Dynamics of Sheared Ellipses and Circular Disks: Effects of Particle Shape

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There has been recent interest in glassy dynamics, jamming[16,18] and rheology[2,11] of particulate systems, such as colloids, emulsions, foams and granular materials. Despite variations in interaction details, particulate systems show similar rheology[11]. Traditionally, rheology concerns aging, rate effects, shear band formation, and failure[7,11]. And, steadily sheared systems provide an alternative route for approaching the jamming transition for particulate systems by moving along the yield stress curve which separates static ( jammed) and flowing (un-jammed) states in a parameter space of shear stress, \( \tau \), and packing fraction, \( \phi \).

Many recent rheology studies involved spherically symmetric particles. The question we pose is, how does breaking of spherical particle symmetry change the rheology of near-jamming granular materials. A complete answer to this question is beyond the scope of a single paper. Here, we take a first step towards understanding the role played by particle shape by considering elliptical particles. We show that steadily sheared elliptical particles show dramatically stronger rate dependence than their circular counterparts. This rate dependence appears to be linked to slow evolution of the particle orientation.

Although we have focused on ellipses because they are easy to characterize, there are key aspects that ellipses share with more complex particle shapes: particle orientation and rotation couple to density and normal contact forces between particles lead to torques.

The ability of particles to exert torques on each other impacts jamming in important ways. For the torque-free case of frictionless spheres, previous simulations[11] reported a fraction, \( \phi_J \), below which the system is always stress-free (both the pressure, \( P \) and \( \tau \) vanish), and the shear modulus vanishes. Above \( \phi_J \), there exist \( \tau = 0 \), \( P > 0 \) mechanically stable states. In the frictionless spherical case, if \( \phi < \phi_J \), \( \dot{\gamma} \) must be non-zero to obtain non-zero \( \tau \). However, Bi et al.[3] recently showed that systems of frictional disks behave differently. There exists a range \( \phi_S \leq \phi \leq \phi_J \) where it is possible to shear jam a system at fixed \( \phi \) by applying shear strain, starting from a \( \tau = P = 0 \) state, and arriving at a jammed (mechanically stable) state. Here, \( \phi_J \) corresponds to the lowest density at which frictional granular systems can be jammed at zero \( \tau \), and \( \phi_S \) corresponds to the lowest density at which it is possible to jam a system by applying shear strain. These observations are consistent with simulations along the yield stress curve by Otsuki and Hayakawa[15], and older experiments by Howell et al.[6] who observed a what is now understood as a lower limit for shear jamming \( \phi_S \approx 0.76 \) for bidisperse frictional disks subject to Couette shear. Thus, quasi-static shear of frictional spheres (disks in 2D) in the range \( \phi_S \leq \phi \leq \phi_J \) is clearly affected by the possibility of shear jamming due to the presence of rotational degrees of freedom. However, for non-spherical particles, such as ellipses, rotational modes are imposed by the geometry even for the frictionless case. This experimental study directly probes the effect of particle asphericity on shear jamming of particulate systems.

A key issue for rheology is the dependence of the shear stress and other properties of a shear system in the strain rate, \( \dot{\gamma} \). Here, we contrast two particular cases: a Newtonian fluid has \( \tau \propto \dot{\gamma} \), and an ideal granular material has \( \tau \) independent of \( \dot{\gamma} \) for very slow \( \dot{\gamma} \[16\]. Although rate-independence for stresses in slowly sheared granular materials has frequently been assumed, there are multiple granular experiments[9,14,17] showing logarithmic rate dependence in \( \dot{\gamma} \). The true origin of this rate dependence is still an open question, although several groups have proposed models in the spirit of soft glassy
rheology \[9, 17\]. However, as with many (although not all\[18, 19\]) studies of jamming, previous work on rate dependence, has typically involved systems of spherical or circular particles, in 3D or 2D, respectively. Here, we show that elliptical particles under sustained shear show stronger rate dependence than their circular counterparts. By inference, this rate dependence is linked to particle orientation.

Here, we directly compare sheared systems of ellipses and of bidisperse disks, and find several crucial differences: 1) Very long transients for \( P \) of ellipses but not for disks; there is a packing fraction range for ellipses where \( P \) builds up relatively quickly, and then relaxes very slowly. For disks, only monotonic evolution occurs. 2) The order parameter, the density, and the velocity profile exhibit very slow evolution for the ellipse systems; 3) The power spectrum of stress fluctuations for ellipses suggests power-law scaling with rate, with an exponent of \( \sim 1/2 \); spectra for disks have much weaker rate-dependence.

**Experiment:** The experimental data of this study were collected from a quasi-2D Couette experiment\[6\]. This geometry allows continuous shear for arbitrary time/strain intervals. Using this feature, we performed shearing experiments for up to 20 revolutions of the inner wheel, and collected two types of data. In one case, we kept the shear rate constant at \( \Omega = 0.01 \text{rpm} \), and varied the global packing fraction, \( \phi \). In the second case, focused on rate dependence, we varied the \( \Omega \) over a bit more than a decade, for representative packing fractions. All the packing fractions were below \( \phi_J \), which is 0.84 for disks, and 0.91 for ellipses \[24\]. Fig. 1 gives schematics of the experimental setup. The particles, rest on a smooth horizontal Plexiglas sheet, and are confined between an inner wheel and a rigid, concentric, outer ring. They are continuously sheared by slowly rotating the inner wheel at constant rates, spanning \( 0.008 \leq \Omega \leq 0.128 \text{rpm} \). Both ellipses and disks are made from the same photoelastic material (Vishay polymer PSM-4), and are machined with similar procedures. Thus, the particle surfaces have similar friction coefficients. The ellipses have semi-minor axis \( b \approx 0.25 \text{cm} \) and aspect ratio \( \sim 2 \). The radius of the small/large disks is \( r_s \approx 0.38 \text{ cm} / r_l \approx 0.44 \text{ cm} \). The number ratio of small-to-large disks is kept at \( \sim 9:2 \). The radii of the inner and outer rings of the Couette apparatus are 10.5 cm and 25 cm respectively. As the system was sheared, two synchronized cameras obtained polarized and unpolarized digital images every \( \sim 10 \text{s} \). We studied systems of either identical ellipses or bi-disperse collections of disks, for packing fractions near the shear-jamming threshold.

We used the green channel of our images to compute \( g^2 \), since the photoelastic response is color-dependent, and the polarizers are optimized for green.

We focus on several key results: 1) Anomalously slow relaxation for the motion of ellipses at densities just below the shear-jamming threshold; 2) Differences in the density profiles of disks and ellipses; 3) Slow evolution of the average orientation of ellipses; and 4) Strong, and previously unreported (to our knowledge) rate dependence for stress fluctuations with ellipses that has not counterpart for disks. The first three of these are clearly linked.

**Velocities:** We show the evolution of velocity profiles for ellipses at two typical \( \phi \)'s in Fig. 1(c). The velocity profile is stable for denser states, but for slightly lower \( \phi \), it relaxes slowly to zero. To quantify the evolution of flow profile, we use the average speed, \( V_{\text{int}} = \frac{1}{A} \int <V > rdr/ \int rdr \), where, \( V \) is the scalar value of the particle velocity vector, \( V = |\ell| \), and \( A \) is the area of the
Couette cell. Here, we first focus on a small range of $\phi$ where transient behavior occurs for ellipses but not for disks. Fig. 2 shows the evolution of $V_{int}$ for ellipses and disks. For ellipses, within a small but non-zero range of ‘meta-stable’ $\phi$’s, $V_{int}$ rises relatively quickly during the first revolution to $V_{int} \approx 0.01$ for all packing fractions, but then drops slowly to zero. For disks, we found no such meta-stable regime: $V_{int}$ has an almost monotonic trend in the logarithmic time of an experiment. We show the changes in the meta-stable state region in a narrow range of densities, $0.85 \lesssim \phi \lesssim 0.86$, in Fig. 2. In this, and other figures throughout the paper, systematic errors are at or below the scale of dots used to indicate data points. The apparent noise on the data is due to statistical fluctuations, not to measurement errors.

Structure: local density and orientational order: The gradual decrease in $V_{int}$ for the meta-stable regime of ellipses is associated with a gradual, and anomalously large, increase in Reynolds dilatancy in the region close to the inner wheel, which is facilitated by the evolution of the ellipse orientation. We measure the local density by determining the Voronoi areas, $V^v$, for individual particles [24]. We take the Voronoi area for a given particle to correspond to the interior area of the particle, plus the area corresponding to all points in the plane that are closer to the boundary of the given particle than to the boundary of any other particle. The local solid fraction in the region enclosed by $V^v$ is $\phi_L = \frac{V^v}{V^r}$, where $V^r$ is the area of a particle. Fig. 3 shows radially and time averaged local density profiles extracted from Voronoi analysis. Close to the inner wheel, both types of particles show a dip in density, i.e. a shear band. Also, the drop in density is roughly twice as big for ellipses (e.g. as a fraction of the mean packing fraction) as for the disks. The small rise in $\phi$ at the inner wheel is due to the carrying of particles by inner shearing wheel. Since the global packing fraction is fixed for each data set, the relative density minimum near the inner wheel induces closer packing in the middle and outer radial regions of the Couette cell. For all measurements of structure, e.g. for Figs. 2-5 we keep the rotation rate of the inner wheel fixed at $\Omega = 0.01$ rpm. At the end of this paper, we consider the effects of varying the rotation rate.

It is known that random packing of ellipses can be denser than random disks [19]. However, compacting a randomly prepared system of ellipses requires reorientation of the particles which leads to locally nematic order, as characterized by the local mean director. The director has a similar sense as for the liquid crystal context. In order to quantify nematic ordering, we determine the local order parameter $q_L$. Here, $q_L$, is computed from the orientational order matrix, $Q$, calculated for small patches of $N \approx 10$ ellipses surrounding the $i$th particle:

$$Q_{xy} = \frac{1}{N} \sum_{n} (u^n_x w^n_y - \frac{1}{2} \delta_{xy})$$

The size of a patch was chosen to give a local average. Here, $x$ and $y$ are two given orthogonal directions, and $u^n$ is the unit vector in the direction of the major axis for a given particle. The summation is made over all particles in the patch. We determine $q_L$, as the maximum eigenvalue of $Q$.

Fig. 4a shows the evolution of $< q_L >^*$, the spatially average of $q_L$, for the outer region of the cell vs. rotations of the inner wheel (on a logarithmic scale). Specifically, we average $q_L$ over a radial region $16a \leq r \leq R_o$, where $a$ is the semi-minor axis of the ellipses and $R_o$ is the radius.
of the outer confining ring of the Couette cell. In this region, the average velocity is about 10% of the average velocity next to the inner wheel. Considering only this region enables us to characterize the orientational ordering of particles which are almost stationary, but at the same time are reoriented gradually by the dynamics generated in the shear band. In Fig. 4, for all densities, including stable and meta-stable values, $< q_L >^*$ increases gradually in time, roughly as $\log(t)$. The evolution of average local densities (from Voronoi analysis) in the same outer region, $< \phi_L >^*$, is shown in Fig 4b. Both density and orientational order increase with time, although the relation between these two may not be strictly linear. This extremely slow reorientation and compaction is driven by allows more dilation in the inner layers. It is then the source of the meta-stability and it subsequently halts the shearing when the compaction in the outer region leads to too weak mechanical contact between the shearing wheel and the ellipses.

**Evolution of average pressure:** Fig. 5 shows the evolution of the system-average of $g^2$ for representative packing fractions of disks and ellipses. As mentioned, $g^2$ quantifies global pressure. For disks, $< g^2 >$ attains its initial value relatively quickly in most cases, except for a slight increase over the course of an experiment. However, for ellipses, $< g^2 >$ steadily increases over a span of about 10 revolutions of the inner wheel. After this long initial increase, the average $g^2$ drops only for meta-stable systems and saturates for densities above this range.

**Rate dependence of pressure fluctuations:** Perhaps the most significant difference in rate effects for ellipses vs. disks occurs for time fluctuations of the pressure in the stable regime. Specifically, the instantaneous pressure of the system is proportional to $g^2$ integrated over the whole field of view. Here, we consider the fluctuations, $\delta g^2(t)$, in time-series of this global $g^2(t)$. Up to this point, the shear rate was held fixed at $\Omega = 0.01$ rpm. Now, we consider the effect of changing $\Omega$, by allowing this quantity to vary over $0.008 \leq \Omega \leq 0.128$ rpm.

The concept of rate-independence is based on the fact that if the strains are slow enough, the system remains very close to a sequence of states in mechanical equilibrium. Then, the system response, for instance the pressure, would depend on the strain, not the rate of strain, $\dot{\gamma}$. By contrast, if $\dot{\gamma}$ were too high, the system would be out of mechanical equilibrium, and the response would explicitly depend on $\dot{\gamma}$ (here, $\Omega$). The power spectrum, $P(\omega)$, for fluctuations of $g^2(t)$ provide a simple test of whether the system is rate independent or not. Normally, $P(\omega) = T^{-1} | \int_0^T \delta g^2(t) \exp(-i\omega t) dt |^2$, where $T$ is the time over which $\delta g^2(t) = g^2(t) - < g^2 >$, is measured, and $\omega$ is the spectral frequency. If the fluctuations were statistically rate independent, then it would be possible to replace time, $t$, with strain, $\gamma = \dot{\gamma} t$, or here, $\Omega$. We would then find the same power spectra regardless of
The present experiments suggest that, more generally, when the density is coupled to another system property, there exists the possibility of a long internal response time as the system responds to shear or other strains by forming more compact structures. We note that complex grain shapes do not have any obvious orientational order. But, they do have the property that rotations of the grains can lead to changes of density or stresses. These changes might be either positive or negative, depending on the initial conditions and the strain history.

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