On one generalization of modular subgroups

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Abstract

Let $G$ be a finite group. If $M_n < M_{n-1} < \ldots < M_1 < M_0 = G$ where $M_i$ is a maximal subgroup of $M_{i-1}$ for all $i = 1, \ldots, n$, then $M_n$ ($n > 0$) is an $n$-maximal subgroup of $G$. A subgroup $M$ of $G$ is called modular if the following conditions are held: (i) $\langle X, M \cap Z \rangle = \langle X, M \rangle \cap Z$ for all $X \leq G$, $Z \leq G$ such that $X \leq Z$, and (ii) $\langle M, Y \cap Z \rangle = \langle M, Y \rangle \cap Z$ for all $Y \leq G$, $Z \leq G$ such that $M \leq Z$.

In this paper, we study finite groups whose $n$-maximal subgroups are modular.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes and the symbol $\pi(G)$ stands for the set of prime divisors of the order of $G$.

We say that $G$ is: nearly nilpotent if $G$ is supersoluble and $G$ induces on any its non-Frattini chief factor $H/K$ (that is, $H/K \not\leq \Phi(G/K)$) an automorphism group of order dividing a prime; strongly nilpotent if $G$ is nearly nilpotent and every chief factor of $G$ is a direct product of $p$-groups for some prime $p$; strong srg (strongly supersoluble group) if $G$ is strongly nilpotent.

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supersoluble if $G$ is supersoluble and $G$ induces on any its chief factor $H/K$ an automorphism group of square free order. We use $\mathfrak{N}_n$ and $\mathfrak{U}_n$ to denote the classes of all nearly nilpotent and of all strongly supersoluble groups, respectively. Nearly nilpotent and strongly supersoluble groups were studied respectively in [1] and [2, 3].

It is clear that: the group $C_7 \rtimes \text{Aut}(C_7)$ is strongly supersoluble but it is not nearly nilpotent; the group $C_{13} \rtimes \text{Aut}(C_{13})$ is supersoluble but it is not strongly supersoluble; the group $S_3$ is nearly nilpotent but it is not nilpotent.

A subgroup $M$ of $G$ is called modular if $M$ is a modular element (in the sense of Kurosh [4, p. 43]) of the lattice $\mathcal{L}(G)$ of all subgroups of $G$, that is,

(i) $\langle X, M \cap Z \rangle = \langle X, M \rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$, and

(ii) $\langle M, Y \cap Z \rangle = \langle M, Y \rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $M \leq Z$.

Recall that a subgroup $H$ of $G$ is called a 2-maximal (second maximal) subgroup of $G$ whenever $H$ is a maximal subgroup of some maximal subgroup $M$ of $G$. Similarly we can define 3-maximal subgroups, and so on.

The relationship between $n$-maximal subgroups (where $n > 1$) of $G$ and the structure of $G$ was studied by many authors (see, in particular, the recent papers [5]–[12] and Chapter 4 in the book [13]). One of the earliest results in this line research was obtained by Huppert in the article [14] who established the supersolubility of the group whose all second maximal subgroups are normal. In the same article Huppert proved that if all 3-maximal subgroups of $G$ are normal in $G$, then the commutator subgroup $G'$ of $G$ is a nilpotent group and the principal rank of $G$ is at most 2. These two results were developed by many authors. In particular, Schmidt proved [11] that: if all 2-maximal subgroups of $G$ are modular in $G$, then $G$ is nearly nilpotent; if all 3-maximal subgroups of $G$ are modular in $G$ and $G$ is not supersoluble, then either $G$ is a group of order $pq^2$ for primes $p$ and $q$ or $G = Q \rtimes P$, where $Q = C_G(Q)$ is a quaternion group of order 8 and $|P| = 3$. Mann proved [15] that if all $n$-maximal subgroups of a soluble group $G$ are subnormal and $n < |\pi(G)|$, then $G$ is nilpotent; but if $n \leq |\pi(G)| + 1$, then $G$ is $\phi$-dispersive for some ordering $\phi$ of $P$. Finally, in the case $n \leq |\pi(G)|$ Mann described $G$ completely.

In this paper, we prove the following modular analogues of the above-mentioned Mann’s results.

**Theorem A.** Suppose that $G$ is soluble and every $n$-maximal subgroup of $G$ is modular. If $n \leq |\pi(G)|$, then $G$ is strongly supersoluble and $G$ induces on any its non-Frattini chief factor $H/K$ an automorphism group of order $p_1 \cdots p_m$ where $m \leq n$ and $p_1, \ldots, p_m$ are distinct primes.

We use $G^{\mathfrak{G}_n}$ to denote the intersection of all normal subgroups $N$ of $G$ with strongly supersoluble quotient $G/N$.

**Theorem B.** Suppose that $G$ is soluble and every $n$-maximal subgroup of $G$ is modular. If $n \leq |\pi(G)| + 1$, then $G^{\mathfrak{G}_n}$ is a nilpotent Hall subgroup of $G$.

Finally, note that the restrictions on $|\pi(G)|$ in Theorems A and B cannot be weakened (see
Section 4 below).

2 Proof of theorem A

A normal subgroup $A$ of $G$ is said to be hypercyclically embedded in $G$ [4, p. 217] if either $A = 1$ or $A \neq 1$ and every chief factor of $G$ below $A$ is cyclic. We use $Z_u(G)$ to denote the product of all normal hypercyclically embedded subgroups of $G$. It is clear that a normal subgroup $A$ of $G$ is hypercyclically embedded in $G$ if and only if $A \leq Z_u(G)$.

Recall that $G$ is said to be a $P$-group [4, p. 49] if $G = A \rtimes \langle t \rangle$ with an elementary abelian $p$-group $A$ and an element $t$ of prime order $q \neq p$ induces a non-trivial power automorphism on $A$.

The following two lemmas collect the properties of modular subgroups which we use in our proofs.

Lemma 2.1 (See Theorems 5.1.14 and 5.2.5 in [4]). Let $M$ be a modular subgroup of $G$.

(i) $M/M_G$ is nilpotent and $M^G/M_G \leq Z_u(G/M_G)$.

(ii) If $M_G = 1$, then

$$G = S_1 \times \cdots \times S_r \times K,$$

where $0 \leq r \in \mathbb{Z}$ and for all $i, j \in \{1, \ldots, r\}$,

(a) $S_i$ is a non-abelian $P$-group,

(b) $(|S_i|, |S_j|) = 1 = (|S_i|, |K|)$ for all $i \neq j$,

(c) $M = Q_1 \times \cdots \times Q_r \times (M \cap K)$ and $Q_i$ is a non-normal Sylow subgroup of $S_i$,

(d) $M \cap K$ is quasinormal in $G$.

Lemma 2.2 (See p. 201 in [4]). Let $A$, $B$ and $N$ be subgroups of $G$, where $A$ is modular in $G$ and $N$ is normal in $G$.

(1) If $B$ is modular in $G$, then $\langle A, B \rangle$ is modular in $G$.

(2) $AN/N$ is modular in $G/N$.

(3) $N$ is modular in $G$.

(4) If $A \leq B$, then $A$ is modular in $B$.

(5) If $\varphi$ is an isomorphism of $G$ onto $\bar{G}$, then $A^\varphi$ is modular in $\bar{G}$.

A subgroup $H$ of $G$ is said to be quasinormal (respectively $S$-quasinormal) in $G$ if $HP = PH$ for all subgroups (for all Sylow subgroups) $P$ of $G$.

Lemma 2.3 (See Chapter 1 in [16]). Let $H \leq K \leq G$.

(1) If $H$ is $S$-quasinormal in $G$, then $H$ is $S$-quasinormal in $K$.

(2) Suppose that $H$ is normal in $G$. Then $K/H$ is $S$-quasinormal in $G/H$ if and only if $K$ is $S$-quasinormal in $G$. 

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(3) If $H$ is $S$-quasinormal in $G$, then $H$ is subnormal in $G$ and $H^G/H_G$ is nilpotent.

Lemma 2.4. Suppose that $G$ is soluble, and let $N \neq G$ be a minimal normal subgroup of $G$. Suppose also that every $n$-maximal subgroup of $G$ is either modular or $S$-quasinormal in $G$, where $n \leq |\pi(G)| + r$ for some integer $r$. Then there is a natural number $m \leq n$ such that every $m$-maximal subgroup of $G/N$ is either modular or $S$-quasinormal in $G/N$ and $m \leq |\pi(G/N)| + r$.

Proof. First assume that $N$ is not a Sylow subgroup of $G$. Then $|\pi(G/N)| = |\pi(G)|$. Moreover, if $H/N$ is an $n$-maximal subgroup of $G/N$, then $H$ is an $n$-maximal subgroup of $G$, so $H$ is either modular or $S$-quasinormal in $G$ by hypothesis. Consequently, $H/N$ is either modular or $S$-quasinormal in $G/N$ by Lemmas 2.2(2) and 2.3(2). On the other hand, if $G/N$ includes no $n$-maximal subgroups, then, by the solubility of $G$, the trivial subgroup of $G/N$ is modular in $G/N$ and is a unique $m$-maximal subgroup of $G/N$ for some $m < n$ with $m < |\pi(G/N)|$. Hence $m < |\pi(G/N)| + r$. Thus the conclusion of the lemma is fulfilled for $G/N$.

Finally, consider the case that $N$ is a Sylow $p$-subgroup of $G$. Let $E$ be a Hall $p'$-subgroup of $G$. It is clear that $|\pi(E)| = |\pi(G)| - 1$ and $E$ is a maximal subgroup of $G$. Therefore, every $(n-1)$-maximal subgroup of $E$ is either modular or $S$-quasinormal in $E$ by Lemmas 2.2(4) and 2.3(1). Thus, by the isomorphism $G/N \simeq E$, Lemma 2.2(5) implies that every $(n-1)$-maximal subgroup of $G/N$ is either modular or $S$-quasinormal in $G/N$, and also we have $n - 1 \leq |\pi(G/N)| + r$. The lemma is proved.

A formation is a class $\mathfrak{F}$ of groups with the following properties: (i) Every homomorphic image of any group in $\mathfrak{F}$ belongs to $\mathfrak{F}$; (ii) $G/N \cap R \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$ and $G/R \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be: saturated if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; hereditary if $H \in \mathfrak{F}$ whenever $H \subseteq G \in \mathfrak{F}$.

Lemma 2.5 (See Theorem A in [3]). The class of all strongly supersoluble groups is a hereditary saturated formation.

Let $\mathfrak{X}$ be a class of groups. A group $G$ is called a minimal non-$\mathfrak{X}$-group [13] or $\mathfrak{X}$-critical group [17] if $G$ is not in $\mathfrak{X}$ but all proper subgroups of $G$ are in $\mathfrak{X}$. An $\mathfrak{X}$-critical group is also called a Schmidt group.

Fix some ordering $\phi$ of $\mathbb{P}$. The record $p\phi q$ means that $p$ precedes $q$ in $\phi$ and $p \neq q$. Recall that a group $G$ of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ is called $\phi$-dispersive whenever $p_1\phi p_2\phi\cdots\phi p_n$ and for every $i$ there is a normal subgroup of $G$ of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_i^{\alpha_i}$. Furthermore, if $\phi$ is such that $p\phi q$ always implies $p > q$ then every $\phi$-dispersive group is called Ore dispersive.

Lemma 2.6 (See [13], I, Propositions 1.8, 1.11 and 1.12]). The following claims hold for every $\Phi$-critical group $G$:

1. $G$ is soluble and $|\pi(G)| \leq 3$.
2. If $G$ is not a Schmidt group, then $G$ is Ore dispersive.
3. $G^\Phi$ is a unique normal Sylow subgroup of $G$.
4. If $S$ is a complement to $G^\Phi$ in $G$, then $S/S \cap \Phi(G)$ is either a cyclic prime power order group
or a Miller-Moreno (that is, a minimal non-abelian) group.

(5) $G^d/\Phi(G^d)$ is a non-cyclic chief factor of $G$.

(6) If $G^d$ is non-abelian, then the center, commutator subgroup, and Frattini subgroup of $G^d$ coincide with one another.

(7) If $p > 2$, then $G^d$ is of exponent $p$; for $p = 2$ the exponent of $G^d$ is at most 4.

Lemma 2.7 (See Lemma 12.8 in [13]). If $H/K$ is an abelian chief factor of $G$ and $M$ is a maximal subgroup of $G$ such that $K \leq M$ and $MH = G$, then

$$G/MG \simeq (H/K) \rtimes (G/C_G(H/K)) \simeq (HM_G/MG) \rtimes (G/C_G(HM_G/MG)).$$

The following lemma is evident.

Lemma 2.8. If $H/K$ and $T/L$ are $G$-isomorphic chief factors of $G$, then $(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L)).$

Recall that a class of soluble groups $\mathfrak{X}$ is a Schunck class [17, III, 2.7] if $G \in \mathfrak{X}$ whenever $G/MG \in \mathfrak{X}$ for all maximal subgroups $M$ of $G$.

Proposition 2.9. The class of all nearly nilpotent groups $\mathfrak{N}_n$ is a Schunck class, and $\mathfrak{N}_n \subseteq \mathfrak{U}_s$. Hence every homomorphic image of any nearly nilpotent group is nearly nilpotent, and $G$ is nearly nilpotent whenever $G/\Phi(G)$ is nearly nilpotent.

Proof. Suppose that for every maximal subgroup $M$ of $G$ we have $G/MG \in \mathfrak{N}_n$. Then $G/\Phi(G)$ is supersoluble, so $G$ is supersoluble. If $H/K$ is a non-Frattini chief factor of $G$ and $M$ is a maximal subgroup of $G$ such that $K \leq M$ and $MH = G$, then $G/MG \simeq (H/K) \rtimes (G/C_G(H/K))$ by Lemma 2.7. Since clearly $C_{(H/K)}\rtimes(G/C_G(H/K))(H/K) = H/K$, it follows that $|G/C_G(H/K)| = p$ is a prime. Hence $G \in \mathfrak{N}_n$. Therefore $\mathfrak{N}_n$ is a Schunck class, so every homomorphic image of any nearly nilpotent group is nearly nilpotent, and $G$ is nearly nilpotent whenever $G/\Phi(G)$ is nearly nilpotent by [17, III, 2.7].

Now we show that every nearly nilpotent group $G$ is strongly supersoluble. Assume that this is false and let $G$ be a counterexample of minimal order. Let $R$ be a minimal normal subgroup of $G$. Then $G/R$ is strongly supersoluble by the choice of $G$ since $G/R$ is nearly nilpotent. Moreover, if $R \leq \Phi(G)$, then $G$ is strongly supersoluble by Lemma 2.5, contrary to the choice of $G$. Therefore $R \not\leq \Phi(G)$, so $G/C_G(R)$ is of prime order since $G$ is nearly nilpotent. Therefore $G$ is strongly supersoluble by the Jordan-Hölder theorem. This contradiction completes the proof of the proposition.

Lemma 2.10. Let $G = R \rtimes M$ be a soluble primitive group, where $R = C_G(R)$ is a minimal normal subgroup of $G$. Let $T \neq 1$ be a subgroup of $G$. Suppose that $G$ is not nearly nilpotent.

(1) If $T < M$, then $T$ is neither modular nor $S$-quasinormal in $G$.

(2) If $T < R$ and $|M|$ is a prime, then some subgroup $V$ of $R$ with $|V| = |T|$ is neither modular nor $S$-quasinormal in $G$. 5
Proof. (1) First assume that $T$ is modular in $G$ but it is not $S$-quasinormal in $G$. Then $T$ is not quasinormal in $G$, so Lemma 2.1(ii) implies that $G$ is a non-abelian $P$-group since $T_G \leq M_G = 1$. But then $G$ is supersoluble. This contradiction shows that $T$ is $S$-quasinormal in $G$, so $T$ is subnormal in $G$ by Lemma 2.3(3). Hence $1 < T^G = T^RM = T^M \leq M_G = 1$ by [17, A, 14.3], a contradiction. Hence we have (1).

(2) Let $V$ be a subgroup of $R$ with $|V| = |T|$ such that $V$ is normal in a Sylow $p$-subgroup of $G$. If $V$ is $S$-quasinormal in $G$, then for every Sylow $q$-subgroup $Q$ of $G$, where $q \neq p$, we have $VQ = QV$ and so $V = R \cap VQ$. Hence $Q \leq N_G(V)$. Thus $V$ is normal in $G$, a contradiction. Hence $V$ is modular in $G$, which implies that $1 < V \leq R \cap Z_G(G)$ by Lemma 2.1(i) and so $R \leq Z_G(G)$. But then $|R| = p$, which implies that $G$ is nearly nilpotent, a contradiction.

The lemma is proved.

Proposition 2.11. If every maximal subgroup of $G$ or every 2-maximal subgroup of $G$ is either modular or $S$-quasinormal in $G$, then $G$ is nearly nilpotent. Hence $G$ is strongly supersoluble.

Proof. Assume this proposition is false and let $G$ be a counterexample of minimal order.

First we show that $G$ is soluble. Indeed, if $M$ is a maximal subgroup of $G$ and either $M$ is modular in $G$ or $M$ is $S$-quasinormal in $G$, then $|G : M|$ is a prime by Lemmas 2.1(i) and 2.3(3). Therefore if every maximal subgroup of $G$ is either modular or $S$-quasinormal in $G$, then $G$ is supersoluble. On the other hand, if every 2-maximal subgroup of $G$ is either modular or $S$-quasinormal in $G$, then every maximal subgroup of $G$ is supersoluble by Lemmas 2.2(4) and 2.3(1) and so $G$ is soluble by Lemma 2.6(1).

Therefore, in view of Proposition 2.9, we need only to show that for every maximal subgroup $M$ of $G$ we have $G/M_G \in \frak{N}_n$. If $M_G \neq 1$, then the choice of $G$ and Lemmas 2.2(2) and 2.3(2) imply that $G/M_G \in \frak{N}_n$. Now assume that $M_G = 1$, so there is a minimal normal subgroup $R$ of $G$ such that $G = R \rtimes M$ and $R = C_G(R)$ by [17, A, 15.6]. Then $M$ is not $S$-quasinormal in $G$ by Lemma 2.3(3). On the other hand, if $M$ is modular in $G$, then $G = M^G$ is a non-abelian $P$-group by Lemma 2.1(ii). It follows that $G$ is nearly nilpotent, a contradiction. Hence every 2-maximal subgroup of $G$ is either modular or $S$-quasinormal in $G$.

Now let $T$ be any maximal subgroup of $M$. Then $T$ is either modular or $S$-quasinormal in $G$, so $T = 1$ and hence $|M| = q$ for some prime $q$. Therefore $R$ is a maximal subgroup of $G$. Then every maximal subgroup of $R$ is either modular or $S$-quasinormal in $G$ and so $|R| = p$ by Lemma 2.10(2), which implies that $|G| = pq$. Hence $G$ is nearly nilpotent, a contradiction.

The proposition is proved.

In fact, Theorem A is a special case of the following

Theorem 2.12. Suppose that $G$ is soluble and every $n$-maximal subgroup of $G$ is either modular or $S$-quasinormal in $G$. If $n \leq |\pi(G)|$, then $G$ is strongly supersoluble and $G$ induces on any its non-Frattini chief factor $H/K$ an automorphism group of order $p_1 \cdots p_m$, where $m \leq n$ and $p_1, \ldots, p_m$
are distinct primes.

**Proof.** Assume this theorem is false and let $G$ be a counterexample of minimal order.

First we show that $G$ is strongly supersoluble. Suppose that this is false. Let $R$ be a minimal normal subgroup of $G$.

(1) $G/R$ is strongly supersoluble. Hence $G$ is primitive and so $R \not\leq \Phi(G)$ and $R = C_G(R) = O_p(G)$ for some prime $p$.

Lemma 2.4 implies that the hypothesis holds for $G/R$, so the choice of $G$ implies that $G/R$ is strongly supersoluble. Therefore, again by the choice of $G$, $R$ is a unique minimal normal subgroup of $G$ and $R \not\leq \Phi(G)$ by Lemma 2.5. Hence $G$ is primitive and so $R = C_G(R) = O_p(G)$ for some prime $p$ by [17, A, 15.6].

(2) Every maximal subgroup $M$ of $G$ is strongly supersoluble.

By hypothesis every $(n - 1)$-maximal subgroup $T$ of $M$ is either modular or $S$-quasinormal in $G$. Hence $T$ is modular in $M$ by Lemma 2.2(4) in the former case, and it is $S$-quasinormal in $M$ by Lemma 2.3(1) in the second case. Since the solubility of $G$ implies that either $|\pi(M)| = |\pi(G)|$ or $|\pi(M)| = |\pi(G)| - 1$, the hypothesis holds for $M$. It follows that $M$ is strongly supersoluble by the choice of $G$.

(3) $G$ is supersoluble.

Suppose that this is false. Since every maximal subgroup $M$ of $G$ is strongly supersoluble by Claim (2), $G$ is a minimal non-supersoluble group. Then Lemma 2.6(1) yields that $|\pi(G)| = 2$ or $|\pi(G)| = 3$. But in the former case $G$ is strongly supersoluble by Proposition 2.11, so $|\pi(G)| = 3$ and every 3-maximal subgroup of $G$ is either modular or $S$-quasinormal in $G$. Claim (1) and Lemma 2.6 imply that $G = R \rtimes S$, where $S$ is a Miller-Moreno group. Moreover, since $|\pi(S)| = 2$ and $S$ is strongly supersoluble, $S$ is not nilpotent and so $S = Q \rtimes T$, where $|Q| = q$, $|T| = t$ and $C_S(Q) = Q$ for some distinct primes $q$ and $t$ by [13, I, Proposition 1.9]. Hence $R$ is a 2-maximal subgroup of $G$, so every maximal subgroup of $R$ is either modular or $S$-quasinormal $G$. Therefore $G$ is supersoluble by Lemma 2.10(2).

(4) $G$ is strongly supersoluble.

From Claims (1) and (3) we get that for some maximal subgroup $M$ of $G$ we have $G = R \rtimes M = C_G(R) \rtimes M$ and $|R| = p$, so $M$ is cyclic. Since $G$ is not strongly supersoluble, for some prime $q$ dividing $|M|$ and for the Sylow $q$-subgroup $Q$ of $M$ we have $|Q| > q$. First assume that $RQ \neq G$, and let $RQ \leq V$, where $V$ is a maximal subgroup of $G$. Then $V$ is strongly supersoluble by Claim (2). Hence $C_Q(R) \neq 1$, contrary to $R = C_G(R)$. Hence $RQ = G$ and so $|\pi(G)| = 2$. Therefore $G$ is strongly supersoluble by Proposition 2.11, a contradiction. Thus we have (4).

(5) $G$ induces on any its non-Frattini chief factor $H/K$ an automorphism group $G/C_G(H/K)$ of order $p_1 \cdots p_m$ where $m \leq n$ and $p_1, \ldots, p_m$ are distinct primes.
If $G$ is nearly nilpotent, it is clear. Now suppose that $G$ is not nearly nilpotent. Let $M$ be a maximal subgroup of $G$ such that $K \leq M$ and $MH = G$. Then $G/M_G \simeq (H/K) \times (G/C_G(H/K))$ by Lemma 2.7. If $M_G \neq 1$, the choice of $G$ implies that $m \leq n$. Now suppose that $M_G = 1$, so $G = H \times M$, where $|H|$ is a prime and $H = C_G(H)$. Then, by Claim (4), $M$ is a cyclic group of order $p_1 \ldots p_m$ for some distinct primes $p_1, \ldots, p_m$. Assume that $n < m$. Then $G$ has an $n$-maximal subgroup $T$ such that $T \leq M$ and $|T|$ is not a prime. But since $G$ is not nearly nilpotent, this is not possible by Lemma 2.10(1). This contradiction completes the proof of the result.

3 Proof of Theorem B

**Lemma 3.1** (See p. 359 in [17]). Given any ordering $\phi$ of the set of all primes, the class of all $\phi$-dispersive groups is a saturated formation.

**Proposition 3.2.** Suppose that every 3-maximal subgroup of $G$ is either $S$-quasinormal or modular in $G$. If $G$ is not supersoluble, then either $G$ is a group of order $pq^2$ for some distinct primes $p$ and $q$, or $G = Q \times P$, where $Q = C_G(Q)$ is a quaternion group of order 8 and $|P| = 3$.

**Proof.** Assume that this proposition is false and let $G$ be a counterexample of minimal order. Lemmas 2.2(4), 2.3(1) and Proposition 2.11 imply that every maximal subgroup of $G$ is strongly supersoluble. Hence $G$ is soluble by Lemma 2.6(1), so $|\pi(G)| = 2$ by Theorem 2.12.

Since $G$ is not supersoluble, $G$ is a $\Omega$-critical group. Let $D = G^{2^l}$ be the supersoluble residual of $G$. Lemma 2.6 implies that the following hold: (a) $D$ is a Sylow $p$-subgroup of $G$ for some prime $p$, and if $Q$ is a Sylow $q$-subgroup of $G$, where $q \neq p$, then $DQ = G$ and $Q/Q \cap \Phi(G)$ is either a cyclic prime power order group or a Miller-Moreno group; (b) $D/\Phi(D)$ is a non-cyclic chief factor of $G$ and if $D$ is non-abelian, then the center, commutator subgroup, and Frattini subgroup of $D$ coincide with one another; (c) if $p > 2$, then $D$ is of exponent $p$, for $p = 2$ the exponent of $D$ is at most 4. From Assertion (b) it follows that $QG = G$.

First we show that $|\Phi(D)| \leq p$. Indeed, assume that $|\Phi(D)| > p$, and let $M$ be a maximal subgroup of $G$ with $G = DM$ and $Q \leq M$. Then $M$ is supersoluble, so $G$ has a 3-maximal subgroup $T$ such that $Q \leq T$. Then $T^G = G$. If $T$ is $S$-quasinormal in $G$, then $G/T_G$ is nilpotent by Lemma 2.3(3). Hence $QT_G/T_G$ is normal in $G/T_G$, which implies that $QT_G = G \leq M$. This contradiction shows that $T$ is modular in $G$. Therefore $G/T_G$ is a $P$-group by Lemma 2.1(ii). But then from the $G$-isomorphism

$$DT_G/T_G\Phi(D) \simeq D/D \cap T_G\Phi(D) = D/\Phi(D)(D \cap T_G) = D/\Phi(D)$$

we get that $D/\Phi(D)$ is cyclic. This contradiction shows that $|\Phi(D)| \leq p$.

Now we show that $|Q| = q$. Assume that $|Q| > q$. Let $M$ be a maximal subgroup of $G$ with $|G : M| = q$. Then $M$ is supersoluble, so $G$ has a 3-maximal subgroup $T$ such that $|G : T| = pq^2$. Then $D \leq T^G$ and also we have $T_G \cap D \leq \Phi(D)$ and $T_G \leq \Phi(D)Q$. Moreover, $T$ is not $S$-quasinormal
in $G$ since $Q$ is a Sylow $q$-subgroup of $G$ and $|T \cap \Phi(D)| > p$. Hence $G/T_G = (T^G/T_G) \times (K/T_G)$ where $T^G/T_G$ is a non-abelian $P$-group of order prime to $|K/T_G|$ by Lemma 2.1(ii). But, clearly, $q$ divides $|(G/T_G) : (T^G/T_G)|$, so $Q \leq K$ and hence

$$Q T_G/T_G \leq C_{G/T_G}(T^G/T_G) \leq C_{G/T_G}(D T_G/T_G \Phi(D)),$$

where $D T_G/T_G \Phi(D)$ is $G$-isomorphic to $D/\Phi(D)$. Therefore $|D/\Phi(D)| = p$, a contradiction. Hence $|Q| = q$.

Note that if $|D/\Phi(D)| > p^2$ and $T$ is a subgroup of $D$ with $|D : T| = p^2$, then $T$ is a 3-maximal subgroup of $G$. But $T$ is neither $S$-quasinormal nor modular in $G$, a contradiction. Hence $|D/\Phi(D)| = p^2$.

Finally, assume that $\Phi(D) \neq 1$, so $|D| = p^3$. Assume also that $p \neq 2$. First note that since $|Q| = q$, $D$ is a maximal subgroup of $G$. On the other hand, from Assertions (b) and (c) we get that some subgroup $T$ of $D$ of order $p$ is not contained in $\Phi(D)$. It is clear that $T$ is a 3-maximal subgroup of $G$ and it is not $S$-quasinormal in $G$. Hence $T$ is modular in $G$, so $T^G = D$ is a non-abelian $P$-group by Lemma 2.1(ii). This contradiction shows that $p = 2$. Hence $q = 3$, since $G/C_G(D/\Phi(D)) \simeq Q$ and $|D/\Phi(D)| = 4$.

The proposition is proved.

**Lemma 3.3.** Suppose that $G$ is soluble and every $n$-maximal subgroup of $G$ is either modular or $S$-quasinormal in $G$. If $n \leq |\pi(G)| + 1$, then $G$ is $\phi$-dispersive for some ordering $\phi$ of $\mathbb{P}$.

**Proof.** Suppose that this lemma is false and let $G$ be a counterexample of minimal order. Let $N$ be a minimal normal subgroup of $G$ and $P$ a Sylow $p$-subgroup of $G$ where $p$ divides $|N|$. Then $N \leq P$.

(1) $C_G(N) = N$ and $G/N$ is strongly supersoluble. Hence $N < P$.

Lemma 2.4 implies that the hypothesis holds for $G/N$. Hence the choice of $G$ implies that $G/N$ is $\phi$-dispersive for some ordering $\phi$ of $\mathbb{P}$, so $N < P$. Therefore the choice of $G$ and Lemma 3.1 imply that $N \not\leq \Phi(G)$. Hence for some maximal subgroup $M$ of $G$ we have $G = N \rtimes M$. Then $\pi(M) = \pi(G)$, so $G/N \simeq M$ is strongly supersoluble by Theorem 2.12. Therefore $N$ is a unique minimal normal subgroup of $G$ by Lemma 2.5. Hence $C_G(N) = N$.

(2) $|\pi(G)| > 2$.

Indeed, assume that $\pi(G) = \{p, q\}$, and let $Q$ be a Sylow $q$-subgroup of $G$. Since $G/N$ is Ore dispersive by Claim (1) and $P$ is not normal in $G$, $NQ/N$ is a normal Sylow subgroup of $G/N$, so for some normal subgroup $V$ of $G$ we have $N \leq V$ and $|G : V| = p$. Then $\pi(V) = \pi(G)$. Hence $V$ is strongly supersoluble by Theorem 2.12. It follows that for the largest prime $r \in \pi(V)$ a Sylow $r$-subgroup $R$ of $V$ is characteristic in $V$ and so $R$ is normal in $G$. Hence $r = p$ is the largest prime in $\pi(G)$. Since $M$ is also Ore dispersive, a Sylow $p$-subgroup $M_p$ of $M$ is normal in $M$, so it is normal in $G$ since $N_G(M_p) \not\leq M$. But then $NM_p$ is a normal Sylow subgroup of $G$. This contradiction shows that $|\pi(G)| > 2$. 

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Take a prime divisor \( q \) of the order of \( G \) distinct from \( p \). Take a Hall \( q' \)-subgroup \( E \) of \( G \), and let \( E \leq W \) where \( W \) is a maximal subgroup of \( G \). Then \( N \leq E \) and since \( G \) is soluble, Lemmas 2.2(4) and 2.3(1) imply that the hypothesis holds for \( W \). Consequently, the choice of \( G \) implies that for some prime \( t \) dividing \( |E| \) a Sylow \( t \)-subgroup \( Q \) of \( E \) is normal in \( E \). Furthermore, since \( C_G(N) = N \) we have \( N \leq Q \). Hence, \( Q \) is a Sylow \( p \)-subgroup of \( E \). It is clear also that \( Q \) is a Sylow \( p \)-subgroup of \( G \) and \( \langle G : N_G(Q) \rangle, r \rangle = 1 \) for every prime \( r \neq q \). Since \( |\pi(G)| > 2 \), it follows that \( Q \) is normal in \( G \), so \( N = Q = P \). This contradiction completes the proof of the lemma.

In fact, Theorem B is a special case of the following

**Theorem 3.4.** Suppose that \( G \) is soluble and every \( n \)-maximal subgroup of \( G \) is either modular or \( S \)-quasinormal in \( G \). If \( n \leq |\pi(G)| + 1 \), then \( G^{\Delta_s} \) is a nilpotent Hall subgroup of \( G \).

**Proof.** Suppose that this theorem is false and let \( G \) be a counterexample of minimal order. Then \( G \) is not strongly supersoluble, so \( D = G^{\Delta_s} \neq 1 \). By Lemma 3.3, \( G \) has a normal Sylow \( p \)-subgroup \( P \) for some prime \( p \) dividing \( |G| \).

1. The conclusion of the theorem holds for every quotient \( G/R \neq G/1 \) (This directly follows from Lemma 2.4).

2. \( D \) is nilpotent.

Assume that this is false. Then, since \( G^{\Delta_s} \leq G' \), \( G \) is not supersoluble.

Let \( R \) be a minimal normal subgroup of \( G \). By Claim (1) and \[13\] 2.2.8, \( (G/R)^{\Delta_s} = DR/D \simeq D/D \cap R \) is nilpotent. If \( G \) has a minimal normal subgroup \( N \neq R \), then \( D/D \cap (R \cap N) \) is nilpotent. Hence \( R \) is a unique minimal normal subgroup of \( G \) and, by \[17\] A, 13.2, \( R \notin \Phi(G) \). Therefore \( R = C_G(R) \) by \[17\] A, 15.6, and \( G = R \rtimes M \) for some maximal subgroup \( M \) of \( G \) with \( M_G = 1 \). Then \( R = P \) is a Sylow \( p \)-subgroup of \( G \) by \[17\] A, 13.8. It is clear that \( M \) is not supersoluble, so \( |R| > p \) since otherwise \( M \simeq G/R = G/C_G(R) \) is cyclic.

Now let \( T \) be any maximal subgroup of \( M \). Then \( RT \) is a maximal subgroup of \( G \) and \( |\pi(RT)| = |\pi(G)| \) or \( |\pi(RT)| = |\pi(G)| - 1 \). Hence, by Lemmas 2.2(4) and 2.3(1), \( RM \) satisfies the same assumptions as \( G \), with \( n - 1 \) replacing \( n \). The choice of \( G \) implies that \( (RT)^{\Delta_s} \leq F(RT) = R \). Therefore \( T \simeq T/(T \cap (RT)^{\Delta_s}) \simeq (RT)^{\Delta_s}T/(RT)^{\Delta_s} \) is strongly supersoluble. Hence \( M \) is a \( \Delta \)-critical group.

By Lemma 2.6(1), \( 1 < |\pi(M)| \leq 3 \). First assume that \( |\pi(M)| = 2 \), then \( n = 4 \) by Theorem 2.12 since \( M \) is not supersoluble. Hence every 3-maximal subgroup of \( M \) is either modular or \( S \)-quasinormal in \( G \). Proposition 3.2 implies that \( M \) is a non-supersoluble group of order \( qr^2 \) for some distinct primes \( q \) and \( r \), or \( M = Q \times L \), where \( Q = C_M(Q) \) is a quaternion group of order 8 and \( |L| = 3 \). Then \( R \) is a 3-maximal subgroup of \( G \). Thus every maximal subgroup of \( R \) is either modular or \( S \)-quasinormal in \( G \) and so \( |R| = p \) by Lemma 2.10(2), a contradiction. Thus \( |\pi(M)| = 3 \), so \( n = 5 \) and hence every 4-maximal subgroup of \( M \) is either modular or \( S \)-quasinormal in \( G \). Let \( |M| = q^ar^br^c \), where \( p, r \) and \( t \) are primes. If \( a + b + c > 4 \), then some member \( T \) of
a composition series of $M$ is a non-identity $4$-maximal subgroup of $M$ since $G$ is soluble, which is impossible by Lemma 2.10. Hence $a+b+c=4$ since $M$ is not supersoluble. Therefore, by Lemma 2.6, $M = Q \times (L \times T)$, where $|Q|=q^2$, $|L|=r$ and $|T|=t$. Then $R$ is a $4$-maximal subgroup of $G$, so every maximal subgroup of $R$ is either modular or $S$-quasinormal in $G$, which is impossible by Lemma 2.10(2). This contradiction completes the proof of Claim (2).

(3) $D$ is a Hall subgroup of $G$.

Suppose that this is false. Then $G$ is not strongly supersoluble. Let $P$ be a Sylow $p$-subgroup of $D$ such that $1 < P < G_p$, where $G_p \in \text{Syl}_p(G)$.

(a) $D = P$ is a minimal normal subgroup of $G$.

Let $R$ be a minimal normal subgroup of $G$ contained in $D$. Then $R$ is a $q$-group for some prime $q$. Moreover, $D/R = (G/R)^{[D]}$ is a Hall subgroup of $G/R$ by Claim (1) and [19, 2.2.8]. Suppose that $PR/R \neq 1$. Then $PR/R \in \text{Syl}_p(G/R)$. If $q \neq p$, then $P \in \text{Syl}_p(G)$. This contradicts the fact that $P < G_p$. Hence $q = p$, so $R \subseteq P$ and therefore $P/R \in \text{Syl}_p(G/R)$ and we again get that $P \in \text{Syl}_p(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is the unique minimal normal subgroup of $G$ contained in $D$. Since $D$ is nilpotent by Claim (2), a $p'$-complement $E$ of $D$ is characteristic in $D$ and so it is normal in $G$. Hence $E = 1$, which implies that $R = D = P$.

(b) $D \not\subseteq \Phi(G)$. Hence for some maximal subgroup $M$ of $G$ we have $G = D \times M$ (This follows from Lemma 2.5 since $G$ is not strongly supersoluble).

(c) If $G$ has a minimal normal subgroup $L \neq D$, then $G_p = D \times L$. Hence $O_{p'}(G) = 1$.

Indeed, $DL/L \simeq D$ is a Hall subgroup of $G/L$ by Claim (1). Hence $G_pL/L = RL/L$, so $G_p = D \times (L \cap G_p)$. But $D < G_p$, so $(L \cap G_p)$ is a non-trivial subgroup of $L$. Since $G$ is soluble, it follows that $L$ is a $p$-group and so $G_p = D \times L$. Thus $O_{p'}(G) = 1$.

(d) $\Phi(G_p) = 1$.

Suppose that $\Phi = \Phi(G_p) \neq 1$. Then, since $G_p$ is normal in $G$ by Claim (c), $\Phi$ is normal in $G$ and so we can take a minimal normal subgroup $L$ of $G$ contained in $\Phi$. But then $G_p = D \times L = D$ by Claim (c), a contradiction. Hence we have (d).

Final contradiction for (3). Claim (d) implies that $G_p$ is an elementary abelian normal subgroup of $G$. By Maschke’s theorem $G_p = N_1 \times N_2$ is the direct product of some minimal normal subgroups of $G$. Claim (a) implies that $N_1 < G_p$. Let $M = N_2E$, where $E$ is a complement to $G_p$ in $G$. Then $M$ is a maximal subgroup of $G$ and $\pi(M) = \pi(G)$. On the other hand, every $(n-1)$-maximal subgroup of $M$ is either modular or $S$-quasinormal in $M$ by hypothesis and Lemmas 2.2(4) and 2.3(1). Thus $M \simeq G/N_1$ is strongly supersoluble by Theorem 2.12. Similarly we get that $G/N_2$ is strongly supersoluble. Hence $G \simeq G/N_1 \cap N_2$ is strongly supersoluble by Lemma 2.5. This contradiction shows that $D = G^{[4\text{ vs.}]}$ a Hall subgroup of $G$.

The proof of the theorem is complete.
4 Final remarks

1. Some preliminary results are of independent significance because they generalize some known results.

From Proposition 2.11 we get the following

**Corollary 4.1** (Schmidt [1]). If Every 2-maximal subgroup $M$ of $G$ is modular, then $G$ is nearly nilpotent.

**Corollary 4.2.** If every 2-maximal subgroup of $G$ is $S$-quasinormal in $G$, then $G$ is nearly nilpotent.

**Corollary 4.3** (Agrawal [20]). If every 2-maximal subgroup of $G$ is $S$-quasinormal in $G$, then $G$ is supersoluble.

From Proposition 3.2 we get the following known result.

**Corollary 4.4** (Schmidt [1]). If every 3-maximal subgroup $M$ of $G$ is modular in $G$ and $G$ is not supersoluble, then either $G$ is a group of order $pq^2$ for some distinct primes $p$ and $q$ or $G = Q \times P$, where $Q = C_G(Q)$ is a quaternion group of order 8 and $|P| = 3$.

2. In closing note that the restrictions on $|\pi(G)|$ in Theorems A and B cannot be weakened. Indeed, for Theorem A this follows from the example of the alternating group $A_4$ of degree 4. For Theorem B this follows from the example of the $A_4 \times C_2$, where $C_2$ is a group of order 2.

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