Towards deformation quantization over a $\mathbb{Z}$-graded base

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To Murray Gerstenhaber, Jim Stasheff and Dennis Sullivan on the occasion of their jubilees

Abstract

The goal of this note is to describe a class of formal deformations of a symplectic manifold $M$ in the case when the base ring of the deformation problem involves parameters of non-positive degrees. The interesting feature of such deformations is that these are deformations “in $A_\infty$-direction” and, in general, their description involves all cohomology classes of $M$ of degrees $\geq 2$.

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1 Introduction

Let $M$ be a real manifold, $\mathcal{O}(M)$ be the algebra of smooth complex-valued functions on $M$, and $\varepsilon, \varepsilon_1, \ldots, \varepsilon_g$ be formal variables of degrees

$$\deg(\varepsilon) = 0, \quad \deg(\varepsilon_1) = d_1, \quad \deg(\varepsilon_2) = d_2, \ldots, \quad \deg(\varepsilon_g) = d_g,$$

where $d_1, d_2, \ldots, d_g$ are non-positive integers.

In this paper, we investigate the problem of deformation quantization \cite{2}, \cite{3}, \cite{4}, \cite{5}, \cite{7}, \cite{14}, \cite{22} of $M$ in the setting when we have several formal deformation parameters $\varepsilon, \varepsilon_1, \ldots, \varepsilon_g$ and some of the (non-positive) integers $d_1, d_2, \ldots, d_g$ are actually non-zero.

A formal deformation of $\mathcal{O}(M)$, in this setting, is a (non-curved) $\mathbb{C}\llbracket[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]\rrbracket$-linear $A_\infty$-structure on $\mathcal{O}(M)\llbracket[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]\rrbracket$ with the multiplications $\{m_n\}_{n \geq 2}$ of the form

$$m_n(a_1, \ldots, a_n) = \begin{cases} a_1a_2 + \sum_{k_0d_0+\cdots+k_gd_g=0} \varepsilon^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g} \mu_{k_0, k_1, \ldots, k_g}(a_1, a_2) & \text{if } n = 2, \\ \sum_{k_0d_0+\cdots+k_gd_g=2-n} \varepsilon^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g} \mu_{k_0, k_1, \ldots, k_g}(a_1, \ldots, a_n) & \text{if } n > 2, \end{cases} \quad (1.2)$$

where each $\mu_{k_0, k_1, \ldots, k_g}$ is a polydifferential operator on $M$ (with complex coefficients) acting on $2 - k_0d_0 - \cdots - k_gd_g$ arguments. Moreover, $\mu_{k_0, k_1, \ldots, k_g} \equiv 0$ if at least one $k_i < 0$ or $k_0 + k_1 + \cdots + k_g = 0$.

Such $A_\infty$-structures are in bijection with Maurer-Cartan (MC) elements of the dg Lie algebra $(\varepsilon, \varepsilon_1, \ldots, \varepsilon_g)\text{PD}^\bullet(M)\llbracket[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]\rrbracket$, \quad (1.3)
where \( PD^\bullet(M) \) denotes the algebra of polydifferential operators on \( M \).

Let us denote by \( \mathfrak{G} \) the group which is obtained by exponentiating the Lie algebra of degree zero elements of (1.3) and recall [6], [16] that this group acts in a natural way on the set of MC elements of (1.3).

By analogy with 1-parameter ("ungraded") formal deformations, we declare that two such formal deformations are equivalent if the corresponding MC elements of (1.3) belong to the same orbit of the action of \( \mathfrak{G} \).

Let us observe that, for every MC element \( \mu \) of (1.3), the coset of \( \mu \) in \( PD^\bullet(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] / (\varepsilon, \varepsilon_1, \ldots, \varepsilon_g)^2 PD^\bullet(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \) (1.4)
is closed with respect to the Hochschild differential \( \partial^{\text{Hoch}} \). Moreover, if two MC elements \( \mu_1 \) and \( \mu_2 \) lie on the same orbit of \( \mathfrak{G} \), then the corresponding cosets in (1.4) are \( \partial^{\text{Hoch}} \)-cohomologous. By analogy with "ungraded" formal deformations, we call the \( \partial^{\text{Hoch}} \)-cohomology class of the coset of \( \mu \) in (1.4) the Kodaira-Spencer class of \( \mu \).

Let us recall that the cohomology space of \( PD^\bullet(M) \) (with respect to \( \partial^{\text{Hoch}} \)) is isomorphic to the space \( PV^\bullet(M) \) of polyvector fields on \( M \). So the Kodaira-Spencer class of every MC element of (1.3) can be identified with a degree 1 vector in the graded space
\[
\varepsilon PV^\bullet(M) \oplus \varepsilon_1 PV^\bullet(M) \oplus \cdots \oplus \varepsilon_g PV^\bullet(M).
\]

Let us now assume that \( M \) has a symplectic structure \( \omega \) and denote by \( \alpha \in PV^1(M) \) the Poisson structure corresponding to \( \omega \).

In this paper, we consider formal deformations (1.2) of \( \mathcal{O}(M) \) which satisfy these two conditions:

1. the Kodaira-Spencer class of this deformation is \( \varepsilon \alpha \) and
2. 
\[
m_n|_{\varepsilon=0} = \begin{cases} a_1a_2 & \text{if } n = 2, \\ 0 & \text{if } n > 2. \end{cases}
\]

We denote by \( TL \) the set of equivalence classes of formal deformations (1.2) satisfying the above conditions and call \( TL \) the topological locus of the triple \((M, \omega, \{\varepsilon, \varepsilon_1, \ldots, \varepsilon_g\})\). Using Kontsevich's formality [22] and a construction inspired by paper [30] due to Sharygin and Talalaev, we give a description of the topological locus \( TL \) in terms of the singular cohomology of \( M \). More precisely,

**Theorem 1.1** For every symplectic manifold \((M, \omega)\), the equivalence classes of formal deformations (1.2) of \( \mathcal{O}(M) \) satisfying the above conditions are in bijections with degree 2 vectors of the graded vector space
\[
\bigoplus_{q \geq 0} (\varepsilon, \varepsilon_1, \ldots, \varepsilon_g) H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]],
\]
where \( H^\bullet(M, \mathbb{C}) \) is the singular cohomology of \( M \) with coefficients in \( \mathbb{C} \) and every vector of \( H^q(M, \mathbb{C}) \) carries degree \( q \).

**Remark 1.2** In the "ungraded" case (i.e. \( g = 0 \)), this result reproduces the classical theorem [5], [7] of Bertelson, Deligne, Cahen, and Gutt on the description of the equivalence classes of star products on a symplectic manifold. In this respect, Theorem 1.1 may be viewed as a generalization of this classification theorem to the case of a \( \mathbb{Z} \)-graded base.

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1 See Section 1.1.2
2 In particular, it means that the dimension of \( M \) is even.
**Remark 1.3** Deformations over a $\mathbb{Z}$-graded (and even differential graded) base were considered in the literature. See, for example, paper [1] by Barannikov and Kontsevich or J. Lurie’s ICM address [28] in which even more sophisticated examples of bases for deformation problems were considered. We should also mention that deformations over a differential graded base naturally show up in the construction of rational homotopy models for classifying spaces of fibrations. For more details, we refer the reader to paper [26] by Lazarev.

**Organization of the paper.** The remainder of the introduction is devoted to the notational conventions and preliminaries. In this part, we give a brief reminder of the Deligne-Getzler-Hinich (DGH) groupoid(s) and fix our conventions related to the sheaf of polydifferential operators and polyvector fields.

Section 2 starts with a brief reminder of 1-parameter formal deformations of an associative algebra (over $\mathbb{C}$). Then we propose a natural generalization of this story to the case when the base ring of the deformation problem is the completion of the free graded commutative algebra $\mathbb{C}[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]$. Next, we consider the case when $A$ is the algebra of functions $\mathcal{O}(M)$ on a smooth real manifold $M$. Finally, we assume that $M$ has a symplectic structure and formulate the main result of this paper (see Theorem 2.4).

The proof of Theorem 2.4 is given in Section 5 and it is based on two auxiliary constructions which are presented in Sections 3 and 4, respectively. In Section 6, we propose two conjectures related to Theorem 2.4. Finally, Appendix A is devoted to the proof of a technical proposition.

1.1 Notational conventions and preliminaries

We assume that the ground field is the field of complex numbers $\mathbb{C}$ and set $\otimes := \otimes_{\mathbb{C}}$, $\text{Hom} := \text{Hom}_{\mathbb{C}}$. For a cochain complex $V$ we denote by $sV$ (resp. by $s^{-1}V$) the suspension (resp. the desuspension) of $V$. In other words,

$$(sV)^* = V^{*-1}, \quad (s^{-1}V)^* = V^{*+1}.$$

The notation $S_n$ is reserved for the symmetric group on $n$ letters and $\text{Sh}_{p_1, \ldots, p_k}$ denotes the subset of $(p_1, \ldots, p_k)$-shuffles in $S_n$, i.e. $\text{Sh}_{p_1, \ldots, p_k}$ consists of elements $\sigma \in S_n$, $n = p_1 + p_2 + \cdots + p_k$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p_1),$$

$$\sigma(p_1 + 1) < \sigma(p_1 + 2) < \cdots < \sigma(p_1 + p_2),$$

$$\ldots$$

$$\sigma(n - p_k + 1) < \sigma(n - p_k + 2) < \cdots < \sigma(n).$$

For a groupoid $\mathcal{G}$, $\pi_0(\mathcal{G})$ denotes the set of isomorphism classes of objects of $\mathcal{G}$. For a graded vector space (or a cochain complex) $V$ the notation $S(V)$ (resp. $S(V)$) is reserved for the underlying vector space of the symmetric algebra (resp. the truncated symmetric algebra) of $V$:

$$S(V) = \mathbb{C} \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \ldots,$$

$$\underline{S}(V) = V \oplus S^2(V) \oplus S^3(V) \oplus \ldots,$$

$\mathbb{C}[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]$ should not be confused with the polynomial algebra because $\varepsilon_i \varepsilon_j = (-1)^{d_i d_j} \varepsilon_j \varepsilon_i$. 
where \( S^n(V) = (V^\otimes n)_{S_n} \) is the space of coinvariants with respect to the obvious action of \( S_n \).

Recall that \( S(V) \) is the vector space of the cofree cocommutative coalgebra (without counit) cogenerated by \( V \). The comultiplication on \( S(V) \) is given by the formula:

\[
\Delta(v_1, v_2, \ldots, v_n) := \sum_{p=1}^{n-1} \sum_{\sigma \in \text{Sh}_{p,n-p}} (-1)^{\varepsilon(\sigma; v_1, \ldots, v_n)} (v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \otimes (v_{\sigma(p+1)}, \ldots, v_{\sigma(n)}),
\]

where \((-1)^{\varepsilon(\sigma; v_1, \ldots, v_n)}\) is the Koszul sign factor

\[
(-1)^{\varepsilon(\sigma; v_1, \ldots, v_n)} := \prod_{i < j} (-1)^{|v_i||v_j|}
\]

and the product in (1.5) is taken over all inversions \((i < j)\) of \( \sigma \in S_n \).

For an associative algebra \( A \), we denote by \( C^\bullet(A) \) the Hochschild cochain complex of \( A \) with coefficients in \( A \) and with the shifted grading:

\[
C^\bullet(A) = \bigoplus_{k \geq -1} C^k(A), \quad C^k(A) := \text{Hom}(A^{\otimes (k+1)}, A).
\]

We denote by \([, ]_G\) the Gerstenhaber bracket\(^{[15]}\) on \( C^\bullet(A) \):

\[
[P_1, P_2]_G(a_0, \ldots, a_{k_1+k_2}) := \sum_{i=0}^k (-1)^{i k_2} P_1(a_0, \ldots, a_{i-1}, P_2(a_i, \ldots, a_{i+k_2}), a_{i+k_2+1}, \ldots, a_{k_1+k_2}) - (-1)^{k_1 k_2} (1 \leftrightarrow 2),
\]

where \( P_j \in C^{k_j}(A) \).

It is convenient to think of the multiplication on \( A \) as the Hochschild cochain \( m_A \in C^1(A) \) and define the Hochschild differential \( \partial^\text{Hoch} \) on \( C^\bullet(A) \) as

\[
\partial^\text{Hoch} := [m_A, ]_G.
\]

We reserve the notation \( HH^\bullet(A) \) for the Hochschild cohomology of \( A \) with coefficients in \( A \), i.e.

\[
HH^\bullet(A) := H^\bullet(C^\bullet(A)).
\]

For example, if \( A \) is the polynomial algebra \( \mathbb{C}[x^1, \ldots, x^m] \) is \( m \) variables then \[27\] Section 3.2

\[
HH^\bullet(A) \cong S_A(\text{sDer}(A)),
\]

where \( S_A(E) \) denotes the symmetric algebra of an \( A \)-module \( E \) and \( \text{Der}(A) \) is the \( A \)-module of \( \mathbb{C} \)-linear derivations.

If \( \varepsilon, \varepsilon_1, \ldots, \varepsilon_g \) are variables of degrees

\[
\deg(\varepsilon) = d_0, \quad \deg(\varepsilon_1) = d_1, \quad \deg(\varepsilon_2) = d_2, \quad \ldots, \quad \deg(\varepsilon_g) = d_g,
\]

then the notation

\[
\mathbb{C}[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]
\]

where

\[
\varepsilon_i := \varepsilon_i(1) \otimes \cdots \otimes \varepsilon_i(n)
\]
is reserved for the free graded commutative algebra over \(\mathbb{C}\) generated by \(\varepsilon, \varepsilon_1, \ldots, \varepsilon_g\). Furthermore,

\[
\mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]
\]

denotes the completion of (1.10) with respect to the ideal \(\mathfrak{m} = (\varepsilon, \varepsilon_1, \ldots, \varepsilon_g) \subset \mathbb{C}[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]\). For example, if \(\varepsilon_1\) carries an odd degree then \(\varepsilon_1^2 = 0\) in (1.10) and in its completion.

For every dg Lie algebra \((L, \partial, [\ , \ ]\)\), the cofree cocommutative coalgebra

\[
\mathcal{S}(s^{-1}L)
\]

(1.11)
is equipped with a degree 1 coderivation \(Q\) which satisfies \(Q^2 = 0\). The composition \(p_{s^{-1}L} \circ Q\) of \(Q\) with the canonical projection \(p_{s^{-1}L} : \mathcal{S}(s^{-1}L) \to s^{-1}L\) is expressed in terms of \(\partial\) and \([\ , \ ]\) as follows:

\[
p_{s^{-1}L} \circ Q(s^{-1}v_1 \ldots s^{-1}v_n) := \begin{cases} 
 s^{-1}(\partial v_1) & \text{if } n = 1, \\
 (-1)^{|v_1|-1}s^{-1}[v_1, v_2] & \text{if } n = 2, \\
 0 & \text{if } n \geq 3.
\end{cases}
\]

(1.12)
The assignment \((L, \partial, [\ , \ ]) \mapsto (\mathcal{S}(s^{-1}L), Q, \Delta)\) is often called the Chevalley-Eilenberg construction.

Let us recall that an \(L_{\infty}\)-morphism \(F\) from a dg Lie algebra \((L, \partial, [\ , \ ])\) to a dg Lie algebra \((\tilde{L}, \tilde{\partial}, [\ , \ ]\)\) is a homomorphism of the corresponding dg cocommutative coalgebras

\[
F : (\mathcal{S}(s^{-1}L), Q) \to (\mathcal{S}(s^{-1}\tilde{L}), \tilde{Q}).
\]

(1.13)

It is not hard to see that every coalgebra homomorphism (1.13) is uniquely determined by its composition

\[
p_{s^{-1}\tilde{L}} \circ F : \mathcal{S}(s^{-1}L) \to s^{-1}\tilde{L}
\]

with the canonical projection \(p_{s^{-1}\tilde{L}} : \mathcal{S}(s^{-1}\tilde{L}) \to s^{-1}\tilde{L}\). In this paper, we denote by \(F_n\) the restriction of \(p_{s^{-1}\tilde{L}} \circ F\) onto \(S^n(s^{-1}L)\):

\[
F_n := p_{s^{-1}\tilde{L}} \circ F|_{S^n(s^{-1}L)} : S^n(s^{-1}L) \to s^{-1}\tilde{L}
\]

and call \(F_1\) the linear term of the \(L_{\infty}\)-morphism \(F\). Recall that, for every \(L_{\infty}\)-morphism \(F\), its linear term \(F_1\) is a chain map \((L, \partial) \to (\tilde{L}, \tilde{\partial})\).

Most of the differential graded (dg) Lie algebras \((L, \partial, [\ , \ ])\), we consider, are equipped with a complete descending filtration

\[
\cdots \supset F_0L \supset F_1L \supset F_2L \supset F_3L \supset \cdots \ L \cong \lim_k L/F_kL.
\]

(1.14)

Furthermore, we tacitly assume that our \(L_{\infty}\)-morphisms are compatible with the filtrations in the following sense

\[
F_n(s^{-1}F_{k_1}L \otimes \cdots \otimes s^{-1}F_{k_n}L) \subset s^{-1}F_{k_1 + \cdots + k_n}\tilde{L}.
\]

(1.15)

For a smooth real manifold \(M\), the notation \(\mathcal{O}_M\) (resp. \(\mathcal{O}(M)\)) is reserved for the sheaf (resp. the algebra) of smooth complex valued functions on \(M\). The symbols \(x^1, x^2, \ldots, x^m\) are often reserved for coordinates on an open subset \(U \subset M\). \(TM\) (resp. \(T^*M\)) denotes the tangent (resp. cotangent) bundle of \(M\). Moreover, \(\{dx^1, dx^2, \ldots, dx^m\}\) and \(\{\theta_1, \theta_2, \ldots, \theta_m\}\) will be the standard local frames.

\(\text{It is not hard to see that every coderivation } Q \text{ of a cofree cocommutative coalgebra (1.11) is uniquely determined by the composition } p_{s^{-1}L} \circ Q : \mathcal{S}(s^{-1}L) \to s^{-1}L.\)

\(\text{In other words, } \theta_i := \partial/\partial x^i.\)
for $T^*M|_U$ and $TM|_U$ corresponding to coordinates $x^1, x^2, \ldots, x^m$, respectively. In particular, the graded commutative algebra $\Omega^*(U)$ (resp. $\wedge TM(U)$) of exterior forms (resp. polyvector fields) on $U$ will be tacitly identified with $\mathcal{O}_M(U)[dx^1, dx^2, \ldots, dx^m]$ (resp. $\mathcal{O}_M(U)[\theta_1, \theta_2, \ldots, \theta_m]$). Occasionally, we will use the (left) “partial derivative”

$$\frac{\partial}{\partial dx^i} : \Omega^*(U) \to \Omega^{*-1}(U)$$

with respect to the degree 1 symbol $dx^i$. This operation is defined by the formula

$$\frac{\partial}{\partial dx^i} \eta_{i_1 \ldots i_k}(x)dx^{i_1}dx^{i_2} \ldots dx^{i_k} := k\eta_{i_1 \ldots i_k}(x)dx^{i_2}dx^{i_3} \ldots dx^{i_k},$$

where the summation over repeated indices is assumed. Equivalently, $\frac{\partial \eta}{\partial dx^i}$ is the contraction of an exterior form $\eta$ with the local vector field $\partial/\partial x^i$.

1.1.1 A reminder of the Deligne-Getzler-Hinich (DGH) groupoid(s)

Let $L$ be a dg Lie algebra equipped with a complete descending filtration (1.14). A Maurer-Cartan element of $L$ is a degree 1 element $\mu \in F^1L$ satisfying the equation

$$\partial \mu + \frac{1}{2}[\mu, \mu] = 0.$$

In this paper, $\text{MC}(L)$ denotes the set of Maurer-Cartan elements of a dg Lie algebra $L$.

Let us recall [6, Appendix B], [16] that the formula

$$e^\xi(\mu) := e^{[\xi, \cdot]} \mu - \frac{e^{[\xi, \cdot]} - 1}{[\xi, \cdot]}(\partial \xi), \quad \xi \in F^1L^0$$

(1.17)

defines an action of the group

$$\exp(F^1L^0)$$

on the set of MC elements of $L$. In (1.17), the expressions $e^{[\xi, \cdot]}$ and

$$\frac{e^{[\xi, \cdot]} - 1}{[\xi, \cdot]}$$

are defined by the Taylor expansion of the functions $e^x$ and $(e^x - 1)/x$, respectively, around the point $x = 0$. We denote by $G(L)$ the transformation groupoid of the action (1.18).

To a dg Lie algebra $L$ (equipped with a complete filtration (1.14)), we can also associate a very useful simplicial set $\mathbf{MC}_\bullet(L)$ [17], [18] with

$$\mathbf{MC}_n(L) := \text{MC}(L \hat{\otimes} \Omega_n),$$

(1.19)

where

$$L \hat{\otimes} \Omega_n := \varprojlim_k ((L/F_kL) \otimes \Omega_n)$$

and $\Omega_n$ is the de Rham-Sullivan algebra of polynomial differential forms on the geometric simplex $\Delta^n$ (with coefficients in $\mathbb{C}$).

6Recall that, since the symbols $dx^i$ and $\theta_i$ carry degree 1, we have $dx^i dx^j = -dx^j dx^i$ and $\theta_i \theta_j = -\theta_j \theta_i$. 6
As the graded commutative algebra (with 1), \( \Omega_n \) is generated by \( n+1 \) symbols \( t_0, t_1, \ldots, t_n \) of degree 0 and \( n+1 \) symbols \( dt_0, dt_1, \ldots, dt_n \) of degree 1 subject to the relations

\[
t_0 + t_1 + \cdots + t_n = 1, \quad dt_0 + dt_1 + \cdots + dt_n = 0.
\]

Furthermore, the differential \( d_t \) on \( \Omega_n \) is defined by the formulas

\[
d_t(t_i) := dt_i, \quad d_t(dt_i) := 0.
\]

For example, \( \Omega_0 = \mathbb{C} \) and \( \Omega_1 \cong \mathbb{C}[t] \oplus \mathbb{C}[t] dt \).

Due to [12, Proposition 4.1], the simplicial set \( \mathbf{MC}_\bullet(L) \) is a Kan complex (a.k.a. an \( \infty \)-groupoid). Moreover, due to [11, Lemma B.2], two MC elements \( \mu \) and \( \tilde{\mu} \) of \( L \) (i.e. 0-cells of \( \mathbf{MC}_\bullet(L) \)) are connected by a 1-cell if and only if they belong to the same orbit of the action \( (1.17) \). In other words, we have the identification:

\[
\pi_0(\mathcal{G}(L)) \cong \pi_0(\mathbf{MC}_\bullet(L)). \tag{1.20}
\]

**Remark 1.4** The Kan complex (a.k.a. a fibrant simplicial set) \( \mathbf{MC}_\bullet(L) \) is called the Deligne-Getzler-Hinich (DGH) \( \infty \)-groupoid. In this paper, we mostly use the “truncation” of \( \mathbf{MC}_\bullet(L) \), i.e. the honest (transformation) groupoid \( \mathcal{G}(L) \).

Recall [11, Section 2] that, for every \( L_\infty \)-morphism \( F : L \rightsquigarrow \tilde{L} \), the formula

\[
F_*(\mu) := \sum_{n=1}^{\infty} \frac{1}{n!} F_n((s^{-1}\mu)^n)
\]

defines a map of sets

\[
F_* : \mathbf{MC}(L) \to \mathbf{MC}(\tilde{L}). \tag{1.21}
\]

Furthermore, since \( F \) naturally extends to an \( L_\infty \)-morphism \( F : L \otimes \Omega_n \rightsquigarrow \tilde{L} \otimes \Omega_n \) for every \( n \geq 1 \), the map \( F_* \) naturally upgrades to the morphism of simplicial sets

\[
F_* : \mathbf{MC}_\bullet(L) \to \mathbf{MC}_\bullet(\tilde{L}) \tag{1.22}
\]

for which we use the same notation. Therefore \( F_* \) gives us a map of sets\(^7\)

\[
\pi_0(F_*) : \pi_0(\mathcal{G}(L)) \to \pi_0(\mathcal{G}(\tilde{L})). \tag{1.23}
\]

**Remark 1.5** It is not hard to see that the assignments \( L \mapsto \mathbf{MC}_\bullet(L) \) and \( F \mapsto F_* \) define a functor from the category of filtered dg Lie algebras to the category of simplicial sets.

### 1.1.2 The sheaf \( \mathbf{PD}^\bullet \) of polydifferential operators and the sheaf \( \mathbf{PV}^\bullet \) of polyvector fields

Let \( U \subset M \) be an open coordinate subset of a manifold \( M \) with coordinates \( x^1, x^2, \ldots, x^m \).

For every \( k \geq -1 \), the space \( \mathbf{PD}^k(U) \) consists of \( \mathbb{C} \)-multilinear maps

\[
P : \mathcal{O}_M(U)^{\otimes k+1} \to \mathcal{O}_M(U)
\]

\[^7\] Here we use the identification \((1.20)\).
which can be written (in local coordinates) as finite sums
\[ P = \sum_{\alpha_0, \alpha_1, \ldots, \alpha_k} P^{\alpha_0, \alpha_1, \ldots, \alpha_k}(x) \partial_{x^{\alpha_0}} \otimes \partial_{x^{\alpha_1}} \otimes \cdots \otimes \partial_{x^{\alpha_k}}, \] (1.24)
where \( \alpha_j \) are multi-indices, \( P^{\alpha_0, \alpha_1, \ldots, \alpha_k}(x) \in \mathcal{O}_M(U) \) and, if \( \alpha = (i_1, \ldots, i_s) \), then
\[ \partial_{x^{\alpha}} = \partial_{x^{i_1}} \partial_{x^{i_2}} \cdots \partial_{x^{i_s}}. \]

For example, \( \mathcal{P}D^{-1} := \mathcal{O}_M \) and \( \mathcal{P}D^0 \) is the sheaf of differential operators on \( M \).

We do consider polydifferential operators which do not necessarily annihilate constant functions. For example, the usual (commutative) multiplication \( m_{\mathcal{O}_M} \) can be viewed as the global section of \( \mathcal{P}D^1 \).

It is easy to see that the Gerstenhaber bracket (1.7) is defined on sections of the sheaf
\[ \mathcal{P}D^\bullet := \bigoplus_{k \geq -1} \mathcal{P}D^k. \] (1.25)
Thus \( \mathcal{P}D^\bullet \) is a sheaf of graded Lie algebras.

It is also easy to see that the formula
\[ \partial^{\text{Hoch}} := [m_{\mathcal{O}_M}, ] \] (1.26)
defines a differential on \( \mathcal{P}D^\bullet \) which is compatible with the Lie bracket \( [\ , ]]_G \). So we view \( \mathcal{P}D^\bullet \) as a sheaf of dg Lie algebras.

Let us denote by \( \mathcal{P}V^k \) the sheaf of local sections of \( \wedge^{k+1}TM \) and set
\[ \mathcal{P}V^\bullet := \bigoplus_{k \geq -1} \mathcal{P}V^k. \]
We call \( \mathcal{P}V^\bullet \) the sheaf of polyvector fields on \( M \).

Since the graded commutative algebra \( \wedge^*TM(U) \) is identified with
\[ \mathcal{O}_M(U)[\theta_1, \theta_2, \ldots, \theta_m], \]
where \( \theta_1, \theta_2, \ldots, \theta_m \) are degree 1 symbols, every polyvector field \( v \) of degree \( k \) has the unique expansion
\[ v = \sum v^{i_0 i_1 \cdots i_k}(x) \theta_{i_0} \theta_{i_1} \cdots \theta_{i_k}, \quad v^{i_0 i_1 \cdots i_k}(x) \in \mathcal{O}_M(U), \]
\[ v^{i_0 \ldots i_{i+i+1} \ldots i_k}(x) = -v^{i_0 \ldots i_{i+i+1} \ldots i_k}(x). \]
The functions \( v^{i_0 i_1 \cdots i_k}(x) \) are called components of \( v \in \mathcal{P}V^k(U) \).

Recall [25] that \( \mathcal{P}V^\bullet \) is a sheaf of Lie algebras. The Lie bracket \( [\ , ]_S \) (known as the Schouten bracket) is defined locally by the equations
\[ [x^i, x^j]_S = [\theta_i, \theta_j]_S = 0, \quad \theta_i, x^j]_S = -[x^j, \theta_i]_S = \delta^j_i \] (1.27)
and the Leibniz compatibility condition with the multiplication
\[ [v, v_1 v_2]_S = [v, v_1]_S v_2 + (-1)^{(|v_1|+1)|v|} v_1 [v, v_2]_S. \]

Let us also recall the every polyvector field \( v \in \mathcal{P}V^k(U) \) can be identified with the polydifferential operator in \( \mathcal{P}D^k(U) \) which acts as
\[ a_0 \otimes a_1 \otimes \cdots \otimes a_k \mapsto \sum v^{i_0 i_1 \cdots i_k}(x)(\partial_{x^{i_0}} a_0)(\partial_{x^{i_{k-1}} a_1}) \cdots (\partial_{x^{i_k}} a_k), \]
where $v^{i_0i_1...i_k}(x)$ are components of $v$ and $x^1, \ldots, x^m$ are coordinates on $U$.

This embedding of sheaves
\[ PV^* \hookrightarrow PD^* \] (1.28)
is often called the Hochschild-Kostant-Rosenberg (HKR) map [19].

Clearly, every polyvector field is a $\partial_{\text{Hoch}}$-closed polydifferential operator. So (1.28) is a chain map from the graded sheaf $PV^*$ with the zero differential to the graded sheaf $PD^*$ with the Hochschild differential $\partial_{\text{Hoch}}$.

We claim that

**Proposition 1.6 (M. Kontsevich, Section 4.6.1.1, [22])** The embedding (1.28) gives us a quasi-isomorphism of cochain complexes
\[ (PV^*(M), 0) \sim \rightarrow (PD^*(M), \partial_{\text{Hoch}}). \] (1.29)

In other words, $H^k(PD^*(M), \partial_{\text{Hoch}}) \cong PV^k(M)$ for all $k$.

**Remark 1.7** A version of Proposition 1.6 in the setting of algebraic geometry is known as the Hochschild-Kostant-Rosenberg theorem [19].

One has to be careful with the above identification of $PV^*$ (resp. $PV^*(M)$) with the corresponding subsheaf of $PD^*$ (resp. subspace of $PD^*(M)$) because the embedding (1.28) is compatible with the Lie brackets only up to homotopy. So in general,
\[ [v, w]_S \neq [v, w]_G, \quad v, w \in PV^*(M). \]

On the other hand, we have celebrated Kontsevich’s formality theorem which states that

**Theorem 1.8 (M. Kontsevich, Section 7.3.5, [22])** There exists a sequence of quasi-isomorphisms of dg Lie algebras which connects $(PV^*(M), 0, [\ , \ ]_S)$ with $(PD^*(M), \partial_{\text{Hoch}}, [\ , \ ]_G)$.

**Remark 1.9** Paper [22] gives us an explicit sequence of quasi-isomorphisms of dg Lie algebras connecting $(PV^*(\mathbb{R}^m), 0, [\ , \ ]_S)$ with $(PD^*(\mathbb{R}^m), \partial_{\text{Hoch}}, [\ , \ ]_G)$. For a detailed proof of Theorem 1.8 for an arbitrary smooth manifold, we refer the reader to [8], [23, Appendix A.3].

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2 Formal deformations of an associative algebra

Let $A$ be an associative algebra over $\mathbb{C}$. Let us recall that formal deformations of $A$ with the base ring $\mathbb{C}[[\varepsilon]]$ are in bijection with MC elements
\begin{equation}
\mu = \sum_{k \geq 1} \varepsilon^k \mu_k \tag{2.1}
\end{equation}
of the dg Lie algebra
\begin{equation}
\varepsilon C^\bullet(A)[[\varepsilon]] \tag{2.2}
\end{equation}
and two deformations are equivalent if the corresponding MC elements are isomorphic, i.e. lie on the same orbit of the action $1.17$ (with $L = \varepsilon C^\bullet(A)[[\varepsilon]]$).

Indeed, every MC element $\mu \in \varepsilon C^1(A)[[\varepsilon]]$ gives us an associative multiplication on $A[[\varepsilon]]$:
\begin{equation}
a \bullet_\mu b := ab + \mu(a, b). \tag{2.3}
\end{equation}
Furthermore, if $\tilde{\mu} = e[\xi, ] \mu - \frac{e[\xi, ] - 1}{[\xi, ]}(\partial^{Hoch} \xi)$ for some $\xi \in \varepsilon C^0(A)[[\varepsilon]]$ then the operator
\[ T_\xi : A[[\varepsilon]] \rightarrow A[[\varepsilon]], \quad T_\xi(a) := a + \sum_{k=1}^{\infty} \frac{1}{k!} \xi^k(a) \]
intertwines the multiplications $\bullet_\mu$ and $\bullet_{\tilde{\mu}}$:
\[ T_\xi(a \bullet_\mu b) = T_\xi(a) \bullet_{\tilde{\mu}} T_\xi(b), \quad \forall a, b \in A[[\varepsilon]]. \]

Thus equivalence classes of 1-parameter formal deformations of $A$ are in bijection with elements in
\[ \pi_0(G(\varepsilon C^\bullet(A)[[\varepsilon]])). \]

MC equation $1.16$ implies that the first term $\mu_1$ of $\mu$ in $2.1$ is necessarily a degree 1 cocycle in $C^\bullet(A)$. The cohomology class $\kappa$ of this cocycle in $HH^1(A)$ depends only on the isomorphism class of the MC element $\mu$. This cohomology class is traditionally $[15], [21]$ called the Kodaira-Spencer class of $\mu$.

MC equation $1.16$ also implies that the Kodaira-Spencer class $\kappa$ of any formal deformation satisfies the “integrability” condition
\[ [\kappa, \kappa] = 0, \tag{2.4}\]
where $[\ , \ ]$ is the induced Lie bracket on $HH^\bullet(A)$.

By analogy with the above “classical” 1-parameter case, we define formal deformations of $A$ with the base ring $k := \mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$
as MC elements
\[ \mu = \sum_{k_0 + k_1 + \cdots + k_g \geq 1} \varepsilon^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g} \mu_{k_0, k_1, \ldots, k_g}, \quad \mu_{k_0, k_1, \ldots, k_g} \in C^{1-(k_0d_0 + \cdots + k_gd_g)}(A) \tag{2.5}\]
\[ \text{The general deformation theory works for any ground field of characteristic zero.}\]
\[ \text{In other words, if } \mu \text{ is isomorphic to } \tilde{\mu} \text{ then the corresponding cocycles in } C^i(A) \text{ are cohomologous.}\]
\[ \text{In this paper, we assume that the formal variables } \varepsilon, \varepsilon_1, \ldots, \varepsilon_g \text{ have non-positive degrees.}\]
of the dg Lie algebra
\[ mC^\bullet(A)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]], \tag{2.6} \]
where \( m \) is the maximal ideal
\[ m = (\varepsilon, \varepsilon_1, \ldots, \varepsilon_g) \subset \mathbb{C}[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]. \]
Furthermore, we declare that two such deformations are equivalent if the corresponding MC elements are isomorphic.

An interesting feature of such deformations is that, if at least one formal parameter carries a non-zero degree, then the resulting MC element \( \mu \) corresponds to (a \( \mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \)-linear) \( A_\infty \)-structure on the graded vector space:
\[ A[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \tag{2.7} \]
with the multiplications:
\[
m_\mu(a_1, \ldots, a_n) := \begin{cases} a_1 a_2 + \sum_{k_0 + \cdots + k_g = 0} \varepsilon^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g} \mu_{k_0,k_1,\ldots,k_g}(a_1, a_2) & \text{if } n = 2, \\ \sum_{k_0 + \cdots + k_g = 2-n} \varepsilon^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g} \mu_{k_0,k_1,\ldots,k_g}(a_1, \ldots, a_n) & \text{if } n > 2. \end{cases}
\]
Since the degrees of all formal parameters are non-positive, all non-zero \( A_\infty \)-multiplications have \( \geq 2 \) inputs. In other words, \( \mu \) gives us a usual (i.e. non-curved) \( A_\infty \)-structure on (2.7) with the zero differential.

Just as in the 1-parameter case, the MC equation for \( \mu \) implies that the element
\[ \varepsilon \mu_{1,0,\ldots,0} + \varepsilon_1 \mu_{0,1,0,\ldots,0} + \cdots + \varepsilon_g \mu_{0,0,\ldots,0,1} \tag{2.8} \]
is a degree 1-cocycle in
\[ \varepsilon C^\bullet(A) \oplus \varepsilon_1 C^\bullet(A) \oplus \cdots \oplus \varepsilon_g C^\bullet(A) \cong \frac{mC^\bullet(A)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]}{m^2 C^\bullet(A)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]}. \tag{2.9} \]
Furthermore, isomorphic MC elements have cohomologous cocycles in (2.9). As in the 1-parameter case, we call the cohomology class \( \kappa \) of (2.8) the Kodaira-Spencer class of \( \mu \).

**The Penkava-Schwarz example**

Let \( A \) be the polynomial algebra \( \mathbb{C}[x^1, \ldots, x^{2n-1}] \) is \( 2n - 1 \) variables (of degree zero) and \( \varepsilon_1 \) be formal parameter of degree \( 3 - 2n \). Then the element
\[ \mu := \varepsilon_1 \partial_{x^1} \cup \partial_{x^2} \cup \cdots \cup \partial_{x^{2n-1}} \in mC^\bullet(A)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \tag{2.10} \]
is \( \partial_{\text{Hoch}} \)-closed. Furthermore,
\[ [\mu, \mu]_G = 0 \]
since \( \varepsilon_1 \) is an odd variable and hence \( \varepsilon_1^2 = 0 \).

Thus \( \mu \) is a MC element of (2.6) which gives an example of a deformation of \( A \) in “the \( A_\infty \)-direction”. It is easy to see that the Kodaira-Spencer class of \( \mu \) is non-zero. So this is an example of non-trivial deformation.\(^{11}\)

\(^{11}\)A very similar example is described in [29, Section 3.2] and [29].
2.1 The case when $A$ is the algebra of functions $\mathcal{O}(M)$ on a smooth manifold $M$

The general story presented above applies to the case $A = \mathcal{O}(M)$ with the minor amendment: instead of the full Hochschild cochain complex, we use the sub-dg Lie algebra

$$\operatorname{PD}^\bullet(M) \subset C^\bullet(\mathcal{O}(M)), \quad (2.11)$$

where $\operatorname{PD}^\bullet$ is the sheaf of polydifferential operators on $M$.

By analogy with the 1-parameter “ungraded” case (when $g = 0$) formal deformations of $\mathcal{O}(M)$ with the base ring $\mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$ are defined as MC elements $\mu$ of the dg Lie algebra

$$\mathcal{L}_{\operatorname{PD}} := m\operatorname{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]. \quad (2.12)$$

Furthermore, the equivalence classes of such deformations are elements of

$$\pi_0(\mathcal{G}(\mathcal{L}_{\operatorname{PD}})), \quad (2.13)$$

where the dg Lie algebra $\mathcal{L}_{\operatorname{PD}}$ is considered with the filtration

$$\mathcal{F}_k \mathcal{L}_{\operatorname{PD}} := m^k\operatorname{PD}^\bullet(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]].$$

The MC equation for $\mu$

$$\partial^\text{Hoch} \mu + \frac{1}{2} [\mu, \mu]_G = 0 \quad (2.14)$$

implies that the coset of $\mu$ in

$$\mathcal{F}_1 \mathcal{L}_{\operatorname{PD}} / \mathcal{F}_2 \mathcal{L}_{\operatorname{PD}} \cong \varepsilon \operatorname{PD}^\bullet(M) \oplus \varepsilon_1 \operatorname{PD}^\bullet(M) \oplus \varepsilon_2 \operatorname{PD}^\bullet(M) \oplus \cdots \oplus \varepsilon_g \operatorname{PD}^\bullet(M)$$

is $\partial^\text{Hoch}$-closed and the corresponding vector in

$$\varepsilon \mathcal{PV}^1(M) \oplus \varepsilon_1 \mathcal{PV}^{1-d_1}(M) \oplus \cdots \oplus \varepsilon_g \mathcal{PV}^{1-d_g}(M) \quad (2.15)$$

does not depend on the choice of a representative in the equivalence class of the deformation. We call the corresponding vector in (2.15) the Kodaira-Spencer class of $\mu$.

Since the degrees of all formal parameters are non-positive, the $A_\infty$-structure on

$$\mathcal{O}(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \quad (2.16)$$

corresponding to $\mu$ has the following $\mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$-multilinear multiplications:

$$m_\mu(a_1, \ldots, a_n) := \begin{cases} 
  a_1 a_2 + \sum_{k_0 d_0 + \cdots + k_g d_g = 0} \varepsilon^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g} \mu_{k_0, k_1, \ldots, k_g}(a_1, a_2) & \text{if } n = 2, \\
  \sum_{k_0 d_0 + \cdots + k_g d_g = 2-n} \varepsilon^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g} \mu_{k_0, k_1, \ldots, k_g}(a_1, \ldots, a_n) & \text{if } n > 2,
\end{cases}$$

where $a_i \in \mathcal{O}(M)$ and $\mu_{k_0, k_1, \ldots, k_g}$ are the coefficients in the expansion of $\mu$

$$\mu = \sum_{k_0 + k_1 + \cdots + k_g \geq 1} \varepsilon^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_g^{k_g} \mu_{k_0, k_1, \ldots, k_g}.$$
2.2 The main result

Let \( M \) be a smooth real manifold equipped with a symplectic structure \( \omega \) and \( \alpha \in PV^1(M) \) be the corresponding (non-degenerate) Poisson structure. In other words, for every open coordinate subset \( U \subset M \), we have

\[
\alpha^{ij}(x)\omega_{jk}(x) = \delta^i_k,
\]

where \( \alpha^{ij}(x) \) (resp. \( \omega_{ij}(x) \)) are components of \( \alpha|_U \) (resp. \( \omega|_U \)).

Let us consider MC elements \( \mu \) in (2.12) satisfying these two conditions:

**Condition 2.1** The Kodaira-Spencer class of \( \mu \) equals \( \varepsilon \alpha \).

**Condition 2.2** The MC element \( \mu \) satisfies the equation

\[
\mu|_{\varepsilon=0} = 0. \tag{2.17}
\]

We denote by \( \tilde{G}(L_{PD}) \) the full subgroupoid of \( G(L_{PD}) \) whose objects are MC elements \( \mu \) satisfying Conditions 2.1 and 2.2. Furthermore, we denote by \( TL \) the set of isomorphism classes of objects of \( \tilde{G}(L_{PD}) \), i.e.

\[
TL := \pi_0(\tilde{G}(L_{PD})). \tag{2.18}
\]

We call \( TL \) the topological locus of \( \pi_0(\tilde{G}(L_{PD})) \).

**Remark 2.3** Note that every MC element \( \mu \) satisfying Condition 2.2 is isomorphic to infinitely many MC elements of \( L_{PD} \) which do not satisfy this condition. Indeed, consider a MC element \( \mu \) which satisfies (2.17) and a polydifferential operator \( P \in PD^{-d_1}(M) \) for which \( \partial^{Hoch} P \neq 0 \). Then the MC element

\[
e^{[\varepsilon_1 P]_G} \mu - e^{[\varepsilon_1 P]_G} - \frac{1}{[\varepsilon_1 P]_G} (\partial^{Hoch} \varepsilon_1 P)
\]

does not satisfy equation (2.17).

Equation (2.17) guarantees that the \( A_\infty \)-multiplications \( \{m_n\}_{n \geq 2} \) corresponding to the MC element \( \mu \) satisfy the property

\[
m_n(a_1, \ldots, a_n)|_{\varepsilon=0} = \begin{cases} a_1 a_2 & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}
\]

In other words, the \( A_\infty \)-algebra

\[
(\mathcal{O}(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]], \{m_n\}_{n \geq 2})
\]

can be viewed as a 1-parameter formal deformation of the graded commutative algebra \( \mathcal{O}(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \).

The goal of this note is to describe the topological locus \( TL \) of equivalence classes of formal deformations of \( \mathcal{O}(M) \) with the base ring \( \mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \):

**Theorem 2.4** For every symplectic manifold \( M \), the isomorphism classes of formal deformations of \( \mathcal{O}(M) \) with the base ring \( \mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \) satisfying Conditions 2.1 and 2.2 are in bijection with elements of the vector space

\[
\bigoplus_{q \geq 0} \frac{1}{\varepsilon^{q-1}} (\mathfrak{m}^{q} H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]])^2. \tag{2.20}
\]
Here \( m \) is the maximal ideal \( (\varepsilon, \varepsilon_1, \ldots, \varepsilon_g) \subset \mathbb{C}[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g] \), \( H^\bullet(M, \mathbb{C}) \) is the singular cohomology of \( M \) with coefficients in \( \mathbb{C} \), and \( \langle m^0 H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \rangle^2 \) is the subspace of degree 2 elements in the graded vector space \( \langle m^0 H^q(M, \mathbb{C})[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \rangle^2 \).

The proof of Theorem 2.4 is given in Section 5 and it is based on two auxiliary constructions. The first construction is presented in Section 3 and its main ingredient is Kontsevich’s formality quasi-isomorphism [22] for polydifferential operators. The second construction is presented in Section 4 and it is inspired by a result [30] due to G. Sharygin and D. Talalaev.

3 Applying Kontsevich’s formality theorem

Let us fix an \( L_\infty \)-quasi-isomorphism
\[
\mathcal{U} : P^\bullet(V) \sim P^\bullet(D)
\]
whose linear term coincides with the embedding (1.29). Let us also extend it by \( \mathbb{C}[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \)-linearity to
\[
\mathcal{U} : m P^\bullet(V)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \sim m P^\bullet(D)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]
\]
and denote by \( \mu_\alpha \) the following MC element of (2.12)
\[
\mu_\alpha := \mathcal{U}_\alpha(\varepsilon \alpha).
\]

Twisting (3.2) by \( \alpha \), we get an \( L_\infty \)-morphism
\[
\mathcal{U}_\alpha : (m P^\bullet(V)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]], [\varepsilon \alpha, ]_S) \sim L^\mu_{PD},
\]
where the dg Lie algebra \( L^\mu_{PD} \) is obtained from \( L_{PD} \) via replacing the differential \( \partial^{Hoch} \) by
\[
\partial^{Hoch} + [\mu_\alpha, ]_G.
\]

Notice that
\[
\partial^{Hoch} + [\mu_\alpha, ]_G = \partial^{Hoch}_s,
\]
where \( \partial^{Hoch}_s \) is the Hochschild differential corresponding to the star product
\[
a * b := ab + \mu_\alpha(a, b).
\]

Furthermore, the dg Lie algebra \( L^\mu_{PD} \) carries the same descending filtration as \( L_{PD} \)
\[
\mathcal{F}_k L^\mu_{PD} := m^k P^\bullet(D)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]].
\]

It is easy to see that the formula
\[
\text{Shift}_{\mu_\alpha}(\mu) := \mu_\alpha + \tilde{\mu}
\]
defines a bijection from the set of MC elements of \( L^\mu_{PD} \) to the set of MC elements of \( L_{PD} \) (2.12). A simple direct computation shows that for every degree zero element \( \xi \in L^\mu_{PD} \) and for every \( \tilde{\mu} \in \text{MC}(L^\mu_{PD}) \)
\[
\text{Shift}_{\mu_\alpha} \left( e[^{\varepsilon, \varepsilon_1, \ldots, \varepsilon_g} \tilde{\mu} - e[^{\varepsilon, \varepsilon_1, \ldots, \varepsilon_g} \partial^{Hoch}_s \xi + [\mu_\alpha, ]_G] \right) = e[^{\varepsilon, \varepsilon_1, \ldots, \varepsilon_g} \mu_\alpha + \tilde{\mu} - e[^{\varepsilon, \varepsilon_1, \ldots, \varepsilon_g} \partial^{Hoch}_s \xi]_G.
\]

\[{}^{12}\text{For the version of this statement in the setting of } L_\infty\text{-algebras and the corresponding DGH } \infty\text{-groupoids, we refer the reader to } [12, \text{Lemma 4.3}].\]
In other words, $\text{Shift}_{\mu} \alpha$ upgrades to a functor

$$\text{Shift}_{\mu} : \mathcal{G}(\mathcal{L}_{PD}^{\mu}) \to \mathcal{G}(\mathcal{L}_{PD})$$

which acts “as identity” on the set of morphisms. It is not hard to see that (3.8) is actually a strict isomorphism of groupoids and the inverse functor operates on objects as

$$\mu \mapsto \mu - \mu_\alpha : \text{MC}(\mathcal{L}_{PD}) \to \text{MC}(\mathcal{L}_{PD}^{\mu}).$$

(3.9)

Let us denote by $\tilde{\mathcal{L}}_{PD}$ the following sub-dg Lie algebra of $\mathcal{L}_{PD}^{\mu}$

$$\mathcal{L}_{PD} := (\varepsilon \text{ m PD}^\bullet (M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]], \partial^\text{Hoch}_G, \cdot)$$

(3.10)

and observe that the filtration from $\mathcal{L}_{PD}^{\mu}$ induces the descending filtration on $\tilde{\mathcal{L}}_{PD}$:

$$F_k \tilde{\mathcal{L}}_{PD} := F_k \mathcal{L}_{PD}^{\mu} \cap \tilde{\mathcal{L}}_{PD} = \varepsilon \text{ m}_{PD}^{k-1}(\text{ m PD}^\bullet (M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]].$$

(3.11)

Next, we denote by $\tilde{\mathcal{G}}(\mathcal{L}_{PD}^{\mu})$ the full subgroupoid of $\mathcal{G}(\mathcal{L}_{PD}^{\mu})$ whose set of objects is $\text{MC}(\tilde{\mathcal{L}}_{PD})$. Moreover, we set

$$\mathcal{T}L^\text{tw} := \pi_0(\tilde{\mathcal{G}}(\mathcal{L}_{PD}^{\mu})).$$

(3.12)

Let us prove that

**Proposition 3.1** The restriction of the functor (3.8) to the subgroupoid $\tilde{\mathcal{G}}(\mathcal{L}_{PD}^{\mu})$ induces a bijection

$$\mathcal{T}L^\text{tw} \cong \mathcal{T}L.$$

Proof. Let $\tilde{\mu}$ be a MC element of $\tilde{\mathcal{L}}_{PD}$. It is clear that the Kodaira-Spencer class of $\mu_\alpha + \tilde{\mu}$ coincides with the Kodaira-Spencer class of $\mu_\alpha$. Hence $\mu_\alpha + \tilde{\mu}$ satisfies Condition 2.1.

Since

$$\mu_\alpha|_{\varepsilon = 0} = \tilde{\mu}|_{\varepsilon = 0} = 0,$$

the MC element $\mu_\alpha + \tilde{\mu}$ also satisfies Condition 2.2.

Thus restricting $\text{Shift}_{\mu_\alpha}$ to the full sub-groupoid $\tilde{\mathcal{G}}(\mathcal{L}_{PD}^{\mu})$, we get a functor

$$\tilde{\mathcal{G}}(\mathcal{L}_{PD}^{\mu}) \to \tilde{\mathcal{G}}(\mathcal{L}_{PD}).$$

(3.13)

Hence we get a map

$$\mathcal{T}L^\text{tw} \to \mathcal{T}L.$$

(3.14)

Using the functor (3.9) from $\mathcal{G}(\mathcal{L}_{PD})$ to $\mathcal{G}(\mathcal{L}_{PD}^{\mu})$, it is easy show that the map (3.14) is one-to-one. So it remains to show that, for every $\mu \in \text{MC}(\mathcal{L}_{PD})$ satisfying Conditions 2.1 and 2.2 there exists $\tilde{\mu} \in \text{MC}(\mathcal{L}_{PD})$ such that $\mu$ is isomorphic to $\mu_\alpha + \tilde{\mu}$.

Since the Kodaira-Spencer classes of $\mu$ and $\mu_\alpha$ coincide, the coset of the difference $\mu - \mu_\alpha$ in

$$\text{m PD}^\bullet (M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] / \text{m}^2\text{PD}^\bullet (M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$$

is $\partial^\text{Hoch}$-exact.

Hence there exists a degree zero vector $\xi \in \text{m PD}^\bullet (M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]$ such that

$$\left( e^\xi, |\xi_\mu = \frac{e^{\xi_\mu}}{|\xi_\mu - \mu_\alpha} (\partial^\text{Hoch} \xi) \right) - \mu_\alpha \in \text{m}^2\text{PD}^\bullet (M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]].$$

(3.15)
Moreover, since \( \mu_{\epsilon=0} = 0 \), the vector \( \xi \), for which (3.15) holds, can be found in 

\[
(e)\text{PD}^\bullet(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]].
\]

Therefore,

\[
\left( e^{[\xi, \cdot]} G \mu - \frac{e^{[\xi, \cdot]} G - 1}{[\xi, \cdot] G} (\partial^\text{Hoch} \xi) \right) - \mu_\alpha \in \epsilon m\text{PD}^\bullet(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]].
\]

In other words, the MC element

\[
\tilde{\mu} := \left( e^{[\xi, \cdot]} G \mu - \frac{e^{[\xi, \cdot]} G - 1}{[\xi, \cdot] G} (\partial^\text{Hoch} \xi) \right) - \mu_\alpha
\]

of \( L^{\mu_\alpha}_{\text{PD}} \) belongs to the sub- dg Lie algebra \( \tilde{L}_{\text{PD}} \).

The MC element \( \mu_\alpha + \tilde{\mu} \) of \( L_{\text{PD}} \) is isomorphic to \( \mu \) by construction.

Thus the proposition is proved.

To describe the set \( TL^{tw} \), we denote by \( L_{PV} \) and \( \tilde{L}_{PV} \) the dg Lie algebra

\[
L_{PV} := (m^{PV^*}(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]], [\epsilon \alpha, ], [ ] S)
\]

and its sub- dg Lie algebra

\[
\tilde{L}_{PV} := \epsilon m^{PV^*}(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]] \subset L_{PV},
\]

respectively.

We consider \( L_{PV} \) and \( \tilde{L}_{PV} \) with the following descending filtrations:

\[
F_k L_{PV} := m^k PV^*(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]], \quad (3.17)
\]

\[
F_k \tilde{L}_{PV} := \tilde{L}_{PV} \cap F_k L_{PV} = \epsilon m^{k-1} PV^*(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]]. \quad (3.18)
\]

Furthermore, we denote by \( \tilde{G}(L_{PV}) \) the full subgroupoid of \( G(L_{PV}) \) whose set of objects is \( \text{MC}(\tilde{L}_{PV}) \) and set

\[
TL_{PV} := \pi_0(\tilde{G}(L_{PV})). \quad (3.19)
\]

Restricting \( U^\alpha (3.4) \) to the sub- dg algebra \( \tilde{L}_{PV} \), we get an \( L_\infty \)-morphism

\[
U^\alpha : \tilde{L}_{PV} \rightarrow \tilde{L}_{PD}. \quad (3.20)
\]

Let us prove that

**Claim 3.2** The \( L_\infty \)-morphism (3.4) (resp. (3.20)) is compatible with the filtrations (3.6), (3.11) (resp. (3.11), (3.15)) in the sense of (1.15). Moreover the linear terms of the \( L_\infty \)-morphisms (3.3) and (3.20) give us quasi-isomorphisms of cochain complexes

\[
F_k L_{PV} \xrightarrow{\sim} F_k L^{\mu_\alpha}_{PD}, \quad F_k \tilde{L}_{PV} \xrightarrow{\sim} F_k \tilde{L}_{PD}
\]

for every \( k \geq 1 \), respectively.
Proof. The $L_\infty$-morphisms (3.4) and (3.20) are compatible with the filtrations by construction. So we proceed to the second statement.

For every fixed $k \geq 1$ the dg Lie algebras $\mathcal{F}_k\mathcal{L}_{PV}$, $\mathcal{F}_k\tilde{\mathcal{L}}_{PV}$, $\mathcal{F}_k\mathcal{L}_{PD}$, and $\mathcal{F}_k\tilde{\mathcal{L}}_{PD}$ are equipped with the complete descending filtrations. For example,

$$\mathcal{F}_k\mathcal{L}_{PV} \supset \mathcal{F}_{k+1}\mathcal{L}_{PV} \supset \mathcal{F}_{k+2}\mathcal{L}_{PV} \supset \ldots$$

The linear term $U_\alpha$ gives us chain maps

$$(\mathcal{F}_k\mathcal{L}_{PV}, [\varepsilon, \alpha], S) \to (\mathcal{F}_k\mathcal{L}_{PD}, \partial_\mathrm{Hoch}) \quad (3.21)$$

and

$$(\mathcal{F}_k\tilde{\mathcal{L}}_{PV}, [\varepsilon, \alpha], S) \to (\mathcal{F}_k\tilde{\mathcal{L}}_{PD}, \partial_\mathrm{Hoch}) \quad (3.22)$$

compatible with these filtrations. As above, $\partial_\mathrm{Hoch}$ is the Hochschild differential corresponding to the star product (3.5).

At the level of associated graded complexes, we get the chain maps

$$J_{HKR} : (\mathfrak{m}^k\mathcal{PV}^*(M)[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g], 0) \to (\mathfrak{m}^k\mathcal{PD}^*(M)[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g], \partial_\mathrm{Hoch}) \quad (3.23)$$

and

$$J_{HKR} : (\mathfrak{m}^{k-1}\mathcal{PV}^*(M)[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g], 0) \to (\mathfrak{m}^{k-1}\mathcal{PD}^*(M)[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g], \partial_\mathrm{Hoch}) \quad (3.24)$$

where $\partial_\mathrm{Hoch}$ is the Hochschild differential corresponding to the usual (commutative) multiplication on $\mathcal{O}_M$ and $J_{HKR}$ is the Hochschild-Kostant-Rosenberg embedding (1.28) extended by linearity with respect to $\mathbb{C}[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]$.

Since the functors $\otimes_{\mathbb{C}} \mathfrak{m}^k$ and $\otimes_{\mathbb{C}} \mathfrak{m}^{k-1}$ preserve quasi-isomorphisms of cochain complexes, Proposition 1.6 implies that (3.23) and (3.24) are quasi-isomorphisms.

Hence, by the Lemma on filtered complexes (see, for example, [13, Lemma D.1, App. D]), the chain maps (3.21) and (3.22) are quasi-isomorphisms of cochain complexes.

Claim 3.2 is proved. $\square$

Let us prove that

**Proposition 3.3** The $L_\infty$-morphism (3.20) induces a map

$$\mathcal{T}_{LPV} \to \mathcal{T}_{L^{tw}}. \quad (3.25)$$

Furthermore, this map is a bijection.

Proof. To prove the first statement, we observe that the $L_\infty$-morphism (3.20) gives a map

$$U_\alpha^a : \text{MC}(\tilde{\mathcal{L}}_{PV}) \to \text{MC}(\tilde{\mathcal{L}}_{PD}). \quad (3.26)$$

Let $\mu_1$ and $\mu_2$ be MC elements of $\tilde{\mathcal{L}}_{PV}$ connected by a 1-cell

$$\eta \in \text{MC}_1(\mathcal{L}_{PV}) = \text{MC}(\mathcal{L}_{PV} \otimes \mathbb{C}[t] \oplus \mathcal{L}_{PV} \otimes \mathbb{C}[t] dt),$$

i.e.

$$\eta\big|_{t=dt=0} = \mu_1, \quad \eta\big|_{t=1,dt=0} = \mu_2.$$

Clearly, the 1-cell

$$U_\alpha^a(\eta) \in \text{MC}_1(\mathcal{L}_{PD})$$

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connects the MC elements
\[ U^\alpha_\epsilon(\mu_1), U^\alpha_\epsilon(\mu_2) \in \mathcal{L}_{PD}. \]

Thus (3.20) descends to the map of sets
\[ TL_{PV} \to TL^{tw}. \]

Let us now show that this map is a bijection.

According to Claim 3.2, both \( L_\infty \)-morphisms (3.4) and (3.20) satisfy conditions of [11, Theorem 1.1]. So, applying this theorem to (3.20), we conclude that the map (3.25) is surjective. Furthermore, applying [11, Theorem 1.1] to (3.4), we conclude that the map (3.25) is injective.

\[ \square \]

4 Passing to exterior forms

Let \( f \in \mathcal{O}(M) \) and \( \theta_i := \partial x^i \) be the local vector field defined in a coordinate chart of \( M \). Since the symplectic structure \( \omega \) is non-degenerate, the formulas
\[ J_\omega(f) := f, \quad J_\omega(\theta_i) := \frac{1}{\epsilon} \omega_{ij}(x) dx^j \]
define an isomorphism of (shifted) graded commutative algebras
\[ J_\omega : \mathfrak{PV}(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]] \to \bigoplus_{q \geq 0} \frac{1}{\epsilon^q} s^{-1} \mathfrak{m} \Omega^q(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]]. \tag{4.1} \]

The inverse of \( J_\omega \) is given by
\[ J_\omega^{-1}(f) = f, \quad J_\omega^{-1}(dx^i) = \epsilon \alpha^{ij}(x) \theta_j, \tag{4.2} \]
where \( \alpha^{ij}(x) \) are components of the corresponding Poisson structure (in local coordinates).

Here, we tacitly assume that every vector \( \eta \in \Omega^q(M) \) carries degree \( q \). The latter means that the vector \( \eta \epsilon_{k_1} \ldots \epsilon_{k_g} \) carries degree \( q + k_1 d_1 + \cdots + k_g d_g \) and the vector \( s^{-1} \eta \epsilon_{k_1} \ldots \epsilon_{k_g} \) carries degree \( q - 1 + k_1 d_1 + \cdots + k_g d_g \).

Using this isomorphism and the dg Lie algebra structure \( ([\epsilon \alpha, \ ]_S, [\ , ]_S) \) on \( \mathcal{L}_{PV} \), we equip the graded vector space
\[ \mathcal{L}_\Omega := \bigoplus_{q \geq 0} \frac{1}{\epsilon^q} s^{-1} \mathfrak{m} \Omega^q(M)[[\epsilon, \epsilon_1, \ldots, \epsilon_g]] \tag{4.3} \]
with the structure of a dg Lie algebra.

A direct computation shows that the differential on \( \mathcal{L}_\Omega \) corresponding to \( [\epsilon \alpha, \ ]_S \) is
\[ -d \]
(where \( d \) is the de Rham differential) and the Lie bracket \( [\ , ]_\omega \) corresponding to \( [\ , ]_S \) is given by the formulas (in local coordinates)
\[ [s^{-1} \eta_1, s^{-1} \eta_2]_\omega := \]
\[ s^{-1} \epsilon dx^k \partial_{x^k} \alpha^{ij}(x) \frac{\partial \eta_1}{\partial dx^i} \frac{\partial \eta_2}{\partial dx^j} - (-1)^{|\eta_1|} s^{-1} \epsilon \alpha^{ij}(x) \frac{\partial \eta_1}{\partial dx^i} \partial_{x^j} \eta_2 + s^{-1} \epsilon \alpha^{ij}(x) \partial_{x^j} \eta_1 \frac{\partial \eta_2}{\partial dx^i}. \tag{4.4} \]

For example,
\[ [s^{-1} f, s^{-1} g]_\omega := 0, \quad [s^{-1} dx^i, s^{-1} f]_\omega := s^{-1} \epsilon \alpha^{ij}(x) (\partial_{x^j} f), \tag{4.5} \]
\[ [s^{-1} \, dx^i, s^{-1} \, dx^j]_\omega := s^{-1} \varepsilon \, dx^k \partial_x^k \alpha^{ij}(x), \]

where \( f, g \in \mathcal{O}(M) \).

It is clear that, restricting \( J_\omega \) to the sub- dg Lie algebra (3.16), we get an isomorphism of dg Lie algebras

\[ J_\omega : \mathcal{L}_{PV} \xrightarrow{\simeq} \mathcal{L}_\Omega, \]

where \( \mathcal{L}_\Omega \) is the sub- dg Lie algebra of \( \mathcal{L}_\omega \):

\[ \mathcal{L}_\omega := \bigoplus_{q \geq 0} \frac{1}{\varepsilon^{q-1}} s^{-1} m \Omega^q(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]. \]

Both dg Lie algebras (4.3) and (4.7) are equipped with the obvious complete descending filtrations

\[ F_k \mathcal{L}_\Omega := \bigoplus_{q \geq 0} \frac{1}{\varepsilon^{q-k}} s^{-1} m \Omega^q(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]], \]

\[ F_k \mathcal{L}_\omega := \bigoplus_{q \geq 0} \frac{1}{\varepsilon^{q-1}} s^{-1} m^{-1} \Omega^q(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]. \]

Moreover, the isomorphisms (4.1) and (4.6) are compatible with these filtrations.

We denote by \( \tilde{\mathcal{G}}(\mathcal{L}_\Omega) \) the full subgroupoid of \( \mathcal{G}(\mathcal{L}_\Omega) \) whose set of objects is \( \text{MC}(\tilde{\mathcal{L}}_\Omega) \) and set

\[ T\mathcal{L}_\Omega := \pi_0(\tilde{\mathcal{G}}(\mathcal{L}_\Omega)). \]

Using the above properties of the isomorphism \( J_\omega \) we easily deduce that

**Claim 4.1** The map \( J_\omega \) gives us an isomorphism of groupoids

\[ (J_\omega)_*: \tilde{\mathcal{G}}(\mathcal{L}_{PV}) \xrightarrow{\simeq} \tilde{\mathcal{G}}(\mathcal{L}_\Omega) \]

and hence a bijection

\[ T\mathcal{L}_{PV} \xrightarrow{\simeq} T\mathcal{L}_\Omega. \]

\[ \square \]

**Remark 4.2** Note that the grading on \( \mathcal{L}_\Omega \) and \( \tilde{\mathcal{L}}_\Omega \) comes with an additional shift. For example, every degree zero exterior form \( \varepsilon \eta \in \varepsilon \Omega^0(M) = \varepsilon \mathcal{O}(M) \) gives us the degree \(-1\) vector \( s^{-1} \varepsilon \eta \in \mathcal{L}_\Omega \). In particular, monomials in \( S^n(s^{-1} \tilde{\mathcal{L}}_\Omega) \) will be written as

\[ s^{-2} \eta_1 s^{-2} \eta_2 \ldots s^{-2} \eta_n, \]

where \( \eta_1, \eta_2, \ldots, \eta_n \) belong to the space

\[ \bigoplus_{q \geq 0} \frac{1}{\varepsilon^{q-1}} \varepsilon \Omega^q(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]. \]

Let us consider the cocommutative coalgebra

\[ S(s^{-1} \tilde{\mathcal{L}}_\Omega) \]
with the degree 1 coderivations $Q_{-d}$ and $Q_{[\cdot, \cdot]}$, where $Q_{-d}$ (resp. $Q_{[\cdot, \cdot]}$) comes from the dg Lie algebra structure $(-d, [\cdot, \cdot] = 0)$ (resp. $(\partial = 0, [\cdot, \cdot])$) on $\tilde{L}_\Omega$ in the sense of (1.12). For example, 

$$Q_{-d}(s^{-2} \eta_1 s^{-2} \eta_2) = -s^{-2} d\eta_1 s^{-2} \eta_2 - (-1)^{|\eta_1|} s^{-2} \eta_1 s^{-2} d\eta_2,$$  \hspace{1cm} (4.13)

and 

$$Q_{[\cdot, \cdot]}(s^{-2} \eta_1 s^{-2} \eta_2) := (-1)^{|\eta_1|} s^{-2} \varepsilon dx^k \partial x^k \alpha^{ij}(x) \frac{\partial \eta_1}{\partial dx^i} \frac{\partial \eta_2}{\partial dx^j},$$  \hspace{1cm} (4.14)

where $|\eta_1|$ is the degree of $\eta_1$ in (4.11).

Using the idea of [30], we consider the coderivation $\Pi$ of the coalgebra (4.12) defined in local coordinates by the formulas 

$$p \circ \Pi(s^{-2} \eta_1 s^{-2} \eta_2) := (-1)^{|\eta_1|} s^{-2} \varepsilon \alpha^{ij}(x) \left( \frac{\partial}{\partial dx^i} \eta_1 \right) \left( \frac{\partial}{\partial dx^j} \eta_2 \right),$$  \hspace{1cm} (4.15)

$$p \circ \Pi(s^{-2} \eta_1 s^{-2} \eta_2 \ldots s^{-2} \eta_n) := 0, \quad \text{if} \quad n \neq 2,$$

where $p$ is the canonical projection $S(s^{-1} \tilde{L}_\Omega) \to s^{-1} \tilde{L}_\Omega$.

Properties of the coderivation $\Pi$ are listed in the following proposition:

**Proposition 4.3** The coderivation $\Pi$ has degree 0. Furthermore, 

$$\Pi \circ Q_{-d} - Q_{-d} \circ \Pi = Q_{[\cdot, \cdot]}$$  \hspace{1cm} (4.16)

and 

$$\Pi \circ Q_{[\cdot, \cdot]} = Q_{[\cdot, \cdot]} \circ \Pi.$$  \hspace{1cm} (4.17)

**Remark 4.4** By keeping track of terms of negative powers of $\varepsilon$, it is easy to see that, in general, 

$$p \circ \Pi(s^{-1} \mathcal{L}_\Omega \otimes s^{-1} \mathcal{L}_\Omega) \not\subset s^{-1} \mathcal{L}_\Omega.$$  \hspace{1cm} (4.18)

However 

$$p \circ \Pi(s^{-1} \tilde{\mathcal{L}}_\Omega \otimes s^{-1} \mathcal{L}_\Omega) \subset s^{-1} \mathcal{L}_\Omega$$  \hspace{1cm} (4.19)

and we will use the inclusion (4.18) to extend the coderivation $\Pi$ to the coalgebra 

$$S(s^{-1} L),$$  \hspace{1cm} (4.20)

where $L$ is the following graded vector space 

$$L := \tilde{\mathcal{L}}_\Omega \hat{\otimes} \mathbb{C}[t] + \mathcal{L}_\Omega \hat{\otimes} \mathbb{C}[t]dt,$$

$\hat{\otimes}$ is the completed tensor product and $\mathbb{C}[t] \hat{\otimes} \mathbb{C}[t]dt$ is the algebra of polynomial de Rham forms on the 1-simplex. We will freely use the obvious generalizations of (4.16) and (4.17) to the corresponding coderivations of the coalgebra (4.19).

The proof of Proposition 4.3 is given in Appendix A. Here we use this proposition to deduce the following statement:
Corollary 4.5 The formula
\[ \exp(\Pi) := 1 + \sum_{m \geq 1} \frac{1}{m!} \Pi^m \]  
(4.21)
defines a strictly invertible \( L_\infty \) quasi-isomorphism
\[ (\tilde{\mathcal{L}}_\Omega, -d, 0) \rightsquigarrow (\tilde{\mathcal{L}}_\Omega, -d, [\ , \ ]_\omega). \]

This isomorphism extends, in the obvious way, to a strictly invertible \( L_\infty \) quasi-isomorphism
\[ (L, -d + d_t, 0) \rightsquigarrow (L, -d + d_t, [\ , \ ]_\omega), \]
where \( L \) is defined in (4.20) and \( d_t \) is the de Rham differential \( dt \partial_t \) on \( \mathbb{C}[t] \oplus \mathbb{C}[t] dt \).

Proof. It is straightforward to show that the equation (4.21) defines automorphisms of the cocommutative coalgebras (4.12) and (4.19) (considered with the zero differentials). The inverse of (4.21) is given by the formula:
\[ \exp(-\Pi) := 1 + \sum_{m \geq 1} \frac{(-1)^m}{m!} \Pi^m. \]  
(4.22)

It remains to prove that
\[ \exp(\Pi) \circ Q_d = (Q_d + Q_{[\ , \ ]}_\omega) \circ \exp(\Pi) \]
or equivalently
\[ \exp(\Pi) \circ Q_d \circ \exp(-\Pi) = Q_d + Q_{[\ , \ ]}_\omega. \]  
(4.23)

For this purpose we introduce two elements
\[ \Psi_L, \Psi_R \in \text{Hom}(\mathbb{S}(s^{-1} \tilde{\mathcal{L}}_\Omega), \mathbb{S}(s^{-1} \tilde{\mathcal{L}}_\Omega)[u]) \]
defined by
\[ \Psi_L := \exp(u \Pi) \circ Q_d \circ \exp(-u \Pi), \quad \Psi_R := Q_d + u Q_{[\ , \ ]}_\omega, \]  
(4.24)
where \( u \) is an auxiliary variable of degree 0.

Using (4.16) and (4.17) we get
\[ \frac{d}{du} \Psi_L = \exp(u \Pi) \circ (\Pi \circ Q_d - Q_d \circ \Pi) \circ \exp(-u \Pi) \]
\[ = \exp(u \Pi) \circ Q_{[\ , \ ]}_\omega \circ \exp(-u \Pi) = Q_{[\ , \ ]}_\omega. \]

Hence both \( \Psi_L \) and \( \Psi_R \) satisfy the same formal ordinary differential equation
\[ \frac{d}{du} \Psi = Q_{[\ , \ ]}_\omega \]
with the same initial condition \( \Psi|_{u=0} = Q_d \).

Therefore, \( \Psi_L = \Psi_R \) and (4.23) follows.

The similar argument shows that the same operator \( \exp(\Pi) \) defines an \( L_\infty \) quasi-isomorphism
\[ (L, -d + d_t, 0) \rightsquigarrow (L, -d + d_t, [\ , \ ]_\omega), \]
Thus the corollary is proved. \( \square \)
5 The proof of Theorem 2.4

Let us denote by \( \tilde{G}(L_\Omega, -d, 0) \) the full subgroupoid of \( G(L_\Omega, -d, 0) \) whose set of objects is \( MC(L_\Omega, -d, 0) \).

Clearly,

\[
\pi_0(\tilde{G}(L_\Omega, -d, 0)) \cong \bigoplus_{q \geq 0} \frac{1}{\varepsilon^{q-1}} \left( m s^q H^q(M, C)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]\right)^2, \tag{5.1}
\]

where \( (m s^q H^q(M, C)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]])^2 \) is the subspace of degree 2 elements in \( m s^q H^q(M, C)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \).

Since the \( L_\infty \) quasi-isomorphism \( \approx \)\) is strictly invertible, it induces a bijection of sets

\[
\exp(\Pi)_* : MC(L_\Omega, -d, 0) \cong MC(L_\Omega, -d, [\ , \ ]_\omega). \tag{5.2}
\]

Moreover the second part of Corollary 4.5 implies that, if \( \mu_1, \mu_2 \in MC(L_\Omega, -d, 0) \) are connected by a 1-cell in \( MC(L_\Omega, -d, [\ , \ ]_\omega) \) then \( \exp(\Pi)_*(\mu_1) \) and \( \exp(\Pi)_*(\mu_2) \) are connected by a 1-cell in \( MC(L_\Omega, -d, [\ , \ ]_\omega) \).

In other words, (5.2) gives us a well defined map

\[
\Theta_\Pi : \pi_0(\tilde{G}(L_\Omega, -d, 0)) \to TL_\Omega. \tag{5.3}
\]

Let us prove that

Claim 5.1 The map \( \Theta_\Pi \) is a bijection of sets.

Proof. The surjectivity of \( \Theta_\Pi \) follows from the fact that the map \( \approx \) is a bijection.

To prove the injectivity, we consider two \( MC \) elements \( \mu_1, \mu_2 \in MC(L_\Omega, -d, 0) \) and assume that the \( MC \) elements \( \exp(\Pi)_*(\mu_1) \) and \( \exp(\Pi)_*(\mu_2) \) are connected by a 1-cell in \( MC(L_\Omega, -d, [\ , \ ]_\omega) \).

Due to the second part of Corollary 4.5 the \( MC \) elements

\[
\exp(-\Pi)_* \circ \exp(\Pi)_*(\mu_1) = \mu_1 \quad \text{and} \quad \exp(-\Pi)_* \circ \exp(\Pi)_*(\mu_2) = \mu_2
\]

are connected by a 1-cell in \( MC(L_\Omega, -d, 0) \).

Thus \( \Theta_\Pi \) is indeed injective. \( \square \)

By putting all the things together, we can now complete the proof of Theorem 2.4.

Indeed, due to Proposition 3.1 we have a bijection

\[ TL \cong TL^{tw}. \]

The map \( \approx \) induced by the \( L_\infty \) quasi-isomorphism \( \approx \) gives us a bijection

\[ TL_{PV} \cong TL^{tw}. \]

The isomorphism \( J_\omega \) from \( \approx \) gives us a bijection

\[ TL_{PV} \cong TL_\Omega \]

and, finally, the map \( \exp(\Pi)_* \) induces a bijection

\[ \pi_0(\tilde{G}(L_\Omega, -d, 0)) \cong TL_\Omega. \]

Since

\[
\pi_0(\tilde{G}(L_\Omega, -d, 0)) \cong \bigoplus_{q \geq 0} \frac{1}{\varepsilon^{q-1}} \left( m s^q H^q(M, C)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]\right)^2,
\]

the proof of Theorem 2.4 complete. \( \square \)
6 Final remarks

In this concluding section, we would like to pose two open questions. It is known \[10\], \[22, 4.6.3\] and \[24\] that there are infinitely many homotopy classes of formality quasi-isomorphisms from the dg Lie algebra of polyvector fields to the dg Lie algebra of polydifferential operators. So it would be interesting to determine whether the bijection from Theorem 2.4 depends on the homotopy type of the formality quasi-isomorphism for polydifferential operators.

We believe that

Conjecture 6.1 There is a construction of a bijection between TL and the set of formal series in (2.20) which bypasses the use of Kontsevich’s formality theorem. This construction comes from an appropriate generalization of the zig-zag of quasi-isomorphisms of dg Lie algebras from paper \[9\].

Conjecture 6.2 The bijection between TL and the set of formal series in (2.20) coming from the above conjectural construction coincides with the bijection produced in this paper for any choice of a formality quasi-isomorphism (3.1).

If true, the statement of Conjecture 6.2 would imply that the constructed bijection between TL and the set of formal series in (2.20) does not depend on the choice of a formality quasi-isomorphism (3.1).

We believe that a solution of Conjecture 6.1 will allow us to produce explicit examples of A\(_{\infty}\)-structures on \(O(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]\) corresponding to formal series in (2.20) for a large class of symplectic manifolds. To our genuine surprise, Kontsevich’s quasi-isomorphism \[22\] is not very helpful for computing these A\(_{\infty}\)-structures even in the case when \(M\) is an even dimensional torus with the standard symplectic structure!

We also believe that Conjecture 6.2 can be tackled using the ideas from \[6\].

A Properties of the coderivation \(\Pi\)

In this appendix, we prove the properties of the coderivation \(\Pi\) (see (4.15)) of the coalgebra (4.12) listed in Proposition 4.3.

It is straightforward to see that \(\Pi\) has degree zero.

Since
\[
p \circ \Pi, Q_{-d} (s^{-2} \eta_1 \ldots s^{-2} \eta_n) = p \circ Q|_{\omega} (s^{-2} \eta_1 \ldots s^{-2} \eta_n) = 0
\]
if \(n \neq 2\), to prove (4.16), it suffices to show that
\[
p \circ (\Pi \circ Q_{-d} - Q_{-d} \circ \Pi) (s^{-2} \lambda s^{-2} \eta) = p \circ Q|_{\omega} (s^{-2} \lambda s^{-2} \eta),
\]
where \(\lambda\) and \(\eta\) are homogeneous vectors in
\[
\frac{1}{\varepsilon r-1} m \Omega^k(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]] \quad \text{and} \quad \frac{1}{\varepsilon r-1} m \Omega^r(M)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]]
\]
respectively.

Let \(U \subset M\) be an open coordinate subset with coordinates \(x^1, x^2, \ldots, x^m\) and
\[
\lambda|_U = dx^{i_1} dx^{i_2} \ldots dx^{i_k} \lambda_{i_1 \ldots i_k}, \quad \eta|_U = dx^{j_1} dx^{j_2} \ldots dx^{j_r} \eta_{j_1 \ldots j_r},
\]
where \(\lambda_{i_1 \ldots i_k} \in C^\infty(U)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]][\varepsilon]^{-1}, \eta_{j_1 \ldots j_r} \in C^\infty(U)[[\varepsilon, \varepsilon_1, \ldots, \varepsilon_g]][\varepsilon]^{-1}\) and summation over repeated indices is assumed.
The direct computation shows that

\[ p \circ \Pi \circ Q_d(s^{-2} \lambda s^{-2} \eta)|_U = \]

\[ (-1)^{|\lambda|} s^{-2} \varepsilon \alpha^{ij}(x)dx^i dx^j \ldots dx^k \partial_{x^i \lambda_{i_1 \ldots i_k}} \frac{\partial \eta}{\partial x^j}(A.2) \]

\[ - s^{-2} \varepsilon \alpha^{ij}(x) \frac{\partial \lambda}{\partial x^i} dx^j dx^{j_2} \ldots dx^{j_r} \partial_{x^j \eta_{j_1 \ldots j_r}} \]

\[ - (-1)^{|\lambda|} s^{-2} \varepsilon k \alpha^{i_1 j}(x)dx^i dx^j \ldots dx^k \partial_{x^i \lambda_{i_1 \ldots i_k}} \frac{\partial \eta}{\partial x^j}(A.3) \]

\[ + s^{-2} \varepsilon r \alpha^{i_1 j}(x) \frac{\partial \lambda}{\partial x^i} dx^j dx^{j_2} \ldots dx^{j_r} \partial_{x^j \eta_{j_1 \ldots j_r}} \]

(A.4)

Furthermore,

\[ -p \circ Q_d \circ \Pi(s^{-2} \lambda s^{-2} \eta)|_U = \]

\[ (-1)^{|\lambda|} s^{-2} \varepsilon dx^i \partial_x \alpha^{ij}(x) \frac{\partial \lambda}{\partial x^i} \frac{\partial \eta}{\partial x^j}(A.6) \]

\[ (-1)^{|\lambda|} s^{-2} \varepsilon k \alpha^{i_1 j}(x)dx^i dx^{i_2} \ldots dx^{i_k} \partial_{x^i \lambda_{i_1 \ldots i_k}} \frac{\partial \eta}{\partial x^j}(A.7) \]

\[ - s^{-2} \varepsilon r \alpha^{i_1 j}(x) \frac{\partial \lambda}{\partial x^i} dx^j dx^{i_2} \ldots dx^{i_r} \partial_{x^j \eta_{i_1 \ldots i_r}} \]

(A.8)

Term (A.7) (resp. term (A.8)) cancels term (A.4) (resp. term (A.5)) in the left hand side of (A.1). Moreover the sum of terms (A.2), (A.3) and (A.6) coincides with

\[ p \circ Q_{(\mu)}(s^{-2} \lambda s^{-2} \eta)|_U. \]

Thus identity (A.1) (and hence (4.16)) is proved.

To prove (4.17), we observe that

\[ p \circ \Pi \circ Q_{(\mu)}(s^{-2} \eta \ldots s^{-2} \eta_n) = p \circ Q_{(\mu)} \circ \Pi(s^{-2} \eta \ldots s^{-2} \eta_n) = 0 \]

if \( n \neq 3 \).

For \( n = 3 \), a direct computation shows that

\[ p \circ \Pi \circ Q_{(\mu)}(s^{-2} \eta \ldots s^{-2} \eta_2 s^{-2} \eta_3) - p \circ Q_{(\mu)} \circ \Pi(s^{-2} \eta \ldots s^{-2} \eta_2 s^{-2} \eta_3) = \]

\[ -2(\varepsilon s^{-2} \alpha^{ij_1}(x)(\partial_{x^{i_1} \alpha^{ij}(x)}) \frac{\partial \eta_1}{\partial x^{j_1}} \frac{\partial \eta_2}{\partial x^1} \frac{\partial \eta_3}{\partial x^j}) \]

\[ + (-1)^{|\eta_1|+|\eta_2|+|\eta_3|} \varepsilon s^{-2} \alpha^{i_1 j_1}(x)(\partial_{x^{i_1} \alpha^{ij}(x)}) \frac{\partial \eta_1}{\partial x^{j_1}} \frac{\partial \eta_2}{\partial x^1} \frac{\partial \eta_3}{\partial x^j} \]

\[ + (-1)^{|\eta_1|+|\eta_2|+|\eta_3|} \varepsilon s^{-2} \alpha^{i_1 j_1}(x)(\partial_{x^{i_1} \alpha^{ij}(x)}) \frac{\partial \eta_1}{\partial x^{j_1}} \frac{\partial \eta_2}{\partial x^1} \frac{\partial \eta_3}{\partial x^j} \]

2(\varepsilon s^{-2}(\alpha^{i_1 j_1}(x)\partial_{x^{i_1} \alpha^{ij}(x)} + \alpha^{i_1 i}(x)\partial_{x^{i_1} \alpha^{ij}(x)} + \alpha^{i_1 j}(x)\partial_{x^{i_1} \alpha^{ij}(x)}) \frac{\partial \eta_1}{\partial x^{j_1}} \frac{\partial \eta_2}{\partial x^1} \frac{\partial \eta_3}{\partial x^j}.

Thus (4.17) is a consequence of the Jacobi identity for the Poisson structure \( \alpha \). Proposition 4.3 is proved.
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