1. Introduction. The natural generalisation to Riemannian manifolds of the classical elastica problem studied by Euler and Bernoulli (see [18] or [28] for a historical survey) is the following: find critical points of the restriction of
\[ F(x) := \int_0^\ell k^2(x) ds = \int_0^1 k^2 \|\dot{x}\| dt \] (1)
to the set \( \Omega_\ell \) of immersed curves which have prescribed initial and final points, initial and final tangent directions, and length \( \ell \). Here \( x : I = [0, 1] \to M \) is a sufficiently regular curve with length \( \ell \) on a complete Riemannian manifold \( M \), \( k(x) = \|\nabla_x T\| \) is the geodesic curvature of \( x \) and \( T = \frac{\dot{x}}{\|\dot{x}\|} \) the unit tangent vector.

We will use the term elastica to refer to critical points of \( F|_{\Omega_\ell} \). Among the special cases of elastica we distinguish pinned elastica and closed elastica as those which are critical subject to zeroth order boundary conditions and first order periodicity conditions respectively. These are special cases in the sense that they satisfy the same differential equation with special boundary conditions (see Section 2), but of course not in the sense that they are special cases of the same variational problem (i.e., the pinned and closed constraints are not special cases of the constraints in the original problem). In the absence of the length constraint, the resulting critical curves are known as free elastica. Note that the terms elastica and elastic curve are often used interchangeably, but the latter is somewhat equivocal so we will avoid it.

In modern times elastica have reappeared in several different contexts. In approximation theory they are known as nonlinear splines: a mathematical model for the drafting tool known as a spline (see eg. [17, 10, 20]). The better known cubic
splines are used for ease of computation, not because they are a good approximation to drafting splines. Elastica also appear as a model for curve completion in computer vision [23], and as an important example of an optimal control problem with nonholonomic constraints [11]. As a consequence of the variety of applications, the problem has been approached from several different perspectives. For example Bryant and Griffiths used the theory of exterior differential systems to prove a partial integrability result in homogeneous spaces and study solutions of the Euler-Lagrange equations for elastica in the Euclidean and hyperbolic planes [4]. At about the same time, Langer and Singer obtained similar results using Frenet frames and elliptic functions [14]. Jurdjevic has shown that the Euler-Lagrange equations are completely integrable in surfaces of constant curvature using techniques from geometric control theory [11, 12]. This includes in particular elastica on $SO(3)$, by way of the double cover by $S^3$. Popiel and Noakes also studied elastica in Lie groups, reducing the problem to the Lie algebra and solving for elastica in $SO(3)$ by quadratures [26].

In [14] Langer and Singer also studied stability properties of the negative gradient flow of $F$, which they later termed the curve straightening flow. They proved that the only stable closed free elastica in $S^2$ are non-trivial closed geodesics. This motivated the study of the curve straightening flow on Riemannian manifolds as a method of finding non-trivial closed geodesics. In [15] they showed that this flow is well behaved on closed curves in $\mathbb{R}^3$ with fixed length. They also proved that for almost all initial curves the flow approaches a circle, i.e. the circles are the only stable closed elastica in $\mathbb{R}^3$. Subsequent work on the curve-straightening flow was carried out by Linnér [21, 22]. Moreover, in [19] Linnér investigates free elastica in the Euclidean plane and gives some conditions for existence and non-existence.

There is also a considerable mathematical physics literature on free elastica and critical points of other functionals depending on geodesic curvature. We mention [24], [2] and references therein. However, these authors are mainly interested in finding explicit solutions to initial value problems in semi-Riemannian manifolds, usually without constraints on the length or speed.

1.1. The Palais-Smale condition for $F$. The natural domain for the total squared geodesic curvature $F$ is the set of $C^1$ immersions with square integrable second covariant derivative along the curve. This set, which we will denote $\text{Imm}^2(I,M)$, is an open submanifold of the Hilbert manifold $H^2(I,M)$ consisting of $C^1$ curves with square integrable second covariant derivative. The space $\Omega^2_F$ of admissible curves can be given the structure of a submanifold of $\text{Imm}^2(I,M)$ defined by the boundary conditions and the length constraint.

Let $(x_i) \subset \Omega^2_F$ be a sequence which is minimizing for $F$, i.e. $F(x_i) \to \inf F$. It is possible to prove\(^1\), as in [12] p. 17, that such a sequence has a weakly convergent subsequence and therefore a limiting curve of class $H^2$ exists.

There are good reasons for wanting to prove stronger convergence results, such as the Palais-Smale (PS) condition. In general, a PS sequence for a smooth real valued function $f$ on a complete Hilbert manifold $X$ is a sequence $(x_i)$ of points on which $f$ is bounded and $|df_{x_i}| \to 0$, and we say $f$ satisfies the PS condition if any PS sequence has a (strongly) convergent subsequence. In particular, for the elastica problem this means that a PS sequence for $F|\Omega^2_F$ must have a subsequence which converges in the $H^2$ metric. If the PS condition holds then the associated

\(^1\)This was pointed out by one of the reviewers.
negative gradient flow is a positive semi-group and has at least one critical point as a limit point (cf. [25] p. 183). We note that this condition provides a constructive proof of the existence: it ensures that the method of gradient descent will locate critical points. Moreover, the PS condition makes available the minimax and Morse theoretic methods of counting critical points.

In this paper we will verify the PS condition for the elastica and pinned elastica variational problems on any complete Riemannian manifold $M$. For closed elastica we will do so under the additional assumption that $M$ is compact. The relationship between these and the earlier results of [16] will be discussed at the end of this section. We will also prove a Morse index theorem for elastica and use the Morse inequalities to give lower bounds for the number of elastica in terms of the Betti numbers of the appropriate path space.

It is not possible to prove that $F$ satisfies the PS condition on its natural domain $\text{Imm}^2(I, M)$ because $F$ is invariant under reparametrization, and the orbits of the action of reparametrization on immersed curves are not compact. It follows that any critical point is contained in a non-compact orbit of critical points at the same level of $F$, which contradicts the PS condition. Several methods of resolving this kind of problem are discussed in [25] p. 245, with regard to the length functional. One of these methods is to find a second function which ‘breaks the symmetry’, meaning it is not invariant under reparametrization, but has the same critical points as the original function in the following sense: each critical point of the symmetry breaking function is a critical point of the original function and each orbit of critical points of the original function contains a critical point of the symmetry breaking function. This is the preferred method for the length function; the energy function $E = \int_0^1 \| \dot{x} \|^2 dt$ is not parametrization invariant and it is well known that the critical points of $E$ (geodesics) are arc-length proportionally parametrized critical points of the length.

An alternative which is also discussed in [25] is to impose a so-called ‘gauge fixing condition’ to define a smooth submanifold of the domain which intersects each of the orbits only once. This method turns out to be the most appropriate for $F$. The condition we choose is that the curves should have constant speed $v$. This leads to a neat simplification of $F$: on the submanifold $\Sigma^v$ of constant speed curves $F$ coincides with the total squared covariant acceleration:

$$J(x) := \frac{1}{2} \int_0^1 \langle \nabla_t \dot{x}, \nabla_t \dot{x} \rangle dt = \frac{v^2}{2} F$$

We show in Lemma 2.2 that a curve $x$ which is parametrized proportional to arc length is a critical point of $F|\Omega^r$ if and only if it is a critical point of $J|\Sigma^v$ with $v = \ell$. We therefore carry out all our analysis on $J|\Sigma^v$.

Langer and Singer have proved related results in [16] but with a different objective. They aim to prove that the curve straightening flow on closed curves is well behaved by showing that $F$ satisfies the PS condition. The parametrization invariance of $F$ makes this impossible, so Langer and Singer restrict $F$ to normalised curves: those parametrized proportional to arc length. They are not interested in fixing the length, because the curve straightening flow is intended to be used to find closed non-trivial geodesics whose length may not be known in advance.

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2We will frequently refer to the speed etc. of a curve, even though strictly speaking this is only appropriate when elastica are considered from a dynamical point of view; classically the elastica problem is one of shape or equilibrium position, not dynamics.
The restriction of $F$ to normalized curves still does not satisfy the PS condition because a subset of curves on which $F$ is bounded does not necessarily have bounded length (in the terminology of Eliasson, see Section 3, this means that $F$ is not weakly proper). Thus we have the counterexamples to the Palais-Smale condition mentioned in [21] §1.7: the sequence $x_n$ of geodesics wrapping around the sphere $n$ times, and circles in the plane with radii increasing without bound. The sequence of geodesics wrapping around the sphere is a counterexample to the Palais-Smale condition but it is not an example of a curve straightening trajectory that has no convergent subsequence. It is still an open question whether such an example exists on a compact manifold. The success of Linér [22] in numerically generating periodic geodesics in sphere-like surfaces seems to suggest that the curve-straightening flow may in fact be convergent in this case.

To circumvent the difficulties outlined above, Langer and Singer consider the modified function

$$F^\alpha := \int_0^1 (k^2 + \alpha) \|\dot{x}\| \, dt$$

instead, with $\alpha$ assumed to be positive so that $F^\alpha$ bounds the length. They prove that $F^\alpha$, $\alpha > 0$ satisfies the PS condition on manifolds of closed, normalized curves on compact Riemannian manifolds. They also remark that their techniques can be used to prove that $F|^\Sigma^\alpha_\nu$ satisfies the PS condition, thus there is some overlap with our Theorem 5.5. Nevertheless it seems worth providing a detailed treatment, particularly since the techniques used in this paper lend themselves to the development of a Morse index theorem for elastica.

2. Lagrange multipliers and elastica. We have already mentioned that the natural domain of $F$ is the set $\text{Imm}^2(I, M)$, which is an open submanifold of the Hilbert manifold $H^2(I, M)$ consisting of $C^1$ curves with square integrable second covariant derivative. Our approach to the geometry of such manifolds of maps aligns closely with the work of Eliasson: see [6, 7], or the summaries in [9, 27]. The length function $L : H^2(I, M) \to \mathbb{R}$, $L(x) = \int_0^1 \|\dot{x}\| \, dt$ and the speed function $\nu : H^2(I, M) \to H^1(I, \mathbb{R})$, $\nu(x) = \|\dot{x}\|$ are both differentiable on $\text{Imm}^2(I, M)$.

Suppose $p, q \in M$, $v \in T_p M$ and $w \in T_q M$. We will make use of the following subsets of $H^2(I, M)$

$$H^2(I, M)_{p,q} := \{ x \in H^2(I, M) : x(0) = p, x(1) = q \}$$
$$H^2(I, M)_{v,w} := \{ x \in H^2(I, M)_{p,q} : \dot{x}(0) = v, \dot{x}(1) = w \}$$
$$H^2(I, M)_c := \{ x \in H^2(I, M) : x(0) = x(1), \dot{x}(0) = \dot{x}(1) \}$$
$$\text{Imm}^2(I, M)_* := \text{Imm}^2(I, M) \cap H^2(I, M)_c$$
$$\Omega^\ell_* := \{ x \in \text{Imm}^2(I, M)_* : L(x) = \ell \}$$
$$\Sigma_*^v := \{ x \in H^2(I, M)_* : \nu(x) = v \}$$
$$\text{Imm}^2(I, M)_T := \{ x \in \text{Imm}^2(I, M) : T(0) = \hat{v}, T(1) = \hat{w} \}$$
$$\Omega^\ell_* := \{ x \in \text{Imm}^2(I, M)_T : L(x) = \ell \}$$

where $\ell, v$ are positive real numbers, $\hat{v}, \hat{w}$ are unit vectors, and $*$ denotes either $(p, q), (v, w), c$ or void. $H^2(I, M)_*$ is a closed submanifold of $H^2(I, M)$ and $\text{Imm}^2(I, M)_*$ is an open submanifold of $H^2(I, M)_*$. For now we will assume that
\( \Omega^\ell_t \) and \( \Sigma^v_t \) are submanifolds of \( \text{Imm}^2(I, M)_s \) and \( H^2(I, M)_s \) obtained as the pre-images of regular values \( \ell \) and \( v \) respectively. In section 4 we will prove that this is true for \( \Sigma^v \) under the assumption that it contains no geodesics.

To begin with we work with the following version of the Lagrange multiplier theorem which is similar to that in [1] p. 211.

**Theorem 2.1.** (Lagrange multiplier theorem) Let \( X \) be a Banach manifold, \( E \) a Banach space and \( f : X \to \mathbb{R}, \phi : X \to E \) differentiable maps. Suppose \( c_0 \in E \) is a regular value of \( \phi \) so that \( \Omega := \phi^{-1}(c_0) \) is a submanifold of \( X \), with \( T_x \Omega = \ker d\phi_x \) split in \( T_x X \), and denote \( \tilde{f} := f|\Omega \). Then the following are equivalent for \( x \in \Omega \):

- \( x \) is a critical point of \( \tilde{f} : \Omega \to \mathbb{R} \)
- there is a critical point of \( f - (\lambda, \phi) : X \to \mathbb{R} \)

**Proof.** Suppose \( x \) is a critical point of \( \tilde{f} \) and let \( \lambda \in E^* \) be such that \( \lambda(e) := df_x e \) for any \( e \in df_x \Omega \). To see that \( \lambda \) is well defined let \( V' \in df_x \Omega \) also, then \( V - V' \in \ker d\phi_x = T_x \Omega \) and therefore \( df_x (V' - V) = 0 \), since \( x \) is a critical point of \( \tilde{f} \). Now \( (df_x - \lambda d\phi_x)V = 0 \) for all \( V \in T_x X \). Conversely, if \( x \) is a critical point of \( f - \lambda \circ \phi \) for some \( \lambda \in E^* \) then \( (df_x - \lambda d\phi_x)V = 0 \) for all \( V \in T_x X \), which implies \( df_x V = 0 \) for all \( V \in T_x \Omega \).

For elastica we have the following

\[
\Omega^\ell_t \xrightarrow{\text{Im}\text{m}^2(I, M)_s} L \xrightarrow{F|\Omega^\ell_t} \mathbb{R}
\]

By the Lagrange multiplier theorem \( x \) is a critical point of \( F|\Omega^\ell_t \) iff there exists \( \beta \in \mathbb{R} \) such that \( \beta \in \mathbb{R} \) such that \( x \) is a critical point of

\[
F^\beta := \int_0^1 k^2 \| \dot{x} \| dt - \beta \int_0^1 \| \dot{x} \| dt = \int_0^1 (k^2 - \beta) ds
\]

with domain \( \text{Im}^2(I, M)_s \). From [14] p. 3 the derivative of \( F^\beta \) in arc length proportional parametrization is (taking account of the different sign given to the Lagrange multiplier)

\[
dF^\beta_x W = \int_0^\ell (2\nabla^3 T + \nabla_T(3k^2 + \beta)T + 2R(\nabla_T T, T)T, W) ds + [2\langle \nabla_T W, \nabla_T T \rangle - \langle W, 2\nabla^2_T T + (3k^2 + \beta)T \rangle]_0^\ell \tag{2}
\]

where the higher derivatives of \( T \) are understood as weak derivatives. As in [14], supposing \( W|_{t=0,1} = \nabla T W|_{0,1} = 0 \), setting \( dF^\beta_x W = 0 \) and using the fundamental lemma of calculus of variations gives the Euler-Lagrange equation

\[
2\nabla^2_{T} T + \nabla_T (3k^2 + \beta) T + 2R(\nabla_T T, T) T = 0 \tag{3}
\]

The fact that weak solutions of this equation are also strong solutions, i.e. the higher derivatives of \( T \) are actually continuous, is a consequence of the regularity theory of elliptic operators. Alternatively, one can use a so-called bootstrap argument based on the Du Bois-Reymond lemma to show inductively that weak solutions of (3) are in fact smooth (cf. [13] p. 21).

Note that the Euler-Lagrange equation (3) is not always equivalent to \( d(F|\Omega^\ell_t)_x = 0 \) or \( d(F|^\beta| \text{Im}^2(I, M)_s)_x = 0 \). When we consider \( F|\Omega^\ell_t \) the boundary terms in (2)
vanish automatically because the tangent space $T_x \Omega^\ell_p$ consists of fields $W$ along $x$ of class $H^2$ which satisfy $W|_{t=0, \ell} = 0$ and $\nabla_T W|_{t=0, \ell} = 0$. However, for $x$ to be a critical point of $F|_{\Omega^\ell_p}$, we require in addition to (3) that $x$ satisfy the natural boundary conditions $\nabla_T T|_{0, \ell} = 0$ in order for the boundary terms in (2), and therefore $dF^\ell_x$, to vanish for all $W \in T_x \Omega^\ell_p$. This is the precise sense in which pinned elastica are a special case of elastica; both satisfy the Euler-Lagrange equation, but pinned elastica necessarily have vanishing acceleration on the boundary, whereas elastica do not.

For $F|\Omega^\ell_p$, the tangent space $T_x \Omega^\ell_p$ consists of those fields $W$ along $x$ which satisfy $W(0) = W'(\ell)$ and $\nabla_T W(0) = \nabla_T W(\ell)$. Then from (2) the natural boundary conditions are $\nabla_T T(0) = \nabla_T T(\ell)$ and $\nabla_T^2 T(0) = \nabla_T^2 T(\ell)$ (note that $T(0) = T(\ell)$ is automatic from the definition of $\Omega^\ell_p$). It then follows from (3) and derivatives thereof that a critical point satisfies $\nabla_T^2 T(0) = \nabla_T^2 T(\ell)$ for any $k$, i.e. the critical points are $C^\infty$-periodic.

Applying the Lagrange multiplier theorem to the restriction of total acceleration $J|\Sigma^v_x$

$$\begin{array}{ccc}
\Sigma^v_x & \longrightarrow & H^2(I, M)_* \longrightarrow \nu \longrightarrow H^1(I, \mathbb{R})_* \\
J|\Sigma^v_x & \downarrow & \mathbb{R}
\end{array}$$

we have that $x \in \Sigma^v_x$ is a critical point of $J|\Sigma^v_x$ iff there is a $\lambda \in H^1(I, \mathbb{R})^*$ such that $x$ is a critical point of $J - (\lambda, \nu)$. Equivalently, changing $\lambda$ to its Riesz representative in $H^1(I, \mathbb{R})$, $x$ is a critical point of

$$J^\lambda := \int_0^1 \frac{1}{2} (\nabla_t \dot{x}, \nabla_t \dot{x}) - \lambda(t) \|\dot{x}\| - \dot{\lambda} \frac{d}{dt} \|\dot{x}\| dt$$

for some $\lambda \in H^1(I, \mathbb{R})$. Writing $\Lambda := \lambda - \dot{\lambda}$ (weakly) we have

$$dJ^\lambda_x V = \int_0^1 (\nabla_t^2 \dot{x} + R(\nabla_t \dot{x}, \dot{x}) \dot{x} + \nabla_t(\Lambda T), V) dt$$

$$+ \left[ (\nabla_t V, \nabla_t \dot{x}) - \langle V, \nabla_t^2 \dot{x} - \Lambda T \rangle \right]_0^1$$

(4)

with Euler Lagrange equation (cf. [26])

$$\nabla_t^2 \dot{x} + R(\nabla_t \dot{x}, \dot{x}) \dot{x} + \nabla_t(\Lambda T) = 0.$$  

(5)

**Lemma 2.2.** A curve $x \in \Omega^\ell_p$ is an elastica parametrized proportional to arc length iff $x$ is a critical point of $J|\Sigma^v_{v, w}$ with $v = \ell$.

**Proof.** If $\|\dot{x}\| = \ell$, then equation (3) becomes

$$\nabla_t^2 \dot{x} + R(\nabla_t \dot{x}, \dot{x}) \dot{x} + \nabla_t(\frac{\ell}{2\pi}(\|\nabla_t \dot{x}\|^2 + \beta) \dot{x}) = 0$$

combining (5) with $\|\dot{x}\| = \ell$ and derivatives thereof forces $\dot{\lambda}(t) = \frac{\ell}{2\pi} \|\nabla_t \dot{x}\|^2$ (cf. [26] for $t = 1$). Integrating and substituting into (5) gives the same equation as above, with $\beta$ the constant of integration of $\dot{\lambda}$. \hfill $\square$

**Corollary 1.** $x \in \Omega^\ell_{p,q}$ is a pinned elastica parametrized proportional to arc length iff it is a critical point of $J|\Sigma^v_{p,q}$ with $v = \ell$, and $x$ is a closed elastica parametrized proportional to arc length iff it is a critical point of $J|\Sigma^v_c$ with $v = \ell$. 

Hessian in the presence of such constraints.

However this doesn’t prove that $\tilde{J}$ itself satisfies the PS condition.

It is tempting now to prove that $J^\lambda$ satisfies the PS condition, since then a sequence $(x_i)$ in $\Sigma_{v,w}$ which is a PS-sequence for $J^\lambda|H^2(I,M)$ has a subsequence which converges in $H^2(I,M)_{v,w}$ to a critical point, and $\Sigma_{v,w}$ is closed in $H^2(I,M)$.

Proof. Follows from the previous lemma as well as the observation that when $x$ is parametrized proportional to arc length the natural boundary conditions obtained from each of (2) and (4) coincide.

3. Lagrange multipliers and the PS condition. This section will serve as an outline of the method we will use to verify the PS condition for elastica. We begin with an excerpt from [27] reviewing the method of Eliason (cf. [8], [9]), and then outline of the method we will use to verify the PS condition for elastica. We begin

Let $X, X_0$ be Banach manifolds modelled on $B, B_0$ respectively, and suppose $X \subset X_0, B \subset B_0$ with the latter a continuous linear inclusion. Then $X$ is a weak submanifold of $X_0$ if for any $x_0$ in the closure of $X$ there is a chart $(\theta_0, U_0)$ for $X_0$ containing $x_0$, such that, setting $U = U_0 \cap X$, we have $\theta_0(U) \subset B$ and the restriction of $\theta_0$ to $U$ is a chart $\theta : U \rightarrow \theta(U)$ for $X$. Any chart for $X$ which arises in this way will be called a weak chart\footnote{When reading these definitions it is helpful to keep in mind the typical example $X = H^k(I,M)$ and $X_0 = C^0(I,M)$, where the weak charts are given by $\exp_h : \xi \rightarrow \xi$, where $h \in C^\infty(I,M)$ and $\xi \in C^0(h^*TM)$.} at $x_0$.

Note that this definition allows the topology of the weak submanifold to be finer than the relative topology.

A Finsler structure on a Banach manifold $X$ is a continuous function $v \mapsto \|v\|$ on the tangent bundle $\tau : TX \rightarrow X$ such that the restriction to each fibre $T_x X$ is a norm, and such that in any local trivialisation $\Theta : \tau^{-1} U \rightarrow \theta(U) \times B$, and for any constant $k > 1$, we have

$$\frac{1}{k} \|\Theta^{-1}(\xi, \eta)\| \leq \|\Theta^{-1}(\xi_0, \eta)\| \leq k\|\Theta^{-1}(\xi, \eta)\| \quad \text{(6)}$$

with $\eta \in B$ and $\xi$ sufficiently close to $\xi_0 = \theta(x)$. That is, the fibre norms are locally equivalent.

Suppose $X$ is a weak submanifold of $X_0$ and let $\|\|_B$ denote the norm for $B$ and $|\|_0$ the norm for $B_0$. We call a Finsler structure on $X$ locally bounded with respect to (weak charts from) $X_0$ if for any $x_0$ in the closure $\bar{X}$ and any constant $L$, there is a local trivialisation $\Theta$ over a weak chart $\theta$ at $x_0$ and a constant $c$ such that

$$\|\Theta^{-1}(\xi, \eta)\| \leq c\|\eta\|_B$$

for all $\xi \in \theta(U)$ with $\|\xi\|_B < L$, and all $\eta \in B$.

A function $f : X \rightarrow \mathbb{R}$ is called weakly proper with respect to $X_0$ if any subset $A \subset X$ on which $f$ is bounded is relatively compact in $X_0$.\footnote{When reading these definitions it is helpful to keep in mind the typical example $X = H^k(I,M)$ and $X_0 = C^0(I,M)$, where the weak charts are given by $\exp_h : \xi \rightarrow \xi$, where $h \in C^\infty(I,M)$ and $\xi \in C^0(h^*TM)$.}
(ii) \textit{locally bounding} with respect to $X_0$ if for any constants $K, L$ and $x_0 \in \bar{X}$
there is a weak chart $(U, \theta)$ at $x_0$ and a constant $\alpha$ such that for all $\xi \in \theta(U)$
with $|\xi|_0 < K$ and $f_\theta(\xi) := f(\theta^{-1}(\xi)) < L$, we have $||\xi||_B < \alpha$.

(iii) \textit{locally coercive} with respect to $X_0$ if it is $C^1$ and for any $x_0 \in \bar{X}$ and any
constant $K$, there is a weak chart $(U, \theta)$ at $x_0$ and there exist constants $c_1 > 0, c_2$ such that
\begin{equation}
(Df_\theta(\xi) - Df_\theta(\eta))(\xi - \eta) \geq c_1||\xi - \eta||^2_B - c_2||\xi - \eta||^2_0
\end{equation}
for all $\xi, \eta \in \theta(U)$ with $||\xi||_B < K, ||\eta||_B < K$. If $f$ is of class $C^2$ we have an
equivalent condition:
\begin{equation}
D^2f_\theta(\xi)(\eta, \eta) \geq c_1||\eta||^2_B - c_2||\eta||^2_0
\end{equation}
for all $\xi \in \theta(U)$ with $||\xi||_B < K$ and all $\eta \in B$.

The assumption of an upper bound for $|\xi|_0$ is not included in the original definition of locally bounding [8] because it does not need to be assumed if $f$ is weakly proper. However we will find it useful to be able to prove that $f$ is locally bounding independently.

**Theorem 3.1.** (Eliašson [9]) Let $X$ be a regular Banach manifold and a weak submanifold of $X_0$ as above, with a locally bounded Finsler structure. If $f : X \to \mathbb{R}$
is of class $C^1$ and weakly proper, locally bounding and locally coercive each with
respect to $X_0$, then $f$ satisfies the Palais-Smale condition.

Suppose now that $\Omega$ is a submanifold of $X$, which is in turn a weak submanifold
of $X_0$. Note that it is not necessarily true that $\Omega$ is also a weak submanifold of $X_0$.
Of course at any point in $\Omega$ there is a chart for $X$ which restricts to a chart for $\Omega$, i.e.
satisfies the submanifold property, but in general it is not necessary that this chart is the restriction of a chart for $X_0$, viz. a weak chart. For example, consider $\Sigma^1$:
the submanifold of $H^1(I, \mathbb{R}^2)$ consisting of unit speed curves in the Euclidean plane.
The natural charts $\exp_{\theta} \circ \xi \to \xi, h \in C^\infty(I, \mathbb{R}^2)$ for $H^1(I, \mathbb{R}^2)$ are weak charts with
respect to $C^0(I, \mathbb{R}^2)$. Suppose $r_1, r_2 \in H^1(I, \mathbb{R}^2)$ are parametrizations of the upper
and lower halves of a circle respectively, with unit speed and the same initial and
terminal points. Then in the natural chart centred at $r_1$, the representative of $r_2$ is
$r_2 - r_1$. However, $\exp_{r_1} \frac{1}{2}(r_2 - r_1)$ will not have unit speed. This means $\frac{1}{2}(r_2 - r_1)$
is not in the local image of $\Sigma^1$, and therefore the natural charts for $H^1(I, \mathbb{R}^2)$ do
not satisfy the submanifold property for $\Sigma^1$.

For this reason the definitions above are not directly applicable and require the
following modifications.

**Remark 1.** In this and subsequent sections it will frequently be the case that we
are interested in bounding some quantity by a constant, but the precise value of
the constant is not important. It will therefore be convenient to use the symbol
$c$ to denote a \textit{floating} constant, i.e. it may change during a calculation but is
nevertheless independent of any variables.

**Definition 3.2.** Let $X$ be a weak submanifold of $X_0$ and $\Omega$ a submanifold of
$X$ (which is not necessarily a weak submanifold of $X_0$), and suppose we have a
smooth projection $\text{pr}^{T\Omega} : TX|\Omega \to T\Omega$. We will say $\text{pr}^{T\Omega}$ is \textit{locally bounded} with
respect to (weak charts from) $X_0$ if for any $\omega_0$ in the $X_0$-closure of $\Omega$, and any
constant $L$, there is a weak chart $(\theta, U)$ for $X$ at $\omega_0$ and a constant $c$ such that
$||\text{pr}^{T\Omega}(\xi)||_B \leq c||\eta||_B$ for all $\xi \in \theta(U \cap \Omega)$ with $||\xi||_B < L$, and all $\eta \in B$ (we
are adopting a standard abuse of notation whereby $\text{pr}^{T\Omega}$ is used to denote both
the map and its local representative). It then follows that $|pr^T\Omega(\xi)| \leq c$. We call $\tilde{f} = f|\Omega$ locally coercive with respect to $(X, X_0)$ if for any $\omega_0$ in the $X_0$-closure of $\Omega$, and any constant $\alpha$, there is a weak chart $(U, \theta)$ at $\omega_0$ and constants $c_+ > 0$ and $c$ such that

$$c_+ \|\xi - \eta\|^2_B \leq c\|\xi - \eta\|_0 + (Df(\xi) pr^T\Omega(\xi) - Df(\eta) pr^T\Omega(\eta))(\xi - \eta)$$

for all $\xi, \eta$ in $\theta(U \cap \Omega)$ with $\|\xi\|_B, \|\eta\|_B < \alpha$. Note the absence, in contrast with (7), of the square on $|\xi - \eta|_0$.

**Theorem 3.3.** Let $f : X \to \mathbb{R}$ be a smooth function, where $X$ is a weak submanifold of $X_0$ with a locally bounded Finsler structure. Suppose also that $\Omega$ is a submanifold of $X$ with a smooth projection $pr^T\Omega : TX|\Omega \to T\Omega$ which is locally bounded with respect to $X_0$. Then if $\tilde{f} := f|\Omega$ is weakly proper with respect to $X_0$, $f$ is locally coercive with respect to $X_0$, and $\tilde{f}$ is locally coercive with respect to $(X, X_0)$, then $\tilde{f}$ satisfies the Palais-Smale condition.

**Proof.** Let $(x_i) \subset \Omega$ be a sequence for which $f(x_i)$ is bounded and $|d\tilde{f}_{x_i}| \to 0$. Since $\tilde{f}$ is weakly proper we can find a subsequence converging in $X_0$ to some $x_0$. We choose a weak chart $\theta : U \to B$ at $x_0$, with corresponding trivialisation $\Theta : \tau^{-1}(U) \to B \times B$ of the tangent bundle, and a subsequence $\xi_i := \theta(x_i)$. Then $\xi_i$ is bounded in $B$ because $f$ is locally bounding. Using the local coercivity of $\tilde{f}$ we have

$$c_+ \|\xi_i - \xi_j\|^2_B \leq c\|\xi_i - \xi_j\|_0 + (Df(\xi_i) pr^T\Omega(\xi_i) - Df(\xi_j) pr^T\Omega(\xi_j))(\xi_i - \xi_j)$$

$$\leq c\|\xi_i - \xi_j\|_0 + (|D\tilde{f}(\xi_i)| + |D\tilde{f}(\xi_j)|)c\|\xi_i - \xi_j\|_B$$

(10)

where we have also used the assumption that the projection is locally bounded. Moreover, since the Finsler structure for $X$ is locally bounded,

$$|D\tilde{f}(\xi_i)\eta| = |d\tilde{f}(x_i)\Theta^{-1}_{x_i}\eta| \leq |d\tilde{f}(x_i)|\|\eta\|_B$$

i.e. $|D\tilde{f}(\xi)| \leq c|d\tilde{f}(x_i)|$. Finally, using the assumption $|d\tilde{f}(x_i)| \to 0$, and the $B_0$-convergence of $(\xi_i)$, we have from (10) that $(\xi_i)$ is Cauchy in $B$ and then the corresponding subsequence $(x_i)$ converges in $\Omega$ because it is closed in $X$. \qed

We consider again the setting of Theorem 2.1:

$$\begin{array}{ccc}
\Omega & \longrightarrow & X \\
\downarrow \phi & & \downarrow \phi \\
\mathbb{R} & \longrightarrow & E
\end{array}$$

where $e_0 \in E$ is a regular value, meaning $T_x\phi$ is surjective and has a split kernel for all $x \in \Omega := \phi^{-1}(e_0)$. Then $\phi^*TE = X \times E$ and we have a short exact sequence of VB morphisms

$$0 \longrightarrow T\Omega \longrightarrow TX|\Omega \longrightarrow T\Omega \times E \longrightarrow 0$$

(11)

Suppose the above sequence admits a right split, i.e. a VB morphism such that $\phi^*T\phi \circ r = I_{\Omega \times E}$, or fibrewise: $d\phi_x \circ r_x = I_E$. Equivalently ([1] p. 183) we have a splitting $TX|\Omega \cong T\Omega \oplus \text{Im}r$ and a smooth projection $pr^T\Omega : TX|\Omega \to T\Omega$ given fibrewise by $pr^T\Omega(x) = 1 - r_x d\phi_x$.

**Lemma 3.4.** The sequence (11) admits a local right split at any $x \in \Omega$. 

Proof. Since \( \phi \) is a submersion at \( x \in \Omega = \phi^{-1}(e_0) \) there is a chart \( \theta : U \to B \) for \( X \) at \( x \) such that:

- \( \theta(U) \ni U_1 \times U_2 \subset B_1 \times B_2 \cong B \) and \( \phi_\theta : U_1 \times U_2 \to E \) can be factored into \( \phi_\theta = \gamma \circ \text{pr}_1 : U_1 \times U_2 \to U_1 \to V \subset E \) where \( \gamma \) is a diffeomorphism, and

- \( \theta[U \cap \Omega \to \{ \gamma^{-1}(e_0) \}] \times U_2 \) is a chart for \( \Omega \).

Then since \( \phi_\theta(\gamma^{-1}(e), w) = e \) for any \( e \in V, w \in U_2 \), we also have

\[
D_1 \phi_\theta(\gamma^{-1}(e), w) D \gamma^{-1}(e) = I_E.
\]

So defining \( r_\theta : U_2 \times E \to U_2 \times B_1 \times B_2 \) by \( r_\theta(w, e) := (w, D \gamma^{-1}(e_0)e, 0) \) gives a local right split.

If \( X \) admits partitions of unity then a split can be constructed from such local splits.

**Lemma 3.5.** Let \( \tilde{f} = f - (\lambda, \phi - e_0) : X \to \mathbb{R} \), where \( \lambda : X \to B^* \) is any smooth map which satisfies \( \lambda_x = df_x \circ r_x \) for all \( x \in \Omega \). Then \( d \tilde{f}_x V = df_x \pi^{T\Omega}_1 V \) for all \( x \in \Omega \) and \( V \in T_x X \), and therefore \( x \) is a critical point of \( f|\Omega \) iff it is a critical point of \( f \).

**Proof.** For any \( x \in \Omega, V \in T_x X \) we have, with \( \phi(x) = e_0 \),

\[
d\tilde{f}_x V = df_x V - (\lambda_x, d\phi_x) - (d\lambda_x V, \phi(x) - e_0)
= df_x (V - r_x d\phi_x V) = df_x \pi^{T\Omega}(x) V \quad (12)
\]

**Remark 2.** Note that in contrast with Theorem 2.1, where the Lagrange multiplier is treated as an extra variable, \( \lambda \) is now a function \( X \to B^* \) and is defined in advance on all of \( \Omega \) by a choice of \( r \). This is not necessary in order to write down Euler-Lagrange equations, but it is needed for Theorem 3.7.

**Lemma 3.6.** Let \( \Omega, X, r \) be as above with \( X \) a weak submanifold of \( X_0 \). Suppose \( f : X \to \mathbb{R} \) is locally coercive with respect to \( X_0 \), and that in a weak chart \( \theta, U \) for \( X \) where (7) holds we also have

\[
|Df(\xi)r(\xi)D\phi(\xi)\eta| \leq \epsilon|\eta|_0 \quad (13)
\]

for any \( \eta \in B \), whenever \( \xi \in \theta(U \cap \Omega) \) with \( \|\xi\|_B \leq \epsilon \). Then \( f|\Omega \) is locally coercive with respect to \( (X, X_0) \).

**Proof.** Since \( \pi^{T\Omega}_2 = I - r_x d\phi_x \), for any \( \eta \in B \) we have

\[
Df(\xi)\eta = Df(\xi)(\pi^{T\Omega}_2(\xi) + r(\xi)D\phi(\xi))\eta
\]

when \( \|\xi\|_B \leq \epsilon \). Then since \( f \) is locally coercive on \( X \) with respect to \( X_0 \), using (13),

\[
eq \epsilon\|\xi - \eta\|^2_0 \leq \epsilon|\xi - \eta|_0^2 + (Df(\xi) - Df(\eta))(\xi - \eta)
\leq \epsilon|\xi - \eta|_0 + (Df(\xi) \pi^{T\Omega}(\xi) - Df(\eta) \pi^{T\Omega}(\eta))(\xi - \eta).
\]
Recall that given a critical point \( x \) of \( f \) and vector fields \( V, W \) on \( X \), the Hessian \( \text{Hess}_x f(V,W) := W_x V f \) is bilinear and symmetric in \( V \) and \( W \), and depends only on the vectors \( V_x, W_x \). The Morse index of a critical point \( x \) of \( f \) is the dimension of the maximal subspace on which \( \text{Hess}_x f \) is negative definite. We say \( f \) is a Morse function if \( \text{Hess}_x f \) is strongly nondegenerate at every critical point, i.e. if the associated self-adjoint operator is an isomorphism.

Suppose \( f \) is a Morse function on a complete Riemannian manifold \( X \) which satisfies the Palais-Smale condition. Let \( m_i \) denote the number of critical points of \( f \) with index \( i \), and \( \beta_i \) the \( i \)th Betti number of \( X \). Then the weak Morse inequalities state that \( \beta_i \leq m_i \) (see [25] p. 220).

For \( f \) to be a Morse function it is necessary that the nullspace of \( \text{Hess}_x f \) be trivial at each critical point \( x \). We observe that \( V_x \in \text{null} \text{Hess}_x f \) iff \( \text{Hess}_x f(V,W) = W_x V f = 0 \) for all \( W_x \in T_x X \), i.e. if \( x \) is also a critical point of \( V f \). Just as it is necessary to introduce a Lagrange multiplier in order to write the condition \( df_x = 0 \) in strong form (i.e. as a differential equation), so it is also necessary to introduce a Lagrange multiplier in order to characterise the nullspace of \( \text{Hess}_x \tilde{f} \).

**Theorem 3.7.** The following are equivalent statements for \( x \in \Omega \) a critical point of \( f \) and \( V_x \in T_x \Omega \):

- \( V_x \) is in the nullspace of \( \text{Hess}_x \tilde{f} \), that is, \( \text{Hess}_x \tilde{f}(V_x,Y) = 0 \) for all \( Y \in T_x \Omega \).
- there exists \( \mu \in B^* \) such that
  \[
  WVf - (\lambda_x, WV\phi) - (\mu, d\phi W) = 0
  \] (14)
  for all \( W \in T_x X \).

**Proof.** Suppose \( x \) is a critical point of \( \tilde{f} \) and \( V_x \in T_x \Omega \), and let \( V \) be an extension of \( V_x \) to \( X \) such that \( V(\Omega) \subset T \Omega \). Then from equation (12) we see that
  \[
  V \tilde{f} = (Vf)|_\Omega = (Vf - (V\lambda, \phi - e_0) - (\lambda, V\phi))|_\Omega = (Vf - \lambda(V\phi))|_\Omega
  \]
Now \( V_x \) is in null \text{Hess}_x \tilde{f} \iff \( x \) is a critical point of \( V \tilde{f} \) if \( x \) is a critical point of \( (Vf - (\lambda, V\phi))_{|\Omega} \), which, by the Lagrange multiplier theorem (2.1), is equivalent to
  \[
  0 = d(Vf - (\lambda, V\phi))_x W - (\mu, d\phi_x W) = WVf - (W\lambda, V\phi) - (\lambda, WV\phi) - (\mu, W\phi)
  \]
for all \( W \in T_x X \), where \( \mu \in B^* \), and the \((W\lambda, V\phi)\) term vanishes because \( V\phi(x) = 0 \).

4. **Manifolds of constant speed curves.** We will focus here on conditions which ensure that \( \Sigma^i \) is the preimage of a regular value of \( \nu : \text{Imm}^2(I, M) \to H^1(I, \mathbb{R}) \), where \( H^1(I, \mathbb{R}) \), is a suitable submanifold of \( H^1(I, \mathbb{R}) \). According to Lemma 2.2 we will only need to work with \( \Sigma^i \), and the proofs for \( \Omega^i \) are similar.

We recall that in order for \( \nu \in \text{Imm}^2(I, M) \) to be a regular point of \( \nu \) the requirement is that \( d\nu_x \) should be surjective and have split kernel. For Banach spaces the latter is not automatic, however the kernel is a closed subspace so in a Hilbert space it has a closed orthogonal complement, i.e. it splits.

Define the endpoint maps
  \[
  P_0 : H^2(I, M) \to M \times M, \quad P_0(x) := (x(0), x(1))
  \]
  \[
  P_1 : H^2(I, M) \to TM \times TM, \quad P_1(x) := (x(0), \dot{x}(0), x(1), \dot{x}(1))
  \]
Each of these maps can be shown to be a submersion and then \( H^2(I, M) \) is a submanifold of \( H^2(I, M) \) with \( T_x H^2(I, M) = \ker dP_0(x) \). For example, \( T_x H^2(I, M)_{\nu,w} \) consists of vector fields along \( x \) which satisfy \( V|_{0,1} = 0 \) and \( \nabla_x V|_{0,1} = 0 \).
Then we can write (15) as

\[ V \] in a more familiar form we work in an orthonormal parallel frame \( \{ V_i \} \) so that \( V \) is a collection of vector fields along \( x \) which span the orthogonal complement of \( \dot{x} \), and the \( u_i \) are functions which we are free to choose. Equation (15) represents a \textit{linear time dependent control system}. Such a system is called \textit{controllable} on \([0,1]\) if for any initial state \( V(0) \) and any \( V_i \) there exist \( u_i \) and a corresponding solution \( V \) such that \( V(1) = V_1 \). If this system is controllable then \( x \) is a regular point of \( \nu^0 \). In order to write (15) in a more familiar form we work in an orthonormal parallel frame \( \{ e_k \} \) along \( x \) so that \( V = V^k e_k, \dot{x} = \dot{x}^k e_k, E_i = E_i^k e_k \) with repeated indices summed and \( V, \dot{x}, E_i \in \mathbb{R}^n \). Then we can write (15) as

\[
\nabla_i V = \frac{w}{\|w\|} \dot{x} + \sum_{i=1}^{n-1} u_i E_i
\]  

(15)

where \( \{ E_i(t) \} \) is a collection of vector fields along \( x \) which span the orthogonal complement of \( \dot{x} \), and the \( u_i \) are functions which we are free to choose.

Equation (15) represents a \textit{linear time dependent control system}. Such a system is called \textit{controllable} on \([0,1]\) if for any initial state \( V(0) \) and any \( V_i \) there exist \( u_i \) and a corresponding solution \( V \) such that \( V(1) = V_1 \). If this system is controllable then \( x \) is a regular point of \( \nu^0 \). In order to write (15) in a more familiar form we work in an orthonormal parallel frame \( \{ e_k \} \) along \( x \) so that \( V = V^k e_k, \dot{x} = \dot{x}^k e_k, E_i = E_i^k e_k \) with repeated indices summed and \( V, \dot{x}, E_i \in \mathbb{R}^n \). Then we can write (15) as

\[
\dot{V} = \frac{w}{\|w\|} \dot{x} + Bu
\]  

(16)

where \( u \in H^1(I, \mathbb{R}^{n-1}) \) and \( B \) is the \( n \times (n-1) \) matrix with the coordinates of \( E_i \) in the \( i \)th column. We address the question of controllability as follows. First consider the linear time dependent control system

\[
\dot{b} = Bu
\]  

(17)

Suppose (17) is controllable and \( a \) is a solution of \( \dot{a} = \frac{w}{\|w\|} \dot{x} \). Then given \( V_0, V_1 \) there exists \( u \) and a corresponding \( b \) such that \( b(0) = V_0 - a(0), b(1) = V_1 - a(1) \), so that \( \dot{V} = a + b \) is a solution to (16) with \( V(0) = V_0 \) and \( V(1) = V_1 \). Thus controllability of (16) is equivalent to controllability of (17). A necessary and sufficient condition for (17) to be controllable on \([0,1]\) is that the matrix

\[
W := \int_0^1 B(t)B(t)^T dt,
\]

should be non-singular, in which case a particular control which drives the solution to \( b(1) = b_1 \) is given by \( u = B^T W^{-1}(b_1 - b_0) \), (see e.g. [3] p. 76). If \( W \) is singular then there exists a non-zero \( y \in \mathbb{R}^n \) such that

\[
y^T W y = \int_0^1 y^T B(y^T B)^T dt = 0
\]

Then \( y^T B(t) = 0 \) almost everywhere on \( I \). This is only possible if there exists a real valued function \( \alpha \) such that \( y = \alpha(t) \dot{x} \), and then since \( y \) is constant \( \dot{x} + \alpha \dot{x} = 0 \), i.e.

\[
\dot{x} + \alpha \nabla_i \dot{x} = 0
\]  

(18)
Since we have assumed $\alpha \neq 0$ it then follows that $x$ is a regular point of $\nu^0$ if it is not a reparametrized geodesic.

In particular if $\nu(x) \equiv v$ then $\langle \nabla_t x, \dot{x} \rangle = \frac{1}{2} \frac{d}{dt} \|\dot{x}\|^2 = 0$ and then (18) holds iff $x$ is a geodesic. Thus if there are no geodesics joining $p, q$ with constant speed $v$ (and therefore length $L(x) = \int_0^1 v dt = v$) then $v$ is a regular value of $\nu^0$ and $\Sigma^r_{p, q}$ is a submanifold of $H^2(I, M)_{p, q}$.

Next we characterise regular points of the restriction $\nu^1 := \nu|\text{Imm}^2(I, M)_{\nu, w} \to H^1(I, \mathbb{R})_{\|\cdot\|}$.

The codomain is a submanifold of $H^1(I, \mathbb{R})$ with tangent space $H^1(I, \mathbb{R})_{0, 0} = \{ w \in H^1(I, \mathbb{R}) : w|_{t=0, 1} = 0 \}$, and an element $V$ of the tangent space of the domain must satisfy $V|_{t=0, 1} = 0$ and $\nabla_t V|_{t=0, 1} = 0$. Therefore instead of (15) we look for solutions of

$$\nabla_t V = \frac{w}{\|w\|} \dot{x} + \sum_{i=1}^{n-1} u_i \beta(t) E_i$$

where $\beta : I \to \mathbb{R}$ is any smooth function which satisfies $\beta(0) = 0 = \beta(1)$ and otherwise $\beta(t) \neq 0$. Then since $w(0) = w(1)$, any solution of (19) automatically satisfies $\nabla_t V|_{0, 1} = 0$. Moreover the system (19) is controllable by precisely the same argument as above, and therefore $x$ is a regular point of $\nu^1$, provided there is no solution to (18).

In particular, it follows that if there are no geodesics in $\Sigma^u_{\nu, w}$ then it is a submanifold.

Finally, let us consider the restriction $\nu^c := \nu|\text{Imm}^2(I, M)_{c} \to H^1(I, \mathbb{R})_{c} = P_0^{-1}(\text{diag} \mathbb{R}^2)$, where the codomain is the submanifold of $H^1(I, \mathbb{R})$ consisting of periodic functions and the tangent space $T_x H^2(I, M)_{c} = \{ V \in T_x H^2(I, M) : V(0) = V(1), \nabla_t V(0) = \nabla_t V(1) \}$. In this case we again look for a solution of (19) but now with $\dot{z}$ and $w$ periodic, and then $\nabla_t V$ is automatically periodic. Moreover, if the system is controllable then we can set $V(0) = V(1)$ and so $x$ is a regular point. Again it follows that if there are no geodesics in $\Sigma^u_{\nu, w}$ then it is a submanifold. We summarize the required results from above in the following Lemma.

**Lemma 4.1.** For $* = \text{void}$ or $p$, the constant $v$ is a regular value of the restriction $\nu|\text{Imm}^2(I, M)_{*}$, and therefore $\Sigma^c_{\nu} = \nu^{-1}(v)$ is a submanifold of $\text{Imm}^2(I, M)_{*}$. If $* = (p, q), (\nu, w)$ or $c$ then the same is true provided there are no geodesics in $\Sigma^u_{\nu}$.

**Proof.** See preceding discussion.

5. **The Palais-Smale condition for elastica.** Our goal in this section is to prove that $J|\Sigma^u_{\nu}$ satisfies the PS condition using Theorem 3.3. We will assume henceforth that $\nu, (p, q), (\nu, w)$ where relevant, are such that $\Sigma^u_{\nu}$ contains no geodesics and is therefore a submanifold by Lemma 4.1.

For $\Sigma^u_{\nu, w}$, the short exact sequence corresponding to (11) is

$$0 \longrightarrow T\Sigma^u_{\nu, w} \longrightarrow T\text{Imm}^2(I, M)_{\nu, w} \Sigma^u_{\nu, w} \overset{\nu^* T_{\nu, w}}{\longrightarrow} \Sigma^u_{\nu, w} \times H^1(I, \mathbb{R})_{0, 0} \longrightarrow 0$$

and we will begin by constructing a right split $r$ for the above sequence. For $x \in \Sigma^u_{\nu, w}$ and $w \in H^1(I, \mathbb{R})$ we will define $r_x w$ as follows. We have already observed that a solution $V$ of (19) will satisfy $d\nu_x V = w$. Setting $r_x w := V$ where $V$ is a solution of (19) with $V|_{t=0, 1} = 0$ will then satisfy the desired property: $d\nu_x r_x = I$. However we haven’t specified the frame $\{ E_i \}$ for $\dot{x}$ or the controls $u_i$ and so $r_x$ is not yet well-defined.
First we will show how to construct a particular frame for $\dot{x}^\perp$ for any $x \in \Sigma^u_{v,w}$. Fix an orthonormal basis $\{\frac{1}{n}v, e_i\}$ for $T_pM$ and solve

$$\nabla_t E_i = -\frac{1}{n}\langle E_i, \nabla_t \dot{x}\rangle \dot{x}, \quad E_i(0) = e_i. \quad (21)$$

Then $0 = \langle \nabla_t E_i, \dot{x} \rangle + \langle E_i, \nabla_t \dot{x} \rangle = \frac{d}{dt}\langle E_i, \dot{x} \rangle$ and therefore $\langle E_i, \dot{x} \rangle = 0$ since $\langle e_i, v \rangle = 0$. It also follows that $\langle \nabla_t E_i, E_j \rangle = 0$, then $\frac{d}{dt}\langle E_i, E_j \rangle = 0$ and $\langle E_i, E_j \rangle = \delta_{ij}$ where $\delta$ is the Kronecker delta function. Hence (21) defines an orthonormal frame $\{\frac{1}{n} \dot{x}, E_i\}$ along $x$.

We will use this adapted frame in (19) and as in (16) we (temporarily) work in an orthonormal parallel frame along $x$ and write (19) as

$$\dot{V} = \frac{\dot{w}}{n} \dot{x} + \sum_{i=1}^{n-1} u_i \beta E_i \quad (22)$$

with $V, \dot{x}, E \in \mathbb{R}^n$. We will also assume that $\beta$ is normalised to $\int_0^1 \beta dt = 1$. To construct a solution to the above with $\dot{V}(0) = 0 = \dot{V}(1)$ we first let $a$ be the solution to $\dot{a} = \frac{\dot{w}}{n} \dot{x}$ with $a(0) = 0$. Then we look for a solution of $\dot{b} = \sum_{i=1}^{n-1} u_i \beta E_i$, such that $b(0) = 0$ and $b(1) = -a(1)$ and let $\dot{V} = a + b$. In matrix form $\dot{b} = B u$ where $B := \beta [E_1 \ldots E_{n-1}]$. According to [3] p. 76, a control which drives $b(0) = 0$ to $b(1) = -a(1)$ is given by $u = B^T \eta$ where $\eta$ is any solution of $\int_0^1 B B^T dt \eta = a(1)$.

Since the $E_i$ are orthonormal and $\beta$ is normalised we have $\int_0^1 B B^T dt = n - 1$. So we let $\eta = \frac{1}{n-1} a(1)$ and then $u = \frac{1}{n-1} B^T a(1)$, i.e. $u_i = \frac{\beta}{n-1} \langle E_i, a(1) \rangle E_i$.

In covariant terms this means we define

$$r_s w := a + b \quad (23)$$

where $a$ and $b$ are the solutions of

$$\nabla_t a = \frac{\dot{w}}{n} \dot{x}, \quad a(0) = 0 \quad (24)$$

$$\nabla_t b = \sum_{i=1}^{n-1} \frac{\beta}{n-1} \langle E_i, P_{t-1} a_1 \rangle E_i, \quad b(0) = 0, a_1 = a(1) \quad (25)$$

and where by $P_t$ we mean parallel translation along $x$ for time $t$ beginning at $p = x(0)$.

In order to apply Theorem 3.3 we will need to prove that the projection induced by $r$ is locally bounded. This will require some estimates for $\|r_s w\|_2$.

First we estimate $\|a\|_2$. From (24) we have $\|\nabla_t a\|^2 = w^2$. Then since $a(0) = 0$, using the fundamental theorem of calculus and the Cauchy-Schwarz and Hölder inequalities gives

$$\|a\|^2 = \int_0^t \frac{d}{dt}\|a\|^2 dt = 2 \int_0^t \langle \nabla_t a, a \rangle dt \leq 2 \int_0^t |w|\|a\| dt \leq 2\|\dot{a}\|_0\|w\|_0 \quad (26)$$

from which we observe $\|a\|_0 \leq 2\|w\|_0$ and also $|a_0| \leq 2\|w\|_0$. Differentiating (24) gives

$$\nabla_t^2 a = \frac{\dot{w}}{n} \nabla_t \dot{x} + \frac{\ddot{w}}{n} \dot{x} \quad (27)$$

and therefore $\|\nabla_t^2 a\|^2 = \left(\frac{\dot{w}}{n}\right)^2 \|\nabla_t \dot{x}\|^2 + \ddot{w}^2$. Now overall we have

$$\|a\|_2 \leq \|\dot{a}\|_0 + \|\nabla_t a\|_0 + \|\nabla_t^2 a\|_0 \leq c\|w\|_1 + c\|w\|_0 \|\nabla_t \dot{x}\|_0 \quad (28)$$
From (25), the Cauchy-Schwarz inequality, and recalling that parallel translation gives isometries
\[ \| \nabla b \|^2 = \left( \frac{\beta}{n-1} \right)^2 \sum_i \langle E_i, \mathcal{P}_i^{-1} a_1 \rangle^2 \leq c \| a(1) \|^2 \leq c \| w \|_0^2 \]
where the last step uses the inequality \( |a|_0 \leq 2 \| w \|_0 \) proved above. Then since \( b(0) = 0 \) the same argument used for \( a \) gives \( \| b \|_0 \leq c \| w \|_0 \). Differentiating (25) gives
\[ \nabla_i^2 b = \left( \frac{\beta}{n-1} \right)^2 \sum_i (\langle \nabla_i E_i, \mathcal{P}_i^{-1} a_1 \rangle E_i + \langle E_i, \mathcal{P}_i^{-1} a_1 \rangle \nabla_i E_i) \]
(29)
From (21) \( \| \nabla_i E_i \|^2 = \langle E_i, \nabla_i \dot{x} \rangle^2 \leq \| \nabla_i \dot{x} \|^2 \), and also \( \langle \nabla_i E_i, E_j \rangle = 0 \), so
\[ \| \nabla_i^2 b \|^2 = \left( \frac{\beta}{n-1} \right)^2 \sum_i \langle \nabla_i E_i, \mathcal{P}_i^{-1} a_1 \rangle^2 + \langle E_i, \mathcal{P}_i^{-1} a_1 \rangle^2 \| \nabla_i E_i \|_0^2 \]
\[ \geq c \| \mathcal{P}_i^{-1} a_1 \|^2_0 \| \nabla_i E_i \|^2 \leq c \| w \|_0^2 \| \nabla_i \dot{x} \|^2 \]
Combining the preceding estimates for \( b \) shows that \( \| b \|_2 \leq c \| w \|_0 + c \| w \|_0 \| \nabla_i \dot{x} \|_0 \), which together with (28) yields
\[ \| r_x w \|_2 \leq c \| w \|_1 + c \| w \|_0 \| \nabla_i \dot{x} \|_0 \]
(30)
The next task is to prove that the projection \( \text{pr}^{T \Sigma^w_{\nu}} = 1 - r_x d\nu_x \), henceforth abbreviated to \( \text{pr} \), is locally bounded with respect to \( (H^2, C^1) \). For this we will need to infer bounds on the local expression for \( \text{pr} \) in a trivialisation induced by a weak chart, from bounds obtained in tangent spaces, such as (30) above. In order to do so we require the following auxiliary lemmas.

**Lemma 5.1.** Let \( \theta_h, U_h \) be the natural chart for \( H^2(I, M) \) centred at \( h \). Then for any constant \( c_1 \) there is a constant \( c_2 \) such that \( x = \theta_h^{-1}(\xi) \) satisfies \( \| \dot{x} \|_1 \leq c_2 \) whenever \( \xi \in \theta_h(U_h) \) with \( \| \xi \|_2 \leq c_1 \).

**Proof.** From [7] Theorem 11 (or see [27] for a summary) the local expressions for \( \dot{x} \) and \( \nabla_i \dot{x} \) with respect to the induced trivialisation \( \Theta_h = \theta_h \xi + Q_1(\xi) \) and \( (\nabla_i \partial_h)_{\theta_h} \xi = 2Q_1(\xi) \) where \( Q_1, Q_2 \) are polynomial differential operators of order 0 and 1 respectively. Using these local expressions, and the fact that the Finsler structure on \( H^0(H^2(I, M)^\ast TM) \) is locally bounded (cf. [27] Lemma 2) :
\[ \| \dot{x} \|^2_0 \leq c \| \Theta_h(\xi, \partial_\xi) \xi \|^2_0 \leq c \| \partial_\xi \xi \|^2_0 \leq c \| \xi \|^2_1 \]
and similarly \( \| \nabla_i \dot{x} \|^2_0 \leq c \| (\nabla_i \theta)_h \xi \|^2_0 \leq c \| \xi \|^2_2 \). \( \Box \)

**Lemma 5.2.** Let \( \theta_h \) be the local trivialisation for \( TH^2(I, M) \) induced by the natural chart \( \theta_h, U_h \). Then for any constant \( c_1 \) there is a positive constant \( c_2 \) such that for any \( \eta \in H^2(h^* TM) \), we have that \( v = \Theta_h^{-1}(\xi, \eta) \) satisfies \( \| v \|_2 \geq c_2 \| \eta \|_2 \) whenever \( \| \xi \|_2 \leq c_1 \).

**Proof.** Similar to the proof of [27] Lemma 2, we have:
\[ \| v \|^2_2 = \sum_{i=0}^2 \int_I g(x)(\nabla_{i} x, \nabla_{i} x) = \sum_{i=0}^2 \int_I g(h)(G(\xi)(\nabla_{i} \xi, \nabla_{i} \xi) \eta, (\nabla_{i} \xi) \eta) \geq c \| \eta \|^2_2 \]
using the fact that \( G(\xi) \) is positive definite, the assumption that \( \| \xi \|_0 \) is bounded, and the local formula ([27] eq. (5)) for \( (\nabla_i)_h \).

**Proposition 1.** The projection \( \text{pr} : TH^2(I, M)_{\nu,w} \to T\Sigma^w_{\nu,w} \), obtained from the right split \( r \) (23) as \( \text{pr} = 1 - r_x d\nu_x \), is locally bounded with respect to \( C^1(I, M) \).
Proof. From $dv_x V = \frac{1}{\nu} \langle \nabla_t V, \dot{x} \rangle$ we have $\frac{d}{dt}(dv_x V) = \frac{1}{\nu} \langle \langle \nabla_t^2 V, \dot{x} \rangle + \langle \nabla_t V, \nabla_t \dot{x} \rangle \rangle$ and then

$$\|dv_x V\| \leq \frac{1}{\nu} \|\nabla_t V\| \|\dot{x}\| = \|\nabla_t V\|$$

$$\|\frac{d}{dt}(dv_x V)\| \leq \|\nabla_t^2 V\| + \frac{1}{\nu} \|\nabla_t V\| \|\nabla_t \dot{x}\|$$

Thus $\|dv_x V\|_1 \leq \|\nabla_t^2 V\|_0 + c\|\nabla_t V\|_0(1 + \|\nabla_t \dot{x}\|_0)$ and using (30):

$$\|r_x dv_x V\|_2 \leq c\|dv_x V\|_1 + c\|dv_x V\|_0 \|\nabla_t \dot{x}\|_0$$

$$\leq c\|\nabla_t^2 V\|_0 + c\|\nabla_t V\|_0(1 + 2\|\nabla_t \dot{x}\|_0)$$

(31)

Let $\theta_h, U_h$ be the natural chart for $H^2(I, M)$ centred at $h$. Then for any $\xi \in \theta_h(U_h)$ with $\|\xi\|_2 \leq c$, writing $x = \theta_h^{-1}(\xi)$, we have by Lemma 5.1 that $\|\dot{x}\|_1 \leq c$ and so from (31) $\|r_x dv_x V\|_2 \leq c\|V\|_2$. Thus

$$\|pr_x V\|_2 = \|(1 - r_x dv_x)\|_2 \leq \|V\|_2 + \|r_x dv_x V\|_2 \leq c\|V\|_2$$

(32)

Now we will write $pr(x)\eta = \Theta_h(x, pr_x V)$, where $\Theta_h$ is the local trivialisation corresponding to $\theta_h$, and $\eta = \Theta_h(x, \dot{x})$, then by Lemma 5.2 and (32) we have

$$\|pr(x)\eta\|_2 \leq c\|pr_x V\|_2 \leq c\|V\|_2 \leq c\|\eta\|_2$$

where we have also used the local boundedness of the Finsler structure on $TH^2(I, M)$. Since the above inequality has been shown to hold for any $\xi \in \theta_h(U_h)$ such that $\|\xi\|_2 \leq c$, we have shown that $pr$ is locally bounded. \hfill \Box

**Proposition 2.** $\mathcal{J}|\Sigma_{v,w}^u$ is locally coercive with respect to $(H^2, C^1)$.

Proof. The derivative of $\mathcal{J}$ at $x \in H^2(I, M)$ is

$$dJ_x V = \langle \nabla_t^2 V, \nabla_t \dot{x}\rangle_0 + \langle R(\nabla_t \dot{x}, \dot{x})\dot{x}, V\rangle_0$$

so for any $V \in T_x H^2(I, M)v,w$ and $x \in \Sigma_{v,w}^u$,

$$dJ_x r_x dv_x V = \langle \nabla_t^2 (r_x dv_x V), \nabla_t \dot{x}\rangle_0 + \langle R(\nabla_t \dot{x}, \dot{x})\dot{x}, r_x dv_x V\rangle_0$$

(33)

From (27) and (29)

$$\nabla_t^2 (r_x w) = \frac{\nu}{\nu} \nabla_t \dot{x} + \frac{\nu}{\nu} \dot{x} + \frac{\beta}{\nu} \sum_i \left( \langle \nabla_t E_i, \mathcal{P}_{1-a_i}^{-1} e_i \rangle E_i + \langle E_i, \mathcal{P}_{1-a_i}^{-1} E_i \rangle \nabla_t E_i \right)$$

and therefore, recalling from (21) that $\langle \nabla_t E_i, \nabla_t \dot{x} \rangle = 0$ and $\langle E_i, \nabla_t \dot{x} \rangle = -\langle \nabla_t E_i, \dot{x} \rangle$

$$\langle \nabla_t^2 (r_x dv_x V), \nabla_t \dot{x}\rangle_0 = \frac{1}{\nu} \langle \nabla_t V, \dot{x} \rangle \|\nabla_t \dot{x}\|^2 - \frac{\beta}{\nu} \sum_i \langle \nabla_t E_i, \mathcal{P}_{1-a_i}^{-1} E_i \rangle \langle \nabla_t E_i, \dot{x} \rangle$$

Now using the estimate for $|a|_0$ from (26), $\|\mathcal{P}_{1-a_i}^{-1} e_i \| = \|a_i \| \leq \|a|_0 \leq c\|dv_x V\|_0 \leq c\|\nabla_t V\|_0$. Moreover from (21) we have $\|\nabla_t E_i\| \leq \|\nabla_t \dot{x}\|$, hence

$$\langle \nabla_t^2 (r_x dv_x V), \nabla_t \dot{x}\rangle_0 \leq \int_0^1 \frac{1}{\nu} \langle \nabla_t V, \dot{x} \rangle \|\nabla_t \dot{x}\|^2 + c\|\nabla_t V\| \|\nabla_t \dot{x}\|^2 dt$$

$$\leq c\|\nabla_t V\|_0 \|\nabla_t \dot{x}\|^2_0$$

Using the bounds obtained for $\|a|_0, \|b\|_0$ we have

$$\|r_x dv_x V\|_0 \leq c\|dv_x V\|_0 \leq c\|\nabla_t V\|_0,$$

and then from (33)

$$|dJ_x r_x dv_x V| \leq c\|\nabla_t V\|_0 \|\nabla_t \dot{x}\|^2_0 + \|R(\nabla_t \dot{x}, \dot{x})\dot{x}\|_0 \|\nabla_t V\|_0$$

(34)
Now suppose we work in a natural chart \((\theta_h, U_h)\) centred at \(h\). Then for any \(\xi \in \phi_h U_h\) with \(\|\xi\|_2 \leq c\) we have from Lemma 5.1 that \(x := \phi_h^{-1}\xi\) satisfies \(\|\dot{x}\|_1 \leq c\). Moreover, \(x(I)\) is contained in a compact subset of \(M\) because the length and \(x(0) = p\) are fixed. Thus, from (34) we have \(|dJ_x r_x d\nu_x V| \leq c|V|_1\), and then locally, for any \(\eta \in H^2(h^* TM)\),

\[
|DJ_h(\xi) r_h(\xi) D\nu_h(\xi)\eta| = |dJ_x r_x d\nu_x \Theta^{-1}_h(\xi, \eta)| \leq c|\Theta^{-1}_h(\xi, \eta)|_1 \leq c|\eta|_1
\]

because the Finsler structure \(|.|_1\) is locally bounded by a very similar argument to the proof of Lemma 2 in [27]. Since \(J\) is locally coercive with respect to \(C^1\) by Theorem 3 in [27], the result now follows from Lemma 3.6.

**Lemma 5.3.** \(\Sigma^v\) is a compact subset of \(C^0(I, M)\) for \(* = (p, q), (v, w), c\), provided \(M\) is compact in the case \(* = c\).

**Proof.** \(\Sigma^v\) is equicontinuous by [27] Lemma 5, and since each \(x \in \Sigma^v\) has length \(v\) and a fixed initial point there exists a closed and bounded \(K \subset M\) such that \(x(I) \subset K\) for all \(x \in \Sigma^v\). \(K\) is compact by the Hopf-Rinow theorem and therefore \(\Sigma^v\) is pointwise relatively compact (i.e. given a sequence \((x_i) \subset \Sigma^v\) and fixed \(t_1, (x_i(t_1))\) has a convergent subsequence). Hence by the Arzel`a-Ascoli theorem \(\Sigma^v\) is a compact subset of \(C^0(I, M)\), which contains \(\Sigma_{v, w}\) as a closed subset. For \(\Sigma_c\) the initial point is not fixed and so we assume that \(M\) is compact in this case.

**Lemma 5.4.** The restriction \(J|\Sigma^v\) is weakly proper with respect to \(C^1(I, M)\), provided \(M\) is compact in the case \(* = c\).

**Proof.** If \(U \subset \Sigma^v\) then \(U\) is relatively compact in \(C^0(I, M)\) by Lemma 5.3. Furthermore if \(J(x)\) is bounded for all \(x \in U\) we have \(\|\dot{x}\|_1^2 = \|v\|^2 + 2J(x)\) also bounded and \(U\) is relatively compact in \(C^1(I, M)\) by [27] Corollary 7.

**Theorem 5.5.** \(J|\Sigma^v\) satisfies the Palais-Smale condition.

**Proof.** Recalling (Lemma 4.1) that \(\Sigma_{v, w}\) is the inverse image of a regular value of \(\nu^1: \text{Imm}^2(I, M)_{v, w} \rightarrow H^1(I, \mathbb{R})\) and \(\text{Imm}^2(I, M)_{v, w}\) is open in \(H^2(I, M)_{v, w}\) which is in turn a weak submanifold of \(C^1(I, M)\) ([27] Lemma 4), we check the conditions of Theorem 3.3:

- By Proposition 1, \(pr\) is locally bounded with respect to \(C^1\)
- \(J|\Sigma^v\) is weakly proper with respect to \(C^1\) by Lemma 5.4.
- \(J\) is locally bounding with respect to \(C^1\) by [27] Theorem 3
- \(J|\Sigma^v\) is locally coercive with respect to \((H^2, C^1)\) by Proposition 2.

We now consider the pinned elastica, i.e. \(J|\Sigma^v\). In this case it is not possible to use exactly the same right split \(r\) because (21) required a fixed initial adapted basis for \(T_p M\) but we are now allowing the direction of \(\dot{x}(0)\) to vary. Moreover, by the hairy ball theorem a global smoothly \(\dot{x}(0)\)-dependent choice of adapted basis for \(T_p M\) may be impossible. Fortunately, as we will see below, a global definition will not be needed. We define \(\nu^0\) in a \(C^1\) neighbourhood of \(x_0 \in \Sigma_{p, q}\) as follows. Suppose \(x_0(0) = v_0\) and let \(U\) be a neighbourhood of \(v_0\) in the sphere of radius \(v\) in \(T_p M\) such that the orthonormal frame bundle is trivial over \(U\). Fix a smooth section \(f\) of the orthonormal frame bundle over \(U\). Then for any \(x \in \Sigma^v\) with \(\dot{x}(0) \in U\) solve (cf. (21))

\[
\nabla_t F_i = -\frac{1}{\ell} (F_i, \nabla_t \dot{x}) \dot{x}, \quad F_i(0) = f_i(\dot{x}(0))
\]
to obtain an adapted orthonormal frame \( \{ \frac{\hat{x}}{w}, F_i \} \) along \( x \) adapted to \( \dot{x} \). Then as before we define \( r^0 \) by (23)-(25) (although \( \beta \) is actually no longer needed), but now using \( F_i \) instead of \( E_i \).

**Theorem 5.6.** \( J|_{\Sigma^u_{p,q}} \) satisfies the PS condition.

**Proof.** Let \( (x_i) \subset \Sigma^u_{p,q} \) be a PS sequence for \( J|_{\Sigma^u_{p,q}} \). Then since \( J|_{\Sigma^u_{p,q}} \) is weakly proper with respect to \( C^1(I,M) \) by Lemma 5.4, there is a subsequence, still denoted \( (x_i) \), such that \( (x_i) \) converges in \( C^1 \) to \( x_0 \in C^1(I,M) \). We may therefore choose a natural chart \( \theta, U \) centred at \( h \in C^\infty(I,M) \) and containing \( x_0 \), and a subsequence \( (x_i) \subset U \) with \( \xi_i := \theta(x_i) \). If necessary we may then further restrict attention (and take a further subsequence) to a subet \( U' \subset U \) such that for any \( x \in U' \), \( \dot{x}(0) \) is contained in a neighbourhood of \( v_0 := \dot{x}_0(0) \) in the sphere of radius \( v \) in \( T\dot{x}_0 \) which has trivial orthonormal frame bundle. We then define \( r^0 \) on \( U' \) as described above. The estimates (26) – (30) are also valid with this definition of \( r^0 \) (on \( U' \)). Moreover, the proofs of Propositions 1 and 2 also carry through to prove that on \( U' \) the corresponding projection \( pr_{T\Sigma^u_{p,q}} = 1 - r^0 \) is locally bounded with respect to \( C^1 \), and \( J|_{U'} \) is almost locally coercive. As in the proof of Theorem 3.3 it follows that \( \xi_i \) is Cauchy and converges in \( H^2(h^*TM) \), and therefore \( x_i \) converges in \( \Sigma^u_{p,q} \). \( \square \)

**Theorem 5.7.** \( J|_{\Sigma^c} \) satisfies the PS condition, provided \( M \) is compact.

**Proof.** For the same reasons as those given above for \( \Sigma^u_{p,q} \), we can only define \( r^c \) locally. We mimic the construction of the adapted orthonormal frame \( F_i \) above. Then again we define \( r^c \) by (23)-(25) using \( F_i \), and the periodicity of \( w \) and \( \dot{x} \) ensures that \( r^c w \) is \( C^1 \)-periodic as required. We may then follow the same argument as in the proof of Theorem 5.6, because \( J|_{\Sigma^c} \) is weakly proper when \( M \) is compact (Lemma 5.4). \( \square \)

**Corollary 2.** Provided \( M \) is compact in the case \( * = c \), \( J|_{\Sigma^c} \) obtains its infimum on \( \Sigma^c \) and in any connected component there is a critical point which minimises \( J \) with respect to the component. Furthermore there are at least \( \text{cat}(\Sigma^c) \) critical points altogether, where \( \text{cat} \) denotes the Lusternik-Schnirelman category.

**Proof.** Both statements are standard consequences of the Palais-Smale condition (see [25] pg. 188-190). \( \square \)

6. **Morse theory.** In this section we prove a Morse index theorem for elastica and use the Morse inequalities to give lower bounds for the number of elastica with each index. There is no hope of proving that either of \( J|_{\Sigma^u_{p,q}} \) or \( J|_{\Sigma^c} \) are Morse functions; in the first case consider \( M = \mathbb{R}^3 \) where any pinned elastica can be varied through critical curves by rotating about the line through the endpoints. As for \( J|_{\Sigma^c} \) there is a degeneracy in the parametrization because the curve will be critical regardless of which point corresponds to \( t = 0 \). We therefore focus on \( J|_{\Sigma^c,v,w} \) which we will denote by \( \tilde{J} \) to correspond with the notation in Theorem 3.7.

We will now use Theorem 3.7 to derive the Jacobi equation for elastica. We proceed by calculating each term in equation (14) separately. For the derivatives of \( \tilde{J} \) we calculate

\[
VJ = \langle \nabla_i^2 V + R(V, \dot{x}) \dot{x}, \nabla_i \dot{x} \rangle_0
\]

\[
WVJ = \langle \nabla_i^2 \nabla_w V + \nabla_i (R(W, \dot{x}) V) + R(W, \dot{x}) \nabla_i V + \nabla_w (R(W, \dot{x}) V), \nabla_i \dot{x} \rangle_0
\]

\[
+ \langle \nabla_i^2 V + R(V, \dot{x}) \dot{x}, \nabla_i^2 W + R(W, \dot{x}) \dot{x} \rangle_0
\]

(35)
For \( \nu \) we have \( V\nu = \frac{1}{\|x\|} (\nabla_1 V, \dot{x}) \), and

\[
WV\nu = \frac{1}{\|x\|} \left( \frac{1}{\|x\|} (\nabla_1 W, \dot{x}) (\nabla_1 V, \dot{x}) + \langle \nabla_1 \nabla W V + R(W, \dot{x}) V, \dot{x} \rangle + \langle \nabla_1 V, \nabla_1 W \rangle \right)
\]

where the first term vanishes if \( V \in T_x \Sigma^v \). Now supposing \( x \) is a critical point of \( J|_{\Sigma^v} \), notice that the only terms in (35) and (36) which depend on the values of \( V \) away from \( x \) are those involving \( \nabla W V \). When we calculate \( WVJ - \lambda WV\nu \) these terms group together to form \( dJ_{\dot{x}} \nabla W V \), which is zero since \( x \) is also a critical point of \( J \). It will be convenient to represent the Lagrange multipliers \( \lambda, \gamma \) as elements of \( H^1(I, \mathbb{R}) \) and write \( \Lambda := \lambda - \dot{\gamma}, \Gamma := \gamma - \dot{\gamma} \) (weakly). After repeated integration by parts and several applications of Bianchi identities we find that

\[
WVJ - (\lambda, WV\nu) - (\gamma, WV) = \langle \nabla_1^2 V + F(V, \dot{x}) + \frac{\Delta}{\rho} R(V, \dot{x}) \dot{x} + \nabla_t (\frac{\Delta}{\rho} \nabla_1 V) + \nabla_t (\frac{\rho}{\Delta} \dot{x}) - W, 0 \rangle
\]

(37)

where \( F(V, \dot{x}) \) is the same large collection of curvature terms that appears in [5] eq. (9). Thus by Lemma 3.7 and the fundamental lemma of calculus of variations, we have that \( V \in T_x \Sigma^v \) is in the nullspace of \( \text{Hess}_x J \) if

\[
\nabla_1^2 V + F(V, \dot{x}) + \frac{\Delta}{\rho} R(V, \dot{x}) \dot{x} + \nabla_t (\frac{\Delta}{\rho} \nabla_1 V) + \nabla_t (\frac{\rho}{\Delta} \dot{x}) = 0
\]

(38)

which we call the Jacobi equation for elastica. From Lemma 2.2 the value of \( \Lambda \) is known. Similarly, if we take the inner product of equation (38) with \( \dot{x} \) and use the constraints \( \|x\| = 1, \langle \nabla_1 V, \dot{x} \rangle = 0 \), derivatives thereof, and the Euler-Lagrange equation (5) to simplify we find (after several manipulations)

\[
\dot{\Gamma} = \frac{3}{\rho} \frac{d}{dt} \langle \nabla_1^2 V + R(V, \dot{x}) \dot{x}, \nabla_1 \dot{x} \rangle
\]

(39)

It then follows that the nullspace of \( \text{Hess}_x J \) at a critical point \( x \) of \( J \), being the intersection of \( T_x \Sigma^v \) with the space of solutions of the system (38),(39), is finite dimensional.

**Lemma 6.1.** If \( x \) is a critical point of \( J \) then \( \text{Hess}_x J \) is strongly nondegenerate iff the associated self-adjoint operator \( \text{hess}_x J : T_x \Sigma^v \rightarrow T_x \Sigma^v \) has trivial kernel.

**Proof.** We have seen above that ker \( \text{hess}_x J \) is finite dimensional. Since it is self-adjoint we have ker \( \text{hess}_x J = \text{coker} \text{hess}_x J \), and therefore if ker \( \text{hess}_x J \) is trivial then \( \text{hess}_x J \) is an isomorphism.

**Corollary 3.** \( \text{hess}_x J \) is Fredholm with Fredholm index zero.

**Theorem 6.2.** (Uhlenbeck [29]) Let \( B \) be a bilinear form on a Hilbert space \( H_t \), and \( H_0 \subset H_1 \subset H_1 = H, 0 \leq t \leq 1 \) an increasing family of closed subspaces. Denote \( B|_{H_t \times H_t} \) by \( B_t \) and let \( N_t \) be the nullspace of \( B_t \). If

(i) the dimension of the maximal subspace on which \( B \) is non-positive is finite
(ii) \( N_t \cap N_k = \{0\} \) for \( t \neq k \)
(iii) \( B \) is Fredholm of finite index
(iv) \( \bigcup_{t \leq k} H_t = H_k = \cap_{t \geq k} H_t \)

then there are only finitely many conjugate points, i.e. \( t \in [0, 1] \) such that \( n(t) := \dim N_t \) is non zero. Furthermore index \( B - \text{index} B_0 = \sum_{0 \leq t \leq 1} n(t) \) where index \( B_t \) is the dimension of the maximal subspace on which \( B_t \) is negative definite.
Note that (i) does not appear in the statement of this theorem in [29] but it is assumed earlier in the paper.

We let $H_1 = T_x \Sigma_{v,w}^e$, $H_t = \{V \in T_x \Sigma_{v,w}^e : \text{supp} \, V \subset (0,t)\}$. Then (iv) is satisfied and (iii) has just been proved. For (ii), suppose there exists $V \in N_t \cap N_k, k > t$, then $V(\tau) = 0$ for all $\tau \in (t,k)$ and $V$ satisfies the Jacobi equation for elastica. But then by local uniqueness of solutions of the Jacobi equation and the compactness of $I, V = 0$ on the entire unit interval. As for (i) we proceed as follows. At a critical point, $\text{Hess}_x \bar{J}$ is equal to the restriction of $\text{Hess}_x J$ to $T_x \Sigma_{v,w}^e \times T_x \Sigma_{v,w}^e$.

**Concluding remarks.** In section 5 we have cat($\Sigma_v^e$) as a lower bound for the total number of critical points. Typically we would compare the homotopy type (and therefore category) of this path space with that of the based loop space. However in the case of elastica it is not clear that any such general statements can be made, since the based loop space may contain homotopy classes of curves which all have length greater than $e$. It might be interesting to study the topology of $\Sigma_v^e$. At the beginning of section 6 it was explained that $J|\Sigma_{p,q}^e$ and $J|\Sigma_v^e$ are not Morse functions. However, we have not excluded the possibility that they are Morse-Bott functions; it may be that the critical sets are nondegenerate critical manifolds. Finally, it is possible that Theorem 3.3, or some variant thereof, will be useful for other constrained variational problems.
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