Some Open Problems in Combinatorial Physics

G H E Duchamp\textsuperscript{a}, H Cheballah\textsuperscript{a} and the CIP team.

\textsuperscript{a} LIPN - UMR 7030
CNRS - Université Paris 13
F-93430 Villetaneuse, France

E-mail:
ghed@lipn-univ.paris13.fr,
hayat.cheballah@lipn-univ.paris13.fr

17-01-2009 12:47

Contents

1 Problem A: Multiplicities in diag. .......................... 2
1.1 Setting ................................................................. 2
1.2 Problem A ............................................................... 3

2 Problem B: Combinatorics of Riordan-Sheffer one-parameter groups. 4
2.1 Problem B ............................................................... 4

3 Problem C: A corpus for combinatorial vector fields. 4
3.1 Problem C ............................................................... 5

4 Problem D Probabilistic study of approximate substitutions 5
4.1 Problem D ............................................................... 6
1. Problem A: Multiplicities in diag.

1.1. Setting

Let \( \mathcal{H}(F, G) \) be the Hadamard exponential product as defined below by

\[
F(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}, \quad G(z) = \sum_{n \geq 0} b_n \frac{z^n}{n!}, \quad \mathcal{H}(F, G) := \sum_{n \geq 0} a_n b_n \frac{z^n}{n!}.
\]

(1)

In the case of free exponentials, that is if we write the functions as

\[
F(z) = \exp\left(\sum_{n=1}^{\infty} L_n \frac{z^n}{n!}\right), \quad G(z) = \exp\left(\sum_{n=1}^{\infty} V_n \frac{z^n}{n!}\right),
\]

(2)

and using the expansion with Bell polynomials in the sets of variables \( L = \{L_n\} \), \( V = \{V_m\} \) (see [6, 10] for details), we obtain

\[
\mathcal{H}(F, G) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{P_1, P_2 \in UP_n} L^{\text{Type}(P_1)} V^{\text{Type}(P_2)}
\]

(3)

where \( UP_n \) is the set of unordered partitions of \([1 \cdots n]\).

An unordered partition \( P \) of a set \( X \) is a subset of \( P \subset \mathcal{P}(X) \setminus \{\emptyset\} \) (that is an unordered collection of blocks, i.e. non-empty subsets of \( X \)) such that

- the union \( \bigcup_{Y \in P} Y = X \) (\( P \) is a covering)
- \( P \) consists of disjoint subsets, i.e.
  \[ Y_1, Y_2 \in P \text{ and } Y_1 \cap Y_2 \neq \emptyset \implies Y_1 = Y_2. \]

The type of \( P \in UP_n \) (denoted above by \( \text{Type}(P) \)) is the multi-index \((\alpha_i)_{i \in \mathbb{N}^+}\) such that \( \alpha_k \) is the number of \( k \)-blocks, that is the number of members of \( P \) with cardinality \( k \).

\( \mathbb{B} \) Let \( P_1, P_2 \) be two unordered partitions of the same set. To each labelling of the blocks

\[
P_r = \{B_i^{(r)}\}_{1 \leq i \leq n_r}, \quad r = 1, 2
\]

(4)

one can associate the intersection matrix

\[
M = \left(\text{card}(B_i^{(1)} \cap B_j^{(2)})\right)_{1 \leq i \leq n_1 ; 1 \leq j \leq n_2}.
\]

(5)

As \((P_1, P_2)\) are, in essence, unlabelled, the arrow so constructed

\[
(P_1, P_2) \mapsto \text{class}(M) = d
\]

(6)

aims at classes of packed matrices \([7]\) under permutations of rows and columns.

These classes have been shown \([2, 3]\) to be in one to one correspondence with Feynman-Bender diagrams \([1]\) which are bicoloured graphs with \( p \) (\(= \text{card}(P_1)\)) black spots, \( q \) (\(= \text{card}(P_2)\)) white spots, no isolated vertex and integer multiplicities. We denote the set of such diagrams by \( \text{diag} \) \([8, 9]\).

Then, the correspondence goes as showed below.

\( \mathbb{D} \) The set of subsets of \( X \) is denoted by \( \mathcal{P}(X) \) (this notation \([4]\) is that of the former German school).
Fig 1. — Diagram from $P_1$, $P_2$ (set partitions of $[1 \cdots 11]$).

$P_1 = \{\{2,3,5\}, \{1,4,6,7,8\}, \{9,10,11\}\}$ and $P_2 = \{\{1\}, \{2,3,4\}, \{5,6,7,8,9\}, \{10,11\}\}$
(respectively black spots for $P_1$ and white spots for $P_2$).

The incidence matrix corresponding to the diagram (as drawn) or these partitions is
\[
\begin{pmatrix}
0 & 2 & 1 & 0 \\
1 & 1 & 3 & 0 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]
But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. The diagram could be represented by the matrix
\[
\begin{pmatrix}
0 & 0 & 1 & 2 \\
0 & 2 & 1 & 0 \\
1 & 0 & 3 & 1
\end{pmatrix}
\]
as well.

Noting $\text{mult}(d)$ the cardinality of each fibre of (6), formula (3) reads
\[
\mathcal{H}(F,G) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{d \in \text{diag} \atop |d| = n} \text{mult}(d)L^{\alpha(d)}V^{\beta(d)}
\]
where $\alpha(d)$ (resp. $\beta(d)$) is the “white spots type” (resp. the “black spots type”) i.e. the multi-index $\alpha_i \in \mathbb{N}^+$ (resp. $\beta_i \in \mathbb{N}^+$) such that $\alpha_i$ (resp. $\beta_i$) is the number of white spots (resp. black spots) of degree $i$ ($i$ lines connected to the spot) and $\text{mult}(d)$ is the number of pairs of unordered partitions of $[1 \cdots |d|]$ (here $|d| = |\alpha(d)| = |\beta(d)|$ is the number of lines of $d$) with associated diagram $d$.

1.2. Problem A

Give a formula (as smart as possible) for $\text{mult}(d)$ as a function of $d$ (in the language of [7], as a function of the class of a packed matrix under the permutation of rows and columns).

Hint. — For practical computations, one of the two partitions may be kept fixed, say $P_1$ and the result of the enumeration multiplied by $\frac{n!}{|\text{stab}(P_1)|}$.
2. Problem B: Combinatorics of Riordan-Sheffer one-parameter groups.

We start with the (vector) space $\mathbb{C}^{N \times N}$ of complex bi-infinite matrices. Let $\mathcal{RF}(N, \mathbb{C}) = (\mathbb{C}^{N})^{N}$ the space of row-finite matrices (i.e., matrices for which every row is finitely supported). To every matrix $T \in \mathcal{RF}(N, \mathbb{C})$, one can associate the sequence transformation

$$(a_k)_{k \in \mathbb{N}} \mapsto (b_n)_{n \in \mathbb{N}}$$

given by

$$b_n = \sum_{k \in \mathbb{N}} T[n, k] a_k$$

this sum is finitely supported as $T \in \mathcal{RF}(N, \mathbb{C})$. One can prove that the set $\mathcal{RF}(N, \mathbb{C})$ is exactly the algebra of continuous endomorphisms of $\mathbb{C}^{N}$ endowed with the topology of pointwise convergence.

This transformation can be transported on EGFs by

$$f = \sum_{k \in \mathbb{N}} a_k \frac{z^k}{k!} \mapsto \hat{f} = \sum_{n \in \mathbb{N}} b_n \frac{z^n}{n!}$$

and, in case $\hat{f}$ is given by

$$\hat{f}(z) = \Phi_{g,\phi}[f](z) = g(z)f(\phi(z)).$$

with

$$g(z) = 1 + \text{higher terms and } \phi(z) = z + \text{higher terms}.$$  

we say that the matrix is a matrix of substitutions with prefunction.

In classical combinatorics (for OGF and EGF), the matrices $M_{g,\phi}(n, k)$ are known as Riordan matrices (see [11, 12] for example). One can prove, using a Zariski-like argument, the following proposition [10, 5].

**Proposition 2.1 [10]** Let $M$ be the matrix of a substitution with prefunction; then so is $M^t$ for all $t \in \mathbb{C}$.

2.1. Problem B

a) Provide a combinatorial proof of the preceding proposition for $t \in \mathbb{Q}$ (without using the "pro-algebraic" structure of the group of substitutions with prefuctions, directly or indirectly).

b) Give a combinatorial interpretation of $M^{1/2}$ for some Sheffer matrices.

3. Problem C: A corpus for combinatorial vector fields.

With the preceding notations one can show that, if $M$ is a matrix of substitution with prefunction, the limit

$$\lim_{q \to +\infty} q(M^{1}_q - I)$$
exists (call it \( L \)) and the associated transformation of sequences (see above) is the sum of a vector field and a scalar field. One can see that

\[ M \in \mathbb{Q}^{N \times N} \implies L \in \mathbb{Q}^{N \times N}. \]  

(14)

in addition, if \( M \) is a matrix of substitution (i.e. the prefunction is \( \equiv 1 \)) then the scalar field is zero and so the associated differential operator is a pure vector field (with coefficients in \( \mathbb{Q} \) if \( M \) is in \( \mathbb{Q}^{N \times N} \)).

On the other hand, if \( C \) is a class of labelled graphs for which the exponential formula applies, the matrix \( M \) such that

\[ M[n,k] = \text{Number of graphs labelled by } [1..n] \text{ and with } k \text{ connected components} \] 

(15)

is a matrix of substitution [10]. For example with the graphs of equivalence relations on finite sets, the substitution is \( z \mapsto e^z - 1 \); for graphs of idempotent endofunctions, the substitution is \( z \mapsto ze^z \).

3.1. Problem C

a) What is the combinatorial interpretation of the coefficients of the vector field for the two preceding examples?

b) Can we give any insight of the form of this vector field for general classes of graphs?

\[ \text{Hint.} \quad M^z = e^{z \log(M)} \] 

where \( \log(M) \) is the matrix of a differential operator of the form \( q(z) \frac{d}{dz} + v(z) \).

4. Problem D Probabilistic study of approximate substitutions

Our motivation, in this section, consists in approximating the matrices of infinite substitutions by finite matrices of (approximate) substitutions. We are then interested in the probabilistic study of these matrices. To this end, we randomly generate unipotent (unitriangular) matrices and observe the number of occurrences of matrices of substitutions.

We start by giving some examples of our experiment which are summarized in the table below:

| Size    | Number of drawings | Range of variables | Probability |
|---------|--------------------|--------------------|-------------|
| \([3 \times 3]\) | 300                | \([1 \cdots 10]\)    | 1           |
|         |                    | \([1 \cdots 100]\)  | 1           |
|         |                    | \([1 \cdots 10000]\) | 1           |
| \([4 \times 4]\) | 275                | \([1 \cdots 10]\)    | 0.0473      |
|         |                    | \([1 \cdots 100]\)  | 0.0001      |
|         |                    | \([1 \cdots 10000]\) | 0+          |
| \([10 \times 10]\) | 1500               | \([1 \cdots 10]\)    | 0.0327      |
|         |                    | \([1 \cdots 100]\)  | 0+          |
|         |                    | \([1 \cdots 10000]\) | 0+          |
According to the results obtained, we observe that the (approximate) substitution matrices are not very frequent. However, in meeting certain conditions such as size, the number of drawings and the range of the variables, we can obtain positive probabilities that these matrices appear.

Let us note that the smaller the size of the matrix the more probable one obtains a matrix of substitution in a reasonable number of drawings.

We also notice that, if we vary the range of variables, and this in an increasing way and by keeping unchanged the number of drawings and size, the probability tends to zero.

We also notice that the unipotent matrices of size 3 are all matrices of approximate substitutions. This is easy to see because the exponential generating series of the 3rd column will always have the form \( c_k = \frac{x^2}{2!} \).

Thus, we can say that the test actually starts from the matrices of size higher or equal to 4.

**Result 4.1** Let \( r \) represent the cardinality of the range of variables and \( n \times n \) be the size of the matrix.

According to the results obtained; we can say that the probability \( p_n \) of appearance of the matrices of substitutions depends on \( r \) and \( n \) and we have the following upper bound:

\[
p_n \leq \frac{r^{n-3}}{\frac{n(n-1)}{2}}
\]

which shows that

\[
p_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty
\]

**4.1. Problem D**

One can conjecture that the effect of the range selection vanishes when \( n \) tends to infinity. More precisely:

\[
p_n \sim \frac{r^{2n-3}}{\frac{n(n-1)}{2}}
\]
References

[1] C. M. Bender, D. C. Brody, and B. K. Meister, Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999)
[2] P. Blasiak, A. Horzela, K. A. Penson, G. H. E. Duchamp, A.I. Solomon, Boson normal ordering via substitutions and Sheffer-Type Polynomials, Phys. Lett. A 338 (2005) 108
[3] P. Blasiak, K. A. Penson, A.I. Solomon, A. Horzela, G. H. E. Duchamp, Some useful formula for bosonic operators, Jour. Math. Phys. 46 052110 (2005).
[4] Bourbaki N., Theory of sets, Springer
[5] H. Cheballah, G. H. E. Duchamp, K. A. Penson, Approximate substitutions and the normal ordering problem, Symmetry and Structural Properties of Condensed Matter, IOP Publishing Journal of Physics: Conference Series, 104 (2008).
  arXiv: quant-ph/0802.1162
[6] G. H. E. Duchamp, P. Blasiak, A. Horzela, K. A. Penson, A. I. Solomon, Feynman graphs and related Hopf algebras, Journal of Physics: Conference Series, SSPCM'05, Myczkowce, Poland. arXiv: cs.SC/0510041
[7] G. Duchamp, F. Hivert, J. Y. Thibon, Non commutative functions VI: Free quasi-symmetric functions and related algebras, International Journal of Algebra and Computation Vol 12, No 5 (2002).
[8] G. H. E. Duchamp, K. A. Penson, P. Blasiak, A. Horzela, A. I Solomon, A Three Parameter Hopf Deformation of the Algebra of Feynman-like Diagrams arXiv:0704.2522 (to be published).
[9] G. H. E. Duchamp, J.-G. Luque, J.-C. Novelli, C. Tollu, F. Toumazet, Hopf algebras of diagrams, FPSAC07.
[10] G. Duchamp, A.I. Solomon, K.A. Penson, A. Horzela and P. Blasiak, One-parameter groups and combinatorial physics, Proceedings of the Symposium Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3) (Porto-Novo, Benin, Nov. 2003), J. Govaerts, M. N. Hounkonnou and A. Z. Msezane (eds.), p.436 (World Scientific Publishing 2004)
  arXiv: quant-ph/04011262
[11] S. Roman, The Umbral Calculus (New York: Academic Press) (1984)
[12] L. W. Shapiro, S. Getu, W.J. Woan and L. Woodson, The Riordan group Discrete Appl. Math. 34 229 (1991)