From Hurwitz to Kontsevich-Witten via Virasoro

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Abstract. In this letter we present our conjecture on the connection between the Kontsevich-Witten and Hurwitz tau-functions. The conjectural formula connects these two tau-functions by means of the $GL(\infty)$ group element, and the important feature is that the corresponding operator is quite simple: it is built of only generators of the Virasoro algebra. If proved, this conjecture would allow to derive the Virasoro constraints for the Hurwitz tau-function, which remain unknown in spite of existence of several matrix model representations, as well as to give an integrable operator description of the Kontsevich-Witten tau-function.

Introduction

While it is commonly assumed that the generating functions in enumerative geometry, a particular subclass of the string theory partition functions, are the tau-functions of integrable hierarchies, possessing nice matrix model representations, the number of cases for which this statement is established is relatively small (see e.g. [1, 2] and references therein). For example, integrable hierarchies behind the full generating functions of Gromov-Witten invariants are known only for the simplest compact manifolds (a point and a two dimensional sphere) and for the certain classes of their deformations/modifications. This is why it is important to establish the common properties of the corresponding tau-functions and to investigate relations between them. In this letter we conjecture a new, integrable, reformulation of the relation between two important tau-functions, which belong to the domain of enumerative geometry: Kontsevich-Witten tau-function and a generating function of single Hurwitz numbers, which we will call the Hurwitz tau-function for the simplicity.

Since the early nineties, when the spectacular Witten’s conjecture [3] was proved by Kontsevich [4], Kontsevich-Witten tau-function, that is the partition function of two-dimensional topological gravity, became an inevitable part of mathematical physics. It can arguably be considered as an elementary building block for more complicated partition functions of the “Generalized Topological String Theory,” which unifies conventional topological strings with other theories of the topological/combinatorial invariants possessing universal integrable properties. As a consequence of its special role the Kontsevich-Witten partition function is very well studied and a lot of elements of universal description, which are still lacking for

1 However see [5], where it is claimed that the partition function of $CP^1$ model should be considered as the most elementary one and Kontsevich-Witten tau-function is just its particular limit.
more complicated models, are known in this case. Among such interrelated elements are Virasoro constraints, integrable properties, moment variables description, random partitions representation, spectral-curve-based description, free fermion representation, a vast net of connections with other models and, of course, Kontsevich matrix integral representation (see, e.g., [1,6,7] and references therein).

Another classical subject of enumerative geometry is the theory of the Hurwitz numbers. The generating function of Hurwitz numbers can be represented in terms of the cut-and-join operator [8,9]. This operator belongs to the (central extension of the) $gl(\infty)$ algebra, acting on the space of KP solutions transitively, that guarantees the KP integrability of the generating function. This generating function can be restored by topological recursion [10,11], and several matrix models representations of this generating function are known [12–14].

In this paper we conjecture a relation between the Kontsevich-Witten tau-function and the Hurwitz tau-function. These two functions are very well known to be deeply related [15–18] and our conjecture is based on these known connections. In particular, the Hurwitz numbers are related to the Hodge integrals via profound Ekedahl-Lando-Shapiro-Vainshtein (ELSV) formula [19,20]. On the other side, a generating function of the Hodge integrals is a deformation of the correspondent Gromov-Witten generating function by a Givental-type operator [21,23]. The first part of this connection is known to be given up to the linear change of variables, by a $GL(\infty)$ operator, which respects integrability. Integrable properties of the second part of this relation are more subtle. Namely, the generating function of the Hodge integrals and Kontsevich-Witten tau-function are connected with each other by a Givental-type operator, which does not belong to the $GL(\infty)$ group, supplemented by a linear change of variables, which is not equivalent to the local change of the spectral parameter. Our conjecture is that while neither of two elements (Givental-type operator and change of variables) respects KP integrability, their combination can be represented as a simple $GL(\infty)$ operator, built of only negative components of Virasoro algebra, which does respect an integrability. Thus a $GL(\infty)$ operator, connecting two tau-functions is particularly simple: it consists of all components of the Virasoro subalgebra of $gl(\infty)$.

$$Z_K(\tau_k = 2^k t_{2k+1}) = e^{\hat{Q}_0} \cdot e^{\hat{Q}_+} \cdot e^{\hat{Q}_-} \cdot e^{-\sum_{k>0} \frac{k^k-1}{k^2} \beta_k} Z_H(t_k; \beta)$$

where

$$\hat{Q}_0 = a_0 \log \beta \ \hat{L}_0$$

$$\hat{Q}_\pm = \sum_{k>0} a_{\pm k} \beta^{\mp k} \hat{L}_{\pm k}$$

are linear combinations of the Virasoro operators [14] and $a_k$ are rational numbers. Let us stress that l.h.s. does not depend on $\beta$ and even times $t_{2k}$.

This type of connection between two KP tau-functions is one of the simplest possible, and it naturally generalizes a notion of equivalent hierarchies [24]. Namely, instead of an operator with positive components of Virasoro algebra, which describes equivalent hierarchies, here there appears an operator with all components.

The organization of this paper is as follows. In the Section 1 we remind the reader some details about Kontsevich-Witten tau-function and a construction of the cut-and-join representation for it. This representation is not quite satisfactory because the operator does not belong to $GL(\infty)$ group. In Section 2 we remind the reader some basic facts about $gl(\infty)$ algebra of the KP solutions symmetry. In Section 3 we remind the definition of the Hurwitz tau-function in terms of the cut-and-join operator as well as some related matrix models. Section 4 is devoted to the formulation of the details of our conjectural formula.
1. Kontsevich-Witten tau-function

In this section we basically follow a matrix model theory point of view and do not give an explanation of the enumerative geometry origin of the Kontsevich-Witten tau-function, which can be found elsewhere. Kontsevich matrix integral over Hermitian matrix $\Phi$

$$Z_K(\tau_k) = \frac{\int [d\Phi] \exp \left(-\frac{1}{g} \text{Tr} \left( \frac{\Phi^3}{3!} + \frac{\Delta \Phi^2}{2} \right) \right)}{\int [d\Phi] \exp \left(-\frac{1}{g} \text{Tr} \frac{\Delta \Phi^2}{2} \right)}$$  (3)

gives the Kontsevich-Witten tau-function with times given by the Miwa variables

$$\tau_k = \frac{g}{(2k + 1)} \text{Tr} \frac{1}{\Lambda^{2k+1}}$$  (4)

The tau-function depends on the infinite set of the independent time variables $\tau_k$, that means that the size of the matrix tends to infinity. This tau-function has a natural genus expansion:

$$Z_K(\tau_k) = \exp \left( \sum_{h=0}^{\infty} g^{2h-2} F_K^{(h)}(\tau_k) \right)$$  (5)

It satisfies to an infinite set of the differential equations known as the Virasoro constraints

$$\hat{\mathcal{L}}_n Z_K = \frac{\partial}{\partial \tau_{n+1}} Z_K, \quad n \geq -1$$  (6)

where the operators

$$\hat{\mathcal{L}}_m = \sum_{k=1}^{\infty} \left( k + \frac{1}{2} \right) \tau_k \frac{\partial}{\partial \tau_{k+m}} + \frac{g^2}{8} \sum_{k=0}^{m-1} \frac{\partial^2}{\partial \tau_k \partial \tau_{m-k-1}} + \frac{\tau_0^2}{2g^2} \delta_{m,-1} + \frac{1}{16} \delta_{m,0}, \quad m \geq -1$$  (7)

constitute a subalgebra of the Virasoro algebra:

$$[\hat{\mathcal{L}}_n, \hat{\mathcal{L}}_m] = (n - m) \hat{\mathcal{L}}_{n+m}$$  (8)

These Virasoro constraints allow to derive a cut-and-join-type representation for the Kontsevich-Witten tau-function [25].

$$Z_K = e^{\hat{W}_K} \cdot 1$$  (9)

where

$$\hat{W}_K = \frac{2}{3} \sum_{k=1}^{\infty} \left( k + \frac{1}{2} \right) \tau_k \hat{\mathcal{L}}_{k-1}$$

$$= \frac{2}{3} \sum_{k,m \geq 0} \left( k + \frac{1}{2} \right) \left( m + \frac{1}{2} \right) \tau_k \tau_m \frac{\partial}{\partial \tau_{k+m-1}}$$

$$+ \frac{g^2}{12} \sum_{k,m \geq 0} \left( k + m + \frac{5}{2} \right) \tau_{k+m+2} \frac{\partial^2}{\partial \tau_k \partial \tau_m} + \frac{1}{g^2} \frac{\tau_0^3}{3!} + \frac{1}{16} \tau_1$$  (10)

The most important disadvantage of this representation is that the operator $\hat{W}_K$ does not belong to the $gl(\infty)$ algebra so that the KdV integrability is hidden. The main result of this note is a conjecture, which represents the Kontsevich-Witten tau-function in terms of the finite number of simple $GL(\infty)$ operators acting on the trivial tau-function.
2. $gl(\infty)$ algebra

$GL(\infty)$ group is generated by normal ordered powers $: \hat{J}(z)^k :$ of the bosonic current

$$ \hat{J}(z) = \sum_{k=1}^{\infty} \left( k t_k z^{k-1} + \frac{g^2}{z^{k+1}} \frac{\partial}{\partial t_k} \right) $$

(11)

For instance for spin $k = 1$

$$ : \hat{J}(z) := \hat{J}(z) $$

(12)

so that the corresponding group elements are given by constant time shifts and by multiplication by $\exp(\alpha_k t_k)$, which obviously preserve the KP equations. The key role in our construction is played by the Virasoro algebra appearing for $k = 2$:

$$ : \hat{J}(z)^2 := 2g^2 \sum_{k=-\infty}^{\infty} \frac{\hat{L}_k}{z^{k+2}} $$

(13)

where

$$ \hat{L}_m = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{g^2}{2} \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b} + \frac{1}{2} \frac{g^2}{2} \sum_{a+b=-m} ab t_a t_b $$

(14)

These operators constitute a central extension of the Virasoro algebra with central charge $c = 1$ [20]:

$$ [\hat{L}_n, \hat{L}_m] = (n-m) \hat{L}_{n+m} + \frac{1}{12} (n^3-n) \delta_{n,-m} $$

(15)

Let us stress that the Virasoro constraints for the Witten-Kontsevich tau function can be given in terms of the current

$$ \hat{\mathcal{J}}(z) = \sum_{k=1}^{\infty} (2k+1) \tau_k z^{2k} + \frac{g^2}{2} z^{2k+2} \frac{\partial}{\partial \tau_k} $$

(16)

which almost coincides with the odd part of (11), namely:

$$ : \hat{\mathcal{J}}(z)^2 := 2g^2 \sum_{k=-\infty}^{\infty} \frac{\hat{W}_k}{z^{2k+2}} - \frac{g^2}{8 z^2} $$

(17)

This is the different pre-factor in front of the second term in this current what is partially responsible for the difference between $\tau_k$ and $t_{2k+1}$ in [25].

Among higher spin operators the most important for us is $k = 3$

$$ : \hat{\mathcal{J}}(z)^3 := 3g^2 \sum_{k=-\infty}^{\infty} \frac{\hat{W}_k}{z^{k+3}} $$

(18)

a component of which

$$ \hat{W}_0 = \sum_{i,j \geq 1} i j t_i t_j \frac{\partial}{\partial t_{i+j}} + g^2 (i + j) t_{i+j} \frac{\partial^2}{\partial t_i \partial t_j} $$

is the cut-and-join operator of the Hurwitz tau-function.
3. Hurwitz tau-function

Here again, as for the Kontsevich-Witten tau-function, we will not give the reader a definition of the Hurwitz tau-function in terms of the enumerative geometry, but we will remind some properties related to the matrix model representations. Namely, it is known \[8, 9\] that the generation function of Hurwitz numbers can be given by the cut-and-join operator \[19\]

\[
Z_H(t_k; \beta) = \exp \left( \frac{\beta}{2} \hat{W}_0 \right) \exp \left( \frac{t_1}{g^2} \right) \tag{20}
\]

For this generating function several matrix model representations are known, among which let us mention the simplest ones \[13, 14\]. Namely, the Hurwitz tau-function can be represented as a Hermitian matrix integral

\[
Z_H(t_k; \beta) = \int_{N \times N} [d\mu(\Phi)] \exp \left( -\frac{1}{2\beta} \text{Tr} \Phi^2 + \text{Tr} \left( e^{\Phi - N\beta/2} \psi \right) \right) \tag{21}
\]

where

\[
[d\mu(\Phi)] = \sqrt{\det \frac{\sinh \left( \frac{Y \otimes 1 - 1 \otimes Y}{2} \right)}{Y \otimes 1 - 1 \otimes Y}} [d\Phi] \tag{22}
\]

and Miwa variables

\[
t_k = \frac{1}{k} \text{Tr} \psi^k \tag{23}
\]

and as a normal matrix integral

\[
Z_H(t_k; \beta) = \int_{N \times N} \frac{[dZ]}{\det (ZZ^\dagger)^{N+\frac{1}{2}}} \exp \left( -\frac{1}{2\beta} \text{Tr} \log^2 ZZ^\dagger + \text{Tr} Z^\dagger + \sum_{k=1}^\infty t_k \text{Tr} Z^k \right) \tag{24}
\]

In both formulas it is assumed that the size of the matrices tends to infinity and unimportant \( t_k \)-independent factors are omitted.

4. Relation between two tau-functions

The main result of this letter is a formula, connecting two tau-functions (Kontsevich-Witten and Hurwitz), defined in the previous sections. Namely, on the basis of the explicit calculations we guess that

\[
Z_K(\tau_k = 2^k t_{2k+1}) = \hat{U} Z_H(t_k; \beta) \tag{25}
\]

with

\[
\hat{U} = e^{\hat{Q}_0} \cdot e^{\hat{Q}_+} \cdot e^{\hat{Q}_-} \cdot e^{-\sum_{k>0} \frac{k^{k-1} \beta^{k-1} t_k}{k! g^2}} \tag{26}
\]

and

\[
\hat{Q}_0 = a_0 \log \beta \hat{L}_0
\]

\[
\hat{Q}_\pm = \sum_{k>0} a_{\pm k} \beta^{\pm k} \hat{L}_{\pm k} \tag{27}
\]
The first part of the relation between two tau-functions is known to be given by a $GL(\infty)$ operator [17], namely

$$Z_{Hodge}(t) = \exp \left( \sum_{k<0} a_k \beta^{-k} L_k \right) \exp \left( -\sum_{k=1}^{\infty} \frac{k^{k-1} \beta^{(k-1)} t_k}{k! g^2} \right) Z_H(t)$$

$$= 1 + \frac{t_2 \beta^3}{6} + \frac{t_1^2 h + 8 t_1^3 + 6 t_3 g^2}{48 g^2} \beta^4 + \frac{1}{120} t_2 \left( 60 t_1 + g^2 \right) \beta^5$$

$$+ \frac{1}{1440} \frac{500 h t_2^2 + 414 t_3 h^2 + 80 t_1^3 h + t_1^2 h^2 + 240 t_1^4 + 2400 t_1 t_3 h}{h}$$

$$+ \frac{1}{10080} \frac{2 t_2 h^3 + 21224 t_4 h^2 + 15155 t_1^2 t_2 h + 7000 t_1^3 t_2 + 5250 t_2 t_3 h + 1400 t_1 t_2 h^2 + 16800 t_1 t_1 h}{h} \beta^7$$

$$+ O(\beta^8)$$

(28)

The coefficients $a_k$ for the negative $k$ can be restored from the equation [17]

$$\exp \left( \sum_{k>0} a_k z^{-k+1} \frac{\partial}{\partial z} \right) \cdot z = \frac{z}{1 + z} e^{-\frac{z}{1+z}}$$

(29)

The few first coefficients $a_k$ are given in the following table:

| $k$ | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | $-6$ | $-7$ | $-8$ | $-9$ |
|-----|------|------|------|------|------|------|------|------|------|
| $a_k$ | $-2$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $3$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

For the positive $k$ the coefficients $a_k$ can be found by a direct calculation, namely

| $k$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|-----|----|----|----|----|----|----|----|----|
| $a_k$ | $-\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $3^2 \cdot 5$ | $-\frac{2}{3}$ | $3^5 \cdot 5$ | $11$ | $3^6 \cdot 5^2$ |

and $a_0 = -\frac{4}{3}$, as it follows from the Theorem 2.3 of [17]. Unfortunately, we do not know any simple recursion relation for the positive $a_k$. The only observation we made is that

$$f(z) = \exp \left( \sum_{k=1}^{\infty} a_k z^{-k} \frac{\partial}{\partial z} \right) \frac{z}{1 + z} e^{-\frac{z}{1+z}}$$

(30)

is symmetric

$$f(z) = f(-z)$$

(31)

which could be related to the fact that the Kontsevich-Witten tau-function satisfies to the KdV equations, that is 2-reduction of the KP hierarchy. Let us stress that the operator, connecting $Z_K$ and $Z_H$ can be represented in various ways, so most probably it can be simplified.

The formula relating two tau-function can be of use for both of them. For the Kontsevich-Witten tau-function it gives an integrable representation in terms of $GL(\infty)$ operators. This
relations also could be used to construct a matrix model representation in terms of times for the Kontsevich-Witten tau-function, contrary to the Kontsevich matrix integral, which depends on the Miwa variables.

For the Hurwitz tau-function this relation allows to derive a set of Virasoro constraints by a simple conjugation of the Virasoro constraints for the Kontsevich-Witten tau-function. Namely,

\[ \hat{L}_n Z_H(t_k; \beta) = 0, \quad n \geq -1 \] (32)

where

\[ \hat{L}_n = \hat{U}^{-1} \hat{L}_n \hat{U} \] (33)

Probably this transformation can be given in terms of the global current on the Lambert spectral curve. Let us stress that in this formulation the change of the variables is performed by the Virasoro operators, thus a problem with non-invertible change of variables does not appear at all.

5. Conclusion and open questions

In this letter we present a conjectural relation between two important generating functions of enumerative geometry: Kontsevich-Witten tau-function and generating function of single Hurwitz numbers. This conjecture, if proved, should help in unification of three overlapping, but different types of operators, which show themselves in modern string theory enumerative geometry. These three types of operators are Givental operators, \( GL(\infty) \) operators and cut-and-join operators.

Givental operators constitute a special class of exponential operators given by quantization of quadratic hamiltonians \[22,23\], and they correspond to particular transformations of the bosonic currents on the spectral curve. They allow to express some of non-trivial partition functions of string theory in terms of elementary building blocks, such as Kontsevich-Witten tau-functions, and they are conjectured to be applicable in more general setup. In particular, the Givental operators appear in the matrix model theory \[27,32\]. Their relation with integrability remains poorly investigated, see, however \[33\].

\( GL(\infty) \)-operators, by definition, preserve the KP-type integrability of the partition functions, that’s why relations in terms of these operators look particularly attractive. They are generated by the powers of the bosonic fields currents on the Riemann surfaces. Some matrix models partition functions can be related with elementary functions by means of \( GL(\infty) \) operators \[13,14\].

The cut-and-join operator, appearing in the description of the single and double Hurwitz numbers, also belongs to this \( gl(\infty) \) algebra. But, some other cut-and-join operators, appearing in the description of more general Hurwitz partition functions, are not of this integrable form \[34,36\].

It would be extremely interesting to understand the meaning of our conjectural relation in terms of matrix integrals. Since both partition functions can be represented as matrix integrals dependent on external matrix this could be possible to clarify this relation in terms of operators, acting on the eigenvalues. This type of relations should generalize the relations between different solutions of the Generalized Kontsevich model, given by equivalent hierarchies (for a review see \[37\]).

Unfortunately, at the moment we do not know how to prove this conjectural relation, not even how to find all the coefficients \( a_k \) in a simple form.
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