Relations of multiple $t$-values of general level∗

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Abstract

We study the relations of multiple $t$-(star) values of general level. The generating function of sums of multiple $t$-(star) values of level $N$ with fixed weight, depth and height is represented by the generalized hypergeometric function $\mathbf{3}F_{2}$, which generalizes the results for multiple zeta-(star) values and multiple $t$-(star) values. As applications, we obtain formulas for the generating functions of sums of multiple $t$-(star) values of level $N$ with height one and maximal height and a weighted sum formula for sums of multiple $t$-(star) values of level $N$ with fixed weight and depth. Using the stuffle algebra, we also get the symmetric sum formulas and Hoffman’s restricted sum formulas for multiple $t$-(star) values of level $N$. Some evaluations of multiple $t$-star values of level 2 with one-two-three indices are given.

Keywords multiple $t$-value, multiple zeta value, generalized hypergeometric function.

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1 Introduction

Let $k = (k_1, \ldots, k_n)$ be a finite sequence of positive integers, which is called an index. The quantities

$$k_1 + \cdots + k_n, \quad n \quad \text{and} \quad |\{j| 1 \leq j \leq n, k_j \geq 2\}|$$

are called the weight, depth and height of the index $k$, respectively. The index $k$ is called admissible if $k_1 > 1$.

The multiple zeta values of any level were introduced by H. Yuan and J. Zhao in [21] and they focused on the double zeta values of level $N$, where $N$ is a fixed positive integer. Set $R = R_N = \mathbb{Z}/N\mathbb{Z}$. For an admissible index $k = (k_1, \ldots, k_n)$ and any $a = (a_1, a_2, \ldots, a_n) \in R^n$, the multiple zeta value of level $N$ is defined by

$$\zeta_N(k; a) = \sum_{m_1 > \cdots > m_n > 0, m_i \equiv a_i \pmod{N}} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$
Z. Li and Z. Wang studied the algebraic framework for the double shuffle relations of multiple zeta values of level $N$ and gave sum formulas and weighted sum formulas of double zeta values of level 2 and 3 in [15]. Two variants of multiple zeta values of level 2 called the multiple $t$-values [8] and the multiple $T$-values [10] have been studied recent years, see for example [3, 8, 10, 14, 16, 20].

In this paper, we consider multiple $t$-(star) values of level $N$, which generalize the multiple $t$-(star) values. For an admissible index $\mathbf{k} = (k_1, \ldots, k_n)$ and any $a \in \{1, 2, \ldots, N\}$, we define the multiple $t$-value of level $N$ and the multiple $t$-star value of level $N$ respectively by

\[
t_{N,a}(\mathbf{k}) = t_{N,a}(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n > 0, \ m_i \equiv a \ (\mod N)} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},
\]

\[
t_{N,a}^*(\mathbf{k}) = t_{N,a}^*(k_1, \ldots, k_n) = \sum_{m_1 \geq \cdots \geq m_n > 0, \ m_i \equiv a \ (\mod N)} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.
\]

Note that if the index is empty, we treat the values $t_{N,a}(\emptyset) = t_{N,a}^*(\emptyset) = 1$. And we have the following iterated integral representations

\[
t_{N,a}(\mathbf{k}) = \int_0^1 \left( \frac{dt}{t} \right)^{k_1-1} \frac{t^{N-1}dt}{1-t^N} \cdots \left( \frac{dt}{t} \right)^{k_n-1} \frac{t^{N-1}dt}{1-t^N}, \tag{1.1}
\]

\[
t_{N,a}^*(\mathbf{k}) = \int_0^1 \left( \frac{dt}{t} \right)^{k_1-1} \frac{dt}{t(1-t^N)} \cdots \left( \frac{dt}{t} \right)^{k_n-1} \frac{dt}{t(1-t^N)} \left( \frac{dt}{t} \right)^{a-1} \frac{t^{a-1}dt}{1-t^N}, \tag{1.2}
\]

where for one forms $w_i = f_i(t)dt$, $i = 1, 2, \ldots, k$, we define that

\[
\int_0^z w_1 w_2 \cdots w_k = \int_{z>t_1>t_2>\ldots>t_k>0} f_1(t)f_2(t) \cdots f_k(t)dt_1dt_2 \cdots dt_k.
\]

Obviously,

\[
t_{1,1}(\mathbf{k}) = \zeta(\mathbf{k}) = \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},
\]

\[
t_{1,1}^*(\mathbf{k}) = \zeta^*(\mathbf{k}) = \sum_{m_1 \geq \cdots \geq m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},
\]

which are the multiple zeta value and the multiple zeta-star value [6] respectively. We also have $t_{N,N}(\mathbf{k}) = N^{-k} \zeta(\mathbf{k})$ and $t_{N,N}^*(\mathbf{k}) = N^{-k} \zeta^*(\mathbf{k})$, where $k$ is the weight of the index $\mathbf{k}$. Taking $N = 2$ and $a = 1$, we get the multiple $t$-value

\[
t_{2,1}(\mathbf{k}) = t(\mathbf{k}) = \sum_{m_1 > \cdots > m_n > 0, \ m_i \text{ odd}} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},
\]

and the multiple $t$-star value

\[
t_{2,1}^*(\mathbf{k}) = t^*(\mathbf{k}) = \sum_{m_1 \geq \cdots \geq m_n > 0, \ m_i \text{ odd}} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.
\]
Y. Ohno and D. Zagier [17] studied the generating function of sums of multiple zeta values with fixed weight, depth and height by using the Gaussian hypergeometric function. T. Aoki, Y. Kombu and Y. Ohno [1] represented the generating function of sums of multiple zeta-star values with fixed weight, depth and height by the generalized hypergeometric function \( {}_3F_2 \). Here for a positive integer \( m \) and complex numbers \( b_1, \ldots, b_{m+1}, c_1, \ldots, c_m \) with none of \( c_i \) is zero or a negative integer, the generalized hypergeometric function \( {}_{m+1}F_m \) is defined by

\[
{}_{m+1}F_m(b_1, \ldots, b_{m+1}; c_1, \ldots, c_m; z) = \sum_{n=0}^{\infty} \frac{(b_1)_n \cdots (b_{m+1})_n}{(c_1)_n \cdots (c_m)_n} \frac{z^n}{n!},
\]

where the Pochhammer symbol \((b)_n\) is defined by

\[
(b)_n = \frac{\Gamma(b + n)}{\Gamma(b)} = \begin{cases} 1 & \text{if } n = 0, \\ b(b+1) \cdots (b+n-1) & \text{if } n > 0. \end{cases}
\]

The above series is absolutely and uniformly convergent for \(|z| < 1\) and the convergence also extends over the unit circle if \( \Re(\sum c_i - \sum b_i) > 0 \). As analogues of Ohno-Zagier relations, the Ohno-Zagier type relations for the multiple T-values were studied by Y. Takeyama in [20] and the Ohno-Zagier type relations for the multiple \( t \)-(star) values were given by Z. Li and Y. Song in [14]. Similarly as in [14], we study the Ohno-Zagier type relations for multiple \( t \)-(star) values of level \( N \) and give some applications in Section 2.

Due to the infinite series representations, it is easy to see that the multiple \( t \)-(star) values of level \( N \) satisfy stuffle relations. Hence we obtain the symmetric sum formulas and Hoffman’s restricted sum formulas for multiple \( t \)-(star) values of level \( N \) by establishing the algebraic setup in Section 3.

D. Zagier found evaluations of \( \zeta(2, \ldots, 2, 3, 2, \ldots, 2) \) and \( \zeta^* (2, \ldots, 2, 3, 2, \ldots, 2) \) by establishing the generating functions in [22]. By similar method, T. Murakami gave the evaluation of \( t(2, \ldots, 2, 3, 2, \ldots, 2) \) in [16] and S. Charlton provided the evaluation of \( t(2, \ldots, 2, 1, 2, \ldots, 2) \) in [3]. We get evaluations of their star version \( t^* (2, \ldots, 2, 3, 2, \ldots, 2) \) and \( t^* (2, \ldots, 2, 1, 2, \ldots, 2) \) in Section 4.

## 2 Ohno-Zagier type relations

For nonnegative integers \( k, n, s \), let \( I_0(k, n, s) \) be the set of admissible indices of weight \( k \), depth \( n \) and height \( s \). Notice that \( I_0(k, n, s) \) is nonempty if and only if \( k \geq n + s \) and \( n \geq s \geq 1 \). Define the sums

\[
G_{0,N,a}(k, n, s) = \sum_{\mathbf{k} \in I_0(k, n, s)} t_{N,a}^{(k)},
\]

\[
G^*_{0,N,a}(k, n, s) = \sum_{\mathbf{k} \in I_0(k, n, s)} t^*_{N,a}^{(k)}.
\]

Then we obtain Ohno-Zagier type relations which represent the generating functions of \( G_{0,N,a}(k, n, s) \) and \( G^*_{0,N,a}(k, n, s) \) by the generalized hypergeometric function \( {}_3F_2 \).
Theorem 2.1. For formal variables \( u, v, w \),

\[
\sum_{k \geq n + s, n \geq s \geq 1} G_{0, N, a}(k, n, s) u^{k-n-s} v^{n-s} w^{s-1} = \frac{1}{a(a-u)} {}_3F_2 \left( \frac{a, \beta, 1}{N}, \frac{1}{a+N-u}; 1 \right),
\]

where \( \alpha, \beta \) are determined by \( \alpha + \beta = \frac{1}{N}(2a-u+v) \) and \( \alpha \beta = \frac{1}{N^2}(a(a-u+v)-uv+w) \).

Theorem 2.2. For formal variables \( u, v, w \),

\[
\sum_{k \geq n + s, n \geq s \geq 1} G^{*}_{0, N, a}(k, n, s) u^{k-n-s} v^{n-s} w^{s-1} = \frac{1}{a(a-u-v)+uv-w} {}_3F_2 \left( \frac{a, \alpha^*, \beta^*}{N}, \frac{a-n}{N}, 1; 1 \right),
\]

where \( \alpha^*, \beta^* \) are determined by \( \alpha^* + \beta^* = \frac{1}{N}(2a+2N-u-v) \) and \( \alpha^* \beta^* = \frac{1}{N^2}((a+N)(a+N-u-v)+uv-w) \).

From the above theorems and using some summation formulas of \( {}_3F_2 \), we obtain some corollaries in Subsection 2.1. For example, Theorem 2.1 and Theorem 2.2 deduce the Ohno-Zagier type relations for multiple zeta values and multiple zeta-star values respectively. Furthermore, we study the generating functions of sums of multiple \( t \)- (star) values of level \( N \) with height one and maximal height and obtain a weighted sum formula of sums of multiple \( t \)-(star) values of level \( N \) with fixed weight, fixed depth and \( N = 2a \). We will prove the above theorems in Subsection 2.2.

2.1 Applications

2.1.1 Level one and level two cases

Setting \( N = a = 1 \) or only \( a = N \) in Theorem 2.1 and Theorem 2.2, we can obtain the Ohno-Zagier type relations for the multiple zeta(-star) values.

Setting \( N = a = 1 \) in Theorem 2.1, we get \( \alpha + \beta = 2-u+v \) and \( \alpha \beta = 1-u+v-w+w \). Let \( \alpha_1 = u+\alpha-1 \) and \( \beta_1 = u+\beta-1 \), then \( \alpha_1 + \beta_1 = u+v \) and \( \alpha_1 \beta_1 = w \). Hence we have

\[
\frac{1}{1-u} {}_3F_2 \left( \frac{\alpha, \beta, 1}{2, 2-u}; 1 \right) = \frac{1}{uv-w} \left[ 1-\frac{1}{w} \sum_{n=1}^{\infty} \frac{\zeta(n)}{n} \left( \frac{\alpha_1}{1-u} \right)^n \right]
\]

Using the expansion

\[
\Gamma(1-z) = \exp \left( \gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} z^n \right),
\]

we obtain the following result, which is the Ohno-Zagier relation for the multiple zeta values.
Corollary 2.3 ([17, Theorem 1]). For formal variables $u, v, w$,

\[
\sum_{k \geq n+s, n \geq s \geq 1} \left( \sum_{k \in I_0(k,n,s)} \zeta(k) \right) u^{k-n-s} v^{n-s} w^{s-1} = \frac{1}{wv - w} \left\{ 1 - \exp \left( \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} (u^n + v^n - \alpha^n - \beta^n) \right) \right\},
\]

where $\alpha$ and $\beta$ are determined by $\alpha + \beta = u + v$ and $\alpha \beta = w$.

Similarly, setting $N = a = 1$ in Theorem 2.2, we get $\alpha^* + \beta^* = 4 - u - v$ and $\alpha^* \beta^* = 4 - 2(u + v) + uv - w$. Let $\alpha_1^* = 2 - \alpha^*$ and $\beta_1^* = 2 - \beta^*$, then $\alpha_1^* + \beta_1^* = u + v$ and $\alpha_1^* \beta_1^* = uv - w$. Using the summation formula [18, 7.4.4.1]

\[
\binom{3}{a, b, c ; d, e ; 1} = \frac{\Gamma(d) \Gamma(d + e - a - b - c) \Gamma(d + e - a - b - c)}{\Gamma(d - c) \Gamma(d + e - a - b) \Gamma(d + e - a - b - c)} \binom{3}{e - a, e - b, c ; d + e - a - b ; 1}
\]

with $a = c = 1, b = 1 - u, d = \alpha^*$ and $e = \beta^*$, we have

\[
\frac{1}{1 - u - v + uv - w} \binom{3}{1, 1 - u; 1} = \binom{3}{1 - \beta^* - 1, \beta^* - 1 + u, 1; 1}
\]

Then we obtain the following result, which is the Ohno-Zagier type relation for the multiple zeta-star values.

Corollary 2.4 ([1, Proposition 3.1]). For formal variables $u, v, w$,

\[
\sum_{k \geq n+s, n \geq s \geq 1} \left( \sum_{k \in I_0(k,n,s)} \zeta^*(k) \right) u^{k-n-s} v^{n-s} w^{s-1} = \frac{1}{(1 - \beta^*)(1 - v)} \binom{3}{1 - \beta^*, 1 - \beta^* + u, 1; 2 - \beta^*, 2 - v ; 1},
\]

where $\alpha^*$ and $\beta^*$ are determined by $\alpha^* + \beta^* = u + v$ and $\alpha^* \beta^* = uv - w$.

For the level two case, setting $N = 2$ and $a = 1$ in Theorem 2.1 and Theorem 2.2, we get the Ohno-Zagier relations for multiple $t$-(star) values [14, Theorems 1.1 and 1.2].

2.1.2 Sum of height one

To save space, denote by $\{k\}^n$ the sequence of $k$ repeated $n$ times.

Setting $w = 0$ in Theorem 2.1, we have $\alpha = \frac{a - u}{N}, \beta = \frac{a + u}{N}$. Therefore, we get the generating function of height one multiple $t$-values of level $N$. 

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Corollary 2.5. For formal variables $u, v$, we have
\[ \sum_{k \geq n+1, n \geq 1} t_{N,a}(k-n, 1)u^{k-n-1}v^{n-1} = \frac{1}{a(a-u)} 3F_2 \left( \frac{a-u}{a+N}, \frac{a+u}{a+N}; \frac{1}{N} \right). \]

Setting $w = 0$ in Theorem 2.2, we have $\alpha^* = \frac{a+N-u}{N}$, $\beta^* = \frac{a+N-v}{N}$. Hence, we get the generating function of height one multiple $t$-star values of level $N$.

Corollary 2.6. For formal variables $u, v$, we have
\[ \sum_{k \geq n+1, n \geq 1} t_{N,a}^*(k-n, 1)u^{k-n-1}v^{n-1} = \frac{1}{(a-u)(a-v)} 3F_2 \left( \frac{a}{a+N}, \frac{a-u}{a+N}; \frac{1}{N} \right). \]

Similar to [8, Lemma 5.2], we have the following lemma.

Lemma 2.7. For integers $a, N$ with $1 \leq a \leq N$ and a formal variable $x$, if
\[ \frac{x}{a-x} 3F_2 \left( p, q, \frac{a-x}{N}; \frac{1}{a+x-N} \right) = \sum_{j=1}^\infty t_j x^j, \]
then
\[ t_n = a^{-n} F_{n+1} \left( p, q, \left\{ \frac{a}{N} \right\}_n ; \frac{1}{a+N-x} \right). \]

Proof. By definition,
\[ \frac{x}{a-x} 3F_2 \left( p, q, \frac{a-x}{N}; \frac{1}{a+x-N} \right) = x \sum_{i=0}^\infty \frac{(p)_i(q)_i}{(s)_i} \frac{1}{i!} \frac{1}{a-Ni-x}. \] (2.2)

Since the $n$th derivative of the right-hand side of (2.2) with respect to $x$ is
\[ n! \sum_{i=0}^\infty \frac{(p)_i(q)_i}{(s)_i} \frac{1}{i!} \frac{1}{(a-Ni-x)^n} + n!x \sum_{i=0}^\infty \frac{(p)_i(q)_i}{(s)_i} \frac{1}{i!} \frac{1}{(a-Ni-x)^{n+1}}, \]
we have
\[ t_n = \frac{1}{n!} \frac{d^n}{dx^n} \bigg|_{x=0} \frac{x}{a-x} 3F_2 \left( p, q, \frac{a-x}{N}; \frac{1}{a+N-x-N} \right) = \sum_{i=0}^\infty \frac{(p)_i(q)_i}{(s)_i} \frac{1}{i!} \frac{1}{(a+N-x)^n}. \]

Therefore, the conclusion follows since $\left( \frac{a}{N} \right)_i / \left( \frac{a+N}{N} \right)_i = \frac{a}{a+N}$. \hfill \Box

Using Lemma 2.7, for any integer $m \geq 2$, we have
\[ \sum_{n=1}^\infty t_{N,a}(m, 1)u^{n-1} = a^{-m} F_{m+1} \left( 1, \frac{a+v}{N}, 1; \left\{ \frac{a}{N} \right\}^{m-1} ; \frac{1}{N} \right), \]
\[ \sum_{n=1}^\infty t_{N,a}^*(m, 1)u^{n-1} = \frac{1}{a^{m-1}(a-v)} F_{m+1} \left( 1, \frac{a+N-v}{N}, 1; \left\{ \frac{a+N}{N} \right\}^{m-1} ; 1 \right). \]
2.1.3 Sum of maximal height

We recall the following expansion formula of the gamma function with a proof for the convenience of the readers.

**Lemma 2.8.** For integers $a, N$ with $1 \leq a \leq N$, we have

$$
\Gamma \left( \frac{a - z}{N} \right) = \Gamma \left( \frac{a}{N} \right) \exp \left\{ -\frac{1}{N} \psi \left( \frac{a}{N} \right) z + \sum_{n=2}^{\infty} \frac{t_{N,a}(n)}{n} z^n \right\},
$$

(2.3)

where $\psi$ denotes the digamma function.

**Proof.** Since $\log \Gamma(z) = -\gamma z - \log z + \sum_{k=1}^{\infty} \left( \frac{1}{k} - \log \left( 1 + \frac{z}{k} \right) \right)$ and $\log \Gamma(z+1) = \log \Gamma(z) + \log z$, we have

$$
\frac{d}{dz} \log \Gamma(z + 1) = -\gamma + \sum_{k=1}^{\infty} \frac{z}{k(z + k)}
$$

and

$$
\frac{d^2}{dz^2} \log \Gamma(z + 1) = \sum_{k=1}^{\infty} \frac{1}{(z + k)^2}.
$$

Therefore, we obtain that

$$
\frac{d^2}{dz^2} \log \Gamma \left( \frac{a - N - z}{N} + 1 \right) = \frac{1}{N^2} \left. \frac{d^2}{d\omega^2} \log \Gamma(\omega + 1) \right|_{\omega = \frac{a - N - z}{N}} = \sum_{n=2}^{\infty} (n - 1) t_{N,a}(n) z^{n-2}.
$$

Hence, we have

$$
\Gamma \left( \frac{a - z}{N} \right) = \exp \left\{ c_2 + c_1 z + \sum_{n=2}^{\infty} \frac{t_{N,a}(n)}{n} z^n \right\},
$$

where $c_2 = \log \Gamma \left( \frac{a}{N} \right)$ and

$$
c_1 = \frac{\gamma}{N} + \frac{1}{N} \sum_{k=1}^{\infty} \frac{N - a}{k((k-1)N + a)}.
$$

Note that

$$
\frac{1}{N} \sum_{k=1}^{m} \frac{N - a}{k((k-1)N + a)} = \frac{1}{N} \sum_{k=1}^{m} \left[ \frac{1}{(k-1)N + a} - \frac{1}{kN} \right] = \frac{1}{N} \left[ \psi \left( m + \frac{a}{N} \right) - \psi \left( \frac{a}{N} \right) - H_m \right],
$$

where $H_m$ denotes the $m$-th harmonic number. Since $H_m = \psi(m + 1) + \gamma$, using the asymptotics of the digamma function, we find that

$$
\frac{1}{N} \sum_{k=1}^{\infty} \frac{N - a}{k((k-1)N + a)} = -\frac{\gamma}{N} - \frac{1}{N} \psi \left( \frac{a}{N} \right).
$$

Thus, we prove the result. \(\square\)
Corollary 2.9. For formal variables \( u, w \), we have

\[
1 + \sum_{k \geq 2n, n \geq 1} G_{0,N,a}(k, n, n) u^{k-2n} w^n = \exp \left\{ \sum_{n=2}^{\infty} \frac{t_{N,a}(n)}{n} (u^n - x^n - y^n) \right\},
\]

where \( x, y \) are determined by \( x + y = u \) and \( xy = w \).

Proof. Setting \( v = 0 \) in Theorem 2.1, we get \( \alpha + \beta = \frac{1}{N}(2a - u) \) and \( \alpha \beta = \frac{1}{N^2}(a(a - u) + w) \). Let \( x = a - Na \) and \( y = a - N\beta \), then \( x + y = u \) and \( xy = w \). Using the summation formula [18, 7.4.4.28]

\[
3F_2 \left( \begin{array}{c} a, b, 1 \\ c, 2 + a + b - c \end{array} : 1 \right) = \frac{1 + a + b - c}{(1 + a - c)(1 + b - c)} \left( 1 - c + \frac{\Gamma(c) \Gamma(1 + a + b - c)}{\Gamma(a) \Gamma(b)} \right) \tag{2.5}
\]

with \( a = \alpha, b = \beta \) and \( c = \frac{a + N}{N} \), we find that

\[
\frac{1}{a(a - u)^3} 3F_2 \left( \begin{array}{c} a + N, a + u - 1 \\ \frac{a + N}{N}, \frac{a + N - u}{N} : 1 \end{array} \right) = -\frac{1}{w} + \frac{1}{\Gamma \left( \frac{1}{N} \right)} \frac{\Gamma \left( \frac{a - u}{N} \right)}{\Gamma \left( \frac{a - y}{N} \right)} \tag{2.6}
\]

Then it is easy to finish the proof by Lemma 2.8.

Corollary 2.10. For formal variables \( u, w \), we have

\[
1 + \sum_{k \geq 2n, n \geq 1} G_{0,N,a}^{*}(k, n, n) u^{k-2n} w^n = \exp \left\{ \sum_{n=2}^{\infty} \frac{t_{N,a}^{*}(n)}{n} ((x^*)^n + (y^*)^n - u^n) \right\},
\]

where \( x^*, y^* \) are determined by \( x^* + y^* = u \) and \( x^* y^* = -w \).

Proof. Setting \( v = 0 \) in Theorem 2.2, we get \( \alpha^* + \beta^* = \frac{1}{N}(2a + 2N - u) \) and \( \alpha^* \beta^* = \frac{1}{N^2}((a + N)(a + N - u) - w) \). Let \( x^* = a + N - Na^* \) and \( y^* = a + N - N\beta^* \), then \( x^* + y^* = u \) and \( x^* y^* = -w \). Using (2.5) with \( a = \frac{a}{N}, b = \frac{a - u}{N} \) and \( c = \alpha^* \), we find that

\[
\frac{1}{a(a - u) - w^3} 3F_2 \left( \begin{array}{c} \frac{a}{N}, \frac{a - u}{N} \\ \alpha^*, \beta^* : 1 \end{array} \right) = -\frac{1}{w} + \frac{1}{\Gamma \left( \frac{1}{N} \right)} \frac{\Gamma \left( \frac{a - u}{N} \right)}{\Gamma \left( \frac{a - y}{N} \right)},
\]

which with Lemma 2.8 implies the result.

By (2.4) and (2.6), we find that

\[
\left( 1 + \sum_{k \geq 2n, n \geq 1} G_{0,N,a}(k, n, n) u^{k-2n} w^n \right) \\
\times \left( 1 + \sum_{k \geq 2n, n \geq 1} (-1)^n G_{0,N,a}^{*}(k, n, n) u^{k-2n} w^n \right) = 1.
\]

Furthermore, setting \( u = 0 \) in (2.4) and (2.6), we have

\[
1 + \sum_{n=1}^{\infty} t_{N,a}(\{2\}^n) w^n = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t_{N,a}(2n)}{n} w^n \right),
\]

\[
1 + \sum_{n=1}^{\infty} t_{N,a}^{*}(\{2\}^n) w^n = \exp \left( \sum_{n=1}^{\infty} \frac{t_{N,a}(2n)}{n} w^n \right).
\]

Using the symmetric sum formulas showed in Proposition 3.1, we will give more general formulas in Corollary 3.2.
2.1.4 A weighted sum formula

Let \( I_0(k, n) \) be the set of admissible indices of weight \( k \) and depth \( n \). Define

\[
G_{0,N,a}(k, n) = \sum_{k \in I_0(k, n)} t_{N,a}(k), \quad G^*_0(k, n) = \sum_{k \in I_0(k, n)} t^*_N(k).
\]

**Proposition 2.11.** For a formal variable \( u \), we have

\[
\sum_{k=2}^{\infty} \left( \sum_{n=1}^{k-1} 2^{n-1} G_{0,2a,a}(k, n) \right) u^{k-2} = \sum_{k=2}^{\infty} \left( \sum_{n=1}^{k-1} (-1)^{k-n-1} 2^{n-1} G^*_0(k, n) \right) u^{k-2}
\]

\[
= a^{-2}2^{2a} t(2) \exp \left( \sum_{n=2}^{\infty} \frac{4(2^{n-1} - 1)}{na^n(2^n - 1)} t(n) u^n \right). \tag{2.7}
\]

**Proof.** Let \( uv = w \) in Theorem 2.1, we have \( \alpha = \frac{a-u+v}{N}, \beta = \frac{a}{N} \). Hence, we get the generating function of sums of multiple \( t \)-values of level \( N \) with fixed weight and depth:

\[
\sum_{k=2}^{\infty} G_{0,N,a}(k, n) u^{k-n-1} v^{n-1} = \frac{1}{a(a - u)} 3F_2 \left( \frac{a}{N}, \frac{a}{2N}; \frac{a}{N}, 1; \frac{1}{2} \right).
\]

Setting \( v = 2u \) and \( N = 2a \), we have

\[
\sum_{k=2}^{\infty} 2^{n-1} G_{0,2a,a}(k, n) u^{k-2} = \frac{1}{a(a - u)} 3F_2 \left( \frac{a}{2a}, \frac{1}{2}; \frac{1}{2}, 1; 1 \right).
\]

Using the summation formula [18, 7.4.4.21]

\[
3F_2 \left( \frac{a}{1+a-b}, \frac{a}{1+a-c}, \frac{1}{1+a}; 1 \right) = \frac{\sqrt{\pi}}{2^a} \frac{\Gamma(1+a-b) \Gamma(1+a-c) \Gamma(1+a-b-c)}{\Gamma(1) \Gamma(1+a) \Gamma(1+a-b-c)}, \quad \Re(a - 2b - 2c) > -2,
\]

with \( a = 1, b = \frac{1}{2} \) and \( c = \frac{a+u}{2a} \), we get

\[
\frac{1}{a(a - u)} 3F_2 \left( \frac{a}{2a}, \frac{1}{2}; \frac{1}{2}, 1; 1 \right) = \frac{1}{a(a - u)} \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{2} - \frac{u}{2a}) \Gamma(\frac{1}{2}) \Gamma(1)}{\Gamma(\frac{3}{2} - \frac{u}{2a}) \Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})} = \frac{\pi}{8a^2} \frac{\Gamma(\frac{1}{2} - \frac{u}{2a})^2}{\Gamma(1 - \frac{u}{2a})^2}.
\]

Note that \( \zeta(n) = (1 - 2^{-n})^{-1} t(n) \) and \( t(2) = \frac{\pi^2}{8} \), by using (2.1) and the duplication formula

\[
\Gamma \left( \frac{1}{2} - \frac{z}{2} \right) = \sqrt{\pi} 2^z \Gamma(1 - z) \frac{\Gamma(1 - z)}{\Gamma(1 - \frac{z}{2})},
\]

we have

\[
\sum_{k=2}^{\infty} \left( \sum_{n=1}^{k-1} 2^{n-1} G_{0,2a,a}(k, n) \right) u^{k-2} = a^{-2}2^{2a} t(2) \exp \left( \sum_{n=2}^{\infty} \frac{4(2^{n-1} - 1)}{na^n(2^n - 1)} t(n) u^n \right).
\]
Similarly, setting $uv = w$ in Theorem 2.2, we have \( \alpha^* = \frac{a + N - u - v}{N}, \beta^* = \frac{a + N}{N} \). Hence

\[
\sum_{k > n \geq 1} G_{0,N,a}^*(k, n) u^{k-n-1} v^{n-1} = \frac{1}{a(a - u - v)} {}_3F_2\left( \frac{\alpha}{a + N}, \frac{a - u}{a + N - u - v}; \frac{1}{a + N}, 1 \right).
\]

Setting \( v = -2u \) and \( N = 2a \), then

\[
\sum_{k > n \geq 1} (-2)^{n-1} G_{0,2a,a}^*(k, n) u^k = \frac{1}{a(a + u)} {}_3F_2\left( \frac{\frac{1}{2}, \frac{a - u}{2a}, 1}{\frac{3}{2}, \frac{3a + u}{2a}}; 1 \right).
\]

Therefore we obtain that

\[
\sum_{k=2}^{\infty} \left( \sum_{n=1}^{k-1} (-2)^{n-1} G_{0,2a,a}^*(k, n) \right) u^{k-2} = a^{-2} 2^{-\frac{2a}{a}} t(2) \exp \left( \sum_{n=2}^{\infty} \frac{(n-1)(2^{n-1} - 1)}{n a^n (2^n - 1)} t(n) u^n \right) t(2) t(n_1) \cdot \cdot \cdot t(n_m) \log^n(2).
\]

Then the result follows easily.

Expanding the right-hand side of (2.7), we get the following weighted sum formula.

**Corollary 2.12.** For any integer \( k \geq 2 \), we have

\[
\sum_{n=1}^{k-1} 2^{n-1} G_{0,2a,a}(k, n) = \sum_{n=1}^{k-1} (-1)^{k-n-1} 2^{n-1} G_{0,2a,a}^*(k, n)
\]

\[
= \sum_{n=1}^{k-1} \frac{2^{n+2} m_1 \cdots m_m (2^{n_1} - 1) \cdots (2^{n_m} - 1)}{a^k n! m! n_1 \cdot \cdot \cdot n_m (2^{n_1} - 1) \cdots (2^{n_m} - 1)} t(2) t(n_1) \cdot \cdot \cdot t(n_m) \log^n(2).
\]

### 2.2 Proofs of Theorems 2.1 and 2.2

For an index \( \mathbf{k} = (k_1, \ldots, k_n) \), any \( a \in \{1, 2, \ldots, N\} \) and any nonnegative integer \( b \), define

\[
\mathcal{L}_{N,a,b}(\mathbf{k}; z) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_q + (n-1)b \geq a + (n-1)b \mod N} \frac{z^m}{m_1^{k_1} \cdots m_q^{k_q}}.
\]

Then \( \mathcal{L}_{N,a,b}(\mathbf{k}; z) \) converges absolutely for \( |z| < 1 \). And we have the following iterated integral representation

\[
\mathcal{L}_{N,a,b}(\mathbf{k}; z) = \int_0^z \left( \frac{dt}{t} \right)^{k_1-1} t^{b-1} \cdots \left( \frac{dt}{t} \right)^{k_{n-1}-1} t^{b-1} \left( \frac{dt}{t} \right)^{k_{n-1}-1} t^{b-1} \cdots \left( \frac{dt}{t} \right)^{k_{n-1}-1} t^{a-1} dt.
\]

For an admissible index \( \mathbf{k} \), from the integral representations (1.1) and (1.2), one can easily get that

\[
\mathcal{L}_{N,a,N}(\mathbf{k}; 1) = t_{N,a}(\mathbf{k}), \quad \mathcal{L}_{N,a,0}(\mathbf{k}; 1) = t_{N,a}^*(\mathbf{k}).
\]
By derivation, it is easy to find that

\[
\frac{d}{dz} \mathcal{L}_{N,a,b}(k; z) = \begin{cases} \frac{1}{z} \mathcal{L}_{N,a,b}(k_1 - 1, k_2, \ldots, k_n; z) & \text{if } k_1 > 1, \\ \frac{z^{b-1}}{1-z^N} \mathcal{L}_{N,a,b}(k_2, \ldots, k_n; z) & \text{if } n \geq 2, k_1 > 1, \\ \frac{z^{a-1}}{1-z^N} & \text{if } n = k_1 = 1. \end{cases} \tag{2.9}
\]

For nonnegative integers \( k, n, s \), we denote by \( I(k, n, s) \) the set of indices of weight \( k \), depth \( n \) and height \( s \), and define the sums

\[
G_{N,a,b}(k, n, s; z) = \sum_{k \in I(k, n, s)} \mathcal{L}_{N,a,b}(k; z), \quad G_{0,N,a,b}(k, n, s; z) = \sum_{k \in I_0(k, n, s)} \mathcal{L}_{N,a,b}(k; z).
\]

If the indices set is empty, the sum is treated as zero. And we also set \( G_{N,a,b}(0, 0, 0; z) = 1 \). For integers \( k, n, s \), using (2.9), we have

1. if \( k \geq n + s \) and \( n \geq s \geq 1 \),

\[
\frac{d}{dz} G_{0,N,a,b}(k, n, s; z) = \frac{1}{z} [G_{N,a,b}(k - 1, n, s - 1; z) + G_{0,N,a,b}(k - 1, n, s; z) - G_{0,N,a,b}(k - 1, n, s - 1; z)],
\tag{2.10}
\]

2. if \( k \geq n + s \), \( n \geq s \geq 0 \) and \( n \geq 2 \),

\[
\frac{d}{dz} [G_{N,a,b}(k, n, s; z) - G_{0,N,a,b}(k, n, s; z)] = z^{b-1} \frac{1}{1-z^N} G_{N,a,b}(k - 1, n - 1, s; z).
\tag{2.11}
\]

Now we define the generating functions

\[
\Phi_{N,a,b}(z) = \Phi_{N,a,b}(u, v, w; z) = \sum_{k, n, s \geq 0} G_{N,a,b}(k, n, s; z) u^{k-n-s} v^n w^s, \\
\Phi_{0,N,a,b}(z) = \Phi_{0,N,a,b}(u, v, w; z) = \sum_{k, n, s \geq 0} G_{0,N,a,b}(k, n, s; z) u^{k-n-s} v^n w^{s-1} = \sum_{\substack{k \geq n+s \atop n \geq s \geq 1}} G_{0,N,a,b}(k, n, s; z) u^{k-n-s} v^n w^{s-1}.
\]

Using (2.9), (2.10) and (2.11), we get that

\[
\frac{d}{dz} \Phi_{0,N,a,b}(z) = \frac{1}{v z} (\Phi_{N,a,b}(z) - 1 - w \Phi_{0,N,a,b}(z)) + \frac{u}{z} \Phi_{0,N,a,b}(z), \\
\frac{d}{dz} (\Phi_{N,a,b}(z) - w \Phi_{0,N,a,b}(z)) = \frac{v z^{b-1}}{1-z^N} (\Phi_{N,a,b}(z) - 1) + \frac{v z^{a-1}}{1-z^N}.
\]

Eliminating \( \Phi_{N,a,b}(z) \), we obtain the differential equation satisfied by \( \Phi_{0,N,a,b}(z) \).

**Proposition 2.13.** \( \Phi_{0,N,a,b} = \Phi_{0,N,a,b}(z) \) satisfies the following differential equation

\[
z(1-z^N) \Phi_{0,N,a,b}' + [(1-u)(1-z^N) - vz^b] \Phi_{0,N,a,b} + (uv - w) z^{b-1} \Phi_{0,N,a,b} = z^{a-1}.
\tag{2.12}
\]

We prove Theorem 2.1 and Theorem 2.2 by solving the differential equation (2.12) in the special cases \( b = N \) and \( b = 0 \).
2.2.1 A proof of Theorem 2.1

Let $b = N$ in (2.8). Set

$$G_{0,N,a}(k, n, s; z) = G_{0,N,a,N}(k, n, s; z), \quad \Phi_{0,N,a}(z) = \Phi_{0,N,a,N}(z).$$

Using (2.12), we get the following differential equation

$$z(1 - z^N)\Phi_{0,N,a}'' + [(1 - u)(1 - z^N) - vz^N]\Phi_{0,N,a}' + (uv - w)z^{N-1}\Phi_{0,N,a} = z^{a-1}. \quad (2.13)$$

We want to find the unique power series solution $\Phi_{0,N,a}(z) = \sum_{n=1}^{\infty} p_n z^n$. By (2.13), we see that $p_1, \ldots, p_{a-1}, p_{a+1}, \ldots, p_N = 0$, $p_a = \frac{1}{a(a-u)}$ and

$$p_{n+N} = \frac{n(n-1) + n(1-u+v) - (uv-w)}{(n+N)(n+N-u)} p_n, \quad n \geq 1.$$  

Hence for any $m \geq 1$ with $m \equiv a \pmod{N}$, we have $p_m = 0$ and for any $n \geq 1$,

$$p_{a+nN} = \frac{(\alpha + n - 1)(\beta + n - 1)}{\left(\frac{a}{N} + n\right)\left(\frac{a-n}{N} + n\right)} p_{a+(n-1)N} = \frac{(\alpha)_{n}(\beta)_{n}}{\left(\frac{a}{N} + 1\right)_{n}\left(\frac{a-n}{N} + 1\right)_{n} a(a-u)}. $$

Therefore we have the following theorem.

**Theorem 2.14.** For formal variables $u, v, w$, we have

$$\sum_{k \geq n+s, n \geq s \geq 1} G_{0,N,a}(k, n, s; z) u^{k-n-s} v^{n-s} w^{s-1} = \frac{z^a}{a(a-u)} \left[ F_2\left( \frac{\alpha, \beta, 1}{a+N} ; \frac{a-n}{N} ; \frac{a-u}{N} ; z^N \right) \right],$$

where $\alpha, \beta$ are determined by $\alpha + \beta = \frac{1}{N}(2a-u+v)$ and $\alpha \beta = \frac{1}{N^2}(a(a-u+v)-uv+w)$.

Setting $z = 1$ in Theorem 2.14, we get Theorem 2.1.

2.2.2 A proof of Theorem 2.2

Let $b = 0$ in (2.8). Set

$$G_{0,N,a}^*(k, n, s; z) = G_{0,N,a,0}(k, n, s; z), \quad \Phi_{0,N,a}^*(z) = \Phi_{0,N,a,0}(z).$$

Using (2.12), we get the following differential equation

$$z^2(1 - z^N)(\Phi_{0,N,a}^*)'' + z[(1 - u)(1 - z^N) - v](\Phi_{0,N,a}^*)' + (uv - w)\Phi_{0,N,a}^* = z^{a}. \quad (2.14)$$

Assume that $\Phi_{0,N,a}^*(z) = \sum_{n=1}^{\infty} q_n z^n$. By (2.14), we see that $q_1, \ldots, q_{a-1}, q_{a+1}, \ldots, q_N = 0$, $q_a = \frac{1}{a(a-u-v)+uv-w}$ and

$$q_{n+N} = \frac{n(n-u)}{(n+N)(n+N-u-v)+uv-w} q_n, \quad n \geq 1.$$  

Hence for any $m \geq 1$ with $m \equiv a \pmod{N}$, we have $q_m = 0$, and for any $n \geq 1$

$$q_{a+nN} = \frac{(n + \frac{a}{N} - 1)(n + \frac{a-u}{N} - 1)}{(\alpha^* + n - 1)(\beta^* + n - 1)} q_{a+(n-1)N} = \frac{(\frac{a}{N})_{n}(\frac{a-u}{N})_{n}}{(\alpha^*)_{n}(\beta^*)_{n} a(a-u-v)+uv-w}. $$

Therefore we have the following theorem.
Theorem 2.15. For formal variables \( u, v, w \), we have
\[
\sum_{k \geq n+s, n \geq s \geq 1} C_{0,N,a}^*(k, n, s; z) u^{k-n-s} v^n w^s = \frac{z^a}{a(a - u - v) + uv - w} 3F_2 \left( \frac{a}{N}, \frac{a-u}{N}, 1; z^N \right),
\]
where \( \alpha^*, \beta^* \) are determined by
\[
\alpha^* + \beta^* = \frac{1}{N}(2a + 2N - u - v) \quad \text{and} \quad \alpha^* \beta^* = \frac{1}{N^2}(a + N)(a + N - u - v) + uv - w).
\]

Setting \( z = 1 \) in Theorem 2.15, we get Theorem 2.2.

3 Symmetric sum formulas and Hoffman’s restricted sum formulas

Recall the construction of stuffle algebra from M. E. Hoffman [7]. See also [11]. Let \( N \) be a fixed positive integer and \( a \in \{1, 2, \ldots, N\} \). Define \( \mathfrak{A}_{N,a} = \mathbb{Q} \langle x, y_a \rangle \) be the non-commutative polynomial algebra generated by the alphabet \( \{x, y_a\} \) over the rational field \( \mathbb{Q} \). Define two subalgebras \( \mathfrak{A}^1_{N,a} = \mathbb{Q} + \mathfrak{A}_{N,a}y a \) and \( \mathfrak{A}^0_{N,a} = \mathbb{Q} + x \mathfrak{A}_{N,a}y a \). For a positive integer \( k \), set \( z_{k,a} = x^{k-1} y_a \). Define two \( \mathbb{Q} \)-linear maps \( t_{N,a} : \mathfrak{A}^0_{N,a} \rightarrow \mathbb{R} \) and \( t^*_{N,a} : \mathfrak{A}^1_{N,a} \rightarrow \mathbb{R} \) by \( t_{N,a}(1_w) = t^*_{N,a}(1_w) = 1 \) and
\[
t_{N,a}(z_{k_1,a} \cdots z_{k_n,a}) = t_{N,a}(k_1, \ldots, k_n), \quad t^*_{N,a}(z_{k_1,a} \cdots z_{k_n,a}) = t^*_{N,a}(k_1, \ldots, k_n),
\]
where \( 1_w \) is the empty word and \( k_1, \ldots, k_n \) are positive integers with \( k_1 > 1 \). Let \( \gamma_{N,a} \) be the algebra automorphism on \( \mathfrak{A}_{N,a} \) characterized by \( \gamma_{N,a}(x) = x \) and \( \gamma_{N,a}(y_a) = x + y_a \), and define the \( \mathbb{Q} \)-linear transformation \( S_{N,a} : \mathfrak{A}^1_{N,a} \rightarrow \mathfrak{A}^1_{N,a} \) by
\[
S_{N,a}(1_w) = 1_w \quad \text{and} \quad S_{N,a}(w y_a) = \gamma_{N,a}(w)y_a
\]
for any word \( w \in \mathfrak{A}_{N,a} \). Then by the integral representations (1.1) and (1.2), we have \( t^*_{N,a} = t_{N,a} \circ S_{N,a} \).

Define two commutative products \( * \) and \( \overline{\ast} \) called the stuffle product and the star-stuffe product respectively on \( \mathfrak{A}^1_{N,a} \) by \( \mathbb{Q} \)-bilinearity and the rules:
\[
1_w * w = w * 1_w = w, \quad z_{k,a}w_1 * z_{l,a}w_2 = z_{k,a}(w_1 * z_{l,a}w_2) + z_{l,a}(z_{k,a}w_1 * w_2) + z_{k+l,a}(w_1 * w_2),
\]
\[
1_w \overline{\ast} w = w \overline{\ast} 1_w = w, \quad z_{k,a}w_1 \overline{\ast} z_{l,a}w_2 = z_{k,a}(w_1 \overline{\ast} z_{l,a}w_2) + z_{l,a}(z_{k,a}w_1 \overline{\ast} w_2) - z_{k+l,a}(w_1 \overline{\ast} w_2),
\]
where \( k, l \) are positive integers and \( w, w_1, w_2 \) are words in \( \mathfrak{A}^1_{N,a} \). Then we get commutative algebras \( \mathfrak{A}^1_{N,a,*} \) and \( \mathfrak{A}^1_{N,a,\overline{\ast}} \), their subalgebras \( \mathfrak{A}^0_{N,a,*} \) and \( \mathfrak{A}^0_{N,a,\overline{\ast}} \). It is easy to see that the maps \( t_{N,a} : \mathfrak{A}^0_{N,a,*} \rightarrow \mathbb{R} \) and \( t^*_{N,a} : \mathfrak{A}^0_{N,a,\overline{\ast}} \rightarrow \mathbb{R} \) are algebra homomorphisms.

Denote by \( \mathcal{P}_n \) the set of all partitions of the set \( \{1, 2, \ldots, n\} \). For a partition \( \Pi = \{P_1, P_2, \ldots, P_l\} \in \mathcal{P}_n \), let
\[
c(\Pi) = \prod_{j=1}^{l}(|P_j| - 1)!, \quad \overline{c}(\Pi) = (-1)^{n-l} c(\Pi),
\]
13
and for any maximal height index \( k = (k_1, \ldots, k_n) \), define

\[
t_{N,a}(k, \Pi) = \prod_{j=1}^{l} t_{N,a} \left( \sum_{i \in \mathcal{P}_j} k_i \right), \quad t^*_{N,a}(k, \Pi) = \prod_{j=1}^{l} t^*_{N,a} \left( \sum_{i \in \mathcal{P}_j} k_i \right).
\]

Then applying the maps \( t_{N,a} \) and \( t^*_{N,a} \) to [12, Lemma 5.1], we get the symmetric sum formulas for multiple \( t \)-\((\text{star})\) values of multiple level \( N \), which generalizes the results for multiple \( \zeta \)-(\(\text{star}\)) values [6, Theorems 2.1 and 2.2] and multiple \( t \)-(\(\text{star}\)) values [8, Theorems 3.2 and 3.5].

**Proposition 3.1.** For an index \( k = (k_1, \ldots, k_n) \) with \( k_1, \ldots, k_n > 1 \), we have

\[
\sum_{\sigma \in S_n} t_{N,a}(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = \sum_{\Pi \in \mathcal{P}_n} \tilde{c}(\Pi) t_{N,a}(k, \Pi),
\]

\[
\sum_{\sigma \in S_n} t^*_{N,a}(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = \sum_{\Pi \in \mathcal{P}_n} c(\Pi) t^*_{N,a}(k, \Pi),
\]

where \( S_n \) is the symmetric group of degree \( n \).

Taking \( k_1 = \cdots = k_n \) in this result, we get a generating function for multiple \( t \)-(\(\text{star}\)) values of level \( N \) with repeated arguments.

**Corollary 3.2.** For any integer \( k > 1 \), we have

\[
1 + \sum_{n=1}^{\infty} t_{N,a}(\{k\}^n)x^{kn} = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t_{N,a}(kn)x^{kn}}{n} \right), \quad (3.1)
\]

\[
1 + \sum_{n=1}^{\infty} t^*_{N,a}(\{k\}^n)x^{kn} = \exp \left( \sum_{n=1}^{\infty} \frac{t_{N,a}(kn)x^{kn}}{n} \right). \quad (3.2)
\]

**Proof.** We prove (3.2) and one can prove (3.1) similarly. Using Proposition 3.1 with \( k_1 = \cdots = k_n = k \), we get

\[
t^*_{N,a}(\{k\}^n) = \frac{1}{n!} \sum_{l=1}^{n} \sum_{\{P_1, \ldots, P_l\} \in \mathcal{P}_n} \prod_{s=1}^{l} (p_s - 1)! t^*_{N,a}(kp_s),
\]

where \( p_s = |P_s| \). For fixed \( p_1, \ldots, p_l \), we have

\[
|\{P_1, \ldots, P_l\} \in \mathcal{P}_n \mid p_s = |P_s|, s = 1, \ldots, l| = \frac{1}{m_1!m_2! \cdots m_n!p_1!p_2! \cdots p_l!},
\]

where \( m_i = |\{s \mid p_s = i\}| \). Then we obtain

\[
t^*_{N,a}(\{k\}^n)
= \sum_{l=1}^{n} \sum_{\{P_1, \ldots, P_l\} \in \mathcal{P}_n} \frac{1}{m_1! \cdots m_n!p_1! \cdots p_l} t^*_{N,a}(kp_1) \cdots t^*_{N,a}(kp_l)
\]

\[
= \sum_{m_1+2m_2+\cdots+n=\infty} \frac{1}{m_1!m_2! \cdots m_n!} (t_{N,a}(k))^{m_1} \left( \frac{t^*_{N,a}(2k)}{2} \right)^{m_2} \cdots \left( \frac{t^*_{N,a}(nk)}{n} \right)^{m_n}.
\]
Hence we easily get the desired generating function.

Setting \( N = a = 1 \) in Corollary 3.2, we get the generating function for multiple zeta-(star) values with repeated arguments [9, Proposition 3]. And setting \( N = 2 \) and \( a = 1 \) in Corollary 3.2, we obtain the generating function for multiple \( t \)-(star) values with repeated arguments. Note that using \( t(k) = (1 - 2^{-k})\zeta(k) \), M. E. Hoffman [8, Theorems 3.4 and 3.6] in fact showed that

\[
1 + \sum_{n=1}^\infty t\left(\{k\}^n\right)x^{kn} = \frac{Z_k(x)}{Z_k\left(\frac{x}{2}\right)},
\]

\[
1 + \sum_{n=1}^\infty t^*(\{k\}^n)x^{kn} = \frac{Z_k\left(e^{\frac{\pi i}{2}}x\right)}{Z_k\left(e^{\frac{\pi i}{2}}x\right)},
\]  

(3.3)

where \( Z_k(x) = 1 + \sum_{n=1}^\infty \zeta(\{k\}^n)x^{kn} \). We need the following generating function for \( t^*(\{2\}^n) \) which is important for the evaluations in Section 4.

**Proposition 3.3.** We have

\[
1 + \sum_{n=1}^\infty t^*(\{2\}^n)x^{2n} = \frac{1}{\cos\left(\frac{\pi x}{2}\right)}.
\]  

(3.4)

**Proof.** Using (3.3) and [2, Eq. (34)]

\[
Z_{2\ell}(x) = \frac{1}{(i\pi x)^\ell} \prod_{j=1}^{\ell} \sin \left( e^{\frac{(2j-1)i\pi}{2\ell}} \pi x \right),
\]

we easily get the result. \(\square\)

**Remark 3.4.** Since \( \sum_{a=1}^N t_{N,a}(k) = \zeta(k) \), for any integer \( k > 1 \), we have

\[
\prod_{a=1}^N \left( 1 + \sum_{n=1}^\infty t_{N,a}(\{k\}^n)x^{kn} \right) = Z_k(x).
\]

Applying the maps \( t_{N,a} \) and \( t^*_{N,a} \) to [13, Proposition 3.27], we get Hoffman’s restricted sum formula for multiple \( t \)-(star) values of level \( N \), which generalizes the results for multiple zeta values [4, Theorem A], multiple zeta-star values [5, Theorem 1.1] and multiple \( t \)-values [19, Theorem 1].

**Proposition 3.5.** For any integer \( m > 1 \), we have

\[
\sum_{k_1 + \cdots + k_n = m \atop k_i \geq 1} t_{N,a}(mk_1, \ldots, mk_n) = \sum_{j=0}^{k-n} (-1)^{k-n-j} \binom{k-j}{n} t^*_{N,a}(\{m\}^j)t_{N,a}(\{m\}^{k-j}),
\]

\[
\sum_{k_1 + \cdots + k_n = m \atop k_i \geq 1} t^*_{N,a}(mk_1, \ldots, mk_n) = \sum_{j=0}^{k-n} (-1)^j \binom{k-j}{n} t_{N,a}(\{m\}^j)t^*_{N,a}(\{m\}^{k-j}).
\]
4 Evaluations with one-two-three indices

In this section, we consider multiple $t$-(star) values of level 2. For simplicity, we denote $\mathfrak{A}^1_{1,1}$ and $S_{2,1}$ defined in Section 3 by $\mathfrak{A}^1$ and $S$, respectively. And for a positive integer $k$, set $z_k = z_{k,1} \in \mathfrak{A}^1$. Recall that for an index $k$, there is a stuffle regularization $t^\star_V(k)$ of multiple $t$-values with regularization parameter $t^\star_V(1) = V$ in [3]. For a word $w = z_{k_1} \cdots z_{k_n} \in \mathfrak{A}^1$, define $t^\star_V(w) = t^\star_V(k_1, \ldots, k_n)$. Then the stuffle regularization map $t^\star_V$ can be extended to $\mathfrak{A}^1$ by $\mathbb{Q}$-linearities. The star-stuffle regularization map $t^\star_V$ on $\mathfrak{A}^1$ is defined by

$$t^\star_V(w) = t^V(S(w)), \text{ for any } w \in \mathfrak{A}^1.$$ 

Similarly, for an index $k = (k_1, \ldots, k_n)$, set $t^\star_V(k) = t^\star_V(z_{k_1} \cdots z_{k_n})$. Then we have $t^\star_V(1) = t^\star_V(y_1) = V$.

In this section, we get two evaluations for the multiple $t$-star values $t^\star(\{2\}^a_1, 3, \{2\}^b_2)$ and $t^\star_V(\{2\}^a_1, 1, \{2\}^b_3)$, which are analogous to the evaluations of the multiple $t$-values established in [16] and [3] respectively.

**Theorem 4.1.** For any integers $a, b \geq 0$, we have

$$t^\star(\{2\}^a_1, 3, \{2\}^b_2) = \sum_{\substack{r=1 \atop a+b+1}} 2^{-2r} \left[ (1 - 2^{-2r}) \left( \frac{2r}{2a+1} \right) + \left( \frac{2r}{2b+1} \right) \right] t^\star(\{2\}^{a+b+1-r}_1) \zeta(2r+1). \tag{4.1}$$

**Theorem 4.2.** For any integers $a, b \geq 0$, we have

$$t^\star_V(\{2\}^a_1, 1, \{2\}^b_3) = \sum_{\substack{r=1 \atop a+b}} 2^{-2r} \left[ \left( \frac{2r}{2a} \right) + (1 - 2^{-2r}) \left( \frac{2r}{2b} \right) \right] t^\star(\{2\}^{a+b-r}_1) \zeta(2r+1)$$

$$+ \delta_{a=0}(V - \log(2)) t^\star(\{2\}^b_3) + \delta_{b=0} \log(2) t^\star(\{2\}^a_1). \tag{4.2}$$

where $\delta$ is the Kronecker symbol.

We will prove Theorem 4.1 and Theorem 4.2 by using the generating functions. Furthermore, we will show the algebraic equivalences between the above evaluation formulas and the corresponding evaluations for multiple $t$-values.

4.1 Evaluation of $t^\star(\{2\}^a_1, 3, \{2\}^b_2)$

To evaluate $t^\star(\{2\}^a_1, 3, \{2\}^b_2)$, we need Murakami’s generating function for $t(\{2\}^a_1, 3, \{2\}^b_2)$ in [16]. For an admissible index $(k_1, \ldots, k_n)$, set $t(k_1, \ldots, k_n) = 2^{k_1+\cdots+k_n} t(k_1, \ldots, k_n)$. Let $K(a, b) = t(\{2\}^a_1, 3, \{2\}^b_2)$ and $G(x, y)$ be the generating function

$$G(x, y) = \sum_{a,b \geq 0} (-1)^{a+b} K(a, b) x^{2a+1} y^{2b+1}.$$

By the definition of $K(a, b)$, $G(x, y)$ can be written as

$$G(x, y) = xy \sum_{m=1}^{\infty} \prod_{k=m+1}^{\infty} \left( 1 - \frac{x^2}{(k - \frac{1}{2})^2} \right) \frac{1}{(m - \frac{1}{2})^3} \prod_{l=1}^{m-1} \left( 1 - \frac{y^2}{(l - \frac{1}{2})^2} \right). \tag{4.3}$$
From \cite[Proposition 13]{16}, \(G(x, y)\) is displayed as
\[
G(x, y) = \cos \pi x [A(x + y) - A(x - y)] + \cos \pi y [B(x + y) - B(x - y)], \tag{4.4}
\]
where \(A(z)\) and \(B(z)\) are the power series
\[
A(z) = \sum_{r=1}^{\infty} \zeta(2r + 1) z^{2r}, \quad B(z) = \sum_{r=1}^{\infty} (1 - 2^{-2r}) \zeta(2r + 1) z^{2r}.
\]

**Proof of Theorem 4.1.** Let \(K^*(a, b) = \tilde{t}^*(\{2\}^a, 3, \{2\}^b)\) and \(G^*(x, y)\) be the generating function
\[
G^*(x, y) = \sum_{a, b \geq 0} K^*(a, b) x^{2a} y^{2b},
\]
where \(\tilde{t}^*(k_1, \ldots, k_n) = 2^{k_1+\cdots+k_n} t^*(k_1, \ldots, k_n)\). By the definitions of \(K^*(a, b)\), we get
\[
G^*(x, y) = \sum_{m=1}^{\infty} \prod_{k=m}^{\infty} \left(1 - \frac{x^2}{(k - \frac{1}{2})^2}\right)^{-1} \prod_{l=1}^{\infty} \left(1 - \frac{y^2}{(l - \frac{1}{2})^2}\right)^{-1}.
\]

Using (4.3) together with (4.4), we have
\[
G^*(x, y) = \frac{G(y, x)}{xy \cos \pi x \cos \pi y}
= \frac{A(x + y) - A(x - y)}{xy \cos \pi x} + \frac{B(x + y) - B(x - y)}{xy \cos \pi y}.
\]
By (3.4) and the identity
\[
2 \sum_{n=0}^{r-1} \binom{2r}{2n+1} x^{2r-2n-1} y^{2n+1} = (x + y)^{2r} - (x - y)^{2r}, \tag{4.5}
\]
we find that
\[
G^*(x, y) = 2 \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \sum_{k=0}^{r-1} \binom{2r}{2k+1} \tilde{t}^*(\{2\}^n) \zeta(2r + 1) x^{2n+2r-2k-2} y^{2k}
+ 2 \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \sum_{l=0}^{r-1} \sum_{k=0}^{r-1} (1 - 2^{-2r}) \binom{2r}{2l+1} \tilde{t}^*(\{2\}^n) \zeta(2r + 1) x^{2r-2l-2} y^{2n+2l}.
\]
Hence we get the desired result by comparing the coefficient of \(x^{2a} y^{2b}\) on both sides. \(\square\)

The evaluation of \(t(\{2\}^a, 3, \{2\}^b)\) in \cite[Theorem 3]{16} shows that
\[
t(\{2\}^a, 3, \{2\}^b)
= \sum_{r=1}^{a+b+1} (-1)^{r-1} 2^{-2r} \left(1 - 2^{-2r}\right) \binom{2r}{2a+1} + \binom{2r}{2b+1} \right) t(\{2\}^{a+b+1-r}) \zeta(2r + 1). \tag{4.6}
\]
We prove the equivalence of (4.1) and (4.6).
Proposition 4.3. (4.1) holds for any integers \(a, b \geq 0\) if and only if (4.6) holds for any integers \(a, b \geq 0\).

Proof. We give an algebraic proof here. Using [11, Eq. (2.4)], we get that

\[
T^*(u, v) = T(v, u) \ast S^*(u) \ast S(v),
\]

(4.7)

where

\[
T(u, v) = \sum_{a,b=0}^\infty (-1)^{a+b}z_2^az_3^bz_2^a v^{2b}, \quad T^*(u, v) = \sum_{a,b=0}^\infty S(z_2^az_3^b)u^{2a}v^{2b}
\]

and

\[
S^*(u) = \sum_{n=0}^\infty S(z_2^n)u^{2n}.
\]

On the other hand, let

\[
\hat{T}(u, v) = \sum_{a,b=0}^\infty (-1)^{a+b}\hat{H}(a, b)u^{2a}v^{2b}, \quad \hat{T}^*(u, v) = \sum_{a,b=0}^\infty \hat{H}^*(a, b)u^{2a}v^{2b},
\]

where

\[
\hat{H}(a, b) = \sum_{r=1}^{a+b+1} (-1)^{r-1}2^{-2r} \left[ (1 - 2^{-2r}) \left( \frac{2r}{2a+1} \right) + \left( \frac{2r}{2b+1} \right) \right] z_{2r+1} \ast z_2^{a+b+1-r},
\]

\[
\hat{H}^*(a, b) = \sum_{r=1}^{a+b+1} 2^{-2r} \left[ (1 - 2^{-2r}) \left( \frac{2r}{2a+1} \right) + \left( \frac{2r}{2b+1} \right) \right] z_{2r+1} \ast S(z_2^{a+b+1-r}).
\]

Set

\[
A(u) = \sum_{r=1}^\infty z_{2r+1}u^{2r}, \quad B(u) = \sum_{r=1}^\infty (1 - 2^{-2r})z_{2r+1}u^{2r}
\]

and

\[
S(u) = \sum_{n=0}^\infty (-1)^nz_2^nu^{2n}.
\]

Then we find that

\[
\hat{T}(u, v) = \frac{S(u)}{2uv} \ast \left[ A \left( \frac{u+v}{2} \right) - A \left( \frac{u-v}{2} \right) \right] + \frac{S(v)}{2uv} \ast \left[ B \left( \frac{u+v}{2} \right) - B \left( \frac{u-v}{2} \right) \right], \quad (4.8)
\]

\[
\hat{T}^*(v, u) = \frac{S^*(u)}{2uv} \ast \left[ A \left( \frac{u+v}{2} \right) - A \left( \frac{u-v}{2} \right) \right] + \frac{S^*(v)}{2uv} \ast \left[ B \left( \frac{u+v}{2} \right) - B \left( \frac{u-v}{2} \right) \right]. \quad (4.9)
\]
We give the proof of (4.8) and one can prove (4.9) similarly. In fact, we have

\[ \hat{T}(u, v) = \sum_{a, b=0}^{\infty} \sum_{r=1}^{\infty} (-1)^{a+b+r-1} 2^{-2r} \left[ (1 - 2^{-2r}) \left( \frac{2r}{2a+1} \right) + \left( \frac{2r}{2b+1} \right) \right] \times z_{2r+1} \ast z_{2}^{a+b+1-r} u^{2a} v^{2b} \]

\[ = \sum_{r \geq 1, s \geq 0} (-1)^{s} 2^{-2r} (1 - 2^{-2r}) \left( \frac{2r}{2a+1} \right) z_{2r+1} \ast z_{2}^{s} u^{2a} v^{2s+2r-2b-2} \]

\[ + \sum_{r \geq 1, s \geq 0} (-1)^{s} 2^{-2r} \left( \frac{2r}{2b+1} \right) z_{2r+1} \ast z_{2}^{s} u^{2s+2r-2a-2} v^{2b} \]

\[ = \sum_{s=0}^{\infty} (-1)^{s} z_{2}^{s} u^{2s} \ast \sum_{r=1}^{\infty} 2^{-2r} (1 - 2^{-2r}) z_{2r+1} \sum_{a=0}^{r-1} \left( \frac{2r}{2a+1} \right) u^{2a} v^{2r-2b-2} \]

\[ + \sum_{s=0}^{\infty} (-1)^{s} z_{2}^{s} u^{2s} \ast \sum_{r=1}^{\infty} 2^{-2r} z_{2r+1} \sum_{b=0}^{r-1} \left( \frac{2r}{2b+1} \right) u^{2r-2a-2} v^{2b}. \]

Hence by (4.5) and the definitions of \( A(u), B(u) \) and \( S(u) \), we get (4.8) easily.

Notice the fact \( S(u) \ast S^{*}(u) = 1 \) (see \([9, \text{Corollary 1}]\)). Then we have

\[ \hat{T}^{*}(u, v) = \hat{T}(v, u) \ast S^{*}(u) \ast S^{*}(v). \] (4.10)

Hence, (4.7) and (4.10) imply the equivalence of the evaluation formulas (4.1) and (4.6).

### 4.2 Evaluation of \( t_{*}^{V}(\{1\}^{a}, 2, \{1\}^{b}) \)

Analogous to [3, Definition 3.4], for \( s_{1}, \ldots, s_{n} \in \mathbb{Z}_{\geq 0} \), the multiple \( t \)-star polylogarithm is defined by

\[ T_{*_{s_{1},\ldots,s_{n}}}(z_{1}, \ldots, z_{n}) = \sum_{m_{1} \geq \cdots \geq m_{n} > 0} \frac{z_{1}^{2m_{1}-1} \cdots z_{n}^{2m_{n}-1}}{(2m_{1}-1)^{s_{1}} \cdots (2m_{n}-1)^{s_{n}}}, \]

which converges when \( |z_{1} \cdots z_{i}| < 1 \) for \( i = 1, \ldots, n \). Note that for \( s_{1} > 1 \) and \( z_{1} = \cdots = z_{n} = 1 \), \( T_{*_{s_{1},\ldots,s_{n}}}(1, \ldots, 1) \) is exactly the multiple \( t \)-star value \( t^{*}(s_{1}, \ldots, s_{n}) \). When \( n = s_{1} = 1 \), we get

\[ T_{*_{1}}(z) = \sum_{m=1}^{\infty} \frac{z^{2m-1}}{2m-1} = \tanh^{-1}(z), \] (4.11)

which tends to infinity when \( z \to 1^{-} \).

We calculate the generating function

\[ \sum_{a, b \geq 0} T_{*_{\{2\}^{a},\{2\}^{b}}}(\{1\}^{a}, z, \{1\}^{b})(2x)^{2a}(2y)^{2b} \]

\[ = \sum_{r=1}^{\infty} \prod_{k=r}^{\infty} \left( 1 - \frac{(2x)^{2}}{(2k-1)^{2}} \right)^{-1} \frac{z^{2r-1}}{2r-1} \prod_{l=1}^{r} \left( 1 - \frac{(2y)^{2}}{(2l-1)^{2}} \right)^{-1} \]
\[
= \frac{1}{\cos \pi x} \sum_{r=1}^{\infty} \prod_{k=1}^{r-1} (2k - 1)^2 - (2x)^2 \frac{z^{2r-1}}{(2k - 1)^2} \prod_{l=1}^{r} (2l - 1)^2 - (2y)^2 \\
= \frac{1}{(1 - 4y^2) \cos \pi x} \sum_{r=1}^{\infty} \left( \frac{1}{2} + x \right) \cdots \left( \frac{r - \frac{3}{2} + x}{2} \right) \cdots \left( \frac{r - \frac{3}{2} - x}{2} \right) \left( \frac{1}{2} - x \right) \cdots \left( \frac{r - \frac{3}{2} - x}{2} \right) (2r - 1) z^{2r-1} \\
= \frac{z}{(1 - 4y^2) \cos \pi x} \left. _1F_3 \right( \begin{array}{c}
\frac{1}{2}, \frac{3}{2} - x, \frac{3}{2} + x \\
\frac{3}{2} - y, \frac{3}{2} + y, z^2
\end{array} ; \frac{1}{2}\right)
\] 

(4.12)

By the stuffle product, we have

\[
\text{Ti}_{1,(2)^b}(z, \{1\}^b) = \text{Ti}_{1}(z) \text{Ti}_{(2)^b}(\{1\}^b) - \sum_{i=1}^{b} \text{Ti}_{(2)^b}(\{1\}^i, z, \{1\}^{b-i}) \\
+ \sum_{j=0}^{b-1} \text{Ti}_{(2)^j,3,(2)^{b-1-j}}(\{1\}^j, z, \{1\}^{b-1-j}).
\]

Hence

\[
t_{V=0}^{*}(1, \{2\}^b) = -\sum_{i=1}^{b} t^{*}(\{2\}^i, 1, \{2\}^{b-i}) + \sum_{j=0}^{b-1} t^{*}(\{2\}^j, 3, \{2\}^{b-1-j}).
\]

Then by (3.4) and (4.11), the divergent part (as \( z \to 1^- \)) of the generating function can be written as

\[
\sum_{b=0}^{\infty} \text{Ti}_{1,(2)^b}(z, \{1\}^b)(2y)^{2b} = \sum_{b=0}^{\infty} \text{Ti}_{1}(z)t^{*}(\{2\}^b)(2y)^{2b} + \sum_{b=0}^{\infty} t_{V=0}^{*}(1, \{2\}^b)(2y)^{2b} \\
= \frac{\tanh^{-1}(z)}{\cos \pi y} + \sum_{b=0}^{\infty} t_{V=0}^{*}(1, \{2\}^b)(2y)^{2b}.
\]

(4.13)

Subtracting (4.13) from (4.12) and letting \( z \to 1^- \), we get the generating function of the star-stuffle regularized (at \( V = 0 \)) multiple \( t \)-star values

\[
\sum_{a,b \geq 0} t_{V=0}^{*}(\{2\}^a, 1, \{2\}^b)(2x)^{2a}(2y)^{2b} \\
= \lim_{z \to 1^-} \left\{ \frac{z}{(1 - 4y^2) \cos \pi x} \left. _1F_3 \right( \begin{array}{c}
\frac{1}{2}, \frac{3}{2} - x, \frac{3}{2} + x \\
\frac{3}{2} - y, \frac{3}{2} + y, z^2
\end{array} ; \frac{1}{2}\right) - \frac{\tanh^{-1}(z)}{\cos \pi y} \right\} \\
= \frac{1}{2 \cos \pi x} \left[ A(x + y) + A(x - y) - 2 \log(2) \right] \\
+ \frac{1}{2 \cos \pi x} \left[ B(x + y) + B(x - y) + 2 \log(2) \right],
\]

(4.14)

where the last step follows from [3, Eqs. (14), (17)].

Then by comparing the coefficient of \( x^{2a} y^{2b} \) on both sides of (4.14), we have

\[
t_{V=0}^{*}(\{2\}^a, 1, \{2\}^b) = \sum_{r=1}^{a+b} 2^{-2r} \left[ \binom{2r}{2a} + (1 - 2^{-2r}) \binom{2r}{2b} \right] t^{*}(\{2\}^{a+b-r}) \zeta(2r + 1)
\]

(4.14)
\[-δ_{a=0} \log(2)t^*({\{2\}^b}) + δ_{b=0} \log(2)t^*({\{2\}^a}). \quad (4.15)\]

Notice that
\[t^V_\tau(1, {\{2\}^b}) = Vt^*({\{2\}^b}) + t^V_\tau(1, {\{2\}^b}),\]
we complete the proof of Theorem 4.2 by giving the necessary correction terms to add to the right-hand side of (4.15).

The evaluation of \(t^V_\tau({\{2\}^a}, 1, {\{2\}^b})\) in [3, Theorem 1.1] shows that
\[t^V_\tau({\{2\}^a}, 1, {\{2\}^b}) = \sum_{r=1}^{a+b} (-1)^r 2^{-2r} \left[ \binom{2r}{2a} + (1 - 2^{-2r}) \binom{2r}{2b} \right] t^*({\{2\}^{a+b-r}}) \zeta(2r + 1)
+ δ_{a=0}(V - \log(2))t({\{2\}^b}) + δ_{b=0} \log(2)t({\{2\}^a}). \quad (4.16)\]

We give an algebraic proof of the equivalence of (4.2) and (4.16).

**Proposition 4.4.** (4.2) holds for any integers \(a, b \geq 0\) if and only if (4.16) holds for any integers \(a, b \geq 0\).

**Proof.** Using [11, Eq. (2.4)], we get that
\[\mathcal{P}^*(u, v) = \mathcal{P}(v, u) \ast \mathcal{S}^*(u) \ast \mathcal{S}^*(v), \quad (4.17)\]
where
\[\mathcal{P}(u, v) = \sum_{a, b=0}^{\infty} (-1)^{a+b} z_2^a z_1^b u^{2a} v^{2b}, \quad \mathcal{P}^*(u, v) = \sum_{a, b=0}^{\infty} S(z_2^a z_1^b) u^{2a} v^{2b}.\]

Let
\[\hat{\mathcal{P}}(u, v) = \sum_{a, b=0}^{\infty} (-1)^{a+b} \hat{J}(a, b) u^{2a} v^{2b}, \quad \hat{\mathcal{P}}^*(u, v) = \sum_{a, b=0}^{\infty} \hat{J}^*(a, b) u^{2a} v^{2b},\]
where
\[\hat{J}(a, b) = \sum_{r=1}^{a+b} (-1)^r 2^{-2r} \left[ \binom{2r}{2a} + (1 - 2^{-2r}) \binom{2r}{2b} \right] z_{2r+1} \ast z_2^{a+b-r}
+ δ_{a=0}(z_1 - \log(2)) \ast z_2^b + δ_{b=0} \log(2) z_2^a,\]
\[\hat{J}^*(a, b) = \sum_{r=1}^{a+b} 2^{-2r} \left[ \binom{2r}{2a} + (1 - 2^{-2r}) \binom{2r}{2b} \right] z_{2r+1} \ast S(z_2^{a+b-r})
+ δ_{a=0}(z_1 - \log(2)) \ast S(z_2^b) + δ_{b=0} \log(2) S(z_2^a).\]

Then, we obtain that
\[\hat{\mathcal{P}}(u, v) = z_1 \ast S(v) + log(2)(S(u) - S(v)) + \frac{S(v)}{2} \ast \left[ A \left( \frac{u + v}{2} \right) + A \left( \frac{u - v}{2} \right) \right]
+ \frac{S(u)}{2} \ast \left[ B \left( \frac{u + v}{2} \right) + B \left( \frac{u - v}{2} \right) \right], \quad (4.18)\]
\[ \hat{P}^*(u, v) = z_1 \ast S^*(v) + \log(2)(S^*(u) - S^*(v)) + \frac{S^*(v)}{2} \ast \left[ A \left( \frac{u + v}{2} \right) + A \left( \frac{u - v}{2} \right) \right] \\
+ \frac{S^*(u)}{2} \ast \left[ B \left( \frac{u + v}{2} \right) + B \left( \frac{u - v}{2} \right) \right]. \]  

(4.19)

We give a proof of (4.18) and one can prove (4.19) similarly. We have \( \hat{P}(u, v) = \hat{P}_1(u, v) + \hat{P}_2(u, v) \) with

\[ \hat{P}_1(u, v) = \sum_{a,b=0}^{\infty} (-1)^{a+b} \left[ \delta_{a=0}(z_1 - \log(2)) \ast z_2^b + \delta_{b=0} \log(2)z_2^a \right] u^{2a}v^{2b}, \]

\[ \hat{P}_2(u, v) = \sum_{a,b=0}^{\infty} \sum_{r=1}^{a+b} (-1)^{a+b+r}2^{-2r} \left[ \left( \frac{2r}{2a} \right) + (1 - 2^{-2r}) \left( \frac{2r}{2b} \right) \right] z_{2r+1} \ast z_2^{a+b-r}u^{2a}v^{2b}. \]

Obviously,

\[ \hat{P}_1(u, v) = \sum_{b=0}^{\infty} (-1)^b(z_1 - \log(2)) \ast z_2^b u^{2b} + \sum_{a=0}^{\infty} (-1)^a \log(2)z_2^a u^{2a} \]

\[ = z_1 \ast S(v) - \log(2)S(v) + \log(2)S(u). \]

Replacing \( a + b - r \) by \( s \), we get

\[ \hat{P}_2(u, v) = \sum_{r=0}^{\infty} \sum_{a=0}^{r} (-1)^s2^{-2r} \left( \frac{2r}{2a} \right) z_{2r+1} \ast z_2^s u^{2s+2r-2a} \]

\[ + \sum_{r=1}^{\infty} \sum_{a=0}^{r} (-1)^s2^{-2r} \left( 1 - 2^{-2r} \right) \left( \frac{2r}{2b} \right) z_{2r+1} \ast z_2^s u^{2s+2r-2b}v^{2b} \]

\[ = \sum_{s=0}^{\infty} (-1)^s z_2^s u^{2s} \ast \sum_{r=1}^{\infty} 2^{-2r} z_{2r+1} \sum_{a=0}^{r} \left( \frac{2r}{2a} \right) u^{2a}v^{2r-2a} \]

\[ + \sum_{s=0}^{\infty} (-1)^s z_2^s u^{2s} \ast \sum_{r=1}^{\infty} 2^{-2r} \left( 1 - 2^{-2r} \right) z_{2r+1} \sum_{b=0}^{r} \left( \frac{2r}{2b} \right) u^{2r-2b}v^{2b}. \]

Using the identity

\[ 2 \sum_{n=0}^{r} \left( \frac{2r}{2n} \right) u^{2n}v^{2r-2n} = (u + v)^{2r} + (u - v)^{2r}, \]

we find that

\[ \hat{P}_2(u, v) = \frac{S(v)}{2} \ast \left[ A \left( \frac{u + v}{2} \right) + A \left( \frac{u - v}{2} \right) \right] + \frac{S(u)}{2} \ast \left[ B \left( \frac{u + v}{2} \right) + B \left( \frac{u - v}{2} \right) \right], \]

which together with the formula of \( \hat{P}_1(u, v) \) deduces (4.18). And finally, we obtain

\[ \hat{P}^*(u, v) = \hat{P}(v, u) \ast S^*(u) \ast S^*(v). \]  

(4.20)

Hence, (4.17) and (4.20) imply the equivalence of the evaluation formulas (4.2) and (4.16).
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