Boundary TBA Equations for a Non-diagonal Theory

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Abstract

We compute the boundary entropies for the allowed boundary conditions of the $SU(2)$-invariant principal chiral model at level $k = 1$. We used the reflection factors determined in a previous work. As a by-product we obtain some miscellaneous results such as the ground-state energy for mixed boundary conditions as well as the degeneracies of the Kondo model in the underscreened and exactly screened cases. All these computations are in perfect agreement with known results.
1 Introduction

Boundary integrable models in (1 + 1)-dimensions have attracted some attention in recent years, especially in view of their successful application to quantum impurity problems. The typical examples are the Kondo model [12], dissipative quantum mechanics [36], quantum Hall liquids with constriction [37] and the Callan-Rubakov model [21]. The common minimal denominator in these situations is the fact that the bulk theory is conformally invariant and it is the boundary that is responsible for the broken scale invariance. Here, our purpose is to consider the alternative situation, where the boundary respects the conformal invariance of the theory and the renormalization group (RG) trajectory is controlled by a bulk perturbation. The model in question is the SU(2) principal chiral model (PCM) at level \( k = 1 \). In a previous work [27], we proposed the set of permissible boundary conditions suggested by the symmetries of the problem and computed the corresponding minimal reflection factors that are compatible with the bulk scale invariant limit (Kondo problem). In this work, we wish to go one step further and study the finite size effects both in the infrared (IR) and ultraviolet (UV) limits, in analogy with the bulk problem [10]. This is done by boundary thermodynamic Bethe ansatz (TBA) [17] - [19], [26]. To motivate our conjecture, we compute the boundary entropies for the quasi-particles of the Kondo model in the exactly screened and underscreened cases. This has already been done previously [43] - [48]. However, this explicit computation will be used as a means of comparison for further generalizations.

For one of the boundary conditions the boundary degeneracy is shown to be noninteger. Furthermore, the typical decrease of entropy along the RG flow from the unstable UV fixed point to a more stable IR one is found to be respected. This \( g \)-theorem was conjectured for theories that are scale invariant in the bulk [20]. However, it also seems to hold here.

Finally, we use our results to conjecture the form of the ground state energy for mixed boundary conditions in the IR limit. It is found to be in perfect agreement with boundary conformal field theory predictions [24].

2 The principal chiral model and the Kondo effect

The integrability of the PCM has been studied by various authors [1], [1], [15], [16]. The RG analysis reveals that it interpolates between two fixed points. The crossover between the two limiting behaviours introduces a mass scale that breaks the conformal invariance. The RG trajectory terminates at the IR fixed point where the theory becomes massless at all distances. It is characterized by a conformal field theory based on two \( SU(2)_k \) Kac-Moody algebras (at level \( k \)). Al.B.Zamolodchikov and A.B.Zamolodchikov [10] proposed the background factorized scattering in terms of massless particles that leads to the correct TBA equations for \( k = 1 \). Zamolodchikov’s \( c \)-function was shown to take the values \( c_{UV} = 3 \) and \( c_{IR} = 1 \) at the fixed points [10], [35]. The latter coincides with...
the central charge of the $SU(2)_1$ conformal field theory $[3]$. The multi-channel Kondo model ($[12], [13], [38]-[48]$), on the other hand, consists of a $k$-tuple of $(1 + 1)$ free massless fermions on the half-line antiferromagnetically coupled to a fixed impurity of spin $S$ sitting at the boundary ($x = 0$) $[3]$. Let us denote the coupling constant by $\lambda$. The RG flow interpolates between an unstable UV fixed point, where the impurity is decoupled ($\lambda = 0$) and a strongly coupled IR one, where the spin of the impurity is screened ($\lambda = \frac{2k+2}{\pi}$). The effective spin thus becomes $q = S - \frac{k}{2}$. As before, the crossover introduces a scale $T_K$ called the Kondo temperature, $[39]-[42]$. For $k = 1$, the bulk spectrum of both theories consists of stable massless particles: left- and right-movers. At level $k = 1$ the kink structure of the particles and boundary impurity is eroded. The on-mass-shell momenta of the particles are parametrised in terms of the rapidity variables $-\infty < \beta, \beta' < \infty$:

$$
\begin{align*}
E = p = \frac{M}{2} e^\beta, & \quad \text{for right-movers,} \\
E = -p = \frac{M}{2} e^{-\beta'}, & \quad \text{for left-movers.}
\end{align*}
$$

The right- and left-movers are represented by the symbols $R_a(\beta)$ and $L_{\bar{b}}(\beta')$, respectively, where $a, \bar{\beta} = \pm$ are the $SU(2)$ isotopic spin indices. The $SU(2)$ invariant $R$-$R$ scattering $[10]$ is given by:

$$
R_a(\beta_1)R_b(\beta_2) = S_{ab}^{cd}(\beta_1 - \beta_2)R_d(\beta_2)R_c(\beta_1),
$$

with

$$
S_{ab}^{cd}(\beta) = \sigma_T(\beta)\delta_a^c\delta_b^d + \sigma_R(\beta)\delta_a^d\delta_b^c.
$$

The transition ($\sigma_T(\beta)$) and reflection amplitudes ($\sigma_R(\beta)$) satisfy:

$$
\begin{cases}
\sigma_T(\beta) = \frac{i}{\pi} \beta \sigma_R(\beta), \\
\sigma_R(\beta) = -\frac{i}{\beta - i\pi} S_V(\beta),
\end{cases}
$$

where

$$
S_V(\beta) \equiv \sigma_T(\beta) + \sigma_R(\beta) = \frac{\Gamma \left( \frac{1}{2} + \frac{\beta}{2\pi} \right) \Gamma \left( -\frac{\beta}{2\pi} \right)}{\Gamma \left( \frac{1}{2} - \frac{\beta}{2\pi} \right) \Gamma \left( \frac{\beta}{2\pi} \right)},
$$

is the 2-particle amplitude in the isovector channel. All the previous formulae hold for the $L$-$L$ scattering as well. In the Kondo model the $R$-$L$ scattering is trivial. However, for the PCM, we have:

$$
R_a(\beta)L_{\bar{b}}(\beta') = U(\beta - \beta')L_{\bar{b}}(\beta')R_a(\beta),
$$

with

$$
U(\beta) = \tanh \left( \frac{\beta}{2} - \frac{i\pi}{4} \right).
$$

\footnote{A summary of the results presented here can be found in ref. [4].}
Notice that when $\beta \to \pm \infty$, this amplitude becomes trivial.

Let us now consider a reflecting boundary that preserves the integrability. In the Kondo problem we distinguish two situations [13]. In the exactly screened case, the impurity spin is completely screened yielding a zero effective spin ($q = 0$). The impurity is thus a $SU(2)$ singlet. In the underscreened case ($S = 1$), the effective spin is $q = 1/2$ and the impurity becomes a $SU(2)$ doublet.

In the former case, we define the operator $B$ which, acting on the vacuum $|0\rangle$, creates a boundary state $|B\rangle$ [24], i.e.:

$$|B\rangle = B|0\rangle.$$  \hspace{1cm} (8)

The reflection matrix $R_a^b(\beta)$ is the amplitude corresponding to a right-mover of rapidity $\beta$ and spin $a$ being reflected into a left-moving state of rapidity $-\beta$ and spin $\bar{b}$:

$$R_a^b(\beta) B = R_{\bar{b}}^a(\beta) L_{\bar{b}}(-\beta) B.$$  \hspace{1cm} (9)

The fact that the boundary impurity is a $SU(2)$ singlet implies the following diagonal form:

$$R_a^b(\beta) = \delta_a^b R_K(\beta),$$  \hspace{1cm} (10)

where

$$R_K(\beta) = U(\beta) = \tanh \left( \frac{\beta}{2} - \frac{i\pi}{4} \right).$$  \hspace{1cm} (11)

Similarly, for the underscreened case, we introduce the operator $B_b$ ($b = \pm$), which acting on the vacuum creates a boundary state with spin $b$. The reflection amplitude $R_{ab}^{cd}(\beta)$ is:

$$R_a(\beta) B_b = R_{ab}^{cd}(\beta) L_c(-\beta) B_d,$$  \hspace{1cm} (12)

with

$$R_{ab}^{cd}(\beta) = f_K(\beta) \delta_a^c \delta_b^d + g_K(\beta) \delta_a^d \delta_b^c;$$  \hspace{1cm} (13)

and

$$\begin{cases} f_K(\beta) = \frac{1}{\pi} \beta g_K(\beta), \\ g_K(\beta) = \sigma_K(\beta). \end{cases}$$  \hspace{1cm} (14)

Here it is understood that $\beta$ stands for $\beta_Q - \beta_K$, where $\beta_Q$ is the rapidity of the quasi-particle and $\beta_K$ is associated with the Kondo temperature, $T_K = e^{\beta_K}$.

In the case of the PCM, we know that in the IR limit there are two boundary conditions compatible with the conformal symmetry [24], [25]. Each one of them is associated with a primary field of the $SU(2)_1$ conformal field theory. These fields have conformal dimensions $\Delta = 0$ and $\Delta = 1/4$ and have been identified with the identity operator and the field $g$ of the Wess-Zumino-Witten action, respectively [14].

2Sum over repeated indices is understood.
The 'fixed' boundary condition is associated with a reflection amplitude of the form (9), (10) except that this time:

\[
R_P(\beta) = \exp \left( -\frac{i\pi}{4} \right) \frac{\sinh \left( \frac{\beta}{2} - i\frac{\pi}{8} \right)}{\sinh \left( \frac{\beta}{2} + i\frac{\pi}{8} \right)}.
\]  (15)

Similarly, there is an amplitude \( R^{cd}_{ab}(\beta) \) associated with the 'free' boundary condition, which is given by (12), (13), (14) and this time:

\[
g_P(\beta) = iR_P(\beta)\sigma_R(\beta).
\]  (16)

3 TBA in the \( L \)-channel

3.1 Fixed boundary conditions

We start by considering a system of \( N \) right-movers with rapidities \( \beta_1, \ldots, \beta_N \) and \( M \) left-movers with rapidities \( \beta'_1, \ldots, \beta'_M \) in an interval of length \( R \) with fixed boundary conditions on both ends. Since the reflection matrix is diagonal on both sides, we can assume that the Bethe wave function vanishes at the two extremes of the interval. We can thus impose a "standing wave quantization condition" [18]. Mathematically, this condition can be expressed in the form:

\[
e^{2iU(\beta_k)R} R^2(\beta_k) \prod_{i=1}^N U(\beta_k + \beta_i) \prod_{j=1}^M U(\beta_k - \beta'_j) \times \\
\sum_{a,b} T^h \left( \left\{ \beta \right\}_{1\ldots N} \right) \times \bar{T}^h \left( \left\{ \beta' \right\}_{1\ldots M} \right) = 1, \quad k = 1, \ldots, N.
\]  (17)

In the above equations it is understood that \( a = \bar{a} \) and \( b = \bar{b} \). \( R(\beta) \) is the reflection amplitude in eqs.(11) or (15). The term \( \frac{1}{U(\beta_k)} \) arises because the particle does not interact with itself. The transfer matrix \( T^h_{a}(u| \left\{ \beta \right\}_{1\ldots N}) \) is the \( 2^N \times 2^N \) matrix defined by:

\[
T^h_{a}(u| \left\{ \beta \right\}_{1\ldots N}) = S^a_{b_1} (u - \beta_1) S^a_{b_2} (u - \beta_2) \cdots S^a_{b_N} (u - \beta_N)
\]  (18)

and is represented in fig.1.

Each dot represents an interaction described by the amplitude (3). Following the method described in ref. [10], we start by defining the pseudo-energies \( \varepsilon_n \) and the functions \( L_n \) in the thermodynamic limit,

\[
\rho_0 = \frac{e^{-\varepsilon_0}}{1 + e^{-\varepsilon_0}},
\]

\[
\rho_n = \frac{e^{-\varepsilon_n}}{1 + e^{-\varepsilon_n}}, \quad n = 1, 2, \ldots, \infty,
\]

\[
L_n = \log \left( 1 + e^{-\varepsilon_n(\beta)} \right), \quad n = 0, 1, \ldots, \infty,
\]  (19)

where \( \rho_n \) is the density of \( n \)-magnons, \( \Lambda_n \) the density of \( n \)-magnon states and \( \tilde{\Lambda}_n = \Lambda_n - \rho_n \) is the density of holes. The scattering matrices for particle-magnon (\( S_n \)) and magnon-magnon (\( S_{nm} \)) interactions are:
\[ S_n(y - \beta) = \frac{y - \beta - i\pi/2}{y - \beta + in\pi/2}, \]

\[ S_{nm}(y) = \frac{y + i\pi(m+n)/2}{y - i\pi(m+n)/2} \times \left[ \frac{y + i\pi(m+n-2)/2}{y - i\pi(m+n-2)/2} \right] \times \frac{y + i\pi(m-n)/2}{y - i\pi(m-n)/2}, \tag{20} \]

with kernels:

\[ \phi_n(y) = -i \frac{\partial}{\partial y} \log S_n(y), \quad \tag{21} \]

\[ \phi_{nm}(y) = -i \frac{\partial}{\partial y} \log S_{nm}(y). \]

The standard procedure leads to the set of Bethe equations:

\[ -LM e^\beta + \epsilon_0 + \frac{1}{2\pi} \varphi \ast L_0 + \frac{1}{2\pi} \varphi \ast L_1 = 0, \]

\[ \epsilon_n + \frac{1}{2\pi} \varphi \ast \sum_{m=0}^{\infty} l_{mn} L_m = 0, \quad n = 1, 2, \ldots, \infty, \tag{22} \]

where

\[ L_0(\beta) = L_0(-\beta), \tag{23} \]

and \( l_{mn} \) (\( m, n = 0, 1, \ldots, \infty \)) is called the incidence matrix. Its elements vanish except for adjacent nodes (fig.2). The unified kernel \( \varphi(y) \) is given by:

\[ \varphi(y) = \frac{1}{\cosh y}. \tag{24} \]
The asymptotic values $x_n \equiv e^{-\varepsilon_n(-\infty)}$, $y_n \equiv e^{-\varepsilon_n(+\infty)}$ are given by:

$$1 + x_n = \frac{\sin^2\left(\frac{\pi(n+2)}{p+4}\right)}{\sin^2\left(\frac{\pi}{p+1}\right)}, \quad n = 0, 1, \ldots, p, \quad (25)$$

$$1 + y_n = \frac{\sin^2\left(\frac{\pi(n+1)}{p+3}\right)}{\sin^2\left(\frac{\pi}{p+3}\right)}, \quad n = 0, 1, \ldots, p. \quad (26)$$

The infinite system has been truncated for some $p$. Eventually, we shall take the limit $p \to \infty$.

The saddle point evaluation of the boundary free energy yields:

$$\log g^{(1)} = \frac{1}{4\pi} \int_{-\infty}^{+\infty} d\beta \Theta(\beta)L_0(\beta) + \text{const}, \quad (27)$$

where

$$g^{(1)} = \langle 0|B \rangle \quad (28)$$

is the boundary degeneracy and

$$\Theta(\beta) \equiv \frac{1}{i} \frac{\partial}{\partial \beta} \log R^2(\beta) - \frac{1}{i} \frac{\partial}{\partial \beta} \log U(2\beta). \quad (29)$$

In the IR limit, when $L \to \infty$, we see from (22) that $\varepsilon_0 \to \infty$ and so the first term on the right-hand side of eq.(27) vanishes. In the UV limit ($L \to 0$) we get:

$$\log(g^{(1)}_{UV}) = \frac{1}{2} \log(1 + e^{-\varepsilon_0(-\infty)}) + \text{const} =$$

$$= \frac{1}{2} \lim_{p \to \infty} \log \left[ \frac{\sin^2\left(\frac{2\pi}{p+4}\right)}{\sin^2\left(\frac{\pi}{p+1}\right)} \right] + \text{const} = \log 2 + \text{const}. \quad (30)$$

Which means that:

$$\frac{g^{(1)}_{UV}}{g^{(1)}_{IR}} = 2 \quad (PCM). \quad (32)$$

For the Kondo problem, the second term in eq.(29) vanishes and $R(\beta)$ is given by eq.(11). And so:

$$\frac{g^{(1)}_{UV}}{g^{(1)}_{IR}} = \lim_{p \to \infty} \left[ \frac{\sin^2\left(\frac{2\pi}{p+4}\right)}{\sin^2\left(\frac{\pi}{p+1}\right)} \right]^{\frac{1}{2}} = 2 \quad (\text{Kondo}). \quad (33)$$

This is in perfect agreement with refs. [43]-[48].
Before we conclude this section, let us make one remark. Notice that, from (27), the magnons do not contribute to the boundary entropy. This is because they account for the isotopic degrees of freedom. Since for fixed boundary condition, the impurity is a singlet, the magnons will not be included. For free boundary condition, on the other hand, the impurity becomes a doublet. Therefore its spin degrees of freedom contribute to the boundary degeneracy. We thus expect the magnons to play a role here. Indeed, this is what happens, as we shall see in the next section.

3.2 Free boundary conditions

Let us assume that our system is now subject to free boundary condition at the two extremes. We can assume that the standing wave condition (17) still holds if we have free boundary conditions at the two extremes of the interval. The following simple argument shows that this is indeed the case.

If the boundary satisfies the fixed boundary condition, the Bethe wave function vanishes there. Consequently, it is reflected off the boundary with a phase shift of $\pi$. The other boundary produces another shift of $\pi$, bringing the Bethe wave function back to its original form. The end result is the standing wave.

For a free boundary the Bethe wave function satisfies a Neumann boundary condition corresponding to zero flux of momentum across the boundary. Consequently, it remains unscathed upon reflection (no phase shift).

In summary, after two reflections, the Bethe wave function finds itself in its original shape both for free as well as fixed boundary conditions, provided the same condition holds at the two boundaries.

Proceeding with the same program that followed eq.(17), we obtain the following set of Bethe equations:

$$-LM e^{\beta} + \epsilon_0 + \frac{1}{2\pi} \varphi * L_0 + \frac{1}{2\pi} \varphi * L_1 = 0,$$

$$\epsilon_n + \frac{1}{2\pi} \varphi * \sum_{m=0}^{\infty} l_{mn} L_m = 0, \quad n = 1, 2, \cdots, \infty.$$  \hspace{1cm} (34)

The evaluation of the boundary free energy leads to

$$\mathcal{F}_{\text{boundary}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\beta \left\{ \Theta_0^{(2)}(\beta) L_0(\beta) + \Theta_1^{(2)}(\beta) L_1(\beta) \right\} + \text{const},$$  \hspace{1cm} (35)

where

$$\Theta_0^{(2)}(\beta) = \Theta^{(2)}(\beta) - \varphi^B(\beta),$$

$$\Theta_1^{(2)}(\beta) = \varphi^B(\beta).$$  \hspace{1cm} (36)

For the PCM:

$$\Theta_P^{(2)}(\beta) = \frac{1}{i} \frac{\partial}{\partial \beta} \log h_P(\beta) - \frac{1}{i} \frac{\partial}{\partial \beta} \log U(2\beta),$$  \hspace{1cm} (37)

7
\[ \phi^B_P(\beta) = -i \frac{\partial}{\partial \beta} \log h_P(\beta), \quad (38) \]

\[ \varphi^B_P(\beta) = \frac{1}{\cosh \beta}, \quad (39) \]

\[ h_P(\beta) = f_P(\beta) + g_P(\beta). \quad (40) \]

We then have:

\[ \frac{g_{UV}^{(2)}}{g_{IR}^{(2)}}(PCM) = \frac{(1 + x_0)^{\frac{1}{2}}(1 + x_1)^{\frac{1}{2}}}{(1 + y_0)^{\frac{1}{2}}} = \frac{3\sqrt{2}}{2}. \quad (41) \]

Let us check that our formulae yield the correct results for the Kondo model. First of all notice that \( h_K(\beta) = S_V(\beta) \), and so:

\[ \Theta^{(2)}_K(\beta) = -i \frac{\partial}{\partial \beta} \log S_V(\beta) = \phi^B_K(\beta). \quad (42) \]

Consequently, from (36), the first term in the integrand of eq.(35) vanishes. We are thus left with:

\[ \frac{g_{UV}^{(2)}}{g_{IR}^{(2)}}(Kondo) = \frac{(1 + x_1)^{\frac{1}{2}}}{(1 + y_1)^{\frac{1}{2}}} = \frac{3}{2}. \quad (43) \]

These results are in perfect agreement with ref.\[13\]-[18] for the underscreened case \( q = S - k/2 = 1/2 \).

4 TBA in the R-Channel

The previous work allows us to conjecture the set of TBA equations in the R-channel. In essence they are not very different from those in the L-channel. They arise from imposing periodic boundary conditions on all Cooper pairs of the form \( R_\alpha(\beta)L_\beta(\beta) \), [17]. If \( \rho_0(\beta) \) is the density of pairs with rapidity in a small vicinity around \( \beta \), then the TBA equations for these pairs and the \( n \)-magnons are:

\[ -\nu_0 + \epsilon_0 + \frac{1}{2\pi} \varphi \ast L_0 + \frac{1}{2\pi} \varphi \ast L_1 = 0, \]

\[ -\nu_1 + \epsilon_1 + \frac{1}{2\pi} \varphi \ast L_0 + \frac{1}{2\pi} \varphi \ast L_2 = 0, \]

\[ \epsilon_n + \frac{1}{2\pi} \varphi \ast \sum_{m=1}^{\infty} \tilde{l}_{mn} L_m = 0; \quad n = 2, 3, \cdots, \infty. \quad (44) \]

The matrix \( \tilde{l}_{mn} \) \( (m, n = 1, 2, \cdots, \infty) \) is obtained from \( l_{mn} \) by omitting the first node. In the L-channel the particles and magnons contribute with \( \Theta_0(\beta) \) and \( \Theta_1(\beta) \), respectively, to the boundary degeneracies. This contribution will translate to \( \nu_0 \) and \( \nu_1 \) in eq.(44). If \( \Theta_i(\beta) \), associated with a particular boundary condition \( \alpha \), is of the form \( \frac{1}{i\pi} \partial_{\beta} \log R_\alpha(\beta) - \frac{1}{2i\pi} \partial_{\beta} \log S(2\beta) \), then its contribution to \( \nu_i(\beta) \) reads \( -\log K_\alpha(\beta) \), where \( K_\alpha(\beta) = R_\alpha(i\pi/2 - \beta) \).
Suppose we have fixed boundary condition on one side and free on the other. The contribution of the free boundary to \( \nu_1 \) is \(-\log K_K(\beta) \) where \( K_K(\beta) = R_K(i\pi/2 - \beta) \) is given by the Kondo reflection amplitude (11). Its contribution to \( \nu_0 \) is \(-\log K_P(\beta) \), where \( K_P(\beta) = R_P(i\pi/2 - \beta) \) is given by eq.(15). In principle its contribution should be \(-\log h_P(\beta) \). However, the amplitude in the logarithm must be the one associated with the kernel in \( \Theta_0^{(2)} \), which was shown to be \( R_P \) at the end of the previous section (cf.eqs.(36), (42)). Finally, the fixed boundary does not contribute to \( \nu_1 \) and its contribution to \( \nu_0 \) amounts to \(-\log K_P(\beta) \).

Altogether, we have:

\[
\begin{align*}
\nu_0(\beta) &= RM e^\beta - \log [K_P(\beta)K_P(\beta)], \\
\nu_1(\beta) &= -\log K_K(\beta).
\end{align*}
\]

Let us define \( \lambda_0(\beta) = K_P(\beta)K_P(\beta), \lambda_1(\beta) = K_K(\beta), \lambda_n(\beta) = 1, n = 2, 3, \ldots, \infty \)
and

\[
\begin{align*}
\dot{\epsilon}_n(\beta) &= \epsilon_n(\beta) + \log \lambda_n(\beta), \\
\dot{L}_n(\beta) &= \log(1 + \lambda_n(\beta)e^{-\epsilon_n}).
\end{align*}
\]

We can then rewrite eq.(44) in the form:

\[
\begin{align*}
-M Re^\beta + \dot{\epsilon}_0 + \frac{1}{2\pi} \varphi \star \tilde{L}_0 + \frac{1}{2\pi} \varphi \star \tilde{L}_1 &= 0, \\
-m_n Re^\beta + \dot{\epsilon}_n + \frac{1}{2\pi} \varphi \star \sum_{m=0}^{\infty} l_{mn} \tilde{L}_m &= 0; \quad n = 1, 2, \ldots, \infty.
\end{align*}
\]

Now notice that \( \lambda_0(\beta) = \tilde{K}_P(\beta)K_P(\beta) = 1 \) and \( \lim_{\beta \to \pm \infty} \lambda_1(\beta) = \lim_{\beta \to \pm \infty} K_K(\beta) = \mp 1 \). If we define \( z_n = e^{-\epsilon_n(\pm \infty)} \), we get the following system in the limit \( R \to \infty \):

\[
\begin{align*}
z_0 &= 0, \\
z_1 &= 1, \\
1 + z_n &= \frac{\sin^2 \left[ \frac{\pi z_n - 1}{\pi \mp} \right]}{\sin^2 \left[ \frac{\pi z_n - 1}{\pi \pm} \right]}, \quad n = 2, 3, \ldots, p.
\end{align*}
\]

Similarly, if we define \( x_n = e^{\epsilon_n(-\infty)} \), in the region \(-RM < \beta\), we obtain the system (25), as \( R \to \infty \).

The standard evaluation of the free energy yields:

\[
\mathcal{F} = \frac{-L}{2\pi R} \sum_{n=0}^{\infty} \left\{ \mathcal{L} \left[ x_n \right] - \mathcal{L} \left[ z_n \right] \right\},
\]

where \( \mathcal{L}(x) \) is Rodger's dilogarithm [32]:

\[
\mathcal{L}(x) = -\frac{1}{2} \int_{0}^{x} dt \left[ \frac{\log t}{1 - t} + \frac{\log(1 - t)}{t} \right].
\]

Using the sum rules of ref. [32] we get for the ground state energy: \( E_0(R) = -\mathcal{F}(R) = \frac{5\pi}{2\mp} \).
On the other hand,

\[ \Delta_{\min} = \frac{R}{\pi} E_0(R) + \frac{c}{24} = \frac{1}{4}, \]  

(51)

which is one of the conformal dimensions of the \( SU(2)_1 \) conformal theory. This is in fact the correct one, according to the predictions of boundary conformal field theory, \([24], [25]\).

5 Conclusions

Let us restate our results. We computed the boundary degeneracies corresponding to the two allowed boundary conditions for the PCM at level \( k = 1 \). This was accomplished by quantizing the system in a finite box. We argued that for the free boundary condition, the Bethe wave function was subject to a Neumann condition preventing the flow of momentum across the boundary. This condition allows the exchange of isospin between the particles and the boundary impurity. Since the magnons are attached to these degrees of freedom, they contribute to the boundary degeneracy. This situation is in contrast with the fixed boundary condition, which does not permit isospin permutation. The free boundary condition was subsequently shown to yield a noninteger degeneracy.

As a consistency check, we compared our results in the bulk scale invariant limit \( (U(\beta) \rightarrow 1) \) with those of the Kondo model.

We stress that in all cases the boundary entropy decreases under renormalization from the more unstable UV fixed point to the more stable IR one.

Finally, we used the previous results to conjecture the form of the TBA equations in the \( R \)-channel and the ground-state energy, when we let our system evolve between two distinct boundary states. The boundary conformal field theory approach of Cardy \([24]\) predicts a partition function of the form \( Z_{\text{free, fixed}} = \chi^{(1)}_k \) in the IR limit. \( \chi^{(1)}_k \) denotes the \( SU(2) \) Kac-Moody character at level \( k = 1 \) associated with the conformal tower labeled by the primary operator with isospin \( l = \frac{1}{2} \). This operator has conformal dimension \([3]\) \( \Delta = \frac{l(l+1)}{k+2} = \frac{1}{2} \), in perfect agreement with our result (52). Some of our results may be tested numerically using the truncated systems (25), (26) and (48). We expect to carry out that program in a future work.

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\[ \text{The central charge for the PCM at level } k = 1 \text{ is } c = 1. \]
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