ABSTRACT We study the geometry of determinant line bundles associated to Dirac operators on compact odd dimensional manifolds. Physically, these arise as (local) vacuum line bundles in quantum gauge theory. We give a simplified derivation of the commutator anomaly formula using a construction based on noncyclic trace extensions and associated nonmultiplicative renormalized determinants.

1. INTRODUCTION

The use of noncyclic renormalized traces has been implicit in perturbative quantum field theory for a very long time. The 1-loop Feynman diagrams can be thought of as traces of pseudodifferential operators (PSDO’s) which are composed of products of inverses of (free) Dirac operators or Laplacians and multiplication operators defined by external fields. However, typically Feynman diagrams are ultraviolet divergent which reflects the fact that the pseudodifferential operator is not of trace class. Instead, one has to introduce an infinite renormalization in the form of cutoff dependent subtractions, dimensional regularization, or some other more or less standard method. This means that one is extending the trace to a family of nontraceclass operators.

For renormalized traces the basic property $\text{tr}(AB) = \text{tr}(BA)$ is usually lost. Likewise, if we try to define a determinant by the use of the formula $\det(e^A) = e^{\text{tr}(A)}$, the renormalized determinant is plagued by the multiplicative anomaly, $\det(AB) \neq \det(A) \cdot \det(B)$. The multiplicative anomaly has been extensively studied in mathematics literature, [KV],[F],[D],[O]; see also [PR] for related matter on renormalized traces and the geometry of determinant line bundles associated to families of Dirac operators. There has been a discussion in physics literature on the role of the multiplicative anomaly for computations of effective actions; see e.g. [CZ], [ECZ], [EFVZ].

Here we shall discuss another application of noncyclic trace extensions in quantum field theory. We shall study the determinant bundles and their relation to hamiltonian gauge anomalies along the lines of the paper [MR]. The important
difference as compared to [MR] is that we shall extensively take advantage of the special features of the trace calculus for pseudodifferential operators. The setting differs from [PR]; here we are dealing with line bundles associated to zero order pseudodifferential operators (signs of Dirac Hamiltonians), in contrast to positive order operators and the determinant calculus is geared to deal with operators having principal symbol equal to a unit matrix; this leads to some changes and simplifications in the calculations.

In [MR] a general abstract formula for the gauge commutator anomaly was derived, and it was only later shown that it was equivalent with a local anomaly formula, [LM]. The main result of the present paper is a construction of the determinant line bundle and the gauge group action in such a way that it leads directly to a local expression, including the local Schwinger terms for current commutators, compatible with standard perturbative QFT calculations.

The results below give a concrete geometric realization of a more general property of BRST cocycles in quantum field theory models based on noncommutative geometry. As shown in [LMR], the BRST cocycles (including commutator anomalies) can typically be expressed as generalized traces of sums of commutators of operators; in the case of PSDO’s this leads always to local expressions involving integrals of de Rham forms.

We want to thank the referee for a careful reading of the manuscript and suggestions leading to improvements in the original version.

2. RENORMALIZED DETERMINANTS

Let \( T = 1 + X \) be a pseudodifferential operator (PSDO) acting on sections of a hermitean vector bundle \( E \) over a compact closed manifold \( M \) such that the order of \( X \) is less or equal to \(-1\). Recall first the definition of generalized Fredholm determinants [S], [MR]. If \( \dim M = d \) then the operator \( X \) belongs to the Schatten ideal \( L_p \) of operators in the Hilbert space \( L^2(E) \) of square integrable sections for any \( p > d \). The ideal \( L_p \) consists of all bounded operators \( A \) such that \(|A|^p \) is a trace-class operator. Fix an integer \( p > d \). We can define a generalized determinant by

\[
\det_p(1 + X) = \det((1 + X)e^{-X + \frac{1}{2}X^2 + \cdots + (-1)^{p-1}\frac{1}{p!}X^{p-1}}),
\]

where the determinant on the right-hand-side is an ordinary Fredholm determinant, which converges since the log of the operator argument is a trace-class operator. This generalized determinant is nonzero if and only if \( 1 + X \) is invertible, but it is not multiplicative; instead, one has

\[
\det_p((1 + X)(1 + Y)) = \det_p(1 + X) \cdot \det_p(1 + Y) \cdot e^{\gamma_p(X,Y)},
\]

where \( \gamma_p \) is a polynomial of order \( p \). In the case \( p = 1 \) we have \( \det_1 = \det \), which is multiplicative, for \( p = 2 \) one has \( \gamma_2(X,Y) = -\text{tr}(XY) \).
The problem with the above determinant is that in general \( \det_p(1+X) \neq \det(1+X) \) even when \( X \) is a trace-class operator.

We introduce an improved renormalized determinant \( \det_{\text{ren}} \) such that
\[
\det_{\text{ren}}(e^X) = e^{\text{TR}(X)},
\]
where \( \text{TR} \) is a generalized (noncyclic) trace. The definition of \( \text{TR} \) involves a choice of a positive PSDO \( Q \) of positive order \( q \). We set
\[
(2.3) \quad \text{TR}(A) = \text{TR}_Q(A) = \lim_{z \to 0} \text{tr} \left( Q^{-z}A - \frac{1}{qz} \text{res}(A) \right),
\]
where \( \text{res}(A) \) is the Wodzicki residue of the PSDO \( A \). The limit at \( z = 0 \) gives the finite part of the singular trace (with poles at integer points \( z \)). The trace \( \text{TR} \) has the important property that it agrees with the ordinary trace \( \text{tr} \) whenever \( A \) is a trace-class pseudodifferential operator. This trace has been studied in detail in [MN], [CDMP]. In particular, they proved the important formula
\[
(2.4) \quad \text{TR}_Q[A, B] = -\frac{1}{q} \text{res}([\log |Q|, A]B).
\]

Here \( \text{res} \) is the Wodzicki residue; it is defined as
\[
\text{res}(A) = \frac{1}{(2\pi)^d} \int_M dx \int_{|p|=1} \text{tr} a_{-d}(x, p),
\]
where \( a_{-d} \) is the homogeneous term of order \(-d\) in the momenta, in the asymptotic expansion of the (matrix valued) symbol of \( A \).

A particular case of this was used earlier in a study of QFT anomalies, [M], [LM2]. In that case it was assumed that the symbols are globally defined, with supports of the \( x \) coordinate in a compact subset of \( \mathbb{R}^d \) so that one can define a cut-off trace
\[
(2.5) \quad \text{tr}_{\Lambda}(A) = \frac{1}{(2\pi)^d} \int_{|p| \leq \Lambda} \text{tr} a(x, p) dx dp,
\]
which has an asymptotic expansion
\[
(2.6) \quad \text{tr}_{\Lambda}(A) = \sum_{i=1}^{k} \Lambda^{i} \alpha_i(A) + \alpha_0(A) + \log(\Lambda)\text{res}(A) + \sum_{i=-1}^{-\infty} \Lambda^{i} \alpha_i(A).
\]

The renormalized trace is then the finite term \( \alpha_0(A) \). It agrees with the earlier definition when we take \( Q = -\Delta \), the Laplace operator in \( \mathbb{R}^d \).

An important consequence of (2.4) is that the renormalized trace of a commutator is local in the sense that it is given in terms of integrals of finite number of symbols in the asymptotic expansion of the pseudodifferential operators involved.
We now set
\[
\text{(2.7)} \quad \det_{\text{ren}}(1 + X) = \det_p(1 + X) \cdot e^{\text{TR}(X - \frac{1}{2}X^2 + \cdots + (-1)^{p-1}X^{p-1})}
\]
with \(p\) any integer strictly larger than \(d\); the value of the determinant does not depend on the choice of \(p\). From the definition follows immediately that
\[
\text{(2.8)} \quad \det_{\text{ren}}(e^X) = e^{\text{TR}(X)}
\]
for any pseudodifferential operator \(X\) of order \(-1\). In particular, when \(\text{ord}X < -d\) we have \(\det_{\text{ren}}(e^X) = \det(e^X)\). The reason we preferred to use (2.7) as a definition instead of the simpler formula (2.8) is that the right-hand-side of (2.7) is manifestly independent of the choice of logarithm of \(A = 1 + X\) whereas when using (2.8) one has give the (not very difficult) proof of it.

**Proposition 1.** (Multiplicative anomaly). Let \(X,Y\) be a pair of PSDO’s of order \(-1\). Then
\[
\det_{\text{ren}}((1 + X)(1 + Y)) = \det_{\text{ren}}(1 + X) \cdot \det_{\text{ren}}(1 + Y) \cdot e^\gamma(X,Y),
\]
where \(\gamma(X,Y) = \text{TR} \left( \log((1 + X)(1 + Y)) - \log(1 + X) - \log(1 + Y) \right) = \text{TR} \left( \frac{1}{2}[X,Y] + \frac{1}{6}[Y,X^2] + \frac{1}{6}[Y,YX] - \frac{1}{6}[X,Y^2] - \frac{1}{6}[X,YX] + \ldots \right)\).

**Proof.** By direct computation from \(\log \det_{\text{ren}}(1 + X) = \text{TR} \log(1 + X)\). Note that we do not need to worry about the convergence of the logarithmic expansions since only a finite number of the commutators contribute to the trace.

**Remark** The \(L_p\) determinants \(\det_p\) are continuous in the \(L_p\) Schatten ideal topology. The renormalized determinant is not continuous with respect to this topology. Instead, from Prop. II.3.4. in [D] follows that it is continuous in a Frechet topology on the pseudodifferential symbols. In the same way, the correction \(\gamma(X,Y)\) is continuous in the Frechet topology; this follows in fact directly from the property that it is a residue of a polynomial in the operators \(X,Y\); all terms of order higher than \(\text{dim} M\) in the variables \(X,Y\) drop out by (2.4) and by the definition of the residue.

Since \(\gamma(X,Y)\) is a renormalized trace of a sum of commutators it follows that the logarithm of the multiplicative anomaly is local. Note that in the case \(d = 1, 2\) the anomaly \(\gamma\) vanishes identically whereas in the case \(d = 3\) we get the simple formula
\[
\text{(2.9)} \quad \det_{\text{ren}}((1 + X)(1 + Y)) = \det_{\text{ren}}(1 + X) \cdot \det_{\text{ren}}(1 + Y) \cdot e^{\frac{1}{2}\text{TR}[X,Y]}.
\]
This follows from the fact that in three dimensions \(\text{TR}[A,B] = 0\) if the sum of orders of \(A, B\) is less or equal to \(-3\), by formula (2.4), whereas in the cases \(d = 1, 2\) the trace \(\text{TR}[A,B]\) vanishes for all operators \(A, B\) of order less or equal to \(-1\).
3. THE DETERMINANT BUNDLE

Let $D$ be the Dirac operator defined in the spinor bundle over an odd dimensional compact closed manifold $M$. The Hilbert space $H$ of square integrable sections of the spin bundle has a spectral decomposition $H = H_+ \oplus H_-$ corresponding to $D \geq 0$ and $D < 0$ in the subspaces $H_{\pm}$. We also allow for the possibility that the spin bundle is tensored with a trivial complex vector bundle of finite rank $N$. The multiplication operators in $H$ are then represented by smooth $N \times N$ matrix valued functions on $M$; they are examples of bounded PSDO’s of order zero and their interest to us lies in the fact that they represent infinitesimal gauge transformation in Yang-Mills theory.

Denote by $\epsilon$ the sign of the Dirac operator, i.e., the grading operator corresponding to the splitting $H = H_+ \oplus H_-$. It is a PSDO of order zero with the property $\epsilon^2 = 1$. The infinitesimal gauge transformations are contained in an associative algebra $A$ which consists of bounded PSDO’s $a$ such that $[D, a]$ is bounded and $[\epsilon, a]$ is order $-1$ for any $a \in A$ (this is of course a part of the definition of a spectral triple in noncommutative geometry, [C]).

Let us define the infinite dimensional Grassmann manifold $Gr = Gr(H_+ \oplus H_-)$ as the set of all subspaces $W \subset H$ with the property that the orthogonal projection $pr_+: W \to H_-$ is a PSDO of order $\leq -1$.

Let $G$ be the gauge group consisting of all smooth functions $M \to U(N)$; note that the $G$ orbit of $H_+$ is contained in $Gr$. The group $G$ is contained in the group $GL_{d+}$ of all invertible elements in $A$; the notation is introduced in order to remind that elements $g$ of $GL_{d+}$ have the property that $[\epsilon, g]^d$ is a PSDO of order $-d$ and is thus contained in the Dixmier ideal $L_{1+}$.

The Grassmannian $Gr$ is a subset of any Schatten Grassmannian $Gr_p$ for $p > d$, where $Gr_p$ is defined by the condition that $pr_+$ belongs to the Schatten ideal $L_p$. For this reason the determinant bundles $DET_p$ studied in [MR] can be restricted to define complex line bundles over $Gr$. However, we want to take advantage of the special properties of $Gr$ in order to get local formulas to describe the geometry and the gauge group action in the determinant bundle.

The Grassmannian $Gr$ splits to connected components $Gr^{(k)}$ labelled by the Fredholm index $k$ of the projection $pr_+$. The operator $pr_+$ is Fredholm since $pr_-$ (the orthogonal projection on $H_-$) is compact. Likewise, the group $GL_{d+}$ is an union of connected components labelled by the index of $pr_+ gpr_+: H_+ \to H_+$.

Let $H^{(k)}_+ = H_+ \oplus V_k$ for $k \geq 0$ and $H^{(k)}_- = H_+ \oplus V_k$ for $k < 0$, where $V_k$ is a $k$ dimensional subspace of $H_-$ (resp. of $H_+$). We denote by $pr^{(k)}_+$ the projection onto $H^{(k)}_+$.

Following [PS], we define the Stiefel manifold $St$ as the set of linear isometries $w: H_+ \to H$ such that $pr^{(k)}_+ \circ w - 1$ is a PSDO of order $\leq -2$. Note that the image $w(H_+)$ is then an element of $Gr^{(k)}$. Thus there is a natural projection $\pi: St \to Gr$ defined by $w \mapsto w(H_+)$. 


If \( w \in \pi^{-1}(W) \in St \) then \( w \circ q \), where \( q : H_+ \to H_+ \) is an invertible pseudodifferential operator, is in the same fiber if and only if \( q - 1 \) is a PSDO of order \( \leq -2 \). Thus \( St \) is a principal bundle over \( Gr \) with fiber the group \( U(2)(H_+) \) of unitary PSDO’s with the above property.

The determinant bundle \( DET \) over \( Gr \) is defined as the complex line bundle \( St \times \mathbb{C}/\sim \), where the equivalence relation is

\[
(w \circ q, \lambda) \sim (w, \omega(w, q)\lambda),
\]

with

\[
\omega(w, q) = \det_{ren}(q) \cdot e^{\gamma(pr^{(k)}_+ \circ w, q)},
\]

where \( w \in St^{(k)} = \pi^{-1}(Gr^{(k)}) \) and \( \lambda \in \mathbb{C} \). This is really an equivalence relation, since

\[
\omega(w, qq') = \omega(wq, q') \cdot \omega(w, q)
\]

which in turn follows directly from the multiplicative anomaly relation, Prop. 1.

By the remark at the end of the Section 2, \( \omega(w, q) \equiv \det_{ren}(q) \) when the dimension \( d = 1, 2, 3 \). The determinant \( \det_{ren} \) is actually multiplicative in these dimensions when the argument \( q \) satisfies \( \text{ord}(q - 1) \leq -2 \).

The dual determinant bundle \( DET^* \) is defined analogously by the relation

\[
(wq, \lambda) \sim^* (w, \lambda \omega(w, q)^{-1}).
\]

A section of the dual determinant bundle is then a complex function \( \psi : St \to \mathbb{C} \) such that

\[
\psi(wq) = \psi(w) \cdot \omega(w, q).
\]

A particular section is given by the determinant function itself, \( \psi(w) = \det_{ren}(pr^{(k)}_+ \circ w) \) for \( w \in St^{(k)} \).

The group \( GL_{d+} \), which contains the gauge group \( \mathcal{G} \), acts naturally on \( Gr \). However, this action does not lift to \( St \) and this leads to group extensions of the type which were discussed in [MR] in the context of Schatten ideal Grassmannians. Taking advantage of special features of the PSDO algebra here we get simplified local expressions for the cocycles describing the gauge group extensions.

We first define the set \( \mathcal{E} \) consisting of triples \( (g, q, \mu) \) where \( g \in GL_{d+}, q \in GL(H_+) \) such that \( pr_+ g pr_+ - q \) is a PSDO of order \(-2\) and \( \mu \) is a smooth \( \mathbb{C}^\times \) valued function on \( Gr \).

The elements of \( \mathcal{E} \) act on \( DET \) (or on \( DET^* \)). The action is given by

\[
(g, q, \mu) \cdot (w, \lambda) = (gwq^{-1}, \mu(\pi(w))\lambda\alpha(g, q, w)).
\]
Here the function $\alpha$ must be chosen such that

$$\frac{\alpha(g, q, wt)}{\alpha(g, q, w)} = \frac{\omega(w, t)}{\omega(gwq^{-1}, qtq^{-1})}$$

in order that $(g, q)$ maps equivalence classes to equivalence classes. We denote by $F$ the grading operator corresponding to the decomposition $H = W \oplus W^\perp$, that is, $F$ is the unit operator in $W = \pi(w)$ and $(-1)$ times the unit operator in the complement $W^\perp$. A particular solution of (3.5) is given by

$$\alpha(g, q, w) = \frac{\det_{ren}(w_+)}{\det_{ren}(gwq^{-1})} \frac{\det_{ren}(\frac{1}{2}q^{-1}(\alpha(F_{11} + 1) + bF_{21}))}{\det_{ren}(\frac{1}{2}(F_{11} + 1))}$$

where $w_+ = pr_+ \circ w$ and we use the block decompositions

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

corresponding to the fixed polarization $H = H_+ \oplus H_-$. In the case $d = 3$ there is a simpler solution:

**Lemma 2.** In the case of dimension $d = 3$ the function

$$\alpha_{d=3}(g, q, w) = e^{-\text{TR}[q, w_+ q^{-1}]}$$

where $w_+ = pr_+ \circ w$, satisfies (3.5).

**Proof.** Since $w_+ - 1$ and $t - 1$ are of order $-2$ we have $\omega(w, t) = \det_{ren}(t)$. On the other hand, the right-hand-side of (3.5) is given by the multiplicative anomaly,

$$\frac{\omega(w, t)}{\omega(gwq^{-1}, qtq^{-1})} = \frac{\det_{ren}(t)}{\det_{ren}(qtq^{-1})}.$$ 

Writing $t = e^z$ we get

$$\frac{\det_{ren}(t)}{\det_{ren}(qtq^{-1})} = e^{\text{TR}[q z q^{-1} - z]} = e^{\text{TR}[q z q^{-1}]} = e^{\text{TR}[q, (t - 1)q^{-1}]}.$$ 

But the logarithm of the left-hand-side of (3.5) is

$$\text{TR}[q, w_+ q^{-1}] - \text{TR}[g, w_+ tq^{-1}] = -\text{TR}[q, tq^{-1}] = -\text{TR}[q, (t - 1)q^{-1}].$$

since $(w_+ - 1)(t - 1)q^{-1}$ is of order $-4$ and therefore its commutator with the bounded operator $q$ has zero trace. This proves the statement. One might wonder why we did not use the multiplicative anomaly formula directly to $\det_{ren}(qtq^{-1})$. The reason is that $q$ is not of type $1+$ negative order operator for which the formula in Proposition 1 is valid.
The action on $\text{DET}$ defines a group structure in $\mathcal{E}$. The multiplication is given by

\begin{equation}
(3.8) \quad (g, q, \mu)(g', q', \mu') = (gg', qq', \mu g' \theta(g, g', q, q'; \cdot))
\end{equation}

where $\theta$ is a function on $\text{Gr}$ defined by

\begin{equation}
(3.9) \quad \theta(g, g', q, q'; W) = \alpha(g, q, g'wq^{-1})\alpha(g', q', w)\alpha(gg', qq', w)^{-1}
\end{equation}

where $W = \pi(w)$. Here $\mu_g$ denotes the translated function $\mu_g(W) = \mu(g^{-1}W)$. The right-hand-side of (3.9) is invariant with respect to the transformation $w \mapsto wt$ with $t \in \text{GL}^{(2)}$, as follows from (3.5), and therefore it is indeed a function of $W = \pi(w)$.

**Proposition 3.** The subset $N$ of elements $(1, q, \mu)$, with $q \in \text{GL}^{(2)}$ and $\mu(W) = (\omega(w, q^{-1})\alpha(1, q, w))^{-1}$ (with $W = \pi(w)$) forms the maximal normal subgroup of $\mathcal{E}$ which acts trivially on $\text{DET}$ and therefore the group $\widehat{\text{GL}_{d+}} = \mathcal{E}/N$ acts on $\text{DET}$.

**Proof.** In order that $(g, q, \mu)$ acts trivially on $\text{DET}$ it has to act trivially on the base $\text{Gr}$ and therefore $g = 1$. Next

\begin{align*}
(1, q, \mu) \cdot (w, \lambda) &= (wq^{-1}, \lambda \mu(\pi(w))\alpha(1, q, w)) \\
&\sim (w, \lambda \mu(\pi(w))\alpha(1, q, w)\omega(w, q^{-1})) = (w, \lambda)
\end{align*}

for all $w \in \text{St}$ if and only if $\mu(\pi(w))\alpha(1, q, w)\omega(w, q^{-1}) = 1$ for all $w$. Note that this expression depends on $W = \pi(w)$ and not on $w$, as follows from the relation (3.5); in fact, using the solution (3.6) we have $\mu \equiv 1$.

The group $\widehat{\text{GL}_{d+}}$ is an extension of $\text{GL}_{d+}$ by the normal abelian subgroup consisting of triples $(1, 1, \mu)$, i.e., by the group $\text{Map}(\text{Gr}, \mathbb{C}^\times)$. Locally, near the unit element, the extension can be given by a 2-cocycle $\xi$ on $\text{GL}_{d+}$ with values in $\text{Map}(\text{Gr}, \mathbb{C}^\times)$. Using the local section $g \mapsto (g, a, 1)$ the 2-cocycle is given by

\begin{equation}
(3.10) \quad \xi(g, g')(W) = \theta(g, g', q, q'; W)\omega(w_+, (aa')^{-1}a'')^{-1}.
\end{equation}

In the important case $d = 3$ we can use the simple formula (3.7) and obtain

\begin{align*}
(3.11) \quad &\xi_{d=3}(g, g')(W) = \det_{\text{ren}}((aa')^{-1}a'') \cdot \exp \left\{-\frac{1}{2} \text{TR} \left( [w_+, (aa')^{-1}a''] \right) + [a''w_+(aa')^{-1}a', a''^{-1}] - [a(g'wa'')^{-1}_+, a^{-1}] - [a'w_+, a''] \right\}.
\end{align*}

Using the standard formula

\begin{equation}
(3.12) \quad [X, Y] = \frac{d^2}{dt ds} \bigg|_{t=s=0} e^{tX} e^{sY} e^{-tX} e^{-sY}
\end{equation}
relating the group multiplication to Lie product, one obtains a formula for the commutator in the Lie algebra extension \( \hat{gl}_{3+} = gl_{3+} \oplus Map(Gr, \mathbb{C}) \), where \( gl_{3+} \) is the Lie algebra of \( GL_{3+} \),

\[
(3.13) \quad [(X, \mu), (Y, \nu)] = ([X, Y], X \cdot \nu - Y \cdot \mu + c(X, Y)).
\]

Here \( X \cdot \nu \) denotes the Lie derivative of the function \( \nu \) defined by the left group action, \( X \cdot \nu = \frac{d}{dt} \nu_{g(t)} |_{t=0} \) for \( g(t) = \exp(tX) \).

In the case \( d = 3 \) we obtain from (3.11) (setting \( g = e^{tX} \) and \( g' = e^{sY} \) and taking the second derivative with respect to \( t, s \)) the formula

\[
(3.14) \quad c_{d=3} (X, Y) = \text{TR} \left( b_Y c_X - b_X c_Y - \frac{1}{2} [a_X, b_Y w] + [a_Y, b_X w] \right)
\]

for the Lie algebra 2 cocycle. We have used the same block decomposition for the operators \( X, Y \) as we have used for the group elements \( g \in GL_{3+} \). The cocycle should be a function of \( F \in Gr \), not of the variable \( w \) in the Stiefel manifold. This is indeed the case; it follows from the fact that \( w - \frac{1}{2} F_{21} \) is of order \(-3\) and \( b_X \) is of order \(-1\). Thus the difference \( b_X w - \frac{1}{2} b_X F_{21} \) is of order \(-4\) and therefore its commutator with any bounded PSDO has a vanishing trace. We can now write

\[
(3.15) \quad c_{d=3} (X, Y; F) = \text{TR} \left( b_Y c_X - b_X c_Y - \frac{1}{2} [a_X, b_Y F_{21}] + \frac{1}{2} [a_Y, b_X F_{21}] \right).
\]

We finish the discussion by giving an important application of formula (3.15) in gauge theory.

In Yang-Mills theory the variable \( F \in Gr \) comes as the sign of the Dirac hamiltonian coupled to a Yang-Mills potential. It specifies the vacuum in the fermionic Fock space parametrized by an external vector potential \( A \), [MR]. The operators \( X, Y \) are infinitesimal gauge transformations acting on the fermion field. Writing \( F = (D_0 + A)/|D_0 + A| \), where \( D_0 \) is the free massless Dirac hamiltonian, acting on right-handed spinors, in three dimensions and \( A = A_k \sigma^k \) is the potential (each component \( A_k \) with \( k = 1, 2, 3 \) is a smooth function on \( M \) with values in a Lie algebra of a compact Lie group). The \( \sigma \)'s are the hermitean complex \( 2 \times 2 \) Pauli matrices. The first two terms on the right in (3.15) can be written as \(-\frac{1}{8} \text{TR}(1 + \epsilon)[[\epsilon, X], [\epsilon, Y]]\). This vanishes by the fact that the Pauli matrices (involved in \( \epsilon \)) are traceless and by a parity argument, odd powers in momenta lead to vanishing integrals in the momentum space.

One can expand the symbol of \( F = F_A \) in powers of \( A \), denoting \( p = p_\epsilon \sigma^k \), as

\[
\text{symb}(F_A - \epsilon) = \frac{1}{|p|} \frac{1}{2} \left( A - \frac{1}{p^2} p A p \right) + O(1/|p|^2).
\]

We need to take into account only a the first two terms in the expansion since higher order terms lead to operators which give vanishing trace in the commutators in (3.15), see [LM2] for similar calculations. The final result is:
**Theorem 4.** The group of smooth gauge transformations acts through the extension (3.11) on sections of the determinant bundle $\text{DET}$ over $\text{Gr}$. The corresponding Lie algebra action leads to an extension of the Lie algebra of infinitesimal gauge transformations by the abelian ideal of smooth functions on $\text{Gr}$. When the point $F \in \text{Gr}$ is parametrized by vector potentials using the sign of the chiral Dirac hamiltonian and $d = 3$ the Lie algebra extension is given by the Mickelsson-Faddeev-Shatasvili cocycle

$$c_{d=3}(X, Y; A) = \frac{i}{24\pi^2} \int_M \text{tr}A[dx, dy].$$

This agrees with [LM2], [M] and the cohomological arguments in [M2], [F-Sh].

**REFERENCES**

[C] A. Connes: *Noncommutative Geometry.* Academic Press, San Diego (1994)

[CDMP] A. Cardona, C. Ducourtioux, J.P. Magnot, and S. Paycha: Weighted traces on the algebra of pseudo-differential operators and geometry of loop groups. [math.OA/0001117](https://arxiv.org/abs/math.OA/0001117)

[CZ] G. Cognola, S. Zerbini: Consistent, covariant and multiplicative anomalies. Lett.Math.Phys.48, 375-383 (1999); [hep-th/9811039](https://arxiv.org/abs/hep-th/9811039)

[D] C. Ducourtioux: Weighted traces of pseudo-differential operators and associated determinants. Ph.D. thesis, Mathematics Department, Université Blaise Pascal, 2001

[ECZ] E. Elizalde, G. Cognola, and S. Zerbini: Applications in physics of the multiplicative anomaly formula involving some basic differential operators. Nucl.Phys.B532, 407-428,1998; [hep-th/9804046](https://arxiv.org/abs/hep-th/9804046)

[EFVZ] E. Elizalde, A. Filippi, L. Vanzo, and S. Zerbini: Is the multiplicative anomaly relevant? [hep-th/9804072](https://arxiv.org/abs/hep-th/9804072)

[F] L. Friedlander: PhD Thesis, Department of Mathematics, MIT (1989)

[F-Sh] L. Faddeev and S. Shatasvili: Algebraic and Hamiltonian methods in the theory of nonabelian anomalies. Theor. Math. Phys. 60, 770 (1985)

[KV] M. Kontsevich and S. Vishik: Determinants of elliptic pseudo-differential operators. [hep-th/9404046](https://arxiv.org/abs/hep-th/9404046)

[LM] E. Langmann and J. Mickelsson: (3 + 1)-dimensional Schwinger terms and non-commutative geometry. Phys. Lett. B 338, 241 (1994)

[LM2] E. Langmann and J. Mickelsson: Elementary derivation of the chiral anomaly. Lett. Math. Phys. 6, 45 (1996)

[LMR] E. Langmann, J. Mickelsson, and S. Rydh: Anomalies and Schwinger terms in NCG field theory models. J.Math.Phys.42,4779, (2001); [hep-th/0103006](https://arxiv.org/abs/hep-th/0103006).
[M] J. Mickelsson: Wodzicki residue and anomalies of current algebras. In: *Integrable Models and Strings*. Springer LNP 436, p. 123 (1994). Ed. by A. Alekseev, A. Hietamäki, K. Huitu, and A. Niemi

[M2] J. Mickelsson: Chiral anomalies in even and odd dimensions. Commun. Math. Phys. **97**, 361 (1985)

[MR] J. Mickelsson and S. Rajeev: Current algebras in \(d+1\) dimensions and determinant bundles over infinite-dimensional Grassmannians. Commun. Math. Phys. **116**, 365 (1988)

[MN] R. Melrose and V. Nistor: Homology of pseudo-differential operators I. Manifolds with boundary. [arXiv:funct-an/9606003](https://arxiv.org/abs/funct-an/9606003)

[O] K. Okikiolu: The multiplicative anomaly for determinants of elliptic operators. Duke Math. J. **79**, 723-750 (1995)

[PR] S. Paycha and S. Rosenberg: Curvature of determinant bundles and first Chern forms. [math.DG/0009172](https://arxiv.org/abs/math.DG/0009172)

[PS] A. Pressley and G. Segal: *Loop Groups*. Oxford University Press (1986)

[S] B. Simon: *Trace Ideals and their Applications*. London Mathematical Society Lecture Notes Series **35**, Cambridge University Press, Cambridge - New York (1979)