Triviality of Entanglement Entropy in the
Galilean Vacuum

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Abstract

We study the entanglement entropy of the vacuum in non-relativistic
local theories with Galilean or Schrödinger symmetry. We clear some
confusion in the literature on the free Schrödinger case. We find that
with only positive $U(1)$ charge particles (states) and a unique zero $U(1)$
charge state (the vacuum) the entanglement entropy must vanish in that
state.

1 Introduction

Entanglement entropy is a property of quantum systems described by a Hilbert
space. Given a state $|\psi\rangle$, one defines the entanglement entropy between two
subsystems $A$ and $B$ to be the von-Neumann entropy of the density matrix
$\rho = |\psi\rangle \langle \psi| \traced over one of the subsystems. Entanglement entropy has been
intensively studied in relativistic theories in which the area law has been demon-
strated explicitly [1] and the holographic interpretation of entangle ment was
founded [2].

Galilean field theories are theories where the spacetime symmetries are Galilean
rather than Lorentzian. Recall that the Galilean algebra contains a central
charge $M$ generating the particle number symmetry. It is related to the other
spacetime symmetries by the commutator

$$[P_i, K_j] = -i\delta_{ij} M.$$ (1)

This central charge is responsible for many Galilean phenomena different from
what we are used to in relativistic field theories [3]. See [4–14] for related recent
Galilean and Schrödinger field theory papers.

Recently, the interest in entanglement entropy in Galilean field theories has
raised and a few works have been published in which the entanglement entropy is
computed in Galilean framework using different methods. In particular, in [15]
a computation using the heat-kernel method and an argument using a Lifshitz

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holographic dual are given for the case of free Schrödinger field theory. We suspect both arguments are not appropriate for the free Schrödinger operator. The first due to an ill-defined Schrödinger operator and the second by using a non-Schrödinger dual. Note that a later work [16] going forward with more free Schrödinger computations, based on the same inappropriate method.

It’s possible that the computations done in [15] could be used to study a different case, but not the free Schrödinger case. The problem with the operator in [15] was mentioned briefly in [17] and some of the fundamental ingredients leading to the correct result for the free case were mentioned there as well, and in this paper we would like to elaborate on that. We present arguments for the triviality of the entanglement entropy under certain conditions in the Galilean vacuum and emphasize the importance of the particle number symmetry generator $M$.

2 Free Schrödinger

We claim that the entanglement entropy of a subset of space in the Schrödinger vacuum state is zero. Recall that given a representation of the Hilbert space as a product of two Hilbert spaces

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

and given a (pure) state $|\psi\rangle$ in $\mathcal{H}$, the entanglement entropy of $|\psi\rangle$ with respect to $A$ (or $B$) is defined to be

$$S(A) = -\text{Tr}(\rho_A \log \rho_A)$$

where $\rho_A$ is the density matrix $\rho = |\psi\rangle \langle \psi|$ reduced to the subspace $\mathcal{H}_A$ by tracing over the complement subspace $\mathcal{H}_B$. Note that the entanglement entropy is defined on a fixed time.

First, we claim that if the two subspaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are completely uncorrelated on the given state $|\psi\rangle$, then the entanglement entropy should vanish. By complete uncorrelation we mean that every correlation function that involves operators defined on either $\mathcal{H}_A$ or $\mathcal{H}_B$ is given by the product of the correlation functions on $\mathcal{H}_A$ and $\mathcal{H}_B$ separately.

Second, in the free Schrödinger theory, the equal time two point function vanishes on separated points, indeed, in the free case it is well known that

$$\langle \phi(\vec{x}_1, t) \phi(\vec{x}_2, t) \rangle \sim \delta(\vec{x}_1 - \vec{x}_2).$$

Third, we claim that every $n$-point function can be factorized to the $A$ part and the $B$ part. We can use Wick’s theorem to write the $n$-point function as a sum of products of two point functions. Every term that involves a two point function that mixes $A$ and $B$ necessarily vanishes because $A \cap B = \emptyset$. Therefore, any $n$-point function may be factorized to the $n_A$- and $n_B$-point function, i.e., $n$-point functions are separable.

Therefore, we conclude that the free Schrödinger vacuum has zero entanglement entropy for every subset $A$.

Actually [17], the fundamental reason for the vacuum state in the free Schrödinger field theory to be entanglement free is that the Hilbert space has a basis in terms of a set of particles localized in space

$$|\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\rangle.$$
The vacuum state is the state with no particles, or with zero $U(1)$ charge, and it is the only such state. In comparison, relativistic field theory doesn’t have states with completely localized particles.

We can look at a subspace $A$ and there we also have such a basis provided that $\vec{x}_i \in A$, and similarly for $B$. Therefore, the vacuum state of the full space can be written as $|0\rangle = |0\rangle_A \otimes |0\rangle_B$, and thus, obviously, the vacuum state is not entangled, tracing over $B$ leaves us with a pure state $|0\rangle_A$.

3 Theories with Galilean Symmetry

From the above argument we should expect that a Schrödinger theory is entanglement free in the vacuum state if the vacuum has, and is the only state to have, zero $U(1)$ charge.

We want to generalize the above correlation functions argument to not-necessarily free theories. Note that the two point function must satisfy

$$\langle \phi(\vec{x}_1, t_1) \phi(\vec{x}_2, t_2) \rangle = e^{i \frac{m}{2} (\vec{x}_2 - \vec{x}_1)^2} f(t_2 - t_1) \tag{6}$$

To prove that, we shall use space and time translation invariance as well as boost invariance (of the theory and of the vacuum) \[18\].

This expression is not well defined when $t_2 = t_1$. To see that this is zero for $t_2 = t_1$ on separated points $\vec{x}_1 \neq \vec{x}_2$ we can regularize the space dependence by integrating over a small region of $\vec{x}_2 - \vec{x}_1$ and take the limit $t_2 - t_1 \to 0$.

When we do that, unless $\vec{x}_2 = \vec{x}_1$ and provided that the function $f$ diverges polynomially, we get zero – since the exponent phase varies rapidly, if one integrates this phase factor around a small region of $x$s (thus, inserting a regulator to find the limit of equal times) multiplied by a (diverging) power function, the integral would vanish (in the limit of equal times).

Generalizing to $n$-point functions isn’t trivial because in the non-free case we have loop integrals over the whole spacetime. Indeed, we must use some knowledge about the particle density in the problem, otherwise, using non-zero chemical potential, one may construct examples with non-zero entanglement entropy.

We’ll therefore adopt a more general approach. Let’s prove that in a local field theory with Galilean symmetry, with only positive $U(1)$ charge particles (states), and a unique state with zero $U(1)$ charge (the vacuum), there is no entanglement entropy in that state.

If the decomposition of the vacuum to a superposition of product states on $A$ and $B$ is $|0\rangle = |0\rangle_A |0\rangle_B$ then clearly there is no entanglement, because after tracing over $B$ we get the pure state $|0\rangle_A$. For that not to be the case one must have a non-trivial decomposition $|0\rangle = \sum |i\rangle_A |j\rangle_B$ where $|i\rangle_A$ and $|j\rangle_B$ some states in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. By charge decomposition one has

$$M_0 = M_A + M_B + M_{\text{boundary}} \tag{7}$$

1It’s important that we first take the limit $t_2 - t_1 \to 0$ and only then we take the regularization parameter, i.e., the space integration region, to zero.

2Here we use charge decomposition which is a direct result of a charge density existence assumption which is common in Galilean field theories. For example, in the free case, $M_A$ and $M_B$ can be easily defined algebraically via their eigenvalues in the basis of localized particles – a localized particles state with $n_A$ particles in region $A$ and $n_B$ particles in region $B$ is an eigenstate of $M_A$ and $M_B$ with eigenvalues $n_A$ and $n_B$ respectively. $M_{\text{boundary}}$ is mentioned
but since $M_0 = 0$ and $M$ is non-negative (it must be non-negative on $A$, $B$ and the boundary as well), one must have $M_i = M_j = 0$, and since there is a unique $U(1)$ charge state, one gets $|0\rangle_A^i |0\rangle_B^j$.

Maybe as an explanatory example we can look again at the free case. When we decompose the Hilbert space to $\mathcal{H}_A$ and $\mathcal{H}_B$, the basis for these spaces is formed of localized particles in $A$ and localized particles in $B$ (the boundary may be taken separately but we will avoid this unnecessary complication here). The $M$ charge for every state can be written as the sum of $M_A$ and $M_B$ (and $M_{\text{boundary}}$) all of which must be non-negative. There is also uniqueness of $M = 0$ states in $A$, $B$ (and the boundary), so the vacuum must be just the product of the two vacua which proves entanglement freedom of the vacuum.

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