Rainbow connectivity of randomly perturbed graphs

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Abstract

In this note we examine the following random graph model: for an arbitrary graph \(H\), with quadratic many edges, construct a graph \(G\) by randomly adding \(m\) edges to \(H\) and randomly coloring the edges of \(G\) with \(r\) colors. We show that for \(m\) a large enough constant and \(r \geq 5\), every pair of vertices in \(G\) are joined by a rainbow path, i.e., \(G\) is rainbow connected, with high probability. This confirms a conjecture of Anastos and Frieze \([J. Graph Theory 92 (2019)]\) who proved the statement for \(r \geq 7\) and resolved the case when \(r \leq 4\) and \(m\) is a function of \(n\).

1 Introduction

The randomly perturbed graph model is the following: for a fixed positive constant \(\delta > 0\), let \(G(n, \delta)\) be the set of graphs on vertex set \([n]\) with minimum degree at least \(\delta n\). A graph \(H = (V, E)\) is chosen arbitrarily from \(G(n, \delta)\) and a set \(R\) of \(m\) edges are chosen uniformly at random from \(\binom{[n]}{2}\) \(\setminus E\) (i.e., those edges not in \(H\)) and added to \(H\). The resulting perturbed graph is

\[ G_{H,m} = (V, E \cup R). \]

This model was introduced by Bohman, Frieze, and Martin \([5]\). As taking \(\delta = 0\) gives the standard random graph, this model can be viewed as a generalization of the classic Erdős-Rényi random graph model. Given a class of graphs \(G\) and a monotone increasing property \(P\), a natural line of questioning is: for a graph \(H \in G\) can we perturb the graph, specifically by adding edges randomly so that the resulting graph satisfies \(P\) with high probability? For example, in \([5]\) it is shown that for an \(n\)-vertex dense graph \(H\), the perturbed graph \(G_{H,m}\) is Hamiltonian with high probability (w.h.p.) if \(m\) is at least linear in \(n\) and that there are dense non-Hamiltonian graphs \(H\) such that it is necessary for \(m\) to be linear in \(n\) for \(G_{H,m}\) to be Hamiltonian w.h.p.

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Various other properties of randomly perturbed graphs have been investigated including clique number, chromatic number, diameter and connectivity [4], spanning trees [6, 7, 14, 16, 17], Ramsey properties [10, 11], tilings [3], and containing some fixed subgraph [18].

Anastos and Frieze [2] examined random edge-colorings of perturbed graphs. Let $G_{rH,m}$ be the graph $G_H,m$ equipped with an $r$-edge-coloring where the colors of the edges are independently and uniformly selected from $[r]$ (in this way the coloring need not be proper). In [2] it was shown that a linear number of edges $m$ and number of colors $r \geq (120 - 20 \log \delta)n$ suffice for $G_{rH,m}$ to have a rainbow-Hamilton cycle w.h.p. Later Aigner-Horev and Hefetz [1] improved the value of $r$ to the asymptotically best-possible $(1 + o(1))n$. Very recently Katsamaktsis and Letzter [15] showed that the optimal number of colors $n$ suffices.

An edge-colored graph is rainbow connected if there is a rainbow path (i.e., each edge has a distinct color) between every pair of vertices. Rainbow connectivity has been studied in graphs in general as well as in the standard Erdős-Rényi random graph model. Typically, the question is to find the minimum $r$ such that a graph $G$ has an $r$-edge-coloring such that $G$ is rainbow connected. This parameter was introduced by Chartrand, Johns, McKeon and Zhang [9]. For a survey of rainbow connectivity in graphs, see Li and Sun [19].

In random graphs the goal is to determine a threshold on the edge probability $p$ such that w.h.p. $G(n, p)$ has rainbow connection number $r$. Caro, Lev, Roditty, Tuza and Yuster [8] showed that $p = \sqrt{\frac{\log n}{n}}$ is a sharp threshold for rainbow connection number $r = 2$ and He and Liang [12] proved that $(\frac{\log n}{n^{1/r}})^r$ is a sharp threshold for $r \geq 3$. A further refinement for $r \geq 3$ was given by Heckel and Riordan [13].

In a perturbed graph $G_{rH,m}$, Anastos and Frieze [2] proved that if $r \geq 7$ and $m = \omega(1)$, then with high probability $G_{rH,m}$ is rainbow connected. Moreover, they showed the following.

**Theorem 1** (Anastos, Frieze [2]).

(i) For every $\delta > 0$, if $m \geq 60\delta^{-2}\log n$, then w.h.p. $G_{3H,m}$ is rainbow connected.

(ii) For every $0.1 \geq \delta > 0$, there exists $H \in \mathcal{G}(n, \delta)$ such that if $m \leq 0.5\log n$, then w.h.p. $G_{4H,m}$ is not rainbow connected.

(iii) If $r \geq 7$ and $m = \omega(1)$, then w.h.p. $G_{rH,m}$ is rainbow connected.

It is not immediately clear that rainbow connectivity is monotone in $r$ for constant $m$. For $r = 3$ colors, (i) shows that if $m$ has order of magnitude $\log n$, then $G_{3H,m}$ is rainbow connected w.h.p., which shows that (ii) is best possible. Anastos and Frieze [2] put forth that likely part (iii) holds for 5 or 6. In this note we prove the result for $r = 5$ and $r = 6$. Note that part (ii) implies that this cannot be improved to $r \geq 4$.

**Theorem 2.** For every $\delta > 0$, if $r \geq 5$ and $m \geq 4\delta^{-2}\log 2500\delta^{-2}$, then w.h.p. $G_{rH,m}$ is rainbow connected.

We make no particular attempt to optimize the constant $m$ in Theorem 2; however proving sharp bounds remains an interesting problem. A natural modification of this model would be to introduce some deterministic part to the edge-coloring which may lead to different values of $m$ and $r$. 

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2 Proof of Theorem 2

Fix \( r \geq 5 \) and \( 0 < \delta < 1 \) and integers \( m \geq 4\delta^{-2} \log 2500\delta^{-2} \) and \( k \) such that \( 6 \leq \delta k \leq 7 \). Let \( H \) be an arbitrary \( n \)-vertex graph with minimum degree at least \( \delta n \). Randomly add \( m \) edges from \( \binom{[n]}{2} \) to \( H \) (ignoring duplicate edges) and randomly 5-color the edges of the resulting graph \( G_{H,m}^r \). Note that this is a slightly weaker assumption than adding \( m \) edges from \( \binom{[n]}{2} \setminus E(H) \) to \( H \), which will simplify arguments later. At this point, we have two sources of randomness: the \( m \) edges added to \( H \) and the edge-coloring of \( G_{H,m}^r \).

We begin by building \( t = 100 \log n \) vertex sets of size \( k \) in \( H \) such that every pair of vertices \( u, v \) are in the neighborhoods of a large proportion of these sets.

**Claim 3.** There exists vertex sets \( S_1, S_2, \ldots, S_t \), each of \( k \) vertices in \( V(H) \) such that for every pair of vertices \( u, v \in V(H) \) there is an index set \( I_{u,v} \subset [t] \) such that \( \left| I_{u,v} \right| \geq 0.6t \) and for all \( i \in I_{u,v} \) we have \( u, v \in N(S_i) \).

**Proof.** Select uniformly at random with replacement \( t \) sets \( S_1, S_2, \ldots, S_t \) each of \( k \) vertices from \( V(H) \).

For distinct vertices \( u, v \) the probability \( \mathbb{P}(u \notin N(S_i) \text{ or } v \notin N(S_i)) \) is at most

\[
\mathbb{P}(u \notin N(S_i)) + \mathbb{P}(v \notin N(S_i)) \leq 2\mathbb{P}(u \notin N(S_i)) = 2\mathbb{P}(S_i \cap N(u) = \emptyset)
\]

\[
\leq 2\binom{(1-\delta)n}{k} \leq 2(1-\delta)^k < 2\exp(-\delta k) < 0.01,
\]

as \( k \geq 6\delta^{-1} \). Thus, \( \mathbb{P}(u, v \in N(S_i)) \geq 0.99 \).

Let \( X_i \) be the indicator random variable for the event that \( u, v \in N(S_i) \). Now \( X = X_1 + X_2 + \cdots + X_t \) is the number of sets \( S_i \) such that \( u, v \in N(S_i) \). Therefore, \( \mathbb{E}[X] \geq 0.99t \).

As each set \( S_i \) is selected uniformly at random with replacement, the random variables \( X_i \) are independent. Therefore, a Chernoff bound (see [20], pg. 66) gives

\[
\mathbb{P}(X \leq (1 - \alpha)\mathbb{E}[X]) \leq \exp \left( -\frac{1}{2} \alpha^2 \mathbb{E}[X] \right)
\]

for \( 0 < \alpha < 1 \). With \( \alpha = 0.39 \) this gives

\[
\mathbb{P}(X \leq 0.6t) \leq \mathbb{P}(X \leq 0.61 \cdot 0.99t) \leq \exp \left( -\frac{1}{2} (0.39)^2 \cdot 0.99 \cdot 100 \log n \right) < n^{-7} < n^{-2}.
\]

Therefore, by the union bound, the probability that some \( u, v \) is not contained in \( 0.6t \) sets \( S_i \) is less than \( 1 \). Thus, with positive probability the desired vertex sets exist which completes the proof. \( \square \)

Put \( S = \bigcup S_i \). A set \( S_i \) is *good* if for every pair \( a, b \in S_i \) there exists an edge of \( G_{H,m}^r \) between \( N(a) \setminus S \) and \( N(b) \setminus S \). Note that these two sets are not necessarily disjoint, so such an edge may be contained in their intersection.

**Claim 4.** \( \mathbb{P}(S_i \text{ is good}) > 0.99 \) for \( n \) large enough.
Proof. Here we only use the randomness of the \( m \) edges added to \( H \). Fix \( a, b \in S_i \). As \( |S| = |\cup S_i| \leq kt = 100k \log n \), we can choose \( n \) large enough such that \( N(a) \setminus S \) and \( N(b) \setminus S \) both have size at least \( \frac{1}{2} \delta n \). Note that \( N(a) \setminus S \) and \( N(b) \setminus S \) are not necessarily disjoint, so the probability that there is no edge between \( N(a) \setminus S \) and \( N(b) \setminus S \) is at most

\[
\left( \frac{\binom{n}{2} - \left( \frac{1}{2} \delta n \right)^2}{\binom{n}{2}} \right)^m \leq \left( 1 - \frac{1}{4} \delta^2 \right)^m < \exp\left( -\frac{1}{4} \delta^2 m \right).
\]

The set \( S_i \) has \( k \) vertices, so by the union bound, the probability that \( S_i \) is not good is less than

\[
\left( \frac{k}{2} \right)^k \exp \left( -\frac{1}{4} \delta^2 m \right) \leq \frac{1}{2} \left( \frac{7}{\delta} \right)^2 \exp \left( -\frac{1}{4} \delta^2 m \right) < 0.01,
\]

as \( m \geq 4 \delta^{-2} \log 2500 \delta^{-2} \).

Fix arbitrary vertices \( u \) and \( v \) in \( G_{H,m} \). We estimate the probability that there is a rainbow \( u-v \) path (i.e. a path with end-vertices \( u \) and \( v \)) of length at most 5. Here we only use the randomness of the edge-coloring of \( G_{H,m} \). If \( uv \) is an edge, then we immediately have a rainbow \( u-v \) path, so assume that \( uv \) is not an edge.

Let \( I_{u,v} \) be the index set guaranteed by Claim 3. For each \( i \in I_{u,v} \), let us estimate the probability that there is a rainbow \( u-v \) path of length at most 5 using vertices in \( S_i \). Let \( a \in S_i \) be a neighbor of \( u \) and let \( b \in S_i \) be a neighbor of \( v \). If \( a = b \), then we have a \( u-v \) path of length 2 which is rainbow with probability \( \frac{r-1}{r} \cdot \frac{r-2}{r} \cdot \frac{r-3}{r} \cdot \frac{r-4}{r} \geq \frac{4}{5} \). Thus, the probability that there is a rainbow \( u-v \) path using vertices \( u, v, a, b \) is at least \( 0.99 \cdot \frac{4}{5} \). (Here we needed \( r \geq 5 \) colors as with fewer colors this path cannot be rainbow.)
Therefore, the probability that there is no rainbow $u-v$ path of length at most 5 is at most

$$\left(1 - 0.99 \cdot \frac{4!}{5^4}\right)^{|I_{uv}|} \leq \left(1 - 0.99 \cdot \frac{4!}{5^4}\right)^{0.6t} \leq \exp\left(-0.99 \cdot \frac{4!}{5^4} \cdot 0.6t\right)$$

$$= \exp\left(-0.99 \cdot \frac{4!}{5^4} \cdot 0.6 \cdot 100 \log n\right) < n^{-2.25} = o(n^{-2}).$$

Now, the union bound implies that the probability that there is a pair $u, v$ not connected by a rainbow path of length at most 5 is $o(1)$, i.e., w.h.p. $G_{H,m}^r$ is rainbow connected. \qed

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