Online Anomaly Detection Systems
Using Incremental Commute Time

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Abstract—Commute Time Distance (CTD) is a random walk based metric on graphs. CTD has found widespread applications in many domains including personalized search, collaborative filtering and making search engines robust against manipulation. Our interest is inspired by the use of CTD as a metric for anomaly detection. It has been shown that CTD can be used to simultaneously identify both global and local anomalies. Here we propose an accurate and efficient approximation for computing the CTD in an incremental fashion in order to facilitate real-time applications. An online anomaly detection algorithm is designed where the CTD of each new arriving data point to any point in the current graph can be estimated in constant time ensuring a real-time response. Moreover, the proposed approach can also be applied in many other applications that utilize commute time distance.

Keywords—commute time distance; incremental commute time; random walk; anomaly detection;

I. INTRODUCTION

Commute Time Distance (CTD) is a random walk based metric on graphs. The $CTD(i,j)$ between two nodes $i$ and $j$ is the expected number of steps a random walk starting at $i$ will take to reach $j$ for the first time and then return back to $i$. The fact that CTD is averaged over all paths (and not just the shortest path) makes it more robust to data perturbations.

CTD has found widespread applications in personalized search [1], collaborative filtering [2], [3] and making search engines robust against manipulation [4]. Our interest is inspired by the use of CTD as a metric for anomaly detection. It has been shown that CTD can be used to simultaneously identify global, local and even collective anomalies in data [5].

More advanced measures generally require more expensive computation. Estimating CTD involves the eigen decomposition of the graph Laplacian matrix and consequently has $O(n^3)$ time complexity which is impractical for large graphs. Saerens, Pirote and Fouss [6] used subspace approximation, and Khoa and Chawla [5] used graph sampling to reduce the complexity. Sarkar and Moore [7] introduced a notion of truncated commute time and a pruning algorithm to find nearest neighbors in commute time. They empirically demonstrated achieving a near-linear running time as a function of graph size. Spielman and Srivastava [8] have proposed a near-linear time algorithm for approximating pairwise CTD in $O(\log n)$ time based on random projections.

However, there are many applications in practice which require the computation of CTD in an online fashion. When a new data point arrives, the application needs to respond quickly without recomputing everything from scratch. The algorithms noted above all work in a batch fashion and have a high computation cost for online applications.

We are interested in the following scenario: a dataset $D$ is given from an underlying domain of interest. For example, data from a network traffic log or environment or climate change monitoring. A new data point arrives and we want to determine if it is an anomaly with respect to $D$ in CTD. Intuitively a new data point is an anomaly if it is far away from its neighbors in CTD.

Example 1: Consider the two graphs shown in Figure 1. While the shortest path distance between node 1 and 2 is the same in both graphs, $CTD(1,2)$ increases after node 5 is added. This property of CTD can be used to great effect to detect both global and local outliers. However, the same property makes it challenging to calculate the CTD in an incremental manner.

![Figure 1: $CTD(1,2)$ increases after addition of node 5 even though shortest path distance remains unchanged. This property of CTD has many applications including anomaly detection.](image-url)

Here we propose and compare two methods for computing CTD in an incremental fashion. The first method is based on an incremental update of the eigen decomposition of a Laplacian matrix. The second method uses the recursive
Definition of CTD based on hitting time. To the best of our knowledge both these methods are novel and their comparison provides revealing insights about CTD which are independent of the application domain.

The contributions of this paper are as follows:

- We make use of the characteristics of random walk to estimate CTD incrementally in constant time. The same approach could be used for estimating the hitting time incrementally.
- We propose a provably fast method to incrementally update the eigenvalues and eigenvectors of the graph Laplacian matrix. This method can be integrated with any technique requiring a graph spectral computation, such as spectral clustering.
- We design an online algorithm for anomaly detection using incremental CTD. The technique is verified by experiments in synthetic and real datasets. The experiments show the effectiveness of the proposed methods in terms of accuracy and performance.

The remainder of the paper is organized as follows. Section II reviews notation and concepts related to random walk and CTD, and a simple example to tie up all the definitions and ideas. In Sections III and IV we present two methods to incrementally approximate the CTD. In Section V we introduce an online anomaly detection algorithm which uses incremental CTD. In Section VI we evaluate our approach using experiments on synthetic and real datasets. Sections VII covers related work. We conclude in Section VIII with a summary and a direction for future research.

II. COMMUTE TIME DISTANCE

We provide a self-contained introduction to random walks with an emphasis on CTD. Assume we are given a connected undirected and weighted graph \( G = (V, E, W) \).

**Definition 1:** Let \( i \) be a node in \( G \) and \( N(i) \) be its neighbors. The degree \( d_i \) of a node \( i \) is \( \sum_{j \in N(i)} w_{ij} \). The volume \( V_G \) of the graph is defined as \( \sum_{i \in V} d_i \).

**Definition 2:** The transition matrix \( M = (p_{ij})_{i,j \in V} \) of a random walk on \( G \) is given by

\[
p_{ij} = \begin{cases} \frac{w_{ij}}{d_i}, & \text{if } (i, j) \in E \\ 0, & \text{otherwise} \end{cases}
\]

**Definition 3:** Let \( P_0 \) be an initial distribution on \( G \). Define \( P_t = (M^T)^t P_0 \) for all \( t \geq 0 \). A distribution \( P_0 \) is stationary if \( P_1 = P_0 \).

**Fact 1:** The distribution \( P_0 \) defined by \( \pi(v) = \frac{d_v}{V_G} \) for all \( v \in V \) is a stationary distribution.

**Definition 4:** A random walk is time-reversible if for every pair of nodes \( i,j \in V \), \( \pi(i) p_{ij} = \pi(j) p_{ji} \).

**Definition 5:** The hitting time \( h_{ij} \) is the expected number of steps that a random walk starting at \( i \) will take before reaching \( j \) for the first time.

**Definition 6:** The hitting time can be defined in terms of the recursion

\[
h_{ij} = \begin{cases} 1 + \sum_{t \in \mathbb{N}(i)} p_{it} h_{tj} & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases}
\]

**Definition 7:** The commute time distance \( c_{ij} \) between two nodes \( i \) and \( j \) is given by \( c_{ij} = h_{ij} + h_{ji} \).

**Fact 2:** CTD is a metric: (i) \( c_{ii} = 0 \), (ii) \( c_{ij} = c_{ji} \) and (iii) \( c_{ij} \leq c_{ik} + c_{kj} \).

Remarkably, CTD can be expressed in terms of the Laplacian of \( G \).

**Definition 8:** Let \( D \) be the diagonal degree matrix and \( A \) be the adjacency matrix of \( G \). The Laplacian of \( G \) is the matrix \( L = D - A \).

**Fact 3:** 1) Let \( e_i \) be the \( V \) dimensional column vector with a 1 at location \( i \) and zero elsewhere.
2) Let \( (\lambda_i, v_i) \) be the eigenpair of \( L \) for all \( i \in V \), i.e., \( L v_i = \lambda_i v_i \).
3) It is well known that \( \lambda_1 = 0 \), \( v_1 = (1, 1, \ldots, 1)^T \) and all \( \lambda_i \geq 0 \).
4) Assume \( 0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{|V|} \).
5) Then the pseudo-inverse of \( L \) denoted by \( L^+ \) is

\[
L^+ = \sum_{i=2}^{|V|} \frac{1}{\lambda_i} v_i v_i^T
\]

**Fact 4:**

\[
c_{ij} = V_G (t_{ij}^1 + t_{jj}^1 - 2t_{ij}^1) = V_G (e_i - e_j)^T L^+ (e_i - e_j) \tag{1}
\]

where \( t_{ij}^1 \) is the \((i, j)\) element of \( L^+ \).

**Example:** Again, consider the graph \( G \) shown in Figure 1a where all the edge weights equal to 1. The sum of the degree of nodes, \( V_G = 8 \). We will calculate the commute time \( c_{12} \) in two different ways:

1) Using random walk: note that the expected number of steps for a random walk starting at node 1 and returning back to it is \( \frac{1}{d_1} = \frac{1}{8} = \frac{1}{8} \). But the walk from node 1 can only go to node 2 and then return from node 2 to 1. Thus \( c_{12} = 8 \).
2) Using algebraic approach: the Laplacian matrix is

\[
L = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 3 & -1 & -1 \\
0 & -1 & 2 & -1 \\
0 & -1 & -1 & 2
\end{pmatrix}
\]

and the pseudo-inverse is

\[
L^+ = \begin{pmatrix}
0.69 & -0.06 & -0.31 & -0.31 \\
-0.06 & 0.19 & -0.06 & -0.06 \\
-0.31 & -0.06 & 0.35 & 0.02 \\
-0.31 & -0.06 & 0.02 & 0.35
\end{pmatrix}
\]

Now \( c_{12} = V_G (e_1 - e_2)^T L^+ (e_1 - e_2) \) and

\[
(1 -1 0 0)(0.69 -0.06 -0.31 -0.31) \begin{pmatrix} 1 \\ -0.06 \end{pmatrix} = 1
\]

\[
(0.31 -0.06 0.35 0.02)(1 -0.06 -0.06 -0.06) = 1
\]
Thus \( c_{12} = V_G \times 1 = 8 \).

Suppose we add a new node (labeled 5) to node 4 with a unit weight as in Figure 11. Then \( e_{12}^{\text{new}} = V_G^{\text{new}}/d_1 = 10/1 = 10 \).

The example in Figure 10 shows that by adding an edge, i.e., making the “cluster” which contains node 2 denser, \( c_{12} \) increases. This shows that CTD between two nodes captures not only the distance between them (as measured by the edge weights) but also their neighborhood densities. For the proof of this claim, see [5]. This property of CTD has been used to simultaneously discover global and local anomalies in data - an important problem in the anomaly detection literature.

In the above example, we exploited the specific topology (degree one node) of the graph to calculate CTD efficiently. This can only work for very special instances. The general, more widely used but slower approach for computing CTD is to use the Laplacian formula. A key contribution of this paper is that for incremental computation of CTD we can use insights from this example to accurately and efficiently compute the CTD in much more general situations.

III. INCREMENTAL EIGEN DECOMPOSITION OF GRAPH LAPLACIAN

In this section, we propose a method to incrementally update the eigensystem (eigenvalues and eigenvectors) of the Laplacian when a new node along with its neighbors is added to the underlying graph. The unique feature of our approach as opposed to that of Ning et. al. [11] are (i) our emphasis is on handling the addition of a new node and the corresponding edges to its nearest neighbors as opposed to just weight updates on existing edges and (ii) simultaneous updating of all weight edges as opposed to one edge at a time.

A. Iterative incremental update of the Laplacian eigensystem

We propose an algorithm based on the following proposition to incrementally update the eigensystem \((\lambda, v)\) of the Laplacian \(L\) when a new node \(i\) is added to the graph. Suppose there are \(k\) edges \(e = (i, j) \in E_n\) with weight \(w_e\) added to the graph from \(i\). Denote \(\Delta L\) and \((\Delta \lambda, \Delta v)\) be changes of \(L\) and \((\lambda, v)\) resulting from the addition of \(i\). Note that the size of matrix \(L\) and its eigenvector \(v\) change as mentioned in the Appendix.

Proposition 1: The solution of the eigensystem

\[
(L + \Delta L)(v + \Delta v) = (\lambda + \Delta \lambda)(v + \Delta v)
\]

can be derived from the solution of the following set of simultaneous equations.

\[
\Delta \lambda = \frac{\sum_{e \in E_n} w_e [v(i) - v(j)][v(i) - v(j) + \Delta v(i) - \Delta v(j)]}{1 + v^T \Delta v}
\]

\[
\Delta v = K^{-1}h
\]  

where

\[
K = L + \Delta L - (\lambda + \Delta \lambda)I,
\]

and

\[
h = (\Delta \lambda I - \Delta L)v.
\]

For the proof, see Appendix.

Note that in general it is not practical to solve the system \(\Delta v = K^{-1}h\) at the arrival of each new data point \(i\). In practice, as noted by Ning, et. al. [11], we can set \(\Delta v(k) = 0\) for all components which are not \(i\), its first or second order neighbors.

Denote \(N_i = \{j | d(i, j) \leq 2\}\), where \(d(i, j)\) is the shortest path between \(i\) and \(j\). Let \(K_N\) be the matrix derived from \(K\) after removing columns which do not correspond to nodes in \(N_i\), \(v_N\) and \(\Delta v_N\) be the vectors derived from \(v\) and \(\Delta v\) after removing elements which do not correspond to nodes in \(N_i\). Since \(K_N\) is not a square matrix, we obtain:

\[
\Delta v = (K_N^T K_N)^{-1} K_N^T h,
\]

\[
\Delta \lambda = \frac{\sum_{e \in E_n} w_e [v(i) - v(j)][v(i) - v(j) + \Delta v(i) - \Delta v(j)]}{1 + v^T \Delta v}
\]

Since \(\Delta \lambda\) in Equation 7 depends on the value of \(\Delta v\) in Equation 6 and vice versa, we can update the values of \(\Delta \lambda\) and \(\Delta v\) as follows. We initialize the values \(\Delta v = 0\) to update the value of \(\Delta \lambda\) and then using that to update \(\Delta v\). The procedure is repeated until convergence. Algorithm 1 gives the details.

Algorithm 1 Incremental update eigenvalues and eigenvectors of Laplacian matrix \(L\)

**Input:** Laplacian matrix \(L\), its eigenvalues \(S\) and eigenvectors \(V\), weights \(w_e\) of all the new edges

**Output:** New eigenvalues \(S_n\) and eigenvectors \(V_n\)

1: for each eigenvalue and eigenvector do
2: \(\Delta v = 0\)
3: Update \(\Delta \lambda\) using Equation 7
4: Update \(\Delta v\) using Equation 6
5: Repeat steps 3 and 4 until there is no significant change in \(\Delta \lambda\) or until the loop reaches a maximum iterations
6: \(v_n = v + \Delta v\), \(\lambda_n = \lambda + \Delta \lambda\)
7: end for

IV. INCREMENTAL ESTIMATION OF COMMUTE TIME DISTANCE

In this section, we derive a new method for computing the CTD in an incremental fashion. This method uses the definition of CTD based on the hitting time. The basic intuition is to expand the hitting time recursion until the random walk has moved a few steps away from the new
node and then use the old values. In Section VII, we will show that this method results in remarkable agreement between the batch and online mode.

We deal with two cases shown in Figure 2.

![Graph G](attachment:image.png)

(a) Rank 1

(b) Rank k

Figure 2: Rank 1 and rank k perturbation

1) Rank one perturbation corresponds to the situation when a new node connects with one other node in the existing graph.

2) Rank k perturbation deals with the situation when the new node has k neighbors in the existing graph.

The term rank is used as it corresponds to the rank of the perturbation matrix $ΔL$.

A. Rank one perturbation

**Proposition 2:** Let $i$ be a new node connected by one edge to an existing node $l$ in the graph $G$. Let $w_{il}$ be the weight of the new edge. Let $j$ be an arbitrary node in the graph $G$. Then

$$c_{ij} ≈ c_{ij}^{old} + \frac{V_G}{w_{il}}$$

where ‘old’ represents the CTD in graph $G$ before adding $i$.

**Proof:** (Sketch) Since the random walk needs to pass $l$ before reaching $j$, the commute distance from $i$ to $j$ is:

$$c_{ij} = c_{il} + c_{lj}.$$  \hspace{1cm} (9)

It is known that:

$$c_{il} = \frac{(V_G + 2w_{il})}{w_{il}}$$  \hspace{1cm} (10)

where $V_G$ is volume of graph $G$. We also know $c_{lj} = h_{jl} + h_{lj}$ and $h_{jl} = h_{jl}^{old}$. The only unknown factor is $h_{lj}$. By definition:

$$h_{lj} = 1 + \sum_{q \in N(l)} p_{lj}h_{qj} = 1 + \sum_{q \in N(l), q \neq i} p_{lj}h_{qj} + p_{li}h_{ij}.$$  

Since $h_{qj} ≈ h_{qj}^{old}$, $p_{lj} = (1 - p_{li})p_{lj}^{old}$, and $h_{ij} = 1 + h_{ij}$,

$$h_{lj} ≈ 1 + \sum_{q \in N(l), q \neq i} (1 - p_{li})p_{lj}^{old}h_{qj}^{old} + p_{li}(1 + h_{ij})$$  

$$= 1 + (1 - p_{li})\sum_{q \in N(l), q \neq i} p_{lj}^{old}h_{qj}^{old} + p_{li}(1 + h_{ij})$$  

$$= 1 + (1 - p_{li})(h_{lj}^{old} - 1) + p_{li}(1 + h_{ij}).$$

After simplification, $h_{lj} = h_{lj}^{old} + \frac{2p_{li}}{1 - p_{li}}$. Then $c_{ij} ≈ h_{jl}^{old} + h_{lj}^{old} + \frac{2p_{li}}{1 - p_{li}}$.

Since there is only one edge connecting from $i$ to $G$, $i$ is likely an isolated point and thus $p_{ii} ≪ 1$. Then

$$c_{ij} ≈ h_{jl}^{old} + h_{lj}^{old} + \frac{V_G}{w_{il}}$$  \hspace{1cm} (11)

As a result from equations 9, 10, and 11:

$$c_{ij} ≈ \frac{(V_G + 2w_{il})}{w_{il}} + c_{ij}^{old} + \frac{V_G}{w_{il}}$$

B. Rank $k$ perturbation

The rank $k$ perturbation analysis is more involved but the final formulation is an extension of the rank one perturbation.

**Proposition 3:** Denote $l ∈ G$ be one of $k$ neighbors of $i$ and $j$ be a node in $G$. The approximate commute distance between nodes $i$ and $j$ is:

$$c_{ij} ≈ \sum_{t \in N(i)} p_{it}c_{ij}^{old} + \frac{V_G}{d_i}$$  \hspace{1cm} (12)

For the proof, see Appendix.

V. ONLINE ANOMALY DETECTION ALGORITHM

We return to our original motivation for computing incremental CTD. We are given a dataset $D$ which is representative of the underlying domain of interest. We want to check if a new data point is an anomaly with respect to $D$. We will use the CTD as a distance metric.

This section describes an online anomaly detection system using the incremental update of the eigensystem of the Laplacian in Section III and the incremental estimation of commute time in Section IV.

Generally CTD is robust against small changes or perturbation in data. Therefore, only the anomaly score of the new data point needs to be estimated and be compared with the anomaly threshold in the training data. This claim will be verified by experiment in Section VII.

A. CTD-based anomaly detection

This section reviews the batch method based on CTD to detect anomalies [5]. The method is described in Algorithm 2. First, a mutual $k_1$-nearest neighbor graph is constructed from the dataset. Then the graph Laplacian matrix $L$, its eigenvectors $V$ and eigenvalues $S$ are computed. Finally, a CTD distance-based anomaly detection with a pruning rule proposed by Bay and Schwabacher [12] is used to find the top $N$ anomalies. The anomaly score used is the average CTD of an observation to its $k_2$ nearest neighbors.

**Pruning Rule [12]:** A data point is not an anomaly if its score (e.g., the average distance to its $k$ nearest neighbors) is less than an anomaly threshold. The threshold can be fixed or be adjusted as the score of the weakest anomaly found so far. Using the pruning rule, many non-anomalies can be pruned without carrying out a full nearest neighbors search.
Algorithm 2 CTD-Based Anomaly Detection

Input: Data matrix $X$, the numbers of nearest neighbors $k_1$ (for building the $k$-nearest neighbor graph) and $k_2$ (for estimating the anomaly score), the number of anomalies to return $N$

Output: Top $N$ anomalies
1: Construct the mutual $k$-nearest neighbor graph from the dataset (using $k_1$)
2: Compute the graph Laplacian matrix $L$, its eigenvectors $V$ and eigenvalues $S$
3: Find top $N$ anomalies using the CTD based technique with pruning rule (using $k_2$). Each CTD query uses Equation $[1]$
4: Return top $N$ anomalies

B. Online Algorithms

Algorithm 3 (denote as iLED) is a method to detect anomalies online using the eigenvalues and eigenvectors of the new Laplacian matrix which is updated incrementally. Algorithm 4 (denote as iECT) on the other hand is a method to detect anomalies online using incremental estimation of commute time based on hitting time.

Algorithm 3 Online Anomaly Detection using incremental Laplacian Eigen Decomposition (iLED)

Input: Graph $G$, Laplacian matrix $L$, its eigenvalues $S$ and eigenvectors $V$, the anomaly threshold $\tau$ of the training set, and a test data point $p$

Output: Determine if $p$ is an anomaly or not

1: Add $p$ to $G$ using the mutual nearest neighbor graph, we have a new graph $G_n$
2: Incrementally compute the new eigenvalues and eigenvectors of the new Laplacian $L_n$ using Algorithm $[1]$
3: Use Gram-Schmidt process $[13]$ to orthogonalize the new eigenvectors
4: Determine if $p$ is an anomaly or not by estimating its anomaly score using CTDs derived from the new eigenpairs. Use pruning rule with threshold $\tau$ to reduce the computation
5: Return whether $p$ is an anomaly or not

When a new data point $p$ arrives, it is connected to graph $G$ created in the training phase. The CTDs are incrementally updated to estimate the anomaly score of $p$ using the approach in sections III and IV. For iLED algorithm, after updating the eigenvalues and eigenvectors, we use Gram-Schmidt process $[13]$ to normalize and orthogonalize the eigenvectors.

Algorithm 4 Online Anomaly Detection using the incremental Estimation of Commute Time (iECT)

Input: Graph $G$, Laplacian matrix $L$, its eigenvalues $S$ and eigenvectors $V$, the anomaly threshold $\tau$ of the training set, and a test data point $p$

Output: Determine if $p$ is an anomaly or not

1: Add $p$ to $G$ using the mutual nearest neighbor graph, we have a new graph $G_n$
2: Determine if $p$ is an anomaly or not by estimating its anomaly score using incremental CTDs mentioned in Section IV. Use pruning rule with threshold $\tau$ to reduce the computation
3: Return whether $p$ is an anomaly or not

C. Analysis

First, we analyse the incremental eigen decomposition of the Laplacian in Section III. Here $n$ is the size of the original graph (Laplacian) and $N$ is the neighborhood size used in $K_N$ (i.e., cardinality of $N_j$). Note $N \ll n$.

It takes constant time to update $\Delta \lambda$ and $O(N^2n)$ to compute $X = K_N^T K_N$, $O(N^3)$ for $X^{-1}$, $O(Nn)$ for $y = K_N^T h$ and $O(N^2)$ for $\Delta v = X^{-1} y$. Since $N \ll n$, we obtain $O(n)$ time for the incremental update of eigenvalues and eigenvectors of the Laplacian.

On the other hand, incremental estimation of commute time update in Section IV requires $O(m)$ for each query of $c^j_{lj}$ for Algorithm $[1]$, and $O(km)$ for Algorithm $[4]$ (iECT). Therefore, iLED takes $O(n + k_2m) = O(n)$, and iECT takes $O(k_2km) = O(1)$ to determine if a new arriving point is an anomaly or not. Note that since $L$ and $K$ are sparse, we can get better than $O(n)$ for iLED.

VI. EXPERIMENTS AND RESULTS

We report on the experiments carried out to determine and compare the effectiveness of the iECT and iLED methods. To recall, iECT uses the recursive definition of hitting time to calculate CTD while iLED uses the Laplacian definition.

**Approach:** We split a data set into two parts: training and test. We use Algorithm 2 to compute the top $N$ anomalies in the training set and use the average distance of a data point to its $k_2$ nearest neighbor (in CTD) as its anomaly score. The weakest anomaly in the top $N$ set is one which has
the smallest average distance to its nearest neighbors and is 
used as the threshold value \( \tau \). Then the anomaly score of 
each instance \( p \) in the test set is calculated based on its \( k_2 \) 
neighbors in the training set. If this score is greater than \( \tau \) 
then the test instance is reported as an anomaly. During the 
time searching for the nearest neighbors of \( p \), if its average 
distance to the nearest neighbors found so far is smaller than 
\( \tau \), we can stop the search as \( p \) is not anomaly (pruning rule).

Data and Parameters: The experiments were carried out 
on synthetic as well as real datasets. We chose the number 
of nearest neighbors \( k_1 = 10 \) to build the mutual nearest 
neighbor graph, \( k_2 = 20 \) to estimate the anomaly score, the 
number of Laplacian eigenvectors \( m = 50 \). In Algorithm 1 
the threshold to estimate the change of \( \Delta \lambda \) was \( 10^{-6} \) and 
the maximum iterations was 5. The choice of parameters 
was determined from the experiments. In all experiments, 
the batch method was used as the benchmark. The anomaly 
threshold \( \tau \) was set based on the training data. It was the 
score of the weakest anomaly in the top \( N = 50 \) anomalies 
found by Algorithm 2 in the training set.

A. Synthetic datasets

We created six synthetic datasets, each of which contained 
several clusters generated from Normal distributions and a 
number of random points generated from uniform distribution 
which might be anomalies. The number of clusters, the 
sizes, and the locations of the clusters were also chosen 
randomly. Each dataset was divided into a training set and 
a test set. There were 100 data points in every test set and 
and half of them were random anomalies mentioned above.

Experiments on Robustness: We first tested the robustness 
of CTD between nodes in an existing set when a new data 
instance is introduced. As \( CTD(i, j) \) between nodes \( i \) and 
\( j \) is a measure of expected path distance, the hypothesis is 
that the addition of a new point will have minimal influence 
on \( CTD(i, j) \) and thus the anomaly scores of data points in 
the existing set are relatively unchanged.

Table I shows the average, standard deviation, minimum, 
and maximum of anomaly scores of points in graph \( G \) before 
and after a new data point was added to \( G \). Graph \( G \) 
was created from the 1,000 point dataset in Figure 3. The result 
with test point was averaged over 100 test points in the 
test set. The result shows that the anomaly scores of data 
instances in \( G \) do not change much. This shows CTD is a 
robust measure, a small change or perturbation in the data 
will not result in large changes in CTD. Therefore, only the 
anomaly score of the new point needs to be estimated.

Experiments on Effectiveness: We applied iECT and 
iLED to all the datasets. The effectiveness of the iECT 
algorithm over iLED is shown in Figure 4. There were 36 
anomalies detected by batch method using CTD which are 
shown in Figure 3. iECT captured all of them and had 8 
false positives whose scores were close to the threshold. 
iLED approach, on the other hand, had better precision with 
no false positives but worse recall with only 15 anomalies 
found. The reason was anomalies have more effect on the 
eigenvectors and eigenvalues of the graph Laplacian and thus 
iLED was unable to capture many of them.

![Figure 3: 1,000 points dataset with training and test sets](image-url)

![Figure 4: Accuracy of iLED and iECT in 1,000 points 
dataset. iECT detects anomalies better than iLED.](image-url)
anomaly, the approximate eigenvalues and eigenvectors were significantly more accurate than those when a new point was an anomaly.

![Figure 5: Accuracy of iLED method. iLED has better approximation when a test point is not an anomaly.](image)

Table II shows the results in accuracy and performance of iLED and iECT in the synthetic datasets. Average score was the average anomaly score over 100 test points. The precision and recall were for the anomalous class. The time was the average time to process each of 100 test points. iECT generally had better approximation than iLED, did not miss any anomaly and had acceptable false alarms. iLED, on the other hand, did not have any false alarms but did miss many anomalies. Both of them were more efficient than the batch method. Note that the scores shown here were the anomaly scores with pruning rule and the scores for anomalies are always much higher than scores for normal points. Therefore the scores were dominated by the scores of anomalies and that is the reason why iLED had much lower scores. In fact, iLED is more accurate than iECT in estimating the scores of normal points.

There is an interesting dynamic at play between the anomaly, pruning rule, iECT, iLED, and the number of anomalies in the data. iECT was slightly slower than iLED in the experiment. The reason is we have many anomalies in the test set. We know that the pruning rule only works for non-anomalies. Moreover, iLED is faster per CTD query compared to iECT. Therefore, for anomalies, iLED is generally faster. Furthermore, because iLED tends to underestimate the scores of anomalies (and that was the reason it missed some anomalies), anomalies are treated as non-anomalies and the pruning kicks-in making it faster. In practice, since most of the test points are not anomalies, iECT will be more efficient than iLED. It is shown in Figure 6 where except a few false alarms, iECT was generally faster than iLED in 50,000 points dataset when test instances were not anomalies. The same tendency also happened in other datasets used in the experiments.

B. Real Datasets

1) DBLP dataset: In this section, we evaluated the iECT method on the DBLP co-authorship network. Nodes are authors and edge weights are the number of collaborated papers between the authors. Since the graph is not fully connected, we extracted its biggest component. It has 344,800 nodes and 1,158,824 edges.

We randomly chose a test set of 50 nodes and removed them from the graph. We ensured that the graph remained connected. After training, each node was added back into the graph along with their associated edges.

Since the size of the graph is very large, normal training using the batch mode in Algorithm 2 is not feasible. Instead we implemented the approximate method proposed by Spielman and Srivastava (SS) [8] and used the underlying linear time CMG solver proposed by Kouts [14]. The SS methods combines random projections with a linear time solver for diagonally dominant matrices to approximate the CTD. The SS method creates a matrix $Z$ from which CTD between two nodes can be computed in $k = O(\log n)$ time with provable accuracy. In practice we can use $k$ to be much smaller than $O(\log n)$ and still attain highly accurate results.

We trained the graph using the SS approach, stored the matrix $Z$ and used $Z$ to query the $c_{ij}^{old}$ in iECT algorithm. The batch method here is the CTD approximation using the matrix $Z_{new}$ created from the new graph after adding each test data point. The parameter for random projection was $k = 200$.  

![Figure 6: Performance of iLED and iECT in 50,000 points dataset. Except a few false alarms, iECT is generally faster than iLED when a test point is not an anomaly.](image)
Table II: Effectiveness of incremental methods. iECT generally does not miss any anomaly and has acceptable false alarms. iLED, on the other hand, does not have any false alarms but misses many anomalies.

| Dataset       | iLED             |          | iECT             |          | Batch            |          |
|---------------|------------------|----------|------------------|----------|------------------|----------|
| Size          | Avg Score        | Precision (%) | Recall (%) | Time (s) | Avg Score        | Precision (%) | Recall (%) | Time (s) | Avg Score        | Precision (%) | Recall (%) | Time (s) |
| 1,000         | 3.84 × 10⁶       | 100      | 41.7            | 0.13     | 1.69 × 10⁵       | 81.8      | 100 | 0.20 | 1.61 × 10⁴       | 91.8      | 100 | 0.17 |
| 10,000        | 7.09 × 10⁵       | 100      | 53.2            | 1.28     | 4.87 × 10⁶       | 95.9      | 100 | 1.42 | 4.99 × 10⁶       | 95.9      | 100 | 2.04 |
| 20,000        | 5.36 × 10⁶       | 100      | 83.3            | 2.64     | 1.75 × 10⁷       | 80.0      | 100 | 2.96 | 1.70 × 10⁷       | 80.0      | 100 | 4.53 |
| 30,000        | 4.81 × 10⁶       | 100      | 39.6            | 3.68     | 1.39 × 10⁸       | 96.0      | 100 | 4.33 | 1.40 × 10⁸       | 96.0      | 100 | 7.13 |
| 40,000        | 3.27 × 10⁶       | 100      | 15.6            | 4.44     | 5.17 × 10⁹       | 71.1      | 100 | 5.19 | 4.88 × 10⁹       | 71.1      | 100 | 9.05 |
| 50,000        | 8.88 × 10⁵       | 100      | 32.5            | 5.70     | 6.15 × 10¹       | 87.0      | 100 | 6.61 | 5.96 × 10⁷       | 92.0      | 100 | 11.60 |

The result shows that it took 0.0066 seconds on average over 50 test data points to detect whether each test point was an anomaly or not. The batch method, which is the fastest approximation of CTD to date, required 944 seconds on average to process each test data point. This dramatically highlights the constant time complexity of iECT algorithm and suggests that iECT is highly suitable for the computation of CTD in an incremental fashion. Since there was no anomaly in the random test set, we cannot report the detection accuracy here. The average anomaly score over all the test points of iECT was 1.1 times higher than the batch method. This shows the relatively high accuracy of iECT approximation even in a very large graph.

2) **KDD Cup 1999 datasets**: We used the 10% dataset from the KDD cup 1999 competition provided by UCI Machine Learning Repository [13]. It was used to build detection tools of network attacks or intrusions. Since the dataset is huge and there are more anomalies than normal instances, we sampled 2,200 data points from it where there were 2,000 normal points and 200 anomalies (network intrusions). Categorical features were ignored and 38 numerical features were used. The dataset was divided into a training set and a test set with 100 data points.

iLED and iECT were applied on this dataset and min-max scaling was used as data normalization. iLED had a precision of 100% and a recall of 66.7% while iECT had a precision of 75% and a recall of 100%. The average anomaly scores of iLED and iECT were 2% and 1% lower than that of the batch method, respectively.

3) **NICTA datasets**: The dataset is from a wireless mesh network which has seven nodes deployed by NICTA at the School of IT, University of Sydney [16]. It used a traffic generator to simulate traffic on the network. Packets were aggregated into one-minute time bins and the data was collected in 24 hours. There were 391 origin-destination flows and 1,270 time bins. Some anomalies were introduced to the network including DOS attacks and ping floods. The dataset was divided into a training set and a test set with 100 data points.

iLED and iECT were applied on this dataset and min-max scaling was used as data normalization. iLED had a precision of 100% and a recall of 27.3% while iECT had a precision of 84.6% and a recall of 100%. The average anomaly scores of iLED and iECT were 20% lower and 2% higher than that of the batch method, respectively. The tendency of the detection here of iLED and iECT are also similar to those of the synthetic datasets.

### C. Summary and Discussion

The experimental results on both synthetic and real datasets show that iECT generally has better detection ability than iLED. iECT has very high recall and acceptable precision. iLED, on the other hand, has very high precision but low recall. Both of them are faster than the batch method. The results on real datasets collected from different sources also have similar tendency showing the reliability and effectiveness of the proposed methods.

The experiments also reveal that iLED tends to underestimate the CTDs for anomalies while iECT tends to overestimate the CTDs for non-anomalies. It leads to a high precision for iLED and a high recall for iECT. If we can come up with a strategy to combine the strengths of the two methods, we can have a more accurate estimation. iECT is faster than iLED but it can only be used in case where a new test point is added and cannot be used when there are weigh updates in the graph. iLED on the other hand can be used in both cases by just changing the perturbation matrix.

### VII. RELATED WORK

Incremental learning using an update on eigen decomposition has been studied for a long time. Early work studied the rank one modification of the symmetric eigen decomposition [17], [18], [19]. The authors reduced the original problem to the eigen decomposition of a diagonal matrix. Though they can have a good approximation of the new eigenpair, they are not suitable for online applications nowsaday since they have at least $O(n^2)$ computation for the update.

More recent approach was based on the matrix perturbation theory [20], [21]. They used the first order perturbation analysis of the rank-one update for a data covariance matrix to compute the new eigenpair. The algorithms have a linear time computation. The advantage of using the covariance matrix is if the perturbation involving an insertion of a new point, the size of the covariance matrix is unchanged. This approach cannot be applied directly to increasing matrix size due to an insertion of a new point. For example, in spectral clustering or CTD-based anomaly detection, the size of the...
graph Laplacian matrix increases when a new point is added to the graph.

Ning et al. [11] proposed an incremental approach for spectral clustering with application to monitor evolving blog communities. It incrementally updates the eigenvalues and eigenvectors of the graph Laplacian matrix based on a change of an edge weight on the graph using the first order error of the generalized eigen system.

VIII. CONCLUSION

The paper shows two novel approaches to compute CTD incrementally. The first one incrementally updates the eigenvectors and eigenvalues of the graph Laplacian matrix. It is linearly scaled and can be applied to the estimation of CTD incrementally or any application involving graph spectral computation. The second approach incrementally estimates CTD in constant time using the property of random walk and hitting time. We design novel anomaly detection algorithms using two approaches to detect anomalies online. The experimental results show the effectiveness of the proposed approaches in terms of performance and accuracy. It took less than 7 milliseconds on average to process a new arriving point in a graph of more than 300,000 nodes and one million edges. Moreover, the idea of this work can be extended in many applications which use the CTD and it is the direction for our future work.

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APPENDIX

We provide relation between perturbation and incidence matrix, and the proof details of Propositions 1 and 3.

A. Incidence matrix and perturbation

It is well known that the Laplacian of a graph can be expressed in terms of an incidence matrix.

Definition 9: Given a weighted graph \(G=(V,E,W)\) and an arbitrary but fixed orientation of the edges, the incidence matrix \(R\) is a \(|V| \times |E|\) matrix where the columns of the matrix are defined as

\[
R_e(v) = \begin{cases} \sqrt{w} & \text{at location } v \text{ if } v \text{ is the head of } e \\ -\sqrt{w} & \text{at location } v \text{ if } v \text{ is the tail of } e \\ 0 & \text{otherwise} \end{cases}
\]

Note that \(R_e\) is a column vector of size \(|V|\). We can also express each \(R_e = \sqrt{w}u_e\) where \(u_e\) is a column vector with entries 1, -1 at locations corresponding to head and tail of \(e\) and 0 at other locations.

Fact 5: If \(L\) is a graph Laplacian then \(L = RR^T\) [22].

Fact 6: If an edge \(e\) undergoes a similarity change \(\Delta w_e\), the new graph Laplacian \(L_n\) is \(L_n = R_nR_n^T\) where \(R_n = [R + R_e(\Delta u_e)]\) [11]. Therefore, a change in an edge weight can be represented by appending an incidence vector to \(R\).

Proposition 4: Denote \(R\) as an incidence matrix of a given graph \(G=(V,E,W)\). Suppose a new node is added to the graph which results in \(k\) new edges. Let \(R_n\) represent the new \(|V+1| \times |E+1|\) matrix and let \(\Delta R\) represent the matrix of new incidence vectors of size \(|V+1| \times k\). Then the new Laplacian \(L_n\) can be expressed in terms of the old Laplacian \(L\) as

\[
L_n = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \Delta L
\]

Also note that if \((\lambda, v)\) is an eigenpair of \(L\) then \((\lambda, \begin{bmatrix} v \\ 0 \end{bmatrix})\) is an eigenpair of \(L\).

Proof: Let \(E_n\) be the set of new edges resulting from the addition of a new node. Then

\[
R_nR_n^T = \begin{bmatrix} (R) & \Delta R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (R) & \Delta R \end{bmatrix}^T
= \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \sum_{e \in E_n} w_eu_eu_e^T = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \Delta L = L_n
\]

B. Incremental Update of Eigenvalues and Eigenvectors

Proof for Proposition 1:

Given the eigen decomposition of the new Laplacian matrix:

\[
(L + \Delta L)(v + \Delta v) = (\lambda + \Delta \lambda)(v + \Delta v)
\]

The perturbation is:

\[
\Delta L = \sum_{e \in E_n} w_eu_eu_e^T
\]

Since \(Lv = \lambda v\),

\[
L(\Delta v + \Delta Lv + \Delta L\Delta v) = \Delta \lambda v + \lambda \Delta v + \Delta \lambda \Delta v
\]

Left multiply both sides by \(v^T\):

\[
v^T \Delta L v + v^T \Delta L v + v^T \Delta L \Delta v = v^T \Delta \lambda v + v^T \lambda \Delta v + v^T \Delta \lambda \Delta v
\]

Since \(v^T L = \lambda v\) (L is symmetric):

\[
v^T \Delta \lambda (v + \Delta v) = v^T \Delta L (v + \Delta v)
\]

Then we have the update of the eigenvalue \(\lambda\):

\[
\Delta \lambda = \frac{v^T \Delta L (v + \Delta v)}{v^T(v + \Delta v)} = \frac{v^T \Delta L (v + \Delta v)}{1 + v^T \Delta v}
\]

\[
= \frac{v^T \sum_{e \in E_n} w_eu_eu_e^T(v + \Delta v)}{1 + v^T \Delta v}
\]

\[
= \sum_{e \in E_n} w_e[v(i)-v(j)][v(i) - v(j) + \Delta v(i) - \Delta v(j)]
\]

\[
= \frac{1}{1 + v^T \Delta v} \frac{1 + v^T \Delta v}{1 + v^T \Delta v}
\]

From equation [16] we have:

\[
[L + \Delta L - (\lambda + \Delta \lambda)I] \Delta v = (\Delta \lambda I - \Delta L)v
\]

Denotes \(K = L + \Delta L - (\lambda + \Delta \lambda)I\) and \(h = (\Delta \lambda I - \Delta L)v\), we have \(\Delta v = K^{-1}h\).

C. Rank k Perturbation

Lemma 1: Denote \(i \in G\) is one of \(k\) neighbors of \(i\) and \(j\) is a node in \(G\). We have:

\[
\sum_{l \in N(i)} p_{il}h_{li} = \frac{V_G}{d_i} + 1.
\]

Proof: Using the reversibility property of the random walk, it is easy to prove that the expected number of steps that a random walk which has just visited node \(i\) will take before returning back to \(i\) is \(graph-volume/d_i\) [9].
In case of $i$, this distance equals to the distance from $i$ to one of its neighbors $l$ (one step) plus the hitting time $h_{li}$. Since the random walk goes from $i$ to $l$ with the probability $p_{il}$, we have $1 + \sum_{l \in N(i)} p_{il} h_{li} = \frac{V_G + 2d_i}{d_i}$. Therefore, 
\[ \sum_{l \in N(i)} p_{il} h_{li} = \frac{V_G}{d_i} + 1. \]

**Proof for Proposition 3**

Proof: (Sketch) By definition, 
\[
h_{ij} = 1 + \sum_{l \in N(i)} p_{il} h_{lj} = 1 + \sum_{l \in N(i)} p_{il} (1 + \sum_{q \in N(i)} p_{iq} h_{qj})
\]

Using the same approach as the rank one case, 
\[
h_{ij} = 1 + \sum_{l \in N(i)} p_{il} [1 + (1 - p_{li}) \sum_{q \in N(i), q \neq i} p_{iq} h_{qj} + p_{li} h_{ij}]
\]
\[
= 1 + \sum_{l \in N(i)} p_{il} [1 + (1 - p_{li})(h_{lji}^{old} - 1) + p_{li} h_{ij}]
\]

After a few manipulations, we have 
\[
h_{ij} = \frac{1 + \sum_{l \in N(i)} p_{il} h_{lj}^{old} - \sum_{l \in N(i)} p_{il} p_{li} h_{lj}^{old} + \sum_{l \in N(i)} p_{il} p_{li} h_{li}}{1 - \sum_{l \in N(i)} p_{il} p_{li}}
\]

Because $\sum_{l \in N(i)} p_{il} p_{li} \ll 1$, 
\[
h_{ij} \approx 1 + \sum_{l \in N(i)} p_{il} h_{lj}^{old}.
\] (18)

Since the commute distance between two nodes is the average of all possible path-length between them, $h_{ji} \approx \frac{1}{k} \sum_{l \in N(i)} (h_{lj} + h_{li})$. Instead of using the normal average, we take into account the probability $p_{il}$: 
\[
h_{ji} \approx \sum_{l \in N(i)} p_{il} (h_{lj} + h_{li}) = \sum_{l \in N(i)} p_{il} h_{lj} + \sum_{l \in N(i)} p_{il} h_{li}
\] (19)

We have $h_{jl} \approx h_{lj}^{old}$. Moreover, from Lemma 1 we have $\sum_{l \in N(i)} p_{il} h_{li} = \frac{V_G}{d_i} + 1$. Then from (19) 
\[
h_{ji} \approx \sum_{l \in N(i)} p_{il} h_{lj}^{old} + \frac{V_G}{d_i} + 1
\] (20)

As a result of equations (18) and (20), 
\[
c_{ij} \approx 1 + \sum_{l \in N(i)} p_{il} c_{lj}^{old} + \frac{V_G}{d_i} + 1 \approx \sum_{l \in N(i)} p_{il} c_{lj}^{old} + \frac{V_G}{d_i}
\]

When $k = 1$ (rank one case), the formula becomes Equation 8.