(x_i), x_{i+1}) < \delta = 2^{-k}$, it follows that $x_{i,1} x_{i,2} \ldots x_{i,k+1} = x_{i+1,0} x_{i+1,1} \ldots x_{i+1,k}$, and recursively we get that $x_{i,\ell} = x_{i+j,\ell-j}$ for all $i \in \mathbb{N}$, $1 \leq \ell \leq k + 1$ and $0 \leq j \leq \ell$. In particular, this implies that

$$x_{i,1} x_{i,2} \ldots x_{i,k+1} = x_{i+1,0} x_{i+2,0} \ldots x_{i+k+1,0} \quad (1)$$

for all $i \in \mathbb{N}$. Let $z = (x_{n,0})_{n \in \mathbb{N}}$. Equation (1), $k + 1 \geq p$, and that $X$ is of order $p$, implies that $z$ lies in $X$. Finally, note that (1) also implies that $d(\sigma^i(z), x_i) \leq 2^{-k} < \varepsilon$.

(ii) $\implies$ (iii) It is direct.

(iii) $\implies$ (i) To prove that $X$ is a shift of finite order, we use Proposition 2.1.5. More precisely, we will show that there exists $p \geq 0$ such that for all $u, v, w \in \mathcal{L}(X)$ with $|v| = p - 1$, and $uv, vw \in \mathcal{L}(X)$, we have that $uvw \in \mathcal{L}(X)$.

Take $0 < \varepsilon < 1$, and let $\delta > 0$ be such that any finite $\delta$-pseudo-orbit is $\varepsilon$-shadowed by some point. Take $p \in \mathbb{N}$ so that $2^{-(p-2)} < \delta$. Suppose $u = u_0 u_1 \ldots u_m$, $v = v_{m+1} v_{m+2} \ldots v_{m+p-1}$, $w = w_{m+p} w_{m+p+1} \ldots w_{m+p+n} \in \mathcal{L}(X)$ such that $uv, vw \in \mathcal{L}(X)$. Consider $(x_i)_{0 \leq i \leq m+p+n}$ defined as follows: $x_0$ is any sequence of $X$ starting with the word $uv$; for $i = 1, \ldots, m$ define $x_i := \sigma^i(x_0)$; $x_{m+1}$ is any sequence of $X$ starting with the word $vw$; for $i = m+2, \ldots, m+p+n$ define $x_i := \sigma^{i-m-1}(x_{m+1})$. It follows that $(x_i)_{0 \leq i \leq m+p+n}$ is a finite $\delta$-pseudo-orbit since for all $i = 0, \ldots, m + p + n - 1$ we have that $\sigma(x_i)$ and $x_{i+1}$ coincide at least in the first $p - 1$ entries, and therefore $d(\sigma(x_i), x_{i+1}) \leq 2^{-(p-2)} < \delta$. Now, let $z \in X$ be a point that $\varepsilon$-shadows
Since \( d(\sigma^i(z), x_i) < \varepsilon < 1 \) for all \( i = 0, \ldots, m + p + n \), it follows that \( z_i = x_{i,0} \) for all \( i = 0, \ldots, m + p + n \), which means that the sequence \( z \) starts with the word \( uvw \), and then \( uvw \in L(X) \).

\[ \square \]

## 3 Defining Sequences, Ultrametric Spaces and \( P \)-adics

In this section we define three notions which are essential for our main results. Our definitions goes from the most general to specific. The notion of tame defining sequence is new as far as we know. Of course, ultrametric spaces and \( p \)-adics are rather well known.

### 3.1 Defining Sequences

Recall that every 0-dimensional Polish space admits a cover of pairwise disjoint clopen sets. This fact motivates the following definition, which will be a key concept in our work.

**Definition 3.1.1.** Let \( X \) be a topological space. A defining sequence of \( X \) is a sequence \( \{U_n\}_{n \in \mathbb{N}} \) satisfying the following conditions.

(i) Each \( U_n \) is a countable clopen partition of \( X \), i.e., a countable collection of pairwise disjoint nonempty clopen sets whose union is \( X \).

(ii) \( U_{n+1} \) is a refinement of \( U_n \), i.e., each element of \( U_{n+1} \) is a subset of some (necessarily unique) element of \( U_n \).

(iii) If \( \{U_n\}_{n \in \mathbb{N}} \) is such that \( U_n \subseteq U_n \) and \( U_{n+1} \subseteq U_n \) for all \( n \), then \( \bigcap_n U_n \) has exactly one element.

(iv) The collection \( \{U : U \in U_n, \text{ for some } n \in \mathbb{N}\} \) is a basis for the topology in \( X \).

It is a well-known fact every 0-dimensional Polish space \( X \) has a defining sequence. Moreover, if \( X \) happens to be compact, then each \( U_n \) is finite.

For our main theorem we will need a "tame" defining sequence. More precisely, we have the following definition.

**Definition 3.1.2.** Let \( (X, d) \) be a metric space and \( \{U_n\}_{n \in \mathbb{N}} \) be a defining sequence of \( X \). For all \( n \in \mathbb{N} \), let

\[ S_n = \sup\{\text{diam}(O) : O \in U_n\} \]
where \( \text{diam}(O) \) stands for the diameter of the set \( O \). We say that \( \{U_n\}_{n \in \mathbb{N}} \) is a tame defining sequence of \( X \) if \( S_n \to 0 \) and, for all \( n \in \mathbb{N} \), there exists \( \rho_n > 0 \) such that if \( O_1, O_2 \) are distinct element of \( U_n \) and \( x_i \in O_i \), then \( d(x_1, x_2) \geq \rho_n \). For such \( \rho_n \), we say that \( U_n \) is \( \rho_n \)-separated.

### 3.2 Ultrametric Spaces

In the applications section we will be interested in ultrametric Polish spaces. Therefore, we recall the relevant concepts here and show that ultrametric Polish spaces admit a tame defining sequence.

**Definition 3.2.1.** An ultrametric space is a metric space \( (X, d) \) where the ultrametric inequality holds, that is,

\[
d(x, z) \leq \max\{d(x, y), d(y, z)\}
\]

for all \( x, y, z \in X \).

The ultrametric inequality has interesting consequences, some of which are listed below.

- \( (UM_1) \) All balls of strictly positive radius are both open and closed in the induced topology.
- \( (UM_2) \) If \( d(x, y) < r \) then \( B(x, r) = B(y, r) \), where \( B(z, r) \) denotes the ball centered at \( z \) of radius \( r \).
- \( (UM_3) \) The intersection of two balls is either empty or one is contained in the other.
- \( (UM_4) \) The distance between any two balls of radius \( r > 0 \) is \( r \) or greater.

Shifts on countable alphabets with the metric of Definition 2.1.1 are ultrametric spaces. For us the key property of ultrametric spaces is that we can build tame defining sequences for them.

**Proposition 3.2.2.** Let \( (X, d) \) be an ultrametric space and let \( U_n = \{B(x, \frac{1}{n}) : x \in X\} \), for \( n \in \mathbb{N}^+ \), and \( U_0 = \{X\} \). Then the sequence \( \{U_n\}_{n \in \mathbb{N}} \) is a tame defining sequence for \( X \).

**Proof.** We have to check that \( \{U_n\}_{n \in \mathbb{N}} \) satisfy the conditions of Definitions 3.1.1 and 3.1.2.

By Properties \( (UM_1) \) and \( (UM_2) \), each \( U_n \) is a clopen partition of \( X \). That \( U_{n+1} \) is a refinement of \( U_n \) follows from Property \( (UM_3) \). The third and fourth conditions in Definition 3.1.1 are straightforward to check. Hence, \( \{U_n\} \) is a defining sequence. To check that it is tame, we notice that the diameter of each ball in \( U_n \) is \( \frac{1}{n} \) (so \( S_n \to 0 \)) and that each \( U_n \) is \( \frac{1}{n} \)-separated by Property \( (UM_4) \). 

\[ \square \]
3.3 $p$-adics

We finish this section by recalling the construction of the $p$-adic integers and the $p$-adic rationals. The standard norms on these spaces generate metrics which are ultrametrics. As such, they admit tame defining sequences. For the sake of concreteness, we also give explicit descriptions of these tame defining sequences. For general information on $p$-adics, we refer the reader to [32].

**Definition 3.3.1.** Let $p$ be a prime number and let $A = \{0, \ldots, p-1\}$, i.e., the field of integers modulo $p$. Formally, we define $\mathbb{Z}_p$ and $\mathbb{Q}_p$ as follows:

\[
\mathbb{Z}_p = \left\{ \sum_{i=0}^{+\infty} a_i p^i : a_i \in A \right\} \quad \mathbb{Q}_p = \left\{ \sum_{i=l}^{+\infty} a_i p^i : l \in \mathbb{Z}, a_i \in A \right\}.
\]

Elements in $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are summed pointwise modulo $p$ with a carryover. Multiplication of an element of $\mathbb{Q}_p$ with a scalar in $A$ is defined pointwise with a carryover. Using addition and multiplication by a scalar, multiplication in $\mathbb{Z}_p$ and $\mathbb{Q}_p$ is defined in a natural way. Equipped with these algebraic operations, we have that $\mathbb{Z}_p$ is a ring and $\mathbb{Q}_p$ is a field.

We now recall $p$-valuation and the metric it induces.

**Definition 3.3.2.** For $x = \sum_{i=0}^{+\infty} a_i p^i \in \mathbb{Q}_p$, define

\[
\|x\|_p := \begin{cases} 
0 & \text{if } a_i = 0 \quad \forall i \in \mathbb{Z} \\
 p^{-l} & \text{if } a_l \neq 0.
\end{cases}
\]

Then, $\| \cdot \|_p$ induces natural metrics on $\mathbb{Z}_p$ and $\mathbb{Q}_p$ by $d(x, y) = \|x - y\|_p$. It is easily verified that this metric is an ultrametric. Moreover, equipped with this metric, $\mathbb{Z}_p$ is a compact completion of $\mathbb{Z}$ homeomorphic to the Cantor space, whereas $\mathbb{Q}_p$ is a locally compact completion of $\mathbb{Q}$.

We next give an explicit description of the tame defining sequences formed by the ultrametric on $\mathbb{Z}_p$ and $\mathbb{Q}_p$.

We denote the set of all words on $A$ of length $n$ by $A^n$, that is, $A^n = \{\sigma_0 \sigma_1 \ldots \sigma_{n-1} : \sigma_i \in A\}$, recalling $A^0 = \{\varepsilon\}$ where $\varepsilon$ is the empty word. For $\sigma \in A^n$, say $\sigma = \sigma_0 \ldots \sigma_{n-1}$, let

\[
\mathbb{Z}_p(\sigma) := \left\{ \sum_{j=0}^{+\infty} a_j p^j : a_i \in A \text{ and } a_i = \sigma_i \text{ for } 0 \leq i \leq n-1 \right\},
\]

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and let $Z_p(\varepsilon) := Z_p$. Then, letting

$$U_n = \{Z_p(\sigma) : \sigma \in A^n\}$$
$$V_n = \{Z_p(\sigma)p^j : \sigma \in A^{n-j}, \sigma_0 \neq 0, j \in \mathbb{Z}, j \leq n\},$$

we have that $\{U_n\}$ and $\{V_n\}$ are tame defining sequences for $Z_p$ and $Q_p$, respectively. Indeed such is the case, as $U_n$ and $V_n$ consist of open balls of radius $p^{-n+1}$ in $Z_p$ and $Q_p$, respectively.

## 4 Inverse limits, Shadowing and Shifts of Finite Order

We begin by introducing the basic terminology of inverse limits.

### 4.1 Inverse Limits and Shadowing

**Definition 4.1.1.** For each $m \in \mathbb{N}$, let $X_m$ be a topological space and let $g_m : X_{m+1} \to X_m$ be a continuous map. The inverse limit of $(g_m, X_m)$ is the space

$$\lim_{\leftarrow} \{g_m, X_m\} := \{(x_m)_{m \in \mathbb{N}} \in \prod X_m : x_m = g_m(x_{m+1}) \forall m \in \mathbb{N}\},$$

with the subspace topology inherited from the product topology on $\prod X_m$. The maps $g_m$’s are called bonding maps.

**Remark 4.1.2.** If each $X_m$ is a Polish space then the inverse limit $\lim_{\leftarrow} \{g_m, X_m\}$ is a closed subspace of the product $\prod X_m$, and hence it is also a Polish space.

The following condition is essential to guarantee that the inverse limit of spaces with the shadowing property has the shadowing property.

**Definition 4.1.3.** The inverse limit of $(g_m, X_m)$ satisfies the Mittag-Leffler Condition if for every $N \in \mathbb{N}$ there is a $k$ such that for all $i \geq k$ the following holds

$$g_N \circ \ldots \circ g_k(X_{k+1}) = g_N \circ \ldots g_i(X_{i+1}).$$

**Remark 4.1.4.** Notice that in an inverse limit that satisfies the Mittag-Leffler Condition, if $k > N$ witnesses the condition for $N$ then for $y \in g_N \circ \ldots \circ g_{k-1}(X_k)$ we have that $\pi_N^{-1}(y) \cap \lim_{\leftarrow} \{g_m, X_m\} \neq \emptyset$, where $\pi_N$ denotes the usual projection on the $N$-coordinate, see [17, Pg. 721].
We next define inverse limit dynamical system.

**Definition 4.1.5.** Let \( \{(X_m, f_m)\} \) be a sequence of dynamical systems. Furthermore, assume that \( \{g_m\} \), \( g_m : X_{m+1} \to X_m \), is a sequence of continuous bonding maps that satisfies the following property:

\[
f_m \circ g_m = g_m \circ f_{m+1}.
\]

The inverse limit of \( (g^n_m, (X_m, f_m)) \) is the dynamical system \( \lim_{\leftarrow} \{g_m, X_m\}, (f_m)^* \) where \( (f_m)^* \) is the induced map given by

\[
(f_m)^* (x_m) = f_m(x_m).
\]

Notice that \( \lim_{\leftarrow} \{g^n_m, X_m\}, (f_m)^* \) is a continuous dynamical system. Furthermore, if each \( f_m \) is uniformly continuous then \( (f_m)^* \) is also uniformly continuous as we show in Lemma 4.1.9.

Our main goal in this subsection is to prove that the inverse limit of Polish spaces with the shadowing property has the shadowing property. However, we need some auxiliary results first.

**Definition 4.1.6.** For each \( i \in \mathbb{N} \), let \( X_i \) be a Polish space with a complete metric \( d_i \) (bounded by 1 by convention). Then, for \( x = (x_i)_{i \in \mathbb{N}}, y = (y_i)_{i \in \mathbb{N}} \in \prod X_i \) we define

\[
d(x, y) = \sum_{i=0}^{\infty} \frac{1}{2^i} d_i(x_i, y_i).
\]

We note that \( d \) is a complete, separable metric on \( \prod X_i \) giving the product topology.

**Remark 4.1.7.** For future use, notice that if \( d(x, y) < \varepsilon \) then \( d(x_i, y_i) < 2^i \varepsilon \) for all \( i \in \mathbb{N} \).

**Lemma 4.1.8.** Suppose we are in the setting of Definition 4.1.5 with each \( X_i \) a Polish space with a complete metric \( d_i \) bounded by 1. Let \( \varepsilon > 0 \) and choose \( N \) such that \( \frac{1}{2^N} < \varepsilon \). If \( x = (x_i)_{i \in \mathbb{N}} \) and \( t = (t_i)_{i \in \mathbb{N}} \) belong to \( \lim_{\leftarrow} \{g^n_m, X_m\} \) then

\[
d(x, t) \leq \sum_{i=0}^{N} \frac{1}{2^i} d_i(x_i, t_i) + \varepsilon.
\]

**Proof.** Just notice that

\[
dl(x, t) = \sum_{i=0}^{\infty} \frac{1}{2^i} d_i(x_i, t_i) = \sum_{i=0}^{N} \frac{1}{2^i} (x_i, t_i) + \sum_{i=N+1}^{\infty} \frac{1}{2^i} d_i(x_i, t_i)
\]

\[
\leq \sum_{i=0}^{N} d_i(x_i, t_i) + \sum_{i=N+1}^{\infty} \frac{1}{2^i} < \sum_{i=0}^{N} d_i(x_i, t_i) + \varepsilon.
\]
Lemma 4.1.9. Suppose we are in the setting of Definition 4.1.5 with the additional hypothesis that $f_m$ is uniformly continuous for each $m$. Then $(f_m)^*$ is uniformly continuous.

Proof. Given $\varepsilon > 0$, let $N$ be such that $\frac{1}{2N} < \frac{\varepsilon}{2}$. For each $0 \leq i \leq N$, choose $\delta_i$ from the uniform continuity of $f_i$ with respect to $\frac{\varepsilon}{N+1}$. Let $\delta = \min_{0 \leq i \leq N} \{\delta_i\}$. Now notice that if $d(x, t) < \frac{\delta}{2N}$ then, by Proposition 4.1.8,

$$d((f_m)^*(x), (f_m)^*(t)) \leq \sum_{i=0}^{N} \frac{1}{2}d_i(f_i(x_i), f_i(t_i)) + \frac{\varepsilon}{2} < \sum_{i=0}^{N} \frac{1}{2} \frac{\varepsilon}{N+1} + \frac{\varepsilon}{2} = \varepsilon.$$

\[\square\]

A straightforward, but useful, result that we will need in the sequel is the following.

Lemma 4.1.10. Under the hypothesis of Definition 4.1.5 if $(x_i)_{i \in \mathbb{N}}$ is a $\delta$-pseudo-orbit in $(\lim \leftarrow \{g_m, X_m\}, (f_m)^*)$ then, for every fixed $m \in \mathbb{N}$, the sequence $(x_{i,m})_{i \in \mathbb{N}}$ is a $2^m \delta$-pseudo-orbit of $f_m$ in $X_m$.

Proof. Notice that, by hypothesis,

$$d((f_j)^*((x_i)), x_{i+1}) = d((f_j(x_{i,j})), x_{i+1}) = \sum_{j \in \mathbb{N}} \frac{1}{2^j}d_j(f_j(x_{i,j}), x_{i+1,j}) < \delta,$$

for all $i$. Then, for each fixed $m \in \mathbb{N}$ and for all $i \in \mathbb{N}$, it follows that $d_m(f_m(x_{i,m}), x_{i+1,m}) < 2^m \delta$ and hence $(x_{i,m})_{i \in \mathbb{N}}$ is a $2^m \delta$-pseudo-orbit in $X_m$.

\[\square\]

We now prove the main result of this subsection.

Theorem 4.1.11. Let $\{(X_m, f_m)\}$ be a sequence of Polish dynamical systems, with the shadowing property, such that each $X_i$ is a Polish space with a complete metric $d_i$ bounded by 1. Let $\{g_m\}$, with $g_m : X_{m+1} \to X_m$, be a sequence of uniformly continuous bonding maps. If the inverse limit $(\lim \leftarrow \{g_m, X_m\}, (f_m)^*)$ satisfies the Mittag-Leffler Condition then it has the shadowing property with respect to the metric given in Definition 4.1.6.

Proof. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{2^{N+1}} < \frac{\varepsilon}{8}$. Let $k > N$ witness the Mittag-Leffler Condition for $N$. 

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Let \( \varepsilon' < \varepsilon \) be such that if two points in \( X_k \) are \( \varepsilon' \) close (that is, their distance is less than \( \varepsilon' \)) then their images under \( g_i \circ \ldots \circ g_{k-1}, \) for \( 1 \leq i \leq k-1, \) is \( \frac{\delta}{2k} \) close. Such \( \varepsilon' \) exists from the uniform continuity of \( g_i \)'s.

Let \( \delta_k > 0 \) be such that every \( \delta_k \)-pseudo-orbit in \( X_k \) is \( \varepsilon' \)-shadowed. Let \( \delta = \frac{\delta_k}{2} \). We will show that every \( \delta \)-pseudo-orbit in \( (\lim\{g_m, X_m\}, (f_m)^*) \) is \( \varepsilon \)-shadowed.

Let \( (x_i) \) be a \( \delta \)-pseudo-orbit in \( (\lim\{g_m, X_m\}, (f_m)^*) \). By Lemma 4.1.10 we have that \( (x_{i,k})_{i=1}^{\infty} \) is a \( 2^k \delta \)-pseudo-orbit in \( X_k \). Hence, it is a \( \delta_k \)-pseudo-orbit in \( X_k \). By our choice of \( \delta_k \), we may choose \( t_k \) in \( X_k \) that \( \varepsilon' \)-shadows \( (x_{i,k})_{i=1}^{\infty} \), i.e., \( d(\{f_k^i(t_k)\}, x_{i,k}) < \varepsilon' \) for all \( i \). Consider the point \( t_N = g_N \circ \ldots \circ g_{k-1}(t_k) \). By Remark 4.1.4 we can find a \( t \) in \( (\lim\{g_m, X_m\}, (f_m)^*) \) such that

\[
\begin{align*}
(0 \ldots g_{N-1}(t_N), g_1 \ldots g_{N-1}(t_N), \ldots, t_N, \ldots, t_k, z_{k+1}, z_{k+2}, \ldots).
\end{align*}
\]

We now show that \( t \) is a point that \( \varepsilon \)-shadows \( (x_i) \). Indeed,

\[
\begin{align*}
d\left((f_j)^i(t), x_i\right) &= \sum_{j=0}^{\infty} \frac{1}{2^j} d_j(f_j^i(t_j), x_{i,j}) \\
&= \sum_{j=0}^{k-1} \frac{1}{2^j} d_j(f_j^i(t_j), x_{i,j}) + \sum_{j=k}^{\infty} \frac{1}{2^j} d_j(f_j^i(t_j), x_{i,j}) \\
&< \sum_{j=0}^{k-1} \frac{1}{2^j} d_j(\{g_j \circ \ldots \circ g_{k-1}(t_k)\}, x_{i,j}) + \frac{\varepsilon}{8} \\
&= \sum_{j=0}^{k-1} \frac{1}{2^j} d_j(g_j \circ \ldots \circ g_{k-1}(f_k^i(t_k)), x_{i,j}) + \frac{\varepsilon}{8} \\
&= \sum_{j=0}^{k-1} \frac{1}{2^j} d_j(g_j \circ \ldots \circ g_{k-1}(f_k^i(t_k)), g_j \ldots g_{k-1}(x_{i,k})) + \frac{\varepsilon}{8} \\
&\leq \sum_{j=0}^{k-1} \frac{1}{2^j} \frac{\varepsilon}{2k} + \frac{\varepsilon}{8} < \varepsilon,
\end{align*}
\]

where the last line follows from the previous one by our choice of \( t_k \) and the uniform continuity of \( g_j \circ \ldots \circ g_{k-1} \). This concludes the proof of the shadowing property of \( (\lim\{g_m, X_m\}, (f_m)^*) \) as desired.

**Remark 4.1.12.** In Theorem 4.1.11, if \( \{X_m, f_m\} \) is a sequence of Polish dynamical systems with the finite shadowing property, then the same proof applies and we have that the inverse limit has the finite shadowing property.
4.2 Shadowing and Shifts of Finite Order

Our aim now will be to prove a converse of the above result. More precisely
we will show that any continuous map with finite shadowing property defined
on a 0-dimensional Polish space is conjugate to an inverse limit of 1-step shift
spaces over countable alphabets.

Throughout the rest of this section, \( X \) is a 0-dimensional Polish space
and \( f : X \to X \) is a continuous map.

Let \( \mathcal{U} \) be a clopen partition of \( X \). Then, we define
\[
\mathcal{O}(\mathcal{U}, f) = \{(O_i)_{i \in \mathbb{N}} \in \mathcal{U}^{\mathbb{N}} : \forall k \in \mathbb{N}, \exists x \in X \text{ s.t. } f^i(x) \in O_i, \ 0 \leq i \leq k\},
\]
and
\[
\mathcal{P}\mathcal{O}(\mathcal{U}, f) = \{(O_i)_{i \in \mathbb{N}} \in \mathcal{U}^{\mathbb{N}} : \forall i \in \mathbb{N}, f(O_i) \cap O_{i+1} \neq \emptyset\}.
\]
When we are working with a single function \( f \), we will often suppress \( f \)
and write \( \mathcal{O}(\mathcal{U}) \) and \( \mathcal{P}\mathcal{O}(\mathcal{U}), \) instead of \( \mathcal{O}(\mathcal{U}, f) \) and \( \mathcal{P}\mathcal{O}(\mathcal{U}, f), \) respectively.

We endow \( \mathcal{U} \) with the discrete metric (so the distance between any two
distinct points is one) and \( \mathcal{U}^{\mathbb{N}} \) with the product topology, which is generated
by the metric given in Definition 4.1.6.

We now prove a sequence of lemmas which lead to the proof of the main
theorem.

**Lemma 4.2.1.** Let \( \mathcal{U} \) be a clopen partition of \( X \) and \( f : X \to X \) be a
continuous function. Then, the following hold.

(i) \( \mathcal{O}(\mathcal{U}) \subseteq \mathcal{P}\mathcal{O}(\mathcal{U}) \)

(ii) \( \mathcal{O}(\mathcal{U}) \) is a subshift of \( \mathcal{U}^{\mathbb{N}}; \)

(iii) \( \mathcal{P}\mathcal{O}(\mathcal{U}) \) is a 1-step subshift of \( \mathcal{U}^{\mathbb{N}}. \)

**Proof.**

(i) This follows directly from the definitions.

(ii) It is straightforward that \( \mathcal{O}(\mathcal{U}) \) is shift invariant. Thus, we just need
to check that \( \mathcal{O}(\mathcal{U}) \) is closed in \( \mathcal{U}^{\mathbb{N}}. \)

Let \( (O_\ell)_{\ell \in \mathbb{N}} \) be a sequence of points of \( \mathcal{O}(\mathcal{U}), \) that is, \( O_\ell = (O_{\ell,i})_{i \in \mathbb{N}} \in \mathcal{O}(\mathcal{U}), \) for each \( \ell \in \mathbb{N}. \) Suppose that \( (O_\ell)_{\ell \in \mathbb{N}} \) converges to some point
\( \tilde{O} = (\tilde{O}_i)_{i \in \mathbb{N}} \in \mathcal{U}^{\mathbb{N}}. \) Then, for all \( k \in \mathbb{N}, \) there exist \( N \in \mathbb{N} \) such that
for all \( \ell \geq N \) we have \( O_{\ell,i} = \tilde{O}_i \) for all \( 0 \leq i \leq k. \) Thus, for any fixed \( \ell \geq N \) we can take \( x \in X \) such that \( f^i(x) \in O_{\ell,i} = \tilde{O}_i \) for all \( 0 \leq i \leq k; \)
which means that \( \tilde{O} \in \mathcal{O}_n. \)
Lemma 4.2.4.

Let \( V \) be the implied assumption that the image of \( A \) continuous map from \( \mathcal{O}(U) \) to the unique element \( (V_i)_{i \in \mathbb{N}} \in \mathcal{V}(n) \) such that \( U_i \subseteq V_i \) for all \( i \). Moreover, if \( A \subseteq \mathcal{U}(n) \) and \( B \subseteq \mathcal{V}(n) \), then \( A \rightarrow B \) denotes the same map restricted to \( A \) with the implied assumption that the image of \( A \) is contained in \( B \). Keeping in mind the metric given in Definition 4.1.6, we have that \( \rightarrow \) is a uniformly continuous map from \( A \) into \( B \). In the specific case that \( A = \{(U_i)_{i \in \mathbb{N}}\} \) and \( B = \{(V_i)_{i \in \mathbb{N}}\} \), we abuse the notation and simply write \( (U_i)_{i \in \mathbb{N}} \rightarrow (V_i)_{i \in \mathbb{N}} \).

We now define a function \( \theta \) which yields a conjugacy between the map \( f \) and the inverse limit representing it.

**Definition 4.2.2.** Let \( \{U_n\}_{n \in \mathbb{N}} \) be a defining sequence of \( X \) and \((X,f)\) a dynamical system. Define \( \theta : X \rightarrow \lim\{\rightarrow, \mathcal{O}(U_n)\} \) by \( \theta(x) = (O_n)_{n \in \mathbb{N}} \), where \( O_{n,l} \) is the unique element of \( U_n \) that contains \( f^i(x) \).

**Lemma 4.2.3.** The map \( \theta \) is well-defined and bijective.

**Proof.** To prove that \( \theta \) is well-defined we need to check that \( O_{n+1} \rightarrow O_n \), i.e., \( O_{n+1,l} \subseteq O_{n,l} \) for all \( n, l \in \mathbb{N} \). But this follows from the fact that \( \{U_n\}_{n \in \mathbb{N}} \) is a defining sequence and, given \( n, l \in \mathbb{N} \), \( f^i(x) \in O_{n,l} \cap O_{n+1,l} \).

We next show that \( \theta \) is bijective. For this we construct \( \theta^{-1} \). Let \( (O_n)_{n \in \mathbb{N}} \in \lim\{\rightarrow, \mathcal{O}(U_n)\} \). We define \( \beta((O_n)) \) as the unique point in \( \bigcap O_{n,0} \) (since \( \{U_n\}_{n \in \mathbb{N}} \) is a defining sequence, \( \bigcap O_{n,0} \) intersects to a point). Next we show that \( \beta = \theta^{-1} \).

Let \( (O_n)_{n \in \mathbb{N}} \in \lim\{\rightarrow, \mathcal{O}(U_n)\} \). Fix \( i \in \mathbb{N} \). Since \( O_n \in \mathcal{O}(U_n) \), there exists \( x_n \in O_{n,0} \) such that \( f^i(x_n) \in O_{n,i} \). Furthermore, from Property (iv) in Definition 3.1.1 we have that \( (x_n) \rightarrow x := \beta((O_n)) \). From the continuity of \( f \) we have that \( f^i(x_n) \rightarrow f^i(x) \). On the other hand \( f^i(x_n) \rightarrow \bigcap O_{n,i} \). Therefore \( f^i(x) = \bigcap O_{n,i} \) for all \( i \). Hence \( \theta \circ \beta = id \).

It remains to show that \( \beta \circ \theta = id \). Let \( x \in X \). Notice that \( \theta(x) = (O_n)_{n \in \mathbb{N}} \), where \( O_{n,0} \) is the unique element of \( U_n \) that contains \( x \). By definition, \( \beta(O_n) = \bigcap O_{n,0} \), which is equal to \( x \) since \( U_n \) is a defining sequence. Hence \( \beta \circ \theta = id \) as desired. \( \square \)

**Lemma 4.2.4.** Let \( \sigma_n \) be the shift map on \( \mathcal{O}(U_n) \). Then, \( \theta \circ f = \sigma^* \circ \theta \), where \( \sigma^* := (\sigma_n)^* \) as defined in Definition 4.1.3.
Proof. Let \( x \in X \). Recall that \( \theta(x) = (O_n)_{n \in \mathbb{N}} \), where \( O_{n,l} \) is the unique element of \( \mathcal{U}_n \) that contains \( f^l(x) \), \( l \in \mathbb{N} \). Notice that \( f^l(f(x)) = f^{l+1}(x) \in O_{n,l+1} = \sigma_n(O_{n,l}) \) for each \( l \in \mathbb{N} \). Hence
\[
\theta(f(x)) = (\sigma_n(O_n))_{n \in \mathbb{N}} = \sigma^*(\theta(x))
\]
as desired. \( \square \)

Lemma 4.2.5. The map \( \theta \) is a homeomorphism.

Proof. Recall that the inverse limit space \( \lim\{\hookrightarrow, \mathcal{O}(\mathcal{U}_n)\} \) has the induced topology from the product topology on \( \prod \mathcal{O}(\mathcal{U}_n) \). Therefore, to show that \( \theta \) is continuous, it is enough to prove that \( \pi_k \circ \theta \) is continuous for all \( k \in \mathbb{N} \), where \( \pi_k : \prod \mathcal{O}(\mathcal{U}_n) \rightarrow \mathcal{O}(\mathcal{U}_k) \) is the projection map. To this end, fix \( k \in \mathbb{N} \).

Let \( x \in X \). Recall that the basic open sets in \( \mathcal{O}(\mathcal{U}_k) \) are cylinder sets. Let \( A_0, \ldots, A_M \in \mathcal{U}_k \), and
\[
A = [A_0A_1 \ldots A_M] := \{(U_i)_{i \in \mathbb{N}} \in \mathcal{O}(\mathcal{U}_k) : U_i = A_i \mbox{ for } i = 1 \ldots M\}
\]
be a cylinder set in \( \mathcal{O}(\mathcal{U}_k) \) containing \( \pi_k(\theta(x)) \). By the continuity of \( f \), there exists an open set \( U \subseteq X \) containing \( x \) such that \( f^i(U) \subseteq A_i \) for all \( 0 \leq i \leq M \). By the fact that \( \mathcal{U}_k \) is a partition of \( X \) we have that, for all \( y \in U \), the initial segment of \( \pi_k \circ \theta(y) \) is \( A_0A_1 \ldots A_M \). Hence we have that \( \pi_k \circ \theta(U) \subseteq A \), verifying the continuity of \( \pi_k \circ \theta \) as well as the continuity of \( \theta \).

Next we prove that \( \theta^{-1} \) is continuous. Let \( (O_n)_{n \in \mathbb{N}} \in \lim\{\hookrightarrow, \mathcal{O}(\mathcal{U}_n)\} \), let \( x = \theta^{-1}((O_n)) \) and let \( V \) be an open set in \( X \) containing \( x \). We need to find an open set in \( \lim\{\hookrightarrow, \mathcal{O}(\mathcal{U}_n)\} \) containing \( (O_n)_{n \in \mathbb{N}} \) whose image under \( \theta^{-1} \) is a subset of \( V \). As \( \mathcal{U}_n \) is a defining sequence, and \( x = \bigcap_n O_{n,0} \), there exists \( k \in \mathbb{N} \) such that \( O_{k,0} \subseteq V \). Let
\[
A = \lim\{\hookrightarrow, \mathcal{O}(\mathcal{U}_n)\} \cap \pi_k^{-1}([O_{k,0}])
\]
(recall that \( [O_{k,0}] \) is a cylinder in \( \mathcal{O}(\mathcal{U}_k) \)). Then, \( A \) is an open set in \( \lim\{\hookrightarrow, \mathcal{O}(\mathcal{U}_n)\} \) containing \( (O_n) \) whose image under \( \theta^{-1} \) is a subset of \( V \).

\( \square \)

From the three lemmas above we get the following result.

Theorem 4.2.6. Let \((X, f)\) be a dynamical system and \( \{\mathcal{U}_n\}_{n \in \mathbb{N}} \) be a defining sequence of \( X \). Then, \((X, f)\) is topologically conjugate to the inverse limit of \( (\hookrightarrow, (\mathcal{O}(\mathcal{U}_n), \sigma_n)) \), equipped with the shift map \( \sigma^* := (\sigma_n)^* \).
For dynamical systems \((X, f)\) with a tame defining sequence \(\{U_n\}_{n \in \mathbb{N}}\) we can relate uniform continuity of \(f\) with uniform continuity of the map \(\theta\). For this, we need to choose a metric for the topology of the inverse limit space \(\operatorname{lim}\{\leftarrow, \mathcal{O}(U_n)\}\). Of course we will choose a metric that is compatible with the earlier metric.

Recall that in \(U_n\) we have the discrete metric, which from now on we denote by \(d_s\). Then on \(\mathcal{O}(U_n)\) we define
\[
d_n((O_i), (V_i)) = \sum_{i \in \mathbb{N}} \frac{1}{2^{i+1}} d_s(O_i, V_i),
\]
and finally, on \(\operatorname{lim}\{\leftarrow, \mathcal{O}(U_n)\}\) we place the metric given in Definition 4.1.6, that is,
\[
d_I((O_{n,i}), (V_{n,i})) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} d_n((O_{n,i})_{i \in \mathbb{N}}, (V_{n,i})_{i \in \mathbb{N}}).
\]

With the above setting we can prove the following.

**Proposition 4.2.7.** Let \((X, d)\) be a metric space with a tame defining sequence \(\{U_n\}_{n \in \mathbb{N}}\) and suppose that \((X, f)\) is a dynamical system. Then, the map \(\theta: X \to \operatorname{lim}\{\leftarrow, \mathcal{O}(U_n)\}\) and its inverse, are uniformly continuous if, and only if, \(f\) is uniformly continuous.

**Proof.** Suppose first that \(f\) is uniformly continuous. We first prove that \(\theta\) is uniformly continuous. Given \(\epsilon > 0\), let \(N > 1\) be such that \(\frac{N+2}{2^N} < \epsilon\), and let \(\epsilon' = \min\{\rho_i : i = 0, \ldots, N\}\), where \(\rho_i\) comes from the definition of a tame defining sequence.

Let \(\delta > 0\) be such that if \(d(x, y) < \delta\) then \(d(f^i(x), f^i(y)) < \epsilon'\), for \(i = 0, \ldots, N\).

Suppose that \(d(x, y) \leq \delta\). Let \(\theta(x) = (O_{n,i})\) and \(\theta(y) = (V_{n,i})\). From the definition of \(\delta\) we have that \(O_{n,i} = V_{n,i}\) for \(n = 0, \ldots, N\) and \(i = 0, \ldots, N\). Now observe that
\[
d_I(\theta(x), \theta(y)) = d_I((O_{n,i}), (V_{n,i})) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} d_n((O_{n,i}), (V_{n,i}))
\]
\[
= \sum_{n=0}^N \frac{1}{2^n} d_n((O_{n,i}), (V_{n,i})) + \sum_{n=N+1}^{\infty} \frac{1}{2^n} d_n((O_{n,i}), (V_{n,i}))
\]
\[
\leq (N + 1) \frac{1}{2^N} + \frac{N+2}{2^N} < \epsilon
\]

Next we prove that \(\theta^{-1}\) is uniformly continuous. Given \(\epsilon > 0\), let \(N > 1\) be such that \(S_N < \epsilon\), where \(S_N\) comes from the definition of a tame defining
sequence. Let \( \delta \) be such that \( 0 < \delta < \frac{1}{2^N + 1} \). Hence, if \( d_f((O_n,i),(V_n,i)) < \delta \) then \( O_{N,0} = V_{N,0} \). Therefore, if \( d_f((O_n,i),(V_n,i)) < \delta \) then \( x : \theta^{-1}((O_n,i)) \) and \( y : \theta^{-1}((V_n,i)) \) both belong to \( O_{N,0} \) (see the definition of \( \theta^{-1} \) in the proof of Lemma 4.2.3). Hence \( d(x,y) \leq \text{diam}(O_{N,0}) < S_N < \epsilon \).

The converse follows from Lemma 4.2.6 as it allow us to write \( f = \theta^{-1} \circ \sigma^* \circ \theta \), a composition of uniformly continuous maps. \( \square \)

**Remark 4.2.8.** Notice that we have proved above that the map \( \theta^{-1} \) is uniformly continuous, regardless whether \( f \) is uniformly continuous or not.

For metric spaces which admit a tame defining sequence we can write the finite shadowing property in terms of the shift spaces \( O(U_n) \) and \( PO(U_n) \).

In fact, we have the following result.

**Proposition 4.2.9.** Let \( (X,d) \) be a metric space that admits a tame defining sequence. Then a map \( f : X \to X \) has the finite shadowing property if, and only if, for every \( m \in \mathbb{N} \), there is \( n > m \) such that \( PO(U_n) \hookrightarrow O(U_m) \).

**Proof.** Fix a tame defining sequence of \( X \), say \( \{U_n\}_{n \in \mathbb{N}} \), and let \( \rho_n > 0 \), \( n \in \mathbb{N} \), be such that \( U_n \) is \( \rho_n \)-separated.

Suppose that \( f : X \to X \) has the finite shadowing property. Given \( m \in \mathbb{N} \) choose \( \epsilon > 0 \) such that \( \epsilon < \rho_m \). Let \( 0 < \delta < \epsilon \) be such that any finite \( \delta \)-pseudo-orbit is \( \epsilon \)-shadowed, and take \( n > m \) such that \( S_n < \delta \).

Let \( (O_i)_{i \in \mathbb{N}} \) be a sequence in \( PO(U_n) \). Then, \( (O_i) \hookrightarrow (V_i) \) where \( V_i \in U_m \).

We have to prove that \( (V_i) \in O(U_m) \).

By the definition of \( PO(U_n) \), there exists a sequence \( (x_i) \) in \( X \) such that \( x_0 \in O_0 \) and \( f(x_i), x_{i+1} \in O_{i+1} \) for all \( i \in \mathbb{N} \) (so \( x_i \in V_i \) for \( i \in \mathbb{N} \)). Since \( \text{diam}(O_i) \leq S_n < \delta \) for every \( i \), we have that \( (x_i) \) is a \( \delta \)-pseudo-orbit. Given \( k > 0 \), let \( z \in X \) be a point that \( \epsilon \)-shadows the sequence \( (x_i)_{i=1}^k \), i.e., such that \( d(f^i(z), x_i) < \epsilon \) for all \( i = 0, \ldots, k \). Since \( x_i \in O_i \subseteq V_i \) for all \( i \), the definition of \( \rho_n \) and choice of \( \epsilon \) imply that \( f^i(z) \in V_i \) for \( i = 1, \ldots, k \). Hence \( (V_i) \in O(U_m) \) as desired.

We now prove the converse. Suppose that for each \( m \in \mathbb{N} \) there is \( n > m \) such that \( PO(U_n) \hookrightarrow O(U_m) \). Given \( \epsilon > 0 \), take \( m \in \mathbb{N} \) such that \( S_m < \epsilon \). Let \( n > m \) be such that \( PO(U_n) \hookrightarrow O(U_m) \) and take \( \delta < \rho_n \).

Let \( (x_i)_{i=1}^k \) be a finite \( \delta \)-pseudo-orbit. Then \( (y_i)_{i \in \mathbb{N}} \), where \( y_i = x_i \) for \( i = 0, \ldots, k \) and \( y_{k+j} = f^j(x_k) \), for \( j = 1, 2, \ldots \), is a \( \delta \)-pseudo-orbit. For each \( i \in \mathbb{N} \), let \( O_i \in U_n \) be such that \( y_i \in O_i \). Since \( d(f(y_i), y_{i+1}) < \delta < \rho_n \)

we have, from the definition of \( \rho_n \), that \( f(y_i), y_{i+1} \in O_i \) for all \( i \). Hence \( (O_i) \in PO(U_n) \). Let \( (V_i) \in O(U_m) \) be such that \( (O_i) \hookrightarrow (V_i) \). Let \( z \in X \) be such that \( f^i(z) \in V_i \) for each \( i = 0, \ldots, k \). Then for all \( i = 0, \ldots, k \) both \( f^i(z) \) and \( x_i \) belong to \( V_i \). Since \( \text{diam}(V_i) < S_m < \epsilon \) we conclude that \( z \) \( \epsilon \)-shadows \( (x_i)_{i=1}^k \) and hence \( f \) has the finite shadowing property.
We now prove our characterization of the finite shadowing property in terms of inverse limits.

**Theorem 4.2.10.** Let \( X \) be a metric space with a tame defining sequence and \((X, f)\) be a dynamical system with the finite shadowing property. Then, \( f \) is conjugate to the inverse limit of a sequence of 1-step shifts on a countable alphabet. More precisely, \((X, f)\) is conjugate to \( (\lim \leftarrow \{\text{PO}(U_n)\}, \sigma^*) \).

Moreover, if \( f \) is uniformly continuous, then the conjugacy can be made uniform.

**Proof.** Let \( \{U_n\}_{n \in \mathbb{N}} \) be a tame defining sequence of \( X \). By Proposition 4.2.9, for each \( m \in \mathbb{N} \), we can choose \( n > m \) such that
\[
\text{PO}(U_n) \leftrightarrow \mathcal{O}(U_m).
\]
Moreover, for each \( k \in \mathbb{N} \), \( \mathcal{O}(U_k) \subseteq \text{PO}(U_k) \), implying that
\[
\mathcal{O}(U_m) \leftrightarrow \text{PO}(U_m).
\]
Hence, we can choose increasing sequences of positive integers
\[
m_1 < n_1 < m_2 < n_2 \ldots
\]
such that
\[
\mathcal{O}(U_{m_1}) \leftrightarrow \text{PO}(U_{n_1}) \leftrightarrow \mathcal{O}(U_{m_2}) \leftrightarrow \text{PO}(U_{n_2}) \ldots \quad (a)
\]
The mapping on this inverse limit space \((a)\) is the shift map on each coordinate space. Now consider the following two factors of the above dynamical system.
\[
\mathcal{O}(U_{m_1}) \leftrightarrow \text{PO}(U_{n_1}) \leftrightarrow \ldots \quad (b)
\]
\[
\text{PO}(U_{n_1}) \leftrightarrow \text{PO}(U_{n_2}) \leftrightarrow \ldots \quad (c)
\]
Each of the dynamical system \((b)\) and the dynamical system \((c)\) is conjugate to the dynamical system \((a)\), and hence \((b)\) is conjugate to \((c)\). By Theorem 4.2.6, the dynamical system \((b)\) is conjugate to \((X, f)\). By Lemma 4.2.1, each \( \text{PO}(U_n) \) is 1-step shift. Hence, we have that the dynamical system \((X, f)\) is conjugate to the dynamical system
\[
\text{PO}(U_{n_1}) \leftrightarrow \text{PO}(U_{n_2}) \leftrightarrow \ldots \quad (c)
\]
consisting of 1-step shift spaces.

Now note that the dynamical system \((b)\) and the dynamical system \((c)\) are actually uniformly conjugate to the dynamical system \((a)\). Hence, they
are uniformly conjugate to each other. Moreover, if \((X, f)\) is uniformly continuous then, by Lemma 4.2.7, we have that \((X, f)\) is uniformly conjugate to the dynamical system \((b)\) and hence uniformly conjugate to the dynamical system \((c)\), completing the proof.

In the context of uniformly continuous Polish dynamical systems with tame defining sequences, we completely describe the relation between the shadowing property and the finite shadowing property below.

**Proposition 4.2.11.** Let \((X, d)\) be a metric space with a tame defining sequence \(\{U_n\}_{n \in \mathbb{N}}\), and \(f : X \to X\) be uniformly continuous map. Then, the following are equivalent:

(i) \(f\) has the finite shadowing property and \(\lim \left\{ \rightsquigarrow, \mathcal{PO}(U_n) \right\}\) satisfies the Mittag-Leffler Condition.

(ii) \(f\) has the shadowing property.

**Proof.** Suppose that (i) holds. Notice that subsystems of inverse limits with the Mittag-Leffler Condition also satisfies the Mittag-Leffler Condition. By Theorem 4.2.10 we have that \(f\) is uniformly conjugate to an inverse limit of 1-step shifts satisfying the Mittag-Leffler Condition. By Proposition 2.3.5, each of these 1-step shifts has the shadowing property. By Theorem 4.1.11, we have that the inverse limit space of these 1-step shifts has the shadowing property. As of now we have that \(f\) is uniformly conjugate to a space with the shadowing property. We conclude, by Proposition 2.2.5, that \(f\) has the shadowing property.

Conversely, suppose that \(f\) has the shadowing property. Let \(N \in \mathbb{N}\) and pick \(\delta\) that witnesses the shadowing property for \(\rho_N\). Choose \(k \geq N\) such that \(S_k < \delta\) and let \(i \geq k\). Let \((O_j) \in \mathcal{PO}(U_k)\) and \((V_j) \in \mathcal{PO}(U_N)\) be such that \((O_j) \hookrightarrow (V_j)\). We have to show that there exists \((W_j) \in \mathcal{PO}(U_i)\) such that \((W_j) \hookrightarrow (V_j)\).

From the definition of \(\mathcal{PO}(U_i)\) we can find an \(S_k\) pseudo-orbit \((x_0, x_1, ...)\) such that \(x_j \in O_j\) for all \(j\). By the shadowing property of \(f\), there is a \(z\) that \(\rho_N\) shadows \((x_0, x_1, ...)\), that is, \(d(f^j(z), x_j) < \rho_N\) for all \(j\). Notice now that the orbit of \(z\) determine an element in \(\mathcal{PO}(U_i)\). More precisely, let \(W_i \in \{U_i\}\) be such that \(f^i(z) \in W_i\). Finally, notice that \(x_j \in V_j\) and, since \(d(f^j(z), x_j) < \rho_N\), we have that \(f^j(z) \in V_j\) and hence \((W_j) \hookrightarrow (V_j)\) as desired.

**Corollary 4.2.12.** Let \((X, d)\) be a metric space with a tame defining sequence and \(f : X \to X\) be a uniformly continuous map. Then, \(f\) has the shadowing property.
property if, and only if, \( f \) is uniformly conjugate to an inverse limit, satisfying Mittag-Leffler Condition, of a sequence of 1-step shifts on a countable alphabet, with bonding maps of the inverse limit uniformly continuous.

**Proof.** If \( f \) has the shadowing property then just apply Theorem 4.2.10 and Proposition 4.2.11.

For the converse, notice that by Proposition 2.3.5 we have that each of the 1-step shifts in the inverse limit has the shadowing property. By Theorem 4.1.11, we have that the inverse limit space of these 1-step shifts has the shadowing property. As of now we have that \( f \) is uniformly conjugate to a space with the shadowing property. We conclude, by Proposition 2.2.5, that \( f \) has the shadowing property.

**Remark 4.2.13.** Example 2.3.4 shows that Corollary 4.2.12 is sharp, i.e., one cannot drop the hypothesis of \( f \) being uniformly continuous.

The following proposition guarantees that Mittag-Leffler Condition is satisfied for certain nice defining sequences.

**Proposition 4.2.14.** Let \((X, d)\) be a metric space with a tame defining sequence \(\{U_n\}_{n \in \mathbb{N}}\), and \(f : X \to X\) be a continuous map with the finite shadowing property. If \(U_n\) is a partition consisting of compact sets, for all \(n\), then \(\lim\{\to, \mathcal{P}O(U_n, f)\}\) satisfies the Mittag-Leffler Condition.

**Proof.** We will commence as in the proof of Proposition 4.2.9. Let \(m \in \mathbb{N}\). Choose \(\varepsilon > 0\) such that \(\varepsilon < \rho_m\). Let \(0 < \delta < \varepsilon\) be such that any finite \(\delta\)-pseudo-orbit is \(\varepsilon\)-shadowed. Let \(n > m\) be such that \(S_n < \delta\). Let \((O_i)_{i \in \mathbb{N}}\) be a sequence in \(\mathcal{P}O(U_n)\) and \((O_i) \hookrightarrow (V_i)\) where \(V_i \in U_m\). We have to prove that for each \(k \geq n\), there exists \((O'_i)_{i \in \mathbb{N}}\) in \(\mathcal{P}O(U_k)\) such that \((O'_i) \hookrightarrow (V_i)\).

By the definition of \(\mathcal{P}O(U_n)\), there exists a sequence \((x_i)\) in \(X\) such that \(x_0 \in O_0\) and \(f(x_i), x_{i+1} \in O_{i+1}\) for all \(i \in \mathbb{N}\) (so \(x_i \in V_i\) for \(i \in \mathbb{N}\)). Since \(diam(O_i) \leq S_n < \delta\) for every \(i\), we have that \((x_i)\) is a \(\delta\)-pseudo-orbit. Given \(j > 0\), let \(z_j \in X\) be a point that \(\varepsilon\)-shadows the sequence \((x_i)_{i=0}^j\), i.e., such that \(d(f^i(z_j), x_i) < \varepsilon\) for all \(i = 0, \ldots, j\). Since \(x_i \in O_i \subseteq V_i\) for all \(i\), the definition of \(p_n\) and choice of \(\varepsilon\) imply that \(f^i(z_j) \in V_i\), for \(i = 0, \ldots, j\). As \(U_m\) is a partition consisting of compact sets, we have that \(V_0\) is compact and a subsequence of \(\{z_j\}_{j=1}^\infty\) converges to some point \(z\). Now, for this particular \(z\) we have that \(f^i(z) \in V_i\) for all \(i \geq 0\). Let \((O'_i)_{i \in \mathbb{N}}\) in \(\mathcal{P}O(U_k)\) be such that \(f^i(z) \in O'_i\). Then, we have that \((O'_i) \hookrightarrow (V_i)\).

The above result allow us to describe a class of functions-spaces where the shadowing property and the finite shadowing property agree. More precisely, we have the following.
Corollary 4.2.15. Let \((X, d)\) be a metric space with a tame defining sequence \(\{U_n\}_{n \in \mathbb{N}}\) such that each partition \(U_n\) consists of compact sets. Then a uniformly continuous map \(f : X \to X\) has the shadowing property if, and only if, it has the finite shadowing property.

Proof. Suppose that \(f\) has the finite shadowing property. By Proposition 4.2.14, we have that \(\lim \{\to, PO(U_n, f)\}\) satisfies the Mittag-Leffler Condition. That \(f\) has the shadowing property now follows from Proposition 4.2.11.

5 Applications

5.1 Shadowing in ultrametric spaces

In this subsection, as applications of the techniques and results developed so far, we prove that various classes of maps in ultrametric spaces have the shadowing property.

Definition 5.1.1. We call \(f : X \to X\) an eventually Lipschitz \(L\) map if there is \(\varepsilon > 0\) such that for all \(x, y \in X\) with \(d(x, y) < \varepsilon\) we have that \(d(f(x), f(y)) \leq L \cdot d(x, y)\).

We call \(f : X \to X\) an eventual similarity if there is \(\varepsilon > 0\) and \(s > 0\) such that for all \(x, y \in X\) with \(d(x, y) < \varepsilon\) we have that \(d(f(x), f(y)) = s \cdot d(x, y)\).

We start by showing the finite shadowing property for maps that are Lipschitz 1 or such that the inverse is Lipschitz 1.

Theorem 5.1.2. Suppose that \(X\) is an ultrametric space and \(f : X \to X\) is continuous. Then, under any one of the following hypothesis the map \(f\) has the finite shadowing property.

(i) \(f\) is eventually Lipschitz 1

(ii) \(f^{-1}\) is eventually Lipschitz 1.

(iii) \(f^{-1}\) is uniformly continuous and \(f\) is an eventual similarity.

Proof. Let \(\{U_n\}\) be the tame defining sequence of Proposition 3.2.2. By Proposition 4.2.9, it suffices to show that for every \(m \in \mathbb{N}\) there is \(n > m\) such that \(PO(U_n, f) \to O(U_m, f)\).

We first prove the result for \(f\) eventually Lipschitz 1. Let \(\varepsilon > 0\) be as in the definition of eventual Lipschitz. Let \(m \in \mathbb{N}\). As \(O(U_m', f) \to O(U_m, f)\) for \(m' \geq m\), we may assume that \(m\) is large enough so that \(1/m < \varepsilon\). Now let
we have that $f(x) \in V_i$. We actually prove more, namely, for any $x \in O_0$, we have that $f^i(x) \in V_i$. We accomplish this by showing that $f^i(O_0) \subseteq O_{i+1}$, $i \geq 1$. This, in turn, is accomplished by showing that $f(O_i) \subseteq O_{i+1}$, $i \geq 0$. Indeed, by definition, $(O_i)_{i \in \mathbb{N}} \in \mathcal{PO}(U_n, f)$ implies that $f(O_i) \cap O_{i+1} \neq \emptyset$. Let $f(x) \in O_{i+1}$ with $x \in O_i$. Then for any $z \in O_i$, we have that $d(f(x), f(z)) \leq d(x, z) < \frac{1}{n}$, implying that $f(O_i) \subseteq B(f(x), \frac{1}{n})$. From Property $[UM_2]$ we have that $B(f(x), \frac{1}{n}) = O_{i+1}$, yielding that $f(O_i) \subseteq O_{i+1}$.

Now we consider the case when $f^{-1}$ is eventually Lipschitz 1. We proceed as earlier and let $n = m + 1$. Take $(O_i)_{i \in \mathbb{N}} \in \mathcal{PO}(U_n, f)$, and let $V_i \in U_m$ be such that $O_i \hookrightarrow V_i$. We have to prove that $\forall \ u \in \mathbb{N}$, $\exists \ \mathbf{x} \in X$ s.t. $f^i(\mathbf{x}) \in V_i$, $0 \leq i \leq k$. This time we observe that for all $i \geq 0$, we have that $f^{-1}(O_{i+1}) \subseteq O_i$. Indeed, as $(O_i)_{i \in \mathbb{N}} \in \mathcal{PO}(U_n, f)$ we have that there is $x \in O_{i+1}$ with $f^{-1}(x) \in (O_i)$. Then for any $z \in O_{i+1}$ we have that $d(f^{-1}(x), f^{-1}(z)) \leq d(x, z) < \frac{1}{n}$, implying that $f^{-1}(O_{i+1}) \subseteq B(f^{-1}(x), \frac{1}{n})$. From Property $[UM_2]$ we have that $B(f^{-1}(x), \frac{1}{n}) = O_i$, and hence $f^{-1}(O_{i+1}) \subseteq O_i$ for all $i \geq 0$. Now this fact and induction implies that for all $k \geq 1$,

$$f^{-k}(O_k) \subseteq f^{k-1}(O_{k-1}) \subseteq \ldots \subseteq f^{-1}(O_1) \subseteq O_0.$$ 

Let $x \in f^{-k}(O_k)$. Then, for all $0 \leq i \leq k$, we have that $f^i(x) \in O_i \subseteq V_i$, completing the proof.

Let us now show (iii). Let $f$ be as in the hypothesis, with the similarity constant $s$. The case of $s \leq 1$ is covered in part (i). Hence let us assume that $s > 1$. Let $\varepsilon$ be as in the definition of eventual similarity. Let $\delta > 0$ witness the uniform continuity of $f^{-1}$ for $\varepsilon$. We will show that $f^{-1}$ is an eventually Lipschitz 1 contraction with eventuality constant $\delta$. By the choice of $\delta$, for all $x, y \in X$ with $d(x, y) < \delta$, we have that $d(f^{-1}(x), f^{-1}(y)) < \varepsilon$. Hence, as $f$ is an eventual similarity, we have that

$$d(f(f^{-1}(x)), f(f^{-1}(y))) = s \cdot d(f^{-1}(x), f^{-1}(y)),$$

or equivalently,

$$d(f^{-1}(x), f^{-1}(y)) = \frac{1}{s} \cdot d(x, y).$$

We have just shown that $f^{-1}$ is an eventually Lipschitz 1. By (ii), we have that $f$ has the finite shadowing property.

We now explore when the shadowing property holds for the above classes.
Corollary 5.1.3. Let $X$ be a Polish ultrametric space. If $f : X \to X$ is an eventually Lipschitz 1 function, then $f$ has the shadowing property.

Proof. In light of Theorem 5.1.2, we already have that $f$ has the finite shadowing property. In order to complete the proof, by Proposition 1.2.11, it suffices to show that $\lim \{\to, \mathcal{PO}(U_n)\}$ satisfies the Mittag-Leffler Condition.

Let $\varepsilon > 0$ witness the fact that $f$ is eventually Lipschitz. Let $j \in \mathbb{N}$ be such that $\frac{1}{j} < \varepsilon$. Consider the tame defined sequence given by $\{U_n\}$ where $U_n$ consists of open balls of radius $\frac{1}{n+j}$.

Given $N \in \mathbb{N}$, let $k = N + 1$ and let $i \geq k$. To verify the Mittag-Leffler Condition, it suffices to show that for $(O_l) \in \mathcal{PO}(U_i)$ such that $(O_l) \hookrightarrow (V_l)$, there is $(W_l) \in \mathcal{PO}(U_l)$ such that $(W_l) \hookrightarrow (V_l)$. To this end, let $(x_l)$ be a sequence such that $x_l \in O_l$ and $f(x_l) \in O_{l+1}$, $l \in \mathbb{N}$.

We will show that $f^l(x_0) \in V_l$ for all $l \in \mathbb{N}$. By hypothesis, $x_0 \in O_l \subseteq V_l$ and $f(x_l) \in O_{l+1} \subseteq V_{l+1}$, for all $l \in \mathbb{N}$. Clearly, $x_0 \in V_0$. Suppose that $f^l(x_0) \in V_l$ for every $l \leq M$. Then,

$$d(f^{M+1}(x_0), f(x_M)) \leq d(f^M(x_0), x_M) < \frac{1}{N+j},$$

verifying that $f^{M+1}(x_0) \in V_{M+1}$ as desired.

Now, let $W_l$ be the element in $\mathcal{PO}(U_l)$ such that $f^l(x_0) \in W_l$. Clearly, $(W_l) \hookrightarrow (V_l)$.

Corollary 5.1.4. Suppose that $X$ is an ultrametric Polish space with the additional property that for some $\varepsilon > 0$, all balls of radius less than $\varepsilon$ are compact. Let $f : X \to X$ be a uniformly continuous map.

(i) If $f^{-1}$ is an eventually Lipschitz 1 map, then $f$ has the shadowing property.

(ii) If $f^{-1}$ is uniformly continuous and $f$ is an eventual similarity, then $f$ has the shadowing property.

Proof. Proof of (i). By Theorem 5.1.2, we have that $f$ has the finite shadowing property and so, by Corollary 1.2.13, $f$ has the shadowing property.

Proof of (ii). The proof is analogous to the proof of (i).

Corollary 5.1.5. Let $X$ be a compact ultrametric space and $f : X \to X$ be an eventual similarity. Then $f$ has the shadowing property.

The next example shows that Theorem 5.1.2 is not valid for general Lipschitz functions.
Example 5.1.6. Let $A$ be a countable alphabet and consider the metric in $A^\mathbb{N}$ given in Definition 4.1.6. This is an ultrametric space. Let $X \subseteq A^\mathbb{N}$ be a shift space which is not of finite order. Then the shift map on $X$ is a Lipschitz 2 map which, by Proposition 2.3.5, does not have the shadowing property.

Motivated by the question (1) left at the end of the paper [2], we finish this subsection with a result regarding two sided shadowing in ultrametric spaces. The definition of two sided shadowing is the same as Definition 2.2.2, with $I = \mathbb{Z}$ in Definition 2.2.1.

Proposition 5.1.7. Let $(X, d)$ be an ultrametric space and $f : X \to X$ be a surjective isometry. Then $(X, f)$ has the two sided shadowing property.

Proof. Given $\varepsilon > 0$, let $0 < \delta < \varepsilon$ and $(x_n)_{n \in \mathbb{Z}}$ be a $\delta$-pseudo-orbit. Notice that for $n \in \mathbb{N}$ we have:

$$d(f^n(x_0), x_n) \leq \max\{d(f^n(x_0), f(x_{n-1})), d(f(x_{n-1}), x_n)\}$$

$$= \max\{d(f^{n-1}(x_0), x_{n-1}), d(f(x_{n-1}), x_n)\}.$$ 

Since $d(f(x_0), x_1) < \delta$ the above and induction imply that $x_0$ shadows $(x_n)_{n \in \mathbb{N}}$. Now notice that we also have, for $n \in \mathbb{N}$, that

$$d(f^{-n}(x_0), x_{-n}) \leq \max\{d(f^{-n}(x_0), f^{-1}(x_{n+1})), d(f^{-1}(x_{n+1}), x_{-n})\}$$

$$= \max\{d(f^{-n+1}(x_0), x_{-n+1}), d(f^{-1}(x_{n+1}), x_{-n})\}.$$ 

Since $d(f(x_i), x_{i+1}) < \delta$ implies that $d(x_i, f^{-1}(x_{i+1})) < \delta$ the above, and another induction, proves that $(x_n)_{n \in \mathbb{Z}}$ is two sided shadowed by $x_0$. \qed

Remark 5.1.8. It follows from the above proposition that the identity map in an ultrametric space always has the two sided shadowing property. This is in contrast with the behavior of the identity map in a nontrivial connected space, where it does not have even the finite shadowing property.

5.2 Shadowing in $P$-adic dynamics

In the recent paper [2] shadowing and structural stability in $p$-adics dynamics is studied. Below we show that some of their main results concerning shadowing follow from our general results in ultrametric spaces.

Before we state the results, recall that a map $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is $(p^{-k}, p^m)$ locally scaling ($1 \leq m \leq k$ integers) if for all $x, y \in \mathbb{Z}_p$ with $\|x - y\|_p \leq p^{-k}$, we have that $\|f(x) - f(y)\|_p = p^m\|x - y\|_p$, see [2, 21].

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Corollary 5.2.1. The following hold.

(i) [2, Theorem 1] If \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) is a \((p^{-k}, p^m)\) locally scaling function, where \( 1 \leq m \leq k \) are integers, then \( f \) has the shadowing property.

(ii) [2, Proposition 18] If \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) is a Lipschitz 1 map, then \( f \) has the shadowing property.

(iii) [2, Remark 19] If \( f : \mathbb{Q}_p \to \mathbb{Q}_p \) is a Lipschitz 1 map, then \( f \) has the shadowing property.

Proof. Since \( \mathbb{Z}_p \) is compact, item (i) follows from Corollary 5.1.5. Items (ii) and (iii) follow from Corollary 5.1.3 and the fact that \( \mathbb{Q}_p \) is an ultrametric space with compact balls.

Finally, at the end of [2] the following question is left open:

Question: Let \( f : \mathbb{Q}_p \to \mathbb{Q}_p \) be an homeomorphism. Assuming that \( f \) is 1-Lipschitz, can \( f \) be (two-sided) shadowing or Lipschitz structurally stable?

The (affirmative) answer to the shadowing part of this question follows directly from our Proposition 5.1.7.

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Udayan B. Darji
ubdarj01@louisville.edu
Department of Mathematics,
University of Louisville,
Louisville, KY 40292, USA.

Daniel Gonçalves
daemig@gmail.com
Departamento de Matemática
Universidade Federal de Santa Catarina
Florianópolis, SC 88040-900, Brazil

Marcelo Sobottka
marcelo.sobottka@ufsc.br
Departamento de Matemática
Universidade Federal de Santa Catarina
Florianópolis, SC 88040-900, Brazil