Variations in driving torque in Couette-Taylor flow subject to a vertical magnetic field

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Abstract. We numerically investigate variations in the required driving torque for Couette-Taylor flow in an electrically-conducting fluid under the influence of a uniform, vertical magnetic field. The radius ratio of the coaxial cylinders ($\eta = R_i/R_o$) is 0.5 and the aspect ratio ($A = \text{height}/\text{gap}$) is 4. The inner cylinder rotates at a constant speed and the outer cylinder is kept at rest. A uniform magnetic field is applied vertically. Without the magnetic field, the azimuthal Couette-Taylor flow induces a secondary meridional flow when the non-dimensional rotation speed of the inner cylinder exceeds the critical Reynolds number. When a moderate magnetic field is applied, Lorentz forces in the fluid lead to first order changes in the meridional flow mode and the required driving torque is reduced by as much as 25%. However, a stronger imposed magnetic field increases the torque due to the suppression of the azimuthal flow by the Lorentz force.

1. Introduction
The study of Couette-Taylor flow is a classical problem in fluid mechanics and applied mathematics. Chandrasekhar [1] theoretically determined the onset of the instability of the secondary flow in terms of the rotating speed of the inner and outer coaxial cylinders. He also estimated the required driving torque on the inner cylinder when the outer one is at rest. Lim and Tan [2] carried out the laboratory experiments and verified the scaling behavior of the dimensionless torque versus the inner cylinder Reynolds number for values both above and below the critical Reynolds number. In addition, they measured the number of Taylor vortices as a function of Reynolds number.

The stability of the magneto-Couette-Taylor flow was theoretically studied by Rüdiger and Shalybkov [3]. They studied the critical Reynolds number for electrically conducting fluids of various Prandtl numbers under a vertical magnetic field as a function of various parameters such as gap width, electric conductivity of the sidewalls, and rotation rate of both the inner and outer cylinders. However, their theoretical studies presumed infinite vertical cylinders and neglected the effects of top and bottom end walls. In this report, the effect of an external vertical magnetic field on the rotating torque of the Couette-Taylor flow with end walls is numerically studied for $Re < 1000$. We focus on variations in the flow mode and the minimum required torque to drive the system.

2. Numerical model and governing equations
In this report, three-dimensional calculations are carried out to obtain more accurate velocity profiles and torque estimates. Figure 1 shows a schematic of our numerical model. In the gap between the two
coaxial cylinders, we simulate the flow of a liquid metal. The dimensionless inner and outer radii are 1 and 2, respectively and thus the radius ratio $\eta$ is 1/2. The aspect ratio $A$, which is the ratio of the height of the cylinders divided by the gap width, is 4. We solve the following dimensionless equations [4.]

The continuity equation:
$$\nabla \cdot \mathbf{U} = 0; \quad (1)$$

The Navier-Stokes equations:
$$\frac{D\mathbf{U}}{D\tau} = -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U} + \frac{Ha}{Re} \left( \mathbf{J} \times \mathbf{B} \right); \quad (2)$$

Ohm’s law:
$$\mathbf{J} = -\nabla \Psi + \mathbf{U} \times \mathbf{B}; \quad (3)$$

and conservation of electric charge:
$$\nabla \cdot \mathbf{J} = 0. \quad (4)$$

The dimensionless variables and parameters are $R = r/r_i$, $Z = z/r_i$, $\mathbf{U} = (U,V,W) = u/r_i \omega$, $\tau = r_i \omega$, $P = p/(r_i^2 \omega^2)$, $\mathbf{J} = j/(\sigma r_i \omega)$, $\Psi = \psi/(\sigma r_i \omega^2)$, where $r_i$, $r_o$, $\rho$, $\sigma$, $\mu$, $\omega$ and $\psi$ denote the dimensional inner and outer radii, density, electric conductivity and fluid viscosity, inner cylinder angular rotation speed and electric scalar potential. In addition to the radius ratio and the aspect ratio, the governing dimensionless parameters are the Hartmann number $Ha = (\sigma \mu)^{1/2} b_0 r_i$ and Reynolds number $Re = r_i \omega (r_o - r_i)/\nu$.

The boundary conditions are as follows. The end walls and the outer cylinder are stationary and subject to a non-slip condition ($\mathbf{U} = \mathbf{0}$). At the inner cylinder, $U = W = 0$ and $V = 1$. All the walls are electrically insulating. The vertical magnetic field, $\mathbf{B} = (0, 0, 1)$, is presumed to remain uniform in the fluid region due to the small magnetic Reynolds number $R_m$. For example, for liquid gallium at $Re = 300$ and $35 \, ^\circ C$, $R_m = 4.38 \times 10^{-4}$. The liquid metal in the gap is at initially at rest ($\mathbf{U} = \mathbf{0}$), and no initial electric currents or electric potentials are induced ($\mathbf{J} = \mathbf{0}$, $\Psi = 0$).

The above partial differential equations are approximated by finite difference equations with staggered meshes of 64, 24 and 256 for $R$, $\theta$ and $Z$ in the gap. Each mesh size is presumed uniform. The inertial terms in Navier-Stokes equations are approximated with a third-order upwind scheme and the system of equations are numerically integrated using the HSMAC scheme developed by Hirt et al. [5].

The dimensionless torque, $G = T/(\rho \nu^2 L)$, where $L$ is the height of the rotating cylinder, is calculated by the integration of the azimuthal velocity gradient, $dV/dR$, over the inner cylinder’s surface.

A validation exercise for our numerical code has been carried out. Figure 2 shows the chord velocity profile at $Z = 3$ for $Re = 124$ without a magnetic field. The symbols with error bars and the line represent, respectively, the experimental results of Aurnou’s group [6] and the present numerical results. The experiment has almost the same configuration as our numerical model ($r_i = \text{gap} = 4.45cm$ and height $= 17.8cm$). Silicone oil was employed as the working fluid. The experimental chord velocity profile was measured using acoustic Doppler velocimetry. The experimental data had a positioning offset uncertainty of the Doppler transducer of about 4.8 mm and was corrected herein. The experimental data at the furthest chord distances have wider error bars and disagrees with the numerical results. This is due to the divergence of the acoustic beam with distance from the transducer. However, the velocity profiles agree very well for chord distances – less than 100mm. Moreover, the peak value of velocity is almost same. Thus, we consider this validation exercise to be a success.

3. Results and discussion

3.1. Effect of the Lorentz force on the velocity field
In experiments with no-slip end walls, meridional flows always exist for non-zero Re. For Re near zero, this meridional flow takes the form of a 2-cell mode. Without the magnetic field, the critical Reynolds number for onset of the secondary meridional flow is observed to be approximately 70 in our configuration. This is almost the same as the data of Di Prima and Swinney (Re$_{\text{critical}} \approx 68.19$) [7].

The secondary flow generates a 4-cell meridional flow pattern at $Re_{\text{critical}} < Re < 1000$. A vertical cross-section (the hatched area in Figure 1) of the secondary flow is shown in figure 3(a) for $Re = 300$ and $Ha = 0$. When the vertical magnetic field is applied, the Lorentz forces induced by the interaction of the induced electric current and the external magnetic field affects the fluid field. Figure 3(b) shows the electric current, the Lorentz force and resulted velocity vectors at for $Re = 300$ and $Ha = 10$. The Lorentz force reduces the secondary flow and the number of the meridional cells decreases from 4 to 2.

Figure 1. Schematic model for the numerical computation.

Figure 2. Chord velocity profile crossing at $Z = 3$ for $Re = 124$ and $Ha = 0$. The symbols with error bars denote the data of [6] and line shows the numerical result.

Figure 3. Cross-sectional view of vector components at $Re = 300$. (a) velocity vectors at $Ha = 0$ (b) electric current, Lorentz force and velocity vectors at $Ha = 10$. Vector number is reduced for the readability.
3.2. Dimensionless torque with and without magnetic field

The plotted data of the dimensionless torque $G$ with and without a magnetic field for the Reynolds number is shown in figure 4. Open circles with a solid line show the case of $Ha = 0$ (without magnetic field) and filled ones with a dashed line are at $Ha = 10$.

When applying the magnetic field at $Ha = 10$, the torque curve is shifted higher for $Re < 70$ and becomes lower for $Re > 70$ in comparison to the non-magnetic case. Thus, the required driving torque decreases to maintain the same rotation speed in the range for $Re > 70$, whereas it increases for $Re < 70$. To see this effect in greater detail, the deviation torque, which we define as the torque at $Ha = 10$ minus that at $Ha = 0$ normalized by the torque at $Ha = 0$, is shown in figure 5. The deviation torque is positive for $Re < 70$. Around $Re = 70$, the deviation decreases to zero and a large negative deviation is obtained for $Re > 70$. The negative deviation increases as $Re$ increases but it may have a convergence at higher $Re$ values.

Figure 6 shows the effect of the Hartmann number on the deviation torque at $Re = 300$. When the Hartmann number is less than 20, the required torque decreases monotonically by about 25 percent. However, for $Ha > 20$, the deviation torque increases again with increasing $Ha$ value.

At $Re < Re_{critical}$, the magnetic field increases the required driving torque. In contrast, at $Re > Re_{critical}$, relatively weak magnetic fields ($Ha < 20$) reduce the required driving torque. These results can be interpreted as follows. When $Re < Re_{critical}$, the 2-cell mode occurs irrespective of the magnetic field. Thus, applied magnetic fields act only to suppress the azimuthal flow and a greater torque is required to produce a given velocity field. This results in the positive torque deviations shown in figures 4 and 5.

When $Re > Re_{critical}$, the Lorentz force acts first to affect the secondary meridional flow. For weak magnetic fields, the number of the secondary cells decreases 4 to 2, which lowers the radial angular momentum transport by the meridional flow and reduces the driving torque. For the case of a strong imposed magnetic field, the background azimuthal flow is also strongly suppressed, which causes the driving torque to increase at $Ha > 20$, as shown in figure 6.

**Figure 4.** Dimensionless torque for $Re$. Open and filled circles show the case at $Ha = 0$ and $Ha = 10$.

**Figure 5.** Normalized deviation torque versus $Re$ at $Ha = 10$. 
3.3. **Flow mode change**

A vertical cross-section of the stream function is shown in figure 7 for \( \text{Re} = 300 \) and \( \text{Ha} = 0, 20 \) and 50. This corresponds to the results in figure 3. The red and blue contours represent clockwise and counterclockwise circulations, respectively. Without a magnetic field, four Taylor vortices are induced due to the rotation of the inner cylinder above the critical Reynolds number. The top and bottom vortices are affected by the end walls and they are larger than the two vortices that are adjacent to the midplane. When the magnetic field is applied at \( \text{Ha} = 20 \), the Lorentz force reduces the secondary flow and the flow mode becomes a two-cell structure and the torque decreases as shown in figure 6. As the Hartmann number increases at \( \text{Ha} = 50 \), the two cells are strongly suppressed and the cells are shifted to near the inner cylinder. The shift of the cells towards the inner cylinder occurs because the Hartmann boundary layer is thinner than the case of zero-magnetic field. Thus, the driving torque on the inner cylinder becomes larger. Figure 8 shows the azimuthally and axially averaged angular momentum plotted as a function of radial position at \( \text{Re} = 300 \) and \( \text{Ha} = 0, 21, 40 \). The minimum radial slope of the angular momentum occurs at \( \text{Ha} = 21 \) and at the slope for the \( \text{Ha} = 40 \) case lies between the cases of \( \text{Ha} = 0 \) and 21. Thus, the change in meridional flow mode relaxes the boundary layer up to \( \text{Ha} = 21 \). Then, for \( \text{Ha} > 21 \), Hartmann layer effects (i.e., suppression of the background azimuthal flow) cause the boundary layer to become thinner.

**Figure 6.** Normalized deviation torque versus \( \text{Ha} \) at \( \text{Re} = 300 \).

**Figure 7.** Stream function at the cross section of the gap at \( \text{Re} = 300 \).

**Figure 8.** Azimuthally and axially averaged angular momentum at \( \text{Re} = 300 \).
4. Conclusion

Magneto-Couette-Taylor flow is numerically studied to determine the required driving torque on the inner cylinder as a function of rotation rate (Re) and vertical magnetic field strength (Ha). When the Reynolds number is lower than its critical value (Re_{critical} ~70), magnetic fields increase the required driving torque at a given Re value. In contrast, for Re values above the critical value, we find that moderate magnetic fields lower the driving torque. When the Reynolds number is greater than its critical value, the effect of moderate magnetic fields is reverse. At Re > Re_{critical} and low Ha values, the cell mode decreases from 4 cells to 2 and this alteration in meridional flow field reduces the driving torque by as much as 25% at Re = 300. At Re > Re_{critical} and higher Ha values, the cell mode remains at 2 but the Lorentz forces now suppress the azimuthal flow strongly enough that the driving torque must increase over that of the minimum value.

References

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