Supergravity, AdS/CFT Correspondence, and Matrix Models

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The recent developments towards the possible non-perturbative formulation of string/M theory using supersymmetric Yang-Mills matrix models (SYMs) are discussed. In the first part, we give a critical review on the status of our present understanding, focusing on the connection of the D0-brane matrix models to supergravity and its relevance to the so-called Matrix-theory conjecture. We also discuss some problems concerning the conjectured relation between supergravity in AdS background and SYM from the viewpoint of D-brane interactions. We present a qualitative argument showing how the boundary condition at AdS boundary dictates the correlators on the large $N$ system of source D-branes. Then, in the final part, we turn to the question how to formulate the condensation of graviton in matrix models, taking the simplest example of type IIB matrix model. We argue the emergence of a hidden symmetry GL(10, R), beyond the manifest Lorentz symmetry SO(9,1), by embedding U($N$) model into models with higher $N$ and by treating the whole recursive series of models simultaneously. This suggests a possible approach toward background independent formulations of matrix models.

§1. Introduction

One of the most remarkable insights gained through the recent developments in the studies of duality symmetries of string theory is the possibility of formulating nonperturbative string theory in terms of supersymmetric Yang-Mills theories. From an ordinary viewpoint of perturbative string theory, Yang-Mills theories are regarded as the low-energy effective theories for describing interactions of gauge-field excitations of strings. The discovery of crucial roles played by Dirichlet branes (D-branes) for realizing string dualities, however, paved a way toward possible reformulations of string theory using new degrees of freedom other than the fundamental strings, on the basis of an entirely different interpretation of Yang-Mills fields. Since string field theories assuming the fundamental strings themselves to be the basic degrees of freedom do not seem to be appropriate for nonperturbative studies of the theory, such an alternative possibility has long been sought, but has never been materialized in concrete form.

In the present report, I would like to review the status of Yang-Mills matrix models from the viewpoint of asking the question, “Why could the models be the theory of quantum gravity?” In this written version of the talk, some of the subjects which I have presented in the YITP workshop, held in succession to the Nishinomiya Yukawa symposium, will also be included and expanded.

We will start from the so-called Matrix theory which was proposed first and has been a focus of most intensive studies. Next we will turn to the so-called AdS/CFT(SYM) correspondence. The purpose of this first part is to review the

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known results critically and provide a few new observations on some unsolved issues. We will then proceed to the issue why Yang-Mills matrix theories could be the models for quantum gravity. A special emphasis will be put on possible hidden symmetry structure which would ensure the emergence of general covariance at long distance regime. The last part discussing the problem of background independence is a preliminary report from a still unfinished project and will be of very speculative character.

§2. Yang-Mills matrix models and supergravity

The D-branes are objects carrying Ramond-Ramond (RR) charges. They are necessary for realizing duality symmetry among various perturbative vacua of string theory, since the transformations associated with duality interchange Neveu-Schwarz-Neveu-Schwarz (NS-NS) and RR charges. In the case of perturbative closed-string theories, it is known that the type IIA or IIB theory allows even or odd (spatial) dimensional D-branes, respectively. In particular, the lowest dimensional objects are D0-brane (D-particle) in IIA and D(-1)-brane (D-instanton) in IIB.

In low-energy effective field theory, namely, IIA or IIB supergravity, D-branes are represented as solitonic classical solutions. From the viewpoint of ordinary world-sheet formulation of the theories, they are described as collective modes of fundamental strings, in which the collective coordinates can be identified with the space-time coordinates at the boundaries of open strings with Dirichlet condition. In old perturbative string theory, it has been thought that open strings cannot be coupled to closed strings consistently, since they necessarily break the N=2 supersymmetry of closed-string sector. In our new understanding, the partial breakdown of supersymmetry just indicates the existence of D-branes as physical objects, and the remaining supersymmetry is reinterpreted as the manifestation of the BPS property of D-branes.

To describe the dynamics of D-branes, we have to therefore study coupled systems of closed strings and open strings with dynamical Dirichlet boundary conditions. Here it is worthwhile to recollect an old but well-known formulation of open strings, namely, Witten’s string field theory. The latter only uses open-string fields as the dynamical degrees of freedom. Nevertheless, it includes the whole dynamics of interacting closed-open strings. Namely, it is possible to describe the dynamics of closed strings in terms of open-string degrees of freedom without explicitly introducing fields corresponding to closed strings. This remarkable property is actually a consequence of the old s-t duality which is the basis for conformal invariance of the world-sheet string dynamics. Furthermore, if there were circumstances where we can neglect the excitation modes of open strings higher than the lowest Yang-Mills degrees of freedom, we can even imagine situations where the whole dynamics including quantum gravity effect can be described by Yang-Mills theories. Let us briefly review some representative proposals along this line.
2.1. D-particle model or Matrix theory

The first such model is called ‘Matrix theory’. The model is based on the 1+0-dimensional Yang-Mills theory with maximum (N=16) supersymmetry, which is obtained by the dimensional reduction from 10 (=1+9) dimensional N=1 supersymmetric Yang-Mills theory.

\[
S = \int dt \text{Tr} \left( \frac{1}{2g_s \ell_s} D_t X_i D_t X_i + i \theta D_t \theta + \frac{1}{4g_s \ell_s^5} [X_i, X_j]^2 - \frac{1}{\ell_s^2} \theta \gamma_i [\theta, X_i] \right) \tag{2.1}
\]

\[
S = \int dt \text{Tr} \left( \frac{1}{2R} D_t X_i D_t X_i + i \theta D_t \theta + \frac{R}{4\ell_P^6} [X_i, X_j]^2 - \frac{R}{\ell_P^3} \theta \gamma_i [\theta, X_i] \right), \tag{2.2}
\]

where \( \ell_s \) and \( g_s \) are string length and coupling constants, respectively. In the second line, we have introduced the 11 dimensional parameters of M-theory, compactification radius \( R = g_s \ell_s \) along the 11th direction (10th spatial direction) and the 11 dimensional Planck length \( \ell_P = g_s^{1/3} \ell_s \). The Higgs field \( X_i (i = 1, 2, \ldots, 9) \) are dimensionally reduced U(\( N \)) gauge-field matrices whose diagonal components are identified with the collective coordinates of \( N \) D-particles, while the off-diagonal components are the fields of lowest open string modes connecting the D-particles. The 16 component Grassmann (hermitian) matrices \( \Psi \) transforming as SO(9) spinor are the super partner of the Higgs fields.

In the matrix theory conjecture proposed in ref.3, this action is interpreted as the effective action of the theory in the Infinite-Momentum Frame (IMF) where the 11th total momentum \( P_{11} \) is taken to be infinitely large. Following the M-theory identification of the 11th momentum with the RR 1-form charges of D-particle, it is assumed that \( P_{11} = N/R \). \tag{2.3}

Thus, for any finite fixed \( R \), the IMF limit corresponds to taking the large \( N \) limit, \( N \to \infty \). In other words, this requires that the IMF Hamiltonian \( P^- = P^0 - P_{11} \) must have nontrivial dynamics in the part which scales as \( 1/N \). One of the reasonings for this conjecture is that in the IMF frame (the part of) the (super) Poincaré symmetry is reduced to (super) Galilean symmetry in the 9 dimensional transverse space and the above action precisely exhibits that symmetry. In particular, to be a theory in 11 dimensional space-time, it should exhibit the \( N = 1 \) supersymmetry in 11 dimensions which amounts to \( N = 2 \) supersymmetry in 1+9 dimensions. Indeed the model has, in addition to the \( N = 16 \) supersymmetry in 1+0 dimensions inherited from 10 dimensional \( N = 1 \) supersymmetry

\[
\delta^{(1)} \theta = \frac{1}{2} \left( \frac{1}{R} D_t X_i \gamma^i + \frac{i}{2\ell_P^2} [X_i, X_j] \gamma^{ij} \right) \epsilon^{(1)}, \tag{2.4}
\]

a trivial supersymmetry under

\[
\delta^{(2)} \theta = \epsilon^{(2)}, \tag{2.5}
\]

where \( \epsilon^{(1)} \) and \( \epsilon^{(2)} \) are two independent constant Majorana spinors. The algebra of these two supersymmetry transformations closes with central charges up to a field-dependent gauge transformation. For a single D-particle state at rest as the simplest
example, the first symmetry is unbroken, corresponding to the BPS property of the state, while the second is broken. Furthermore, the dimension of the multiplet of single particle states fits to the desired multiplet corresponding to the first Kaluza-Klein mode of the 11 dimensional multiplet containing massless graviton and gravitino: The 16 component Grassmann coordinate $\theta$ leads to $2^{16/2} = 256 = 128 + 128$ dimensional representation of transverse $\text{SO}(9) \simeq \text{Spin}(9)$ group, which is precisely the physical dimension of the 11 dimensional supergravity multiplet.

Of course, the Galilean symmetry is not sufficient to justify the decoupling of the higher string modes. That is the crucial dynamical assumption of the model. A piece of evidence for this conjecture comes from the old observation made long time ago in connection with the theory of membrane. Namely, the same model can be interpreted as a special regularized version of the membrane action\(^4\) in the light-cone gauge in 11 dimensions. The large $N$ limit in this interpretation is nothing but the continuum limit. The fundamental string of 10 dimensional type IIA theory is identified with the membrane which is wrapped along the compactified 11th direction. If this really works, it is quite natural to expect that after taking the appropriate large $N$ limit the model would reproduce the whole dynamics including the effect corresponding to higher string modes. The crucial new observation here is that the model should be interpreted as describing the arbitrary multi-body systems of membranes or D-particles. This solves the long-standing problem in the formulations of supersymmetric membranes, namely, the difficulty of continuous energy spectrum. The energy spectrum of the system must be continuous from zero, to be the theory of multi-body system including the massless particles. For the validity of this interpretation, it is necessary that there exists one and only one threshold bound state which is identified with the single-particle graviton supermultiplet for each fixed $N$ of $\text{U}(N)$. At least for $N = 2$, this is consistent with the Witten index of the model\(^5\).

Another impetus for this model is the proposal that the model might be meaningful even for finite $N$. That is, Susskind\(^6\) suggested that the model for finite $N$ should be interpreted following the framework of the so-called discrete light-cone quantization (DLCQ), in which the compactification is made along the light-like direction $x^- = x^{11} - x^0$ instead of the space-like direction. Such a formalism has often been discussed to regularize gauge field theories. In fact, this proposal can be related to the IMF interpretation by considering the limit of small $R$ keeping $N$ fixed, which is another way of making $P_{11}$ large. If we boost the system simultaneously with taking this limit, we can keep the longitudinal momentum $P_-$ finite and the condition of compactification is imposed on the $x^-$ direction with finite compactification radius in the small $R$ limit. This is essentially the argument given in ref.\(^7\). Equivalently, using the original frame with sufficiently small $R$, the compactification condition $x^\pm \sim x^\pm + 2\pi R$ in the space-like 11th dimension can be approximated by the light-like condition $(x^-, x^+) \sim (x^- + 2\pi R, x^+)$, since in the limit we are only interested in the small longitudinal energy $P_+ \sim P_i^2/2P_-$ proportional to $R$ while $P_- \sim P_{11} \sim O(1/R)$ becomes large.

Now since the limit forces the 11 dimensional Planck length small compared to the string length, $\ell_P \ll \ell_s$, we expect that the interaction of D-particles can be
described by lowest open string modes at least at distance scales shorter than the string length, according to the result of ref. \(8\). Also, the 11 dimensional Newton constant \(G_{11} \sim g_s^3 \ell_s^3\) becomes small in this limit. Thus from the viewpoint of closed strings or membranes, the interaction of D-particles should be approximated well by classical supergravity at least for distance scales much larger than the string length \(\ell_s\). If one naively assumes that Matrix theory is correct and that the justification of Matrix theory comes solely from the infinite momentum limit, one might expect that Matrix theory for sufficiently small \(R\) with finite \(N\) must reproduce the classical supergravity at all distance scales which are larger than the 11 dimensional Planck length. This would in particular require that the lowest open string mode alone describes correctly the gravitational interaction of D-particles for such wide ranges of distance scales, namely, from the infinite large distances all the way down to near the 11D Planck length which is far below the string scale. This would be quite a surprising conclusion, since we usually think that the duality between open and closed strings is due to the existence of full tower of higher string modes on both sides of closed and open strings. In particular, the effective dynamics near the string length after eliminating the higher-string modes would necessarily be non-local either in terms of the lowest graviton fields alone or of the lowest gauge modes alone.

Before discussing further the meaning of this and where this naive expectation may be invalidated, let us briefly review known results related to this issue.

### 2.2. Matrix theory vs. supergravity

In fact, as discussed in ref. \(8\), supersymmetry ensures that the above conclusion is indeed true at least in the one-loop approximation in terms of open string computation. The two-body interaction, \(v^4/r^7\), of D-particles in the leading approximation with respect to the expansion in velocity is correctly reproduced by only the lowest open string modes. Namely, the same expression is valid for the large \(r\) region where the only lowest modes of the closed string couple, as described by supergravity. Furthermore, at least for two-body interactions, a non-renormalization theorem \(9\) is established demanding that the one-loop result for the leading term is not renormalized by higher order effects. This theorem can be generalized to the next order \(v^6\) for the two-body interaction and is consistent with the result of explicit two-loop computation \(10\) of the two-body interactions.

Whether similar non-renormalization theorem is valid for more general multi-body interactions is not known. Extension of the argument given in \(10\) to general \(N\)-body interactions \(10\) is difficult. In general, however, we hope that some symmetry together with certain additional inputs would fix the theory of gravity completely. For example, we believe that general covariance and locality uniquely lead to General Relativity at sufficiently large distances. So the question is whether the supersymmetry of the matrix model (2.2) is sufficient to ensure the general coordinate invariance at large distances as interpreted in 11 dimensional space-time. If we assume the existence of massless graviton supermultiplet and Lorentz symmetry in the flat background in 11 dimensions, only consistent low-energy effective theory is believed to be supergravity. Establishing Lorentz invariance of the model in the limit \(R, N \to \infty\) would thus be most desirable. At least for membrane approxima-
tion, this is very plausible. However, the membrane approximation is not sufficient to establish the Lorentz symmetry, since the interpretation of the matrix model is really very different from membrane as emphasized already. Unfortunately no concrete proposal for general case has been given. It is thus desirable to perform explicit computations for multi-body interactions.

Let us here briefly review the result of explicit computations of 3-body interaction of D-particles at finite $N$. A scaling argument shows that the effective lagrangian of the 3-body interaction must take the following form

$$L_3 \sim \frac{G_{11}^2}{R^5} \frac{v^6}{r^{14}},$$  \hspace{1cm} (2.6)

where the factor $v^6/r^{14}$ only indicates power behaviors with respect to relative velocities ($v$) and to relative distances ($r$). The power $R^{-5}$ with respect to the compactification radius is required by boost invariance along the 11th direction. Note that in terms of the Yang-Mills coupling $g_{YM}^2 \propto g_s^2$ the factor $G_{11}^2/R^5 \propto g_{YM}^2$ corresponds to the two-loop contribution. For small $g_s$, the compactification radius is small, but the Newton constant is also vanishing such that the expansion parameter $G_{11}^2/R^5$ is arbitrarily small. This indicates that the regions of validity of classical supergravity and perturbative computation in the matrix model might overlap, if only the parameters are concerned neglecting the real roles of the dynamical variables. Therefore it is not unreasonable if the matrix model with finite $N$ is able to reproduce supergravity results to some finite orders with respect to the Newton constant.

In classical supergravity, we can derive the following explicit form for the interaction lagrangian

$$L_3 = L_V + L_Y.$$  \hspace{1cm} (2.7)

where

$$L_V = -\sum_{a,b,c} \frac{(15)^2 N_a N_b N_c}{64 R^5 M^{18}} v_{ab}^2 v_{ca}^2 (v_{ca} \cdot v_{ab}) \frac{1}{r_{ab}^7 r_{ca}^7}.$$  \hspace{1cm} (2.8)

$$L_Y = -\sum_{a,b,c} \frac{(15)^3 N_a N_b N_c}{96(2\pi)^4 R^5 M^{18}} \left[ -v_{bc}^2 v_{ca}^2 (v_{cb} \cdot \nabla_c)(v_{ca} \cdot \nabla_c) \\
+\frac{1}{2} v_{ca}^4 (v_{cb} \cdot \nabla_c)^2 + \frac{1}{2} v_{bc}^4 (v_{ca} \cdot \nabla_c)^2 \\
-\frac{1}{2} v_{ba}^2 v_{bc}^2 (v_{cb} \cdot \nabla_c)(v_{bc} \cdot \nabla_b) \\
+\frac{1}{4} v_{bc}^4 (v_{ba} \cdot \nabla_b)(v_{ca} \cdot \nabla_c) \right] \Delta(a,b,c)$$  \hspace{1cm} (2.9)

and

$$\Delta(a,b,c) = \int d^3 y \frac{1}{|x_a - y| |x_b - y| |x_c - y|}$$

$$= \frac{64(2\pi)^3}{(15)^3} \int_0^{\infty} d^3 \sigma (\sigma_1 + \sigma_2 + \sigma_3) \frac{1}{|x_a - x_b| |x_b - x_c| |x_c - x_a|^{3/2}} \exp \left(-\sigma_1 |x_a - x_b|^2 - \sigma_2 |x_b - x_c|^2 - \sigma_3 |x_c - x_a|^2 \right).$$  \hspace{1cm} (2.10)
and the indices $a, b, c, \ldots$ label the D-particles whose masses are $N_a/R, N_b/R, N_c/R, \ldots$. The Planck mass $M = 1/\ell_P$ is defined by $G_{11} = 2\pi^5/M^9$. The above separation into V-part and Y-part roughly corresponds to the contributions from the seagull-type diagrams and the diagrams with one 3-point self-interaction of graviton, respectively. Because of the BPS property, the contribution from the Y-part vanishes whenever any two D-particles have parallel velocities.

On the side of Matrix theory, we compute the scattering phase shift in the eikonal approximation. Each of the D-particles with masses $N_a/R, N_b/R, N_c/R, \ldots$ is approximated as a cluster of corresponding number $(N_a, N_b, N_c, \ldots)$ of D-particles, moving parallel within each cluster, with the smallest unit of mass $1/R$. We can again separate the two-loop contributions into V- and Y-types. The Y-type contribution only comes from the diagrams with two 3-point vertices. The V-type contribution comes from the diagrams with one 4-point vertex and also from the diagrams with two 3-point vertices in which one of the propagators is canceled by the derivatives acting on the 3-point vertices. For more details, the reader should consult our original papers.\cite{12,13}. It turns out that the V-type contribution to the eikonal phase shift can be written as the time integral of the above lagrangian $L_V$. For the Y-type contribution which is vastly more complicated, we have confirmed that the result of explicit time integration of the lagrangian $L_Y$ precisely agrees with the phase shift obtained from the matrix model. We can also show the precise correspondence at the level of the equations of motion on both sides including the effect of recoil\cite{13} to the present order of approximation.

In the absence of general arguments which may guarantee the agreement between matrix theory with finite $N$ and supergravity in the long-distance limit, the above 3-body computation is the strongest evidence so far for the validity of Matrix theory conjecture in its DLCQ interpretation. A related computation involving the nonlinear graviton interaction has also been done for graviton scattering in an orientifold background\cite{14} and exact agreement is verified. Extension of these computations to higher-loop/body interactions is not difficult at least conceptually, but is technically formidable and no complete computations for higher cases have been reported yet, except for a partial computation\cite{15} which indicates some signal for a possible discrepancy with supergravity at 3-loop order.

It should be emphasized here that given only the connection between the matrix model as the low-energy effective theory for D-particles in type IIA theory, on one hand, and the connection of supergravity and closed strings on the other hand, the agreement of D-particle scatterings between supergravity and matrix model at arbitrary large distances is in no sense a logical consequence. Remember that the argument of ref.\cite{7} is not applicable at distances near and larger than the string scale. Suppose that the original BFSS conjecture that in the large $N$ limit (and for fixed $R$) the agreement is achieved is true. Then the disagreement between supergravity and matrix theory at finite $N$ with small $R$, if it indeed occurs, must be due to the neglect of bound-state effect in forming the states of D-particles with large longitudinal momentum. Namely, no matter how $R$ is small, only for sufficiently large $N$ we would expect that the effect of higher string modes which would ensure the validity of the $s$-$t$ duality between open and closed strings is correctly taken into
account. To check whether matrix theory can give sensible results in this way is, however, a very difficult problem, since for large $N$ the size of graviton is known to grow indefinitely in the limit, and hence we have to deal with complicated many body dynamics of D-particles (or partons).

If that is the case, it is desirable to have definite criteria on the basis of which we can assess various situations such that agreements or disagreements between supergravity and matrix theory can be predicted by general arguments. For example, if we assume the correspondence between supergravity and D0-matrix model following Maldacena’s general conjecture which will be the subject of the next subsection, the validity of the classical 10 dimensional supergravity description is expected for the distance scales $\ell_P N^{1/7} \ll r \ll \ell_P N^{1/3}$, where the first and the second inequalities come from the weak coupling condition and the small curvature condition, respectively. At the lower end, the effective radius along the 11th direction becomes of the same order as $\ell_P$. Thus from the viewpoint of 11 dimensions, it should not be regarded as the limit of the supergravity description as long as the curvature radius is much larger than the Planck length, although 10 dimensional description is no more valid. However the upper limit indicates that the agreement with classical supergravity at arbitrarily large distances can only be achieved in an appropriate large $N$ limit. Namely, the parameter $N$ plays effectively a role of infrared cutoff for the theory, not only with respect to the 11th direction but also to the transverse space in the bulk, as we have argued before from the correspondence between the matrix model and a regularized theory of membranes. In other words, the low-energy long distance physics of supergravity is governed by the high-energy physics of open strings where in general we cannot neglect higher string (or membrane) modes.

However, we should also keep in mind that the near-horizon limit, approximating the factor $1 + q/r^7$ by $q/r^7$, is only valid if $r \ll (g_s N)^{1/7}$ where $q \propto g_s N$, and hence the argument cannot be extended to the upper limit $\ell_P N^{1/3}$ in the strong coupling region. Thus strictly speaking we can not be sure whether Matrix theory reproduces the large distance behavior of classical supergravity in the large $N$ limit for fixed $g_s$ when $g_s N \gg 1$, even if we assume the validity of the Maldacena correspondence in its original form. Of course, the Maldacena conjecture only proposes sufficient conditions, and hence does not necessarily exclude the possibility that the region of validity extends beyond these conditions because of some (hidden) symmetry constraints depending on the type of physical quantities in question. We should also expect from a more general viewpoint that some (but perhaps already ‘built-in’) symmetry must be responsible for the matrix model to reproduce supergravity in spite of the rapid growth of the size of graviton in the large $N$ limit. Perhaps the precise agreement of 3-body interaction in the above finite $N$ calculation should be interpreted as a partial indication for the existence of such higher symmetry.

Concerning the question on why the matrix model can be the theory of gravity, one of the other crucial unsolved problems is how to extend the model to general curved backgrounds. It has been argued that any simple modification of the quantum mechanical lagrangian (2.2) for the curved space cannot reproduce the supergravity result even at the order $r^4$ for the D-particle interaction in curved space for finite $N$. This seems to indicate that the curved background cannot, in general, be described
by finite $N$ models. Indeed, this is not unreasonable since to really modify the background in a self-consistent fashion within the framework of M-theory, we must consider the condensation of gravitons. It is difficult to treat finite condensation of graviton in the present framework of Matrix theory which assumes fixed $N$ however it is large. In the last part of this talk, I will give a preliminary consideration on the graviton condensation in a simpler case of type IIB matrix model \cite{I}. For the possibility of modifying the action to curved backgrounds, an axiomatic approach called D-geometry \cite{J} has been suggested. We have to await to see whether this approach can resolve the above issues. We also mention a recent important work \cite{K} discussing the change of the background in Matrix theory, on the basis of one-loop computations of the interactions between an arbitrary pair of extended objects in the theory. For an earlier approach from the viewpoint of membrane dynamics in curved background, see \cite{L} and references therein.

2.3. AdS/SYM correspondence

As already mentioned, another recent development closely related to Matrix theory is the conjectured correspondence \cite{M} between supergravity in anti de Sitter background on one hand and super Yang-Mills theory of D-branes on the other. This is essentially based on the following two observations. Firstly, the low-energy (low-velocity) dynamics of many D-branes which are situated almost on top each other is well described by the effective super Yang-Mills theory for any finite \(N\), since we can assume the decoupling of higher modes of open strings for the same reason as we have argued in the case of Matrix theory. Secondly, the field-theory description in terms of supergravity is expected to be simultaneously effective when the curvature near the horizon becomes sufficiently small compared with the string length. In the case of D3-brane, in particular, the curvature radius at the horizon is of order

\[ R_c \propto (g_{YM}^2 N)^{1/4} \ell_s. \] (2.11)

D3-brane is special in that the background dilaton is constant, and thus can be made arbitrarily small by keeping \( R_c \) large \((g_{YM}^2 N \gg 1)\) if \(N\) is sufficiently large. Then by the duality between open and close strings, we naturally expect that the descriptions of the dynamics of D3-branes in terms of supersymmetric Yang-Mills theory or type IIB supergravity are both valid. In other words, super Yang-Mills theory in the large \(N\) limit is expected to be ‘dual’ to supergravity with the D3-background in the near horizon limit. The D3-brane metric in the near horizon limit is the direct product, \(\text{AdS}_5 \times S^5\), of the five dimensional anti de Sitter space-time \(\text{AdS}_5\) and five dimensional sphere \(S^5\). Thus the metric has isometric symmetry under the group \(SO(4,2) \times SO(6)\). Correspondingly, Yang-Mills theory for D3 branes are the \(N=4\) superconformal Yang-Mills theory which has the same conformal symmetry \(SO(4,2)\) and global R-symmetry \(SO(6)\). Using this correspondence, we can for example predict the spectrum of the superconformal Yang-Mills theory in the large \(N\) limit by analyzing the Kaluza-Klein spectrum around the \(\text{AdS}_5 \times S^5\).

A more concrete prescription which allows us to connect correlators of both sides has been proposed in \cite{N}. It essentially says that the effective action of supergravity for the supergravity fields which satisfy appropriate boundary condition at the
boundary of the AdS space (opposite to the horizon) is the generating functional for the correlators of super Yang-Mills theory. The external fields for the latter generating functional coupled to operators of the Yang-Mills theory are nothing but the boundary value of the bulk fields in supergravity. Many computations of correlators have been done based on this conjecture. However, it seems that this prescription has never been derived logically from the duality between open and closed strings. For example, it is not clear why the boundary condition at the boundary of the AdS space-time can dictate the choice of operators of the large \( N \) Yang-Mills theory, since naively the D-branes as the heavy source producing the AdS background seem to be situated at the opposite ‘boundary’ of the AdS space. In the following, we will first discuss some interesting aspect related to the correspondence of conformal symmetries on both sides, and then come back again to the issue of correlators later.

The metric of the AdS\(_5\)×S\(_5\) is given by

\[
d s^2 = \alpha' \left( \frac{R^2}{U^2} (dU^2 + U^2 d\Omega_5^2) + \frac{U^2}{R_c^2} dx_4^2 \right),
\]

(2.12)

where \( U = r/\alpha' \) (\( \alpha' \propto \ell_s^2 \)) is the energy of an open string stretched from the source D3-branes at the origin to a probe D3-brane. The four dimensional flat metric \( dx_4^2 \) is interpreted as describing the world-volume of the source consisting of \( N \) D3-branes which are almost coincident to each other. The special conformal transformation in the SO(4,2) isometry is

\[
\delta_K x^a = -2 \epsilon \cdot x x^a + \epsilon^a R^4 c U^2,
\]

(2.13)

\[
\delta_K U = 2 \epsilon \cdot x U.
\]

(2.14)

As noted originally in ref.\( \text{[19]} \), the existence of the last term \( R_c^4 / U^2 \) leads to a nonlinear and field-dependent transformation for the dynamical coordinates of the probe D3-brane. The latter property constrains the action to be the Dirac-Born-Infeld action for the probe D3-brane in the background of the source D3-branes, with a help of a supersymmetric nonrenormalization theorem. If the conjectured relation between Yang-Mills theory and supergravity is valid, it must be possible to derive the same property on the side of D3-brane Yang-Mills theory. However, the special conformal transformation of the world-volume Yang-Mills theory is the standard one,

\[
\delta_K x^a = -2 \epsilon \cdot x x^a + \epsilon^a x^2
\]

(2.15)

without the last term of eq. (2.13). On the Yang-Mills side, the coordinate \( U \) is the radial component of the diagonal part of the 6 Higgs fields \( X_i \) (\( i = 1 \sim 6 \)) which, on the side of supergravity, correspond to the space described by \( \{ U, S^5 \} \).

The solution of this puzzle is the following. To study the dynamics of a probe D3-brane in the background of the source D3-branes, we have to derive effective theory for the diagonal Higgs fields by integrating over the off-diagonal components corresponding to the elements of the quotient group \( U(N)/U(N - 1) \times U(1) \). In performing this integration, we have to impose the gauge condition, most conveniently, the familiar background gauge condition. However, it turns out that the
background gauge condition (or any other reasonable gauge condition) is not invariant under the special conformal transformation, and therefore we have to make a field-dependent gauge transformation which compensates the violation of the conformal invariance. Thus the transformation law of the diagonal Higgs fields receives a correction in a field-dependent manner. In the large \( U \) approximation, we can easily evaluate the correction by performing a one-loop calculation. The final result precisely takes the form (2.13) including the numerical coefficient. For details, we refer the reader to ref. \( ^{23} \). That the correct transformation law is obtained in the one-loop approximation including the precise coefficient suggests that some sort of the non-renormalization theorem is at work here, demanding that the lowest order result for the metamorphosed transformation law on the Yang-Mills side gets no higher order corrections at least in the large \( N \) limit.

Since the derivation of the isometry almost amounts to the derivation of the background metric of the \( \text{AdS}_5 \), the above result provides a strong support to the conjecture on the general relation between supergravity and supersymmetric Yang-Mills theory. In particular, the corrected transformation law explains the appearance of the natural scale \((g_s N)^{1/4}\) from the side of Yang-Mills theory. If this result is not corrected by the higher order effect, it would be the first derivation of the scale which goes as \( N^{1/4} \) in the large \( N \) limit from a purely Yang-Mills point of view.

This result also provides further evidence on our view about the relation between the source and the probe. Namely, the probe D3-brane is at somewhere far away from the horizon, while the \( \text{AdS} \) space itself is produced by the large number (=\( N - 1 \)) of the source D-branes at somewhere near (or inside?) the horizon. From this viewpoint, the operators corresponding to the boundary value of the bulk field for \( U \to \infty \) are not, at least directly, the operators of the world-volume theory which corresponds to the large \( N \) Yang-Mills theory. This raises a puzzle mentioned in the beginning of this subsection: Why and how does the boundary condition for large \( U \) dictate the operators of the large \( N \) Yang-Mills theory which corresponds to the source of the \( \text{AdS} \) space-time? In the following, we suggest a simple argument which justifies the prescription of \( ^{21} \) under the assumption that the Maldacena’s conjecture is true.

The breaking of the gauge group \( \text{U}(N) \) into \( \text{U}(N-M) \times \text{U}(M) \) (\( N \gg M \)) by assigning the large vacuum expectation value for the Higgs field corresponding to the radial direction amounts to introducing a heavy source and a light probe at a distance scale \( U \) in the energy unit. We assume that the position of probe is at somewhere outside the near horizon limit, \( U > R_c/\alpha' \). On the supergravity side, in the limit of large \( N \) with fixed \( M \), we can treat the effect of the probe as a small perturbation around the background of the heavy source. We thus decompose the metric as

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}
\]

where the first term \( \bar{g}_{\mu\nu} \) is the classical metric produced by the source D3-branes and \( h_{\mu\nu} \) is the metric produced by the probe in the background \( \bar{g}_{\mu\nu}(N) \). The perturbative metric \( h_{\mu\nu} \) satisfies the linearized equation in the lowest order approximation.

\[
\mathcal{D}_N h_{\mu\nu}(u) = 2\kappa^2_{10} T_{\mu\nu}^p(u)
\]
where \( T^p_{\mu\nu} \) is the energy-momentum tensor of the probe

\[
T^p_{\mu\nu}(u, x) \propto \frac{1}{\sqrt{-g}} \delta^{(6)}(u - U) T^p_{\mu\nu}(x) \tag{2.18}
\]

and \( D_N \) is the kinetic operator for the linearized theory in the background of the source D3-branes. We denote by \( u \) the variable corresponding to the radial transverse coordinate in the bulk, while the coordinate along the D3-branes is denoted by \( x \).

For notational simplicity, we suppress the angle variable corresponding to \( S^5 \). For the perturbative metric polarized along the direction parallel to the world volume, the kinetic operator essentially takes the following form

\[
-D_N = (1 + 2g^2_{YM} N a'^2 / r^4)^{1/2} \Delta^\parallel + (1 + 2g^2_{YM} N a'^2 / r^4)^{-1/2} \Delta^R_6 , \tag{2.19}
\]

where \( \Delta^\parallel \) is the laplacian for the flat four dimensions along the world volume and \( \Delta^R_6 \) is the flat six dimensional laplacian corresponding to the six dimensional transverse space. Note that the laplacian for the transverse part is proportional to the flat space laplacian as noted in \( ^25 \) even before taking the near horizon limit. The boundary condition for the linearized field (in the Euclidean formulation \( ^* \)) is that it vanishes as \( u \to 0 \), since otherwise the solution diverges at the origin. By assuming that the states of the probe D3-brane can be chosen arbitrarily, the perturbative metric can also be assumed to induce an arbitrary boundary value \( f_{\mu\nu}(x) \) at some large value of \( u \). It is natural to set the boundary at \( u = R_c / \alpha' \) where the near-horizon limit loses its validity:

\[
h_{\mu\nu}(u, x) \to 0 \quad (u \to 0) \quad h_{\mu\nu}(u, x) \to f_{\mu\nu}(x) \quad (u \to R_c / \sqrt{\alpha'}) . \tag{2.20}
\]

In the low-energy limit along the direction of the world-volume, we neglect the laplacian along the D3-brane (‘quasi static’ approximation) and we can approximate the boundary value as

\[
f_{\mu\nu}(x) \propto \kappa^2_{10} \left. \left( T^p_{\mu\nu}(x) \right|_{u=R_c/\alpha'} \right) , \tag{2.21}
\]

since the laplacian for the transverse part of six dimensions is proportional to the flat space laplacian even outside the near horizon limit. The effective action for the boundary value is obtained by substituting the perturbed metric into the supergravity action,

\[
S^\text{eff}_{\text{sg}} = S[\mathcal{F} + h] , \tag{2.22}
\]

using the AdS metric for the background. \( \mathcal{F}_{\mu\nu} \). What we have done is essentially to replace the effect of the probe in arbitrary given states at somewhere outside the near horizon region by the boundary condition \( f_{\mu\nu} \) for the perturbation \( h_{\mu\nu} \) around the background of the source at the boundary of the AdS space-time.

Now, on the Yang-Mills side, we construct the effective theory for the unbroken part \( U(N - M) \times U(M) \) after integrating over the heavy Higgs and W-bosons corresponding to the off-diagonal matrix elements whose ‘mass’ is of order \( U \) for large

\[^{(*)} \text{For a Lorentzian formulation, see} ^{[3].} \]
U corresponding to the broken part of the gauge group. This leads, again in the
low-energy limit, to the effective action

$$S_{YM}^{\text{eff}} \sim \kappa_{10}^2 \int d^4x \frac{T_{\mu\nu}^s(x)T_{\mu\nu}^p(x)}{(u' - U)^4}$$

(2.23)

where $T_{\mu\nu}^s \sim \text{Tr}_{N-M}(F_{\mu\sigma}F_{\nu\sigma})$ is the energy-momentum tensor of the source D3-branes on the Yang-Mills side. This one-loop result is exact because of the non-renormalization theorem, and hence is valid even for the probe at somewhere outside the near horizon region. Note that this is consistent with the fact that the laplacian is proportional to that of the flat space in supergravity. Here we have assumed that the distance between the source and probe is sufficiently large and is order of $|u' - U|$. It seems natural to assume that $u'$ is of the same order as $R_c/\alpha'$. The source D3-branes cannot be considered to be at rest at the origin. They are expected to extend to the whole range of the near-horizon region. Then the average position of the source D3-branes would be determined by the scale $R_c$ which is the only scale in this region. Apart from a proportinal factor, we can replace the above expression by

$$S_{YM}^{\text{eff}} \sim \int d^4x f_{\mu\nu}(x)T_{\mu\nu}^s(x).$$

(2.24)

The equivalence between (2.22) and (2.24), $e^{-S_{YM}^{\text{eff}}[f]} \propto \langle e^{-S_{YM}^{\text{eff}}[f]} \rangle$, is essentially the statement of the usual prescription. We have only discussed the metric perturbation, but the general idea can be easily extended to other massless fields. Details remain to be seen.

Our discussion clearly shows that the correlators we compute using the prescription are those of the unbroken part $U(N - M)$ corresponding to the source, in spite of the fact that we use boundary conditions at the boundary of the AdS space-time. It is not correct to think that the large $N$ Yang-Mills system is literally on the ‘boundary’. In order to derive the correlators from the boundary, we need in general definite rules which allow us to connect the operator insertions at the probe and the source. This is somewhat analogous to the LSZ relation between S-matrix elements and the corresponding Green functions. In our argument above, the heavy ‘Higgs and W bosons’ play the role of a ‘mediator’ for connecting the probe and the source. A similar reasoning also justifies the method for computing the Wilson loop expectation value, proposed in which naturally treats heavy W-bosons as ‘quarks’ by utilizing the breaking $U(N) \to U(N - 1) \times U(1)$ of gauge group as in our argument. In this case, the fundamental open string corresponding to the infinitely heavy W-boson, treated as a heavy point-like test particle, is playing the role of mediator.

Our argument is based on the quasi-static approximation. This is justified for sufficiently large distances between the source and the probe, since the mass scale in terms of the world volume theory is very large and hence the characteristic distance scale with respect to the world volume is small. This is a manifestation of the spacetime uncertainty relation explained in the next subsection. The correction to the

\footnote{During writing the present manuscript, several works which are related to this issue appeared. Note, however, that the context of these recent works is slightly different from ours.}
quasi-static approximation can also be interpreted on the basis of the uncertainty relation, as discussed in [25]: An uncertainty in the momenta along the world-volume is proportional to the uncertainty with respect to the transverse positions of the probe. Including this effect, more precise understanding on the correlators and also the extension to general D-branes are very important, since they may provide otherwise scarce information on the physics of the matrix models in the large $N$ limit. For example, it would be extremely interesting if we can obtain some useful information on the large $N$ behavior of Matrix theory in this way [33]. In our argument, it is very crucial that the lowest order interaction between the source and the probe is equivalently described by both supergravity and matrix model even outside the near horizon region. For the validity of this property, the supersymmetric nonrenormalization theorem is important on the matrix side, while the laplacian must be essentially proportional to the flat space laplacian on the supergravity side. It is not difficult to see that, from the viewpoint of 10 dimensions, the latter is satisfied for D-particles after taking into account the nontrivial behavior of dilaton. From the viewpoint of 11 dimensions, we have already seen this in the previous subsection.

2.4. Generalized conformal symmetry and space-time uncertainty principle

Next let us consider the question whether the conformal symmetry which is so important in the AdS/SYM relation has any generality beyond the special case of D3-branes. One of the characteristics of Yang-Mills theories interpreted as the dynamical theory of D-branes is of course that the fields on the world-volume now represent the collective motion of D-branes in the bulk space-time. This in particular implies that the dimensionalities of the fields on the world-volume and of the base-space coordinates are opposite, as is seen from the transformation law (2.13) and (2.14), or more simply from the scale transformation

$$X_i(x_a) \rightarrow X'_i(x'_{a}) = \lambda X_i(x_a),$$

$$x_a \rightarrow x'_a = \lambda^{-1} x_a.$$  \hspace{1cm} (2.25)

(2.26)

As is emphasized in ref. [30], this indicates a general qualitative property that the long-distance phenomena in the (transverse) target space is dual to the short distance phenomena in the world volume and \textit{vice versa}. This property has also been emphasized independently (named as ‘UV-IR correspondence’) from the context of establishing the holographic bound for the entropy using the AdS/SYM correspondence in [33]. For a recent discussion on holography, we refer the reader to [29].

Qualitatively, such a dual correspondence between the two different distance scales is precisely the prediction of the ‘space-time uncertainty principle’ [4] [9] which has been proposed long ago as a possible space-time interpretation of the world-sheet conformal symmetry of perturbative string theory. As already reviewed in some previous publications [33] [4], to which I would like to refer the reader for the explanation of the original motivation and examples, the statement can be summarized as follows:

Let

1. $\Delta T$ : uncertainty in probing the distance scales in the longitudinal directions along the world volumes of D-branes including time.
If the world-volume coordinates of D-branes in the static gauge are denoted by $x_a (a = 0, \ldots, p)$,

$$\Delta T = |\Delta x|$$

where $| \cdot |$ is the length in the Euclidean metric. If we use the Minkowski metric, the original derivation of the relation requires that $\Delta T$ should be measured along the time-like direction along the world volume.

2. $\Delta X$: uncertainty in probing the distance scales in the bulk along the transverse directions orthogonal to D-branes.

Then the following uncertainty relation is universally valid,

$$\Delta T \Delta X > \alpha' \quad \text{(2.27)}$$

Note that this relation survives in the Maldacena limit ($\Delta T \Delta U > 1$), since

$$\Delta X \sim |\Delta r| = \alpha'|\Delta U|.$$

This explains the dual relation between the two different length scales determined by the mass of the open strings stretched between D-branes, on one hand, and the transverse distance between the branes, on the other. As discussed in [24] and [26], this elementary property is responsible for explaining some important qualitative aspects of D-brane dynamics in connection with the AdS/CFT(SYM) correspondence and holography. Furthermore, if this relation is applied to D-particles, we can immediately derive the characteristic Planck scale $\ell_P = g_s^{1/3} \ell_s$ of 11 dimensions given only that the mass of a D-particle is of order $1/g_s \ell_s$ by combining with the ordinary quantum mechanical uncertainty relations. This also leads to the holographic property that the minimum bit of information of the quantum state of a D-particle is stored in a cell of the order of the Planck volume in the transverse space in 11 dimensions.

In particular, as is suggested in [24], the space-time uncertainty relation can be regarded as an underlying principle behind the ultraviolet-infrared relation [30] which is on the basis of the AdS/CFT(SYM) correspondence. Since the previous account given in [24] was ambiguous in some point, I would like to repeat the discussion here very briefly by taking into account the important observation made in [38]. Our discussion on the correspondence between AdS$_3 \times$S$^3$ and the type IIB string theory suggests that the uncertainty of the positions of the D3-branes in the radial direction $U$ is of order the AdS radius $R_c \sim (g_{YM}^2 N)^{1/4} \sim \Delta X$ in the string unit $\ell_s = 1$. The space-time uncertainty relation then demands that the uncertainty with respect to the time-like length scale along the world volume is of order $\Delta T \sim 1/R_c$. Thus as the AdS radius increases, the dynamics of the D3-branes are probing the high-energy region of the order $R_c$ of the AdS space-time. This is due to the fact that the typical mass scale of the open strings mediating the source D3-branes are growing as $R_c$. Does this imply that the length scale along the space-like directions on the world volume also decreases? Naively it might look so if we assume Lorentz invariance on the world volume. However, the AdS/CFT relation leads to a contrary conclusion. From the viewpoint of the D-brane dynamics, the energy of the open strings mediating the interaction among the D3-branes can also be regarded as the self-energy of the heavy charged fields (U(1)). Let the uncertainty of the spatial
position of such a charged field be $\Delta X_s$. The self energy of the field in the large $N$ strong coupling region can be estimated from the behavior of the Wilson loop, which tells us that it is of order $R^2_c/\Delta X_s$. Note that this is different from the weak coupling behavior which would be proportional to $R^4_c$. Coulomb force is still there corresponding to conformal symmetry but with the different effective charge. Equating this result with the energy scale determined by the space-time uncertainty relation, we have

$$\Delta X_s \sim R_c.$$  \hspace{1cm} (2.28)

Thus the world volume of the source D3-branes can be regarded as the collection of cells with volume $R^3_c$ in the space-like directions with a continuous flow of time, and hence the degrees of freedom of the theory is given, in terms of the 10 dimensional Newton constant $G_{10} \sim g_s^3 \sim g_{YM}^4$, as

$$N_{\text{dof}} \sim N^2 \frac{L^3}{R^3_c} = \frac{L^3 R^5_c}{G_{10}}$$

where we have assumed that the source D3-branes wrap around a 3-torus of the length $L$ and also that the degrees of freedom is proportional to $N^2$ even in the strong coupling regime in view of the result for the entropy at finite temperature. The final result is consistent with the Beckenstein-Hawking formula and is equivalent with the original result derived in \cite{39}. The relation (2.28) is at first sight quite surprising, but is an essential property ensuring holography. The above argument is consistent with the recent analysis \cite{28}, \cite{29} of holography in the flat space limit. It should however be kept in mind that the space-time uncertainty relation or ultraviolet-infrared relation alone is not sufficient to derive holography. We have to combine it with some dynamical information, as exemplified by the assumptions needed in the argument.

Although the space-time uncertainty relation might look at first sight too simple in order to characterize the short-distance space-time structure, it indeed captures the most important characteristics of quantum string theory including D-branes. We hope that it plays some role as one of the guiding principles toward nonperturbative formulation of string/M theory.

The conformal symmetry can be regarded as a mathematical structure which characterizes the space-time uncertainty relation (2.27): Clearly, the relation is invariant under the scale transformations $\Delta T \rightarrow \lambda \Delta T$, $\Delta X \rightarrow \lambda^{-1} \Delta X$. The invariance can be extended to full conformal symmetry for general $Dp$-branes, if we identify the uncertainty with the infinitesimal variations of the coordinate and fields, as

$$\delta_K \Delta T = -2\epsilon \cdot x \Delta T, \quad \delta_K \Delta X = 2\epsilon \cdot x \Delta X$$  \hspace{1cm} (2.29)

using the relation

$$\Delta(x_a + \delta_K x_a) = -2\epsilon \cdot x \Delta x_a + 2(\epsilon_a x \cdot dx - x_a \epsilon \cdot \Delta x),$$

where the second term is orthogonal to the first term and therefore we have the first equality of eq. (2.29). Thus it seems that the conformal symmetry plays an
analogous role in the target space-time as that of the canonical structure in the phase space of classical mechanics. This strongly suggests that the noncommutative nature of the space-time coordinates which characterizes the matrix models should be understood as a realization of the quantization of space-time and the conformal symmetry is a signature of certain unknown symmetry structure behind it.

These considerations motivate us to generalize the conformal symmetry of D3-brane to general D-branes. Let us first consider the D-particle model. The action (2.2) is invariant under the scale transformation

\[ X_i(t) \rightarrow X_i'(t') = \lambda X_i(t), \quad t \rightarrow t' = \lambda^{-1} t \]

(2.30)

\[ g_s \rightarrow g_s' = \lambda^3 g_s. \]

(2.31)

One might wonder whether this can be regarded as symmetry since we transformed the coupling constant simultaneously. But this is not unreasonable if we remember that the string coupling constant, being given by the vacuum expectation value of dilaton at infinity, is not really a constant supplied by hand. Ultimately the string coupling should be eliminated from the theory. From the viewpoint of 11 dimensions, the string coupling is replaced by the compactification radius, and the scale transformation can be reinterpreted as the boost transformation along the 11th direction as follows. In the above scale transformation, we have assumed that the string length is invariant. However, from the point of view of M-theory, we should fix the 11 dimensional Planck length instead of the string length. This is achieved by redefining the unit of length as

\[ \ell_s \rightarrow \lambda^{-1} \ell_s, \quad t \rightarrow \lambda^{-1}, \quad X_i \rightarrow \lambda^{-1} X_i, \quad A \rightarrow \lambda^{-1} A \]

where \( A \) is the gauge U(\( N \)) gauge field. Combining the change of unit, which does not change the action, with the above scaling transformation, the net transformation becomes

\[ t \rightarrow \lambda^{-2} t, \quad R \rightarrow \lambda^2 R, \]

(2.32)

while the transverse coordinates and gauge field are scalar. This is precisely the boost transformation provided we identify the time as the light-cone time \( x^+ \) and the compactification radius as that along the light-like direction \( x^- \). From this 11 dimensional viewpoint, it is more appropriate to express the space-time uncertainty relation in the form \( R \Delta T \Delta X > \ell_p^2 \) which suggests some ‘tripod’-like interpretation, possibly connected with the membrane structure, of the relation as already emphasized in [37].

Once we allow the variation of the string coupling, we can easily extend the symmetry to SO(2,1) group by considering the trivial time translation and the ‘special conformal’ transformation whose infinitesimal form is

\[ \delta_K X_i = 2t X_i, \quad \delta_K A = 2t A, \quad \delta_K t = -t^2, \quad \delta_K g_s = 6tg_s. \]

(2.33)

In all these transformations, the fermionic variables are assumed to be scalar.

The above transformation property of the string coupling is essentially equivalent to the fact that the characteristic spatial and temporal scale of the dynamics
of D-particle is proportional to $g_s^{1/3} \ell_s$ and $g_s^{-1/3} \ell_s$, respectively. Of course the inverse powers with respect to $g_s$ in these length scales just reflects the space-time uncertainty relation. In contrast with this, there is no fixed characteristic scale in the case of D3-brane, because the dynamics is conformal invariant and all scales are equally important with respect to both $\Delta T$ and $\Delta X$. Of course, if we assume some particular background, the dynamics around the background can have characteristic scales. The scales which appeared in the case of D3-branes should be interpreted as such. We emphasize that this dual nature of two different scales in time and space explains the simultaneous appearance of the short distance scale $\Delta X$ and small energy scale $\Delta E \sim 1/\Delta T$ in the weak coupling dynamics of D-particles, ensuring the decoupling of the higher string modes in the short distance regime contrary to the naive intuition.

The argument discussed in the previous section connecting the D-brane Yang-Mills theory and supergravity should equally be valid for D-particles. Then we expect that the conformal symmetry of D-particle Yang-Mills theory must be reflected in the metric produced by a heavy source of D-particles. The 10 dimensional metric around the D-particle is given, in the Maldacena limit $\alpha' \to 0, U = r/\alpha'$, by

$$ds_{10}^2 = \alpha' \left( -\frac{U^{7/2}}{\sqrt{Q}} dt^2 + \frac{\sqrt{Q}}{U^{7/2}} \left( dU^2 + U^2 d\Omega_8^2 \right) \right),$$

(2.34)

where

$$Q = 60\pi^3 (\alpha')^{-3/2} g_s N = 240\pi^5 g_{YM}^2 N.$$  

(2.35)

In the case of D-particle, the dilaton is not constant and is given as

$$e^\phi = g_s \left( \frac{q}{\alpha' U^4} \right)^{3/4} = g_{YM}^2 \left( \frac{Q}{U^4} \right)^{3/4}.$$  

(2.36)

Both the metric and the dilaton are invariant under the dilatation

$$U \to \lambda U, \quad t \to \lambda^{-1} t, \quad g_s \to \lambda^3 g_s.$$  

(2.37)

Furthermore, they are also invariant under the infinitesimal special conformal transformation

$$\delta_K t = -\epsilon (t^2 + \frac{g_{YM}^2 N}{96\pi^5 U^5}),$$  

(2.38)

$$\delta_K U = 2\epsilon t U, \quad \delta_K g_s = 6\epsilon t g_s.$$  

(2.39)

Just as in the Yang-Mills case, these transformations with the time translation form an SO(2,1) algebra. The additional term $g_{YM}^2 N/96\pi^5 U^5$ in the special conformal transformation plays the similar role as in the case of D3-brane: The nonlinear field dependence is equally powerful to determine the effective action of the probe D-particle in the background of source D-particles. We can derive this modification of the transformation law in the bulk, extending the similar mechanism as we have discussed for D3-brane in the previous subsection. For details about this, we refer the reader to refs. [40]. It is straightforward to extend the conformal transformations of the above type to general Dp-branes ($0 \leq p \leq 4$), as discussed in the second of
the latter references. The case of D-instanton matrix model, the so-called type IIB model, is very special in this respect, since here all of the space-time coordinates are treated as matrices. For an interpretation of the model from the point of view of the space-time uncertainty relation and conformal symmetry, we refer the reader to\cite{42}.

Finally, one might wonder what is the relation of the space-time uncertainty relation and the associated conformal symmetries to supersymmetry. We can perhaps say that the supersymmetry is necessary to ensure some of prerequisites for applying the principle. For example, to discuss the scattering of D-branes meaningfully, it is necessary that the clusters far apart from each other should be free except for the weak gravitational forces among them. If the supersymmetry is not there, the quantum zero-point energy induces the forces which do not decay at large distances.

§3. Graviton condensation in type IIB matrix model

As the final topic of this report, I would like to present some preliminary considerations on the treatment of graviton condensation in matrix models. We have already seen some evidence that supersymmetric Yang-Mills models indeed describe the gravitational interactions of D-branes to certain extent. However, it is clear that we do not have definite general principles which might explain the emergence of gravity from Yang-Mills theory. From the viewpoint of symmetry, the existence of $N = 2$ supersymmetry in space-time in 10 dimensions is the strongest argument for the existence of supermultiplet containing graviton, since only massless representation of the maximal $N = 2$ supersymmetry in 10 dimensions is indeed the supergravity multiplet. However, it is difficult to decide the presence of massless particles only from the logical structure of the Yang-Mills theory. In other words, without making concrete computations of D-brane scattering, we cannot decide whether the $N = 2$ global symmetry is really elevated to the consistent local symmetry ensuring the emergence of gravity in the long-distance regime. Since in general we expect that the matrix models are only sensible after taking appropriate large $N$ limit and then various questions can only be answered by solving complicated dynamics, it is very important to establish the symmetries of the models as far as possible.

Now after seeing some evidence for the emergence of gravity in the Yang-Mills matrix models, we should be able to identify the local space-time supersymmetry directly within the models. The purpose of the following preliminary consideration is to start an initial discussion toward such a possibility taking the simplest toy example of the type IIB matrix model. We hope that our discussion will be a useful starting point for exploring possible higher symmetry structure in matrix-model approaches to non-perturbative string/M theory from a more general viewpoint.

In the case of usual perturbative string theory, that the theory is indeed a dynamical theory of space-time geometry is reflected on the fact that we can deform an allowed space-time background by insertion of the vertex operator corresponding to physical graviton modes of strings. Or, if we use the language of string field theory, the change of background is compensated by an appropriate redefinition of the string field corresponding to a shift of its graviton component. In particular, the general
coordinate transformation is compensated by such a field redefinition. That is how the string theory can be generally covariant and in principle be a background independent formulation even if the theory is formulated without introducing the space-time metric explicitly as an independent degree of freedom. In the case of general Yang-Mills matrix models, on the other hand, we cannot identify graviton modes directly in the classical action of the model. They only appear as a part of loop effect in the ‘t-channel’. For this reason, they can neither be treated as ordinary bound states, in general. In the special case of Matrix theory, only the Kaluza-Klein mode with non-zero 11th momentum can be directly treated, and the graviton with zero 11th momentum can only appear as the loop effect.

Let us now concentrate on the case of matrix model of D-instantons. The model is already Lorentz invariant and thus we can immediately ask a question, “How is the symmetry extended to general coordinate invariance?”. The action of the model is

\[ S_N = \text{Tr}_N \left( \frac{1}{4g^2} [X^\mu, X^\nu]^2 + \frac{1}{2} \overline{\Psi} T^\mu [X^\mu, \Psi] \right). \] (3.1)

Of course, it is not obvious what we mean by the general coordinate transformation for the matrix variables \(X^\mu, \Psi\). In the usual interpretation, only the diagonal components of \(X^\mu\) have the meaning of the space-time coordinates and the off-diagonal components are really fields corresponding to the lowest open string modes. In general, the space-time coordinate and the fields of open strings can have different transformation property. However, at least for the general linear transformations \(\text{GL}(10, \mathbb{R})\) which is globally defined, it is natural to suppose that the transformation law is the standard one

\[ X^\mu \rightarrow a^\mu_\nu X^\nu, \] (3.2)

where \(a^\mu_\nu\) are arbitrary \(10 \times 10\) coefficients. The action is manifestly invariant under the subgroup \(\text{SO}(9,1)\), if the spinor matrix \(\Psi\) transformed as usual. Our question is then whether it can be made invariant under the transformations belonging to the remaining broken quotient group \(\text{GL}(10, \mathbb{R})/\text{SO}(9,1)\). Of course, the standard procedure is to introduce the metric (or viel-bein) degree of freedom which absorbs the noninvariant piece of the action. But as the metric degrees of freedom is supposed to be contained in the loop effect, there must exist different way of compensating the transformation without introducing the metric explicitly.

Now we will present briefly an argument showing[4] that this can be achieved by embedding a model with fixed \(N\) into models with larger \(N\). The idea is to add more instantons to the model with appropriate information on the ‘state’ of the added instantons such that they effectively produce the metric insertion for the original action with lower \(N\). If we perform the embedding for all \(N\) recursively, we naturally expect that the set of all such models as a whole, which we denote as \(\{\ldots, U(N), U(N+1), \ldots\}\), can in principle describe all possible backgrounds of the model. In this way, it should ultimately be possible to reconstruct the model in a background independent fashion.

\[ )\] For an initial discussion of this phenomenon, see[3].
Let us study the simplest embedding from $N$ to $N+1$. We will use the following notations for the embedded matrices.

\[
\begin{align*}
X^\mu_{a,N+1} &= \phi^\mu_a, \quad X^\mu_{N+1,a} = \overline{\phi}_a, \quad X^\mu_{N+1,N+1} = x^\mu, \\
\psi_{a,N+1} &= \theta_a, \quad \overline{\psi}_{N+1,a} = \overline{\theta}_a, \quad \psi_{N+1,N+1} = \psi.
\end{align*}
\]

Thus the $N \times N$ matrices $X^\mu_{N \times N}, \psi_{N \times N}$ are embedded into the corresponding $(N+1) \times (N+1)$ matrices as

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
& X^\mu & |\phi^\mu| & \vdots \\
& \langle \phi^\mu | & x^\mu & \vdots \\
\vdots & \vdots & \vdots & \vdots
\end{pmatrix}_{(N+1) \times (N+1)}
\]

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
& \psi & |\theta| & \vdots \\
& \langle \theta | & \psi & \vdots \\
\vdots & \vdots & \vdots & \vdots
\end{pmatrix}_{(N+1) \times (N+1)}
\]

Here we use Dirac’s bra-ket notation for the vector part. Since the information on the state of the added instanton is specified by using the $N$-th diagonal elements $x, \psi$ (collective coordinates of the added instanton), we can first integrate over the vector parts to derive the effective action for the insertion of graviton. Then we can further integrate over the collective coordinates by inserting an appropriate function $\Phi(x, \psi)$ whose form is determined later. Let us call the result of this $\Delta \Gamma_N(X, \Psi; \Phi)$. Then by combining with the original $\text{U}(N)$ model $\exp(S_N) \to \exp(S_N) + \Delta \Gamma_N(X, \Psi; \Phi)$, we can define the new partition function of the $(N, N+1)$ system.

\[
Z_{N}[\Phi] = N^{-1} \int d^{10N^2}X d^{16N^2}\Psi \exp\left(S_N[X, \Psi] + \sum_i c_i \Delta Z_N[X, \Psi; \Phi_i]\right)
\]

to the first order in the strengths $\{c_i\}$ of the insertion, where the sum is over all independent ‘wave functions’ of added instanton. It would be more appropriate to regard the wave functions as the scalar products of two wave functions.

In the one-loop approximation, we can show that the following special choice of $\Phi$, which is the simplest candidate corresponding to the degree of freedom of symmetric tensor, gives the infinitesimal (first order) deformation of the action $S_N$ which compensates the change of the action by the infinitesimal $\text{GL}(10,\mathbb{R})/\text{SO}(9,1)$ coordinate transformation $a_{\mu \nu} = \eta_{\mu \nu} + S_{\mu \nu}$ where $S_{\mu \nu}$ is an arbitrary infinitesimal symmetric tensor.

\[
\Phi(x, \psi) \propto (\Gamma^{\mu \beta \gamma} \Gamma^0)_{ab}(\Gamma^{\nu \beta \gamma} \Gamma^0)_{cd} S_{\mu \nu} \frac{\partial}{\partial \psi_a} \frac{\partial}{\partial \psi_b} \frac{\partial}{\partial \psi_c} \frac{\partial}{\partial \psi_d} \Phi_0
\]
with
\[ \Phi_0 = \prod_{a=1}^{16} \psi_a, \quad \left( \int d^{16}\psi \Phi_0 = \langle 0|0 \rangle = 1 \right). \] (3.7)

Namely, apart from the proportional constants, the first order deformations of the bosonic and fermionic part of the action are, respectively,
\[ S_{\mu\beta} \text{Tr}_N \left( [X^\mu, X^\nu] [X^\mu, X^\beta] \right), \quad S^{\mu\alpha} \text{Tr}_N \left( \overline{\psi} \Gamma^\mu [X^\alpha, \psi] \right) \]
which are nothing but the change of the bosonic and fermionic actions corresponding to the quotient group \( \text{GL}(10,\mathbb{R})/\text{SO}(9,1) \). The one-loop approximation is justified for the wave functions which are constant with respect to the space-time coordinates, since then the infrared region \( x \to \infty \) is dominant in the integral over the collective coordinate of the added instanton implying the infinite mass limit in the propagator of the fluctuating fields: The coefficients of the above deformation are proportional to an infrared-divergent integral \( \int d^{10}x/x^{10} \), which is cancelled by choosing the normalization of the wave function.

Our result suggests how to describe the change of background using only the degrees of freedom of the model itself, if we treat all possible embeddings simultaneously. In particular, we have seen that the model indeed has full \( \text{GL}(10,\mathbb{R}) \) symmetry. Thus the metric degrees of freedom appearing as a loop effect can be regarded as the Goldstone boson associated with the spontaneously broken part \( \text{GL}(10,\mathbb{R})/\text{SO}(9,1) \) of the infinitesimal symmetry \( \text{GL}(10,\mathbb{R}) \) of the recursively embedded model \( \{ \cdots, U(N), U(N+1), \cdots \} \). Together with the space-time supersymmetry, this explains how the model can indeed be the theory of gravity.

§4. Concluding remarks

In principle, the formalism suggested in the last section should be extended to arbitrary changes of background and hence to the definition of the model for general curved space-times. For example, the ‘wave function’
\[ (\Gamma^{\mu\beta\gamma} \Gamma^0)_{ab} H_{\mu\beta\gamma} \frac{\partial}{\partial \psi_a} \frac{\partial}{\partial \psi_b} \Phi_0 \]
describes the infinitesimal condensation of the antisymmetric tensor field \( B_{\mu\nu} \) with constant field strength \( H_{\mu\nu\gamma} \). Also, it is not difficult to derive the susy transformation law corresponding to the shift of the background. But, technically, computations required for such a generalization become increasingly difficult. I feel that we need some entirely new framework for developing the idea in a tractable way. What we are pursuing amounts to investigating the condensation of Goldstone bosons using the configuration space formalism. Something which can play the role of the field-theory like formalism must be a desired language, by which we can treat the matrix models with different sizes of matrices in a much more unified and dynamical manner. Only by using such a formalism, we would be able to discuss the major questions related to the present approach, such as the proof of S-duality symmetry, the background
independent formulation, and so on. If we symbolically represent the whole recursive series \{\cdots, U(N), U(N + 1), \cdots\} by \(\mathcal{H}[\Phi]\) as the functional of all possible background fields \(\Phi\), background independence of the theory amounts to something like

\[
\delta \mathcal{H}[\Phi] \delta \Phi = 0
\]

(4.1)

which should simultaneously play the role of the field equation in a perturbative approximation. We also note that our idea is intimately related to that\(\ast\) of large-\(N\) renormalization group, in which we try to derive the equation of motion for the background by imposing the fixed point condition in the sense of the renormalization group with respect to \(N\). In the latter, it is not clear how to define the model in curved space-time, and also how to treat the zero modes in formulating the renormalization group. Unless we insert ‘wave functions’ as we have done, the result of embedding would only lead to a null result. In the approach suggested here, it may, at least in principle, be possible to develop the procedure in a more constructive fashion.

Finally, it may be worthwhile to mention possible connection\(\ast\ast\) of the present idea with the K-theory formulation\(\ast\) of bound states of brane-anti-brane systems, which has been discussed to describe the stable non-BPS states\(\ast\). We should generalize the above construction such that the formalism includes not only variable number of D-instantons but also of anti-D-instantons simultaneously. Obviously, the system with a fixed number of both D-branes and anti-D-branes cannot be supersymmetric. It would be extremely interesting if we could recover supersymmetry by some similar mechanism as we have suggested for recovering the full GL(10,\(\mathbb{R}\)) symmetry beyond manifest Lorentz symmetry. Such must also be crucial for developing covariant or background-independent formulation of Matrix theory. I hope to report some progress along this line in near future.

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\(\ast\) For example, such a possibility has been suggested in\(\ast\).

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