Gauss-Bonnet Inflation

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We consider a pure scalar-Gauss-Bonnet gravitational theory without the Ricci scalar. We demonstrate that such a theory, with a quadratic coupling function between the scalar field and the Gauss-Bonnet term, naturally supports inflationary – de Sitter – solutions. During inflation, the scalar field decays exponentially and its effective potential remains always bounded. The theory contains also solutions where these de Sitter phases possess a natural exit mechanism and are replaced by linearly expanding – Milne – phases.

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I. INTRODUCTION

For consideration of gravity in higher dimensions, what should be the right gravitational equation? A basic requirement is that it should be of second order to ward off undesirable features like ghosts. Then, there are only two possibilities: Einstein’s equation itself, and its Lovelock generalization for which action is a homogeneous polynomial of degree $N$ of Riemann curvature [1]. It includes Einstein’s term for $N = 1$, the Gauss-Bonnet term for $N = 2$, and so on, each order coming with a dimensionful coupling constant. Is there a gravitational property that has so far remained unexplored and which could discern between Einstein and Lovelock theories in higher dimensions?

It may be mentioned that pure Lovelock gravity has several other universal features like existence of bound orbits around a static source in all even $d = 2N + 2$ dimensions (in contrast to Einstein’s gravity where it happens only for $d = 4$) [3], and thermodynamical parameters like temperature and entropy bearing the same universal relation to horizon radius in all $d = 2N + 1, 2N + 2$ dimensions [6].

However, in $d = 2N$, Lovelock Lagrangian is topological, and to make it non-trivial it should be coupled to a scalar field. In particular, for the 4-dimensional universe we live in, a gravitational theory usually contains the linear Einstein-Hilbert term $R$ and a scalar field coupled to the quadratic Gauss-Bonnet term

$$R_{GB}^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \quad (1)$$

Motivated by the fact that pure Lovelock equation seems to be the right gravitational equation in higher dimensions [7], here we consider a theory of a scalar field coupled to the Gauss-Bonnet term without the Einstein term. Such a theory could be investigated as the valid action for a gravitational theory even in four dimensions or considered to be a particular limit of a more general theory following from a higher-dimensional fundamental theory. Here, we limit our discussion to the cosmological consequences of such a theory and, in particular, we explore whether inflationary scenarios with more attractive characteristics – compared to the existing models – emerge.
II. THE SCALAR-GAUSS-BONNET THEORY

We consider the following gravitational theory of a scalar field $\phi$

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{8} f(\phi) R_{GB}^2 - \frac{1}{2} (\nabla \phi)^2 \right] + \mathcal{S}_m \tag{2}$$
coupled non-minimally to gravity via the Gauss-Bonnet Lagrangian [1] by a general coupling function $f(\phi)$. The last term $\mathcal{S}_m$ denotes the action functional of any additional distribution of matter in the universe. The variation of the action $\mathcal{S}_m$ with respect to the metric tensor and scalar field leads to the field equations

$$P_{\mu\nu\beta} \nabla^\alpha f - \phi_{,\mu} \phi_{,\nu} + g_{\mu\nu} \frac{(\nabla \phi)^2}{2} = T_{\mu\nu}, \tag{3}$$

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} \partial^{\mu} \phi \right] + \frac{1}{8} f' R_{GB}^2 = 0, \tag{4}$$

where $f' \equiv df/d\phi$, $P_{\mu\nu\beta}$ is defined as

$$P_{\mu\nu\beta} = R_{\mu\nu\beta} + 2 g_{\mu[\beta} R_{\nu]\alpha] + 2 g_{\alpha[\mu} R_{\beta]n} + R g_{\mu[\nu} g_{\beta]a}, \tag{5}$$

and $T_{\mu\nu}$ is the energy momentum tensor associated with $\mathcal{S}_m$.

We assume that the metric has the Friedmann-Robertson-Walker form

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \tag{6}$$

with $k = 0, \pm 1$ denoting the curvature of the 3-dimensional space. Assuming also that the matter content of the universe has the form of a perfect fluid with energy density $\rho$ and isotropic pressure $P$, the field equations take the explicit form

$$3 \dot{H} (H^2 + \frac{k}{a^2}) - \frac{\dot{\phi}^2}{2} = \rho, \tag{7}$$

$$-2 H \dot{f}(H^2 + \dot{H}) - \frac{\dot{\phi}^2}{2} - \ddot{\phi} (H^2 + \frac{k}{a^2}) = P, \tag{8}$$

$$\ddot{\phi} + 3 \dot{H} \dot{\phi} - 3 f' (H^2 + \frac{k}{a^2}) (H^2 + \dot{H}) = 0, \tag{9}$$

where $\dot{H} \equiv \dot{a}/a$ is the Hubble parameter and the dot denotes derivative with respect to time.

III. INFLATIONARY EXPANSION

We will now search for inflationary solutions supported by the system of Eqs. (7)-(9). To this end, we will assume that any other distribution of matter is negligible, and thus set $\rho = P = 0$. In this case, Eq. (7) can be straightforwardly solved for $\phi$ to give

$$\dot{\phi} = 6 f' (H^2 + \frac{k}{a^2}), \tag{10}$$

for a general coupling function $f(\phi)$. Then, combining Eqs. (7)-(9) and using Eq. (10), we end up with the constraint

$$(5 H^2 + \frac{k}{a^2}) (H^2 + \dot{H}) + 12 H^2 f'' (H^2 + \frac{k}{a^2})^2 = 0. \tag{11}$$

In the above, we have ignored a multiplicative factor proportional to $(k + \dot{a}^2)^2$: as $R_{GB}^2 = 24 \ddot{a} (k + \dot{a}^2)/a^3$, both $k + \dot{a}^2$ and $f'$ are not allowed to be zero in order to keep the GB term in the theory.

A variety of cosmological solutions may be sought for by assuming different forms of the coupling function $f(\phi)$ - for an extensive analysis, see [3]. However, it is only for a quadratic coupling function $f(\phi) = \lambda \phi^2$, where $\lambda$ is a constant, that inflationary expansion can be supported by the field equations. It is also for this choice that the constraint (11) becomes a differential equation for the scale factor $a(t)$. However, for $k \neq 0$, Eq. (11), upon integrated once with respect to time, leads to a transcendental equation that cannot be solved for $\dot{a}$ [3] - it is thus only for $k = 0$ that an explicit solution may be derived for the scale factor of the universe.

Therefore, specializing to the case of a quadratic coupling function and for a flat universe, Eq. (11) reduces to

$$H + H^2 + \frac{24 \lambda}{5} H^4 = 0. \tag{12}$$

The above implies that a de Sitter solution, with $H$ a constant and thus $\dot{H} = 0$, indeed exists but only when $\lambda < 0$. The corresponding accelerating solution with $H_{de S}^2 = -5/24 \lambda$ is a repeller, while the second solution $\dot{H} = 0$, describing a Minkowski spacetime, is an attractor. Therefore, in order to have a natural exit from the inflationary phase we have to consider an initial condition $(H_i)$ such that $H_i < H_{de S}$. Integrating Eq. (12) once, we obtain

$$H = \frac{H_{de S}}{\sqrt{1 - \frac{C_1}{4 \lambda} a^2}}, \tag{13}$$

where $C_1$ is an arbitrary integration constant. Depending on the values of the two parameters $\lambda$ and $C_1$, the above equation results to different types of cosmological solutions. The existence of de Sitter implies $\lambda < 0$ and the natural exit implies $C_1 \geq 0 (H < H_{de S})$. 


Here, we will present only these two cases that support a non-eternal inflationary phase of expansion for the universe. For a more comprehensive analysis, please refer to [8].

A. **The case with** $C_1 = 0$ **and** $\lambda < 0$

If $C_1 = 0$, then integrating once more Eq. (13), we find

$$a(t) = a_0 \exp \left( \sqrt{\frac{5}{24|\lambda|}} t \right). \tag{14}$$

The above solution may be considered as an alternative to the usual inflationary solutions following from the Einstein-Hilbert action which require an appropriate potential for the scalar field while here the GB term itself provides a potential for the scalar field.

The solution for the scalar field can be found via Eq. (10) that, for $k = 0$, takes the form

$$\frac{\dot{\phi}}{\phi} = 12\lambda \frac{\dot{a}^3}{a^3}, \tag{15}$$

and leads to the solution

$$\phi(t) = \phi_0 \exp \left( \frac{5}{4} \sqrt{\frac{5}{6|\lambda|}} t \right). \tag{16}$$

The above describes a regular, exponentially decaying scalar field. One may easily verify that the combination of solutions (14) and (16) satisfies the complete set of field equations.

The profiles of both the scale factor and the scalar field are shown in Figs. 1 and 2, respectively, for the values $|\lambda| = 0.1, 0.2, 0.5, 1, 2, 10$. We observe that the smaller the value of the coupling constant $\lambda$ the faster both quantities evolve with time. Note that the scalar field starts from an arbitrary value $\phi_0$ – normalized to unity in Fig. 2 – and quickly reduces to zero. As a result, the coupling function $f(\phi) = \lambda \phi^2$ remains always bounded. The effective potential of the scalar field receives contributions from both the GB term and the coupling function and has the explicit form

$$V_{eff} = -\frac{1}{8} f(\phi) R_{GB}^2 = \frac{25}{24} \frac{\phi^2}{8|\lambda|}. \tag{17}$$

Therefore, $V_{eff}$ also remains bounded as the field evolves; at the same time, it may become arbitrarily large at early times by appropriately choosing the value of the coupling constant $\lambda$. This is in sharp contrast to the behavior seen in more traditional inflationary models, such as chaotic [9], where the effective potential blows up for the super-Planckian initial values of the field unless a fine-tuning is imposed on the parameters of the model – as we will see in Sec. IV, such an unnatural initial value is not needed in our case for the necessary number of e-foldings to be obtained. It is also a significant improvement compared to models for inflation [10] where the scalar potential remains bounded but its value is too small to justify its dominance over any other distribution of matter in the universe and thus allow inflation to be realised.

B. **The case with** $C_1 > 0$ **and** $\lambda < 0$

We will now demonstrate that the pure de Sitter solution of the previous case is the early-time limit of a more interesting class of cosmological solutions that follow when $\lambda < 0$ and $C_1$ positive. In this case,
obtain a linear dependence of the scale factor on time, while, at later times, it reduces to a constant. Its effective potential is also decreasing, exponentially at early times, as seen in Fig. 3. In the limit \( \epsilon \to 0 \), the acceleration is eternal.

We may finally define the slow-roll parameters \( \epsilon_1 \equiv -\dot{H}/H^2 \) and \( \epsilon_{i+1} \equiv -\epsilon_i/H\epsilon_i \) as follows

\[
\epsilon_{2n+1} = \epsilon_1 = 1 - H^2/H_{dS}^2, \quad \epsilon_{2n} = \epsilon_2 = 2H^2/H_{dS}^2. \tag{20}
\]

During inflation, the odd slow-roll parameters approach zero \( \epsilon_{2n-1} \to 0 \) while the even parameters are of order unity \( \epsilon_{2n} \approx 2 \). The slow-roll approximation is thus violated. A very similar behavior exists in the so called ultra-slow roll inflation \( [11] \) (also known as fast-roll inflation \( [12] \)).

At later times, the slow-roll parameters move away from the above values to mark the end of inflation. In Fig. 4 we depict the evolution of \( \epsilon_1 \), for different initial conditions: inflation is realized at early times, when \( \epsilon_1 \approx 0 \), and a natural exit takes place at later times as \( \epsilon_1 \) moves to values closer to unity.

**IV. E-FOLDINGS AND SLOW-ROLL**

The early de Sitter phase should have a long-enough duration in order to resolve the problems of the standard cosmology. Using the form \( [13] \), we demand the initial condition to satisfy \( H_i \approx H_{dS}(1 - e^{-2N}) \) which implies \( C_1 \approx |\lambda|e^{-2N} \), where \( N \) is the number of e-folds \(( N > 60 )\). According to this, the smaller the value of \( C_1 \), the longer the acceleration: this was expected since for \( C_1 = 0 \), the acceleration is eternal. Therefore the number of e-foldings during inflation depends on the initial condition \( H_i \).

We may finally define the slow-roll parameters \( \epsilon_1 \equiv -\dot{H}/H^2 \) and \( \epsilon_{i+1} \equiv -\epsilon_i/H\epsilon_i \) as follows

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**FIG. 3:** Cosmological solutions with \( C_1 > 0 \), \( \lambda < 0 \), and for \( \nu = 0.1, 1, 10, 100 \).

**FIG. 4:** Evolution of the slow-roll parameter \( \epsilon_1 \) as function of \( N = \log(a) \) for different initial conditions \( C_1 = |\lambda|e^{-2N} \) with \( N = (40, 60, 80) \).
V. CONCLUSIONS

We have considered a pure scalar-Gauss-Bonnet gravitational theory without the Ricci scalar. We have demonstrated that such a theory with a quadratic coupling function between the scalar field and the Gauss-Bonnet term naturally supports inflationary expanding – de Sitter – cosmological solutions. The theory contains also solutions where these de Sitter phases possess a natural exit mechanism and are replaced by linearly expanding – Milne – phases. Two important questions that still need to be answered are whether these solutions survive in the context of a more realistic cosmological set-up where the presence of matter is taken into account and whether this type of inflation gives a perturbation spectrum in accordance to observations – we plan to report on both of these questions in a forthcoming work [8].

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