A Framework for Analyzing Resparsification Algorithms

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Abstract

A spectral sparsifier of a graph $G$ is a sparser graph $H$ that approximately preserves the quadratic form of $G$, i.e., for all vectors $x$, $x^T L_G x \approx x^T L_H x$, where $L_G$ and $L_H$ denote the respective graph Laplacians. Spectral sparsifiers generalize cut sparsifiers, and have found many applications in designing graph algorithms. In recent years, there has been interest in computing spectral sparsifiers in semi-streaming and dynamic settings. Natural algorithms in these settings often involve repeated sparsification of a graph, and in turn accumulation of errors across these steps. We present a framework for analyzing algorithms that perform repeated sparsifications that only incur error corresponding to a single sparsification step, leading to better results for many of these resparsification-based algorithms.

As an application, we show how to maintain a spectral sparsifier in the semi-streaming setting: We present a simple algorithm that, for a graph $G$ on $n$ vertices and $m$ edges, computes a spectral sparsifier of $G$ with $O(n \log n)$ edges in a single pass over $G$, using only $O(n \log n)$ space, and $O(m \log^2 n)$ total time. This improves on previous best semi-streaming algorithms for both spectral and cut sparsifiers by a factor of $\log n$ in both space and runtime. The algorithm extends to semi-streaming row sampling for general PSD matrices. We also use our framework to combine a spectral sparsification algorithm by Koutis with improved spanner constructions to give a parallel algorithm for constructing $O(n \log^2 n \log \log n)$ sized spectral sparsifiers in $O(m \log^2 n \log \log n)$ time. This is the best combinatorial graph sparsification algorithm to date, and the size of the sparsifiers produced is only a factor $\log n \log \log n$ more than ones produced by numerical routines.

1 Introduction

Graph Sparsifiers. Consider an undirected graph $G$ with vertices $V$, and edges $E$, such that $|V| = n$ and $|E| = m$. A sparsifier of $G$ is a, hopefully sparser, graph $H$ that approximates $G$ in a meaningful way. Benczúr and Karger [BK96] introduced the notion of a combinatorial sparsifier (also called a cut-sparsifier), where $H$ is also a graph on $V$ such that for every cut of $V$, its value in $H$ is within a $1 \pm \epsilon$ factor of the same cut in $G$. Their result [BK96] gave an algorithm to construct a cut-sparsifier $H$ with $O(m \epsilon^{-2} \log n)$ edges in $O(m \log^2 n)$ time, and a better running time of $O(n \log^2 n + m)$ was shown in [FHHP11]. Cut sparsifiers led to faster algorithms for approximating $s$-$t$ min-cuts [BK96], sparsest cuts [BK96, KRV06], and undirected maximum-flows [KL15].

Spielman and Teng [ST04] generalized the notion of graph sparsification to the spectral setting. We define the Laplacian of $G$ to be the unique symmetric matrix $L_G$ such that for all vectors $x \in \mathbb{R}^n$, we have $x^T L_G x = \sum_{(i,j) \in E} (x_i - x_j)^2$, the natural quadratic form on $G$. $H$ is an $\epsilon$-spectral-sparsifier of $G$ if it approximately preserves the natural quadratic form, i.e., for all $x \in \mathbb{R}^V$, we have $(1-\epsilon)x^T L_G x \leq x^T L_H x \leq (1+\epsilon)x^T L_G x$. By considering $x \in \{0,1\}^n$, it is immediate that a spectral-sparsifier is also a cut-sparsifier. Spielman and Teng [ST04] gave an algorithm for constructing spectral sparsifiers with $O(n \epsilon^{-2} \log \log n)$ edges in $O(\log n \log \log n)$ time, and utilized them to design a nearly-linear time solver for linear systems.

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in graph Laplacians. These nearly-linear time solvers have been used as primitives in the design of fast algorithms [KM09, OSV12, KMP12, KLP15, KRS15]. Spielman and Srivastava [SS08] showed that sampling edges proportional to their leverage scores and rescaling them appropriately, produces a spectral sparsifier with $O(\sqrt{n\epsilon^{-2}\log n})$ edges with high probability.

The above works [KLP15, LS15, ACK+16] were motivated by the desire to develop better sparsification algorithms. Progress on sampling-based constructions include more robust sampling methods [KL13], and faster estimations of sampling probabilities [KLP15]. When combined with improved solvers for linear systems in graph Laplacians [KMP11, Pen13, KLP+16], these works lead to the current state-of-the-art running time of $O(m\log^2 n)$ for finding spectral sparsifiers with $O(\sqrt{n\epsilon^{-2}\log n})$ edges.

On the other hand, when such routines are used to analyze large graphs, there appears to be a significant gap between linear system solving oriented approaches motivated by these theoretical studies [MGLKT15], and the local, often combinatorial approaches taken in many practical studies [JS16, KFS13]. In this paper, we build upon recent developments in local steps for solving linear systems [KS16] to address a major obstacle for analyzing these local sparsification routines: the accumulation of error across randomized steps. We give a systematic way to analyze the error accumulation, and a versatile sufficient condition for showing that these local steps are no worse than global sampling steps. This result has direct consequences for sparsification in semi-streaming settings, as well as combinatorial constructions of graph sparsifiers.

**Semi-streaming algorithms.** For analyzing massive graphs, it is often prohibitive to even store the entire graph in memory. One model for describing such graphs is the streaming computational model, where the graph is presented as a stream of edges, and the algorithm is limited to a few passes over the stream, and space that is polylogarithmic in the input size. This model turns out to be too restrictive for many graph problems, as even simple problems such as $s$-$t$ connectivity require $\Omega(n/k)$ space with $k$ passes [HRR99]. The semi-streaming model, where the algorithm is permitted $O(n\text{polylog } n)$ space has been more fruitful for designing graph algorithms [FKM*05].

The problem of constructing graph sparsifiers in the semi-streaming model was first studied by Ahn and Guha [AG09]. They construct a cut-sparsifier for $G$ with $O(n\epsilon^{-2}\log n \log \frac{n}{\epsilon})$ edges using $O(n\epsilon^{-2}\log n \log \frac{n}{\epsilon})$ space. Kelner and Levin [KL13] gave a simple single-pass algorithm for constructing a spectral sparsifier. This algorithm maintains a sparsifier at every step by adding incoming edges to it, and resparsifying once the edge count reaches a certain threshold. The immediate concern is that we can not compute the sampling probabilities with respect to the final graph. The key idea in [KL13] is to compute the sampling probabilities with respect to the current sparsifier and perform rejection sampling using the ratio between these and the earlier probabilities. Assuming we have a good sparsifier of the graph that we have seen so far, we can obtain upper bounds for sampling probabilities for the final graph. Unfortunately, this introduces dependencies between the sampled edges, and the argument in [KL13] has difficulties handling these dependencies (this is discussed in more detail in [CMP16]). Our algorithm is closely related to that of Kelner and Levin, but only works with sampling probabilities computed based on the current graph.

1.1 Our Results: Analyzing Resparsification.

There are two main challenges in analyzing resparsification routines such as the one from [KL13]. The first is the dependencies between the edge samples. These dependencies prevent us from analyzing the entire sampling process as a single sparsification step, or invoke matrix concentration inequalities. The second is understanding the accumulation of errors: if we resparsify a graph $k$ times with an error of $1 \pm \epsilon$ at each step, we are only guaranteed an $(1 \pm \epsilon)^k \approx 1 \pm k\epsilon$ approximation to the original graph by the end. However, if the resparsifications are independent, an adversarial accumulation of error seems too pessimistic.

In this paper, we present a framework for analyzing resparsification routines, that allows us to handle the dependencies and avoids pessimistic accumulation of error, resulting in improved parameters. Applying our framework, we obtain the following result in the semi-streaming setting, for an algorithm very similar to the one in [KL13].

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1The primary difference between the algorithms is that we decide whether to keep each edge independently, while the
Theorem 1.1 (Semi-Streaming Sparsification). For all $\epsilon > 0$, the algorithm \textsc{StreamSparsify}, given as input a graph $G$ with $n$ vertices and $m$ edges, outputs a graph $H$ such that $H$ is an $(1+\epsilon)$-spectral sparsifier of $G$ with probability $1 - \frac{1}{\text{poly}(n)}$. \textsc{StreamSparsify} requires just one pass over $G$, and runs in $O(m \log^2 n)$ total time using $O(ne^{-2} \log n)$ space.

This result can viewed as reducing the memory usage of sparsification by leverage scores [SS08] to the size of its output, $O(ne^{-2} \log n)$, without any additional caveats. In the semi-streaming setting, it gives a tight analysis of the spectral sparsifier construction proposed in [KL13], and also improves on the previous best constructions of cut sparsifiers [AG09]. The fact that the error due to resparsifications does not accumulate is a priori surprising!

As another application of our framework, we show that it results in a better analysis of an algorithm of Koutis for constructing sparsifiers in parallel [Kou14]. Combining with a better sparsifier construction, we obtain the following result:

Theorem 1.2. For all $\epsilon > 0$, the algorithm \textsc{ParallelSparsify} given as input a graph $G$ with $n$ vertices and $m$ edges, outputs a graph $H$ with $O(ne^{-2} \log^2 n \log \log n)$ edges such that $H$ is a $(1+\epsilon)$-spectral sparsifier of $G$ with probability at least $1 - \frac{1}{\text{poly}(n)}$. The total work of the algorithm $O(m e^{-2} \log^2 n \log \log n)$ and the depth is $O(e^{-2} \log^4 n \log^* n)$.

This result gives the best combinatorial construction of spectral sparsifiers, improving over the work of Kapralov and Panigrahy [KP12] that constructs a sparsifier with $O(n e^{-3} \log^2 n)$ edges in $O(m e^{-3} \log^3 n)$ time. It also improves on the previous best result for parallel construction of spectral sparsifiers due to Koutis [Kou14] that constructs a sparsifier with $O(n e^{-2} \log^6 n)$ edges using $O(m e^{-2} \log^5 n)$ work and $O(e^{-2} \log^6 n)$ depth. A detailed comparison to the other sparsification algorithms is in Figure 2 of Section 4.

The question of whether similar or better guarantees can be obtained using combinatorial graph algorithms is important for deciding how graph sparsifiers can be used as an algorithmic tool. Works in this direction have centered around using sparsifiers either as direct effective resistance estimators [KP12] or as part of resparsification schemes [Kou14].

Framework and Techniques. Our framework for analyzing resparsification algorithms has several key components. We describe these components here with the semi-streaming algorithm as our running example. Our algorithm can be described roughly as follows: Let $S = O(ne^{-2} \log n)$ be the target size of our sparsifier. We start with the empty graph, and keep adding incoming edges until the sparsifier has size $S$. We then use the current sparsifier to estimate the leverage scores of the edges, and use them to sparsify the current sparsifier down to size $S/2$, and continue.

The first key conceptual idea is that we formulate the entire algorithm as a matrix martingale. In the semi-streaming setting, this corresponds to studying the Laplacian of the current sparsifier plus the remaining edges in the stream. Three simple observations illustrate the usefulness of this view: 1. Initially, this matrix is the Laplacian of the whole graph. 2. At the end, this matrix is the Laplacian of the sparsifier outputted by the algorithm, and 3. Each sparsification step preserves the expectation of this matrix.

The key tool we use for analyzing the matrix martingale is Freedman’s inequality [Tro11]. Freedman’s inequality is the martingale counterpart to Bernstein’s inequality. It allows us to control the deviation in the martingale via bounding the predictable quadratic variation, which is roughly the sum of the variances over the steps. In our applications, this predictable quadratic variation is then easily bounded using matrix Chernoff bounds [Tro12].

Our improved constructions of spectral sparsifiers in Section 4 are based on analyzing the resparsification steps of a sparsification algorithm due to Koutis [Kou14] in our framework. This routine uses bundles of spanners to repeatedly compute upper bounds of effective resistances that suffice for reducing edge count by a constant factor. We also provide improved parallel algorithms for finding these estimates in Section 4.1.

algorithm in [KL13] partially employs sampling with replacement.
2 Preliminaries

For symmetric matrices, we write $A \preceq B$ iff $B - A$ is a positive-semidefinite matrix. Throughout the paper, we use ‘with high probability’ for events that happen with probability at least $1 - \frac{1}{\text{poly}(n)}$, where $\text{poly}(n)$ can be set to an arbitrarily large polynomial by only adjusting constant factors.

Laplacians and Sparsifiers. We consider connected undirected graphs $G = (V, E)$, with vertices $V$ and edges $E$. We assume the edges have positive weights $w : E \to \mathbb{R}_+$. Let $n = |V|$ and $m = |E|$. Let $e_i$ denote the $i$th standard basis vector with a 1 in the $i$th-coordinate and 0 otherwise. For every edge $e$, we assign an arbitrary order to its endpoints, and for $e = (u, v)$, we define $b_e = e_u - e_v$. Finally, we define the Laplacian of $G$ as $L_G = \sum_{e \in E} w(e) b_e b_e^\top$. For every $x \in \mathbb{R}^n$, we have,

$$x^\top L_G x = \sum_{e \in E} w(e) x^\top b_e b_e^\top x = \sum_{e = (u, v) \in E} w(e) (x_u - x_v)^2.$$ 

It is immediate that $L_G \succeq 0$. Note that $L_G$ is independent of the choice of the ordering for each edge.

Definition 2.1 (Sparsifier). A graph $H(V, E')$ is said to be a $(1 \pm \epsilon)$-spectral-sparsifier of $G(V, E)$ if we have $E' \subseteq E$, and for all $x \in \mathbb{R}^V$,

$$(1 - \epsilon) x^\top L_G x \leq x^\top L_H x \leq (1 + \epsilon) x^\top L_G x.$$ 

Matrix Concentration. We will use the following two theorems due to Tropp [Tro11, Tro12]:

Theorem 2.2 (Matrix Chernoff). Consider a finite sequence $\{X_k\}$ of independent, random, symmetric $n \times n$-matrices. Assume that each random matrix satisfies

$$X_k \succeq 0 \text{ and } \|X_k\| \leq R \text{ almost surely.}$$

Define

$$\mu_{\text{max}} \overset{\text{def}}{=} \left\| \sum_k \mathbb{E}[X_k] \right\|.$$

Then, we have, for every $\delta > 0$,

$$\mathbb{P} \left[ \left\| \sum_k X_k \right\| \geq (1 + \delta) \mu_{\text{max}} \right] \leq n \cdot \left( \frac{e^\delta}{(1 + \delta)^{1 + \delta}} \right)^{\mu_{\text{max}}/R}.$$ 

Recall, a set of random variables $Y_0, Y_1, Y_2, \ldots$ that take values over symmetric $n \times n$ matrices is said to be a matrix martingale, if (informally), each $Y_j$ only depends on the previous variables $Y_0, \ldots, Y_{j-1}$ and $\mathbb{E}_{j-1}[Y_j] = Y_{j-1}$, where $\mathbb{E}_{j-1}[\cdot]$ denotes expectation conditional on $Y_0, \ldots, Y_{j-1}$.

Theorem 2.3 (Matrix Freedman). Let $Y_0, Y_1, Y_2, \ldots$ be a matrix martingale whose values are symmetric $n \times n$ matrices, and let $X_1, X_2, \ldots$ be the difference sequence $X_i = Y_i - Y_{i-1}$. Assume that the difference sequence is uniformly bounded in the sense that

$$\|X_k\| \leq R \text{ almost surely, for all } k.$$ 

Define the predictable quadratic variation process of the martingale:

$$W_k \overset{\text{def}}{=} \sum_{j=1}^k \mathbb{E}_{j-1} \left[ X_j^2 \right], \text{ for all } k.$$ 

Then, for all $t > 0$ and $\sigma^2 > 0$,

$$\mathbb{P} \left[ \exists k : \|Y_k\| \geq t \text{ and } \|W_k\| \leq \sigma^2 \right] \leq n \cdot \exp \left( -\frac{t^2}{2 \sigma^2 + R \delta / 3} \right).$$
3 Resparsification Game

We introduce a resparsification game that is an abstraction of a large class of resparsification algorithms. We will make minimal assumptions about the algorithm (or adversary) and prove that with the right choice of parameters, resparsification does not lead to error accumulation. In the next few sections, we will present algorithms for semi-streaming sparsification and parallel sparsification, and prove that they satisfy the assumptions of our resparsification game.

The Game. Say we’re given \( \epsilon \in (0, \frac{1}{2}) \), and \( m \) vectors \( \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n \), such that

\[
\sum_{i=1}^{m} \mathbf{a}_i \mathbf{a}_i^\top = M.
\]

Set \( \alpha := O(\log n \epsilon^{-2}) \), and initialize \( w_i \) to 1 for each \( 1 \leq i \leq m \). We will analyze a game played by an adversary on the weights \( w_i \). The game consists of a single move, repeated while the game is not over:

1. The adversary picks any \( i \in \{1, \ldots, m\} \) and \( p \in (0, 1] \) such that \( w_i \neq 0 \) and \( \frac{w_i}{p} \mathbf{a}_i^\top M \mathbf{a}_i \leq \frac{1}{\alpha} \). If there’s no such pair, the game ends and the adversary loses.
2. With probability \( p \), set \( w_i \leftarrow \frac{w_i}{p} \); otherwise, set \( w_i \leftarrow 0 \).

The adversary wins if at some point in the game, the matrix \( \sum w_i \mathbf{a}_i \mathbf{a}_i^\top \) is not a \((1 \pm \epsilon)\)-approximation to \( M \). More formally, the adversary wins if at any point in the game, the following condition fails to hold:

\[
(1 - \epsilon)M \preceq \sum w_i \mathbf{a}_i \mathbf{a}_i^\top \preceq (1 + \epsilon)M.
\]

Note that the main power gained by the adversary in this setting compared to static graph sparsification is that it can pick \( i \) and \( p \) based on the matrices and weights chosen so far. We will show that with appropriate settings of constants in \( \alpha \), the probability that the adversary wins the game can be controlled at \( O(n^{-c}) \).

Theorem 3.1. With high probability, the adversary will not win the game defined above.

For the analysis, we will make the following assumptions without loss of generality:

- \( \mathbf{M} = \mathbf{I} \). The case of arbitrary \( \mathbf{M} \) can be easily reduced to this case by multiplying all \( \mathbf{a}_i \) by \( \mathbf{M}^{-1/2} \) and projecting out the space corresponding to \( \ker(\mathbf{M}) \). This is a standard reduction, see e.g. \cite{BSS12}.

- For all \( i \), \( \mathbf{a}_i^\top \mathbf{a}_i \leq \frac{1}{\alpha} \). Any rows with higher leverage score can simply be excluded from the game.

- The adversary can only pick \( p \geq \frac{1}{2} \). Note that the adversary can still simulate an arbitrarily small value of \( p \) through multiple moves.

We will also assume without loss of generality that the randomness in the game is generated in the following way. Let \( x_1, \ldots, x_m \sim \text{Exp}(1) \) be independent random variables drawn from the exponential distribution with parameter 1. Whenever the adversary picks a pair \((i, p)\), the weight \( w_i \) is not set to zero iff

\[
\frac{w_i}{p} \leq e^{x_i}.
\]

The probability of this event, conditioning on the history of the game, is

\[
P \left[ e^{x_i} \geq \frac{w_i}{p} \mid e^{x_i} \geq w_i \right] = p
\]

because of the memoryless property of the exponential distribution.

\[2\text{Here, } \mathbf{M}^\dagger \text{ denotes the Moore-Penrose pseudoinverse of } \mathbf{M} \].
3.1 Bounding the Predictable Quadratic Variation

We consider a martingale with the difference \( X_j \) corresponding to the \( j \)-th move by the adversary, or 0 if the adversary has made fewer than \( j \) moves. Assume the adversary chooses row \( i \) and keeping probability \( p \); then we have

\[
X_j \triangleq \begin{cases} 
\frac{1-p}{p} \cdot w_i a_i a_i^\top & \text{with probability } p \\
-w_i a_i a_i^\top & \text{otherwise.}
\end{cases}
\]

Let \( \{W_k\} \) be the predictable quadratic variation process of the martingale given by the \( \{X_j\} \):

\[
W_k \triangleq \sum_{j=1}^{k} E_{j-1} \left[ X_j^2 \right].
\]

In order to bound \( \|W_k\| \), we need an auxiliary lemma. For all \( i \in \{1, \ldots, m\} \) define

\[
w_i \triangleq \min(e^{x_i}, \frac{1}{\alpha a_i a_i^\top}).
\]

Note that the \( w_i \) are independent random variables, and, throughout the entire game, we have, \( w_i \leq \bar{w}_i \).

**Lemma 3.2.** With high probability, we have that

\[
\sum_{i=1}^{m} \bar{w}_i^2 (a_i a_i^\top)^2 \preceq \frac{4}{\alpha}. \]

**Proof:** Let \( Z_i \triangleq \bar{w}_i^2 (a_i a_i^\top)^2 \), for \( i = 1, \ldots, m \). Our goal is to bound the norm of the sum of the independent matrices \( Z_i \). First of all, note that we always have \( \|Z_i\| \leq \frac{1}{\alpha} \). Moreover,

\[
E[Z_i] = \left( P \left[ e^{x_i} > \frac{1}{\alpha a_i a_i^\top} \right] \left( \frac{1}{\alpha a_i a_i^\top} \right)^2 + \int_{0}^{-\log \alpha a_i a_i^\top} e^{-x} (e^x)^2 dx \right) (a_i a_i^\top)^2 = \left( \frac{2}{\alpha a_i a_i^\top} - 1 \right) (a_i a_i^\top)^2 \leq \frac{2}{\alpha} (a_i a_i^\top)^2.
\]

Thus, we get \( \sum_{i=1}^{m} E[Z_i] \leq \frac{2}{\alpha} \). Theorem 2.2 with \( \mu_{\text{max}} = \frac{2}{\alpha}, \delta = \frac{2}{\alpha \mu_{\text{max}}}, R = \frac{1}{\alpha} \) then gives the lemma. ■

We can now bound the total quadratic variation \( W_k = \sum_{j=1}^{k} E_{j-1} \left[ X_j^2 \right] \) as specified in the Matrix Freedman inequality in Theorem 2.3.

**Lemma 3.3.** With high probability, for all \( k \) we have that

\[
\|W_k\| \leq \frac{16}{\alpha}.
\]

**Proof:** Suppose the adversary picks \( i \) and \( p \) in the \( j \)-th move. In what follows, \( w_i \) refers to the value of \( w_i \) in the \( j \)-th move. We have

\[
E_{j-1} \left[ X_j^2 \right] = \frac{1-p}{p} \cdot w_i^2 (a_i a_i^\top)^2 = \left( \frac{w_i^2}{p} - w_i^2 \right) (a_i a_i^\top)^2 \leq \left( \frac{w_i^2}{p^2} - w_i^2 \right) (a_i a_i^\top)^2.
\]
In order to bound $W_k$, we sum the above expression for all $j$, grouping them by the index $i$ picked in the $j$th round. For every fixed $i$, we have a telescoping sum. Since $p \leq \frac{1}{2}$, and $w_i$ is always less than $\max_i$, the sum for $i$ is spectrally upper bounded by $4w_i^2(a_i a_i^\top)^2$. Hence, we have

$$W_k \leq \sum_{i=1}^m 4 \cdot w_i^2 (a_i a_i^\top)^2.$$

The claim now follows from Lemma 3.2.

**Proof of Theorem 3.1:** The condition on $p_i$ ensure that $\|X_k\| \leq \frac{1}{\alpha}$. Together with Lemma 3.3, applying Theorem 2.3 with $t = \epsilon$, $\alpha = \frac{16}{\alpha}$, $R = \frac{1}{\alpha}$ yields the theorem.

3.2 Application to Streaming Sparsification

Consider a sparsification algorithm that reads edges of the graph one by one and adds them to the sparsifier, and resparsifies when too many edges have been accumulated. Such an algorithm can be implemented in $O(n \log n \epsilon^{-2})$ space and nearly linear time (see Figure 1).

Theorem 3.1 gives that any such algorithm will end up with a sparsifier of the original graph. Some additional care is required to show that we can maintain a sparsifier of the current graph at all times.

**Lemma 3.4.** Let $\tilde{A}$ be the matrix returned by STREAMSPARSIFY($A, \epsilon$). Then, with high probability, $(1 - \epsilon)A^\top A \preceq \tilde{A}^\top \tilde{A} \preceq (1 + \epsilon)A^\top A$.

**Proof:** To apply Theorem 3.1, we consider coupling the algorithm with an adversary in a resparsification game. If the algorithm picks an $i$ such that $\beta \cdot \tilde{\tau}_i < 1$, and samples the row, the adversary pick the same $i$, and picks $p = \beta \tilde{\tau}_i$. We argue by induction that the probability that the algorithm fails is upper bounded by the probability that the adversary wins. At any step in the algorithm, assuming that the adversary has not won so far, Since $\tilde{\tau}_i$ is an upper bound on the leverage score, we have

$$p \geq \beta \tilde{a}_i^\top (\tilde{A}^\top \tilde{A})^\dagger \tilde{a}_i^\top \geq \beta \tilde{a}_i^\top \left( \sum_{i=1}^j \tilde{a}_i^\top + \sum_{i=j+1}^m a_i a_i^\top \right)^\dagger \tilde{a}_i^\top \geq \frac{\beta}{1 + \epsilon} \tilde{a}_i^\top (A^\top A)^\dagger \tilde{a}_i^\top \geq \alpha \tilde{a}_i^\top (A^\top A)^\dagger \tilde{a}_i^\top,$$
as required by the game. Hence, the adversary can make a legal move identical to the algorithm. Here, the third inequality uses the fact that adversary has not won so far. If the adversary continues to play the game till the end of the algorithm, the algorithm fails only if the adversary wins before the last step.

Thus, the algorithm succeeds with probability at least as much as the probability that the adversary loses, which it does with high probability. ■

**Proof of Theorem 1.1:** For a graph $G(V,E)$ with edge weights given by $w$, we can write the Laplacian as $L_G = \sum_e (\sqrt{w_e} b_e)(\sqrt{w_e} b_e)^\top$, and hence apply lemma 3.4. It remains to show that step 3(b) of STREAMSPARSIFY can be implemented in $O(n,\beta \log^2 n)$ time. This can be done by combining a fast solver for linear systems in graph Laplacians [KMP11, Pen13], with a Johnson-Lindenstrauss based estimation procedure, as in [SS08, KLP15, Pen13]. ■

## 4 Improved Combinatorial Algorithms for (Parallel) Spectral Sparsification

In [Kou14], Koutis gives a simple parallel algorithm for graph sparsification. Unlike other sparsification algorithms that rely on effective resistances [SS08], this algorithm is combinatorial. It finds, via a collection of spanners, upper bounds of leverage scores that suffice for reducing edge counts. The running time of this method, as well as a previous spanner based sparsification algorithm by Kapralov and Panigrahy [KP12] are several log factors away from the numerical methods. A summary of the best known bounds is presented in Figure 2.

Compared to the algebraic sparsification algorithms, all previous spanner based sparsification routines have overheads of either $\log n$ or $\log \log^2 n$ over the underlying spanner constructions. This is due to the more gradual size reductions offered by combinatorial routines for estimating sampling probabilities. Specifically, the spanner based routines from [Kou14] output a set of probabilities that upper bound the true probabilities, but sum to $O(nh) + m/k$ where $h$ is an overhead related to the combinatorial algorithm and $k$ is a ‘reduction factor’ that can be picked. As a result, such routines need to be invoked repeatedly for up to $O(\log n)$ times to bring $m$ close to $n$. This is handled either by doing some sampling ahead of the estimation steps [KP12], or by having the errors compound during these steps [Kou14]. In each of these invocations, the $O(nh)$ term is accumulated in the edge count, leading to an overhead in edge count of at least $\log n$. \(^4\)

The guarantees we give for resparsification immediately imply that these errors no longer accumulate. Algorithmically it means it suffices to set the error threshold in each of these steps to the desired final accuracy, instead of one that’s smaller by a factor of $\log n$. This leads to the following result:

\(^3\)These bounds take into account the improvements in [Pen13]. The nearly $m \log^{1/2}$ time solvers in [CKM+14] are for vector-based guarantees. Converting them into operator guarantees would incur an additional factor of $O(\log n)$, leading to a higher total than directly using the algorithm from [KMP11]

\(^4\)It is possible to reduce this overhead to polylog $\log n$ using steps similar to those in [KLP15], but we’re not aware of an explicit statement of this.

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**Table:**

| Algorithm / Reference | Sparsifier Size | Runtime (work) |
|-----------------------|-----------------|----------------|
| Effective Resistances + Solvers [SS08, KMP11] \(^3\) | $O(n \log ne^{-2})$ | $O(m \log^2 n)$ |
| Barrier functions [LS15] | $O(\theta n)$ | $O(m^{1+1/\theta})$ |
| Graph partitioning [ST11, OV11] | $O(n \log^6 ne^{-2})$ | $O(m \log^6 n)$ |
| Random spanners [KP12] | $O(n \log^3 ne^{-3})$ | $O(m \log^3 ne^{-3})$ |
| Effective resistance sampling by spanners [Kou14] | $O(n \log^2 n \log^3 (m/n)e^{-2})$ | $O(m \log^3 n \log^3 (m/n)e^{-2})$ |
| Resparsification (this paper) | $O(n \log^2 n \log ne^{-2})$ | $O(m \log^2 ne^{-2})$ |

**Figure 2:** Runtime Bounds of Efficient Sparsification Algorithms

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8
Theorem 4.1. Given a graph $G$ with $n$ vertices, $m$ edges, and an error parameter $\epsilon > 0$, we can compute w.h.p. in $\mathcal{O}(m \log^2 n \epsilon^{-2})$ work and $\mathcal{O}(\epsilon^{-2} \log^3 n \log^* n)$ depth a $(1 \pm \epsilon)$-spectral-sparsifier $H$ of $G$ with $\mathcal{O}(n \log^2 n \log \log n \epsilon^{-2})$ edges.

Pseudocode of the algorithm is given in Figure 3. It relies on a spanner based subroutine for estimating total sampling probabilities.

In addition to this refined analysis of sparsification, we also improve upon the leverage score estimation from [Kou14]. Directly applying the parallel spanner algorithm from [MPVX15] leads to a depth dependent on $\log U$, where $U$ is the ratio between the maximum and minimum edge weights in the graph. In Section 4.1, we remove this factor by relaxing the requirements needed for spanners. This in fact further simplifies the algorithms. In particular, we obtain an algorithm for estimating leverage scores of the edges with the following guarantees:

Lemma 4.2. There exists a routine SpannerEstimate that takes a graph $G$ with $n$ edges, $m$ vertices, and a parameter $\alpha \geq 1$ and computes in total work $\mathcal{O}(m \alpha \log^2 n \epsilon^{-2})$ and depth $\mathcal{O}(\log^3 n \epsilon^{-2} \log^* n)$ estimates $\hat{\tau}_e$ for all the edges such that with high probability:

1. For each edge $e$, we have $\tau_e \leq \hat{\tau}_e \leq 1$ where $\tau_e$ is the true leverage score of $e$ in $G$.
2. $\sum_e \min\{1, \alpha \hat{\tau}_e\} \leq \mathcal{O}(\alpha \cdot n \log n \log \log n) + m/10$.

We use $\hat{\tau}$ to denote the probability estimates because $\tau$ is often used to denote exact statistical leverage scores in the randomized numerical linear algebra literature. Such a routine then meets the properties of the probability estimation algorithm with $h = \mathcal{O}(\alpha \log n \log \log n)$ and $k = 10$. Applying the resparsification game as described in Theorem 3.1 with this routine as the adversary then leads to the sparsification algorithm shown in Figure 3.

**Lemma 4.2.** There exists a routine SpannerEstimate that takes a graph $G$ with $n$ edges, $m$ vertices, and a parameter $\alpha \geq 1$ and computes in total work $\mathcal{O}(m \alpha \log^2 n \epsilon^{-2})$ and depth $\mathcal{O}(\log^3 n \epsilon^{-2} \log^* n)$ estimates $\hat{\tau}_e$ for all the edges such that with high probability:

1. For each edge $e$, we have $\tau_e \leq \hat{\tau}_e \leq 1$ where $\tau_e$ is the true leverage score of $e$ in $G$.
2. $\sum_e \min\{1, \alpha \hat{\tau}_e\} \leq \mathcal{O}(\alpha \cdot n \log n \log \log n) + m/10$.

We use $\hat{\tau}$ to denote the probability estimates because $\tau$ is often used to denote exact statistical leverage scores in the randomized numerical linear algebra literature. Such a routine then meets the properties of the probability estimation algorithm with $h = \mathcal{O}(\alpha \log n \log \log n)$ and $k = 10$. Applying the resparsification game as described in Theorem 3.1 with this routine as the adversary then leads to the sparsification algorithm shown in Figure 3.

**Figure 3: Spectral Sparsification Algorithm**

The correctness of this sparsification routine follows from combining the estimation guarantees with the resparsification game.

**Proof.** (of Theorem 4.1) It is easy to check that the resampling of edges in step 2b) of PARALLELSPARSIFY follows the rules of the game defined in Section 3, unless the game is already lost. Thus, the fact that we obtain a sparsifier follows in a similar fashion to that in proof of Lemma 3.4.

The edge count is given by the termination condition given in Figure 3, so it only remains to bound running time. By an ordinary Chernoff bound, we have that w.h.p. that as long as the edge count of $G^{(i)}$ is
\( \omega(n \log^2 n \log \log n \epsilon^{-2}) \), it decreases by a constant factor in step 2b) with high probability. As \( m \leq n^2 \), this process terminates in \( \mathcal{O}(\log n) \) steps.

In each of these steps, the cost is dominated by the computation of leverage score estimates \( \hat{\tau}^{(i)} \) using \textsc{SpannerEstimate} from Lemma 4.2. As the edge counts are geometrically decreasing, this total is within a constant factor of the first step, giving a total cost of \( \mathcal{O}(m \log^2 n \epsilon^{-2}) \).

One further consequence of this leverage score oracle is that the sparsifier is a union of forests. This property is crucial to some uses of sparsifiers in combinatorial algorithms, such as their recent incorporation in data structures [ADK+16]. By accounting for the structure of the output of \textsc{ProbabilisticSpanner} given in Section 4.1, we have a similar property.

**Corollary 4.3.** The output of Theorem 4.1 can be written as a sum of \( \mathcal{O}(\log^2 n \log \log n \epsilon^{-2}) \) forests.

This result is similar to the ones from [CCPP14], and improves the dynamic sparsification algorithms in [ADK+16] by a factor of \( \log^2 n \). Furthermore, its running time is within \( \mathcal{O}(\log \log n) \) factor of numerically oriented routines based on random projections and solving linear systems [SS08]. As a result, we’re optimistic about the practical potential of this sparsification approach.

On the other hand, this improvement over [Kou14] translates to a fairly minor improvement for constructing spanner chains [Kou14, CCL+15, KLP+16]. This is because the running time of those routines directly depend on running the sparsification algorithm on its own output. In our case, as the runtime depends on \( m \), the two factors of \( \log^2 n \) accumulate to \( \log^4 n \), a rather large overhead. Further improvements in this direction would require lowering the dependencies in \( m \), in ways similar to the \( \mathcal{O}(m + n \log^2 n) \) time sparsification algorithms in [KLP15, FHHP11, HP10, JK15].

### 4.1 Faster Probabilistic Spanners

Koutis [Kou14] obtained estimates of effective resistance estimates as needed in Lemma 4.2 through combinatorial combinations of resistors in series and parallel. Here the resistance of a single resistor can be viewed as the inverse of its weight. Low resistance estimates can be obtained via the following two facts:

**Fact 4.4.**

1. A sequence of resistors in series with resistances \( r_1, r_2, \ldots, r_k \) has effective resistance \( r_1 + r_2 + \ldots + r_k \).
2. A set of \( k \) parallel, non-overlapping paths, each with effective resistance at most \( r \), has resistance at most \( r/k \).

Part 1 suggests that short paths can give reasonable bounds, while Part 2 suggests that these bounds can be further improved by having multiple disjoint paths. Koutis [Kou14] combines these by repeatedly removing spanners from a graph. Each spanner is a sparse graph that guarantees that all remaining edges have ‘shortcuts’ that are longer by a factor of \( \mathcal{O}(\log n) \). Specifically, it guarantees that the stretch of an edge \( e = uv \),

\[
\text{str}_H(e = uv) \equiv \frac{\text{dist}_H(u, v)}{l_e},
\]

where \( \text{dist}_H(u, v) \) is the shortest path distance in \( l \) in \( H \) between \( u \) and \( v \), is bounded by \( \mathcal{O}(\log n) \).

If the lengths are set to resistances, specifically \( l_e = \frac{1}{r_e} \), each spanner provides a path of resistance \( \Omega(\log n) \) meeting condition Part 1 for the edge \( e \). Removing this spanner and building another one then gives another such path, and repeating then gives the \( k \) disjoint paths need in Part 2. Our primary improvement is demonstrating that probabilistic spanners also suffice for these purposes.

**Definition 4.5.** The random graph \( H \) is a probabilistic \( k \)-spanner for \( G \) if \( H \subseteq G \) and every edge in \( G \) has probability at least \( 1/2 \) to have stretch at most \( k \) in \( H \).
In the rest of this section, we show that probabilistic $O(\log n)$-spanners of weighted graphs of size $O(n \log \log n)$ can be constructed in $O(m)$ work and $O(\log n \log^* n)$ parallel depth. Compared to the parallel spanner construction from [MPVX15], our routine has depth that’s lower by a factor of $\log U$, where $U$ is the ratio of the maximum to minimum edge weights. Our algorithm is based on directed invocations of an exponential start time clustering routine, whose guarantees as given in [MPX13] and [MPVX15] \(^5\) are:

**Lemma 4.6.** There is a routine \(\text{ESTCluster} \) that given a graph \(G = (V, E)\) with \(n\) vertices, \(m\) edges, and all weights \(w_e \geq 1\), along with a parameter \(\beta > 0\), \(\text{ESTCluster}(G, \beta)\) generates in \(O(\beta^{-1} \log n \log^* n)\) depth and \(O(m)\) work a partition of the vertices into clusters \(\mathcal{X} = (X_1 \cup X_2 \cup \ldots \cup X_k)\) such that:

1. \(X_1 \ldots X_k\) form a disjoint partition of \(V\).
2. With high probability, the combinatorial (unweighted) diameter of each \(X_i\) is certified by a spanning tree on \(X_i\) with diameter \(O(\beta^{-1} \log n)\).
3. For any neighboring vertices \(u\) and \(v\), the probability that \(u\) and \(v\) are in different clusters is at most \(\beta\).

Although this routine can be extended to weighted graphs, spanner constructions using it also work with the unweighted variant [MPVX15]. We will invoke it by first partitioning the graphs by weight scales (which we denote with superscripts \((i)\)), applying this clustering routine to each scale separately, and then combining the results using minimum spanning trees. Pseudocode of this algorithm is given in Figure 4.

\[
S = \text{ProbSpanner}(G, l), \text{ where } G \text{ is a graph with } n \text{ vertices and } m \text{ edges, and } l \text{ are lengths on the edges.}
\]

1. For integers \(i\) in parallel:
   
   a. Let \(E^{(i)}\) be the set of edges in \(G\) of length in \([2^i, 2^{i+1})\), and \(G^{(i)}\) be the unweighted graphs formed from \(E^{(i)}\).
   
   b. Let \(\mathcal{X}^{(i)} \leftarrow \text{ExpCluster}(G^{(i)}, 1/3)\), and \(C^{(i)}\) be the union of the BFS trees certifying the diameters of the clusters in \(\mathcal{X}^{(i)}\), but with the original edge weights from \(G\).

2. Let \(t := O(\log \log n)\).

3. For each \(0 \leq j < t\), let \(F^{(j)}\) be a minimum spanning forest of \(\bigcup_{i \equiv j \pmod{t}} C^{(i)}\).

4. Return the union of \(F^{(j)}\).

Figure 4: Probabilistic Spanner Algorithm

**Theorem 4.7.** The random graph \(S := \text{ProbabilisticSpanner}(G)\) is a probabilistic \(O(\log n)\)-spanner of \(G\). Moreover, it is the sum of \(O(\log \log n)\) trees, and can be computed in \(O(m)\) total work and \(O(\log^2 n \log^* n)\) depth.

**Proof:** The work and depth bounds follow from Lemma 4.6 and the cost of finding minimum spanning trees in parallel [JaJ92, Lei92]. So it suffices to show that each edge has stretch \(O(\log n)\) with probability at least \(1/2\).

Consider an edge \(e = uv \in E^{(i)}\). Part 3 of Lemma 4.6 gives that the probability that \(u\) and \(v\) belong to the same cluster of \(\mathcal{X}^{(i)}\) with probability at least \(1 - \beta = \frac{2}{3}\). Combining this with Part 2 of Lemma 4.6 gives that with probability at least \(1/2\) we have that \(e\) is contained in some cluster \(X_k^{(i)}\), and that the combinatorial

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\(^5\)We omit a precise pointer as these manuscripts are still undergoing edits.
diameter of all clusters is \( c_0 \cdot \log n \) for some absolute constant \( c_0 \). We will now use these two conditions to bound the stretch of \( e \) in \( F^{(i)} \), where \( 0 \leq j < t \) and \( j \equiv i \) (mod \( t \)).

Let \( i_1 < i_2 < \ldots < i_k \) be the indices \( \equiv i \) (mod \( t \)) for which \( E^{(i)} \) is nonempty. We will show by induction that if \( 2^i \geq 8c_0 \cdot \log n \), then the diameter of any connected component of any minimum spanning forest \( T^{(q)} \) of \( C^{(i_1)} \cup \ldots \cup C^{(i_k)} \) is bounded by

\[
4c_0 \cdot 2^{i_k} \log n.
\]

The base case of \( q = k \) follows directly from the diameter bound.

For the inductive case, the inductive hypothesis gives that all connected components of \( T^{(q-1)} \) have diameter at most \( 4c_0 \cdot 2^{i_q} \log n \), while we also have that the length of each edge in \( E^{(i_q)} \) is at most \( 2^{i_q+1} \). Since the combinatorial diameter of each connected component of \( C^{(i_q)} \) is \( c_0 \log n \), the diameter of \( T^{(q)} \) is bounded by the lengths of these edges / components alternating, giving:

\[
c_0 \log n \cdot 2^{i_q+1} + (c_0 \log n + 1) \left( 4c_0 \cdot 2^{i_q-1} \log n \right).
\]

The condition of \( 2^i \geq 8c_0 \log n \) and \( i_q \geq i_{q-1} + t \) then imply \( 2^{i_q} \geq 2^{i_{q-1}} \), and the total is bounded by \( 4c_0 2^{i_q} \log n \), so the inductive hypothesis holds for \( q \) as well.

The bound on stretch then follows from \( e \) having length at least \( 2^i \).

Repeatedly invoking this probabilistic spanner routine then leads to the leverage score estimation algorithm. Its pseudocode is given in Figure 5.

\[
\tilde{\tau} = \text{SpannerEstimate}(G, \alpha), \text{ where } G \text{ is a graph with } n \text{ vertices and } m \text{ edges.}
\]

1. Let \( H^{(0)} := \emptyset, k := \Theta(\alpha \cdot \log n) \).
2. For \( i := 1, \ldots, k \):
   - Let \( H^{(i)} := H^{(i-1)} \cup \text{ProbSpanner}(G \setminus H^{(i-1)}, 1/w) \).
3. Let
   \[
   \tilde{\tau}_e := \begin{cases} 
   1 & \text{for } e \in H^{(k)} \\
   \frac{1}{\alpha} & \text{for all other } e \in G
   \end{cases}
   \]
4. Return \( \tilde{\tau} \).

Figure 5: Probabilistic Spanner Based Leverage Score Estimation

**Proof of Lemma 4.2:** The work and depth guarantees follow from the cost of invoking \textsc{ProbabilisticSpanner} \( O(\alpha \cdot \log n) \) times.

For the stretch bounds, by Chernoff bounds for scalars, since \( k \geq \Omega(\log n) \), we have that with high probability every remaining edge will have stretch at most \( O(\log n) \) w.r.t. at least \( k/3 \) of the removed spanners. This means that for every remaining edge \( e \) there are \( \Omega(\alpha \cdot \log n) \) edge disjoint paths each of which has length at most \( O(\log n) \) times the length of \( e \). Since the length of each edge was set to its resistance, Fact 4.4 then gives that these paths upper bound the statistical leverage score of the edge by \( 1/(10\alpha) \). As the total number of edges in each spanner is \( O(n \log \log n) \), we get \( \sum_e \min\{1, \alpha \tilde{\tau}_e\} \leq O(\alpha \cdot n \log n \log \log n) + m/10 \).

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