Robust Matrix Completion with Heavy-tailed Noise

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Abstract

This paper studies low-rank matrix completion in the presence of heavy-tailed and possibly asymmetric noise, where we aim to estimate an underlying low-rank matrix given a set of highly incomplete noisy entries. Though the matrix completion problem has attracted much attention in the past decade, there is still lack of theoretical understanding when the observations are contaminated by heavy-tailed noises. Prior theory falls short of explaining the empirical results and is unable to capture the optimal dependence of the estimation error on the noise level. In this paper, we adopt an adaptive Huber loss to accommodate heavy-tailed noise, which is robust against large and possibly asymmetric errors when the parameter in the loss function is carefully designed to balance the Huberization biases and robustness to outliers. Then, we propose an efficient nonconvex algorithm via a balanced low-rank Burer-Monteiro matrix factorization and gradient decent with robust spectral initialization. We prove that under merely bounded second moment condition on the error distributions, rather than the sub-Gaussian assumption, the Euclidean error of the iterates generated by the proposed algorithm decrease geometrically fast until achieving a minimax-optimal statistical estimation error, which has the same order as that in the sub-Gaussian case. The key technique behind this significant advancement is a powerful leave-one-out analysis framework. The theoretical results are corroborated by our simulation studies.

Keywords: Huber loss, nonconvex optimization, gradient descent, leave-one-out analysis

Contents

1 Introduction 2
  1.1 Comparison with prior theory .................................................. 3
  1.2 Paper organization and notation .............................................. 5

2 Robust matrix completion and main results 5
  2.1 Model and algorithm .......................................................... 5
  2.2 Theoretical guarantees ......................................................... 6

3 Prior arts 10

4 Numerical experiments 11

5 Proof sketch 14
  5.1 Local geometry .................................................................... 14
  5.2 Leave-one-out sequences ........................................................ 15
  5.3 Spectral Initialization ............................................................. 18

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1 Introduction

In a diverse array of real-world applications such as collaborative filtering (Rao et al., 2015), quantum-state tomography (Gross, 2011), spectrum sensing (Corroy et al., 2011) and recommender system (Ramlatchan et al., 2018), we are interested in recovering a large-scale low-rank data matrix from noisy and highly incomplete observations. This problem, usually termed as matrix completion, has attracted a lot of attention over a decade (Candes and Plan, 2010; Keshavan et al., 2010; Candes and Plan, 2011; Ma et al., 2017; Chi et al., 2019; Chen et al., 2020a,b).

Suppose the matrix of interest is $M^\star = \{M^\star_{i,j}\} \in \mathbb{R}^{n \times n}$ of rank $r$ \footnote{Here we assume $M^\star$ is a square matrix for simplicity of presentation. It is straightforward to extend our results to the case of a rectangular matrix $M^\star \in \mathbb{R}^{n_1 \times n_2}$.}, and we can observe a subset of noisy entries

$$M_{i,j} = M^\star_{i,j} + \varepsilon_{i,j}, \quad (i,j) \in \Omega,$$

where $\varepsilon_{i,j}$ denotes the additive noise at index $(i,j)$, and $\Omega \subseteq \{1, \cdots, n\} \times \{1, \cdots, n\}$ represents sampling set. A variety of algorithms has been proposed for estimating $M^\star$, among which two paradigms have received much attention: convex relaxation and nonconvex optimization. Both of them minimize a form of regularized loss function (Candes and Plan, 2010; Keshavan et al., 2010; Candes and Plan, 2011; Ma et al., 2017; Chi et al., 2019; Chen et al., 2020a,b). To enforce low-rank structure, the convex relaxation approach usually adds a penalty term to the loss function and then implements well-developed convex programming algorithms to obtain the estimates. There is vast literature regarding this and Section 3 provides a more detailed coverage on this part. However, a major drawback of convex approach is that it suffers from high computational costs. To remedy this issue, one turns to the nonconvex approach
with a good initialization, which often enjoys better computational performance and thus can be applied to data of larger scale. This will also be the focus of this paper.

For both convex and nonconvex methods, the vast majority of prior literature relies heavily on the sub-Gaussian assumption of the noise (Candes and Plan, 2010; Negahban and Wainwright, 2012; Klopp, 2014; Chen and Wainwright, 2015; Ma et al., 2017; Chen et al., 2020b). Under this assumption, regularized least-squares methods have been proposed for nonconvex regularization and widely studied over the past decade, which usually takes the form of

$$\min_{X,Y} \frac{1}{2p} \sum_{(i,j) \in \Omega} \left( (XY^\top)_{i,j} - M_{i,j} \right)^2 + \text{regularization terms,}$$

(1.1)

where the factorization $XY^\top$ is used to model rank-$r$ matrix $M^*$. Theoretical studies are often conducted under the sub-Gaussian assumption, which can easily fail in many modern applications. In fact, heavy-tailed data is ubiquitous and can be encountered in various domains such as functional magnetic resonance imaging (Eklund et al., 2016), financial markets (Cont, 2001), gene microarray analysis (Wang et al., 2015), to name just a few. See also Fan et al. (2021) for additional examples where they also argue that by chance alone, some noises will have heavy tails in high dimensions. Therefore, it is compelling to address the robustness issue in the matrix completion.

In this paper, we focus on recovering a low-rank matrix from its highly incomplete subset of entries contaminated by heavy-tailed noise. To accommodate the new challenges here, we cannot stick to the least-squares formulation (1.1), since it is well-known that square loss can be vulnerable when dealing with heavy-tailed noise (Huber, 1973; Catoni, 2012). To address this issue, a natural solution is to resort to more robust loss functions such as $\ell_1$-loss (Bassett Jr and Koenker, 1978; Huber, 2004), Huber loss (Huber, 1973) and quantile loss (Koenker and Hallock, 2001). As in Fan et al. (2017) and Sun et al. (2020), we allow the distributions of $\varepsilon_{i,j}$ to be asymmetric so that $\tau$ should diverge appropriately in order to control the bias due to Huberization. For this reason, the loss function is also referred to as the adaptive Huber loss, which will be our main focus. We shall adopt the following nonconvex minimization problem

$$\min_{X,Y} \frac{1}{2p} \sum_{(i,j) \in \Omega} \rho_\tau \left( (XY^\top)_{i,j} - M_{i,j} \right) + \frac{1}{8} \left\| X^\top X - Y^\top Y \right\|_F^2,$$

(1.2)

where $\rho_\tau(\cdot)$ is the Huber loss function which will be defined formally later and $\tau$ diverges at an appropriate rate. Here, the penalty term is intended to control the balance between two low-rank factors $X$ and $Y$ which is crucial to the establishments of our theoretical guarantees as we shall present later. Due to the fact that (1.2) is a highly nonconvex function, the objective function has numerous local minima which prevents us from solving it easily by applying some standard algorithm. This calls for a carefully designed algorithm with provable performance guarantees. The constant $1/8$ is sufficient to guarantee the locally strong convexity of the objective function (1.2) around the ground truth (Zheng and Lafferty, 2016; Tu et al., 2016).

1.1 Comparison with prior theory

Inadequacy of prior works. Among prior literature, matrix completion with heavy-tailed noise has been studied by a series of papers. Elsener and van de Geer (2018) assumes a constant lower bound of the density function and adopts Huber loss with nuclear norm penalty. Minsker (2018) deals with the case that the noise has finite second moment and preprocesses the data before feeding into the nuclear norm penalized square loss function. Fan et al. (2021) truncates the data before
Table 1: Comparison of our theoretical guarantees to prior theory, where we hide all logarithmic factors. Here, the Euclidean estimation error refers to $\|XY^\top - M^*\|_F$.

| Algorithm | Euclidean estimation error |
|-----------|---------------------------|
| Minsker (2018) | $\sqrt{\left(\sigma^2 + \|M^*\|_2^2\right)\frac{r_n}{p}}$ |
| Elsener and van de Geer (2018) | $\sqrt{\left(\tau^2 + \|M^*\|_\infty^2\right)\frac{r_n}{p}}$ |
| Fan et al. (2021) | $\sqrt{\left(\mu^2 r + \sigma^2\right)\frac{r_n}{p}}$ |
| Shen et al. (2022) | $\max\left\{\tau + E|\varepsilon|, 1 + E|\varepsilon|\tau^{-1}\right\}\sqrt{\frac{r_n}{p}}$ |
| This paper | nonconvex GD | $\sigma \sqrt{\frac{r_n}{p}}$ |

passing into the estimation method and it requires $2 + \varepsilon$ moment of the noise to be finite. Despite the various assumptions and somewhat different formulations, their statistical estimation errors behave similarly (cf. Table 1). They have a trailing term and prevent the upper bound from being proportional to the noise level, even for sub-Gaussian noise. Consequently, when the noise level is small, there would be a considerable gap between their results and the optimal results available on sub-Gaussian noise (Ma et al., 2017; Chen et al., 2020b).

Furthermore, a recent work (Shen et al., 2022) proposes a nonconvex Riemannian sub-gradient algorithm under the condition that the noise is symmetric (which transfers means to medians) and some regularity conditions hold. Their algorithm has improved over the convex approach in terms of computational costs. However, their estimation error still has a trailing term as listed in Table 1, which summarized the state-of-art theoretical results discussed above.

In view of these prior theories, the following questions arise naturally.

1. *Is it possible to complete the matrix under only bounded second moment condition with the same rate of convergence as the sub-Gaussian case?*

2. *Is it possible to close the theoretical gap by incorporating the techniques of robust statistics into nonconvex optimization?*

3. *Is it possible to design an efficient nonconvex algorithm to achieve the desired statistical accuracy?*

These questions are important but poorly understood. They form the subject of this paper.

**Our contribution.** The current paper is devoted to providing a satisfactory answer to the aforementioned questions. In a nutshell, we propose a two-stage nonconvex gradient descent algorithm with a robust spectral initialization and establish theories to guarantee the optimality of its iterates after running for a sufficient number (logarithmically dependent on the model parameters) of iterations. Our result is also listed in Table 1, which, to the best of our knowledge, is the first one that achieves the optimal error rate under only bounded second moment condition (without assuming symmetric distribution).
1.2 Paper organization and notation

The outline of the paper is as follows. Section 2 provides a formal statement of the model assumptions and presents our main results. Section 3 gives a review on prior literature of matrix completion. Section 4 conducts numerical experiments that verify our theoretical results. Section 5 gives a sketch of the proof techniques. We conclude the paper in Section 6 by discussing several future directions. All the proof details are deferred to the Appendix.

Throughout the paper, for two functions $f(\cdot)$ and $g(\cdot)$, we use the notations $f(n) \lesssim g(n)$ and $f(n) = O(g(n))$ to indicate that there exists some constant $C_1 > 0$ such that $f(n) \leq C_1 g(n)$ holds when $n$ is sufficiently large. Analogously, we adopt the notation $f(n) \gtrsim g(n)$ to indicate that $f(n) \geq C_2 g(n)$ for some constant $C_2 > 0$ for all $n$ that are large enough. Moreover, $f(n) \asymp g(n)$ means that $f(n) \lesssim g(n)$ and $f(n) \gtrsim g(n)$ hold simultaneously. In our proof, $C$ and $C'$ serve as constants whose value might change from line to line.

Additionally, the matrix notations $X, Y \in \mathbb{R}^{n \times r}$ and $M \in \mathbb{R}^{n \times n}$ shall be frequently used. The notation $\|v\|_2$ represents the $\ell_2$ norm of an vector $v$, and we let $\|M\|$ and $\|M\|_F$ represent the spectral norm and the Frobenius norm of $M$, respectively. Moreover, we define $\|M\|_{\infty} := \max_{i,j} |M_{i,j}|$ and $\|X\|_{2,\infty} := \max_i \|X_i,\|_2$. We use $P_{\Omega}(\cdot) : \mathbb{R}^{n \times n} \mapsto \mathbb{R}^{n \times n}$ to stand for the projection onto the subspace of matrices whose support is $\Omega$, i.e.

$$[P_{\Omega}(M)]_{i,j} = \begin{cases} M_{i,j}, & \text{if } (i,j) \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

for any matrix $M \in \mathbb{R}^{n \times n}$. Furthermore, for any matrix $Z$, we denote by $Z_{i,l}$ (resp. $Z_{i,:}$) the $l$th column (resp. column) of $Z$. For a function $f(X,Y)$, we use $\nabla f_X(X,Y)$ (resp. $\nabla f_Y(X,Y)$) to denote the gradient of $f(\cdot)$ with respect to $X$ (resp. $Y$). For a non-singular matrix $R \in \mathbb{R}^{r \times r}$ with SVD $R = U_R \Sigma_R V_R^T$, we define the orthogonal matrix $\text{sgn}(R)$ by

$$\text{sgn}(R) =: U_R V_R^T.$$  

2 Robust matrix completion and main results

2.1 Model and algorithm

Model. As elucidated in Section 1, we are interested in recovering a rank-$r$ matrix $M^*$. Let $M^* = U^* \Sigma^* V^*^T$ be the SVD of $M^*$ where $U^*$, $V^* \in \mathbb{R}^{n \times r}$ consists of orthogonal columns and $\Sigma^* \in \mathbb{R}^{r \times r}$ is a diagonal matrix with decreasing singular values $\sigma_1^* \geq \sigma_2^* \geq \cdots \geq \sigma_r^* > 0$. Denote by $\kappa := \sigma_{\max}^*/\sigma_{\min}^*$ the condition number of $M^*$, where $\sigma_{\max}^* := \sigma_1^*$ and $\sigma_{\min}^* := \sigma_r^*$. In addition, let $X^* = U^*(\Sigma^*)^{1/2}$ and $Y^* = V^*(\Sigma^*)^{1/2}$ be the balanced low-rank factors of $M^*$, namely $X^* Y^*^T = M^*$ and $X^T X^* = Y^T Y^*$. We consider the following assumptions regarding the highly incomplete and noisy observations of $M^*$.

Assumption 1. We assume

1. (Random sampling) The entry at each index $(i,j)$ can be observed independently with probability $p$, namely the entry missing at random with probability $1-p$.

2. (Heavy-tailed noise) The noise matrix $E = \{\varepsilon_{i,j}\}_{1 \leq i, j \leq n}$ is composed of independent heteroskedastic noise with zero mean and bounded variance:

$$\mathbb{E}[\varepsilon_{i,j}] = 0, \quad \mathbb{E}[\varepsilon_{i,j}^2] = \sigma_{i,j}^2 \leq \sigma^2.$$
Note that the heavy-tailed noise should be contrasted to the sub-Gaussian assumption in high-dimension. In particular, the bounded second moment can include distribution such as the mixture normal \((1 - \delta) \cdot N(0, 1) + \delta N(0, \delta^{-1})\) with \(\delta \to 0\). It can contain data points with outliers of order \(n^{1/2}\) among \(n\) data points by taking \(\delta \approx n^{-1}\) or \(O(n^{1/4})\) numbers of outliers of order \(n^{3/8}\) by taking \(\delta \approx n^{-3/4}\). As introduced before, the robust nonconvex problem studied here is

\[
\min_{X, Y \in \mathbb{R}^{n \times r}} f(X, Y) = \frac{1}{2p} \sum_{(i, j) \in \Omega} \rho_\tau \left( (XY^\top)_{i,j} - M_{i,j} \right) + \frac{1}{8} \|X^\top X - Y^\top Y\|_F^2, \tag{2.1}
\]

where the Huber loss function with parameter \(\tau\) is defined as

\[
\rho_\tau(x) := \begin{cases} 
  x^2/2, & \text{if } |x| \leq \tau, \\
  \tau |x| - \tau^2/2, & \text{if } |x| > \tau. 
\end{cases}
\tag{2.2}
\]

The regularization term in (2.1) is widely employed in the literature (Zheng and Lafferty, 2016; Tu et al., 2016; Chen et al., 2020a) to control the discrepancy or balance between \(X\) and \(Y\). It accommodates an unavoidable scaling issue underlying this model, since there is no hope to distinguish between \((XR, YR^{-\top})\) with \(R \in \mathbb{R}^{r \times r}\) being any invertible matrix and \((X, Y)\) given only observations based on \(XY^\top\).

**Algorithm.** This paper considers an algorithm consists of two stages: (i) robust spectral initialization which would generate a consistent yet not optimal initial estimate, (ii) a gradient descent (GD) algorithm which update the estimate iteratively. It can be seen momentarily that the initial estimate given by (i) would fall into a local region in the neighborhood of the global minimum where restricted strong convexity holds true, and then the GD algorithm can iteratively refine the estimates within the local region. The complete algorithm is summarized in Algorithm 1.

- **Spectral initialization.** Due to the nonconvex landscape, nonconvex algorithms typically require initialization point with good properties to avoid getting stuck into some highly sub-optimal local minima. To achieve this goal, in the first stage of Algorithm 1, we initialize the algorithm by the top-\(r\) SVD of (2.4) where

\[
\psi_\tau(t) = \frac{\partial}{\partial t} \rho_\tau(t), \tag{2.3}
\]

is the truncation (Winsorization) operator. Under this definition, \(M^0\) given by (2.4) is a nearly unbiased estimator of \(M^*\) when \(\tau\) is sufficiently large, suggesting that the top-\(r\) SVD of \(M^0\) shall be a proper estimate for the low rank factors of \(M^*\).

- **Gradient descent.** In the second stage, we proceed by performing gradient descent iteratively to refine our estimates. As we shall see momentarily, the number of iterations \(t_0\) is logarithmically dependent on the model parameters. This implies superior computational performance of Algorithm 1, since in each iteration, we only need to compute the gradient \(\nabla f(\cdot)\) and update the estimates \(X^t\) and \(Y^t\). In addition, the step size \(\eta\) is fixed throughout iterations. We will state how to choose its value shortly.

**2.2 Theoretical guarantees**

In this section, we present our theory for Algorithm 1 and elaborate on the implications of our results.
Algorithm 1 Gradient descent for robust matrix completion (with spectral initialization)

**Input**: data matrix $\mathbf{M}$, sampling set $\Omega$, rank $r$, observation probability $p$, and maximum number of iterations $t_0$.

**Spectral initialization**: let $\mathbf{U}^0 \Sigma^0 \mathbf{V}^{0\top}$ be the top-$r$ SVD of $\mathbf{M}^0 := \frac{1}{p} \mathcal{P}_\Omega (\psi_\tau (\mathbf{M}))$, (2.4)

(see (2.3) for definition of $\psi_\tau (\cdot)$) and set $\mathbf{X}^0 = \mathbf{U}^0 (\Sigma^0)^{1/2}$, $\mathbf{Y}^0 = \mathbf{V}^0 (\Sigma^0)^{1/2}$.

**Gradient updates**: for $t = 0, 1, \ldots, t_0 - 1$

\[
\begin{align*}
\mathbf{X}^{t+1} &= \mathbf{X}^t - \eta \nabla f_X (\mathbf{X}^t, \mathbf{Y}^t) \\
\mathbf{Y}^{t+1} &= \mathbf{Y}^t - \eta \nabla f_Y (\mathbf{X}^t, \mathbf{Y}^t)
\end{align*}
\] (2.5a, 2.5b)

where $\nabla f_X (\cdot)$ and $\nabla f_Y (\cdot)$ represent the gradient of $f (\cdot)$ given by (2.1) w.r.t. $\mathbf{X}$ and $\mathbf{Y}$, respectively.

Before proceeding to the main results, we introduce a crucial condition on $\mathbf{M}^*$, which allows for reliable estimation schemes. It is standard and widely adopted in the literature of matrix completion (Candès and Recht, 2009; Candès and Plan, 2010; Chen, 2015; Chen and Wainwright, 2015; Sun and Luo, 2016).

**Definition 1. (Incoherence)**. A rank-$r$ matrix $\mathbf{M}^*$ with SVD $\mathbf{M}^* = \mathbf{U}^* \Sigma^* \mathbf{V}^{*\top}$ is said to satisfy the incoherence condition with parameter $\mu$ if

\[
\| \mathbf{U}^* \|_{2,\infty} \leq \sqrt{\frac{\mu}{n}} \| \mathbf{U}^* \|_F = \sqrt{\frac{\mu r}{n}}, \quad \text{and} \quad \| \mathbf{V}^* \|_{2,\infty} \leq \sqrt{\frac{\mu}{n}} \| \mathbf{V}^* \|_F = \sqrt{\frac{\mu r}{n}}.
\]

We note an identifiability issue underlying this problem, namely, given any orthonormal matrix $\mathbf{R} \in \mathbb{R}^{r \times r}$, there always holds $\mathbf{X}^* \mathbf{Y}^{*\top} = \mathbf{X}^* \mathbf{R} (\mathbf{Y}^* \mathbf{R})^{\top}$. In view of this, when measuring the discrepancy between $\mathbf{F}^t := \begin{bmatrix} \mathbf{X}^t \\ \mathbf{Y}^t \end{bmatrix}$ and $\mathbf{F}^* := \begin{bmatrix} \mathbf{X}^* \\ \mathbf{Y}^* \end{bmatrix}$, we shall consider the distance metric modulo the global rotation matrix $\mathbf{H}^t$ which best align $\mathbf{F}^t$ and $\mathbf{F}^*$ in the sense that

\[
\mathbf{H}^t =: \arg \min_{\mathbf{R} \in O^{r \times r}} \| \mathbf{F}^t \mathbf{R} - \mathbf{F}^* \|_F = \text{sgn}(\mathbf{F}^{t\top} \mathbf{F}^*) \equiv \mathbf{F}^{t\top} \mathbf{F}^*,
\] (2.6)

where $\text{sgn}(\cdot)$ is defined in (1.3). With these notions in hand, we are ready to present the main theorems. To begin with, the first theorem below presents the theoretical results when the condition number $\kappa$, the incoherence parameter $\mu$, and the rank $r$ of $\mathbf{M}^*$ are all constantly bounded. It makes the requirement of sample size and noise level clearer to recognize. It also presents the property of the robust special spectral method as the initialization.

**Theorem 1.** Let $\mathbf{M}^*$ be rank-$r$ and $\mu$-incoherent with condition number $\kappa$. Suppose $\kappa, \mu, r \sim O(1)$ and Assumption 1 holds. Take $\tau = C_\tau (\| \mathbf{M}^* \|_\infty + \sigma \sqrt{\frac{np}{p}})$ for some large enough constant $C_\tau > 0$.

Assume the sample size and the noise level satisfy

\[
n^2 p \geq C n \log^2 n \quad \text{and} \quad \sigma \sqrt{\frac{n}{p}} \leq c \frac{\sigma_{\min}}{\log n},
\] (2.7)
where $C > 0$ is some large enough constant and $c > 0$ is some sufficiently small constant. Then with probability exceeding $1 - O(n^{-3})$, the iterates of Algorithm 1 obey

$$
\| F^0 H^0 - F^* \|_F \leq C_0 \left( \frac{\sigma}{\sigma_{\min}} + \frac{\| M^* \|_\infty}{\sigma_{\min}} \right) \sqrt{\frac{n}{p}} \| F^* \|_F, \tag{2.8}
$$

$$
\| F^t H^t - F^* \|_F \leq \rho^t \| F^0 H^0 - F^* \|_F + C_1 \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \| F^* \|_F. \tag{2.9}
$$

for all $0 \leq t \leq t_0 = O(n^5)$, where $C_0$ and $C_1$ are some absolute constants and $\rho = 1 - \frac{\sigma_{\min}}{20} \eta$, as long as $0 \leq \eta \leq c'/(\sigma_{\max} \log n)$ for some small constant $c' > 0$.

**Remark 1.** The analysis behind Theorem 1 remains valid as long as the total number of iterations $t_0$ is polynomially dependent on the problem dimension, i.e. $t_0 = O(n^c)$ for some constant $c > 0$. To make the contraction term in (2.9) negligible, the number of iterations should be at least of order $\log(\| M^* \| / \sigma)$.

**Remark 2.** Theorem 1 contains both the results for the robust spectral initialization and its subsequence iterates. On the way of our proof, we also establish

$$
\| F^t H^t - F^* \|_{2,\infty} \lesssim \kappa^{1.5} \sqrt{T} \left( \frac{\sigma}{\sigma_{\min}} + \frac{\| M^* \|_\infty}{\sigma_{\min}} \right) \sqrt{\frac{n}{p}} \log n \| F^* \|_{2,\infty}. \tag{2.10}
$$

We expect this result can be improved further.

Despite its simplicity, Theorem 1 reveals deep insights into the core idea of our newly-developed theoretical understanding towards robust matrix completion. As can be seen from (2.9), Theorem 1 guarantees that the robust spectral initialization point $F^0$ falls close enough to the ground truth $F^*$ and the estimation error of the iterates $\{ X^t, Y^t \}_{t>0}$ generated by the gradient descent step decay geometrically fast until reaching some error floor. This behavior will be further illustrated numerically in Section 4. A few remarks are in order.

- **Minimax optimality.** An immediate consequence of the theorem is that

$$
\| X^t Y^{t\top} - M^* \|_F \leq \tilde{C} \rho^t (\sigma + \| M^* \|_\infty) \sqrt{\frac{n}{p}} + C_1 \sigma \sqrt{\frac{n}{p}}.
$$

Consequently, as $t$ increases, $\| X^t Y^{t\top} - M^* \|_F$ converges to $\sigma \sqrt{\frac{n}{p}}$, matching the lower bound developed in Kolchinskie et al. (2011); Negahban and Wainwright (2012) in the presence of sub-Gaussian noise. This confirms the minimaxity of nonconvex optimization for matrix completion with heavy-tailed noise. Furthermore, this implies that when addressed properly, heavy-tailed noise can even behave analogously to sub-Gaussian noise in matrix completion (Chen et al., 2020b). Compared with the other methods listed in Table 1, this error level gets rid of the trailing term and is proportional to the noise level $\sigma$ even as $\sigma$ becomes vanishingly small, which also coincides with our intuition.

- **Fast convergence.** In stark contrast to the convex approaches, which usually suffer from high computational costs, the gradient descent algorithm here is easy to implement and demonstrates linear convergence with contraction rate $\rho$, resulting in an iteration complexity scaling logarithmically with the model parameters. Hence it is straightforward to see that to reach the error floor, the computational complexity is $O(n)$ (up to some log factors) under our sample size condition, which is almost the best we can expect since the time spent loading the data is $O(n^2)$ in our case. Note that previous work (Shen et al., 2022) also achieves geometric convergence, while its estimation error is not optimal.
• **Minimal sample size and noise conditions.** As stated in Theorem 1, when \( \kappa, \mu \) and \( r \) are all \( O(1) \), the sample size requirement scales as

\[ n^2 p \gtrsim np \log(n), \tag{2.11} \]

which matches the information-theoretic lower limit even in the absence of noise (Candès and Recht, 2009; Candès and Plan, 2010).

Under the same conditions, the noise level requirement (2.7) in our main theorem is \( \sigma \sqrt{n \log^2 n / p} \lesssim \sigma_{\text{min}} \). Therefore, if we adopt the following definition of signal-to-noise ratio (SNR)

\[ \text{SNR} := \frac{\mathbb{E} \left[ \left\| P_\Omega \left( M^* \right) \right\|_F^2 / |\Omega| \right]}{\sigma^2}, \]

the noise level requirement (2.7) implies that our theory will work as long as

\[ \text{SNR} = \frac{\left\| M^* \right\|_F^2}{n^2 \sigma^2} \gtrsim \frac{\log n}{np}. \]

The lower bound in the above equation can be vanishingly small in view of the sample size condition (2.11). This shows that our theory works even in the low-SNR regime.

Furthermore, it is worth noting that Theorem 1 removes the symmetric noise assumption which is required in previous works (Elsener and van de Geer, 2018; Shen et al., 2022). This will also be verified shortly by the numerical experiments reported in Section 4. Note that the symmetric noise assumption makes Huberization bias zero and problem becomes easier.

• **Implicit regularization.** On closer inspection of the works listed in Table 1, the convex problems need either the constraint \( \| M \|_\infty \leq \alpha \) (Elsener and van de Geer, 2018; Fan et al., 2021) or the nuclear norm penalty \( \| M \|_* \) in loss function (Elsener and van de Geer, 2018; Minsker, 2018; Fan et al., 2021), while these can be removed in our study by a powerful entrywise control, as we shall elaborate on shortly in Section 5. This indicates that gradient descent can implicitly bound the spikiness of the estimates.

Next, we present the more general case where \( \kappa, \mu, r \) are allowed to grow with \( n \). It allows us to examine the explicit dependence on \( \kappa, \mu, r \), which is not available in Theorem 1.

**Theorem 2.** Let \( M^* \) be rank- \( r \) and \( \mu \)-incoherent with condition number \( \kappa \). Suppose Assumption 1 holds and take \( \tau = C_\tau (\| M^* \|_\infty + \sigma \sqrt{np}) \) for some large enough constant \( C_\tau > 0 \). Assume the sample size and the noise level satisfy

\[ n^2 p \geq C \kappa^6 \mu^2 r^4 n \log^2 n \quad \text{and} \quad \sigma \sqrt{\frac{n}{p}} \leq c \frac{\sigma_{\text{min}}}{\sqrt{\kappa^4 \mu r^2 \log^2 n}}, \tag{2.12} \]

where \( C > 0 \) is some large enough constant and \( c > 0 \) is some sufficiently small constant. Then with probability exceeding \( 1 - O(n^{-3}) \), the iterates of Algorithm 1 obey

\[ \| F^t H^t - F^* \|_F \leq \rho^t \| F^0 H^0 - F^* \|_F + C_1 \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \| F^* \|_F, \tag{2.13} \]

\[ \| F^0 H^0 - F^* \|_F \leq C_0 \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\text{min}}} + \frac{\| M^* \|_\infty}{\sigma_{\text{min}}} \right) \sqrt{\frac{n}{p}} \| F^* \|_F. \tag{2.14} \]

for all \( 0 \leq t \leq t_0 = O(n^{-5}) \), where \( C_0 \) and \( C_1 \) are some absolute constants and \( \rho = 1 - \frac{\sigma_{\text{min}}}{20} \eta \), as long as \( 0 \leq \eta \leq c'/(\mu \kappa^3 r^2 \sigma_{\text{max}} \log n) \) for some small constant \( c' > 0 \).
In an analogy to Theorem 1, Theorem 2 also exhibits that Algorithm 1 starts from a proper initialization and then the iterates \( \{F_t\}_{t=0}^\infty \) demonstrates geometric convergence to some error floor. Furthermore, there are several aspects of Theorem 2 calling for future improvement. For example, the sample size condition (2.12) requires that the sample complexity to scale as \( O(\kappa^4nr^3) \). In contrast, in the noiseless setting, Gross (2011) and Chen (2015) have shown that the sample complexity needed to recover the low-rank matrix scales as \( O(nr) \). Moreover, in the presence of sub-Gaussian noise, Chen et al. (2020b) has established minimax optimal estimation error with sample complexity \( O(\kappa^4nr^2) \). Hence, there might still exist much room for improvement of the dependency on \( r \) and \( \kappa \). To put it into perspective, this sub-optimal scaling in \( r \) and \( \kappa \) appears frequently in theory of nonconvex low-rank matrix recovery (Chen and Wainwright, 2015; Sun and Luo, 2016; Ding and Chen, 2020; Shen et al., 2022). New analysis techniques shall be explored to sharpen the results.

3 Prior arts

Due to its superior computational advantage over the convex approach, the nonconvex approach has been employed to study a diverse array of high dimensional statistical estimation problems with low-rank structure, including matrix sensing, matrix completion, phase retrieval, blind deconvolution, robust PCA, to name just a few. The readers are referred to Chi et al. (2019) for an overview of this topic and references therein. Among these problems, matrix completion is the focus of this paper and the recent decade has witnessed a flurry of research activities under this topic since the seminal work by Candès and Recht (2009). A variety of nonconvex algorithms have been analyzed, such as projected gradient descent (Chen and Wainwright, 2015; Zheng and Lafferty, 2016; Sun and Luo, 2016), alternating minimization (Jain et al., 2013; Hardt, 2014; Hardt and Wootters, 2014), Riemannian gradient descent (Wei et al., 2016) and gradient descent with Burer-Monteiro factorization (Zheng and Lafferty, 2016; Burer and Monteiro, 2003). Besides, nuclear norm minimization has also attracted much attention (Candes and Plan, 2010; Gross, 2011; Candès and Plan, 2011; Koltchinskii et al., 2011; Negahban and Wainwright, 2012; Chen, 2015). All these papers consider either noiseless setting or sub-Gaussian noise, while heavy-tailed noise is not allowed for.

Heavy-tailed noise is an ubiquitous and widely studied issue arising in a variety of modern statistical problems. A number of papers have been dedicated to resolving this problem. For instance, Catoni (2012) proposes using a robust loss function to estimate the mean and variance of data with bounded variance; Brownlees et al. (2015) investigates empirical risk minimization based on the robust estimator proposed by Catoni (2012). In high-dimensional linear regression problem, Fan et al. (2017); Loh (2017) and Sun et al. (2020) study the usage of robust loss functions and analyze the theoretical properties of the proposed robust estimators. Charisopoulos et al. (2019); Tong et al. (2021); Li et al. (2020) study the nonsmooth and nonconvex formulation of low-rank matrix recovery with the help of \( \ell_1 \) loss, which subsumes many well-known problems including phase retrieval, matrix completion, blind deconvolution, etc. Alquier et al. (2019) studies the applications of general Lipschitz loss functions in a series of statistical problems including matrix completion, logistic LASSO and kernel methods. Another line of research follows the “median of means” approach (Nemirovskij and Yudin, 1983; Minsker, 2015; Hsu and Sabato, 2016) to attenuate the effects of heavy-tailed data.

Taking a closer look at robust matrix completion, a variety of papers have been devoted to studying the scenarios when observations are contaminated by outliers or heavy-tailed noises. In this regime, the proposed methods can also be roughly categorized as convex and nonconvex ones. Regarding the convex approach, to mitigate the effects of heavy-tailed noises, Fan et al. (2021)
proposes to first shrink the data to construct robust covariance estimators and then minimizes \( \ell_2 \) risk with nuclear norm penalty under the assumption of finite \( 2 + \varepsilon \) moment of noise. Elsener and van de Geer (2018) assumes a constant lower bound of the density function and a regularity condition on the distribution function of the errors. Under such conditions, it then studies the performance of \( \ell_1 \) and Huber loss with nuclear norm penalty, and obtains estimation error rates for approximately low-rank matrices. Minsker (2018) introduces a robust estimator inspired by Catoni (2012) and proves a similar estimation error bound to Fan et al. (2021) under the finite second moment condition. Turning to robust nonconvex optimization, Shen et al. (2022) introduces a nonconvex Riemannian sub-gradient algorithm to study matrix completion with heavy-tailed noise and \( \ell_1 \) loss, Huber loss and quantile loss respectively, under some regularity conditions on the density function and distribution function of the errors similar to that of Elsener and van de Geer (2018). Another collection of works focuses on studying robust matrix completion in the presence of outliers. For example, Cambier and Absil (2016) considers the case where the observed entries are corrupted by random outliers and studies \( \ell_1 \) loss function with the help of Riemannian optimization. Klopp et al. (2017) and Chen et al. (2021b) extend the model setting to incorporate both outliers and sub-Gaussian noise, and the latter achieves optimal estimation error.

### 4 Numerical experiments

In this section, we conduct a variety of numerical experiments to corroborate the validity of our theory established in Section 2. Throughout the experiments, we fix the dimension to be \( n = 1000 \) and the rank \( r = 5 \). The observation probability is \( p = 0.3 \). The ground truth matrix \( \mathbf{U}^*, \mathbf{V}^* \in \mathbb{R}^{n \times r} \) are generated by sampling from standard Gaussian distribution and then orthogonalizing their columns. The diagonal of \( \Sigma^* \) is set to be equidistant from \( \sigma_1^* = r \) to \( \sigma_r^* = 1 \).

In the first series of experiments, we report the numerical convergence of gradient descent (cf. Algorithm 1) as the noise level \( \sigma \) varies from \( 10^{-6} \) to \( 10^{-3} \). The step size \( \eta \) is set to be 0.05 and the threshold parameter in Huber loss function (2.2) is taken to be \( \tau = 3 \left( \| \mathbf{M}^* \|_\infty + \sigma \sqrt{np} \right) \). Let \( \mathbf{M}_{ncvx} = \mathbf{X}_{ncvx} \mathbf{Y}_{ncvx}^\top \) be the nonconvex solution from Algorithm 1 and \( \mathbf{M}^* \) be the ground truth. Figure 1 displays the relative Euclidean estimation errors \( \left( \| \mathbf{M}_{ncvx} - \mathbf{M}^* \|_F / \| \mathbf{M}^* \|_F \right) \) vs. the iteration count respectively. In (a) and (b), the noises are generated from Gaussian distribution and Student’s \( t \)-distribution with 3 degrees of freedom respectively. In (c), we adopt the noise...
distribution defined by the following probability mass function for a trinomial distribution:

\[
f(x) = \begin{cases} 
\frac{1}{2}\delta, & x = \frac{\sigma}{\sqrt{\delta}} \\
\frac{1}{2}\delta, & x = -\frac{\sigma}{\sqrt{\delta}} \\
1 - \delta, & x = 0
\end{cases}
\] (4.1)

and take \(\delta = 0.01\). In this case, only a fraction of observed entries are corrupted by noise and the magnitude of noise can be much larger (10 times) than \(\sigma\). Here, (b) and (c) focus on heavy-tailed noise distribution with finite second moments, while (a) considers Gaussian distribution which does not have heavy tail and serves as a benchmark to be compared with. As can be seen from the plots, the nonconvex gradient descent algorithm studied here converges linearly (in fact, within around 200 iterations) before it hits an error floor. In addition, the relative error of matrix completion increases as the noise level \(\sigma\) increases, which is consistent with Theorem 1.

In the second series of experiments, we report the statistical estimation errors as the noise level \(\sigma\) varies. The parameter \(\tau\) in Huber loss function (2.2) is also chosen to be \(\tau = 3(\|M^*\|_\infty + \sigma \sqrt{n/p})\). For each value of \(\sigma\), we conduct 50 random trials and use their average as the reported estimation error. In each trial, we run the nonconvex algorithm (cf. Algorithm 1) until convergence or the maximum number of iterations is reached. Figure 2 depicts the relative Euclidean error \(\|M_{ncvx}^* - M^*\|_F / \|M^*\|_F\) vs. the noise level \(\sigma\). It captures the behavior of (2.1) as \(\sigma\) varies from \(10^{-6}\) to \(10^{-3}\). The results suggest that the relative Euclidean estimation error scales linearly with \(\sigma\), providing empirical evidence for the theories developed in Theorems 1 and 2.

Next, we study how the estimation error depends on the choice of parameter \(\tau\) in Huber loss function (2.2). The noise level \(\sigma\) is fixed to be \(10^{-3}\) and the parameter \(\tau\) is varied from \(10^{-5}\) to \(10^2\). To demonstrate the capacity of our theory to incorporate asymmetric distribution, we adopt a highly asymmetric noise distribution which is defined as follows:

\[
f(x) = \begin{cases} 
\delta, & x = \sigma \sqrt{\frac{1-\delta}{\delta}} \\
1 - \delta, & x = -\sigma \sqrt{\frac{\delta}{1-\delta}},
\end{cases}
\] (4.2)

and choose \(\delta = 0.0001\). It is straightforward to check that the distribution defined above is zero-mean with variance \(\sigma^2\). The other two distributions used in Figure 3 are (b) Student’s \(t\)-distribution with shape parameter \(\nu = 2.1\) and scale parameter \(\sigma\); (c) Gaussian distribution with variance \(\sigma^2\).
Figure 3 displays the relative Euclidean error \( \|M_\tau - M^*_\|_F / \|M^*_\|_F \) vs. \( \tau \). Here, we denote the estimator associated with parameter \( \tau \) by \( M_\tau \). As can be seen in (a) and (b), adopting Huber loss with proper choice of \( \tau \) can indeed significantly improve the estimation error, with the minimum is achieved approximately by \( \tau = 0.01 \). This corresponds to the constant \( C_\tau \) defined in Theorems 1 and 2 being roughly 0.15, much smaller than our choice of \( C_\tau = 3 \) in Figures 1 and 2. For all distributions, there are truncation biases in the spectral initialization: the smaller \( \tau \), the bigger the bias. This gives various qualities of spectral initializations, which clearly have adverse impact on the convergence of the gradient decent for the non-convex loss. That explains the poor performance of the estimator when \( \tau \) is small even for symmetric error distributions. There is additional Huberization bias for the error distribution (4.2) that makes the performance for small \( \tau \) much worse than the optimally chosen one in Figure 3. As \( \tau \) increases, the biases get smaller, but the impact of heavy tails gradually becomes dominant (except for the Gaussian noise) and this is why we observe the shapes in Figure 3 (the fluctuations are probably due to large value of variance for \( t(2.1) \) distribution and relatively small number of simulations). In contrast, Figure 3(c) plots the results of Gaussian distribution, which remains roughly the same after \( \tau \) becomes large enough so that the bias in the initialization is small. These intriguing observations lend further support to our theoretical results and highlight the benefits of adopting Huber loss function when encountering heavy-tailed noise and also provide some guidance on how to choose \( \tau \) in real applications.

Finally, we investigate the improvement of (2.1) over the regularized least-squares estimator defined by

\[
\minimize_{X,Y \in \mathbb{R}^{n \times r}} \frac{1}{2p} \sum_{(i,j) \in \Omega} \left( (XY^\top)_{i,j} - M_{i,j} \right)^2 + \frac{1}{8} \|X^\top X - Y^\top Y\|_F^2, \tag{4.3}
\]

as the noise level \( \sigma \) varies from \( 10^{-6} \) to \( 10^{-3} \). Note that (4.3) is equivalent to (2.1) with \( \tau = \infty \). For each value of \( \sigma \), we experiment with a series of \( \tau \) ranging from \( 10^{-4} \) to \( 10^{-1} \). Specifically, for each pair of \( \sigma \) and \( \tau \), we conduct 50 random trials and calculate their average estimation error. Then for each value of \( \sigma \), we record the minimum estimation error across different values of \( \tau \). In addition, we also calculate the estimation error of the regularized least-squares estimator (4.3). We denote the minimizer of (4.3) by \((X_{LS},Y_{LS})\) and define \( M_{LS} = X_{LS}Y_{LS}^\top \). Figure 4 depicts the ratio between the estimation error of (2.1) with best choice of \( \tau \) and the estimation error of \( M_{LS} \) (\( \min_\tau \|M_\tau - M^*_\|_F / \|M_{LS} - M^*_\|_F \)) vs. the noise level \( \sigma \). For Gaussian distribution, adopting the Huber loss with the best choice of \( \tau \) has barely improved over the least-squares estimator,
Figure 4: The ratio between the minimum estimation error of (2.1) with respect to $\tau$ (i.e. $\min_{\tau} \| M_{\tau} - M^* \|_F$) and the estimation error of (4.3) (i.e. $\| M_{LS} - M^* \|_F$) vs. the noise level $\sigma$.

as expected. In contrast, for asymmetric distribution (4.2) and Student’s $t$-distribution, we can observe considerable improvement over least-squares estimator when $\sigma$ is not exceedingly small. When $\sigma = 10^{-3}$, the estimation error of adopting square loss in the objective function can be almost 10 times larger than using the best Huber loss. This impressive result emphasizes the superior advantage of the Huber loss over square loss when dealing with heavy-tailed distribution.

5 Proof sketch

In this section, we sketch the proof of Theorem 1. The proof details are all deferred to the Appendix. We would establish the following set of induction hypotheses for all $t \geq 0$:

\[
\| F^t H^t - F^* \|_F \lesssim \left( 1 - \frac{\sigma_{\min}}{20} \eta \right)^t \| F^0 H^0 - F^* \|_F + C_1 \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \| F^* \|_F, \tag{5.1a}
\]

\[
\| F^0 H^0 - F^* \|_F \leq C_0 \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^* \|_F, \tag{5.1b}
\]

\[
\| F^t H^t - F^* \|_{2,\infty} \lesssim \kappa^{1.5} \sqrt{\tau} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \log n \| F^* \|_{2,\infty}, \tag{5.1c}
\]

for all $t \geq 0$. With these in hand, Theorem 1 follows immediately. In what follows, Section 5.1 proves the hypothesis (5.1a) by a careful investigation of the landscape. Section 5.2 is devoted to justifying (5.1c) for all $t > 0$. Finally, Section 5.3 verifies (5.1b) and (5.1) for the base case, i.e. $t = 0$. (5.1b), the base case with $t = 0$.

5.1 Local geometry

In this section, we start from the following lemma which characterizes the region where the empirical loss function $f(\cdot)$ enjoys both restricted strong convexity and smoothness, and then establish the contraction of the error $\| F^t H^t - F^* \|_F$ by use of Lemma 1.

Lemma 1. (Restricted strong convexity and smoothness). Set $\tau = C_\tau \left( \| M^* \|_\infty + \sigma \sqrt{np} \right)$ for some constant $C_\tau > 0$. Suppose the sample size obeys $n^2 p \geq C \mu^2 \tau^2 \kappa n \log n$ for some sufficiently large constant $C > 0$ and the noise satisfies $\frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n \log n}{p}} \leq c$ for some sufficiently small constant $c > 0$. Then with probability exceeding $1 - O(n^{-10})$, one has

\[
\text{vec} (\Delta)^\top \nabla^2 f (X, Y) \text{vec} (\Delta) \geq \frac{\sigma_{\min}}{20} \| \Delta \|_F^2,
\]
generated by Algorithm 2 with the following auxiliary loss function for each 

\[ \forall l \leq n, \{F_{t}^{l}\}_{t \geq 0} \text{ are constructed to be the gradient descent iterates generated by Algorithm 2 with the following auxiliary loss function} \]

\[ \left\| \nabla^{2} f (X, Y) \right\| \leq 10\sigma_{\max}, \]

hold uniformly over all \( X, Y \in \mathbb{R}^{n \times r} \) obeying

\[ \left\| \begin{bmatrix} X - X^{*} \\ Y - Y^{*} \end{bmatrix} \right\|_{2, \infty} \leq \frac{c}{\kappa \sqrt{n}} \| F^{*} \|, \quad (5.2) \]

and all \( \Delta = \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} \in \mathbb{R}^{2n \times r} \) lying in the set

\[ \left\{ \begin{bmatrix} X_{1} \\ Y_{1} \end{bmatrix} \right\} = \hat{H} - \left[ \begin{bmatrix} X_{2} \\ Y_{2} \end{bmatrix} \right], \quad \left\| \begin{bmatrix} X_{2} - X^{*} \\ Y_{2} - Y^{*} \end{bmatrix} \right\| \leq c \| F^{*} \|, \quad \hat{H} := \arg \min_{R \in \mathbb{O}^{n \times r}} \left\{ \begin{bmatrix} X_{1} \\ Y_{1} \end{bmatrix} \right\} R - \left[ \begin{bmatrix} X_{2} \\ Y_{2} \end{bmatrix} \right], \quad (5.3) \]

where \( c \) and \( c' \) are some sufficiently small constants.

In words, Lemma 1 shows that when restricted to points close to the ground truth \( F^{*} \) in the sense of \( \ell_{2} / \ell_{\infty} \) norm, the Hessian \( \nabla^{2} f (\cdot) \) is well-conditioned along directions defined in (5.3). Armed with this lemma, we are ready to establish the first induction hypothesis (5.1a) as follows.

**Lemma 2.** (Frobenius and spectral norm errors). Set \( \tau = C_{\tau} (\| M^{*} \|_{\infty} + \sigma \sqrt{np}) \) for some constant \( C_{\tau} > 0 \). Suppose the sample size obeys \( np \geq \kappa^{6} \mu^{2} r^{4} n \log^{2} n \) for some sufficiently large constant \( C > 0 \) and the noise satisfies \( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \leq \frac{c}{\sqrt{\kappa^{4} \mu r^{4} \log^{2} n}} \) for some sufficiently small constant \( c > 0 \). If the iterates satisfy at the \( t \)th iteration, then with probability over \( 1 - O(n^{-100}) \), one has

\[ \| F^{t+1} H^{t+1} - F^{*} \|_{F} \leq \left( 1 - \frac{\sigma_{\min}}{20} \eta \right) \| F^{t} H^{t} - F^{*} \|_{F} + C \eta \sigma \sqrt{\frac{n}{p}} \| F^{*} \|_{F} \]

\[ \leq \left( 1 - \frac{\sigma_{\min}}{20} \eta \right)^{t+1} \| F^{0} H^{0} - F^{*} \|_{F} + C \eta \sigma \sqrt{\frac{n}{p}} \| F^{*} \|_{F}, \]

for some given step size \( \eta \) such that \( 0 \leq \eta \leq c' / (\mu \kappa^{6} r^{4} \sigma_{\max} \log n) \) with some small constant \( c' > 0 \).

As an immediate consequence of Lemma 2, one has

\[ \| F^{t+1} H^{t+1} - F^{*} \|_{F} \leq \| F^{t+1} H^{t+1} - F^{*} \|_{F} \leq \| F^{0} H^{0} - F^{*} \|_{F} + C \eta \sigma \sqrt{\frac{n}{p}} \| F^{*} \|_{F} \]

\[ \lesssim \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^{*} \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^{*} \|_{F}, \quad (5.4) \]

where we make use of (2.8) in the last inequality. This crude inequality will be used in our proof.

### 5.2 Leave-one-out sequences

In this section, we introduce a powerful leave-one-out analysis framework, which assists us to decouple the dependence between noise and iterates. This has already been employed to study various statistical problems (El Karoui, 2018; Zhong and Boumal, 2018; Li et al., 2019; Chen et al., 2019a; Ding and Chen, 2020; Chen et al., 2021c).

In what follows, we shall introduce a collection of auxiliary leave-one-out sequences \( \{F_{t}^{l(l)}\}_{t \geq 0} \) for each \( 1 \leq l \leq 2n \) to decouple the complicated dependency structure and thus establish (5.1c). Specifically, for each \( 1 \leq l \leq n \), \( \{F_{t}^{l(l)}\}_{t \geq 0} \) are constructed to be the gradient descent iterates generated by Algorithm 2 with the following auxiliary loss function.
\[ f^{(l)}(X, Y) = \frac{1}{2p} \sum_{(i,j) \in \Omega, i \neq l} \rho_{\tau} \left( \left( XY^\top \right)_{i,j} - M_{i,j} \right) + \frac{1}{2} \sum_{j=1}^{n} \rho_{\tau} \left( \left( XY^\top \right)_{i,j} - M_{i,j}^* \right) + \frac{1}{8} \left\| X^\top X - Y^\top Y \right\|_F^2. \]  

(5.5)

For \( n+1 \leq l \leq 2n \), \( \{ F^{t,(l)} \}_{t \geq 0} \) are generated similarly by running Algorithm 2 with the loss function
\[ f^{(l)}(X, Y) = \frac{1}{2p} \sum_{(i,j) \in \Omega, i \neq l-n} \rho_{\tau} \left( \left( XY^\top \right)_{i,j} - M_{i,j} \right) + \frac{1}{2} \sum_{j=1}^{n} \rho_{\tau} \left( \left( XY^\top \right)_{i,j-l} - M_{i,j-l}^* \right) + \frac{1}{8} \left\| X^\top X - Y^\top Y \right\|_F^2. \]  

(5.6)

When constructing the auxiliary loss functions (5.5) and (5.6), we drop the error term in each single row (or column) respectively, and thus the resulting loss function is independent of the randomness in that row (or column). In this way, at the cost of a small perturbation, we are able to eliminate the dependence between \( \{ F^{t,(l)} \}_{t \geq 0} \) and \( F^{t,(l)}_{l,*} \), which plays a key role in our analysis of \( \ell_2/\ell_\infty \) norm.

**Algorithm 2** Construction of the \( l \)-th leave-one-out sequence

**Input:** \( M, r, p \).

**Spectral initialization:** let \( U^{0,(l)}_r \Sigma^{0,(l)} V^{0,(l)^\top} \) be the top-\( r \) SVD of
\[ M^{0,(l)} := \frac{1}{2p} \mathcal{P}_{\Omega \setminus l} (\psi_r(M)) + \mathcal{P}_l(M^*), \]  

(5.7)

and set \( X^{0,(l)} = U^{0,(l)} (\Sigma^{0,(l)})^{1/2}, Y^{0,(l)} = V^{0,(l)} (\Sigma^{0,(l)})^{1/2}. \)

**Gradient updates:** for \( t = 0, 1, \ldots, t_0-1 \) do
\[ X^{t+1,(l)} = X^{t,(l)} - \eta \nabla f_{X}^{(l)} \left( X^{t,(l)} \right) \]  

(5.8a)

\[ Y^{t+1,(l)} = Y^{t,(l)} - \eta \nabla f_{Y}^{(l)} \left( Y^{t,(l)} \right) \]  

(5.8b)

where \( \nabla f_X(\cdot) \) and \( \nabla f_Y(\cdot) \) represent the gradient of \( f(\cdot) \) w.r.t. \( X \) and \( Y \), respectively.

To facilitate our analysis, we define the rotation matrices
\[ H^{t,(l)} \triangleq \arg \min_{R \in \mathcal{O}^{r \times r}} \left\| F^{t,(l)} R - F^* \right\|_F, \]  

(5.9)

\[ R^{t,(l)} \triangleq \arg \min_{R \in \mathcal{O}^{r \times r}} \left\| F^{t,(l)} R - F^t H^t \right\|_F. \]  

(5.10)

In the sequel, in order to justify (5.1c), we shall establish the following set of hypotheses:
\[ \max_{1 \leq t \leq 2n} \left\| F^{t,(l)} H^{t,(l)} - F^* \right\|_{2,\infty} \leq 2 \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n \log n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right), \]  

(5.11a)

\[ \max_{1 \leq t \leq 2n} \left\| F^t H^t - F^{t,(l)} R^{t,(l)} \right\|_F \leq \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^* \|_{2,\infty} \log n, \]  

(5.11b)
\[ \|F^t H^t - F^*\|_{2,\infty} \lesssim \kappa^{3/2} \sqrt{T} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \right) \log n \|F^*\|_{2,\infty}. \quad (5.11c) \]

The results are summarized in the following three lemmas.

**Lemma 3.** \((\ell_2/\ell_\infty \text{ norm error of leave-one-out sequences})\). Set \(\tau = C_\tau (\|M^*\|_\infty + \sigma \sqrt{np})\) for some constant \(C_\tau > 0\). Suppose the sample size obeys \(n^2 p \geq C \kappa^4 \mu^2 r^2 n \log n\) for some sufficiently large constant \(C > 0\) and the noise satisfies \(\sigma \sqrt{\frac{n}{p}} \leq \frac{c}{\sqrt{\kappa^2 \mu r^2 n}}\) for some sufficiently small constant \(c > 0\). If the iterates satisfy at the \(t\)th iteration, then with probability over \(1 - O(n^{-100})\), one has

\[
\max_{1 \leq i \leq 2n} \left\| \left( F^{t+1,(l)} H^{t+1,(l)} - F^* \right)_{l,t} \right\|_2 \lesssim \kappa \sqrt{T} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n \log n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \right) \|F^*\|_{2,\infty}. \]

This lemma justifies (5.11a) and establishes the incoherence of \(\{F^{t+1,(l)}\}_{l=1}^{2n}\). Next, we turn to show that up to some orthogonal transformation, \(F^{t+1}\) can indeed be well approximated by \(\{F^{t+1,(l)}\}_{l=1}^{2n}\).

**Lemma 4.** \((\text{Leave-one-out perturbation})\). Set \(\tau = C_\tau (\|M^*\|_\infty + \sigma \sqrt{np})\) for some constant \(C_\tau > 0\). Suppose the sample size obeys \(n^2 p \geq C \kappa^6 \mu^2 r^4 n \log^2 n\) for some sufficiently large constant \(C > 0\) and the noise satisfies \(\sigma \sqrt{\frac{n}{p}} \leq \frac{c}{\sqrt{\kappa^4 \mu^2 r^2 n}}\) for some sufficiently small constant \(c > 0\). If the iterates satisfy at the \(t\)th iteration, then with probability over \(1 - O(n^{-100})\), one has

\[
\max_{1 \leq i \leq 2n} \left\| F^{t+1} H^{t+1} - F^{t+1,(l)} R^{t+1,(l)} \right\|_F \leq \left( 1 - \frac{\sigma_{\text{min}}}{20} \eta \right)^{t+1} \left\| F^0 H^0 - F^{0,(l)} R^{0,(l)} \right\|_F + \frac{C}{\sigma_{\text{min}}} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \right) \|F^*\|_{2,\infty} \log n
\]

\[
\lesssim \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \right) \|F^*\|_{2,\infty} \log n,
\]

provided that \(0 \leq \eta \leq c' / (\mu \kappa^3 r^2 \sigma_{\text{max}} \log n)\) with some small constant \(c' > 0\).

Now we are positioned to prove the induction hypothesis (5.11c) by combining the previous two lemmas.

**Lemma 5.** \((\ell_2/\ell_\infty \text{ norm error})\). Set \(\tau = C_\tau (\|M^*\|_\infty + \sigma \sqrt{np})\) for some constant \(C_\tau > 0\). Suppose the sample size obeys \(n^2 p \geq C \mu^2 r^2 n \log n\) for some sufficiently large constant \(C > 0\) and the noise satisfies \(\sigma \sqrt{\frac{n}{p}} \leq \frac{c}{\sqrt{\kappa^4 \mu r^2 n}}\) for some sufficiently small constant \(c > 0\). If the iterates satisfy at the \(t\)th iteration, then with probability over \(1 - O(n^{-100})\), one has

\[
\|F^{t+1} H^{t+1} - F^*\|_{2,\infty} \lesssim \kappa^{1.5} \sqrt{T} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \right) \log n \|F^*\|_{2,\infty}. \]

The incoherence of \(F^{t+1}\) has thus been established. Finally, we introduce another induction hypothesis which demonstrates the approximate balancedness between \(X^t\) and \(Y^t\) through iterations.

**Lemma 6.** \((\text{Approximate balancedness})\). Set \(\tau = C_\tau (\|M^*\|_\infty + \sigma \sqrt{np})\) for some constant \(C_\tau > 0\). Suppose the sample size obeys \(n^2 p \geq C \mu^2 n \log n\) for some sufficiently large constant \(C > 0\) and
the noise satisfies $\sqrt{\frac{n}{p}} \leq \frac{c\sigma}{\sigma_{\min}} \sqrt{\log n}$ for some sufficiently small constant $c > 0$. If the iterates satisfy at the $t$th iteration, then with probability over $1 - O(n^{-100})$, one has \[
\left\| X_{t+1}^T X_{t+1} - Y_{t+1}^T Y_{t+1} \right\|_F \lesssim \eta \mu_\kappa^4 r^3.5 \sigma_{\max}^2 \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^\star\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \log^2 n, \]
given that $\eta \ll 1/(\mu_\kappa^2 r^2 \sigma_{\max} \log n)$.

Until now, we have finished verifying the induction hypotheses for $t > 0$. It remains to justify the base case $t = 0$ in the next section.

### 5.3 Spectral Initialization

According to Algorithm 1, the robust spectral method initializes the algorithm by top-$r$ SVD of the matrix \[ M^0 = \frac{1}{n} \mathcal{P}_{\Omega}(\psi_\tau(M)), \] with $\psi_\tau(\cdot)$ is defined in (2.3). Now we are ready to present the following several lemmas justifying (2.8) and (5.11) with $t = 0$.

**Lemma 7.** Suppose the sample size obeys $n^2 p \geq C_\kappa^3 \mu_\kappa^2 r^3 n \log n$ for some sufficiently large constant $C > 0$, the noise satisfies $\frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \leq c$ for some sufficiently small constant $c > 0$. Then with probability over $1 - O(n^{-10})$, one has

\[
\| F^0 H^0 - F^\star \| \lesssim \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^\star\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^\star \|, \tag{5.13}
\]

and

\[
\| F^0 H^0 - F^\star \|_F \lesssim \sqrt{\tau} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^\star\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^\star \| \\lesssim \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^\star\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^\star \|_F. \tag{5.14}
\]

This lemma verifies induction hypothesis (2.8). Then we focus on establishing the incoherence condition of spectral initialization.

**Lemma 8.** Suppose the sample size obeys $n^2 p \geq C_\mu^2 r^2 \kappa n \log n$ for some sufficiently large constant $C > 0$, the noise satisfies $\frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \leq \frac{c_\sigma_{\min}}{\sqrt{\kappa^3 \mu^2 r^3 \log n}}$ for some sufficiently small constant $c > 0$. Then with probability over $1 - O(n^{-10})$, one has

\[
\| F^0 H^0 - F^\star \|_{2,\infty} \lesssim \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^\star\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \log n \| F^\star \|_{2,\infty}. \tag{5.15}
\]

Next, we move on to proving the incoherence property of spectral initialization in the leave-one-out algorithm (cf. Algorithm 2).

**Lemma 9.** Suppose the sample size obeys $n^2 p \geq C\mu^2 r^2 \kappa^2 n \log n$ for some sufficiently large constant $C > 0$, the noise satisfies $\frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \leq c$ for some sufficiently small constant $c > 0$. Then with probability over $1 - O(n^{-10})$, one has

\[
\left\| (F^{0,(t)} H^{0,(t)} - F^\star)_{t} \right\|_2 \lesssim \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^\star\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^\star \|_{2,\infty}. \tag{5.16}
\]
Finally, we turn to justify (5.11b) for \( t = 0 \), demonstrating the proximity between \( F^0 \) and \( \{ F^{0,(l)} \}_{l=1}^{2n} \) up to some orthonormal transformation.

**Lemma 10.** Suppose the sample size obeys \( n^2 p \geq C \kappa^2 \mu^2 r^3 n \log n \) for some sufficiently large constant \( C > 0 \), the noise satisfies \( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{\kappa \mu^2 n \log^2 n}{p}} \leq c \) for some sufficiently small constant \( c > 0 \). Then with probability over \( 1 - O(n^{-10}) \), one has

\[
\left\| F^0 H^0 - F^{0,(l)} R^{0,(l)} \right\|_F \lesssim \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_\infty}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \right) \log n \| F^* \|_{2,\infty}. \tag{5.17}
\]

### 6 Discussion

This paper establishes the convergence guarantees for gradient descent of robust matrix completion with second moment condition. Going beyond this, there are a few interesting directions for future study, as elaborated below.

- **Tightening the dependency on \( r \) and \( \kappa \).** As discussed below Theorem 2, the dependence of the sample size requirement on \( r \) and \( \kappa \) is sub-optimal, calling for the application of novel analysis techniques for improvement.

- **Incorporating outliers.** As stated in Section 3, outlier is another important source of contamination, while as far as we are concerned, there is no existing theory of model incorporating heavy-tailed noise and outliers simultaneously. The most similar setting might be robust PCA which includes outliers and sub-Gaussian noise Agarwal et al. (2012); Klopp et al. (2017); Chen et al. (2021b). The techniques and insights of this paper may enlighten studies on the more general setting.

- **Approximate low-rank structure.** This present paper requires the matrix of interest \( M^* \) to be exactly low-rank, while in many real applications, \( M^* \) may be only approximately low-rank. Specifically, papers Fan et al. (2021); Elsener and van de Geer (2018) allowing for approximate low-rank structure typically suppose the \( \ell_q \) norm of the singular values of \( M^* \) is bounded, which reduces to Assumption 1 by setting \( q = 0 \). It remains largely unclear whether the nonconvex approach still works in the approximate low-rank scenario.

- **Convex estimator for robust matrix completion.** As elucidated in Table 1, existing theoretical guarantees of convex approach have a trailing term which is not proportional to noise, creating a considerable gap from optimal results when noise level vanishes. To handle this problem, the idea of connecting convex relaxation and nonconvex optimization in Chen et al. (2020b) might be inspiring and worth future exploration.

- **Valid inference procedures.** This present paper focuses on estimation of robust matrix completion and establishes a minimax optimal statistical error. To move forward, we note it is vastly under-explored how to assess the uncertainty of the estimates obtained from Algorithm 1. Methods and techniques in Chen et al. (2019b); Xia and Yuan (2021); Yan et al. (2021) may shed light on the procedure to perform valid inference on these matrices. However, the focal point of Chen et al. (2019b); Xia and Yuan (2021) is matrix completion with sub-Gaussian noise, while inference in the presence of heavy-tailed noise has been a long-standing open question.
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A Proofs for gradient descent

It is straightforward to obtain the gradients of \( f(\cdot) \):

\[
\nabla_X f(X, Y) = \frac{1}{2p} \mathcal{P}_\Omega \left( \left\{ \psi_\tau \left( (XY^T)_{i,j} - M_{i,j} \right) \right\}_{i,j} \right) Y + \frac{1}{2} X \left( X^T X - Y^T Y \right), \tag{A.1}
\]

\[
\nabla_Y f(X, Y) = \frac{1}{2p} \mathcal{P}_\Omega \left( \left\{ \psi_\tau \left( (XY^T)_{i,j} - M_{i,j} \right) \right\}_{i,j} \right)^T X + \frac{1}{2} Y \left( Y^T Y - X^T X \right). \tag{A.2}
\]

We start with an auxiliary lemma which is immediate consequence of Lemma 2.5.

**Lemma 11.** Instate the notation and assumptions in Theorem 1. For an integer \( t > 0 \), suppose that the hypotheses \((5.1)\) and \((5.11)\) hold in the \( t \)th iteration. Then under the assumptions that \( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \ll 1/\sqrt{\kappa r \log^2 n} \) and \( np \gg \mu^2 \kappa^3 r^3 \log^2 n \), one has

\[
\left\| F^{t, (l)} R^{t, (l)} - F^* \right\|_{2, \infty} \lesssim \kappa^{3/2} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{n} \frac{1}{p} + \frac{\| M^* \|_{\infty}}{\sigma_{\text{min}}^{p}} \frac{n}{p} \right)^{\frac{1}{2}} \log n \left\| F^* \right\|_{2, \infty}, \tag{A.3a}
\]

\[
\left\| F^{t, (l)} R^{t, (l)} - F^* \right\| \lesssim \sqrt{r} \left( \frac{\sigma_{\text{min}}}{\sqrt{n} \frac{1}{p} + \frac{\| M^* \|_{\infty}}{\sigma_{\text{min}}^{p}} \frac{n}{p} \right) \left\| F^* \right\|, \tag{A.3b}
\]

\[
\left\| F^{t, (l)} H^{t, (l)} - F^* \right\| \leq \left( \frac{\sigma_{\text{min}}}{\frac{n}{p} + \frac{\| M^* \|_{\infty}}{\sigma_{\text{min}}^{p}} \frac{n}{p} \right) \sqrt{\tau} \left\| F^* \right\|, \tag{A.3c}
\]

\[
\left\| F^{t, (l)} H^{t, (l)} - F^* H^t \right\|_F \leq 5 \left\| F^{t, (l)} R^{t, (l)} - F^* H^t \right\|_F, \tag{A.3d}
\]

\[
\left\| F^t \right\| \leq 2 \left\| F^* \right\|, \quad \left\| F^t \right\|_F \leq 2 \left\| F^* \right\|_F, \quad \left\| F^t \right\|_{2, \infty} \leq 2 \left\| F^* \right\|_{2, \infty}, \tag{A.3e}
\]

\[
\left\| F^{t, (l)} \right\| \leq 2 \left\| F^* \right\|, \quad \left\| F^{t, (l)} \right\|_F \leq 2 \left\| F^* \right\|_F, \quad \left\| F^{t, (l)} \right\|_{2, \infty} \leq 2 \left\| F^* \right\|_{2, \infty}, \tag{A.3f}
\]

\[
\frac{\sigma_{\text{min}}}{2} \leq \sigma_{\text{min}} \left[ \left( Y^{t, (l)} H^{t, (l)} \right)^T Y^{t, (l)} H^{t, (l)} \right] \leq \sigma_{\text{max}} \left[ \left( Y^{t, (l)} H^{t, (l)} \right)^T Y^{t, (l)} H^{t, (l)} \right] \leq 2 \sigma_{\text{max}}. \tag{A.3g}
\]

**A.1 Proof of Lemma 1**

In view of \((2.1)\), simple calculation yields

\[
\vec{(\Delta)}^T \nabla^2 f(X, Y) \vec{(\Delta)} = \frac{2}{p} \left\langle \mathcal{P}_\Omega \left( \left( (XY^T - M)^{i,j} \right)_{i,j} \mathbb{1} \left( \left| (XY^T)_{i,j} - M_{i,j} \right| \leq \tau \right) \right) \mathcal{P}_\Omega \left( \Delta X \Delta Y^T \right) \right\rangle
\]

\[
+ \frac{1}{p} \left\langle \mathcal{P}_\Omega \left( \left( \Delta X Y^T + X \Delta Y^T \right)^{i,j} \mathbb{1} \left( \left| (XY^T)_{i,j} - M_{i,j} \right| \leq \tau \right) \right) \right\|_F^2
\]

\[
+ \frac{1}{2} \left( \left\langle X^T X - Y^T Y, \Delta X \Delta X - \Delta Y \Delta Y \right\rangle + \frac{1}{4} \left\| \Delta X^T X + X^T \Delta X - Y^T \Delta Y - \Delta Y^T Y \right\|_F^2 \right)
\]

\[
+ \frac{2}{p} \left\langle \mathcal{P}_\Omega \left( \left( \tau \text{sgn} \left( (XY^T)^{i,j} \right) \right) \mathbb{1} \left( \left| (XY^T)_{i,j} - M_{i,j} \right| > \tau \right) \right) \right\rangle, \mathcal{P}_\Omega \left( \Delta X \Delta Y^T \right) \right\rangle.
\]

Chen et al. (2020a, Lemma 3.1) has proved that

\[
\mathcal{E}_0 := \frac{2}{p} \left\langle \mathcal{P}_\Omega \left( XY^T - M^* \right) \mathcal{P}_\Omega \left( \Delta X \Delta Y^T \right) \right\rangle + \frac{1}{p} \left\| \mathcal{P}_\Omega \left( \Delta X Y^T + X \Delta Y^T \right) \right\|_F^2
\]
\[
+ \frac{1}{2} \left< X^T X - Y^T Y, \Delta_X \Delta_X - \Delta_Y \Delta_Y \right> + \frac{1}{4} \left\| \Delta_X X + X^T \Delta_X - Y^T \Delta_Y - \Delta_Y Y \right\|_F^2 \geq \frac{\sigma_{\min}}{10} \left\| \Delta \right\|^2_F. \tag{A.4}
\]

We are left with considering
\[
\text{vec} \left( \Delta \right)^T \nabla^2 f \left( X, Y \right) \text{vec} \left( \Delta \right) - \varepsilon_0
\]
\[
= -\frac{2}{p} \left\langle P_\Omega \left( \left( X Y^T - M^* \right)_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right), P_\Omega \left( \Delta X \Delta_Y^T \right) \right\rangle
\]
\[
= -\frac{2}{p} \left\langle P_\Omega \left( \epsilon_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| \leq \tau \right), P_\Omega \left( \Delta X \Delta_Y^T \right) \right\rangle
\]
\[
+ \frac{1}{p} \left\| P_\Omega \left( \left( \Delta_X Y^T + X \Delta_Y^T \right)_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right) \right\|_F^2
\]
\[
+ \frac{2}{p} \left\langle P_\Omega \left( \left\{ \tau \text{sgn} \left( (X Y^T)_{i,j} \right) 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right\}_{i,j}, P_\Omega \left( \Delta X \Delta_Y^T \right) \right\rangle \right\rangle. \tag{A.5}
\]

1. Regarding \( \alpha_1 \), it can be further decomposed as
\[
\alpha_1 = \frac{2}{p} \left\langle P_\Omega \left( \left( X Y^T - X^* Y^* \right)_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right), P_\Omega \left( \Delta X \Delta_Y^T \right) \right\rangle
\]
\[
\leq \frac{2}{p} \left\langle P_\Omega \left( \left( X (Y - Y^*)^T \right)_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right), P_\Omega \left( \Delta X \Delta_Y^T \right) \right\rangle
\]
\[
+ \frac{2}{p} \left\langle P_\Omega \left( \left( X - X^* \right) Y^* \right)_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right), P_\Omega \left( \Delta X \Delta_Y^T \right) \right\rangle. \tag{A.6}
\]

For \( \beta_1 \), we have
\[
\left\langle P_\Omega \left( \left( X (Y - Y^*)^T \right)_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right), P_\Omega \left( \Delta X \Delta_Y^T \right) \right\rangle
\]
\[
= \left\langle \frac{1}{p} \left\{ \delta_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right\}_{i,j}, X (Y - Y^*)^T \circ \Delta X \Delta_Y^T \right\rangle
\]
\[
\leq \left\{ \frac{1}{p} \left\{ \delta_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau \right\}_{i,j} \right\} \left\| X (Y - Y^*)^T \circ \Delta X \Delta_Y^T \right\|_p
\]
\[
\leq \left\| \frac{1}{p} \left\{ \delta_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau / 2 \right\}_{i,j} \right\| \left\| X \right\|_{2,\infty} \left\| Y - Y^* \right\|_{2,\infty} \left\| \Delta \right\|_F^2
\]
\[
\leq \left\| \frac{1}{p} \left\{ \delta_{i,j} 1 \left| (X Y^T)_{i,j} - M_{i,j} \right| > \tau / 2 \right\}_{i,j} \right\| \left\| X \right\|_{2,\infty} \left\| Y - Y^* \right\|_{2,\infty} \left\| \Delta \right\|_F^2
\]
\[
\leq n^p \left( \left| \delta_{i,j} \right| > \tau / 2 \right) \frac{\epsilon \mu r \sigma_{\max}}{n} \left\| \Delta \right\|_F^2
\]
\[
\leq \mu r \sigma_{\max} n^p \left\| \Delta \right\|_F^2. \tag{A.7}
\]
Here (i) comes from the fact that

\[ |(XY^\top)_{i,j} - M_{i,j}| > \tau = |(XY^\top)_{i,j} - M_{i,j}^* - \varepsilon_{i,j}| > \tau \leq \mathbb{1}_{|\varepsilon_{i,j}| > \tau}(XY^\top)_{i,j} - M_{i,j}^*| \leq \mathbb{1}_{|\varepsilon_{i,j}| > \tau/2}, \]

where the last inequality is due to

\[
\max_{i,j} \left| (XY^\top)_{i,j} - M_{i,j}^* \right| \leq \|F - F^*\|_{2,\infty} \|F^*\|_{2,\infty} < \|F^*\|_{2,\infty}^2 \leq \frac{\mu r \sigma_{\max}}{n} \leq \frac{\tau}{2}; \quad (A.8)
\]

(ii) comes from Lemma 17, (5.2) and the fact that \( \|F^*\|_{2,\infty} \leq \sqrt{\mu r \sigma_{\max}/n} \); (iii) applies Markov inequality to obtain that

\[
\mathbb{P}(|\varepsilon_{i,j}| > \tau/2) \leq \frac{\sigma_{\max}^2}{(\tau/2)^2} \leq \frac{\sigma^2}{(\tau/2)^2}.
\]

Analogously, one has

\[
\beta_2 \leq \frac{\mu r \sigma_{\max}}{np} \|\Delta\|_F^2.
\]

Hence, plugging (A.7) and (A.9) into (A.6) yields

\[
\alpha_1 \leq \frac{\mu r \sigma_{\max}}{np} \|\Delta\|_F^2. \quad (A.10)
\]

2. Turning attention to \( \alpha_2 \), one has

\[
\frac{2}{p} \mathbb{P}_\Omega \left( \ve_{i,j} \mathbb{1}_{|(XY^\top)_{i,j} - M_{i,j}| \leq \tau} \right) \leq \frac{2}{p} \mathbb{P}_\Omega \left( \ve_{i,j} \mathbb{1}_{|(XY^\top)_{i,j} - M_{i,j}| \leq \tau} \right) \|\Delta X \Delta Y^\top\|_F
\]

where the last inequality can be easily obtained from the elementary fact of the nuclear norm that

\[
\|Z\|_\ast = \inf_{U, V \in \mathbb{R}^{n \times r}, UV^\top = Z} \left\{ \frac{1}{2} \|U\|_F^2 + \frac{1}{2} \|V\|_F^2 \right\}. \quad (A.11)
\]

To bound \( \frac{2}{p} \|\mathbb{P}_\Omega(\ve_{i,j} \mathbb{1}_{|(XY^\top)_{i,j} - M_{i,j}| \leq \tau})\| \), one has

\[
\frac{2}{p} \mathbb{P}_\Omega \left( \ve_{i,j} \mathbb{1}_{|(XY^\top)_{i,j} - M_{i,j}| \leq \tau} \right) \leq \frac{2}{p} \mathbb{P}_\Omega \left( \ve_{i,j} \mathbb{1}_{|\varepsilon_{i,j}| \leq \tau} - \mathbb{E} \left[ \ve_{i,j} \mathbb{1}_{|\varepsilon_{i,j}| \leq \tau} \right] \right) =: \beta_1
\]

\[
+ \frac{2}{p} \mathbb{P}_\Omega \left( \varepsilon_{i,j} \left( \mathbb{1}_{|(XY^\top)_{i,j} - M_{i,j}| \leq \tau} - \mathbb{1}_{|\varepsilon_{i,j}| \leq \tau} \right) \right) \quad =: \beta_2
\]

To bound \( \beta_1 \), we intend to apply (C.1), which needs the following quantities:

\[
\mathbb{V} \left[ \frac{1}{p} \left( \ve_{i,j} \mathbb{1}_{|\varepsilon_{i,j}| \leq \tau} - \mathbb{E} \left[ \ve_{i,j} \mathbb{1}_{|\varepsilon_{i,j}| \leq \tau} \right] \right) \right] = \mathbb{V} \left[ \frac{1}{p} \ve_{i,j} \mathbb{1}_{|\varepsilon_{i,j}| \leq \tau} \right] \leq \mathbb{E} \left[ \frac{1}{p^2} \sigma_{\max}^2 \mathbb{1}_{|\varepsilon_{i,j}| \leq \tau} \right]
\]

26
and

\[ B := \max_{i,j} \frac{1}{p} \left( \varepsilon_{i,j} \mathbb{I}_{|\varepsilon_{i,j}| \leq \tau} - \mathbb{E} \left[ \varepsilon_{i,j} \mathbb{I}_{|\varepsilon_{i,j}| \leq \tau} \right] \right) \leq \frac{2\tau}{p}. \]

Therefore, applying (C.1) to \( \beta_1 \) yields

\[ \beta_1 \leq \sigma \sqrt{n} + B \sqrt{\log n} \leq \frac{\sigma \sqrt{n} + \sqrt{\log n} \|M^*\|_\infty}{\sqrt{p}}. \tag{A.13} \]

For \( \beta_2 \), one has

\[
\beta_2 \overset{(i)}{=} \frac{2}{p} \left[ \mathbb{P} \left( \left| \frac{1}{p} \mathbb{E} \left[ \varepsilon_{i,j} \mathbb{I}_{\tau - \|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau + \|XY^\top - M\|_\infty} \right] \right) \right] \]

\[
\quad + \frac{2}{p} \left[ \mathbb{P} \left( \left| \frac{1}{p} \mathbb{E} \left[ \varepsilon_{i,j} \mathbb{I}_{\tau - \|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau + \|XY^\top - M\|_\infty} \right] \right) \right] \]

\[
\overset{(ii)}{=} \frac{2}{p} \left[ \mathbb{P} \left( \left| \frac{1}{p} \mathbb{E} \left[ \varepsilon_{i,j} \mathbb{I}_{\tau - \|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau + \|XY^\top - M\|_\infty} \right] \right) \right] \]

\[
\quad + \frac{2}{p} \left[ \mathbb{P} \left( \left| \frac{1}{p} \mathbb{E} \left[ \varepsilon_{i,j} \mathbb{I}_{\tau - \|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau + \|XY^\top - M\|_\infty} \right] \right) \right] \]

where (i) is due to

\[
\left\| \varepsilon_{i,j} \left( \mathbb{I}_{\|XY^\top\|_\infty \leq \varepsilon_{i,j} \leq \tau + \|XY^\top - M\|_\infty} \right) \right\| \leq \left\| \varepsilon_{i,j} \left( \mathbb{I}_{\|XY^\top\|_\infty \leq \varepsilon_{i,j} \leq \tau} \right) \right\| + \left\| \varepsilon_{i,j} \left( \mathbb{I}_{\|XY^\top\|_\infty \leq \varepsilon_{i,j} \leq \tau} \right) \right\|
\]

\[
\leq \left\| \varepsilon_{i,j} \left( \mathbb{I}_{\|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau + \|XY^\top - M\|_\infty} \right) \right\| \leq \left\| \varepsilon_{i,j} \left( \mathbb{I}_{\|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau + \|XY^\top - M\|_\infty} \right) \right\|
\]

and (ii) comes from (A.8). Another application of (C.1) gives rise to

\[
\frac{2}{p} \left[ \mathbb{P} \left( \left| \frac{1}{p} \mathbb{E} \left[ \varepsilon_{i,j} \mathbb{I}_{\tau - \|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau} \right] \right) \right] \right) \leq \frac{\sigma \sqrt{n} + \sqrt{\log n} \|M^*\|_\infty}{\sqrt{p}},
\]

\[
\frac{2}{p} \left[ \mathbb{P} \left( \left| \frac{1}{p} \mathbb{E} \left[ \varepsilon_{i,j} \mathbb{I}_{\tau - \|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau} \right] \right) \right] \right) \leq \frac{\sigma \sqrt{n} + \sqrt{\log n} \|M^*\|_\infty}{\sqrt{p}}.
\]

Regarding \( \theta_3 \), we have

\[
\theta_3 \overset{(i)}{=} \frac{2}{p} \left[ \mathbb{P} \left( \mathbf{1}^\top \right) \right] \cdot \max_{i,j} \mathbb{E} \left[ \varepsilon_{i,j} \mathbb{I}_{\tau - \|XY^\top - M\|_\infty \leq \varepsilon_{i,j} \leq \tau} \right].
\]
\[ \sum_{i} \frac{2}{p} \left\| \mathcal{P}_{\Omega} \left( \mathbf{1}^{\top} \right) \right\| \sqrt{E \left[ \varepsilon_{i,j}^2 \right] E \left[ 1_{\tau/2 \leq \varepsilon_{i,j} \leq 3\tau/2} \right]} \]
\[ \leq \sigma \frac{\sqrt{n}}{p}, \quad (A.14) \]

where (i) applies the observation that for any matrix \( \mathbf{W} \), it holds that
\[ \left\| \mathbf{W} \right\| \leq \left\{ \left| \mathbf{W}_{i,j} \right| \right\}_{i,j} \leq \left\{ 1_{\mathbf{W}_{i,j} \neq 0} \right\}_{i,j} \max_{i,j} \left| \mathbf{W}_{i,j} \right| ; \quad (A.15) \]

(ii) arises from the Cauchy-Schwartz inequality; (iii) follows from Lemma 17 and the application of Markov inequality
\[ E \left[ 1_{\tau/2 \leq \varepsilon_{i,j} \leq 3\tau/2} \right] \leq P \left[ \tau/2 \leq \varepsilon_{i,j} \leq 3\tau/2 \right] \leq \sigma \frac{\sqrt{n}}{p}, \quad (A.16) \]

The bound of \( \theta_{3} \) follows analogously from (A.9). Finally, we have
\[ \beta_{3} \leq \frac{2}{p} \left\| \mathcal{P}_{\Omega} \left( \mathbf{1}^{\top} \right) \right\| \cdot \max_{i,j} \left[ E \left[ \varepsilon_{i,j} 1_{|\varepsilon_{i,j}| > \tau} \right] \right] \]
\[ \leq \sigma \sqrt{\frac{n}{p}}, \quad (A.17) \]

where (i) holds due to (A.15); (ii) comes from Lemma (17) and the Cauchy-Schwartz inequality; (iii) relies on Markov inequality
\[ E \left[ 1_{|\varepsilon_{i,j}| > \tau} \right] \leq P \left[ |\varepsilon_{i,j}| > \tau \right] \leq \frac{\sigma^2}{\tau^2}. \]

Plugging the bounds of \( \{\beta_{i}\}_{i=1}^{3} \) into (A.12) yields
\[ \left\| \mathcal{P}_{\Omega} \left( \varepsilon_{i,j} 1_{\left| (\mathbf{X}^{\top})_{i,j} - M_{i,j} \right| \leq \tau} \right) \right\| \leq \sigma \sqrt{\frac{n}{p}}, \quad (A.17) \]

and therefore,
\[ \alpha_{2} \leq \frac{2}{p} \left\| \mathcal{P}_{\Omega} \left( \varepsilon_{i,j} 1_{\left| (\mathbf{X}^{\top})_{i,j} - M_{i,j} \right| \leq \tau} \right) \right\| \left\| \Delta \right\|_{F}^2 \leq \left( \frac{\sigma \sqrt{n} + \sqrt{\log n} \left\| M^{*} \right\|_{\infty}}{\sqrt{p}} \right) \left\| \Delta \right\|_{F}^2. \quad (A.18) \]

3. Next, \( \alpha_{3} \) can be decomposed as
\[
\frac{1}{p} \left\| \mathcal{P}_{\Omega} \left( \left( \Delta \mathbf{X}^{\top} + \mathbf{X} \Delta^{\top} \right)_{i,j} 1_{\left| (\mathbf{X}^{\top})_{i,j} - M_{i,j} \right| \leq \tau} \right) \right\|_{F}^2
\leq \frac{2}{p} \left\| \mathcal{P}_{\Omega} \left( \left( \Delta \mathbf{X}^{\top} \right)_{i,j} 1_{\left| (\mathbf{X}^{\top})_{i,j} - M_{i,j} \right| > \tau} \right) \right\|_{F}^2
\]
\[ + \frac{2}{p} \left\| \mathcal{P}_{\Omega} \left( \left( \mathbf{X} \Delta^{\top} \right)_{i,j} 1_{\left| (\mathbf{X}^{\top})_{i,j} - M_{i,j} \right| > \tau} \right) \right\|_{F}^2. \quad (A.19) \]
The first term on the right-hand side can be bounded by

\[
\frac{2}{p} \left\| P_\Omega \left( \left\{ (\Delta X Y^\top)_{i,j} \mathbb{1}_{|M_{i,j}| > \tau} \right\} \right) \right\|_F^2 \leq \frac{2}{p} \left\| P_\Omega \left( \left\{ (\Delta X Y^\top)_{i,j} \mathbb{1}_{|\xi_{i,j}| > \tau/2} \right\} \right) \right\|_F^2 \leq 4n \min \left\{ \| \Delta X \|_2^2, \| Y \|_2^2, \| \Delta X \|_2^2, \frac{\| F \|_2}{2} \right\} \mathbb{P} \left( |\xi_{i,j}| > \frac{\tau}{2} \right) \\
\leq \frac{\mu r \sigma_{\max}}{np} \| \Delta \|_F^2. \quad (A.20)
\]

Here (i) comes from the fact that

\[
\mathbb{1}_{|M_{i,j}| > \tau} = \mathbb{1}_{|M^*_{i,j} - \xi_{i,j}| > \tau} - \mathbb{1}_{|\xi_{i,j}| > \tau/2} \leq \mathbb{1}_{|\xi_{i,j}| > \tau/2}, \quad (A.21)
\]

where the last inequality is due to

\[
\max_{i,j} \left| (XY^\top)_{i,j} - M^*_{i,j} \right| \leq \| F - F^* \|_{2,\infty} \| F^* \|_{2,\infty} \ll \| F^* \|_{2,\infty} \leq \frac{\mu r \sigma_{\max}}{n} \leq \frac{\tau}{2};
\]

(ii) invokes Chen et al. (2020a, Lemma A.3) and holds uniformly for all matrices \( \Delta X \in \mathbb{R}^{n \times r} \) and \( Y \in \mathbb{R}^{n \times r} \) with probability over \( 1 - O(n^{-10}) \); (iii) applies Markov inequality to obtain that

\[
\mathbb{P} \left( |\xi_{i,j}| > \frac{\tau}{2} \right) \leq \frac{\sigma_{\max}^2}{(\tau/2)^2} \leq \frac{\sigma^2}{(\tau/2)^2}. \quad (A.22)
\]

Analogously, one has

\[
\frac{2}{p} \left\| P_\Omega \left( \left\{ (\Delta X Y^\top + X \Delta Y^\top)_{i,j} \mathbb{1}_{|M_{i,j}| > \tau} \right\} \right) \right\|_F^2 \leq \frac{\mu r \sigma_{\max}}{np} \| \Delta \|_F^2. \quad (A.23)
\]

Taking (A.20), (A.23) together with (A.19) yields

\[
\frac{1}{p} \left\| P_\Omega \left( (\Delta X Y^\top + X \Delta Y^\top)_{i,j} \mathbb{1}_{|M_{i,j}| > \tau} \right) \right\|_F^2 \leq \frac{\mu r \sigma_{\max}}{np} \| \Delta \|_F^2,
\]

and hence,

\[
\alpha_3 \leq \frac{\mu r \sigma_{\max}}{np} \| \Delta \|_F^2. \quad (A.24)
\]

4. Finally, we turn to consider \( \alpha_4 \). Simple calculation reveals that

\[
\left| \frac{1}{p} P_\Omega \left( \tau \mathbb{sgn} \left( (XY^\top)_{i,j} - M_{i,j} \right) \mathbb{1}_{|M_{i,j}| > \tau} \right), P_\Omega \left( \Delta X \Delta Y^\top \right) \right| \approx \left| \frac{1}{p} P_\Omega \left( \tau \mathbb{sgn} \left( (XY^\top)_{i,j} - M_{i,j} \right) \mathbb{1}_{|M_{i,j}| > \tau} \right), \Delta X \Delta Y^\top \right|
\]

29
\[
\leq \left\| \frac{1}{p} \mathcal{P}_\Omega \left( \left\{ \tau \text{sgn} \left( \left( (XY)^\top \right)_{i,j} - M_{i,j} \right) \right\}_{i,j} \mathbb{1}_{(XY)^\top_{i,j} - M_{i,j} > \tau} \right) \right\| \| \Delta X \Delta Y^\top \|_* .
\]

One has (A.11) suggests that \( \| \Delta X \Delta Y^\top \|_* \leq \| \Delta \|_F^2 \). In addition, we have

\[
\left\| \frac{1}{p} \mathcal{P}_\Omega \left( \left\{ \tau \text{sgn} \left( \left( (XY)^\top \right)_{i,j} - M_{i,j} \right) \right\}_{i,j} \mathbb{1}_{(XY)^\top_{i,j} - M_{i,j} > \tau} \right) \right\| \\
\leq \frac{\tau}{p} \left\| \mathcal{P}_\Omega \left( \left\{ \mathbb{1}_{(XY)^\top_{i,j} - M_{i,j} > \tau} \right\}_{i,j} \right) \right\| \\
\leq \frac{\tau}{p} \left\| \mathbb{1}_{|\epsilon_{i,j}| > \tau / 2} \right\|_{i,j} \\
\lesssim \tau n \mathbb{P} (|\epsilon_{i,j}| > \tau / 2) \lesssim \tau n \frac{\sigma^2}{(\tau / 2)^2} \lesssim \sigma \sqrt{n} / p .
\]

Here, (i) holds due to (A.15); (ii) follows from (A.21); (iii) applies Lemma 17; (iv) arises from (A.22). Therefore, one has

\[
\alpha_4 \lesssim \sigma \sqrt{n} / p \| \Delta \|_F^2 .
\]

Plugging (A.10), (A.18), (A.24), and (A.27) into (A.5) yields

\[
\left\| \text{vec} (\Delta)^\top \nabla^2 f (X, Y) \text{vec} (\Delta) - \mathcal{E}_0 \right\| \lesssim \left( \frac{\sigma \sqrt{n} + \sqrt{\log n} \| M^* \|_\infty}{\sqrt{p}} \right) \| \Delta \|_F^2 .
\]

Combining this with (A.4) gives

\[
\text{vec} (\Delta)^\top \nabla^2 f (X, Y) \text{vec} (\Delta) \sigma \geq \frac{\sigma_{\min}}{10} \| \Delta \|_F^2 - \left( \frac{\sigma \sqrt{n} + \sqrt{\log n} \| M^* \|_\infty}{\sqrt{p}} \right) \| \Delta \|_F^2 \\
\geq \frac{\sigma_{\min}}{20} \| \Delta \|_F^2 ,
\]

where the last inequality holds provided that \( \frac{\sigma}{\sigma_{\min}} \sqrt{n} / p \ll 1 \) and \( np \gg \mu^2 r^2 \kappa^2 \).

### A.2 Proof of Lemma 2

The definition of \( H^{t+1} \) (cf. (2.6)) and the update rule (2.5) give

\[
\| F^{t+1} H^{t+1} - F^* \|_F \leq \| F^{t+1} H^t - F^* \|_F = \| \left[ F^t - \eta \nabla f (F^t) \right] H^t - F^* \|_F \\
\overset{(i)}{=} \| F^t H^t - \eta \nabla f (F^t H^t) - F^* \|_F \\
\leq \| F^t H^t - \eta \nabla f (F^t H^t) - [F^* - \eta \nabla f (F^*')] \|_F + \eta \| \nabla f (F^*) \|_F ,
\]

where (i) holds due to the fact that \( \nabla f (FR) = \nabla f (F) R \) for all \( R \in \mathcal{O}^{r \times r} \). \( \alpha_1 \) is exactly the term \( \alpha_1 \) in Chen et al. (2020b, Section D.3) with \( f_{\text{aug}} \) replaced by \( f \). Reusing the results therein, we obtain

\[
\alpha_1 \leq \left( 1 - \frac{\sigma_{\min}}{20} \eta \right) \| F^t H^t - F^* \|_F ,
\]

30
holds as long as \( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \ll \frac{1}{\sqrt{\kappa \mu^2 \log^2 n}} \) and \( np \gg \kappa^6 \mu^2 \sigma^4 \log^2 n \). Regarding \( \alpha_2 \), one has

\[
\| \nabla f(\mathbf{F}^*) \|_F = \left\| \frac{1}{2p} \mathcal{P}_\Omega \left( \left\{ \psi_\tau (-\epsilon_{i,j}) \right\}_{i,j} \right) Y^* \right\|_F \leq \frac{1}{2p} \left\| \mathcal{P}_\Omega \left( \left\{ \psi_\tau (-\epsilon_{i,j}) \right\}_{i,j} \right) \mathbf{F}^* \right\|_F.
\]

Substitution of the definition of \( \psi_\tau(\cdot) \) (cf. (5.12)) into the equation above gives

\[
\frac{1}{2p} \left\| \mathcal{P}_\Omega \left( \left\{ \psi_\tau (-\epsilon_{i,j}) \right\}_{i,j} \right) \right\| \leq \frac{1}{p} \left\| \mathcal{P}_\Omega \left( \left\{ 2\epsilon_{i,j} \mathbf{1}_{|\epsilon_{i,j}| \leq \tau} + \tau \text{sgn} (\epsilon_{i,j}) \mathbf{1}_{|\epsilon_{i,j}| > \tau} \right\}_{i,j} \right) \right\|
\]

\[
\leq \frac{1}{p} \left\| \mathcal{P}_\Omega \left( \left\{ \epsilon_{i,j} \mathbf{1}_{|\epsilon_{i,j}| \leq \tau} \right\}_{i,j} \right) \right\| + \frac{1}{2p} \left\| \mathcal{P}_\Omega \left( \left\{ \tau \text{sgn} (\epsilon_{i,j}) \mathbf{1}_{|\epsilon_{i,j}| > \tau} \right\}_{i,j} \right) \right\|.
\]

\( \beta_1 \) can be controlled as

\[
\frac{1}{p} \left\| \mathcal{P}_\Omega \left( \left\{ \epsilon_{i,j} \mathbf{1}_{|\epsilon_{i,j}| \leq \tau} \right\}_{i,j} \right) \right\| \leq \frac{1}{p} \left\| \mathcal{P}_\Omega \left( \left\{ \epsilon_{i,j} \mathbf{1}_{|\epsilon_{i,j}| \leq \tau} - \mathbb{E} \left[ \epsilon_{i,j} \mathbf{1}_{|\epsilon_{i,j}| \leq \tau} \right] \right\}_{i,j} \right) \right\|
\]

\[
+ \frac{1}{p} \left\| \mathcal{P}_\Omega \left( \left\{ \mathbb{E} \left[ \epsilon_{i,j} \mathbf{1}_{|\epsilon_{i,j}| \leq \tau} \right] \right\}_{i,j} \right) \right\|
\]

\[
\overset{(i)}{\leq} \sigma \sqrt{\frac{n}{p}},
\]

where (i) comes from the bounds of \( \beta_1 \) and \( \beta_4 \) in (A.13) and (A.16).

For the next, one has

\[
\frac{1}{2p} \left\| \mathcal{P}_\Omega \left( \left\{ \tau \text{sgn} (\epsilon_{i,j}) \mathbf{1}_{|\epsilon_{i,j}| > \tau} \right\}_{i,j} \right) \right\| \leq \frac{\tau}{2p} \left\| \mathcal{P}_\Omega \left( \left\{ \mathbf{1}_{|\epsilon_{i,j}| > \tau} \right\}_{i,j} \right) \right\|
\]

\[
\overset{(ii)}{\leq} \tau \sqrt{np} \| \epsilon_{i,j} > \tau \|
\]

\[
\overset{(iii)}{\leq} \sigma \sqrt{\frac{n}{p}},
\]

where (i) is due to (A.21); (ii) comes from Lemma 17; (iii) comes from (A.22). Therefore, combining (A.30), (A.32) and (A.33) yields

\[
\| \nabla f(\mathbf{F}^*) \|_F \leq \frac{1}{2p} \left\| \mathcal{P}_\Omega \left( \left\{ \psi_\tau (-\epsilon_{i,j}) \right\}_{i,j} \right) \right\| \| \mathbf{F}^* \|_F \leq \sigma \sqrt{\frac{n}{p}} \| \mathbf{F}^* \|_F.
\]

Plugging this result into (A.28) reveals that

\[
\| \mathbf{F}^{t+1} H^{t+1} - \mathbf{F}^* \|_F \leq \left( 1 - \frac{\sigma_{\min}}{20} \eta \right) \| \mathbf{F}^t H^t - \mathbf{F}^* \|_F + \bar{C} \eta \sigma \sqrt{\frac{n}{p}} \| \mathbf{F}^* \|_F
\]

\[
\overset{(i)}{\leq} \left( 1 - \frac{\sigma_{\min}}{20} \eta \right)^{t+1} C \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_F}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| \mathbf{F}^* \|_F + C_1 \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \| \mathbf{F}^* \|_F
\]

\[
\overset{(i)}{\leq} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_F}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \sqrt{\kappa} \| \mathbf{F}^* \|_F,
\]

where (i) arises from the induction hypothesis.
A.3 Proof of Lemma 3

For $1 \leq l \leq n$, the update rule (5.8) implies the decomposition

$$
\left( F^{t+1,l} H^{t+1,l} - F^* \right)_{l,r} = \left\{ X^{t,l} - \eta \nabla f_X \left( X^{t,l} \right) \right\}_{l,r} H^{t+1,l} - X^*_{l,r}
$$

$$
= \left\{ X^{t,l} - \eta \nabla f_X \left( X^{t,l} \right) \right\}_{l,r} H^{t,l} - X^*_{l,r}
$$

$$
= h_1 + \left\{ X^{t,l} - \eta \nabla f_X \left( X^{t,l} \right) \right\}_{l,r} \left( H^{t,l} \right)^{-1} H^{t+1,l} - I_r \right)
$$

(A.34)

where the gradient is

$$
\left\{ \nabla f_X \left( X^{t,l} \right) \right\}_{l,r} = \left\{ \sum_{j=1}^{n} \psi_{r\tau} \left( \left( X^{t,l} Y^{t,l} \right)_{l,j} - M^*_{l,j} \right) \right\} Y^{t,l}
$$

$$
+ \frac{1}{2} X^{t,l}_{l,r} \left( X^{t,l} \right)^T \left( X^{t,l} - Y^{t,l} \right)^T Y^{t,l}.
$$

Then we proceed by controlling $h_1$ and $h_2$ separately. For notational simplicity we denote

$$
\Delta^{t,l} := \begin{bmatrix} \Delta^{t,l} \\ \Delta^{t,Y} \end{bmatrix} = \begin{bmatrix} X^{t,l} H^{t,l} - X^* \\ Y^{t,l} H^{t,l} - Y^* \end{bmatrix}.
$$

1. Regarding the first term $h_1$, one has

$$
h_1 = X^{t,l}_{l,r} H^{t,l} - X^*_{l,r} - \eta \left\{ \psi_{r\tau} \left( \left( X^{t,l} Y^{t,l} \right)_{l,j} - M^*_{l,j} \right) \right\} Y^{t,l} H^{t,l}
$$

$$
- \eta \frac{1}{2} X^{t,l}_{l,r} \left( X^{t,l} \right)^T X^{t,l} - Y^{t,l} \left( Y^{t,l} \right)^T Y^{t,l} \right) H^{t,l}
$$

$$
\overset{(i)}{=} X^{t,l}_{l,r} H^{t,l} - X^*_{l,r} - \eta \left\{ \left( X^{t,l} Y^{t,l} \right)_{l,j} - M^*_{l,j} \right\} \left( \left( X^{t,l} \right)^T \left( X^{t,l} \right) \right)_{l,j} H^{t,l}
$$

$$
- \eta \left\{ \tau \left( X^{t,l} Y^{t,l} \right)_{l,j} - M^*_{l,j} \right\} \left( \left( X^{t,l} \right)^T \left( X^{t,l} \right) \right)_{l,j} H^{t,l}
$$

$$
\overset{(ii)}{=} X^{t,l}_{l,r} H^{t,l} - X^*_{l,r} - \eta \left( X^{t,l} Y^{t,l} - M^* \right)_{l,r} Y^{t,l} H^{t,l}
$$

$$
- \eta \frac{1}{2} X^{t,l}_{l,r} \left( X^{t,l} \right)^T X^{t,l} - Y^{t,l} \left( Y^{t,l} \right)^T Y^{t,l} \right) H^{t,l}
$$

$$
= (X^{t,l}_{l,r} H^{t,l} - X^*_{l,r}) \left( I_r - \eta \left( Y^{t,l} H^{t,l} \right)^T Y^{t,l} H^{t,l} \right)
$$

$$
- \eta X^*_{l,r} \left( Y^{t,l} H^{t,l} - Y^* \right)^T Y^{t,l} H^{t,l}
$$

32
- \eta \frac{1}{2} \mathbf{X}_t^{(l)} \left( \mathbf{X}_t^{(l)} \mathbf{X}_t^{(l)\top} - \mathbf{Y}_t^{(l)} \mathbf{Y}_t^{(l)\top} \right) \mathbf{H}_t^{(l)}.

(A.36)

Here (i) makes use of the definition of \psi_\tau(\cdot) (cf. (2.3)); (ii) follows from the fact that

\[ 1 \left| (\mathbf{X}_t^{(l)} \mathbf{Y}_t^{(l)\top})_{t,j} - M_{t,j}^* \right| > \tau = 0, \]

which can be verified by

\[
\max_j \left| \left( \mathbf{X}_t^{(l)} \mathbf{Y}_t^{(l)\top} \right)_{t,j} - M_{t,j}^* \right| \\
= \max_{i,j} \left| \left( \mathbf{X}_t^{(l)} \mathbf{H}_t^{(l)} \right)_{i,j} \left( \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} \right)_{j,j} - \mathbf{X}_t^* \left( \mathbf{Y}_j^* \right)_{j,j} \right| \\
\leq \max_{i,j} \left| \left( \mathbf{X}_t^{(l)} \mathbf{H}_t^{(l)} \right)_{i,j} \left( \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} \right)_{j,j} - \mathbf{X}_t^* \left( \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} \right)_{j,j} \right| \\
+ \max_{i,j} \left| \mathbf{X}_t^* \left( \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} \right)_{j,j} - \mathbf{X}_t^* \left( \mathbf{Y}_j^* \right)_{j,j} \right| \\
\leq 2 \left\| \left( \mathbf{F}_t^{(l)} \mathbf{H}_t^{(l)} - \mathbf{F}^* \right) \right\|_{t,\|l\|_2} + 2 \left\| \mathbf{F}^* \right\|_{t,\|l\|_\infty} \left\| \mathbf{F}_t^{(l)} \mathbf{H}_t^{(l)} - \mathbf{F}^* \right\|_{t,\|l\|_\infty}
\]

\[
\overset{(i)}{\lesssim} \kappa \sqrt{r} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n \log n}{p}} + \frac{\|\mathbf{M}^*\|_{\infty}}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \right) \|\mathbf{F}^*\|_{t,\|l\|_\infty}^2,
\]

\[ \ll \tau, \]

where (i) relies on Lemma 3 and the result holds provided \( np \gg \mu^2 k^4 r^3 \log n \). Hence, the bound of \( h_1 \) follows from (A.36) that

\[
\|h_1\|_2 \leq \left\| \mathbf{I}_r - 2\eta \left( \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} \right) \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} \right\|_2 + 2\eta \|\mathbf{X}_t^*\|_2 \left\| \Delta_{\mathbf{X}}^{(l)} \right\|_2 + 8\eta \left\| \Delta_{\mathbf{Y}}^{(l)} \right\|_2 \|\mathbf{F}^*\| \|\mathbf{X}^*\|_{2,\|l\|_\infty},
\]

(A.37)

where the last line utilizes (A.3g) and the fact that

\[
\left\| \mathbf{X}_t^* \left( \mathbf{X}_t^{(l)} \mathbf{X}_t^{(l)\top} - \mathbf{Y}_t^{(l)} \mathbf{Y}_t^{(l)\top} \right) \mathbf{H}_t^{(l)} \right\|_2
\]

\[
\overset{(i)}{\leq} 2 \|\mathbf{X}^*\|_{2,\|l\|_\infty} \left\| \mathbf{X}_t^{(l)} \mathbf{X}_t^{(l)\top} - \mathbf{Y}_t^{(l)} \mathbf{Y}_t^{(l)\top} \right\|
\]

\[
\overset{(ii)}{\leq} \|\mathbf{X}^*\|_{2,\|l\|_\infty} \left( \left\| \mathbf{X}_t^{(l)} \mathbf{X}_t^{(l)\top} - \mathbf{X}^* \mathbf{X}^* \right\| + \left\| \mathbf{Y}^* \mathbf{Y}^* - \mathbf{Y}_t^{(l)} \mathbf{Y}_t^{(l)\top} \right\| \right)
\]

\[
\leq \|\mathbf{X}^*\|_{2,\|l\|_\infty} \left( \left\| \mathbf{X}_t^{(l)} \mathbf{H}_t^{(l)} - \mathbf{X}^* \right\| \left\| \mathbf{X}_t^{(l)} \right\| + \left\| \mathbf{X}_t^{(l)} \mathbf{H}_t^{(l)} - \mathbf{X}^* \right\| \left\| \mathbf{X}^* \right\| \right)
\]

\[
+ \|\mathbf{X}^*\|_{2,\|l\|_\infty} \left( \left\| \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} - \mathbf{Y}^* \right\| \left\| \mathbf{Y}_t^{(l)} \right\| + \left\| \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} - \mathbf{Y}^* \right\| \left\| \mathbf{Y}^* \right\| \right)
\]

\[
\overset{(iii)}{\leq} \|\mathbf{X}^*\|_{2,\|l\|_\infty} \left( 3 \left\| \mathbf{X}_t^{(l)} \mathbf{H}_t^{(l)} - \mathbf{X}^* \right\| \left\| \mathbf{X}^* \right\| + 3 \left\| \mathbf{Y}_t^{(l)} \mathbf{H}_t^{(l)} - \mathbf{Y}^* \right\| \left\| \mathbf{Y}^* \right\| \right)
\]

\[
\leq 6 \|\Delta_{\mathbf{X}}^{(l)}\| \|\mathbf{F}^*\| \|\mathbf{X}^*\|_{2,\|l\|_\infty},
\]
where (i) and (iii) are due to (A.3f); (ii) relies on the fact that $X^* \top X^* = Y^* \top Y^*$. Consequently, it is easy to obtain that
\[
\|h_1\|_2 \leq (1 - \eta \sigma_{\min}) C \sqrt{F} \|F^*\|_{2,\infty} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n \log n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) + 8\eta C \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \log n
\]
\[
\leq \|F^*\|_{2,\infty},
\]
where the last link utilizes the induction hypothesis (5.11a) and (A.3c).

2. For the next, making use of (A.38) reveals that
\[
\|h_2\|_2 \leq \left\| \left( H^{t,(l)} \right)^{-1} H^{t+1,(l)} - I_r \right\| \|h_1 + \Delta^{t,(l)}\|_2 \leq 2 \left\| \left( H^{t,(l)} \right)^{-1} H^{t+1,(l)} - I_r \right\| \|F^*\|_{2,\infty} \|
\]
\[
(A.39)
\]
To bound the right-hand side, we invoke the following claim whose proof is deferred to Section A.3.1.

**Claim 1.** With probability exceeding $1 - O(n^{-100})$, one has
\[
\left\| \left( H^{t,(l)} \right)^{-1} H^{t+1,(l)} - I_r \right\| \lesssim \eta \sqrt{t} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \|F^*\|^2 \log n
\]
\[
+ \frac{\eta}{2} \kappa^{1.5} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \|F^*\|_{2,\infty} \|F^*\| \log n,
\]
as long as $0 \leq \eta \leq c/(\mu \kappa^5 \sigma_{\max} \log n)$ for some sufficiently small constant $c > 0$.

Finally plugging (A.37) and (A.39) into (A.34) gives
\[
\left\| \left( \Delta^{t+1,(l)} \right)_{l,r} \right\|_2 \leq (1 - \eta \sigma_{\min}) \left\| \left( \Delta^{t,(l)} \right)_{l,r} \right\|_2 + 8\eta \|\Delta^{t,(l)}\| \|F^*\| \|X^*\|_{2,\infty}
\]
\[
+ \bar{C} \eta \sqrt{t} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \|F^*\|^2 \|F^*\|_{2,\infty} \log n
\]
\[
+ \bar{C} \frac{\eta}{2} \kappa^{1.5} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \|F^*\|_{2,\infty} \|F^*\| \log n
\]
\[
\lesssim \kappa \sqrt{t} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \|F^*\|_{2,\infty},
\]
where the last line utilizes the hypothesis (5.11a) and (A.3c).

### A.3.1 Proof of Claim 1

To start with, we introduce an auxiliary sequence
\[
\tilde{F}^{t+1,(l)} = \begin{bmatrix} \tilde{X}^{t+1,(l)} \\ \tilde{Y}^{t+1,(l)} \end{bmatrix},
\]
with
\[
\begin{align*}
\hat{X}^{t+1,(l)} &= X^{t,(l)} H^{t,(l)} - \eta \frac{1}{p} P_{\Omega_{t-1}} \left( \psi_r \left( \left( X^{t,(l)} Y^{t,(l)} \right)_{i,j} - M_{i,j} \right) \right) Y^* \\
&\quad - \eta P_{\Omega_r} \left( \psi_r \left( \left( X^{t,(l)} Y^{t,(l)} \right)_{i,j} - M_{i,j}^{*} \right) \right) Y^* \\
&\quad - \frac{\eta}{2} X^* H^{t,(l)} \left( X^{t,(l)} \right)^\top X^{t,(l)} - Y^{t,(l)} Y^{t,(l)} H^{t,(l)}, \\
\tilde{Y}^{t+1,(l)} &= Y^{t,(l)} H^{t,(l)} - \eta \left[ \frac{1}{p} P_{\Omega_{t-1}} \left( \psi_r \left( \left( X^{t,(l)} Y^{t,(l)} \right)_{i,j} - M_{i,j} \right) \right) \right]^\top X^* \\
&\quad - \eta P_{\Omega_r} \left( \psi_r \left( \left( X^{t,(l)} Y^{t,(l)} \right)_{i,j} - M_{i,j}^{*} \right) \right]^\top X^* \\
&\quad - \frac{\eta}{2} X^* H^{t,(l)} \left( Y^{t,(l)} \right)^\top Y^{t,(l)} - X^{t,(l)} X^{t,(l)} H^{t,(l)}. 
\end{align*}
\]

Then we turn attention to \( \| (H^{t,(l)})^{-1} H^{t+1,(l)} - I_r \| \). We begin with a claim (Chen et al., 2020b, Claim 4) showing that \( I_r \) aligns \( F^{t+1,(l)} \) with \( F^* \).

**Claim 2.** One has
\[
I_r = \arg \min_{R \in \mathbb{O}^{\tau \times r}} \left\| F^{t+1,(l)} R - F^* \right\|_F \quad \text{and} \quad \sigma_{\min} \left( F^{t+1,(l)}^\top F^* \right) \geq \sigma_{\min}/2.
\]

Invoking Ma et al. (2017, Lemma 36) yields
\[
\left\| (H^{t,(l)})^{-1} H^{t+1,(l)} - I_r \right\| = \left\| \text{sgn} \left( \left( F^{t+1,(l)} H^{t,(l)} \right)^\top F^* \right) - \text{sgn} \left( F^{t+1,(l)}^\top F^* \right) \right\| \\
\leq \frac{1}{\sigma_{\min} \left( F^{t+1,(l)}^\top F^* \right)} \left\| \left( F^{t+1,(l)} H^{t,(l)} - F^{t+1,(l)} \right)^\top F^* \right\| \\
\leq \frac{2}{\sigma_{\min}} \left\| F^{t+1,(l)} H^{t,(l)} - F^{t+1,(l)} \right\| \left\| F^* \right\|. \tag{A.40}
\]

To control \( F^{t+1,(l)} H^{t,(l)} - F^{t+1,(l)} \), we can decompose it into the sum of three terms as follows,
\[
F^{t+1,(l)} H^{t,(l)} - F^{t+1,(l)} = \eta \left[ \begin{array}{cc}
B & 0 \\
0 & B^\top
\end{array} \right] \left[ \begin{array}{c}
\Delta_{Y^{t,(l)}} \\
\Delta_{X^{t,(l)}}
\end{array} \right] + \frac{\eta}{2} \left[ \begin{array}{c}
X^* \\
- Y^*
\end{array} \right] H^{t,(l)} CH^{t,(l)}
\]
\[
+ \frac{\eta}{2} \left[ \begin{array}{c}
\Delta_{X^{t,(l)}}^\top \\
- \Delta_{Y^{t,(l)}}^\top
\end{array} \right] H^{t,(l)}^\top CH^{t,(l)}, \tag{A.41}
\]

where
\[
B := -\frac{1}{p} P_{\Omega_{t-1}} \left( \psi_r \left( \left( X^{t,(l)} Y^{t,(l)} \right)_{i,j} - M_{i,j} \right) \right) - P_{\Omega_r} \left( \psi_r \left( \left( X^{t,(l)} Y^{t,(l)} \right)_{i,j} - M_{i,j}^{*} \right) \right)
\]
\[
C := X^{t,(l)}^\top X^{t,(l)} - Y^{t,(l)}^\top Y^{t,(l)}.
\]

Regarding \( B \), one has
\[
B = -\frac{1}{p} P_{\Omega} \left( \psi_r \left( \left( X^{t,(l)} Y^{t,(l)} \right)_{i,j} - M_{i,j} \right) \right) + \frac{1}{p} P_{\Omega} \left( \psi_r \left( \left( X^{t,(l)} Y^{t,(l)} \right)_{i,j} - M_{i,j}^{*} \right) \right)
\]

35
\[-\mathcal{P}_t\left(\psi_{\tau}\left(\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j}\right)\right)\]

\[= \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5,\]

where

\[\Theta_1 := \frac{1}{p} \mathcal{P}_\Omega \left( \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j}\right| \leq \tau} \right),\]

\[\Theta_2 := \frac{1}{p} \mathcal{P}_\Omega \left( \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j}\right| \leq \tau} \right) - \mathcal{P}_t\left( \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j}\right| \leq \tau} \right),\]

\[\Theta_3 := \frac{1}{p} \mathcal{P}_\Omega \left( \tau \text{sgn} \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M_{i,j}\right| > \tau} \right),\]

\[\Theta_4 := \frac{1}{p} \mathcal{P}_\Omega \left( \tau \text{sgn} \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M_{i,j}\right| > \tau} \right) - \mathcal{P}_t\left( \tau \text{sgn} \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j}\right| > \tau} \right),\]

\[\Theta_5 := \frac{1}{p} \mathcal{P}_\Omega (-\varepsilon_{i,j} \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M_{i,j}\right| \leq \tau}).\]

We shall bound these terms separately as follows.

1. In terms of $\Theta_1$, one has

\[\left\| \frac{1}{p} \mathcal{P}_\Omega \left( \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j}\right| \leq \tau} \right) \right\| \leq (i) \left\| \frac{1}{p} \mathcal{P}_\Omega \left( 11^T \right) \right\| \cdot \max_{i,j} \left\| \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M^*_{i,j}\right| \leq \tau} \right\| \leq (ii) n \left\| X^{t,(l)}Y^{t,(l),T} - M^* \right\|_\infty \leq (iii) n \left\| F^{t,(l)}R^{t,(l)} - F^* \right\|_{2,\infty} \left( \left\| F^{t,(l)} \right\|_{2,\infty} + \left\| F^* \right\|_{2,\infty} \right) \leq (ii) n \left\| F^{t,(l)} - F^* \right\|_{2,\infty} \left\| F^* \right\|_{2,\infty},\]

where (i) is due to (A.15); (ii) comes from Lemma 17; (iii) utilizes (A.3f).

2. Next, we turn attention to $\Theta_3$, which can be bounded as

\[\left\| \frac{1}{p} \mathcal{P}_\Omega \left( \tau \text{sgn} \left( \left( X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M_{i,j} \right) \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M_{i,j}\right| > \tau} \right) \right\| \leq \tau \left\| \frac{1}{p} \mathcal{P}_\Omega \left( \mathbb{1}_{\left|\left(X^{t,(l)}Y^{t,(l),T}\right)_{i,j} - M_{i,j}\right| > \tau} \right) \right\| \leq \tau \left\| \frac{1}{p} \mathcal{P}_\Omega \left( \mathbb{1}_{|\varepsilon_{i,j}| > \tau/2} \right) \right\| \leq \frac{\sigma^2}{(\tau/2)^2} \leq \sigma \sqrt{\frac{n}{p}}.\]
Here (i) arises from (A.15); (ii) comes from the observation that

\[ \mathbb{1}_{\left| (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}\right| > \tau} = \mathbb{1}_{\left| (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}^* - \varepsilon_{i,j}\right| > \tau} \leq \mathbb{1}_{\left| \varepsilon_{i,j}\right| > \tau/2}, \] (A.42)

where the last inequality is due to (A.3a) resulting in

\[ \max_{i,j} \left| (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}^* \right| \leq \left\| F^{t,(l)}R^{t,(l)} - F^* \right\|_{2,\infty} \left\| F^* \right\|_{2,\infty} \ll \left\| F^* \right\|^2_{2,\infty} \leq \frac{\mu \sigma_{\max}}{n} \leq \frac{\tau}{2}; \] (A.43)

(iii) follows from Lemma 17.

3. Next we turn attention to \( \Theta_5 \) which clearly obeys

\[ \left\| \frac{1}{p} \mathcal{P}_{\Omega} \sum_{i,j} (\varepsilon_{i,j} \mathbb{1}_{\left| (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}\right| \leq \tau} \right\| \leq \frac{1}{p} \mathcal{P}_{\Omega} \left( \varepsilon_{i,j} \mathbb{1}_{\left| (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}\right| \leq \tau} \right). \]

Similarly as (A.18), one has

\[ \left\| \frac{1}{p} \mathcal{P}_{\Omega} \left( \varepsilon_{i,j} \mathbb{1}_{\left| (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}\right| \leq \tau} \right) \right\| \leq \sigma \sqrt{\frac{n}{p}}. \]

4. Regarding \( \Theta_2 \), it can be expressed as

\[ \Theta_2 = \frac{1}{p} \sum_{j=1}^{p} v_j, \]

where

\[ v_j := \delta_{l,j} \left( (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}^* \right) \mathbb{1}_{\left| (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}\right| \leq \tau} e_j \]

\[ - p \left( (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}^* \right) \mathbb{1}_{\left| (X^{t,(l)}Y^{t,(l)})_{ij} - M_{i,j}\right| \leq \tau} e_j. \]

It is easy to verify that

\[ L := \max_{1 \leq j \leq n} \left\| v_j \right\|_2 \leq \left\| X^{t,(l)}Y^{t,(l)} - M^* \right\|_\infty, \]

\[ V := \left\| \sum_{j=1}^{n} \mathbb{E} (v_j \top v_j) \right\| \leq np \left\| X^{t,(l)}Y^{t,(l)} - M^* \right\|^2_\infty. \]

We are positioned to apply the matrix Bernstein inequality (Tropp, 2015, Theorem 6.1.1). One has

\[ \frac{1}{p} \left\| \sum_{j=1}^{n} v_j \right\|_2 \leq \frac{1}{n} \left( \sqrt{V \log n} + L \log n \right) \]

\[ \leq \frac{1}{p} \sqrt{np \left\| X^{t,(l)}Y^{t,(l)} - M^* \right\|^2_\infty \log n} + \frac{1}{p} \left\| X^{t,(l)}Y^{t,(l)} - M^* \right\|_\infty \log n \]
\[ \lesssim \sqrt{\frac{n \log n}{p}} \left\| F^{t,(l)} R^{t,(l)} - F^* \right\|_{2,\infty} \| Y^* \|_{2,\infty} + \frac{1}{p} \left\| F^{t,(l)} R^{t,(l)} - F^* \right\|_{2,\infty} \| Y^* \|_{2,\infty} \log n \]

\[ \lesssim \sqrt{\frac{\mu \tau \log n}{p} \sigma_{\max}} \left\| F^{t,(l)} R^{t,(l)\top} - F^* \right\|_{2,\infty}, \]

where the last inequality holds as long as \( np \geq 1 \).

5. Finally for \( \Theta_4 \), since (A.43) implies

\[ \frac{1}{p} \mathcal{P}_{\Omega_l} \left( \tau \text{sgn} \left( (X^{t,(l)} Y^{t,(l)\top})_{l,j} - M_{l,j} \right) \mathbb{I} \left( (X^{t,(l)} Y^{t,(l)\top})_{l,j} - M_{l,j} \right) > \tau \right) e_j^\top e_j = \mathbb{I}_{u_j} \]

It is easy to check that

\[ L := \max_{1 \leq j \leq n} \| u_j \| \leq 1 \]

\[ V := \left\| \sum_{j=1}^n \mathbb{E} \left[ u_j^\top u_j \right] \right\| = \left\| \sum_{j=1}^n \mathbb{E} \left[ \delta_{\tau \text{sgn} \left( (X^{t,(l)} Y^{t,(l)\top})_{l,j} - M_{l,j} \right) \mathbb{I} \left( (X^{t,(l)} Y^{t,(l)\top})_{l,j} - M_{l,j} \right) > \tau \right] e_j^\top e_j \right\| \]

\[ \leq \max \left\{ \sum_{j=1}^n \mathbb{E} \left[ \delta_{\tau \text{sgn} \left( (X^{t,(l)} Y^{t,(l)\top})_{l,j} - M_{l,j} \right) \mathbb{I} \left( (X^{t,(l)} Y^{t,(l)\top})_{l,j} - M_{l,j} \right) > \tau \right] \right\} \left\| \sum_{j=1}^n e_j^\top e_j \right\| \]

\[ \overset{(i)}{\lesssim} \max_j \sum_{j=1}^n \left\| \delta_{\tau \text{sgn} \left( (X^{t,(l)} Y^{t,(l)\top})_{l,j} - M_{l,j} \right) \mathbb{I} \left( (X^{t,(l)} Y^{t,(l)\top})_{l,j} - M_{l,j} \right) > \tau / 2 \right\| \left\| \sum_{j=1}^n e_j^\top e_j \right\| \overset{(ii)}{\lesssim} \frac{\tau p}{p} \sigma^2 (\tau/2)^2 \ll 1, \]

where (i) applies (A.42) and (ii) invokes Markov inequality. Apply the matrix Bernstein inequality (Tropp, 2015, Theorem 6.1.1) to discover that with probability over \( 1 - O(n^{-100}) \),

\[ \frac{\tau}{p} \left( \sqrt{V \log n} + L \log n \right) \lesssim \sigma \sqrt{\frac{n}{p} \log n} + \frac{\| M^* \|_{\infty}}{p} \log n. \]

Combining all the bounds above gives

\[ \| B \| \lesssim n \left\| F^{t,(l)} R^{t,(l)} - F^* \right\|_{2,\infty} \| F^* \|_{2,\infty} + \sigma \sqrt{\frac{n}{p} \log n} + \sqrt{\frac{\mu \tau \log n}{p} \sigma_{\max}} \left\| F^{t,(l)} R^{t,(l)} - F^* \right\|_{2,\infty} \]

\[ + \sigma \sqrt{\frac{n}{p} \log n} + \frac{\| M^* \|_{\infty}}{p} \log n \]

\[ \lesssim \left\| F^{t,(l)} R^{t,(l)} - F^* \right\|_{2,\infty} \sqrt{\mu \tau n \sigma_{\max}} + \sigma \sqrt{\frac{n}{p} \log n} + \frac{\| M^* \|_{\infty}}{p} \log n, \quad (A.44) \]
where the last line invokes Lemma 11 and holds provided that \( np \gg \log n \).

Turning attention to \( C \), we have

\[
\left\| X^{t,(l)} \!^\top X^{t,(l)} - Y^{t,(l)} \!^\top Y^{t,(l)} \right\|_F
\]

\[\overset{(i)}{=} \left\| \left( X^{t,(l)} \!^\top X^{t,(l)} - Y^{t,(l)} \!^\top Y^{t,(l)} \right) R^{t,(l)} \right\|_F\]

\[\overset{(ii)}{\leq} \left\| R^{t,(l)} \!^\top X^{t,(l)} X^{t,(l)} - H^{t} \!^\top X^{t} X^{t} H^{t} \right\|_F + \left\| H^{t} \!^\top Y^{t} Y^{t} H^{t} - R^{t,(l)} \!^\top Y^{t,(l)} Y^{t,(l)} R^{t,(l)} \right\|_F\]

\[\overset{(iii)}{\leq} \left\| R^{t,(l)} \!^\top X^{t,(l)} X^{t,(l)} R^{t,(l)} - H^{t} \!^\top X^{t} X^{t} R^{t,(l)} \right\|_F\]

\[+ \left\| H^{t} \!^\top Y^{t} Y^{t} R^{t,(l)} - H^{t} \!^\top Y^{t} Y^{t} H^{t} \right\|_F + \left\| X^{t} X^{t} - Y^{t} Y^{t} \right\|_F\]

\[+ \left\| Y^{t} Y^{t} R^{t,(l)} - Y^{t} Y^{t} H^{t} \right\|_F\]

\[\leq \left( \| X^* \| + \| X^{t,(l)} \| \right) \left\| X^{t,(l)} R^{t,(l)} - X^{t} H^{t} \right\|_F + \left\| X^{t} X^{t} - Y^{t} Y^{t} \right\|_F\]

\[+ \left( \| Y^* \| + \| Y^{t,(l)} \| \right) \left\| Y^{t,(l)} R^{t,(l)} - Y^{t} H^{t} \right\|_F\]

\[\overset{(iv)}{\leq} \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^* \|_{2,\infty} \| F^* \| \log n\]

\[+ \eta \mu^4 \kappa^3 \sigma_{\max}^2 \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \log^2 n,\]

(A.45)

where (i) follows from the fact that \( \| X \|_F = \| X O \|_F \) for any \( O \in \mathcal{O}^{r \times r} \); (ii) and (iii) come from the triangle inequality; (iv) arises from Lemma 11, 6 and (5.11b). where (i) follows from the fact that \( \| X \|_F = \| X O \|_F \) for any \( O \in \mathcal{O}^{r \times r} \); (ii) is due to \( X^* X = Y^* Y^* \); (iii) comes from (A.3f); (iv) arises from Lemma 4 and 2.

Combining (A.44) and (A.45) with (A.41) yields

\[
\left\| F^{t+1,(l)} H^{t,(l)} - \tilde{F}^{t+1,(l)} \right\|
\]

\[\overset{(i)}{\leq} \eta \| B \| \left\| \Delta^{t,(l)} \right\| + \eta \left\| F^* \right\| \left\| C \right\|_F + \eta \left\| \Delta^{t,(l)} \right\| \left\| C \right\|
\]

\[\overset{(i)}{\leq} \eta \left( \left\| F^{t,(l)} R^{t,(l)} - F^* \right\|_{2,\infty} \sqrt{\kappa \rho n \sigma_{\max}} + \sigma \sqrt{\frac{n}{p}} \log n + \frac{\| M^* \|_{\infty}}{p} \log n \right)
\]

\[\times \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \left\| F^* \right\|
\]

\[+ \frac{\eta}{2} \mu^4 \kappa^3 \sigma_{\max}^2 \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \| F^* \| \log^2 n
\]

\[+ \frac{\eta}{2} \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \left\| F^* \right\|_{2,\infty} \| F^* \| \log n
\]

\[\overset{(i)}{\leq} \eta \sqrt{\kappa} \sigma_{\min} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \left\| F^* \right\| \log n
\]

39
+ \frac{\eta}{2} \sqrt{k} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \|F^*\|_{2,\infty} \|F^*\|^2 \log n,$

where (i) is due to A.3e and the last inequality holds provided that $0 \leq \eta \leq c/(\mu k^5 r^3 \sigma_{\max} \log n)$ for some small constant $c > 0$. Plugging this inequality into (A.40) gives

$$\left\| \left( H^{t,(l)} \right)^{-1} H^{t+1,(l)} - I_r \right\| \leq \frac{2}{\sigma_{\min}} \left\| F^{t+1,(l)} H^{t,(l)} - \tilde{F}^{t+1,(l)} \right\| \left\| F^* \right\| \leq \eta \sqrt{r} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \left\| F^* \right\|^2 \log n \leq \frac{\eta}{2} \kappa^{1.5} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \left\| F^* \right\|_{2,\infty} \left\| F^* \right\| \log n. $$

### A.4 Proof of Lemma 4

We only consider $1 \leq l \leq n$ here. When $n + 1 \leq l \leq 2n$, the bound can be derived analogously. The definition of $R^{t,(l)}$ (cf. (5.10)) implies

$$\left\| F^{t+1} H^{t+1} - F^{t+1,(l)} R^{t+1,(l)} \right\|_F \leq \left\| F^{t+1} H^t - F^{t+1,(l)} R^{t,(l)} \right\|_F. $$

Then the gradient update rules (2.5) reveal that

$$F^{t+1} H^t - F^{t+1,(l)} R^{t,(l)} = F^t H^t - F^t R^{t,(l)} - \eta \left[ \nabla f \left( F^t H^t \right) - \nabla f \left( F^t R^{t,(l)} \right) \right] =: D_1 \quad \text{and} \quad \nabla f \left( F^t R^{t,(l)} \right) - \nabla f \left( F^t R^{t,(l)} \right) =: D_2. \quad \text{(A.46)}$$

To start with, $D_1$ can be controlled similarly as the proof of Lemma 2 and gives

$$\left\| D_1 \right\|_F \leq \left( 1 - \frac{\sigma_{\min}}{20} \eta \right) \left\| F^t H^t - F^t R^{t,(l)} \right\|_F,$$

provided that Then we are left to consider $D_2$. In view of the definition of gradients (cf. (2.1) and (5.5)) and (A.43), one has

$$\nabla f^{(l)} \left( F^{t,(l)} R^{t,(l)} \right) - \nabla f \left( F^{t,(l)} R^{t,(l)} \right) = \begin{bmatrix} P_{1^c} \left( \psi_{t^c} \left( X^{t,(l)} Y^{t,(l)^\top} - M^* \right) \right) - \frac{1}{p} P_{1^c} \left( \psi_{t^c} \left( X^{t,(l)} Y^{t,(l)^\top} - M \right) \right) \end{bmatrix} \left( X^{t,(l)} Y^{t,(l)^\top} - M^* \right) \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

where

$$W_1 := P_{1^c} \left( \left( X^{t,(l)} Y^{t,(l)^\top} \right)_{i,j} - M^*_{i,j} \right) Y^{t,(l)} R^{t,(l)}$$
We would control these terms one by one.

1. The first term $W_1$ obeys

$$p \|W_1\|_F = \left\| \sum_{j=1}^{n} u_j \right\|_F,$$

with

$$u_j := \left( p - \delta_{i,j} \mathbb{1}_{|\langle X^{t(l)} Y^{t(l)} \rangle_{i,j} - M_{i,j}^*| \leq \tau} \right) \left( \left( X^{t(l)} Y^{t(l)} \right)_{i,j} - M_{i,j}^* \right) Y^{t(l)}_{j,\cdot}.$$

We note that $\{u_j\}_{j=1}^{n}$ is a set of independent vectors when conditional on $X^{t(l)}$ and $Y^{t(l)}$. It follows that

$$L \triangleq \max_{1 \leq j \leq n} \|u_j\|_2 \leq \left\| X^{t(l)} Y^{t(l)} - M^* \right\|_\infty \left\| Y^{t(l)} \right\|_{2,\infty},$$

and

$$V \triangleq \left\| \sum_{j=1}^{n} \mathbb{E} \left[ u_j^\top u_j \right] \right\|_2.$$

$$\leq 2 \left\| \sum_{j=1}^{n} \mathbb{E} \left[ p^2 + \delta_{i,j} \mathbb{1}_{|\langle X^{t(l)} Y^{t(l)} \rangle_{i,j} - M_{i,j}^*| \leq \tau} \right] \right\|_2 \left\| X^{t(l)} Y^{t(l)} - M_{i,j}^* \right\|_{2,\infty} \left\| Y^{t(l)} \right\|_{2,\infty}$$

$$\leq 4p \left\| X^{t(l)} Y^{t(l)} - M^* \right\|_\infty \left\| Y^{t(l)} \right\|_F^2.$$
The matrix Bernstein’s inequality gives

\[
\|W_1\|_F \lesssim \frac{1}{p} \left( \sqrt{V \log n} + L \log n \right)
\]

\[
\lesssim \sqrt{\frac{\log n}{p}} \left\| X_{t,(l)} Y_{t,(l)\top} - M^* \right\|_\infty \left\| Y_{t,(l)} \right\|_F^2 \log n + \frac{1}{p} \left\| X_{t,(l)} Y_{t,(l)\top} - M^* \right\|_\infty \left\| Y_{t,(l)} \right\|_2 \log n
\]

\[
\lesssim \sqrt{\frac{\log n}{p}} \left\| F_{t,(l)} R_{t,(l)} - F^* \right\|_{2,\infty} \left\| F^* \right\|_2 \log n + \frac{1}{p} \left\| F_{t,(l)} R_{t,(l)} - F^* \right\|_{2,\infty} \left\| F^* \right\|_2 \log n
\]

\[
\lesssim \sqrt{\frac{\mu r^2 \log n}{np} \sigma_{\max} \left\| F_{t,(l)} R_{t,(l)} - F^* \right\|_{2,\infty}}.
\]

where the second line follows from Lemma 11.

2. Regarding \(W_2\), one has

\[
p \left\| W_2 \right\|_F = \left\| \sum_{j=1}^n v_j \right\|_F = \left\| \sum_{j=1}^n v_j \right\|_2 \left\| X_{t,(l)} \right\|_F
\]

with

\[
v_j := e_j \left( p - \delta_{l,j} 1_{\left\| X_{t,(l)\top} Y_{t,(l)\top} \right\|_{1,j} - M_{l,j} \leq \tau} \right) \left( \left( X_{t,(l)} Y_{t,(l)\top} \right)_{l,j} - M_{l,j}^* \right).
\]

It is easy to obtain that

\[
L \triangleq \max_{1 \leq j \leq n} \left\| v_j \right\|_2 \leq \left\| X_{t,(l)} Y_{t,(l)\top} - M^* \right\|_\infty,
\]

\[
V \triangleq \sum_{j=1}^n \mathbb{E} \left[ \left( p - \delta_{l,j} 1_{\left\| X_{t,(l)\top} Y_{t,(l)\top} \right\|_{1,j} - M_{l,j} \leq \tau} \right) \left( \left( X_{t,(l)} Y_{t,(l)\top} \right)_{l,j} - M_{l,j}^* \right) e_j e_j \right]
\]

\[
\leq 2 \sum_{j=1}^n \mathbb{E} \left[ \left( p^2 + \delta_{l,j} 1_{\left\| X_{t,(l)\top} Y_{t,(l)\top} \right\|_{1,j} - M_{l,j} \leq \tau} \right) \left( \left( X_{t,(l)} Y_{t,(l)\top} \right)_{l,j} - M_{l,j}^* \right) e_j e_j \right]
\]

\[
\leq 4p \left\| X_{t,(l)} Y_{t,(l)\top} - M^* \right\|_\infty^2 \left\| \sum_{j=1}^n e_j e_j \right\| = 4np \left\| X_{t,(l)} Y_{t,(l)\top} - M^* \right\|_\infty^2.
\]

Invoke matrix Bernstein’s inequality (Tropp, 2015, Theorem 6.1.1) gives

\[
\|W_2\|_F \lesssim \frac{1}{p} \left( \sqrt{V \log n} + L \log n \right) \left\| X_{t,(l)} \right\|_2
\]

\[
\lesssim \frac{1}{p} \left( \sqrt{np \left\| X_{t,(l)} Y_{t,(l)\top} - M^* \right\|_\infty^2 \log n + \left\| X_{t,(l)} Y_{t,(l)\top} - M^* \right\|_\infty \log n} \right) \left\| X_{t,(l)} \right\|_2
\]

\[
\lesssim \sqrt{\frac{n \log n}{p}} \left\| X_{t,(l)} Y_{t,(l)\top} - M^* \right\|_{\infty} \left\| X_{t,(l)} \right\|_2
\]

\[
\lesssim \sqrt{\frac{\mu r^2 \sigma_{\max}^2 \log n}{np}} \left\| F_{t,(l)} R_{t,(l)} - F^* \right\|_{2,\infty},
\]

where the third line holds as long as \(np \gg \log n\) and the last inequality holds due to Lemma 11.
3. Turn attention to $Z_1$, we have

$$p \|Z_1\|_F = \tau \left\| \sum_{j=1}^{n} r_j \right\|_F,$$

with

$$r_j := -\delta_{t,j} \text{sgn} \left( \left( X^{t,(l)} Y^{t,(l)\top} \right)_{t,j} - M_{t,j} \right) \mathbb{1}_{\left| (X^{t,(l)} Y^{t,(l)\top})_{t,j} - M_{t,j} \right| > \tau} Y^{t,(l)}_{j,\cdot},$$

where the last equality is due to (A.43). Conditional on $X^{t,(l)}$ and $Y^{t,(l)}$, we can easily show that

$$L \triangleq \max_{1 \leq j \leq n} \| r_j \|_2 \leq \left\| Y^{t,(l)} \right\|_{2,\infty},$$

$$V \triangleq \left\| \sum_{j=1}^{n} \mathbb{E} \left[ r_j r_j^\top \right] \right\| = \left\| \sum_{j=1}^{n} \mathbb{E} \left[ \delta_{t,j} \text{sgn} \left( \left( X^{t,(l)} Y^{t,(l)\top} \right)_{t,j} - M_{t,j} \right) \mathbb{1}_{\left| (X^{t,(l)} Y^{t,(l)\top})_{t,j} - M_{t,j} \right| > \tau} Y^{t,(l)} \left( Y^{t,(l)} \right)^\top \right] \right\| \leq \max_j \left\{ \delta_{t,j} \mathbb{E} \left[ \left( \left( X^{t,(l)} Y^{t,(l)\top} \right)_{t,j} - M_{t,j} \right) \mathbb{1}_{\left| (X^{t,(l)} Y^{t,(l)\top})_{t,j} - M_{t,j} \right| > \tau} \right]^2 \right\} \left\| Y^{t,(l)} \right\|_F^2 \leq \frac{p \sigma^2}{(\tau/2)^2} \left\| Y^{t,(l)} \right\|_F^2,$$

where (i) follows from (A.42) and (A.43) and (ii) applies Markov inequality. The matrix Bernstein inequality (Tropp, 2015, Theorem 6.1.1) implies that

$$\| Z_1 \|_F \lesssim \frac{\tau}{p} \left( \sqrt{p \log n} + L \log n \right) \lesssim \frac{\tau}{p} \left( \sqrt{\frac{p \sigma^2}{\tau^2} \log n} \left\| Y^{t,(l)} \right\|_F + \left\| Y^{t,(l)} \right\|_{2,\infty} \log n \right).$$

4. Similarly, in view of (A.43), we have

$$\| Z_2 \|_F = \left\| \left( \frac{-1}{p} P_{Y_{t,\cdot}^\top} (\tau \text{sgn} \left( \left( X^{t,(l)} Y^{t,(l)\top} \right)_{i,j} - M_{i,j} \right) \mathbb{1}_{\left| (X^{t,(l)} Y^{t,(l)\top})_{i,j} - M_{i,j} \right| > \tau} \right)^\top X^{t,(l)} R^{t,(l)} \right\|_F \lesssim \tau \left\| \sum_{j=1}^{n} b_j \right\|_2 \left\| X^{t,(l)} \right\|_2^\top, \quad (A.47)$$

with

$$b_j := e_j \left( \frac{1}{p} \delta_{i,j} \text{sgn} \left( \left( X^{t,(l)} Y^{t,(l)\top} \right)_{i,j} - M_{i,j} \right) \mathbb{1}_{\left| (X^{t,(l)} Y^{t,(l)\top})_{i,j} - M_{i,j} \right| > \tau} \right).$$

It is easy to verify that

$$L \triangleq \max_{1 \leq j \leq n} \| b_j \|_2 \lesssim \frac{1}{p},$$

43
\[ V \triangleq \left\| \sum_{j=1}^{n} \mathbb{E} \left[ \frac{1}{p} \delta_{l,j} \text{sgn} \left( (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j} \right) \mathbb{I}_{\left| (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j} \right| \geq \tau} \right] e_{j}^{T} e_{j} \right\| \]

\[ \leq 2 \left\| \sum_{j=1}^{n} \mathbb{E} \left[ \frac{1}{p^{2}} \delta_{l,j} \mathbb{I}_{\left| (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j} \right| \geq \tau} \right] e_{j}^{T} e_{j} \right\| \]

\[ \leq 2 \max_{j} \mathbb{E} \left[ \frac{1}{p^{2}} \delta_{l,j} \mathbb{I}_{|\varepsilon_{l,j}| \geq \tau/2} \right] \left\| \sum_{j=1}^{n} e_{j}^{T} e_{j} \right\| \leq 2n \left( \frac{1}{p} \cdot \frac{\sigma^{2}}{(\tau/2)^{2}} \right) \lesssim \frac{1}{p^{2}}, \]

where (i) arises from (A.42) and (ii) is due to Markov inequality. The matrix Bernstein inequality (Tropp, 2015, Theorem 6.1.1) reveals that

\[ \left\| \sum_{j=1}^{n} b_{j} \right\| \lesssim \sqrt{V \log n} + L \log n \lesssim \frac{\log n}{p}. \]

Plugging this bound into (A.42) yields

\[ \|Z_{2}\| \lesssim \frac{\tau \log n}{p} \|X_{l,\ast}\|_{2,\infty} \lesssim \frac{\tau \log n}{p} \|F^{\ast}\|_{2,\infty}, \]

where the last inequality comes from (A.3e).

5. In terms of \( V_{1} \), one has

\[ \frac{p}{2} \left\| V_{1} \right\|_{F} = \mathcal{P}_{\Omega_{l,\ast}} \left( \varepsilon_{l} \mathbb{I}_{\left| (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j} \right| \leq \tau} \right) \|Y^{t,(l)}\|_{F} \]

\[ = \sum_{j=1}^{n} \delta_{l,j} \varepsilon_{l,j} \mathbb{I}_{\left| (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j} \right| \leq \tau} \|Y^{t,(l)}_{j,\ast}\|_{\psi_{1}}. \]

Conditional on \( Y^{t,(l)} \), one has

\[ \left\| \delta_{l,j} \varepsilon_{l,j} \mathbb{I}_{\left| (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j} \right| \leq \tau} \|Y^{t,(l)}_{j,\ast}\|_{\psi_{1}} \right\|_{2,\infty} \lesssim \|Y^{\ast}\|_{2,\infty} \left\| \delta_{l,j} \varepsilon_{l,j} \mathbb{I}_{|\varepsilon_{l,j}| \leq 2\tau} \right\|_{\psi_{1}} \lesssim \tau \|Y^{\ast}\|_{2,\infty}. \]

Here (i) uses (A.3f) and the fact that

\[ \mathbb{I}_{\left| (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j} \right| \leq \tau} = \mathbb{I}_{\left| (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j}^{\ast} - \varepsilon_{l,j} \right| \leq \tau} \]

\[ \leq \mathbb{I}_{|\varepsilon_{l,j}| \leq \tau + \left| (X^{t,(l)}Y^{t,(l)})_{l,j} - M_{l,j}^{\ast} \right| \leq \mathbb{I}_{|\varepsilon_{l,j}| \leq 2\tau}, \]

44
where the last inequality is due to the consequence of (5.11a) and (A.3e) as below,
\[
\max_j \left| \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j^* \right| \leq \left\| \left( F^{t,(l)} H^{t,(l)} - F^* \right) \right\|_{1,2} \left( \left\| F^{t,(l)} \right\|_{2,\infty} + \left\| F^* \right\|_{2,\infty} \right)
\leq 3 \left\| \left( F^{t,(l)} H^{t,(l)} - F^* \right) \right\|_{1,2} \left\| F^* \right\|_{2,\infty}
\lesssim \kappa \sqrt{\tau} \left\| F^* \right\|_{2,\infty} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n \log n}{p}} + \frac{\left\| M^* \right\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \leq \tau.
\]
In addition, one has
\[
V \triangleq \left\| \mathbb{E} \left[ \sum_{j=1}^n \delta_{t,j} \varepsilon_{l,j}^2 I \left( \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j \right) \leq \tau \right] Y_{t,j}^{t,(l)^T} \right\|_F \leq \rho \sigma^2 \left\| Y^{t,(l)} \right\|_F^2.
\]
Hence, invoking the matrix Bernstein inequality (Koltchinskii et al., 2011, Proposition 2) gives that
\[
\left\| \sum_{j=1}^n \delta_{t,j} \varepsilon_{l,j}^2 I \left( \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j \right) \leq \tau \right\|_2 \lesssim \sqrt{V \log n + \tau \left\| Y^* \right\|_{2,\infty} \log n}
\lesssim \tau \left\| Y^* \right\|_{2,\infty} \log n.
\]
6. For the last term $V_2$, one has
\[
\frac{p}{2} \left\| V_2 \right\|_F = \left\| \mathcal{P}_{\Omega_\ell} \left( \varepsilon_{i,j} I \left( \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j \right) \leq \tau \right) X^{t,(l)}_{t,\ell} \right\|_F
= \left\| \mathcal{P}_{\Omega_\ell} \left( \varepsilon_{i,j} I \left( \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j \right) \leq \tau \right) \right\|_2 \left\| X^{t,(l)}_{t,\ell} \right\|_2
= \left\| \sum_{j=1}^n e_j \delta_{t,j} \varepsilon_{l,j} I \left( \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j \right) \leq \tau \right\|_2 \left\| X^{t,(l)}_{t,\ell} \right\|_2.
\]
Similar to the bound of $V_1$, one has
\[
L := \left\| \sum_{j=1}^n e_j \delta_{t,j} \varepsilon_{l,j} I \left( \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j \right) \leq \tau \right\|_2 \lesssim \tau,
\]
\[
V = \left\| \sum_{j=1}^n \mathbb{E} \left[ \delta_{t,j} \varepsilon_{l,j}^2 I \left( \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j \right) \leq \tau \right] e_j e_j^\top \right\| \lesssim \sigma^2 pn.
\]
The matrix Bernstein inequality (Koltchinskii et al., 2011, Proposition 2) reveals that
\[
\left\| \sum_{j=1}^n e_j \delta_{t,j} \varepsilon_{l,j} I \left( \left( X^{t,(l)} Y^{t,(l)^T} \right)_{t,j} - M_j \right) \leq \tau \right\|_2 \lesssim \sigma \sqrt{pm \log n + \tau \log n} \lesssim \tau \log n.
\]
and consequently
\[
\left\| V_2 \right\|_F \lesssim \frac{\tau \log n}{p} \left\| \sum_{j=1}^n e_j^\top \right\|_2 \lesssim \frac{\tau \log n}{p} \left\| F^* \right\|_{2,\infty},
\]
where the last inequality is due to (A.3f).
where (i) follows from (A.3d) and (ii) is due to Lemma 3 and 4.

Taking all the bounds above together, we arrive at

\[
\| \nabla f^{(l)} \left( F^{t,(l)} R^{t,(l)} \right) - \nabla f \left( F^{t,(l)} R^{t,(l)} \right) \|_{F} \\
\lesssim \sqrt{\frac{\mu r^2 \log n}{n p} \sigma_{\max} \| F^{t,(l)} R^{t,(l)} - F^* \|_{2,\infty} + \sqrt{\frac{\mu r^2 \sigma_{\max}^2 \log n}{n p} \| F^{t,(l)} R^{t,(l)} - F^* \|_{2,\infty}} \\
+ \frac{\tau}{p} \left( \sqrt{\frac{\sigma_{\min}^2}{\tau}} \log n \right) \| Y^{t,(l)} \|_{F} + \| Y^{t,(l)} \|_{2,\infty} \log n \right) + \frac{\tau}{p} \| F^* \|_{2,\infty} \log n \\
\lesssim \left( \sigma \sqrt{\frac{n}{p} + \| M^* \|_{\infty} \sqrt{\frac{n}{p}}} \right) \log n \| F^* \|_{2,\infty},
\]

where the second line uses Lemma 11. Plugging the equation above into (A.46) yields

\[
\| F^{t+1} H^t - F^{t+1,(l)} R^{t,(l)} \|_{F} \\
\leq \left( 1 - \frac{\sigma_{\min}}{20 \eta} \right) \| F^t H^t - F^{t,(l)} R^{t,(l)} \|_{F} + C \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \| M^* \|_{\infty} \sqrt{\frac{n}{p}} \right) \| F^* \|_{2,\infty} \log n \\
\lesssim \sqrt{\kappa} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \| M^* \|_{\infty} \sqrt{\frac{n}{p}} \right) \| F^* \|_{2,\infty} \log n.
\]

### A.5 Proof of Lemma 5

For any \( 1 \leq l \leq 2n \), we have

\[
\| (F^{t+1} H^{t+1} - F^*)_{L^2} \|_2 \\
\leq \left\| (F^{t+1} H^{t+1} - F^{t+1,(l)} H^{t+1,(l)})_{L^2} \right\|_2 + \left\| (F^{t+1,(l)} H^{t+1,(l)} - F^*)_{L^2} \right\|_2 \\
\leq \| F^{t+1} H^{t+1} - F^{t+1,(l)} H^{t+1,(l)} \|_{F} + \left\| (F^{t+1,(l)} H^{t+1,(l)} - F^*)_{L^2} \right\|_2 \\
\overset{(i)}{\leq} 5\kappa \| F^{t+1} H^{t+1} - F^{t+1,(l)} R^{t+1,(l)} \|_{F} + \left\| (F^{t+1,(l)} H^{t+1,(l)} - F^*)_{L^2} \right\|_2 \\
\overset{(ii)}{\leq} \kappa^{1.5} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \| M^* \|_{\infty} \sqrt{\frac{n}{p}} \right) \| F^* \|_{2,\infty} \log n + \kappa \sqrt{r} \| F^* \|_{2,\infty} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n \log n}{p}} + \| M^* \|_{\infty} \sqrt{\frac{n}{p}} \right) \\
\lesssim \kappa^{1.5} \sqrt{r} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \| M^* \|_{\infty} \sqrt{\frac{n}{p}} \right) \| F^* \|_{2,\infty} \log n,
\]

where (i) follows from (A.3d) and (ii) is due to Lemma 3 and 4.

### A.6 Proof of Lemma 6

For notational simplicity, we define

\[
A^t := X^t X^t - Y^{t^T} Y^t, \quad \text{and} \quad D^t := \frac{1}{2p} \Omega \left\{ \psi_{\tau} \left( (XY^\top)_{i,j} - M_{i,j} \right) \right\}_{i,j}.
\]
The gradient descent update rules \((2.5)\) reveals that
\[
A^{t+1} = A^t - \eta B^t + \eta^2 C^t
\]  
(A.49)
where
\[
B^t := X^t \nabla_X f(X^t, Y^t) + \nabla_X f(X^t, Y^t) ^\top X^t - Y^t \nabla_Y f(X^t, Y^t) - \nabla_Y f(X^t, Y^t) ^\top Y^t
\]
\[
C^t := \nabla_X f(X^t, Y^t) ^\top \nabla_X f(X^t, Y^t) - \nabla_Y f(X^t, Y^t) ^\top \nabla_Y f(X^t, Y^t).
\]

Due to the definition of \(D^t\) (cf. (A.48)), one has
\[
\nabla_X f(X^t, Y^t) = D^t Y^t + \frac{1}{2} X^t (X^t X^t - Y^t Y^t),
\]
\[
\nabla_Y f(X^t, Y^t) = D^t X^t + \frac{1}{2} Y^t (Y^t Y^t - X^t X^t).
\]

Simple calculation gives that
\[
B^t = X^t \left[ D^t Y^t + \frac{1}{2} X^t (X^t X^t - Y^t Y^t) \right] ^\top X^t
\]
\[
- Y^t \left[ D^t X^t + \frac{1}{2} Y^t \left( Y^t Y^t - X^t X^t \right) \right] ^\top Y^t
\]
\[
= \frac{1}{2} \left( X^t X^t + Y^t Y^t \right) \left( X^t X^t - Y^t Y^t \right) + \frac{1}{2} \left( X^t X^t - Y^t Y^t \right) \left( X^t X^t + Y^t Y^t \right),
\]
\[
C^t = \left[ D^t Y^t + \frac{1}{2} X^t \left( X^t X^t - Y^t Y^t \right) \right] ^\top \left[ D^t Y^t + \frac{1}{2} X^t \left( X^t X^t - Y^t Y^t \right) \right]
\]
\[
- \left[ D^t X^t + \frac{1}{2} Y^t \left( Y^t Y^t - X^t X^t \right) \right] ^\top \left[ D^t X^t + \frac{1}{2} Y^t \left( Y^t Y^t - X^t X^t \right) \right]
\]
\[
= Y^t D^t D^t Y^t - X^t D^t D^t X^t + \frac{1}{4} (A^t)^3
\]
\[
+ \frac{1}{2} \left( Y^t D^t D^t X^t + X^t D^t D^t Y^t \right) A^t + \frac{1}{2} A^t \left( X^t D^t Y^t + Y^t D^t X^t \right).
\]

Plugging these into (A.49), one has
\[
A^{t+1} = A^t - \eta B^t + \eta^2 C^t
\]
\[
= \left[ \frac{1}{2} - \eta \left( X^t X^t + Y^t Y^t \right) + \frac{\eta^2}{8} (A^t)^2 + \frac{\eta^2}{2} \left( Y^t D^t D^t X^t + X^t D^t D^t Y^t \right) \right] A^t
\]
\[
+ A^t \left[ \frac{1}{2} - \eta \left( X^t X^t + Y^t Y^t \right) + \frac{\eta^2}{8} (A^t)^2 + \frac{\eta^2}{2} \left( Y^t D^t D^t X^t + X^t D^t D^t Y^t \right) \right] + \eta^2 E^t,
\]
and thus
\[
\|A^{t+1}\|_F \leq \left\| \frac{1}{2} I_r - \eta \left( X^t X^t + Y^t Y^t \right) + \frac{\eta^2}{8} (A^t)^2 + \frac{\eta^2}{2} \left( Y^t D^t D^t X^t + X^t D^t D^t Y^t \right) \right\| \|A^t\|_F
\]
\[
+ \eta^2 \|E^t\|_F.
\]

To control this upper bound, one has Lemma 11 shows that
\[
\sigma_{\text{min}} \left( X^t X^t + Y^t Y^t \right) \geq \sigma_{\text{min}} \left( X^t X^t \right) \geq \frac{3\sigma_{\text{min}}}{4}.
\]
Furthermore, one has

\[ \|D^t\| = \left\| \frac{1}{2p} \mathcal{P}_\Omega \left( \left\{ \psi_T \left( (XY^\top)_{i,j} - M_{i,j} \right) \right\}_{i,j} \right) \right\| \]

\[ \leq \left\| \frac{1}{p} \mathcal{P}_\Omega \left( \left( (XY^\top)_{i,j} - M_{i,j} \right) \mathbb{1}_{|(XY^\top)_{i,j} - M_{i,j}| \leq \tau} \right) \right\| + \left\| \frac{1}{2p} \mathcal{P}_\Omega \left( \varepsilon_{i,j} \mathbb{1}_{|(XY^\top)_{i,j} - M_{i,j}| > \tau} \right) \right\| \]

\[ \overset{(i)}{\lesssim} \left\| \frac{1}{p} \mathcal{P}_\Omega \left( I \right) \right\|_{\infty} \left( \max_{i,j} \left( (XY^\top)_{i,j} - M_{i,j} \right) \mathbb{1}_{|(XY^\top)_{i,j} - M_{i,j}| \leq \tau} \right) + \sigma \sqrt{\frac{n}{p}} + \sigma \sqrt{\frac{n}{p}} \]

\[ \overset{(ii)}{\lesssim} n \left\| X^{\top} - M^* \right\|_{\infty} + \sigma \sqrt{\frac{n}{p}} \]

\[ \lesssim n \left\| F^t H^t - F^* \right\|_{2,\infty} \left\| F^* \right\|_{2,\infty} + \sigma \sqrt{\frac{n}{p}} \]

\[ \lesssim \mu \kappa^{3/2} r^{3/2} \sigma_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\left\| M^* \right\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \log n, \]

where (i) is due to (A.15), (A.17) and (A.26); (ii) comes from Lemma 17; the last line relies on Lemma 5. Then we turn back to consider (A.50). We have

\[ \left\| \frac{\eta^2}{8} (A^t)^2 + \frac{\eta^2}{2} \left( Y^{Tt} D^{tt} X^t + X^{Tt} D^t Y^t \right) \right\| \]

\[ \lesssim \eta^2 \left( \left\| A^t \right\|^2 + \left\| F^t \right\|^2 \left\| D^t \right\| \right) \]

\[ \lesssim \eta^2 \left\{ \eta \sqrt{\kappa} \sigma^2_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n \log n}{p}} + \frac{\mu r}{\sqrt{n p}} \right)^2 + \mu \kappa^{3/2} r^{3/2} \sigma^2_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\left\| M^* \right\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \log n \right\} \]

\[ \ll \eta \sigma_{\min}, \]

provided \(\eta \mu \kappa r^2 \sigma_{\max} \log n \ll 1\), and then we arrive at

\[ \sigma_{\min} \left( \frac{\eta}{2} (X^{Tt} X^t + Y^{Tt} Y^t) - \frac{\eta^2}{8} (A^t)^2 - \frac{\eta^2}{2} \left( Y^{Tt} D^{tt} X^t + X^{Tt} D^t Y^t \right) \right) \]

\[ \geq \sigma_{\min} \left( \frac{\eta}{2} (X^{Tt} X^t + Y^{Tt} Y^t) - \left\| \frac{\eta^2}{8} (A^t)^2 + \frac{\eta^2}{2} \left( Y^{Tt} D^{tt} X^t + X^{Tt} D^t Y^t \right) \right\| \right) \]

\[ \geq \frac{3\eta \sigma_{\min}}{8} - \frac{\eta \sigma_{\min}}{8} \geq \frac{\eta \sigma_{\min}}{4}. \]

Combine (A.50) with the previous bound to obtain

\[ \|A^{t+1}\|_F \leq 2 \left( \frac{1}{2} - \frac{\eta \sigma_{\min}}{4} \right) \|A^t\|_F + \eta^2 \|E^t\|_F. \]

It remains to bound \(E^t\). We have

\[ \|E^t\|_F \leq \left\| Y^{Tt} D^{tt} D^t Y^t - X^{Tt} D^{tt} D^t X^t \right\|_F. \]

48
\[ \leq 8 \|D^t\|^2 \|F^*\| \|F^*\|_F, \]

which utilizes (A.3e). Plugging this back into (A.51) yields
\[
\|A^{t+1}\|_F \lesssim \frac{\eta}{\sigma_{\min}} \|D^t\|^2 \|F^*\| \|F^*\|_F \lesssim \eta \sqrt{\kappa} \|D^t\|^2 \\
\lesssim \eta \mu^{4r^{3.5} \sigma_{\max}^2} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{n \frac{p}{p}} + \frac{\|M^\ast\|_\infty}{\sigma_{\min}} \sqrt{n \frac{p}{p}} \right)^2 \log^2 n.
\]

## B Proofs for spectral initialization

Before embarking on the proof, let us introduce a few matrices as follows
\[
H_X^0 := \mathop{\text{sgn}}\left((X^0)^\top X^*\right) \quad \text{and} \quad H_Y^0 := \mathop{\text{sgn}}\left((Y^0)^\top Y^*\right),
\]
\[
Q_U := \mathop{\text{sgn}}\left((U^0)^\top U^*\right) \quad \text{and} \quad Q_V := \mathop{\text{sgn}}\left((V^0)^\top V^*\right),
\]
\[
H_U := (U^0)^\top U^* \quad \text{and} \quad H_V := (V^0)^\top V^*. \tag{B.1}
\]

In addition, we give an useful lemma which will facilitate our proof.

**Lemma 12.** Suppose the sample size obeys \(n^2 p \geq C \mu^2 r^2 \kappa^2 n \log n\) for some sufficiently large constant \(C > 0\), the noise satisfies \(\sigma_{\max} \sqrt{n \frac{p}{p}} \leq c\) for some sufficiently small constant \(c > 0\). Then with probability at least \(1 - O(n^{-10})\), one has
\[
\|M^0 - M^*\| \lesssim \sigma \sqrt{n \frac{p}{p}} + \sqrt{n \frac{p}{p}} \|M^*\|_\infty, \tag{B.3a}
\]
\[
\max_{1 \leq l \leq n} \|M^{0,(l)} - M^*\| \lesssim \sigma \sqrt{n \frac{p}{p}} + \sqrt{n \frac{p}{p}} \|M^*\|_\infty \tag{B.3b}
\]
\[
\max \{\|UQ_U - U^*\|, \|VQ_V - V^*\|\} \lesssim \frac{\sigma}{\sigma_{\min}} \sqrt{n \frac{p}{p}} + \frac{\|M^\ast\|_\infty \sqrt{n \frac{p}{p}}}{\sigma_{\min}}, \tag{B.3c}
\]
\[
\max \{\|Q_U - H_U\|, \|Q_V - H_V\|\} \lesssim \frac{n \frac{p}{p}}{\sigma_{\min}^2} \|M^\ast\|_\infty^2 + \sigma^2, \tag{B.3d}
\]
\[
\frac{1}{2} \leq \min \{\|H_U\|, \|H_V\|\} \leq \max \{\|H_U\|, \|H_V\|\} \leq 2. \tag{B.3e}
\]

### B.1 Proof of Lemma 7

Let us introduce the symmetric versions of \(M^*\) and \(M^0\), denoted by \(\tilde{M}^*\) and \(\tilde{M}^0\)
\[
\tilde{M}^* := \begin{bmatrix} 0 & M^* \\ (M^*)^\top & 0 \end{bmatrix}, \quad \tilde{M}^0 := \begin{bmatrix} 0 & M^0 \\ (M^0)^\top & 0 \end{bmatrix}.
\]

Recall that the top-\(r\) SVD of \(M^0\) is \(U^0 \Sigma^0 V^0\). It follows that the top-\(r\) SVD of \(\tilde{M}^0\) would be
\[
\left(\frac{1}{\sqrt{2}} \begin{bmatrix} U^0 \\ V^0 \end{bmatrix}\right) \Sigma^0 \left(\frac{1}{\sqrt{2}} \begin{bmatrix} U^0 \\ V^0 \end{bmatrix}\right)^\top.
\]
Define
\[
Z^0 := \frac{1}{\sqrt{2}} \begin{bmatrix} U^0 \\ V^0 \end{bmatrix}, \quad Z^* := \frac{1}{\sqrt{2}} \begin{bmatrix} U^* \\ V^* \end{bmatrix},
\]
\[
Q := \arg \min_{R \in \mathbb{O}^d \times F} \| Z^0 R - Z^* \|_F.
\]

To prove Lemma 7, we start from an application of Ma et al. (2017, Lemma 45, 46, 47) on \( \tilde{M}^0 \) and obtain:
\[
\| H^0 - Q \| \leq \frac{1}{\sigma_{\min}} \| \tilde{M}^0 - \tilde{M}^* \|, \quad (B.4a)
\]
\[
\| (\Sigma^0)^{1/2} Q - Q (\Sigma^*)^{1/2} \| \leq \frac{1}{\sqrt{\sigma_{\min}}} \| \tilde{M}^0 - \tilde{M}^* \|, \quad (B.4b)
\]
\[
\| Z^0 Q - Z^* \| \leq \frac{1}{\sigma_{\min}} \| \tilde{M}^0 - \tilde{M}^* \|. \quad (B.4c)
\]

Then, we turn attention to the decomposition
\[
\begin{bmatrix} X^0 H^0 - X^* \\ Y^0 H^0 - Y^* \end{bmatrix} = \begin{bmatrix} U^0 \\ V^0 \end{bmatrix} (\Sigma^0)^{1/2} (H^0 - Q) + \begin{bmatrix} U^0 \\ V^0 \end{bmatrix} [(\Sigma^0)^{1/2} Q - Q (\Sigma^*)^{1/2}]
\]
\[
+ \left( \begin{bmatrix} U^0 \\ V^0 \end{bmatrix} Q - \begin{bmatrix} U^* \\ V^* \end{bmatrix} \right) (\Sigma^*)^{1/2}.
\] (B.5)

Taking (B.5) collectively with (B.4) reveals that
\[
\| \begin{bmatrix} X^0 H^0 - X^* \\ Y^0 H^0 - Y^* \end{bmatrix} \| \leq \| (\Sigma^0)^{1/2} \| \| H^0 - Q \| + \| (\Sigma^0)^{1/2} Q - Q (\Sigma^*)^{1/2} \|
\]
\[
+ \| \begin{bmatrix} U^0 \\ V^0 \end{bmatrix} Q - \begin{bmatrix} U^* \\ V^* \end{bmatrix} \| \| (\Sigma^*)^{1/2} \|
\]
\[
\leq \sqrt{\sigma_{\max}} \| H^0 - Q \| + \| (\Sigma^0)^{1/2} Q - Q (\Sigma^*)^{1/2} \| + \sqrt{\sigma_{\max}} \| Z^0 Q - Z^* \|
\]
\[
\leq \left( \frac{\sigma_{\max}}{\sigma_{\min}} + \frac{1}{\sqrt{\sigma_{\min}}} + \frac{\sqrt{\sigma_{\max}}}{\sigma_{\min}} \right) \| \tilde{M}^0 - \tilde{M}^* \|
\]
\[
\leq \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| X^* \|,
\]

where the last line utilizes the immediate consequence of Lemma 12 that
\[
\| \tilde{M}^0 - \tilde{M}^* \| = \| M^0 - M^* \| \leq \sigma \sqrt{\frac{n}{p}} + \sqrt{\frac{n}{p}} \| M^* \|_{\infty}. \quad (B.6)
\]

B.2 Proof of Lemma 8

To start with, we prove an useful lemma.

Lemma 13. Suppose \( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{\nu \mu n \log n}{p}} \leq c \) for some small enough constant \( c > 0 \) and \( n^2 p \geq Ck^3 \mu^2 r^3 n \log n \) for some large enough constant \( C > 0 \). Then with probability at least \( 1 - O(n^{-10}) \), one has
\[
\| U^0 Q_U - U^* \|_{2, \infty} \leq \| U^* \|_{2, \infty} \left[ \kappa \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 + \sqrt{\frac{1}{p} \left( \| M^* \|_{\infty}^2 + \sigma^2 \right) \frac{\sqrt{r \log n}}{\sigma_{\min}}} \right]
\]
\[
\left\| X^0 H^0 - X^* \right\|_{2, \infty} \leq \left\| U^0 \right\|_{2, \infty} \left\{ \left( \Sigma^0 \right)^{1/2} \left\| H^0 - Q_U \right\| + \left( \Sigma^0 \right)^{1/2} Q_U \left( \Sigma^* \right)^{1/2} \right\} + \sqrt{\sigma_{\max}} \left\| U^0 Q_U - U^* \right\|_{2, \infty}
\]

\[
\lesssim \left\| U^* \right\|_{2, \infty} \left( \sqrt{\sigma_{\max}} \frac{1}{\sigma_{\min}} \left\| \tilde{M}^0 - \tilde{M}^* \right\| + \frac{1}{\sqrt{\sigma_{\min}}} \left\| \tilde{M}^0 - \tilde{M}^* \right\| \right)
\]

\[
+ \sqrt{\sigma_{\max}} \left( \sqrt{\sigma_{\max}} \frac{1}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\sigma_{\max}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \log n \right).
\]

### B.2.1 Proof of Lemma 13

To prove this, we begin from a collection of useful lemmas.

**Lemma 14.** Suppose that \( n^2 p \geq C \mu^2 r n \log n \) for some sufficiently large constant \( C > 0 \). Then with probability at least \( 1 - O(n^{-10}) \), one has

\[
\left\| Q_U^0 \Sigma^0 Q_V - \Sigma^* \right\| \lesssim \sigma_{\max} \left( \frac{1}{\sigma_{\min}^2} \sqrt{\frac{\log n}{p} + \frac{\sigma^2}{\tau}} \right) + \sqrt{\frac{1}{p} \left( \left\| M^* \right\|_\infty^2 + \sigma^2 \right) \log n + \frac{\sigma^2}{\tau}},
\]

\[
\left\| H_U^0 \Sigma^0 H_V - \Sigma^* \right\| \lesssim \left( \sqrt{\frac{n}{p}} + \sqrt{\frac{\sigma_{\max}}{\sigma_{\min}^2}} \left\| M^* \right\|_\infty \right)^3 \frac{1}{\sigma_{\min}^2} + \sqrt{\frac{1}{p} \left( \left\| M^* \right\|_\infty^2 + \sigma^2 \right) \log n + \frac{\sigma^2}{\tau}}.
\]

**Lemma 15.** Suppose the sample size obeys \( n^2 p \geq C \mu \kappa r n \log n \) for some sufficiently large constant \( C > 0 \), the noise satisfies \( \frac{\sigma_{\min}}{\sigma_{\max}^2} \sqrt{\frac{n}{p}} \leq \frac{c}{\log n} \) for some sufficiently small constant \( c > 0 \). Then with probability at least \( 1 - O(n^{-10}) \), one has

\[
\left\| U^0 \Sigma^0 H_V - M^0 V^* \right\|_{2, \infty} \lesssim \frac{\log^2 n}{\sigma_{\min}^2} \sqrt{\mu \kappa n \sigma^2 + \frac{\left\| M^* \right\|_\infty^2}{p} + \frac{\tau + \left\| M^* \right\|_\infty}{p} \log n \left\| V^0 H_V - V^* \right\|_{2, \infty}}
\]

\[
+ \frac{n}{\sigma_{\min}^2} \left\| M^* \right\|_\infty \frac{\log n}{p} \left\| U^0 H_U - U^* \right\|_{2, \infty}
\]

\[
+ \left\| U^* \right\|_{2, \infty} \sigma_{\max} \left( \frac{1}{\sigma_{\min}^2} \sqrt{\frac{n}{p}} + \left\| M^* \right\|_\infty \sqrt{\frac{n}{p}} \right)^2.
\]

**Lemma 16.** Suppose the sample size obeys \( np \geq C \kappa^2 \mu^2 r^2 \log^3 n \) for some sufficiently large constant \( C > 0 \), the noise satisfies \( \frac{\sigma_{\min}}{\sigma_{\max}^2} \sqrt{\frac{n}{p}} \leq \frac{c}{\sqrt{\mu \log n}} \) for some sufficiently small constant \( c > 0 \). Then with probability at least \( 1 - O(n^{-10}) \), one has

\[
\left\| U^0 H_U - U^* \right\|_{2, \infty} \lesssim \left( \frac{1}{\sigma_{\min}^2} \sqrt{\frac{\log n}{p}} + \left\| U^* \right\|_{2, \infty} \right)^2 \kappa \left( \frac{1}{\sigma_{\min}^2} \sqrt{\frac{n}{p}} + \left\| M^* \right\|_\infty \sqrt{\frac{n}{p}} \right)^2
\]

51
For the first term \( \beta_1 \), one has
\[
\left\| (M^0 - M^*) \mathbf{V}^* (\Sigma^*)^{-1} \right\|_{2, \infty} \leq \left\| (\mathbf{E} - \mathbf{E}[\mathbf{E}]) \mathbf{V}^* (\Sigma^*)^{-1} \right\|_{2, \infty} + \left\| \mathbf{E}[\mathbf{E}] \mathbf{V}^* (\Sigma^*)^{-1} \right\|_{2, \infty},
\]
where we denote
\[
\mathbf{E} = M^0 - M^*.
\]
Note that the entries of \( \mathbf{E} \) are independent and
\[
E_{i,j} = \left[ \frac{1}{p} \delta_{i,j} (M^*_{i,j} + \epsilon_{i,j}) - M^*_{i,j} \right] \mathbb{I}_{|M^0_{i,j}| \leq \tau} + \left[ \frac{\tau}{p} \delta_{i,j} \text{sign} (M^0_{i,j}) - M^*_{i,j} \right] \mathbb{I}_{|M^0_{i,j}| > \tau}.
\]
It then follows that
\[
\mathbb{E}[E_{i,j}] = \mathbb{E} \left[ \epsilon_{i,j} \mathbb{I}_{|M^0_{i,j}| \leq \tau} + \left( \tau \text{sign} (M^0_{i,j}) - M^*_{i,j} \right) \mathbb{I}_{|M^0_{i,j}| > \tau} \right]
\]
\[
= \mathbb{E} \left( -\epsilon_{i,j} \mathbb{I}_{|M^0_{i,j}| > \tau} + \left( \tau \text{sign} (M^0_{i,j}) - M^*_{i,j} \right) \mathbb{I}_{|M^0_{i,j}| > \tau} \right)
\]
\[
\leq \mathbb{E} \left( \epsilon_{i,j} \mathbb{I}_{|M^0_{i,j}| > \tau} \right) + \mathbb{E} \left( \tau |\text{sign} (M^0_{i,j}) - M^*_{i,j}| \mathbb{I}_{|M^0_{i,j}| > \tau} \right)
\]
\[
\leq \sqrt{\mathbb{E} [\epsilon_{i,j}^2] \mathbb{E} \left[ \mathbb{I}_{|M^0_{i,j}| > \tau} \right]} + \tau + \|M^*\|_{\infty} \mathbb{E} \left[ \mathbb{I}_{|M^0_{i,j}| > \tau} \right]
\]
\[
\leq \frac{\sigma^2}{\tau},
\]
where (i) comes from the fact that
\[
0 = \mathbb{E} [\epsilon_{i,j}] = \mathbb{E} \left[ \epsilon_{i,j} \mathbb{I}_{|M^0_{i,j}| \leq \tau} \right] + \mathbb{E} \left[ \epsilon_{i,j} \mathbb{I}_{|M^0_{i,j}| > \tau} \right],
\]
and (ii) is due to an application of Markov inequality,
\[
\mathbb{E} \left[ \mathbb{I}_{|M^0_{i,j}| > \tau} \right] \leq \mathbb{E} \left[ \mathbb{I}_{|\epsilon_{i,j}| > \tau - \|M^*\|_{\infty}} \right] \leq \mathbb{E} \left[ \mathbb{I}_{|\epsilon_{i,j}| > \tau/2} \right] \leq \frac{\sigma^2}{(\tau/2)^2}.
\]
Furthermore, one has
\[
\mathbb{V} [E_{i,j}] = \mathbb{V} \left[ \frac{1}{p} \delta_{i,j} \left( (M^*_{i,j} + \epsilon_{i,j}) \mathbb{I}_{|M^0_{i,j}| \leq \tau} + \tau \text{sign} (M^0_{i,j}) \mathbb{I}_{|M^0_{i,j}| > \tau} \right) \right]
\]
\[
\leq \frac{1}{p} \mathbb{E} \left[ \left( (M^*_{i,j} + \epsilon_{i,j}) \mathbb{I}_{|M^0_{i,j}| \leq \tau} + \tau \text{sign} (M^0_{i,j}) \mathbb{I}_{|M^0_{i,j}| > \tau} \right)^2 \right].
\]
Next, we turn our attention to combining Lemma 15 and Lemma 16. This yields the bound of

\[ (i) \leq \frac{2}{p} \mathbb{E} \left[ (M^*_i + \epsilon_{i,j})^2 1_{|M^*_i| \leq \tau} + \tau^2 1_{|M^*_i| > \tau} \right] \]

\[ \leq \frac{2}{p} \left( \mathbb{E} \left[ (M^*_i + \epsilon_{i,j})^2 \right] + \tau^2 \mathbb{E} \left[ 1_{|M^*_i| > \tau} \right] \right) \]

\[ \leq \frac{6}{p} \left( \|M^*\|_\infty^2 + \sigma^2 \right) =: \tilde{\sigma}^2, \quad (B.11) \]

where (i) comes from the elementary fact that \((a + b)^2 \leq 2(a^2 + b^2)\) and (ii) makes use of (B.10).

Additionally, we have a simple upper bound that

\[ B := \max_{i,j} |E_{i,j} - \mathbb{E}[E_{i,j}]| \leq \frac{\tau + 2 \|M^*\|_\infty}{p}. \quad (B.12) \]

Therefore, Lemma C.2 gives rise to

\[ \left\| (E - \mathbb{E}[E]) V^* (\Sigma^*)^{-1} \right\|_{2,\infty} \leq \tilde{\sigma} \left\| V^* (\Sigma^*)^{-1} \right\|_F \sqrt{\log n} + B \left\| V^* (\Sigma^*)^{-1} \right\|_{2,\infty} \log n \]

\[ \leq \tilde{\sigma} \sqrt{\tau} \sqrt{\log n} + \frac{B}{\sigma_{\min}} \left\| V^* \right\|_{2,\infty} \log n. \]

Regarding \(\|\mathbb{E}[E] V^*(\Sigma^*)^{-1}\|_{2,\infty}\), one has

\[ \left\| \mathbb{E}[E] V^* (\Sigma^*)^{-1} \right\|_{2,\infty} \leq \left\| \mathbb{E}[E] \right\|_{2,\infty} \left\| V^* (\Sigma^*)^{-1} \right\|_{2,\infty} \leq \frac{\sqrt{n}}{\sigma_{\min}} \max_{i,j} |E_{i,j}| \leq \frac{1}{\sigma_{\min}} \frac{\sigma^2 \sqrt{n}}{\tau}, \]

where the last inequality follows from (B.9).

2. Next, we turn our attention to \(\beta_2\), which can be further decomposed as

\[ \|\Delta U\|_{2,\infty} \leq \|U^0 \Sigma^0 H_V - M^0 V^*\|_{2,\infty} + \|U^0 \Sigma^0 (H_V - Q_V)\|_{2,\infty} \]

\[ \leq \left\| U^0 \Sigma^0 H_V - M^0 V^* \right\|_{2,\infty} + \left\| U^0 \right\|_{2,\infty} \left\| \Sigma^0 \right\| \left\| H_V - Q_V \right\|. \]

Combining Lemma 15 and Lemma 16 yields the bound of \(\gamma_1\)

\[ \|U^0 \Sigma^0 H_V - M^0 V^*\|_{2,\infty} \leq \log^2 n \frac{\sigma_{\min}^2}{\mu r n} \frac{\sigma^2 + \|M^*\|_\infty^2}{p} \]

\[ + \left\| U^* \right\|_{2,\infty} \sigma_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2. \quad (B.13) \]

Regarding \(\gamma_2\), one has

\[ \|U^0\|_{2,\infty} \leq \|U^0 H_U - U^*\|_{2,\infty} + \|U^*\|_{2,\infty} \]

\[ \leq C \left( \sigma + \|M^*\|_\infty \right) \sqrt{\frac{r \log n}{p}} + C \|U^*\|_{2,\infty} \kappa \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \]

\[ + C \left( \sigma \sqrt{np} + \|M^*\|_\infty \right) \log n \sqrt{\frac{\mu r}{n}} + \|U^*\|_{2,\infty} \]

\[ \leq 2 \|U^*\|_{2,\infty}, \quad (B.14) \]
as long as \( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{\kappa_{min} \log n}{p}} \ll 1 \) and \( n^2 p \gg \kappa^2 \mu^2 r^3 n \log n \). Furthermore, we have

\[
\| \Sigma^0 \| = \| Q_U^T \Sigma^0 Q_V - \Sigma^* \| \leq \| Q_U^T \Sigma^0 Q_V - \Sigma^* \| + \| \Sigma^* \| \\
\leq \tilde{C} \sigma_{\max} \left( \frac{n}{\sigma_{\min}^2} \| M^* \|_\infty^2 + \sigma^2 \right) + \tilde{C} \sqrt{\frac{1}{p} \left( \| M^* \|_\infty^2 + \sigma^2 \right)^2 r \log n + \tilde{C} n \sigma^2 \tau + \| \Sigma^* \|} \\
\leq 2 \| \Sigma^* \|, \tag{B.15}
\]

provided \( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \ll 1 \) and \( n^2 p \gg \kappa^2 \mu^2 r^2 n \log n \). Taking (B.14), (B.15) and Lemma 12 collectively gives

\[
\gamma_2 \lesssim \sigma_{\max} \| U^* \|_{2,\infty} \frac{n}{\sigma_{\min}^2} \| M^* \|_\infty^2 + \sigma^2. \tag{B.16}
\]

Consequently, (B.16) combined with (B.13) reveals that

\[
\| \Delta U \|_{2,\infty} \lesssim \frac{\log^2 n}{\sigma_{\min}} \frac{\sqrt{\mu r n} \sigma^2 + \| M^* \|_\infty^2}{p} + \| U^* \|_{2,\infty} \sigma_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \| M^* \|_\infty \sqrt{\frac{n}{p}} \right)^2 \\
+ \sigma_{\max} \| U^* \|_{2,\infty} \frac{n}{\sigma_{\min}^2} \| M^* \|_\infty^2 + \sigma^2 \frac{p}{p}
\]

\[
\lesssim \frac{\log^2 n}{\sigma_{\min}} \frac{\sqrt{\mu r n} \sigma^2 + \| M^* \|_\infty^2}{p} + \| U^* \|_{2,\infty} \sigma_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \| M^* \|_\infty \sqrt{\frac{n}{p}} \right)^2.
\]

3. The last term \( \beta_3 \) can be controlled by utilizing (B.14) and Lemma 14:

\[
\left\| U^0 Q_U \Delta \Sigma (\Sigma^*)^{-1} \right\|_{2,\infty} \\
\leq \| U^0 \|_{2,\infty} \| Q_U \| \| \Delta \Sigma \| \left\| (\Sigma^*)^{-1} \right\| \\
\lesssim \| U^* \|_{2,\infty} \frac{1}{\sigma_{\min}} \left[ \sigma_{\max} \left( \frac{n}{\sigma_{\min}^2} \| M^* \|_\infty^2 + \sigma^2 \right) + \sqrt{\frac{1}{p} \left( \| M^* \|_\infty^2 + \sigma^2 \right)^2 r \log n + \frac{\sigma^2}{\tau}} \right].
\]

Plugging all these bounds into the decomposition (B.7), we arrive at

\[
\left\| U^0 Q_U - U^* \right\|_{2,\infty} \\
\leq \left\| (M^0 - M^*) V^* (\Sigma^*)^{-1} \right\|_{2,\infty} + \left\| \Delta U (\Sigma^*)^{-1} \right\|_{2,\infty} + \left\| U^0 Q_U \Delta \Sigma (\Sigma^*)^{-1} \right\|_{2,\infty} \\
\lesssim \frac{\tilde{C} \sqrt{r} \sqrt{\log n}}{\sigma_{\min}} + \frac{B}{\sigma_{\min}} \| V^* \|_{2,\infty} \log n + \frac{1}{\sigma_{\min}} \frac{\sigma^2 \sqrt{n}}{\tau} \\
+ \frac{1}{\sigma_{\min}} \left[ \log^2 n \frac{\sqrt{\mu r n} \sigma^2 + \| M^* \|_\infty^2}{p} + \| U^* \|_{2,\infty} \sigma_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \| M^* \|_\infty \sqrt{\frac{n}{p}} \right)^2 \right] \\
+ \| U^* \|_{2,\infty} \frac{1}{\sigma_{\min}} \left[ \sigma_{\max} \left( \frac{n}{\sigma_{\min}^2} \| M^* \|_\infty^2 + \sigma^2 \right) + \frac{1}{p} \left( \| M^* \|_\infty^2 + \sigma^2 \right)^2 r \log n + \frac{\sigma^2}{\tau} \right] \\
\lesssim \| M^* \|_\infty \frac{\sqrt{\mu r n}}{p} \log n + \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{\mu r}{p}} \log n
\]

54
whereas the last inequality holds as long as \( np \gg \mu^2 \kappa^2 r^2 \log^2 n \).

### B.2.2 Proof of Lemma 14

To begin with, we can decompose \( \|Q_U^T \Sigma^0 Q_V - \Sigma^*\| \) as

\[
\|Q_U^T \Sigma^0 Q_V - \Sigma^*\| \leq \left\| \left( Q_U - H_U \right)^T \Sigma^0 Q_V \right\| + \left\| H_U^T \Sigma^0 H_V \right\| + \left\| H_U^T \Sigma^0 H_V - U^T M^0 V^* \right\|
\]

\[
\approx_{\alpha_1} \left\| (Q_U - H_U)^T \Sigma^0 Q_V \right\| + \left\| H_U^T \Sigma^0 (Q_V - H_V) \right\|
\]

\[
\leq \left( \|Q_U - H_U\| \|\Sigma^0\| \|H_V\| + \|H_U\| \|\Sigma^0\| \|Q_V - H_V\| \right)
\]

\[
\leq \sigma_{\max} \left( n \sigma_{\min}^2 \|M^*\|_\infty^2 + \sigma^2 \right),
\]

where the last line follows from Lemma 12 and (B.15).

1. Regarding the first term \( \alpha_1 \), one has

\[
\alpha_1 \leq \left( \frac{n}{\sigma_{\min}^2} \right)^3 \frac{1}{\sigma_{\min}},
\]

where the last line arises from Lemma 12.

2. Next, since \( \alpha_2 \) is exactly the same as the term in Yan et al. (2021, Section C.3.2), we can invoke the results therein to obtain

\[
\alpha_2 \lesssim \left( \frac{n}{\sigma_{\min}} \right)^3 \frac{1}{\sigma_{\min}},
\]

where the last line arises from Lemma 12.

3. Turning to the last term \( \alpha_3 \), one has

\[
\alpha_3 = \left\| U^T (M^0 - M^*) V^* \right\| \leq \left\| U^T (M^0 - M^* - \mathbb{E} [M^0 - M^*]) V^* \right\| + \left\| \mathbb{E} [M^0 - M^*] V^* \right\|
\]

The bound of \( \alpha_{31} \) can be derived in the same way as Yan et al. (2021, Equation (C.11))

\[
\alpha_{31} \lesssim \left( \frac{n}{\sigma_{\min}} \right)^3 \frac{1}{\sigma_{\min}},
\]

where \( \tilde{\sigma} \) and \( B \) are defined in (B.11) and (B.12), and the last inequality holds as long as \( n \gg \mu^2 r \log n \) and \( np \gg 1 \). Regarding \( \alpha_{32} \), one has

\[
\alpha_{32} \leq \left\| U^* \right\| \left\| V^* \right\| \left\| \mathbb{E} [M^0 - M^*] \right\| \leq \left\| M^0 - M^* \right\|_F \lesssim nB.
\]
Finally, taking all the results above together, one has
\[
\|\Delta \Sigma\| \lesssim \sigma_{\max} \left( \frac{n}{\sigma_{\min}^2} \frac{\|M^*\|^2}{p} + \frac{1}{p} \left( \|M^*\|^2 + \sigma^2 \right) \sqrt{r \log n + n \frac{\sigma^2}{\tau}} \right).
\]

Additionally, we have
\[
\|H_U^\top \Sigma H_V - \Sigma^*\| \leq \|H_U^\top \Sigma H_V - U^\top M^0 V^*\| + \|U^\top M^0 V^* - \Sigma^*\| = \alpha_2 + \alpha_3 \leq \sigma_{\max} \left( \frac{n}{\sigma_{\min}^2} \frac{\|M^*\|^2}{p} + \frac{1}{p} \left( \|M^*\|^2 + \sigma^2 \right) \sqrt{r \log n + n \frac{\sigma^2}{\tau}} \right).
\]

**B.2.3 Proof of Lemma 15**

Applying the triangle inequality enables us to obtain
\[
\|U^0 \Sigma^0 H_V - M^0 V^*\|_{2,\infty} \leq \|M^0 - M^*\|_{2,\infty} + \|M^* (V^0 H_V - V^*)\|_{2,\infty} \leq \|M^* (V^0 H_V - V^*)\|_{2,\infty} + \|0 - M^* - E [M^0 - M^*]\| \|V^0 H_V - V^*\|_{2,\infty} + \|E [M^0 - M^*]\| \|V^0 H_V - V^*\|_{2,\infty} =: \beta_1 + \beta_2 + \beta_3.
\]

In what follows, we shall control these three terms separately.

1. We start from \(\beta_2\). In view of the leave-one-out sequences defined in Algorithm 2, one has
\[
\|M^0 - M^* - E [M^0 - M^*]\| \|V^0 H_V - V^*\|_{2,\infty} \leq \|M^0 - M^*\| \|V^0 H_V - V^*\|_{2,\infty} \leq \|M^0 - M^*\| \|V^0 H_V - V^*\|_{2,\infty} + \|E [M^0 - M^*]\| \|V^0 H_V - V^*\|_{2,\infty} =: \alpha_1 + \alpha_2.
\]

Recall the definitions of \(B\) and \(\tilde{\sigma}\) in (B.11) and (B.12). Conditional on \(V^{(l)}\), invoking Lemma C.2 yields
\[
\alpha_1 \lesssim \tilde{\sigma} \sqrt{\log n} \|V^{0,(l)} H^{(l)}_V - V^*\|_F + B \log n \|V^{0,(l)} H^{(l)}_V - V^*\|_{2,\infty} \lesssim \tilde{\sigma} \sqrt{\log n} \|V^0 H_V - V^*\|_F + B \log n \|V^0 H_V - V^*\|_{2,\infty} + \left( \tilde{\sigma} \sqrt{\log n} + B \log n \right) \|V^{0,(l)} H^{(l)}_V - V^0 H_V\|_F.
\]

Regarding \(\alpha_2\), we have
\[
\alpha_2 \leq \|(M^0 - M^* - E [M^0 - M^*])\|_2 \|V^{0,(l)} H^{(l)}_V - V^0 H_V\|_F
\]
56
where the second line makes use of Chen et al. (2021a, Theorem 3.4). To bound \( \| V^{0,(l)} H_V^{(l)} - V^0 H_V \| \), one has

\[
\| V^{0,(l)} H_V^{(l)} - V^0 H_V \|_F = \left\| \left( V^{0,(l)} V^{0,(l)^T} - V^0 V^{0^T} \right) V^* \right\|_F \\
\leq \left\| V^{0,(l)} V^{0,(l)^T} - V^0 V^{0^T} \right\|_F.
\]

(B.21)

Invoking Wedin’s sin\( \Theta \) Theorem Chen et al. (2021a, Theorem 2.9) yields

\[
\max \left\{ \| U^{0,(l)} U^{0,(l)^T} - U^0 U^{0^T} \|_F, \| V^{0,(l)} V^{0,(l)^T} - V^0 V^{0^T} \|_F \right\} \\
\lesssim \max \left\{ \| (M^{0,(l)} - M^0) V^{0,(l)} \|_F, \| (M^{0,(l)} - M^0)^T U^{0,(l)} \|_F \right\} \\
\lesssim \max \left\{ \| (M^{0,(l)} - M^0) V^{0,(l)} \|_F, \| (M^{0,(l)} - M^0)^T U^{0,(l)} \|_F \right\} \\
\lesssim \max \left\{ \| (M^{0,(l)} - M^0) V^{0,(l)} \|_F, \| (M^{0,(l)} - M^0)^T U^{0,(l)} \|_F \right\},
\]

(B.22)

where the last inequality utilizes Lemma 12. Then we turn attention to \( \| (M^{0,(l)} - M^0) V^{0,(l)} \|_F \) and \( \| (M^{0,(l)} - M^0)^T U^{0,(l)} \|_F \). In view of the definition of \( M^{0,(l)} \) (cf. (5.7)), we can deduce that (B.11)

\[
\left( M^{0,(l)} - M^0 \right) V^{0,(l)} \|_F \\
\leq \left( \frac{1}{2p} P_l \left( \psi_r \left( M^0 \right) \right) - P_l \left( M^* \right) \right) V^{0,(l)} \\
\overset{(i)}{\leq} \left( E_l V^{0,(l)} H^{(l)}_V \right) \|_2 \\
\leq \left( E - E [E] \right)_{l^*} V^* \|_2 + \left( E - E [E] \right)_{l^*} \left( V^{0,(l)} H^{(l)}_V - V^* \right) \|_2 \\
\overset{(ii)}{\lesssim} \tilde{\sigma} \sqrt{\log n} \| V^* \|_F + B \log n \| V^* \|_{2,\infty} + \alpha_1 + \left( \frac{\sigma^2}{\tau} \right) \sqrt{n} \| V^* \|,
\]

(B.23)

where (i) is due to (B.3e); (ii) comes from (B.9), Lemma C.2 and the fact that

\[
\| V^{0,(l)} H^{(l)}_V \| \lesssim \| V^{0,(l)} \| \lesssim \| V^* \|.
\]

In terms of \( \| (M^{0,(l)} - M^0)^T U^{0,(l)} \|_F \), one has

\[
\left\| (M^{0,(l)} - M^0)^T U^{0,(l)} \right\|_F = \left\| E_l^T U^{0,(l)}_{l^*} \right\|_F = \left\| E_l \right\|_2 \left\| U^{0,(l)}_{l^*} \right\|_2 \\
\overset{(i)}{\leq} 2 \left( \left\| (E - E [E])_{l^*} \right\|_2 + \left\| (E [E])_{l^*} \right\|_2 \right) \left\| U^{0,(l)}_{l^*} H^{(l)}_U \right\|_2
\]

57
\[
\begin{align*}
\text{(i)} & \quad \lesssim 2 \left( \|E - E^\top E\| + \frac{\sigma^2}{\tau} \sqrt{n} \right) \left( \|U^*\|_{2,\infty} + \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \right) \\
\text{(ii)} & \quad \lesssim 2 \left( \bar{\sigma} \sqrt{n} + \frac{\sigma^2}{\tau} \sqrt{n} \right) \left( \|U^*\|_{2,\infty} + \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \right) \\
\text{(iii)} & \quad \lesssim \bar{\sigma} \sqrt{n} \left( \|U^*\|_{2,\infty} + \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \right),
\end{align*}
\]

(B.24)

where (i) arises from Lemma 12; (ii) is due to (B.9); (iii) follows from the standard matrix tail bounds \citep[Theorem 3.4]{chen2021}. Plugging (B.23) and (B.24) into (B.22) and (B.21) gives

\[
\max \left\{ \left\| U^{0,(l)} H^{(l)}_U - U^0 H_U \right\|_F, \left\| V^{0,(l)} H^{(l)}_V - V^0 H_V \right\|_F \right\}
\]

\[
\lesssim \frac{\bar{\sigma}}{\sigma_{\min}} \sqrt{\log n} \left\| V^* \right\|_F + \frac{B}{\sigma_{\min}} \log n \left\| V^* \right\|_{2,\infty} + \frac{\alpha_1}{\sigma_{\min}} + \left( \frac{\sigma^2}{\bar{\sigma} \sigma_{\min}} \right) \sqrt{n} \left\| V^* \right\|
\]

\[
+ \frac{\bar{\sigma} \sqrt{n}}{\sigma_{\min}} \left( \|U^*\|_{2,\infty} + \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \right). \tag{B.25}
\]

Then we turn to bound \( \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \). One has

\[
\left\| U^{0,(l)} H^{(l)}_U - U^* \right\|_{2,\infty} \lesssim \|U^0 H_U - U^*\|_{2,\infty} + \left\| U^{0,(l)} H^{(l)}_U - U^0 H_U \right\|_F
\]

\[
\lesssim \|U^0 H_U - U^*\|_{2,\infty} + \frac{\bar{\sigma}}{\sigma_{\min}} \sqrt{\log n} \left\| V^* \right\|_F + \frac{B}{\sigma_{\min}} \log n \left\| V^* \right\|_{2,\infty} + \frac{\alpha_1}{\sigma_{\min}}
\]

\[
+ \left( \frac{\sigma^2}{\bar{\sigma} \sigma_{\min}} \right) \sqrt{n} \left\| V^* \right\| + \frac{\bar{\sigma} \sqrt{n}}{\sigma_{\min}} \left( \|U^*\|_{2,\infty} + \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \right). \tag{B.26}
\]

Rearrange the terms containing \( \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \) to obtain

\[
\left\| U^{0,(l)} H^{(l)}_U - U^* \right\|_{2,\infty} \lesssim \|U^0 H_U - U^*\|_{2,\infty} + \frac{\bar{\sigma} \sqrt{\log n}}{\sigma_{\min}} \left\| V^* \right\|_F + \frac{B \log n \left\| V^* \right\|_{2,\infty}}{\sigma_{\min}} + \frac{\alpha_1}{\sigma_{\min}} + \left( \frac{\sigma^2}{\bar{\sigma} \sigma_{\min}} \right) \sqrt{n} \left\| V^* \right\|
\]

\[
+ \frac{\bar{\sigma} \sqrt{n}}{\sigma_{\min}} \left( \|U^*\|_{2,\infty} + \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \right). \tag{B.26}
\]

Substitution of (B.26) into (B.20) yields

\[
\max \left\{ \left\| U^{0,(l)} H^{(l)}_U - U^0 H_U \right\|_F, \left\| V^{0,(l)} H^{(l)}_V - V^0 H_V \right\|_F \right\}
\]

\[
\lesssim \frac{\bar{\sigma} \sqrt{\log n}}{\sigma_{\min}} \left\| V^* \right\|_F + \frac{B \log n \left\| V^* \right\|_{2,\infty}}{\sigma_{\min}} + \frac{\alpha_1}{\sigma_{\min}} + \left( \frac{\sigma^2}{\bar{\sigma} \sigma_{\min}} \right) \sqrt{n} \left\| V^* \right\|
\]

\[
+ \frac{\bar{\sigma} \sqrt{n}}{\sigma_{\min}} \left( \|U^*\|_{2,\infty} + \|U^{0,(l)} H^{(l)}_U - U^*\|_{2,\infty} \right), \tag{B.27}
\]

provided that \( \frac{\sigma}{\sigma_{\min}} \sqrt{n} \ll 1 \). Plugging this into (B.20) gives

\[
\alpha_2 \lesssim \frac{\left( \sqrt{n} + B \sqrt{\log n} \right)}{\sigma_{\min}} \left( \frac{\sigma^2}{\bar{\sigma} \sigma_{\min}} \right) \left( \bar{\sigma} \sqrt{\log n} \left\| V^* \right\|_F + B \log n \left\| V^* \right\|_{2,\infty} + \alpha_1 + \left( \frac{\sigma^2}{\tau} \right) \sqrt{n} \left\| V^* \right\| \right)
\]

58
Furthermore, substitution of (B.27) into (B.19) yields

\[
\alpha_1 \lesssim \tilde{\sigma} \sqrt{\log n} \| V^0 U - V^* \|_F + B \log n \| V^0 H_U - V^* \|_{2,\infty}
+ \frac{(\tilde{\sigma} \sqrt{\log n} + B \log n)}{\sigma_{\min}} \left( \tilde{\sigma} \sqrt{\log n} \| V^* \|_F + B \log n \| V^* \|_{2,\infty} + \alpha_1 + \left( \frac{\sigma^2}{\tau} \right) \sqrt{n} \| V^* \| \right)
+ \frac{(\tilde{\sigma} \sqrt{\log n} + B \log n)}{\sigma_{\min}} \left( \tilde{\sigma} \sqrt{n} \| U^* \|_{2,\infty} + \tilde{\sigma} \sqrt{n} \| U^0 H_U - U^* \|_{2,\infty} \right).
\]

We can rearrange terms to derive the bound of \(\alpha_1\)

\[
\alpha_1 \lesssim \tilde{\sigma} \sqrt{\log n} \| V^0 H_U - V^* \|_F + B \log n \| V^0 H_U - V^* \|_{2,\infty}
+ \frac{(\tilde{\sigma} \sqrt{\log n} + B \log n)}{\sigma_{\min}} \left( \tilde{\sigma} \sqrt{\log n} \| V^* \|_F + B \log n \| V^* \|_{2,\infty} + \left( \frac{\sigma^2}{\tau} \right) \sqrt{n} \| V^* \| \right)
+ \frac{(\tilde{\sigma} \sqrt{\log n} + B \log n)}{\sigma_{\min}} \left( \tilde{\sigma} \sqrt{n} \| U^* \|_{2,\infty} + \tilde{\sigma} \sqrt{n} \| U^0 H_U - U^* \|_{2,\infty} \right)
\lesssim \frac{\tilde{\sigma} \sqrt{\log n}}{\sigma_{\min}} \left( \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty} \sqrt{\frac{n}{p}}}{\sigma_{\min}} + B \log n \left( B \sqrt{\frac{\log n}{n}} \| V^* \| + \tilde{\sigma} \sqrt{\mu r} \right) \right)
+ B \log n \| V^0 H_U - V^* \|_{2,\infty}.
\]

Here the second line utilizes the fact that

\[
\| V^0 H_U - V^* \|_F = \| V^0 V^0^T V^* - V^* \|_F = \left\| V^0 V^0^T - V^* V^*^T \right\|_F
\leq \| V^0 V^0^T - V^* V^*^T \|_F \leq \| V^0 Q_V - V^* \|_F
\lesssim \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty} \sqrt{n}}{\sigma_{\min}},
\]

where the last inequality follows from Lemma 12. Combining (B.29) and (B.28), we obtain

\[
\left\| (M^0 - M^* - E [M^0 - M^*]) (V^0 H_U - V^*) \right\|_{2,\infty}
\lesssim \frac{(\tilde{\sigma} \sqrt{\log n} + B \log n)}{\sigma_{\min}} \left( \tilde{\sigma} \sqrt{\log n} \| V^* \|_F + B \log n \| V^* \|_{2,\infty} + \alpha_1 + \left( \frac{\sigma^2}{\tau} \right) \sqrt{n} \| V^* \| \right)
+ \frac{(\tilde{\sigma} \sqrt{n} + B \log n)}{\sigma_{\min}} \left( \tilde{\sigma} \sqrt{n} \| U^* \|_{2,\infty} + \tilde{\sigma} \sqrt{n} \| U^0 H_U - U^* \|_{2,\infty} \right)
+ \frac{\tilde{\sigma} \sqrt{\tau \log n}}{\sigma_{\min}} \left( \sigma \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty} \sqrt{n}}{\sigma_{\min}} \right) + B \log n \left( B \sqrt{\frac{\log n}{n}} \| V^* \| + \tilde{\sigma} \sqrt{\mu r} \right)
+ B \log n \| V^0 H_U - V^* \|_{2,\infty}
\lesssim \tilde{\sigma}^2 \sqrt{\mu r \log n} \frac{\log^2 n}{\sigma_{\min}} + B \log n \| V^0 H_U - V^* \|_{2,\infty} + \frac{\tilde{\sigma}^2 n \sqrt{\log n}}{\sigma_{\min}} \| U^0 H_U - U^* \|_{2,\infty},
\]

where the last inequality holds provided \(\frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \ll \frac{1}{\log n}\) and \(np \gg \mu \kappa r \log n\).
2. Next, we turn attention to $\beta_3$, which satisfies
\[
\| \mathbb{E} [M^0 - M^*] (V H V^* - V^*) \|_{2,\infty} \\
\leq \| \mathbb{E} [M^0 - M^*] \|_{2,\infty} \| V^0 H V - V^* \|
\]
\[
\lesssim \sqrt{n} \sigma^2 \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right),
\]
where the third line follows from (B.9) and a consequence of Lemma 12.

\[
\| V^0 H V - V^* \| \leq \| V^0 Q V - V^* \| + \| V^0 H V - V^0 Q V \|
\leq \| V^0 Q V - V^* \| + \| H V - Q V \|
\]
\[
\lesssim \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}}.
\]

(B.31)

3. Lastly, we can reuse the results in Yan et al. (2021, Section 3.3 Step 3) to bound $\beta_1$ as
\[
\| M^* (V^0 H V - V^*) \|_{2,\infty} \lesssim \| U^* \|_{2,\infty} \sigma_{\max} \| V^0 R V - V^* \|^2
\]
\[
\lesssim \| U^* \|_{2,\infty} \sigma_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2.
\]

(B.32)

Finally, plugging the bounds (B.30), (B.31) and (B.32) into (B.18) reveals that
\[
\| U^0 \Sigma^0 H V - M^0 V^* \| \lesssim \frac{\log^2 n}{\sigma_{\min}} \sqrt{\mu \sigma^2 + \| M^* \|_{\infty}^2} + \frac{\tau + \| M^* \|_{\infty}}{p} \log n \| V^0 H V - V^* \|_{2,\infty}
\]
\[
+ \frac{1}{\sigma_{\min}} \frac{\sigma^2 + \| M^* \|_{\infty}^2}{n} \| U^0 H V - U^* \|_{2,\infty}
\]
\[
+ \| U^* \|_{2,\infty} \sigma_{\max} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2.
\]

(B.2.4 Proof of Lemma 16)

We start from a crucial decomposition
\[
\| U^0 H V - U^* \|_{2,\infty} \leq \frac{1}{\sigma_{\min}} \| U^0 H V \Sigma^* - U^* \Sigma^* \|_{2,\infty}
\]
\[
\leq \frac{1}{\sigma_{\min}} \| U^0 H V \Sigma^* - U^0 \Sigma^0 H V \|_{2,\infty} + \frac{1}{\sigma_{\min}} \| U^0 \Sigma^0 H V - M^0 V^* \|_{2,\infty}
\]
\[
+ \frac{1}{\sigma_{\min}} \| (M^0 - M^*) V^* \|_{2,\infty}.
\]

In the sequel, we shall bound these three terms separately.

1. To bound $\omega_1$, we note that it is exactly the term $\beta_3$ in Yan et al. (2021, Section C.3.4). Invoking the results therein, one has
\[
\omega_1 \leq \frac{1}{\sigma_{\min}} \left( \| U^* \|_{2,\infty} + \| U^0 H V - U^* \|_{2,\infty} \right) \left( \| H_U \Sigma^0 H V - \Sigma^* \| + \| \Sigma^* \| \| H_U - Q V \| \right).
\]

60
2. Regarding $\omega_2$, Lemma 15 reveals that

$$\frac{1}{\sigma_{\min}} \|U^0 \Sigma^0 H_V - M^0 V^*\|_{2,\infty}$$

$$\lesssim \frac{\log^2 n}{\sigma_{\min}^2} \frac{\sigma^2 + \|M^*\|_{\infty}^2}{p} + \frac{\tau + \|M^*\|_{\infty}}{p \sigma_{\min}} \log n \|V^0 H_V - V^*\|_{2,\infty}$$

$$+ \frac{n \sigma^2 + \|M^*\|_{\infty}^2}{p} \|U^0 H_U - U^*\|_{2,\infty} + \|U^*\|_{2,\infty} \kappa \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2.$$

3. Turning attention to the last term $\omega_3$, we have

$$\frac{1}{\sigma_{\min}} \|(M^0 - M^*) V^*\|_{2,\infty}$$

$$\lesssim \frac{1}{\sigma_{\min}} \| (M^0 - M^* - \mathbb{E} [M^0 - M^*]) V^* \|_{2,\infty} + \frac{1}{\sigma_{\min}} \| \mathbb{E} [M^0 - M^*] V^* \|_{2,\infty}$$

$$\lesssim \frac{1}{\sigma_{\min}} \left( \overline{\sigma} \|V^*\|_F \sqrt{\log n} + B \|V^*\|_{2,\infty} \log n \right) + \frac{\sqrt{n}}{\sigma_{\min}} B$$

$$\lesssim \frac{(\sigma + \|M^*\|_{\infty})}{\sigma_{\min}} \sqrt{\frac{r \log n}{p}} + \frac{(\tau + \|M^*\|_{\infty})}{\sigma_{\min}} \log n \|V^*\|_{2,\infty},$$

where (i) is due to Lemma C.2 and (ii) plugs in the definitions of $\overline{\sigma}$ and $B$ (cf. (B.11) and (B.12)).

Finally, taking the above bounds in $\omega_1, \omega_2$ and $\omega_3$ collectively, we arrive at

$$\|U^0 H_U - U^*\|_{2,\infty}$$

$$\lesssim \frac{1}{\sigma_{\min}} \left( \|U^*\|_{2,\infty} + \|U^0 H_U - U^*\|_{2,\infty} \right) \left[ \|H_U^\top \Sigma^0 H_V - \Sigma^*\| + \|V^0 H_V - V^*\|_{2,\infty} \right]$$

$$+ \frac{\log^2 n}{\sigma_{\min}^2} \frac{\sigma^2 + \|M^*\|_{\infty}^2}{p} + \frac{\tau + \|M^*\|_{\infty}}{p \sigma_{\min}} \log n \|V^0 H_V - V^*\|_{2,\infty}$$

$$+ \frac{n \sigma^2 + \|M^*\|_{\infty}^2}{p} \|U^0 H_U - U^*\|_{2,\infty} + \|U^*\|_{2,\infty} \kappa \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2$$

$$+ \frac{(\sigma + \|M^*\|_{\infty})}{\sigma_{\min}} \sqrt{\frac{r \log n}{p}} + \frac{(\tau + \|M^*\|_{\infty})}{\sigma_{\min}} \log n \|V^*\|_{2,\infty}.$$
\begin{align*}
\leq & \left( \frac{\sigma + \|M^*\|_{\infty}}{\sigma_{\min}} \right) \sqrt{\frac{r \log n}{p}} + \frac{\tau + \|M^*\|_{\infty}}{p \sigma_{\min}} \log n \|U^0H_U - U^*\|_{2,\infty} \\
& + \|V^*\|_{2,\infty} \kappa \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 + \frac{\tau + \|M^*\|_{\infty}}{\sigma_{\min} p} \|U^*\|_{2,\infty}.
\end{align*}
\tag{B.34}

Plugging (B.34) into (B.33) and rearranging terms reveal that

\begin{align*}
\|U^0H_U - U^*\|_{2,\infty} \leq & \left( \frac{\sigma + \|M^*\|_{\infty}}{\sigma_{\min}} \right) \sqrt{\frac{r \log n}{p}} + \|U^*\|_{2,\infty} \kappa \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right)^2 \\
& + \left( \frac{\sigma \sqrt{n p} + \|M^*\|_{\infty}}{\sigma_{\min} p} \right) \log n \|V^*\|_{2,\infty}.
\end{align*}

The bound of \( \|V^0H_V - V^*\|_{2,\infty} \) can be established by a similar argument.

**B.3 Proof of Lemma 9**

For any fixed \( 1 \leq l \leq 2n \), the triangle inequality enables us to obtain

\begin{align*}
\left\| \begin{bmatrix} X^{0,(l)}H^{0,(l)} - X^* \\ Y^{0,(l)}H^{0,(l)} - Y^* \end{bmatrix} \right\|_2 & \leq \left\| \begin{bmatrix} X^{0,(l)} \left( H^{0,(l)} - Q^{(l)} \right) \\ Y^{0,(l)} \left( H^{0,(l)} - Q^{(l)} \right) \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} X^{0,(l)}Q^{(l)} - X^* \\ Y^{0,(l)}Q^{(l)} - Y^* \end{bmatrix} \right\|_2 \\
& \leq \left\| \begin{bmatrix} X^{0,(l)} \\ Y^{0,(l)} \end{bmatrix} \right\|_2 \left\| H^{0,(l)} - Q^{(l)} \right\| + \left\| \begin{bmatrix} X^{0,(l)}Q^{(l)} - X^* \\ Y^{0,(l)}Q^{(l)} - Y^* \end{bmatrix} \right\|_2.
\end{align*}
\tag{B.35}

In view of

\[ \begin{bmatrix} X^{0,(l)} \\ Y^{0,(l)} \end{bmatrix} = \begin{bmatrix} 0 & M^{0,(l)}_0 \\ (M^{0,(l)}_0)^\top & 0 \end{bmatrix} \begin{bmatrix} U^{0,(l)} \\ V^{0,(l)} \end{bmatrix} \left( \Sigma^{(l)} \right)^{-1/2} \]

and the definition of \( \widetilde{M}^* \) (cf. (B.6)), we can obtain

\[ \begin{bmatrix} X^{0,(l)}Q^{(l)} - X^* \\ Y^{0,(l)}Q^{(l)} - Y^* \end{bmatrix} = \widetilde{M}^{0,(l)}_l \begin{bmatrix} U^{0,(l)} \\ V^{0,(l)} \end{bmatrix} \left( \Sigma^{(l)} \right)^{-1/2}Q^{(l)} - \widetilde{M}^* \begin{bmatrix} U^* \\ V^* \end{bmatrix} \left( \Sigma^* \right)^{-1/2}. \]

Due to the fact that \( \widetilde{M}^{0,(l)}_l = \widetilde{M}^*_l \), one has

\[ \begin{bmatrix} X^{0,(l)}Q^{(l)} - X^* \\ Y^{0,(l)}Q^{(l)} - Y^* \end{bmatrix} \bigg|_{l^*} = \widetilde{M}^*_l \begin{bmatrix} U^{0,(l)} \\ V^{0,(l)} \end{bmatrix} \left( \Sigma^{(l)} \right)^{-1/2}Q^{(l)} - \begin{bmatrix} U^* \\ V^* \end{bmatrix} \left( \Sigma^* \right)^{-1/2} \]

\[ + \widetilde{M}^*_l \begin{bmatrix} U^{0,(l)}Q^{(l)} - U^* \\ V^{0,(l)}Q^{(l)} - V^* \end{bmatrix} \left( \Sigma^* \right)^{-1/2}. \]

Therefore, it follows that

\begin{align*}
\left\| \begin{bmatrix} X^{0,(l)}Q^{(l)} - X^* \\ Y^{0,(l)}Q^{(l)} - Y^* \end{bmatrix} \bigg|_{l^*} \right\|_2 & \leq \|M^*\|_{2,\infty} \left\| \left( \Sigma^{(l)} \right)^{-1/2}Q^{(l)} - Q^{(l)} \left( \Sigma^* \right)^{-1/2} \right\|_2 \\
& + \frac{\|M^*\|_{2,\infty}}{\sqrt{\sigma_{\min}}} \left\| \begin{bmatrix} U^{0,(l)}Q^{(l)} - U^* \\ V^{0,(l)}Q^{(l)} - V^* \end{bmatrix} \right\|.
\end{align*}
\tag{B.36}
To control this bound, Ma et al. (2017, Lemma 46) gives

\[
\left\| \left( \Sigma^{(l)} \right)^{-1/2} Q^{(l)} - Q^{(l)} \left( \Sigma^{*} \right)^{-1/2} \right\| = \left\| \left( \Sigma^{(l)} \right)^{-1/2} \left[ Q^{(l)} \left( \Sigma^{*} \right)^{1/2} - \left( \Sigma^{(l)} \right)^{1/2} Q^{(l)} \right] \left( \Sigma^{*} \right)^{-1/2} \right\|
\leq \frac{1}{\sigma_{\min}} \left\| Q^{(l)} \left( \Sigma^{*} \right)^{1/2} - \left( \Sigma^{(l)} \right)^{1/2} Q^{(l)} \right\|
\leq \frac{1}{3\sigma_{\min}} \left\| \tilde{M}^{0,(l)} - \tilde{M}^{*} \right\|.
\] (B.37)

Furthermore, Ma et al. (2017, Lemma 45, 47) imply that

\[
\left\| \begin{bmatrix} U^{0,(l)} & V^{0,(l)} \\ V^{0,(l)} & -\Sigma^{*} \end{bmatrix} \right\| \leq \frac{1}{\sigma_{\min}} \left\| \tilde{M}^{0,(l)} - \tilde{M}^{*} \right\|, \quad (B.38)
\]

\[
\left\| H^{0,(l)} - Q^{(l)} \right\| \leq \frac{1}{\sigma_{\min}} \left\| \tilde{M}^{0,(l)} - \tilde{M}^{*} \right\|. \quad (B.39)
\]

Substitution of (B.37)-(B.39) into (B.36) yields

\[
\left\| \begin{bmatrix} X^{0,(l)}Q^{(l)} - X^{*} \\ Y^{0,(l)}Q^{(l)} - Y^{*} \end{bmatrix} \right\|_{2,*} \leq \frac{1}{3\sigma_{\min}} \left\| \tilde{M}^{0,(l)} - \tilde{M}^{*} \right\| \| M^{*} \|_{2,\infty}. \quad (B.40)
\]

Plugging (B.39) and (B.40) into (B.35) gives

\[
\left\| \begin{bmatrix} X^{0,(l)}H^{0,(l)} - X^{*} \\ Y^{0,(l)}H^{0,(l)} - Y^{*} \end{bmatrix} \right\|_{2,*} \leq \frac{1}{\sigma_{\min}} \left\| \tilde{M}^{0,(l)} - \tilde{M}^{*} \right\| \left\| \begin{bmatrix} X^{0,(l)} \\ Y^{0,(l)} \end{bmatrix} \right\|_{2,\infty} + \frac{1}{3\sigma_{\min}} \left\| \tilde{M}^{0,(l)} - \tilde{M}^{*} \right\| \| M^{*} \|_{2,\infty}
\leq \frac{\sqrt{k}}{\sigma_{\min}} \left( \sigma \sqrt{\frac{n}{p}} + \sqrt{\frac{n}{p}} \| M^{*} \|_{\infty} \right) \| X^{*} \|_{2,\infty},
\]

where the last inequality makes use of (A.3f) and (B.3b)

\[
\left\| \tilde{M}^{0,(l)} - \tilde{M}^{*} \right\| = \left\| M^{0,(l)} - M^{*} \right\| \lesssim \sigma \sqrt{\frac{n}{p}} + \sqrt{\frac{n}{p}} \| M^{*} \|_{\infty}.
\]

### B.4 Proof of Lemma 10

To start with, we define

\[
Q^{0,(l)} = \arg \min_{R \in \mathcal{O}_{*p,r}} \left\| U^{0,(l)}R - U^0 \right\|_F.
\]

The definition of $R^{0,(l)}$ (cf. (5.10)) implies that

\[
\left\| X^0H^0 - X^{0,(l)}R^{0,(l)} \right\|_F \leq \left\| X^0 - X^{0,(l)}Q^{0,(l)} \right\|_F.
\] (B.41)

Then one has the following decomposition of $X^{0,(l)}Q^{0,(l)} - X^0$,

\[
X^{0,(l)}Q^{0,(l)} - X^0 = U^{0,(l)} \left[ \left( \Sigma^{0,(l)} \right)^{1/2} Q^{0,(l)} - Q^{0,(l)} \left( \Sigma^{0} \right)^{1/2} \right] + \left( U^{0,(l)}Q^{0,(l)} - U^0 \right) \left( \Sigma^{0} \right)^{1/2}.
\]

The triangle inequality reveals that

\[
\left\| X^{0,(l)}Q^{0,(l)} - X^0 \right\|_F \leq \left\| \left( \Sigma^{0,(l)} \right)^{1/2} Q^{0,(l)} - Q^{0,(l)} \left( \Sigma^{0} \right)^{1/2} \right\|_F + \left\| U^{0,(l)}Q^{0,(l)} - U^0 \right\|_F \left\| \left( \Sigma^{0} \right)^{1/2} \right\|.
\]
Invoking Ma et al. (2017, Lemma 46) gives
\[
\left\| \left( \Sigma^{0,(l)} \right)^{1/2} Q^{0,(l)} - Q^{0,(l)} \left( \Sigma^{0} \right)^{1/2} \right\|_F \\
\lesssim \frac{1}{\sqrt{\sigma_{\min}}} \left\| \left( \tilde{M}^0 - \tilde{M}^{0,(l)} \right) \begin{bmatrix} U^{0,(l)} \\ V^{0,(l)} \end{bmatrix} \right\|_F \\
\lesssim \frac{1}{\sqrt{\sigma_{\min}}} \left\| \left( M^0 - M^{0,(l)} \right) V^{0,(l)} \right\|_F + \frac{1}{\sqrt{\sigma_{\min}}} \left\| \left( M^0 - M^{0,(l)} \right)^T U^{0,(l)} \right\|_F.
\]

Furthermore, Davis-Kahan’s \( \sin \Theta \) theorem (Davis and Kahan, 1970) implies that
\[
\left\| U^{0,(l)} Q^{0,(l)} - U^0 \right\|_F \lesssim \frac{1}{\sigma_{\min}} \left\| \left( M^0 - M^{0,(l)} \right) \right\|_F.
\]

Taking the results above together gives rise to
\[
\left\| X^{0,(l)} Q^{0,(l)} - X^0 \right\|_F \lesssim \frac{1}{\sqrt{\sigma_{\min}}} \left\| \left( M^0 - M^{0,(l)} \right) \right\|_F + \frac{1}{\sqrt{\sigma_{\min}}} \left\| \left( M^0 - M^{0,(l)} \right)^T U^{0,(l)} \right\|_F \\
+ \frac{1}{\sigma_{\min}} \left\| \left( M^0 - M^{0,(l)} \right)^T U^{0,(l)} \right\|_F \left\| \left( \Sigma^{0} \right)^{1/2} \right\| \\
\lesssim \frac{1}{\sqrt{\sigma_{\min}}} \left\| \left( M^0 - M^{0,(l)} \right) \right\|_F + \sqrt{\frac{\kappa}{\sigma_{\min}}} \left\| \left( M^0 - M^{0,(l)} \right)^T U^{0,(l)} \right\|_F,
\]

where the last inequality utilizes (B.15). Then we resort to the following claim to control the right-hand side.

**Claim 3.** With probability exceeding \( 1 - O(n^{-100}) \), one has
\[
\left\| \left( M^0 - M^{0,(l)} \right)^T U^{0,(l)} \right\|_F \lesssim \sqrt{\frac{n \left( \| \mathbf{M}^* \|_\infty^2 + \sigma^2 \right)}{p} \log n + \frac{(\tau + \| \mathbf{M}^* \|_\infty)}{p} \log n} \left\| U^{0,(l)} \right\|_{2,\infty},
\]

and
\[
\left\| \left( M^0 - M^{0,(l)} \right) V^{0,(l)} \right\|_F \lesssim \sqrt{\frac{n \left( \| \mathbf{M}^* \|_\infty^2 + \sigma^2 \right)}{p} \log n + \frac{(\tau + \| \mathbf{M}^* \|_\infty)}{p} \log n} \left\| V^{0,(l)} \right\|_{2,\infty}.
\]

In terms of \( \| U^{0,(l)} \|_{2,\infty} \), applying similar derivation as Lemma 13 enables us to obtain that
\[
\left\| U^{0,(l)} Q_U^{(l)} - U^* \right\|_{2,\infty} \lesssim \frac{\| \mathbf{M}^* \|_\infty}{\sigma_{\min}} \sqrt{\frac{r \log n}{p}} + \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{\mu r}{p}} \log n.
\]
It then follows that
\[ V := \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}}^2 + \sqrt{\frac{1}{p} \left( \frac{\|M\|_{\infty}^2 + \sigma^2}{\sigma_{\min}} \right) \sqrt{\log n}} \]
\[ \ll \|U^*\|_{2,\infty}, \]
as long as \( \frac{\sigma}{\sigma_{\min}} \sqrt{\kappa n \log^2 n} \ll 1 \) and \( n^2 p \gg \kappa^2 \mu^2 \tau^3 n \log n \). Analogously we have
\[ \left\| V^{0,(l)} Q_V^{(l)} - V^* \right\|_{2,\infty} \lesssim \|V^*\|_{2,\infty}. \]

It then follows that
\[ \left\| U^{0,(l)} \right\|_{2,\infty} = \left\| U^{0,(l)} Q_U^{(l)} \right\|_{2,\infty} \lesssim \|U^*\|_{2,\infty} + \left\| U^{0,(l)} Q_U^{(l)} - U^* \right\|_{2,\infty} \lesssim 2 \|U^*\|_{2,\infty}, \]
and similarly \( \|V^{0,(l)}\|_{2,\infty} \leq 2 \|V^*\|_{2,\infty} \). Therefore, combining (B.41), (B.42) and Claim 3 yields
\[ \left\| X^0 H^0 - X^{0,(l)} R^{0,(l)} \right\|_F \lesssim \sqrt{\frac{\kappa}{\sigma_{\min}} \left( \left\| (M^0 - M^{0,(l)})^\top U^{0,(l)} \right\|_F + \left\| (M^0 - M^{0,(l)}) V^{0,(l)} \right\|_F \right)} \]
\[ \lesssim \sqrt{\frac{\kappa}{\sigma_{\min}} \left( n \left( \frac{\|M^*\|_{\infty}^2 + \sigma^2}{\|M^*\|_{\infty}} \right) \log n + \frac{(\tau + \|M^*\|_{\infty})}{p} \log n \right) \left\| \left[ \begin{array}{c} U^* \\ V^* \end{array} \right] \right\|_{2,\infty} \]
\[ \lesssim \sqrt{\frac{\kappa}{\sigma_{\min}} \left( \frac{n}{p} + \frac{\|M^*\|_{\infty}}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \log n \|F^*\|_{2,\infty}. \]

This completes the proof.

Proof of Claim 3. We prove (B.44) here, and (B.43) would follow in an analogous way. From the definition of \( M^{0,(l)} \) (cf. (5.7)), one has
\[ p \left( M^0 - M^{0,(l)} \right) V^{0,(l)} = (P_{\Omega_l} (\psi_\tau (M)) - p P_l (M^*)) V^{0,(l)} \]
\[ = \sum_{j=1}^n (\delta_{l,j} \psi_\tau (M_{l,j}) - p M^*_{l,j}) V^{0,(l)}_{j,.}. \]

It is then easy to check that
\[ L := \max_{1 \leq j \leq n} \left\| (\delta_{l,j} (M_{l,j} I_{|M_{l,j}| \leq \tau} + \tau I_{|M_{l,j}| > \tau}) - p M^*_{l,j}) V^{0,(l)}_{j,.} \right\|_2 \leq (\tau + \|M^*\|_{\infty}) \|V^*\|_{2,\infty} \]
\[ V := \left\| \sum_{j=1}^n E \left[ (\delta_{l,j} \psi_\tau (M_{l,j}) - p M^*_{l,j})^2 \right] V^{0,(l)}_{j,.} V^{0,(l)}_{j,.}^\top \right\|_F \]
\[ \leq \max_j \left[ E \left[ (\delta_{l,j} - p) M^*_{l,j} I_{|M_{l,j}| \leq \tau} + \delta_{l,j} \epsilon_{l,j} \tau I_{|M_{l,j}| > \tau} + (\tau \delta_{l,j} - p M^*_{l,j}) I_{|M_{l,j}| > \tau} \right]^2 \right] V^{0,(l)}_{j,.} V^{0,(l)}_{j,.}^\top \]
\[ \lesssim \left[ E \left[ (\delta_{l,j} - p)^2 (M^*_{l,j})^2 \right] + E \left[ \delta_{l,j} \epsilon_{l,j}^2 \tau^2 \right] + E \left[ (\tau \delta_{l,j} - p M^*_{l,j})^2 \right] \right] \left\| V^{0,(l)} \right\|_F^2 \]
\[ \lesssim p \left( \|M^*\|_{\infty}^2 + \sigma^2 \right) \left\| V^{0,(l)} \right\|_F^2. \]
Here the last line is an application of Markov inequality due to
\[ P(|M_{i,j}| > \tau) \leq P(|\varepsilon_{i,j}| > \tau - |M^*_i|) \leq P(|\varepsilon_{i,j}| > \tau/2) \leq \frac{\sigma^2}{(\tau/2)^2}. \]

We are now ready to apply the matrix Bernstein inequality (Tropp, 2015, Theorem 6.1.1):
\[
\| (M^0 - M^{0, (l)}) V^{0, (l)} \|_F \\
\lesssim \frac{1}{p} \left( \sqrt{V \log n + L \log n} \right) \\
\lesssim \sqrt{\frac{\|M^*\|^2 + \sigma^2}{p} \log n} \|V^{0, (l)}\|_F + \frac{\tau + \|M^*\|_\infty}{p} \|V^{0, (l)}\|_{2, \infty} \log n \\
\lesssim \sqrt{\frac{n (\|M^*\|^2 + \sigma^2)}{p} \log n} \|V^{0, (l)}\|_{2, \infty} + \frac{\tau + \|M^*\|_\infty}{p} \|V^{0, (l)}\|_{2, \infty} \log n.
\]

\[\Box\]

B.5 Proof of Lemma 11

For (A.3a), one has
\[
\| F^{t, (l)} R^{t, (l)} - F^* \|_{2, \infty} \\
\leq \| F^{t, (l)} R^{t, (l)} - F^t H^t \|_{2, \infty} + \| F^t H^t - F^* \|_{2, \infty} \\
\leq \| F^{t, (l)} R^{t, (l)} - F^t H^t \|_F + \| F^t H^t - F^* \|_{2, \infty} \\
\overset{(i)}{\leq} \kappa^{3/2} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \log n \|F^*\|_{2, \infty},
\]
where (i) follows from Lemma 4 and 5.

Turning attention to (A.3b), we have
\[
\| F^{t, (l)} R^{t, (l)} - F^* \| \leq \| F^{t, (l)} R^{t, (l)} - F^t H^t \|_F + \| F^t H^t - F^* \| \\
\lesssim \sqrt{r} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \|F^*\|,
\]
where the second line makes use of Lemma 4 and 2.

Moreover,
\[
\| F^{t, (l)} H^{t, (l)} - F^* \| \\
\leq \| F^{t, (l)} H^{t, (l)} - F^t H^t \|_F + \| F^t H^t - F^* \| \\
\leq 5\kappa \| F^{t, (l)} R^{t, (l)} - F^t H^t \|_F + \| F^t H^t - F^* \| \\
\lesssim \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\|M^*\|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \sqrt{r} \|F^*\|,
\]

66
which is due to (A.3d), Lemma 4 and 2. (A.3d) can be proved analogously as Chen et al. (2020b, Lemma 18) and thus omitted here for brevity.

Finally, one has

\[
\| F^t \| \leq \| F^t H^t - F^* \| + \| F^* \|
\]

\[
\leq C \sqrt{n} \left( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}} \right) \| F^* \|_F + \| F^* \|
\]

\[
\leq 2 \| F^* \|
\]

where (i) is due to (5.4), and (ii) holds as long as \( \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} \ll \frac{1}{\sqrt{k r}} \) and \( np \gg \mu^2 k^3 r^3 \). In a similar way, we can prove (A.3e) and (A.3f).

B.6 Proof of Lemma 12

Recalling the definition of \( E \) (cf. (B.8)), one has

\[
\| E \| \leq \| E - E [E] \| + \| E [E] \|_F
\]

\[
\leq \tilde{\sigma} \sqrt{n} + n \frac{\sigma^2}{\tau}
\]

\[
\leq \sigma \sqrt{\frac{n}{p}} + \sqrt{\frac{n}{p}} \| M^* \|_\infty,
\]

where (i) makes use of standard matrix tail bounds (Chen et al., 2021a, Theorem 3.1.4) and (ii) follows from the definition of \( \tilde{\sigma} \) in (B.11). In addition, we note that Weyl’s inequality implies

\[
\sigma_r (M^0) \geq \sigma_{\min} - \| E \| \geq \frac{1}{2} \sigma_{\min}.
\]

Applying Wedin’s sin\( \Theta \) Theorem (Chen et al., 2021a, Theorem 2.3.1) then reveals

\[
\max \{ \| UQ_U - U^* \|, \| VQ_V - V^* \| \} \leq \frac{\sqrt{2} \| E \|}{\sigma_r (M^0) - \sigma_{r+1} (M^*)} \leq \frac{\sigma}{\sigma_{\min}} \sqrt{\frac{n}{p}} + \frac{\| M^* \|_\infty}{\sigma_{\min}} \sqrt{\frac{n}{p}}.
\]

Next, we turn to control \( \| H_V - Q_V \| \). The definition of \( H_V \) (cf. (B.2)) enables us to write the SVD of \( H_V \) as

\[
H_V = L_1 \cos \Theta L_2^T,
\]

where \( \Theta \) is a diagonal matrix consisting of the principal angles between the subspaces \( V \) and \( V^* \). Then (B.1) gives \( Q_V = L_1 L_2^T \) and it follows that

\[
\| H_V - Q_V \| = \left\| L_1 (\cos \Theta - I_r) L_2^T \right\| = \| 2 \sin^2 (\Theta/2) \| \lesssim \| \sin \Theta \|^2 \lesssim \frac{n}{\sigma_{\min}^2} \frac{\| M^* \|_\infty^2 + \sigma^2}{p}, \quad (B.45)
\]

where the last inequality invokes Wedin’s sin\( \Theta \) Theorem (Chen et al., 2021a, Theorem 2.3.1) again. The results of \( H_U \) can be obtained similarly.

The last inequality is a direct consequence of (B.3d)

\[
\frac{1}{2} \leq \| Q_U \| - \| H_U - Q_U \| \leq \| H_U \| \leq \| H_U - Q_U \| + \| Q_U \| \leq 2.
\]
C Technical lemmas

Lemma 17. Suppose \( \{g_{i,j}\}_{i,j} \) are i.i.d. Bernoulli random variables with parameter \( p \). Then with probability at least \( 1 - O(n^{-10}) \), one has

\[
\left\| \{g_{i,j}\}_{i,j} \right\| \lesssim np.
\]

Proof. Triangle inequality gives

\[
\left\| \{g_{i,j}\}_{i,j} \right\| \leq \left\| \{g_{i,j}\}_{i,j} - p11^\top \right\| + \left\| p11^\top \right\| \lesssim \sqrt{np} + np \lesssim np.
\]

where the second line makes use of Keshavan et al. (2010, Lemma 3.2) and holds provided that \( np \gg 1 \).

Lemma 18. Assume the matrix \( E := \{E_{i,j}\}_{i,j} \in \mathbb{R}^{n \times n} \) consists of independent random variables obeying that for any \( 1 \leq i, j \leq n \),

\[
\mathbb{E}[E_{i,j}] = 0, \quad \mathbb{E}[E_{i,j}^2] = \sigma_{i,j}^2 \leq \sigma^2, \quad |E_{i,j}| \leq B.
\]

1. With probability exceeding \( 1 - O(n^{-10}) \), one has

\[
\left\| \mathcal{P}_\Omega (E) \right\| \lesssim \sigma \sqrt{n} + B \sqrt{\log n}. \tag{C.1}
\]

2. For any fixed matrix \( A \), one has

\[
\left\| EA \right\|_{2,\infty} \lesssim \sigma \left\| A \right\|_F \sqrt{\log n} + B \left\| A \right\|_{2,\infty} \log n, \tag{C.2}
\]

with probability over \( 1 - O(n^{-100}) \).

Proof. Chen et al. (2021a, Equation 3.9) gives (C.1). This is the same as Yan et al. (2021, Lemma 5).