On-off intermittency in small-world networks of chaotic maps *

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Abstract

Small-world networks are highly clustered networks with small average distance among the nodes. There are many natural and technological networks that present this kind of connections. The on-off intermittency is investigated in small-world networks of chaotic maps in this paper. We show how the small-world topology would affect the on-off intermittency behavior. The distributions of the laminar phase are calculated numerically. The results show that the laminar phases obey power-law distributions.

A great deal of research interest on the theory and applications of small-world networks have been aroused [1] since the pioneering work of Watts and Strogatz [2]. Some common properties of complex networks, such as the Internet, power grids, forest fires, and disordered porous media, are mainly determined by the way of connections among their vertices or occupied sites. One extremal case is a regular network that has a high degree of local clustering and a large average distance, while the other extremal case is a random network with negligible local clustering and a small average distance. In between, a small-world network is a special case of complex networks with a high degree of local clustering as well as a small average distance. Recently, the dynamics of small-world networks were studied in different fields. For example, many authors have investigated the synchronization of small-world networks of phase oscillators, coupled map lattices and general dynamical systems [3]. Bifurcation, fractals and chaos in small-world networks were studied in [4]. Self-organized criticality on small-world networks was studied in [5]. Turbulence

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in small-world networks was investigated in [6]. Oscillators death phenomenon on small-world networks was reported in [7]. Stochastic resonance in small-world networks was studied in [8]. And dynamical behaviors of small-world neural networks were investigated in [9].

Recently, the so called on-off intermittency has been reported in literatures [10-12]. This type of intermittency is characterized by a two state nature. The “off” state, which is nearly constant, and remains so for long periods of time and is suddenly changed by a burst, the so called “on” state, which departs quickly, and return quickly to the “off” state. Moreover, the power-law distributions of the laminar phases was also observed and discussed in these literatures. In [10], on-off intermittency in low-dimensional systems were studied. In [11, 12], the authors studied on-off intermittencies in regular coupled systems. In [11], on-off intermittency was investigated in a nearest-neighbor coupled-map lattice by applying noise at a single node. In [12], on-off intermittency in globally coupled map lattices were studied.

However, connection topology in real-world networks are usually not completely regular. And it is well-known that usually topology structure affects network dynamics critically. It is interesting to investigate how small-world topology would affect the on-off intermittency of coupled chaotic map networks. In this paper, we study this topic numerically. We fix the coupling coefficient to a constant so that the globally coupled lattice is synchronous. Then we decrease the connection-adding probability gradually. We found when the connection-adding probability slightly less than a critical value, the synchronous chaos is no longer stable and on-off intermittency appears. When further decrease this connection-adding probability, the intermittent dynamics is eventually replaced by fully developed asynchronous chaos.

The network model of $N$ coupled chaotic maps studied in this paper is described by the following equations:

$$x_i(t+1) = (1-\epsilon)f(x_i(t)) + \frac{\epsilon}{N_i} \sum_{j=1}^{N} a_{ij} f(x_j(t))$$  \hspace{1cm} (1)

where $f(x)$ is a chaotic map, $\epsilon > 0$ is the coupling coefficient, $N_i$ is the number of maps connected to the $i$th map. The matrix $A = \{a_{ij}\}$ encodes the connection topology: if there is a connection between maps $i$ and $j$, $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = 0$. $a_{ii} = 0$ for all $i$, and $N_i = \sum_{j=1}^{N} a_{ij}$. We are interested in small-world connection topology of network (1) in this paper.

The original small-world (SW) model introduced in [2] can be described as follows. Take a one-dimensional lattice of $N$ vertices arranged in a ring with connections only between nearest neighbors. We “rewire” each connection with probability $p$. Rewiring in this context means reconnecting randomly the whole lattice, with the constraint that no two different vertices can have more than one connection in between, and no vertex can have a connection with itself.

Note, however, that there is a possibility for the SW model to be broken into unconnected clusters. This
problem can be resolved by a slight modification of the SW model, suggested by Newman and Watts (NW) [13]. In the NW model, one does not break any connection between any two nearest neighbors. Instead, one adds with probability $p$ a connection between each unconnected pair of vertices. Likewise, one does not allow a vertex to be coupled to another vertex more than once, or to couple with itself. For $p = 0$, it reduces to the originally nearest-neighbor coupled network; for $p = 1$, it becomes a globally coupled network. From a coupling-matrix point of view, network (1) with small-world connections will evolve according to the rule that, in the nearest-neighbor coupling matrix $A$, if $a_{ij} = 0$, then set $a_{ij} = a_{ji} = 1$ with probability $p$.

In [12], the authors showed, for globally coupled networks, there is a critical value $\epsilon_c$ of the coupling coefficient $\epsilon$. When $\epsilon > \epsilon_c$, the network is synchronous, and when $\epsilon$ slightly less than $\epsilon_c$, on-off intermittency appears. In this paper, we are interested in the NW small-world connection topology of network (1). It is natural to ask if we let $\epsilon > \epsilon_c$, whether on-off intermittency can occur by decreasing the probability $p$ so as to make the network be a small-world. In this paper, we study this topic and provide a positive answer to this question.

In order to discuss the behavior of on-off intermittency, we will work in the variable difference space

$$\Delta x_i(t) = x_i(t) - \bar{x}(t) = x_i(t) - \frac{1}{N} \sum_{j=1}^{N} x_j(t), \quad i = 1, \ldots, N$$

(2)

By subtracting the average, all variable differences $\Delta x_i$ keep their desynchronized parts and the synchronous chaos is eliminated.

For simplicity, we consider only 1-D chaotic maps in this paper. Let $f(x) = 1 - ax^2$ be the logistic map. For $a = 1.9$, this map is chaotic. Consider a globally coupled network of $N = 100$ such maps. From [12], we can get a critical value $\epsilon_c = 0.4225$. When $\epsilon > \epsilon_c$, the globally coupled network is synchronous. In the following we let $\epsilon = 0.6$. Obviously, with this coupling coefficient, the globally coupled network is synchronous. We gradually decrease the connection-adding probability $p$. We found when $p$ less than a critical value $p_c \approx 0.29$, the synchronous states lose their stability. Fig.1 (a) shows the time evolution dynamics of $\Delta x_i(t)$ of 30 randomly selected maps for $p = 0.3 > p_c$. For the 30 maps displayed in the figure, we observe synchronized chaos in which the plotted quantity is uniformly zero. For $p = 0.27$, which is slightly below $p_c$, the time evolution dynamics of $\Delta x_i(t)$ of 30 randomly selected maps is shown in Fig.1 (b), which is interspersed with bursts away from the synchronization attractor, suggesting the occurrence of on-off intermittency. This intermittent dynamics is eventually replaced by fully developed asynchronous chaos as the probability $p = 0.12$ is far removed from the critical value, as shown in Fig. 1 (c). In Fig. 2, we show the intermittent dynamics of $\Delta x_i(t)$ of a randomly selected map $i$ of the network for $p = 0.27$. Further, we define a variable to measure the distance between the system state and the
Figure 1: The time series of $\Delta x_i(t)$ of 30 randomly selected maps. (a) $p = 0.3$, synchronous; (b) $p = 0.27$, intermittency; and (c) $p = 0.12$, asynchronous chaos.

synchronization manifold as follows [12]:

$$d(t) = \frac{1}{N} \sum_{j=1}^{N} |x_j(t) - \bar{x}(t)|$$  \hspace{1cm} (3)

Fig. 3 show the value of $d$ as a function of time $t$ for $p = 0.27$, which is also a on-off intermittent time series.

A commonly used characteristic of on-off intermittent time series is the laminar length distribution. Let $\tau$ denote the threshold value of $|\Delta x_i|$ such that for $|\Delta x_i| \geq \tau$ the signal is considered “on” and for $|\Delta x_i| < \tau$ the signal is considered “off”. The length of the laminar phase is defined as the length of the off state. We use $P_n$ to represent the probability of the laminar phase of length $n$, namely, $P_n = M_n/N$.

Figure 2: The time series of $\Delta x_i(t)$ of a randomly selected map $i$ for $p = 0.27$
where $N$ is the total number of segments of the laminar phase, and $M_n$ the number of those of length $n$. We let $\tau = 0.001$ and $p = 0.27$ in our simulations. We collect 10000 iterations for each map. We plot the numerically calculated distribution for the length of laminar phase for 100 coupled logistic maps in Fig. 4, which is a power-law distribution with a heavy-tail. In Fig. 5, we plot the distribution of laminar length of the time series $d$. The time series used here is constructed in the same way as that shown in Fig.3. In Fig. 6, we plot the distribution of laminar length of the time series $|\Delta x_i|$ of a randomly selected map $i$. As we can see from these two figures, the distributions are also power-laws with heavy-tails.

In summary, in this paper, we have studied how the small-world topology affects on-off intermittency of networks of coupled chaotic maps. We found that with a fixed coupling coefficient $\epsilon > \epsilon_0$, by decreasing the connection-adding probability gradually, when $p$ slightly less than a critical value, the synchronous chaos is no longer stable and on-off intermittency appears. When further decrease $p$, the intermittent

![Figure 3: The time series of $d(t)$ for $p = 0.27$](image)

![Figure 4: The probability distribution $P_n$ of laminar length of on-off intermittent time series for 100 maps.](image)
dynamics is eventually replaced by fully developed asynchronous chaos. The probability distributions of the length of laminar phase are also presented to characterize the on-off intermittent time series. As in may intermittent time series [10-12], the length of laminar phases also obey power-law distributions. For some other values of $N$ and $\epsilon > \epsilon_c$, and some other chaotic maps, similar phenomena were also observed. We omitted them.

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