Calculation of ensquared energy of the diffraction-limited optical system with Higher-order parabolic filter

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Abstract. Mathematical properties of the ensquared energy functions for apodized point-spread function (PSF) are presented. An expression of ensquared energy for the apodized point spread function of the optical system with a circular aperture was derived using a parabolic apodized filter with a different arrangement N = 1, 2, 3, 4. The results obtained were discussed graphically.

Keywords: ensquared energy, point spread function, parabolic filter

1. Introduction

Image performance of optical systems is specified by optical transfer function, wavefront error, Strehl ratio, point spread function, and encircled energy. Encircled energy measures the relative amount of intensity in the diffraction PSF contained within a circle of specific size centered on the PSF centroid [1]. or we can say The encircled energy of the resulting image gives the distribution of energy in that PSF. With the advent of detectors with square pixels, ensquared energy is being used as an alternative to encircled energy. Flux in a square of the specific radius (ensquare energy) is one of the significant parameters in quantifying the performance of optical without aberrations.

Lord Rayleigh [2] first pointed out the importance of the studies on the ensquared energy in the diffraction pattern as an image quality assessment parameter. [3]. It can be considered as one of the significant parameters that can serve as an index of the performance of an optical system [4-5]. We will designate this important parameter by the symbol EE(Z)p. Although the encircled energy is a good test for the quality of an optical system, it has not been used much in the early years because of the difficulties in its calculations. To overcome this difficulty, the calculating of ensquared energy has been an important part of the system requirements for various imaging instruments [7-8]. In this paper, we present mathematical derivative for ensquared energy functions for apodized PSF that facilitate the accurate calculation of these functions, the apodized function used is a parabolic mask to achieve a desired distribution of illuminance over a given optical plane in the image field

2. Related work:

We will present a brief review of the previous works done on the EE(Z)p by various authors in what follows now. WELFORD [9] has studied the focal depth with annual apertures and discussed the encircled energy. TSCHUNKO [10] has shown that there will be increases in the resolution with the obscured apertures. STAMNES, HEIER, and LJUNGGREN [11] have calculated the encircled energy for large no. of aberration-free annular aperture systems using HOPKIN'S algorithm [12]. They have also
assessed the focal shift tolerances. TSCHUNKO [10] derived the total energy function and determined the system's partial energy integrals with an annular aperture apodized with different types of apodizing functions. BISWAS and BOIVIN [13] derived a general formula to study the influence of wave aberrations on optimum apodizers' performance, particularly in the encircled energy values. In their study, they have used Straubel class and Lansraux-Boivin apodisers. DEVARAYALU, RAO, and MONDAL [14] discussed the possibility of identifying super-resolving and apodizing properties of an optical system having shaded circular apertures from encircled energy considerations. VISWANATHAM, RAO, and MONDAL [15] have studied the fraction of the encircled energy about the dispersion factor in the diffraction pattern of circular apertures. LUNEBERG [16] proposed four apodization problems and suggested a method to solve them by using the calculus of variations. The third Lunenburg problem is to find the optimum pupil function which gives maximum energy in a given area in the image field at the receiving plane. LANSRAUX and BOIVIN [6], BARAKAT [17] also investigated this problem based on the calculation of variations. UENO and ASAKURA [18] have solved this problem of maximum encircled energy for an aberration-free, rotationally symmetric optical system apodized with a specified overall transmittance. CLEMNTS and WILLIKINS [19] investigated the problem of finding the diffraction pattern and corresponding pupil function having the maximum possible encircled energy ratio for an arbitrarily fixed radius and a fixed Raleigh limit of resolution. MONDAL [20] has derived an expression for the encircled energy within the circle of a specified radius in the Fraunhofer diffraction pattern due to an elliptical aperture. When a converging monochromatic spherical wave is diffracted at a circular aperture, the classical theory predicts that the focal region's intensity distribution will be symmetrical about the focal plane. But subsequent studies show that the diffraction pattern's principal maximum may not be at the geometrical focuses and moves towards the aperture depending upon the Fresnel number (N) of the system. This effect is referred to as focal shift. BARAKAT [17] has studied the variation of encircled energy with Fresnel number N of the system experimentally. MAHAJAN [21] has discussed the encircled energy of systems with non-centrally obscured apertures and has shown that non-central obscuration yields an equal or higher encircled energy value than with central obscuration.

VENKAT REDDY, PRASAD, and MONDAL [22] have computed optical systems' encircled energy with non-centrally apodized pupils.

3. Mathematical formula for ensquared energy EE(Z)_p:

Although the point spread function is one of the important means in evaluating optical systems' performance, it loses its relevance and properties when aberrations increase and hence cannot be relied upon in determining the efficiency of the system. Therefore, another criterion depends on the point propagation function but loses its general shape when the deviations increase. This parameter is the enclosed energy which is defined as part of the overall system image energy. To find the relationship of the closed energy function, we perform numerical integration for the point spread function equation:

$$ EE(Z)_p = N \cdot F \int_{-m}^{m} \int_{-z}^{z} \text{PSF} \; dz \; dm $$  \hspace{1cm} (1) \\
$$ \text{PSF} = |A(z, m)|^2 $$  \hspace{1cm} (2) \\
$$ EE(Z)_p = N \cdot F \int_{-m}^{m} \int_{-z}^{z} \int \int f(x, y, f^*(x1, y1)e^{i(2\pi+my)}e^{-i(2\pi+my)} \; dx \; dy \; dz \; dm $$  \hspace{1cm} (3)
Where \( f^*(x_1, y_1) \) represents the complex conjugate of the pupil function, and each of \((x_1, y_1)\) represents its coordinates so that we can say that:

\[
\text{Where: } f^*(x_1, y_1) = \tau(x_1, y_1)e^{ikw(x_1, y_1)}
\]

\( \tau(x_1, y_1) \): It represents the true amplitude distribution and is often set equal to one unit.

\( e^{ikw(x_1, y_1)} \): The function represents the aberration of the wavefront.

\( (x_1, y_1) \): It is the coordinates of the output pupil.

\( w(x_1, y_1) \): Represent the coefficient of aberration.

Simplifying equation 3 we get:

\[
\text{EE}(Z)_p = N \cdot F \iint f(x, y) \cdot f^*(x_1, y_1)dx_1dy_1 \iint_{-\infty}^{\infty} e^{iz(x-x_1)} \frac{\sin z(x-x_1)}{(x-x_1)} \cdot e^{im(y-y_1)}dm
\]

By Using: \( \sin \phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} \)

And

\[
\left( \int_{-\infty}^{\infty} e^{iz(x-x_1)} \frac{\sin z(x-x_1)}{(x-x_1)} \right) \cdot \left( \int_{-\infty}^{\infty} e^{im(y-y_1)}dm \right)
\]

Getting:

\[
\text{EE}(Z)_p = N \cdot F \iint f(x, y) \cdot f^*(x_1, y_1) \frac{\sin z(x-x_1)}{(x-x_1)} \cdot \frac{\sin m(y-y_1)}{(y-y_1)} dx_1dy_1dy
\]

The normalizing factor \( N \cdot F = \frac{1}{\pi^3} \)

the N.F Normalizing factor in equation 6 makes the value of the total luminance function equal to one when the total energy of the point spread function PSF is contained when \( R\rightarrow\infty \) and when assuming a perfect optical system \( W(x, y)=0 \) and by \( R=z=m \) by substituting the value of the normalizing factor \( N \), the relationship 6 after substitute the apodized filter becomes as follows:

\[
\text{EE}(Z)_p = \frac{1}{\pi^3} \int_{-1}^{1} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[ \alpha + \beta(x^2 + y^2)^N \right] \frac{\sin z(x-x_1)}{(x-x_1)} \cdot \frac{\sin m(y-y_1)}{(y-y_1)} dx_1dy_1dy
\]

(7)

Where:

\[ [\alpha + \beta(x^2 + y^2)^N] \]: Is the parabolic filter.

\( \beta \): The apodization parameter is known and indicates the degree of non-uniformity of pupil transmission.

\( \alpha \): is a numerical constant less than one.

\( N=1,2,3,4 \)
4. Results and Discussions:
The ensquared energy for increasing values of δ starting from z=0.0 to 9 has been computed using the expression (7) the computed results have been presented graphically in figures. Figures (1.1, 1.2, 1.3, 1.4) represent the ensquared energy curves in the point spread function of the finite diffraction system of the circular aperture using a parabola filter of different classes (N = 1,2,3,4). Math CAD software was used to simulate the ensquared energy equation. We have shown the variation of ensquared Energy EE(Z)p with specified values of radii Z for a smaller value of β = 0.3 an intermediate value of β = 0.6 and a higher value of β = 0.9 for α = 0,0.25,0.50 & 0.75. It is observed from the figures that for a particular value of β the effects of increasing the α values are to increase the bias or the average value of the EE(Z)p curves, as it should be expected. So far as the effects of β, for a particular value of α are concerned, increasing the value of β increases the individual values of EE(Z)p. Thus, for the relative EE(Z)p the effects of both α and β are similar. Note that the ensquared energy decreases with increasing the N and increasing the distance. From these figures we indicated that the ensquared energy function is the flow function compared to the point spread function is an oscillatory function. figure (5) indicate that the best result is for the value of β =0.9

**Fig:1.1** First order: Variation of ensquared Energy EE(Z)p with β for α=0,0.25,0.50,0.75
Fig. 1.2 Second order: Variation of ensquared Energy $EE(Z)_p$ with $\beta$ for $\alpha=0, 0.25, 0.50, 0.75$

Fig. 1.3 Third order: Variation of ensquared Energy $EE(Z)_p$ with $\beta$ for $\alpha=0, 0.25, 0.50, 0.75$
**Fig: 1.4** Fourth order: Variation of ensquared Energy $EE(Z)_{p}$ with $\beta$ for $\alpha=0, 0.25, 0.50, 0.75$

**Fig: 5** Variation of ensquared Energy $EE(Z)_{p}$ with $\beta$

**Conclusions:** The ensquared energy, was calculated for different values of $\beta$ and $\alpha$ with different orders which can be used to analyze the accuracy of the optical system. Studies on ensquared energy and its corollaries reveal that in most of the cases, It is observed from the figures that for a given percentage of light flux within the diffraction pattern 50% of the value of $z$ increases gradually with $\beta$. we notice that the value of encircled energy $EE(Z)_{p}$ decreasing with increasing $(N)$, while increment ensquared energy $EE(Z)_{p}$ with increment the distance $(Z)$. 
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