The Secondary Stars of Cataclysmic Variables

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Abstract
I review what we know about the donor stars in cataclysmic variables (CVs), focusing particularly on the close link between these binary components and the overall secular evolution of CVs. I begin with a brief overview of the “standard model” of CV evolution and explain why the key observables this model is designed to explain – the period gap and the period minimum – are intimately connected to the properties of the secondary stars in these systems.

CV donors are expected to be slightly inflated relative to isolated, equal-mass main-sequence (MS) stars, and this “donor bloating” has now been confirmed observationally. The empirical donor mass-radius relationship also shows a discontinuity at $M_2 \approx 0.2 M_\odot$, which neatly separates long- and short-period CVs. This is strong confirmation of the basic disrupted magnetic braking scenario for CV evolution. The empirical $M_2 - R_2$ relation can be combined with stellar models to construct a complete, semi-empirical donor sequence for CVs. This sequence provides all physical and photometric properties of “normal” CV secondaries along the standard CV evolution track.

The observed donor properties can also be used to reconstruct the complete evolution track followed by CVs, i.e. the mass-transfer rate and angular-momentum-loss rate as a function of orbital period. Such a reconstruction suggests that angular momentum loss rates below the period gap are too high to be driven solely by gravitational radiation.

1. Introduction
The last decade or so has seen tremendous progress in our understanding of cataclysmic variables (CVs), particularly as it relates to the evolution of these binary systems. Several of the key breakthroughs have been connected to the properties of the secondary stars in these systems. In this brief review, I will take stock of what we expect theoretically from and know observationally about CV donor stars. I will also explain how theory and observation can be powerfully combined to define a benchmark semi-empirical “CV donor sequence” and even to reconstruct the entire evolutionary path followed by CVs.

2. The Evolution of Cataclysmic Variables: A Primer
Figure 1 shows the orbital period distribution of CVs as of 2006. Two key features are immediately obvious: (i) the famous CV “period gap” between $P_{gap,-} \approx 2$ hrs and $P_{gap,+} \approx 3$ hrs; (ii) the period minimum around $P_{min} \approx 80$ min. The most important requirement for any successful model of CV evolution is that it must provide a natural explanation for the origin and location of these features.

In what has become the “standard model” of CV evolution, the period gap is interpreted as signalling a switch in the dominant angular momentum loss (AML) mechanism. More specifically, the idea is that, above the period gap, CV evolution is driven mainly by “magnetic braking” (MB), i.e. by AML associated with a weak stellar wind from the donor star. This ionized wind is forced to co-rotate with the donor’s magnetic field out to the Alfvén radius, where the ram pressure in
Fig. 1. Differential and cumulative orbital period distribution of CVs, based on data taken from Edition 7.6 of the Ritter & Kolb catalogue [1]. Estimated values for the minimum period and the period gap edges are shown as vertical lines. The shaded regions around them indicate our estimate of the errors on these values. Figure reproduced from [2].

the outflow becomes equal to the pressure associated with the magnetic field. It turns out that the donor mass at the upper edge of the period gap corresponds roughly to the mass where the donor is expected to transition from a star with a radiative core to a fully convective object \( M_2 \approx 0.2 - 0.3 \, M_\odot \), see Section 3.1). The standard model thus posits that this transition effectively shuts down the magnetic field on the secondary and hence also disrupts MB. The physical justification for this idea is that the transition region between the radiative core and the convection zone – the so-called “tachocline” – is the location where the magnetic fields are anchored in many magnetic dynamo models for low-mass stars.

In the “vanilla-flavoured” standard model, MB ceases completely at the upper edge of the period gap, leaving only gravitational radiation (GR) to drive the further evolution of CVs. Why does such a switch in the AML rate produce a period gap? As it turns out, the answer is entirely associated with the properties of the donor star at the time of the switch and will be discussed in detail in Section 3.3. For now, let us just note that the key difference between MB and GR is that, at least according to the standard model, the former produces much higher AML and mass-transfer rates, by a factor of \( \geq 10 \) in the vicinity of the period gap.

Turning to the minimum period, this is often described as being associated with another transition of the donor, this time from a Hydrogen-burning star to a sub-stellar object. The point here is that stars generally have a positive mass-radius index, whereas sub-stellar objects with masses below the hydrogen-burning limit \( M_{H} \approx 0.07 \, M_\odot \) have a negative one. Thus, so long as \( M_2 > M_H \), the donor radius, binary orbit, and orbital period are all expected to decrease in response to the mass loss the donor experiences, but all three quantities are expected to increase once \( M_2 < M_H \). We can therefore expect the condition \( M_2 \approx M_H \) to set the minimum period a CV can reach.
One of the key points to take away from the discussion above is just how intimately CV evolution is tied up with the properties of the donor stars in these systems. Consider: evolution above the gap is thought to be driven by a magnetic stellar wind from the secondary, the gap itself is thought to be associated with the transition of the secondary to a fully convective state, and the period minimum is set by the transition of the secondary into a sub-stellar object. In the following section, I will take a closer look at the physical link between CV secondaries and binary evolution. This will allow us to understand more precisely how the period gap is produced and also remind us that the period minimum is not necessarily associated with the stellar to sub-stellar transition of the donor.

3. The Physics of CV Secondaries

3.1. Fundamentals

The radius of a Roche-lobe-filling star depends only the binary separation, \( a \), and the mass ratio, \( q = M_2/M_1 \). A particularly convenient approximation for the Roche-lobe radius is

\[
\frac{R_L}{a} = \frac{2}{3^{1/3}} \left[ \frac{q}{1 + q} \right]^{1/3},
\]

which can be combined with Kepler’s third law

\[
P_{\text{orb}}^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}
\]

to yield the well-known period-density relation for Roche-lobe-filling stars with \( R_2 \approx R_L \)

\[
\langle \rho_2 \rangle = \frac{M_2}{(4\pi/3)R_2^3} \approx 100 G^{-1} P_{\text{orb}}^{-2}.
\]

If we are allowed to assume that donors are mostly low-mass, near-MS stars, we expect that their mass-radius relationship will be roughly

\[
R_2/R_\odot = f(M_2/M_\odot)^{\alpha}
\]

with \( f \approx \alpha \approx 1 \). Combining this with the period-density relation immediately gives us approximate mass-period and radius-period relations for CV donors

\[
M_2/M_\odot = M_2/M_\odot \approx 0.1 P_{\text{orb,hr}},
\]

where \( P_{\text{orb,hr}} \) is the orbital period in units of hours. This shows that, indeed, the period gap between 2 hrs and 3 hrs corresponds roughly to the expected transition of the secondary from partly radiative to fully convective, i.e. \( M_2 \approx 0.2 - 0.3 M_\odot \).

3.2. Are CV Donors on the Main Sequence?

So far, we have assumed that CV donors can be thought of as ordinary main-sequence stars. But is this actually true? The key question here is whether the mass loss the donor experiences should be expected to affect its overall stellar properties. The answer to this question depends on the competition between two time scales. First, there is the mass-loss time scale,

\[
\tau_{M_2} \approx \frac{M_2}{\dot{M}_2}
\]
Fig. 2. The mass-radius relation for mass-losing stars for several ratios of $\tau_{th}/\tau_{\dot{M}}$, as indicated. Note that the notation in the figure legend is slightly different from that used in the text, but should be obvious: $\tau_{KH} = \tau_{th}$ and $\tau_{M} = \tau_{\dot{M}}$. Figure reproduced from [4].

which is the time scale on which the ongoing mass transfer reduces the donor mass. Second, we have the donor’s thermal time scale,

$$\tau_{th} \simeq \frac{GM_2^2}{L_2 R_2} \simeq 10^8 (M_2/M_\odot)^{-3/2} \text{yrs}, \quad (7)$$

which is the time scale on which the donor can correct deviations from thermal equilibrium (TE).

If mass loss is slow, in the sense that $\tau_{\dot{M}_2} >> \tau_{th}$, the donor always has time to adjust itself to attain the appropriate TE structure for its current mass. It therefore remains on the MS, with a mass-radius index $\alpha \simeq 1$, and is essentially indistinguishable from an isolated MS star.

Conversely, if mass loss is fast, i.e. $\tau_{\dot{M}_2} << \tau_{th}$, the donor cannot adjust its structure quickly enough to remain in TE. Instead, the mass loss is effectively adiabatic. The response of low-mass stars with at least a substantial convective envelope to such mass loss is to expand, with $\alpha \simeq -1/3$.

So which of these limits is appropriate for CVs? Neither, as it turns out. Let us take some typical parameters suggested by the standard model for CVs above and below the gap, say $M_2 \simeq 0.4$ with $M_2 \simeq 1 \times 10^{-9}$ and $M_2 \simeq 0.1$ with $M_2 \simeq 3 \times 10^{-11}$, respectively. Plugging these values into Equations 6 and 7 we find $\tau_{\dot{M}_2} \simeq \tau_{th} \simeq 4 \times 10^{8}$ yrs above the gap and $\tau_{\dot{M}_2} \simeq \tau_{th} \simeq 3 \times 10^{9}$ yrs below. Thus the thermal and mass-loss time scales are expected to be comparable for CV donors, both above and below the period gap.

What does this mean for the response of the secondary to the mass loss it experiences? The answer is simply that the donor cannot shrink quite fast enough to keep up with the rate at which mass is removed from its surface. As a result, it is driven slightly out of thermal equilibrium and becomes somewhat oversized for its mass. This is nicely illustrated by Figure 2, taken from [4], which shows how the donor mass-radius relationship depends on $\tau_{th}/\tau_{\dot{M}_2}$, when this ratio is assumed to be constant along the evolution track.
3.3. The Importance of Being Slightly Disturbed

So CV donors are almost, but not quite, in TE. Does this slight deviation from TE actually matter? Yes it does. In fact, it is this deviation that is responsible for producing both the period gap and the period minimum. Let us see how this works.

We will take as given, for the moment, that the period gap is “somehow” associated with a sudden cessation of (or at least reduction in) MB at $P_{\text{orb}} \approx 3$ hrs.

Recall that the donor star is slightly out of thermal equilibrium – i.e. slightly bloated – as it encounters the upper edge of the period gap. Now, since mass transfer in CVs is driven entirely by AML, a sudden reduction in AML will also result in a sudden reduction in the mass-loss rate the donor experiences. This lower mass-loss rate cannot sustain the same degree of thermal disequilibrium and inflation in the secondary star. The donor therefore responds to this change by shrinking closer to its thermal equilibrium radius. However, this shrinkage almost immediately causes a total loss of contact between the stellar radius and the Roche lobe. The reason for this is that, in the semi-detached configuration, the stellar and Roche-lobe radii match extremely closely, to within $|R_2 - R_L| \approx H$, where $H/R_2 \approx 10^{-4}$ is the exponential scale-height in the donor’s envelope.

As a result, even a small reduction in radius (so long as it is greater than $\approx H$), will cause total loss of contact on a time-scale of roughly $(H/R_2)\tau_{th} \sim 10^4$ yrs.

The origin of the period gap in the standard model is now clear: a CV approaches the upper edge of the gap with a slightly bloated donor star. The (assumed) cessation of MB at $P_{\text{orb}} \approx 3$ hrs – associated, perhaps, with the transition of the donor to a fully convective state – then leads to a reduction in the mass-loss rate from the donor, which in turn causes the donor to shrink and lose contact with the Roche lobe altogether. The upper edge of the gap thus marks a cessation of mass transfer in CVs. According to the standard model, CVs then evolve through the period gap as detached systems. During this detached phase, the binary orbit and Roche lobe continue to shrink, since there is still ongoing AML due to GR. However, provided the thermal relaxation of the donor is faster than the shrinkage of the Roche lobe, the donor manages to relax all the way back to its TE radius. In practice, this condition is met, so long as the AML rate is reduced by at least a factor 5-10 at the upper gap edge. The bottom edge of the period gap then corresponds to the location where the Roche lobe radius catches up once again to the TE radius of the donor. At this point, mass transfer restarts, and the system emerges from the gap as an active CV once more.

How bloated must CV donors be to account for the observed size of the period gap? Since there is no mass transfer in the gap, the donor mass just above and below the gap must be the same, $M_2(P_{\text{gap,}+}) = M_2(P_{\text{gap,}^-})$. From the period-density relation (Equation 3), we then get

$$\frac{R_2(P_{\text{gap,}+})}{R_2(P_{\text{gap,}^-})} = \left(\frac{P_{\text{gap,}+}}{P_{\text{gap,}^-}}\right)^{2/3} \approx \left[\frac{3}{2}\right]^{2/3} \approx 1.3.$$ (8)

We also know that the donor at the bottom edge is in or near equilibrium, so we conclude that donors at the upper edge of the period gap must be oversized by $\approx 30\%$ relative to equal-mass, isolated MS stars.

Let us now take a closer look at the minimum period for CVs. As it turns out, this, too, is closely connected to the properties of the donor stars in these sytems. If we combine the period-density relation (Equation 3) with the simple power-law approximation to the donor mass-radius relation (Equation 4), we find

$$P_{\text{orb}}^{-2} \propto M_2^{1-3\alpha}.$$ (9)
Differentiating this logarithmically yields a simple expression for the orbital period derivative, i.e.
\[ \frac{\dot{P}_{\text{orb}}}{P_{\text{orb}}} = \frac{3\alpha - 1}{2} \frac{M_2}{M_2} . \]  

Since the period minimum must correspond to \( \dot{P}_{\text{orb}} = 0 \), Equation 10 tells us that \( P_{\text{min}} \) occurs when the donor has been driven so far out of thermal equilibrium that its mass-radius index along the evolution track has been reduced from its near-MS value of \( \alpha \simeq 1 \) to \( \alpha = 1/3 \). So, as already noted above, \( P_{\text{min}} \) does not necessarily have to coincide with the orbital period at which the donor mass reaches \( M_H \). In fact, recall that we noted in Section 3.1 that, for any donor with at least a substantial convective envelope, the mass-radius index in the limit of fast (adiabatic) mass-transfer is \( \alpha \simeq -1/3 \). Thus the period evolution of a CV can in principle be made to turn around at any donor mass, provided only that mass loss becomes sufficiently fast compared to the donor’s thermal time scale. The significance of \( M_H \) in this context is that period bounce becomes inevitable when the donor reaches this limit. This is because sub-stellar objects are out of TE by definition and respond even to slow mass loss by increasing in radius, i.e. \( \alpha \leq 0 \). And, in practice, \( P_{\text{min}} \) does, in fact, correspond roughly to \( M_2 \simeq M_H \).

4. Are CV Donors Observationally Distinguishable from MS Stars?

We have seen in the previous section that we expect CV donors to be significantly larger than equal-mass, isolate MS stars. Can we actually observe this difference?

Yes, we can. Figure 3 shows the empirical mass-radius relationship of CV donors, as first constructed by [6]. The figure shown is actually taken from [2].
but this is only a minor modification of the original relation in [6] and is still based on the same data.

Figure 3 represents a major break-through in our understanding of CV donors and evolution. First, it definitively confirms the theoretical expectation that CV donors are larger than ordinary MS stars, both above and below the period gap. Second, the donor mass-radius indices above and below the gap are just what the doctor ordered: they are less than the MS-based (TE) values, but greater than the critical value of \( \alpha = 1/3 \). Third, the period bouncer regime is very poorly constrained by the data, but we do find \( M_2 \approx M_H \) near \( P_{\text{min}} \), as well as \( \alpha < 1/3 \) for the lowest mass donors below this limit. Again, this is nicely in line with our theoretical expectations.

However, the single most important aspect of Figure 3 is that it reveals a discontinuity in the donor radii at \( M_2 \approx 0.2 M_\odot \) that neatly separates long-period CVs from short-period ones. Moreover, on the low-mass (short-period) side of the discontinuity, \( R_2 \) is very close to the MS (thermal equilibrium) radius for this mass. This is by far the strongest evidence to date that the basic “disrupted MB” scenario for CV evolution is fundamentally correct.

5. A Complete Semi-Empirical Donor Sequence for CVs

Virtually all observable properties of CV donors depend on just three physical parameters: \( M_2, R_2 \) and \( T_{\text{eff},2} \). For example, the total luminosity of the donor is \( L_2 = 4\pi R_2^2 \sigma T_{\text{eff},2}^4 \); its spectral energy distribution depends primarily on \( T_{\text{eff},2} \) and \( \log g_2 = \log \left( \frac{GM_2}{R_2^2} \right) \), and, with the SED fixed, its flux in any particular wave-band depends only on \( R_2 \). Now the empirical \( M_2 - R_2 \) relation in Figure 3 gives us a unique relationship between two of these three key donor parameters. If we could find just one additional relationship between either of these two parameters and \( T_{\text{eff},2} \), we would essentially know all there is to know about “ordinary” CV donors.

As it turns out, there is such a relationship, albeit a theoretical one. The key insight is that the low-mass donors we care about all have large convective envelopes. As a result, their effective temperature is almost completely independent of luminosity and instead depends only on mass [8, 9]. Thus CV donors are expected to obey a standard MS \( M_2 - T_{\text{eff},2} \) relationship, which can be taken from state-of-the-art stellar models. By combining the empirical \( M_2 - R_2 \) relation with the theoretical \( M_2 - T_{\text{eff},2} \) one, we can then construct a complete, semi-empirical donor sequence for CVs that defines all physical and photometric properties of CV secondaries as a function of \( P_{\text{orb}} \).

This program was carried out in [2], and some key aspects of the resulting donor sequence are shown in Figure 4. This sequence is useful for at least two reasons: first, it helps to define precisely what we mean by a “normal” CV, i.e. it provides a useful benchmark for theoretical and observational studies of CV evolution. Second, it provides an immediate estimate of the expected donor brightness for any CV in any desired waveband, given only an estimate of the system’s \( P_{\text{orb}} \). This can be used, for example, to set limits on distances to CVs, via the method of photometric parallax.

6. Reconstructing CV Evolution from Donor Properties

The ultimate goal of essentially all work on CV evolution is to determine the AML rate as a function of \( P_{\text{orb}} \), i.e. to find the correct form of \( \dot{J}(P_{\text{orb}}) \), or, equivalently, \( \dot{M}_2(P_{\text{orb}}) \). The reason this is the “holy grail” is that such a recipe would allow us to calculate/predict almost everything there is to know about CVs, from the complete set of binary properties of individual systems at given \( P_{\text{orb}} \), to the population properties of large samples of CVs (e.g. their orbital period distribution). In reality, there would, of course, still be complications –
for example, we would still have to worry about selection effects \cite{16} – but it is nevertheless true that “understanding CV evolution” is broadly synonymous with “knowing $\dot{J}(P_{\text{orb}})$ and/or $\dot{M}_2(P_{\text{orb}})$”.

It may seem that we have already extracted a lot of information from the empirical $M_2 - R_2$ relation in constructing our semi-empirical donor sequence. However, we can actually push things even further and use the $M_2 - R_2$ relation to reconstruct the full evolution path of CVs, i.e. to determine $\dot{M}_2(P_{\text{orb}})$ and hence $\dot{J}(P_{\text{orb}})$. It is actually easy to see that this should be possible: after all, the degree of thermal disequilibrium and radius inflation a donor experiences increases with increasing mass-loss rate (see, for example, Figure 2.). Thus $R_2(P_{\text{orb}})$ is a direct tracer of $\dot{M}_2(P_{\text{orb}})$.

The great advantage of donor-based $\dot{M}_2(P_{\text{orb}})$ and $\dot{J}(P_{\text{orb}})$ determinations is that they are very likely to trace the true secular (i.e. long-term) rates. This is primarily because the time scale on which the donor can adjust its radius is long compared to the averaging time scales inherent in essentially all other methods \cite{10, 11, 12, 13}. The main difficulty, on the other hand, is that one has to carefully correct for other effects that might cause (or masquerade as) donor-bloating, such as stellar activity, imperfect stellar models, tidal/rotational deformation and irradiation.

Figure 6. (taken from \cite{14}) shows the results of an attempt to deal with these complications and derive a complete, purely donor-based evolution track for CVs. The observed $M_2 - R_2$ points are the same as in Figure 5, but the figure now also shows two self-consistent CV evolution tracks superposed on this data set. The thin black line shows the track predicted by the standard model, in which
Fig. 5. The observed donor $M_2 - R_2$ relation compared to the theoretical relations predicted by two self-consistent CV evolution tracks. The same symbols are used for the data points as in Figure 3, except that eclipsers are now indicated by open diamonds. The black dash-dotted line shows the $M_2 - R_2$ relation predicted by the standard model for CV. In this model, AML above the period gap is assumed to follow a standard “RVJ-like” MB law with $\gamma = 3$ [15], while AML below the gap is assumed to be driven solely by GR. By contrast, the red line shows the best-fitting $M_2 - R_2$ relation, if we allow the strength of AML to deviate from the standard prescription. Figure adapted from [14].

MB is assumed to be at full strength above the gap, and switches off completely at $P_{\text{gap,}+}$. By contrast, the thick red line shows the optimal fit to the data, which requires approximately $0.65 \times$ the standard MB rate above the gap, but $2.5 \times$ the GR-driven AML rate below.

Depending on one’s point of view, this best-fit model is either a minor modification of the standard model (in the sense that it is still a straightforward implementation of the basic disrupted MB scenario) or a major departure from conventional CV wisdom (in the sense that it requires AML rates significantly in excess of GR below the period gap). What is clear, however, is that the revised model fits the donor data substantially better than the standard one.

The revised model may also resolve two long-standing problems in our understanding of CV evolution. First, the minimum period predicted by the standard model is considerably shorter than is observed [17]. By contrast, we find in [14] that the revised model does an excellent job of matching the observed location of $P_{\text{min}}$ [18]. Second, observations suggest that the number of long-period CVs relative to short-period, pre-bounce CVs is higher than predicted by the standard model, by at least a factor of 3 [16, 19, 20]. In the revised model, the combination of enhanced AML rates below the gap and reduced rates above the gap increases the predicted ratio by just this factor [14].
7. Summary and Conclusions

Our understanding of CV secondary stars, as well as their relation to CV evolution, has improved dramatically over the last decade or so. Perhaps most importantly, we now know empirically that CV donors are oversized relative to equal-mass MS stars, and also that their mass-radius relation has a discontinuity at $M_2 \simeq 0.2M_\odot$ that separates short-period and long-period CVs. All of this is strong confirmation of the basic disrupted MB scenario for CV evolution.

By combining the observed $M_2 - R_2$ relation with a theoretical $M_2 - T_{eff,2}$ one, we have also been able to construct a complete, semi-empirical “donor sequence” for CVs that provides all physical and photometric parameters of CV secondaries as a function of only $P_{orb}$.

Finally, we can even use the observed $M_2 - R_2$ relation to reconstruct the entire evolutionary path of CVs, that is to say $J(P_{orb})$ and $M_2(P_{orb})$. This is possible because the degree of donor inflation is a direct measure of its mass-loss rate (and hence of the AML rate from the system). A first attempt to implement this idea suggests that the MB-driven AML rate above the period gap is slightly lower than is usually assumed, but that an AML rate in excess of GR (by a factor of $\simeq 2.5$) is required to match the observed mass-radius relation. This revised model may also resolve two long-standing problems in CV evolution: the mis-match between the observed and predicted location of $P_{min}$, and the higher-than-expected ratio of long-period CVs to short-period, pre-bounce CVs.

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