STRUCTURE THEORY OF SET ADDITION III. RESULTS AND PROBLEMS.

GREGORY A. FREIMAN

1. INTRODUCTION

This review is motivated by a powerful and breathtaking development in “Additive Combinatorics”, the direction of study connected with the names of Timothy Gowers, Jean Bourgain, Terence Tao and Ben Green.

The results and methods of Inverse Additive Number Theory have been used substantially in this new field.

Additive Combinatorics now looks towards algebra and computation. Results in these areas existed in Inverse Additive Number Theory and they will be discussed below.

This paper is a continuation of the reviews [32] and [33].

2. THE MAIN THEOREM

We consider a finite subset $A$ of $\mathbb{Z}^n$ (and often $\mathbb{Z}$) with cardinality $|A| = k$. We define

$$2A = A + A = \{x : x = a + b, a \in A, b \in A\}.$$

We say that $A$ has the small doubling property if

$$|2A| < Ck,$$

where $C$ is a positive constant or a slowly increasing function. The number

$$\frac{|2A|}{k}$$

is called the doubling coefficient of $A$.

We define the notion of additive isomorphism of $A \subseteq \mathbb{Z}^n$ onto $B \subseteq \mathbb{Z}^m$ as a bijection $\varphi : A \to B$ such that, for all $a, b, c, d$ in $A$, we have

$$\varphi(a) + \varphi(b) = \varphi(c) + \varphi(d) \tag{1}$$

if and only if

$$a + b = c + d.$$
We write $A \sim B$ and note that $\sim$ is an equivalence relation. We name a $d$-dimensional parallelepiped the subset $D \subseteq \mathbb{Z}^d$

$$D = \{(x_1, x_2, \ldots, x_d : x_i \in \mathbb{Z}, 0 \leq x_i < h_i, h_i \geq 2, 1 \leq i \leq d)\},$$

where $|D| = h_1h_2 \ldots h_d$.

**Main Theorem.** For all finite subsets $A \in \mathbb{Z}$ for which $|2A| < Ck$, where $C$ is a positive constant or $C = C(k)$ is a slowly increasing function, there exist $c = c(C)$ and a parallelepiped $D$ with $d \leq [C - 1]$ and $|D| < ck$ such that $A \subseteq \varphi(D)$, where $\varphi$ is an isomorphism.

For a given $A \in \mathbb{Z}$, we consider the images $\varphi(A)$ under all isomorphisms $\varphi : A \rightarrow \mathbb{Z}^n$, for all $n \geq 1$. The dimension of any $\varphi(A)$ is the dimension of the smallest affine subspace containing it. The maximum dimension of $\varphi(A)$ for all such $\varphi$ is called the dimension of $A$ and denoted $d(A)$.

We will define the notion of volume of $A$. Let $A \in \mathbb{Z}^n$, where $n$ is the dimension of $A$. For each isomorphism $\varphi : A \rightarrow \mathbb{Z}^n$, let us take the convex hull of $\varphi(A)$ and let $|\varphi(A)|$ be the number of integer points in it. Then we define

$$V(A) = \min_{\varphi} |\varphi(A)|.$$

The proof of the Main Theorem was obtained gradually (see [14]–[19]). The final text of the proof may be found in [13] and Yuri Bilu’s paper [2]. Somewhat different proofs were given by Ruzsa [41], [42], Mei-Chu Chang [11] and Sanders [43], with better estimates of volume.

Now, some remarks on estimates of $V(A)$, the volume of $A$. We can also estimate $V(A)$ by the volume

$$|D| = h_1h_2 \ldots h_d$$

of a $d$-dimensional parallelepiped $D$ with a minimal $|D|$ and $A \subseteq \varphi(D)$, where $d$ is the dimension of $A$ and $\varphi$ is an isomorphism. The set $\varphi(D) \subseteq \mathbb{Z}$ is called a generalized proper arithmetic progression of dimension $d$.

Existing estimates were obtained with the help of $\varphi(D)$, where $\varphi$ is not an isomorphism but a homomorphism giving a generalized arithmetic progression which is not proper. As a result, we obtain an estimate of the volume $V(A)$ and dimension of $A$, which wait to be improved. Below, we will give a hypothetical best possible estimate.

### 3. Hypothetical value of $V(A)$

We begin with an example which does not cover all possible values of $T = |2A|$ but is very simple and representative.

Let

$$A = \{0, 1, \ldots, m - 1, 2(m - 1), 4(m - 1), \ldots, 2^{c-2}(m - 1)\}$$
where, as usual, \(|A| = k\), and
\[3 \leq m \leq k, \quad (2)\]
\[c = k + 2 - m, \quad (3)\]
so that \(2 \leq c \leq k - 1\).

In this case we have \(d(A) = 1\) and the values of \(V\) and \(T\) have a very simple form for \(c\) which is not very large:
\[V = 2^{c-2}(m - 1) + 1 = \frac{1}{4}2^c(k + 1 - c) + 1 = \frac{1}{4}2^ck + O(c2^c) \quad (4)\]
and
\[T = (k - 1) + (k - 2) + \cdots + m + 2m - 1 = \frac{k - 1 + m}{2}(k - m) + 2m - 1 =\]
\[= \frac{k - 1 + k + 2 - c}{2}(c - 2) + 2(k + 2 - c) - 1 =\]
\[= \frac{ck - (c - 1)(c - 2)}{2} - 2(c - 2) - 1 = ck - \frac{c^2 + c - 4}{2} =\]
\[= ck + O(c^2). \quad (5)\]

If \(c = O(\log^\delta k), \delta < 1\), then, for \(V\) and \(T\), (4) and (5) give very simple expressions, sufficient for today’s use.

Now, we will give examples of the sets \(A\), for which \(T\) takes all possible values. We will see, that the “net” of sets \(A\), given earlier, represents the general situation in a very good way.

For values \(m\) and \(c\) as in (2) and (3), let
\[a_{m-1} = m - 1 + b, \quad (6)\]
where
\[0 \leq b \leq m - 3 = k - c - 1, \quad (7)\]
and denote
\[A = \{0, 1, \ldots, m - 2, a_{m-1}, 2a_{m-1}, 4a_{m-1}, \ldots, 2^{c-2}a_{m-1}\}, \quad (8)\]
where conditions (6) and (7) are valid. We get
\[V = 2^{c-2}a_m + 1 = \frac{1}{4}2^c(m - 1 + b) + 1 =\]
\[= \frac{1}{4}2^c(k - c + 1 + b) + 1 = \frac{1}{4}2^c(k + b) + O(c2^c), \quad (9)\]
and
\[T = (k - 1) + (k - 2) + \cdots + m + 2m - 1 + b = ck + b + O(c^2). \quad (10)\]
If \( d(A) > 1 \), we build the set \( A \) in the following manner. Let
\[
A = A_d = \{ A'e_1, e_2, \ldots, e_d \}
\]
where \( A' \subseteq \mathbb{Z} \) will be defined a bit later.

For a given value \( T \), we define
\[
|2A'| = T - (k + (k - 1) + (k - 2) + \cdots + (k - d + 2)).
\]
We have
\[
k' = |A'| = k - d + 1.
\]

For values \( k' \) and \(|2A'|\) given in (12) and (13), we can build the set \( A' \) as in (8) and the set \( A_d = A \) with the help of (11).

In the case where the values of \( k \) and \( T \) are given, we can also get the corresponding values of \( V(A) \) as a function of \( k \) and \( T \). It was done in [13, p. 37], where the hypothesis about the best estimate of \( V(A) \) was formulated. We may reproduce it now in a nicer form.

**Hypothesis.** The inequality
\[
V(A) \leq \frac{1}{4} 2^{c}(k - c + 1 + b) + 1
\]
gives an estimate from above for \( V(A) \) for a given \( k \) and \( T \) which cannot be improved.

If this hypothesis were proved to be true, the proof would be important not only as an improvement of an estimate of \( V(A) \) for a given doubling coefficient \( T \), but also the range of values \( T \) would cover all possible values at the time when existing estimates make sense only for rather small values of \( T \).

The set \( A \) may, because of (11), be put in a parallelepiped \( D \) with edges
\[
h_1 = V(A'), \quad h_2 = h_3 = \ldots h_d = 2
\]
and
\[
V(D) = 2^{d-1}V(A').
\]
We may define the volume of a set \( A \) with the help of the volume of \( D \supseteq A \) for a minimal \(|D|\) for such \( D \) (a proper \( d \)-dimensional arithmetical progression) and formulate a hypothesis about minimal volume in a simpler though less exact way.

4. **On a family of extremal sets**

We would now like to go one step deeper into the study of the structure of \( A \). We have used a concrete example of \( A \) with given values \( k \) and \( T \) for which the maximal value given in (14) is achieved.

Let us ask about a list of such \( A \). Here, as is usual in solving an inverse additive problem, see [32, p. 5], we study the structure of the set when its characteristic is equal to its extremal value.
As an example of such sets, let us build a family of sets \( \{B\} \) which are “friendly” to \( A \) defined in (8) in the following manner:

\[
|B| = |A| = k, \\
B = \{b_0, b_1, \ldots, b_{k-1}\}.
\]

Define \( B(s) \) as a subset of \( B \) containing the first \( s \) elements of \( B \):

\[
B(s) = \{b_0, b_1, \ldots, b_{s-1}\}.
\]

Let \( B(m) \) be one of the sets described in Theorem 1.9 of [13], and in the formulation given at the end of p.11, we have

\[
b_0 = 0, b_{m-1} = a_{m-1} = m - 1 + b, \quad |2B(m)| = 2m - 1 + b.
\]

Let \( p_m \) be the number of such sets. Define the sets \( B(s) \) for \( m \leq s \leq k \) in the following manner: Let the sets for some value which is less than or equal to some chosen value of \( s \) be already known. Take \( B(s) \) which is one of them. Define

\[
sym(B(s)) = \max B(s) - B(s),
\]

and, for a chosen \( B(s) \), take two possibilities

\[
B'(s + 1) = \{B(s) \cup \{2^{s-m+1}a_{m-1}\}\}
\]

or

\[
B''(s + 1) = \{\text{sym } B'(s + 1)\}.
\]

Note that for \( s > m \) we have

\[
\max B(s) = 2^{s-m}a_{m-1}
\]

and so

\[
\max B(k) = 2^{k-m}a_{m-1} = 1/4 * 2^c(k - c + 1 + b). \tag{17}
\]

The number of sets in the family \( \{B\} \) is equal to \( 2^{c-2}p_m \).

The values \( V \) and \( T \), for each set \( B \) just built, are the same as in (9) and (10).

Let us give a numerical example. Take \( k = 8, m = 6 \) and because of (3) we get \( c = 4 \). Further, take in \( a_{m-1} = m - 1 + b, b = 2 \) and \( a_5 = 7 \).

For sets \( \{B(6)\} \) we obtain the following list:

\[
\{B(6)\} = \{0, 1, 2, 3, 4, 7\}, \{0, 1, 2, 3, 5, 7\}, \{0, 3, 4, 5, 6, 7\}, \\
\{0, 2, 4, 5, 6, 7\}, \{0, 2, 3, 4, 5, 7\}. \tag{18}
\]

Therefore, the number of these sets is \( p_6 = 5 \).

From (9) and (10) we get

\[
T = 7 + 6 + 12 - 1 + 2 = 26, \tag{19}
\]

\[
V = \frac{1}{4}2^4(6 - 1 + 2) + 1 = 29. \tag{20}
\]
Take, for example, the set \( \{0, 1, 2, 3, 4, 7\} \) from \( \{B(6)\} \). Using (9) and (10), we get two sets \( \{0, 1, 2, 3, 4, 7, 14\} \) and \( \{0, 7, 10, 11, 12, 13, 14\} \) from \( \{B(7)\} \) and four sets \( \{0, 1, 2, 3, 4, 7, 14, 28\} \), \( \{0, 14, 21, 25, 26, 27, 28\} \), \( \{0, 7, 10, 11, 12, 13, 14, 28\} \) and \( \{0, 14, 15, 16, 17, 18, 21, 28\} \) from \( \{B(8)\} \).

As a result we will get 20 sets from \( \{B(8)\} \). We gave a partial list of extremal sets which gives some idea of how to construct them.

5. Partial case: \( T \leq 4k - 7 \)

Let us use the following lemma, see [13, p. 24].

**Lemma.** Let \( A \subseteq E^n \) be a finite set of dimension \( n \) with \( |A| = k \). Then

\[
T \geq (n + 1)k - \frac{n(n + 1)}{2}.
\]

We see that, if \( n = 2 \), then

\[
T \geq 3k - 3
\]

and if \( n = 3 \) then

\[
T \geq 4k - 6. \tag{22}
\]

From (21) and (22), we see that if \( T \leq 3k - 4 \) then \( d(A) = 1 \). This case was studied in detail in [13, p. 11–14]. As to the case,

\[
3k - 3 \leq T \leq 4k - 7,
\]

it gives a good example of the Hypothesis for a very simple case. In this case, (8) takes the form

\[
A = 0, 1, 2, \ldots, k - 3, k - 2 + b, 2(k - 2 + b)
\]

where

\[
T = k - 1 + 2(k - 1) - 1 + b = 3k - 4 + b,
\]

\[
0 \leq b \leq k - 3.
\]

Then, according to the hypothesis, \( A \) is part of a segment of length \( 2k - 3 + 2b \) or has dimension two. This case is much simpler than the general case (on the second page of this article); I had formulated it immediately after Theorem 1 was proved, but even in this simpler case, the problem is still open (for some progress, see Jin [38]).
Over the last few years, there has been exciting development in group theory (Tao [44], Breuillard and Green [7], [8], Green [36], Fisher, Katz and Peng [12], Hrushovski [37]) connected with sets with small doubling. I’m referring to the results in approximation groups.

An approximation group is a finite subset of a group which, together with the property of small doubling, contains a unit and has a symmetry property.

Here are some of my old results in this direction:

**Theorem I.** Let $G$ be a torsion-free group, $K$ and $M$ subsets of $G$, for which

$$2 \leq |K|, |M|$$

and

$$|KM| = |K| + |M| - 1,$$

then $K$ and $M$ have the form

$$K = \{a, aq, \ldots , aq^{l-1}\}$$

and

$$M = \{b, qb, \ldots , q^{h-1}b\},$$

where

$$a, b, q \in G, q \neq 1.$$

And what about $K = M$?

**Theorem II.** If $|K^2| = 2|K| - 1$ then $K$ has the form (23) and either $aq = qa$ or $qaq = a$ holds.

The proof is in [28].

The immediate problem here is to describe the structure of the subset $A \subset G$ in a case when $|A^2| \leq 3k - 4$ and the group $G$ is torsion-free. In the study of approximate groups, the study of subsets of groups of special kinds take place, and this is a very natural and rewarding turn. If $G$ is not torsion free then the doubling coefficient may be less than two. In this case, the structure of $A$ was described in [20] for the doubling coefficient $8/5$. What about bigger coefficients?

7. **All small subsets have small doubling**

Now, we will describe the study of group structure, which is different from the fast growing study of approximate groups, but which also uses the idea of small doubling as a main tool, and sometimes arrives at similar results. They show that some subsets of a group have the structure of a subgroup, a coset or a union of cosets for some normal subgroup. Let us translate the notion of isomorphic subsets to the notion of similarity of multiplication tables (Latin squares). We will now call an algebraic
operation a multiplication, and take a product $ab$ instead of the sum $a + b$. We will enter the elements of the sets $D$ and $F$ into the following tables:

\[
\begin{array}{ccc}
\text{a} & \text{b} & \text{c} \\
\text{a} & \text{A} & \text{B} & \text{C} \\
\text{b} & \text{c} \\
\end{array}
\]  

(24)

and

\[
\begin{array}{ccc}
\text{x} & \text{y} & \text{z} \\
\text{x} \\
\text{y} \\
\text{z} \\
\end{array}
\]

(25)

Here $a, b, c, \ldots$ in $D$ and $x, y, z, \ldots$ in $F$.

Entries in a multiplication table may be put in some freely chosen order. So, let us change the order of columns in (25), and correspondingly the order of lines in (25), in such a way that we will get the following table instead of (25):

\[
\begin{array}{ccc}
\varphi(a) & \varphi(b) & \varphi(c) \\
\varphi(a) \\
\varphi(b) \\
\varphi(c) \\
\end{array}
\]

(26)

As to notation inside the multiplication table (24), the elements of $D^2$ and $D$ may be different, and we denote elements of $D$ by lower-case Latin letters and elements of $D^2$ by upper-case ones, and we denote equal elements of $D^2$ by the same letter and different ones by different letters. And now, how is condition (1) in multiplicative form

\[
ab = cd \iff \varphi(a)\varphi(b) = \varphi(c)\varphi(d)
\]

formulated in the case of Latin squares? The condition $ab = cd$ means the element of the table on the line with entry $a$ and on column $b$ is equal to the element with line $c$ and column $d$. The condition $\varphi(a)\varphi(b) = \varphi(c)\varphi(d)$ tells us that elements in the same places in the table (26) are equal and have the same notation. If the equality is not valid in table (25), it is not valid for corresponding places in table (26), as well. So, as formulated in [22, p.141], let $D$ and $F$ be Latin squares of the same order; also, let $\theta$ be a bijection of $D$ onto $F$. Applying $\theta$ to $D$, we have a new Latin square $\theta[D]$. If $F$ can be obtained from $\theta[D]$ by a permutation of rows and the same permutation of columns of $\theta[D]$, then $D$ and $F$ are said to be isomorphic. In paper [22] the classification of all isomorphic classes of three-element sets was presented. The number of commutative classes is six, and of noncommutative classes is 45. Having these and similar results in mind, we can now ask plenty questions.

Some examples.
1) Let us have, for each three-element sets of a group \( G \), \(|A^2| \leq a\), \(a \leq 8\). All such groups were described in [1], [39]. What if \(|A| = 4\) and \(|A^2| \leq a\), \(a \leq 15\)?

2) Take one subset of three elements with a given multiplication table. Describe all groups without this Latin square. Do the same for a given family of Latin squares. Looking at groups in an artistic way, we can think of a group as a complete building. The subgroups will then be, say, storeys of this building, and small subsets, described with the help of their multiplication tables, as the bricks used to construct the building. The kind of material used in the course of construction has an influence on the properties of the whole building. We may begin to analyze kinds and numbers of different kinds of small subsets which are used in groups of some known kinds, say, crystallographic groups. It would be important, though not that simple, to find some connection between the properties of such groups and some properties of their composition.

8. SUBSETS OF SPECIAL ELEMENTS

Let us introduce a notion of an \((n, m)\)-special element in a group \( G \). Element \( a \in G \) is called \((n, m)\) special if \(|K^n| \leq m < 2^n\), for all \( K = \{a, g\}, g \in G \), where \( n, m \) are positive integers \( m, n \geq 2 \). By \( K^n \) we mean the set of elements \( g \in G \) having at least one representation of the form \( g = k_1, k_2, \ldots, k_n \). The set of all the \((n, m)\)-special elements is denoted by \( S_{m,n}(G) \). The notion of a special element is attributed to J.G. Berkovich. In [6], it was shown that \( S_{2,3}(G) \) and \( S_{3,5}(G) \) are normal subgroups of \( G \). Some further results and questions in this direction can be found in [6]. The main problem here: Let \( n \) be fixed. For which \( m \) the set \( S \) is a normal subgroup of \( G \), union of cosets,...?

9. Computation complexity

Results in Integer Programming were obtained mainly between the years 1987 and 1990, see [9], [10], [21], [25], [26], [27], [30], [31]. The main problem which was studied there: Let an equation

\[
a_{1}x_{1} + a_{2}x_{2} + \cdots + a_{m}x_{m} = b
\]

be given, where \( a_{i} \) are different large positive integers. Do solutions exist of this equation or, perhaps, there are no solutions? The unknowns \( x_{i} \) take the values 0 or 1. Direct computation, see, for example [40], gave a solution to the problem for \( m \) equal to two-three hundred, and \( a_{i} \) of order \( 10^{10} \). The use of analytical methods of Additive Number Theory enabled us to formulate general conditions when a solution and, in fact, many solutions exist. Methods of Inverse Additive Number Theory enabled us to build the structure of the set \( A = a_{i} \) in the case where there are no solutions. Algorithms and programs were built for values of \( m \) of order \( 10^{7} \) allowing us to produce computations in a split second (see [10]). A problem for further study
is to go from one equation to a system of several equations (see [31] and the thesis of Alain Plagne).

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G. A. Freiman
School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel.
E-mail address: grisha@post.tau.ac.il