Supercritical stratified flow over an uneven bottom

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Abstract. The solvability of nonlinear problem on steady internal waves in supercritical flow of a heavy stratified fluid over rough topography is investigated. Existence of solution in the Hardy-type function classes is proven by the perturbation technique using small parameter which characterizes typical height of obstacles. It is shown that the solution is analytic with respect to this parameter.

1. Introduction

In this paper, we study the nonlinear problem on stationary flows of stratified fluid affected by the gravity over an uneven bottom. Such a flows are of interest due to their importance in meteorology and oceanology \cite{1}. There are two different regimes of steady-state flow which are called sub- and supercritical flows. In the case of subcritical flow over the obstacle, the most known structure is stationary configuration with attached wave train having periodic asymptotics in the downstream direction. Respectively, the supercritical regime is characterized by that the wave structures form in the vicinity of the obstacle, but these disturbances are damped far downstream. Mathematical problem on stationary free surface flow of homogeneous ideal fluid over an uneven bottom was studied in \cite{2,3}. The boundary value problem for the Dubreil–Jacotin–Long (DJL) equation \cite{4,5}, which models the motion of stratified fluid over the obstacles, was considered in \cite{6,7}. These papers contain the proof of existence of solutions for sufficiently fast upstream flow (the Froude numbers should be much greater than the critical value). In present work, we show the solvability of the nonlinear problem for the DJL equation in the whole range of supercritical values of the Froude numbers.

2. Statement of the problem

We consider steady two-dimensional flow of an inhomogeneous incompressible fluid in a finite-depth layer over an uneven bottom. Basic dimensionless constants are the Boussinesq parameter $\sigma$ and the squared inverse densimetric Froude number $\lambda$ defined by the formulæ

$$\sigma = \frac{N^2 H}{g}, \quad \lambda = \frac{\sigma g H}{U^2}.$$ 

Here $N$ is a constant buoyancy frequency, $H$ is the thickness of undisturbed fluid layer (Fig. 1), $g$ is the gravity acceleration, and $U$ is the velocity of upstream flow. Constancy of the frequency $N$ means that the fluid density depends exponentially on height at rest. The flow over the obstacles
is modelled by a boundary value problem for the DJL equation which has the following form in dimensionless variables:

\[ \psi_{xx} + \psi_{yy} + \lambda (\psi - y) = \frac{\sigma}{2} (\psi_x^2 + \psi_y^2 - 1), \] (1)

\[ \psi\big|_{y=ah(x)} = 0, \quad \psi\big|_{y=1} = 1, \] (2)

\[ \psi(x, y) \to y \quad (x \to -\infty), \] (3)

where \( \psi(x, y) \) is unknown stream function. The parameter \( \sigma \) defines the slope of the density profile, and the parameter \( \lambda \) indicates whether the upstream flow is sub- or supercritical. Namely, the flow is supercritical if the parameters \( \lambda \) and \( \sigma \) satisfy the inequality

\[ \lambda < \frac{\sigma^2}{4} + \pi^2, \] (4)

and the flow is subcritical in opposite case. The value \( \lambda = \sigma^2/4 + \pi^2 \) is noted as the critical value of the Froude number \( \lambda \). In what follows, the condition (4) is assumed to be valid.

The equation (1) is equivalent to the system of Euler equations of inhomogeneous fluid flow [4, 5]. It is assumed that the flow domain is bounded from above by the rigid lid, and the bottom shape is defined by smooth function \( h(x) \) which decays as \( |x| \to \infty \). Kinematic boundary conditions at the bottom and at the lid have the form (2). In this work, we use the dimensionless height of the obstacle \( a = d/H \) as a natural small parameter, where \( d \) is typical height of the obstacles. The radiation condition (3) means that there are no waves in the upstream flow.

**Figure 1.** The motion of a stratified flow over obstacles.

It is assumed that all the streamlines \( \psi = \text{const} \) have one-to-one projection on the \( x \)-axis, so the flow has no overturns. By that, we can use semi-Lagrangian independent variables \( (x, \psi) \) known as the von Mises variables. In this case, the shape of streamlines can be found in the form \( y = \psi + w(x, \psi) \), and the boundary value problem (1)–(3) can be rewritten with new unknown function \( w \) as follows:

\[ w_{xx} + w_{\psi\psi} - \sigma w_\psi + \lambda w = f(w), \]

\[ w(x, 0) = ah(x), \quad w(x, 1) = 0, \]

\[ w(x, \psi) \to 0 \quad (x \to -\infty), \] (5)

where the nonlinear operator \( f(w) \) has the form

\[ f(w) = -\frac{\partial}{\partial x} \frac{w_x w_\psi}{1 + w_\psi} + \frac{1}{2} \frac{\partial}{\partial \psi} \frac{w_x^2 + 3w_\psi^2 + w_\psi^3}{(1 + w_\psi)^2} - \frac{\sigma}{2} \frac{w_x^2 + 3w_\psi^2 + w_\psi^3}{(1 + w_\psi)^2}. \]

Thus, the original boundary value problem (1)–(3), formulated in the flow region \( ah(x) < y < 1 \), reduces to the problem (5) in the rectilinear strip \( (x, \psi) \in \Pi = \mathbb{R} \times [0, 1] \).
3. The analysis of linearized problem
We consider here nonhomogeneous boundary value problem arising from linearized equation (5) with a given right-hand side function \( f(x, \psi) \), and with fixed parameters \( \sigma, \lambda, a \):

\[
\begin{align*}
  w_{xx} + w_{\psi\psi} - \sigma w_{\psi} + \lambda w &= f(x, \psi), \\
  w(x, 0) &= ah(x), \\
  w(x, 1) &= 0, \\
  w(x, \psi) &\to 0 \quad (x \to -\infty).
\end{align*}
\]

(6)

Using partial Fourier transform of \( w \) with respect to horizontal variable \( x \),

\[
\hat{w}(\xi, \psi) = F[w](\xi, \psi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} w(x, \psi) dx,
\]

we represent the solution \( w(x, \psi) \) to the problem (6) by explicit formula

\[
w = F^{-1} \left[ G(\hat{f}) + K(\hat{h}) \right],
\]

(7)

where \( F^{-1} \) denotes inverse Fourier transformation in \( x \). Integral operator \( G \) has the form

\[
G(\xi, \psi, s) = \int_{0}^{1} G(\xi, \psi, s) \hat{f}(\xi, s) ds,
\]

(8)

where the Green function \( G(\xi, \psi, s) \) is given by the formula

\[
G(\xi, \psi, s) = e^{\frac{\xi^2}{2}(\psi-s)} \frac{\sqrt{\xi^2 - \nu^2}}{\sqrt{\xi^2 - \nu^2} \sin \sqrt{\nu^2 - \xi^2}} \begin{cases} 
\text{sh} \sqrt{\xi^2 - \nu^2}(s-1) \text{sh} (\sqrt{\xi^2 - \nu^2} \psi), & \psi \leq s, \\
\text{sh} (\sqrt{\xi^2 - \nu^2}s) \text{sh} \sqrt{\xi^2 - \nu^2}(\psi-1), & \psi > s,
\end{cases}
\]

as \( \xi^2 > \nu^2 = \lambda - \frac{\sigma^2}{4} \), and

\[
G(\xi, \psi, s) = e^{\frac{\xi^2}{2}(\psi-s)} \frac{\sqrt{\nu^2 - \xi^2}}{\sqrt{\nu^2 - \xi^2} \sin \sqrt{\nu^2 - \xi^2}} \begin{cases} 
\sin \sqrt{\nu^2 - \xi^2}(s-1) \sin (\sqrt{\nu^2 - \xi^2} \psi), & \psi \leq s, \\
\sin (\sqrt{\nu^2 - \xi^2}s) \sin \sqrt{\nu^2 - \xi^2}(\psi-1), & \psi > s,
\end{cases}
\]

as \( \xi^2 < \nu^2 \). The operator \( K \) is defined by the relation

\[
K(\hat{h}) = a \hat{h}(\xi) e^{\frac{\xi^2}{2} \psi} K(\xi, \psi),
\]

(9)

where the multiplier \( K(\xi, \psi) \) has the form

\[
K(\xi, \psi) = \begin{cases} 
\text{sh} \sqrt{\xi^2 - \nu^2}(1 - \psi), & \xi^2 > \nu^2, \\
\frac{\text{sh} \sqrt{\xi^2 - \nu^2}}{\sqrt{\nu^2 - \xi^2}}, & \xi^2 < \nu^2.
\end{cases}
\]

The function \( G \) and \( K \) are defined at \( \xi^2 = \nu^2 \) by continuity.

The formulae (7)–(9) provide a formal solution, so we need to specify it in more details. For instance, an analytic extension of the Fourier transform \( \hat{w}(\xi, \psi) \) on the complex plane \( \zeta = \xi + i\eta \) is important here. The analyticity strip \( |Im \zeta| < \alpha \) of the function \( \hat{w} \) is evidently restricted by zeros.
of the functions $\sqrt{\xi^2 - \nu^2}$, $\sin \sqrt{\nu^2 - \xi^2}$. These square roots possess branching points, but the map $\hat{w}$ is not multivalent since their representation involves only even analytic functions of the variable $\kappa = \sqrt{\xi^2 - \nu^2}$. In accordance with the condition (4), we are interested in the solution which satisfies inequality $\nu^2 < \pi^2$. In this case, zeros of functions $\sqrt{\xi^2 - \nu^2}$, $\sin \sqrt{\nu^2 - \xi^2}$ lie on the imaginary axis and form the set $\{\pm \sqrt{|k^2\pi^2 - \nu^2|} : k \in \mathbb{Z}\}$. Hence, the quantity

$$\alpha_* = \sqrt{\pi^2 - \nu^2} = \sqrt{\pi^2 + \frac{\sigma^2}{4} - \lambda}$$

(10)

presents natural bound for decay exponent $\alpha < \alpha_*$ of the solution $w(x, \psi)$ as $|x| \to \infty$.

Let us introduce the function spaces where we are looking for a solution to the boundary value problem (6). Suppose that real numbers $\rho > 0$, $\alpha > 0$ are given, and real interval $[a, b]$ is chosen. We denote by $\Pi_{[a, b]}$ a strip $\mathbb{R} \times [a, b]$. The class $E^\alpha_{\rho} (\Pi_{[a, b]})$ consists of the functions $w(x, \psi)$ that are defined in the strip $\Pi_{[a, b]}$, and these functions admit the Fourier transform $\hat{w}(\xi, \psi)$ which is analytic with respect to $\xi = \xi + i\eta$ in the complex domain $|Im \xi| < \alpha$, and $\hat{w}(\xi, \psi)$ is jointly continuous in closed set $\{(\zeta, \psi) : |\eta| \leq \alpha_1, a \leq \psi \leq b\}$ for any $\alpha_1 < \alpha$. The norm of the function $w(x, \psi)$ is defined as follows:

$$\|w\|_{\rho, \alpha} = \sup_{|\eta| < \rho} \int_{-\infty}^{\infty} e^{\rho|\xi|} \sup_{\psi \in [a, b]} |\hat{w}(\xi + i\eta, \psi)|d\xi.$$ (11)

Since this norm is finite, the function $w(z, \psi)$ is analytic with respect to $z = x + iy$ in the domain $|y| < \rho$, and their Fourier transform $\hat{w}$ is jointly continuous for $|y| \leq \rho$. By the definition of the norm (11), both functions $\hat{w}(\xi, \psi) = \exp(\pm \rho \xi) \hat{w}(\xi, \psi)$ belong to the Hardy class $H^1$ for every fixed $\psi \in [a, b]$ in the strip $|Im \xi| < \alpha$. Therefore, in accordance with properties of the Hardy class $H^1$ [8], the function $w \in E^\alpha_{\rho} (\Pi_{[a, b]})$ decays exponentially as $|x| \to \infty$ in such a way that

$$|w(x + iy, \psi)| \leq \|w\|_{\rho, \alpha} e^{-\alpha|x|}$$

for $x \in \mathbb{R}$, $|y| \leq \rho$.

The class of the Fourier transforms $\hat{w}(\xi, \psi)$ mentioned above is a special case of generalized Hardy classes [9]. In addition to the Banach space $E^\alpha_{\rho} (\Pi_{[a, b]})$, we consider also the spaces $E^{\alpha, 2}_{\rho, 2} (\Pi_{[a, b]})$ of functions that belong to $E^\alpha_{\rho} (\Pi_{[a, b]})$ together with their derivatives up to $n$-th order, equipped with the norm

$$\|w\|_{E^{\alpha, 2}_{\rho, 2}} = \sum_{0 \leq m + k \leq 2} \|D^m_x D^k_\psi w\|_{\rho, \alpha}.$$ 

Also, we define the class $E^\alpha_{\rho}(\mathbb{R})$ of the function of a single variable $x$, equipped with the norm (11) (the spaces $E^{\alpha, 2}_{\rho, 2}(\mathbb{R})$ are defined in a similar way). In accordance with the notation for the strip $\Pi = \Pi_{[0, 1]}$, we find the solutions of the problem (6) in the class $E^{\alpha, 2}_{\rho, 2} (\Pi)$ with exponent $\alpha < \alpha_*$, where $\alpha_*$ is determined in (10). The condition that the solution derivatives should belong to the spaces $E^\alpha_{\rho} (\Pi)$ is caused by the fact that the nonlinear boundary value problem (5) has the right-hand side operator $f$ depending on the second-order derivatives $D^m_x D^k_\psi w$ ($m + n \leq 2$).

In this section we apply the techniques of estimates developed in the paper [10]. The following Lemma gives us preliminary estimates of the Green function $G$. For this purpose, we introduce the cut-off functions $M_j : \mathbb{R}_+ \to \mathbb{R}_+ (j = 0, 1, 2)$ by setting

$$M_j(t) = 1 \quad (0 \leq t \leq 1), \quad M_j(t) = t^{-j} \quad (t \geq 1).$$
The subscript \( j \) indicates here the decay rate of the Fourier transforms as \( |Re \zeta| \to \infty \).

**Lemma 1.** The following estimates hold in the strip \( |Im \zeta| \leq \alpha < \alpha_* \) where \( \alpha_* \) is given by (10):

\[
\int_{0}^{1} |D^j_\psi G(\zeta, \psi, s)|ds \leq C_1(\alpha, \alpha_*, \nu)M_{2-j}(Re|\zeta|) \quad (j = 0, 1).
\]

**Proof.** At first, we choose a real number \( L > 0 \), and consider separately the cases \( |Re \zeta| \leq L \) and \( |Re \zeta| > L \). Using elementary inequalities for hyperbolic functions of complex- and real variable

\[
|\text{sh} \xi| \leq |\text{sh} \zeta| \leq |\text{ch} \xi|,
\]

(12)

we obtain that continuous function \( G(\zeta, \psi, s) \) is uniformly bounded on the compact set \( |Re \zeta| \leq L \). Hence we have the estimate

\[
\int_{0}^{1} |G(\zeta, \psi, s)|ds \leq \frac{\sqrt{\pi^2 + \alpha^2 - \alpha_*^2}}{\sin \sqrt{\pi^2 + \alpha^2 - \alpha_*^2}}C_2(L).
\]

Further, applying inequalities (12) and (13) by \( |Re \zeta| > L \), we find that the source integral has the second-order decay with respect to the quantity \( |\gamma| = |Re \sqrt{\zeta^2 - \nu^2}| \):

\[
\int_{0}^{1} |G(\zeta, \psi, s)|ds \leq \frac{C_3(L)}{|\gamma|^2}.
\]

Finally, using trigonometric representation \( \zeta^2 - \nu^2 = \tau e^{i\theta} \), which implies \( |\gamma| = \sqrt{\tau} |\cos \frac{\theta}{2}| \) and \( \theta = \arg(\zeta^2 - \nu^2) = \arg(\zeta - \nu) + \arg(\zeta + \nu) \), we estimate the value of \( |\gamma| \) by \( L > 2|\nu| \) as follows:

\[
\frac{1}{|\gamma|} \leq \frac{C_4(\alpha, \nu)}{|\xi|}.
\]

The estimate of integral, mentioned in the Lemma for \( j = 1 \), is obtained in a similar way. \( \square \)

The following Theorem gives the sufficient condition for existence of the solution to linear problem (6) in the space \( E^{0}_{\rho, 2}(\Pi) \).

**Theorem 1.** Suppose that the upstream flow is supercritical in the sense of (4), and the bottom shape is such that \( h(x) \in E_{\rho, 2}(\mathbb{R}) \), \( f(x, \psi) \in E^{0}_{\rho}(\Pi) \) with \( \alpha < \alpha_* \), where \( \alpha_* \) is given by (10). Then the formula (7)–(9) represents the solution to the problem (6) belonging to the class \( E^{0}_{\rho, 2}(\Pi) \). Furthermore, for \( 0 \leq m + n \leq 2 \) the following estimates hold:

\[
\|D^m_x D^n_\psi w\|_{E^0_{\rho}(\Pi)} \leq C_5(\alpha, \alpha_*, \nu) \left( a\|h\|_{E_{\rho, 2}(\mathbb{R})} + \|f\|_{\rho, 0} \right).
\]

**Proof.** According to the formulae (7)–(9), we have:

\[
\|D^m_x D^n_\psi w\|_{E^0_{\rho}(\Pi)} = \sup_{|\eta| < \alpha} \int_{-\infty}^{\infty} e^{i\eta \xi} \sup_{\psi \in [0,1]} \left| \zeta^m D^n_\psi \tilde{w}(\xi + i\eta, \psi) \right| d\xi \leq J_1 + J_2,
\]
Thus, the problem (5) reduces to the finding a fixed point of the operator Φ. Note that
where is denoted

\[ J_1 = \sup_{|\eta| < \alpha} \int_{-\infty}^{\infty} e^{\rho|\xi|} \sup_{\psi \in [0,1]} \left| \zeta^m \int_0^1 D_0^m G(\zeta, \psi, s) \tilde{f}(\zeta, s) ds \right| d\xi \]

and

\[ J_2 = \sup_{|\eta| < \alpha} \int_{-\infty}^{\infty} e^{\rho|\xi|} \sup_{\psi \in [0,1]} \left| \zeta^m \alpha \tilde{h}(\zeta) D_0^m (e^{2\psi K(\zeta, \psi)}) \right| d\xi. \]

Let us consider the case \( n \neq 2 \). Using Lemma 1, we obtain for \( J_1 \) the estimate

\[ J_1 \leq \frac{\sqrt{\pi^2 + \alpha^2 - \alpha_s^2}}{\sin \sqrt{\pi^2 + \alpha^2 - \alpha_s^2}} C_6 \| f \|_{\rho, \alpha}. \]

Since the derivatives \( h'(x) \) and \( h''(x) \) belong to the class \( E_\rho^\alpha(\mathbb{R}) \), we have for \( J_2 \) the estimate

\[ J_2 \leq \frac{\sqrt{\pi^2 + \alpha^2 - \alpha_s^2}}{\sin \sqrt{\pi^2 + \alpha^2 - \alpha_s^2}} C_7 \| h \|_{E_\rho^\alpha}. \]

In the case \( n = 2 \), the estimate of the Theorem 1 follows, due to the equations (6), from above-obtained solution estimates. \( \square \)

4. The analysis of the nonlinear problem

According to results of the Section 3, the nonlinear problem (5) is equivalent to the nonlinear operator equation

\[ w = \Phi(w, a, h, \lambda, \sigma), \]

where is denoted

\[ \Phi(w, a, h, \lambda, \sigma) = F^{-1} \left[ G(\tilde{f}(w)) + K(\tilde{h}) \right]. \]

Thus, the problem (5) reduces to the finding a fixed point of the operator \( \Phi \). Note that the norm in \( E_\rho^\alpha(\Pi_{[a,b]}) \) is multiplicative due to the convolution theorem for the Fourier transform:

\[ \| wv \|_{\rho, \alpha + \beta} \leq \| w \|_{\rho, \alpha} \| v \|_{\rho, \beta} \]

(15)

where \( w \in E_\rho^\alpha, v \in E_\rho^\beta \). The nonlinear operator \( f \) can be presented by convergent Taylor series in powers \( w \), similarly to the geometric progression series, so the map \( \tilde{f}(w) \) is analytic at \( w \) in a neighborhood of the element \( w = 0 \). Using this fact, we obtain by (15) the estimate

\[ \| f(w) - f(v) \|_{\rho, \alpha} \leq (28 + 6\sigma) \frac{\| w \|_{E_\rho^\alpha} + \| v \|_{E_\rho^\alpha}}{(1 - \| w \|_{E_\rho^\alpha} - \| v \|_{E_\rho^\alpha})^4} \| w - v \|_{E_\rho^\alpha}. \]

(16)

Fixe the real parameters \( \lambda, \sigma, a \) and the function \( h \), we obtain from (16) under the estimates of Theorem 1 the following inequality which is valid for \( \| w \|_{E_\rho^\alpha} \leq r, \| v \|_{E_\rho^\alpha} \leq r \):

\[ \| \Phi(w, a, h, \lambda, \sigma) - \Phi(v, a, h, \lambda, \sigma) \|_{E_\rho^\alpha} \leq C_K(\alpha, \alpha_s, \nu) r \frac{\| w - v \|_{E_\rho^\alpha}}{(1 - 2r)^4}. \]

Therefore, the mapping \( \Phi(w, a, h, \lambda, \sigma) \) is the contraction in the ball \( B_r = \{ w \in E_\rho^\alpha(\Pi) : \| w \|_{E_\rho^\alpha} < r \} \), where the radius \( r > 0 \) is limited as follows:

\[ r < C_9 \frac{\sin \sqrt{\pi^2 + \alpha^2 - \alpha_s^2}}{\sqrt{\pi^2 + \alpha^2 - \alpha_s^2}}. \]

(17)
For \( a = 0 \), the problem (14) has the trivial solution \( w = 0 \). Then, using that mapping \( \Phi \) depends continuously on \( a \), we can select the radius \( r \) such that the mapping \( \Phi \) is contractor on the closed ball \( \bar{B}_r \). According to the contraction mapping principle, there is a unique fixed point \( w = w(a,h,\lambda,\sigma) \in \bar{B}_r \) which gives unknown solution. Due to the point uniqueness we have \( w(0,h,\lambda,\sigma) = 0 \). As \( \Phi(w,a,h,\lambda,\sigma) \) is jointly continuous of the \( \lambda, \sigma, a, h \), the constructed solution is continuous of these parameters. Moreover, since \( \Phi \) is analytic with respect to \( w \) and \( a \), the solution is analytic in \( a \). Thus, we have proved the following theorem.

**Theorem 2.** Let the upstream flow be supercritical in the sense of (4). Suppose that the function \( h(x) \), which defines the bottom shape, is in \( E^{\alpha}_{p,2}(\mathbb{R}) \) with \( \alpha < \alpha_* \), where \( \alpha_* \) is given by (10). Then the boundary value problem (5) has the solution \( w \in E^{\alpha}_{p,2}(\Pi) \) which is analytic with respect to the parameter \( a \).

Note that the restrictions, formulated for the parameters \( \lambda, \sigma, a \) and the function \( h \), have clear physical meaning. The conditions (4) means that the velocity \( U \) of upstream flow should be not small. The fact that the derivatives of the function \( h \) belong to the spaces \( E^{\alpha}_{p,2}(\mathbb{R}) \) suggests that the bottom shape is nearly flat. In particular, the quantity \( a\|h\|_{E^{\alpha}_{p,2}} \) is limited due to (17). It means that the height of obstacles, as well as their steepness, should not exceed a certain value. Otherwise, as it is known from the theory of stratified fluid [5], the flow may be blocked in the front of the obstacle. Constructed solution has limited derivatives by small \( a \), so the streamlines have bijective projection on the \( x \)-axis. In addition, analyticity in \( a \) justifies, at least in the supercritical case, the formal perturbation procedure used in [11] by the construction of approximate solutions describing non-linear lee wave structures. According to (4) and (10), we obtain that the parameters \( \alpha, \alpha_* \rightarrow 0 \) as the Froude number \( \lambda \) tends to the critical value \( \lambda = \sigma^2/4 + \pi^2 \). Therefore, the ball, which contains the solution \( w \), vanishes in this limit case due to the estimate (17). Possible reason may be the change of the flow regime by the transition to the subcritical flow, so the nonlinear wave structures can bifurcate at critical value of \( \lambda \).

5. Conclusion
The nonlinear boundary value problem on steady 2D flows of an inhomogeneous incompressible fluid in a finite-thickness layer over an uneven bottom is studied. The existence of exact solution is proved under the assumption that upstream flow satisfies the super-criticality condition \( \lambda < \sigma^2/4 + \pi^2 \), and the bottom shape is nearly flat. The solution of inhomogeneous linearized problem is constructed in explicit form (with the corresponding estimates). It is shown that the nonlinear solution is analytic with respect to the parameter of the obstacle height.

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