Spinorial Wheeler–DeWitt wave functions inside a horizon

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We revisit the solution of the Wheeler–DeWitt (WDW) equation inside the horizon of spherical black holes and planar topological black holes in arbitrary dimensions. For these systems, the solutions of the equations are found to have the same form. Therefore, Yeom’s Annihilation-to-nothing interpretation can be applied to each case. We have introduced the Dirac-type WDW equations into quantum cosmology in a recent paper, so we also apply our formulation to the quantum theory of the interior of the black hole in order to obtain the solution of the spinorial wave function. The shape of the wave packet of the spinorial WDW wave function indicates that the variation of Yeom’s interpretation holds in this scheme.

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I. INTRODUCTION

Theoretical study on the black hole singularity problem continues with various approaches. Although the quantum theory of gravity [1, 2] itself has not been completed, many authors apply the quantum cosmological approach (see, for example, Refs. [3–5] and references therein), which is a method of conducting quantum nature of gravity. This is because inside a symmetric horizon, the roles of the time ($t$) and the radial coordinates ($r$) are exchanged, so that the spacetime of the interior of a black hole is regarded as a time-dependent universe and can be described by the Kantowski–Sachs metric [6], for instance.

Recently, many works using the minisuperspace Wheeler–DeWitt (WDW) equation inside the horizon have appeared [7–13]. In particular, Yeom and his collaborators [7–10] have constructed a concrete wave packet and proposed the Annihilation-to-nothing interpretation based on its typical behavior. One of the motivations for the present study is to consider further various types of wave packets and to reveal the properties of the quantum spacetime inside the horizon.

In our previous paper, the authors of the present paper introduced the Dirac wave equation into quantum cosmology on the basis of the extended minisuperspace [14, 15]. We now apply our formulation to the quantum theory inside the horizon. Since this spinorial formulation leads to a positive definite probability density, it can be applied to further consideration of the singularity problem.

The rest of the paper is organized as follows. In Sec. II, we review the WDW equations inside the horizon of spherically symmetric black holes and planar topological black holes in general dimensions. We obtain their fundamental solutions and construct wave packets using them. In Sec. III, we introduce the Dirac-type wave equation as an alternative to the WDW equation. We construct the wave packets and compare them with the wave packets of scalar wave functions obtained in the previous section. We conclude the present paper with a summary of results and a discussion in the last section.
II. A REVIEW OF THE WDW EQUATION INSIDE A BLACK HOLE

In this section, we review the derivation of the minisuperspace WDW equation inside a black hole. First, we consider the Einstein–Hilbert action in \(D\)-dimensional \((D \geq 4)\) spacetime,

\[
S = \frac{1}{D-2} \int d^D x \sqrt{-g} R ,
\]

where \(g\) is the determinant of the metric tensor \(g_{\mu\nu}\), and \(R\) denotes the scalar curvature constructed from \(g_{\mu\nu} (\mu, \nu = 0, 1, \ldots, D-1)\).

As the metric of the black-hole interior, we assume the Kantowski–Sachs-type metric \([6]\), which represents a homogeneous space with spatial section of topology \(\approx S^{D-2} \times \mathbb{R}\),

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2n(t)} dt^2 + e^{2\alpha(t)} dr^2 + r_s^2 e^{2\beta(t)} d\Omega_{D-2}^2 ,
\]

where \(d\Omega_{D-2}^2\) is the line element of the \((D-2)\)-sphere with unit radius, and \(r_s\) is a constant with dimension of length. The metric (2.2) is related to the Schwarzschild–Tangherlini interior metric \([18]\) by, for example,

\[
e^{2\beta} = \frac{r^2}{r_s^2} , \quad e^{2\alpha} = \frac{r_s^{D-3} - 1}{r_s^{D-3} - 1} , \quad e^{2n} = \left( \frac{r_s^{D-3} - 1}{r_s^{D-3} - 1} \right)^{-1} ,
\]

where we should notice that the time and radial coordinates are mutually exchanged from the definition on the exterior solution.

Now, we derive the WDW equation for the interior of a spherical black hole. We should note that the lapse function \(n(t)\) can be arbitrarily chosen because of the reparametrization invariance on \(t\). Thus, we here choose

\[
n(t) = \alpha(t) + (D - 2)\beta(t) .
\]

Substituting the metric (2.2) with the gauge choice (2.4), we find

\[
S \propto \int dt \left[ -2\dot{\alpha} \dot{\beta} - (D - 3)\dot{\beta}^2 + (D - 3)r_s^{2(D-3)} e^{2\alpha + 2(D-3)\beta} \right] .
\]

Therefore, we obtain the reduced action

\[
S \propto \int dt \left[ \frac{1}{D - 3} (\dot{X}^2 - \dot{Y}^2) + (D - 3)r_s^{2(D-3)} e^{2Y} \right] ,
\]

1 Strictly speaking, the Gibbons–York terms \([16, 17]\) should be noted here, but is omitted as usual.

2 The overall constant factor can be absorbed in the redefinition of the time.
where \( X(t) = \alpha(t) \) and \( Y(t) = \alpha(t) + (D - 3)\beta(t) \), and consequently we obtain the following Hamiltonian of the system:

\[
H(X, Y, \Pi_X, \Pi_Y) = (D - 3) \left[ \frac{1}{4}(\Pi_X^2 - \Pi_Y^2) - r_s^{2(D-3)} e^{2Y} \right],
\]

where \( \Pi_X \) and \( \Pi_Y \) are the canonical momenta conjugate to \( X \) and \( Y \), respectively.

Now, the Hamiltonian constraint leads to the (minisuperspace) WDW equation for the scalar wave function \( \Phi(X, Y) \),

\[
\left( \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} + 4r_s^{2(D-3)} e^{2Y} \right) \Phi(X, Y) = 0,
\]

is obtained after the replacement of \( \Pi_X \rightarrow \hat{\Pi}_X = -i\frac{\partial}{\partial X} \) \( \Pi_Y \rightarrow \hat{\Pi}_Y = -i\frac{\partial}{\partial Y} \) in the quantum Hamiltonian constraint \( \hat{H}\Phi = 0 \) [1, 2, 19, 20]. Then, we just reproduced the equation presented by Yeom et al. [7–10]. Note that this equation is independent of the spacetime dimensions up to the power of \( r_s \).

Before discussing the solutions of the WDW equation, we consider another system, the interior of a planar topological black hole [21–25]. The action taken here has a negative cosmological term added,

\[
S_\lambda = \frac{1}{D-2} \int d^D x \sqrt{-g} \left[ R + 2\lambda \right].
\]

The metric of the interior of a planar topological black hole reads

\[
ds^2 = - \left( \frac{r_s^{D-3}}{t^{D-3} - \frac{t^2}{L^2}} \right)^{-1} dt^2 + \left( \frac{r_s^{D-3}}{t^{D-3} - \frac{t^2}{L^2}} - \frac{t^2}{L^2} \right) dr^2 + t^2 d\bar{\Omega}^2_{D-2},
\]

which is the classical solution of the Einstein equation obtained from varying the action (2.9). Here, the length scale \( L \) is given by \( L \equiv \left( \frac{2\lambda}{(D-1)(D-2)} \right)^{-1/2} \), and \( d\bar{\Omega}^2_{D-2} \) denotes the line element of a flat \((D-2)\) dimensional torus.

Substituting the previous metric (2.2), but with \( d\bar{\Omega}^2_{D-2} \) replaced by \( d\bar{\Omega}^2_{D-2} \), into the action (2.9), we obtain

\[
S \propto \int dt \left[ -2\dot{\alpha} \dot{\beta} - (D - 3)\dot{\beta}^2 + \frac{D-1}{L^2} r_s^{2(D-2)} e^{2\alpha + 2(D-2)\beta} \right].
\]

Further setting \( X(t) = \alpha(t) - \beta(t) \) and \( Y(t) = \alpha(t) + (D - 2)\beta(t) \) gives

\[
S \propto \int dt \left[ \frac{1}{D-1} (\dot{X}^2 - \dot{Y}^2) + \frac{D-1}{L^2} r_s^{2(D-2)} e^{2Y} \right].
\]
Therefore, the Hamiltonian for this system is found to be

\[
H = (D - 1) \left[ \frac{1}{4}(\Pi_X^2 - \Pi_Y^2) - \frac{1}{L^2} r_s^{2(D-2)} e^{2Y} \right],
\]

(2.13)

and the corresponding WDW equation reads [13]

\[
\left( \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} + \frac{4}{L^2} r_s^{2(D-2)} e^{2Y} \right) \Phi(X, Y) = 0.
\]

(2.14)

Note that the translation \( Y - \ln(L/r_s) \rightarrow Y \) just turns the equation (2.14) into the equation (2.8).³ Thus, we have only to study the WDW equation (2.8) for both systems, the interior of a spherical black hole and that of a planar topological black hole.

Note that the event horizon is located at \( X, Y \rightarrow -\infty \), while the singularity is located at \( X \rightarrow \infty \) and \( Y \rightarrow -\infty \), in both cases.

The fundamental solution of (2.8) is [7–10]

\[
\phi_k(X, Y) = e^{-ikX} K_{ik}(2r_s e^Y),
\]

(2.15)

and general solutions are written by

\[
\Phi(X, Y) = \int_{-\infty}^{\infty} f(k) \phi_k(X, Y) \, dk,
\]

(2.16)

where the function \( f(k) \) denotes the amplitude.

Asymptotics of the modified Bessel function of the second kind with complex order are known to be [26]

\[
K_{ik}(z) \sim \sqrt{2\pi} e^{-\frac{\pi}{2} (k^2 - z^2)^{-\frac{1}{4}}} \sin \left( \frac{\pi}{4} - \sqrt{k^2 - z^2} + k \cosh^{-1} \frac{k}{z} \right),
\]

(2.17)

for \( k \rightarrow \infty \). According to the analysis of Kiefer [27], the peak of the wave packet tracing the classical path [7–10]

\[
e^Y \cosh X = \text{const},
\]

(2.18)

can be constructed with the Gaussian amplitude, whose central value in \( k \) takes a relatively large value, owing to the asymptotic form (2.17).

Nevertheless, we can arbitrarily choose the amplitude \( f(k) \), so as to reflect boundary conditions. The Gaussian or rectangular amplitudes are the other candidates.⁴ Another

³ At the classical level, the redefinition of time can also absorb the constant factor in front of the potential term.

⁴ Graphical representation of wave packets of similar wave functions in quantum cosmology are dealt with Refs. [28–30] et al.
type of the wave packet with an analytical form is considered by Refs. [7–10], where the simple choice \( f(k) = ik \) leads to

\[
\Phi_1(X, Y) = \int_{-\infty}^{\infty} k \sin kX K_{ik}(2r_s^{D-3}e^Y) \, dk = 2\pi r_s^{D-3}e^Y \sinh Xe^{-2r_s^{D-3}e^Y} \cosh X. \tag{2.19}
\]

The peak of the absolute square of the wave packet \( \Phi_1(X, Y) \), which can be regarded as the probability density [1, 2, 19, 20], exhibits the classical behavior (2.18) at the boundaries \( X \to \pm \infty \), as shown in Fig. 1.

![Absolute square of the scalar wave function for \( r_s = 1 \).](image)

**FIG. 1.** Absolute square \(|\Phi_1|^2\) of the scalar wave function for \( r_s = 1 \).

It is worth observing that the quantum effect seems significant in the vicinity of \( X \approx 0 \), where classically there is neither a singularity nor a horizon: The probability density vanishes here \( |\Phi_1(0, 0)|^2 \approx 0 \). One can interpret this wave packet as an annihilation process of the trajectory from the horizon (\( X = -\infty \)) and that from the singularity (\( X = \infty \)). This interpretation is the so-called *Annihilation-to-nothing* interpretation proposed by Yeom et al. [7–10].

In this section, we have reviewed the scalar wave packet of the WDW equation for the interior of the black hole, leading to Yeom’s interpretation. We have confirmed that the same interpretation is possible for spherically symmetric and topological black holes in arbitrary dimensions.
III. SPINOR WAVE FUNCTION INSIDE A BLACK HOLE

In the previous work [14, 15], the present authors introduced the extended minisuperspace description with an additional degree of freedom by a method of the Eisenhart–Duval lift [32–41]. Using the covariance in the extended minisuperspace, we also constructed an associated Dirac equation for spinor wave function. The introduction of the Dirac-type WDW equation (dubbed spinorial WDW equation) is originally motivated for obtaining a positive-definite probability density [42–50]. The Dirac equation in the extended minisuperspace has a unique form and conformal covariance.

In this section, we apply the formulation to the present model for a black-hole interior. In short, the realization of the plan is achieved by finding the metric of the minisuperspace manifold such that the WDW equation is written by Laplace equation ($\nabla^2 \Phi = 0$), and drawing down the Dirac equation ($\slashed{D} \Psi = 0$) on that manifold.

The essential idea of the Eisenhart–Duval lift is to introduce the metric $G_{MN}$ of the extended space with an auxiliary dimension $Z$, that is, in the present case,

$$G_{MN} dX^M dX^N = 4r_s^{2(D-3)} e^{2Y} (-dX^2 + dY^2) + dZ^2,$$

(3.1)

where $X^M = (X,Y,Z)$, and consider the Laplace equation in the extended minisuperspace:

$$\frac{1}{\sqrt{-G}} \partial_M (\sqrt{-G} G^{MN} \partial_N \Phi) = \left[ \frac{1}{4r_s^{2(D-3)} e^{2Y}} \left( -\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) + \frac{\partial^2}{\partial Z^2} \right] \Phi = 0.$$  

(3.2)

Here, $G^{MN}$ is the inverse of $G_{MN}$, $G = -(4r_s^{2(D-3)} e^{4Y})^2$ is the determinant of $G_{MN}$, and the derivatives are expressed as $\partial_M \equiv \frac{\partial}{\partial X^M}$. If we further impose an additional (first-order) constraint

$$\frac{\partial^2 \Phi}{\partial Z^2} = -\Phi,$$

(3.3)

we can exactly reproduce the WDW equation (2.8).

The WDW equation of Dirac-type for a spinor wave function $\Psi$ in the extended minisuperspace is now naturally introduced by [14, 15]

$$\slashed{D} \Psi \equiv \gamma^M D_M \Psi \equiv \gamma^A e_A^M D_M \Psi = 0,$$

(3.4)

where the constant gamma matrices in the flat spacetime $\gamma^A$ ($A = 1, 2, 3$) are $\gamma^1 = \sigma^1$, $\gamma^2 = i\sigma^2$, and $\gamma^3 = i\sigma^3$ ($\sigma^1, \sigma^2, \sigma^3$ denote the Pauli matrices). Note that $\{\gamma^A, \gamma^B\} = -2\eta^{AB}$,  

5 A similar method has been developed by Robles-Pérez [31].
where \( \eta^{AB} = \eta_{AB} = \text{diag.}(1, 1, 1) \). Here, the dreibein \( e^A_M = \text{diag.}(2r_s^{D-3}e^Y, 2r_s^{D-3}e^Y, 1) \) is defined through \( \eta_{AB}e^A_Me^B_N = G_{MN} \), and \( e^M_A = \text{diag.}((2r_s^{D-3})^{-1}e^{-Y}, (2r_s^{D-3})^{-1}e^{-Y}, 1) \) is its inverse matrix. Subsequently, we find that \( \{\hat{\gamma}^M, \hat{\gamma}^N\} = -2G^{MN} \). The covariant derivative \( D_M \) for the spin connection \( \omega_{MAB} \) is defined as \( D_M \equiv \partial_M + \frac{1}{4}\omega_{MAB}\Sigma^{AB} \), where \( \Sigma^{AB} \equiv -\frac{1}{2}[\gamma^A, \gamma^B] \). The spin connection \( \omega_{MAB} \) is given by

\[
\omega_{MAB} = \frac{1}{2}e^N_A(\partial_M e_{NB} - \Gamma^L_{MN}e_{LB}) - (A \leftrightarrow B),
\]

(3.5)

where the Christoffel symbol \( \Gamma^L_{MN} \) is given by

\[
\Gamma^L_{MN} = \frac{1}{2}G^{LP}(\partial_M G_{PN} + \partial_N G_{PM} - \partial_P G_{MN}).
\]

(3.6)

In the present case, one can find \( \omega_{X12} = -\omega_{X21} = -1 \).

In the extended minisuperspace presently considered, we find that the Dirac equation (3.4) is equivalent to

\[
\left[ \sigma^1 \frac{\partial}{\partial X} + i\sigma^2 \left( \frac{\partial}{\partial Y} + \frac{1}{2} \right) + i\sigma^3 (2r_s^{D-3})^2 e^Y \frac{\partial}{\partial Z} \right] \Psi = 0.
\]

(3.7)

In order to reduce the equation to that of physical variables \( X \) and \( Y \), we choose the additional constraint on \( \Psi \):

\[
\frac{1}{i} \frac{\partial}{\partial Z} \Psi = \Psi.
\]

(3.8)

Now, the Dirac equation reads in the matrix form,

\[
\begin{pmatrix}
-2r_s^{D-3}e^Y & \frac{\partial}{\partial X} + \frac{1}{2} + \frac{\partial}{\partial Y} \\
2r_s^{D-3}e^Y & -\frac{\partial}{\partial X} - \frac{\partial}{\partial Y}
\end{pmatrix}
\begin{pmatrix}
\Psi_+ \\
\Psi_-
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

(3.9)

where \( \Psi = (\Psi_+ \Psi_-) \). The fundamental solutions of the equation are found to be

\[
\psi_{\pm,k} \propto \pm K_{ik\pm\frac{1}{2}}(2r_s^{D-3}e^Y)e^{-ikX},
\]

(3.10)

and general solutions can be written as

\[
\Psi_{\pm}(X,Y) = \int_{-\infty}^{\infty} f(k)\psi_{\pm,k}(X,Y) \, dk,
\]

(3.11)

where \( f(k) \) denotes the amplitude.

Since the inner product of two spinors in the curved minisuperspace can be defined by

\[
(\vartheta, \varphi) \equiv \int dY \sqrt{-G} |G^{XX}|^{1/2} \vartheta^\dagger \varphi = 2r_s^{D-3} \int dY e^Y \vartheta^\dagger \varphi,
\]

(3.12)
the (positive-definite) probability density $\rho(X, Y)$ should takes the form

$$\rho(X, Y) \propto e^Y (|\Psi_+(X, Y)|^2 + |\Psi_-(X, Y)|^2).$$  \hfill (3.13)

Asymptotics of the modified Bessel function of the second kind with complex order are known to be [51]

$$K_{ik \pm \frac{1}{2}}(z) \sim \sqrt{2\pi} e^{-\frac{1}{2} - \frac{i}{2} \pi} (k^2 - z^2)^{-\frac{1}{4}} \left[ \frac{\sqrt{k + z}}{2z} \sin \left( \frac{\pi}{4} - \frac{1}{2} \sqrt{y^2 + k \cosh^{-1} \frac{1}{z}} \right) \right. \left. \pm i \sqrt{\frac{k - z}{2z}} \cos \left( \frac{\pi}{4} - \sqrt{k^2 - z^2 + k \cosh^{-1} \frac{1}{z}} \right) \right],$$  \hfill (3.14)

as $k \to \infty$. This function has a similar oscillatory behavior as $K_{ik}(z)$ (2.17). The similarity in asymptotics of $K_{ik}(z)$ and $\sqrt{2} K_{ik \pm \frac{1}{2}}(z)$ also helps to construct wave packet solutions for the Dirac-type WDW equation as well as the usual WDW equation. The different behaviors of the functions $K_{ik}(z)$ and $\sqrt{2} K_{ik \pm \frac{1}{2}}(z)$ seems to be especially revealed in the vicinity of $X \approx 0$ in the form of the wave packets. Therefore, the difference between wave-packet solutions of the usual WDW equation and the Dirac-type WDW equation is expected to be found in the region of turning point of the classical trajectory.

We find that the simple choice $f(k) = ik$ in (3.11) leads to

$$(\Psi)_\pm(X, Y) = \pm \int_{-\infty}^{\infty} ike^{-ikX} K_{ik \pm \frac{1}{2}}(2r_s^{D-3}e^Y) \, dk$$

$$= \pm \pi \left[ 2r_s^{D-3}e^Y \sinh X \mp \frac{1}{2} \right] e^{\pm \frac{X}{2}} e^{-2r_s^{D-3}e^Y \cosh X}. \hfill (3.15)$$

The probability density $\rho_1(X, Y) = e^Y (|\Psi_+(X, Y)|^2 + |\Psi_-(X, Y)|^2)$ is exhibited in Fig. 2. The peak of the probability density reproduces the classical path where $|X|$ is large, but a local hump of the density appears around $X = 0$. It would be appropriate to interpret this feature of the probability density as “Annihilation-to-Something” process.

On the other hand, another simple choice $f(k) = 1$ in (3.11) leads to

$$(\Psi)_0\pm(X, Y) = \pm \int_{-\infty}^{\infty} e^{-ikX} K_{ik \pm \frac{1}{2}}(2r_s^{D-3}e^Y) \, dk$$

$$= \pm \pi e^{\pm \frac{X}{2}} e^{-2r_s^{D-3}e^Y \cosh X}. \hfill (3.16)$$

The probability density $\rho_0(X, Y) = e^Y (|\Psi_+(X, Y)|^2 + |\Psi_-(X, Y)|^2)$ is exhibited in Fig. 3. The peak of the density rather exactly reproduces the classical path, even where $X \approx 0$. It can be said that there is “no annihilation” process.
Although only simple amplitude functions have been assumed here, it turns out that the behavior of the density function in the region where the quantum effect is expected ($X \approx 0$) has more diverse for spinor waves than for scalar waves.

In this section, we have introduced the spinorial WDW equation for the interior of the black hole and constructed the typical wave packets. We have found that, in our spinorial formulation, we can create wave packets with various interpretations.
IV. SUMMARY AND DISCUSSION

We have solved the spinorial wave function (as well as the scalar wave function as a comparison) of the interior of the Schwarzschild–Tangherlini and planar topological black holes in $D$-dimensional spacetime, by using the extended minisuperspace. We have constructed the wave packets of the spinor wave function, which correspond to the classical trajectories in the asymptotic region near the horizon and the singularity.

The behavior of the scalar wave packet near $X \approx 0$ suggests the “Annihilation-to-nothing” interpretation proposed by Yeom et al. [7–10]. The probability density of the spinor wave function is slightly different from that of the scalar wave function, in the vicinity of $X = 0$ for typical examples of the packets. Thus, we speculate that alternative interpretation of “no annihilation” or “annihilation-to-something” is possible.

Since our spinorial formulation leads to a positive definite probability density, it will be applied to further analysis of the expectation values [13] near the singularity (even without using wave packets). We shall leave the issue for future study.

We should also incorporate possible noncommutativity in the variables of the minisuperspace (see, for example, Ref. [52] and references therein). The approach by considering loop quantum cosmology (see Refs. [4, 5] and Ref. [53] for references) (or polymer quantization) may be effective for the analysis at the singularity. We hope that the study in the aforementioned directions will be advanced in future.

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