INTERSECTION $K$-THEORY

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Abstract. For a proper map $f : X \to S$ between varieties over $\mathbb{C}$ with $X$ smooth, we introduce increasing filtrations $P^c_f \subset P^\leq f \subset P^\geq f$ on $\text{gr} \, K(X)$, the associated graded on $K$-theory with respect to the codimension filtration, both sent by the cycle map to the perverse filtration on cohomology $^pH^*_f(X)$. The filtrations $P^c_f$ and $P^\leq f$ are functorial with respect to proper pushforward and $P^\geq f$ is functorial with respect to pullback.

We use the above filtrations to propose two definitions of (graded) intersection $K$-theory $\text{gr} \, IK^\cdot(S)$ and $\text{gr} \, IK^\cdot(S)$. Both have cycle maps to intersection cohomology $IH^\cdot(S)$. We conjecture a version of the decomposition theorem for semismall surjective maps and prove it in some particular cases.

1. Introduction

For a complex variety $X$, intersection cohomology $IH^\cdot(X)$ coincides with singular cohomology with rational coefficients $H^\cdot(X)$ when $X$ is smooth and has better properties than $H^\cdot(X)$ when $X$ is singular, for example it satisfies Poincaré duality and the Hard Lefschetz theorem. Many applications of intersection cohomology, for example in representation theory [18], [7, Section 4], are through the decomposition theorem of Beilinson–Bernstein–Deligne–Gabber [3].

A construction of intersection $K$-theory is expected to have applications in computations of $K$-theory via a $K$-theoretic version of the decomposition theory, and in representation theory, for example in the construction of representations of vertex algebras using (framed) Uhlenbeck spaces [5]. The Goresky–MacPherson construction of intersection cohomology [15] does not generalize in an obvious way to $K$-theory.

1.1. The perverse filtration and intersection cohomology. For $S$ a variety over $\mathbb{C}$, intersection cohomology $IH^\cdot(S)$ is a subquotient of $H^\cdot(X)$ for any resolution of singularities $f : X \to S$. The decomposition theorem implies that $IH^\cdot(S)$ is a (non-canonical) direct summand of $H^\cdot(X)$. Consider the perverse filtration

$$^pH^c^\leq f(X) := H^\cdot(S, ^p\tau^c_!Rf_*IC_X) \hookrightarrow H^\cdot(S, Rf_*IC_X) = H^\cdot(X).$$

For $V \hookrightarrow S$, denote by $X_V := f^{-1}(V)$. Let $A_V$ be the set of irreducible components of $X_V$ and let $c^n_V$ be the codimension on $i_V^n : X^n_V \hookrightarrow X$ for $a \in A_V$. Consider a
3.5. In (1), we further impose that \( \Gamma \) is a quasi-smooth scheme surjective over \( X \). Let \( g^\alpha_a : Y^\alpha_a \to V \). Define

\[
\begin{align*}
\hat{p}H^\leq_i|_{f,V} & := \bigoplus_{a \in A_V} \iota^a_{V*} \pi^\alpha_a : \hat{p}H^\leq_i(Y^\alpha_a) \subset \hat{p}H^\leq_i(X), \\
\hat{p}H^\leq_i & := \bigoplus_{V \leq S} \hat{p}H^\leq_i \subset \hat{p}H^\leq_i(X).
\end{align*}
\]

The decomposition theorem implies that

\[
IH^i(S) \cong \hat{p}H^\leq_0(X)^i/\hat{p}H^\leq_0(X).
\]

1.2. **Perverse filtrations in \( K \)-theory.** Inspired by the above characterization of intersection cohomology via the perverse filtration, we propose two \( K \)-theoretic perverse filtrations \( \mathcal{F}_f \subset \mathcal{P}_f \) on \( \mathcal{P}_K(X) \) for a proper map \( f : X \to S \) of complex varieties with \( X \) smooth. Here, the associated graded \( \mathcal{P}_K(X) \) is with respect to the codimension of support filtration on \( K(X) \) [13, Section 5.4].

The precise definition of the filtration \( \mathcal{P}_f \mathcal{P}_K(X) \) is given in Subsection 3.3, roughly, it is generated by (subspaces of) images

\[
(1) \quad \Phi_T : \mathcal{P}_K(T) \to \mathcal{P}_K(X)
\]

induced by correspondences \( \Gamma \) on \( X \times T \) of restricted dimension, see [3], for \( T \) a smooth variety with a generically finite map onto a subvariety of \( S \). These subspaces of the images of \( \Phi_T \) are required to satisfy certain conditions when restricted to the subvarieties \( Y^\alpha_a \) from Subsection 1.1.

The definition of the filtration \( \mathcal{P}_f \mathcal{P}_K(X) \subset \mathcal{P}_f \mathcal{P}_K(X) \) is given in Subsection 3.3. In [1], we further impose that \( \Gamma \) is a quasi-smooth scheme surjective over \( T \). This further restricts the possible dimension of the cycles \( \Gamma \), see Proposition 3.9, and allows for more computations.

**Theorem 1.1.** Let \( f : X \to S \) be a proper map with \( X \) smooth. Then the cycle map \( \mathcal{P}_f \mathcal{P}_K(X)_Q \to \mathcal{P}_f \mathcal{P}_K(X) \) respects the perverse filtration

\[
\mathcal{P}_f \mathcal{P}_K(X)_Q \subset \mathcal{P}_f \mathcal{P}_K(X)_Q \to \mathcal{P}_f \mathcal{P}_K(X).
\]

Perverse filtrations in \( K \)-theory have the following functorial properties. Let \( X \) and \( Y \) be smooth varieties with \( c = \dim X - \dim Y \). Consider proper maps

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & X \\
\downarrow{g} & & \downarrow{f} \\
S & \xrightarrow{f} & \end{array}
\]

There are induced maps

\[
\begin{align*}
& h_* : \mathcal{P}_g^{\leq i-c} \mathcal{P}_K(Y) \to \mathcal{P}_f^{\leq i} \mathcal{P}_K(X), \\
& h_* : \mathcal{P}_g^{\leq i-c} \mathcal{P}_K(Y) \to \mathcal{P}_f^{\leq i} \mathcal{P}_K(X), \\
& h_* : \mathcal{P}_f^{\leq i-c} \mathcal{P}_K(X) \to \mathcal{P}_g^{\leq i} \mathcal{P}_K(Y).
\end{align*}
\]
If $h$ is surjective, then there is also a map
\[ h^* : P^\leq_{f-c} \text{gr} K(X) \to P^\leq_g \text{gr} K(Y). \]

Let $f : X \to S$ be a resolution of singularities. We define $\tilde{P}^\leq_{f} \text{gr} K(X)$ and $P^\leq_{f} \text{gr} K(X)$ similarly to $\tilde{P}^\leq_{f} \text{gr} K(X)$. Inspired by the discussion in cohomology from Subsection 1.1, define
\[ \text{gr} I\mathcal{K} \cdot (S) := P^\leq_{f} \text{gr} K(X)/\left( \tilde{P}^\leq_{f} \text{gr} K(X) \cap \ker f_* \right). \]

Note that we do not construct $\text{gr} I\mathcal{K} \cdot (S)$ and $\text{gr} I\mathcal{K} \cdot (S)$ as associated graded of spaces $I\mathcal{K} \cdot (S)$ or $I\mathcal{K} \cdot (S)$, but we hope that such a construction is possible, see Subsection 1.6.

**Theorem 1.2.** The definitions of $\text{gr} I\mathcal{K} \cdot (S)$ and $\text{gr} I\mathcal{K} \cdot (S)$ do not depend on the resolution of singularities $f : X \to S$ with the properties mentioned above. Further, there are cycle maps
\[ c : \text{gr}^j I\mathcal{K} \cdot 0 (S) \to IH^{2j}(S) \]
\[ c : \text{gr}^j I\mathcal{K} \cdot 0 (S) \to IH^{2j}(S). \]

We also propose definitions for $\text{gr} I\mathcal{K} \cdot (S, L)$ and $\text{gr} I\mathcal{K} \cdot (S, L)$ for $L$ a local system on $U$ open in $S$ of the form $L \cong h_* (\mathbb{Z}_V)$ for an étale map $h : V \to U$. There are cycle maps
\[ c : \text{gr}^j I\mathcal{K} \cdot (S, L) \to IH^{2j}(S, L \otimes \mathbb{Q}) \]
\[ c : \text{gr}^j I\mathcal{K} \cdot (S, L) \to IH^{2j}(S, L \otimes \mathbb{Q}). \]

**1.3. Properties of the perverse filtrations and intersection $K$-theory.** The perverse filtrations in $K$-theory and intersection $K$-theory have similar properties to their counterparts in cohomology.

For a map $f : X \to S$, let $s := \dim X \times_S X - \dim X$ be its defect of semismallness. In Theorem 3.11 we show that
\[ P^\leq_{f-s-1} \text{gr} K_0(X) = 0, \]
\[ P^\leq_{f-s} \text{gr} K_0(X) = P^\leq_{f-s} \text{gr} K_0(X) = \text{gr} K_0(X). \]

This implies that
\[ \text{gr} I\mathcal{K} \cdot (S) = \text{gr} I\mathcal{K} \cdot (S) = \text{gr} K \cdot (S) \text{ for } S \text{ smooth}, \]
\[ \text{gr} I\mathcal{K} \cdot (S) = \text{gr} K \cdot (S) \text{ if } S \text{ has a small resolution } f : X \to S. \]

For more computations of perverse filtrations in $K$-theory and intersection $K$-theory, see Subsections 3.7 and 1.4.
Let $d = \dim S$. In cohomology, there are natural maps

$$H^i(S) \to IH^i(S) \to H_{BM}^{2d-i}(S)$$

$$IH^i(S) \otimes IH^j(S) \to H_{BM}^{2d-i-j}(S).$$

The composition in the first line is the natural map $H^i(S) \to H_{BM}^{2d-i}(S)$. The second map is non-degenerate for cycles of complementary dimensions. In Subsection 4.3 we explain that there exist natural maps

$$gr_i IK(S) \to gr_i G(S)$$

$$gr_i IK(S) \times gr_j IK(S) \to gr_{d-i-j} G(S)$$

and their analogues for $IK$. The above filtration on $G$-theory is by dimension of supports, see [13, Section 5.4].

1.4. The decomposition theorem for semismall maps. As mentioned above, many applications of intersection cohomology are based on the decomposition theorem. When the map

$$f : X \to S$$

is semismall, the statement of the decomposition theorem is more explicit, which we now explain. Let $\{S_a | a \in I\}$ be a stratification of $S$ such that $f_a : f^{-1}(S_a^0) \to S_a^0$ is a locally trivial fibration, where $S_a^0 = S_a - \bigcup_{b \in I} (S_a \cap S_b)$. Let $A \subset I$ be the set of relevant strata, that is, those strata such that for $x_a \in S_a^0$:

$$\dim f^{-1}(x_a) = \frac{1}{2}(\dim S - \dim S_a).$$

For $x_a \in S_a^0$, the monodromy group $\pi_1(S_a^0, x_a)$ acts on the set of irreducible components of $f^{-1}(x_a)$ of top dimension; let $L_a$ be the corresponding local system. Let $c_a$ be the codimension of $X_a = f^{-1}(S_a)$ in $X$. The decomposition theorem for the map $f : X \to S$ says that there exists a canonical decomposition [7, Theorem 4.2.7]:

$$H^j(X) \cong \bigoplus_{a \in A} IH^{j-c_a}(S_a, L_a).$$

We conjecture the analogous statement in $K$-theory.

**Conjecture 1.3.** Let $f : X \to S$ be a semismall map and consider $\{S_a | a \in I\}$ a stratification as above, and let $A \subset I$ be the set of relevant strata. There is a decomposition for any integer $j$:

$$gr^j K(X) \cong \bigoplus_{a \in A} gr^{j-c_a} IK(S_a, L_a).$$

See Conjecture 5.1 for a more precise statement. In Theorem 5.4, we check the above conjecture for $K_0$ under the extra condition that for any $a \in A$, there are small maps $\pi_a : T_a \to S_a$ such that $\pi_a^{-1}(S_a^0) \to S_a^0$ is étale with associated local system $L_a$. The proof of the above result is based on a theorem of de Cataldo–Migliorini [6, Section 4]. In Subsection 4.4.4, we prove the statement for $K_0$ when $f : X \to S$ is a resolution of singularities of a surface.
1.5. **Intersection Chow groups.** When $X$ is smooth, $\text{gr}^i K_0(X)_\mathbb{Q} = CH^i(X)_\mathbb{Q}$. Thus $\text{gr}^i IK_0(S)_\mathbb{Q}$ is a candidate for an intersection Chow group of $S$. Corti–Hanamura already defined (rational) intersection Chow groups (or Chow motives) in \cite{10}, \cite{11} inspired by the decomposition theorem. One proposed definition assumes conjectures of Grothendieck and Murre and proves a version of the decomposition theorem for Chow groups; the other approach defines a perverse-type filtration on Chow groups by induction on level $i$ of the filtration and via correspondences involving all varieties $W \to S$. These correspondences need to satisfy certain vanishings for the perverse filtration in cohomology. The advantage of the definition we propose is that one uses fewer correspondences to define $P_f^{\leq i}$ and $P_f^{< i}$ and this allows for computations, see Subsection 4.4 and Theorem 5.3.

For varieties $S$ with a semismall resolution of singularities $f : X \to S$, de Cataldo–Migliorini \cite{6} proposed a definition of Chow motives $ICH(S)$ and proved a version of the decomposition theorem for semismall maps.

1.6. **Intersection $K$-theory.** Our approach to define (graded) intersection $K$-theory uses functoriality of the perverse filtration in an essential way. To obtain this functoriality, it is essential to pass to $\text{gr} K_*(X)$. It is a very interesting problem to find a definition of the perverse filtration on $K_*(X)$. We hope that such a definition will provide a version of equivariant intersection $K$-theory with applications to geometric representation theory, for example in understanding the $K$-theoretic version of \cite{5}.

There are proposed definitions of intersection $K$-theory in some particular cases. Cautis \cite{8}, Cautis–Kamnitzer \cite{9} have an approach for categorification of intersection sheaves for certain subvarieties of the affine Grassmannian. Eberhardt defined intersection $K$-theoretic sheaves for varieties with certain stratifications \cite{12}. In \cite{19}, we proposed a definition of intersection $K$-theoretic for good moduli spaces of smooth Artin stacks which has applications to the structure theory of Hall algebras of Kontsevich–Soibelman \cite{20}.

1.7. **Outline of the paper.** In Section 2 we discuss preliminary material. In Section 3 we introduce the two perverse filtrations $P_f^{\leq *} \subset P_f^{< *}$. In Section 4 we define the two versions of intersection $K$-theory $\text{gr} IK_*(S)$ and $\text{gr} IK_*(S)$. In Section 5 we discuss the decomposition theorem for semismall maps.

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## 2. Preliminary material

2.1. **Notations and conventions.** All schemes considered in this paper are finite type and defined over $\mathbb{C}$. The definition of the filtration in Subsection 3.1 works over any field, but to define intersection $K$-theory we use resolution of singularities. The definition of intersection $K$-theory works over any field of characteristic
zero. A variety is an irreducible reduced scheme. We use quasi-smooth schemes in Subsections 3.3 and 3.4 which are the natural extension of lci schemes in derived geometry.

For \( S \) a scheme, let \( D^b\text{Coh}(S) \) be the derived category of bounded complexes of coherent sheaves and let \( \text{Perf}(S) \) be its subcategory of bounded complexes of locally free sheaves on \( S \). For their analogues for quasi-smooth schemes, see [16, Subsections 2.1 and 3.1]. The functors used in the paper are derived; we sometimes drop \( R \) or \( L \) from notation, for example we write \( f_* \) instead of \( Rf_* \). When \( S \) is smooth, the two categories coincide. Define

\[
G_i(S) := K_\big( D^b\text{Coh}(S)^{\geq i} \big),
\]

\[
K_i(S) := K_\big( \text{Perf}(S)^{\geq i} \big).
\]

For \( Y \) a subvariety of \( X \), let \( D^b\text{Coh}_Y(X) \) be the subcategory of \( D^b\text{Coh}(X) \) of complexes supported on \( Y \), and define

\[
G_Y, i(X) := K_\big( D^b\text{Coh}_Y(X)^{\geq i} \big).
\]

When \( X \) is smooth, we also use the notation \( K_Y, i(X) \) for the above. We will usually drop the subscript \( \cdot \) from the notation.

The local systems used in this paper are of the form \( L \cong h_* (\mathbb{Z}_V) \) (or \( L \cong h_* (\mathbb{Q}_V) \)) for an étale morphism \( h : V \to U \). We call these local systems integer (or rational) finite local systems.

Singular and intersection cohomology and Borel-Moore homology are used only with coefficients in a rational finite local system, usually \( \mathbb{Q} \).

For \( f \) a morphism of varieties, we denote by \( \mathbb{L}_f \) its cotangent complex.

### 2.2. Filtrations in K-theory

A reference for the following is [13], especially Section 5 in loc. cit. Let \( F^iG_i(S) \) be the filtration on \( G_i(S) \) by sheaves with support of codimension \( \geq i \); it induces a filtration on \( K_i(S) \). The associated graded will be denoted by \( \text{gr} G_i(S) \), \( \text{gr} K_i(S) \). A morphism \( f : X \to Y \) of smooth varieties induces maps:

\[
f^* : F^iK_i(Y) \to F^iK_i(X)
\]

\[
f^* : \text{gr}^iK_i(Y) \to \text{gr}^iK_i(X).
\]

Further, let \( F^\dim_iG_i(S) \) be the filtration on \( G_i(S) \) by sheaves with support of dimension \( \leq i \); it induces a filtration on \( K_i(S) \). The associated graded will be denoted by \( \text{gr}^\dim G_i(S) \), \( \text{gr}^\dim K_i(S) \). A proper morphism \( f : X \to Y \) of schemes induces maps:

\[
f_* : F^\dim_iG_i(X) \to F^\dim_iG_i(Y)
\]

\[
f_* : \text{gr}^\dim_iG_i(X) \to \text{gr}^\dim_iG_i(Y).
\]

There are similar filtrations and associated graded on \( G_Y(X) \) for \( Y \hookrightarrow X \) a subvariety. If \( X \) is smooth of dimension \( d \), then \( \text{gr}^\dim G_Y(X) = \text{gr}^{d-i}G_Y(X) \).
Proposition 2.1. Let $S \xrightarrow{\alpha} \text{Spec} \mathbb{C}$ be a variety of dimension $d$. Then
\[
\left( a^*, \bigoplus_{T \subseteq S} \iota_{T^*} \right) : G_0(\text{Spec} \mathbb{C}) \oplus \bigoplus_{T \subseteq S} \text{gr}_i G_0(T) \to \text{gr}_0 G_0(S),
\]
where the sum is taken over all proper subvarieties $T$ of $S$.

Proof. For $i < d$, the map
\[
\bigoplus_{T \subseteq S} \iota_{T^*} : \bigoplus_{T \subseteq S} \text{gr}_i G_0(T) \to \text{gr}_i G_0(S)
\]
is surjective by definition of the filtration $F^i_{\dim}$. Finally, the following map is an isomorphism
\[
a^* : G_0(\text{Spec} \mathbb{C}) \xrightarrow{\sim} \text{gr}_d G_0(S).
\]
\[\square\]

Proposition 2.2. Let $S$ be a singular variety of dimension $d$, and let $f : X \to S$ be a resolution of singularities. The following map is surjective:
\[
f_* : \text{gr}_i G_0(X) \to \text{gr}_i G_0(S).
\]

Proof. We use induction on $d$. By Proposition 2.1, the following is an isomorphism
\[
f_* : \text{gr}_d G_0(X) \xrightarrow{\sim} \text{gr}_d G_0(S) \xrightarrow{\sim} G_0(\text{Spec} \mathbb{C}).
\]
For $V \subseteq S$ a subvariety, consider $g$ a resolution of singularities as follows:
\[
Y \xrightarrow{g} X \xrightarrow{f} S.
\]
The surjectivity of $f_*$ for $i < d$ follows using Proposition 2.1 and the induction hypothesis. \[\square\]

2.3. Quasi-smooth schemes.

2.3.1. A morphism $f : X \to Y$ of derived schemes is quasi-smooth if it is locally of finite presentation and the cotangent complex $\mathcal{L}_f$ has Tor-amplitude $\leq 1$. Alternatively, there is a factorization
\[
X \xrightarrow{\iota} X' \xrightarrow{\pi} Y
\]
with $\pi$ smooth and $\iota$ a quasi-smooth immersion, that is, the complex $\mathcal{L}_\iota[-1]$ is a vector bundle, see [17, Section 2], [2, Subsections 2.1 and 2.2]. A proper map between smooth varieties is quasi-smooth. Any quasi-smooth map has a well defined relative dimension.

A derived scheme $X$ is quasi-smooth if the structure morphism $X \to \text{Spec} \mathbb{C}$ is quasi-smooth. Any quasi-smooth scheme has a well defined dimension.
2.3.2. We list some properties satisfied by quasi-smooth morphisms and schemes. We are using them only in the proof of an excess intersection formula, see [1]. Consider a cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array}
\]

If \( f \) is quasi-smooth, then \( f' \) is quasi-smooth and

\[
\text{rel.dim}(f) = \text{rel.dim}(f').
\]

Composition of quasi-smooth maps is quasi-smooth.

Let \( Y \) be a quasi-smooth scheme. Define a filtration \( F_i \) on \( G \cdot Y \) generated by images of \( f^* : G \cdot X \rightarrow G \cdot Y \) for \( f : Y \rightarrow X \) a quasi-smooth morphism with \( \text{rel.dim}(f) \leq -i \). It induces a filtration on \( K \cdot Y \). One can define similarly a filtration \( F_i \) on \( G \cdot Y \) when \( Y \) is a classical scheme, this definitions recover the filtrations introduced in Subsection 2.2, see [16, Theorem 6.21]. By (2), pullback respects the filtrations \( F_i \). Pushforward clearly respects the filtrations \( F_i \).

2.3.3. Let \( X \) be a quasi-smooth scheme. There exists an open set \( U \subset X \) of codimension \( \geq 1 \) and a vector bundle \( E \) on \( U^{\text{cl}} \) such that \( \mathcal{O}_U \cong \mathcal{O}_{U^{\text{cl}}}[\mathcal{E}[1]; d] \), where \( d : \mathcal{E} \rightarrow \mathcal{O}_U \) is the zero map. The bundle \( \mathcal{E} \) has rank \( \dim X^{\text{cl}} - \dim X \).

2.4. The perverse filtration in cohomology. Let \( S \) be a scheme over \( \mathbb{C} \). Let \( D^b_c(S) \) be the derived category of bounded complexes of rational constructible sheaves [7, Section 2]. Consider the perverse \( t \)-structure \((\mathcal{P}^{\leq i}, \mathcal{P}^{\geq i})_{i \in \mathbb{Z}}\) on this category. There are functors:

\[
\begin{align*}
\tau^{\leq i} : D_c^b(S) &\rightarrow \mathcal{P}^{\leq i}, \\
\tau^{\geq i} : D_c^b(S) &\rightarrow \mathcal{P}^{\geq i}
\end{align*}
\]

such that for \( F \in D_c^b(S) \) there is a distinguished triangle in \( D_c^b(S) \):

\[
\tau^{\leq i} F \rightarrow F \rightarrow \tau^{\geq i} F[1].
\]

For a proper map \( f : X \rightarrow S \) and \( F \in D_c^b(X) \), the perverse filtration on \( H^r(X,F) \) is defined as the image of

\[
\tau^{\leq i} F \rightarrow F \rightarrow \tau^{\geq i} F[1].
\]

For \( F = \mathcal{I}C_X \), the decomposition theorem implies that

\[
\tau^{\leq i} \mathcal{I}C_X \cong \mathcal{I}H^{\leq i}(X).
\]
Let \( f : X \to S \) be a generically finite morphism from \( X \) smooth, let \( U \) be a smooth open subset of \( S \) such that \( f^{-1}(U) \to U \) is étale, and let \( L = f_* (\mathbb{Q}_{f^{-1}(U)}) \).

For \( V \hookrightarrow S \), denote by \( X_V := f^{-1}(V) \). Let \( A_V \) be the set of irreducible components of \( X_V \). Let \( c^a_V \) be the codimension on \( V^a \hookrightarrow X \) for \( a \in A_V \). Consider a resolution of singularities \( \pi^a_V : Y^a \to V^a \). Let \( g^a_V := f \pi^a_V : Y^a \to V \). Then

\[
p_{\tau \leq 0} Rf_* IC_X \cong \ker \left( Rf_* IC_X \to \bigoplus_{V \subseteq S} \bigoplus_{a \in A_V} (p_{\tau > c^a_V} Rg^a_V IC_{Y^a}) [c^a_V] \right).
\]

Define the subspace

\[
p_{\tau \leq 0} Rf_* IC_X := \operatorname{image} \left( \bigoplus_{V \subseteq S} \bigoplus_{a \in A_V} (p_{\tau \leq -c^a_V} Rg^a_V IC_{Y^a}) [-c^a_V] \to p_{\tau \leq 0} Rf_* IC_X \right).
\]

By a computation of Corti–Hanamura [11, Proposition 1.5, Theorem 2.4], there is an isomorphism:

\[
IC_S(L) \cong p_{\tau \leq 0} Rf_* IC_X / p_{\tau \leq 0} Rf_* IC_X.
\]

Further, consider a more general morphism \( f : X \to S \) with \( X \) smooth. Let \( V \subsetneq S \) be a subvariety. For \( i \in \mathbb{Z} \), denote by \( p^{\mathcal{H}}(Rf_* IC_X)_V \) the direct sum of simple summands of \( p^{\mathcal{H}}(Rf_* IC_X) \) with support equal to \( V \). A computation of Corti–Hanamura [11, Proposition 1.5] shows that:

\[
p^{\mathcal{H}}_i (Rf_* IC_X)_V \hookrightarrow \bigoplus_{a \in A_V} p^{\mathcal{H}}_{i+c^a_V} (Rg^a_V IC_{Y^a}).
\]

3. The perverse filtration in \( K \)-theory

3.1. Definition of the filtration \( P_{\tau \leq i} \). Let \( f : X \to S \) be a proper map between varieties. We define an increasing filtration

\[
P_f^{\tau \leq i} \operatorname{gr} G.(X) \subset \operatorname{gr} G.(X).
\]

It induces a filtration on \( \operatorname{gr} K.(X) \). We use the notations from Subsection 2.4. Let \( Y \hookrightarrow X \) be a subvariety and let \( T \xrightarrow{\pi} S \) be a map generically finite onto its image from \( T \) smooth. Consider the diagram:

\[
\begin{array}{ccc}
T \times X & \xrightarrow{p} & X \\
\downarrow q & & \downarrow f \\
T & \xrightarrow{\pi} & S.
\end{array}
\]

For a correspondence \( \Gamma \in \operatorname{gr}_{\dim X - s} G_{T \times_s Y, 0}(T \times X) \), define

\[
\Phi_\Gamma := p_*(\Gamma \otimes q^*(-)) : \operatorname{gr} K_i(T) \to \operatorname{gr}^{-s} G_{Y,s}(X).
\]
We usually drop the shift by \( s \) in the superscript of \( \text{gr}G_Y(X) \). We define the subspace of \( \text{gr}G_Y(X) \):

\[
P_{f,T}^{\leq i} := \text{span}_T (\Phi_T : \text{gr}K(T) \to \text{gr}G_Y(X))
\]

\[
P_{f}^{\leq i} := \text{span} \left( P_{f,T}^{\leq i} \text{ for all maps } \pi \text{ as above} \right),
\]

where the dimension of the correspondence satisfies

\[
\left\lfloor \frac{i + \dim X - \dim T}{2} \right\rfloor \geq s.
\]

We also define a quotient of \( \text{gr}G_Y(X) \):

\[
P_{f}^{i <} \text{gr}G_Y(X) \hookrightarrow \text{gr}G_Y(X) \twoheadrightarrow P_{f}^{i >} \text{gr}G_Y(X).
\]

### 3.2. Functoriality of the filtration \( P_{i \leq} \).

**Proposition 3.1.** Let \( X \) and \( Y \) be smooth varieties with \( c = \dim X - \dim Y \).

Consider proper maps

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & X \\
\downarrow g & & \downarrow f \\
S & & \\
\end{array}
\]

There are induced maps

\[
h^* : P_{f}^{i \leq i - c} \text{gr}K(X) \to P_{g}^{i \leq i - c} \text{gr}K(Y).
\]

**Proof.** Let \( T \to S \) be a generically finite map onto its image with \( T \) smooth. It suffices to show that

\[
h^* : P_{f,T}^{i \leq i - c} \text{gr}K(X) \to P_{g,T}^{i \leq i - c} \text{gr}K(Y)
\]

Consider the diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & X \\
\uparrow p_Y & & \uparrow p_X \\
Y \times T & \xrightarrow{\bar{h}} & X \times T \\
\downarrow q_Y & & \downarrow q_X \\
T & & \\
\end{array}
\]

Let \( \Theta \in \text{gr}_{\dim X - s}G_{T \times S,X,0}(T \times X) \) be a correspondence such that

\[
i \geq 2s - \dim X + \dim T.
\]

For \( j \in \mathbb{Z} \), we have that:

\[
\begin{array}{ccc}
\text{gr}^j K(T) & \xrightarrow{\Phi_{\Theta}} & \text{gr}^{j - s} K(X) \\
\downarrow \Phi_{h^* \Theta} & & \downarrow h^* \\
\text{gr}^{j - s} K(Y) & & \\
\end{array}
\]
To see this, we compute:

\[ h^*\Phi_{\Theta}(F) = h^* p_{X*}(\Theta \otimes q_X^* F) = p_{Y*}\tilde{h}^*(\Theta \otimes q_X^* F) = p_{Y*}(\tilde{h}^*\Theta \otimes q_Y^* F) = \Phi_{\tilde{h}^*\Theta}(F). \]

The correspondence \( \tilde{h}^*\Theta \in \text{gr}_{\dim Y-s} G_{T \times S}Y(T \times Y) \) satisfies

\[ i + c \geq 2s - \dim Y + \dim T, \]

and this implies the desired conclusion. \( \square \)

**Proposition 3.2.** Let \( X \) and \( Y \) be varieties with proper maps

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & X \\
\downarrow{g} & & \downarrow{f} \\
S
\end{array}
\]

Let \( c = \dim X - \dim Y \). There are induced maps

\[ h_* : P^i_{g,T} \text{gr}.G.(Y) \to P^i_{f,T} \text{gr}.G.(X). \]

*Proof.* Let \( T \to S \) be a generically finite map onto its image from \( T \) smooth. We first explain that

\[ h_* : P^i_{g,T} \text{gr}.G.(Y) \to P^i_{f,T} \text{gr}.G.(X). \]

We use the notation from the proof of Theorem 3.1. Consider a correspondence \( \Gamma \in \text{gr}_{\dim Y-s} G_{T \times S}Y(T \times Y) \) such that

\[ i \geq 2s - \dim Y + \dim T. \]

For \( j \in \mathbb{Z} \), we have that:

\[
\begin{array}{ccc}
\text{gr}_{\dim T-j}K(T) & \xrightarrow{\Phi_{\Gamma}} & \text{gr}_{\dim Y-j+s}G.(Y) \\
& \xrightarrow{\Phi_{\tilde{h}^*\Gamma}} & \text{gr}_{\dim Y-j+s}G.(X). \\
\end{array}
\]

To see this, we compute:

\[ h_* p_{Y*}(\Gamma \otimes q_Y^* F) = p_{X*}\tilde{h}^*(\Gamma \otimes h^* q_X^* F) = p_{X*}(\tilde{h}^*\Gamma \otimes q_X^* F). \]

The correspondence

\[ \tilde{h}^*\Gamma \in \text{gr}_{\dim Y-s} G_{T \times S}X(T \times X) = \text{gr}_{\dim X-(c+s)} G_{T \times S}X(T \times X) \]

satisfies

\[ i + c \geq 2(s + c) - \dim X + \dim T, \]

and thus the conclusion follows. \( \square \)

We continue with some further properties of the filtration \( P^i_{\leq \cdot} \). The following is immediate:

**Proposition 3.3.** Let \( f : X \to S \) be a proper map. Let \( U \) be an open subset of \( S \), \( X_U := f^{-1}(U) \), \( \iota : X_U \to X \), and \( f_U : X_U \to U \). Then

\[ \iota^* : P^i_{f,T} \text{gr}.G.(X) \to P^i_{f_U,T} \text{gr}.G.(X_U). \]
Proposition 3.4. Let $f : X \to S$ be a proper map from $X$ smooth and consider $e \in \text{gr}^i K_0(X)$. Then
\[ e \cdot P_f^{\leq i} \text{gr}^i K(X) \subset P_f^{\leq i+2j} \text{gr}^{i+j} K(X). \]

Proof. Let $T \to S$ be a generically finite map onto its image with $T$ smooth and let $\Theta \in \text{gr}_{a_G T \times S, 0}(T \times X)$. Let $p : T \times X \to X$ be the natural projection. Then
\[ p^*(\cdot) \cdot \Theta \in \text{gr}^{-j} G T \times S, 0((T \times X)). \]

For $x \in \text{gr} \cdot K \cdot (T)$, we have that
\[ e \cdot \Phi(x) = \Phi(p^*(\cdot) \cdot \Theta(x)), \]
and the conclusion thus follows. □

Proposition 3.5. Let $X$ and $Y$ be smooth varieties with proper maps $f : X \to S$, $g : Y \to S$ such that $h$ is surjective. Let $c = \dim X - \dim Y$. Then
\[ h^* \left( P_f^{\leq i} \text{gr}^i K(X) \right) = P_g^{\leq i+c} \text{gr}^i K(Y) \]
\[ h^* \left( \text{gr}^i K(X) \right) \cap P_g^{\leq i+c} \text{gr}^i K(Y) = h^* \left( P_f^{\leq i} \text{gr}^i K(X) \right). \]

If there exists $X' \to Y$ such that the induced map $X' \to X$ is birational, then the above isomorphisms hold integrally.

Proof. The statement and its proof are similar to [11, Proposition 3.11].

Let $i : X' \to Y$ be a map such that $hi : X' \to X$ is generically finite and surjective. Let $f' := fi : X' \to S$. Then, by Proposition 3.2
\[ P_f^{\leq i+c} \text{gr}^i K(X') \xrightarrow{i_*} P_g^{\leq i+c} \text{gr}^i K(Y) \xrightarrow{h_*} P_f^{\leq i+c} \text{gr}^i K(X). \]
The map $h_* i_* : \text{gr}^i K(X') \to \text{gr}^i K(X)$ is multiplication by the degree of the map $hi$, so it is an isomorphism rationally. It is an isomorphism integrally if $X' \to X$ has degree 1. The pullback statement is similar. □

3.3. The filtration $P^{\leq i}$. Let $f : X \to S$ be a proper map from $X$ smooth. Let $V \hookrightarrow S$ be a subvariety, and let $\mathcal{A}_V$ the set of irreducible components of $f^{-1}(V)$. For an irreducible component $X^a_V$ of $f^{-1}(V)$, consider a resolution of singularities $\pi^a_V$ as follows:

\[
\begin{array}{c}
\tilde{X}^a_V \xrightarrow{\pi^a_V} X^a_V \xrightarrow{i^a_V} X \\
\downarrow \quad \quad \downarrow \quad \downarrow \\
\overline{f^a} \quad \quad \quad \quad \quad \quad \quad f
\end{array}
\]

where $\overline{f^a} : \mathcal{V} \hookrightarrow S$. 

\[
\mathcal{A}_V = \{ \pi^a_V : V \to \overline{f^a} \}
\]

The map $\pi_* i_* : \text{gr}^i K(X') \to \text{gr}^i K(X)$ is multiplication by the degree of the map $hi$, so it is an isomorphism rationally. It is an isomorphism integrally if $X' \to X$ has degree 1. The pullback statement is similar. □
Let $c_V^a$ be the codimension of $X_V^a$ in $X$. Denote by $\tau_V^a = e_V^a \pi_V^a$. Consider a subvariety $Y \hookrightarrow X$. Define

$$P_f^{\leq i} \gr G_Y(X) := \bigcap_{V \supseteq S} \bigcap_{a \in A_V} \ker \left( \tau_V^{a*} : P_f^{\leq i} \gr G_Y(X) \to P_{f_V}^{\leq i+c_V^a} \gr K \left( \overline{X}_V^a \right) \right).$$

The definition is independent of the resolutions $\pi_V^a$ chosen. Indeed, consider two different resolutions $\overline{X}_V^a$, $\overline{X}_V'^a$. There exists $W$ such that

$$\pi : \overline{X}_V^a \to X \quad \pi' : \overline{X}_V'^a \to X,$$

where the maps $\pi$ and $\pi'$ are successive blow-ups along smooth subvarieties of $\overline{X}_V^a$ and $\overline{X}_V'^a$, respectively. Let $\tau_V^a : \overline{X}_V'^a \to X$ as above. Then $\tau_V^a \pi = \tau_V^a \pi'$. By Proposition 3.4

$$\ker \left( \tau_V^{a*} : P_f^{\leq i} \gr G_Y(X) \to P_{f_V}^{\leq i+c_V^a} \gr K \left( \overline{X}_V^a \right) \right) \cong$$

$$\ker \left( \pi^* \tau_V^{a*} : P_f^{\leq i} \gr G_Y(X) \to P_{f_V}^{\leq i+c_V^a} \gr K(W) \right) \cong$$

$$\ker \left( \tau_V'^{a*} : P_f^{\leq i} \gr G_Y(X) \to P_{f_V'}^{\leq i+c_V^a} \gr K \left( \overline{X}_V'^a \right) \right).$$

**Theorem 3.6.** Let $X$ and $Y$ be smooth varieties with $c = \dim X - \dim Y$. Consider proper maps

$$Y \xrightarrow{h} X \quad \xleftarrow{g} \quad \xrightarrow{f} S.$$

There are induced maps

$$h^* : P_f^{\leq i-c} \gr K.(X) \to P_g^{\leq i} \gr K.(Y)$$

$$h_* : P_g^{\leq i-c} \gr K.(Y) \to P_f^{\leq i} \gr K.(X).$$

**Proof.** The functoriality of $h^*$ follows from Proposition 3.1 and induction on dimension of $S$.

We discuss the statement for $h_*$. We use induction on the dimension of $S$. The case of $S$ a point is clear as $P_f^{\leq i} = P_f^{\leq i}$. We use the notation from the beginning of Subsection 3.3. Let $V$ be a subvariety of $S$. Let $X_V^a$ be an irreducible component of $f^{-1}(V)$ with a resolution of singularities $\overline{X}_V^a \to X_V^a$. Let $B$ be the set of irreducible
component of $Y_V$ over $X_V^a$. For $b \in B$, consider a resolution of singularities $\widetilde{Y}^b_V \to Y^b_V$ and maps such that

$$
\begin{array}{ccc}
\bigcup_{b \in B} \widetilde{Y}^b_V & \xrightarrow{\Theta_B h^b_V} & \widetilde{X}^a_V \\
\downarrow \Theta_B \tau^b_V & & \downarrow \tau^a_V \\
Y & \xrightarrow{h} & X.
\end{array}
$$

Consider the cartesian diagram

$$
\begin{array}{ccc}
Y^\text{der}_V & \xrightarrow{\tilde{h}} & \tilde{X}^a_V \\
\downarrow \tau & & \downarrow \tau^a_V \\
Y & \xrightarrow{h} & X.
\end{array}
$$

The scheme $Y^\text{der}_V$ is quasi-smooth, see Subsection 2.3, and \text{reldim} $\tilde{h} = \text{reldim} h$. For $b \in B$, there is a map $p_b : \widetilde{Y}^b_V \to Y^\text{der}_V$. Let $d_b = \dim \widetilde{Y}^b_V - \dim Y^\text{der}_V$ and define

$$
e_b = \det \left( \mathbb{L}^{b^*}_V / h^*_V \mathbb{L}^{\tau^*_V} \right) \in \text{gr}^{d_b} K^0 \left( \widetilde{Y}^b_V \right).
$$

By a version of the excess intersection formula, the following diagram commutes:

$$
\begin{array}{ccc}
\text{gr.} K(Y) & \xrightarrow{h^*} & \text{gr.} K(X) \\
\downarrow \Theta_B \tau^*_V & & \downarrow \\
\bigoplus_B \text{gr.} K(Y^b_V) & & \bigoplus_B \text{gr.} K(X^a_V)
\end{array}
$$

where we have ignored shifts in the above gradings. We now explain that (6) commutes. Consider the diagram

$$
\begin{array}{ccc}
\bigcup_B \widetilde{Y}^b_V & \xrightarrow{\bigcup_B h^b_V} & \widetilde{X}^a_V \\
\downarrow \bigcup_B p_b & & \downarrow \\
Y^\text{der}_V & \xrightarrow{\tilde{h}} & \tilde{X}^a_V \\
\downarrow \tau & & \downarrow \tau^a_V \\
Y & \xrightarrow{h} & X.
\end{array}
$$

Then

$$
\sum_{b \in B} h^b_V \left( e_b \cdot \tau^a_V \right) = \sum_{b \in B} \tilde{h}_* p^*_b \left( e_b \cdot \tau^*_V \right) = \tilde{h}_* \left( \left( \sum_{b \in B} p^*_b e_b \right) \cdot \tau^* \right).
$$
For $M$ a quasi-smooth scheme, denote by $1 := [\mathcal{O}_M] \in \text{gr}^0K_0(M)$. It suffices to show that
\[
\sum_{b \in B} p_b^*(e_b) = 1 \in \text{gr}^0K_0(Y^{\text{der}}_V).
\]

The underlying scheme $Y^{\text{cl}}_V$ has irreducible components indexed by $B$ and these components are birational to $\widetilde{Y}_V$. Recall the discussion in Subsection 2.3.3. There exist open sets
\[
W = \bigsqcup_{b \in B} W^b \subset Y^{\text{der}}_V,
\]
\[
U^b \subset \widetilde{Y}_V
\]
whose complements have codimension $\geq 1$ and such that for any $b \in B$:
\[
W_b^{\text{cl}} \cong U^b,
\]
\[
W^b \times_{Y^{\text{der}}_V} \widetilde{Y}_V \cong U^b,
\]
\[
\mathcal{O}_{W^b} \cong \mathcal{O}_{U^b}[E^b[1]; d],
\]
where $E^b$ is a vector bundle on $U^b$ of rank $d^b$ and the differential $E^b \to \mathcal{O}_{U^b}$ is zero.

Let $e_b \in \text{gr}^{d^b}K_0(U^b)$ be the Euler class of $E^b$. Then $p_b^*(e_b) = 1 \in \text{gr}^0K_0(W^b)$ and the restriction map sends
\[
\text{res} : \text{gr}^{d^b}K_0(\widetilde{Y}_V) \to \text{gr}^{d^b}K_0(U^b)
\]
\[
e_b \mapsto e_b.
\]

Back to proving (7), we have that $\text{gr}^0K_0(Y^{\text{der}}_V) \cong \bigoplus_{b \in B} \text{gr}^0K_0(W^b)$. Consider the diagram
\[
\begin{array}{ccc}
\text{gr}^{d^b}K_0(\widetilde{Y}_V) & \xrightarrow{\text{res}} & \text{gr}^{d^b}K_0(U^b) \\
\downarrow{p_b^*} & & \downarrow{p_b^*} \\
\text{gr}^0K_0(Y^{\text{der}}_V) & \xrightarrow{\text{res}} & \text{gr}^0K_0(W^b),
\end{array}
\]
where the horizontal maps are restriction to open sets maps. Then
\[
\text{res} p_b^*(e_b) = p_b^*(e_b) = 1 \text{ in } \text{gr}^0K_0(W^b).
\]
The diagram (8) thus commutes. The conclusion now follows from Propositions 3.2 and 3.3.

3.4. Towards the filtration $P_i^j$. We continue with the notation from Subsection 3.3. Let $X$ be a smooth variety with a proper map $f : X \to S$. Let $T \xrightarrow{\pi} S$ be a generically finite map onto its image from $T$ smooth.
We say that $\Gamma$ is a $(f, \pi)$-quasi-smooth scheme if $\Gamma$ is a derived scheme with maps

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\iota} & X' \\
\downarrow^{\iota} & & \downarrow^{t} \\
T & \xrightarrow{\pi} & S
\end{array}
$$

such that $\iota$ is a quasi-smooth immersion in a smooth variety $X'$ (i.e. the cotangent complex $L_{\iota}[-1]$ is a vector bundle on $\Gamma$), $t$ is smooth, and $q^{cl}$ is surjective. The conditions on the maps $\iota$ and $t$ imply that $\Gamma$ is quasi-smooth. Let

$$
gr K^q_{T \times S \times X}(T \times X) \subset gr K_{T \times S}(T \times X)
$$

be the subspace generated by classes $[\Gamma]$ for $(f, \pi)$-quasi-smooth schemes as above.

**Proposition 3.7.** Let $h$ be a proper map:

$$
\begin{array}{ccc}
Y & \xrightarrow{h} & X \\
\downarrow^{g} & & \downarrow^{f} \\
S
\end{array}
$$

There are induced maps

$$
h_* : gr K^q_{T \times S \times Y}(T \times Y) \rightarrow gr K^q_{T \times S \times X}(T \times X).
$$

If $h$ is surjective, then there are induced maps

$$
h^* : gr K^q_{T \times S \times X}(T \times X) \rightarrow gr K^q_{T \times S \times Y}(T \times Y).
$$

**Proof.** We discuss the statement about pullback. Consider the diagram:

$$
\begin{array}{ccc}
\Theta & \xrightarrow{\iota} & Y' \\
\downarrow^{r} & & \downarrow^{t_Y} \\
\Gamma & \xrightarrow{\iota} & X' \\
\downarrow^{q} & & \downarrow^{t_X} \\
T & \xrightarrow{\pi} & S
\end{array}
$$

where $\Gamma$ is a $(f, \pi)$-quasi-smooth scheme with $q^{cl}$ is surjective, $t_X$ is smooth, and the upper squares are cartesian. Then the map $\Theta \hookrightarrow Y'$ is a quasi-smooth immersion and $t_Y$ is smooth. The map $h$ is surjective, so $r^{cl} : \Theta^{cl} \rightarrow \Gamma^{cl}$ is surjective, and thus $(gr)^{cl} : \Theta^{cl} \rightarrow T$ is surjective as well, so $\Theta$ is a $(g, \pi)$-quasi-smooth scheme.
We next discuss the statement about pushforward. Consider

\[
\begin{array}{ccc}
Y' & \xrightarrow{t'} & Y \\
\downarrow & \downarrow & \downarrow \\
T & \xrightarrow{q} & S
\end{array}
\]

such that \( t \) is a closed immersion, \( t' \) is smooth, and \( q^{cl} \) is surjective. The map \( Y' \to X \) is a proper map of smooth quasi-projective varieties, so we can choose \( X' \) with maps

\[
\begin{array}{ccc}
Y' & \xrightarrow{t'} & X' \\
\downarrow & \downarrow & \downarrow \\
T & \xrightarrow{q} & S
\end{array}
\]

such that \( t' \) is a closed immersion and \( t' \) is smooth. Then

\[
\begin{array}{ccc}
X' & \xrightarrow{\iota'} & X \\
\downarrow & \downarrow & \downarrow \\
T & \xrightarrow{p} & S
\end{array}
\]

such that \( \iota' \) is a closed immersion, \( t' \) is smooth, and \( q^{cl} \) is surjective.

Consider a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{t'} & X \\
\downarrow & \downarrow & \downarrow \\
T & \xrightarrow{p} & S
\end{array}
\]

as above, with \( t \) a smooth map and with \( \iota \) a closed immersion. Let

\[
T \times_S X = Z_1 \cup Z_2,
\]

where \( Z_1 \) is the union of irreducible components of \( T \times_S X \) dominant over \( T \) and \( Z_2 \) is the union of the other irreducible components. Denote by \( Z_1^{\alpha} := Z_1 \setminus (Z_1 \cap Z_2) \). Similarly define \( Z_1' \) and \( Z_2' \) for \( T \times_S X' \). Let \( b = \text{reldim } q \) and \( a = b + \dim T = \dim \Gamma \).

**Proposition 3.8.** The class \([\Gamma] \in gr_{a}K_{T \times_S X'}(T \times X')\) is not supported on \( Z_2' \).

**Proof.** Let \( \ell \) be an \( ft \)-ample divisor. Denote by \( \text{pr}_1 : T \times X' \to T \). Then

\[
\text{pr}_{1*} \left( [\Gamma] \cdot \ell^b \right) = d[T] \in \text{gr}_{d}K_{T}(T)
\]

for \( d \) a non-zero integer. Indeed, let \( \eta \) be the generic point of \( T \). By abuse of notation, we denote by \( \eta \) its image in \( S \). It suffices to show \( 9 \) after restricting to \( \eta \). In this case, \( d \) is the intersection number \([\Gamma_{\eta}] \cdot \ell^b \) in \( X'_{\eta} \).
Further, let $x \in \text{gr}_a K_{Z_2}(T \times X')$. We have that
\[
\text{pr}_{1*} (x \cdot \ell^b) = 0 \in \text{gr}_{\dim T} K(T)
\]
because the support on $x \cdot \ell^b$ is not dominant over $T$. The conclusion thus follows.

**Proposition 3.9.** Let $T \xrightarrow{\tilde{\pi}} X$ be a generically finite map from $T$ smooth with image $V$. Let $a > \dim X_V$. Then $\text{gr}_a K^q_{T \times S} (T \times X) = 0$. Further, $\text{gr}_{\dim X_V} K^q_{T \times S} (T \times X)$ is generated by irreducible components of $T \times S X$ dominant over $T$ of dimension $X_V$.

**Proof.** Suppose we are in the setting of (8) and let $s : X \to X'$ be a section of $t$. We write $\tilde{p} : \Gamma \to T \times X$, $\tilde{r} : \Gamma \to T \times X'$ etc. Assume that
\[
\tilde{r}_s \tilde{r}_s [\Gamma] = \tilde{p}_s [\Gamma] \neq 0 \in \text{gr}_a K^q_{T \times S} (T \times X).
\]
Then there exists a non-zero $x \in \text{gr}_a K^q_{T \times S} (T \times X)$ such that
\[
\tilde{p}_s [\Gamma] = \tilde{s}_s (x) \in \text{gr}_a K^q_{T \times S} (T \times X').
\]
Consider the diagram
\[
\begin{array}{ccc}
\text{gr}_a K_{T \times S} (T \times X') & \xrightarrow{\text{res}} & \text{gr}_a K_{Z_1^a} (T \times X') \\
\tilde{s}^* & & \tilde{s}^* \\
\text{gr}_a K_{T \times S} (T \times X) & \xrightarrow{\text{res}} & \text{gr}_a K_{Z_1^a} (T \times X \setminus Z_2).
\end{array}
\]
By Proposition 3.8, we have that $\text{res}(x) \neq 0 \in \text{gr}_a K_{Z_1^a} (T \times X \setminus Z_2)$. We have that $\dim Z_1^a = \dim X_V$, and the conclusion follows from here.

3.5. **The perverse filtration $P^{\leq i}_f$.** We now define a smaller filtration $P^{\leq i}_f \subset P^{\leq i}_f$.

We use the notation from Subsection 3.3.

Let $X$ be a smooth variety with a proper map $f : X \to S$ and let $T \xrightarrow{\pi} S$ be a generically finite map onto its image from $T$ smooth. Consider a subvariety $Y \hookrightarrow X$.

Define the subspaces of $\text{gr} G_Y(X)$:
\[
P^{\leq i}_{f,T} := \text{span}_T \left( \Phi_T : \text{gr} K(T) \to \text{gr} G_Y(X) \right)
\]
\[
P^{\leq i}_{f,V} := \text{span} \left( P^{\leq i}_{f,T} \text{ for all maps } \pi \text{ as above } V \right),
\]
where $\Gamma \in \text{gr}_{\dim X - s} K^q_{T \times S, Y \hookrightarrow} (T \times X)$ and
\[
\left\lfloor \frac{i + \dim X - \dim T}{2} \right\rfloor \geq s.
\]

Using the notation from Subsection 3.3 define
\[
P^{\leq i}_f \text{gr} G_Y(X) := \bigcap_{V \subset S} \bigcap_{a \in A_V} \ker \left( \tau_{V,T}^{\alpha} : P^{\leq i}_f \text{gr} G_Y(X) \to P^{\leq i+e_{a,T}}_{f,V} \text{gr} K \left( X^u \right) \right).
\]

The definition is independent of the resolutions $\tilde{X}^u_V$ chosen, see Subsection 3.3.
Theorem 3.10. Let $X$ and $Y$ be smooth varieties with $c = \dim X - \dim Y$. Consider proper maps

$$
Y \xrightarrow{h} X
\quad
\xrightarrow{g} \quad
\xrightarrow{f} S
$$

There are induced maps

$$
h_* : P_g^{\leq c} \cdot \gr K(Y) \to P_f^{\leq c} \cdot \gr K(X)
$$

$$
h_* : P_g^{\leq c} \cdot \gr K(Y) \to P_f^{\leq c} \cdot \gr K(X).
$$

If $h$ is surjective, then there are induced maps

$$
h^* : P_f^{\leq c} \cdot \gr K(X) \to P_g^{\leq c} \cdot \gr K(Y)
$$

$$
h^* : P_f^{\leq c} \cdot \gr K(X) \to P_g^{\leq c} \cdot \gr K(Y).
$$

Proof. The functoriality follow as in Propositions 3.1, 3.2, and Theorem 3.6, using Proposition 3.7. □

3.6. Properties of the perverse filtration. Consider a proper map $f : X \to S$ with $X$ smooth. Define the defect of semismallness of $f$ by

$$
s := s(f) = \dim X \times S X - \dim X.
$$

Further, define $s' = \max (\dim X + \dim S - 4, \dim X)$. The perverse filtration in cohomology satisfies

$$
p^{\leq -s-1} H_f(X) = 0 \quad \text{and} \quad p^{\leq s}_f(X) = H_f(X),
$$

see [7, Section 1.6]. We prove an analogous result in $K$-theory:

Theorem 3.11. For $f$ as above,

$$
P_f^{\leq -s' - 1} \cdot \gr K(X) = P_f^{\leq -s - 1} \cdot \gr K(X) = 0
$$

$$
P_f^{\leq s} \cdot \gr K_0(X) = P_f^{\leq s} \cdot \gr K_0(X) = \gr K_0(X).
$$

Proposition 3.12. Let $f : X \to S$ be a surjective map from $X$ smooth with $\text{rerdim } f > 0$ and consider a subvariety $Z \hookrightarrow X$ of codimension $\geq 2$. Then there exists a subvariety $\iota : Y \hookrightarrow X$ of codimension 1 such that $Z \subset Y$ and $f \iota : Y \to S$ is surjective.

Proof. It suffices to pass to the generic point of $Z$, and we can thus assume that $Z$ is a point and is given by a complete intersection of smooth hypersurfaces $H_1, \cdots, H_r$ in $X$ with $r \geq 2$. Localizing at the generic point of $Z$, we can assume that $Z$ is a point. Further restricting to an open set of $X$, we can assume that the fibers of $f$ are irreducible. Assume that none of the maps

$$
f_i : H_i \to Z
$$

are surjective. Let $S_i$ be the image of $f_i$. Let $S' := \bigcap_{i=1}^r S_i$. Then $S'$ is not empty because it contains $f(Z)$. We have $\pi^{-1}(S_i) = H_i$ and so $\bigcap_{i=1}^r H_i$ contains $\pi^{-1}(S')$. 

This means that \( \dim (\bigcap_{i=1}^r H_i) \geq \text{reldim } f \). This bound contradicts that \( \bigcap_{i=1}^r H_i \) is a point \( Z \).

**Proposition 3.13.** Let \( f : X \to S \) be a proper surjective map from \( X \) smooth of relative dimension \( d \). Then

\[
P_f^{\leq d} \text{gr} K_0(X) = \text{gr} K_0(X).
\]

**Proof.** We use induction on \( d \). Assume that \( f \) is generically finite. Consider the correspondence \( \Delta \cong X \leftarrow X \times_S X \):

\[
\begin{array}{ccc}
\Delta & \sim \rightarrow & X \\
\downarrow & & \downarrow f \\
X & \sim \rightarrow & S.
\end{array}
\]

This implies that \( P_f^{\leq 0} \text{gr} K_0(X) = \text{gr} K_0(X) \).

Consider \( f \) with \( d > 0 \). Let \( \iota : Z \hookrightarrow X \) be a subvariety of codimension \( \geq 2 \).
By Proposition 3.12 there exists \( Y \hookrightarrow X \) of codimension 1 such that \( Z \subset Y \) and \( Y \to S \) has image \( W \) of codimension \( \leq 1 \) in \( S \). Let \( Y' \to Y \) be a resolution of singularities and denote the resulting map by \( g : Y' \to W \). By induction,

\[
P_g^{d-1} \text{gr} K_0(Y') = \text{gr} K_0(Y').
\]

By Proposition 2.2

\[
\text{image} \left( \iota_* : \text{gr} K_0(Z) \to \text{gr} K_0(X) \right) \subseteq \text{image} \left( g_* : \text{gr} K_0(Y') \to \text{gr} K_0(X) \right).
\]

Finally, assume that \( Z \hookrightarrow Y \) has codimension 1. By Proposition 2.1 it suffices to show that

\[
\text{image} \left( \text{gr}_{\dim Z} G_0(Z) \to \text{gr}_{\dim Z} G_0(X) \right) \subseteq P_f^{\leq d} \text{gr} K_0(X)
\]

because \( \text{gr}_i G_0(Z) \) for \( i < \dim Z \) is generated by varieties of smaller dimension than \( Z \). If \( Z \to S \) is surjective, then it has relative dimension \( d - 1 \) and we can treat it as above. If \( Z \to S \) is not surjective, let \( W \subset S \) be its image. Choose a resolution of singularities \( T \to W \) and a smooth variety \( \Gamma \) with surjective maps \( p \) and \( q \):

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{p} & Z & \xrightarrow{q} & X \\
\downarrow & & \downarrow & & \downarrow f \\
T & \xrightarrow{q} & W & \xrightarrow{g} & S.
\end{array}
\]

Then \([\Gamma] \in \text{gr}_{\dim X-1} K^d_T \times_S X(T \times X)\) and its image \( \Phi_\Gamma \) is in \( P_f^{\leq d} \text{gr} K_0(X) \). Then

\[
\text{image} \left( \text{gr}_{\dim Z} G_0(Z) \to \text{gr}_{\dim Z} K_0(X) \right) \subseteq \text{image} \Phi_\Gamma \subseteq P_f^{\leq d} \text{gr} K_0(X).
\]

The conclusion now follows from Proposition 2.1. \( \square \)
Proof of Theorem 3.11. We first show that \( P^{\leq -s'} \) gr \( K_0(X) = 0 \). Consider a map \( \pi : T \to X \) generically finite onto its image \( V \subseteq S \) with \( T \) smooth and consider a correspondence

\[ \Gamma \in \text{gr}_{\dim X-b}G_{T \times S}(T \times S). \]

Then \( \dim X-b \leq \dim T \times S X \leq \max(\dim X, \dim X + \dim T - 2) \), and so

\[ b \geq \min(0, -\dim T + 2). \]

By the bound (5), it suffices to show that

\[ \left\lfloor -s' - 1 + \dim X - \dim T \right\rfloor \leq 0, \]

or, alternatively, that

\[ \max(\dim X - \dim T - 1, \dim X + \dim T - 5) < s', \]

which is true because \( 0 \leq \dim T \leq \dim S \).

We next explain that \( P^{\leq s-1} \) gr \( K_0(X) = 0 \). We keep the notation from the previous paragraph. Let \( [\Gamma] \in \text{gr}_{\dim X-b}K_T^{\omega}(T \times S) \). By Proposition 3.9 we have that

\[ b \geq \dim X - \dim X_V. \]

It suffices to show that

\[ \left\lfloor -s - 1 + \dim X - \dim T \right\rfloor < \dim X - \dim X_V, \]

or, alternatively, that

\[ 2 \dim X_V - \dim V \leq s - \dim X = \dim X \times_S X, \]

which is true because \( 2 \dim X_V - \dim V \leq \dim X_V \times_V X_V \leq \dim X \times_S X \).

We next show that \( P^{\leq s} \) gr \( K_0(X) = 0 \). We can assume that \( f \) is surjective of relative dimension \( d \). Use the notation from Subsection 3.3. We have that

\[ P^{\leq s} \text{gr} K_0(X) := \bigcap_{V \subseteq S} \bigcap_{a \in A_V} \ker \left( \tau^{\omega_s}_V : P^{\leq s} \text{gr} K_0(X) \to P^{\leq s+c^a_V} \text{gr} K_0 \left( \widetilde{X}_V \right) \right). \]

We claim that

\[ \text{reldim} \left( \widetilde{X}_V \to V \right) = \text{reldim} \ (X^a_V \to V) \leq s + c^a_V. \]

Indeed,

\[ \dim X^a_V - \dim V \leq (\dim X \times_S X - \dim X) + (\dim X - \dim X^a_V) \]

\[ 2 \dim X^a_V - \dim V \leq \dim X^a_V \times_V X^a_V \leq \dim X \times_S X, \]

which is true. By Proposition 3.13 this implies that \( P^{s+c^a_V} \) gr \( K_0 \left( \widetilde{X}_V \right) = 0 \). Furthermore, \( s \geq d \), so Proposition 3.13 implies that \( P^{\leq s} \) gr \( K_0(X) = \text{gr} K_0(X) \), and
thus $P_f^{\leq s} \gr K_0(X) = \gr K_0(X)$. This also implies that $P_f^{\leq s} \gr K_0(X) = \gr K_0(X)$.

\[\square\]

3.7. Examples of perverse filtration in $K$-theory.

3.7.1. Let $X$ be a smooth variety of dimension $d$, and let $f : X \to \Spec \mathbb{C}$. Then

$$P_f^{\leq i} \gr^j K_0(X) = \begin{cases} \gr^j K_0(X) & \text{if } j \leq \frac{i+d}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

This also implies that $P_f^{\leq s} \gr^j K_0(X) = \gr^j K_0(X)$.

3.7.2. Let $X$ be a smooth variety and let $E$ be a vector bundle on $X$ of rank $d + 1$. Let $Y := \mathbb{P}_X(E)$. Denote by $h := c_1(O_Y(1)) \in \gr^2 K_0(Y)$. Consider the projection map $f : Y \to X$. We have that $s(f) = d$. For $i \leq d$, there exists an isomorphism

$$\bigoplus_{0 \leq j \leq \frac{i+d}{2}} \gr^{a-2j} K_0(X) \cong P_f^{\leq i} \gr^a K_0(Y)$$

$$(x_0, \cdots, x_{\frac{i+d}{2}}) \mapsto \sum_{j \leq \frac{i+d}{2}} h^j f^*(x_j).$$

The condition for $P^{\leq i}$ is checked using projective bundles over varieties of smaller dimension, and we obtain that

$$\bigoplus_{0 \leq j \leq \frac{i+d}{2}} \gr^{a-2j} K_0(X) \cong P_f^{\leq i} \gr^a K_0(Y).$$

3.7.3. Let $X$ be a smooth variety and let $Z$ be a smooth subvariety of codimension $d + 1$. Consider the blow-up diagram for $Y = \text{Bl}_Z X$:

$$\begin{array}{ccc} E & \xleftarrow{i} & Y \\ \downarrow p & & \downarrow f \\ Z & \xleftarrow{j} & X. \end{array}$$

Let $h := c_1(O_E(1)) \in \gr^2 K_0(E)$. We have that $s(f) = d - 1$. For $i \leq d - 1$, there is an isomorphism:

$$\gr^a K_0(X)^{\varepsilon} \oplus \bigoplus_{0 \leq j \leq \frac{i+d}{2} - 1} \gr^{a-2-2j} K_0(Z) \cong P_f^{\leq i} \gr^a K_0(Y)$$

$$(x, z_0, \cdots, z_{\frac{i+d}{2} - 1}) \mapsto f^*(x) + \sum_{j \leq \frac{i+d}{2} - 1} \iota_\ast (h^j q^s(z_j)).$$

Here $\varepsilon$ is 0 if $i < 0$ and is 1 otherwise. This follows from the computation in Subsection 3.7.2 and Proposition 4.4.

One can check that in the above examples, we have that $P_f^{\leq s} = P_f^{\leq s}$.
3.8. Compatibility with the perverse filtration in cohomology. Consider a proper map $f : X \to S$ with $X$ smooth. Define filtrations $P_f^{\leq i}, P_f^{< i}, P_f^{> i}$ on $H^\cdot(X), H^\cdot(X)_{\text{alg}}$ as in Subsections 3.1 and 3.5. We have that
\[
\text{image} \left( \zeta : P_f^{\leq i} \text{gr}^j K_0(X)_Q \to P_f^{\leq i} H^{2j}(X) \right) = P_f^{\leq i} H^{2j}(X)_{\text{alg}}.
\]
We use the notation $P_f^{\leq i} H^\cdot(X)_{\text{full}}$ for the cohomology of summands of $P_f^{\leq i} R^\cdot f_! IC_X$ with support $S$.

**Proposition 3.14.** There exist natural inclusions
\[
P_f^{\leq i} H^\cdot(X) \subset P_f^{< i} H^\cdot(X) \subset P_f^{\leq i} H^\cdot(X)_{\text{alg}} \subset P_f^{\leq i} H^\cdot(X)_{\text{alg}} \subset p_h^{\leq i} H^\cdot(X)_{\text{alg}}.
\]
Thus the cycle map restricts to
\[
\zeta : P_f^{\leq i} \text{gr}^j K_0(X)_Q \to p_h^{\leq i} H^{2j}(X)_{\text{alg}}.
\]

**Proof.** Let $\pi : T \to S$ be a generically finite map with $T$ smooth. Consider a correspondence
\[
\Gamma \in \text{gr}^{\dim X - s} K_{T \times S, 0}(T \times X)
\]
such that
\[
\left\lfloor \frac{i + \dim X - \dim T}{2} \right\rfloor \geq s.
\]
The correspondence $\Gamma$ induces a map of constructible sheaves on $S$:
\[
R\pi_* Q_T[-2s] \xrightarrow{\Phi_\Gamma} Rf_* Q_X.
\]
\[
R\pi_* IC_T[\dim X - \dim T - 2s] \xrightarrow{\Phi_\Gamma} Rf_* IC_X.
\]
If $\pi$ is not surjective, $R\pi_* IC_T$ has summands with support $W \subseteq S$. If $\pi$ is surjective, the complex $R\pi_* IC_T$ has summands $IC_S(\mathcal{L})$ of full support and of perverse degree zero, and other summands with support $W \subsetneq S$. The perverse degree of the sheaf with support $S$ in the image of $\Phi_\Gamma$ is
\[
\dim X - \dim T - 2s \leq i.
\]
Thus $P_f^{\leq i} H^\cdot(X)$ contains cohomology of sheaves $IC_S(\mathcal{L})[j]$ with $j \leq i$ which appear as summands of $Rf_* IC_X$ and of other sheaves with support $W \subsetneq S$. Thus
\[
P_f^{\leq i} H^\cdot(X) \to p_h^{\geq i} H^\cdot(X)_{\text{full}}.
\]
Using the notation in Subsection 3.3, we have that
\[
P_f^{\leq i} H^\cdot(X) := \bigcap_{V \subsetneq S} \bigcap_{a \in A_V} \ker \left( \tau'^* : P_f^{\leq i} H^\cdot(X) \to P_f^{> i + e\nu} H^\cdot \left( \widetilde{X^\nu} \right) \right).
\]
In particular,
\[ P_{f}^{<i} H^{i}(X) \subset \bigcap_{V \subseteq S} \bigcap_{a \in A_{V}} \ker \left( \tau_{V}^{a*} : P_{f}^{<i} H^{i}(X) \to P_{f}^{<1+i} \left( X_{V}^{a} \right) \right). \]

Using (4), we obtain that \( P_{f}^{<i} H^{i}(X) \subseteq P_{f}^{<i} H^{i}(X). \)

\[ \square \]

**Remark.** We expect equalities \( P_{f}^{<i} H^{i}(X)_{al} = P_{f}^{<i} H^{i}(X)_{alg} = P_{f}^{<i} H^{i}(X)_{alg} \) in the above Proposition.

4. **Intersection \( K \)-theory**

4.1. **Definition of intersection \( K \)-theory.** Let \( S \) be a variety, let \( U \) be an open subset, let \( f : X \to S \) be such \( f^{-1}(U) \to U \) is étale, and let \( L = f_{*} \left( Z_{f^{-1}(U)} \right) \).

Recall the notation of Subsection 3.3. Define
\[ \tilde{P}_{f}^{<i} \operatorname{gr} K(X) := \text{image} \left( \bigoplus_{V \subseteq S} \bigoplus_{a \in A_{V}} P_{f}^{<i} \operatorname{gr} K_{X_{V}^{a}}(X) \to P_{f}^{<i} \operatorname{gr} K(X) \right). \]

Define
\[ \operatorname{gr} IK(S, L) := P_{f}^{<0} \operatorname{gr} K(X)/ \left( \tilde{P}_{f}^{<0} \operatorname{gr} K(X) \cap \ker f_{*} \right). \]

**Theorem 4.1.** The definitions of \( \operatorname{gr} IK(S, L) \) and \( \operatorname{gr} IK(S, L) \) do not depend on the choice of the map \( f : X \to S \) with \( f^{-1}(U) \to U \) étale such that \( L \cong f_{*} \left( Z_{f^{-1}(U)} \right) \).

Further, let \( U^{o} \subset U \) be an open set and let \( L^{o} := L|_{U^{o}}. \) Then
\[ \operatorname{gr} IK(S, L) \cong \operatorname{gr} IK(S, L^{o}), \]
\[ \operatorname{gr} IK(S, L) \cong \operatorname{gr} IK(S, L^{o}). \]

We start with some preliminary results. Let \( f : X \to S \) be a proper map with \( X \) smooth. Let \( Z \) be a smooth subvariety of \( X \) with normal bundle \( N, Y = \text{Bl}_{Z} X, \) and \( E = \mathbb{P}_{Z}(N) \) the exceptional divisor
\[ E \xrightarrow{\iota} Y \xrightarrow{\pi} X. \]

Consider the proper maps
\[ E \leftarrow^{\iota} Y \xrightarrow{\pi} X \]
\[ \xrightarrow{h} Z \xrightarrow{j} X. \]
Let \( X' \hookrightarrow X \) be a closed subset, and denote its preimages in \( Y, Z, E \) by \( Y', Z', E' \) respectively. Denote by
\[
\text{gr} \, K_{Y'}(Y)^0 = \ker (\pi_* : \text{gr} \, K_{Y'}(Y) \to \text{gr} \, K_{X'}(X)).
\]

**Proposition 4.2.** Let \( T \to S \) be a map with \( T \) smooth which is generically finite onto its image. Then
\[
\text{gr} \, K_{T \times_S Y'}(T \times Y) = \pi^* \text{gr} \, K_{T \times_S X'}(T \times X) \oplus \text{gr} \, K_{T \times_S E'}(T \times Y)^0
\]
\[
\text{gr} \, K_{T \times_S Y'}^q(T \times Y) = \pi^* \text{gr} \, K_{T \times_S X'}^q(T \times X) \oplus \text{gr} \, K_{T \times_S E'}^q(T \times Y)^0.
\]

**Proof.** Let \( c + 1 \) be the codimension of \( Z \) in \( X \). Denote by \( O(1) \) the canonical line bundle on \( E \) and let \( h = c_1(O(1)) \in \text{gr}^2 K_0(E) \). There is a semi-orthogonal decomposition [4, Theorem 4.2] with \( \pi^* \) fully faithful on \( D^b(X) \):
\[
D^b(Y) = \left\langle \pi^* D^b(X), t_* \left( p^* D^b(Z) \otimes O(-1) \right), \cdots, t_* \left( p^* D^b(Z) \otimes O(-c) \right) \right\rangle,
\]
which implies that
\[
\text{gr}^j K_{Y'}(Y) = \pi^* \text{gr}^j K_{X'}(X) \oplus \bigoplus_{0 \leq k \leq c - 1} t_* \left( h^k \cdot p^* \text{gr}^j - 2k K_{Z'}(Z) \right).
\]
Using the analogous decomposition for \( Y' \setminus Y'' = \text{Bl}_{Z'}(X \setminus X') \) and the localization sequence in K-theory [21, V.2.6.2], we obtain that
\[
\text{gr}^j K_{Y'}(Y) = \pi^* \text{gr}^j K_{X'}(X) \oplus \bigoplus_{0 \leq k \leq c - 1} t_* \left( h^k \cdot p^* \text{gr}^j - 2k K_{Z'}(Z) \right).
\]
In particular, we have that
\[
\text{gr}^j K_{T \times_S Y'}(T \times Y) = \pi^* \text{gr}^j K_{T \times_S X'}(T \times X) \oplus \bigoplus_{0 \leq k \leq c - 1} t_* \left( h^k \cdot p^* \text{gr}^j - 2k K_{T \times_S E'}(T \times Z) \right)
\]
and thus that
\[
\text{gr} \, K_{T \times_S Y'}(T \times Y) = \pi^* \text{gr} \, K_{T \times_S X'}(T \times X) \oplus \text{gr} \, K_{T \times_S E'}(T \times Y)^0.
\]
By Proposition 3.7, we also have that
\[
\text{gr} \, K_{T \times_S Y'}^q(T \times Y) = \pi^* \text{gr} \, K_{T \times_S X'}^q(T \times X) \oplus \text{gr} \, K_{T \times_S E'}^q(T \times Y)^0.
\]

An immediate corollary of Proposition 4.2 is:

**Corollary 4.3.** We continue with the notation from Proposition 4.2. There are decompositions
\[
P^{\leq i}_g \text{gr} \, K_{Y'}(Y) = \pi^* \! P^{\leq i}_f \text{gr} \, K_{X'}(X) \oplus P^{\leq i}_g \text{gr} \, K_{E'}(Y)
\]
\[
P^{< i}_g \text{gr} \, K_{Y'}(Y) = \pi^* \! P^{< i}_f \text{gr} \, K_{X'}(X) \oplus P^{< i}_g \text{gr} \, K_{E'}(Y).
\]
We next prove:
Proposition 4.4. We continue with the notation from Proposition 4.2. There are decompositions

\[
\begin{align*}
P_g^i \text{gr}' K_Y(Y) &= \pi^* P_f^i \text{gr}' K_X'(X) \oplus P_g^i \text{gr}' K_E(Y) \\
P_g^i \text{gr}' K_Y(Y) &= \pi^* P_f^i \text{gr}' K_X'(X) \oplus P_g^i \text{gr}' K_E(Y).
\end{align*}
\]

Proof. We use the notation from Subsection 3.3. For \(V \subseteq S\), let \(A_V\) be the set of irreducible components of \(f^{-1}(V)\). Let \(X_V^a\) be such a component.

If \(X_V^a \subset Z\), then there is only one irreducible component \(Y_V^a = \mathbb{P}_{X_V^a}(N)\) of \(g^{-1}(V)\) above it.

If \(X_V^a\) is not in \(Z\), then there is one component \(Y_V^a\) of \(g^{-1}(V)\) birational to \(X_V^a\). The other components are \(\mathbb{P}_{W_V^b}(N)\), where \(W_V^b\) is an irreducible component of \(X_V^a \cap Z\). Denote by \(B_a\) the set of such components. For \(a \in A\) and \(b \in B_a\), consider resolutions of singularities \(r\) such that

\[
\begin{array}{cccccc}
\widetilde{Y}_V^a & \xrightarrow{r} & Y_V^a & \xrightarrow{r} & Y \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{X}_V^a & \xrightarrow{r} & X_V^a & \xrightarrow{r} & X \\
& & X_V^a \cap Z & & \\
\widetilde{W}_V^b & \xrightarrow{r} & W_V^b
\end{array}
\]

Denote by \(\tau\) maps as in Subsection 3.3 for example \(\tau_V^a : \widetilde{X}_V^a \to X\), and by \(\mu\) the map

\[
(10) \quad \tau_V^b : \widetilde{W}_V^b \xrightarrow{\mu} \widetilde{X}_V^a \xrightarrow{\tau_V^a} X.
\]

We consider the proper maps

\[
\begin{align*}
\widetilde{f}_V^a : \widetilde{X}_V^a & \to X_V^a \to V \\
\widetilde{g}_V^a : \widetilde{Y}_V^a & \to Y_V^a \to V \\
\widetilde{f}_V^b : \widetilde{W}_V^b & \to W_V^b \to V \\
\widetilde{g}_V^b : \mathbb{P}_{\widetilde{W}_V^b}(N) & \to \mathbb{P}_{W_V^b}(N) \to V.
\end{align*}
\]

Denote by

\[
\begin{align*}
\phi_V^a &= \text{codim } \langle X_V^a \text{ in } X \rangle = \text{codim } \langle Y_V^a \text{ in } Y \rangle \\
\phi_V^b &= \text{codim } \langle W_V^b \text{ in } X \rangle \\
\phi_V^b &= \text{codim } \langle \mathbb{P}_{W_V^b}(N) \text{ in } Y \rangle
\end{align*}
\]
Any two such varieties \( f : X \to S \) and \( f' : X' \to S \) are birational, so by (10) there is a smooth variety \( W \) with \( \overline{\tau_V^a} : \pi^* P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{W}_V^a) \), so that \( \ker (\tau_V^a : \pi^* P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{W}_V^a)) \).

By Proposition 3.5 and Proposition 3.1 for the map \( \mu \) in (10), we have that

\[
\ker (\tau_V^a : \pi^* P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{W}_V^a)) \supset \ker (\tau_V^a : \pi^* P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{X}_V^a)).
\]

Let \( B_V \) be the set of irreducible components of \( g^{-1}(V) \). For \( d \in B_V \), denote by \( \overline{g}_d^V : \overline{Y}_d^V \to V \) and let \( c_d^V := \operatorname{codim} (Y_d^V \text{ in } Y) \). We have that \( B_V = A \cup \bigcup_{a \in A} B_a \).

The statements in (11) and (12) imply that

\[
\pi_* \left( \bigcap_{V \subseteq S} \bigcap_{d \in B_V} \ker (\tau_V^a : \pi^* P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{Y}_d^V)) \right) = \bigcap_{V \subseteq S} \bigcap_{a \in A_V} \ker (\tau_V^a : P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{X}_V^a)).
\]

Using Corollary 4.3 we obtain that

\[
P_g^i \operatorname{gr} K_{Y^j}(Y) = \pi^* P_{f^j}^i \operatorname{gr} K_{X^j}(X) \oplus P_g^i \operatorname{gr} K_{E^j}(Y)^0.
\]

The analogous statement for \( P_{\leq i}^j \) follows similarly.

**Proof of Theorem 4.4** Any two such varieties \( f : X \to S \) and \( f' : X' \to S \) are birational, so by (11) there is a smooth variety \( W \) such that

\[
\begin{array}{ccc}
X & \xleftarrow{\pi} & W \\
\downarrow{f} & \quad & \uparrow{\pi'} \\
S & \xrightarrow{f'} & X'
\end{array}
\]

and the maps \( \pi \) and \( \pi' \) are successive blow-ups along smooth subvarieties of \( X \) and \( X' \), respectively. It thus suffices to show that

\[
P_f^{i_0} \operatorname{gr} K. (X) / \left( \overline{P}_f^{i_0} \operatorname{gr} K. (X) \cap \ker f_* \right) \cong P_g^{i_0} \operatorname{gr} K. (Y) / \left( \overline{P}_g^{i_0} \operatorname{gr} K. (Y) \cap \ker g_* \right),
\]

the codimensions as in Subsection 3.3. By Proposition 3.5 we have that

\[
(11) \quad \ker (\tau_V^a : \pi^* P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{Y}_V^a)) \cong \\
\ker (\tau_V^a : P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{X}_V^a)).
\]

By Proposition 3.5 and Proposition 3.1 for the map \( \mu \) in (10), we have that

\[
(12) \quad \ker (\tau_V^a : \pi^* P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{W}_V^a)) \supset \\
\ker (\tau_V^a : \pi^* P_{f^a}^i \operatorname{gr} K. (X) \to P_{f^a}^{i+\varepsilon} \operatorname{gr} K. (\overline{X}_V^a)).
\]
where \( \pi : Y \to X \) is the blow up along smooth subvariety \( Z \hookrightarrow X \) and

\[
Y \xrightarrow{\pi} X \quad \text{with support} \quad \pi^{-1}(S) \subset f^{-1}(U) \to U \text{ étale, and let } L = f_*\left(Z_{f^{-1}(U)}\right).
\]

By Proposition 4.4, we have that

\[
P_g^{\leq i} \text{gr}^r K(Y) = \pi^* P_f^{\leq i} \text{gr}^r K(X) \oplus P_g^{\leq i} \text{gr}^r K_E(Y)^0
\]

\[
\tilde{P}_g^{\leq i} \text{gr}^r K(Y) = \pi^* \tilde{P}_f^{\leq i} \text{gr}^r K(X) \oplus P_g^{\leq i} \text{gr}^r K_E(Y)^0.
\]

The map \( \pi^* \) is injective. Taking the quotients we thus obtain the isomorphism (13). The analogous statement for \( IK \) is similar. \( \square \)

4.2. Cycle map for intersection K-theory. Let \( S \) be a variety, let \( U \) be an open subset, let \( f : X \to S \) be such \( f^{-1}(U) \to U \) is étale, and let \( L = f_*\left(Z_{f^{-1}(U)}\right) \).

**Proposition 4.5.** The cycle map \( \text{ch} : \text{gr}^d K_0(X)_Q \to H^{2j}(X) \) induces cycle maps independent of the map \( f : X \to S \) as in Subsection 4.1:

\[
\epsilon : \text{gr}^d IK_0(S, L)_Q \to IH^{2j}(S, L \otimes Q)
\]

\[
\epsilon : \text{gr}^d IK_0(S, L)_Q \to IH^{2j}(S, L \otimes Q).
\]

**Proof.** Define \( P_f^{\leq i} H_{X^p}^j(X) \) as in Subsection 3.1 and denote by

\[
\tilde{P}_f^{\leq 0} H^j(X) := \text{image} \left( \bigoplus_{V \subseteq S, a \in A_T} P_f^{\leq i} H_{X^p}^j(X) \to H^j(X) \right) \cap P_f^{\leq 0} H^j(X).
\]

Denote by \( \tilde{P}_f^{\leq 0} H^j(X) \subset P_f^{\leq 0} H^j(X) \) the sum of summands of \( \text{gr}^d Rf_*IC_X \) with support strictly smaller than \( S \). By Proposition 3.14, the cycle map respects the perverse filtrations in K-theory and cohomology

\[
\epsilon : P_f^{\leq 0} \text{gr}^d K_0(X)_Q \to P_f^{\leq 0} H^{2j}(X) \hookrightarrow \tilde{P}_f^{\leq 0} H^j(X)
\]

\[
\epsilon : \tilde{P}_f^{\leq 0} \text{gr}^d K_0(X)_Q \to \tilde{P}_f^{\leq 0} H^{2j}(X) \hookrightarrow \tilde{P}_f^{\leq 0} H^j(X).
\]

Taking the quotient and using (3), we obtain a map

\[
\epsilon : \text{gr}^d IK_0(S, L)_Q \to IH^{2j}(S, L \otimes Q).
\]

The proof that the above cycle map is independent of the map \( f \) chosen follows as in Theorem 4.1. The argument for \( IK \) is similar. \( \square \)

4.3. Further properties of intersection K-theory. Intersection cohomology satisfies the following properties, the second one explaining its name [10]. Motivation:

- The natural map \( H^i(S) \to H^{BM}_{2d-i}(S) \) factors through

\[
H^i(S) \to IH^i(S) \to H^{BM}_{2d-i}(S).
\]
• There is a natural intersection map

\[ IH^i(S) \otimes IH^j(S) \to H_{2d-i-j}^{BM}(S) \]

which is non-degenerate for \( i + j = 2d \).

We prove analogous, but weaker versions of the above properties in \( K \)-theory.

**Proposition 4.6.** (a) There are natural maps

\[ gr^j IK.(S) \to gr_{d-i} G.(S) \]

\[ gr^j IK.(S) \to gr_{d-i} G.(S). \]

(b) There are natural intersection maps

\[ gr^j IK.(S) \otimes gr^j IK.(S) \to gr_{d-i-j} G.(S) \]

\[ gr^j IK.(S) \otimes gr^j IK.(S) \to gr_{d-i-j} G.(S). \]

**Proof.** Let \( f : X \to S \) be a resolution of singularities. We discuss the claims for \( IK_\cdot \), the ones for \( IK \) are similar. We construct maps as above using \( f \). They are independent by \( f \) by an argument as in Theorem 4.4.

(a) There is a natural map \( gr^j K.(X) = gr_{d-i} G.(X) \xrightarrow{f_*} gr_{d-i} G.(S) \), and we thus obtain a map

\[ gr^j IK.(S) = P_f^{\leq 0} gr^j K.(X) / (\tilde{P}_f^{\leq 0} gr^j K.(X) \cap \ker f_*) \to gr_{d-i} G.(S). \]

(b) Consider the composite map

\[ P_f^{\leq 0} gr^j K.(X) \otimes P_f^{\leq 0} gr^j K.(X) \to gr^{i+j} K.(X \times X) \xrightarrow{\Delta_*} gr^{i+j} K.(X) \xrightarrow{f_*} gr_{d-i-j} G.(S). \]

The subspaces

\[ (\tilde{P}_f^{\leq 0} gr^j K.(X) \cap \ker f_*) \otimes P_f^{\leq 0} gr^j K.(X) \]

\[ P_f^{\leq 0} gr^j K.(X) \otimes (\tilde{P}_f^{\leq 0} gr^j K.(X) \cap \ker f_*) \]

are in the kernel of \( f_* \Delta^* = \Delta^* (f_* \otimes f_*) \). We thus obtain the desired map. \( \square \)

4.4. **Computations of intersection \( K \)-theory.**

4.4.1. If \( S \) is smooth, then \( gr^j IK.(S) = gr^j IK.(S) = gr^j K.(S) \).

4.4.2. Let \( f : X \to S \) be a small resolution of singularities. Then

\[ gr^j IK_0(S) = gr^j K_0(X). \]

Let \( T \xrightarrow{q} S \) be a generically surjective finite map from \( T \) smooth. By Proposition 3.9, \( gr_{dim.X} K^q_{T \times S X}(T \times X) \) is generated by the irreducible components of \( T \times S X \) dominant over \( S \). This means that the cycles in \( gr^a K^q_{T \times S X}(T \times X) \) supported on the exceptional locus have \( a < \dim X \), and thus they only contribute in perverse degrees \( \geq 1 \), see [3].
Next, say that $T \xrightarrow{\pi} S$ has image $V \subsetneq S$. Let $[\Gamma] \in \text{gr}_{\dim X - a} K^q_{T \times S X}(T \times X)$. By Proposition 3.9 $a \leq \dim X - \dim X_V$. If it contributes in perverse degree $i$, then 
\[
\left\lfloor \frac{i + \dim X - \dim V}{2} \right\rfloor \geq \dim X - \dim X_V,
\]
and thus that
\[
i \geq \dim X + \dim V - 2 \dim X_V \geq 1.
\]
This means that $\widetilde{P}^0_f \text{gr} K.(X) = 0$. By Theorem 3.11 $P^0_f \text{gr} K_0(X) = \text{gr} K_0(X)$, and thus $\text{gr} \text{K}_0(S) = \text{gr} K_0(X)$.

4.4.3. Let $S$ be a surface. Consider a resolution of singularities $f : X \to S$. Let $B$ be the set of singular points of $S$. For each $p$ in $B$, let $A_p = \{C^a_p\}$ be the set of irreducible components of $X_p := f^{-1}(p)$. For each such curve, consider the diagram
\[
\begin{array}{ccc}
C^a_p & \xrightarrow{g^a_p} & X \\
h^a_p \downarrow & & f \\
p \xleftarrow{} & & S
\end{array}
\]
Consider the maps
\[
m^a_p := g^a_p h^{a*}_p : K.(p) \to \text{gr}^1 K.(X)
\]
\[
\Delta^a_p := h^a_p g^{a*} : \text{gr}^1 K.(X) \to K.(p).
\]
We claim that
\[
\widetilde{P}^0_f \text{gr} K.(X) = \text{image} \left( \bigoplus_{p \in B} \bigoplus_{a \in A_p} m^a_p : K.(p) \to \text{gr}^1 K.(X) \right).
\]

The correspondences which contribute to $\widetilde{P}^0_f$ are in $\text{gr}_{2 - s} K^q_{T \times S X}(T \times X)$ for $\pi : T \to S$ a generically finite map onto its image $V \subsetneq S$ with $T$ smooth. By Proposition 3.9
\[
\left\lfloor \frac{2 - \dim V}{2} \right\rfloor \geq s \geq \dim X - \dim X_V.
\]
So the map $T \to S$ is the inclusion of a point $p \hookrightarrow S$ for $p \in B$ and $\Gamma$ is in $\text{gr}_1 G_{X_p}(X)$. Further, for $p, q \in B, a \in A_p, b \in A_q$:
\[
\Delta^b_q m^a_p = \delta_{pq} \delta_{ab} \text{id}.
\]
This means that:
\[
\bigoplus_{p \in B} \bigoplus_{a \in A_p} m^a_p : \bigoplus_{p \in B} K.(p)^{|A_p|} \cong \widetilde{P}^0_f \text{gr}^1 K.(X).
\]
The map \( f \) is semismall, so by Theorem 3.11 we obtain a form of the decomposition theorem for the map \( f \):

\[
\text{gr}^* K_0(X) \cong \text{gr}^* I K_0(S) \oplus \bigoplus_{p \in B} K_0(p)^{[A_p]}.
\]

See Section 5 for further discussions of the decomposition theorem for semismall maps.

4.4.4. Let \( Y \) be a smooth projective variety of dimension \( d \) and let \( \mathcal{L} \) be a line bundle on \( Y \). Consider the cone \( X := \text{Tot}_Y \mathcal{L} \to S \).

Let \( o \) be the vertex of the cone \( X \). There is only one fiber with nonzero dimension \( Y \times \{ o \} \cong S \).

Using the correspondence \( X \cong \Delta \leftarrow X \times_S X \), we see that

\[
P_{f}^{P \leq 0} \text{gr}^i K(X) = \ker (\iota^* : \text{gr}^i K(X) \to P_{f}^{P > 1} \text{gr}^i K(Y)).
\]

By the computation in Subsection 3.7.1

\[
P_{f}^{P > 0} \text{gr}^i K(Y) = \begin{cases} 
\text{gr}^i K(Y) & \text{if } j > |\frac{d+1}{2}|, \\
0 & \text{otherwise}.
\end{cases}
\]

The map \( \iota^* : \text{gr}^i K(X) \to \text{gr}^i K(Y) \) is an isomorphism, so we have that

\[
P_{f}^{P \leq 0} \text{gr}^i K(X) = \begin{cases} 
\text{gr}^i K(Y) & \text{if } j \leq |\frac{d+1}{2}|, \\
0 & \text{otherwise}.
\end{cases}
\]

Further, \( \tilde{P}_{f}^{P \leq 0} \text{gr}^i K(X) \) is generated by the cycles over \( X_o \cong Y \) of codimension between 0 and \( |\frac{d-1}{2}| \). The map

\[
\iota_* : \text{gr}^i K(Y) \to \text{gr}^{i+2} K(X) \cong \text{gr}^{i+2} K(Y)
\]

is multiplication by the class \( h := c_1(\mathcal{L}|_Y) \in \text{gr}^2 K_0(Y) \). As a vector space, we thus have that

\[
\text{gr}^i I K(S) = \begin{cases} 
\text{gr}^i K(S)/h \text{gr}^{i-2} K(Y) & \text{if } j \leq |\frac{d+1}{2}|, \\
0 & \text{otherwise}.
\end{cases}
\]
The computation in cohomology is similar, see [7, Example 2.2.1].

5. The decomposition theorem for semismall maps

We will be using the notation from Subsection 1.4. For \(a, b \in A\), we write \(b < a\) if \(S_b \subset S_a\). Denote by \(\iota_{ba}: X_b \rightarrow X_a\). For \(a \in A\), define

\[
\tilde{P}_f^{< 0}gr^j K_{X_a}(X) = \text{image} \left( \bigoplus_{b < a} \iota_{ba}^* : P_{fgr}^{< 0}gr^j K_{X_b}(X) \rightarrow P_{fgr}^{< 0}gr^j K_{X_a}(X) \right).
\]

First, we state a more precise version of Conjecture 1.3.

Conjecture 5.1. Let \(f: X \rightarrow S\) be a semismall map and consider \(\{S_a| a \in I\}\) a stratification as in Subsection 1.4, denote by \(A \subset I\) the set of relevant strata. For \(a \in A\), consider generically finite maps \(\pi_a: T_a \rightarrow S_a\) with \(T_a\) is smooth such that \(\pi_a^{-1}(S_a) \rightarrow S_a\) is smooth and \(L_a \cong f_*(\mathbb{Z}_{S_a})\). For each \(a\), there exists a rational map \(X_a \rightarrow T_a\), and let \(\Gamma_a\) be the closure of its graph

\[
\begin{array}{ccc}
\varGamma_a & \rightarrow & X_a \\
\downarrow & & \downarrow f_a \\
T_a & \xrightarrow{\pi_a} & S_a \\
\end{array}
\]

The correspondence \(\varGamma_a\) induces an isomorphism

\[
(14) \quad \iota_{a*}\Phi_{\varGamma_a} : P_{\pi_a}^{< 0}gr^j - c_a K(T_a)_\mathbb{Q} \rightarrow P_{\pi_a}^{< 0}gr^j K(T_a)_\mathbb{Q} \cong \iota_{a*} \left( P_{fgr}^{< 0}gr^j K_{X_a}(X)_\mathbb{Q} / \tilde{P}_f^{< 0}gr^j K_{X_a}(X)_\mathbb{Q} \right)
\]

and a decomposition

\[
\bigoplus_{a \in A} gr^j - c_a IK(S_a, L_a)_\mathbb{Q} \cong gr^j K(X)_\mathbb{Q}
\]

\[
(x_a)_{a \in A} \mapsto \sum_{a \in A} \iota_{a*}\Phi_{\varGamma_a}(x_a).
\]

In relation to (14), we propose the following:

Conjecture 5.2. Let \(f: X \rightarrow S\) be a surjective map of relative dimension \(d\) with \(X\) is smooth. Let \(U\) be a smooth open subset of \(S\) such that \(f^{-1}(U) \rightarrow U\) is smooth. For \(y \in U\), \(\pi_1(U, y)\) acts on the irreducible components of \(f^{-1}(y)\) of top dimension and let \(L\) be the associated local system. Then \(L\) is an integer finite local system. There is an isomorphism

\[
P_{fgr}^{< -d}gr^j K(X)_\mathbb{Q} / \tilde{P}_f^{< -d}gr^j K(X)_\mathbb{Q} \cong gr^j IK(S, L)_\mathbb{Q}.
\]

The analogous statement in cohomology follows from the decomposition theorem. In this section, we prove the following:

Theorem 5.3. We use the notation from Conjecture 5.1. Assume that the maps \(\pi_a: T_a \rightarrow S_a\) are small. Then Conjecture 5.1 holds for \(K_0\).
We first note a preliminary result.

**Proposition 5.4.** Consider varieties $S$ and $X$, and a smooth variety $Y$ with surjective maps $f : X \to S$ of relative dimension $d$ and $g : Y \to S$ of relative dimension $0$. Assume there exists an open subset $U$ of $S$ and a map $h$ such that:

$$ g^{-1}(U) \xleftarrow{h} f^{-1}(U) \xrightarrow{g} U \xrightarrow{f} Y $$

Denote also by $h$ the rational map $h : X \dasharrow Y$. Consider a resolution of singularities $\pi : X' \to X$ such that there exists a regular map $h'$ as follows:

$$ X' \xrightarrow{h'} X \xrightarrow{\pi} Y \xrightarrow{g} S. $$

Let $\Gamma$ be the closure of the graph of $h$ in $Y \times X$ and let $\Gamma'$ be the graph of $h'$ in $Y \times X'$. Then the following diagram commutes:

$$ \begin{array}{ccc} gr^r K(Y) & \xrightarrow{\Phi_{\Gamma'}} & gr^r K(X') \\ \Phi_{\Gamma} \downarrow & & \downarrow \pi_* \\ & gr^r G(X). & \end{array} $$

**Proof.** Consider the maps:

$$ \begin{array}{ccc} Y \times X' & \xrightarrow{\pi'} & X' \\ \pi' \downarrow & & \downarrow \pi \\ Y \times X & \xrightarrow{\pi} & X \\ \pi \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & S. \end{array} $$

Let $x \in gr^r K(Y)$. We want to show that:

$$ \pi_* p'_*(\Gamma' \otimes \pi'^* q^*(x)) = p_*(\Gamma \otimes q^*(x)). $$

It suffices to show that

(15) \[ \pi'_* [\Gamma'] = [\Gamma] \text{ in } gr G(X \times Y). \]

Both $\Gamma$ and $\Gamma'$ have dimension equal to the dimension of $X$. The map $\pi' : \Gamma' \to \Gamma$ is birational, so the cone of

$$ \mathcal{O}_\Gamma \to \pi'_* \mathcal{O}_{\Gamma'} $$

is supported on a proper set of $\Gamma$, which implies the equality in (15). \qed
Proof of Theorem 5.3. Let \( a \in A \) and consider the diagram:

\[
\begin{array}{c}
Y_a \\
\downarrow \tau_a \\
X_a \\
\downarrow f \\
T_a \xrightarrow{\pi_a} S_a \\
\end{array}
\]

where the map \( \tau_a \) is a resolution of singularities. Let \( \Gamma_a \) be the closure of the natural rational map \( X_a \rightarrow T_a \). By Proposition 5.4 and Theorem 3.10, the map \( \Phi_{\Gamma_a} \) factors as:

\[
\Phi_{\Gamma_a} : \text{gr}^j K(T_a) \xrightarrow{h_a} \text{gr}^j K(Y_a) \xrightarrow{\tau_a} \text{gr}^j G(X) \rightarrow \text{gr}^{j+c_a} K_{X_a}(X).
\]

By Theorems 3.6 and 3.11, the map \( \Phi_{\Gamma_a} \) factors as:

\[
\Phi_{\Gamma_a} : \text{gr}^j K_0(T_a) = P_{h_a} \text{gr}^j K_0(T_a) \xrightarrow{h_a} P_{f_a} \text{gr}^j K_0(Y_a) \xrightarrow{\tau_a} P_{f_a} \text{gr}^j K_0(X_a) \rightarrow P_{f} \text{gr}^{j+c_a} K_{X_a,0}(X) \rightarrow P_{f} \text{gr}^{j+c_a} K_{X_a,0}(X) = \text{gr}^{j+c_a} K_0(X).
\]

We thus obtain a map of vector spaces

\[
\bigoplus_{a \in A} \Phi_{\Gamma_a} : \bigoplus_{a \in A} \text{gr}^{j-c_a} K_0(T_a) \rightarrow \bigoplus_{a \in A} \text{gr}^{j-c_a} K_{X_a,0}(X) \rightarrow \text{gr}^j K_0(X).
\]

A theorem of de Cataldo–Migliorini [6, Theorem 4.0.4] says that there is an isomorphism:

\[
\bigoplus_{a \in A} \Phi_{\Gamma_a} : \bigoplus_{a \in A} \text{gr}^{j-c_a} K_0(T_a)_Q \xrightarrow{\sim} \text{gr}^j K_0(X)_Q.
\]

Combining with \((16)\), we see that in this case

\[
\Phi_{\Gamma_a} : \text{gr}^{j-c_a} K_0(T_a)_Q \xrightarrow{\sim} \text{gr}^{j-c_a} K_0(T_a)_Q \rightarrow \text{gr}^{j-c_a} K_{X_a,0}(X)_Q \rightarrow \text{gr}^{j-c_a} K_{X_a,0}(X)_Q.
\]

This implies the claim of Theorem 5.3. □

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