Analytical Calculation for slope stability based on the Combination of Horizontal Slice Method and Oblique Slice Method

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Abstract. An analytical method for analyzing the stability of the homogeneous slope based on horizontal slice method is presented. A combination model of horizontal slice and oblique slice are used for analysis the inter-slice forces. The analytical expression of factor of safety is obtained with the assumption that the shear strength of soil meets the Mhor-Coublum strength criterion. The position of critical slip surface and its corresponding factor of safety computed by the development of the optimization technique are in good agreement with Bishop’s method. The advantages of presented method are straightforward in analyzing the stability of slope.

1. Introduction

The equilibrium methods for analysis of slope stability are mostly vertical slice methods [1-3]. For homogeneous rock and soil, the variation of soil mass is less affected by the internal variation of the vertical slice. However, when the slope is layered soil, it will bring a great inconvenience to the calculation, and horizontal slice method is more applicable. The idea of horizontal slice method is proposed for stability analysis of reinforced slope by Lo and Xu [4]. Shangholi et al.[5] applied the horizontal slice method in analyzing the reinforced slopes seismic stability based on pseudo-static method. Nouri et al.[6-7] extended the latter work of reinforced slope stability by developing a number of formulations and studying the effect of earthquake. Chandaluri et al.[8] used the horizontal slice method suggested by Shangholi et al.[5] to Study the influence of various slope parameters on slope stability. These approaches for analysis of reinforced soil slope are based on the assumption that the failure surface pass through the toe of slope. Therefore, it is necessary to carry out stability analysis of slope with general sliding surface based on horizontal slice method. Furthermore, when the slope sliding surface pass below the slope toe, the use of horizontal slice method for analysis of slope stability will bring a problem that two sliding bottom surfaces will appear, which will bring a trouble to analysis of the inter-slice forces. Chen et al.[9] proposed a method for analysis of soil nailing that the failure wedge of soil is divided into a number of oblique slices. Ge et al.[10] analyzed the slope stability using the oblique slice method. The presented paper proposed analytical method for analysis of slope stability based on the combination of horizontal slice method and oblique slice method. And related optimization technique is developed for obtaining the slip sliding surface and its corresponding factor of safety.

2. Basic assumptions

In the study, the following assumptions are made:
(1) The vertical stress is equal to the overburden pressure for both horizontal soil slice and oblique soil slice.

(2) The slip sliding surface is circular, and the factor of safety for all slices along the critical slip surface is equal.

(3) The shear strength of soil meets the Mhor-Coublum strength criterion, that is

$$\tau_j = \sigma \tan \phi + c$$

Where $\tau_j$ and $\sigma$ is the shear stress and normal stress along the failure surface at failure, respectively.

Figure 1 shows a critical slip surface with centre $O$ and radius $r_0$ for a homogeneous slope with the height $H$, inclination angle $\theta$, unit weight $\gamma$, internal friction angle $\phi$ and cohesion $c$. $x_0, y_0$ are the centre of the abscissa and ordinate coordinates of the circular sliding surface, respectively. $D, F$ are the intersections of the vertical line at $C, A$ and the failure surface, respectively. $\mu$ is the abscissa of point $G, \alpha, \alpha_2, \eta, \eta, \beta$ are the vertical centre angles of the points $G, F, E, D, B, B$, respectively, with respect to the centre $O$. $\phi$ is the central angle of any point on the circular sliding surface, and the vertical centre angle is positive counter clockwise.

Horizontal slice method and oblique slice method are combined to analysis of slope stability. The soil above the toe of slope is divided in horizontal slices, and the soil below the toe of slope is divided into oblique slices. Forces diagram of the horizontal slice are shown in Figure 2a. According to the assumption used by Shangholi et al.[5]. The inter-slice forces have the following relationship:

$$\begin{align*}
V_j - V_{j-1} &= G_j - G_{j-1} \\
H_j - H_{j-1} &= 0
\end{align*}$$

Where $V_j$ and $V_{j-1}$ and $H_j$, $H_{j-1}$ are the vertical inter-slice and the horizontal inter-slice forces on the horizontal slice, respectively. $G_j, G_{j-1}$ are the gravity of block def and abe, respectively.

Forces diagram of the oblique slice are shown in Figure 2b. In order to obtain the relationship of inter-slice forces, the block $GhqA$ and $sghq$ are analyzed shown in Figure 3a and 3b. $X_k, X_{k-1}$ and $E_k, E_{k-1}$ are the horizontal inter-slice forces and the vertical inter-slice forces on the vertical slice, respectively. According to the equilibrium conditions, the inter-slice forces have the following relationship:

$$\begin{align*}
-V_k &= (G_k + E_k) \cos \delta_k + X_k \sin \delta_k \\
-H_k &= -(G_k + E_k) \sin \delta_k + X_k \cos \delta_k
\end{align*}$$

After the conversion of equation (3), $X_k, E_k$ can be represented by:
\[ \begin{align*}
X_k &= -V_i \sin \delta_i - H_i \cos \delta_i \\
E_k &= H_i \sin \delta_i - V_i \cos \delta_i - G_i
\end{align*} \tag{4} \]

For the traditional method, the relationship between the inter-slice forces has different assumptions. Bishop\cite{2} ignored the inter-slice forces, and Morgenstern NR\cite{1} considered the inter-slice forces in great detail. According to relationship between the inter-slice forces assumed by Bishop, and it has the following relationship:

\[ \begin{align*}
E_k - E_{k-1} &= 0 \\
X_k - X_{k-1} &= 0
\end{align*} \tag{5} \]

**Figure 3a** Forces acting on the overburden soil  
**Figure 3b** Forces acting on the vertical slice

Substituting equation (5) into equation (4):

\[ \begin{align*}
V_i \cos \delta_i - H_i \sin \delta_i - V_{k-1} \cos \delta_{k-1} + H_{k-1} \sin \delta_{k-1} &= G_k - G_{k-1} \\
V_i \sin \delta_i + H_i \cos \delta_i - V_{k-1} \sin \delta_{k-1} - H_{k-1} \cos \delta_{k-1} &= 0
\end{align*} \tag{6} \]

Where \( V_i \) and \( V_{k-1} \) and \( H_k \), \( H_{k-1} \) are the vertical inter-slice and the horizontal inter-slice forces on the oblique slice, respectively. \( G_k \), \( G_{k-1} \) are the gravity of block \( GhqA \) and \( GgsA \), respectively. \( \delta_i \) is the angle between the upper side of oblique slice and the horizontal direction. \( \delta_{k-1} \) is the angle between the lateral side of oblique slice and the horizontal direction, and clockwise is positive. The following conclusions that the resultant force of the vertical forces is equal to the overburden pressure, and the resultant force of the horizontal force is equal to zero obtained are the same as the assumption presented by Shangholi et al\cite{5}.

**3. Analytical method for safety factor**

Horizontal and vertical equilibrium are considered for individual slice, no account being taken of moment equilibrium. The following equilibrium equations can be obtained from Figure 2a and 2b:

\[ \begin{align*}
N_j &= (W_j + V_j - V_{j-1}) \cos \phi \\
T_j &= (W_j + V_j - V_{j-1}) \sin \phi \\
N_k &= H_{k-1} \sin (\delta_{k-1} + \phi) - V_{k-1} \cos (\delta_{k-1} + \phi) - H_k \sin (\delta_k + \phi) + V_i \cos (\delta_k + \phi) + W_k \cos \phi \\
T_k &= -H_{k-1} \cos (\delta_{k-1} + \phi) - V_{k-1} \sin (\delta_{k-1} + \phi) + H_k \cos (\delta_k + \phi) + V_i \sin (\delta_k + \phi) + W_k \sin \phi
\end{align*} \tag{7,8} \]

Where \( W_j \) and \( W_k \) are the gravity of horizontal and oblique slice, respectively. \( N_j \), \( T_j \) are the tangential and normal forces on the slip surface of horizontal slice. \( N_k \), \( N_{k-1} \) are the tangential and normal forces on the slip surface of oblique slice. The expressions of \( x, y \) can be expressed as:

\[ \begin{align*}
x &= x_0 + r_n \sin \phi \\
y &= y_0 - r_n \cos \phi
\end{align*} \tag{9} \]

And the expressions of \( x_0 \), \( y_0 \) and \( r_n \) can be represented by:
\[
\begin{align*}
\begin{cases}
x_0 &= -\frac{H \sin \alpha}{\cos \alpha - \cos \beta} \\
y_0 &= -\frac{H \cos \alpha}{\cos \alpha - \cos \beta} \\
r_0 &= \frac{H}{\cos \alpha - \cos \beta}
\end{cases}
\tag{10}
\end{align*}
\]

Furthermore, equations (2) and (6) are introduced into the equations (7) and (8), respectively. The following relationship can be obtained, that is
\[
\begin{align*}
\begin{cases}
N_i &= (W_i + G_i - G_{i-1}) \cos \phi \\
T_i &= (W_i + G_i - G_{i-1}) \sin \phi
\end{cases}
\tag{i = j, k}
\end{align*}
\]

The slope soils meet the Mohr-Coublum strength criterion, and the safety factor can be represented by:
\[
F_s = \frac{\sum (c l_i + N_i \tan \varphi)}{\sum T_i} = \frac{\sum c l_i + \tan \varphi \sum (W_i + G_i - G_{i-1}) \cos \phi}{\sum (W_i + G_i - G_{i-1}) \sin \phi}
\tag{12}
\]

Where, \(l_i \) is the length of the bottom surface of the slice, \(n \) is the number of slices. In order to analyze the slope stability more accurately, the slope is presented in the terms of integral rather than the sums of terms for all slices. The following expressions can be derived from Figure.1:
\[
\begin{align*}
G_i - G_{i-1} &= \begin{cases}
\gamma (H - y) dx - \gamma (x - y \cot \theta) dy (\eta \leq \phi \leq \beta) \\
\gamma (x \tan \theta - \gamma) (dx - \cot \theta dy) (\eta \leq \phi < \eta) \\
\gamma x \tan \theta dx - 0.5 \gamma (x - \mu) dy - 0.5 \gamma y dx (\alpha_2 \leq \phi < \eta_1) \\
-0.5 \gamma (x - \mu) dy - 0.5 \gamma y dx (\alpha_1 \leq \phi < \alpha_2) \\
W_i &= \begin{cases}
\gamma (x - y \cot \theta) dy (\eta \leq \phi \leq \beta) \\
-0.5 \gamma [-y dx + (x - \mu) dy] (\alpha_1 \leq \phi < \eta_1)
\end{cases}
\end{cases}
\tag{13}
\end{align*}
\]

After the conversion and simplification of equations (9), (10), (12),(13)and(14), the analytical expression of \(F_s \) is derived, that is
\[
F_s = \frac{c (\beta - \alpha_1) (\cos \alpha_1 - \cos \beta) + \tan \varphi \cdot I_c}{I_c}
\tag{15}
\]

Where :
\[
\begin{align*}
I_c &= (\cos \alpha_1 - \cos \beta) [\frac{1}{2} (\beta - \eta) + \frac{1}{4} (\sin 2 \beta - \sin 2 \eta)] - \tan \theta \sin \alpha_1 \frac{1}{2} (\eta - \alpha_2) + \frac{1}{4} (\sin 2 \eta - \sin 2 \alpha_2) \\
&- \cos \alpha_1 [\frac{1}{2} (\beta - \alpha_1) + \frac{1}{4} (\sin 2 \beta - \sin 2 \alpha_1)] + \frac{1}{3} \tan \theta (\cos^3 \alpha_2 - \cos^3 \eta) + \frac{1}{3} (\sin^3 \alpha_1 - \sin^3 \beta) + \sin \beta \\
&- \sin \alpha_1
\end{align*}
\tag{16}
\]

\[
\begin{align*}
I_c &= \frac{1}{2} (\cos \alpha_1 - \cos \beta) (\sin^2 \beta - \sin^2 \eta) - \frac{1}{2} \tan \theta \sin \alpha_1 (\sin^2 \eta - \sin^2 \alpha_2) - \frac{1}{2} \cos \alpha_1 (\sin^2 \beta - \sin^2 \alpha_1) \\
&+ \frac{1}{3} \tan \theta (\sin^3 \eta - \sin^3 \alpha_2) + \frac{1}{3} (\cos^3 \alpha_1 - \cos^3 \beta)
\end{align*}
\tag{17}
\]

It is a commonly used method to determine the critical slip surface with the minimum factor of safety for analysis of slope stability. In this paper, the safety factor is a function of four variables. MATLAB software is used to optimize the function (15), which is a nonlinear constrained optimization problem, to obtain the minimum value of \(F_s \). The minimum factor of safety can be
quickly obtained with the related optimization procedures (presented in Appendix), and the constraint conditions are:

\[
\begin{align*}
&\cos \alpha_1 - \cos \alpha_2 \leq 0 \\
&(\cos \alpha_1 - \cos \beta) \cot \theta - \sin \eta + \sin \alpha_2 = 0
\end{align*}
\]

(18)

4. Verification

The cases are analyzed in order to verify the reasonableness of the proposed analytical method in this study for a homogeneous slope with \( \gamma = 18.5 \text{kN/m}^3 \). Slope height is 8m with the configurations of 1:1 slope and 2:1 slope. In order to analyze the influence of slope soil strength on slope stability more comprehensively, cohesions ranging from 5 to 30kPa and angles of internal friction ranging from 10° to 20°. The rationality of proposed analytical method is verified with two methods of slopes stability analysis, Bishop’s method and the local minimum factor of safety method. The results are represented in Table 1.

| Slope (radio) | Cohesion \( c(kPa) \) | Friction angle \( \varphi \) (deg.) | Safety factor by proposed method | Safety factor by Bishop’s method | Safety factor by Local minimum factor of safety method(1993) [11] |
|--------------|-------------------|---------------------------------|---------------------------------|-------------------------------|-------------------------------------------------------------|
| 1:1          | 25                | 20                              | 1.68                            | 1.74                          | 1.87                                                        |
| 1:1          | 20                | 20                              | 1.46                            | 1.50                          | 1.68                                                        |
| 1:1          | 15                | 20                              | 1.24                            | 1.29                          | 1.46                                                        |
| 1:1          | 10                | 20                              | 1.00                            | 1.05                          | 1.00                                                        |
| 1:1          | 30                | 15                              | 1.73                            | 1.75                          | 1.85                                                        |
| 1:1          | 25                | 15                              | 1.52                            | 1.53                          | 1.65                                                        |
| 1:1          | 20                | 15                              | 1.30                            | 1.32                          | 1.45                                                        |
| 1:1          | 15                | 15                              | 1.09                            | 1.11                          | 1.24                                                        |
| 1:1          | 25                | 10                              | 1.35                            | 1.35                          | 1.42                                                        |
| 1:1          | 20                | 10                              | 1.15                            | 1.15                          | 1.23                                                        |
| 2:1          | 20                | 20                              | 1.96                            | 2.09                          | 2.05                                                        |
| 2:1          | 15                | 20                              | 1.71                            | 1.82                          | 1.85                                                        |
| 2:1          | 10                | 20                              | 1.44                            | 1.54                          | 1.60                                                        |
| 2:1          | 5                 | 20                              | 1.15                            | 1.21                          | 1.23                                                        |
| 2:1          | 25                | 15                              | 1.78                            | 2.05                          | 1.87                                                        |
| 2:1          | 20                | 15                              | 1.55                            | 1.78                          | 1.72                                                        |
| 2:1          | 15                | 15                              | 1.31                            | 1.53                          | 1.54                                                        |
| 2:1          | 10                | 15                              | 1.06                            | 1.29                          | 1.29                                                        |
| 2:1          | 15                | 10                              | 1.21                            | 1.27                          | 1.19                                                        |

It is found that the safety factors obtained by the proposed analytical method are in good agreement with those determined by the Bishop’s method and the local minimum factor of safety. From the above analysis, it is reasonable for assumption of the inter-slice forces on the oblique slices, and it is safer for analysis of slope stability. The presented method solves the problem that the failure surface goes down the toe of slope by using the combination of horizontal method and oblique method, and it is a development of the horizontal slice method for analysis of slope stability.
5. Examples
Two cases[12] are analyzed to verify the reasonableness of the proposed analytical method. The material properties are shown in Table 2. The position of the critical slip surface and its corresponding factors of safety calculated by proposed analytical method and Bishop’s method for a circle slip surface are shown in Figure 4a and 4b.

In addition, the position of critical slip surface and its corresponding factor of safety are in good agreement with each other for 1:1 slope, and they all pass through the toe of slopes as expected. Although the proposed analytical method defined the critical slip surface passing the toe of slope that is different from the critical slip surface passing below the toe of slope that determined by Bishop's method, the position of slip surface and its corresponding factor of safety can be considered to be substantially the same for 1:2 slope. Moreover, the safety factor calculated by the analytical method is relatively safer compared with the safety factor calculated by the Bishop’s method. It is noted that critical slip surfaces calculated by presented method are deeper than the computed by Bishop’s method.

| Table 2 Material properties |
|-----------------------------|
| examples | H / m | Slope(radio) | γ(kN/ m³) | c(kPa) | φ(deg.) |
|----------|-------|--------------|-----------|-------|--------|
| 1        | 20    | 1:1.5        | 18.82     | 41.65 | 15.0   |
| 2        | 5     | 1:2.0        | 17.64     | 9.80  | 10.0   |

Figure. 4a The critical slip surface in example 1  Figure. 4b The critical slip surface in example 2

6. Conclusions
The situation that the critical slip surface does pass below the toe of slope was considered in the study of slope stability. The analytical method is proposed based on the combination model of horizontal slice and oblique slice. The location of critical slip surface and its corresponding factor of safety are in good agreement for both proposed analytical method and Bishop’s method. The advantages of proposed analytical method are straightforward, ease to use, less time-consuming in analyzing the stability of slope. The proposed analytical method is for homogeneous slopes, this is a limitation.

7. References
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APPENDIX
Optimization procedure for equation (15):

For example: $\theta = 45^\circ$, $c/\gamma H = 0.20$, $\varphi = 20^\circ$

$x(1) = \alpha_1$, $x(2) = \alpha_2$, $x(3) = \eta$, $x(4) = \beta$

fun='abs((0.2*(x(4)-x(1))*(cos(x(1))-cos(x(4)))+tan(pi/9)*((cos(x(1))-cos(x(4)))*(0.5*(x(4)-x(3)))+(0.25*(sin(2*x(3))-sin(2*x(4)))*cos(x(1))]*(0.5*(x(4)-x(3))+0.25*(sin(2*(x(3))-sin(2*(x(4))))-tan(pi/4)*sin(x(2)))+(0.5*(x(3)-x(2)))+0.5*tan(pi/4)*((cos(x(2)))*3-\(cos(x(3)))^3)\}

% x0=[0 0.1 0.5 1.3];
% A=[1 -1 0 0; 0 1 -1 0; 0 0 1 -1];
% b=[0;0;pi/2];
% options=optimset('TolX',1e-12);
% [x,fval]=fmincon(fun,x0,A,b,[],[],[],[],comfun,options)

Comfun:

function [c,ceq]=comfun(x)

c=(tan(x(2))^2-(tan(x(1))^2);

ceq=(cos(x(1))-cos(x(4))))*sin(x(3))+sin(x(2));