Entanglement Properties of Localized States in 1D Topological Quantum Walks

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The symmetries associated with discrete-time quantum walks (DTQWs) and the flexibilities in controlling their dynamical parameters allow to create a large number of topological phases. An interface in position space, which separates two regions with different topological numbers, can, for example, be effectively modelled using different coin parameters for the walk on either side of the interface. Depending on the neighbouring numbers, this can lead to localized states in one-dimensional configurations and here we carry out a detailed study into the strength of such localized states. We show that it can be related to the amount of entanglement created by the walks, with minima appearing for strong localizations. This feature also persists in the presence of small amounts of σx (bit flip) noise.

I. INTRODUCTION

Quantum walks1,2 can be used to efficiently create non-classical states and have therefore been of large interest for designing quantum algorithms3–9 and realizing universal quantum computation10,11. However, in recent years, quantum walks have also been employed to understand the dynamics of a considerable range of other physical processes, for example dielectric breakdown in driven electron system12, transport in biological or chemical systems13–15 or effects in relativistic quantum dynamics16–19. One of the more recent topics of interest is the creation of topological phases using quantum walks. Topological properties of materials have recently been recognised as a rich source of interesting physics and have led to a new class known as topological insulators (TIs)20–22. However, only a small number of natural TIs are known and therefore interest in creating artificial materials with non-trivial topological states is a prime research activity. Discrete-time quantum walks (DTQWs) are a method to create such states, as they can simulate time-independent lattice Hamiltonians with the required symmetries. At the same time they possess additional degrees of freedom, for example the possibility for varying coin operations, which can lead to much richer system23–28. Progress in the theoretical understanding of these systems is going hand in hand with current advances in experimental implementations and engineering of quantum walks in various physical systems29. Exploring topological phases using DTQWs has therefore emerged as a promising approach to realizing TIs in artificial materials.

The nontrivial topological phases of TIs are intricately linked to the presence or absence of certain symmetries, namely, time-reversal symmetry, particle-hole symmetry, and chiral symmetry30. For one-dimensional DTQWs with the all three symmetries present (belonging to class BDI), the topological properties have recently been studied using a split-step24 and a double split-step DTQW27. Due to the 2π periodicity of the quasi-energies, the topological numbers of a 1D DTQW are defined not only for 0 but also for π quasi-energies, which means that they become $\mathbb{Z} \times \mathbb{Z}$ winding numbers27. Consequently, at the interface where two domains with different winding numbers are connected, topologically protected surface states appear at the two specific quasi-energies. Because of the one-dimensionality and the particle-hole symmetry, these surface states are the localized Majorana edge states, which have recently been experimentally observed25.

As the winding number is a function of the angle θi used in the quantum coin operation for each split-step, the phase diagram of a TI can be written in terms of the angle on either side of the interface. This allows to identify the combinations that lead to the appearance of localized states at the interface, but does not give any information about the strength of the localization, i.e. the probability of finding the particle at the interface. While for some configurations the localization is very strong and only a small probability of finding the particle away from the interface exist, for other configurations it can be weaker. Knowing which configurations result in strongly topologically localized states is necessary to identify parameters that lead to TIs with a strongly insulating bandstructure.

In the following we will show that strong localization at the interface due to topological effects can be signaled by a minimum in entanglement generated during split-step and double split-step DTQWs. This is in contrast to the properties found for localized states due to disordered coin operations in DTQWs, where for the standard DTQW an enhancement of entanglement is seen for temporal31,32 and spatio-temporal disorder and only a small decrease is seen for purely spatial disorder31. In addition, we will also discuss the effect of noise on topologically localized states and show that they are robust against σx (bit flip) noise.

II. TOPOLOGICAL QUANTUM WALKS

A 1D DTQW is defined for a system composed of a particle space and a position space. The basis states of the particle space can be any two internal states represented by $|0\rangle$ and $|1\rangle$ and the basis states of the
position space are defined on $|x\rangle$, where $x$ is an integer. If the initial state is given by a particle in state $|\Psi_{in}\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |x = 0\rangle$, which is located at the origin, each step of the walk is composed of a quantum coin operation

$$R_{\theta} = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \otimes I,$$

followed by a position shift operation

$$S = |0\rangle |0\rangle \otimes |x - 1\rangle + |1\rangle |1\rangle \otimes |x + 1\rangle.$$

The unitary operator $W(\theta) = S R_{\theta}$ therefore defines one step of the standard DTQW and the state after $t$ steps is given by $|\Psi_t\rangle = [S R_{\theta}]^t |\Psi_{in}\rangle$.

The eigenstates of the single time step operator can be written as

$$W(\theta)|\psi_x\rangle = e^{-i\varepsilon}|\psi_x\rangle,$$

where the quasi-energies $\varepsilon$ are real and have $2\pi$ periodicity. These spectral properties of $W(\theta)$ give insight into the long time behavior of the walk and therefore also the behavior of topologically protected localized states.

Nontrivial topological phases in DTQW can be found when the evolution operator $W(\theta)$ indicates the presence of certain specific symmetries, such as time-reversal, particle-hole, or chiral symmetries. For 1D systems particle-hole or chiral symmetries are known to be important to lead to two different topological numbers, $\nu_0$ and $\nu_\pi$, for the quasi-energies $\varepsilon = 0$ and $\pi$, which in turn leads to edge states that are localized at the interfaces across which the topological numbers change.

As all the matrix elements of the operator $W(\theta)$ used for defining the DTQW above are real, particle-hole symmetry is automatically guaranteed. To ensure chiral symmetry for $W(\theta)$ one needs to ensure the existence of a chiral operator $\Gamma$ which satisfy the relation

$$\Gamma W(\theta) \Gamma^{-1} = W(\theta)^{-1},$$

For this we will first decompose the operator $W(\theta)$ as

$$W(\theta) = F \cdot G,$$

where $F$ and $G$ are two sub-steps with each being a composition of coin ($R_{\theta}$) and shift operator ($S$). They are related by

$$\Gamma F \Gamma^{-1} = G^{-1},$$

and the above expression is guaranteed if the components of both, $F$ and $G$, satisfy

$$\Gamma R_{\theta} \Gamma^{-1} = R_{\theta}^{-1} \quad \text{and} \quad \Gamma S \Gamma^{-1} = S^{-1}.$$  

This leads to a chiral symmetry operator of the form

$$\Gamma \equiv \sigma_x \otimes I, \quad \text{with} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

FIG. 1: Schematic of the position space showing the boundary created by $\theta_{2-}$ and $\theta_{2+}$, which leads to the appearance of topological properties.

The topological numbers $(\nu_0, \nu_\pi)$ of the 1D DTQW stemming from this kind of chiral symmetry have been calculated recently. If $W(\theta)$ satisfies chiral symmetry (Eq. (4)), a counterpart state with opposite sign of the quasi-energy is guaranteed,

$$W(\theta)|\psi_{-\varepsilon}\rangle = e^{i\varepsilon}|\psi_{-\varepsilon}\rangle,$$

where $|\psi_{-\varepsilon}\rangle \equiv \Gamma |\psi_\varepsilon\rangle.$

Taking into account the $2\pi$ periodicity of $\varepsilon$, the above relation for the edge states at $\varepsilon = 0$ and $\pi$ is therefore identical to the eigenstate equation of the chiral symmetry operator $\Gamma$,

$$\Gamma |\psi_{0,\pi}\rangle = \pm |\psi_{0,\pi}\rangle,$$

with the eigenvalues $\pm 1$.

In the following we will focus on two specific DTQW with chiral symmetry and discuss their topological properties. First we consider a DTQW with each step split into two with different coin parameters $\theta_1$ and $\theta_2$ as

$$W(\theta_1, \theta_2) = S_+ R_{\theta_2} S_- R_{\theta_1},$$

and for which the position shift operators are

$$S_- = |0\rangle \langle 0| \otimes |x - 1\rangle \langle x + 1| + |1\rangle \langle 1| \otimes |x\rangle \langle x|;$$

$$S_+ = |0\rangle \langle 0| \otimes |x\rangle \langle x| + |1\rangle \langle 1| \otimes |x + 1\rangle \langle x|.$$  

To create a real space boundary between topologically distinct phases and reveal non-trivial topological properties at the interface, one can choose different $\theta_2$ to the left ($R_{\theta_2}$) and right side ($R_{\theta_2}$) of a point in the position space as shown in Fig. 1, while defining the coin operation $R_{\theta_1}$ uniformly on the entire position space.

The topological numbers $(\nu_0, \nu_\pi)$ for this split-step DTQW as a function of the coin parameters $\theta_1$ and $\theta_2$ are shown in Fig. 2. Using this phase diagram one can identify combinations of $\theta_{2-}$ and $\theta_{2+}$ that are located in regions with different topological numbers and therefore lead to an interface where a localized state can exist.

Examples of the behaviour can be seen in Figs. 3(b) and 3(d), where we show the spatial probability distribution after 100 steps for $(\theta_1, \theta_{2+}, \theta_{2-}) =
The four pairs of parameters \( (\theta_1, \theta_2, \theta_3, \theta_4) \) associated with the split-step DTQW as a function of the coin parameters \( \theta_1 \) and \( \theta_2 \). Topologically distinct gapped phases are separated by lines where the gap closes at either \( \varepsilon = 0 \) (solid lines) or \( \varepsilon = \pi \) (dashed lines). Fours pairs of circles with different colours and labeled as (a), (b), (c) and (d) are marked to identify different parameters corresponding to the probability distributions shown in Fig. 3.

\[(\pi/2, -3\pi/4, 3\pi/4) \text{ and } (-3\pi/2, -\pi, \pi).\] In both cases localization is clearly visible. If, on the other hand, \( \theta_{2+} \) and \( \theta_{2-} \) correspond to regions with the same topological number, the probability at \( x = 0 \) decreases with time, indicating the absence of a localized state. Examples of this are shown in Figs. 3(a) and 3(c) for values of \( (\theta_1, \theta_{2+}, \theta_{2-}) = (\pi/2, -\pi/4, \pi/4) \) and \( (-3\pi/2, 5\pi/4, 3\pi/4) \), respectively.

A second class of DTQWs with rich topological features are double split-step evolutions \(^{27}\). These are described by four parameters \( \theta_1 \), leading to an effective Hamiltonian with long range hopping that results in higher values for winding numbers and topological numbers. Each step in the double split-step walk is a composition of the operators

\[
W(\theta_1, \theta_2, \theta_3, \theta_4) = S_+ R_{\theta_1} S_+ R_{\theta_2} S_+ R_{\theta_3} S_+ R_{\theta_4},
\]

where setting \( \theta_2 = \theta_4 \) ensures chiral symmetry (CS). For simplicity we will also set \( \theta_1 = 0 \) in the following and in Fig. 4 we show the phase diagram as function of \( \theta_2 \) and \( \theta_3 \). Regions with different topological numbers can again be clearly identified \(^{27}\) and in Fig. 5 examples of spatial probability distributions for the situation in which different coin parameters have been used on the left and the right of the initial position, creating a boundary at the origin, are shown. The four pairs of parameters \( (\theta_2, \theta_3) \) used to generate these probability distributions are marked with circles of different color in the phase diagram (Fig. 4). For the parameters \( (\theta_{2-}, \theta_{2+}; \theta_{3-}, \theta_{3+}) = (-\pi/8, \pi/8; -\pi, \pi) \) and \( (-3\pi/8, \pi/8; -3\pi/2, -\pi/2) \), chosen from regions with different topological numbers, the probabilities of finding the particle at \( x = 0 \) remains high, indicating the presence of localized state (see Figs. 5(b) and 5(d)). For the parameters \( (\theta_{2-}, \theta_{2+}; \theta_{3-}, \theta_{3+}) = (-\pi/4, \pi/4; \pi, \pi) \) and \( (-\pi/4, 3\pi/4; \pi/4, \pi/4) \), chosen from regions with same topological numbers, the probability of finding the particle at \( x = 0 \) is very low, indicating the dominance of diffusion (see Figs. 5(a) and 5(c)). One should note that it is possible to generate localized states for certain sets of parameters from the regions with the same topological invariant. Those, however, have energies different from 0 or \( \pi \) and are therefore distinguishable from localized states originating from topological effects.

### III. Entanglement Properties

DTQW are known to entangle the particle and the position space. The degree of entanglement depends on the parameters that define the evolution operators \(^{33,34}\) and it is intriguing to explore this quantity for topological quantum walks. While in the previous section localized states were shown to appear when choosing coin parameters for the left and right regions from areas with different topological numbers in the phase diagram, no indication could be drawn from this about the strength of the localized state. As often the probability of the diffusing component can be higher than the one of the localized part, it is important to identify the parameters that lead

![Fig. 2: Phase diagram of the topological numbers (ν0, ν1) associated with the split-step DTQW as a function of the coin parameters θ1 and θ2.](image-url)

![Fig. 3: Probability distributions of the split-step DTQW after 100 steps with a particle initially in the state |Ψin⟩ = 1/√2(|0⟩ + |1⟩) ⊗ |x = 0⟩.](image-url)
FIG. 4: Phase diagram of the topological numbers associated with double split-step DTQW as a function of the coin parameters $\theta_2$ and $\theta_3$. The parameters $\theta_2$ and $\theta_4$ are set equal to $\frac{\pi}{2}$ to ensure CS and we have chosen $\theta_1 = 0$. Topologically distinct gapped phases are separated by lines where the gap closes at either $\varepsilon = 0$ (solid lines) or $\varepsilon = \pi$ (dashed lines). Four pairs of circles with different shades labeled as (a), (b), (c) and (d) are marked to identify the different parameters used to generate probability distributions shown in Fig. 5.

To the highest probability for finding a strongly topologically localized state at $x = 0$, in order to create artificially synthesized TIs. In this section we ask and answer the question if entanglement is an effective measure to identify the configurations of parameters that result in strongly localized states. For this we calculate the entanglement generated by different topological quantum walks and identify the regions which lead to strongly localized states.

To quantify the entanglement between the particle and the position space we will use negativity, which is the absolute sum of the negative eigenvalues of the partial transpose of the density operator, $\rho = |\Psi_t\rangle\langle\Psi_t|$. It is given by

$$N(\rho) = \sum_i \frac{|\lambda_i| - \lambda_i}{2}$$

(16)

where the $\lambda_i$ are the eigenvalues of $\rho$.

To reduce the number of free parameters we initially fix $\theta_1 = -3\pi/2$ for the split-step DTQW and show in Fig. 6 the negativity as function of $\theta_2$ and $\theta_3$. A varied landscape is clearly visible and to interpret the structure, one can map the diagram to the one for the topological numbers.

FIG. 5: Probability distributions of the double split-step DTQW after 50 steps of the walk for a particle with the initial state $|\Psi_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |x = 0\rangle$. The chosen parameters for $(\theta_{2-}, \theta_{2+}; \theta_{3-}, \theta_{3+})$ are marked with circles in Fig. 4, (a) $(-\pi/4, \pi/4; \pi, \pi)$, (b) $(-\pi/8, \pi/8; -\pi, \pi)$, (c) $(-\pi/4, -3\pi/4; \pi/4, \pi/4)$ and (d) $(-3\pi/8, \pi/8; -3\pi/2, -\pi/2)$. The parameters in (a) and (c) are from regions with the same topological number and the absence of a localized state is evident, whereas the parameters in (b) and (d) are from regions with different topological numbers, which leads to localization.

FIG. 6: Negativity as a function of $\theta_{2-}$ and $\theta_{2+}$ for the split-step DTQW with $\theta_1 = -3\pi/2$. The initial state was given by $|\Psi_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |x = 0\rangle$ and the results are shown after 100 steps of walk.

For this we show in Fig. 7 the phase diagram again and the vertical line at $\theta_1 = -3\pi/2$ indicates the parameters for which the negativity in displayed in Fig. 6. If $\theta_{2+} = \pi$ (marked with a filled circle in region D) and $\theta_{2-}$ is ranging from $-2\pi$ to $2\pi$, one can see that the values of negativity have two clear trends. They are high if $\theta_{2-}$ is in a region with the same topological invariant as $\theta_{2+}$ (here the gray area around D) and low if $\theta_{2-}$ is in a region that has a different topological invariant. In fact, one can see that a minimum appears when $\theta_{2-}$ is from region $B$ and we find that in general lower values of negativity indicate the presence of localized states with...
FIG. 7: Phase diagram of the split-step DTQW. The vertical line corresponds to the results shown in Fig. 6 and the horizontal lines correspond to the negativity presented in Fig. 8.

FIG. 8: Negativity of the split-step DTQW as a function of $\theta_1$ and $\theta_2$ for $\theta_2 = 3\pi/4$. The initial state was $|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |x = 0\rangle$ and the results shown are after 100 steps of walk.

smaller fractions of the particle’s amplitude diffused. (see Fig. 3(d)).

In Fig. 8 we show the negativity as a function of $\theta_1$ and $\theta_2$ by fixing $\theta_2 = 3\pi/4$. When $\theta_2 = -\pi$ the values of $\theta_2$ and $\theta_2$ will be in regions with different topological numbers for all values of $\theta_1$, except for $\theta_1 = \pm \pi$, where regions of different topological numbers meet. This can be seen from Fig. 7 where the two horizontal lines indicate the topological regions in which $\theta_2$ and $\theta_2$ lie when $\theta_1$ is ranging from $-2\pi$ to $2\pi$. The corresponding values of negativity correspond to the vertical line in Fig. 8 and one can clearly see low values of negativity for all values of $\theta_1$, except at the points $\theta_1 = \pm \pi$. A general comparison of the phase diagram (Fig. 7) and the negativity profiles (Figs. 6 and 8) for different combinations of $\theta_1$, $\theta_2$, and $\theta_2$ shows that low values for the negativity appear whenever the combination of $\theta_2$ and $\theta_2$ is chosen from regions with different topological invariants. This indicates that a low area in the negativity landscape can be effectively used to identify the combinations that result in localized states, with the minima corresponding to localized states with zero or minimal diffusion component.

Similarly, a negativity plot for the double split-step DTQW can be effectively used to identify the combination of parameters which lead to strongly localized states. In Fig. 9 we show the negativity as a function of $\theta_2$ and $\theta_2$ for $\theta_1 = 0$, $\theta_3 = -7\pi/8$ and $\theta_4 = \theta_2$. The visible valleys of entanglement corresponds to parameter ranges in which a strongly localized state is obtained.

This observation is in contrast to the behavior of entanglement for localization in 1D DTQW using disordered quantum coin operations. With spatially disordered coin operations, only a small decrease in entanglement is seen when compared to the entanglement due to standard DTQW and with temporally and spatio-temporally disordered coin operations, enhancement of entanglement is seen. Though the states are localized, the degree of entanglement is not significantly affected because of the longer localization length of the disordered localized state when compared to the short localization length of topologically localized states.
FIG. 10: Effect of $\sigma_x$ noise on the split-step DTQW for different configurations of $\theta_{2\pm}$ when $\theta_1 = -\pi/4$. (a), (c) and (e) show the probability distribution after 100 steps of walk in the absence and in the presence of noise. (b), (d) and (f) show the negativity as a function of the number of steps in absence and in presence of noise. The probability distributions show that the noise affects the diffusing part of the walk significantly but hardly influences the topologically localized part.

IV. STRENGTH OF THE LOCALIZED STATE IN THE PRESENCE OF NOISE

The application of noise to DTQWs is known to result in decoherence\cite{33, 35}, however small amounts of noise can also be advantageous for quantum algorithms and quantum transport. Here we will look into the effect of $\sigma_x = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$ (bit flip) noise on the topological quantum walk and show its effect on the localized and diffusive components.

The operation used for describing the two split-step DTQW evolution with $\sigma_x$ noise is given by

$$\rho(t) = P \left[ f_1 W(\theta_1, \theta_2) \rho(t-1) W(\theta_1, \theta_2) d^{\dagger} f_1 \right] + (1-P) \left[ W(\theta_1, \theta_2) \rho(t-1) W(\theta_1, \theta_2) d^{\dagger}, \right. \quad \left(17\right)$$

where $\rho(0) = |\Psi_{in}\rangle \langle \Psi_{in}|$, $W(\theta_1, \theta_2)$ is same as Eq. (12), $f_1 \equiv \sigma_x \otimes I$ and $P$ is the magnitude of noise. No noise is described by $P = 0$ and due to the fact that symmetries are not affected by bit flip noise, the maximum noise corresponds to $P = 0.5$ \cite{35}.

In Fig. 10 we show the probability distributions and the corresponding values of negativity for the split-step DTQW for different configurations of $\theta_{2\pm}$ in the presence and absence of noise. Applying noise to an evolution which in the absence of noise leads to a delocalized state (see Fig. 10(a), blue line), now leads to a state that is located around the origin. For a combination of $\theta_{2\pm}$ parameters resulting in a probability distribution with both, localized and a diffusive components (see Fig. 10(c) and (e)) the effect of noise results in a reduction of the probability for spreading in position space away from the origin, while the effect on the localized part is very small. This indicates a robustness of topologically localized states to noise, which is absent for diffusive states. This behaviour is also reflected in the negativity and in Figs. 10(b) and (d) and can see that the non-zero value of negativity in the absence of noise, indicative of a diffusive component in the probability distribution, decreases fast when noise is present. Even for noise levels as small as $P = 0.02$, the effect of noise on the delocalized probability distribution is very strong, see Fig. 10(e).

In Fig. 11 we show the negativity as function of $\theta_{2-}$ and $\theta_{2+}$ for $\theta_1 = -\pi/4$ for the the noiseless ($P = 0.00$, upper) and the noisy ($P = 0.02$, lower) split-step DTQW. Note the difference in the color scale.

FIG. 11: Negativity as a function of $\theta_{2-}$ and $\theta_{2+}$ for $\theta_1 = -\pi/4$ for the the noiseless ($P = 0.00$, upper) and the noisy ($P = 0.02$, lower) split-step DTQW. Note the difference in the color scale.
localized state against $\sigma_x$ noise.

However, topologically localized states in the split-step DTQW are not robust to other forms of noise. In Fig. 12 we show the probability distributions for an evolution without (first row) and with noise (rest of the rows) after 100 (first column) and 200 (second column) steps. The $\sigma_x$ noise is the same as given in Eq. (17) and the $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ noises are obtained by replacing $\sigma_x$ by $\sigma_y$ and $\sigma_z$ in the Eq. (17). The operation to describe depolarizing noise is given by

$$\rho(t) = \frac{1}{3} \left[ f_1 W(\theta_1, \theta_2) \rho(t-1) W(\theta_1, \theta_2)^\dagger f_1^\dagger + f_2 W(\theta_1, \theta_2) \rho(t-1) W(\theta_1, \theta_2)^\dagger f_2^\dagger + f_3 W(\theta_1, \theta_2) \rho(t-1) W(\theta_1, \theta_2)^\dagger f_3^\dagger \right] + (1 - P) W(\theta_1, \theta_2) \rho(t-1) W(\theta_1, \theta_2)^\dagger. \quad (18)$$

Comparing the probability distributions of the localized states after evolution in the presence of these different kinds of noises, one can clearly see that the topological states are robust only against $\sigma_x$ noise and a significant decrease in localized probability at the interface is visible for all other forms of noise. From Eqs. (8) and (11) we can see that the edge states at $\varepsilon = 0$ and $\pi$ are the eigenstates of the chiral symmetry operator $\Gamma$, which is identical to $f_1$ noise operator. Therefore this symmetry is preserved in the presence of $f_1$ noise and the edge state protected. However, this is not true for other forms of noise and the localized state can decay.

### V. CONCLUSION

Engineering DTQWs with different combination of variable quantum coin and position shift operations allows to create a wide range of rich, topological phases. Choosing parameters $\theta_i$ with different topological numbers to the left and right side of an interface of the position space, leads to topology induced localized states, which are sometimes accompanied by a diffusing component. Identifying combinations resulting strong localization with minimal or completely absent diffusing components is important for simulations of artificial TIs and in this work we have shown that the negativity of a state can be used for such an identification. By exploring the negativity landscape as function of the quantum coin parameters we have linked the strength of the topologically localized states to the appearance of low values of the negativity.

These topology induced localized states are different from the localized states originating from disordered DTQWs, where the presence of entanglement is usually robust against disorder. This therefore allows to differentiate between topologically localized states and localized state due to spatial and dynamic disorder in 1D DTQW. Finally, we have demonstrated that the topologically localized component of a state is robust against $\sigma_x$ noise, whereas the diffusing component decays. We strongly believe that studies like this can lead to better engineering of the artificial materials to realize TIs.

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