Rare Kaon Decays in the $1/N_c$-Expansion

C. Bruno and J. Prades

Centre de Physique Théorique, C.N.R.S. - Luminy, Case 907
F-13288 Marseille Cedex 9, France

Abstract

We study the unknown coupling constants that appear at $\mathcal{O}(p^4)$ in the Chiral Perturbation Theory analysis of $K \to \pi \gamma^* \to \pi l^+ l^-$, $K^{+\,-} \to \pi^+^- \gamma \gamma$ [1] and $K \to \pi \pi \gamma$ [2] decays. To that end, we compute the chiral realization of the $\Delta S = 1$ Hamiltonian in the framework of the $1/N_c$-expansion of the low-energy action proposed in Ref. [3]. The phenomenological implications are also discussed.
1 Introduction

In this work we shall study the unknown coupling constants that govern the analysis of $K \to \pi\gamma^{*} \to \pi l^{+} l^{-}$, $K^{+} \to \pi^{+} \gamma \gamma$ \cite{1} and $K \to \pi \pi \gamma$ \cite{2} decays at $O(p^4)$ in the framework of Chiral Perturbation Theory (ChPT). We shall restrict ourselves to the non-anomalous sector of the theory. (For a recent discussion of anomalous non-leptonic decays see Ref. \cite{2}.) These transitions have become particularly interesting because of possible CP-violation effects in rare $K$-decays. The various tests on this subject discussed in Ref. \cite{1} depend on the values of these coupling constants. These are not fixed by chiral symmetry requirements alone but are, in principle, determined by the dynamics of the underlying theory.

Recently there have been attempts to derive the low energy effective chiral action of QCD \cite{3,4} as well as some of the coupling constants of the effective chiral Lagrangian of strangeness-changing four-quark Hamiltonian \cite{3}. Here we shall calculate the above mentioned couplings in the framework of this approach.

It is well known that gauge invariance together with chiral symmetry forbid the $O(p^2)$ contribution \cite{1} to $K$-decays with at most one pion in the final state. The first non-vanishing contributions to the transition amplitudes $K \to \pi \gamma^{*}$ and $K \to \pi \gamma \gamma$ appear at $O(p^4)$. The contributions from chiral loops have been calculated in Ref. \cite{1}. The $K \to \pi \pi \gamma$ decays have been analysed in Ref. \cite{2} to the same order in the chiral expansion. In Refs. \cite{1,4,5}, some of the couplings constants we are interested in have been calculated within the context of various models.

The effective non-leptonic chiral Lagrangian consistent with the octet structure of the dominant $\Delta S = 1$ weak Hamiltonian has been classified in Refs. \cite{3,6}. This Lagrangian, when restricted to the rare kaon decays processes we are interested in can be parametrized in terms of the set of coupling constants $\omega_1$, $\omega_2$, $\omega_4$, $\omega'_1$ and $\omega'_2$, introduced in Refs. \cite{1,2} as follows,

\[
L_{\text{eff}}^{(4)} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{f_0^2}{4} g_8 \left[ i \omega_1 (f^{\mu \nu}_{(+)} \{ \Delta_{32}, \xi_{\mu} \xi_{\nu} \}) + 2 i \omega_2 (f^{\mu \nu}_{(+)} \xi_{\mu} \Delta_{32} \xi_{\nu}) \\
+ \frac{4}{3} \omega_4^{++} (f^{\mu \nu}_{(-)} f^{(+)\mu \nu} \Delta_{32}) - 4 \omega_4^{++} (f^{\mu \nu}_{(-)} f^{(+\mu \nu)} \Delta_{32}) - 2 \omega_4^{++} (\{ f^{\mu \nu}_{(-)} , f^{(+\mu \nu)} \} \Delta_{32}) \\
+ i \omega'_1 (f^{\mu \nu}_{(-)} \{ \Delta_{32}, \xi_{\mu} \xi_{\nu} \}) + 2 i \omega'_2 (f^{\mu \nu}_{(-)} \xi_{\mu} \Delta_{32} \xi_{\nu}) \right] + \text{h.c.} \\
+ \text{non - octet operators,}
\]

with $\omega_4 = \omega_4^{--} + \omega_4^{++} + \omega_4^{+-}$, where $\langle \rangle$ denotes a trace in flavour. The notation here is defined below:
\[ f_{(\pm)}^{\mu\nu} = \xi F_{L}^{\mu\nu} \xi^\dagger \pm \xi^\dagger F_{R}^{\mu\nu} \xi, \]
\[ \Delta_{32} = \xi \lambda_{32} \xi^\dagger, \]
\[ \xi_{\mu} = i \xi^\dagger D_{\mu} U \xi^\dagger = -i \xi D_{\mu} U^\dagger \xi, \]

where \( \lambda_{ab} \) is the 3 \times 3 flavour matrix \( (\lambda_{ab})_{ij} = \delta_{ai} \delta_{bj} \) and \( \xi \) is chosen such that
\[ U = \xi^2. \]

\( U \equiv \exp \left( -\frac{\sqrt{2}i \Phi}{f} \right) \) is a SU(3) matrix incorporating the octet of pseudoscalar mesons,
\[ \Phi(x) \equiv \frac{\lambda}{\sqrt{2}} \bar{\varphi} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}, \]

and \( f \approx f_{\pi} = 93.2 \text{ MeV} \) is the pion decay constant at lowest order. \( D_\mu \) is a covariant derivative which acts on \( U \). In the presence of external electromagnetic fields only, it is defined as
\[ D_\mu U \equiv \partial_\mu U - i|e| A_\mu [Q, U], \]

where \( e \) is the electron charge and \( Q \) is a diagonal 3 \times 3 matrix that takes into account the electromagnetic (u,d,s)-light-quark charges, \( Q = \frac{1}{3} \text{diag}(2, -1, -1) \). In this case we also have
\[ F_{L}^{\mu\nu} = F_{R}^{\mu\nu} = |e| Q F^{\mu\nu} = |e| Q (\partial_\mu A_\nu - \partial_\nu A_\mu). \]

In Eq. (1) \( G_F \) is the Fermi constant and \( V_{ij} \) are the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. The \( g_8 \)-coupling introduced in Eq. (1) is the constant that modulates the octet operator in the \( \Delta S = 1 \) effective Lagrangian of order \( p^2 \); i.e.,
\[ \mathcal{L}_{\text{eff}}^{(2)} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* f_\pi^4 g_8 \langle \Delta_{32} \xi_\mu \xi^\mu \rangle + \text{h.c.} + \text{non-octet operators}. \]

We want to perform a calculation of the coupling constants which appear in \( \mathcal{L}_{\text{eff}}^{(4)} \) in Eq. (1) in the context of an expansion in powers of \( 1/N_c \), \( (N_c = \text{number of colours}) \). The paper is organized as follows. In Section 2 we shall introduce the \( \Delta S = 1 \) effective Hamiltonian. In Section 3 we shall study the large-\( N_c \) limit result for the counterterms defined in Eq. (1). In Section 4, the effective realization of the
\( \Delta S = 1 \) four-fermionic Hamiltonian is studied including terms of \( \mathcal{O}(N_c(\alpha_s N_c)) \) as in Ref. [3]. Phenomenological implications of our results are presented in Section 5 and finally, in Section 6, the conclusions are given.

2 The \( \Delta S = 1 \) effective Hamiltonian

The \( \Delta S = 1 \) chiral Lagrangians in Eq. (1) and (7) are part of the effective realization of the corresponding Standard Model (SM) sector in terms of low-energy degrees of freedom. In the SM with three flavours, once the heaviest particles (W-boson, t-, b- and c-quarks) have been integrated out, the complete basis of operators of weak and strong origin that induce strangeness changing processes with \( \Delta S = 1 \) via W-exchange is given by,

\begin{align}
Q_1 &= 4(\bar{s}_L\gamma^\mu d_L)(\bar{u}_L\gamma_\mu u_L) \\
Q_2 &= 4(\bar{s}_L\gamma^\mu u_L)(\bar{u}_L\gamma_\mu d_L) \\
Q_3 &= 4(\bar{s}_L\gamma^\mu d_L) \sum_{q=u,d,s} (\bar{q}_L\gamma_\mu q_L) \\
Q_4 &= 4 \sum_{q=u,d,s} (\bar{s}_L\gamma^\mu q_L)(\bar{q}_L\gamma_\mu d_L) \\
Q_5 &= 4(\bar{s}_L\gamma^\mu d_L) \sum_{q=u,d,s} (\bar{q}_R\gamma_\mu q_R) \\
Q_6 &= -8 \sum_{q=u,d,s} (\bar{s}_L q_R)(\bar{q}_R d_L).
\end{align}

where \( \Psi^R_L \equiv \frac{1}{2} (1 \pm \gamma_5) \Psi(x) \), and summation over colour indices is understood inside each bracket.

The reduction of the SM Lagrangian to an effective electroweak Hamiltonian has been discussed in the literature [11]. The structure of the effective non-leptonic low-energy Hamiltonian is the following,

\begin{align}
\mathcal{H}^{\Delta S = 1}_{\text{eff}} &= \frac{G_F}{\sqrt{2}} V_{ud} V^*_{us} \left\{ \frac{1}{2} C_+(\mu^2) (Q_2 + Q_1) + \frac{1}{2} C_- (\mu^2) (Q_2 - Q_1) + C'_3 (\mu^2) Q_3 + C_4 (\mu^2) (Q_3 + Q_2 - Q_1) + C_5 (\mu^2) Q_5 + C_6 (\mu^2) Q_6 \right\} + \text{h.c.}
\end{align}

The Wilson coefficients \( C_\pm \) and \( C_i, \ i = 3, 4, 5, 6 \) are known functions of the heavy masses and the renormalization scale \( \mu \) beyond the leading logarithmic approximation [12]. In the limit we are working here we shall use these coefficients in the leading logarithmic approximation. Of course, the matrix elements of the \( Q_{i=1,\ldots,6} \) operators must depend on the \( \mu \) scale in such a way that physical amplitudes do not depend on it. We shall be only interested in the octet components of the four-quark
operators that induce \( \Delta I = 1/2 \) transitions (due to the octet dominance of the \( \Delta I = 1/2 \) enhancement); namely, \( Q_- \equiv Q_2 - Q_1 \), the octet part of \( Q_+ \equiv Q_2 + Q_1 \) and \( Q_3, Q_4, Q_5, Q_6 \) which are pure octet operators.

To these \( \Delta S = 1 \) operators in (8) of weak and strong origin we have to add two more operators that induce \( \Delta S = 1 \) transitions, coming from the so-called electroweak penguins \[13\],

\[
Q_7^V = \frac{e^2}{2\pi} (\bar{s}_L \gamma^\mu d_L) \not\! \! \! \! \! \! \mu l \quad \text{and} \quad Q_7^A = \frac{e^2}{2\pi} (\bar{s}_L \gamma^\mu d_L) \not\! \! \! \! \! \! \mu \gamma_5 l, \quad (10)
\]

with their corresponding Wilson coefficients, \( C_7^V (\mu^2) \) and \( C_7^A (\mu^2) \). So the matrix elements of the \( Q_{i=1,...,6} \) operators have to be evaluated to order \( \alpha = e^2/4\pi \) whereas the matrix element of \( Q_7^{A,V} \) must be calculated to order \( \alpha^0 \). The effective Hamiltonian for \( Q_7^{A,V} \) can be written as

\[
\mathcal{H}^{Q_7}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left\{ C_7^V (\mu^2) Q_7^V + C_7^A (\mu^2) Q_7^A \right\} + \text{h.c.} \quad (11)
\]

The Wilson coefficient \( C_7^A \) only receives contributions from the so-called \( Z \) penguin and \( W \) box diagrams. In the present paper we are just interested in transitions that are mediated by external photons, therefore we are not going to consider the electroweak penguin operator modulated by \( C_7^A \) and we have \( C_7^V = C_7^A \). The expression for the electromagnetic penguin Wilson coefficient \( C_7^V (\mu^2) \) can be found in Ref. \[13\]. This Wilson coefficient contains a complex phase coming from the CKM matrix elements that can give rise to “direct” CP-violation effects. In the rest of the paper when we refer to the \( Q_7 \) operator we would mean \( Q_7^V \).

The other two \( \Delta S = 1 \) operators which arise from electromagnetic penguin-like diagrams, see Ref. \[14\], start to contribute at order \( \alpha^2 \).

3 The \( \Delta S = 1 \) effective Hamiltonian in the large-\( N_c \) limit

In the large-\( N_c \) limit, the \( \Delta S = 1 \) effective Hamiltonian is given by,

\[
\mathcal{H}^{\Delta S=1}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* Q_2 + \text{h.c.} \quad (12)
\]

In this limit we just need to calculate the effective action for the factorizable pattern of the four-quark operator \( Q_2 \). This can be performed in a model independent way by doing appropriate products of the low-energy realization of quark currents in the framework of effective chiral Lagrangians.
To obtain the large-\( N_c \) limit effective realization at \( \mathcal{O}(p^4) \) we need to know the effective realization of quark currents up to \( \mathcal{O}(p^3) \). These can be easily derived from the \( \mathcal{O}(p^2) \) and \( \mathcal{O}(p^4) \) strong effective chiral Lagrangian given in Refs. [8, 10] in the presence of external sources. In addition, we want to keep only the octet component of the isospin-1/2 operators. In this limit the value of the \( g_8 \)-coupling in Eq. (7) is \( \left( g_8 \right)_{1/N_c} = \frac{3}{5} \), to be compared with the experimental value from \( K \rightarrow \pi\pi \) decays, \( |g_8|_{\text{exp}} \approx 5.1 \). For the coupling constants \( \omega_{1,2,4} \) and \( \omega_1'_{1,2} \) of the \( \mathcal{O}(p^4) \) effective Lagrangian in (1) we then obtain the following results,

\[
\begin{align*}
\omega_1 &= 0, \\
g_8 \omega_2 &= 8 \left( g_8 \right)_{1/N_c} L_9, \\
g_8 \omega_4 &= 12 \left( g_8 \right)_{1/N_c} L_{10}, \\
g_8 \omega_1' &= 8 \left( g_8 \right)_{1/N_c} (L_9 - 2 L_{10}), \\
\text{and} \\
\omega_2' &= 0.
\end{align*}
\]

Here \( L_9 \) and \( L_{10} \) are two of the couplings that appear modulating local terms in the chiral Lagrangian at order \( p^4 \). In the notation of Gasser and Leutwyler [8], they read as follows,

\[
- \ i \ L_9 \langle F^\mu_\nu D_\mu U D_\nu U^\dagger + F^\mu_\nu D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle UF^{\mu\nu}U^\dagger F_{L,\mu\nu} \rangle.
\]

The present experimental results on the \( K^+ \rightarrow \pi^+e^+e^- \) process [15] allow for the determination of the combination of coupling constants,

\[
g_8(\omega_1 + 2 \omega_2) = 0.41^{+0.10}_{-0.05}. \tag{15}
\]

Numerically, in the large-\( N_c \) limit we find,

\[
g_8(\omega_1 + 2 \omega_2) = 24 \left( g_8 \right)_{1/N_c} L_9 = 0.10; \tag{16}
\]

i.e., a factor four lower than the experimental value.

If one looks at Eq. (13), the naive approach would be to consider that \( g_8 \) factorizes in the r.h.s. to all orders in the \( 1/N_c \)-expansion and therefore write down the following result,

\[
\begin{align*}
\omega_1 &= \omega_2 = 8 L_9, \\
\omega_4 &= 12 L_{10}, \\
\omega_1' &= 8 (L_9 - 2 L_{10}), \\
\text{and} \\
\omega_2' &= 0.
\end{align*}
\]
Then

\[ \omega_1 + 2 \omega_2 = 24 L_9 = 0.16, \]  

(18)

which is about twice the experimental value. In Section 4 we shall come back to the question of whether or not factorization of \( g_8 \) in Eq. (13) remains valid after including the next-to-leading corrections in the \( 1/N_c \)-expansion.

It can be seen from the results above that we do not obtain the octet dominance relation \( \omega_2 = 4 L_9 \) which was assumed in Ref. [1]. Our results differ from those found in Ref. [16], where it is claimed the numerical relation \( \omega_1 = \omega_2 = 4 L_9 \) in the large-\( N_c \) limit. In addition we get \( \omega_4 = 12 L_{10} \) which also differs from Ref. [16] where \( \omega_4 = 0 \). The results found in Ref. [7] for \( \omega_{1,2,4} \) from the “weak deformation model” are different to those we find in the large-\( N_c \) limit.

The amplitude for the process \( K^{++} \to \pi^+\gamma\gamma \) to lowest non-trivial order in ChPT was calculated in Ref. [1]. The result turns out to depend on the following combination of coupling constants which is renormalization scale independent,

\[
\hat{c} = 32 \pi^2 \left[ 4(L_9 + L_{10}) - \frac{1}{3} (\omega_1 + 2 \omega_2 + 2 \omega_4) \right] = -\frac{32 \pi^2}{3} \left[ (\omega_1 - \omega_2) + 3 (\omega_2 - 4L_9) + 2 (\omega_4 - 6L_{10}) \right].
\]  

(19)

The combination \( L_9 + L_{10} \) is a renormalization scale invariant quantity that is determined from the so-called structure term in \( \pi \to e\nu\gamma \) [8, 17] to be

\[ L_9 + L_{10} = (1.39 \pm 0.38) \times 10^{-3}. \]  

(20)

The combination \( \omega_1 + 2 \omega_2 + 2 \omega_4 \) is, of course, also renormalization scale invariant. In the large-\( N_c \) limit we find for the coupling constant \( \hat{c} \) the following value,

\[
\hat{c} = -128 \pi^2 \frac{(g_8)_{1/N_c}}{g_8} (L_9 + L_{10}) = -1.8 \frac{(g_8)_{1/N_c}}{g_8}.
\]  

(21)

Again, if we assume that factorization of \( g_8 \) in the r.h.s. of Eq. (13) is valid, we would obtain

\[
\hat{c} = -128 \pi^2 (L_9 + L_{10}) = -1.8.
\]  

(22)
The “weak deformation model” of Ref. [7] predicts \( \hat{c} = 0 \).

In the transition \( K^+ \to \pi^+ \pi^0 \gamma \) at \( \mathcal{O}(p^4) \) there appears the following combination of coupling constants [2]: \( \omega_1 + 2 \omega_2 - \omega'_1 + 2 \omega'_2 \). The combinations \( \omega_1 + 2 \omega_2 - \omega'_1 \) and \( \omega'_2 \) are scale independent separately. For these combinations, we get the following results in the large-\( N_c \) limit:

\[
g_8 (\omega_1 + 2 \omega_2 - \omega'_1) = 16 (g_8)_{1/N_c} (L_9 + L_{10}) = 0.01
\]

and

\[
\omega'_2 = 0.
\]

In the following Section we shall estimate the next-to-leading corrections in the \( 1/N_c \)-expansion to the couplings \( g_8 \omega_{1,2,4} \) and \( g_8 \omega'_{1,2} \) in the same spirit as the calculation done for \( g_8 \) in Ref. [3].

4 The effective action of four-quark operators at \( \mathcal{O}(N_c(\alpha_s N_c)) \)

In order to calculate the counterterms we are interested in, we shall need the effective realization of the \( Q_{i=1,\ldots,7} \) operators at \( \mathcal{O}(p^4) \). The procedure we shall follow up is the one described in Ref. [3] where the corresponding calculation to \( \mathcal{O}(p^2) \) has been reported. As there, we shall work in the chiral limit; i.e., we neglect the (u,d,s)-light-quark masses. Now, it is the complete set of strangeness changing \( \Delta S = 1 \) operators \( Q_{i=1,\ldots,7} \) that has to be taken into account. Since we want to do our analysis in the context of the \( 1/N_c \)-expansion, it is worth to recall the behaviour in the large-\( N_c \) limit of the various parameters we have introduced: \( f_\pi^2, L_9, L_{10}, \omega_{1,2,4} \) and \( \omega'_{1,2} \) are order \( N_c \); \( g_8, C_+, C_- \) and \( C_7 \) are order 1 and \( C_{3,4,5,6} \) are order \( 1/N_c \). We want to perform a calculation of the effective Hamiltonian in (9) and (11) up to order \( N_c(\alpha_s N_c) \). Only the \( Q_+, Q_- \) and \( Q_7 \) operators are then needed at this order, since all the other four-quark operators are modulated by Wilson coefficients that are already order \( 1/N_c \). The \( Q_{i=1,\ldots,5} \) operators and \( Q_{i=6,7} \) operators have different spinorial structure, thus we are going to study them separately. This will be done in Section 4.1 and Section 4.2, respectively. We shall summarize the corresponding results for the couplings \( g_8 \omega_{1,2,4} \) and \( g_8 \omega'_{1,2} \) in Section 4.3.

4.1 The effective action of the \( Q_{i=1,\ldots,5} \) operators

As it has been stated in the introduction of this Section, we need the effective action of \( Q_- \) and \( Q_+ \) to order \( N_c(\alpha_s N_c) \); and to order \( N_c^2 \) for the \( Q_{3,4,5} \) operators.
The results to $\mathcal{O}(p^4)$ for the terms which are relevant to the decays $K \to \pi\gamma^*$, $K^{+-} \to \pi^{+-}\gamma\gamma$ and $K \to \pi\pi\gamma$ are the following:

\[
\langle Q^- \rangle \Rightarrow \\
- f_\pi^2 \left\{ 2i L_9 \left( 1 - \frac{9\eta}{N_c} - \gamma_-(\mu) \right) \left[ \{ f_{(+)}^{\mu\nu} \{ \Delta_{23}, \xi_\mu \xi_\nu \} \} + 2 \{ f_{(+)}^{\mu\nu} \xi_\mu \Delta_{32} \xi_\nu \} \right] \\
+ 4 L_{10} \left( 1 - \frac{9\eta}{N_c} - \gamma_-(\mu) \right) \left[ \{ f_{(-)}^{\mu\nu} f_{(-)}^{(-\mu\nu) \Delta_{32}} \right] \\
+ 2i \left[ L_9 \left( 1 - \frac{9\eta}{N_c} - \gamma_-(\mu) \right) - 2 L_{10} \left( 1 - \frac{9\eta}{N_c} - \gamma_-(\mu) \right) \right] \left[ \{ f_{(-)}^{\mu\nu} \{ \Delta_{32}, \xi_\mu \xi_\nu \} \} \right] \\
+ 4 \pi (2 H_1 + L_{10}) \left[ \frac{\alpha}{N_c} (\langle Q_6 \rangle + \langle Q_4 \rangle) + \frac{8}{3} \left( 1 - \frac{9\eta}{N_c} - \gamma_-(\mu) \right) \langle Q_7 \rangle \right];
\]

and

\[
\langle Q^+ \rangle \Rightarrow \\
- f_\pi^2 \left\{ 2i L_9 \left( 1 + \frac{9\eta}{N_c} - \gamma_+(\mu) \right) \left[ \{ f_{(+)}^{\mu\nu} \{ \Delta_{23}, \xi_\mu \xi_\nu \} \} + 2 \{ f_{(+)}^{\mu\nu} \xi_\mu \Delta_{32} \xi_\nu \} \right] \\
+ 4 L_{10} \left( 1 + \frac{9\eta}{N_c} - \gamma_+(\mu) \right) \left[ \{ f_{(-)}^{\mu\nu} f_{(-)}^{(-\mu\nu) \Delta_{32}} \right] \\
+ 2i \left[ L_9 \left( 1 + \frac{9\eta}{N_c} - \gamma_+(\mu) \right) - 2 L_{10} \left( 1 + \frac{9\eta}{N_c} - \gamma_+(\mu) \right) \right] \left[ \{ f_{(-)}^{\mu\nu} \{ \Delta_{32}, \xi_\mu \xi_\nu \} \} \right] \\
+ 4 \pi (2 H_1 + L_{10}) \left[ \frac{\alpha}{N_c} (\langle Q_6 \rangle + \langle Q_4 \rangle) - \frac{8}{3} \left( 1 + \frac{9\eta}{N_c} - \gamma_+(\mu) \right) \langle Q_7 \rangle \right] \\
+ \text{non-octet terms.}
\]

The relevant terms for the effective action of the $Q_3$, $Q_4$, $Q_5$ penguin operators are:

\[
\langle Q_3 \rangle \Rightarrow \mathcal{O}(N_c);
\]

\[
\langle Q_4 \rangle \Rightarrow - f_\pi^2 \left\{ 2i L_9 \left[ \{ f_{(+)}^{\mu\nu} \{ \Delta_{32}, \xi_\mu \xi_\nu \} \} + 2 \{ f_{(+)}^{\mu\nu} \xi_\mu \Delta_{32} \xi_\nu \} \right] \\
+ 4 L_{10} \langle f_{(+)}^{\mu\nu} f_{(-)}^{(-\mu\nu) \Delta_{32}} \rangle + 2i \left( L_9 - 2 L_{10} \right) \langle f_{(-)}^{\mu\nu} \{ \Delta_{32}, \xi_\mu \xi_\nu \} \rangle \right\};
\]

\[
\langle Q_5 \rangle \Rightarrow \mathcal{O}(N_c).
\]

In the expressions above there appear three coupling constants of the chiral Lagrangian at order $p^4$ \[\Box\]. Two of them, $L_9$ and $L_{10}$, have been already introduced in
Eq. (14). The coupling constant $H_1$ is the constant that modulates a contact term between external sources in the chiral Lagrangian at order $p^4$ \[8\] as follows,

$$H_1 \langle F_{\mu\nu}^R F_{R,\mu\nu} + F_{\mu\nu}^L F_{L,\mu\nu} \rangle,$$

(29)

this constant is $O(N_c)$ in the large-$N_c$ limit. We have identified the coupling constants $L_9$, $L_{10}$ and $H_1$ that appear in the effective action of $Q_-$, $Q_+$ and $Q_4$ by comparing our results with their respective expressions found in the context of the mean-field approximation to the Nambu Jona-Lasinio model discussed in Ref. \[5\]. Thus, whenever their numerical values are needed, we shall use the values found in this model.

In Eqs. (24) and (25) we use the following short-hand notation for the leading non-perturbative gluonic corrections,

$$g_0 = 1 - \frac{1}{2} \mathcal{G};$$
$$g_1 = 1 - \frac{1}{6} \frac{f_\pi^2}{12M_Q^2L_9} \mathcal{G};$$
$$g_2 = 1 + \frac{1}{2} \frac{f_\pi^2}{24M_Q^2L_{10}} \mathcal{G};$$
$$g_3 = 1 + \frac{1}{3} \frac{f_\pi^2}{12M_Q^2(2H_1 + L_{10})} \mathcal{G},$$

(30)

with

$$\mathcal{G} \equiv \frac{N_c \langle \frac{\alpha_s}{\pi} G^2 \rangle}{16\pi^2 f_\pi^4},$$

(31)

which is $O(1)$ in the large-$N_c$ limit and the constituent quark mass $M_Q$, which arises from the following term,

$$-M_Q(\bar{q}_R U q_L + \bar{q}_L U^\dagger q_R).$$

(32)

This term is equivalent to the mean-field approximation of the Nambu Jona-Lasinio mechanism discussed in Ref. \[5\]. There are also perturbative gluonic corrections which we have gathered in the terms,

$$\gamma_-(\mu) = -\frac{3}{4} \frac{\alpha_s}{\pi} \ln \left( \frac{\mu^2}{M_Q^2} \right);$$
$$\gamma_+(\mu) = \frac{3}{4} \frac{\alpha_s}{\pi} \ln \left( \frac{\mu^2}{M_Q^2} \right).$$

(33)
From these expressions, it can be seen that the anomalous dimensions of the effective action for the $Q_-$ and $Q_+$ four-quark operators are the needed ones to compensate the scale dependence of the Wilson coefficients.

4.2 The penguin operators $Q_6$ and $Q_7$

Let us first calculate the effective action of $Q_6$ since, as discussed in the introduction of this Section, we only need to know its result to leading $\mathcal{O}(N_c^2)$. The calculation is rather straightforward and the result is

$$
\langle Q_6 \rangle \Rightarrow -\frac{\langle \bar{\Psi}\Psi \rangle}{M_Q} \frac{N_c}{48\pi^2} \left[ 3 i \langle f^{\mu\nu}_{(+)} \{ \Delta_{32} , \xi_\mu \xi_\nu \} \rangle + 2 i \langle f^{\mu\nu}_{(+)} \xi_\mu \Delta_{32} \xi_\nu \rangle 
+ 3 \langle f^{\mu\nu}_{(-)} f_{(-)\mu\nu} \Delta_{32} \rangle + 2 \langle f^{\mu\nu}_{(+)} f_{(+))\mu\nu} \Delta_{32} \rangle \right].
$$

(34)

Here $\langle \bar{\Psi}\Psi \rangle$ is a scale-dependent quantity which, at one-loop level, is defined by

$$
\langle \bar{\Psi}\Psi \rangle_{\mu^2} = \left( \frac{1}{2} \ln \left( \frac{\mu^2}{\Lambda_{MS}^2} \right) \right)^{4/9} \langle \bar{q}q \rangle.
$$

(35)

where $\langle \bar{q}q \rangle$ is the scale invariant quark vacuum condensate. In the large-$N_c$ limit the quark condensate is order $N_c$.

For the electromagnetic penguin operator $Q_7$ we obtain the following effective action,

$$
\langle Q_7 \rangle \Rightarrow -\frac{3}{32\pi} f_\pi^2 \left[ 2 i \langle f^{\mu\nu}_{(+)} \{ \Delta_{32} , \xi_\mu \xi_\nu \} \rangle + 2 i \langle f^{\mu\nu}_{(-)} \{ \Delta_{32} , \xi_\mu \xi_\nu \} \rangle 
- 2 \langle f^{\mu\nu}_{(-)} f_{(-)\mu\nu} \Delta_{32} \rangle - \langle \{ f^{\mu\nu}_{(-)} , f_{(+))\mu\nu} \} \Delta_{32} \rangle \right].
$$

(36)

which is valid to all orders in the $1/N_c$-expansion.

The terms in Eqs. (34) and (36) break the relation $\omega_1 = \omega_2$ which was found to leading $\mathcal{O}(N_c^2)$.

4.3 Results

The coupling constants $g_8 \omega_{1,2,4}$ and $g_8 \omega'_{1,2}$ in Eq. (11) can now be read off from the expression of the low-energy $\Delta S = 1$ Hamiltonian, by inserting the effective action of the four-quark operators $Q_-, Q_+, Q_3$, $Q_4$, $Q_5$, $Q_6$ and the electroweak penguin operator $Q_7$ that are in Eqs. (24)-(28) and (34)-(36), in Eqs. (9) and (11). To evaluate these coupling constants, we have fixed the matching renormalization
scale $\Lambda_\chi$ of the Wilson coefficients and the effective action of four-quark operators to be of the order of the first vector meson resonance mass, ($\Lambda_\chi = M_\rho = 770$ MeV).

We also need to know the coupling constant $g_8$ defined in Eq. (7). This coupling, within this framework was already computed in Ref. [3]. The result we get at the $\Lambda_\chi$ scale is the following,

$$g_8 = \frac{1}{2} C_-(\Lambda_\chi) \left( 1 - \frac{90}{N_c} - \gamma_-(\Lambda_\chi) \right) + \frac{1}{16} C_+(\Lambda_\chi) \left( 1 + \frac{90}{N_c} - \gamma_+(\Lambda_\chi) \right)$$

$$+ C_4(\Lambda_\chi) + \frac{2\pi \alpha_s(\Lambda_\chi)}{N_c} (2 H_1 + L_{10}) (C_+(\Lambda_\chi) + C_-(\Lambda_\chi))$$

$$- 16 L_5 \frac{\langle \overline{\Psi} \Psi \rangle^2}{f_\pi^2} \left[ C_6(\Lambda_\chi) + \frac{2\pi \alpha_s(\Lambda_\chi)}{N_c} (2 H_1 + L_{10}) \right.$$

$$\times (C_+(\Lambda_\chi) + C_-(\Lambda_\chi)) \left. + O(\alpha_s N_c) \right]. \quad (37)$$

We recall that $L_5$ is one of the $O(p^4)$ constants needed to renormalize the UV-behaviour of the lowest order chiral loops [8]. In the large-$N_c$ limit $L_5$ is order $N_c$.

It turns out that the measurable quantities in the transitions we are interested in only depend on the combinations: $g_8(\omega_1 - \omega_2)$, $g_8(\omega_2 - 4L_9)$, $g_8(\omega_4 - 6L_{10})$ and $g_8(\omega_1 + 2\omega_2 - \omega'_1 + 2\omega'_2)$, [1, 2]. In the large-$N_c$ limit the combination $(\omega_1 - \omega_2)/L_9$ is order $1/N_c$ whereas the combinations $(\omega_2 - 4L_9)/L_9$ and $(\omega_4 - 6L_{10})/L_{10}$ are order 1. From our calculation we find that the difference $\omega_1 - \omega_2$ depends only on the penguin operators $Q_6$ and $Q_7$.

At this point, it is worth coming back to the question of factorization of the coupling constant $g_8$ in the r.h.s of Eq. (13). The expression for the effective action of the Hamiltonian in Eqs. (9) and (11) calculated at order $p^4$ in the chiral expansion and at order $N(c_s N_c)$ in the $1/N_c$-expansion together with the expression of the $g_8$ coupling constant in Eq. (37) lead us to the conclusion that the approach of factorizing out $g_8$ in Eq. (13) is not valid when one considers next-to-leading corrections in the $1/N_c$-expansion. Therefore in the rest of the paper we shall give results for the combinations $g_8 \omega_{1,2,4}$ and $g_8 \omega'_{1,2}$.

5 Analysis of the results

Let us now analyse some phenomenological implications which follow from our calculation of the coupling constants $g_8 \omega_{1,2,4}$ and $g_8 \omega'_{1,2}$. In Ref. [4], the decay amplitudes of $K \rightarrow \pi \gamma^*$ were calculated at the one-loop level with the following results:
\begin{align}
A(K^+ \to \pi^+ \gamma^*) &= \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{g_8}{16\pi^2} q^2 \tilde{\Phi}_+(q^2) \epsilon^\mu(p+p')_\mu; \quad (38) \\
A(K_S^0 \to \pi^0 \gamma^*) &= \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{g_8}{16\pi^2} q^2 \tilde{\Phi}_S(q^2) \epsilon^\mu(p+p')_\mu; \quad (39)
\end{align}

with

\begin{align}
\tilde{\Phi}_+(q^2) &= \frac{16\pi^2}{3} \left[ (\omega_1^r - \omega_2^r) + 3(\omega_2^r - 4L_0) \right] (\nu^2) \\
&- \left[ \Phi_K(q^2) + \Phi_\pi(q^2) - \frac{1}{3} \ln \left( \frac{m_\pi m_K}{\nu^2} \right) \right]; \\
\tilde{\Phi}_S(q^2) &= -\frac{16\pi^2}{3} (\omega_1^r - \omega_2^r) (\nu^2) + 2 \Phi_K(q^2) - \frac{1}{3} \ln \left( \frac{m_\pi^2}{\nu^2} \right). \quad (40)
\end{align}

Here,

\begin{align}
\Phi_K(\pi)(q^2) &= - \frac{4m_{K(\pi)}^2}{3q^2} + \frac{5}{18} + \frac{1}{3} \left( \frac{4m_{K(\pi)}^2}{q^2} - 1 \right)^{3/2} \arctan \left( \frac{1}{\sqrt{\frac{4m_{K(\pi)}^2}{q^2} - 1}} \right) \quad \text{for } q^2 \leq 4m_{K(\pi)}^2. \quad (42)
\end{align}

The coupling constants \( \omega_1, \omega_2, \omega_4, \omega_1', \) and \( \omega_2' \) are scale dependent quantities. In our approach, the explicit scale dependence of \( \omega_{1,2,4} \) and \( \omega_{1,2}' \) comes from next-to-leading terms which we have not calculated. We identify the values we get for \( \omega_{1,2,4} \) and \( \omega_{1,2}' \) with those of \( \omega_{1,2,4}' \) and \( \omega_{1,2}' \) renormalized at the \( \rho \)-resonance mass. We shall also identify the constant \( L_9 \) in Eq. (40) with the coupling \( L_9 \) renormalized at this same scale. Following Ref. [1] we define the constants,

\begin{align}
\omega_+ &\equiv -\frac{16\pi^2}{3} \left[ (\omega_1^r - \omega_2^r) + 3(\omega_2^r - 4L_0) \right] (M_\rho^2) - \frac{1}{3} \ln \left( \frac{m_{K(\pi)} m_\pi}{M_\rho^2} \right) \quad (43) \\
\omega_+ &\equiv -\frac{16\pi^2}{3} (\omega_1^r - \omega_2^r) (M_\rho^2) - \frac{1}{3} \ln \left( \frac{m_\pi^2}{M_\rho^2} \right). \quad (44)
\end{align}

Thus

\begin{align}
\tilde{\Phi}_+(q^2) &= - \left[ \Phi_K(q^2) + \Phi_\pi(q^2) + \omega_+ \right]; \quad (45) \\
\tilde{\Phi}_S(q^2) &= 2\Phi_K(q^2) + \omega_S; \quad (46)
\end{align}

and the decay rates for \( K \to \pi l^+ l^- \) can be written in the following way.
\[
\Gamma(K \rightarrow \pi l^+l^-) = \Gamma \int_{4\epsilon}^{(1-\sqrt{3})^2} dz \lambda^{3/2}(1, z, \delta) \left(1 - 4\frac{\xi}{z}\right)^{1/2} \left(1 + 2\frac{\xi}{z}\right) |\tilde{\Phi}|^2,
\]
(47)

where
\[
z = \frac{q^2}{m_K^2}, \quad \epsilon = \frac{m^2}{m_K^2}, \quad \delta = \frac{m^2}{m_K^2},
\]
(48)
\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx,
\]
and \(\Gamma\) is an overall normalization factor,
\[
\Gamma = \left|\frac{G_F V_{ud} V_{us}^*}{\sqrt{2}}\right|^2 \frac{\alpha^2 m_K^5 |g_8|^2}{12\pi (4\pi)^4}.
\]
(49)

In our numerical estimates we shall use the following set of input values:
\[
\langle \bar{q}q \rangle = -((190 - 210) \text{MeV})^3; \quad \langle \frac{G_F}{\pi} G^2 \rangle = ((330 - 390) \text{MeV})^4;
\]
\[
\Lambda_{\chi} = (700 - 900) \text{MeV}; \quad \Lambda_{\overline{\text{MS}}} = (100 - 200) \text{MeV}
\]
and
\[
M_Q = (250 - 350) \text{MeV}.
\]

Then we have the following result for \(g_8 \text{Re} \omega_+, g_8 \text{Re} \omega_S, g_8 (\omega_1 + 2 \omega_2)\) and \(g_8 (\omega_1 - \omega_2)\),
\[
g_8 \text{Re} \omega_+ = 7.5^{+5.5}_{-3.3};
\]
\[
g_8 \text{Re} \omega_S = 5^{+4}_{-2};
\]
\[
g_8 (\omega_1 + 2 \omega_2) = 0.12^{+0.02}_{-0.01};
\]
\[
g_8 (\omega_1 - \omega_2) = -0.08^{+0.04}_{-0.08},
\]
where the central value corresponds to the input values \(\langle \bar{q}q \rangle = -(200 \text{MeV})^3; \langle \frac{G_F}{\pi} G^2 \rangle = (360 \text{MeV})^4, \Lambda_{\overline{\text{MS}}} = 150 \text{MeV}\) and \(M_Q = 300 \text{MeV}\) with \(\Lambda_{\chi} = 800 \text{MeV}\). Experimentally [15] we know that,
\[
g_8 \text{Re} \omega_+ = 4.6^{+1.2}_{-0.7};
\]
\[
g_8 (\omega_1 + 2 \omega_2) = 0.41^{+0.10}_{-0.05}.
\]
(51)
(52)

In view of the these experimental results a value of the quark condensate lower than \(-(210 \text{MeV})^3\) turn out to be not favoured.

Our results tell us that the combination of counterterms for \(K^+ \rightarrow \pi^+ l^+l^-\), \(K^0 \rightarrow \pi^0 l^+l^-\) and \(\eta \rightarrow \bar{K}^0 l^+l^-\) decay amplitudes, \(\omega_S\) and \(\omega_+\) [1], depend strongly
on the penguin diagrams (both hadronic and electroweak). In addition, it turns 
out that the coupling $g_8 \omega_S$ only depends on the penguin operators whereas $g_8 \omega_+$ 
depends also on the non-penguin Wilson coefficients. This fact could be used for 
measuring their respective strength.

With the value of $\omega_S$ in Eq. (51) we can predict the following branching ratio,

$$\Gamma(K_S^0 \to \pi^0 e^+ e^-) \simeq 2.6 \times 10^{-9} [\text{Re} \omega_S^2 - 0.66 \text{Re} \omega_S + 0.11]$$

$$\simeq (0.5 - 5) \times 10^{-9},$$

for which there is an experimental upper bound of the order of $10^{-5}$ [17]. We can also 
give a prediction for the ratio of decay rates of the $K^+ \to \pi^+ e^+ e^-$ and $K_S^0 \to \pi^0 e^+ e^-$ 
transitions;

$$\frac{\Gamma(K_S^0 \to \pi^0 e^+ e^-)}{\Gamma(K^+ \to \pi^+ e^+ e^-)} \simeq \frac{\text{Re} \omega_S^2 - 0.66 \text{Re} \omega_S + 0.11}{\text{Re} \omega_+^2 - 0.59 \text{Re} \omega_+ + 0.09} = 0.30^{+0.50}_{-0.25}. \quad (54)$$

The coupling constant $\hat{c}$ introduced in Eq. (19) can also be determined from 
the results above. It turns out that there is no contribution from the electroweak 
penguin $Q_7$ to the $\hat{c}$ coupling constant. For its real part we find

$$\text{Re} \hat{c} = -0.7 \pm 0.5. \quad (55)$$

which translates in the following prediction for the branching ratio of the transition 
$K^+ \to \pi^+ \gamma \gamma$ [1]:

$$\text{BR}(K^+ \to \pi^+ \gamma \gamma) = (5.2 \pm 0.7) \times 10^{-7}. \quad (56)$$

The experimental upper limit depends very much on the $\pi^+$ energy spectrum, giving 
a wide range of allowed values [18],

$$\text{BR}(K^+ \to \pi^+ \gamma \gamma) \leq 1.5 \times 10^{-4}. \quad (57)$$

The imaginary part of $\hat{c}$ vanishes since there is no contribution of the electromagnetic penguin $C_\gamma^7$ to this coupling. This implies that there is no charge asymmetry 
$\Gamma(K^+ \to \pi^+ \gamma \gamma) - \Gamma(K^- \to \pi^- \gamma \gamma)$ from the CP-violating phase of the CKM mixing 
matrix [1] appearing in the electromagnetic penguin Wilson coefficient.

Finally, for the combinations $\omega_1 + 2 \omega_2 - \omega'_1$ and $\omega'_2$ introduced in Eq. (1) we 
get the results,
\[ g_8 \left( \omega_1 + 2 \omega_2 - \omega'_1 \right) = 0.02 \pm 0.01 , \]
\[ \omega'_2 = 0 . \] 

These two combinations turn out to be independent of the electroweak penguin operator \( Q_7 \) and therefore real. They can be used in the theoretical prediction of the electric-type amplitude of the \( K^+ \to \pi^+\pi^0\gamma \) transitions, see Ref. \[2\].

6 Conclusions

In the framework of the effective action approach for four-quark operators \[3\], we have calculated the various coupling constants that enter in the chiral perturbation theory prediction for the \( K \to \pi\gamma^* \to \pi l^+ l^- \), \( K \to \pi \gamma \gamma \) and \( K \to \pi \pi \gamma \) decay rates, \[1, 2\]. These constants are not determined by symmetry requirements alone. They turn out to depend strongly on the effective action of the penguin operators \( Q_6 \) and \( Q_7 \).

We have given a prediction for the phenomenological constants \( \omega_+, \omega_S \) and \( \hat{c} \) defined in Ref. \[1\], which fix the decay rates for \( K \to \pi \gamma^* \) and \( K^+ \to \pi^- \gamma \gamma \). In particular we have predicted the branching ratio for \( K^0_S \to \pi^0 e^+ e^- \) in \(53\) and the ratio of decay rates of \( K^0_S \to \pi^0 e^+ e^- \) and \( K^+ \to \pi^+ e^+ e^- \) in \(54\). We have given the branching ratio for \( K^+ \to \pi^+ \gamma \gamma \) in \(53\) and found that there is no charge asymmetry \( \Gamma(K^+ \to \pi^+ \gamma \gamma) - \Gamma(K^- \to \pi^- \gamma \gamma) \) coming from the electromagnetic penguin Wilson coefficient \( C_7^\gamma \). In the transition \( K \to \pi \pi \gamma \), further counterterms are possible. They have been classified in Ref. \[2\]. The coupling constants that modulate these new counterterms \( \omega'_1 \) and \( \omega'_2 \) have also been predicted in \(58\).

Acknowledgements

We wish to thank Eduardo de Rafael for suggesting this calculation to us and for many useful discussions. We have also benefited from discussions with Gerhard Ecker, Toni Pich and Josep Taron. We would like to thank Lars Hörnfeldt for his help with the algebraic manipulating program STENSOR. The work of one of us (J.P.) has been supported in part by CICYT, Spain, under Grant No. AEN90-0040. J.P. is also indebted to the Spanish Ministerio de Educación y Ciencia for a fellowship.
References

[1] G. Ecker, A. Pich and E. de Rafael, Nucl. Phys. B291 (1987) 692; Phys. Lett. B189 (1987) 363; Nucl. Phys. B303 (1988) 665.

[2] G. Ecker, H. Neufeld and A. Pich, Phys. Lett. B278 (1992) 337;
   J. Bijnens, G. Ecker and A. Pich, “The Chiral Anomaly in Non-Leptonic Weak Interactions” CERN preprint CERN-TH.6444/92 (1992).

[3] A. Pich and E. de Rafael, Nucl. Phys. B358 (1991) 311.

[4] D. Espriu, E. de Rafael and J. Taron, Nucl. Phys. B345 (1990) 22; erratum ibid. B355 (1991) 278.

[5] J. Bijnens, C. Bruno and E. de Rafael, “Nambu Jona-Lasinio Like Models and the Low Energy Effective Action of QCD” CERN and Marseille preprint CERN-TH.6521/92, CPT-92/P.2710 (1992).

[6] G. Esposito-Farèse, Z. Phys. C50 (1991) 255.

[7] G. Ecker, A. Pich and E. de Rafael, Phys. Lett. B237 (1990) 481.

[8] J. Gasser and H. Leutwyler, Ann. of Phys. 158 (1984) 142; Nucl. Phys. B250 (1985) 465, 517, 539.

[9] J. Kambor, J. Missimer and D. Wyler, Nucl. Phys. B346 (1990) 17;
   G. Ecker, “Geometrical Aspects of the Non-Leptonic Weak Interactions of Mesons”, in Proc. IX Int. Conf. on the Problems of Quantum Field Theory, ed. M.K. Volkov, Dubna (1990).

[10] S. Weinberg, Physica A96 (1979) 327.

[11] F.J. Gilman and M.B. Wise, Phys. Rev. D20 (1979) 1216.

[12] A.J. Buras, M. Jamin, M.E. Lautenbacher and P.H. Weisz, Nucl. Phys. B370 (1992) 69.

[13] C.O. Dib, I. Dunietz and F.J. Gilman, Phys. Rev. D39 (1989) 2639;
    F.J. Gilman and M.B. Wise, Phys. Rev. D21 (1980) 3150.

[14] J. Bijnens and M.B. Wise, Phys. Lett. 137B (1984) 245.

[15] C. Alliegro et al., Phys. Rev. Lett. 68 (1992) 278.

[16] H.-Y. Cheng, Phys. Rev. D42 (1990) 72.
[17] Particle Data Group, J.J. Hernández et al., Phys. Lett. B239 (1990) 1.

[18] M.S. Atiya et al., Phys. Rev. Lett. 65 (1990) 1188.