A NEW APPROACH TO POSTERIOR CONTRACTION RATES VIA WASSERSTEIN DYNAMICS

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ABSTRACT. This paper presents a new approach to the classical problem of quantifying posterior contraction rates (PCRs) in Bayesian statistics. Our approach relies on Wasserstein distance, and it leads to two main contributions which improve on the existing literature of PCRs. The first contribution exploits the dynamic formulation of Wasserstein distance, for short referred to as Wasserstein dynamics, in order to establish PCRs under dominated Bayesian statistical models. As a novelty with respect to existing approaches to PCRs, Wasserstein dynamics allows us to circumvent the use of sieves in both stating and proving PCRs, and it sets forth a natural connection between PCRs and three well-known classical problems in statistics and probability theory: the speed of mean Glivenko-Cantelli convergence, the estimation of weighted Poincaré-Wirtinger constants and Sanov large deviation principle for Wasserstein distance. The second contribution combines the use of Wasserstein distance with a suitable sieve construction to establish PCRs under full Bayesian nonparametric models. As a novelty with respect to existing literature of PCRs, our second result provides with the first treatment of PCRs under non-dominated Bayesian models. Applications of our results are presented for some classical Bayesian statistical models, e.g., regular parametric models, infinite-dimensional exponential families, linear regression in infinite dimension and nonparametric models under Dirichlet process priors.

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1. Introduction

This paper presents a new approach to the classical problem of quantifying posterior contraction rates (PCRs) in Bayesian statistics. Although the problem of PCRs in finite-dimensional models were studied earlier by various authors (Ibragimov and Has'minskiǐ [34], LeCam [38]), it was only with the seminal works of Ghosal et al. [29] and Shen and Wasserman [47] that the problem was solved within a systematic framework, capable to include both finite-dimensional and infinite-dimensional Bayesian statistical models. Since then, different approaches have been proposed and investigated in order to obtain explicit PCRs. Among them, here we recall approaches based on: metric entropy, in combination with the definition of specific tests (Schwartz [45], Ghosal et al. [29]); bracketing numbers and entropy integrals (Shen and Wasserman [47]); martingales (Walker [53], Walker et al. [54]); Hausdorff entropy (Xing [56]); Wasserstein distance (Chae [13]). At the ground of all these approaches there is the explicit construction of a sieve in the space of the parameters or, at least, the existence of a suitable sieve is implied. Furthermore, these approaches to PCRs rely on the critical assumption of an underlying dominated Bayesian model for the observations, that is, the posterior distribution is expressed directly via Bayes formula.

In this paper, we present two main contributions which improve on the existing literature of PCRs. At the basis of our contributions there is the use of Wasserstein distance. The first result of the present work relies on the dynamic formulation of Wasserstein distance, for short referred to as Wasserstein dynamics, discovered by Benamou and Brenier [7]. See also Chapter 8 of Ambrosio et al. [4] and references therein. In particular, we exploit Wasserstein dynamics to provide with a general strategy to evaluate PCRs under dominated Bayesian models for the observations. Wasserstein dynamics allows us to circumvent the use of sieves in both stating and proving PCRs and, critical to our study, it sets forth a natural connection between PCRs and three well-known classical problems in statistic and probability theory: the speed of mean Glivenko-Cantelli convergence (Ajtai et al. [1], Ambrosio et al. [6], Dobrić and Yukic [21], Dolera and Regazzini [24], Fournier and Guillin [27], Talagrand [48, 49], Bobkov and Ledoux [10], Weed and Bach [55], Jing [36]); the estimation of weighted Poincaré-Wirtinger constants (Chapter 4 of Bakry et al. [9], Chapter 15 of Heinonen et al. [33]); Sanov large deviation principle under Wasserstein distance (Bolley et al. [11], Jing [36]). Establishing such a novel connection is at the core of our first main result on PCRs.

As a complement to our first contribution for dominated Bayesian models, we then consider the problem of defining a general strategy to evaluate PCRs in a full Bayesian nonparametric setting. Bayesian nonparametric priors, such as the Dirichlet process prior of Ferguson [26]
and generalizations thereof (Ghosal and van der Vaart [30]), have already appeared in the literature of PCRs (Section 6 of Ghosal et al. [29]). However, their use has been limited to the role of nonparametric hyperpriors, and the actual underlying Bayesian model for the observations is dominated. That is, the posterior distribution can be written directly via Bayes formula. See also Ghosal and van der Vaart [30], and references therein, for a comprehensive account on PCRs and Bayesian nonparametric priors. In this paper, we consider posterior distributions which are not obtained via Bayes formula, but rather via a more general disintegration argument. In particular, the second result of the present work combines the use of Wasserstein distance with a suitable sieve construction to establish PCRs under non-dominated Bayesian models for the observations. To the best of our knowledge, this is the first comprehensive treatment of PCRs under non-dominated Bayesian models for the observations.

The paper is structured as follows. In Section 2 we recall the definition of Bayesian consistency and PCR, we present a reformulation of the definition of PCR in terms of Wasserstein distance, and we outline the key arguments of our approach to PCRs in connection with the speed of mean Glivenko-Cantelli convergence, the estimation of weighted Poincaré-Wirtinger constants and Sanov large deviation principle under Wasserstein distance. These preliminaries pave the way to our main contributions in Section 3. In particular, in Section 3 we recall the dynamic formulation of Wasserstein distance, and we state our main results: Theorem 1 provides with PCRs under dominated Bayesian models for the observations; Theorem 2 provides with PCRs under a full Bayesian nonparametric setting, and hence under non-dominated Bayesian models for the observations. In Section 4 we present applications of our results to classical Bayesian statistical models: regular parametric models, multinomial models, infinite-dimensional exponential families, linear regression in infinite dimension, hierarchical models based on the Dirichlet process priors, and nonparametric models under Dirichlet process priors. Section 5 contains a discussion of our results in connection with some lines of future work. Proofs of main results and corollaries are deferred to appendices.

2. Preliminaries, and a new approach to PCRs

For ease in exposition, we confine ourselves to treat observations that are modeled, from the Bayesian point of view, as a part of a sequence $X^{(\infty)} := \{X_i\}_{i \geq 1}$ of exchangeable random variables, each $X_i$’s taking values in a measurable space $(X, \mathcal{X})$. Let $(\Theta, \mathcal{T})$ be measurable space, referred to as the parameter space, let $\pi$ be a probability measure on $(\Theta, \mathcal{T})$, referred to as the prior measure, and let $\kappa(\cdot|\cdot) : \mathcal{X} \times \Theta \to [0, 1]$ be a probability kernel, referred to as the statistical model for the single observation. The Bayesian approach hinges on modeling the parameter as a $\Theta$-valued random variable, say $\tilde{\theta}$, with probability distribution $\pi$. Then, the core of Bayesian inferences is the posterior distribution, which is defined as the conditional distribution of $\tilde{\theta}$ given a random sample $(X_1, \ldots, X_n)$ of observations, that is $P[\tilde{\theta} \in \cdot | X_1, \ldots, X_n]$, whenever both $\tilde{\theta}$ and the sequence $X^{(\infty)}$ are supported on a common probability space $(\Omega, \mathcal{F}, P)$. The minimal regularity conditions, which will be maintained (or possibly strengthened) throughout the present paper, are the following: the set $X$ is thought of as a separable topological space, with $\mathcal{X}$ coinciding with the ensuing Borel $\sigma$-algebra, and $(\Theta, \mathcal{T})$ is a standard Borel space.
In the above Bayesian setting, the posterior can be represented by means of a probability kernel \( \pi_n(\cdot) : \mathcal{T} \times \mathbb{X}^n \rightarrow [0,1] \) satisfying the disintegration problem
\[
P[X_1 \in A_1, \ldots, X_n \in A_n; \hat{\theta} \in B] = \int_{A_1 \times \cdots \times A_n} \pi_n(B|x^{(n)}) \mu_n(dx^{(n)})
\]
for all \( A_1, \ldots, A_n \in \mathcal{T} \) and \( B \in \mathcal{T} \) and \( n \geq 1 \), where \( x^{(n)} := (x_1, \ldots, x_n) \) and
\[
\mu_n(A_1 \times \cdots \times A_n) := \int_{\Theta} \left[ \prod_{i=1}^n \kappa(A_i|\theta) \right] \pi(d\theta),
\]
so that \( P[\hat{\theta} \in B|X_1, \ldots, X_n] = \pi_n(B|X_1, \ldots, X_n) \) is valid P-a.s. for any \( B \in \mathcal{T} \). Another useful notion is that of predictive distribution, namely the conditional law \( P \kappa \) of Definition 1 in terms of the so-called Wasserstein distance. To recall this concept in full generality, given \( \Theta \) of Definition 1, we refer to Ghosal and van der Vaart [30].

We say that the posterior distribution is (weakly) consistent at \( \theta \) if
\[
P[\theta | X_1, \ldots, X_n] \rightarrow \pi(\theta)
\]
for all \( A_1, \ldots, A_n \in \mathcal{T} \) and \( B \in \mathcal{T} \) and \( n \geq 1 \), where \( x^{(n)} := (x_1, \ldots, x_n) \) and
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Now, we recall the notion of Bayesian consistency (see Definition 6.1 in Ghosal and van der Vaart [30]). We say that the posterior distribution is (weakly) consistent at \( \theta_0 \in \Theta \) if, as \( n \rightarrow +\infty \), \( \pi_n(U_0|\xi_1, \ldots, \xi_n) \rightarrow 0 \) holds in probability for any neighborhood \( U_0 \) of \( \theta_0 \), where \( \xi^{(\infty)} := \{\xi_i\}_{i \geq 1} \) stands for a sequence of \( \mathbb{X} \)-valued independent random variables identically distributed as \( \kappa_0(\cdot) := \kappa(\cdot|\theta_0) \). The non uniqueness of \( \pi_n \) requires some additional regularity assumptions in order that the random measure \( \pi_n(\cdot|\xi_1, \ldots, \xi_n) \) is well-defined. The notion of PCR, which is critical in the present work, strengthens the notion of Bayesian consistency. A PCR provides with a precise quantification of Bayesian consistency. In particular, after the specification of a distance \( d_\Theta \) on \( \Theta \), that yields \( \mathcal{T} \) as relative Borel \( \sigma \)-algebra, the definition of PCR can be stated as follows (see Definition 8.1 in Ghosal and van der Vaart [30]).

**Definition 1 (PCR).** A sequence \( \{\epsilon_n\}_{n \geq 1} \) of positive numbers is a PCR at \( \theta_0 \) if, as \( n \rightarrow +\infty \),
\[
\pi_n \left( \{\theta \in \Theta : d_\Theta(\theta, \theta_0) \geq M_n \epsilon_n \} \big| \xi_1, \ldots, \xi_n \right) \rightarrow 0
\]
holds in probability for every sequence \( \{M_n\}_{n \geq 1} \) of positive numbers such that \( M_n \rightarrow \infty \).

To summarize, a PCR quantifies the speed at which a \( d_\Theta \)-neighborhood of the true parameter \( \theta_0 \) is allowed to shrink while maintaining, nevertheless, very high posterior probability. We refer to Ghosal and van der Vaart [30], and references therein, for a detailed account on Bayesian consistency and PCRs.

### 2.1. A new approach to PCRs

Our approach to PCRs originates from a reformulation of Definition 1 in terms of the so-called \( p \)-Wasserstein distance. To recall this concept in full generality, given \( p \geq 1 \), we let \( (\mathbb{S}, d_\mathbb{S}) \) denote an abstract separable metric space, and we let \( \mathcal{P}(\mathbb{S}) \) denote the relative space of all probability measures on \( (\mathbb{S}, \mathcal{B}(\mathbb{S})) \). Then, the \( p \)-Wasserstein distance is defined as
\[
W_p(\mathcal{P}(\mathbb{S}))(\gamma_1; \gamma_2) := \inf_{\eta \in \mathcal{F}(\gamma_1, \gamma_2)} \left( \int_{\mathbb{S}^2} [d_\mathbb{S}(x, y)]^p \eta(dx dy) \right)^{1/p}
\]
for any \( \gamma_1, \gamma_2 \in \mathcal{P}_p(S) \), where
\[
\mathcal{P}_p(S) := \left\{ \gamma \in \mathcal{P}(S) : \int_S [d_S(x,x_0)]^p \gamma(dx) < +\infty \text{ for some } x_0 \in S \right\}
\]
and \( \mathcal{F}(\gamma_1, \gamma_2) \) denotes the class of all probability measures on \((S^2, \mathcal{B}(S^2))\) with \(i\)-the marginal \(\gamma_i\), \(i = 1, 2\). See Chapter 7 of Ambrosio et al. [4] and, in particular, Proposition 7.1.5. Now, the reformulation of Definition 1 exploits the fact that \(\gamma\) and \(F\) of Regazzini and Sazonov [43], we introduce a parameter \(\delta > 1\) for any
\[
\text{constant is instead hidden in the constant } L_0^{(n)} \text{. The weighted Poincaré-Wirtinger hints at an application of Sanov’s large deviation principle. The weighted Poincaré-Wirtinger constant is instead hidden in the constant } L_0^{(n)}. \text{ Theorem 1 provides with a further detailing of the right-hand side of (9), while Corollary 1 shows that, under suitable assumptions, the same right-hand side is even bounded by } C n^{-\alpha}, \text{ for some } \alpha \in (0, \frac{1}{2}) \text{ and positive constant } C. \]

Theorem 2 of Section 3.3 is our second main result on PCRs, which deals with the Bayesian nonparametric framework. That is, we assume \(\Theta = \mathcal{P}(\mathcal{X})\) and \(\kappa(\cdot|p) = p(\cdot)\). Following Regazzini and Sazonov [43], we introduce a parameter \(\delta > 0\) and a suitable finite partition \(\{A_{j,\delta}\}_{j=1,...,N}\) of the metric space \((\mathcal{X}, d_\mathcal{X})\), that we assume to be totally bounded. We will show that the partition induces a sequence of random variables \(\{e_n\}_{n \geq 1}\), \(\eta_n\) being an approximation of \(\xi_n\), and a random probability measure \(\tilde{p}_\theta\), which is a discretized version of the directing (de Finetti) measure \(\tilde{p}\) of the sequence \(X(\infty)\) (see Aldous [2]). Then, starting again from (6) and
shortening the notation \( \mathcal{P}(\Theta) = \mathcal{P}(\mathcal{P}(X)) \) to \( \mathcal{P} \), we consider, in alternative to (9), the upper bound

\[
\epsilon_n = E \left[ \mathcal{W}_p^{(\mathcal{P})}(\pi_n(\cdot | \xi_1, \ldots, \xi_n); \delta_{p_0}) \right] \\
\leq E \left[ \mathcal{W}_p^{(\mathcal{P})}(\pi_n(\cdot | \xi_1, \ldots, \xi_n); \Gamma_N^*(\cdot | \eta_1, \ldots, \eta_n)) \right] \\
+ E \left[ \mathcal{W}_p^{(\mathcal{P})}(\Gamma_N^*(\cdot | \eta_1, \ldots, \eta_n); \Sigma_N^*(\cdot | \eta_1, \ldots, \eta_n)) \right] \\
+ E \left[ \mathcal{W}_p^{(\mathcal{P})}(\Sigma_N^*(\cdot | \eta_1, \ldots, \eta_n); \delta_{\epsilon_n^{(\eta)}}) \right] \\
+ E \left[ \mathcal{W}_p^{(\mathcal{P}(X))}(\epsilon_n^{(\eta)}; \epsilon_n^{(\xi)}) \right] + E \left[ \mathcal{W}_p^{(\mathcal{P}(X))}(\epsilon_n^{(\xi)}; p_0) \right]
\]

where \( \epsilon_n^{(\eta)} := \frac{1}{n} \sum_{i=1}^{n} \delta_{\eta_i} \), while \( \Gamma_N^* (\Sigma_N^*) \) is a probability kernel that stands for the posterior distribution of \( \bar{p} \) (\( p_0 \), respectively) evaluated at the discretized (hypothetical) data \( \eta_1, \ldots, \eta_n \). Again, since \( \kappa_0 = p_0 \) in the new setting, we notice the occurrence of \( \varepsilon_{n,p}(X, \kappa_0) \), defined in (10), in the right-most term of the last inequality. Theorem 2 provides a new refined upper bound for the right-hand side of the inequality in (11), which will be obtained by re-expressing the various summands in terms of \( \delta \) and \( N \). Finally, Corollary 2 exploits a functional dependence between \( \delta \) and \( N \) (of course, related to the concept of dimension of \( (X, d_X) \)) to get more concrete upper bounds.

3. Main results on PCRs

3.1. Metric issues and Wasserstein dynamics. We start with a schematic representation of the various metric spaces, as well as their interplay, that appear within the Bayesian framework described in Section 2. Links between these metric spaces are not established solely by the definition of the fundamental objects of the theory, such as the statistical model and the posterior distribution, but also by the presence of the \( p \)-Wasserstein distance, whose definition is strongly influenced by the base metric. In Figure 1, solid arrows point out the presence of a specific mapping. Furthermore, dotted arrows indicate that the Wasserstein distance corresponding to the head of the arrow is built on the metric space corresponding to the nocl of the same arrow. Dashed arrows denote some metric construction, like the product or the quotient.

In particular, \( X^n / \sim \) stands for the quotient of the product space \( X^n \) by the action of the symmetric group \( \mathfrak{S}_n \). More precisely, upon specifying that we write

\[
d^n_X((x_1, \ldots, x_n); (y_1, \ldots, y_n)) := \left( \frac{1}{n} \sum_{i=1}^{n} d_X(x_i, y_i) \right)^{1/p},
\]

the quotient metric \( d^n_X \) is given by

\[
d^n_X([x], [y]) := \inf_{(x_1, \ldots, x_n) \in [x]} \inf_{(y_1, \ldots, y_n) \in [y]} d^n_X((x_1, \ldots, x_n); (y_1, \ldots, y_n))
\]

\[
= \inf_{\tau_\eta \in \mathfrak{S}_n} d^n_X((x_1, \ldots, x_n); (y_{\tau_\eta(1)}, \ldots, y_{\tau_\eta(n)}))
\]

for all \([x], [y] \in X^n / \sim \). Now, a well-known theorem by Birkhoff (see Theorem 6.0.1 of Ambrosio et al. [4]), entails that \( d^n_X([x], [y]) = \mathcal{W}_p^{(\mathcal{P}(X))}(x^n; y^n) \). This particular construction will play a relevant role in the proof of Theorem 2. Let us also recall the distinguished case of the
nonparametric model, in which the two metric spaces \((\Theta, d_\Theta)\) and \((\mathcal{P}_p(X), W_p^{\mathcal{P}(X)})\) coincide, and \(\kappa(\cdot|p) = p(\cdot)\).

We conclude by recalling some analytical concepts that we will use in the sequel. First, given a regular statistical model characterized by the set of densities \(\{f(x|\theta)\}_{\theta \in \Theta}\), we fix the symbol \(K(\theta|\theta_0)\) to denote the Kullback-Leibler divergence, i.e.

\[
K(\theta|\theta_0) := \int_X \left[ \ln \frac{f(x|\theta_0)}{f(x|\theta)} \right] f(x|\theta_0) \, dx .
\]

Then, we connect the abstract definition (5) of Wasserstein distance on the space \(\mathcal{P}_p(S)\), for \(1 < p < +\infty\), with the dynamical formulation due to Benamou and Brenier [7], which is based on the continuity equation. For short, this is referred to as Wasserstein dynamics. Here, we assume that \(S\) is the norm-closure of some nonempty, open and connected subset of a separable Hilbert space \(\mathcal{V}\) with scalar product \(\langle \cdot, \cdot \rangle\) and norm \(\| \cdot \|\). In particular, from Benamou and Brenier [7], for any \(\gamma_0, \gamma_1 \in \mathcal{P}_p(S)\), we have that

\[
\left[ W_p^{\mathcal{P}(S)}(\gamma_0; \gamma_1) \right]^p = \inf_{\{\gamma_t\}_{t \in [0,1]} \in AC^p[\gamma_0; \gamma_1]} \int_0^1 \int_S \|v_t(x)\|^p \gamma_t(dx) \, dt
\]

where \(AC^p[\gamma_0; \gamma_1]\) is the space of all absolutely continuous curves in \(\mathcal{P}_p(S)\) with \(L^p(0,1)\) metric derivative (w.r.t. \(W_p\)) connecting \(\gamma_0\) to \(\gamma_1\), and \([0,1] \times S \ni (t, x) \mapsto v_t(x) \in \mathcal{V}\) is a Borel function such that for almost every \(t \in (0,1)\)

\[
\frac{d}{ds} \int_S \psi(x) \gamma_s(dx) \bigg|_{s=t} = \int_S \langle v(x), D\psi(x) \rangle \gamma_t(dx) \quad \forall \psi \in C^1_b(S).\]

(12)
Here, $D\psi$ denotes the (Riesz representative of) the Fréchet differential of $\psi$, and $\psi \in C^1_b(S)$ means that the function $\psi$ is the restriction to $S$ of a function in $C^1_b(V)$, that is, of a bounded continuous function with bounded continuous Fréchet derivative on $V$. We refer to Chapter 8 of Ambrosio et al. [4] or Chapter 4 of [44] for a detailed account on the partial differential equation (12). For any fixed $t$ and given $\gamma_t$, it is natural to look for $v_t(\cdot)$ solving to (12) in the form of a gradient, and thus we may interpret (12) as an abstract elliptic equation, for which the fundamental role of Poincaré inequalities in proving both existence and regularity of a solution is well-known.

We say that a probability measure $\mu$ on $(S, B(S))$ satisfies a weighted Poincaré inequality of order $p$ if there exists a constant $C_p$ such that it holds true
\[
\inf_{a \in \mathbb{R}} \left( \int_S |\psi(x) - a|^p \mu(dx) \right)^{\frac{1}{p}} \leq C_p \left( \int_S \|D\psi(x)\|^p \mu(dx) \right)^{\frac{1}{p}}
\]
for every $\psi \in C^1_b(S)$. In particular, hereafter, we denote by $C_p[\mu]$ the the best constant in such inequality. In particular, $C_2[\mu]$ enjoys the following characterization:
\[
\left( \frac{1}{C_2[\mu]} \right)^2 = \inf \left\{ \int_S \|D\psi(x)\|^2 \mu(dx) : \psi \in C^1_b(S), \int_S \psi(x) \mu(dx) = 0, \int_S |\psi(x)|^2 \mu(dx) = 1 \right\}.
\]

3.2. A first result on PCRs. We start with a precise formulation of the fact that (6) provides with a PCR at $\theta_0 \in \Theta$. Although very simple, the following statement is the starting point of our work. See Subsection A.1 for the proof.

**Lemma 1.** Assume that $\pi \in \mathcal{P}_p(\Theta)$ and that, for any $n \in \mathbb{N}$, $\kappa_0^\otimes n \ll \mu_n$. Then, $\pi_n(\cdot|\xi_1, \ldots, \xi_n)$ is a well-defined random probability measure belonging to $\mathcal{P}_p(\Theta)$ with $\mathbb{P}$-probability one, and (6) gives a PCR at $\theta_0$.

In order to exploit the Wasserstein dynamics, we assume that both the sample space $X$ and parameter spaces $\Theta$ possess richer analytical structures. Therefore, on the basis of the mathematical theory developed in Chapter 8 of Ambrosio et al. [4], we fix the following set of assumptions for our approach to PCRs.

**Assumptions 1.** The set $X$, the parameter space $\Theta$ and the statistical model $\kappa(\cdot|\cdot)$ are such that

i) $X$ coincides with an open, connected subset of $\mathbb{R}^m$ with Lipschitz boundary. (With minor changes of notation, $X$ could also coincide with a smooth Riemannian manifold without boundary of dimension $m \in \mathbb{N}$).

ii) $\Theta$ coincides with the closure of an open, connected subset of a separable Hilbert space of dimension $d \in \mathbb{N} \cup \{+\infty\}$.

iii) $\kappa(\cdot|\cdot)$ is dominated by the $m$-dimensional Lebesgue measure, i.e. $\kappa(A|\theta) = \int_A f(x|\theta)dx$ for every $A \in \mathcal{B}(\mathbb{R}^m)$, where $x \mapsto f(x|\theta)$ is a probability density function for any $\theta \in \Theta$.

The above preliminaries on metric spaces and Wasserstein dynamics, combined with Assumptions 1, pave the way to state our first main results on PCRs of a dominated Bayesian model for the observations. See Subsection A.2 for the proof.
Theorem 1. Within the setting specified by Assumptions 1, let assume that

i) \((x, \theta) \mapsto f(x|\theta) \in C^2(\mathbb{X} \times \Theta)\);

ii) the model \(\{f(\cdot|\theta)\}_{\theta \in \Theta}\) is \(C^2\)-regular at \(\theta_0\);

iii) for any \(\theta\), there exist positive constants \(b(\theta), c(\theta)\) for which

\[
|\log f(x|\theta)| \leq b(\theta)(1 + |x|^2) \quad \text{and} \quad |\nabla_x \log f(x, \theta)| \leq c(\theta)(1 + |x|) \tag{13}
\]

hold for every \(x \in \mathbb{X}\);

iv) \(\pi \in P_2(\Theta)\), with full support;

v) \(\kappa_0 \in P_2(\mathbb{X})\).

Then, (7) is fulfilled with

\[
\pi_n^*(d\theta|\gamma) := \frac{\exp\{n \int_X \log f(y|\theta)\gamma(dy)\} \pi(d\theta)}{\int_\Theta \exp\{n \int_X \log f(y|\tau)\gamma(dy)\} \pi(d\tau)} \tag{14}
\]

Moreover, (8) holds relatively to a suitable choice of a \(W_2^{\mathcal{P}(\mathbb{X})}\)-neighborhood \(V_0^{(n)}\) of \(\kappa_0\), provided that

\[
L_0^{(n)} := n \sup_{\gamma \in V_0^{(n)}} \|\mathcal{C}_2(\pi_n^*(\cdot|\gamma))\|^2 \left(\int_\Theta \int_X \left\|D_{\theta} \frac{\nabla_x f(x|\theta)}{f(x|\theta)}\right\|^2 \gamma(dx) \pi_n^*(d\theta|\gamma)\right)^{1/2} < +\infty \tag{15}
\]

for any \(n \in \mathbb{N}\). Therefore, the assumption of Lemma 1 are fulfilled and a PCR at \(\theta_0\) is given by

\[
\epsilon_n = 3 \left(\frac{\int_\Theta \|\theta - \theta_0\|^2 e^{-\kappa_0(\theta|\theta_0)} \pi(d\theta)}{\int_\Theta e^{-\kappa_0(\theta|\theta_0)} \pi(d\theta)}\right)^{1/2} \tag{16}
\]

\[
+ L_0^{(n)} \epsilon_{n,2}(\Theta, \kappa_0) + 2\|\theta_0\| \mathbb{P}[\xi_0 \notin V_0^{(n)}] + E \left(\frac{2 \int_\Theta \|\theta\|^{2} \prod_{i=1}^n f(\xi_i|\theta) \pi(d\theta)}{\int_\Theta \prod_{i=1}^n f(\xi_i|\theta) \pi(d\theta)}\right)^{1/2} \mathbb{1}\{\xi_0 \notin V_0^{(n)}\}.
\]

In particular, if the choice \(V_0^{(n)} = P_2(\mathbb{X})\) makes the constant \(L_0^{(n)}\) finite for every \(n \in \mathbb{N}\), then the expression on the right-hand side of (16) reduces to the first two terms.

The posterior distribution, i.e. the kernel \(\pi_n^*\), appears in (15) and (16). Now, we explain how to get rid of the dependence of \(\pi_n^*\), and to reduce (15) and (16) to expressions involving only the prior distribution and the statistical model. Critical is the employment of well-known quantities such as \(\epsilon_{n,2}(\mathbb{X}, \kappa_0)\) and \(\mathbb{P}[\xi_0 \notin V_0^{(n)}]\). With regards to \(\epsilon_{n,2}(\mathbb{X}, \kappa_0)\), we recall from Theorem 1 of Fournier and Guillin [27] that, if \(\int_\mathbb{X} |x|^q \kappa_0(dx) < +\infty\) for some \(q > 2\), then

\[
\epsilon_{n,2}(\mathbb{X}, \kappa_0) \leq C(q, m) \left(\int_\mathbb{X} |x|^q \kappa_0(dx)\right)^{1/q}
\]

\[
\times \begin{cases} 
  n^{-1/4} + n^{-(q-2)/(2q)} & \text{if } m = 1, 2, 3, \text{ and } q \neq 4 \\
  n^{-1/4} \sqrt{\log(1 + n) + n^{-(q-2)/(2q)}} & \text{if } m = 4, \text{ and } q \neq 4 \\
  n^{-1/m} + n^{-(q-2)/(2q)} & \text{if } m > 4, \text{ and } q \neq m/(m-2)
\end{cases}
\]

with some positive constant \(C(q, m)\). Under more restrictive assumptions than the sole \(q\)-moment condition, \(\epsilon_{n,2}(\mathbb{X}, \kappa_0)\) is of order \(O(n^{-1/2})\), which is optimal in the one-dimensional
setting. See Section 5 of Bobkov and Ledoux [10] for sharp results. In dimension 2, the optimal rate is \(\sqrt{\log n}/n\) (Ambrosio et al. [6]), while when \(m \geq 3\), the optimal rate is \(n^{-1/m}\) (Talagrand [49]). Lastly, when \(X\) has infinite dimension, logarithmic rates have been obtained in Jing [36]. With regards to \(\mathcal{P}(\epsilon_n^q \notin V_0^{(n)})\), we refer to Theorem 2.7 of Bolley et al. [11]. In particular, if \(\int_X |x|^q \kappa_0(dx) < +\infty\) for some \(q \geq 1\), then
\[
P[\mathcal{W}_2(\epsilon_n^q; \kappa_0) > t] \leq B(q, m)t^{-q} \times \left\{ \begin{array}{ll}
\frac{n^{-q/4}}{1} & \text{if } q > 4 \\
\frac{n^{1-q/2}}{1} & \text{if } q \in [2, 4) \end{array} \right.
\]
for any \(t > 0\) and \(n \in \mathbb{N}\), with some positive constant \(B(q, m)\). Exponential bounds can be also obtained upon requiring that \(\int_X \epsilon^{\alpha|x|}\kappa_0(dx) < +\infty\) for some \(\alpha > 0\). See Theorem 2.8 of Bolley et al. [11]. In the next corollary we show that, under additional assumptions, similar bounds hold true for the other terms appearing on the right-hand side of (16). See Subsection A.3 for the proof.

**Corollary 1.** In addition to the hypotheses of Theorem 1, suppose that there exist constants some \(C > 0\) and \(\beta \geq 2\) for which
\[
K(\theta|\theta_0) \geq C \min\{\|\theta - \theta_0\|^\beta, 1\}
\]
holds for all \(\theta \in \Theta\). Moreover, assume that
\[
\int_X |x|^q \kappa_0(dx) < +\infty
\]
holds for some constants \(q > 4\), and
\[
\mathcal{M}_{n, r} := E\left[\int_{\Theta} \|\theta\|^r \pi_n(d\theta|\xi_1, \ldots, \xi_n)\right] < +\infty
\]
for all \(n \in \mathbb{N}\) and some \(r > 2\). Then, if the neighborhood \(V_0^{(n)}\) has the form \(\{\gamma \in \mathcal{P}_2(X) \mid \mathcal{W}_2(\gamma; \kappa_0) \leq Kn^{-a}\}\) for some \(K > 0\) and \(a \in [0, 1/4)\), for the PCR given in (16) we obtain the new bound
\[
\epsilon_n \leq C_1 n^{-1/\beta} + L_0^{(n)} \epsilon_{n, 2}(X, \kappa_0) + 2\|\theta_0\|K^{-q}\mathcal{B}(q, m)n q(a-1/4) + C_2 n^{[q(r-1)(a-1/4)]/r} \mathcal{M}_{n, r}
\]
with suitable positive constants \(C_1\) and \(C_2\).

In (19), the posterior distribution appears in \(L_0^{(n)}\) and in the posterior moment (18). With regards to (18), this term is often available in an explicit form, even if the posterior is not explicit. Here, we do not further analyze it and we will make it explicit in connection with applications in Section 4. With regards to \(L_0^{(n)}\), we recall some bounds for the Poincaré constant \(\mathcal{C}_2(\pi_n^\ast(\cdot|\gamma))\) with the aim of getting rid of the direct dependence on the posterior distribution. A very popular bound for \(\mathcal{C}_2(\pi_n^\ast(\cdot|\gamma))\) is the Bakry-Emery criterion on a convex set \(\Theta \subseteq \mathbb{R}^d\). In particular, suppose that the prior \(\pi\) has a positive \(C^2\) density with respect to the Lebesgue measure \(\mathcal{L}^d\) and that the mapping \(\theta \mapsto -\int_X \log f(y|\theta)\gamma(dy)\) is in \(C^2(\Theta)\) and strictly convex. Then, upon denoting by \(\lambda(\gamma)\) the minimum eigenvalue of the Hessian of the last mapping, it holds
\[
[\mathcal{C}_2(\pi_n^\ast(\cdot|\gamma))]^2 = \frac{1}{n\lambda(\gamma) + l(\pi)}
\]
for all sufficiently large \(n\), where \(l(\pi)\) is a real constant depending only on the prior distribution, under suitable assumptions on \(\pi\) (like log-concavity). See Bakry et al. [9], and references therein, for a comprehensive account on (20).
Variants of (20) exist in infinite-dimensional Hilbert spaces, and they can be found in Chapter 10 and Chapter 11 of Da Prato [16]. Other more abstract conditions can lead to similar conclusions. Furthermore, an interesting approach to evaluate Poincaré constant \( \mathcal{V}_2(\pi_n(\cdot|\gamma)) \), which is based on the choice of a Lyapunov function, is described in Bakry et al. [8]. In regular situations,

\[
n \sup_{\gamma \in V_0} [\mathcal{V}_2(\pi_n^*(\cdot|\gamma))]^2
\]

is expected to be a bounded quantity with respect to \( n \), for some fixed neighborhood \( V_0 \) of \( \kappa_0 \). Therefore, it remains to handle with the following quantity

\[
\sup_{\gamma \in V_0(n)} \int_\Theta \int_X \| D_\theta \nabla_x f(x|\theta) \|^2 \gamma(dx) \pi_n^*(d\theta|\gamma),
\]

which is also expected to be bounded with respect to \( n \), in regular situations. Here, we do not further manipulate this expression, and we will make it explicit in connection with applications in Section 4. For example, as we will see in Subsection 4.3, for models that can be expressed as an element of the exponential family, in canonical form, the norm \( \| D_\theta \nabla_x f(x|\theta) \| \) turns out to be even independent of \( \theta \), yielding that the entire double integral does not depend on \( \pi_n^* \).

3.3. A second result on PCRs. As a complement to Theorem 1, we consider the problem of defining a general strategy to evaluate PCRs in a full Bayesian nonparametric setting. This general strategy, in turn, provides with PCRs under non-dominated Bayesian models for the observations. We start by fixing a parameter \( \delta > 0 \), and by considering the \( \delta \)-covering number \( N(\delta, \mathcal{X}, d_\mathcal{X}) \) of the metric space \( (\mathcal{X}, d_\mathcal{X}) \), which is supposed to be totally bounded. Then, we consider the full Bayesian nonparametric setting, in which the two metric spaces \( (\Theta, d_\Theta) \) and \( (\mathcal{P}_p(\mathcal{X}), W_p^{(\mathcal{P}(\mathcal{X}))}) \) coincide, and \( \kappa(\cdot|p) = p(\cdot) \). We observe that in the nonparametric setting it is not straightforward to directly fulfill (7), because of the lack of a general representation formula. Such a task could be successfully carried out when starting from distinguished classes of nonparametric priors, although this is not relevant at the level of a general theory. Instead, here we resort to another version of (8), which consists in finding a solution of (1), again denoted by \( \pi_n^*(\cdot|\cdot) \), satisfying

\[
W_p^{(\mathcal{P}(\mathcal{X}))}(\pi_n^*(\cdot|x^{(n)}); \pi_n^*(\cdot|y^{(n)})) \leq L_n \ W_p^{(\mathcal{P}(\mathcal{X}))}(\epsilon_n(x); \epsilon_n(y))
\]

for all \( x^{(n)} := (x_1, \ldots, x_n) \) and \( y^{(n)} := (y_1, \ldots, y_n) \) in \( \mathcal{X}^n \), with some positive constant \( L_n \). This paves the way to state our second main results on PCRs under a full Bayesian nonparametric setting. See Subsection A.4 for the proof.

Theorem 2. Assume that \( (\mathcal{X}, d_\mathcal{X}) \) is a totally bounded metric space and that there exists a distinguished solution of (1) that fulfills (21) for all \( x^{(n)}, y^{(n)} \in \mathcal{X}^n \). Then, for any infinitesimal sequence \( \delta_n \) of positive numbers, we can find two other infinitesimal sequences \( M_n(N_{\delta_n}, \delta_n) \) and \( V_n(N_{\delta_n}, \delta_n) \), depending only on the nonparametric prior through its finite-dimensional laws, such that

\[
\epsilon_n = \epsilon_{n,p}(\mathcal{X}, p_0) + 2(2 + L_n)\delta_n + \text{diam}(\mathcal{X}) \left\{ \frac{1}{2} [M_n(N_{\delta_n}, \delta_n) + V_n(N_{\delta_n}, \delta_n)] \right\}^{1/p}
\]

gives a PCR at \( p_0 \).
The precise definition of the function $M_n$ and the function $V_n$ is given in Subsection A.4. See Equation (48). Essentially, these functions are obtained from posterior means and posterior variances, respectively, of Bernoulli models with prior distributions of the form $P[p(A) \in \cdot]$ and data coinciding with i.i.d. Bernoulli variables with common parameter $\kappa_0(A)$, for some subsets $A \in \mathcal{X}$.

Although Theorem 2 does not aim at providing the sharpest PCR, it has several merits. First, Theorem 2 deals with the full Bayesian nonparametric setting. In particular, to the best of our knowledge, this is the first result on PCRs under non-dominated Bayesian models for the observations. Moreover, the PCR (22) is appropriate to obtain more explicit PRCs, whenever the metric structure of the space $(\mathcal{X}, d_\mathcal{X})$ is known along with the system of finite-dimensional distributions of the prior. Under additional assumptions, the next corollary quantifies the order of decaying of $M_n(N_\delta, \delta)$ and $V_n(N_\delta, \delta)$ and makes the PCR (22) more explicit. See Subsection A.5 for the proof.

**Corollary 2.** In addition to the hypotheses of Theorem 2, we assume that:

i) $N_\delta(\mathcal{X}, d_\mathcal{X}) \sim (1/\delta)^d$ for some $d > 0$, as $\delta \to 0$;

ii) $L_n \sim n^s$ for some $s \geq 0$, as $n \to +\infty$;

iii) for any $\delta > 0$, there holds

$$M_n(N_\delta, \delta) + V_n(N_\delta, \delta) \leq C(\pi) N n^{-\alpha}$$

for some positive constant $C(\pi)$ depending only on the prior $\pi$ and $\alpha > sd$.

Then, the expression in (22) has an upper bound which enjoys the asymptotic expansion, as $n \to +\infty$,

$$\varepsilon_{n,p}(\mathcal{X}, p_0) + n^{-\left(\frac{\alpha-ds}{\delta + s}\right)}$$

Checking inequality (23) is usually very simple if the prior $\pi$ has 1-dimensional laws, i.e. $P[p(A) \in \cdot]$ for $A \in \mathcal{X}$, which coincide with beta distributions. We will prove it explicitly in Subsection 4.6, where we will deal with the Dirichlet process prior of Ferguson [26]. On the other hand, we will also provide a more general estimation of the term $M_n(N_\delta, \delta) + V_n(N_\delta, \delta)$ based on an explicit concentration inequality due to Diaconis and Freedman. See Subsection A.6.

We conclude our study of PCRs with a proposition that is particularly useful when the posterior distribution in known explicitly. For instance, this is the case of a Bayesian non-parametric model with a Dirichlet process prior. In particular, under this assumption, it is possible to deal with predictive distributions induced by the posterior distribution. See Subsection A.7 for the proof.

**Proposition 1.** For any fixed $n \in \mathbb{N}$, the validity of (21) for $p = 1$ and $L_n = L$ for every $n \in \mathbb{N}$ is equivalent to having

$$W_1^{(P(\mathcal{X}))}(\alpha^*_n(\cdot|x^{(n)}); \alpha^*_n(\cdot|y^{(n)})) \leq L W_1^{(P(\mathcal{X}))}(\varepsilon_n; \varepsilon'_n)$$

for all $x^{(n)}, y^{(n)} \in \mathcal{X}^n$, $\alpha^*_n$ being a distinguished solution of (3).

4. **Applications**

4.1. **Regular parametric models.** The simplest application is represented by dominated Bayesian statistical models with a finite-dimensional parameter $\theta \in \Theta \subset \mathbb{R}^d$. Accordingly, we start by considering the set of Assumptions 1, with $d \in \mathbb{N}$, along with the hypotheses of Theorem 1. In particular, in this setting, the Kullback-Leibler divergence $K(\theta|\theta_0)$ is a
$C^2$ function, whose Hessian at $\theta_0$ just coincides with the Fisher information matrix at $\theta_0$. Whence, we write

$$K(\theta|\theta_0) = \frac{1}{2} \left\{ \begin{array}{c} \theta - \theta_0 \right\} I[\theta_0] (\theta - \theta_0) + o(\|\theta - \theta_0\|^2) \end{array} \right.$$

(26)
as $\theta \to \theta_0$. Another property of the Kullback-Leibler divergence, which we assume, is

$$\inf_{\theta \in \Theta} K(\theta|\theta_0) > 0$$

(27)
for any $\delta > 0$. Therefore, we can apply Theorem 41 in Breitung [12], leading to

$$\int_\Theta e^{-nK(\theta|\theta_0)} \pi(d\theta) \sim \left( \frac{2\pi}{n} \frac{1}{\sqrt{I[\theta_0]}} \right)^d$$

(28)
while Theorem 43 of Breitung [12] gives

$$\int_\Theta |\theta - \theta_0|^p e^{-nK(\theta|\theta_0)} \pi(d\theta) \sim \left( \frac{2}{n} \frac{1}{2} \Gamma \left( \frac{d + p}{2} \right) \frac{1}{\sqrt{I[\theta_0]}} \right)^p$$

(29)
for any $p > 0$. Then, by considering the ratio between the expressions displayed in (28)-(29), for first term on the right-hand side of (16), for any $p \geq 1$, we get

$$\mathcal{W}_p(\pi_n^*(d\theta|\kappa_0), \delta_{\theta_0}) = \left( \frac{\int_\Theta |\theta - \theta_0|^p e^{-nK(\theta|\theta_0)} \pi(d\theta)}{\int_\Theta e^{-nK(\theta|\theta_0)} \pi(d\theta)} \right)^{1/p} \sim \frac{1}{\sqrt{n}}$$

(30)
as $n \to +\infty$, which is the well-known optimal rate for regular parametric models.

Now, we discuss the behavior of the constant $L_0(n)$, as $n$ goes to infinity. In particular, first, we would like to stress that there are plenty of conditions that entail

$$[\mathcal{C}_2(\pi_n^*(\cdot|\gamma))]^2 \leq \frac{C(\gamma)}{n}$$

for every $n \in \mathbb{N}$ and some positive constant $C(\gamma)$. We have already mentioned the hypothesis that the mapping $G : \theta \mapsto -\int_\mathcal{X} [\log f(y|\theta)] \gamma(dy)$ is in $C^2(\Theta)$ and strictly convex, which leads to (20). For example, always supposing that $G \in C^2(\Theta)$, if $\pi(d\theta) = e^{-U(\theta)}d\theta$ for some $U \in C^2(\Theta)$, we can quote the following result from Dolera and Mainini [23], which specifies some results in Bakry et al. [8].

**Proposition 2.** Suppose that $U$ and $G$ are bounded from below, and that $\text{Hess}(V(\theta)) \geq \alpha I$ and $\text{Hess}(U(\theta)) \geq h I$ in the sense of quadratic forms whenever $|\theta| \leq R$, for some $\alpha > 0$, $R > 0$ and $h \in \mathbb{R}$.

1. If, in addition, there exist $c > 0$ and $\ell \in \mathbb{R}$ such that $\theta \cdot \nabla G(\theta) \geq c|\theta|$ and $\theta \cdot \nabla U(\theta) \geq \ell|\theta|$ whenever $|\theta| \geq R$, then

$$[\mathcal{C}_2(\pi_n^*(\cdot|\gamma))]^2 \leq \frac{cn + h + (c + \ell - d_R + nG_R + U_R)C_R}{(cn + h)(cn + \ell - 1 - d_R)} \sim \frac{1}{n}$$

for every $n > (-h/\alpha) \vee ((d_R + 1 - \ell)/c)$, where $d_R := (d - 1)/R$, $G_R := \sup_{B_R} |\nabla G|$, $U_R := \sup_{B_R} |\nabla U|$ and $C_R$ is an explicit universal constant only depending on $R$.

2. If, in addition, there exist $c_1 > 0$, $c_2 > 0$ such that

$$|\nabla G(\theta)|^2 \geq 2c_1 + c_2 [\Delta G(\theta) + \nabla G(\theta) \cdot \nabla U(\theta)]_+$$
whenever $|\theta| \geq R$, then,
\[
[\mathcal{C}_2(\pi_n^*(|\gamma|))]^2 \leq \frac{\alpha n + h + e^{\omega R}(c_1 n + G^*_R + W_R)}{(\alpha n + h)c_1 n} \sim \frac{1}{n}
\]
for every $n > (1 + 1/c_2) \vee (-h/\alpha)$, where $G^*_R := \sup_{B_R} |\Delta G|$, $W_R := \sup_{B_R} |\nabla U| |\nabla G|$ and $\omega_R := \sup_{B_R} G - \inf_{\Theta} G$.

Next, we consider the double integral
\[
\int_{\Theta} \int_{\mathcal{X}} \|D_\theta \nabla_x f(x|\theta) \|_2^2 \pi_n^*(d\theta | \gamma) \sim \|D_\theta \nabla_x f(x|\theta) \|_2^2 \mid_{\theta = \theta^*(\gamma)}
\]
as $n \to +\infty$, where $\theta^*(\gamma)$ denotes a maximum point of the mapping $\theta \mapsto \int_{\mathcal{X}} \log f(y|\theta) \gamma(dy)$. Therefore, if the above right-hand side proves to be positive, a reasonable plan to prove global boundedness of $L_0^{(n)}$ with respect to $n$ can be based on the following to steps. First, we check the validity of an inequality like
\[
\sup_{n \in \mathbb{N}} \int_{\Theta} \|D_\theta \nabla_x f(x|\theta) \|_2^2 \pi_n^*(d\theta | \gamma) \leq C \|D_\theta \nabla_x f(x|\theta) \|_2^2 \mid_{\theta = \theta^*(\gamma)}
\]
for every $\gamma$ belonging to a $\mathcal{W}_2^{(p(\mathcal{X}))}$-neighborhood of $\kappa_0$, where $C$ is a positive constant possibly depending on the fixed neighborhood. Second, we prove global boundedness (for $\gamma$ varying in the neighborhood) of the following integral
\[
\int_{\mathcal{X}} \|D_\theta \nabla_x f(x|\theta) \|_2^2 \mid_{\theta = \theta^*(\gamma)} \gamma(dx) < +\infty
\]
To fix the ideas in a more concrete way, we consider the Gaussian case, where $\theta = (\mu, \Sigma)$ and
\[
f(x|\theta) = (2\pi)^{-m/2} \frac{1}{\sqrt{\det(\Sigma)}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \quad (x \in \mathbb{R}^m)
\]
It is straightforward to notice that the mapping $\theta \mapsto \int_{\mathcal{X}} \log f(y|\theta) \gamma(dy)$ depends on $\gamma$ only through its moments of order 1 and 2. Thus, the above strategy reduces to an ordinary finite-dimensional maximization problem, very similar to the question of finding the maximum likelihood estimator. Finally, the last term in on the right-hand side of (16) can be treated as in Corollary 1, by studying the asymptotic behavior of some posterior $r$-moment as in (18).
4.2. Multinomial models. In this subsection, we consider the case in which our observations (i.e. both the sequences \( \{X_i\}_{i\geq 1} \) and \( \{\xi_i\}_{i\geq 1} \) take values in the finite set \( \{a_1, \ldots, a_N\} \). In this case, it is straightforward to check that \( \Theta \) can be assumed to coincide with the \((N-1)\)-dimensional simplex

\[
\Delta_{N-1} := \left\{ \theta = (\theta_1, \ldots, \theta_{N-1}) \in [0, 1]^{N-1} \mid \sum_{i=1}^{N-1} \theta_i \leq 1 \right\}
\]

and the posterior distribution is given by

\[
\pi_n(d\theta|x_1, \ldots, x_n) = \frac{\prod_{i=1}^{N} \theta_i^{n_{x_i}(x)}}{\int_{\Delta_{N-1}} \prod_{i=1}^{N} \nu_i^{n_{t_i}}(x) \pi(d\theta)}
\]

where \( \theta = (\theta_1, \ldots, \theta_{N-1}), t = (t_1, \ldots, t_{N-1}), \theta_N := 1 - \sum_{i=1}^{N-1} \theta_i, t_N := 1 - \sum_{i=1}^{N-1} t_i \) and

\[
\nu_{n,i}(x) := \sum_{j=1}^{n} [x_j = a_i] \quad i = 1, \ldots, N.
\]

Of course, if we put \( \Omega = \{a_1, \ldots, a_N\} \), we can not directly apply Theorem 1, as it is based on the PDE approach described in Subsection 3.1. Nonetheless, we can resort to a reinterpretation of the data, in terms of the frequencies \( \nu_{n,i} \), that we now explain, that allows the use of our theorem. Indeed, we can consider

\[
\pi_n^*(d\theta|p) := \frac{\prod_{i=1}^{N} \theta_i^{np_i}}{\int_{\Delta_{N-1}} \prod_{i=1}^{N} \nu_i^{np_i} \pi(d\theta)}
\]

defined for \( p = (p_1, \ldots, p_{N-1}) \in \Delta_{N-1} \) with the usual proviso that \( p_N := 1 - \sum_{i=1}^{N-1} p_i \).

Whence,

\[
\pi_n(d\theta|x_1, \ldots, x_n) = \pi_n^*\left(d\theta\left|\left(\frac{\nu_{n,1}(x)}{n}, \ldots, \frac{\nu_{n,N-1}(x)}{n}\right)\right\right).
\]

The problem of consistency, and the allied question of finding a PCR, can be now reformulated as follows. After fixing \( \theta_0 \in \Delta_{N-1} \), we consider the sequence \( \{\xi_i\}_{i\geq 1} \) of i.i.d. random variables, each taking values in \( \{a_1, \ldots, a_N\} \), with \( P[\xi_1 = a_i] = \theta_{0,i}, \) for \( i = 1, \ldots, N \). An analogous version of Lemma 1 states that

\[
\epsilon_n = \mathbb{E}\left[\mathcal{W}_2(P(\theta))\left(\pi_n(d\theta|\xi_1, \ldots, \xi_n); \delta_{\theta_0}\right)\right]
\]

provides a PCR at \( \theta_0 \). Now, we reformulate Theorem 1 as follows. First of all, we notice that

\[
\mathbb{E}\left[\left(\frac{\nu_{n,1}(\xi)}{n}, \ldots, \frac{\nu_{n,N-1}(\xi)}{n}\right) - \theta_0\right] \leq \sum_{i=1}^{N-1} \mathbb{E}\left[\frac{\nu_{n,1}(\xi)}{n} - \theta_{0,i}\right] \leq \frac{1}{\sqrt{n}}
\]

replaces the speed of mean Glivenko-Cantelli convergence. Then, we have that

\[
K(\theta|\theta_0) = \sum_{i=1}^{N} \theta_{0,i} \log \left(\frac{\theta_{0,i}}{\theta_i}\right).
\]
The relation analogous to that in (16), which gives a PCR at $\theta_0$, reads as follows

$$
\epsilon_n = 3 \left( \frac{\int_{\Delta_{N-1}} |\theta - \theta_0|^2 e^{-nK(\theta|\theta_0)} \pi(d\theta)}{\int_{\Theta} e^{-nK(\theta|\theta_0)} \pi(d\theta)} \right)^{1/2}
$$

$$
+ L^{(n)}_0(\delta_n) E \left[ \left( \frac{\nu_{n,1}(\xi)}{n}, \ldots, \frac{\nu_{n,N-1}(\xi)}{n} \right) - \theta_0 \right] 
$$

$$
+ 2|\theta_0| \mathbb{P} \left[ \left( \frac{\nu_{n,1}(\xi)}{n}, \ldots, \frac{\nu_{n,N-1}(\xi)}{n} \right) - \theta_0 \right] > \delta_n 
$$

$$
+ \mathbb{E} \left[ 2 \left( \frac{\int_{\Delta_{N-1}} |\theta|^2 \left( \prod_{i=1}^N \theta_i^{\nu_{n,i}(\xi)} \right) \pi(d\theta)}{\int_{\Delta_{N-1}} \left( \prod_{i=1}^N \theta_i^{\nu_{n,i}(\xi)} \right) \pi(d\theta)} \right)^{1/2} \times \right. 
$$

$$
\times \mathbb{1} \left\{ \left( \frac{\nu_{n,1}(\xi)}{n}, \ldots, \frac{\nu_{n,N-1}(\xi)}{n} \right) - \theta_0 \right\} > \delta_n \right\} 
$$

where $\{\delta_n\}_{n \geq 1}$ is a sequence of positive numbers and $L^{(n)}_0$ is defined as follows

$$
L^{(n)}_0 := n \sup_{|p - \theta_0| \leq \delta_n} \left[ \mathcal{C}_2(\pi_n^* (\cdot | p)) \right]^2 \left( \int_{\Delta_{N-1}} \left| \nabla_p \nabla \mathbf{K}(\theta|p) \right|^2 \pi_n^*(d\theta|p) \right)^{1/2}
$$

Despite its complex form, we show that, actually, the PCR in (31) is easily reducible to a much simpler expression. Indeed, the first term on the right-hand side of (31) is similar to the one already studied in the previous subsection. In particular, by resorting to the same theorems from Breitung [12], recalling that the mapping $\theta \mapsto \mathbf{K}(\theta|p)$ is minimum when $\theta = \theta_0$, we get

$$
\left( \frac{\int_{\Delta_{N-1}} |\theta - \theta_0|^2 e^{-nK(\theta|\theta_0)} \pi(d\theta)}{\int_{\Delta_{N-1}} e^{-nK(\theta|\theta_0)} \pi(d\theta)} \right)^{1/2} \sim \frac{1}{\sqrt{n}}
$$

provided that $\pi$ has full support. As regards the second terms on the right-hand side of (31), we have already shown that the expectation is controlled by $1/\sqrt{n}$. Apropos of the constant $L^{(n)}_0(\delta_n)$, we can easily show that it is bounded, at least whenever $\theta_0$ is fixed in the interior of $\Delta_{N-1}$. In fact, $\delta_n$ can be chosen equal to any positive constant $\delta$ less than the distance between $\theta_0$ and the boundary of $\Delta_{N-1}$. In particular, by exploiting the convexity of the mapping $\theta \mapsto \mathbf{K}(\theta|p)$, we can resort to Proposition 2 (upon assuming more regularity on the prior distribution $\pi$), to obtain $\mathbb{C}_2(\pi_n^*(\cdot | p))^2 \leq C(\delta)/n$, with a positive constant $C(\delta)$ which is independent of $p$. Then, we can show by direct computation that, under the above conditions on $\theta_0$ and $\delta$, the integral

$$
\int_{\Delta_{N-1}} \left| \nabla_p \nabla \mathbf{K}(\theta|p) \right|^2 \pi_n^*(d\theta|p)
$$

can be bounded uniformly in $n$. To conclude the analysis of the terms on the right-hand side of (31), we only need to exploit the boundedness of $|\theta|$, as $\theta$ varies in $\Delta_{N-1}$, to show that the third and the fourth terms are both bounded by a multiple

$$
\mathbb{P} \left[ \left( \frac{\nu_{n,1}(\xi)}{n}, \ldots, \frac{\nu_{n,N-1}(\xi)}{n} \right) - \theta_0 \right] > \delta_n \right] .
$$
Thus, if $\theta_0$ is in the interior of $\Delta_{N-1}$ and $\delta_n = \delta$, for the same $\delta$ as above, it is well-known (this being perhaps the simplest application of the theory of large deviations) that this probability goes to zero exponentially fast. See, e.g., Dembo and Zeitouni [18, Chapter 2] and references therein for an account.

4.3. **Infinite-dimensional exponential families.** This subsection deals with a class of dominated models specified by density functions of the following form

$$f(x|\theta) = \frac{e^{\theta(x)}}{\int_0^{1} e^{\theta(y)}dy}$$

where for simplicity we have fixed $\mathcal{X} = [0, 1]$. Here, $\Theta$ coincides with a Hilbert space of regular functions on $[0, 1]$, which is continuously embedded in $C^0[0, 1]$, such as a Sobolev space $H^s(0, 1)$ with $s > \frac{1}{2}$. See, e.g., Di Nezza et al. [20, Chapters 3 and 8]. The last structural assumptions is that $\pi$ is a Gaussian measure on $\Theta$, with zero mean and covariance operator $Q : \Theta \to \Theta$, recalling that $Q$ is a trace operator with eigenvalues $\{\lambda_k\}_{k \geq 0}$ that satisfy $\sum_{k=0}^{\infty} \lambda_k < +\infty$. See Da Prato [16] for a review on Gaussian measures on Hilbert spaces.

This model has been thoroughly investigated to understand the question of density estimation, from both classical (see Crain [14, 15]) and Bayesian (see Lenk [39, 40]) point of view. Moreover, various results on PCRs can be found in Giné and Nickl [32], Scricciolo [46], van der Vaart and van Zanten [50]. The distinguished, novel feature of our approach is that, in agreement with Definition 1, our PCRs are relative to the original norm on $\Theta$, a metric which is generally stronger with respect to those considered so far in the literature. Indeed, a Sobolev norm of $\theta - \theta_0$ is, for suitable exponents, grater that the $L^r$ norm considered in Giné and Nickl [32] and, in turn, greater than the (squared) Hellinger distance between $f(\cdot|\theta)$ and $f(\cdot|\theta_0)$, as proved in Lemma A.1 of Scricciolo [46]. Therefore, despite the exact coherence with Definition 1, our PCRs are unavoidably slower than those already obtained. Here, our aim is just to show how Theorem 1 and Corollary 1 can be applied to this statistical model. The complete analysis, which is too complex to be reported here in full detail, is deferred to a forthcoming paper entirely focused on this task.

On the technical side, we start by considering the Riesz representative $T_x$ of the delta operator $\delta_x$, $x \in [0, 1]$, so that $\theta(x)$ can be viewed as the scalar product $\langle \theta, T_x \rangle$. In this notation, the model can be written in canonical form, i.e.

$$f(x|\theta) = \exp\{\langle \theta, T_x \rangle - M(\theta)\}$$

where $M(\theta) := \log \int_0^{1} e^{\theta(y)}dy$ is a convex map. Then, we can write the following expansion

$$K(\theta|\theta_0) = \int_0^{1} [\theta_0(y) - M(\theta_0) - \theta(y) + M(\theta)]f(x|\theta_0)dx$$

$$= \int_0^{1} \langle \theta_0 - \theta, T_y \rangle f(y|\theta_0)dy + M(\theta) - M(\theta_0)$$

$$= (DM(\theta_0), \theta_0 - \theta) + M(\theta) - M(\theta_0)$$

$$= \frac{1}{2} \langle \text{Hess}[M](\theta_0)(\theta_0 - \theta), \theta_0 - \theta \rangle + o(\|\theta_0 - \theta\|^2)$$

valid as $\theta \to \theta_0$. As to the assumptions of Theorem 1, we notice that points $ii) - v)$ are trivially in force, by virtue of the elementary features of the model and the prior. We also notice that the Fisher information at $\theta$ coincides with $\text{Hess}[M](\theta)$. Lastly, to guarantee hypothesis $i)$, it is enough to assume that $\Theta = H^s(0, 1)$ with $s \geq 3$, by virtue of the classical Sobolev embedding.
As to the operator $\text{Hess}[M]$, it proves to be unbounded, if it is seen as defined on $L^2(0,1)$. The usual way to see this fact is to regard the well-known Karhunen-Lo`eve representation theorem to write $\theta(x) = \sum_{k=0}^{\infty} Z_k e_k(x)$, where, in general, the $e_k$’s are continuous real-valued functions on $[0,1]$ that are pairwise orthogonal in $L^2(0,1)$ and the sequence $\{Z_k\}_{k \geq 1}$ belongs to $\ell^2$. See Scricciolo [46, Section 1] for the application of the Karhunen-Lo`eve theorem and the works of Crain [14] and Crain [15] for the above-mentioned unboundedness.

This way, the parameter $\theta$ is identified with $\{Z_k\}_{k \geq 1}$ which, recalling that we are assuming a Gaussian prior, can be thought of as a sequence of independent, centered Normal random variables, i.e. $Z_k \sim \mathcal{N}(0,\tau_k)$. At this stage, upon denoting by $\{\gamma_k\}_{k \geq 1}$ the eigenvalues of $\text{Hess}[M](\theta_0)$, a crucial quantity is represented by the exponent $\sigma > 0$ for which $\gamma_k \sim k^\sigma$ as $k \to \infty$. If $\sigma > 2s$, we can prove the validity of (17). Indeed, for $\|\theta - \theta_0\| \leq 1$, writing $\{Z_k^{(0)}\}_{k \geq 1}$ for the sequence in $\ell^2$ that represents $\theta_0$, Hölder’s inequality entails

\[
\langle \text{Hess}[M](\theta_0)(\theta_0 - \theta), \theta_0 - \theta \rangle = \sum_{k=0}^{\infty} \gamma_k [Z_k - Z_k^{(0)}]^2 \geq \left[ \sum_{k=0}^{\infty} k^{2s}[Z_k - Z_k^{(0)}]^2 \right]^p = C \left[ \sum_{k=0}^{\infty} k^{2s}[Z_k - Z_k^{(0)}]^2 \right]^p = C\|\theta - \theta_0\|^{2p}
\]

where $p, q \geq 1$ are conjugate exponents, chosen in such a way that the series $\sum_{k=0}^{\infty} k^{2s}q^{-q/p} \gamma_k$ is convergent. Thus, with reference to Corollary 1, we can put $\beta = 2p$. Actually, we could manage also values of $\sigma$ in $(1,2s)$, but this would require a reformulation of Theorem 1 in which the regularity of $f(x|\theta)$ in the $x$-variable is slightly reduced, and it is not discussed here. To conclude the treatment of this example, we just observe that the other assumptions of Corollary 1, namely that $\int_{\mathcal{X}} |x|^q \kappa_0(dx) < +\infty$ for some $q > 4$ and the inequality (18), are trivially in force. See Lenk [39, 40] for the analysis of the posterior moments. It remains only to deal with the asymptotic behavior of the constant $L_0^{(n)}$. The validity of the Poincaré inequality is discussed, e.g., in Da Prato [16, Chapter 10]. The asymptotic behavior of $[\mathcal{C}_2(\pi_n^*(\cdot|\gamma))]^2$, as $n \to +\infty$, is determined by the behavior of the two sequences of eigenvalues $\{\lambda_k\}_{k \geq 1}$ and $\{\gamma_k\}_{k \geq 1}$. In particular, the asymptotic behavior of $[\mathcal{C}_2(\pi_n^*(\cdot|\gamma))]^2$ is the same of the series $\sum_{k=0}^{\infty} 1/q k^{\lambda_k}$, which is worse than $O(1/n)$. Cfr. Proposition 2. Finally, the discussion of the double integral

\[
\int_{\Theta} \int_{\mathcal{X}} \left\| D_{\theta} \frac{\partial_x f(x|\theta)}{f(x|\theta)} \right\|^2 \gamma(dx) \pi_n^*(d\theta|\gamma)
\]

is relatively simple to deal with because, due to the exponential structure, the term $D_{\theta} \frac{\partial_x f(x|\theta)}{f(x|\theta)}$ reduces to $\partial_x T(x)$, which is independent of $\theta$. Therefore, the double integral at issue does not depend on $n$. The analysis is now complete.

4.4. Linear regression in infinite dimension. We study the statistical model that arises from the very popular model of linear regression. In particular, the observed data, say $x_1, \ldots, x_n$, are couples, say $(u_1, v_1), \ldots, (u_n, v_n)$. For simplicity, we assume that $u_1, \ldots, u_n$ varies in an interval $[a, b] \subset \mathbb{R}$, and that they are modeled as i.i.d. random variables, say $U_1, \ldots, U_n$, with a known distribution, say $\pi(du) = h(u)du$, on $([a, b], \mathcal{B}([a, b]))$. Instead, the data $v_1, \ldots, v_n$ vary in $\mathbb{R}$, and they are modeled as i.i.d. random variables $V_1, \ldots, V_n$. Each
Lastly, as for the Kullback-Leibler divergence, a straightforward computation yields

\[ C f \]

\[ \text{On the other hand, from the Bayesian point of view, upon fixing a prior } \pi \]

\[ \theta \text{ is chosen, as in the previous subsection, as a Sobolev space } \Theta \]

\[ \text{where } \]

\[ \]  

\[ \text{V}_i \text{ is stochastically dependent on } U_i, \text{ for any } i, \text{ according to the well-known relation} \]

\[ V_i = \theta(U_i) + E_i, \quad i = 1, \ldots, n, \]

\[ \text{where } E_1, \ldots, E_n \text{ are i.i.d. random variables with Normal } \mathcal{N}(0, \sigma^2) \text{ distribution, while } \theta : [a, b] \to \mathbb{R} \text{ is an unknown continuous function. Assuming for simplicity that } \sigma^2 > 0 \text{ is known, the statistical model is characterized by probability densities } f(\cdot | \theta) \text{ on } [a, b] \times \mathbb{R}, \text{ with respect to the Lebesgue measure, given by} \]

\[ f(x | \theta) = f((u, v) | \theta) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left\{ - \frac{[\theta(u) - v]^2}{2\sigma^2} \right\} h(u). \]

\[ \text{The space } \Theta \text{ is chosen, as in the previous subsection, as a Sobolev space } H^s(a, b) \text{ with } s > 1/2, \]

\[ \text{which is continuously embedded in } C^0[a, b]. \text{ Whence, upon fixing } \theta_0 \in \Theta, \]

\[ \kappa_0(dudv) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left\{ - \frac{[\theta_0(u) - v]^2}{2\sigma^2} \right\} h(u)dudv. \]

\[ \text{On the other hand, from the Bayesian point of view, upon fixing a prior } \pi \text{ on } (\Theta, \mathcal{F}) \]

\[ \text{and resorting to the Bayes formula, the posterior takes on the form} \]

\[ \pi_n(d\theta | x_1, \ldots, x_n) = \pi_n(d\theta | (u_1, v_1), \ldots, (u_n, v_n)) \]

\[ \quad = \exp \left\{ - \frac{1}{2\sigma^2} \sum_{i=1}^n [\theta(u_i) - v_i]^2 \right\} \pi(d\theta) \]

\[ \int_{\Theta} \exp \left\{ - \frac{1}{2\sigma^2} \sum_{i=1}^n [\tau(u_i) - v_i]^2 \right\} \pi(d\tau). \]

\[ \text{Whence, for any probability measure } \gamma \in \mathcal{P}_2([a, b] \times \mathbb{R}), \text{ we can write the following} \]

\[ \pi_n^* (d\theta | \gamma) = \exp \left\{ - \frac{n}{2\sigma^2} \int_{[a, b] \times \mathbb{R}} [\theta(u) - v]^2 \gamma(dudv) \right\} \pi(d\theta) \]

\[ \int_{\Theta} \exp \left\{ - \frac{n}{2\sigma^2} \int_{[a, b] \times \mathbb{R}} [\tau(u) - v]^2 \gamma(dudv) \right\} \pi(d\tau). \]

\[ \text{Lastly, as for the Kullback-Leibler divergence, a straightforward computation yields} \]

\[ K(\theta | \theta_0) = \frac{1}{2\sigma^2} \int_a^b [\theta(u) - \theta_0(u)]^2 h(u)du. \]

\[ \text{Again, our aim is just to show how Theorem 1 and Corollary 1 can be applied to this statistical model. As to the assumptions of Theorem 1, we can easily prove their validity if, for instance,} \]

\[ h \text{ belongs to } C^0[a, b] \cap C^2(a, b) \text{ and it is bounded away from zero. Moreover, we can also discuss the application of Corollary 1. We can check the validity of (17) as a direct consequence of the Gagliardo-Nirenberg interpolation inequality. See, e.g., Maz’ya [41, Section 12.3]. Indeed, being } K(\theta | \theta_0) \text{ substantially equivalent to the squared } L^2\text{-norm, we have} \]

\[ \| \theta - \theta_0 \|_{H^s(a, b)} < \| \theta - \theta_0 \|_{L^2(a, b)} \]

\[ \text{for any } s' > s, \text{ where } \alpha = s/s'. \text{ Therefore, choosing a prior (for example of Gaussian type) which is supported on } H^{s'}(a, b), \text{ and recalling that } H^{s'}(a, b) \text{ is dense in } H^s(a, b), \text{ it is enough to consider the neighborhood} \]

\[ \| \theta - \theta_0 \|_{H^{s'}(a, b)} \leq 1 \]

\[ \text{and check that the interpolation inequality immediately yields (17). Whence, } \beta = 2/(1 - \alpha). \text{ Finally, the assumption} \int_X |x|^q \kappa_0(dx) < +\infty \]

\[ \text{is valid for any } q > 0 \text{ and (18) holds if we assume, for instance, a Gaussian prior } \pi. \text{ To complete the treatment of the regression model, we are left to discuss the asymptotic behavior of the constant } L^{(n)}. \text{ Apropos of the Poincaré constant } E_2(\pi_n^*(|\cdot|)) \|

\[ \text{here it is trivial to notice that the mapping } \theta \mapsto \int_{[a, b] \times \mathbb{R}} [\theta(u) - v]^2 \gamma(dudv) \text{ is twice Fréchet-differentiable with respect to } \theta. \text{ Therefore, the Bakry-Emery criterion applies and, if } \pi \text{ is Gaussian, the results in Da Prato} \]
of the first vector is given by the vector \( Y \). Status censoring model. To describe the model in detail, let \( Y \) be an i.i.d. sample with common, unknown distribution function \( \theta \), and \( C_1, \ldots, C_n \) be an independent i.i.d. sample with common, known distribution function \( G \), both the distribution functions being supported on \([0, +\infty)\). We suppose to observe \( X_i = (\Delta_i, C_i) \) for \( i = 1, \ldots, n \), where \( \Delta_i = \mathbb{I}\{Y_i \leq C_i\} \). If we assume that \( G \) has a density \( g \), supported in \([a, b]\) \( \subseteq [0, +\infty) \), then the model is dominated, with density function of the form

\[
   f(x|\theta) = f((\delta, c)|\theta) = [\theta(c)]^{\delta} [1 - \theta(c)]^{1-\delta} g(c)
\]

with respect to the product measure \((\delta_0 + \delta_1) \otimes \mathcal{L}^1_{[a,b]}\). From the classical point of view, we proceed by fixing a distribution function \( \theta_0 \) and then by putting

\[
   \kappa_0(\{1\} \times A) = \int_{a}^{b} \left( \int_{y}^{b} \mathbb{1}_A(u) g(u) du \right) d\theta_0(y)
\]

for any \( A \in \mathcal{B}([a,b]) \). On the Bayesian point of view, it is customary to fix a Dirichlet prior \( \pi \) on the parameter, to obtain the following posterior distribution

\[
   \pi_n(d\theta|x_1, \ldots, x_n) = \pi_n(d\theta|\delta_1, c_1, \ldots, (\delta_n, c_n))
\]

\[
   = \frac{\prod_{i=1}^{n} [\theta(c_i)]^{\delta_i} [1 - \theta(c_i)]^{1-\delta_i}}{\int_{\Theta} \left( \prod_{i=1}^{n} \tau(c_i)^{\delta_i} [1 - \tau(c_i)]^{1-\delta_i} \right) \pi(d\tau)}
\]

where \( \Theta \) denotes the space of all distribution functions on \([0, +\infty)\). Now, to evaluate a PCR in this setting, it is not convenient to directly resort to Theorem 1, but rather to consider Lemma 1 with \( p = 1 \). Upon endowing \( \Theta \) with the \( W_1 \)-distance, we can take advantage of the Dall’Aglio identity (see Gibbs and Su [31, pag. 424]), to write (after exchanging integrals and expectaion)

\[
   \epsilon_n = \int_{0}^{+\infty} E \left[ \int_{\Theta} |\theta(t) - \theta_0(t)| \pi_n(d\theta|\xi_1, \ldots, \xi_n) \right] dt .
\]

Thus, we can fix \( t \in [0, +\infty) \), and then we can study the inner expectation with the methods proposed within our approach. Indeed, here, we are studying the posterior distribution of \( \theta(t) \) with respect to the random sample \( \xi_1, \ldots, \xi_n \).
From now on, we confine ourself to set forth the main line of reasoning. The ensuing computations are complex and, for this reason, they will be deferred to a forthcoming paper, entirely dedicated to this model. Here, the aim is to show that, even for this model, the theory presented in this paper can be applied.

First, we fix the values of $\delta_1, \ldots, \delta_n$. If they are all equal to 1, the posterior $\pi_n(\cdot|\xi_1, \ldots, \xi_n)$ can be written as a function of the empirical measure of the variable $c_i$’s, namely $\xi^{(c)}_n = \frac{1}{n} \sum_{i=1}^n \delta_{c_i}$. In this case, we have the posterior distribution

$$
\pi_n(d\theta|\xi_1, \ldots, \xi_n) = \frac{\exp\{n \int_a^b \log \theta(u) \xi^{(c)}_n(du)\}}{\int_{\Theta} \exp\{n \int_a^b \log \tau(u) \xi^{(c)}_n(du)\} \pi(d\tau)}
$$

and the posterior distribution of $\theta(t)$ can be explicitly found after computing the moments $E[(\theta(t))^m \theta(c_1) \ldots \theta(c_n)]$ for all $m \in \mathbb{N}$. Analogously, the same procedure can be pursued when $\delta_1, \ldots, \delta_n$ are all equal to 0, again finding a posterior distribution that depends on the empirical measure $\xi^{(c)}_n$. Hence, the posterior probability that $\theta(t) \in B$ can be rewritten as $\pi_n(B|\xi^{(c)}_n)$ for any $B \in \mathcal{B}([0,1])$ and a suitable probability kernel $\pi_n^* : \mathcal{B}([0,1]) \times \mathcal{P}_1([a,b]) \rightarrow [0,1]$. The theory already discussed in Subsection 4.1 can be now applied to this kernel.

Finally, we study the case in which $\delta_1, \ldots, \delta_k$ are equal to 1 and $\delta_{k+1}, \ldots, \delta_n$ are equal to zero, for some $k \in \{1, \ldots, n-1\}$. In particular, in this case, it is useful to think of two independent samples $c_1, \ldots, c_k$ and $c_{k+1}, \ldots, c_n$, with respective distributions given by $\mathbb{P}[C_1 \in \cdot|Y_1 \leq C_1]$ and $\mathbb{P}[C_1 \in \cdot|Y_1 > C_1]$. Upon putting $\xi^{(c)}_{1,k} := \frac{1}{k} \sum_{i=1}^k \delta_{c_i}$ and $\xi^{(c)}_{k+1,n} := \frac{1}{n-k} \sum_{i=k+1}^n \delta_{c_i}$, we can write that

$$
\pi_n(d\theta|\xi_1, \ldots, \xi_n)
= \frac{\exp\{k \int_a^b \log \theta(u) \xi^{(c)}_{1,k}(du) + (n-k) \int_a^b \log(1 - \theta(u)) \xi^{(c)}_{k+1,n}(du)\} \pi(d\theta)}{\int_{\Theta} \exp\{k \int_a^b \log \tau(u) \xi^{(c)}_{1,k}(du) + (n-k) \int_a^b \log(1 - \tau(u)) \xi^{(c)}_{k+1,n}(du)\} \pi(d\tau)}.
$$

Again, we can compute the posterior distribution of $\theta(t)$ by evaluating the moments $E[(\theta(t))^m \theta(c_1) \ldots \theta(c_k)(1 - \theta(c_{k+1})) \ldots (1 - \theta(c_n))]$ for all $m \in \mathbb{N}$. The resulting conditional probability that $\theta(t) \in B$ is therefore presentable as $\pi_n^*\{B|\xi^{(c)}_{1,k}, \xi^{(c)}_{k+1,n}\}$ for any $B \in \mathcal{B}([0,1])$ and a suitable probability kernel $\pi_n^* : \mathcal{B}([0,1]) \times \mathcal{P}_1([a,b]) \rightarrow [0,1]$. Again, the theory already discussed in Subsection 4.1 can be now applied to this particular kernel.

4.6. Nonparametric models. Let us consider the Dirichlet process on the space $(\mathcal{X}, \mathcal{B})$. Let $q > 0$ and $H \in \mathcal{P}(\mathcal{X})$. It is well-known that the predictive distributions of a Dirichlet process with total mass $q$ and mean p.m. $H$ is given by

$$
\alpha_n(\cdot| x_1, \ldots, x_n) = \frac{q}{q+n} H(\cdot) + \frac{n}{q+n} \xi_n(x) .
$$

Whence,

$$
\mathcal{W}_1^{(\mathcal{P}(\mathcal{X}))}(\alpha_n(\cdot| x_1, \ldots, x_n); \alpha_n(\cdot| y_1, \ldots, y_n)) = \frac{n}{q+n} \mathcal{W}_1^{(\mathcal{P}(\mathcal{X}))}(\xi_n(x); \xi_n(y))
$$

for all $n \in \mathbb{N}$ and $x^{(n)}, y^{(n)} \in \mathcal{X}^n$, entailing that our Proposition 1 is directly applicable, with $L = 1$. Therefore, Theorem 2 applies to this noteworthy models. Finally, Corollary 2 can also be applied, after a preliminary analysis on the finite-dimensional distributions. Indeed,
condition (23) can be directly checked, as we do next. In particular, coming back to Equation (48), we have that

$$m_{j,n}(\pi_{N,j}) = \frac{qH(A_{j,\delta}) + n\phi_{j,n}^{(\xi)}}{q + n}$$

$$v_{j,n}(\pi_{N,j}) = \frac{[qH(A_{j,\delta}) + n\phi_{j,n}^{(\xi)}] \cdot [qH(A_{c,j,\delta}) + n(1 - \phi_{j,n}^{(\xi)})]}{(q + n)^2(q + n + 1)}$$

for \(j = 1, \ldots, N\). Whence,

$$|m_{j,n}(\pi_{N,j}) - \phi_{j,n}^{(\xi)}| = \frac{q}{q + n} |H(A_{j,\delta}) - \phi_{j,n}^{(\xi)}| \leq \frac{q}{q + n}$$

yielding that

$$M_n(N_\delta, \delta) \leq \frac{qN_\delta}{n}.$$ 

On the other hand, it is straightforward to show that \(v_{j,n}(\pi_{N,j}) \leq 1/n\), so that

$$V_n(N_\delta, \delta) \leq \frac{N_\delta}{\sqrt{n}}.$$ 

In conclusion, (23) assumes the form

$$M_n(N_\delta, \delta) + V_n(N_\delta, \delta) \leq \frac{(q + 1)N_\delta}{\sqrt{n}}.$$ 

Now, if the relation \(i)\) of Corollary 2 holds for some \(d > 0\), then the asymptotic relation (24) is in force with \(s = 0\) and \(\alpha = 1/2\). Finally, another example that we deem worth examining within this approach is the extended Gamma process prior considered in Example 2 of the work of James et al. [35]. However, since this requires cumbersome computations, we defer the task to a forthcoming work, focused on the class of Bayesian nonparametric priors obtained by normalizing completely random measures. See James et al. [35].

5. Discussion

We conclude by presenting and discussing some possible future developments of our work. The flexibility of Wasserstein distance is promising when considering non-regular Bayesian statistical models, even in a finite-dimensional setting. In this regards, one may consider the problem of dealing with dominated statistical models that have moving supports, i.e. supports that depends on \(\theta\). The prototypical example is the family of Pareto distributions, which is characterized by a density function of the form

$$f(x|\alpha, x_0) = \frac{\alpha x_0^{\alpha}}{x^{1+\alpha}} I\{x \geq x_0\}$$

where \(\theta = (\alpha, x_0) \in (0, +\infty)^2\). Under this model, when rewriting the posterior distribution for obtaining the representation (7), we observe that the empirical distribution can be replaced by the minimum of the observations, which is actually the maximum likelihood estimator. In doing this, we expect to parallel the proof of Theorem 1 with the minimum as sufficient statistics, instead of the empirical measure. In particular, we expect that the term \(\varepsilon_{n,p}(X, \kappa_0)\) should be replaced by other rates typically involved in limit theorems of order statistics. The theoretical framework for such an extension of our results is developed in Dolera and Mainini.
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Another promising line of research consists in extending Theorem 1 to the case of metric measure spaces. The theoretical ground for this development may be found in the seminal works of Ambrosio et al. [5], Otto and Villani [42] and von Renesse and Sturm [52]. In such a context, it is of interest the treatment of the relative entropy-functional in Wasserstein space. It is well-known that the Hessian of the relative entropy-functional, i.e. the Kullback-Leibler divergence, generalizes by using techniques from infinite-dimensional Riemannian geometry. See Otto and Villani [42]. From the statistical side, the possibility of choosing a parameter space that coincides with a space of measures allows to consider popular Bayesian statistical models such as Dirichlet process mixture models. These models can be generically specified by the formula

$$f(x|p) = \int \tau(x|y)p(dy)$$

where $\tau$ is a kernel parameterized by $y$, and $p$ is a random probability measure with a Dirichlet process prior. See Chapter 5 of Ghosal and van der Vaart [30]. We conjecture that the emergence, in the study of Bayesian consistency, of critical cases denominated by smooth or super smooth functions leads to consider very refined bounds for the Kullback-Leibler divergence, such as the so-called HWI inequality. See Bakry et al. [9, Chapter 9] and Otto and Villani [42]. Furthermore, the involvement of the weighted Sobolev spaces on metric measure space is particularly appealing from a mathematical point of view.

Finally, another interesting line of research that emerges from our work concerns with the full Bayesian nonparametric setting of Theorem 2. In this respect, we hope to obtain new refined versions of Theorem 2, for example that consider posterior (or predictive) distributions that share the common form

$$\pi_n(\cdot|x_1, \ldots, x_n) = \gamma_0(\cdot) + \sum_{k=1}^{+\infty} \int_{\mathcal{X}^k} \gamma_k(\cdot|y_1, \ldots, y_k)\mathcal{E}_n^k(dy_1) \cdots \mathcal{E}_n^k(dy_k),$$

which can be interpreted as a Taylor expansion. Expressions of this type are common in the theory of $U$-statistics and in the framework of von Mises calculus. The advantage of such a representation can be a parallel with the representation 7. However, at the moment, it is not clear which Bayesian nonparametric prior, beyond the Dirichlet process prior, enjoys such a representation.

Appendix A. Proofs

A.1. Proof of Lemma 1. Recalling that $(\Theta, \mathcal{F})$ is a standard Borel space, we have that any two solutions $\pi_n(\cdot|\cdot)$ and $\pi_n'(\cdot|\cdot)$ of (1) satisfy $\pi_n(\cdot|x^{(n)}) = \pi_n'(\cdot|x^{(n)})$, as elements of $\mathcal{P}(\mathcal{X})$, for all $x^{(n)} \in \mathcal{X}^n \setminus \mathcal{N}_n$, where $\mathcal{N}_n$ is a $\mu_n$-null set. Thus, the assumption $\mu_n^2 \ll \mu_n$ entails that $\xi^{(n)} := (\xi_1, \ldots, \xi_n)$ takes values in $\mathcal{N}_n$ with $\mathcal{P}$-probability zero, yielding the desired well-definiteness. In addition, if $\pi \in \mathcal{P}_p(\Theta)$, any solution $\pi_n(\cdot|\cdot)$ of (1) satisfies $\pi_n(\mathcal{P}_p(\Theta)|x^{(n)}) = 1$ for almost every $x^{(n)} \in \mathcal{X}^n$. Whence, $\pi_n(\mathcal{P}_p(\Theta)|\xi^{(n)}) = 1$ $\mathcal{P}$-a.s., yielding that

$$\mathcal{W}_p(\pi_n(\cdot|\xi^{(n)}); \delta_{\theta_0}) = \left( \int_{\Theta} [d\Theta(\theta, \theta_0)]^p \pi_n(d\theta|\xi^{(n)}) \right)^{1/p}$$
is a random variable which is $\mathbb{P}$-a.s. finite. Then, by Markov inequality, we write
\[ \pi_n \left( \{ \theta \in \Theta : d_{\Theta}(\theta, \theta_0) \geq M_n \epsilon_n \} \big| \xi^{(n)} \right) \leq \frac{\int_{\Theta} (d_{\Theta}(\theta, \theta_0))^p \pi_n(d\theta|\xi^{(n)})}{M_n^p e_n^p} \]
holds $\mathbb{P}$-a.s. Since
\[ \epsilon_n^p \leq \mathbb{E} \left[ \int_{\Theta} (d_{\Theta}(\theta, \theta_0))^p \pi_n(d\theta|\xi^{(n)}) \right] \]
follows from Lyapunov’s inequality, then we conclude that the following holds
\[ \mathbb{E} \left[ \pi_n \left( \{ \theta \in \Theta : d_{\Theta}(\theta, \theta_0) \geq M_n \epsilon_n \} \big| \xi^{(n)} \right) \right] \leq \frac{1}{M_n^p} \rightarrow 0 . \]
Therefore, Equation (4) holds in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and, hence, it holds in $\mathbb{P}$-probability.

A.2. Proof of Theorem 1. To establish (14), we start from the Bayes formula
\[ \pi_n(d\theta|x^{(n)}) = \frac{\prod_{i=1}^n f(x_i|\theta)}{\int_{\Theta} \prod_{i=1}^n f(x_i|\tau)} \pi(d\tau) \]
and we observe that the regularity of the mapping $x \mapsto f(x|\theta)$ allows us to write $\prod_{i=1}^n f(x_i|\theta)$ as
\[ \exp \left\{ \sum_{i=1}^n \log f(x_i|\theta) \right\} = \exp \left\{ n \int_X \log f(y|\theta) \left( \frac{1}{n} \sum_{i=1}^n \delta_{x_i}(dy) \right) \right\} . \]
Then, the bound (13) entails that the integral $\int_X \log f(y|\theta)\gamma(dy)$ is well-defined and finite for any $\gamma \in \mathcal{P}_2(X)$ and $\theta \in \Theta$. Now, recalling that $\kappa_0 \in \mathcal{P}_2(X)$, let $V_0^{(n)}$ be the $\mathcal{W}_2(\mathcal{P}(X))$-neighborhood of $\kappa_0$ for which (15) is in force. Let $\zeta$ be a fixed element of such a neighborhood and let $\{\zeta_t\}_{t \in [0,1]}$ be a $\mathcal{W}_2$-constant speed geodesic connecting $\kappa_0$ with $\zeta$. In particular, $[0,1] \ni t \mapsto \zeta_t$ is an absolutely continuous curve in $\mathcal{P}_2(X)$. The map $\pi_n^{*}$ allows the construction of a lifting of this path, in the sense that $\{\pi_n^{*}(\cdot|\zeta_t)\}_{t \in [0,1]}$ is a path in $\mathcal{P}_2(\Theta)$ connecting $\pi_n^{*}(\cdot|\kappa_0)$ with $\pi_n^{*}(\cdot|\zeta_1)$, with $\pi_n^{*}(\cdot|\zeta_1)$ having full support in $\Theta$ for any $t \in [0,1]$. The Benamou-Brenier formula discussed in Section 3 shows that
\[ \left[ \mathcal{W}_2(\mathcal{P}_2(\Theta)) \right](\pi_n^{*}(\cdot|\kappa_0), \pi_n^{*}(\cdot|\zeta_1)) \leq \int_0^1 \int_{\Theta} \|D_{\theta}u(\theta, t)\|^2 \pi_n^{*}(d\theta|\zeta_t)dt \] (34)
where, for almost every $t \in (0,1)$, $u(\cdot, t)$ denotes the solution of the elliptic problem. The weak formulation of the elliptic problem reads as the following problem
\[ \int_{\Theta} \langle D_{\theta} u(\theta, t), D_{\theta} \psi(\theta) \rangle \pi_n^{*}(d\theta|\zeta_t) = \frac{d}{ds} \int_{\Theta} \psi(\theta) \pi_n^{*}(d\theta|\zeta_s) \bigg|_{s=t} , \quad \forall \psi \in C^1_b(\Theta) . \] (35)
The right space for the solution of this problem is, for fixed $t \in (0,1)$, the weighted Sobolev space $H^1_m(\Theta; \pi_n^{*}(\cdot|\zeta_t))$, defined as the completion of the space
\[ \left\{ \psi \in C^1_b(\Theta) \bigg| \int_{\Theta} \psi(\theta) \pi_n^{*}(d\theta|\zeta_t) = 0 \right\} \]
with respect to the norm
\[ \|\psi\|_{1,t} := \left( \int_{\Theta} \|D_{\theta} \psi(\theta)\|^2 \pi_n^{*}(d\theta|\zeta_t) \right)^{1/2} . \]
associated with the scalar product
\[ \langle \varphi, \psi \rangle_{1,t} := \int_{\Theta} (D_{\theta} \varphi(\theta), D_{\theta} \psi(\theta)) \pi^*_n(d\theta|\zeta_t). \]

Now, since the equation displayed in (35) can be re-written in the abstract form as
\[ T_t[\psi] = \langle u(\cdot, t), \psi \rangle_{1,t}, \]
where
\[ T_t[\psi] := \frac{d}{ds} \int_{\Theta} \psi(\theta) \pi^*_n(d\theta|\zeta_s) \bigg|_{s=t}, \]
existence, uniqueness and regularity for the solution of (35) would follow from the Riesz representation theorem, provided that the functional \( T_t \) belongs to the dual of \( H^1_{\pi}(\Theta; \pi^*_n(\cdot|\zeta)) \). Whence, again by Riesz theorem, we have
\[ \|u(\cdot, t)\|_{1,t} = \left( \int_{\Theta} \|D_\theta u(\theta, t)\|^2 \pi^*_n(d\theta|\zeta_t) \right)^{1/2} \sup_{\psi \in H^1_{\pi}(\Theta; \pi^*_n(\cdot|\zeta))} \frac{\|T_t[\psi]\|}{\|\psi\|_{1,t}}. \]

Accordingly, by combining the Riesz representation with (34), we obtain that
\[ \left[ W_2(\pi^*_n(\cdot|\kappa_0), \pi^*_n(\cdot|\zeta)) \right]^2 \leq \int_0^1 \left[ \sup_{\psi \in H^1_{\pi}(\Theta; \pi^*_n(\cdot|\zeta))} \frac{T_t[\psi]}{\|\psi\|_{1,t}} \right]^2 dt. \]

Now, in order to obtain further estimates, we introduce the following function
\[ G(\theta, t) := \int_{\mathcal{X}} \log f(y|\theta) \zeta_t(dy) \]
and we indicate by \( G'(\theta, t) \) the partial derivative of \( G(\theta, t) \) with respect to \( t \). Coming back to the expression of the operator \( T_t \), after justifying the exchange of derivatives with integrals by the regularity assumptions on the mapping \((x, \theta) \mapsto f(x|\theta)\), the Leibniz rule for the derivative of a quotient gives
\[ T_t[\psi] = n \int_{\Theta} \psi(\theta) G'(\theta, t) \frac{e^{\pi^*_G(d\pi)}}{\int_{\Theta} e^{\pi^*_G(d\pi)}} - \int_{\Theta} \psi(\theta) e^{\pi^*_G(d\pi)} \int_{\Theta} G'(\theta, t) \pi^*_n(d\theta|\zeta_t) \int_{\Theta} e^{\pi^*_G(d\pi)} \frac{(\int_{\Theta} e^{\pi^*_G(d\pi)})^2}{2} \]
\[ = n \int_{\Theta} \psi(\theta) \pi^*_n(d\theta|\zeta_t) \frac{G'(\theta, t)}{\int_{\Theta} G'(\theta, t) \pi^*_n(d\theta|\zeta_t)} \]
\[ = n \left[ \int_{\Theta} \psi(\theta) - \int_{\Theta} \psi(\tau) \pi^*_n(d\tau|\zeta_t) \right] \left[ G'(\theta, t) - \int_{\Theta} G'(\tau, t) \pi^*_n(d\tau|\zeta_t) \right] \pi^*_n(d\theta|\zeta_t). \]

The last term in the above chain of inequalities can be interpreted as a covariance operator, so that the Cauchy-Schwartz inequality entails the following
\[ \{T_t[\psi]\}^2 \leq n^2 \int_{\Theta} \left[ \psi(\theta) - \int_{\Theta} \psi(\tau) \pi^*_n(d\tau|\zeta_t) \right]^2 \pi^*_n(d\theta|\zeta_t) \times \]
\[ \times \int_{\Theta} \left[ G'(\theta, t) - \int_{\Theta} G'(\tau, t) \pi^*_n(d\tau|\zeta_t) \right]^2 \pi^*_n(d\theta|\zeta_t). \]
In order to obtain further bounds, we now recall the definition of the Poincaré-Wirtinger constant $C_2[\cdot]$, which is given in Subsection 3.1. Thus, (39) directly gives

$$\{T_t[\psi]\}^2 \leq n^2 \{C_2[\pi_n^*(\cdot|\zeta_t)]\}^4 \int_\Theta \|D_\theta \psi(\theta)\|^2 \pi_n^*(d\theta|\zeta_t) \int_\Theta \|D_\theta G'(\theta, t)\|^2 \pi_n^*(d\theta|\zeta_t).$$

(40)

Now, we provide another expression for $G'(\theta, t)$, exploiting the fact that $\zeta_t$ is a Wasserstein constant speed geodesic: indeed, applying again the Benamou-Brenier representation, in this case there exist $w \in L^1((0, 1); L^2(\mathbb{X}))$ such that

$$\left\{W_2(P(\mathbb{X}))(\kappa_0, \zeta_t)\right\}^2 = \int_0^1 \int_{\mathbb{X}} |w(x, t)|^2 \zeta_t(dx)dt$$

(41)

where, for almost every $t \in (0, 1)$, $w(\cdot, t)$ satisfies

$$\frac{d}{ds} \int_{\mathbb{X}} \phi(x) \zeta_s(dx) \bigg|_{s=t} = \int_{\mathbb{X}} w(x, t) \cdot \nabla_x \phi(x) \zeta_t(dx) \quad \forall \phi \in C_0^1(\mathbb{X}).$$

(42)

See Ambrosio and Gigli [3, Proposition 3.30]. At this stage, in view of a density argument and in view of (13), by replacing $\phi$ by $\log f(\cdot|\theta)$ in (42) yields

$$G'(\theta, t) = \int_{\mathbb{X}} w(x, t) \cdot \nabla_x \frac{f(x|\theta)}{f(x|\theta)} \zeta_t(dx).$$

(43)

Thus, in view of (40), the squared supremum in (37) can be bounded as follows

$$\sup_{\psi \in H_0^1(\Theta, \pi_n^*(\cdot|\zeta_t)) \atop \|\psi\|_{L^1} \leq 1} \{T_t[\psi]\}^2 = n^2 \{C_2[\pi_n^*(\cdot|\zeta_t)]\}^4 \int_\Theta \|D_\theta G'(\theta, t)\|^2 \pi_n^*(d\theta|\zeta_t).$$

(44)

By (43), after justifying the exchange of the gradient with the integral, we can write

$$\nabla_\theta G'(\theta, t) = \int_{\mathbb{X}} w(x, t) \cdot D_\theta \frac{\nabla_x f(x|\theta)}{f(x|\theta)} \zeta_t(dx)$$

so that, again by Cauchy-Schwartz,

$$\|D_\theta G'(\theta, t)\|^2 \leq \int_{\mathbb{X}} |w(x, t)|^2 \zeta_t(dx) \int_{\mathbb{X}} \left\|D_\theta \frac{\nabla_x f(x|\theta)}{f(x|\theta)} \right\|^2 \zeta_t(dx).$$

Then, by a direct combination of (37) and (44) with this last inequality we obtain

$$\left\{W_2(P(\mathbb{X}))(\pi_n^*(\cdot|\kappa_0), \pi_n^*(\cdot|\zeta_t))\right\}^2 \leq n^2 \int_0^1 \{C_2[\pi_n^*(\cdot|\zeta_t)]\}^4 \left(\int_{\mathbb{X}} |w(x, t)|^2 \zeta_t(dx) \right) \times \left(\int_\Theta \int_{\mathbb{X}} \left\|D_\theta \frac{\nabla_x f(x|\theta)}{f(x|\theta)} \right\|^2 \zeta_t(dx) \pi_n^*(d\theta|\zeta_t)\right) dt.$$

(45)

We invoke assumption (15) to conclude that

$$\left\{W_2(P(\mathbb{X}))(\pi_n^*(\cdot|\kappa_0), \pi_n^*(\cdot|\zeta_t))\right\}^2 \leq \{L_0(n)\}^2 \int_0^1 \int_{\mathbb{X}} |w(x, t)|^2 \zeta_t(dx).$$
which, in view of (41), coincides with (8) to be proved. To get (16), we start from considering the right-hand side of the last inequality in (9). For the first summand,

$$\mathcal{W}_2^{(P(\cdot))}(\pi^*_n(\cdot|\kappa_0); \delta_{\theta_0})$$

$$= \left( \frac{\int_\Theta \|\theta - \theta_0\|^2 \exp \left\{ n \int_X \log f(y|\theta) f(y|\theta_0) dy \right\} \pi(\theta) }{\int_\Theta \exp \left\{ n \int_X \log f(y|\theta) f(y|\theta_0) dy \right\} \pi(\theta) } \right)^{1/2}$$

so that it is enough to observe that

$$\int_X \log f(y|\theta) f(y|\theta_0) dy$$

$$= \int_X \log f(y|\theta_0) f(y|\theta_0) dy + \int_X \left[ \log \left( \frac{f(y|\theta)}{f(y|\theta_0)} \right) \right] f(y|\theta_0) dy$$

$$= H(\theta_0) - K(\theta|\theta_0)$$.

Whence,

$$\mathcal{W}_2^{(P(\cdot))}(\pi^*_n(\cdot|\kappa_0); \delta_{\theta_0}) = \left( \frac{\int_\Theta \|\theta - \theta_0\|^2 e^{n[H(\theta_0) - K(\theta|\theta_0)]}\pi(\theta) }{\int_\Theta e^{n[H(\theta_0) - K(\theta|\theta_0)]}\pi(\theta) } \right)^{1/2}$$

Then, the second summand on the right-hand side of (16) is already provided by the second summand on the the right-hand side of the last inequality in (9). Finally, the last two terms on the the right-hand side of (16) comes from the last summand on the the right-hand side of (9), after noticing that we have

$$\mathcal{W}_2^{(P(\cdot))}(\pi^*_n(\cdot|\kappa_0); \pi^*_n(\cdot|\xi^{(i)})) \leq \sqrt{2 \int_\Theta \|\theta\|^2 \pi^*_n(\theta|\kappa_0) + \sqrt{2 \int_\Theta \|\theta\|^2 \pi^*_n(\theta|\xi^{(i)}))}.$$
of Bolley et al. [11]. For the last term, we start from applying, in combination, Hölder and Lyapunov’s inequalities to get

\[
E \left[ \left( \int_{\Theta} \|\theta\|^2 \prod_{i=1}^{\eta} f(\xi_i|\theta) \pi(d\theta) \right)^{1/2} \mathbb{1}\{c_{n}^{(}\xi_{n}^{(}\not\in V_{0}^{(n)}\}) \right] \\
\leq \left\{ E \left[ \int_{\Theta} \|\theta\|^r \pi_{n}(d\theta|\xi_{1}, \ldots, \xi_{n}) \right] \right\}^{1/r} \cdot \left\{ P[\xi_{n}^{(}\not\in V_{0}^{(n)}\}] \right\}^{(r-1)/r}.
\]

The conclusion of the proof then follows by using (18), again by a direct combination with respect to the bound borrowed from Theorem 2.7 of Bolley et al. [11].

A.4. Proof of Theorem 2. For fixed \( \delta > 0 \), put \( N := N_{\delta}(\mathbb{X}, d_{\mathbb{X}}) \) and indicate by \( \{ A_{j,\delta} \}_{j=1,\ldots,N} \) a partition of \( \mathbb{X} \) such that \( \text{diam}(A_{j,\delta}) \leq 2\delta \) and \( \mu_{1}(\partial A_{j,\delta}) = 0 \) for all \( j \in \{1, \ldots, N\} \), where \( \mu_{1} \) is defined in (2). Then, define \( l_{\delta} : \mathbb{X} \to \{1, \ldots, N\} \) by the rule that, for any \( x \in \mathbb{X} \), \( l_{\delta}(x) = j \) iff \( x \in A_{j,\delta} \). Choose a point \( a_{j,\delta} \in A_{j,\delta} \) for any \( j \in \{1, \ldots, N\} \), and put \( \mathbb{X}_{\delta} := \{ a_{j,\delta} \}_{j=1,\ldots,N} \) and \( T_{\delta}(x) := a_{l_{\delta}(x),\delta} \), for any \( x \in \mathbb{X} \). Consider the new sequences \( \{ Y_{i} \}_{i \geq 1} \) and \( \{ \eta_{i} \}_{i \geq 1} \) of \( \mathbb{X}_{\delta} \)-valued random variables, given by \( Y_{i} := T_{\delta}(X_{i}) \) and \( \eta_{i} := T_{\delta}(\xi_{i}) \) for any \( i \in \mathbb{N} \), respectively. Of course, the sequences \( \{ Y_{i} \}_{i \geq 1} \) is formed by exchangeable random variables, with directing measure

\[
\tilde{\nu} \circ T_{\delta}^{-1} = \sum_{j=1}^{N} \tilde{\nu}(A_{j,\delta})\delta_{a_{j,\delta}}.
\]

This fact actually follows from the mapping theorem for weak convergence, recalling that \( \mu_{1}(\partial A_{j,\delta}) = 0 \) for all \( j \in \{1, \ldots, N\} \). Now, for any \( n \in \mathbb{N} \), for any \( y_{1}, \ldots, y_{n} \in \mathbb{X}_{\delta} \) and for any \( j \in \{1, \ldots, N\} \), introduce the following notation

\[
\nu_{\delta}(j; y_{1}, \ldots, y_{n}) := \sum_{i=1}^{n} \mathbb{1}\{y_{i} = a_{j,\delta}\},
\]

by which

\[
P[\tilde{\nu} \in B \mid Y^{(n)}] = \frac{\int_{B} \prod_{j=1}^{N} [\tilde{\nu}(A_{j,\delta})]^{\nu_{\delta}(j; Y^{(n)})} \pi(d\tilde{\nu})}{\int_{\mathcal{P}(\mathbb{X})} \prod_{j=1}^{N} [\tilde{\nu}(A_{j,\delta})]^{\nu_{\delta}(j; Y^{(n)})} \pi(d\tilde{\nu})} \quad (P - a.s.)
\]

holds for all \( B \in \mathcal{B} := \mathcal{B}(\mathcal{P}(\mathbb{X})) \), where \( Y^{(n)} := (Y_{1}, \ldots, Y_{n}) \). In particular, this identity suggests the introduction of the probability kernel \( \Gamma_{N} \) given by

\[
\Gamma_{N}(B|\nu_{1}, \ldots, \nu_{N}) := \frac{\int_{B} \prod_{j=1}^{N} [\tilde{\nu}(A_{j,\delta})]^{\nu_{j}} \pi(d\tilde{\nu})}{\int_{\mathcal{P}(\mathbb{X})} \prod_{j=1}^{N} [\tilde{\nu}(A_{j,\delta})]^{\nu_{j}} \pi(d\tilde{\nu})}
\]

for \( (B; \nu_{1}, \ldots, \nu_{N}) \in \mathcal{B} \times \mathbb{N}_{0}^{N} \), where \( \mathbb{N}_{0} := \{0\} \cup \mathbb{N} \). Now, define the random probability measure

\[
\Gamma_{N}(B|\eta_{1}, \ldots, \eta_{n}) := \Gamma_{N}(B|\nu_{6}(1; \eta_{(n)}^{(1)}), \ldots, \nu_{6}(N; \eta_{(n)}))
\]

for any \( B \in \mathcal{B} \), where \( \eta^{(n)} := (\eta_{1}, \ldots, \eta_{n}) \). Now, let consider the finite-dimensional distribution of the prior \( \pi \) relative to the partition \( \{ A_{j,\delta} \}_{j=1,\ldots,N} \), i.e.

\[
\pi_{N}(D; A_{1,\delta} \ldots A_{N,\delta}) := P[(\tilde{\nu}(A_{1,\delta}), \ldots, \tilde{\nu}(A_{N-1,\delta})) \in D]
\]
for any $D \in \mathcal{B}(\Delta_{N-1})$. Then, define
\[
\Upsilon_N(D \mid \nu_1, \ldots, \nu_N) := \frac{\int_D \prod_{j=1}^{N-1} u_j^{\nu_j} \left( 1 - \sum_{j=1}^{N-1} u_j \right)^{\nu_N} \pi_N(du; A_{1,\delta} \ldots A_{N,\delta})}{\int_{\Delta_{N-1}} \prod_{j=1}^{N-1} u_j^{\nu_j} \left( 1 - \sum_{j=1}^{N-1} u_j \right)^{\nu_N} \pi_N(du; A_{1,\delta} \ldots A_{N,\delta})}
\]
for any $(D; \nu_1, \ldots, \nu_N) \in \mathcal{B}(\Delta_{N-1}) \times \mathbb{N}_0^N$. Denoting by $G_\delta : \Delta_{N-1} \to \mathcal{P}(\mathcal{X}_\delta)$ the one-to-one mapping which sends the probability masses $u \in \Delta_{N-1}$ into the probability measure $\sum_{j=1}^N u_j \delta_{u_j}$, where $u_N := 1 - \sum_{j=1}^{N-1} u_j$, observe that
\[
P[p \circ T_\delta^{-1} \in F \mid Y^{(n)}] = \Upsilon_N(G_\delta^{-1}(F) \mid \nu_\delta(1; Y^{(n)}), \ldots, \nu_\delta(N; Y^{(n)})) \quad (P - a.s.)
\]
holds for any $F \in \mathcal{B}(\mathcal{P}(\mathcal{X}_\delta))$. Now, if $\iota_\delta : \mathcal{X}_\delta \to \mathcal{X}$ indicates the inclusion map, notice that $I_\delta : p \mapsto p \circ \iota_\delta^{-1}$ induces the inclusion of $\mathcal{P}(\mathcal{X}_\delta)$ into $\mathcal{P}(\mathcal{X})$. Lastly, set
\[
\Sigma_N(B; \nu_1, \ldots, \nu_N) := \Upsilon_N(G_\delta^{-1}(I_\delta^{-1}(B)) \mid \nu_1, \ldots, \nu_N)
\]
for any $(B; \nu_1, \ldots, \nu_N) \in \mathcal{P} \times \mathbb{N}_0^N$ and
\[
\Sigma_N(B; \eta^{(n)}) := \Sigma_N(B; \nu_\delta(1; \eta^{(n)}), \ldots, \nu_\delta(N; \eta^{(n)})) .
\]
Upon noticing that
\[
\mathcal{W}_p^{(F)}(\delta_{\epsilon_n}; \delta_{\epsilon_\nu}) = \mathcal{W}_p^{(\mathcal{P}(\mathcal{X}))}(\epsilon_n; \epsilon_\nu)
\]
and
\[
\mathcal{W}_p^{(F)}(\delta_{\epsilon_\nu}; \delta_{p_0}) = \mathcal{W}_p^{(\mathcal{P}(\mathcal{X}))}(\epsilon_n; p_0) ,
\]
there are now all the elements to rigorously take account of the inequality displayed in (11).
The remaining part of the present proof shows how to manipulate each of the five terms on the right-hand side of (11) in order to get (22).

For the first term on the right-hand side of (11), observe preliminarily that
\[
P[p \in B \mid Y^{(n)}] = \frac{\int_{C^{(n)}[Y^{(n)}]} \pi_n(B \mid x^{(n)}) \mu_n(dx^{(n)})}{\mu_n(C^{(n)}[Y^{(n)]})} \quad (P - a.s.)
\]
is valid by de Finetti’s representation theorem, where
\[
C^{(n)}[Y^{(n)}] = \times_{j=1}^N A_{\nu_\delta(1; Y^{(n)})-times}^{\nu_\delta(N; Y^{(n)})-times} .
\]
Whence,
\[
\Gamma_N^{\ast}(B; \eta_1, \ldots, \eta_n) = \frac{\int_{C^{(n)}[\eta^{(n)}]} \pi_n(B \mid x^{(n)}) \mu_n(dx^{(n)})}{\mu_n(C^{(n)}[\eta^{(n)]})} \quad (P - a.s.)
\]
and, by convexity of $\mathcal{W}_p^{(F)}$ (see Villani [51, Chapter 7]),
\[
\left[ \mathcal{W}_p^{(F)}(\pi_n(\cdot \mid \xi_1, \ldots, \xi_n); \Gamma_N^{\ast}(\cdot \mid \eta_1, \ldots, \eta_n)) \right]^p \leq \frac{\int_{C^{(n)}[\eta^{(n)}]} \mathcal{W}_p^{(F)}(\pi_n(\cdot \mid x^{(n)}); \pi_n(\cdot \mid \xi^{(n)})) \mu_n(dx^{(n)})}{\mu_n(C^{(n)}[\eta^{(n)]})} .
\]
At this stage, an application of (21) yields
\[ \mathcal{W}_p^{(\mathcal{P}(\mathcal{X}))}(\pi_n(\cdot|\xi^{(n)}); \pi_n(\cdot|\eta^{(n)})) \leq L_n \mathcal{W}_p^{(\mathcal{P}(\mathcal{X}))}(\epsilon_n^{(x)}; \epsilon_n^{(\xi)}) \quad (\mathcal{P} - \text{a.s.}) \]
for any \( x^{(n)} \in C(n)[\eta^{(n)}] \). In particular, the definition of the relation \( x^{(n)} \in C(n)[\eta^{(n)}] \) entails
\[ \sum_{i=1}^n \mathbb{1}\{x_i \in A_{j,\delta}\} = \sum_{i=1}^n \mathbb{1}\{x_i \in A_{j,\delta}\} \]
for all \( j \in \{1, \ldots, N\} \). Therefore, a direct application of the Birkhoff theorem (see Subsection 3.1) shows that \( \mathcal{W}_p^{(\mathcal{P}(\mathcal{X}))}(\epsilon_n^{(x)}; \epsilon_n^{(\xi)}) \leq 2\delta, \mathcal{P}-\text{a.s.} \). Whence, we write
\[ \left[ \mathcal{W}_p^{(\mathcal{P}(\mathcal{X}))}(\pi_n(\cdot|\xi_1, \ldots, \xi_n); \Gamma_N(\cdot|\eta_1, \ldots, \eta_n)) \right]^p \leq \frac{\int_{C(n)[\eta^{(n)}]} (2L_n\delta)^p \mu_n(dx^{(n)})}{\mu_n(C(n)[\eta^{(n)}])} = (2L_n\delta)^p \quad \mathcal{P} - \text{a.s.} \]

As for the second term on the right-hand side of Equation (11), notice preliminarily that, for any bounded and measurable function \( h: \mathcal{P}(\mathcal{X}) \to \mathbb{R} \), we have
\[ \int_{\mathcal{P}(\mathcal{X})} h(p) \Sigma_N(dp | \nu_1, \ldots, \nu_N) = \int_{\mathcal{P}(\mathcal{X})} h(p \circ T_\delta^{-1}) \Gamma_N(dp | \nu_1, \ldots, \nu_N) \]
holds for all \((\nu_1, \ldots, \nu_N) \in \mathbb{N}^N_0\). Accordingly, there exists a distinguished coupling between the probability kernels \( \Sigma_N(\cdot | \nu_1, \ldots, \nu_N) \) and \( \Gamma_N(\cdot | \nu_1, \ldots, \nu_N) \) that yields
\[ \mathcal{W}_p^{(\mathcal{P}(\mathcal{X}))}(\Sigma_N(\cdot | \nu_1, \ldots, \nu_N); \Gamma_N(\cdot | \nu_1, \ldots, \nu_N)) \leq \left( \int_{\mathcal{P}(\mathcal{X})} \left[ \mathcal{W}_p^{(\mathcal{P}(\mathcal{X}))}(p; p \circ T_\delta^{-1}) \right]^p \Gamma_N(dp | \nu_1, \ldots, \nu_N) \right)^{1/p} \]
for all \((\nu_1, \ldots, \nu_N) \in \mathbb{N}^N_0\). At this stage, again by choosing a distinguished coupling, one has
\[ \left[ \mathcal{W}_p(p; p \circ T_\delta^{-1}) \right]^p \leq \int_{\mathcal{X}} [\delta \mathcal{X}(x, T_\delta(x))]^p p(dx) . \]
Then, by means of the law of total probability, the above right-hand side is equal to
\[ \sum_{j=1}^N p(A_{j,\delta}) \int_{\mathcal{X}} [\delta \mathcal{X}(x, T_\delta(x))]^p p(dx)|A_{j,\delta}| , \]
where \( p(|A_{j,\delta}|) \) denotes the probability measure \( A \mapsto p(A \cap A_{j,\delta})/p(A_{j,\delta}) \) in the case that \( p(A_{j,\delta}) > 0 \) and any other probability measure \( (\cdot, p_0(\cdot)) \) in the case that \( p(A_{j,\delta}) = 0 \). As far as those \( A_{j,\delta} \)’s for which \( p(A_{j,\delta}) > 0 \), one gets
\[ \int_{\mathcal{X}} [\delta \mathcal{X}(x, T_\delta(x))]^p p(dx)|A_{j,\delta}| \leq (2\delta)^p \]
for all \( j \in \{1, \ldots, N\} \), because \( p(\cdot|A_{j,\delta}) \) is supported in \( A_{j,\delta} \). Therefore, conclude that the second term on the right-hand side of (11) is globally bounded by \( 2\delta \).
In view of the definition of the probability kernel $\Sigma_N(\cdot|\nu_1,\ldots,\nu_N)$, the argument of the expectation in the third term on the right-hand side of (11), can be rewritten as follows:

\[
\mathcal{W}_p^{(\mathbb{P})} \left( \delta_{\epsilon_n}; \Sigma_N(\cdot|\nu_1,\ldots,\nu_N) \right)
= \left( \int_{\mathcal{P}(\mathcal{X})} \mathcal{W}_p^{(\mathbb{P}(\mathcal{X}))}(\epsilon_n; p)^p \Sigma_N(dp|\nu_1,\ldots,\nu_N) \right)^{1/p}
= \left( \int_{\Delta_{N-1}} \mathcal{W}_p^{(\mathbb{P}(\mathcal{X}))}(\epsilon_n; \sum_{j=1}^N u_j \delta_{\xi_j})^p \mathcal{Y}_N(du|\nu_1,\ldots,\nu_N) \right)^{1/p}.
\]

Now, observe that

\[
\epsilon_n(q) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^N 1\{\eta_i = a_{j,\delta}\} \delta_{\xi_j} = \sum_{j=1}^N \phi_{j,n}(\xi_j),
\]

where $\phi_{j,n}(\xi_j) := \frac{1}{n} \sum_{i=1}^n 1\{\xi_i \in A_{j,\delta}\}$. Thus, upon noticing that $\epsilon_n(q)$ and $\sum_{j=1}^N u_j \delta_{\xi_j}$ are both supported on $X_\delta$, the fact the total variation distance is given by an optimal coupling (see, e.g., Gibbs and Su [31, pag. 424]) yields the following

\[
\left\lVert \mathcal{W}_p^{(\mathbb{P}(\mathcal{X}))}(\epsilon_n; \sum_{j=1}^N u_j \delta_{\xi_j})^p \leq \frac{1}{2} \text{diam}(\mathcal{X})^p \sum_{j=1}^N |\phi_{j,n}(\xi_j) - u_j| \right. \quad .
\]

Therefore, $\mathbb{P}$-a.s., it holds

\[
\mathcal{W}_p^{(\mathbb{P})} \left( \delta_{\epsilon_n}; \Sigma_N(\cdot|\nu_1,\ldots,\nu_N) \right)
\leq \text{diam}(\mathcal{X})^p \left( \frac{1}{2} \int_{\Delta_{N-1}} \left| \sum_{j=1}^N \phi_{j,n}(\xi_j) - u_j \right| \mathcal{Y}_N(du|\nu_1,\ldots,\nu_N) \right)^{1/p}
\]

so that it is worth studying the integral

\[
\int_0^1 |\phi_{j,n}(\xi_j) - u_j| \mathcal{Y}_{N,j}(du_j|\nu_1,\ldots,\nu_N) = \int_0^1 |\phi_{j,n}(\xi_j) - t l^{n\phi_{j,n}}(1-t)^n(1-\phi_{j,n}) \pi_{N,j}(dt) - \int_0^1 l^{n\phi_{j,n}}(1-t)^n(1-\phi_{j,n}) \pi_{N,j}(dt) |
\]

where $\mathcal{Y}_{N,j}(\cdot|\nu_1,\ldots,\nu_N)$ denotes the $j$-th marginal of $\mathcal{Y}_{N}(\cdot|\nu_1,\ldots,\nu_N)$, and $\pi_{N,j}(\cdot) = \mathbb{P}[^{\mathcal{P}}(A_{j,\delta} \in \cdot)]$. Upon putting

\[
m_{j,n}(\pi_{N,j}) := \int_0^1 l^{n\phi_{j,n}}(1-t)^n(1-\phi_{j,n}) \pi_{N,j}(dt)
\]

and

\[
u_{j,n}(\pi_{N,j}) := \int_0^1 |t - m_{j,n}(\pi_{N,j})| l^{n\phi_{j,n}}(1-t)^n(1-\phi_{j,n}) \pi_{N,j}(dt)
\]

we obtain

\[
m_{j,n}(\pi_{N,j}) := \frac{1}{2} \int_0^1 l^{n\phi_{j,n}}(1-t)^n(1-\phi_{j,n}) \pi_{N,j}(dt)
\]

and

\[
u_{j,n}(\pi_{N,j}) := \frac{1}{2} \int_0^1 |t - m_{j,n}(\pi_{N,j})| l^{n\phi_{j,n}}(1-t)^n(1-\phi_{j,n}) \pi_{N,j}(dt)
\]
one has
\[
\mathbb{E} \left[ W_p^{(p)} \left( \delta_n^{(\eta)}; \Sigma_N(\cdot \mid \nu_\delta(1; \eta^{(m)}), \ldots, \nu_\delta(N; \eta^{(m)})) \right) \right] \\
\leq \text{diam}(X) \left\{ \frac{1}{2} \sum_{j=1}^{N} \mathbb{E} \left[ |\phi_{j,n}^{(\xi)} - m_{j,n}^{(\xi)}(\pi_{N,j})| + \sqrt{\nu_{j,n}^{(\xi)}(\pi_{N,j})} \right] \right\}^{1/p} \\
=: \text{diam}(X) \left\{ \frac{1}{2} \left[ M_n(N, \delta) + V_n(N, \delta) \right] \right\}^{1/p} 
\tag{48}
\]

The asymptotic evaluation of the terms \(|\phi_{j,n}^{(\xi)} - m_{j,n}^{(\xi)}(\pi_{N,j})|\) and \(\nu_{j,n}^{(\xi)}(\pi_{N,j})\) is a longstanding topic in classical probability theory and Bayesian theory, going back to Laplace. Note that a precise asymptotic expansion can be found in Johnson [37, Section 3] and DasGupta [17, Chapter 20], where it is shown that, under suitable regularity assumptions on \(\pi_{N,j}\), both of these terms are of order 1/\(p\).

The fourth term on the right-hand side of Equation (11) contains the term \(W_p^{(P(X))}(\xi_n^{(\eta)}; \xi_n^{(\eta)})\), which can be bounded in view of the Birkhoff theorem as follows
\[
W_p^{(P(X))}(\xi_n^{(\eta)}; \xi_n^{(\eta)}) \leq \left( \frac{1}{n} \sum_{i=1}^{n} [d_X(\xi_i; \eta_i)]^p \right)^{1/p}.
\]

By definition of \(\eta_i\), notice that \(d_X(\xi_i; \eta_i) \leq \text{max}_{j=1,\ldots,N} \text{diam}(A_{j,\delta}) \leq 2\delta\), yielding
\[
W_p^{(P(X))}(\xi_n^{(\eta)}; \xi_n^{(\eta)}) \leq 2\delta \quad \text{P-a.s.}
\]

Finally, recalling that \(\kappa_0 = p_0\), observe that the fifth term on the right-hand side of Equation (11) just coincide with \(\varepsilon_{n,p}(X, \kappa_0)\). This concludes the proof.

A.5. **Proof of Corollary 2.** To prove the corollary, put \(\delta \sim n^{-\beta}\) for some \(\beta > 0\), so that \(N_{\delta_n} \sim n^{d\beta}\). Therefore, leaving the Glivenko-Cantelli-term \(\varepsilon_{n,p}\) out of these computations, check that the term 2(2 + \(L_n\))\(\delta_n\) is asymptotic to \(n^{s-\beta}\), as \(n \to +\infty\). Taking account of (23), the last term on the right-hand side of (22) has an upper bound which is asymptotic to \(n^{-(\alpha-d\beta)/p}\). At this stage, by equalizing the two exponents, that is by imposing
\[
\beta - s = \frac{\alpha - d\beta}{p},
\]
yields \(\beta = \frac{\alpha + pd}{d + p}\). The assumption \(\alpha > sd\) entails that \(\frac{\alpha + pd}{d + p} \in (s, \alpha/d)\). Therefore, the sum of the last two terms on the right-hand side of (22) is bounded by a term which is globally asymptotic to \(n^{-(\alpha-sd)/(d+p)}\). This concludes the proof.

A.6. **An inequality by Diaconis and Freedman.** Let \(\chi\) be a probability measure on \([0, 1]\). For any \(h \in (0, \frac{1}{4})\), put
\[
\phi(h) := \inf_{p \in [0, 1]} \chi([0, 1] \cap [p, p + h]) \quad (49)
\]
and, for any \(p \in [0, 1]\),
\[
R(n, p, h) := \frac{\int_{[0, 1] \cap [p-h, p+h]} t^{np}(1-t)^{n(1-p)} \chi(dt)}{\int_{[0, 1] \cap [p-h, p+h]} t^{np}(1-t)^{n(1-p)} \chi(dt)}. 
\]
Note that, if \( \inf_{h \in (0, \frac{1}{4})} \phi(h) > 0 \), then the above ratio is well-defined since the denominator is positive. In addition, put

\[
H(p, t) := -p \log t - (1 - p) \log(1 - t)
\]

and

\[
g(h) := \inf_{p, t \in [0, 1], \ |p - t| \geq h} \{H(p, t) - H(p, p)\}.
\]

Finally, setting

\[
h^* := \min \{h, \frac{1}{2} g(h) \} - \frac{1}{2} g(h) g(h) - 2h^2
\]

and

\[
\psi(h) := \phi(h^*) \tag{50}
\]

there holds

\[
R(n, p, h) \geq \psi(h) e^{2nh^2} \tag{51}
\]

for all \( n \in \mathbb{N} \), all \( p \in [0, 1] \) and all \( h \in (0, \frac{1}{4}) \).

We now show how to use inequality (51) to get a bound as in (23). Actually, it is enough to deal with the terms \( M_n(N, \delta) \), the treatment of the other term \( V_n(N, \delta) \) being analogous. For fixed \( j \in \{1, \ldots, N\} \), we have that

\[
|\phi_{j,n}(\xi) - m_{j,n}(\pi_{N,j})| \leq h + \frac{1}{R(n, \phi_{j,n}(\xi), h)} \leq h + \frac{e^{-2nh^2}}{\psi(h)}
\]

for any \( h \in (0, \frac{1}{4}) \), so that we can minimize the last expression with respect to \( h \). If \( \psi(h) \sim h^s \) as \( h \to 0^+ \), for some \( s > 0 \) (possibly depending on \( N \)), then the minimization of the last expression turns out to be asymptotic to \( \sqrt{(s+1) \log n} \).

A.7. Proof of Proposition 1. First, if there exists a distinguished solution \( \pi^*_{n}(\cdot|\cdot) \) of (1) satisfying (21) for all \( x^{(n)}, y^{(n)} \in \mathcal{X}^n \), then we can define

\[
\alpha^*_{n}(A|x^{(n)}) := \int_{\mathcal{P}(\mathcal{X})} p(A) \pi^*_{n}(dp|x^{(n)})
\]

and notice that the above probability kernel \( \alpha^*_{n}(\cdot|\cdot) \) fulfills both (3), by Fubini’s theorem, and (25), by the convexity of \( \mathcal{W}_1 \). On the other hand, we now suppose that (25) is in force for some (unique) kernel \( \alpha^*_{n}(\cdot|\cdot) \) satisfying (3). The key observation is that the posterior distribution can be obtained as weak limit of

\[
P \left[ \frac{1}{m} \sum_{i=1}^{m} \delta_{X_{i+n}} \in \cdot | X_1, \ldots, X_n \right]
\]

as \( m \to +\infty \), by an application of de Finetti’s theorem. Therefore, in view of the well-known Kantorovic-Rubinstein dual representation (see, e.g., Dudley [25, Chapter 11]), the thesis to
be proved is implied by the validity of inequality

\[
\sup_{\varphi: \mathcal{P}_1(\mathbb{X}) \to \mathbb{R}} 1 - \text{Lipschitz}(\mathcal{W}_1) \left| E \left[ \varphi \left( \frac{1}{m} \sum_{i=1}^{m} \delta_{X_{i+n}} \right) \right] \right| X^{(n)} = x^{(n)} \right| - E \left[ \varphi \left( \frac{1}{m} \sum_{i=1}^{m} \delta_{X_{i+n}} \right) \right] \left| X^{(n)} = y^{(n)} \right| \leq L \mathcal{W}_1^{(P(\mathbb{X}))}(\epsilon^{(x)}_n; \epsilon^{(y)}_n) \tag{52}
\]

for any \( n, m \in \mathbb{N} \) and \( x^{(n)}, y^{(n)} \in \mathbb{X}^n \), where the notation \( 1 - \text{Lipschitz}(\mathcal{W}_1) \) indicates that \( |\varphi(p_1) - \varphi(p_2)| \leq \mathcal{W}_1^{(P(\mathbb{X}))}(p_1; p_2) \) for all \( p_1, p_2 \in \mathcal{P}(\mathbb{X}) \). We observe that an equivalent formulation could be given in terms of the \( m \)-predictive distributions on the quotient metric space \((\mathbb{X}^m \setminus \sim, d^X_\sim)\) (as presented in Subsection 3.1), say \( \alpha_n^{(m)}(\cdot|X_1, \ldots, X_n) \), which is given by the expression

\[
\alpha_n^{(m)}(C_m|X_1, \ldots, X_n) := P \left[ \frac{1}{m} \sum_{i=1}^{m} \delta_{X_{i+n}} \in \epsilon_m \right. \left. \left( \text{quotient}^{-1}(C_m) \right) \right| X^{(n)} \]
\]

for any \( C_m \in \mathcal{B}(\mathbb{X}^m \setminus \sim) \). Now, we refer to the abstract scheme in Subsection 3.1, i.e. Figure 1, for the explanation of the symbols. Hereafter, we prove (52) by induction on \( m \). First, we notice that, for \( m = 1 \), inequality (52) boils down to (25), where we have chosen just \( \alpha_n^{(1)}(\cdot|\cdot) \) as representative, according to

\[
E \left[ \varphi(\delta_{X_{n+1}}) \right] \left| X^{(n)} = x^{(n)} \right| = \int_{\mathbb{X}} \varphi(\delta_z)\alpha_n^{*}(dz|x^{(n)}) .
\]

Then, we assume that (52) is true for some \( m \) and we prove that it is true for \( m+1 \). We denote by \( \alpha_{n,m}^{*}(\cdot|\cdot) \) a version of the \( m \)-predictive distribution (i.e., the conditional distributions of \((X_{n+1}, \ldots, X_{n+m}) \) given \((X_1, \ldots, X_n)\)) for which (52) is true for any \( x^{(n)}, y^{(n)} \in \mathbb{X}^n \). Hence, we notice that we can exploit the tower property of the predictive distributions to select \( \alpha_{n,m+1}^{*}(\cdot|\cdot) \) so that

\[
\begin{align*}
E \left[ \varphi \left( \frac{1}{m+1} \sum_{i=1}^{m+1} \delta_{X_{i+n}} \right) \right] \left| X^{(n)} = x^{(n)} \right] &= \int_{\mathbb{X}^{m+1}} \varphi \left( \frac{1}{m+1} \sum_{i=1}^{m+1} \delta_{z_i} \right) \alpha_{n,m+1}^{*}(dz_1, \ldots, dz_{m+1} | x^{(n)}) \\
&= \int_{\mathbb{X}^m} \int_{\mathbb{X}} \varphi \left( \frac{1}{m+1} \sum_{i=1}^{m+1} \delta_{z_i} \right) \alpha_{n+1,m}^{*}(dz_1, \ldots, dz_m | x^{(n)}, z_{m+1}) \alpha_{n,1}^{*}(dz_{m+1} | x^{(n)}) .
\end{align*}
\]
At this stage, the left-hand side of (52) with $m + 1$ in place of $m$ can be bounded by the supremum, over those $\varphi : \mathcal{P}_1(\mathbb{X}) \to \mathbb{R}$ which are 1-Lipschitz($\mathcal{W}_1$), of

$$
\left| \int_{\mathbb{X}^m} \int_{\mathbb{X}} \varphi \left( \frac{1}{m + 1} \sum_{i=1}^{m+1} \delta_{z_i} \right) \alpha_{n+1,m}^*(dz_1, \ldots, dz_m \mid x^{(n)}, z_{m+1}) \alpha_{n,1}^*(dz_{m+1} \mid x^{(n)}) \right.
$$

$$
- \left| \int_{\mathbb{X}^m} \int_{\mathbb{X}} \varphi \left( \frac{1}{m + 1} \sum_{i=1}^{m+1} \delta_{z_i} \right) \alpha_{n+1,m}^*(dz_1, \ldots, dz_m \mid x^{(n)}, z_{m+1}) \alpha_{n,1}^*(dz_{m+1} \mid y^{(n)}) \right|
$$

$$
+ \left| \int_{\mathbb{X}^m} \int_{\mathbb{X}} \varphi \left( \frac{1}{m + 1} \sum_{i=1}^{m+1} \delta_{z_i} \right) \alpha_{n+1,m}^*(dz_1, \ldots, dz_m \mid x^{(n)}, z_{m+1}) \alpha_{n,1}^*(dz_{m+1} \mid y^{(n)}) \right|
$$

$$
- \left| \int_{\mathbb{X}^m} \int_{\mathbb{X}} \varphi \left( \frac{1}{m + 1} \sum_{i=1}^{m+1} \delta_{z_i} \right) \alpha_{n+1,m}^*(dz_1, \ldots, dz_m \mid y^{(n)}, z_{m+1}) \alpha_{n,1}^*(dz_{m+1} \mid y^{(n)}) \right|
$$

which, for notational ease, we shorten as $|A| + |B|$. As to the term $|B|$, we observe that

$$
|B| \leq \int_{\mathbb{X}^m} \int_{\mathbb{X}} \varphi \left( \frac{1}{m + 1} \sum_{i=1}^{m+1} \delta_{z_i} \right) \alpha_{n+1,m}^*(dz_1, \ldots, dz_m \mid x^{(n)}, z_{m+1})
$$

$$
- \int_{\mathbb{X}^m} \varphi \left( \frac{1}{m + 1} \sum_{i=1}^{m+1} \delta_{z_i} \right) \alpha_{n+1,m}^*(dz_1, \ldots, dz_m \mid y^{(n)}, z_{m+1}) \alpha_{n,1}^*(dz_{m+1} \mid y^{(n)})
$$

and, since the mapping $\mathbb{X}^m \ni \sum_{i=1}^{m+1} \delta_{z_i}$ is $\frac{m}{m+1}$-Lipschitz for any fixed $z_{m+1}$,

$$
|B| \leq L \frac{m}{m + 1} \mathcal{W}_1 \left( \frac{1}{n + 1} \left[ \delta_{z_{m+1}} + \sum_{i=1}^{n} \delta_{x_i} \right] ; \frac{1}{n + 1} \left[ \delta_{z_{m+1}} + \sum_{i=1}^{n} \delta_{y_i} \right] \right)
$$

$$
= L \frac{m}{m + 1} \frac{n}{n + 1} \mathcal{W}_1 (\alpha_n^{(x)}; \alpha_n^{(y)})
$$

(53)

by the convexity of $\mathcal{W}_1$. As to the term $A$, we notice that it can be written as follows

$$
\int_{\mathbb{X}} \Phi_{x^{(n)},n}(z) \alpha_{n,1}^*(dz \mid y^{(n)}) - \int_{\mathbb{X}} \Phi_{x^{(n)},n}(z) \alpha_{n,1}^*(dz \mid x^{(n)})
$$

with

$$
\Phi_{x^{(n)},n}(z) := \int_{\mathbb{X}^m} \varphi \left( \frac{1}{m + 1} \left[ \delta_{z} + \sum_{i=1}^{m} \delta_{z_i} \right] \right) \alpha_{n+1,m}^*(dz_1, \ldots, dz_m \mid x^{(n)}, z).
$$
If we write
\[
\left| \Phi_{x^{(n)},n}(u) - \Phi_{x^{(n)},n}(v) \right| \\
\leq \left| \int_{X^m} \varphi \left( \frac{1}{m+1} \left[ \delta_u + \sum_{i=1}^{m} \delta_{z_i} \right] \right) \alpha_{n+1,m}^* (d z_1, \ldots, d z_m \mid x^{(n)}, u) \\
- \int_{X^m} \varphi \left( \frac{1}{m+1} \left[ \delta_u + \sum_{i=1}^{m} \delta_{z_i} \right] \right) \alpha_{n+1,m}^* (d z_1, \ldots, d z_m \mid x^{(n)}, v) \\
+ \int_{X^m} \left| \varphi \left( \frac{1}{m+1} \left[ \delta_u + \sum_{i=1}^{m} \delta_{z_i} \right] \right) - \varphi \left( \frac{1}{m+1} \left[ \delta_v + \sum_{i=1}^{m} \delta_{z_i} \right] \right) \right| \right|
\]
\[
\alpha_{n+1,m}^* (d z_1, \ldots, d z_m \mid x^{(n)}, u),
\]
we conclude that, uniformly in \( x^{(n)} \in X^n \), the function \( z \mapsto \Phi_{x^{(n)},n}(z) \) is \( \left( \frac{m}{m+1} \frac{1}{n+1} + \frac{1}{m+1} \right) \)-Lipschitz, so that
\[
|A| \leq L \left( \frac{m}{m+1} \frac{1}{n+1} + \frac{1}{m+1} \right) W_1(\epsilon_n^{(x)}; \epsilon_n^{(y)}).
\]
Combining this last inequality with (53) yields (52) for any \( m \in \mathbb{N} \) and, hence, (21).

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