NONLINEAR STABILITY OF PLANAR TRAVELING WAVES IN A CHEMOTAXIS MODEL OF TUMOR ANGIOGENESIS WITH CHEMICAL DIFFUSION

MYEONGJU CHAE AND KYUDONG CHOI

Abstract. We consider a simplified chemotaxis model of tumor angiogenesis, described by a Keller-Segel system on the two dimensional infinite cylindrical domain \((x, y) \in \mathbb{R} \times S^\lambda\), where \(S^\lambda\) is the circle of perimeter \(\lambda > 0\). The domain models a virtual channel where newly generated blood vessels toward the vascular endothelial growth factor will be located. The system is known to allow planar traveling wave solutions of an invading type. In this paper, we establish the nonlinear stability of these traveling invading waves when chemical diffusion is present if \(\lambda\) is sufficiently small. The same result for the corresponding system in one-dimension was obtained by Li-Li-Wang (2014) [16]. Our result solves the problem remained open in [3] at which only linear stability of the waves was obtained under certain artificial assumption.

1. INTRODUCTION

1.1. A Keller-Segel system. The formation of new blood vessels from pre-existing vessels, which is so-called angiogenesis, is the essential mechanism for tumour progression and metastasis. Focusing on the interaction between endothelial cells and growth factor, a simplified model of tumor angiogenesis can be described by the following Keller-Segel system [7, 14, 22]:

\[
\begin{align*}
\partial_t n - \Delta n &= -\nabla \cdot (n\chi(c)\nabla c) \\
\partial_t c - \epsilon \Delta c &= -cn.
\end{align*}
\]

(1.1)

We consider the above system in two-dimension with a front boundary condition in \(x\) and a periodic condition in \(y\), both specified later, with \(m > 0\) and \(\epsilon > 0\). In a general Keller-Segel context, the unknown \(n(x, y, t) > 0\) is the bacterial density while the unknown \(c(x, y, t) > 0\) is the concentration of chemical nutrient consumed by bacteria at position \((x, y)\) and time \(t\). Considering formation of new blood vessels, \(n\) denotes the density of endothelial cells while \(c\) does the concentration of the protein known as the vascular endothelial growth factor (VEGF). The chemosensitivity function \(\chi(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is a given decreasing function, reflecting that the chemosensitivity gets lower as the concentration of the chemical gets higher. The positive constant \(\epsilon > 0\) is the diffusion rate constant for the chemical substance \(c\) while \(m\) indicates the consumption rate of nutrient \(c\).

When we model endothelial angiogenesis, we interpret that the endothelial cells behave as an invasive species, responding to signals produced by the hypoxic tissue. Accordingly, we choose the \(x\)-axis by the propagating direction and the system (1.1) is given the front condition at left-right ends such that

\[
\begin{align*}
\lim_{x \to -\infty} n(x, y, t) &= n_- > 0, \quad \lim_{x \to \infty} n(x, y, t) = 0, \\
\lim_{x \to -\infty} c(x, y, t) &= 0, \quad \lim_{x \to \infty} c(x, y, t) = c_+ > 0.
\end{align*}
\]

(1.2)

(1.3)

To all functions in this paper, we impose the periodic condition in \(y\)-variable of period \(\lambda > 0\).
A planar traveling wave solution of (1.1) is a traveling wave solution independent of the transversal direction $y$:

$$n(x,y,t) = N(x - st), \quad c(x,y,t) = C(x - st)$$

with a given wave speed $s > 0$ which we always assume positive in this paper without loss of generality. We consider only waves $(N,C)$ satisfying the boundary conditions (1.2) and (1.3) which means

$$\lim_{x \to -\infty} N(z) = n_+ > 0, \quad \lim_{x \to +\infty} C(z) = c_+ > 0, \quad \lim_{x \to -\infty} N(z) = \lim_{x \to +\infty} C(z) = 0.$$  

We also assume that

$$\lim_{x \to \pm\infty} N'(z) = \lim_{x \to \pm\infty} C'(z) = 0.$$  

To have a traveling wave, it is known that the chemosensitivity function $\chi(c)$ needs to be singular near $c = 0$ (e.g. see [13, 25]). In the paper [13], $\chi(c) = c - 1$, which yields the logarithmic singularity ($\chi(c) \nabla c = \nabla \ln(c)$), is assumed, which choice of $\chi(c)$ is then adopted on modeling the formation of the vascular network toward cancerous cells (e.g. see [7, 14, 22]). The existence of traveling wave solution with an invading front might be an evidence of the tumor encapsulation (e.g. see [1, 2, 26]).

In this paper, we consider only the case $\chi(c) = c - 1$ and $m = 1$ of (1.1):

$$\partial_t n - \Delta n = -\nabla \cdot \left( n \frac{\nabla c}{c} \right),$$

$$\partial_t c - \epsilon \Delta c = -cn, \quad (x,y,t) \in \mathbb{R} \times S^\lambda \times \mathbb{R}_+$$

where $S^\lambda$ is the circle of perimeter $\lambda > 0$. This 2D cylindrical domain would be understood as a virtual channel where newly generated blood vessels toward the chemical (VEGF) will be located. We focus on establishing the time asymptotic stability of a planar traveling wave solution $(N,C)$ of (1.7). The restriction on $m = 1$ is required for treating the singularity of $1/c$ by the Cole-Hopf transformation

$$p := -\nabla \ln c = -\nabla c = -\left( \frac{\partial_x c}{c}, \frac{\partial_y c}{c} \right).$$

A well-written explanation of the system including biological interpretation can be found in [23, 24] (also refer to [20] and the references therein).

1.2. A parabolic system of conservation laws. By the Cole-Hopf transform, we translate the singular Keller-Segel system (1.7) into the following system of $(n,p) = (n,(p_1,p_2))$ without singularity:

$$\partial_t n - \Delta n = \nabla \cdot (np),$$

$$\partial_t p - \epsilon \Delta p = -2\epsilon (p \cdot \nabla)p + \nabla n, \quad (x,y,t) \in \mathbb{R} \times S^\lambda \times \mathbb{R}_+$$

with the notation $((p \cdot \nabla)p)_i = \sum_{k=1,2} p_k \partial_k p_i$.

By denoting

$$\mathcal{P} := -C'/C \quad \text{and} \quad P := (\mathcal{P},0),$$

we have a planar traveling wave solution $(N,P) = (N,(\mathcal{P},0))$ of (1.9) of speed $s$ with the boundary conditions inherited from those of $(N,C)$. The existence and some properties of those waves $(N,C)$ and $(N,\mathcal{P})$ can be found in [30, 17]. We put some of the results on the waves we need in Subsection 2.1.
The study on the existence of traveling wave solutions of a Keller-Segel model was initiated by the paper [13], then many works followed (see [3] and the references therein). We also refer to the survey paper [31], which is an excellent exposition of the topic. The existence of traveling waves with the front conditions (1.2) and (1.3) can be found in [32] for $\epsilon = 0$, and [17, 30] for $\epsilon > 0$. When considering the one dimensional system (i.e., no $y$-dependency in (1.7)), the nonlinear stability results were shown in a weighted Sobolev space in [11] for $\epsilon = 0$, and [16] when $\epsilon > 0$ is small (also see [20]). The weighted Sobolev space has commonly appeared when studying nonlinear stability of viscous shocks of conservation laws since [12] (also see [19]).

The study of higher dimensional traveling waves is a very interesting topic and remains open for many questions including existence and stability of such waves as indicated in [31]. As a special case in 2D, planar waves for an infinite cylinder $\mathbb{R} \times S^1$ was considered by [3] following the spirit of the nonlinear energy estimates developed in [11] for the whole line $\mathbb{R}$ case. In angiogenesis, one may consider that a blood vessel in our body has a 2D cylinder structure.

The previous result [3] mainly proved two things: one is the nonlinear stability for $\epsilon = 0$ and the other is the stability of the linearized equation for small $\epsilon > 0$ under the additional mean-zero assumption in transversal direction $y$ for some technical reason. In addition to these two results, Theorem 1.6 in [3] gives a hint why studying planar waves is natural instead of doing general 2D traveling waves by showing that the $y$-derivative of any smooth solution $(n, p)$ decays to zero in $L^2$-sense under certain additional assumption.

In this paper, we show that traveling wave solutions $(N, P)$ of the nonlinear system (1.9) are globally stable under the smallness assumption on the parameters $\epsilon > 0$ and $\lambda > 0$ without the artificial mean-zero assumption in transversal direction $y$, which was needed in [3] even for the corresponding linearized system. Indeed, the main estimate (2.10) holds uniformly for small $\epsilon > 0$ when the antiderivative $(\varphi, \psi)$ of a perturbation of the form $(n - N, p - P) = (\nabla \cdot \varphi, \nabla \psi)$ is sufficiently small in a weighted Sobolev space (see (2.8) and (2.9)). Our result can be considered as an extension of [3] into $\epsilon > 0$ case and an extension of [10] into 2D case. See Theorem 2.9 and Subsection 2.2 for the precise set-up. We state the stability result in terms of the perturbation of $(n, p)$ in Theorem 2.9, then explain the implication of the theorem for the perturbation of $(n, c)$ to $(N, C)$ in Remark 2.10.

At first glance, the transformed $(n, p)$-system (1.9) seems simpler than the $(n, c)$-system (1.7) to analyze since this parabolic system (1.9) of conservation laws does not have the logarithmic singularity. As a price for this, however, the quadratic nonlinear term $2\epsilon (p \cdot \nabla)p$ appears, and it is not clear at all if the linear term $2\epsilon P \cdot \nabla \psi$ in the main perturbation equation (2.13) produced by the nonlinear term $2\epsilon (p \cdot \nabla)p$ in (1.9) can be controlled by the diffusion term $\epsilon \Delta \psi$ in (2.13) produced by the diffusion term $\epsilon \Delta p$ in (1.9).

In this regard, the main obstacle is to handle the quantity $\epsilon \int_0^t \| \sqrt{F} \psi \|^2$ in (3.3), which is the time integral of a localized $L^2$-norm of $\psi$ multiplied by $\epsilon$. We overcome the difficulty thanks to certain dissipations of a localized $L^2$-norm of $\varphi$ (not of $\psi$) together with a careful manipulation done in Lemma 3.3 (see Remark 3.2). In doing so, we need the smallness assumption on $\epsilon > 0$. This idea was first used in [15] for the one-dimensional system while for our two dimensional system, it becomes more delicate due to the non-symmetric nature of the main perturbation equation (2.13) on the propagating direction $x$ and the transversal direction $y$. For instance, when we denote $\varphi = (\varphi^1, \varphi^2)$, we see the non-symmetric term $\int \frac{\partial}{\partial t} \varphi^1 (\varphi^2)_y$ in (3.5). The smallness condition on the chemical diffusion constant $\epsilon$ might be understood in the sense that the chemical in angiogenesis often diffuses in the dense network of extracellular matrix and tissues which are almost static as mentioned in [31].

Unfortunately, we also need the smallness assumption on the perimeter $\lambda > 0$ of a 2D infinite cylinder, and it appears due to a technical reason in our proof. In fact, with wave speed $s > 0$, we ask the product $s \cdot \lambda$ to be smaller than a given absolute constant (see (3.2)). This condition
enables us to employ Poincaré inequality (3.1) in the transversal direction $y$ in order to control a non-localized $L^2$-norm of $\varphi$ (see (3.5) and (3.6))). In our opinion, it is very challenging to remove this technical smallness assumption on $\lambda > 0$.

For the Cauchy problem of (1.1), we refer to [4, 5, 6, 7, 15], where [4, 5] prove the existence of a global weak solution, and [15] proves the existence of a global classical solution considering the zero chemical diffusion case in a multi-dimension. When a bounded domain is considered, a boundary layer may appear. We refer to [10] and [21] for the stability questions of the layer.

The remaining parts of the paper are organized as follows. In Subsection 2.1, we introduce background materials including the existence and some properties of traveling wave solutions and state the main result (Theorem 2.9) with its set-up in Subsection 2.2. In Subsection 2.3, we state the local existence of a perturbative solution and its a priori uniform-in-time estimate. In Section 3, we prove the uniform-in-time estimate. The zero-th and first order estimates (Subsection 3.1 and 3.2) are the essential part. Then, the higher order estimate (Lemma 3.9) can be obtained in a similar way. We present its proof for completeness in Subsection 3.3.

2. Main theorem and background materials

2.1. Existence and Properties of traveling wave solutions. We collect some results on traveling wave solutions $(N, C)$ and $(N, P) = (N, (P, 0))$ with the front conditions introduced in Section 1.

We first observe that a traveling wave solution $(N, C)$ defined by (1.4) solve the following ODE system by plugging the expression (1.4) into (1.7):

\[-sN' - N'' = -(\frac{C'}{C} N)',\]
\[-sC' - \epsilon C'' = -CN.\]

(2.1)

**Theorem 2.1** ([30] Lemma 3.2, Lemma 3.4). A monotone solution of (2.1) for $\epsilon > 0$ with the boundary conditions (1.5) and (1.6) exists if the relation

\[n_-(1 + \epsilon) s^2\]

holds. More precisely it holds that

1. $N'< 0$ and $C'> 0$,
2. $N(z) \sim e^{-sz}$, as $z \to \infty$, and
3. $\lim_{z \to -\infty} \frac{C'(z)}{C(z)} = s$ and $\lim_{z \to \infty} \frac{C'(z)}{C(z)} = 0$.

In [30], the author used the results of the KPP-Fisher equation to establish the above theorem.

The relation $P = -C'/C$ gives the system

\[-sN' - N'' = (PN)',\]
\[-sP' - \epsilon P'' = (N - \epsilon P^2)'.\]

(2.2)

We observe that $(N, P) = (N, (P, 0))$ is a traveling wave solution of (1.9). From (1.5) with the above theorem, the wave $(N, P)$ is given the boundary condition

\[N(-\infty) = (1 + \epsilon) s^2, N(+\infty) = 0, P(-\infty) = -s, P(+\infty) = 0, N'(\pm\infty) = P'(\pm\infty) = 0.\]

(2.3)

We abbreviate $\lim_{z \to \pm\infty} f(z)$ by $f(\pm\infty)$ for any function $f$ on $\mathbb{R}$. Moreover, the following theorem holds.
Lemma 2.3. For \( s > 0 \), there exist constants \( \epsilon_1 > 0 \) and \( L \in \mathbb{R} \) such that if \((N, \mathcal{P})\) is a solution of (2.2) and (2.3) for some \( \epsilon \in (0, \epsilon_1) \) given by Theorem 2.2, then

\[
\left| N^{(k)} \right| \leq L, \quad \left| \mathcal{P}^{(k)} \right| \leq L, \quad \text{for } 0 \leq k \leq 2, \quad \text{and}
\]

\[
\left| \left( \frac{1}{N} \right)' \right| + \left| \left( \frac{1}{N} \right)'' \right| \leq \frac{L}{N}, \quad \left| \left( \frac{1}{\sqrt{N}} \right)' \right| \leq \frac{L}{\sqrt{N}}.
\]

Proof. The estimate for \( k \leq 1 \) in the first line was proved in [16] while the proof of the rest can be found in Lemma 4.3 in [3]. \( \square \)

Lastly, we need the following lemma which gives a point in \( \mathbb{R} \) contained both in the transition layer of \( N \) and in that of \( \mathcal{P} \).

Lemma 2.4. For any \( s > 0 \), there exists a constant \( \epsilon_1 > 0 \) such that if \((N, \mathcal{P})\) is a solution of (2.2) and (2.3) for some \( \epsilon \in (0, \epsilon_1) \) given by Theorem 2.2, then there exists a point \( z_0 \in \mathbb{R} \) satisfying

\[
\mathcal{P}(z_0) = -\frac{s}{2} \quad \text{and} \quad N(z_0) \geq \frac{s^2}{4}.
\]

Proof. Since \( \mathcal{P} \) is continuous on \( \mathbb{R} \) and \( \mathcal{P}(-\infty) = -s, \mathcal{P}(+\infty) = 0 \), there exists a point \( z_0 \in \mathbb{R} \) such that \( \mathcal{P}(z_0) = -\frac{s}{2} \). To show \( N(z_0) \geq \frac{s^2}{4} \) for sufficiently small \( \epsilon \), recall the equation (2.2). From \( N(+\infty) = \mathcal{P}(+\infty) = 0 \), we have

\[
-s\mathcal{P} - \epsilon\mathcal{P}' = (N - \epsilon\mathcal{P}^2)
\]

Assume that \( \epsilon_1 > 0 \) is smaller than \( \epsilon_1 \) in Lemma 2.3. Then for any \( \epsilon \in (0, \epsilon_1) \), we get

\[
N(z_0) = -s\mathcal{P}(z_0) + \epsilon((\mathcal{P}(z_0))^2 - \mathcal{P}'(z_0)) \geq \frac{s^2}{2} - \epsilon|\mathcal{P}'(z_0)| \geq \frac{s^2}{2} - \epsilon L
\]

where \( L \) is the constant from Lemma 2.3. We take \( \epsilon_1 > 0 \) small enough to have \( \epsilon_1 L \leq \frac{s^2}{4} \). Then \( N(z_0) \geq \frac{s^2}{4} \) for any \( \epsilon \in (0, \epsilon_1) \). \( \square \)

Remark 2.5. The lemma is due to the fact \( N = -s\mathcal{P} + \epsilon\mathcal{O}(1) \), which means the transition layers of \( N \) is overlapped with that of \( \mathcal{P} \) in some extent when \( \epsilon \) is small enough.

Remark 2.6. The first equation in (2.2) with (1.6) and (2.3) gives the simple relation between \( N \) and \( \mathcal{P} \):

\[
(2.4) \quad \frac{-N'}{N} = s + \mathcal{P}.
\]

Remark 2.7. If we denote \( w(\cdot) = \frac{1}{N(\cdot)} \), then the above lemma implies

\[
(2.5) \quad \frac{w'(z)}{w(z)} \geq \frac{s}{2} \quad \text{for } z \geq z_0 \quad \text{and} \quad w(z) \leq \frac{4}{s^2} \leq \frac{16}{s^4} N \quad \text{for } z \leq z_0.
\]

Indeed, for \( z \geq z_0 \), we have

\[
\frac{w'}{w} = \frac{(1/N)'}{1/N} = \frac{-N'/N^2}{1/N} = \frac{-N'}{N} = s + \mathcal{P} \geq s + \mathcal{P}(z_0) = \frac{s}{2}
\]
thanks to (2.4) and $P' > 0$. For $z \leq z_0$, we have

$$w = \frac{1}{N} \leq \frac{1}{N(z_0)} \leq \frac{4}{s^2} = \frac{16}{s^2} \cdot \frac{s^2}{4} \leq \frac{16}{s^4} N(z_0) \leq \frac{16}{s^4} N(z)$$

due to $N' < 0$. We will use (2.3) in the proof of Lemma 3.7.

Figure 1 describes the above discussions including monotonicity of waves.

2.2. Main theorem. We recall (1.9):

$$\partial_t n - \Delta n = \nabla \cdot (np)$$
$$\partial_t p - \epsilon \Delta p = -2\epsilon (p \cdot \nabla)p + \nabla n,$$

$(x, y, t) \in \mathbb{R} \times S^\lambda \times \mathbb{R}_+.$

Let $(N, P) = (N, (P, 0))$ be a traveling wave solution of (2.6) with (2.3). In the below we introduce a weighted Sobolev space where our perturbative functions are constructed. We use the weight function $w(z)$ (only in the horizontal direction) defined by

$$w(z) = \frac{1}{N(z)}, \quad z \in \mathbb{R}$$

where this unbounded weight was essentially introduced in [16] to handle the difficulty coming from the vacuum state $n_+ = N(+\infty) = 0$. Note that $w$ is monotonically increasing, $w(-\infty) = \frac{1}{(1+\epsilon)s^2}$ and $w(z) \sim e^{sz}$ when $z \gg 1$ by (2.3) and Theorem 2.2.

For an integer $k \geq 0$ and for any $\lambda > 0$, we define the Sobolev spaces $H^k$ and a weighted Sobolev space $H^k_w$ for functions periodic in $y$ with period $\lambda$ as follows;

$$H^k := \{ \varphi \in H^k_{loc}(\mathbb{R}^2) \mid \sum_{i+j\leq k} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} n^{2j}|\partial^j_z \varphi_n(z)|^2 dz < \infty, \varphi(z, \cdot + \lambda) = \varphi(z, \cdot y) \},$$

$$H^k_w := \{ \varphi \in H^k_{loc}(\mathbb{R}^2) \mid \sum_{i+j\leq k} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} n^{2j}|\partial^j_z \varphi_n(z)|^2 w(z) dz < \infty, \varphi(z, \cdot + \lambda) = \varphi(z, \cdot y) \}$$

where for each $n \in \mathbb{Z}$ and for each $z \in \mathbb{R}$, $\varphi_n(z)$ is the $n$th Fourier coefficient of the $(\lambda\cdot)$ periodic (in $y$) function $\varphi(z, \cdot y)$. 
We define the norms by

\[ \| \varphi \|_{H^k_w}^2 := \sum_{i+j \leq k} \int_{\mathbb{R} \times [0,\lambda]} |\partial_z^i \partial_y^j \varphi(z, y)|^2 dzdy, \]

\[ \| \varphi \|_{H^k_w}^2 := \sum_{i+j \leq k} \int_{\mathbb{R} \times [0,\lambda]} |\partial_z^i \partial_y^j \varphi(z, y)|^2 w(z)dzdy. \]

Note that for any \( f \in H^k_w \), we know

\[ (2.7) \quad \| f \|_{H^k_w}^2 \leq (1+\epsilon)s^2 \| f \|_{H^k_b}^2 \]
due to \( w \geq \frac{1}{(1+\epsilon)s^2} \).

We perturb the equation (2.6) around the wave

\[ (2.8) \quad n(x, y, t) = N(x - st) + \nabla \cdot \varphi(x - st, y, t) \quad \text{and} \quad p(x, y, t) = P(x - st) + \nabla \psi(x - st, y, t). \]

With \( z := x - st \) in the moving frame, we expect that for each time \( t \), the perturbation \( (\varphi(\cdot, \cdot, t), \psi(\cdot, \cdot, t)) \) lies on the following function class:

\[ (2.9) \quad \varphi = (\varphi^1, \varphi^2) \in (H^3_w)^2 \quad \text{and} \quad \psi \in H^3 \quad \text{with} \quad \nabla \psi \in H^2_w. \]

Remark 2.8. Such a one-sided decaying function (in the weighted Sobolev space) appears typically with respect to the solvability of \( \nabla \cdot \varphi = u \) in the infinite cylinder \( \mathbb{R} \times S^1 \) (e.g. see [25]). An explanation relevant to the perturbation (2.8) is given in Remark 1.2 in [3].

Now we state the main theorem:

**Theorem 2.9.** For any \( s > 0 \) and for any \( \lambda > 0 \) such that the product \( s \cdot \lambda \) is sufficiently small, there exist constants \( \epsilon_0 > 0, K_0 > 0, \) and \( C_0 \geq 1 \) such that if \( (N, P) \) is a solution of (2.2) for some \( \epsilon \in (0, \epsilon_0) \) with (2.3) given by Theorem 2.2, then for any initial data \( (n_0, p_0) \) of (2.6) in the form of \( n_0 = N + \nabla \cdot \varphi_0 \) and \( p_0 = P + \nabla \psi_0 \) satisfying

\[ M_0 := (\| \varphi_0 \|_{H^3_w}^2 + \| \psi_0 \|_{H^3_w}^2 + \| \nabla \psi_0 \|_{H^2_w}^2) \leq K_0, \]

there exists a unique global solution \( (n, p) \) of (2.6) in the form of

\[ n(x, y, t) = N(x - st) + \nabla \cdot \varphi(x - st, y, t), \quad p(x, y, t) = P(x - st) + \nabla \psi(x - st, y, t), \]

where \( \varphi|_{t=0} = \varphi_0 \) and \( \psi|_{t=0} = \psi_0 \), and \( (\varphi, \psi) \) satisfies the following inequality:

\[ (2.10) \quad \sup_{t \in [0,\infty)} (\| \varphi \|_{H^3_w}^2 + \| \psi \|_{H^3_w}^2 + \| \nabla \psi \|_{H^2_w}^2)(t) + \int_0^\infty \left( \| \nabla \varphi \|_{H^3_w}^2 + \| \nabla \psi \|_{H^2_w}^2 + \epsilon \| \nabla^4 \psi \|_{H^2_w}^2 \right) dt \leq C_0 M_0. \]

Remark 2.10. From (2.8), (1.10) and (1.8), we have \( c(\cdot + st)/C = e^{-\psi} \). Together with \( n(\cdot + st) - N = \nabla \cdot \varphi, \) the above theorem implies

\[ \sup_{t \in [0,\infty)} \left( \| n(\cdot + st, \cdot, t) - N(\cdot) \|_{H^2_w}^2 + \| \nabla \left( \log c(\cdot + st, \cdot, t) - \log C(\cdot) \right) \|_{H^2_w}^2 \right) \]

\[ + \int_0^\infty \left( \| n(\cdot + st, \cdot, t) - N(\cdot) \|_{H^2_w}^2 + \| \nabla \left( \log c(\cdot + st, \cdot, t) - \log C(\cdot) \right) \|_{H^2_w}^2 \right) dt \leq C_0 M_0. \]

1 The two quantities used to define \( \| \cdot \|_{H^k} \) are equivalent up to the transversal length scale \( \lambda > 0 \). In this paper, we do not pursue any estimate which needs to hold uniformly on \( \lambda \).
Before closing this subsection we give a summary on notations used in the paper.

\[ \Omega = \mathbb{R} \times [0, \lambda], \]

\[ w(z) := \frac{1}{N(z)}, \]

\[ M(t) := \sup_{s \in [0, t]} (\|\varphi(s)\|_{H^3_w}^2 + \|\psi(s)\|_{H^3_w}^2 + \|\nabla \psi(s)\|_{H^2_w}^2), \]

\[ M_0 := (\|\varphi_0\|_{H^3_w}^2 + \|\psi_0\|_{H^3_w}^2 + \|\nabla \psi_0\|_{H^2_w}^2) \]

\[ \|f\| := \|f\|_{L^2(\Omega)}, \]

\[ \|f\|_k := \|f\|_{H^k} = \sum_{|\alpha|=0}^k \int_{\Omega} |D^\alpha f|^2 dz dy, \]

\[ \|f\|_{H^k_w}^2 := \|f\|_{H^k_w}^2 = \sum_{|\alpha|=0}^k \int_{\Omega} |D^\alpha f(z, y)|^2 w(z) dz dy, \]

\[ \int f := \int_\Omega f(z, y) dz dy, \]

\[ \int_0^t g := \int_0^t g(s) ds. \]

Here we use the notation \( \| \cdot \| \) to indicate certain norm in space only. For instance, when \( f \) is time-dependent then \( \|f\| \) means \( \|f(t)\| \) in the sequel.

2.3. Perturbation equation. In this subsection, we derive the system on \((\varphi, \psi)\) first. Next we state the main propositions including results on the local existence and the uniform estimates of \((\varphi, \psi)\).

From (2.6) and (2.8), by setting \( u = \nabla \cdot \varphi \) and \( v = \nabla \psi \) temporarily in (2.8), we obtain

\[ \begin{align*}
  u_t - su_z - \Delta u &= \nabla \cdot (Np + Pu + up), \\
  v_t - sv_z - \epsilon \Delta v &= -2\epsilon \left( ((P + v) \cdot \nabla)(P + v) - (P \cdot \nabla)P \right) + \nabla u.
\end{align*} \]

Plugging the relation (2.11) in (2.12) and taking off derivatives, we find that the antiderivative \((\varphi, \psi) = ((\varphi^1, \varphi^2), \psi)\) of \((u, v)\) satisfies the system

\[ \begin{align*}
  \varphi_t - s\varphi_z - \Delta \varphi &= N\nabla \psi + P \nabla \cdot \varphi + \varphi \cdot \nabla \psi, \\
  \psi_t - s\psi_z - \epsilon \Delta \psi &= -2\epsilon P \cdot \nabla \psi - \epsilon |\nabla \psi|^2 + \nabla \cdot \varphi
\end{align*} \]

for \((z, y, t) \in \mathbb{R} \times S^\lambda \times \mathbb{R}_+\). In doing so, we use the curl free property of \( v \) and \( p \). Here the term \( P \nabla \cdot \varphi \) means the vector \((P \nabla \cdot \varphi, 0)\). The multidimensional setting (2.11) was proposed in [3].

Looking for a perturbation in our system as an antiderivative follows the setting in one dimensional works [11], [16]. This method can be found in the study on the nonlinear stability of shock profiles of viscous conservation laws under the mean zero condition with a weight function since the papers [12] and [8]. Without the mean zero condition, we refer to [18], [29], [27] and references therein.

We obtain Theorem 2.9 immediately without any difficulty once we prove the proposition below.
Proposition 2.11. For any $s > 0$ and any $\lambda > 0$ such that the product $(s \cdot \lambda)$ is sufficiently small, there exist constants $\epsilon_0 > 0$, $K_0 > 0$, and $C_0 \geq 1$ such that if $(N, \mathcal{P})$ is a solution of (2.2) for some $\epsilon \in (0, \epsilon_0)$ with (2.3) given by Theorem 2.2, then we have the following:

For any initial data $(\varphi_0, \psi_0)$ of (2.13) satisfying

$$M_0 := \|\varphi_0\|_{L^3_w}^2 + \|\psi_0\|_{L^3}^2 + \|\nabla \psi_0\|_{L^2_w}^2 \leq K_0,$$

there exists a unique global solution $(\varphi, \psi)$ of (2.13) where $\varphi|_{t=0} = \varphi_0$ and $\psi|_{t=0} = \psi_0$, and $(\phi, \psi)$ satisfies the following inequality:

$$\sup_{t \in [0, \infty)} \left(\|\varphi\|_{L^3_w}^2 + \|\psi\|_{L^3}^2 + \|\nabla \psi\|_{L^2_w}^2\right) + \int_0^\infty \left(\|\nabla \varphi\|_{L^3_w}^2 + \|\nabla \psi\|_{L^2_w}^2 + \epsilon\|\nabla^4 \psi\|_{L^2_w}^2\right) dt \leq C_0 M_0.$$

Proposition 2.11 is a consequence of the following two propositions: Proposition 2.12 which gives a local-in-time existence result and Proposition 2.13 which shows an a priori uniform-in-time estimate.

Proposition 2.12. Let $s > 0$ and $\lambda > 0$. For sufficiently small $\epsilon > 0$, if $(N, \mathcal{P})$ is a solution of (2.2) with (2.3) given by Theorem 2.2, then for any $M > 0$, there exists $T_0 > 0$ such that for any data $(\varphi_0, \psi_0)$ with $\|\varphi_0\|_{L^3_w}^2 + \|\psi_0\|_{L^3}^2 + \|\nabla \psi_0\|_{L^2_w}^2 \leq M$, the system (2.13) has a unique solution $(\varphi, \psi)$ on $[0, T_0]$ satisfying

$$\varphi \in L^\infty(0, T_0; H^3_w), \quad \psi \in L^\infty(0, T_0; H^3), \quad \nabla \psi \in L^\infty(0, T_0; H^2),$$

with $\varphi|_{t=0} = \varphi_0$, $\psi|_{t=0} = \psi_0$ and

$$\sup_{t \in [0, T_0]} \left(\|\varphi\|_{L^3_w}^2 + \|\psi\|_{L^3}^2 + \|\nabla \psi\|_{L^2_w}^2\right) \leq 2M.$$

The local solution of (2.13) can be obtained by the usual contraction method and by a similar computation as in the proof of Proposition 2.13, for which we omit its proof (or see [3]).

The following proposition gives a uniform-in-time estimate, which is the main heart of this paper.

Proposition 2.13. For any $s > 0$ and any $\lambda > 0$ such that the product $(s \cdot \lambda)$ is sufficiently small, there exist constants $\epsilon_0 > 0$, $\delta_0 > 0$, and $C_0 \geq 1$ such that if $(N, \mathcal{P})$ is a solution of (2.2) for some $\epsilon \in (0, \epsilon_0)$ with (2.3) given by Theorem 2.2, then we have the following:

If $(\varphi, \psi)$ is a local solution of (2.13) on $[0, T]$ for some $T > 0$ with $M(T) \leq \delta_0$, then we have

$$M(T) + \int_0^T \sum_{i=1}^4 \|\nabla^i \varphi\|_{L^2_w}^2 dt + \int_0^T \sum_{i=1}^3 \|\nabla^i \psi\|_{L^2_w}^2 + \epsilon \int_0^T \|\nabla^4 \psi\|_{L^2_w}^2 dt \leq C_0 M(0).$$

Note that $C_0$ does not depend on $T > 0$.

Proof of Proposition 2.13 from Proposition 2.12 and Proposition 2.13. We include the proof here for readers’ convenience even if this continuation argument is now standard (or see [3]). Let’s take $M := \delta_0/2$ and $K_0 := M/C_0$ where $\delta_0 > 0$ and $C_0 \geq 1$ are the constants in Proposition 2.13. Due to $C_0 \geq 1$, we know $K_0 \leq M$. Consider the initial data $(\varphi_0, \psi_0)$ with $M_0 \leq K_0$. By using the constant $M$ to the local-existence result (Proposition 2.12), there exist $T_0 > 0$, and there is the unique local solution $(\varphi, \psi)$ on $[0, T_0]$ with $M(T_0) \leq 2M$. Due to $M(T_0) \leq M \leq \delta_0$, we can use the result of Proposition 2.13 to obtain $M(T_0) \leq C_0 M(0) = C_0 M_0$, which implies $M(T) \leq C_0 K_0 \leq M$. Hence we can extend the solution from the time $T_0$ up to the time $2T_0$ by Proposition 2.12 and we obtain $M(2T_0) \leq 2M \leq \delta_0$. Again by Proposition 2.13, it implies $M(2T_0) \leq C_0 M_0 \leq M$. Thus we can repeat this process of the extension to get $M(kT_0) \leq C_0 M_0$ for any $k \in \mathbb{N}$.

□

In the rest of the paper, we focus on proving Proposition 2.13.
3. Uniform-in-time estimate: Proof of Proposition 2.13

Let $s > 0$ and $\lambda > 0$. Recall the Poincaré inequality on intervals which says that there is a constant $C_p > 0$ such that for any $\lambda > 0$ and for any $f \in W^{1,2}(0,\lambda)$, the inequality

$$
\|f - \overline{f}\|_{L^2(0,\lambda)} \leq \lambda C_p \|f'\|_{L^2(0,\lambda)}
$$

holds. Here the mean value $\overline{f}$ of $f$ is defined by $\overline{f} := \frac{1}{\lambda} \int_0^\lambda f(y)dy$. We assume that the product $s \cdot \lambda$ is small to have

$$
s \cdot \lambda \cdot C_p \leq \frac{1}{16}.
$$

From now on, these values $s > 0$ and $\lambda > 0$ are fixed until the end of the proof. Let’s assume $0 < \epsilon_0 \leq 1$ and $0 < \delta_0 \leq 1$ which will be taken sufficiently small later in the proof several times.

We suppose first that $\epsilon_0 > 0$ is sufficiently small so that any $\epsilon \in (0, \epsilon_0]$ meets the assumption of Theorem 2.2. Let $(N, P)$ be a solution of (2.2) for some $\epsilon \in (0, \epsilon_0]$ with (2.3) given by Theorem 2.2. Let $(\varphi, \psi)$ be a local solution of (2.13) on $[0, T]$ for some $T > 0$ with $M(T) \leq \delta_0$.

In the sequel, $C$ denotes a positive constant which may change from line to line, but which stays independent on ANY choice of $\epsilon \in (0, \epsilon_0)$ and $T > 0$ as long as the positive parameters $\epsilon_0$ and $\delta_0$ are sufficiently small.

3.1. Zero-th order estimate.

Lemma 3.1. If the positive constants $\epsilon_0, \delta_0$ are sufficiently small, then there exists a constant $C_1 > 0$ such that for any $t \in [0, T],$

$$
\|\psi\|^2 + \|\varphi\|^2 + \int_0^t \|\nabla \varphi\|^2_w + \epsilon \int_0^t \|\nabla \psi\|^2 + \int_0^t \left( \frac{(N')^2}{N^3} |\varphi|^2 + \frac{P'}{N} (\varphi^1)^2 + \frac{PN'}{N^2} (\varphi^2)^2 \right) 
$$

$$
\leq C_1 \cdot (\|\psi_0\|^2 + \|\varphi_0\|^2_w) + \epsilon \cdot C_1 \int_0^t \|P'\|_w^2 + C_1 \cdot M(t) \cdot \int_0^t \|\nabla \psi\|^2_w.
$$

Remark 3.2. We note that the term

$$
\int_0^t \left( \frac{(N')^2}{N^3} |\varphi|^2 + \frac{P'}{N} (\varphi^1)^2 + \frac{PN'}{N^2} (\varphi^2)^2 \right)
$$

in the left-hand side of (3.3) plays a role of dissipation on the zero-th order. This is non-symmetric for $\varphi_1$ and $\varphi_2$ due to the non-symmetric structure of the main equation (2.13). In Lemma 3.3 these localized $L^2$-norms of $\varphi$ will be used to control

$$
\epsilon \int_0^t \|P'\|_w^2
$$

in the right-hand side of (3.3), which is a localized $L^2$-norm of $\psi$ multiplied by $\epsilon$.

Proof. We multiply $\frac{\varphi}{N}$ to the $\varphi$ equation and $\psi$ to the $\psi$ equation:

$$
\frac{1}{N} \varphi \cdot (\varphi_t - s\varphi_z - \Delta \varphi) + \psi(\psi_t - s\psi_z - \epsilon \Delta \psi)
$$

$$
= \frac{1}{N} \varphi \cdot (N \nabla \psi + P \nabla \cdot \varphi + \nabla \cdot \varphi \nabla \psi) + \psi(-2\epsilon P \cdot \nabla \psi - \epsilon |\nabla \psi|^2 + \nabla \cdot \varphi).
$$
Thus we get
\[
\left(\frac{1}{N}|\varphi|^2/2\right)_t - s\left(\frac{1}{N}|\varphi|^2/2\right)_z + s\left(\frac{1}{N}|\varphi|^2/2 + \frac{1}{N}\varphi \cdot (-\Delta \varphi) + (|\psi|^2/2)_t - s(|\psi|^2/2)_z + \psi(-\epsilon \Delta \psi)
\]
\[
= \varphi \cdot \nabla \psi + \frac{P}{N}\varphi^1 \nabla \cdot \varphi + \frac{1}{N}(\nabla \cdot \varphi)\varphi \cdot \nabla \psi - 2\epsilon P \psi \psi_z - \epsilon \psi|\nabla \psi|^2 + \psi \nabla \cdot \varphi.
\]

By integrating in space \(\Omega\), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \int \frac{|\varphi|^2}{N} + \int |\psi|^2 \right) + \int \frac{\nabla \varphi^2}{N} + \epsilon \int |\nabla \psi|^2 + s \int |\varphi|^2 \left(\frac{1}{N}\right)^t
\]
\[
= - \int \varphi \cdot \varphi_z \left(\frac{1}{N}\right)^t + \int \frac{P}{N}\varphi^1 \nabla \cdot \varphi + \int \frac{\varphi \cdot \nabla \psi}{N} \nabla \cdot \varphi - 2\epsilon \int P \psi \psi_z - \epsilon \int |\nabla \psi|^2 \psi.
\]

Here we use the notation
\[
|\nabla \varphi|^2 := \sum_{i=1}^{2} |\nabla \varphi^i|^2.
\]

Recall the Sobolev embedding which gives us a constant \(C_{SV} > 0\) such that for any \(f \in H^2(\Omega)\), the inequality
\[
\|f\|_{L^\infty} \leq C_{SV} \|f\|_{H^2}
\]
holds. We control the cubic term:
\[
\int \frac{\varphi^1 \nabla \cdot \varphi}{N} \leq \left( 4\|\varphi\|_{L^\infty}^2 \int \frac{|\nabla \psi|^2}{N} + \frac{1}{16} \int \frac{|\nabla \varphi|^2}{N} \right)
\]
\[
\leq C \cdot M(t) \int \frac{|\nabla \psi|^2}{N} + \frac{1}{16} \int \frac{|\nabla \varphi|^2}{N}
\]
where we used \(\|\varphi\|_{L^\infty} \leq C_{SV}\|\varphi\|_{H^2} \leq C_{SV} \cdot s \sqrt{1 + \epsilon_0\|\varphi\|_{H^2}} \leq C \sqrt{M(t)}\) due to (2.7) and \(0 < \epsilon \leq \epsilon_0 \leq 1\).

We control the quadratic term:
\[
\int \frac{P}{N}\varphi^1 \nabla \cdot \varphi = \int \frac{P}{N} \left(\frac{\varphi^1}{2}\right)_z + \int \frac{P}{N}\varphi^1 \nabla \cdot \varphi
\]
\[
= - \int \left( \frac{P}{N} \right)' \left(\frac{\varphi^1}{2}\right)_z + \int \frac{P}{N} \int_{0}^{\lambda} (\varphi^1(z,y) - \bar{\varphi}^1(z))(\varphi^2)_y(z,y) dydz
\]
where \(\bar{\varphi}^1(z) := \frac{1}{\lambda} \int_{0}^{\lambda} \varphi^1(z,y) dy\). Note \(\frac{P}{N} \leq \frac{s}{N}\) and the Poincaré inequality (3.1) on an interval \((0, \lambda)\). Thus we get
\[
\int \frac{P}{N} \int_{0}^{\lambda} (\varphi^1(z,y) - \bar{\varphi}^1(z))(\varphi^2)_y(z,y) dydz \leq s_{C} \int \left| \frac{\varphi^1}{\lambda} \right|_{L^2(0,\lambda)} \left| \frac{\varphi^2}{\lambda} \right|_{L^2(0,\lambda)} \int dz
\]
\[
\leq s_{C} \lambda \int \frac{|\nabla \varphi^2|^2}{N} \leq \frac{1}{16} \int \frac{|\nabla \varphi|^2}{N}
\]
where we used the assumption (3.2). Then we have
\[
\int \frac{P}{N}\varphi^1 \nabla \cdot \varphi \leq - \int \left( \frac{P}{N} \right)' \left(\frac{\varphi^1}{2}\right)^2 + \frac{1}{16} \int \frac{|\nabla \varphi|^2}{N}.
\]
For the $\epsilon$-terms, we assume that $\delta_0 > 0$ is small enough to get
\[
C_{SV} s\sqrt{2}\sqrt{\delta_0} \leq 1/4.
\]
Then, we have
\[
-2\epsilon \int \mathcal{P}\psi_z \psi = \epsilon \int \mathcal{P}'|\psi|^2 \quad \text{and}
\-
\epsilon \int |\nabla \psi|^2 \psi \leq \frac{1}{4}\epsilon \|\nabla \psi\|^2
\]
where we used $\|\psi\|_{L^\infty} \leq C_{SV} \|\psi\|_{H^2} \leq C_{SV} s\sqrt{2}\|\psi\|_{H^2} \leq C_{SV} s\sqrt{2}\sqrt{\delta_0} \leq \frac{1}{4}$.

Up to now, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \int \frac{|\varphi|^2}{N} + \int |\psi|^2 \right) + \frac{7}{8} \int \frac{|\nabla \varphi|^2}{N} + \frac{3}{4} \epsilon \int |\nabla \psi|^2
\leq - \int \varphi \cdot \varphi_z \left( \frac{1}{N} \right)' - \frac{8}{3} \int |\varphi|^2 \left( \frac{1}{N} \right)' - \left( \frac{\mathcal{P}}{N} \right)' \frac{(\varphi^1)^2}{2} + \epsilon \int \mathcal{P}'|\psi|^2 + C \cdot M(t) \int \frac{|\nabla \psi|^2}{N}.
\]
We observe
\[
(*) = \int \varphi \cdot \varphi_z \frac{N'}{N^2} - \frac{1}{2} \int \left( \frac{s + \mathcal{P}}{N^2} \right) |\varphi|^2 - \int \left( \frac{\mathcal{P}}{N} \right)' \frac{(\varphi^2)^2}{2}
\leq \int |\varphi||\varphi_z| \frac{N'}{N^2} - \frac{1}{2} \int \left( \frac{s + \mathcal{P}}{N^2} \right) |\varphi|^2 - \int \left( \frac{\mathcal{P}}{N} \right)' \frac{(\varphi^2)^2}{2} - \int \mathcal{P}' \frac{N'}{N^2} \frac{(\varphi^2)^2}{2}
\leq \frac{3}{4} \int \frac{|\varphi|^2}{N} + \frac{1}{3} \int \frac{|\varphi|^2 (N')^2}{N^3} - \frac{1}{2} \int \left( \frac{N'}{N^2} \right) |\varphi|^2 - \int \mathcal{P}' \frac{(\varphi^1)^2}{2} - \int \mathcal{P}' \frac{(\varphi^2)^2}{2} - \int \mathcal{P}' \frac{N'}{N^2} \frac{(\varphi^2)^2}{2}
\leq \frac{3}{4} \int \frac{|\nabla \varphi|^2}{N} - \frac{1}{6} \int \left( \frac{N'}{N^3} \right) |\varphi|^2 + \int \mathcal{P}' \frac{(\varphi^1)^2}{2} + \int \mathcal{P}' \frac{(\varphi^2)^2}{2} - \int \mathcal{P}' \frac{N'}{N^2} \frac{(\varphi^2)^2}{2}.
\]

Thanks to Theorem 2.2 we observe
\[
\frac{(N')^2}{N^3} > 0, \quad \frac{\mathcal{P}}{N} > 0 \quad \text{and} \quad \frac{\mathcal{P} N'}{N^2} > 0.
\]
Thus we get
\[
\frac{1}{2} \frac{d}{dt} \left( \int \frac{|\varphi|^2}{N} + \int |\psi|^2 \right) + \frac{1}{8} \int \frac{|\nabla \varphi|^2}{N} + \frac{3}{4} \epsilon \int |\nabla \psi|^2
\leq \epsilon \int \mathcal{P}'|\psi|^2 + C \cdot M(t) \int \frac{|\nabla \psi|^2}{N}.
\]
Integrating in time gives the lemma. \hfill \Box

**Lemma 3.3.** If the positive constants $\epsilon_0, \delta_0$ are sufficiently small, then there exists a constant $C_2 > 0$ such that for any $t \in [0, T]$,
\[
\int_0^t \int \mathcal{P}'|\psi|^2 \leq C_2 \cdot \left[ \text{LHS of (3.3)} \right] + C_2 \cdot M(t) \int_0^t \int \frac{|\nabla \psi|^2}{N}.
\]
Here LHS is shorthand for the left-hand side.
Proof. We multiply \( \frac{P}{N} \varphi \) to the \( \varphi \) equation and \( P \psi \) to the \( \psi \) equation:

\[
\frac{P}{N} \varphi \cdot (\varphi_t - s\varphi_z - \Delta \varphi) + P \psi (\psi_t - s\psi_z - \epsilon \Delta \psi) = \frac{P}{N} \varphi \cdot (N \nabla \psi + N \varphi + \nabla \cdot \varphi \nabla \psi) + P \psi (-2\epsilon P \cdot \nabla \psi - \epsilon |\nabla \psi|^2 + \nabla \cdot \varphi).
\]

Thus we get

\[
\frac{d}{dt} \int \left( \frac{P}{N} |\varphi|^2 / 2 + P |\psi|^2 / 2 \right) + s \int P |\psi|^2 / 2 = - \int \left( \frac{P}{N} \right)' \varphi_z \cdot \varphi - s \int \left( \frac{P}{N} \right)' |\varphi|^2 / 2 - \int P' \varphi \psi_z - \int \frac{P}{N} |\nabla \varphi|^2 - s \int P |\nabla \psi|^2 - \int P \psi |\nabla \psi|^2.
\]

For the cubic term as in (3.4) with \( |\mathcal{P}| \leq s \), we have

\[
\left| \int \frac{P}{N} (\nabla \cdot \varphi) \varphi \cdot \nabla \psi \right| \leq s \int \left| \frac{\varphi \cdot \nabla \psi}{N} \nabla \cdot \varphi \right| \leq C \cdot M(t) \int \frac{|\nabla \varphi|^2}{N} + C \int \frac{|\nabla \psi|^2}{N}.
\]

For the quadratic term, we get

\[
- \int \frac{P}{N} |\nabla \varphi|^2 \leq C \int \frac{|\nabla \varphi|^2}{N} \quad \text{and}
\]

\[
- \int P' \varphi^1 \leq \frac{8}{s} \int P |\psi|^2 + \frac{2}{s} \int P |\varphi|^2 \leq \frac{s}{8} \int P |\psi|^2 + C \int \frac{P |\varphi|^2}{N}.
\]

As we did in and after (3.5),

\[
\int \frac{P}{N} |\varphi|^2 \nabla \cdot \varphi \leq - \int \left( \frac{P}{N} \right)' \left( \frac{\varphi^1}{2} \right)^2 + C \int \frac{|\nabla \varphi|^2}{N}
\]

by using (3.2).

For \( \epsilon \)-terms, we assume that \( \epsilon_0 > 0 \) is smaller than \( \epsilon_1 \) in Lemma 2.3. Then we estimate

\[
-2\epsilon \int P^2 \psi_z = \epsilon \int (P^2)' |\psi|^2 = 2\epsilon \int P P' |\psi|^2 \leq 0,
\]

\[
- \epsilon \int P |\nabla \psi|^2 \leq C \epsilon \int |\nabla \psi|^2,
\]

\[
- \epsilon \int P \psi |\nabla \psi|^2 \leq s \int |\psi||\nabla \psi|^2 \leq C_{SV} \cdot s \sqrt{M(t)} \int |\nabla \psi|^2 \leq C \cdot \epsilon \int |\nabla \psi|^2, \quad \text{and}
\]

\[
- \epsilon \int P' \psi_z \leq \frac{s}{4} \int P |\psi|^2 + \frac{2}{s} \int P' |\psi|^2 \leq \frac{s}{4} \int P |\psi|^2 + \frac{2}{s} \int P' |\nabla \psi|^2 \leq \frac{s}{4} \int P |\psi|^2 + C \cdot \epsilon \int |\nabla \psi|^2
\]

where we used \( \delta_0 \leq 1 \), \( \epsilon_0 \leq 1 \), and \( |\mathcal{P}'| \leq L \) where \( L \) is the constant in Lemma 2.3.
Up to now, we have

\[ \frac{d}{dt} \int \frac{P}{N} |\varphi|^2/2 + \frac{P|\psi|^2}{2} + \frac{s}{8} \int P'|\varphi|^2 \leq -s \int (\frac{P}{N})'|\varphi|^2/2 - \frac{2}{N} \int \frac{P'|(\varphi^2)^2}{2} - \int (\frac{P}{N})' \varphi \cdot \varphi \]

\[ \therefore(\ast \ast) \]

\[ + C \cdot \epsilon \int |\nabla \psi|^2 + C \int \frac{|\nabla \varphi|^2}{N} \]

\[ + C \int \frac{P'}{N} |\varphi|^1 + C \cdot M(t) \int \frac{|\nabla \psi|^2}{N}. \]

For the first two terms in (\ast \ast), we have

\[ -s \int (\frac{P}{N})'|\varphi|^2/2 - \int (\frac{P^2}{N})'(\varphi^1)^2 \]

\[ = -s \int \frac{P'}{N} |\varphi|^2/2 + s \int \frac{P}{N} |\varphi|^2/2 - \int \frac{2P P'}{N^2} \frac{(\varphi^1)^2}{2} - \int \frac{P^2}{N} \varphi (\varphi^1)^2 \]

\[ \leq -s \int \frac{P'}{N} (\varphi^2)^2/2 + s \int \frac{P}{N} (\varphi^1)^2 + s \int \frac{P N'}{N^2} (\varphi^2)^2/2 + \frac{(s + P) P N'}{N^2} (\varphi^1)^2/2 \]

\[ \leq -s \int \frac{P'}{N} (\varphi^2)^2/2 + C \int \frac{P'}{N} (\varphi^1)^2 + C \int \frac{P N'}{N^2} (\varphi^2)^2/2 + C \int \frac{(N')^2}{N^3} (\varphi^1)^2/2. \]

For the last term in (\ast \ast), we have

\[ - \int (\frac{P}{N})' \varphi \cdot \varphi = - \int (\frac{P'}{N} - \frac{P N'}{N^2}) \varphi \cdot \varphi \]

\[ = - \int \frac{P'}{N} (\varphi^1) \cdot \varphi^1 - \int \frac{P'}{N} (\varphi^2) \cdot \varphi^2 + \int \frac{P N'}{N^2} \varphi \cdot \varphi \]

\[ \leq \left( \frac{1}{2} + \frac{1}{s} \right) \int \varphi^2 + \frac{1}{2} \int \frac{P'}{N} (\varphi^1)^2 + \left( \frac{s}{4} \right) \int \frac{P'}{N} (\varphi^2)^2 + \frac{1}{2} \int \frac{|\varphi|^2}{N} + \frac{1}{2} \int \frac{(N')^2}{N^3} |\varphi|^2 \]

\[ \leq \left( \left( \frac{1}{2} + \frac{1}{s} \right) L + \frac{L^2}{2} \right) \int \frac{|\nabla \varphi|^2}{N} + \frac{1}{2} \int \frac{P'}{N} (\varphi^1)^2 + \left( \frac{s}{4} \right) \int \frac{P'}{N} (\varphi^2)^2 \]

\[ \leq C \int \frac{|\nabla \varphi|^2}{N} + C \int \frac{P'}{N} (\varphi^1)^2 + C \int \frac{P'}{N} (\varphi^2)^2 + C \int \frac{(N')^2}{N^3} |\varphi|^2 + \frac{1}{4} \int \frac{P'}{N} (\varphi^2)^2. \]

We combine the above two computations to get

\[ (\ast \ast) \leq -s \int \frac{P'}{N} (\varphi^2)^2/2 + C \int \frac{|\nabla \varphi|^2}{N} + C \int \frac{P'}{N} (\varphi^1)^2 + C \int \frac{(N')^2}{N^3} |\varphi|^2 + C \int \frac{P N'}{N^2} (\varphi^2)^2. \]
Proof. We differentiate (2.13) in \( z \) to get

\[
\varphi_{zz} - s\varphi_{zzz} - \Delta \varphi_z = N' \nabla \psi + N \nabla \psi_z + P' \nabla \cdot \varphi_z + P' \nabla \cdot \varphi + P' \nabla \cdot \varphi_z + \nabla \psi \nabla \cdot \varphi + \nabla \psi \nabla \cdot \varphi_z,
\]

\[
\psi_{zz} - s\psi_{zzz} - \epsilon \Delta \psi_z = -2\epsilon(P \cdot \nabla \psi)_{zz} - \epsilon(\nabla \psi^2)_{zz} + \nabla \cdot \varphi_z.
\]

In sum, we have

\[
\frac{s}{8} \int P' |\psi|^2 \leq -\frac{d}{dt} \int \left( \frac{P}{N} \varphi^2/2 + P|\psi|^2/2 \right) \leq 0
\]

\[
+ C \int \frac{\nabla \varphi}{N} + C \int \frac{P'}{N}(\varphi^1)^2 + C \int \frac{(N')^2}{N^3} |\varphi|^2 + C \int \frac{P N'}{N^2} (\varphi^2)^2
\]

\[
+ C \cdot \epsilon \int |\nabla \psi|^2 + C \cdot M(t) \int \frac{|\nabla \psi|^2}{N}.
\]

By taking integral in time, we have the lemma since \( \mathcal{P} < 0 \) and \( |\mathcal{P}| \leq s \) implies

\[
- \int_0^t \frac{d}{dt} \int \left( \frac{P}{N} \varphi^2/2 + P|\psi|^2/2 \right) = \int \left( \frac{P}{N} \varphi^2/2 + P|\psi|^2/2 \right) \leq C(||\varphi(t)||^2_w + ||\psi(t)||^2).
\]

We combine Lemma 3.1 with Lemma 3.3 in the following way: We assume \( \epsilon_0 > 0 \) small enough to have

\[
\epsilon_0 C_1 C_2 \leq \frac{1}{2}
\]

where \( C_1 \) is from Lemma 3.1 and \( C_2 \) is from Lemma 3.3. Then add \[\epsilon C_1 \cdot \text{(the resulting estimate of Lemma 3.3)}\] to (3.3) to get

\[
\frac{1}{2} \cdot \text{(LHS of (3.3))} \leq C_1 \cdot (||\psi_0||^2 + ||\varphi_0||^2_w) + (C_1 + \epsilon_0 C_1 C_2) \cdot M(t) \cdot \int_0^t ||\nabla \psi||^2_w.
\]

In sum, we have the following zero-th order estimate which hasn’t been closed yet:

\[
||\psi||^2 + ||\varphi||^2_w + \int_0^t ||\nabla \varphi||^2_w + \epsilon \int_0^t ||\nabla \psi||^2 \leq C \cdot (||\psi_0||^2 + ||\varphi_0||^2_w) + C \cdot M(t) \cdot \int_0^t ||\nabla \psi||^2_w.
\]

3.2. First order estimate.

From now on, we estimate the derivatives of \( \varphi \) and \( \psi \).

**Lemma 3.4.** If the positive constants \( \epsilon_0, \delta_0 \) are sufficiently small, then there exists a constant \( C > 0 \) such that for any \( t \in [0, T] \),

\[
||\nabla \varphi||^2_w + ||\nabla \psi||^2 + \int_0^t ||\nabla^2 \varphi||^2_w + \epsilon \int_0^t ||\nabla^2 \psi||^2 \leq C(||\nabla \varphi_0||^2_w + ||\psi_0||^2 + ||\varphi_0||^2_w) + C \int_0^t \int N||\nabla \psi||^2 + CM(t) \int_0^t \frac{||\nabla \psi||^2}{N}.
\]

**Proof.** We differentiate (2.13) in \( z \) to get

\[
\varphi_{zz} - s\varphi_{zzz} - \Delta \varphi_z = N' \nabla \psi + N \nabla \psi_z + P' \nabla \cdot \varphi_z + P' \nabla \cdot \varphi + P' \nabla \cdot \varphi_z + \nabla \psi \nabla \cdot \varphi + \nabla \psi \nabla \cdot \varphi_z,
\]

\[
\psi_{zz} - s\psi_{zzz} - \epsilon \Delta \psi_z = -2\epsilon(P \cdot \nabla \psi)_{zz} - \epsilon(\nabla \psi^2)_{zz} + \nabla \cdot \varphi_z.
\]
We multiply \( \frac{\varphi}{N} \) to \( \varphi \) equation and \( \psi_z \) to the \( \psi \) equation from above and do integration by parts to get

\[
\frac{1}{2} \frac{d}{dt} \left( \int \frac{\varphi_z^2}{N} + |\psi_z|^2 \right) + \sum_{ij} \frac{|\partial_j \varphi_i^j|^2}{N} + \epsilon \int |\nabla \psi_z|^2 \\
= \frac{1}{2} \int |\varphi_z|^2 \left( \frac{1}{N} \right)^\prime - \frac{s}{2} \int |\varphi_z|^2 \left( \frac{1}{N} \right)\prime \\
+ \int \frac{N'}{N} \nabla \psi \varphi_z + \int \frac{P'}{N} \nabla \cdot \varphi \varphi_z + \int \frac{P}{N} \nabla \cdot \varphi_z \varphi_z \\
+ \int \nabla \psi_z \nabla \cdot \frac{\varphi_z}{N} + \int \nabla \psi \nabla \cdot \frac{\varphi_z}{N} \\
- 2\epsilon \int \frac{P \cdot \nabla \psi}{N} \varphi_z - \epsilon \int |\nabla \psi_z|^2 \psi_z.
\]

Similarly, we get

\[
\frac{1}{2} \frac{d}{dt} \left( \int \frac{\varphi_y^2}{N} + |\psi_y|^2 \right) + \sum_{ij} \frac{|\partial_j \varphi_y^j|^2}{N} + \epsilon \int |\nabla \psi_y|^2 \\
= \frac{1}{2} \int |\varphi_y|^2 \left( \frac{1}{N} \right)^\prime - \frac{s}{2} \int |\varphi_y|^2 \left( \frac{1}{N} \right)\prime \\
+ \int \frac{P}{N} \nabla \cdot \varphi_y \varphi_y \\
+ \int \nabla \psi_y \nabla \cdot \frac{\varphi_y}{N} + \int \nabla \psi \nabla \cdot \frac{\varphi_y}{N} \\
- 2\epsilon \int \frac{P \cdot \nabla \psi}{N} \varphi_y - \epsilon \int |\nabla \psi_y|^2 \psi_y.
\]

First, we observe

\[
\frac{1}{2} \int |\nabla \varphi|^2 \left( \frac{1}{N} \right)^\prime - \frac{s}{2} \int |\nabla \varphi|^2 \left( \frac{1}{N} \right)\prime \leq - \int \nabla \varphi \cdot \nabla \varphi \left( \frac{1}{N} \right)\prime \leq C \int |\nabla \varphi| |\nabla \varphi_z| \left( \frac{1}{N} \right) \\
\leq \frac{1}{8} \int |\nabla^2 \varphi|^2 \left( \frac{1}{N} \right) + C \int |\nabla \varphi|^2 \left( \frac{1}{N} \right).
\]

We estimate the quadratic terms as follows:

\[
\int \frac{P}{N} \nabla \cdot \varphi_z \varphi_z + \int \frac{P}{N} \nabla \cdot \varphi_y \varphi_y \leq ||P||_{L^\infty} (||\frac{\nabla \varphi_z}{\sqrt{N}}|| ||\frac{\varphi_z}{\sqrt{N}}|| + ||\frac{\nabla \varphi_y}{\sqrt{N}}|| ||\frac{\varphi_y}{\sqrt{N}}||) \leq \frac{1}{4} ||\nabla \varphi||^2 + C||\nabla \varphi||^2,
\]

\[
\int \frac{N'}{N} \nabla \psi \varphi_z \leq ||s + P||_{L^\infty} ||\sqrt{N} \nabla \psi|| ||\frac{\varphi_z}{\sqrt{N}}|| \leq C ||\sqrt{N} \nabla \psi||^2 + C||\nabla \varphi||^2 \\
\text{and}
\]

\[
\int \frac{P'}{N} \nabla \cdot \varphi_z \leq ||P'||_{L^\infty} ||\frac{\nabla \varphi}{\sqrt{N}}||^2 \leq C||\nabla \varphi||^2.
\]
The sum of all cubic terms is bounded by

\[ C \int |\nabla \psi||\nabla^2 \varphi| \frac{[\nabla^2 \varphi]}{N} + C \int |\nabla \psi||\nabla^2 \varphi| (1/N) | \leq C \sqrt{M(t)} \left( \frac{\nabla^2 \varphi}{\sqrt{N}} \right)^2 + \left( \frac{\nabla \varphi}{\sqrt{N}} \right)^2 \]

\[ \leq \frac{1}{4} ||\nabla \varphi||^2 + C ||\nabla \varphi||^2 \]

by \( \|\nabla \psi\|_{L^2} \leq C \sqrt{M(t)} \leq C \sqrt{\delta_0} \) and by assuming \( \delta_0 > 0 \) small enough.

For the \( \epsilon \)-terms, we estimate

\[ -2\epsilon \int (P \psi)_z \psi_z - 2\epsilon \int P \psi_y \psi_y = -2\epsilon \left( \int P' \psi_z \psi_z + \int P \psi_z \psi_z + \int P \psi_y \psi_y \right) \]

\[ \leq C\epsilon \|P\|_{L^\infty} ||\nabla \psi|| \|\nabla^2 \psi\| + C\epsilon \|P'\|_{L^\infty} ||\nabla \psi||^2 \]

\[ \leq C\epsilon \|\nabla \psi\|^2 + \frac{\epsilon}{4} \|\nabla^2 \psi\|^2 \]

and

\[ -\epsilon \int (|\nabla \psi|^2)_z \psi_z - \epsilon \int (|\nabla \psi|^2)_y \psi_y = -\epsilon \left( \int (|\nabla \psi|^2)_z \psi_z + \int (|\nabla \psi|^2)_y \psi_y \right) \]

\[ \leq C\epsilon \sqrt{M(t)} \int |\nabla \psi||\nabla^2 \psi| \leq C\epsilon \|\nabla \psi\|^2 + \frac{\epsilon}{4} \|\nabla^2 \psi\|^2. \]

Adding up all the estimates above, we have

\[ \frac{1}{2} \frac{d}{dt} \left( \int \frac{|\nabla \varphi|^2}{N} + \int |\nabla \psi|^2 \right) + \frac{1}{4} \int \frac{|\nabla^2 \psi|^2}{N} + \frac{\epsilon}{4} \int \|\nabla^2 \psi\|^2 \leq C\|\sqrt{N} \nabla \psi\|^2 + C\frac{\|\nabla \varphi\|^2}{\sqrt{N}} + C\epsilon \|\nabla \psi\|^2. \]

After integration in time, thanks to (3.7), we can control the last two terms above so that we arrive at (3.8).

\[ \square \]

**Lemma 3.5.** If the positive constants \( \epsilon_0, \delta_0 \) are sufficiently small, then there exists a constant \( C > 0 \) such that for any \( t \in [0, T] \),

\[ \int_0^t \int N|\nabla \psi|^2 + \epsilon \int_0^t \int |\nabla^2 \psi|^2 \leq C \left( \frac{||\nabla \psi_0||^2 + ||\psi_0||^2}{N} + ||\psi_0||^2 \right) + C \sqrt{M(t)} \int_0^t \frac{|\nabla \psi|^2}{N}. \]

**Proof.** Multiplying \( \nabla \psi \) to the \( \varphi \)-equation, we have

\[ N|\nabla \psi|^2 = \varphi_t \cdot \nabla \psi - s \varphi_z \cdot \nabla \psi - \Delta \varphi \cdot \nabla \psi - (\nabla \cdot \varphi) P \cdot \nabla \psi - \nabla \cdot \varphi |\nabla \psi|^2 \]

\[ = (\varphi \cdot \nabla \psi)_t - \varphi \cdot \nabla \psi_t - s \varphi_z \cdot \nabla \psi - \Delta \varphi \cdot \nabla \psi - (\nabla \cdot \varphi) P \cdot \nabla \psi - \nabla \cdot \varphi |\nabla \psi|^2. \]

For the second term \( \varphi \cdot \nabla \psi_t \), we use the \( \psi \)-equation (after taking \( \nabla \)):

\[ \nabla \psi_t = s \nabla \psi + \nabla (\nabla \cdot \varphi) + \Delta \nabla \psi - 2\epsilon \nabla (P \cdot \nabla \psi) - \epsilon \nabla (|\nabla \psi|^2) \]

For (\( \ast \)), we observe that

\[ (\ast) = (\nabla \cdot \varphi)_z \psi_z + (\varphi_{yy} - \varphi_{zy}^2) \psi_z + (\nabla \cdot \varphi)_y \psi_y + (\varphi_{zz}^2 - \varphi_{zy}^2) \psi_y. \]

Integrating by parts for \( \int (\ast) \) gives

\[ \int (\ast) = \int (\nabla \cdot \varphi)_z \psi_z + (\nabla \cdot \varphi)_y \psi_y = \int (\nabla \cdot \varphi) \cdot \nabla \psi. \]
By using the \( \psi \)-equation which have \( \nabla \cdot \varphi \) on its right-hand side, we get
\[
\int \nabla (\nabla \cdot \varphi) \cdot \nabla \psi = \int \nabla (\psi_t - s \psi_z - \epsilon \Delta \psi + 2 \epsilon P \cdot \nabla \psi + \epsilon |\nabla \psi|^2) \cdot \nabla \psi
\]
\[
= \frac{1}{2} \frac{d}{dt} \int |\nabla \psi|^2 + \int \nabla (-\epsilon \Delta \psi + 2 \epsilon P \cdot \nabla \psi + \epsilon |\nabla \psi|^2) \cdot \nabla \psi.
\]
Thus integration on \( (3.10) \) gives us
\[
\int \n |\nabla \psi|^2 = \frac{d}{dt} \int \varphi \cdot \nabla \psi - \int s \varphi \cdot \nabla \psi_z - \int \varphi \cdot (\nabla (\nabla \cdot \varphi) + \epsilon \Delta \nabla \psi - 2 \epsilon \nabla (P \cdot \nabla \psi) - \epsilon \nabla (|\nabla \psi|^2))
\]
\[
- \int s \varphi_z \cdot \nabla \psi
\]
\[
- \frac{1}{2} \frac{d}{dt} \int |\nabla \psi|^2 + \int \nabla (\epsilon \Delta \psi - 2 \epsilon P \cdot \nabla \psi - \epsilon |\nabla \psi|^2) \cdot \nabla \psi
\]
\[
- \int (\nabla \cdot \varphi) P \cdot \nabla \psi - \int \nabla \cdot \varphi |\nabla \psi|^2.
\]
We rearrange the above to get
\( (3.11) \)
\[
\int \n |\nabla \psi|^2 = \frac{d}{dt} \int \varphi \cdot \nabla \psi - \frac{1}{2} \frac{d}{dt} \int |\nabla \psi|^2
\]
\[
+ \int |\nabla \cdot \varphi|^2 - \int (\nabla \cdot \varphi) P \cdot \nabla \psi - \int \nabla \cdot \varphi |\nabla \psi|^2
\]
\[
+ \epsilon \int \nabla (\Delta \psi - 2 P \cdot \nabla \psi - |\nabla \psi|^2) \cdot \nabla \psi - \epsilon \int \varphi \cdot (\Delta \nabla \psi - 2 \nabla (P \cdot \nabla \psi) - \nabla (|\nabla \psi|^2))
\]
\[
= (I) + (II) + (III).
\]
For the first term \( (I) \), after integrating in time, we get
\[
\int_0^t (I) = \int_0^t \left( \frac{d}{dt} \int \varphi \cdot \nabla \psi - \frac{1}{2} \frac{d}{dt} \int |\nabla \psi|^2 \right) \leq C(\|\varphi(t)\|^2 + \|\nabla \psi_0\|^2 + \|\varphi_0\|^2)
\]
by \( \int \varphi \nabla \psi \leq C \|\varphi\|^2 + \frac{1}{2} \|\nabla \psi\|^2 \).
For the second term \( (II) \), we estimate
\[
\int |\nabla \cdot \varphi|^2 - \int (\nabla \cdot \varphi) P \cdot \nabla \psi \leq C \int \frac{|\nabla \varphi|^2}{N} + \frac{1}{4} \int \n |\nabla \psi|^2,
\]
\[
\int |\nabla \cdot \varphi||\nabla \psi|^2 \leq C \sqrt{M(t)} \int |\nabla \psi|^2 \leq C \sqrt{M(t)} \int \frac{|\nabla \psi|^2}{N}
\]
by bounding \( \|\nabla \cdot \varphi\|_{L^\infty} \leq C \|\nabla \cdot \varphi\|_{L^2} \leq C \sqrt{M(t)} \).
For the \( \epsilon \)-term \( (III) \), by bounding \( \|\nabla \psi\|_{L^\infty} \leq C \|\nabla \psi\|_{L^2} \leq C \sqrt{M(t)} \leq C \sqrt{\sigma_0} \leq C \), we estimate
\[
\epsilon \int \nabla (\Delta \psi - 2 P \cdot \nabla \psi - |\nabla \psi|^2) \cdot \nabla \psi = -\epsilon \int |\nabla^2 \psi|^2 - \epsilon \int \nabla (2 P \cdot \nabla \psi + |\nabla \psi|^2) \cdot \nabla \psi
\]
\[
\leq -\epsilon \|\nabla^2 \psi\|^2 + C \epsilon \int (|\nabla \psi|^2 + |\nabla^2 \psi||\nabla \psi| + |\nabla \psi|^2 |\nabla^2 \psi|) \leq C \|\nabla \psi\|^2
\]
\[
\leq -\epsilon \|\nabla^2 \psi\|^2 + C \epsilon \|\nabla \psi\|^2 + \frac{\epsilon}{4} \|\nabla^2 \psi\|^2 + C \epsilon \|\nabla \psi\|^2
\]
\[
\leq -\frac{3}{4} \epsilon \|\nabla^2 \psi\|^2 + C \epsilon \|\nabla \psi\|^2
\]
\[-\epsilon \int \varphi \cdot (\Delta \nabla \psi - 2\nabla (P \cdot \nabla \psi) - \nabla (|\nabla \psi|^2)) \leq C\epsilon \int (|\nabla \varphi| |\nabla^2 \psi| + |\nabla \varphi| |\nabla \psi| + |\nabla \varphi| |\nabla^2 \psi|) \leq C|\nabla \psi| \]
\[\leq \frac{\epsilon}{4} |\nabla^2 \psi|^2 + C\epsilon |\nabla \varphi|^2 + C\epsilon |\nabla \psi|^2 \]
\[\leq \frac{\epsilon}{4} |\nabla^2 \psi|^2 + C\epsilon |\nabla\varphi|^2 + C\epsilon |\nabla \psi|^2.\]

Remark 3.6. The key idea is to observe that the chemical \(c\) is consumed by the cells \(n\) in the system (1.1). More precisely, the negative sign of the term \(cn\) in the right-hand side of the \(c\)-equation in (1.7), which is related to the positive sign of the term \(\nabla n\) in the right-hand side of the \(p\)-equation in (1.9), is passed down to the signs of terms \(-\frac{1}{2}\int_0^t \frac{d}{dt} |\nabla \psi|^2\) in (I) and \(-\epsilon |\nabla^2 \psi|^2\) in (III).

Integrating (3.11) in time, we get

\[\int_0^t \int N|\nabla \psi|^2 \leq C(||\varphi(t)||^2 + ||\nabla \psi_0||^2 + ||\varphi_0||^2)\]
\[+ \int_0^t \left(C||\nabla\varphi||_w^2 + \frac{1}{4} \int N|\nabla \psi|^2 + C\sqrt{M(t)} \int \frac{|\nabla \psi|^2}{N} - \frac{\epsilon}{2} |\nabla^2 \psi|^2 + C\epsilon |\nabla \psi|^2 \right).\]

By the estimate (3.7), we have (3.9). \(\square\)

Lemma 3.7. If the positive constants \(\epsilon_0, \delta_0\) are sufficiently small, then there exists a constant \(C > 0\) such that for any \(t \in [0, T]\),

\[(3.12) \int \frac{|\nabla \psi|^2}{N} + \int_0^t \int \frac{|\nabla \psi|^2}{N} + \epsilon \int_0^t \int \frac{|\nabla^2 \psi|^2}{N} \leq C(||\nabla \psi_0||_w^2 + ||\psi_0||^2 + ||\varphi_0||_w^2) + C \int_0^t \int \frac{|\nabla^2 \varphi|^2}{N}.\]

Proof. First we take \(\nabla\) to the \(\psi\)-equation then multiply by \(w\nabla \psi\) to get

\[\frac{1}{2}(w|\nabla \psi|^2)_t - \frac{s}{2}(w|\nabla \psi|^2)_z + \frac{s}{2} w'|\nabla \psi|^2\]
\[= w \nabla (\nabla \cdot \varphi) \cdot \nabla \psi + \epsilon w \nabla \psi \cdot (\Delta \nabla \psi - 2\nabla (P \cdot \nabla \psi) - \nabla (|\nabla \psi|^2)) \text{ (}\epsilon\text{-terms}\text{)}.

We assume that \(\epsilon_0 > 0\) is smaller than \(\epsilon_1 > 0\) in Lemma [2.4]. Then, by (2.5), there exists a point \(z_0 \in \mathbb{R}\) such that

\[\frac{w'(z)}{w(z)} \geq \frac{s}{2} \quad \text{for} \quad z \geq z_0 \quad \text{and} \quad w(z) \leq \frac{4}{s^2} \leq \frac{16}{s^4} N \quad \text{for} \quad z \leq z_0.\]
Integrating on each half strip (notation: $\int_{z>z_0} f := \int_{z_0}^{\infty} f(z, y, t) dydz$) and in time, we get

\[
\frac{1}{2} \int_{z>z_0} w|\nabla \psi|^2 \\leq \frac{1}{2} \int_{z>z_0} w|\nabla \psi_0|^2 + \int_0^t \int_{z>z_0} w(\nabla \cdot \varphi) \cdot \nabla \psi - \frac{s}{2} \int_0^t \int_{z>z_0} w' |\nabla \psi|^2 \\
- \frac{s}{2} \int_0^t \int_0^\lambda w |\nabla \psi|^2(z_0, y) dy + \int_0^t \int_{z>z_0} e\text{-terms}
\]

\[
\leq \frac{1}{2} \int_{z>z_0} w|\nabla \psi_0|^2 + \int_0^t \int_{z>z_0} w|\nabla(\nabla \cdot \varphi)||\nabla \psi| - \frac{s^2}{4} \int_0^t \int_{z>z_0} w|\nabla \psi|^2 \\
- \frac{s}{2} \int_0^t \int_0^\lambda w |\nabla \psi|^2(z_0, y) dy + \int_0^t \int_{z>z_0} e\text{-terms}
\]

\[
\leq \frac{1}{2} \int_{z>z_0} w|\nabla \psi_0|^2 - \frac{s^2}{8} \int_0^t \int_{z>z_0} w|\nabla \psi|^2 + C \int_0^t \int_{z>z_0} w|\nabla(\nabla \cdot \varphi)|^2 \\
- \frac{s}{2} \int_0^t \int_0^\lambda w |\nabla \psi|^2(z_0, y) dy + \int_0^t \int_{z>z_0} e\text{-terms}
\]

and

\[
\frac{1}{2} \int_{z<z_0} w|\nabla \psi|^2 \\leq \frac{1}{2} \int_{z<z_0} w|\nabla \psi_0|^2 + \int_0^t \int_{z<z_0} \underbrace{w|\nabla(\nabla \cdot \varphi)||\nabla \psi|}_{\leq C} - \frac{s}{2} \int_0^t \int_{z<z_0} w' |\nabla \psi|^2 \\
+ \frac{s}{2} \int_0^t \int_0^\lambda w |\nabla \psi|^2(z_0, y) dy + \int_0^t \int_{z<z_0} e\text{-terms}
\]

\[
\leq \frac{1}{2} \int_{z<z_0} w|\nabla \psi_0|^2 + C \int_0^t \int_{z<z_0} |\nabla(\nabla \cdot \varphi)||\nabla \psi| \\
+ \frac{s}{2} \int_0^t \int_0^\lambda w |\nabla \psi|^2(z_0, y) dy + \int_0^t \int_{z<z_0} e\text{-terms}
\]

\[
\leq \frac{1}{2} \int_{z<z_0} w|\nabla \psi_0|^2 + C \int_0^t \int_{z<z_0} N|\nabla \psi|^2 + C \int_0^t \int_{z<z_0} \frac{|\nabla^2 \varphi|^2}{N} \\
+ \frac{s}{2} \int_0^t \int_0^\lambda w |\nabla \psi|^2(z_0, y) dy + \int_0^t \int_{z<z_0} e\text{-terms}.
\]

Adding the above two estimates, we get

\[
\frac{1}{2} \int w|\nabla \psi|^2 + \frac{s^2}{8} \int_0^t \int_{z>z_0} w|\nabla \psi|^2 \\
\leq \frac{1}{2} \int w|\nabla \psi_0|^2 + C \int_0^t \int N|\nabla \psi|^2 + C \int_0^t \int w|\nabla^2 \varphi|^2 + \int_0^t \int e\text{-terms}.
\]
Adding $\frac{c^2}{8} \int_0^t \int_{z < z_0} w|\nabla \psi|^2$ to the both sides and noting $w \leq CN$ on $\{z < z_0\}$, we get

$$
\frac{1}{2} \int w|\nabla \psi|^2 + \frac{s^2}{8} \int_0^t \int w|\nabla \psi|^2 \\
\leq \frac{1}{2} \int w|\nabla \psi|^2 + C \int_0^t \int N|\nabla \psi|^2 + C \int_0^t \int w|\nabla^2 \varphi|^2 + \int_0^t \int \epsilon\text{-terms} \\
\leq C(\|\nabla \psi\|_w^2 + \|\psi\|_w^2 + \|\varphi_0\|_w^2) + C\sqrt{M(t)} \int_0^t \int \frac{|\nabla \varphi|^2}{N} + \int_0^t \int \epsilon\text{-terms}
$$

where we used the previous estimate [3.9] for the last inequality.

For the $\epsilon$-terms, we estimate

$$
\int \epsilon\text{-terms} = \epsilon \int w \nabla \psi \cdot (\Delta \psi - 2\nabla (P \cdot \nabla \psi) - \nabla (|\nabla \psi|^2)) \\
= \epsilon \int (w|\nabla^2 \psi|^2 - w' \psi \cdot \nabla \psi + w \nabla \psi \cdot (\Delta \psi + (w |\psi|^2 + w \Delta \psi)|\nabla \psi|) \\
\leq -\epsilon \int w|\nabla^2 \psi|^2 + C \epsilon \int (w|\nabla \psi||\nabla^2 \psi| + w|\nabla \psi||\nabla \psi| + w|\nabla \psi||\nabla \psi|^2 + w|\nabla^2 \psi||\nabla \psi|^2) \\
\leq -\frac{\epsilon}{4} \int w|\nabla^2 \psi|^2 + C \epsilon \int w|\nabla \psi|^2 + C \epsilon \sqrt{M(t)} \int w|\nabla \psi|^2 \leq -\frac{\epsilon}{4} \int w|\nabla^2 \psi|^2 + C \epsilon \int w|\nabla \psi|^2
$$

where we used the estimate $|\frac{w}{w}| = |s + P| \leq s$.

In sum, we have

$$
\frac{1}{2} \int w|\nabla \psi|^2 + \left(\frac{s^2}{8} - C(\epsilon_0 + \sqrt{\delta_0})\right) \int_0^t \int w|\nabla \psi|^2 + \frac{\epsilon}{4} \int_0^t \int w|\nabla^2 \psi|^2 \\
\leq C(\|\nabla \psi\|_w^2 + \|\psi\|_w^2 + \|\varphi_0\|_w^2) + C \int_0^t \int \frac{|\nabla^2 \varphi|^2}{N}.
$$

Then, by making $\epsilon_0 > 0$ and $\delta_0 > 0$ small enough, it proves the estimate [3.12].

□

Up to now, we have proved the following first order energy estimate, which is closed except that we assumed that higher order norms are small by $M(T) \leq \delta_0$:

**Lemma 3.8.** If the positive constants $\epsilon_0, \delta_0$ are sufficiently small, then there exists a constant $C > 0$ such that for any $t \in [0, T]$,

$$
\| \varphi \|_{1,w}^2 + \| \psi \|_w^2 + \| \nabla \psi \|_w^2 + \int_0^t \sum_{i=1,2} \| \nabla^i \varphi \|_w^2 + \int_0^t \| \nabla \psi \|_w^2 + \epsilon \int_0^t \| \nabla^2 \psi \|_w^2 \\
\leq C(\| \varphi_0 \|_{1,w}^2 + \| \psi_0 \|_w^2 + \| \varphi_0 \|_{1,w}^2).
$$

**Proof.** Plugging the estimates [3.9] and [3.12] into (3.8), we have

$$
\| \nabla \varphi \|_w^2 + \| \nabla \psi \|_w^2 + \int_0^t \| \nabla^2 \varphi \|_w^2 + \epsilon \int_0^t \| \nabla^2 \psi \|_w^2 \\
\leq C(\| \nabla \psi \|_w^2 + \| \psi_0 \|_w^2 + \| \varphi_0 \|_{1,w}^2) + C \sqrt{M(t)} \int_0^t \int \frac{|\nabla^2 \varphi|^2}{N}
$$

which gives us

$$
\| \nabla \varphi \|_w^2 + \| \nabla \psi \|_w^2 + \int_0^t \| \nabla^2 \varphi \|_w^2 + \epsilon \int_0^t \| \nabla^2 \psi \|_w^2 \leq C(\| \nabla \psi \|_w^2 + \| \psi_0 \|_w^2 + \| \varphi_0 \|_{1,w}^2)
$$

(3.14)
if we assume $\delta_0 > 0$ small enough. In addition, from the estimate (3.12) together with the above estimate (3.14), we get

$$(3.15) \int \frac{\|\nabla \psi\|^2}{N} + \int_0^t \int \frac{\|\nabla \psi\|^2}{N} + \varepsilon \int_0^t \int \frac{\|\nabla^2 \psi\|^2}{N} \leq C(\|\varphi_0\|^2_w + \|\nabla \psi_0\|^2_w + \|\psi_0\|^2_w).$$

By adding the estimate (3.7) to the above estimates (3.14) and (3.15) and by assuming $\delta_0$ small enough, we have (3.13).

\[\square\]

### 3.3. Higher order estimate.

To finish the proof of Proposition 2.13, we need to do similar energy estimates up to the third order derivatives. We collect all the higher order estimates into the following SINGLE lemma, which can be proved in a similar way as we did for the first order estimate in Lemma 3.4, 3.5, 3.7 and 3.8 in the last subsection. Here we present its proof in detail for readers’ convenience.

**Lemma 3.9.** If the positive constants $\varepsilon_0, \delta_0$ are sufficiently small, then there exists a constant $C > 0$ such that for any $t \in [0, T]$ and for $k = 2, 3$, we have

$$\|\nabla^k \varphi\|^2_w + \|\nabla^k \psi\|^2_w + \int_0^t \|\nabla^{k+1} \varphi\|^2_w + \int_0^t \|\nabla^k \psi\|^2_w + \varepsilon \int_0^t \|\nabla^{k+1} \psi\|^2_w \leq C(\|\varphi_0\|^2_w + \|\nabla \psi_0\|^2_w + \|\psi_0\|^2_w).$$

**Proof of Lemma 3.9** Differentiating the $\varphi, \psi$ equations $i + j$ times in $y$ or $z$, we have

$$\begin{align*}
\partial_y^i \partial_z^j \varphi_t - s \partial_y^i \partial_z^j \varphi_z - \Delta \partial_y^i \partial_z^j \varphi &= (\partial_y^i \partial_z^j (N \nabla \psi) - N \partial_y^i \partial_z^j \nabla \psi) + N \nabla \partial_y^i \partial_z^j \psi + \partial_y^i \partial_z^j (P \nabla \varphi) + \partial_y^i \partial_z^j (\nabla \cdot \varphi \nabla) \\
\partial_y^i \partial_z^j \psi_t - s \partial_y^i \partial_z^j \psi_z - \varepsilon \Delta \partial_y^i \partial_z^j \psi &= -2\varepsilon \partial_y^i \partial_z^j (P \cdot \nabla \psi) - \varepsilon \partial_y^i \partial_z^j (|\nabla \psi|^2) + \nabla \cdot \partial_y^i \partial_z^j \varphi.
\end{align*}$$

Thus we get

$$\begin{align*}
(3.16) \quad &\frac{1}{2} \frac{d}{dt} \left( \int \frac{\|\partial_y^i \partial_z^j \varphi\|^2}{N} + \|\partial_y^i \partial_z^j \psi\|^2 \right) + \int \left( \frac{\|\partial_y^i \partial_z^j \varphi\|^2}{N} + \|\partial_y^i \partial_z^j \psi\|^2 \right) \\
&= \frac{1}{2} \int \|\partial_y^i \partial_z^j \varphi\|^2 \left( \frac{1}{N} \right) + s \int \|\partial_y^i \partial_z^j \psi\|^2 \left( \frac{1}{N} \right) \\
&+ \int (\partial_y^i \partial_z^j (N \nabla \psi) - N \partial_y^i \partial_z^j \nabla \psi) \cdot \frac{\partial_y^i \partial_z^j \varphi}{N} + \partial_y^i \partial_z^j (P \nabla \varphi) \cdot \frac{\partial_y^i \partial_z^j \varphi}{N} \\
&+ \partial_y^i \partial_z^j (\nabla \cdot \varphi \nabla) \frac{\partial_y^i \partial_z^j \varphi}{N} \\
&\text{Quadratic term} \\
&\text{Cubic term} \\
&\text{\varepsilon-term} \\
&\int \partial_y^i \partial_z^j (P \psi_0) \partial_y^i \partial_z^j \psi - \varepsilon \int \partial_y^i \partial_z^j (|\nabla \psi|^2) \partial_y^i \partial_z^j \psi.
\end{align*}$$

- Case $k = i + j = 2$
First, we estimate
\[ \frac{1}{2} \int |\nabla^2 \phi|^2 \left( \frac{1}{N} \right)^\prime - \frac{s}{2} \int |\nabla^2 \phi|^2 \left( \frac{1}{N} \right)^\prime \leq - \int \nabla^2 \phi \cdot \nabla^2 \varphi \left( \frac{1}{N} \right)^\prime \]
\[ \leq C \int |\nabla^2 \phi| |\nabla^2 \varphi| \left( \frac{1}{N} \right)^\prime \leq \frac{1}{4} \int |\nabla^3 \varphi|^2 \left( \frac{1}{N} \right) + C \int |\nabla^2 \varphi|^2 \left( \frac{1}{N} \right) . \]

What it follows, we do not distinguish \( \partial_y \) and \( \partial_z \) derivatives.
The quadratic terms are symbolically
\[ \frac{P''}{N} \nabla \psi \nabla^2 \varphi, \quad \frac{P'}{N} \nabla^2 \varphi \nabla^2 \varphi, \quad \text{and} \quad \frac{P}{N} \nabla^3 \varphi \nabla^2 \varphi. \]

We recall \( P = (P, 0) , \) Lemma \( 2.3 \) and \( (2.4) \) :
\[ |N(k)| < C, \quad |P(k)| < C, \quad \text{for} \quad 0 \leq k \leq 2, \quad \left| \frac{N'}{N} \right| = |P + s| \leq C, \quad \text{and} \quad \left| \left( \frac{N'}{N} \right)' \right| = |P'| \leq C. \]

So the quadratic terms are estimated by
\[ C \int \left[ \left| \frac{N''}{N} \right| |\nabla \psi| |\nabla^2 \varphi| + \left| \frac{P''}{N} \right| |\nabla \varphi| |\nabla^2 \varphi| + \left| \frac{P'}{N} \right| |\nabla^2 \varphi| |\nabla^2 \varphi| \right] \]
\[ \leq C \frac{\nabla^2 \varphi}{\sqrt{N}} \left( \| \nabla \psi \| \sqrt{N} + \| \nabla \varphi \| \sqrt{N} + \| \nabla^2 \varphi \| \sqrt{N} \right) \]
\[ \leq C \left( \| \nabla \psi \|^2 + \| \nabla \varphi \|^2 + \| \nabla^2 \varphi \| ^2 \right) , \]
\[ C \int \left| \frac{P}{N} \right| |\nabla^3 \varphi| |\nabla^2 \varphi| \leq C \left\| \frac{\nabla^3 \varphi}{\sqrt{N}} \right\| \left\| \frac{\nabla^2 \varphi}{\sqrt{N}} \right\| \leq \frac{1}{8} \left\| \frac{\nabla^3 \varphi}{\sqrt{N}} \right\|^2 + C \left\| \frac{\nabla^2 \varphi}{\sqrt{N}} \right\|^2 \]
and
\[ C \left| \int \frac{N'}{N} \nabla^2 \psi \nabla^2 \varphi \right| \leq C \int |\nabla \psi| |\nabla^2 \varphi| + C \int |\nabla \psi| |\nabla^3 \varphi| \]
\[ \leq C \| \nabla \psi \| \left( \| \nabla^2 \varphi \| + \| \nabla^3 \varphi \| \right) \leq C \| \nabla \psi \|^2 + C \| \nabla^2 \varphi \|^2 + \frac{1}{8} \| \nabla^3 \varphi \|^2 \]
where we used integration by parts for the last estimate.
The cubic terms are symbolically written as \( \nabla^2 (\nabla \varphi \nabla \psi) \frac{\nabla^2 \varphi}{N} \). By using integration by parts once, it can be written as
\[ \nabla (\nabla \varphi \nabla \psi) \frac{\nabla^3 \varphi}{N} \quad \text{and} \quad \nabla (\nabla \varphi \nabla \psi) \nabla^2 \varphi (\frac{1}{N})' . \]

So by assuming \( M(t) \) small enough, these terms are estimated by
\[ C \int \left( |\nabla^2 \varphi| |\nabla \psi| + |\nabla \varphi| |\nabla^2 \psi| \right) \frac{|\nabla^3 \varphi|}{N} \leq C \sqrt{M(t)} \int \left( |\nabla^2 \varphi| + |\nabla^2 \psi| \right) \frac{|\nabla^3 \varphi|}{N} \]
\[ \leq C \left\| \frac{\nabla^2 \varphi}{\sqrt{N}} \right\|^2 + C \sqrt{M(t)} \left\| \frac{\nabla^2 \psi}{\sqrt{N}} \right\|^2 + \frac{1}{8} \left\| \frac{\nabla^3 \varphi}{\sqrt{N}} \right\|^2 \]
and
\[ C \int (|\nabla^2 \varphi||\nabla \psi| + |\nabla \varphi||\nabla^2 \psi|) \frac{|\nabla^2 \varphi|}{N} \leq C \frac{|\nabla^2 \varphi|^2}{\sqrt{N}} + C \sqrt{M(t)} \frac{|\nabla^2 \psi|^2}{\sqrt{N}}. \]

Note that we used \( M(t) \) for \( \|\nabla \varphi\|_{L^\infty} \leq M(t) \) and \( \|\nabla \psi\|_{L^\infty} < M(t) \).

For the \( \epsilon \)-terms, we can write them symbolically:
\[ \epsilon \int \nabla^2 (P \psi_z) \nabla^2 \psi \text{ and } \epsilon \int \nabla^2 (|\nabla \psi|^2) \nabla^2 \psi. \]

After integration by parts, we can estimate them by
\[ C \epsilon \int \nabla (P \psi_z) \nabla^3 \psi \leq C \epsilon \int |\nabla \psi| + |\nabla^2 \psi| \leq C \epsilon (\|\nabla \psi\|^2 + \|\nabla^2 \psi\|^2) + \frac{\epsilon}{4} \|\nabla^3 \psi\|^2 \]
and
\[ C \epsilon \int |\nabla (|\nabla \psi|^2)| \nabla^3 \psi | \leq C \epsilon \int |\nabla \psi| |\nabla^2 \psi| \|\nabla^3 \psi| \]
\[ \leq C \epsilon \sqrt{M(t)} \int |\nabla^2 \psi| \|\nabla^3 \psi| \leq C \epsilon \|\nabla^2 \psi\|^2 + \frac{\epsilon}{4} \|\nabla^3 \psi\|^2. \]

Up to now, by Lemma 3.8, (3.16) becomes
\[ \int \frac{|\nabla^2 \varphi|^2}{N} + \int \nabla^2 \psi^2 + \int_0^t \int \frac{|\nabla^3 \varphi|^2}{N} + \epsilon \int_0^t \int |\nabla^3 \psi^2| \]
\[ \leq C(\|\varphi_0\|_{2, w}^2 + \|\nabla \varphi_0\|_{1, w}^2 + \|\psi_0\|_{1, w}^2) + C \sqrt{M(t)} \int_0^t \int \frac{|\nabla^2 \psi|^2}{N}. \]

This estimate is the second order version of Lemma 3.4.

Now we claim the following two estimates which are the second order versions of Lemmas 3.5 and 3.7:
\[ \int_0^t \int N |\nabla^2 \psi|^2 + \epsilon \int_0^t \int |\nabla \Delta \psi|^2 \]
\[ \leq C \left( \|\varphi_0\|_{1, w}^2 + \|\nabla \varphi_0\|_{1, w}^2 + \|\psi_0\| + \|\Delta \psi_0\| \right) + C \sqrt{M(t)} \int_0^t \int \frac{|\nabla^2 \psi|^2}{N}. \]

As we did in Lemma 3.5 and 3.7, our plan is to prove (3.18) first and to use the result in order to get (3.19). Then we will close the estimate (3.17) by using them.

- **Proof of (3.18)**

Taking \( \nabla \cdot \phi \) to \( \phi \) equation, we have
\[ \nabla \cdot \phi_t - s \nabla \cdot \phi_z - \Delta \nabla \cdot \phi \]
\[ = N \Delta \psi + \nabla N \cdot \nabla \psi + P' \nabla \cdot \phi + P(\nabla \cdot \phi)_z + \nabla \cdot (\nabla \cdot \phi \nabla \psi) \]
\[ \quad + \int_{R_1} \]
We multiply $\Delta \psi$ on the both sides to get
\[
N |\Delta \psi|^2 = (\nabla \cdot \varphi - s \nabla \cdot \varphi_z - \Delta \nabla \cdot \varphi) \Delta \psi - R_1 \Delta \psi
= (\nabla \cdot \varphi \Delta \psi)_t - \nabla \cdot \varphi \Delta \psi_t - s \nabla \cdot \varphi_z \Delta \psi - \Delta \nabla \cdot \varphi \Delta \psi - R_1 \Delta \psi.
\]

For the second term $\nabla \cdot \varphi \Delta \psi_t$, we use the $\psi$ equation (after taking $\Delta$):
\[
\Delta \psi_t = s \Delta \psi_z + \epsilon \Delta \Delta \psi + \Delta (-2\epsilon P \cdot \nabla \psi - \epsilon |\nabla \psi|^2) + \Delta (\nabla \cdot \varphi)
\]
in order to get
\[
N |\Delta \psi|^2 = (\nabla \cdot \varphi \Delta \psi)_t - \nabla \cdot \varphi (s \Delta \psi_z + \Delta \nabla \cdot \varphi) - s \nabla \cdot \varphi_z \Delta \psi - \Delta \nabla \cdot \varphi \Delta \psi - \Delta \nabla \cdot \varphi (\Delta \psi - 2P \cdot \nabla \psi - |\nabla \psi|^2).
\]

For $(*)$, we get
\[
\int (*) = \int \Delta (\nabla \varphi) \Delta \psi = \int \Delta (\psi_t - s \psi_z) \Delta \psi - \int \Delta (\epsilon \Delta \psi - 2\epsilon P \cdot \nabla \psi - \epsilon |\nabla \psi|^2) \Delta \psi
= \frac{1}{2} \frac{d}{dt} \int |\nabla \psi|^2 - \epsilon \int \Delta (\Delta \psi - 2P \cdot \nabla \psi - |\nabla \psi|^2) \Delta \psi = \frac{1}{2} \frac{d}{dt} \int |\nabla \psi|^2 - \epsilon \int \Delta R_2 \Delta \psi.
\]

So, integrating on the strip, we have
\[
\int N |\Delta \psi|^2 = \frac{d}{dt} \int \nabla \cdot \varphi \Delta \psi + \int |\nabla \varphi|^2 \int \Delta \psi|^2 - \frac{1}{2} \frac{d}{dt} \int |\nabla \psi|^2
- \int R_1 \Delta \psi + \epsilon \int \Delta (R_2) \Delta \psi - \epsilon \int \nabla \cdot \varphi (\Delta R_2).
\]

We observe that
\[
\int_0^t \left( \frac{d}{dt} \int \nabla \cdot \varphi \Delta \psi - \frac{1}{2} \frac{d}{dt} \int |\nabla \psi|^2 \right) \leq C (\|\nabla \varphi(t)\|^2 + \|\Delta \psi_0\|^2 + \|\nabla \varphi_0\|^2).
\]

The terms in $\int R_1 \Delta \psi$ are estimated as follows;
\[
\int |(\nabla \nabla \cdot \nabla \psi) \Delta \psi| \leq C \int \frac{|\nabla \psi|^2}{N} + \frac{1}{4} \int N |\Delta \psi|^2,
\]
\[
\int |P' \nabla \cdot \varphi \Delta \psi| + \int |P (\nabla \cdot \varphi_z) \Delta \psi| \leq C \left( \int \frac{|\nabla \varphi|^2}{N} + \int \frac{|\nabla^2 \varphi|^2}{N} \right) + \frac{1}{8} \int N |\Delta \psi|^2;
\]
\[
\int |\nabla \cdot (\nabla \cdot \varphi \nabla \psi) \Delta \psi| \leq C (\|\nabla \psi\|_{L^\infty} + \|\nabla \varphi\|_{L^\infty}) \left( C \int \frac{|\nabla \varphi|^2}{N} + C \int \frac{|\Delta \psi|^2}{N} + \frac{1}{4} \int N |\Delta \psi|^2 \right)
\leq C \int \frac{|\nabla \psi|^2}{N} + C \sqrt{M(t)} \int \frac{|\Delta \psi|^2}{N} + \frac{1}{4} \int N |\Delta \psi|^2
\]
by assuming $\delta_0$ small enough.

For the $\epsilon$ terms, we estimate them by
\[
\epsilon \int (\Delta (R_2) \Delta \psi - \nabla \cdot \varphi (\Delta R_2)) \leq -\epsilon \int \nabla \Delta \psi|^2
+ \epsilon \int \Delta (-2P \cdot \nabla \psi - |\nabla \psi|^2) \Delta \psi - \epsilon \int \nabla \cdot \varphi (\Delta \psi - 2P \cdot \nabla \psi - |\nabla \psi|^2).
\]
Thus we get
\[- \epsilon \int \Delta (2P \cdot \nabla \psi + |\nabla \psi|^2) \Delta \psi = \epsilon \int \nabla (2P \cdot \nabla \psi + |\nabla \psi|^2) \nabla \Delta \psi \]
\[\leq C \epsilon (\|\nabla \psi\|^2 + \|\nabla^2 \psi\|^2) + \frac{\epsilon}{4} \|\nabla \Delta \psi\|^2 \]
and
\[- \epsilon \int \nabla \cdot \varphi (\Delta \psi - 2(P \cdot \nabla \psi) - |\nabla \psi|^2) = \epsilon \int \nabla \nabla \cdot \varphi (\Delta \psi - 2(P \cdot \nabla \psi) - |\nabla \psi|^2) \]
\[\leq C \epsilon \int |\nabla^2 \varphi| (|\nabla \Delta \psi| + |\nabla \psi| + |\nabla^2 \psi|) \leq \frac{\epsilon}{4} \|\nabla \Delta \psi\|^2 + C \epsilon \|\nabla \psi\|^2 + C \epsilon \|\nabla^2 \psi\|^2. \]

As a result, we get
\[\epsilon \int (\Delta (R_2) \Delta \psi - \nabla \cdot \varphi (\Delta R_2)) \leq - \frac{\epsilon}{2} \|\nabla \Delta \psi\|^2 + C \epsilon \|\nabla^2 \varphi\|^2 + C \epsilon \|\nabla \psi\|^2 + C \epsilon \|\nabla^2 \psi\|^2. \]

Collecting the above estimates and using Lemma 3.8 we have
\[\int_0^t \int N|\Delta \psi|^2 + \epsilon \int_0^t \int |\nabla \Delta \psi|^2 \leq C (\|\varphi_0\|^2_{L^2} + \|\nabla \psi_0\|^2_{L^2} + \|\psi_0\|^2 + \|\Delta \psi_0\|^2) \]
\[+ C \sqrt{M(t)} \int_0^t \int \frac{|\Delta \psi|^2}{N}. \]

To get (3.18) from the above estimate, we have to control \(\int N|\nabla^2 \psi|^2\) by \(\int N|\Delta \psi|^2\) (possibly with lower order terms). Observe
\[\int N|\Delta \psi|^2 = \int N((\partial_{zz} \psi)^2 + (\partial_{yy} \psi)^2 + 2\partial_{zyz} \psi \partial_{yy} \psi) \]
\[= \int N((\partial_{zz} \psi)^2 + (\partial_{yy} \psi)^2 + 2(\partial_{zy} \psi)^2) - 2 \int N\partial_{z} \psi \partial_{yy} \psi \]
and
\[\left| \int N' \partial_{z} \psi \partial_{yy} \psi \right| = \left| - \int (P + s) \partial_{z} \psi \partial_{yy} \psi \right| \leq \frac{1}{4} \int N(\partial_{yy} \psi)^2 + C \int N|\nabla \psi|^2. \]

Thus we get
\[(3.20) \int N|\nabla^2 \psi|^2 \leq 2 \int N((\partial_{zz} \psi)^2 + \frac{1}{2} (\partial_{yy} \psi)^2 + 2(\partial_{zy} \psi)^2) \]
\[= 2 \int N|\Delta \psi|^2 + 4 \int N\partial_{z} \psi \partial_{yy} \psi - \int N(\partial_{yy} \psi)^2 \]
\[\leq 2 \int N|\Delta \psi|^2 + \int N(\partial_{yy} \psi)^2 + C \int N|\nabla \psi|^2 - \int N(\partial_{yy} \psi)^2 \]
\[\leq 2 \int N|\Delta \psi|^2 + C \int N|\nabla \psi|^2. \]
So we have
\[
\int_0^t \int N|\nabla^2 \psi|^2 \leq C(\|\varphi_0\|^2_{L^2} + \|\nabla^2 \psi_0\|^2_{L^2} + \|\psi_0\|^2) + C \int_0^t \int N|\Delta \psi|^2
\]
by Lemma 3.8. Thus we proved (3.18).

- **Proof of (3.19)**

Multiplying \(w \nabla^2 \psi\) to the equation
\[
\nabla^2 \psi_t - s \nabla^2 \psi_z - \epsilon\Delta \nabla^2 \psi = \nabla^2 (\nabla \cdot \varphi) - 2\epsilon \nabla^2 (P \cdot \nabla \psi) - \epsilon \nabla^2 (|\nabla \psi|^2),
\]
we have
\[
\frac{1}{2} (w|\nabla^2 \psi|^2)_t - \frac{s}{2} (w|\nabla^2 \psi|^2)_z + \frac{s}{2} w' |\nabla^2 \psi|^2
\]
\[
= w \nabla^2 (\nabla \cdot \varphi) \cdot \nabla^2 \psi + \epsilon w \nabla^2 \psi \cdot (\Delta \nabla^2 \psi_2 - 2 \nabla^2 (P \cdot \nabla \psi) - \nabla^2 (|\nabla \psi|^2)).
\]

Recall that there exists a point \(z_0 \in \mathbb{R}\) such that
\[
\frac{w'(z)}{w(z)} \geq \frac{s}{2} \quad \text{for } z \geq z_0 \quad \text{and} \quad w(z) \leq \frac{4}{s^2} \leq \frac{16}{s^4}N \quad \text{for } z \leq z_0.
\]

by (2.5).

Integrating on each half strip (notation : \(\int_{z>z_0} f := \int_{z_0}^{\infty} \int_0^\lambda f(z, y, t) dy dz\)) and in time, we get
\[
\frac{1}{2} \int_{z>z_0} w|\nabla^2 \psi|^2 \leq \frac{1}{2} \int_{z>z_0} w|\nabla^2 \psi_0|^2 + \int_0^t \int_{z>z_0} w\nabla^2 (\nabla \cdot \varphi) \cdot \nabla^2 \psi - \frac{s^2}{4} \int_0^t \int_{z>z_0} w|\nabla^2 \psi|^2
\]
\[
- \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^2 \psi|^2 (z_0, y) dy + \int_0^t \int_{z>z_0} \epsilon\text{-terms}
\]
\[
\leq \frac{1}{2} \int_{z>z_0} w|\nabla^2 \psi_0|^2 - \frac{s^2}{8} \int_0^t \int_{z>z_0} w|\nabla^2 \psi|^2 + C \int_0^t \int_{z>z_0} w|\nabla^2 (\nabla \cdot \varphi)|^2
\]
\[
- \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^2 \psi|^2 (z_0, y) dy + \int_0^t \int_{z>z_0} \epsilon\text{-terms}
\]

and
\[
\frac{1}{2} \int_{z<z_0} w|\nabla^2 \psi|^2 \leq \frac{1}{2} \int_{z<z_0} w|\nabla^2 \psi_0|^2 + \int_0^t \int_{z<z_0} w\nabla^2 (\nabla \cdot \varphi) \cdot \nabla^2 \psi
\]
\[
+ \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^2 \psi|^2 (z_0, y) dy + \int_0^t \int_{z<z_0} \epsilon\text{-terms}
\]
\[
\leq \frac{1}{2} \int_{z<z_0} w|\nabla^2 \psi_0|^2 + C \int_0^t \int_{z<z_0} |\nabla^3 \varphi|||\nabla^2 \psi|
\]
\[
+ \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^2 \psi|^2 (z_0, y) dy + \int_0^t \int_{z<z_0} \epsilon\text{-terms}
\]
\[
\leq \frac{1}{2} \int_{z<z_0} w|\nabla^2 \psi_0|^2 + C \int_0^t \int_{z<z_0} N|\nabla^2 \psi|^2 + C \int_0^t \int_{z<z_0} \frac{|\nabla^3 \varphi|^2}{N}
\]
\[
+ \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^2 \psi|^2 (z_0, y) dy + \int_0^t \int_{z<z_0} \epsilon\text{-terms}.
\]
As in the proof of Lemma 3.7, we get
\[
\frac{1}{2} \int w|\nabla^2 \psi|^2 + \frac{s^2}{8} \int_0^t \int w|\nabla^2 \psi|^2 \leq \frac{1}{2} \int w|\nabla^2 \psi_0|^2 + C \int_0^t \int (N|\nabla^2 \psi|^2 + w|\nabla^3 \varphi|^2 + \epsilon\text{-terms}).
\]

For the \(\epsilon\)-terms, as before, we estimate
\[
\int \epsilon\text{-terms} = \epsilon \int w|\nabla^2 \psi \cdot (\Delta \nabla^2 \psi - 2\nabla^2(P \cdot \nabla \psi) - \nabla^2(|\nabla \psi|^2))
= \epsilon \int (w'\nabla^3 \psi|^2 - w'\nabla^3 \psi \cdot \nabla^2 \psi_z - w\nabla^3 \psi \cdot (2\nabla^2(P \cdot \nabla \psi)) + (w'\nabla \psi_z + w\Delta \nabla \psi) \nabla(|\nabla \psi|^2))
\leq -\epsilon \int w|\nabla^3 \psi|^2
+ C \epsilon \int (w|\nabla^2 \psi||\nabla^3 \psi| + w|\nabla^2 \psi||\nabla^2 \psi| + |\nabla \psi| + w|\nabla \psi|(|\nabla^2 \psi||\nabla^2 \psi| + |\nabla^3 \psi||\nabla^2 \psi|))
\leq -\frac{\epsilon}{4} \int w|\nabla^3 \psi|^2 + C \epsilon \int w|\nabla^2 \psi|^2 + C \epsilon \int w|\nabla \psi|^2
\]
where we used the estimate \(|w'| \leq C|w|\) and for the last inequality, we assumed \(\delta_0\) small enough.

Collecting the above estimates, and using Lemma 3.8 and the previous claim (3.18), we have
\[
\frac{1}{2} \int w|\nabla^2 \psi|^2 + \frac{s^2}{8} \int_0^t \int w|\nabla^2 \psi|^2 + \epsilon \int_0^t \int w|\nabla^3 \psi|^2
\leq C(\|\nabla \psi_0\|_{1,w}^2 + \|\psi_0\|^2 + \|\varphi_0\|_{1,w}^2) + C \int_0^t \int \frac{|\nabla^3 \varphi|^2}{N}.
\]

Then, by making \(\epsilon_0\) and \(\delta_0\) small enough, it proves the claim (3.19).

Now we are ready to finish this proof for Lemma 3.9 for the second order (the case \(k = 2\)).
Plugging (3.19) into (3.17) with small \(\delta_0\), we have
\[
\|\nabla^2 \varphi\|_{w}^2 + \|\nabla^2 \psi\|_{w}^2 + \int_0^t \|\nabla^3 \varphi\|_{w}^2 + \epsilon \int_0^t \|\nabla^3 \psi\|^2
\leq C(\|\nabla \psi_0\|_{1,w}^2 + \|\psi_0\|^2 + \|\varphi_0\|_{2,w}^2).
\]

In turns, we have
\[
\int \frac{|\nabla^2 \psi|^2}{N} + \int_0^t \int \frac{|\nabla^2 \psi|^2}{N} + \epsilon \int_0^t \int \frac{|\nabla^3 \psi|^2}{N} \leq C(\|\varphi_0\|_{2,w}^2 + \|\nabla \psi_0\|_{1,w}^2 + \|\psi_0\|^2).
\]
This proves Lemma 3.9 for case \(k = i + j = 2\).

Remark 3.10. Together with Lemma 3.8, we have proved
\[
(3.21) \quad \|\varphi\|_{2,w}^2 + \|\nabla \psi\|_{1,w}^2 + \|\psi\|^2 + \int_{l=1,2,3} \|\nabla^l \varphi\|_{w}^2 + \int_{l=1,2} \|\nabla^l \psi\|_{w}^2 + \epsilon \int_{l=1,2,3} \|\nabla^l \psi\|_{w}^2
\leq C(\|\varphi_0\|_{2,w}^2 + \|\nabla \psi_0\|_{1,w}^2 + \|\psi_0\|^2).
\]

- Case \(k = i + j = 3\)
For $k = 3$, we present its proof for completeness even if there is almost no new idea. First, we recall the equation \([3.16]\). As before, we estimate

$$
\frac{1}{2} \int |\nabla^3 \varphi|^2 \left( \frac{1}{N} \right)^{n} - \frac{s}{2} \int |\nabla^3 \varphi|^2 \left( \frac{1}{N} \right)^{n} \leq \frac{1}{8} \int |\nabla^4 \varphi|^2 \left( \frac{1}{N} \right) + C \int |\nabla^3 \varphi|^2 \left( \frac{1}{N} \right).
$$

Observe that the quadratic terms are symbolically

$$
\frac{N(l)}{N} |\nabla^3 \varphi|^2 \quad \text{for } l = 1, 2, 3 \quad \text{and}
$$

$$
\frac{P(l)}{N} \nabla \cdot (\nabla^{3-l} \varphi) \nabla^{3} \varphi \quad \text{for } l = 0, 1, 2, 3.
$$

After integration by parts, the terms with $l = 3$ are bounded by

$$
|\int \frac{N''}{N} \nabla \psi \cdot \phi_{zzz}| \leq C \int \left[ \frac{N''}{N} |\nabla \psi| |\nabla^3 \varphi| + \frac{N''}{N} |\nabla^2 \psi| |\nabla^3 \varphi| + \frac{N''}{N} |\nabla \psi| |\nabla^4 \varphi| \right]
$$

$$
\leq C \left( \frac{\| \nabla \psi \|}{\sqrt{N}}^2 + \| \nabla^3 \varphi \| \| \nabla^2 \psi \| + \frac{\| \nabla \psi \|}{\sqrt{N}}^2 \right) + \frac{1}{8} \| \nabla^4 \varphi \|^2,
$$

$$
\int \frac{P''}{N} (\nabla \cdot \psi) (\phi_{1zzz}) \leq C \int \left[ \frac{P''}{N} |\nabla \varphi| |\nabla^3 \varphi| + \frac{P''}{N} |\nabla^2 \varphi| |\nabla^3 \varphi| + \frac{P''}{N} |\nabla \varphi| |\nabla^4 \varphi| \right]
$$

$$
\leq C \left( \frac{\| \nabla \varphi \|}{\sqrt{N}}^2 + \| \nabla^3 \varphi \| \| \nabla^2 \varphi \| + \frac{\| \nabla \varphi \|}{\sqrt{N}}^2 \right) + \frac{1}{8} \| \nabla^4 \varphi \|^2.
$$

All the other quadratic terms are estimated by

$$
C \int \left[ \frac{N''}{N} \right] |\nabla^2 \psi| |\nabla^3 \varphi| + \frac{P''}{N} |\nabla^2 \varphi| |\nabla^3 \varphi| + \frac{P'}{N} |\nabla^3 \varphi| |\nabla^3 \varphi| \right]
$$

$$
\leq C \left( \frac{\| \nabla^2 \psi \|}{\sqrt{N}}^2 + \frac{\| \nabla \varphi \|}{\sqrt{N}}^2 + \frac{\| \nabla^2 \varphi \|}{\sqrt{N}}^2 + \| \nabla^3 \varphi \| \right),
$$

$$
C \int \left| \frac{P}{N} \right| |\nabla^4 \varphi| |\nabla^3 \varphi| \leq \frac{1}{8} \| \nabla^4 \varphi \| \| \nabla^3 \varphi \| + C \| \nabla^3 \varphi \| \| \nabla^3 \varphi \|
$$

and

$$
C \left| \int \frac{N''}{N} \nabla^3 \psi \nabla^3 \varphi \right| \leq C \int |\nabla^2 \psi| |\nabla^3 \varphi| + C \int |\nabla^2 \psi| |\nabla^4 \varphi| \right]
$$

$$
\leq C \| \nabla^2 \psi \| \left( \| \nabla^3 \varphi \| + \| \nabla^4 \varphi \| \right) \leq C \| \nabla^2 \psi \|^2 + C \| \nabla^3 \varphi \|^2 + \frac{1}{8} \| \nabla^4 \varphi \|^2
$$

where we used integration by parts for the last estimate.

So by \([3.21]\), we have

$$
\int \frac{|\nabla^3 \varphi|^2}{N} + \int |\nabla^3 \varphi|^2 + \int \frac{|\nabla^4 \varphi|^2}{N} + C \int |\nabla^4 \varphi|^2 \\
\leq C(\| \varphi_0 \|_{L^2}^2 + \| \nabla^2 \psi_0 \|^2 + \| \nabla^2 \psi_0 \|_{L^2}^2 + \| \varphi_0 \|^2) + \text{the cubic terms} + \epsilon \text{-terms}.
$$
The cubic terms are estimated by
\[ C \left| \int \nabla^3 (\nabla \varphi \nabla \psi) \frac{\nabla^3 \varphi}{N} \right| \leq C \left| \int \nabla^2 (\nabla \varphi \nabla \psi) \left( \frac{\nabla^4 \varphi}{N} + \nabla^3 \varphi (\frac{1}{N})' \right) \right| \leq C \left| \int \nabla^2 (\nabla \varphi \nabla \psi) \frac{\nabla^4 \varphi}{N} \right| + C \left| \int \nabla^2 (\nabla \varphi \nabla \psi) \nabla^3 \varphi (\frac{1}{N})' \right|. \]

From \(|(\frac{1}{N})'| + |(\frac{1}{N})''| \leq \frac{C}{N}\) (Lemma 2.3), we can estimate each term. Indeed, recall
\[ \|\nabla \varphi\|_{L^\infty} + \|\nabla \varphi\|_{L^\infty} \leq C M(t). \]

So we get
\[ C \left| \int \nabla^3 \varphi \nabla \psi \nabla^4 \varphi (\frac{1}{N}) \right| + C \left| \int \nabla^3 \varphi \nabla^3 \varphi (\frac{1}{N})' \right| \leq C \left| \int \nabla^3 \varphi ||\nabla \psi|| \frac{\nabla^3 \varphi}{N} + ||\nabla \varphi\| ||\nabla^3 \psi\| \frac{\nabla^3 \varphi}{N} \right| \leq C \sqrt{M(t)} \left( \int \|\nabla^3 \varphi\|^2 + \|\nabla^3 \psi\|^2 \right) + \frac{1}{8} \int |\nabla^4 \varphi|^2. \]

Then the term \(\int \nabla^2 \varphi \nabla^2 \psi \frac{\nabla^4 \varphi}{N}\) remains. Note that by the Sobolev embedding,
\[ \|f\|_{L^4} \leq C (\|f\|_{L^2} + \|\nabla f\|_{L^2}). \]

So we estimate
\[ C \left| \int \nabla^2 \varphi \nabla^2 \psi \frac{\nabla^4 \varphi}{N} \right| \leq C \|\nabla^2 \varphi\|_{L^2} \frac{\|\nabla^3 \varphi\|}{\sqrt{N}} \frac{\|\nabla^4 \varphi\|}{\sqrt{N}} \right| \leq \frac{1}{8} \|\nabla^4 \varphi\|_{L^2} + C \left( \|\nabla^2 \varphi\|_{L^2} \frac{\|\nabla^3 \varphi\|}{\sqrt{N}} \frac{\|\nabla^4 \varphi\|}{\sqrt{N}} \right) \cdot (\|\nabla^3 \varphi\|_{L^2} + \|\nabla^2 \varphi\|_{L^2})^2 \leq \frac{1}{8} \|\nabla^4 \varphi\|_{L^2} + C \left( \|\nabla^2 \varphi\|_{L^2} + \|\nabla^2 \varphi\|_{L^2} \right)^2 \cdot (\|\nabla^3 \varphi\|_{L^2} + \|\nabla^2 \varphi\|_{L^2})^2. \]

Using \(|(\frac{1}{N})'| \leq \frac{C}{N}\) (Lemma 2.3), we get
\[ \left( \frac{\|\nabla^3 \varphi\|}{\sqrt{N}} + \|\nabla^2 \varphi\| (\frac{1}{N})' \right) \right|_{L^2} + C \left( \frac{\|\nabla^2 \varphi\|}{\sqrt{N}} \right) \leq C \left( \frac{\|\nabla^3 \varphi\|}{\sqrt{N}} \right) \leq CM(t). \]

For the \(\epsilon\)-terms, we can write them symbolically:
\[ \epsilon \int \nabla^3 (P \psi_\epsilon) \nabla^3 \psi \quad \text{and} \quad \epsilon \int \nabla^3 (||\nabla \psi||^2) \nabla^3 \psi. \]
After integration by parts, we can estimate these terms by

\[
C \epsilon \int |\nabla^2 (P \psi_z)| |\nabla^4 \psi| \leq C \epsilon \int (|\nabla \psi| + |\nabla^2 \psi| + |\nabla^3 \psi|)|\nabla^4 \psi|
\]
\[
\leq C \epsilon (\|\nabla \psi\|^2 + \|\nabla^2 \psi\|^2 + \|\nabla^3 \psi\|^2) + \frac{\epsilon}{4} \|\nabla^4 \psi\|^2
\]

and

\[
C \epsilon \int |\nabla^2 (|\nabla \psi|^2)| |\nabla^4 \psi| \leq C \epsilon \int (|\nabla \psi| |\nabla^3 \psi| + |\nabla^2 \psi| |\nabla^2 \psi|)|\nabla^4 \psi|
\]
\[
\leq C \epsilon \sqrt{M(t)} \int |\nabla^3 \psi| |\nabla^4 \psi| + C \epsilon \|\nabla^2 \psi\|^2 \cdot \|\nabla^4 \psi\|_{L^2}
\]
\[
\leq C \epsilon \sqrt{M(t)} \|\nabla^3 \psi\|^2 + \frac{\epsilon}{8} \|\nabla^4 \psi\|^2 + C \epsilon (|\nabla^3 \psi|^2_{L^2} + \|\nabla^2 \psi\|^2_{L^2}) \cdot \|\nabla^4 \psi\|_{L^2}
\]
\[
\leq C \epsilon \sqrt{M(t)} \|\nabla^3 \psi\|^2 + \frac{\epsilon}{4} \|\nabla^4 \psi\|^2
\]

by assuming \( \delta_0 \) small enough.

Collecting the above estimates and using (3.21), we get the third order version of (3.17):

(3.22) \[
\int \frac{|\nabla^3 \varphi|^2}{N} + \int |\nabla^3 \psi|^2 + \int_0^t \int \frac{|\nabla^4 \varphi|^2}{N} + \epsilon \int_0^t \int |\nabla^4 \psi|^2
\]
\[
\leq C (\|\varphi_0\|^2_{3,w} + \|\nabla^3 \psi_0\|^2 + \|\nabla \psi_0\|^2_{1,w} + \|\psi_0\|^2) + C \sqrt{M(t)} \int_0^t \int \frac{|\nabla^3 \psi|^2}{N}.
\]

As before, we claim the following two estimates for the third order derivatives:

(3.23) \[
\int \frac{|\nabla^3 \varphi|^2}{N} + \int \frac{|\nabla^3 \psi|^2}{N} + \epsilon \int_0^t \int |\nabla^2 \Delta \psi|^2
\]
\[
\leq C (\|\varphi_0\|^2_{2,w} + \|\nabla \psi_0\|^2_{1,w} + \|\psi_0\|^2 + \|\nabla \Delta \psi_0\|^2) + C \sqrt{M(t)} \int_0^t \int \frac{|\nabla^3 \psi|^2}{N},
\]

(3.24) \[
\int \frac{|\nabla^3 \varphi|^2}{N} + \int \frac{|\nabla^3 \psi|^2}{N} + \epsilon \int_0^t \int \frac{|\nabla^4 \varphi|^2}{N} \leq C (\|\nabla \psi_0\|^2_{2,w} + \|\psi_0\|^2 + \|\varphi_0\|^2_{2,w}) + C \int_0^t \int \frac{|\nabla^4 \psi|^2}{N}.
\]

We will prove (3.23) below and we will use the result in order to get (3.24). Then we will apply (3.24) to close (3.22).

- **Proof of (3.23)**

Taking \( D \nabla \cdot \) to \( \varphi \) equation where \( D \) is either \( \partial_z \) or \( \partial_y \), we have

\[
D \nabla \cdot \varphi_t - s D \nabla \cdot \varphi_z - \Delta D \nabla \cdot \varphi
\]
\[
= ND \Delta \psi + DN \Delta \psi + D \left( \nabla N \cdot \nabla \psi + \nabla \cdot P \nabla \cdot \varphi + P \cdot \nabla (\nabla \cdot \varphi) + \nabla \cdot (\nabla \cdot \varphi \nabla \psi) \right). \quad \text{\( R_1 \)}
\]
We multiply \( D\Delta \psi \) on the both sides to get
\[
N|D\Delta \psi|^2 = D(\nabla \cdot \varphi_t - s \nabla \cdot \varphi_z - \Delta \nabla \cdot \varphi)D\Delta \psi - R_1 D\Delta \psi
\]
\[
= (D\nabla \cdot \varphi D\Delta \psi)_t - D\nabla \cdot \varphi D\Delta \psi_t - s D\nabla \cdot \varphi_z D\Delta \psi - D\Delta \nabla \cdot \varphi D\Delta \psi - R_1 D\Delta \psi. \tag{*}
\]

For the second term \( D\nabla \cdot \varphi D\Delta \psi_t \), we use the \( \psi \) equation (after taking \( D\Delta \)):
\[
D\Delta \psi_t = s D\Delta \psi_z + \epsilon \Delta D\Delta \psi + \Delta \left(-2\epsilon P \cdot \nabla \psi - \epsilon|\nabla \psi|^2\right) + D\Delta(\nabla \cdot \varphi)
\]
in order to get
\[
N|D\Delta \psi|^2 = (D\nabla \cdot \varphi D\Delta \psi)_t - D\nabla \cdot \varphi(s D\Delta \psi_z + D\Delta \nabla \cdot \varphi) - s D\nabla \cdot \varphi_z D\Delta \psi - \epsilon \Delta D\Delta \psi - \epsilon|\nabla \psi|^2 - (\nabla \cdot \varphi)\Delta \psi - \epsilon\Delta^2 D\Delta \psi - R_1 D\Delta \psi.
\]

We observe that
\[
\int (\epsilon \Delta D\Delta \psi - \epsilon|\nabla \psi|^2)D\Delta \psi = \int D\Delta(\nabla \cdot \varphi)D\Delta \psi \quad \text{and} \quad \int \left| D\Delta \psi \right|^2 - \epsilon \int D\Delta(\nabla \cdot \varphi)D\Delta \psi
\]
\[
= \frac{1}{2} \frac{d}{dt} \int |D\Delta \psi|^2 - \epsilon \int R_1 D\Delta \psi + \epsilon \int D\Delta(R_2)D\Delta \psi - \epsilon \int D\nabla \cdot \varphi(D\Delta R_2).
\]

So, integrating on the strip, we have
\[
\int \left( d \frac{d}{dt} \int |D\nabla \cdot \varphi D\Delta \psi - \frac{1}{2} \frac{d}{dt} \int |D\Delta \psi|^2 \right) \leq C \left( \|\nabla^2 \varphi(t)\|^2 + \|D\Delta \psi_0\|^2 + \|\nabla^2 \varphi_0\|^2 \right).
\]

Note that
\[
\int \left( \frac{d}{dt} \int |D\nabla \cdot \varphi D\Delta \psi - \frac{1}{2} \frac{d}{dt} \int |D\Delta \psi|^2 \right) \leq C \left( \frac{1}{2} \int |\nabla \varphi + |\nabla^2 \varphi|^2 + |\nabla^3 \varphi|^2} N \right) + \frac{1}{8} \int N|D\Delta \psi|^2.
\]

The integral containing \( R_1 \) is estimated as follows;
The quadratic terms:
\[
C \left( \frac{1}{2} \int |\nabla \varphi + |\nabla^2 \varphi|^2 + |\nabla^3 \varphi|^2} N \right) + \frac{1}{8} \int N|D\Delta \psi|^2.
\]

The cubic term:
\[
C \left( \frac{1}{2} \int |\nabla \varphi + |\nabla^2 \varphi|^2 + |\nabla^3 \varphi|^2} N \right) + \frac{1}{8} \int N|D\Delta \psi|^2.
\]

for small \( \delta_0 \). The last term in the above can be estimated:
\[ C \int |\nabla^2 \varphi| |\nabla^2 \psi| |D \Delta \psi| = C \int \frac{|\nabla^2 \varphi|}{\sqrt{N}} |\nabla^2 \psi| \sqrt{N} |D \Delta \psi| \leq C ||\nabla^2 \psi||_{L^4} ||\nabla^2 \varphi||_{L^4} ||\sqrt{N} D \Delta \psi||_{L^2} \]

\[ \leq \frac{1}{8} ||\sqrt{N} D \Delta \psi||_{L^2}^2 + C \left( \frac{\int \nabla^2 \varphi}{\sqrt{N}} |||L^2 + \frac{\int \nabla^2 \varphi}{\sqrt{N}}|||_{L^2}^2 \right) \cdot \left( ||\nabla^3 \varphi||_{L^2} + ||\nabla^2 \psi||_{L^2} \right)^2 \]

\[ \leq \frac{1}{8} ||\sqrt{N} D \Delta \psi||_{L^2}^2 + C \left( \frac{\int \nabla^3 \varphi}{\sqrt{N}} + ||\nabla^2 \varphi|| \left( \frac{1}{\sqrt{N}} \right)|||L^2 + C ||\nabla^2 \varphi||_{L^2} \right)^2 \cdot \left( ||\nabla^3 \varphi||_{L^2} + ||\nabla^2 \psi||_{L^2} \right)^2 \cdot \sum_{CM(t)} \]

Up to now, using (3.21), we have

\[ \int_{0}^{t} \int N |D \Delta \psi|^2 \leq C \left( ||\varphi_0||_{2,w}^2 + ||\nabla \varphi_0||_{1,w}^2 + ||\psi_0||^2 + ||D \Delta \psi_0||^2 \right) + C \sqrt{M(t)} \int_{0}^{t} \int \frac{|\nabla^3 \psi|^2}{N} \]

\[ + \epsilon \int (D \Delta (R_2) D \Delta \psi - D \nabla \cdot \varphi(D \Delta R_2)) \]

where \( R_2 = (\Delta \psi - 2P \cdot \nabla \psi - |\nabla \psi|^2) \).

For \( \epsilon \) terms, we estimate them by

\[ \epsilon \int (D \Delta (R_2) D \Delta \psi - D \nabla \cdot \varphi(D \Delta R_2)) \leq -\epsilon \int |\nabla D \Delta \psi|^2 \]

\[ + \epsilon \int D \Delta (-2P \cdot \nabla \psi - |\nabla \psi|^2) D \Delta \psi - \epsilon \int D \nabla \cdot \varphi(D \Delta (\Delta \psi - 2P \cdot \nabla \psi - |\nabla \psi|^2)) \]

For the last two terms, thanks to

\[ ||\nabla^2 \psi||_{L^4}^4 \leq C \left( ||\nabla^2 \psi|| + ||\nabla^3 \varphi|| \right)^4 \leq CM(t) \left( ||\nabla^2 \psi|| + ||\nabla^3 \varphi|| \right)^2, \]

we estimate

\[ -\epsilon \int \Delta D(2P \cdot \nabla \psi + |\nabla \psi|^2) D \Delta \psi = \epsilon \int \nabla D (2P \cdot \nabla \psi + |\nabla \psi|^2) \nabla D \Delta \psi \]

\[ \leq C \epsilon \left( ||\nabla \psi||^2 + ||\nabla^2 \psi||^2 + ||\nabla^3 \varphi||^2 + ||\nabla^2 \psi||_{L^4}^4 \right) + \frac{\epsilon}{4} ||\nabla D \Delta \psi||^2 \]

\[ \leq C \epsilon \left( ||\nabla \psi||^2 + ||\nabla^2 \psi||^2 + ||\nabla^3 \varphi||^2 \right) + \frac{\epsilon}{4} ||\nabla D \Delta \psi||^2 \]

and

\[ -\epsilon \int D \nabla \cdot \varphi(\Delta D (\Delta \psi - 2P \cdot \nabla \psi) - |\nabla \psi|^2)) = \epsilon \int D \nabla D \nabla \cdot \varphi(D \Delta \psi - 2(P \cdot \nabla \psi) - |\nabla \psi|^2)) \]

\[ \leq C \epsilon \int ||\nabla \psi||^2 + ||\nabla^2 \psi||^2 + C \epsilon ||\nabla^3 \varphi||^2 + C \epsilon ||\nabla^2 \psi||^2 + C \epsilon ||\nabla^3 \varphi||^2 + C \epsilon ||\nabla^2 \psi||_{L^4}^4 \]

\[ \leq \frac{\epsilon}{4} ||\nabla D \Delta \psi||^2 + C \epsilon ||\nabla \psi||^2 + C \epsilon ||\nabla \psi||^2 + C \epsilon ||\nabla^2 \psi||^2 + C \epsilon ||\nabla^3 \varphi||^2. \]

As a result, we get

\[ \epsilon \int (D \Delta (R_2) D \Delta \psi - D \nabla \cdot \varphi(D \Delta R_2)) \leq -\frac{\epsilon}{2} ||\nabla D \Delta \psi||^2 \]

\[ + C \epsilon \left( ||\nabla^3 \varphi||^2 + ||\nabla \psi||^2 + ||\nabla^2 \psi||^2 + ||\nabla^3 \varphi||^2 \right). \]
We use (3.21) again to get
\[
\int_0^t \int N |\Delta D\psi|^2 + \epsilon \int_0^t \int |\nabla D\Delta \psi|^2 \\
\leq C(\|\varphi_0\|_{L^2}^2 + \|\nabla\psi_0\|_{L^1}^2 + \|\psi_0\|^2 + \|D\Delta \psi_0\|^2) + C \sqrt{M(t)} \int_0^t \int \frac{|\nabla^3 \psi|^2}{N}.
\]

We replace \(D\) with \(\partial_z\) and \(\partial_y\) and add these two estimates to get

\[
(3.25) \quad \int_0^t \int N |\Delta \nabla^3 \psi|^2 + \epsilon \int_0^t \int |\nabla^2 \Delta \psi|^2 \\
\leq C(\|\varphi_0\|_{L^2}^2 + \|\nabla\psi_0\|_{L^1}^2 + \|\psi_0\|^2 + \|\nabla \Delta \psi_0\|^2) + C \sqrt{M(t)} \int_0^t \int \frac{|\nabla^3 \psi|^2}{N}.
\]

To get (3.23) from the above estimate, we have to estimate
\[
\int N |\nabla^3 \psi|^2 \leq 2 \int N |\Delta (\nabla \psi)|^2 + C \int N |\nabla (\nabla \psi)|^2.
\]

So we have
\[
\int_0^t \int N |\nabla^3 \psi|^2 \leq C \int_0^t \int N |\nabla \psi|^2 + C(\|\varphi_0\|_{L^2}^2 + \|\nabla\psi_0\|_{L^1}^2 + \|\psi_0\|^2)
\]
by (3.21). Together with (3.25), it proves (3.23).

- Proof of (3.24)

Multiplying \(w \nabla^3 \psi\) to the equation
\[
\nabla^3 \psi_t - s \nabla^3 \psi_z - \epsilon \nabla \nabla^3 \psi = \nabla^3 (\nabla \cdot \varphi) - 2\epsilon \nabla^3 (P \cdot \nabla \psi) - \epsilon \nabla^3 (|\nabla \psi|^2),
\]

we have
\[
\frac{1}{2} (w|\nabla^3 \psi|^2)_t - \frac{s}{2} (w|\nabla^3 \psi|^2)_z + \frac{s}{2} w'|\nabla^3 \psi|^2 \\
= w \nabla^3 (\nabla \cdot \varphi) \cdot \nabla^3 \psi + \epsilon w \nabla^3 \psi \cdot \left(\Delta \nabla^3 \psi - 2 \nabla^3 (P \cdot \nabla \psi) - \nabla^3 (|\nabla \psi|^2)\right),
\]

as before, we use the point \(z_0 \in \mathbb{R}\) satisfying
\[
\frac{w'(z)}{w(z)} \geq \frac{s}{2} \quad \text{for } z \geq z_0 \quad \text{and} \quad w(z) \leq \frac{4}{s^2} \leq \frac{16}{s^4} N \quad \text{for } z \leq z_0.
\]
by (2.5).
Integrating on each half strip and in time, we get

\[
\frac{1}{2} \int_{z>z_0} w|\nabla^2 \psi|^2 \leq \frac{1}{2} \int_{z>z_0} w|\nabla^3 \psi_0|^2 + \int_0^t \int_{z>z_0} w \nabla^3 (\nabla \cdot \varphi) \cdot \nabla^3 \psi - \frac{s^2}{4} \int_0^t \int_{z>z_0} w|\nabla^3 \psi|^2 \\
- \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^3 \psi|^2 (z_0, y) dy + \int_0^t \int_{z>z_0} \epsilon \text{-terms}
\]

\[
\leq \frac{1}{2} \int_{z>z_0} w|\nabla^3 \psi_0|^2 - \frac{s^2}{8} \int_0^t \int_{z>z_0} w|\nabla^3 \psi|^2 + C \int_0^t \int_{z>z_0} w|\nabla^3 (\nabla \cdot \varphi)|^2 \\
- \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^3 \psi|^2 (z_0, y) dy + \int_0^t \int_{z>z_0} \epsilon \text{-terms}
\]

and

\[
\frac{1}{2} \int_{z<z_0} w|\nabla^3 \psi|^2 \leq \frac{1}{2} \int_{z<z_0} w|\nabla^3 \psi_0|^2 + \int_0^t \int_{z<z_0} w \nabla^3 (\nabla \cdot \varphi) \cdot \nabla^3 \psi \\
+ \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^3 \psi|^2 (z_0, y) dy + \int_0^t \int_{z<z_0} \epsilon \text{-terms}
\]

\[
\leq \frac{1}{2} \int_{z<z_0} w|\nabla^3 \psi_0|^2 + C \int_0^t \int_{z<z_0} |\nabla^4 \varphi| |\nabla^3 \psi| \\
+ \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^3 \psi|^2 (z_0, y) dy + \int_0^t \int_{z<z_0} \epsilon \text{-terms}
\]

\[
\leq \frac{1}{2} \int_{z<z_0} w|\nabla^3 \psi_0|^2 + C \int_0^t \int_{z<z_0} N|\nabla^3 \psi|^2 + C \int_0^t \int_{z<z_0} \frac{|\nabla^4 \varphi|^2}{N} \\
+ \frac{s}{2} \int_0^t \int_0^\lambda w|\nabla^3 \psi|^2 (z_0, y) dy + \int_0^t \int_{z<z_0} \epsilon \text{-terms}.
\]

As before, we get

\[
\frac{1}{2} \int w|\nabla^3 \psi|^2 + \frac{s^2}{8} \int_0^t \int w|\nabla^3 \psi|^2 \leq \frac{1}{2} \int w|\nabla^3 \psi_0|^2 + C \int_0^t \int \left( N|\nabla^3 \psi|^2 + w|\nabla^4 \varphi|^2 + \epsilon \text{-terms} \right).
\]

For \( \epsilon \text{-terms} \), as before, we estimate

\[
\int \epsilon \text{-terms} = \epsilon \int w \nabla^3 \psi \cdot (\Delta \nabla^3 \psi - 2 \nabla^3 (P \cdot \nabla \psi) - \nabla^3 (|\nabla \psi|^2))
\]

\[
= \epsilon \int \left( -w|\nabla^4 \psi|^2 - w' \nabla^3 \psi \cdot \nabla \psi_2 - w \nabla^3 \psi \cdot (2 \nabla^3 (P \cdot \nabla \psi)) + (w' \nabla^2 \psi_2 + w \Delta \nabla^2 \psi) \nabla^2 (|\nabla \psi|^2) \right)
\]

\[
\leq -\epsilon \int w|\nabla^4 \psi|^2 + C \epsilon \int w \psi \cdot \nabla \psi_2 + C \epsilon \int \left[ w|\nabla^3 \psi||\nabla^4 \psi| + w|\nabla^3 \psi|(|\nabla^4 \psi| + |\nabla^3 \psi| + |\nabla^2 \psi|) \\
+ w|\nabla^3 \psi| (|\nabla^3 \psi||\nabla \psi| + |\nabla^2 \psi||\nabla^2 \psi|) + w|\nabla^4 \psi| (|\nabla^3 \psi||\nabla \psi| + |\nabla^2 \psi||\nabla^2 \psi|) \right]
\]

\[
\leq -\frac{\epsilon}{4} \int w|\nabla^4 \psi|^2 + C \epsilon \int w \sum_{k=1}^3 |\nabla^k \psi|^2 + C \epsilon \int \left( w|\nabla^3 \psi||\nabla^2 \psi|^2 + w|\nabla^4 \psi||\nabla^2 \psi|^2 \right).
\]
For the last term (the cubic term), we estimate

\[ C\epsilon \int (w|\nabla^3\psi||\nabla^2\psi|^2 + w|\nabla^4\psi||\nabla^2\psi|^2) \leq C\epsilon \left( \|\sqrt{w}\nabla^3\psi\| + \|\sqrt{w}\nabla^4\psi\| \right) \|\sqrt{w}\nabla^2\psi\|_{L^4} \|\nabla^2\psi\|_{L^4} \]

\[ \leq C\epsilon \left( \|\sqrt{w}\nabla^3\psi\| + \|\sqrt{w}\nabla^4\psi\| \right) \left( \|\sqrt{w}\nabla^3\psi\| + (\sqrt{w})^\prime \|\nabla^2\psi\| + \|\sqrt{w}\nabla^2\psi\| \right) \left( \|\nabla^3\psi\| + \|\nabla^2\psi\| \right) \]

\[ \leq C\epsilon \left( \|\sqrt{w}\nabla^3\psi\| + \|\sqrt{w}\nabla^4\psi\| \right) \cdot \sqrt{M(t)} \cdot (\|\nabla^3\psi\| + \|\nabla^2\psi\|) \]

\[ \leq \frac{C}{8} \int w|\nabla^4\psi|^2 + C\epsilon \int w \sum_{k=2}^{3} |\nabla^k\psi|^2 \]

for small \( \delta_0 \).

In sum, using the estimate \((3.21)\) and the previous claim \((3.23)\), we obtain

\[ \frac{1}{2} \int w|\nabla^3\psi|^2 + (\frac{\epsilon^2}{8} - C(\epsilon_0 + \sqrt{\delta_0})) \int_0^t \int w|\nabla^3\psi|^2 + C\epsilon \int w|\nabla^4\psi|^2 \]

\[ \leq C(\|\nabla\psi_0\|^2_{2,w} + \|\psi_0\|^2 + \|\varphi_0\|^2_{2,w}) + C \int_0^t \int \frac{|\nabla^4\varphi|^2}{N}. \]

By making \( \epsilon_0 \) and \( \delta_0 \) small enough, we proved the claim \((3.24)\).

Now we can prove Lemma \(3.9\) fully. Indeed plugging \((3.24)\) into \((3.22)\) with \( \delta_0 \) small, we have

\[ \|\nabla^3\varphi\|^2_{2,w} + \|\nabla^3\psi\|^2 + \int_0^t \|\nabla^4\varphi\|^2_{2,w} + \epsilon \int_0^t \|\nabla^4\psi\|^2 \]

\[ \leq C(\|\nabla\psi_0\|^2_{2,w} + \|\psi_0\|^2 + \|\varphi_0\|^2_{3,w}). \]

So from \((3.24)\), we have

\[ \int \frac{|\nabla^3\psi|^2}{N} + \int_0^t \int \frac{|\nabla^3\psi|^2}{N} + \epsilon \int_0^t \int \frac{|\nabla^4\psi|^2}{N} \leq C(\|\varphi_0\|^2_{3,w} + \|\nabla\psi_0\|^2_{2,w} + \|\psi_0\|^2). \]

This proves Lemma \(3.9\) for the case \( k = 3 \).

Finally, we obtain Proposition \(2.13\). Indeed, by adding Lemma \(3.8\) and Lemma \(3.9\) for \( k = 2, 3 \), we have

\[ M(T) + \int_0^T \sum_{l=1}^{4} \|\nabla^l\varphi\|^2_{2,w} + \int_0^T \sum_{l=1}^{3} \|\nabla^l\psi\|^2_{2,w} + \epsilon \int_0^T \|\nabla^4\psi\|^2_{2,w} \leq CM(0). \]

**ACKNOWLEDGEMENT**

The work of MC was supported by NRF-2018R1A1A3A04079376. The work of KC was supported by NRF-2018R1D1A1B07043065 and by the POSCO Science Fellowship of POSCO TJ Park Foundation.

**REFERENCES**

[1] R. A. Anderson and M. A. J. Chaplain, *Modelling the growth and form of capillary networks in On Growth and Form: Spatio-Temporal Pattern formation in Biology*, Wiley, 225–249, 1999.

[2] P. K. Brazhnik and J. J. Tyson, *On traveling wave solutions of fisher’s equation in two spatial dimensions*, SIAM J. Appl. Math., 60, 371–391, 2000.

[3] M. Chae, K. Choi, K. Kang and J. Lee, *Stability of planar traveling waves in a Keller-Segel equation on an infinite strip domain*, J. Differential Equations, 265, 237–279, 2018.
[4] L. Corrias, B. Perthame and H. Zaag, *A chemotaxis model motivated by angiogenesis*, C. R. Math. Acad. Sci. Paris, 336(2), 141–146, 2003.
[5] L. Corrias, B. Perthame and H. Zaag, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math., 72(1), 1–28, 2004.
[6] M. A. Fontelos, A. Friedman and B. Hu, *Mathematical analysis of a model for the initiation of angiogenesis*, SIAM. J. Math. Anal., 33, 1330–1355, 2002.
[7] A. Friedman and J. I. Tello, *Stability of solutions of chemotaxis equations in reinforced random walks*, J. Math. Anal. Appl., 272, 136–163, 2002.
[8] J. Goodman, *Remarks on the stability of viscous shock waves*, in: Viscous profiles and numerical methods for shock waves (Raleigh, NC, 1990), 66-72, SIAM, Philadelphia, PA, 1991.
[9] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I*, Jahresber. Deutsch. Math.-Verein., 105, 103-165, 2003.
[10] Q. Hou, C. Liu, Y. Wang and Z. Wang, *Stability of Boundary Layers for a Viscous Hyperbolic System Arising from Chemotaxis: One-Dimensional Case*, SIAM J. Math. Anal., 50, 3058–3091, 2018.
[11] H-Y. Jin, J. Li and Z. A. Wang, *Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity*, J. Differential equations, 225, 193–210, 2013.
[12] S. Kawashima and A. Matsumura, *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Comm. Pure Appl. Math., 47, 1547-1569, 1994.
[13] E. F. Keller and L. A. Segel, *Traveling bands of chemotactic bacteria: A Theoretical Analysis*, J. theor. Biol. 30, 235-248, 1971.
[14] H. A. Levine, B. D. Sleeman and M. Nilson-Hamilton, *A mathematical model for the roles of pericytes and macrophages in the onset of angiogenesis: I. the role of protease inhibitors in preventing angiogenesis*, Math. Biosci., 168, 77–115, 2000.
[15] D. Li and T. Li, *On a hyperbolic-parabolic system modeling chemotaxis*, Math. Models Methods Appl. Sci. 21, 1631–1650, 2011.
[16] J. Li, T. Li and Z. A. Wang, *Stability of traveling waves of the Keller-Segel system with logarithmic sensitivity*, Math. Models and Methods in Appl. Sci., 24, 2819–2849, 2014.
[17] T. Li and Z. A. Wang, *Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis*, J. Differential equations, Vol. 250, 1310–1333, 2011.
[18] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, Berlin, 1994.
[19] V. A. Solonnikov, *On the solvability of boundary and initial boundary value problems for the Navier-Stokes system in domains with noncompact boundaries*, Pacific J. Math., 93, 443–458, 1981.
[20] A. Szepessy and Z.P. Xin, *Nonlinear stability of viscous shock waves*, Arch. Ration. Mech. Anal. 122, 53-103, 1993.
Department of Mathematics, Hankyong University, Anseong-si, Gyeonggi-do, Republic of Korea
E-mail address: mchae@hknu.ac.kr

Department of Mathematical Sciences, Ulsan National Institute of Science and Technology, Ulsan, Republic of Korea
E-mail address: kchoi@unist.ac.kr