COLORING AND MAXIMUM WEIGHT INDEPENDENT SET OF RECTANGLES

PARINYA CHALERMSOOK\(^a\) AND BARTOSZ WALCZAK\(^b\)

ABSTRACT. In 1960, Asplund and Grünbaum proved that every intersection graph of axis-parallel rectangles in the plane admits an \(O(\omega^2)\)-coloring, where \(\omega\) is the maximum size of a clique. We present the first asymptotic improvement over this six-decade-old bound, proving that every such graph is \(O(\omega \log \omega)\)-colorable and presenting a polynomial-time algorithm that finds such a coloring. This improvement leads to a polynomial-time \(O(\log \log n)\)-approximation algorithm for the maximum weight independent set problem in axis-parallel rectangles, which improves on the previous approximation ratio of \(O(\frac{\log n}{\log \log n})\).

1. INTRODUCTION

Coloring of Rectangles. Let \(\mathcal{R}\) be a family of axis-parallel rectangles in the plane. The chromatic number of \(\mathcal{R}\), denoted by \(\chi(\mathcal{R})\), is the minimum number of colors that can be assigned to the rectangles so that any two intersecting rectangles receive different colors. The clique number of \(\mathcal{R}\), denoted by \(\omega(\mathcal{R})\), is the maximum size of a set \(\mathcal{C} \subseteq \mathcal{R}\) such that any two rectangles in \(\mathcal{C}\) intersect. These two terms are equivalent to the chromatic number \(\chi(G)\) and the clique number \(\omega(G)\) of the intersection graph \(G\) of \(\mathcal{R}\). Since \(\chi(G) \geq \omega(G)\), a natural question is whether \(\chi(G)\) can be bounded from above in terms of \(\omega(G)\). This question in various graph classes has received a lot of attention from discrete mathematics community, and it has also played crucial roles in the theory of algorithms and mathematical programming.

In general, it is well known that triangle-free graphs (that is, graphs with clique number 2) can have arbitrarily large chromatic number [15]. Classes of graphs \(\mathcal{G}\) that admit a function bounding \(\chi(G)\) in terms of \(\omega(G)\) for every \(G \in \mathcal{G}\) are called \(\chi\)-bounded. There has been immense progress in the study of \(\chi\)-bounded classes of graphs in recent years—see the survey by Scott and Seymour [30] and the references therein. In particular, various classes of geometric intersection graphs are known to be \(\chi\)-bounded or not \(\chi\)-bounded—see e.g. [23, 28, 29]. The history of this question for rectangle intersection graphs dates back to 1948, when Bielecki [5] asked whether triangle-free rectangle graphs have bounded chromatic number. Asplund and Grünbaum [4] not only answered this question in the positive but also showed a more general bound of \(\chi(\mathcal{R}) \leq 4\omega(\mathcal{R})^2 - 4\omega(\mathcal{R}) - 1\). This bound was later improved by Hendler [21] to \(\chi(\mathcal{R}) \leq 3\omega(\mathcal{R})^2 - 2\omega(\mathcal{R}) - 1\). Kostochka [24] constructed rectangle families \(\mathcal{R}\) with \(\chi(\mathcal{R}) = 3\omega(\mathcal{R})\), and this remains the best known lower bound. Chalermsook [7] proved the bound \(\chi(\mathcal{R}) = O(\omega(\mathcal{R}) \log \omega(\mathcal{R}))\) for the special case that \(\mathcal{R}\) contains no nested pair of rectangles. Closing or even narrowing down the gap between the linear lower bound and the quadratic upper bound for the general families of rectangles has been a long-standing open problem.

Maximum Weight Independent Set of Rectangles (MWISR). In MWISR, we are given a family of \(n\) axis-parallel rectangles in the plane together with weights assigned to them, and we

\(^a\)Aalto University, Finland (parinya.chalermsook@aalto.fi). Supported by European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 759557) and by Academy of Finland Research Fellowship, under grant number 310415.

\(^b\)Jagiellonian University, Kraków, Poland (walczak@tcs.uj.edu.pl). Partially supported by National Science Center of Poland grant 2015/17/D/ST1/00585.
aim at finding a maximum weight subfamily (called an independent set or a packing) that contains no two intersecting rectangles. Besides being a fundamental problem in geometric optimization, MWISR is interesting from several perspectives. First, it arises in various applications, including map labeling \[ \text{[3, 14]} \], resource allocation \[ \text{[27]} \], data mining \[ \text{[26, 22, 19]} \], and unsplittable flow routing \[ \text{[6]} \]. Second, it is one of the “somewhat tractable” special cases of the general maximum weight independent set problem: given an \( n \)-vertex graph with weights on the vertices, find a maximum weight subset of the vertices containing no two vertices connected by an edge. This problem for general graphs is NP-hard to approximate to within a factor of \( n^{1-\varepsilon} \) for every \( \varepsilon > 0 \) \[ \text{[20, 31]} \], with the best known approximation factor being \( O(n(\log \log n)^{3} / \log^{3} n) \) \[ \text{[17]} \]. In special graph classes defined by intersections of geometric objects (such as disks, squares, and more generally—fat objects), polynomial-time approximation schemes (PTASes) are known \[ \text{[10, 16]} \]. Rectangles are perhaps the simplest natural objects for which the maximum independent set problem is not known to admit a PTAS. MWISR is NP-hard \[ \text{[18]} \] and there have been active attempts in the past decade from various groups of researchers on obtaining approximation algorithms. The best known approximation factor is \( O\left(\frac{\log n}{\log \log n}\right) \) by Chan and Har-Peled \[ \text{[11]} \]. In the unweighted case, Chalermsook and Chuzhoy \[ \text{[8]} \] presented an \( O(\log \log n) \)-approximation. Recently, a quasi-polynomial-time approximation scheme (QPTAS) was presented by Adamaszek, Har-Peled, and Wiese \[ \text{[1, 2]} \] (see also an improvement on the unweighted case by Chuzhoy and Ene \[ \text{[13]} \]). Obtaining a PTAS or even a polynomial-time constant-factor approximation for MWISR remains an elusive open problem.

**Connections between Coloring and MWISR.** These two problems are related through the perspective of mathematical programming. In particular, consider the clique-constrained independent set polytope of a graph \( G \):

\[
\text{QSTAB}(G) = \left\{ x \in \mathbb{R}^{V(G)} : x \geq 0 \text{ and } \sum_{v \in Q} x_v \leq 1 \text{ for every clique } Q \text{ in } G \right\}.
\]

For a graph \( G \) and a weight vector \( w \in \mathbb{R}^{V(G)} \), let \( \text{FRAC}(G, w) = \max\{w \cdot x : x \in \text{QSTAB}(G)\} \) and \( \text{INT}(G, w) = \max\{w \cdot x : x \in \text{QSTAB}(G) \cap \{0, 1\}^{V(G)}\} \), the latter being the maximum weight of an independent set in \( G \) with respect to the weights \( w \). Clearly, \( \text{INT}(G, w) \leq \text{FRAC}(G, w) \). The integrality ratio (or integrality gap) \( \text{gap}(G, w) \) is the ratio \( \frac{\text{FRAC}(G, w)}{\text{INT}(G, w)} \). Since a fractional solution \( x \in \text{QSTAB}(G) \) with value \( w \cdot x \geq \text{INT}(G, w) \) can be found efficiently \[ \text{[4]} \], rounding this LP solution is a natural algorithmic paradigm for approximating the maximum weight independent set problem, especially in restricted graph classes.

The integrality ratio of QSTAB has a strong connection to certain Ramsey-type bounds. More formally, let \( \mathcal{G} \) be any graph class that is closed under clique replacement operation \[ \text{[2]} \]. When \( w = 1 \) (the unweighted case), proving the upper bound \( \text{gap}(G, w) \leq \gamma \) for all \( G \in \mathcal{G} \), is equivalent to proving the upper bound \( R(s, t) \leq \gamma s(t - 1) \) on the Ramsey numbers \[ \text{[9]} \] for all graphs in the same graph class \( \mathcal{G} \). When allowing an arbitrary weight function \( w \), proving \( \text{gap}(G, w) \leq \gamma \) for all \( G \in \mathcal{G} \), is equivalent to upper bounding the ratio \( \frac{\chi_f(G)}{\chi(G)} \leq \gamma \) for all \( G \in \mathcal{G} \). These connections are constructive \[ \text{[9]} \]. Therefore, one way to design an efficient approximation algorithm for the maximum independent set problem in any graph class \( \mathcal{G} \) is to prove an (algorithmic) upper bound on \( \frac{\chi_f(G)}{\chi(G)} \) for graphs in the same graph class.

The polytope QSTAB(\( G \)) has played crucial roles from both algorithms and mathematical optimization perspectives; a notable example is its application to finding maximum cliques and independent sets in perfect graphs. It is particularly appealing for rectangle intersection graphs \( G \),

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1. In general graphs, it can be computed via SDP, as a solution optimizing \( w \cdot x \) over the Lovász theta body of \( G \).
2. This holds for various natural graph classes such as perfect graphs and geometric intersection graphs.
3. The Ramsey number \( R(s, t) \) is the minimum integer \( n \) such that every \( n \)-vertex graph contains a clique of size \( s \) or an independent set of size \( t \).
which have only $O(n^2)$ maximal cliques. For these graphs, an LP over $\text{QSTAB}(G)$ can be explicitly written and solved by a near-linear-time algorithm [12]. Therefore, it is an interesting question on its own to pinpoint the value of $\text{gap}(G,w)$ for rectangle graphs.

**Our Contributions.** First, we present the following improvement on the $O(\omega^2)$ coloring bound of Asplund and Grünbaum [4].

**Theorem 1.** Every family of axis-parallel rectangles in the plane with clique number $\omega$ is $O(\omega \log \omega)$-colorable, and an $O(\omega \log \omega)$-coloring of it can be computed in polynomial time.

Second, via a simple reduction, we obtain the following result for MWISR. We remark that the reduction was used implicitly in the paper of Chalermsook and Chuzhoy [5].

**Theorem 2.** There is a polynomial-time $O(\log \log n)$-approximation algorithm for MWISR, and the integrality ratio of the clique-constrained LP for rectangle graphs is at most $O(\log \log n)$.

This result improves upon the $O(\log n \frac{\log n}{\log \log n})$-approximation by Chan and Har-Peled [11] for MWISR. It also substantially simplifies and derandomizes the known $O(\log \log n)$-approximation in the unweighted setting [8]. The bound on the integrality ratio combined with a fast LP solver from [12] imply that an $O(\log n)$ estimate on the value of MWISR can be computed in $O(n^2 \polylog n)$ time.

The main new technical ingredient of this paper is a “hierarchical decomposition” of a family of rectangles, inspired by the work of Kierstead and Trotter [23]. In section 3, we present a small “warm-up” result that highlights the main idea behind this decomposition. In section 4 we define the decomposition and use it to prove Theorem 1. We present the proof of Theorem 2 in section 5.

2. Preliminaries

**Definitions.** A **rectangle** is a closed set of the form $[a,b] \times [c,d]$ in the plane, where $a < b$ and $c < d$. The **width** and the **height** of such a rectangle are the values $b - a$ and $d - c$, respectively.

Let $R$ be a family of rectangles. The **intersection graph** of $R$ has vertex set $R$ and edge set defined as follows: two rectangles $R, R' \in R$ are connected by an edge if they intersect, that is, if $R \cap R' \neq \emptyset$. A subfamily $S$ of $R$ is a **clique** if the intersection of all rectangles in $S$ is non-empty. This is the same as to say that every pair of rectangles in $S$ intersects, so this notion of a clique corresponds to a clique in the intersection graph of $R$. We say that a clique $C$ in $R$ contains a point $p$ if $p$ belongs to every rectangle in $C$. We let $\omega(R)$ denote the **clique number** of $R$, that is, the maximum size of a clique in $R$. A subfamily $S$ of $R$ is an **independent set** if the rectangles in $S$ are pairwise disjoint. A coloring of $R$ is an assignment of colors to the rectangles in $R$ such that the rectangles of any given color form an independent set. These notions correspond to independent sets and colorings of the intersection graph of $R$. We say that $R$ is $k$-**colorable** if there is a coloring of $R$ using $k$ colors. We let $\chi(R)$ denote the **chromatic number** of $R$, that is, the minimum number $k$ such that $R$ is $k$-colorable.

Let $R$ and $R'$ be two rectangles in the plane that intersect ($R \cap R' \neq \emptyset$). We distinguish several possible types of intersections; see Figure 1. If $R$ contains at least one corner of $R'$ or vice versa, then we have a **corner intersection** between $R$ and $R'$. Otherwise, we have a **crossing intersection** between $R$ and $R'$, and we say that $R$ and $R'$ **cross**. We have a **containment intersection** when one rectangle contains the other (which is a particular case of a corner intersection). We have a **vertical intersection** if one rectangle intersects both the top and the bottom sides of the other.

Let us fix a family $R$ of $n$ rectangles that is the input to our problem. For each rectangle $R \in R$, let $V(R)$ denote the rectangles in $R \setminus \{R\}$ that intersect both the bottom and the top sides of $R$, and let $X(R)$ denote the rectangles in $V(R)$ that cross $R$. Thus, if $R, R' \in R$ and the height of $R'$ is greater than the height of $R$, then the following holds:

- There is a vertical intersection between $R$ and $R'$ if and only if $R' \in V(R)$;
- There is a crossing intersection between $R$ and $R'$ if and only if $R' \in X(R)$.
Figure 1. All possible ways a pair of rectangles can intersect: (a) a crossing intersection, (b)–(h) corner intersections (each involving at least two corners), (b) a containment intersection, (a)–(d) vertical intersections.

It is important to observe that if \( R' \in \mathcal{X}(R) \), then \( V(R') \subseteq V(R) \).

**Preliminary Results.** A family of rectangles \( \mathcal{R} \) is \( s \)-sparse if one can fix \( s \) points \( p_1^R, \ldots, p_s^R \) in each rectangle \( R \in \mathcal{R} \) so that the intersection \( R \cap R' \) of any two crossing rectangles \( R, R' \in \mathcal{R} \) contains at least one of the points \( p_1^R, \ldots, p_s^R, p_1^{R'}, \ldots, p_s^{R'} \). For instance, a family of squares is 0-sparse, because no pair of squares can cross. The following lemma is slightly modified from [27, 7].

**Lemma 3.** For each \( s \in \mathbb{N} \), every \( s \)-sparse family of rectangles with clique number \( \omega \) is \((2s+4)(\omega - 1)\)-colorable, and such a coloring can be computed in polynomial time.

**Proof.** Let \( \mathcal{R} \) be an \( s \)-sparse family of rectangles with clique number \( \omega \). It suffices to show that the number of edges in the intersection graph of \( \mathcal{R} \) is strictly less than \((s + 2)(\omega - 1)|\mathcal{R}|\), because this implies that there is a vertex of degree less than \((2s + 4)(\omega - 1)\) in every induced subgraph, which leads to a \((2s + 4)(\omega - 1)\)-coloring by straightforward induction.

For each edge \( RR' \) in the intersection graph, if \( R \) and \( R' \) cross, we give one token to one of the points \( p_1^R, \ldots, p_s^R, p_1^{R'}, \ldots, p_s^{R'} \) that lies in \( R \cap R' \). Otherwise, \( RR' \) corresponds to a corner intersection, which involves at least two corners (four in case of containment and two otherwise; see Figure 1), and we give half of a token to any two corners involved in the intersection. Clearly, the total number of tokens handed out is equal to the number of edges in the intersection graph. Moreover, for each \( R \), each point \( p_i^R \) receives at most \( \omega - 1 \) tokens, and each corner of a rectangle receives at most \( \frac{\omega - 1}{2} \) tokens (with some corners receiving strictly fewer tokens). Therefore, the number of edges is less than \((s + 2)(\omega - 1)|\mathcal{R}|\). \( \square \)

**Corollary 4.** Every family of rectangles with no crossing intersections and with clique number \( \omega \) is \(4(\omega - 1)\)-colorable, and such a coloring can be computed in polynomial time.

The following result of Asplund and Grünbaum [3] will be used as a subroutine.

**Lemma 5.** Every family of rectangles with clique number \( \omega \) is \(4\omega(\omega - 1)\)-colorable, and such coloring can be computed in polynomial time.

**Proof.** Let \( \mathcal{R} \) be a family of rectangles with clique number \( \omega \). Let \( < \) be the strict partial order on \( \mathcal{R} \) defined so that \( R < R' \) if and only if \( R' \in \mathcal{X}(R) \). Every chain in the poset \((\mathcal{R}, <)\) is a clique in \( \mathcal{R} \), so the height of \((\mathcal{R}, <)\) is at most \( \omega \). Therefore, \( \mathcal{R} \) can be partitioned into \( \omega \) antichains in the poset \((\mathcal{R}, <)\), that is, families \( \mathcal{R}_1, \ldots, \mathcal{R}_\omega \) with no crossing intersections. By Corollary 4, each of the families \( \mathcal{R}_1, \ldots, \mathcal{R}_\omega \) is \(4(\omega - 1)\)-colorable, so the entire family is \(4\omega(\omega - 1)\)-colorable. \( \square \)

3. Warm-Up

In this section, we present two simple coloring results that capture some of the key ideas behind our general \(O(\omega \log \omega)\) bound.

**Proposition 6.** Let \( \mathcal{R} \) be a family of rectangles.

1. If there are only crossing and containment intersections within \( \mathcal{R} \), then \( \chi(\mathcal{R}) = \omega(\mathcal{R}) \).
Let \( \mathcal{R} \) be a family of rectangles with only vertical intersections, and let \( \omega = \omega(\mathcal{R}) \). We process the rectangles in \( \mathcal{R} \) in the decreasing order of their heights and put each rectangle into one of the sets \( S_1, \ldots, S_\omega \) as follows. We put a rectangle \( R \in \mathcal{R} \) into \( S_i \) where \( i \) is the maximum integer such that there is an \( i \)-witnessing clique for \( R \), that is, a clique in \( V(R) \) containing at least one rectangle from each of the sets \( S_1, \ldots, S_{i-1} \). This is well defined—when a rectangle \( R \) is being processed, the rectangles in \( V(R) \) have been already processed and distributed to \( S_1, \ldots, S_\omega \), and \( i \) is always at most \( \omega \) because the aforesaid clique in \( V(R) \) together with \( R \) forms a clique in \( \mathcal{R} \) of size at most \( \omega \). Let \( \mathcal{C}(R) \) denote any \( i \)-witnessing clique for a rectangle \( R \in \mathcal{R} \) where \( i \) is such that \( R \in S_i \).

First, we show that, for each \( i \), there can be no crossing or containment intersection between rectangles from a single set \( S_i \). Suppose for the sake of contradiction that two rectangles \( R, R' \in S_i \) form a crossing or containment intersection, where \( R' \) is taller than \( R \), that is, \( R' \in V(R) \). If \( R' \) contains \( R \), then \( \mathcal{C}(R) \cup \{ R' \} \) is an \( (i+1) \)-witnessing clique for \( R \), contradicting the assumption that \( R \notin S_i \). If \( R \) and \( R' \) cross, then \( V(R') \cup \{ R' \} \subseteq V(R) \), and \( \mathcal{C}(R') \cup \{ R' \} \) is an \( (i+1) \)-witnessing clique for \( R \), again contradicting the assumption that \( R \notin S_i \). See Figure 2.

If there are only containment and crossing intersections between the rectangles in \( \mathcal{R} \), then \( S_1, \ldots, S_\omega \) are independent sets. This completes the proof of statement (1).

To prove statement (2), we argue below that there is no crossing or containment intersection in each set \( S_i \). On the left, the dotted rectangles are the rectangles in \( \mathcal{C}(R') \); on the right, the dotted rectangles are those in \( \mathcal{C}(R) \).

\[ \text{Figure 2. An illustration of the proof that there is no crossing or containment intersection in each set } S_i. \text{ On the left, the dotted rectangles are the rectangles in } \mathcal{C}(R'); \text{ on the right, the dotted rectangles are those in } \mathcal{C}(R). \]

(2) If there are only vertical intersections within \( \mathcal{R} \), then \( \chi(\mathcal{R}) \leq 3\omega(\mathcal{R}) - 2 \).

Proof. Let \( \mathcal{R} \) be a family of rectangles with only vertical intersections, and let \( \omega = \omega(\mathcal{R}) \). We process the rectangles in \( \mathcal{R} \) in the decreasing order of their heights and put each rectangle into one of the sets \( S_1, \ldots, S_\omega \) as follows. We put a rectangle \( R \in \mathcal{R} \) into \( S_i \) where \( i \) is the maximum integer such that there is an \( i \)-witnessing clique for \( R \), that is, a clique in \( V(R) \) containing at least one rectangle from each of the sets \( S_1, \ldots, S_{i-1} \). This is well defined—when a rectangle \( R \) is being processed, the rectangles in \( V(R) \) have been already processed and distributed to \( S_1, \ldots, S_\omega \), and \( i \) is always at most \( \omega \) because the aforesaid clique in \( V(R) \) together with \( R \) forms a clique in \( \mathcal{R} \) of size at most \( \omega \). Let \( \mathcal{C}(R) \) denote any \( i \)-witnessing clique for a rectangle \( R \in \mathcal{R} \) where \( i \) is such that \( R \in S_i \).

Now, we present a 3-coloring algorithm for each set \( S_i \). We process the rectangles in \( S_i \) in the decreasing order of their heights and color them greedily. When a rectangle \( R \in S_i \) is being
processed, at most two other rectangles in \( S_i \) have been assigned colors (one intersecting the left side and one intersecting the right side of \( R \)), so the greedy coloring uses at most three colors. □

Proposition 6 (1) is a strengthening of the observation by Asplund and Grünbaum [3] (used in the proof of Lemma 5) that families \( R \) with only crossing intersections satisfy \( \chi(R) = \omega(R) \).

Proposition 6 (2) is closely related to the result of Kierstead and Trotter [23] that families of intervals on the real line with clique number \( \omega \) can be colored on-line using at most \( 3\omega - 2 \) colors. Specifically, by a correspondence described in [25], for every deterministic strategy of the adversary in the on-line coloring problem for intervals, there is an ‘equivalent’ family of axis-parallel rectangles (with only vertical intersections) whose chromatic number is equal to the number of colors forced by that strategy against any on-line coloring algorithm. The decomposition of the family \( R \) into sets \( S_1, \ldots, S_\omega \) used in the proof above is an adaptation of a decomposition used by Kierstead and Trotter in their proof of the upper bound of \( 3\omega - 2 \) for the on-line problem. Kierstead and Trotter also showed a deterministic strategy of the adversary forcing any on-line coloring algorithm to use at least \( 3\omega - 2 \) using on a family of intervals with clique number \( \omega \). That strategy gives rise to a family of axis-parallel rectangles \( R \) with only vertical intersections and with \( \chi(R) = 3\omega(R) - 2 \) [25, Proposition 3.1], which shows that the bound in Proposition 6 (2) is sharp.

4. An \( O(\omega \log \omega) \)-Coloring Algorithm

Let \( R \) be a family of rectangles and let \( k = \lceil \log_2 \omega(R) \rceil \). Thus \( \omega(R) \leq 2^k \). We show how to construct a coloring of \( R \) using \( O(2^k : k) \) colors (in polynomial time), which yields Theorem 1.

The argument consists of two steps. In the first step, we construct a “hierarchical decomposition” of \( R \) similar to the decomposition into sets \( S_1, \ldots, S_\omega \) used in the proof of Proposition 6, but defined with a “divide and conquer” approach rather than a simple linear induction. This modification is essential to make it work with the second step—a “clique reduction” argument, which is an adaptation of an argument used before in [7, 8].

**Step 1: Hierarchical Decomposition.** For \( i \in \mathbb{N} \), let \( B_i \) denote the set of binary words of length \( i \), and let \( \varepsilon \) denote the empty binary word, so that \( B_0 = \{ \varepsilon \} \) and \( B_i = \{0,1\}^i \) for \( i \geq 1 \). For each \( i = 0, \ldots, k \), by induction, we construct a partition \( \{ S_i(w) : w \in B_i \} \) of \( R \) and a non-empty set \( P_i(R) \) of witness points for every rectangle \( R \in R \).

For \( i = 0 \), let \( S_0(\varepsilon) = R \), and for every rectangle \( R \in R \), let \( P_0(R) \) be the set of intersection points of the top side of \( R \) with the left and right sides of the rectangles in \( V(R) \cup \{ R \} \) (which includes the two top corners of \( R \)). See Figure 4 for an illustration.

Now, let \( i \in \{1, \ldots, k\} \), and let \( u \in B_{i-1} \). We partition the set \( S_{i-1}(u) \) into two subsets \( S_i(u0) \) and \( S_i(u1) \), and we define the witness sets \( P_i(R) \) for the rectangles \( R \in S_i(u0) \cup S_i(u1) \), as follows. We consider the rectangles \( R \in S_{i-1}(u) \) in the order decreasing by height, so that all rectangles in \( V(R) \cap S_{i-1}(u) \) are considered before \( R \). For each rectangle \( R \in S_{i-1}(u) \) (in that order), if some witness point \( p \in P_{i-1}(R) \) belongs to at least \( 2^{k-i} \) rectangles from \( V(R) \) that have been already
Corollary 10. For each \( i = 0, \ldots, k \), each \( w \in B_i \), and any two rectangles \( R, R' \in S_i(w) \) such that \( R' \in \mathcal{X}(R) \), there is a witness point \( p \in \mathcal{P}_i(R) \) that belongs to \( R' \).

\[ \text{Figure 4. An illustration of the initial set } \mathcal{P}_0(R) \text{ of witness points.} \]
Proof. Immediate from Lemma 9, as the witness set $\mathcal{P}_i(R')$ is always non-empty.

**Lemma 11.** For each $i = 0, \ldots, k$ and each rectangle $R \in \mathcal{S}_i(w)$, every clique in $\mathcal{X}(R) \cap \mathcal{S}_i(w)$ has size at most $2^{k-i+1}$.

Proof. For the sake of contradiction, suppose that there is a clique $C$ of size greater than $2^{k-i+1}$ in $\mathcal{X}(R) \cap \mathcal{S}_i(w)$. Let $q$ be a point on the top side of $R$ that lies in all rectangles in $C$. Let $C_L$ be the $2^{k-i}$ rectangles from $C$ with leftmost left sides. Let $C_R$ be the $2^{k-i}$ rectangles from $C$ with rightmost right sides. Let $R'$ be a rectangle in $C \setminus (C_L \cup C_R)$, which exists as $|C| > |C_L| + |C_R|$. By Corollary 10 applied to $R$ and $R'$, there is a witness point $p \in \mathcal{P}_i(R)$ such that $p \in R'$. If $p$ is to the left of $q$, then $p$ belongs to all rectangles in $C_L$ (as they contain $q$ and their left sides are more to the left than the left side of $R'$), which contradicts Lemma 8. An analogous contradiction is reached for $C_R$ if $p$ is to the right of $q$.

**Step 2: Clique Reduction.** For $\alpha \in \mathbb{N}$, an $\alpha$-covering of a rectangle $R$ is a clique that contains $R$, at least $\alpha$ rectangles intersecting the top side of $R$, and at least $\alpha$ rectangles intersecting the bottom side of $R$ (not necessarily different from those intersecting the top side).

**Lemma 12.** Every clique in $\mathcal{R}$ of size greater than $2\alpha$ is an $\alpha$-covering of one of its rectangles.

Proof. Let $C$ be a clique in $\mathcal{R}$ of size greater than $2\alpha$. Let $C_T$ be the $\alpha$ rectangles in $C$ with top-most top sides. Let $C_B$ be the $\alpha$ rectangles in $C$ with bottom-most bottom sides. Let $R$ be a rectangle in $C \setminus (C_T \cup C_B)$, which exists as $|C| > |C_T| + |C_B|$. The rectangles in $C_T$ and $C_B$ witness that $C$ is an $\alpha$-covering of $R$.

For each $i = 0, \ldots, k$ and each $w \in B_i$, let $\mathcal{T}_i(w)$ be the set of rectangles $R \in \mathcal{S}_i(w)$ such that there is a $2^{k-i+2}$-covering of $R$ in $\mathcal{S}_i(w)$. Observe that $\mathcal{T}_i(w) = \emptyset$ when $i \leq 2$ and that it is easy to compute the sets $\mathcal{T}_i(w)$ in polynomial time. By Lemma 12, at least one rectangle from every clique in $\mathcal{S}_i(w)$ of size greater than $2^{k-i+3}$ belongs to $\mathcal{T}_i(w)$, so $\omega(\mathcal{S}_i(w) \setminus \mathcal{T}_i(w)) \leq 2^{k-i+3}$.

**Lemma 13.** For each $i = 0, \ldots, k$ and each $w \in B_i$, the set $\mathcal{T}_i(w)$ is $3$-sparse.

Proof. Let $R$ be a rectangle in $\mathcal{T}_i(w)$. We define three points $p_1^R, p_2^R, p_3^R \in R$ as follows. Let $p_1^R$ be the leftmost point and $p_2^R$ be the rightmost point in the witness set $\mathcal{P}_i(R)$. Recall that these points lie on the top side of $R$. Choose a clique $C$ that forms a $2^{k-i+2}$-covering of $R$ in $\mathcal{S}_i(w)$, and let $p_3^R$ be a point in the intersection of all rectangles in $C$ (which include $R$).

We verify that these points capture all intersecting pairs of rectangles, as in the definition of sparseness. Consider two crossing rectangles $R, R' \in \mathcal{T}_i(w)$ such that $R' \in \mathcal{X}(R)$. We claim that at least one point of $p_1^R, p_2^R, p_3^R, p_1^{R'}, p_2^{R'}, p_3^{R'}$ lies in $R' \cap R$. Suppose, for the sake of contradiction, that this is not the case.
Assume that the maximum weight independent set problem to the coloring problem have been already used in the literature of approximation algorithms \cite{7,8,27}. We also show that the reduction can be made polynomial-time.

First, observe that \( R' \) cannot lie to the left of \( p_i^R \): due to Corollary \ref{cor:10}, this would mean that some witness point in \( \mathcal{R}_i(R) \) lies to the left of \( p_i^R \), contradicting to the choice of \( p_i^R \). For the same reason, \( R' \) cannot lie to the right of \( p_i^R \), so it must lie between \( p_i^R \) and \( p_i^R \).

Since \( p_i^R \notin R \), the point \( p_i^R \) must lie either above or below \( R \). Assume that it lies below \( R \) (see Figure \ref{fig:6} the other case is analogous, by symmetry). Since \( R' \in \mathcal{T}_i(w) \), there is a \( 2^{k-i+2} \)-covering \( C \) of \( R' \) in \( \mathcal{S}_i(w) \). Let \( C' \) be the rectangles in \( C \) that intersect the top side of \( R' \). Thus \( |C'| \geq 2^{k-i+2} \), by the definition of \( 2^{k-i+2} \)-covering. Since \( R \) lies above \( p_i^R \) and below every point on the top side of \( R' \), we have \( C' \subseteq V(R) \). Lemma \ref{lem:11} yields \( |C' \cap \mathcal{X}(R)| \leq 2^{k-i+1} \), so \( |C' \setminus \mathcal{X}(R)| \geq 2^{k-i+1} \). Each rectangle in \( C' \setminus \mathcal{X}(R) \) fully contains the left or the right side of \( R \) (or both). Let \( C_1' \) be the rectangles in \( C' \setminus \mathcal{X}(R) \) containing the left side of \( R \), and let \( C_2' \) be those containing the right side of \( R \), so that \( C_1' \cup C_2' = C' \setminus \mathcal{X}(R) \). It follows that \( |C_j'| \geq 2^{k-i} \) for some \( j \in \{1,2\} \). This contradicts Lemma \ref{lem:8} because \( C_j' \) is a clique in \( V(R) \cap \mathcal{S}_i(w) \) of size at least \( 2^{k-i} \) containing the witness point \( p_j^R \in \mathcal{P}_i(R) \).

Finally, we present the coloring algorithm. It proceeds in \( k \) rounds. Initially, we have \( \mathcal{R}_0 = \mathcal{R} \). At round \( i = 1, \ldots, k \), we obtain \( \mathcal{R}_i \) by removing \( \bigcup_{u \in B_i} \mathcal{T}_i(w) \) from \( \mathcal{R}_{i-1} \). After the last round, we are left with the set \( \mathcal{R}_k \). Let \( \mathcal{T}'_i(w) = \mathcal{T}_i(w) \cap \mathcal{R}_{i-1} \) and \( \mathcal{R}'_i = \bigcup_{u \in B_i} \mathcal{T}'_i(w) \) be the set of rectangles removed in round \( i \). Hence, the families \( \mathcal{R}'_1, \ldots, \mathcal{R}'_k \) and \( \mathcal{R}_k \) form a partition of \( \mathcal{R} \). We argue that each family in this partition can be colored using \( O(2^k) \) colors.

For each \( i = 1, \ldots, k \) and each \( w \in B_i \) by Lemma \ref{lem:13} the family \( \mathcal{T}'_i(w) \) is \( 3 \)-sparse, and since \( \mathcal{T}'_i(w) \subseteq \mathcal{S}_i(w) \cap \mathcal{R}_{i-1} \subseteq \mathcal{S}_{i-1}(u) \setminus \mathcal{T}_{i-1}(u) \) (where \( u \) is the prefix of \( w \) in \( B_{i-1} \)) and \( \omega(\mathcal{S}_{i-1}(u) \setminus \mathcal{T}_{i-1}(u)) \leq 2^{k-i+4} \) (by the remark before Lemma \ref{lem:13}), the maximum size of a clique in \( \mathcal{T}'_i(w) \) is at most \( 2^{k-i+4} \). Therefore, by Lemma \ref{lem:3} each set \( \mathcal{T}'_i(w) \) is \( O(2^{k-i}) \)-colorable. Since \( |B_i| = 2^i \), by coloring each set \( \mathcal{T}'_i(w) \) where \( w \in B_i \) using a separate bunch of \( O(2^{k-i}) \) colors, we color \( \mathcal{R}'_i \) using \( O(2^k) \) colors. Finally, for each \( w \in B_k \), we have \( \mathcal{S}_k(w) \cap \mathcal{R}_k \subseteq \mathcal{S}_k(w) \setminus \mathcal{R}_k(w) \), so (by the remark before Lemma \ref{lem:13}) \( \omega(\mathcal{S}_k(w) \cap \mathcal{R}_k) \leq \omega(\mathcal{S}_k(w) \setminus \mathcal{R}_k(w)) \leq 8 \), and therefore (by Lemma \ref{lem:5}) the set \( \mathcal{S}_k(w) \cap \mathcal{R}_k \) can be colored using \( O(1) \) colors. Again, using a separate bunch of \( O(2^{k-i}) \) colors on each set \( \mathcal{S}_k(w) \cap \mathcal{R}_k \) where \( w \in B_k \), we color the family \( \mathcal{R}_k \) using \( O(2^k) \) colors.

5. An \( O(\log \log n) \)-Approximation Algorithm for MWISR

In this section, we present a reduction from MWISR to the coloring problem, which leads to a polynomial-time \( O(\log \log n) \)-approximation algorithm for MWISR. Similar reductions from the maximum weight independent set problem to the coloring problem have been already used in the literature of approximation algorithms \cite{7,8,27}. We also show that the reduction can be made deterministic via a derandomization trick similar to the one used by Chan and Har-Peled \cite{11}.

Let \( \mathcal{R} \) be a family of \( n \) rectangles, and for each \( R \in \mathcal{R} \), let \( w_R \) be the weight associated with \( R \). Assume that \( w_R > 0 \) for each \( R \in \mathcal{R} \) (otherwise \( R \) can be disregarded). Let \( \mathfrak{M} \) be the family of
inclusion-maximal cliques in $\mathcal{R}$. Thus $|\mathcal{M}| \leq n^2$, because the intersection of every such clique is a rectangle whose top left corner is the intersection point of the top side of some rectangle in $\mathcal{R}$ with the left side of some (possibly the same) rectangle in $\mathcal{R}$. Consider the following clique-constrained LP relaxation of the maximum weight independent set problem:

$$\begin{align*}
\text{maximize} & \quad \sum_{R \in \mathcal{R}} w_R x_R \\
\text{subject to} & \quad \sum_{R \in \mathcal{C}} x_R \leq 1 \quad \text{for every } \mathcal{C} \in \mathcal{M}, \\
& \quad x_R \geq 0 \quad \text{for every } R \in \mathcal{R}.
\end{align*}$$

Let $(x^*_R)_{R \in \mathcal{R}}$ be an optimal fractional solution to the LP, and let $w^*$ be the optimum value, which is therefore an upper bound on the maximum weight of an independent set in $\mathcal{R}$. Let $m = \lceil 9 \ln n \rceil$.

**Claim 14.** There is an integral vector $(y_R)_{R \in \mathcal{R}} \in \{0, \ldots, m\}^{\mathcal{R}}$ such that

$$\sum_{R \in \mathcal{C}} y_R \leq 2m \quad \text{for every } \mathcal{C} \in \mathcal{M} \quad \text{and} \quad \sum_{R \in \mathcal{R}} w_R y_R \geq \frac{mw^*}{2}.$$ 

Moreover, such a vector can be computed by a polynomial-time deterministic algorithm.

**Proof.** If $w_R \geq w^*/2$ for some $R \in \mathcal{R}$, then it is enough to set $y_R = m$ and $y_{R'} = 0$ for $R' \in \mathcal{R} \setminus \{R\}$. Therefore, assume henceforth that $w_R < w^*/2$ for every $R \in \mathcal{R}$.

For each $R \in \mathcal{R}$, let $x'_R = \lfloor mx^*_R \rfloor$, let $x''_R$ be a random variable in $\{0, 1\}$ such that $E x''_R = P(x''_R = 1) = mx^*_R - x'_R$, and let $y_R = x'_R + x''_R$. It follows that $0 \leq x'_R \leq y_R \leq \lfloor mx^*_R \rfloor \leq m$ for $R \in \mathcal{R}$. The LP inequality for a clique $\mathcal{C} \in \mathcal{M}$ yields

$$m \geq \sum_{R \in \mathcal{C}} mx^*_R = \sum_{R \in \mathcal{C}} x'_R + E\left(\sum_{R \in \mathcal{C}} x''_R\right),$$

which implies

$$P\left(\sum_{R \in \mathcal{C}} y_R > 2m\right) \leq P\left(\sum_{R \in \mathcal{C}} x'_R + \sum_{R \in \mathcal{C}} x''_R > 2 \sum_{R \in \mathcal{R}} x'_R + 2E\left(\sum_{R \in \mathcal{C}} x''_R\right)\right) \leq P\left(\sum_{R \in \mathcal{C}} x''_R > 2E\left(\sum_{R \in \mathcal{C}} x''_R\right)\right) \leq \exp\left(-\frac{1}{3}E\left(\sum_{R \in \mathcal{C}} x''_R\right)\right) \leq \exp\left(-\frac{m}{3}\right) \leq n^{-3},$$

where the strict inequality is the following form of the Chernoff bound for a sum of independent zero-one random variables $z$: $P(z > 2Ez) < \exp(-\frac{1}{3}Ez)$. For each clique $\mathcal{C} \in \mathcal{M}$, let a random variable $\zeta_\mathcal{C}$ be defined as follows:

$$\zeta_\mathcal{C} = \begin{cases} 
1 & \text{if } \sum_{R \in \mathcal{C}} y_R > 2m, \\
0 & \text{otherwise},
\end{cases}$$

so that $E \zeta_\mathcal{C} < n^{-3}$. Let a random variable $\xi$ be defined as follows:

$$\xi = \sum_{R \in \mathcal{R}} w_R y_R - \frac{mw^*}{2} \sum_{\mathcal{C} \in \mathcal{M}} \zeta_\mathcal{C}.$$

It follows that

$$E\xi = \sum_{R \in \mathcal{R}} w_R E y_R - \frac{mw^*}{2} \sum_{\mathcal{C} \in \mathcal{M}} E \zeta_\mathcal{C} > m \sum_{R \in \mathcal{R}} w_R x^*_R - \frac{mw^*}{2n^3} |\mathcal{M}| \geq mw^* - \frac{mw^*}{2} = \frac{mw^*}{2}.$$
Therefore, there is a choice of \((y_R)_{R \in \mathcal{R}}\) where \(\xi > mw^*/2\). It can be computed by a polynomial-time deterministic algorithm using the conditional expectation method, because \(E(\xi)\) can be computed in polynomial time for every clique \(\mathcal{C} \in \mathcal{M}\) by dynamic programming. Moreover, whenever \(\sum_{R \in \mathcal{C}} y_R > 2m\) for some \(\mathcal{C} \in \mathcal{M}\), then \(\xi < 0\) (because \(w_R < w^*/2\) and \(y_R \leq m\) for every \(R \in \mathcal{R}\)), so the resulting choice of \((y_R)_{R \in \mathcal{R}}\) satisfies the conditions of the claim. \(\square\)

Now, let \((y_R)_{R \in \mathcal{R}}\) be as in the claim. Let \(\mathcal{R}'\) be a multiset of rectangles where each rectangle \(R\) from \(\mathcal{R}\) occurs in \(y_R\) copies. The first condition of the claim implies that \(\mathcal{R}'\) has clique number at most \(2m\), so it has chromatic number \(O(m \log m)\), and moreover, a proper \(O(m \log m)\)-coloring of \(\mathcal{R}'\) can be computed in polynomial time. The second condition of the claim implies that the rectangles in \(\mathcal{R}'\) have total weight at least \(mw^*/2\), so some color class in the aforesaid coloring of \(\mathcal{R}'\) has total weight \(O(w^*/\log m) = O(w^*/\log \log n)\). That color class can be returned as a requested \(O(\log \log n)\)-approximation of the maximum weight independent set in \(\mathcal{R}\).

References

[1] Anna Adamaszek, Sariel Har-Peled, and Andreas Wiese. Approximation schemes for independent set and sparse subsets of polygons. *Journal of the ACM*, 66(4):Article 29, 2019.

[2] Anna Adamaszek and Andreas Wiese. Approximation schemes for maximum weight independent set of rectangles. In *Proceedings of the 54th Annual IEEE Symposium on Foundations of Computer Science*, pages 400–409. IEEE, 2013.

[3] Pankaj K. Agarwal, Marc van Kreveld, and Subhash Suri. Label placement by maximum independent set in rectangles. *Computational Geometry*, 11(3–4):209–218, 1998.

[4] Edgar Asplund and Branko Grünbaum. On a coloring problem. *Mathematica Scandinavica*, 8(1):181–188, 1960.

[5] Adam Bielecki. Problem 56.

[6] Paul Bonsma, Jens Schulz, and Andreas Wiese. A constant-factor approximation algorithm for unsplittable flow on paths. *SIAM journal on computing*, 43(2):767–799, 2014.

[7] Parinya Chalermsook. Coloring and maximum independent set of rectangles. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 123–134. Springer, 2011.

[8] Parinya Chalermsook and Julia Chuzhoy. Maximum independent set of rectangles. In *Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 892–901. SIAM, 2009.

[9] Parinya Chalermsook and Daniel Vaz. A note on fractional coloring and the integrality gap of LP for Maximum Weight Independent Set. *Electronic Notes in Discrete Mathematics*, 55:113–116, 2016.

[10] Timothy M. Chan. Polynomial-time approximation schemes for packing and piercing fat objects. *Journal of Algorithms*, 46(2):178–189, 2003.

[11] Timothy M. Chan and Sariel Har-Peled. Approximation algorithms for maximum independent set of pseudo-disks. *Discrete and Computational Geometry*, 48(2):373–392, 2012.

[12] Chandra Chekuri, Sariel Har-Peled, and Kent Quanrud. Fast LP-based approximations for geometric packing and covering problems. In *Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1019–1038. SIAM, 2020.

[13] Julia Chuzhoy and Alina Ene. On approximating maximum independent set of rectangles. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 820–829. IEEE, 2016.

[14] Jeffery S. Doerschler and Herbert Freeman. A rule-based system for dense-map name placement. *Communications of the ACM*, 35(1):68–80, 1992.

[15] Paul Erdős. Graph theory and probability. *Canadian Journal of Mathematics*, 11:34–38, 1959.

[16] Thomas Erlebach, Klaus Jansen, and Eike Seidel. Polynomial-time approximation schemes for geometric intersection graphs. *SIAM Journal on Computing*, 34(6):1302–1323, 2005.

[17] Uriel Feige. Approximating maximum clique by removing subgraphs. *SIAM Journal on Discrete Mathematics*, 18(2):219–225, 2004.

[18] Robert J. Fowler, Michael S. Paterson, and Steven L. Tanimoto. Optimal packing and covering in the plane are NP-complete. *Information Processing Letters*, 12(3):133–137, 1981.

[19] Takeshi Fukuda, Yasukiko Morimoto, Shinichi Morishita, and Takeshi Tokuyama. Data mining using two-dimensional optimized association rules: Scheme, algorithms, and visualization. *Acm Sigmod Record*, 25(2):13–23, 1996.

[20] Johan Håstad. Clique is hard to approximate within \(n^{1-\varepsilon}\). *Acta Mathematica*, 182(1):105–142, 1999.
[21] Clemens Hendler. Schranken für Färbungs- und Cliquenüberdeckungszahl geometrisch repräsentierbarer Graphen. Master’s thesis, Freie Universität Berlin, 1998.

[22] Sanjeev Khanna, Shan Muthukrishnan, and Mike Paterson. On approximating rectangle tiling and packing. In Proceedings of the ninth annual ACM-SIAM Symposium on Discrete Algorithms, pages 384–393. SIAM, 1998.

[23] Henry A. Kierstead and William T. Trotter. An extremal problem in recursive combinatorics. In Frederic Hoffman, editor, 12th Southeastern Conference on Combinatorics, Graph Theory, and Computing (CGTC 1981), volume 33 of Congressus Numerantium, pages 143–153. Utilitas Mathematica, Winnipeg, 1981.

[24] Alexandr Kostochka. Coloring intersection graphs of geometric figures with a given clique number. In János Pach, editor, Towards a Theory of Geometric Graphs, volume 342 of Contemporary Mathematics, pages 127–138. American Mathematical Society, Providence, RI, 2004.

[25] Tomasz Krawczyk and Bartosz Walczak. On-line approach to off-line coloring problems on graphs with geometric representations. Combinatorica, 37(6):1139–1179, 2017.

[26] Brian Lent, Arun Swami, and Jennifer Widom. Clustering association rules. In Proceedings 13th International Conference on Data Engineering, pages 220–231. IEEE, 1997.

[27] Liane Lewin-Eytan, Joseph Seffi Naor, and Ariel Orda. Routing and admission control in networks with advance reservations. In International Workshop on Approximation Algorithms for Combinatorial Optimization, pages 215–228. Springer, 2002.

[28] Arkadiusz Pawlik, Jakub Kozik, Tomasz Krawczyk, Michał Lasoń, Piotr Micek, William T. Trotter, and Bartosz Walczak. Triangle-free intersection graphs of line segments with large chromatic number. Journal of Combinatorial Theory, Series B, 105:6–10, 2014.

[29] Alexandre Rok and Bartosz Walczak. Outerstring graphs are \(\chi\)-bounded. SIAM Journal on Discrete Mathematics, 33(4):2181–2199, 2019.

[30] Alex Scott and Paul Seymour. A survey of \(\chi\)-boundedness. arXiv:1812.07500.

[31] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. Theory of Computing, 3:109–128, 2007.