ITERATED INTEGRALS ON $\mathbb{P}^1 \setminus \{0, 1, \infty, z\}$ AND A CLASS OF RELATIONS AMONG MULTIPLE ZETA VALUES

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Abstract. In this paper we consider iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty, z\}$ and define a class of $\mathbb{Q}$-linear relations among them, which arises from the differential structure of the iterated integrals with respect to $z$. We then define a new class of $\mathbb{Q}$-linear relations among the multiple zeta values by taking their limits as $z \to 1$, which we call confluence relations (i.e., the relations obtained by the confluence of two punctured points). One of the significance of the confluence relations is that it gives a rich family and seems to exhaust all the linear relations among the multiple zeta values. As a good reason for this, we show that confluence relations imply both the regularized double shuffle relations and the duality relations.

1. Introduction

The multiple zeta values, MZVs in short, are the real numbers defined by the multiple Dirichlet series

$$\zeta(k_1, \ldots, k_d) = \sum_{0 < m_1 < \cdots < m_d} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}$$

or equivalent iterated integral

$$\zeta(k_1, \ldots, k_d) = (-1)^d I(0; 1, \{0\}^{k_1-1}, \ldots, 1, \{0\}^{k_d-1}; 1)$$

where

$$I(0; a_1, \ldots, a_n; 1) = \int_{0 < t_1 < \cdots < t_n < 1} \prod_{i=1}^n \frac{dt_i}{t_i - a_i}.$$ 

Here $(k_1, \ldots, k_d)$ is called the index, $k := k_1 + \cdots + k_d$ is called the weight and $d$ is called the depth of the MZV. We assume that $k_1, \ldots, k_{d-1} \geq 1$ and $k_d \geq 2$ for convergence of the series/integral. It is known that there are many $\mathbb{Q}$-linear relations among the multiple zeta values, and their structure is one of the main interest of the study of multiple zeta values.

There are three known classes of linear relations among the MZVs which are conjectured to exhaust all the linear relations among them:

- Associator relations
- Regularized double shuffle relations (RDS)
- Kawashima’s relations [7] (when the product is expanded by the shuffle relation)

All three families above come from different backgrounds. The associator relations come from geometric relations of Drinfel’d associator of KZ-equation. The regularized double shuffle relations come from the series and iterated integral expression of multiple zeta values. Kawashima’s relations come from the Newton series which interpolate the multiple harmonic sums.

The first purpose of this paper is to define a new class of linear relations among MZVs which we call confluence relations based on the theory of iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty, z\}$, and the second purpose is to show that this class includes the regularized double shuffle relations (and the duality relations). Clearly, the latter result implies that “confluence relations” are also expected to exhaust all linear relations of MZVs. Together with the known inclusion between Kawashima relations, Associator relations, RDS plus duality relations, the implication relations of the classes can be summarized as follows.

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Here, the implication (Associator) ⇒ (RDS) was proved by Furusho [2], and the implication (RDS + duality) ⇒ (Kawashima) was proved by Kaneko-Yamamoto [6].

The sketch of our method is as follows. To introduce the confluence relation, we first consider the holomorphic function of \( z \in \mathbb{C} \setminus [0,1] \) defined by the iterated integral

\[
I(0; a_1, \ldots, a_n; 1) \quad (a_1, \ldots, a_n \in \{0,1,z\}).
\]

Then the derivatives of these functions are again expressible in terms of the functions of the same form. The standard relations is a class of linear relations of such holomorphic functions which arise from this differential structure. Since the limit \( z \to 1 \) of any relation among these functions should give a relation among MZVs, we define the “confluence relation” by the limit \( z \to 1 \) of the “standard relations”, which is the basic idea of the confluence relation. However, as the limit does not always exist, we need to invent some tricks/techniques to deal with the behavior of divergence and to obtain a suitable meaningful substitute of the limit. Next we prove that the confluence relations includes RDS and the duality. The key to these proofs are the differential formulas of the shuffle and the stuffle products and the duality (Theorem 8, [4]).

In Section 2 we introduce some algebraic settings and define the standard relations of iterated integrals (Definition 13, Theorem 15). Then in Section 3 we formulate the confluence relations of MZVs (Definition 21, Theorem 20). In Section 4 we show that the confluence relations includes RDS (Theorem 26) and duality relations (Theorem 28). At the end, in Appendix A we give proofs of complementary propositions and in Appendix B we give a table of the confluence relations up to weight 4.

1.1. Notations. We denote by \((A, \circ)\) an algebra \(A\) equipped with the bilinear product \(\circ\). For \(\mathbb{Z}\)-modules \(A\) and \(B\), we denote by \(\text{Hom}(A, B)\) the \(\mathbb{Z}\)-module formed by \(\mathbb{Z}\)-linear maps from \(A\) to \(B\). Also, we denote \(\text{End}(A) := \text{Hom}(A, A)\).

2. Standard relations of iterated integrals

2.1. Algebraic settings and differential formulas for iterated integrals. Let \(A_z := \mathbb{Z}\langle e_0, e_1, e_z \rangle\) (resp. \(A := \mathbb{Z}\langle e_0, e_1 \rangle \subset A_z\)) be a non-commutative ring generated by formal symbols \(e_0, e_1, e_z\) (resp. \(e_0, e_1\)). We define the subspaces \(A^0_z \subset A^1_z \subset A_z\) and \(A^0 \subset A^1 \subset A\) by

\[
A^0_z := \mathbb{Z} \oplus e_z + \bigoplus_{a \in \{1,z\}} e_a A_z e_b,
\]

\[
A^0 := \mathbb{Z} \oplus e_1 A e_0 \quad (= A \cap A^0) ,
\]

\[
A^1_z := \mathbb{Z} \oplus \bigoplus_{a \in \{1,z\}} e_a A_z,
\]

\[
A^1 := \mathbb{Z} \oplus e_1 A \quad (= A \cap A^1) .
\]

For \(z \in \mathbb{C} \setminus [0,1]\), we define a \(\mathbb{Z}\)-linear map \(L : A^0_z \to \mathbb{C}\) by

\[
L(e_{a_1} \cdots e_{a_m}) := \int_{0 < t_1 < \cdots < t_m < 1} \prod_{i=1}^m \frac{d t_i}{t_i - a_i}.
\]
Note that, for \( w \in \mathcal{A}_z^0 \), \( L(w) \) can be regarded as a holomorphic function of \( z \) on \( \mathbb{C} \setminus [0, 1] \). In particular, the multiple zeta value is expressed as
\[
\zeta(k_1, \ldots, k_d) = L((-e_1) e_0^{k_1-1} \cdots (-e_1) e_0^{k_d-1})
\]
by \( L \).

**Definition 1.** For \( \alpha, \beta \in \{0, 1, z\} \), we define a linear operator \( \partial_{\alpha, \beta} \) on \( \mathcal{A}_z \) by
\[
\partial_{\alpha, \beta} (e_{a_1} \cdots e_{a_n}) := \sum_{i=1}^{n} (\delta_{\{a_i, a_{i+1}\}, \{\alpha, \beta\}} - \delta_{\{a_{i-1}, a_i\}, \{\alpha, \beta\}}) e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_n}
\]
where \( a_0 = 0, a_{n+1} = 1 \) and \( \delta_{S, T} \) denotes the Kronecker delta i.e.,
\[
\delta_{S, T} = \begin{cases} 1 & S = T \\ 0 & S \neq T, \end{cases}
\]
for sets \( S \) and \( T \).

It is easy to check that \( \partial_{\alpha, \beta}(\mathcal{A}_0^0) \subset \mathcal{A}_0^0 \). Note that \( \partial_{z, 0} + \partial_{z, 1} + \partial_{1, 0} = 0 \) on \( \mathcal{A}_z^0 \) (see Proposition 5).

The operator \( \partial_{z, a} \) is related to the actual differentiation \( d/dz \) as follows.

**Proposition 2** ([3 Theorem 2.1]). \( \frac{d}{dz} L(w) = \sum_{a \in \{0, 1\}} \frac{1}{z-a} L(\partial_{z, a}(w)) \).

### 2.2. Shuffle product, stuffle product, duality, and their algebraic differential formulas.

**Definition 3.** The shuffle product \( 
\odot : \mathcal{A}_z \times \mathcal{A}_z \rightarrow \mathcal{A}_z\) is a bilinear map defined by \( w \odot 1 = 1 \odot w = w \) for \( w \in \mathcal{A}_z \) and
\[
e_{a} u \odot e_{b} v = e_{a}(u \odot e_{b} v) + e_{b}(e_{a} u \odot v) \quad (u, v \in \mathcal{A}_z).
\]

Note that, for \( ? \in \{0, 1\}, \mathcal{A}_z^? \) and \( \mathcal{A}^? \) become commutative rings by the shuffle product.

**Definition 4** ([4]). The (generalized) stuffle product \( \ast : \mathcal{A} \times \mathcal{A}_z \rightarrow \mathcal{A}_z \) is a bilinear map defined by \( u \ast 1 = u, 1 \ast v = v \) for \( u \in \mathcal{A}, v \in \mathcal{A}_z \) and
\[
e_{a} u \ast e_{b} v = e_{ab}(u \ast e_{b} v + e_{a} u \ast v) \quad (u \in \mathcal{A}, v \in \mathcal{A}_z).
\]

Note that, for \( ? \in \{0, 1\}, \mathcal{A}_z^? \) becomes a commutative ring by the stuffle product, and \( \mathcal{A}_z^1 \) becomes an \( (\mathcal{A}_z^1, \ast) \) module by the stuffle product (it follows from the associativity of more generalized stuffle products shown in [4 Proposition 6]). See [3] for the compatibility with the standard definition of the stuffle product.

**Definition 5.** The duality map \( \tau : \mathcal{A}_z \rightarrow \mathcal{A}_z \) is an anti-automorphism (i.e., \( \tau_z(wv) = \tau_z(v) \tau_z(u) \)) defined by \( \tau_z(e_0) = e_z - e_1, \tau_z(e_1) = e_z - e_0 \) and \( \tau_z(e_z) = e_z \).

Note that \( \tau_z(\mathcal{A}_z^0) \subset \mathcal{A}_z^0 \) from the definition.

**Definition 6.** For \( w \in \mathcal{A}_z^1 \), the shuffle regularization \( \text{reg}_{\odot w}(w) \in \mathcal{A}_z^0 \) is defined by \( \text{reg}_{\odot w}(w) := w_0 \) where \( w_0 \) is defined by the unique expression
\[
w = \sum_{i=0}^{\deg(w)} w_i \odot e_i^1 \quad (w_i \in \mathcal{A}_z^0).
\]

The following proposition is a fundamental property of the shuffle and stuffle products and the duality map.

**Proposition 7.** We have:

1. For \( u, v \in \mathcal{A}_z^0 \),
\[L(u \odot v) = L(u)L(v).\]

2. For \( u \in \mathcal{A}_z^0 \) and \( v \in \mathcal{A}_z^0 \),
\[L(u \ast v) = L(u)L(v).\]

3. For \( u \in \mathcal{A}_z^0 \)
\[L(\tau_z(u)) = L(u).\]
Proof. (1) and (2) is just a special case of the shuffle and stuffle product identities of hyperlogarithms \[1\]. For (3), see \[3\] Theorem 1.1.

The following theorem gives fundamental relations between the operators $\shuffle$, $\ast$, $\tau_z$ and $\partial_{z,c}$.

**Theorem 8.** \[4\] Theorem 10: Let $c \in \{0, 1\}$.

1. For $u, v \in \mathcal{A}_z$, 
   \[\partial_{z,c}(u \shuffle v) = (\partial_{z,c} u) \shuffle v + u \shuffle (\partial_{z,c} v).\]

2. For $u \in \mathcal{A}^1$ and $v \in \mathcal{A}_z$, 
   \[\partial_{z,c}(u \ast v) = u \ast (\partial_{z,c} v).\]

3. For $u \in \mathcal{A}_z^0$, 
   \[\tau_z^{-1} \circ \partial_{z,c} \circ \tau_z(u) = \partial_{z,c} u.\]

**Remark 9.** It is also possible to prove Proposition \[ \[4\] from Theorem \[8\].

2.3. Standard relations. We define $\text{Const} \in \text{Hom}(\mathcal{A}_z, \mathcal{A})$ by

\[\text{Const}(e_{a_1} \cdots e_{a_n}) := \begin{cases} e_{a_1} \cdots e_{a_n} & \text{if } a_i \in \{0, 1\} \text{ for all } i, \\ 0 & \text{if } a_j = z \text{ for some } i. \end{cases}\]

Note that $\lim_{z \to \infty} L(w) = L(\text{Const}(w))$ for $w \in \mathcal{A}_z^0$ since $\lim_{z \to 0} L(w) = 0$ for $w' \in \mathcal{A}_z^0 \cap \mathcal{A}_z e_z \mathcal{A}_z$. Note that $\text{Const}(u \shuffle v) = \text{Const}(u) \shuffle \text{Const}(v)$ for $u, v \in \mathcal{A}_z$ and $\text{Const}(u \ast v) = u \ast \text{Const}(v)$ for $u \in \mathcal{A}, v \in \mathcal{A}_z$ (for the latter identity, see Proposition \[31\]).

We define $\varphi_\shuffle \in \text{Hom}(\mathcal{A}_z^0, \mathcal{A}^0 \otimes \mathbb{Z}(e_0, e_z))$ by

\[\varphi_\shuffle(w) := \sum_{r \in \mathbb{Z}^\geq 0, b_1, \ldots, b_r \in \{0, z\}} \text{Const}(\partial_{1,b_1} \cdots \partial_{1,b_r} w) \otimes e_{b_1} \cdots e_{b_r}.\]

Then we can show that $\varphi_\shuffle(\mathcal{A}_z^0) \subset \mathcal{A}^0 \otimes (\mathbb{Z}(e_0, e_z) \cap \mathcal{A}_z^0)$ (see Proposition \[32\]). We define $\varphi_\shuffle, \varphi_\ast \in \text{End}(\mathcal{A}_z^0)$ by composing $\varphi_\shuffle$ and shuffle or stuffle product, i.e.

\[\varphi_\shuffle(w) := \sum_{r \in \mathbb{Z}^\geq 0, b_1, \ldots, b_r \in \{0, z\}} \text{Const}(\partial_{1,b_1} \cdots \partial_{1,b_r} w) \shuffle e_{b_1} \cdots e_{b_r},\]

\[\varphi_\ast(w) := \sum_{r \in \mathbb{Z}^\geq 0, b_1, \ldots, b_r \in \{0, z\}} \text{Const}(\partial_{1,b_1} \cdots \partial_{1,b_r} w) \ast e_{b_1} \cdots e_{b_r}.\]

**Remark 10.** The map $\varphi_\shuffle$ is a ring endomorphism of $(\mathcal{A}_z^0, \shuffle)$, and the map $\varphi_\ast$ is a homomorphism as $(\mathcal{A}_z^0, \ast)$-module (see Proposition \[33\] and \[34\]).

**Lemma 11.** We have $\varphi_\shuffle \circ \varphi_\shuffle = \varphi_\shuffle \circ \varphi_\ast = \varphi_\shuffle$.

**Proof.** The claim follows from (1) and (2) of Theorem \[8\] since $\partial_{1,b}(e_{b_1} \cdots e_{b_r}) = \delta_{b,b_1} e_{b_1} \cdots e_{b_{r-1}}$ for $b, b_1, \ldots, b_r \in \{0, z\}$. \[\square\]

**Proposition 12.** The following six subspaces of $\mathcal{A}_z^0$ are equal.

1. The image of $(\text{id} - \varphi_\shuffle) : \mathcal{A}_z^0 \to \mathcal{A}_z^0$.
2. The kernel of $\varphi_\shuffle : \mathcal{A}_z^0 \to \mathcal{A}_z^0$.
3. The image of $(\text{id} - \varphi_\ast) : \mathcal{A}_z^0 \to \mathcal{A}_z^0$.
4. The kernel of $\varphi_\ast : \mathcal{A}_z^0 \to \mathcal{A}_z^0$.
5. The kernel of $\varphi_\shuffle : \mathcal{A}_z^0 \to \mathcal{A}_z^0 \otimes \mathbb{Z}(e_0, e_z)$.
6. The set $\{ w \in \mathcal{A}_z^0 \mid \text{Const}(\partial_{z,\alpha_1} \cdots \partial_{z,\alpha_r} w) = 0 \text{ for } r \geq 0, \alpha_1, \ldots, \alpha_r \in \{0, 1\} \}$.

**Proof.** The equality of (5) and (6) follows from $\partial_{1,0} + \partial_{z,0} + \partial_{z,1} = 0$. Since $\varphi_\shuffle$ and $\varphi_\ast$ factor through $\varphi_\shuffle$, we have ker$(\varphi_\shuffle) \subset$ ker$(\varphi_\shuffle)$ and ker$(\varphi_\shuffle) \subset$ ker$(\varphi_\ast)$. Since $(\text{id} - \varphi_\shuffle)(w) = w$ for $w \in$ ker$(\varphi_\shuffle)$, we have ker$(\varphi_\shuffle) \subset$ Im$(\text{id} - \varphi_\shuffle)$. Similarly, since $(\text{id} - \varphi_\ast)(w) = w$ for $w \in$ ker$(\varphi_\ast)$, we have ker$(\varphi_\ast) \subset$ Im$(\text{id} - \varphi_\ast)$. From Lemma \[11\] we have Im$(\text{id} - \varphi_\shuffle) \subset$ ker$(\varphi_\shuffle)$ and Im$(\text{id} - \varphi_\ast) \subset$ ker$(\varphi_\ast)$. Thus all these subspaces are equal. \[\square\]
Definition 13. We define the set of standard relations \( \mathcal{I}_{ST} \) as the subspace of \( \mathcal{A}_z^0 \) given in Proposition [12].

Remark 14. \( \mathcal{I}_{SD} \) forms an ideal of \( (\mathcal{A}_z^0, \omega) \) since \( \mathcal{I}_{SD} = \ker(\varphi_\omega) \) and \( \varphi_\omega \) is a ring endomorphism of \( (\mathcal{A}_z^0, \omega) \). By the similar reasoning, \( \mathcal{I}_{SD} \) is also \( (\mathcal{A}_\ast^0, \ast) \)-module.

The next theorem states that “standard relations” are in fact relations of iterated integrals.

Theorem 15. For \( w \in \mathcal{I}_{ST} \), \( L(w) = 0 \).

Proof. Since \( \mathcal{I}_{ST} = \{ w \in \mathcal{A}_z^0 \mid \text{Const}(\partial_{z,a_1} \cdots \partial_{z,a_r} w) = 0 \text{ for } r \geq 0, a_1, \ldots, a_r \in \{0, 1\} \} \), we have \( v \in \mathcal{I}_{ST} \iff (\text{Const}(v) = 0, \partial_{z,0} v \in \mathcal{I}_{ST} \text{ and } \partial_{z,1} v \in \mathcal{I}_{ST}) \) for \( v \in \mathcal{A}_z^0 \). We prove the theorem by induction on the degree of \( w \). Let \( w \in \mathcal{I}_{ST} \). Since \( \text{Const}(w) = 0 \), we have

\[
\lim_{z \to \infty} L(w) = 0.
\]

From the induction hypothesis, \( L(\partial_{z,0} w) = L(\partial_{z,1} w) = 0 \). Thus

\[
\frac{d}{dz} L(w) = \frac{1}{z} L(\partial_{z,0} w) + \frac{1}{z - 1} L(\partial_{z,1} w) = 0.
\]

Thus the claim \( L(w) = 0 \) is proved. \( \square \)

3. Confluence relations

From now on, we consider the limit \( z \to 1_{+0} \) of the standard relations. We define the subspaces \( \mathcal{A}_z^{-2} \subset \mathcal{A}_z^{-1} \subset \mathcal{A}_z^0 \)

by

\[
\mathcal{A}_z^{-2} := (\mathbb{Z} \oplus e_1 A_z e_0) \oplus e_z A_z e_0
\]

and

\[
\mathcal{A}_z^{-1} := \mathcal{A}_z^{-2} \langle e_z \rangle.
\]

We define an endomorphism \( \mathcal{A}_z^{-1} \ni a \mapsto b : \mathcal{A}_z \to \mathcal{A}_z \) \((a, b \in \{0, 1, z\})\) by

\[
e_x|_{a \mapsto b} = \begin{cases} e_b & x = a \\ e_x & x \neq a \end{cases}
\]

For \( w \in \mathcal{A}_z^{-2} \), \( \lim_{z \to 1_{+1}} L(w) \) exists and equals to \( L(w|_{z \to 1}) \). For general \( w \in \mathcal{A}_z^0 \), \( \lim_{z \to 1_{+1}} L(w) \) does not exist, but the following lemma holds.

Lemma 16. For \( w \in \mathcal{A}_z^0 \), there exists a natural number \( m \) and a unique polynomial \( P_w(T) \in \mathbb{R}[T] \) such that

\[
L(w) - P_w(\log(z - 1)) = O((z - 1) \log^m(z - 1)).
\]

Proof. The uniqueness is obvious. By Theorem [5] \( L(w) = (\varphi_\omega(w)) \) which is a sum of \( L(u)L(v) \) with \( u \in \mathcal{A}_z^0 \) and \( v \in \mathcal{A}_z^0 \cap \langle e_0, e_z \rangle \), and thus the proof of the lemma can be reduced to the case \( w \in \mathcal{A}_z^0 \cap \langle e_0, e_z \rangle \). Put \( w = e_z e_0^{k_1 - 1} \cdots e_z e_0^{k_d - 1} \). Then we have

\[
L(w) = (-1)^d \text{Li}_{k_1, \ldots, k_d}(z^{-1})
\]

where \( \text{Li}_{k_1, \ldots, k_d} \) is the multiple polylogarithm. Thus, the lemma follows from the following well-known fact (c.f. [5], p311) that there exists \( m \in \mathbb{Z}_{>0} \) and \( Q \in \mathbb{R}[T] \) such that

\[
\text{Li}_{k_1, \ldots, k_d}(t) = Q(\log(1 - t)) + O((1 - t) \log^m(1 - t)).
\]

Thus, for general \( w \in \mathcal{A}_z^0 \), we consider \( P_w(0) \) instead of \( \lim_{z \to 1_{+1}} L(w) \). We define \( \lambda' : \mathcal{A}_z^{-2} \to \mathcal{A}_z^0 \) by

\[
\lambda'(w) = w|_{z \to 1_{+1}}.
\]

Then obviously \( P_w(0) = L(\lambda'(w)) \) for \( w \in \mathcal{A}_z^{-2} \). Now we construct a extension \( \lambda : \mathcal{A}_z^0 \to \mathcal{A}_z^0 \) of \( \lambda' \) such that \( P_w(0) = L(\lambda(w)) \) for \( w \in \mathcal{A}_z^0 \).

Note that there exists an isomorphism

\[
\mathcal{A}_z^{-2} \oplus (\mathcal{A}_z^0 \cap \langle e_1, e_z \rangle) \simeq \mathcal{A}_z^0 ; u \oplus v \mapsto u \cup v
\]

(see Proposition [35] for bijectivity). We denote the inverse of this isomorphism by \( \text{reg}_{\{z,1\}} \).
Definition 17. We define a homomorphism $N : \mathcal{A}_z^0 \rightarrow \mathcal{A}_z^{-1}$ by the composition
\[
\mathcal{A}_z^0 \xrightarrow{\sim} \mathcal{A}_z^{-2} \otimes (\mathcal{A}_z^0 \cap \mathbb{Z} \langle e_1, e_2 \rangle) \xrightarrow{\sim} \mathcal{A}_z^{-2} \otimes (\mathcal{A}_z^0 \cap \mathbb{Z} \langle e_0, e_2 \rangle) \xrightarrow{\sim} \mathcal{A}_z^{-1}
\]
where
\[
f := \tau_z |_{\mathcal{A}_z^0 \cap \mathbb{Z} \langle e_1, e_2 \rangle} = \varphi \varphi^* |_{\mathcal{A}_z^0 \cap \mathbb{Z} \langle e_1, e_2 \rangle}.
\]
From the definition, we have $L(N(w)) = L(w)$. Note that there exists an isomorphism $\mathcal{A}_z^{-2} \otimes \mathbb{Z} \langle e_1 \rangle \simeq \mathcal{A}_z^{-1}$; $u \otimes v \mapsto u \uplus v$.
We denote the inverse of this isomorphism by $\text{reg}_{(z)}$.

Definition 18. We define a homomorphism $\lambda : \mathcal{A}_z^0 \rightarrow \mathcal{A}_z^0$ by the composition
\[
\mathcal{A}_z^0 \xrightarrow{\lambda} \mathcal{A}_z^{-1} \xrightarrow{\text{reg}_{(z)}} \mathcal{A}_z^{-2} \otimes \mathbb{Z} \langle e_2 \rangle \xrightarrow{\text{id} \otimes \text{const}} \mathcal{A}_z^{-2} \otimes \mathbb{Z} = \mathcal{A}_z^{-2} \xrightarrow{\lambda} \mathcal{A}_z^0.
\]

Proposition 19. We have $P_w(0) = L(\lambda(w))$ for $w \in \mathcal{A}_z^0$.

Proof. Put $\text{reg}_{(z)}' = (\text{id} \otimes \text{const}) \circ \text{reg}_{(z)}$. Since $L(N(w)) = L(w)$, it is enough to prove that $P_w(0) = L(\lambda'(\text{reg}_{(z)} w))$ for $w \in \mathcal{A}_z^{-1}$. Define $w_0, w_1, \ldots \in \mathcal{A}_z^{-2}$ by
\[
\text{reg}_{(z)} w = \sum_{k \geq 0} w_k \otimes e_z^k.
\]
Since $P_{w_0} = P_u(T) \frac{T^k}{k!}$ for $u \in \mathcal{A}_z^0$, we have $P_w(T) = \sum k \geq 0 \lambda'(w_k) \frac{T^k}{k!}$.
Thus
\[
P_w(0) = L(\lambda'(w_0)) = L(\lambda'(\text{reg}_{(z)} w)). \quad \square
\]

Theorem 20. For $w \in \mathcal{I}_{ST}$, we have
\[
L(\lambda(w)) = 0.
\]

Proof. It follows from Theorem 15 and Proposition 13. \quad \square

Definition 21. We define a $\mathbb{Z}$-module $\mathcal{I}_{CF}$ by
\[
\mathcal{I}_{CF} = \{ \lambda(w) \mid w \in \mathcal{I}_{ST} \} \subset \mathcal{A}_z^0
\]
and say $w \in \ker (L : \mathcal{A}_z^0 \rightarrow \mathbb{R})$ is a confluence relation if $w \in \mathcal{I}_{CF}$.

Remark 22. Since $\mathcal{I}_{ST} = \text{Im}(\text{id} - \varphi_{\parallel})$, $\mathcal{I}_{CF} = \{ \lambda(w - \varphi_{\parallel}(w)) \mid w \in \mathcal{A}_z^0 \}$. Since there are $4 \cdot 3^{k-2}$ words of length $k$ for $k \geq 2$, we can obtain $4 \cdot 3^{k-2}$ relations of multiple zeta values of weight $k$ for $k \geq 2$. For example, let $w = e_z e_1 e_0$. Then we have
\[
w - \varphi_{\parallel}(w) = e_z e_1 e_0 - (e_1 e_0 \uplus e_2 + 1 \uplus e_0 e_2 - 1 \uplus e_0 e_2).
\]
Thus
\[
\text{reg}_{(1,z)}(w - \varphi_{\parallel}(w)) = (e_z e_1 e_0 - e_0^3 e_2^2 e_0) \otimes 1 + (-e_1 e_0 e_2 + e_0 e_2) \otimes e_z.
\]
Since $\tau_z(1) = 1$ and $\tau_z(\langle e_z \rangle) = e_z$, $N(w - \varphi_{\parallel}(w)) = (e_z e_1 e_0 - e_0^3 e_2^2 e_0) \uplus 1 + (-e_1 e_0 e_2 + e_0 e_2) \uplus e_z$.
Thus
\[
\lambda(w - \varphi_{\parallel}(w)) = (e_z e_1 e_0 - e_0^3 e_2^2 e_0) \uplus e_z.
\]
Therefore $-e_0^3 e_2^2 e_0 - e_1 e_0^2 \in \mathcal{I}_{CF} \subset \ker(L)$. This gives $L(-e_0^3 e_2^2 e_0 - e_1 e_0^2) = -\zeta(1,2) + \zeta(3) = 0$.

Remark 23. Since $\lambda(N(w)) = \lambda(w)$ and $\varphi_{\parallel}(N(w)) = \varphi_{\parallel}(w)$, we have $\mathcal{I}_{CF} = \{ \lambda(w - \varphi_{\parallel}(w)) \mid w \in \mathcal{A}_z^{-1} \}$. Since $\lambda = \text{reg}_{(z-1)}$ on $\mathcal{A}_z^{-1}$ and $\varphi_{\parallel}(\mathcal{A}_z^0) \subset \mathcal{A}_z^{-1}$, we have
\[
\mathcal{I}_{CF} = \{ \text{reg}(w - \varphi_{\parallel}(w)) \mid w \in \mathcal{A}_z^{-1} \}.
\]
Conjecture 24. The confluence relations exhaust all the relations of the multiple zeta values, i.e.,
\[ \mathcal{I}_{\text{CF}} \otimes \mathbb{Q} = \ker(L : \mathcal{A}^0 \to \mathbb{R}) \otimes \mathbb{Q}. \]

4. Regularized double shuffle and duality relations are in \( \mathcal{I}_{\text{CF}} \).

In this section we shall prove that the regularized double shuffle relations and the duality relations of the multiple zeta values are confluence relations. The results in this section give theoretical supports of our Conjecture 24.

4.1. The confluence relations imply the regularized double shuffle relations. We denote by \( \mathcal{I}_{\text{RDS}} \) the ideal of \( (\mathcal{A}^0, \omega) \) generated by the regularized double shuffle relations, i.e.,
\[ \{ \text{reg} \omega(u \uplus v - u \ast v) \mid u \in \mathcal{A}^1, v \in \mathcal{A}^0 \}. \]

Lemma 25. For \( c \in \{0, 1\} \),
\[ \partial_{z,c} \text{reg} \omega(w) = \text{reg} \omega(\partial_{z,c}w). \]

Proof. From \( \partial_{z,c}(e_1^c) = 0 \) and (1) of Theorem 8 we have
\[ \partial_{z,c}(u \uplus e_1^c) = \partial_{z,c}(u) \uplus e_1^c \quad (c \in \{0, 1\}). \]
Thus the lemma is proved. \( \square \)

Theorem 26. \( \mathcal{I}_{\text{RDS}} \subset \mathcal{I}_{\text{CF}} \).

Proof. Fix \( u \in \mathcal{A}^1 \) and \( v \in \mathcal{A}^0 \). Since \( \mathcal{I}_{\text{RDS}} \) is an ideal, it suffices to prove that
\[ \text{reg} \omega(u \uplus v - u \ast v) \in \mathcal{I}_{\text{CF}}. \]
Put
\[ f(w) := \text{reg} \omega(u \uplus w - u \ast w) \quad (w \in \mathcal{A}_1^0 \cap \mathbb{Z}\langle e_0, e_2 \rangle). \]
Then we have \( \partial_{z,c}f(w) = f(\partial_{z,c}w) \) for all \( c \in \{0, 1\} \) by Lemma 25 and (1), (2) of Theorem 8. Therefore, we can show that \( f(w) \in \mathcal{I}_{\text{ST}} \) by the induction on the degree of \( w \) since \( \text{Const}(f(w)) = 0 \). Especially, we have
\[ f(v_z) = \text{reg} \omega(u \uplus v_z - u \ast v_z) \in \mathcal{I}_{\text{ST}} \]
where \( v_z := v|_{1 \to z} \in \mathbb{Z} \oplus e_2 \mathbb{Q} \langle e_0, e_2 \rangle e_0 \). Since \( f(v_z) \in \mathcal{A}_z^{-2} \), we have
\[ \mathcal{I}_{\text{CF}} \ni \lambda(f(v_z)) = f(v_z)|_{z \to 1} = \text{reg} \omega(u \uplus v - u \ast v). \]
\( \square \)

4.2. The confluence relations imply the duality relations. Let \( \tau_\infty \) be an antiautomorphism of \( \mathcal{A} \) defined by \( \tau_\infty(e_0) = -e_1, \tau_\infty(e_1) = -e_0 \). Set \( \Delta(w) := w - \tau_\infty(w) \) and \( \Delta_z(w) := w - z(w) \). Let \( \mathcal{I}_\Delta \) denote the ideal of \( \mathcal{A}^0 \) generated \( \{ \Delta(w) \mid w \in \mathcal{A}^0 \} \). In this section we prove \( \mathcal{I}_\Delta \subset \mathcal{I}_{\text{ST}} \).

Lemma 27. For \( w \in \mathcal{A}_1^0 \), \( \phi(\Delta_z(w)) \in \mathcal{I}_\Delta \otimes \mathcal{I}_{\text{ST}} \).

Proof. For \( r \in \mathbb{Z}_{\geq 0} \) and \( a_1, \ldots, a_r \in \{0, 1\} \),
\[ \text{Const}(\partial_{z,a_1} \cdots \partial_{z,a_r} \Delta_z(w)) = \text{Const}(\Delta_z(\partial_{z,a_1} \cdots \partial_{z,a_r}w)) \]
for \( w \in \mathcal{A}_0^1 \) by (3) of Theorem 8. Since \( \text{Const}(\Delta_z(w)) = \Delta(\text{Const}(w)) \in \mathcal{I}_\Delta \) for \( w \in \mathcal{A}_z \), this proves the corollary. \( \square \)

Theorem 28. \( \mathcal{I}_\Delta \subset \mathcal{I}_{\text{ST}} \).

Proof. Put \( \mathcal{I}_\Delta^{(k)} := \{ \Delta(u) \mid u \in \mathcal{A}_1^0, \text{deg } u \leq k \} \). We prove the claim \( \mathcal{I}_\Delta^{(k)} \subset \mathcal{I}_{\text{ST}} \) by the induction on \( k \). Take \( u \in \mathcal{A}_1^0 \) such that \( \text{deg } u \leq k \). Since \( u \in \mathcal{A}_1^0 \), we can assume that \( u = e_1u'e_0 \) where \( u' \in \mathcal{A} \). Put \( w = \Delta(u) + \Delta((e_2 - e_1)u'e_0) \in \mathcal{A}_z^{-2} \). Then we have \( \text{Const}(w) = 0 \) and \( \phi(\Delta_z(w)) \in \mathcal{I}_\Delta^{(k-1)} \otimes \mathcal{A} \) from Lemma 27. Thus \( \text{reg} \omega(\phi(\Delta_z(w))|_{z \to 1}) \in \mathcal{I}_\Delta^{(k)} \otimes \mathcal{A} \subset \mathcal{I}_{\text{ST}} \) from the induction assumption. On the other hand, we have \( \text{reg} \omega(w|_{z \to 1}) = \Delta(u) \). Thus the theorem is proved since \( \Delta(u) = \text{reg} \omega((w - \phi(w))|_{z \to 1}) + \text{reg} \omega(\phi(w)|_{z \to 1}) \in \mathcal{I}_{\text{ST}} \). \( \square \)

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Appendix A. Proofs of Some Complementary Propositions

**Lemma 29.** For \( \alpha \in \{0,1,z\} \) and \( u \in A_2^0 \), \( \partial_{\alpha,a}(u) = 0 \).

**Proof.** Let \( u = e_{a_1} \cdots e_{a_n} \). Put \((a_0, a_{n+1}) = (0,1)\), \( S = \{1 \leq i \leq n \mid a_i = a_{i-1} = \alpha\} \) and \( T = \{1 \leq i \leq n \mid a_i = a_{i+1} = \alpha\} \). Then \( S = \{i+1 \mid i \in T\} \) since \( a_1 \not= a_0 \) and \( a_n \not= a_{n+1} \). From the definition of \( \partial_{\alpha,a} \), we have

\[
\begin{align*}
\partial_{\alpha,a} u &= \sum_{i \in S} e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_n} - \sum_{i \in T} e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_n} \\
&= \sum_{i \in S} e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_n} - \sum_{i \in T} e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_n} \\
&= 0.
\end{align*}
\]

\( \Box \)

**Proposition 30.** For \( u \in A_2^0 \), \( \partial_{z,0}(u) + \partial_{z,1}(u) + \partial_{1,0}(u) = 0 \).

**Proof.** Put \( u = e_{a_1} \cdots e_{a_n} \) and \( X = \{(0,0),(1,1),(z,z),(z,0),(z,1),(1,0)\} \). From the definition, we have

\[
\begin{align*}
\sum_{(\alpha,\beta) \in X} \partial_{\alpha,\beta}(u) &= \sum_{i=1}^{n} \sum_{(\alpha,\beta) \in X} \left( \delta_{\{a_i,a_{i+1}\},\{\alpha,\beta\}} - \delta_{\{a_{i-1},a_i\},\{\alpha,\beta\}} \right) e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_n} \\
&= \sum_{i=1}^{n} (1-1) e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_n} \\
&= 0.
\end{align*}
\]

Thus \( \partial_{z,0}(u) + \partial_{z,1}(u) + \partial_{1,0}(u) = 0 \) by Lemma 29.

\( \Box \)

**Proposition 31.** For \( u \in A \) and \( v \in A_2 \), \( \text{Const}(u \ast v) = u \ast \text{Const}(v) \).

**Proof.** Put \( I := A_z e_z A_z = \ker(\text{Const}) \). It suffices to show that \( \text{Const}(u \ast v) \in I \) for all monomials \( u \in A \) and \( v \in I \). We prove this by induction on the sum of degrees of \( u \) and \( v \). Put \( u = e_a u' \) and \( v = e_b v' \). Then from the induction hypothesis, we have \( u' \ast v \in I \), and \( u \ast v', u' \ast v' \in I \) if \( b \neq \not z \). If \( a = 0 \) then we have \( u \ast v \in I \) since \( e_0 u' \ast v = e_0(u' \ast v) \). Assume that \( a = 1 \). Then we have

\[
u \ast v = e_b (u' \ast v + u \ast v' - e_0(u' \ast v')) \in I
\]

since either \( e_b = e_z \) or \( u' \ast v + u \ast v' - e_0(u' \ast v') \in I \) holds.

\( \Box \)

**Proposition 32.** \( \varphi_z(A_2^0) \subset A_2 \otimes (\mathbb{Z}(e_0, e_z) \cap A_2^0) \).

**Proof.** From the definition of \( \varphi_z \), it is enough to prove that \( \text{Const}(\partial_{1,0}(u)) = 0 \) for \( u \in A_2^0 \). For a word \( w = e_{a_1} \cdots e_{a_n} \), put

\[
\deg_z(w) = \#\{1 \leq i \leq n \mid a_i = z\}.
\]
Then $\partial_{1,0}(A_2^{0,k}) \subset A_2^{0,k}$. Therefore $\text{Const}(\partial_{1,0}(u)) = 0$ for all $u \in \bigoplus_{k \geq 1} A_2^{0,k}$. Since $\partial_{1,0} = -\partial_{z,0} - \partial_{z,1}$, $\partial_{1,0}(u) = 0$ for $u \in A_2^{0,0}$. This proves the claim.

**Proposition 33.** For $u, v \in A_2^0$,

$$\varphi_\shuffle(u \shuffle v) = \varphi_\shuffle(u) \shuffle \varphi_\shuffle(v).$$

**Proof.** By (1) of Theorem 8,

$$\partial_{1,b_1} \cdots \partial_{1,b_r}(u \shuffle v) = \sum_{k=0}^r \sum_{\sigma \in S_{k,r-k}} \partial_{1,b_{\sigma(1)}} \cdots \partial_{1,b_{\sigma(k)}}(u) \shuffle \partial_{1,b_{\sigma(k+1)}} \cdots \partial_{1,b_{\sigma(r)}}(v)$$

where $S_{k,r-k} := \left\{ \sigma \in \mathfrak{S}_r \mid \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(r) \right\}$. Since

$$\text{Const}(w \shuffle w') = \text{Const}(w) \shuffle \text{Const}(w')$$

for $w, w' \in A_2$, we have

$$\varphi_\shuffle(u \shuffle v) = \sum_{k,l \in \mathbb{Z}_{\geq 0}} \sum_{\sigma \in S_{k,l} \cap \{0,z\}} \text{Const}(\partial_{b_{\sigma(1)}} \cdots b_{\sigma(k)}(u)) \shuffle \text{Const}(\partial_{b_{\sigma(k+1)}} \cdots b_{\sigma(r)}(v))$$

$$= \sum_{\sigma \in S_{k,l}} \text{Const}(\partial_{a_1} \cdots a_k(u)) \shuffle \text{Const}(\partial_{a_{k+1}} \cdots a_{k+l}(v))$$

For each $\sigma \in S_{k,l}$, we change the labeling of the suffixes by $b_{\sigma(i)} = a_i$ for $1 \leq i \leq k + l$. Thus,

$$\sum_{\sigma \in S_{k,l}} \text{Const}(\partial_{b_{\sigma(1)}} \cdots b_{\sigma(k)}(u)) \shuffle \text{Const}(\partial_{b_{\sigma(k+1)}} \cdots b_{\sigma(r)}(v))$$

$$= \sum_{\sigma \in S_{k,l}} \text{Const}(\partial_{a_1} \cdots a_k(u)) \shuffle \text{Const}(\partial_{a_{k+1}} \cdots a_{k+l}(v))$$

Since $\sum_{\sigma \in S_{k,l}} e_{a_{\sigma(1)}} \cdots e_{a_{\sigma(k+l)}} = e_{a_1} \cdots e_{a_k} \shuffle e_{a_{k+1}} \cdots e_{a_{k+l}}$, it follows that $\varphi_\shuffle(u \shuffle v) = \varphi_\shuffle(u) \shuffle \varphi_\shuffle(v)$. □

**Proposition 34.** For $u \in A^0$ and $v \in A_2^0$,

$$\varphi_\#(u \ast v) = u \ast \varphi_\#(v).$$

**Proof.** By (2) of Theorem 8 we have

$$\varphi_\#(u \ast v) = \sum_{r \geq 0} \sum_{b_1, \ldots, b_r \in \{0,z\}} \text{Const}(u \ast \partial_{1,b_1} \cdots \partial_{1,b_r}) \ast e_{b_1} \cdots e_{b_r}$$

$$= u \ast \sum_{r \geq 0} \sum_{b_1, \ldots, b_r \in \{0,z\}} \text{Const}(\partial_{1,b_1} \cdots \partial_{1,b_r}) \ast e_{b_1} \cdots e_{b_r}$$

$$= u \ast \varphi_\#(v).$$

□

**Proposition 35.** The homomorphism $f : A_2^{-2} \otimes (A_2^0 \cap \mathbb{Z}\langle e_1, e_2 \rangle) \simeq A_2^0$ defined by

$$f(u \otimes v) = u \shuffle v$$

is bijective.

**Proof.** Let $X_k = \left\{ u \in A_2^0 \cap \mathbb{Z}\langle e_1, e_2 \rangle \mid \deg u \leq k \right\}$. Put $A = A_2^{-2} \otimes (A_2^0 \cap \mathbb{Z}\langle e_1, e_2 \rangle)$. Define filtrations $F_k$ on $A$ and $A_2^0$ by $F_k A = A_2^{-2} \otimes X_k$ and $F_k A_2^0 = A_2^{-2} \otimes X_k$. Put $\text{gr}_k A = F_k A / F_{k-1} A$ and $\text{gr}_k A_2^0 = F_k A_2^0 / F_{k-1} A_2^0$. Then the induced map $\text{gr}_k f : \text{gr}_k A \to \text{gr}_k A_2^0$ of each graded piece is obviously bijective since the shuffle product map $u \otimes v \mapsto u \shuffle v$ is equal to just a concatenation map $u \otimes v \mapsto u v$. □
Appendix B. A Table of confluence relations

By virtue of the explicit representation of the standard relations (Proposition 12 (1)), the confluence relations can be explicitly given as
\[ \{ \lambda (w - \varphi_\omega (w)) \mid w \in A_\omega^0 \}. \]

Thus, we give a table of the confluence relations up to weight 4 as follows. We omit the case \( w \in \mathbb{Z} \langle e_0, e_1 \rangle \cup \mathbb{Z} \langle e_0, e_2 \rangle \cup \mathbb{Z} \langle e_1, e_2 \rangle \) since \( \lambda (w - \varphi_\omega (w)) = 0 \) in this case. For each \( w \in A_\omega^0 \), \( L(\lambda (w - \varphi_\omega (w))) = 0 \) where
\[ L(e_1 e_0^{k_1-1} \cdots e_1 e_0^{k_d-1}) = (-1)^d \zeta (k_1, \ldots, k_d). \]

For example, the table below says that we can obtain a relation
\[ -3 \zeta (4) + 5 \zeta (2, 2) + 13 \zeta (1, 3) - 4 \zeta (1, 1, 2) = 0 \]
from the case \( w = e_1 e_2 e_1 e_0 \).

| weight | \( w \) | \( \lambda (w - \varphi_\omega (w)) \) | weight | \( w \) | \( \lambda (w - \varphi_\omega (w)) \) |
|--------|---------|--------------------------------|--------|---------|--------------------------------|
| 3      | \( e_2 e_1 e_0 \) | \( -e_1 e_0^2 - e_1^2 e_0 \) | 4      | \( e_1 e_2 e_0 \) | \( 2 e_1 e_0 e_1 e_0 + 6 e_1^2 e_0^2 + 3 e_1 e_0^3 \) |
|        | \( e_1 e_2 e_0 \) | \( 2 e_1 e_0^2 + 2 e_1^2 e_0 \) |        | \( e_1 e_2 e_0 \) | \( -2 e_1 e_0 e_1 e_0 - 6 e_1^2 e_0^2 - 3 e_1 e_0^3 \) |
|        | \( e_2 e_0 e_1 e_0 \) | \( 3 e_1 e_0 + 5 e_1 e_0 e_1 e_0 + 13 e_1^2 e_0^2 + 4 e_1^3 e_0 \) |        | \( e_2 e_0 e_1 e_0 \) | \( 3 e_1 e_0 + 5 e_1 e_0 e_1 e_0 + 13 e_1^2 e_0^2 + 4 e_1^3 e_0 \) |
|        | \( e_2 e_0 e_1 e_0 \) | \( -e_1 e_0^2 - 4 e_1^2 e_0 \) |        | \( e_2 e_0 e_1 e_0 \) | \( -3 e_1 e_0 - 2 e_1 e_0 e_1 e_0 - 6 e_1^2 e_0^2 \) |
|        | \( e_2 e_1 e_2 e_0 \) | \( -2 e_1 e_0 e_1 e_0 - 6 e_1^2 e_0^2 - 3 e_1 e_0^3 \) |        | \( e_2 e_1 e_2 e_0 \) | \( e_1 e_0 e_1 e_0 + e_1^2 e_0^2 \) |
|        | \( e_2 e_1 e_0 e_0 \) | \( 4 e_1^2 e_0^2 + e_1 e_0^3 \) |        | \( e_2 e_1 e_0 e_0 \) | \( e_1 e_0 e_1 e_0 + e_1^2 e_0^2 \) |
|        | \( e_2 e_1 e_0 e_0 \) | \( -e_1 e_0^2 - e_1 e_0 e_1 e_0 - e_1^2 e_0 \) |        | \( e_2 e_1 e_0 e_0 \) | \( e_1 e_0 e_1 e_0 + e_1^2 e_0^2 \) |
|        | \( e_2 e_1 e_0 e_0 \) | \( -e_1 e_0^2 - e_1 e_0 e_1 e_0 - e_1^2 e_0 \) |        | \( e_2 e_1 e_0 e_0 \) | \( e_1 e_0 e_1 e_0 + e_1^2 e_0^2 \) |

Table 1. The table of \( \lambda (w - \varphi_\omega (w)) \) for a monomial \( w \).

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