Matrix logarithms and range of the exponential maps for the symmetry groups $SL(2, \mathbb{R})$, $SL(2, \mathbb{C})$, and the Lorentz group

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Abstract
Physicists know that covering the continuously connected component $L^\perp_+$ of the Lorentz group can be achieved through two Lie algebra exponentials, whereas one exponential is sufficient for compact symmetry groups like $SU(N)$ or $SO(N)$. On the other hand, both the general Baker–Campbell–Hausdorff formula for the combination of matrix exponentials in a series of higher order commutators, and the possibility to define the logarithm $\ln(M)$ of a general matrix $M$ through the Jordan normal form, seem to naively suggest that even for non-compact groups a single exponential should be sufficient. We provide explicit constructions of $\ln(M)$ for all matrices $M$ in the fundamental representations of the non-compact groups $SL(2, \mathbb{R})$, $SL(2, \mathbb{C})$, and $SO(1, 2)$. The construction for $SL(2, \mathbb{C})$ also yields logarithms for $SO(1, 3)$ through the spinor representations. However, it is well known that single Lie algebra exponentials are not sufficient to cover the Lie groups $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$. Therefore we revisit the maximal neighbourhoods $\mathcal{N}_+ \subset SL(2, \mathbb{R})$ and $\mathcal{N}^+_{1, \mathbb{C}} \subset SL(2, \mathbb{C})$ which can be covered through single exponentials $\exp(X)$ with $X \in sl(2, \mathbb{R})$ or $X \in sl(2, \mathbb{C})$, respectively, to clarify why $\ln(M) \notin sl(2, \mathbb{R})$ or $\ln(M) \notin sl(2, \mathbb{C})$ outside of the corresponding domains $\mathcal{N}_+$ or $\mathcal{N}^+_{1, \mathbb{C}}$. On the other hand, for the Lorentz groups $SO(1, 2)$ and $SO(1, 3)$, we confirm through construction of the logarithm $\ln(\Delta)$ that every transformation $\Delta$ in the connectivity component $L^\perp_+$ of the identity element can be represented in the form $\exp(X)$ with $X \in so(1, 2)$ or $X \in so(1, 3)$, respectively. We also examine why the proper orthochronous Lorentz group can be covered by single Lie algebra exponentials, whereas this property does not hold for its covering group $SL(2, \mathbb{C})$: The logarithms $\ln(\Delta)$ in $L^\perp_+$ correspond to logarithms on the first sheet of the covering map $SL(2, \mathbb{C}) \to L^\perp_+$, which is contained in $\mathcal{N}^+_{1, \mathbb{C}}$. The special linear groups and the Lorentz group therefore provide instructive examples for different global behaviour of non-compact Lie groups under the exponential map.

1. Introduction
Physicists are well versed in the use of Lie groups and Lie algebras for the description of continuous symmetries in nature, and indeed some of the most extensive and finest textbooks on applications of Lie groups have been written by physicists, see e.g. [1–3]. It is also well known that at least in a neighbourhood $\mathcal{N}_+$ of the identity element, we can represent every linear symmetry transformation through a matrix

$$M = \exp(X) \approx 1 + X, \quad X = \ln(M),$$

(1)

see e.g. Proposition 1.6, Chapter II, in [4], and the properties of the matrices $M$ in the Lie group $G$ can be inferred from the properties of the matrices $X$ in the corresponding Lie algebra $\mathfrak{g}$. The Lie algebra elements have an expansion in a finite basis $X_i$ (we use summation convention),

$$X = \alpha^i X_i,$$

(2)
with characteristic commutation relations

$$[X_i, X_j] = C_{ij}^k X_k.$$  \hfill (3)

The use of matrix notation already indicates that we identify Lie groups and their Lie algebras with their defining matrix representations for simplicity.

The commutation relations (3) are in one-to-one correspondence to the composition law in the group in a neighbourhood of the identity,

$$M \cdot N = \exp(X) \cdot \exp(Y),$$ \hfill (4)

through the iterative Baker-Campbell-Hausdorff procedure to combine two matrix exponentials into a series of higher order commutators,

$$\exp(X) \cdot \exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [X, Y]] + \frac{1}{24}[X, [Y, [X, Y]]] + \ldots\right).$$ \hfill (5)

On the other hand, we can write every invertible matrix in terms of its polar decomposition [3] in a positive definite hermitian matrix and a unitary matrix,

$$M = H \cdot U = (M \cdot M^\dagger)^{1/2} \cdot [(M \cdot M^\dagger)^{-1/2} \cdot M],$$ \hfill (6)

and both of these factors can be represented in terms of single matrix exponentials, using e.g. the general construction of matrix logarithms given in equations (21), (22) below. This tells us that we can write every matrix in the connected component of a matrix symmetry group as the product of at most two exponentials. Another argument based on the decomposition into compact and non-compact generators is reviewed in [5].

A famous example for an everyday use of the polar decomposition is the Lorentz group, for which every element (up to reflections) can be written as the product of a boost with velocity $v = c\beta = c \tanh(u)\hat{u}$ and a rotation with axis $\hat{\varphi}$ and angle $0 \leq \varphi = |\varphi| \leq \pi$ (with $\pi\hat{\varphi}$ and $-\pi\hat{\varphi}$ identified), see e.g. Theorem 8.2 in [6],

$$\Lambda(u, \varphi) = \exp(u \cdot K) \cdot \exp(\varphi \cdot L) = \begin{pmatrix} \gamma & -\beta \hat{\beta}^T \\ -\gamma\beta & 1 - \hat{\beta} \otimes \hat{\beta} + \gamma\beta \otimes \hat{\beta} \end{pmatrix} \cdot R(\varphi).$$ \hfill (7)

(8)

The rotation matrix is given by the Rodrigues formula

$$\frac{L(\varphi)}{\varphi} = \exp(\varphi \cdot L) = \begin{pmatrix} 1 & 0 \\ 0 & L(\varphi) \end{pmatrix}$$ \hfill (9)

$$L(\varphi) = \hat{\varphi} \otimes \hat{\varphi} + (1 - \hat{\varphi} \otimes \hat{\varphi}) \cos \varphi + \hat{\varphi} \cdot \hat{\varphi} \sin \varphi,$$ \hfill (10)

with rotation generators

$$L_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \ell_i \end{pmatrix}, \quad (L_i)_{jk} = \epsilon_{ijk}.$$ \hfill (11)

The generator of the boost matrix is

$$u \cdot K = \begin{pmatrix} 0 & -u_1 & -u_2 & -u_3 \\ -u_1 & 0 & 0 & 0 \\ -u_2 & 0 & 0 & 0 \\ -u_3 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -u^T \\ -u & 0 \end{pmatrix},$$ \hfill (12)

with the rapidity $u,

$$\gamma = \cosh(u), \quad \beta = \tanh(u), \quad u = \frac{1}{2} \ln \left(1 + \frac{\beta}{1 - \beta}\right) \hat{\beta}.$$ \hfill (13)

The generators of the Lorentz group satisfy commutation relations

$$[L_i, L_j] = -\epsilon_{ijk} L_k, \quad [L_i, K_j] = -\epsilon_{ijk} K_k, \quad [K_i, K_j] = \epsilon_{ijk} L_k.$$ \hfill (14)

This implies in particular,

$$[u \cdot K, \varphi \cdot L] = (\varphi \times u) \cdot L,$$ \hfill (15)

and the Baker-Campbell-Hausdorff (BCH) formula (5) therefore implies that the exponents in (8) can be combined trivially into a single exponent corresponding to the Lie algebra element $u \cdot K + \varphi \cdot L$, if $\varphi \times u = 0$. However, the Baker-Campbell-Hausdorff formula also seems to suggest that we can always combine the Lie algebra elements $u \cdot K$ and $\varphi \cdot L$ into a single Lie algebra element.
\[
\ln \Delta = u'(u, \varphi) \cdot K + \varphi'(u, \varphi) \cdot L,
\]
or can we? The reach of the exponential map is a non-trivial question for non-compact groups, and groups where the identity-connected component cannot be covered by single Lie algebra exponentials include e.g. \( SL(n, \mathbb{R}) \) and \( GL(n, \mathbb{R}) \).

While the general representation (8) of proper orthochronous Lorentz transformations in terms of two Lie algebra exponentials is standard textbook knowledge in relativity, electrodynamics and quantum mechanics (see e.g. [7–9]), definitive statements about the question whether every proper orthochronous Lorentz transformation can be expressed in the form \( \Delta = \exp(X) \) with \( X = \ln \Delta \in so(1, 3) \), do not seem to have made it into the textbook literature, although experts on the Lorentz group know that this property holds [10–12]. Riesz proved it using the decomposition of Minkowski space into different invariant subspaces under Lorentz transformations. Furthermore, exponentials of general \( so(1, 3) \) elements

\[
\Delta = \exp(w \cdot K + \chi \cdot L)
\]
have been evaluated in closed form (generalized Rodrigues formulae) both in the vector and the spinor representations of the Lorentz group [11–14], and both Zeni and Rodrigues, as well as Özdemir and Erdoğan, have pointed out that these formulae can in principle be used to answer the question of Lie algebra coverage for \( L_1 \), affirmatively by comparing the closed forms of (16) with equation (8) and demonstrating that the requirement \( \Delta = \Delta \) leads to bijective maps \( (w, \chi) \Leftrightarrow (u, \varphi) \). We would like to confirm this observation from a different instructive angle, because the question for the maximal reach of single Lie algebra exponentials in a non-compact group can also be phrased in terms of the general definition of the logarithm of a matrix.

Suppose \( M \) is an invertible square matrix which is related to its Jordan canonical form through

\[
M = \mathcal{V} \cdot \oplus_{i} \lambda_{i} \mathcal{V}^{-1}.
\]

Each of the smaller square matrices \( I_{n} \) has the form

\[
I = \lambda_{i}
\]
or the form

\[
I = \begin{pmatrix}
\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & \lambda
\end{pmatrix},
\]

and \( \det(M) \neq 0 \) implies that none of the eigenvalues \( \lambda \) can vanish. We do not presume whether the matrix \( M \) is real or complex. However, the Jordan canonical form may require that we allow for complex eigenvalues \( \lambda_{n} \) and complex transformation matrices \( \mathcal{V} \) to ensure that the characteristic equation \( \det(M - \lambda I) = 0 \) for the eigenvalues and the corresponding eigenvector conditions can be solved.

In the case (18) we have

\[
\mathcal{I} = \exp[\ln(\lambda_{i})], \quad \ln(\mathcal{I}) = \ln(\lambda_{i}).
\]

However, it is also possible to construct the logarithm of a Jordan block matrix (19), see e.g. [15, 16] for the general construction of matrix functions \( f(A) \) through the Jordan normal form.

Suppose the Jordan matrix (19) is a \((\nu + 1) \times (\nu + 1)\) matrix. Its logarithm can then be defined through

\[
\mathcal{X} = \ln(\mathcal{I}) = \begin{pmatrix}
\ln \lambda & \lambda^{-1} - \lambda^{-2} + \cdots & \lambda^{-3} + \cdots & (-)^{\nu-1} & \lambda^{-\nu} \\
0 & \ln \lambda & -\lambda^{-2} + \cdots & \lambda^{-3} + \cdots & (-)^{\nu-2} & \lambda^{-\nu-1} & (-)^{\nu-1} (-)^{\nu-1} & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\lambda^{-2} & \cdots & \lambda^{-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \ln \lambda
\end{pmatrix},
\]

and it is possible to verify \( \mathcal{I} = \exp(\mathcal{X}) \) through direct calculation, see e.g. Appendix F in [9].

The direct sum of the logarithms of all the matrices \( I_{n} \) then yields the logarithm of the matrix \( M \),

\[
\ln(M) = \mathcal{V} \cdot \oplus_{i} \ln(I_{n}) \cdot \mathcal{V}^{-1}, \quad \ln(M) = \mathcal{V} \cdot \oplus_{i} \ln(I_{n}) \cdot \mathcal{V}^{-1}.
\]

This indicates that we can always construct \( \ln(M) \) (at least over \( \mathbb{C} \)) for every element \( M \) of a matrix Lie group. How can it then be that we cannot represent every group element as a single exponential of a Lie algebra element for some groups like \( SL(2, \mathbb{R}) \)? An explicit counterexample for \( SL(2, \mathbb{R}) \) is e.g. provided by the diagonal matrices \( \text{diag}(a, 1/a) \) with \( 0 > a > -1 \) or \( a < -1 \) [2], and Gilmore reviews the criterion \( \text{tr}([\exp(\mathcal{X})]) \geq -2 \) if \( \mathcal{X} \in sl(2, \mathbb{R}) \) [5].
Can the transformation with $V$ in equation (22) introduce singularities in $\ln(M)$ if $M$ is too far away from the identity element? Or can we have negative or complex eigenvalues? And can the imaginary parts of the logarithms of those eigenvalues survive the transformation with $V$ in equation (22) if $M$ is too far away from the identity element? What else could destroy the property $M \in G \Rightarrow \ln(M) \in g$? And how does the Lorentz group behave under the exponential map of its Lie algebra?

We will study these questions from the point of view of the matrix logarithm. Specifically, we will revisit the group $SL(2, \mathbb{R})$ in section 2, the group $SL(2, \mathbb{C})$ in section 3, and the Lorentz group in section 4. The covering maps $SL(2, \mathbb{R}) \rightarrow \mathbb{R}^1$ and $SL(2, \mathbb{C}) \rightarrow \mathbb{R}^1$ onto the identity-connected components of $SO(1, 2)$ and $SO(1, 3)$ are then revisited in section 5 to clarify why the components $\mathbb{R}^1$ are covered by single Lie algebra exponentials when their double covers do not share that property. Section 6 summarizes our conclusions.

2. The range of the exponential map in $SL(2, \mathbb{R})$

The group $SL(2, \mathbb{R})$ is the group of all real $2 \times 2$ matrices with unit determinant,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$  \hspace{1cm} (23)

The inverse matrix is

$$M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$ \hspace{1cm} (24)

If the logarithm $\ln(M)$ of a matrix exists, transformation into canonical form (17) proves that

$$\det(M) = \exp[\text{tr}(\ln(M))].$$ \hspace{1cm} (25)

Real-valued logarithms of matrices with unit determinant must therefore satisfy

$$\text{tr}[\ln(M)] = 0,$$ \hspace{1cm} (26)

such that $\ln(M)$ must have the general form

$$\ln(M) = \begin{pmatrix} a_3 & a_+ \\ a_- & -a_3 \end{pmatrix} = a_3 \sigma_3 + a_+ \sigma_+ + a_- \sigma_-,$$ \hspace{1cm} (27)

with the matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ \hspace{1cm} (28)

These matrices provide a basis for the Lie algebra $sl(2, \mathbb{R})$ of real traceless $2 \times 2$ matrices with commutation relations

$$[\sigma_+, \sigma_-] = 2 \sigma_3, \quad [\sigma_3, \sigma_\pm] = \pm \sigma_\pm.$$ \hspace{1cm} (29)

These matrices also satisfy the relations

$$\sigma_3^2 = 1, \quad \sigma_\pm^2 = 0$$ \hspace{1cm} (30)

and the anti-commutation relations,

$$[\sigma_+, \sigma_-] = 1, \quad [\sigma_3, \sigma_\pm] = 0.$$ \hspace{1cm} (31)

Therefore the exponential of $\ln(M)$ in equation (27) can be readily evaluated to yield

$$M = \exp\{a_3 \sigma_3 + a_+ \sigma_+ + a_- \sigma_-\}$$ \hspace{1cm} (32)

in the form (see also [5])

$$M = \cosh[\sqrt{a_3^2 + a_+a_-}] + \frac{a_3}{a_-} - a_3 \frac{a_3}{\sqrt{a_3^2 + a_+a_-}}.$$ \hspace{1cm} (33)

if $a_3^2 + a_+a_- \geq 0$, and in the form

$$M = \cosh[\sqrt{-a_3^2 - a_+a_-}] + \frac{a_3}{a_-} + a_3 \frac{a_3}{\sqrt{-a_3^2 - a_+a_-}}.$$ \hspace{1cm} (34)

if $a_3^2 + a_+a_- \leq 0$.

Equations (33), (34) tell us that the single exponential representation (32) in terms of an $sl(2, \mathbb{R})$ element always holds if $\text{tr}(M) > -2$:

$$\text{tr}(M) > -2 \Rightarrow \ln(M) \in sl(2, \mathbb{R}).$$ \hspace{1cm} (35)
Furthermore, the relation between $M \in SL(2, \mathbb{R})$ and $\ln(M) \in sl(2, \mathbb{R})$ is unique for $\text{tr}(M) > -2$ due to

$$a^2 + a_+ a_- = \begin{cases} -\arccos^2[\text{tr}(M)/2] & \text{if } |\text{tr}(M)| < 2, \\ \arccosh^2[\text{tr}(M)/2] & \text{if } |\text{tr}(M)| \geq 2, \end{cases}$$

(36)

and the elements $a - d, b, c$ of $M - (\text{tr}(M)/2)I$ then determine the elements $a_3$ and $a_{\pm}$ of $\ln(M)$. Here we used the inverse hyperbolic cosine to emphasize the connection between the results through $\arccosh x = \pm i \arccos x$. However, in the following we will use the representation through the logarithm for the inverse hyperbolic cosine function,

$$0 \leq \arccosh x = \ln(x + \sqrt{x^2 - 1}),$$

(37)

because this facilitates the discussion of limiting cases.

The single exponential representation (32) in terms of an $sl(2, \mathbb{R})$ element holds in the case $\text{tr}(M) = -2$ if and only if $M = -1$, with the infinitely degenerate set of logarithms

$$\ln(-I) = \begin{pmatrix} a_3 \\ -(a_3^2 + \pi^2)/a_+ \end{pmatrix}.$$  

(38)

Indeed, we can replace $\pi \to (2n + 1)\pi, n \in \mathbb{N}_0$, in this equation, i.e. $\ln(-1) \in sl(2, \mathbb{R})$ has a degeneracy of order $\mathbb{R} \times \mathbb{N}$. For $\text{tr}(M) > -2$ the relation between the matrix elements of $M$ and the Lie algebra expansion coefficients $a_3$ and $a_{\pm}$ of $\ln(M)$ was invertible, whereas the Lie algebra expansion coefficients for $\ln(-1)$ are only restricted by the relation $a_3^2 + a_+ a_- = -(2n + 1)^2\pi^2$.

Direct construction of the neighbourhood

$$N_i: X \in SL(2, \mathbb{R}) \to M = \exp[X] \in SL(2, \mathbb{R}),$$

(39)

of the unit element in $SL(2, \mathbb{R})$ from exponentiation of the Lie algebra $sl(2, \mathbb{R})$ revealed that the neighbourhood $U_i$, where this map is bijective, is given by $\text{tr}(M) > -2$. Furthermore, the total neighbourhood in $SL(2, \mathbb{R})$ where the map exists is

$$N_i = U_i \cup \{-1\} = \{M \in SL(2, \mathbb{R}) | \text{tr}(M) > -2\} \cup \{-1\}.$$  

(40)

To reveal what happens with the exponential map outside of the region $N_i$, we proceed in reverse direction and explicitly construct the logarithm $\ln(M)$ for the general $SL(2, \mathbb{R})$ matrix $M$ (23) by following through with the general procedure through the canonical form (21), (22). The characteristic equation

$$\det(M - \lambda I) = \lambda^2 - (a + d)\lambda + 1 = 0$$

(41)

yields eigenvalues $\lambda_+ = 1/\lambda_-$,

$$\lambda_{\pm} = \frac{a + d}{2} \pm \frac{1}{2} \sqrt{(a + d)^2 - 4} = \frac{1}{2} \left(\text{tr}(M) \pm \sqrt{[\text{tr}(M) - 2][\text{tr}(M) + 2]}\right).$$

(42)

The conditions

$$(M - \lambda_{\pm}I) \cdot v_{\pm} = 0$$

(43)

then yield the eigenvectors up to overall factors,

$$v_{\pm} = \begin{pmatrix} \frac{1}{2}(a - d) \pm \frac{1}{2} \sqrt{(a + d)^2 - 4} \\ -b \end{pmatrix}.$$  

(44)

The equations (42), (44) imply that we need to discuss the cases $a + d = \text{tr}(M) > -2$, $\text{tr}(M) < -2$, and $\text{tr}(M) = -2$ separately.

2.1. The logarithm of SL(2, R) matrices with $\text{tr}(M) > -2$

The results (44) imply that the transformation matrix $V$ in equation (17) can be chosen as

$$V = (v_+, \ v_-).$$  

(45)

The inverse matrix $V^{-1}$ contains the dual vectors $\tilde{v}_{\pm}$ as adjoint row vector. The orthogonality condition $\tilde{v}_{\pm}^T \cdot v_{\pm} = 0$ implies

$$\tilde{v}_{\pm} = \frac{1}{N_{\pm}} \begin{pmatrix} \frac{1}{2}(a - d) \pm \frac{1}{2} \sqrt{(a + d)^2 - 4} \\ b \end{pmatrix}.$$  

(46)
and the normalization conditions \( \tilde{\psi}^\dagger \cdot \tilde{\psi} = 1 \) determine the normalizing factors

\[
N_\pm = \left( \frac{1}{2} (a - d) \pm \frac{1}{2} \sqrt{(a + d)^2 - 4} \right) \times \left( \frac{1}{2} (a - d) \mp \frac{1}{2} \sqrt{(a + d)^2 - 4} \right) = \mp b \sqrt{(a + d)^2 - 4}.
\]

(47)

The inverse transformation matrix is therefore

\[
V^{-1} = \begin{pmatrix} \tilde{\psi}^\dagger \\ \tilde{\psi} \end{pmatrix} = \frac{1}{b \sqrt{(a + d)^2 - 4}} \begin{pmatrix} 1/2 (d - a) - 1/2 \sqrt{(a + d)^2 - 4} & -b \\ 1/2 (a - d) - 1/2 \sqrt{(a + d)^2 - 4} & b \end{pmatrix}.
\]

(48)

and the transformation equation into canonical form,

\[
V^{-1} \cdot M \cdot V = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix},
\]

(49)

implies

\[
\ln(M) = V \cdot \left( \begin{pmatrix} \ln(\lambda_+) & 0 \\ 0 & \ln(\lambda_-) \end{pmatrix} \right) \cdot V^{-1}.
\]

(50)

Evaluation of the products then yields

\[
\ln(M) = \begin{pmatrix} a_3 & -a_5 \\ a_{-3} & a_5 \end{pmatrix} = \frac{\ln [(a + d + \sqrt{(a + d)^2 - 4})/2]}{\sqrt{(a + d)^2 - 4}} \left( a - d \begin{pmatrix} 2b \\ 2c & d - a \end{pmatrix} \right)
\]

(51)

if \( a + d \geq 2 \), and

\[
\ln(M) = \begin{pmatrix} a_3 & -a_5 \\ a_{-3} & a_5 \end{pmatrix} = \arccos[(a + d)/2] \begin{pmatrix} a - d \begin{pmatrix} 2b \\ 2c & d - a \end{pmatrix} \end{pmatrix}
\]

(52)

if \(-2 < a + d \leq 2\). Of course, equations (51), (52) comply with the results (33), (34) if we substitute the matrix elements \( a, b, c, d \) for \( a_3 \) and \( a_{-3} \).

Note that \( \ln(\lambda_\pm) \) pick up complex phases for \(-2 < a + d < 2\), but this does not invalidate \( \ln(M) \in sl(2, \mathbb{R}) \) because the complex phases in \( \ln(\lambda_\pm) \) cancel with the complex phases in \( V \) and \( V^{-1} \).

Equations (51), (52) confirm the results (33)–(35) for the requirement \( \ln(M) \in sl(2, \mathbb{R}) \). The boundary case \( \text{tr}(M) = a + d = 2 \) between equations (51), (52) is continuous with

\[
\ln(M)|_{\text{tr}(M)=2} = M - 1.
\]

(53)

Indeed, it is easily confirmed that the matrices

\[
M = \begin{pmatrix} 1 + \alpha & b \\ c & 1 - \alpha \end{pmatrix}, \quad a^2 + bc = 0,
\]

(54)

satisfy \((M - 1)^2 = 0\) and therefore \(\exp(M - 1) = M\).

We also note that we can express the results (51), (52) in the form

\[
\ln(M) = \ln(\lambda_+) - \ln(\lambda_-) \left( \frac{M - 1}{2 \text{tr}(M)} \right)
\]

(55)

Furthermore, we can recover \( \ln(-1) \in sl(2, \mathbb{R}) \) from equation (38) in this framework in the following way: Rescaling the eigenvectors (44) yields eigenvectors

\[
u_\pm = \begin{pmatrix} \ln(\lambda_+) \\ \lambda_+ - \lambda_- \end{pmatrix} = \begin{pmatrix} -a_+ \\ a_3 - \ln(\lambda_+) \end{pmatrix}
\]

(56)

such that instead of the transformation matrices \( V \) and \( V^{-1} \) we can also use

\[
U = (u_+, u_-) = \begin{pmatrix} -a_+ \\ a_3 - \ln(\lambda_+) \end{pmatrix} \begin{pmatrix} -a_+ \\ a_3 + \ln(\lambda_+) \end{pmatrix}
\]

(57)
and
\[
U^{-1} = \frac{1}{2a_i \ln(\lambda_i)} \begin{pmatrix} -a_3 - \ln(\lambda_+) - a_1 \\ a_3 - \ln(\lambda_+) a_1 \end{pmatrix}. 
\]  
(58)

In the singular case \( M = 1 \) we have \( \lambda_+ = \lambda_- = -1 \), and we choose \( \ln(\lambda_+) = -\ln(\lambda_-) = i\pi \) (or more generally \( (2n + 1)i\pi \)). We then recover the result (38) in the following form,
\[
\ln(-1) = U \cdot \begin{pmatrix} i\pi & 0 \\ 0 & -i\pi \end{pmatrix} \cdot U^{-1}.
\]  
(59)

2.2. The logarithm of \( SL(2, \mathbb{R}) \) matrices with \( \text{tr}(M) < -2 \)

We can also go beyond the previous \( s(2, \mathbb{R}) \) constructions by using \( \ln(x) = (2n + 1)i\pi + \ln(-x) \), \( n \in \mathbb{Z} \), for \( x < 0 \). This yields
\[
\ln(M)|_{\text{tr}(M)<-2} = \frac{(2n + 1)i\pi + \ln[(-a - d - \sqrt{(a + d)^2 - 4})/2]}{\sqrt{(a + d)^2 - 4}} \begin{pmatrix} a - d & 2b \\ 2c & d - a \end{pmatrix}.
\]  
(60)

This equation demonstrates that \( \ln(M) \) also exists for \( \text{tr}(M) < -2 \). However, in this case \( \ln(M) \not\in s(2, \mathbb{R}) \) because \( \ln(M) \) picks up a complex phase. Therefore we have an exponential representation for \( M \) also in the domain \( \text{tr}(M) < -2 \), but not in terms of a Lie algebra element. If we want to use exponentials of Lie algebra elements also for \( SL(2, \mathbb{R}) \) matrices \( M \) with \( \text{tr}(M) < -2 \), we can instead use two exponentials e.g. in the form
\[
M|_{\text{tr}(M)<-2} = \exp[\ln(-1) \cdot \exp[\ln(-M)]].
\]  
(61)

The Baker–Campbell–Hausdorff series for this combination of exponentials apparently does not converge.

We also note that equation (60) still complies with the form (55) (which was initially only found for \( \text{tr}(M) > -2 \)), if we agree to phase conventions
\[
\arg[\ln(\lambda_+)] = -\arg[\ln(\lambda_-)] = \left( n + \frac{1}{2} \right)\pi
\]  
(62)

if \( \lambda_- = 1/\lambda_+ < -2 \).

2.3. The logarithm of \( SL(2, \mathbb{R}) \) matrices with \( \text{tr}(M) = -2 \)

Finally, we note that equation (52) seems to indicate that we cannot define the logarithm of an \( SL(2, \mathbb{R}) \) matrix \( M \neq -1 \) with \( \text{tr}(M) = -2 \). However, in this case we have degenerate eigenvalues \( \lambda_+ = \lambda_- = -1 \) without a corresponding two-dimensional subspace of eigenvectors. Degenerate eigenvalues without a corresponding number of independent eigenvectors is exactly the case where the canonical form of a matrix is the Jordan form, and we have to proceed through the construction (21), (22) to construct the logarithm \( \ln(M) \). Therefore we parametrize the set of \( SL(2, \mathbb{R}) \) matrices with \( \text{tr}(M) = -2 \) in the form
\[
M = \begin{pmatrix} \alpha & b \\ c & -\alpha - 1 \end{pmatrix}, \quad \alpha^2 + bc = 0.
\]  
(63)

These matrices are transformed into Jordan canonical form through the transformation matrices
\[
V = \begin{pmatrix} \alpha & 1 \\ c & 0 \end{pmatrix}, \quad V^{-1} = \frac{1}{c} \begin{pmatrix} 0 & 1 \\ c & -\alpha \end{pmatrix},
\]  
(64)
\[
V^{-1} \cdot M \cdot V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.
\]  
(65)

We can therefore use (21), (22) to construct the logarithm also for the matrices (63) in the form (using again \( i\pi \) where in general we could use \( (2n + 1)i\pi \))
\[
\ln(M)|_{\text{tr}(M)=-2,M=-1} = V \cdot \begin{pmatrix} i\pi & -1 \\ 0 & i\pi \end{pmatrix} \cdot V^{-1} = \begin{pmatrix} i\pi - \alpha & -b \\ -c & i\pi + \alpha \end{pmatrix} = (i\pi - 1)I - M.
\]  
(66)

Indeed, the matrices (63) satisfy \( (M + 1)^2 = 0 \), and therefore one can immediately verify
\[
\exp(-M - 1) = -M.
\]  
(67)

In principle, we have to go through a Jordan normal form construction of \( \ln(M) \) also in the case \( \text{tr}(M) = 2, M = 1 \), when we also have two-fold degeneracy of eigenvalues with only one corresponding
eigenvector. However, in this case the calculation through the Jordan form recovers the result (53) which we already found from the limit of the non-singular cases.

In summary we observe that we can represent every \( SL(2, \mathbb{R}) \) matrix through single exponentials \( M = \exp[\ln(M)] \). However, we can choose \( \ln(M) \in sl(2, \mathbb{R}) \) only in the domain \( \mathcal{N}_1 \), see equation (40). Outside of this domain we have to use combinations of two matrix exponentials, e.g. in the form (61), if we want to represent \( M \) through exponentials of Lie algebra elements.

3. The range of the exponential map in \( SL(2, \mathbb{C}) \)

All the calculations from section 2 go through in the same way for \( M \in SL(2, \mathbb{C}) \). The only difference is that if the eigenvalues (42) are now generically complex, \( \lambda_+ , \lambda_- \in \mathbb{C} \). With the definition of the complex logarithm \( \ln(z), z \in \mathbb{C} \), equation (51) now covers all cases except \( \text{tr}(M) = -2 \), while equation (38) will be generalized in equations (75), (76) below. This implies \( \text{tr}[\ln(M)] = 0 \Rightarrow \ln(M) \in sl(2, \mathbb{C}) \) in all cases except if \( \text{tr}(M) = -2, M \neq -1 \), when the results (66), (67) still hold and therefore \( \text{tr}[\ln(M)] = (2n + 1)2\pi i \). The range of the exponential map \( \exp:sl(2, \mathbb{C}) \rightarrow SL(2, \mathbb{C}) \) is therefore

\[
\mathcal{N}_{1,C} = \{ M \in SL(2, \mathbb{C}) | \text{tr}(M) = -2 \} \cup \{-1\}. \tag{68}
\]

Note that this is a star shaped neighbourhood of the identity element, as required by general Lie group properties. If \( \text{tr}[\exp(Y)] = \lambda_+ + \lambda_- = \lambda_+^2 + \lambda_- < -2 \), then \( \lim_{t \to 0} \text{tr}[\exp(tY)] = 2 \) will go through complex values \( \text{tr}[\exp(tY)] = \lambda_+^t + \lambda_-^t \), \( t \in \mathbb{R} \), without going through the singular value \( \text{tr}(M) = -2 \).

The observation \( \ln(M) \notin sl(2, \mathbb{C}) \) if \( \text{tr}(M) = -2, M \neq -1 \), can also be inferred from

\[
\exp(z \cdot \sigma) = \exp[(x + iy) \cdot \sigma] = \cosh(X) \cdot \cos(Y) + i \cdot \sinh(X) \cdot \sin(Y) + \frac{z \cdot \sigma}{\sqrt{z^2}} [\sinh(X) \cdot \cos(Y) + i \cdot \cosh(X) \cdot \sin(Y)]. \tag{69}
\]

Here \( \sigma \) is the vector of Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{70}
\]

The functions \( X(x, y) \) and \( Y(x, y) \) are given by

\[
\sqrt{z^2} = \sqrt{x^2 - y^2 + 2ix \cdot y} = X + iY, \tag{71}
\]

\[
|z|^2 = (x^2 + y^2)^2 + 2x^2 y^2 \cos(2\alpha), \quad \cos \alpha = \frac{x \cdot y}{|x||y|}, \tag{72}
\]

\[
X = \frac{|z|^2 + x^2 - y^2}{2}, \quad Y = \text{sign}(x \cdot y) \sqrt{|z|^2 - x^2 + y^2}, \tag{73}
\]

and the realization of \( SL(2, \mathbb{R}) \) in section 2 corresponds to

\[
x = \begin{pmatrix} (a_+ + a_-)/2 \\ 0 \\ a_3 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ (a_+ - a_-)/2 \\ 0 \end{pmatrix}. \tag{74}
\]

The requirement \( \text{tr}[\exp(z \cdot \sigma)] = -2 \) implies \( X = 0 \) and \( Y = (2m + 1)\pi, m \in \mathbb{Z} \), and therefore \( \exp(z \cdot \sigma) = -1 \). This confirms that the only \( SL(2, \mathbb{C}) \) matrix \( M \) with \( \text{tr}(M) = -2 \) and logarithms \( \ln(M) \in sl(2, \mathbb{C}) \) is \( M = -1 \). The general Lie algebra element \( \ln(-1) = X_{-1, C} \in sl(2, \mathbb{C}) \) is

\[
X_{-1, C} = (x + iy) \cdot \sigma \tag{75}
\]

with

\[
x \cdot y = 0, \quad |z|^2 = y^2 - x^2 = (2m + 1)^2 \pi^2. \tag{76}
\]

If \( \text{tr}(M) \neq -2 \), then \( \text{tr}(M) \) determines \( X \) and \( Y \) such that \( \sinh(X) \cdot \cos(Y) + i \cdot \cosh(X) \cdot \sin(Y) = 0 \), and the off-diagonal components and the traceless component of \( M \) determine \( z \).

4. The range of the exponential map in the Lorentz group

We will see in section 5 that application of equation (51) to \( SL(2, \mathbb{C}) \) provides logarithms for \( SO(1, 3) \) through the spinor representations, and this construction provides another proof of the fact that the Lorentz group is covered by single Lie algebra exponentials. However, it is also instructive to examine logarithms in the vector representation. The calculation of matrix logarithms for general \( SO(1, 3) \) transformations in the vector representation.
representation is algebraically much more involved, but we will see from the BCH formula (and also from equation (119) below) that for a demonstration of complete coverage of SO(1, 3) in the vector representation, it is sufficient to calculate the logarithms for the subgroup SO(1, 2), where the calculation is not harder than in the spinor representations.

The BCH formula (5) and the commutator (15) yield

\[ \Delta(u, \varphi) = \exp \left( \frac{u \cdot K}{2} + \frac{1}{12} (u \times (\varphi \times u)) \cdot L - \frac{1}{12} (\varphi \times (u \times \varphi)) \cdot L + \ldots \right) \]

(77)

It is a direct consequence of the commutation relations (14) that the higher order expansion coefficients are all determined through cross products of \(u\) and \(\varphi\), and for given magnitudes \(|u|\) and \(|\varphi|\), we face the greatest obstacle in combining the boost and the rotation of the general proper orthochronous Lorentz transformation \(\hat{u}\) into a single exponential if the rapidity \(u\) and the rotation axis \(\hat{\varphi}\) are perpendicular. This is the case which tells us whether the logarithm \(\ln(\Delta)\) of a proper orthochronous Lorentz transformation is always an element in the Lie algebra of the Lorentz group, or whether there are exceptions. Furthermore, the commutation relations (14) imply the conjugation properties

\[ \exp(\psi \cdot L) \cdot \exp(u \cdot K) \cdot \exp(\psi \cdot L) = \exp[u \cdot \mathcal{L}(\psi) \cdot K], \]

(78)

\[ \exp(\psi \cdot L) \cdot \exp(\varphi \cdot L) \cdot \exp(\psi \cdot L) = \exp[\varphi \cdot \mathcal{L}(\psi) \cdot L]. \]

(79)

Equation (78) implies that it does not matter whether proper orthochronous Lorentz transformations are encoded as a boost followed by a rotation, or in the form (8) as a rotation followed by a boost. Furthermore, equations (78), (79) together also imply that we can write the transformation (8) in the form

\[ \Delta(u, \varphi) = \exp(\psi \cdot L) \cdot \exp(u \cdot \mathcal{L}(\psi) \cdot K) \cdot \exp[\varphi \cdot \mathcal{L}(\psi) \cdot L] \]

\[ \times \exp(-\psi \cdot L), \]

(80)

and this implies

\[ \ln[\Delta(u, \varphi)] = \exp(\psi \cdot L) \cdot \ln(\exp[u \cdot \mathcal{L}(\psi) \cdot K]) \cdot \exp[\varphi \cdot \mathcal{L}(\psi) \cdot L] \]

\[ \times \exp(-\psi \cdot L). \]

(81)

The property

\[ \exp(\psi \cdot L) \cdot (w \cdot K + \chi \cdot L) \cdot \exp(-\psi \cdot L) = w \cdot \mathcal{L}(-\psi) \cdot K \]

\[ + \chi \cdot \mathcal{L}(-\psi) \cdot L \]

(82)

then implies together with equation (81) that we can calculate \(\ln[\Delta(u, \varphi)]\) as a Lie algebra element if and only if we can calculate it as a Lie algebra element for a special direction of the boost vector \(u\). In particular, we can rotate the boost direction \(\hat{u}\) into the \(x\)-direction through the rotation vector

\[ \psi(\hat{u} \to \hat{x}) = \hat{u} \times \hat{x}, \quad \arcsin(|\hat{u} \times \hat{x}|), \quad u \cdot \mathcal{L}(\psi(\hat{u} \to \hat{x})) = |u|\hat{x}. \]

(83)

The determination of \(\ln[\Delta(u, \varphi)]\) for \(u \perp \varphi\) is equivalent to the construction of the general logarithm of the identity-connected component \(\mathcal{L}_{1+}\) of the Lorentz group SO(1, 2) in the \((x, y)\) plane. After rotation of the boost direction into the \(x\)-axis, a proper orthochronous Lorentz transformation in SO(1, 2) has the form

\[ \Delta = \begin{pmatrix} \gamma & -\gamma \beta & 0 \\ -\gamma \beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{pmatrix} \]

(84)

\[ \mathcal{L}_{1+} = \begin{pmatrix} \gamma & -\gamma \beta \cos \varphi & -\gamma \beta \sin \varphi \\ -\gamma \beta \cos \varphi & \gamma & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{pmatrix} \]

i.e. we use generators for \(so(1, 2)\)

\[ K_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \]

(85)
The eigenvalue condition is

\[
\det(\Delta - \lambda I) = (1 - \lambda)(\lambda^2 - \xi \cdot \lambda + 1),
\]

where we introduced

\[
\xi = \gamma + \gamma \cos \varphi + \cos \varphi - 1 = 2(\gamma + 1)\cos^2(\varphi/2) - 2 \geq -2.
\]

The eigenvalues are therefore \(\lambda_1 = 1\) and \(\lambda_\pm = 1/\lambda_+\),

\[
\lambda_\pm = \frac{1}{2}(\xi \pm \sqrt{\xi^2 - 4}).
\]

We have either

\[
|\xi| \leq 2 \Leftrightarrow -1 \leq \cos \varphi \leq \frac{3 - \gamma}{1 + \gamma}
\]

and

\[
|\lambda_\pm| = 1, \quad \ln(\lambda_+) = -\ln(\lambda_-) = i \cdot \arccos(\xi/2),
\]

or

\[
\xi \geq 2 \Leftrightarrow \cos \varphi \geq \frac{3 - \gamma}{1 + \gamma}
\]

and

\[
\ln(\lambda_+) = -\ln(\lambda_-) \geq 0.
\]

We denote the case (90), (91) with \(-2 \leq \xi < 2\) as rotation dominated and the case (92), (93) with \(\xi > 2\) as boost dominated, see figures 1 and 2.

The eigenvectors are

\[
\mathbf{v}(\lambda) = \begin{pmatrix}
\gamma \beta (1 - \lambda \cos \varphi) \\
(\gamma - \lambda)(\cos \varphi - \lambda) \\
(\gamma - \lambda)\sin \varphi
\end{pmatrix}
\]

The eigenvector for \(\lambda_1 = 1\) satisfies

\[
\mathbf{v}^2(1) = (\gamma - 1)(1 - \cos \varphi)(\xi - 2),
\]

i.e. the invariant vector is spacelike for boost domination, timelike for rotation domination, and a lightlike vector in the transition case \(\xi = 2\).
We find the logarithm \( \ln(\Delta) \) of the Lorentz transformation (84) using the transformation matrix

\[
V = (v(\lambda_+), v(\lambda_-), v(1)).
\]

in the form

\[
\ln(\Delta) = V \cdot \left( \begin{array}{ccc}
\ln(\lambda_+) & 0 & 0 \\
0 & \ln(\lambda_-) & 0 \\
0 & 0 & 0
\end{array} \right) \cdot V^{-1}.
\]

This yields

\[
\ln(\Delta) = -\frac{1}{2} \frac{\ln(\lambda_+) - \ln(\lambda_-)}{\lambda_+ - \lambda_-} \gamma \beta (1 + \cos \varphi) \\
\gamma \beta \gamma \beta \sin \varphi (\gamma + 1) \sin \varphi
\]

with boost-like parameters

\[
w_1 = \frac{1}{2} \frac{\ln(\lambda_+) - \ln(\lambda_-)}{\lambda_+ - \lambda_-} \gamma \beta (1 + \cos \varphi),
\]

\[
w_2 = \frac{1}{2} \frac{\ln(\lambda_+) - \ln(\lambda_-)}{\lambda_+ - \lambda_-} \gamma \beta \sin \varphi,
\]

and a rotation parameter

\[
\chi = \frac{1}{2} \frac{\ln(\lambda_+) - \ln(\lambda_-)}{\lambda_+ - \lambda_-} (\gamma + 1) \sin \varphi.
\]

The relations (98)–(102) also hold in the degenerate limits \( \xi \to \pm 2 \Rightarrow \lambda_+ \to \lambda_- \to \pm 1 \). The limit \( \xi \to 2 \) corresponds to the blue line in figure 2 and yields \( [\ln(\lambda_+) - \ln(\lambda_-)]/(\lambda_+ - \lambda_-) \to 1 \). This limit yields \( w_1^2 + w_2^2 = \chi^2 \) and \( \ln^2(\Delta) = 0 \), i.e. it corresponds exactly to the class of ‘parabolic transformations’ in the terminology of Riesz [10],

\[
\Delta = 1 + \ln(\Delta) + \frac{1}{2} \ln^2(\Delta).
\]

The limit \( \xi \to -2 \) corresponds to the limits \( \varphi = \pm \lim_{\xi \to -2}(\pi - \epsilon) \). These limits of rotations by \( \pm \pi \) yield

\[
\ln(\Delta) = \pm \frac{\pi}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \sqrt{\gamma - 1} \\ 0 & 0 & -\sqrt{\gamma + 1} \\ \sqrt{\gamma - 1} & \sqrt{\gamma + 1} & 0 \end{pmatrix}.
\]
Furthermore, the relations \((100)-(102)\) are invertible,

\[
\beta = 2\sqrt{x^2 + y^2}, \quad \varphi = 2\arctan(x/y),
\]

and the representation

\[
\Delta = \exp(w_1 K_1 + w_2 K_2 + \chi L_3)
\]

indeed covers the identity-connected component of \(SO(1, 2)\) completely. The Lorentz group is therefore an example of a non-compact symmetry group where the identity-connected component is completely covered with single Lie algebra exponentials.

If \(\gamma\) and \(|\varphi|\) are too large then \(|\chi| > \pi\), see figure 3. Furthermore, contrary to the angle \(\varphi\), the relations \((105)\) and \((106)\) of \(\Delta\) in terms \(\chi\) are not periodic in \(\chi\). Therefore \(\chi\) can only be interpreted as a rotation angle in the limit \(\gamma \to 1\).

5. The ranges of the exponential maps in the covering groups \(SL(2, \mathbb{R})\) and \(SL(2, \mathbb{C})\) versus the ranges in the corresponding Lorentz groups

We have reconfirmed through the matrix logarithms that the identity connected components of the Lorentz groups, i.e. the proper orthochronous Lorentz groups \(L_+\) and \(L_{++}\) in three or four dimensions, are covered by single Lie algebra exponentials, while their corresponding double covers \(SL(2, \mathbb{R})\) and \(SL(2, \mathbb{C})\) cannot be covered by single Lie algebra exponentials. How can that be? The point is that the surjective mappings from the covering groups onto the proper orthochronous Lorentz groups are not injective, and therefore we cannot infer that both elements \(\hat{U}\) and \(-\hat{U}\) of a covering group, which map into the same Lorentz transformation \(\Delta\) in \(L_+\) or \(L_{++}\), have logarithms in \(sl(2, \mathbb{R})\) or \(sl(2, \mathbb{C})\), respectively. On the other hand, the fact that the proper orthochronous Lorentz groups are covered by single Lie algebra exponentials must also hold in the spinor representations, and therefore the maximal reaches \(N_+ \subset SL(2, \mathbb{R})\) and \(N_{++} \subset SL(2, \mathbb{C})\) of the exponential maps in the covering groups must encompass complete sheets of the corresponding covering maps onto the proper orthochronous Lorentz groups. We will confirm this by calculating the matrix logarithm of the spinor representation of the Lorentz transformation \((84)\) in \((x, y, z)\) subspace (with \(z' = z\)) in the \((1/2, 0)\) representation of \(SL(2, \mathbb{C})\). However, the standard embedding of spinor representations of \(SO(1, 2)\) in the right-handed \((1/2, 0)\) representation or the left-handed \((0, 1/2)\) representation of \(SL(2, \mathbb{C})\) does not respect \(SL(2, \mathbb{R})\) since the rotation generator \(\ell_3' = i\sigma_3/2\) is complex in those representations. Therefore we will also confirm that \(N_+ \subset SL(2, \mathbb{R})\) encompasses a complete sheet of the covering map \(SL(2, \mathbb{R}) \to L_+\) in a separate calculation in section 5.3.

5.1. The relation between \(L_{++}\) and \(SL(2, \mathbb{C})\) revisited

It is well known that \(SL(2, \mathbb{C})\) is the universal double cover of the proper orthochronous Lorentz group \(L_{++}\) through the mappings

\[
\pm \hat{U}(u, \varphi) \to \Delta(u, \varphi)
\]
The SL(2, C) matrix

\[ \mathcal{U}(u, \varphi) = \exp(-u \cdot \sigma/2) \cdot \exp(i\varphi \cdot \sigma/2) \]

is the spin 1/2 representation of a proper orthochronous Lorentz transformation on the right-handed components of a Dirac spinor, also known as the (1/2,0) representation, while

\[ \mathcal{U}(u, \varphi) = \exp(u \cdot \sigma/2) \cdot \exp(i\varphi \cdot \sigma/2) \]

yields the (0,1/2) representation on the left-handed components. The mapping SL(2, C) \( \rightarrow \mathcal{L}_+ \) is encoded in a mapping from the spin 1/2 representations into the vector representation with the help of four-dimensional Pauli matrices \( \sigma^\mu \) and \( \sigma^\nu \),

\[ \sigma_0 = \bar{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{\sigma}_i = -\sigma_i. \]

The overbar in the notation is related to complex conjugation through

\[ \bar{\sigma}_{\mu} = \xi \cdot \sigma^\mu \cdot \xi^T, \]

where \( \xi \) is the two-dimensional anti-symmetric tensor with \( \epsilon^{12} = 1 \). The explicit mappings are then

\[ N_{\nu}(u, \varphi) = -\frac{1}{2} \text{tr} [ \mathcal{U}^+(u, \varphi) \cdot \sigma^\mu \cdot \mathcal{U}(u, \varphi) \cdot \sigma_\nu ] \]

\[ = -\frac{1}{2} \text{tr} [ \mathcal{U}^+(u, \varphi) \cdot \sigma^\mu \cdot \mathcal{U}(u, \varphi) \cdot \sigma_\nu ] \cdot \]

The double cover property \( \pm \mathcal{U} \rightarrow \Delta \) is again manifest in (114). However, from a geometrical or physical perspective, the double cover arises from the fact that the SO(1, 3) rotation vector \( \varphi \) can be limited to \( 0 \leq \varphi \leq \pi \). This applies because \( -\pi \varphi = \pi \cdot (-\hat{\varphi}) \) is the same rotation as \( \pi \hat{\varphi} \), or more generally, the rotation \( R(\varphi) \) is the same transformation of Minkowski space as the rotation by the angle \( 2\pi - \varphi \) with axis \( \hat{\varphi} \), and therefore

\[ \Delta(u, \varphi - 2\pi \hat{\varphi}) = \Delta(u, \varphi), \]

On the other hand, the complete cover of SL(2, C) in the representations (109), (110) requires extension of the range to \( 0 \leq \varphi \leq 2\pi \). We have for \( \varphi \rightarrow \varphi - 2\pi \hat{\varphi} \), \( \cos(\varphi/2) \rightarrow \cos[\pi - (\varphi/2)] = -\cos(\varphi/2) \), \( \hat{\varphi} \rightarrow -\hat{\varphi} \), and therefore

\[ \mathcal{U}(u, \varphi - 2\pi \hat{\varphi}) = -\mathcal{U}(u, \varphi), \]

and those two SL(2, C) transformations yield the same Lorentz transformation \( \Delta \) (114). We denote the range \( 0 \leq \varphi \leq \pi \) as the first sheet of the covers (114), (115) and the range \( \pi < \varphi \leq 2\pi \) as the second sheet.

5.2. The matrix logarithm in the (1/2,0) spinor representation

The surjective mappings (114), (115) are mappings between matrix elements, but they are not direct mappings between matrix logarithms in the two groups. Therefore it is not possible to draw a direct link between the facts that the exponential map \( \text{Exp} : \text{SO}(1, 3) \rightarrow \text{SO}(1, 3) \) is surjective, whereas the exponential map \( \text{Exp} : \text{sl}(2, \mathbb{C}) \rightarrow \text{SL}(2, \mathbb{C}) \) only covers \( N_{\nu} \subseteq \text{SL}(2, \mathbb{C}) \).

1 For the cognoscenti in van der Waerden notation, these are the complex conjugate Pauli matrices with raised spinor indices,

\[ (\sigma_3)_{ab} = \epsilon^{a_3} \epsilon^{b_3} (\sigma_3)_{ab} = \epsilon^{a_3} \epsilon^{b_3} (\sigma_3)_{ab}. \]

However, we do not have to use explicit van der Waerden notation in the following.
In the parametrizations (109), (110) for the $SL(2, \mathbb{C})$ matrices we have

$$\text{tr} \left[ U(u, \varphi) \right] = \text{tr} \left[ U(-u, \varphi) \right] = \frac{\sqrt{\gamma + 1} \cos(\varphi/2) - \sqrt{\gamma - 1} \sin(\varphi/2)}{2} \bigl( \sqrt{\gamma + 1} \cos(\varphi/2) - \sqrt{\gamma - 1} \sin(\varphi/2) \bigr),$$

and therefore the singular $SL(2, \mathbb{C})$ cases $\text{tr} \left[ U \right] = -2$, $U \propto -1$, occur for

$$\sqrt{\gamma + 1} \cos(\varphi/2) = -\sqrt{2}, \quad \hat{u} \cdot \varphi = 0.$$

This observation independently confirms that it is sufficient for the Lorentz group to construct matrix logarithms for Lorentz transformations with $\hat{u} \cdot \varphi = 0$, in order to demonstrate that the proper orthochronous Lorentz groups are covered by single Lie algebra exponentials. We have previously inferred this from the BCH formula.

The cases (119) occur on the second sheet $\pi < \varphi \leq 2\pi$ of $SL(2, \mathbb{C}) \to L^\dagger_{4+}$, and according to equation (88) they correspond to $\xi = 2$, i.e. to the ‘second’ spinor representation of the Lorentz transformations with an invariant lightlike vector in Minkowski space. The fact that $\{ U(u, \varphi) | \varphi | \pi \} \subset N_{\mathbb{C}}$ is the reason why the surjective property of $\text{Exp} : \text{SO}(1, 3) \to \mathcal{L}_{4+}$ onto the connected component $L^\dagger_{4+}$ can also be found using the spinor representations of the Lorentz group, although the exponential map $\text{Exp} : \text{SL}(2, \mathbb{C}) \to \text{SL}(2, \mathbb{C})$ is not surjective.

These observations also imply that equation (109) or equation (110) for $0 \leq \varphi \leq \pi$, combined with equation (51) (through the identification of the $SL(2, \mathbb{C})$ matrix elements $a(u, \varphi)$, $b(u, \varphi)$ etc in the spinor representations) provides complete logarithms for general proper orthochronous Lorentz transformations $L^\dagger_{4+}$.

We can confirm this explicitly through a calculation of logarithms of $SL(2, \mathbb{C})$ matrices e.g. in the right-handed spinor representation (109) and comparison with our previous results in the vector representation. The right-handed spinor representation corresponding to the Lorentz transformation (84) is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\gamma + 1} \exp(i\varphi/2) & -\sqrt{\gamma - 1} \exp(-i\varphi/2) \\ -\sqrt{\gamma - 1} \exp(i\varphi/2) & \sqrt{\gamma + 1} \exp(-i\varphi/2) \end{pmatrix},$$

(120)

The corresponding eigenvalues $\kappa_\pm = 1/\kappa_\mp$ and eigenvectors are

$$\kappa_\pm = \sqrt{\frac{\gamma + 1}{2} \cos \frac{\varphi}{2} \pm \sqrt{\frac{\gamma + 1}{2} \cos^2 \frac{\varphi}{2} - 1}},$$

(121)

and

$$\psi_\pm = \begin{cases} \frac{\sqrt{\gamma - 1} \exp(-i\varphi/2)}{\sqrt{\gamma + 1} \sin(\varphi/2) + \sqrt{\gamma + 1} \cos^2(\varphi/2) - 2} \\ \frac{\sqrt{\gamma + 1} \exp(i\varphi/2)}{\sqrt{\gamma - 1} \sin(\varphi/2) + \sqrt{\gamma + 1} \cos^2(\varphi/2) - 2} \end{cases}.$$

(122)

The diagonalization matrices for (120) are then $V = (\psi_+, \psi_-)$ and

$$V^{-1} = \frac{\exp(i\varphi/2)}{2\sqrt{\gamma - 1} \sqrt{\gamma + 1} \cos^2(\varphi/2) - 2} \begin{pmatrix} \sqrt{\gamma + 1} \sin(\varphi/2) + \sqrt{\gamma + 1} \cos^2(\varphi/2) - 2 \\ -\sqrt{\gamma - 1} \sin(\varphi/2) + \sqrt{\gamma + 1} \cos^2(\varphi/2) - 2 \end{pmatrix} \begin{pmatrix} \sqrt{\gamma - 1} \exp(-i\varphi/2) \\ \sqrt{\gamma - 1} \exp(i\varphi/2) \end{pmatrix}.$$

(123)

and this yields the logarithm of $U$ in the form

$$\ln(U) = V \cdot \begin{pmatrix} \ln(\kappa_+) & 0 \\ 0 & \ln(\kappa_-) \end{pmatrix} V^{-1} = \frac{\ln(\kappa_+)}{\sqrt{\gamma + 1} \cos^2(\varphi/2) - 2} \begin{pmatrix} i\sqrt{\gamma + 1} \sin(\varphi/2) & -\sqrt{\gamma - 1} \exp(-i\varphi/2) \\ -\sqrt{\gamma - 1} \exp(i\varphi/2) & i\sqrt{\gamma + 1} \sin(\varphi/2) \end{pmatrix}.$$

(124)

This corresponds to composition laws

$$w_{1/2}^{(1/2)} = \frac{2\ln(\kappa_+)}{\sqrt{\gamma + 1} \cos^2(\varphi/2) - 2} \sqrt{\gamma - 1} \cos(\varphi/2),$$

(125)
\[
\eta_0^{1/21} = \frac{2 \ln(k_+)}{\sqrt{(\gamma + 1) \cos^2(\varphi/2)} - 2} \sqrt{\gamma - 1} \sin(\varphi/2),
\]
and
\[
\chi^{(1/2)} = \frac{2 \ln(k_+)}{\sqrt{(\gamma + 1) \cos^2(\varphi/2)} - 2} \sqrt{\gamma + 1} \sin(\varphi/2).
\]

These results compare to the vector results (100)–(102) if we use that
\[
\lambda_+ = \kappa_+^2,
\]
and
\[
\lambda_+ - \lambda_- = 2 \sqrt{\gamma + 1} \cdot \cos(\varphi/2) \cdot \sqrt{(\gamma + 1) \cos^2(\varphi/2)} - 2.
\]

This shows that the spinor results (125)–(127) and the vector results (100)–(102) coincide on the first sheet 0 ≤ \(\varphi\) ≤ \(\pi\). The logarithm \(\ln(\Delta)\) in the Lie algebra of the Lorenz group corresponds to the logarithm \(\ln(U)\) in the spinor representation on the first sheet of the covering map (114), and therefore the fact (cf (66))
\[
\ln(U)_{|\gamma U = -2, U = -1} = ((\pi - 1) \frac{1}{2} - U \not\in s(2, \mathbb{C})
\]
does not affect the Lorentz group.

The results (125)–(127) reflect the property (130) through the singularity for \(\sqrt{(\gamma + 1) \cos(\varphi/2)} = -\sqrt{2}\), \(\kappa_+ \rightarrow -1\), while the limit for \(\sqrt{(\gamma + 1) \cos(\varphi/2)} = \sqrt{2}\), \(\kappa_+ \rightarrow 1\), is regular.

### 5.3. The covering map and matrix logarithm in the \(SL(2, \mathbb{R})\) case

The previous \(SL(2, \mathbb{C})\) cover of \(SO(1, 3)\) does not directly reduce to a covering map \(SL(2, \mathbb{R}) \rightarrow SO(1, 2)\) through restrictions to Lorentz transformations in the \(\{x, y\}\) plane, because the rotation generator \(L_z = i\sigma_3/2\) is complex in the \((1/2, 0)\) and \((0, 1/2)\) representations. Instead, we can also use boost generators
\[
\xi_3 = -\gamma/2, \quad \xi_2 = -\gamma/2,
\]
and a rotation generator
\[
\xi_0 = i\gamma z/2.
\]

In terms of embeddings in \(SL(2, \mathbb{C})\), this corresponds to the realization of \(SO(1, 2)\) through boosts and rotations in the \(\{z, x\}\) plane in the \((1/2, 0)\) representation, i.e. we have performed a similarity transformation with a rotation vector \(\varphi = 2\pi(1, 1, 1)/\sqrt{27}: x \rightarrow y, y \rightarrow z, z \rightarrow x,\)
\[
\Delta = U(0, \varphi) = \frac{1}{2} (1 + i\xi_1 + i\xi_2 + i\xi_3),
\]
\[
\xi_1 = -\frac{1}{2} \Delta \cdot \xi_1 \cdot \Delta^{-1}, \quad \xi_2 = -\frac{1}{2} \Delta \cdot \xi_2 \cdot \Delta^{-1}, \quad \xi_0 = \frac{i}{2} \Delta \cdot \xi_3 \cdot \Delta^{-1}.
\]

The covering map \(SL(2, \mathbb{R}) \rightarrow SO(1, 2)\) can then also be directly inferred from (with the convention \(\eta_{00} = -1\))
\[
\text{tr}(\xi_0 \cdot \xi_0) = \frac{1}{2} \eta_{00}
\]
which implies for the matrix representations \(\xi \equiv x^\mu \xi_\mu\) of vectors \(x^\mu\) the property
\[
\text{tr}(x \cdot y) = \frac{1}{2} x \cdot y,
\]
whence every \(SL(2, \mathbb{R})\) transformation \(M\),
\[
x \rightarrow x' = M \cdot x \cdot M^{-1}
\]
induces a Lorentz transformation
\[
A_\mu^\nu = 2 \text{tr}(\xi_\mu \cdot M \cdot \xi_\nu \cdot M^{-1}).
\]

We can now compare with the results of sections 2 and 4 by calculating the logarithm of the \(SO(1, 2)\) transformation (84) in spinor representation,
\[ M = \exp(u_2) \cdot \exp(\varphi u_0) \]
\[ = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
(\sqrt{\gamma + 1} - \sqrt{\gamma - 1}) \cos(\varphi/2) \\
-(\sqrt{\gamma + 1} + \sqrt{\gamma - 1}) \sin(\varphi/2)
\end{array} \right) \]
\[ = \Delta \cdot U \cdot \Delta^{-1}, \quad (139) \]

where \( \Delta \) is the standard \((1/2,0)\) representation matrix from equation (120). The logarithm of \( M \) can therefore be directly inferred from our previous result (124),

\[
\ln(M) = \Delta \cdot \ln(U) \cdot \Delta^{-1} = \frac{\ln(\kappa_+)}{\sqrt{(\gamma + 1)\cos^2(\varphi/2) - 2}} \times \\
\left( \begin{array}{c}
-\sqrt{\gamma - 1} \cos(\varphi/2) \\
-(\sqrt{\gamma + 1} - \sqrt{\gamma - 1}) \sin(\varphi/2)
\end{array} \right) \]
\[= -\frac{w_1^{(1/2)}}{2} - \frac{w_2^{(1/2)}}{2} + i\chi^{(1/2)} \frac{\alpha_2}{2}, \quad (140) \]

with the boost and rotation parameters from equations (125)--(127). We have \( \ln(M) \in s/l(2, \mathbb{R}) \) if \((\gamma + 1)\cos^2(\varphi/2) > 2\), i.e. if

\[
\text{tr}(M) = \sqrt{2(\gamma + 1)} \cos(\varphi/2) > -2, \quad (141)
\]

while for \( M = -1 \) the construction (38) applies, i.e. we find of course the same conditions for \( \ln(M) \in s/l(2, \mathbb{R}) \) as in section 2, and the first sheet \( 0 \leq \varphi \leq \pi \) of the cover \( SL(2, \mathbb{R}) \rightarrow SO(1, 2) \) is a subset of the maximal range

\[ \mathcal{N}_1 = \{ M \in SL(2, \mathbb{R}) | \text{tr}(M) > -2 \} \cup \{-1\} \quad (142) \]

of single \( s/l(2, \mathbb{R}) \) exponentials in \( SL(2, \mathbb{R}) \).

The representation (131), (132) of \( s/l(2, \mathbb{R}) \) is related to the representation (28) through

\[
\xi_z = -(\xi_+ + \xi_-)/2, \quad \xi_0 = (\xi_+ - \xi_-)/2,
\]

i.e. the Lie algebra expansion coefficients are related by

\[
a_3 = -\frac{w_1^{(1/2)}}{2}, \quad a_+ = \frac{1}{2}(\chi^{(1/2)} - w_2^{(1/2)}), \quad a_- = -\frac{1}{2}(\chi^{(1/2)} + w_2^{(1/2)}). \quad (143)
\]

### 6. Summary and conclusions

The range of \( \exp(X) \), \( X \in s/l(2, \mathbb{R}) \), in \( SL(2, \mathbb{R}) \) is \( \mathcal{N}_1 = U_4 \cup \{-1\} \). Here \( U_4 = \{ M \in SL(2, \mathbb{R}) | \text{tr}(M) > -2 \} \) is the region where the exponential map \( \exp : s/l(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R}) \) is bijective, whereas the general \( s/l(2, \mathbb{R}) \) element \( X_{-1} \), with \( \exp(X_{-1}) = -1 \) is given by

\[
X_{-1} = \left( -(a_3^2 + [(2n + 1)\pi]^2)/a_+ \right)^{a_3}, \quad n \in \mathbb{Z}. \quad (145)
\]

The logarithm \( \ln(M) \) is still defined on \( SL(2, \mathbb{R}) \setminus \mathcal{N}_1 \), but does not yield elements in \( s/l(2, \mathbb{R}) \) because \( \ln(M) \) becomes complex (60), (66). The elements \( M \in SL(2, \mathbb{R}) \setminus \mathcal{N}_1 \) can be represented in terms of Lie algebra elements through products of exponential maps, \( M = \exp(X_{-1}) \cdot \exp(\ln(-M)) \). The BCH series for this product does not converge.

The range of \( \exp(X) \), \( X \in s/l(2, \mathbb{C}) \), in \( SL(2, \mathbb{C}) \) is \( \mathcal{N}_{1,\mathbb{C}} = U_{4,\mathbb{C}} \cup \{-1\} \). Here \( U_{4,\mathbb{C}} = \{ M \in SL(2, \mathbb{C}) | \text{tr}(M) > -2 \} \) is the region where the exponential map \( \exp : s/l(2, \mathbb{C}) \rightarrow SL(2, \mathbb{C}) \) is bijective, whereas the general \( s/l(2, \mathbb{C}) \) element \( X_{-1,\mathbb{C}} \), with \( \exp(X_{-1,\mathbb{C}}) = -1 \) is given by equations (75), (76).

The logarithms of matrices \( M \) with \( \text{tr}(M) = -2, M \neq -1 \), are both for \( SL(2, \mathbb{R}) \) and \( SL(2, \mathbb{C}) \) given by

\[
\ln(M) |_{\text{tr}(M) = -2, \mathcal{N}_{1,\mathbb{C}} = -1} = [(2n + 1)i\pi - 1] - M, \quad n \in \mathbb{Z}. \quad (146)
\]

The logarithm of an \( SO(1, 2) \) transformation can always be found as an element of \( so(1, 2) \),

\[
\ln[\Delta(u, \varphi)] = w(u, \varphi) \cdot \mathbb{K} + \chi(u, \varphi) \mathbb{L}_3, \quad (147)
\]

with \( w(u, \varphi) \) and \( \chi(u, \varphi) \) determined through equations (100)--(102) and (80)--(83). The range of \( \exp : so(1, 2) \rightarrow SO(1, 2) \) therefore covers the whole group.

The logarithms of \( SO(1, 3) \) transformations correspond to logarithms on the first sheet of the covering group \( SL(2, \mathbb{C}) \), and therefore the singularity (146), which limits the coverage of \( SL(2, \mathbb{C}) \) under
Exp : sl(2, \mathbb{C}) \rightarrow SL(2, \mathbb{C})$, does not affect the coverage of the Lorentz group through single Lie algebra exponentials. The corresponding coefficients $w(u, \varphi), \chi(u, \varphi)$ for SO(1, 3),

$$\ln[A(u, \varphi)] = w(u, \varphi) \cdot \mathbf{K} + \chi(u, \varphi) \cdot \mathbf{L}$$

(148)

can be read off from the right-handed or left-handed spinor representations (109), (110) and equation (51) for $\mathbf{M} \in SL(2, \mathbb{C})$.

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