Snarks with resistance $n$ and flow resistance $2n$

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Abstract
We examine the relationship between two measures of uncolourability of cubic graphs – their resistance and flow resistance. The resistance of a cubic graph $G$, denoted by $r(G)$, is the minimum number of edges whose removal results in a 3-edge-colourable graph. The flow resistance of $G$, denoted by $r_f(G)$, is the minimum number of zeroes in a 4-flow on $G$. Fiol et al. [Electron. J. Combin. 25 (2018), #P4.54] made a conjecture that $r_f(G) \leq r(G)$ for every cubic graph $G$. We disprove this conjecture by presenting a family of cubic graphs $G_n$ of order $34n$, where $n \geq 3$, with resistance $n$ and flow resistance $2n$. For $n \geq 5$ these graphs are nontrivial snarks.

Mathematics Subject Classifications: 05C

1 Introduction
Snarks are 2-connected cubic graphs whose edges cannot be properly coloured with three colours. The significance of this class of graphs derives mainly from the fact that it may contain counterexamples to several important and long-standing conjectures in graph
theory, such as the cycle double cover conjecture, the 5-flow conjecture, Fulkerson’s conjecture, and others. While most of these conjectures are easy for 3-edge-colourable graphs, they are exceedingly difficult for snarks in general. On the other hand, a number of recent results confirm that some of these conjectures become tractable for snarks that are in a certain sense close to 3-edge-colourable graphs, see for example [6, 10, 15]. In this situation it is natural to focus on the study of invariants of cubic graphs that express – in various ways – to what extent a graph differs from a 3-edge-colourable graph. Such invariants are called measures of uncolourability. A deeper examination of relations between various uncolourability measures may provide new insights into the studied conjectures and lead to interesting partial results. An excellent survey on this topic, by Fiol et al. [4], is available and highly recommended.

One of the most prominent uncolourability measures is the resistance of a cubic graph. It is defined as the smallest number of edges (or, equivalently, vertices) whose removal from a graph results in a 3-edge-colourable graph [13, 14]; we denote the resistance of graph $G$ by $r(G)$. Clearly, $r(G) = 0$ if and only if $G$ is 3-edge-colourable. Moreover, $r(G) \geq 2$ whenever $G$ is not 3-edge-colourable.

A similar measure of uncolourability is obtained by regarding snarks as cubic graphs that do not admit a nowhere-zero 4-flow. The flow resistance of a cubic graph $G$, denoted by $r_f(G)$, is the smallest number of zero-valued edges that any integer 4-flow on $G$ can have. Like resistance, $r_f(G) = 0$ if and only if $G$ is 3-edge-colourable. The similarity of these two measures is underscored by the fact that a cubic graph admits an integer nowhere-zero 4-flow if and only if it admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, and the latter coincides with a proper 3-edge-colouring where colours are the non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In contrast to resistance, however, flow resistance can take value 1. In fact, cubic graphs with flow resistance 1 are very common, while those with flow resistance greater than 1, introduced by Jaeger [7, 8] as strong snarks, appear to be rare. The complete list of all cyclically 4-edge-connected snarks with girth at least 5 on up to 36 vertices generated by Brinkmann et al. [2] contains 64 326 024 items, only 32 of which are strong snarks, all with flow resistance 2. At the same time, all snarks in the list have resistance 2 (these facts easily follow from [2, Observations 4.10 and 4.14] and the definition of a strong snark).

The importance of flow resistance for the study of snarks is quite obvious because it offers a natural approach to Tutte’s 5-flow-conjecture. A similar approach to Tutte’s 3-flow conjecture has recently been taken by DeVos et al. [3] where the authors proved that every 3-edge-connected graph admits a 3-flow in which at most one sixth of the edges carries value zero.

In spite of these facts, flow resistance has so far attracted surprisingly little attention. The only explicit mention of flow resistance in the literature occurs in the cited survey [4], with Section 4.1 being completely devoted to this invariant. The authors of [4] note that flow resistance can be equivalently defined as the minimum number of edges that have to be contracted in order to obtain a graph that admits a nowhere-zero 4-flow [4, Theorem 33]. Moreover, they show in [4, Proposition 29] that $r_f(G)$ is bounded above by
the minimum number of edges that any two perfect matchings of $G$ can have, which is another useful uncolourability measure (denoted in [4] by $\gamma_2(G)$). They also propose the following interesting conjecture (Conjecture 51).

**Conjecture 1.** (Fiol et al. [4]) If $G$ is a bridgeless cubic graph, then $r_f(G) \leq r(G)$.

Intuitively, every individual zero-valued edge $e$ of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on a cubic graph $G$ generates two faulty vertices of the corresponding 3-edge-colouring, the end-vertices of $e$. After removing one edge at each of the vertices a proper 3-edge-colouring is obtained. This observation seems to speak in favour of the conjecture. Nevertheless, the conjecture is false, as follows from the main result of the present paper.

**Theorem 2.** For every integer $n \geq 3$ there exists a cubic graph $G_n$ on $34n$ vertices with $r(G_n) = n$ and $r_f(G_n) = 2n$. Moreover, if $n \geq 5$, then $G_n$ is cyclically 4-edge-connected and has girth 5.

2 Preliminaries

A semi-graph $G$ is a pair $G = (V, E)$ which consists of a set of vertices $V = V(G)$ and a set $E = E(G) \subseteq P_2(V) \cup P_1(V)$ consisting of edges and semi-edges; here $P_i(A)$ denotes the set of all $i$-element subsets of a set $A$. In $E(G)$, the 2-element sets are called edges (as expected) while the 1-element sets are called semi-edges. Note that if $E$ contains no elements from $P_1(V)$, then $G$ is simply a graph.

We denote the edge $\{u, v\}$ as $uv$ and the semi-edge $\{u\}$ as $(u)$. Furthermore, we define the join between two semi-edges $(u)$ and $(v)$ as the removal of semi-edges $(u)$ and $(v)$, and the addition of the edge $uv$. A semi-edge $(u)$ and a vertex $v$ may also join to form an edge $uv$, with semi-edge $(u)$ being removed. The degree of a vertex $v$ in a semi-graph $G$ is defined as the combined total number of edges and semi-edges incident with $v$. Thus a cubic semi-graph is a semi-graph with each vertex having degree 3.

Essentially, semi-edges behave like edges except that they are associated with one vertex instead of two, with each vertex having at most one semi-edge. We say that a semi-graph $G$ contains a semi-graph $G'$ if $V(G') \subseteq V(G)$, $uv \in E(G')$ implies that $uv \in E(G)$, semi-edge $(u) \in E(G')$ implies semi-edge $(u) \in E(G')$ or there is an edge $uv \in E(G)$, and for every vertex $u \in V(G')$ the degree of $u$ in $G$ is greater than or equal to the degree of $u$ in $G'$.

Let $G$ be a semi-graph. An edge colouring of $G$ is an assignment of colours to the elements of $E(G)$ such that adjacent elements receive distinct colours; such colourings are often termed proper. An edge colouring that uses $k$ colours is a $k$-edge-colouring.

It is well known that every cubic graph can be properly edge-coloured with three or four colours. A snark is a 2-connected cubic graph that admits no proper 3-edge-colouring. Snarks with small edge cuts and short circuits are usually considered trivial. A snark is called nontrivial if it is cyclically 4-edge-connected and has no circuits of length smaller than 5. Recall that a connected graph is cyclically $k$-edge-connected if it contains no
subset $S \subseteq E(G)$ of size $|S| < k$ such that $G - S$ is disconnected and has at least two components containing a circuit.

Let $A$ be an abelian group. An $A$-flow on $G$ is a pair $(D, \phi)$ where $\phi$ is an assignment of elements of $A$ to the elements of $E(G)$, and $D$ is an assignment of one of two directions to the elements of $E(G)$ such that, for every vertex $v$ in $G$, the sum of values flowing into $v$ equals the sum of values flowing out of $v$ (Kirchhoff’s law). A nowhere-zero $A$-flow is one which does not assign the zero element of $A$ to any edge or semi-edge of $G$. Note that the choice of $D$ is immaterial since one can reverse the orientation of any edge $e$ and replace the value $\phi(e)$ with $-\phi(e)$ without violating the Kirchhoff law. Moreover, if each element $x \in A$ satisfies $x = -x$, then the assignment of an orientation can be omitted from the definition altogether. It is well known that the latter condition is satisfied if and only if $A \cong \mathbb{Z}_2^n$ for some $n \geq 1$.

An $A$-flow $(D, \phi)$ where $A = \mathbb{Z}$ and $\phi(e) \in \{0, \pm 1, \ldots, \pm (k - 1)\}$ for all $e \in E(G)$ is called an integer $k$-flow. A well-known useful result on flows is the following: A graph admits a nowhere-zero $A$-flow, for some finite abelian group $A$, if and only if it admits a nowhere-zero integer $|A|$-flow [11, 16]. In particular, a cubic semi-graph admits an integer 4-flow with $m$ zeros if and only if it admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow with $m$ zeros. This observation offers an additional advantage that in the study of flow resistance we do not need to bother with orientations of the graph in question.

It is easy to see that a nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on a cubic graph $G$ corresponds exactly to a 3-edge-colouring. That is, if the assignment of elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the edges of $G$ is interpreted as an edge colouring, and no edge is assigned the zero element, then the edge colouring is proper. This is easily seen to be true for semi-graphs as well. Furthermore, given any $A$-flow on a semi-graph, the sum of the flow values of its semi-edges must be zero. If $G$ is cubic and $A = \mathbb{Z}_2 \times \mathbb{Z}_2$, the latter amounts to what is generally known as the Parity Lemma. These facts will be used implicitly throughout this paper.

3 Main result

Let $X$ denote the semi-graph created from the Petersen graph by the removal of two adjacent vertices; it is shown in Figure 1. The semi-edges of $X$ occur in two pairs $a, b$ and $c, d$, each of them arising by the removal of the same vertex of the Petersen graph. The following properties of $X$, stated in Lemma 1, are well known. In fact, they easily follow from the Kirchhoff law and the fact that the Petersen graph is the smallest snark.

![Figure 1: The semi-graph X](image-url)
Lemma 3. The following statements hold for the semi-graph $X$ depicted in Figure 1.

(i) $r(X) = 0$.

(ii) If $f$ is a proper 3-edge-colouring of $X$, then $f(a) = f(b)$ and $f(c) = f(d)$.

Figure 2 depicts a semi-graph $Y$ constructed from two instances $X_1$ and $X_2$ of $X$ and from three additional vertices $p$, $q$, and $r$ as follows. Denoting by $x_i$ the semi-edge of $X_i$ corresponding to the semi-edge $x$ of $X$, with $x \in \{a, b, c, d\}$, we join $c_1$ and $a_2$ to $p$, next we join $d_1$ and $b_2$ to $q$, and finally we add the edges $pr$, $qr$, and the semi-edge $(r) = e$.

We relabel the semi-edges $a_1$ and $b_1$ as $a$ and $b$, respectively, and relabel the semi-edges $c_2$ and $d_2$ as $c$ and $d$, respectively.

![Figure 2: The semi-graph $Y$](image)

Lemma 4. The following statements hold for the semi-graph $Y$ depicted in Figure 2.

(i) $r(Y) = 1$.

(ii) $r_f(Y) = 1$.

Proof. (i): We first show that $r(Y) \geq 1$. It is clearly sufficient to prove that $Y$ is not 3-edge-colourable. Suppose the contrary. Recall that $Y$ contains two instances $X_1$ and $X_2$ of $X$, where $X_1$ is the one containing the semi-edges $a$ and $b$ and $X_2$ is the one containing the semi-edges $c$ and $d$. The other two semi-edges of $X_1$ are $c_1$ and $d_1$ and those of $X_2$ are $a_2$ and $b_2$. By Lemma 3, every proper 3-edge-colouring of $Y$ assigns the same colour to $c_1$ and $d_1$ and the same colour to $a_2$ and $b_2$. As a consequence, the end-vertex of the semi-edge $e$ of $Y$ has two edges with the same colour, a contradiction. Thus $Y$ is not 3-edge-colourable. On the other hand, $Y$ becomes 3-edge-colourable after removing either of the two edges adjacent to $e$, which implies that $r(Y) = 1$.

(ii): Suppose to the contrary that $r_f(Y) = 0$. Then $Y$ admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Since $Y$ is cubic, the flow is also a proper 3-edge-colouring, and therefore $r(Y) = 0$, contradicting Statement (i). Hence, $r_f(Y) \geq 1$. It is easy to see that there exists a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on $Y$ with $e$ being the only zero edge. Therefore $r_f(Y) = 1$. \qed
Figure 3 displays a semi-graph $Z$ constructed from an instance $X_1$ of $X$, an instance $Y_2$ of $Y$, and four additional vertices $p$, $q$, $r$, and $s$ in a similar manner as $Y$ was constructed from two instances of $X$. If we use the same convention for indices as before, then $p$, $q$, and $r$ get the same incidences as previously. The semi-edges $e_1$ of $X_1$ and $e_2$ of $Y_2$ are then joined to the vertex $s$ and a new semi-edge $(s) = e$ is added. As before, we relabel the semi-edges of $Z$ inherited from $X_1$ and $Y_2$ as $a$, $b$, $c$, and $d$. The instance of $X$ which contains the edge $f$, as seen in Figure 3, is called the central instance of $X$.

Figure 3: The semi-graph $Z$

**Lemma 5.** The following statements hold for the semi-graph $Z$ depicted in Figure 3.

(i) $r(Z) = 1$.

(ii) $r_f(Z) = 2$.

**Proof.** (i): Since $Z$ contains a copy of $Y$ and $r(Y) = 1$, from Lemma 4 (i) we readily obtain that $r(Z) \geq 1$. Moreover, $Z - f$ is 3-edge-colourable for the edge $f$ indicated in Figure 3, whence $r(Z) = 1$.

(ii): Lemma 4 (ii) implies that $r_f(Z) \geq 1$. Suppose that $r_f(Z) = 1$ and let $\phi$ be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on $Z$ with one zero. The zero-valued element of $E(Z)$ must be contained in the central instance of $X$, otherwise there would be an instance of $Y$ which contains no zero edges, contradicting Lemma 4 (ii). By Lemma 3 and the fact that a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on a semi-graph corresponds to a proper 3-edge-colouring, we have that $\phi(a) = \phi(b)$ and $\phi(c) = \phi(d)$. Consequently, $\phi(a) + \phi(b) = 0$ and $\phi(c) + \phi(d) = 0$. Since the sum of the values assigned to the five semi-edges in $Z$ must be zero, we conclude that $\phi(e) = 0$. This contradicts the assumption that $r_f(Z) = 1$. Furthermore, it is not difficult to find a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on $Z$ with $f$ and $e$ being the only zero edges. Hence, $r_f(Z) = 2$. □

Following on from this, it is straightforward to construct nontrivial snarks with flow resistance equal to twice their resistance. For each integer $n \geq 3$ we define a graph $G_n$ on $34n$ vertices as follows: Let $G_n$ contain $n$ instances of $Z$, called $Z_0, Z_1, \ldots, Z_{n-1}$. In $Z_i$, let
The semi-edges $a_i$, $b_i$, $c_i$, $d_i$, and $e_i$ denote the semi-edges corresponding, respectively, to the semi-edges $a$, $b$, $c$, $d$, and $e$ of $Z$, and let $f_i$ be the edge of $Z_i$ corresponding to the edge $f$ of $Z$ shown in Figure 3. For each $Z_i$ we also add the vertices $u_i$, $v_i$, and $w_i$ to the graph $G_n$. The semi-edge $a_i$ is then joined to the vertex $u_i$, the semi-edge $b_i$ is joined to the vertex $v_i$, and the semi-edge $e_i$ is joined to the vertex $w_i$. The edge $u_iv_i$ is added. Finally, the semi-edge $c_i$ is joined to $u_{i+1}$, the semi-edge $d_i$ is joined to $v_{i+1}$, and the vertex $w_i$ is joined to the vertex $w_{i+1}$, where the subscripts are reduced modulo $n$. The graph $G_4$ is illustrated in Figure 4.

\[\text{Figure 4: Graph } G_4.\]

**Theorem 6.** For every integer $n \geq 3$ there exists a cubic graph $G_n$ on $34n$ vertices with $r(G_n) = n$ and $r_f(G_n) = 2n$. If $n \geq 5$, then $G_n$ is a nontrivial snark.

**Proof.** Clearly, $G_n$ has girth 5 unless $n \leq 4$. It is straightforward to check that if $n \geq 4$, then $G_n$ is cyclically 4-edge-connected. Indeed, from the construction of $G_n$ it is clear that $G_n$ has no bridges, 2-edge-cuts, and nontrivial 3-edge-cuts. Thus the smallest cycle-separating edge-cut is of size at least 4 (and one of size 4 can be easily identified). The details are left to the reader.

We now prove that each $G_n$ has the stated values of resistance and flow resistance.

Since $G_n$ contains $n$ disjoint instances of $Z$, Lemma 5 (i) implies that $r(G_n) \geq n$. Moreover, since $Z_i-f_i$ is 3-edge-colourable for each $i$, it is easily seen that $G_n-\{f_1, \ldots, f_n\}$ is 3-edge-colourable, whence $r(G_n) = n$. 

\[\text{Figure 4: Graph } G_4.\]
To prove that $r_f(G_n) \geq 2n$, observe that between any two instances of $Z$ there exists at least one vertex. Therefore $r_f(G_n) \geq 2n$, otherwise there would be an instance of $Z$ with fewer than two zero edges, contradicting Lemma 5. Furthermore, since each $Z_i$ admits a 4-flow with the only zero edges being $f_i$ and $e_i$ for each $i$, it is easy to find a 4-flow on $G_n$ with the only zero edges being from $\{f_1, \ldots, f_n\}$ and from the set of $n$ edges which join $w_i$ to $Z_i$ for each $i$. Therefore, $r_f(G_n) = 2n$.

4 Remarks

4.1. Flow resistance is an uncolourability measure which certainly merits further study. One possible direction, motivated by an obvious approach to the 5-flow conjecture of Tutte [16], is bounding the number of zeros in a 4-flow. This line of research relates the 5-flow conjecture to other important conjectures in the area, in particular to the celebrated conjecture of Fulkerson [5]. Recall that Fulkerson’s conjecture suggests that every bridgeless cubic graph admits a list of six perfect matchings that together cover every edge exactly twice. It is easy to see that once a bridgeless cubic graph fulfils the conjecture, it has a pair of perfect matchings whose intersection covers at most $1/15$ of the number of edges. Now we can use the inequality between $r_f$ and $\gamma_2$ proved Proposition 29 of [4] to conclude that

$$r_f(G) \leq \gamma_2(G) \leq m/15$$

where $m$ is the number of edges. Moreover, the Petersen graph certifies that this bound is the best that one can hope for general bridgeless cubic graphs.

On the other hand, Kaiser et al. in [9] employ Edmonds’ perfect matching polytope theorem [12, Theorem 25.1] to prove that every bridgeless cubic graph contains two perfect matchings $M_1$ and $M_2$ that together cover at least $3m/5$ edges. It follows that

$$r_f(G) \leq \gamma_2(G) \leq |M_1 \cap M_2| \leq 2m/3 - 3m/5 = m/15$$

without assuming Fulkerson’s conjecture to be true. Thus the bound (1) holds for every bridgeless cubic graph, a fact in support of the validity of Fulkerson’s conjecture.

4.2. Let $G$ be a cubic graph. A 1-reduction of $G$ is a graph obtained by the removal of adjacent vertices $u$ and $v$ from $G$, and the subsequent addition of edges to restore 3-regularity. In [1, Conjecture 1], it was conjectured that there exists a 1-reduction $G'$ of every bridgeless cubic graph $G$ with $r(G) > 2$, such that $r(G) > r(G')$. The graphs $G_n$ for $n \geq 3$ as defined in this paper are counterexamples to this conjecture. Consider $G_3$, for example. Each of the three instances of $Z$ contribute to the resistance by 1. Then, any 1-reduction of $G_3$ which could potentially reduce resistance must be contained in an instance of $Z$. Furthermore, the edge in the instance of $Z$ which is being 1-reduced must be contained in the central instance of $X$, otherwise there remains an instance of $Y$, and resistance would not be affected. However, assuming that resistance is reduced after a 1-reduction in the central instance of $X$ in an instance of $Z$, a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, denoted by $\phi$, would violate Kirchhoff’s Law. Indeed, from Lemma 3 we would have
Figure 5: A counterexample to Conjecture 1 on 50 vertices, with cyclic connectivity 3, resistance 2, and flow resistance 3.

\[ \phi(a) = \phi(b) \text{ and } \phi(c) = \phi(d), \]

implying that \( \phi(e) = 0 \), which contradicts the assumption that resistance has been reduced in that instance of \( Z \). Therefore, \( G_n \) is a counterexample to the conjecture for \( n \geq 3 \).

4.3. The smallest nontrivial snark that provides a counterexample to Conjecture 1 and arises from the construction preceding Theorem 6 is the graph \( G_5 \), which has 170 vertices. A significantly smaller counterexample can be constructed in a manner similar to \( G_n \) for \( n = 2 \), with the only difference that the semi-edges \( e_1 \) of \( Z_1 \) and \( e_2 \) of \( Z_2 \) are not joined to the vertices \( w_1 \) and \( w_2 \), respectively, but instead are joined directly to each other. The resulting graph has 66 vertices, resistance 2, flow resistance 3, and is clearly a nontrivial snark. The number of vertices can be further decreased to 50 if we do not insist that the counterexample be a nontrivial snark, see Figure 5. Arguments are similar to those in the proof of Theorem 6.

Problem 7. Determine the smallest order of a snark \( G \) (trivial or nontrivial) with \( r_f(G) > r(G) \).

Another related question to ask is the following.

Problem 8. Can the ratio of \( r_f(G) \) to \( r(G) \) be arbitrarily large?
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