Capillary Rise - Jurin’s Height vs Spherical Cap

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When liquid rises in a capillary, the interface has a shape close to a spherical cap when the Eotvos number is sufficiently small. This assumption is typically used when computing the stationary rise height of the liquid. Considering a continuum mechanical approach it can be shown that the exact shape of a spherical cap is not a solution to the rise problem yet a good approximation. Consequently, the assumption of a spherical cap used in Washburn-type rise models is justified only for small Eotvos numbers. In this limit, Jurin's height has the largest error.

1 Introduction

The rise of liquid in capillary due to surface tension and against gravitational forces is a prototypical case for wetting phenomena and a suitable benchmark for continuum mechanical simulations [1]. A standard description for this problem is a model family of ordinary differential equations originating from [2] and considered in [3]. This model family is based on the assumption that the interface has the form of a spherical cap [1, 4].

2 Model

For simplicity we consider the 2D case for a liquid between two planar plates with a distance of $2R$, while the approach is also applicable in the 3D case. In the stationary state, the liquid in a capillary is at rest, i.e. $v = 0$. Using the standard continuum mechanical model with outer pressure $p_0 = 0$ gives

$$\nabla p = -\rho g \kappa \eta \mathbf{y} \text{ in } \Omega, \quad -p = \sigma \kappa \eta \text{ on } \Sigma, \quad \mathbf{n}_\Sigma \cdot \mathbf{n}_{\partial \Omega} = -\cos \theta \text{ in } \Gamma, \quad (1)$$

where $\Omega$, $\Sigma$ and $\Gamma$ are the liquid domain, the interface, and the contact line in the stationary state, respectively. The outwards pointing normal of the interface and the domain are denoted by $\mathbf{n}_\Sigma$ and $\mathbf{n}_{\partial \Omega}$, respectively. We can deduce from (1) that $\partial_x p = 0$, concluding that the pressure $p_0$ at the inflow/outflow boundary of the liquid domain is constant. This allows to integrate the bulk equation in $y$-direction and combine the result with the transmission condition at the interface yielding

$$\rho g H(x) = \sigma \kappa(x) + p_0, \quad (2)$$

Here, $H(x)$ is the distance between the inflow/outflow boundary and the interface, where the coordinate system is located on the symmetry axis at the inflow/outflow boundary. Evaluating (2) at $x = 0$ and assuming that the interface has the shape of a circular section (constant curvature) implies $\kappa = \cos \theta / R$ and hence $H = \sigma \cos \theta / (\rho g R) + p_0 / (\rho g)$. This, is the stationary rise height also known as Jurin’s height. This however, implies that $H(x)$ is constant which is a contradiction (for $0 < \theta < \pi/2$) if the interface is curved (as $H(x) > H(0)$ for $0 < x \leq R$ in this case). Hence, there exists no stationary solution for which the interface has the shape of a spherical cap.

3 Spherical Cap - A Good Approximation

Evaluating (2) at the symmetry axis gives

$$\rho g H(0) = \sigma \kappa(0) + p_0, \quad (3)$$

where $H(0)$ is the height of the apex which we consider as the rise height of the capillary. As all remaining parameters and the pressure $p_0$ are constants, the curvature at the interface can not be constant. The following computation shows how the interface is shaped at this point.

Plugging in the free surface boundary conditions and expressing the curvature term with a graph representation yields a non-linear boundary value problem. Scaling the radius as well as the rise height with the radius of the capillary gives

$$h'' = (1 + h^2)^{\frac{3}{2}} (p_0^* - \text{Eo}h), \quad h'(0) = 0, \quad h'(1) = \cot \theta \quad (4)$$

with a scaled height $h := H(x)/R$, and Eotvos number $\text{Eo} := \rho g R^2 / \sigma$, a contact angle $\theta$, and a non-dimensional (capillary) pressure $p_0^* := p R / \sigma$. Hence, the pressure at the inflow shifts the height by $p_0 / (\rho g)$. Note that it is always possible to obtain a general form without $p_0^*$, since the equation is invariant to the transformation $\tilde{h} := h - p_0 / (\rho g)$. Also note that (4) is a singularly perturbed problem regarding the Eo-number.

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The boundary value problem (4) has been solved numerically using python, version 3.6.5, in combination with the numpy package, version 1.14.3. The results of the computations are shown in Figure 1 for a contact angle of $45^\circ$. The profiles are shown between the symmetry axis at $x^* = 0$ and the wall boundary at $x^* = 1$. The surfaces are aligned by adjusting their apex height in order to compare their shapes displayed using the variable $h^*$. The plot is scaled equally on both axis. The profiles are shown for $Eo = 10^{-2}, \ldots, 10^4$. As a reference, a circular section with the same contact angle as the curves from the boundary value problem is illustrated by black markers. For vanishing $Eo$-numbers, the surface shapes approach the circular reference curve. For increasing $Eo$-numbers the interface shape becomes increasingly flat and the region where the interface is curved is more and more restricted close to the contact line. Equation (2) can be non-dimensionalized giving

$$H^*(0) = \frac{\kappa^*_\text{cap}}{Eo} - \alpha + \bar{p}_0, \quad \alpha := \frac{\kappa^*_\text{cap} - \kappa^*(0)}{Eo},$$

with $H^* = H/R$, $\kappa^*_\text{cap} = \cos \theta$ and $\bar{p}_0 = p_0/(R \rho g)$. Here, the first term corresponds to a non-dimensional Jurin’s height and the second term gives a correction listed in Table 1 for the surfaces in Figure 1. The upper limit on $\alpha$ can be computed from the case with circular interface, yielding $\lim_{Eo \to \infty} \alpha = 0.1288$.

### Table 1: Non-dimensional corrections (rounded) obtained from the numerical solution of (4).

| $Eo$   | $10^{-2}$ | $10^{-1}$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ |
|--------|-----------|-----------|--------|--------|--------|--------|
| $\alpha$ | 1.286 · $10^{-1}$ | 1.269 · $10^{-1}$ | 1.125 · $10^{-1}$ | 5.236 · $10^{-2}$ | 7.065 · $10^{-3}$ | 7.071 · $10^{-4}$ |

### 4 Conclusion

To obtain a unique stationary solution for the capillary rise problem, it is sufficient to assume a constant pressure boundary condition at the inflow boundary. While rise models for a liquid in a capillary assume an interface with the shape of a spherical cap, this condition is only satisfied in the limit of a vanishing $Eo$ number. While an interface with the shape of a spherical cap is not a stationary solution of the capillary rise problem, it does provide an excellent approximation in the limit case. Computations show that for the stationary state $Eo$ numbers smaller than 0.1 are a sufficient criterion to assume a spherical cap, yet the error in the rise height quantified by Jurin’s height is maximal in this limit.

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