Bulk and surface magnetoinductive breathers in binary metamaterials

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(Dated: May 27, 2009)

We study theoretically the existence of bulk and surface discrete breathers in a one-dimensional magnetic metamaterial comprised of a periodic binary array of split-ring resonators. The two types of resonators differ in the size of their slits and this leads to different resonant frequencies. In the framework of the rotating-wave approximation (RWA) we construct several types of breather excitations for both the energy-conserved and the dissipative-driven systems by continuation of trivial breather solutions from the anticontinuous limit to finite couplings. Numerically-exact computations that integrate the full model equations confirm the quality of the RWA results. Moreover, it is demonstrated that discrete breathers can spontaneously appear in the dissipative-driven system as a result of a fundamental instability.

PACS numbers: 41.20.Jb, 63.20.Pw, 75.30.Kz, 78.20.Ci

I. INTRODUCTION

Discrete breathers (DBs) or intrinsic localized modes, are time-periodic and spatially localized excitations that may be produced generically in discrete arrays of weakly coupled nonlinear elements [1, 2]. A large body of theoretical work has produced means of precise numerical analysis of coupled oscillator systems with breathers both in the Hamiltonian as well as dissipative regimes [3, 4, 5, 6, 7]. Breathers have been experimentally observed in several diverse systems, including optical waveguides systems [8], solid-state systems [9, 10], antiferromagnetic chains [11], Josephson junction arrays [12], and micromechanical oscillators [13, 14], among others. Discrete breathers can be generated spontaneously in a lattice either through stochastic mechanisms [15, 16] or by purely deterministic mechanisms [17, 18, 19, 20] in a process by which energy, initially evenly distributed in a nonlinear lattice, can be localized into large amplitude nonlinear excitations. Indeed, it has been demonstrated experimentally [13, 20] that the standard modulation instability (MI) mechanism in dissipative systems driven by an alternating term can initiate that process by the formation of low-amplitude breathers. The energy exchange between those low-amplitude DBs favors the higher-amplitude ones, resulting eventually in the formation of a few high-amplitude DBs.

A few years ago a whole new class of artificially structured materials, referred to as metamaterials, was discovered: the latter are comprised of discrete elements and exhibit electromagnetic properties not available in naturally occurring materials. A subclass of those metamaterials, the magnetic metamaterials (MMs), exhibit significant magnetic properties and sometimes even negative magnetic response up to Terahertz (THz) and optical frequencies [21, 22]. The most common realization of a MM is composed of periodically-arranged electrically small sub-wavelength particles, referred to as split-ring resonators (SRRs) [23, 24]. In its simplest form, each of those resonators is just a highly conducting metallic ring with one slit. The MM thus built can become nonlinear either by the insertion of a nonlinear dielectric [25] or a nonlinear electronic component (e.g., a varactor diode) [26, 27] in the slit of each SRR, resulting in voltage-dependent SRR capacitance. In microwave frequencies, such a MM has been realized [28] and is dynamically tunable by varying the input power. The combination of nonlinearity and the discreteness that is inherent in those metamaterials, makes possible the generation of nonlinear excitations in the form of DBs. The existence and stability of DBs in nonlinear SRR-based MM models, that are localized either in the bulk [29, 30] or at the surface [31, 32] of the MM, have been demonstrated numerically. Moreover, domain walls [33] and envelope solitons [34] may also be excited in such systems. The surface DBs in MMs are very similar with the surface modes observed in discrete waveguide arrays [35, 36].

Recently, a novel MM comprised of two types of SRRs was investigated theoretically [37], and it was demonstrated that in the nonlinear regime such binary MMs are perfectly suited for the observation of phase-matched parametric interaction and enhanced second harmonic generation (SHG). In the present work we extend the previous studies on DB generation in model SRR-based MMs to the case of a one-dimensional (1D) binary MM with on-site cubic nonlinearity. In practice, a binary MM can be constructed in many different ways, by changing for example one or more of the material and/or the geometrical parameters of the SRRs belonging to one type (say type a), with respect to the same parameters of the SRRs belonging to the other type (say type b). Here we allow for different slit-widths for the two types of SRRs, which makes their resonance frequencies differ by a factor that we call resonance frequency mismatch (RFM).
The considered binary MM is formed by type \( a \) SRRs at the even-numbered sites and type \( b \) SRRs at the odd-numbered sites of a periodic 1D array. In the next section we give the model equations that describe the dynamics of the binary MM and we obtain the corresponding linear dispersion relation for magnetoinductive waves in that medium \cite{45, 46}. In Sections III and IV we construct, using the rotating wave approximation (RWA), several types of Hamiltonian and dissipative breathers (DDBs), respectively. The dynamic stability of those DBs is discussed in Section V where the full model equations are integrated numerically. In most of the investigated cases the numerics confirm the quality of the RWA results. Moreover, in that Section we also demonstrate the spontaneous generation of DDBs induced by MI, using frequency chirping of the driving field. That procedure has simultaneous generation of DDBs induced by MI, using frequency chirping of the driving field. That procedure has

\begin{equation}
M \approx \frac{\mu_0 r_a^2 r_b^2}{4 D^3}
\end{equation}

where \( D \) is their center-to-center distance, and \( r_a, r_b \) are their average radii. In order to construct a binary array, we have to change one or more of the material and/or geometrical parameters of the SRRs that are going to be of one type, with respect to the same parameters of the SRRs that are going to be of the other type. As it can be observed from Eqs. (1) and (2):

- A change in the SRR radius \( r \) affects \( L, M \) and \( R \).
- A change in \( h \) affects \( L, C_l \) and \( R \).
- A change in \( \epsilon \) affects \( C_l \) which in turn, implies a change in the nonlinear response.
- A change in \( d \) affects \( C_l \) and slightly \( R \) and \( L \).
- A change in \( \rho \) affects \( R \).

Obviously there are many possibilities for constructing two types of SRRs and consequently a binary array. Here we make the relatively simple choice to create two types of SRRs by considering different slit-widths, i.e., \( d_a \) for type \( a \) and \( d_b \) for type \( b \). Thus, the linear capacitances of type \( a \) and \( b \) SRRs become respectively \( C_a \) and \( C_b \), resulting in different resonance frequencies \( \omega_a = 1/\sqrt{LC_a} \) and \( \omega_b = 1/\sqrt{LC_b} \).

Now let us return to the nonlinear problem, and assume that the slits of all the SRRs in the array are filled with a Kerr-type dielectric. Then, the charge \( Q_n \) accumulated in the capacitor of the \( n \)th SRR is \cite{40}

\begin{equation}
Q_n = C_n \left( 1 + \frac{U_n^2}{U_c^2} \right) U_n
\end{equation}

where \( C_n = C_a \) (\( C_n = C_b \)) for SRRs at even- (odd-) numbered sites of the array, and \( \chi = \alpha/(3\epsilon_l) \) is the dimensionless nonlinearity coefficient, with \( \alpha = +1 \) (\( \alpha = -1 \)) for a self-focusing (self-defocusing) dielectric. The above equation leads to the following approximate form for the voltage \( U_n \) across the slit of the \( n \)th SRR

\begin{equation}
U_n \approx \frac{Q_n}{C_n} \left[ 1 - \frac{1}{U_c^2} \left( \frac{Q_n}{C_n} \right)^2 \right]
\end{equation}

where \( U_c \) is a characteristic (large) voltage. Then, the coupled equations describing the charge dynamics in the nonlinear MM that is placed in an alternating magnetic...
field are generally written as
\[
\frac{d^2}{dt^2} [MQ_{2n-1} + L_2nQ_{2n} + MQ_{2n+1}] + R_2n \frac{d}{dt} Q_{2n} + \frac{1}{C_{2n}} Q_{2n} - \chi_0 \left( \frac{Q_{2n}}{C_{2n}} \right)^3 = F(t),
\]
where we assume that \(R_m = R, L_m = L\), and
\[
F(t) = E_0 \sin(\omega t).
\]

The above equation gives the electromotive force (emf) of amplitude \(E_0\) and frequency \(\omega\) that is excited in each SRR due to the action of the field. Let us define the quantities
\[
Q_c = \sqrt{C_u C_b U_c}, \quad \tau = \sqrt{\omega d_0} = \omega_0 t, \quad \gamma = R \sqrt{\frac{\sqrt{C_u C_b}}{L}}.
\]
With the above definitions, Eqs. (5) and (6) can be written in normalized form as
\[
\frac{d^2}{d\tau^2} [\lambda q_{2n-1} + q_{2n} + \lambda q_{2n+1}] + \delta q_{2n} - \chi^3 q_{2n}^3 = -\gamma \frac{dq_{2n}}{d\tau} + \epsilon_0 \sin(\Omega \tau), \quad (9)
\]
\[
\frac{d^2}{d\tau^2} [\lambda q_{2n+1} + q_{2n} + \lambda q_{2n+2}] + \frac{q_{2n+2}}{\delta} - \chi^3 q_{2n+1}^3 = -\gamma \frac{dq_{2n+1}}{d\tau} + \epsilon_0 \sin(\Omega \tau), \quad (10)
\]
where \(\lambda \equiv M/L\) is the dimensionless coupling parameter, \(\Omega = \omega_0/\omega\) is the dimensionless driving frequency, \(\delta \equiv \omega_n/\omega\) is the RFM ratio, \(\epsilon_0 = E_0/U_c\), and \(n\) is an integer. From Eqs. (9) and (10), we can see that the change in the linear capacitances also affects the nonlinear terms, and that, actually, the latter are affected much more than the linear ones. The changes that are caused to the terms proportional to \(R\) and \(L\) are of higher order and thus they are neglected. As the RFM \(\delta\) increases, the resonance frequency as well as the nonlinear term of the even-numbered site SRRs increase, while at the same time the resonance frequency and the nonlinear term of the odd-numbered site SRRs decrease. The inductive coupling parameter \(\lambda\) can be either positive or negative depending on whether the array geometry is of the “axial” or “planar” type.\(^{39}\)

Such MMIs and other systems with magnetically coupled elements, support in the low-amplitude (linear) limit a new kind of waves, the magnetoinductive waves.\(^{38,39}\) In the present case, the frequency dispersion for linear magnetoinductive waves is obtained by substituting
\[
q_{2n} = a e^{i(2n\kappa - \Omega \tau)}, \quad q_{2n+1} = b e^{i((2n+1)\kappa - \Omega \tau)}, \quad (11)
\]
where \(\kappa\) is the normalized wavenumber, into the linearized Eqs. (9) and (10), without the dissipative and the external driving terms (\(\gamma = 0\) and \(\epsilon_0 = 0\))
\[
\Omega^2 \pm \left[ \delta + \left( \frac{1}{\delta} \right) \pm \sqrt{\frac{\delta + \left( \frac{1}{\delta} \right)}{2(1 - 4\lambda^2 \cos^2 \kappa)}} \right.
\]
\[
\pm \left( \frac{1}{\delta} \right) \pm \sqrt{\frac{\delta + \left( \frac{1}{\delta} \right)}{2(1 - 4\lambda^2 \cos^2 \kappa)}} \right] + V_n.
\]

The dispersion curves for a particular choice of RFM \(\delta = 0.8\) and \(\lambda = -0.2\).

\[
\text{FIG. 2: (Color online) Linear dispersion relation for } \delta = 0.8 \text{ and } \lambda = -0.2.
\]

Even though we can think of each linear SRR as an RLC circuit, i.e., an electromagnetic oscillator, we notice that the dispersion curves for the coupled array do not contain any acoustic-like branch; rather both curves are of the “optical” type: \(\lim_{k \to 0} w(k) \neq 0\). This is due to the particular (inductive) nature of the coupling between the SRRs. There are now three regions where we can look for breathers in the nonlinear case: Above and below the bands, and in the Bragg gap (BG) in between. In the absence of RFM (\(\delta = 1\)) we recover the single band for the ‘monoatomic’ MM.\(^{30}\) As \(\delta\) diverges from unity the BG increases, pushing the \(\Omega_+\) branch upwards and the \(\Omega_-\) branch downwards.

Eqs. (9) and (10) can be written conveniently in the compact form
\[
\frac{d^2}{d\tau^2} [\lambda q_{n-1} + q_n + \lambda q_{n+1}] + \omega_n^2 q_n - \chi^3 q_n^3 = -\gamma \frac{dq_n}{d\tau} + \epsilon_0 \sin(\Omega \tau), \quad (13)
\]
where \(\omega_n = \delta (\omega_n^2 = 1/\delta)\) for even (odd) \(n\). Without dissipation and external driving, the earlier equation can be obtained from the Hamiltonian \(\mathcal{H} = \sum_n \mathcal{H}_n\), where the discrete Hamiltonian density \(\mathcal{H}_n\) is given by
\[
\mathcal{H}_n = \frac{1}{2} \left( \dot{q}_n^2 + \lambda \dot{q}_n (\dot{q}_{n-1} + \dot{q}_{n+1}) \right) + V_n. \quad (14)
\]

The last term on the right-hand side of the earlier equation is the nonlinear on-site potential which is given by
\[
V_n \equiv V(q_n) = \frac{1}{2} (\omega_n q_n)^2 \left[ 1 - \frac{1}{2} \chi \omega_n^2 (\omega_n q_n)^2 \right]. \quad (15)
\]
The Hamiltonian $\mathcal{H}$ is actually the conserved energy of the lossless system in the absence of any driving terms system. For the dissipative system, $\mathcal{H}$ is also useful since its time-average per period gives correctly the average energy per period for that system.

### III. HAMILTONIAN BREATHERS IN THE ROTATING-WAVE APPROXIMATION

A standard method of DB construction in Hamiltonian systems, that gives numerically exact results up to arbitrary precision, uses the Newton’s method [4, 5], which has been applied successfully for DB generation in MMs [29, 30]. In this Section, we use the RWA method that keeps a simple physical picture and moreover can produce quite accurate results. According to the simplest version of that method, one looks for stationary solutions of the system that are separable with an assumed time-dependence (e.g., sinusoidal) of the form $q_n(\tau) = q_n \sin(\Omega \tau)$. Direct substitution of $q_n(\tau)$ into Eq. (13) with the approximation $\sin(x)^3 \approx (3/4)\sin(x)$ gives an algebraic system of nonlinear equations for the $q_n$s that reads

$$-\Omega^2(\lambda q_{n+1} + q_n + \lambda q_{n-1}) + \omega_n^2 q_n - (3/4)\chi \omega_n^3 q_n = 0 \quad (16)$$

In the anticontinuous limit ($\lambda \to 0$) the earlier equation has the solutions

$$q_n = 0 \quad \text{or} \quad q_n^2 = \frac{\omega_n^2}{(3/4)\chi} \quad (17)$$

According to Eqs. (17), we have the following interesting possible scenarios:

(i) $\alpha > 0$. Then, $q_n^2 > 0$ for all $n$, if $\Omega^2 < \min\{\delta, 1/\delta\}$.

(ii) $\alpha < 0$. Then $q_n^2 > 0$ for all $n$ provided $\Omega^2 > \max\{\delta, 1/\delta\}$.

(iii) In the intermediate case where $\min\{\delta, 1/\delta\} < \Omega^2 < \max\{\delta, 1/\delta\}$, what happens is that $q_{2n} > 0$ and $q_{2n+1} = 0$, or the converse, depending upon the sign of $\alpha$.

The RWA method can be used for the construction of DBs both on the ‘surface’ and the bulk of the energy-conserved binary MM. For a 1D MM, a surface localized DB obviously corresponds to an edge state, i.e., a state with maximum amplitude at either of the two ends of the array. A bulk DB, on the other hand, is meant to be a DB whose maximum amplitude is far from the end-points of the array. However, the procedure of obtaining DBs by the RWA method, both at the surface or in the bulk, proceeds in the same way. The first step is to set up a trivial DB. We first choose its central site, i.e., the site where the DB shall have its maximum amplitude. Suppose that the central site is taken at $n = n_B$ where the ‘coordinate’ $q_n = q_{n_B}$ and set all the $q_n$ for $n \neq n_B$ equal to zero. The value of $q_{n_B}$ is calculated from Eq. (17), with an appropriately chosen $\Omega$. That solution is subsequently continued for finite couplings up to a maximum value $\lambda = \lambda_{max}$ where DBs cease to exist. Usually we consider a DB to be localized around the site where it exhibits its maximum amplitude. For a finite array, the boundary conditions that should be imposed to the dynamic equations resulting from the Hamiltonian (14) should be specified. For DBs excited in the bulk one may use either periodic or open-ended boundary conditions, since DBs are highly localized entities and are not affected by the boundaries. However, for surface DBs the termination of the structure should be taken into account, and for that purpose we should use open-ended boundary conditions. In the following, we always use that type of boundary conditions, i.e., $q_0 = 0$, $q_{N+1} = 0$, where $N$ is the total number of SRRs in the binary array.

**Surface breathers.** Typical surface-localized, single-site DBs, that are generally very similar to the ones examined for a discrete nonlinear Schrödinger (DNLS) model for a semi-infinite binary waveguide array [41], are displayed in Fig. 3. The untagged modes shown there originate in the lower gap region $0 < \Omega < \min[\delta, 1/\delta]$. There are also staggered modes (not shown) originating from the Bragg gap region that constitute magnetoinductive Tamm states [31, 32]. From the surface DBs shown in Fig. 3, only one of them (Fig. 3a) corresponds to a truly surface state, since it is localized exactly at the left end of the array ($n = 1$). The next two (Figs. 3b and 3c) can also be characterized as surface DBs, since they are localized very close to the surface ($n = 2$ and $n = 3$, respectively), but actually they are cross-over states between surface and bulk DBs. Since the DBs shown here are highly localized, they obtain their bulk form within a distance of only a few sites from the surface, so that the DB shown in Fig. 3d (localized at $n = 4$) can be considered as a bulk DB. With the appropriate choice of the initial conditions we may also construct multi-site surface DBs such as those shown in Fig. 4, that remind four-site antisymmetric DB excitations. Of course, close to the

![FIG. 3: (Color online) Typical Hamiltonian surface breather profiles in a magnetoinductive binary chain for $\delta = 0.9$, $\lambda = 0.1$, $\Omega = 0.77$, $\chi = 1/6$, obtained by the RWA.](image-url)
surface, we cannot obtain exact antisymmetric DBs.

* bulk breathers.* Typical bulk Hamiltonian DB profiles obtained with the RWA method are shown in Figs. 5 and 6. In Fig. 5, the two single-site symmetric DBs differ in that the first one (Fig. 5(a)) is staggered, while the other one (Fig. 5(b)) is unstaggered. The staggered/unstaggered character of those Hamiltonian DBs depends on the sign of the product of the coupling parameter and the nonlinearity parameter, \( \sigma = \text{sgn}(\alpha \lambda) \). For \( \sigma > 0 \), that implies either \( \alpha = +1 \) and \( \lambda > 0 \) or \( \alpha = -1 \) and \( \lambda < 0 \), the excited DBs are unstaggered. On the other hand, for \( \sigma < 0 \), that implies either \( \alpha = +1 \) and \( \lambda < 0 \) or \( \alpha = -1 \) and \( \lambda > 0 \), the excited DBs are staggered. In that figure the DB frequency \( \Omega_B = 2\pi/T_B \), with \( T_B \) the DB period, was chosen to ensure that the DB amplitude at all sites is nonzero in the anticontinuous limit. In Fig. 6, that shows two-site antisymmetric bulk DBs, the nonlinearity parameter is negative (\( \alpha = -1 \)) implying that in the anticontinuous limit, only the even sites of the array can have a nonzero amplitude. Again, with appropriate choice of the initial conditions and by changing the sign of the coupling parameter \( \lambda \) and/or the nonlinearity parameter \( \alpha \) we may also construct multi-site bulk DBs with different symmetry. It should be also possible to generate a large variety of surface and bulk Hamiltonian DBs in two dimensional binary arrays, just like in planar arrays of identical SRRs [30]. Increased dimensionality offers more possibilities for generating different DB types.

**IV. DISSIPATIVE BREATHERS IN THE ROTATING WAVE APPROXIMATION**

We consider the more realistic case of dissipative discrete breathers (DDBs), that can be excited either on the ‘surface’ or the bulk of a 1D binary MM. In typical experiments that involve MMs, the metamaterial is driven by an applied electromagnetic field of appropriate polarization. In 1D there are two possible geometries for the arrangement of the SRRs [30]: the planar geometry (as shown in Fig. 1), for which the coupling parameter is negative, and the axial geometry for which the coupling parameter is positive. That polarization can be chosen such that, for example, the magnetic component of the field is perpendicular to the SRR planes, while the electric component is parallel to the sides of the SRRs that do not have a slit. This choice simplifies physically the situation, since only the magnetic field excites an emf in the SRRs. Thus, in the equivalent circuit picture, a binary SRR-based MM in an electromagnetic field can be described by an array of nonlinear RLC circuits driven by an alternating emf that are coupled through their mutual inductances. The losses of the SRRs can be described in this picture by an equivalent resistance. That effective resistance \( R \) may actually describe both Ohmic losses of the SRR as well as radiative losses, if these are relatively low [34]. Under those assumptions, the dynamics of the charges \( q_n(\tau) \), \( n = 1, 2, ... , N \), is given by Eq. (13). In the framework of the RWA method, we look for stationary solutions of that equation in the form \( q_n(\tau) = q_n \sin(\Omega \tau + \phi_n) \). By direct substitution of \( q_n(\tau) \) into Eq. (13) and by making the RWA replacement
In the anticontinuous limit Eqs. (18) and (19) become

\[ P(q_n) = q_0^2 \left\{ \left( \omega_n^2 - \Omega^2 - \frac{3}{4} \chi \omega_n^2 q_n^2 \right)^2 + \gamma^2 \Omega^2 \right\} = \varepsilon_0^2 \]

(20)

\[ \phi_n = \tan^{-1} \left\{ \frac{-\gamma \Omega}{\omega_n^2 - \Omega^2 - \frac{3}{4} \chi \omega_n^2 q_n^2} \right\} \]

(21)

where we kept the subscript \( n \) to distinguish between oscillators located either at an odd-numbered site \( (n = \text{odd integer}) \) or even-numbered site \( (n = \text{even integer}) \). The polynomial \( P(q_n) \) is cubic in \( q_n^2 \) for general values of the parameters \( \delta, \chi, \Omega \) and \( \gamma \), with \( P(0) = 0 \). Thus, there can be at most three real roots that correspond to attractors of the SRR oscillator (see Fig. 7), from which two are stable (unstable) and one is unstable (stable). However, by varying a parameter in that four-dimensional parameter space, two of these solutions may disappear through a pitchfork bifurcation, leaving behind only one single attractor. The boundary in parameter space that separates those two cases can be found implicitly by computing the values of \( q_n \) denoted by \( q_1^n \) for which \( dP(q_n)/dq_n = 0 \) (i.e., the values of \( q_n \) that correspond to the local extrema of \( P(q_n) \)). Obviously, if \( q_1^n = q_2^n \) corresponds to a local minimum (maximum) then for \( P(q_1^n) > \varepsilon_0^2 \) \( P(q_2^n) < \varepsilon_0^2 \) there is only one attractor left. Thus, the earlier inequalities determine different regions in the parameter space where, depending on the values of the parameters, we may have either one or three attractors. Thus, we may distinguish on such a diagram two different ‘phases’ that correspond to either three or one real solutions. A typical example is shown in the reduced \( \varepsilon_0 - \Omega \) parameter space in Fig. 8, for \( \delta = 1 \) (no RFM) and opposite values of \( \chi \), while the values of the driving field strength \( \varepsilon_0 \) and the driving frequency \( \Omega \) are varying. There we observe that the colored area, corresponding to three attractors, expands with increasing \( \delta \). The

![FIG. 7: The intersection of \( P(q_n) \) (solid line) and \( \varepsilon_0^2 \) (dashed line) provides the nonlinear attractor(s) for a single SRR. Case displayed here corresponds to \( \delta = 1, \chi = 1/6, \gamma = 0.02, \Omega = 0.92, \varepsilon_0 = 0.04 \). The black (red) circle represents a stable (unstable) attractor.](image)

![FIG. 8: (Color online) ‘Phase diagram’ in the reduced parameter space \( \varepsilon_0 - \Omega \) for a single driven-damped SRR oscillator showing the regions with different number of attractors, for \( \delta = 1, \gamma = 0.1 \), and \( \chi = +1/6 \) (left); \( \chi = -1/6 \) (right). Inside (outside) the colored region we have three (one) attractors.](image)

![FIG. 9: (Color online) Same as in Fig. 8 for a single driven-damped SRR oscillator with \( \gamma = 0.1, \chi = +1/6 \), and (from left to right) \( \delta = 0.5; \delta = 0.75; \delta = 1; \delta = 1.5 \).](image)
dicts three attractors at $q_0^1 = \pm 1.23531$, $q_0^2 = \pm 1.33055$ and $q_0^3 = \pm 0.0996811$, of which the latter two are stable. The unstable attractor at $q_0^1$ is not reachable through simple numerical integration of the dynamical equation. For the stable attractors, we have checked with direct numerical integration that their values are practically the same with those obtained from the RWA approach. In general, the presence of dissipation and driving severely limit the possible spatial profile of the breathers. The structures tend now to be either strongly localized ones, or rather extended, like domain walls [27, 29]. The situation is similar for DDBs in the bulk (Figs. 10 and 11). As soon as $\delta$ deviates from unity, that is, when we are dealing with a bona fide binary chain, the area in phase space with two stable attractors reduce (increase) as $\delta$ decreases (increases) from unity, if the site chosen is an even-numbered one, as can be seen in Fig. 8. The opposite behavior occurs for an odd-numbered site.

In order to illustrate how the RWA method works in this case, we calculate some of typical surface DDBs. The calculation of bulk DDBs proceeds in the same way, by simply choosing the central site of the corresponding trivial DB that is located at $n = n_B$ somewhere in the bulk. For a given value of the RFM, we first determine first the attractors available for each single SRR oscillator, which is located either at odd- or even-numbered site. Then, we set up a trivial surface DB which is subsequently continued for finite values of $\lambda$. The continuation procedure proceeds in exactly the same way as that for Hamiltonian DBs, except that the relevant equations are now Eqs. (18) and (19). Thus, we can obtain several types of surface DDBs for an interval of $\lambda$ up to a maximum, i.e., up to $\lambda = \lambda_{\text{max}}$. For example, for the parameter set $\chi = +1/6, \gamma = 0.01, \Omega = 0.5, \delta = 2$, and $\varepsilon_0 = 0.04$, we have stable attractors $q_1^2 = \pm 4.06719$ and $q_2^2 = \pm 0.160225$ at odd-numbered sites, and $q_0^3 = \pm 1.334$ and $q = \pm 0.0228639$ at even-numbered sites. We can set up a trivial surface DB localized at $n_B = 1$ as $q_1 = 4.06719$, $q_2 = 0.0228639$ and $q_{n-1} = 0.160225$ ($n > 1$). For a trivial surface DB localized at $n_B = 2$ we may choose $q_2 = 1.334$. $q_{n-1} = -0.160225$ and $q_2 = -0.0228639$ ($n > 1$). Or, for a trivial surface DB localized at $n_B = 3$ we may choose $q_{n-1} = -0.160225$, $q_{n-1} = -0.160225$ ($n \neq 2$), and $q_3 = 4.06719$. Continuation of those trivial DDBs up to $\lambda = 0.025$ gives the surface DDB profiles shown in Fig. 10. Of course there also other trivial DDB profiles that we could choose. Similar bulk DDBs can be obtained from the trivial DDBs given above only by changing $n_B$ to a value relatively far from the end-points. An illustrative example of a bulk DDB localized at an odd-numbered site is shown in the left panel of Fig. 11, while in the middle and right panels of Fig. 11 are shown two multi-site DDBs also localized at odd-numbered sites. The latter two DDBs have been obtained with appropriate choice of a trivial DDB profile.

![FIG. 10: (Color online) Dissipative surface breather profiles for $\delta = 2, \Omega = 0.5, \lambda = 0.025, \chi = +1/6, \gamma = 0.01, \varepsilon_0 = 0.04$, that are localized at $n = 1$ (left panel); $n = 2$ (middle panel); $n = 3$ (right panel).](image1)

![FIG. 11: (Color online) Dissipative bulk breather profiles for $\delta = 2, \Omega = 0.5, \lambda = 0.025, \chi = +1/6, \gamma = 0.01, \varepsilon_0 = 0.04$, that are localized at $n = 1$ (left panel); $n = 2$ (middle panel); $n = 3$ (right panel).](image2)

V. NUMERICALLY EXACT CALCULATIONS

In this Section we construct several types of both energy-conserved and dissipative DBs which are localized either in the bulk or at the surface, using standard numerical algorithms [4, 5]. Moreover, in the case of a dissipative binary MM we also generate DDB excitations through a procedure that can be used for parameter values where the homogeneous solution is modulationally unstable [13]. In other words, we exploit MI to initiate spontaneous localization of energy in the binary array [17].

Hamiltonian breathers.- For the Hamiltonian binary MM, DBs can be constructed from the anticontinuous limit of Eqs. (13) with $\varepsilon_0 = 0$ and $\gamma = 0$ where all the SRRs are decoupled [29, 31]. Using Newton’s method we have constructed several types of Hamiltonian, numerically exact DBs for the 1D binary MM, for different parameter sets. The obtained Hamiltonian DB profiles are in excellent agreement with those obtained with the RWA method.

Dissipative Breathers.- In order to generate DDBs we start from the anticontinuous limit of Eq. (13), where dissipation and driving are included. We identify stable attractors of each SRR oscillator, that is either located at odd- or even-numbered sites. For constructing a trivial DDB profile we need to find, for at least one of the two types of oscillators, two different amplitude stable attractors. For example, for the parameter set $\delta = 2.0, \Omega = 0.5, \gamma = 0.01, \varepsilon_0 = 0.04, \chi = +1/6$, we obtain stable attractors $q_1^2 = \pm 4.06719$ and $q_2^2 = \pm 0.160225$ at odd-numbered sites, and $q_0^3 = \pm 1.334$ and $q = \pm 0.0228639$.
Typical profiles for several values of $\lambda$ localized at $\Omega = 0$ we may choose even-numbered site oscillators, while it predicts three attractors at $\Omega$ localized at the single SRR oscillators. We set up a trivial surface DB localized at $n = 1$ as $q_1 = 4.06719$, $q_{2n} = 0.0228639$ and $q_{2n-1} = 0.160225$ ($n > 1$). For a trivial surface DB localized at $n_B = 3$ we may choose $q_1 = 4.06719$, $q_{2n} = 0.0228639$, and $q_{2n-1} = -0.160225$ ($n \neq 2$). Continuation of those trivial DDBs gives surface DDB profiles up to $\lambda_{\text{max}} \simeq 0.19$. Typical profiles for several values of $\lambda$, both for DDBs localized at $n = 1$ and $n = 3$, are shown in Fig. 12.

Another example is given for the parameter set $\delta = 0.8$, $\Omega = 0.92$, $\gamma = 0.01$, $\epsilon_0 = 0.04$, $\chi = +1/6$, where the RWA method predicts a single attractor at $q_1^a = \pm 0.582163$ at even-numbered site oscillators, while it predicts three attractors at $q_1^a = \pm 1.23531$, $q_2^a = \pm 1.33055$ and $q_3^a = \pm 0.996811$, of which the latter two are stable, for an odd-numbered site oscillators. These values are also in agreement with those obtained by direct integration of the single SRR oscillators. We set up a trivial surface DDB localized at $n_B = 1$ as $q_1 = 1.33055$, $q_{2n} = 0.582163$ and $q_{2n-1} = 0.0996811$ ($n > 1$), and continue it up to $\lambda_{\text{max}} \sim 0.07$ where DDBs cease to exist. Typical DDB profiles are shown in Fig. 13 for several values of $\lambda$ shown on the figure. A profile for $\lambda = 0.072$ which is greater than $\lambda_{\text{max}}$, where the homogeneous solution is restored, is also shown in Fig. 13d. The frequency of the DDBs shown here is the same with that of the driver, i.e., $\Omega_B = 2\pi / T_B = \Omega$. However, the phase differences of the SRR oscillators in the array with respect to the driving field are generally different for each oscillator, as can be observed in Fig. 14, where the time-evolution of $q_1 - q_4$ is followed for approximately two periods $T_B$ of the DDB oscillation. Note also that the time-evolution seems practically sinusoidal (harmonic), that may not be necessarily true for DDBs obtained with some other parameter set.

**Dissipative Breathers by frequency chirping.** For a frequency gapped linear spectrum, some of the linear modes become unstable at large amplitude. If the curvature of

![Figure 12: Dissipative breather profiles at maximum amplitude for several values of the coupling parameter as shown on the figure which are localized at $n = 1$ and $n = 3$. The parameters are $\Omega = 0.5$, $\chi = +1/6$, $\gamma = 0.01$, $\epsilon_0 = 0.04$, and $\delta = 2$.](image1)

![Figure 13: Dissipative breather profiles at maximum amplitude for several values of the coupling parameter as shown on the figure which are localized at the surface (at $n = 1$), along with an almost uniform solution for $\lambda = 0.072$ just above the value of $\lambda_{\text{max}}$ for this particular parameter set. The parameters are $\Omega = 0.92$, $\chi = +1/6$, $\gamma = 0.01$, $\epsilon_0 = 0.04$, and $\delta = 0.8$.](image2)

![Figure 14: (Color online) Time-dependence of $q_1$ (black solid curve), $q_2$ (red dotted curves), $q_3$ (green short-dashed curve), and $q_4$ (blue dashed curves), for a surface breather localized at $n = 1$ and $\lambda = 0.03$. The other parameters are the same with those in the caption of Fig. 14.](image3)
cause of that, they are trapped at particular SRRs. After that, the driver frequency is kept constant and the high amplitude DDBs (and even some low-amplitude ones) continue to receive energy falling into a stationary state. When the driver is switched off all DDBs die out in a short time interval.

In Figs. 15 and 16, the contours of the energy density $\mathcal{H}_n$ on the $\tau - n$ plane identify the evolution of the DDBs formed by that procedure. The chirping phase lasts for $2000 T_0$ time units ($T_0 = 2\pi/\Omega$), where the frequency varies linearly from $\Omega_i = 0.997\Omega$ to $\Omega_f = 1.020\Omega$. The driver is subsequently kept at constant frequency $\Omega_f$ until it is switched off after another $2000 T_0$ time units. Figs. 15 and 16 correspond to the regions of the binary MMs where several DDBs have survived after the chirping phase. In Fig. 15 we clearly observe a high amplitude DDB at $n = 251$, along with some other DDBs of considerably lower amplitude, that survive until the end of the constant frequency phase. In Fig. 16, where the binary MM is driven not as strongly as that in Fig. 15, we observe one relatively low-amplitude DDB which however survives until the end of the constant frequency phase. It is possible that the procedure described above, which relies on the MI of the large amplitude linear modes, can be used for the generation of DDBs in other magnetoinductive systems as well [42, 43], in their binary versions.

**VI. CONCLUSION**

We presented detailed analysis for induced nonlinear localization in binary nonlinear magnetic metamaterials. The systems we analyzed are one dimensional and consist of two types of SRRs; this configuration leads to a linearized magnetoinductive system with two optical bands separated by a gap. When nonlinearity is also taken into account nonlinear localized modes of discrete breather type may be generated in the gaps. As in the case where all units are identical [29, 30] depending on the parameter regime these breathers may be left-handed in a right-handed background or the opposite. We focussed on both the Hamiltonian as well as the dissipative case; the latter is clearly the most interesting physically since it corresponds to true propagation of waves in the medium. We used two approaches, one based on the rotating wave approximation while the second on exact numerics using the breather analysis from the anticontinuous limit. The comparison of the two showed that the RWA is in most cases a good approximation for a relatively accurate breather construction.

In the Hamiltonian case we found two types of breathers with even or odd local symmetry depending on the sign of the product of the coupling parameter and the nonlinearity parameter. Both types may exist in the bulk but also in the boundary of the chain; the latter form surface breathers. A similar situation occurs also in the dissipative case where depending on the number of attractors of the single driven nonlinear oscillator system we have different type of dissipative breathers. Both bulk and surface breathers appear with corresponding symmetries.

The binary structure of the lattice allows for generation of breathers through direct external induction. This is accomplished through frequency chirping to the desired frequency. In the process of frequency modulation induction of plane wave instability occurs that leads to breather generation. These breathers move around in the lattice, collide, some decay and eventually a single “large” breather is left in the metamaterial. This method of breather generation is direct and may be used for experimental breather investigation in metamaterials.

**Acknowledgments**

One of us (MIM) acknowledges support from Fondecyt Grant 1080374 of Chile.
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