The Equation of State Approach to Cosmological Perturbations in $f(R)$ Gravity

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The discovery of apparent cosmological acceleration has spawned a huge number of dark energy and modified gravity theories. The $f(R)$ models of gravity are obtained when one replaces the Ricci scalar in the Einstein-Hilbert action by an arbitrary function $f(R)$,

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( R + f(R) \right) + S_m,$$

where $\kappa \equiv 8\pi G$ is the rescaled Newton’s constant, $R$ is the Ricci scalar and $S_m$ is the action describing the standard matter fields. In this work we provide expressions for the equations of state (EoS) for perturbations which completely characterize the linearized perturbations in $f(R)$ gravity, including the scalar, vector, and tensor modes. The EoS formalism is a powerful and elegant parametrization where the modification to General Relativity are treated as a dark-energy-fluid. The perturbed dark-energy-fluid variables such as the anisotropic stress or the entropy perturbation are explicitly given in terms of parameters of the model of interest.

Keywords: Modified gravity; Dark Energy; Cosmological perturbations.

1. Field Equations

Varying the action with respect to the metric yields the field equations,

$$G_{\mu\nu} = \kappa \left( T_{\mu\nu} + D_{\mu\nu} \right),$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor of the standard matter fields. All contributions due to $f(R)$ are packaged into the extra-term $D_{\mu\nu}$, which we call the stress-energy tensor of the dark sector, explicitly formulated as

$$\kappa D_{\mu\nu} \equiv \frac{1}{2} fg_{\mu\nu} - (R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f_R,$$

where $R_{\mu\nu}$ is the Ricci tensor, and $f_R \equiv \frac{df}{dR}$. A direct calculation shows that $D_{\mu\nu}$ is covariantly conserved, that is, $\nabla^\alpha D_{\mu\nu} = 0$. The background geometry is assumed to be $ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$, where $a(t)$ is the scale factor. Instead of the first and second order time derivative of the Hubble parameter, $H$, we use
the dimensionless parameters

\[ \epsilon_H \equiv -\frac{H'}{H}, \quad \bar{\epsilon}_H \equiv -\frac{R'}{6H^2}, \]

where the prime denotes derivative with respect to \( d/d\ln a \). The dark sector can be viewed as a fluid, with energy density \( \rho_{de} \equiv \frac{1}{a^2}D_{00} \) and pressure \( P_{de} \equiv \frac{1}{3a^2}\delta_{ij}D_{ij} \).

The field equations (2) read

\[ \Omega_m + \Omega_{de} = 1, \quad w_m \Omega_m + w_{de} \Omega_{de} = \frac{2}{3}(\epsilon_H - 1), \]

where \( \Omega_i = \frac{\rho_i}{3H^2} \) is the density parameter and \( w_i \equiv P_i/\rho_i \). From (3). For the \( f(R) \) fluid, they are explicitly given by

\[ \Omega_{de} = -fR + (1 - \epsilon_H)R' - f'R, \]

\[ w_{de} + 1 = -\frac{1}{3H_{de}}(2\epsilon_H fR + (1 + \epsilon_H)R' - f'R). \]

2. Dynamics of Linear Perturbations

The dynamics of linear perturbations is presented in Fourier space. Instead of the coordinate wavenumber, \( k \), a reduced dimensionless wavenumber is introduced,

\[ K \equiv \frac{k}{aH}, \]

so that \( K \) can easily recognize the sub-(super)-horizon regimes. In the synchronous gauge, the non-zero metric perturbations are \( \delta g_{ij} = a^2h_{ij} \). In an orthonormal basis \( \{k, l, m\} \) in \( k \)-space, the spatial matrix \( h_{ij} \) is further decomposed as \( h_{ij} = \frac{1}{2}h\delta_{ij} + h_{ij}\sigma_{ij} + h_{ij}^\sigma v_{ij} + h_{ij}^\tau e_{ij} \), where the notations \( h^\sigma \) and \( h^\tau \) contain the two vector and the two tensor polarization states respectively, and the dot product has to be understood as a sum over the polarization states. Instead of \( h_{ij} \), we use the combination \( 6\eta \equiv h_{ij} - \eta \). The basis matrices are \( \sigma_{ij} = k_i\bar{k}_j - \frac{1}{2}\delta_{ij} \) for the longitudinal traceless mode, \( v_{ij}^{(1)} = 2k_i\bar{\ell}_j + \frac{1}{2}(\delta_{ij} - \ell_i\ell_j) \) for the vector modes, and \( e_{ij}^{\tau} = \gamma_i\bar{\gamma}_j \) for the tensor modes. In the conformal Newtonian gauge, \( \delta g_{00} = -2a^2\psi \) and \( \delta g_{ij} = -2a^2\dot{\phi}\delta_{ij} \) (the tensor and vector modes remain the same in both gauges). An additional scalar degree of freedom arises at the perturbative level from the non-vanishing \( f_R' \), given by

\[ \chi \equiv \frac{f_R'}{\epsilon_H} \frac{\delta R}{6H^2}. \]

This feature is a manifestation of the well-known connection between \( f(R) \) theories and non-minimally coupled scalar-tensor theories. The expression of a generic

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\textsuperscript{a}With these notation the Ricci scalar reads \( R = 12H^2(1 - \frac{1}{2}\epsilon_H) \).

\textsuperscript{b}With the gauge invariant notation the perturbed Ricci scalar reads \( \delta R = -6H^2(W + 4X - \frac{3}{2}K^2(Y - 2Z) - \epsilon_H T) \). The fact that \( T \) appears explicitly in the expression of \( \delta R \) indicates that this is not a gauge invariant quantity.
perturbed stress-energy tensor, $U^\mu_\nu$, is

$$
\delta U^\mu_\nu = (\rho \delta + \delta P) u^\mu u_\nu + (\rho + P) (u_\nu \delta u^\mu + u^\mu \delta u_\nu) + \delta P \delta u^\mu u_\nu + P \Pi^\mu_\nu, \tag{9}
$$

where the density contrast is $\delta \equiv \delta \rho / \rho$, the Hubble flow is parametrized by $u_\nu = (-1, \vec{0})$ in coordinate time, and $\delta u_\nu = (0, \delta u_i)$ is the perturbed velocity field whose scalar mode is $\theta \equiv \frac{i k^2 \delta u_i}{k^2}$.

Our results are are obtained in both the synchronous and conformal Newtonian gauges thanks to a new set of variables presented below (quantities denoted with the subscript ‘S’ (‘C’) are evaluated in the synchronous (conformal Newtonian) gauges respectively).

| Symbol | Synchronous gauge | Conformal Newtonian gauge |
|--------|-------------------|---------------------------|
| $T$    | $\frac{\nu_i}{K}$ | 0                         |
| $Y$    | $T' + \epsilon_\nu T$ | $\psi$          |
| $Z$    | $\eta - T$        | $\phi$                   |
| $X$    | $Z' + Y$          | $Z' + Y$                 |
| $W$    | $X' - \epsilon_\nu (X + Y)$ | $X' - \epsilon_\nu (X + Y)$ |
| $\Theta$ | $\Theta_s + 3(1 + w) T$ | $\Theta_c$             |
| $\delta P$ | $\delta P_s + P_s' T$ | $\delta P_c$     |
| $\chi$ | $\chi_s + f'_{\nu} T$ | $\chi_c$              |

Instead of $\delta$ and $\theta$, we make an extensive use of the dimensionless variables

$$
\Delta \equiv \delta + 3(1 + w) H \theta, \quad \Theta \equiv 3(1 + w) H \theta. \tag{11}
$$

The perturbed pressure is packaged into the gauge invariant entropy perturbation,

$$
\omega \Gamma = \frac{\delta P}{\rho} \frac{dp}{d\rho} (\Delta - \Theta). \tag{12}
$$

The anisotropic stress is the spatial traceless part of the stress-energy tensor. In the same way as the metric perturbation, it decomposes into one scalar, $\Pi^S$, two vector, $\Pi^V$, and two tensor modes, $\Pi^T$. The generic perturbed fluid equation, $\delta (\nabla^\mu U_\mu) = 0$, are

$$
\Delta' - 3 w \Delta - 2 w \Pi^S + g_k \epsilon_n \hat{\Theta} = 3(1 + w) X, \tag{13a}
$$

$$
\hat{\Theta}' + \left( \epsilon_n - \frac{w'}{1 + w} \right) \hat{\Theta} - \left( 3 w - \frac{w'}{1 + w} \right) \Delta - 2 w \Pi^S - 3 w \Gamma = 3(1 + w) Y. \tag{13b}
$$

The prime denote derivative with respect to $d/d \ln a$ and $g_k \equiv 1 + \frac{\kappa^2}{a^2}$. The field equations expanded to linear order in perturbations are

$$
\frac{2}{3} K^2 Z = \Omega_m \Delta_m + \Omega_{de} \Delta_{de}, \tag{14a}
$$

$$
2 X = \Omega_m \Theta_m + \Omega_{de} \Theta_{de}, \tag{14b}
$$

$$
\frac{2}{3} W + 2 X - \frac{2}{3} K^2 (Y - Z) = \Omega_m (\delta P_m / \rho_m) + \Omega_{de} (\delta P_{de} / \rho_{de}), \tag{14c}
$$

$$
\frac{1}{3} K^2 (Y - Z) = \Omega_m w_m \Pi_m^V + \Omega_{de} w_{de} \Pi_{de}^V, \tag{14d}
$$

$$
\frac{1}{6} h^{V''} + \left( \frac{1}{2} - \frac{1}{3} \epsilon_n \right) h^{V'} = \Omega_m w_m \Pi_m^V + \Omega_{de} w_{de} \Pi_{de}^V, \tag{14e}
$$

$$
\frac{1}{6} h^{T''} + \left( \frac{1}{2} - \frac{1}{3} \epsilon_n \right) h^{T'} + \frac{1}{3} K^2 h^T = \Omega_m w_m \Pi_m^T + \Omega_{de} w_{de} \Pi_{de}^T. \tag{14f}
$$
3. Equation of State for Perturbations

In the $f(R)$ scenario, the expansion to first order in perturbations of the dark sector stress-energy tensor is

$$\kappa \delta D_{\mu \nu} = -f_{R} \delta R_{\mu \nu} + \frac{1}{2} f \delta g_{\mu \nu} + \frac{1}{2} g_{\mu \nu} f R \delta R - f R_{\mu \nu} R_{\rho \sigma} \delta R + \delta (\nabla_{\mu} \nabla_{\nu} f R) - (\Box f R) \delta g_{\mu \nu} - g_{\mu \nu} \delta (\Box f R).$$  \hspace{1cm} (15)

This allows us to isolate the perturbed fluid variables for the $f(R)$ dark sector theory. The tensor and vector projections of (15) readily constitute the EoS for $\Pi_{de}$ and $\Pi_{T}$,

$$\Omega_{de} w_{de} \Pi_{de}^{V} = -\frac{1}{2} f R h^{\nu \nu} - \frac{1}{2} \left\{ (3 - \epsilon_{n}) f R + f'_{R} \right\} h^{\nu' \nu'},$$  \hspace{1cm} (16a)

$$\Omega_{de} w_{de} \Pi_{de}^{T} = -\frac{1}{2} R f R h^{\tau \tau} - \frac{1}{2} \left\{ (3 - \epsilon_{n}) f R + f'_{R} \right\} h^{\eta \eta'} - \frac{1}{2} f R K^{2} h^{\tau \tau}. $$  \hspace{1cm} (16b)

The time-time projection provides a useful formulae that enables to write $\hat{\chi}$ in terms of $\Delta_{de}$, $X$ and $Z$,

$$\Omega_{de} \Delta_{de} = -g_{KL} \epsilon_{n} \hat{\chi} + f'_{R} (X + \frac{1}{3} K^{2} Z). $$  \hspace{1cm} (17)

The longitudinal spatial traceless projection is

$$\Omega_{de} w_{de} \Pi_{de}^{S} = -\frac{1}{2} K^{2} \hat{\chi} - \frac{1}{2} f R K^{2} (Y - Z). $$  \hspace{1cm} (18)

Using (18) and the perturbed field equations (14), it is possible to write $\Gamma_{de}$ and $\Pi_{de}^{S}$ in terms of the other perturbed fluid variables:

$$w_{de} \Gamma_{de} = (\hat{\zeta}_{de} - \frac{2}{3} \frac{\epsilon_{n}}{g_{R} \epsilon_{n}} \frac{1}{f R}) \Delta_{de} - \hat{\zeta}_{de} \hat{\Theta}_{de} + \frac{2}{3} \frac{\Omega_{de}}{\Omega_{de}} \hat{\zeta}_{de} \left( \Delta_{de} - \hat{\Theta}_{de} \right) $$  \hspace{1cm} (19a)

$$w_{de} \Pi_{de}^{S} = \frac{2}{3} \frac{\epsilon_{n}}{g_{R} \epsilon_{n}} \frac{1}{f R} \left\{ (1 + \frac{1}{2} f'_{R}) \Delta_{de} - \frac{1}{2} f'_{R} \hat{\Theta}_{de} + \frac{1}{2} \frac{\Omega_{de}}{\Omega_{de}} f'_{R} (\Delta_{de} - \hat{\Theta}_{de}) \right\} $$  \hspace{1cm} (19b)

where $\hat{\zeta} \equiv \frac{2(5 - \epsilon_{n})}{3} \frac{\epsilon_{n}}{g_{R} \epsilon_{n}} - \frac{4 f f'_{R}}{f R}$. These expressions constitute the EoS for perturbations in $f(R)$ gravity. From here, the whole dynamics of the (scalar) perturbations is illustrated the EoS approach with $f(R)$ gravity, but it should apply to any modified gravity theory. The EoS approach can be an alternative (or complementary) to the “parametrized post Friedmann framework” developped by Ferreira-Baker-Skordis. It enables to simply characterize intricate modified gravity theories through their equations of state for perturbations. In the quest for understanding the nature of dark energy, this approach seems to us particularly attractive as it links formal modification to General Relativity to phenomenology in a straightforward way.

\[^{\ddagger}Z' = X - Y\text{, where } X \text{ and } Y \text{ are written in terms of the perturbed fluid variables.}\]

\[^{\ddagger\ddagger}\text{In fact, the system of five differential equations is overdetermined: when } K = 0, \text{ the field equation (19a) can be used to express } \Delta_{de} \text{ in terms of } \Delta_{m} \text{; and when } K \neq 0, \text{ the same equation gives } Z \text{ in terms of the } \Delta_{i}'s\text{.}\]
References

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