Generalized Lotka–Volterra systems connected with simple Lie algebras

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Abstract. We devise a new method for producing Hamiltonian systems by constructing the corresponding Lax pairs. This is achieved by considering a larger subset of the positive roots than the simple roots of the root system of a simple Lie algebra. We classify all subsets of the positive roots of the root system of type $A_n$ for which the corresponding Hamiltonian systems are transformed, via a simple change of variables, to Lotka–Volterra systems. For some special cases of subsets of the positive roots of the root system of type $A_n$, we produce new integrable Hamiltonian systems.

1. Introduction
The Volterra model, also known as the KM system is a well-known integrable system defined by

\[ \dot{x}_i = x_i (x_{i+1} - x_{i-1}), \quad i = 1, 2, \ldots, n, \]

where $x_0 = x_{n+1} = 0$. It was studied by Lotka in [1] to model oscillating chemical reactions and by Volterra in [2] to describe population evolution in a hierarchical system of competing species. It was first solved by Kac and van Moerbeke in [3], using a discrete version of inverse scattering due to Flaschka [4]. In [5] Moser gave a solution of the system using the method of continued fractions and in the process he constructed action-angle coordinates. Equations (1) can be considered as a finite-dimensional approximation of the Korteweg–de Vries (KdV) equation. The Poisson bracket for this system can be thought as a lattice generalization of the Virasoro algebra [6]. The Volterra system is associated with a simple Lie algebra of type $A_n$. Bogoyavlensky generalized this system for each simple Lie algebra and showed that the corresponding systems are also integrable. See [7, 8] for more details. The generalization in this paper is different from the one of Bogoyavlensky.

The KM-system given by equation (1) is Hamiltonian (see [9, 10]) and can be written in Lax pair form in various ways. We will focus on the symmetric version due to Moser, where the
matrices $L$ and $B$ have the form

$$L = \begin{pmatrix}
0 & a_1 & 0 & \cdots & \cdots & 0 \\
& a_1 & 0 & a_2 & \ddots & \vdots \\
& & a_2 & 0 & \ddots & \vdots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & a_n \\
0 & \cdots & \cdots & 0 & a_n & 0
\end{pmatrix}, \quad (2)$$

and

$$B = \begin{pmatrix}
0 & 0 & a_1a_2 & \cdots & \cdots & 0 \\
0 & 0 & 0 & \ddots & \vdots & \vdots \\
-\alpha_1\alpha_2 & 0 & 0 & a_2a_3 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & -a_{n-1}a_n & 0 & 0
\end{pmatrix}.$$ 

In terms of the simple root vectors $X_{\alpha_i}$, $i = 1, 2, \ldots, n$, the matrices $L$ and $B$ are written as $L = \sum_{i=1}^n a_i (X_{\alpha_i} + X_{-\alpha_i})$ and $B = \sum_{i=1}^{n-1} a_i a_{i+1} (X_{\alpha_i + \alpha_{i+1}} - X_{-\alpha_i - \alpha_{i+1}})$. The matrix equation $\dot{L} = [B, L]$ gives a polynomial (in fact cubic) system of differential equations. The change of variables $x_i = 2a_i^2$ gives equations (1). The purpose of this paper is to show that this Lax pair can be generalized and produce a larger class of Hamiltonian systems which we call generalized Volterra systems.

It is evident from the form of $L$ in the Lax pair, that the position of the variables $a_i$ corresponds to the simple root vectors of a root system of type $A_n$. On the other hand a non-zero entry of the matrix $B$ occurs at a position corresponding to the sum of two simple roots $\alpha_i$ and $\alpha_j$. In this paper we generalize the Lax pair of Moser (2) as follows.

Instead of considering the set of simple roots $\Pi$, we begin with a subset $\Phi$ of the positive roots $\Delta^+$ which contains $\Pi$, i.e. $\Pi \subset \Phi \subset \Delta^+$. For each such choice of a set $\Phi$ we produce a Lax pair and thus a new Hamiltonian system. We call these cubic systems generalized Volterra systems since in some cases by a simple change of variables we produce Lotka–Volterra systems.

In this paper we restrict our attention to some examples in the $A_n$ case. However, this algorithm applies, more generally for each complex simple Lie algebra. In dimension 3 this procedure produces only two systems, the KM system and the periodic KM system. In dimensions 4 and 5 (i.e. the cases of $A_3$ and $A_4$) and by allowing the use of complex coefficients this method works in all possible cases and in fact we have verified using Maple that all the resulting systems are Liouville integrable. After the definition of Lotka–Volterra systems in Section 2, we describe our algorithm in Section 3. In Section 4 we give a classification of all cases which give rise to Lotka–Volterra systems via the transformation $a_i \rightarrow 2a_i^2$. Finally in Section 5 we present some examples where complex coefficients are used to produce new systems.

2. Lotka–Volterra systems

The KM-system belongs to a large class of the so called Lotka–Volterra systems. The most general form of the Lotka–Volterra equations is

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad i = 1, 2, \ldots, n.$$
We may assume that there are no linear terms ($\varepsilon_i = 0$). We also assume that the matrix $A = (a_{ij})$ is skew-symmetric. All these systems can be written in Hamiltonian form using the Hamiltonian function

$$H = x_1 + x_2 + \cdots + x_n.$$ 

Hamilton’s equations take the form

$$\dot{x}_i = \{x_i, H\} = \sum_{j=1}^{n} \pi_{ij} x_j, \quad i, j = 1, 2, \ldots, n. \quad (3)$$

From the skew symmetry of the matrix $A = (a_{ij})$ it follows that the Jacobi identity is satisfied.

The Poisson tensor (3) is Poisson isomorphic to the constant Poisson structure defined by the constant matrix $A$, see [11]. If $k = (k_1, k_2, \ldots, k_n)$ is a vector in the kernel of $A$ then the function

$$f = x_{k_1}^1 x_{k_2}^2 \cdots x_{k_n}^n$$

is a Casimir. Indeed for an arbitrary function $g$ the Poisson bracket $\{f, g\}$ is

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} a_{ij} k_i \right) x_j f \frac{\partial g}{\partial x_j} = 0.$$ 

If the matrix $A$ has rank $r$, then there are $n - r$ functionally independent Casimirs. This type of integral can be traced back to Volterra [2]; see also [11–13].

3. The procedure

We recall the following procedure from [14]. Let $\mathfrak{g}$ be any simple Lie algebra equipped with its Killing form $\langle \cdot | \cdot \rangle$. One chooses a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a basis $\Pi$ of simple roots for the root system $\Delta$ of $\mathfrak{h}$ in $\mathfrak{g}$. The corresponding set of positive roots is denoted by $\Delta^+$. To each positive root $\alpha$ one can associate a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of vectors in $\mathfrak{g}$ which generate a Lie subalgebra isomorphic to $sl_2(\mathbb{C})$. The set $(X_\alpha, X_{-\alpha})_{\alpha \in \Delta^+} \cup (H_\alpha)_{\alpha \in \Pi}$ is a basis of $\mathfrak{g}$, called a root basis. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ and let $X_{\alpha_1}, \ldots, X_{\alpha_\ell}$ be the corresponding root vectors in $\mathfrak{g}$. Define

$$L = \sum_{\alpha_i \in \Pi} a_i (X_{\alpha_i} + X_{-\alpha_i}).$$

To find the matrix $B$ we use the following procedure. For each $i, j$ form the vectors $[X_{\alpha_i}, X_{\alpha_j}]$. If $\alpha_i + \alpha_j$ is a root then include a term of the form $a_i a_j [X_{\alpha_i}, X_{\alpha_j}]$ in $B$. We make $B$ skew-symmetric by including the corresponding negative root vectors $a_i a_j [X_{-\alpha_i}, X_{-\alpha_j}]$. Finally, we define the system using the Lax pair equation

$$\dot{L} = [B, L].$$

For a root system of type $A_n$ we obtain the KM system.

In this paper we generalize this algorithm as follows. Consider a subset $\Phi$ of $\Delta^+$ such that

$$\Pi \subset \Phi \subset \Delta^+.$$ 

The Lax matrix is easy to construct

$$L = \sum_{\alpha_i \in \Phi} a_i (X_{\alpha_i} + X_{-\alpha_i}).$$

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Here we use the following enumeration of $\Phi$ which we assume to have $m$ elements. The variables $a_j$ correspond to the simple roots $\alpha_j$ for $j = 1, 2, \ldots, \ell$. We assign the variables $a_j$ for $j = \ell + 1, \ell + 2, \ldots, m$ to the remaining roots in $\Phi$. To construct the matrix $B$ we use the following algorithm. Consider the set $\Phi \cup \Phi^-$ which consists of all the roots in $\Phi$ together with their negatives. Let

$$\Psi = \{\alpha + \beta | \alpha, \beta \in \Phi \cup \Phi^-, \alpha + \beta \in \Delta^+\}.$$ 

Define

$$B = \sum_{i,j} c_{ij}a_ia_j(X_{\alpha_i+\alpha_j} - X_{-\alpha_i-\alpha_j}), \quad (4)$$

where $c_{ij} = \pm 1$ if $\alpha_i + \alpha_j \in \Psi$ with $\alpha_i, \alpha_j \in \Phi \cup \Phi^-$ and 0 otherwise. In all eight in $A_3$ cases we are able to make the proper choices of the sign of the $c_{ij}$ so that we can produce a Lax pair. This method produces a Lax pair in all but five out of sixty four cases in $A_4$. However, when we allow the $c_{ij}$ to take the complex values $\pm i$ we are able to produce a Lax pair in all sixty four cases. By using Maple we were able to check that all these examples in $A_3$ and $A_4$ are in fact Liouville integrable. We will not attempt to prove the integrability of these systems in general due to the complexity of their definition.

We have to point out that in [15] there is a similar construction for the case of the Toda lattice where Hamiltonian systems are defined which interpolate between the classical Kostant–Toda lattice and the full Kostant–Toda lattice. In that case there is a simple criterion on the set $\Phi$ which ensures the construction of the Lax pair. In our case there is no such simple criterion. In the next proposition we present a sufficient (but not necessary) condition on the subset $\Phi$ which gives a consistent Lax pair.

**Proposition 1.** Let $\Pi \subset \Phi \subset \Delta^+$ be a subset of the positive roots with the property that whenever $\alpha, \beta, \gamma \in \Phi \cup \Phi^-$ then $\alpha + \beta + \gamma \neq 0$ and if $\alpha + \beta + \gamma \in \Delta^+$ then $\alpha + \beta + \gamma \in \Phi$. Also let $B$ be the matrix constructed using the algorithm described in (4). Then for any choice of the signs $c_{i,j}$ the pair $L, B$ is a Lax pair.

**Proof.** Let $K$ be the following subset of the positive roots

$$K = \{\alpha + \beta + \gamma : \alpha, \beta, \gamma \in \Phi \cup \Phi^-, \alpha + \beta + \gamma \in \Delta^+\}.$$ 

It easily follows from the construction of the matrix $B$ that for all possible choices of the signs $c_{i,j}$, the nonzero entries of the bracket $[B, L]$ appear in the positions corresponding to the root vectors $X_\alpha$, $\alpha \in K$. The condition $\alpha, \beta, \gamma \in \Phi \cup \Phi^- \Rightarrow \alpha + \beta + \gamma \neq 0$ implies that there are no variables in the diagonal of $[B, L]$ while the condition $\alpha, \beta, \gamma \in \Phi \cup \Phi^-$ and $\alpha + \beta + \gamma \in \Delta^+ \Rightarrow \alpha + \beta + \gamma \in \Phi$ implies that $K \subset \Phi$. Since we also have $\Phi \subset K$ we deduce that $\Phi = K$ and therefore the pair $L, B$ is a Lax pair.

This condition is of course not necessary. For example the KM and the periodic KM systems do not fall in this class. In Theorem 1 we find several other families of subsets $\Phi$ which give consistent Lax pairs.

**Example 1.** Let $\Phi$ be the subset of the positive roots of the root system $A_n$ containing all the roots of odd height. We immediately see that if $K$ is the subset of the positive roots described in Proposition 1 then $K = \Phi$ and therefore for all possible choices of the signs $c_{i,j}$ we have a consistent Lax pair. Take for example the case $n = 3$. In that case the subset $\Phi$ is given by

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_1 + \alpha_2 + \alpha_3\}$$

and this choice gives rise to the Lax matrix

$$L = \sum_{i=1}^4 a_i (X_{\alpha_i} + X_{-\alpha_i}).$$
The skew symmetric matrix $B$ constructed using (4) is
\[ B = (c_{1,2}a_1a_2 + c_{3,4}a_3a_4) (X_{\alpha_1 + \alpha_2} - X_{-\alpha_1 - \alpha_2}) + (c_{1,4}a_1a_4 + c_{2,3}a_2a_3) (X_{\alpha_2 + \alpha_3} - X_{-\alpha_2 - \alpha_3}) \].

Now we easily verify that all 16 possible choices of the signs $c_{i,j}$ give a consistent Lax pair. Of course only half of them give possibly non-isomorphic systems and only one of them gives a Lotka–Volterra system (see Theorem 1), the well known periodic KM system.

**Example 2** ($A_3$ root system). Let $E$ be the hyperplane of $\mathbb{R}^4$ for which the coordinates sum to 0 (i.e. vectors orthogonal to $(1, 1, 1, 1)$). Let $\Delta$ be the set of vectors in $E$ of length $\sqrt{2}$ with integer coordinates. There are 12 such vectors in all. We use the standard inner product in $\mathbb{R}^4$ and the standard orthonormal basis $\{e_1, e_2, e_3, e_4\}$. Then, it is easy to see that $\Delta = \{\epsilon_i - \epsilon_j \mid i \neq j\}$. The vectors
\[ \alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \quad \alpha_3 = \epsilon_3 - \epsilon_4 \]
form a basis of the root system in the sense that each vector in $\Delta$ is a linear combination of these three vectors with integer coefficients, either all nonnegative or all nonpositive. For example, $\epsilon_1 - \epsilon_3 = \alpha_1 + \alpha_2$, $\epsilon_2 - \epsilon_4 = \alpha_2 + \alpha_3$ and $\epsilon_1 - \epsilon_4 = \alpha_1 + \alpha_2 + \alpha_3$. Therefore $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ and the set of positive roots $\Delta^+$ is given by
\[ \Delta^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \].

If we take $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$, then
\[ \Phi \cup \Phi^{-} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2, -\alpha_3, -\alpha_1 - \alpha_2\} \]
and $\Psi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. In this example the variables $a_i$ for $i = 1, 2, 3$ correspond to the three simple roots $\alpha_1$, $\alpha_2$, $\alpha_3$. We associate the variable $a_4$ to the root $\alpha_4 = \alpha_1 + \alpha_2$. We obtain the following Lax pair
\[ L = \sum_{i=1}^{4} a_i (X_{\alpha_i} + X_{-\alpha_i}) \],
\[ B = -a_4a_2 (X_{\alpha_1} + X_{-\alpha_1}) - a_1a_4 (X_{\alpha_2} + X_{-\alpha_2}) + a_1a_2 (X_{\alpha_4} + X_{-\alpha_4}) + a_2a_3 (X_{\alpha_2 + \alpha_3} + X_{-\alpha_2 - \alpha_3}) - a_3a_4 (X_{\alpha_1 + \alpha_2 + \alpha_3} + X_{\alpha_1 - \alpha_2 - \alpha_3}) \].

The Lax pair (using the substitution $x_i = a_i^2$ followed by scaling) becomes equivalent to the following equations of motion
\[ \dot{x}_1 = x_1 x_2 - x_1 x_4, \]
\[ \dot{x}_2 = -x_1 x_2 + x_2 x_3 + x_2 x_4, \]
\[ \dot{x}_3 = -x_2 x_3 + x_3 x_4, \]
\[ \dot{x}_4 = x_1 x_4 - x_2 x_4 - x_3 x_4. \]

The system is integrable. There exist two functionally independent Casimir functions $F_1 = x_1 x_3 = \det L$ and $F_2 = x_1 x_2 x_4$. The additional integral is the Hamiltonian $H = x_1 + x_2 + x_3 + x_4 = \text{tr} L^2$.

The standard quadratic Poisson bracket (3) is given by
\[ \pi = \begin{pmatrix} 0 & x_1 x_2 & 0 & -x_1 x_4 \\ -x_2 x_1 & 0 & x_2 x_3 & x_2 x_4 \\ 0 & -x_3 x_2 & 0 & x_3 x_4 \\ x_4 x_1 & -x_4 x_2 & -x_4 x_3 & 0 \end{pmatrix}. \]
One can find the Casimirs by computing the kernel of the matrix

\[ A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}. \]

The two eigenvectors with eigenvalue 0 are \((1, 0, 1, 0)\) and \((1, 1, 0, 1)\). We obtain the two Casimirs

\[ F_1 = x_1^2 x_2 x_3 x_4^2 = x_1 x_3 \quad \text{and} \quad F_2 = x_1^4 x_2 x_3^2 x_4 = x_1 x_2 x_4. \]

**Example 3.** A Lax pair \(L, B\) corresponding to the subset \(\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_1 + \alpha_2 + \alpha_3\}\) of the root system of type \(A_3\) is

\[ L = \sum_{i=1}^{4} a_i (X_{\alpha_i} + X_{-\alpha_i}), \]
\[ B = (a_1 a_2 - a_3 a_4) (X_{\alpha_1 + \alpha_2} + X_{-\alpha_1 - \alpha_2}) + (a_2 a_3 - a_1 a_4) (X_{\alpha_2 + \alpha_3} + X_{-\alpha_2 - \alpha_3}). \]

Using the substitution \(x_i = 2a_i^2\) we obtain the periodic KM-system

\[ \begin{aligned} x_1' &= x_1 x_2 - x_1 x_4, \\ x_2' &= -x_2 x_1 + x_2 x_3, \\ x_3' &= x_3 x_4 - x_3 x_2, \\ x_4' &= x_4 x_1 - x_4 x_3. \end{aligned} \]

(5)

The Poisson matrix (which can be read from the right hand side of (5)) is

\[ \pi = \begin{pmatrix} 0 & x_1 x_2 & 0 & -x_1 x_4 \\ -x_1 x_2 & 0 & x_2 x_3 & 0 \\ 0 & -x_2 x_3 & 0 & x_3 x_4 \\ x_1 x_4 & 0 & -x_3 x_4 & 0 \end{pmatrix} \]

of rank 2. In addition to the Hamiltonian

\[ H = x_1 + x_2 + x_3 + x_4 \]

it possesses two Casimirs \(C_1 = x_1 x_3\) and \(C_2 = x_2 x_4\).

**Example 4.** Consider the subset

\[ \Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 = \alpha_2 + \alpha_3\} \]

of the positive roots of the root system of type \(A_4\). The Lax pair corresponding to the subset \(\Phi\) is given by

\[ L = \sum_{i=1}^{5} a_i (X_{\alpha_i} + X_{-\alpha_i}) \]

and

\[ \begin{aligned} B &= -a_2 a_5 (X_{\alpha_3} - X_{-\alpha_3}) - a_3 a_5 (X_{\alpha_2} - X_{-\alpha_2}) a_1 a_2 (X_{\alpha_1 + \alpha_2} - X_{-\alpha_1 - \alpha_2}) \nonumber \\ &\quad + a_2 a_3 (X_{\alpha_5} - X_{-\alpha_5}) + a_3 a_4 (X_{\alpha_3 + \alpha_4} - X_{-\alpha_3 - \alpha_4}) \nonumber \\ &\quad - a_3 a_5 (X_{\alpha_1 + \alpha_2 + \alpha_3} - X_{-\alpha_1 - \alpha_2 - \alpha_3}) - a_4 a_5 (X_{\alpha_2 + \alpha_3 + \alpha_4} - X_{-\alpha_2 - \alpha_3 - \alpha_4}). \nonumber \end{aligned} \]
Using the change of variables \( x_i = 2a_i^2 \) the corresponding Lotka–Volterra system becomes

\[
\begin{align*}
\dot{x}_1 &= x_1x_2 - x_1x_5, \\
\dot{x}_2 &= -x_2x_5 + x_2x_3 - x_2x_1, \\
\dot{x}_3 &= x_3x_5 + x_3x_4 - x_3x_2, \\
\dot{x}_4 &= x_4x_5 - x_4x_3, \\
\dot{x}_5 &= -x_5x_4 - x_5x_3 + x_5x_1 + x_5x_2.
\end{align*}
\]

The associated Poisson matrix is of rank 4.

The constants of motion are

\[
\begin{align*}
H &= x_1 + x_2 + x_3 + x_4 + x_5 \text{ (Hamiltonian)}, \\
F &= x_1x_3 + x_1x_4 + x_2x_4, \\
C &= x_2x_3x_5 \text{ (Casimir)}.
\end{align*}
\]

4. Classification of Lotka–Volterra systems

In the previous section we presented several examples of cubic systems which (after a simple change of variables) are equivalent to Lotka–Volterra systems. In this section we classify all subsets \( \Phi \) of the positive roots of \( A_n \) which produce, after a suitable change of variables, Lotka–Volterra systems. We also explicitly describe the corresponding Lotka–Volterra systems. We have the following Theorem.

**Theorem 1.** Consider the root system of type \( A_n \) with simple roots \( \Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) and positive roots

\[
\Delta^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j : 1 \leq i \leq j \leq n\}.
\]

The only choices for the subset \( \Phi \) of \( \Delta^+ \) so that the corresponding generalized Volterra system transforms into a Lotka–Volterra system, using the substitution \( x_i = 2a_i^2 \), are the following five:

1. \( \Phi = \Pi \).
2. \( \Phi = \Pi \cup \{\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}\} \).
3. \( \Phi = \Pi \cup \{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\} \).
4. \( \Phi = \Pi \cup \{\alpha_2 + \alpha_3 + \cdots + \alpha_n\} \).
5. \( \Phi = \Pi \cup \{\alpha_1 + \alpha_2 + \cdots + \alpha_n\} \).

Case (1) gives rise to the KM system while case (5) gives rise to the periodic KM system. Case (2) corresponds to the Lax equation \( L = [B, L] \) with

\[
L = \sum_{i=1}^{n} a_i \left( X_{\alpha_i} + X_{-\alpha_i} \right) + a_{n+1} \left( X_{\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}} + X_{-\alpha_2 - \alpha_3 - \cdots - \alpha_{n-1}} \right).
\]

The skew symmetric matrix \( B \) is defined using the method described in Section 3 (see also Proposition 2).

After substituting \( x_i = 2a_i^2 \) for \( i = 1, \ldots, n + 1 \), the Lax pair \( L, B \) becomes equivalent to the following equations of motion

\[
\begin{align*}
\dot{x}_1 &= x_1(x_2 - x_{n+1}), \\
\dot{x}_2 &= x_2(x_3 - x_1 - x_{n+1}), \\
\dot{x}_i &= x_i(x_{i+1} - x_{i-1}), \quad i = 3, 4, \ldots, n - 2, n, \\
\dot{x}_{n-1} &= x_{n-1}(x_n - x_{n-2} + x_{n+1}), \\
\dot{x}_{n+1} &= x_{n+1}(x_1 + x_2 - x_{n-1} - x_n).
\end{align*}
\]
It is easily verified that for \( n \) even, the rank of the Poisson matrix is \( n \) and the function \( f = x_2 x_3 \cdots x_{n-1} x_{n+1} \) is the Casimir of the system, while for \( n \) odd, the rank of the Poisson matrix is \( n - 1 \) and the functions \( f_1 = x_1 x_3 \cdots x_n = \sqrt{\det L} \) and \( f_2 = x_2 x_3 \cdots x_{n-1} x_{n+1} \) are the Casimirs.

Case (3) corresponds to the Lax pair whose Lax matrix \( L \) is given by

\[
L = \sum_{i=1}^{n} a_i (X_{\alpha_i} + X_{-\alpha_i}) + a_{n+1} (X_{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}} + X_{-\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1}}).
\]

After substituting \( x_i = 2a_i^2 \) for \( i = 1, \ldots, n+1 \), we obtain the following equivalent equations of motion

\[
\begin{align*}
\dot{x}_1 &= x_1(x_2 - x_{n+1}), \\
\dot{x}_i &= x_i(x_{i+1} - x_{i-1}), \quad i = 2, 3, \ldots, n - 2, n, \\
\dot{x}_{n-1} &= x_{n-1}(x_n - x_{n-2} + x_{n+1}), \\
\dot{x}_{n+1} &= x_{n+1}(x_1 - x_n - x_{n-1}).
\end{align*}
\]

For \( n \) even, the rank of the Poisson matrix is \( n \) and the function \( f = x_1 x_2 \cdots x_{n-1} x_{n+1} \) is the Casimir, while for \( n \) odd, the rank of the Poisson matrix is \( n - 1 \) and the functions \( f_1 = x_1 x_3 x_5 \cdots x_n = \sqrt{\det L} \) and \( f_2 = x_2 x_3 \cdots x_{n-1} x_{n+1} \) are Casimirs.

The system obtained in case (4) turns out to be isomorphic to the one in case (3). In fact, the change of variables \( u_{n+1-i} = -x_i \) for \( i = 1, 2, \ldots, n \) and \( u_{n+1} = -x_{n+1} \) in case (3) gives the corresponding system of case (4).

Since subsystems of Lotka–Volterra systems are also Lotka–Volterra, in order to prove Theorem 1 it is enough to consider the case where the subset \( \Phi \) contains the simple roots and only one extra root. The following proposition shows that we have only four possible choices for the extra root in \( \Phi \) which give rise to a Lotka–Volterra system. Therefore the proof of Theorem 1 is a case by case verification of the 16 possible subsets \( \Phi \) containing the simple roots and roots given in the following proposition.

**Proposition 2.** Let \( \Phi = \{\alpha_1, \ldots, \alpha_{n+1}\} \) be the subset of the positive roots of the root system \( A_n \) containing the simple roots and the additional extra root \( \alpha_{n+1} \). Suppose that \( \alpha_{n+1} = \alpha_k + \alpha_{k+1} + \cdots + \alpha_m \) for some \( 1 \leq k < m \leq n \). Then the corresponding generalized Volterra system is transformed, with the substitution \( x_i = 2a_i^2 \), to a Lotka–Volterra system if and only if the integers \( k, m \) satisfy

\[
k \leq 2 \quad \text{and} \quad m \geq n - 1.
\]

We outline the proof of this proposition. We explicitly write down the bracket \([B, L]\) as a linear combination of the root vectors \( \{X_\alpha : \alpha \in \Delta^+\} \). Note that there are some signs, \( c_{i,j} \), involved in the construction of \( B \) which we have to determine so that the corresponding system is transformed to a Lotka–Volterra system. In order to have a Lax pair we set the coefficients of the root vectors \( \{X_\alpha : \alpha \notin \Phi\} \) equal to zero and in order to have a Lotka–Volterra system we set the coefficients of the form \( c_{i,j} a_i a_j a_k \) for \( i \neq j \neq k \neq i \) equal to zero. We end up with a linear system of the coefficients \( c_{i,j} \) which in order to have a solution the integers \( k, m \) must be of the required form. For a detailed proof see [16].

**5. Using complex coefficients**

As we noted in the introduction, we may produce more Lotka–Volterra systems by changing the matrix \( L \) from symmetric to Hermitian. The aim of this section is to describe this idea of using complex coefficients and give some examples of new Lotka–Volterra systems produced by this new method. Let us begin with an example.
Example 5. Consider the case of a root system of type \( A_4 \) and the subset of positive roots \( \Phi = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 = \alpha_1 + \alpha_2 \} \). It turns out that if

\[
L = \sum_{i=1}^{4} a_i (X_{\alpha_i} + X_{-\alpha_i}) + a_5 (X_{\alpha_5} + X_{-\alpha_5}),
\]

the corresponding linear system of signs (described in the comments after Proposition 2) does not have a solution while if

\[
L = \sum_{i=1}^{4} a_i (X_{\alpha_i} + X_{-\alpha_i}) + ia_5 (X_{\alpha_5} - X_{-\alpha_5}),
\]

the corresponding system of signs does have a solution and gives rise to the system

\[
\begin{align*}
\dot{a}_1 &= a_1 a_2^2 + a_1 a_5^2, \\
\dot{a}_2 &= a_2 a_3^2 - a_2 a_4^2, \\
\dot{a}_3 &= a_3 a_4^2 - a_3 a_5^2, \\
\dot{a}_4 &= -a_4 a_3^2, \\
\dot{a}_5 &= a_5 a_3 a_5^2 - a_1 a_5^2.
\end{align*}
\] (6)

This system can be easily transformed to a Lotka–Volterra system which is integrable with one rational Casimir, \( a_{1/5} \), and an extra constant of motion, \( \text{tr}(L^4) \).

In general, the idea is to make the matrix \( L \) Hermitian by replacing some terms of the form \( a_i (X_{\alpha_i} + X_{-\alpha_i}) \) with \( i a_i (X_{\alpha_i} - X_{-\alpha_i}) \) where \( i \) is the imaginary unit. The construction of the matrix \( B \) is the same as in Section 3 with the only difference of \( B \) being skew Hermitian. Example 5 suggest that we may replace with \( i a_i (X_{\alpha_i} - X_{-\alpha_i}) \) all variables in \( L \) corresponding to the roots of height 2. But doing so we see that the only possible way to have a consistent Lax pair is to replace with \( i a_i (X_{\alpha_i} - X_{-\alpha_i}) \) all variables in \( L \) corresponding to roots of even height. Therefore we end up with the following alternative method of constructing Lax pairs.

We begin with a subset \( \Phi \) of the positive roots containing the simple roots. We write \( \Phi = \Phi_1 \cup \Phi_2 \) where \( \Phi_1 \) are the roots in \( \Phi \) of odd height and \( \Phi_2 \) are the roots in \( \Phi \) of even height. The Lax matrix is constructed as

\[
L = \sum_{\alpha_i \in \Phi_1} a_i (X_{\alpha_i} + X_{-\alpha_i}) + \sum_{\alpha_i \in \Phi_2} ia_i (X_{\alpha_i} - X_{-\alpha_i}) = \sum_{\alpha_i \in \Phi} b_i (X_{\alpha_i} \pm X_{-\alpha_i}),
\] (7)

where the variables \( b_i \) are defined as \( b_i = a_i \) if \( \alpha_i \in \Phi_1 \) and \( b_i = ia_i \) if \( \alpha_i \in \Phi_2 \). Consider the set \( \Phi \cup \Phi^- \) which consists of all the roots in \( \Phi \) together with their negatives. Let

\[
\Psi = \{ \alpha + \beta | \alpha, \beta \in \Phi \cup \Phi^-, \alpha + \beta \in \Delta^+ \}.
\]

We define the upper triangular part of the skew-Hermitian matrix \( B \) as

\[
\sum c_{ij} b_j X_{\alpha_i + \alpha_j},
\] (8)

where \( c_{ij} = \pm 1 \) if \( \alpha_i + \alpha_j \in \Psi \) with \( \alpha_i, \alpha_j \in \Phi \cup \Phi^- \) and 0 otherwise.

An easy consequence of the construction of the matrices \( L \) and \( B \) is the following lemma.

Lemma 1. Let \( \Phi \) be a subset of the positive roots containing the simple roots and \( L, B \) the matrices constructed in (7) and (8). Also let \( K \) be the subset of the positive roots defined by

\[
K = \{ \alpha + \beta + \gamma | \alpha, \beta, \gamma \in \Phi \cup \Phi^-, \alpha + \beta + \gamma \in \Delta^+ \}.
\]
Let’s write \( K = K_1 \cup K_2 \) where \( K_1 \) are the roots in \( K \) of odd height and \( K_2 \) are the roots in \( K \) of even height. Then the bracket \([B, L]\) is decomposed into \([B, L] = A_1 + iA_2 \) for a symmetric matrix \( A_1 \) and a skew-symmetric matrix \( A_2 \) where the nonzero entries of \( A_1 \) and \( A_2 \) appear in the “correct” positions; i.e. those of \( A_1 \) in positions corresponding to root vectors \( X_\alpha, \alpha \in K_1 \) while those of \( A_2 \) in positions corresponding to root vectors \( X_\alpha, \alpha \in K_2 \).

Next we show that this method gives more Lotka–Volterra systems and in general produces more Lax pairs than the one described in Section 3. Since for this method we don’t have to worry about the diagonal entries of the bracket \([B, L]\) (see Lemma 1). We end up with the following proposition.

**Proposition 3.** Let \( \Pi \subset \Phi \subset \Delta^+ \) be a subset of the positive roots containing the simple roots with the property that whenever \( \alpha, \beta, \gamma \in \Phi \cup \Phi^- \) and \( \alpha + \beta + \gamma \in \Delta^+ \) then \( \alpha + \beta + \gamma \in \Phi \). Also let \( L, B \) be the matrices constructed using the algorithm described in (7) and (8). Then for any choice of the signs \( c_{1,i} \) the pair \( L, B \) is a Lax pair.

**Example 6.** Let \( k, n \in \mathbb{N} \) with \( 1 \leq k < n \). If \( \Phi \) is the subset of the positive roots of the root system \( A_n \) containing the simple roots and all the roots of height larger than \( k \) then for all possible choices of the signs \( c_{1,i} \) we have a consistent Lax pair.

**Example 7.** Consider the root system of type \( A_3 \) with simple roots \( \Pi = \{\alpha_1, \alpha_2, \alpha_3\} \) and positive roots
\[
\Delta^+ = \Pi \cup \{\alpha_4 = \alpha_1 + \alpha_2, \alpha_5 = \alpha_2 + \alpha_3, \alpha_6 = \alpha_1 + \alpha_2 + \alpha_3\}.
\]

Then all Lax pairs corresponding to \( \Phi = \Pi \cup \{\alpha_5, \alpha_6\} \) are given by
\[
L = \sum_{i=1}^{3} a_i (X_{\alpha_i} + X_{-\alpha_i}) + ia_4 (X_{\alpha_4} - X_{-\alpha_4}) + a_5 (X_{\alpha_5} - X_{-\alpha_5})
\]
and
\[
B = ic_{4,4}a_4a_5 (X_{\alpha_1} + X_{-\alpha_1}) + ic_{3,4}a_3a_4 (X_{\alpha_2} + X_{-\alpha_2}) + ic_{4,5}a_2a_4 (X_{\alpha_3} + X_{-\alpha_3}) + (c_{1,2}a_1a_2 + c_{3,5}a_3a_5) (X_{\alpha_4} - X_{-\alpha_4}) + (c_{1,5}a_1a_5 + c_{2,3}a_2a_3) (X_{\alpha_5} - X_{-\alpha_5}) - ic_{3,4}a_1a_4 (X_{\alpha_6} + X_{-\alpha_6}).
\]

We can verify that for all 32 choices of the signs \( c_{1,i} \) no one of the corresponding systems is the same as the one produced by the method of Section 3. Therefore this procedure produces systems which in general are different from the ones of Section 3.

**Example 8.** For the root system of type \( A_5 \) with simple roots \( \Pi = \{\alpha_i : i = 1, 2, 3, 4, 5\} \) and positive roots
\[
\Delta^+ = \Pi \cup \{\alpha_{3+i} = \alpha_i + \alpha_i+1 : i = 1, 2, 3, 4\} \cup \{\alpha_9+i = \alpha_i + \alpha_i+1 + \alpha_i+2 : i = 1, 2\} \cup \{\alpha_{12+i} = \alpha_i + \alpha_i+1 + \alpha_i+2 + \alpha_i+3 : i = 1, 2\} \cup \{\alpha_{15} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\},
\]
consider the Lax pair
\[
L = \sum_{i=1}^{5} a_i (X_{\alpha_i} + X_{-\alpha_i}) + ia_6 (X_{\alpha_7} - X_{-\alpha_7}) + ia_7 (X_{\alpha_9} - X_{-\alpha_9}) + ia_8 (X_{\alpha_{14}} - X_{-\alpha_{14}}) + a_9 (X_{\alpha_{15}} + X_{-\alpha_{15}})
\]
and

\[
B = i a_8 a_9 (X_{\alpha_1} + X_{-\alpha_1}) - ia_3 a_6 (X_{\alpha_2} + X_{-\alpha_2}) - ia_2 a_6 (X_{\alpha_3} + X_{-\alpha_3})
\]

\[
- ia_5 a_7 (X_{\alpha_4} + X_{-\alpha_4}) - ia_4 a_7 (X_{\alpha_5} + X_{-\alpha_5}) - a_1 a_2 (X_{\alpha_6} - X_{-\alpha_6})
\]

\[
+ (-a_2 a_8 + a_2 a_9) (X_{\alpha_7} - X_{-\alpha_7}) - a_3 a_4 (X_{\alpha_8} - X_{-\alpha_8}) + (-a_4 a_5 + a_6 a_8) (X_{\alpha_9} - X_{-\alpha_9})
\]

\[
- (ia_1 a_6 + ia_7 a_9) (X_{\alpha_10} + X_{-\alpha_10}) + (-ia_4 a_6 + ia_5 a_8) (X_{\alpha_11} + X_{-\alpha_11})
\]

\[
+ (ia_2 a_8 - ia_3 a_7) (X_{\alpha_{12}} + X_{-\alpha_{12}}) + a_5 a_9 (X_{\alpha_{13}} - X_{-\alpha_{13}}) + (a_1 a_9 + a_6 a_7) (X_{\alpha_{14}} - X_{-\alpha_{14}})
\]

\[
+ ia_1 a_8 (X_{\alpha_{15}} + X_{-\alpha_{15}}).
\]

This Lax pair corresponds to the subset \( \Phi = \Pi \cup \{\alpha_7, \alpha_9, \alpha_{14}, \alpha_{15}\} \) and gives rise to a Lotka–Volterra system. Note that according to Theorem 1 if we construct the matrices \( L, B \) using the algorithm of Section 3, the subset \( \Phi \) does not give a Lotka–Volterra system.

The Poisson matrix for the associated system is given by

\[
\pi = \begin{pmatrix}
0 & -a_1 a_2 & 0 & 0 & 0 & -a_1 a_6 & 0 & a_1 a_8 & a_1 a_9 \\
(a_1 a_2 & 0 & -a_2 a_3 & 0 & 0 & 0 & a_2 a_8 & 0 \\
0 & a_2 a_3 & 0 & -a_3 a_4 & 0 & a_3 a_6 & -a_3 a_7 & 0 & 0 \\
0 & 0 & a_3 a_4 & 0 & -a_4 a_5 & a_4 a_6 & -a_4 a_7 & 0 & 0 \\
0 & 0 & 0 & a_4 a_5 & 0 & 0 & a_5 a_7 & -a_5 a_8 & -a_5 a_9 \\
(a_1 a_6 & a_2 a_6 & -a_2 a_6 & -a_4 a_6 & 0 & 0 & -a_6 a_7 & a_6 a_8 & 0 \\
0 & 0 & a_3 a_7 & a_3 a_7 & -a_5 a_7 & a_5 a_7 & 0 & -a_7 a_8 & -a_7 a_9 \\
-a_1 a_8 & -a_2 a_8 & 0 & 0 & a_5 a_8 & -a_6 a_8 & a_7 a_8 & 0 & a_8 a_9 \\
-a_1 a_9 & 0 & 0 & a_5 a_9 & 0 & a_7 a_9 & -a_8 a_9 & 0 & 0
\end{pmatrix}
\]

which has the following Casimirs

\[
a_{2a_4 a_9,} \frac{a_{2a_4 a_8}}{a_1}, \frac{a_{1a_3 a_7}}{a_4}, \frac{a_6}{a_{2a_3}}, a_1 a_3 a_5.
\]

The additional integral is given by \( H_4 = \text{tr} L^4 \).

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