Momentum space topology of QCD

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Abstract

We discuss the possibility to consider quark matter as the topological material. We consider hadronic phase (HP), the quark - gluon plasma phase (QGP), and the hypothetical color - flavor locking (CFL) phase. In those phases we identify the relevant topological invariants in momentum space. The formalism is developed, which relates those invariants and massless fermions that reside on vortices and at the interphases. This formalism is illustrated by the example of vortices in the CFL phase.

1. Introduction

QCD has the reach phase structure, which has not been yet investigated exhaustively. Various external conditions (temperature, baryonic chemical potential, external magnetic field, pressure, isospin chemical potential, chiral chemical potential, etc) cause the appearance of different phases (for the review see [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] and references therein). Some of those phases are connected continuously, while the others are separated by the true phase transitions. The major known phases have different symmetries. At the same time, within each phase the topological phase transitions [20] are possible, which are characterized by the same symmetry but different values of momentum space topological invariants. While the breakdown of various symmetries across the transitions between the phases has been discussed in details in many cases, the consideration of the possibility for the topological phase transitions is still in its infancy.

Momentum space topology (i.e. the theory of topological invariants in momentum space) has been widely discussed recently in the context of relativistic theories (see [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35]). Originally momentum space topology was developed in the condensed matter theory [36] [37] [38] [39] [40] [41], where it is nowadays used widely for the description of various systems among which the most well known are the topological insulators. The first step towards the prediction of the appearance of the new topological sub - phases within each phase of QCD is the identification of the relevant topological invariant.

The different sub - phases of the given phase would correspond to different values of a topological invariant. The continuous deformation of the system between the states with different values of the topological invariant is impossible. The only possibility to move between such phases is to cross a phase transition, which is accompanied by the appearance of the singularity of the fermionic Green function, and the change of the value of the topological invariant composed of this Green function. Typically, such a topological phase transition is accompanied by the discontinuities in thermodynamical quantities.

The present paper is devoted to the identification of the relevant topological invariants in the following phases of QCD:

1. Hadronic phase (HP). This is the most well - known phase of QCD, which corresponds to the observed hadronic and nuclear matter [1] [2] [3] [4] [5] [6] [7] [8]. The latter corresponds to the nonzero chemical potential, which is typical for the matter inside the nuclei. There are the two sub - phases of the HP, which are separated by the vapor - gas first order phase transition, having its endpoint, so that those two sub - phases are connected continuously.
2. Quark - gluon plasma phase (QGP). It appears at the temperatures above the confinement - deconfinement transition \[1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\]. Notice, that recent data of lattice numerical simulations \[17\] indicate, that at the realistic values of quark masses this transition may appear to be a crossover, and that the low temperature Hadronic phase is connected continuously with the QGP phase. That means, that those two phases actually have the same symmetry. In the limit of massless \(u\) and \(d\) quarks however, those two phases are indeed different and the transition between them corresponds to the breakdown of chiral symmetry.

3. When the quark chemical potential is increased, the hadronic phase undergoes first the gas - liquid phase transition, after that the other transitions are possible \[1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\]. It is widely believed, that the further increase of chemical potential finally leads to the formation of color superconductivity \[12\ 13\ 14\ 15\ 16\]. According to this hypothesis at extremely high values of chemical potential the CFL phase appears, which is characterized by the condensation of the quark - quark pairs composed of \(u\), \(d\) and \(s\) quarks.

Those three phases are represented in Fig. 1 where the expected phase diagram is given schematically. The hypothetical existence of the topological phase transitions within various phases of QCD follows the analogy with the topological insulators \[36\ 37\ 38\ 39\], which have reach topological structure caused by the similar topological invariants. But this is not the only consequence of the existence of the momentum space topological invariants. The analogy to the condensed matter physics also prompts that another consequence would be related to the existence of the specific massless boundary states. Recall, that in condensed matter physics this is the massless boundary state, which identifies bulk topological matter (either topological semimetal \[42\ 43\ 44\ 45\ 46\] or a topological insulator \[47\ 48\ 49\ 50\]). In the same way, in QCD the existence of nonzero value of the topological invariant might be related to the existence of massless fermions at the boundary of the region of space filled by the quark matter of the given state. To find the massless boundary states using the topological invariant in momentum space through the bulk - boundary correspondence \[20\ 21\ 22\ 23\ 24\] we need the finite regions of space filled by the given state of quark matter, outside of these regions the state of matter is different while the symmetry is the same. In this case one may use the same topological invariant at both sides of the boundary.

The appearance of the finite region of space filled by the given state of quark matter is typical, for example, for the heavy ion collisions \[55\], or inside the stars \[56\]. Out of such regions of space the quark matter is typically in the state with different symmetry. This does not allow to use directly the topological invariants protected by symmetry for the description of the fermion zero modes at the interphases between the phases of different symmetries. Such a description would become possible if the topological phase transition is found inside a certain phase so that the two sub - phases with the same symmetry but different values of the topological invariant may coexist. Then at the interphase between them one would find the massless fermions through the bulk - boundary correspondence.

The power of the topological invariants in momentum space is demonstrated through the consideration of vortices that are formed inside the CFL phase. Then the bulk topological invariant is intimately related to the zero modes existing at the topological defects. Using the ideas of the representation of such a correspondence \[54\] (see also \[52\]) in terms of the mixed \(r - p\) invariant (the invariant in mixed coordinate - momentum space), we apply the methodology of \[35\] that operates with the Wigner transformation \[58\ 59\ 60\ 61\] of the Green function and describe the fermion zero modes that reside on vortices in the CFL phase.

2. Quark - gluon plasma (QGP) phase

The discussion of this section is to a certain extent similar to the consideration of the symmetric high temperature phase of the whole Standard Model presented in the Second section of \[33\]. For vanishing quark chemical potential the quark gluon plasma phase appears at the temperatures above the confinement - deconfinement phase transition. In the approximation when the \(u\) and \(d\) quarks are considered as massless while the \(s\) quark mass is kept equal to its physical value close to 100 MeV the transition between the two phases is the true phase transition, which is accompanied by the restoration of the chiral \(SU(2)\) symmetry. The chiral \(U(1)\) symmetry remains broken due to the instantons, but its subgroup \(Z_2\) is already not broken. This subgroup corresponds to the transformations \(q = (u, d, s) \rightarrow e^{i\pi \Pi_{u,d} \gamma^5/2} q\), where \(\Pi_{u,d}\) is the projector.
Figure 1: The expected phase diagram of QCD at finite baryon chemical potential $\mu$ and finite temperature $T$ is represented schematically (assuming the validity of the hypothesis that at large values of $\mu$ the color superconductivity appears).

to the states of $u$ and $d$ quarks:

$$u \rightarrow e^{i\pi\gamma/2}u, \quad d \rightarrow e^{i\pi\gamma/2}d, \quad s \rightarrow s$$

This is because the corresponding transformation of measure gives $D\bar{u}Du \rightarrow e^{\nu i}D\bar{u}Du$ and $D\bar{d}Dd \rightarrow e^{\nu i}D\bar{d}Dd$, and therefore,

$$D\bar{u}Du \bar{d}Dd \rightarrow e^{2\nu i}D\bar{u}Du \bar{d}Dd$$

Here $\nu$ is the integer number of $SU(3)$ instantons.

First of all, let us consider the following topological invariant expressed through the fermion two point Green function $G$ determined in the 4 dimensional momentum-frequency space $[40, 20]$ :

$$N_3 = tr \mathcal{N}, \quad \mathcal{N} = \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\gamma} \int_{\sigma} dS^\gamma \, G\partial_{p_\mu}G^{-1}G\partial_{p_\nu}G^{-1}G\partial_{p_\lambda}G^{-1}. \quad (1)$$

The integral here is over the $S^3$ surface $\sigma$ embracing the point in the 4D space $p = 0, p_4 = 0$, where $p_4$ is the frequency along the imaginary axis; $tr$ is the trace over the fermionic indices. We imply the fixing of Landau gauge, in which the functional $\int d^4x Tr A^2$ is minimized with respect to local gauge transformations, where $A \in su(3)$ is the gauge field.

The topological invariant Eq. (1) given by the integral over the closed 3D surface in momentum space is well defined only if the values of imaginary frequency are continuous. At finite temperatures they are discrete. In order to define such an invariant we need the analytical continuation of the Green function to the continuous values $p_4$ of imaginary frequency. In this case the invariant protects the pole at $p_4 = 0$ (if any) of the resulting analytically continued fermion propagator. At zero temperature this pole appears when the system is gapless. At finite temperature the discrete Matsubara frequency never vanishes, and the physical propagator depending on it is not singular even if the analytically continued propagator is singular at $p_4 = 0$.

In the absence of the exchange by the $SU(3)$ gauge bosons for the right handed Weyl fermions representing $u$ and $d$ quarks one has $N_3 = 1$, and $N_3 = -1$ for the left handed Weyl fermions. Both are gapless in the absence of interactions in the chiral limit. At the same time for the massive $s$ quarks the Fermi point is absent, and the corresponding components of the Green function give vanishing contribution to $N_3$. Therefore, altogether the value of $N_3$ is vanishing.

At the same time due to the mentioned above $Z_2$ symmetry the following expression gives the value of the topological invariant protected by symmetry

$$N_3^{Z_2} = tr \left[ e^{i\frac{\pi}{2} \left( \Pi_{u,d}\gamma^5 + 1 \right)} \mathcal{N} \right] \quad (2)$$
Again, suppose first that there are no interactions due to the color gauge bosons. Then the \( u \) and \( d \) quarks are gapless even at finite temperature. And the topological invariant of Eq. (2) protects the \( u \) and \( d \) quarks from appearance of mass. Smooth continuous transformations of the system cannot change the value of a topological invariant. If the region of the phase diagram where chiral symmetry is not broken, and which is connected continuously to the system of noninteracting \( u, d, s \) quarks, we may substitute to Eq. (2)

\[
G^{-1} \sim p \gamma, \quad p^2 \to 0
\]

This gives \( N_f^2 = 2N_f N_c = 12 \). (In this case the number of quark flavors is \( N_f = 2 \).)

The so-called thermal mass appears in QED at high temperatures \(^{65}\), which may be demonstrated using the technique of HTL (the hard thermal loop). This thermal mass appears precisely in the mentioned above analytically continued fermion propagator, and it actually means the absence of the pole in it. For such a system the topological invariant similar to that of Eq. (2) should be equal to zero. The similar behavior of the fermion propagator has been also predicted in the framework of perturbative QCD at finite temperatures \(^{65}\). Namely, the one loop results give the fermion propagator

\[
\mathcal{G}(p, p_4) = \frac{1}{2} (\gamma^0 - \gamma \mathbf{p}) \Delta_-(p_4, \mathbf{p}) + \frac{1}{2} (\gamma^0 + \gamma \mathbf{p}) \Delta_+(p_4, \mathbf{p})
\]

Here \( \gamma^k \) are the usual Dirac matrices (in Minkowski spacetime), while

\[
\Delta_\pm = ip_4 \mp p - \frac{m_f^2}{2p} \left[ (1 \mp \frac{ip_4}{p}) \log \frac{ip_4 + p}{ip_4 - p} \pm 2 \right]
\]

with \( p = |\mathbf{p}| \) and the thermal mass

\[
m_f^2 = \frac{1}{8} g_{SU(3)}^2 C_F (T^2 + \mu^2)
\]

Here \( g_{SU(3)} \) is the QCD coupling constant, while \( C_F = 4/3 \).

If the nonperturbative effects do not alter this prediction, then the value of the topological invariant of Eq. (1) equals to zero in the QGP phase at high temperatures. At the same time without interactions it is equal to \( N_f^2 = 12 \). This indicates the appearance of the topological phase transition in QCD, which occurs when exchange by the color gauge bosons is turned on. We do not exclude, that this or similar transitions occur in the QGP phase. At least, it is reasonable to suppose, that at the asymptotically large values of temperature the QGP exists in the topological state with \( N_f^2 = 0 \). The state with \( N_f^2 = 12 \) appears when \( \alpha_s \to 0 \) and it could also appear at some values of temperature and baryon chemical potential inside the QGP.

Recent investigations indicate, that the QGP just above the phase transition is strongly coupled. Therefore, we also do not exclude, that the state with the value of \( N_f^2 \) that differs both from 0 and from 12 might appear as well.

### 3. Hadronic phase (HP)

As it was mentioned above, the hadronic phase is separated from the QGP phase if masses of \( u \) and \( d \) quarks are disregarded. According to the so-called Columbia plot \(^{66}\) the transition between the confinement and deconfinement phases is of the first order for sufficiently small \( m_u, m_d, m_s \), and becomes a crossover at large enough values of masses (unfortunately, the present data obtained by different lattice regularizations contradict each other on the corresponding critical values of masses). Nevertheless, even if the real values of the current masses \( m_u, m_d, m_s \) belong to the lower left corner of the Columbia plot, where the transition is of the first order, we may deform the theory smoothly at \( T < T_c \) without a phase transition increasing the values of masses. Next, we come to the high temperature deconfinement phase via a crossover, and decrease the values of masses to original values. Thus if nonzero current masses are taken into account, then the two phases are connected continuously. Actually, taking into account those small but finite current masses we may consider these two phases as the one phase. The relevant topological invariant is the one given by \(^{55}\)

\[
N_f^2 = \frac{1}{24 \pi^2} \epsilon_{\mu \nu \lambda} \text{tr} \left[ K \int d^3p \, G \partial_{p_\mu} G^{-1} G \partial_{p_\nu} G^{-1} G \partial_{p_\lambda} G^{-1} \right].
\]
Here \( p_4 = 0 \); the integral is over 3-momentum space; and \( K \) is the proper symmetry operation (it should either commute of anti-commute with \( G \)). As above, at finite temperatures we need to continue the Green function to the vanishing value of imaginary frequency \( \omega = -ip_0 = p_4 \). In the hadronic phase (and in the considered further CFL phase) the quark condensate appears that gives rise to the scale that affects strongly the Green function. Therefore, unlike the QGP phase we prefer not to consider the finite temperatures and starting from here we restrict ourselves by the case of vanishing temperature, where the Green function is well defined ab initio at \( p_4 = 0 \).

Parity is not broken by QCD with \( u, d, s \) quarks. Therefore, we take following \([33]\) \( K \) as the composition of \( C \) and \( T \) transformations. At zero temperature and vanishing chemical potential the Green’s function has the form

\[
G(p) = \frac{1}{A(-p^2)\gamma^\mu p_\mu - B(-p^2)},
\]

where \( \gamma^\mu \) are the Dirac matrices in Minkowski space, \( p^2 = -p^2 + p_0^2 \), while \( A \) and \( B \) are diagonal hermitian matrices. The fermion mass matrix \( m \) is given by the solution of equation

\[
A(-m^2)m = B(-m^2)
\]

Here it is the matrix \( K = i\gamma^5\gamma^0 \), which commutes with the Green’s function at \( p_4 = \omega = -ip_0 = 0 \). This is the matrix of the combination of \( C \) and \( T \). If we take into account only three light quarks, then

\[
N^{\text{C}T}_3 = N_c \times N_f,
\]

if vacuum is connected continuously to the vacuum of massive noninteracting Dirac fermions. Here \( N_c = 3 \) is the number of colors while \( N_f = 3 \) is the number of flavors. To demonstrate this we take the fermion Green function of the form

\[
G^{-1} = \gamma^\mu p_\mu - m
\]

and obtain

\[
N^{\text{C}T}_3 = \frac{\epsilon_{ijk}}{24\pi^2} \text{tr} \left[ \int_{\omega=0} \frac{d^3p}{(p^2 + m^2)^2} \gamma^0 \gamma^5 m \gamma^i \gamma^j \gamma^k \right] = N_c \times N_f
\]

Each fermion mode (enumerated by flavor and color) contributes 1 to this quantity.

As it was mentioned above, according to the lattice numerical data the hadronic phase is connected continuously to the QGP phase. The latter, in turn, is assumed to be connected continuously with the system of noninteracting fermions (this statement is to be verified by direct calculations, though). At finite values of chemical potential the form of the Green function is more complicated than Eq. \([7]\). However, the invariance under the combination of \( C \) and \( T \) is still present, and therefore, \( K \) still commutes with the Green function at \( \omega = 0 \).

It is worth mentioning, that if we would consider two flavors of light quarks instead of three then the different values of \( N^{\text{C}T}_3 \) may mark the same topological classes. This is because \( B(-p^2) \) is defined up to the global \( Z_2 \) transformation \( B(-p^2) \rightarrow B(-p^2)e^{i\pi \tau^3} \), where \( \tau^3 = \text{diag}(1, -1) \). The values of the \( N^{\text{C}T}_3 \) with opposite signs mark the same topological classes, and the actual classification in this case is \( Z/Z_2 \) rather than \( Z \). However, for the three quark flavors this is not so, and the opposite values of \( N^{\text{C}T}_3 \) do represent different topological classes of vacua.

We may consider the phase diagram of QCD with the axes: temperature, baryonic chemical potential, external magnetic field, external electric potential, angular velocity of rotation, etc. Thus we describe quark matter that may be hot, dense, may experience the external magnetic field and external electric potential, may be rotated. Such states might appear both in the heavy ion collisions and inside the stars. We may also introduce the isotopic and strangeness chemical potentials, which are conserved in heavy ion collisions. This enriches further the phase diagram. In addition, we may consider the smooth deformation of the theory turning off slowly interactions between gauge fields and quarks. This adds an extra axis to the phase diagram. Although certain efforts we made during the last year to investigate the phase structure of QCD \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]\), the behavior of quark matter in many regions
of the mentioned above extended phase diagram is not known. We expect, that there might be the place in this extended phase diagram, where the topological phase transition to the phase with the value of \( N_{3}^{CT} \) different from 9 is present.

To demonstrate, how this phase may be organized in principle, let us consider the form of the Green function of Eq. \((7)\). The terms with the transitions between the quarks are absent here. Therefore, after diagonalization we think of \( A \) and \( B \) as of usual functions specific for each quark. Suppose then, that for the given quark \( A(-p^{2}) \) has the form with \( A(\infty) = 1 \), \( A(0) = -A_{0} < 0 \), and \( A(P^{2}) = 0 \) for a certain value of Euclidean momentum \( P_{c} \). We also suppose, that \( B(0) < 0 \) while \( B(P^{2}) > 0 \).

Let us denote \( g_{i}(P) = A(P^{2})P_{i} \) for \( i = 1,2,3 \) and \( g_{4}(P) = B(P^{2}) \). The general expression for the contribution of the given quark \( q \) to the topological invariant may be reduced to

\[
N_{3}^{(q)CT} = N_{c} \frac{1}{3!\pi^{2}} \epsilon^{abcd}e_{\mu\nu\lambda} \int d^{3}p \hat{g}_{a} \partial_{\mu} \hat{g}_{b} \partial_{\nu} \hat{g}_{c} \partial_{\lambda} \hat{g}_{d}.
\]

(11)

where integral is taken at \( \omega = 0 \). Also we denoted here \( \hat{g}_{a} = \frac{g_{a}}{\sqrt{\sum_{b=1,2,3,4}g_{b}^{2}}} \). Similar to the calculations presented in \([32,33]\) we are able to represent the last expression as

\[
N_{3}^{(q)CT} = N_{c} \sum_{i} \text{sign}(g_{4}(P_{i})) \text{Res}(P_{i})
\]

(12)

Here the sum is over the positions of the common zeros of functions \( g_{i}(P) = A(P^{2})P_{i} \) for \( i = 1,2,3 \) while

\[
\text{Res}(P_{i}) = \frac{1}{8\pi} \epsilon^{ijk} \int_{\Sigma(P_{i})} \hat{g}_{i} d\hat{g}_{j} \wedge d\hat{g}_{k}
\]

(13)

is the integer number, the corresponding integral is along the infinitely small surface \( \Sigma \) placed in 3D momentum space at \( \omega = 0 \), which is wrapped around the given zero of \( g_{i} \) (\( i = 1,2,3 \)). We denoted \( \hat{g}_{i} = \frac{g_{i}}{\sqrt{\sum_{j=1,2,3}g_{j}^{2}}} \).

If some of the zeros form the two dimensional surface, this does not change the above expression. In our particular case we come to the expression

\[
N_{3}^{(q)CT} = N_{c} \text{sign}(B(0)) \text{Res}(0) + N_{c} \text{sign}(B(P_{c})) \text{Res}(P_{c}) = -N_{c} \text{sign}(B(0)) + 2N_{c} \text{sign}(B(P_{c})) = 3M_{A}
\]

Therefore, if the functions \( A \) and \( B \) for \( q = u,d,s \) would behave in the given way, we obtain

\[
N_{3}^{CT} = 3N_{f}N_{c}
\]

(15)

Notice once again that we did not demonstrate that the considered form of the Green function corresponds to any particular external conditions. However, such a possibility is not excluded, which means, that the topological phase transition to the state with \( N_{3}^{CT} \neq N_{f}N_{c} \) may appear. It is obvious, that the interactions must be strong to provide such a transition. In Hadronic phase the interactions are strong, which suggests to search for the existence of the new topological sub - phases within this phase. The whole story may be related to the infrared physics, where strong interactions are especially strong, and therefore, the functions \( A(P^{2}) \) and \( B(P^{2}) \) may, possibly, become negative at \( P = 0 \) passing through zero at the values of \( P \) that are not equal to each other and may be much smaller than \( \Lambda_{QCD} \).

4. Hypothetical domain wall fermions in QCD

In this section we consider the possibility that the phases of quark matter with different values of \( N_{3}^{CT} \) coexist (one of them may be metastable). Notice, that the consideration of this section has the general character and it does not refer to the quark matter only. One may, actually, consider the derivation of the present section for any field theoretical or condensed matter system with the topological invariant of the form of Eq. \((6)\) with appropriate matrix \( K \). The examples of such systems are given by the gapped phases of superfluid \( \nu He_{c} \) (see \([20]\) and by the topological insulators of various types \([36,37,38,39,40,41]\).

The boundary between the two phases is the domain wall. It should carry the massless fermions. Their number is equal to the jump of the topological invariant \( N_{CT} \) across the interface. This may be proved as follows (for the description of the method see also \([20]\) and \([52]\)).
Let us consider the case, when the domain wall forms the plane $yz$. Momenta $p^2, p^3$ along the plane remain the good quantum numbers, and they may be considered as parameters on which the Green function $\hat{G}$ depends. Thus in this case the Green function $\hat{G}$ represents the operator depending functionally on the momentum operator $\hat{p}^1$, and on the coordinate operator $r_1$ as well as on the parameters $p^2, p^3$, which represent conserved momenta. Below we imply $p^4 = 0$ and consider the following expression

$$ N_1(C) = \frac{1}{2\pi i} \int_{C} \text{Tr} K \hat{G}(p^2, p^3) d\hat{G}^{-1}(p^2, p^3) $$

(16)

Here $K = K_{CT}$, contour $C$ embraces zero in the plane $p^2, p^3$. This quantity counts the number of the zero modes incident at the domain wall. This may be demonstrated as follows. At $\omega = 0$ we denote

$$ \hat{G} = -\sum_n \mathcal{E}_n(p_2, p_3) |n, p_2, p_3\rangle \langle n, p_2, p_3| $$

where the sum is over the eigenstates of the Hamiltonian enumerated by index $n$ and the values of conserved momenta $p_2, p_3$. Let us denote $G_n = -\mathcal{E}_n(p_2, p_3)$. Matrix $K$ commutes with $\mathcal{G}$, therefore, they are diagonalized simultaneously, and the corresponding eigenvalues of $K$ are denoted by $K_n$. Then Eq. (16) may be represented as

$$ N_1(C) = \frac{1}{2\pi i} \int_{C} \sum_n K_n G_n dG_n^{-1} $$

(17)

This expression gives the number of the massless states protected by $CT$ symmetry. Only those states which are localized at the domain wall may have gapless excitations because in the bulk of the given system there are no poles of the Green function. Thus we come to the conclusion, that $N_1$ enumerates the gapless fermion modes.

We use the Wigner transform of the Green function in order to describe the correspondence between the bulk and the domain wall. Wigner transform $[58]$ of the Green function now depends on the parameters $p^2, p^3$

$$ \hat{G}(R, p, p^2, p^3) = \int d\epsilon e^{-ip\epsilon} G(R + r/2, R - r/2) $$

(18)

where $G(r_1, r_2)$ represents matrix elements of $\hat{G}(p^2, p^3)$. Here $R$ and $p$ correspond to the $x$ coordinate, that
is \( R = x, p = p_1 \). One may rewrite expression for \( N_1 \) as follows (for the details see [35])

\[
N_1(C) = \frac{1}{2\pi i} \int dR \frac{dp}{(2\pi)} \int_C \text{Tr} K \tilde{G}(R,p,p^2,p^3) d\left[\tilde{G}^{(0)}(R,p,p^2,p^3)\right]^{-1}
\]

(19)

Here we denote by \( \tilde{G}^{(0)}(R,p,p^2,p^3) \) the zero order approximation to \( \hat{G}(R,p,p^2,p^3) \) in the derivative expansion given by

\[
\tilde{G}^{(0)}(R,p,p^2,p^3) = \frac{1}{Q(R,p,p^2,p^3)}
\]

Here function \( Q \) represents the Weyl symbol of operator \( \hat{Q} \equiv \hat{G}^{-1} \) being the Wigner transform of its matrix elements [60, 61]. The mentioned derivative expansion gives the iterative solution for the Groenewold equation (we omit parameters \( p^2, p^3 \))

\[
1 = Q(R,p) \ast \tilde{G}(R,p)
\]

\[
≡ Q(R,p)e^{\frac{i}{2}(\tilde{G}_n \tilde{\sigma}_p - \tilde{\sigma}_p \tilde{G}_n)} \hat{G}(R,p)
\]

(20)

We use Eq. [12] and expand exponent in powers of its arguments, which gives

\[
\tilde{G}(R,p) = \tilde{G}^{(0)}(R,p) + \tilde{G}^{(1)}(R,p) + \tilde{G}^{(2)}(R,p) + ...
\]

(21)

\[
\tilde{G}^{(1)} = \frac{i}{2} \tilde{G}^{(0)} \left( \frac{\partial}{\partial p} \tilde{G}^{(0)} \right) \left( \frac{\partial}{\partial R} \tilde{G}^{(0)} \right)
\]

\[
- \frac{i}{2} \tilde{G}^{(0)} \left( \frac{\partial}{\partial R} \tilde{G}^{(0)} \right) \left( \frac{\partial}{\partial p} \tilde{G}^{(0)} \right)
\]

(22)

Next, let us substitute Eq. (43) into Eq. (41). Since Eq. (38) is the topological number, in the whole series of terms entering our expression only the topological term contributes. The only topological term we are able to find in this derivative expansion is the following one

\[
N_1(C) \equiv N_3 = \frac{1}{3! (2\pi)^2} \int K \tilde{G}^{(0)} d\left(\tilde{G}^{(0)}\right)^{-1} \wedge d\tilde{G}^{(0)} \wedge d\left(\tilde{G}^{(0)}\right)^{-1}
\]

(23)

Here the integral is over the surface \( C \otimes R \otimes R \), where the first \( R \) corresponds to the values of \( p \) while the second \( R \) corresponds to the values of \( x \).

Since there are no poles of \( \hat{G} \) inside the bulk we are able to deform this hyper - surface into the form of the closed surface, which embraces the line \( x = 0, p_1 = p_2 = p_3 = 0 \). Such a surface may be chosen in the form of the two planes placed at \( x = \pm \epsilon \), where \( \epsilon \) is small. This transformation is illustrated by Fig. 2. This gives the difference between the values of \( N_{CT} \) at the two sides of the domain wall.

5. Color Flavor Locking (CFL) phase

It is widely believed that the ground state of the quark - gluon matter at extremely large baryon chemical potential is the color - flavor locking phase (CFL). In the simplest phenomenological models of this phase the three quarks \( u, d, s \) are supposed to be massless. The condensate is formed [12, 13]

\[
\langle \psi^i \rangle^* \gamma^5 \gamma^\alpha \psi^j \sim \Phi^I_{\alpha \beta} \epsilon^{ijI} \sim \epsilon_{\alpha \beta I} \epsilon^{ijI}
\]

(24)

The condensate of Eq. [24] appears in [12] in the form of \( \langle \psi^i \rangle^* C \gamma^5 \psi^j \), for the definition of charge conjugation matrix \( C \) the authors of [12] refer to [68], where in Eqs. (1.27), (1.33) it is defined as \( C = -i\gamma^0 \gamma^2 \).

We are not aware of the exhaustive proof that this phase indeed appears at large baryon chemical potential. The most popular explanations of the appearance of the CFL phase at asymptotically large values of baryon chemical potential \( \mu_B \) are based on the consideration of the one - gluon exchange [67].

However, even at very large values of \( \mu_B \) the strong interaction might remain strong. Strictly speaking, we are not aware of neither the proof that at the large chemical potential the nonperturbative effects disappear,
nor of the proof that they remain. Therefore, we may rely on the known results at vanishing potential and large temperature. Naively, one would expect, that in the running coupling constant $\alpha$, the scale equals to temperature is to be substituted. If it is sufficiently large, then the coupling constant is sufficiently small. This indicates that the nonperturbative effects may disappear. However, numerical lattice simulations show that this is incorrect. Even at large temperatures the spacial string tension survives [63], which is a purely nonperturbative effect. At large chemical potential and vanishing temperature one should substitute $\mu_B$ to the running coupling constant $\alpha$, and naively come to the conclusion that the nonperturbative effects disappear. Although the QCD at zero temperature and large $\mu_B$ is understood much weaker, one might suppose, that here the spatial string tension survives as well. This means that the nonperturbative effects still may be important. Unfortunately, this region of the phase diagram is not accessible for the lattice numerical simulations, which complicates the situation. Nevertheless, following the authors of [12, 13, 14, 15, 16] we assume, that the CFL phase appears at least at the asymptotically large values of baryonic chemical potential.

There are 18 scalar fluctuations of $\Phi$ around this condensate (there are also 18 pseudo - scalar fluctuations with the same masses [13]). The symmetry breaking pattern is $SU(3)_L \otimes SU(3)_R \otimes SU(3)_C \otimes U(1)_V \rightarrow SU(3)_{CF}/Z_3$. That’s why there are $8 + 8 + 1$ massless Goldstone modes. Among the remaining $9 + 9 + 1$ Higgs modes there are two octets of the traceless modes and two singlet trace modes as well as one state that becomes massive due to the instantons. Correspondingly, the quark excitations also form singlets and octets. The singlet fermionic gap $\Delta_1$ is twice larger than the octet fermionic gap $\Delta_0$ (see Sect. 5.1.2. of [13]).

In terms of the left - handed $q_L$ and the right - handed $q_R$ components of quarks one may represent Eq. (24) using chiral representation of Dirac matrices as follows

$$\langle q^A_{L,\alpha} \epsilon_{AB} q^B_{L,\beta} \rangle = -\langle \bar{q}^A_{L,\alpha} \epsilon_{AB} q^B_{L,\beta} \rangle = \langle q^A_{R,\alpha} \epsilon_{AB} q^B_{R,\beta} \rangle = -\langle \bar{q}^A_{R,\alpha} \epsilon_{AB} q^B_{R,\beta} \rangle \sim \delta^I_{\alpha \beta} \epsilon_{\alpha \beta J} \epsilon^{ij}$$  (25)

Spinor notations and relative signs in those expressions correspond to those of Eq. (2.1) in [12], where $\epsilon_{ab} = \left( \begin{array} {cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ while $\epsilon_{\hat{a} \hat{b}} = \left( \begin{array} {cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ according to the notations of Eqs. (1.34), (1.35) in [68], to which the authors of [12] refer for the explanation of their own notations. It is worth mentioning, that

$$\bar{q}^B_{L,\beta} \epsilon_{BA} q^A_{L,\alpha} = -\bar{q}^B_{L,\beta} \epsilon_{BA} q^A_{L,\alpha}$$

(26)

Instead of $q_R, q_L$ one may use the CP - conjugated spinors $q^c_R = i\sigma^2 \bar{q}_R$, $q^c_L = -i\sigma^2 \bar{q}_L$. In terms of these spinors we get

$$- \langle \bar{q}^c_{L,\alpha} \epsilon_{AB} q^c_{L,\beta} \rangle = -\langle \bar{q}^c_{R,\alpha} \epsilon_{AB} q^c_{R,\beta} \rangle \sim \delta^I_{\alpha \beta} \epsilon_{\alpha \beta J} \epsilon^{ij}$$  (26)

We use the conventional definition of the corresponding Nambu - Gorkov spinors: $Q_L = (\bar{q}^c_{L,\alpha})^T = (\bar{q}^c_{R,\alpha})^T = (\bar{q}^c_{R,\alpha})^T T = (\bar{q}^c_T, i\sigma^2 \bar{q}_L)^T$, $Q_R = (\bar{q}^c_{R,\alpha})^T = (\bar{q}^c_{R,\alpha})^T T = (\bar{q}^c_T, i\sigma^2 \bar{q}_R)^T$. The condensate may be written as

$$- \langle \bar{Q}^{c,i}_{L,\alpha} Q^{c,j}_{L,\beta} \rangle = -\langle \bar{Q}^{c,i}_{R,\alpha} Q^{c,j}_{R,\beta} \rangle \sim \delta^I_{\alpha \beta} \epsilon_{\alpha \beta J} \epsilon^{ij}$$  (27)

where

$$\bar{Q}_L = Q_L^T i\gamma^2 \gamma^0, \quad \bar{Q}_R = Q_R^T i\gamma^2 \gamma^0$$  (28)

In the simplest Nambu - Jona - Lasinio model of CFL phase [13] the effective interaction 4 - fermion lagrangian has the form

$$L_{qq} = -\text{const} \sum_{A,A' = 2,5,7} \left( \bar{q}^i_{L,\alpha} \gamma^2 \gamma^0 \lambda^A_F \lambda^{A'}_C \bar{q}^j_{L,\beta} \right) \left( \bar{q}^i_{R,\alpha} \gamma^2 \gamma^0 \lambda^A_F \lambda^{A'}_C \bar{q}^j_{R,\beta} \right)$$

$$= \sum_{A,A' = 2,5,7} \left( \bar{q}^i_{L,\alpha} \gamma^2 \gamma^0 \lambda^A_F \lambda^{A'}_C \bar{q}^j_{L,\beta} \right) \left( \bar{q}^i_{R,\alpha} \gamma^2 \gamma^0 \lambda^A_F \lambda^{A'}_C \bar{q}^j_{R,\beta} \right) + \sum_{A,A' = 2,5,7} \left( \bar{q}^i_{L,\alpha} \gamma^2 \gamma^0 \lambda^{A'}_F \lambda^{A}_C \bar{q}^j_{L,\beta} \right) \left( \bar{q}^i_{R,\alpha} \gamma^2 \gamma^0 \lambda^{A'}_F \lambda^{A}_C \bar{q}^j_{R,\beta} \right)$$

(29)

where $\lambda^A_F$ and $\lambda^A_C$ are the generators of $SU(3)_C$ and $SU(3)_F$ respectively. Here $i\lambda^A_{ij} = \epsilon_{ij \beta}$, $i\hat{\lambda}^A_{ij} = \epsilon_{ij \beta}$, $i\hat{\lambda}^A_{ij}$ is the one more form of the CFL condensate:

$$-\langle \bar{q}^i_{L,\alpha} \gamma^2 \gamma^0 \lambda^A_F \lambda^{A'}_C \bar{q}^j_{L,\beta} \rangle + \langle \bar{q}^i_{R,\alpha} \gamma^2 \gamma^0 \lambda^{A'}_F \lambda^{A}_C \bar{q}^j_{R,\beta} \rangle \sim \delta^{AA'}$$

$A,A' = 2,5,7$
In the presence of this condensate the mean field lagrangian for the left-handed fermionic excitations may be written as follows:

\[ L_L = \frac{1}{2} \tilde{Q}_{\alpha}^L \left( (\gamma \mu p_\mu - \mu \gamma_5) \delta_{ij} \delta^{\alpha \beta} - \epsilon_{ij} \epsilon^{\alpha \beta} M \right) Q_{\beta}^L \]  

Here \( M \) is the parameter of the dimension of mass, that is proportional to the condensate (coefficient of proportionality is the constant entering the four fermion interaction term of Eq. (29)). \( \mu \) is the quark chemical potential. In the basis of the Nambu-Gorkov spinors the time reversal transformation \( q_{L,R}(t) \rightarrow i\sigma^2 \tilde{q}_{L,R}(-t) \) corresponds to matrix

\[ K_T = -\gamma^0 \gamma^5 \]  

In the following we will use the operator that differs from Eq. (31) by factor \( i \):

\[ K_T = iK_T = -i\gamma^0 \gamma^5 \]  

It also commutes with the Green function at \( \omega = 0 \) and therefore may be used to compose the topological invariants.

The similar situation takes place for the right-handed quarks. In terms of the Nambu-Gorkov spinors the mean field lagrangian for the massive right-handed quarks may be written as:

\[ L_R = \frac{1}{2} \tilde{Q}_{\alpha}^R \left( (\gamma \mu p_\mu + \mu \gamma_5) \delta_{ij} \delta^{\alpha \beta} - \epsilon_{ij} \epsilon^{\alpha \beta} M \right) Q_{\beta}^R \]  

Therefore, the matrix of time reversal transformation is given by

\[ K_T = i\gamma^0 \gamma^5 \gamma^5 \]  

where \( \gamma^5 = +1 \) for the right-handed spinors and \( \gamma^5 = -1 \) for the left-handed.

Adding the Dirac mass matrix \( m \) we arrive at the tree level lagrangian

\[ L = \frac{1}{2} \tilde{Q}_{\alpha} \left( (\gamma \mu p_\mu + \mu \gamma_5 T^0 \gamma_5 - m \gamma^5 T^0 \gamma^5) \delta_{ij} \delta^{\alpha \beta} - \epsilon_{ij} \epsilon^{\alpha \beta} M \right) Q_{\beta} \]  

If \( m \) would be the same for all fermions, the dispersion of quasiparticles described by this lagrangian is given by

\[ \mathcal{E} = \pm \sqrt{M^2 + (\mu \pm \sqrt{p^2 + m^2})^2}, \pm \sqrt{4M^2 + (\mu \pm \sqrt{p^2 + m^2})^2} \]  

One can see, that for \( M = 0, \mu < m \) the dispersion matches qualitatively the situation that takes place in hadronic phase, where the fermions are gapped, and the Dirac mass is present. For \( M \neq 0 \) the system is gapped both for \( \mu < m \) and \( m < \mu \). This case qualitatively corresponds to the CFL phase.

The lagrangian of Eq. (35) appears if we omit all interactions and take into account only the contribution of the quark-quark condensate to the Majorana mass term of quarks. The real effective lagrangian is much more complicated than Eq. (35), it contains extra terms due to interactions. However, we may assume, that the system may be deformed continuously to the one having the complete lagrangian of the form of Eq. (35).

We may consider in the CFL phase the topological invariant protected by \( T \) symmetry. (Notice, that in the basis of the considered here Nambu-Gorkov spinors the charge conjugation symmetry acts as follows: \( Q \rightarrow \gamma^5 T^0 Q \), the composition of \( C \) and \( T \) acts as \( Q \rightarrow K_{CT} Q = -i\gamma^0 T^0 \gamma^5 \) while the \( P \) transformation is given by matrix \( \gamma^0 T^0 \). Here \( \Gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).) Let us calculate the corresponding topological invariant

\[ N^T_3 = \frac{\epsilon_{ijk}}{48\pi^2} \text{tr} \left[ \int_{\omega=0} d^3 p \; K_T \; G\partial_{p_i} G^{-1} G\partial_{p_j} G^{-1} G\partial_{p_k} G^{-1} \right]. \]  

For the Green function of the form

\[ G^{-1} = \gamma^\mu p_\mu - m \gamma^5 T^0 \gamma^5 + \mu \gamma^0 \gamma^5 T^0 - \mathbf{E} M \]
with $\mathbf{E}^{\alpha \beta}_{ij} = \epsilon_{ij} \epsilon^{\alpha \beta}$ we easily obtain

$$\mathcal{N}_3^T = 0$$

One can also easily check that for the noninteracting fermions with Dirac mass the topological invariant protected by the time reversal symmetry vanishes as well. Further, in Sect. 8 we will introduce the topological invariant that takes nontrivial values in both cases.

It is worth mentioning, that the topological properties of the CFL phase have been considered in [57], where bulk topological invariant (Eqs. (15) and (16) of [57]) has been considered for the case when the system may be deformed continuously to the noninteracting one (that topological invariant, though, is not expressed directly through the Green function).

6. Relation between the fermion zero modes that reside on vortices and the topological invariant in mixed $r - p$ space

![Diagram showing the form of the surface and the tube](image)

Figure 3: Left: The form of the surface $C \otimes R^2 \otimes R^2$ in the coordinates $x, p_3, p_4$. Contour $C$ embraces zero in the plane $p_3, p_4$, while the first $R^2$ corresponds to the values of $x, y$ and the second $R^2$ corresponds to the values of $p_1, p_2$. Singularity of the Green function is situated at the vortex ($x = y = 0$) inside the tube. Right: The tube $C \otimes R^2 \otimes R^2$ may be rotated first such that the circle $C$ is transformed to the circle in the plane $x, p_4$. This is visualized here as the result of the rotation around the axis $p_4$.

To reveal the relation of momentum space topological invariants with the zero modes of the fermions that reside on vortices we develop the method of Chapter 23 of [20]. In this section we consider the system of general type, not necessarily the one of the QCD in the CFL phase. Let us consider the case, when the vortex forms the straight line directed along the $z$ axis. Momentum $p^3$ along the vortex remains the good quantum number, and it may be considered as parameter on which the Green function $\hat{G}$ depends. The same is valid in the static case for the imaginary frequency $p^4$. Thus in this case the Green function $\hat{G}$ represents the operator depending functionally on the momentum operators $\hat{p}^1, \hat{p}^2$, and on the coordinate operators $r_1, r_2$ as well as on the parameters $p^3, p^4$, which represent conserved momentum and imaginary frequency. We then consider the following expression

$$\mathcal{N}_1(C) = \frac{1}{2\pi i} \int_C \text{Tr} \hat{G}(p^3, p^4) d\hat{G}^{-1}(p^3, p^4)$$

(38)

Here contour $C$ embraces zero in the plane $p^3, p^4$. This quantity encodes the chirality of the zero modes incident at the vortex core. This may be demonstrated as follows. For simplicity we may consider here the case of the noninteracting system. The generalization to the interacting system is straightforward. Now we have $\hat{G} = i\omega - \hat{H}$, where $\omega = p^4$ is imaginary frequency, and

$$\hat{G} = i\omega - \sum_n \mathbf{c}_n(p_3) \langle n, p_3 \rangle \langle n, p_3 \rangle$$
where the sum is over the eigenstates of the Hamiltonian enumerated by index \( n \) and the values of conserved momentum \( p_3 \). Let us denote \( G_n = i\omega - \xi_n(p_3) \). Then Eq. (38) may be represented as

\[
\mathcal{N}_1(C) = \frac{1}{2\pi i} \sum_n \int_C G_n dG_n^{-1}
\]

(39)

This expression gives the sum of the chiralities of the states enumerated by \( n \). Only those states which are localized in the vortex core may have gapless excitations because in the bulk of the given system there are no poles of the Green function. Thus we come to the conclusion, that \( \mathcal{N}_1 \) enumerates the gapless fermion modes residing at the vortex core, and its sign corresponds to their chirality.

Let us use the Wigner transform of the Green function in order to describe the correspondence between the bulk and the vortex sheet (for the description of the method see Chapter 23 in [20]). For the description of the properties of the Wigner transformation of Green functions see [34, 55] and references therein. Wigner transform [58] of the Green function is defined as

\[
\tilde{G}(\mathbf{r}, \mathbf{p}, p^3, p^4) = \int d^D r e^{-i p r} G(\mathbf{R} + r/2, \mathbf{R} - r/2)
\]

(40)

where \( G(\mathbf{r}_1, \mathbf{r}_2) \) represents matrix elements of \( \hat{G}(p^3, p^4) \). Here \( \mathbf{R} \) and \( \mathbf{p} \) are two dimensional and correspond to \( x, y \) coordinates, that is \( \mathbf{R} = (x, y) \) and \( \mathbf{p} = (p_1, p_2) \). One may rewrite expression for \( \mathcal{N}_1 \) as follows

\[
\mathcal{N}_1(C) = \frac{1}{2\pi i} \int d^2 \mathbf{R} \frac{d^2 \mathbf{p}}{(2\pi)^2} \int_C \text{Tr} \tilde{G}(\mathbf{R}, \mathbf{p}, p^3, p^4) d\left[\tilde{G}^{(0)}(\mathbf{R}, \mathbf{p}, p^3, p^4)\right]^{-1}
\]

(41)

Here we denote by \( \tilde{G}^{(0)}(\mathbf{R}, \mathbf{p}, p^3, p^4) \) the zero order approximation to \( \tilde{G}(\mathbf{R}, \mathbf{p}, p^3, p^4) \) in the derivative expansion. The latter expansion gives the iterative solution for the Groenewold equation (we omit parameters \( p^4, p^3 \))

\[
1 = \mathcal{Q}(\mathbf{R}, \mathbf{p}) \ast \tilde{G}(\mathbf{R}, \mathbf{p})
= \mathcal{Q}(\mathbf{R}, \mathbf{p}) e^{\frac{i}{2} \left[ \mathcal{G}_R(p_1,p_2) - \mathcal{G}_p(p_1,p_2) \right]} \tilde{G}(\mathbf{R}, \mathbf{p})
\]

(42)

Here function \( \mathcal{Q} \) represents the so-called Weyl symbol of operator \( \mathcal{G} = \mathcal{G}^{-1} \) being the Wigner transform of its matrix elements [60, 61]. We use Eq. (42) and expand exponent in powers of its arguments, which gives

\[
\tilde{G}(\mathbf{R}, \mathbf{p}) = \tilde{G}^{(0)}(\mathbf{R}, \mathbf{p}) + \tilde{G}^{(1)}(\mathbf{R}, \mathbf{p}) + \tilde{G}^{(2)}(\mathbf{R}, \mathbf{p}) + ...
\]

(43)

\[
\tilde{G}^{(1)} = \frac{i}{2} \tilde{G}^{(0)} \frac{\partial}{\partial \mathbf{p}} \tilde{G}^{(0)} - \frac{\partial}{\partial \mathbf{R}} \tilde{G}^{(0)}
\]

\[
\tilde{G}^{(2)} = \frac{i}{2} \tilde{G}^{(0)} \frac{\partial}{\partial \mathbf{p}} \tilde{G}^{(1)} - \frac{\partial}{\partial \mathbf{R}} \tilde{G}^{(0)}
\]

(44)
Here $\tilde{G}^{(0)}(\mathbf{R}, \mathbf{p})$ is given by the solution of equation

\[
\tilde{G}^{(0)}(\mathbf{R}, \mathbf{p}) \mathcal{Q}(\mathbf{R}, \mathbf{p}) = 1
\]  

(45)

Next, let us substitute Eq. (43) into Eq. (41). Since Eq. (38) is the topological number, in the whole series of terms entering our expression only the topological term contributes. The only topological term we are able to find in our expression is the following one

\[
\mathcal{N}_1(\mathcal{C}) \equiv \mathcal{N}_5 = \frac{2}{5!} \frac{4\pi}{3} \int \text{Tr} \left[ \tilde{G}^{(0)}(\mathbf{R}, \mathbf{p}) \tilde{G}(\mathbf{R}, \mathbf{p}) \right] - 1 \wedge d\tilde{G}(\mathbf{R}, \mathbf{p}) \wedge d\tilde{G}(\mathbf{R}, \mathbf{p}) - 1 \wedge d\tilde{G}(\mathbf{R}, \mathbf{p}) \wedge d\tilde{G}(\mathbf{R}, \mathbf{p}) - 1 \wedge d\tilde{G}(\mathbf{R}, \mathbf{p}) - 1 \right]
\]  

(46)

Here the integral is over the surface $\mathcal{C} \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$, where the first $\mathbb{R}^2$ corresponds to the values of $p_1, p_2$ while the second $\mathbb{R}^2$ corresponds to the values of $x, y$. The topological invariant of the form of Eq. (46) has been discussed earlier for the particular case of vortices in $^3$He-B in [54], where instead of our well defined (through the Wigner transformation) $G^{(0)}(\mathbf{R}, \mathbf{p})$ the Green function depending both on coordinates and on momenta was introduced intuitively. Actually our construction may be considered as the precise formulation of the one discussed in [54].

Since there are no poles of $\tilde{G}$ inside the bulk we are able to deform this hyper-surface into the form of the closed surface, which embraces the line $x = y = 0$, $\mathbf{p} = 0$. Such a surface may be chosen in the form of the product $S^1 \otimes \mathbb{R}^1 \otimes \mathbb{R}^3$, where circle $S^1$ is in the $xy$ plane with the center at $x = y = 0$, $\mathbb{R}^1$ is the line of imaginary frequency while $\mathbb{R}^3$ is the three-dimensional momentum space $(p_1, p_2, p_3)$. Geometrically this transformation of the surface looks like the rotation of the tube $\mathcal{C} \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ in such a way, that the circle $\mathcal{C}$ in the plane $p_3, p_4$ is transformed to the circle in the plane $x, y$. This rotation may be visualized as a sequence of two steps. First, the tube is rotated in the plane $(x, p_3)$. This step is illustrated by Fig. 3. As a result the circle $\mathcal{C}$ is rotated to the plane $(p_4, x)$. Next, the tube is rotated in the plane $(y, p_4)$. As a result $\mathcal{C}$ belongs to the plane $(x, y)$ - see Fig. 4. Both rotations are performed in such a way, that the singularity of the Green function placed at the vortex $x = y = 0$ remains inside the tube.

Thus we came to conclusion, that the chirality of the fermion zero modes residing on the vortex (given by Eq. (38)) is equal to the topological invariant in mixed $r-p$ space of Eq. (46), where the integration surface is $\mathcal{C} \otimes \mathbb{R}^4$, where $\mathcal{C}$ is the circle in the plane $x, y$ that embraces the position of the vortex, while $\mathbb{R}^4$ is the 4-dimensional momentum space.

![Figure 4](image_url)

Figure 4: The final form of the integration surface for Eq. (46): $\mathcal{C} \otimes \mathbb{R}^4$, where $\mathcal{C}$ is the circle in the plane $x, y$ that embraces the position of the vortex. $\mathbb{R}^4$ is the 4-dimensional momentum space.

7. Fermion zero modes on the vortices in the CFL phase

In the CFL phase the symmetry breaking pattern is $SU(3)_L \otimes SU(3)_R \otimes SU(3)_C \otimes U(1)_V \rightarrow SU(3)_{CF} / \mathbb{Z}_3$. The first homotopic group of the factor space $\mathcal{H} = SU(3)_L \otimes SU(3)_R \otimes U(1)_V \otimes \mathbb{Z}_3$ is nontrivial due to
the $U(1)$ constituent. This allows the existence of topologically stable vortices. Also there exist the ones, which are topologically stable only in the limit, when the contribution of instantons is neglected. The latter is believed to be suppressed at high densities [5] as we mentioned above. These vortexes correspond to the approximate $U(1)_A$. Below we consider first of all these approximately stable vortices. Let us denote

$$
\Phi^I_{L,J} = \epsilon_{\alpha\beta I} e^{ijI} (q_{L,\alpha}^A \epsilon_{ABq_{L,\beta}^B}), \quad \Phi^I_{R,J} = -\epsilon_{\alpha\beta I} e^{ijI} (q_{R,\alpha}^A \epsilon_{ABq_{R,\beta}^B})
$$

(47)

The simplest configuration of the $U(1)_B$ vortex corresponds to [10]

$$
\Phi_L(x, y, z) = -\Phi_R(x, y, z) = e^{i\theta n} f(\rho) M
$$

where $\theta$ and $\rho$ are the polar coordinates in the $xy$ plane. Function $f(\rho)$ tends to unity at $\rho \to \infty$. Similarly the simplest configuration of the $U(1)_A$ vortex is [16]

$$
\Phi_L(x, y, z) = -\Phi_R(x, y, z) = e^{i\theta n} f(\rho) M
$$

(48)

We suppose, that the system in the presence of the order parameter field may be continuously deformed to the one with the tree level propagator (since we are considering the CFL phase, where $\alpha_S(\mu_B)$ is essentially smaller, than the typical values of $\alpha_S$ in Hadronic phase, it is generally implied, that this continuous deformation may indeed be performed). Then the mass term for the quarks in the presence of the condensate of the form of Eq. [48] is

$$
\mathcal{M} = e^{-i\theta n} f(\rho) M \epsilon_{\alpha\beta I} e^{ijI} q_{L,\alpha}^A \epsilon_{ABq_{L,\beta}^B} - e^{-i\theta n} f(\rho) M \epsilon_{\alpha\beta I} e^{ijI} q_{R,\alpha}^A \epsilon_{ABq_{R,\beta}^B} + e^{i\theta n} f(\rho) M \epsilon_{\alpha\beta I} e^{ijI} q_{L,\alpha}^A \epsilon_{ABq_{L,\beta}^B} - e^{-i\theta n} f(\rho) M \epsilon_{\alpha\beta I} e^{ijI} q_{R,\alpha}^A \epsilon_{ABq_{R,\beta}^B} = -f(\rho) M \epsilon_{\alpha\beta I} e^{ijI} q_{L,\alpha}^A \epsilon_{ABq_{L,\beta}^B} - f(\rho) M \epsilon_{\alpha\beta I} e^{ijI} q_{R,\alpha}^A \epsilon_{ABq_{R,\beta}^B}
$$

(49)

Here $\gamma^5 = \text{diag}(1, 1, -1, -1)$. Then the Green function in the background of the Abelian $U(1)_A$ vortex is

$$
\left[\tilde{G}^{(0)}\right]^{-1} = \gamma^\mu p_\mu - \mu \gamma^\mu \gamma^5 - E e^{i\gamma^\mu f(\rho) M} = e^{i\theta_1/2} \left(\gamma^\mu p_\mu - \mu \gamma^\mu \gamma^5 - E f(\rho) M\right) e^{i\theta_1/2}
$$

(50)

Here

$$
E_{\alpha\beta} = \epsilon_{ijI} e^{ijI}
$$

We substitute this expression to Eq. [46] and denote $M = f(\rho) M$. (Notice that instead of $p_0$ we should substitute into the Green function $ip_4$, where $p_4$ is the imaginary frequency.) Recall, that the energy gap does not disappear if we smoothly turn off the chemical potential. Therefore, such a deformation of theory does not lead to phase transition. During the decreasing of $\mu$ the value of any topological invariant is not changed, and as in Sect. [5] we may simply set $\mu = 0$ in order to calculate $N_1(C)$ of Eq. [46]. The integral over $\theta$ gives the factor $2\pi n$ while the integral over the imaginary frequency gives

$$
2 \int_0^{\infty} \frac{d\omega}{(\omega^2 + \omega^2 + E^2 M^2)^3} = \frac{3}{8\pi} \frac{E^2 M^2}{(p^2 + E^2 M^2)^{3/2}}
$$

We come to

$$
N_1(C) = \frac{2 \times 5 \times 4 \times 2\pi n}{5! (2\pi)^3} \int \text{Tr} \left[ E i \gamma^0 \gamma^5 \tilde{G}_0^{(0)} d(\tilde{G}_0^{(0)})^{-1} \wedge d\tilde{G}_0^{(0)} \wedge d(\tilde{G}_0^{(0)})^{-1} \right] \times \frac{\tilde{M}}{(p^2 + E^2 M^2)^{1/2}} \times \frac{3}{8\pi}
$$

(51)

Here the integral is over 3D momentum space while

$$
\tilde{G}_0^{(0)} = \tilde{G}_0^{(0)}\bigg|_{\theta=0} = \gamma^\mu p_\mu - E f(\rho) M
$$

(52)

For the considered case of the noninteracting fermions with the propagator $\tilde{G}_0^{(0)}$ we have

$$
N_1(C) = 2n N f N_c
$$

(53)
One can see, that the Abelian vortex with the topological number \( n \) carries \( 2nN_fN_c \) chiral fermions (this answer does not depend on the signs of the Majorana masses of the bulk fermions). Those fermions are Majorana because each of them corresponds to the single branch of spectrum. That means that each such fermion in the 1 + 1 D space of vortex sheet contains only one independent component and is described by one rather than two independent pair of Grassmann variables.

One may also consider the following non - Abelian vortices (it is similar to M1 in the terminology of [16]) with the order parameter field of the form

\[
\Phi_L^+(x, y, z) = -\Phi_R(x, y, z) = e^{i\theta n/3}e^{-i\lambda^8 \theta n} f(\rho) M
\]

where

\[
\lambda^8 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\]

(In [16] the M1 vortex corresponds to \( \Phi_L(x, y, z) = -\Phi_R(x, y, z) = e^{i\theta n/3}e^{-i\lambda^8 \theta n} f(\rho) M \).) Now

\[
\tilde{G}^{-1} = e^{i\theta \gamma^5 n(1/6 + \lambda^8_0/2 + \lambda^8_2)/2} \left( \gamma^\mu p_\mu - \mu \gamma^0 \gamma^5 \Gamma^5 - \mathbf{E} f(\rho) M \right) e^{i\theta \gamma^5 n(1/6 + \lambda^8_0/2 + \lambda^8_2)/2}
\]

In this situation the terms proportional to \( \lambda^8 \) vanish in the expression for the topological invariant and when all masses are positive, we come to

\[
\mathcal{N}_1(C) = 2nN_f
\]

which means, that the number of the corresponding zero modes is equal to \( 2nN_f \).

Notice, that the fermion zero modes obtained here for the mentioned two types of vortices were discussed earlier (see, for example, [16]) using the direct solutions of the wave equations for the fermions in the presence of the particular vortex configurations.

In the same way one may consider the topologically stable \( U(1)_B \) vortices. Then the topological invariant of Eq. (46) already does not protect the number of the massless fermions localized at these vortices (its expression is equal to zero). However, one may consider the invariant protected by chiral \( Z_2 \) symmetry

\[
\mathcal{N}'_5 = \frac{2}{5!(2\pi)^3} \int \text{Tr} \left[ \Gamma^5 \tilde{G}^{(0)} d(\tilde{G}^{(0)})^{-1} \wedge d\tilde{G}^{(0)} \wedge d(\tilde{G}^{(0)})^{-1} \wedge d(\tilde{G}^{(0)})^{-1} \right]
\]

This expression still may be considered as the topological invariant in the approximation when instantons are neglected. It will protect the number of the massless fermions localized on such vortices just like the considered above Eq. (46).

Thus we come to the conclusion that the approximately stable \( U(1)_A \) vortices (and their non - Abelian analogues) carry the exactly massless Majorana fermions, the number of which is given by the topological invariant of Eq. (46). At the same time the exactly stable \( U(1)_V \) vortices (and their non - Abelian analogues) carry the approximately massless Majorana fermions, the number of which is given by the topological invariant of Eq. (57). In the other words, these latter fermions localized on the vortex sheets have masses that appear only thanks to the instantons. Those masses are supposed to be suppressed as well as the other quantities existing in the CFL phase due to the instantons [16, 64].

8. The common topological invariant for the gapped phases

The expression obtained above for the case of the fermions at the background of a vortex Eq. (46) does not contain the symmetry matrix specific for the given phase. This prompts that there exists the formulation of the topological invariant in momentum space of the gapped phases, which has the common form for all such phases. We already encountered such a situation in [31] where the two topological invariants were discussed: \( \mathcal{N}_5 \) and \( \mathcal{N}_5' \). The \( \mathcal{N}_5 \) is the analogue of the symmetry protected topological invariants discussed above while the expression for \( \mathcal{N}_5' \) is similar to that of Eq. (46). Here we propose the construction similar to that of [31], which allows to construct the topological invariants of the same form for all gapped phases of
We substitute this expression to Eq. (59). Instead of $p$, where $p$ is the imaginary frequency. The dispersion of quasiparticles is given by

$$\varepsilon = \pm \sqrt{M^2 + \left(\mu \pm \sqrt{p^2 + m^2}\right)^2}$$

Let us suppose first, that the Majorana mass $M$ remains nonzero. Again, the energy gap does not disappear if we smoothly turn off the chemical potential. It also does not disappear if we turn off Dirac mass $m$. During the decreasing of $\mu$ and $m$ the value of Eq. (59) is not changed, and we are able to set $\mu = m = 0$ in order to calculate it. The integral over $\theta$ gives the factor $2\pi$ while the integral over the imaginary frequency gives

$$2 \int_{0}^{\infty} \frac{d\omega}{(\omega^2 + M^2)^{3/2}} = \frac{3}{8\pi} \frac{M^2}{(p^2 + M^2)^{3/2}}$$

We obtain

$$N_{5}^{\gamma_{5}} = -\frac{2 \times 5 \times 4 \times 2\pi}{5! (2\pi)^3} \int \text{Tr} \left[ i\gamma^0 \gamma^5 G_0 dG_0^{-1} \wedge dG_0 \wedge dG_0^{-1} \right] \times \frac{M}{(p^2 + M^2)^{1/2}} \times \frac{3}{8\pi}$$

Here the integral is over 3D momentum space while

$$G_0 = \gamma^\mu p_\mu - M$$

We have

$$N_{5}^{\gamma_{5}} = \frac{1}{3! (2\pi)^2} \int \text{Tr} \left[ K_T \Gamma^5 G_0 d(G_0)^{-1} \wedge dG_0 \wedge d(G_0)^{-1} \right] \times \zeta$$

where

$$\zeta = \frac{\int p^2 dp \int \frac{M^2}{p^2 + M^2} \times \frac{3}{4\pi} \times \frac{M}{p^2 + M^2}}{\int p^2 dp} = \text{sign} \; M$$

This gives

$$N_{5}^{\gamma_{5}} = 1$$
By construction Eq. (59) is well defined in any gapped phase of the theory. Let us calculate it for the one non-interacting fermion with positive Dirac mass. We consider the Green function of the form of Eq. (61) with $M = \mu = 0$. As a result we come to the following expression for the topological invariant:

$$\mathcal{N}_5^{\gamma} = \frac{2 \times 5 \times 4 \times 2\pi}{5! (2\pi)^3} \int \text{Tr} \left[ \hat{G}_0(-i\gamma^0\gamma^5\Gamma^3)d\hat{G}_0^{-1} \wedge d\hat{G}_0 \wedge d\hat{G}_0^{-1} \right] \times \frac{m}{(p^2 + m^2)^{1/2}} \times \frac{3}{8\pi} \quad (66)$$

Here the integral is over 3D momentum space while

$$\hat{G}_0 = \gamma^\mu p_\mu - m\gamma^5\Gamma^0\Gamma^5$$

Thus we obtain $\mathcal{N}_5^{\gamma} = 1$ for one massive Dirac fermion. In the case of several massive noninteracting Dirac fermions with positive masses we get

$$\mathcal{N}_5^{\gamma} = \mathcal{N}_3^{CT} \quad (68)$$

Thus the topological invariant $\mathcal{N}_5^{\gamma}$ defined above has the same value in the CFL phase and in the domains of the HP or QGP, where the theory is connected continuously with the system of non-interacting $u, d, s$ fermions with Dirac masses. This property may be illustrated by the consideration of the interphase between those two phases of quark matter. Let us consider the dispersion of the fermions corresponding to the simplified Green function of the form

$$G^{-1} = \gamma^\mu p_\mu - m\gamma^5\Gamma^0\Gamma^5 + \mu\gamma^0\gamma^5\Gamma^5 - EM \quad (69)$$

It is given by

$$\mathcal{E} = \pm \sqrt{M^2 + (\mu \pm \sqrt{p^2 + m^2})^2}, \pm \sqrt{4M^2 + (\mu \pm \sqrt{p^2 + m^2})^2} \quad (70)$$

This dispersion describes the case when the phase of massive Dirac fermions may coexist with the CFL phase. If $M \neq 0$ we are inside the CFL phase. Let us consider first the case, when $\mu > m$ inside this phase. Then if $M$ is decreased and reaches the value $M = 0$, the system drops to the phase with the Fermi surface. The further decreasing of $\mu$ brings us to the gapped phase with the Dirac masses at $\mu < m$. In this pattern we pass through the gapless phase while going between the two different gapped phases. There is another choice of parameters, when from the very beginning in the CFL phase $\mu < m$. Then it does not follow from Eq. (70) that at the interphase between the two gapped phases the massless fermions appear. When $M$ is decreased and reaches the value $M = 0$ the CFL phase is transformed smoothly to the gapped phase with Dirac masses of fermions. There will be no such possibility if the values of $\mathcal{N}_5^{\gamma}$ would be different in the mentioned two gapped phases.

9. Conclusions

In this paper we discussed the topological invariants in momentum space in quantum chromodynamics with three flavors of quarks and their relevance for the description of physical phenomena. Below we summarize our results, enumerate the considered topological invariants and mention their relation to the potentially observed phenomena.

1. $\mathcal{N}_3^{2Z}$ is defined in the quark-gluon plasma phase in the approximation of vanishing current $u$ and $d$ quark masses. It is given by Eq. (2). For the definition of this invariant we need the continuation of the Green function from the discrete values of imaginary frequencies to the continuous values. This invariant protects the pole of this analytically continued fermion propagator corresponding to $u$ and $d$ quarks. If this invariant is nonzero, such poles (and the thermal masses of $u$ and $d$) do not appear. The HTL (hard thermal loop) approximation predicts the appearance of the thermal masses of quarks, at least, at asymptotically large temperature. Therefore $\mathcal{N}_3^{2Z} = 0$ in the corresponding part of the phase diagram. On the other hand, when the interactions are turned off, the value of $\mathcal{N}_3^{2Z}$ is nonzero. That means that the topological phase transition should be present somewhere in the extended phase diagram of QCD, which includes the value of $\alpha_s$ (on the certain particular scale) as an extra axis.
2. $N^{CT}_3$ is defined in any gapped phase of QCD at vanishing temperature. It is given by Eq. (59). $N^{CT}_3$ is defined in hadronic phase at zero temperature. It is given by Eq. (6). If the system may be reduced by continuous deformation to the noninteracting fermion system with positive masses (which is the most reasonable possibility for the hadronic phase), then $N^{CT}_3 = N^{\gamma}_5$.

In general case the phase diagram of QCD may contain the axes: temperature, baryonic chemical potential, external magnetic field, external electric potential, angular velocity of rotation, the isotopic and strangeness chemical potentials, etc. As it was mentioned above, we may also add an axis corresponding to the value of $\alpha_s$ at the certain particular scale. Thus we describe quark matter that may be hot, dense, may experience the external magnetic field and external electric potential, may be rotated, etc. In spite of the efforts made during the last years in order to investigate the phase structure of QCD [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], the behavior of quark matter in many regions of the mentioned above extended phase diagram is not known.

We point out the hypothetical possibility that somewhere in this phase diagram the topological phase transitions may occur between the sub-phases that differ by the values of the topological invariant $N^{CT}_3$. This may occur due to the strong nature of $SU(3)$ interaction between quarks in these phases. The detailed investigation of this possibility is out of the scope of the present paper and it will most likely require the extensive lattice numerical simulations.

3. In the hypothetical CFL phase at zero temperature $N^{CT}_3 = N_c N_f = 9$ if the system may be reduced by continuous deformation to the noninteracting fermion system with Majorana masses (which is almost the only real possibility for the CFL phase).

According to [10] vortices may appear in rotated quark matter being in the CFL phase. In particular, this may occur in the cores of the dense stars. The number of chiral fermions on vortices in the CFL phase is described by the specific topological invariant existing in the mixed $r-p$ space, which is proportional to the product of the vortex topological number $n$ and the value of $N^{CT}_3$. For the considered non-Abelian vortexes the number of fermion zero modes is $N_c = 3$ times smaller than for the Abelian vortexes.

Massless fermions always invent the important new physics to the description of the real processes. Thus we expect, that the investigation of those objects may become important for the description of quark matter at extreme conditions. It is worth mentioning, that the topological properties of the other color superconductor phases (like 2SC phase, for example) are in many aspects similar to the considered here CFL phase and may be considered on the same grounds.

In each of the considered above cases the smooth deformation of the system does not cause the change of the values of the topological invariants unless a phase transition is encountered. This provides the stability of vacua under small perturbations.

To conclude, momentum space topology being applied to QCD allows to investigate the stability of vacuum, it helps in the systematization of the potentially existing phases, and allows to describe successfully the fermion zero modes existing at the topological defects (vortices, etc) and at the surfaces that separate various phases of quark matter.

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