When $\text{LOCC}$ offers no advantage over finite $\text{LOCC}$

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Abstract

We consider bipartite LOCC, the class of operations implementable by local quantum operations and classical communication between two parties. Surprisingly, there are operations that cannot be implemented with finitely many messages but can be approximated to arbitrary precision with more and more messages. This significantly complicates the analysis of what can or cannot be approximated with LOCC. Towards alleviating this problem, we exhibit two scenarios in which allowing vanishing error does not help. The first scenario involves implementation of measurements with projective product measurement operators. The second scenario is the discrimination of unextendible product bases on two 3-dimensional systems.

1 Introduction

We consider bipartite finite dimensional quantum systems, and what state transformations can be achieved given arbitrary quantum operations on each system and classical communication between them. This class of quantum operations is known as LOCC. LOCC arises in many natural settings. For example, it is much easier to transmit classical data than quantum data over long distances. For another example, quantum gates involving multiple registers are much harder to implement, and methods to effect them using entangled states, measurements, and classical feedback hold high promise. The study of LOCC also provides insights on the nature of quantum information, leading to discoveries including teleportation [BBC+93], quantum error correcting codes [BDSW96], and security proofs for quantum key exchange [LC99, SP00].

Unfortunately, LOCC, as an operationally defined class, does not have a succinct mathematical description. Traditionally, one turns to relaxations of LOCC such as SEP or PPT instead of analysing LOCC operations. This method has proved fruitful for many problems such as data hiding [DLT02] and state discrimination [Wat05]. Yet this approach fails to answer other interesting questions such as whether more rounds of communication, or infinitely many intermediate measurement outcomes can make a difference, or whether there are operations that can be approximated arbitrarily closely with LOCC but do not belong to LOCC (i.e., whether LOCC is equal to its topological closure $\overline{\text{LOCC}}$ or not). Recent investigation of the LOCC class itself has resolved these questions; for example, more communication rounds can be helpful [Chi11] and $\text{LOCC} \neq \overline{\text{LOCC}}$ [Chi11, CCL12b, CCL12a, CLM+12].
A common technique to prove that a certain task cannot be accomplished perfectly by a finite LOCC protocol is to start by assuming the contrary. Then the properties of steps taken in any perfect implementation of the task are shown to be incompatible with the structure of an LOCC protocol. However, it could still be possible to accomplish the task with vanishing error using LOCC protocols. Excluding the possibility of approaching perfect implementation is much harder and few results have been established [BDF+99, KTYI07, KKB11, CLMO13, CH13b, CH13a].

In this paper, we focus on two problems concerning LOCC. Our first problem is, are there natural classes of measurements in LOCC that are closed? In other words, are there sets of measurements such that it does not help to allow vanishing error and more and more rounds of communication? Our second problem concerns the possibility of discriminating sets of orthogonal product states called UPBs (for unextendible product bases though these states are not bases) in LOCC.

We first summarize prior works in the two problems of interest. The first example of a task for which LOCC gives no advantage over finite LOCC concerns perfect discrimination of the so-called domino states [BDF+99]. They cannot be discriminated by either set. Over a decade later a generalization was reached by establishing that the set of full basis measurements implementable by LOCC is closed [KKB11]. For our second question, [Ter99, DMS+03] established that UPBs cannot be distinguished with finite LOCC. Reference [Rin04] studied discrimination of UPBs in LOCC. However, as pointed out in [KKB11], the proof in [Rin04] is incomplete and a particular claim in the proof contradicts other proven results.

We make partial progress in the two problems posed above. First, we show that the set of LOCC implementable projective measurements with tensor product operators is closed (see Theorem 3). Second, we prove that LOCC cannot be used to perfectly discriminate states from a UPB in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \) (see Theorem 9). Both results are applications of a necessary condition from [KKB11].

The rest of the paper is organized as follows. In Section 2 we set up the notation and give basic definitions. In Section 3 we identify a closed class of projective measurements that can be implemented with finite LOCC protocols and provide an application of this result. In Section 4 we establish that two-qutrit UPBs cannot be perfectly discriminated even with LOCC. We conclude in Section 5.

2 Preliminaries

Finite and asymptotic LOCC

We say that a measurement \( \mathcal{M} \) can be implemented with finite LOCC, and write \( \mathcal{M} \in \text{LOCC}_N \), if \( \mathcal{M} \) can be implemented exactly using a finite LOCC protocol (i.e., an LOCC protocol with finitely many communication rounds). We say that \( \mathcal{M} \) can be implemented using asymptotic LOCC, and write \( \mathcal{M} \in \text{LOCC} \), if there exists a sequence of finite LOCC protocols \( \mathcal{P}_1, \mathcal{P}_2, \ldots \) that implement \( \mathcal{M} \) with vanishing error. Note that asymptotic LOCC is the (topological) closure of finite LOCC. It represents the set of operations that can be implemented by LOCC protocols with arbitrary precision. For more detailed explanation of classes \( \text{LOCC}_N \) and \( \text{LOCC} \) see [CLM+12, Man13].

State discrimination problem

Let \( [n] := \{1, \ldots, n\} \), \( \pi : [n] \to [0,1] \) be a probability distribution, and \( S = \{\rho_i : i \in [n]\} \subseteq \mathbb{C}^d_A \otimes \mathbb{C}^d_B \) be a set of bipartite states. In the state discrimination problem an index \( i \) is chosen with
probability $\pi(i)$ and Alice and Bob are given their respective registers of $\rho_i$. Their task is to find
the index $i$ without error. The scenario where they are allowed to err is also of interest but will
not be studied in this paper. In the error free case the probability distribution $\pi$ is not relevant and
will therefore be chosen to be uniform. We say that the states from $S$ can be discriminated with
finite LOCC (asymptotic LOCC), if there exists a measurement $M \in \text{LOCC}_N$ ($M \in \text{LOCC}$) that
discriminates the states perfectly.

There is a close connection between discrimination of mutually orthogonal states and implement-
tation of projective measurements. In particular, the states from $S$ can be discriminated by finite
LOCC (asymptotic LOCC) if and only if finite LOCC (asymptotic LOCC) can be used to implement
the projective measurement onto the supports of the states $\rho_i$ [CLMO13].

**Non-disturbing measurements**

We now introduce the concept of non-disturbing operators which is central for state discrimina-
tion with LOCC.

**Definition 1.** Let $S \subseteq \text{Pos}(\mathbb{C}^d)$ be a set of orthogonal states. We say that $E \in \text{Pos}(\mathbb{C}^d)$ is non-
disturbing for $S$, if

$$\text{Tr}(E\rho E\sigma) = 0$$

(1)

for all distinct $\rho, \sigma \in S$. We say that a measurement $M$ is non-disturbing for $S$ if each of its POVM
elements of $M$ is non-disturbing for $S$.

Let $\text{supp}(M)$ denote the support of $M$. Then Condition (1) is equivalent to requiring that for all
distinct $\rho, \sigma \in S$ and all $|\psi\rangle \in \text{supp}(\rho)$ and $|\phi\rangle \in \text{supp}(\sigma)$

$$\langle \psi | E | \phi \rangle = 0.$$ (2)

Note that any measurement protocol transforms $S$ to a new set conditioned on the culmulative
measurement outcome. In a perfect discrimination protocol for $S$, at any point, the next measure-
ment applied to this conditioned set must not disturb it. In particular, the protocol must start with
a measurement that is non-disturbing for $S$. In an LOCC protocol each measurement must be
local. For finite LOCC, each measurement has to be non-trivial. Hence, the states from a set $S$ can
be perfectly discriminated with finite LOCC only if $S$ admits a non-disturbing product operator $a \otimes b$ where exactly one of the matrices $a, b$ is the identity matrix. If such an operator does not
exist, the states from $S$ cannot be discriminated with finite LOCC. Non-disturbing operators also
provide a necessary condition for state discrimination with asymptotic LOCC.

**Theorem 1 ([KKB11]).** Consider a set of states $S = \{\rho_1, \ldots, \rho_n\} \subseteq \text{Pos}(\mathbb{C}^d_A \otimes \mathbb{C}^d_B)$ such that $\bigcap_i \ker \rho_i$
does not contain any nonzero product vector. Then $S$ can be discriminated with asymptotic LOCC only if
for all $\chi$ with $1/n \leq \chi \leq 1$ there exists a positive semidefinite product operator $E = a \otimes b$ satisfying all of
the following:

1. $\sum_{\rho \in S} \text{Tr}(E\rho) = 1$,
2. $\max_{\rho \in S} \text{Tr}(E\rho) = \chi$,
3. $E$ is non-disturbing for $S$.

Theorem 1 implies that the set of LOCC implementable full basis measurements is closed.
Corollary 2 ([KKB11]). If a full orthogonal basis measurement can be implemented using asymptotic LOCC then it can already be implemented with finite LOCC.

In the next section we generalize Corollary 2 for a larger class of projective measurements.

3 Projective measurements with tensor product operators

In light of the findings of [KKB11] presented in the previous section it is natural to ask whether a similar result holds for the class of all projective measurements, or even for the class of all POVM measurements. To this end, in this section we show that the class of all projective measurements with tensor product operators that can be implemented with finite LOCC is indeed closed.

3.1 Results

Theorem 3. Let \( M = \{P^A_i \otimes P^B_i\}_{i \in [n]} \subseteq \text{Pos}(\mathbb{C}^d_A \otimes \mathbb{C}^d_B) \) be a projective measurement. Then \( M \in \text{LOCC} \) implies that \( M \in \text{LOCC}^N \).

We say that a set of states \( S = \{\rho_i\}_{i \in [n]} \subseteq \text{Pos}(\mathbb{C}^d_A \otimes \mathbb{C}^d_B) \) is a full orthogonal set if the states \( \rho_i \) are mutually orthogonal, and \( \sum_i \rho_i \) has full rank. We now rephrase Theorem 3 in terms of state discrimination.

Theorem 4. Let \( S = \{\rho_i := \tau_i \otimes \sigma_i\}_{i \in [n]} \subseteq \text{Pos}(\mathbb{C}^d_A \otimes \mathbb{C}^d_B) \) be a full orthogonal set. If the states from \( S \) can be discriminated with asymptotic LOCC then they can be discriminated with finite LOCC.

We first prove two lemmas that help establish Theorem 4. The main ingredient for proving Theorem 4 is the construction of an operator \( m \) such that either \( m \otimes I \) or \( I \otimes m \) is non-disturbing for \( S \). A matrix \( M \) is non-disturbing for a complete orthogonal set \( S = \{\rho_i\}_{i \in [n]} \) if and only if the row spaces \( \mathcal{H}_i \) of \( \rho_i \) are \( M \)-invariant for each \( i \). Here, a subspace \( \mathcal{H} \) is said to be \( M \)-invariant, if \( M|\psi\rangle \in \mathcal{H} \) for all \( |\psi\rangle \in \mathcal{H} \).

Our first lemma provides useful characterization of \( M \)-invariance.

Lemma 5. Let \( M \in \text{Herm}(\mathbb{C}^d) \), \( \mathcal{H} \) be an \( h \)-dimensional subspace of \( \mathbb{C}^d \), \( \{|v_i\}_{i \in [h]} \) be a fixed orthonormal basis of \( \mathcal{H} \), \( |i\rangle \in \mathbb{C}^h \), and \( Q := \sum_{i \in [h]} |v_i\rangle \langle i| \). Then the following are equivalent:

1. \( \mathcal{H} \) is \( M \)-invariant;
2. \( MQ = QX \) for an \( h \times h \) matrix \( X \);
3. \( \mathcal{H} \) has an orthonormal basis consisting of eigenvectors of \( M \).

Proof. Note that the operator \( Q \) maps \( \mathbb{C}^d \) to the subspace \( \mathcal{H} \) and the action of \( X \) on \( \mathbb{C}^d \) (in the basis \( |v_i\rangle \)) represents the action of \( M \) on \( \mathcal{H} \) (in the basis \( |v_i\rangle \)).

We first prove (1) \( \Rightarrow \) (2). If \( \mathcal{H} \) is \( M \)-invariant, then for all \( i \in [h] \)

\[
M|v_i\rangle = \sum_{j \in [h]} x_{ji}|v_j\rangle
\]
for some $x_{ji} \in \mathbb{C}$. If we let $X := (x_{ij})$, then

$$MQ = \sum_{i,j \in [h]} x_{ji} |v_j\rangle \langle i| = \sum_{j \in [h]} \left( |v_j\rangle \sum_{i \in [h]} \langle i|x_{ji}\rangle \right) = \sum_{j \in [h]} |v_j\rangle \langle j|X = QX. \quad (4)$$

Next, let us show that (2) $\Rightarrow$ (3). If $MQ = QX$, then $X = Q^\dagger MQ$ as $Q^\dagger Q = I_h$. Since $M$ is Hermitian, $X$ is also Hermitian and it has a spectral decomposition $X = \sum_{i \in [h]} \lambda_i |w_i\rangle \langle w_i|$. Then for all $i \in [h]$

$$MQ|w_i\rangle = QX|w_i\rangle = \lambda_i Q|w_i\rangle. \quad (5)$$

Therefore, the vectors $Q|w_i\rangle \in \mathcal{H}$ are eigenvectors of $M$. Finally, for all $i, j \in [h]$ we have

$$\langle w_i|Q^\dagger Q|w_j\rangle = \langle w_i|w_j\rangle = \delta_{ij}. \quad \text{So the set}$$

$$\{Q|w_i\rangle\}_{i \in [h]} \quad (6)$$

is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $M$.

Last, we prove that (3) $\Rightarrow$ (1). Let $\{|u_i\rangle\}_{i \in [h]} \subseteq \mathcal{H}$ be a set of orthogonal eigenvectors of $M$ with corresponding eigenvalues $\mu_i$. Then any vector $u \in \mathcal{H}$ can be expressed as $|u\rangle = \sum_{i \in [h]} c_i |u_i\rangle$ for some $c_i \in \mathbb{C}$. Now we have

$$M|u\rangle = \sum_{i \in [h]} \mu_i c_i |u_i\rangle \in \mathcal{H} \quad (7)$$

as desired. \hfill \Box

We now show that whenever $a \otimes b \in \text{Pos}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is non-disturbing for a full orthogonal set of product states, so are $a \otimes I$ and $I \otimes b$.

**Lemma 6.** Let $a \otimes b \in \text{Pos}(\mathbb{C}^d \otimes \mathbb{C}^d)$ and $P = P_A \otimes P_B \in \text{Pos}(\mathbb{C}^d \otimes \mathbb{C}^d)$ be a projector onto a subspace $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. If $(a \otimes b)P \neq 0$ and $\mathcal{H}$ is $(a \otimes b)$-invariant, then $\mathcal{H}$ is also $(a \otimes I)$ and $(I \otimes b)$-invariant.

**Proof.** Let $h_A, h_B$ be the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Fix some orthonormal basis $\{|\alpha_i\rangle\}_{i \in [h_A]}$ of $\mathcal{H}_A$ and let $Q_A = \sum_{i \in [h_A]} |\alpha_i\rangle \langle i|$, where $|i\rangle \in \mathbb{C}^{h_A}$. Define $Q_B$ similarly. Then $P_A = Q_A Q_A^\dagger$ and $P_B = Q_B Q_B^\dagger$. If $\mathcal{H}$ is $(a \otimes b)$-invariant, then

$$(a \otimes b)(Q_A \otimes Q_B) = (Q_A \otimes Q_B)X \quad (8)$$

for an $(h_A h_B) \times (h_A h_B)$ matrix $X$. Note that $X$ is a tensor product, since $X = (Q_A^\dagger aQ_A) \otimes (Q_B^\dagger bQ_B)$. Since $(a \otimes b)P \neq 0$ we also have that $(a \otimes b)(Q_A \otimes Q_B) \neq 0$. Hence, Equation (8) together with the fact that $X$ is a tensor product implies that

$$aQ_A = Q_A X_A \text{ and } bQ_B = Q_B X_B \quad (9)$$

for some $X_A$ and $X_B$ such that $X = X_A \otimes X_B$. By Lemma 5, Equation (9) implies that $\mathcal{H}_A$ is $a$-invariant and $\mathcal{H}_B$ is $b$-invariant. Since any subspace is invariant under the identity operation, the lemma follows. \hfill \Box

We are ready to prove Theorem 4 using Lemma 5 and Lemma 6.
Proof of Theorem 4. We prove by induction on \(d_A + d_B\). Clearly the states in \(S\) can be discriminated with both finite and asymptotic LOCC if \(d_A + d_B \leq 3\). We assume that the theorem statement holds for all values \(d_A + d_B < m\) for some \(m \in \mathbb{N}\).

Suppose \(d_A + d_B = m\) and the \(n\) states in \(S\) can be discriminated with asymptotic LOCC. Then for every \(1/n \leq \chi \leq 1\) there exists a product operator \(E = a \otimes b \in \text{Pos}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})\) satisfying the three conditions in Theorem 1. Our goal is to choose appropriate value of \(\chi\) and use Lemma 6 to conclude that both \(a \otimes I\) and \(I \otimes b\) are non-disturbing for \(S\).

Pick any \(\chi \in \left(\frac{1}{n}, \frac{1}{n-1}\right)\) and let \(a \otimes b\) be the corresponding operator. Let us now check that \(a \otimes b\) is nontrivial and satisfies the hypothesis of Lemma 6. The range of \(\chi\) is chosen so that Conditions (1) and (2) together imply that

- \(a \otimes b\) cannot be proportional to the identity matrix (from now on we assume that \(a\) is not proportional to the identity matrix as the other case is similar);
- for all \(\rho \in S\) we have \(E\rho \neq 0\).

For each \(i \in [n]\), let \(\mathcal{H}^{(i)} = \mathcal{H}^{(i)}_a \otimes \mathcal{H}^{(i)}_b\) be the column space of \(\rho_i\) and \(P_i\) be the projector onto \(\mathcal{H}^{(i)}\). Then the last item implies that \((a \otimes b)P_i \neq 0\). Since \(a \otimes b\) is non-disturbing for \(S\), the subspace \(\mathcal{H}^{(i)}\) is \((a \otimes b)\)-invariant. Due to Lemma 6, \(\mathcal{H}^{(i)}\) is \((a \otimes I)\)-invariant.

Let \(a_\lambda\) be the projector onto the \(\lambda\)-eigenspace of \(a\). Due to the equivalence of (1) and (3) in Lemma 5, the subspaces \(\mathcal{H}^{(i)}\) are \((a_\lambda \otimes I)\)-invariant for all \(\lambda \in \text{spec}(a)\). So if \(\mathcal{I}_B\) is the identity measurement on Bob, the nontrivial local projective measurement

\[
\{a_\lambda : \lambda \in \text{spec}(a)\} \otimes \mathcal{I}_B =: \mathcal{M}_A \otimes \mathcal{I}_B
\]

is non-disturbing for \(S\).

Suppose we measure the states in \(S\) using \(\mathcal{M}_A \otimes \mathcal{I}_B\) and obtain outcome \(\lambda \in \text{spec}(a)\). If we restrict the unnormalized post-measurement states to the column space of \(a_\lambda \otimes I\), we have the following set:

\[
S_\lambda := \{(Q_\lambda \otimes I)^\dagger \rho_i (Q_\lambda \otimes I)\}_{i \in [n]} = \left\{(Q_\lambda^i \tau_i Q_\lambda) \otimes \sigma_i\right\}_{i \in [n]} \subseteq \mathbb{C}^{\text{rank}(a_\lambda)} \otimes \mathbb{C}^{d_B}.
\]

Here,

\[
Q_\lambda := \sum_{i \in [\text{rank}(a_\lambda)\]} |\lambda_i\rangle \langle i|,
\]

\(|i\rangle \in \mathbb{C}^{\text{rank}(a_\lambda)}\), and \(\{|\lambda_i\rangle\}_{i} \subseteq \mathbb{C}^{d_A}\) is some orthonormal basis of the \(\lambda\)-eigenspace of \(a\). We now want to use the induction hypothesis to conclude that the states in \(S_\lambda\) can be discriminated with finite LOCC. To do so, we have to check that \(S_\lambda\) is a set of mutually orthogonal states that can be discriminated with asymptotic LOCC and that \(\sum_{\rho \in S_\lambda} \rho\) has full rank.

First, since \(\sum_{i \in [n]} \rho_i\) is positive semidefinite and has full rank and \(Q_\lambda\) has full column rank, the matrix

\[
\sum_{\rho \in S_\lambda} \rho = (Q_\lambda \otimes I)^\dagger \left(\sum_{i \in [n]} \rho_i\right) (Q_\lambda \otimes I)
\]

has full rank. Suppose that a sequence \(\mathcal{R}_1, \mathcal{R}_2, \ldots\) of finite LOCC protocols can be used to certify that the states in \(S\) can be discriminated with asymptotic LOCC. Let \(\mathcal{R}_i\) be the finite LOCC protocol in which Alice first embeds her input space \(\mathbb{C}^{\text{rank}(a_\lambda)}\) in \(\mathbb{C}^{d_A}\) by applying the isometry \(Q_\lambda\) and then the two parties proceed with the protocol \(\mathcal{R}_i\). After the embedding, Alice and Bob have the states \((a_\lambda \otimes I)\rho_i(a_\lambda \otimes I)\) up to a normalization. Since the column space, \(\mathcal{H}^{(i)}\), of \(\rho_i\) is \((a_\lambda \otimes I)\)-invariant,
the column space of \((a_\lambda \otimes I)\) is contained in \(H^{(i)}\). Therefore, the sequence \(R'_1, R'_2, \ldots\) can be used to certify the asymptotic distinguishability of the states from \(S_\lambda\).

Since \(M_A \otimes I_B\) is non-disturbing for \(S\), the states in \(S_\lambda\) are mutually orthogonal. Finally, as \(\text{rank}(a_\lambda) + d_B < d_A + d_B\), the states from \(S_\lambda\) can be discriminated by a finite LOCC protocol \(P\) by induction hypothesis. Combining the measurement \(M_A \otimes I_B\) with the finite LOCC protocol \(P\) gives a finite LOCC protocol for discriminating the states in \(S\) (See Figure 1).

We can lift the tensor product requirement for one of the states in Theorem 4.

**Corollary 7.** Let \(S = \{\rho_i\}_{i \in [n]} \subseteq \text{Pos}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})\) be a full orthogonal set and assume that all but one \(\rho_i\) can be expressed as \(\rho_i = \sigma_i \otimes \tau_i\). If the states from \(S\) can be discriminated with asymptotic LOCC then they can be discriminated with finite LOCC.

**Proof.** Suppose that \(\rho_1\) is the state that is not a tensor product. The proof is similar to that of Theorem 4, except we cannot use Lemma 6 to conclude that \(H^{(1)}\) is \((a \otimes I)\)-invariant. Instead we use the fact that the orthogonal subspaces \(H^{(2)}, \ldots, H^{(n)}\) are all \((a \otimes I)\)-invariant, \(\bigoplus_{i \in \{2, \ldots, n\}} H^{(i)} = (H^{(1)})^\perp\), and \(a\) is Hermitian, to conclude that \(H^{(1)}\) is \((a \otimes I)\)-invariant.

The main obstacle in generalizing Theorem 3 to all separable projective measurements is the lack of an analogue of Lemma 6 for separable projectors \(P\). For example, consider

\[
P := (|0\rangle\langle 0| \otimes |1\rangle\langle 1|) + (|1\rangle\langle 1| \otimes |0 - 1\rangle\langle 0 - 1|) + (|0 - 1\rangle\langle 0 - 1| \otimes |2\rangle\langle 2|)
\]

and \(a \otimes b := |1\rangle\langle 1| \otimes |0 - 1\rangle\langle 0 - 1|\), where \(|0 - 1\rangle := (|0\rangle - |1\rangle)/\sqrt{2}\). Let \(\mathcal{H}\) be the space onto which \(P\) projects. Although \(\mathcal{H}\) is \((a \otimes b)\)-invariant, it is neither \((a \otimes I)\)-nor \((I \otimes b)\)-invariant, since \((a \otimes I)|0 - 1, 2\rangle \notin \mathcal{H}\) and \((I \otimes b)|0, 1\rangle \notin \mathcal{H}\).

Therefore, the general question of whether the set of POVM measurements implementable by LOCC is closed remains open, despite partial progress presented by Theorem 3.

3.2 Applications

Although Corollary 7 is only a slight generalization of Theorem 3, it provides answers to natural questions. For example, let us consider the following orthonormal product basis is known as the
domino basis \([BDF^+99]\):

\[
|\psi_0\rangle = |1\rangle|1\rangle, \quad |\psi_1^+\rangle = |0\rangle|0\rangle \pm |1\rangle, \quad |\psi_2^+\rangle = |0\rangle|1\rangle \pm |2\rangle, \\
|\psi_3^+\rangle = |2\rangle|1\rangle, \quad |\psi_4^+\rangle = |1\rangle|2\rangle,
\]

where \(|i \pm j \rangle := (|i\rangle \pm |j\rangle)/\sqrt{2}\). It is known that the domino states cannot be discriminated by asymptotic LOCC \([BDF^+99]\). However, as soon as we modify the problem slightly the answer becomes unclear. For example, it is not known whether the states from

\[
S := \left\{ |\psi_i^+\rangle \langle \psi_i^+| : i \in [4]\right\} \cup \left\{ \rho := |\psi_0\rangle \langle \psi_0| + \frac{1}{4} \sum_{i \in [4]} |\psi_i^-\rangle \langle \psi_i^-| \right\}, \quad (15)
\]

can be discriminated with asymptotic LOCC. Questions about asymptotic LOCC can be difficult to answer. In some settings, Theorem 3 and Corollary 7 allows us to reduce such questions to those about finite LOCC which usually are more tractable, as demonstrated in the following example.

**Lemma 8.** Let \(S = \{|\phi_i\rangle \langle \phi_i| : i \in [4]\} \cup \{|\rho\rangle\} \) be such that \(|\phi_i\rangle = |\psi_i^+\rangle \) or \(|\phi_i\rangle = |\psi_i^-\rangle \) and \(|\rho\rangle\) is the uniform mixture of the remaining 5 domino states. Then the states from \(S\) cannot be discriminated with LOCC.

**Proof.** Because of Corollary 7, it suffices to prove that \(S\) cannot be discriminated with LOCC. To do so, we only need to disprove the existence of nontrivial (i.e., not proportional to the identity) positive semidefinite operators of the form \(a \otimes I\) and \(I \otimes b\).

Let \(|\alpha_i\rangle|\beta_i\rangle = |\phi_i\rangle\) and assume \(a \otimes I \in \text{Pos}(C^3 \otimes C^3)\) is non-disturbing for the set of states \(S\). Then

\[
0 = \langle 1, 1| (a \otimes I)|\phi_1\rangle = \langle 1|\alpha|0\rangle \langle 0|\beta_1\rangle \quad \iff \quad a_{10} = 0 \iff a_{01} = 0 \quad (16)
\]

since \(|\beta_1\rangle \neq 0\) both for \(|\beta_1\rangle = |0 + 1\rangle\) and \(|\beta_1\rangle = |0 - 1\rangle\) and \(a\) is Hermitian. Similarly, from \(0 = \langle 1, 1| (a \otimes I)|2, \beta_3\rangle\) we obtain \(a_{12} = a_{21} = 0\). Either \(|\psi_2^-\rangle\) or \(|\psi_2^+\rangle\) belongs to the support of a different state from \(S\) than \(|\psi_3^+\rangle\). Hence, either \(\langle 0 - 1|a|2\rangle = 0\) or \(\langle 0 + 1|a|2\rangle = 0\). In both cases we obtain \(a_{02} = a_{20} = 0\). To see that all the diagonal elements have to be equal, note that \(0 = \langle 0 + 1, 2|(a \otimes I)|0 - 1, 2\rangle = a_{00} - a_{11}\) and \(0 = \langle 1 + 2, 0|(a \otimes I)|1 - 2, 0\rangle = a_{11} - a_{22}\). Thus, \(a\) is proportional to the identity matrix. Via similar analysis, one can reach the same conclusion for \(I \otimes b\). Therefore, all the non-disturbing operators for \(S\) are proportional to the identity matrix and our lemma follows. \hfill \Box

Building on \([KKB11]\), \([CH13a]\) presents a necessary condition for discriminating two states with asymptotic LOCC. When the support of these two states cover the whole space this condition yields a simple criterion. This allows the authors to show that

\[
\rho_+ := \frac{1}{4} \sum_{i \in [4]} |\psi_i^+\rangle \langle \psi_i^+| \quad \text{and} \quad \rho_- := \frac{1}{5} \left( |1, 1\rangle \langle 1, 1| + \sum_{i \in [4]} |\psi_i^-\rangle \langle \psi_i^-| \right) \quad (17)
\]

cannot be discriminated with asymptotic LOCC. Since our set \(S\) is a refinement of \(\{\rho_+, \rho_-\}\) the impossibility to distinguish the states from \(S\) with asymptotic LOCC also follows from the result of \([CH13a]\). We illustrate Corollary 7 on the domino states because they are well-known. There are cases where Corollary 7 applies but the criterion from \([CH13a]\) cannot be used to conclude that a set of states cannot be discriminated with asymptotic LOCC.
4 \quad \textbf{UBP’s in }3 \otimes 3\textbf{ cannot be discriminated by LOCC}

In this section, we turn our attention to the problem of discrimination of unextendible product bases (UPBs). It has been shown in \cite{Ter99, DMS+03} that UPBs cannot be perfectly discriminated by finite LOCC. Here, we show that any UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ cannot be discriminated by asymptotic LOCC.

Reference \cite{DMS+03} establishes that any UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ has exactly 5 states of the form $|\psi_i\rangle = |\alpha_i\rangle \otimes |\beta_i\rangle$ which, up to local unitary transformations, can be parametrized by six angles $\theta_{A}, \gamma_{A}, \phi_{A}, \theta_{B}, \gamma_{B}, \phi_{B}$:

$$
|\alpha_0\rangle = |0\rangle \\
|\alpha_1\rangle = |1\rangle \\
|\alpha_2\rangle = \cos \theta_{A}|0\rangle + \sin \theta_{A}|2\rangle \\
|\alpha_3\rangle = \sin \gamma_{A} \sin \theta_{A}|0\rangle + \cos \gamma_{A} e^{i\phi_{A}}|1\rangle - \sin \gamma_{A} \cos \theta_{A}|2\rangle \\
|\alpha_4\rangle = \frac{1}{N_{A}} (\sin \gamma_{A} \cos \theta_{A} e^{i\phi_{A}}|1\rangle + \cos \gamma_{A}|2\rangle) \\
|\beta_0\rangle = |1\rangle \\
|\beta_1\rangle = \sin \gamma_{B} \sin \theta_{B}|0\rangle + \cos \gamma_{B} e^{i\phi_{B}}|1\rangle - \sin \gamma_{B} \cos \theta_{B}|2\rangle \\
|\beta_2\rangle = |0\rangle \\
|\beta_3\rangle = \cos \theta_{B}|0\rangle + \sin \theta_{B}|2\rangle \\
|\beta_4\rangle = \frac{1}{N_{B}} (\sin \gamma_{B} \cos \theta_{B} e^{i\phi_{B}}|1\rangle + \cos \gamma_{B}|2\rangle)
$$

(18)

where $N_{A,B} = \sqrt{\cos^{2} \gamma_{A,B} + \sin^{2} \gamma_{A,B} \cos^{2} \theta_{A,B}}$, and unextendibility implies that none of $\cos \gamma_{A,B}, \sin \gamma_{A,B}, \cos \theta_{A,B}, \sin \theta_{A,B}$ vanish.

**Theorem 9.** The set of states $S = \{|\psi_i\rangle\}_{i=0,1,...,4}$ cannot be discriminated by asymptotic LOCC.

**Proof.** First, we show why it suffices to prove the case for $\phi_{A,B} = 0$. We can replace $|1\rangle$ by $|\tilde{1}\rangle = e^{i\phi_{A}}|1\rangle$ in the choice of the local basis on Alice’s system. Then, $\phi_{A}$ only appears in $|\alpha_1\rangle = e^{-i\phi_{A}}|\tilde{1}\rangle$. Apply a similar change of basis on Bob’s system. Clearly, replacing $|\alpha_1\rangle$ and $|\beta_0\rangle$ by $|\tilde{1}\rangle$ does not affect distinguishability.

We now proceed to prove the theorem by contradiction. Suppose $S$ can be discriminated by asymptotic LOCC.

Then, Theorem 1 applies to $S$ since the kernel condition holds automatically when $S$ is a UPB. Here, $n = 5$ and we take $\chi = 0.22$ (any $1/n < \chi < 1/(n-1)$ will do). The theorem states that $\exists E = a \otimes b \geq 0$ that is non-disturbing for $S$. Furthermore, $E \neq 0$ (else $\sum_{|\psi\rangle \in S} \text{Tr}(E|\psi\rangle \langle \psi|) = 0 \neq 1$), $E \not\in I$ (else, $\max_{|\psi\rangle \in S} \text{Tr}(E|\psi\rangle \langle \psi|) = \frac{1}{5} \sum_{|\psi\rangle \in S} \text{Tr}(E|\psi\rangle \langle \psi|) = 1/5 \neq 0.22$, and $\min_{|\psi\rangle \in S} \text{Tr}(E|\psi\rangle \langle \psi|) > 0.12$. From these we derive constraints for $a$ and $b$. We have $a,b \neq 0$, $a$ cannot have two nonzero eigenvalues of opposite signs (else the same holds for $E = a \otimes b$) and similarly for $b$, so, without loss of generality, $a \geq 0$, $b \geq 0$. Finally, $\min_i \text{Tr}(a|\alpha_i\rangle \langle \alpha_i|) > 0$ and $\min_i \text{Tr}(b|\beta_i\rangle \langle \beta_i|) > 0$.

We will show that the non-disturbing property of $E$ is inconsistent with the conditions on $a,b$.

Our main tool in the analysis is an extension of the orthogonality graph for UPBs defined in \cite{DMS+03}. Given $a,b \in \text{Herm}(\mathbb{C}^3)$, define two graphs $G_a, G_b$ as follows. They both have vertex set $V = \{0,1,\ldots,4\}$. $G_a$ ($G_b$) has an edge $(i,j)$ whenever $\langle \alpha_i|a|\alpha_j\rangle = 0 (\langle \beta_i|b|\beta_j\rangle = 0)$. Since
E is non-disturbing for S, the pair \((i, j)\) is an edge in \(G_a\) or \(G_b\) for all distinct \(i, j\). Since any three \(|\alpha_i\rangle\) span \(\mathbb{C}^3\), no vertex in \(G_a\) has degree more than 2, and similarly for \(G_b\). So, the only possible \(G_a, G_b\) are 5-cycles with complementary sets of edges.

Denote the 12 possible 5-cycles as \(O_{1, \ldots, 12}\). We analyse the 12 possible cases for which \(G_a = O_{1, \ldots, 12}\) (\(G_b\) is then fixed) in detail in the appendix. We will see that in one case, \(E \propto I\) which is a contradiction. For all other cases, \(\min_i \text{Tr}(a|\alpha_i\rangle\langle\alpha_i|) = 0\) which is also a contradiction.

We note as a side remark that, using Theorems 2 and 3 in [DMS+03], any bipartite UPB with 5 states can be perfectly discriminated by a separable measurement. Theorem 9 thus provides another example of the phenomenon nonlocality without entanglement, in which a set of unentangled states cannot be discriminated by asymptotic LOCC, but can be discriminated by separable operations.

5 Discussions

To ease the analysis of asymptotic LOCC we have introduced two scenarios in which no new task can be accomplished (information theoretically) collapse. by allowing vanishing error. The first scenario is the implementation of projective measurements with tensor product operators. The second is the discrimination of the states from an unextendible product basis in \(\mathbb{C}^3 \otimes \mathbb{C}^3\). On the first subject, an obvious next step is to investigate whether asymptotic LOCC can be helpful for implementing any projective or general measurement. A second question is whether asymptotic LOCC can help for perfect state discrimination. On the second subject, it is likely that a general UPB cannot be discriminated by asymptotic LOCC. It will be nice to obtain a rigorous proof.

Another very poorly understood subject in LOCC is round complexity. Almost nothing is known about how many messages the parties need to exchange in order to accomplish a task in the LOCC setting, especially when a small probability of error is allowed.

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### A Case analysis for Theorem 9

For concreteness, we denote the 12 possible 5-cycles as follows:

\[
\begin{align*}
O_1: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_2: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_3: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_4: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_5: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_6: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_7: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_8: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_9: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_{10}: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_{11}: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \\
O_{12}: & \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{align*}
\]

Omitting the subscript \(A\) (which is irrelevant here), and using the shorthands \(c_\theta, s_\theta\) for \(\cos \theta, \sin \theta\), \(c_\gamma, s_\gamma\) for \(\cos \gamma, \sin \gamma\), we have

\[
\begin{align*}
|\alpha_0\rangle &= |0\rangle \\
|\alpha_1\rangle &= |1\rangle \\
|\alpha_2\rangle &= c_\theta |0\rangle + s_\theta |2\rangle \\
|\alpha_3\rangle &= s_\theta s_\gamma |0\rangle + c_\gamma |1\rangle - s_\gamma c_\theta |2\rangle = s_\gamma |\alpha_2^\perp\rangle + c_\gamma |1\rangle = s_\gamma s_\theta |0\rangle + N |\alpha_4^\perp\rangle \\
|\alpha_4\rangle &= \frac{1}{N} (s_\gamma c_\theta |1\rangle + c_\gamma |2\rangle)
\end{align*}
\]  
\text{(19)}

where \(N = \sqrt{c_\gamma^2 + s_\gamma^2 c_\theta^2}\), and we rephrase \(|\alpha_3\rangle\) in terms of \(|\alpha_2^\perp\rangle = s_\theta |0\rangle - c_\theta |2\rangle\), and \(|\alpha_4^\perp\rangle = (c_\gamma |1\rangle - s_\gamma c_\theta |2\rangle)/N\), two states that appear frequently in the analysis.

If we swap \(|0\rangle\) with \(|1\rangle\), Eq. (19) becomes

\[
\begin{align*}
|\alpha'_0\rangle &= |1\rangle \\
|\alpha'_1\rangle &= |0\rangle \\
|\alpha'_2\rangle &= c_\theta |1\rangle + s_\theta |2\rangle = \frac{1}{N} (s_\gamma', c_{\theta'} |1\rangle + c_\gamma' |2\rangle) \\
|\alpha'_3\rangle &= s_\gamma s_\theta |1\rangle + c_\gamma |0\rangle - s_\gamma c_\theta |2\rangle = c_\gamma' |1\rangle + s_\gamma' s_{\theta'} |0\rangle - c_{\theta'} s_\gamma' |2\rangle \\
|\alpha'_4\rangle &= \frac{1}{N} (s_\gamma c_\theta |0\rangle + c_\gamma |2\rangle) = c_{\theta'} |0\rangle + s_{\theta'} |2\rangle
\end{align*}
\]  
\text{(20)}
where \( s_\theta = c_\gamma/N \), \( c_\theta = s_\gamma c_\theta / N \), \( c_\gamma = s_\gamma s_\theta \), \( s_\gamma = N \), and \( N' = s_\gamma \). So the local change of basis \(|0\rangle \leftrightarrow |1\rangle\) swaps \(|a_0\rangle\) with \(|a_1\rangle\) and swaps \(|a_2\rangle\) with \(|a_4\rangle\) with modified angles. Thus the analysis for \( G_a = O_j \) applies to the case \( G_a = O_{j+1} \) for \( j = 2, 4, 8, 10 \).

Here, we summarize the methodology in the analysis. In each case, \( a|a_0\rangle \) lives in a one-dimensional subspace because it is orthogonal to two (linearly independent) \(|a_i\rangle\)'s where \( i \) is adjacent to \( 0 \) in \( G_a \). Similarly for \( a|a_1\rangle \). We thus obtain the first two columns of \( a \) in terms of \( \theta, \gamma \) and two scalar multipliers \( r_{1,2} \) for \( a|a_{0,1}\rangle \). We often use the original orthogonality conditions between the \(|a_i\rangle\)'s to deduce the form of \( a|a_{0,1}\rangle \). With the first two columns of \( a \) fixed, we use hermiticity of \( a \) to fix all but one entry of \( a \) (which we call \( r_3 \)). We use the remaining orthogonality conditions to relate \( r_{1,2,3} \) until we either obtain \( a \propto I \) (in which case we show \( b \propto I \) as well and thus \( E \propto I \)) or use \( a \geq 0 \) to force \( r_1 = 0 \) or \( r_2 = 0 \) thereby contradicting \( \text{Tr}(a|a_i\rangle\langle a_i|) > 0 \).

**Case I:** If \( G_a = O_1 \), then
- \( a|a_0\rangle \perp |a_1\rangle, |a_4\rangle \), so \( a|a_0\rangle = r_1|a_0\rangle \).
- \( a|a_1\rangle \perp |a_0\rangle, |a_2\rangle \), so \( a|a_1\rangle = r_2|a_1\rangle \).

By hermiticity, \( a = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \). So, \( a|a_3\rangle = r_1 s_\gamma |s_\theta\rangle + r_2 c_\gamma |1\rangle + r_3 s_\gamma c_\theta |2\rangle \).

Impose \( 0 = \langle a_2|a|a_3\rangle = c_\theta r_1 s_\gamma s_\theta - s_\theta r_3 s_\gamma c_\theta \) gives \( r_1 = r_3 \), and \( 0 = N\langle a_4|a|a_3\rangle = s_\gamma c_\theta r_2 c_\gamma - c_\theta r_3 s_\gamma c_\theta \) gives \( r_2 = r_3 \), so \( a \propto I \).

When \( G_a = O_2 \), \( G_b = O_7 \). Up to different choices for the angles, the states \(|\beta_{0,3,1,4,2}\rangle\) are the same as \(|a_{1,2,3,4,0}\rangle\), so, the above analysis applies to \( G_b = O_7 \) and \( b \propto I \). Thus \( E = a \otimes b \propto I \) a contradiction.

**Case II:** If \( G_a = O_2 \), then
- \( a|a_0\rangle \perp |a_1\rangle, |a_3\rangle \), so \( a|a_0\rangle = r_1|a_2\rangle \).
- \( a|a_1\rangle \perp |a_0\rangle, |a_2\rangle \), so \( a|a_1\rangle = r_2|a_1\rangle \).

By hermiticity, \( a = \begin{pmatrix} r_1 c_\theta & 0 & 0 \\ 0 & r_2 & 0 \\ r_1 s_\theta & 0 & r_3 \end{pmatrix} \). So, \( a(N|a_4\rangle) = r_1 s_\theta c_\gamma |0\rangle + r_2 s_\gamma c_\theta |1\rangle + r_3 c_\gamma |2\rangle \).

Impose \( 0 = N\langle a_2|a|a_4\rangle = c_\theta r_1 s_\gamma c_\theta + s_\theta r_3 c_\gamma \) gives \( c_\theta r_1 = -r_3 \). But \( a \geq 0 \) implies non-negativity of all diagonal elements, so \( r_1 = 0 \) and thus \( \text{Tr}(a|a_0\rangle\langle a_0|) = 0 \) a contradiction.

The same analysis applies to \( G_a = O_3 \).

**Case III:** If \( G_a = O_4 \), then
- \( a|a_0\rangle \perp |a_1\rangle, |a_2\rangle \), so \( a|a_0\rangle = r_1|a_2\rangle \).
- \( a|a_1\rangle \perp |a_0\rangle, |a_3\rangle \), so \( a|a_1\rangle = r_2(N|a_4\rangle) \).

By hermiticity, \( a = \begin{pmatrix} r_1 s_\theta & 0 & -r_1 c_\theta \\ 0 & r_2 s_\gamma c_\theta & r_2 c_\gamma \\ -r_1 c_\theta & r_2 c_\gamma & r_3 \end{pmatrix} \).

So, \( a(N|a_4\rangle) = -r_1 c_\theta c_\gamma |0\rangle + r_2 (s_\gamma^2 c_\theta^2 + c_\gamma^2) |1\rangle + (r_2 c_\gamma s_\gamma c_\theta + r_3 c_\gamma) |2\rangle \).

Now \( 0 = N\langle a_2|a|a_4\rangle = c_\gamma (-r_1^2 c_\theta^2 + s_\theta (r_2 s_\gamma c_\theta + r_3)) \) so \( s_\theta r_3 = r_1^2 c_\theta^2 - s_\theta r_2 s_\gamma c_\theta \).

Since \( a \geq 0 \), the minors \( r_1 s_\theta r_2 s_\gamma c_\theta \) \( \geq 0 \), \( r_1 s_\theta r_3 - r_1^2 c_\theta^2 \geq 0 \).

Eliminating \( r_3 \) in the latter, \( r_1 (r_1^2 c_\theta^2 - s_\theta r_2 s_\gamma c_\theta) - r_1^2 c_\theta^2 = -r_1 s_\theta r_2 s_\gamma c_\theta \geq 0 \). So \( r_1 r_2 = 0 \), which means \( \text{Tr}(a|a_i\rangle\langle a_i|) = 0 \) either for \( i = 0 \) or \( 1 \) which is a contradiction.

The same analysis applies to \( G_a = O_5 \).

**Case IV:** If \( G_a = O_6 \), then
- \( a|a_0\rangle \perp |a_1\rangle, |a_2\rangle \), so \( a|a_0\rangle = r_1|a_2\rangle \).
- \( a|a_1\rangle \perp |a_0\rangle, |a_4\rangle \), so \( a|a_1\rangle = r_2(N|a_4\rangle) \).
By hermiticity, \( a = \begin{pmatrix} r_1 s_\theta & 0 & -r_1 c_\theta \\ -r_1 c_\theta & r_2 c_\gamma & -r_2 s_\gamma c_\theta \\ -r_1 s_\theta & r_2 s_\gamma & r_3 \end{pmatrix} \).

So, \( a(\alpha_3) = r_1 s_\theta \langle 0 \rangle + r_2 (s_\gamma^2 c_\theta^2 + c_\gamma^2) \langle 1 \rangle - c_\theta s_\gamma (r_1 s_\theta + r_2 c_\gamma + r_3) \langle 2 \rangle \).

Now \( 0 = (a_4 | a_3) = s_\gamma c_\theta r_2 (s_\gamma^2 c_\theta^2 + c_\gamma^2) - c_\gamma c_\theta s_\gamma (r_1 s_\theta + r_2 c_\gamma + r_3) = s_\gamma c_\theta (r_2 s_\gamma^2 c_\theta^2 - c_\gamma r_1 s_\theta - c_\gamma r_3) \).

So, \( c_\gamma r_3 = r_2 s_\gamma^2 c_\theta^2 - c_\gamma r_1 s_\theta \).

Since \( a \geq 0 \), the minors \( r_1 s_\theta r_2 c_\gamma \geq 0, r_2 c_\gamma r_3 - r_2^2 s_\gamma^2 c_\theta^2 \geq 0 \).

Eliminating \( r_3 \) in the latter, \( r_2 (r_2 s_\gamma^2 c_\theta^2 - c_\gamma r_1 s_\theta) - r_2^2 s_\gamma^2 c_\theta^2 \geq 0 \). Simplifying, \( r_2 (-c_\gamma r_1 s_\theta) \geq 0 \). So, \( r_1 r_2 = 0 \), which is a contradiction (see case III).

**Case V:** If \( G_a = O_7 \), then

- \( a(\alpha_0) \perp \langle 0, 2 \rangle, \langle 3 \rangle, \) so, \( a(\alpha_0) = r_1 (s_\gamma \langle 1 \rangle - c_\gamma | \alpha_4^2 \rangle) = r_1 (-c_\gamma s_\theta | 0 \rangle + s_\gamma | 1 \rangle + c_\gamma c_\theta | 2 \rangle) \).
- \( a(\alpha_1) \perp \langle 3 \rangle, \langle 4 \rangle, \) so, \( a(\alpha_1) = r_2 N(N| 0 \rangle - s_\gamma s_\theta | \alpha_4^4 \rangle) = r_2 ((c_\gamma^2 + s_\gamma^2 c_\theta^2) | 0 \rangle - s_\gamma s_\theta c_\gamma | 1 \rangle + s_\gamma^2 c_\theta c_\theta | 2 \rangle) \).

Thus \( a = \begin{pmatrix} -r_1 c_\gamma s_\theta & r_2 (c_\gamma^2 + s_\gamma^2 c_\theta^2) & * \\ r_1 s_\gamma & -r_2 s_\gamma s_\theta c_\gamma & * \\ r_1 c_\gamma c_\theta & r_2 s_\gamma^2 s_\theta c_\theta & * \end{pmatrix} \) where we omit the third column which does not enter the analysis.

By hermiticity, \( r_1 s_\gamma = r_2 (c_\gamma^2 + s_\gamma^2 c_\theta^2) \). Multiplying both sides by \( r_2 \), we get \( r_1 s_\gamma r_2 \geq 0 \).

The minor after deleting the second row and the third column is \( r_1 r_2 s_\gamma (r_2 c_\gamma^2 s_\theta^2 - c_\gamma^2 - s_\gamma^2 c_\theta^2) \geq 0 \). The expression in the parenthesis is negative, so, \( r_1 r_2 s_\gamma \leq 0 \).

Together, \( r_1 r_2 = 0 \), but that gives a contradiction.

**Case VI:** If \( G_a = O_8 \), then

- \( a(\alpha_0) \perp \langle 0, 2 \rangle, \langle 4 \rangle, \) so, \( a(\alpha_0) = r_1 | \alpha_3 \rangle = r_1 (s_\gamma | 0 \rangle + c_\gamma | 1 \rangle - s_\gamma c_\theta | 2 \rangle) \).
- \( a(\alpha_1) \perp \langle 3 \rangle, \langle 4 \rangle \) (which is same as in case V).

Thus \( a = \begin{pmatrix} r_1 s_\gamma s_\theta & r_2 (c_\gamma^2 + s_\gamma^2 c_\theta^2) & * \\ r_1 c_\gamma & -r_2 s_\gamma s_\theta c_\gamma & * \\ -r_1 s_\gamma c_\theta & r_2 s_\gamma^2 s_\theta c_\theta & * \end{pmatrix} \) where we omit the third column which does not enter the analysis.

By hermiticity, \( r_1 c_\gamma = r_2 (c_\gamma^2 + s_\gamma^2 c_\theta^2), \) so, \( r_1 r_2 c_\gamma \geq 0 \).

The determinant of the \( | 0 \rangle, | 1 \rangle \) block is \( -r_1 r_2 c_\gamma (s_\gamma^2 c_\theta^2 + s_\gamma^2 c_\theta^2 + c_\gamma^2) = -r_1 r_2 c_\gamma \geq 0 \).

Together, \( r_1 r_2 = 0 \), giving a contradiction.

The same analysis applies to \( G_a = O_9 \).

**Case VII:** If \( G_a = O_{10} \), then \( a(\alpha_0) \perp \langle 0, 3 \rangle, \langle 4 \rangle \) and \( a(\alpha_1) \perp \langle 2 \rangle, \langle 4 \rangle \).

So, part of \( a \) can be obtained from that in case VI with the first two columns interchanged:

\[
a = \begin{pmatrix} r_2 (c_\gamma^2 + s_\gamma^2 c_\theta^2) & r_1 s_\gamma s_\theta & * \\ -r_2 s_\gamma s_\theta c_\gamma & r_1 c_\gamma & * \\ r_2 s_\gamma^2 s_\theta c_\theta & -r_1 s_\gamma c_\theta & * \end{pmatrix}.
\]

Hermiticity now implies \( -r_2 c_\gamma = r_1 \).

The determinant of the \( | 0 \rangle, | 1 \rangle \) block is minus that in case VI. So, \( r_1 r_2 c_\gamma \geq 0 \).

Together, \( -r_1^2 \geq 0 \), so, \( r_1 = 0 \) giving a contradiction.

The same analysis applies to \( G_a = O_{11} \).
Case VIII: If $G_a = O_{12}$, then $a |\alpha_0\rangle \perp |\alpha_3\rangle, |\alpha_4\rangle$ and $a |\alpha_1\rangle \perp |\alpha_2\rangle, |\alpha_3\rangle$.

So, part of $a$ can be obtained from that in case V with the first two columns interchanged:

$$a = \begin{pmatrix}
  r_2(c_\gamma^2 + s_\gamma^2c_\theta^2) & -r_1c_\gamma s_\theta & r_2s_\gamma^2 s_\theta c_\theta \\
  -r_2s_\gamma s_\theta c_\gamma & r_1s_\gamma & r_1c_\gamma c_\theta \\
  r_2s_\gamma^2 s_\theta c_\theta & r_1c_\gamma c_\theta & r_3
\end{pmatrix}$$

where we fill in part of the third column using hermiticity. Hermiticity also implies $r_2s_\gamma = r_1$.

Since $a |\alpha_2\rangle \perp |\alpha_1\rangle, |\alpha_4\rangle$, so $a |\alpha_2\rangle \propto |0\rangle$ and

$$0 = \langle 2 | a | 2 \rangle = \langle 2 |c_\theta|0\rangle + s_\theta\langle 2 | 2 \rangle = c_\theta \langle 2 | a | 0 \rangle + s_\theta \langle 2 | a | 2 \rangle = c_\theta r_2s_\gamma^2 s_\theta c_\theta + s_\theta r_3$$

so, $r_3 = -r_2s_\gamma^2 c_\theta$.

Using the relation between $r_{1,2}$, we get $r_3 = -r_1s_\gamma c_\theta$.

Since $a \geq 0$, product of the last two diagonal elements is non-negative. So, $r_1s_\gamma(-r_1s_\gamma^2 c_\theta) \geq 0$, and $r_1 = 0$ a contradiction.