Szegö kernels and asymptotic expansions for Legendre polynomials

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Abstract

We present a geometric approach to the asymptotics of the Legendre polynomials $P_{k,n+1}$, based on the Szegö kernel of the Fermat quadric hypersurface, and leading to complete asymptotic expansions holding on expanding subintervals of $[-1,1]$.

1 Introduction

The goal of this paper is to develop a geometric approach to the asymptotics of the Legendre polynomial $P_{k,n+1}(t)$ for $k \to +\infty$, with $t = \cos(\vartheta) \in [-1,1]$ and $n \geq 1$ fixed; as is well-known, $P_{k,n+1}(t)$ is the restriction to $S^n$ of the Legendre harmonic, expressed in polar coordinates on the sphere. We follow here the terminology of [M], [M1] and [AH].

There is a tight relation between $P_{k,n+1}(t)$ and the orthogonal projector

$$\mathcal{P}_{k,n} : L^2(S^n) \to V_{k,n},$$

where $V_{k,n}$ is the space of level-$k$ spherical harmonics on $S^n$; equivalently, $V_{k,n}$ is the eigenspace of the (positive) Laplace-Beltrami operator on functions on $S^n$, corresponding to its $k$-th eigenvalue $\lambda_{k,n} = k(k + n - 1)$.

Namely, for any choice of an orthonormal basis $(\varphi_{kj})_{j=1}^{N_{k,n}}$ of $V_{k,n}$ the distributional kernel $\mathcal{P}_{k,n}(\cdot, \cdot) \in C^\infty(S^n \times S^n)$ satisfies

$$\mathcal{P}_{k,n}(q, q') = \sum_{j=1}^{N_{k,n}} \varphi_{kj}(q) \cdot \overline{\varphi_{kj}(q')} ,$$

where $\varphi_{kj}$ is the distributional kernel

$\mathcal{P}_{k,n}(q, q') = \sum_{j=1}^{N_{k,n}} \varphi_{kj}(q) \cdot \overline{\varphi_{kj}(q')} ,$


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where \( q \cdot q' = q' q' \) (we think of \( q \) and \( q' \) as columns vectors), and \( N_{k,n} \) is the dimension of \( V_{k,n} \). By symmetry considerations, \( \mathcal{P}_{k,n}(q,q') \) only depends on \( q \cdot q' \). In fact, with the normalization \( \mathcal{P}_{k,n+1}(1) = 1 \),

\[
\mathcal{P}_{k,n}(q,q') = \frac{N_{k,n}}{\text{vol}(S^n)} \mathcal{P}_{k,n+1}(q \cdot q').
\]

Thus it equivalent to give asymptotic expansions for \( \mathcal{P}_{k,n+1}(\cos(\vartheta)) \) and for \( \mathcal{P}_{k,n}(q,q') \) with \( q \cdot q' = \cos(\vartheta) \).

Since for any \( (q,q') \in S^n \times S^n \) we have

\[
\mathcal{P}_{k,n}(q,q) = \frac{N_{k,n}}{\text{vol}(S^n)}, \quad \mathcal{P}_{k,n}(q,-q') = (-1)^k \mathcal{P}_{k,n}(q,q'),
\]

we may assume \( q \neq \pm q' \). Then there is a unique great circle parametrized by arc length going from \( q \) to \( q' \) in a time \( \vartheta \in (0, \pi) \), and \( q' q' = \cos(\vartheta) \).

Our geometric approach uses on the one hand the specific relation between spherical harmonics on \( S^n \) and the Hardy space of the Fermat quadric hypersurface in \( \mathbb{P}^n \) ([L], [G]), and the other hand the off-diagonal scaling asymptotics of the level-\( k \) Szegö kernel of polarized projective manifold ([BSZ], [SZ]).

The following asymptotic expansions involve a sequence of constants \( C_{k,n} > 0 \) with a precise geometric meaning [G]. There is a natural conformally unitary isomorphism between the level-\( k \) Szegö kernel of the Fermat quadric \( F_n \subset \mathbb{P}^n \) and \( V_{k,n} \), given by a push-forward operation, and \( C_{k,n} \) is the corresponding conformal factor.

An asymptotic expansion for \( C_{k,n} \) is discussed in [G], building on the theory of [L]; an alternative derivation is given in Proposition 1.1 (with an explicit computation of the leading order term).

In the following, the symbol \( \sim \) stands for ‘has the same asymptotics as’.

**Theorem 1.1.** There exists smooth functions \( A_{nl} \) and \( B_{nl} \) (\( l = 1, 2, \ldots \)) on \([0, \pi]\) such that the following holds. Let us fix \( C > 0 \) and \( \delta \in [0, 1/6) \). Then, uniformly in \( (q,q') \in S^n \times S^n \) satisfying \( q' q' = \cos(\vartheta) \) with

\[
C k^{-\delta} < \vartheta < \pi - C k^{-\delta},
\]

we have for \( k \to +\infty \) an asymptotic expansion of the form

\[
\mathcal{P}_{k,n}(q,q') = \frac{2\pi}{C_{k,n}^2} \left( \frac{1}{\sin(\vartheta)} \right)^{(n-1)/2} \cdot [\cos(\alpha_{k,n}(\vartheta)) \cdot \mathcal{A}_n(\vartheta, k) + \sin(\alpha_{k,n}(\vartheta)) \cdot \mathcal{B}_n(\vartheta, k)],
\]

where

\[
[\mathcal{A}_n(\vartheta, k) = \cdots, \mathcal{B}_n(\vartheta, k) = \cdots]
\]
where

\[ \alpha_{k,n}(\vartheta) =: k\vartheta + \left(\frac{\vartheta}{2} - \frac{\pi}{4}\right)(n - 1), \]

and

\[ A_n(\vartheta, k) \sim 1 + \sum_{l=1}^{+\infty} k^{-l} \frac{A_{nl}(\vartheta)}{\sin(\vartheta)^{6l}}, \quad B_n(\vartheta, k) \sim \sum_{l=1}^{+\infty} k^{-l} \frac{B_{nl}(\vartheta)}{\sin(\vartheta)^{6l}}. \]

At the \( l \)-th step, we have for some constant \( C_l > 0 \)

\[ \left| \frac{A_{nl}(\vartheta)}{\sin(\vartheta)^{6l}} \right|, \left| \frac{B_{nl}(\vartheta)}{\sin(\vartheta)^{6l}} \right| \leq C_l k^{-l(1-6\delta)}, \]

and a similar estimate holds for the error term. Hence the previous is an asymptotic expansion for \( \delta \in [0, 1/6) \).

As mentioned, the same techniques yield an asymptotic expansion for \( C_{n,k} \) (see (6.18) in [G]).

**Proposition 1.1.** For \( k \to +\infty \) we have an asymptotic expansion of the form:

\[ C_{k,n} \sim \left[ \frac{(n - 1)!}{2\sqrt{2}} \cdot \frac{\text{vol}(S^n) \cdot \text{vol}(S^{n-1})}{\pi k} \right]^{1/2} (n - 1)^{(n-1)/4} \cdot \left[ 1 + \sum_{j \geq 1} k^{-j} a_j \right]. \]

If we insert the latter expansion in the one provided by Theorem 1.1 we obtain the following:

**Corollary 1.1.** With the assumptions and notation of Theorem 1.1, for \( k \to +\infty \) there is an asymptotic expansion

\[ P_{k,n}(q, q') = \frac{2^{n+1}}{(n - 1)! \cdot \text{vol}(S^n) \cdot \text{vol}(S^{n-1})} \left( \frac{\pi k}{\sin(\vartheta)} \right)^{(n-1)/2} \cdot \left[ \cos(\alpha_{k,n}(\vartheta)) \cdot C_n(\vartheta, k) + \sin(\alpha_{k,n}(\vartheta)) \cdot D_n(\vartheta, k) \right], \]

where \( C_n(\vartheta, k) \) and \( D_n(\vartheta, k) \) admit asymptotic expansions similar to those of \( A_n(\vartheta, k) \) and \( B_n(\vartheta, k) \), respectively (of course, with different functions \( C_{nl} \) and \( D_{nl}, l \geq 1 \)).

Pairing Corollary 1.1 with (2), we obtain:

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Corollary 1.2. In the same situation as in Theorem 1.1, for \( k \to +\infty \) there is an asymptotic expansion

\[
P_{k,n+1}(\cos(\vartheta)) = \frac{2^{n+1}}{\text{vol}(S^{n-1})} \left( \frac{\pi}{\sin(\vartheta) k} \right)^{(n-1)/2} \left[ \cos(\alpha_{k,n}(\vartheta)) \cdot \mathcal{E}_n(\vartheta, k) + \sin(\alpha_{k,n}(\vartheta)) \cdot \mathcal{F}_n(\vartheta, k) \right],
\]

where again \( \mathcal{E}_n(\vartheta, k) \) and \( \mathcal{F}_n(\vartheta, k) \) admit asymptotic expansions similar to those of \( A_n(\vartheta, k) \) and \( B_n(\vartheta, k) \), respectively.

Let us verify that Corollary 1.2 fits with the classical asymptotics. For example, when \( n = 1 \) we obtain

\[
P_{k,2}(\cos(\vartheta)) \sim \cos(k \vartheta) + \cdots,
\]

so that the leading order term is the \( k \)-th Chebychev polynomial. Since it is known that in this case the Legendre polynomial is the Chebychev polynomial ([M], page 11), this is in fact the only term of the expansion.

For \( n = 2 \), we obtain the formula of Laplace (cfr [Leb], §4.6; [O], (8.01) of Ch. 4; [S], Theorem 8.21.2), but as a full asymptotic expansion holding uniformly on expanding subintervals converging to \([-1,1]\) at a controlled rate, as above:

\[
P_{k,3}(\cos(\vartheta)) \sim \sqrt{\frac{2}{\pi k \sin(\vartheta)}} \cos \left( \left( k + \frac{1}{2} \right) \vartheta - \frac{\pi}{4} \right) + O(\sqrt{k^{-3/2+6\delta}}).
\]

For arbitrary \( n \), \( P_{k,n+1} \) is a multiple of a Gegenbauer polynomial ([B]; [M], page 16):

\[
P_k^{(n/2-1,n/2-1)}(\cos(\vartheta)) = r_{k,n} P_{k,n+1}(\cos(\vartheta)). \tag{3}
\]

Given the standardization for \( P_k^{(n/2-1,n/2-1)} \) ([H], §10.8)

\[
r_{k,n} = P_k^{(n/2-1,n/2-1)}(1) = \binom{k+n/2-1}{k} = \frac{(n/2)_k}{k!} \frac{\Gamma(k+n/2)}{k! \Gamma(n/2)},
\]

where \( \Gamma \) is of course the Gamma function. By (35.31) in [M], for \( k \to +\infty \) we have

\[
\Gamma(k+n/2) \sim k^{n/2} \Gamma(k) = k^{n/2} (k-1)!.
\]
Therefore,
\[ r_{k,n} \sim \frac{k^{n/2} (k-1)!}{k! \Gamma(n/2)} = \frac{k^{n/2-1}}{\Gamma(n/2)}. \]

If we use the well-known formula (see e.g. (2) of [M])
\[ \text{vol} \left( S^{n-1} \right) = \frac{2 \pi^{n/2}}{\Gamma(n/2)}, \]
we obtain for \( F_k^{(n/2-1, n/2-1)} (\cos(\vartheta)) \) as asymptotic expansion with leading order term
\[
2^{n+1} \frac{k^{n/2-1}}{\Gamma(n/2)} \frac{\Gamma(n/2)}{2 \pi^{n/2}} \left( \frac{\pi}{\sin(\vartheta) k} \right)^{(n-1)/2} \cos \left( \alpha_{k,n}(\vartheta) \right)
= \frac{1}{\sqrt{\pi k}} \cos(\vartheta/2)^{(n-1)/2} \frac{1}{\sin(\vartheta/2)^{(n-1)/2}} \cos \left( \alpha_{k,n}(\vartheta) \right),
\]
in agreement with (10) on page 198 of [B].

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2 Preliminaries

2.1 The geometric picture

For the following, see [G], [L].

Let \( S^n_1 \subset \mathbb{R}^{n+1} \) be the unit sphere, and let us identify the tangent and cotangent bundles of \( S^n_1 \) by means of the standard Riemannian metric. The unit (co)sphere bundles of \( S^n_1 \) is given by the incidence correspondence
\[ S^* (S^n_1) \cong S_1^n = \left\{ (\mathbf{q}, \mathbf{p}) \in S^n_1 \times S^n_1 : \mathbf{q}^t \mathbf{p} = 0 \right\}. \tag{4} \]

The Fermat quadric hypersurface in complex projective space is
\[ F_n =: \{ [\mathbf{z}] \in \mathbb{P}^n : \mathbf{z}^t \mathbf{z} = 0 \}; \]
let \( A \) be the restriction to \( F_n \) of the hyperplane line bundle. Given the standard Hermitian product on \( \mathbb{C}^{n+1} \), \( A \) is naturally a positive Hermitian line bundle, \( F_n \) inherits a Kähler structure \( \omega_{F_n} \) (the restriction of the Fubini-Study metric), and the spaces of global holomorphic sections of higher powers of \( A \), \( H^0 \left( F_n, A^\otimes k \right) \), have an induced hermitian structure.
The affine cone over $F_n$ is $C_n = \{z'z = 0\} \subset \mathbb{C}^{n+1}$; the intersection $X_1 = C_n \cap S^{2n+1}_1$ may be viewed as the unit circle bundle in the dual line bundle $A^\vee$. More generally, for any $r > 0$ the intersection

$$X_r = C_n \cap S^{2n+1}_r$$

(5)

with the sphere of radius $r$ is naturally identified with the circle bundle of radius $r$ in $A^\vee$. In particular, $X_{\sqrt{2}} = \{q + i p : \|q\|^2 = \|p\|^2 = 1, \quad q'p = 0\}$

(6)

is diffeomorphic to $S^*(S^n)$ by the map $\beta : (q, p) \mapsto q + i p$; furthermore, $\beta$ is equivariant for the natural actions of $O(n+1)$ on $S^*(S^n)$ and $X_{\sqrt{2}}$ defined by, respectively,

$$B \cdot (q, p) = (Bq, Bp), \quad B \cdot (q + i p) = Bq + i Bp \quad (B \in O(n+1)).$$

We shall identify $S^*(S^n)$ and $X_{\sqrt{2}}$, and denote the projection by

$$\nu : S^*(S^n) \cong X_{\sqrt{2}} \rightarrow S^n, \quad q + i p \mapsto q.$$

(7)

There is also a standard structure action of $S^1$ on $X_{\sqrt{2}}$, induced by fibre-wise scalar multiplication in $A^\vee$, or equivalently in $\mathbb{C}^{n+1}$. The latter action is intertwined by $\beta$ with the ‘reverse’ geodesic flow on $S^*(S^n) \cong S(S^n)$. The $S^1$-orbits are the fibers of the circle bundle projection

$$\pi_{\sqrt{2}} : q + i p \in X_{\sqrt{2}} \mapsto [q + i p] \in F_n.$$

(8)

This holds for any $r > 0$; we shall denote by $\pi_r : X_r \rightarrow F_n$ the projection for general $r > 0$.

2.2 The metric on $X_r$

Let us dwell on the metric aspect of (5); there are two natural choices of a Riemannian metric on $X_r$, hence of a Riemannian density, and we need to clarify the relation between the two.

There is an obvious choice of a Riemannian metric $g'_r$ on $X_r$, induced by the standard Euclidean product on $\mathbb{C}^{n+1}$. With respect to $g'_r$, the $S^1$ orbits on $X_r$ have length $2\pi r$. Clearly, $g'_r$ is homogeneous of degree 2 with respect to the dilation $\mu_r : x \in X \mapsto r x \in X_r$, and therefore the corresponding volume form $\mathcal{V}'_{X_r}$ on $X_r$ is homogeneous of degree $\dim(X) = 2n - 1$. That is,

$$\mu_r^*(\mathcal{V}'_{X_r}) = r^{2n-1} \mathcal{V}'_X.$$

(9)
An alternative and common choice of a Riemannian structure $g_1$ on $X_1$ comes from its structure of a unit circle bundle over $F_n$. Let $\alpha \in \Omega^1(X_1)$ be the connection 1-form associated to the unique compatible covariant derivative on $A$, so that $d\alpha = 2\pi^*_1(\omega_{F_n})$. Also, let

$$H(X_1/F_n) = \ker(\alpha), \quad V(X_1/F_n) = \ker(d\pi_1) \subseteq TX$$

denote the horizontal and vertical tangent bundles for $\pi_1$, respectively. There is a unique Riemannian metric $g_1$ on $X_1$ such that $\pi_1$ a Riemannian submersion, and the $S^1$-orbits on $X_1$ have unit length. The corresponding volume form on $X_1$ is given by

$$\Upsilon_{X_1} = \frac{1}{(n-1)!} \pi^*_1 \left( \omega_{F_n}^{\wedge (n-1)} \right) \wedge \frac{1}{2\pi} \pi^*_1(\Upsilon_{F_n}) \wedge \alpha,$$

where $\Upsilon_{F_n} = \omega_{F_n}^{\wedge (n-1)}/(n-1)!$ is the symplectic volume form on $F_n$.

We wish to compare the two Riemannian metrics $g_1$ and $g'_1$, the corresponding volume forms, $\Upsilon_{X_1}$ and $\Upsilon_{X_1}$, and densities, $dV_X$ and $d'V_X$.

**Lemma 2.1.** $\Upsilon_{X_1} = \frac{1}{2\pi} \Upsilon'_{X_1}$ and $dV_X = \frac{1}{2\pi} d'V_X$.

**Proof of Lemma 2.1.** The connection 1-form for the Hopf map $S^{2n+1} \to \mathbb{P}^n$ is

$$\theta = \frac{i}{2} (z^i \, dz^i - z^i \, d\bar{z}^i)$$

thus $\alpha$ is the restriction of $\theta$ to $X_1$. Let $\omega_0$ be the standard symplectic structure on $\mathbb{C}^{n+1}$. Since $\theta_z(w) = \omega_0(z, w)$, we have $\ker(\theta_z) = z^i \omega_0$ (symplectic annihilator). In other words,

$$\ker(\theta_z) = \left( \text{span}_\mathbb{R}(z) \oplus z^{\perp_{\omega_0}} \right) \cap T_zS^{2n+1} = z^{\perp_{\omega_0}},$$

where $z^{\perp_{\omega_0}}$ is the Hermitian orthocomplement of $z$ for the standard Hermitian product.

Thus, if $z \in X_1$ then

$$H_z(X_1/F_n) = \ker(\alpha_z) = z^{\perp_{\omega_0}} \cap T_z\mathbb{C}^n = z^{\perp_{\omega_0}} \cap \bar{z}^{\perp_{\omega_0}}.$$

On the other hand, $V_z(X_1/F_n) = \text{span}_\mathbb{R}(iz)$. Thus $V(X_1/F_n)$ and $H(X_1/F_n)$ are orthogonal with respect to both $g_1$ (by construction) and $g'_1$ (by the previous considerations). Hence we may compare $g_1$ and $g'_1$ separately on $H(X_1/F_n)$ and $V(X_1/F_n)$.

On the complex vector bundle $H(X_1/F_n)$, $g'_1$ and $g_1$ are, respectively, the Euclidean scalar products associated to the restrictions of the $(1,1)$-forms

$$\omega_0 = \frac{i}{2} \partial \bar{\partial} \|z\|^2, \quad \omega_1 = \frac{i}{2} \partial \bar{\partial} \ln \left( \|z\|^2 \right).$$

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Given that $\omega_0$ and $\omega_1$ agree on $T S^{2n+2}_1$, $g_1 = g_1'$ on $H(X_1/F_n)$. 

On the other hand, both $g_1$ and $g_1'$ are $S^1$-invariant, but $S^1$-orbits on $X_1$ have length $2\pi$ for $g_1'$ and 1 for $g_1$. Thus $g_1 = g_1'/2\pi$ on $V(X_1/F_n)$.

The claim follows directly from this.

\[ \square \]

### 2.3 The Szegő kernel on $X_r$

For every $r > 0$, $X_r$ is the boundary of a strictly pseudoconvex domain, and as such it carries a CR structure, a Hardy space $H(X_r)$, and a Szegő projector $\Pi_r : L^2(X_r) \to H(X_r)$. We aim to relate the various $\Pi_r$'s.

Let $\mathcal{O}(\mathbb{C}_n \setminus \{0\})$ be the ring of holomorphic functions on the conic complex manifold $\mathbb{C}_n \setminus \{0\}$. Let $\mathcal{O}_k(\mathbb{C}_n \setminus \{0\}) \subset \mathcal{O}(\mathbb{C}_n \setminus \{0\})$ be the subspace of holomorphic functions of degree of homogeneity $k$.

For every $r > 0$ and $k = 0, 1, 2, \ldots$ let $H_k(X_r) \subset H(X_r)$ be the finite-dimensional $k$-th isotypical component of $H(X_r)$ with respect to the standard $S^1$-action. Restriction induces an algebraic isomorphism $\mathcal{O}_k(\mathbb{C}_n \setminus \{0\}) \to H_k(X_r)$; with a slight abuse of language, we shall denote by the same symbol an element of $H_k(X_r)$ and the corresponding element of $\mathcal{O}_k(\mathbb{C}_n \setminus \{0\})$.

Suppose that $(s_{kj})^N_{j=0} \subseteq \mathcal{O}_k(\mathbb{C}_n \setminus \{0\})$ restricts to an orthonormal basis of $H_k(X_1)$:

\[
\int_{X_1} s_{kj}(x) \overline{s_{kl}(x)} \, dV_{X_1}(x) = \delta_{jl}.
\]

Setting $y = r \, x$, and using (9) together with Lemma 2.1 we get

\[
\int_{X_r} s_{kj}(y) \overline{s_{kl}(y)} \frac{1}{2\pi} \, d'V_{X_r}(y) = r^{2n+2k-1} \int_{X_1} s_{kj}(x) \overline{s_{kl}(x)} \frac{1}{2\pi} \, d'V_{X_1}(x) = r^{2n+2k-1} \delta_{jl}.
\]

Therefore we have:

**Lemma 2.2.** If $(s_{kj})^N_{j=0} \subseteq \mathcal{O}_k(\mathbb{C}_n \setminus \{0\})$ restricts to an orthonormal basis of $H_k(X_1)$ with respect to $d'V_{X_1}$, then for every $r > 0$

\[
\left( r^{-(k+n-1/2)} s_{kj} \right)^N_{j=0}
\]

restricts to an orthonormal basis of $H_k(X_r)$, with respect to $d'V_{X_r}/2\pi$.

Let now $\Pi_{r,k}$ be the level-$k$ Szegő kernel on $X_r$, that is, the orthogonal projector

\[
\Pi_{r,k} : L^2(X_r, d'V_{X_r}/2\pi) \to H_k(X_r).
\]
By Lemma 2.2, its Schwartz kernel $\Pi_{r,k} \in \mathcal{C}^\infty(X_r \times X_r)$ is given by

$$\Pi_{r,k}(y, y') = r^{-(2k+2n-1)} \sum_{j=0}^{N_k} s_{kj}(y) \cdot \overline{s_{kj}(y')} \quad (y, y' \in X_r).$$

(13)

When pulled-back to $X_1$, this is (here $x, x' \in X_1$)

$$\Pi_{r,k}(r x, r x') = r^{-(2k+2n-1)} \sum_{j=0}^{N_k} s_{kj}(r x) \cdot \overline{s_{kj}(r x')}$$

$$= r^{-(2n-1)} \sum_{j=0}^{N_k} \hat{s}_{kj}(x) \cdot \overline{\hat{s}_{kj}(x')} = r^{1-2n} \Pi_{1,k}(x, x').$$

(14)

In particular,

$$\Pi_{\sqrt{2},k}(\sqrt{2} x, \sqrt{2} x') = \left(\frac{\sqrt{2}}{2^m}\right) \Pi_{1,k}(x, x').$$

(15)

We shall make repeated use of the following asymptotic property of $\Pi_{1,k}$, which follows from the microlocal description of $\Pi$ as an FIO (explicit exponential estimates are discussed in [C]).

**Theorem 2.1.** Let $\text{dist}_{F_n}$ be the distance function on $F_n$ associated to the Kähler metric. Given any $C, \epsilon > 0$, uniformly for $x, x' \in X$ satisfying

$$\text{dist}_{F_n}(\pi(x), \pi(x')) \geq C k^{\epsilon - 1/2},$$

we have

$$\Pi_{1,k}(x, x') = O(k^{-\infty})$$

when $k \to +\infty$.

### 2.4 Heisenberg local coordinates

There are two unit circle bundles in our picture: the Hopf fibration $\pi : S^{2n+1}_1 \to \mathbb{P}^n$, and $\pi_1 : X_1 \to F_n$. Clearly, $\pi_1$ is the pull-back of $\pi$ under the inclusion $F_n \hookrightarrow \mathbb{P}^n$. Both $S^{2n+1}_1$ and $X_1$ are boundaries of strictly pseudo-convex domains, and carry a CR structure.

On both $S^{2n+1}_1$ and $X_1$, we may consider privileged systems of coordinates called *Heisenberg local coordinates* (HLC). In these coordinates, Szegő kernel asymptotics exhibit a ‘universal’ structure [SZ]; we refer to *ibidem* for a detailed discussion.
Given \( z_0 \in X_1 \), a HLC system on \( X_1 \) centered at \( z_0 \) will be denoted in additive notation:

\[
(\theta, v) \in (-\pi, \pi) \times B_{2n-2}(0, \delta) \mapsto z_0 + (\theta, v) \in X_1.
\]

Here \( \theta \in (-\pi, \pi) \) is an ‘angular’ coordinate measuring displacement along the \( S^1 \)-orbit through \( z_0 \) (the fiber through \( z_0 \) of \( \pi_1 : X_1 \to F_n \)); instead \( v \in B_{2n-2}(0, \delta) \subseteq \mathbb{R}^{2n-2} \cong \mathbb{C}^{n-1} \) descends to a local coordinate on \( F_n \) centered at \( m_0 = \pi(x_0) \), inducing a unitary isomorphism \( T_{[z_0]}F_n \cong \mathbb{C}^{n-1} \). We may thus think of \( v \) as a tangent vector in \( T_{[z_0]}F_n \).

Here this additive notation might be misleading, since \( X_1 \subset \mathbb{C}^{n+1} \). Therefore we shall write \( z_0 + x_1 (\theta, v) \) for HLC’s on \( X_1 \) centered at \( z_0 \). We shall generally abridge notation by writing \( z_0 + x_1 v \) for \( z_0 + x_1 (0, v) \).

Similarly, \( (\theta, w) \in (-\pi, \pi) \times B_{2n}(0, \delta) \mapsto z_0 + s_{2n+1}^2 (\theta, v) \) will denote a system of Heisenberg local coordinates on \( s_{2n+1}^1 \) centered at \( z_0 \). There is in fact a natural choice of HLC on \( s_{2n+1}^1 \) centered at any \( z_0 \in s_{2n+1}^1 \).

Namely, let \( (a_1, \ldots, a_n) \) be an orthonormal basis of the Hermitian orthocomplement \( z_0^{\perp} \subseteq \mathbb{C}^{n+1} \), and for \( w = (w_j) \in \mathbb{C}^n \) let us set

\[
z_0 + s_{2n+1}^1 (\theta, w) := \frac{e^{i\theta}}{\sqrt{1 + \|w\|^2}} \left( z_0 + \sum_{j=1}^n w_j a_j \right).
\]

Since there is a canonical unitary identification \( z_0^{\perp} \cong T_{[z_0]}F_n \), we shall also write this as \( z_0 + s_{2n+1}^1 (\theta, v) \) with \( (\theta, v) \in (-\pi, \pi) \times T_{[z_0]}F_n \).

If \( z_0 \in X_1 \), HLC’s on \( X_1 \) centered at \( z_0 \) can be chosen so that they agree to second order with the former HLC’s on \( s_{2n+1}^1 \). More precisely, we may assume that for any \( v \in T_{[z_0]}F_n \subset T_{[z_0]}F_n \) we have

\[
z_0 + x_1 (\theta, w) = z_0 + s_{2n+1}^1 (\theta, v + R_2(v)),
\]

where \( R_2 \) is a function vanishing to second order at the origin.

Given \( v, w \in \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2} \), let us define

\[
\psi_2(v, w) =: -i\omega_0(v, w) - \frac{1}{2}\|v - w\|^2;
\]

here \( \omega_0 \) is the standard symplectic structure, and \( \| \cdot \| \) is the standard Euclidean norm. We shall make use of the following asymptotic expansion, for which we refer again to [SZ].

**Theorem 2.2.** Let us fix \( C > 0 \) and \( \epsilon \in (0, 1/6) \). Then for any \( z \in X_1 \), and for any choice of HLC’s on \( X_1 \) centered at \( z \), there exists polynomials
$P_j$ of degree $\leq 3j$ and parity $j$ on $T_{[z]}F_n \times T_{[z]}F_n \cong \mathbb{R}^{2n-2} \times \mathbb{R}^{2n-2}$, such that following holds. Uniformly in $v_1, v_2 \in T_{[z]}F_n$ with $\|v_j\| \leq C k^\epsilon$ for $j = 1, 2$, and $\theta_1, \theta_2 \in (-\pi, \pi)$, one has for $k \to +\infty$ the following asymptotic expansion:

$$\Pi_{1,k} \left( z + \left( \theta_1, \frac{v_1}{\sqrt{k}} \right), z + \left( \theta_2, \frac{v_2}{\sqrt{k}} \right) \right) \sim \left( \frac{k}{\pi} \right)^{n-1} e^{ik(\theta_1 - \theta_2) + \varphi_2(v_1, v_2)} \left[ 1 + \sum_{j=1}^{+\infty} k^{-j/2} P_j(v_1, v_2) \right].$$

In the given range the above is an asymptotic expansion, since

$$|k^{-j/2} R_j(v_1, v_2)| \leq C_j k^{-\frac{j}{2}(1-6\epsilon)}.$$

### 2.5 $\mathcal{P}_k$ and $\Pi_{\sqrt{2},k}$

As discussed in [G], the push-forward operator $\nu_* : C^\infty(X_{\sqrt{2}}) \to C^\infty(S^n)$ restricts to an algebraic isomorphism

$$C^\infty(X_{\sqrt{2}}) \cap H(X_{\sqrt{2}}) \to C^\infty(S^n);$$  \tag{19}

for every $k$, (19) restricts to a conformally unitary isomorphism

$$H_k(X_{\sqrt{2}}) \to V_k,$$

with a scalar conformal factor $C_{k,n} > 0$. Thus we have

$$\|\nu_*(s)\|_{L^2(S^n)} = C_{k,n} \|s\|_{H(X_{\sqrt{2}})} \quad (s \in H_k(X_{\sqrt{2}})).$$  \tag{20}

Therefore, if $(\sigma_{kj})_{j=0}^{N_k}$ is an orthonormal basis of $H_k(X_{\sqrt{2}})$, then

$$(C_{k,n}^{-1} \cdot \nu_*(\sigma_{kj}))_{j=0}^{N_k}$$

is an orthonormal basis of $V_k$. It follows that $\mathcal{P}_{k,n}$ in (1) is given by

$$\mathcal{P}_{k,n} = \frac{1}{C_{k,n}^2} (\nu \times \nu)_* \left( \Pi_{\sqrt{2},k} \right),$$  \tag{21}

where $\nu \times \nu : X_{\sqrt{2}} \times X_{\sqrt{2}} \to S^n \times S^n$ is the product projection.

More explicitly, for $q \in S^n$ let $S(q^\perp) \cong S^{n-1}$ be the unit sphere centered at the origin in the orthocomplement $q^\perp$, and let $dV_{S(q^\perp)}$ be the Riemannian density on $S(q^\perp)$; then

$$\mathcal{P}_{k,n}(q_0, q_1) = \frac{1}{C_{k,n}^2} \int_{S(q_0^\perp)} \int_{S(q_1^\perp)} \Pi_{\sqrt{2},k}(q_0 + \sqrt{k} p, q_1 + \sqrt{k} p') dV_{S(q_0^\perp)}(p) dV_{S(q_1^\perp)}(p').$$  \tag{22}
2.6 \( \Pi_{r,k} \) and conjugation

Conjugation \( \sigma : z \mapsto \overline{z} \) in \( \mathbb{C}^{n+1} \) leaves invariant the affine cone \( C_n \) and every \( X_r \). Furthermore, it yields a Riemannian isometry of \( X_r \) into itself. For \( f \in \mathcal{O}(C_n \setminus \{0\}) \), let us set

\[
f^\sigma(z) =: \overline{f(z)}.
\]

If \( f \in \mathcal{O}_k(C_n \setminus \{0\}) \), then \( f^\sigma \in \mathcal{O}_k(C_n \setminus \{0\}) \).

Hence, if \( (s_{kj})_j \subseteq \mathcal{O}_k(C_n \setminus \{0\}) \) restricts to an orthonormal basis of \( H_k(X_r) \), then so does \( (s^\sigma_{kj})_j \). Thus for any \( z_0, z_1 \in X_r \) we have

\[
\Pi_{r,k}(z_0, z_1) = \sum_j s_{kj}(z_0) \cdot s_{kj}(z_1) = \Pi_{r,k}(z_1, z_0) = \overline{\Pi_{r,k}(z_0, z_1)}. \quad (23)
\]

3 Proof of Theorem 1.1

Proof of Theorem 1.1. Given \( q_0, q_1 \in S^{n-1} \) with \( q_1 \neq \pm q_0 \), let \( \gamma_+ \) be the unique unit speed geodesic on \( S^n \) such that \( \gamma_+(0) = q_0 \) and \( \gamma_+(\vartheta) = q' \) for some \( \vartheta \in (0, \pi) \). Then

\[
p_0 := \dot{\gamma}_+(0) \in S^{n-1}(q_0^\perp), \quad p_1 := \dot{\gamma}_+(\vartheta) \in S^{n-1}(q_1^\perp).
\]

The reverse geodesic \( \gamma_-(\vartheta) := \gamma(-\vartheta) \) satisfies \( \gamma_-(0) = q_0, \gamma_-(\vartheta) = -p_0 \) and \( \gamma_-(\vartheta') = q' \) for a unique \( \vartheta' = -\vartheta \in (-\pi, 0) \).

Although they project down to the same locus in \( S^n, \gamma_+ \) and \( \gamma_- \) correspond to distinct fibers of the circle bundle projection \( \pi : X(\sqrt{2}) \to F_n \). Let us express the (co)tangent lift \( \tilde{\gamma}_\pm \) of the geodesics \( \gamma_\pm \) in complex coordinates, and set \( p_1 = \dot{\gamma}_+(\vartheta) \). Then

\[
\tilde{\gamma}_\pm(\theta) = \gamma_\pm(\theta) + i \dot{\gamma}_\pm(\theta) = e^{-i\theta}(q_0 \pm i p_0) = q_1 \pm i p_1, \quad (24)
\]

In view of (3), we have:

\[
q_0 \pm i p_0, \quad q_1 \pm i p_1 \in \pi^{-1}_\sqrt{2}(\{q_0 \pm i p_0\}). \quad (25)
\]

On the other hand, \( [q_0 + i p_0] \neq [q_0 - i p_0] \in F_n \), since \( q_0 + i p_0 \) and \( q_0 - i p_0 \) are linearly independent in \( \mathbb{C}^{n+1} \).

Thus we have:
Lemma 3.1. Suppose \( q_0, q_1 \in S^n \) and \( q_1 \neq \pm q_0 \). Then the only points \( [z] \in F_n \) such that

\[
\nu^{-1}(q_0) \cap \pi^{-1}(\sqrt{2}[z]) \neq \emptyset \quad \text{and} \quad \nu^{-1}(q_1) \cap \pi^{-1}(\sqrt{2}[z]) \neq \emptyset
\]

are

\[
[z_+] = [q_0 + i p_0], \quad [z_-] = [q_0 - i p_0].
\]

By Theorem 2.1 for fixed \( p \) and \( p' \) and \( k \to +\infty \) we have

\[
\Pi_{\sqrt{2},k}(q_0 + i p, q_1 + i p') = O(k^{-\infty}),
\]

unless \( p = \pm p_0 \) and \( p' = \pm p_1 \). Therefore, for a fixed \( \vartheta \in (0, \pi) \) integration in (22) may be localized in a small neighborhood of \((\pm p_0, \pm p_1)\), perhaps at the cost of disregarding a negligible contribution to the asymptotics.

Since however we are allowing \( \vartheta \) to approach 0 or \( \pi \) at a controlled rate, we need to give a more precise quantitative estimate of how small the previous neighborhood may be chosen when \( k \to +\infty \).

To this end, let us introduce some further notation. Given linearly independent \( a, b \in S^n \), let us set

\[
R(a, b) =: \text{span}_R(a, b) \subseteq \mathbb{R}^{n+1},
\]

and

\[
R(a, b)_\mathbb{C} =: R(a, b) \otimes \mathbb{C} = \text{span}_\mathbb{C}(a, b) \subseteq \mathbb{C}^{n+1},
\]

Furthermore, for \( \|v\| \leq 1 \) we shall set

\[
S_\pm(v) =: -1 \pm \sqrt{1 - \|v\|^2}.
\]

A straightforward computation yields the following:

**Lemma 3.2.** Assume that \( q_0 + i p_0 \in X_{\sqrt{2}} \) and \( q_1 + i p_1 = e^{-i\vartheta}(q_0 + i p_0) \) with \( \vartheta \in (0, \pi) \). Then any \( p \in S^{n+1}(q_0) \) with \( p_0^t p \geq 0 \), respectively, \( p_0^t p \leq 0 \), may be written uniquely in the form

\[
p = (1 + S_+(v)) p_0 + v,
\]

respectively

\[
p = (1 + S_-(v)) p_0 + v,
\]

where \( v \in q_0^\perp \cap p_0^\perp = R(q_0, q_1)^\perp \) (the Euclidean orthocomplement) has norm \( \|v\| \leq 1 \), and

\[
S_\pm(v) =: -1 \pm \sqrt{1 - \|v\|^2}.
\]
Proposition 3.1. Let us fix $C > 0$, $\delta \in (0, 1/6)$ and $\epsilon > \delta$. Then there exist constants $D, \epsilon_1 > 0$ such that the following holds. Suppose that

1. $C k^{-\delta} < \vartheta < \pi - C k^{-\delta}$;
2. $q_j + i p_j \in X_{\sqrt{2}}$ for $j = 0, 1$;
3. $q_1 + i p_1 = e^{-i\vartheta} (q_0 + i p_0)$;
4. $v_j \in q_0^\perp \cap q_1^\perp$ for $j = 0, 1$;
5. $1 \geq \max \{ \|v_0\|, \|v_1\| \} \geq C k^{e_1 - 1/2}$;
6. $p_j' = (1 + S_j(v_j)) p_j + v_j \in S^{n-1}(q_j^\perp)$ for $j = 0, 1$, where $S_j$ can be either one of $S_{\pm}$ (Lemma 3.2).

Then
\[
\text{dist}_{F_n}([q_0 + i p_0'], [q_1 + i p_1']) \geq D k^{e_1 - 1/2}
\]
for every $k \gg 0$.

In view of Theorem 2.1, Proposition 3.1 implies:

Corollary 3.1. Uniformly in the range of Proposition 3.1, we have
\[
\Pi_{\sqrt{2}, k}(q_0 + i p_0', q_1 + i p_1') = O \left( k^{-\infty} \right).
\]

Proof of Proposition 3.1. Let us set for $\gamma \in [-\pi, \pi]$:
\[
\Phi(\gamma, p_0', p_1') =: e^{-i\gamma} (q_0 + i p_0') - (q_1 + i p_1').
\] (28)

Let $\text{dist}_{F_n}$ be the restriction to $F_n$ of the distance function on $\mathbb{P}^n$. Then
\[
\text{dist}_{F_n}([q_0 + i p_0'], [q_1 + i p_1'])
\] (29)
\[
= \frac{1}{\sqrt{2}} \min \left\{ \|\Phi(\gamma, p_0', p_1')\| : \gamma \in [0, 2\pi] \right\}.
\]

The factor in front is needed because while the Hopf map $S^{2n+1}_1 \to \mathbb{P}^n$ is a Riemannian submersion, the projection $S^{2n+1}_{\sqrt{2}} \to \mathbb{P}^n$ is so only in a conformal sense.

We are reduced to proving that in the given range there exist constants $D, \epsilon_1 > 0$ such that for every $k \gg 0$ and $\gamma \in [0, 2\pi]$
\[
\|\Phi(\gamma, p_0', p_1')\| \geq D k^{e_1 - 1/2}.
\] (30)
We have
\[
\Phi(\gamma, \mathbf{p}_0', \mathbf{p}_1') = e^{-i\gamma} (\mathbf{q}_0 + i\mathbf{p}_0 + iS_0(\mathbf{v}_0) \mathbf{p}_0 + i\mathbf{v}_0) - (e^{-i\theta} (\mathbf{q}_0 + i\mathbf{p}_0 + iS_1(\mathbf{v}_1) \mathbf{p}_1 + i\mathbf{v}_1)) \\
= (A \mathbf{q}_0 + B \mathbf{p}_0) + i [e^{-i\gamma} \mathbf{v}_0 - \mathbf{v}_1],
\]
where
\[
A := (e^{-i\gamma} - e^{-i\theta}) + i S_1(\mathbf{v}_1) \sin(\theta) \\
= \cos(\gamma) - \cos(\theta) + i \left[ -\sin(\gamma) + \sin(\theta)(1 + S_1(\mathbf{v}_1)) \right],
\]
\[
B := i (e^{-i\gamma} - e^{-i\theta}) + i (e^{-i\gamma} S_0(\mathbf{v}_0) - S_1(\mathbf{v}_1) \cos(\theta)) \\
= \sin(\gamma) (1 + S_0(\mathbf{v}_0)) - \sin(\theta) \\
+ i \left( \cos(\gamma) S_0(\mathbf{v}_0) - S_1(\mathbf{v}_1) \cos(\theta) + \cos(\gamma) - \cos(\theta) \right).
\]
Regarding the two summands on the last line of (31), we have
\[
A \mathbf{q}_0 + B \mathbf{p}_0 \in R(\mathbf{q}_0, \mathbf{q}_1) \perp_h, \quad i [e^{-i\gamma} \mathbf{v}_0 - \mathbf{v}_1] \in R(\mathbf{q}_0, \mathbf{q}_1) \perp_h,
\]
where \(\perp_h\) denotes the Hermitian orthocomplement. Hence
\[
\|\Phi(\gamma, \mathbf{p}_0', \mathbf{p}_1')\|^2 \geq \|e^{-i\gamma} \mathbf{v}_0 - \mathbf{v}_1\|^2 \\
\geq (1 - |\cos(\gamma)|) \left[ \|\mathbf{v}_0\|^2 + \|\mathbf{v}_1\|^2 \right].
\]
Since \(1 - |\cos(\gamma)|\) vanishes exactly to second order at \(\gamma = 0, \pi, 2\pi\), there exists \(D > 0\) such that for \(\gamma \in [0, 2\pi]\) we have
\[
1 - |\cos(\gamma)| \geq D^2 \min \left\{ \gamma^2, (\gamma - \pi)^2, (\gamma - 2\pi)^2 \right\}.
\]
Given this and (31), we conclude that, under the present hypothesis,
\[
\|\Phi(\gamma, \mathbf{p}_0', \mathbf{p}_1')\| \geq D \min \{ \gamma, |\gamma - \pi|, 2\pi - \gamma \} \max \{ \|\mathbf{v}\|, \|\mathbf{v}'\| \} \\
\geq CD \min \{ \gamma, |\gamma - \pi|, 2\pi - \gamma \} k^{\epsilon - 1/2}.
\]
Let us now pick \(\delta'\) with \(\epsilon > \delta' > \delta\), and assume
\[
\min \{ \gamma, |\gamma - \pi|, 2\pi - \gamma \} \geq k^{-\delta'}.
\]
Then
\[
\|\Phi(\gamma, \mathbf{p}, \mathbf{p}')\| \geq CD k^{(\epsilon - \delta') - 1/2}.
\]
This establishes (30) with $\epsilon_1 = \epsilon - \delta'$, in the case where (36) holds. Thus we are reduced to assuming

$$\min \{ \gamma, |\gamma - \pi|, 2\pi - \gamma \} \leq k^{-\delta'}.$$  \hspace{1cm} (38)

Then we also have $|\sin(\gamma)| \leq k^{-\delta'}$. Let us then look at the first summand on the last line of (31). We have an Hermitian orthogonal direct sum

$$ R(q_0, q_1)_C = R(q_0, p_0)_C = \text{span}_C(q_0) \oplus \text{span}_C(p_0). $$

On the other hand, since $\sin(\vartheta)$ vanishes exactly to first order at $\vartheta = 0$ and $\vartheta = \pi$, there exists $E > 0$ such that for $\vartheta \in (0, \pi)$ under the assumptions of the Lemma we have

$$ \sin(\vartheta) \geq E \min \{ \vartheta, \pi - \vartheta \} \geq EC k^{-\delta}. $$

Hence, in view of (33), we have for some $D_1 > 0$ and $k \gg 0$

$$ \| \Phi(\gamma, p, p') \| \geq |A q_0 + B p_0| \geq |B| \geq |\Re(B)| $$

$$ = |\sin(\gamma) (1 + S_0(v_0)) - \sin(\vartheta)| \geq |\sin(\vartheta)| - k^{-\delta'} $$

$$ \geq EC k^{-\delta} - k^{-\delta'} \geq \frac{1}{2} EC k^{-\delta} \geq \frac{1}{2} EC k^{-1/6}, \hspace{1cm} (39) $$

since $\delta' > \delta$ and $\delta < 1/6$. This establishes (30) with $\epsilon_1 = 1/3$ when (38) holds.

The proof of Proposition 3.1 is complete. \hfill $\square$

Equations (26) and (27) parametrize neighborhoods of $p_0$ and $-p_0$, respectively. Therefore, Proposition 3.1 implies that in (22) only a negligible contribution to the asymptotics is lost, if integration in $p$ and $p'$ is restricted to shrinking neighborhoods of $\pm p_0$ and $\pm p_1$, of radii $O(k^{-1/2})$.

This may be rephrased as follows. Let $\varrho \in C_0^\infty(\mathbb{R}^{n+1})$ be even, supported in a small neighborhood of the origin, and identically equal to one in a smaller neighborhood of the origin. Then the asymptotics of (22) are unchanged, if the integrand is multiplied by

$$ \left[ \varrho \left( k^{1/2-\epsilon} (p - p_0) \right) + \varrho \left( k^{1/2-\epsilon} (p + p_0) \right) \right] \left[ \varrho \left( k^{1/2-\epsilon} (p' - p_1) \right) + \varrho \left( k^{1/2-\epsilon} (p' + p_1) \right) \right]. $$

(40)

In this way the integrand splits into four summands. In fact, only two of these are non-negligible for $k \to +\infty$. Namely, consider the summand containing the factor

$$ \varrho \left( k^{1/2-\epsilon} (p - p_0) \right) \varrho \left( k^{1/2-\epsilon} (p' + p_1) \right). $$

(41)
On its support, \( p \) lies in a shrinking neighborhood of \( p_0 \), and \( p' \) in a shrinking neighborhood of \(-p_1\). Therefore, on the same support \( q_0 + i \ p \) lies in a shrinking neighborhood of \( q_0 + i \ p_0 \), and \( q_1 - i \ p' \) lies in a shrinking neighborhood of \( q_1 - i \ p_1 \). Since

\[
\frac{1}{\sqrt{2}} (q_0 + i \ p_0) \wedge \frac{1}{\sqrt{2}} (q_1 - i \ p_1) = \frac{1}{2} (q_0 + i \ p_0) \wedge e^{i\theta} (q_0 - i \ p_0) = -i q_0 \wedge p_0
\]

has unit norm, on the support of \( [q_0 + i \ p] \) and \([q_1 + i \ p']\) remain at a distance \( \geq 2/3 \), say, in projective space. This implies that as \( k \to +\infty \)

\[
\Pi_{\sqrt{2},k}(q_0 + i \ p, q_1 + i \ p') = O(k^{-\infty})
\]

uniformly in \((p, p')\) in the support of \( [q_0 + i \ p] \). A similar argument applies to the summand containing the factor

\[
\varrho \left( k^{1/2-\epsilon} (p + p_0) \right) \varrho \left( k^{1/2-\epsilon} (p' - p_1) \right).
\]

Thus we may rewrite \((22)\) as follows:

\[
P_{k,n}(q_0, q_1) \sim P_{k,n}(q_0, q_1)_+ + P_{k,n}(q_0, q_1)_-,
\]

where

\[
P_{k,n}(q_0, q_1)_\pm := \frac{1}{C_{k,n}^2} \int_{S(q_0^\perp)} \int_{S(q_1^\perp)} \varrho \left( k^{1/2-\epsilon} (p \mp p_0) \right) \varrho \left( k^{1/2-\epsilon} (p' \mp p_1) \right) \\
\cdot \Pi_{\sqrt{2},k}(q_0 + i \ p, q_1 + i \ p') \, dV_{S(q_0^\perp)}(p) \, dV_{S(q_1^\perp)}(p').
\]

As a further reduction, we need only deal with one of \( P_{k,n}(q_0, q_1)_\pm \).

**Lemma 3.3.** \( P_{k,n}(q_0, q_1)_\pm = \overline{P_k(q_0, q_1)_\mp} \).

**Proof of Lemma 3.3.** Let us apply the change of integration variable \( p \mapsto -p \) and \( p' \mapsto -p' \), and apply \((23)\). Since \( \varrho \) is even, we get

\[
P_{k,n}(q_0, q_1)_- = \frac{1}{C_{k,n}^2} \int_{S(q_0^\perp)} \int_{S(q_1^\perp)} \varrho \left( k^{1/2-\epsilon} (p - p_0) \right) \varrho \left( k^{1/2-\epsilon} (p' - p_1) \right) \\
\cdot \Pi_{\sqrt{2},k}(q_0 + i \ p, q_1 + i \ p') \, dV_{S(q_0^\perp)}(p) \, dV_{S(q_1^\perp)}(p') = \overline{P_k(q_0, q_1)_+}.
\]

\(\square\)
Lemma 3.3 and (43) imply
\[ P_{k,n}(q_0, q_1) \sim 2 \Re(P_k(q_0, q_1^+) + P_k(q_0^+, q_1)). \] (45)

In the definition of \( P_k(q_0, q_1^+) \), integration is over a shrinking neighborhood of \((p_0, p_1) \in S(q_0^+) \times S(q_1^+)\). We can thus make use of the parametrization (26), and write in (44):
\[ p = p_0 + A(v_0), \quad p' = p_1 + A(v_1), \]
where we have set \( A(v_j) =: v_j + S_+\). It is also harmless to replace \( p - p_j \) by \( v_j \) in the rescaled cut-offs in (44). Let us also set \( z_j = q_j + i p_j \), and recall that \( z_1 = e^{-i\vartheta} z_0 \). We then obtain
\[ P_{k,n}(q_0, q_1^+) = \frac{1}{C_{k,n}} \int_{q_0^+} \int_{q_1^+} \varphi(k^{1/2} \nu_0) \varphi(k^{1/2} \nu_1) \cdot \Pi_{\sqrt{k}, k}(z_0 + i A_0(v_0), z_1 + i A_1(v_1)) \cdot \mathcal{V}(v_0, v_1) d\nu_0 d\nu_1, \]
where \( \mathcal{V}(0, 0) = 1 \).

Let us pass to rescaled integration variables \( v_j \mapsto v_j/\sqrt{k} \) in (46). Then
\[ P_{k,n}(q_0, q_1^+) = \frac{k^{1-n}}{C_{k,n}} \int_{q_0^+} \int_{q_1^+} \varphi(k^{-\epsilon} \nu_0) \varphi(k^{-\epsilon} \nu_1) \cdot \Pi_{\sqrt{k}, k}(z_0 + i A_0(v_0), z_1 + i A_1(v_1)) \cdot \mathcal{V}(v_0, v_1) d\nu_0 d\nu_1, \]
with
\[ A_{jk}(v) := v + \sqrt{k} \cdot S_+\left(\frac{v}{\sqrt{k}}\right) p_j. \] (48)

Let us consider the Szegő term in the integrand. In view of (15), this is
\[ \Pi_{\sqrt{k}, k} \left( z_0 + \frac{i}{\sqrt{k}} A_0(v_0), z_1 + \frac{i}{\sqrt{k}} A_1(v_1) \right) = \frac{\sqrt{2}}{2^n} e^{ik\vartheta} \Pi_{L, k} \left( \frac{z_0}{\sqrt{2}} + \frac{1}{\sqrt{k}} \frac{i A_0(v_0)}{\sqrt{2}}, \frac{z_0}{\sqrt{2}} + \frac{1}{\sqrt{k}} \frac{i e^{i\vartheta} A_1(v_1)}{\sqrt{2}} \right) \]
Now the sums in the previous expression are just algebraic sums in \( \mathbb{C}^{n+1} \); in order to apply the scaling asymptotics of Theorem 2.2, we need to first express the argument of (49) in terms of local Heisenberg coordinates on \( X_1 \) centered at \( z_0/\sqrt{2} \).
Lemma 3.4. Suppose $z = q + ip \in X_1$ and choose a system of HLC’s on $X_1$ centered at $z$. Then for $\delta p \sim 0 \in \mathbb{R}^{n+1}$ and $e^{i\theta} \in S^1$ such that $z + e^{i\theta} \delta p \in X_1$ we have

$$z + i e^{i\theta} \delta p = z + X_1 (0, i e^{i\theta} \delta p + R_2(\theta; \delta p)),$$

for a suitable smooth function $R_2(\theta; \cdot)$ vanishing to second order at the origin (in $v$).

Proof of Lemma 3.4. In view of (17), it suffices to prove the statement on $S^{2n+1}$, working with the HLC’s (16). Since $z, z + ie^{i\theta} \delta p \in C^{n}$, we have

$$0 = z^t z + 2i e^{i\theta} z^t \delta p - e^{2i\theta} \delta p^t \delta p = 2i e^{i\theta} z^t \delta p - e^{2i\theta} \|\delta p\|^2,$$

so that $i z^t \delta p = e^{i\theta} \|\delta p\|^2/2$. (50)

Let us look for $\beta > 0$ and $h \in z^\perp$ (Hermitian orthocomplement) such that

$$z + ie^{i\theta} \delta p = \beta (z + h).$$

(51)

If this is possible at all, then necessarily $\beta = 1/\|z + h\|$, as $\|z + ie^{i\theta} \delta p\| = 1$. Then

$$z + ie^{i\theta} \delta p = z + S^{2n+1}_{e^{i\theta}} (0, h).$$

(52)

Assuming that (51) may be solved, then, taking the Hermitian product with $z$ on both sides of (51) and using (50) we get

$$\beta = z^t (z - ie^{-i\theta} \delta p) = 1 - ie^{-i\theta} z^t \delta p = 1 - 1/2 \|\delta p\|^2 > 0.$$ (53)

With this value of $\beta$, let us set

$$h =: \frac{1}{\beta} (z + ie^{i\theta} \delta p) - z,$$

(54)

so that (51) is certainly satisfied. We need to verify that $h \in z^\perp$. Indeed we have

$$h^t z = \frac{1}{\beta} (1 + ie^{i\theta} \delta p^t z) - 1 = \frac{1}{\beta} \left(1 - \frac{1}{2} \|\delta p\|^2\right) - 1 = 0.$$ (55)

Since $h = ie^{i\theta} \delta p + R_2(\delta p)$, the proof of the Lemma is complete. □

Notice that $h$ is given for $\delta p \sim 0$ by an asymptotic expansion in homogeneous polynomials of increasing degree in $\delta p$ of the form

$$h \sim ie^{i\theta} \delta p + \frac{1}{2} \|\delta p\|^2 z + \frac{i}{2} e^{i\theta} \|\delta p\|^2 \delta p + \frac{1}{4} \|\delta p\|^4 z + \cdots$$ (55)
This holds on $S_1^{2n+1}$, but a similar expansion obviously holds on $X_1$, possibly with modified terms in higher degree.

Let us apply Lemma 3.4 with $z = z_0/\sqrt{2}$ and $\delta p_j = e^{i\theta} (A_{jk}(v_j)/\sqrt{2})/\sqrt{k}$ (we’ll set $\theta = 0$ for $j = 0$ and $\theta = \vartheta$ for $j = 1$). To this end, let us note that in view of (48) for $k \to +\infty$ there is an asymptotic expansion of the form

$$A_{jk}(v) \sim \sum_{j \geq 0} \frac{1}{k^{l/2}} P_{j,l+1}(v),$$

(56)

where $P_{j,l}$ is a homogeneous (vector valued) polynomial function of degree $l$, and $P_{j,1}(v) = v$. Hence

$$\delta p_j = \frac{e^{i\theta}}{\sqrt{2k}} A_{jk}(v_j) \sim \frac{e^{i\theta}}{\sqrt{2}} \sum_{j \geq 0} \frac{1}{k^{(l+1)/2}} P_{j,l+1}(v_j) = \frac{e^{i\theta}}{\sqrt{k}} \frac{v_j}{\sqrt{2}} + \cdots$$

(57)

Making use of (57) in (55) we obtain

$$h_{kj} \sim \sum_{l \geq 1} \frac{1}{k^{l/2}} Q_{jl}(\theta; v) = \frac{1}{\sqrt{k}} \left( i e^{i\theta} \frac{v_j}{\sqrt{2}} + \sum_{l \geq 1} \frac{1}{k^{l/2}} Q_{j,l+1}(\theta; v_j) \right)$$

(58)

where $Q_{jl}(\theta; \cdot)$ is a homogeneous polynomial function of degree $l$, and we have emphasized the dependence on $k$.

Thus we obtain for $j = 0$ (with $\theta = 0$) that

$$\frac{z_0}{\sqrt{2}} + \frac{1}{\sqrt{k}} \frac{i A_{0k}(v_0)}{\sqrt{2}} = \frac{z_0}{\sqrt{2}} + x_1(0, h_{k0}),$$

(59)

where

$$h_{k0} \sim \frac{1}{\sqrt{k}} \left( i \frac{v_0}{\sqrt{2}} + \sum_{l \geq 1} \frac{1}{k^{l/2}} Q_{0,l+1}(0; v_0) \right) = \frac{1}{\sqrt{k}} a_{k0},$$

(60)

with $a_{k0}$ defined by the latter equality. Similarly, for $j = 1$ (with $\theta = \vartheta$) we have

$$\frac{z_0}{\sqrt{2}} + \frac{1}{\sqrt{k}} \frac{i e^{i\theta} A_{1k}(v_1)}{\sqrt{2}} = \frac{z_0}{\sqrt{2}} + x_1(0, h_{k1}),$$

where

$$h_{k1} \sim \frac{1}{\sqrt{k}} \left( i e^{i\theta} \frac{v_1}{\sqrt{2}} + \sum_{l \geq 1} \frac{1}{k^{l/2}} Q_{1,l+1}(\vartheta; v_1) \right) = \frac{1}{\sqrt{k}} a_{k1}.$$
Let us return to (49). In view of Theorem 2.2 we get

\[
\Pi_{1,k} \left( \frac{z_0}{\sqrt{2}} + \frac{1}{\sqrt{k}} \frac{i A_{0k}(v_0)}{\sqrt{2}}, \frac{z_0}{\sqrt{2}} + \frac{1}{\sqrt{k}} \frac{i e^{i\theta} A_{1k}(v_1)}{\sqrt{2}} \right) = \Pi_{1,k} \left( \frac{z_0}{\sqrt{2}} + x_1 \frac{1}{\sqrt{k}} a_{k0}, \frac{z_0}{\sqrt{2}} + x_1 \frac{1}{\sqrt{k}} a_{k1} \right) \sim \left( \frac{k}{\pi} \right)^{n-1} e^{\psi_2(a_{k0},a_{k1})} \left[ 1 + \sum_{b=1}^{+\infty} k^{-b/2} P_b(a_{k0},a_{k1}) \right].
\]

We have

\[
\psi_2(a_{k0},a_{k1}) \sim \frac{1}{2} \psi_2(v_0,e^{i\theta} v_1) + \sum_{l \geq 1} \frac{1}{k^{l/2}} \tilde{Q}_{l+2}(\theta;v_0,v_1),
\]

where \( \tilde{Q}_l(\theta;\cdot,\cdot) \) is a homogeneous \( \mathbb{C} \)-valued polynomial of degree \( l \). For any \( r \geq 1 \) and \( l_1, \ldots, l_r \geq 1 \), we have

\[
\prod_{j=1}^{r} \frac{1}{k^{l_j/2}} \tilde{Q}_{l_j+2}(\theta;v_0,v_1) = \frac{1}{k^{\sum_{j=1}^{r} l_j/2}} \tilde{Q}_{\sum_{j=1}^{r} l_j+2r}(\theta;v_0,v_1),
\]

where \( \tilde{Q}_{\sum_{j=1}^{r} l_j+2r}(\theta;\cdot,\cdot,\cdot) \) is homogeneous of degree \( \sum_{j=1}^{r} l_j + 2r \). Since \( l_j \geq 1 \) for every \( j \), we have \( \sum_{j=1}^{r} l_j + 2r \leq 3 \sum_{j=1}^{r} l_j \).

One can see from this that

\[
e^{\psi_2(a_{k0},a_{k1})} \sim e^{\frac{1}{2} \psi_2(v_0,e^{i\theta} v_1)} \left[ 1 + \sum_{l \geq 1} \frac{1}{k^{l/2}} B_l(\theta;v_0,v_1) \right],
\]

where \( B_l(\theta;\cdot,\cdot,\cdot) \) is a polynomial of degree \( \leq 3l \), and having the same parity as \( l \).

Similarly, recalling that \( P_b \) has the same parity as \( b \) and degree \( \leq 3b \), each summand \( k^{-b/2} P_b(a_{k0},a_{k1}) \) in (61) gives rise to an asymptotic expansion in terms of the form

\[
\frac{1}{k^{b/2}} \prod_{a=1}^{r} \frac{1}{k^{l_a/2}} R_{l_a+1}(\theta;v_0,v_1) = \frac{1}{k^{(b+\sum_{a=1}^{r} l_a)/2}} \tilde{R}_{\sum_{a=1}^{r} l_a+r}(\theta;v_0,v_1),
\]

where \( R_l(\theta;\cdot,\cdot,\cdot) \) and \( \tilde{R}_l(\theta;\cdot,\cdot,\cdot) \) are homogeneous polynomials of the given degree, \( r \leq 3b \), and \( b - r \) is even. Then \( 3 \left( b + \sum_{a=1}^{r} l_a \right) \geq \sum_{a=1}^{r} l_a + r \), and \( b + \sum_{a=1}^{r} l_a \) is also even. Hence each summand \( k^{-b/2} P_b(a_{k0},a_{k1}) \) (\( b \geq 1 \)) yields an asymptotic expansion of the form

\[
\sum_{l \geq 1} k^{-l/2} T_{bl}(\theta;v_0,v_1),
\]
where again each $T_l$ has the same parity as $l$ and degree $\leq 3l$.

Putting this all together, we obtain an asymptotic expansion for the integrand in (47):

**Lemma 3.5.** For $l \geq 0$, there exist polynomials $Z_l(\vartheta; \cdot, \cdot)$ of degree $\leq 3l$ and parity $(-1)^l$, with $Z_0(\vartheta; \cdot, \cdot) = 1$, such that

$$
\prod_{k \mid z_0 + \frac{i}{k} A_0(k, v_0), z_1 + \frac{i}{k} A_1(k, v_1)} \sqrt{2^{2^{n-1}}} e^{ik\vartheta} \left(\frac{k}{\pi}\right)^{n-1} e^{\frac{i}{2} \psi_2(v_0, e^{i\vartheta} v_1)} \sum_{l \geq 0} \frac{1}{k^{3l/2}} Z_l(\vartheta; v_0, v_1).
$$

**Proof of Lemma 3.5.** The previous arguments yield an asymptotic expansion of the given form for the first factor. We need only multiply the latter expansion by the Taylor expansion of the second factor.

Since integration in (47) takes place over a poly-ball or radius $O(k^\epsilon)$ in $(q_0^\perp \cap q_1^\perp)^2$, the expansion may be integrated term by term. In addition, given that the exponent and the cut-offs are even functions of $(v_0, v_1)$, only terms of even parity yield a non-zero integral. Hence we may discard the half-integer powers and obtain

$$
P_k(q_0, q_1)_+ \sim \frac{k^{1-n}}{C_{k,n}^{2^n}} \sqrt{2^{2^{n-1}}} e^{ik\vartheta} \left(\frac{k}{\pi}\right)^{n-1} \sum_{l \geq 0} k^{-l} \hat{P}_l(\vartheta)_+, \quad (64)
$$

where

$$
\hat{P}_l(\vartheta)_+ := \int_{q_0^\perp \cap q_1^\perp} \int_{q_0^\perp \cap q_1^\perp} \varrho(k^{-\epsilon} v_0) \varrho(k^{-\epsilon} v_1) \cdot e^{\frac{i}{2} \psi_2(v_0, e^{i\vartheta} v_1)} Z_{2l}(\vartheta; v_0, v_1) \, dv_0 \, dv_1. \quad (65)
$$

We can slightly simplify the previous asymptotic expansion, as follows. First, as emphasized the dependence on $(q_0, q_1)$ is of course only through the angle $\vartheta$. In particular, in (65) nothing is lost by assuming that $q_0$ and $q_1$ span the 2-plane $\{0\} \times \mathbb{R}^2 \subseteq \mathbb{R}^{n+1}$, and therefore that $q_0^\perp \cap q_1^\perp = \mathbb{R}^{n-1} \times \{0\}$.

Furthermore, given (18), we have

$$
\psi_2(v_0, e^{i\vartheta} v_1) = -i \sin(\vartheta) v_0^t v_1 - \frac{1}{2} \|v_0 - \cos(\vartheta) v_1\|^2 - \frac{1}{2} \sin(\vartheta)^2 \|v_1\|^2. \quad (66)
$$

With the change of variables

$$
\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} b_0 + \cot(\vartheta) b_1 \\ (1/\sin(\vartheta)) b_1 \end{bmatrix}
$$

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we obtain
\[ \psi_2(\mathbf{v}_0, e^{i\vartheta} \mathbf{v}_1) = -\frac{1}{2} \| \mathbf{b}_0 \|^2 - i \mathbf{b}_0^\dagger \mathbf{b}_1 - \frac{1}{2} \left( 1 + 2i \cot(\vartheta) \right) \| \mathbf{b}_1 \|^2. \] (67)

Since \( Z_{2l}(\vartheta, \cdot, \cdot) \) is even and has degree \( \leq 6l \), we can write
\[ Z_{2l}(\vartheta; \mathbf{b}_0 + \cot(\vartheta) \mathbf{b}_1, \frac{\mathbf{b}_1}{\sin(\vartheta)}) = \frac{1}{\sin(\vartheta)^{6l}} T_l(\vartheta; \mathbf{b}_0, \mathbf{b}_1), \]
where \( T_l(\vartheta, \cdot, \cdot) \) is an even polynomial of degree \( \leq 6l \), with smooth bounded coefficients for \( \vartheta \in [0, \pi] \). Thus
\[
\tilde{P}_l(\vartheta) + \\
= \left( \frac{2}{\sin(\vartheta)} \right)^{n-1} \frac{1}{\sin(\vartheta)^{6l}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2} \| \mathbf{b}_0 \|^2 - i \mathbf{b}_0^\dagger \mathbf{b}_1 - \frac{1}{2} \left( 1 + 2i \cot(\vartheta) \right) \| \mathbf{b}_1 \|^2} \phi \left( k^{1-\epsilon} \sqrt{2} (\mathbf{b}_0 + \cot(\vartheta) \mathbf{b}_1) \right) \phi \left( k^{1-\epsilon} \sqrt{2} \sin(\vartheta)^{-1} \mathbf{b}_1 \right) T_l(\vartheta; \mathbf{b}_0, \mathbf{b}_1) d\mathbf{b}_0 d\mathbf{b}_1. \] (68)

There is a constant \( C > 0 \) such that the support of
\[ 1 - \phi \left( k^{1-\epsilon} \sqrt{2} (\mathbf{b}_0 + \cot(\vartheta) \mathbf{b}_1) \right) \phi \left( k^{1-\epsilon} \sqrt{2} \sin(\vartheta)^{-1} \mathbf{b}_1 \right) \]
is contained in the locus where \( \| (\mathbf{b}_0, \mathbf{b}_1) \| \geq C k^{\epsilon} \sin(\vartheta) \). Under the assumptions of the Theorem, this implies, perhaps for a different constant \( C > 0 \), that \( \| (\mathbf{b}_0, \mathbf{b}_1) \| \geq C k^{\epsilon-\delta} \). On the other hand, the exponent in (68) satisfies
\[
\left| -\frac{1}{2} \| \mathbf{b}_0 \|^2 - i \mathbf{b}_0^\dagger \mathbf{b}_1 - \frac{1}{2} \left( 1 + 2i \cot(\vartheta) \right) \| \mathbf{b}_1 \|^2 \right| \leq -\frac{1}{2} \left( \| \mathbf{b}_0 \|^2 + \| \mathbf{b}_1 \|^2 \right). \]

Given that \( \epsilon > \delta \) (statement of Proposition 3.1), we conclude that only a negligible contribution to the asymptotics is lost, if the cut-off function is omitted and integration is now extended to all of \( \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \).

We can thus rewrite (68) as follows:
\[ \mathcal{P}_k(\mathbf{q}_0, \mathbf{q}_1) + \\
\sim \frac{1}{C_{k,n}^2} \frac{\sqrt{2}}{2} e^{i k \vartheta} \left( \frac{1}{\pi \sin(\vartheta)} \right)^{n-1} \sum_{l \geq 0} k^{-l} \frac{1}{\sin(\vartheta)^{6l}} \tilde{P}_l(\vartheta) +, \] (69)
where
\[
\tilde{P}_l(\vartheta) + \\
= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2} \| \mathbf{b}_0 \|^2 - i \mathbf{b}_0^\dagger \mathbf{b}_1 - \frac{1}{2} \left( 1 + 2i \cot(\vartheta) \right) \| \mathbf{b}_1 \|^2} T_l(\vartheta; \mathbf{b}_0, \mathbf{b}_1) d\mathbf{b}_0 d\mathbf{b}_1.
\]
Let us set \( B_\vartheta = (1 + i \cot(\vartheta)) I_{n-1} \). The leading order coefficient is

\[
\tilde{P}_0(\vartheta)_+ = \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2} \left( 1 + 2i \cot(\vartheta) \right) \|b_1\|^2} \left[ \int_{\mathbb{R}^{n-1}} e^{-i b_0^* b_1 - \frac{1}{2} \|b_0\|^2} db_0 \right] db_1
= (2\pi)^{(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2} \left( 2 + 2i \cot(\vartheta) \right) \|b_1\|^2} db_1
= 2^{(n-1)/2} \pi^{n-1} \frac{1}{\sqrt{\det(B_\vartheta)}}
= \left( \sqrt{2\pi} \right)^{n-1} \sin(\vartheta)^{(n-1)/2} e^{i \left( \frac{\vartheta}{2} - \frac{\pi}{2} \right) (n-1)}.
\]

Given (70), (69) and (45), \( \mathcal{P}_k(q_0, q_1) \) has an asymptotic expansion for \( k \to +\infty \) with leading order term

\[
\frac{2^{n/2}}{C_{k,n}^2} \frac{1}{\sin(\vartheta)^{(n-1)/2}} \cos \left( k \vartheta + \left( \frac{\vartheta}{2} - \frac{\pi}{2} \right) (n-1) \right).
\]

For any \( l \), we can write

\[
\tilde{P}_l(\vartheta)_+ = \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2} \left( 1 + 2i \cot(\vartheta) \right) \|b_1\|^2} \left[ \int_{\mathbb{R}^{n-1}} e^{-i b_0^* b_1 - \frac{1}{2} \|b_0\|^2} T_l(\vartheta; b_0, b_1) db_0 \right] db_1
= \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2} \left( 2 + 2i \cot(\vartheta) \right) \|b_1\|^2} T_l(\vartheta; b_1) db_1,
\]

where \( T_l(\vartheta; \cdot) \) is an even polynomial of degree \( \leq 6l \).

Let us introduce the Fourier transform

\[
\mathcal{F}(c) = \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2} \left( 2 + 2i \cot(\vartheta) \right) \|b_1\|^2 - i b_1^* c} db_1
= (2\pi)^{(n-1)/2} \sin(\vartheta)^{(n-1)/2} e^{i \left( \frac{\vartheta}{4} - \frac{\pi}{4} \right) (n-1)} e^{-\frac{1}{2} \left( 2 + 2i \cot(\vartheta) \right) \|c\|^2}.
\]

Then (72) is the result of applying an even differential polynomial \( P_l(D_c) \) of degree \( \leq 6l \) to \( \mathcal{F}(c) \), and then evaluating the result at \( c = 0 \).

Given this and (71), we conclude that

\[
\mathcal{P}_k(q_0, q_1) = \frac{2^\frac{n}{2}}{C_{k,n}^2} \left( \frac{1}{\sin(\vartheta)} \right)^{(n-1)/2}
\cdot \left[ \cos \left( k \vartheta + \left( \frac{\vartheta}{2} - \frac{\pi}{4} \right) (n-1) \right) \cdot A(\vartheta) + \sin \left( k \vartheta + \left( \frac{\vartheta}{2} - \frac{\pi}{4} \right) (n-1) \right) \cdot B(\vartheta) \right],
\]

(74)
where
\[ A(\vartheta) \sim 1 + \sum_{l=1}^{+\infty} k^{-l} \frac{A_l(\vartheta)}{\sin(\vartheta)^l}, \quad B(\vartheta) \sim \sum_{l=1}^{+\infty} k^{-l} \frac{B_l(\vartheta)}{\sin(\vartheta)^l}, \]

with \(A_l\) and \(B_l\) smooth functions of \(\vartheta\) on \([0, 2\pi]\).

4 Proof of Proposition 1.1

Proof of Proposition 1.1. The diagonal restriction \(P_{k,n}(q,q)\) may be computed in two different ways. On the one hand, since \(P_{k,n}(q,q)\) is constant we have
\[
P_{k,n}(q,q) = N_{k,n} \frac{\text{vol}(S^n)}{\text{vol}(S^n)} = 2 \frac{k^{n-1}}{(n-1)!} + O\left(k^{n-2}\right). \tag{75}\]

On the other hand, (22) with \(q_0 = q_1 = q\) yields
\[
P_{k,n}(q,q) = \frac{1}{C_{k,n}^2} \int_{S(q^+)} \int_{S(q^+)} \Pi_{\sqrt{2} k}(q + i p, q + i p') \ dV_{S(q^+)}(p) \ dV_{S(q^+)}(p') \tag{76}\]
where \(F_k(q,p) = \int_{S(q^+)} \Pi_{\sqrt{2} k}(q + i p, q + i p') \ dV_{S(q^+)}(p')\). \tag{77}\]

Again, integration in \(dV_{S(q^+)}(p')\) localizes in a shrinking neighborhood of \(p\). Hence we may let
\[
p' = p + A(v), \quad A(v) = v + S_+(v) p, \]
where \(v \in q^+ \cap p^\perp\), and introduce the cut-off \(\varrho\left(k^{1/2-\varepsilon} v\right)\). Passing to rescaled coordinates, and setting \(z = q + i p\), we get
\[
F_k(q,p) = \frac{1}{k^{(n-1)/2}} \int_{q^+ \cap p^\perp} \varrho\left(k^{-\varepsilon} v\right) \Pi_{\sqrt{2} k} \left(z, z + \frac{i}{\sqrt{k}} A_k(v)\right) \mathcal{V}\left(\frac{v}{\sqrt{k}}\right) \ dv. \tag{78}\]

where
\[
A_k(v) = v + \sqrt{k} S_+ \left(\frac{v}{\sqrt{k}}\right) p, \quad \mathcal{V}(0) = 1.
\]
By Lemma 3.5 (with \( z = z_0 = z_1, \ v_0 = 0, \ v_1 = v, \ \vartheta = 0 \)), we have

\[
\Pi_{\sqrt{k}, k} \left( z, z + \frac{i}{\sqrt{k}} A_k(v) \right) \cdot V \left( \frac{v}{\sqrt{k}} \right) \sim \sqrt{\frac{2}{2^n}} \left( \frac{k}{\pi} \right)^{n-1} \sum_{l \geq 0} \frac{1}{k^{l/2}} Z_l(v),
\]

for certain polynomials \( Z_l \) of degree \( \leq 3l \) and parity \((-1)^l\), with \( Z_0(\cdot) = 1 \).

As before, the expansion may be integrated term by term and, by parity, only the summands with \( l \) even yield a non-zero contribution. In addition, only a negligible contribution is lost if the cut off is omitted and integration is extended to all of \( q^\perp \cap p^\perp \cong \mathbb{R}^{n-1} \). Therefore

\[
F_k(q, p) \sim \frac{\sqrt{2}}{2^n} \frac{k^{(n-1)/2}}{\pi^{n-1}} \sum_{l \geq 0} k^{-l} \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{4} \|v\|^2} Z_{2l}(v) \, dv
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{k}{\pi} \right)^{(n-1)/2} + \cdots \quad (79)
\]

Inserting this in (76), we obtain an asymptotic expansion

\[
\mathcal{P}_{k, n}(q, q) \sim \frac{\text{vol}(S^{n-1})}{C_{k, n}^2} \frac{1}{\sqrt{2}} \left( \frac{k}{\pi} \right)^{(n-1)/2} + \cdots \quad (80)
\]

Comparing (75) and (80), we obtain an asymptotic expansion in descending powers of \( k \), of the form

\[
C_{k, n} \sim \left[ \frac{\text{vol}(S^n) \text{vol}(S^{n-1})}{2 \sqrt{2}} \cdot (n - 1)! \right]^{1/2} (\pi k)^{-(n-1)/4} + \cdots
\]

\[
\square
\]

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