SELECTIVELY SEQUENTIALLY PSEUDOCOMPACT GROUP TOPOLOGIES
ON TORSION AND TORSION-FREE ABELIAN GROUPS

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Abstract. A space \( X \) is selectively sequentially pseudocompact if for every family \( \{ U_n : n \in \mathbb{N} \} \) of non-empty open subsets of \( X \), one can choose a point \( x_n \in U_n \) for every \( n \in \mathbb{N} \) in such a way that the sequence \( \{ x_n : n \in \mathbb{N} \} \) has a convergent subsequence. Let \( G \) be a group from one of the following three classes: (i) \( V \)-free groups, where \( V \) is an arbitrary variety of Abelian groups; (ii) torsion Abelian groups; (iii) torsion-free Abelian groups. Under the Singular Cardinal Hypothesis SCH, we prove that if \( G \) admits a pseudocompact group topology, then it can also be equipped with a selectively sequentially pseudocompact group topology. Since selectively sequentially pseudocompact spaces are strongly pseudocompact in the sense of García-Ferreira and Ortiz-Castillo, this provides a strong positive (albeit partial) answer to a question of García-Ferreira and Tomita.

The symbol \( \mathbb{N} \) denotes the set of natural numbers and \( \mathbb{N}^+ = \mathbb{N} \setminus \{0\} \) denotes the set of positive integer numbers. The symbol \( \mathbb{Z} \) denotes the group of integer numbers, \( \mathbb{R} \) the set of real numbers and \( \mathbb{T} \) the circle group \( \{ e^{i\theta} : \theta \in \mathbb{R} \} \subseteq \mathbb{R}^2 \). The symbols \( \omega, \omega_1 \) and \( c \) stand for the first infinite cardinal, the first uncountable cardinal and the cardinality of the continuum, respectively. For a set \( X \), the set of all finite subsets of \( X \) is denoted by \( [X]^{|\omega} \), while \( [X]^\omega \) denotes the set of all countably infinite subsets of \( X \).

If \( X \) is a subset of a group \( G \), then \( \langle X \rangle \) is the smallest subgroup of \( G \) that contains \( X \).

For groups which are not necessary Abelian we use the multiplication notation, while for Abelian groups we always use the additive one. In particular, \( e \) denotes the identity element of a group and 0 is used for the zero element of an Abelian group. Recall that an element \( g \) of a group \( G \) is torsion if \( g^n = e \) for some positive integer \( n \). A group is torsion if all of its elements are torsion. A group \( G \) is bounded torsion if there exists \( n \in \mathbb{N} \) such that \( g^n = e \) for all \( g \in G \).

Let \( G \) be an Abelian group. The symbol \( t(G) \) stands for the subgroup of torsion elements of \( G \). For each \( n \in \mathbb{N} \), note that \( nG = \{ ng : g \in G \} \) is a subgroup of \( G \) and the map \( g \mapsto ng \, (g \in G) \) is a homomorphism of \( G \) onto \( nG \). For a cardinal \( \tau \) we denote by \( G(\tau) \) the direct sum of \( \tau \) copies of the group \( G \).

We will say that a sequence \( \{ x_n : n \in \mathbb{N} \} \) of points in a topological space \( X \):
- converges to a point \( x \in X \) if the set \( \{ n \in \mathbb{N} : x_n \not\in W \} \) is finite for every open neighbourhood \( W \) of \( x \) in \( X \);
- is convergent if it converges to some point of \( X \).

1. Introduction

In this paper, we continue the study of the class of selectively sequentially pseudocompact spaces introduced by the authors in [9].
Definition 1.1. \[9\] A topological space \(X\) is selectively sequentially pseudocompact if for every sequence \(\{U_n : n \in \mathbb{N}\}\) of non-empty open subsets of \(X\), one can choose a point \(x_n \in U_n\) for every \(n \in \mathbb{N}\) in such a way that the sequence \(\{x_n : n \in \mathbb{N}\}\) has a convergent subsequence.

A nice feature of the class of selectively sequentially pseudocompact spaces is that it is closed under taking arbitrary Cartesian products:

Proposition 1.2. \[9, \text{Corollary 4.4}\] Arbitrary products of selectively sequentially pseudocompact spaces are selectively sequentially pseudocompact.

García-Ferreira and Ortiz-Castillo \[13\] have recently introduced the class of strongly pseudocompact spaces. The original definition involves the notion of a \(p\)-limit for an ultrafilter \(p\) on \(\mathbb{N}\). The authors have shown in \[9, \text{Theorem 2.1}\] that the original definition is equivalent to the following one:

Definition 1.3. A topological space \(X\) is called strongly pseudocompact provided that for each sequence \(\{U_n : n \in \mathbb{N}\}\) of pairwise disjoint non-empty open subsets of \(X\), one can choose a point \(x_n \in U_n\) for every \(n \in \mathbb{N}\) such that the set \(\{x_n : n \in \mathbb{N}\}\) is not closed in \(X\).

In the class of topological groups, this property has been investigated by García-Ferreira and Tomita in \[14\].

The following implications are clear from Definitions 1.1 and 1.3

(1) selectively sequentially pseudocompact \(\rightarrow\) strongly pseudocompact \(\rightarrow\) pseudocompact.

It easily follows from Definition 1.3 that countably compact spaces are strongly pseudocompact. Since there exists a countably compact space \(X\) such that \(X \times X\) is not even pseudocompact, an analogue of Proposition 1.2 for strongly pseudocompact spaces is false. Whether strong pseudocompactness is productive in the class of topological groups remains an open problem \[14\].

Neither of the implications in (1) can be reversed even in the class of topological (Abelian) groups. An example of a pseudocompact Abelian group which is not strongly pseudocompact was constructed by García-Ferreira and Tomita in \[14\]. The authors gave a consistent example of a strongly pseudocompact Abelian group which is not selectively sequentially pseudocompact in \[9, \text{Example 5.7}\]. A ZFC example of such a group is given in \[17\], thereby answering \[9, \text{Question 8.3(i)}\].

García-Ferreira and Tomita asked if the last implication in (1) can be reversed when the existential quantifier is added to both sides of it.

Question 1.4. \[14, \text{Question 2.7}\] If an Abelian group admits a pseudocompact group topology, does it admit a strongly pseudocompact group topology?

In view of the implications in (1), the following version of this question is even harder to answer in the positive.

Question 1.5. If an Abelian group admits a pseudocompact group topology, does it also admit a selectively sequentially pseudocompact group topology?

The goal of this paper is to provide a positive answer to both questions for the following classes of groups:

(a) \(V\)-free groups, where \(V\) is a precompact variety of groups (Corollary 3.8);
(b) \(V\)-free groups, where \(V\) is any variety of Abelian groups (Corollary 3.12);
(c) torsion Abelian groups (Theorem 5.2);
(d) torsion-free Abelian groups (Corollary 6.4; see also Theorem 6.3 for a more general result).

Our results depend on some additional set-theoretic assumptions which we outline below.

Let \(\tau\) be a cardinal. As usual, symbol \(\tau^\omega\) denotes the cardinality of the set \([X]_\omega\) of all countably infinite subsets of some (equivalently, every) set \(X\) such that \(|X| = \tau\), while \(2^\tau\) denotes the
cardinality of the set of all subsets of some (equivalently, every) set $X$ such that $|X| = \tau$. Symbol $\text{cf}(\tau)$ denotes the cofinality of $\tau$, i.e. the smallest cardinal $\lambda$ for which there exists a representation $\tau = \sum \{ \tau_\alpha : \alpha < \lambda \}$ where $\tau_\alpha < \tau$ for all $\alpha < \lambda$. Finally, $\tau^+$ denotes the smallest cardinal bigger than $\tau$.

We fix the symbol SCH for denoting the following condition on cardinals:

\begin{equation}
\text{if } \tau \geq \kappa \text{ is a cardinal and } \text{cf}(\tau) \neq \omega, \text{ then } \tau^\omega = \tau.
\end{equation}

We choose the abbreviation symbol SCH for denoting the condition (2) since it is known to be equivalent to the Singular Cardinals Hypothesis; see [15, Chapter 8].

The Generalized Continuum Hypothesis (abbreviated GCH) states that $2^\tau = \tau^+$ for every infinite cardinal $\tau$. It is well known that GCH implies SCH and that GCH is consistent with the usual axioms ZFC of set theory.

Prikry and Silver have constructed models of set theory violating SCH assuming the consistency of supercompact cardinals; see [15, chapter 36]. Devlin and Jensen proved that the failure of SCH implies the existence of an inner model with a large cardinal; see [4]. Therefore, large cardinals are needed in order to violate SCH, and so SCH can be viewed as a “mild” additional set-theoretic assumption beyond the axioms of ZFC.

Our results mentioned in items (b), (c) and (d) above hold under SCH, while we were able to prove item (a) only under GCH.

It is worth mentioning that our results are close in spirit to those obtained by Galindo and Macario [12]. Indeed, they prove, under SCH, that if an Abelian group has a pseudocompact group topology, then it also admits a pseudocompact group topology without infinite compact subsets. This additional property is important in the Pontrjagin duality theory. Clearly, group topologies we construct are substantially different from those in [12], as selectively sequentially pseudocompact spaces have many non-trivial convergent sequences [9, Proposition 3.1].

We refer the interested reader to [9] for a detailed survey of connections of selective pseudocompactness and strong pseudocompactness to other classical compactness-like notions, including the notion of sequential pseudocompactness of Artico, Marconi, Pelant, Rotter and Tkachenko [1].

We finish this introduction with the simple proposition from [9] which shall be used later:

**Proposition 1.6.** [9, Corollary 3.4] If a space $X$ has a dense selectively sequentially pseudocompact (strongly pseudocompact) subspace, then $X$ is selectively sequentially pseudocompact (respectively, strongly pseudocompact).

2. **Building selectively sequentially pseudocompact $\mathcal{V}$-independent sets for a variety $\mathcal{V}$ of groups**

Recall that a *variety of groups* is a class of (abstract) groups closed under Cartesian products, subgroups and quotients [16].

**Definition 2.1.** [7, Definition 2.1] Let $\mathcal{V}$ be a variety of groups and let $G$ be a group. A subset $X$ of $G$ is said to be $\mathcal{V}$-independent if

(a) $(X) \in \mathcal{V}$, and

(b) for each map $f : X \rightarrow H \in \mathcal{V}$ there exists a unique homomorphism $\tilde{f} : (X) \rightarrow H$ extending $f$.

The cardinal

$$r_\mathcal{V}(G) = \sup \{ |X| : X \text{ is a } \mathcal{V}\text{-independent subset of } G \}$$

is called the $\mathcal{V}$-rank of $G$.

We shall need the following useful fact from [7].

**Lemma 2.2.** Let $\mathcal{V}$ be a variety of groups and $X$ be a subset of a group $G$. 

(i) \( X \) is \( \mathcal{V} \)-independent if and only if each finite subset of \( X \) is \( \mathcal{V} \)-independent.

(ii) If \( H \) is a group and \( f : G \to H \) is a homomorphism such that \( f(X) \) is a \( \mathcal{V} \)-independent subset of \( H \), \( \langle X \rangle \in \mathcal{V} \) and \( f \upharpoonright X : X \to H \) is an injection, then \( f \upharpoonright \langle X \rangle : X \to H \) is an injection and \( X \) is a \( \mathcal{V} \)-independent subset of \( G \).

Proof. Item (i) is \([7] \text{ Lemma 2.3}\). The first statement in the conclusion of item (ii) is \([7] \text{ Lemma 2.4}\) and the second statement can be proved similarly to the proof of \([7] \text{ Lemma 2.4}\). \( \square \)

The next two lemmas are the main technical tool in this paper.

Lemma 2.3. Let \( \mathcal{V} \) be a variety of groups. Suppose that \( H \in \mathcal{V} \) is a compact metric group with \( r_\mathcal{V}(H) \geq \omega \). Let \( \sigma \) be an infinite cardinal such that \( \sigma^\omega = \sigma \). Then there exist a dense \( \mathcal{V} \)-independent selectively sequentially pseudocompact subset \( Y \) of \( H^\sigma \) and a set \( C \subseteq \sigma \) such that \( |Y| = |C| = |\sigma \setminus C| = \sigma \) and \( \langle Y \rangle \cap H^C = \{e\} \), where

\[
H^C = \{ f \in H^\sigma : f(\gamma) = e \text{ for all } \gamma \in \sigma \setminus C \}.
\]

Proof. A subset \( B \) of \( H^\sigma \) shall be called basic if \( B = \prod_{\gamma \in \sigma} B_\gamma \), where \( B_\gamma \subseteq H \) for all \( \gamma \in \sigma \) and the set \( \text{supp}(B) = \{ \gamma \in \sigma : B_\gamma \neq H \} \) is finite.

Let \( B \) be a base for \( H^\sigma \) of size \( \sigma \) consisting of basic open sets. Since \( |B^\omega| = |B|^\omega = \sigma^\omega = \sigma \), we can enumerate

\[
B^\omega = \{ (U_{\alpha,n} : n \in \mathbb{N}) : \alpha \in \sigma \}.
\]

Let \( \alpha \in \sigma \). Since \( \text{supp}(U_{\alpha,n}) \) is finite for every \( n \in \mathbb{N} \), the set \( S_\alpha = \bigcup_{n \in \mathbb{N}} \text{supp}(U_{\alpha,n}) \) is a countable subset of \( \sigma \). For every \( n \in \mathbb{N} \) take a point \( x_{\alpha,n} \in U_{\alpha,n} \). Since \( H^{S_\alpha} \) is compact metric, there are an infinite set \( J_\alpha \subseteq \mathbb{N} \) and a point \( x_{\alpha,\omega} \in H^\sigma \) such that \( \{ x_{\alpha,n} |_{S_\alpha} : n \in J_\alpha \} \) converges to \( x_{\alpha,\omega} |_{S_\alpha} \).

Since \( \sigma \) is infinite, there exists a subset \( C \) of \( \sigma \) such that \( |C| = |D| = \sigma \), where \( D = \sigma \setminus C \). Since the set \( [\sigma \times (\omega + 1)]^{<\omega} \) of all finite subsets of \( \sigma \times (\omega + 1) \) has cardinality \( \sigma = |D| \), we can fix an enumeration

\[
[\sigma \times (\omega + 1)]^{<\omega} = \{ F_\gamma : \gamma \in D \}
\]

such that

\[
|\{ \gamma \in D : F_\gamma = F \}| = \sigma \text{ for every } F \in [\sigma \times (\omega + 1)]^{<\omega}.
\]

Since \( r_\mathcal{V}(H) \geq \omega \), for every \( \gamma \in \sigma \) we can fix an injection \( h_\gamma : F_\gamma \to H \) such that \( h_\gamma(F_\gamma) \) is a \( \mathcal{V} \)-independent subset of \( H \).

For every \( (\alpha,n) \in \sigma \times (\omega + 1) \), define \( y_{\alpha,n} \in H^\sigma \) by

\[
y_{\alpha,n}(\gamma) = \begin{cases} x_{\alpha,n}(\gamma) & \text{if } \gamma \in S_\alpha; \\ h_\gamma(\alpha,n) & \text{if } \gamma \in D \setminus S_\alpha \text{ and } (\alpha,n) \in F_\gamma; \\ h_\gamma(\alpha,\omega) & \text{if } \gamma \in D \setminus S_\alpha \text{ and } (\alpha,n) \notin F_\gamma \text{ and } (\alpha,\omega) \in F_\gamma; \\ e & \text{otherwise} \end{cases}
\]

for all \( \gamma \in \sigma \). Finally, let

\[
Y = \{ y_{\alpha,n} : (\alpha,n) \in \sigma \times (\omega + 1) \}.
\]

Claim 1. \( y_{\alpha,n} \in U_{\alpha,n} \) for every \( (\alpha,n) \in \sigma \times \omega \).

Proof. Since \( \text{supp}(U_{\alpha,n}) \subseteq S_\alpha \), \( x_{\alpha,n} \in U_{\alpha,n} \) and \( y_{\alpha,n} |_{S_\alpha} = x_{\alpha,n} |_{S_\alpha} \) by (6), we get \( y_{\alpha,n} \in U_{\alpha,n} \). \( \square \)

Claim 2. \( Y \) is \( \mathcal{V} \)-independent and \( \langle Y \rangle \cap H^C = \{e\} \).

Proof. By Lemma 2.2 (i), in order to prove that \( Y \) is \( \mathcal{V} \)-independent, it suffices to show that every finite subset \( X \) of \( Y \) is \( \mathcal{V} \)-independent. Similarly, in order to show that \( \langle Y \rangle \cap H^C = \{e\} \), it suffices to check that \( \langle X \rangle \cap H^C = \{e\} \) for every finite subset \( X \) of \( Y \). Therefore, we fix an arbitrary finite subset \( X \) of \( Y \), and we are going to prove that \( X \) is \( \mathcal{V} \)-independent and satisfies \( \langle X \rangle \cap H^C = \{e\} \).
Since $X$ is a finite subset of $Y$, we can use (7) to fix a finite set $F \subseteq \sigma \times (\omega + 1)$ such that

\[
X = \{y_{\alpha,n} : (\alpha, n) \in F\}.
\]

Since $F$ is finite, so is the set $A = \{\alpha \in \sigma : (\alpha, n) \in F \text{ for some } n \in \omega + 1\}$. Therefore, the set $S = \bigcup_{\alpha \in A} S_{\alpha}$ is at most countable. Since $\sigma^\omega = \sigma \geq \omega$, the cardinal $\sigma$ is uncountable. Since $|\{\gamma \in D : F_{\gamma} = F\}| = \sigma$ by (5), there exists $\gamma \in D \setminus S$ such that $F_{\gamma} = F$. Let $\pi_\gamma : H^\sigma \to H$ be the projection on the $\gamma$’s coordinate.

To prove that $X$ is $\mathcal{V}$-independent, it suffices to check that $G = H^\sigma$, $X$ and $f = \pi_\gamma$ satisfy the assumptions of Lemma 2.12 (ii).

Since $\mathcal{V}$ is a variety of groups, it is closed under taking arbitrary products and subgroups. Since $H \in \mathcal{V}$ and $\langle X \rangle$ is a subgroup of $H^\sigma$, this implies $\langle X \rangle \in \mathcal{V}$.

Let $g : F \to X$ be the map defined by $g(\alpha, n) = y_{\alpha,n}$ for $(\alpha, n) \in F$. Suppose that $(\alpha, n) \in F = F_{\gamma}$. Since $S_{\alpha} \subseteq S$ and $\gamma \in D \setminus S$, we get $\gamma \in D \setminus S_{\alpha}$, so

\[
\pi_\gamma(g(\alpha, n)) = \pi_\gamma(y_{\alpha,n}(\gamma)) = h_\gamma(\alpha, n)
\]
by (6). This shows that $\pi_\gamma \circ g = h_\gamma$. Since $h_\gamma$ is an injection, so is $\pi_\gamma \circ g = \pi_\gamma \mid X \circ g$. Since $g$ is a surjection, this implies that $\pi_\gamma \mid X$ is an injection. Finally, $\pi_\gamma(X) = \pi_\gamma(g(F)) = h_\gamma(F)$ is a $\mathcal{V}$-independent subset of $H$ by the choice of $h_\gamma$.

Applying Lemma 2.12 (ii), we conclude that $X$ is $\mathcal{V}$-independent and $\pi_\gamma \mid \langle X \rangle$ is an injection.

Let $x \in \langle X \rangle \setminus \{e\}$ be arbitrary. Then $x(\gamma) = \pi_\gamma(x) \neq e$, because $\pi_\gamma$ is injective on $\langle X \rangle$. Since $\gamma \in D = \sigma \setminus C$, it follows from (3) that $x \notin H^C$. This proves the equation $\langle X \rangle \cap H^C = \{e\}$. \qed

Claim 3. $Y$ is selectively sequentially pseudocompact.

Proof. Let $\{U_n : n \in \mathbb{N}\} \subseteq \mathcal{B}^\omega$. By (3), there exists $\alpha \in \sigma$ such that $U_n = U_{\alpha,n}$ for every $n \in \mathbb{N}$. Since $y_{\alpha,n} \in U_{\alpha,n} = U_n$ for every $n \in \mathbb{N}$ by Claim 1 it suffices to prove that the subsequence $\{y_{\alpha,n} : n \in J_\alpha\}$ of the sequence $\{y_{\alpha,n} : n \in \mathbb{N}\} \subseteq Y$ converges to $y_{\alpha,\omega} \in Y$. Note that the sequence $\{y_{\alpha,n} \mid S_\alpha : n \in J_\alpha\} = \{x_{\alpha,n} \mid S_\alpha : n \in J_\alpha\}$ converges to $x_{\alpha,\omega} = y_{\alpha,\omega} \mid S_\alpha$ by (3) and the choice of $\{x_{\alpha,n} : n \in \mathbb{N}\}$ and $J_\alpha$. Therefore, it suffices to show that the sequence $\{y_{\alpha,n}(\gamma) : n \in \mathbb{N}\}$ converges to $y_{\alpha,\omega}(\gamma)$ for every $\gamma \in \sigma \setminus S_\alpha$. We consider two cases.

Case 1. $\gamma \in D \setminus S_\alpha$. If $(\alpha, \omega) \in F_{\gamma}$, then (3) implies that $y_{\alpha,\omega}(\gamma) = h_\gamma(\alpha, \omega)$ and $\{n \in \mathbb{N} : y_{\alpha,n}(\gamma) \neq h_\gamma(\alpha, \omega)\} \subseteq \{n \in \mathbb{N} : (\alpha, n) \in F_{\gamma}\}$. Since $F_{\gamma}$ is finite, the sequence $\{y_{\alpha,n}(\gamma) : n \in \mathbb{N}\}$ converges to $y_{\alpha,\omega}(\gamma)$. Suppose now that $(\alpha, \omega) \notin F_{\gamma}$. Then $y_{\alpha,\omega}(\gamma) = e$ and $\{n \in \mathbb{N} : y_{\alpha,n}(\gamma) \neq e\} \subseteq \{n \in \mathbb{N} : (\alpha, n) \in F_{\gamma}\}$ by (6). Since $F_{\gamma}$ is finite, the sequence $\{y_{\alpha,n}(\gamma) : n \in \mathbb{N}\}$ converges to $y_{\alpha,\omega}(\gamma)$.

Case 2. $\gamma \in \sigma \setminus (D \cup S_\alpha)$. In this case (3) implies that $y_{\alpha,n}(\gamma) = e$ for all $n \in \omega + 1$. Therefore, the sequence $\{y_{\alpha,n}(\gamma) : n \in \mathbb{N}\}$ converges to $y_{\alpha,\omega}(\gamma)$. \qed

Claim 4. $Y$ is dense in $H^\sigma$.

Proof. Let $V$ be a non-empty open subset of $H^\sigma$. We need to show that $Y \cap V \neq \emptyset$. By taking a smaller set $V$ if necessary, we may assume that $V$ is a basic open subset of $H^\sigma$. Let $U_n = V$ for every $n \in \mathbb{N}$. Then $\{U_n : n \in \mathbb{N}\} \subseteq \mathcal{B}^\omega$, so by (3), there exists $\alpha \in \sigma$ such that $U_n = U_{\alpha,n}$ for every $n \in \mathbb{N}$. Since $y_{\alpha,1} : U_{\alpha,1} = V$ by Claim 1 and $y_{\alpha,1} \in Y$ by (7), we get $Y \cap V \neq \emptyset$. \qed

Lemma 2.4. Let $\sigma$ and $\tau$ be infinite cardinals satisfying $\sigma^\omega = \sigma \leq \tau \leq 2^\omega$. If $\mathcal{V}$ is a variety of Abelian groups and $H \in \mathcal{V}$ is a compact metric group with $r_\mathcal{V}(H) \geq \omega$, then there exist a dense $\mathcal{V}$-independent selectively sequentially pseudocompact subset $X$ of $H^\sigma$ and a set $D \subseteq \sigma$ such that $|X| = \tau, |D| = |\sigma \setminus D| = \sigma$ and $\langle X \rangle \cap H^D = \{0\}$, where

\[
H^D = \{f \in H^\sigma : f(\gamma) = 0 \text{ for all } \gamma \in \sigma \setminus D\}.
\]
**Proof.** By Lemma 2.3 there exist a dense $\mathcal{V}$-independent selectively sequentially pseudocompact subset $Y$ of $H^\sigma$ and a set $C \subseteq \sigma$ such that $|Y| = |C| = |\sigma \setminus C| = \sigma$ and $(Y) \cap H^C = \{0\}$.

By [2] Lemmas 3.4 and 4.3, there exist a $\mathcal{V}$-independent subset $Z$ of $H^C$ and a set $D \subseteq C$ such that $|Z| = \tau, |D| = |C \setminus D| = \sigma$ and $(Z) \cap H^D = \{0\}$. Since $Z \subseteq H^C$ and $H^C$ is a subgroup of $H^\sigma$, we have $(Z) \subseteq H^C$. Since $(Y) \cap H^C = \{0\}$, this implies $(Y) \cap (Z) = \{0\}$.

We claim that $X = Y \cup Z$ and $D$ are as required. Indeed, $|X| = |Y| + |Z| = \sigma + \tau = \tau$. Since $Y$ is dense in $H^\sigma$ and $Y \subseteq X$, $X$ is dense in $H^\sigma$ as well. Since $Y$ is dense in $H^\sigma$ and $Y \subseteq X$, $Y$ is dense in $X$. Since $Y$ is selectively sequentially pseudocompact, Proposition 1.16 implies that $X$ is selectively sequentially pseudocompact as well.

Next, we shall prove that $(X) \cap H^D = \{0\}$. Since $(Z) \cap H^D = \{0\}$, it suffices to check that $(X) \cap H^D \subseteq (Z)$. Let $x \in (X) \cap H^D$ be arbitrary. We are going to show that $x \in (Z)$. Since $x \in (X) \cap H^D$, $X = Y \cup Z \subseteq H^\sigma$ and the latter group is Abelian, we can find elements $y \in (Y)$ and $z \in (Z)$ such that $x = y + z$.

Consider an arbitrary $\gamma \in \sigma \setminus C$. Since $D \subseteq C$, we have $\gamma \in \sigma \setminus D$, and so $x(\gamma) = 0$ by $x \in H^D$ and (3). Since $z \in H^D$ and $(Z) \cap H^D = \{0\}$, we also get $z(\gamma) = 0$. Therefore, $0 = x(\gamma) = y(\gamma) + z(\gamma) = y(\gamma)$. For all $\gamma \in \sigma \setminus C$, from (3) we conclude that $y \in H^C$. Since $y \in (Y)$, we get $y \in (Y) \cap H^C = \{0\}$, which implies $y = 0$. Therefore, $x = z \in (Z)$.

Finally, we are going to check that the set $X = Z \cup Y$ is $\mathcal{V}$-independent. To show this, we need to check items (a) and (b) of Definition 2.1.

(a) Since $(X)$ is a subgroup of $H^\sigma$, $H \in \mathcal{V}$ and $\mathcal{V}$ is a variety of groups, $(X) \in \mathcal{V}$.

(b) Let $f : X \to A \in \mathcal{V}$ be an arbitrary map. Since $Y$ is $\mathcal{V}$-independent, there exists a unique homomorphism $g : (Y) \to A$ such that $g |_Y = f |_Y$. Similarly, since $Z$ is $\mathcal{V}$-independent, there exists a unique homomorphism $h : (Z) \to A$ such that $h |_Z = f |_Z$. Since $(Y) \cap (Z) = \{0\}$ and $H^\sigma$ is an Abelian group, $(X) = (Y) \oplus (Z)$, so there exists a unique homomorphism $\tilde{f} : (X) \to A$ extending both $g$ and $h$. Clearly, $\tilde{f} |_X = f$. $\square$

### 3. Selectively Sequentially Pseudocompact Topologies on $\mathcal{V}$-free Groups

**Definition 3.1.** [7] An infinite cardinal $\tau$ is called admissible if there exists a pseudocompact group of cardinality $\tau$.

**Definition 3.2.** We shall say that a cardinal $\tau$ is selectively admissible if either $\tau$ is finite or there exists an infinite cardinal $\sigma$ such that $\sigma^{\omega} = \sigma \leq \tau \leq 2^\sigma$.

We shall need the following lemma describing properties of (selectively) admissible cardinals.

**Lemma 3.3.** [7] Lemma 3.4

(i) Infinite selectively admissible cardinals are admissible.

(ii) Under SCH, every admissible cardinal is selectively admissible.

(iii) Under GCH, $\sigma = \sigma^{\omega}$ holds for every admissible cardinal $\sigma$.

**Definition 3.4.** [7] Definitions 2.1 and 2.2] Let $\mathcal{V}$ be a variety of groups.

(i) A subset $X$ of a group $G$ is a $\mathcal{V}$-base of $G$ if $X$ is $\mathcal{V}$-independent and $(X) = G$.

(ii) A group is $\mathcal{V}$-free if it contains a $\mathcal{V}$-base.

(iii) For every cardinal $\tau$, we denote by $F_\tau(\mathcal{V})$ the unique (up to isomorphism) $\mathcal{V}$-free group having a $\mathcal{V}$-base of cardinality $\tau$.

**Lemma 3.5.** Let $\mathcal{V}$ be a variety of groups. Assume that $H$ is a compact group, $\sigma, \tau$ are infinite cardinals and $X$ is a dense $\mathcal{V}$-independent selectively sequentially pseudocompact subset of $H^\sigma$ such that $|X| = \tau$. Then $G = (X) \cong F_\tau(\mathcal{V})$ is a dense selectively sequentially pseudocompact $\mathcal{V}$-free subgroup of $H^\sigma$. Moreover,

(i) if $H$ is connected, then so is $G$;

(ii) if $H$ is locally connected, then so is $G$;
(iii) if $H$ is zero-dimensional, then so is $G$.

Proof. The isomorphism $\langle X \rangle \cong F_\sigma(V)$ is clear. Since $X \subseteq \langle X \rangle$ and $X$ is dense in $H^\sigma$, so is $G = \langle X \rangle$. Since $X$ is dense in $H^\sigma$, it is also dense in $G$. Since $X$ is selectively sequentially pseudocompact, Proposition 1.6 implies that $G$ is selectively sequentially pseudocompact as well. Note that the completion of $G$ coincides with $H^\sigma$, so the rest of the statements follows from [7, Fact 2.10].

Definition 3.6. [7, Definition 5.2] A variety $V$ is precompact if there exists a compact zero-dimensional metric group $H \in V$ with $r_V(H) \geq \omega$.

Non-precompact varieties are not easy to come by, as many of the known varieties are precompact. Indeed, any variety consisting of Abelian groups, the variety of all groups, the variety of all nilpotent groups, the variety of all polynilpotent groups, the variety of all soluble groups are known to be precompact [6]. For every prime number $p > 665$, the Burnside variety $B_p$ consisting of all groups satisfying the identity $x^p = e$ is not precompact [5].

Theorem 3.7. Let $V$ be a precompact variety of groups. Then for every infinite cardinal $\sigma$ such that $\sigma^\omega = \sigma$, the group $F_\sigma(V)$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Proof. Since $V$ is a precompact variety, by Definition 3.6, there is a compact zero-dimensional metric group $H \in V$ with $r_V(H) \geq \omega$. By Lemma 2.3, there exists a dense $V$-independent selectively sequentially pseudocompact subset $Y$ of $H^\sigma$ such that $|Y| = \sigma$. Applying Lemma 3.3 (with $X = Y$ and $\tau = \sigma$), we conclude that $\langle Y \rangle \cong F_\sigma(V)$ is a zero-dimensional selectively sequentially pseudocompact (dense) subgroup of $H^\sigma$. The subgroup topology that $\langle Y \rangle \cong F_\sigma(V)$ inherits from $H^\sigma$ is the required group topology on $F_\sigma(V)$.

Corollary 3.8. Let $V$ be a precompact variety of groups. Under GCH, the following conditions are equivalent for every infinite cardinal $\sigma$:

(i) the group $F_\sigma(V)$ admits a pseudocompact group topology;
(ii) the group $F_\sigma(V)$ admits a strongly pseudocompact group topology;
(iii) the group $F_\sigma(V)$ admits a selectively sequentially pseudocompact group topology;
(iv) the group $F_\sigma(V)$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Proof. The implication (iv)$\Rightarrow$(iii) is trivial, while the implications (iii)$\Rightarrow$(ii)$\Rightarrow$(i) follow from equation (7).

(i)$\Rightarrow$(iv) Let $\sigma$ be an arbitrary infinite cardinal. Assume (i); that is, $F_\sigma(V)$ admits a pseudocompact group topology. Since $\sigma$ is infinite, $|F_\sigma(V)| = \sigma$ is an admissible cardinal by Definition 3.1. Since GCH holds, Lemma 3.3 (iii) implies that $\sigma^\omega = \sigma$. Applying Theorem 3.7, we conclude that $F_\sigma(V)$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

We use $G$ for denoting the variety of all groups.

Theorem 3.9. For every infinite cardinal $\sigma$ such that $\sigma^\omega = \sigma$, the group $F_\sigma(G)$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Proof. The group $H = SO(3,\mathbb{R})$ of all rotations around the origin of the three-dimensional Euclidean space $\mathbb{R}^3$ under the operation of composition is a compact metric group satisfying $r_G(SO(3,\mathbb{R})) \geq \omega$; see [2]. Moreover, $H$ (trivially) belongs to the variety $G$. By Lemma 2.3, there exists a dense $G$-independent selectively sequentially pseudocompact subset $Y$ of $H^\sigma$ such that $|Y| = \sigma$. Since $H$ is both connected and locally connected, applying Lemma 3.3 (with $X = Y$ and $V = G$), we conclude that $\langle Y \rangle \cong F_\sigma(G)$ is a connected, locally connected, selectively sequentially pseudocompact (dense) subgroup of $H^\sigma$. The subgroup topology that $\langle Y \rangle \cong F_\sigma(G)$ inherits from $H^\sigma$ is the required group topology on $F_\sigma(G)$. 

\[\square\]
Corollary 3.10. Under GCH, the following conditions are equivalent for every infinite cardinal $\tau$:

(i) $F_\tau(\mathcal{G})$ admits a pseudocompact group topology;
(ii) $F_\tau(\mathcal{G})$ admits a strongly pseudocompact group topology;
(iii) $F_\tau(\mathcal{G})$ admits a selectively sequentially pseudocompact group topology;
(iv) $F_\tau(\mathcal{G})$ admits a zero-dimensional selectively sequentially pseudocompact group topology;
(v) $F_\tau(\mathcal{G})$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Proof. By [7, Lemma 5.3], the variety $\mathcal{G}$ is precompact. Hence, the equivalence of items (i)–(iv) follows from Corollary 3.8. The implication (i)$\Rightarrow$(v) was proved in Theorem 3.9. The implication (v)$\Rightarrow$(iii) is trivial. \hfill $\Box$

We use $\mathcal{A}$ for denoting the variety of all Abelian groups.

Theorem 3.11. Let $\mathcal{V}$ be a variety satisfying $\mathcal{V} \subseteq \mathcal{A}$. Then for every selectively admissible cardinal $\tau \geq \omega$, the $\mathcal{V}$-free group $F_\tau(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Proof. Since $\mathcal{V} \subseteq \mathcal{A}$, the variety $\mathcal{V}$ is precompact [10]. By Definition 3.6, there exists a zero-dimensional compact metric group $H \in \mathcal{V}$ with $r_\mathcal{V}(H) \geq \omega$. Let $\tau \geq \omega$ be a selectively admissible cardinal. By Definition 3.2, there exists an infinite cardinal $\sigma$ satisfying $\sigma^\omega = \sigma \leq \tau \leq 2^\sigma$. By Lemma 2.4, there exists a dense $\mathcal{V}$-independent selectively sequentially pseudocompact subset $X$ of $H^\sigma$ such that $|X| = \tau$. By Lemma 3.3, $\langle X \rangle \cong F_\tau(\mathcal{V})$ is a zero-dimensional selectively sequentially pseudocompact subgroup of $H^\sigma$. The subgroup topology that $\langle X \rangle \cong F_\tau(\mathcal{V})$ inherits from $H^\sigma$ is the required group topology on $F_\tau(\mathcal{V})$. \hfill $\Box$

Corollary 3.12. Let $\mathcal{V}$ be a variety satisfying $\mathcal{V} \subseteq \mathcal{A}$. Under SCH, the following conditions are equivalent for every infinite cardinal $\tau$:

(i) $F_\tau(\mathcal{V})$ admits a pseudocompact group topology;
(ii) $F_\tau(\mathcal{V})$ admits a strongly pseudocompact group topology;
(iii) $F_\tau(\mathcal{V})$ admits a selectively sequentially pseudocompact group topology;
(iv) $F_\tau(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Proof. (i)$\Rightarrow$(iv) By (i), $F_\tau(\mathcal{V})$ admits a pseudocompact group topology. Since $\tau$ is infinite, $|F_\tau(\mathcal{V})| = \tau$ is an admissible cardinal by Definition 3.1. Since SCH holds, $\tau$ is selectively admissible by Lemma 3.3 (ii). Applying Theorem 3.11, we conclude that $F_\tau(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

The implication (iv)$\Rightarrow$(iii) is trivial. The implications (iii)$\Rightarrow$(ii)$\Rightarrow$(i) follow from equation (11). \hfill $\Box$

Theorem 3.13. For every selectively admissible cardinal $\tau \geq \omega$, the $\mathcal{A}$-free group $F_\tau(\mathcal{A})$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Proof. Let $\tau \geq \omega$ be a selectively admissible cardinal. By Definition 3.2, there exists an infinite cardinal $\sigma$ satisfying $\sigma^\omega = \sigma \leq \tau \leq 2^\sigma$. Since $r_\mathcal{A}(\mathbb{T}) = \omega \geq \omega$, by Lemma 2.3, there exists a dense $\mathcal{A}$-independent selectively sequentially pseudocompact subset $X$ of $\mathbb{T}^\sigma$ such that $|X| = \tau$. Since $\mathbb{T}$ is both connected and locally connected, by Lemma 3.3, $\langle X \rangle \cong F_\tau(\mathcal{A})$ is a connected, locally connected, selectively sequentially pseudocompact subgroup of $\mathbb{T}^\sigma$. The subgroup topology that $\langle X \rangle \cong F_\tau(\mathcal{A})$ inherits from $\mathbb{T}^\sigma$ is the required group topology on $F_\tau(\mathcal{A})$. \hfill $\Box$

Corollary 3.14. Under SCH, the following conditions are equivalent for every infinite cardinal $\tau$:

(i) $F_\tau(\mathcal{A})$ admits a pseudocompact group topology;
(ii) $F_\tau(\mathcal{A})$ admits a strongly pseudocompact group topology;
(iii) $F_\tau(\mathcal{A})$ admits a selectively sequentially pseudocompact group topology;
(iv) $F_\tau(\mathcal{A})$ admits a zero-dimensional selectively sequentially pseudocompact group topology;
(v) $F_r(A)$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Proof. The equivalence of items (i)–(iv) follows from Corollary 3.12. The implication (i)$\Rightarrow$(v) was proved in Theorem 3.13. The implication (v)$\Rightarrow$(iii) is trivial. $\square$

4. The algebraic structure of pseudocompact torsion Abelian groups

The main result in this section is Theorem 4.7 which describes the algebraic structure of pseudocompact torsion Abelian groups. One can deduce this theorem from [7, Theorem 6.2] but we decided to give a separate proof of it. Another description of the algebraic structure of pseudocompact torsion Abelian groups can be found in [3, Theorem 3.19] and [7, Theorem 6.2].

Definition 4.1. For $m, n \in \mathbb{N}^+$ with $m \leq n$, a finite sequence $(\tau_m, \tau_{m+1}, \ldots, \tau_n)$ of cardinals will be called:

(i) good provided that, for every integer $k$ such that $m \leq k \leq n$, the cardinal $\max_{k \leq i \leq n} \tau_i$ is either finite or admissible;

(ii) very good if $\tau_n$ is either finite or admissible and $\tau_i < \tau_n$ for every $i \in \mathbb{N}^+$ with $m \leq i < n$.

Remark 4.2. Assume that $m, l, n \in \mathbb{N}^+$ and $m \leq l \leq n$. If $(\tau_m, \tau_{m+1}, \ldots, \tau_n)$ is a good sequence of cardinals, then the sequence $(\tau_l, \tau_{l+1}, \ldots, \tau_n)$ is good as well.

The next lemma shows that each good sequence of cardinals can be partitioned into finitely many adjacent very good subsequences.

Lemma 4.3. Assume that $m, n \in \mathbb{N}^+$, $m \leq n$ and $(\tau_m, \tau_{m+1}, \ldots, \tau_n)$ is a good sequence of cardinals. Then there exists a strictly increasing sequence $(s_0, s_1, \ldots, s_k)$ of integers such that $s_0 = m - 1$, $s_k = n$ and the sequence $(\tau_{s_1}, \tau_{s_1+2}, \ldots, \tau_{s_k+1})$ is very good for every $i = 0, \ldots, k - 1$.

Proof. We shall prove this lemma by induction on $n - m$.

Basis of induction. Suppose that the sequence $(\tau_n)$ is good. By item (i) of Definition 4.1, $\tau_n$ is either finite or admissible. It follows from item (ii) of Definition 4.1 that the sequence $(\tau_n)$ is very good.

Inductive step. Suppose that $j \in \mathbb{N}^+$ and our lemma has been already proved for all good sequences $(\tau_m, \tau_{m+1}, \ldots, \tau_n)$ such that $n - m < j$. Fix a good sequence $(\tau_m, \tau_{m+1}, \ldots, \tau_n)$ with $n - m = j$.

The set $K = \{k \in \mathbb{N}^+ : m \leq k \leq n, \tau_k = \max_{m \leq i \leq n} \tau_i\}$ is non-empty, so we can define $l = \min K$.

We claim that the sequence $(\tau_m, \ldots, \tau_l)$ is very good. Since the sequence $(\tau_m, \tau_{m+1}, \ldots, \tau_n)$ is good by our assumption, the cardinal $\max_{m \leq i \leq n} \tau_i$ is either finite or admissible by Definition 4.1(i). Since $l \in K$, this means that $\tau_l = \max_{m \leq i \leq n} \tau_i$ is either finite or admissible. Since $l = \min K$, it follows from the definition of $K$ that $\tau_l < \tau_i$ whenever $m \leq i < l$. Therefore, the sequence $(\tau_m, \ldots, \tau_l)$ is very good by Definition 4.1(ii). Now we consider two cases.

Case 1. $l = n$. In this case, the original sequence $(\tau_m, \tau_{m+1}, \ldots, \tau_n)$ is very good.

Case 2. $l < n$. By Remark 4.2, the sequence $(\tau_{l+1}, \tau_{l+2}, \ldots, \tau_n)$ is good. Since $n - (l + 1) < n - m = j$, we can apply our inductive assumption to this sequence to get a strictly increasing sequence $(s_1, s_2, \ldots, s_k)$ of integers such that $s_1 = l$, $s_k = n$ and the sequence $(\tau_{s_1}, \tau_{s_1+2}, \ldots, \tau_{s_k+1})$ is very good for every $i = 1, \ldots, k - 1$. Let $s_0 = m - 1$. Now $(s_0, s_1, \ldots, s_k)$ is the desired sequence. $\square$

Good sequences of cardinals typically appear in the situation described by our next lemma. Even though this lemma can be derived from [7, Theorem 6.2], we include its simple proof for convenience of the reader.
Lemma 4.4. If \( G \) is a pseudocompact group such that \( G \cong \bigoplus_{i=m}^{n} \mathbb{Z}(p^i)(\tau_i) \) for some prime number \( p \), positive integers \( m, n \in \mathbb{N}^+ \) and cardinals \( \tau_m, \tau_{m+1}, \ldots, \tau_n \), then the sequence \( (\tau_m, \tau_{m+1}, \ldots, \tau_n) \) is good.

Proof. It suffices to show that, for every integer \( k \) such that \( m \leq k \leq n \), the cardinal \( \mu_k = \max_{k \leq i \leq n} \tau_i \) is either finite or admissible. Fix an integer \( k \) satisfying \( m \leq k \leq n \). If all cardinals \( \tau_k, \tau_{k+1}, \ldots, \tau_n \) are finite, then \( \mu_k \) is finite. Assume now that at least one of the cardinals \( \tau_k, \tau_{k+1}, \ldots, \tau_n \) is infinite. Then the group

\[
p^{k-1}G = \bigoplus_{i=k}^{n} \mathbb{Z}(p^{i-k+1})(\tau_i)
\]

is infinite, so \( |p^{k-1}G| = \max_{k \leq i \leq n} \tau_i = \mu_k \). Observe that \( p^{k-1}G \) is a continuous image of the pseudocompact group \( G \) under the map \( g \mapsto p^{k-1}g \), so \( p^{k-1}G \) is pseudocompact as well. Since the cardinality \( \mu_k \) of the pseudocompact group \( p^{k-1}G \) is infinite, it must be admissible by Definition 3.11.

Definition 4.5. We shall say that a group \( G \) is nice provided that there exist a prime number \( p \), positive integers \( m, n \in \mathbb{N}^+ \) and a very good sequence \( (\tau_m, \tau_{m+1}, \ldots, \tau_n) \) of cardinals such that \( G \cong \bigoplus_{i=m}^{n} \mathbb{Z}(p^i)(\tau_i) \).

Let \( p \) be a prime number. Recall that an Abelian group \( G \) is called a \( p \)-group provided that for every \( g \in G \) one can find \( k \in \mathbb{N} \) such that \( p^k g = 0 \).

Lemma 4.6. A non-trivial pseudocompact bounded torsion Abelian \( p \)-group can be decomposed into a finite direct sum of nice groups.

Proof. Let \( G \) be a non-trivial pseudocompact bounded torsion \( p \)-group. Then

\[
G \cong \bigoplus_{j=1}^{n} \mathbb{Z}(p^j)(\tau_j)
\]

for a suitable \( n \in \mathbb{N}^+ \) and cardinals \( \tau_1, \tau_2, \ldots, \tau_n \); see [11, Theorem 17.2]. By Lemma 4.4, the sequence \( (\tau_1, \tau_2, \ldots, \tau_n) \) is good. Applying Lemma 4.3 to the sequence \( (\tau_1, \tau_2, \ldots, \tau_n) \), we obtain a strictly increasing sequence \( (s_0, s_1, \ldots, s_k) \) of integers such that \( s_0 = 0 \), \( s_k = n \) and the sequence \( (\tau_{s_i+1}, \tau_{s_i+2}, \ldots, \tau_{s_{i+1}}) \) is very good for every \( i = 0, \ldots, k-1 \). By Definition 4.3, the group

\[
G_i = \bigoplus_{j=s_i+1}^{s_{i+1}} \mathbb{Z}(p^j)(\tau_j)
\]

is nice for every \( i = 0, \ldots, k-1 \). Clearly, \( G = \bigoplus_{i=0}^{k-1} G_i \).

Theorem 4.7. Every non-trivial pseudocompact torsion Abelian group is isomorphic to a finite direct sum of nice groups.

Proof. Let \( G \) be a non-trivial pseudocompact torsion Abelian group. Then \( G \) is bounded torsion [7, Corollary 3.9], and so there exists a finite set \( P \) of prime numbers such that \( G = \bigoplus_{p \in P} G_p \), where \( G_p \) is a non-trivial (bounded torsion) \( p \)-group for every \( p \in P \); see [11, Theorem 17.2].

Let \( p \in P \) be arbitrary. For every \( q \in P \setminus \{p\} \), \( G_q \) has order \( q^{k_q} \) for some \( k_q \in \mathbb{N}^+ \); that is, \( q^{k_q} G_q = \{0\} \). Define \( n = \prod_{q \in P \setminus \{p\}} q^{k_q} \). Since \( G_p \) is \( q \)-divisible for every prime number \( q \) different from \( p \), it follows that \( n G = n G_p = G_p \). Since \( n G \) is an image of \( G \) under the continuous map \( g \mapsto nG \) and \( G \) is pseudocompact, we conclude that \( G_p \) is also pseudocompact. Now we can apply Lemma 4.6 to \( G_p \) to get a decomposition of \( G_p \) into a finite direct sum of nice groups.
5. Selectively sequentially pseudocompact topologies on torsion Abelian groups

Lemma 5.1. Assume that $p$ is a prime number, $m, n \in \mathbb{N}^+$, $\tau_m, \tau_{m+1}, \ldots, \tau_n$ are cardinals such that $\tau_i \leq \tau_j$ for all $i = m, \ldots, n$, and $\tau_n$ is selectively admissible. Then $G = \bigoplus_{i=m}^{n} \mathbb{Z}(p^i)(\tau_i)$ can be equipped with a selectively sequentially pseudocompact group topology.

Proof. Suppose that $\tau_n$ is finite. Since $\tau_i \leq \tau_j$ for all $i = m, \ldots, n$, the group $G$ is finite. The discrete topology on $G$ is compact, so also selectively sequentially pseudocompact.

Suppose now that $\tau_n$ is infinite. Since $\tau_n$ is selectively admissible, Definition 3.2 implies that $\sigma^\omega = \sigma \leq \tau_n \leq 2^\sigma$ for some cardinal $\sigma$.

Let $H = \mathbb{Z}(p^n)^\omega$. By Lemma 2.4 applied to this $H$, the variety $A_{p^n} = \{G : G$ is an Abelian group such that $p^ng = 0$ for all $g \in G\}$ taken as $\mathcal{V}$ and the cardinal $\tau_n$ taken as $\tau$, we can find a selectively sequentially pseudocompact dense $A_{p^n}$-independent subset $X$ of $H^\sigma$ and a set $D \subseteq \sigma$ such that $|X| = \tau_n$, $|D| = |\sigma \setminus D| = \sigma$ and $(X) \cap H^D = \{0\}$, where $H^D = \{f \in H^\sigma : f(\alpha) = 0$ for all $\alpha \in \sigma \setminus D\}$. Since $X$ is $A_{p^n}$-independent and $|X| = \tau_n$, we have $(X) \cong \mathbb{Z}(p^n)(\tau_n)$. Let $\varphi : (\mathbb{Z}(p^n)(\tau_n)) \to (X)$ be an isomorphism. Define

\begin{equation}
G' = \begin{cases}
\{0\} & \text{if } n = m \\
\bigoplus_{i=m}^{n-1} \mathbb{Z}(p^i)(\tau_i) & \text{if } n > m.
\end{cases}
\end{equation}

Clearly, $G = G' \oplus \mathbb{Z}(p^n)(\tau_n)$.

We claim that there is a monomorphism $\psi : G' \to H^D$. This is clear for $n = m$, as $G' = \{0\}$ by (12). Suppose now that $n > m$. Then $G' = \bigoplus_{i=m}^{n-1} \mathbb{Z}(p^i)(\tau_i)$ by (12). Let $D = \bigcup_{i=m}^{n-1} D_i$ be a decomposition of $D$ into pairwise disjoint sets $D_i$, such that $|D_i| = |D|$ for all $i = m, \ldots, n - 1$.

Let $i = m, \ldots, n - 1$ be arbitrary. Note that $H^{D_i} \cong H^D \cong \mathbb{Z}(p^n)^D$, so $r_{A_{p^n}}(H^{D_i}) \geq 2^{|D|} = 2^\sigma$ by [7; Lemma 4.1]. Since $\tau_i \leq \tau_n \leq 2^\sigma \leq r_{A_{p^n}}(H^{D_i})$, we can fix a monomorphism $\psi_i : \mathbb{Z}(p^n)(\tau_i) \to H^{D_i}$.

Let $\psi : G' = \bigoplus_{i=m}^{n-1} \mathbb{Z}(p^i)(\tau_i) \to \bigoplus_{i=m}^{n-1} H^{D_i} \cong H^D$ be the unique monomorphism extending each of $\psi_i$. Since $\psi(G') \subseteq H^D$, $\varphi(\mathbb{Z}(p^n)(\tau_n)) = \langle X \rangle$ and $H^D \cap \langle X \rangle = \{0\}$, there is a unique monomorphism $\chi : G = G' \oplus \mathbb{Z}(p^n)(\tau_n) \to H^D \oplus \langle X \rangle \subseteq H^\sigma$ extending both $\varphi$ and $\psi$.

Since $(X)$ is a dense selectively sequentially pseudocompact subgroup of $H^\sigma$ and $\varphi(\mathbb{Z}(p^n)(\tau_n)) \subseteq \chi(G)$, the subgroup $(X)$ of $\chi(G)$ is dense in $\chi(G)$. Since $(X)$ is selectively sequentially pseudocompact, Proposition 1.6 implies that $\chi(G)$ is selectively sequentially pseudocompact as well. Since $\chi : G \to \chi(G)$ is an isomorphism, $G$ admits a selectively sequentially pseudocompact group topology.

\[ \square \]

Theorem 5.2. Under SCH, the following conditions are equivalent for every torsion Abelian group $G$:

(i) $G$ has a pseudocompact group topology;
(ii) $G$ has a strongly pseudocompact group topology;
(iii) $G$ has a selectively sequentially pseudocompact group topology.

Proof. The implications (iii)⇒(ii)⇒(i) follow from equation (11).

(i)⇒(iii). Without loss of generality, we shall assume that $G$ is non-trivial. By Theorem 4.7, $G \cong \bigoplus_{i=1}^{j} G_k$, where $j \in \mathbb{N}^+$ and each $G_k$ is a nice group. By Proposition 1.2, it suffices to prove that each $G_k$ admits a selectively sequentially pseudocompact group topology.

Fix $k = 1, \ldots, j$. Since $G_k$ is nice, we can apply Definition 4.5 to fix a prime number $p$, positive integers $m, n \in \mathbb{N}^+$ and a very good sequence $(\tau_m, \tau_{m+1}, \ldots, \tau_n)$ of cardinals such that $G_k \cong \bigoplus_{i=m}^{n} \mathbb{Z}(p^i)(\tau_i)$. By Definition 4.5 (ii), $\tau_n$ is either finite or admissible and $\tau_i < \tau_n$ for every $i \in \mathbb{N}^+$ with $m \leq i < n$. If $\tau_n$ is finite, then it is selectively admissible by Definition 3.2. Suppose now that $\tau_n$ is admissible. Since SCH holds, $\tau_n$ is selectively admissible by Lemma 3.3 (ii). By Lemma 5.1, $G_k$ admits a selectively sequentially pseudocompact group topology.

\[ \square \]
6. Selectively sequentially pseudocompact topologies on torsion-free Abelian groups

Recall that a subgroup $H$ of an Abelian group $G$ is said to be essential in $G$ provided that $N \cap H \neq \{0\}$ for every non-zero subgroup $N$ of $G$.

Lemma 6.1. If $F$ is a maximal $A$-free subgroup of an Abelian group $G$, then $H = t(G) + F = t(G) \oplus F$ is an essential subgroup of $G$.

Proof. Since $F$ is torsion-free and $t(G)$ is torsion, $t(G) \cap F = \{0\}$, which means that the sum $H = t(G) + F$ is direct.

Let $N$ be a non-zero subgroup of $G$. Fix a non-zero element $x \in N$. If $x$ is torsion, then $x \in t(G)$, so $x \in t(G) \cap N \subseteq H \cap N$, which implies $H \cap N \neq \{0\}$. Suppose now that $x$ is non-torsion. Then $\langle x \rangle$ is an infinite cyclic subgroup of $G$. If $F \cap \langle x \rangle = \{0\}$, then $F' = F + \langle x \rangle = F \oplus \langle x \rangle$ would be an $A$-free subgroup of $G$ containing $F$ as a proper subgroup, contradicting the maximality of $F$. Therefore, $F \cap \langle x \rangle \neq \{0\}$, which implies $H \cap N \neq \{0\}$. \qed

Theorem 6.2. Suppose that $G$ is an Abelian group and $\sigma^\omega = \sigma \leq r_A(G) \leq |G| \leq 2^\sigma$ for some infinite cardinal $\sigma$. Then $G$ is isomorphic to a dense connected, locally connected, selectively sequentially pseudocompact subgroup of $\mathbb{T}^\sigma$.

Proof. Let $F$ be a maximal $A$-free subgroup of the group $G$; the existence of such $F$ follows from Zorn’s lemma. Then $r_A(F) = r_A(G) \geq \sigma^\omega \geq \omega$, as $\sigma$ is infinite. Since $r_A(F) = r_A(G) \leq |G| \leq 2^\sigma$, the cardinal $r_A(F)$ is selectively admissible by Definition 3.2. Arguing as in the proof of Theorem 3.13 we can fix a monomorphism $f : F \to \mathbb{T}^\sigma$ such that $f(F)$ is a dense selectively sequentially pseudocompact subgroup of $\mathbb{T}^\sigma$.

Since $\mathbb{T}^\sigma$ is divisible, $r_A(t(G)) = 0 \leq 2^\omega = r_A(\mathbb{T}^\sigma)$ and $r_A(\mathbb{Z}_p)(t(G)) \leq |G| \leq 2^\sigma = r_A(\mathbb{Z}_p)(\mathbb{T}^\sigma)$ for every prime number $p$, we can use Lemma 3.16 to find a monomorphism $g : t(G) \to \mathbb{T}^\sigma$.

Let $h : H = t(G) \oplus F \to \mathbb{T}^\sigma$ be the unique homomorphism extending both $g$ and $f$. Since $h(t(G))$ is torsion and $h(F) = f(G)$ is torsion-free, $h(t(G)) \cap h(F) = \{0\}$. Since both $g$ and $f$ are monomorphisms, so is $h$; see Lemma 3.10 (ii). Since $\mathbb{T}^\sigma$ is divisible, there exists a homomorphism $\varphi : G \to \mathbb{T}^\sigma$ extending $h$; see Theorem 21.1. Since $H$ is essential in $G$ by Lemma 6.1 and $h$ is a monomorphism, $\varphi$ is a monomorphism as well; see Lemma 3.5. Therefore, $G$ and $\varphi(G)$ are isomorphic.

Since $\varphi(F) = f(F)$ is a dense selectively sequentially pseudocompact subgroup of $\mathbb{T}^\omega$ and $\varphi(F) \subseteq \varphi(G)$, the group $\varphi(G)$ is selectively sequentially pseudocompact by Proposition 1.6. Finally, since $\mathbb{T}$ is connected and locally connected, Fact 2.11] implies that $\varphi(G)$ is also connected and locally connected. \qed

Theorem 6.3. Under SCH, the following conditions are equivalent for every Abelian group $G$ satisfying $r_A(G) = |G|$:

(i) $G$ admits a pseudocompact group topology;
(ii) $G$ admits a strongly pseudocompact group topology;
(iii) $G$ admits a selectively sequentially pseudocompact group topology;
(iv) $G$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Proof. The implication (iv)⇒(iii) is trivial, while the implications (iii)⇒(ii)⇒(i) follow from equation (1).

(i)⇒(iv) Suppose that $G$ admits a pseudocompact group topology. Observe that the equality $r_A(G) = |G|$ implies that $G$ is infinite, so $|G|$ is admissible by Definition 3.1. Since SCH is assumed, $|G|$ is selectively admissible by Lemma 3.3 (ii). By Definition 3.2 there exists an infinite cardinal $\sigma$ such that $\sigma^\omega = \sigma \leq |G| \leq 2^\sigma$. Since $r_A(G) = |G|$, the assumptions of Theorem 6.2 are satisfied.
Applying this theorem, we conclude that $G$ is isomorphic to a dense connected, locally connected, selectively sequentially pseudocompact subgroup of $\mathbb{T}^\sigma$. □

**Corollary 6.4.** Under SCH, the following conditions are equivalent for every torsion-free Abelian group $G$:

(i) $G$ admits a pseudocompact group topology;
(ii) $G$ admits a strongly pseudocompact group topology;
(iii) $G$ admits a selectively sequentially pseudocompact group topology;
(iv) $G$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

**Proof.** Since $G$ is torsion-free, $G$ must be infinite.

(i)⇒(iv) Suppose that $G$ admits a pseudocompact group topology. Since $G$ is infinite, $|G| \geq r_A(G) \geq c$ by [7, Theorem 3.8]. Since $G$ is an uncountable, torsion-free Abelian group, $r_A(G) = |G|$. By the implication (i)⇒(iv) of Theorem 6.3, $G$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

The implication (iv)⇒(iii) is trivial, while the implications (iii)⇒(ii)⇒(i) follow from equation (1). □

**Corollary 6.5.** Under SCH, the following conditions are equivalent for every Abelian group $G$:

(i) $G/t(G)$ admits a pseudocompact group topology;
(ii) $G/t(G)$ admits a strongly pseudocompact group topology;
(iii) $G/t(G)$ admits a selectively sequentially pseudocompact group topology;
(iv) $G/t(G)$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

**Proof.** If $G$ is torsion, then $G/t(G)$ is the trivial group, and so all four conditions are equivalent for $G/t(G)$. If $G$ is non-torsion, then $G/t(G)$ is a torsion-free group, so Corollary 6.4 applies to $G/t(G)$. □

**Theorem 6.6.** Under GCH, the following conditions are equivalent for every Abelian group $G$ satisfying the inequality $|G| \leq 2^{r_A(G)}$:

(i) $G$ admits a pseudocompact group topology;
(ii) $G$ admits a strongly pseudocompact group topology;
(iii) $G$ admits a selectively sequentially pseudocompact group topology;
(iv) $G$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

**Proof.** If $r_A(G) = 0$, then $|G| \leq 2^{r_A(G)} = 1$ by our assumption, so $G$ is trivial, and all items (i)–(iv) are equivalent for the trivial group. Therefore, from now on we shall assume that $r_A(G) \geq 1$; in particular, $G$ is infinite and non-torsion.

(i)⇒(iv) Suppose that $G$ admits a pseudocompact group topology. Since $G$ is non-torsion, $\sigma = r_A(G)$ is admissible by [6, Corollary 1.19]. Since GCH is assumed, $\sigma^\omega = \sigma$ by Lemma 3.3 (iii). Since $|G| \leq 2^\sigma$, the assumptions of Theorem 6.2 are satisfied. Applying this theorem, we conclude that $G$ is isomorphic to a dense connected, locally connected, selectively sequentially pseudocompact subgroup of $\mathbb{T}^\sigma$.

The implication (iv)⇒(iii) is trivial, while the implications (iii)⇒(ii)⇒(i) follow from equation (1). □

7. Two open questions

We finish with two concrete versions of general Question 1.5 related to our results.
Question 7.1. Can Corollaries 3.8 and 3.10 be proved in ZFC? Or at least, can GCH be weakened to SCH in the assumption of these corollaries?

Question 7.2. Can Theorems 5.2, 6.3, 6.6 and Corollaries 3.12, 3.14, 6.4, 6.5 be proved in ZFC?

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