A combinatorial algorithm to compute presentations of mapping-class groups of orientable surfaces with one boundary component

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Abstract

We give an algorithm which computes a presentation for a subgroup, denoted \( \mathcal{A}M_{g,1,p} \), of the automorphism group of a free group. It is known that \( \mathcal{A}M_{g,1,p} \) is isomorphic to the mapping-class group of an orientable genus-\( g \) surface with one boundary component and \( p \) punctures. We define a variation of Auter space.

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1 Introduction

Let \( S \) be an orientable genus-\( g \) surface with \( b \) boundary components and \( p \) punctures. We denote by \( M(S) \) the group of isotopy classes of orientation-preserving homeomorphisms of \( S \) which permute the set of punctures and pointwise fix the boundary components. Since the group \( M(S) \) only depends, up to isomorphism, on the genus \( g \) of \( S \), the number \( b \) of boundary components of \( S \) and the number \( p \) of punctures of \( S \), we denote \( M(S) \) by \( M_{g,b,p} \). We call \( M_{g,b,p} \) the mapping-class group of \( S \).

Presentations for \( M_{g,b,p} \) were obtained after a sequence of papers started by Hatcher and Thurston [11], and followed by Harer [10], Wajnryb [17, 18], Matsumoto [14] and, Labruère and Paris [13]. For \( p = 0 \), Gervais [9] used the\footnote{The research was funded by MIC (Spain) through Project MTM2008-01550.} to deduce another presentations for \( M_{g,b,0} \). Before [11] very little was known about...
the presentation of $M_{g,b,p}$. Birman and Hilden [4] gave a presentation for $M_{2,0,0}$, and McCool [15] proved that $M_{g,b,p}$ is finitely presented.

Benvenuti [3] uses a variation of the curve complex, called ordered curve complex, to obtain presentations for $M_{g,b,p}$ from an inductive process. This inductive process starts from presentation for the sphere and the torus with “few” boundary components and/or punctures. Hirose [12] uses the curve complex and induction on $g$ and $b$ to deduce Gervais presentation. Both of these papers are independent of [11].

Our algorithm is independent of [11]. We feel that our point of view goes back to McCool [15]. Section 7 contains the presentation given by our algorithm.

This presentation has generators $ze_{i}, ze_{i}e_{j}$ where $z$ ranges over a finite set $L$ and $e_{i}, e_{j}$ range over $z$. There are three type of relations:

(a). $ze_{i} = 1, ze_{i}e_{j} = 1$, for some generators $ze_{i}, ze_{i}e_{j}$.
(b). $z_{1}e_{i}e_{j} = z_{2}e_{i}e_{j}$, for some generators $z_{1}e_{i}e_{j}, z_{2}e_{i}e_{j}$.
(c). $ze_{i} \cdot ze_{i}e_{j} = ze_{j} \cdot ze_{j}e_{i}$, for every generator $ze_{i}e_{j}$.

Armstrong, Forrest and Vogtmann [1] give a new presentation for $Aut(F_{n})$, the automorphism group of the free group of rank $n$. This presentation for $Aut(F_{n})$ is obtained by studying the action of $Aut(F_{n})$ on a subcomplex of the spine of Auter space. Following Armstrong, Forrest and Vogtmann [1], we obtain our algorithm by studying the action of an algebraic analogous of $M_{g,1,p}$ on a subcomplex of the spine of a variation of Auter space.

## 2 Preliminaries

Throughout the paper $n$ will be a non-negative integer, $F_{n}$ will be a free group of rank $n$, $Aut(F_{n})$ will be the automorphism group of $F_{n}$ and $Out(F_{n})$ will be the automorphism group of $F_{n}$ modulo inner automorphisms. Given $v, w \in F_{n}$, we denote by $[v, w]$ the element $v^{-1}w^{-1}vw$ of $F_{n}$. We denote by $[w]$ the conjugacy class of $w$.

Let $S$ be an orientable genus-$g$ surface with $b$ boundary components and $p$ punctures. A homeomorphism $f$ of $S$ which fixes the basepoint of $\pi_{1}(S)$ and permutes the set of punctures of $S$ induces an automorphism $f_{*} \in Aut(\pi_{1}(S))$. The isotopy class of $f$ defines an automorphism of $\pi_{1}(S)$ up to inner automorphisms, that is, an element of $Out(\pi_{1}(S))$. For $(b, p) = (0, 0)$, by Dehn-Nielsen-Baer Theorem, $M_{g,0,0}$ is isomorphic to a index 2 subgroup of $Out(\pi_{1}(S))$. For $(g, p) \neq (0, 0)$ by a modification of Dehn-Nielsen-Baer Theorem, $M_{g,b,p}$ is isomorphic to an infinite index subgroup of $Out(\pi_{1}(S))$.

Suppose now $b = 1$, that is, $S$ has exactly one boundary component. If we choose the basepoint of $\pi_{1}(S)$ to be a boundary point of $S$ and we restrict
ourselves to homeomorphisms of $S$ which pointwise fix the boundary, then the isotopy class of a homeomorphism of $S$ defines an element of $\text{Aut}(\pi_1(S))$. Since $S$ has one boundary component, the fundamental group of $S$ is a free group. We denote by

$$\Sigma_{g,1,p} = \langle x_1, y_1, x_2, y_2, \ldots, x_g, y_g, t_1, t_2, \ldots, t_p \mid \rangle$$

a presentation of $\pi_1(S,*)$ where $*$ is a boundary point of $S$, for every $1 \leq k \leq p$ the generator $t_k$ represents a loop around the $k$-th puncture of $S$ and the word $[x_1, y_1][x_2, y_2]\cdots[x_g, y_g]t_1t_2\cdots t_p$ represents a loop around the boundary component of $S$.

### 2.1 Definition

We denote by $\mathcal{AM}_{g,1,p}$ the subgroup of $\text{Aut}(\Sigma_{g,1,p})$ consisting of automorphisms of $\Sigma_{g,1,p}$ which fix the word $[x_1, y_1][x_2, y_2]\cdots[x_g, y_g]t_1t_2\cdots t_p$ of $\Sigma_{g,1,p}$ and fix the set of conjugacy classes $[t_1^{-1}], [t_2^{-1}], \ldots, [t_p^{-1}]$ of $\Sigma_{g,1,p}$.

Using a modification of Dehn-Nielsen-Baer Theorem, it can be proved that $\mathcal{M}_{g,1,p}$ is isomorphic to $\mathcal{AM}_{g,1,p}$, see [8] with some changes of notation and some different conventions. We call $\mathcal{AM}_{g,1,p}$ the \textit{algebraic mapping-class group} of an orientable genus-$g$ surface with one boundary component and $p$ punctures.

### 3 Auter space $A_n$

#### 3.1 Definition

Let $(\Gamma, v_0, \phi)$ be a 3-tuple such that

1. $\Gamma$ is a finite connected graph with no separating edges.
2. $\Gamma$ is a metric graph with total volume 1.
3. $v_0$ is a distinguished vertex of $\Gamma$.
4. Every vertex of $\Gamma$ but $v_0$ has valence at least 3; $v_0$ has valence at least 2.
5. $\phi : \pi_1(\Gamma, v_0) \to F_n$ is an isomorphism called “marking”.

A point in $A_n$ is an equivalence class of 3-tuples $(\Gamma, v_0, \phi)$, where $(\Gamma, v_0, \phi)$ is equivalent to $(\Gamma', v_0', \phi')$ if there exists an isometry $h : \Gamma \to \Gamma'$ such that $h(v_0) = v_0'$ and the isomorphism $h_* : \pi_1(\Gamma, v_0) \to \pi_1(\Gamma', v_0')$ satisfies $\phi = \phi' \circ h_*$. We call $A_n$ Auter space.

Auter space $A_n$ was introduced by Hatcher and Vogtmann [2] as an analogous for $\text{Aut}(F_n)$ of Outer space. Often in the literature the marking is defined as $\phi^{-1} : F_n \to \pi_1(\Gamma, v_0)$.

If $\Gamma$ has $k + 1$ edges, then $(\Gamma, v_0, \phi)$ defines an open $k$-simplex of $A_n$ denoted $\sigma(\Gamma, v_0, \phi)$. We can obtain $\sigma(\Gamma, v_0, \phi)$ by varying the length of the edges of $\Gamma$. The $k$-simplex $\sigma(\Gamma, v_0, \phi)$ can be parametrized by $\Delta^k$, the standard open $k$-simplex of $\mathbb{R}^k$, as follows: $(\Gamma_s, v_0, \phi) \in \sigma(\Gamma, v_0, \phi)$ is the point of $A_n$ such that...
the length of the edges of \( \Gamma \) equal the barycentric coordinates of \( s \in \Delta^k \). It is important that \( \Delta^k \) is open. Since a non-trivial isometry of \( \Gamma \) permutes same edges of \( \Gamma \), such an isometry gives a non-trivial element of \( H_1(\Gamma) \). Hence, every \( s \in \Delta \) defines a different point of \( \sigma(\Gamma, v_0, \phi) \).

Some faces of \( \sigma(\Gamma, v_0, \phi) \) belong to \( A_n \). If an edge of \( \Gamma \) is incident to two different vertices, then we can reduce the length of that edge to zero, and increase the length of the other edges, to obtain a new graph \( \Gamma' \) with one edge minus. We say that we have collapsed one edge of \( \Gamma \). We have a quotient map \( \Gamma \to \Gamma' \) which defines a point \( (\Gamma', v'_0, \phi') \) of \( A_n \). We say that \( \sigma(\Gamma', v'_0, \phi') \) is a face of \( \sigma(\Gamma, v_0, \phi) \). Faces of \( \sigma(\Gamma', v'_0, \phi') \) are faces of \( \sigma(\Gamma, v_0, \phi) \). We cannot collapse an edge which is incident to a unique vertex. Hence, some face of \( \sigma(\Gamma, v_0, \phi) \) are missing. In particular, \( A_n \) is not a simplicial complex.

There exists a deformation retract, denoted \( SA_n \), of \( A_n \) which is a simplicial complex. We can define \( SA_n \) as follows. For every simplex of \( A_n \), there exists a vertex of \( SA_n \). Two vertices of \( SA_n \) expand an edge if the simplex of \( A_n \) which defines one of the two vertices of \( SA_n \) is a face of the simplex of \( A_n \) which defines the other vertex of \( SA_n \); \( i+1 \) vertices of \( SA_n \) expand a \( i \)-simplex of \( SA_n \) if every pair of vertices expand an edge.

There exists a natural inclusion of \( SA_n \) into \( A_n \) by sending every vertex of \( SA_n \) to the barycenter of the corresponding simplex, and every \( i \)-simplex of \( SA_n \) to the convex hull of the corresponding barycenters. This inclusion is a deformation retract. See [2].

Collapsing an edge of \( \Gamma \) has an inverse process which splits a vertex of \( \Gamma \) into two new vertices, and the two new vertices are joined by a new edge. Often in the literature splitting of a vertex is called blowing up an edge. If \( \tilde{\Gamma} \) is obtained from \( \Gamma \) by splitting a vertex, then we can identify, in a natural way, every edge of \( \Gamma \) with an edge of \( \tilde{\Gamma} \). Collapsing the only edge of \( \tilde{\Gamma} \) which is not identified with an edge of \( \Gamma \) we obtain \( \Gamma \). There exists a quotient map \( \tilde{\Gamma} \to \Gamma \). If \( \tilde{\Gamma} \) is obtained from \( \Gamma \) by splitting a vertex different from \( v_0 \), then the quotient map \( \tilde{\Gamma} \to \Gamma \) defines a point \( (\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}) \) of \( A_n \). If \( \tilde{\Gamma} \) is obtained from \( \Gamma \) by splitting \( v_0 \), then the point \( (\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}) \) of \( A_n \) depends of the election, between the two possibilities, of the new distinguished vertex \( \tilde{v}_0 \). The simplex \( \sigma(\Gamma, v_0, \phi) \) is a face of \( \sigma(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}) \).

We give a combinatorial definition of the topological type of \( \Gamma \), that is, \( \Gamma \) when we forget its metric. When we forget the metric of \( \Gamma \) we can see \( (\Gamma, v_0, \phi) \) as a vertex of \( SA_n \), in fact, the vertex of \( SA_n \) defined by the simplex \( \sigma(\Gamma, v_0, \phi) \) of \( A_n \). We translate to our combinatorial definition the processes of collapsing an edges and splitting a vertex. Our combinatorial definition of the topological type of \( \Gamma \) is different from the one given in [3].

### 3.2 Definition
Let

1. \( V(\Gamma) \) be the set of vertices of \( \Gamma \).
2. \( E(\Gamma) \) be the set of edges of \( \Gamma \).
3. \( E(\Gamma) = \{ \tau \mid e \in E(\Gamma) \} \) be a set disjoint with \( E(\Gamma) \).

We extend \( \tau \) to an involution of \( E(\Gamma) \cup E(\Gamma) \). We fix an orientation of every edge of \( \Gamma \). We say that \( e \in E(\Gamma) \) starts at \( v_1 \in V(\Gamma) \) and finishes at \( v_2 \in V(\Gamma) \) if \( e \) is incident to \( v_1 \) and \( v_2 \); and \( e \) is oriented from \( v_1 \) to \( v_2 \). In this case we say that \( e \) starts at \( v_2 \) and finishes at \( v_1 \).

Given \( v \in V(\Gamma) \), we define the following subset of \( E(\Gamma) \cup E(\Gamma) \).

\[
v^* = \{ a \in E(\Gamma) \cup E(\Gamma) \mid a \text{ starts at } v \}.
\]

We set \( V^*(\Gamma) = \{ v^* \mid v \in V(\Gamma) \} \).

The topological type of \( \Gamma \) is completely determined by \( (V(\Gamma), E(\Gamma), V^*(\Gamma)) \).

Notice that \( v^* \) is the star of \( v \in V(\Gamma) \) and \( V^*(\Gamma) \) is a partition of \( E(\Gamma) \cup E(\Gamma) \).

Condition 1 of Definition 3.1 can be translated by saying that \( E(\Gamma) \) is finite and, for every \( v \in V(\Gamma) \), there exist \( a, b \in v^* \) such that \( a \neq b \) and \( a, b \notin v^* \).

Condition 2 of Definition 3.1 can be translated by saying that for every \( v \in V(\Gamma) - \{ v_0 \}, v^* \) has at least 3 elements; \( v_0^* \) has at least 2 elements.

3.3 Definition. Let \( e \in E(\Gamma) \) such that \( e \in v_1^*, \tau \in v_2^* \), where \( v_1, v_2 \in V(\Gamma) \) and \( v_1 \neq v_2 \). We can collapse \( e \). When we collapse the edge \( e \) we have a graph with topological type

\[
(V(\Gamma) \cup \{ v \} - \{ v_1, v_2 \}, E(\Gamma) - \{ e \}, V^*(\Gamma) \cup \{ v^* \} - \{ v_1^*, v_2^* \})
\]

where \( v \notin V(\Gamma) \) and \( v^* = v_1^* \cup v_2^* - \{ e, \tau \} \).

3.4 Definition. Let \( v \in V(\Gamma) - \{ v_0 \} \) and \( A, B \) a partition of \( v^* \) such that both \( A \) and \( B \) have at least two elements, there exists \( a \in A \) such that \( \tau \notin A \) and there exists \( b \in B \) such that \( \tau \notin B \). When we split the vertex \( v \) with respect to \( A \) and \( B \) we have a graph with topological type

\[
(V(\Gamma) \cup \{ v_1, v_2 \} - \{ v \}, E(\Gamma) \cup \{ e \}, V^*(\Gamma) \cup \{ v_1^*, v_2^* \} - \{ v^* \})
\]

where \( v_1, v_2 \notin V(\Gamma), e \notin E(\Gamma), v_1^* = A \cup \{ e \} \) and \( v_2^* = B \cup \{ \tau \} \). To split \( v_0 \) we have to choose between \( v_1 \) or \( v_2 \) as the new distinguished vertex. Since the distinguished vertex can have valence two, the subset of \( v_0^* \) corresponding to the new distinguished vertex may have only one element.

4 Ordered Auter space \( \text{ord} \mathbb{A}_{g,p} \)

Our motivation for defining ordered Auter space is that when a graph is embedded into an orientable surface, the star of every vertex of the graph which is mapped to an interior point of the surface gets a cyclic order, and, the star of a vertex which is mapped to a boundary point of the surface gets a linear order. When we want to collapse an edge or to split a vertex we have to do it respecting the orders of the stars.
4.1 Definition. Let \((\Gamma, v_0, \phi, \text{ord})\) be a 4-tuple where \((\Gamma, v_0, \phi)\) satisfies conditions in Definition 3.1, \text{ord} is a linear order of \(v_0^*\) and a cyclic order of \(v^*\) for every \(v \in V(\Gamma) - \{v_0\}\).

Suppose \(V(\Gamma) = \{v_0, v_1, v_2, \ldots, v_q\}\) and

\[
\begin{align*}
\text{ord}(v_0^*) &= (a_0^0, a_0^1, \ldots, a_0^{r_0}), \\
\text{ord}(v_1^*) &= (a_1^1, a_1^2, \ldots, a_1^{r_1}), \\
\text{ord}(v_2^*) &= (a_2^2, a_2^3, \ldots, a_2^{r_2}), \\
& \vdots \\
\text{ord}(v_q^*) &= (a_q^q, a_q^q, \ldots, a_q^{r_q}).
\end{align*}
\]

(4.1.1)

For \(i \neq 0\), since \(\text{ord}(v_i^*)\) is cyclically ordered, the subindices of \(\text{ord}(v_i^*)\) are modulo \(r_i\).

We consider the following element of \(\pi_1(\Gamma, v_0)\) and the following conjugacy classes of \(\pi_1(\Gamma, v_0)\).

\[
\begin{align*}
w_0 &= b_1^0 b_2^0 \ldots b_{l_0}^0, \\
[w_1] &= [b_1^1 b_2^1 \ldots b_{l_1}^1], \\
[w_2] &= [b_1^2 b_2^2 \ldots b_{l_2}^2], \\
& \vdots \\
[w_p] &= [b_1^p b_2^p \ldots b_{l_p}^p],
\end{align*}
\]

where \(b_{l_i}^i = a_{l_i}^i\), for every \(1 \leq i \leq p, 1 \leq j \leq l_i\) the subsequence \((b_j^i, b_{j+1}^i)\) appears in (4.1.1), \(b_{l_0}^0 = \pi_{r_0}^y\), and every element of \(E(\Gamma) \cup E(\Gamma)\) appears exactly once in (4.1.2).

We denote by \(w(\Gamma, v_0, \text{ord})\) the set \(\{w_0, [w_1], [w_2], \ldots, [w_p]\}\).

4.2 Example. Let \((\Gamma, v_0, \phi)\) be a 3-tuple where \(V(\Gamma) = \{v_0, v_1, v_2\}\), \(E(\Gamma) = \{e_1, e_2, e_3, e_4, e_5\}\) and

\[
\begin{align*}
\text{ord}(v_0^*) &= (e_1, e_2), \\
\text{ord}(v_1^*) &= (\overline{e}_1, e_3, \overline{e}_3, e_4, e_5), \\
\text{ord}(v_2^*) &= (\overline{e}_2, \overline{e}_5, e_4).
\end{align*}
\]

Then \(w(\Gamma, v_0, \text{ord}) = \{w_0, [w_1], [w_2], [w_3]\}\) where

\[
\begin{align*}
w_0 &= e_1 e_3 e_4 \overline{e}_2, \\
[w_1] &= [\overline{e}_1 e_2 \overline{e}_5], \\
[w_2] &= [\overline{e}_3], \\
[w_3] &= [\overline{e}_4 e_5].
\end{align*}
\]

Notice that for every \(v \in V(\Gamma)\), \(\text{ord}(v^*)\) is completely determined by \(w(\Gamma, v_0, \text{ord})\).
4.3 Definition. Let $(\Gamma, v_0, \phi, \ord)$ be a 4-tuple such that $w(\Gamma, v_0, \ord)$ has $p$ conjugacy classes.

We denote $\frac{n - p}{2}$ by $g$. We will see that $n - p$ is even. Hence, $g$ is a non-negative integer.

We define $\ord A_{g,p}$ as the space of equivalence classes of 4-tuples $(\Gamma, v_0, \phi, \ord)$ such that $\phi : \pi_1(\Gamma, v_0) \to \Sigma_{g,1,p}$, $w(\Gamma, v_0, \ord) = \{w_0, [w_1], [w_2], \ldots, [w_p]\}$ and

$$\phi(w_0) = [x_1, y_1][x_2, y_2] \cdots [x_g, y_g][t_1^{-1}t_2 \cdots t_p].$$

$$\{\phi([w_1]), \phi([w_2]), \ldots, \phi([w_p])\} = \{[t_1^{-1}], [t_2^{-1}], \ldots, [t_p^{-1}]\}.$$

The 4-tuples $(\Gamma, v_0, \phi, \ord)$ and $(\Gamma', v_0', \phi', \ord')$ represent the same point of $\ord A_{g,p}$ if there exists an isometry $h : \Gamma \to \Gamma'$ such that $h(v_0) = v_0'$, the isomorphism $h_* : \pi_1(\Gamma, v_0) \to \pi_1(\Gamma', v_0')$ satisfies $\phi = \phi' \circ h_*$, and $h : \Gamma \to \Gamma'$ preserves the orders, that is, $\ord(v^*) = (a_1, a_2, \ldots, a_r)$ implies $\ord'(h(v)^*) = (h(a_1), h(a_2), \ldots, h(a_r))$ for every $v \in V(\Gamma)$.

We call $\ord A_{g,p}$ ordered Anisouter space.

We define $\ord S\mathbb{A}_{g,p}$ for $\ord A_{g,p}$ as we defined $S\mathbb{A}_n$ for $A_n$. In particular, $\ord S\mathbb{A}_{g,p}$ is a simplicial complex, and, $\ord S\mathbb{A}_{g,p}$ is a deformation retract of $\ord A_{g,p}$.

The following definitions are based on Definition 3.3 and Definition 3.4 respectively.

4.4 Definition. Let $e \in E(\Gamma)$. Suppose $e = a_{k_1}^i, \overline{e} = a_{k_2}^i$, where $i \neq j$ and $1 \leq k_1 \leq r_1, 1 \leq k_2 \leq r_2$. Since $i \neq j$, we can collapse $e$. We can suppose $j \neq 0$. To adapt Definition 3.3 to $\ord S\mathbb{A}_{g,p}$ we set

$$\ord(v^*) = (a_{k_1-1}^i, a_{k_2}^i, \ldots, a_{k_1-1}^i),$$

$$a_{k_2+1}^j, a_{k_2+2}^j, \ldots, a_{r_j}^j, a_{k_2}^i, a_{k_2-1}^i, \ldots, a_{r_2}^i,$$

$$a_{k_1+1}^i, a_{k_1+2}^i, \ldots, a_{r_1}^i).$$


definition 4.4 end

4.5 Example. Let $(\Gamma, v_0, \ord)$ be as in Example 4.2. When we collapse $e_1$ we obtain $(\Gamma', v_0', \ord')$ such that $v_0' = v_0$, $V(\Gamma') = \{v_0, v_2\}$, $E(\Gamma') = \{e_2, e_3, e_4, e_5\}$ and

$$\ord'(v_0^*) = (e_3, \overline{e}_3, e_4, e_5, e_2),$$

$$\ord'(v_2^*) = (\overline{e}_2, \overline{e}_5, \overline{e}_4).$$

We have $w(\Gamma', v_0', \ord') = \{w_0', [w_1], [w_2], [w_3]\}$ where

$w_0' = e_3e_4\overline{e}_2,$

$[w_1'] = [e_2\overline{e}_5],$

$[w_2'] = [\overline{e}_3],$

$[w_3'] = [\overline{e}_4e_5].$
Let $\Gamma, v_0, \phi, \text{ord}$ be a vertex of $\text{ordS}_A g,p$. When we collapse an edge of $\Gamma$ according to Definition 4.4 we obtain $(\Gamma', v'_0, \phi', \text{ord}')$. As it is seen in Example 4.5, $w(\Gamma', v'_0, \phi', \text{ord}')$ has $p$ conjugacy classes. Hence, $(\Gamma', v'_0, \phi', \text{ord}')$ is a vertex of $\text{ordS}_A g,p$.

4.6 Definition. Let $v \in V(\Gamma)$. Let $A, B$ be a partition of $v^*$. Suppose $\text{ord}(v^*) = (a_1, a_2, \ldots, a_r)$ and $A = (a_k, a_{k+1}, \ldots, a_k)$, where $1 \leq k_1 < k_2 \leq r$. We can split the vertex $v$ with respect to $A, B$. To adapt Definition 3.4 to $\text{ordS}_A g,p$ we set

$$
\text{ord}(v_1^*) = (e, a_k, a_{k+1}, \ldots, a_k), \\
\text{ord}(v_2^*) = (a_1, a_2, \ldots, a_{k-1}, e, a_{k+1}, a_{k+2}, \ldots, a_r).
$$

\[
\square
\]

4.7 Example. Let $(\Gamma, v_0, \text{ord})$ be as in Example 4.2. When we split $v_1$ with respect to $\{e_3, e_4\}, \{e_1, e_3, e_5\}$ we obtain $(\tilde{\Gamma}, \tilde{v}_0, \text{ord})$ such that $\tilde{v}_0 = v_0$, $V(\tilde{\Gamma}) = \{v_0, v_{1,1}, v_{1,2}, v_2\}$, $E(\tilde{\Gamma}) = \{e, e_1, e_2, e_3, e_4, e_5\}$ and

$$
\tilde{\text{ord}}(v_0^*) = (e_1, e_2), \\
\tilde{\text{ord}}(v_{1,1}^*) = (e, e_3, e_4), \\
\tilde{\text{ord}}(v_{1,2}^*) = (e_1, e_3, e_5), \\
\tilde{\text{ord}}(v_2^*) = (e_2, e_5, e_4).
$$

We have $w(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}, \tilde{\text{ord}}) = \{\tilde{w}_0, [\tilde{w}_1], [\tilde{w}_2], [\tilde{w}_3]\}$ where

$$
\tilde{w}_0 = e_1 e_3 e_4 e_2, \\
[\tilde{w}_1] = [e_1 e_2 e_3], \\
[\tilde{w}_2] = [e_3 e_4], \\
[\tilde{w}_3] = [e_4 e_5].
$$

\[
\square
\]

Let $(\Gamma, v_0, \phi, \text{ord})$ be a vertex of $\text{ordS}_A g,p$. When we split a vertex of $\Gamma$ according to Definition 4.6 we obtain $(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}, \tilde{\text{ord}})$. As it is seen in Example 4.7, $w(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}, \tilde{\text{ord}})$ has $p$ conjugacy classes. Hence, $(\tilde{\Gamma}, \tilde{v}_0, \tilde{\phi}, \tilde{\text{ord}})$ is a vertex of $\text{ordS}_A g,p$.

Since a graph satisfying Definition 3.1 can have at most $3n - 2$ edges, the dimension of $A_n$ is $3n - 3$. On the other hand, $A_n$ is not a manifold. The dimension of $\text{ordS}_A g,p$ is $6g + 3p - 3$; $\text{ordA}_n$ is a manifold.

By [2] PROPOSITION 2.1 $A_n$ is contractible. Since $S A_n$ is a deformation retract of $A_n$, we see that $S A_n$ is contractible.
4.8 Proposition. \( \text{ord} \mathbb{A}_{g,p} \) is contractible.

Hatcher and Vogtmann proof [2, PROPOSITION 2.1] using spheres complexes. It is not clear how to translate to the context of spheres complexes an ordered graph. On the other hand, the proof of Culler and Vogtmann [6] that Outer space is contractible can be applied to \( \text{ord} \mathbb{A}_{g,p} \): adding a basepoint is straightforward, all the geometric arguments in [6] can be applied to \( \text{ord} \mathbb{A}_{g,p} \) respecting the orders as Definition 4.4 and Definition 4.6 and McCool [16], [7] proved that \( \mathbb{A}M_{g,1,p} \) is generated by Nielsen automorphism which “respect” the orders (recall that Nielsen automorphisms are a special case of Whitehead automorphisms).

Recall \( n = 2g + p \). There exists a natural map \( \text{ord} \mathbb{A}_{g,p} \rightarrow \mathbb{A}_n \) which “forgets” the ordering, that is, \( (\Gamma, v_0, \phi, \text{ord}) \mapsto (\Gamma, v_0, \phi) \). Recall that \( \text{ord} \) is completely determined by \( w(\Gamma, v_0, \text{ord}) \). Since \( \phi : \pi_1(\Gamma, v_0) \rightarrow \Sigma_{g,1,p} \) is an isomorphism, we have

\[
w(\Gamma, v_0, \text{ord}) = \{ \phi^{-1}(x_1, y_1)[x_2, y_2] \cdots [x_g, y_g]t_1t_2 \cdots t_p), \]
\[
[\phi^{-1}(t_1^{-1})], [\phi^{-1}(t_2^{-1})], \cdots, [\phi^{-1}(t_p^{-1})]\}
\]

Hence, the natural map \( \text{ord} \mathbb{A}_{g,p} \rightarrow \mathbb{A}_n \) is injective.

We want to see that \( n - p \) is even.

For \( n = 1 \), we have \( p = 1 \) and \( n - p = 0 \) is even. We do induction on \( n \).

By Definition 4.4 we can collapse a maximal subtree of \( (\Gamma, v_0, \text{ord}) \). Hence, we can suppose that \( V(\Gamma) = \{v_0\} \). Put \( \text{ord}(v_0^*) = (a_1, a_2, \ldots, a_{2n}) \) and \( w(\Gamma, v_0, \text{ord}) = \{w_0, [w_1], [w_2], \ldots, [w_p]\} \). Let \( 1 \leq j \leq 2n \) such that \( a_j = a_1 \).

Then \( w_0 = a_1u \) in reduced form, for some \( u \in F_n \). Let \( (\Gamma', v_0', \text{ord}') \) be obtained from \( (\Gamma, v_0, \text{ord}) \) by deleting the edges \( a_1, a_j \). We have \( \text{ord}'(v_0') = (a_2, a_3, \ldots, a_{j-1}, a_{j+1}, a_{j+2}, \ldots, a_{2n}) \) and \( p' \) the number of conjugacy classes of \( w(\Gamma', v_0', \text{ord}') \). By induction hypothesis \( n' - p' \) is even.

If \( w_0 = a_1u'a_ju'' \) cyclically reduced, then \( w(\Gamma', v_0', \text{ord}') = \{u'', [u'], [w_1], [w_2], \ldots, [w_p]\} \). Notice that \( u' \neq 1, u'' \neq 1 \) because \( a_1u'a_ju'' \) is cyclically reduced. Hence, \( p' = p + 1 \) and \( n - p = (n' + 1) - (p' - 1) = n' - p' + 2 \) is even.

If there exists \( 1 \leq k \leq p \) such that \( [w_k] = [a_j w_k'] \), then \( w(\Gamma', v_0', \text{ord}') = \{w_k'u', [w_1], [w_2], \ldots, [w_{k-1}], [w_{k+1}], \ldots, [w_p]\} \). Hence, \( p' = p - 1 \) and \( n - p = (n' + 1) - (p' + 1) = n' - p' \) is even.

5 The Degree Theorem

Recall \( \pi_1(\Gamma, v_0) \simeq F_n \).

We denote the valence of \( v \in V(\Gamma) \) by \( |v^*| \).
5.1 Definition. The degree of \((\Gamma, v_0)\) is \(2n - |v_0^*|\). Equivalently, the degree of \((\Gamma, v_0)\) is \(\sum_{v \in V(\Gamma) - \{v_0\}} (|v^*| - 2)\).

To see the equivalence of the two definitions see [2, p. 636].

From the first definition of the degree of \((\Gamma, v_0)\) we see that when we collapse an edge of \(\Gamma\) which is not incident with \(v_0\) the degree is preserved, and, when we collapse an edge of \(\Gamma\) which is incident with \(v_0\) the degree decreases. Hence, graphs of degree at most \(i\) expand a subcomplex \(D_i\) of \(SA_n\). Hatcher and Vogtmann [2] proof the following.

5.2 Theorem. \(D_i\) is \(i\)-dimensional and \((i - 1)\)-connected.

In particular, \(D_2\) is a simply-connected 2-complex.

We define \(ordD_i\) for \(ordSA_{g,p}\) as we define \(D_i\) for \(SA_n\).

All the arguments of Hatcher and Vogtmann to proof [2, THEOREM 3.3] can be applied to \(ordSA_{g,p}\). In particular, what they call “canonical splitting” and “sliding in the \(\epsilon\)-cone” are combinations of splitting vertices and collapsing edges. We have the following.

5.3 Theorem. \(ordD_i\) is \(i\)-dimensional and \((i - 1)\)-connected.

In particular, \(ordD_2\) is a simply-connected 2-complex.

6 The action of \(AM_{g,1,p}\) on \(ordA_n\)

Recall that \(Aut(F_n)\) acts on \(A_n\) by “changing” the markings: for every \(\varphi \in Aut(F_n)\) we define \(\varphi \cdot (\Gamma, v_0, \phi) = (\Gamma, v_0, \varphi \circ \phi)\). This action restricts to \(SA_n\) and to \(D_2\). The stabilizer of a vertex of \(SA_n\) by this action is a finite group which permutes some edges and invert some edge orientations. The quotient complex \(Aut(F_n)\backslash SA_n\) is finite. See [1, Section 3], [2 Section 5]. Armstrong, Forrest and Vogtmann [1] apply a result of Brown [5] to \(Aut(F_n)\backslash D_2\) to compute a new presentation of \(Aut(F_n)\). Following this argument, we want to obtain a presentation of \(AM_{g,1,p}\).

We can define an action of \(AM_{g,1,p}\) on \(ordA_{g,p}\) by “changing” the markings: for every \(\varphi \in AM_{g,1,p}\) we define \(\varphi \cdot (\Gamma, v_0, \phi, ord) = (\Gamma, v_0, \varphi \circ \phi, ord)\). This action restricts to \(ordSA_{g,p}\) and to \(ordD_2\). The stabilizer of a vertex of \(ordSA_{g,p}\) by this action is trivial and the quotient complex \(AM_{g,1,p}\backslash ordSA_{g,p}\) is finite, but much bigger than \(Aut(F_n)\backslash SA_n\). By Theorem 5.3, \(ordD_2\) is simply-connected. Hence, \(AM_{g,1,p}\) is isomorphic to the fundamental group of \(AM_{g,1,p}\backslash ordD_2\). In the next section we give an algorithm which computes a presentation of the fundamental group of \(AM_{g,1,p}\backslash ordD_2\).
7 The algorithm

Recall $n = 2g + p$.

Vertices of $\text{ordSA}_{g,p}$ are represented by 4-tuples $(\Gamma, v_0, \phi, \text{ord})$ such that $w(\Gamma, v_0, \text{ord})$ has $p$ conjugacy classes. Recall that $\varphi \in \mathcal{AM}_{g,1,p}$ acts on $\text{ordSA}_{g,p}$ by “changing” the marking, that is, $\varphi \cdot (\Gamma, v_0, \phi, \text{ord}) = (\Gamma, v_0, \varphi \circ \phi, \text{ord})$. Hence, the quotient map $\text{ordSA}_{g,p} \to \mathcal{AM}_{g,1,p} \setminus \text{ordSA}_{g,p}$, $(\Gamma, v_0, \phi, \text{ord}) \mapsto (\Gamma, v_0, \text{ord})$ “forgets” the marking. We can represent vertices of $\mathcal{AM}_{g,1,p} \setminus \text{ordD}_2$ by 3-tuples $(\Gamma, v_0, \text{ord})$ such that $(\Gamma, v_0)$ has degree at most 2.

We want to compute a presentation for the fundamental group of complex $\mathcal{AM}_{g,1,p} \setminus \text{ordD}_2$. Recall that the degree of $(\Gamma, v_0)$ can be defined as $\sum_{v \in V(\Gamma) - \{v_0\}} (|v^*| - 2)$. Hence, if $(\Gamma, v_0)$ has degree 2 then $\Gamma$ has at most three vertices: $v_0$ and two more vertices of valence 3.

Let $\mathcal{L}$ be a list of vertices $(\Gamma, v_0, \text{ord})$ of $\mathcal{AM}_{g,1,p} \setminus \text{ordD}_2$ such that $\Gamma$ has 3 vertices.

Let $z = (\Gamma, v_0, \text{ord})$ be an element of $\mathcal{L}$. Suppose $E(\Gamma) = \{e_1, e_2, \ldots, e_k\}$.

We construct a tree $T(z)$ as follows. There exists a vertex $z$ of $T(z)$. Let $e_i$ be an edge of $\Gamma$ which can be collapsed, that is, $e_i$ is incident to two different vertices. When we collapse $e_i$ we have a quotient 3-tuple $z^i = (\Gamma^i, v_0^i, \text{ord}^i)$. There exists a vertex $z^i$ of $T(z)$ and an edge $ze_i$ of $T(z)$ from $z$ to $z^i$. We identify edges of $z^i$ with edges of $z$. Let $e_j$ be an edge of $\Gamma^i$ which can be collapsed. When we collapse $e_j$ in $\Gamma^i$ we have a quotient 3-tuple $z^{i(j)} = (\Gamma^{(i,j)}, v_0^{(i,j)}, \text{ord}^{(i,j)})$. There exists a vertex $z^{(i,j)}$ of $T(z)$ and an edge $ze_i e_j$ from $z^i$ to $z^{(i,j)}$. We repeat this process for every edge which can be collapsed.

Our generating set for the fundamental group of $\mathcal{AM}_{g,1,p} \setminus \text{ordD}_2$ is the set of edges of $T(z)$, where $z$ ranges over $\mathcal{L}$.

The group $\text{Sym}_k \times C_2^{\times k}$ acts on $E(\Gamma) \cup \overline{E}(\Gamma)$ by permuting edges ($C_2^{\times k}$ is the Cartesian product of $k$ copies of the cyclic group of order 2). Hence, $\text{Sym}_k \times C_2^{\times k}$ acts on the set of 3-tuples $(\Gamma, v_0, \text{ord})$ by permuting edges and inverting edge orientations. Two 3-tuples $(\Gamma, v_0, \text{ord})$ and $(\Gamma', v_0', \text{ord}')$ represent the same vertex of $\mathcal{AM}_{g,1,p} \setminus \text{ordSA}_{g,p}$ if and only if they are in the same orbit by the action of $\text{Sym}_k \times C_2^{\times k}$. Since every vertex of $T(z)$ is a 3-tuple $(\Gamma, v_0, \text{ord})$, we see that $\text{Sym}_k \times C_2^{\times k}$ acts on $T(z)$. We can identify $(\text{Sym}_k \times C_2^{\times k}) \setminus T(z)$ with the 1-skeleton of a subcomplex of $\mathcal{AM}_{g,1,p} \setminus \text{ordD}_2$. We can identify

$$(\text{Sym}_k \times C_2^{\times k}) \setminus \left( \bigcup_{z \in \mathcal{L}} T(z) \right)$$

with the 1-skeleton of a subcomplex of $\mathcal{AM}_{g,1,p} \setminus \text{ordD}_2$.

We attach some 2-cells to $(\text{Sym}_k \times C_2^{\times k}) \setminus \left( \bigcup_{z \in \mathcal{L}} T(z) \right)$. If there exists the generator $ze_i e_j$, we attach a 2-cell though the edge-path $ze_i, ze_i e_j, z e_j e_i, z e_j$. With
these attached 2-cells, the 2-complex \((\text{Sym}_k \times C_2^{\times k})\setminus (\bigcup_{z \in \mathcal{L}} T(z))\) is homeomorphic to \(\mathcal{AM}_{g,1,p} \setminus \text{ordD}_2\). We fix a maximal subtree of \((\text{Sym}_k \times C_2^{\times k})\setminus (\bigcup_{z \in \mathcal{L}} T(z))\).

Our presentation for the fundamental group of \(\mathcal{AM}_{g,1,p} \setminus \text{ordD}_2\) has three types of relations:

(a). \(ze_i = 1, ze_i e_j = 1\) if the edges \(ze_i, ze_i e_j\) are in our maximal subtree.

(b). \(z_1 e_i e_j = z_2 e_i e_j\) if the generator \(z_1 e_i e_j\) exists and \(g \cdot z_i^j = z_1^j\) for some \(g \in \text{Sym}_k \times C_2^{\times k}\) such that either \(g \cdot e_j = e_j\) or \(g \cdot e_j = e_j\).

(c). \(ze_i \cdot ze_i e_j = ze_j \cdot ze_j e_i\) if there exists the generator \(ze_i e_j\).

We illustrate the algorithm with two easy examples. The main difficulty of the algorithm is to find \(\mathcal{L}\). Once \(\mathcal{L}\) is known our, it is straightforward to apply the algorithm. Example 7.2 shows that the algorithm can be applied in "pieces", each piece corresponding to an element of \(\mathcal{L}\).

7.1 Example. We take \((g, p) = (1, 0)\). The list \(\mathcal{L}\) has a single element. We can represent the element of \(\mathcal{L}\) by

\[ z = (V(\Gamma), E(\Gamma), V^*(\Gamma), \text{ord}) = \{\{v_0, v_1, v_2\}, \{e_1, e_2, e_3, e_4\}, \{v_0^*, v_1^*, v_2^*\}, \text{ord}\}, \]

where \(\text{ord}(v_0^*) = (e_1, e_2)\), \(\text{ord}(v_1^*) = (\overline{e}_1, e_3, e_4)\) and \(\text{ord}(v_2^*) = (\overline{e}_2, \overline{e}_3, \overline{e}_4)\). To simplify the notation we put

\[ z = \text{ord}(v_0^*); \text{ord}(v_1^*), \text{ord}(v_2^*) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, \overline{e}_4). \]

We can collapse all 4 edges of \(z\) and we have

\[ z^1 = (e_3, e_4, e_2); (\overline{e}_2, \overline{e}_3, \overline{e}_4), \]
\[ z^2 = (e_1, \overline{e}_3, \overline{e}_4); (\overline{e}_1, e_3, e_4), \]
\[ z^3 = (e_1, e_2); (\overline{e}_1, \overline{e}_4, \overline{e}_2, e_4), \]
\[ z^4 = (e_1, e_2); (\overline{e}_1, e_3, \overline{e}_2, \overline{e}_3). \]

We see \(z^1 = g^{2,1} \cdot z^2, z^3 = g^{4,3} \cdot z^4\), where \(g^{2,1}, g^{4,3} \in \text{Sym}_4 \times C_2^{\times 4}\) and

\[ g^{2,1} = \begin{cases} e_1 \mapsto e_3, \\ e_3 \mapsto \overline{e}_4, \\ e_4 \mapsto \overline{e}_2, \end{cases} \quad \text{and} \quad g^{4,3} = \begin{cases} e_1 \mapsto e_1, \\ e_2 \mapsto e_2, \\ e_3 \mapsto \overline{e}_4. \end{cases} \]

We can collapse some edges of \(z^1\) and \(z^3\) and we have

\[ z^{(1,3)} = (\overline{e}_4, \overline{e}_2, e_4, e_2), \quad z^{(1,4)} = (e_3, \overline{e}_2, \overline{e}_3, e_2), \quad \text{and} \quad z^{(3,1)} = (\overline{e}_4, \overline{e}_2, e_4, e_2), \]
\[ z^{(3,2)} = (e_1, e_4, \overline{e}_1, \overline{e}_4). \]
We see $z^{(1,2)}$, $z^{(1,3)}$, $z^{(1,4)}$, $z^{(3,1)}$ and $z^{(3,2)}$ are in the same orbit by $\text{Sym}_4 \times C_2 \times 4$. Hence, they represent the same vertex of $(\text{Sym}_4 \times C_2)\setminus T(z)$.

We take the maximal subtree of $(\text{Sym}_4 \times C_2)\setminus T(z)$ with edges $ze_1, ze_3$ and $ze_1e_2$. Then $\mathcal{A}M_{1,0}$ has presentation with generators:

$$ze_1, ze_2, ze_3, ze_4,$$
$$ze_1e_3, ze_1e_4, ze_1e_2,$$
$$ze_2e_1, ze_2e_3, ze_2e_4,$$
$$ze_3e_1, ze_3e_2,$$
$$ze_4e_1, ze_4e_2;$$

and relations:

$$ze_1 = 1, ze_3 = 1, ze_1e_2 = 1,$$
$$ze_2e_1 = ze_1e_3, ze_2e_3 = ze_1e_4, ze_2e_4 = ze_1e_2, ze_4e_1 = ze_3e_1, ze_4e_2 = ze_3e_2,$$
$$ze_1 \cdot ze_1e_2 = ze_2 \cdot ze_1e_1, ze_1 \cdot ze_1e_3 = ze_3 \cdot ze_3e_1, ze_1 \cdot ze_1e_4 = ze_4 \cdot ze_4e_1,$$
$$ze_2 \cdot ze_2e_3 = ze_3 \cdot ze_3e_2, ze_2 \cdot ze_2e_4 = ze_4 \cdot ze_4e_2.$$

An easy simplification shows $\mathcal{A}M_{1,1,0} = \langle ze_2, ze_4 \mid ze_2 \cdot ze_2 = ze_4 \cdot ze_2 \cdot ze_4 \rangle$. □

7.2 Example. We take $(g, p) = (0, 3)$. The list $L$ is

$$z_1 = (e_1, e_2, e_3, e_4); (\overline{e}_1, e_5, e_2), (\overline{e}_3, \overline{e}_5, \overline{e}_4),$$
$$z_2 = (e_1, e_2, e_3, e_4); (\overline{e}_1, e_5, e_2), (\overline{e}_2, e_5, \overline{e}_3),$$
$$z_3 = (e_1, e_1, e_2, e_3); (\overline{e}_2, e_4, e_6), (\overline{e}_3, e_5, \overline{e}_4),$$
$$z_4 = (e_1, e_2, e_3, e_3); (\overline{e}_1, e_4, e_5), (\overline{e}_3, e_5, \overline{e}_4),$$
$$z_5 = (e_1, e_2, e_3, e_3); (\overline{e}_1, e_4, e_5), (\overline{e}_2, e_5, \overline{e}_4),$$
$$z_6 = (e_1, e_2, e_3, e_3); (\overline{e}_2, e_4, e_5), (\overline{e}_3, e_5, \overline{e}_4).$$

For $z_1$ we have

$$z_1 = (e_1, e_2, e_3, e_4); (\overline{e}_1, e_5, e_2), (\overline{e}_3, \overline{e}_5, \overline{e}_4),$$
$$z_1^1 = (e_5, e_2, e_3, e_4); (\overline{e}_3, e_5, \overline{e}_4),$$
$$z_1^2 = (e_1, e_1, e_5, e_4); (\overline{e}_3, e_5, \overline{e}_4),$$
$$z_1^3 = (e_1, e_2, e_5, e_4); (\overline{e}_1, e_5, \overline{e}_2),$$
$$z_1^4 = (e_1, e_2, e_3, e_3); (\overline{e}_1, e_5, \overline{e}_2),$$
$$z_1^5 = (e_1, e_2, e_3, e_4); (\overline{e}_1, e_4, \overline{e}_3, \overline{e}_2).$$

Generators for $z_1$ are:

$$z_1e_1, z_1e_2, z_1e_3, z_1e_4, z_1e_5,$$
$$z_1e_1e_3, z_1e_1e_4, z_1e_1e_4,$$
$$z_1e_2e_3, z_1e_2e_4, z_1e_2e_4,$$
$$z_1e_3e_1, z_1e_3e_2, z_1e_3e_5,$$
$$z_1e_4e_1, z_1e_4e_2, z_1e_4e_5,$$
$$z_1e_5e_1, z_1e_5e_2, z_1e_5e_3, z_1e_5e_4.$$
We have

\[ z_1^{(1,5)} = (\overline{e}_4, \overline{e}_3, \overline{e}_2, e_2, e_3, e_4), \]
\[ z_1^{(1,3)} = (e_5, \overline{e}_2, e_2, \overline{e}_3, \overline{e}_4, e_4), \]
\[ z_1^{(1,4)} = (e_5, \overline{e}_2, e_2, e_3, \overline{e}_3, \overline{e}_5), \]
\[ z_1^{(2,5)} = (e_1, \overline{e}_1, \overline{e}_4, \overline{e}_3, e_3, e_4), \]
\[ z_1^{(2,3)} = (e_1, \overline{e}_1, e_5, \overline{e}_4, e_4, e_4), \]
\[ z_1^{(2,4)} = (e_1, \overline{e}_1, e_5, e_3, \overline{e}_3, \overline{e}_5), \]
\[ z_1^{(3,5)} = (e_1, e_2, \overline{e}_2, \overline{e}_1, \overline{e}_4, e_4), \]
\[ z_1^{(4,5)} = (e_1, e_2, e_3, \overline{e}_3, \overline{e}_2, \overline{e}_1). \]

We see that \( z_1^{(2,4)} = g \cdot z_1^{(2,5)}, z_1^{(3,5)} = g' \cdot z_1^{(1,3)}, z_1^{(4,5)} = g'' \cdot z_1^{(1,5)} \) for some \( g, g', g'' \in \text{Sym}_5 \times C_2^x \).

Relations for \( z_1 \) are:

\[ z_1 e_1 = 1, \ z_1 e_2 = 1, \ z_1 e_3 = 1, \ z_1 e_4 = 1, \ z_1 e_5 = 1, \]
\[ z_1 e_1 e_5 = 1, \ z_1 e_1 e_3 = 1, \ z_1 e_1 e_4 = 1, \ z_1 e_2 e_5 = 1, \ z_1 e_2 e_3 = 1, \]
\[ z_1 e_1 \cdot z_1 e_1 e_5 = z_1 e_3 \cdot z_1 e_1 e_3 = z_1 e_3 \cdot z_1 e_1 e_1, \]
\[ z_1 e_1 \cdot z_1 e_1 e_4 = z_1 e_4 \cdot z_1 e_4 e_1, \]
\[ z_1 e_2 \cdot z_1 e_2 e_5 = z_1 e_3 \cdot z_1 e_2 e_3 = z_1 e_3 \cdot z_1 e_2 e_2, \]
\[ z_1 e_2 \cdot z_1 e_2 e_4 = z_1 e_4 \cdot z_1 e_4 e_2, \]
\[ z_1 e_3 \cdot z_1 e_3 e_5 = z_1 e_3 \cdot z_1 e_3 e_3, \]
\[ z_1 e_4 \cdot z_1 e_4 e_5 = z_1 e_5 \cdot z_1 e_5 e_4. \]

An easy simplification shows that for \( z_1 \) generators are \( z_1 e_2 e_4, \ z_1 e_3 e_5, \ z_1 e_4 e_5 \) and for \( z_1 \) there are no relations.

From \( z_2 \) we have

\[ z_2 = (e_1, e_2, e_3, e_4); (\overline{e}_1, \overline{e}_4, e_5), (\overline{e}_2, \overline{e}_5, \overline{e}_3), \]
\[ z_2^1 = (\overline{e}_4, e_5, e_2, e_3, e_4); (\overline{e}_2, \overline{e}_5, \overline{e}_3), \]
\[ z_2^2 = (e_1, \overline{e}_5, \overline{e}_3, e_4); (\overline{e}_1, \overline{e}_4, e_5), \]
\[ z_2^3 = (e_1, e_2, \overline{e}_3, e_4); (\overline{e}_1, e_4, e_5), \]
\[ z_2^4 = (e_1, e_2, e_3, e_5); (\overline{e}_2, \overline{e}_5, \overline{e}_3), \]
\[ z_2^5 = (e_1, e_2, e_3, e_4); (\overline{e}_1, \overline{e}_4, e_3, \overline{e}_2). \]
Generators for $z_2$ are:

$$z_2 e_1, z_2 e_2, z_2 e_3, z_2 e_4, z_2 e_5,$$

$$z_2 e_1 e_3, z_2 e_1 e_2, z_2 e_1 e_3,$$

$$z_2 e_2 e_1, z_2 e_2 e_5, z_2 e_2 e_4,$$

$$z_2 e_3 e_1, z_2 e_3 e_5, z_2 e_3 e_4,$$

$$z_2 e_4 e_2, z_2 e_4 e_3, z_2 e_4 e_5,$$

$$z_2 e_5 e_1, z_2 e_5 e_2, z_2 e_5 e_3, z_2 e_5 e_4.$$

We see $z_1^4 = g_{2,1}^{2,4} \cdot z_2^2$, $z_1^3 = g_{2,1}^{3,1} \cdot z_2^3$, $z_1^2 = g_{2,2}^{4,1} \cdot z_2^4$, $z_1^5 = g_{2,2}^{5,5} \cdot z_2^5$, where

$$g_{2,1}^{2,4} = \begin{cases}
  e_1 &\mapsto e_1, \\
  e_3 &\mapsto \overline{e}_3, \\
  e_4 &\mapsto \overline{e}_4, \\
  e_5 &\mapsto \overline{e}_5;
\end{cases}$$

$$g_{2,1}^{3,1} = \begin{cases}
  e_1 &\mapsto e_5, \\
  e_2 &\mapsto \overline{e}_2, \\
  e_4 &\mapsto e_4, \\
  e_5 &\mapsto \overline{e}_3;
\end{cases}$$

$$g_{2,2}^{4,1} = \begin{cases}
  e_1 &\mapsto \overline{e}_4, \\
  e_2 &\mapsto e_5, \\
  e_3 &\mapsto e_2, \\
  e_5 &\mapsto e_3;
\end{cases}$$

$$g_{2,2}^{5,5} = \begin{cases}
  e_1 &\mapsto e_1, \\
  e_2 &\mapsto e_2, \\
  e_3 &\mapsto e_3, \\
  e_4 &\mapsto e_4.
\end{cases}$$

Relations for $z_2$ are:

$$z_2 e_1 = 1, z_2 e_2 = 1,$$

$$z_2 e_2 e_1 = z_1 e_4 e_1, z_2 e_2 e_5 = z_1 e_4 e_2, z_2 e_2 e_4 = z_1 e_4 e_3,$$

$$z_2 e_3 e_1 = z_1 e_1 e_3, z_2 e_3 e_5 = z_1 e_1 e_3, z_2 e_3 e_4 = z_1 e_1 e_4,$$

$$z_2 e_4 e_2 = z_2 e_1 e_5, z_2 e_4 e_3 = z_2 e_1 e_2, z_2 e_4 e_4 = z_2 e_1 e_3,$$

$$z_2 e_5 e_1 = z_1 e_3 e_1, z_2 e_5 e_2 = z_1 e_3 e_2, z_2 e_5 e_3 = z_1 e_3 e_3, z_2 e_5 e_4 = z_1 e_3 e_4,$$

$$z_2 e_1 \cdot z_2 e_1 e_5 = z_2 e_5 \cdot z_2 e_5 e_1, z_2 e_1 \cdot z_2 e_1 e_2 = z_2 e_2 \cdot z_2 e_2 e_1,$$

$$z_2 e_1 \cdot z_2 e_1 e_3 = z_2 e_3 \cdot z_2 e_3 e_1,$$

$$z_2 e_2 \cdot z_2 e_2 e_5 = z_2 e_5 \cdot z_2 e_5 e_2, z_2 e_2 \cdot z_2 e_2 e_4 = z_2 e_4 \cdot z_2 e_4 e_2,$$

$$z_2 e_3 \cdot z_2 e_3 e_5 = z_2 e_5 \cdot z_2 e_5 e_3, z_2 e_3 \cdot z_2 e_3 e_4 = z_2 e_4 \cdot z_2 e_4 e_3,$$

$$z_2 e_4 \cdot z_2 e_4 e_5 = z_2 e_5 \cdot z_2 e_5 e_4.$$

An easy simplification shows no new generators are needed, the relations

$$z_1 e_4 e_5 = z_1 e_2 e_4 \cdot z_1 e_3 e_5 \cdot z_1 e_2 e_4, z_1 e_4 e_5 \cdot z_1 e_3 e_5 = z_1 e_2 e_4 \cdot z_1 e_4 e_5$$

are needed.

From $z_3$ we have

$$z_3 = (e_1, \overline{e}_1, e_2, e_3); (\overline{e}_2, e_4, e_5), (\overline{e}_3, e_5, \overline{e}_4),$$

$$z_3^2 = (e_1, \overline{e}_1, e_4, e_5, e_3); (\overline{e}_3, \overline{e}_5, \overline{e}_4),$$

$$z_3^3 = (e_1, \overline{e}_1, e_2, \overline{e}_5, \overline{e}_4); (\overline{e}_2, e_4, e_5),$$

$$z_3^4 = (e_1, \overline{e}_1, e_2, e_3); (\overline{e}_2, \overline{e}_3, \overline{e}_5, e_5),$$

$$z_3^5 = (e_1, \overline{e}_1, e_2, e_3); (\overline{e}_2, e_4, \overline{e}_4, \overline{e}_3).$$

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Generators for $z_3$ are:

$$
\begin{align*}
&z_3e_2, z_3e_3, z_3e_4, z_3e_5, \\
z_3e_2e_4, z_3e_2e_5, z_3e_2e_3, \\
z_3e_3e_2, z_3e_3e_5, z_3e_3e_4, \\
z_3e_4e_2, z_3e_4e_3, \\
z_3e_5e_2, z_3e_5e_3.
\end{align*}
$$

We see $z_4^2 = g_{3,1}^{2,2} \cdot z_3^2$, $z_4^3 = g_{3,1}^{3,2} \cdot z_3^3$, where

$$
\begin{align*}
g_{3,1}^{2,2} &= \begin{cases}
e_1 &\mapsto e_1, \\
e_3 &\mapsto e_4, \\
e_4 &\mapsto e_5, \\
e_5 &\mapsto e_3.
\end{cases}
&
g_{3,1}^{3,2} &= \begin{cases}
e_1 &\mapsto e_1, \\
e_2 &\mapsto e_5, \\
e_4 &\mapsto \overline{e}_4, \\
e_5 &\mapsto \overline{e}_3.
\end{cases}
\end{align*}
$$

Relations for $z_3$ are:

\begin{align*}
z_3e_2, z_3e_4 &= 1, z_3e_5 = 1, \\
z_3e_2e_4 &= z_1e_2e_5, z_3e_2e_5 = z_1e_2e_3, z_3e_2e_3 = z_1e_2e_4, \\
z_3e_3e_2 &= z_1e_2e_5, z_3e_3e_5 = z_1e_2e_3, z_3e_3e_4 = z_1e_2e_4, \\
z_3e_2 \cdot z_3e_2e_4 &= z_3e_4 \cdot z_3e_4e_2, z_3e_2 \cdot z_3e_2e_5 = z_3e_5 \cdot z_3e_5e_2, \\
z_3e_2 \cdot z_3e_2e_3 &= z_3e_3 \cdot z_3e_3e_2, \\
z_3e_3 \cdot z_3e_3e_5 &= z_3e_5 \cdot z_3e_5e_3, z_3e_3 \cdot z_3e_3e_4 = z_3e_4 \cdot z_3e_4e_3.
\end{align*}

An easy simplification shows that neither new generators nor new relations are needed.

From $z_4$ we have

\begin{align*}
z_4 &= (e_1, e_2, \overline{e}_2, e_3); (\overline{e}_1, e_4, e_5), (\overline{e}_4, e_5, \overline{e}_4), \\
z_4^1 &= (e_4, e_5, e_2, \overline{e}_2, e_3); (\overline{e}_3, \overline{e}_5, \overline{e}_4), \\
z_4^2 &= (e_1, e_2, \overline{e}_2, e_3); (\overline{e}_1, e_4, e_5), \\
z_4^4 &= (e_1, e_2, \overline{e}_2, e_3); (\overline{e}_1, \overline{e}_5, e_5), \\
z_4^5 &= (e_1, e_2, \overline{e}_2, e_3); (\overline{e}_1, e_4, \overline{e}_4, \overline{e}_3).
\end{align*}

Generators for $z_4$ are:

\begin{align*}
z_4e_1, z_4e_3, z_4e_4, z_4e_5, \\
z_4e_1e_4, z_4e_1e_5, z_4e_1e_3, \\
z_4e_3e_1, z_4e_3e_5, z_4e_3e_4, \\
z_4e_4e_1, z_4e_4e_3, \\
z_4e_5e_1, z_4e_5e_3.
\end{align*}
We see $z_1^4 = g_{4,1}^{1,4} \cdot z_1^4$, $z_1^3 = g_{4,1}^{3,1} \cdot z_4^3$, where

$$g_{4,1}^{1,4} = \begin{cases} e_2 & \mapsto e_3, \\ e_3 & \mapsto e_5, \\ e_4 & \mapsto e_1, \\ e_5 & \mapsto e_2; \end{cases} \quad g_{4,1}^{3,1} = \begin{cases} e_1 & \mapsto e_5, \\ e_2 & \mapsto e_2, \\ e_4 & \mapsto e_4, \\ e_5 & \mapsto e_3. \end{cases}$$

Relations for $z_4$ are:

- $z_4 e_4 = 1$, $z_4 e_4 = 1$, $z_4 e_5 = 1$,
- $z_4 e_1 e_4 = z_1 e_4 e_1$, $z_4 e_1 e_5 = z_1 e_4 e_2$, $z_4 e_1 e_3 = z_1 e_4 e_5$,
- $z_4 e_2 e_1 = z_1 e_1 e_5$, $z_4 e_5 e_5 = z_1 e_1 e_3$, $z_4 e_5 e_4 = z_1 e_1 e_4$,
- $z_4 e_1 \cdot z_4 e_1 e_4 = z_4 e_4 \cdot z_4 e_1 e_1$, $z_4 e_1 \cdot z_4 e_1 e_5 = z_4 e_5 \cdot z_4 e_1 e_1$,
- $z_4 e_1 \cdot z_4 e_1 e_3 = z_4 e_3 \cdot z_4 e_3 e_1$,
- $z_4 e_3 \cdot z_4 e_3 e_5 = z_4 e_5 \cdot z_4 e_5 e_3$, $z_4 e_3 \cdot z_4 e_3 e_4 = z_4 e_4 \cdot z_4 e_4 e_3$.

An easy simplification shows that neither new generators nor new relations are needed.

From $z_5$ we have

$$z_5 = (e_1, e_2, e_3, \overline{e}_3); (\overline{e}_1, e_4, e_5); (\overline{e}_2, \overline{e}_5, \overline{e}_4),$$

$$z_5^1 = (e_4, e_5, e_2, e_3, \overline{e}_3); (\overline{e}_2, \overline{e}_5, \overline{e}_4),$$

$$z_5^2 = (e_1, e_5, \overline{e}_4, e_3, \overline{e}_3); (\overline{e}_1, e_4, e_5),$$

$$z_5^4 = (e_1, e_2, e_3, \overline{e}_3); (\overline{e}_1, \overline{e}_2, \overline{e}_5, e_5),$$

$$z_5^5 = (e_1, e_2, e_3, \overline{e}_3); (\overline{e}_1, e_4, \overline{e}_4, \overline{e}_2).$$

Generators for $z_5$ are:

- $z_5 e_1$, $z_5 e_2$, $z_5 e_4$, $z_5 e_5$,
- $z_5 e_1 e_4$, $z_5 e_1 e_5$, $z_5 e_1 e_2$,
- $z_5 e_2 e_1$, $z_5 e_2 e_5$, $z_5 e_2 e_4$,
- $z_5 e_4 e_1$, $z_5 e_4 e_5$,
- $z_5 e_5 e_1$, $z_5 e_5 e_2$.

We see $z_1^3 = g_{5,1}^{1,3} \cdot z_1^1$, $z_1^3 = g_{5,1}^{2,3} \cdot z_5^2$, where

$$g_{5,1}^{1,3} = \begin{cases} e_2 & \mapsto \overline{e}_5, \\ e_3 & \mapsto \overline{e}_4, \\ e_4 & \mapsto e_1, \\ e_5 & \mapsto e_2; \end{cases} \quad g_{5,1}^{2,3} = \begin{cases} e_1 & \mapsto e_1, \\ e_3 & \mapsto \overline{e}_2, \\ e_4 & \mapsto e_5, \\ e_5 & \mapsto \overline{e}_4. \end{cases}$$
Relations for $z_5$ are:
\[
\begin{align*}
  z_5 e_1 &= 1, 
  z_5 e_4 = 1, 
  z_5 e_5 = 1, \\
  z_5 e_1 e_4 &= z_1 e_3 e_1, 
  z_5 e_1 e_5 = z_1 e_3 e_2, 
  z_5 e_1 e_2 = z_1 e_2 e_4, \\
  z_5 e_2 e_1 &= z_1 e_3 e_1, 
  z_5 e_2 e_5 = z_1 e_3 e_4, 
  z_5 e_2 e_4 = z_1 e_3 e_5, \\
  z_5 e_1 \cdot z_5 e_1 e_4 &= z_5 e_4 \cdot z_5 e_4 e_1, 
  z_5 e_1 \cdot z_5 e_1 e_5 &= z_5 e_5 \cdot z_5 e_5 e_1, \\
  z_5 e_1 \cdot z_5 e_1 e_2 &= z_5 e_2 \cdot z_5 e_3 e_1, \\
  z_5 e_2 \cdot z_5 e_2 e_5 &= z_5 e_5 \cdot z_5 e_5 e_2, 
  z_5 e_2 \cdot z_5 e_2 e_4 = z_5 e_4 \cdot z_5 e_4 e_2.
\end{align*}
\]

An easy simplification shows that neither new generators nor new relations are needed.

From $z_6$ we have
\[
\begin{align*}
  z_6 &= (e_1, e_2, e_3, \bar{e}_1); (e_2, e_4, e_5), (\bar{e}_3, \bar{e}_5, \bar{e}_4), \\
  z_6^2 &= (e_1, e_4, e_5, e_3, \bar{e}_1); (\bar{e}_3, \bar{e}_5, \bar{e}_2), \\
  z_6^3 &= (e_1, e_2, \bar{e}_5, \bar{e}_4); (\bar{e}_2, e_4, e_5), \\
  z_6^4 &= (e_1, e_2, e_3, \bar{e}_1); (\bar{e}_2, \bar{e}_3, \bar{e}_5, e_5), \\
  z_6^5 &= (e_1, e_2, e_3, \bar{e}_1); (\bar{e}_2, e_4, \bar{e}_4, \bar{e}_3).
\end{align*}
\]

Generators for $z_6$ are:
\[
\begin{align*}
  z_6 e_2, 
  z_6 e_3, 
  z_6 e_4, 
  z_6 e_5, \\
  z_6 e_2 e_4, 
  z_6 e_2 e_5, 
  z_6 e_2 e_3, \\
  z_6 e_3 e_2, 
  z_6 e_3 e_5, 
  z_6 e_3 e_4, \\
  z_6 e_4 e_2, 
  z_6 e_4 e_3, \\
  z_6 e_5 e_2, 
  z_6 e_5 e_3.
\end{align*}
\]

We see $z_6^4 = g_{6,2}^4 \cdot z_6^2$, $z_6^4 = g_{6,2}^3 \cdot z_6^3$, where
\[
\begin{align*}
  g_{6,2}^4 &= \begin{cases}
    e_1 &\mapsto e_1, \\
    e_3 &\mapsto e_5, \\
    e_4 &\mapsto e_2, \\
    e_5 &\mapsto e_3
  \end{cases}, \\
  g_{6,2}^3 &= \begin{cases}
    e_1 &\mapsto e_1, \\
    e_2 &\mapsto e_2, \\
    e_4 &\mapsto \bar{e}_5, \\
    e_5 &\mapsto \bar{e}_3
  \end{cases}.
\end{align*}
\]

Relations for $z_6$ are:
\[
\begin{align*}
  z_6 e_2, 
  z_6 e_4 = 1, 
  z_6 e_5 = 1, \\
  z_6 e_2 e_4 &= z_2 e_4 e_2, 
  z_6 e_2 e_5 = z_2 e_4 e_3, 
  z_6 e_2 e_3 = z_2 e_4 e_5, \\
  z_6 e_3 e_2 &= z_2 e_4 e_2, 
  z_6 e_3 e_5 = z_2 e_4 e_3, 
  z_6 e_3 e_4 = z_2 e_4 e_5, \\
  z_6 e_2 \cdot z_6 e_2 e_4 &= z_6 e_4 \cdot z_6 e_4 e_2, 
  z_6 e_2 \cdot z_6 e_2 e_5 = z_6 e_5 \cdot z_6 e_5 e_2, \\
  z_6 e_2 \cdot z_6 e_2 e_3 &= z_6 e_3 \cdot z_6 e_3 e_2, \\
  z_6 e_3 \cdot z_6 e_3 e_5 &= z_6 e_5 \cdot z_6 e_5 e_3, 
  z_6 e_3 \cdot z_6 e_3 e_4 = z_6 e_4 \cdot z_6 e_4 e_3.
\end{align*}
\]
An easy simplification shows that neither new generators nor new relations are needed.

We conclude

\[
\mathcal{AM}_{0,1,3} = \left\langle z_1 e_2 e_4, z_1 e_3 e_5, z_1 e_4 e_5 \mid z_1 e_4 e_5 = z_1 e_2 e_4 \cdot z_1 e_3 e_5 \cdot z_1 e_2 e_4, \\
z_1 e_4 e_5 \cdot z_1 e_3 e_5 = z_1 e_2 e_4 \cdot z_1 e_4 e_5 \right\rangle
\]

\[
= \left\langle z_1 e_2 e_4, z_1 e_3 e_5 \mid z_1 e_3 e_5 \cdot z_1 e_2 e_4 \cdot z_1 e_3 e_5 = z_1 e_2 e_4 \cdot z_1 e_3 e_5 \cdot z_1 e_2 e_4 \right\rangle.
\]

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