Uncertainty-based weighted least squares density integration for background-oriented schlieren

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Abstract
We propose an improved density integration methodology for Background-Oriented Schlieren (BOS) measurements that overcomes the noise sensitivity of the commonly used Poisson solver. The method employs a weighted least-squares (WLS) optimization of the 2D integration of the density gradient field by solving an over-determined system of equations. Weights are assigned to the grid points based on density gradient uncertainties to ensure that a less reliable measurement point has less effect on the integration procedure. Synthetic image analysis with a Gaussian density field shows that WLS constrains the propagation of random error and reduces it by 80% in comparison to Poisson for the highest noise level. Using WLS with experimental BOS measurements of flow induced by a spark plasma discharge shows a 30% reduction in density uncertainty in comparison to Poisson, thereby increasing the overall precision of the BOS density measurements.

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1 Introduction and methodology

Background-Oriented Schlieren (BOS) is an optical technique used to measure density gradients by tracking the apparent distortion of a target dot pattern (Raffel 2015). The apparent displacement is obtained by comparing the distorted image and a reference image without the density gradients, and the estimation can be performed by cross-correlation, tracking, or optical flow algorithms (Raffel 2015; Atcheson et al. 2009; Rajendran et al. 2019). This displacement is related to the density gradient field and the optical layout parameters as given by

\[
\Delta \vec{X} = M Z_D K \frac{n_0}{\omega_0} \int \nabla \rho dz,
\]

where \(\Delta \vec{X}\) is the apparent displacement, \(M\) is the magnification of the dot pattern, \(Z_D\) is the distance between the dot pattern and the mid-plane of the density gradient field, \(K\) is the Gladstone–Dale constant (= \(0.225 \times 10^{-3}\) kg/m\(^3\) for air), \(n_0\) is the ambient refractive index, \(\nabla \rho\) is the density gradient field and \(z\) is the co-ordinate along the viewing axis. The integral is over the depth/thickness of the density gradient field.

Given the apparent displacement from the image processing algorithms, Eq. (1) can be used to calculate the projected density gradient field, and then the density field can be obtained by spatial integration (Venkatakrishnan and Meier 2004). The BOS method provides a robust and simple experimental setup, yields quantitative density information, and can be extended to large scale flows.

The density gradient integration is traditionally performed by solving Eq. (2) using a Poisson solver,

\[
\rho = (\nabla^2)^{-1} (\nabla \cdot \nabla \rho) = (G^T G)^{-1} (G^T \nabla \rho),
\]

where \(\rho\) is the density field, \(\nabla \rho\) is the density gradient field, and \(G\) is the gradient operator used for discretizing the derivative (Venkatakrishnan and Meier 2004). However, this procedure is sensitive to measurement noise, and the noise can spread from one part of the measurement domain to contaminate other regions (Charonko et al. 2010). There can

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*Graphic abstract*

Weighted Least Squares (WLS) Density Integration for Background Oriented Schlieren (BOS)

Formulate Integration as Least Squares Optimization

\[ \nabla \rho \rightarrow \| W (G \rho - \nabla \rho) \| \rightarrow \rho \]

Weights based on Displacement/Density Gradient Uncertainty

\[ \rho = (G^T W G)^{-1} (G^T W \nabla \rho) \]

80% reduction in density random error

Poisson WLS

30% reduction in density uncertainty

Poisson WLS
be several sources of noise in BOS measurements. For example, non-uniform illumination in the field of view that can increase the effect of image noise, unreliable measurements due to a failure in the displacement estimation algorithm, as well as noise in the boundary condition such as to a pressure/temperature measurement from a probe. Therefore, a robust integration method is required that can withstand and constrain the effect of propagation of measurement noise.

Least Squares (LS) Optimization is an alternate approach that is robust and customizable. It involves formulating the integration as an optimization problem, with the aim of minimizing a pre-defined cost function. For example, the cost function can be defined as the difference between the measured density gradient field and a finite difference approximation of the unknown density field, and the density field can be calculated by minimizing this cost function subject to constraints imposed by the boundary conditions. While solving the least squares problem with the particular cost function defined above is mathematically equivalent to solving the Poisson equation in Eq. (2) given the same stencil, the LS-optimization based approach allows the introduction of additional information/constraints about the flow field and flow measurement to improve the density integration procedure.

For example, non-uniform weights can be assigned to grid points to form the Weighted Least Squares (WLS) problem, which can be solved by Eq. (3), where $W$ is the “weight matrix”:

$$\rho = (G^TWG)^{-1}(G^TW\nabla \rho).$$

A common approach is to assign weights based on the inverse variance of the measurement error for each point, to ensure that more precise measurements have a greater effect on the result (Fessler 1994; Ghiglia and Pritt 1998). Recently Zhang et al. (Ghiglia and Pritt 1998; Zhang et al. 2019) showed that WLS can significantly improve the performance of velocity-based pressure integration in incompressible flow, when the weights are assigned based on the accuracy of pressure gradient estimated from velocity error or velocity uncertainty. However, this approach is not applicable to planar BOS in general, due to the compressibility of the flow. Instead, the weights can be assigned based on the density uncertainty of the BO measurement which is directly related to the density gradient.

Recently, Rajendran et al. (Rajendran et al. 2020; Rajendran et al. 2020) made advancements in uncertainty quantification methods for BOS measurements by developing a method to report local, instantaneous, a-posteriori density uncertainty across all points in the field of view. To achieve this, PIV-based displacement uncertainty quantification methods are used to estimate displacement uncertainties from cross-correlation BOS and propagated through the density integration procedure. One of the findings is that displacement uncertainty schemes from PIV are also applicable for cross-correlation BOS, and that result will be used here. Further, the methodology to estimate the density uncertainty will also be utilized in this work.

Therefore, we propose a WLS-based density integration methodology for BOS wherein the displacement/density gradient uncertainty will be used to assign weights for the integration procedure. For a grid point $k$, the weight is given by,

$$W_k = \left(\sigma_{\Delta X,k}\right)^{-2} = \left(\frac{1}{MKZ_0 \Delta z} \sigma_{\Delta X,k}\right)^{-2},$$

where $\sigma_{\Delta X,k}$ is the density gradient uncertainty at this point and $\sigma_{\Delta X,k}$ is the displacement uncertainty. The pre-multiplying term involves the optical layout parameters described earlier in Eq. (1), with the additional parameter $\Delta z$ denoting the depth/ thickness of the density gradient field. This weight matrix is used along with Eq. (3) to perform the WLS density integration for BOS. In this manner, the unreliable density gradient data points (with greater uncertainty) are assigned lower weights as defined in Eq. (4), and thus have a lower effect on the density integration procedure. Since the state-of-the-art displacement uncertainty estimation methods are sensitive to a wide variety of error and uncertainty sources (Sciacchitano 2019), it is possible to identify unreliable measurements in a robust manner, if a reliable uncertainty quantification method is utilized. If the errors in the density gradient are unbiased and uncorrelated, the weight matrix is the inverse of the covariance matrix of the density gradient error, and WLS provides the best unbiased linear estimator for the density integration problem (Aitken 1936).

The following sections detail the assessment of this methodology with synthetic and experimental BOS images, and show that WLS can reduce the density random error/uncertainty and improve the overall precision of the measurement.

### 2 Analysis with synthetic BOS images

We performed error analysis using synthetic BOS images with a known density field, and assessed the performance of three density integration algorithms: (1) Poisson solver, (2) WLS with weights based on the random displacement error, and (3) WLS with weights based on the displacement uncertainty. Further, a patch of high noise was added to one part of the BOS image to assess how the error due this patch propagates to the surrounding field during the density integration procedure.

The synthetic BOS images were generated using a ray tracing-based image generation methodology (Rajendran et al. 2020).
et al. 2019). In this method, light rays are launched from source points on the BOS target, propagated through density gradients by numerically solving Fermat’s equation with a 4th order Runge–Kutta method (Sharma et al. 1982), and then through the camera lens until the final intersection with the camera sensor, to generate the dot pattern images.

The density field chosen for the error analysis is a Gaussian density field, described by Eq. (5),

$$\rho(X, Y) = \rho_0 + \Delta \rho_0 \exp \left\{ -\frac{(X - X_0)^2 + (Y - Y_0)^2}{2\sigma_0^2} \right\}, \quad (5)$$

where $\rho_0$ is the ambient density, $\Delta \rho_0$ is the peak density difference and $\sigma_0$ is the standard deviation of the Gaussian field. For the simulations reported in this paper, $\rho_0$ was set to be 1.225 kg/m$^3$, $\Delta \rho_0$ was set to be 0.3 kg/m$^3$, and $\sigma_0$ was set to be 0.03/4 of the field of view (= 2.41 mm). The dimensions of the density gradient field were 10 × 10 × 10 mm, and it was located at a distance of 0.25 m from the dot pattern. The optical layout used to image the dot pattern and the density field consisted of a 105 mm lens at a distance of 0.5 m from the dot pattern to provide a magnification of about 40 mm field.

The displacement error (defined as the standard deviation of the error distribution at each grid point) is shown in Fig. 1e, showing, as expected, higher values in the noisy patch.

In addition, the displacement uncertainty is also estimated during the correlation processing using the Moment of Correlation (MC) algorithm developed by Bhattacharya et al. (Bhattacharya et al. 2018). In this algorithm, the uncertainty is estimated directly from the PDF of displacements that contribute to the cross-correlation plane. The PDF of displacements is obtained by an inverse fast Fourier transform of the phase plane (Eckstein et al. 2008; Knapp and Carter 1976) [also referred to as the Generalized Cross Correlation (GCC)], filtered by a Gaussian to improve subsequent calculations of the peak diameter (Eckstein and Vlachos 2009), scaled by the number of correlating particles using the Mutual Information (MI) (Xue et al. 2015), and corrected for displacement gradients in the interrogation window (Westerveel 2008; Scharnowski et al. 2012). The instantaneous displacement uncertainty field is shown in Fig. 1d, and also shows a large increase in the noisy patch, which is consistent with the increase in the random error. Thus, the weights chosen based on the random error/uncertainties are lower in the noisy patch, and thus, measurements in the patch would have less effect on the density integration.

However, it is also seen that MC uncertainty is lower than the random error in the noisy patch, and higher in the rest of the field. This is because the displacement uncertainty predicted by MC is an instantaneous estimate of the random error from just a single snapshot, while the true random error is a statistically averaged estimate from several (here 1000) error fields. Despite many recent advances in the development of displacement uncertainty quantification methods, there is no universally accepted method with superior performance across all flow fields (Sciacchitano et al. 2015; Boomsma et al. 2016), and the development of PIV/BOS uncertainty quantification is a young and active research area. While the results reported in this paper utilize MC to estimate the uncertainty, the WLS method is general and can be integrated with any uncertainty quantification method.

Next, the displacements were used to calculate the density gradients, which were then spatially integrated to calculate the density field using the three integration methods: (a) Poisson, (b) WLS with weights based on the random error, and (c) WLS with weights based on MC uncertainty. A 2nd-order central difference scheme is used for spatial discretization, with Neumann boundary conditions on all four boundaries and the Dirichlet boundary condition at the midpoint of the top boundary. The density error was then calculated by comparing the integrated density field with the reference field used to render the images, and a snapshot of the instantaneous density field as well as a profile along a vertical line passing through X = 2.5 mm is shown in Fig. 2. All three methods show an under-prediction in the density.
field, potentially due to spatial discretization and truncation errors in the numerical integration. This deviation appears to increase with the noise amplification ratio.

The error statistics from 1000 such fields are calculated, and the resulting random error in the density field obtained from the three integration methods are shown in Fig. 3. The WLS method (middle and right columns) is able to constrain the spread of the random error from the patch, while the error has a wider spread with Poisson integration (left column), and this difference increases with the patch amplification ratio. Finally, the results show that WLS with weights based on the displacement uncertainty (middle column) performs just as well as the case with weights based on the random error (right column), thereby justifying the use of the displacement/density gradient uncertainty to estimate the weights. This demonstrates the practical value of WLS
since the error is not known and only the uncertainty can be estimated in the vast majority of experiments. The variation of the random error along a vertical line located at X = 2.5 mm is shown in the right most column. With increasing amplification ratio, the WLS method is able to constrain the spread of error and limit the peak error in the patch. The profile of the density random error is seen to be asymmetric with respect to Y = 0 mm because the Dirichlet boundary condition is imposed at the top mid-point of the field of view, with the rest of the boundaries featuring Neumann boundary conditions.

In addition, the probability density function (PDF) and cumulative density function (CDF) of the density random error distribution were calculated from 250,000 grid points and are shown in Fig. 4, along with the RMS error for the PDF plot and the 90th percentile error for the CDF plot. As suggested by the PDFs of the random errors in the left column, the modes of the error distributions by WLS methods are less than one-third of the error mode by the Poisson (blue) for an amplification ratio of 20. The RMS of the error distributions are shown by dashed lines as a metric to compare the distributions. The RMS values of the WLS methods are about 50% lower than the corresponding RMS for the Poisson.

The right column shows the CDF of the random error distributions for the three integration methods. The dashed lines represent the 90th percentile of the error distribution which is an indication of the noise spreading characteristics.
For WLS methods, 90% of the points have errors are less than 3e-4 kg/m³, whereas the errors by Poisson have a much wider spread and yield a 500% increase in terms of the 90th percentile compared to WLS methods.

It is also seen that the 90th percentile errors from WLS methods are less affected by the noise level. Moreover, the WLS based on MC uncertainty results in a similar error distribution with the WLS based on the random error.

In summary, the error analysis shows that the WLS integration can significantly improve the precision of the density integration procedure in comparison to the traditional Poisson solver.

### 3 Experimental demonstration

The methodology was also tested using experimental BOS data of the flow induced by a nanosecond spark plasma discharge. A spark discharge of nanosecond duration leads to the rapid deposition of heat in the electrode gap leading to the development of a complex flow field with large thermal gradients. The experimental details corresponding to the dataset used in this assessment are reported in the work by Singh et al. (Singh et al. 2020). The BOS measurements were performed by imaging a random dot pattern (fabricated from sand-blasted aluminum) in the presence of flow induced by a spark across a 5-mm electrode gap. The dot pattern and flow were imaged at a magnification of 0.8 and a frame rate of 20 kHz with a 3.18-cm separation between the target and the electrodes. More details of the experimental setup can be found in Singh et al. (Singh et al. 2020).

The dot pattern images were processed using the standard cross-correlation (SCC) method with multi-grid window deformation (Scarano 2002). The window sizes were varied from 64 to 48–32 pixels over 4 passes with an overlap of 50% resulting in a final grid resolution of 16 pixels. The displacement uncertainties were calculated using the MC method described earlier, and the vector field is validated using Universal Outlier Detection (UOD) (Westerweel and Scarano 2005). A median-based UOD filter was used with a 3 × 3 grid point neighborhood, and performed in two passes with a normalized residual threshold set to be 3 and 2 respectively. The minimum normalization level was of 0.1 pixels. The detected outliers were replaced by a weighted average of the inverse distance between the invalid vector and the neighboring grid points.

In the case of the experimental data, the extent of the path integration in Eq. (1) is not known, as the flow is three-dimensional and only one view is presently...
available. Therefore, the displacements were used to calculate the projected density gradients, \( \nabla \rho_p = \int \nabla \rho \, dz \), thereby constituting a 2D simplification of the 3D field. The gradients were then spatially integrated using the Poisson and WLS methods, with the weights assigned based on the projected density gradient uncertainty for the latter, to yield the projected density field \( \rho_p = \int \rho \, dz \). Dirichlet boundary conditions were imposed at the mid-points of the left and right boundaries and Neumann conditions were imposed elsewhere. Further, the Dirichlet density values were set to zero to calculate the ‘relative’ projected density field with respect to the ambient. During the experiment, the field of view was large enough (= 1 electrode gap on either side of the spark) to ensure that the left and right boundaries were far from the induced flow and truly in the ambient. While the analysis of Singh et al. employed an Abel inversion procedure to further extract the radial density field, that was not performed here, and instead a direct comparison was performed on the projected density field to avoid introducing additional downstream steps/variables in the comparison.

The dot pattern displacements are shown in Fig. 5a, b for two time instants, and the largest displacements occur at the boundary of the hot gas kernel as this corresponds to the largest temperature/density gradients. The kernel is initially cylindrical and deforms into a more complex shape at later times. The corresponding density field calculated using Poisson is shown in (c) and (d), and the density from WLS is shown in (e) and (f) for the same time instants. It is seen that the density is lower inside the gas kernel, corresponding

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**Fig. 4** PDF (left) and CDF (right) of the density random error associated with the three integration schemes. The dashed lines indicate the RMS error for the left column, and the 90th error percentile for the right column. Each row corresponds to the same patch noise amplification ratio, denoted on the left.
to a higher gas temperature, as expected, and further that the two schemes result in similar density fields. However, it will be seen that there is a significant difference in the density uncertainty field.

The corresponding uncertainties in the displacement fields are shown in Fig. 6a, b, and the displacement uncertainty is highest within the region occupied by the hot gas kernel, with the noise amplification ratio varying from 5 to 10 in this region. This is expected because in addition to higher displacements, the displacement gradients are also expected to be higher in this region, leading to a rise in the uncertainty. This uncertainty in the displacement is then propagated through the density integration process using the methodology described in (Rajendran et al. 2020), and the density uncertainty fields shown in Fig. 6c–f. The uncertainty propagation method is as follows: the displacement uncertainties are used along with the parameters of the optical layout to estimate the density gradient uncertainties. Then the density gradient uncertainties are propagated to the density field through a matrix representation of the density integration process. The computations are performed using sparse linear operators for speed and efficiency. For more details on the method along with the assessment, please refer (Rajendran et al. 2020). Since the true density is not known for this experiment, the density uncertainty will be used in place of the density random error as the metric to compare the integration methods. The uncertainty provides the range for the error at a specified confidence level, and is therefore a measure of the ‘sensitivity’ of the density integration procedure to upstream noise. While estimating the error requires an independent measurement (a ground truth), it is not required for uncertainty quantification. Similar to the analysis with synthetic images, it is again seen that the density uncertainty from WLS is again lower and confined to the region within the hot gas kernel as opposed to that from Poisson where the uncertainty spreads to all regions in the domain. Therefore, WLS integration yields a more robust estimate of the density which is less sensitive noise in the density gradient measurement.

Fig. 5 Flow induced by a spark discharge at two time instants. a, b Instantaneous displacement fields and c, d Density fields obtained using Poisson, and e, f Density fields obtained using WLS. Plots (a), (c) and (e) correspond to the same time instant, as do (b), (d) and (f).
In addition, the density uncertainty fields from 20 such snapshots of the flow were used to calculate the PDF and CDF distributions for the Poisson and WLS methods. The PDF with RMS of the uncertainty is shown in Fig. 7a and the CDF with the 90th percentile is shown in Fig. 7b. It is seen that WLS reduces the RMS of the density uncertainty by 30% and the 90th percentile of the density by

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**Fig. 6** Instantaneous spatial distribution of the displacement uncertainty (a), (b) and the density uncertainty fields obtained from Poisson (c), (d) and WLS (e), (f) methods. Plots (a), (c) and (e) correspond to the same time instant, as do (b), (d), and (f).

**Fig. 7** a PDF and b CDF of the density uncertainty associated with the two integration schemes for flow induced by the spark discharge.
about 25%, thereby improving the overall precision in the density estimation.

### 4 Conclusion

In conclusion, we presented a weighted least squares (WLS)-based density integration procedure in which weights are assigned to density gradient measurements based on the corresponding measurement uncertainty to improve the overall precision of the density integration procedure. Results from the synthetic image analysis showed that WLS was able to constrain the spread of the density random error compared to the Poisson solver and reduced the RMS error by 80% for the highest noise level. It was also seen that weights based on the Moment of Correlation uncertainty quantification scheme performed just as well as when weights were based on the displacement random error, thereby demonstrating that the displacement/density gradient uncertainty is a valid reference for assigning the weights. From the experimental images of flow induced by a spark plasma discharge, it was seen that WLS reduced the RMS uncertainty by 30% in comparison to Poisson, thus producing more precise density estimation. However, a limitation of this work is that it does not show that the WLS reduced the actual density measurement error in the experiment. An independent measure of density with a benchmark experiment is required in order to perform this assessment. Such an experiment is beyond the scope of this work and will be considered in a future publication.

Further improvement of the method can be achieved by accounting for the covariance in the displacement estimation procedure, to be used for assigning weights in a Generalized Least Squares (GLS) integration framework. Recent work in pressure integration has shown that GLS can further reduce the errors and uncertainties when the covariance information is available (Zhang et al. 1912). For BOS, the covariance in the density gradient between neighboring grid points arises from the window overlap used in the cross-correlation based displacement estimation. There is currently no method to estimate this covariance in an automated and reliable manner, and this is an avenue for future work. Finally, the WLS method can also be combined with tracking based displacement estimation methods for BOS using recent developments on estimating the displacement uncertainty using the ratio of dot diameters in the reference and gradient images (Rajendran et al. 2020). The codes implementing this WLS methodology can be downloaded from https://github.rcac.purdue.edu/lrajendr/wls-bos.git.

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