Robust parameter design with covariates of multiple noise factors

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Abstract:

The intent of robust parameter design is to make a system insensitive to noise factors by choosing optimum levels for the controllable factors. This is conducted by finding the interactions between noise factors and control factors. In general, it is necessary to control the levels of noise factors; however, there are some that cannot be controlled. On the other hand, there are some which can be observed as covariates.

Hirano and Miyakawa (2007) proposed a method based on linear regression to analyze the interactions between a single noise factor and the control factors, when the noise factor is observed as a covariate. This paper discusses robust parameter designs when multiple noise factors are observed as covariates. We propose two extensions of the Hirano and Miyakawa method: extended methods 1 and 2. We perform Monte Carlo simulations under several data models to estimate the accuracy of these methods. It is shown that the extended methods are able to analyze the interactions between each of the control factors and the noise factors, when multiple noise factors are observed as covariates. Since the noise factors are combined in extended method 1, it cannot detect which noise factor influences the control factors. On the other hand, extended method 2 distinguishes between noise factors, and thus it can detect the interactions between the control factors and each of the noise factors.

Keywords
Robust parameter design, Interaction analysis, Linear regression model, Powers

1. Introduction

The intent of robust parameter design is to make a system insensitive to noise factors by choosing optimum levels for the controllable factors (see Miyakawa, 2000; Taguchi, 1976, 1977, 1988, 1993; Tatebayashi, 2004). This is conducted by finding the interactions between the noise factors and the control factors. In robust parameter designs, a product array experiment is often used. In this method, the control factors are assigned inside of a suitable orthogonal array, and the noise factors are observed outside of that orthogonal array. Thus, any interaction between the noise factors and the control factors can be estimated. However, Joseph (2003) suggested that there exist noise factors for which the level cannot be controlled in an experiment. On the other hand, there are noise factors which can be observed as covariates. For situations in which there is more than a single noise factor, Taguchi (1976, 1977) proposed a method of compounding the noise factors and considering their overall impact. However, this method is not available for noise factors that are not controllable, such as those observed as covariates.

Hirano and Miyakawa (2007) proposed a method based on linear regression to analyze the interactions between a single noise factor that is observed as a covariate and the control factors. This paper discusses robust parameter designs for situations in which multiple noise factors are observed as covariates. We propose two extensions of the Hirano and Miyakawa method: extended method 1, in which the noise factors are combined, and extended method 2, which distinguishes between the noise factors. We performed Monte Carlo simulations under several data models to estimate the accuracy of these methods. We show that extended method 1 cannot determine which noise factor influences the control factors. Furthermore, if noise factors which have no effect are included, then the influence of the existing interactions is weakened, and they might not be detected accurately. On the other hand, extended method 2 can detect the interaction of each individual noise factor. We
also investigated the effect of the order in which the testing procedures are performed.

This paper consists of five sections. In Section 2, we explain the Hirano and Miyakawa method. In Section 3, we propose two extensions to the Hirano and Miyakawa method. In Section 4, we report the results of Monte Carlo simulations, compare the accuracies of the two extended methods, and investigate the effect of the order of the testing procedures. Finally, in Section 5, we present our conclusions.

2. Hirano and Miyakawa method

In this section, we briefly summarize the method proposed by Hirano and Miyakawa (2007). They assumed that there is a single noise factor and that it is observed as a covariate. Let us suppose that there are four control factors and that each has two levels. These factors are assigned inside of an $L_8$ orthogonal array. In addition, one noise factor is observed outside of the same $L_8$ orthogonal array. Experiments are repeated $r$ times for each combination of the levels of the control factors. Therefore, we obtain an $N = 8 \times r$ table of the quality characteristics, as shown in Table 1.

Table 1: When a Noise Factor is Observed Outside of the $L_8$ Orthogonal Array

| No. | A  | B  | C  | D  | Noise Factors and Quality Characteristics $(z_{ij}, y_{ij})$ |
|-----|----|----|----|----|--------------------------------------------------|
| 1   | 1  | 1  | 1  | 1  | $(z_{11}, y_{11})$, $(z_{12}, y_{12})$, $\ldots$, $(z_{1r}, y_{1r})$ |
| 2   | 1  | 1  | 2  | 2  | $(z_{21}, y_{21})$, $(z_{22}, y_{22})$, $\ldots$, $(z_{2r}, y_{2r})$ |
| 3   | 1  | 2  | 1  | 2  | $\ldots$, $\ldots$, $\ldots$ |
| 4   | 1  | 2  | 2  | 1  | $\ldots$, $\ldots$, $\ldots$ |
| 5   | 2  | 1  | 1  | 2  | $\ldots$, $\ldots$, $\ldots$ |
| 6   | 2  | 1  | 2  | 1  | $\ldots$, $\ldots$, $\ldots$ |
| 7   | 2  | 2  | 1  | 1  | $\ldots$, $\ldots$, $\ldots$ |
| 8   | 2  | 2  | 2  | 2  | $(z_{81}, y_{81})$, $(z_{82}, y_{82})$, $\ldots$, $(z_{8r}, y_{8r})$ |

We assume that the interaction effects of the noise factor on the quality characteristics are linear. Thus, each quality characteristic $y_{ij}$ is described by a linear model, as follows:

$$y_{ij} = \mu_i + \beta_i z_{ij} + \epsilon_{ij} \quad (i = 1,2,\ldots,8; j = 1,2,\ldots,r),$$

where $\beta_i$ is the regression coefficient which represents the effect of the noise factor $z_{ij}$ on the quality characteristics, and the $\epsilon_{ij}$'s are errors which are independently distributed as $N(0, \sigma^2)$. It is not assumed that the control factors have additive effects on the regression coefficients.

For an example, we will illustrate a test of the interaction between control factor $A$ and the noise factor, based on Table 1. We will consider the following hypotheses:

Null hypothesis, $H_0$: There is no interaction effect between control factor $A$ and the noise factor:

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8;$$

(2)

Alternative hypothesis, $H_1$: There is an interaction effect between control factor $A$ and the noise factor:

$$\beta_1 = \beta_2 = \beta_3 = \beta_4, \quad \beta_5 = \beta_6 = \beta_7 = \beta_8 .$$

(3)

This test corresponds to the test for factorial effects in multiple regression analysis. Therefore, we can use the test statistic $F$, as follows:
where \( \text{RSS}_0 \) is the residual sum of squares under the null hypothesis \( H_0 \), and \( \text{RSS}_1 \) is the residual sum of squares under the alternative hypothesis (see Ryan, 1997). Under the null hypothesis, the \( F \) statistic \((4)\) is distributed according to the \( F \) distribution with \((\phi_{c(t_{H_0})} - \phi_{c(t_{H_1})}, \phi_{c(t_{H_1})})\) degrees of freedom. For example, let \( r \) be 4, which implies \( N = 32 \). Then, under the null hypothesis \((2)\), \( \phi_{c(t_{H_0})} = N - (\text{number of intercepts}) - (\text{number of coefficients}) = 32 - 8 - 1 = 23 \). On the other hand, under the alternative hypothesis \((3)\), \( \phi_{c(t_{H_1})} = N - (\text{number of intercepts}) - (\text{number of coefficients}) = 32 - 8 - 2 = 22 \). Thus, \( (\phi_{c(t_{H_0})} - \phi_{c(t_{H_1})}, \phi_{c(t_{H_1})}) = (22,20) \). If the \( P \) value of the \( F \) statistic is less than or equal to 0.05, we conclude that this is significant, and thus we conclude that control factor \( A \) has an effect on the regression coefficients (the \( \beta_i \)'s). This means the interaction between control factor \( A \) and the noise factor exists. The other control factors \((B, C, \text{ and } D)\) are each tested in the same way.

Next, we describe a two-stage testing procedure; this was not described explicitly by Hirano and Miyakawa (2007). This procedure is based on the stepwise forward selection method of multiple regression analysis (see, for example, Nagata and MunECHIKA, 2001).

First stage: Using the \( F \) statistic \((4)\) and the corresponding \( P \) value, we test each of the control factors individually. If all factors are non-significant, then the testing procedure ends. If at least one \( P \) value indicates significance, we proceed to the second stage.

Second stage: Suppose that the minimum significant \( P \) value in the first stage corresponds to control factor \( A \). In this case, we test for an effect due to the interaction between the other control factors and the noise factor, under the assumption that such an effect exists between control factor \( A \) and the noise factor. We will illustrate a test of the interaction effect between the control factor \( B \) and the noise factor, as follows. We will consider the following hypotheses:

Null hypothesis, \( H_0 \): There is no effect of the control factor \( B \) on the regression coefficients \((\beta_i \)'s) under the assumption that the interaction effect of control factor \( A \) exists:

\[
\beta_1 = \beta_2 = \beta_3 = \beta_4, \quad \beta_5 = \beta_6 = \beta_7 = \beta_8; \quad (5)
\]

Alternative hypothesis, \( H_1 \): There exists an effect of the control factor \( B \) on the regression coefficients \((\beta_i \)'s) under the assumption that the interaction effect of control factor \( A \) exists:

\[
\beta_1 = \beta_2, \quad \beta_3 = \beta_4, \quad \beta_5 = \beta_6, \quad \beta_7 = \beta_8. \quad (6)
\]

The test statistic is \( F \) \((4)\); when \( r \) is 4, the degrees of freedom are \((\phi_{c(t_{H_0})} - \phi_{c(t_{H_1})}, \phi_{c(t_{H_1})}) = (22,20) \). Note that the degrees of freedom of the test statistics for the first and second stages are different. The other control factors \((C \text{ and } D)\) are also tested in the same way. In this stage, all of the remaining factors \((B, C, \text{ or } D)\) are tested for significance.

When there are only three factors \((A, B, \text{ and } C)\) to be tested, we can consider a third stage. If both \( B \) and \( C \) are found to be non-significant in the second stage, then the testing procedure ends. If the \( P \) value of either \( B \) or \( C \) indicates significance, we move to a third stage.

Third stage: Let us assume that control factor \( A \) was found to be significant in the first stage, and \( B \) was found to be significant in the second stage. We will then conduct a test for an interaction effect between the remaining control factor \( C \) and the noise factor, under the assumption that interaction effects of the control factors exist for both \( A \) and \( B \). We will consider the following hypotheses:

Null hypothesis, \( H_0 \): There is no effect of the control factor \( C \) on the regression coefficients \((\beta_i \)'s), under the assumption that interaction effects exist for control factors \( A \) and \( B \):

\[
\beta_1 = \beta_2, \quad \beta_3 = \beta_4, \quad \beta_5 = \beta_6, \quad \beta_7 = \beta_8; \quad (7)
\]
Alternative hypothesis, \( H_1 \): There exists an effect of the control factor \( C \) on the regression coefficients \( (\beta_i)'s \), under the assumption that interaction effects exist for control factors \( A \) and \( B \):

Equation (7) does not hold. (8)

The test statistic is \( F(4) \); when \( r \) is 4, the degrees of freedom are \( (\phi_{e(1)} - \phi_{e(2)}, \phi_{e(2)}) = (4,16) \).

When only three factors are assigned in \( L_8 \), the experiment is equivalent to a three-way factorial design and a third stage is available. However, when more than three factors are assigned in \( L_8 \), the third stage cannot be conducted, and the significance of the remaining factors must be determined in the second stage. Note that if \( L_{16} \) can be used and the design has resolution \( V \), then the third stage is always available, even if there are more than three factors.

3. Extensions of the Hirano and Miyakawa method

In this section, we discuss the interaction between the noise factors and the control factors, when multiple noise factors are observed as covariates. We propose two extensions to the Hirano and Miyakawa method: extended method 1, which is a simple extension of the Hirano and Miyakawa method, and extended method 2, in which the noise factors can be distinguished.

3.1 Extended method 1

Extended method 1 is defined by combining the noise factors as a vector. For example, we suppose four control factors, each with two levels. These factors are assigned inside of an \( L_8 \) orthogonal array. In addition, \( q \) noise factors are observed outside of the same \( L_8 \) orthogonal array. Experiments are repeated \( r \) times for each combination of the levels of the control factors. Thus, we obtain a \( 8 \times r \) table of the quality characteristics, as shown in Table 2. Here, the multiple noise factors are represented by the vector \( z_{ij} = (z_{ij1}, \cdots, z_{ijq})' \) \( (i = 1,2, \cdots, 8; j = 1,2, \cdots, r) \).

Table 2: When Multiple Noise Factors are Observed Outside of the \( L_8 \) Orthogonal Array

| No. | A | B | C | D | Noise Factors and Quality Characteristics \( (z_{ij}, y_{ij}) \) |
|-----|---|---|---|---|------------------------------------------------------------|
| 1   | 1 | 1 | 1 | 1 | \( (z_{11}, y_{11}) \) \( (z_{12}, y_{12}) \) \( \cdots \) \( (z_{1r}, y_{1r}) \) |
| 2   | 1 | 1 | 2 | 2 | \( (z_{21}, y_{21}) \) \( (z_{22}, y_{22}) \) \( \cdots \) \( (z_{2r}, y_{2r}) \) |
| 3   | 1 | 2 | 1 | 2 | \( \vdots \) \( \vdots \) \( \cdots \) \( \vdots \) |
| 4   | 1 | 2 | 2 | 1 | \( \vdots \) \( \vdots \) \( \cdots \) \( \vdots \) |
| 5   | 2 | 1 | 1 | 2 | \( \vdots \) \( \vdots \) \( \cdots \) \( \vdots \) |
| 6   | 2 | 1 | 2 | 1 | \( \vdots \) \( \vdots \) \( \cdots \) \( \vdots \) |
| 7   | 2 | 2 | 1 | 1 | \( \vdots \) \( \vdots \) \( \cdots \) \( \vdots \) |
| 8   | 2 | 2 | 2 | 2 | \( (z_{81}, y_{81}) \) \( (z_{82}, y_{82}) \) \( \cdots \) \( (z_{8r}, y_{8r}) \) |

The interaction effects of the noise factors on the quality characteristics are assumed to be linear. Thus, the quality characteristic \( y_{ij} \) is described as a linear combination model, as follows:

\[
y_{ij} = \mu_i + \beta_{i1}z_{ij1} + \beta_{i2}z_{ij2} + \cdots + \beta_{iq}z_{ijq} + \epsilon_{ij}
\]

\[
= \mu_i + \beta_i'z_{ij} + \epsilon_{ij}
\]

\( (i = 1,2, \cdots, 8; j = 1,2, \cdots, r) \),

where \( \beta_i = (\beta_{i1}, \beta_{i2}, \cdots, \beta_{iq})' \), \( \beta_{ik} \) is the regression coefficient which represents the effect of the noise factor \( z_k \) on the quality characteristics, and the \( \epsilon_{ij} \)'s are the errors, which are independently distributed as \( N(0, \sigma^2) \).

We now illustrate a test of the interaction between control factor \( A \) and the noise factors. We will consider the following hypotheses:

Null hypothesis, \( H_0 \): There is no interaction effect between the control factor \( A \) and the noise factors:
Alternative hypothesis, $H_1$: There is an interaction effect between control factor $A$ and the noise factors:

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8.$$  

(11)

The test statistic is calculated under the null hypothesis that there is no interaction between the other noise factors and the other control factors. The test statistic is $F(4)$, which is the same as in the original Hirano and Miyakawa method. The other control factors ($B, C,$ and $D$) are also tested in the same way, and the two-stage testing procedure is conducted as described in Section 2.

### 3.2 Extended method 2

In extended method 2, the noise factors are distinguishable, and this method uses the definitions in Table 2. The interaction effects of the noise factors which are observed as covariates on the quality characteristics are assumed to be linear. The quality characteristic $y_{ij}$ is described as a linear model, as follows:

$$y_{ij} = \mu_i + \beta_{1i}z_{i1} + \beta_{12}z_{i2} + \cdots + \beta_{iq}z_{iq} + \epsilon_{ij} \quad (i = 1, 2, \cdots, 8; j = 1, 2, \cdots, r),$$  

(12)

where $\beta_{ik}$ is the regression coefficient that represents the effect of the noise factor $z_k$ on the quality characteristics, and the $\epsilon_{ij}$'s are the errors, which are independently distributed as $N(0, \sigma^2)$.

Extended method 1 does not distinguish between the $q$ noise factors, whereas extended method 2 tests the effects of the interactions between each control factor and each noise factor.

We now illustrate a test of the interaction between control factor $A$ and the first noise factor $z_1$. We will consider the following hypotheses:

Null hypothesis, $H_0$: There is no interaction effect between the control factor $A$ and the first noise factor $z_1$:

$$\beta_{11} = \beta_{21} = \beta_{31} = \beta_{41} = \beta_{51} = \beta_{61} = \beta_{71} = \beta_{81};$$  

(13)

Alternative hypothesis, $H_1$: There exists an interaction effect between the control factor $A$ and the first noise factor $z_1$:

$$\beta_{11} = \beta_{21} = \beta_{31} = \beta_{41}, \quad \beta_{51} = \beta_{61} = \beta_{71} = \beta_{81}.$$  

(14)

The test statistic is calculated under the null hypothesis that there is no interaction between the other noise factors and the other control factors. The test statistic is $F(4)$, which is the same as in the original Hirano and Miyakawa method.

The other control factors ($B, C,$ and $D$) and the noise factors ($z_1, z_2, \cdots, z_q$) are also tested in the same way.

We now explain the sequential testing procedure used in extended method 2. This procedure is based on the two-stage testing procedure presented in Section 2.

To illustrate this, let us consider four control factors $A, B, C,$ and $D$, and two noise factors $z_1$ and $z_2$. We will define some notation. Set $x_1 = A, x_2 = B, x_3 = C,$ and $x_4 = D$. Calculate the $F$ statistic (4) and its $P$ value for the control factor $x_1$ and the first noise factor $z_1$, which we will denote as $P(1)(x_1, z_1)$, as shown in Table 3. Similarly, calculate the $F$ statistic and its $P$ value for the control factor $x_1$ and noise factor $z_j$, which we will denote as $P(1)(x_1, z_j)$. These $P$ values are the same statistics as those calculated in the first stage in Section 2. Next, we prepare the $P$ value $P(1)(x_1, z_j)$, which is based on the $F$ statistic for the control factor $x_1$ and the noise factor $z_j$, as in the second stage in Section 2. These values are shown in Table 4.

The algorithm for the sequential testing procedure is as follows.
1. Find the minimum \( P^{(1)}(x_i, z_j) \) in Table 3. If this value is non-significant, the testing procedure ends. Suppose \( P^{(1)}(x_i, z_j) \) is the minimum value. If it is significant, then go to 2.

2. Find the minimum value among the set \{ \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \) \}. Note that first three terms are elements of Table 4, and the remaining four terms are elements of Table 3. Since the \( P \) values in algorithm 2 are calculated under the condition that \( P^{(1)}(x_i, z_j) \) is significant, the values of \{ \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \) \} in algorithm 1 are different from the values of \{ \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \), \( P^{(2)}(x_i, z_j) \) \} in algorithm 2. If this minimum value is non-significant, the testing procedure ends. If one of these \( P \) values is significant, go to 2.

3. Find the minimum value among the set \{ \( P^{(1)}(x_i, z_j) \), \( P^{(1)}(x_i, z_j) \), \( P^{(1)}(x_i, z_j) \), \( P^{(1)}(x_i, z_j) \) \}. If this value is non-significant, the testing procedure ends. If one of these \( P \) values, say, \( P^{(1)}(x_i, z_j) \) is minimum and significant, then go to 4.

4. Judge the significance of \( (x_2, z_j) \), \( (x_3, z_j) \), and \( (x_4, z_j) \) by computing the \( P \) values \( P^{(2)}(x_2, z_j) \), \( P^{(2)}(x_3, z_j) \), and \( P^{(2)}(x_4, z_j) \), respectively.

| \( x_i = A \) | \( P^{(1)}(x_i, z_j) \) | \( P^{(1)}(x_i, z_j) \) |
| \( x_i = B \) | \( P^{(1)}(x_i, z_j) \) | \( P^{(1)}(x_i, z_j) \) |
| \( x_i = C \) | \( P^{(1)}(x_i, z_j) \) | \( P^{(1)}(x_i, z_j) \) |
| \( x_i = D \) | \( P^{(1)}(x_i, z_j) \) | \( P^{(1)}(x_i, z_j) \) |

Table 3: \( P \) Values in the First Stage

| \( x_i = A \) | \( P^{(2)}(x_i, z_j) \) | \( P^{(2)}(x_i, z_j) \) |
| \( x_i = B \) | \( P^{(2)}(x_i, z_j) \) | \( P^{(2)}(x_i, z_j) \) |
| \( x_i = C \) | \( P^{(2)}(x_i, z_j) \) | \( P^{(2)}(x_i, z_j) \) |
| \( x_i = D \) | \( P^{(2)}(x_i, z_j) \) | \( P^{(2)}(x_i, z_j) \) |

Table 4: \( P \) Values in the Second Stage

4. Simulation to Investigate the Properties of the Proposed Methods

4.1 Models and Settings

In the simulations, for each method, we investigated whether the interactions between the noise factors and the control factors could be detected when multiple noise factors were observed as covariates. The model was assumed to be

\[
y_{ij} = \mu(x_i, \ldots, x_p) + \sum_{n=1}^{q} w_n z_{ijn} + \sum_{m=1}^{r} \gamma_{mn} x \cdot z_{ijn} + \epsilon_{ij} = \mu(x) + w' z_i + x' C z_i + \epsilon_{ij}
\]

where \( x = (x_1, \ldots, x_p)' \) is the vector of \( p \) control factors; \( z_{ij} = (z_{ij1}, \ldots, z_{ijq})' \) is the vector of \( q \) noise factors; \( \mu(x) \) is a function of \( x \) and represents the main response of the control factors; \( w = (w_1, \ldots, w_q)' \) is a vector of the main effects of the noise factors; \( C = (\gamma_{mn}) \) is a \( p \times q \) matrix which represents the interaction effects of the noise factors and the control factors; and the \( \epsilon_{ij} \)’s are independently distributed as \( N(0, \sigma^2) \).

We defined the first set of simulation conditions as follows:

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Simulation 1

- Control factors: \( x = (x_1, x_2, x_3, x_4)' \)
- Levels of the control factors: 2 levels (1, 2)
- Function which represents the response of the control factors: \( \mu_i(x) = 0 \) (without loss of generality)
- Main effects of the noise factors: \( w = 0 \) (without loss of generality)
- Error variance: \( \sigma^2 = 0.5^2 \)
- Noise factors: \( z_{ij} = (z_{y1}, \cdots, z_{yk})', z_{yk} \sim N(0,1^2) \)
- Number of replications \( r \) of the experiments: 6
- Number of replications of the simulations: 5000

We set the number of noise factors 6 to be larger than the number of control factors 4, because we wanted to investigate whether the power of extended method 1 is weakened when the number of the noise factors is increased.

The control factors and the noise factors were assigned in an \( L_8 \) orthogonal array, as shown in Table 5.

Table 5: Assignment in an \( L_8 \) Orthogonal Array

| No. | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | Noise Factors and Quality Characteristics (\( z_{ij}, y_{ij} \)) |
|-----|----------|----------|----------|----------|-------------------------------------------------|
| 1   | 1        | 1        | 1        | 1        | \((z_{11}, y_{11}), (z_{12}, y_{12}), \ldots, (z_{1r}, y_{1r})\) |
| 2   | 1        | 1        | 2        | 2        | \((z_{21}, y_{12}), (z_{22}, y_{22}), \ldots, (z_{2r}, y_{2r})\) |
| 3   | 1        | 2        | 1        | 2        | \ldots |
| 4   | 1        | 2        | 2        | 1        | \ldots |
| 5   | 2        | 1        | 1        | 2        | \ldots |
| 6   | 2        | 1        | 2        | 1        | \ldots |
| 7   | 2        | 2        | 1        | 1        | \ldots |
| 8   | 2        | 2        | 2        | 2        | \((z_{81}, y_{81}), (z_{82}, y_{82}), \ldots, (z_{8r}, y_{8r})\) |

We simulated various patterns of interactions between the noise factors and the control factors. Three patterns of the interaction effect matrix \( C \) are defined as follows.

1. Pattern 1
   These are the interaction effects of \( x_1 \times z_1, x_2 \times z_2 \) and \( x_3 \times z_3 \):

   \[
   C = \begin{pmatrix}
   1.5 & 0 & 0 & 0 & 0 \\
   0 & 1.0 & 0 & 0 & 0 \\
   0 & 0 & 0.5 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0
   \end{pmatrix}
   \]  \hspace{1cm} (16)

2. Pattern 2
   These are the interaction effects of \( x_1 \times z_1, x_2 \times z_2, x_3 \times z_3, x_4 \times z_4, x_2 \times z_3, x_3 \times z_4 \) and \( x_3 \times z_4 \):

   \[
   C = \begin{pmatrix}
   1.0 & 0 & 0 & 0 & 0 \\
   0 & 1.0 & -1.0 & 0 & 0 \\
   0 & 0 & 1.0 & -1.0 & 0 \\
   0 & 0 & 0 & 1.0 & 0
   \end{pmatrix}
   \]  \hspace{1cm} (17)

3. Pattern 3
   These are the interaction effects of \( x_1 \times z_1, x_1 \times z_2, x_1 \times z_3, x_1 \times z_4, x_2 \times z_1, x_2 \times z_2, x_2 \times z_3, x_3 \times z_1, x_3 \times z_2 \) and \( x_4 \times z_1 \):
We defined the second set of simulation conditions as follows:

<Simulation 2>

- Control factors: \( x = (x_1, x_2, x_3)' \)
- Levels of the control factors: 2 levels (1, 2)
- Function which represents the response of the control factors: \( \mu_i(x) = 0 \) (without loss of generality)
- Main effects of the noise factors: \( w = 0 \) (without loss of generality)
- Error variance: \( \sigma^2 = 0.5^2 \)
- Noise factors: \( z_{ij} = (z_{i1}, z_{i2})', z_{ijk} ~ N(0,1^2) \)
- Number of replications \( r \) of the experiments: 4
- Number of replications of the simulations: 5000

In this simulation, there were two noise factors. Under the simple structure of the model, we wanted to investigate whether there was an order effect for the interactions between each control factor and each noise factor.

The control factors and the noise factors were assigned to an \( L_8 \) orthogonal array, as shown in Table 6.

Table 6: Assignment in an \( L_8 \) Orthogonal Array

| No. | \( x_1 \) | \( x_2 \) | \( x_3 \) | Noise Factors and Quality Characteristics \( (z_{ij}, y_{ij}) \) |
|-----|--------|--------|--------|---------------------------------|
| 1   | 1      | 1      | 1      | \((z_{11}, y_{11}) \) \((z_{12}, y_{12})\) \cdots \((z_{1r}, y_{1r})\) |
| 2   | 1      | 1      | 2      | \((z_{21}, y_{21}) \) \((z_{22}, y_{22})\) \cdots \((z_{2r}, y_{2r})\) |
| 3   | 1      | 2      | 1      | \cdots \cdots \cdots \cdots |
| 4   | 1      | 2      | 2      | \cdots \cdots \cdots \cdots |
| 5   | 2      | 1      | 1      | \cdots \cdots \cdots \cdots |
| 6   | 2      | 1      | 2      | \cdots \cdots \cdots \cdots |
| 7   | 2      | 2      | 1      | \cdots \cdots \cdots \cdots |
| 8   | 2      | 2      | 2      | \((z_{81}, y_{81}) \) \((z_{82}, y_{82})\) \cdots \((z_{8r}, y_{8r})\) |

The interaction effect matrices \( C_1 \) and \( C_2 \) are defined as follows:

\[
C_1 = \begin{pmatrix} 1.0 & 0 \\ 0 & a \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1.0 & 1.0 \\ 0 & a \\ 0 & 0 \end{pmatrix},
\]

where interaction effects \( x_1 \times z_1 \) and \( x_2 \times z_2 \) are considered in \( C_1 \) and \( x_1 \times z_1, x_1 \times z_2 \) and \( x_2 \times z_2 \) in \( C_2 \). We let \( a \) vary from 0.5 to 2.0 in increments of 0.1.

4.2 Sizes and Powers Under Fixed Interaction Effects

Tables 7 through 12 show the powers of the two methods. Note that each row of a matrix \( C \) corresponds to a control factor \( x \), and each column corresponds to a noise factor \( z \), but each row in Tables 8, 10, and 12 corresponds to a noise factor \( z \), and each column corresponds to a control factor \( x \).

Table 7 shows that extended method 1 was able to detect a large interaction effect, but the power diminished as the effect became smaller. Table 9 shows that when we included noise factors which had no effect, this weakened the influence of the existing interaction effects. Extended method 2 performed better than did extended method 1. Tables 8 and 10 show that extended method 2 was able to detect interactions when they...
occurred in different combinations. Tables 11 and 12 show that when there were many noise factors interacting with the control factors, extended method 1 performed better than did extended method 2. We note that multiple testing inflates the size of each test for null effects (see, for example, Hochberg and Tamhane, 1987, and Nagata and Yoshida, 1997), which should have a nominal significance level of 0.05. However, we also observe that this inflation was small due to the sequential testing procedure.

Table 7: Performance under Pattern 1
(Extended Method 1)

|          | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|----------|-------|-------|-------|-------|
| Power of the test (%) | 95.86 | 61.16 | 13.90 | 5.90  |

Table 8: Performance under Pattern 1
(Extended Method 2)

|          | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|----------|-------|-------|-------|-------|
| Power of the test (%) | 99.94 | 6.14  | 5.98  | 5.72  |
| $z_1$    | 5.66  | 98.88 | 6.18  | 5.40  |
| $z_2$    | 6.30  | 6.28  | 71.04 | 6.32  |
| $z_3$    | 6.14  | 6.22  | 5.78  | 5.56  |
| $z_4$    | 5.98  | 5.78  | 5.50  | 5.76  |
| $z_5$    | 5.18  | 6.34  | 5.78  | 5.66  |

Table 9: Performance under Pattern 2
(Extended Method 1)

|          | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|----------|-------|-------|-------|-------|
| Power of the test (%) | 26.34 | 59.00 | 59.54 | 26.86 |

Table 10: Performance under Pattern 2
(Extended Method 2)

|          | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|----------|-------|-------|-------|-------|
| Power of the test (%) | 87.68 | 7.62  | 7.14  | 7.80  |
| $z_1$    | 7.50  | 87.66 | 7.44  | 7.50  |
| $z_2$    | 8.26  | 85.88 | 86.14 | 8.34  |
| $z_3$    | 7.86  | 7.44  | 86.00 | 85.68 |
| $z_4$    | 7.22  | 7.12  | 6.72  | 6.64  |
| $z_5$    | 7.32  | 6.48  | 7.44  | 7.18  |
Table 11: Performance under Pattern 3
(Extended Method 1)

|      | X_1  | X_2  | X_3  | X_4  |
|------|------|------|------|------|
| Power of the test (%) | 77.88 | 60.60 | 36.16 | 23.60 |

Table 12: Performance under Pattern 3
(Extended Method 2)

|      | X_1  | X_2  | X_3  | X_4  |
|------|------|------|------|------|
| Power of the test (%) | 45.06 | 45.30 | 45.74 | 47.28 |
| z_1  | 48.58 | 47.76 | 48.44 | 12.24 |
| z_2  | 53.54 | 54.70 | 8.62  | 9.38  |
| z_3  | 57.08 | 8.56  | 12.46 | 8.46  |
| z_4  | 8.02  | 7.64  | 7.60  | 7.30  |
| z_5  | 8.00  | 7.98  | 8.42  | 7.28  |

4.3 Order Effects Under Varying Interaction Effects

Figures 1 and 2 show the power curves for extended methods 1 and 2, respectively, when C_1 is used. Figure 1 shows that, as a increases, the power for the interaction x_2*z (z is the vector of noise factors) becomes larger and the power for x_1*z becomes a little smaller. This occurs because, when a is small, x_1*z is tested in the first stage, but when a is larger, it is tested in the second stage. As explained in Section 2, the two stages have different degrees of freedom, and this results in a decrease in the power for x_1*z, as shown in Figure 1. Figure 2 shows that, as a increases, the power for x_2*z becomes larger, and the power for x_1*z is almost constant. The powers for the other interaction effects are close to the level of significance 0.05. The pattern of values in C_1 does not show much effect due to order, when using extended method 2.

Figures 3 and 4 show the power curves for extended methods 1 and 2, respectively, when C_2 is used. The power curve for x_2*z in Figure 3 is very similar to that shown in Figure 1. On the other hand, the power curve for x_1*z in Figure 3 shows that the power is greater than that shown in Figure 1. We can see in Figure 4 that the power for x_1*z is slightly smaller than that for x_1*z, even though x_1*z and x_1*z have the same effects. This is also induced by the difference between degrees of freedom for the first- and second-stage tests.

Figure 1: Power when C_1 was used (Extended Method 1)
5. Conclusions

In this paper, we proposed two extensions to the method of Hirano and Miyakawa, for situations in which multiple noise factors are observed as covariates. Extended method 1 requires fewer tests, because it is a simple extension of the Hirano and Miyakawa method. However, we found that extended method 1 is not able to detect which of the individual noise factors influence the control factors. In addition, if we include noise factors which
have no effect, this weakens the interaction effects that were originally present. Although extended method 2 requires more tests, it can detect the interaction effect of each noise factor. We recommend the use of extended method 2 when there are multiple noise factors that are observed as covariates.

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