Internal-time and dilatations in classical relativity

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Abstract. The proper-time of any classical (non-quantum) relativistic system, derived from its centre-of-mass (CM) coordinate, is defined and identified as an observable – the system’s internal time observable, measuring its aging and internal evolution. This is a Lorentz-invariant observable, and its definition requires that the system be symmetric not only under Lorentz-Poincaré transformations but also under uniform scaling, with the associated existence of a dilatation function $D$. These dilatation functions are required to be varying – not conserved – quantities, due to the existence of masses, and they exist at least for a very large family of systems. They split into a CM-part and an internal part, which is in general non-vanishing. The proper-time is expressed, as an observable, as a combination of the global dilatation function $D$ and the internal dilatation function $D_{\text{int}}$, where $D$ brings into it the generic time-like variation while $D_{\text{int}}$ brings in the time-like variation that involves the details of the internal evolution of the system.

1. Introduction

The purpose of the talk was to present the internal-time observable of relativistic systems and its relationship with the dilatation function associated with the system. The talk was by large based on a recently published article [1], and some necessary details will be repeated here for completeness.

The idea that an internal-time observable should appear naturally in any description of relativistic systems leans on two arguments. First, if the existence of a central coordinate in the role of some centre-of-mass (CM) is assumed [2], which must necessarily be an observable (functional of the dynamic parameters of the constituents of the system), then the proper-time which corresponds to the trajectory of the centre-of-mass must also be an observable. Then, a more essential argument comes from the concept of the age of a system: Age is a temporal manifestation of the internal evolution of the system – a manifestation of the correspondence between the internal evolution, on the one hand, and the external time measurement on the other hand. The external time measurement necessarily uses, for this purpose, the system’s CM proper-time. Therefore, the association of the proper-time of a system with an observable allows us to regard it as providing the age of the system. Since the proper-time depends, as an observable, on the internal dynamics, it is appropriate to regard it as an internal-time observable. A well-known association of proper-time with age appears of course in the so-called "Twins’ paradox" [3].

The systems in question are assumed to be Lorentz-Poincaré symmetric, with conserved total linear momentum $P^{\mu}$ and angular momentum $J^{\mu\nu}$. The proper-time observable ($\tau$ in the following) is obviously Lorentz-invariant. In the following it is shown how the definition
of the proper-time as an observable requires that a dilatation function $D$ be also definable, implying that these systems must be symmetric also under uniform scaling of the Minkowski space-time, together with corresponding scaling of the particles’ masses, whose generator is the dilatation function. This is in accordance with the result by Zeeman [4] that the global symmetry transformations that leave causal relations invariant are uniform translations, homogeneous space-time rotations, space inversion, and dilatations. Therefore, the minimal symmetry group – which may be regarded as the ”causal group” – is not 10- but 11-dimensional, with $P^\mu$, $J^{\mu
u}$ and $D$ as generators. However, in contrast to $P^\mu$ and $J^{\mu\nu}$, the association of $\tau$ with $D$ requires that $D$ be a varying quantity, which is certainly the case when particle masses are included (in contrast to field theory, where $D$ is expected to be conserved like the other global observables and generators of symmetries).

This relationship between the internal-time and the dilatation function is of particular interest, because it offers a clear physical meaning to the dilation function. Since the realization of the conformal symmetry of Maxwell’s equations ([5], cited in [6]) about a century ago, the physical meaning of the generators of the conformal transformations, in particular in classical (non-quantum) physics, remained obscure [6], unlike the very clear meaning associated with, say, linear and angular momenta. An explicit expression of the internal-time in terms of the dilatation function, equivalent to equation (6) below, was suggested few years ago by Jaekel and Reynaud [7]. However, as is shown in [1] and further elaborated in the following, this relation is incomplete and holds only in very particular cases. In the general case a new observable needs to be introduced – the so-called internal dilatation function, which is a moment of dilatations relative to the centre-of-mass. It is this function which is shown to contain, and therefore to introduce into the proper-time, the information regarding the internal evolution of the system.

**Notation.** In the following we consider dynamics described in a Minkowski space-time $\{x^\mu\}$, $\mu = 0, 1, 2, 3$ with metric tensor $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. A dot implies scalar product, $a \cdot b = a_\mu b^\mu$. $c = 1$ is assumed throughout.

## 2. Internal time and dilatations

The internal time or age of a system is thus defined as the system’s proper-time $\tau$, which in turn requires the existence of a CM coordinate $X^\mu$. The CM trajectory is defined as an observable as follows: Any system of particles is described, classically, by the bundle of trajectories they traverse in space-time. Picking arbitrarily an event $x_a^\mu$ on each particle’s trajectory, any such set of events defines an event on the CM trajectory (Figure 1). The prescription by which this association is made makes $X^\mu$ a functional of these events, $X^\mu = X^\mu\{x_a\}$, i.e., an observable. The proper time which corresponds to this CM event is then identified as the corresponding age of the system. In the following we don’t consider any particular such prescription, only general properties which should be satisfied by all.

Given a closed relativistic system with conserved total linear momentum $P^\mu$ and total angular momentum $J^{\mu\nu}$, its CM trajectory in Minkowski space-time should be subject to the following conditions:

(i) The space-time trajectory of the CM coordinate is a straight line in the direction of $P^\mu$.
(ii) It should behave like an ordinary space-time coordinate under the global symmetry transformations of the Minkowski space-time.
(iii) Its definition should depend only on the dynamical variables of the constituents of the system. Clearly, the definition should be independent of the choice of any particular observer.

It may always be written as [8]

$$X^\mu = R^\mu + \frac{P^\mu}{M} \tau$$

(1)
where $M = \sqrt{-P^\mu P_\mu}$, $\tau$ is the proper-time and $R^\mu$ is a constant 4-vector. Appropriately fixing the zero of $\tau$, $R^\mu$ may be assumed orthogonal to $P^\mu$ without loss of generality, $R \cdot P = 0$. Then, contracting equation (1) with $P_\mu$ yields

$$\tau = -\frac{X \cdot P}{M} \quad (2)$$

Although relation (2) was already realized many years ago [9, 10], no attempt has been done to make it an observable until recently [7]. However, $X^\mu$ being an observable makes $\tau$ an observable as well, and since it must be a time-varying observable it cannot be constructed from the invariant $P^\mu$ and $J^{\mu\nu}$ alone. This is how dilatation functions enter the picture.

Using continuum-like notation, the dilatation function $D$ is defined, like in field theory, from the conserved energy-momentum tensor of the system $T^\mu_\nu$ as an integral over a space-like hypersurface $\Sigma$ which contains the given system of particles,

$$D(\Sigma) = \int_\Sigma x^\mu T^\nu_\mu d\Sigma_\nu \quad (3)$$

For a single free particle $D = x \cdot p$, and this definition can be extended to systems of point particles with direct interactions [1] (see also [11] for a corresponding conserved, but not an
observable, quantity). Even when not conserved, these functions can serve as generators of uniform dilatations of coordinates, masses and linear momenta in Minkowski space-time,

\[ x^\mu \rightarrow e^{\lambda} x^\mu, \quad m_a \rightarrow e^{-\lambda} m_a, \quad p^\mu_a \rightarrow e^{-\lambda} p^\mu_a \]  

(4)

The association of the proper-time with the dilatation function then follows from the observation that \( X \cdot P \) and \( D \) behave in the same way under all causal symmetry transformations of the Minkowski space-time: Both transform under uniform translations as

\[ x^\mu \rightarrow x^\mu + a^\mu, \quad X \cdot P \rightarrow X \cdot P + a \cdot P, \quad D \rightarrow D + a \cdot P \]  

(5)

and are invariant under homogeneous Lorentz transformations and scaling. This identical behaviour may seem to suggest, apparently in a natural way, that \( X \cdot P \) and \( D \) should be identified so that the proper-time is given by [7]

\[ \tau = -\frac{D}{M} \]  

(6)

as is indeed the case for a single free particle. However, the following argument points to the contrary: The dilatation function \( D \) is a contracted first-order moment of the energy-momentum tensor relative to the origin of the reference frame. Using the continuum-like definition (3), with the linear momentum given by \( P^\mu = \int_\Sigma T^\mu_{\nu} d\Sigma^\nu \), the difference between \( D \) and \( X \cdot P \) turns out to be the moment of dilatations relative to the centre-of-mass,

\[ D(\Sigma) - X \cdot P = \int_\Sigma (x^\mu - X^\mu) T^\nu_{\mu} d\Sigma^\nu \]  

(7)

This quantity vanishes for a single point-like particle, but there is no reason to expect that it is zero for many-body or extended systems. The difference (7) may therefore be regarded as an internal dilatation function or moment, and is denoted accordingly \( D_{\text{int}} \). Its non vanishing is further corroborated using the so-called condition of simultaneity, introduced in [1] and discussed in the following. Therefore, \( X \cdot P \) and \( D \) are not identical (except for very particular cases), and \( D \) is found to split into CM- and internal parts,

\[ D = D_{CM} + D_{\text{int}} = X \cdot P + D_{\text{int}} \]  

(8)

like the splitting of the total angular momentum (another moment of the energy-momentum tensor) into an orbital (CM) part and an internal part.

Making use of the decomposition (8), the proper-time (2) is thus defined via a combination of both dilatation functions in the form

\[ \tau = \frac{D_{\text{int}} - D}{M} \]  

(9)

As has been indicated in [1], the separate contributions of \( D \) and \( D_{\text{int}} \) to \( \tau \) are of a very distinct, but complementary, nature. Let us elaborate now this point.

Consider a system of \( N \) point particles with masses \( m_a \) moving on the trajectories \( x^\mu_a = x^\mu_a(\tau_a) \), \( a = 1, ..., N \), \( \tau_a \) being the proper time of the \( a \)-th particle. Any arbitrary set of particles’ events \( \{ x^\mu_a(\tau_a) \} \) with the corresponding set of proper-times \( \{ \tau_a \} \) defines the corresponding CM event \( X^\mu(\{ \tau_a \}) = X^\mu[\{ x^\mu_a(\tau_a) \}] \), as in Figure 1. This then defines, via equations (2) or (9), a unique value of the CM proper-time \( \tau(\{ \tau_a \}) \) as a function of the proper-times of the particles. In particular, all the particle’s events may be chosen so that they are simultaneous in the CM reference frame. A-priori, the corresponding CM event, defined by these particles’ events, could
Figure 2. The simultaneity condition for a system of particles in the CM frame.

fall anywhere on the CM trajectory. However, it would seem unnatural to the concept of centre-of-mass if the CM-event thus defined is not simultaneous with the CM-simultaneous particles’ events that define it. Thus we introduce the *simultaneity condition*, that states that the events that constitute τ as internal time are expected to be simultaneous, in the CM reference frame, with the time τ as measured along the CM time-axis (Figure 2).

Consider the hyperplane \( x^o = t \) in the CM reference frame. Each particle’s trajectory (as observed in the CM frame) \( x^\mu = x^\mu_a(\tau_a) \) cuts this hyperplane once, defining for each \( \tau_a \) a unique function \( \tau_a^{CM}(t) \) via the identity

\[
x_a^o(\tau_a) = t \iff \tau_a = \tau_a^{CM}(t)
\] (10)

These CM-simultaneous particles’ events define the corresponding proper-time observable as a unique function of the CM-observer time \( t \),

\[
\tau = \tau(\{\tau_a^{CM}(t)\}) = \tau(t)
\] (11)

The simultaneity condition is then satisfied if and only if \( \tau(t) = t \), at least up to an additive constant.

Combining now the simultaneity condition together with the proper-time observable definition (9), provides us with means to get some insight into the properties of the internal dilatation function. On common-CM-time hyperplanes \( \tau = t \), it is \( D_{int}(t) = Mt + D \). To compute its variation, we use the fact proven in [1] that if the particles interact via interactions which are by themselves \( D \)-conserving (such as interactions carried by a massless field), the variation of \( D \) is generically found to be

\[
dD(\{\tau_a\}) = - \sum_a m_a d\tau_a,
\] (12)

the same as for free particles, independent of the details of the interaction. Assuming this is the case, the variation of \( D_{int}(t) \) with respect to \( t \) is easily computed:

\[
\frac{dD_{int}(t)}{dt} = M + \frac{dD}{dt} = M - \sum_a m_a \frac{d\tau_a^{CM}(t)}{dt} = M - \sum_a m_a \gamma^{-1}(v_a)
\] (13)
where \( v_a \) is the \( a \)-th particle’s velocity as measured in the CM reference frame and the relation \( d\tau_C^C M(t) = \gamma^{-1}(v_a) dt = \sqrt{1 - v_a^2} dt \) is used. Equation (13) may now be used to compare some characteristic systems:

(i) Circular motion – for many-particle systems in action-at-a-distance electrodynamics with circular orbits around a common centre the rhs of (13) vanishes [12, 13], so that \( D_{\text{int}}(t) = \text{const} \).

(ii) Bound systems – for general bound systems with non-circular orbits \( \langle dD_{\text{int}}/dt \rangle = 0 \) \((< > = \text{average})\) so that \( D_{\text{int}}(t) \) oscillates around a constant value [11, 14].

(iii) Unbound systems – here \( M > \sum_a m_a \), thus it follows from (13) that \( D_{\text{int}}(t) \) increases monotonically. In particular, for a system of free particles with \( M = \sum_a m_a \gamma(v_a) \) we obtain

\[
\frac{dD_{\text{int}}(t)}{dt} = \sum_a m_a v_a^2 \gamma(v_a) = \sum_a \frac{p_a^2}{E_a(p_a)} \quad (p_a = |\vec{p}_a|)
\]

As an example, for a two-body system, either in the Newtonian limit or in the absence of interactions, \( D_{\text{int}}(t) = \vec{r} \cdot \vec{p} \), where \( \vec{r} \) is the relative vector between the particles and \( \pm \vec{p} \) are their momenta in the CM-frame. It vanishes if the orbit is circular, and oscillates around zero for otherwise bound systems. For unbound systems (scattering) it grows monotonically from \(-\infty\) to \(\infty\), vanishing at the point of closest approach.

3. Dilatations and the distinction between time translation and evolution in time

We turn now to the discussion of the preceding results. The concept of age associates external time measurement with internal evolution. From (12) it follows that \( D \) cannot provide this association because its variation does not depend on the details of the interaction – the variation of \( D \) provides only a generic time-like variation. It must be therefore the internal dilatation function \( D_{\text{int}} \) that introduces into \( \tau \) its dependence upon the evolution of the system, a measure of the changes that the internal configuration of the system undergoes during its evolution. The discussion above, following (13), does indeed demonstrate that \( D_{\text{int}} \) has this property and reflects, in the nature of its time variation, the fine and particular details of the internal configuration of the system and its evolution: Its constancy for circular motion, which is associated with the uniformity of this motion; its oscillations around a constant value for noncircular bound orbits; and its monotonic growth for unbound systems.

It is easy to verify another essential and characteristic property of \( D_{\text{int}} \): Its invariance under all the causal global space-time symmetries of uniform translations, rotations and scaling. Hence, being time-dependent, yet ”transparent” for these geometrical transformations, implies that \( D_{\text{int}} \) is affected only by dynamic time translations.

Concluding, \( D \) brings into \( \tau \) the generic time-like variation, while \( D_{\text{int}} \) brings in the time-like variation that involves the details of the internal evolution of the system. Thus, while \( D \) may be regarded as bringing in the geometrical aspect of time translation, \( D_{\text{int}} \) brings in the dynamical aspect.

The outcome of these results is that all the causal global symmetries, even the time-like ones, are pure (Minkowski space-time-) geometrical transformations. None of their generators \( - P^\mu, J^{\mu\nu} \) and \( D \) – or any combination thereof, can serve as a generator of internal dynamic evolution. This refers, in particular, to the time-like components of \( P^\mu \) or \( J^{\mu\nu} \). Consequently, the Hamiltonian or Hamiltonians – generators of internal dynamic evolution – must be separate observables, containing elements that are different than \( P^\mu, J^{\mu\nu} \) and \( D \). The requirement that the Hamiltonian be distinct from the Lorentz-Poincaré generators has already been recognized in the past (see, e.g., [15]). Here we arrived at this conclusion from a different approach, focusing on the internal evolution and showing that the dilatation functions (which are missing in previous discussions) are a necessary part of the picture.
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