Bergman projections on weighted Fock spaces in several complex variables

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Abstract

Let $\phi$ be a real-valued plurisubharmonic function on $\mathbb{C}^n$ whose complex Hessian has uniformly comparable eigenvalues, and let $\mathcal{F}^p(\phi)$ be the Fock space induced by $\phi$. In this paper, we conclude that the Bergman projection is bounded from the $L^p$ space $L^p(\phi)$ to $\mathcal{F}^p(\phi)$ for $1 \leq p \leq \infty$. As a remark, we claim that Bergman projections are also well defined and bounded on Fock spaces $\mathcal{F}^p(\phi)$ with $0 < p < 1$. We also obtain the estimates for the distance induced by $\phi$ and the $L^p(\phi)$-norm of Bergman kernel for $\mathcal{F}^2(\phi)$.

Keywords: Bergman kernel; Bergman projection; reverse-Hölder inequality

1 Introduction

The symbol $dv$ denotes the Lebesgue volume measure on $\mathbb{C}^n$, and

$$B(z, r) = \{ w \in \mathbb{C}^n : |w - z| < r \} \quad \text{for } z \in \mathbb{C}^n \text{ and } r > 0.$$ 

Suppose $\phi : \mathbb{C}^n \to \mathbb{R}$ is a $C^2$ plurisubharmonic function. We say that $\phi$ belongs to the weight class $W$ if $\phi$ satisfies the following statements:

(I) There exists $c > 0$ such that for $z \in \mathbb{C}^n$

$$\inf_{z \in \mathbb{C}^n} \sup_{w \in B(z, c)} \Delta \phi(w) > 0; \quad \text{(1)}$$

(II) $\Delta \phi$ satisfies the reverse-Hölder inequality

$$\| \Delta \phi \|_{L^\infty(B(z, r))} \leq Cr^{-2n} \int_{B(z, r)} \Delta \phi \, dv, \quad \forall z \in \mathbb{C}^n, r > 0 \quad \text{(2)}$$

for some $0 < C < +\infty$;

(III) The eigenvalues of $H_\phi$ are comparable, i.e., there exists $\delta_0 > 0$ such that

$$(H_\phi(z) u, u) \geq \delta_0 \Delta \phi(z) |u|^2, \quad \forall z, u \in \mathbb{C}^n,$$

where

$$H_\phi = \left( \frac{\partial^2 \phi}{\partial z_j \partial \overline{z_k}} \right)_{j,k}.$$
Suppose $0 < p < \infty$, $\phi \in W$. The space $L^p(\phi)$ consists of all Lebesgue measurable functions $f$ on $\mathbb{C}^n$ for which
\[ \|f\|_{p,\phi} = \left( \int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(z)} \, dv(z) \right)^{\frac{1}{p}} < \infty. \]

$L^\infty(\phi)$ is the set of all Lebesgue measurable functions $f$ on $\mathbb{C}^n$ with
\[ \|f\|_{\infty,\phi} = \sup_{z \in \mathbb{C}^n} |f(z)| e^{-\phi(z)} < \infty. \]

Let $H(\mathbb{C}^n)$ be the family of all holomorphic functions on $\mathbb{C}^n$. The weighted Fock space is defined as
\[ \mathcal{F}^p(\phi) = L^p(\phi) \cap H(\mathbb{C}^n) \]
with the same norm $\| \cdot \|_{p,\phi}$. It is easy to check that $\mathcal{F}^p(\phi)$ is a Banach space under $\| \cdot \|_{p,\phi}$ if $1 \leq p < \infty$, and $\mathcal{F}^p(\phi)$ is a Fréchet space with the metric $\rho(f, g) = \|f - g\|_{p,\phi}$ whenever $0 < p < 1$. Taking $\phi(z) = \frac{1}{2} |z|^2$, $\mathcal{F}^p(\phi)$ is the classical Fock space which has been studied by many authors, see [1–3] and the references therein. Notice that the weight function $\varphi$ on $\mathbb{C}^n$ with the restriction that $dd^c \varphi \approx dd^c |z|^2$ in [4] and [5] belongs to $W$.

In the one-dimensional case, an important contribution to weighted Fock spaces was given by Christ [6] (but see also [7, 8]). They work under the assumption that $\phi$ is subharmonic and that $\Delta \phi \, dA$ is a doubling measure, where $dA$ is the area measure on $\mathbb{C}$. Notice that the hypotheses on $\Delta \phi \, dA$ are a sort of finite-type assumption and are automatically verified when $\phi$ is a subharmonic non-harmonic polynomial.

The result of Christ was extended by Delin to several complex variables under the assumption of strict plurisubharmonicity of the weight in [9]. Dall’Ara [10] tried to extend Christ’s approach to $n \geq 2$. Given $\phi \in W$, let $K(\cdot, \cdot)$ be the weighted Bergman kernel for $\mathcal{F}^2(\phi)$. In particular, Theorem 20 of [10] proves that there is a constant $C, \epsilon > 0$ such that
\[ |K(z, w)| \leq Ce^{\phi(z) + \phi(w)} \frac{e^{-\epsilon \rho(z,w)}}{\rho(z)^n \rho(w)^n} \tag{3} \]
for $z, w \in \mathbb{C}^n$, where $d(\cdot, \cdot)$, $\rho_\varphi(\cdot)$ described in Section 2.

In the setting of Bergman spaces, the Bergman projection is bounded on $p$-Bergman spaces for $1 < p < \infty$, it also maps $L^\infty$ into Bloch spaces, see [11] for details. With the Bergman kernel $K(\cdot, \cdot)$ for $\mathcal{F}^2(\phi)$, the Bergman projection $P$ can be represented as
\[ Pf(z) = \int_{\mathbb{C}^n} K(z, w) f(w) e^{-\phi(w)} \, dv(w), \quad z \in \mathbb{C}^n. \]

It is well known that $P(f) = f$ for $f \in \mathcal{F}^2(\phi)$. The purpose of this work is to discuss the boundedness of Bergman projection acting on $\mathcal{F}^p(\phi)$ for general $p$. Section 2 is devoted to some basic estimates, including the distance $d(\cdot, \cdot)$ and the $L^p(\phi)$-norm of the Bergman kernel. In Section 3, we will discuss the boundedness of Bergman projections from $L^p(\phi)$ to $\mathcal{F}^p(\phi)$ with $1 \leq p \leq \infty$. We also show that the Bergman projection is well defined and bounded on $\mathcal{F}^p(\phi)$ for $p < 1$. 
In what follows, we always suppose $\phi \in W$ and use $C$ to denote positive constants whose values may change from line to line but do not depend on the functions being considered. Two quantities $A$ and $B$ are called equivalent, denoted by $A \simeq B$, if there exists some $C$ such that $C^{-1}A \leq B \leq CA$.

2 Some basic estimates
In this section, we are going to give some estimates, which will be useful in the following section. At the beginning, we will give some notations.

For $z \in \mathbb{C}^n$, set
\[ \rho_\phi(z) = \sup\{ r > 0 : \sup_{w \in B(z,r)} \Delta \phi(w) \leq r^{-2} \}. \] (4)

By (1), there exist $c, s > 0$ such that for $z \in \mathbb{C}^n$
\[ \sup_{w \in B(z,c)} \Delta \phi(w) \geq s. \]

We then have some $M > 0$ such that
\[ \sup_{z \in \mathbb{C}^n} \rho_\phi(z) \leq M. \]

Moreover, there are some positive constants $C, M_1$ and $M_2$ such that for all $z, w \in \mathbb{C}^n$, we have
\[ C^{-1} \theta^{-M_1} \rho_\phi(w) \leq \rho_\phi(z) \leq C \theta^{M_2} \rho_\phi(w), \] (5)

where $\theta = \max(1, \frac{|z-w|}{\rho_\phi(w)})$. We can see this in Proposition 10 of [10].

Given $r > 0$, write
\[ B'(z) = B(z, r \rho_\phi(z)) \quad \text{and} \quad B(z) = B^1(z). \]

Then (5) implies that there is some $C$ such that for $z \in \mathbb{C}^n$
\[ C^{-1} \rho_\phi(w) \leq \rho_\phi(z) \leq C \rho_\phi(w) \quad \text{for} \quad w \in B(z). \] (6)

By (6) and the triangle inequality, we have $m_1, m_2 > 0$ such that
\[ B(z) \subseteq B^{m_1}(w), \quad B(w) \subseteq B^{m_2}(z) \quad \text{whenever} \quad w \in B(z). \] (7)

Given a sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{C}^n$, we say that $\{a_k\}_{k=1}^\infty$ is a lattice if $\{B(a_k)\}_{k=1}^\infty$ covers $\mathbb{C}^n$ and the balls of $\{B^2(a_k)\}_{k=1}^\infty$ are pairwise disjoint. This lattice exists by a standard covering lemma, see Theorem 2.1 in [12], or Proposition 7 in [10] as well. Moreover, for the lattice $\{a_k\}$ and any $m > 0$, there exists some integer $N$ such that each $z \in \mathbb{C}^n$ can be in at most $N$ balls of $\{B^m(a_k)\}$. Equivalently,
\[ \sum_{k=1}^\infty \chi_{B^m(a_k)}(z) \leq N \quad \text{for} \quad z \in \mathbb{C}^n. \] (8)
To the radius function $\rho_\phi$ defined as (4), we associate the Riemannian metric $\rho_\phi(z)^{-2} dz \otimes d\bar{z}$. In fact, we are interested only in the associated Riemannian distance, which we describe explicitly. If $\gamma : [0, 1] \to \mathbb{C}^n$ is piecewise $C^1$ curves, we define

$$L_{\rho_\phi}(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{\rho_\phi(\gamma(t))} dt.$$ 

Given $z, w \in \mathbb{C}^n$, we put

$$d(z, w) = \inf_{\gamma} L_{\rho_\phi}(\gamma),$$

where the inf is taken as $\gamma$ varies over the collection of curves with $\gamma(0) = z$ and $\gamma(1) = w$. We then have the estimate for this distance as follows.

**Lemma 1** There exist $\alpha, \beta, C > 0$ such that for $z, w \in \mathbb{C}^n$

$$\frac{1}{C} \left( \frac{|z - w|}{\rho_\phi(z)} \right)^\alpha \leq d(z, w) \leq C \left( \frac{|z - w|}{\rho_\phi(z)} \right)^\beta.$$

**Proof** First, we claim that there is some $C > 0$ such that

$$d(z, w) \geq C \left( \frac{|z - w|}{\rho_\phi(z)} \right)^\alpha. \tag{9}$$

In fact, set $\mu$ to be

$$\mu(B(z, r)) = r^2 \|\Delta \phi\|_{L^\infty(B(z, r))}, \quad z \in \mathbb{C}^n, r > 0. \tag{10}$$

By (2), it is easy to check that there is some $M > 2$ such that

$$\mu(B(z, 2r)) \leq M \mu(B(z, r)). \tag{11}$$

Moreover,

$$\mu(B(z, \rho_\phi(z))) = 1 \tag{12}$$

because of (4). Given any $r \leq R$, it is easy to check that

$$\mu(B(z, r)) \leq \left( \frac{R}{r} \right)^2 \mu(B(z, R)) \leq \mu(B(z, R)) \tag{13}$$

for $z \in \mathbb{C}^n$ because of (10). Also, there is a positive integer $m$ such that $2^{m-1} r < R \leq 2^m r$. Hence, (11) and (12) tell us

$$\mu(B(z, R)) \leq \mu(B(z, 2^m r)) \leq M \mu(B(z, 2^{m-1} r)) \leq \cdots \leq M^m \mu(B(z, r)).$$

Since $M^{m-1} = 2^{(m-1) \log_2 M} \leq \left( \frac{R}{r} \right)^{\log_2 M}$, we get

$$\mu(B(z, R)) \leq M \left( \frac{R}{r} \right)^{\log_2 M} \mu(B(z, r)). \tag{14}$$
For \(z, w \in \mathbb{C}^n\), notice that \(B(w, |w - z|) \subset B(z, 2|w - z|)\). If \(|w - z| < \rho_\phi(z)\), take any piecewise \(C^1\) curve \(\gamma : [0, 1] \to \mathbb{C}^n\) connecting \(z\) and \(w\), and let \(T_0\) be the minimum time such that \(|z - \gamma(T_0)| = \rho_\phi(z)\). By (6), \(\rho_\phi(\gamma(t)) \simeq \rho_\phi(z)\) for \(t \in [0, T_0]\). This implies

\[
\int_0^1 \frac{|\gamma'(t)|}{\rho_\phi(\gamma(t))} \, dt \geq \frac{C}{\rho_\phi(z)} \int_0^{T_0} |\gamma'(t)| \, dt \geq C \frac{|z - w|}{\rho_\phi(z)}.
\]

If \(|z - w| \geq \rho_\phi(w)\), then (11), (10), (13) and (12) give

\[
\mu\left(B\left(\frac{1}{4}|z - w|\right)\right) \geq C \mu\left(B(z, 2|z - w|)\right) \geq C \mu\left(B(w, |z - w|)\right)
\]

\[
\geq C \left(\frac{|z - w|}{\rho_\phi(w)}\right)^2 \mu\left(B(w, \rho_\phi(w))\right)
\]

\[
= C \left(\frac{|z - w|}{\rho_\phi(w)}\right)^2.
\]

On the other hand, for \(\zeta \in B(z, \frac{1}{4}|z - w|)\), there are

\[
B\left(\frac{1}{4}|z - w|\right) \subset B\left(\frac{1}{2}|z - w|\right)
\]

and

\[
B\left(\frac{1}{4}|z - w|\right) \subset B\left(\frac{1}{2}|z - w|\right).
\]

Combining the above with (11), we know

\[
\mu\left(B\left(\frac{1}{4}|z - w|\right)\right) \simeq \mu\left(B\left(\frac{1}{2}|z - w|\right)\right).
\]

By the fact \(\log_2 M > 0\), (13), (14) and (12), there exists \(t > 0\) such that

\[
\mu\left(B\left(\frac{1}{4}|z - w|\right)\right) \simeq \mu\left(B\left(\frac{1}{2}|z - w|\right)\right)
\]

\[
\leq C \left[\frac{|z - w|}{\rho_\phi(\zeta)}\right]^t \mu\left(B(\zeta, \rho_\phi(\zeta))\right)
\]

\[
\simeq \left[\frac{|z - w|}{\rho_\phi(\zeta)}\right]^t.
\]

Hence, \(\left[\frac{|z - w|}{\rho_\phi(w)}\right]^2 \leq C \left[\frac{|z - w|}{\rho_\phi(\zeta)}\right]^t\). This implies

\[
\rho_\phi(\zeta) \leq C |z - w| \left(\frac{|z - w|}{\rho_\phi(w)}\right)^{-\alpha}, \quad \zeta \in B\left(\frac{1}{4}|z - w|\right),
\]
where \( \alpha = \frac{\beta}{2} > 0 \). For any piecewise \( C^1 \) curves \( \Gamma \), defined as \( \gamma : [0,1] \to \mathbb{C}^n \) with \( \gamma(0) = z \) and \( \gamma(1) = w \), we have

\[
\int_{\Gamma} |\gamma'(t)| \, dt \geq \int_{\Gamma \cap \partial \{z \in \mathbb{C}^n : |z-w| \leq \frac{|z-w|}{\rho_\phi(w)}\} \rho_\phi(\gamma(t)) \, dt
\]

\[
\geq \frac{1}{|z-w|} \int_{\Gamma \cap \partial \{z \in \mathbb{C}^n : |z-w| \leq \frac{|z-w|}{\rho_\phi(w)}\}} |\gamma'(t)| \, dt
\]

\[
\geq C \left( \frac{|z-w|}{\rho_\phi(w)} \right)^\alpha.
\]

This yields (9) is true. Now, we are going to prove the other direction. For \( z, w \in \mathbb{C}^n \), take \( \gamma(t) = z + t(w - z) \) and \( \gamma(t_0) \in \partial B(z) \) (set \( t_0 = 1 \) if \( w \in B(z) \)). Then (5) gives

\[
d(z, w) \leq |w - z| \int_0^1 \frac{dt}{\rho_\phi(\gamma(t))}
\]

\[
\leq C |w - z| \left( \int_0^{t_0} + \int_{t_0}^1 \frac{dt}{\rho_\phi(\gamma(t))} \right)
\]

\[
\leq C \frac{|w - z|}{\rho_\phi(z)} \int_0^1 dt + C \left( \frac{|w - z|}{\rho_\phi(z)} \right)^{1+M_1} \int_0^1 t^{M_1} \, dt
\]

\[
\leq C \left( \frac{|w - z|}{\rho_\phi(z)} \right)^\beta,
\]

where \( \beta > 0 \). The proof is completed. \( \Box \)

Now, we can estimate the following integral.

**Lemma 2** Given \( p > 0 \) and \( k \in \mathbb{R} \), we have

\[
\int_{\mathbb{C}^n} \rho_\phi(\xi)^k e^{-p d(z, \xi)} \, dv(\xi) \leq C \rho_\phi(z)^{k+2n},
\]

where \( C > 0 \) is a constant depending only on \( n, p \) and \( k \).

**Proof** By (6), it is easy to check that

\[
\int_{B(z)} \rho_\phi(\xi)^k e^{-p d(z, \xi)} \, dv(\xi) \leq \int_{B(z)} \rho_\phi(\xi)^k \, dv(\xi) \leq C \rho_\phi(z)^{k+2n}.
\]

Estimate (9) gives

\[
\int_{\mathbb{C}^n \setminus B(z)} \rho_\phi(\xi)^k e^{-p d(z, \xi)} \, dv(\xi) \leq \int_{\mathbb{C}^n \setminus B(z)} \rho_\phi(\xi)^k e^{-p \left( \frac{|\xi|}{\rho_\phi(z)} \right)^{\frac{1}{2n}} w} \, dv(\xi)
\]

\[
\leq \int_{\mathbb{C}^n \setminus B(z)} \rho_\phi(\xi)^k \int_{\rho_\phi(z)^{\frac{1}{2n}}}^{\infty} e^{-s} \, ds \, dv(\xi)
\]

\[
\leq \int_{\rho_\phi(z)^{\frac{1}{2}}}^{\infty} e^{-s} \int_{\rho_\phi(z)^{\frac{1}{2n}}}^{\infty} \rho_\phi(\xi)^k \, dv(\xi) \, ds.
\]
By (5), the inequality above is no more than

\[
\int_{p_{C_1}}^{\infty} \sup_{\frac{1}{2} < \frac{1}{p} < 1} \rho_\phi(\zeta) \nu(B^{1-p_{C_1}}(\zeta)) e^{-s} ds \\
\leq C \rho_\phi(z)^{k+2n} \int_{p_{C_1}}^{\infty} s^{2n+\max(\|\lambda_1\|,\|\lambda_2\|)} e^{-s} ds = C \rho_\phi(z)^{k+2n}.
\]

Therefore,

\[
\int_{C^n} \rho_\phi(w)^k e^{-\rho_\phi(d(z,w))} dv(w) \leq C \rho_\phi(z)^{k+2n}.
\]

The proof is completed. \(\square\)

Next, we will give the \(L^p(\phi)\)-norm of the Bergman kernel \(K(\cdot, \cdot)\) for \(F^2(\phi)\).

**Proposition 3** For \(0 < p < \infty\), we have

\[
\|K(\cdot, z)\|_{p, \phi} \leq C e^{\phi(z)} \rho_\phi(z)^{2n(\frac{1}{p} - 1)}, \quad z \in \mathbb{C}^n.
\]

**Proof** By (3) and Lemma 2, we obtain

\[
\int_{C^n} |K(w, z)|^p e^{-\rho_\phi(d(z,w))} dv(w) \leq C e^{\phi(z)} \rho_\phi(z)^{2n(\frac{1}{p} - 1)} \int_{C^n} \rho_\phi(w)^{-pn} e^{-\rho_\phi(d(z,w))} dv(w)
\]

\[
\leq C e^{\phi(z)} \rho_\phi(z)^{2n(1-p)}.
\]

The proof is completed. \(\square\)

**Lemma 4** For \(0 < p < \infty\), there is a constant \(C > 0\) such that for all \(r \in (0,1]\), \(f \in H(\mathbb{C}^n)\) and \(z \in \mathbb{C}^n\), we have

\[
|f(z)| e^{-\phi(z)} \leq \frac{C}{r^p \rho_\phi(z)^{2p}} \left( \int_{p(\zeta)} |f(w) e^{-\phi(w)}|^p dv(w) \right)^{\frac{1}{p}}.
\]

**Proof** If \(p = 2\), (15) is just Lemma 13 in [10]. For \(p \neq 2\), we borrow the idea in Lemma 19 of [7] and Lemma 13 in [10]. The details are omitted. \(\square\)

### 3 Boundedness of Bergman projections

Recall that the Bergman projection \(P\) on \(L^p(\phi)\) is defined as

\[
Pf(z) = \int_{C^n} K(z, w) f(w) e^{-2\phi(w)} dv(w), \quad z \in \mathbb{C}^n.
\]

In this section, we focus on the boundedness of Bergman projections \(P\) from \(L^p(\phi)\) to \(F^p(\phi)\) for \(1 \leq p \leq \infty\).

**Theorem 5** Let \(1 \leq p \leq \infty\). Then the Bergman projection \(P\) is bounded as a map from \(L^p(\phi)\) to \(F^p(\phi)\).
Proof  By the definition of $P$, we can conclude $Pf$ is holomorphic on $\mathbb{C}^n$. Fubini’s theorem and Proposition 3 yield

$$
\|Pf\|_{1,\phi} \leq \int_{\mathbb{C}^n} e^{-\phi(z)} \frac{d\nu(z)}{C} \int_{\mathbb{C}^n} |K(z, w)f(w)| e^{-2\phi(w)} \frac{d\nu(w)}{C} \\
= \int_{\mathbb{C}^n} |f(w)| e^{-2\phi(w)} \frac{d\nu(w)}{C} \int_{\mathbb{C}^n} |K(z, w)| e^{-\phi(z)} \frac{d\nu(z)}{C} \\
\leq C \|f\|_{1,\phi}
$$

for $f \in L^1(\phi)$. Given $f \in L^\infty(\phi)$, we obtain

$$
\|Pf\|_{\infty,\phi} \leq \sup_{z \in \mathbb{C}^n} e^{-\phi(z)} \int_{\mathbb{C}^n} |K(z, w)f(w)| e^{-2\phi(w)} \frac{d\nu(w)}{C} \\
\leq \|f\|_{\infty,\phi} \sup_{z \in \mathbb{C}^n} e^{-\phi(z)} \int_{\mathbb{C}^n} |K(z, w)| e^{-\phi(w)} \frac{d\nu(w)}{C} \\
\leq C \|f\|_{\infty,\phi}.
$$

If $1 < p < \infty$, Hölder’s inequality and Fubini’s theorem give

$$
\|Pf\|_{\infty,\phi}^p = \int_{\mathbb{C}^n} e^{-p\phi(z)} \frac{d\nu(z)}{C} \left( \int_{\mathbb{C}^n} |K(z, w)f(w)| e^{-2\phi(w)} \frac{d\nu(w)}{C} \right)^p \\
\leq \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |f(w)|^p e^{-p\phi(w)} |K(z, w)| e^{-\phi(w)} \frac{d\nu(w)}{C} \|K(z, w)\|_{1,\phi} \|1,\phi\|_{p-1} e^{-p\phi(z)} \frac{d\nu(z)}{C} \\
\leq C \int_{\mathbb{C}^n} e^{-\phi(z)} \frac{d\nu(z)}{C} \left( \int_{\mathbb{C}^n} |f(w)|^p e^{-p\phi(w)} |K(z, w)| e^{-\phi(w)} \frac{d\nu(w)}{C} \right) \\
\leq C \int_{\mathbb{C}^n} |f(w)|^p e^{-p\phi(w)} e^{-\phi(w)} \frac{d\nu(w)}{C} \int_{\mathbb{C}^n} |K(z, w)| e^{-\phi(z)} \frac{d\nu(z)}{C} \\
\leq C \|f\|_{p,\phi}^p
$$

for $f \in L^p(\phi)$. Thus, $P$ is bounded from $L^p(\phi)$ to $L^p(\phi)$ for $1 \leq p < \infty$. The proof is ended. □

In addition, we observe that the Bergman projection is also well defined and bounded on the weighted Fock space $F^p(\phi)$ with $p < 1$.

Remark 6  For $p < 1$, the Bergman projection $P$ is bounded on $F^p(\phi)$.

Proof  First, we claim that $P$ is well defined on $F^p(\phi)$. In fact, given any $f \in F^p(\phi)$, by (3), (15) and Lemma 2, we obtain

$$
\int_{\mathbb{C}^n} |K(z, w)f(w)| e^{-2\phi(w)} \frac{d\nu(w)}{C} \\
\leq C \|f\|_{p,\phi} \int_{\mathbb{C}^n} \rho(\phi(w))^{\frac{-n}{p}} |K(z, w)| e^{-\phi(w)} \frac{d\nu(w)}{C}
$$
Now, we deal with the boundedness of $P$. In fact, let $\{a_k\}_k$ be the lattice. For $f \in \mathcal{F}^p(\phi)$, we get

\[
|Pf(z)|^p \leq \left( \sum_{k=1}^{\infty} \int_{B(a_k)} |f(w)K(w,z)|^p e^{-2p\phi(w)} \, dv(w) \right)^{\frac{1}{p}} \\
\leq \sum_{k=1}^{\infty} \left( \int_{B(a_k)} |f(w)K(w,z)|^p e^{-2p\phi(w)} \, dv(w) \right)^{\frac{1}{p}} \\
\leq \sum_{k=1}^{\infty} v(B(a_k))^p \left( \sup_{w \in B(a_k)} |f(w)K(w,z)|^p e^{-2p\phi(w)} \right)^{\frac{1}{p}}.
\]

Notice that the associated function $\rho_{2\phi} = \frac{e^{\phi}}{2} \rho_{\phi}$, which follows from (4). Applying Lemma 4 with weight $2\phi$ instead of $\phi$, there then is some constant $C > 0$ such that $|Pf(z)|^p$ is no more than $C$ times

\[
\sum_{k=1}^{\infty} \rho_{\phi}(a_k)^{2np-2n} \sup_{w \in B(a_k)} \int_{B(a_k)} |f(u)|^p |K(u,z)|^p e^{-2p\phi(u)} \, dv(u).
\]

Combining (7) with (8), we obtain

\[
|Pf(z)|^p \leq C \sum_{k=1}^{\infty} \int_{B^2(a_k)} \rho_{\phi}(u)^{2np-2n} |f(u)|^p |K(u,z)|^p e^{-2p\phi(u)} \, dv(u) \\
\leq CN \int_{\mathbb{C}^n} \rho_{\phi}(u)^{2np-2n} |f(u)|^p |K(u,z)|^p e^{-2p\phi(u)} \, dv(u).
\]

Therefore, applying Fubini’s theorem and Proposition 3, we get

\[
\int_{\mathbb{C}^n} |Pf(z)|^p e^{-p\phi(z)} \, dv(z) \\
\leq C \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |K(u,z)|^p e^{-p\phi(z)} \, dv(z) \rho_{\phi}(u)^{2np-2n} |f(u)|^p e^{-2p\phi(u)} \, dv(u) \\
\leq C \int_{\mathbb{C}^n} |f(u)|^p e^{-p\phi(u)} \, dv(u).
\]

This means that $P$ is bounded on $\mathcal{F}^p(\phi)$. The proof is ended. \qed

4 Conclusion

In this paper, we show the boundedness of Bergman projection from the $p$th Lebesgue space $L^p(\phi)$ to the weighted Fock space $\mathcal{F}^p(\phi)$ for $1 \leq p \leq \infty$. We also remark that the Bergman projection is bounded on $\mathcal{F}^p(\phi)$ with $p < 1$. Meanwhile, we get the estimates for the distance induced by $\phi$ and the $L^p(\phi)$-norm of Bergman kernel for $\mathcal{F}^2(\phi)$.
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Competing interests
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Authors’ contributions
The author wrote this paper by herself. She has read and approved the final manuscript.

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