In this talk I shall introduce our recent works on general pairing interactions and pair truncation approximations for fermions in a single-$j$ shell, including the spin zero dominance, features of eigenvalues of fermion systems in a single-$j$ shell interacting by a $J-$pairing interaction.
1 Introduction

It is my great pleasure to talk to you here. I would like to thank the organizers, especially Dr. Bruno Gruber. I am extremely glad to see many of my friends again today in this beautiful city Bregenz.

My talk consists of four subjects.

(1) Spin $0^+$ ground state dominance
(2) Pair approximations for fermions in a single-$j$ shell
(3) Regularities of states in the presence of $J_{\text{max}}$-pairing
(4) Solutions for cases of $n = 3$ and 4 with $H_J$

2 $0^+$ ground state dominance

A preponderance of $0^+$ ground states was discovered by Johnson, Bertsch and Dean in 1998 [1] using the two-body random ensemble (TBRE), and was related to a reminiscence of generalized seniority by Johnson, Bertsch, Dean and Talmi in 1999 [2]. These phenomena have been confirmed in different systems [3, 4].

Let us take a simple systems consisting of four particles in a single-$j$ shell. The Hamiltonian that we use is as follows.

$$H_J = \sum J G_J A_J^{\dagger} \cdot A_J^{\dagger} \equiv \sum J \sqrt{2J + 1} \left( A_J^{\dagger} \times A_J^{\dagger} \right)^{(0)},$$

$$A_J^{\dagger} = \frac{1}{\sqrt{2}} \left( a_J^{\dagger} \times a_J^{\dagger} \right)^{(J)}, A_J = (-1)^M \frac{1}{\sqrt{2}} [\tilde{a}_J \times \tilde{a}_J]^{(J)}.$$  \hspace{1cm} (1)

where $G_J$ is given by

$$G_J = \langle j^2 | V | j^2 J \rangle.$$  

Here $V$ is a two-body interaction.

We have used a two-body random ensemble to confirm this interesting phenomenon and discovered an empirical method to predict the probability of a ground
state to have a spin $I$ [5]. We keep only one $G_J$ to be $-1$ and all others 0:

$$G_{J'} = -\delta_{JJ'}.$$}

We then diagonalize the Hamiltonian to find the angular momenta which give the lowest eigenvalues. They are shown in Table I. We count how many different $G_J$’s give the largest eigenvalue to an angular momentum $I$. The number is denoted as $N_I$. For example for $j = \frac{21}{2}$ and $n = 4$, $N_0=5$, $N_2=N_4=N_8 = N_{20}=N_{28}=N_{36}=1$ and all others are equal to 0. The total number of different $G_J$’s is $N = \frac{2j+1}{2}$. Then the $I$ g.s. probability is approximately predicted as

$$P_{\text{pred}}(I) = \frac{N_I}{N}.$$  \tag{2}$$

Fig. 1 shows a comparison between $P_{\text{pred}}(0)$ and $P_{\text{TBRE}}(0)$ which is obtained by diagonalization of a TBRE Hamiltonian for four fermions in a single-$j$ shells. Fig. 2 shows a comparison of between $P_{\text{pred}}(I)$ and $P_{\text{TBRE}}(I)$ for examples of various systems.

One can see that the agreements between the $P_{\text{pred}}(I)$ and $P_{\text{TBRE}}(I)$ are very good. It is therefore important to diagonalize $H$ with $G_{J'} = -\delta_{JJ'}$. For this purpose we introduce the $J$–pair approximation for low-lying states.

3 Pair Approximation for Fermions in a single-$j$ shell

Our Hamiltonian is defined as

$$H_J = -A^{J\dagger} \cdot A^J.$$  \tag{3}$$

We first point out that the low-lying eigenvalues of $H_J$ can be approximated by wavefunctions of pairs with spin $J$.

$$\Phi(I) = \frac{1}{\sqrt{N}} \left[A^{J\dagger} \times A'^{J\dagger} \times \cdots \times A^{J\dagger}\right]^{(I)} |0\rangle,$$  \tag{4}$$
where $\frac{1}{\sqrt{N}}$ is the normalization factor. It is very easy to prove that the $J$-pair truncation describes the low-lying states exactly in three body systems.

Fig. 3(a) shows the spin of the ground state of $j^4$ configuration for $G_6 = -1$. The ground states with spin 0 are obtained by exact shell model calculations and by the $J$-pair approximation. Fig. 3 (b) shows the similar thing for $G_{14} = -1$.

Fig. 4 shows energy levels obtained by shell model calculation and by the $J$-pair approximation when $j = 25/2$, $J = 14$ and $n = 4$. For the low-lying states, the pair approximation is very good. Giving the four low-lying states, two of them compete to be the ground state. These energies are almost the same in both exact shell model calculation and the pair approximation. This is why we failed to predict the ground state in this case.

For the $n=5$ and 6 cases that we have examined, the low-lying states are reasonably well approximated by the $J$-pair truncation.

So far $J$ is general, between 0 and $2j - 1$. Now let us take a very special value, $J_{\text{max}} = 2j - 1$. For $H = H_{\text{max}} = H_{2j-1}$, the $I = I_{\text{max}} = 4j - 6$ is the lowest, and $I = I_{\text{rmmax}} - 2$ state is the second lowest. These two states can be constructed by using pairs with angular momentum either $J_{\text{max}}$ or $J_{\text{max}} - 2$.

However, pairs with angular momentum $J_{\text{max}} - 2$ do not present a good approximation of the other $I$ states, while those with angular momentum $J_{\text{max}}$ do. For example, for $n = 4$, $|J_{\text{max}}^2, I = 0\rangle$ is exact but $|(J_{\text{max}} - 2)^2, I = 0\rangle$ is not exact, $|J_{\text{max}}^2, I(\leq j)\rangle$ is almost exact ($\sim -2$) but $|(J_{\text{max}} - 2)^2, I(\leq j)\rangle$ are not.

4 Regularities of states in the presence of $H_{J_{\text{max}}}$

We first point out that eigenvalues of low $I$ states ($n = 3, 4, 5$) are approximately integers. This can be proved in terms of six-$j$ symbols for $n = 3$ [6]. For $n = 4$, one can prove this in terms of nine-$j$ symbols [7].

Another regularity may be examplified below by $j = 21/2$ and $n = 3$ and 4. Among many states of $n = 4$ with the same $I$, the lowest eigenvalue is expressed as $E_I$ (obtained by a shell model diagonalization). The $E_I$ of four fermions in a single-$j$
(\(j = 21/2\)) shell with \(I\) between 18 to 25 are as follows. (When \(I\) is smaller than 18 there is no eigenvalue lower than \(-2\).) The eigenvalue of the \(I_{\text{max}}^{(3)}\) state with three fermions in the same single-\(j\) shell is \(-\frac{60}{26} = -2.26923076923077\). From Table II, one sees that the \(E_I\)'s of \(n = 4\) with \(18 \leq I \leq 25\) are very close to \(E_{I_{\text{max}}^{(3)}}\) and also very close to that of an \(I\) state constructed by 

\[
\Psi_I = \left[ a_j^\dagger \times a_j^\dagger \times a_j^\dagger \right]^{(I_{\text{max}}^{(3)})}\left[ I \right]
\]

We have calculated overlaps between the above states of \(n = 4\) and the \(\Psi_I\). They are almost 1 within a precision of \(10^{-5}\). This phenomenon have been confirmed for \(n\) up to 6 (\(j \geq 11/2\)).

5 Solutions for the case of \(n = 3\)

We take the following basis for three fermions

\[
|j^3[jJ]\rangle = \frac{1}{\sqrt{N_{jjj}^{(I)}}} \left( a_j^\dagger \times A^{j^I} \right)_{M} |0\rangle,
\]

where \(N_{jjj}^{(I)}\) is the diagonal matrix element of the normalization matrix

\[
N_{jjj}^{(I)} = \langle 0 | \left( a_j \times A^{j'} \right)_{M} (a_j^\dagger \times A^{j^I})_{M} |0\rangle.
\]

In general this basis is over complete and the normalization matrix may have zero eigenvalues for a given \(I\). Here \(J\) is not necessarily equal to \(J_{\text{max}}\).

The \(N_{jjj}^{(I)}\) and \(\langle j^3[jK']I, M|H_J|j^3[jK]I, M\rangle\) can be evaluated analytically:

\[
N_{jjj}^{(I)} = \delta_{J',J} + 2\hat{J}_{J',j} \left\{ \begin{array}{ccc} J & j' & I \\ J & j & J \end{array} \right\},
\]

\[
\langle j^3[jK']I, M|H_J|j^3[jK]I, M\rangle = -\frac{1}{\sqrt{N_{jK'I,jK'}^{(I)}N_{jjj}^{(I)}}} N_{jK'I,jK}^{(I)} N_{jjj}^{(I)}.
\]

where \(\hat{L}\) is a short hand notation of \(\sqrt{2L + 1}\).

For a fixed \(J\) and any \(I\), we construct one state \(|j^3J : I\rangle = |j^3[jJ]I\rangle\) and all other states \(|j^3K : I\rangle\), which are orthogonal to \(|j^3J : I\rangle\), as follows:

\[
|j^3K : I\rangle = |j^3[jK]I\rangle - \frac{N_{jjj}^{(I)}}{\sqrt{N_{jjj}^{(I)}N_{jjj}^{(I)}}} |j^3[jJ]I\rangle, \ (K \neq J),
\]

\[
|j^3J : I\rangle = |j^3[jJ]I\rangle.
\]
One easily confirms that all matrix elements of the Hamiltonian, $\langle j^3K' : I|H_J|j^3K : I \rangle$, are zero, except for $K' = K = J$:

$$\langle j^3[ j]I|H_J|j^3[j]I \rangle = -N_{j,j,j}^{(I)} = -1 - 2(2J + 1) \left\{ J \quad j \quad I \right\}.$$

Thus, all the eigenvalues of $H_J$ for $n = 3$ with any angular momentum $I$ are zero except for the state with one pair of spin $J$, which has the eigenvalue $E_I^{(j)} = -N_{j,j,j}^{(I)}$.

As by-products, we obtain a number of sum rules for six-$j$ symbols. The procedure to derive these sum rules is straightforward. As is well known, the summation of all eigenvalues with a fixed $I$ is equal to $\frac{n(n-1)}{2}$ times the number of $I$ states, where $n$ is the particle number. For $n = 3$, the number of states can be expressed in a compact formula [8].

We use $E_I^{(j)}$ to denote the non-zero eigenvalue of $H = H_J$ for any $I$, we have that

$$\sum_J E_I^{(j)} = \sum_J \left[ -1 - 2(2J + 1) \left\{ j \quad I \quad J \right\} \right]$$

$$= -\frac{n(n-1)}{2} D(j^3, I).$$

For $I \leq j$ ($j$ is a half integer),

$$\sum_{J=\text{even}} 2(2J + 1) \left\{ j \quad I \quad J \right\} = 3 \left[ \frac{2I+3}{6} \right] - I - \frac{1}{2},$$

where $[\ ]$ means to take the largest integer not exceeding the value inside.

Our new sum rules of six-$j$ symbols will be given in [7] in details.

6 Summary

In this talk, I have discussed four interesting aspects concerning general pairing interactions and pair truncation approximations for fermions in a single-$j$ shell. I first discuss an empirical rule to predict the spin $I$ ground state probability. I then
show that pairs with spin $J$ are reasonable building blocks for the low-lying states of a Hamiltonian with an attractive $J$-pairing interaction only. I also present two interesting regularities of eigenvalues of Hamiltonian with $J_{\text{max}}$-pairing interaction: for low $I$ states of $n$ up to 5 we found that the eigenvalues are asymptotic integers; some of $n = 4$ states may be tracted back to $n = 3$. Finally I prove for the case of $n = 3$ the eigenvalues are written in terms of six-$j$ symbols. This result presents new sum rules of six-$j$ symbols.

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Figure Captions:

Figure 1  Comparison between $P^{\text{pred}}(0)$ and $P^{\text{TBRE}}(0)$ of four fermions in a single-$j$ shell. The solid squares are obtained by 1000 runs of a TBRE Hamiltonian and the open squares are predicted by Eq. (2).

Figure 2  Comparison between $P^{\text{pred}}(I)$ and $P^{\text{TBRE}}(I)$ for more complicated systems. The solid squares are obtained by 1000 runs of a TBRE Hamiltonian and the open squares are predicted by Eq. (2).

Figure 3  Ground state spin $I$ for four fermions in a single-$j$ shell for $J = 6$ in (a) and 14 in (b) as a function of $j$. The solid squares are obtained by diagonalized the $H_J$ in the full shell model space and open squares are obtained by truncating the space with two pairs with spin $J$ only.

Figure 4  A comparison of low-lying spectra with two pairs with spin $J = 14$ (the column on the left hand side) and by a diagonalization of the full space (the column in the middle and the column on the right hand side) for the case of four nucleons in a single-$j$ ($j = 25/2$) shell. The middle column plots the shell model states which are well reproduced by the two $J = 14$ pairs, and the right column plots the shell model states which are not well reproduced by two $J = 14$ pairs. All the levels below $0^+_1$ in the full shell model space are included. One sees that the low-lying states with $I = 2^+_1$, $6^+_1$, $12^+_1$, and $10^+_1$ are well reproduced.
Table I  The angular momenta which give the lowest eigenvalues when $G_J = -1$
and all other parameters are 0 for 4 fermions in single-$j$ shells.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $G_{20}$ |
|------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|
| 7    | 0     | 4     | 2     | 8     |       |         |         |         |         |         |         |
| 9    | 0     | 4     | 0     | 0     | 12    |         |         |         |         |         |         |
| 11   | 0     | 4     | 0     | 4     | 8     | 16      |         |         |         |         |         |
| 13   | 0     | 4     | 0     | 2     | 2     | 12      | 20      |         |         |         |         |
| 15   | 0     | 4     | 0     | 2     | 0     | 0       | 16      | 24      |         |         |         |
| 17   | 0     | 4     | 6     | 0     | 4     | 2       | 0       | 20      | 28      |         |         |
| 19   | 0     | 4     | 8     | 0     | 2     | 8       | 2       | 16      | 24      | 32      |         |
| 21   | 0     | 4     | 8     | 0     | 2     | 0       | 0       | 0       | 20      | 28      | 36      |

Table II  A comparison between eigen-energies obtained by diagonalizing $H_{J_{\text{max}}}$
in the full shell model space (the column “(SM)” and matrix elements $\langle \Psi_I | H | \Psi_I \rangle$ (column “$F_I$”).

| $I$  | $E_I$ (SM)          | $F_I$ (coupled)       |
|------|---------------------|-----------------------|
| 18   | -2.26923076925915   | -2.26923076923498     |
| 19   | -2.26923076930701   | -2.26923076930702     |
| 20   | -2.26923078555239   | -2.26923077167687     |
| 21   | -2.26923078386646   | -2.26923078385375     |
| 22   | -2.26923245245008   | -2.26923102362432     |
| 23   | -2.26923165420128   | -2.26923165276669     |
| 24   | -2.26930608933736   | -2.26924197057701     |
| 25   | -2.26925701778767   | -2.26925692933680     |
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$0_{\text{g.s.}}$ Probabilities (%)
a) $TBR_E$, pred. $j=9/2$ shell with 4 fermions

b) $TBR_E$, pred. $j=9/2$ shell with 5 fermions

c) $TBR_E$, pred. 7 fermions in the $j_1=7/2, j_2=5/2$ orbits

d) $TBR_E$, pred. 10 $sd$ bosons system
Angular momentum \( I \) in the ground state.

\( G_6 \) and \( G_{14} \) are compared in the plots.

**Figure a)**

**Figure b)**
pair truncation

shell model

energy levels

-2.5

-2.0

-1.5

0+

18+

28+

0+

21+

15+

18+

13+

17+20+

8+

15+

19+

23+

11+

17+11+25+

19+

16+

24+

14+

26+

20+

18+

22+

14+

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