BLACK HOLES AND CLOSED TRAPPED SURFACES:
A REVISION OF A CLASSIC THEOREM

CLARISSA-MARIE CLAUDEL

ABSTRACT. It is standard assertion in relativity that, subject to an energy condition and the cosmic censorship hypothesis, closed trapped surfaces are not visible from $\mathcal{I}^+$. A proof given by Hawking & Ellis in The Large Scale Structure of Space-Time is flawed since it is formulated in terms of an inadequate definition of a weakly asymptotically simple and empty space-time. A new proof is given based on a more restrictive definition of a weakly asymptotically simple and empty space-time.

1. INTRODUCTION

Weakly asymptotically simple and empty (WASE) space-times provide a setting for the analysis of isolated gravitational objects in classical general relativity. There is sufficient freedom and generality to allow for the presence of singularities and black holes, but there is also sufficient structure, in the form of a well-behaved asymptotic region, to permit the proof of significant results. Examples of WASE space-times include the Schwarzschild, Kerr and Reissner-Nordström space-times.

Despite the evident physical importance of WASE space-times, it is often the case that scant attention is paid to their precise definition. Indeed, as will be seen, the definition proposed and used by Hawking & Ellis [1], henceforth H&E, is inadequate for their purposes. A more restrictive definition was subsequently proposed in [2] as a foundation for a class of censorship theorems (see also [3, 4, 5, 6]). One purpose of the present paper is to provide basic results for this definition further to those given in [2].

As has been well-known since [7], the concept of a closed trapped surface is central to the understanding of black holes. In the context of WASE space-times it is a standard assertion, stated as Proposition 9.2.1 in H&E, that subject to a form of weak cosmic censorship and a suitable energy condition, it is not possible for closed trapped surfaces to be seen from future null infinity. In itself this is of no direct physical significance because there need be nothing remarkable about the geometry at a space-time point which lies on a closed trapped surface. The assertion does however imply, again subject to weak cosmic censorship and an energy condition, that the presence of a closed trapped surface in a WASE space-time implies the presence of a black hole. The assertion also leads to an elementary censorship theorem according to which, in a WASE satisfying the same energy condition, if every future singularity is sufficiently strong as to be preceded by a closed trapped surface, then weak cosmic censorship must hold. The second purpose of this paper is to obtain a rigorous proof of a modified statement of H&E Proposition 9.2.1 on the basis of the definition of a WASE space-time given in [2].
2. Preliminary concepts

The following two basic definitions are taken from [2].

**Definition 2.1.** An asymptote of a space-time \((\bar{M}, \bar{g}, \Omega, \psi)\) is a quadruple \((\bar{M}, \bar{g}, \Omega, \psi)\), where \((\bar{M}, \bar{g})\) is a space-time-with-boundary, \(\Omega\) is a \(C^\infty\) real-valued function on \(\bar{M}\), and \(\psi: \bar{M} \to \bar{M}\) is a \(C^\infty\) embedding such that

(i) \(\psi(M) = M \setminus \partial M\);
(ii) \((\psi^* \Omega) \bar{g} = \psi^* \bar{g}\);
(iii) one has \(\Omega(p) = 0\) and \(d\Omega(p) \neq 0\) for all \(p \in \partial \bar{M}\).

**Definition 2.2.** An asymptote \((\bar{M}, \bar{g}, \Omega', \psi')\) of a space-time \((M', \bar{g}')\) is asymptotically simple and empty (ASE) if

(i) \(\bar{M} \setminus \text{supp}(\psi'\text{Ricc}(\bar{g}')) \supset \partial \bar{M}\);
(ii) \((\bar{M}, \bar{g}, \Omega', \psi')\) is strongly causal;
(iii) every inextendible null geodesic \(\gamma'\) of \((M', \bar{g}')\) is such that \(\psi' \circ \gamma'\) has two endpoints in \(\bar{M}\), both of which lie in \(\partial \bar{M}\).

A space-time is asymptotically simple and empty (ASE) if it admits an asymptotically simple and empty asymptote.

Standard arguments give that any ASE space-time \((M, g)\) is globally hyperbolic, and that for any ASE asymptote \((\bar{M}, \bar{g}, \Omega', \psi')\) of \((M', \bar{g}')\), the boundary \(\partial \bar{M}\) of \(\bar{M}\) is the union of two disjoint connected null hypersurfaces \(\bar{J}^+ := I^+((\bar{M}, \bar{g}, \bar{M}) \cap \partial \bar{M}\) and \(\bar{J}^- := I^-((\bar{M}, \bar{g}; \bar{M}) \cap \partial \bar{M}\). By means of condition (ii) of Definition 2.2 one can show that \(\bar{J}^+\) and \(\bar{J}^-\) are diffeomorphic to \(S^2 \times \mathbb{R}\).

In the terminology of [8], a slice of \(\bar{J}^+\) is a non-empty locally acausal compact connected topological 2-submanifold of \(\bar{J}^+\). Theorem 5.1 of [8] gives that every slice of \(\bar{J}^+\) is homeomorphic to \(S^2\). Since, by assumption here, strong causality holds at every point of \(\bar{J}^+\) in \((\bar{M}, \bar{g})\), Proposition 7.1 of [8] gives that every null geodesic generator of \(\bar{J}^+\) cuts every slice of \(\bar{J}^+\), and Theorem 7.4 of [8] gives that \(\bar{J}^+\) is acausal in \((\bar{M}, \bar{g})\). Any given slice of \(\bar{J}^+\) may be mapped along the generators of \(\bar{J}^+\) to yield a foliation of \(\bar{J}^+\) by slices of \(\bar{J}^+\). Similar assertions apply to \(\bar{J}^-\).

Note that \((\bar{M}, \bar{g})\) need not be causally simple. For example, let \((M', \bar{g}')\) be Minkowski space. Let \(\tilde{\nu}: \mathbb{R} \to \tilde{J}^-\) be a future-directed null geodesic generator of \(\tilde{J}^-\) and let \(\tilde{\nu}: \mathbb{R} \to \tilde{J}^+\) be the antipodal future-directed null geodesic generator of \(\tilde{J}^+\). Let \(\tilde{q} \in \tilde{J}^-\) and \(\bar{q} \in \tilde{J}^+\) with \(\bar{q} = \tilde{q}^-\). One then has \(J^+(\tilde{q}, \bar{q}; \tilde{M}) \cap \bar{J}^+ = \tilde{J}^+ \setminus [\lambda_\bar{q}]\) for \(a := \tilde{\nu}^{-1}(\bar{q}) \in \mathbb{R}\). Since \(\tilde{J}^+ \setminus [\lambda_\bar{q}]\) is not relatively closed in \(\tilde{J}^+\), the set \(J^+(\tilde{q}, \bar{q}; \tilde{M})\) cannot be closed in \((\bar{M}, \bar{g})\).

Condition (ii) of Definition 2.2 may be decomposed into two parts, first that \((M', \bar{g}')\) is strongly causal and second that \((\bar{M}, \bar{g})\) is strongly causal at every point of \(\bar{J}^+\). Since the physical interpretation of the latter is unclear one is led, as in [8], to the more general concept of a “simple” space-time for which only the chronology condition is imposed on \((M', \bar{g}')\), with no additional causality conditions imposed on \(\bar{J}^+\) or \(\bar{J}^-\) in \((M, \bar{g})\). Simple space-times are globally hyperbolic with Cauchy surfaces which, subject to the truth of the Poincaré conjecture, are diffeomorphic to \(\mathbb{R}^3\). But the topological and causal structure of \(\tilde{J}^+\) and \(\tilde{J}^-\)
may exhibit new complications and subtleties. Despite the possible interest of this additional generality, \((\tilde{M}, \tilde{g})\) will for present purposes be assumed to be strongly causal as expressed in condition (ii) if Definition 2.2.

In H&E, a weakly asymptotically simple and empty (WASE) space-time is introduced with a definition which, when re-expressed in terms of asymptotes, assumes the following form.

**Provisional Definition 2.3** (c.f. H&E p.225). A space-time \((M, g)\) is weakly asymptotically simple and empty (WASE) if there exists an asymptote \((\bar{M}, \bar{g}, \Omega, \psi)\) of \((M, g)\), a space-time \((M', g')\), an asymptote \((\tilde{M}, \tilde{g}, \Omega', \psi')\) of \((M', g')\) and open sets \(U\) and \(U'\) of \(M\) and \(M'\) respectively such that

1. \((\tilde{M}, \tilde{g}, \Omega', \psi')\) is ASE;
2. \(\psi(U) \cup \partial \bar{M}\) is an open neighbourhood of \(\partial \bar{M}\) in \(\bar{M}\);
3. \(\psi'(U') \cup \partial \tilde{M}\) is an open neighbourhood of \(\partial \tilde{M}\) in \(\tilde{M}\);
4. \((\Omega, g|_U)\) and \((\Omega', g'|_{U'})\) are globally isometric.

This definition of a weakly asymptotically simple and empty space-time is not as restrictive as its authors seem to have intended. In particular, it allows the future and past null infinities \(\mathcal{I}^+ := I^+(\bar{M}, \bar{g}; M) \cap \partial M\) and \(\mathcal{I}^- := I^-(\bar{M}, \bar{g}; M) \cap \partial M\) of \((\bar{M}, \bar{g})\) to be separated from one another and not join up at a spatial infinity (see Figure 1). (To add an assumption that \(U\) is connected would not help.) The deficiency undermines several of their results, and in particular their proposed proof (Proposition 9.2.1) that closed trapped surfaces are necessarily confined to black holes.

**Figure 1.** The H&E definition of a WASE space-time has the unwanted feature that the future and past null infinities \(\mathcal{I}^+\) and \(\mathcal{I}^-\) of \((M, g)\) need not join up at a spatial infinity.

In order to overcome the problems in Definition 2.3, one may adopt the following, more restrictive definition of a WASE space-time proposed in [2].

**Definition 2.4.** An asymptote \((\bar{M}, \bar{g}, \Omega, \psi)\) of a space-time \((M, g)\) is weakly asymptotically simple and empty (WASE) if there exists an open set \(U\) of \(M\), an extension \((M', g')\) of \((U, g|_U)\), an asymptote \((\tilde{M}, \tilde{g}, \Omega', \psi')\) of \((M', g')\) and a topological embedding \(\xi : \psi(U) \cup \partial M \to \tilde{M}\) such that
(i) \((\tilde{M}, \tilde{g}, \Omega', \psi')\) is ASE;

(ii) for every \(p' \in M'\), the set \(M' \setminus (\mathcal{U} \cup I(p', g'; M'))\) is compact;

(iii) one has \(\xi(\partial \bar{M}) = \partial \tilde{M}\) and \(\xi \circ \psi|\mathcal{U} = \psi'|\mathcal{U}\);

(iv) for all \(q \in \mathcal{U}\) and all future-pointing timelike vectors \(v\) of \((M, g)\), the vectors \(\psi_*v, v, \text{ and } \psi'\) are future-pointing in \((\tilde{M}, \tilde{g})\), \((M', g')\) and \((\bar{M}, g)\) respectively.

A space-time is weakly asymptotically simple and empty (WASE) if it admits a weakly asymptotically simple and empty asymptote.

**Remark.** For any Cauchy surface \(S'\) of \((M', g')\) one has \(S' \setminus \mathcal{U} \subset M' \setminus (\mathcal{U} \cup I(p', g'; M'))\) for any \(p' \in \mathcal{I}\) and hence that \(\mathcal{I} \setminus \mathcal{U}\) is compact.

The future and past null infinities of a WASE asymptote \((\bar{M}, \bar{g}, \Omega, \psi)\) are defined by \(\mathcal{I}^+ := I^+(\bar{M}, \bar{g}; \bar{M}) \cap \partial \bar{M}\) and \(\mathcal{I}^- := I^-(\bar{M}, \bar{g}; \bar{M}) \cap \partial \bar{M}\) respectively. It is clear that \(\partial \bar{M}\) is the disjoint union of \(\mathcal{I}^+\) and \(\mathcal{I}^-\).

The mappings involved in Definition 2.4 are shown in Figure 2. In Figure 3 one can see how the pathologies inherent in Definition 2.3 are eliminated by condition (ii) of Definition 2.4. This condition may be regarded as a way to require that \(\mathcal{U}\) is a neighbourhood of spatial infinity without reference to the geometrical structure of spatial infinity.

\[
\begin{array}{ccc}
(M, \bar{g}) & \supset & \psi(\mathcal{U}) \cup \partial \tilde{M} \\
\downarrow \psi & & \downarrow \psi|\mathcal{U} \\
(M, \tilde{g}) & \supset & \mathcal{U} \subset (M', \bar{g}')
\end{array}
\]

\[
\begin{array}{ccc}
\xi & \longrightarrow & (\tilde{M}, \tilde{g}) \\
\end{array}
\]

**FIGURE 2.** The mappings in the Definition 2.4 of a WASE space-time. The left and right squares commute in the category of topological spaces and continuous mappings.

The following two lemmas are basic in the analysis of WASE space-times. Their proofs are given in [2].

**Lemma 2.5.** Within the context of Definition 2.4 one has

(i) \(\psi(\mathcal{U}) \cup \partial \bar{M}\) is open in \(\bar{M}\);

(ii) \(\psi(\mathcal{U}) \cup \partial \bar{M}\) is open in \(\bar{M}\).

**Lemma 2.6.** Within the context of Definition 2.4 one has

(i) a subset of \(\mathcal{U}\) is open in \(M\) if and only if it is open in \(M'\);

(ii) a subset of \(\mathcal{U}\) is compact in \(M\) if and only if it is compact in \(M'\);

(iii) a subset of \(\mathcal{U}\) is closed in \(M\) if (but not only if) it is closed in \(M'\).
Similarly the definition of $\xi$ pointwise to curve of $(\tilde{\mathcal{I}} \cup \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}})$ in Definition 2.4 is such that $\xi|_{\psi(\mathcal{U})}$ is a diffeomorphism onto $\psi'(\mathcal{U})$. Moreover $\xi|_{\psi(\mathcal{U})}$ is a conformal isometry onto $\psi'(\mathcal{U})$ in the sense of $(\xi|_{\psi(\mathcal{U})})^*\bar{g} = (\Omega/\Omega')^2g|_{\psi(\mathcal{U})}$. Note however that $\xi$ need not be differentiable at points of $\partial \tilde{\mathcal{M}}$.

**Proposition 2.7.** One has

(i) $\xi(\mathcal{I}^+) = \tilde{\mathcal{I}}^+$ and $\xi(\mathcal{I}^-) = \tilde{\mathcal{I}}^-$;

(ii) $\xi$ maps the null geodesic generators of $\mathcal{I}^+$ and $\mathcal{I}^-$ onto null geodesic generators of $\tilde{\mathcal{I}}^+$ and $\tilde{\mathcal{I}}^-$ respectively.

**Proof.** Let $p \in \mathcal{I}^+$. Lemma 2.7 gives that $\psi(\mathcal{U}) \cup \partial \tilde{\mathcal{M}}$ is an open neighbourhood of $p$ in $\tilde{\mathcal{M}}$. Hence, by the definition of $\mathcal{I}^+$, there exists a smooth timelike curve $\alpha : (0,1) \rightarrow \psi(\mathcal{U})$ of $(\tilde{\mathcal{M}}, \bar{g})$ having a future endpoint at $p$ in $(\tilde{\mathcal{M}}, \bar{g})$. One has $\xi(p) \in \partial \tilde{\mathcal{M}}$ by condition (iii) of Definition 2.4. Since $\xi \circ \alpha$ is a smooth timelike curve of $(\tilde{\mathcal{M}}, \bar{g})$ with a future endpoint at $\xi(p)$, one therefore has $\xi(p) \in \tilde{\mathcal{I}}^+$ by the definition of $\tilde{\mathcal{I}}^+$. There follows $\xi(\mathcal{I}^+) \subset \tilde{\mathcal{I}}^+$. A similar argument gives $\xi^{-1}(\tilde{\mathcal{I}}^+) \subset \mathcal{I}^+$ which implies $\tilde{\mathcal{I}}^+ \subset \xi(\mathcal{I}^+)$. Hence one has $\xi(\mathcal{I}^+) = \tilde{\mathcal{I}}^+$ and similarly $\xi(\mathcal{I}^-) = \tilde{\mathcal{I}}^-$. This establishes (i).

Let $\gamma : \mathbb{R} \supset I \rightarrow \mathcal{I}^+$ be a future-directed null geodesic generator of $\mathcal{I}^+$ and let $\tilde{\gamma} := \xi \circ \gamma$. Let $[a,b] \subset I$ for $a < b$ and let $\mu := \gamma|[a,b)$. One has $|\mu| \subset \mathcal{I}^+$. By Lemma 2.5 and the definition of $\mathcal{I}^+$ there exists a smooth future-directed timelike curve $\mu_0 : [a,b) \rightarrow \psi(\mathcal{U})$ of $(\tilde{\mathcal{M}}, \bar{g})$ with a future endpoint at $\gamma(b)$. Indeed there exists a sequence of smooth future-directed timelike curves $\mu_i : [a,b) \rightarrow \psi(\mathcal{U})$ of $(\tilde{\mathcal{M}}, \bar{g})$ converging pointwise to $\mu$ in $\tilde{\mathcal{M}}$, each with a future endpoint at $\gamma(b)$.

Since the smooth timelike curves $\tilde{\mu}_i := \xi \circ \mu_i : [a,b) \rightarrow \psi(\mathcal{U})$ of $(\tilde{\mathcal{M}}, \bar{g})$ converge pointwise to $\tilde{\mu} := \xi \circ \mu$ in $\tilde{\mathcal{M}}$ one has that $\tilde{\mu}$ is a causal curve of $(\tilde{\mathcal{M}}, \bar{g})$ with a future endpoint at $\tilde{\gamma}(b)$. Since part (i) gives $|\tilde{\mu}| = \xi(|\mu|) \subset \tilde{\mathcal{I}}^+$ and since $\tilde{\mathcal{I}}^+$ is a null hypersurface of $(\tilde{\mathcal{M}}, \bar{g})$ it follows that $\tilde{\mu} = \xi \circ \mu$ is a null geodesic generating segment.
of $\mathcal{J}^+$. Since $a$ and $b$ were arbitrary in $\mu := \gamma|[a,b]$ it follows that $\tilde{\gamma} = \xi \circ \gamma$ is a null geodesic generating segment of $\tilde{\mathcal{J}}^+$. Clearly $\tilde{\gamma}$ cannot have a future endpoint $q \in \tilde{\mathcal{J}}^+$ otherwise $\gamma$ would have a future endpoint at $\xi^{-1}(q) \in \mathcal{J}^+$ and so would not be a generator of $\mathcal{J}^+$, contrary to hypothesis. Similarly $\tilde{\gamma}$ cannot have a past endpoint in $\tilde{\mathcal{J}}^-$. Hence $\tilde{\gamma}$ is a null geodesic generator of $\tilde{\mathcal{J}}^+$. The corresponding result for $\mathcal{J}^-$ is similar. This establishes (ii).

Proposition 2.7 shows that the structure of $\mathcal{I}^+$ and $\mathcal{I}^-$ for a WASE asymptote $(\bar{M}, \bar{g}, \Omega, \psi)$ is directly analogous to the structure of $\tilde{\mathcal{J}}^+$ and $\tilde{\mathcal{J}}^-$ for an ASE asymptote $(\bar{M}, \bar{g}, \Omega', \psi')$. In particular one may, following [3], define a slice of $\mathcal{I}^+$ (respectively a slice of $\mathcal{I}^-$) as a non-empty locally acausal compact connected topological 2-submanifold of $\mathcal{I}^+$ (respectively $\mathcal{I}^-$). Then $\mathcal{I}^+$ and $\mathcal{I}^-$ are acausal in $(\bar{M}, \bar{g})$, every null geodesic generator of $\mathcal{I}^+$ cuts every slice of $\mathcal{I}^+$ and every null geodesic generator of $\mathcal{I}^-$ cuts every slice of $\mathcal{I}^-$. Slices of $\mathcal{I}^+$ and $\mathcal{I}^-$ are mapped by $\xi$ to slices of $\tilde{\mathcal{J}}^+$ and $\tilde{\mathcal{J}}^-$ respectively. Slices of $\tilde{\mathcal{J}}^+$ and $\tilde{\mathcal{J}}^-$ are mapped by $\xi^{-1}$ to slices of $\mathcal{I}^+$ and $\mathcal{I}^-$ respectively.

The following is a useful restriction on the causal structure of a WASE space-time. It is equivalent to a definition of asymptotic simplicity in [2] but is re-expressed here in a form more convenient for present purposes. The change of terminology seems appropriate because the term “simple” has become overworked.

**Definition 2.8.** A WASE asymptote $(\bar{M}, \bar{g}, \Omega, \psi)$ is asymptotically chronologically consistent if $\mathcal{W}, (M', g')$ and $(\bar{M}, \bar{g}, \Omega', \psi')$ in Definition 2.4 may be chosen such that for any achronal set $\mathcal{A}$ of $(\bar{M}, \bar{g})$ such that $\mathcal{A} \subset \psi(\mathcal{W}) \cup \partial \bar{M}$ one has that $\xi^{-1}(\mathcal{A}) \subset \psi(\mathcal{W}) \cup \partial \bar{M}$ is achronal in $(\bar{M}, \bar{g})$.

### 3. The main result

A form of weak cosmic censorship hypothesis will be required. The H&E concept of future asymptotic predictability is suitable for this purpose. The following definition formulates future asymptotic predictability in terms of asymptotes and provides a weaker concept of partial future asymptotic predictability that is also well-established in the literature.

**Definition 3.1.** Let $\mathcal{I}$ be a closed achronal set without edge in a WASE space-time $(M, g)$ and let $(\bar{M}, \bar{g}, \Omega, \psi)$ be a WASE asymptote of $(M, g)$. Then $(\bar{M}, \bar{g}, \Omega, \psi)$ is future asymptotically predictable from $\psi(\mathcal{I})$ if one has $\mathcal{I}^+ \subset \bar{D}^+(\psi(\mathcal{I}), \bar{g}; \bar{M})$. One says that $(\bar{M}, \bar{g}, \Omega, \psi)$ is partially future asymptotically predictable from $\psi(\mathcal{I})$ if there exists a slice $\Sigma^+$ of $\mathcal{I}^+$ such that $\bar{J}^- (\Sigma^+, \bar{g}; \bar{M}) \cap \mathcal{I}^+ \subset \bar{D}^+(\psi(\mathcal{I}), \bar{g}; \bar{M})$.

The main result is the following:

**Theorem 3.2.** Let $(M, g)$ be a WASE space-time and let $(\bar{M}, \bar{g}, \Omega, \psi)$ be a WASE asymptote of $(M, g)$. Suppose

1) there exists a closed, edgeless achronal set $\mathcal{I}$ in $(M, g)$ such that $(\bar{M}, \bar{g}, \Omega, \psi)$ is future asymptotically predictable from $\psi(\mathcal{I})$;

2) $(\bar{M}, \bar{g}, \Omega, \psi)$ is asymptotically chronologically consistent;

3) one has $R_{ab}k^ak^b \geq 0$ for all null vectors $k^a$,

then for any closed trapped surface $\mathcal{F}$ of $(M, g)$ in $\bar{I}^+(\mathcal{I}, \bar{g}; \bar{M})$ one has $\psi(\mathcal{I}) \cap \bar{J}^-(\mathcal{I}^+, \bar{g}; \bar{M}) = \emptyset$. 

Conditions (1) and (3) of Theorem 3.2 coincide with conditions in the statement of H&E Proposition 9.2.1. However condition (2), which would seem to be necessary, makes no appearance in H&E. Note also that Theorem 3.2 requires only \( \mathcal{I} \subset \bar{I}^+(\mathcal{I}, g; M) \) whereas H&E impose the stronger condition \( \mathcal{I} \subset D^+(\mathcal{I}, g; M) \).

The basic idea of the H&E argument in support of the statement of their Proposition 9.2.1 is to show that if \( \mathcal{I} \) is visible from \( \mathcal{I}^+ \) then there must be a null geodesic generator of \( I^+(\mathcal{I}, g; M) \) which reaches from \( \mathcal{I} \) to \( \mathcal{I}^+ \) and which is therefore of infinite affine length. A contradiction then follows by means of the Raychaudhuri equation and the null convergence condition. The argument fails though because the H&E definition of a WASE space-time is not sufficiently strong. Specifically, things begin to go wrong when they claim that, in the associated ASE space-time \( (M', g') \), for a Cauchy surface \( \mathcal{I}' \) chosen such that \( \mathcal{I}' \cap \mathcal{U}' = \mathcal{I} \cap \mathcal{U} \) it is necessarily the case that \( \mathcal{I}' \setminus \mathcal{U}' \) is compact. (At this point H&E are tacitly identifying \( \mathcal{U} \) and \( \mathcal{U}' \), as is explicitly done in the present formalism.) In the first place it is unclear that there need be any Cauchy surface \( \mathcal{I}' \) of \( (M', g') \) such that \( \mathcal{I}' \cap \mathcal{U}' = \mathcal{I} \cap \mathcal{U} \). For example \( \mathcal{I} \cap \mathcal{U} \) might not be achronal in \( (M', g') \). And second, since \( \mathcal{I} \) need not even intersect \( \mathcal{U} \) and every Cauchy surface of \( (M', g') \) is non-compact, the set \( \mathcal{I}' \setminus \mathcal{U}' \) could be non-compact. Definition 2.4 directly overcomes the second difficulty, as was indicated in the Remark that followed Definition 2.4. In order to overcome the first it will be necessary to employ different techniques.

*Figure 4.* A space-time which is WASE in the sense of H&E and future asymptotically predictable from a partial Cauchy surface \( \mathcal{I} \). The closed trapped surface \( \mathcal{I} \) is visible from \( \mathcal{I}^+ \) but no null geodesic generator of the boundary of the causal future of \( \mathcal{I} \) meets \( \mathcal{I}^+ \).
At a pictorial level one might seek a counterexample to H&E Proposition 9.2.1 by arranging to have \( \mathcal{J}^+ \) both separated from \( \mathcal{J}^- \) and contained in the chronological future of a closed trapped surface \( \mathcal{T} \) (see Figure 4). Since none of the null geodesic generators of \( \mathcal{J}^+(\psi(\mathcal{T}), \tilde{g}; \tilde{M}) \) then meet \( \mathcal{J}^+ \), the central contradiction in the proof of H&E Lemma 9.2.1 is avoided. However to obtain a full counterexample to H&E Proposition 9.2.1 one also needs to arrange for the null convergence condition to be satisfied. Even though it is not clear how this might be done, it seems unlikely that H&E Proposition 9.2.1 is correct.

The following three lemmas and a proposition are the key to the proof of the revised trapped surfaces theorem.

**Lemma 3.3.** Let \( (M, g) \) be a WASE space-time with a asymptotically chronologically consistent WASE asymptote \( (\bar{M}, \tilde{g}, \Omega, \psi) \) and let \( \Sigma^+ \) be a slice of \( \mathcal{J}^- \). Then there exists a slice \( \Sigma^+ \) of \( \mathcal{J}^+ \) such that \( J^-(\Sigma^+, \tilde{g}; \tilde{M}) \cap J^+(\Sigma^-, \tilde{g}; \tilde{M}) \) is a closed set of \( \bar{M} \) contained in \( \psi(\mathcal{U}) \cup \partial \bar{M} \) and \( J^-(\Sigma^+, \tilde{g}; \tilde{M}) \cap J^+(\Sigma^-, \tilde{g}; \tilde{M}) \) is a closed set of \( \bar{M} \) contained in \( \psi(\mathcal{U}) \cup \partial \bar{M} \) for \( \Sigma^- := \xi(\Sigma^-) \) and \( \Sigma^+ := \xi(\Sigma^+) \).

**Proof.** The time reverse of Lemma 3.6 of [8] gives that \( \bar{M} \setminus \partial \bar{M} \) cannot be contained entirely in \( I^-(\Sigma^+, \tilde{g}; \tilde{M}) \). Let \( \bar{p}_+ \in \bar{M} \setminus (I^-(\Sigma^+, \tilde{g}; \tilde{M}) \cup \partial \bar{M}) \) and let \( \bar{p} \in J^-(\bar{p}_+; \tilde{g}; \tilde{M}) \). Then there exists a causal curve \( \xi \) of \( (\bar{M}, \tilde{g}; \tilde{M}) \) which does not intersect \( J^-(\Sigma^+, \tilde{g}; \tilde{M}) \). Hence \( J^-(\Sigma^+, \tilde{g}; \tilde{M}) \cap J^+(\Sigma^-, \tilde{g}; \tilde{M}) \) is contained in \( \psi(\mathcal{U}) \cup \partial \bar{M} \).

The set \( \Sigma^+_0 := \xi^{-1}(\Sigma^+_{\bar{p}_+}) \) is a slice of \( \mathcal{J}^+ \). Suppose there exists a point \( x \in (I^-\Sigma^+_0, \tilde{g}; \tilde{M}) \cap J^+(\Sigma^-, \tilde{g}; \tilde{M})) \setminus (\psi(\mathcal{U}) \cup \partial \bar{M}) \). Then there exists a causal curve \( \alpha : [0, 1] \rightarrow \bar{M} \) of \( (M, \tilde{g}) \) from \( x \) to \( \Sigma^+_0 \). Let \( a := \text{sup}\{t \in [0, 1] : \beta(t) \notin \psi(\mathcal{U}) \cup \partial \bar{M} \} \). Then \( \beta \) lies in the topological boundary of \( \psi(\mathcal{U}) \cup \partial \bar{M} \). Then \( \beta(a) \) lies in \( I^+(\Sigma^-, \tilde{g}; \tilde{M}) \cap J^+(\Sigma^+, \tilde{g}; \tilde{M}) \) for all \( t \in [a, 1] \). For each \( t \in [a, 1] \) the set \( \Sigma^+ \cup \{\beta(t)\} \) is non-achronal in \( (\bar{M}, \tilde{g}, \Omega, \psi) \) and contained in \( \psi(\mathcal{U}) \cup \partial \bar{M} \). So, by the asymptotic chronologcal consistency of \( (M, g, \Omega, \psi) \), one has that \( \xi(\Sigma^- \cup \{\beta(t)\}) = \Sigma^- \cup \xi(\Sigma^- \cup \{\beta(t)\}) \) is non-achronal in \( (\bar{M}, \tilde{g}) \) for each \( t \in [a, 1] \). Since \( \xi \circ \beta : [a, 1] \rightarrow \bar{M} \) is a timelike curve of \( (M, g) \) from \( \xi \circ \beta(a) \) to \( \xi \circ \beta(t) \) one thus has \( \xi \circ \beta(t) \in I^+(\Sigma^+, \tilde{g}; \tilde{M}) \cap J^-(\Sigma^+, \tilde{g}; \tilde{M}) \) for all \( t \in [a, 1] \).
Let \( \Sigma^+ := \xi^{-1}(\bar{\Sigma}^+) \). One has \( J^-(\Sigma^+, \bar{\hat{g}}; M) \cap J^+(\Sigma^-, \bar{\hat{g}}; \bar{M}) \subset (I^- (\Sigma^+; \bar{g}; \bar{M}) \cap J^+(\Sigma^-, \bar{g}; \bar{M})) \cap \partial M \subset \psi(\mathscr{U}) \cup \partial M \). Hence a point of \( \bar{M} \) lies in \( J^-(\Sigma^+, \bar{\hat{g}}; \bar{M}) \cap J^+(\Sigma^-, \bar{g}; \bar{M}) \) if and only if it lies on a causal curve of \((\bar{M}, \bar{g})\) in \( \psi(\mathscr{U}) \cup \partial \bar{M} \) from \( \Sigma^- \) to \( \Sigma^+ \).

One thus has \( J^-(\Sigma^+, \bar{\hat{g}}; \bar{M}) \cap J^+(\Sigma^-, \bar{g}; \bar{M}) = \xi^{-1}(J^- (\Sigma^+, \bar{g}; \bar{M}) \cap J^+(\Sigma^-, \bar{g}; \bar{M}) \cap \partial M \subset \psi(\mathscr{U}) \cup \partial M \).

Since \( J^-(\Sigma^+, \bar{\hat{g}}; \bar{M}) \cap J^+(\Sigma^-, \bar{g}; \bar{M}) \) has been shown to be closed in \( \bar{M} \) it follows that \( J^-(\Sigma^+, \bar{\hat{g}}; \bar{M}) \cap J^+(\Sigma^-, \bar{g}; \bar{M}) \) is closed in \( \bar{M} \).

**Corollary.** Let \( \mathscr{K} \) be a compact set of \( M \). Then there exists a slice \( \Sigma^+_1 \) of \( \mathscr{K}^+ \) such that \( J^-(\Sigma^+_1; \bar{\hat{g}}; \bar{M}) \cap J^+(\Sigma^-, \bar{g}; \bar{M}) \) does not intersect \( \psi(\mathscr{K}) \).

**Proof.** One may assume \( \mathscr{U} \subset M \setminus \mathscr{K} \) otherwise one may redefine \( \mathscr{U} \) as \( \mathscr{U} \setminus \mathscr{K} \).

The Lemma then gives that there exists a slice \( \Sigma^+_1 \) of \( \mathscr{K}^+ \) such that \( J^-(\Sigma^+_1; \bar{\hat{g}}; \bar{M}) \cap J^+(\Sigma^-, \bar{g}; \bar{M}) \subset \psi(\mathscr{U}) \cup \partial \bar{M} \subset M \setminus \psi(\mathscr{K}) \).

**Lemma 3.4.** Let \( \Sigma^- \) be a slice of \( \mathscr{K}^- \) and let \( \mathcal{N} \) be an open neighbourhood of \( \Sigma^- \) in \( \bar{M} \). Then there exists a slice \( \Sigma^+_2 \) of \( \mathscr{K}^+ \) such that, for every \( q \in \Sigma^+_2 \), the set \( \mathcal{N} \) is cut by every past endless timelike curve of \((\bar{M}, \bar{g})\) to \( q \).

**Proof.** It suffices to assume \( \mathcal{N} \subset \psi(\mathscr{U}) \cup \partial \bar{M} \) since one may otherwise redefine \( \mathcal{N} \) as \( \mathcal{N} \cap (\psi(\mathscr{U}) \cup \partial \bar{M}) \).

The set \( \bar{\mathcal{N}} := \xi(\mathcal{N}) \) is open in \( \bar{M} \) and \( \bar{\Sigma}^- := \xi(\Sigma^-) \) is a slice of \( \bar{\mathcal{K}}^- \). By the time reverse of Lemma 4.12 of [8], one has that \( J^+ (\bar{\Sigma}^-, \bar{g}; \bar{M}) \) is compact in \((\bar{M}, \bar{g})\) and such that \( J^+(\bar{\Sigma}^-, \bar{g}; \bar{M}) \cap \bar{\mathcal{K}}^- = \bar{\Sigma}^- \). The time reverse of Proposition 7.2 of [8] gives \( \bar{\mathcal{K}}^+ = I^+ (\bar{\Sigma}^-, \bar{g}; \bar{M}) \cap \partial \bar{M} \) whereby one has \( \bar{\mathcal{K}}^+ (\bar{\Sigma}^-, \bar{g}; \bar{M}) \cap \partial \bar{M} = \bar{\Sigma}^- \). Hence one has \( \bar{\mathcal{K}}^+ (\bar{\Sigma}^-, \bar{g}; \bar{M}) \cap \partial \bar{M} = \bar{\Sigma}^- \). Let \( \bar{\mathcal{N}} \) be the compact set \( J^+ (\bar{\Sigma}^-, \bar{g}; \bar{M}) \setminus \bar{\mathcal{N}} \subset M \setminus \partial \bar{M} \).

The set \( \Sigma^+_2 := J^+ (\bar{\mathcal{K}}^+, \bar{g}; \bar{M}) \cap \partial \bar{M} \) is a slice of \( \bar{\mathcal{K}}^+ \). Since \( \bar{\mathcal{K}}^+ = I^+ (\bar{\Sigma}^-, \bar{g}; \bar{M}) \), \( \bar{\mathcal{N}} \cap \partial \bar{M} \subset \psi(\mathscr{U}) \cup \partial \bar{M} \in M \setminus \psi(\mathcal{N}) \).

Let \( \Sigma^+ \) be as in the statement of Lemma 3.3 and let \( \Sigma^+_3 := \xi (\Sigma^+) \). Let \( \Sigma^+_2 \) be a slice of \( \mathcal{K}^+ \) lying strictly to the past of both \( \Sigma^+_2 \) and \( \Sigma^+ \) along the generators of \( \mathcal{K}^+ \).

One then has \( J^- (\Sigma^+_2; \bar{\hat{g}}; \bar{M}) \cap J^+ (\Sigma^-, \bar{g}; \bar{M}) \subset \psi(\mathscr{U}) \cup \partial \bar{M} \) and \( J^+ (\Sigma^-; \bar{g}; \bar{M}) \cap J^- (\Sigma^+_2; \bar{g}; \bar{M}) \subset J^+ (\Sigma^-, \bar{g}; \bar{M}) \cap J^- (\Sigma^+_2; \bar{g}; \bar{M}) \setminus \mathcal{N} \subset \partial \bar{M} \).

The set \( \Sigma^+_2 := \xi^{-1}(\Sigma^+_2) \) is a slice of \( \mathcal{K}^+ \). Let \( q \in \Sigma^+_2 \) and let \( \sigma : (-\infty, 0] \to \bar{M} \setminus \partial \bar{M} \) be a future-directed, past timelike curve of \((\bar{M}, \bar{g})\) having a future endpoint at \( \sigma(0) = q \in \Sigma^+_2 \). Let \( \nu : [b, 0] \to \bar{M} \) be the maximal segment of \( \sigma \) to \( \bar{q} := \xi(q) \in \Sigma^+_2 \). One clearly has \( |\bar{\nu}| \subset J^- (\Sigma^+_2; \bar{\hat{g}}; \bar{M}) \). In order to show that \( \bar{\nu} \) cuts \( \mathcal{N} \) it therefore suffices to show that \( \bar{\nu} \) cuts \( J^+ (\Sigma^-, \bar{g}; \bar{M}) \).

Suppose first that \( \bar{\nu} \) is past endless in \((\bar{M}, \bar{g})\). The time reverse of Lemma 3.6 of [8] gives that \( I^+ (\Sigma^-, \bar{g}; \bar{M}) \) cannot contain all of \( \bar{M} \setminus \partial \bar{M} \), whilst the time reverse of Lemma 4.2 of [8] gives \( \bar{M} \setminus \partial \bar{M} \subset I^+ (\bar{\nu}, \bar{g}; \bar{M}) \). Hence \( \bar{\nu} \) cuts \( \bar{M} \setminus I^+ (\Sigma^-, \bar{g}; \bar{M}) \).

Since \( \bar{\nu} \) is past endless and timelike in \((\bar{M}, \bar{g})\) it follows that \( \bar{\nu} \) cuts \( \bar{M} \setminus I^+ (\Sigma^-, \bar{g}; \bar{M}) \). Because \( \bar{\nu} \) has a future endpoint at \( q \in \Sigma^+_2 \subset \mathcal{K}^+ \subset I^+ (\Sigma^-, \bar{g}; \bar{M}) \) one thus has that \( \bar{\nu} \) cuts both \( I^+ (\Sigma^-, \bar{g}; \bar{M}) \subset J^+ (\Sigma^-, \bar{g}; \bar{M}) \) and \( \bar{M} \setminus I^+ (\Sigma^-, \bar{g}; \bar{M}) \) and so cuts \( J^+ (\Sigma^-, \bar{g}; \bar{M}) \).
Now suppose that \( \tilde{\nu} \) has a past endpoint \( \tilde{z} \) in \((\bar{M}, \bar{g})\). The point \( \tilde{z} \) must lie in the topological boundary of the open set \( \psi'(U) \cup \partial M \) in \( M \) otherwise it would lie in \( \psi'(U) \cup \partial M \), in which case \( \xi^{-1}(\tilde{z}) \in \psi'(U) \cup \partial M \) would be a past endpoint to \( \nu \) in \((M, g)\) and \( \nu \) would be past extendible in \( \psi'(U) \cup \partial M \). Because \( \psi'(U) \cup \partial M \) is open in \( M \) the set \( J^{-}(\bar{\Sigma}^{+}, \bar{g}; M) \cap J^{+}(\bar{\Sigma}^{-}, \bar{g}; M) \subseteq \psi'(U) \cup \partial M \) cannot contain \( \tilde{z} \). Since \( J^{-}(\bar{\Sigma}^{+}, \bar{g}; M) \cap J^{+}(\bar{\Sigma}^{-}, \bar{g}; M) \) is closed in \( \bar{M} \) the set \( \bar{M} \setminus (J^{-}(\bar{\Sigma}^{+}, \bar{g}; M) \cap J^{+}(\bar{\Sigma}^{-}, \bar{g}; M)) \) is an open neighbourhood of \( \tilde{z} \) in \( \bar{M} \) and so is cut by \( \tilde{\nu} \). In view of \( |\tilde{\nu}| \subset J^{-}(\bar{\Sigma}^{+}_{2}, \bar{g}; M) \subset J^{-}(\bar{\Sigma}^{+}, \bar{g}; M) \) one thus has that \( \tilde{\nu} \) cuts \( \bar{M} \setminus J^{+}(\bar{\Sigma}^{-}, \bar{g}; M) \).

Since \( \tilde{\nu} \) has a future endpoint at \( \tilde{q} \in \tilde{\Sigma}^{+}_{2} \subset \tilde{\mathcal{I}}^{+} \subset \bar{I}^{+}(\bar{\Sigma}^{-}, \bar{g}; M) \) it follows that \( \tilde{\nu} \) cuts both \( I^{+}(\bar{\Sigma}^{-}, \bar{g}; M) \) and \( M \setminus J^{+}(\bar{\Sigma}^{-}, \bar{g}; M) \) and so cuts \( \tilde{\mathcal{I}}^{+}(\bar{\Sigma}^{-}, \bar{g}; M) \).

Since \( \tilde{\nu} \coloneqq \xi \circ \nu \) cuts \( \tilde{I}^{+}(\bar{\Sigma}^{-}, \bar{g}; M) \) and therefore cuts \( \tilde{\mathcal{I}} \coloneqq \xi(\mathcal{I}) \) one has that \( \nu \) cuts \( \mathcal{I} \). Hence \( \sigma \) cuts \( \mathcal{I} \). \( \square \)

The final lemma will require the use of the following result.

**Proposition 3.5.** Let \((\bar{M}, \bar{g}, \bar{\Omega}, \psi)\) be a WASE asymptote of a WASE space-time \((M, g)\) and let \( \mathcal{I} \) be a closed edgeless achronal set of \((M, g)\). If \((\bar{M}, \bar{g}, \bar{\Omega}, \psi)\) is partially future asymptotically predictable from \( \psi(\mathcal{I}) \) then \( \psi(\mathcal{I}) \) is closed edgeless and achronal in \((\bar{M}, \bar{g})\).

**Proof.** Let \((M', g')\) be the associated ASE space-time and \((\bar{M}, \bar{g}, \bar{\Omega}, \psi')\) an ASE asymptote of \((M', g')\).

If \( \lambda \) was a timelike curve of \((\bar{M}, \bar{g})\) from \( \psi(\mathcal{I}) \subset \bar{M} \setminus \partial \bar{M} \) to \( \psi(\mathcal{I}) \subset \bar{M} \setminus \partial \bar{M} \) then, because \( \partial \bar{M} \) is a null hypersurface of \((\bar{M}, \bar{g})\), one would have \( |\lambda| \subset \bar{M} \setminus \partial \bar{M} \). So there would exist a timelike curve \( \lambda \) of \((M, g)\) such that \( \lambda = \psi \circ \lambda \). But then \( \lambda \) would be a timelike curve of \((M, g)\) from \( \mathcal{I} \) to \( \mathcal{I} \). This would contradict the achronality of \( \mathcal{I} \) in \((M, g)\). Hence \( \psi(\mathcal{I}) \) is achronal in \((\bar{M}, \bar{g})\).

Suppose there exists \( s \in \psi(\mathcal{I}) \cap \mathcal{I}^{+} \). Since \((\bar{M}, \bar{g}, \bar{\Omega}, \psi)\) is partially future asymptotically predictable from \( \psi(\mathcal{I}) \) there exists a slice \( \Sigma^{+} \) of \( \mathcal{I}^{+} \) such that \( J^{-}(\Sigma^{+}, \bar{g}; M) \cap J^{+}(\Sigma^{+}, \bar{g}; M) \subset D^{+}(\psi(\mathcal{I}), \bar{g}; M) \). In the case \( s \in J^{-}(\Sigma^{+}, \bar{g}; M) \cap J^{+}(\Sigma^{+}, \bar{g}; M) \), every past end time timelike curve of \((\bar{M}, \bar{g})\) to \( s \) would cut \( \psi(\mathcal{I}) \), and so \( I^{+}(\psi(\mathcal{I}), \bar{g}; M) \) would be an open neighbourhood of \( s \in \psi(\mathcal{I}) \) and so would intersect \( \psi(\mathcal{I}) \). This is impossible since \( \psi(\mathcal{I}) \) is achronal in \((\bar{M}, \bar{g})\). So suppose \( s \in J^{+}(\psi(\mathcal{I}), \bar{g}; M) \setminus \Sigma^{+} \). There exists \( r \in \Sigma^{+} \) lying strictly to the past of \( s \) on the null geodesic generator of \( \mathcal{I}^{+} \) through \( s \). In view of \( r \in \tilde{D}^{+}(\psi(\mathcal{I}), \bar{g}; M) \), every past end time timelike curve of \((\bar{M}, \bar{g})\) to \( r \) must cut \( \psi(\mathcal{I}) \). But then \( I^{+}(\psi(\mathcal{I}), \bar{g}; M) \) is an open neighbourhood in \( \bar{M} \) of \( r \in J^{-}(s, \bar{g}; M) \setminus \{s\} \) and therefore of \( s \in \mathcal{I}^{+} \cap \psi(\mathcal{I}) \) and so must intersect \( \psi(\mathcal{I}) \). So again one has a contradiction to the achronality of \( \psi(\mathcal{I}) \). One thus has \( \psi(\mathcal{I}) \cap \mathcal{I}^{+} = \emptyset \).

Suppose there exists \( p \in \psi(\mathcal{I}) \cap \mathcal{I}^{-} \). Let \( \Sigma^{-} \) be a slice of \( \mathcal{I}^{-} \) such that \( p \in \Sigma^{-} \). By Lemma 3.3 there exists a slice \( \Sigma^{+} \) of \( \mathcal{I}^{+} \) such that \( J^{-}(\Sigma^{+}, \bar{g}; M) \cap J^{+}(\Sigma^{-}, \bar{g}; M) \subset \psi'(\mathcal{I}) \cup \partial M \) for \( \Sigma^{-} \coloneqq \xi(\Sigma^{-}) \) and \( \Sigma^{+} \coloneqq \xi(\Sigma^{+}) \). Since \((\bar{M}, \bar{g}, \bar{\Omega}, \psi)\) is partially future asymptotically predictable from \( \psi(\mathcal{I}) \) one may assume that \( \Sigma^{+} \) is taken sufficiently far to the past in \( \mathcal{I}^{+} \) to give \( \Sigma^{+} \subset \tilde{D}^{+}(\psi(\mathcal{I}), \bar{g}; M) \). Let \( \tilde{p} \coloneqq \xi(p) \in \Sigma^{-} \). By Lemma 7.2 of [3] one has \( \tilde{p} \in \mathcal{I}^{-} \subset \mathcal{I}^{-}(\Sigma^{+}, \bar{g}; M) \). Hence there is a timelike curve \( \tilde{\alpha} \) of \((\bar{M}, \bar{g})\) from \( \tilde{p} \in \Sigma^{-} \) to some point \( \tilde{q} \in \Sigma^{+} \). In view of \( |\tilde{\alpha}| \subset J^{-}(\Sigma^{+}, \bar{g}; M) \cap J^{+}(\Sigma^{-}, \bar{g}; M) \subset \psi'(\mathcal{I}) \cup \partial M \) one has that \( \alpha \coloneqq \xi^{-1} \circ \tilde{\alpha} \) \( \square \)
is a timelike curve of $(\bar{M}, \bar{g})$ from $p \in \psi(\mathcal{I}) \cap \mathcal{I}^-$ to $q := \xi^{-1}(\bar{q}) \in \Sigma^+$. The set $I^+(p, \bar{g}; \bar{M})$ cannot intersect $\psi(\mathcal{I})$ otherwise $I^-(\psi(\mathcal{I}), \bar{g}; \bar{M})$ would be an open neighbourhood of $p \in \psi(\mathcal{I})$ and so $\psi(\mathcal{I})$ would not be achronal in $(\bar{M}, \bar{g})$. Since the past endless null geodesic generating segment of $\mathcal{I}^-$ to $p$ clearly does not cut $\psi(\mathcal{I})$ it follows that $I^+(p, \bar{g}; \bar{M})$ does not intersect $D^+(\psi(\mathcal{I}), \bar{g}; \bar{M})$. But $I^+(p, \bar{g}; \bar{M})$ is a neighbourhood of $q \in \Sigma^+ \subset \bar{D}^+(\psi(\mathcal{I}), \bar{g}; \bar{M})$ in $\bar{M}$ so one has a contradiction.

One now has $\bar{\psi}(\mathcal{I}) \cap \partial \bar{M} = \emptyset$. Since $\psi : M \to \bar{M}$ is a diffeomorphism onto its image it follows that $\psi(\mathcal{I})$ is relatively closed in $\bar{\psi}(M) = \bar{M} \setminus \partial \bar{M}$. Hence $\bar{\psi}(\mathcal{I})$ is closed in $\bar{M}$.

One has $\text{edge}(\psi(\mathcal{I}), \bar{g}; \bar{M}) \cap \partial \bar{M} = \emptyset$ because $\psi(\mathcal{I})$ is closed in $\bar{M}$ and does not intersect $\partial \bar{M}$. And, because $\psi : (M, \bar{g}) \to (\bar{M}, \bar{g})$ is a conformal isometry onto its image, one has $\text{edge}(\psi(\mathcal{I}), \bar{g}; \bar{M}) \cap \psi(M) = \psi(\text{edge}(\mathcal{I}, \bar{g}; \bar{M})) = \emptyset$. Hence $\text{edge}(\psi(\mathcal{I}), \bar{g}; \bar{M})$ is empty. 

\textbf{Lemma 3.6.} Let $(M, \bar{g})$ be a WASE space-time and let $(M, \bar{g}, \Omega, \psi)$ be a WASE asymptote of $(M, \bar{g})$. Suppose

1) there exists a closed edgeless achronal set $\mathcal{I}$ in $(M, \bar{g})$ such that $(M, \bar{g}, \Omega, \psi)$ is partially future asymptotically predictable from $\psi(\mathcal{I})$;

2) $(M, \bar{g}, \Omega, \psi)$ is asymptotically chronologically consistent.

Then for any compact set $\mathcal{K} \subset \bar{I}^+(\mathcal{I}, \bar{g}; \bar{M})$ of $M$ there exists a slice $\Sigma^+_3$ of $\mathcal{I}^+$ such that $\bar{I}^+(\psi(\mathcal{I}), \bar{g}; \bar{M}) \cap \mathcal{I}^+ \subset J^+(\Sigma^+_3, \bar{g}; \bar{M})$.

\textbf{Proof.} It suffices to assume $\mathcal{K} \cap \mathcal{W} = \emptyset$ since one may otherwise redefine $\mathcal{W}$ as $\mathcal{W} \setminus \mathcal{K}$.

Let $\Sigma^+_0$ be a slice of $\mathcal{I}^-$ and let $\Sigma^-_0$ be a slice of $\mathcal{I}^-$ lying strictly to the past of $\Sigma^+_0$ along the null geodesic generators of $\mathcal{I}^-$. Let $\Sigma^-_0 := \xi(\Sigma^+_0)$ and $\Sigma^- := \xi(\Sigma^-)$. By Lemma 3.3 there exists a slice $\Sigma^+_0$ of $\mathcal{I}^+$ such that $J^+(\Sigma^+_0, \bar{g}; \bar{M}) \cap J^-((\Sigma^+_0, \bar{g}; \bar{M}) \subset \psi(\mathcal{I}) \cup \partial \bar{M}$ and $J^+(\Sigma^-_0, \bar{g}; \bar{M}) \cap J^-((\Sigma^-_0, \bar{g}; \bar{M}) \subset \psi(\mathcal{I}) \cup \partial \bar{M}$ for $\Sigma^+_0 := \xi(\Sigma^+_0)$. Lemma 4.12 of [8] gives $J^+(\Sigma^-_0, \bar{g}; \bar{M}) \cap \mathcal{I}^+ = \Sigma^-$ which, since $\Sigma^+_0$ lies strictly to the future of $\Sigma^-_0$ along the null geodesic generators of $\mathcal{I}^-$, implies that $J^+(\Sigma^-_0, \bar{g}; \bar{M})$ is a neighbourhood of $\Sigma^+_0$ in $\bar{M}$. Proposition 7.2 of [8] gives $\Sigma^-_0 \subset \mathcal{I}^- \subset I^-((\Sigma^+_0, \bar{g}; \bar{M})$ whereby one has that $J^-((\Sigma^+_0, \bar{g}; \bar{M}) \subset I^-((\Sigma^+_0, \bar{g}; \bar{M})$ is a neighbourhood of $\Sigma^+_0$ in $\bar{M}$. Thus there exists an open neighbourhood $\mathcal{N} \subset J^+(\Sigma^-_0, \bar{g}; \bar{M}) \cap J^-((\Sigma^+_0, \bar{g}; \bar{M}) \subset \bar{M}$ in $\bar{M}$. The set $\mathcal{N} := \xi^{-1}(\mathcal{N}) \subset \mathcal{N} \subset J^+(\Sigma^-_0, \bar{g}; \bar{M}) \cap J^-((\Sigma^+_0, \bar{g}; \bar{M})$ is an open neighbourhood of $\Sigma^-_0$ in $\bar{M}$. In view of Proposition 3.3 one may, by passing to a subset of $\mathcal{N}$ if necessary, assume $\mathcal{N} \cap (\psi(\mathcal{I}) \cup \mathcal{I}^+) = \emptyset$. By passing to a further subset of $\mathcal{N}$ if necessary, one may arrange that each point of $\mathcal{N} \setminus \mathcal{I}^+$ is a future endpoint of a timelike curve of $(\bar{M}, \bar{g})$ in $\mathcal{N}$ from $\mathcal{I}^- \cap \mathcal{N}$. Lemma 1.4 gives that there exists a slice $\Sigma^+_2$ of $\mathcal{I}^+$ such that every past endless timelike curve of $(\bar{M}, \bar{g})$ to $\Sigma^+_2$ cuts $\mathcal{N}$. One may assume that $\Sigma^+_2$ lies strictly to the past of $\Sigma^+_0$ along the null geodesic generators of $\mathcal{I}^+$. Since $(\bar{M}, \bar{g}, \Omega, \psi)$ is partially future asymptotically predictable from $\psi(\mathcal{I})$ there exists a slice $\Sigma^+_3$ of $\mathcal{I}^+$ lying strictly to the past of $\Sigma^+_2$ along the generators of $\mathcal{I}^+$ such that $\Sigma^+_3 \subset D^+(\psi(\mathcal{I}), \bar{g}; \bar{M})$.

Suppose there exists a timelike curve $\alpha : [0, 1] \to \bar{M}$ of $(\bar{M}, \bar{g})$ from $\psi(\mathcal{I}) \subset \bar{M} \setminus (\psi(\mathcal{I}) \cup \partial \bar{M})$ to $\Sigma^+_3$. In view of $J^+(\Sigma^-_0, \bar{g}; \bar{M}) \cap J^-((\Sigma^+_3, \bar{g}; \bar{M}) \subset \psi(\mathcal{I}) \cup \partial \bar{M}$ the
set $I^-(\alpha(0), \bar{g}; \bar{M})$ cannot intersect $J^+(\Sigma^+, \bar{g}; \bar{M})$. So, because every past endless timelike curve of $(\bar{M}, \bar{g})$ to $\Sigma^+ \subset J^+(\Sigma^-, \bar{g}; \bar{M})$, there exists $\alpha \in (0, 1)$ such that $\alpha(\alpha) \in \mathcal{N} \cap \mathcal{J}^-$. By the construction of $\mathcal{N}$ there exists $\tilde{x} \in I^-(\alpha, \bar{g}; \mathcal{N}) \cap \mathcal{J}^-$. If $\alpha(0, 1)$ did not cut $\psi(\mathcal{J})$ one could concatenate the past endless null geodesic generating segment of $\mathcal{J}^- \subset \bar{M} \setminus \psi(\mathcal{J})$ to $\tilde{x}$, a timelike curve in $\mathcal{N} \subset \bar{M} \setminus \psi(\mathcal{J})$ from $\tilde{x}$ to $\alpha(a)$, and the segment $\alpha[0, 1]$ of $\alpha$ from $\alpha(0)$ to $\alpha(1)$ to obtain a past endless causal curve $\beta$ of $(\bar{M}, \bar{g})$ to $\alpha(1) \in \Sigma^+_3$ which did not cut $\psi(\mathcal{J})$. For an open neighbourhood $\mathcal{O}_{\alpha(1)} \subset \bar{M} \setminus \psi(\mathcal{J})$ of $\alpha(1) \in \bar{M} \setminus \psi(\mathcal{J})$ in $\bar{M}$ there would exist $\tilde{c} \in (\alpha, 1)$ such that $\alpha(c) \in \mathcal{O}_{\alpha(1)}$. But then $I^+(\alpha(c), \bar{g}; \mathcal{O}_{\alpha(0)})$ would be an open neighbourhood of $\alpha(1) \in \Sigma^+_3$ in $\bar{M}$ not intersecting $D^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$, which gives a contradiction. Thus $\alpha[0, 1]$ must cut $\psi(\mathcal{J})$ and indeed there must exist $b \in (\alpha, 1)$ such that $\alpha(b) \in \psi(\mathcal{J})$. One now has that $\alpha[0, b]$ is a timelike curve of $(\bar{M}, \bar{g})$ from $\alpha(0) \in \psi(\mathcal{J})$ to $\alpha(b) \in \psi(\mathcal{J})$. Hence $I^-(\psi(\mathcal{J}), \bar{g}; \bar{M})$ intersects $\psi(\mathcal{J}) \subset I^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$ and so $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$ intersects the causal boundary of $\psi(\mathcal{J})$ in $(\bar{M}, \bar{g})$. Hence there can be no timelike curve of $(\bar{M}, \bar{g})$ from $\psi(\mathcal{J})$ to $\Sigma^+_3$. One thus has $\psi(\mathcal{J}) \cap I^-(\Sigma^+_3, \bar{g}; \bar{M}) = \emptyset$.

Suppose there exists $\tilde{y} \in I^+(\psi(\mathcal{J}), \bar{g}; \bar{M}) \setminus J^+(\Sigma^+_3, \bar{g}; \bar{M})$. Since $\mathcal{J}^+ \setminus J^+(\Sigma^+_3, \bar{g}; \bar{M})$ is relatively open in $\mathcal{J}^+$ one can construct an open neighbourhood $\mathcal{O}_{\bar{y}}$ of $\tilde{y}$ in $\bar{M}$ such that every point of $\mathcal{O}_{\bar{y}} \setminus \mathcal{J}^+$ is a past endpoint of a timelike curve of $(\bar{M}, \bar{g})$ in $\mathcal{O}_{\bar{y}}$ to $\mathcal{J}^+ \setminus J^+(\Sigma^+_3, \bar{g}; \bar{M}) \subset J^+(\Sigma^+_3, \bar{g}; \bar{M})$. Then $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$ intersects $\mathcal{O}_{\bar{y}} \setminus \mathcal{J}^+ \subset I^-(\Sigma^+_3, \bar{g}; \bar{M})$ and so $I^-(\Sigma^+_3, \bar{g}; \bar{M})$ intersects $\psi(\mathcal{J})$, which is impossible. Hence $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$ does not intersect $\mathcal{J}^+ \setminus J^+(\Sigma^+_3, \bar{g}; \bar{M})$. There follows $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M}) \cap \mathcal{J}^+ \subset J^+(\Sigma^+_3, \bar{g}; \bar{M})$.

It is now possible to give the proof of Theorem 3.2.

**Proof of Theorem 3.2.** One may, by passing to a subset of $\mathcal{U}$ if necessary, assume $\mathcal{U} \cap \mathcal{J} = \emptyset$.

Suppose, for the purpose of obtaining a contradiction, that $\psi(\mathcal{J}) \cap J^-(\mathcal{J}^+, \bar{g}; \bar{M})$ is non-empty. Then $J^-(\psi(\mathcal{J}), \bar{g}; \bar{M}) \cap \mathcal{J}^+$ is non-empty and so is $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M}) \cap \mathcal{J}^+$. By Lemma 3.4 there exists a slice $\Sigma^+$ of $\mathcal{J}^+$ such that $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M}) \cap \mathcal{J}^+ \subset J^+(\Sigma^+, \bar{g}; \bar{M})$. Since $J^+(\Sigma^+, \bar{g}; \bar{M})$ is a non-empty proper subset of $\mathcal{J}^+$ it follows that $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M}) \cap \mathcal{J}^+$ is a non-empty proper subset of $\mathcal{J}^+$. Hence there exists $\tilde{q} \in I^+(\psi(\mathcal{J}), \bar{g}; \bar{M}) \cap \mathcal{J}^+ \subset J^+(\Sigma^+, \bar{g}; \bar{M})$. There exists a null geodesic generator $\tilde{\gamma}$ of $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$ to $\tilde{q}$ having either a past endpoint in $\psi(\mathcal{J})$ or no past endpoint in $\bar{M}$. In the former case $\tilde{\gamma}$ could not be a null geodesic generating segment of $\mathcal{J}^+$ because it would have a past endpoint in $\psi(\mathcal{J})$ or no past endpoint in $\bar{M}$. In the latter case $\tilde{\gamma}$ could not be a null geodesic generating segment of $\mathcal{J}^+$ because it would then cut $\mathcal{J}^+ \setminus J^+(\Sigma^+, \bar{g}; \bar{M})$. Hence one has $|\tilde{\gamma}| \setminus \{\tilde{q}\} \subset \bar{M} \cap \partial \bar{M}$.

Suppose $\tilde{\gamma}$ were past endless in $(\bar{M}, \bar{g})$. Then $\tilde{\gamma}$ would be a past endless causal curve of $(\bar{M}, \bar{g})$ in $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$. Let $\tilde{r} \in \mathcal{J}^+$ lie strictly to the future of $\tilde{q}$ along the null geodesic generator of $\mathcal{J}^+$ though $\tilde{q}$. Then one could deform $\tilde{\gamma}$ to the future in $(\bar{M}, \bar{g})$ so as to give a past endless timelike curve $\tilde{\gamma}_+ \subset (\bar{M}, \bar{g})$ to $\tilde{r}$ in $I^+(\psi(\mathcal{J}), \bar{g}; \bar{M}) \subset I^+(\psi(\mathcal{J}), \bar{g}; \bar{M}) \subset I^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$. Clearly $\tilde{\gamma}_+$ could not intersect $\psi(\mathcal{J})$ because $\psi(\mathcal{J})$ is achronal in $(\bar{M}, \bar{g})$. Since $\tilde{\gamma}_+$ is timelike curve of $(\bar{M}, \bar{g})$ to $\tilde{r} \in \bar{M} \setminus \psi(\mathcal{J})$ in $(\bar{M}, \bar{g})$ it follows that there would exist a neighbourhood of $\tilde{r}$ in $\bar{M}$ that did not intersect $D^+(\psi(\mathcal{J}), \bar{g}; \bar{M})$. This would be contrary to the future asymptotic predictability of $(\bar{M}, \bar{g}, \Omega, \psi)$ from $\psi(\mathcal{J})$. Thus $\tilde{\gamma}$ must have a
past endpoint in \((\tilde{M}, \tilde{g})\) at \(\psi(\mathcal{T})\). Consequently there exists a null geodesic \(\gamma\) of 
\((M, g)\) such that \(\psi \circ \gamma\) is the unique maximal segment of \(\tilde{\gamma}\) in \(\tilde{M} \setminus \partial \tilde{M}\).

For each \(p \in [\gamma]\), every open neighbourhood of \(\tilde{p} := \psi(p)\) in \(\tilde{M}\) intersects both \(I^+(\psi(\mathcal{T}), \mathcal{T}, g; M)\) \(\setminus \partial M\) = \(\psi(I^+(\mathcal{T}, \mathcal{T}, g; M))\) and \(M \setminus I^+(\psi(\mathcal{T}), \mathcal{T}, g; M) = \psi(M \setminus I^+(\mathcal{T}, \mathcal{T}, g; M))\). One thus has \(|\gamma| \subset I^+(\mathcal{T}, \mathcal{T}, g; M)\) and hence that \(\gamma\) is a null geodesic generator of \(I^+(\mathcal{T}, \mathcal{T}, g; M)\). Since \(\tilde{\gamma}\) has a past endpoint at \(\psi(\mathcal{T})\) in \((\tilde{M}, \tilde{g})\) it follows that \(\gamma\) has a past endpoint at \(\mathcal{T}\) in \((M, g)\). Since \(\gamma\) has a future endpoint at \(\tilde{q} \in \mathcal{T}^+\) in \((\tilde{M}, \tilde{g})\) it follows that \(\gamma\) is future-ended and future complete in \((M, g)\). One may assume that \(\gamma\) is an affine future-directed null geodesic of \((M, g)\) of the form \(\gamma : [0, \infty) \to M\).

Let \(k\) and \(l\) be normal fields to \(\mathcal{T}\) along a relative open neighbourhood \(\mathcal{V}_{\gamma(0)}\) of \(\gamma(0)\) in \(\mathcal{T}\), normalised such that \(g_{ab}k^a l^b = -1\), with \(k(\gamma(0)) = \gamma(0) \in T_{\gamma(0)}M\).

The induced metric on \(\mathcal{T}\) is given by \(h_{ab} = g_{ab} + 2k(a)l(b)\), whilst \((1) \chi^a_a := h^a_c h^b_d k_{cd}\) and \((2) \chi^a_b := h_a c h^b_c d_{cd}\) are null second fundamental forms of \(\mathcal{T}\) along \(\mathcal{V}_{\gamma(0)} \subset \mathcal{T}\).

By the definition of a closed trapped surface one has \((1) \chi^a_a < 0\) and \((2) \chi^a_b < 0\) along \(\mathcal{V}_{\gamma(0)}\). The vector field \(k\) along \(\mathcal{V}_{\gamma(0)} \subset \mathcal{T}\) defines a congruence of future-ended affine null geodesics of \((M, g)\) from \(\mathcal{T}\) with tangents that coincide with \(k\) along \(\mathcal{V}_{\gamma(0)} \subset \mathcal{T}\). Let \(l\) also denote the tangents to these null geodesics. Each tangent vector to \(\mathcal{T}\) at \(\gamma(0)\) may be Lie propagated along \(\gamma\) with respect to \(k\) to yield a vector field \(Z\) along \(\gamma\). From vanishing torsion one has \(Z^a b k^b = k^a b Z^b\) and hence that \(Z\) satisfies the defining equation \((Z^a b k^b)_c k^c = R^a_{bcd} k^b Z^d\) for a Jacobi field along \(\gamma\). Note that \(Z\) is orthogonal to \(k\) at \(\gamma(0)\) and satisfies \((k_a Z^a)_b k^b = 0\) along \(\gamma\), and so is orthogonal to \(k\) along \(\gamma\).

One may parallelly propagate the vector \(l\) along the integral curves of \(k\) and so define \(h_{ab} = g_{ab} + 2k(a)l(b)\) along these curves. Then \(h^a_b\) is a projective operator such that \(h^a_{bc} k^c = 0\). One thus has that \(Z^a b = h^a_b Z^b\) satisfies \(Z^a b k^b = h^a_b k^c = (1) \chi^a_b Z^b\) for \((1) \chi^a_b := h^a_c h^b_d k_{cd}\) now defined all along \(\gamma\). One also has \((Z^a b k^b)_c k^c = h^a_b R^b_{ced} k^d Z^e\).

The expansion and shear tensors of the vector fields \(Z\) along \(\gamma\) may be expressed as \(\vartheta_{ab} = h^a_c h^b_d k_{cd}\) and \(\varsigma_{ab} = \vartheta_{ab} - \frac{1}{2} \vartheta h_{ab}\) respectively, where \(\vartheta := h^{ab} \vartheta_{ab}\) is the scalar expansion. Then, defining \(\varsigma^2 := 2 \varsigma_{ab} \varsigma^{ab}\), one has that \(\vartheta(\lambda)\) satisfies the Raychaudhuri equation

\[
\frac{d}{d\lambda} \vartheta(\lambda) = -R_{ab} k^a k^b - 2\varsigma^2(\lambda) - \frac{1}{2} \vartheta^2(\lambda)
\]

and is subject to the initial condition

\[
\vartheta(0) = (\chi^a_a(\gamma(0))) < 0
\]

By means of condition (3) of Theorem 1.2 one thus has

\[
\frac{d}{d\lambda} \left( \frac{2}{-\vartheta(\lambda)} \right) \leq -1
\]

for all \(\lambda \in [0, \infty)\) such that \(\vartheta(\lambda) \neq 0\). Hence there exists \(\lambda_0 \in (0, 2/(\varsigma^2(\gamma(0))))\) such that \(\lim_{\lambda \to \lambda_0} \vartheta(\lambda) = -\infty\). Thus \(\gamma(\lambda)\) is conjugate to \(\mathcal{T}\) at \(\lambda = \lambda_0\) and so there exists a Jacobi field \(Z\) along \(\gamma\) which is non-zero and tangent to \(\mathcal{T}\) at \(\gamma(0) \in \mathcal{T}\) and such that \(Z(\lambda_0) = 0\). One may, by an adaptation of the technique of H&E Proposition 4.5.12, use the vector field \(Z\) to construct a timelike curve of \((M, g)\) from \(\mathcal{T}\) to \(\gamma(\lambda)\) for any \(\lambda > \lambda_0\). But this is impossible because \(\gamma\) is a generator of...
\( \hat{I}^+(\mathcal{T}, g; M) \). This establishes the required contradiction. 

This then is the revised proof of the familiar assertion that, subject to the null convergence condition and weak cosmic censorship, closed trapped surfaces are not visible from \( \mathcal{T}^+ \). The key idea of the H&E argument in support of this assertion was evidently sound, but additional constraints and analysis have been seen to be necessary to make the detailed theory of WASE space-times match intuitive expectations.

One could, as discussed in \( \S 2 \), consider weakening the assumed strong causality of the reference ASE asymptote \( (\hat{M}, \hat{g}, \Omega', \psi') \) to a the chronology condition on the underlying ASE space-time \( (M', g') \). This would lead to a more general definition of a WASE space-time. It would be of some interest to find whether or not Theorem 3.2 remains true in this more general setting.

4. Concluding remarks

The definition of a WASE space-time, proposed in \( \S 2 \) as the foundation for certain types of cosmic censorship theorems, has been seen also to provide the basis for a rigorous proof of standard relativity folklore concerning the invisibility of closed trapped surfaces from \( \mathcal{T}^+ \). Other approaches to the definition of WASE space-time have been attempted \( \S 3, \S 4 \), but a comparison will not be attempted here.

Whilst this paper was in preparation, Chruściel et al. \( [11] \) have been reconsidering another piece of relativity folklore, namely the area theorem for black holes. This states that, subject to weak cosmic censorship and an energy condition, the area of a black hole cannot decrease. A formalised statement to this effect appears as Proposition 9.2.7 in H&E, but the proof there is flawed because it is based on unsubstantiated assumptions concerning the smoothness of the event horizon. A proof is provided in \( [11] \) of a weaker area theorem, but with full attention to matters of differentiability. A central hypothesis is one of \( \mathcal{H} \)-regularity which, in present terminology, requires that there exists a neighbourhood \( \mathcal{O} \) of the event horizon \( \mathcal{H} := \hat{J}^- (\mathcal{T}^+, g; \hat{M}) \) in \( \hat{M} \) such that, for any compact set \( \mathcal{C} \subset \mathcal{O} \) which intersects \( I^- (\mathcal{T}^+, g; \hat{M}) \), there is a null geodesic generator of \( \mathcal{T}^+ \) which cuts both \( \hat{I}^+(\mathcal{C}, g; \hat{M}) \) and \( \hat{M} \setminus \hat{I}^+(\mathcal{C}, g; \hat{M}) \). The authors consider various ways to derive \( \mathcal{H} \)-regularity from seemingly more natural hypotheses. However, Lemma 3.6 of the present paper shows that for space-times which are WASE in the sense of \( \S 2 \) and asymptotically chronologically consistent in the sense of Definition 2.8, \( \mathcal{H} \)-regularity is in fact a consequence of partial future asymptotic predictability.

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MATHEMATICAL INSTITUTE, UNIVERSITY OF UMEÅ, S-901 87 UMEÅ, SWEDEN.