Conformal Field Theories on the Two-Torus and Quotients of $SL2(Z)$

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Abstract

We present remarkable properties of the groups $SL2(Z/NZ)$ which might be useful in detailed studies of some quotients appearing in Conformal Field Theories (CFTs).

Introduction

The main object underlying this study is a finite dimensional representation $(\rho, V)$ of the group $SL2(Z)$ of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Our (writing) efforts are justified by the high interest for physics and mathematics of the space $V$; indeed there exist an infinity of such representations, a countable subset of explicit examples being provided by integrable representations of infinite dimensional affine algebras, ("Kac-Moody" algebras) and Virasoro algebra representations called “minimal models”. Any of the rational conformal field theories (whenever their classification will be achieved) will provide such a representation.

In the following paragraphs we look at subgroups or quotients of $SL2(Z)$ expressing elements in terms of words in the two generators $T$ and $S$. Physical motivation for this is that $T$ is represented by a unitary diagonal matrix whose eigenvalues are related to the physical dimension of fundamental excitations, whereas $S$ describes the effect of putting a box on one of its sides, or permuting the role of “space” and “time” in some hamiltonian description.

Since this field of research is very popular, we refer the reader to monographies (like the one by J.M. Drouffe and C. Itzykson) and go directly into:

Preliminary formulae:
The best to get acquainted with beauties of matrix groups over rings is maybe to let oneself play; therefore let us consider products of matrices of the form:

$T^x S := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$

$T^{x_n} S \cdots T^{x_1} S := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} = \begin{pmatrix} x_n A_{n-1} - C_{n-1} & x_n B_{n-1} - D_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix}$

For arbitrary elements of $SL2(Z)$, we know the minimal $n$ required is not bounded, according to the famous Farey’s enumeration of rational numbers between 0 and 1.
For $n = 3$ we get:

$$T^x \ ST^{-d} \ ST^y \ S = \begin{pmatrix} -(xdy + x + y) & 1 + xd \\ -(dy + 1) & d \end{pmatrix}$$

and also

$$T^u \ ST^c \ ST^v = \begin{pmatrix} uc - 1 & ucv - u - v \\ c & cv - 1 \end{pmatrix}$$

and

$$T^u \ ST^{-c} \ S^{-1} \ T^v = \begin{pmatrix} uc + 1 & ucv + u + v \\ c & cv + 1 \end{pmatrix}$$

We will always denote integers with lower case letters: $a,b,c,d$; and by $A,B,C,D$ their residues modulo $N$, a fixed integer equal to the order of the matrix $\rho(T)$. Quite often $ad - bc = 1$ and when $(c,N) = 1$ we will denote by $C^{-1}$ the inverse of $C$ in $\mathbb{Z}/N\mathbb{Z}$. In the case $(d,n) = 1$ we will denote its inverse also by $D^{-1}$.

In the next paragraphs, we will use some formulae valid for elements $A, B, C, D$ satisfying $AD - BC \equiv 1$ in a commutative ring. (in view of $\mathbb{Z}/N\mathbb{Z}$ we will denote equality by $\equiv$, but the formulae are also true in $\mathbb{Z}$ with equality of integers).

Proposition: If there exists $U$ such that $UC \equiv A + 1$, then

$$T^U \ ST^C \ ST^{U-D-B} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Proof: is straightforward from above formula. Similarly we also have:

Proposition: If there exists $U$ such that $UC \equiv A - 1$, then

$$T^U \ ST^{-C} \ S^{-1} \ T^{B-DU} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Proposition: If there exists $X$ such that $XD \equiv B - 1$, then

$$T^X \ ST^{-D} \ ST^{X-C-A} \ S \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Proposition: If there exists $X$ such that $XD \equiv B + 1$, then

$$T^X \ ST^D \ S^{-1} \ T^{A-XC} \ S \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Proposition: If there exists $X$ such that $XA \equiv -(1 + C)$, then

$$ST^X \ ST^{-A} \ ST^{-(D+BX)} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Proposition: If there exists $X$ such that $XA \equiv 1 - C$, then

$$ST^X \ ST^A \ S^{-1} \ T^{D+BX} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} = T^2 \ ST^2 \ ST^2 \ in \ Z$$

2
\[
\begin{pmatrix}
5 & -8 \\
2 & -3
\end{pmatrix} = T^2 S T^{-2} S^{-1} T^{-2} \quad \text{in } \mathbb{Z}
\]

If \((c, N) = 1\), we have both solutions \((U_+, V_+) \equiv (A + 1C^{-1}, (D + 1C^{-1})\),
and \((U_-, V_-) \equiv (A - 1C^{-1}, (D - 1C^{-1})\), where \(C^{-1}\) is the inverse of \(C\) mod \(N\):

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \equiv T^AC^{-1} S_C TDC^{-1}
\]

where \(S_C := T^{-C-1} STC^{-1} STC^{-1} \equiv \)

\[
\equiv T^{-C-1} STC^{-1} S^{-1} T^{-C-1} \equiv \begin{pmatrix} 0 & -C^{-1} \\ C & 0 \end{pmatrix} \equiv (\sigma_C(S), \text{ see below })
\]

These equalities reflect some relations in the group \(SL2(\mathbb{Z}/N\mathbb{Z})\):

\[
T^2C^{-1} S T^{C} S = S T^{-C} S^{-1} T^{-2C^{-1}}
\]

Since \(S \begin{pmatrix} A & B \\ C & D \end{pmatrix} S^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\),

only when the four residues \(A, B, C, D\) are non invertible mod \(N\), could we have complicated expressions
for elements of \(SL2(\mathbb{Z}/N\mathbb{Z})\). This could happen only when \(N\) is not a prime power.

**Results from Physical Approaches**

In the sequel, we will use an abelian group of automorphisms (which we call the \(\sigma_L\)'s ) of the group
\(SL2(\mathbb{Z}/N\mathbb{Z})\) defined by:

\[
\sigma_L \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \equiv \begin{pmatrix} A & BL \\ CL^{-1} & D \end{pmatrix}
\]

Obviously, this group of automorphisms is isomorphic to the multiplicative group of invertible residues mod \(N\):

Furthermore these morphisms satisfy:

\[
\sigma_K(S) \sigma_L(S) \equiv \sigma_{K/L}(S) S \equiv S \sigma_{L/K}(S) \equiv \begin{pmatrix} -K/L & 0 \\ 0 & -L/K \end{pmatrix}
\]

**Theorems** (de Boer, Goeree, A.C., Gannon, Lascoux, Bantay):

Any Rational Conformal Field Theory defines a representation \(\rho\) of \(SL2(\mathbb{Z})\) whose matrix elements
\(\rho(M)_{pp'}\) are in a cyclotomic field of \(N\)-th roots of unity, \(\rho(T^N)\) is the identity, \(\Gamma(N)\) lies in its kernel , and
for any matrix \(M\) of \(SL2(\mathbb{Z}/N\mathbb{Z})\), the above morphisms go into the cyclotomic characters:

\[
\sigma_L(\rho(M)_{pp'}) = \rho(\sigma_L(M))_{pp'}
\]

where on the l.h.s \(\sigma_L(\xi_N) = \xi_N^L\)

In previous texts, explicit computations were given for the example of \(sl(2)\) affine Lie algebra (so called
Wess Zumino Witten models), Furthermore it was exposed how these theorems can lead to very compact
formulae for the representation on Virasoro characters.
Relations

In this paragraph we give examples of constructions of quotient groups as announced above:
We define the relations $R_N$ to be:

\[ R_N : S^4 = T^N = 1, \ (ST)^3 = S^2 \]

The group generated by $S$, $T$ and these relations, is for $N \geq 6$, associated to a tessellation by triangles. The question of which relations can be added in order to obtain a quotient group which is finite, or a triangulated surface of finite genus (and finite area) naturally arises. Note that for some authors a Riemann surface is compact and this is not the case as long as we keep the hyperbolic metric structure for which the punctures are at (logarithmically) infinite distance.

Other questions are: what are the relations which lead to the group $SL_2(Z/NZ)$? At given $N$, what are the relations which give a surface of minimal genus?

Here we address only the first question for which we need the following:

Lemma: when $BC \equiv -2$,

\[ ST^C ST^{-B} \equiv T^B ST^{-C} S^{-1} = \begin{pmatrix} -1 & B \\ C & 1 \end{pmatrix} \]

when $BC \equiv 0$,

\[ ST^C ST^{-B} \equiv T^{-B} ST^C S = \begin{pmatrix} -1 & B \\ C & -1 \end{pmatrix} \]

\[ ST^{-C} S^{-1} T^B \equiv T^B ST^{-C} S^{-1} = \begin{pmatrix} 1 & B \\ C & 1 \end{pmatrix} \]

Proposition:

\[ ST^U ST^{-A} S T^V = T^X ST^{-D} S T^Y S \]

If and only if

\[ A \equiv -(X + BY), \ D \equiv -(V + BU), \ UX \equiv VY \]

where necessarily $B \equiv 1 + XD \equiv 1 + AV$.

Proposition: let $p$ be a prime

If $N = p^n > 2$, $SL_2(Z/NZ)$ is presented with extra relations:

\[ H_A := T^A ST^{1/A} ST^A S^{-1} \text{ for invertible } A' s, \ H_A H_B = H_{AB} \]

\[ H_A T = T^{A^2} H_A, \ H_A S = S^{-1} H_{1/A} \]

Proof: define for each $(C, D)$ such that there exist $A, B$, $AD - BC \equiv 1$, a word $X_{(C, D)}$ in $S$ and $T$ which corresponds to both an $SL_2$ matrix and an hyperbolic triangle. We can enumerate elements of the group mod $N$ as words of the form $T^x X_{(C, D)}$.

Then one proves that $SL_2(Z/NZ)/ \pm 1$ is generated by relations given above (with $S^2 = 1$) by a careful study of glueing formulae at the boundary of the connected triangulated domain. This is achieved by checking by use of the above relations that for each $X_{(C, D)}$, $X_{(C, D)} T^\pm 1$ and $X_{(C, D)} S$ are of the form $T^{L'} X_{(C', D')}$ for some $L, C', D' \in Z/NZ$. That the relations do not give a smaller quotient comes from the fact easily checked that the matrix group $SL_2(Z/NZ)$ explicitly do satisfy these relations.

Since it deserves some time, let us give some explicit steps in a constructive proof of the above presentation:
$X_{(0,1)} = \text{identity.}$

$X_{(1,D)} = ST^D$

For $(c, N) = 1, 2 \leq c \leq \frac{N-1}{2}$ and for any $D$: $X_{(C,D)} = ST^C ST^{(D+1)/C}$. Note that for more general $N$ one could also take $X_{(C,-1)} = ST^C S$.

For $c$ not coprime with $p$ (thus with $N$), $0 \leq c \leq \frac{N}{2}$ and $(d, N) = 1, 2 \leq d \leq \frac{N-1}{2}$ we take $X_{(C,D)} = ST^d ST^{(1-c)/d} S^{-1}$.

With the same $c$ and $(d, N) = 1, -\frac{N}{2} < d \leq -2$

$X_{(C,D)} = ST^{-D} ST^{-(1+C)/D} S \equiv \left( \begin{array}{cc} (1+C)/D & 1 \\ C & 1 \end{array} \right)$

Finally for $C$ non invertible, $2 \leq c < N/2$: $X_{(C,1)} = ST^{-C} S^{-1} = \left( \begin{array}{cc} 1 & 0 \\ C & 1 \end{array} \right)$

This presentation by generators and relations being established, a few remarks are useful:

A dual point of view which is also useful is to construct the surface by gluing $N$-gons which are collections of triangles labelled by words $Y_{(A,C)} T^z$. Then the centers of the corresponding $N$-gon can be seen as having coordinate $\tau = \frac{a}{c}$ on the real axis (boundary of the upper half plane). A more rigorous formulation is of course to identify the center of the $N$-gon to the orbit of $\frac{a}{c}$ under homographic $\Gamma(N)$ transformations.

Of course, the above relations in terms of the $H_{A_s}$ are redundant, it suffices to have them for generators of this abelian group (Cartan torus) isomorphic to the group of invertible residues mod $N$. Even one can find in literature various relations, which we already collected in a previous electronic text with T. Gannon (arXiv/math.QA/9909080). According to various authors their relations do in fact imply the above relations. (Such implication may come from interesting constructions and from the congruence subgroup property).

We give below for small values of $N$, such simple and compact looking presentations.

Note the genus of the Riemann surface is

\[
g \equiv 1 + \frac{p^{3n} - p^{3n-2} - 6p^{2n} + 6p^{2n-2}}{24}
\]

The above construction allows us to enumerate explicitly elements of $SL_2(\mathbb{Z}/N\mathbb{Z})$, when $N$ is a prime power. Of course an explicit use of the chinese remainder theorem gives us a description of the general group $SL_2(\mathbb{Z}/NZ)$ as direct product of its primary factors.

Let us give explicitly formulae for the generators in terms of decomposition of $1 \mod N$ into orthogonal sum of idempotents: This is textbook result for commutative semisimple algebras, called in french “algèbres réduites” by Bourbaki:

\[
1 \equiv \sum_{p, p|N} c_p \mod N \quad \quad T_p := T^{c_p}
\]

\[
S_p := S^2 (ST^{1-c_p})^3 = S^2 (T^{1-c_p} S)^3
\]

But a decomposition into primary factors gives a much too complicated description of elements of $SL_2(\mathbb{Z}/NZ)$ as words in $S$ and $T$. There is a much smarter approach!:

**Proposition:**

Any element of $SL_2(\mathbb{Z}/NZ)$ can be written with at most four powers of $T$, i.e. as a word like:

\[T^{x_3} ST^{x_2} ST^{x_1} ST^{x_0}\]

**Proof:** We start with the following lemma:
Let \(a,b,c,d, \ N\) be five integers satisfying \(ad - bc = 1\), \(N > 0\). Then there exists an integer \(m\) such that \(d' := d - mc\) is coprime to \(N\). Then denote \(D'\) its residue mod \(N\), and \(D'^{-1}\) its inverse.

Dirichlet has even proven that one could find an infinity of values of \(m\) such that \(d'\) is a prime number not dividing \(N\); but here the requirement is much weaker, so that one can find a convenient \(m\) again with help of the chinese remainders, because that means there exist \(u,v\) such that \(u(d - mc) - vN = 1\) which is equivalent to the existence of a residue \(M_p\) modulo each \(p\)' factor of \(N\) such that the residue \(D_p - M_pC_p\) is invertible mod \(p\). If \(D_p\) is invertible, \(M_p = 0\) works, and if \(D_p\) is not invertible, \(C_p\) is and therefore any \(M_p\) which is not divisible by \(p\) will do the job.

**Then** we have \((b' = b - ma \ , \ d' = d - mc)\):

\[
\begin{pmatrix} \ a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \ a & b - ma \\ c & d - mc \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}
\]

Thus \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv T^{(b'-1)/D'} ST^{-D'} ST^{-(1+C')/D'} ST^M \ mod \ N\)

Physicists’ intuition would find it natural since \(SL2(R)\) is a three dimensional manifold, and indeed here we have succeeded in decomposing any group element into an expression with four parameters so there is some kind of one-parameter degree of arbitrariness. Nevertheless this is too naive a picture because \(SL2(Z)\) is really exceptional: one needs unbounded words to express any matrix, the continued fraction of any rational \(a/c\) can be of any arbitrary length. This is even exceptional, compared to \(SL3(Z)\).

As a **conclusion** we could say that the groups \(SL2(Z/NZ)\) appear in fact simpler than what could be anticipated from \(SL2(Z)\). This allowed us to improve slightly upon prior works, bringing our little stone to the building.

Nevertheless there is a fascinating interplay between geometry and number theory, as usual due to the apparently chaotic occurrence of primes and congruences when one decomposes an integer, \(N\), into its prime factors. If we were considering strings, membranes or black holes we could claim that Conformal Field Theories bring more pieces into some cosmic hyperbolic puzzle! We prefer to let the reader appreciate the intrinsic beauty of mathematics and rigorous physics.

**Examples:**

We finally give explicit presentations for small values of \(N\): \(N = 5\) is the famous F. Klein’s icosahedron (or dodecaedron). For \(N = 6\) we have a torus, which can be equivalently defined by the two presentations below. Another very interesting approach from a geometric point of view is to identify fundamental domains as done in places like Bonn by Kulkarni (see refs ). Examples of quotients from conformal theories are detailed in previous texts by A. C.

\[
SL2(Z/5Z) = < S,T \mid R_5 > \\
SL2(Z/6Z) = < S,T \mid S^4 = T^6 = 1, \ (ST)^3 = S^2, \ ST^2 ST^{-2} = T^2 \ ST^{-2} S > \\
SL2(Z/6Z) = < S,T \mid S^4 = T^6 = 1, \ (ST)^3 = S^2, \ ST^3 ST^2 = T^2 \ ST^3 S > \\
SL2(Z/8Z) = < S,T \mid S^4 = T^8 = 1, \ (ST)^3 = S^2, \ ST^2 ST^4 = T^4 \ ST^2 S > \\
SL2(Z/9Z) = < S,T \mid S^4 = T^9 = 1, \ (ST)^3 = S^2, \ ST^3 ST^{-2} S^{-1} = T^{-4} \ ST^{-2} ST^{-2} \\
, \ (ST^3)^2 = (T^3 S)^2, \ ST^4 ST^{-4} = T^4 \ ST^{-4} S^{-1}, \\
ST^{-2} ST^4 ST^{-2} = T^{-4} \ ST^2 ST^{-4} S^{-1},
\]
\[ ST^2 ST^{-2} ST^4 = T^2 ST^{-4} ST^3 S^{-1} > \]

\[ SL_2(\mathbb{Z}/10\mathbb{Z}) = \langle S,T \mid S^4 = T^{10} = 1, (ST)^3 = S^2, ST^2 ST^5 = T^5 ST^2 S \]

\[ , \ ST^3 ST^4 = T^{-4} S^{-1} T^{-3} S, \ ST^3 ST^{-3} ST = T^{-1} ST^3 ST^{-3} S \]

\[ , \ ST^4 ST^5 = T^5 ST^4 S > \text{ genus 13} \]

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