Abstract: Let $M$ be a connected generic real-analytic CR-submanifold of a finite-dimensional complex vector space $E$. Suppose that for every $a \in M$ the Lie algebra $\mathfrak{hol}(M, a)$ of germs of all infinitesimal real-analytic CR-automorphisms of $M$ at $a$ is finite-dimensional and its complexification contains all constant vector fields $\alpha \partial / \partial z, \alpha \in E$, and the Euler vector field $z \partial / \partial z$. Under these assumptions we show that: (I) every $\mathfrak{hol}(M, a)$ consists of polynomial vector fields, hence coincides with the Lie algebra $\mathfrak{hol}(M)$ of all infinitesimal real-analytic CR-automorphisms of $M$; (II) every local real-analytic CR-automorphism of $M$ extends to a birational transformation of $E$, and (III) the group $\text{Bir}(M)$ generated by such birational transformations is realized as a group of projective transformations upon embedding $E$ as a Zariski open subset into a projective algebraic variety. Under additional assumptions the group $\text{Bir}(M)$ is shown to have the structure of a Lie group with at most countably many connected components and Lie algebra $\mathfrak{hol}(M)$. All of the above results apply, for instance, to Levi non-degenerate quadrics, as well as a large number of Levi degenerate tube manifolds.

1. Introduction and Preliminaries

Let $h = (h_1, \ldots, h_k)$ be a $C^k$-valued Hermitian form on $\mathbb{C}^n$, with $n, k \geq 1$. The form $h$ is called non-degenerate if the following two conditions are satisfied:

(i) the scalar Hermitian forms $h_1, \ldots, h_k$ are linearly independent over $\mathbb{R}$;

(ii) $h(z, z') = 0$ for all $z' \in \mathbb{C}^n$ implies $z = 0$.

For a non-degenerate $h$ one has $k \leq n^2$. Note that many authors define a non-degenerate Hermitian form as a form satisfying condition (ii) alone.

To any $C^k$-valued Hermitian form $h$ on $\mathbb{C}^n$ one associates the quadric $Q_h \subset \mathbb{C}^{n+k}$ of CR-dimension $n$ and CR-codimension $k$ as follows:

$$Q_h := \{(z, w) \in \mathbb{C}^{n+k} : \text{Im} \, w = h(z, z)\},$$
where \( z = (z_1, \ldots, z_n) \) is a point in \( \mathbb{C}^n \), and \( w = (w_1, \ldots, w_k) \) is a point in \( \mathbb{C}^k \). The CR-manifold \( Q_h \) is called the \textit{quadric associated to} \( h \).

If \( h \) is non-degenerate, then any \( C^1 \)-smooth CR-isomorphism between domains in \( Q_h \) extends to a birational map of \( \mathbb{C}^{n+k} \) (see the classical papers [Po], [Tan1], [A] for \( k = 1 \) and the papers [KT], [F], [Tum], [Ka1], [Su], [B1], [B2] for \( 1 < k \leq n^2 \)). These birational maps form a group (this is not obvious at all and requires a justification – see Remark 1.1). We denote this group by \( \text{Bir}(Q_h) \) and call the \textit{group of birational transformations of} \( Q_h \).

For \( k = 1 \) every element of \( \text{Bir}(Q_h) \) is a linear fractional transformation induced by an automorphism of \( \mathbb{C}P^{n+1} \) (see [Po], [Tan1], [A]). For some Hermitian forms \( h \) with \( 1 < k \leq n^2 \) formulas for elements of certain subgroups of \( \text{Bir}(Q_h) \) were given in [ES2], [ES3]. It was shown in [Tum] that the group \( \text{Bir}(Q_h) \) can be endowed with the structure of a Lie group (possibly with uncountably many connected components) with Lie algebra isomorphic to the Lie algebra of all infinitesimal CR-automorphisms of \( Q_h \), where a smooth vector field on \( Q_h \) is called an infinitesimal CR-automorphism if in a neighborhood of every point of \( Q_h \) translations along the integral curves of the vector field form a local group of local CR-automorphisms. Every infinitesimal CR-automorphism of \( Q_h \) is known to be polynomial. We will see below that \( \text{Bir}(Q_h) \) can be embedded in a natural way into the complex group \( \text{PGL}_N(\mathbb{C}) \) as a closed real subgroup (see Corollary 1.5 and Remark 1.6).

We are interested in regularizing the elements of the group \( \text{Bir}(Q_h) \) as stated in Definition 1.2 below. This definition applies to more general CR-submanifolds \( M \) of a finite-dimensional complex vector space \( E \) than quadrics, and we will first introduce \( \text{Bir}(M) \), the \textit{group of birational transformation of} \( M \). Throughout the paper \( M \) is assumed to be connected, locally closed, real-analytic and generic in \( E \).

For a rational map \( g \) of \( E \), we denote by \( \text{reg}(g) \) the subset of all regular points of \( g \). Let \( \text{Bir}(E) \) be the group of all birational transformations of \( E \). The restriction of \( g \in \text{Bir}(E) \) to the Zariski open subset \( \text{reg}(g) \) defines a biholomorphic map \( \text{reg}(g) \rightarrow \text{reg}(g^{-1}) \). Denote by \( \text{BR}(M) \) the collection of all elements \( g \in \text{Bir}(E) \) with the following property: there exists a non-empty domain \( V \subset M \) with \( V \subset \text{reg}(g) \) and \( g(V) \subset M \). Note that in general elements of \( \text{BR}(M) \) are not defined on all of \( M \): they may have poles and points of indeterminacy on \( M \). It is clear that \( (\text{BR}(M))^{-1} = \text{BR}(M) \). In general, however, \( \text{BR}(M) \) is a proper subset of \( \text{BR}(M) \cdot \text{BR}(M) \). We define \( \text{Bir}(M) \) to be the subgroup of \( \text{Bir}(E) \) generated by \( \text{BR}(M) \).

One can give a sufficient condition that guarantees that \( \text{Bir}(M) = \text{BR}(M) \). Recall, first of all, that \( M \) is called \textit{minimal at a point} \( a \in M \), if there
does not exist a CR-submanifold $M_0 \subset M$, with $\dim M_0 < \dim M$ and $\text{CR-dim} M_0 = \text{CR-dim} M$, passing through $a$. The manifold $M$ is called \emph{minimal} if it is minimal at its every point.

Let $M$ be a connected real-analytic generic CR-submanifold of $E$. For such $M$ we introduce the following

\textbf{Condition (\ast):}

(a) $M$ is minimal,

(b) $M_1 \subset M$ holds for every connected real-analytic submanifold $M_1 \subset E$ such that $W \cap M = W \cap M_1 \neq \emptyset$ for some domain $W$ in $E$.

In Proposition 2.5 in Section 2 we show that if $M$ satisfies Condition (\ast) then $\text{Bir}(M)$ coincides with $\text{BR}(M)$. This condition is satisfied, for example, if $M$ is minimal and closed in $E$. In particular, Condition (\ast) is satisfied for any quadric $Q_h$ (note that part (i) of the definition of the non-degeneracy of an Hermitian form $h$ given at the beginning of Section 1 is equivalent to $Q_h$ being minimal). There are also a large number of examples of non-closed everywhere Levi degenerate CR-submanifolds satisfying Condition (\ast). An interesting family of such CR-submanifolds is presented in Example 5.4 in Section 5.

\textbf{Remark 1.1.} Proposition 2.5 plays a key role in understanding the group $\text{Bir}(Q_h)$, but it appears to have been overlooked in the literature on quadrics so far. Indeed, many authors seem to assume without proof that the set of maps $\text{BR}(Q_h)$ is a group.

We will now give an exact definition of what we mean by regularization. For a complex manifold $Y$ we denote by $\text{Aut}(Y)$ the group of all biholomorphic automorphisms of $Y$.

\textbf{Definition 1.2.} Let $M$ be a connected real-analytic generic CR-submanifold $M$ of a finite-dimensional complex vector space $E$. A subgroup $G \subset \text{Bir}(M)$ is said to be

(i) \emph{regularizable} on a complex manifold $Y$, if there exists an open holomorphic embedding $\varphi : E \to Y$ and a group homomorphism $\tau : G \to \text{Aut}(Y)$, such that for every $g \in G$ one has $\varphi \circ g = \tau(g) \circ \varphi$ on $\text{reg}(g)$;

(ii) \emph{projectively regularizable} if for a suitable integer $N$ there exists an irreducible complex algebraic subvariety $X \subset \mathbb{CP}^N$, a group homomorphism $\tau : G \to \text{PGL}_{N+1}(\mathbb{C})$, and an algebraic isomorphism $\varphi : E \to X_0$, where $X_0$
is a Zariski open subset of $X$, such that $\varphi \circ g = \tau(g) \circ \varphi$ on $\text{reg}(g)$ for every $g \in G$.

The map $\varphi$ is called a regularization map.

Clearly, if $G$ is projectively regularizable, it is regularizable on the connected Zariski open subset

$$\hat{E} := \bigcup_{g \in G} \tau(g)\varphi(E),$$

(1.1)

of the non-singular part $X_{\text{reg}}$ of $X$. The set $\hat{E}$ is the smallest $\tau(G)$-invariant domain in $X$ that contains $\varphi(E)$. Note also that one can assume that $X$ is not contained in any projective hyperplane in $\mathbb{CP}^N$.

Regularization results for certain groups of birational transformations can be found in [HZ], [Z1]. If $Q_h$ is a hyperquadric (i.e. $k = 1$), the group $\text{Bir}(Q_h)$ is known to be projectively regularizable with $N = n + 1$ due to the classical work [Po], [Tan1], [A]. Further, it was shown in [ES1] (see also [B2], [Mi]) that for $2 \leq k \leq n^2 - 1$, excluding the situation $k = n = 2$, a quadric in general position has only affine automorphisms, in which case $\text{Bir}(Q_h)$ is projectively regularizable with $N = n + k$ for trivial reasons.

In fact, we show in Section 3 that $\text{Bir}(Q_h)$ is projectively regularizable for any non-degenerate form $h$. This is a consequence of our main theorem, which applies to much more general CR-manifolds than quadrics. In order to state the theorem we need to introduce some notation and give necessary definitions.

Let $M$ be a real-analytic generic CR-submanifold of a complex manifold $Z$. In what follows all local CR-automorphisms and infinitesimal CR-automorphism of $M$ are assumed to be real-analytic (note that a $C^1$-smooth CR-isomorphism between Levi non-degenerate real-analytic CR-manifolds is in fact real-analytic – see Theorem 3.1 in [BJT]). We denote by $\mathfrak{hol}(M)$ the real Lie algebra of all real-analytic infinitesimal CR-automorphisms of $M$. A vector field $\xi$ on $M$ lies in $\mathfrak{hol}(M)$ if and only if $\xi$ extends to a holomorphic vector field on a neighborhood $U$ of $M$ in $Z$. [We think of holomorphic vector fields on $U$ as holomorphic sections over $U$ of the tangent bundle $TU$. In particular if $Z = E$, a holomorphic vector field $f(z) \partial / \partial z$ is just given by a holomorphic map $f : U \to E$.]

For $a \in M$ we denote by $\mathfrak{hol}(M, a)$ the real Lie algebra of all germs at $a$ of vector fields in $\mathfrak{hol}(V)$, with $V$ running over all open neighborhoods of $a$ in $M$. Clearly, $\mathfrak{hol}(M, a)$ is a real Lie subalgebra of the complex Lie algebra $\mathfrak{hol}(Z, a)$. By Proposition 12.5.1 of [BER] the finite-dimensionality of $\mathfrak{hol}(M, a)$ implies that $M$ is holomorphically non-degenerate at $a$, i.e. the Lie algebra $\mathfrak{hol}(M, a)$ is totally real in $\mathfrak{hol}(Z, a)$ for all $a \in M$. Indeed, if
\[ \xi \] lies in \( \mathfrak{hol}(M, a) \cap i\mathfrak{hol}(M, a) \), then \( \psi \cdot \xi \in \mathfrak{hol}(M, a) \) for any germ \( \psi \) of a holomorphic function near \( a \). Thus the formal complexification of \( \mathfrak{hol}(M, a) \) is isomorphic to \( \mathfrak{hol}(M, a) + i\mathfrak{hol}(M, a) \subset \mathfrak{hol}(Z, a) \) if \( \dim \mathfrak{hol}(M, a) < \infty \).

Let \( M \) be a real-analytic generic CR-submanifold of a finite-dimensional complex vector space \( E \). For such \( M \) and a point \( a \in M \) we introduce the following

**Property (P) at \( a \):**

(a) the Lie algebra \( \mathfrak{hol}(M, a) \) is finite-dimensional,

(b) the complex Lie algebra \( \mathfrak{hol}(M) + i\mathfrak{hol}(M) \) contains the complex solvable Lie algebra

\[ s := \{ (\alpha + cz) \partial/\partial z : \alpha \in E, c \in \mathbb{C} \} . \tag{1.2} \]

Further, we say that \( M \) has **Property (P)** if it has Property (P) at its every point. In Section 5 we give sufficient conditions for \( M \) to have Property (P) (see Proposition 5.1), and discuss several examples. In particular, every non-degenerate quadric \( Q_h \) has Property (P).

We now state our main result that provides projective regularization of \( \text{Bir}(M) \) for a large class of CR-submanifolds.

**THEOREM 1.3.** Let \( M \) be a connected real-analytic generic CR-submanifold of a finite-dimensional complex vector space \( E \). Assume further that \( M \) has Property (P). Then the following holds:

(I) for every \( a \in M \) the Lie algebra \( \mathfrak{hol}(M, a) \) consists of polynomial vector fields, hence \( \mathfrak{hol}(M, a) = \mathfrak{hol}(M) \);

(II) every real-analytic CR-isomorphism \( g \) between non-empty domains in \( M \) extends to a map lying in \( \text{Bir}(M) \) of the form \( q(z)^{-1}p(z) \), where \( p : E \to E, q : E \to \text{End}(E) \) are polynomial maps, and \( \text{reg}(g) = \text{reg}(q^{-1}) = \{ z \in E : \det q(z) \neq 0 \} \);

(III) \( \text{Bir}(M) \) is projectively regularizable.

Our next theorem provides information on the extension of \( \varphi(M) \) into \( \mathbb{CP}^N \). Recall that a real-analytic CR-manifold \( M \) is called **locally homogeneous at a point \( a \in M \)** if the evaluation map \( \mathfrak{hol}(M, a) \to T_aM, \xi \mapsto \xi_a \), is surjective, and \( M \) is called **locally homogeneous** if \( M \) is locally homogeneous at every point (see \( \text{Z2} \) for equivalent definitions of local homogeneity). In the theorem to follow we assume that \( M \) has Property (P) at some point, satisfies part (b) of Condition (\( \ast \)), and is locally homogeneous. Observe
that these assumptions imply that $M$ has Property (P) and satisfies Condition (*). Indeed, local homogeneity implies that $M$ has Property (P). Further, by Proposition 4.2 of \[Z2\] the finite-dimensionality of $\mathfrak{hol}(M,a)$ and local homogeneity at $a$ for all points $a \in M$ yield that $M$ is minimal. Hence $M$ satisfies Condition (*). For such a manifold $M$, we denote by $\hat{M}$ the unique Bir$(M)$-orbit in $\mathbb{CP}^N$ containing $\varphi(M)$. Clearly, $\hat{M}$ is a connected immersed generic CR-submanifold of $\hat{E}$ (see (1.1)), and we denote by Aut$(\hat{M})$ the group of all real-analytic CR-automorphisms of $\hat{M}$.

We now state our next result.

**THEOREM 1.4.** Let $M$ be a connected real-analytic generic CR-submanifold of a finite-dimensional complex vector space $E$. Assume that $M$ has Property (P) at some point, satisfies part (b) of Condition (*), and is locally homogeneous. Then for the regularization map $\varphi$ and homomorphism $\tau$ arising in Theorem [1.3] the set $\varphi(M)$ is open and dense in $\hat{M}$, and Aut$(\hat{M}) = \tau$(Bir$(M)$). Furthermore, if $M \setminus M$ does not contain a CR-submanifold of $E$ locally CR-equivalent to $M$, then $\tau$(Bir$(M)$) is closed in $\text{PGL}_{N+1}(\mathbb{C})$, and the Lie algebra of $\tau$(Bir$(M)$) is canonically isomorphic to $\mathfrak{hol}(M)$.

For $M$ satisfying the assumptions of Theorem [1.4] we now introduce a Lie group structure on Bir$(M)$ by pulling back the Lie group structure from $\tau$(Bir$(M)$) by means of $\tau$. In this Lie group topology Bir$(M)$ has at most countably many connected components. In Section 4 we give another sufficient condition for the existence of a Lie group structure on Bir$(M)$ with this property (see Theorem [4.1]). It comes from the natural faithful representation of Bir$(M)$ on $\mathfrak{hol}(M)$.

Applying Theorems [1.3] [1.4] [4.1] to any quadric $Q_h$ we obtain the following corollary.

**Corollary 1.5.** If $h$ is non-degenerate, then Bir$(Q_h)$ is projectively regularizable, and for the regularization map $\varphi$ the set $\varphi(Q_h)$ is open and dense in a Bir$(Q_h)$-orbit in $\mathbb{CP}^N$. The corresponding homomorphism $\tau$ maps Bir$(Q_h)$ onto a closed real subgroup of $\text{PGL}_{N+1}(\mathbb{C})$, and Bir$(Q_h)$ admits the structure of a Lie group with at most countably many connected components and Lie algebra isomorphic to $\mathfrak{hol}(Q_h)$.

For the case when $Q_h$ is the Šilov boundary of a Siegel domain, the regularization statement of Corollary [1.5] is essentially contained in Theorem 9 of [KMO].

**Remark 1.6.** For quadrics the degrees of the polynomial maps $p$ and $q$ arising in statement (II) of Theorem [1.3] do not exceed 2. The rationality property for local automorphisms of quadrics can be derived already from
the results of [Ka1] (see Satz 2, p. 134). This property was also obtained in [Tum], but our arguments are simpler even for more general CR-manifolds. In addition, a Lie group structure on Bir($Q_h$) with Lie algebra $\mathfrak{hol}(Q_h)$ has been constructed in [Tum] by means of considering the natural faithful representation $\rho$ of Bir($Q_h$) on $\mathfrak{hol}(Q_h)$ that maps every $g \in$ Bir($Q_h$) into the corresponding push-forward transformation $g_*$ of vector fields in $\mathfrak{hol}(Q_h)$. By a general theorem due to Palais (see [Pa], Theorem VII, p. 103) the image $\rho(\text{Bir}(Q_h)) \subset \text{GL}(\mathfrak{hol}(Q_h))$ has the structure of a Lie group with Lie algebra $\mathfrak{hol}(Q_h)$, but this Lie group may a priori have uncountably many connected components, if $\rho(\text{Bir}(Q_h))$ is not closed in $\text{GL}(\mathfrak{hol}(Q_h))$. No proof of closedness was given in [Tum]. Our construction of a Lie group structure on Bir($M$) in Theorem 1.4 relies on the algebraic regularization map $\varphi : E \rightarrow \mathbb{C}P^N$, while the Lie group structure arising in Theorem 4.1 comes from the natural representation $\rho$ of Bir($M$) on $\mathfrak{hol}(M)$. In Theorem 1.4 we show that Bir($M$) embeds as a closed subgroup into $\text{PGL}_{N+1}(\mathbb{C})$, whereas in Theorem 4.1 we prove that $\rho(\text{Bir}(M))$ is closed in $\text{GL}(\mathfrak{hol}(M))$. The Lie group structures on Bir($Q_h$) arising from Theorems 1.4 and 4.1 for $M = Q_h$ are identical. We also note that since the extension $\hat{Q}_h$ of $Q_h$ is Levi non-degenerate and has pairwise equivalent Levi forms at all points, the existence of the structure of a Lie group on Aut($\hat{Q}_h$) (and hence on Bir($Q_h$)) with Lie algebra $\mathfrak{hol}(Q_h)$ in a certain topology follows from the results of [Tan2]. We refer the reader to [BRWZ], [LMZ] and references therein for results on the existence of Lie group structures on the groups of CR-automorphisms of more general CR-manifolds.

If one does not insist on finding a projective regularization, the group Bir($Q_h$) (in fact, the group Bir($M$) for much more general $M$) can be regularized on some complex manifold in the sense of part (i) of Definition 1.2 as follows. Consider the complexification $\mathfrak{l}$ of $\mathfrak{hol}(Q_h)$. The complex Lie algebra $\mathfrak{l}$ consists of polynomial vector fields of degree not exceeding 2 and has a natural grading $\mathfrak{l} = \mathfrak{l}^{-1} \oplus \mathfrak{l}^0 \oplus \mathfrak{l}^1$, where the Lie subalgebra $\mathfrak{l}^{-1}$ consists of all constant vector fields on $E$, and all vector fields in $\mathfrak{l}_0 := \mathfrak{l}^0 \oplus \mathfrak{l}^1$ vanish at the origin (see e.g. Section 3). Since $[\xi, \mathfrak{l}^0]$ is not contained in $\mathfrak{l}^0$ for every non-zero $\xi \in \mathfrak{l}^{-1}$, the normalizer of $\mathfrak{l}_0$ in $\mathfrak{l}$ coincides with $\mathfrak{l}_0$. Let $\mathfrak{L}$ be the connected simply-connected group with Lie algebra $\mathfrak{l}$. The stabilizer $\mathfrak{L}_0$ of $\mathfrak{l}_0$ under the adjoint representation of $\mathfrak{L}$ is a closed complex subgroup of $\mathfrak{L}$. Since the normalizer of $\mathfrak{l}_0$ in $\mathfrak{l}$ coincides with $\mathfrak{l}_0$, the Lie algebra of $\mathfrak{L}_0$ coincides with $\mathfrak{l}_0$. Thus $\mathfrak{L}_0$ is a closed complex connected subgroup of $\mathfrak{L}$ with Lie algebra $\mathfrak{l}_0$, and we consider the simply-connected complex homogeneous manifold $Y_h := \mathfrak{L}/\mathfrak{L}_0$. One can show that the vector group
$E^+ := (E, +)$ naturally lies in $\mathfrak{L}$, and therefore $E$ embeds into $Y_h$ as an an open (and dense) subset. Let $\text{Bir}(Q_h)^\circ$ denote the connected component of the identity of $\text{Bir}(Q_h)$ with respect to the Lie group topology on $\text{Bir}(Q_h)$ provided, say, by the results of Tum. It can be easily shown that $\text{Bir}(Q_h)^\circ$ is regularizable on the manifold $Y_h$.

Further, let $\text{Bir}_0(Q_h) := \{ g \in \text{Bir}(Q_h) : 0 \in \text{reg}(g) \text{ and } g(0) = 0 \}$. The full group $\text{Bir}(Q_h)$ is generated by $\text{Bir}(Q_h)^\circ$ and $\text{Bir}_0(Q_h)$. For an element $g \in \text{Bir}_0(Q_h)$ the corresponding push-forward map $g_*$ is a Lie algebra automorphism of $\mathfrak{l}$ leaving $\mathfrak{l}_0$ invariant. This automorphism induces an automorphism of $\mathfrak{L}$ leaving $\mathfrak{L}_0^\circ$ invariant, and therefore gives rise to an element of $\text{Aut}(Y_h)$. Hence the full group $\text{Bir}(Q_h)$ is regularizable on $Y_h$.

While the approach that we have just outlined solves the regularization problem for $\text{Bir}(Q_h)$ in principle (in the sense of part (i) of Definition 1.2), our Theorem 1.3 contains a much stronger result. It provides an algebraic solution to this problem and applies to a large class of CR-manifolds.

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2. Birational Transformations of a Vector Space

In this section we state two general propositions on birational maps of a finite-dimensional complex vector space $E$.

The first proposition will be used in the proofs of Theorems 1.3, 1.4 but is also of independent interest (cf. Ko1, Ko2, Ka2). For every $\alpha \in E$ we consider the constant holomorphic vector field $\alpha \partial/\partial z$, and denote by $\eta$ the Euler vector field $z \partial/\partial z$.

**Proposition 2.1.** Let $D_1, D_2 \subset E$ be non-empty domains and $g : D_1 \to D_2$ a biholomorphic map with induced Lie algebra isomorphism $g_* : \mathfrak{hol}(D_1) \to \mathfrak{hol}(D_2)$. With $g^* := g_*^{-1}$ define the holomorphic maps

$$p_g : D_1 \to E \quad \text{and} \quad q_g : D_1 \to \text{End}(E)$$

by

$$g^*(\eta) = p_g(z) \partial/\partial z, \quad g^*(\alpha \partial/\partial z) = (q_g(z)\alpha) \partial/\partial z \quad (2.1)$$

for all $\alpha \in E$. Then $g_1(D_1) \subset \text{GL}(E)$ and

$$g(z) = q_g(z)^{-1}p_g(z), \text{ with } g'(z) = q_g(z)^{-1}, \text{ for all } z \in D_1. \quad (2.2)$$

**Proof:** For every $h(z) \partial/\partial z \in \mathfrak{hol}(D_2)$ we have by definition

$$g^*(h(z) \partial/\partial z) = \left( g'(z)^{-1}h(g(z)) \right) \partial/\partial z \in \mathfrak{hol}(D_1),$$
where $g'(z) \in \text{GL}(E)$ for $z \in D_1$, is the derivative of $g$ at $z$. For $h(z) \equiv \alpha$, with $\alpha \in E$, this implies $g'(z)^{-1}\alpha = q_g(z)\alpha$, and for $h(z) \equiv z$ we get $g'(z)^{-1}g(z) = p_g(z)$. Formula (2.2) follows from these two relations. □

Recall that $\mathfrak{s}$ is the complex solvable Lie subalgebra of $\mathfrak{hol}(E)$ spanned by all constant vector fields $\alpha \partial/\partial z$ and the Euler vector field $\eta$ (see (1.2)). Proposition 2.1 yields the following corollary.

**Proposition 2.2.** Suppose that for the biholomorphic map $g : D_1 \to D_2$ from Proposition 2.1 all vector fields in both $g^*(\mathfrak{s})$ and $g_*(\mathfrak{s})$ extend to rational vector fields on $E$. Then $g$ extends to an element of Bir($E$), with $\text{reg}(g) = \text{reg}(g') = \text{reg}(q^{-1})$.

**Proof:** We only need to show that $\text{reg}(g) = \text{reg}(g')$. Clearly, we have $\text{reg}(g) \subset \text{reg}(g')$. To obtain the opposite inclusion, we suppose that $\text{reg}(g') \setminus \text{reg}(g)$ is non-empty. We let $n := \dim E$, identify $E$ with $\mathbb{C}^n$, and write $g$ as $g = (g_1, \ldots, g_n)$. Then there exists $j$ such that $A := \text{reg}(g') \setminus \text{reg}(g_j)$ is non-empty. It then follows that one can find a point $a \in A$, which is not an indeterminacy point of $g_j$, that is, $g_j = r_j/s_j$, where $r_j$ and $s_j$ are polynomials with $r_j(a) \neq 0$, $s_j(a) = 0$. Hence for some $k$ the order of vanishing of $s_j\partial r_j/\partial z_k - r_j\partial s_j/\partial z_k$ at $a$ is finite and strictly less than that of $s_j^2$. Therefore, $a$ is not a regular point of $\partial g_j/\partial z_k$, which contradicts our choice of $a$. □

**Remark 2.3.** We will use Proposition 2.2 in Section 3 in the case when all vector fields in $g^*(\mathfrak{s})$ and $g_*(\mathfrak{s})$ extend to polynomial vector fields on $E$. In this situation $\text{reg}(g) = \text{reg}(q_g^{-1}) = \{z \in E : \det q_g(z) \neq 0\}$. In fact, $\det q_g$ is a denominator of the rational map $g$, that is, $(\det q_g)g$ is a polynomial map. As the following example shows, $\det q_g$ need not be an exact denominator (a denominator of minimal degree) of $g$.

**Example 2.4.** Let $E := \mathbb{C}^{n \times m}$, $b \in \mathbb{C}^{m \times n}$ a fixed matrix, and $g(z) := (\mathbb{I} - zb)^{-1}z$, where $\mathbb{I}$ is the $n \times n$ identity matrix. Then $g \in \text{Bir}(E)$ (indeed $g^{-1}(w) = (\mathbb{I} + wb)^{-1}w$). Differentiation yields $g'(z)\alpha = (\mathbb{I} - zb)^{-1}\alpha(\mathbb{I} - bz)^{-1}$ for all $\alpha \in E$. In particular, for the functions $p, q$ from Proposition 2.1 we have $q_g(z)\alpha = (\mathbb{I} - zb)\alpha(\mathbb{I} - bz)$ and $p_g(z) = z - zbz$. 


for all \( \alpha \in E \). Thus \( \det q_{\alpha} \) is not an exact denominator of \( g \). Further, a moment’s thought gives \( \det q_{\alpha}(z) = \det(\mathbb{I} - zb)^m \det(\mathbb{I} - bz)^n \), hence \( \text{reg}(g) = \{ z \in E : \det(\mathbb{I} - zb) \neq 0 \} \).

In the next proposition we relate the group \( \text{Bir}(M) \) of birational transformations of a CR-submanifold \( M \subset E \) to the subset \( \text{BR}(M) \subset \text{Bir}(E) \) by means of Condition \((*)\), as stated in Section 1.

**Proposition 2.5.** Let \( M \) be a connected real-analytic generic CR-submanifold of \( E \). If Condition \((*)\) is satisfied for \( M \), then \( \text{Bir}(M) = \text{BR}(M) \).

Moreover, for every \( g \in \text{BR}(M) \) we have \( g(M \cap \text{reg}(g)) = (M \cap \text{reg}(g^{-1})) \).

**Proof:** Fix \( g \in \text{BR}(M) \), and let \( V \subset M \) be a non-empty domain such that \( V \subset \text{reg}(g) \) and \( g(V) \subset M \). By Lemma 2.2 of [FK2] the non-empty set \( M \cap \text{reg}(g) \) is connected, and therefore \( M_1 := g(M \cap \text{reg}(g)) \) is a real-analytic connected submanifold of \( E \). Since \( W := g(V) \) is a non-empty domain in \( M \) such that \( W \cap M_1 = W \), Condition \((*)\) implies that \( M_1 \subset M \cap \text{reg}(g^{-1}) \). Interchanging the roles of \( g \) and \( g^{-1} \) gives \( g(M \cap \text{reg}(g)) = M \cap \text{reg}(g^{-1}) \).

Now for any \( g_1, g_2 \in \text{BR}(M) \) we choose a non-empty domain \( V \subset M \) with \( V \subset \text{reg}(g_1) \) and \( g_1(V) \subset \text{reg}(g_2) \). Then \( g_2 \circ g_1 \in \text{BR}(M) \). Therefore, \( \text{BR}(M) = \text{Bir}(M) \), as required. \( \square \)

We stress the importance of Proposition 2.5 for the correct understanding of \( \text{BR}(M) \) and \( \text{Bir}(M) \). In particular, if \( M \) does not satisfy the assumptions of Proposition 2.5 then the set \( \text{BR}(M) \) may not be a group.

As we stated in Section 1 a connected real-analytic generic submanifold \( M \subset E \) satisfies Condition \((*)\) if \( M \) is minimal and closed. There is, however, a large class of examples of non-closed CR-submanifolds satisfying Condition \((*)\). An interesting family of such manifolds is given in Example 5.4 in Section 5.

3. **Proof of Theorems 1.3 and 1.4**

We will first prove Theorem 1.3.

Without loss of generality we assume that \( M \) contains the origin, and let \( I \) be the complexification of \( \text{hol}(M, 0) \). Arguing as in the proof of Proposition 4.2 of [FK1], we obtain that \( I \) admits a \( \mathbb{Z} \)-grading

\[
I = \bigoplus_{m \in \mathbb{Z}} I^m, \quad [I^m, I^{\ell}] \subset I^{m+\ell}, \tag{3.1}
\]

where \( I^m \) is the \( m \)-eigenspace of \( \text{ad} \eta \) in \( I \), and \( I^m = 0 \) for \( m < -1 \), as well as for \( m \) large enough. Every \( I^m \) consists of polynomial vector fields
homogeneous of degree $m + 1$, with
\[ \Gamma^{-1} = \{ \alpha \partial/\partial z : \alpha \in E \} \]
being the Lie algebra of all constant vector fields on $E$. Thus every vector field in $\mathfrak{hol}(M, 0)$ is polynomial. Arguing in this way for every $a \in M$ we see that all Lie algebras $\mathfrak{hol}(M, a)$ are polynomial and hence coincide with $\mathfrak{hol}(M)$. Thus we have obtained statement (I).

For a non-empty domain $D \subset E$ we identify $\mathfrak{l}$ with a Lie subalgebra of $\mathfrak{hol}(D)$ by restriction. Let $V_1, V_2$ be non-empty domains in $M$, and $g : V_1 \to V_2$ a real-analytic CR-isomorphism. Then there exist domains $D_1, D_2 \subset E$ and a biholomorphic extension $g : D_1 \to D_2$ with $g_*(\mathfrak{l}) = \mathfrak{l}$. Since all vector fields in $\mathfrak{l}$ are polynomial, Proposition 2.2 yields that $g$ extends to an element of Bir$(M)$ of the form $q^{-1}p$, where $p : E \to E$ and $q : E \to \text{End}(E)$ are polynomial maps (see (2.1)). By Remark 2.3 we have \( \text{reg}(g) = \text{reg}(q^{-1}) = \{ z \in E : \det q(z) \neq 0 \} \). Thus we have obtained statement (II).

Further, for every $a \in E$ the isotropy Lie subalgebra
\[ \mathfrak{l}_a := \{ \xi : \xi_a = 0 \} \]
has codimension $n := \dim E$ in $\mathfrak{l}$, and $\mathfrak{l}$ is the direct sum of subspaces $\mathfrak{l} = \Gamma^{-1} \oplus \mathfrak{l}_a$, with $\mathfrak{l}_a \neq \mathfrak{l}_b$ for all $a, b \in E, a \neq b$. Let $\mathcal{G}$ be the Grassmannian of all complex linear subspaces $\Lambda \subset \mathfrak{l}$ of codimension $n$. Then $\mathcal{G}$ is a rational projective algebraic complex manifold on which the complex linear group $\text{GL}(\mathfrak{l})$ acts transitively and algebraically by means of the canonical projection $\text{GL}(\mathfrak{l}) \to \text{PGL}(\mathfrak{l}) \subset \text{Aut}(\mathcal{G})$.

The subset
\[ U := \{ \Lambda \in \mathcal{G} : \mathfrak{l} = \Gamma^{-1} \oplus \Lambda \} \]
is Zariski open in $\mathcal{G}$ and is algebraically equivalent to the complex vector space of all linear operators $\lambda : \mathfrak{l}_0 \to \Gamma^{-1}$ (just identify every $\lambda$ with its graph $\{ \xi + \lambda(\xi) : \xi \in \mathfrak{l}_0 \} \subset \mathcal{G}$). In this coordinate chart every automorphism of $\mathcal{G}$ arising from the action of $\text{GL}(\mathfrak{l})$ can be written as a matrix linear fractional transformation.

Consider the injective holomorphic map
\[ \varphi : E \to \mathcal{G}, \quad a \mapsto \mathfrak{l}_a. \]
Then $\varphi(E) \subset U$, and since all vector fields in $\mathfrak{l}$ are polynomial, the map $\varphi$ is an algebraic morphism. As a consequence, the set $\varphi(E)$ is constructible. Let $X$ be the Zariski closure of $\varphi(E)$. Clearly, $X$ is an irreducible algebraic subvariety in $\mathcal{G}$ and $\varphi(E)$ contains a Zariski open (and dense) subset of $X$, hence the topological closure of $\varphi(E)$ in $\mathcal{G}$ coincides with $X$. 

Define
\[ \text{Bir}(E, l) := \{ g \in \text{Bir}(E) : g_*(l) = l \} . \]
Observe that Bir\((E, l)\) contains the set BR\((M)\). Since every element of Bir\((M)\) is the composition of a finite number of elements of BR\((M)\), it follows that Bir\((M) \subset \text{Bir}(E, l)\).

For any \( g \in \text{Bir}(E, l) \), we regard the push-forward map \( g_* \) as an element of Aut\((l) \subset \text{GL}(l)\), where Aut\((l)\) is the complex algebraic subgroup of GL\((l)\) that consists of all Lie algebra automorphisms of \( l \). Define \( \nu \) to be the homomorphism
\[ \nu : \text{Bir}(E, l) \to \text{Aut}(l), \quad g \mapsto g_* . \] (3.2)

By formula (2.2) the homomorphism \( \nu \) is injective. Note that the canonical homomorphism \( \pi : \text{Aut}(l) \to \text{PGL}(l) \) is injective as well.

Since for \( g \in \text{Bir}(E, l) \) we have \( g_*(l_a) = l_{g(a)} \) for all \( a \in \text{reg}(g) \), the map \( \pi(g_*) \) preserves \( X \) and the following holds:
\[ \varphi \circ g = \sigma(g) \circ \varphi \quad \text{on } \text{reg}(g), \] (3.3)
where \( \sigma := \pi \circ \nu \). Formula (3.3) applies, in particular, to every translation \( g(z) = z + \beta, \beta \in E \) (note that every translation is an element of Bir\((E, l)\)). It is straightforward to see that the action of the complex vector group \( E^+ := (E, +) \) on \( \mathbb{G} \) through the homomorphism \( \sigma \) is algebraic, and formula (3.3) implies that \( \varphi(E) \) is an orbit of this action. It then follows that \( \varphi(E) \) is a Zariski open subset of \( X \) lying in the non-singular part \( X_{\text{reg}} \) of \( X \). By Zariski’s Main Theorem (see [Mu], III.9.1), \( \varphi : E \to \varphi(E) \) is an algebraic isomorphism.

We now embed \( \mathbb{G} \) into \( \mathbb{CP}^N \) for a sufficiently large integer \( N \) by the Plücker map, and regard \( X \) as an algebraic subvariety of \( \mathbb{CP}^N \), and PGL\((l)\) as a subgroup of PGL\(_{N+1}(\mathbb{C})\). Formula (3.3) then yields that Bir\((M)\) is projectively regularizable with \( \tau := \sigma|_{\text{Bir}(M)} \). This proves statement (III).

The proof of Theorem 1.3 is complete. \( \square \)

We will now prove Theorem 1.4.

Recall that \( \hat{M} \) is a connected generic immersed submanifold of the Zariski open subset \( \hat{E} \) of the non-singular part \( X_{\text{reg}} \) of \( X \) (see (1.1)). The minimality of \( M \) implies that \( \hat{M} \) is minimal as well. Therefore, it follows from Lemma 2.2 of [FK2] that \( \hat{M} \cap \varphi(E) \) is connected. Since \( \hat{M} \) contains \( \varphi(M) \) as an open subset, part (b) of Condition (*) yields that \( \hat{M} \cap \varphi(E) = \varphi(M) \) and that \( \varphi(M) \) is dense in \( \hat{M} \).
To show that $\text{Aut}(\hat{M}) = \tau(\text{Bir}(M))$, observe that for every $g \in \text{Aut}(\hat{M})$ there exist domains $V_1, V_2 \subset \varphi(M)$ such that $g(V_1) = V_2$. Then the composition $\varphi^{-1} \circ g \circ \varphi$ is a real-analytic CR-diffeomorphism between the domains $\varphi^{-1}(V_1), \varphi^{-1}(V_2)$ in $M$. By statement (II) of Theorem 1.3 the map $\varphi^{-1} \circ g \circ \varphi$ extends to an element $g_0$ of $\text{Bir}(M)$, hence $g = \tau(g_0)$. Thus $\text{Aut}(\hat{M}) = \tau(\text{Bir}(M))$.

Assume now that $\overline{M \setminus M}$ does not contain a CR-submanifold of $E$ locally CR-equivalent to $M$. Let $g_n$ be a sequence in $\tau(\text{Bir}(M))$ converging to an element $g \in \text{PGL}_{N+1}(\mathbb{C})$. We claim that $g(\varphi(M)) \cap \varphi(E) \neq \emptyset$. Indeed, otherwise $g(\varphi(M))$ lies in a Zariski closed subset of $X_{\text{reg}}$, which is impossible since $g(\varphi(M))$ is generic in $X_{\text{reg}}$. Thus for some domain $V \subset \varphi(M)$ we have $g(V) \subset \varphi(E)$. Clearly, $g(V)$ is a CR-submanifold of $\varphi(E)$ locally equivalent to $\varphi(M)$ and contained in $\varphi(M)$. It then follows that there exists $x_0 \in V$ for which $g(x_0) \in \varphi(M)$. Since $M$ is locally closed in $E$, for some neighborhood $V' \subset V$ of $x_0$ in $\varphi(M)$ we have $g(V') \subset \varphi(M)$. Hence $g \in \tau(\text{Bir}(M))$, and therefore $\tau(\text{Bir}(M))$ is closed in $\text{PGL}_{N+1}(\mathbb{C})$.

Let $\mathfrak{a}$ be the Lie subalgebra of the Lie algebra of $\text{PGL}_{N+1}(\mathbb{C})$ corresponding to the closed subgroup $\tau(\text{Bir}(M)) \subset \text{PGL}_{N+1}(\mathbb{C})$. Every element $v \in \mathfrak{a}$ is a holomorphic vector field on $\mathbb{CP}^N$ tangent to $\hat{M}$ and gives rise to a holomorphic vector field on $E$ tangent to $M$.

Conversely, consider a vector field $\xi \in \mathfrak{hol}(M)$ and fix $a \in M$. Near $a$ the vector field $\xi$ can be integrated to a local 1-parameter group $t \mapsto g_t$, with $|t| < \varepsilon$, of local real-analytic CR-isomorphisms of $M$. By statement (II) of Theorem 1.3 every $g_t$ extends to an element of $\text{Bir}(M)$. Further, one can define by composition a map $g_t \in \text{Bir}(M)$ for every $t \in \mathbb{R}$ and obtain a 1-parameter subgroup of $\text{Bir}(M)$. Then $t \mapsto \tau(g_t)$ is a continuous 1-parameter subgroup of $\tau(\text{Bir}(M))$ and hence $g_t = \exp(tv)$ for some $v \in \mathfrak{a}$.

Thus we have established an isomorphism between $\mathfrak{a}$ and $\mathfrak{hol}(M)$, and the proof of Theorem 1.3 is complete. \qed

For the remainder of the article we set $\mathfrak{g} := \mathfrak{hol}(M)$. Let $M$ satisfy the assumptions of Theorem 1.3 and set $\rho := \nu|_{\text{Bir}(M)}$, with $\nu$ defined in (3.2). The homomorphism $\rho$ is injective, and the image $\rho(\text{Bir}(M))$ lies in the group $\text{Aut}(\mathfrak{g})$ of all Lie algebra automorphisms of $\mathfrak{g}$, which is a real algebraic subgroup of each of $\text{GL}(\mathfrak{g})$ and $\text{Aut}(\mathfrak{l})$. Thus $\text{Bir}(M)$ can be endowed with the structure of a Lie group (possibly with uncountably many connected components). An argument identical to that at the end of the proof of Theorem 1.3 shows that the Lie algebra of $\text{Bir}(M)$ with respect to this Lie group structure is isomorphic to $\mathfrak{g}$. For $\text{Bir}(M)$ with this Lie group structure
the map
\[ \rho : \text{Bir}(M) \to \text{Aut}(g) \]
is just the adjoint representation. In the next section we will show that under additional assumptions the image \( \rho(\text{Bir}(M)) \) is closed in \( \text{Aut}(g) \).

4. Closed Embedding of \( \text{Bir}(M) \) into \( \text{Aut}(g) \)

Throughout this section we suppose that \( M \) has Property (P) and satisfies Condition \((\ast)\). In particular, by Proposition 2.5 we have \( \text{Bir}(M) = \text{BR}(M) \). Under these assumptions, which are weaker than those of Theorem 1.4, we obtain the existence of a Lie group structure on \( \text{Bir}(M) \) with at most countably many connected components and Lie algebra \( g \). Instead of investigating the closedness of \( \tau(\text{Bir}(M)) \) in \( \text{PGL}_{N+1}(\mathbb{C}) \), as we did in the proof of Theorem 1.4, we investigate the closedness of \( \rho(\text{Bir}(M)) \) in \( \text{Aut}(g) \). Note that a simple example shows that there is always a closed subgroup of \( \text{GL}(l) \) whose image in \( \text{PGL}(l) \) under the canonical projection \( \text{GL}(l) \to \text{PGL}(l) \) is not closed.

The result of this section is the following theorem.

**THEOREM 4.1.** Let \( M \) be a connected real-analytic generic CR-submanifold of a finite-dimensional complex vector space \( E \). Assume that \( M \) has Property (P) and satisfies Condition \((\ast)\). Then \( \rho(\text{Bir}(M)) \) is closed in \( \text{Aut}(g) \).

**Proof:** Let \( g_n \) be a sequence in \( \text{Bir}(M) \) such that the sequence \( f_n := \rho(g_n) \) converges in \( \text{Aut}(g) \) to an element \( f \). By Proposition 2.1 every map \( g_n \) can be written as \( g_n = q_n^{-1} p_n \), where \( p_n := p_{g_n} \) and \( q_n := q_{g_n} \) are polynomial maps on \( E \) found from formulas (2.1). Since the maps \( f_n^{-1} \) converge in \( \text{Aut}(g) \) to \( f^{-1} \), the sequences \( p_n, q_n \) converge (uniformly on compact subsets of \( E \)) to polynomial maps \( p : E \to E \) and \( q : E \to \text{End}(E) \), respectively, and we have \( f^{-1}(\alpha \partial/\partial z) = (q(z)\alpha) \partial/\partial z \) for all \( \alpha \in E \).

Similarly, every map \( g_n^{-1} \) can be written as \( g_n^{-1} = \tilde{q}_n^{-1} \tilde{p}_n \), where \( \tilde{p}_n := p_{g_n^{-1}} \) and \( \tilde{q}_n := q_{g_n^{-1}} \) are polynomial maps. The sequences \( \tilde{p}_n \) and \( \tilde{q}_n \) converge to polynomial maps \( \tilde{p} : E \to E \) and \( \tilde{q} : E \to \text{End}(E) \), respectively, and we have \( f(\alpha \partial/\partial z) = (\tilde{q}(z)\alpha) \partial/\partial z \) for all \( \alpha \in E \).

Since \( f_k^{-1} f \to \text{id} \in \text{Aut}(g) \), for every neighborhood \( \mathcal{V} \) of the identity in \( \text{Aut}(g) \) one can find an element \( \hat{f} \in \rho(\text{Bir}(M)) \) with \( \hat{f} \in \mathcal{V} \). Choosing \( \mathcal{V} = \mathcal{V}^{-1} \) one can also assume that \( f^{-1} \hat{f}^{-1} \in \mathcal{V} \). Hence by replacing \( g_n \) by \( \rho^{-1}(\hat{f}) g_n \) and \( f \) by \( \hat{f} f \) we can assume without loss of generality that \( \det q \neq 0 \) and \( \det \tilde{q} \neq 0 \). We then define the rational maps \( g := q^{-1} p \) and \( \tilde{g} := \tilde{q}^{-1} \tilde{p} \).
Let
\[ A := \{ z \in E : \det q(z) = 0 \}, \quad B := \{ z \in E : \det \tilde{q}(z) = 0 \}. \]
Since \( \det q_n \rightarrow \det q \) and \( \text{reg}(g_n) = \{ z \in E : \det q_n(z) \neq 0 \} \) (see Proposition 2.2), it follows that \( g_n \rightarrow g \) uniformly on compact subsets of \( E \setminus A \). Similarly, \( g_n^{-1} \rightarrow \tilde{g} \) uniformly on compact subsets of \( E \setminus B \). For every \( \xi \in \mathfrak{l} \), \( \xi = h(z) \partial/\partial z \), we have
\[ f(\xi)(z) = \tilde{q}(z)h(\tilde{g}(z)) \partial/\partial z, \quad f^{-1}(\xi)(z) = q(z)h(g(z)) \partial/\partial z. \]
Applying the identity
\[ \xi = f(f^{-1}(\xi)) = \tilde{q}(z)q(\tilde{g}(z))h(\tilde{g}(\tilde{g}(z))) \partial/\partial z \]
to constant vector fields \( \xi = \alpha \partial/\partial z \), we see that \( \tilde{q}(z)q(\tilde{g}(z)) \equiv \text{id} \). Applying this identity to the Euler vector field \( \eta \) and interchanging the roles of \( f \) and \( f^{-1} \) we obtain \( \tilde{g} = g^{-1} \), thus \( g \in \text{Bir}(E) \). By Proposition 2.5 we have \( g_n \in \text{BR}(M) \) and \( g_n(M \cap \text{reg}(g_n)) = M \cap \text{reg}(g_n^{-1}) \), which yields that \( g \in \text{Bir}(M) \). Finally, since \( g_n \rightarrow g \) on \( E \setminus A \) and \( g_n^{-1} \rightarrow g^{-1} \) on \( E \setminus B \), it follows that \( (g_n)_* \rightarrow g_* \). Hence \( f = \rho(g) \).

The proof of the theorem is complete. \( \square \)

5. Property (P)

In this section we give sufficient conditions for a CR-submanifold to have Property (P) and discuss examples of manifolds with this property (for the statement of Property (P) see Section 1).

We call a CR-submanifold \( M \) of a complex manifold \( Z \) \textit{semi-homogeneous} at a point \( a \in M \), if the span over \( \mathbb{C} \) of the values \( \xi_a \) of elements \( \xi \in \mathfrak{hol}(M,a) \) at \( a \) contains \( T_a(M) \). Also, we call \( M \) \textit{semi-homogeneous} if \( M \) is semi-homogeneous at every point. Clearly, local homogeneity implies semi-homogeneity for a real-analytic CR-submanifold.

Recall that \( \mathfrak{g} = \mathfrak{hol}(M) \). With this notation we state the following proposition.

\textbf{Proposition 5.1.} Let \( M \) be a connected real-analytic generic CR-submanifold of a finite-dimensional complex vector space \( E \). Let \( a \) be a point in \( M \). Assume that:

1. \( M \) is holomorphically non-degenerate at \( a \);
2. \( M \) is minimal at \( a \);
3. \( M \) is semi-homogeneous at \( a \);
4. the complex Lie algebra \( \mathfrak{g} + i\mathfrak{g} \) contains the vector field \( (z - a) \partial/\partial z \).
Then $M$ has property (P).

**Proof:** By Theorem 12.5.3 in [BER], assumptions (1) and (2) imply that $\mathfrak{hol}(M,b)$ is finite-dimensional for all $b \in M$.

Without loss of generality we assume that $a = 0$. Then using assumption (4) we obtain, as at the beginning of Section 3, that $\mathfrak{l} := \mathfrak{hol}(M,0) + \mathfrak{i}\mathfrak{hol}(M,0)$ admits grading (3.1) where $\mathfrak{l}^m$ is the $m$-eigenspace of $\text{ad} \eta$ in $\mathfrak{l}$, and $\mathfrak{l}^m = 0$ for $m < -1$, as well as for $m$ large enough. Every $\mathfrak{l}^m$ consists of polynomial vector fields homogeneous of degree $m + 1$. Since all vector fields in $\mathfrak{l}^m$ for $m > -1$ vanish at the origin, assumption (3) implies that $\mathfrak{l}^{-1}$ is the space of all constant vector fields on $E$. The proof is complete. □

We now give examples of CR-submanifolds that have Property (P).

**Example 5.2.** Every quadric $Q_h \subset \mathbb{C}^{n+k}$ associated to a non-degenerate Hermitian form $h$ has Property (P). Indeed, $Q_h$ is homogeneous under the action of the group of maps

$$
z \mapsto z + \alpha, \quad w \mapsto w + 2i h(z, \alpha) + \beta,$$

where $(\alpha, \beta) \in Q_h$. Thus all Lie algebras $\mathfrak{hol}(Q_h, a)$ coincide. In fact, they coincide with the Lie algebra $\mathfrak{hol}(Q_h)$, which is finite-dimensional (see [B1], [B2], [Tum]). Furthermore, $\mathfrak{hol}(Q_h)$ clearly contains the vector fields $s \partial/\partial w$, $r \partial/\partial z + 2i h(z, r) \partial/\partial z$, $z \partial/\partial z + 2w \partial/\partial w$, $iz \partial/\partial z$, with $s \in \mathbb{R}^k$, $r \in \mathbb{C}^n$. Therefore, the complexification of $\mathfrak{hol}(Q_h)$ contains all constant vector fields and the Euler vector field $\eta$. Hence the quadric $Q_h$ has Property (P).

**Example 5.3.** Let $F \subset \mathbb{R}^n$ be an arbitrary connected real-analytic submanifold and

$$M := F + i\mathbb{R}^n \subset \mathbb{C}^n$$

the corresponding tube submanifold with base $F$. Then $M$ is a generic semi-homogeneous CR-submanifold of $E$, and $\mathfrak{g} + i\mathfrak{g}$ contains all constant holomorphic vector fields on $E$. Furthermore (see Lemma 4.1 of [FK2]), the tube manifold $M$ is minimal at a point if and only if

$$F \text{ is not contained in any affine hyperplane of } \mathbb{R}^n,$$

hence a tube manifold is minimal if it is minimal at one point. Next (see Proposition 4.3 of [FK2]), $M$ is holomorphically non-degenerate at a point if and only if

$$\text{the only constant vector field } \xi = \alpha \partial/\partial x \text{ tangent to } F \text{ is } \xi = 0,$$
hence a tube manifold is holomorphically non-degenerate if it is holomorphically non-degenerate at one point.

To meet conditions (1), (2), (4) of Proposition 5.1 it is therefore sufficient to require besides (5.1), (5.2) that \( F \) is a cone, that is, \( tF = F \) for every real \( t > 0 \). For every cone \( F \) the Levi form of \( M \) is degenerate at every point (condition (ii) stated at the beginning of Section 1 does not hold).

From the large class of tube manifolds that have Property (P) we single out the following special one.

**Example 5.4.** Fix integers \( p \geq q \geq 1 \) with \( n = p + q \geq 3 \). Then
\[
H_{p,q} := \{ x \in \mathbb{R}^n : x_1^2 + \cdots + x_p^2 = x_{p+1}^2 + \cdots + x_n^2 \}
\]
(5.3)
is a real hyperquadric with 0 as the only singularity. Let \( F \) be a connected component of the non-singular part of \( H_{p,q} \) and \( M := F + i\mathbb{R}^n \) the corresponding tube submanifold. Then \( M \) is a homogeneous CR-submanifold of \( \mathbb{C}^n \) that has Property (P) and satisfies Condition (*). The Levi form of \( M \) is degenerate at every point. For \( q = 1 \) the non-singular part of \( H_{p,q} \) has two connected components (given by \( x_n > 0 \) and \( x_n < 0 \)), the future light cone and the past light cone. In this case the group \( \text{Bir}(M) \) can be canonically identified with an open subgroup (having two connected components) of \( O(n,2) \). For \( q > 1 \) the non-singular part of \( H_{p,q} \) is connected, and \( \text{Bir}(M) \) is a real algebraic group.

For every \( a \in M \) the Lie algebra \( \mathfrak{hol}(M,a) \) is isomorphic to \( \mathfrak{so}(p+1,q+1) \) (cf. [FK1], p. 21).

**Example 5.5.** Let \( D \subset E \) be an irreducible bounded symmetric domain of rank \( r \) given in its Harish-Chandra realization, and \( Z \) its compact dual containing \( E \) as a Zariski open subset. Then \( D \) is convex and invariant under the circle group \( \exp(i\mathbb{R}\eta) \subset \text{GL}(E) \). The complex manifold \( Z \) is a compact rational algebraic variety on which the simple complex Lie group \( L := \text{Aut}(Z) \) acts transitively. All transformations in \( G := \text{Aut}(D) \) extend to elements of \( L \) and, in this way, \( G \) is a real form of \( L \) and also acts on \( Z \). On \( Z \) the group \( G \) has exactly \( \binom{r+2}{2} \) orbits. Among these there are exactly \( r+1 \) open orbits (including \( D \)) and a unique closed orbit, the Šilov boundary of \( D \), which coincides with the extremal boundary \( \partial_e D \) of the convex set \( \mathcal{D} \). Every \( G \)-orbit is a generic homogeneous CR-submanifold of \( Z \) invariant under the circle group \( \exp(i\mathbb{R}\eta) \subset G \). Furthermore, for every orbit \( G(b), b \in Z \), the intersection \( M := G(b) \cap E \) is a connected CR-submanifold that has Property (P), and for every \( a \in M \) the Lie algebra \( \mathfrak{hol}(M,a) \) is isomorphic to the Lie algebra of \( G \), provided neither \( G(b) \) is open in \( Z \) nor \( G(b) = \partial_e D \) in the case when \( D \) is of tube type (in this last case \( \partial_e D \) is totally real). The Šilov boundary (except when \( D \) is of tube type) can be locally
realized as a standard quadric in $E$ and, in particular, has non-degenerate Levi form. Every $G$-orbit that is neither open nor closed in $Z$ is Levi degenerate (more precisely 2-nondegenerate) and hence cannot be locally realized as a standard quadric in $E$. The group Bir($M$) is regularizable on the simply-connected complex manifold $Z$, and this is the only possibility up to isomorphism.

**Example 5.6.** We specialize Example 5.5 to the case

$$E = \{z \in \mathbb{C}^{2 \times 2} : z' = z\} \quad \text{and} \quad D = \{z \in E : zz^* < I\},$$

where $z'$ is the transpose and $z^*$ is the transpose conjugate of a matrix $z$. Then $D$ is irreducible symmetric of rank 2, and $Z$ can be identified with a complex projective quadric in $\mathbb{CP}^4$. Also, $G$ is isomorphic to an open subgroup of $O(2,3)$ of index 2. The boundary $\partial D$ of $D$ decomposes into two $G$-orbits: the totally real Šilov boundary $\partial_e D \simeq \mathbb{RP}^3$ and the smooth part $M$ of $\partial D$ that has Property (P). The manifold $M$ is locally CR-equivalent to the tube submanifold over the future the light cone (see (5.3) for $p = 2$, $q = 1$). Here $Z$ is simply-connected while the homogeneous manifold $M$ has fundamental group $\mathbb{Z}_2$.

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