Factor Representations of Diffeomorphism Groups

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October 25, 2018

Abstract

General semifinite factor representations of the diffeomorphism group of euclidean space are constructed by means of a canonical correspondence with the finite factor representations of the inductive limit unitary group. This construction includes the quasi-free representations of the canonical commutation and anti-commutation relations. To establish this correspondence requires a non-linear form of complete positivity as developed by Arveson. We also compare the asymptotic character formula for the unitary group with the thermodynamic \((N/V)\) limit construction for diffeomorphism group representations.

1 Introduction

The purpose of this work is to explore connections between the representation theory of inductive limit classical groups and the representation theory of the group \(\text{Diff}_c(\mathbb{R}^d)\) of compactly supported diffeomorphisms.

We need to extend the method of “holomorphic extension” of Ol’shanskii \({\textcite{13}}\) from the full unitary group to certain inductive limit groups. Our approach also uses a non-linear form of complete positivity given by Arveson \({\textcite{1}}\). The corresponding representations of the diffeomorphism group are semifinite factor representations, typically, of type II\(_{\infty}\). Further, certain subfamilies of these representations may be described in terms of the quasi-free representations of the commutation relations \({\textcite{16}}\). The overall framework for constructing representations of the diffeomorphism group as inductive limits was first described in \({\textcite{10}}\).

See \({\textcite{6}}\) for a survey of the representation theory of the diffeomorphism group.

2 Diffeomorphism Group and Its Standard Representation

Let \(X\) denote the euclidean space \(\mathbb{R}^d\), where \(d \geq 2\). Let \(Y\) denote an open subset of \(X\). We let \(\text{Diff}_c(Y)\) denote the group of all compactly supported diffeomorphisms of \(Y\) with the usual topology. That is, a diffeomorphism \(\psi\) has
compact support if it is equal to the identity on the complement of some relatively compact open subset of $X$. Further, a sequence of compactly supported diffeomorphisms $\{\psi_n\}$ converges to $\psi$ if there is a common relatively compact open set on whose complement all diffeomorphisms are equal to the identity and, on the open set itself, the diffeomorphisms converge in the $C^\infty$ topology.

Let $F$ denote the increasing sequence $X_1 \subset X_2 \subset \ldots$ of connected open subsets of $X$, such that $X_N$ is a proper subset of $X_{N+1}$, where we allow the possibility that $m(X_1)$ is infinite. Then $\text{Diff}^c(X) = \bigcup_{N=0}^\infty \text{Diff}^c(X_N)$.

The usual orthogonal action of $\text{Diff}^c(X_N)$ on the real Hilbert space $L^2_{\text{R}}(X_N)$, formed relative to Lebesgue measure $m$, gives rise to the standard representation $T$ of the diffeomorphism group on the complex Hilbert space $L^2(X_N)$, where

$$[T(\psi)f](x) = J_{\psi}(x)^{1/2}(x)f(\psi^{-1}x), \quad \text{where } f \in L^2(X_N),$$

and $J_{\psi}(x)$ denotes the Radon-Nikodym derivative $dm(\psi^{-1}x)/dm$.

3 Inductive Limit Groups and Their Representations

Let $L$ be a real separable infinite-dimensional Hilbert space with an increasing sequence $\{L_N\}_{N=0}^\infty$. We let $\mathcal{O} = \mathcal{O}_L = \lim_{\rightarrow} O(L_N)$ be the inductive limit group where each $O(L_N)$ is the usual orthogonal group with the strong operator topology. Then the embedding $T$ given above gives a natural map from $\text{Diff}^c(X)$ into $\mathcal{O}$ where $L_N = L^2(X_N)$ and $\text{Diff}^c(X_N)$ maps into $O(L_N)$.

We make the convention that by a representation of a topological group we mean a strongly continuous unitary representation.

Let $H_N$ be the complexification of $L_N$. We form $\mathcal{U} = \lim_{\rightarrow} U(H_N)$ so $\mathcal{O}$ is a natural subgroup of $\mathcal{U}$. Recall that a strongly continuous unitary representation of $U(H_N)$ is called holomorphic if its decomposition into irreducibles whose signatures are all non-negative. Now if $\pi_N$ is a representation of $U(H_N)$ then its restriction to $O(L_N)$ has the same commutant. In particular, restriction preserves irreducibility \[17\], p 24. Then by elementary reasoning about generating nests of von Neumann algebras, we find that if $\pi$ is a representation of $\mathcal{U}$, then its restriction to $\mathcal{O}$ has the same commutant. Moreover, by the results of \[7\], the further restriction of $\pi$ to the image under $T$ of $\text{Diff}^c(X)$ has the same commutant. In particular, the restriction of a factor representation $\pi$ of $\mathcal{U}$ to $T[\text{Diff}^c(X)]$ is factor.

We now indicate how the KMS positive definite functions introduced in \[10\] extend continuously to $\mathcal{U}$ and are semifinite factor representations. In general, it is a difficult problem to decide if a such a KMS positive definite function is factor, see \[10, 4\].

The holomorphic finite characters of $U(\infty)$ are naturally parametrized by $\mathcal{M}$, a class of meromorphic functions, where the character $\chi_f(V) = \det[f(V)]$ is
given by \( f \in M \) with \( f(z) \) in the form: 
\[
e^{\lambda(z-1)} \prod_{k=1}^{\infty} (1-\beta_k + \beta_k z)/(1+\alpha_k - \alpha_k z),
\]
where \( 0 \leq \lambda \), \( \alpha_k, \beta_k \) and \( \sum_{k=1}^{\infty} (\alpha_k + \beta_k) < \infty \).

Given the increasing sequence \( \{H_N\}_{N=1}^{\infty} \), we choose unit vectors \( f_N \in H_N \oplus H_{N-1}, \) \( N \geq 1 \). Note that the two collections of vectors are mutually orthornormal. Let \( H_- \) be the closed Hilbert space generated by \( \langle f_1, f_2, \ldots \rangle \) and \( H_+ = H_\perp \).

As a convention, we now fix \( U(\infty) \) to be the inductive limit unitary group formed relative to the subspaces generated by \( \langle f_1, \ldots, f_N \rangle \) and \( U(2\infty) \) relative to \( \langle e_1, \ldots, e_N, f_1, \ldots, f_N \rangle \). Let \( F \) denote the natural projection of \( H \) onto \( H_- \).

The function \( \phi_{F,F}(V) = \det[f(FVF)] \), for \( V \in U(2\infty) \), is positive definite. It extends continuously to \( \mathcal{U} \) and then through restriction gives a positive definite function on \( \text{Diff}(X) \). Let \( \pi_{F,F} \) denote the corresponding cyclic representation.

**Theorem 3.1** \( \pi_{F,F} \) is a semifinite factor representation of \( \text{Diff}_c(X) \). It is irreducible if and only if \( f(z) = z^m \), otherwise it is a \( II_\infty \) factor representation.

By [1], theorem 1.2, if \( m(X_1) = +\infty \), the result of our theorem holds for \( \text{SDiff}_c(X) \), the subgroup of measure-preserving diffeomorphisms.

We can give an explicit expression for this positive definite function on \( \text{Diff}_c(X) \) since \( FT(\psi)F \) can be written as a matrix relative to the basis \( \{f_n\} \) as \( \langle (T(\psi)f_i, f_j) \rangle \).

The method of proof is the use the notion of generalized characters [12] which are described fully in section 8. This approach will give another proof of the positive definiteness of \( \phi_{F,F} \).

We note that the representations \( \pi_{F,F_+} \) or \( \pi_{F,F_-} \), where \( f_-(z) = 1 - \beta + \beta z \), respectively \( f_+(z) = (1 + \alpha - \alpha z)^{-1} \), are given in terms of the quasi-free representations of the CAR, respectively, CCR algebras. See [2, 18].

### 4. \( N/V \) Limit and the Orthogonal Group

We include this section because it is hard to locate these results and they are needed in our discussion in section 5.

Let \( L' \) be a real Hilbert space with a non-zero vector \( \omega \), then the subgroup \( K' \) of \( O(L') \) of orthogonal transformation \( W \) such that \( W\omega = \omega \) can be identified with the orthogonal group of \( L' \odot \langle \omega \rangle \). The irreducible spherical functions \( \phi \) for \( (O(L'), K') \) are classified as

\[
\phi^{(n)}(W) = \frac{1}{||\omega||^{2n}}(S^n(W)\omega^{\otimes n}, \omega^{\otimes n}),
\]

where \( S^n(W) \) denotes the \( n \)-th symmetric power of \( W \), which is a classic result of I.J. Schoenberg.

More generally, for an increasing sequence \( L_1 \subset L_2 \subset \ldots \) of infinite dimensional real Hilbert spaces whose union has completion \( L \), we set \( \Omega_k = f_1 + f_2 + \cdots + f_k \), with \( ||\Omega_k||^2 = V_k \). Here, \( \{f_j\} \) is an orthogonal system (not necessarily orthonormal) such that \( f_j \in L_j \ominus L_{j-1}, \) \( \{L_0 = \{0\} \) \). As usual,
Proposition 4.1 The irreducible spherical functions of the pair $(O, K)$ have the form: $E_\lambda(W) = e^{\lambda[(W-I)[\Omega, \Omega]]}$, $\lambda \in \mathbb{R}^+ \cup \{0\}$, where $\Omega$ is the formal vector $f_1 + f_2 + \ldots$. $E_\lambda$ is given as the limit of irreducible spherical functions of the pairs $(O(L_k), K_k)$, where $\lambda = \lim_{k \to \infty} \frac{N_k}{\|\Omega_k\|}$ and where $\phi_k^{(N_k)}$ is the spherical function with index $N_k$ of the group $K_k$ given in equation 1.

If we restrict these irreducible spherical functions $E_\lambda$ to $\text{Diff}_c(X)$ for representations constructed from an increasing sequence $F$ of subsets $X_1 \subset X_2 \subset \ldots$ with $f_k = \chi_{X_k} - \chi_{X_k-1}$ (take $X_0 = \emptyset$), so $\Omega_k = \chi_{X_k}$ and its norm $\|\Omega_k\|$ gives the volume $V_k$. We find that

$$\lim_{k \to \infty} \phi_k^{(N_k)}(T(\psi)) = \exp[(T(\psi) - I)[\Omega, \Omega]] = \exp \left[ \int_X \{\det \psi(x)\}^{1/2} - 1 \, dx \right],$$

which is the functional for the free Boson gas given in [5, 9, 17].

5 Generalized Characters

We recall that for a pair $(L, K)$, where $L$ is a topological group with closed subgroup $K$, that a $K$-central positive definite function on $L$ is one that is invariant under conjugation by $K$. The set of all such normalized $K$-central positive definite functions forms a convex set whose extreme points are called generalized characters whose corresponding representations are semifinite factor representations of type I or II. Their importance for infinite dimensional classical groups and their basic results were given by Ol’shanskii [12].

6 Complete Positivity and Positive Definite Functions

Let $\chi$ be a holomorphic normalized positive definite function on the unitary group $U(H)$, where $H$ is an infinite dimensional separable Hilbert space, continuous in the strong operator topology. Then $\chi$ is the restriction to $U(H)$ of the non-linear state of $B(H)$ (in the sense of Arveson [1]) of the form

$$\sum_{n=0}^{\infty} c_n \text{Tr}[A_n \pi_0^\otimes n(\cdot)],$$

where $A_n$ is a positive self-adjoint operator on $H^\otimes n$ with trace 1 and $c_n \geq 0$ with $\sum_{n=0}^{\infty} c_n = 1$, see [15]. Further, such non-linear states are completely positive, that is, if $x_{ij} \in B(H)$ with $|x_{ij}| \geq 0$, then $|\chi(x_{ij})| \geq 0$. Further, following
Ol’shanskii \[13\], \(\chi\) has a natural extension, written \(\hat{\chi}\), to \(\Gamma(H)\), the \(*\)-semigroup of contractions on \(H\); if \(\pi_\chi\) is the corresponding \(*\)-representation of \(\Gamma(H)\), then \(\{\pi_\chi(\Gamma(H))\}'=\{\pi_\chi(U(H))\}'\).

We let \(U=\lim_\to U(H_N)\), where \(H_0=H_1\subset H_2\subset\cdots\) are all separable infinite dimensional Hilbert spaces, as in \[8\]. Again any holomorphic positive definite function \(\chi\) on \(U\) extends to \(\Gamma=\lim_\to \Gamma(H_N)\) with \(\{\pi_\chi(U)\}''=\{\pi_\chi(\Gamma)\}''\).

As in section \[8\] we let \(H\) have the orthonormal basis \(\{e_j\}_{j=1}^\infty\cup\{f_j\}_{j=1}^\infty\). Set \(H_0=H_+\) be the closed span of \(\{e_j\}_{j=1}^\infty\) and \(H_-\) the closed span of \(\{f_j\}_{j=1}^\infty\). So, \(H=H_+\oplus H_-\). Set \(H_N=H_0\oplus\langle f_1, f_2, \ldots, f_N\rangle\) and \(U(\infty)=\lim_\to U(\langle f_1, f_2, \ldots, f_N\rangle)\), \(U(2\infty)=\lim_\to U(\langle e_1, e_2, \ldots, f_1, f_2, \ldots, f_N\rangle)\). If \(F\) denotes the projection of \(H\) onto \(H_-\) and \(\Phi\) the compression map \(\Phi(x)=Fx\), then \(\Phi\) is a (linear) completely positive map.

Through \(\Phi\), we obtain a canonical correspondence between positive definite functions on \(U(\infty)\) and \(U(2\infty)\) given as follows:

If \(\chi\) is a holomorphic positive definite function on \(U(\infty)\), let \(\tilde{\chi}\) denote its extension to \(*\)-semigroup \(\Gamma(\infty)\) as a completely positive function. Then \(\tilde{\chi}\circ\Phi\) on \(U(2\infty)\) is positive definite since it is the composition of completely positive functions and a completely positive function is automatically positive definite. Through continuity, we can interpret \(\tilde{\chi}\circ\Phi\) as a positive definite function on \(U\) as well.

If \(\chi\) is, moreover, a holomorphic finite character on \(U(\infty)\), we find that \(\tilde{\chi}\circ\Phi\) are \(U(\infty)\)-conjugate invariant positive definite functions on \(U(2\infty)\). In section \[8\] we find that they are, in fact, generalized characters, that is, extremal in the convex set of \(U(\infty)\)-invariant positive definite functions.

### 7 The Asymptotic Character Formula as an \(N/V\) Limit

As an application of the ideas of section \[8\] we extend the asymptotic representation of finite characters to generalized characters and relate it the \(N/V\)-limit.

The asymptotic character formula \[8, 13\] implies that for a fixed finite holomorphic character \(\chi\) and for a fixed \(W\in U(\mathbb{C}^k)\) (here \(V_0\) is an integer) there exists a sequence of normalized characters \(\tilde{\chi}_V\) of \(U(\mathbb{C}^V)\) such that \(\tilde{\chi}_V(W)\to\chi(W)\). In fact, such a sequence can be chosen relative to the statistics of the signature \(\lambda_V\). Let \(r_j(\lambda_V)\) denote the length of its \(j\)-th row; \(c_j(\lambda_V)\) of its \(j\)-th column; \(|\lambda_V|\) denote the sum of the entries of the signature. Then \(\tilde{\chi}_V\) converges if and only if

\[
\lim_{V\to\infty} \frac{r_j(\lambda_V)}{V} = \alpha_j, \quad \lim_{V\to\infty} \frac{c_j(\lambda_V)}{V} = \beta_j, \quad \text{and} \quad \lim_{V\to\infty} \frac{|\lambda_V|}{V} = \gamma. \tag{2}
\]

where the parameters \(\gamma, \alpha_j, \text{and} \beta_j\) determine the mereomorph function \(f\in M\) that gives \(\chi(W)=\det[f(W)]\).

Since characters are central functions, we may view this limiting process as taking place over the set of eigenvalues \(S^1\times\cdots\times S^1\) instead of \(U(\mathbb{C}^k)\). Further, we know that \(\tilde{\chi}_V\) is a sequence of uniformly bounded holomorphic polynomials
in $V_0$ complex variables and $\chi$ itself is a meromorphic function in $V_0$ complex variables.

Therefore, we also have pointwise convergence of the holomorphic extensions of the normalized characters; that is, $\tilde{\chi}_V$ to $\chi$ over the set $D^1 \times \cdots \times D^1$, where $D^1$ denotes the unit disk in the complex plane. In particular, we find that for fixed $W \in \Gamma(C^{V_0})$, the set of contractions, that $\tilde{\chi}_V(W) \to \det[f(W)]$, where $f = f_\chi$ is the meromorphic function of one complex variable that defines the finite character $\chi$.

Finally, we let $\tilde{\chi} \circ \Phi$ be a generalized character of $\mathcal{U}$. Consider a fixed element $W \in \mathcal{U}$, so that $W \in U(H_{V_0})$. Then $\Phi(W) \in \Gamma(C^{V_0})$. But we know already that there exists a sequence of normalized characters $\tilde{\chi}_V$ such that $\tilde{\chi}_V(\Phi(W)) \to \tilde{\chi}(\Phi(W))$.

For easy comparison with the $N/V$ limit, we use the notation in section 4 so let $U = \lim_{-\rightarrow} U(L^2(X_V))$. Further we consider the special case $\chi(W) = \det[f_+(W)]$, for $f_+(z) = 1/(1 + \beta_1 - \beta_1 z)$. Then the corresponding finite factor representation is an inductive limit of symmetric algebras and can be interpreted in terms of a quasi-free representation of the Weyl (CCR) algebra, see [2]. Let $\beta_1 = \lim_{V \to \infty} \frac{N}{V}$, where $V$ and $N_V$ are positive integers which correspond to the degree of the symmetric tensor power and the dimension of the representation space, respectively. (Note that the other limits in equation (1) are all zero.) Set

$$\tilde{\chi}_V(W) = \text{Tr}[S^{N_V}(\Phi_V(W))] / \text{dim}(S^{N_V}(C^V)),$$

where $W \in U(L^2(X_V))$ and $V$ gives the volume of $X_V$. Then $\tilde{\chi}_V(W) \to \tilde{\chi}(\phi(W))$; in particular, $\tilde{\chi}_V(T(\psi)) \to \tilde{\chi}(\phi(T(\psi)))$.

Hence, the translation of $V$ in the asymptotic formula to the $N/V$ limit is to change its role as the dimension of $C^V$ to the volume of the $X_V$ for the Hilbert space $L^2(X_V)$.

## 8 Proof of the Main Theorem 3.1

The method of proof is to identify the functions $\phi_{F,f}$ as extreme points in the convex set of all generalized characters. Then by [12] we know that the corresponding representation is semifinite and factor. Although these functions were introduced in [10], there are no general results concerning the structure of their corresponding representations.

**Lemma 8.1** Let $L = \lim_{-\rightarrow} U(2\infty) = \lim_{-\rightarrow} U(\langle e_1, \ldots, e_N, f_1, \ldots, f_N \rangle)$, with subgroups $K_1 = U(\langle e_1, e_2, \ldots \rangle) \times I$, $K_2 = I \times U(\langle f_1, f_2, \ldots \rangle)$. Then every holomorphic generalized character $\phi$ of the pair $(L, K_2)$ such that $\phi|_{K_1} = 1$ can be written uniquely as $\phi = \tilde{\chi} \circ \Phi$, where $\tilde{\chi}$ is the holomorphic extension of a holomorphic finite character $\chi$ from $U(\infty)$ to $\Gamma(\infty)$.

**Proof** Let $\phi$ be a generalized character of $(L, K_2)$ whose restriction to $K_1$ is trivial. Set $\phi_{2n} = \phi|U(2n)$. Then we make the claim that
Proposition 8.1

(1) The convex sets

\[ \sum_{j=1}^{\infty} e_j^n \geq 0 \]

\[ \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} c_{e_n}^{(j)} e_n \geq 0 \]

where \( E_n \) is the projection of \( \langle e_1, \ldots, e_n, f_1, \ldots, f_n \rangle \) onto \( \langle f_1, \ldots, f_n \rangle \).

To see this, write \( \phi_{2n} \) as \( \sum_{\alpha} \text{Tr}[P_{2n}^{(2n)}(\pi_{\alpha})] \), where \( P_{2n}^{(2n)} \) is a positive operator on the representation space \( H(\pi_{\alpha}(2n)) \) and \( \sum_{\alpha} \text{Tr}[P_{2n}^{(2n)}] = 1 \), by section 1. The unitary branching law: \( \pi_{\alpha}(2n) \mid U(n) \times U(n) \simeq \sum_{\beta} \pi_{\beta}^{(n)} \times \pi_{\alpha/\beta}^{(n)} \) together with the condition \( \phi_{2n} \mid U(n) \times I \) is trivial implies that the support of the operator \( P_{2n}^{(2n)} \) must be contained in the projection onto the subspace where \( \pi_{\beta}^{(n)} \times \pi_{\alpha/\beta}^{(n)} \) acts as \( \pi_0^{(n)} \times \pi_{\alpha/\beta}^{(n)} \). In particular, \( P_{2n}^{(2n)} = c_{\alpha}^{(2n)}(\pi_{\alpha}(2n)) \), for some \( c_{\alpha}^{(2n)} \geq 0 \). Further, the signatures \( \alpha \) that contribute to \( \phi_{2n} \) have at most \( n \) non-zero entries.

As usual, we let \( \pi_{\alpha}^{(2n)} \) denote the holomorphic extension of \( \pi_{\alpha}(2n) \) from \( U(2n) \) to \( \Gamma(2n) \). Then, by the centrality of the trace, we have:

\[ \phi_{2n}(W) = \sum_{\alpha} c_{\alpha}^{(2n)} \text{Tr}[\pi_{\alpha}^{(2n)}(E_n W)] = \sum_{\alpha} c_{\alpha}^{(2n)} \text{Tr}[\pi_{\alpha}^{(2n)}(\Phi(W))] \]

By (3), we find that \( \phi_{2n} \) is uniquely determined by its restriction to \( I \times U(n) \). Moreover, the map \( \phi \mapsto \phi|I \times U(\infty) \) preserves convex combinations.

Conversely, let \( \hat{\chi} \) denote the extension to \( \Gamma(\infty) \) of a holomorphic finite character \( \chi \) of \( U(\infty) \). Then \( \hat{\chi} \circ \Phi \) is trivial on \( K_1 \) and conjugate invariant by elements of \( K_2 \). Further, this correspondence is the inverse of the map given above: \( \phi \mapsto \phi|K_2 \), since \( \hat{\chi} \circ \Phi|K_2 = \chi \).

Lemma 8.2

Given an orthonormal basis \( \{e_j\} \) for \( H_+ \) such that, if \( S_N = \{j : e_j \in H_N\} \cup \{j : j \in S_N\} \cup \{f_j\}_{j=1}^{N} \) is an orthonormal basis for \( H_N \), where \( \{f_N\} \) is the orthonormal basis for \( H_+ \) as above, we form \( U(2\infty) = \lim_{\rightarrow} U(\langle e_1, \ldots, e_N, f_1, \ldots, f_N \rangle) \). Then \( U(2\infty) \) is dense in \( U \).

Proof Let \( W \in U \) be given so \( W \in U(H_{N_0}) \), for some \( N_0 \). Let \( \epsilon > 0 \) be given, together with vectors \( \xi_1, \ldots, \xi_k \). Then we need to find \( W' \in U(\langle e_1, \ldots, e_{n'}, f_1, \ldots, f_{n'} \rangle) \), for some \( n' \) such that \( \|W' \xi_j - W' \xi_j\| < \epsilon \), for \( 1 \leq j \leq k \).

We consider for each \( j \) the orthonormal expansion \( \xi_j = \sum_{n=1}^{\infty} c_n^{(j)} e_n + \sum_{n=1}^{N_0} d_n^{(j)} f_n \). Choose the index \( n' \) so that \( \|\sum_{n=1}^{\infty} c_n^{(j)} e_n\| < \epsilon/2 \), for \( 1 \leq j \leq k \). We note that since \( \xi_j \in H_{N_0} \) the coefficients \( c_n^{(j)} = 0 \), if \( e_n \notin H_{N_0} \). For this reason, we find that \( W(\sum_{n=1}^{n'} c_n^{(j)} e_n) \in H_{N_0} \). Hence, there exists \( W' \in U(\langle e_1, \ldots, e_{n'}, f_1, \ldots, f_{n'} \rangle) \) such that \( \|W' \xi_j - W' \xi_j\| < \epsilon/2 \), for \( 1 \leq j \leq n' \), where \( \xi_j' = \sum_{n=1}^{n'} c_n^{(j)} e_n \). This is the desired \( W' \), and the Lemma is proven.

Proposition 8.1

(1) The convex sets \( C_1 \) and \( C_2 \) are affinely isomorphic, where \( C_1 \) is the set of all holomorphic positive definite functions on \( U \) that are \( I \times U(\infty) \)-conjugate invariant and whose restriction to \( (U(H_+) \cap U) \times I \) is trivial,
and where \( C_2 \) is the set of all central holomorphic positive definite functions on \( U(\infty) \).

(2) Let \( \phi \) be any holomorphic generalized character from \( C_2 \). Then there exists a unique holomorphic finite character \( \chi \) of \( U(\infty) \) such that \( \phi = \hat{\chi} \circ \phi \).

**Proof** Let \( \phi \) be a generalized character of the pair \( (U, I \times U(\infty)) \). Then \( \phi|U(2\infty) \) is a generalized character as well of \( (U(2\infty), I \times U(\infty)) \), for a version of \( U(2\infty) \) that satisfies Lemma 8.2. Further, since \( U(2\infty) \) is dense in \( U \), \( \phi \) is uniquely determined by its restriction to \( U(2\infty) \). By Lemma 8.1, \( \phi = \hat{\chi} \circ \Phi \) on \( U(2\infty) \). But \( \hat{\chi} \circ \Phi \) is continuous on \( U(2\infty) \) relative to the \( U \)-topology. Hence, \( \phi = \hat{\chi} \circ \Phi \) on \( U \) uniquely.

**Corollary 8.1** Let \( \chi \) denote a holomorphic finite character of \( U(\infty) \). Then \( \hat{\chi} \circ \phi \) is a generalized character of \( U \), which generates either an irreducible representation or a type \( II_\infty \) factor representation. In particular, the generalized character is irreducible if and only if its restriction to \( U(\infty) \) is a power of the determinant; it is type \( II_\infty \) if and only if its restriction to \( U(\infty) \) is type \( II_1 \).

**Proof** We make use of the identification of the generalized characters as spherical functions of the pair \( (L \times K, K \times K) \) [12]. If \( f \) is the generalized character with corresponding cyclic representation \( T = \pi \times \pi' \) of \( L \times K \), then \( T \) is irreducible, \( \pi' \) is finite factor, and \( \pi \) and \( \pi' \) generate each other commutants. Hence, \( \pi \) is irreducible \( \iff \pi' \) is irreducible. This occurs if and only if \( \pi' \) is equivalent to a power of the determinant. Otherwise, \( \pi' \) is always equivalent to a type \( II_1 \) factor representation. So, \( \pi \) must be type \( II \) as well. Since \( U(H_N) \), so \( U \) itself, has no infinite dimensional finite factor representations, we find that \( \pi \) is type \( II_\infty \) if and only if \( \pi' \) is type \( II_1 \).

We now turn to the proof of the main theorem 3.1. By the discussion in Section 3 the von Neumann algebras generated by \( \pi_{F,F}(U) \) and \( \pi_{F,F}(\text{Diff}_c(X)) \) are identical. Hence, the main theorem follows from 8.1 and its corollary.

We close with indicating two possible further extensions of this work.

1. It would be interesting to incorporate other classes of representations into the scheme presented in this paper. For example, the tensor product of the semifinite representations of 3.1 with the standard representations of the diffeomorphism group that come from restricting the tame representations of \( U(L^2(X)) \) as well as the tensor product of the spherical representations discussed in [14]. This would be a much larger family than the representations presented in [17].

2. Further, we would like to explore the connection of our construction of the generalized characters by the condition expectation operator with the Rieffel approach to induced representations. See [7, 8].
References

[1] W. Arveson, Nonlinear states on C*-algebras, in “Operator Algebras and Mathematical Physics,” (P. Jorgensen and P. Muhly, editors), Contemporary Mathematics, Vol. 62, Amer. Math. Soc., 1987, pp. 283-343.

[2] R.P. Boyer, Representations of $U_1(H)$ in symmetric tensors, J. Functional Analysis 78 (1988), 13-23.

[3] R.P. Boyer, Characters and factor representations of the infinite classical groups, J. Operator Theory 28 (1992), 281-307.

[4] R.P. Boyer, Generalized characters of $U(\infty)$, in “Algebraic Methods in Operator Theory,” (P. Jorgensen and R. Curto editors), Birkhauser, 1994, pp. 225-235.

[5] G.A. Goldin, J. Grodnik, R.T. Powers, and D.H. Sharp, Nonrelativistic current algebra in the N/V limit, J. Math. Physics 15 (1974), 88-98.

[6] G.A. Goldin and D.H. Sharp, Diffeomorphism groups and local symmetries: some applications in quantum physics, in (B. Gruber, F. Iachello, editors), Symmetries in Science 111, Plenum Publishing (1989), 181-205.

[7] H. Grundling, A group algebra for inductive limit groups, continuity problems of the canonical commutation relations, Acta Appl. Math. 46 (1997), 107-145.

[8] N. Landsman, Representations of the infinite unitary group from constrained quantization, J. Nonlin. Math. Phys. 6 (1999), 1-20.

[9] R. Menikoff, Generating functionals determining representations of a nonrelativistic local current algebra in the N/V limit, J. Math. Physics 15 (1974), 1394-1408.

[10] R. Menikoff and D.H. Sharp, Approximate representations of a local current algebra, J. Math. Physics 16 (1975), 2353-2360.

[11] A. Okounkov and G. Ol’shanskii, Asymptotics of Jack polynomials as the number of variables goes to infinity, International Mathematics Research Notices 13 (1998), 641-682.

[12] G.I. Ol’shankii, Caractères généralisés de $U(\infty)$ et functions intérieures, C. R. Acad. Sci. Paris, t. 313, Série I (1991), 9-12.

[13] G.I. Ol’shankii, Representations of infinite dimensional classical groups, limits of enveloping algebras, and Yangians, in Topics in Representation Theory, (A.A. Kirillov, editor), Advances in Soviet Math., vol. 2, 1991, pp. 1-66: 67-101.
[14] G.I. Ol’shankii, *Unitary representations of (G, K) pairs and the formalism of R. Howe*, “Representations of Lie Groups and Related Topics,” (A.M. Vershik and D.P. Zhelobenko, editors), Gordon and Breach Publ., New York, 1990, pp. 269-463.

[15] D. Pickrell, *The separable representations of U(H)*, Proc. Amer. Math. Soc. **102** (1988), 416-240.

[16] Ş. Strătilă and D. Voiculescu, *On a class of KMS states for the unitary group U(∞)*, Math. Ann. **235** (1978), 87-110.

[17] A.M. Vershik, I.M. Gel’fand, and M.I. Graev, *Representations of the group of diffeomorphisms*, Russian Math. Surveys **30** (1975), 1-50.

[18] A.M. Vershik and S.V. Kerov, *Characters and factor representations of the infinite dimensional unitary group*, Soviet Math. Dokl. **26** (1982), 570-574.

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**Keywords:** Diffeomorphism group, factor representation, unitary group

**Mathematics Subject Classification:** Primary 22E65, 81R10; Secondary 46L99